INTERPOLATING AND SAMPLING SEQUENCES FOR ENTIRE FUNCTIONS

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ABSTRACT. We characterise interpolating and sampling sequences for the spaces of entire functions f such that $fe^{-\phi} \in L^p(\mathbb{C})$, $p \geq 1$ (and some related weighted classes), where ϕ is a subharmonic weight whose Laplacian is a doubling measure. The results are expressed in terms of some densities adapted to the metric induced by $\Delta\phi$. They generalise previous results by Seip for the case $\phi(z) = |z|^2$, and by Berndtsson & Ortega-Cerdà and Ortega-Cerdà & Seip for the case when $\Delta\phi$ is bounded above and below.

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1. Introduction

In this paper we provide Beurling-type density conditions for sampling and interpolation in certain generalised Fock spaces. We consider a rather general situation, with only mild regularity conditions on the possible growth. Let ϕ be a (nonharmonic) subharmonic function whose Laplacian $\Delta\phi$ is a doubling measure (see definition and properties in Section 2.1), and let ω denote a flat weight, that is, a positive measurable function with slow growth (see details in Section 2.2). The spaces we deal with are parametrised by an index $p \in [1, \infty]$, as follows:

$$\begin{aligned} \mathcal{F}^p_{\phi,\omega} &= \Big\{ f \in H(\mathbb{C}) : \|f\|^p_{\mathcal{F}^p_{\phi,\omega}} = \int_{\mathbb{C}} |f|^p e^{-p\phi} \omega^p \rho^{-2} < \infty \Big\} \qquad 1 \leq p < \infty, \\ \mathcal{F}^\infty_{\phi,\omega} &= \Big\{ f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^\infty_{\phi,\omega}} = \sup_{z \in \mathbb{C}} \omega(z) |f(z)| e^{-\phi(z)} < \infty \Big\}. \end{aligned}$$

The function ρ^{-2} is a regularised version of $\Delta\phi$, as described in [Chr91]. More precisely, if $\mu = \Delta\phi$ and $z \in \mathbb{C}$, then $\rho_\phi(z)$ (or simply $\rho(z)$ if no confusion can arise) denotes the positive radius such that $\mu(D(z, \rho(z))) = 1$. Such a radius exists because doubling measures have no mass on circles.

Canonical examples of the weights considered are $\phi(z) = |z|^\beta$, with $\beta > 0$, and $\omega = \rho^\alpha$, $\alpha \in \mathbb{R}$.

Two particular families of spaces seem of special interest. The first one are the usual weighted L^p-spaces of entire functions, obtained with $\omega = \rho^{2/p}$. The second case arises when $\omega = 1$; then the spaces $\mathcal{F}^p_{\phi,\omega}$ coincide with

$$\{ f \in H(\mathbb{C}) : \int_{\mathbb{C}} |f|^p e^{-p\phi} \Delta\phi < \infty \}.$$

Since functions f in the spaces $\mathcal{F}^p_{\phi,\omega}$ are determined by the growth of $|f|$, their restriction to a sequence should be described as well in terms of growth.

Let $\Lambda \subset \mathbb{C}$ be a sequence and let $v = \{v_\lambda\}_{\lambda \in \Lambda}$ be an associated sequence of values.

Definition 1. A sequence Λ is an *interpolating sequence for* $\mathcal{F}^p_{\phi,\omega}$, $1 \le p < \infty$ (denoted $\Lambda \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$), if for every sequence of values v such that

$$\|v\|^p_{\ell^p_{\phi,\omega}(\Lambda)} = \sum_{\lambda\in\Lambda} \omega^p(\lambda)|v_\lambda|^p e^{-p\phi(\lambda)} < \infty$$

there exists $f \in \mathcal{F}^p_{\phi,\omega}$ such that $f|\Lambda = v$.

Also, $\Lambda \in \operatorname{Int} \mathcal{F}^\infty_{\phi,\omega}$ if for every sequence of values v such that

$$\|v\|_{\ell^\infty_{\phi,\omega}(\Lambda)} = \sup_{\lambda\in\Lambda} \omega(\lambda)|v_\lambda| e^{-\phi(\lambda)} < \infty$$

there exists $f \in \mathcal{F}^\infty_{\phi,\omega}$ such that $f|\Lambda = v$.

An application of the open mapping theorem shows that when $\Lambda \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$ there is $M > 0$ such that for any $v \in \ell^p_{\phi,\omega}(\Lambda)$, there exists $f \in \mathcal{F}^p_{\phi,\omega}$ with $f|\Lambda = v$ and

$$||f||_{\mathcal{F}^p_{\phi,\omega}} \le M||v||_{\ell^p_{\phi,\omega}(\Lambda)}. \tag{1}$$

The least possible M in (1) is called the *interpolating constant* of Λ and is denoted by $M^p_{\phi,\omega}(\Lambda)$, or $M(\Lambda)$ if no confusion is possible.

Definition 2. A sequence Λ is a *sampling sequence for* $\mathcal{F}^p_{\phi,\omega}$, $1 \le p < \infty$ (denoted $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$), if there exists $C > 0$ such that for every $f \in \mathcal{F}^p_{\phi,\omega}$

$$C^{-1}\|f|\Lambda\|_{\ell^p_{\phi,\omega}(\Lambda)} \le \|f\|_{\mathcal{F}^p_{\phi,\omega}} \le C\|f|\Lambda\|_{\ell^p_{\phi,\omega}(\Lambda)}. \tag{2}$$

Also, $\Lambda \in \operatorname{Samp} \mathcal{F}^\infty_{\phi,\omega}$ if there exists $C > 0$ such that for every $f \in \mathcal{F}^\infty_{\phi,\omega}$

$$\|f\|_{\mathcal{F}^\infty_{\phi,\omega}} \le C\|f|\Lambda\|_{\ell^\infty_{\phi,\omega}(\Lambda)}. \tag{3}$$

The least constant C verifying these inequalities is called the *sampling constant* of Λ and is denoted $L^p_{\phi,\omega}(\Lambda)$, or simply $L(\Lambda)$.

The definitions of interpolating and sampling sequences in the spaces defined by L^∞ norms reflect the maximal growth for functions in the space, and are natural. The definition for $p < \infty$ can be motivated in the following way. Consider for instance the case $p = 2$. The estimates of the normalised Bergman kernel $k_{\phi,\omega}(\lambda, z)$ in $\mathcal{F}^2_{\phi,\omega}$ (see Lemma 20) show that $\langle k_{\phi,\omega}(\lambda,\cdot), f\rangle \simeq f(\lambda)\omega(\lambda)e^{-\phi(\lambda)}$ for all $f \in \mathcal{F}^2_{\phi,\omega}$. Thus $\Lambda \in \operatorname{Samp} \mathcal{F}^2_{\phi,\omega}$ if and only if

$$\|f\|_{\mathcal{F}^2_{\phi,\omega}} \simeq \sum_{\lambda\in\Lambda} |\langle k_{\phi,\omega}(\lambda,\cdot), f\rangle|^2 \quad \text{for all } f \in \mathcal{F}^2_{\phi,\omega},$$

that is, if and only if $\{k_{\phi,\omega}(\lambda,\cdot)\}_{\lambda\in\Lambda}$ is a frame in $\mathcal{F}^2_{\phi,\omega}$. Similarly, $\Lambda \in \operatorname{Int} \mathcal{F}^2_{\phi,\omega}$ if and only if $\{k_{\phi,\omega}(\lambda,\cdot)\}_{\lambda\in\Lambda}$ is a Riesz basis in its closed linear span in $\mathcal{F}^2_{\phi,\omega}$. These are the standard problems of interpolation and sampling in Hilbert spaces of functions with reproducing kernels [SS61]. For $p \ne 2$ the previous definitions give the appropriate notions of interpolation and sampling as well, in view of the pointwise growth of functions in the spaces (see Lemma 18 and Remark 5).

Our description of interpolating and sampling sequences is expressed in terms of certain Beurling-type densities adapted to the metric induced by $\Delta\phi$, or more precisely, by its regularisation $\rho^{-2}(z)dz \otimes d\bar{z}$. Before introducing the densities we need the notion of ρ-separation.

Definition 3. A sequence Λ is *ρ-separated* if there exists $\delta > 0$ such that

$$|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda)) \qquad \lambda \neq \lambda',$$

This is equivalent to saying that the points in Λ are separated by a fixed distance in the metric above (Lemma 4).

Definition 4. Assume that Λ is a ρ-separated sequence and denote $\mu = \Delta\phi$.

The *upper uniform density of* Λ with respect to $\Delta\phi$ is

$$\mathcal{D}^+_{\Delta\phi}(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#\Big(\Lambda \cap \overline{D(z, r\rho(z))}\Big)}{\mu(D(z, r\rho(z)))}.$$

The *lower uniform density of* Λ with respect to $\Delta\phi$ is

$$\mathcal{D}^-_{\Delta\phi}(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#\Big(\Lambda \cap \overline{D(z, r\rho(z))}\Big)}{\mu(D(z, r\rho(z)))}.$$

The main theorems are the following. Let Ω_ϕ denote the class of flat weights.

Theorem A. *A sequence Λ is sampling for $\mathcal{F}^p_{\phi,\omega}$, $p \in [1, \infty)$, $\omega \in \Omega_\phi$, if and only if Λ is a finite union of ρ-separated sequences containing a ρ-separated subsequence Λ' such that $\mathcal{D}^-_{\Delta\phi}(\Lambda') > 1/2\pi$. A sequence Λ is sampling for $\mathcal{F}^\infty_{\phi,\omega}$ if and only if Λ contains a ρ-separated subsequence Λ' such that $\mathcal{D}^-_{\Delta\phi}(\Lambda') > 1/2\pi$.*

Theorem B. *A sequence Λ is interpolating for $\mathcal{F}^p_{\phi,\omega}$, $p \in [1, \infty]$, $\omega \in \Omega_\phi$, if and only if Λ is ρ-separated and $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$.*

In particular, there are no sequences which are simultaneously sampling and interpolating (it should be mentioned that this is not obtained as a corollary of the theorems; it is actually an important ingredient of the proofs).

These results generalise previous work, beginning with the papers by Seip and Seip-Wallstén [Sei92], [SW92]. They described the interpolating and sampling sequences for the classical Fock space in terms of the so-called Nyquist densities. In the notation above this corresponds to $\phi(z) = |z|^2$ and $\omega \equiv 1$. This was extended in [LS94], [BOC95] and [OCS98] to the case of entire functions f such that $fe^{-\phi} \in L^p(\mathbb{C})$, where ϕ is subharmonic with bounded Laplacian $\varepsilon < \Delta\phi < M$. The description was given again in terms of some Nyquist type densities. In these cases the function ρ is bounded above and below, hence the metric $\rho^{-2}(z)dz \otimes d\bar{z}$ is equivalent to the Euclidean metric. In particular, $\rho(z)$ can be replaced by the constant 1 in the definition of the uniform densities.

There are also some partial results in several complex variables. The classical Fock space has been studied in [MT00] and the weighted scenario in [Lin01]. In this context there exist necessary or sufficient density conditions, which do not completely characterise the sampling or interpolating sequences.

Interpolation problems for other spaces of functions related to these weights have been considered by Squires and Berenstein and Li (see for instance [Squ83], [BL95] and the references therein).

The results mentioned above relied on the remarkable work by Beurling [Beu89] and on Hörmander's weighted L^2-estimates for the $\bar{\partial}$ equation [Hör94]. In our proofs we first extend Beurling's tools to the context of certain spaces which are non-invariant under translations. We need as well a Hörmander type theorem giving precise estimates for the $\bar{\partial}$ equation in Banach norms other than L^2.

The plan of the paper is the following: In Section 2 we study the properties of doubling measures and introduce the flat weights. Recall that the only assumption on our subharmonic weight ϕ is that the measure $\Delta\phi$ is doubling. We will need a regularisation of ϕ and the construction of a multiplier associated to ϕ (that is, an entire function f such that $|f|$ approximates e^{ϕ}), very much in the spirit of [LM01] and [OC99].

In Section 3 we state and prove some basic properties of functions in $\mathcal{F}^p_{\phi,\omega}$. The main result in this section is the following Hörmander type theorem.

Theorem C. *Let ϕ be a subharmonic function such that $\Delta\phi$ is a doubling measure. For any $\omega \in \Omega_\phi$, there is a solution u to the equation $\bar{\partial} u = f$ such that $\|ue^{-\phi}\omega\|_{L^p(\mathbb{C})} \lesssim \|fe^{-\phi}\omega\rho\|_{L^p(\mathbb{C})}$ for any $1 \leq p \leq \infty$.*

We also include the estimates of the Bergman kernel that justify the notion of interpolating and sampling sequences we have considered. Finally , we study the invariance of our spaces under some appropriate scaled translations. This leads to the notion of weak limit and the corresponding analysis analogous to Beurling's.

Section 4 is devoted to some preliminary (but important) properties of interpolating and sampling sequences, including their behaviour under weak limits. The main results in this section are some inclusion relations between various spaces of interpolating and sampling sequences, and the fact that there are no sequences which are simultaneously interpolating and sampling for the same space of functions $\mathcal{F}^p_{\phi,\omega}$.

In Section 5 we prove the sufficiency part of Theorem A. We use again an approach similar to that of Beurling.

Section 6 includes the proof of the necessity part of Theorem A. For this we need once more Beurling's analysis, plus the non-existence of sampling and interpolating sequences. We use some theorems that relate the densities of sampling and interpolating sequences, following the ideas by Ramanathan and Steger [RS95].

Section 8 is devoted to the proof of the necessity part of Theorem B. We use Ramanathan and Steger's theorem plus an original argument that shows that the density inequality is actually strict.

Finally, in Section 7 we deal with the sufficiency part of Theorem B. In the course of the proof, whose main tool is the multiplier, we need to express the density in terms of rectangles instead of disks. The usual argument of Landau [Lan67] does not work, in view of the inhomogeneity of our measures. Theorem 42 takes care of this.

A final word on notation: C denotes a finite constant that may change in value from one occurrence to the next.The expression $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Subharmonic functions with doubling Laplacian

In this chapter we recap some results on doubling measures and subharmonic functions ϕ whose Laplacian $\Delta\phi$ is doubling. We start with regularity and integrability conditions on doubling measures. Next we show that ϕ can be regularised, in the sense that there exists ψ subharmonic and regular for which the interpolation and sampling problems for $\mathcal{F}^p_{\phi,\omega}$ and $\mathcal{F}^p_{\psi,\omega}$ are equivalent. The final part is dedicated to the construction of the multiplier. A useful application of this is the existence of holomorphic "peak functions" with controlled growth.

Definition 5. A nonnegative Borel measure μ in $\mathbb{C}$ is called *doubling* if there exists $C > 0$ such that

$$\mu(D(z, 2r)) \leq C\mu(D(z, r))$$

for all $z \in \mathbb{C}$ and $r > 0$. We denote by C_μ the minimum constant C for which the inequality holds.

Recall that when ϕ is subharmonic $\Delta\phi$ is a nonnegative Borel measure, finite on compact sets.

For convenience we write $D^r(z) = D(z, r\rho(z))$ and $D(z) = D^1(z)$. We will write $D^r_\phi(z)$ when we need to stress that the radius depends on ϕ.

Henceforth dm denotes the Lebesgue measure in $\mathbb{C}$. We also use the measure $d\sigma = dm/\rho^2$, which should be thought of as a doubling regularisation of $\Delta\phi$ (see Theorem 14).

2.1. Doubling measures.

Throughout this section we assume that μ is a positive doubling measure non-identically zero. We begin with a result of Christ [Chr91, Lemma 2.1].

Lemma 1. *Let μ be a doubling measure in $\mathbb{C}$. There exists $\gamma > 0$ such that for any disks D, D' of respective radius $r(D) > r(D')$ with $D \cap D' \neq \emptyset$:*

$$\left(\frac{\mu(D)}{\mu(D')}\right)^\gamma \lesssim \frac{r(D)}{r(D')} \lesssim \left(\frac{\mu(D)}{\mu(D')}\right)^{1/\gamma}.$$

In particular, the support of μ has positive Hausdorff dimension.

Remark 1. This implies that there exist $k, \varepsilon > 0$ such that

$$r^\varepsilon \lesssim \mu(D^r(z)) \lesssim r^k \qquad z \in \mathbb{C}\,,\ r > 1. \tag{4}$$

Also, applying Lemma 1 and (4) to $D(0,|z|)$ and $D(z)$ we have, for $\rho(z)\le|z|$

$$\frac{1}{|z|^{k/\gamma}}\lesssim\Big(\frac{1}{\mu(D(0,|z|))}\Big)^{1/\gamma}\lesssim\frac{\rho(z)}{|z|}\lesssim\Big(\frac{1}{\mu(D(0,|z|))}\Big)^{\gamma}\lesssim\frac{1}{|z|^{\varepsilon\gamma}}.$$

On the other hand, if $|z|<\rho(z)$, then $0\in D(z)$. Thus Lemma 1 implies $\rho(z)\simeq\rho(0)$, hence $|z|<C$. Therefore, there exist $\eta, C_0>0$ and $\beta\in(0,1)$ such that

$$C_0^{-1}|z|^{-\eta}\le\rho(z)\le C_0|z|^{\beta}\qquad |z|>1. \tag{5}$$

Let us study in more detail the relationship between $\rho(z)$ and $\rho(\zeta)$ for various $z,\zeta\in\mathbb{C}$. A first observation is that $\rho(z)$ is a Lipschitz function. More precisely

$$|\rho(z)-\rho(\zeta)|\le|z-\zeta|\qquad z,\zeta\in\mathbb{C}. \tag{6}$$

To see this there is no loss of generality in assuming that $z,\zeta\in\mathbb{R}$, $\zeta<z$ and $\rho(\zeta)<\rho(z)$. Then $\zeta-\rho(\zeta)<z-\rho(z)$, since otherwise $D(\zeta)\subset D(z)$, contradicting the fact that $\mu(D(z))=\mu(D(\zeta))=1$.

Lemma 2. [Chr91, p.205]. *If $\zeta\notin D(z)$ then*

$$\frac{\rho(\zeta)}{\rho(z)}\lesssim\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^{1-\delta}$$

for some $\delta\in(0,1)$ depending only on the doubling constant C_μ.

As a consequence of Lemma 1 and (5) we have

Corollary 3. *For every $r>1$ there exists $\gamma>0$ such that if $\zeta\in B(z,r)$ then*

$$\frac{1}{r^{\gamma}}\lesssim\frac{\rho(z)}{\rho(\zeta)}\lesssim r^{\gamma}.$$

It will be convenient to express some of the results in terms of the distance d_ϕ induced by the metric $\rho^{-2}(z)dz\otimes d\bar z$.

Lemma 4. *There exists $\delta\in(0,1)$ such that for every $r>0$ there exists $C_r>0$ such that*

(a) $C_r^{-1}\dfrac{|z-\zeta|}{\rho(z)}\le d_\phi(z,\zeta)\le C_r\dfrac{|z-\zeta|}{\rho(z)}$ *if* $|z-\zeta|\le r\rho(z)$.

(b) $C_r^{-1}\left(\dfrac{|z-\zeta|}{\rho(z)}\right)^{\delta}\le d_\phi(z,\zeta)\le C_r\left(\dfrac{|z-\zeta|}{\rho(z)}\right)^{2-\delta}$ *if* $|z-\zeta|>r\rho(z)$.

This shows, in particular, that a sequence Λ is ρ-separated if and only if there exists $\delta>0$ such that $\inf_{\lambda\ne\lambda'}d_\phi(\lambda,\lambda')>\delta$.

Proof. By definition

$$d_\phi(z,\zeta)=\inf\int_0^1|\gamma'(t)|\rho^{-1}(\gamma(t))dt,$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = \zeta$.

The lower inequalities are contained in [Chr91, Lemma 3.1] and its proof.

The upper estimate in case (a) is immediate from Corollary 3. In case (b) take $\gamma(t) = z + t(\zeta - z)$ and use Lemma 2; then

$$d_\phi(z,\zeta) \le |\zeta - z| \int_0^1 \frac{dt}{\rho(\gamma(t))} \lesssim \int_0^1 \frac{(t|\zeta - z|)^{1-\delta}}{(\rho(z))^{2-\delta}} dt \lesssim \left(\frac{|z-\zeta|}{\rho(z)}\right)^{2-\delta}.$$

■

From now on, given $z \in \mathbb{C}$ and $r > 0$, we denote

$$B(z,r) = \{\zeta \in \mathbb{C} : d_\phi(z,\zeta) < r\}.$$

Doubling measures satisfy certain integrability conditions.

Lemma 5. *Let μ be a doubling measure. There exist $C > 0$ and $m \in \mathbb{N}$ depending on C_μ such that for any $r > 0$*

(a) $\displaystyle\int_{D(z,r)} \log\Big(\frac{2r}{|z-\zeta|}\Big) d\mu(\zeta) \le C\, \mu(D(z,r)) \qquad z \in \mathbb{C}.$

(b) $\displaystyle\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} \frac{d\mu(\zeta)}{1 + d_\phi^m(z,\zeta)} < \infty\,.$

Proof. (a) is [Chr91, Lemma 2.3].

(b) According to Lemma 4 it is enough to consider the integral on $|z - \zeta| \ge r\rho(z)$. Applying Fubini's theorem we see that

$$\begin{aligned}\int_{\zeta \notin D^r(z)} \Big(\frac{\rho(z)}{|z-\zeta|}\Big)^m d\mu(\zeta) &= \int_{\zeta\notin D^r(z)} m \int_0^{\rho(z)/|z-\zeta|} t^{m-1} dt d\mu(\zeta) \\ &= m\int_0^{1/r} t^{m-1} \int_{t<\rho(z)/|z-\zeta|<1/r} d\mu(\zeta) dt \le m \int_0^{1/r} t^{m-1} \mu(D^{1/rt}(z))\, dt.\end{aligned}$$

Let $x_0 = \log_2 C_\mu$, and for a given t denote $k(t) = \inf\{k \in \mathbb{N} : 1/rt \le 2^k\}$. Then

$$\mu(D^{1/rt}(z)) \le \mu(D^{2^{k(t)}}(z)) \le 2^{x_0 k(t)} \le \Big(\frac{2}{rt}\Big)^{x_0},$$

hence the integral is bounded if $m > x_0$.

This and Lemma 4(b) show that the result holds for m big enough. ■

Remark 2. It is clear from the proof that

(b') $\displaystyle\lim_{r\to\infty} \sup_{z\in\mathbb{C}} \int_{\zeta\notin B(z,r)} \frac{d\mu(\zeta)}{d_\phi^m(z,\zeta)} = 0.$

There is a discrete version of the previous Lemma.

Lemma 6. *Let Λ be a ρ-separated sequence. There exists $m \in \mathbb{N}$ such that*

$$\sup_{z\in\mathbb{C}} \sum_{\lambda\in\Lambda} \frac{1}{1+d_\phi^m(z,\lambda)} < \infty.$$

Proof. By the separation and Lemma 4, it is enough to see that for m big enough

$$\sup_{z\in\mathbb{C}} \sum_{\lambda\notin B(z,r)} \Big(\frac{\rho(\lambda)}{|z-\lambda|}\Big)^m < \infty.$$

Take $\delta > 0$ such that the balls $\{B(\lambda,\delta)\}_{\lambda\in\Lambda}$ are pairwise disjoint. By Corollary 3

$$\sum_{\lambda\notin B(z,r))} \Big(\frac{\rho(\lambda)}{|z-\lambda|}\Big)^m \lesssim \sum_{\lambda\notin B(z,r)} \int_{B(\lambda,\delta)} \Big(\frac{\rho(\zeta)}{|z-\zeta|}\Big)^m d\mu(\zeta) \lesssim \int_{\lambda\notin B(z,r)} \Big(\frac{\rho(\zeta)}{|z-\zeta|}\Big)^m \, d\mu(\zeta).$$

Lemma 5(b) implies that the integral is bounded. ∎

For later use, we state a refinement that follows similarly from Remark 2.

Corollary 7. *Let Λ be a ρ-separated sequence. There exists $m \in \mathbb{N}$ such that*

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}} \sum_{\lambda\notin B(z,r)} \frac{1}{d_\phi^m(z,\lambda)} = 0.$$

We will need to partition the plane in rectangles of constant mass. We do that by adapting a general result of [Yul85] to our setting (see also [Dra01, Theorem 2.1]).

Theorem 8. *Let μ be a positive doubling measure non-identically zero. There exists a "partition" of $\mathbb{C}$ in rectangles R_k with sides parallel to the coordinate axis such that:*

(a) $\mu = \sum_k \mu_k$, *where* $\mu_k := \mu_{|R_k}$ *satisfy* $\mu_k(\mathbb{C}) = 1$.
(b) R_k *are quasi-squares: there exists* $e > 1$ *depending only on* C_μ *such that the ratio of sides of each* R_k *lies in the interval* $[1/e, e]$.
(c) *There exists* $C < 0$ *such that* $C^{-1}\rho(a_k) \leq diam(R_k) \leq C\rho(a_k)$, *where* a_k *denotes the centre of* R_k.
(d) $\bigcup_k \overline{R_k} = \mathbb{C}$ *and the interiors of* R_k *are distinct.*

Remark 3. Dividing the original measure by $s \in \mathbb{R}^+$ we obtain a partition of $\mathbb{C}$ into quasi-squares of mass s.

Proof. It is enough to partition the plane in quasi-squares of constant entire mass, because by an stopping-time argument of [OC99] these can then be split into quasi-squares of mass 1.

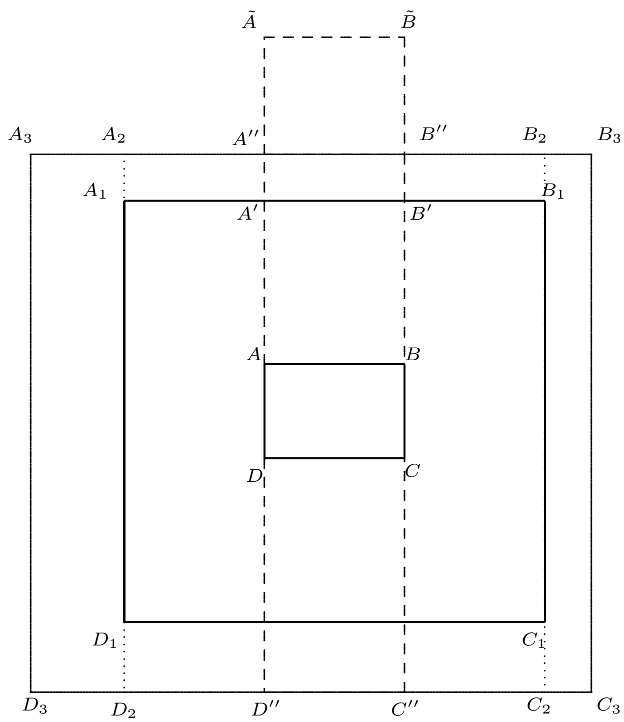
We construct our partition recursively. We start with a rectangle centred at 0 of entire mass, and with sidelengths $l \leq L$ so that $l \geq L/2$ and $l^{1-\beta} \geq 12\sqrt{2}C_0$, where β and C_0 are given in (5) (rectangle $ABCD$ in the picture). Consider next a square Q_1 centred at 0 of sidelength $3L$ ($A_1B_1C_1D_1$ in the picture) and define R as the quasi-square with vertices $ABB'A'$, where A'

and B' are points on the same side of Q_1 taken so that $0 \notin R$. We want to make R a little bigger, to make sure that its mass is entire, and we want to do that keeping control on the ratio of sides. Consider the rectangle $AB\tilde{B}\tilde{A}$, where $\tilde{A}, \tilde{B}$ are taken with $|A\tilde{A}| = |B\tilde{B}| = 2|AA'|$. Denote by R' the rectangle $A'B'\tilde{B}\tilde{A}$ added to R. For $\lambda \in R'$,

$$\frac{\rho(\lambda)}{l} \leq \frac{6\sqrt{2}\rho(\lambda)}{|\lambda|} \leq \frac{6\sqrt{2}C_0}{|\lambda|^{1-\beta}} \leq \frac{6\sqrt{2}C_0}{l^{1-\beta}} \leq \frac{1}{2}.$$

Since the sides of R' have length bigger or equal than l we deduce that R' contains a disk of centre λ and radius $\rho(\lambda)$, hence its mass is at least 1. This shows that there exists a rectangle R_1 ($AA''B''B$ in the picture) of entire mass between the original R and the "doubled" R'.

We finish the first step of the process by constructing the analogous quasi-square R_2 of entire mass at the opposite side of R ($CC''D''D$ in the picture).



Consider next the rectangle Q_2 limited by the segments $(A''B'')$, $(C''D'')$, (B_1C_1), (D_1A_1) (the rectangle $A_2B_2C_2D_2$ in the picture). We iterate the process above to each of the rectangles $B''B_2C_2C''$ and $D''D_2A_2A''$, thus obtaining two new quasi-squares $R_3 = B''B_3C_3C''$ and $R_4 = D''D_3A_3A''$ of entire mass.

All in all, we obtain a new quasi-square $Q_3 := A_3B_3C_3D_3$ with ratio of sides lying in $[1/2, 2]$ which is a disjoint union of 5 quasi-squares of entire mass. From here we repeat the process, taking Q_3 in place of the original R, and continue recursively to obtain the "partition" of $\mathbb{C}$. By construction we have (a), (b) and (d).

To prove (c) assume that R is a quasi-square of mass 1, centre a and sidelengths l, L. Here $R \subset D(a, L\sqrt{2})$, hence $\rho(a) \gtrsim L \gtrsim diam(R)$. Also, $D(a,l) \subset R$ and $diam(R) \lesssim l \lesssim \rho(a)$. ∎

Lemmas 1 and 2 give control on how big a disc $D^r(\zeta)$ can be when $\zeta \in D^R(z)$. We will need another result along the same lines.

Given a doubling measure μ and given $z \in \mathbb{C}$ and $0 < r < R$, consider the associated regions

$$F_r(z,R) = \{\zeta : D^r(\zeta) \subset D^R(z)\} \qquad \text{and} \qquad G_r(z,R) = \bigcup_{\zeta \in D^R(z)} D^r(\zeta)\ .$$

By definition $F_r(z,R) \subset D^R(z) \subset G_r(z,R)$. Let γ be the constant given by Lemma 1, and ε, k the constants in (4).

Lemma 9. *Let $r > 0$ be fixed. There exists $c > 0$ such that if $\epsilon(R) = c(r^k/R^\varepsilon)^\gamma$, for all $z \in \mathbb{C}$ and $R > r$ we have*

(a) $G_r(z,R) \subset D^{R+\epsilon(R)}(z)$.
(b) $D^{R-\epsilon(R)}(z) \subset F_r(z,R)$.

Proof. Applying Lemma 1 to $D^r(\zeta)$ and $D^R(z)$, and using (4), we have

$$\left(\frac{R^\varepsilon}{r^k}\right)^\gamma \lesssim \frac{R\rho(z)}{r\rho(\zeta)} \lesssim \left(\frac{R^k}{r^\varepsilon}\right)^{1/\gamma}.$$

(a) If $\zeta \in D^R(z)$ we have $R\rho(z) + r\rho(\zeta) \le R\rho(z)(1 + c(r^k/R^\varepsilon)^\gamma)$ for some $c > 0$.

(b) $D^r(\zeta) \subset D^R(z)$ when $|\zeta - z| + r\rho(\zeta) \le R\rho(z)$. For $\zeta \in D^{R-\epsilon(R)}(z)$

$$|\zeta - z| + r\rho(\zeta) \le (R - \epsilon(R))\rho(z) + c_1 R\rho(z)\left(\frac{r^k}{R^\varepsilon}\right)^\gamma.$$

Thus if $(R - \epsilon(R))\rho(z) + cR\rho(z)(r^k/R^\varepsilon)^\gamma \le R\rho(z)$ we have $D^{R-\epsilon(R)}(z) \subset F_r(z,R)$. ∎

Corollary 10. *Let $\{R_k\}_k$ be a partition of $\mathbb{C}$, as in Theorem 8. Define*

$$F(z,R) = \bigcup_{k:R_k \subset D^R(z)} R_k \qquad \textit{and} \qquad G(z,R) = \bigcup_{k:R_k \cap D^R(z) \neq \emptyset} R_k\ .$$

There exists a positive function $\epsilon(R)$ with $\lim_{R\to\infty} \epsilon(R)/R = \infty$ and such that for all $z \in \mathbb{C}$ and $R > 0$

(a) $G(z,R) \subset D^{R+\epsilon(R)}(z)$.
(b) $D^{R-\epsilon(R)}(z) \subset F(z,R)$.

Proof. As the previous Lemma, using Theorem 8(c). ∎

We finish with a result showing that the measure of a disk cannot be too concentrated near its border.

Lemma 11. *Let* $\epsilon(r)$ *be a positive function such that* $\lim_{r\to\infty} \epsilon(r)/r = 0$. *Then*

$$\lim_{r\to\infty} \frac{\mu(D^{r+\epsilon(r)}(z))}{\mu(D^r(z))} = \lim_{r\to\infty} \frac{\mu(D^{r-\epsilon(r)}(z))}{\mu(D^r(z))} = 1$$

uniformly in $z \in \mathbb{C}$.

The proof is based in the following projection of the measure μ.

Lemma 12. *For every* $z \in \mathbb{C}$ *define the measure* ν_z *on* $\mathbb{R}^+$ *by*

$$\nu_z(A) = \mu(\{\zeta = z + re^{i\theta} : r \in A\}) \qquad A \subset \mathbb{R}^+.$$

Then ν_z *is doubling and there exists* K *independent of* z *such that* $C_{\nu_z} \leq KC_\mu$.

Proof. Given $x \in \mathbb{R}^+$ and $r > 0$ let $I^r(x) = (x - r, x + r) \cap \mathbb{R}^+$. We want to see that

$$\nu_z(I^{2r}(x)) \leq KC_\mu \; \nu_z(I^r(x))$$

for all $z \in \mathbb{C}$, $x \in \mathbb{R}^+$ and $r > 0$.

Let $A_z^r(x) = \{\zeta = z + se^{i\theta} : s \geq 0\,, |s - x| < r\}$. By definition $\nu_z(I^{2r}(x)) = \mu(A_z^{2r}(x))$. Split $A_z^{2r}(x)$ into $k := [\frac{2\pi}{4r}]$ sectors

$$S_j = \Big\{\zeta = z + se^{i\theta} : s \geq 0\,, |s - x| < 2r\ ,\ (j-1)\frac{2\pi}{k} \leq \theta < j\frac{2\pi}{k}\Big\} \qquad j = 1, \dots, k.$$

Being μ doubling there exists $K > 0$ such that $\mu(S_j) \leq KC_\mu \; \mu(\tilde{S}_j)$, where $\tilde{S}_j$ is half the sector S_j, i.e.

$$\tilde{S}_j = \Big\{\zeta = z + se^{i\theta} : s \geq 0\,, |s - x| < r\ ,\ (j-1)\frac{2\pi}{k} + \frac{2\pi}{4k} < \theta < j\frac{2\pi}{k} - \frac{2\pi}{4k}\Big\}.$$

Since the $\tilde{S}_j$'s are disjoint and $\cup_j \tilde{S}_j \subset A_z^r(x)$, we get

$$\begin{aligned} \nu_z(I^{2r})(x) &= \mu(A_z^{2r}(x)) = \sum_{j=1}^k \mu(S_j) \leq KC_\mu \sum_{j=1}^k \mu(\tilde{S}_j) \leq KC_\mu \; \mu(A_z^r(x)) \\ &= KC_\mu \; \nu_z(I^r(x)). \end{aligned}$$

■

Proof of Lemma 11. It is enough to see that

$$\lim_{r\to\infty} \frac{\mu(D^{r+\epsilon(r)}(z) \setminus D^r(z))}{\mu(D^r(z))} = 0$$

uniformly in z. By definition of ν_z we have

$$\frac{\mu(D^{r+\epsilon(r)}(z) \setminus D^r(z))}{\mu(D^r(z))} = \frac{\nu_z\Big((r\rho(z), (r + \epsilon(r))\rho(z)\Big)}{\nu_z((0, r))},$$

and by the corresponding version of Lemma 1 for doubling measures in $\mathbb{R}^+$, and by Lemma 12, there exists $K > 0$ independent of z such that

$$\frac{\nu_z\Big((r\rho(z),(r+\epsilon(r))\rho(z)\Big)}{\nu_z((0,r))} \le K\left(\frac{\epsilon(r)}{r}\right)^{\gamma}.$$

■

Remark 4. An analogous result is true if in the definition of ν_z we use, instead of a radial projection with respect to z, a projection associated to quasi-squares of a fixed ratio $\alpha \in [e^{-1}, e]$ (e is the constant of Theorem 8(b)). Let $Q_\alpha^r(z)$ denote the rectangle with vertices $z + r(1 + i\alpha)$, $z + r(1 - i\alpha)$, $z - r(1 + i\alpha)$ and $z - r(1 - i\alpha)$. Given $z \in \mathbb{C}$ consider the measure ν_z in $\mathbb{R}$ such that

$$\nu_z(I^r(x)) = \mu(Q_\alpha^{x+r}(z) \setminus Q_\alpha^{x-r}(z))$$

on any interval $I^r(x)$. As before, there exists $K > 0$ independent of $z \in \mathbb{C}$ and $\alpha \in [e^{-1}, e]$ such that ν_z is doubling with $C_\nu \le KC_\mu$. Therefore, if $R_\alpha^r(z) := Q_\alpha^{r\rho(z)}(z)$,

$$\lim_{r\to\infty} \frac{\mu(R_\alpha^{r+\epsilon(r)}(z))}{\mu(R_\alpha^r(z))} = \lim_{r\to\infty} \frac{\mu(R_\alpha^{r+\epsilon(r)}(z))}{\mu(R_\alpha^r(z))} = 1$$

uniformly in z.

2.2. **Flat weights.**

In this section we describe the weights ω appearing in the spaces $\mathcal{F}^p_{\phi,\omega}$.

Definition 6. A positive measurable function ω is called a *flat weight for* ϕ if there exists $C > 0$ such that for all $z, \zeta \in \mathbb{C}$

$$|\log\omega(z) - \log\omega(\zeta)| \le C(1 + \log^+ d_\phi(z,\zeta)). \tag{7}$$

The class of flat weights associated to ϕ will be denoted by Ω_ϕ.

Notice that the product $\omega\widehat{\omega}$ of two weights $\omega, \widehat{\omega} \in \Omega_\phi$ belongs to Ω_ϕ as well. Also, if $\omega \in \Omega_\phi$ then $\omega^\alpha \in \Omega_\phi$, for all $\alpha \in \mathbb{R}$.

Besides the obvious $\omega = 1$, important examples of flat weights for ϕ are the functions $\omega = \rho^\alpha$, $\alpha \in \mathbb{R}$. This is seen applying Lemma 2 and Lemma 4.

Furthermore, the weights $\omega \in \Omega_\phi$ can be assumed to satisfy

$$\Big|1 - \frac{\omega(z)}{\omega(\zeta)}\Big| \le C d_\phi(z,\zeta) \qquad \text{if} \quad d_\phi(z,\zeta) \le 1\,. \tag{8}$$

If the original weight ω does not satisfy this condition, replace it by the regularisation

$$\tilde{\omega}(z) = \frac{1}{\rho^2(z)} \int_{D(z)} \omega.$$

It is clear, by (7), that there exists $C>0$ such that $C^{-1}\leq|\omega/\tilde\omega|\leq C$, hence the spaces of functions and sequences associated to the weights ω and $\tilde\omega$ are the same. On the other hand

$$\Big|\frac{\tilde\omega(\zeta)-\tilde\omega(z)}{\tilde\omega(\zeta)}\Big|\leq\Big|\frac{1}{\rho^2(\zeta)}\Big[\int_{D(\zeta)}\frac{\omega}{\tilde\omega(\zeta)}-\int_{D(z)}\frac{\omega}{\tilde\omega(\zeta)}\Big]\Big|+\Big|\frac{1}{\rho^2(\zeta)}-\frac{1}{\rho^2(z)}\Big|\int_{D(z)}\frac{\omega}{\tilde\omega(\zeta)}.$$

Assuming that $d_\phi(z,\zeta)\leq 1$, from (7), (6) and Lemma 4(a) we deduce that

$$\begin{aligned}\Big|\frac{\tilde\omega(\zeta)-\tilde\omega(z)}{\tilde\omega(\zeta)}\Big| &\lesssim \frac{\sigma[(D(\zeta)\cup D(z))\setminus(D(\zeta)\cap D(z))]}{\rho^2(\zeta)}+\frac{|\rho(z)-\rho(\zeta)||\rho(z)+\rho(\zeta)|}{\rho^2(\zeta)}\\ &\lesssim \frac{\rho(z)|\zeta-z|}{\rho^2(\zeta)}+\frac{|\zeta-z|}{\rho(\zeta)}\lesssim d_\phi(\zeta,z).\end{aligned}$$

2.3. **Local behaviour and regularisation of ϕ.** Let us start with a result comparing the values of ϕ in a disk with the value on its centre.

Lemma 13. *For every $K>0$ there exists $A=A(K)>0$ such that for all $z\in\mathbb{C}$*

$$\sup_{w\in D^K(z)}|\phi(w)-\phi(z)-h_z(w)|\leq A,$$

where h_z is a harmonic function in $D^K(z)$ with $h_z(z)=0$.

Proof. The proof is as in [OCS98, Lemma 1]. On each $D^K(z)$ decompose

$$\phi(w)=\phi(z)+h_z(w)+\int_{D^K(z)}(G(w,\eta)-G(z,\eta))\,\Delta\phi(\eta), \tag{9}$$

where G is the Green function of the disc $D^K(z)$ and h_z is a harmonic function in $D^K(z)$ such that $h_z(z)=0$. By Lemma 5(a)

$$\sup_{z\in\mathbb{C}}\int_{D^K(z)}\log\frac{K\rho(z)}{|z-\eta|}\Delta\phi(\eta)<\infty$$

and the result holds. ■

We have seen in the previous section that $\rho_\phi(z)$ is Lipschitz (see (6)). Also, because of Lemma 1, ϕ is Hölder continuous of some positive order on every bounded subset of $\mathbb{C}$ (see [Chr91, Lemma 2.8]). More regularity can be attained by taking a suitable weight ψ equivalent to ϕ.

Theorem 14. *Let ϕ be subharmonic with $\Delta\phi$ doubling. There exist $\psi\in\mathcal{C}^\infty(\mathbb{C})$ subharmonic and $C>0$ such that $|\psi-\phi|\leq C$, $\Delta\psi$ is a doubling measure and $\Delta\psi\simeq 1/\rho_\psi^2\simeq 1/\rho_\phi^2$. Moreover $|\nabla(\Delta\psi)|\lesssim 1/\rho_\phi^3$.*

Since the spaces of functions and sequences considered do not change if ϕ is replaced by ψ, from now on we will assume that $\phi\in\mathcal{C}^\infty(\mathbb{C})$, $\Delta\phi\simeq 1/\rho^2$ and $|\nabla(\Delta\phi)|\lesssim 1/\rho^3$.

In the proof of this result we will need to partition $\mathbb{C}$ and discretize the measure.

Lemma 15. *Let μ be a positive doubling measure in $\mathbb{C}$. Fix $m \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and $C > 0$ such that for any partition $\{R_p\}_p$ as in Theorem 8 with $\mu(R_p) = mk$ there are points $\lambda_1^{(p)}, \dots, \lambda_{mk}^{(p)} \in CR_p$ such that*

(a) $\mu_p = \mu_{|R_p}$ *and* $\nu_p = \sum\limits_{j=1}^{mk} \delta_{\lambda_j^{(p)}}$ *have the same first m moments.*

(b) $\Lambda = \{\lambda_j^{(p)}\}_{p,j}$ *is a ρ-separated sequence.*

Proof. By Lemma 5 of [OC99], there exists $k \in \mathbb{N}$ such that for all measure μ_p supported in a rectangle R_p with total mass mk, there are points $\sigma_1^{(p)}, \dots \sigma_k^{(p)} \in R_p$ such that μ_p and $m\sum_{j=1}^k \delta_{\sigma_j^{(p)}}$ have the same first m moments.

In order to get a separated sequence replace each $\sigma_j^k(p)$ by m points $\gamma_{j,l}^{(p)} = \sigma_j^{(p)} + \tau_j^{(p)} e^{i2\pi l/m}$, $l = 0, \dots, m-1$, lying on a circle around $\sigma_j^{(p)}$. Since for all polynomials p of degree less than $m-1$

$$m\, p(\sigma_j^{(p)}) = \sum_{l=0}^{m-1} p(\gamma_{j,l}^{(p)}),$$

the measures μ_p and $\sum_{j,l} \delta_{\gamma_{j,l}^{(p)}}$ have still the same first m moments. We will be done as soon as we see that the $\tau_j^{(p)}$ can be chosen uniformly bounded and so that $\Lambda = \{\gamma_{j,l}^{(p)}\}$ is ρ-separated. For this we use a Besicovitch's lemma: the family $\{R_p\}_p$ can be split in q families $\{R_p^1\}_{p\in I_1}, \dots \{R_p^q\}_{p\in I_q}$ such that two rectangles of the same family are far apart, in the sense that $MR_p^l \cap MR_{p'}^l = \emptyset$, $p \neq p'$, for some large constant M. For the first family $\{R_p^1\}_{p\in I_1}$, it is easy to choose $\tau_j^{(p)}$ such that the resulting sequence $\Gamma_1 = \{\gamma_{j,l}^{(p)} : p \in I_1; j = 1, \dots, k; l = 0, \dots m-1\}$ is ρ-separated. Next we choose $\tau_j^{(p)}$, $p \in I_2$, so that $\Gamma_2 \cap \Gamma_1$ is ρ-separated, where $\Gamma_2 = \{\gamma_{j,l}^{(p)} : p \in I_2; j = 1, \dots, k; l = 0, \dots, m-1\}$. Choosing $\tau_j^{(p)}$ recursively in this way we obtain $\Lambda = \Gamma_1 \cup \dots \cup \Gamma_q$ ρ-separated. ■

Proof of Theorem 14. For any M (to be chosen later) consider $k \in \mathbb{N}$ as in Lemma 15 and a partition $\{R_p\}_p$ as in Theorem 8. Take then the sequence $\Lambda = \{\lambda_j^{(p)}\}_{j,p}$ given by Lemma 15. Recall that $\lambda_j^{(p)} \in CR_p$, $\mu(R_p) = Mk$ and that the measures μ_p and $\nu_p = M\sum\limits_{j=1}^{k} \delta_{\lambda_j^{(p)}}$ have the same first M moments.

By Theorem 8(c) there exists $r > 0$ such that $CR_p \subset D^r(\lambda_j^{(p)})$ for any $p \in \mathbb{N}$ and $i \leq k$. Furthermore, by construction of $\{R_p\}_p$ there exists $q \in \mathbb{N}$ such that any $z \in \mathbb{C}$ lies in at most q disks $D^r(\lambda_j^{(p)})$.

We now regularise ν_p by setting

$$\tilde{\nu}_p = \sum_{j=1}^{Mk} \frac{\mathcal{X}\Big(\frac{|z-\lambda_j^{(p)}|}{r\rho(\lambda_j^{(p)})}\Big)}{\int \mathcal{X}\Big(\frac{|z-\lambda_j^{(p)}|}{r\rho(\lambda_j^{(p)})}\Big)},$$

where $\mathcal{X}$ is a smooth non-negative cut-off function of one real variable such that $\mathcal{X}(t) = 1$ if $|t| < 1$, $\mathcal{X}(t) = 0$ if $|t| > 2$ and $|\mathcal{X}'|$ is bounded.

Notice that $\tilde{\nu}_p$ and μ_p have the first M moments. Indeed, by the mean value property

$$\int_{\mathbb{C}} z^l d\tilde{\nu}_p = \sum_{i=1}^{Mk} (\lambda_j^{(p)})^l \qquad l = 0, \dots, M-1.$$

Define $\tilde{\nu} = \sum\limits_{p=1}^{\infty} \tilde{\nu}_p$ and

$$\psi(z) = \phi(z) + \frac{1}{2\pi} \int_{\mathbb{C}} \log|z-\zeta| (\tilde{\nu} - \Delta\phi)(\zeta).$$

We claim that $\tilde{\nu}$ is a doubling measure. The proof of this fact is a bit technical and will be deferred to the end.

By definition $\Delta\psi = \tilde{\nu}$. Also, $\tilde{\nu}(z)$ is a sum of at most q terms of order $1/\rho^2(\lambda_j^{(p)})$, with $z \in D^r(\lambda_j^{(p)})$. Therefore $\Delta\psi \simeq 1/\rho_\phi^2$ and $|\nabla(\Delta\psi)| \lesssim 1/\rho_\phi^3$. In particular

$$\int_{D_\phi(z)} \Delta\psi(\zeta) \simeq \int_{D_\phi(z)} \frac{dm(\zeta)}{\rho_\phi^2(\zeta)} \simeq 1,$$

hence $\rho_\phi \simeq \rho_\psi$.

Let us show next that $|\psi - \phi| \leq C$ for some $C > 0$.

Let a_p denote the centre of R_p. Assume $z \in R_{p_0}$ and let $I_{p_0} = \{p \in \mathbb{N} : d_\phi(a_p, a_{p_0}) \leq 10r\}$. Remark that for $p \notin I_{p_0}$, $\zeta \in supp(\tilde{\nu}_p)$ and $z \in supp(\tilde{\nu}_{p_0})$ we have $d_\phi(z,\zeta) \simeq d_\phi(a_p, a_{p_0})$. Indeed, this follows from

$$|\zeta - a_p| \leq 3r\rho(a_p) \leq \frac{3}{10}|a_p - a_{p_0}|,$$

the analogous estimate for $|z - a_{p_0}|$ and Lemma 4. This yields

$$\int_{\mathbb{C}} d_\phi^{-M}(z,\zeta)\tilde{\nu}_p(\zeta) \lesssim \int_{\mathbb{C}} d_\phi^{-M}(z,\zeta)\mu_p(\zeta) \qquad z \in I_{p_0}\ ,\ p \notin I_{p_0}. \tag{10}$$

We split

$$2\pi(\psi(z) - \phi(z)) = \sum_{p \in I_{p_0}} \int_{\mathbb{C}} \log|z-\zeta|(\tilde{\nu}_p - \mu_p)(\zeta) + \sum_{p \notin I_{p_0}} \int_{\mathbb{C}} \log|z-\zeta|(\tilde{\nu}_p - \mu_p)(\zeta)$$

and estimate each sum separately.

Let p_M denote the M-th Taylor polynomial of $\log(|z-\zeta|/\rho(z))$. Since $\tilde{\nu}_p - \mu_p$ have vanishing moments of order less or equal to M, we can estimate

$$\begin{aligned} I_1 : &= \Big|\sum_{p\notin I_{p_0}} \int_{\mathbb{C}} \log|z-\zeta|(\tilde{\nu}_p - \mu_p)(\zeta)\Big| \\ &= \Big|\sum_{p\notin I_{p_0}} \int_{\mathbb{C}} (\log\frac{|z-\zeta|}{\rho(z)} - p_M(\zeta))(\tilde{\nu}_p - \mu_p)(\zeta)\Big| \le \sum_{p\notin I_{p_0}} \int_{\mathbb{C}} \Big(\frac{\rho(z)}{|z-\zeta|}\Big)^M (\tilde{\nu}_p + \mu_p)(\zeta). \end{aligned}$$

Taking M big enough and using (10) and Lemmas 4(a) and 5(b),

$$I_1 \lesssim \int_{\mathbb{C}\setminus D^r(z)} \Big(\frac{\rho(z)}{|z-\zeta|}\Big)^M \mu(\zeta) \lesssim \int_{\mathbb{C}\setminus B(z,Cr)} \frac{d\mu(\zeta)}{d_\phi^{\delta M}(z,\zeta)} \le C.$$

For the remaining term we use again the moment condition together with the fact that for $p \in I_{p_0}$ there exists γ such that $\cup\{supp(\tilde{\nu}_p),\ p\in I_{p_0}\} \subset D_\phi^\gamma(z)$. Thus

$$\begin{aligned} I_2 : &= \Big|\sum_{p\in I_{p_0}} \int_{\mathbb{C}} \log|z-\zeta|(\tilde{\nu}_p - \mu_p)(\zeta)\Big| = \Big|\sum_{p\in I_{p_0}} \int_{\mathbb{C}} \log\Big(\frac{2\gamma\rho(z)}{|z-\zeta|}\Big)(\tilde{\nu}_p - \mu_p)(\zeta)\Big| \\ &\lesssim \int_{D^\gamma(z)} \log\Big(\frac{2\gamma\rho(z)}{|z-\zeta|}\Big)(\tilde{\nu} + \mu)(\zeta). \end{aligned}$$

By Lemma 5(a) this is finite.

We prove now that $\tilde{\nu}$ is doubling. We first show that it is doubling for big balls, i.e. there exist $R_0 > 0$ and a constant C depending only on the doubling constant $C_{\Delta\phi}$ of $\Delta\phi$ such that for all $R > R_0$ we have $\tilde{\nu}(D^R(a)) \le C\tilde{\nu}(D^{R/2}(a))$.

As in Corollary 10, define

$$F(a,R) = \bigcup_{p: R_p\subset D^R(a)} R_p \qquad \text{and} \qquad G(a,R) = \bigcup_{p: R_p\cap D^R(a)\neq\emptyset} R_p.$$

Since $\tilde{\nu}(R_p) \simeq \int_{R_p} d\sigma/\rho^2 \simeq \mu(R_p)$, we see that $\tilde{\nu}(F(a,R)) \simeq \mu(F(a,R))$ and $\tilde{\nu}(G(a,R)) \simeq \mu(G(a,R))$. By Corollary 10, also $D^{R-\epsilon(R)}(a) \subset F(a,R)$ and $D^{R+\epsilon(R)}(a) \supset G(a,R)$. This and the fact that μ is doubling yield

$$\tilde{\nu}(D^R(a)) \le \tilde{\nu}(G(a,R)) \simeq \mu(G(a,R)) \le \mu(D^{R+\epsilon(R)}(a)) \le C_{\Delta\phi}\, \mu(D^{1/2(R+\epsilon(R))}(a)),$$

and

$$\tilde{\nu}(D^{R/2}(a)) \ge \tilde{\nu}(F(a,R/2)) \simeq \mu(F(a,R/2)) \ge \mu(D^{R/2-\epsilon(R/2)}(a)).$$

Therefore

$$\tilde{\nu}(D^R(a)) \le C_{\Delta\phi}\, \tilde{\nu}(D^{R/2}(a)) \frac{\mu(D^{1/2(R+\epsilon(R))}(a))}{\mu(D^{R/2-\epsilon(R/2)}(a))}.$$

Lemma 11 shows that the quotient converges to 1 as R goes to infinity uniformly in a, so there exists R_0 such that $\tilde{\nu}(D^R(a)) \le 2C_{\Delta\phi}\, \tilde{\nu}(D^{R/2}(a))$ for all $R \ge R_0$.

Corollary 3 implies that $\tilde{\nu} \simeq 1/\rho^2(a)$ on $D^R(a)$ when $R \le R_0$, so we deduce that $\tilde{\nu}(D^R(a)) \lesssim \tilde{\nu}(D^{R/2}(a))$. ■

2.4. **The multiplier.** A basic tool in our approach is the use of the so-called multiplier: an entire function g such that $|g| \simeq e^{\phi}$ outside a neighbourhood of the zeros of g.

Theorem 16. *Let ϕ be a subharmonic function such that $\Delta\phi$ is a doubling measure. There exists an entire function g such that*

(a) *The zero-sequence $\mathcal{Z}(g)$ of g is ρ_ϕ-separated and $\sup_{z\in\mathbb{C}} d_\phi(z, \mathcal{Z}(g)) < \infty$.*
(b) $|g(z)| \simeq e^{\phi(z)} d_\phi(z, \mathcal{Z}(g))$ *for all $z \in \mathbb{C}$.*

The function g can be chosen so that, moreover, it vanishes on a prescribed $z_0 \in \mathbb{C}$. We say that g is a multiplier associated to ϕ.

Proof. Take a partition $\{R_p\}$ of $\mathbb{C}$ with $\mu(R_p) = 2\pi m N$ and consider the sequence Λ given by Lemma 15. For the sake of clarity we write R_p instead of CR_p (C is the constant of Lemma 15). Note that now $\{R_p\}_p$ is not a partition, although there exists a uniform constant q such that all points of $\mathbb{C}$ lie in at most q quasi-squares R_p. As in Lemma 15, denote $\mu_p = (1/2\pi)\mu_{|R_p}$ and let ν_p be the sum of the $\lambda \in \Lambda$ associated to R_p. Recall that μ_p and ν_p have the same first m moments.

Let g be a holomorphic function satisfying

$$\log|g| = \phi - \frac{1}{2\pi}\int_{\mathbb{C}} \log|z-\zeta|(\Delta\phi - 2\pi\sum_{\lambda\in\Lambda}\delta_\lambda),$$

which exists because the Laplacian of the term at the right hand side is a sum of Dirac masses. By definition $\mathcal{Z}(g) = \Lambda$, and the previous construction ensures that (a) holds.

Let us prove (b). Assume that $z \in R_{p_0}$ and let I_{p_0} denote the set of indices p such that $R_p \cap R_{p_0} \neq \emptyset$. As in the previous proof, split

$$\log|g(z)| - \phi(z) = -\int_{\mathbb{C}} \log|z-\zeta|(\frac{\Delta\phi}{2\pi} - \sum_{\lambda\in\Lambda}\delta_\lambda) = S_1(z) + S_2(z),$$

where

$$S_1(z) := \sum_{p\notin I_{p_0}} \int_{\mathbb{C}} \log|z-\zeta|(\nu_p - \mu_p)$$

and

$$S_2(z) := \sum_{p\in I_{p_0}} \int_{\mathbb{C}} \log|z-\zeta|(\nu_p - \mu_p).$$

Again as in the proof of Theorem 14, using the Taylor expansion of $\log|z-\zeta|$ together with the moment condition one sees that $|S_1(z)|$ is bounded.

For the second sum notice that there exits $\gamma > 0$ such that $\cup_{p\in I_{p_0}} R_p \subset D^\gamma(z)$. Hence, denoting $|z-\Lambda| = \inf_{\lambda\in\Lambda} |z-\lambda|$, we get

$$\begin{aligned} S_2(z) &= \sum_{p\in I_{p_0}} \int_{\mathbb{C}} \log \frac{2\gamma\rho(z)}{|z-\zeta|}(\mu_p - \nu_p) \le \int_{D^\gamma(z)} \log \frac{2\gamma\rho(z)}{|z-\zeta|} d\mu(\zeta) - \log \frac{2\gamma\rho(z)}{|z-\Lambda|} \\ &\le C_2 + \log \frac{|z-\Lambda|}{\rho(z)}. \end{aligned}$$

On the other hand, using the ρ-separation of Λ

$$-S_2(z) \le \sum_{p\in I_{p_0}} \sum_{\lambda\in R_p} \log \frac{2\gamma\rho(z)}{|z-\lambda|} \le \log \frac{\rho(z)}{|z-\Lambda|} + C(\delta)\cdot \#\Big(\Lambda \cap \cup_{p\in I_{p_0}} R_p\Big).$$

Since $\# I_{p_0}$ is uniformly bounded, this and the estimate of S_1 give:

$$\log \frac{|z-\Lambda|}{\rho(z)} - C \le \log|g(z)| - \phi(z) \le \log \frac{|z-\Lambda|}{\rho(z)} + C'.$$

The result is then immediate from Lemma 4(a). ■

Next we state a useful application of the multiplier. It is a result about peak functions. These functions attain value 1 at a given point and decay very fast away from the point. They are very useful in the estimates of the Bergman kernel and in the construction of solutions to the $\bar\partial$ equation. Another proof of the following Lemma, using estimates for the $\bar\partial$-equation, can be found in an Appendix. This second proof is along the lines of [FS89, Theorem 2.1], where a related result is proved.

Theorem 17. *Take $\varepsilon > 0$, $\omega \in \Omega_\phi$ and $m \in \mathbb{N}$. There exists $C > 0$ such that for all $\eta \in \mathbb{C}$ there is an entire function P_η with $P_\eta(\eta) = 1$ and*

$$|P_\eta(z)| \le C e^{\varepsilon(\phi(z)-\phi(\eta))} \frac{\omega(\eta)}{\omega(z)} \frac{1}{1+d_\phi^m(z,\eta)}.$$

Proof. Let h be a multiplier for $\varepsilon\phi$ (constructed as in Theorem 16) with zero sequence $\Sigma = \{\sigma_k\}_k$ and such that $\{\eta\}\cup\Sigma$ is ρ-separated. In particular $|h(z)| \simeq e^{\varepsilon\phi(z)} d_\phi(z,\Sigma)$. It follows from the construction of the multiplier that for each $M \in \mathbb{N}$ there exists $r > 0$ such that $\#(\Sigma \cap B(\lambda, r)) \gtrsim M$ for all $\eta \in \mathbb{C}$. Given $\sigma_1,\dots,\sigma_M \in \Sigma \cap B(\lambda, r)$ define

$$P_\eta(z) = c_\eta \frac{h(z)}{(z-\sigma_1)\cdots(z-\sigma_M)} \frac{\rho^M(\eta)}{e^{\varepsilon\phi(\eta)}},$$

where c_η is chosen so that $P_\eta(\eta) = 1$.

Let us see first that there exists $c > 0$ independent of η with $c^{-1} \le c_\eta \le c$. Since $|\eta - \sigma_i| \simeq \rho(\eta)$, then

$$\frac{1}{c_\eta} = \frac{h(\eta)}{(\eta-\sigma_1)\cdots(\eta-\sigma_M)} \frac{\rho^M(\eta)}{e^{\varepsilon\phi(\eta)}} \simeq \frac{e^{\varepsilon\phi(\eta)} d_\phi(\eta,\Sigma)}{\rho^M(\eta)} \frac{\rho^M(\eta)}{e^{\varepsilon\phi(\eta)}} = d_\phi(\eta,\Sigma) \simeq 1.$$

We split the estimate of $|P_\eta(z)|$ into several regions. Let $\varepsilon > 0$ be such that that the balls $B(\sigma_i, \varepsilon)$ and $B(\eta, \varepsilon)$ are pairwise disjoint. Consider $K > 0$ with $\cup_{i=1}^M B(\sigma_i, \varepsilon) \subset B(\eta, K)$.

i) $z \in \bigcup_{i=1}^M B(\sigma_i, \varepsilon)$. For $z \in B(\sigma_i, \varepsilon)$ we have $\rho(z) \simeq \rho(\eta) \simeq \rho(\sigma_i)$, $d_\phi(z, \Sigma) \simeq |z-\sigma_i|/\rho(\sigma_i)$ and $d_\phi(z, \sigma_j) \gtrsim 1$, $j \neq i$. Thus

$$|P_\eta(z)| \lesssim \left|\frac{h(z)}{z-\sigma_i}\right| \rho(\eta) e^{-\varepsilon\phi(\eta)} \simeq e^{\varepsilon(\phi(z)-\phi(\eta))}.$$

ii) $z \in B(\eta, K) \setminus \bigcup_{i=1}^M B(\sigma_i, \varepsilon)$. Here $\rho(z) \simeq \rho(\eta)$ and $|z-\sigma_i| \gtrsim \rho(\eta)$, so

$$|P_\eta(z)| \lesssim \frac{e^{\varepsilon\phi(z)} d_\phi(z, \Sigma)}{\rho^M(\eta)} \frac{\rho^M(\eta)}{e^{\varepsilon\phi(\eta)}} \lesssim e^{\varepsilon(\phi(z)-\phi(\eta))}.$$

iii) $z \notin B(\eta, K)$. Here $d_\phi(z, \sigma_i) \simeq d_\phi(z, \eta)$, so

$$|P_\eta(z)| \lesssim \frac{e^{\varepsilon\phi(z)} d_\phi(z, \Sigma)}{|z-\eta|^M} \frac{\rho^M(\eta)}{e^{\varepsilon\phi(\eta)}} \lesssim e^{\varepsilon(\phi(z)-\phi(\eta))} \Big(\frac{\rho(\eta)}{|z-\eta|}\Big)^M.$$

This and Lemma 4(b) solve the case $\omega = 1$.

For arbitrary $\omega \in \Omega_\phi$ there exists $\gamma > 1$ such that if $d_\phi(z, \eta) \geq 1$ then

$$d_\phi^{-\gamma}(z, \eta) \lesssim \frac{\omega(\eta)}{\omega(z)} \lesssim d_\phi^\gamma(z, \eta).$$

Thus the result follows from the previous construction taking M big enough and using again Lemma 4(b). ∎

3. Basic properties of functions in $\mathcal{F}^p_{\phi,\omega}$

Here we study the behaviour of functions in $\mathcal{F}^p_{\phi,\omega}$ and related topics. We prove the estimates with norms $\|\cdot\|_{\mathcal{F}^p_{\phi,\omega}}$ on the solutions to the $\bar\partial$ equation (Theorem C) and provide estimates of the Bergman Kernel of $\mathcal{F}^2_{\phi,\omega}$ on the diagonal. We also introduce a scaled translation in the plane that gives rise to a translated weight and to an isometry between the spaces of functions for the original and the translated weight. This will be used when studying the properties of weak limits (Section 3.5).

3.1. Pointwise estimates.

Let us first see what is the natural growth of functions in $\mathcal{F}^p_{\phi,\omega}$. Recall that $d\sigma = dm/\rho^2$.

Lemma 18. *Let $1 \leq p < \infty$ and $\omega \in \Omega_\phi$. For any $r > 0$ there exists $C > 0$ such that for any $f \in H(\mathbb{C})$ and $z \in \mathbb{C}$:*

(a) $|f(z)|^p e^{-p\phi(z)} \leq \dfrac{C}{\omega^p(z)} \displaystyle\int_{D^r(z)} |f|^p e^{-p\phi} \omega^p d\sigma.$

(b) $|\nabla(|f|e^{-\phi})(z)| \leq \dfrac{C}{\omega(z)\rho(z)} \Big(\displaystyle\int_{D^r(z)} |f|^p e^{-p\phi} \omega^p d\sigma\Big)^{1/p}.$

(c) *If* $R > r$ *then* $|f(z)|^p e^{-p\phi(z)} \leq \dfrac{C_R}{\omega^p(z)} \displaystyle\int_{D^R(z)\setminus D^r(z)} |f|^p e^{-p\phi}\omega^p d\sigma.$

Proof. Let H_z be a holomorphic function with $Re\, H_z = h_z$, where h_z is the harmonic function in $D^r(z)$ given in Lemma 13.
(a) is proved as in [OCS98, Lemma 1]:

$$\begin{aligned} |f(z)|^p e^{-p\phi(z)} &= |f(z)e^{-H_z(z)}|^p e^{-p\phi(z)} \\ &\leq \frac{C}{\rho^2(z)} \int_{D^r(z)} |f(\zeta)|^p e^{-p(h_z(\zeta)+\phi(z))} \simeq \frac{1}{\omega^p(z)} \int_{D^r(z)} |f|^p e^{-p\phi}\omega^p d\sigma. \end{aligned}$$

(b) First let us see that $|\partial\phi/\partial\zeta - \partial h_z/\partial\zeta| \lesssim 1$ on $D^r(z)$. By (9), if $\zeta \in D^r(z)$

$$\Big|\frac{\partial\phi}{\partial\zeta}(\zeta) - \frac{\partial h_z}{\partial\zeta}(\zeta)\Big| = \Big|\frac{\partial}{\partial\zeta}\int_{D^r(z)} G(\zeta,\eta)\Delta\phi(\eta)\Big| \leq \int_{D^r(z)} \frac{2r\rho(z)}{|\zeta-\eta|}\Delta\phi(\eta).$$

Take R (depending on r) such that $D^r(z) \subset D^R(\zeta)$. From $\Delta\phi \simeq 1/\rho^2$ we deduce

$$\int_{D^r(z)} \frac{2r\rho(z)}{|\zeta-\eta|}\Delta\phi(\eta) \lesssim \frac{1}{\rho(\zeta)} \int_{D^R(\zeta)} \frac{dm(\eta)}{|\zeta-\eta|} \simeq 1.$$

Since $|\nabla(|f|e^{-\phi})| = |f' - 2f\partial\phi/\partial z|e^{-\phi}$, we have

$$|\nabla(fe^{-H_z})(z)| = |f'(z) - 2f(z)h_z'(z)| \simeq |\nabla(|f|e^{-\phi})(z)|e^{\phi(z)}. \tag{11}$$

On the other hand,

$$|\nabla(fe^{-H_z})(z)| \lesssim \Big|\int_{|z-\zeta|=\rho(z)} \frac{f(\zeta)e^{-H_z(\zeta)}}{(z-\zeta)^2}\, d\zeta\Big| \simeq \frac{1}{\rho^2(z)} \int_{|z-\zeta|=\rho(z)} |f(\zeta)|e^{-h_z(\zeta)}|d\zeta|.$$

From (a), for $|z-\zeta| = \rho(z)$

$$|f(\zeta)|e^{-\phi(\zeta)} \lesssim \frac{1}{\omega^p(z)}\Big(\int_{D^r(z)} |f|^p e^{-p\phi}\omega^p d\sigma\Big)^{1/p}.$$

By Lemma 13 we have then

$$|\nabla(fe^{-H_z})(z)| \lesssim \frac{1}{\omega(z)\rho(z)}\Big(\int_{D^r(z)} |f|^p e^{-p\phi}\omega^p d\sigma\Big)^{1/p} e^{\phi(z)},$$

which together with (11) concludes the proof.

(c) As (a), using the subharmonicity of $|fe^{-H_z}|^p$. ■

Lemma 19. *Let* $1 \leq p < \infty$ *and* $\omega \in \Omega_\phi$*. For any entire function* g *with* $g(\lambda) = 0$ *we have*

$$|g'(\lambda)|e^{-\phi(\lambda)} \lesssim \frac{1}{\omega(\lambda)\rho(\lambda)} \left(\int_{D(\lambda)} |g|^p e^{-p\phi}\omega^p d\sigma\right)^{1/p}.$$

Proof. Lemma 18(c) with $r = 1/2$ and $R = 1$ applied to the function $g(z)/(z-\lambda)$ yields

$$\begin{aligned} |g'(\lambda)|^p e^{-p\phi(\lambda)} &\lesssim \frac{1}{\omega^p(\lambda)} \int_{D(\lambda)\setminus D^{1/2}(\lambda)} \frac{|g(z)|^p}{|z-\lambda|^p} e^{-p\phi(z)} \omega^p d\sigma \\ &\lesssim \frac{1}{\omega^p(\lambda)\rho^p(\lambda)} \int_{D(\lambda)} |g(z)|^p e^{-p\phi(z)} \omega^p d\sigma. \end{aligned}$$

∎

3.2. **Hörmander type estimates.** This section is devoted to the proof of the $\bar\partial$-estimates of Theorem C in the introduction.

Theorem C. *Let ϕ be a subharmonic function such that $\Delta\phi$ is a doubling measure. For any $\omega \in \Omega_\phi$, there is a solution u to the equation $\bar\partial u = f$ such that $\|ue^{-\phi}\omega\|_{L^p(\mathbb{C})} \lesssim \|fe^{-\phi}\omega\rho\|_{L^p(\mathbb{C})}$ for any $1 \le p \le \infty$.*

Proof. By Lemma 18(b), there exists $r > 0$ such that $|P_\eta(z)| \gtrsim e^{\varepsilon(\phi(z)-\phi(\eta))}$ on $D^r(\eta)$, for all $\eta \in \mathbb{C}$. Take a sequence Λ such that $\{D^r(\lambda)\}_{\lambda\in\Lambda}$ covers $\mathbb{C}$ and the disks $\{D^{r/5}(\lambda)\}_{\lambda\in\Lambda}$ are pairwise disjoint, which exist by a standard covering Lemma, see [Mat95, Theorem 2.1]. Let $\{\chi_\lambda\} \subset \mathcal{C}_0^\infty$ be a partition of unity associated to $\{D^r(\lambda)\}_\lambda$.

Decompose the datum $f = \sum f_\lambda$, with $f_\lambda(z) = f(z)\chi_\lambda(z)$. By Theorem 17, for any λ there exists an entire function $m_\lambda(z) = P_\lambda(z)e^{-\phi(\lambda)}$ such that

$$|m_\lambda(z)| \lesssim e^{\phi(z)} \frac{1}{d_\phi^M(z,\lambda)+1} \frac{\omega(\lambda)}{\omega(z)}.$$

The radius r has been chosen so that $|m_\lambda(\zeta)| \gtrsim e^{\phi(\zeta)}$ if $\zeta \in D^r(\lambda)$. Define

$$u_\lambda(z) = m_\lambda(z) \frac{1}{\pi} \int_{D^r(\lambda)} \frac{f_\lambda(\zeta)/m_\lambda(\zeta)}{\zeta - z}\, dm(\zeta).$$

Clearly $\bar\partial u_\lambda = f_\lambda$, thus $u = \sum_{\lambda\in\Lambda} u_\lambda$ is as a solution to $\bar\partial u = f$. We must prove the size estimates. As we have used a linear operator to construct u from the datum f, we only need to check that $\|ue^{-\phi}\omega\|_{L^\infty} \lesssim \|fe^{-\phi}\omega\rho\|_{L^\infty}$ and $\|ue^{-\phi}\omega\|_{L^1} \lesssim \|fe^{-\phi}\omega\rho\|_{L^1}$. The estimates for $1 < p < \infty$ follow then by Marcinkiewicz interpolation theorem.

Assume that $z \in D^r(\lambda)$ and take $K, K' > 0$ such that $D^K(z) \subset D^{K'}(\lambda)$. Then

$$\begin{aligned} |u_\lambda(z)e^{-\phi(z)}\omega(z)| &\lesssim \int_{D^K(z)} \frac{|f(\zeta)|e^{-\phi(\zeta)}\omega(\zeta)\rho(\zeta)}{\rho(z)|\zeta - z|}\, dm(\zeta) \\ &\lesssim \int_{D^{K'}(\lambda)} \frac{|f(\zeta)|e^{-\phi(\zeta)}\omega(\zeta)\rho(\zeta)}{\rho(\lambda)|\zeta - z|}\, dm(\zeta). \end{aligned}$$

On the other hand, if $z \notin D^K(\lambda)$

$$\begin{aligned} |u_\lambda(z)e^{-\phi(z)}\omega(z)| &\lesssim d_\phi^{-M}(z,\lambda)\int_{D^r(\lambda)} \frac{|f(\zeta)|e^{-\phi(\zeta)}}{|\zeta - z|}\omega(\zeta)\,dm(\zeta) \\ &\lesssim \frac{d_\phi^{-M}(z,\lambda)}{\rho^2(\lambda)}\int_{D^r(\lambda)} |f(\zeta)|e^{-\phi(\zeta)}\omega(\zeta)\rho(\zeta)\,dm(\zeta). \end{aligned}$$

Therefore, applying Lemma 6

$$\|ue^{-\phi}\omega\|_{L^\infty} \lesssim \|fe^{-\phi}\omega\rho\|_{L^\infty} \sup_{z\in\mathbb{C}}\Big(\int_{D^K(z)} \frac{dm(\zeta)}{\rho(z)|z-\zeta|} + \sum_{\lambda: z\notin D^r(\lambda)} d_\phi^{-M}(z,\lambda)\Big) \lesssim \|fe^{-\phi}\omega\rho\|_{L^\infty}.$$

In the L^1 norm we get

$$\begin{aligned} \|ue^{-\phi}\omega\|_{L^1} \lesssim \sum_{\lambda\in\Lambda}\Big(&\int_{z\in D^r(\lambda)}\int_{\zeta\in D^{K'}(\lambda)} \frac{|f(\zeta)|e^{-\phi(\zeta)}\omega(\zeta)\rho(\zeta)}{\rho(\lambda)|\zeta - z|}\,dm(\zeta)\,dm(z) + \\ &\int_{z\notin D^r(\lambda)} \frac{d_\phi^{-M}(z,\lambda)}{\rho(\lambda)^2}\int_{D^r(\lambda)} |f(\zeta)|e^{-\phi(\zeta)}\omega(\zeta)\rho(\zeta)\,dm(\zeta)\,dm(z)\Big). \end{aligned}$$

Reversing the order of integration we immediately get $\|ue^{-\phi}\omega\|_{L^1} \lesssim \|fe^{-\phi}\omega\rho\|_{L^1}$. ∎

3.3. **Bergman kernel estimates.** Let $K_{\phi,\omega}(z,\zeta)$ denote the Bergman kernel for $\mathcal{F}^2_{\phi,\omega}$, i.e, for any $f \in \mathcal{F}^2_{\phi,\omega}$

$$f(z) = \langle K_{\phi,\omega}(z,\cdot), f\rangle = \int_{\mathbb{C}} K_{\phi,\omega}(z,\zeta)f(\zeta)e^{-2\phi(\zeta)}\omega^2(\zeta)d\sigma(\zeta).$$

By definition

$$K_{\phi,\omega}(z,z) = \int_{\mathbb{C}} |K_{\phi,\omega}(z,\zeta)|^2 e^{-2\phi(\zeta)}\omega^2(\zeta)d\sigma(\zeta).$$

Lemma 20. *There exists $C > 0$ such that*

$$C^{-1}(e^{\phi(z)}/\omega(z))^2 \le K_{\phi,\omega}(z,z) \le C(e^{\phi(z)}/\omega(z))^2 \qquad z \in \mathbb{C}.$$

Proof. We use the identity

$$\sqrt{K_{\phi,\omega}(z,z)} = \sup\{|f(z)| : f \in \mathcal{F}^2_{\phi,\omega}\ ,\ \|f\|_{\mathcal{F}^2_{\phi,\omega}} \le 1\}.$$

The estimate $\sqrt{K_{\phi,\omega}(z,z)} \lesssim e^{\phi(z)}/\omega(z)$ is immediate from Lemma 18(a). In order to prove the reverse estimate we construct $f \in \mathcal{F}^2_{\phi,\omega}$ with $\|f\|_{\mathcal{F}^2_{\phi,\omega}} \le 1$ and $|f(z)| \ge Ce^{\phi(z)}/\omega(z)$, for some constant C independent of z.

By Theorem 17, for every $m \in \mathbb{N}$ there exists P_z entire such that

$$|P_z(\zeta)| \le Ce^{\phi(\zeta)-\phi(z)}\frac{\omega(z)}{\omega(\zeta)}\frac{1}{1 + d_\phi^m(z,\zeta)},$$

with C independent of z. Define $f_z(\zeta) = c_0\, e^{\phi(z)}/\omega(z)\; P_z(\zeta)$, where c_0 is a positive constant to be chosen later. Now $f_z(z) = c_0 e^{\phi(z)}/\omega(z)$ and

$$|f_z(\zeta)|^2 e^{-2\phi(\zeta)}\omega^2(\zeta)\rho^{-2}(\zeta) \leq \frac{c_0\; C}{1+d_\phi^m(z,\zeta)}\Delta\phi(\zeta),$$

hence by Lemma 5(b) there exist c_0 and C independent of z so that $\|f_z\|_{\mathcal{F}^2_{\phi,\omega}} \leq 1$. ■

Remark 5. This argument and Lemma 18(a) show that for any $p \in [1,\infty]$

$$\sup\{|f(z)| : f \in \mathcal{F}^p_{\phi,\omega}\ ,\ \|f\|_{\mathcal{F}^p_{\phi,\omega}} \leq 1\} \simeq e^{\phi(z)}/\omega(z).$$

3.4. **Scaled translations and invariance.** In this section we introduce the scaled translation and its main properties.

Given ϕ consider the class W_ϕ of subharmonic functions ψ such that

(i) $\Delta\psi$ doubling with $C_{\Delta\psi} \leq C_{\Delta\phi}$.
(ii) $\int_{D_\phi(0)} \Delta\psi \simeq 1$.
(iii) $\psi(0) = 0$.

An important property of W_ϕ is that there exists η such that $\Delta\psi(z) \lesssim |z|^{2\eta}$ for all regular $\psi \in W_\phi$. This is a consequence of (5) and the fact that $\Delta\psi \simeq 1/\rho_\psi^2$.

Fix $q > 2\eta + 1$ and consider the kernel

$$\kappa(z,\zeta) := \frac{1}{2\pi}\left[\log|1-\frac{z}{\zeta}| - Re(P_q(\frac{z}{\zeta}))\chi_{\mathbb{C}\setminus D(0,1)}(\zeta)\right],$$

where P_q is the Taylor polynomial of degree q of $\log(1+x)$ around $x = 0$, and its associated integral operator

$$K[f](z) = \int_{\mathbb{C}} \kappa(z,\zeta) f(\zeta)\; dm(\zeta).$$

This operator solves the Poisson equation, that is $\Delta K[f] = f$.

For every $x \in \mathbb{C}$, consider the scaled translation

$$\tau_x(z) = x + z\rho_\phi(x),$$

the associated subharmonic function

$$\phi_x(z) = K[\Delta(\phi\circ\tau_x)](z) - K[\Delta(\phi\circ\tau_x)](0),$$

and the associated weight

$$\omega_x(z) = \omega(\tau_x(z))/\omega(x).$$

Define also $\mathsf{h}_x := \phi\circ\tau_x - \phi_x$. It is clear that h_x is harmonic. Take then H_x holomorphic having h_x as real part and consider the scaled translation operator

$$T_x^{\phi,\omega} f(z) = f(\tau_x(z))e^{-H_x(z)}\omega(x).$$

Lemma 21. *For every $x \in \mathbb{C}$,*

(a) *The subharmonic function* ϕ_x *belongs to* W_ϕ*, and the weight* ω_x *satisfies* $\omega_x(0) = 1$ *and* $\omega_x \in \Omega_{\phi_x}$.
(b) $T_x^{\phi,\omega}$ *is an isometry from* $\mathcal{F}^p_{\phi,\omega}$ *to* $\mathcal{F}^p_{\phi_x,\omega_x}$*, for* $1 \le p \le \infty$.

Proof. Note first that from the identity

$$1 = \int_{D_{\phi_x}(z)} \Delta\phi_x = \int_{D_{\phi_x}(z)} \rho^2_\phi(x)\Delta\phi(\tau_x(\zeta)) = \int_{D(\tau_x(z),\rho_{\phi_x}(z)\rho_\phi(x))} \Delta\phi$$

it follows that

$$\rho_\phi(\tau_x(z)) = \rho_{\phi_x}(z)\rho_\phi(x). \tag{12}$$

This implies that the mapping τ_x is actually an isometry between $\mathbb{C}$ endowed with the distance d_{ϕ_x} and $\mathbb{C}$ with d_ϕ, that is

$$d_{\phi_x}(z,\zeta) = d_\phi(\tau_x(z),\tau_x(\zeta)) \quad \forall z,\zeta \in \mathbb{C}. \tag{13}$$

(a) By definition $\phi_x(0) = 0$, and by (12) $\rho_{\phi_x}(0) = 1$. This gives properties (ii) and (iii) of W_ϕ.

It is also clear that $\Delta\phi_x$ is doubling and $C_{\Delta\phi_x} = C_{\Delta\phi}$, since for any $a \in \mathbb{C}$ and $r > 0$

$$\int_{D(a,2r)} \Delta\phi_x = \int_{D(\tau_x(a),2r\rho_\phi(x))} \Delta\phi \le C_{\Delta\phi}\int_{D(\tau_x(a),r\rho_\phi(x))} \Delta\phi \le C_{\Delta\phi}\int_{D(a,r)} \Delta\phi_x.$$

That $\omega_x(0) = 1$ and $\omega_x(\zeta)/\omega_x(z) = \omega(\tau_x(z))/\omega(\tau_x(\zeta))$ follows from the definition. This and (13) imply that $\omega_x \in \Omega_{\phi_x}$.

(b) For $p < \infty$ we use the change of variable $\zeta = \tau_x(z)$ and (12):

$$\begin{aligned}\int_{\mathbb{C}} |T_x^{\phi,\omega}(f)(z)|^p e^{-p\phi_x(z)}\omega_x^p(z)\frac{dm(z)}{\rho^2_{\phi_x}(z)} \\ = \int_{\mathbb{C}} |f(\tau_x(z))|^p e^{-p\phi(\tau_x(z))}\omega^p(x)\Big(\frac{\omega(\tau_x(z))}{\omega(x)}\Big)^p\Big(\frac{\rho_\phi(\tau_x(z))}{\rho_\phi(x)}\Big)^{-2} dm(z) \\ = \int_{\mathbb{C}} |f(\zeta)|^p e^{-p\phi(\zeta)}\omega^p(\zeta)\frac{dm(\zeta)}{\rho^2_\phi(\zeta)}.\end{aligned}$$

The case $p = \infty$ is straightforward from (12). ∎

Given a sequence Λ and $x \in \mathbb{C}$ let

$$\Lambda_x := (\tau_x)^{-1}(\Lambda).$$

Given a sequence Λ and $z \in \mathbb{C}$, denote $n_\Lambda(z,r) = \#(\Lambda \cap \overline{D(z,r)})$, for any $r > 0$.

Lemma 22. *Let* Λ *be a sequence in* $\mathbb{C}$.

(a) Λ *is* ρ*-separated if and only if* Λ_x *is* ρ_{ϕ_x}*-separated.*
(b) $\Lambda \in \operatorname{Int}\mathcal{F}^p_{\phi,\omega}$ *if and only if* $\Lambda_x \in \operatorname{Int}\mathcal{F}^p_{\phi_x,\omega_x}$*. Similarly,* $\Lambda \in \operatorname{Samp}\mathcal{F}^p_{\phi,\omega}$ *if and only if* $\Lambda_x \in \operatorname{Samp}\mathcal{F}^p_{\phi_x,\omega_x}$*. Furthermore, the interpolation and sampling constants remain the same.*
(c) *The densities are stable:* $\mathcal{D}^+_{\Delta\phi}(\Lambda) = \mathcal{D}^+_{\Delta\phi_x}(\Lambda_x)$*, and* $\mathcal{D}^-_{\Delta\phi}(\Lambda) = \mathcal{D}^-_{\Delta\phi_x}(\Lambda_x)$.

Proof. (a) is an immediate consequence of (12).

(b) is a consequence of Lemma 21 and the identity $\|f|\Lambda\|_{\ell^p_{\phi,\omega}(\Lambda)} = \|T_x^{\phi,\omega} f|\Lambda_x\|_{\ell^p_{\phi_x,\omega_x}(\Lambda_x)}$.

(c) Define

$$\mathcal{D}_{\Delta\phi}(z,r,\Lambda) = \frac{n_\Lambda(z, r\rho(z))}{\int_{D^r_\phi(z)} \Delta\phi}. \tag{14}$$

By a change of variables, it is clear that

$$\mathcal{D}_{\Delta\phi}(z,r,\Lambda) = \mathcal{D}_{\Delta\phi_x}((\tau_x)^{-1}(z), r, \Lambda_x).$$

Taking the supremum over $z \in \mathbb{C}$ and passing to the limsup we get the result for the upper density. The lower density is dealt with similarly. ∎

3.5. **Weak limits.** In this section we study weak limits of sequences Λ and their properties.

Definition 7. A sequence of closed sets Q_j *converges strongly* to Q, denoted $Q_j \to Q$ if $[Q, Q_j] \to 0$; here $[Q, R]$ denotes the Fréchet distance between Q and R. We say that Q_j *converges compactwise* to Q, denoted $Q_j \rightharpoonup Q$, if for every compact set K we have $(Q_j \cap K) \cup \partial K \to (Q \cap K) \cup \partial K$.

Definition 8. A set Λ^* is a *weak limit* of Λ if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that $\Lambda_{x_n} \rightharpoonup \Lambda^*$.

Given a ρ-separated sequence Λ, and a sequence $\{x_n\}_{n\in\mathbb{N}}$ it is always possible to extract a subsequence of $\Lambda_{x_{n_j}}$ such that $\Lambda_{x_{n_j}} \rightharpoonup \Lambda^*$ for some Λ^*. We need also a normal family argument for the translated weights that define the space.

Lemma 23. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. There exist a subharmonic function ϕ^*, a weight $\omega^* \in \Omega_{\phi^*}$ and a subsequence $\{x_{n_k}\}_k$ such that $\{\phi_{x_{n_k}}\}_k$, $\{\omega_{x_{n_k}}\}_k$ and $\{\Delta\phi_{x_{n_k}}\}_k$ converge uniformly on compact sets to ϕ^*, ω^* and $\Delta\phi^*$ respectively. Furthermore, $\Delta\phi^*$ is a doubling measure and $C_{\Delta\phi^*} \le C_{\Delta\phi}$.*

Proof. Take η and $q > 2\eta + 1$ as in the definition of the kernel κ (see previous section). Denote $\mu_n = \Delta\phi_{x_n}$.

Since $|\nabla\mu_n| \lesssim \rho_{\phi_{x_n}}^{-3}$ (Theorem 14) and $\rho_{\phi_{x_n}}(0) = 1$, for any compact set K there exits $C_K > 0$ such that $|\nabla\mu_n(z)| \le C_K$. By the Arzelà-Ascoli theorem, we can extract a subsequence $\{\mu_{n_k}\}_k$ converging uniformly on compact sets of $\mathbb{C}$ to a function μ^*. It follows immediately that the measure with density μ^* is doubling and $C_{\mu^*} \le C_{\mu_n} = C_{\Delta\phi}$. Furthermore, this implies that $\rho_{\phi_{x_n}} \to \rho^*$ uniformly on compacts.

Let now $\phi^* = K[\mu^*] - K[\mu^*](0)$, and denote $\phi_k := \phi_{x_{n_k}}$, $\mu_k := \mu_{n_k}$. We will show that $\{\phi_k\}_k$ converges uniformly on compact sets to ϕ^*.

By definition $\phi_p(z) = K[\mu_p](z) - K[\mu_p](0)$, thus we only have to prove that $K[\mu_p]$ converges uniformly on compacts set to $K[\mu^*]$. Take $z \in D(0,R)$ and $t > R$. Then

$$|K[\mu_p](z) - K[\mu^*](z)| \le \left|\int_{\mathbb{C}\setminus D(0,t)} \kappa(z,\zeta)(\mu_p(\zeta) - \mu^*(\zeta))dm(\zeta)\right| + \left|\int_{D(0,t)} \kappa(z,\zeta)(\mu_p(\zeta) - \mu^*(\zeta))dm(\zeta)\right|.$$

Let I_1 be the first integral. By construction of κ we have

$$|\kappa(z,\zeta)| \lesssim \left(\frac{R}{|\zeta|}\right)^q.$$

Also, $(|\mu_p(\zeta)| + |\mu^*(\zeta)|)dm(\zeta)$ is a doubling measure with doubling constant less than $C_{\Delta\phi}$. By (5) $|\mu_p(\zeta)| + |\mu^*(\zeta)| \lesssim |\zeta|^{2\eta}$, and therefore

$$I_1 \lesssim \int_{|\zeta|>t} \left(\frac{R}{|\zeta|}\right)^q |\zeta|^{2\eta} dm(\zeta).$$

This is smaller than ε for t big enough.

Let I_2 be the second integral in the estimate above. We have

$$I_2 \lesssim \int_{D(0,1)} \left|\log\left|\frac{z-\zeta}{\zeta}\right|\right| |\mu_p(\zeta) - \mu^*(\zeta)| dm(\zeta) + \int_{D(0,t)\setminus D(0,1)} |P_q(\frac{z}{\zeta})| |\mu_p(\zeta) - \mu^*(\zeta)| dm(\zeta)$$

For all $z \in D(0,R)$ and $\zeta \in D(0,t) \setminus D(0,1)$ we have $|P_q(z/\zeta)| \le C(R,t)$, hence the uniform convergence of μ_p implies that for p big enough the second integral here is smaller than ε. It remains to prove the convergence of the first term. Take $C(t)$ such that$\int_{D(0,t)} |\log|z - \zeta/\zeta||dm(\zeta) \le C(t)$ and choose p big enough so that $|\mu_p(\zeta) - \mu^*(\zeta)| \le \varepsilon/C(t)$ uniformly on $D(0,t)$. Then the estimate follows.

We know that the sequence of distance functions $d_{\phi_{x_n}}$ has a subsequence converging to d_{ϕ^*} uniformly on compact sets of $\mathbb{C} \times \mathbb{C}$, because the $\rho_{x_{n_k}}$ converge uniformly. By construction $\omega_{x_n}(0) = 1$. On the other hand, the definition of flat weight implies that they are equibounded on any compact. Moreover, the regularity given by (8) makes them equicontinous on compact sets. We can thus extract again a convergent subsequence. ∎

Corollary 24. *Given a subharmonic function ϕ with doubling Laplacian, Λ a ρ-separated subsequence, $\omega \in \Omega_\phi$ and $\{z_n\}_{n\in\mathbb{N}}$ a sequence of complex numbers, there exist a subharmonic function ϕ^*, a ρ_{ϕ^*}-separated sequence Λ^*, a weight $\omega^* \in \Omega_{\phi^*}$ and a subsequence $\{x_n\}_{n\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $\Lambda_{x_n} \rightharpoonup \Lambda^*$, and $\phi_{x_n} \to \phi^*$, $\omega_{x_n} \to \omega^*$ and $\Delta\phi_{x_n} \to \Delta\phi^*$ uniformly on compact sets.*

We will write $(\Lambda_{x_n}, \phi_{x_n}, \omega_{x_n}) \to (\Lambda^*, \phi^*, \omega^*)$. The set of all such weak limits will be denoted by $W(\Lambda, \phi, \omega)$.

Let us prove now the stability of the upper and lower densities with respect to weak limits.

Lemma 25. *Let Λ be a ρ-separated sequence, $\{x_n\}_n \subset \mathbb{C}$, and assume that $(\Lambda_{x_n}, \phi_{x_n}, \omega_{x_n}) \to (\Lambda^*, \phi^*, \omega^*)$. Then*

(a) $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$ *implies* $\mathcal{D}^+_{\Delta\phi^*}(\Lambda^*) < 1/2\pi$.
(b) $\mathcal{D}^-_{\Delta\phi}(\Lambda) > 1/2\pi$ *implies* $\mathcal{D}^-_{\Delta\phi^*}(\Lambda^*) > 1/2\pi$.

Proof. Denote $\Lambda_n = \Lambda_{x_n}, \phi_n = \phi_{x_n}$ and $\rho_n = \rho_{x_n}$. By hypothesis $\{\Delta\phi_n\}_n \to \Delta\phi^*$ uniformly on compact sets, and therefore $\{\rho_n\}_n \to \rho^*$ also uniformly on compact sets. Thus, for any $\epsilon(r) > 0$,

$$\frac{n_{\Lambda^*}(z,(r-\epsilon(r))\rho_{\phi^*}(z))}{\int_{D^r_{\phi^*}(z)} \Delta\phi^*} \le \liminf_{n\to\infty} \frac{n_{\Lambda_n}(z, r\rho_n(z))}{\int_{D^r_{\phi_n}(z)} \Delta\phi_n} \le$$
$$\le \limsup_{n\to\infty} \frac{n_{\Lambda_n}(z, r\rho_n(z))}{\int_{D^r_{\phi_n}(z)} \Delta\phi_n} \le \frac{n_{\Lambda^*}(z,(r+\epsilon(r))\rho_{\phi^*}(z))}{\int_{D^r_{\phi^*}(z)} \Delta\phi^*}.$$

(a) Since $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$, there exist $\varepsilon, R_0 > 0$ such that, if $w = \tau^{-1}_{x_n}(z)$

$$\frac{n_{\Lambda_n}(w, r\rho_n(w))}{\int_{D^r_{\phi_n}(w)} \Delta\phi_n} = \frac{n_\Lambda(z, r\rho(z))}{\int_{D^r_\phi(z)} \Delta\phi} \le 1/2\pi - \varepsilon \qquad \forall r > R_0\ ,\ \forall n \in \mathbb{N}\ ,\ \forall w \in \mathbb{C}.$$

Taking limits as $n \to \infty$ and picking $\epsilon(r)$ so that $\epsilon(r)/r \to 0$ we see, using Lemma 11, that $\mathcal{D}^+_{\Delta\phi^*}(\Lambda^*) < 1/2\pi$.

(b) is proved similarly. ∎

4. Preliminary properties of sampling and interpolating sequences

This section is devoted to prove auxiliary results on interpolating and sampling sequences. A main result is that there do not exist sequences which are simultaneously sampling and interpolating. We also prove some results on inclusions between spaces of sampling and interpolating sequences for various weights.

An easy consequence of Lemma 18 is that we only need to deal with ρ-separated sequences.

Lemma 26. *Let* $\Lambda \subset \mathbb{C}$.

(a) *If* $\Lambda \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$, *then* Λ *is* ρ*-separated.*
(b) *If* $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$, *there exists a* ρ*-separated subsequence* $\Lambda' \subset \Lambda$ *such that* $\Lambda' \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$.
(c) *If* $p < \infty$ *and* $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$, *then* Λ *is a finite union of* ρ*-separated sequences.*
(d) *Let* $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$ *be* ρ*-separated. There exists* $r > 0$ *such that* $\mathbb{C} = \cup_{\lambda\in\Lambda} D^r(\lambda)$.

Proof. (a) Assume that $\lambda, \mu \in \Lambda$ with $|\lambda - \mu| \le \rho(\lambda)$ and take $f \in \mathcal{F}^p_{\phi,\omega}$ such that $f(\lambda) = e^{\phi(\lambda)}/\omega(\lambda)$, $f(\mu) = 0$ and $\|f\|_{\mathcal{F}^p_{\phi,\omega}} \lesssim 1$. Then

$$\omega^{-1}(\lambda) = \Big| |f(\lambda)| e^{\phi(\lambda)} - |f(\mu)| e^{-\phi(\mu)} \Big| \lesssim |\nabla(|f|e^{-\phi})(\zeta)||\mu - \lambda|.$$

The result follows then from Lemma 18(b).

(b) As in the proof of [Beu89, Theorem 2, p. 344], using here Lemma 18(b) instead of Bernstein's theorem, we get

$$\left| \frac{1}{L^p_{\phi,\omega}(\Lambda)} - \frac{1}{L^p_{\phi,\omega}(\Lambda')} \right| \leq C[\Lambda, \Lambda'].$$

(c) It is enough to show that there exists $r > 0$ and M such that $\#(D^\eta(z) \cap \Lambda) \leq M$ for all $z \in \mathbb{C}$. To this end, consider the function $f_z(\zeta) = e^\phi(z)/\omega(z) P_z(\zeta)$, where P_z is given by Theorem 17 (with $\varepsilon = 1$ and $\omega = 0$). We have $\|f_z\|_{\mathcal{F}^p_{\phi,\omega}} \leq C$, and for r small enough $|f_z(\zeta)| \gtrsim e^\phi(\zeta)/\omega(\zeta)$ in $D^r(z)$. So the left sampling inequality (see (2)) yields

$$\#(D^r(z) \cap \Lambda) \leq \|f_z|\Lambda\|_{\ell^p_{\phi,\omega}(\Lambda)} \leq C L^p_{\phi,\omega}(\Lambda).$$

(d) It is enough to see that for R big enough $\Lambda \cap D^R(z) \neq \emptyset$ for all $z \in \mathbb{C}$.

Take f_z as in (c). Let $\varepsilon > 0$ be the ρ-separation of Λ. Since

$$|f_z(\zeta)|^p e^{-p\phi(\zeta)} \omega^p(\zeta) \rho^{-2}(\zeta) \lesssim \frac{\Delta\phi(\zeta)}{1 + d^m_\phi(z,\zeta)},$$

Lemma 18(a) and Lemma 9 lead to

$$\begin{aligned} \sum_{\lambda \notin D^R(z)} \omega^p(\lambda) |f_z(\lambda)|^p e^{-p\phi(\lambda)} &\lesssim \sum_{\lambda \notin D^R(z)} \int_{D^\varepsilon(\lambda)} \frac{\Delta\phi(\zeta)}{1 + d^m_\phi(z,\zeta)} \\ &\lesssim \int_{\zeta \notin D^{R-\epsilon(R)}(z)} \frac{\Delta\phi(\zeta)}{1 + d^m_\phi(z,\zeta)} \end{aligned}$$

According to Remark 2 this tends to 0 uniformly in z as R goes to ∞. Thus, for R big enough the sampling inequality gives

$$1 \leq C \sum_{\lambda \in \Lambda \cap D^R(z)} \omega^p(\lambda) |f_z(\lambda)|^p e^{-p\phi(\lambda)}.$$

In particular $\Lambda \cap D^R(z) \neq \emptyset$, as desired. ■

4.1. **Weak limits and interpolating and sampling sequences.** In this section τ^ϕ_x will denote the scaled translation associated to the weight ϕ, as described in Section 3.4. The main result is as follows.

Proposition 27. *Let ϕ a subharmonic function with doubling Laplacian, $\omega \in \Omega_\phi$ and Λ be a ρ-separated sequence. Assume $(\Lambda^*, \phi^*, \omega^*) \in W(\Lambda, \phi, \omega)$.*

(a) *If $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$ then $\Lambda^* \in \operatorname{Samp} \mathcal{F}^p_{\phi^*,\omega^*}$.*

(b) *If $\Lambda \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$ then $\Lambda^* \in \operatorname{Int} \mathcal{F}^p_{\phi^*,\omega^*}$.*

Proof. (a) We argue by contradiction. Otherwise there exist $\varepsilon_n > 0$ decreasing to zero and functions $f_n \in \mathcal{F}^p_{\phi^*,\omega^*}$ such that $\|f_n\|_{\mathcal{F}^p_{\phi^*,\omega^*}} = 1$ and $\|f_n|\Lambda^*\|_{\ell^p_{\phi^*,\omega^*}(\Lambda^*)} \leq \varepsilon_n$.

By Corollary 24 there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathbb{C}$ such that $(\phi_j, \omega_j, \Lambda_j) \to (\phi^*, \omega^*, \Lambda^*)$, where we denote $\Lambda_j := \Lambda_{x_j}$, $\omega_j := \omega_{x_j}$ and $\phi_j := \phi_{x_j}$.

For every n consider R_n big enough so that if $D_n := D_{\phi^*}^{R_n}(0)$ then $\|f_n|D_n\|_{\mathcal{F}^p_{\phi^*,\omega^*}} \geq 1 - \varepsilon_n$. Set $\widetilde{D}_n := D_{\phi^*}^{R_n^2}(0)$.

We claim that there exists a smooth cut-off function $\mathcal{X}_n$ such that $\mathcal{X}_n(\zeta) = 1$ in D_n, $\mathcal{X}_n(\zeta) = 0$ in $\mathbb{C} \setminus \widetilde{D}_n$ and $|\bar\partial \mathcal{X}_n| \leq \varepsilon_n/\rho_{\phi^*}$. To see this start with a smooth $\mathcal{X}_n$ depending linearly on $|\zeta|$ on $R_n \leq |\zeta| \leq R_n^2$. Then

$$|\bar\partial \mathcal{X}_n(\zeta)| \leq \frac{1}{\rho_{\phi^*}(0)(R_n^2 - R_n)}.$$

By Lemma 2 $\rho_{\phi^*}(\zeta)/\rho_{\phi^*}(0) \leq R_n^{2(1-\delta)}$ for some $\delta \in (0,1)$. Thus, if R_n is big enough

$$|\bar\partial \mathcal{X}_n(\zeta)| \leq \frac{R_n^{2(1-\delta)}}{\rho_{\phi^*}(\zeta)(R_n^2 - R_n)} \leq \frac{\varepsilon_n}{\rho_{\phi^*}(\zeta)}.$$

Take now j_n big enough so that $\rho_{\phi_{j_n}}/\rho_{\phi^*} \leq 2$ on $\widetilde{D}_n$ and

$$\begin{aligned} \left| \|f_n|\widetilde{D}_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} - \|f_n|\widetilde{D}_n\|_{\mathcal{F}^p_{\phi^*,\omega^*}} \right| &\leq \varepsilon_n, \\ \left| \|f_n|\Lambda_{j_n} \cap \widetilde{D}_n\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}} - \|f_n|\Lambda^* \cap \widetilde{D}_n\|_{\ell^p_{\phi^*,\omega^*}} \right| &\leq \varepsilon_n. \end{aligned}$$

Define $g_n = f_n \mathcal{X}_n$. Then $\bar\partial g_n$ is supported on $C_n := \{R_n \leq |\zeta| \leq R_n^2\}$ and $|\bar\partial g_n(\zeta)| \leq \varepsilon_n |f_n(\zeta)|/\rho_{\phi^*}(\zeta)$, so by Theorem 1 there exists u_n solution to $\bar\partial u_n = \bar\partial g_n$ with

$$\|u_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \lesssim \|\bar\partial g_n \rho_{\phi_{j_n}}\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \lesssim \varepsilon_n \|f_n|\widetilde{D}_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \lesssim \varepsilon_n.$$

The function $G_n = g_n - u_n$ is holomorphic and satisfies

$$\|G_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \geq \|f_n|D_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} - \|u_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \geq 1 - C\varepsilon_n \simeq 1.$$

We will check now that $G_n|\Lambda_{j_n}$ is small. Split Λ_{j_n} into $\widetilde{\Lambda}_{j_n} = \Lambda_{j_n} \cap \{D_n \cup (\mathbb{C} \setminus \widetilde{D}_n))\}$ and $\widehat{\Lambda}_{j_n} = \Lambda_{j_n} \setminus \widetilde{\Lambda}_{j_n}$. On the one hand

$$\|G_n|\widetilde{\Lambda}_{j_n}\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}(\widetilde{\Lambda}_{j_n})} \leq \|f_n|\widetilde{D}_n \cap \widetilde{\Lambda}_{j_n}\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}(\widetilde{\Lambda}_{j_n})} + \|u_n|\widetilde{\Lambda}_{j_n}\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}(\widetilde{\Lambda}_{j_n})}.$$

From $\|u_n|\widetilde{\Lambda}_{j_n}\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}(\widetilde{\Lambda}_{j_n})} \leq \|u_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \leq \varepsilon_n$ (by Lemma 18 for the case $p < \infty$, since u is holomorphic in $D_n \cup (\mathbb{C} \setminus \widetilde{D}_n$)) we deduce that $\|G_n|\widetilde{\Lambda}_{j_n}\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}(\widetilde{\Lambda}_{j_n})} \lesssim \varepsilon_n$. On the other hand

$$\|G_n|\widehat{\Lambda}_{j_n}\|_{\ell^p_{\phi_{j_n},\omega_{j_n}}(\widehat{\Lambda}_{j_n})} \lesssim \|G_n|(\widetilde{D}_n \setminus D_n)\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \lesssim \||f_n| + |u_n||(\widetilde{D}_n \setminus D_n)\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \lesssim \varepsilon_n.$$

This together with the above and the fact that the sampling constants of Λ and Λ_{j_n} coincide (Lemma 22(b)) leads to contradiction.

(b) Assume that $\Lambda^* = \{\lambda_k^*\}_k$, and let $v \in \ell^p_{\phi,\omega}(\Lambda^*)$ with $\|v\|_{\ell^p_{\phi,\omega}(\Lambda^*)} \leq 1$. Let also $\Lambda_j = \{\lambda_k^j\}_k$ be such that $\Lambda_j \to \Lambda^*$ uniformly on compact sets. For ε_n decreasing to zero and R_n big enough (to be chosen later) there exists j_n such that $\|v\|_{\ell^p_{\phi_{j_n}}(\Lambda_{j_n} \cap D^{R_n}_{\phi^*}(0))} \leq 2$ and

$$\frac{e^{-\phi^*}\omega^*\rho_{\phi^*}^{-2/p}}{e^{-\phi_{j_n}}\omega^*_{j_n}\rho_{\phi_{j_n}}^{-2/p}} \leq 2 \qquad \text{on} \qquad D^{R_n^2}_{\phi^*}(0). \tag{15}$$

Since the interpolation constant $M(\Lambda_j)$ does not depend on j there exist $f_n \in \mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}$ with $\|f_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \leq 2M(\Lambda)$ and

$$f_n(\lambda_k^{j_n}) = \begin{cases} v_k & \text{if } \lambda_k^{j_n} \in D^{R_n}_{\phi^*}(0) \\ 0 & \text{otherwise.} \end{cases}$$

We will now use the same technique as in (a) to modify f_n so that it falls in $\mathcal{F}^p_{\phi^*,\omega^*}$. Take the cut-off function $\mathcal{X}_n$ constructed above, define $g_n = f_n\mathcal{X}_n$ and consider a solution u_n to $\bar\partial u_n = f_n\bar\partial(\mathcal{X}_n)$ such that:

$$\begin{aligned} \|u_n\|_{\mathcal{F}^p_{\phi^*,\omega^*}} &\lesssim \|f_n\bar\partial(\mathcal{X}_n)\rho_{\phi^*}\|_{\mathcal{F}^p_{\phi^*,\omega^*}} \lesssim \varepsilon_n \|f_n|D^{R_n^2}_{\phi^*}(0)\|_{\mathcal{F}^p_{\phi^*,\omega^*}} \lesssim \varepsilon_n\|f_n\|_{\mathcal{F}^p_{\phi_{j_n},\omega_{j_n}}} \lesssim \varepsilon_n, \\ \|u_n\|_{\mathcal{F}^\infty_{\phi^*,\omega^*}} &\lesssim \|f_n\bar\partial(\mathcal{X}_n)\rho_{\phi^*}\|_{\mathcal{F}^\infty_{\phi^*,\omega^*}} \lesssim \varepsilon_n\|f_n\|_{\mathcal{F}^p_{\phi^*,\omega^*}} \lesssim \varepsilon_n. \end{aligned}$$

According to Theorem C and (15) such a solution always exists.

The entire function $G_n = f_n\bar\partial(\mathcal{X}_n) - u_n$ is $\mathcal{F}^p_{\phi^*,\omega^*}$ and $\|G_n\|_{\mathcal{F}^p_{\phi^*,\omega^*}} \leq CM$. By Montel's theorem we may assume that G_n converges to a function $G \in \mathcal{F}^p_{\phi^*,\omega^*}$. Notice that $G_n(\lambda_k^{j_n}) = v_k - u_n(\lambda_k^{j_n})$ for $\lambda_k^{j_n} \in D^{R_n}_{\phi^*}(0)$, and by the L^∞ estimates, $|u_n(\lambda_k^{j_n})|$ tends to zero as n goes to infinity. Therefore G interpolates v. ∎

Lemma 28. *Suppose that for every weak limit $(\Lambda^*,\phi^*,\omega^*) \in W(\Lambda,\phi,\omega)$ the sequence Λ^* is a uniqueness set for $\mathcal{F}^\infty_{\phi^*,\omega^*}$. Then there exists $\varepsilon > 0$ such that Λ is sampling for $\mathcal{F}^\infty_{(1+\varepsilon)\phi,\omega}$.*

Proof. If this is not the case there exist $\varepsilon_n > 0$ decreasing to 0, $f_n \in \mathcal{F}^\infty_{(1+\varepsilon_n)\phi,\omega}$ and $z_n \in \mathbb{C}$ such that $|f_n(z_n)|e^{-(1+\varepsilon_n)\phi(z_n)}\omega(z_n) = 1$, $\|f_n\|_{\mathcal{F}^\infty_{(1+\varepsilon_n)\phi,\omega}} \leq 2$ and $\|f_n|\Lambda\|_{\ell^\infty_{(1+\varepsilon_n)\phi,\omega}(\Lambda)} \leq \varepsilon_n$.

Denote $\psi_n = (1+\varepsilon_n)\phi$. Let $\Lambda_n = (\tau^{\psi_n}_{z_n})^{-1}(\Lambda)$, $\omega_n = \omega_{z_n}$ and $g_n = T^{\psi_n,\omega_n}_{z_n} f_n$. Then, denoting $\psi_{n,z_n} = (1+\varepsilon_n)\phi_{z_n}$, we have $|g_n(0)| = 1$ and $\|g_n|\Lambda_n\|_{\ell^\infty_{\psi_{n,z_n},\omega_n}(\Lambda_n)} = \|f_n|\Lambda\|_{\ell^\infty_{\psi_n,\omega_n}(\Lambda)} \leq \varepsilon_n$. Taking a subsequence if necessary, we can assume that Λ_n converges weakly to Λ^*, $\psi_{n,z_n} \to \phi^*$, $\omega_n \to \omega^*$ uniformly on compact sets and $g_n \to g^* \in \mathcal{F}^\infty_{\phi^*,\omega^*}$ (by Montel's Theorem). So g^* vanishes on Λ^* and $|g^*(0)| = 1$, contradicting the fact that Λ^* is a uniqueness sequence. ∎

Corollary 29. *Let ϕ a subharmonic function with doubling Laplacian, let $\omega \in \Omega_\phi$ and let Λ be a ρ-separated sequence. The sequence Λ is in $\operatorname{Samp}\mathcal{F}^\infty_{\phi,\omega}$ if and only if for all weak limit $(\Lambda^*,\phi^*,\omega^*) \in W(\Lambda,\phi,\omega)$, the sequence Λ^* is a uniqueness set for $\mathcal{F}^\infty_{\phi^*,\omega^*}$.*

4.2. **Non-existence of simultaneously sampling and interpolating sequences.** An important result in the proof of Theorems A and B is the following.

Theorem 30. *There is no sequence Λ both sampling and interpolating for $\mathcal{F}^p_{\phi,\omega}$, $p \in [1,\infty]$.*

Proof. Assume that such sequence Λ exists. We claim that

$$\sup_{\lambda^*\in\Lambda} \sum_{\lambda\in\Lambda\setminus\lambda^*} \frac{\rho(\lambda)\rho(\lambda^*)}{|\lambda-\lambda^*|^2} < \infty. \tag{16}$$

Let $p \in [1,\infty)$. Given any $\lambda^* \in \Lambda$ take a function g such that $g(\lambda^*) = 1$, $g(\lambda) = 0$ for $\lambda \neq \lambda^*$ and $\|g\|_{\mathcal{F}^p_{\phi,\omega}} \lesssim e^{-p\phi(\lambda^*)}\omega^p(\lambda^*)$. Such g exists because Λ is interpolating. Consider the function

$$F(z) = \sum_{\lambda\in\Lambda\setminus\lambda^*} \rho(\lambda)\frac{g(z)(z-\lambda^*)}{(z-\lambda)\overline{(\lambda^*-\lambda)}}.$$

The sampling inequality shows that $F \in \mathcal{F}^p_{\phi,\omega}$. Moreover, since $|F(\lambda)| = |g'(\lambda)|\rho(\lambda)$ for all $\lambda \in \Lambda\setminus\lambda^*$ and $F(\lambda^*) = 0$, we have

$$\|F\|^p_{\mathcal{F}^p_{\phi,\omega}} \lesssim \sum_{\lambda\in\Lambda\setminus\lambda^*} |g'(\lambda)|^p\rho^p(\lambda)e^{-p\phi(\lambda)}\omega^p(\lambda).$$

We use now Lemma 19 and the fact that Λ is ρ-separated (since it is interpolating):

$$\|F\|^p_{\mathcal{F}^p_{\phi,\omega}} \lesssim \sum_{\Lambda\setminus\lambda^*} \int_{D(\lambda)} |g|^p e^{-p\phi}\omega^p d\sigma \lesssim \|g\|^p_{\mathcal{F}^p_{\phi,\omega}} \lesssim e^{-p\phi(\lambda^*)}\omega^p(\lambda^*).$$

We want to estimate $|F'(\lambda^*)|$. Using again Lemma 19

$$|F'(\lambda^*)|^p e^{-p\phi(\lambda^*)}\omega^p(\lambda^*)\rho^p(\lambda^*) \lesssim \int_{D(\lambda)} |F|^p e^{-p\phi}\omega^p d\sigma \lesssim e^{-p\phi(\lambda^*)}\omega^p(\lambda^*).$$

Therefore $|F'(\lambda^*)|\rho(\lambda^*) \lesssim 1$. On the other hand

$$F'(\lambda^*) = \sum_{\lambda\in\Lambda\setminus\lambda^*} \frac{\rho(\lambda)}{|\lambda-\lambda^*|^2}.$$

This yields (16). The obvious modifications give (16) in the case $p = \infty$.

According to Lemma 26(d) there exists $r > 0$ with $\mathbb{C} = \cup_{\lambda\in\Lambda} D^r(\lambda)$. Also, there exists $r_0 > 0$ depending on r such that,

$$\int_{D^r(\lambda)\setminus D^{r_0}(\lambda^*)} \frac{dm(z)}{1+|z-\lambda^*|^2} \leq C(r)\frac{\rho^2(\lambda)}{|\lambda-\lambda^*|^2} \qquad \forall\lambda \notin D^{r_0}(\lambda^*).$$

We may now finish by taking a big disk $D(0,M)$ and $\lambda^*_M \in D(0,M)$ in such a way that $\rho(\lambda^*_M) \geq \rho(\lambda)$ for all $\lambda \in \Lambda\cap D(0,M)$. In this case

$$\int_{D(0,M)\setminus D^{r_0}(\lambda^*_M)} \frac{dm(z)}{1+|z-\lambda^*_M|^2} \lesssim \sum_{\substack{\lambda\in\Lambda\\ \lambda\notin D^{r_0}(\lambda^*)}} \int_{D^r(\lambda)} \frac{dm(z)}{1+|z-\lambda^*_M|^2} \lesssim \sum_{\lambda\in\Lambda\setminus\lambda^*_M} \frac{\rho(\lambda)\rho(\lambda^*_M)}{|\lambda-\lambda^*_M|^2} < C.$$

This is a contradiction, since $\lim_{M\to\infty} \rho(\lambda^*_M)/M = 0$ and the left hand side of the previous inequality tends to ∞ as M goes to ∞. ∎

Corollary 31. *Any sequence obtained by deleting a finite number of points of* $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$ *is still in* $\operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$.

We want to prove next an analogue for interpolating sequences: adding a finite number of points to an interpolating sequence gives again an interpolating sequence.

Given Λ and a point z define, following [Beu89, p.352–354]

$$\sigma^p_{\phi,\omega}(z,\Lambda) := \sup\Big\{|f(z)|e^{-\phi(z)}\omega(z),\ \|f\|_{\mathcal{F}^p_{\phi,\omega}} \le 1,\ f|\Lambda \equiv 0\Big\}.$$

Notice first that if Λ is interpolating and $z \notin \Lambda$ this is strictly positive. Indeed, Λ is not a uniqueness sequence, otherwise Λ would be also sampling, contradicting Theorem 30. Thus there exists $f \in \mathcal{F}^p_{\phi,\omega}$, $f \neq 0$ with $f|\Lambda \equiv 0$ and, eventually dividing f by a power of $(\zeta - z)$, $f(z) \neq 0$. Hence $\sigma^p_{\phi,\omega}(z,\Lambda) > 0$.

Lemma 32. *Let* $\Lambda \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$. *Then* $\Lambda \cup \{z\} \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$ *for all* $z \notin \Lambda$. *Furthermore, for all* $\varepsilon > 0$ *there exists* $C > 0$ *such that* $d_\phi(\Lambda, z) \ge \varepsilon$ *implies* $M^p_{\phi,\omega}(\Lambda \cup \{z\}) \le C M^p_{\phi,\omega}(\Lambda)$.

Proof. As in the proof of [Beu89, Lemma 4, p.233], we have

$$M^p_{\phi,\omega}(\Lambda \cup \{z\}) \le \frac{1 + 2M^p_{\phi,\omega}(\Lambda)}{\sigma^p_{\phi,\omega}(z,\Lambda)}.$$

Thus we will be done if we prove that there exists $A > 0$ such that $d_\phi(z,\Lambda) \ge \varepsilon$ implies $\sigma^p_{\phi,\omega}(z,\Lambda) \ge A$.

If this is not true, there exists a sequence $\{z_n\} \in \mathbb{C}$ with $d_\phi(z_n,\Lambda) \ge \varepsilon$ and $\sigma^p_{\phi,\omega}(z_n,\Lambda) \le 1/n$. Transferring z_n to the origin by $\tau^{-1}_{z_n}$ (see Section 21), we get a sequence $\Lambda_n := \Lambda_{z_n}$ such that $|\lambda| \ge \varepsilon$ for all $\lambda \in \Lambda_n$ and $\sigma^p_{\phi_n,\omega_n}(0,\Lambda_n) \le 1/n$, where $\phi_n = \phi_{z_n}$ and $\omega_n = \omega_{z_n}$.

Taking a subsequence if necessary, assume that $(\Lambda_n, \phi_n, \omega_n)$ converges to $(\Lambda^*, \phi^*, \omega^*)$. By Proposition 27, $\Lambda^* \cup \{0\} \in \operatorname{Int} \mathcal{F}^p_{\phi^*,\omega^*}$, so there exists $f \in \mathcal{F}^p_{\phi^*,\omega^*}$ with $f|\Lambda^* = 0$ and $|f(0)| = 1$. Arguing as in the proof of Proposition 27 we see that there exist $f_n \in \mathcal{F}^p_{\phi_n,\omega}$ and ε_n decreasing to zero such that

$$\|f_n|\Lambda_n\|_{\ell^p_{\phi_n,\omega_n}(\Lambda_n)} \le \varepsilon_n, \qquad |f_n(0)| \ge c \qquad \text{and} \qquad \|f_n\|_{\mathcal{F}^p_{\phi_n,\omega_n}} \le C.$$

Since Λ_n is interpolating, there exist also $g_n \in \mathcal{F}^p_{\phi_n,\omega_n}$ with

$$g_n|\Lambda_n = f_n|\Lambda_n \qquad \text{and} \qquad \|g_n\|_{\mathcal{F}^p_{\phi_n,\omega_n}} \le M^p_{\phi_n,\omega_n}(\Lambda_n)\|f_n|\Lambda_n\|_{\ell^p_{\phi_n,\omega_n}(\Lambda_n)} \le \varepsilon_n M(\Lambda).$$

Then $h_n := f_n - g_n \in \mathcal{F}^p_{\phi_n,\omega_n}$ vanishes on Λ_n and $\|h_n\|_{\mathcal{F}^p_{\phi_n,\omega_n}} \le 2C$, therefore $|h_n(0)| \lesssim 1/n$. On the other hand $|g_n(0)| \lesssim \varepsilon_n$ and therefore $|h_n(0)| \ge c/2$, thus contradicting the previous estimate. ■

4.3. **Inclusions between various spaces of interpolating sequences.** We want to study next the relationship between the spaces of interpolating sequences for various weights. We will use the techniques already exploited in [MT00].

We start with the construction of a sort of peak-functions associated to an interpolating sequence. Let $\delta_\lambda^{\lambda'}$ denote the Kroenecker indicator, i.e. $\delta_\lambda^{\lambda'}=1$ if $\lambda=\lambda'$ and $\delta_\lambda^{\lambda'}=0$ otherwise.

Lemma 33. *Let* $\Lambda\in\operatorname{Int}\mathcal{F}^p_{\phi,\omega}$, $1\leq p\leq\infty$. *Given* $\varepsilon>0$ *and* $\tilde\omega\in\Omega_\phi$ *there exist* $m\in\mathbb{N}$, $C>0$ *and functions* $g_\lambda\in\mathcal{F}^p_{(1+\varepsilon)\phi,\tilde\omega}$ *such that*

(a) $g_\lambda(\lambda')=\delta_\lambda^{\lambda'}$ *for all* $\lambda,\lambda'\in\Lambda$.
(b) $\|g_\lambda\|_{\mathcal{F}^p_{(1+\varepsilon)\phi,\tilde\omega}}\simeq\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}$.
(c) $|g_\lambda(z)|\lesssim\dfrac{\tilde\omega(\lambda)}{\tilde\omega(z)}e^{(1+\varepsilon)(\phi(z)-\phi(\lambda))}\dfrac{1}{1+d_\phi^m(z,\lambda)}$.
(d) *For all* $v\in\ell^p_{(1+\varepsilon)\phi,\tilde\omega}(\Lambda)$, $\|v\|_{\ell^p_{(1+\varepsilon)\phi,\tilde\omega}(\Lambda)}\lesssim\left\|\sum_{\lambda\in\Lambda}v_\lambda g_\lambda\right\|_{\mathcal{F}^p_{(1+\varepsilon)\phi,\tilde\omega}}\lesssim\|v\|_{\ell^p_{(1+\varepsilon)\phi,\tilde\omega}(\Lambda)}$.
(e) $\displaystyle\lim_{r\to\infty}\sup_{\lambda\in\Lambda}\frac{e^{p(1+\varepsilon)\phi(\lambda)}}{\tilde\omega^p(\lambda)}\int\limits_{\mathbb{C}\setminus D^r(\lambda)}|g_\lambda(z)|^pe^{-p(1+\varepsilon)\phi(z)}\tilde\omega^p(z)d\sigma(z)=0$.

Proof. By hypothesis, there exist functions $f_\lambda\in\mathcal{F}^p_{\phi,\omega}$ such that $f_\lambda(\mu)=\delta_\lambda^\mu$ for all $\lambda,\mu\in\Lambda$ and $\|f_\lambda\|_{\mathcal{F}^p_{\phi,\omega}}\leq M(\Lambda)\omega(\lambda)e^{-\phi(\lambda)}$. Consider the weights P_λ given by Theorem 17 for the weight $\tilde\omega/\omega$, and define $g_\lambda=f_\lambda P_\lambda$. By construction we have (a) and (c).

(b) When $p=\infty$ we have $\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}=\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}|g_\lambda(\lambda)|\leq\|g_\lambda\|_{\mathcal{F}^\infty_{(1+\varepsilon)\phi,\tilde\omega}}$. The remaining inequality is immediate from (c).

Let $p<\infty$. On the one hand, Lemma 18(a) gives

$$\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}=\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}|g_\lambda(\lambda)|\lesssim\Big(\int_{D(\lambda)}|g_\lambda|^pe^{-p(1+\varepsilon)\phi}\tilde\omega^pd\sigma\Big)^{1/p}\lesssim\|g_\lambda\|_{\mathcal{F}^p_{(1+\varepsilon)\phi,\tilde\omega}}.$$

On the other hand, (c) and Lemma 5(b) show that for m big enough

$$\begin{aligned}\int_{\mathbb{C}}|g_\lambda|^pe^{-p(1+\varepsilon)\phi}\tilde\omega^pd\sigma&\lesssim\tilde\omega^p(\lambda)e^{-p(1+\varepsilon)\phi(\lambda)}\left[\int_{D(\lambda)}d\sigma(z)+\int_{\mathbb{C}\setminus D(\lambda)}\frac{\Delta\phi(z)}{d_\phi^{pm}(z,\lambda)}\right]\\&\lesssim\tilde\omega^p(\lambda)e^{-p(1+\varepsilon)\phi(\lambda)}\end{aligned}$$

(d) Denote $f=\sum_\lambda v_\lambda g_\lambda$. The left inequalities are proved similarly to (b), for

$$\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}|v_\lambda|=\tilde\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}|f(\lambda)|.$$

For $p=\infty$ and $v\in\ell^{\infty,}_{(1+\varepsilon)\phi,\tilde\omega}(\Lambda)$ Lemma 6 and (c) yield

$$\tilde\omega(z)e^{-(1+\varepsilon)\phi(z)}\Big(\sum_{\lambda\in\Lambda}|v_\lambda||g_\lambda(z)|\Big)\lesssim\|v\|_{\ell^\infty_{(1+\varepsilon)\phi,\tilde\omega}(\Lambda)}\sum_{\lambda\in\Lambda}\frac{1}{1+d_\phi^m(\lambda,z)}\lesssim\|v\|_{\ell^\infty_{(1+\varepsilon)\phi,\tilde\omega}(\Lambda)}.$$

Let now $p < \infty$. Using the estimate (c) and Jensen's inequality for convex functions (which is legitimate thanks to Lemma 6) we have

$$\begin{aligned} |f(z)|^p e^{-p(1+\varepsilon)\phi(z)} \widetilde{\omega}^p(z)\rho^{-2}(z) &\lesssim \frac{1}{\rho^2(z)} \Big[\sum_{\lambda\in\Lambda} \widetilde{\omega}(\lambda)|v_\lambda| e^{-(1+\varepsilon)\phi(\lambda)} \frac{1}{1+d_\phi^m(z,\lambda)} \Big]^p \\ &\lesssim \frac{1}{\rho^2(z)} \sum_{\lambda\in\Lambda} \widetilde{\omega}^p(\lambda)|v_\lambda|^p e^{-p(1+\varepsilon)\phi(\lambda)} \frac{1}{1+d_\phi^m(z,\lambda)}. \end{aligned}$$

Now we apply Lemma 5(b) and obtain

$$\int_{\mathbb{C}} |f|^p e^{-p(1+\varepsilon)\phi} \widetilde{\omega}^p \, d\sigma \lesssim \sum_{\lambda\in\Lambda} \widetilde{\omega}^p(\lambda)|v_\lambda|^p e^{-p(1+\varepsilon)\phi(\lambda)} \int_{\mathbb{C}} \frac{\Delta\phi(z)}{1+d_\phi^m(z,\lambda)} \lesssim \|v\|^p_{\ell^p_{(1+\varepsilon)\phi,\tilde{\omega}}(\Lambda)}.$$

(e) This follows from (c) and Remark 2, since

$$\frac{e^{p(1+\varepsilon)\phi(\lambda)}}{\widetilde{\omega}^p(\lambda)} \int_{\mathbb{C}\setminus D^r(\lambda)} |g_\lambda(z)|^p e^{-p(1+\varepsilon)\phi(z)} \widetilde{\omega}^p(z) d\sigma(z) \lesssim \int_{\mathbb{C}\setminus D^r(\lambda)} \frac{\Delta\phi(z)}{d_\phi^m(z,\lambda)}.$$

■

Theorem 34. *For all $\varepsilon > 0$, $1 \le p, p' \le \infty$ and $\omega, \widetilde{\omega} \in \Omega_\phi$, the following inclusions hold*

$$\operatorname{Int} \mathcal{F}^p_{\phi,\omega} \subset \operatorname{Int} \mathcal{F}^{p'}_{(1+\varepsilon)\phi,\tilde{\omega}}.$$

Proof. It will be enough to prove that for all $\varepsilon > 0$, $1 \le p \le \infty$ and $\omega, \widetilde{\omega} \in \Omega_\phi$,

$$\text{(a)} \quad \operatorname{Int} \mathcal{F}^p_{\phi,\omega} \subset \operatorname{Int} \mathcal{F}^\infty_{(1+\varepsilon)\phi,\tilde{\omega}} \qquad \text{(b)} \quad \operatorname{Int} \mathcal{F}^\infty_{\phi,\omega} \subset \mathcal{F}^p_{(1+\varepsilon)\phi,\tilde{\omega}}.$$

(a) Take the functions g_λ given by Lemma 33. For $v \in \ell^\infty_{(1+\varepsilon)\phi,\tilde{\omega}}(\Lambda)$ we consider the interpolating function

$$f(z) = \sum_{\lambda\in\Lambda} v_\lambda g_\lambda(z)$$

A direct estimate using Lemma 33(c) yields

$$\widetilde{\omega}(z)|f(z)|e^{-(1+\varepsilon)\phi(z)} \lesssim \sum_{\lambda\in\Lambda} \frac{1}{1+d_\phi^m(z,\lambda)},$$

which is bounded, by Lemma 6.

(b) Given $v \in \ell^p_{(1+\varepsilon)\phi,\tilde{\omega}}(\Lambda)$, take $f = \sum_\lambda v_\lambda g_\lambda$ as before and estimate as in the proof of Lemma 33(d). ■

4.4. **Inclusions between various spaces of sampling sequences.** In this section we want to prove some inclusions between various spaces of sampling sequences. Unlike in the corresponding result for interpolating sequences, for the spaces of sampling sequences there is a gain, in the sense that any sampling sequence is actually sampling for a slightly bigger space. This will allow us to pass from the non-strict to the strict inequality of Theorem A.

Theorem 35. *Let $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$ be ρ-separated. There exists $\varepsilon > 0$ such that for all $p' \in [1,\infty]$ and $\widetilde{\omega} \in \Omega_\phi$, the sequence $\Lambda \in \operatorname{Samp} \mathcal{F}^{p'}_{(1+\varepsilon)\phi,\tilde{\omega}}$.*

Proof. The proof is divided in three steps.

(a) *If $\Lambda \in \operatorname{Samp} \mathcal{F}^p_{\phi,\omega}$, then $\Lambda \in \operatorname{Samp} \mathcal{F}^\infty_{\phi,\omega}$.* We know from Proposition 27 that for all weak limit $(\Lambda^*, \phi^*, \omega^*)$ the sequence Λ^* is in $\operatorname{Samp} \mathcal{F}^p_{\phi^*,\omega^*}$, and by Lemma 29 it will be enough to see that all weak limit Λ^* is a uniqueness set for $\mathcal{F}^\infty_{\phi^*,\omega^*}$.

If this is not the case, there exists $f \in \mathcal{F}^\infty_{\phi^*,\omega^*}$ with $f|\Lambda^* \equiv 0$, $f \neq 0$.

We claim that for m large enough

$$g(z) := \frac{f(z)}{(z-\lambda_1^*)\dots(z-\lambda_m^*)} \in \mathcal{F}^p_{\phi^*,\omega^*}.$$

It is clear that Lemma 19 gives the p-integrability on $\cup_{j=1}^m D(\lambda_j^*)$. On the other hand, by Lemma 18

$$\int_{z\notin\cup_j D(\lambda_j^*)} \frac{|f|^p e^{-p\phi^*}\omega^{*p}\rho_{\phi^*}^{-2}}{|z-\lambda_1^*|^p\dots|z-\lambda_m^*|^p} \leq C\int_{z\notin\cup_j D(\lambda_j^*)} \frac{\|f\|^p_{\mathcal{F}^\infty_{\phi^*,\omega^*}}\Delta\phi^*}{|z-\lambda_1^*|^p\dots|z-\lambda_m^*|^p}.$$

Since $\Delta\phi^*$ is doubling there exists m such that this integral converges (Lemma 5(b)).

By Corollary 31, $\Lambda^* \setminus \{\lambda_1^* \dots \lambda_m^*\} \in \operatorname{Samp} \mathcal{F}^p_{\phi^*,\omega^*}$. As f vanishes on this sequence we deduce that $f \equiv 0$, which is a contradiction.

(b) *If $\Lambda \in \operatorname{Samp} \mathcal{F}^\infty_{\phi,\omega}$ there exists $\varepsilon > 0$ such that $\Lambda \in \operatorname{Samp} \mathcal{F}^\infty_{(1+\varepsilon)\phi,\omega}$.* If this is not the case for any sequence $\{\varepsilon_n\} \searrow 0$ there exist functions $f_n \in \mathcal{F}^\infty_{(1+\varepsilon_n)\phi,\omega}$ and $\delta_n > 0$ decreasing to 0 with $\|f_n|\Lambda\|_{\ell^\infty_{(1+\varepsilon_n)\phi,\omega}(\Lambda)} \leq \delta_n$ and $|f_n(z_n)| = 1$.

Let $\Lambda_n = \tau_{z_n}^{-1}(\Lambda)$, $\phi_n = (1+\varepsilon_n)\phi_{z_n}$, $\omega_n = \omega_{z_n}$ and $\tilde{f}_n = T^{\phi,\omega}_{z_n} f_n$. Then $|\tilde{f}_n(0)| = 1$, $\|\tilde{f}_n|\Lambda_n\|_{\ell^\infty_{\phi_n,\omega_n}} \leq \delta_n$, and there exist a sequence Λ^* and functions ϕ^*, f^*, ω^* such that

$$(\Lambda_n, \phi_n, \omega_n) \to (\Lambda^*, \phi^*, \omega^*) \in W(\Lambda, \phi, \omega)$$

and $\{f_n\}_n \to f^* \in \mathcal{F}^\infty_{\phi^*,\omega^*}$ uniformly on compact sets. So we have $|f^*(0)| = 1$ and $f^*|\Lambda^* = 0$, i.e. Λ^* is not a uniqueness sequence for $\mathcal{F}^\infty_{\phi^*,\omega^*}$, a contradiction with Lemma 29.

(c) *If $\Lambda \in \operatorname{Samp} \mathcal{F}^\infty_{(1+\varepsilon)\phi,\omega}$ for some $\varepsilon > 0$, then $\Lambda \in \operatorname{Samp} \mathcal{F}^{p'}_{\phi,\tilde{\omega}}$, for all $\tilde{\omega} \in \Omega_\phi$, $1 \leq p' \leq \infty$.* Consider the spaces

$$\begin{aligned}
\mathcal{F}^{\infty,0}_{(1+\varepsilon)\phi,\omega} &= \{f \in \mathcal{F}^\infty_{(1+\varepsilon)\phi,\omega} : \lim_{|z|\to\infty} \omega(z)|f(z)|e^{-(1+\varepsilon)\phi(z)} = 0\}, \\
\ell^{\infty,0}_{(1+\varepsilon)\phi,\omega}(\Lambda) &= \{v \in \ell^\infty_{(1+\varepsilon)\phi,\omega} : \lim_{|\lambda|\to\infty} \omega(z)|v_\lambda|e^{-(1+\varepsilon)\phi(\lambda)} = 0\}.
\end{aligned}$$

There is a sequence of functions $\{g(z,\lambda)\}_{\lambda\in\Lambda}$ such that for all $f \in \mathcal{F}^{\infty,0}_{(1+\varepsilon)\phi,\omega}$

$$\omega(z)e^{-(1+\varepsilon)\phi(z)}f(z) = \sum_{\lambda\in\Lambda} \omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}f(\lambda)\, g(z,\lambda),$$

and $\sum_\lambda |g(z,\lambda)| \le K$ uniformly in z. This is so by a duality argument, because

$$\{f(\lambda)\}_{\lambda\in\Lambda} \mapsto \omega(z)e^{-(1+\varepsilon)\phi(z)}f(z) \qquad \text{with } f \in \mathcal{F}^{\infty,0}_{(1+\varepsilon)\phi,\omega}$$

is a bounded linear functional from a closed subspace of $\ell^{\infty,0}_{(1+\varepsilon)\phi,\omega}(\Lambda)$ whose norm is bounded independently of z. This is an argument from [Beu89, p.348–358] (see also [Sei93, p.36]).

Consider now $f \in \mathcal{F}^p_{\phi,\tilde\omega} \subset \mathcal{F}^{\infty,0}_{\phi,\tilde\omega}$. Given $z \in \mathbb{C}$ take the function P_z of Theorem 17, with weight $\omega/\widetilde{\omega} \in \Omega_\phi$. Then $fP_z \in \mathcal{F}^{\infty,0}_{(1+\varepsilon)\phi,\omega}$, and by the representation above

$$\omega(z)e^{-(1+\varepsilon)\phi(z)}f(z) = \sum_{\lambda\in\Lambda}\omega(\lambda)e^{-(1+\varepsilon)\phi(\lambda)}f(\lambda)\ P_z(\lambda)g(z,\lambda).$$

Hence

$$\begin{aligned}\omega(z)|f(z)|e^{-\phi(z)} &\lesssim \sum_{\lambda\in\Lambda}\omega(\lambda)|f(\lambda)|e^{-\phi(\lambda)}|P_z(\lambda)|e^{\varepsilon(\phi(z)-\phi(\lambda))}|g(z,\lambda)| \\ &\lesssim \frac{\omega(z)}{\widetilde{\omega}(z)}\sum_{\lambda\in\Lambda}\widetilde{\omega}(\lambda)|f(\lambda)|e^{-\phi(\lambda)}\frac{|g(z,\lambda)|}{1+d^m_\phi(z,\lambda)}.\end{aligned}$$

The case $p=\infty$ is clear, so assume that $p<\infty$. Since $\sum_\lambda |g(z,\lambda)| \le K$, we may apply Jensen's inequality and obtain

$$\widetilde{\omega}^p(z)|f(z)|^p e^{-p\phi(z)}\rho^{-2}(z) \lesssim \rho^{-2}(z)\sum_{\lambda\in\Lambda}\widetilde{\omega}^p(\lambda)|f(\lambda)|^p e^{-p\phi(\lambda)}\frac{|g(z,\lambda)|}{1+d^{mp}_\phi(z,\lambda)}.$$

Now integrate, use that $|g(z,\lambda)|\le K$ and apply Lemma 5(b) to finally obtain the sampling inequality

$$\int_{\mathbb{C}}|f(z)|^p e^{-p\phi(z)}\widetilde{\omega}^p(z)d\sigma(z) \lesssim \sum_{\lambda\in\Lambda}\widetilde{\omega}^p(\lambda)|f(\lambda)|^p e^{-p\phi(\lambda)}.$$

■

4.5. **Nets.** We finish this section by giving useful examples of interpolating and sampling sequences.

Lemma 36. *Let f be the multiplier associated to ϕ, as constructed in the proof of Theorem 16, and let $\Lambda = \mathcal{Z}(f)$. Then $\mathcal{D}^+_{\Delta\phi}(\Lambda) = \mathcal{D}^-_{\Delta\phi}(\Lambda) = 1/2\pi$. We say that Λ is a* net *associated to ϕ.*

Proof. The construction of f is made with quasi-squares R_p of $\mu(R_p) = 2\pi mN$ and mN associated points in a dilated CR_p that made up Λ. Thus, for $z\in\mathbb{C}$ and $t>0$:

$$\begin{aligned}n(z,t\rho(z)) &\ge mN\#\{p : CR_p \subset D^t(z)\} = \frac{1}{2\pi}\mu(\bigcup_{p:CR_p\subset D^t(z)} R_p), \\ n(z,t\rho(z)) &\le mN\#\{p : CR_p \cap D^t(z) \neq \emptyset\} = \frac{1}{2\pi}\mu(\bigcup_{p:CR_p\cap D^t(z)\neq\emptyset} R_p).\end{aligned}$$

By Corollary 10,

$$D^{t-\epsilon(t)}(z) \subset \bigcup_{p:CR_p\subset D^t(z)} R_p \subset D^t(z) \subset \bigcup_{p:CR_p\cap D^t(z)\neq\emptyset} R_p \subset D^{t+\epsilon(t)}(z),$$

whence

$$\frac{1}{2\pi}\mu(D^{t-\epsilon(t)}(z)) \leq n(z, t\rho(z)) \leq \frac{1}{2\pi}\mu(D^{t+\epsilon(t)}(z)).$$

The result is then an application of Lemma 11. ∎

Lemma 37. *Let Λ be a net associated to ϕ. Then $\Lambda \in \operatorname{Int}\mathcal{F}^p_{(1+\varepsilon)\phi,\omega}$ and $\Lambda \in \operatorname{Samp}\mathcal{F}^p_{(1-\varepsilon)\phi,\omega}$ for all $\varepsilon > 0$, $1 \leq p \leq \infty$ and $\omega \in \Omega_\phi$.*

Proof. By Theorems 34 and 35, it is enough to consider the case $\omega = \rho$.

Let f be a multiplier associated to ϕ such that $\Lambda = \mathcal{Z}(f)$.

Let us start by proving that Λ is interpolating. By Theorem 34 it is enough to prove that $\Lambda \in \operatorname{Int}\mathcal{F}^p_{(1+\varepsilon)\phi,\rho}$ for all $\varepsilon > 0$. For each $\lambda \in \Lambda$ define

$$g_\lambda(z) = \frac{f(z)}{z-\lambda}\frac{1}{f'(\lambda)}.$$

We want to see that these functions play a similar role to the peak-functions of Lemma 33. Clearly $g_\lambda(\lambda') = \delta^{\lambda'}_\lambda$. The growth condition of the multiplier gives $|f'(\lambda)| \simeq e^{\phi(\lambda)}/\rho(\lambda)$, and then

$$\rho(z)|g_\lambda(z)|e^{-\phi(z)} \lesssim \frac{|z-\Lambda|}{|z-\lambda|}\frac{1}{|f'(\lambda)|} \lesssim \rho(\lambda)e^{-\phi(\lambda)}.$$

Hence $\|g_\lambda\|_{\mathcal{F}^\infty_{\phi,\rho}} \lesssim \rho(\lambda)e^{-\phi(\lambda)}$.

As seen in the proof of Theorem 34 this is enough to construct, for any $\varepsilon > 0$, an interpolation operator for $\mathcal{F}^p_{(1+\varepsilon)\phi,\rho}$.

Let us see next that $\Lambda \in \operatorname{Samp}\mathcal{F}^p_{(1-\varepsilon)\phi,\omega}$. By Theorem 35 it is enough to consider the case $p = \infty$ and $\omega = 1$, and by Corollary 29 it will be enough to see that every weak limit $(\Lambda^*, (1-\varepsilon)\phi^*, \omega^*) \in W(\Lambda, (1-\varepsilon)\phi, \omega)$ is a uniqueness sequence for $\mathcal{F}^\infty_{(1-\varepsilon)\phi^*,1}$.

Let $(\Lambda_{z_n}, \phi_{z_n}, \omega_{z_n}) \to (\Lambda^*, \phi^*, \omega^*)$ and let f_{z_n} be the corresponding multipliers, with $\mathcal{Z}(f_{z_n}) = \Lambda_{z_n}$ and $|f_{z_n}(z)| \simeq e^{\phi_{z_n}(z)}d_{\phi_{z_n}}(z, \Lambda_{z_n})$. By Montel's theorem let $\{f_{z_n}\}_n \to f^*$ with $\mathcal{Z}(f^*) = \Lambda^*$ and $|f^*(z)| \simeq e^{\phi^*(z)}d_{\phi^*}(z, \Lambda^*)$, i.e, f^* is a multiplier for ϕ^*.

Consider also a multiplier g associated to $\varepsilon\phi^*$. In particular $|g(z)| \simeq e^{\varepsilon\phi^*(z)}d_{\phi^*}(z, \mathcal{Z}(g))$. In order to see that Λ^* is a uniqueness sequence assume that $h \in \mathcal{F}^\infty_{(1-\varepsilon)\phi^*,1}$ and $h|\Lambda^* = 0$. Then $hg \in \mathcal{F}^\infty_{\phi^*,1}$, by construction. On the other hand, the function $F := hg/f^*$ is entire, because h vanishes on Λ^*. It is also bounded when z is far from Λ^*, since $|hg| \lesssim e^{\phi^*}$ and $|f^*| \gtrsim e^{\phi^*}$. By the maximum principle F is bounded globally, and by Liouville's theorem there exists $c \in \mathbb{C}$ such that $hg = cf^*$. Since g vanishes in some points outside Λ^* we have $c = 0$, hence $h \equiv 0$. ∎

5. Sufficient conditions for sampling

We prove here the sufficiency part of Theorem A. Assume that $\mathcal{D}^-_{\Delta\phi}(\Lambda) > 1/2\pi$. By Lemma 26 we can assume that Λ is ρ-separated, and according to Theorem 35 it will be enough to prove that $\Lambda \in \mathcal{F}^\infty_{\phi,\omega}$. By Corollary 29 this will be done as soon as we show that every weak limit Λ^* is a uniqueness sequence for $\mathcal{F}^\infty_{\phi^*,\omega^*}$.

Recall the notation $n_\Lambda(z,r) = \#[\Lambda \cap \overline{D(z,r)}]$.

Assume thus that we have $f \in \mathcal{F}^\infty_{\phi^*,\omega^*}$ with $f|\Lambda^* \equiv 0$ and $\|f\|_{\mathcal{F}^\infty_{\phi^*,\omega^*}} = 1$. There is no loss of generality in assuming that $f(0) \neq 0$. Applying Jensen's formula to f on $D_{\phi^*}(0)$

$$\begin{aligned}\int_0^{r\rho_{\phi^*}(0)} \frac{n_{\Lambda^*}(0,t)}{t}dt &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(r\rho_{\phi^*}(0)e^{i\theta})|d\theta - \log|f(0)| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}(\phi^*(r\rho_{\phi^*}(0)e^{i\theta}) - \log\omega^*(r\rho_{\phi^*}(0)e^{i\theta}))d\theta - \log|f(0)| \\ = \Big[\frac{1}{2\pi}\int_0^{2\pi}\phi^*(r\rho_{\phi^*}(0)e^{i\theta})d\theta - \phi^*(0)\Big] &+ \Big[\log\omega^*(0) - \frac{1}{2\pi}\int_0^{2\pi}\log\omega^*(r\rho_{\phi^*}(0)e^{i\theta})d\theta\Big] \\ &+ \phi^*(0) - \log\omega^*(0) - \log|f(0)|.\end{aligned}$$

By definition of flat weight and by Lemma 4, $\omega^*(r\rho_{\phi^*}(0)e^{i\theta})/\omega^*(0) \lesssim r^\gamma$ for some $\gamma > 0$. Then, Green's identity yields

$$\begin{aligned}\int_0^{r\rho_{\phi^*}(0)} \frac{n_{\Lambda^*}(0,t)}{t}dt &\le \frac{1}{2\pi}\int_{D(0,r\rho_{\phi^*}(0))} \log\frac{r\rho_{\phi^*}(0)}{|\zeta|}\Delta\phi^*(\zeta) + \mathsf{O}(\log r) \\ &= \frac{1}{2\pi}\int_0^{r\rho_{\phi^*}(0)} \Delta\phi^*(D(0,t))\frac{dt}{t} + \mathsf{O}(\log r),\end{aligned}$$

for all r big enough. This contradicts the hypothesis, which implies in particular that for some $\varepsilon > 0$ and all t big enough $n_{\Lambda^*}(0,t) \ge (1/2\pi + \varepsilon)\Delta\phi^*(D(0,t))$.

6. Necessary conditions for sampling

This section contains the proof of the necessity part of Theorem A. By Lemma 26(b) and Theorem 35 it will be enough to prove the following result.

Theorem 38. *Let Λ be ρ-separated. If $\Lambda \in \operatorname{Samp}\mathcal{F}^2_{\phi,\omega}$ then $\mathcal{D}^-_{\Delta\phi}(\Lambda) \ge 1/2\pi$.*

We use a result comparing the densities between interpolating and sampling sequences, as in [RS95]. We do that by adapting Lemma 4 in [OCS98] to our setting.

Lemma 39. *Let $\varepsilon > 0$. Assume $I \in \operatorname{Int}\mathcal{F}^2_{(1-\varepsilon)\phi,\omega}$ and $S \in \operatorname{Samp}\mathcal{F}^2_{\phi,\omega}$ is ρ-separated. There exists a positive function $\epsilon(R)$ such that $\lim\limits_{R\to\infty}\epsilon(R)/R = 0$ and for every $\epsilon > 0$ there is $R_0 > 0$ with*

$$(1-\epsilon)\, n_I(z, R\rho(z)) \le n_S(z, (R+\epsilon(R))\rho(z)) \qquad z \in \mathbb{C}.$$

Proof. The proof is as in [OCS98, Lemma 4] with minor modifications, so we keep it short.

According to our definition, if S is sampling then $\{k(z,s) = K_{\phi,\alpha}(z,s)e^{-\phi(s)}\omega(s)\}_{s\in S}$ is a frame in $\mathcal{F}^2_{\phi,\omega}$ ($K_{\phi,\omega}$ denotes the Bergman kernel, as in Section 3.3). That is, for $f \in \mathcal{F}^2_{\phi,\omega}$

$$\|f\|^2_{\mathcal{F}^2_{\phi,\omega}} \simeq \sum_{s\in S} |\,\langle k(z,s), f(z)\rangle\,|^2.$$

A consequence is that

$$f(z) = \sum_{s\in S}\langle k(\xi,s), f(\xi)\rangle \tilde{k}(z,s) = \sum_{s\in S} f(s)e^{-\phi(s)}\omega(s)\tilde{k}(z,s),$$

where $\tilde{k}(z,s)$ is the dual frame of $k(z,s)$.

Consider also the functions g_i given by Lemma 33 for the weight $(1-\varepsilon)\phi$. Lemma 33(d) implies that the normalised functions $\kappa(i,z) := g_i(z)e^{\phi(i)}/\omega(i)$ form a Riesz basis in the closed linear span H of $\{\kappa(i,z)\}_{i\in I}$ in $\mathcal{F}^2_{\phi,\omega}$.

Given $z \in \mathbb{C}$ and $R, r > 0$ (R much bigger that r) consider the following two finite dimensional subspaces of $\mathcal{F}^2_{\phi,\omega}$:

$$\begin{aligned} W_S &= \ < \tilde{k}(\xi,s) : s \in S \cap D^{R+r}(z) > \\ W_I &= \ < \kappa(\xi,i) : i \in I \cap D^{R}(z) > . \end{aligned}$$

Let P_S and P_I denote the orthogonal projections of $\mathcal{F}^2_{\phi,\omega}$ on W_S and W_I respectively. We estimate the trace of the operator $T = P_I P_S$ in two different ways. To begin with

$$\text{tr}(T) \le \text{rank } W_S \le \#\{S \cap D^{R+r}(z)\}.$$

On the other hand

$$\text{tr}(T) = \sum_{i\in I\cap D^R(z)} \langle T(\kappa(\xi,i)), P_I\kappa^*(\xi,i)\rangle,$$

where $\{\kappa^*(\xi,i)\}$ is the dual basis of $\kappa(\xi,i)$ in H. Using that P_I and P_S are projections one deduces that

$$\text{tr}(T) \ge \#\{i \in I \cap D^R(z)\}\Big(1 - \sup_i |\langle P_S(\kappa(\xi,i)) - \kappa(\xi,i), \kappa^*(\xi,i)\rangle|\Big).$$

Since $\|\kappa(\xi,i)\|_{\mathcal{F}^2_{\phi,\omega}} \simeq 1$, also $\|\kappa^*(\xi,i)\|_{\mathcal{F}^2_{\phi,\omega}} \simeq 1$. Thus we will be done as soon as we show that $\|P_S(\kappa(\xi,i)) - \kappa(\xi,i)\|_{\mathcal{F}^2_{\phi,\omega}} \le \varepsilon$ for a suitable $r \simeq \epsilon(R)$.

We have

$$\|P_S(\kappa(\xi,i)) - \kappa(\xi,i)\|^2_{\mathcal{F}^2_{\phi,\omega}} \lesssim \sum_{s\notin D^{R+r}(z)} |\langle\tilde{k}(\xi,s), \kappa(\xi,i)\rangle|^2 = \sum_{s\notin D^{R+r}(z)} \Big|\kappa(s,i)e^{-\phi(s)}\omega(s)\Big|^2.$$

Since S is ρ-separated, there exists $\eta > 0$ such that the disks $D^\eta(s)$ are pairwise disjoint. Using Lemma 18(a) we get, for some $c > 0$ depending on ϕ and η

$$\|P_S(\kappa(\xi,i)) - \kappa(\xi,i)\|^2_{\mathcal{F}^2_{\phi,\omega}} \lesssim \int_{\bigcup_{s\notin D^{R+r}(z)} D^\eta(s)} |\kappa(\zeta,i)|^2 e^{-2\phi(\zeta)}\omega^2(\zeta)d\sigma(\zeta).$$

Applying Lemma 9 with $r^k = R^\tau$ and τ so that $0 < (\varepsilon - \tau)\gamma < 1$, we see that there exist $\delta \in (0,1)$, $c > 0$ and a function $\epsilon(R) = cR^{1-\delta}$ such that

$$\bigcup_{s\notin D^{R+c\epsilon(R)}(z)} D^\eta(s) \subset \mathbb{C} \setminus D^{\varepsilon\epsilon(R)}(i).$$

Finally, for R is big enough Lemma 33(e) yields

$$\|P_S(k(\xi,i)) - \kappa(\xi,i)\|^2_{\mathcal{F}^2_{\phi,\omega}} \lesssim \int_{\mathbb{C}\setminus D^{\varepsilon\epsilon(R)}(i)} |\kappa(\xi,i)|^2 e^{-2\phi(\xi)}\omega^2(\xi)d\sigma(\xi) \lesssim \epsilon.$$

■

Proof of Theorem 38. Given $\varepsilon > 0$ consider a net I associated to $(1-2\varepsilon)\phi$. By Lemma 36 $I \in \operatorname{Int} \mathcal{F}^{2,\alpha}_{(1-\varepsilon)\phi}$, and by Lemma 37 $\mathcal{D}^+_{\Delta\phi}(I) = \mathcal{D}^+_{\Delta\phi}(I) = (1-2\varepsilon)/2\pi$. Apply now Lemma 39: there exist R_0 and $\epsilon(R)$ such that for $R > R_0$

$$n_\Lambda(z, R\rho(z)) \geq (1-\varepsilon)\; n_I(z, (R-\epsilon(R))\rho(z)) \geq \frac{(1-\varepsilon)^3}{2\pi}\; \mu(D^{R-\delta(R)}(z)),$$

where $\delta(R) = R - \epsilon(R) - \epsilon(R - \epsilon(R))$. This estimate together with Lemma 11 finish the proof. ■

7. Sufficient conditions for interpolation

Taking into account Theorem 34 and Lemma 37, in order to prove the sufficiency part of Theorem B it is enough to prove the following.

Theorem 40. *If Λ is ρ-separated and $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$ there exist $\varepsilon > 0$ and a sequence Σ such that $\Lambda \cup \Sigma$ is a ρ-separated net associated to $(1-\varepsilon)\phi$.*

In the proof of this result we need to express the density condition in terms of the quasi-squares appearing in Theorem 8. this will be done in Theorem 42; before we need some preliminaries.

Denote $\phi_r = e^{-r}\phi$.

Lemma 41. *Let*

$$I_r(\zeta) = \int_{|z-\zeta|<\rho_{\phi_r}(z)/r} \frac{r^2\, dm(z)}{\pi\rho^2_{\phi_r}(z)}.$$

Then $\sup_{\zeta\in\mathbb{C}} |I_r(\zeta) - 1| < 1/r$.

Proof. We estimate I_r using the change of variable $w = (z-\zeta)/\rho_{\phi_r}(z)$, whose Jacobian is

$$\rho_{\phi_r}^{-2}(z)\Big|1 - \frac{\langle \nabla\rho_{\phi_r}(z), z-\zeta\rangle}{\rho_{\phi_r}(z)}\Big|.$$

From (6) it follows that $|\nabla\rho_{\phi_r}| \le 1$, hence the Jacobian is bounded above by $\rho_{\phi_r}^{-2}(z)(1+1/r)$ and below by $\rho_{\phi_r}^{-2}(z)(1-1/r)$. Then

$$1-\frac{1}{r}\int_{|w|\le 1/r}\frac{r^2}{\pi}(1-\frac{1}{r})\,dm(w) \le I_r(\zeta) \le \int_{|w|\le 1/r}\frac{r^2}{\pi}(1+\frac{1}{r})\,dm(w) = 1+\frac{1}{r}.$$

■

It follows immediately from (4) that there exist $0<\varepsilon<m$ such that

$$t^{\varepsilon}\rho_\phi \lesssim \rho_{\phi/t} \lesssim t^m\rho_\phi.$$

This implies, with $t = e^r$,

$$\lim_{r\to\infty}\frac{\rho_{\phi_r}(z)}{r\rho_\phi(z)} = \infty \tag{17}$$

uniformly in $z\in\mathbb{C}$.

Let $R_\alpha^s(z)$ denote the rectangle with vertices $z+s\rho(z)(1+i\alpha)$, $z+s\rho(z)(1-i\alpha)$, $z-s\rho(z)(1+i\alpha)$ and $z-s\rho(z)(1-i\alpha)$, where $\alpha\in[e^{-1},e]$ and e is the constant of Theorem 8(b).

Theorem 42. *Let $\mu=\Delta\phi$ and let ν be a positive measure such that*

$$\nu(D_\phi^r(z)) \le (1-\varepsilon)\mu(D_\phi^r(z)) \quad \forall r\ge r_0,\ \forall z\in\mathbb{C}. \tag{18}$$

There exists $s_0>0$ such that for any $\alpha\in[e^{-1},e]$

$$\nu(R_\alpha^s(z)) \le (1-\frac{\varepsilon}{2})\mu(R_\alpha^s(z)) \quad \forall s\ge s_0,\ \forall z\in\mathbb{C}.$$

Proof. Fix r big enough so that $\rho_{\phi_r}/r > r_0\rho_\phi$ and $(1+1/r)(1-\varepsilon) < (1-1/r)(1-3\varepsilon/4)$. This can be done because of (17). By hypothesis

$$\nu(D_{\phi_r}^{1/r}(z)) \le (1-\varepsilon)\ \mu(D_{\phi_r}^{1/r}(z)) \quad \forall z\in\mathbb{C},$$

and if s is much bigger than r we get

$$\int_{z\in R_s^\alpha(w)}\frac{r^2}{\pi\rho_{\phi_r}^2(z)}\nu(D_{\phi_r}^{1/r}(z))\,dm(z) \le (1-\varepsilon)\int_{z\in R_s^\alpha(w)}\frac{r^2}{\pi\rho_{\phi_r}^2(z)}\mu(D_{\phi_r}^{1/r}(z))\,dm(z).$$

Denote

$$\begin{aligned}
\Omega_r(\zeta) &= \{z\in\mathbb{C},\ |z-\zeta|<\rho_{\phi_r}(z)/r\}\\
F_r(w,s) &= \{\zeta\in\mathbb{C},\ \Omega_r(\zeta)\subset R_\alpha^s(w)\}\\
G_r(w,s) &= \bigcup_{\zeta\in R_\alpha^s(w)}\Omega_r(\zeta).
\end{aligned}$$

Reversing the order of integration and using the previous Lemma we deduce that

$$\nu(F_r(w,s)) \le (1-\frac{3}{4}\varepsilon)\ \mu(G_r(w,s)).$$

It is clear that $F_r(w,s) \subset R^s_\alpha(w) \subset G_r(w,s)$. Similarly to the proof of Lemma 9, there exists $\epsilon(s)$ such that $R^{s-\epsilon(s)}_\alpha(w) \subset F_r(w,s)$ and $G_r(w,s) \subset R^{s+\epsilon(s)}_\alpha(w)$.

By Remark 4

$$\lim_{s\to\infty} \frac{\mu(R^{s+\epsilon(s)}_\alpha(w))}{\mu(R^{s-\epsilon(s)}_\alpha(w))} = 1$$

uniformly in z, and therefore there exists s_0 such that for $s > s_0$

$$\begin{aligned} \nu(R^{s-\epsilon(s)}_\alpha(w)) &\le (1-\frac{3}{4}\varepsilon)\ \mu(G_r(w,s)) \le (1-\frac{3}{4}\varepsilon)\ \mu(R^{s+\epsilon(s)}_\alpha(w)) \\ &\le (1-\frac{\varepsilon}{2})\ \mu(R^{s-\epsilon(s)}_\alpha(w)). \end{aligned}$$

■

Proof of Theorem 40. Take an entire function g vanishing exactly on Λ. We will construct a sequence Σ and an entire function h such that for some $\varepsilon > 0$,

(i) $\Lambda \cup \Sigma$ is ρ-separated.
(ii) h vanishes exactly on Σ.
(iii) For any $\tau > 0$, $|\log|h(z)| - (1-\varepsilon)\phi(z) + \log|g(z)|| \le C_\tau$ if $D^\tau(z) \cap (\Lambda\cup\Sigma) = \emptyset$.

Accepting this we reach the result by taking $f = gh$. This is so because the separateness of $\Lambda \cup \Sigma$ and (iii) imply that f is a multiplier for $(1-\varepsilon)\phi$. ■

Construction of Σ and h. To avoid the repetition of the factors 2π and $1-\varepsilon$, denote here $\mu = (1-\varepsilon)\Delta\phi/2\pi$. Let

$$\tilde\mu = \mu - \sum_{\lambda\in\Lambda} \delta_\lambda = \frac{1}{2\pi}\Delta\Big((1-\varepsilon)\phi - \log|g|\Big).$$

Following Theorem 8 and the Remark thereafter, given $n, M \in \mathbb{N}$ we can take a system of quasi-squares $\{R_k\}_k$ such that, denoting $\mu_k = \mu_{|R_k}$, we have $\mu = \sum_k \mu_k$ and $\mu_k(\mathbb{C}) = Mn$. Then $\tilde\mu = \sum_k \tilde\mu_k$, being

$$\tilde\mu_k = \mu_k - \sum_{\lambda\in\Lambda\cap R_k} \delta_\lambda.$$

By hypothesis there exists $\varepsilon > 0$ such that $\mathcal{D}_{\Delta\phi}(\Lambda) < 1/2\pi - 4\varepsilon$. Therefore, there exists $r_0 > 0$ such that

$$\tilde\mu(D^r(z)) \ge 3\varepsilon\mu(D^r(z)) \qquad \text{for all} \quad z\in\mathbb{C}, r \ge r_0.$$

Also, Theorem 42 implies that for $M \ge m/(2\varepsilon)$ and n big enough:

$$Mn \ge \tilde\mu(R_k) \ge 2\varepsilon\mu(R_k) = 2\varepsilon Mn \ge mn.$$

Let $\tilde{\mu}(R_k) = m_k n$, with $m \le m_k \le M$. Notice that $m_k \in \mathbb{N}$, since $\mu(R_k) \in \mathbb{N}$. Applying Lemma 15 we obtain a sequence Σ made of points $\sigma_1^k, \dots, \sigma_{m_k n}^k \in CR_k$ so that the first m moments of the measures $\nu_k = \tilde{\mu}_k - \sum_{j=1}^{m_k n} \delta_{\sigma_j^k}$ vanish. Furthermore, it is clear that we can choose the τ_j^k in the proof of Lemma 15 so that $\Lambda \cup \Sigma$ is ρ-separated.

Let

$$\nu = \sum_k \nu_k = \frac{1}{2\pi} \Delta((1-\varepsilon)\phi - \log|g|) - \sum_{\sigma \in \Sigma} \delta_\sigma.$$

In order to prove (iii) consider $v = (1-\varepsilon)\phi - \log|g| - w$, where

$$w(z) = \int_{\mathbb{C}} \log|z-\zeta| \, d\nu(\zeta).$$

Since

$$\Delta v = 2\pi \sum_{\sigma \in \Sigma} \delta_\sigma,$$

there exists h entire (vanishing exactly on Σ) such that $\log|h| = v$.

We need to estimate $|w(z)|$ when $|z - \Lambda \cup \Sigma| \ge \tau\rho(z)$. Given $z \in \mathbb{C}$, let $k_0 \in \mathbb{N}$ be such that $z \in R_{k_0}$. By Theorem 8(c), there exists $r_0 > 0$ independent of z such that $R_{k_0} \subset D^{r_0}(z) \subset CR_{k_0}$. We have

$$w(z) = \int_{\mathbb{C}} \log|z-\zeta| d\nu(\zeta) = \int_{\mathbb{C}} \log|z-\zeta| d\nu_{k_0}(\zeta) + \sum_{k: k \neq k_0} \int_{\mathbb{C}} \log|z-\zeta| d\nu_k(\zeta),$$

and we estimate the two terms separately.

Let $C > 0$ be the constant of Lemma 15. Since the first m moments of ν_{k_0} vanish,

$$\begin{aligned} \left| \int_{\mathbb{C}} \log|z-\zeta| d\nu_{k_0}(\zeta) \right| &= \left| \int_{\mathbb{C}} \log \frac{|z-\zeta|}{r_0 \rho(z)} d\nu_{k_0}(\zeta) \right| \lesssim \left| \int_{CR_{k_0}} \log \frac{r_0\rho(z)}{|z-\zeta|} d\mu(\zeta) \right| + K|\log\tau| \\ &\lesssim \int_{D^{cr_0}(z)} \log \frac{cr_0\rho(z)}{|z-\zeta|} d\mu(\zeta) + K|\log\tau| \le C_\tau. \end{aligned}$$

The other integral is estimated using the moment condition for each ν_k, as in the estimate of I_1 in Theorem 14. ■

8. Necessary conditions for interpolation

Let us start by proving the non-strict density inequality. By Theorem 34, it is enough to consider the case $p = 2$.

Theorem 43. *If* $\Lambda \in \operatorname{Int} \mathcal{F}^2_{\phi,\omega}$ *then* $\mathcal{D}^+_{\Delta\phi}(\Lambda) \le 1/2\pi$.

Proof. Given $\varepsilon > 0$, take a net S associated to $(1+2\varepsilon)\phi$, as described in Lemma 36. Lemma 37 implies that $S \in \operatorname{Samp} \mathcal{F}^2_{(1+\varepsilon)\phi,\omega}$, and by Lemma 39, there exists $R_0 > 0$ such that

$$n_\Lambda(z, R\rho(z)) \leq (1+\varepsilon)\, n_S(z, (R+\epsilon(R))\rho(z)) \qquad z \in \mathbb{C}\ ,\ R \geq R_0.$$

Since S is a net of density $(1+2\varepsilon)/2\pi$, the radius R_0 can be taken so that for $R \geq R_0$

$$n_S(z, (R+\epsilon(R))\rho(z)) \leq \frac{1+3\varepsilon}{2\pi}\, \mu(D^{R+\epsilon(R)}(z)).$$

This and Corollary 10 give the result. ∎

Let us see now that the inequality is strict.

Proof of the necessity part in Theorem B. Assume that $\Lambda \in \operatorname{Int} \mathcal{F}^p_{\phi,\omega}$. We know that $\mathcal{D}^+_{\Delta\phi}(\Lambda) \leq 1/2\pi$. In order to see that $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$ consider, given $\varepsilon > 0$, a net Σ associated to $2\varepsilon\phi$ such that $Z := \Lambda \cup \Sigma$ is ρ-separated.

Lemma 44. *Denote $Z = \{z_k\}_k$. For every $m \in \mathbb{N}$ and $\varepsilon > 0$ there exist $C > 0$ and functions $f_k \in \mathcal{F}^\infty_{\phi,\omega}$ such that*

(a) $f_k(z_k) = 1$.
(b) $f_k(z_j) = 0$ *for all* $z_j \in D^{1/\varepsilon}(z_k)$.
(c) $|f_k(z)| \leq CM(\Lambda)e^{\phi(z)-\phi(z_k)}\dfrac{\omega(z_k)}{\omega(z)}\dfrac{1}{1+d^m_\phi(z,z_k)}$.
(d) $\|f_k\|_{\phi,\infty} \leq CM(\Lambda)e^{-\phi(z_k)}\omega(z_k)$.

Proof. Assume first that $z_k = \lambda_k \in \Lambda$. By hypothesis there exists $g_k \in \mathcal{F}^p_{\phi,\omega} \subset \mathcal{F}^\infty_{\phi,\omega}$ with $g_k(\lambda_k) = 1$, $g(\lambda_j) = 0$, and $\|g_k\|_{\mathcal{F}^\infty_{\phi,\omega}} \leq M(\Lambda)e^{-\phi(\lambda_k)}\omega(\lambda_k)$. Since Λ plus a finite number of points is still in $\operatorname{Int} \mathcal{F}^p_{\phi,\omega}$ (Lemma 32), we can take g_k so that moreover $g_k(\sigma_j) = 0$ if $|\lambda_k - \sigma_j| \leq 1/\varepsilon\rho(\lambda_k)$ and $g_k(c_j) = 0$, $j = 1, \ldots, M$, where $c_j = \lambda_k + 2\delta\rho(\lambda_k)e^{j\frac{2\pi i}{M}}$ and $\delta > 0$ is taken so that the balls $\{B(\lambda, 10\delta)\}_\lambda$ are pairwise disjoint.

By construction of the nets there exists C independent of z and ε such that $\#\Sigma \cap D^{1/\varepsilon}(z) \leq C$ for any Σ net of density ε/π.

Define then

$$f_k(z) = (2\delta)^{-M}\frac{g_k(z)}{(z-c_1)\cdots(z-c_M)}\rho^M(z_k).$$

It is clear that $f_k \in \mathcal{F}^\infty_{\phi,\omega}$ satisfies (a) and (b).

For $z \notin \cup_{j=1}^M D^\delta(c_j)$,

$$|f_k(z)| \leq C|g_k(z)|\Big(\frac{\rho(z_k)}{|z-z_k|}\Big)^M \leq CM(\Lambda)e^{\phi(z)-\phi(z_k)}\frac{\omega(z_k)}{\omega(z)}\Big(\frac{\rho(z_k)}{|z-z_k|}\Big)^M,$$

and the estimate follows from Lemma 4.

For $z \in D^\delta(c_j)$ we have

$$|f_k(z)| \leq C\Big|\frac{g_k(z)}{z-c_j}\Big|\rho(z_k).$$

Estimating like in (iii) in the proof of Theorem 17 we get $|f_k(z)| \leq CM(\Lambda)e^{\phi(z)-\phi(z_n)}$, as desired.

In case $z_k = \sigma_k \in \Sigma$, use again that Λ plus one point is $\mathcal{F}^2_{\phi,\omega}$-interpolating and start with $g_k \in \mathcal{F}^2_{\phi,\omega} \subset \mathcal{F}^\infty_{\phi,\omega}$ such that $g_k(\sigma_k) = 1$, $g_k(\lambda_j) = 0$ for all j. Then proceed as before. ■

Lemma 45. $Z \in \operatorname{Int} \mathcal{F}^\infty_{\phi,\omega}$.

Proof. Given $v = \{v_k\}_k \in \ell^{\infty,\alpha}_\phi(Z)$ consider the pseudo-extension

$$E(v)(z) = \sum_{k=1}^\infty v_k f_k(z).$$

Let us see first that $E(v) \in \mathcal{F}^\infty_{\phi,\omega}$. By (c) above and Lemma 6 we see that for any $z \in \mathbb{C}$

$$\omega(z)e^{-\phi(z)}|E(v)(z)| \lesssim \sum_{k=1}^\infty \omega(z_k)|v_k|e^{-\phi(z_k)}\frac{1}{1+d^M_\phi(z,z_k)} \lesssim \|v\|_{\ell^{\infty,\alpha}_\phi(Z)}.$$

Let R denote the restriction operator from $\mathcal{F}^\infty_{\phi,\omega}$ to $\ell^\infty_{\phi,\omega}(Z)$. In order to see that Z is in $\operatorname{Int} \mathcal{F}^\infty_{(1+\varepsilon)\phi,\omega}$ it will be enough to prove that $\|RE - I\|_{op} < 1$, since then $(RE)^{-1} = I + \sum_{k=1}^\infty (RE-I)^k$ converges and $E(RE)^{-1}$ defines an inverse to R.

By Lemma 44(b) and (c)

$$\|RE(v) - v\|_{\ell^\infty_{\phi,\omega}(Z)} = \Big\|\Big\{\sum_{k:k\neq j} v_k f_k(z_j)\Big\}_{j\in\mathbb{N}}\Big\|_{\ell^\infty_{\phi,\omega}(Z)}$$

$$\leq \sup_{j\in\mathbb{N}} \omega(z_j)e^{-\phi(z_j)} \sum_{k:z_j\notin D^{1/\varepsilon}(z_k)} |v_k||f_k(z_j)| \leq CM(\Lambda)\|v\|_{\ell^\infty_{\phi,\omega}(Z)} \sum_{k:z_j\notin D^{1/\varepsilon}(z_k)} \frac{1}{d^m_\phi(z_j,z_k)}$$

By Lemma 4 and Corollary 7, if m is big and ε is small enough we have

$$\|RE(v) - v\|_{\ell^\infty_{\phi,\omega}(Z)} \leq 1/2\ \|v\|_{\ell^\infty_{\phi,\omega}(Z)},$$

thus $\|RE - I\|_{op} < 1/2$, as desired. ■

By this Lemma and the results above we have $\mathcal{D}^+_{\Delta\phi}(Z) \leq 1/2\pi$, i.e for all $\delta > 0$ there exists R_0 such that for all $z \in \mathbb{C}$ and $R > R_0$

$$n_\Lambda(z, R\rho(z)) + n_\Sigma(z, R\rho(z)) \leq (1/2\pi + \delta)\mu(D^R(z)).$$

By Lemma 36, $\mathcal{D}^-_{\Delta\phi}(\Sigma) = \varepsilon/\pi$, thus for all $\delta > 0$ there exists R_0 such that for all $R > R_0$

$$n_\Sigma(z, R\rho(z)) \geq (\varepsilon/\pi - \delta)\mu(D^R(z)) \qquad z \in \mathbb{C}\,.$$

This shows that for $\delta > 0$ and R big enough

$$n_\Lambda(z, R\rho(z)) \le (\frac{1-2\varepsilon}{2\pi} + 2\delta)\, \mu(D^R(z)) \qquad z \in \mathbb{C},$$

hence $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$. ■

APPENDIX. ALTERNATIVE CONSTRUCTION OF PEAK FUNCTIONS.

As seen at the end of the proof of Theorem 33, it is enough to consider the case $\omega = \rho$. Also, it will be enough to prove that for any ϕ there exist $C, \delta > 0$ such that for all $\eta \in \mathbb{C}$ there is P_η holomorphic with $P_\eta(\eta) = 1$ and

$$|P_\eta(z)| \le C e^{\phi(z)-\phi(\eta)} \min\Big\{1, \Big(\frac{\rho(\eta)}{|z-\eta|}\Big)^\delta\Big\},$$

since then we can apply this to $\varepsilon\delta/m\ \phi(z)$, take the m-th power and use Lemma 4 to conclude.

We claim that there exists h_η holomorphic with $h_\eta(\eta) = 0$, $h'_\eta(\eta) = 1$ and $|h_\eta(z)| \lesssim e^{\phi(z)-\phi(\eta)}\rho^2(\eta)/\rho(z)$.

Once this is proved we take $w_\eta(z) = h_\eta(z)/(z-\eta)$ and use Lemma 2 to deduce that

$$|P_\eta(z)| \lesssim e^{\phi(z)-\phi(\eta)} \frac{\rho(\eta)}{|z-\eta|}\Big(\frac{|z-\eta|}{\rho(\eta)}\Big)^{1-\delta} = e^{\phi(z)-\phi(\eta)}\Big(\frac{\rho(\eta)}{|z-\eta|}\Big)^\delta \qquad z \notin D(\eta).$$

In order to construct the function h_η define first

$$F(z) = (z-\eta)\mathcal{X}\Big(\frac{|z-\eta|^2}{\rho^2(\eta)}\Big)e^{H_\eta(z)},$$

where H_η is a holomorphic function such that $Re\, H_\eta = h_\eta$ (see Lemma 13) and $\mathcal{X}$ is a smooth cut-off function with $\mathcal{X} \equiv 1$ for $|\zeta| < 1$, $\mathcal{X} \equiv 0$ for $|\zeta| \ge 2$ and $|\mathcal{X}'|$ bounded.

Notice that by construction and by Lemma 13, we have

$$\rho(z)|F(z)|e^{-\phi(z)} \lesssim \rho^2(\eta)e^{-\phi(\eta)}.$$

Lemma 46. *There exists u solution to $\bar\partial u = \bar\partial F$ such that $u(\eta) = \partial u(\eta) = 0$ and $\|u\|_{\mathcal{F}^\infty_{\phi,\rho}} \le C\rho^2(\eta)e^{-\phi(\eta)}$*

Once this is proved we take $h_\eta = F - u$ and we are done.

Proof. First we show that there exists a solution u as in the statement but satisfying an analogous L^2 estimate instead of the L^∞ one. We use Hörmander's theorem [Hör94]: for every ψ subharmonic in $\mathbb{C}$ there exists a solution u to $\bar\partial u = \bar\partial F$ such that

$$\int_{\mathbb{C}} |u|^2 e^{-2\psi} \le C \int_{\mathbb{C}} |\bar\partial F|^2 \frac{e^{-2\psi}}{\Delta\psi}.$$

Define $\psi = \phi + 2v$, where

$$v(z) = \log|z-\eta| - \frac{1}{\mu(D^s(\eta))}\int_{D^s(\eta)} \log|z-\zeta|\Delta\phi(\zeta)dm(\zeta).$$

Take s so that $\mu(D^s(\eta)) = 8\pi$. By the doubling condition there exists c depending only on the doubling constant $C_{\Delta\phi}$ such that $s \leq c$. Then

$$\Delta\psi \geq \Delta\phi - \frac{4\pi}{\mu(D^s(\eta))}\Delta\phi = \frac{1}{2}\Delta\phi \simeq \rho^{-2}.$$

By construction v is bounded above. Notice also that there exists $C > 0$ (independent of η) such that $-v(z) \leq C$ for all $z \in \text{supp}(\bar\partial F)$. Since $|\bar\partial F| \lesssim e^{h_\eta}$, we deduce from Hörmander's estimate and Lemma 13 that

$$\|u\|_{\mathcal{F}^2_{\phi,\rho}} \leq \int_{\mathbb{C}} |u|^2 e^{-2\psi} \leq C\int_{D^2(\eta)\setminus D(\eta)} e^{2h_\eta}e^{-2\psi}\rho^2 \lesssim \rho^4(\eta)e^{-2\phi(\eta)}.$$

On the other hand

$$e^{-2\psi(z)} \simeq |z-\eta|^{-4} \qquad \text{for} \quad |z-\eta| \leq \epsilon\rho(\eta),$$

thus necessarily $u(\eta) = \partial u(\eta) = 0$.

Let us see now that u satisfies also the L^∞ estimate. For any $z \in supp(\bar\partial F)$ define

$$U(\zeta) = \frac{K\rho(z)}{\rho^2(\eta)e^{-\phi(\eta)}}u(\zeta),$$

where $K > 0$ will be chosen later on. Then

$$\int_{D(z)} |U(\zeta)|^2 e^{-2\phi(\zeta)} \leq \frac{\rho^2(z)}{\rho^4(\eta)e^{-2\phi(\eta)}}\|u\|^2_{L^2(e^{-\phi})} \lesssim \rho^2(z).$$

Also, since $\rho(\zeta) \simeq \rho(\eta)$ on $supp(\bar\partial F)$, we have

$$\rho(z)\sup_{\zeta\in D(z)} |\bar\partial U(\zeta)|e^{-\phi(\zeta)} = \sup_{\zeta\in D(z)} \frac{K\rho^2(z)}{\rho^2(\eta)e^{-\phi(\eta)}}|\bar\partial F(\zeta)|e^{-\phi(\zeta)} \lesssim 1.$$

We choose K (independent of z) so that

(a) $\displaystyle\frac{1}{\rho^2(z)}\int_{D(z)} |U(\zeta)|^2 e^{-2\phi(\zeta)} \leq 1$,

(b) $\displaystyle\rho(z)\sup_{\zeta\in D(z)} |\bar\partial U(\zeta)|e^{-\phi(\zeta)} \leq 1.$

We will be done as soon as we prove that

$$|U(z)|e^{-\phi(z)} \leq C.$$

This is consequence of [Ber97, Lemma 3.1] applied to the function $V(\zeta) = u(\rho(z)\zeta + z)$. Defining $\phi_z(\zeta) = \phi(\rho(z)\zeta + z)$ and changing to the variable $w = \rho(z)\zeta + z$ we see that

$$\int_{\mathbb{D}} |V(\zeta)|^2 e^{-2\phi_z(\zeta)}dm(\zeta) = \int_{D(z)} |U(w)|^2 e^{-2\phi(w)}\frac{dm(w)}{\rho^2(z)} \leq 1$$

and

$$\sup_{\zeta\in\mathbb{D}} |\bar\partial V(\zeta)|^2 e^{-2\phi_z(\zeta)} = \sup_{w\in D(\eta)} |\bar\partial U(w)|^2 e^{-2\phi(w)}\rho(z) \leq 1.$$

Thus, by [Ber97, Lemma 3.1] $|V(0)|^2 e^{-\phi_z(0)} \leq Ce^{-a_{\phi_z}}$, where

$$a_{\phi_z} = \sup\{\psi(0) : \psi \leq 0 \ , \Delta\psi = \Delta\phi_\eta\}.$$

Defining v so that $\psi(z) = v(\rho(z)\zeta + z)$ we see that

$$a_{\phi_z} = \sup\{v(z) : v \leq 0 : \Delta v = \Delta\phi\}.$$

The function $v(w) = \phi(w) - h_z(w) - \phi(z) - A$ is negative in $D(z)$ if A is big enough (Lemma 13) and $v(z) = -A$. Hence $a_{\phi_z} \geq -A$ and $|U(z)|^2 e^{-2\phi(z)} = |V(0)|^2 e^{-2\phi_z(0)} \leq Ce^A$, as desired. ■

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