pp. XX–XX

CLASSICAL CONVERSE THEOREMS IN LYAPUNOV'S SECOND METHOD

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ABSTRACT. Lyapunov's second or direct method is one of the most widely used techniques for investigating stability properties of dynamical systems. This technique makes use of an auxiliary function, called a Lyapunov function, to ascertain stability properties for a specific system without the need to generate system solutions. An important question is the converse or reversability of Lyapunov's second method; i.e., given a specific stability property does there exist an appropriate Lyapunov function? We survey some of the available answers to this question.

1. Introduction. Over the last 100 years, Lyapunov's second, or direct, method has arguably been the most widely used technique for analyzing stability properties of various types of mathematically described dynamical systems, including differential and difference equations, hybrid differential-difference equations, stochastic differential equations, and many others. In his original monograph [71], Lyapunov studied ordinary differential equations and provided two results in particular that would have wide-ranging impact:

Theorem 1.1. [71, Section 16, Theorem I] If the differential equations of the disturbed motion are such that it is possible to find a definite function V, of which the derivative V' is a function of fixed sign which is opposite to that of V, or reduces identically to zero, the undisturbed motion is stable.

Theorem 1.2. [71, Section 16, Remark II] If the function V, while satisfying the conditions of the theorem, admits an infinitely small upper limit, and if its derivative represents a definite function, we can show that every disturbed motion, sufficiently near the undisturbed motion, approaches it asymptotically.

For simplicity, "undisturbed motion" can be taken to be an equilibrium point of a differential equation while "disturbed motion" refers to solutions of a differential equation originating from a point other than the equilibrium point; i.e., motions that are initially disturbed or perturbed away from the equilibrium. However, any non-trivial solution of an ordinary differential equation can be considered where a desired "reference" solution is the undisturbed motion and the disturbed motion refers to solutions perturbed away from the reference solution. Similarly, one may

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consider general attractors whereby a disturbed motion is one that originates outside the attractor.

The strength of Lyapunov's second method as encapsulated in Theorems 1.1 and 1.2 is that it is possible to ascertain stability without solving the underlying differential equation. However, the difficulty of Theorems 1.1 and 1.2 lies in finding an appropriate function V. Therefore, the converse or existence question arises; i.e., if an "undisturbed motion" or equilibrium point is stable or asymptotically stable, does an appropriate function V exist? A related question is: how can such a function be constructed? The first question is the subject of this survey paper, while the second is very much the subject of ongoing research.

In addition to their intrinsic mathematical interest, converse Lyapunov theorems are important in that they indicate which stability properties can always be established by an appropriate Lyapunov function. In fact, the study of the converse question was crucial in discovering that Theorem 1.2 implies more than asymptotic stability and, in fact, implies *uniform* (with respect to initial time) asymptotic stability. Converse Lyapunov theorems are also a useful tool when considering perturbed systems, where the perturbations may be additive to the system equations [10], [72], time-delays in the system equations [58], or as the result of a linearization [71]. It was, in fact, this latter concern that motivated Lyapunov to develop his second method. Finally, in the development of numerical, constructive techniques for Lyapunov functions, converse Lyapunov theorems provide a gold-standard by which these techniques can be measured¹. For a thorough survey of computational methods for Lyapunov functions, see [25], which is in the same special issue as this paper.

This survey is organized as follows. In Section 2 we provide the basic theory of Lyapunov's second method. In Section 3 we describe how Lyapunov functions can be constructed for linear systems via an algebraic approach and in Section 4 we discuss extensions to linear systems in feedback with static nonlinearities; i.e., so-called Lur'e systems. In Section 5 we present a constructive technique for autonomous systems based on solution of a partial differential equation. In the results of both Section 3 and Section 5, the assumption of asymptotic stability guarantees that the described techniques yield a Lyapunov function. In Section 6 we trace the historical development of converse Lyapunov theorems and briefly describe some of the approaches used. In Section 7 we present some specific converse Lyapunov theorems for so-called \mathcal{KL} -stability of differential and difference inclusions; a concept equivalent to uniform global asymptotic stability. In Section 8 we present results for unstable equilibrium points. Some concluding remarks are contained in Section 9.

2. Lyapunov's second method. Initially, we will consider dynamical systems described by ordinary differential equations

$$\dot{x} = f(x, t) \tag{1}$$

where, for simplicity, we assume that $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is locally Lipschitz in x and continuous in t so that (local) existence and uniqueness of solutions is guaranteed.

¹This is similar to the role played by Shannon's channel coding theorem in information theory [98] (see also [22, Section 7.7]). Shannon's theorem provides a fundamental limit for communication over a noisy channel by showing that a capacity-achieving channel coding scheme must exist, but does not constructively provide a coding scheme that achieves that limit. Nonetheless, Shannon's limit has been an invaluable idealized goal for information and coding theorists for over 60 years.

Forward completeness will generally follow from assumed stability properties or the presence of a Lyapunov function. We denote a solution to (1) from the initial state $x \in \mathbb{R}^n$ and initial time $t_0 \in \mathbb{R}_{\geq 0}$ at the time $t \geq t_0$ by $\phi(t, t_0, x)$. In other words, $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\frac{d}{dt}\phi(t,t_0,x) = f(\phi(t,t_0,x),t),$$
(2)

and we write $\phi \in \mathcal{S}_{t_0}(x)$ where $\mathcal{S}_{t_0}(x)$ denotes the set of solutions from initial time $t_0 \in \mathbb{R}_{\geq 0}$ and initial state $x \in \mathbb{R}^n$. As we initially assume uniqueness of solutions for (1), $\mathcal{S}_{t_0}(x)$ contains a single function. In the sequel when we consider difference or differential inclusions, or (1) where the righthand side is only continuous, the set $\mathcal{S}_{t_0}(x)$ will, in general, be larger than merely a singleton. We further assume that f(0,t) = 0 for all $t \geq t_0$ so that the origin is an equilibrium point.

Lyapunov precisely defined the notion of stability. However, many types of stability are possible in general and the most useful notions, presented below, are largely due to Chetaev [16], Malkin [72], Massera [75], and Barbashin and Krasovskii [11].

Throughout this survey we will make use of the comparison functions introduced by Massera [75] and Hahn [32]. The use of such functions simplifies many statements and proofs in the area of systems theory. To denote the class of positive definite functions with domain $\mathbb{R}_{\geq 0}$ we use the notation \mathcal{P} ($\rho \in \mathcal{P}$); i.e., functions $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that are continuous, zero at zero, and strictly positive elsewhere. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class- \mathcal{K} ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to be of class- \mathcal{K}_{∞} ($\alpha \in \mathcal{K}_{\infty}$) if, in addition, it approaches infinity as its argument approaches infinity. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class- \mathcal{L} ($\sigma \in \mathcal{L}$) if it is continuous, strictly decreasing, and approaches zero as its argument approaches infinity. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class- $\mathcal{K}\mathcal{L}$ ($\beta \in \mathcal{K}\mathcal{L}$) if it is of class- \mathcal{K} in its first argument and of class- \mathcal{L} in its second argument. For a more extensive introduction to such comparison functions, see [42].

In what follows, for a set $\mathcal{D} \subset \mathbb{R}^n$ we denote its boundary by $\partial \mathcal{D}$ and its closure by $\overline{\mathcal{D}}$. We denote the open ball of radius $\varepsilon \in \mathbb{R}_{>0}$, centered at the origin, by

$$\mathcal{B}_{\varepsilon} := \{ x \in \mathbb{R}^n : |x| < \varepsilon \}$$

and we write $\mathcal{B} = \mathcal{B}_1$.

Definition 2.1. The origin is said to be *stable for* (1) if there exists a neighborhood of the origin $\mathcal{N} \subset \mathbb{R}^n$ so that for each $t_0 \in \mathbb{R}_{\geq 0}$ there exists $\alpha_{t_0} \in \mathcal{K}$ so that, for all $x \in \mathcal{N}$ and all $t \geq t_0$,

$$|\phi(t, t_0, x)| \le \alpha_{t_0}(|x|).$$
(3)

The origin is said to be uniformly stable for (1) if the function $\alpha_{t_0} = \alpha \in \mathcal{K}$ can be chosen independent of the initial time $t_0 \in \mathbb{R}_{\geq 0}$.

Definition 2.2. The origin is said to be asymptotically stable for (1) if there exists a neighborhood of the origin $\mathcal{N} \subset \mathbb{R}^n$ so that for each $x \in \mathcal{N}$ and $t_0 \in \mathbb{R}$ there exists $\sigma_{x,t_0} \in \mathcal{L}$ so that, for all $t \geq t_0$,

$$|\phi(t, t_0, x)| \le \sigma_{x, t_0}(t - t_0). \tag{4}$$

The origin is said to be *equiasymptotically stable for* (1) if, for every $t_0 \in \mathbb{R}$, there exists a function $\beta_{t_0} \in \mathcal{KL}$ so that, for all $t \geq t_0$,

$$|\phi(t, t_0, x)| \le \beta_{t_0}(|x|, t - t_0), \quad \forall x \in \mathcal{N}.$$
(5)

The origin is said to be uniformly asymptotically stable for (1) if the function $\beta_{t_0} =$ $\beta \in \mathcal{KL}$ can be chosen independent of the initial time $t_0 \in \mathbb{R}_{\geq 0}$.

To clarify, the difference between asymptotic stability and equiasymptotic stability lies in the fact that the latter is uniform with respect to the size of the initial state, while uniform asymptotic stability requires that the stability property is uniform with respect to both the size of the initial state and the initial time. One can also define the property of *uniformly attractive* where asymptotic stability is uniform with respect to the initial time but not the initial state; i.e., $\sigma_{x,t_0} = \sigma_x \in \mathcal{L}$ in (4). However, this property appears to have found limited use and we do not consider it further.

Note that in the above definitions, the existence condition for the neighborhood $\mathcal{N} \subset \mathbb{R}^n$ leads to these stability properties sometimes being referred to as *local*. By contrast, for an open set $\mathcal{G} \subset \mathbb{R}^n$ fixed a priori and containing the origin, the above stability properties are said to hold "in the large" if they hold for all $x \in \mathcal{G}$. For example, the origin is said to be asymptotically stable in the large on $\mathcal{G} \subset \mathbb{R}^n$ if for every $x \in \mathcal{G}$ and $t_0 \in \mathbb{R}$ there exists $\sigma_{t_0} \in \mathcal{L}$ so that (4) holds. In the event that $\mathcal{G} = \mathbb{R}^n$, the above stability properties are said to be "global".

Massera [75, Theorem 7] provided several relationships amongst the above stability properties. Of particular interest, uniform stability implies stability and uniform asymptotic stability implies equiasymptotic stability which, in turn, implies asymptotic stability. Furthermore, [75, Theorem 7] demonstrates that if the righthand side of (1) is periodic in, or independent of, the time t, then the converses hold; i.e., stability implies uniform stability and asymptotic stability implies uniform asymptotic stability which, in turn, implies equiasymptotic stability.

The classical definitions for the various stability concepts are given in ε - δ terms for stability and as a combination of stability and limiting behavior as time approaches infinity for the asymptotic stability concepts. That these definitions are equivalent to the comparison function formulations presented above was shown by Hahn [32].

The following theorem summarizes Lyapunov's second method as it relates to (1) and the stability definitions described above; see [32], [58], or [93].

Theorem 2.3. Let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{>0}$ be a continuously differentiable function and consider the following conditions on V:

(i) Suppose there exists $\alpha_1 \in \mathcal{K}$ so that, for all $x \in \mathbb{R}^n$ and $t \geq t_0$

$$\alpha_1(|x|) \le V(x,t). \tag{6}$$

(ii) Suppose there exists a continuous function $\kappa : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\kappa(\cdot,t) \in \mathcal{K}$ for fixed $t \in \mathbb{R}$ and $\kappa(s,\cdot)$ is continuous, positive, monotone increasing, and unbounded for each fixed $s \in \mathbb{R}_{>0}$. Furthermore, suppose that, for all $x \in \mathbb{R}^n$ and $t \ge t_0$,

$$\kappa(|x|, t) \le V(x, t). \tag{7}$$

(iii) Suppose there exists $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous and, for all $x \in \mathbb{R}^n$ and $t \geq t_0$ $\frac{d}{dt} V(\phi(t, t_0, x), t) \Big|_{t=t_0} = \frac{\partial}{\partial t} V + \left\langle \frac{\partial}{\partial x} V, f(x, t) \right\rangle \leq -\rho(|x|). \tag{8}$

$$\left\| \left\langle \phi(\iota,\iota_0,x),\iota \right\rangle \right\|_{t=t_0} = \frac{1}{\partial t} v + \left\langle \frac{1}{\partial x} v, f(x,\iota) \right\rangle \le -\rho(|x|).$$
(8)

(iv) Suppose there exists $\alpha_2 \in \mathcal{K}$ so that, for all $x \in \mathbb{R}^n$ and $t \geq t_0$

$$V(x,t) \le \alpha_2(|x|). \tag{9}$$

The following statements hold with respect to (1):

- (a) If items (i) and (iii) hold, then the origin is stable;
- (b) If items (i), (iii), and (iv) hold, then the origin is uniformly stable;
- (c) If items (ii) and (iii) hold, then the origin is equiasymptotically stable;
- (d) If items (i), (iii), and (iv) hold with $\rho \in \mathcal{P}$, then the origin is uniformly asymptotically stable;
- (e) If items (i), (iii), and (iv) hold with $\rho \in \mathcal{P}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$, then the origin is uniformly globally asymptotically stable.

We will refer to (8) as the "derivative of V along solutions of (1)" or simply as the "total derivative of V". We note that continuous differentiability of the Lyapunov function V is not critical to the development of the theory as the key idea is that the Lyapunov function should decrease along solutions of (1). This property can be stated without any requirements on the regularity of V, but then may require explicit knowledge of the solutions. While continuous differentiability leads to the simple criterion of (8), decrease conditions involving nonsmooth derivatives (such as Dini derivatives or subgradients) for functions with weaker regularity properties can be used (see, e.g., [20], [77], [95], and the references therein).

Remark 1. A sufficient condition for (asymptotic) stability in the large on an open set $\mathcal{G} \subset \mathbb{R}^n$ containing the origin in the above theorem is the requirement that $\lim_{x\to\partial\mathcal{G}} V(x) = \infty$, where, in directions in which \mathcal{G} is unbounded this is interpreted as $\lim_{|x|\to\infty} V(x) = \infty$. Define

$$\omega(x) := \max\left\{ |x|, \frac{1}{|x|_{\partial \mathcal{G}}} - \frac{1}{|0|_{\partial \mathcal{G}}} \right\}$$

where $|x|_{\partial \mathcal{G}} := \min_{y \in \partial \mathcal{G}} |x - y|$ denotes the closest distance to the boundary of \mathcal{G} . Then, to guarantee in-the-large stability properties, we can replace the lower bound (6) by

$$\alpha_1(\omega(x)) \le V(x,t), \quad \forall x \in \mathcal{G}, \ t \in \mathbb{R}_{>0}$$

and with the requirement that $\alpha_1 \in \mathcal{K}_{\infty}$. If $\mathcal{G} = \mathbb{R}^n$ then this implies $\omega(x) = |x|$ and stability in the large coincides with global stability. This property of V is referred to as *radially unbounded* (on \mathcal{G}) by Hahn [32] and as *infinitely large* by Barbashin and Krasovskii [11].

The property of V described in item (iv) was termed *decrescent* by Hahn [32] and as an infinitely small upper bound by Lyapunov [71] as in Theorem 1.2, where the fact that the upper bound is infinitely small clearly only holds near the origin. Note that if V is continuous, independent of t, and satisfies V(0) = 0, then the decrescent bounds hold trivially. Hahn referred to the property described in Theorem 2.3.ii as strongly positive definite [32, Definition 41.5].

The converse question, then, is which of the statements in Theorem 2.3 can be reversed? For systems described by ordinary differential equations and for stability properties related to the origin, this question was largely answered by the end of the 1950's (see Section 6 below). For more general systems and more general stability properties, research is still ongoing (see Section 7 below). Prior to addressing the general existence result, we discuss three constructive techniques.

3. Linear systems. For general systems such as (1), finding an explicit closed-form Lyapunov function is known to be a difficult task. However, for linear systems,

finding such a Lyapunov function is essentially an algebraic problem. In what follows, we denote a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ by P > 0.

The first converse theorem was demonstrated in Lyapunov's original monograph for the case of linear systems

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{10}$$

Stated in modern terms, the following is [71, Section 20, Theorem II]:

Theorem 3.1. Given any Q > 0 there exists P > 0 satisfying

$$A^T P + P A = -Q \tag{11}$$

if and only if the origin is asymptotically stable for (10).

Equation (11) is referred to as the Lyapunov equation and straightforward calculations show that the quadratic $V(x) = x^T P x$ is a Lyapunov function for (10).

A similar result holds for linear discrete time systems described by

$$x^+ = Ax, \quad x \in \mathbb{R}^n. \tag{12}$$

The following result is due to Stein [107, Theorem 1] and was first mentioned in a systems theoretic context in [31] and [39].

Theorem 3.2. Given any Q > 0 there exists P > 0 satisfying

$$A^T P A - P = -Q \tag{13}$$

if and only if the origin is asymptotically stable for (12).

As in the continuous time case, (13) is called the *discrete time Lyapunov equation* (or the Stein equation after [107]) and the quadratic $V(x) = x^T P x$ is a Lyapunov function for (12).

It has proved difficult to generate converse theorems by directly constructing Lyapunov functions for systems more general than linear time-invariant systems. This can be seen in the conditions available for linear time-varying systems where (11) or (13) are replaced by matrix differential or difference equations [2, Theorem 5], [3, Theorem 4.3], for which closed form solutions generally do not exist. General nonlinear systems, naturally, present an even greater challenge.

4. Lur'e systems. Due to its importance in engineering applications, significant efforts were made to extend the converse theorems for linear systems described above to the case of so-called *absolute stability*; i.e., for asymptotic stability of the origin for linear systems with a static, memoryless, sector-bounded nonlinearity in a feedback loop. Such systems have come to be called $Lur'e^2$ systems and can be written as

$$\begin{aligned} \dot{x} &= Ax - B\psi(y) \\ y &= Cx \end{aligned}$$
 (14)

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $\psi : \mathbb{R}^p \to \mathbb{R}^p$ satisfies the sector bound

$$\langle \psi(y), y - K\psi(y) \rangle \ge 0$$
 (15)

for some positive definite symmetric matrix $K \in \mathbb{R}^{p \times p}$. Note that, with the above definitions, the system is assumed to have the same number of inputs as outputs; i.e., $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{p \times n}$.

 $^{^2 {\}rm The}$ Cyrillic-to-Latin transliteration of Jypbe has led to four distinct spellings in the western literature: Lure, Lur'e, Lur'e, Luré.

The study of such systems was instigated by Lur'e and Postnikov [70] with the motivation of studying the stability of controlled systems subject to common nonlinear actuators. In particular, the matrix A is assumed to be Hurwitz (i.e., to have all its eigenvalues in the open left half plane), where this property may have been imposed by a (linear) feedback control, and the unknown but sector bounded nonlinearity may correspond to the feedback being implemented by an actuator with unmodeled characteristics such as friction or deadzones.

Despite the importance of such systems, and the significant effort expended in trying to derive a converse theorem for absolute stability, a constructive converse has yet to be found. However, for single-input single-output systems, an existence result based on a frequency domain condition was provided in three fundamental papers by Popov [85], Yakubovich [118], and Kalman [37]. The modern form of this result is called the Popov-Yakubovich-Kalman Lemma or the Positive Real Lemma (e.g., [60, Lemma D.6] or the thorough discussion in [100, Appendix H]). It is beyond the scope of this survey to deal with the Popov-Yakubovich-Kalman Lemma and its significant ramifications³. The extension from single-input single-output systems to multi-input multi-output systems was provided independently by Popov [87, Lemma 9.3.1] (also [86]) and Anderson [1].

Theorem 4.1. Suppose that, for (14), A is Hurwitz, (A, B) is controllable, (C, A) is observable. Let $G(s) := C(sI - A)^{-1}B$. Suppose the positive definite symmetric matrix K is such that $K\psi(y)$ is the gradient of a positive semidefinite scalar function; i.e.,

$$\int_{\Gamma(0,y)} \psi^T(s) K ds \ge 0, \quad \forall y \in \mathbb{R}^p,$$
(16)

where Γ is any smooth curve in \mathbb{R}^p connecting 0 and y = Cx. For any $\eta \in \mathbb{R}_{>0}$ such that $1/\eta$ is not an eigenvalue of A, if $Z(s) := I + (1 + \eta s)KG(s)$ is strictly positive real, then there exists a positive definite symmetric matrix P so that

$$V(x) = x^T P x + \int_{\Gamma(0,y)} \psi^T(s) K ds$$
(17)

is a Lyapunov function.

Note that since $K\psi(y)$ is the gradient of a scalar function, the integral (16) is path-independent [6, Theorem 10-37].

As in [1], an example K and ψ satisfying the conditions of Theorem 4.1 is when K is diagonal and the sector-bounded nonlinearities are decoupled so that each of p nonlinearities only depends on one element of y. Other examples are provided in [49, Section 10.1].

We observe that Theorem 4.1 is not a converse theorem in the usual sense in that it does not start from a stability property and then provide a Lyapunov function. Rather, it starts from the requirement that the transfer function matrix Z(s) be strictly positive real (see [49, Definition 10.3] for a definition of strict positive realness), which by the Popov-Yakubovich-Kalman Lemma implies solvability of the matrix equations

$$P\hat{A} + \hat{A}^{T}P = -L^{T}L - \varepsilon P$$

$$P\hat{B} = \hat{C}^{T} - L^{T}W$$

$$W^{T}W = \hat{D} + \hat{D}^{T}$$
(18)

³The interested reader is directed to [100, Appendix H], [49, Section 10.1], [54], and [99].

CHRISTOPHER M. KELLETT

for matrix L, positive definite symmetric matrix P, and constant $\varepsilon \in \mathbb{R}_{>0}$, where

 $\hat{A} := A, \quad \hat{B} := B, \quad \hat{C} := KC + \eta KCA, \quad \hat{D} := I + \eta KCB.$

The matrix P from (18) is then the P of the Lur'e-Postnikov Lyapunov function (17). The lack of a standard converse result highlights the difficulty inherent in finding Lyapunov functions for general nonlinear systems.

Similar results on absolute stability and Lyapunov functions are available for discrete time systems with the original study of such systems performed in a sampleddata context by Tsypkin [111], [112]. The discrete time version of the matrix conditions (18) was provided independently by Popov [87, Theorem 10.1.1] and Hitz and Anderson [33] and is similar to the difference between the Lyapunov equation (11) and the Stein equation (13):

$$\begin{aligned} \hat{A}^T P \hat{A} - P &= -L^T L \\ \hat{A}^T P \hat{B} &= \hat{C}^T - L^T W \\ W^T W &= \hat{D} + \hat{D}^T - \hat{B}^T P \hat{B} \end{aligned}$$

The study of controlled systems, and systems subject to external disturbances, that is indicated by the structure of (14), was generalized to the notion of dissipativity [115, 116] and later to input-to-state stability [104, 106], input-output-to-state stability [59] and measurement-to-error stability [35]. While Lyapunov methods are of significant importance in these topics, and existence results are available, the study of such systems is beyond the scope of this survey.

5. **Zubov's method.** Zubov [121] presented a method for estimating the domain of attraction of an autonomous ordinary differential equation

$$\dot{x} = f(x) \tag{19}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and f(0) = 0. In particular, Zubov's method constructs a Lyapunov function that guarantees asymptotic stability in the large on the domain of attraction.

If the origin is uniformly asymptotically stable for (19), then the domain of attraction $\mathcal{D} \subset \mathbb{R}^n$ for the origin is

$$\mathcal{D} := \left\{ x \in \mathbb{R}^n : \lim_{t \to \infty} \phi(t, x) = 0 \right\}.$$
 (20)

We observe that \mathcal{D} is an open set. For $\mathcal{G} \subset \mathbb{R}^n$, a function $V : \mathcal{G} \to \mathbb{R}$ is positive definite if V(0) = 0 and V(x) > 0 for all $x \in \mathcal{G} \setminus \{0\}$.

The following theorem combines [32, Theorem 34.1] and [32, Theorem 51.1].

Theorem 5.1. The origin is asymptotically stable for (19) on a domain of attraction $\mathcal{D} \subset \mathbb{R}^n$ if and only if there exist functions $V, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

- (i) V is continuous and positive definite in \mathcal{D} , $0 \leq V(x) < 1$, and $\lim_{|x|\to\partial\mathcal{D}} V(x) = 1$;
- (ii) h is continuous and positive definite; and
- (iii) the following partial differential equation is satisfied

$$\left\langle \frac{\partial}{\partial x} V(x), f(x) \right\rangle = -h(x)(1 - V(x))\sqrt{1 + |f(x)|^2}.$$
(21)

In other words, given a uniformly asymptotically stable equilibrium point, it is always possible to find a Lyapunov function, defined on the domain of attraction, that satisfies the partial differential equation (21). To construct an appropriate Lyapunov function, we start from the characterization of uniform asymptotic stability given by $\alpha \in \mathcal{K}_{\infty}$, $\sigma \in \mathcal{L}$, and

$$|\phi(t,x)| \le \alpha(|x|)\sigma(t), \quad \forall x \in \mathcal{D}, \ t \in \mathbb{R}_{\ge 0}.$$
(22)

Define

$$\sigma^{\dagger}(s) := \begin{cases} \sigma^{-1}(s) & , \quad s \in (0, \sigma(0)] \\ 0 & , \quad s \ge \sigma(0) \end{cases}$$

and

$$\varphi(s) := \begin{cases} s \exp(-\sigma^{\dagger}(s)) &, s > 0 \\ 0 &, s = 0. \end{cases}$$

$$(23)$$

That $\varphi \in \mathcal{K}_{\infty}$ follows from basic compositional properties of comparison functions; see [42]. The functions

$$V(x) := 1 - \exp\left(-\int_0^\infty \varphi(|\phi(\tau, x)|)d\tau\right), \quad \text{and}$$
(24)

$$h(x) := \frac{\varphi(|x|)}{\sqrt{1 + |f(x)|^2}}$$
(25)

can then be shown to satisfy the necessary properties of Theorem 5.1. See [32, Theorem 51.1] for a complete proof.

We observe that, by definition, the origin is uniformly asymptotically stable in the large on its domain of attraction. In Remark 1 we observed that a Lyapunov function that is radially unbounded on \mathcal{D} can be used to conclude in-the-large stability properties. However, as we see in Zubov's method, the derived Lyapunov function approaches the value 1 on the boundary of the domain of attraction. The critical observation is that this leads to V being *proper* on the domain of attraction, where the term proper refers to preimages of compact sets being compact. In other words, for any compact set $[0,c] \subset [0,1)$, the set defined by $V^{-1}([0,c])$ must also be compact in \mathbb{R}^n . If V is proper and the time derivative of V along solutions is negative definite, then trajectories necessarily move from larger (compact) level sets to smaller (compact) level sets. The radially unbounded on \mathcal{D} condition of Remark 1 is a sufficient condition for V to be proper.⁴

Furthermore, we note that $\mu(s) := -\log(1-s) \text{ maps } [0,1) \mapsto [0,\infty)$, is continuously differentiable, and strictly increasing. Consequently, the function $W(x) := \mu(V(x))$ will be a Lyapunov function for asymptotic stability of the origin in the large on \mathcal{D} as defined by Barbashin and Krasovskii where $\lim_{x\to\partial\mathcal{D}} W(x) = \infty$. Straightforward manipulations of (24) yield that

$$W(x) = \int_0^\infty \varphi(|\phi(\tau, x)|) d\tau.$$
(26)

We will see this Lyapunov function candidate again in (28) below.

Zubov [121, Theorems 19 and 78] extended the above result to dynamical systems on metric spaces including time-varying systems and systems described by PDEs that admit classical solutions, as well as accounting for asymptotic stability of closed invariant sets as opposed to merely the origin.

While Zubov's method is not constructive in the same sense that solving the Lyapunov equation (11) is constructive, the freedom to choose the function h in (21)

⁴ In order to differentiate between a definition of *proper* that requires that the inverse of all compact sets to be compact (i.e., $[0,c] \subset \mathbb{R}_{\geq 0}$) and one that requires that the inverse of compact sets on the range of the function V to be compact (i.e., $[0,c] \subset [0,1)$ as above), the terminology *proper on its range* or *semiproper* is sometimes used.

CHRISTOPHER M. KELLETT

has enabled useful constructions in many cases. See [28], [30] and the references therein for recent applications and extensions of Zubov's method.

6. Historical developments. Though the converse question was answered by Lyapunov in the linear case, the converse of Lyapunov's second method in the more general case represented by (1) remained open throughout the early 1900's.

6.1. Early results - pre-1950. Persidskii [84] provided the first general converse theorem when he demonstrated that, under the assumption that the origin is a stable equilibrium point, the function

$$V(x,t) = \min_{t_0 \le \tau \le t} |\phi(\tau,t,x)|$$
(27)

is in fact a Lyapunov function.

In [74], Massera precisely defined stability, asymptotic stability, and equiasymptotic stability and compared them via examples as well as by sufficient Lyapunov function properties that guaranteed them. In the case when (1) is periodic or autonomous, and the origin is asymptotically stable, Massera showed that the semi-infinite integral

$$V(x,t) = \int_{t}^{\infty} \alpha(|\phi(\tau,t,x)|) d\tau$$
(28)

where $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is an appropriately chosen continuous function, is in fact a continuously differentiable Lyapunov function. Furthermore, Massera demonstrated that if (1) is periodic in t or independent of t then V has this same property.

Massera's manuscript [74] would have a significant impact on the study of the converse question. Not only did Massera provide the first converse theorem for asymptotic stability, but [74] left open the converse question for systems that were neither periodic nor autonomous (a problem that required the notion of uniform stability as described below). The question of the existence of a *smooth* Lyapunov function remained, as did whether or not the assumption in [74] of continuous differentiability of the righthand side of (1) was necessary. Finally, the proof technique used by Massera became the standard approach in much subsequent work. In particular, most subsequent authors have proposed Lyapunov function candidates similar to the semi-infinite integral of (28) and, frequently, the choice of the scaling α is done either directly from, or similar to, that from what is now called "Massera's Lemma" [74, p. 716].

Contemporaneously with [74], and using a different proof technique, Barbashin [10] demonstrated that, for an autonomous system (1), there exists a Lyapunov function with the same regularity as that of the vector field f.

6.2. Fundamental theory - 1950's. Malkin [72] recognized that the important, and more general, property that allowed Massera to derive converse theorems for periodic and autonomous systems with an asymptotically stable equilibrium point is that of uniformity with respect to time. Furthermore, Malkin demonstrated that, as it was originally written in [71, Section 16, Remark 2] (Theorem 1.2 in Section 1), Lyapunov's second method in fact guarantees *uniform* asymptotic stability. In particular, the uniformity follows from the assumed decrescent property of the Lyapunov function

Around the same time, Barbashin and Krasovskii [11] demonstrated that a sufficient condition for (asymptotic) stability in the large on a set $\mathcal{G} \subseteq \mathbb{R}^n$ is that the Lyapunov function be radially unbounded on \mathcal{G} . Subsequently, following Malkin and Massera's proof techniques, Barbashin and Krasovskii [12] demonstrated that a radially unbounded (on \mathcal{G}) and decrescent Lyapunov function is necessary and sufficient for uniform (asymptotic) stability in the large (on \mathcal{G}) of the origin.

Converse results for stability and uniform stability were initially developed by Krasovskii [56] and Kurzweil [61]. At the same time, without assuming uniqueness of solutions, Yoshizawa presented a continuous Lyapunov function assuming stability of the origin in [119]. Smooth Lyapunov functions for stability and uniform stability of the origin, without the assumption of unique trajectories, were provided by Kurzweil and Vrkoc [63].

Almost simultaneously⁵, Kurzweil [62] and Massera [75] demonstrated that, when the righthand side of (1) is continuous, if the origin is uniformly globally asymptotically stable⁶ then there exists a smooth (infinitely differentiable) Lyapunov function. While Massera assumed unique solutions to (1), Kurzweil did not⁷. In order to accommodate the lack of a unique solution, Kurzweil extended Massera's construction (28) by taking the supremum over all solutions from the initial condition, $x \in \mathbb{R}^n$:

$$V(x,t) = \sup_{\phi \in \mathcal{S}(x)} \int_{t}^{\infty} \alpha(|\phi(\tau,t,x)|) d\tau,$$
(29)

where, as in (28), $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is an appropriately chosen continuous function. The functions defined by (28) and (29) can be shown to be locally Lipschitz, after which a transfinite smoothing procedure that maintains the desired Lyapunov function properties, is applied. As in [74], both [75] and [62] show that if the righthand side of (1) is periodic in t or independent of t, then so is the derived Lyapunov function. It is worth noting that for (uniform) stability this does not hold; i.e., even for systems (1) that are independent of t and that possess a (uniformly) stable equilibrium point, it is not always possible to find a Lyapunov function that is independent of t (see [58, p. 46]).

By the end of the 1950's, answers to most of the converse questions for Theorem 2.3, including its in-the-large variants, had thus been obtained. Subsequent research focused on more general systems and on more general stability concepts. The one remaining converse from Theorem 2.3 relates to equiasymptotic stability of the origin. This converse appears to have been originally derived by Hahn in [32, Theorem 49.1], where, similar to the above observation on (uniform) stability, even for systems independent of t it is not always possible to find a Lyapunov function that is independent of t.

6.3. Extensions and consolidation - the 1960's. Lyapunov's second method was extended to so-called "general dynamical systems", namely dynamical systems axiomatically defined based on the attainability sets of differential equations without unique solutions on metric spaces. Research on such systems was initiated by Barbashin [9] and converse Lyapunov theorems for systems with unique solutions

⁵Twice in the 1950's, similar results were submitted almost simultaneously. Similar results were published by Krasovskii [56] (submitted 12 November 1954) and Kurzweil [61] (submitted 2 December 1954). Again involving Kurzweil, similar results were published by Kurzweil [62] (submitted 6 July 1955) and Massera [75] (submitted 30 August 1955).

⁶Kurzweil's result is actually for uniform asymptotic stability in the large on the domain of attraction $\mathcal{D} \subseteq \mathbb{R}^n$.

⁷Despite the similar results on the existence of a smooth Lyapunov function, Massera's assumption of unique solutions allowed a much shorter proof. In fact, [75] grew out of a short course Massera provided in Varenna, Italy in 1954 and contains a nice survey of many topics in stability theory.

were provided by Zubov [121]. The extension to systems without unique solutions was provided in a series of papers by Roxin [95], [96], and [97], where a distinction was made between the stability behavior of all solutions and the stability behavior of at least one solution. As we describe precisely in Definition 7.4 below, these stability properties are termed *strong stability* and *weak stability*, respectively. An excellent summary of the initial work on general dynamical systems can be found in [51].

As a particular case of both partial stability [113] and stability with respect to two measures [82], Hoppensteadt [34] derived a continuously differentiable Lyapunov function for asymptotic stability of the origin for a parametrized non-autonomous differential equation where the parameters take values in an unbounded set. Wilson [117] then extended this by deriving a smooth Lyapunov function under the assumption of uniform asymptotic stability of a closed, but not necessarily bounded, set and Lakshmikantham and Salvadori [66] provided a continuous Lyapunov function under the assumption of stability with respect to two measures.

In the 1960's, five books appeared in English that summarized much of the available theory on Lyapunov's second method, including results on the converse question [31, 58, 121, 120, 32]. Along with the survey papers [5] and [38] and the very readable [65], these texts made Lyapunov's methods widely accessible to the West⁸. Of particular note, similar to the candidate Lyapunov function he used in [119], Yoshizawa proved his converse theorems in [120] based on a function defined as

$$V(x,t) = \sup_{\phi \in \mathcal{S}(x), \ \tau \ge t} \alpha(|\phi(\tau,t,x)|)e^{c\tau}$$
(30)

where $\alpha \in \mathcal{K}_{\infty}$ and $c \in \mathbb{R}_{>0}$ are chosen appropriately.

Krasovskii [58], on the other hand, used a different proof technique to that initially proposed by Massera in [74]. Of particular note is that Krasovskii's technique allowed him to derive a converse theorem not just for asymptotic stability, but also for Lyapunov's first instability theorem (see Section 8). The core of this technique rests on what Krasovskii labels "Property A" [58, Definition 4.1]:

Property A: Let $\{h_k\}_{k=0}^{\infty}$ be a monotonically decreasing sequence satisfying

$$h_0 = |0|_{\mathcal{G}}, \qquad \lim_{k \to \infty} h_k = 0. \tag{31}$$

 $^{^{8}}$ It is tempting to speculate that the wealth of translated material in the 1960's was directly due to the initial accomplishments of the Soviet Union in the space race with the launch of the Sputnik satellite in October 1957 and Yuri Gagarin's orbital flight in April 1961. In particular, what appear to be hasty translations by government departments were made of the 1951 text of Lur'e [69] in 1957 in the United Kingdom by Her Majesty's Stationery Office, and of the 1956 text of Malkin [73] in 1959 by the United States Atomic Energy Commission. Other important Soviet texts that were translated around this time include the 1955 text of Letov [67] (translated 1961), the 1956 text of Chetaev [16] (translated 1961), the 1957 text of Zubov [121] (translated 1964), and the 1959 text of Krasovskii [58] (translated 1963). Furthermore, the preeminent Russian language control journal published since 1936, Прикладная Математика и Механика (Prikladnaya Matematika i Mekhanika), was regularly translated as the Journal of Applied Mathematics and Mechanics beginning in 1958. While Cold War tensions, and the space race in particular, is likely one driver behind this rush of translations, a level of translation activity that has not been seen since, it is also worth noting that the International Federation of Automatic Control was formed in 1957 with a goal of international scientific exchange and whose first two presidents came from the United States of America (H. Chestnut, 1957–1959) and the Soviet Union (A. L. Letov, 1959–1961). Historical narratives are rarely simple.

Suppose that for every closed bounded region $H \subset \mathbb{R}^n$ satisfying $0 \in \overline{H} \subset \mathcal{G}$, and for every k > 0 there is a number T_k such that whenever $t_0 \geq T_k, x \in \mathcal{G} \setminus \mathcal{B}_{h_k}$, there exists $t \in [t_0 - T_k, t_0 + T_k]$ so that $\phi(t, t_0, x) \notin H \setminus \{0\}$.

In words, for any neighborhood of the origin in \mathcal{G} and for a sequence of decreasing balls (centered at the origin), solutions cannot stay in H indefinitely. Intuitively, this property is satisfied for both asymptotically stable and unstable equilibria – though not necessarily for stable equilibria. Krasovskii then demonstrated [58, Theorem 4.3] that Property A is equivalent to the existence of a function V that is decrescent and such that its derivative along solutions of (1) is sign-definite. This provides one of the main technical tools in [58] to derive converse theorems for asymptotic stability and for instability (see Section 8 below for the latter). An interesting result, apparently not available elsewhere, is a converse theorem for equiasymptotic stability of the origin [58, Theorem 10.2] where the total derivative of the Lyapunov function is negative semidefinite and the supremum of the total derivative integrates to negative infinity.

6.4. **1970's onward.** In the late 1970s researchers began to examine differential and difference inclusions. Roxin [94] demonstrated how differential inclusions (also called contingent equations) give rise to the general dynamical systems of Barbashin [9], and so the specific results for differential inclusions could be seen as a special case of Roxin's results. However, the strength of Lyapunov's second method has generally been that one need not generate system trajectories whereas the general systems approach of Roxin requires knowledge of the attainability function or the set of solutions. The specialization to difference and differential inclusions allows the formulation of decrease conditions that depend on the set-valued mapping defining the inclusion rather than requiring knowledge of solutions.

Meilakhs [76] derived a continuously differentiable Lyapunov function given uniform asymptotic stability of the origin for all solutions of a differential inclusion derived from a parametrized differential equation where the parameters vary over a closed bounded linearly connected set. Molchanov and Pyatnitskii studied the problem of absolute stability described in Section 4. In [78] and [79] they formulated the Lur'e problem as a stability problem for a differential inclusion and demonstrated the existence of a Lyapunov function of an approximate quadratic form. Similar to the algebraic criteria of Section 3, in [80] and [81] Molchanov and Pyatnitskii then derived necessary and sufficient criteria for a Lyapunov function in terms of solvability of certain matrix equations.

A result on the existence of a so-called *control Lyapunov function* under the assumption of asymptotic controllability to the origin was provided by Sontag [103]. This is closely related to the converse question for weak asymptotic stability of the origin for differential inclusions, which Smirnov answered in [101] and [102] for differential inclusions described by convex processes. Converse theorems for both weak and strong stability of time-varying differential inclusions defined on a real Banach space were provided by Deimling [23, Propositions 14.1 and 14.2].

A converse theorem for uniform global asymptotic stability of a compact set for a differential inclusion under fairly weak assumptions was provided by Lin et al. [68] and converse theorems for both uniform global strong and weak asymptotic stability were provided by Clarke et al. [19] where a particular impediment in the weak case was identified. We discuss this impediment in Section 7.2. The first converse Lyapunov theorems for discrete time systems described by nonautonomous ordinary difference equations were derived by Gordon [27] for stability and uniform asymptotic stability of the origin. A converse theorem for strong uniform global asymptotic stability of a difference inclusion was provided by Jiang and Wang [36].

Similar to previous results that time-independent or periodic systems yield timeindependent or periodic Lyapunov functions, Rosier [92] demonstrated that for homogeneous systems with non-unique solutions, an asymptotically stable origin implies the existence of a smooth homogeneous Lyapunov function.

Converse theorems for both strong and weak stability with respect to two measures for both differential and difference inclusions were provided by Teel and Praly [110] and by the author and Teel in [45], [46], [47], and [48]. As these general results subsume much previous work as special cases, we specifically survey these results in Section 7.

Recently, Karafyllis and Tsinias [41] and Karafyllis [40] developed converse theorems for strong equiasymptotic stability of the origin for differential and difference inclusions arising from perturbed difference and differential equations. Rather than equiasymptotic stability they use the terminology non-uniform in time stability.

Also recently, Kloeden and co-authors [52, 53, 29] noted that nonautonomous systems naturally give rise to nonautonomous invariant sets. This then leads to three notions of attractor and stability, refered to as pullback, forward, and uniform attractor/stability. Similar to Massera's result [75, Theorem 7] for systems periodic in, or independent of, time the definitions of pullback, forward, and uniform attractor coincide. Appropriate Lyapunov functions were defined and converse results presented in [29, Theorem 29] for pullback, forward, and uniform attractors/stability of nonautonomous differential equations, while [53] presents a converse result for pullback attraction of a nonautonomous difference equation. The constructions used in these references are similar to that proposed by Yoshizawa (30).

7. \mathcal{KL} -stability with respect to two measures for difference and differential inclusions. For a set-valued map $F(\cdot)$ we use the notation $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to denote that $F(\cdot)$ maps points in \mathbb{R}^n to subsets of \mathbb{R}^n . In this section, for comparative purposes, we present some specific converse theorems for difference inclusions

$$x^+ \in F(x) \tag{32}$$

and differential inclusions

$$\dot{x} \in F(x) \tag{33}$$

where $F : \mathcal{G} \Rightarrow \mathcal{G}$ for (32) and $F : \mathcal{G} \Rightarrow \mathbb{R}^n$ for (33), and where $\mathcal{G} \subset \mathbb{R}^n$. In an abuse of notation, in order to avoid unnecessary duplication in the results that follow, we use t both as $t \in \mathbb{R}_{\geq 0}$ when refering to the continuous time system (33) and $t \in \mathbb{Z}_{\geq 0}$ when refering to the discrete time system (32).

For completeness, we provide here regularity definitions for set-valued maps. Note that, for sets $A, B \subset \mathbb{R}^n$, $A + B \subset \mathbb{R}^n$ denotes the Minkowski sum.

Definition 7.1. Let $\mathcal{O} \subset \mathbb{R}^n$ be open. The set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is:

- upper semicontinuous on \mathcal{O} if for each $x \in \mathcal{O}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\xi \in \mathcal{O}$ satisfying $|x - \xi| < \delta$ we have $F(\xi) \subset F(x) + \mathcal{B}_{\varepsilon}$;
- continuous on \mathcal{O} if, in addition to being upper semicontinuous on \mathcal{O} , for each $x \in \mathcal{O}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that, for $\xi \in \mathcal{O}$ satisfying $|\xi x| < \delta$ we have $F(x) \subset F(\xi) + \mathcal{B}_{\varepsilon}$; and

• locally Lipschitz on \mathcal{O} if for each $x \in \mathcal{O}$ there exists a neighborhood $\mathcal{U} \subset \mathcal{O}$ of x and L > 0 such that $x_1, x_2 \in \mathcal{O}$ implies $F(x_1) \subset F(x_2) + L|x_1 - x_2|\overline{\mathcal{B}}$.

Note that the concept of upper semicontinuity for a set-valued map is not the same as that for extended real-valued functions. In fact, for $f : \mathbb{R}^n \to \mathbb{R}^n$, the set-valued map $x \mapsto \{f(x)\}$ is upper semicontinuous if and only if the extended real-valued function $x \mapsto f(x)$ is continuous.

The results of this section generally require a common set of assumptions with regards to the set-valued map defining the difference or differential inclusion.

Definition 7.2. The set-valued map $F : \mathcal{G} \Rightarrow \mathcal{G}$ satisfies the *discrete time basic conditions* if, on \mathcal{G} , it has nonempty and compact values, and is upper semicontinuous.

Definition 7.3. The set-valued map $F : \mathcal{G} \rightrightarrows \mathbb{R}^n$ satisfies the *continuous time* basic conditions if, on \mathcal{G} , it has nonempty, compact, and convex values, and is upper semicontinuous.

The continuous time basic conditions are essentially required in order to guarantee existence of solutions to the differential inclusion (see [24]). These conditions also provide certain technical properties on the solution sets. By contrast, solutions to the difference inclusion (32) will exist so long as the mapping is nonempty. However, the discrete time basic conditions enable certain technical results such as closeness of solutions properties (see [47]).

For systems that do not give rise to unique solutions there are two natural stability notions that were identified by Roxin [95]. The first is the property that *all* solutions must satisfy a desired stability estimate while the second is the property that *at least one* solution must satisfy a desired stability estimate. Roxin termed these properties "strong stability" and "weak stability", respectively.

In the framework of difference and differential inclusions, the subsequent results subsume many commonly encountered system models including ordinary difference and differential equations, such systems with discontinuous righthand sides, and controlled or perturbed systems. To further extend the reach of these results, we can consider a generalization of uniform global asymptotic stability that was introduced by Movchan [82] refered to as stability with respect to two measures or stability with respect to two metrics.

Definition 7.4. Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, i = 1, 2, be continuous functions. We say that (32) (or (33)) is strongly \mathcal{KL} -stable with respect to (ω_1, ω_2) if ((33) is forward complete and) there exists $\beta \in \mathcal{KL}$ such that for every initial condition $x \in \mathcal{G}$ all solutions $\phi \in \mathcal{S}(x)$ satisfy

$$\omega_1(\phi(t,x)) \le \beta(\omega_2(x),t), \quad \forall t \in \mathbb{Z}_{>0} \ (\forall t \in \mathbb{R}_{>0}).$$
(34)

We say that (32) (or (33)) is weakly \mathcal{KL} -stable with respect to (ω_1, ω_2) if the above property holds for at least one solution $\phi \in \mathcal{S}(x)$.

Note that, in some sense, both "stability with respect to two measures" and "stability with respect to two metrics" are unsatisfactory terminology as the functions ω_i are neither measures nor metrics in the usual mathematical sense of measure or metric. Nonetheless, the usage has become standard and we will use the terminology "stability with respect to two measures".

Observe that forward completeness is only explicitly required for continuous time strong \mathcal{KL} -stability with respect to two measures as this is not guaranteed *a priori*

by the stability estimate. Forward completeness of difference inclusions is guaranteed by virtue of the set-valued mapping taking points in \mathcal{G} to subsets of \mathcal{G} . Finally, in the case of continuous time weak \mathcal{KL} -stability, since we are not necessarily interested in the behavior of all solutions, it may in fact be the case that some solutions cannot be continued for all time.

 \mathcal{KL} -stability with respect to two measures is a generalization of uniform asymptotic stability of the origin in the large on \mathcal{G} . This is the case where $\omega_1(x) = \omega_2(x) = |x|$ and, hence, when additionally $\mathcal{G} = \mathbb{R}^n$, we see that \mathcal{KL} -stability with respect to $(|\cdot|, |\cdot|)$ is, in fact, uniform global asymptotic stability of the origin. Additionally, this stability property also encompasses uniform asymptotic stability of a closed set $\mathcal{A} \subset \mathcal{G}$ by taking $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}$ where $|x|_{\mathcal{A}} = \min_{y \in \mathcal{A}} |x - y|$. Many other examples, such as output stability and stability of a particular trajectory are possible.

As a particular example, it is possible to deal with non-autonomous systems (1) as autonomous systems by the technique of state augmentation; that is, consider states $x = (z, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and

$$\dot{z} = f(z,t)
\dot{t} = 1.$$
(35)

Then uniform global asymptotic stability of the origin is equivalent to \mathcal{KL} -stability with respect to (ω, ω) where $\omega(z, t) = |z|$ for all $(z, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. A similar approach can be taken for systems (1) parametrized by a parameter vector, $\theta \in \mathbb{R}^m$, where the system equations can be augmented by $\dot{\theta} = 0$. Note that a limitation of this technique is that it is necessary to impose regularity conditions on the t or θ -dependence of f than are strictly required.

As another example, consider the second order system

$$\dot{x}_1 = x_2 + (1 - x_1^2 - x_2^2)x_1
\dot{x}_2 = -x_1 + (1 - x_1^2 - x_2^2)x_2.$$
(36)

This system has the origin as an unstable equilibrium point and the unit circle as a uniformly asymptotically stable periodic orbit. For $x \in \mathcal{G} = \mathbb{R}^2 \setminus \{0\}$, define the function

$$\omega(x) := \begin{cases} \frac{1-|x|}{|x|}, & x \in \mathcal{B} \setminus \{0\}\\ |x|-1, & x \in \mathbb{R}^2 \setminus \mathcal{B}. \end{cases}$$
(37)

Then (36) is \mathcal{KL} -stable with respect to (ω, ω) which captures the system behavior both in terms of the unstable equilibrium as well as the asymptotically stable periodic orbit.

7.1. Converse theorems for strong \mathcal{KL} -stability. In the context of differential equations with unique solutions, Massera observed that certain stability properties, namely equiasymptotic stability and uniform asymptotic stability, have an inherent robustness property in that the set of solutions is an open set [75, Theorem 8]. In other words, near any solution that satisfies an equiasymptotic or uniform asymptotic stability estimate, there are other solutions that also satisfy that estimate. In the case of strong \mathcal{KL} -stability with respect to two measures for both difference and differential inclusions, as a first step towards various converse theorems, we make a connection between robust stability and the existence of a smooth Lyapunov function. Then, to complete a converse Lyapunov theorem, we present various conditions that guarantee robust stability.

For both difference and differential inclusions we define robust stability in terms of stability of a perturbed inclusion. Define

$$\mathcal{A} := \left\{ x \in \mathcal{G} : \sup_{t \in \mathbb{T}, \phi \in \mathcal{S}(x)} \omega_1(\phi(t, x)) = 0 \right\}$$

where $\mathbb{T} = \mathbb{Z}_{\geq 0}$ for (32) and $\mathbb{T} = \mathbb{R}_{\geq 0}$ for (33). For continuous functions $\sigma, \delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$, such that $\sigma(x), \delta(x) > 0$, for all $x \in \mathcal{G} \setminus \mathcal{A}$, and

$$\{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{G}, \quad \{x\} + \delta(x)\overline{\mathcal{B}} \subset \mathcal{G}$$

we define the perturbed inclusions

$$x^{+} \in F_{\sigma}(x) := \left\{ v \in \mathbb{R}^{n} : v \in \{\eta\} + \sigma(\eta)\overline{\mathcal{B}}_{n}, \ \eta \in F\left(x + \sigma(x)\overline{\mathcal{B}}_{n}\right) \right\}, \quad \text{and} \quad (38)$$
$$\dot{x} \in F_{\delta}(x) := \overline{\operatorname{co}}F\left(\{x\} + \delta(x)\overline{\mathcal{B}}\right) + \delta(x)\overline{\mathcal{B}}. \tag{39}$$

Note that since differential inclusions deal with infinitesimals it is possible to define the inner and outer perturbations of (39) on the basis of the same point $x \in \mathcal{G}$. This is in contrast to the perturbed difference inclusion (38) where the outer perturbation needs to be a superset of the set-valued map applied to the inner perturbation. Also note that it is necessary to take the closed convex hull in (39) to ensure that F_{δ} satisfies the continuous time basic conditions.

If (38), respectively (39), is \mathcal{KL} -stable with respect to (ω_1, ω_2) then we say that (32), respectively (33), is robustly \mathcal{KL} -stable with respect to (ω_1, ω_2) .

Theorem 7.5. [47, Theorem 2.7] Let $F : \mathcal{G} \Rightarrow \mathcal{G}$ satisfy the discrete time basic conditions on \mathcal{G} . The difference inclusion (32) is robustly \mathcal{KL} -stable with respect to (ω_1, ω_2) if and only if there exists a smooth Lyapunov function with respect to (ω_1, ω_2) on \mathcal{G} ; i.e., a smooth function $V : \mathcal{G} \rightarrow \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that for all $x \in \mathcal{G}$

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)) \tag{40}$$

$$\max_{f \in F(x)} V(f) \le V(x)e^{-1} \tag{41}$$

where e^{-1} is the exponential function evaluated at -1.

Theorem 7.6. [110, Theorem 1] Let $F : \mathcal{G} \Rightarrow \mathbb{R}^n$ satisfy the basic conditions on \mathcal{G} . The differential inclusion (33) is robustly \mathcal{KL} -stable with respect to (ω_1, ω_2) if and only if (33) is forward complete on \mathcal{G} and there exists a smooth Lyapunov function with respect to (ω_1, ω_2) on \mathcal{G} ; i.e., a smooth function $V : \mathcal{G} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathcal{G}$

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)) \tag{42}$$

$$\max_{w \in F(x)} \left\langle \frac{\partial}{\partial x} V(x), w \right\rangle \le -V(x). \tag{43}$$

Observe that forward completeness is explicitly required for differential inclusions whereas this is guaranteed for difference inclusions by virtue of the fact that the set-valued map takes points in \mathcal{G} to subsets of \mathcal{G} ; i.e., by definition, solutions to (32) cannot escape \mathcal{G} . Demonstrating forward completeness for differential inclusions can be accomplished via Lyapunov methods [4].

The decrease conditions (41) and (43) guarantee that the Lyapunov functions decrease exponentially along solutions of (32) and (33), respectively, where the constant e^{-1} is chosen to mirror the exponential decrease implied by (43). Given any Lyapunov function that does not decrease exponentially, it is always possible to find a nonlinear scaling such that the nonlinear scaling of the Lyapunov function is

also a Lyapunov function and decreases exponentially. It is important to note that this does not imply that system solutions decrease exponentially. In fact, a sufficient condition for the exponential decrease of solutions (i.e., exponential stability) is that the decrease condition (43) is satisfied and that the upper and lower bounds (42) be quadratic.

In order to compare the Lyapunov function given by (42)-(43) and that in Theorem 2.3 we briefly examine (35). For uniform global asymptotic stability of the origin we observed that, for all $x = (z,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, $\omega_1(x) = \omega_2(x) = |z|$. Therefore, for (35), (42) is

$$\alpha_1(|z|) \le V(z,t) \le \alpha_2(|z|)$$

which is precisely items (i) and (iv) of Theorem 2.3. Furthermore, (43) is

$$\max_{w = [f(z,t)^T \ 1]^T} \left\langle \left[\frac{\partial V}{\partial z}^T \ \frac{\partial V}{\partial t} \right]^T, w \right\rangle = \frac{\partial V}{\partial z}^T f(z,t) + \frac{\partial V}{\partial t} \le -V(x) \le -\alpha_1(|z|)$$

which is precisely item (iii) of Theorem 2.3 where $\rho = \alpha_1 \in \mathcal{K}_{\infty}$. Therefore, item (e) of Theorem 2.3 implies that z = 0 is uniformly globally asymptotically stable for (35).

It remains an open question for both difference and differential inclusions as to whether or not \mathcal{KL} -stability with respect to (ω_1, ω_2) is generally robust. However, sufficient conditions for robustness have been demonstrated in several special cases. Many of these conditions are similar between discrete and continuous time. One such condition is related to the regularity of the set-valued map.

Theorem 7.7. [47, Theorem 2.10] Let $F : \mathcal{G} \Rightarrow \mathcal{G}$ satisfy the discrete time basic conditions on \mathcal{G} and be continuous on an open set containing $\mathcal{G} \setminus \mathcal{A}$. If (32) is strongly \mathcal{KL} -stable with respect to (ω_1, ω_2) on \mathcal{G} , then it is robustly \mathcal{KL} -stable with respect to (ω_1, ω_2) on \mathcal{G} .

Theorem 7.8. [110, Theorem 2] Let $F : \mathcal{G} \Rightarrow \mathbb{R}^n$ satisfy the continuous time basic conditions on \mathcal{G} and be locally Lipschitz on an open set containing $\mathcal{G} \setminus \mathcal{A}$. If (33) is strongly \mathcal{KL} -stable with respect to (ω_1, ω_2) on \mathcal{G} , then it is robustly \mathcal{KL} -stable with respect to (ω_1, ω_2) on \mathcal{G} .

Note that Theorems 7.6 and 7.8 imply the existence of a Lyapunov function on $\mathbb{R}^2 \setminus \{0\}$ for (36) with respect to the measurement function of (37). Such a Lyapunov function is closely related to Conley's complete Lyapunov functions [21] which are defined on the entire space.

Other sufficient conditions for robust \mathcal{KL} -stability of difference and differential inclusions have been provided in [47], [48], and [110]. In particular, for a compact attractor $\mathcal{A} \subset \mathbb{R}^n$, when $\omega_1 = \omega_2 = \omega$ is a proper indicator function⁹ for \mathcal{A} on its domain of attraction $\mathcal{D} \subset \mathbb{R}^n$, the basic conditions are sufficient to guarantee robustness; i.e., the extra regularity of Theorems 7.7 and 7.8 is not required. An additional sufficient condition for robustness in the case of a single measurement function is related to how solutions behave in reverse time; see [48, Theorem 4] and [110, Theorem 3]. Finally, for difference inclusions where the set-valued mapping has no specific regularity requirement but is compact and nonempty, the existence of a continuous Lyapunov function as described in Theorem 7.5 is sufficient to guarantee robust \mathcal{KL} -stability of (32) (see [47, Theorem 2.8]).

⁹A proper indicator for \mathcal{A} on \mathcal{D} is a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\omega(x) = 0$ for $x \in \mathcal{A}, \ \omega(x) > 0$ for $x \in \mathcal{D} \setminus \mathcal{A}$, and $\lim_{x \to \partial \mathcal{D}} \omega(x) = c$ for some $c \in \mathbb{R}_{>0} \cup \infty$.

In concluding our discussion on converse theorems for strong \mathcal{KL} -stability with respect to two measures, we mention that results similar to those described above are available in the framework of hybrid systems that are defined by both difference and differential inclusions and particular rules about when solutions evolve according to the difference inclusion (32) or the differential inclusion (33); see [14], [15], and [26, Section 7.5].

7.2. Converse theorems for weak \mathcal{KL} -stability. As described above, there are many cases in which strong \mathcal{KL} -stability with respect to two measures is in fact robust and, consequently, several converse Lyapunov theorems are possible. Fewer results have been obtained in the case of weak \mathcal{KL} -stability with respect to two measures. In fact, currently available results are limited to the case where both measurement functions are given by the distance to a closed (possibly unbounded) set $\mathcal{A} \subset \mathbb{R}^n$; i.e., $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}$. In part, this is due to the fact that the currently available proofs depend on the measures being the same and on the measures satisfying the triangle inequality.

Theorem 7.9. [45, Theorem 6] Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the discrete time basic conditions, is continuous on $\mathbb{R}^n \setminus \mathcal{A}$, and that (32) is weakly \mathcal{KL} -stable with respect to $(| \cdot |_{\mathcal{A}}, | \cdot |_{\mathcal{A}})$. Then there exists a weak discrete time Lyapunov function; that is, a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that, for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}}), \quad \text{and} \tag{44}$$

$$\min_{f \in F(x)} V(f) \le V(x)e^{-1}.$$
(45)

The following theorem first appeared in [43] and requires the following assumption:

Assumption 1. For each $r \in \mathbb{R}_{>0}$ there exists $M_r \in \mathbb{R}_{>0}$ such that $|x|_{\mathcal{A}} \leq r$ implies $\sup_{w \in F(x)} |w| \leq M_r$.

Theorem 7.10. [46, Theorem 2.1] Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the continuous time basic conditions, is locally Lipschitz on $\mathbb{R}^n \setminus \mathcal{A}$, satisfies Assumption 1, and that (33) is weakly \mathcal{KL} -stable with respect to $(| \cdot |_{\mathcal{A}}, | \cdot |_{\mathcal{A}})$. Then there exists a weak continuous time Lyapunov function; that is, a locally Lipschitz function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that, for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}}), \quad \text{and} \tag{46}$$

$$\min_{w \in F(x)} DV(x;w) \le -V(x),\tag{47}$$

where DV(x; w) denotes the Dini derivative at $x \in \mathbb{R}^n$ in the direction $w \in F(x)$.

Independent to [43] and [46], using techniques from optimal control, Rifford [90], under similar assumptions to those in Theorem 7.10 plus a linear growth condition on the set-valued map, derived a locally Lipschitz weak continuous time Lyapunov function for weak uniform asymptotic stability of a compact set.

There is an interesting contrast between the continuous time case of Theorem 7.10 where a locally Lipschitz Lyapunov function is obtained, versus the other presented cases for strong stability in continuous time and both weak and strong stability in discrete time, where smooth Lyapunov functions are obtained. In fact, Clarke et al. [19] demonstrated that, in general, it is not possible to find a smooth weak continuous time Lyapunov function for weak asymptotic stability of (33). However,

they provided a necessary condition in the form of a restriction on the set-valued map satisfying a covering condition near the origin for such a smooth Lyapunov function.

Theorem 7.11. [19, Theorem 6.1] Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the continuous time basic conditions and there exists a continuously differentiable weak Lyapunov function; i.e., a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that, for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|), \quad \text{and} \tag{48}$$

$$\min_{w \in F(x)} \left\langle \frac{\partial}{\partial x} V(x), w \right\rangle \le -V(x).$$
(49)

Then, for any $\gamma \in \mathbb{R}_{>0}$ there exists $\Delta \in \mathbb{R}_{>0}$ such that

$$\mathcal{B}_{\Delta} \subset F\left(\mathcal{B}_{\gamma}\right) := \cup_{x \in \mathcal{B}_{\gamma}} F(x). \tag{50}$$

A simple example demonstrates that, in the case of continuous time weak \mathcal{KL} stability, even if the above covering condition is satisfied, a continuously differentiable weak Lyapunov function may fail to exist. Consider a system defined on \mathbb{R}^2 by

$$\dot{x} \in \overline{\mathcal{B}}, \quad x \in \mathbb{R}^2.$$
 (51)

It is straightforward to see that $\mathcal{A} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is weakly \mathcal{KL} -stable with respect to $(| \cdot |_{\mathcal{A}}, | \cdot |_{\mathcal{A}})$ for (51). Suppose it were possible to find a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ so that

$$\alpha_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}}), \quad \text{and}$$
(52)

$$\min_{w \in \mathbb{Z}} \left\langle \frac{\partial}{\partial x} V(x), w \right\rangle \le -V(x).$$
(53)

The above implies that $\frac{\partial}{\partial x}V(x) \neq 0$ for all $x \in \mathbb{R}^2 \setminus \mathcal{A}$. On the other hand, since V is continuously differentiable, it obtains its minimum and maximum on $\overline{\mathcal{B}}$. Equation (52) implies that V obtains its minimum everywhere on the boundary of $\overline{\mathcal{B}}$. Consequently, V must obtain its maximum on the interior of $\overline{\mathcal{B}}$, contradicting that $\frac{\partial}{\partial x}V(x) \neq 0$ for all $x \in \mathbb{R}^2 \setminus \mathcal{A}$. Hence, a continuously differentiable weak Lyapunov function cannot exist for (51).

The covering condition of Theorem 7.11 is related to a similar covering condition derived by Brockett [13] in the context of designing continuous feedback stabilizers for controlled differential equations. An example of a system that does not satisfy this covering condition is a tricycle that needs to be steered to the origin where there are clearly initial configurations that require a discontinuous decision to be made in terms of turning the handlebars left or right. This example belongs to the general class of systems referred to as nonholonomic systems.

Such discontinuous feedback stabilizers suffer from a lack of robustness. In particular, for the initial configurations of the tricycle where a discontinuous decision must be made, arbitrarily small errors in measuring the configuration can lead to a so-called chattering phenomenon near the point of discontinuity and the system therefore never approaches the origin (see [105] for an extended discussion). Given the connections between smooth Lyapunov functions and robustness, it is not surprising then that a smooth Lyapunov function is not possible for weak asymptotic stability of (33).

However, having made that observation, it is perhaps surprising that a smooth weak discrete time Lyapunov function is possible for weak asymptotic stability of (32). This stems from the fact that a sampled-data or discrete time implementation

20

can circumvent the lack of robustness just described. In the tricycle example, a decision is made near the point of discontinuity and that decision is adhered to for a set period of time. This moves the system far enough away from the point of discontinuity that the aforementioned chattering phenomenon does not occur, where there is clearly a relationship between how large the measurement errors are and how long the set period of time is. In some generality, the construction of robust discontinuous feedback stabilizers was dealt with in [17], [18], [44], [46], and [91].

8. Instability theorems. In addition to the stability considerations so far described, Lyapunov proposed using similar energy-inspired functions for the study of instability. In intuitive terms, if the system energy is described by a positive function with a minimum at the origin, then stability, or asymptotic stability, follows from the system energy not increasing, or decreasing, respectively. This is clearly captured by the Lyapunov functions discussed in previous sections. In the case of instability, one sufficient condition has the system energy (as described by a Lyapunov function) increasing in a neighborhood of the equilibrium point. However, this can be refined to allow that the system energy is increasing for points arbitrarily close to the equilibrium point, but not necessarily in an entire neighborhood. This is the underlying premise of Chetaev's refinement to Lyapunov's instability theorems. We first present the three instability theorems and then describe their converses. The material in this section is drawn from [58] and [32].

An unstable equilibrium is one that is not stable.

Definition 8.1. The origin is unstable for (1) if, for any sufficiently small $\varepsilon > 0$ there exist sequences $\{x_k\}_{k=0}^{\infty}$ and $\{t_k\}_{k=0}^{\infty}$ such that $x_k \in \mathbb{R}^n \setminus \{0\}$ for all $k, x_k \to 0$ as $k \to \infty, t_k > t_0$ for all k, and

$$|\phi(t_k, t_0, x_k)| \ge \varepsilon, \quad \forall k$$

Note that an equilibrium point can be both unstable and attractive; i.e., systems exist such that solutions from initial conditions arbitrarily close to the origin leave every small neighborhood of the origin but eventually approach the origin (see [32, Section 40]).

Definition 8.2. Let $\mathcal{G} \subset \mathbb{R}^n$ contain the origin. We say that the origin is *unstable* in the region \mathcal{G} for (1) if, for every open, bounded set $H \subset \mathbb{R}^n$ satisfying $0 \in H$ and $\overline{H} \subset \mathcal{G}$, and for every $t_0 \in \mathbb{R}_{\geq 0}$ there exists a sequence of points $\{x_k\}_{k=0}^{\infty}$ satisfying $x_k \in H$, $\lim_{k\to\infty} x_k = 0$, and $\phi(t, t_0, x_k) \notin H$ for some $t > t_0$.

In Lyapunov's First Theorem on Instability [71, Section 16, Theorem II] Lyapunov demonstrated that if a sign-definite function is such that, in a neighborhood containing the origin, its time derivative along solutions of (1) is also sign-definite and of the same sign as the function itself, then the origin is unstable.

Theorem 8.3 (Lyapunov's First Theorem on Instability). Let $\mathcal{G} \subset \mathbb{R}^n$ contain a neighborhood of the origin. Suppose the function $V : \mathcal{G} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuously differentiable, is such that the derivative of V along solutions of (1) is positive definite, and that there exists $\alpha \in \mathcal{K}$ such that $V(x,t) \leq \alpha(|x|)$ for all $(x,t) \in \mathcal{G} \times \mathbb{R}_{\geq 0}$. If for every $\varepsilon > 0$ and $t_0 \geq 0$ there exists a $T \geq t_0$ such that, for all $|x| \leq \varepsilon, x \neq 0, V(x,t) > 0$ for all $t \geq T$, then the origin is unstable.

It is possible to obtain a converse to Theorem 8.3 if the origin is unstable in the region \mathcal{G} and \mathcal{G} satisfies Property A (see [58, Theorem 6.1]).

In Lyapunov's second theorem on instability, [71, Section 16, Theorem III], an extra degree of freedom is allowed in the form of a second function in the decrease condition.

Theorem 8.4 (Lyapunov's Second Theorem on Instability). Let $\mathcal{G} \subset \mathbb{R}^n$ contain a neighborhood of the origin. Suppose $W : \mathcal{G} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Suppose $V : \mathcal{G} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuously differentiable, that there exists an L > 0 such that $|V(x,t)| \leq L$ for all $(x,t) \in \mathcal{G} \times \mathbb{R}_{>0}$, and the derivative of V along solutions of (1) satisfies

$$\frac{dV}{dt} = \lambda V + W \tag{54}$$

for some $\lambda > 0$. Furthermore, if W(x,t) = 0 for all $(x,t) \in \mathcal{G} \times \mathbb{R}_{\geq 0}$ assume that for every $\varepsilon > 0$ and $t_0 \geq 0$ there exists a $T \geq t_0$ such that, for all $|x| \leq \varepsilon$, V(x,t) > 0for all $t \geq T$. Then the origin is unstable.

It is possible to derive a converse to Theorem 8.4 without the requirement that the region \mathcal{G} satisfy Property A (see [58, Theorem 7.2]).

Finally, Chetaev [16, Theorem, p. 27] revised the above to only require the function V to be sign definite in a region containing the origin rather than in a neighborhood containing the origin.

Theorem 8.5 (Chetaev's Theorem). Suppose $V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuously differentiable. If there exists $\varepsilon > 0$ such that, for all $|x| \leq \varepsilon$, the derivative of V along solutions of (1) satisfies dV/dt > 0 on the region where V(x,t) > 0 then the origin is unstable.

The first converse theorems for the above instability theorems were derived by Krasovskii in the case of autonomous systems [55]. General converses for the instability theorems were derived independently¹⁰ by Vrkoč [114] and Krasovskii [57].

It is possible to derive a result that is slightly stronger than the direct converse to Theorem 8.5. To do this, we require the following definition.

Definition 8.6. Let $H \subset \mathbb{R}^n$ be a bounded sub-domain of \mathcal{G} that contains the origin and satisfies $\overline{H} \subset \mathcal{G}$. A set $I(t_0) \subset H$, depending on the initial time t_0 , is called the *domain of instability in* H for $t = t_0$ if for each $x \in I(t_0)$ there exists a finite time $t^* \in [t_0, \infty)$ so that $\phi(t^*, t_0, x) \notin H$.

Note that the domain of instability is an open set in \mathbb{R}^n .

The following converse of Theorem 8.5 includes the result that the region where the function V is positive coincides with the region of instability.

Theorem 8.7. [58, Theorem 7.1] Let the origin be unstable in the region \mathcal{G} and let $H \subset \mathbb{R}^n$ be a bounded region satisfying $0 \in \overline{H} \subset \mathcal{G}$. Then there exists a function $V : \mathcal{G} \times \mathbb{R}_{>0} \to \mathbb{R}$ such that

- 1. If $x \in \overline{H}$ is in the region V(x,t) > 0 for all $t \ge t_0$ then the function dV/dt is positive definite;
- 2. The function V is bounded and continuous in the region \overline{H} and the partial derivatives $\partial V/\partial t$ and $\partial V/\partial x_i$ are bounded uniformly in time; and
- 3. For every value of $t = t_0$ the region of instability $I(t_0)$ coincides with the region V(x,t) > 0.

22

¹⁰Vrkoč's manuscript [114] was submitted on 7 January 1955 while Krasovskii's manuscript [57] was submitted on 3 May 1955.

As in the results of Kurzweil and Massera on uniform asymptotic stability, if (1) is periodic in t or independent of t, then the function of Theorem 8.7 can also be chosen to be periodic in t or independent of t, respectively (see [58, p. 43]).

We note that [121, Theorem 79] extends Lyapunov's second theorem on instability to dynamical systems defined on metric spaces.

9. Concluding remarks. As we have seen, the converse question for Lyapunov's second method has been successfully answered in a wide variety of contexts. The study of the converse question has helped to clarify not only the relationship between different stability concepts, but has helped to identify useful stability concepts as in the case of the important role played by uniformity in various stability definitions. The answers have proved important in the study of robustness to various system perturbations such as persistent disturbances, time delays, and in the role that sampled-data controllers can play in providing robust feedback.

Of necessity, we have restricted our attention to certain specific topics, specifically converse theorems for (uniform asymptotic) stability of differential and difference inclusions, where the results for differential and difference equations can be recovered as special cases. However, a contributing factor in the success and popularity of Lyapunov's second method has been its applicability in many different contexts and, in many of those contexts, converse results are also available. For example, Lyapunov's second method can be extended to the notion of Lagrange stability [8]. An approach to almost global asymptotic stability, refered to as a "dual" to Lyapunov's second method, was presented in [88] with a converse theorem in [89]. So-called complete Lyapunov functions for dynamical systems with multiple equilibria have been defined in [21] with converse theorems provided in [21]. Chapter 2, Section 6.4] and [83]. Lyapunov's second method has also been adapted to the study of stochastic systems with converse theorems available for stochastic differential equations [64], [50, Theorem 5.4, p. 153], discrete-time multivalued systems [108], [109], and random dynamical systems for asymptotic stability of random compact sets [7].

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CHRISTOPHER M. KELLETT

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CHRISTOPHER M. KELLETT

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