

SOLUTIONS MANUAL

to accompany

**Digital Signal Processing:
A Computer-Based Approach**

Third Edition

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Chapter 2

2.1 (a) $\|x_1\|_1 = 22.85, \|x_1\|_2 = 9.1396, \|x_1\|_\infty = 4.81,$

(b) $\|x_2\|_1 = 18.68, \|x_2\|_2 = 7.1944, \|x_2\|_\infty = 3.48.$

2.2 $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$ Hence, $\mu[-n-1] = \begin{cases} 1, & n < 0, \\ 0, & n \geq 0. \end{cases}$ Thus, $x[n] = \mu[n] + \mu[-n-1].$

2.3 (a) Consider the sequence defined by $x[n] = \sum_{k=-\infty}^n \delta[k].$ If $n < 0,$ then $k = 0$ is not included

in the sum and hence, $x[n] = 0$ for $n < 0.$ On the other hand, for $n \geq 0, k = 0$ is included in the sum, and as a result, $x[n] = 1$ for $n \geq 0.$ Therefore,

$$x[n] = \sum_{k=-\infty}^n \delta[k] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} = \mu[n].$$

(b) Since $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$ it follows that $\mu[n-1] = \begin{cases} 1, & n \geq 1, \\ 0, & n < 1. \end{cases}$ Hence,

$$\mu[n] - \mu[n-1] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases} = \delta[n].$$

2.4 Recall $\mu[n] - \mu[n-1] = \delta[n].$ Hence,

$$\begin{aligned} x[n] &= \delta[n] + 3\delta[n-1] - 2\delta[n-2] + 4\delta[n-3] \\ &= (\mu[n] - \mu[n-1]) + 3(\mu[n-1] - \mu[n-2]) - 2(\mu[n-2] - \mu[n-3]) + 4(\mu[n-3] - \mu[n-4]) \\ &= \mu[n] + 2\mu[n-1] - 5\mu[n-2] + 6\mu[n-3] - 4\mu[n-4]. \end{aligned}$$

2.5 (a) $c[n] = x[-n+2] = \{2 \underset{\uparrow}{0} \quad -3 \quad -2 \quad 1 \quad 5 \quad -4\},$

(b) $d[n] = y[-n-3] = \{-2 \quad 7 \quad 8 \quad 0 \quad -1 \quad -3 \quad 6 \quad 0 \quad 0\},$

(c) $e[n] = w[-n] = \{5 \quad -2 \quad 0 \quad -1 \quad 2 \quad 2 \quad 3 \quad 0 \quad 0\},$

(d) $u[n] = x[n] + y[n-2] = \{-4 \quad 5 \quad 1 \quad \underset{\uparrow}{-2} \quad 3 \quad -3 \quad 1 \quad 0 \quad 8 \quad 7 \quad -2\},$

(e) $v[n] = x[n] \cdot w[n+4] = \{0 \quad 15 \quad 2 \quad \underset{\uparrow}{-4} \quad 3 \quad 0 \quad -4 \quad 0\},$

(f) $s[n] = y[n] - w[n+4] = \{-3 \quad 4 \quad \underset{\uparrow}{-5} \quad 0 \quad 0 \quad 10 \quad 2 \quad -2\},$

(g) $r[n] = 3.5 y[n] = \{21 \quad -10.5 \quad \underset{\uparrow}{-3.5} \quad 0 \quad 2.8 \quad 24.5 \quad -7\}.$

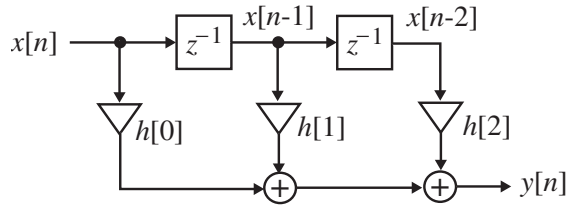
2.6 (a) $x[n] = -4\delta[n+3] + 5\delta[n+2] + \delta[n+1] - 2\delta[n] - 3\delta[n-1] + 2\delta[n-3],$
 $y[n] = 6\delta[n+1] - 3\delta[n] - \delta[n-1] + 8\delta[n-3] + 7\delta[n-4] - 2\delta[n-5],$
 $w[n] = 3\delta[n-2] + 2\delta[n-3] + 2\delta[n-4] - \delta[n-5] - 2\delta[n-7] + 5\delta[n-8],$

(b) Recall $\delta[n] = \mu[n] - \mu[n-1].$ Hence,

$$\begin{aligned} x[n] &= -4(\mu[n+3] - \mu[n+2]) + 5(\mu[n+2] - \mu[n+1]) + (\mu[n+1] - \mu[n]) \\ &\quad - 2(\mu[n] - \mu[n-1]) - 3(\mu[n-1] - \mu[n-2]) + 2(\mu[n-3] - \mu[n-4]) \end{aligned}$$

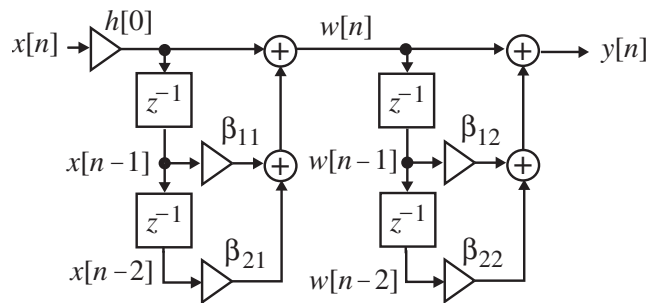
$$= -4\mu[n+3] + 9\mu[n+2] - 4\mu[n+1] - 3\mu[n] - \mu[n-1] + 3\mu[n-2] + 2\mu[n-3] - 2\mu[n-4],$$

2.7 (a)



From the above figure it follows that $y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$.

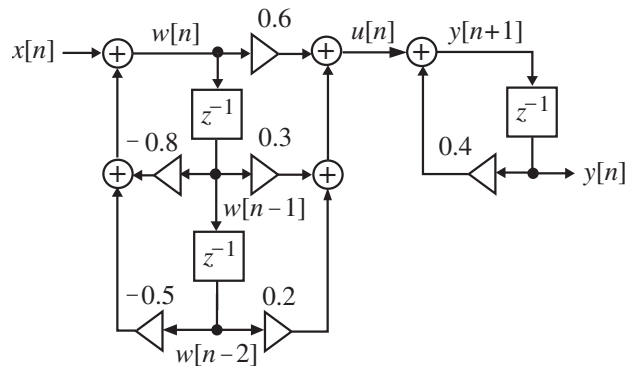
(b)



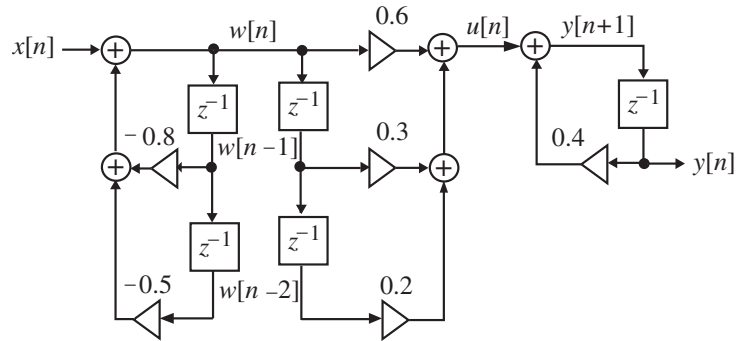
From the above figure we get $w[n] = h[0](x[n] + \beta_{11}x[n-1] + \beta_{21}x[n-2])$ and $y[n] = w[n] + \beta_{12}w[n-1] + \beta_{22}w[n-2]$. Making use of the first equation in the second we arrive at

$$\begin{aligned} y[n] &= h[0](x[n] + \beta_{11}x[n-1] + \beta_{21}x[n-2]) \\ &\quad + \beta_{12}h[0](x[n-1] + \beta_{11}x[n-2] + \beta_{21}x[n-3]) \\ &\quad + \beta_{22}h[0](x[n-2] + \beta_{11}x[n-3] + \beta_{21}x[n-4]) \\ &= h[0](x[n] + (\beta_{11} + \beta_{12})x[n-1] + (\beta_{21} + \beta_{12}\beta_{11} + \beta_{22})x[n-2] \\ &\quad + (\beta_{12}\beta_{21} + \beta_{22}\beta_{11})x[n-3] + \beta_{22}\beta_{21}x[n-4]). \end{aligned}$$

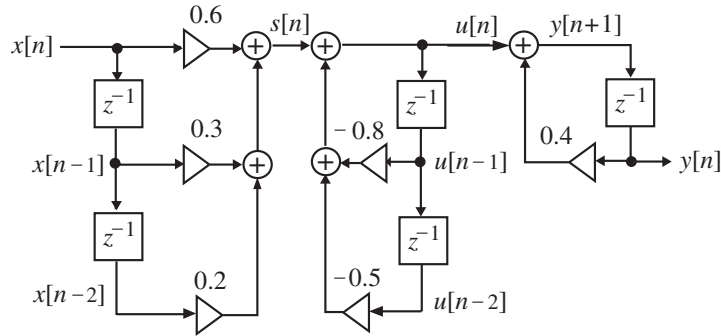
(c) Figure P2.1(c) is a cascade of a first-order section and a second-order section. The input-output relation remains unchanged if the ordering of the two sections is interchanged as shown below.



The second-order section can be redrawn as shown below without changing its input-output relation.



The second-order section can be seen to be cascade of two sections. Interchanging their ordering we finally arrive at the structure shown below:



Analyzing the above structure we arrive at

$$s[n] = 0.6x[n] + 0.3x[n-1] + 0.2x[n-2],$$

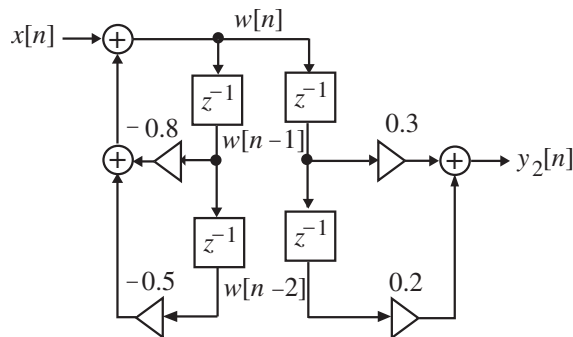
$$u[n] = s[n] - 0.8u[n-1] - 0.5u[n-2],$$

$$y[n+1] = u[n] + 0.4y[n].$$

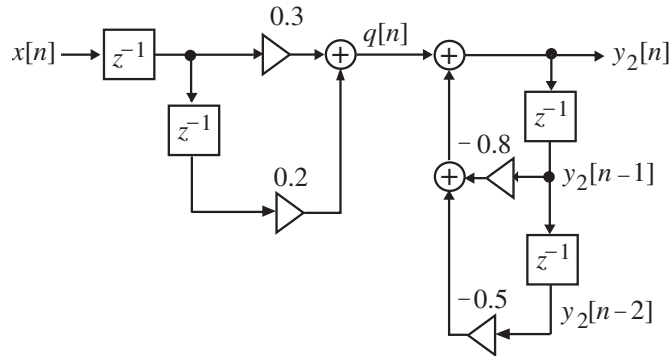
From $u[n] = y[n+1] - 0.4y[n]$. Substituting this in the second equation we get after some algebra $y[n+1] = s[n] - 0.4y[n] - 0.18y[n-1] + 0.8y[n-2]$. Making use of the first equation in this equation we finally arrive at the desired input-output relation

$$y[n] + 0.4y[n-1] + 0.18y[n-2] - 0.2y[n-3] = 0.6x[n-1] + 0.3x[n-2] + 0.2x[n-3].$$

(d) Figure P2.19(d) is a parallel connection of a first-order section and a second-order section. The second-order section can be redrawn as a cascade of two sections as indicated below:



Interchanging the order of the two sections we arrive at an equivalent structure shown below:



Analyzing the above structure we get

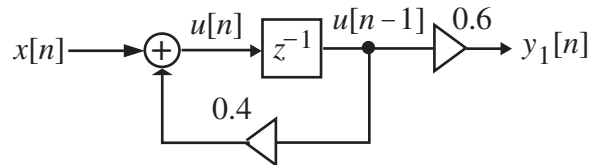
$$q[n] = 0.3x[n-1] + 0.2x[n-2],$$

$$y_2[n] = q[n] - 0.8y_2[n-1] - 0.5y_2[n-2].$$

Substituting the first equation in the second we have

$$y_2[n] + 0.8y_2[n-1] + 0.5y_2[n-2] = 0.3x[n-1] + 0.2x[n-2]. \quad (2-1)$$

Analyzing the first-order section of Figure P2.1(d) given below



we get

$$u[n] = x[n] + 0.4u[n-1],$$

$$y_1[n] = 0.6u[n-1].$$

Solving the above two equations we have

$$y_1[n] - 0.4y_1[n-1] = 0.6x[n-1]. \quad (2-2)$$

The output $y[n]$ of the structure of Figure P2.19(d) is given by

$$y[n] = y_1[n] + y_2[n]. \quad (2-3)$$

From Eq. (2-2) we get $0.8y_1[n-1] - 0.32y_1[n-2] = 0.48x[n-2]$ and

$0.5y_1[n-2] - 0.2y_1[n-3] = 0.3x[n-3]$. Adding the last two equations to Eq. (2-2) we

arrive at $y_1[n] + 0.4y_1[n-1] + 0.18y_1[n-2] - 0.2y_1[n-3]$

$$= 0.6x[n-1] + 0.48x[n-2] + 0.3x[n-3]. \quad (2-4)$$

Similarly, from Eq. (2-1) we get

$-0.4y_2[n-1] - 0.32y_2[n-2] - 0.2y_2[n-3] = -0.12x[n-2] - 0.08x[n-3]$. Adding this

equation to Eq. (2-1) we arrive at

$$y_2[n] + 0.4y_2[n-1] + 0.18y_2[n-2] - 0.2y_2[n-3]$$

$$= 0.3x[n-1] + 0.08x[n-2] - 0.08x[n-3]. \quad (2-5)$$

Adding Eqs. (2-4) and (2-5), and making use of Eq. (2-3) we finally arrive at the input-output relation of Figure P2.1(d) as:

$$y[n] + 0.4y[n-1] + 0.18y[n-2] - 0.2y[n-3] = 0.9x[n-1] + 0.56x[n-2] + 0.22x[n-3].$$

2.8 (a) $x_1^*[n] = \{1 - j4 \quad -2 - j5 \quad \underset{\uparrow}{3 + j2} \quad -7 - j3 \quad -1 - j\},$

$x_1^*[-n] = \{-1 - j \quad -7 - j3 \quad \underset{\uparrow}{3 + j2} \quad -2 - j5 \quad 1 - j4\}.$ Therefore

$x_{1,cs}[n] = \frac{1}{2}(x_1[n] + x_1^*[-n]) = \{j1.5 \quad -4.5 + j \quad \underset{\uparrow}{3} \quad -4.5 - j \quad -j1.5\},$

$x_{1,ca}[n] = \frac{1}{2}(x_1[n] - x_1^*[-n]) = \{1 + j2.5 \quad 2.5 + j4 \quad \underset{\uparrow}{-j2} \quad -2.5 + j4 \quad -1 + j2.5\}.$

(b) $x_2[n] = e^{j\pi n/3}.$ Hence, $x_2^*[n] = e^{-j\pi n/3}$ and thus, $x_2^*[-n] = e^{j\pi n/3} = x_2[n].$

Therefore, $x_{2,cs}[n] = \frac{1}{2}(x_2[n] + x_2^*[-n]) = e^{j2\pi n/3} = x_2[n],$ and

$x_{2,ca}[n] = \frac{1}{2}(x_2[n] - x_2^*[-n]) = 0.$

(c) $x_3[n] = j e^{-j\pi n/5}.$ Hence, $x_3^*[n] = -j e^{j\pi n/5}$ and thus,

$x_3^*[-n] = -j e^{-j\pi n/5} = -x_3[n].$ Therefore, $x_{3,cs}[n] = \frac{1}{2}(x_3[n] + x_3^*[-n]) = 0,$ and

$x_{3,ca}[n] = \frac{1}{2}(x_3[n] - x_3^*[-n]) = x_3[n] = j e^{-j\pi n/5}.$

2.9 (a) $x[n] = \{-4 \quad 5 \quad 1 \quad \underset{\uparrow}{-2} \quad -3 \quad 0 \quad 2\}.$ Hence, $x[-n] = \{2 \quad 0 \quad -3 \quad \underset{\uparrow}{-2} \quad 1 \quad 5 \quad -4\}.$

Therefore, $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n]) = \frac{1}{2}\{-2 \quad 5 \quad -2 \quad \underset{\uparrow}{-4} \quad -2 \quad 5 \quad -2\}$
 $= \{-1 \quad 2.5 \quad -1 \quad \underset{\uparrow}{-2} \quad -1 \quad 2.5 \quad -1\}$

and $x_{od}[n] = \frac{1}{2}(x[n] - x[-n]) = \frac{1}{2}\{-6 \quad 5 \quad 4 \quad \underset{\uparrow}{0} \quad -4 \quad -5 \quad 6\}$
 $= \{-3 \quad 2.5 \quad 2 \quad \underset{\uparrow}{0} \quad -2 \quad -2.5 \quad 3\}.$

(b) $y[n] = \{0 \quad 0 \quad 0 \quad 0 \quad 6 \quad \underset{\uparrow}{-3} \quad -1 \quad 0 \quad 8 \quad 7 \quad -2\}.$ Hence,

$y[-n] = \{-2 \quad 7 \quad 8 \quad 0 \quad -1 \quad \underset{\uparrow}{-3} \quad 6 \quad 0 \quad 0 \quad 0 \quad 0\}.$

Therefore, $y_{ev}[n] = \frac{1}{2}(y[n] + y[-n]) = \{-1 \quad 3.5 \quad 4 \quad 0 \quad 2.5 \quad \underset{\uparrow}{-3} \quad 2.5 \quad 0 \quad 4 \quad 3.5 \quad -1\}$

and $y_{od}[n] = \frac{1}{2}(y[n] - y[-n]) = \{1 \quad -3.5 \quad -4 \quad 0 \quad 3.5 \quad \underset{\uparrow}{0} \quad -3.5 \quad 0 \quad 4 \quad 3.5 \quad -1\}.$

(c) $w[n] = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 3 \quad 2 \quad 2 \quad -1 \quad 0 \quad -2 \quad 5\}.$ Hence,

$w[-n] = \{5 \quad -2 \quad 0 \quad -1 \quad 2 \quad 2 \quad 3 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}.$ Therefore

$w_{ev}[n] = \frac{1}{2}(w[n] + w[-n])$

$$= \{2.5 \quad -1 \quad 0 \quad -0.5 \quad 1 \quad 1 \quad 1.5 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 1.5 \quad 1 \quad 1 \quad -0.5 \quad 0 \quad -1 \quad 2.5\} \text{ and}$$

$$w_{od}[n] = \frac{1}{2}(w[n] - w[-n])$$

$$= \{-2.5 \quad 1 \quad 0 \quad 0.5 \quad -1 \quad -1 \quad -1.5 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 1.5 \quad 1 \quad 1 \quad -0.5 \quad 0 \quad -1 \quad 2.5\}.$$

2.10 (a) $x_1[n] = \mu[n+2]$. Hence, $x_1[-n] = \mu[-n+2]$. Therefore,

$$x_{1,ev}[n] = \frac{1}{2}(\mu[n+2] + \mu[-n+2]) = \begin{cases} 1/2, & n \geq 3, \\ 1, & -2 \leq n \leq 2, \text{ and} \\ 1/2, & -3 \leq n, \end{cases}$$

$$x_{1,od}[n] = \frac{1}{2}(\mu[n+2] - \mu[-n+2]) = \begin{cases} 1/2, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ -1/2, & -3 \leq n. \end{cases}$$

(b) $x_2[n] = \alpha^n \mu[n-3]$. Hence, $x_2[-n] = \alpha^{-n} \mu[-n-3]$. Therefore,

$$x_{2,ev}[n] = \frac{1}{2}(\alpha^n \mu[n-3] + \alpha^{-n} \mu[-n-3]) = \begin{cases} \frac{1}{2} \alpha^n, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \text{ and} \\ \frac{1}{2} \alpha^{-n}, & -3 \leq n, \end{cases}$$

$$x_{2,od}[n] = \frac{1}{2}(\alpha^n \mu[n-3] - \alpha^{-n} \mu[-n-3]) = \begin{cases} \frac{1}{2} \alpha^n, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ -\frac{1}{2} \alpha^{-n}, & -3 \leq n. \end{cases}$$

(c) $x_3[n] = n \alpha^n \mu[n]$. Hence, $x_3[-n] = -n \alpha^{-n} \mu[-n]$. Therefore,

$$x_{3,ev}[n] = \frac{1}{2}(n \alpha^n \mu[n] + (-n) \alpha^{-n} \mu[-n]) = \frac{1}{2} |n| \alpha^{|n|} \text{ and}$$

$$x_{3,od}[n] = \frac{1}{2}(n \alpha^n \mu[n] - (-n) \alpha^{-n} \mu[-n]) = \frac{1}{2} n \alpha^{|n|}.$$

(d) $x_4[n] = \alpha^{|n|}$. Hence, $x_4[-n] = \alpha^{|-n|} = \alpha^{|n|} = x_4[n]$. Therefore,

$$x_{4,ev}[n] = \frac{1}{2}(x_4[n] + x_4[-n]) = \frac{1}{2}(x_4[n] + x_4[n]) = x_4[n] = \alpha^{|n|} \text{ and}$$

$$x_{4,od}[n] = \frac{1}{2}(x_4[n] - x_4[-n]) = \frac{1}{2}(x_4[n] - x_4[n]) = 0.$$

2.11 $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$. Thus, $x_{ev}[-n] = \frac{1}{2}(x[-n] + x[n]) = x_{ev}[n]$. Hence, $x_{ev}[n]$ is an even sequence. Likewise, $x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$. Thus,

$$x_{od}[-n] = \frac{1}{2}(x[-n] - x[n]) = -x_{od}[n]. \text{ Hence, } x_{od}[n] \text{ is an odd sequence.}$$

2.12 (a) $g[n] = x_{ev}[n]x_{ev}[n]$. Thus, $g[-n] = x_{ev}[-n]x_{ev}[-n] = x_{ev}[n]x_{ev}[n] = g[n]$. Hence, $g[n]$ is an even sequence.

(b) $u[n] = x_{ev}[n]x_{od}[n]$. Thus, $u[-n] = x_{ev}[-n]x_{od}[-n] = x_{ev}[n](-x_{od}[n]) = -u[n]$. Hence, $u[n]$ is an odd sequence.

(c) $v[n] = x_{od}[n]x_{od}[n]$. Thus, $v[-n] = x_{od}[-n]x_{od}[-n] = (-x_{od}[n])(-x_{od}[n]) = x_{od}[n]x_{od}[n] = v[n]$. Hence, $v[n]$ is an even sequence.

2.13 (a) Since $x[n]$ is causal, $x[n] = 0, n < 0$. Also, $x[-n] = 0, n > 0$. Now,

$$x_{ev}[n] = \frac{1}{2}(x[n] + x[-n]). \text{ Hence, } x_{ev}[0] = \frac{1}{2}(x[0] + x[0]) = x[0] \text{ and}$$

$$x_{ev}[n] = \frac{1}{2}x[n], n > 0. \text{ Combining the two equations we get } x[n] = \begin{cases} 2x_{ev}[n], & n > 0, \\ x_{ev}[n], & n = 0, \\ 0, & n < 0. \end{cases}$$

Likewise, $x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$. Hence, $x_{od}[0] = \frac{1}{2}(x[0] - x[0]) = 0$ and

$$x_{od}[n] = \frac{1}{2}x[n], n > 0. \text{ Combining the two equations we get } x[n] = \begin{cases} 2x_{ev}[n], & n > 0, \\ 0, & n \leq 0. \end{cases}$$

(b) Since $y[n]$ is causal, $y[n] = 0, n < 0$. Also, $y[-n] = 0, n > 0$. Let

$$y[n] = y_{re}[n] + jy_{im}[n], \text{ where } y_{re}[n] \text{ and } y_{im}[n] \text{ are real causal sequences.}$$

Now, $y_{ca}[n] = \frac{1}{2}(y[n] - y^*[-n])$. Hence, $y_{ca}[0] = \frac{1}{2}(y[0] - y^*[0]) = jy_{im}[0]$ and

$y_{ca}[n] = \frac{1}{2}y[n], n > 0$. Since $y_{re}[0]$ is not known, $y[n]$ cannot be fully recovered from $y_{ca}[n]$.

Likewise, $y_{cs}[n] = \frac{1}{2}(y[n] + y^*[-n])$. Hence, $y_{cs}[0] = \frac{1}{2}(y[0] + y^*[0]) = y_{re}[0]$ and

$y_{cs}[n] = \frac{1}{2}y[n], n > 0$. Since $y_{im}[0]$ is not known, $y[n]$ cannot be fully recovered from $y_{cs}[n]$.

2.14 Since $x[n]$ is causal, $x[n] = 0, n < 0$. From the solution of Problem 2.13 we have

$$x[n] = \begin{cases} 2x_{ev}[n], & n > 0, \\ x_{ev}[n], & n = 0, \\ 0, & n < 0, \end{cases} = \begin{cases} 2 \cos(\omega_o n), & n > 0, \\ 1, & n = 0, \\ 0, & n < 0, \end{cases} = 2 \cos(\omega_o n) \mu[n] - \delta[n].$$

2.15 (a) $\{x[n]\} = \{A\alpha^n\}$ where A and α are complex numbers with $|\alpha| < 1$. Since for $n < 0, |\alpha|^n$ can become arbitrarily large, $\{x[n]\}$ is not a bounded sequence.

(b) $y[n] = A\alpha^n \mu[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0, \end{cases}$ where A and α are complex numbers with

$|\alpha| < 1$. Here, $|\alpha|^n \leq 1, n \geq 0$. Hence $|y[n]| \leq |A|$ for all values of n . Hence, $\{y[n]\}$ is a bounded sequence.

(c) $\{h[n]\} = C\beta^n \mu[n]$ where C and β are complex numbers with $|\beta| > 1$. Since for $n > 0, |\beta|^n$ can become arbitrarily large, $\{h[n]\}$ is not a bounded sequence.

(d) $\{g[n]\} = 4 \cos(\omega_o n)$. Since $|g[n]| \leq 4$ for all values of $n, \{g[n]\}$ is a bounded sequence.

(e) $v[n] = \begin{cases} \left(1 - \frac{1}{n^2}\right), & n \geq 1, \\ 0, & n \leq 0. \end{cases}$ Since $\frac{1}{n^2} < 1$ for $n > 1$ and $\frac{1}{n^2} = 1$ for $n = 1, |v[n]| < 1$ for all values of n . Thus $\{v[n]\}$ is a bounded sequence.

2.16 $x[n] = \frac{(-1)^{n+1}}{n} \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Hence $\{x[n]\}$ is not absolutely summable.

2.17 (a) $x_1[n] = \alpha^n \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x_2[n]| = \sum_{n=1}^{\infty} |\alpha^n| = \sum_{n=1}^{\infty} |\alpha|^n = \frac{|\alpha|}{1-|\alpha|} < \infty$, since $|\alpha| < 1$. Hence, $\{x_1[n]\}$ is absolutely summable.

(b) $x_2[n] = \alpha^n \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x_2[n]| = \sum_{n=1}^{\infty} |n\alpha^n| = \sum_{n=1}^{\infty} n|\alpha|^n = \frac{|\alpha|}{(1-|\alpha|)^2} < \infty$, since $|\alpha|^2 < 1$. Hence, $\{x_2[n]\}$ is absolutely summable.

(c) $x_3[n] = n^2 \alpha^n \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x_3[n]| = \sum_{n=1}^{\infty} |n^2 \alpha^n| = \sum_{n=1}^{\infty} n^2 |\alpha|^n$
 $= |\alpha| + 2^2 |\alpha|^2 + 3^2 |\alpha|^3 + 4^2 |\alpha|^4 + \dots$
 $= (|\alpha| + |\alpha|^2 + |\alpha|^3 + |\alpha|^4 + \dots) + 3(|\alpha|^2 + |\alpha|^3 + |\alpha|^4 + \dots) + 5(|\alpha|^3 + |\alpha|^4 + |\alpha|^5 + \dots)$

$$\begin{aligned}
& + 7(|\alpha|^4 + |\alpha|^5 + |\alpha|^6 + \dots) = \frac{|\alpha|}{1-|\alpha|} + \frac{3|\alpha|^2}{1-|\alpha|} + \frac{5|\alpha|^3}{1-|\alpha|} + \frac{7|\alpha|^4}{1-|\alpha|} + \dots \\
& = \frac{1}{1-|\alpha|} \left(\sum_{n=1}^{\infty} (2n-1)|\alpha|^n \right) = \frac{1}{1-|\alpha|} \left(2 \sum_{n=1}^{\infty} n|\alpha|^n - \sum_{n=1}^{\infty} |\alpha|^n \right) = \frac{1}{1-|\alpha|} \left(\frac{2|\alpha|}{(1-|\alpha|)^2} - \frac{|\alpha|}{1-|\alpha|} \right) \\
& = \frac{|\alpha|(1+|\alpha|)}{(1-|\alpha|)^3} < \infty. \text{ Hence, } \{x_3[n]\} \text{ is absolutely summable.}
\end{aligned}$$

2.18 (a) $x_a[n] = \frac{1}{2^n} \mu[n]$. Now $\sum_{n=-\infty}^{\infty} |x_a[n]| = \sum_{n=0}^{\infty} \left| \frac{1}{2^n} \right| = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$. Hence, $\{x_a[n]\}$ is absolutely summable.

(b) $x_b[n] = \frac{1}{(n+1)(n+2)} \mu[n]$. Now $\sum_{n=-\infty}^{\infty} |x_b[n]| = \sum_{n=0}^{\infty} \left| \frac{1}{(n+1)(n+2)} \right|$
 $= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \dots = 1 < \infty$. Hence, $\{x_b[n]\}$ is absolutely summable.

2.19 (a) A sequence $x[n]$ is absolutely summable if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. By Schwartz inequality we have $\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty} |x[n]| \right) \left(\sum_{n=-\infty}^{\infty} |x[n]| \right) < \infty$. Hence, an absolutely summable sequence is square summable and has thus finite energy.

Now consider the sequence $x[n] = \frac{1}{n} \mu[n-1]$. The convergence of an infinite series can be shown via the integral test. Let $a_n = f(x)$, where a continuous, positive and decreasing function is for all $x \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x) dx$ both converge or both diverge. For $a_n = \frac{1}{n}, f(x) = \frac{1}{x}$. But $\int_1^{\infty} \frac{1}{x} dx = (\ln x) \Big|_1^{\infty} = \infty - 0 = \infty$.

Hence, $\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. As a result, $x[n] = \frac{1}{n} \mu[n-1]$ is not absolutely summable.

(b) To show that $\{x[n]\}$ is square-summable, we observe that here $a_n = \frac{1}{n^2}$, and thus,

$f(x) = \frac{1}{x^2}$. Now, $\int_1^\infty \frac{1}{x^2} dx = \left(-\frac{1}{x}\right)\Big|_1^\infty = -\frac{1}{\infty} + 1 = 1$. Hence, $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, or in other

words, $x[n] = \frac{1}{n} \mu[n-1]$ is square-summable.

2.20 See Problem 2.19, Part (a) solution.

2.21 $x_2[n] = \frac{\cos \omega_c n}{\pi n} \mu[n-1]$. Now, $\sum_{n=-\infty}^\infty |x_2[n]|^2 = \sum_{n=1}^\infty \left(\frac{\cos \omega_c n}{\pi n}\right)^2 \leq \sum_{n=1}^\infty \frac{1}{\pi^2 n^2}$. Since, $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^\infty \left(\frac{\cos \omega_c n}{\pi n}\right)^2 \leq \frac{1}{6}$. Therefore $x_2[n]$ is square-summable.

Using the integral test (See Problem 2.19, Part (a) solution) we now show that $x_2[n]$ is

not absolutely summable. Now, $\int_1^\infty \left| \frac{\cos \omega_c x}{\pi x} \right| dx = \frac{1}{\pi} \cdot \frac{\left| \frac{\cos \omega_c x}{\pi x} \right|}{\cos \omega_c x} \cdot \cos \text{int}(\omega_c x) \Big|_1^\infty$ where

$\cos \text{int}$ is the cosine integral function. Since $\int_1^\infty \left| \frac{\cos \omega_c x}{\pi x} \right| dx$ diverges, $\sum_{n=1}^\infty \left| \frac{\cos \omega_c n}{\pi n} \right|$ also diverges. Hence, $x_2[n]$ is not absolutely summable.

2.22 $\mathcal{P}_x = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K x^2[n] = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{K=-\infty}^\infty (x_{ev}[n] + x_{od}[n])^2$
 $= \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K (x_{ev}^2[n] + x_{od}^2[n] + 2x_{ev}[n]x_{od}[n])$
 $= \mathcal{P}_{x_{ev}} + \mathcal{P}_{x_{od}} + \frac{1}{2} \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K (x[n] + x[-n])(x[n] - x[-n])$
 $= \mathcal{P}_{x_{ev}} + \mathcal{P}_{x_{od}} + \lim_{K \rightarrow \infty} \frac{1}{2K+1} \cdot \frac{1}{2} \left(\sum_{n=-K}^K x^2[n] - \sum_{n=-\infty}^\infty x^2[-n] \right) = \mathcal{P}_{x_{ev}} + \mathcal{P}_{x_{od}}$

as $\sum_{n=-K}^K x^2[n] = \sum_{n=-K}^K x^2[-n]$. Now for the given sequence,

$\mathcal{P}_{x_{od}} = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K x_{od}^2[n] = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=0}^K \left(\frac{1}{3}\right)^6 = \left(\frac{1}{3}\right)^6 \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=0}^K 1$
 $= \left(\frac{1}{3}\right)^6 \lim_{K \rightarrow \infty} \frac{K+1}{2K+1} = \frac{1}{2} \left(\frac{1}{3}\right)^6$. Hence, $\mathcal{P}_{x_{ev}} = \mathcal{P}_x - \mathcal{P}_{x_{od}} = 10 - \frac{1}{2} \left(\frac{1}{3}\right)^6$.

2.23 $x[n] = \sin(2\pi kn/N)$, $0 \leq n \leq N-1$. Now $\mathcal{E}_x = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} \sin^2(2\pi kn/N)$
 $= \frac{1}{2} \sum_{n=0}^{N-1} (1 - \cos(4\pi kn/N)) = \frac{N}{2} - \frac{1}{2} \sum_{n=0}^{N-1} \cos(4\pi kn/N)$. Let $C = \sum_{n=0}^{N-1} \cos(4\pi kn/N)$ and
 $S = \sum_{n=0}^{N-1} \sin(4\pi kn/N)$. Then $C + jS = \sum_{n=0}^{N-1} e^{-j4\pi kn/N} = \frac{1 - e^{-j4\pi kN/N}}{1 - e^{-j4\pi k/N}} = 0$. This implies
 $C = 0$. Hence $\mathcal{E}_x = \frac{N}{2}$.

2.24 (a) $x[n] = A^\alpha \mu[n]$. Then $\mathcal{E}_{x_a} = \sum_{n=-\infty}^{\infty} |x_a[n]|^2 = A^2 \sum_{n=0}^{\infty} \alpha^{2n} = \frac{A^2}{1 - \alpha^2}$.

(b) $x_b[n] = \frac{1}{n^2} \mu[n-1]$. Then $\mathcal{E}_{x_b} = \sum_{n=-\infty}^{\infty} |x_b[n]|^2 = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right|^2 = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

2.25 (a) $x_1[n] = (-1)^n$. Then average power

$$\mathcal{P}_{x_1} = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K |x_1[n]|^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} (2K+1) = 1, \text{ and energy}$$

$$\mathcal{E}_{x_1} = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=-\infty}^{\infty} 1 = \infty.$$

(b) $x_2[n] = \mu[n]$. Then average power

$$\mathcal{P}_{x_2} = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K |x_2[n]|^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=0}^K 1 = \lim_{K \rightarrow \infty} \frac{K+1}{2K+1} = \frac{1}{2}, \text{ and energy}$$

$$\mathcal{E}_{x_2} = \sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \sum_{n=0}^{\infty} 1 = \infty.$$

(c) $x_3[n] = n\mu[n]$. Then average power

$$\mathcal{P}_{x_3} = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K |x_3[n]|^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=1}^K n^2 = \lim_{K \rightarrow \infty} \frac{K(K+1)(2K+1)}{6} = \infty,$$

and energy $\mathcal{E}_{x_3} = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{n=0}^{\infty} n^2 = \infty$.

(d) $x_4[n] = A_0 e^{j\omega_0 n}$. Then average power $\mathcal{P}_{x_4} = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K |x_4[n]|^2$

$$= \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K \left| A_0 e^{j\omega_0 n} \right|^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K A_0^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \cdot A_0^2 (2K+1) = A_0^2,$$

$$\text{and energy } \mathcal{E}_{x_3} = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{n=-\infty}^{\infty} \left| A_0 e^{j\omega_0 n} \right|^2 = \sum_{n=-\infty}^{\infty} A_0^2 = \infty.$$

(e) $x_5[n] = A \cos\left(\frac{2\pi n}{M} + \phi\right)$. Note $x_5[n]$ is a periodic sequence. Then average power

$$P_{x_5} = \frac{1}{M} \sum_{n=0}^{M-1} |x_5[n]|^2 = \frac{1}{M} \sum_{n=0}^{M-1} \left| A \cos\left(\frac{2\pi n}{M} + \phi\right) \right|^2 = \frac{1}{M} \cdot \frac{A^2}{2} \sum_{n=0}^{M-1} \left(\cos\left(\frac{4\pi n}{M} + 2\phi\right) + 1 \right).$$

Let $C = \sum_{n=0}^{M-1} \cos\left(\frac{4\pi n}{M} + 2\phi\right)$ and $S = \sum_{n=0}^{M-1} \cos\left(\frac{4\pi n}{M} + 2\phi\right)$. Then

$$C + jS = \sum_{n=0}^{M-1} e^{j\left(\frac{4\pi n}{M} + 2\phi\right)} = e^{j2\phi} \sum_{n=0}^{M-1} e^{j4\pi n/M} = e^{j2\phi} \cdot \frac{1 - e^{j4\pi}}{1 - e^{j4\pi/M}} = 0.$$

Hence $C = 0$. Therefore $P_{x_5} = \frac{1}{M} \cdot \frac{A^2}{2} \sum_{n=0}^{M-1} 1 = \frac{A^2}{2}$.

Since $x_5[n]$ is a periodic sequence, it has infinite energy.

2.26 In each of the following parts, N denotes the fundamental period and r is a positive integer.

(a) $\tilde{x}_1[n] = 4 \cos(2\pi n/5)$. Here N and r must satisfy the relation $\frac{2\pi}{5} \cdot N = 2\pi r$.

Among all positive solutions for N and r , the smallest values are $N = 5$ and $r = 1$. Hence the average power is given by

$$P_{x_1} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_1[n]|^2 = \frac{1}{5} \sum_{n=0}^4 \left| 4 \cos\left(\frac{2\pi n}{5}\right) \right|^2 = 8.$$

(b) $\tilde{x}_2[n] = 3 \cos(3\pi n/5)$. Here N and r must satisfy the relation $\frac{3\pi}{5} \cdot N = 2\pi r$.

Among all positive solutions for N and r , the smallest values are $N = 10$ and $r = 3$. Hence the average power is given by

$$P_{x_2} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_2[n]|^2 = \frac{1}{10} \sum_{n=0}^9 \left| 3 \cos\left(\frac{3\pi n}{5}\right) \right|^2 = 4.5.$$

(c) $\tilde{x}_3[n] = 2 \cos(3\pi n/7)$. Here N and r must satisfy the relation $\frac{3\pi}{7} \cdot N = 2\pi r$.

Among all positive solutions for N and r , the smallest values are $N = 14$ and $r = 3$. Hence the average power is given by

$$P_{x_3} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_3[n]|^2 = \frac{1}{14} \sum_{n=0}^{13} \left| 2 \cos\left(\frac{3\pi n}{7}\right) \right|^2 = 2.$$

(d) $\tilde{x}_4[n] = 4 \cos(5\pi n/3)$. Here N and r must satisfy the relation $\frac{5\pi}{3} \cdot N = 2\pi r$.

Among all positive solutions for N and r , the smallest values are $N = 6$ and $r = 5$. Hence the average power is given by

$$P_{x_4} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_4[n]|^2 = \frac{1}{6} \sum_{n=0}^5 \left| 4 \cos\left(\frac{5\pi n}{3}\right) \right|^2 = 8.$$

(e) $\tilde{x}_5[n] = 4 \cos(2\pi n/5) + 3 \cos(3\pi n/5)$. We first determine the fundamental period N_1 of $\cos(2\pi n/5)$. Here N_1 and r must satisfy the relation $\frac{2\pi}{5} \cdot N_1 = 2\pi r$. Among all positive solutions for N_1 and r , the smallest values are $N_1 = 5$ and $r = 1$. We next determine the fundamental period N_2 of $\cos(3\pi n/5)$. Here N_2 and r must satisfy the relation $\frac{3\pi}{5} \cdot N_2 = 2\pi r$. Among all positive solutions for N_2 and r , the smallest values are $N_2 = 10$ and $r = 3$. The fundamental period of $\tilde{x}_5[n]$ is then given by $LCM(N_1, N_2) = LCM(5, 10) = 10$.

Hence the average power is given by

$$\begin{aligned} P_{x_5} &= \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_5[n]|^2 = \frac{1}{10} \sum_{n=0}^9 \left| 4 \cos\left(\frac{2\pi n}{5}\right) + 3 \cos\left(\frac{3\pi n}{5}\right) \right|^2 \\ &= \frac{1}{10} \left(\sum_{n=0}^{11} 16 \cos^2\left(\frac{2\pi n}{5}\right) + \sum_{n=0}^{11} 9 \cos^2\left(\frac{3\pi n}{5}\right) + \sum_{n=0}^{11} 24 \left| \cos\left(\frac{2\pi n}{5}\right) \cos\left(\frac{3\pi n}{5}\right) \right| \right) \cong 8 + 4.5 + 0 = 12.5. \end{aligned}$$

(f) $\tilde{x}_6[n] = 4 \cos(5\pi n/3) + 3 \cos(3\pi n/5)$. We first determine the fundamental period N_1 of $\cos(5\pi n/3)$. Here N_1 and r must satisfy the relation $\frac{5\pi}{3} \cdot N_1 = 2\pi r$. Among all positive solutions for N_1 and r , the smallest values are $N_1 = 6$ and $r = 5$. We next determine the fundamental period N_2 of $\cos(3\pi n/5)$. Here N_2 and r must satisfy the relation $\frac{3\pi}{5} \cdot N_2 = 2\pi r$. Among all positive solutions for N_2 and r , the smallest values are $N_2 = 10$ and $r = 3$. The fundamental period of $\tilde{x}_6[n]$ is then given by $LCM(N_1, N_2) = LCM(6, 10) = 30$.

Hence the average power is given by

$$\begin{aligned} P_{x_6} &= \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_6[n]|^2 = \frac{1}{30} \sum_{n=0}^{29} \left| 4 \cos\left(\frac{5\pi n}{3}\right) + 3 \cos\left(\frac{3\pi n}{5}\right) \right|^2 \\ &= \frac{1}{30} \left(\sum_{n=0}^{29} 16 \cos^2\left(\frac{5\pi n}{3}\right) + \sum_{n=0}^{30} 9 \cos^2\left(\frac{3\pi n}{5}\right) + \sum_{n=0}^{29} 24 \left| \cos\left(\frac{5\pi n}{3}\right) \cos\left(\frac{3\pi n}{5}\right) \right| \right) \cong 8 + 4.5 + 0 = 12.5. \end{aligned}$$

2.27 Now, from Eq. (2.38) we have $\tilde{y}[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$. Therefore

$\tilde{y}[n+N] = \sum_{k=-\infty}^{\infty} x[n+kN+N]$. Substituting $r = k+1$ we get

$\tilde{y}[n+N] = \sum_{r=-\infty}^{\infty} x[n+rN] = \tilde{y}[n]$. Hence $\tilde{y}[n]$ is a periodic sequence with a period N .

2.28 (a) $N = 5$. Now $\tilde{x}_p[n] = \sum_{n=-\infty}^{\infty} x[n+k5]$. The portion of $\tilde{x}_p[n]$ in the range $0 \leq n \leq 4$ is

$$\begin{aligned} & \text{given by } x[n-5] + x[n] + x[n+5] = \{0 \ 0 \ -4 \ 5 \ 1\} \\ & + \{-2 \ -3 \ 0 \ 2 \ 0\} + \{0 \ 0 \ 0 \ 0 \ 0\} = \{-2 \ -3 \ -4 \ 7 \ 1\}, 0 \leq n \leq 4. \end{aligned}$$

Hence, one period of $\tilde{x}_p[n]$ is given by $\{-2 \ -3 \ -4 \ 7 \ 1\}, 0 \leq n \leq 4$.

Now $\tilde{y}_p[n] = \sum_{n=-\infty}^{\infty} y[n+k5]$. The portion of $\tilde{y}_p[n]$ in the range $0 \leq n \leq 4$ is given by

$$\begin{aligned} & y[n-5] + y[n] + y[n+5] = \{0 \ 0 \ 0 \ 0 \ 6\} \\ & + \{-3 \ -1 \ 0 \ 8 \ 7\} + \{-2 \ 0 \ 0 \ 0 \ 0\} = \{-5 \ -1 \ 0 \ 8 \ 13\}, 0 \leq n \leq 4. \end{aligned}$$

Hence, one period of $\tilde{y}_p[n]$ is given by $\{-5 \ -1 \ 0 \ 8 \ 13\}, 0 \leq n \leq 4$.

Now $\tilde{w}_p[n] = \sum_{n=-\infty}^{\infty} w[n+k5]$. The portion of $\tilde{w}_p[n]$ in the range $0 \leq n \leq 4$ is given by

$$\begin{aligned} & w[n-5] + w[n] + w[n+5] = \{0 \ 0 \ 0 \ 0 \ 0\} \\ & + \{0 \ 0 \ 3 \ 2 \ 2\} + \{-1 \ 0 \ -2 \ 5 \ 0\} = \{-1 \ 0 \ 1 \ 7 \ 2\}, 0 \leq n \leq 4. \end{aligned}$$

Hence, one period of $\tilde{w}_p[n]$ is given by $\{-1 \ 0 \ 1 \ 7 \ 2\}, 0 \leq n \leq 4$.

(b) $N = 7$. Now $\tilde{x}_p[n] = \sum_{n=-\infty}^{\infty} x[n+k7]$. The portion of $\tilde{x}_p[n]$ in the range $0 \leq n \leq 6$ is

$$\begin{aligned} & \text{given by } x[n-7] + x[n] + x[n+7] = \{0 \ 0 \ 0 \ 0 \ -4 \ 5 \ 1\} \\ & + \{-2 \ -3 \ 0 \ 2 \ 0 \ 0 \ 0\} + \{0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0\} \\ & = \{-2 \ -3 \ 0 \ 2 \ -4 \ 5 \ 1\}, 0 \leq n \leq 6. \end{aligned}$$

Hence, one period of $\tilde{x}_p[n]$ is given by $\{-2 \ -3 \ 0 \ 2 \ -4 \ 5 \ 1\}, 0 \leq n \leq 6$.

Now $\tilde{y}_p[n] = \sum_{n=-\infty}^{\infty} y[n+k7]$. The portion of $\tilde{y}_p[n]$ in the range $0 \leq n \leq 6$ is given by

$$\begin{aligned} & x[n-7] + x[n] + x[n+7] = \{0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 6\} \\ & + \{-3 \ -1 \ 0 \ 8 \ 7 \ -2 \ 0\} + \{0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0\} \\ & = \{-3 \ -1 \ 0 \ 8 \ 7 \ -2 \ 6\}, 0 \leq n \leq 6. \end{aligned}$$

Hence, one period of $\tilde{y}_p[n]$ is given by $\{-3 \ -1 \ 0 \ 8 \ 7 \ -2 \ 6\}, 0 \leq n \leq 6$.

Now $\tilde{w}_p[n] = \sum_{n=-\infty}^{\infty} w[n+k7]$. The portion of $\tilde{w}_p[n]$ in the range $0 \leq n \leq 6$ is given by

$$\begin{aligned} & w[n-7] + w[n] + w[n+7] = \{0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0\} \\ & + \{0 \ 0 \ 3 \ 2 \ 2 \ -1 \ 0\} + \{-2 \ 5 \ 0 \ 0 \ 0 \ 0 \ 0\} \end{aligned}$$

$= \{-2 \ 5 \ 3 \ 2 \ 2 \ -1 \ 0\}$, $0 \leq n \leq 6$. Hence, one period of $\tilde{w}_p[n]$ is given by $\{-2 \ 5 \ 3 \ 2 \ 2 \ -1 \ 0\}$, $0 \leq n \leq 6$.

2.29 $\tilde{x}[n] = A \cos(\omega_o n + \phi)$.

(a) $\tilde{x}[n] = \{1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1\}$. Hence $A = \sqrt{2}$, $\omega_o = \pi/2$, $\phi = \pi/4$.

(b) $\tilde{x}[n] = \{0 \ -\sqrt{3} \ 0 \ \sqrt{3} \ 0 \ -\sqrt{3} \ 0 \ \sqrt{3}\}$. Hence $A = \sqrt{3}$, $\omega_o = \pi/2$, $\phi = \pi/2$.

(c) $\tilde{x}[n] = \{1 \ -0.366 \ -1.366 \ -1 \ 0.366 \ 1.366\}$. Hence $A = \sqrt{2}$, $\omega_o = \pi/3$, $\phi = \pi/4$.

(d) $\tilde{x}[n] = \{2 \ 0 \ -2 \ 0 \ 2 \ 0 \ -2 \ 0\}$. Hence $A = 2$, $\omega_o = \pi/2$, $\phi = 0$.

2.30 The fundamental period N of a periodic sequence with an angular frequency ω_o satisfies Eq. (2.47a) with the smallest value of N and r .

(a) $\omega_o = 0.5\pi$. Here Eq. (2.47a) reduces to $0.5\pi N = 2\pi r$ which is satisfied with $N = 4, r = 1$.

(b) $\omega_o = 0.8\pi$. Here Eq. (2.47a) reduces to $0.8\pi N = 2\pi r$ which is satisfied with $N = 5, r = 2$.

(c) We first determine the fundamental period N_1 of $\text{Re}\{e^{j\pi n/5}\} = \cos(0.2\pi n)$. In this case, Eq. (2.47a) reduces to $0.2\pi N_1 = 2\pi r_1$ which is satisfied with $N_1 = 10, r_1 = 1$. We next determine the fundamental period N_2 of $\text{Im}\{e^{j\pi n/10}\} = j \sin(0.1\pi n)$. In this case, Eq. (2.47a) reduces to $0.1\pi N_2 = 2\pi r_2$ which is satisfied with $N_2 = 20, r_2 = 1$. Hence the fundamental period N of $\tilde{x}_c[n]$ is given by

$$LCM(N_1, N_2) = LCM(10, 20) = 20.$$

(d) We first determine the fundamental period N_1 of $3 \cos(1.3\pi n)$. In this case, Eq. (2.47a) reduces to $1.3\pi N_1 = 2\pi r_1$ which is satisfied with $N_1 = 20, r_1 = 13$. We next determine the fundamental period N_2 of $4 \sin(0.5\pi n + 0.5\pi)$. In this case, Eq. (2.47a) reduces to $0.5\pi N_2 = 2\pi r_2$ which is satisfied with $N_2 = 4, r_2 = 1$. Hence the fundamental period N of $\tilde{x}_4[n]$ is given by $LCM(N_1, N_2) = LCM(20, 4) = 20$.

(e) We first determine the fundamental period N_1 of $5 \cos(1.5\pi n + 0.75\pi)$. In this case, Eq. (2.47a) reduces to $1.5\pi N_1 = 2\pi r_1$ which is satisfied with $N_1 = 4, r_1 = 3$. We next determine the fundamental period N_2 of $4 \cos(0.6\pi n)$. In this case, Eq. (2.47a) reduces to $0.6\pi N_2 = 2\pi r_2$ which is satisfied with $N_2 = 10, r_2 = 3$. We finally determine the fundamental period N_3 of $\sin(0.5\pi n)$. In this case, Eq. (2.47a) reduces to $0.5\pi N_3 = 2\pi r_3$ which is satisfied with $N_3 = 4, r_3 = 1$. Hence the fundamental period N of $\tilde{x}_5[n]$ is given by $LCM(N_1, N_2, N_3) = LCM(4, 10, 4) = 20$.

2.31 The fundamental period N of a periodic sequence with an angular frequency ω_o satisfies Eq. (2.47a) with the smallest value of N and r .

(a) $\omega_o = 0.6\pi$. Here Eq. (2.47a) reduces to $0.6\pi N = 2\pi r$ which is satisfied with $N = 10, r = 3$.

(b) $\omega_o = 0.28\pi$. Here Eq. (2.47a) reduces to $0.28\pi N = 2\pi r$ which is satisfied with $N = 50, r = 7$.

(c) $\omega_o = 0.45\pi$. Here Eq. (2.47a) reduces to $0.45\pi N = 2\pi r$ which is satisfied with $N = 40, r = 9$.

(d) $\omega_o = 0.55\pi$. Here Eq. (2.47a) reduces to $0.55\pi N = 2\pi r$ which is satisfied with $N = 40, r = 11$.

(e) $\omega_o = 0.65\pi$. Here Eq. (2.47a) reduces to $0.65\pi N = 2\pi r$ which is satisfied with $N = 40, r = 13$.

2.32 $\omega_o = 0.08\pi$. Here Eq. (2.47a) reduces to $0.08\pi N = 2\pi r$ which is satisfied with $N = 25, r = 1$. For a sequence $\tilde{x}_2[n] = \sin(\omega_2 n)$ with a fundamental period of $N = 25$, Eq. (2.47a) reduces to $25\omega_2 = 2\pi r$. For example, for $r = 2$ we have $\omega_2 = 4\pi/25 = 0.16\pi$. Another sequence with the same fundamental period is obtained by setting $r = 3$ which leads to $\omega_3 = 6\pi/25 = 0.24\pi$. The corresponding periodic sequences are therefore $\tilde{x}_2[n] = \sin(0.16\pi n)$ and $\tilde{x}_3[n] = \sin(0.24\pi n)$.

2.33 The three parameters A, Ω_o , and ϕ of the continuous-time signal $x_a(t)$ can be determined from $x[n] = x_a(nT) = A \cos(\Omega_o nT + \phi)$ by setting 3 distinct values of n . For example

$$x[0] = A \cos \phi = \alpha,$$

$$x[-1] = A \cos(-\Omega_o T + \phi) = A \cos(\Omega_o T) \cos \phi + A \sin(\Omega_o T) \sin \phi = \beta,$$

$$x[1] = A \cos(\Omega_o T + \phi) = A \cos(\Omega_o T) \cos \phi - A \sin(\Omega_o T) \sin \phi = \gamma.$$

Substituting the first equation into the last two equations and then adding them we get $\cos(\Omega_o T) = \frac{\beta + \gamma}{2\alpha}$ which can be solved to determine Ω_o . Next, from the second

equation we have $A \sin \phi = \beta - A \cos(\Omega_o T) \cos \phi = \beta - \alpha \cos(\Omega_o T)$. Dividing this

equation by the last equation on the previous page we arrive at $\tan \phi = \frac{\beta - \alpha \cos(\Omega_o T)}{\alpha \sin(\Omega_o T)}$

which can be solved to determine ϕ . Finally, the parameter is determined from the first equation of the last page.

Now consider the case $\Omega_T = \frac{2\pi}{T} = 2\Omega_o$. In this case $x[n] = A \cos(n\pi + \phi) = \beta$ and $x[n+1] = A \cos((n+1)\pi + \phi) = A \cos(n\pi + \phi) = \beta$. Since all sample values are equal, the three parameters cannot be determined uniquely.

Finally consider the case $\Omega_T = \frac{2\pi}{T} < 2\Omega_o$. In this case $x[n] = A \cos(\Omega_o nT + \phi) = A \cos(\omega_o n + \phi)$ implying $\omega_o = \Omega_o T > \pi$. As explained in Section 2.2.1, a digital sinusoidal sequence with an angular frequency ω_o greater than π assumes the identity of a sinusoidal sequence with an angular frequency in the range $0 \leq \omega < \pi$. Hence, Ω_o cannot be uniquely determined from $x[n] = A \cos(\Omega_o nT + \phi)$.

2.34 $x[n] = \cos(\Omega_o nT)$. If $x[n]$ is periodic with a period N , then $x[n+N] = \cos(\Omega_o nT + \Omega_o NT) = x[n] = \cos(\Omega_o nT)$. This implies $\Omega_o NT = 2\pi r$ with r any nonzero positive integer. Hence the sampling rate must satisfy the relation $T = 2\pi r / \Omega_o N$. If $\Omega_o = 20$, i.e., $T = \pi/8$, then we must have $20N \cdot \frac{\pi}{8} = 2\pi r$. The smallest value of N and r satisfying this relation are $N = 4$ and $r = 5$. The fundamental period is thus $N = 4$.

2.35 (a) For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = b_0 x_i[n] + b_1 x_i[n-1] + b_2 x_i[n-2] + a_1 y_i[n-1] + a_2 y_i[n-2], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = b_0(Ax_1[n] + Bx_2[n]) + b_1(Ax_1[n-1] + Bx_2[n-1]) + b_2(Ax_1[n-2] + Bx_2[n-2]) + a_1(Ay_1[n-1] + By_2[n-1]) + a_2(Ay_1[n-2] + By_2[n-2]) = A(b_0 x_1[n] + b_1 x_1[n-1] + b_2 x_1[n-2] + a_1 y_1[n-1] + a_2 y_1[n-2]) + B(b_0 x_2[n] + b_1 x_2[n-1] + b_2 x_2[n-2] + a_1 y_2[n-1] + a_2 y_2[n-2]) = Ay_1[n] + By_2[n]$. Hence, the system of Eq. (2.18) is linear.

(b) For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = \begin{cases} x_i[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise.} \end{cases}$

For an input $x[n] = Ax_1[n] + Bx_2[n]$, the output for $n = 0, \pm L, \pm 2L, \dots$ is $y[n] = x[n/L] = Ax_1[n/L] + Bx_2[n/L] = Ay_1[n] + By_2[n]$. For all other values of n , $y[n] = A \cdot 0 + B \cdot 0 = 0$. Hence the system of Eq. (2.20) is linear.

(c) For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = x_i[n/M], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = Ax_1[n/M] + Bx_2[n/M] = Ay_1[n] + By_2[n]$. Hence the system of Eq. (2.21) is linear.

(d) For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = \frac{1}{M} \sum_{k=0}^{M-1} x_i[n-k], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = \frac{1}{M} \sum_{k=0}^{M-1} (Ax_1[n-k] + Bx_2[n-k])$
 $= A \left(\frac{1}{M} \sum_{k=0}^{M-1} x_1[n-k] \right) + B \left(\frac{1}{M} \sum_{k=0}^{M-1} x_2[n-k] \right) = Ay_1[n] + By_2[n]$. Hence the system of Eq. (2.61) is linear.

(e) The first term on the RHS of Eq. (2.65) is the output of a factor-of-2 up-sampler. The second term on the RHS of Eq. (2.65) is simply the output of an unit delay followed by a factor-of-2 up-sampler, whereas, the third term is the output of an unit advance operator followed by a factor-of-2 up-sampler. We have shown in Part (b) that the up-sampler is a linear system. Moreover, the unit delay and the unit advance operator are linear systems. A cascade of two linear systems is linear and the linear combination of linear systems is also linear. Hence, the factor-of-2 interpolator of Eq. (2.65) is a linear system.

(f) Following the arguments given in Part (e), we can similarly show that the factor-of-3 interpolator of Eq. (2.66) is a linear system.

2.36 (a) $y[n] = n^3 x[n]$. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = n^3 x_i[n], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = n^3 (Ax_1[n] + Bx_2[n]) = Ay_1[n] + By_2[n]$. Hence the system is linear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = n^3 \delta[n]$. As $h[n] = 0$ for $n < 0$, and the system is causal.

Let $x[n] = 1$ for all values of n . Then $|y[n]| = |n^3|$ and $y[n] \rightarrow \infty$ as $n \rightarrow \infty$. Since a bounded input results in an unbounded output, the system is not BIBO stable. Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$ then $y_1[n] = n^3 x_1[n] = n^3 x[n - n_o]$. However, $y[n - n_o] = (n - n_o)^3 x[n - n_o]$. Since $y_1[n] \neq y[n - n_o]$, the system is not time-invariant.

(b) $y[n] = (x[n])^5$. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = (x_i[n])^5, i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = (Ax_1[n] + Bx_2[n])^5 \neq A(x_1[n])^5 + B(x_2[n])^5$. Hence the system is nonlinear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = (\delta[n])^5$. As $h[n] = 0$ for $n < 0$, and the system is causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are $|y[n]| = |(x[n])^5| = |x[n]|^5 \leq B^5 < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$ then $y_1[n] = (x_1[n])^5 = (x[n - n_o])^5 = y[n - n_o]$. Hence, the system is time-invariant.

(c) $y[n] = \beta + \sum_{\ell=0}^3 x[n - \ell]$ with β a nonzero constant. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = \beta + \sum_{\ell=0}^3 x_i[n - \ell], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = \beta + \sum_{\ell=0}^3 (Ax_1[n - \ell] + Bx_2[n - \ell]) = \beta + \sum_{\ell=0}^3 Ax_1[n - \ell] + \sum_{\ell=0}^3 Bx_2[n - \ell] \neq Ay_1[n] + By_2[n]$. Hence the system is nonlinear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \beta + \sum_{\ell=0}^{\infty} \delta[n - \ell]$. As

$h[n] \neq 0$ for $n < 0$, the system is noncausal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are $|y[n]| \leq \beta + 4B < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$ then $y_1[n] = \beta + \sum_{\ell=0}^3 x[n - n_o - \ell] = y[n - n_o]$. Hence, the system is time-invariant.

(d) $y[n] = \ln(2 + |x[n]|)$. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = \ln(2 + |x_i[n]|), i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = \ln(2 + |Ax_1[n] + Bx_2[n]|) \neq Ay_1[n] + By_2[n]$. Hence the system is nonlinear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] + \ln(2 + |\delta[n]|)$.

For $n < 0, h[n] = \ln(2) \neq 0$. Hence, the system is noncausal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are $|y[n]| \leq \ln(2 + B) < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If

$x_1[n] = x[n - n_o]$ then $y_1[n] = \ln(2 + |x[n - n_o]|) = y[n - n_o]$. Hence, the system is time-invariant.

(e) $y[n] = \alpha x[-n + 2]$, with a nonzero constant. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = \alpha x_i[-n + 2], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = A\alpha x_1[-n + 2] + B\alpha x_2[-n + 2] = Ay_1[n] + By_2[n]$. Hence the system is linear. For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \alpha\delta[-n + 2]$. For $n < 0$, $h[n] = 0$. Hence, the system is causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are

$|y[n]| = |\alpha B| < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$ then $y_1[n] = \alpha x_1[-n + 2] = \alpha x[-(n - n_o) + 2] = y[n - n_o]$. Hence, the system is time-invariant.

(f) $y[n] = x[n - 4]$. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = x_i[n - 4], i = 1, 2$.

Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = Ax_1[n - 4] + Bx_2[n - 4] = Ay_1[n] + By_2[n]$. Hence the system is linear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \delta[n - 4]$. For $n < 0$, $h[n] = 0$. Hence, the system is causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are

$|y[n]| = B < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$ then $y_1[n] = x_1[n - 4] = x[n - n_o - 4] = y[n - n_o]$. Hence, the system is time-invariant.

2.37 Let $y[n]$ and $y_1[n]$ be the outputs of a median filter of length $2K + 1$ for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$, then

$$\begin{aligned} y_1[n] &= \text{med}\{x_1[n - K], \dots, x_1[n - 1], x_1[n], x_1[n + 1], \dots, x_1[n + K]\} \\ &= \text{med}\{x[n - n_o - K], \dots, x[n - n_o - 1], x[n - n_o], x[n - n_o + 1], \dots, x[n - n_o + K]\} \\ &= y[n - n_o]. \end{aligned}$$

Hence, the system is time-invariant.

2.38 $y[n] = x[n + 1] - 2x[n] + x[n - 1]$. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = x_i[n + 1] - 2x_i[n] + x_i[n - 1], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is

$$\begin{aligned} y[n] &= Ax_1[n + 1] + Bx_2[n + 1] - 2Ax_1[n] - 2Bx_2[n] + Ax_1[n - 1] + Bx_2[n - 1] \\ &= Ay_1[n] + By_2[n]. \end{aligned}$$

Hence the system is linear.

If $x_1[n] = x[n - n_o]$, then $y_1[n] = x_1[n + 1] - 2x_1[n] + x_1[n - 1] = y[n - n_o]$.

Hence, the system is time-invariant.

The impulse response of the system is $h[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$. Now $h[-1] = \delta[0] = 1$. Since $h[n] \neq 0$ for all values of $n < 0$, the system is noncausal.

2.39 $y[n] = x^2[n] - x[n-1]x[n+1]$. For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = x_i^2[n] - x_i[n-1]x_i[n+1], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = (Ax_1[n] + Bx_2[n])^2 - (Ax_1[n-1] + Bx_2[n-1])(Ax_1[n+1] + Bx_2[n+1]) \neq Ay_1[n] + By_2[n]$. Hence the system is nonlinear.

If $x_1[n] = x[n - n_o]$, then $y_1[n] = x_1^2[n] - x_1[n-1]x_1[n+1] = x^2[n - n_o] - x[n - n_o - 1]x[n - n_o + 1] = y[n - n_o]$. Hence, the system is time-invariant.

The impulse response of the system is $h[n] = \delta^2[n] - \delta[n-1]\delta[n+1] = \delta[n]$. Since $h[n] = 0$ for all values of $n < 0$, the system is causal.

2.40 $y[n] = \frac{1}{2} \left(y[n-1] + \frac{x[n]}{y[n-1]} \right)$. Now for an input $x[n] = \alpha\mu[n]$, the output $y[n]$ converges to some constant K as $n \rightarrow \infty$. The input-output relation of the system as $n \rightarrow \infty$ reduces to $K = \frac{1}{2} \left(K + \frac{\alpha}{K} \right)$ from which we get $K^2 = \alpha$ or in other words $K = \sqrt{\alpha}$.

For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = \frac{1}{2} \left(y_i[n-1] + \frac{x_i[n]}{y_i[n-1]} \right), i = 1, 2$. Then,

for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = \frac{1}{2} \left(y[n-1] + \frac{Ax_1[n] + Bx_2[n]}{y[n-1]} \right)$.

On the other hand,

$$Ay_1[n] + By_2[n] = \frac{1}{2} \left(Ay_1[n-1] + \frac{Ax_1[n]}{y_1[n-1]} \right) + \frac{1}{2} \left(By_2[n-1] + \frac{Bx_2[n]}{y_2[n-1]} \right) \neq y[n].$$

Hence the system is nonlinear.

If $x_1[n] = x[n - n_o]$, then $y_1[n] = \frac{1}{2} \left(y_1[n-1] + \frac{x[n - n_o]}{y_1[n-1]} \right) = y[n - n_o]$. Hence, the system is time-invariant.

2.41 $y[n] = x[n] - y^2[n-1] + y[n-1]$.

For an input $x_i[n], i = 1, 2$, the output is $y_i[n] = x_i[n] - y_i^2[n-1] + y_i[n-1], i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is

$y[n] = Ax_1[n] + Bx_2[n] - y^2[n-1] + y[n-1]$. On the other hand, $Ay_1[n] + By_2[n]$

$= Ax_1[n] - Ay_1^2[n-1] + Ay_1[n-1] + Bx_2[n] - By_2^2[n-1] + By_2[n-1] \neq y[n]$. Hence the system is nonlinear.

Let $y[n]$ be the output for an input $x[n]$, i.e., $y[n] = x[n] - y^2[n] + y[n-1]$. Then, for an input $x[n-n_o]$ the output is given by $y[n-n_o] = x[n-n_o] - y^2[n-n_o-1] + y[n-n_o-1]$, or in other words, the system is time-invariant.

Now, for an input $x[n] = \alpha\mu[n]$, the output $y[n]$ converges to some constant K as $n \rightarrow \infty$. The difference equation describing the system as $n \rightarrow \infty$ reduces to $K = \alpha - K^2 + K$, or $K^2 = \alpha$, i.e., $K = \sqrt{\alpha}$.

2.42 The impulse response of the factor-of-3 interpolator of Eq. (2.66) is the output for an input $x_u[n] = \delta[n]$ and is given by

$$h[n] = \delta[n] + \frac{2}{3}(\delta[n-1] + \delta[n+1]) + \frac{1}{3}(\delta[n-2] + \delta[n+2]) \text{ or equivalently by}$$

$$\{h[n]\} = \left\{ \frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3} \right\} - 2 \leq n \leq 2.$$

2.43 The input-output relation of a factor-of- L interpolator is given by

$$y[n] = x_u[n] + \sum_{k=1}^{L-1} \frac{L-k}{L} (x_u[n-k] + x_u[n+k]).$$
 Its impulse response is the output for

$$\text{an input } x_u[n] = \delta[n] \text{ and is thus given by } h[n] = \delta[n] + \sum_{k=1}^{L-1} \frac{L-k}{L} (\delta[n-k] + \delta[n+k])$$

or equivalently by

$$\{h[n]\} = \left\{ \frac{1}{L}, \frac{2}{L}, \dots, \frac{L-2}{L}, \frac{L-1}{L}, 1, \frac{L-1}{L}, \frac{L-2}{L}, \dots, \frac{2}{L}, \frac{1}{L} \right\} - L+1 \leq n \leq L-1.$$

2.44 The impulse response $h[n]$ of a causal discrete-time system satisfies the difference equation $h[n] - ah[n-1] = \delta[n]$. Since the system is causal, we have $h[n] = 0$ for $n < 0$. Evaluating the above difference equation for $n = 0$, we arrive at $h[0] - ah[-1] = 1$ and thus $h[0] = 1$. Next, for $n = 1$, we have $h[1] - ah[0] = 0$ and thus $h[1] = a$. Continuing we get for $n = 2$, $h[2] - ah[1] = 0$, i.e., $h[2] = ah[1] = a^2$. Assume $h[n-1] = a^{n-1}$ with $n > 0$. From the difference equation we then have $h[n] - ah[n-1] = 0$, i.e., $h[n] = ah[n-1] = a^n$. Since the last equation holds for $n = 0, 1, 2$, by induction, it holds for $n \geq 3$.

2.45 As $x[n]$ and $h[n]$ are right-sided sequences, assume $x[n] = 0$ for all $n < N_1$ and $h[n] = 0$ and $n < N_2$. Hence, $y[n] = h[n] \otimes x[n] = 0$ for all $n < N_1 + N_2$ and thus

$y[n]$ is also a right-sided sequence. Therefore,

$$\begin{aligned} \sum_{n=N_1+N_2}^{\infty} y[n] &= \sum_{n=N_1+N_2}^{\infty} h[n] \circledast x[n] \\ &= \sum_{n=N_1+N_2}^{\infty} \sum_{k=N_2}^{\infty} h[k] x[n-k] = \sum_{k=N_2}^{\infty} \sum_{n=N_1+N_2}^{\infty} h[k] x[n-k] = \sum_{k=N_2}^{\infty} h[k] \sum_{n=N_1+N_2}^{\infty} x[n-k] \\ &= \sum_{k=N_2}^{\infty} h[k] \sum_{m=N_1+N_2-k}^{\infty} x[m] = \sum_{k=N_2}^{\infty} h[k] \sum_{m=N_1}^{\infty} x[m] \text{ as } x[m] = 0 \text{ for all } m < N_1. \text{ Hence,} \\ \sum_n y[n] &= \left(\sum_n h[n] \right) \left(\sum_n x[n] \right). \end{aligned}$$

2.46 (a) $\alpha^n \mu[n] \circledast \mu[n] = \sum_{k=-\infty}^{\infty} \alpha^k \mu[k] \mu[n-k] = \sum_{k=0}^{\infty} \alpha^k \mu[n-k] = \begin{cases} \sum_{k=0}^n \alpha^k, & n \geq 0, \\ 0, & n < 0, \end{cases}$

$$= \left(\frac{1 - \alpha^{n+1}}{1 - \alpha} \right) \mu[n].$$

(b) $n \alpha^n \mu[n] \circledast \mu[n] = \sum_{k=-\infty}^{\infty} k \alpha^k \mu[k] \mu[n-k] = \sum_{k=0}^{\infty} k \alpha^k \mu[n-k] = \begin{cases} \sum_{k=0}^n k \alpha^k, & n > 0, \\ 0, & n \leq 0. \end{cases}$

2.47 Now from Eq. (2.72) an arbitrary input $x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \text{ which can be rewritten using Eq. (2.41b) as}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] (\mu[n-k] - \mu[n-k-1]) = \sum_{k=-\infty}^{\infty} x[k] \mu[n-k] - \sum_{k=-\infty}^{\infty} x[k] \mu[n-k-1].$$

Since $s[n]$ is the response of an LTI system for an input $\mu[n]$, $s[n-k]$ is the response for an input $\mu[n-k]$ and $s[n-k-1]$ is the response for an input $\mu[n-k-1]$. Hence,

the output for an input $\sum_{k=-\infty}^{\infty} x[k] \mu[n-k] - \sum_{k=-\infty}^{\infty} x[k] \mu[n-k-1]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] s[n-k] - \sum_{k=-\infty}^{\infty} x[k] s[n-k-1] = x[n] \circledast s[n] - x[n-1] \circledast s[n-1].$$

2.48 $y[n] = \sum_{m=-\infty}^{\infty} h[m] \tilde{x}[n-m]$. Hence,

$$y[n+kN] = \sum_{m=-\infty}^{\infty} h[m] \tilde{x}[n+kN-m] = \sum_{m=-\infty}^{\infty} h[m] x[n-m] = y[n]. \text{ Thus, } y[n] \text{ is also a periodic sequence with a period } N.$$

2.49 In this problem we make use of the identity $\delta[n-m] \circledast \delta[n-r] = \delta[n-m-r]$.

$$\begin{aligned}
\text{(a)} \quad y_1[n] &= x_1[n] \otimes h_1[n] = (3\delta[n-2] - 2\delta[n+1]) \otimes (-\delta[n+2] + 4\delta[n] + 2\delta[n-1]) \\
&= -3\delta[n-2] \otimes \delta[n+2] + 12\delta[n-2] \otimes \delta[n] - 6\delta[n-2] \otimes \delta[n-1] + 2\delta[n+1] \otimes \delta[n+2] \\
&\quad - 8\delta[n+1] \otimes \delta[n] + 4\delta[n+1] \otimes \delta[n-1]. \quad \text{Hence}
\end{aligned}$$

$$\begin{aligned}
y_1[n] &= -3\delta[n] + 12\delta[n-2] - 6\delta[n-3] + 2\delta[n+3] - 8\delta[n+1] + 4\delta[n] \\
&= 2\delta[n+3] - 8\delta[n+1] + \delta[n] + 12\delta[n-2] - 6\delta[n-3].
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad y_2[n] &= x_2[n] \otimes h_2[n] = (5\delta[n-3] + 2\delta[n+1]) \otimes (3\delta[n-4] + 1.5\delta[n-2] - \delta[n+1]) \\
&= 15\delta[n-3] \otimes \delta[n-4] + 7.5\delta[n-3] \otimes \delta[n-2] - 5\delta[n-3] \otimes \delta[n+1] + 6\delta[n+1] \otimes \delta[n-4] \\
&\quad + 3\delta[n+1] \otimes \delta[n-2] - 2\delta[n+1] \otimes \delta[n+1] = 15\delta[n-7] + 7.5\delta[n-5] - 5\delta[n-2] \\
&\quad + 6\delta[n-3] + 3\delta[n-1] - 2\delta[n+2].
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad y_3[n] &= x_1[n] \otimes h_2[n] = (-3\delta[n-2] - 2\delta[n+1]) \otimes (3\delta[n-4] + 1.5\delta[n-2] - \delta[n+1]) \\
&= 9\delta[n-2] \otimes \delta[n-4] + 4.5\delta[n-2] \otimes \delta[n-2] - 3\delta[n-2] \otimes \delta[n+1] - 6\delta[n+1] \otimes \delta[n-4] \\
&\quad - 3\delta[n+1] \otimes \delta[n-2] + 2\delta[n+1] \otimes \delta[n+1] = 9\delta[n-6] + 4.5\delta[n-4] - 3\delta[n-1] \\
&\quad - 6\delta[n-3] - 3\delta[n-1] - 3\delta[n-1] + 2\delta[n+2] = 2\delta[n+2] - 6\delta[n-1] - 6\delta[n-3] \\
&\quad + 4.5\delta[n-4] + 9\delta[n-6].
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad y_4[n] &= x_2[n] \otimes h_1[n] = (5\delta[n-3] + 2\delta[n+1]) \otimes (-\delta[n+2] + 4\delta[n] - 2\delta[n-1]) \\
&= -5\delta[n-3] \otimes \delta[n+2] + 20\delta[n-3] \otimes \delta[n] - 10\delta[n-3] \otimes \delta[n-1] - 2\delta[n+1] \otimes \delta[n+2] \\
&\quad + 8\delta[n+1] \otimes \delta[n] - 4\delta[n+1] \otimes \delta[n-1] = -5\delta[n-1] + 20\delta[n-3] - 10\delta[n-4] - 2\delta[n+3] \\
&\quad + 8\delta[n+1] - 4\delta[n] = -2\delta[n+3] + 8\delta[n+1] - 4\delta[n] - 5\delta[n-1] + 20\delta[n-3] - 10\delta[n-4].
\end{aligned}$$

2.50 (a) $u[n] = x[n] \otimes y[n]$
 $= \{-24, 42, -5, -20, -45, 23, 66, -25, -42, -17, 22, 14, -4\}, -4 \leq n \leq 8.$

(b) $v[n] = x[n] \otimes w[n]$
 $= \{-12, 7, 5, 10, -16, -3, -28, 30, 13, -6, -15, -4, 10\}, -1 \leq n \leq 11.$

(c) $g[n] = w[n] \otimes y[n]$
 $= \{18, 3, 3, -14, 25, 26, 60, -11, -16, -14, 26, 39, -10\}, 1 \leq n \leq 13.$

2.51 $y[n] = \sum_{m=N_1}^{N_2} g[m]h[n-m]$. Now, $h[n-m]$ is defined for $M_1 \leq n-m \leq M_2$. Thus, for $m = N_1$, $h[n-m]$ is defined for $M_1 \leq n - N_1 \leq M_2$, or equivalently, for $M_1 + N_1 \leq n \leq M_2 + N_1$. Likewise, for $m = N_2$, $h[n-m]$ is defined for $M_1 \leq n - N_2 \leq M_2$, or equivalently, for $M_1 + N_2 \leq n \leq M_2 + N_2$. For the specified sequences $N_1 = -3, N_2 = 4, M_1 = 2, M_2 = 6$. **(a)** The length of $y[n]$ is $M_2 + N_2 - M_1 - N_1 + 1 = 6 + 4 - 2 - (-3) + 1 = 12$. **(b)** The range of n for $y[n] \neq 0$ is $\min(M_1 + N_1, M_2 + N_2) \leq n \leq \max(M_1 + N_1, M_2 + N_2)$, i.e., $M_1 + N_1 \leq n \leq M_2 + N_2$. For the specified sequences the range of n is $-1 \leq n \leq 10$.

2.52 $y[n] = x_1[n] \otimes x_2[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]x_2[k]$. Now,
 $v[n] = x_1[n - N_1] \otimes x_2[n - N_2] = \sum_{k=-\infty}^{\infty} x_1[n - N_1 - k]x_2[k - N_2]$. Let $k - N_2 = m$.
Then $v[n] = \sum_{m=-\infty}^{\infty} x_1[n - N_1 - N_2 - m]x_2[m] = y[n - N_1 - N_2]$.

2.53 $g[n] = x_1[n] \otimes x_2[n] \otimes x_3[n] = y[n] \otimes x_3[n]$ where $y[n] = x_1[n] \otimes x_2[n]$. Now
 $v[n] = x_1[n - N_1] \otimes x_2[n - N_2]$. Define $h[n] = v[n] \otimes x_3[n - N_3]$. Then from the results of Problem 2.52, $v[n] = y[n - N_1 - N_2]$. Hence,
 $h[n] = y[n - N_1 - N_2] \otimes x_3[n - N_3]$. Therefore, making use of the results of Problem 2.52 again we get $h[n] = y[n - N_1 - N_2 - N_3]$.

2.54 $y[n] = x[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$. Substituting k by $n-m$ in this expression, we get $y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = h[n] \otimes x[n]$. Hence the convolution operation is commutative.

Let $y[n] = x[n] \otimes (h_1[n] + h_2[n]) = \sum_{k=-\infty}^{\infty} x[n-k](h_1[k] + h_2[k])$
 $= \sum_{k=-\infty}^{\infty} x[n-k]h_1[k] + \sum_{k=-\infty}^{\infty} x[n-k]h_2[k] = x[n] \otimes h_1[n] + x[n] \otimes h_2[n]$. Hence the convolution operation is also distributive.

2.55 $x_3[n] \otimes x_2[n] \otimes x_1[n] = x_3[n] \otimes (x_2[n] \otimes x_1[n])$. As $x_2[n] \otimes x_1[n]$ is an unbounded sequence, the result of this convolution cannot be determined. But

$x_2[n] \otimes x_3[n] \otimes x_1[n] = x_2[n] \otimes (x_3[n] \otimes x_1[n])$. Now $x_3[n] \otimes x_1[n] = 0$ for all values of n , and hence the overall result is zero. As a result, for the given sequences $x_3[n] \otimes x_2[n] \otimes x_1[n] \neq x_2[n] \otimes x_3[n] \otimes x_1[n]$

2.56 $w[n] = x[n] \otimes h[n] \otimes g[n]$. Define $y[n] = x[n] \otimes h[n] = \sum_k x[k]h[n-k]$ and $f[n] = h[n] \otimes g[n] = \sum_k g[k]h[n-k]$. Consider $w_1[n] = (x[n] \otimes h[n]) \otimes g[n] = y[n] \otimes g[n] = \sum_m g[m] \sum_k x[k]h[n-m-k]$. Now consider $w_2[n] = x[n] \otimes (h[n] \otimes g[n]) = x[n] \otimes f[n] = \sum_k x[k] \sum_m g[m]h[n-k-m]$. The difference between the expressions for $w_1[n]$ and $w_2[n]$ is that the order of the summations is changed.

A) Assumptions: $h[n]$ and $g[n]$ are causal sequences, and $x[n] = 0$ for $n < 0$. This

implies $y[m] = \begin{cases} 0, & \text{for } m < 0, \\ \sum_{k=0}^m x[k]h[m-k], & \text{for } m \geq 0. \end{cases}$ Thus, $w[n] = \sum_{m=0}^n g[m]y[n-m]$

$= \sum_{m=0}^n g[m] \sum_{k=0}^{n-m} x[k]h[n-m-k]$. All sums have only a finite number of terms. Hence, the interchange of the order of the summations is justified and will give correct results.

B) Assumptions: $h[n]$ and $g[n]$ are stable sequences, and $x[n]$ is a bounded sequence

with $|x[n]| \leq B < \infty$. Here, $y[m] = \sum_{k=-\infty}^{\infty} h[k]x[m-k] = \left(\sum_{k=k_1}^{k_2} h[k]x[m-k] \right)$

$+ \varepsilon_{k_1, k_2}[m]$ with $|\varepsilon_{k_1, k_2}[m]| \leq \varepsilon_n B$. In this case, all sums have effectively only a finite number of terms and the error $|\varepsilon_{k_1, k_2}[m]|$ can be reduced by choosing k_1 and k_2 sufficiently large. As a result, in this case the problem is again effectively reduced to that of the one-sided sequences. Thus, the interchange of the order of the summations is again justified and will give correct results.

Hence, for the convolution to be associative, it is sufficient that the sequences be stable and single-sided.

2.57 $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$. Since $h[k]$ is of length M and defined for $0 \leq k \leq M-1$, the convolution sum reduces to $y[n] = \sum_{k=0}^{M-1} x[n-k]h[k]$. $y[n]$ will be nonzero for all those values of n and for k which $n-k$ satisfies $0 \leq n-k \leq N-1$. Minimum value of $n-k = 0$ and occurs for lowest n at $n=0$ and $k=0$. Maximum value of $n-k = N-1$ and occurs for maximum value of k at $M-1$. Thus $n-k = M-1 \Rightarrow n = N + M - 2$. Hence the total number of nonzero samples = $N + M - 1$.

2.58 $y[n] = \sum_{k=0}^{N-1} x[n-k]x[k]$. The maximum value of $y[n]$ occurs at $n = N - 1$ when all product terms are present. The maximum value is given by
 $y[N - 1] = \sum_{k=0}^{N-1} a_{N-1-k}a_k$.

2.59 $y[n] = \sum_{k=0}^{N-1} x[n-k]h[k]$. The maximum value of $y[n]$ occurs at $n = N - 1$ when all product terms are present. The maximum value is given by
 $y[N - 1] = \sum_{k=0}^{N-1} a_{N-1-k}b_k$.

2.60 (a) $y[n] = g_{ev}[n] \otimes h_{ev}[n] = \sum_{k=-\infty}^{\infty} h_{ev}[n-k]g_{ev}[k]$. Now,

$y[-n] = \sum_{k=-\infty}^{\infty} h_{ev}[-n-k]g_{ev}[k]$. Replace k by $-k$. Then the summation on the left becomes $y[-n] = \sum_{k=-\infty}^{\infty} h_{ev}[-n+k]g_{ev}[-k] = \sum_{k=-\infty}^{\infty} h_{ev}[-(n-k)]g_{ev}[-k] = y[n]$. Hence $g_{ev}[n] \otimes h_{ev}[n]$ is an even sequence.

(b) $y[n] = g_{ev}[n] \otimes h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[n-k]g_{ev}[k]$. Now,

$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[-n-k]g_{ev}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-n+k]g_{ev}[-k]$
 $= \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)]g_{ev}[-k] = -\sum_{k=-\infty}^{\infty} h_{od}[n-k]g_{ev}[k] = -y[n]$.
Hence $g_{ev}[n] \otimes h_{od}[n]$ is an odd sequence.

(c) $y[n] = g_{od}[n] \otimes h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[n-k]g_{od}[k]$. Now,

$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[-n-k]g_{od}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-n+k]g_{od}[-k]$
 $= \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)]g_{od}[-k] = \sum_{k=-\infty}^{\infty} h_{od}[n-k]g_{od}[k] = y[n]$.
Hence $g_{od}[n] \otimes h_{od}[n]$ is an even sequence.

2.61 The impulse response of the cascade is given by $h[n] = h_1[n] \otimes h_2[n]$ where

$$h_1[n] = \alpha^n \mu[n] \text{ and } h_2[n] = \beta^n \mu[n]. \text{ Hence, } h[n] = \left(\sum_{k=0}^n \alpha^k \beta^{n-k} \right) \mu[n].$$

2.62 Now $h[n] = \alpha^n \mu[n]$. Therefore $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} \alpha^k x[n-k]$
 $= x[n] + \sum_{k=1}^{\infty} \alpha^k x[n-k] = x[n] + \alpha \sum_{k=0}^{\infty} \alpha^k x[n-1-k] = x[n] + \alpha y[n-1]$.

Hence, $x[n] = y[n] - \alpha y[n-1]$. Thus the inverse system is given by

$y[n] = x[n] - \alpha x[n-1]$. The impulse response of the inverse system is given by
 $h[n] = \{1, \alpha\}, 0 \leq n \leq 1$.

2.63 From the results of Problem 2.62 we have $h[n] = \left(\sum_{k=0}^n \alpha^k \beta^{n-k} \right) \mu[n]$. Now,

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} \left(\sum_{m=0}^k \alpha^m \beta^{k-m} \right) x[n-k] = \sum_{k=0}^{\infty} \left(\sum_{m=0}^k \alpha^m \beta^{k-m} \right) x[n-k] \\
&= x[n] + \sum_{k=1}^{\infty} \left(\sum_{m=0}^k \alpha^m \beta^{k-m} \right) x[n-k]. \text{ Substituting } r = k-1 \text{ in the last expression we get} \\
y[n] &= x[n] + \sum_{r=0}^{\infty} \left(\sum_{m=0}^{r+1} \alpha^m \beta^{r+1-m} \right) x[n-r-1] = x[n] + \sum_{r=0}^{\infty} \left(\sum_{m=0}^r \alpha^m \beta^{r+1-m} + \alpha^{r+1} \right) x[n-r-1] \\
&= x[n] + \beta \sum_{r=0}^{\infty} \left(\sum_{m=0}^r \alpha^m \beta^{r-m} \right) x[n-r-1] + \sum_{r=0}^{\infty} \alpha^{r+1} x[n-r-1] \\
&= x[n] + \beta y[n-1] + \alpha x[n-1] + \alpha^2 x[n-2] + \alpha^3 x[n-3] + \dots. \text{ The inverse system is} \\
&\text{therefore given by } x[n] = y[n] - (\alpha + \beta)y[n-1] + \alpha\beta y[n-2].
\end{aligned}$$

2.64 (a) $h[n] = h_1[n] \otimes h_2[n] \otimes h_3[n] \otimes h_3[n] + h_1[n] \otimes h_2[n] + h_3[n] \otimes h_4[n].$

(b)
$$h[n] = h_4[n] + \frac{h_1[n] \otimes h_2[n] \otimes h_3[n]}{1 - h_1[n] \otimes h_2[n] \otimes h_5[n]}.$$

2.65 $h[n] = h_1[n] \otimes h_2[n] + h_3[n].$ Now

$$h_1[n] \otimes h_2[n] = (2\delta[n-2] - 3\delta[n+1]) \otimes (\delta[n-1] + 2\delta[n+2])$$

$$= 2\delta[n-2] \otimes \delta[n-1] - 3\delta[n+1] \otimes \delta[n-1] + 2\delta[n-2] \otimes 2\delta[n+2] - 3\delta[n+1] \otimes 2\delta[n+2]$$

$$= 2\delta[n-3] - 3\delta[n] + 4\delta[n] - 6\delta[n+3] \text{ and}$$

$$h_3[n] = 5\delta[n-5] + 7\delta[n-3] + 2\delta[n-1] - \delta[n] + 3\delta[n+1]. \text{ Therefore,}$$

$$\begin{aligned}
h[n] &= 2\delta[n-3] - 3\delta[n] + 4\delta[n] - 6\delta[n+3] + 5\delta[n-5] + 7\delta[n-3] + 2\delta[n-1] - \delta[n] + 3\delta[n+1] \\
&= 5\delta[n-5] + 9\delta[n-3] + 2\delta[n-1] + 3\delta[n+1] - 6\delta[n+3].
\end{aligned}$$

2.66 (a) The length of $x[n]$ is $8 - 4 + 1 = 5$. Using $x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=0}^3 h[k]x[n-k] \right\}$ we

arrive at $\{x[n]\} = \{3, -2, 0, 1, 2\}, 0 \leq n \leq 4$.

(b) The length of $x[n]$ is $7 - 4 + 1 = 4$. Using $x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=0}^3 h[k]x[n-k] \right\}$ we

arrive at $\{x[n]\} = \{1, 2, 3, 4\}, 0 \leq n \leq 3$.

(c) The length of $x[n]$ is $8 - 5 + 1 = 4$. Using $x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=0}^4 h[k]x[n-k] \right\}$ we

arrive at $\{x[n]\} = \{1, -2, 3, -1\}, 0 \leq n \leq 3$.

2.67 $y[n] = ay[n-1] + bx[n]$. Hence, $y[0] = ay[-1] + bx[0]$. Next, $y[1] = ay[0] + bx[1]$
 $= a(ay[-1] + bx[0]) + bx[1] = a^2y[-1] + abx[0] + bx[1]$. Continuing further in a similar
way we obtained $y[n] = a^{n+1}y[-1] + \sum_{k=0}^n a^{n-k}bx[k]$.

(a) Let $y_1[n]$ be the output due to an input $x_1[n]$. Then

$$y_1[n] = a^{n+1}y[-1] + \sum_{k=0}^n a^{n-k}bx_1[k]. \text{ If } x_1[n] = x[n - n_o], \text{ then}$$

$$y_1[n] = a^{n+1}y[-1] + \sum_{k=0}^n a^{n-k}bx[k - n_o] = a^{n+1}y[-1] + \sum_{r=0}^{n-n_o} a^{n-n_o-r}bx[r].$$

However,

$$y[n - n_o] = a^{n+1}y[-1] + \sum_{k=0}^n a^{n-k}bx[k - n_o] = a^{n-n_o+1}y[-1] + \sum_{r=0}^{n-n_o} a^{n-n_o-r}bx[r].$$

Hence $y_1[n] \neq y[n - n_o]$ if $y[-1] \neq 0$, i.e., the system is time-variant. The system is
time-invariant if and only if $y[-1] = 0$, as then $y_1[n] = y[n - n_o]$.

(b) Let $y_1[n]$ and $y_2[n]$ be the outputs due to inputs $x_1[n]$ and $x_2[n]$, respectively.

Let $y[n]$ be the output due to an input $\alpha x_1[n] + \beta x_2[n]$. However, $\alpha y_1[n] + \beta y_2[n] =$

$$\alpha a^{n+1}y[-1] + \beta a^{n+1}y[-1] + \alpha \sum_{k=0}^n a^{n-k}bx_1[k] + \beta \sum_{k=0}^n a^{n-k}bx_2[k], \text{ whereas,}$$

$$y[n] = a^{n+1}y[-1] + \alpha \sum_{k=0}^n a^{n-k}bx_1[k] + \beta \sum_{k=0}^n a^{n-k}bx_2[k]. \text{ Hence, the system is}$$

nonlinear if $y[-1] \neq 0$ and is linear if and only if $y[-1] = 0$.

(c) Generalizing the above result it can be shown that an N -th order causal discrete-
time system is linear and time-invariant if and only if $y[-r] = 0, 1 \leq r \leq N$.

2.68 $y[n] = p_0x[n] + p_1x[n-1] - d_1y[n-1]$ leads to $x[n] = \frac{1}{p_0}y[n] + \frac{d_1}{p_0}y[n-1] - \frac{p_1}{p_0}x[n-1]$,
which is the difference equation characterizing the inverse system. In other words, simply
solve the equation for $x[n]$ in terms of present and past values of $y[n]$ and $x[n]$.

2.69 $s[n] = \sum_{k=0}^n h[k]\mu[n-k] = \sum_{k=0}^n h[k], n \geq 0$, and $s[n] = 0, n < 0$. Since $h[k]$ is
nonnegative, $s[n]$ is a monotonically increasing function of n for $n \geq 0$, and is not
oscillatory. Hence, there is no overshoot.

2.70 (a) $f[n] = f[n-1] + f[n-2]$. Let $f[n] = \alpha r^n$, then the difference equation reduces to
 $\alpha r^n - \alpha r^{n-1} - \alpha r^{n-2} = 0$ which reduces further to $r^2 - r - 1 = 0$ resulting in

$$r = \frac{1 \pm \sqrt{5}}{2}. \text{ Thus, } f[n] = \alpha_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

As $f[0] = 0$, hence $\alpha_1 + \alpha_2 = 0$. Also $f[1] = 1$, and hence

$$\left(\frac{\alpha_1 + \alpha_2}{2} \right) + \sqrt{5} \left(\frac{\alpha_1 - \alpha_2}{2} \right) = 1. \text{ Solving for } \alpha_1 \text{ and } \alpha_2, \text{ we get } \alpha_1 = -\alpha_2 = \frac{1}{\sqrt{5}}. \text{ Hence,}$$

$$f[n] = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

(b) $y[n] = y[n-1] + y[n-2] + x[n-1]$. As the system is LTI, the initial conditions are equal to zero. Let $x[n] = \delta[n]$. Then $y[n] = y[n-1] + y[n-2] + \delta[n-1]$. Hence, $y[0] = y[-1] + y[-2] = 0$ and $y[1] = y[0] + y[-2] + \delta[0] = 1$. For $n > 1$, the corresponding difference equation is $y[n] = y[n-1] + y[n-2]$ with initial conditions $y[0] = 0$ and $y[1] = 1$, which are the same as those for the solution of the Fibonacci's

sequence. Hence $y[n] = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$. Thus denotes the impulse

response of a causal LTI system described by the difference equation $y[n] = y[n-1] + y[n-2] + x[n-1]$.

2.71 $y[n] = \alpha y[n-1] + x[n]$. Denoting $y[n] = y_{re}[n] + j y_{im}[n]$, and $\alpha = a + j b$, we get

$y_{re}[n] + j y_{im}[n] = (a + j b)(y_{re}[n-1] + j y_{im}[n-1]) + x[n]$. Equating the real and the imaginary parts, and noting that $x[n]$ is real, we get

$y_{re}[n] = a y_{re}[n-1] - b y_{im}[n-1] + x[n]$, $y_{im}[n] = b y_{re}[n-1] + a y_{im}[n-1]$. From the second equation we have $y_{im}[n-1] = \frac{1}{a} y_{im}[n] - \frac{b}{a} y_{re}[n-1]$. Substituting this

equation in the top left equation we arrive at

$y_{re}[n] = a y_{re}[n-1] - \frac{b}{a} y_{im}[n] + \frac{b^2}{a} y_{re}[n-1] + x[n]$, from which we get

$b y_{im}[n-1] = -a y_{re}[n-1] + (a^2 + b^2) y_{re}[n-2] + a x[n-1]$. Substituting this equation in the equation $y_{re}[n] = a y_{re}[n-1] - b y_{im}[n-1] + x[n]$ we arrive at

$y_{re}[n] = 2a y_{re}[n-1] - (a^2 + b^2) y_{re}[n-2] + x[n] - a x[n-1]$ which is a second-order difference equation representing $y_{re}[n]$ in terms of $x[n]$.

2.72 The first-order causal LTI system is characterized by the difference equation

$y[n] = p_0 x[n] + p_1 x[n-1] - d_1 y[n-1]$. Letting $x[n] = \delta[n]$ we obtain the difference equation representation of its impulse response $h[n] = p_0 \delta[n] + p_1 \delta[n-1] - d_1 h[n-1]$.

Solving it for $n = 0, 1, 2$, we get $h[0] = p_0$, $h[1] = p_1 - d_1 h[0] = p_1 - d_1 p_0$, and

$h[2] = -d_1 h[1] = -d_1 p_0 (p_1 - d_1 p_0)$. Solving these equations we get $p_0 = h[0]$,

$d_1 = -\frac{h[2]}{h[1]}$, and $p_1 = h[1] - \frac{h[2]h[0]}{h[1]}$.

2.73 $\sum_{k=0}^M p_k x[n-k] = \sum_{k=0}^N d_k y[n-k]$. Let $x[n] = \delta[n]$. Then $\sum_{k=0}^M p_k \delta[n-k] = \sum_{k=0}^N d_k h[n-k]$.

Thus, $p_r = \sum_{k=0}^N d_k h[r-k]$. Since the system is assumed to be causal, $h[r-k] = 0$

for all $k > r$. Hence, $p_r = \sum_{k=0}^N d_k h[r-k] = \sum_{k=0}^N h[k] d_{r-k}$.

2.74 For a filter with a complex-valued impulse response, the first part of the proof is the same as that for a filter with a real-valued impulse response. From

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \text{ we get } |y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|.$$

Since the input is bounded $|x[n]| \leq B_x$. Therefore $|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|$. So if $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$, then $|y[n]| \leq B_x S$ indicating that $y[n]$ is also bounded.

To prove the converse we need to show that if a bounded input is produced by a bounded input then $S < \infty$. Consider the following bounded input defined by

$$x[n] = \frac{h^*[-n]}{|h[-n]|}. \text{ Then } y[n] = \sum_{k=-\infty}^{\infty} \frac{h^*[-k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = S. \text{ Now since the output}$$

is bounded, $S < \infty$. Thus for a filter with a complex impulse response is BIBO stable if and only if $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$.

2.75 The impulse response of the cascade is $g[k] = \sum_{r=-\infty}^{\infty} h_1[k-r]h_2[r]$. Thus

$$\sum_{k=-\infty}^{\infty} |g[k]| = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} |h_1[k-r]| |h_2[r]| \leq \left(\sum_{k=-\infty}^{\infty} |h_1[k]| \right) \left(\sum_{r=-\infty}^{\infty} |h_2[r]| \right). \text{ Since}$$

$h_1[n]$ and $h_2[n]$ are stable, $\sum_k |h_1[k]| < \infty$ and $\sum_k |h_2[k]| < \infty$. Hence $\sum_k |g[k]| < \infty$ and as a result, a cascade of two stable LTI systems is also stable.

2.76 The impulse response of the parallel structure is $g[n] = h_1[n] + h_2[n]$. Now,

$$\sum_{k=-\infty}^{\infty} |g[k]| = \sum_{k=-\infty}^{\infty} |h_1[k] + h_2[k]| \leq \sum_{k=-\infty}^{\infty} |h_1[k]| + \sum_{k=-\infty}^{\infty} |h_2[k]|. \text{ Since } h_1[n] \text{ and}$$

$h_2[n]$ are stable, $\sum_k |h_1[k]| < \infty$ and $\sum_k |h_2[k]| < \infty$. Hence $\sum_k |g[k]| < \infty$ and as a result, a parallel connection of two stable LTI systems is also stable.

2.77 Consider a cascade connection of two passive LTI systems with an input $x[n]$ and an output $y[n]$. Let $y_1[n]$ and $y_2[n]$ be the outputs of the two systems for the input

$$x[n]. \text{ Now } \sum_{n=-\infty}^{\infty} |y_1[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 \text{ and } \sum_{n=-\infty}^{\infty} |y_2[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2. \text{ Let}$$

$$y_1[n] = y_2[n] = x[n] \text{ satisfying the above inequalities. Then } y[n] = y_1[n] + y_2[n]$$

$= 2x[n]$ and as a result, $\sum_{n=-\infty}^{\infty} |y[n]|^2 = 4 \sum_{n=-\infty}^{\infty} |x[n]|^2 > \sum_{n=-\infty}^{\infty} |x[n]|^2$. Hence, the parallel connection of two passive LTI systems may not be passive.

2.78 Consider a parallel connection of two passive LTI systems with an input $x[n]$ and an output $y[n]$. Let $y_1[n]$ and $y_2[n]$ be the outputs of the two systems for the input

$x[n]$. Now $\sum_{n=-\infty}^{\infty} |y_1[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2$ and $\sum_{n=-\infty}^{\infty} |y_2[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2$. Let $y_1[n] = y_2[n] = x[n]$ satisfying the above inequalities. Then $y[n] = y_1[n] + y_2[n] = 2x[n]$ and as a result, $\sum_{n=-\infty}^{\infty} |y[n]|^2 = 4\sum_{n=-\infty}^{\infty} |x[n]|^2 > \sum_{n=-\infty}^{\infty} |x[n]|^2$. Hence, the parallel connection of two passive LTI systems may not be passive.

2.79 Let the difference equation $\sum_{k=0}^M p_k x[n-k] = y[n] + \sum_{k=1}^N d_k y[n-k]$ represents the causal IIR digital filter. For an input $x[n] = \delta[n]$, the corresponding output is then $y[n] = h[n]$, the impulse response of the filter. As the number of coefficients $\{p_k\}$ is $M+1$ and the number of coefficients $\{d_k\}$ is N , there are a total of $N+M+1$ unknowns. To determine these coefficients from the impulse response samples, we compute only the first $N+M+1$ impulse response samples. To illustrate the method, without any loss of generality, we assume $N=M=3$. Then, from the difference equation we arrive at the following $N+M+1=7$ equations:

$$\begin{aligned} h[0] &= p_0, \\ h[1] + h[0]d_1 &= p_1, \\ h[2] + h[1]d_1 + h[0]d_2 &= p_2, \\ h[3] + h[2]d_1 + h[1]d_2 + h[0]d_3 &= p_2, \\ h[4] + h[3]d_1 + h[2]d_2 + h[1]d_3 &= 0, \\ h[5] + h[4]d_1 + h[3]d_2 + h[2]d_3 &= 0, \\ h[6] + h[5]d_1 + h[4]d_2 + h[3]d_3 &= 0. \end{aligned}$$

Writing the last three equations in matrix form we arrive at

$$\begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix} = \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and hence, } \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = - \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}^{-1} \begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix}.$$

Substituting these values in the first four equations written in matrix form we get

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

2.80 $y[n] = y[-1] + \sum_{\ell=0}^n x[\ell] = y[-1] + \sum_{\ell=0}^n \ell \mu[\ell] = y[-1] + \sum_{\ell=0}^n \ell = y[-1] + \frac{n(n+1)}{2}$.

(a) For $y[-1] = 0$, $y[n] = \frac{n(n+1)}{2}$.

(b) For $y[-1] = -2$, $y[n] = -2 + \frac{n(n+1)}{2} = \frac{n^2 + n - 4}{2}$.

2.81 $y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau) d\tau = y((n-1)T) + T \cdot x((n-1)T)$. Therefore, the difference equation representation is given by $y[n] = y[n-1] + T \cdot x[n-1]$ where $y[n] = y(nT)$ and $x[n] = x(nT)$.

2.82 $y[n] = \frac{1}{n} \sum_{\ell=1}^n x[\ell] = \frac{1}{n} \sum_{\ell=1}^{n-1} x[\ell] + \frac{1}{n} x[n]$, $n \geq 1$. Now $y[n-1] = \frac{1}{n-1} \sum_{\ell=1}^{n-1} x[\ell]$, $n \geq 1$, i.e., $\sum_{\ell=1}^{n-1} x[\ell] = (n-1)y[n-1]$. Thus, the difference equation representation is given by $y[n] = \left(\frac{n-1}{n}\right)y[n-1] + \frac{1}{n} x[n]$.

2.83 $y[n] - 0.35y[n-1] = 2.4\mu[n]$ with $y[-1] = 3$. The total solution is given by $y[n] = y_c[n] + y_p[n]$, where $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution.

$y_c[n]$ is obtained by solving $y_c[n] - 0.35y_c[n-1] = 0$. To this end we set $y_c[n] = \lambda^n$, which yields $\lambda^n - 0.35\lambda^{n-1} = 0$ resulting in the solution $\lambda = 0.35$. Hence $y_c[n] = \alpha(0.35)^n$.

For the particular solution we choose $y_p[n] = \beta$. Substituting this solution in the difference equation representing the system we get $\beta - 0.35\beta = 2.4\mu[n]$. For $n = 0$ we get $\beta - 0.35\beta = 2.4$, i.e., $(1 - 0.35)\beta = 2.4$ and hence $\beta = 2.4/0.65 = 48/13$.

Therefore $y[n] = y_c[n] + y_p[n] = \alpha(0.35)^n + \frac{48}{13}$, $n \geq 0$. For $n = -1$, we thus have

$y[-1] = 3 = \alpha(0.35)^{-1} + \frac{48}{13}$ implying $\alpha = -0.2423$. The total solution is thus given by

$y[n] = -0.2423(0.35)^n + \frac{48}{13}$, $n \geq 0$.

2.84 $y[n] - 0.3y[n-1] - 0.04y[n-2] = 3^n \mu[n]$ with $y[-1] = 2$ and $y[-2] = 1$. The total solution is given by $y[n] = y_c[n] + y_p[n]$, where $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution.

$y_c[n]$ is obtained by solving $y_c[n] - 0.3y_c[n-1] - 0.04y_c[n-2] = 0$. To this end we set $y_c[n] = \lambda^n$, which yields $\lambda^n - 0.3\lambda^{n-1} - 0.04\lambda^{n-2} = 0$ resulting in the solutions $\lambda = 0.4$ or $\lambda = -0.1$. Hence $y_c[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n$.

For the particular solution we choose $y_p[n] = \beta(3)^n$. Substituting this solution in the difference equation representing the system we get

$\beta(3)^n - 0.3\beta(3)^{n-1} - 0.04\beta(3)^{n-2} = 3^n \mu[n]$. For $n = 0$ we have

$\beta - 0.3\beta(3)^{-1} - 0.04\beta(3)^{-2} = 1$ which yields $\beta = 1.1166$. Therefore

$y[n] = y_c[n] + y_p[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n + 1.1166(3)^n, n \geq 0$. For $n = -1$ and $n = -2$ we thus have $y[-1] = \alpha_1(0.4)^{-1} + \alpha_2(-0.1)^{-1} + 1.1166(3)^{-1} = 2$ and $y[-2] = \alpha_1(0.4)^{-2} + \alpha_2(-0.1)^{-2} + 1.1166(3)^{-2} = 1$. Solving these two equations we get $\alpha_1 = 0.5489$ and $\alpha_2 = -0.0255$. Hence,

$$y[n] = 0.5489(0.4)^n - 0.0255(-0.1)^n + 1.1166(3)^n, n \geq 0.$$

2.85 $y[n] - 0.3y[n-1] - 0.04y[n-2] = x[n] + 2x[n-1]$ with $x[n] = 3^n \mu[n]$, $y[-1] = 2$ and $y[-2] = 1$. The total solution is given by $y[n] = y_c[n] + y_p[n]$, where $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution. From the solution of Problem 2.84, the complementary solution is of the form $y_c[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n$.

To determine $y_p[n]$ we observe that it is given by the sum of the particular solution $y_{p1}[n]$ of the difference equation $y_1[n] - 0.3y_1[n-1] - 0.04y_1[n-2] = x[n] = 3^n \mu[n]$ and the particular solution $y_{p2}[n]$ of the difference equation

$y_2[n] - 0.3y_2[n-1] - 0.04y_2[n-2] = 2x[n-1] = 2 \cdot 3^{n-1} \mu[n-1]$. From the solution of Problem 2.84, we have $y_{p1}[n] = \beta(3)^n$. Hence, $y_{p2}[n] = 2y_{p1}[n-1] = 2\beta(3)^{n-1}$.

Therefore, $y_p[n] = y_{p1}[n] + 2y_{p2}[n] = \beta(3)^n + 2\beta(3)^{n-1} = 3^n \mu[n] + 2 \cdot 3^{n-1} \mu[n-1]$.

For $n = 1$ the above equation reduces to $3\beta + 2\beta = 3 + 2$. Thus, $\beta = 1$. Therefore, the total solution is given by

$y[n] = y_c[n] + y_p[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n + (3)^n + 2(3)^{n-1}, n \geq 0$. For $n = -1$ and $n = -2$ we thus have $y[-1] = \alpha_1(0.4)^{-1} + \alpha_2(-0.1)^{-1} + (3)^{-1} + 2(3)^{-2} = 2$ and $y[-2] = \alpha_1(0.4)^{-2} + \alpha_2(-0.1)^{-2} + (3)^{-2} + 2(3)^{-3} = 1$. Solving these two equations we get $\alpha_1 = 0.4883$ and $\alpha_2 = -0.0224$. Hence,

$$y[n] = 0.4883(0.4)^n - 0.0224(-0.1)^n + (3)^n + 2(3)^{n-1}, n \geq 0.$$

2.86 $h[n] - 0.35h[n-1] = \delta[n]$. The solution is given by $h[n] = h_c[n] + h_p[n]$, where $h_c[n]$ is the complementary solution and $h_p[n]$ is the particular solution. If $h[n]$ is the impulse response, then $h_p[n] = 0$. From Problem 2.83 we note that $h_c[n] = \alpha(0.35)^n$. Thus, $h[0] - 0.35h[-1] = h[0] = 1$. This implies $\alpha = 1$. Hence, $h[n] = (0.35)^n, n \geq 0$.

2.87 The overall system can be regarded as the cascade of two causal LTI systems:
S1: $y[n] - 0.3y[n-1] - 0.04y[n-2] = x_1[n]$ and **S2:** $x_1[n] = x[n] + 2x[n-1]$.

The impulse response $h_1[n]$ of the system S1 can be found by solving the complementary solution of $h_1[n] - 0.3h_1[n-1] - 0.04h_1[n-2] = \delta[n]$. Let the complementary solution be $h_{1c}[n] = \lambda^n$, we have $\lambda^n - 0.3\lambda^{n-1} - 0.04\lambda^{n-2} = 0$ hence $\lambda = \{0.4, -0.1\}$. Therefore, the impulse response $h_1[n]$ is given by $h_1[n] = h_{1c}[n] = A(0.4)^n + B(-0.1)^n, n \geq 0$. Solving constants A, B , we get $A = 0.8$ and $B = 0.2$. Hence $h_1[n] = 0.8(0.4)^n + 0.2(-0.1)^n, n \geq 0$.

The impulse response $h_2[n]$ of the system S2 is given by $h_2[n] = \delta[n] + 2\delta[n-1]$.

The impulse response $h[n]$ of the overall system is $h[n] = h_1[n] * h_2[n] = (0.8(0.4)^n + 0.2(-0.1)^n)\mu[n] + 2(0.8(0.4)^{n-1} + 0.2(-0.1)^{n-1})\mu[n-1] = \delta[n] + 1.92(0.4)^{n-1}\mu[n-1] + 0.38(-0.1)^{n-1}\mu[n-1]$.

2.88 $h[n] = (-\alpha)^n \mu[n], 0 < \alpha < 1$. Step response is then given by $s[n] = h[n] \otimes \mu[n]$

$$= (-\alpha)^n \mu[n] \otimes \mu[n] = \sum_{k=-\infty}^{\infty} (-\alpha)^k \mu[k] \mu[n-k] = \begin{cases} \sum_{k=0}^n (-\alpha)^k, & n \geq 0, \\ 0, & n < 0 \end{cases}$$

$$= \begin{cases} \frac{1 - (-\alpha)^{n+1}}{1 + \alpha}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

2.89 Let $A_n = |n^K (\lambda_i)^n|$. Then $\frac{A_{n+1}}{A_n} = \left| \frac{n+1}{n} \right|^K \lambda_i$. Now $\lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right|^K = 1$. Since there exists a positive integer N_o such that for all $n > N_o, 0 < \frac{A_{n+1}}{A_n} < \frac{1 + \lambda_i}{2} < 1$. Hence $\sum_{n=0}^{\infty} A_n$ converges.

2.90 $\{x[n]\} = \{-4, 5, 1, -2, -3, 0, 2\}, -3 \leq n \leq 3$,
 $\{y[n]\} = \{6, -3, -1, 0, 8, 7, -2\}, -1 \leq n \leq 5$,
 $\{w[n]\} = \{3, 2, 2, -1, 0, -2, 5\}, 2 \leq n \leq 8$.

(a) $r_{xx}[\ell] = \sum_{n=-3}^3 x[n]x[n-\ell], -6 \leq \ell \leq 6$.
 $\{r_{xx}[\ell]\} = \{-8, 10, 14, -11, -23, -11, 59, -11, -23, -11, 14, 10, -8\}, -6 \leq \ell \leq 6$,
 $r_{yy}[\ell] = \sum_{n=-5}^5 y[n]y[n-\ell], -6 \leq \ell \leq 6$.
 $\{r_{yy}[\ell]\} = \{-12, 48, 29, -31, -30, 27, 163, 27, -30, -31, 29, 48, -12\}, -6 \leq \ell \leq 6$,
 $r_{ww}[\ell] = \sum_{n=-6}^6 w[n]w[n-\ell], -6 \leq \ell \leq 6$.
 $\{r_{ww}[\ell]\} = \{15, 4, 6, -12, 6, -2, 47, -2, 6, -12, 6, 4, 15\}, -6 \leq \ell \leq 6$,

$$(b) \quad r_{xy}[\ell] = \sum_{n=-5}^3 x[n]y[n-\ell], -8 \leq \ell \leq 4.$$

$$\{r_{xy}[\ell]\} = \{8, -38, 1, 51, 4, -30, -68, 43, 31, -3, -20, -6, 12\}, -8 \leq \ell \leq 4,$$

$$r_{xw}[\ell] = \sum_{n=-8}^3 x[n]y[n-\ell], -11 \leq \ell \leq 1.$$

$$\{r_{xw}[\ell]\} = \{-20, 33, -5, -8, -24, 7, 12, 12, -7, -14, -5, 4, 6\}, -11 \leq \ell \leq 1,$$

$$2.91 \quad (a) \quad x_1[n] = \alpha^n \mu[n]. \quad r_{x_1 x_1}[\ell] = \sum_{n=-\infty}^{\infty} x_1[n]x_1[n-\ell] = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] \alpha^{n-\ell} \mu[n-\ell]$$

$$= \sum_{n=0}^{\infty} \alpha^{2n-\ell} \mu[n-\ell] = \begin{cases} \sum_{n=0}^{\infty} \alpha^{2n-\ell}, & \ell < 0, \\ \sum_{n=\ell}^{\infty} \alpha^{2n-\ell}, & \ell \geq 0, \end{cases} = \begin{cases} \frac{\alpha^{-\ell}}{1-\alpha^2}, & \ell < 0, \\ \frac{\alpha^{\ell}}{1-\alpha^2}, & \ell \geq 0. \end{cases}$$

Note for $\ell \geq 0, r_{x_1 x_1}[\ell] = \frac{\alpha^{\ell}}{1-\alpha^2}$, and for $\ell < 0, r_{x_1 x_1}[\ell] = \frac{\alpha^{-\ell}}{1-\alpha^2}$. Replacing ℓ with $-\ell$

in the second expression we get $r_{x_1 x_1}[-\ell] = \frac{\alpha^{-(-\ell)}}{1-\alpha^2} = \frac{\alpha^{\ell}}{1-\alpha^2} = r_{x_1 x_1}[\ell]$. Hence, $r_{x_1 x_1}[\ell]$

is an even function of ℓ . Maximum value of $r_{x_1 x_1}[\ell]$ occurs at $\ell = 0$ since α^{ℓ} is a decaying function for increasing when $|\alpha| < 1$.

$$(b) \quad x_2[n] = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases} \quad \text{Now } r_{x_2 x_2}[\ell] = \sum_{n=0}^{N-1} x_2[n-\ell], \text{ where}$$

$$x_2[n-\ell] = \begin{cases} 1, & \ell \leq n \leq N-1+\ell, \\ 0, & \text{otherwise.} \end{cases} \quad \text{Therefore, } r_{x_2 x_2}[\ell] = \begin{cases} 0, & \text{for } \ell < -(N-1), \\ N+\ell, & \text{for } -(N-1) \leq \ell \leq 0, \\ N, & \text{for } \ell = 0, \\ N-\ell, & \text{for } 0 < N-\ell \leq N-1, \\ 0, & \text{for } \ell > N-1. \end{cases}$$

It follows from the above that $r_{x_2 x_2}[\ell]$ is a triangular function of ℓ , and hence is an even function with a maximum value of N at $\ell = 0$.

$$2.92 \quad (a) \quad x_1[n] = \cos\left(\frac{\pi n}{M}\right) \text{ where } M \text{ is a positive integer. Period of } x_1[n] \text{ is } 2M, \text{ and}$$

$$\begin{aligned} \text{hence } r_{x_1 x_1}[\ell] &= \frac{1}{2M} \sum_{n=0}^{2M-1} x_2[n]x_2[n+\ell] = \frac{1}{2M} \sum_{n=0}^{2M-1} \cos\left(\frac{\pi n}{M}\right) \cos\left(\frac{\pi(n+\ell)}{M}\right) \\ &= \frac{1}{2M} \sum_{n=0}^{2M-1} \cos\left(\frac{\pi n}{M}\right) \left\{ \cos\left(\frac{\pi n}{M}\right) \cos\left(\frac{\pi \ell}{M}\right) - \sin\left(\frac{\pi n}{M}\right) \sin\left(\frac{\pi \ell}{M}\right) \right\} \\ &= \frac{1}{2M} \cos\left(\frac{\pi \ell}{M}\right) \sum_{n=0}^{2M-1} \cos^2\left(\frac{\pi n}{M}\right). \end{aligned}$$

Now

$$\sum_{n=0}^{2M-1} \cos^2\left(\frac{\pi n}{M}\right) = \sum_{n=0}^{N-1} \cos^2\left(\frac{4\pi n}{N}\right) = \frac{1}{2} \sum_{n=0}^{N-1} \left\{1 + \cos\left(\frac{4\pi n}{N}\right)\right\} = \frac{N}{2} + \frac{1}{2} \sum_{n=0}^{N-1} \cos\left(\frac{4\pi n}{N}\right).$$

Let $C = \sum_{n=0}^{N-1} \cos\left(\frac{4\pi n}{N}\right)$ and $S = \sum_{n=0}^{N-1} \sin\left(\frac{4\pi n}{N}\right)$. Then $C + jS = \sum_{n=0}^{N-1} e^{j(4\pi n/N)}$

$$= \frac{e^{j4\pi} - 1}{e^{j4\pi/N} - 1} = 0. \text{ This implies } C = 0. \text{ Thus } \sum_{n=0}^{2M-1} \cos^2\left(\frac{\pi n}{M}\right) = \frac{N}{2} = M.$$

Hence, $r_{x_1 x_1}[\ell] = \frac{M}{2M} \cos\left(\frac{\pi \ell}{M}\right) = \frac{1}{2} \cos\left(\frac{\pi \ell}{M}\right)$.

(b) $\{x_2[n]\} = \langle n \rangle_6 = \{0, 1, 2, 3, 4, 5\}$, $0 \leq n \leq 5$. It is a periodic sequence with a period 6. Thus, $r_{x_2 x_2}[\ell] = \frac{1}{6} \sum_{n=0}^5 x_2[n] \langle x_2[n+\ell] \rangle_6$, $0 \leq \ell \leq 5$. $r_{x_2 x_2}[\ell]$ is also a periodic sequence with a period 6.

$$r_{x_2 x_2}[0] = \frac{1}{6} (x_2[0]x_2[0] + x_2[1]x_2[1] + x_2[2]x_2[2] + x_2[3]x_2[3] + x_2[4]x_2[4] + x_2[5]x_2[5]) = \frac{55}{6},$$

$$r_{x_2 x_2}[1] = \frac{1}{6} (x_2[0]x_2[1] + x_2[1]x_2[2] + x_2[2]x_2[3] + x_2[3]x_2[4] + x_2[4]x_2[5] + x_2[5]x_2[0]) = \frac{40}{6},$$

$$r_{x_2 x_2}[2] = \frac{1}{6} (x_2[0]x_2[2] + x_2[1]x_2[3] + x_2[2]x_2[4] + x_2[3]x_2[5] + x_2[4]x_2[0] + x_2[5]x_2[1]) = \frac{32}{6},$$

$$r_{x_2 x_2}[3] = \frac{1}{6} (x_2[0]x_2[3] + x_2[1]x_2[4] + x_2[2]x_2[5] + x_2[3]x_2[0] + x_2[4]x_2[1] + x_2[5]x_2[2]) = \frac{28}{6},$$

$$r_{x_2 x_2}[4] = \frac{1}{6} (x_2[0]x_2[4] + x_2[1]x_2[5] + x_2[2]x_2[0] + x_2[3]x_2[1] + x_2[4]x_2[2] + x_2[5]x_2[3]) = \frac{31}{6},$$

$$r_{x_2 x_2}[5] = \frac{1}{6} (x_2[0]x_2[5] + x_2[1]x_2[0] + x_2[2]x_2[1] + x_2[3]x_2[2] + x_2[4]x_2[3] + x_2[5]x_2[4]) = \frac{40}{6}.$$

(c) $x_3[n] = (-1)^n$ is a periodic sequence with a period 2. Thus,

$$r_{x_3 x_3}[\ell] = \frac{1}{2} \sum_{n=0}^1 x_3[n] x_3[n+\ell], \quad 0 \leq \ell \leq 1. \text{ Hence,}$$

$$r_{x_3 x_3}[0] = \frac{1}{2} (x_3[0]x_3[0] + x_3[1]x_3[1]) = 1, \quad r_{x_3 x_3}[1] = \frac{1}{2} (x_3[0]x_3[1] + x_3[1]x_3[0]) = -1.$$

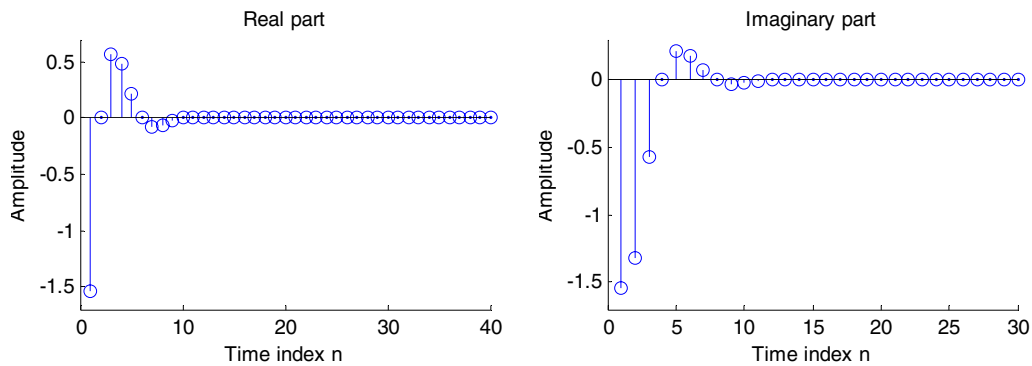
$r_{x_3 x_3}[\ell]$ is also a periodic sequence with a period 2.

M2.1 (a) The input data entered during the execution of Program 2_2.m are:

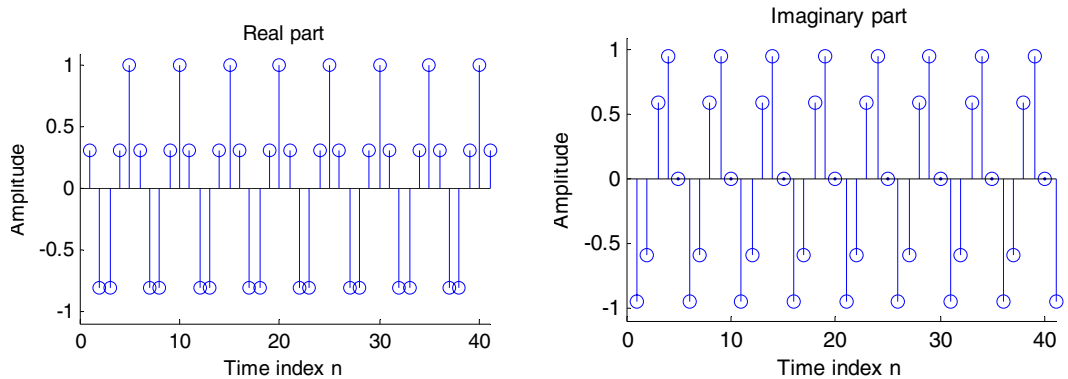
```
Type in real exponent = -1/12
Type in imaginary exponent = pi/6
Type in gain constant = 1
Type in length of sequence = 41
```

(b) The input data entered during the execution of Program 2_2.m are:

```
Type in real exponent = -1/12
Type in imaginary exponent = pi/6
Type in gain constant = 1
Type in length of sequence = 41
```



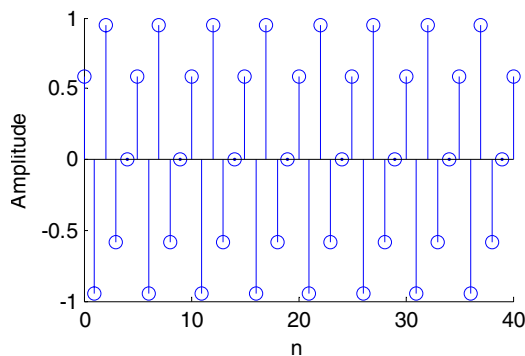
M2.2 (a) $\tilde{x}_a[n] = e^{-j0.4\pi n}$. The plots generated using Program 2_2.m are shown below:



(b) The code fragment used to generate $\tilde{x}_b[n] = \sin(0.8\pi n + 0.8\pi)$ is as follows:

```
x = sin(0.8*pi*n + 0.8*pi);
```

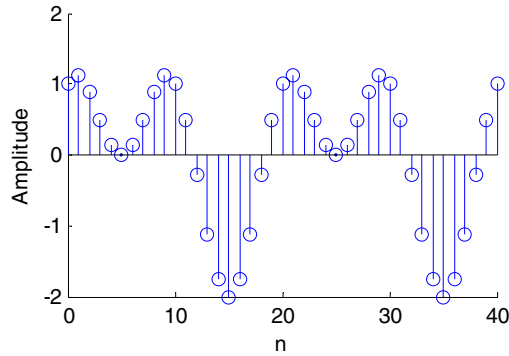
The plot of the periodic sequence is given below:



(c) The code fragment used to generate $\tilde{x}_c[n] = \text{Re}(e^{j\pi n/5}) + \text{Im}(e^{j\pi n/10})$ is as follows:

```
x = real(exp(i*pi*n/5) + imag(exp(i*pi*n/10)));
```

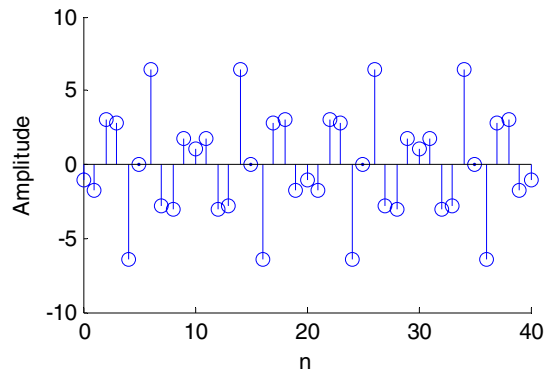
The plot of the periodic sequence is given below:



(d) The code fragment used to generate $\tilde{x}_d[n] = 3 \cos(1.3\pi n) - 4 \sin(0.5\pi n + 0.5\pi)$ is as follows:

```
x = 3*cos(1.3*pi*n) - 4*sin(0.5*pi*n + 0.5*pi);
```

The plot of the periodic sequence is given below:

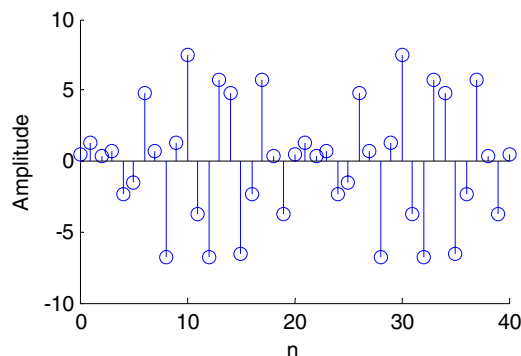


(e) The code fragment used to generate

$\tilde{x}_e[n] = 5 \cos(1.5\pi n + 0.75\pi) + 4 \cos(0.6\pi n) - \sin(0.5\pi n)$ is as follows:

```
x = 5*cos(1.5*pi*n + 0.75*pi) + 4*cos(0.6*pi*n) - sin(0.5*pi*n);
```

The plot of the periodic sequence is given below:



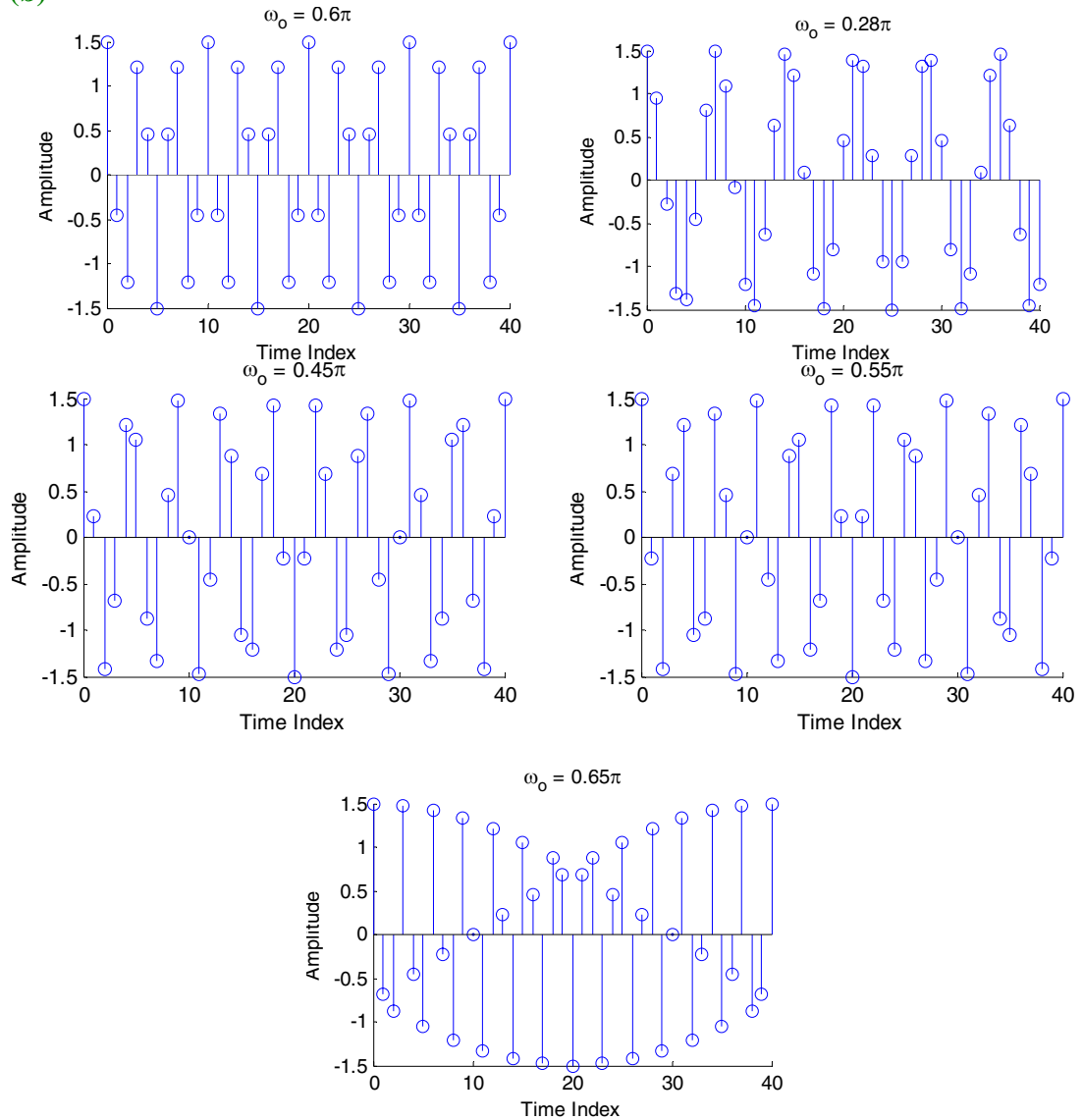
M2.3 (a) `L = input('Desired length = ');`
`A = input('Amplitude = ');`
`omega = input('Angular frequency = ');`
`phi = input('Phase = ');`
`n = 0:L-1;`


```

x = A*cos(omega*n + phi);
stem(n,x);
xlabel('Time Index'); ylabel('Amplitude');
title(['\omega_{o} = ',num2str(omega/pi),'\pi']);

```

(b)



```

M2.4 t = 0:0.001:1;
fo = input('Frequency of sinusoid in Hz = ');
FT = input('Sampling frequency in Hz = ');
g1 = cos(2*pi*fo*t);
plot(t,g1,'-');
xlabel('time'); ylabel('Amplitude'); hold
n = 0:1:FT;
gs = cos(2*pi*fo*n/FT);
plot(n/FT,gs,'o'); hold off

```

```

M2.5 t = 0:0.001:0.85;

```

```

g1 = cos(6*pi*t); g2 = cos(14*pi*t); g3 =
cos(26*pi*t);
plot(t/0.85,g1,'-', t/0.85, g2, '--', t/0.85, g3,':');
xlabel('time'); ylabel('Amplitude'); hold
n = 0:1:8; gs = cos(0.6*pi*n); plot(n/8.5,gs,'o');
hold off

```

M2.6 As the length of the moving average filter is increased, the output of the filter gets more smoother. However, the delay between the input and the output sequences also increases (This can be seen from the plots generated by Program 2_4.m for various values of the filter length.)

```

M2.7 alpha = input('Alpha = ');
y0 = 1; y1 = 0.5*(y0 + (alpha/y0));
while abs(y1-y0)>0.00001
y2 = 0.5*(y1+(alpha/y1));
y0 = y1; y1 = y2;
end
disp('Square root of alpha is'); disp(y1);

```

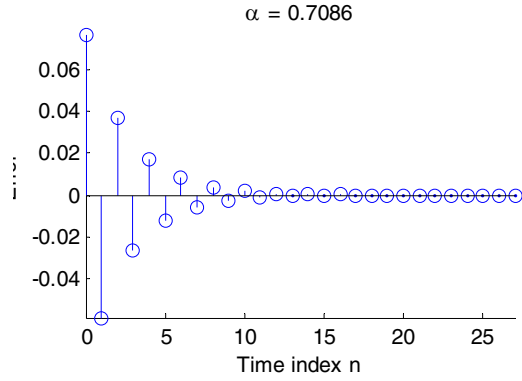
```

M2.8 format long
alpha = input('Alpha = ');
y0 = 0.3; y = zeros(1,61);
L = length(y) - 1;
y(1) = alpha - y0*y0 + y0; n = 2;

while abs(y(n-1) - y0) > 0.00001
    y2 = alpha - y(n-1)*y(n-1) + y(n-1);
    y0 = y(n-1); y(n) = y2;
    n = n+1;
end
disp('Square root of alpha is');disp(y(n-1));
m = 0:n-2;
err = y(1:n-1) - sqrt(alpha);
stem(m,err);
axis([0 n-2 min(err) max(err)]);
xlabel('Time index n'); ylabel('Error');
title(['\alpha = ',num2str(alpha)]);

```

The displayed out is
Square root of alpha is
0.84178104293115

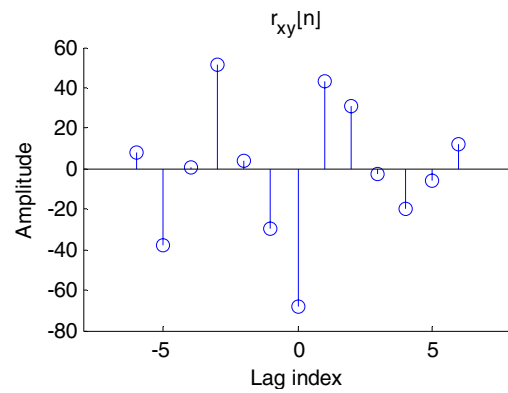
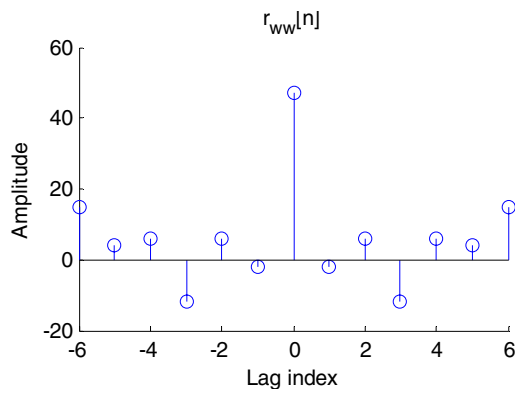
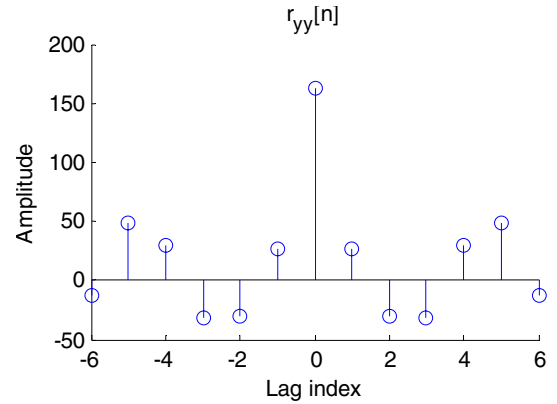
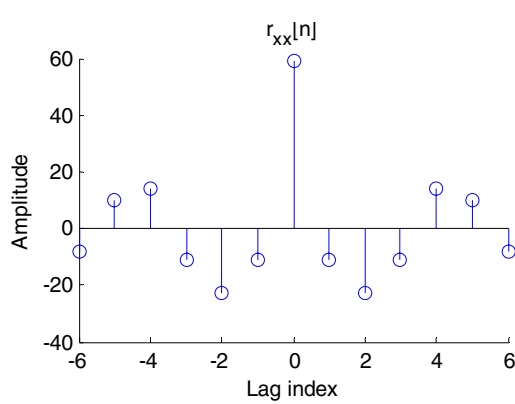


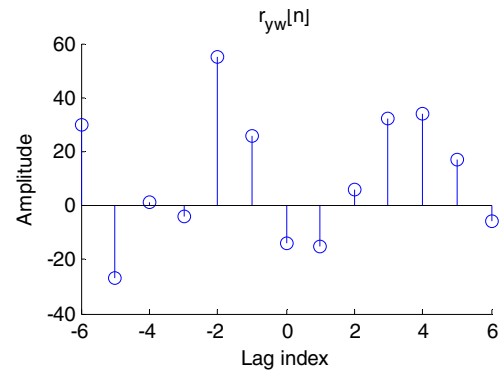
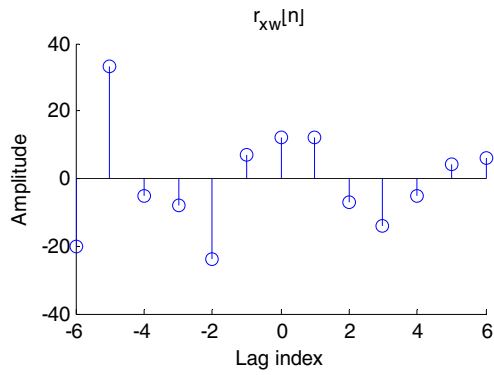
M2.9 $\{x[n]\} = \{-4, 5, 1, -2, -3, 0, 2\}, -3 \leq n \leq 3,$
 $\{y[n]\} = \{6, -3, -1, 0, 8, 7, -2\}, -1 \leq n \leq 5,$
 $\{w[n]\} = \{3, 2, 2, -1, 0, -2, 5\}, 2 \leq n \leq 8.$

$\{r_{xx}[n]\} = \{-8, 10, 14, -11, -23, -11, 59, -11, -23, -11, 14, 10, -8\}, -6 \leq n \leq 6.$

$\{r_{yy}[n]\} = \{-12, 48, 29, -31, -30, -27, 163, -27, -30, -31, 29, 48, -12\}, -6 \leq n \leq 6.$

$\{r_{ww}[n]\} = \{15, 4, 6, -12, 6, -2, 47, -2, 6, -12, 6, 10, 15\}, -6 \leq n \leq 6.$

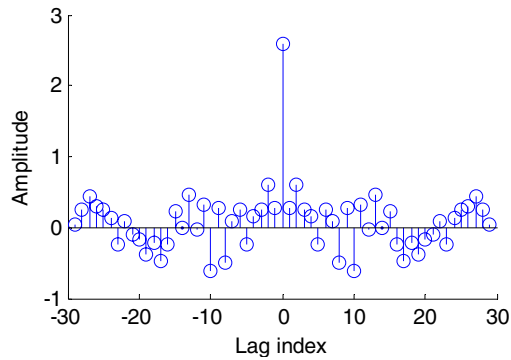




```

M2.10 N = input('Length of sequence = ');
n = 0:N-1;
x = exp(-0.8*n);
y = rand(1,N)-0.5+x;
n1 = length(x)-1;
r = conv(y,fliplr(y));
k = (-n1):n1;
stem(k,r);
xlabel('Lag_index'); ylabel('Amplitude');

```



Chapter 3

3.1 $X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$. Now, $\int_{-\infty}^{\infty} |x_a(t)| dt < \infty$. Hence,

$$|X_a(j\Omega)| = \left| \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt \right| \leq \int_{-\infty}^{\infty} |x_a(t) e^{-j\Omega t}| dt \leq \int_{-\infty}^{\infty} |x_a(t)| dt < \infty.$$

3.2 (a) $Y_a(j\Omega) = \int_{-\infty}^{\infty} \cos(\Omega_o t) e^{-j\Omega t} dt = \frac{1}{2} \int_{-\infty}^{\infty} (e^{j\Omega_o t} + e^{-j\Omega_o t}) e^{-j\Omega t} dt$
 $= \frac{1}{2} (\delta(\Omega - \Omega_o) + \delta(\Omega + \Omega_o)).$

(b) $U_a(j\Omega) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-j\Omega t} dt = \int_{-\infty}^0 e^{\alpha t} e^{-j\Omega t} dt + \int_0^{\infty} e^{-\alpha t} e^{-j\Omega t} dt$
 $= \frac{1}{\alpha - j\Omega} \cdot \left[e^{(\alpha - j\Omega)t} \right]_{-\infty}^0 - \frac{1}{\alpha + j\Omega} \cdot \left[e^{-(\alpha + j\Omega)t} \right]_{-\infty}^0 = \frac{1}{\alpha - j\Omega} + \frac{1}{\alpha + j\Omega} = \frac{2\alpha}{\alpha^2 + \Omega^2}.$

(c) $V_a(j\Omega) = \int_{-\infty}^{\infty} e^{j\Omega_o t} e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{-j(\Omega - \Omega_o)t} dt = \delta(\Omega - \Omega_o).$

(d) $P_a(j\Omega) = \int_{-\infty}^{\infty} \left(\sum_{\ell=-\infty}^{\infty} \delta(t - \ell T) \right) e^{-j\Omega t} dt = \sum_{\ell=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \delta(t - \ell T) e^{-j\Omega t} dt \right)$ using the

linearity property of the CTFT. Next, using the shifting property of the CTFT we get

$P_a(j\Omega) = \sum_{\ell=-\infty}^{\infty} e^{-j\Omega \ell T}$ which can be alternately expressed in the form

$P_a(j\Omega) = \frac{2\pi}{T} \sum_{\ell=-\infty}^{\infty} \delta(\Omega - \frac{2\pi\ell}{T})$ making use of the results of Problem 3.2(c).

3.3 (a) $V_a(j\Omega) = \int_{-\infty}^{\infty} e^{-j\Omega t} dt = \delta(\Omega).$

(b) $\mu(j\Omega) = \int_{-\infty}^{\infty} \mu(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^t \delta(\tau) d\tau \right) e^{-j\Omega t} dt = \frac{1}{j\Omega} + \pi\delta(\Omega).$

(c) The function $x_a(t)$ is also denoted by $\text{rect}(t)$. Thus, $X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$
 $= \int_{-1/2}^{1/2} e^{-j\Omega t} dt = -\frac{1}{j\Omega} (e^{-j\Omega/2} - e^{j\Omega/2}) = \frac{2 \sin(\Omega/2)}{\Omega} = \frac{\sin(\Omega/2)}{\Omega/2} = \text{sinc}(\Omega/2).$

(d) $y_a(t) = \begin{cases} 1 - 2|t|, & |t| < 0.5, \\ 0, & |t| \geq 0.5. \end{cases}$ A more convenient way to determine the CTFT of $y_a(t)$

is to differentiate it twice with respect to t , determine the CTFT of $\frac{d^2 y_a(t)}{dt^2}$ and then

make use of the time-differentiation property given in Problem 3.6(e) and time-shifting

property given in Problem 3.6(a). Now, $\frac{dy_a(t)}{dt} = \begin{cases} 2, & |t| < 0.5, \\ -2, & |t| \geq 0.5. \end{cases}$ As $\frac{dy_a(t)}{dt}$ has jump

discontinuities with a positive jump of value 2 at $t = \pm 0.5$, a negative jump of value -4

at $t = 0$, and zero everywhere else, $\frac{d^2 y_a(t)}{dt^2}$ has only impulses of strength 2 at $t = \pm 0.5$,

and an impulse of strength -4 at $t = 0$, i.e., $\frac{d^2 y_a(t)}{dt^2} = 2\delta(t + 0.5) - 4\delta(t) + 2\delta(t - 0.5)$. If

$Y_a(j\Omega)$ denotes the CTFT of $y_a(t)$, then using of the time-differentiation property we

have $\frac{d^2 y_a(t)}{dt^2} \xleftrightarrow{\text{CTFT}} (j\Omega)^2 Y_a(j\Omega)$. Using the time-shifting property, we arrive at the

CTFT of $\frac{d^2 y_a(t)}{dt^2}$ given by $2e^{j\Omega/2} - 4 + 2e^{-j\Omega/2}$. Therefore $(j\Omega)^2 Y_a(j\Omega) =$

$-\Omega^2 Y_a(j\Omega) = 2e^{j\Omega/2} - 4 + 2e^{-j\Omega/2} = 4(\cos(\Omega/2) - 1)$, i.e.,

$Y_a(j\Omega) = -\frac{4}{\Omega^2}(\cos(\Omega/2) - 1) = \frac{8}{\Omega^2} \sin^2\left(\frac{\Omega}{4}\right) = \frac{1}{2} \text{sinc}^2\left(\frac{\Omega}{4}\right)$.

3.4 $h(t) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(t-\mu)^2/2\sigma^2}$. Thus, $H(j\Omega) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(t-\mu)^2/2\sigma^2} e^{-j\Omega t} dt$.

Making a change of variable $t - \mu = \tau$ we get $H(j\Omega) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\tau^2/2\sigma^2} e^{-j\Omega(\tau+\mu)} d\tau$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-j\Omega\mu} \int_{-\infty}^{\infty} e^{-\tau^2/2\sigma^2} e^{-j\Omega\tau} d\tau = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-j\Omega\mu} \cdot \sigma\sqrt{2\pi} \cdot e^{-\sigma^2\Omega^2/2}$

$= e^{-\left(\frac{\sigma^2\Omega^2}{2} + j\Omega\mu\right)}$. For a zero mean impulse response, we then have the CTFT pair

$e^{-t^2/2\sigma^2} \xleftrightarrow{\text{CTFT}} \sigma\sqrt{2\pi} e^{-\sigma^2\Omega^2/2}$.

3.5 $x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-1}^1 e^{j\Omega t} d\Omega = \frac{1}{j2\pi t} (e^{j\Omega} - e^{-j\Omega}) = \frac{\sin(t)}{\pi t}$.

3.6 (a) $\int_{-\infty}^{\infty} x_a(t-t_o)e^{-j\Omega t} dt = \int_{-\infty}^{\infty} x_a(\tau)e^{-j\Omega(\tau+t_o)} d\tau$ obtained using a change of variable

$t-t_o = \tau$. Therefore $\int_{-\infty}^{\infty} x_a(t-t_o)e^{-j\Omega t} dt = e^{-j\Omega t_o} \int_{-\infty}^{\infty} x_a(\tau)e^{-j\Omega\tau} d\tau = e^{-j\Omega t_o} X_a(j\Omega)$.

(b) $\int_{-\infty}^{\infty} x_a(t)e^{j\Omega_o t} e^{-j\Omega t} dt = \int_{-\infty}^{\infty} x_a(t)e^{-j(\Omega-\Omega_o)t} dt = X(j(\Omega-\Omega_o))$.

(c) $x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t} d\Omega$. Therefore $2\pi x_a(-t) = \int_{-\infty}^{\infty} X_a(j\Omega)e^{-j\Omega t} d\Omega$.

Interchanging t and Ω we get $2\pi x_a(-\Omega) = \int_{-\infty}^{\infty} X_a(jt)e^{-j\Omega t} dt$.

(d) For a positive real constant a the CTFT of $x_a(at)$ is given by

$\int_{-\infty}^{\infty} x_a(at)e^{-j\Omega t} dt = \frac{1}{a} \int_{-\infty}^{\infty} x_a(\tau)e^{-j(\Omega/a)\tau} d\tau = \frac{1}{a} X_a(j\frac{\Omega}{a})$. In a similar manner we

can show that for a negative constant a the CTFT of $x_a(at)$ is given by $-\frac{1}{a} X_a(j\frac{\Omega}{a})$.

Therefore $x_a(at) \stackrel{CTFT}{\leftrightarrow} \frac{1}{|a|} X_a\left(j\frac{\Omega}{a}\right)$.

(e) Differentiating both sides of $x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t} d\Omega$ get

$\frac{dx_a(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\Omega X_a(j\Omega)e^{j\Omega t} d\Omega$. Therefore $\frac{dx_a(t)}{dt} \stackrel{CTFT}{\leftrightarrow} j\Omega X_a(j\Omega)$.

3.7 $X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t} dt = |X_a(j\Omega)|e^{j\theta_a(\Omega)}$, where $\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$. Thus,

$X_a(-j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{j\Omega t} dt$. If $x_a(t)$ is a real function of then it follows from the

definition of $X_a(j\Omega)$ and the expression for $X_a(-j\Omega)$ that $X_a(j\Omega)$ and $X_a(-j\Omega)$ are complex conjugates. Therefore $|X_a(-j\Omega)| = |X_a(j\Omega)|$ and $\theta_a(-\Omega) = -\theta_a(\Omega)$. Or in other words, for a real , the magnitude spectrum $|X_a(j\Omega)|$ is an even function of Ω and the phase spectrum $\theta_a(\Omega)$ is an odd function of Ω .

3.8

3.9 $\hat{x}(t) = \int_{-\infty}^{\infty} h_{HT}(t-\tau)x(\tau)d\tau$. where $h_{HT}(t)$ is the impulse response of the Hilbert

transformer. Taking the CTFT of both sides we get $\hat{X}(j\Omega) = H_{HT}(j\Omega)X(j\Omega)$ where $\hat{X}(j\Omega)$ and $H_{HT}(j\Omega)$ denote the CTFTs of $\hat{x}(t)$ and $h_{HT}(t)$, respectively. Rewriting $\hat{X}(j\Omega) = H_{HT}(j\Omega)(X_p(j\Omega) + X_n(j\Omega)) = -jX_p(j\Omega) + jX_n(j\Omega)$. As the magnitude and phase of $\hat{X}(j\Omega)$ are an even and odd function, $\hat{x}(t)$ is seen to be real signal. Consider the complex signal $y(t) = x(t) + j\hat{x}(t)$. Its CTFT is then given by $Y(j\Omega) = X(j\Omega) + j\hat{X}(j\Omega) = 2X_p(j\Omega)$.

3.10 The total energy $\epsilon_x = \int_{-\infty}^{\infty} |e^{-\alpha t}|^2 dt = \int_{-\infty}^{\infty} e^{-2\alpha t} dt = \frac{1}{-2\alpha} [e^{-2\alpha t}]_0^{\infty} = \frac{1}{2\alpha} \Big|_{\alpha=1/2} = 1$.

The total energy can also be computed using using the Parsevals' theorem

$$\epsilon_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \Omega^2} d\Omega.$$

Therefore, the 80% bandwidth Ω_c can be found by evaluating $\frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} \frac{1}{\alpha^2 + \Omega^2} d\Omega$

$$= \frac{1}{2\pi} \left[\frac{1}{\alpha} \cdot \tan^{-1}\left(\frac{\Omega}{\alpha}\right) \right]_{-\Omega_c}^{\Omega_c} = \frac{1}{2\pi\alpha} \left[\tan^{-1}\left(\frac{\Omega_c}{\alpha}\right) - \tan^{-1}\left(-\frac{\Omega_c}{\alpha}\right) \right] = \frac{1}{\pi\alpha} \tan^{-1}\left(\frac{\Omega_c}{\alpha}\right) \Big|_{\alpha=1/2}$$

$$= \frac{2}{\pi} \tan^{-1}(2\Omega_c) = 0.8. \text{ Therefore, } \Omega_c = \frac{1}{2} \cdot \tan\left(\frac{0.8\pi}{2}\right) = 1.5388.$$

3.11 $y[n] = \mu[n] = y_{ev}[n] + y_{od}[n]$, where $y_{ev}[n] = \frac{1}{2}(y[n] + y[-n]) = \frac{1}{2} + \frac{1}{2}\delta[n]$ and

$$y_{od}[n] = \frac{1}{2}(y[n] - y[-n]) = \frac{1}{2}(\mu[n] - \mu[-n]) = \mu[n] - \frac{1}{2} - \frac{1}{2}\delta[n]. \text{ Now,}$$

$$Y_{ev}(e^{j\omega}) = \frac{1}{2} \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k) \right] + \frac{1}{2} = \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k) + \frac{1}{2}. \text{ Since}$$

$$y_{od}[n] = \mu[n] - \frac{1}{2} - \frac{1}{2}\delta[n], \quad y_{od}[n-1] = \mu[n-1] - \frac{1}{2} - \frac{1}{2}\delta[n-1]. \text{ As a result,}$$

$$y_{od}[n] - y_{od}[n-1] = \mu[n] - \mu[n-1] - \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n] = \frac{1}{2}(\delta[n] + \delta[n-1]).$$

Taking the DTFT of both sides of the above equation, we get

$$Y_{od}(e^{j\omega}) - e^{-j\omega}Y_{od}(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega}) \text{ or } Y_{od}(e^{j\omega}) = \frac{1}{2} \left(\frac{1+e^{-j\omega}}{1-e^{-j\omega}} \right) = \frac{1}{1-e^{-j\omega}} - \frac{1}{2}.$$

$$\text{Hence, } Y(e^{j\omega}) = Y_{ev}(e^{j\omega}) + Y_{od}(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k).$$

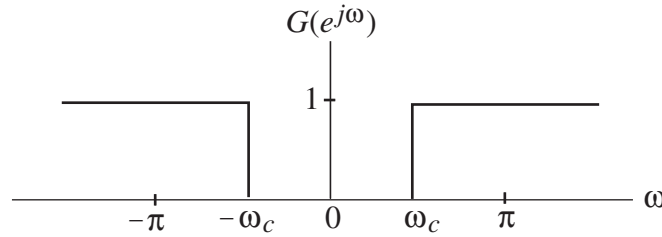
3.12 The inverse DTFT of $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$ is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\omega) e^{j\omega n} d\omega = \frac{2\pi}{2\pi} = 1.$$

3.13 $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} e^{-j\omega n}$ with $|\alpha| < 1$. Rewriting we get

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{-1} \alpha^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=1}^{\infty} (\alpha e^{j\omega})^n + \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n \\ &= \frac{\alpha e^{j\omega}}{1 - \alpha e^{j\omega}} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1 - \alpha^2}{1 - 2\alpha \cos \omega + \alpha^2}. \end{aligned}$$

3.14 $G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\delta[n] - \frac{\sin(\omega_c n)}{\pi n} \right) e^{-j\omega n} = 1 - H_{LP}(e^{j\omega})$.



3.15 $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$. Hence, $x^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) e^{-j\omega n} d\omega$.

(a) Since $x[n]$ is real and even, we have $X(e^{j\omega}) = X^*(e^{j\omega})$. Thus

$$x[-n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n} d\omega. \text{ Therefore,}$$

$$x[n] = \frac{1}{2} (x[n] + x[-n]) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cos(\omega n) d\omega. \text{ As } x[n] \text{ is even, } X(e^{j\omega}) = X(e^{-j\omega}).$$

As a result, the term $X(e^{j\omega}) \cos n(\omega n)$ inside the above integral is even, and hence

$$x[n] = \frac{1}{\pi} \int_0^{\pi} X(e^{j\omega}) \cos(\omega n) d\omega.$$

(b) Since $x[n]$ is real and odd, we have $x[n] = -x[-n]$ and $X(e^{j\omega}) = -X(e^{-j\omega})$. Thus,

$$x[n] = \frac{1}{2} (x[n] - x[-n]) = \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin(\omega n) d\omega. \text{ As a result, the term } X(e^{j\omega}) \sin(\omega n)$$

inside the above integral is even, and hence $x[n] = \frac{j}{2\pi} \int_0^{\pi} X(e^{j\omega}) \sin(\omega n) d\omega$.

3.16 $x[n] = A\alpha^n \sin(\omega_0 n + \phi)\mu[n] = A\alpha^n \left(\frac{e^{j\omega_0 n} e^{j\phi} - e^{-j\omega_0 n} e^{-j\phi}}{2j} \right) \mu[n]$

$= \frac{A}{2j} e^{j\phi} \left(\alpha e^{j\omega_0} \right)^n \mu[n] - \frac{A}{2j} e^{-j\phi} \left(\alpha e^{-j\omega_0} \right)^n \mu[n]$. Therefore, the DTFT of $x[n]$ is given by $X(e^{j\omega}) = \frac{A}{j2} e^{j\phi} \frac{1}{1 - \alpha e^{j\omega} e^{j\omega_0}} - \frac{A}{j2} e^{-j\phi} \frac{1}{1 - \alpha e^{j\omega} e^{-j\omega_0}}$.

3.17 Let $x[n] = \alpha^n \mu[n]$ with $|\alpha| < 1$. Its DTFT was computed in Example 3.6 and is given by $X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$.

(a) $x_1[n] = \alpha^n \mu[n-1]$ with $|\alpha| < 1$. Its DTFT is given by $X_1(e^{j\omega}) = \sum_{n=1}^{\infty} \alpha^n e^{-j\omega n}$

$= \sum_{n=1}^{\infty} (\alpha e^{-j\omega})^n = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n - 1 = \frac{1}{1 - \alpha e^{-j\omega}} - 1 = \frac{\alpha e^{-j\omega}}{1 - \alpha e^{-j\omega}}$.

(b) $x_2[n] = n\alpha^n \mu[n]$ with $|\alpha| < 1$. Note $x_2[n] = n x[n]$. Therefore, using the differentiation-in-frequency property in Table 3.4 we get $X_2(e^{j\omega}) = j \frac{dX(e^{j\omega})}{d\omega} = j \frac{d}{d\omega} \left(\frac{1}{1 - \alpha e^{-j\omega}} \right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$.

(c) $x_3[n] = \alpha^n \mu[n+1]$ with $|\alpha| < 1$. Its DTFT is given by $X_3(e^{j\omega}) = \sum_{n=-1}^{\infty} \alpha^n e^{-j\omega n} = \alpha^{-1} e^{j\omega} + \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \alpha^{-1} e^{j\omega} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{\alpha} \left(\frac{e^{j\omega} - \alpha}{1 - \alpha e^{-j\omega}} \right)$.

(d) $x_4[n] = n\alpha^n \mu[n+2]$ with $|\alpha| < 1$. Its DTFT is given by $X_4(e^{j\omega}) = \sum_{n=-2}^{\infty} n\alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} n\alpha^n e^{-j\omega n} - 2\alpha^{-2} e^{j2\omega} - \alpha^{-1} e^{j\omega}$. From the results of Part

(b) we observe that $\sum_{n=0}^{\infty} n\alpha^n e^{-j\omega n} = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$. Hence,

$X_4(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} - 2\alpha^{-2} e^{j2\omega} - \alpha^{-1} e^{j\omega}$.

(e) $x_5[n] = n\alpha^n \mu[-n-1]$ with $|\alpha| > 1$. Its DTFT is given by

$$X_5(e^{j\omega}) = \sum_{n=-\infty}^{-1} \alpha^n e^{-j\omega n} = \sum_{m=1}^{\infty} \alpha^{-m} e^{j\omega m} = \sum_{m=0}^{\infty} \alpha^{-m} e^{j\omega m} - 1 = \frac{1}{1 - \alpha^{-1} e^{j\omega}} - 1 = \frac{e^{j\omega}}{\alpha - e^{j\omega}}.$$

(f) $x_6[n] = \begin{cases} \alpha^{|n|}, & |n| \leq M, \\ 0, & \text{otherwise.} \end{cases}$ Its DTFT is given by

$$X_6(e^{j\omega}) = \sum_{n=0}^M \alpha^n e^{-j\omega n} + \sum_{n=-M}^{-1} \alpha^{-n} e^{-j\omega n} = \frac{1 - \alpha^{M+1} e^{-j\omega(M+1)}}{1 - \alpha e^{-j\omega}} + \alpha^M e^{j\omega M} \cdot \frac{1 - \alpha^{M+1} e^{-j\omega(M+1)}}{1 - \alpha e^{-j\omega}}.$$

3.18 (a) $x_a[n] = \mu[n] - \mu[n-5]$. Let denote $\mu(e^{j\omega})$ the DTFT of $\mu[n]$. Using the time-shifting property of the DTFT given in Table 3.4, the DTFT of $x_a[n]$ is thus given by

$X_a(e^{j\omega}) = (1 - e^{-j5\omega}) \mu(e^{j\omega})$. From Table 3.3, we have

$$\mu(e^{j\omega}) = \frac{1}{1 - e^{j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k). \text{ Therefore, } X_a(e^{j\omega}) = \frac{1 - e^{-j5\omega}}{1 - e^{j\omega}}.$$

(b) $x_b[n] = \alpha^n (\mu[n] - \mu[n-8])$. Let $x[n] = \alpha^n \mu[n]$ with $|\alpha| < 1$. Its DTFT was

computed in Example 3.6 and is given by $X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$. Now

$x_b[n] = x[n] - x[n-8]$. Using the time-shifting property of the DTFT given in Table 3.4, the DTFT of $x_b[n]$ is thus given by $X_b(e^{j\omega}) = (1 - e^{-j8\omega}) X(e^{j\omega}) = \frac{1 - e^{-j8\omega}}{1 - \alpha e^{-j\omega}}$.

(c) $x_c[n] = (n+1)\alpha^n \mu[n] = n\alpha^n \mu[n] + \alpha^n \mu[n]$ with $|\alpha| > 1$. We can rewrite it as

$x_c[n] = x_2[n] + x[n]$. The DTFT of $x_2[n]$ was computed in Problem 3.17(b) and is

given by $X_2(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$ and the DTFT of $x[n]$ was computed in Example 3.6

and is given by $X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$. Therefore $X_c(e^{j\omega}) = X_2(e^{j\omega}) + X(e^{j\omega})$

$$= \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}.$$

3.19 (a) $y_1[n] = \begin{cases} 1, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Then $Y_1(e^{j\omega}) = \sum_{n=-N}^N e^{-j\omega n} = e^{-j\omega N} \left(\frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}} \right)$

$$= \frac{\sin\left(\omega\left[N + \frac{1}{2}\right]\right)}{\sin(\omega/2)}.$$

(b) $y_2[n] = \begin{cases} 1, & 0 \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Then $Y_2(e^{j\omega}) = \sum_{n=0}^N e^{-j\omega n} = \frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}}$
 $= e^{-j\omega N/2} \left(\frac{\sin(\omega[N+1]/2)}{\sin(\omega/2)} \right).$

(c) $y_3[n] = \begin{cases} 1 - \frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Assume N to be odd. Then we can express

$y_3[n] = \frac{1}{N} y_0[n] \otimes y_0[n]$ where $y_0[n] = \begin{cases} 1, & -\frac{N-1}{2} \leq n \leq \frac{N-1}{2}, \\ 0, & \text{otherwise.} \end{cases}$ Therefore,

$Y_3(e^{j\omega}) = \frac{1}{N} Y_0(e^{j\omega}) Y_0(e^{j\omega}) = \frac{1}{N} Y_0^2(e^{j\omega}).$ Now, from the results of Part (a), we have

$Y_0(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$. Hence, $Y_3(e^{j\omega}) = \frac{1}{N} \cdot \frac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$.

Note: The above result also holds for N even.

(d) $y_4[n] = \begin{cases} N+1-|n|, & -N \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases} = y_1[n] + N y_3[n]$, where $y_1[n]$ is the sequence

considered in Part (a) and $y_3[n]$ is the sequence considered in Part (c). Hence,

$Y_4(e^{j\omega}) = Y_1(e^{j\omega}) + N \cdot Y_3(e^{j\omega}) = \frac{\sin\left(\omega\left[N + \frac{1}{2}\right]\right)}{\sin(\omega/2)} + \frac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$.

(e) $y_5[n] = \begin{cases} \cos(\pi n/2N), & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Then

$Y_5(e^{j\omega}) = \frac{1}{2} \sum_{n=-N}^N e^{-j(\pi n/2N)} e^{-j\omega n} + \frac{1}{2} \sum_{n=-N}^N e^{j(\pi n/2N)} e^{-j\omega n}$
 $= \frac{1}{2} \sum_{n=-N}^N e^{-j\left(\omega - \frac{\pi}{2N}\right)n} + \frac{1}{2} \sum_{n=-N}^N e^{-j\left(\omega + \frac{\pi}{2N}\right)n} = \frac{1}{2} \cdot \frac{\sin\left(\left(\omega - \frac{\pi}{2N}\right)\left(N + \frac{1}{2}\right)\right)}{\sin\left(\left(\omega - \frac{\pi}{2N}\right)/2\right)} + \frac{1}{2} \cdot \frac{\sin\left(\left(\omega + \frac{\pi}{2N}\right)\left(N + \frac{1}{2}\right)\right)}{\sin\left(\left(\omega + \frac{\pi}{2N}\right)/2\right)}$.

3.20 Denote $x_m[n] = \frac{(n+m-1)!}{n!(m-1)!} \alpha^n \mu[n]$ with $|\alpha| < 1$. We shall prove by induction that the

DTFT of $x_m[n]$ is given by $X_m(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^m}$. From Table 3.3, it follows that

it holds for $m = 1$. Let $m = 2$. Then

$x_2[n] = \frac{(n+1)!}{n!} \alpha^n \mu[n] = (n+1)x_1[n] = nx_1[n] + x_1[n]$. Therefore,

$$X_2(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + 1 - \alpha e^{-j\omega} = \frac{1}{(1 - \alpha e^{-j\omega})^2}, \text{ and it also holds for } m = 2.$$

Now, assume that it holds for m . Consider next $x_{m+1}[n] = \frac{(n+m)!}{n!(m)!} \alpha^n \mu[n]$

$$= \left(\frac{n+m}{m} \right) \frac{(n+m-1)!}{n!(m-1)!} \alpha^n \mu[n] = \left(\frac{n+m}{m} \right) x_m[n] = \frac{1}{m} \cdot n \cdot x_m[n] + x_m[n]. \text{ Hence,}$$

$$X_{m+1}(e^{j\omega}) = \frac{1}{m} j \frac{d}{d\omega} \left(\frac{1}{(1 - \alpha e^{-j\omega})^m} \right) + \frac{1}{(1 - \alpha e^{-j\omega})^m} = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^{m+1}} + \frac{1}{(1 - \alpha e^{-j\omega})^m}$$

$$= \frac{1}{(1 - \alpha e^{-j\omega})^{m+1}}.$$

3.21 (a) $X_a(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$. Hence, $x_a[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega = 1$.

(b) $X_b(e^{j\omega}) = \frac{e^{j\omega}(1 - e^{j\omega N})}{1 - e^{j\omega}} = e^{j\omega} \sum_{n=0}^{N-1} e^{j\omega n}$. Let $m = -n$. $X_b(e^{j\omega}) = e^{j\omega} \sum_{m=0}^{-N+1} e^{-j\omega m}$.

Consider the DTFT $X(e^{j\omega}) = \sum_{m=0}^{-N+1} e^{-j\omega m}$. Its inverse is given by

$$x[n] = \begin{cases} 1, & -(N-1) \leq n \leq 0, \\ 0, & \text{otherwise.} \end{cases} \text{ Therefore, by the time-shifting property of the DTFT, the}$$

inverse DTFT of $X_b(e^{j\omega}) = e^{j\omega} X(e^{j\omega})$ is given by $x_b[n] = x[n+1] = \begin{cases} 1, & -N \leq n \leq -1, \\ 0, & \text{otherwise.} \end{cases}$

(c) $X_c(e^{j\omega}) = 1 + 2 \sum_{\ell=0}^N \cos(\omega \ell) = 2 + \sum_{\ell=-N}^N e^{-j\omega \ell}$. Hence, $x_c[n] = \begin{cases} 3, & n = 0, \\ 1, & 0 < |n| \leq N, \\ 0, & \text{otherwise.} \end{cases}$

(d) $X_d(e^{j\omega}) = \frac{-\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$ with $|\alpha| < 1$. We can rewrite $X_d(e^{j\omega})$ as

$$X_d(e^{j\omega}) = \frac{dX_o(e^{j\omega})}{d\omega} \text{ where } X_o(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}. \text{ From Table 3.3, the inverse}$$

DTFT of $X_d(e^{j\omega})$ is given by $x_o[n] = \alpha^n \mu[n]$. From Table 3.4, using the differentiation-in-frequency property the inverse DTFT of $X_d(e^{j\omega})$ is thus given by $x_d[n] = n x_o[n] = n \alpha^n \mu[n]$.

3.22 (a) $H_a(e^{j\omega}) = \sin(4\omega) = \frac{e^{j4\omega} - e^{-j4\omega}}{2j} = \frac{1}{2j}e^{j4\omega} - \frac{1}{2j}e^{-j4\omega}$. Therefore,
 $h_a[n] = \{-j0.5, 0, 0, 0, 0, 0, 0, 0, j0.5\}, -4 \leq n \leq 4$.

(b) $H_b(e^{j\omega}) = \cos(4\omega) = \frac{e^{j4\omega} + e^{-j4\omega}}{2} = 0.5e^{j4\omega} + 0.5e^{-j4\omega}$. Therefore,
 $h_b[n] = \{0.5, 0, 0, 0, 0, 0, 0, 0, 0.5\}, -4 \leq n \leq 4$.

(c) $H_c(e^{j\omega}) = \sin(5\omega) = \frac{e^{j5\omega} - e^{-j5\omega}}{2j} = \frac{1}{2j}e^{j5\omega} - \frac{1}{2j}e^{-j5\omega}$. Therefore,
 $h_c[n] = \{-j0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0, j0.5\}, -5 \leq n \leq 5$.

(d) $H_d(e^{j\omega}) = \cos(5\omega) = \frac{e^{j5\omega} + e^{-j5\omega}}{2} = 0.5e^{j5\omega} + 0.5e^{-j5\omega}$. Therefore,
 $h_d[n] = \{0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.5\}, -5 \leq n \leq 5$.

3.23 (a) $H_1(e^{j\omega}) = 1 + 2\cos(\omega) + 3\cos(2\omega) = 1 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 3\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)$
 $= 1 + e^{j\omega} + e^{-j\omega} + 1.5e^{j2\omega} + 1.5e^{-j2\omega}$. Therefore,
 $\{h_1[n]\} = \{1.5, 1, 1, 1, 1.5\}, -2 \leq n \leq 2$.

(b) $H_2(e^{j\omega}) = (3 + 2\cos(\omega) + 4\cos(2\omega))\cos\left(\frac{\omega}{2}\right)e^{-j\omega/2}$
 $= \left[3 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 4\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)\right]\left(\frac{e^{j\omega/2} + e^{-j\omega/2}}{2}\right)e^{-j\omega/2}$
 $= \frac{1}{2}\left(3 + e^{j\omega} + e^{-j\omega} + 2e^{j2\omega} + 2e^{-j2\omega}\right)\left(1 + e^{-j\omega}\right)$
 $= 2 + 1.5e^{j\omega} + 2e^{-j\omega} + e^{j2\omega} + 1.5e^{-j2\omega} + e^{-j3\omega}$. Hence,
 $\{h_2[n]\} = \{1, 1.5, 2, 2, 1.5, 1\}, -2 \leq n \leq 3$.

(c) $H_3(e^{j\omega}) = j[3 + 4\cos(\omega) + 2\cos(2\omega)]\sin(\omega)$
 $= j\left[3 + 4\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 2\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)\right]\left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right)$
 $= \frac{1}{2}\left(3 + 2e^{j\omega} + 2e^{-j\omega} + e^{j2\omega} + e^{-j2\omega}\right)\left(e^{j\omega} - e^{-j\omega}\right)$
 $= 3 + 2e^{j\omega} + 2e^{-j\omega} + 0.5e^{j2\omega} + 0.5e^{-j2\omega}$. Hence,
 $\{h_c[n]\} = \{0.5, 2, 3, 2, 0.5\}, -2 \leq n \leq 2$.

$$\begin{aligned}
\text{(d)} \quad H_4(e^{j\omega}) &= j[4 + 2 \cos(\omega) + 3 \cos(2\omega)] \sin(\omega/2) e^{j\omega/2} \\
&= j \left[4 + 2 \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 3 \left(\frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) \right] \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} \right) e^{-j\omega/2} \\
&= \frac{1}{2} \left(4 + e^{j\omega} + e^{-j\omega} + 1.5 e^{j2\omega} + 1.5 e^{-j2\omega} \right) (1 - e^{-j\omega}) \\
&= 1.5 - 0.25 e^{j\omega} - 1.5 e^{-j\omega} + 1.5 e^{j2\omega} - 0.5 e^{-j2\omega} - 0.75 e^{-j3\omega}. \text{ Hence,} \\
\{h_4[n]\} &= \{-1.5, -0.5, -3, 3, -0.5, -3\}, -3 \leq n \leq 2.
\end{aligned}$$

3.24 Let $H(e^{j\omega})$ and $G(e^{j\omega})$ denote the DTFTs of the sequences $h[n]$ and $g[n]$, respectively.

$$\begin{aligned}
\text{(a)} \quad \text{Linearity Theorem: } \mathcal{F} \{ \alpha h[n] + \beta g[n] \} &= \sum_{n=-\infty}^{\infty} (\alpha h[n] + \beta g[n]) e^{-j\omega n} \\
&= \alpha \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n} = \alpha H(e^{j\omega}) + \beta G(e^{j\omega}).
\end{aligned}$$

$$\text{(b)} \quad \text{Time-reversal Theorem: } \sum_{n=-\infty}^{\infty} h[-n] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} h[m] e^{j\omega m} = H(e^{-j\omega}).$$

$$\begin{aligned}
\text{(c)} \quad \text{Time-shifting Theorem: } \sum_{n=-\infty}^{\infty} h[n - n_o] e^{-j\omega n} &= \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega(m+n_o)} \\
&= e^{-j\omega n_o} \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} = e^{-j\omega n_o} H(e^{j\omega}).
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad \text{Frequency-shifting Theorem: } \sum_{n=-\infty}^{\infty} (e^{j\omega_o n} h[n]) e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega - \omega_o)n} \\
&= H(e^{j(\omega - \omega_o)}).
\end{aligned}$$

3.25 Let $H_1(e^{j\omega}) = \mathcal{F} \{h_1[n]\}$ and $H_2(e^{j\omega}) = \mathcal{F} \{h_2[n]\}$. From Example 3.8 we have

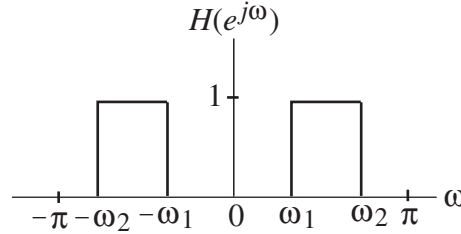
$$\begin{aligned}
H_2(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(\frac{\sin(\omega_2 n)}{\pi n} \right) \\
&= \begin{cases} 1, & 0 \leq |\omega| \leq \omega_2, \\ 0, & \omega_2 < |\omega| \leq \pi. \end{cases} \text{ From the result of Problem 3.14 we get}
\end{aligned}$$

$$H_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\delta[n] - \frac{\sin(\omega_1 n)}{\pi n} \right) e^{-j\omega n} = \begin{cases} 0, & 0 \leq |\omega| \leq \omega_1, \\ 1, & \omega_1 < |\omega| \leq \pi. \end{cases}$$

As the impulse response of the cascade is given by $h[n] = h_1[n] \otimes h_2[n]$, using the

convolution theorem we obtain the DTFT of the cascade: $H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$

$$= \begin{cases} 0, & 0 \leq |\omega| < \omega_1, \\ 1, & \omega_1 \leq |\omega| \leq \omega_2, \\ 0, & \omega_2 < |\omega| \leq \pi. \end{cases}$$



3.26 $Y(e^{j\omega}) = X(e^{j4\omega}) = X((e^{j\omega})^4)$ Now, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. Hence,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = X((e^{j\omega})^4) = \sum_{n=-\infty}^{\infty} x[n](e^{-j\omega n})^4 = \sum_{m=-\infty}^{\infty} x[m/4]e^{-j\omega m}.$$

$$\text{Therefore } y[n] = \begin{cases} x[n], & n = 0, \pm 4, \pm 8, \pm 16, \dots \\ 0, & \text{otherwise.} \end{cases}$$

3.27 $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. Therefore, $X(e^{j\omega/2}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega/2)n}$ and

$$X(-e^{j\omega/2}) = \sum_{n=-\infty}^{\infty} x[n](-1)^n e^{-j(\omega/2)n}. \text{ Thus, we can write}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \frac{1}{2} \left\{ X(e^{j\omega/2}) + X(-e^{j\omega/2}) \right\} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(x[n] + x[n](-1)^n \right) e^{-j(\omega/2)n}.$$

$$\text{Thus, } y[n] = \begin{cases} \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega/2)n}, & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd.} \end{cases} \text{ Hence, } y[n] = \begin{cases} x[2n], & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd.} \end{cases}$$

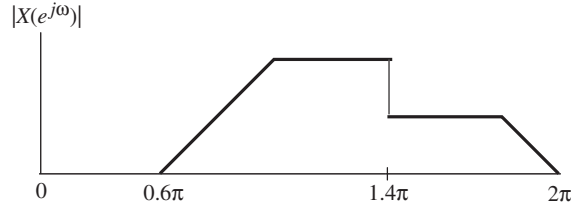
3.28 $\mathcal{F}\{x^*[-n]\} = \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x^*[n]e^{j\omega n} = \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right)^* = X^*(e^{j\omega})$.

3.29 $X_{ca}(e^{j\omega}) = \frac{1}{2} \left[X(e^{j\omega}) - X^*(e^{-j\omega}) \right] =$

$$= \frac{1}{2} \left(\sum_{n=-\infty}^{\infty} x_{re}[n]e^{-j\omega n} + j \sum_{n=-\infty}^{\infty} x_{im}[n]e^{-j\omega n} - \sum_{n=-\infty}^{\infty} x_{re}[n]e^{-j\omega n} - j \sum_{n=-\infty}^{\infty} x_{im}[n]e^{-j\omega n} \right)$$

$$\omega, = j \sum_{n=-\infty}^{\infty} x_{im}[n]e^{-j\omega n} = \mathcal{F}\{j x_{im}[n]\}.$$

3.30



3.31 From Table 3.2 we observe that an even real-valued sequence has a real-valued DTFT and an odd real-valued sequence has an imaginary-valued DTFT.

(a) Since $x_1[n]$ is an odd sequence, it has an imaginary-valued DTFT.

(b) Since $x_2[n]$ is an even sequence, it has a real-valued DTFT.

(c) $x_3[-n] = \frac{\sin(-\omega_c n)}{-\pi n} = \frac{-\sin(\omega_c n)}{-\pi n} = \frac{\sin(\omega_c n)}{\pi n} = x_3[n]$. Since, $x_3[n]$ is an even sequence, it has a real-valued DTFT.

(d) Since $x_4[n]$ is an odd sequence, it has an imaginary-valued DTFT.

(e) Since $x_5[n]$ is an odd sequence, it has an imaginary-valued DTFT.

3.32 From Table 3.2 we observe that an even real-valued sequence has a real-valued DTFT and an odd real-valued sequence has an imaginary-valued DTFT.

(a) Since $Y_1(e^{j\omega})$ is a real-valued function of ω , its inverse is an even sequence.

(b) Since $Y_2(e^{j\omega})$ is an imaginary-valued function of ω , its inverse is an odd sequence.

(c) Since $Y_3(e^{j\omega})$ is an imaginary-valued function of ω , its inverse is an odd sequence.

3.33 (a) Since $H_1(e^{j\omega})$ is a real-valued function of ω , its inverse is an even sequence.

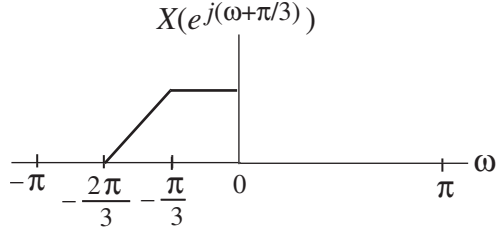
(b) Since $H_2(e^{j\omega})$ is a real-valued function of ω , its inverse is an even sequence.

3.34 Let $u[n] = x^*[-n]$, and let $X(e^{j\omega})$ and $U(e^{j\omega})$ denote the DTFTs of $x[n]$ and $u[n]$, respectively. From the convolution property of the DTFT given in Table 3.4, the DTFT of $y[n] = x[n] \otimes u[n]$ is given by $Y(e^{j\omega}) = X(e^{j\omega})U(e^{j\omega})$. From Table 3.1, $U(e^{j\omega})$

$= X^*(e^{j\omega})$. Therefore, $Y(e^{j\omega}) = X(e^{j\omega})X^*(e^{j\omega}) = |X(e^{j\omega})|^2$ which is a real-valued function of ω .

3.35 From the frequency-shifting property of the DTFT given in Table 3.4,

$\mathcal{F}\{x[n]e^{-j\pi n/3}\} = X(e^{j(\omega+\pi/3)})$. A sketch of this DTFT is shown below



3.36 $\{-\alpha^n \mu[-n-1]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{-1} -\alpha^n e^{-j\omega n} = -\sum_{n=1}^{\infty} \alpha^{-n} e^{j\omega n} = -\alpha^{-1} e^{j\omega} \sum_{n=0}^{\infty} \left(\frac{e^{j\omega}}{\alpha}\right)^n$.

For $|\alpha| > 1$, $X(e^{j\omega}) = -\alpha^{-1} e^{j\omega} \frac{1}{1-(e^{j\omega}/\alpha)} = \frac{1}{1-\alpha e^{j\omega}}$. $\Rightarrow |X(e^{j\omega})|^2 = \frac{1}{1+\alpha^2-2\alpha \cos(\omega)}$.

From Parseval's relation, $\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2$.

(a) $|X(e^{j\omega})|^2 = \frac{1}{5+4 \cos(\omega)}$. Hence $\alpha = -2$. Therefore, $x[n] = (-2)^n \mu[-n-1]$.

Now, $4 \int_0^{\pi} |X(e^{j\omega})|^2 d\omega = 2 \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 4\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 4\pi \sum_{n=-\infty}^{-1} |(-2)^n|^2$
 $= 4\pi \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \pi \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4\pi}{3}$.

(b) $|X(e^{j\omega})|^2 = \frac{1}{3.25-3 \cos(\omega)}$. Hence $\alpha = 1.5$. Therefore, $x[n] = -(1.5)^n \mu[-n-1]$.

Now, $\int_0^{\pi} |X(e^{j\omega})|^2 d\omega = \frac{1}{2} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = \pi \sum_{n=-\infty}^{-1} |(1.5)^n|^2$
 $= \pi \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n = \frac{4\pi}{9} \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n = \frac{4\pi}{9} \cdot \frac{9}{5} = \frac{4\pi}{5}$.

(c) Using the differentiation-in-frequency property of the DTFT as given in Table 3.4, the inverse DTFT of $X(e^{j\omega}) = j \frac{d}{d\omega} \left(\frac{1}{1-\alpha e^{-j\omega}} \right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$ is $x[n] = -n\alpha^n \mu[-n-1]$.

Hence, the inverse DTFT of $\frac{1}{(1-\alpha e^{-j\omega})^2}$ is $-(n+1)\alpha^n \mu[-n-1]$.

$|Y(e^{j\omega})|^2 = \frac{1}{(5-4e^{-j\omega})^2}$. Hence $\alpha = 2$. Therefore, $y[n] = -(n+1)2^n \mu[-n-1]$.

Now, $4 \int_0^{\pi} |X(e^{j\omega})|^2 d\omega = 2 \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 4\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 4\pi \sum_{n=-\infty}^{-1} (n+1)^2 \cdot 2^{2n}$

$$= \pi \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \cdot n^2 = \pi \frac{9/4}{9/16} = 4\pi.$$

3.37 $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \sum_{-\infty}^{\infty} |nx[n]|^2 = 2\pi \sum_{-3}^6 |nx[n]|^2 = 152\pi.$ (Using Parseval's relation with differentiation-in-frequency property)

3.38 (a) $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-2}^6 x[n] = 10.$

(b) $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n] e^{j\pi n} = \sum_{n=-2}^6 x[n] e^{j\pi n} = -6.$

(c) $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi x[0] = -2\pi.$

(d) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 2\pi \sum_{n=-2}^6 |x[n]|^2 = 120\pi.$

(e) $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \sum_{-\infty}^{\infty} |nx[n]|^2 = 2\pi \sum_{-2}^6 |nx[n]|^2 = 1724\pi.$

3.39 (a) $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-6}^2 x[n] = 12.$

(b) $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n] e^{j\pi n} = \sum_{n=-6}^2 x[n] e^{j\pi n} = -12.$

(c) $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi x[0] = -4\pi.$

(d) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 2\pi \sum_{n=-6}^2 |x[n]|^2 = 160\pi.$

(e) $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \sum_{-\infty}^{\infty} |nx[n]|^2 = 2\pi \sum_{-6}^2 |nx[n]|^2 = 136\pi.$

3.40 From the differentiation-in-frequency property of the DTFT given in Table 3.4 we

have $\sum_{n=-\infty}^{\infty} nx[n] e^{-j\omega n} = j \frac{dX(e^{j\omega})}{d\omega}$ where $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$. Therefore,

$$\sum_{n=-\infty}^{\infty} nx[n] = j \left. \frac{dX(e^{j\omega})}{d\omega} \right|_{\omega=0}. \quad \text{From the definition of the DTFT } X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

it follows that $\sum_{n=-\infty}^{\infty} x[n] = X(e^{j0})$. Therefore, $C_g = \frac{j \frac{dX(e^{j\omega})}{d\omega} \Big|_{\omega=0}}{X(e^{j0})}$.

From Table 3.3, $X(e^{j\omega}) = \mathcal{F}\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha e^{-j\omega}}$, $|\alpha| < 1$. As a result,

$$j \frac{dX(e^{j\omega})}{d\omega} \Big|_{\omega=0} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} \Big|_{\omega=0} = \frac{\alpha}{(1-\alpha)^2} \quad \text{and} \quad X(e^{j0}) = \frac{1}{1-\alpha}. \quad \text{Hence, } C_g = \frac{\alpha}{1-\alpha}.$$

3.41 Let $G_1(e^{j\omega}) = \mathcal{F}\{g_1[n]\}$.

(b) Note $g_2[n] = g_1[n] + g_1[n-4]$. Hence, $\mathcal{F}\{g_2[n]\} = G_2(e^{j\omega}) = G_1(e^{j\omega}) + e^{-j4\omega} G_1(e^{j\omega}) = (1 + e^{-j4\omega}) G_1(e^{j\omega})$.

(c) Note $g_3[n] = g_1[-(n-3)] + g_1[n-4]$. Now, $\mathcal{F}\{g_1[-n]\} = G_1(e^{-j\omega})$. Hence, $\mathcal{F}\{g_3[n]\} = G_3(e^{j\omega}) = e^{-j3\omega} G_1(e^{-j\omega}) + e^{-j4\omega} G_1(e^{j\omega})$.

(d) Note $g_4[n] = g_1[n] + g_1[-(n-7)]$. Hence, $\mathcal{F}\{g_4[n]\} = G_4(e^{j\omega}) = G_1(e^{j\omega}) + e^{-j7\omega} G_1(e^{-j\omega})$.

3.42 $Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$, i.e., $\sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \left(\sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} \right) \left(\sum_{n=-\infty}^{\infty} x_2[n]e^{-j\omega n} \right)$.

(a) Setting $\omega = 0$ in the above we get $\sum_{n=-\infty}^{\infty} y[n] = \left(\sum_{n=-\infty}^{\infty} x_1[n] \right) \left(\sum_{n=-\infty}^{\infty} x_2[n] \right)$.

(b) Setting $\omega = \pi$ we get $\sum_{n=-\infty}^{\infty} (-1)^n y[n] = \left(\sum_{n=-\infty}^{\infty} (-1)^n x_1[n] \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n x_2[n] \right)$.

3.43 $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$. From Table 3.3, $\mathcal{F}\{x[n]\} = X(e^{j\omega}) = \frac{1}{1-\alpha e^{j\omega}}$. The total energy

of $x[n]$ is $\mathcal{E}_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{1-\alpha e^{-j\omega}} \right|^2 d\omega = \sum_{n=0}^{\infty} (\alpha^2)^n = \frac{1}{1-\alpha^2} \Big|_{\alpha=1/2} = \frac{4}{3}$. To determine the

80% bandwidth of the signal, we set $\mathcal{E}_{x,80} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \left| \frac{1}{1-\alpha e^{-j\omega}} \right|^2 d\omega = 0.8 \mathcal{E}_x = 0.8 \cdot \frac{4}{3}$

and solve for ω_c , i.e., set $\mathcal{E}_{x,80} = \left[\frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{1}{(1-\alpha \cos \omega)^2 + \alpha^2 \sin^2 \omega} d\omega \right]_{\alpha=1/2} = \frac{3.2}{3}$.

A numerical solution of the above equation yields $\omega_c = 0.5081\pi$.

3.44 Recall $x[n] = x_{ev}[n] + x_{od}[n]$, where $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$ and As $x[n]$ is causal, $x[n] = 0$ for $n < 0$ and $x[-n] = 0$ for $n > 0$. Hence, there is no overlap between the nonzero portions of $x[n]$ and $x[-n]$ except at $n = 0$, and we have $x[n] = 2x_{ev}[n]\mu[n] - x_{ev}[0]\delta[n]$ and $x[n] = 2x_{od}[n]\mu[n] + x_{od}[0]\delta[n]$. Moreover, since $x[n]$ is real, it follows from Table 3.2 that $\mathcal{F}\{x_{ev}[n]\} = X_{re}(e^{j\omega})$ and $\mathcal{F}\{x_{od}[n]\} = jX_{im}(e^{j\omega})$. Taking the DTFT of $x[n] = 2x_{ev}[n]\mu[n] - x_{ev}[0]\delta[n]$ we arrive at

$$X(e^{j\omega}) = \frac{1}{\pi} \int_{-\pi}^{\pi} X_{re}(e^{j\nu}) \mu(e^{j(\omega-\nu)}) d\nu - x[0].$$

From Table 3.3 we have

$$\mu(e^{j\omega}) = \mathcal{F}\{\mu[n]\} = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k) = \frac{1}{2} \left(1 - j \cot\left(\frac{\omega}{2}\right) \right) + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k).$$

Substituting the above in the equation preceding it we get

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{\pi} \int_{-\pi}^{\pi} X_{re}(e^{j\nu}) \left[\frac{1}{2} \left(1 - j \cot\left(\frac{\omega-\nu}{2}\right) \right) \right] d\nu \\ &\quad + \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} X_{re}(e^{j\nu}) \delta((\omega-\nu) + 2\pi k) d\nu - x[0] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{re}(e^{j\nu}) d\nu - \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{re}(e^{j\nu}) \cot\left(\frac{\omega-\nu}{2}\right) d\nu + X_{re}(e^{j\omega}) - x[0]. \end{aligned}$$

Comparing the last equation with $X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$, we arrive at

$$X_{im}(e^{j\omega}) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} X_{re}(e^{j\nu}) \cot\left(\frac{\omega-\nu}{2}\right) d\nu.$$

Likewise, taking the DTFT of $x[n] = 2x_{od}[n]\mu[n] + x_{od}[0]\delta[n]$ we get

$$X(e^{j\omega}) = \frac{j}{\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\nu}) \mu(e^{j(\omega-\nu)}) d\nu + x[0].$$

Substituting the expression for $\mu(e^{j\omega})$ given earlier in the above equation we get

$$\begin{aligned} X(e^{j\omega}) &= \frac{j}{\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\nu}) \left[\frac{1}{2} \left(1 - j \cot\left(\frac{\omega-\nu}{2}\right) \right) \right] d\nu \\ &\quad + j \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} X_{im}(e^{j\nu}) \delta((\omega-\nu) + 2\pi k) d\nu + x[0] \\ &= \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\nu}) d\nu + \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\nu}) \cot\left(\frac{\omega-\nu}{2}\right) d\nu + jX_{im}(e^{j\omega}) + x[0]. \end{aligned}$$

Comparing the last equation with $X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$, we arrive at

$$X_{re}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\nu}) \cot\left(\frac{\omega-\nu}{2}\right) d\nu + x[0].$$

3.45 If $u[n] = z^n$ is the input to the LTI discrete-time system, then its output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = z^n H(z).$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$. Hence is an eigenfunction of the system.

If $v[n] = z^n \mu[n]$ is the input to the LTI discrete-time system, then its output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]v[n-k] = z^n \sum_{k=-\infty}^{\infty} h[k]\mu[n-k]z^{n-k} = z^n \sum_{k=-\infty}^n h[k]z^{-k}.$$

Since in this case the summation depends upon n is not an eigenfunction of the system.

3.46 $\mathcal{F}\{h_1[n]\} = H_1(e^{j\omega}) = \mathcal{F}\{\delta[n] + \frac{1}{2}\delta[n-1]\} = 1 + 0.5e^{-j\omega},$

$\mathcal{F}\{h_2[n]\} = H_2(e^{j\omega}) = \mathcal{F}\{\frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]\} = 0.5 - 0.25e^{-j\omega},$

$\mathcal{F}\{h_3[n]\} = H_3(e^{j\omega}) = \mathcal{F}\{2\delta[n]\} = 2,$

$\mathcal{F}\{h_4[n]\} = H_4(e^{j\omega}) = \mathcal{F}\{-2\left(\frac{1}{2}\right)^n \mu[n]\} = \frac{-2}{1 - 0.5e^{-j\omega}}.$

The overall frequency response of the structure of Figure 2.35 is given by

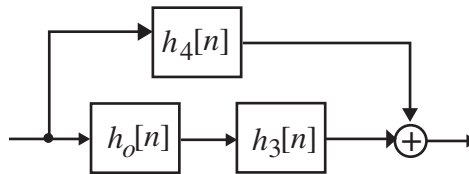
$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega}) + H_2(e^{j\omega})H_3(e^{j\omega}) + H_2(e^{j\omega})H_4(e^{j\omega}) \\ &= 1 + 0.5e^{-j\omega} + 2(0.5 - 0.25e^{-j\omega}) - \frac{2(0.5 - 0.25e^{-j\omega})}{1 - 0.5e^{-j\omega}} = 1. \end{aligned}$$

3.46 Denote $H_i(e^{j\omega}) = \mathcal{F}\{h_i[n]\}, 1 \leq i \leq 5.$

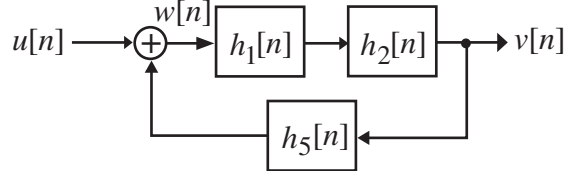
(a) The overall frequency of Figure P2.2(a) is then given by

$$H_i(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) + H_3(e^{j\omega})H_4(e^{j\omega}) + H_1(e^{j\omega})H_2(e^{j\omega})H_3(e^{j\omega})H_5(e^{j\omega}).$$

(b) The structure of Figure P2.2(b) can be redrawn as shown below



where the block with an impulse response represents the part of Figure P2.2(b) with a feedback loop as shown below



Let $U(e^{j\omega}) = \mathcal{F}\{u[n]\}$, $V(e^{j\omega}) = \mathcal{F}\{v[n]\}$, and $W(e^{j\omega}) = \mathcal{F}\{w[n]\}$. Then we have $W(e^{j\omega}) = U(e^{j\omega}) + H_3(e^{j\omega})V(e^{j\omega})$ and $V(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})W(e^{j\omega})$.

Eliminating $W(e^{j\omega})$ from these two equations we get

$$\left(1 - H_1(e^{j\omega})H_2(e^{j\omega})H_5(e^{j\omega})\right)V(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})U(e^{j\omega})$$

which leads to the frequency response of the feedback structure given by

$$H_o(e^{j\omega}) = \frac{V(e^{j\omega})}{U(e^{j\omega})} = \frac{H_1(e^{j\omega})H_2(e^{j\omega})}{1 - H_1(e^{j\omega})H_2(e^{j\omega})H_5(e^{j\omega})}$$

The overall frequency of Figure P2.2(a) is thus given by

$$H(e^{j\omega}) = H_4(e^{j\omega}) + H_o(e^{j\omega})H_3(e^{j\omega}) = H_4(e^{j\omega}) + \frac{H_1(e^{j\omega})H_2(e^{j\omega})}{1 - H_1(e^{j\omega})H_2(e^{j\omega})H_5(e^{j\omega})}$$

3.48 $\mathcal{F}\{h_1[n]\} = H_1(e^{j\omega}) = \mathcal{F}\{2\delta[n-2] - 3\delta[n+1]\} = 2e^{-j2\omega} - 3e^{j\omega}$,

$\mathcal{F}\{h_2[n]\} = H_2(e^{j\omega}) = \mathcal{F}\{\delta[n-1] + 2\delta[n+2]\} = e^{-j\omega} + 2e^{j2\omega}$,

$\mathcal{F}\{h_3[n]\} = H_3(e^{j\omega}) = \mathcal{F}\{5\delta[n-5] + 7\delta[n-3] + 2\delta[n-1] - \delta[n] + 3\delta[n+1]\}$
 $= 5e^{-j5\omega} + 7e^{-j3\omega} + 2e^{-j\omega} - 1 + 3e^{j\omega}$.

The overall frequency of Figure P2.3 is given by

$$H(e^{j\omega}) = H_3(e^{j\omega}) + H_1(e^{j\omega})H_2(e^{j\omega}) = 5e^{-j5\omega} + 7e^{-j3\omega} + 2e^{-j\omega} - 1 + 3e^{j\omega} + (2e^{-j2\omega} - 3e^{j\omega})(e^{-j\omega} + 2e^{j2\omega}) = 5e^{-j5\omega} + 9e^{-j3\omega} + 2e^{-j\omega} + 3e^{j\omega} - 6e^{j3\omega}$$

3.49 Now, $h_{ev}[n]$ is the inverse DTFT of $H_{re}(e^{j\omega})$. Rewriting we get

$$H_{re}(e^{j\omega}) = 1 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 3\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right) + 4\left(\frac{e^{j3\omega} + e^{-j3\omega}}{2}\right)$$

$$= 1 + e^{j\omega} + e^{-j\omega} + 1.5e^{j2\omega} + 1.5e^{-j2\omega} + 2e^{j3\omega} + 2e^{-j3\omega}$$
. Its inverse DTFT is $h_{ev}[n] = \delta[n] + \delta[n+1] + \delta[n-1] + 1.5\delta[n+2] + 1.5\delta[n-2] + 2\delta[n+3] + 2\delta[n-3]$.

Since $h[n]$ is real and causal, and its DTFT $H(e^{j\omega})$ exists, it is also absolutely summable. Hence, we can reconstruct $h[n]$ from $h_{ev}[n]$ as

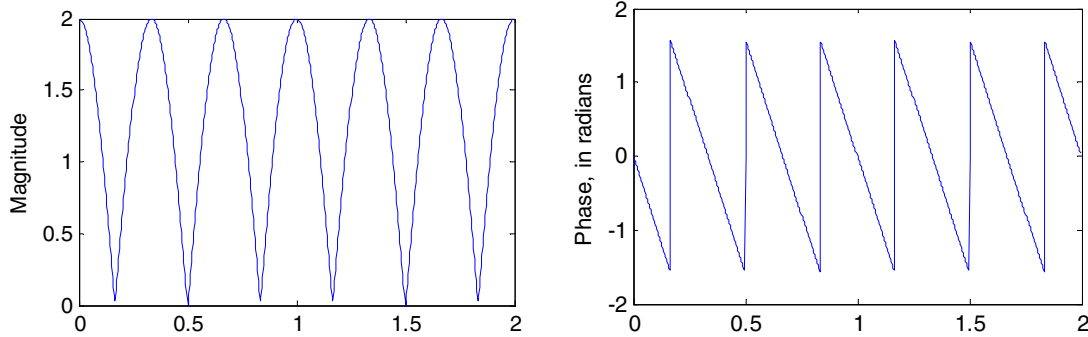
$$h[n] = 2h_{ev}[n]\mu[n] - h_{ev}[0]\delta[n] = 2(\delta[n] + \delta[n-1] + 1.5\delta[n-2] + 2\delta[n-3]) - \delta[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

3.50 (a) $y_a[n] = \sin\left(\frac{\pi n}{3}\right)\cos(\omega_o n) = \frac{1}{2}\sin\left((\omega_o + \frac{\pi}{3})n\right) - \frac{1}{2}\sin\left((\omega_o - \frac{\pi}{3})n\right)$. Hence, the angular frequencies present in the output are $(\omega_o \pm \frac{\pi}{3})n$.

(b) $y_b[n] = \cos^3(\omega_o n) = \cos^2(\omega_o n)\cos(\omega_o n) = \left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_o n)\right]\cos(\omega_o n)$
 $= \frac{1}{2}\cos(\omega_o n) + \frac{1}{2}\cos(2\omega_o n)\cos(\omega_o n) = \frac{1}{2}\cos(\omega_o n) + \frac{1}{4}[\cos(3\omega_o n) + \cos(\omega_o n)]$
 $= \frac{3}{4}\cos(\omega_o n) + \frac{1}{4}\cos(3\omega_o n)$. Hence, the angular frequencies present in the output are $3\omega_o$ and ω_o .

(c) $y_c[n] = \cos(3\omega_o n)$. Hence, the angular frequency present in the output is $3\omega_o$.

3.51 $\mathcal{F}\{\delta[n] - \alpha\delta[n - R]\} = H(e^{j\omega}) = 1 - \alpha e^{-j\omega R}$. Let $\alpha = |\alpha|e^{j\phi}$. Then the maximum value of $|H(e^{j\omega})|$ is $1 + |\alpha|$ and the minimum value is $1 - |\alpha|$. There are R peaks of $|H(e^{j\omega})|$ located at $\omega = 2\pi k / R$, $0 \leq k \leq R - 1$, and R dips located at $\omega = (2k + 1)\pi / R$, $0 \leq k \leq R - 1$ in the frequency range $0 \leq \omega < 2\pi$.



3.52 $G(e^{j\omega}) = \sum_{n=0}^{M-1} \alpha^n e^{-j\omega n} = \frac{1 - \alpha^M e^{-j\omega M}}{1 - \alpha e^{-j\omega}}$. Note $G(e^{j\omega}) = H(e^{j\omega})$ for $\alpha = 1$. In order

to have $G(e^{j0}) = 1$, the impulse response should be multiplied by a factor K , where

$$K = \left| \frac{1 - \alpha}{1 - \alpha^M} \right|.$$

3.53 $H(e^{j\omega}) = [a_1 \cos(2\omega) + (a_2 + a_3)\cos(\omega) + a_3] + j[-a_1 \sin(2\omega) + (a_2 - a_3)\sin(\omega)]$

The frequency response will have zero-phase for $a_2 = a_3$ and $a_1 = 0$.

3.54 $H(e^{j\omega}) = a_1 + a_2 e^{-j\omega} + a_3 e^{-j2\omega} + a_4 e^{-j3\omega} + a_5 e^{-j4\omega}$
 $= (a_1 e^{j2\omega} + a_5 e^{-j2\omega})e^{-j2\omega} + (a_2 e^{j\omega} + a_4 e^{-j\omega})e^{-j2\omega} + a_3 e^{-j2\omega}$.

If $a_1 = a_5$ and $a_2 = a_4$, then we can rewrite the above equation as

$$H(e^{j\omega}) = [2a_1 \cos(2\omega) + 2a_2 \cos(\omega) + a_3] e^{-j2\omega} \text{ which is seen to have a linear phase.}$$

3.55 $H(e^{j\omega}) = a_1 e^{jk\omega} + a_2 e^{j(k-1)\omega} + a_2 e^{-j(k-2)\omega} + a_1 e^{-j(k-3)\omega}$
 $= e^{jk\omega} (a_1 + a_2 e^{-j\omega} + a_2 e^{-j2\omega} + a_1 e^{-j3\omega})$
 $= e^{j(k-\frac{3}{2})\omega} [a_1 (e^{j3\omega/2} + e^{-j3\omega/2}) + a_2 (e^{j\omega/2} + e^{-j\omega/2})]$ Hence, $H(e^{j\omega})$ will be a real function of ω if $k = 3/2$, in which case we have
 $H(e^{j\omega}) = a_1 (e^{j3\omega/2} + e^{-j3\omega/2}) + a_2 (e^{j\omega/2} + e^{-j\omega/2}).$

3.56 $\mathcal{F}\{a\delta[n] + b\delta[n-1] + \delta[n-2]\} = H_1(e^{j\omega}) = a + b e^{-j\omega} + e^{-j2\omega},$
 $\mathcal{F}\{c^n \mu[n]\} = H_2(e^{j\omega}) = \frac{1}{1 - c e^{-j\omega}},$ and $\mathcal{F}\{d^n \mu[n]\} = H_3(e^{j\omega}) = \frac{1}{1 - d e^{-j\omega}}.$

The overall frequency response is then $H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})H_3(e^{j\omega})$

$$= \frac{a + b e^{-j\omega} + e^{-j2\omega}}{(1 - c e^{-j\omega})(1 - d e^{-j\omega})}. \text{ Therefore,}$$

$$\begin{aligned} |H(e^{j\omega})|^2 &= \frac{a + b e^{-j\omega} + e^{-j2\omega}}{(1 - c e^{-j\omega})(1 - d e^{-j\omega})} \frac{a + b e^{j\omega} + e^{j2\omega}}{(1 - c e^{j\omega})(1 - d e^{j\omega})} \\ &= \frac{(a^2 + b^2 + 1) + 2b(a+1)\cos(\omega) + 2a\cos(2\omega)}{(1 - 2c\cos(\omega) + c^2)(1 - 2d\cos(\omega) + d^2)} \\ &= \frac{(a^2 + b^2 + 1) + 2b(a+1)\cos(\omega) + 2a\cos(2\omega)}{(1 + c^2 + d^2 + c^2 d^2 + 2cd) - 2(c+d)(1+cd)\cos(\omega) + 2cd\cos(2\omega)}. \end{aligned} \text{ Hence,}$$

$$|H(e^{j\omega})|^2 = 1, \text{ if } a^2 + b^2 + 1 = 1 + c^2 + d^2 + c^2 d^2 + 2cd,$$

$b(a+1)\cos(\omega) = -(c+d)(1+cd),$ and $a = cd.$ Substituting $a = cd$ in the equation on the left we get $b = -(c+d).$

3.57 $Y(e^{j\omega}) = |X(e^{j\omega})|^\alpha e^{j \arg X(e^{j\omega})}.$ Therefore, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = |X(e^{j\omega})|^{\alpha-1}.$

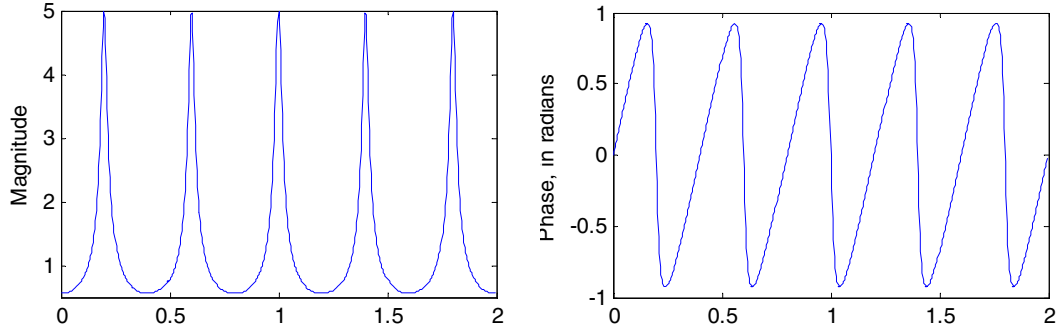
Since $H(e^{j\omega})$ is real function of ω , it has zero-phase.

3.58 $y[n] = x[n] - \alpha y[n-R].$ Taking the DTFT of both sides we get

$$Y(e^{j\omega}) = X(e^{j\omega}) - \alpha e^{-j\omega R} Y(e^{j\omega}). \text{ Hence, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 + \alpha e^{-j\omega R}}.$$

The maximum value of $H(e^{j\omega})$ is $\frac{1}{1-|\alpha|}$ and the minimum value is $\frac{1}{1+|\alpha|}$. There are peaks and dips in the range $0 \leq \omega < 2\pi$. The locations of the peaks and the dips are given by $1 - \alpha e^{-j\omega R} = 1 \pm |\alpha|$ or $e^{-j\omega R} = \pm \frac{|\alpha|}{\alpha}$. The locations of the peaks are given by $\omega = \omega_k = \frac{2\pi k}{R}$ and the locations of the dips are given by $\omega = \omega_k = \frac{(2\pi+1)k}{R}, 0 \leq k \leq R-1$.

Plots of the magnitude and the phase responses of $H(e^{j\omega})$ for $\alpha = 0.8$ and $R = 5$ are shown below:



In this case the maximum value is $\frac{1}{1-0.8} = 5$ and the minimum value is $\frac{1}{1+0.8} = 0.5556$.

3.59 $A(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}}$. Thus, we set $|A(e^{j\omega})|^2 = A(e^{j\omega})A^*(e^{j\omega}) = A(e^{j\omega})A(e^{-j\omega})$

$$= \frac{(b_0 + b_1 e^{-j\omega})(b_0 + b_1 e^{j\omega})}{(1 + a_1 e^{-j\omega})(1 + a_1 e^{j\omega})} = \frac{b_0^2 + b_1^2 + 2b_0 b_1 \cos(\omega)}{1 + a_1^2 + 2a_1 \cos(\omega)} = 1.$$

Solution #1: $b_0 = \pm 1$ and $b_1 = \text{sgn}(b_0)a_1$. In which case $A(e^{j\omega}) = \pm \frac{1 + a_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}} = 1$, a trivial solution.

Solution #2: $b_1 = \pm 1$ and $b_0 = \text{sgn}(b_1)a_1$. In which case $A(e^{j\omega}) = \pm \frac{a_1 + e^{-j\omega}}{1 + a_1 e^{-j\omega}}$.

3.60 $A(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}} = \frac{B_0 e^{j\phi_0} + B_1 e^{j\phi_1} e^{-j\omega}}{1 + A_1 e^{j\theta} e^{-j\omega}}$. Thus, we set

$$|A(e^{j\omega})|^2 = A(e^{j\omega})A^*(e^{j\omega}) = \frac{B_0 e^{j\phi_0} + B_1 e^{j\phi_1} e^{-j\omega}}{1 + A_1 e^{j\theta} e^{-j\omega}} \cdot \frac{B_0 e^{-j\phi_0} + B_1 e^{-j\phi_1} e^{j\omega}}{1 + A_1 e^{-j\theta} e^{j\omega}}$$

$$= \frac{B_0^2 + B_1^2 + 2B_0 B_1 \cos(\omega - \phi_1 + \phi_0)}{1 + A_1^2 + 2A_1 \cos(\omega - \theta)} = 1.$$

Solution #1: $B_0 = \pm 1$, $B_1 = \text{sgn}(B_0)A_1$, $\phi_1 - \phi_0 = \theta$. In which case

$$A(e^{j\omega}) = \frac{\pm e^{j\phi_0} \pm A_1 e^{j(\phi_0 + \theta)} e^{-j\omega}}{1 + A_1 e^{j\theta} e^{-j\omega}} = \pm e^{j\phi_0} \left(\frac{1 + A_1 e^{j\theta} e^{-j\omega}}{1 + A_1 e^{j\theta} e^{-j\omega}} \right) = \pm e^{j\phi_0} \text{ implying}$$

$\phi_0 = \pm\pi$. A trivial solution.

Solution #2: $B_1 = \pm 1$, $B_0 = \text{sgn}(B_1)A_1$, $\phi_1 - \phi_0 = \theta$. In which case

$$A(e^{j\omega}) = \frac{\pm A_1 e^{j(\phi_1 - \theta)} \pm e^{-j\omega}}{1 + A_1 e^{j\theta} e^{-j\omega}} = \pm e^{j\phi_1} \left(\frac{A_1 e^{-j\theta} + e^{-j\omega}}{1 + A_1 e^{j\theta} e^{-j\omega}} \right) = \pm e^{j\phi_1} \text{ implying } \phi_1 = \pm\pi.$$

Hence, a non-trivial solution is $A(e^{j\omega}) = e^{j\pi} \left(\frac{a_1^* + e^{-j\omega}}{1 + a_1 e^{-j\omega}} \right)$.

3.61 (a) $H_a(e^{j\omega}) = \text{cosec}(\omega) = \cot\left(\frac{\omega}{2}\right) - \cot(\omega) = \frac{\cos(\omega/2)}{\sin(\omega/2)} - \frac{\cos(\omega)}{\sin(\omega)}$

$$= \frac{j(e^{j\omega/2} + e^{-j\omega/2})}{(e^{j\omega/2} - e^{-j\omega/2})} - \frac{j(e^{j\omega} + e^{-j\omega})}{(e^{j\omega} - e^{-j\omega})}$$

$$= j \frac{(e^{j\omega/2} + e^{-j\omega/2})(e^{j\omega} - e^{-j\omega}) - (e^{j\omega/2} - e^{-j\omega/2})(e^{j\omega} + e^{-j\omega})}{(e^{j\omega/2} - e^{-j\omega/2})(e^{j\omega} - e^{-j\omega})}$$

$$= j2 \left(\frac{e^{-j\omega} - e^{-j2\omega}}{1 - e^{-j\omega} - e^{-j4\omega} + e^{-j3\omega}} \right). \text{ Therefore, the input-output relation is given by}$$

$$y[n] - y[n-1] - y[n-2] + y[n-3] = j2 x[n-1] - j2 x[n-2].$$

(b) $H_b(e^{j\omega}) = \sec(\omega) = \frac{1}{\cos(\omega)} = \frac{2}{e^{j\omega} + e^{-j\omega}} = \frac{2e^{-j\omega}}{1 + e^{-j2\omega}}$. Therefore, the input-output

relation is given by $y[n] + y[n-2] = 2x[n-1]$.

(c) $H_c(e^{j\omega}) = \cot(\omega) = \frac{\cos(\omega)}{\sin(\omega)} = j \left(\frac{e^{j\omega} + e^{-j\omega}}{e^{j\omega} - e^{-j\omega}} \right) = \frac{j(e^{j\omega} + e^{-j\omega})}{e^{j\omega} - e^{-j\omega}} = \frac{j e^{-j\omega} (e^{j\omega} + e^{-j\omega})}{1 - e^{-j2\omega}}$

$$= \frac{j(1 + e^{-j2\omega})}{1 - e^{-j2\omega}}. \text{ Therefore, the input-output relation is given by}$$

$$y[n] - y[n-2] = j x[n] + j x[n-2].$$

(d) $H_d(e^{j\omega}) = \tan\left(\frac{\omega}{2}\right) = \frac{\sin(\omega/2)}{\cos(\omega/2)} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j(e^{j\omega/2} + e^{-j\omega/2})} = \frac{1 - e^{-j\omega}}{j(1 + e^{-j\omega})} = \frac{-j + j e^{-j\omega}}{1 + e^{-j\omega}}$.

Therefore, the input-output relation is given by $y[n] + y[n-1] = -j x[n] + j x[n-1]$.

3.62 From Eq. (2.20), the input-output relation of a factor-of- L up-sampler is given by

$$y[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \pm 3L, \dots \\ 0, & \text{otherwise.} \end{cases} \text{ The DTFT of } y[n] \text{ is thus given by}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{\substack{n=-\infty \\ n=mL}}^{\infty} x[n/L]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega mL} = X(e^{jL\omega}), \text{ where}$$

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}.$$

3.63 $G(e^{j\omega}) = \frac{1}{1 - \alpha e^{-jL\omega}}$, $|\alpha| < 1$. Thus, we can write $G(e^{j\omega}) = X(e^{jL\omega})$, where

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}. \text{ From Table 3.3, the inverse DTFT of } X(e^{j\omega}) \text{ is } x[n] = \alpha^n \mu[n].$$

Hence, from the results of Problem 3.62, it follows that

$$g[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \pm 3L, \dots \\ 0, & \text{otherwise.} \end{cases}$$

3.64 From Table 3.3, $H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$. Thus, $|H(e^{j\omega})| = \frac{1}{\sqrt{1.25 - \cos(\omega)}}$ and

$$\arg\{H(e^{j\omega})\} = \theta(\omega) = \tan^{-1}\left(\frac{-0.5 \sin(\omega)}{1 - 0.5 \cos(\omega)}\right). \quad H(e^{\pm j\pi/5}) = 1.3504 \mp j0.6664. \text{ Therefore}$$

$$|H(e^{\pm j\pi/5})| = 1.5059 \text{ and } \theta(\pm\pi/5) = \mp 0.4585 \text{ radians.}$$

Now, for an input $x[n] = \sin(\omega_o n)\mu[n]$, the steady-state output is given by

$$y[n] = |H(e^{j\omega_o})| \sin(\omega_o n + \theta(\omega_o)). \text{ For } \omega_o = \pi/5, \text{ the steady-state output is therefore given by } y[n] = |H(e^{j\pi/5})| \sin\left(\frac{\pi}{5}n + \theta\left(\frac{\pi}{5}\right)\right) = 1.5059 \sin\left(\frac{\pi}{5}n - 0.4585\right).$$

3.65 $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[0]e^{-j2\omega} = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega}$

$$= e^{-j\omega}(2h[0]\cos(\omega) + h[1]). \text{ We require } |H(e^{j0.3})| = 2h[0]\cos(0.3) + h[1] = 0$$

$$\text{and } |H(e^{j0.6})| = 2h[0]\cos(0.6) + h[1] = 1. \text{ Solving these two equations we get } h[0] = 3.8461 \text{ and } h[1] = -6.3487.$$

3.66 $H(e^{j\omega}) = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega} = e^{-j\omega}(2h[0]\cos(\omega) + h[1])$. We require

$$|H(e^{j0.3})| = 2h[0]\cos(0.3) + h[1] = 1 \text{ and } |H(e^{j0.6})| = 2h[0]\cos(0.6) + h[1] = 0. \text{ Solving these two equations we get } h[0] = -3.8461 \text{ and } h[1] = 7.3487.$$

3.67

3.68 $H(e^{j\omega}) = h[0](1 + e^{-j4\omega}) + h[1](e^{-j\omega} + e^{-j3\omega}) + h[2]e^{-j2\omega}$

$= e^{-j2\omega}(2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2])$. We require

$$\left|H(e^{j0.2})\right| = 2\cos(0.4)h[0] + 2\cos(0.2)h[1] + h[2] = 0,$$

$$\left|H(e^{j0.5})\right| = 2\cos(1.0)h[0] + 2\cos(0.5)h[1] + h[2] = 1,$$

$$\left|H(e^{j0.8})\right| = 2\cos(1.6)h[0] + 2\cos(0.8)h[1] + h[2] = 0. \text{ Solving these three equations we}$$

get $h[0] = -13.4866$, $h[1] = 45.228$, $h[2] = -63.8089$, i.e.,

$$\{h[n]\} = \{-13.4866, 45.228, -63.8089\}, 0 \leq n \leq 2.$$

3.69 $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$. Therefore,

$$H(e^{j0}) = h[0] + h[1] + h[2] + h[3] = 2,$$

$$\begin{aligned} H(e^{j\pi/2}) &= h[0] + h[1]e^{-j\pi/2} + h[2]e^{-j\pi} + h[3]e^{-j3\pi/2} \\ &= h[0] - jh[1] - h[2] + jh[3] = 7 - j3, \end{aligned}$$

$H(e^{j\pi}) = h[0] - h[1] + h[2] - h[3] = 0$. Since the impulse response is real, the value of

$H(e^{j\omega})$ at $\omega = 3\pi/2$ is the conjugate of its value at $\omega = \pi/2$, i.e.,

$$H(e^{j3\pi/2}) = H^*(e^{j\pi/2}) = h[0] + jh[1] - h[2] - jh[3] = 7 + j3. \text{ Writing the four}$$

equations in matrix form we get

$$\begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 7 - j3 \\ 0 \\ 7 + j3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 7 - j3 \\ 0 \\ 7 + j3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \\ -1 \end{bmatrix}, \text{ and}$$

hence $\{h[n]\} = \{4, 2, -3, -1\}, 0 \leq n \leq 3$.

3.70 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$

$$= h[0] + h[1]e^{-j\omega} - h[1]e^{-j2\omega} - h[0]e^{-j3\omega}. \text{ Therefore,}$$

$$\begin{aligned} H(e^{j\pi/2}) &= h[0] + h[1]e^{-j\pi/2} - h[1]e^{-j\pi} - h[0]e^{-j3\pi/2} \\ &= h[0] - jh[1] + h[1] - jh[0] = -2 + j2, \end{aligned}$$

$H(e^{j\pi}) = h[0] - h[1] - h[1] + h[0] = 8$. Solving these two equations we get

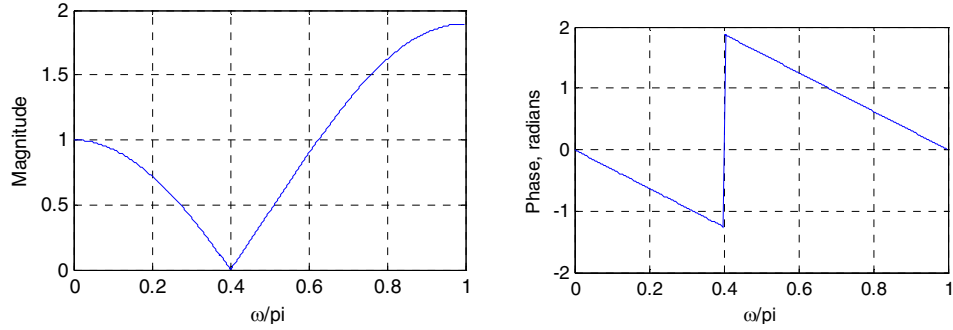
$h[0] = 1$ and $h[1] = -3$. Hence, $\{h[n]\} = \{1, -3, 3, -1\}, 0 \leq n < 3$.

3.71 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} = h[0] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}$. The two conditions to be satisfied by the filter are:

$$H(e^{j0.4\pi}) = h[0] + h[1]e^{-j0.4\pi} + h[0]e^{-j0.8\pi} = 0 \text{ and}$$

$H(e^{j0}) = h[0] + h[1] + h[0] = 1$. Solving these two equations we get $h[0] = 0.7236$ and $h[1] = -0.4472$.

(b) $H(e^{j\omega}) = 0.7236 - 0.4472e^{-j\omega} + 0.7236e^{-j2\omega}$.



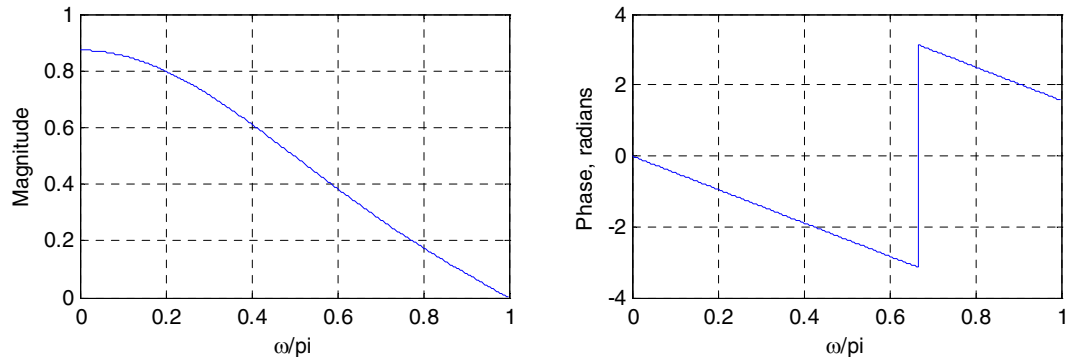
3.72 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$
 $= h[0] + h[1]e^{-j\omega} + h[1]e^{-j2\omega} + h[0]e^{-j3\omega} = e^{-j3\omega/2} (2h[0]\cos(3\omega/2) + 2h[1]\cos(\omega/2)).$

The two conditions to be satisfied by the filter are:

$$\left| H(e^{j0.2\pi}) \right| = 2h[0]\cos(0.3\pi) + 2h[1]\cos(0.1\pi) = 0.8,$$

$$\left| H(e^{j0.5\pi}) \right| = 2h[0]\cos(0.75\pi) + 2h[1]\cos(0.25\pi) = 0.5. \text{ Solving these two equations we get } h[0] = 0.0414 \text{ and } h[1] = 0.395.$$

(b) $H(e^{j\omega}) = 0.0414 + 0.395e^{-j\omega} + 0.395e^{-j2\omega} + 0.0414e^{-j3\omega}.$



3.73 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$

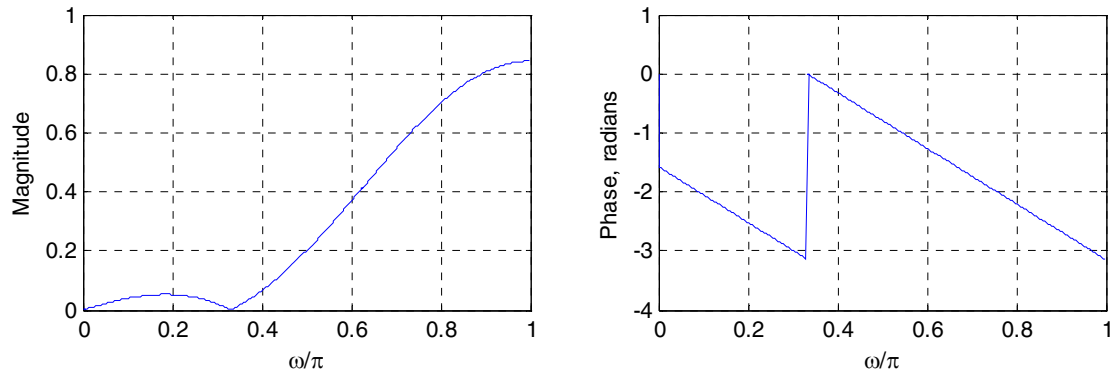
$$= h[0] + h[1]e^{-j\omega} - h[1]e^{-j2\omega} - h[0]e^{-j3\omega} = j e^{-j3\omega/2} (2h[0]\sin(3\omega/2) + 2h[1]\sin(\omega/2)).$$

The two conditions to be satisfied by the filter are:

$$\left| H(e^{j0.5\pi}) \right| = 2h[0]\sin(0.75\pi) + 2h[1]\sin(0.25\pi) = 0.2,$$

$$\left| H(e^{j0.8\pi}) \right| = 2h[0]\sin(1.2\pi) + 2h[1]\sin(0.4\pi) = 0.7. \text{ Solving these two equations we get } h[0] = -0.14 \text{ and } h[1] = 0.2815.$$

(b) $H(e^{j\omega}) = -0.14 + 0.2815e^{-j\omega} - 0.2815e^{-j2\omega} + 0.14e^{-j3\omega}.$



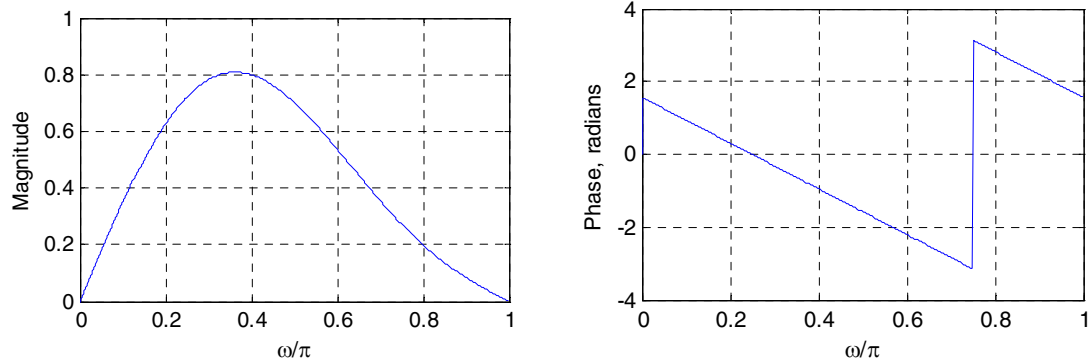
3.74 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$
 $= h[0] + h[1]e^{-j\omega} - h[1]e^{-j3\omega} - h[0]e^{-j4\omega} = j e^{-j2\omega} (2h[0]\sin(2\omega) + 2h[1]\sin(\omega)).$

The two conditions to be satisfied by the filter are:

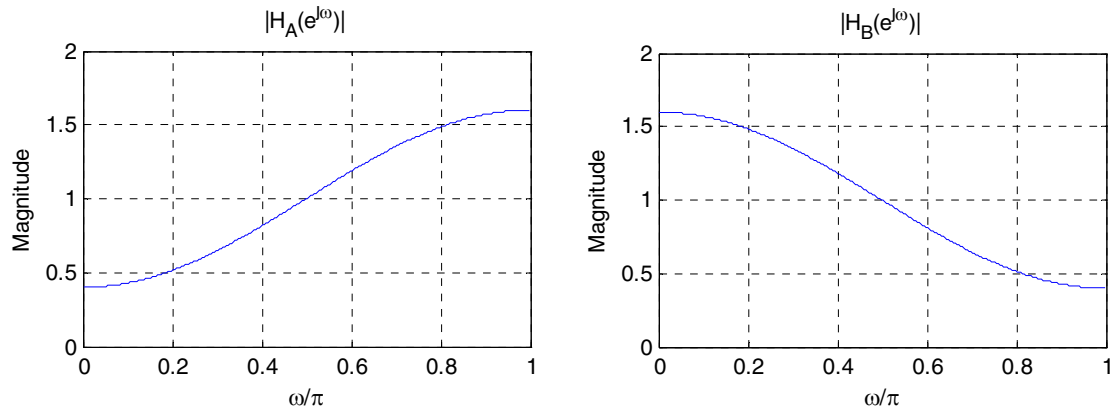
$$|H(e^{j0.5\pi})| = 2h[0]\sin(0.8\pi) + 2h[1]\sin(0.4\pi) = 0.8,$$

$$|H(e^{j0.8\pi})| = 2h[0]\sin(1.6\pi) + 2h[1]\cos(0.8\pi) = 0.2. \text{ Solving these two equations we get } h[0] = 0.112 \text{ and } h[1] = 0.3514.$$

(b) $H(e^{j\omega}) = 0.112 + 0.3514e^{-j\omega} - 0.3514 e^{-j3\omega} - 0.112e^{-j4\omega}.$



3.75 (a) $H_A(e^{j\omega}) = 0.3 - e^{-j\omega} + 0.3e^{-j2\omega}, H_B(e^{j\omega}) = 0.3 + e^{-j\omega} + 0.3e^{-j2\omega}.$



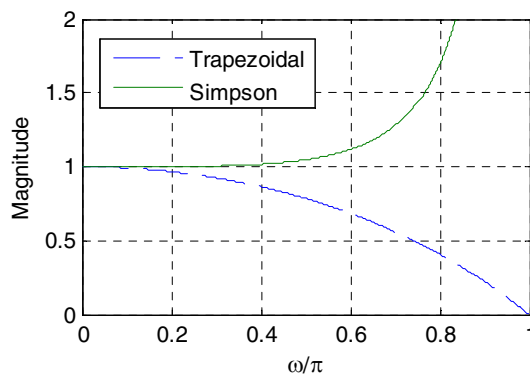
It can be seen from the above plots that $H_A(e^{j\omega})$ is a highpass filter, whereas $H_B(e^{j\omega})$ is a lowpass filter.

(b) $H_C(e^{j\omega}) = H_B(e^{j\omega}) = H_A(e^{j(\omega+\pi)})$.

3.76 $y[n] = y[n-1] + 0.5(x[n] + x[n-1])$. Taking the DTFT of both sides we get $Y(e^{j\omega}) = e^{-j\omega} Y(e^{j\omega}) + 0.5(X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}))$. Hence, the frequency response is given by $H_{trap}(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{2} \left(\frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} \right)$.

3.77 $y[n] = y[n-2] + \frac{1}{3}(x[n] + 4x[n-1] + x[n-2])$. Hence,

$$H_{simpsn}(e^{j\omega}) = \frac{1}{3} \left(\frac{1 + 4e^{-j\omega} + e^{-j2\omega}}{1 - e^{-j2\omega}} \right).$$



Note: To compare the performances of the Trapezoidal numerical integration formula with that of the Simpson's formula, we first observe that if the input is $x_a(t) = e^{j\omega t}$, then the result of integration is $y_a(t) = \frac{1}{j\omega} e^{j\omega t}$. Thus, the desired ideal frequency response is

$H(e^{j\omega}) = \frac{1}{j\omega}$. Hence, we take the ratio of the frequency responses of the approximation to the ideal, and plot the two curves as indicated on the previous page. From this plot, it is evident that the Simpson's formula amplifies high frequencies, whereas, the trapezoidal formula attenuates them. In the very low frequency range, both formulae yield results close to the ideal. However, Simpson's formula is reasonably accurate for frequencies close to the midband range.

$$\begin{aligned}
 \mathbf{3.78} \quad G(e^{j\omega}) &= g_0 + g_1 e^{-j\omega} + g_2 e^{-j2\omega} + g_3 e^{-j3\omega} \\
 &= e^{-j3\omega/2} \left(g_0 e^{j3\omega/2} + g_1 e^{j\omega/2} + g_2 e^{-j\omega/2} + g_3 e^{-j3\omega/2} \right) \\
 &= e^{-j3\omega/2} [g_0 \cos(3\omega/2) + jg_0 \sin(3\omega/2) + g_3 \cos(-3\omega/2) + jg_3 \sin(-3\omega/2) \\
 &\quad + g_1 \cos(\omega/2) + jg_1 \sin(\omega/2) + g_2 \cos(-\omega/2) + jg_2 \sin(-\omega/2)] \\
 &= e^{-j3\omega/2} [(g_0 + g_3) \cos(3\omega/2) + (g_1 + g_2) \cos(\omega/2) \\
 &\quad + j(g_0 - g_3) \sin(3\omega/2) + j(g_1 - g_2) \cos(\omega/2)].
 \end{aligned}$$

Thus, if $g_0 = g_3$ and $g_1 = g_2$, then

$$\begin{aligned}
 G(e^{j\omega}) &= e^{-j3\omega/2} [(g_0 + g_3) \cos(3\omega/2) + (g_1 + g_2) \cos(\omega/2)] \text{ which has a linear phase} \\
 \theta(\omega) &= -\frac{3\omega}{2}, \text{ and hence, a constant group delay.}
 \end{aligned}$$

Alternately, if $g_0 = -g_3$ and $g_1 = -g_2$, then

$$\begin{aligned}
 G(e^{j\omega}) &= j e^{-j3\omega/2} [(g_0 + g_3) \sin(3\omega/2) + (g_1 + g_2) \sin(\omega/2)] \text{ which has a linear phase} \\
 \theta(\omega) &= -\frac{3\omega}{2} + \frac{\pi}{2}, \text{ and hence, a constant group delay.}
 \end{aligned}$$

$$\mathbf{3.79 (a)} \quad H_a(e^{j\omega}) = a + b e^{-j\omega} = a + b \cos \omega - j b \sin \omega. \text{ Thus, } \theta_{H_a}(\omega) = \tan^{-1} \left(\frac{-b \sin \omega}{a + b \cos \omega} \right).$$

$$\begin{aligned}
 \text{Hence, } \tau_{H_a}(\omega) &= -\frac{d\theta_{H_a}(\omega)}{d\omega} = -\frac{1}{1 + \left(\frac{-b \sin \omega}{a + b \cos \omega} \right)^2} \cdot \frac{d}{d\omega} \left(\frac{-b \sin \omega}{a + b \cos \omega} \right) \\
 &= -\frac{(a + b \cos \omega)^2}{(a + b \cos \omega)^2 + (b \sin \omega)^2} \cdot \frac{-(a + b \cos \omega)b \cos \omega - (-b \sin \omega)(-b \sin \omega)}{(a + b \cos \omega)^2} \\
 &= -\frac{-ab \cos \omega - b^2 \cos^2 \omega - b^2 \sin^2 \omega}{a^2 + b^2 \cos^2 \omega + 2ab \cos \omega + b^2 \sin^2 \omega} = \frac{b^2 + ab \cos \omega}{a^2 + b^2 + 2ab \cos \omega}.
 \end{aligned}$$

(b) Let $G_b(e^{j\omega}) = 1 + ce^{-j\omega} = 1 + c \cos \omega - jc \sin \omega$. From the results of Part (a) we have $\tau_{G_b}(\omega) = \frac{c^2 + c \cos \omega}{1 + c^2 + 2c \cos \omega}$. Since $H_b(e^{j\omega}) = \frac{1}{G_b(e^{j\omega})} = \frac{1}{1 + ce^{-j\omega}}$,

$$\theta_{H_b}(\omega) = -\theta_{G_b}(\omega), \text{ we have } \tau_{H_b}(\omega) = -\tau_{G_b}(\omega) = -\frac{c^2 + c \cos \omega}{1 + c^2 + 2c \cos \omega}.$$

(c) $H_c(e^{j\omega}) = \frac{a + be^{-j\omega}}{1 + ce^{-j\omega}} = H_a(e^{j\omega})H_b(e^{j\omega})$, where $H_a(e^{j\omega})$ is the frequency

response of Part (a) and $H_b(e^{j\omega})$ is the frequency response of Part (b). Thus, $\theta_{H_c}(\omega) = \theta_{H_a}(\omega) + \theta_{H_b}(\omega)$, and therefore, $\tau_{H_c}(\omega) = \tau_{H_a}(\omega) + \tau_{H_b}(\omega)$

$$= \frac{b^2 + ab \cos \omega}{a^2 + b^2 + 2ab \cos \omega} - \frac{c^2 + c \cos \omega}{1 + c^2 + 2c \cos \omega}.$$

(d) $H_d(e^{j\omega}) = \frac{1}{1 + ce^{-j\omega}} \cdot \frac{1}{1 + de^{-j\omega}} = H_b(e^{j\omega})H_e(e^{j\omega})$, where $H_b(e^{j\omega})$ is the

frequency response of Part (b) and $H_e(e^{j\omega})$ is similar in form to $H_b(e^{j\omega})$. Thus, $\theta_{H_d}(\omega) = \theta_{H_b}(\omega) + \theta_{H_e}(\omega)$, and therefore, $\tau_{H_d}(\omega) = \tau_{H_b}(\omega) + \tau_{H_e}(\omega)$

$$= \frac{c^2 + c \cos \omega}{1 + c^2 + 2c \cos \omega} + \frac{d^2 + d \cos \omega}{1 + d^2 + 2d \cos \omega}.$$

3.80 The group delay of a causal LTI discrete-time system with a frequency response

$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$ is given by $\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$. Now,

$$\frac{dH(e^{j\omega})}{d\omega} = e^{j\theta(\omega)} \frac{d|H(e^{j\omega})|}{d\omega} + j|H(e^{j\omega})|e^{j\theta(\omega)} \frac{d\theta(\omega)}{d\omega}. \text{ Hence,}$$

$$-j|H(e^{j\omega})|e^{j\theta(\omega)} \frac{d\theta(\omega)}{d\omega} = e^{j\theta(\omega)} \frac{d|H(e^{j\omega})|}{d\omega} - \frac{dH(e^{j\omega})}{d\omega}, \text{ or, equivalently,}$$

$$\frac{d\theta(\omega)}{d\omega} = \frac{e^{j\theta(\omega)}}{j|H(e^{j\omega})|e^{j\theta(\omega)}} \cdot \frac{d|H(e^{j\omega})|}{d\omega} - \frac{1}{j|H(e^{j\omega})|e^{j\theta(\omega)}} \cdot \frac{dH(e^{j\omega})}{d\omega}$$

$$= \frac{1}{j|H(e^{j\omega})|} \cdot \frac{d|H(e^{j\omega})|}{d\omega} + j \frac{1}{H(e^{j\omega})} \cdot \frac{dH(e^{j\omega})}{d\omega}. \text{ The first term on the right-}$$

hand side of the above equation is purely imaginary. Hence,

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = \operatorname{Re} \left\{ \frac{j \frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})} \right\}.$$

3.81 Since $G(e^{j\omega})$ is the Fourier transform of $ng[n]$, $G(e^{j\omega}) = j \frac{dH(e^{j\omega})}{d\omega}$. Rewriting Eq.

$$\begin{aligned} (3.127) \text{ we get } \tau_g(\omega) &= \frac{1}{2} \left\{ \frac{j \frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})} + \left(\frac{j \frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})} \right)^* \right\} = \frac{1}{2} \left\{ \frac{j \frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})} + \frac{\left(j \frac{dH(e^{j\omega})}{d\omega} \right)^*}{H^*(e^{j\omega})} \right\} \\ &= \frac{1}{2} \left\{ \frac{j \frac{dH(e^{j\omega})}{d\omega} H^*(e^{j\omega}) + \left(j \frac{dH(e^{j\omega})}{d\omega} \right)^* H(e^{j\omega})}{|H(e^{j\omega})|^2} \right\} \\ &= \frac{1}{2|H(e^{j\omega})|^2} \left[\left(G_{re}(e^{j\omega}) + j G_{im}(e^{j\omega}) \right) \left(H_{re}(e^{j\omega}) + j H_{im}(e^{j\omega}) \right)^* \right. \\ &\quad \left. + \left(G_{re}(e^{j\omega}) + j G_{im}(e^{j\omega}) \right)^* \left(H_{re}(e^{j\omega}) + j H_{im}(e^{j\omega}) \right) \right] \\ &= \frac{1}{2|H(e^{j\omega})|^2} \left[2G_{re}(e^{j\omega})H_{re}(e^{j\omega}) + 2G_{im}(e^{j\omega})H_{im}(e^{j\omega}) \right] \\ &= \frac{G_{re}(e^{j\omega})H_{re}(e^{j\omega}) + G_{im}(e^{j\omega})H_{im}(e^{j\omega})}{|H(e^{j\omega})|^2}. \end{aligned}$$

3.82 (a) $H_a(e^{j\omega}) = 1 + 0.4e^{-j\omega} = 1 + 0.4\cos\omega - j0.4\sin\omega$ and thus,

$$\operatorname{Re}\{H_a(e^{j\omega})\} = 1 + 0.4\cos\omega \text{ and } \operatorname{Im}\{H_a(e^{j\omega})\} = -0.4\sin\omega.$$

$$G_a(e^{j\omega}) = j \frac{dH_a(e^{j\omega})}{d\omega} = j(-j0.4e^{-j\omega}) = 0.4\cos\omega - j0.4\sin\omega. \text{ Thus,}$$

$\operatorname{Re}\{G_a(e^{j\omega})\} = 0.4\cos\omega$ and $\operatorname{Im}\{G_a(e^{j\omega})\} = -0.4\sin\omega$. Therefore, using Eq. (3.128) we

$$\begin{aligned} \text{get } \tau_{H_a}(\omega) &= \frac{(1 + 0.4\cos\omega)(0.4\cos\omega) + (-0.4\sin\omega)^2}{(1 + 0.4\cos\omega)^2 + (-0.4\sin\omega)^2} \\ &= \frac{0.16\cos^2\omega + 0.16\sin^2\omega + 0.4\cos\omega}{1 + 0.16\cos^2\omega + 0.16\sin^2\omega + 0.8\cos\omega} = \frac{0.16 + 0.4\cos\omega}{1.16 + 0.8\cos\omega}. \end{aligned}$$

(b) Let $G_b(e^{j\omega}) = \frac{1}{H_b(e^{j\omega})} = 1 + 0.6e^{-j\omega}$. Then $\tau_{G_b}(\omega) = -\tau_{H_b}(\omega)$. Then using the

same procedure as in Part (a) we get $\tau_{G_b}(\omega) = \frac{0.36 + 0.6 \cos \omega}{1.36 + 1.2 \cos \omega}$. Therefore,

$$\tau_{H_b}(\omega) = -\frac{0.36 + 0.6 \cos \omega}{1.36 + 1.2 \cos \omega}.$$

(c) Let $H_c(e^{j\omega}) = (1 - 0.5e^{-j\omega}) \left(\frac{1}{1 + 0.3e^{-j\omega}} \right) = G_a(e^{j\omega})G_b(e^{j\omega})$, where

$G_a(e^{j\omega}) = 1 - 0.5e^{-j\omega}$ and $G_b(e^{j\omega}) = \frac{1}{1 + 0.3e^{-j\omega}}$. Therefore,

$\tau_{H_c}(\omega) = \tau_{G_a}(\omega) + \tau_{G_b}(\omega)$. Then using the same procedure as in Part (a) we get

$\tau_{G_a}(\omega) = \frac{0.25 - 0.5 \cos \omega}{1.25 - \cos \omega}$ and using the same procedure as in Part (b) we get

$\tau_{G_b}(\omega) = -\frac{0.09 + 0.3 \cos \omega}{1.09 + 0.6 \cos \omega}$. Hence, $\tau_{H_c}(\omega) = \frac{0.25 - 0.5 \cos \omega}{1.25 - \cos \omega} - \frac{0.09 + 0.3 \cos \omega}{1.09 + 0.6 \cos \omega}$.

(d) Let $H_d(e^{j\omega}) = \left(\frac{1}{1 - 0.3e^{-j\omega}} \right) \left(\frac{1}{1 + 0.5e^{-j\omega}} \right) = G_a(e^{j\omega})G_b(e^{j\omega})$, where

$G_a(e^{j\omega}) = \frac{1}{1 - 0.3e^{-j\omega}}$ and $G_b(e^{j\omega}) = \frac{1}{1 + 0.5e^{-j\omega}}$. Therefore,

$\tau_{H_d}(\omega) = \tau_{G_a}(\omega) + \tau_{G_b}(\omega)$. Then using the same procedure as in Part (b) we get

$\tau_{G_a}(\omega) = \frac{-0.09 + 0.3 \cos \omega}{1.09 - 0.6 \cos \omega}$ and $\tau_{G_b}(\omega) = \frac{-0.25 - 0.5 \cos \omega}{1.25 + \cos \omega}$. Hence,

$$\tau_{H_d}(\omega) = -\left(\frac{0.09 - 0.3 \cos \omega}{1.09 - 0.6 \cos \omega} + \frac{0.25 + 0.5 \cos \omega}{1.25 + \cos \omega} \right).$$

3.83 From Table 3.3, $H(e^{j\omega}) = \frac{1}{1 + 0.5e^{-j\omega}} = \frac{1}{1 + 0.5 \cos \omega - j0.5 \sin \omega}$. Thus,

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1 + 0.5 \cos \omega)^2 + (0.5 \sin \omega)^2}} = \frac{1}{\sqrt{1.25 + \cos \omega}} \text{ and}$$

$\arg\{H(e^{j\omega})\} = \theta(\omega) = -\tan^{-1}\left(\frac{-0.5 \sin \omega}{1 + 0.5 \cos \omega}\right) = \tan^{-1}\left(\frac{0.5 \sin \omega}{1 + 0.5 \cos \omega}\right)$. Now

$H(e^{j\pi/5}) = \frac{1}{1 + 0.5 \cos(\pi/5) - j0.5 \sin(\pi/5)} = 0.6821 + j 0.1427$. Therefore,

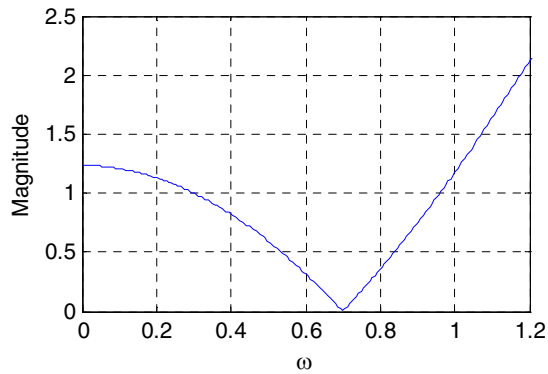
$$\left|H(e^{j\pi/5})\right| = 0.6969 \text{ and } \arg\{H(e^{j\pi/5})\} = \theta(\pi/5) = \tan^{-1}\left(\frac{0.1427}{0.6821}\right) = 0.2063 \text{ radians.}$$

Since for a frequency response with real coefficient impulse response, $\left|H(e^{j\omega})\right|$ is an even function of ω and $\theta(\omega)$ is an odd function of ω , we have $\left|H(e^{-j\pi/5})\right| = 0.6969$ and $\theta(-\pi/5) = -0.2063$.

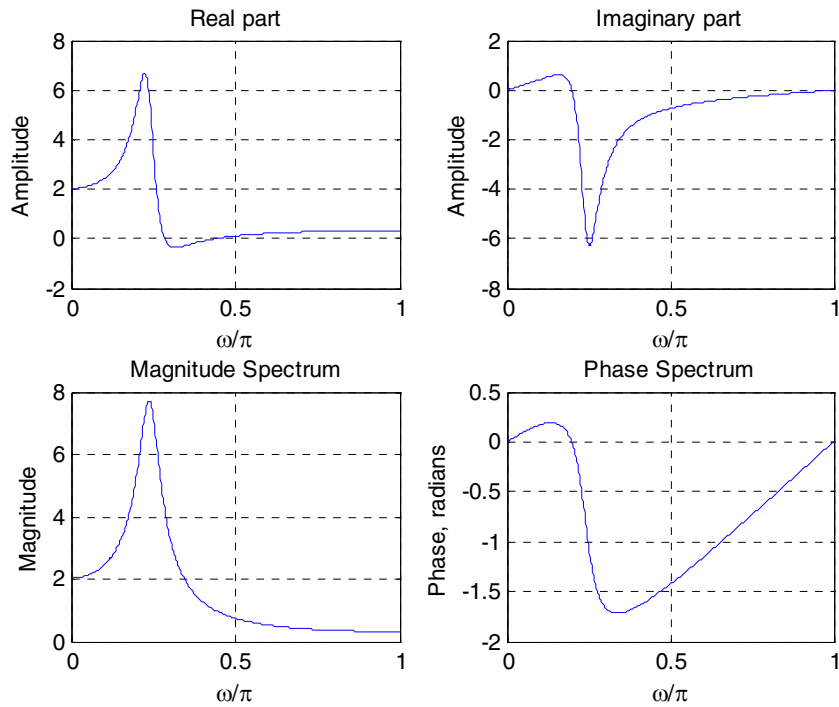
Now, for an input $x[n] = \sin(\omega_o n)\mu[n]$, the steady-state output is given by

$$y[n] = \left|H(e^{j\omega_o})\right| \sin(\omega_o n + \theta(\omega_o)). \text{ Thus, for } \omega_o = \pi/5, \text{ the steady-state output is given by } y[n] = \left|H(e^{j\pi/5})\right| \sin\left(\frac{\pi}{5}n + \theta(\pi/5)\right) = 0.6969 \sin\left(\frac{\pi}{5}n + 0.2063\right).$$

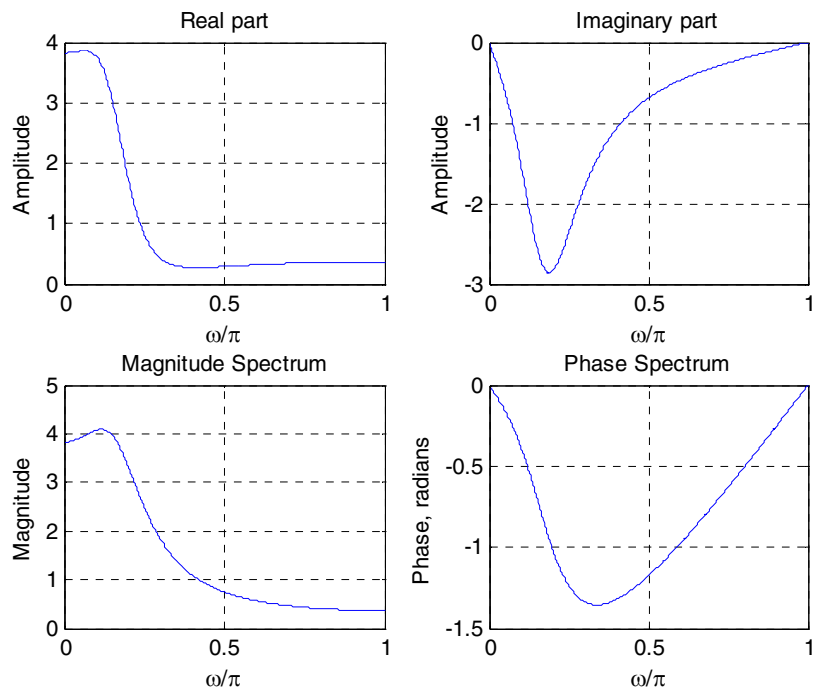
3.84 $H(e^{j\omega}) = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega} = e^{-j\omega}(2h[0]\cos\omega + h[1])$. We require $2h[0]\cos(0.3) + h[1] = 1$ and $2h[0]\cos(0.7) + h[1] = 0$. Solving these two equations we get $h[0] = 2.6248$ and $h[1] = -4.015$.



M3.1 $r = 0.9, \theta = 0.75$



$r = 0.7, \theta = 0.5$



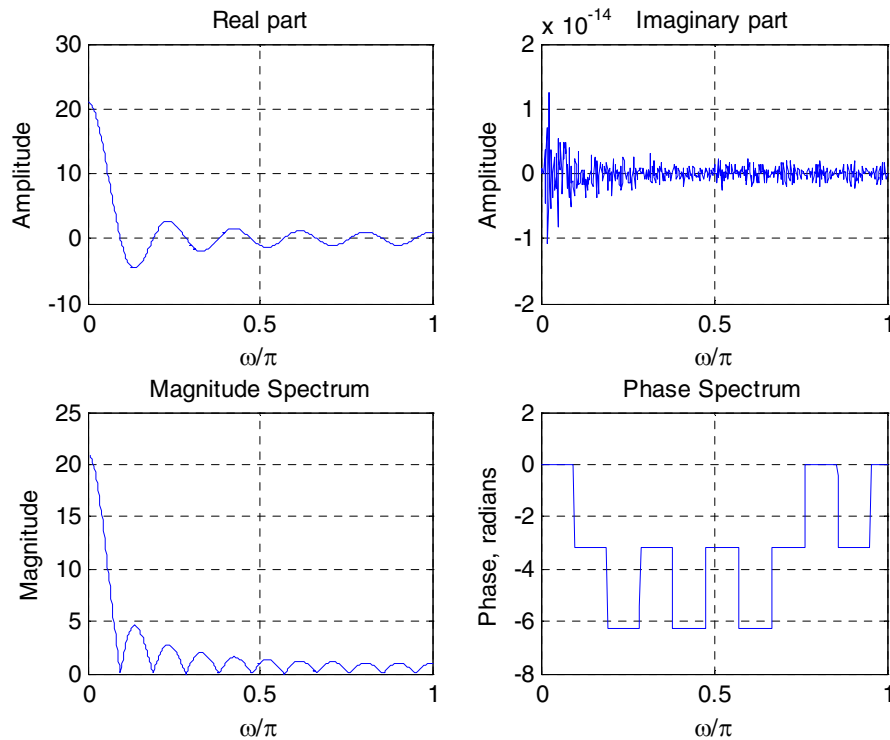
M3.2 It should be noted that Program 3_1.m uses the function `freqz` to determine the samples of a DTFT that is rational function in $e^{-j\omega}$, i.e., a ratio of polynomials in $e^{-j\omega}$. Their inverse DTFTs are two-sided sequences. However, all sequences of

Problem 3.19 except that in Part (b) are two-sided finite-length sequences of length $2N + 1$, and their DTFTs have both positive and negative powers of $e^{j\omega}$. As a result, the frequency sample computed using `freqz` should be multiplied by the vector $e^{j\omega_n N}$ evaluated at the frequency points ω_n used in `freqz`. In Parts (a), (c) and (d), the phase spectra are the plots of the unwrapped phase obtained using the function `unwrap`.

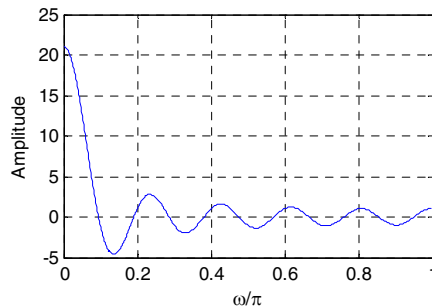
Moreover, the DTFTs of the sequences in Parts (a), (c) and (d) are real functions of ω and thus have zero phase. More accurate plots of the DTFTs are obtained using the function `zerophase`.

(a) $y_1[n] = \begin{cases} 1, & -10 \leq n \leq 10, \\ 0, & \text{otherwise,} \end{cases} \quad Y_1(e^{j\omega}) = \frac{\sin(21\omega/2)}{\sin(\omega/2)}$. The plots obtained using

Program 3_1.m are shown below:

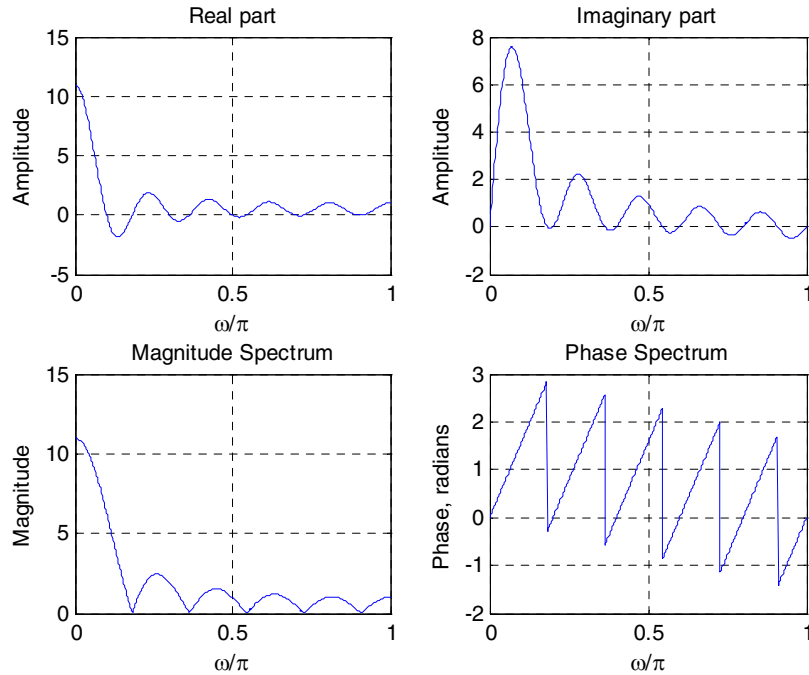


The plot obtained using the function `zerophase` is shown below:



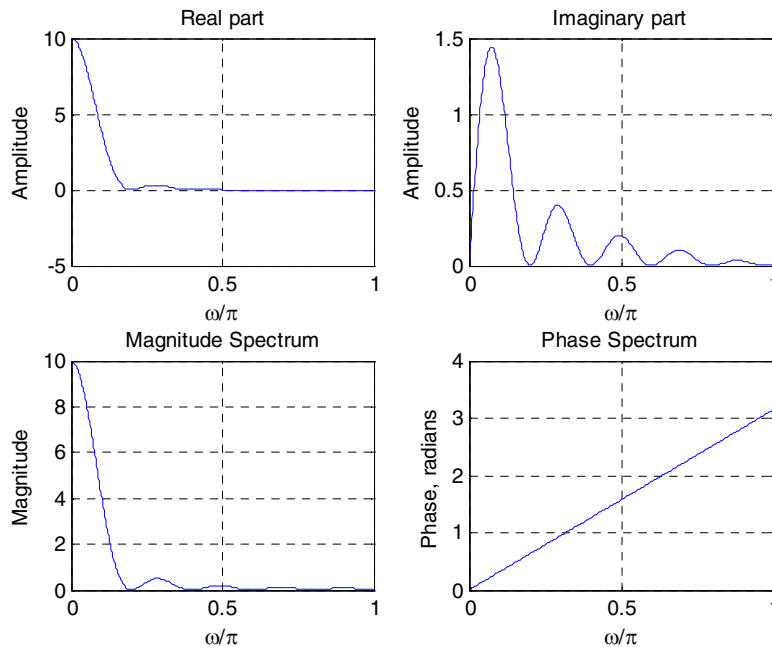
(b) $y_2[n] = \begin{cases} 1, & 0 \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Then $Y_2(e^{j\omega}) = e^{-j\omega N/2} \left(\frac{\sin(\omega[N+1]/2)}{\sin(\omega/2)} \right)$. The plots

obtained using Program 3_1.m are shown below:

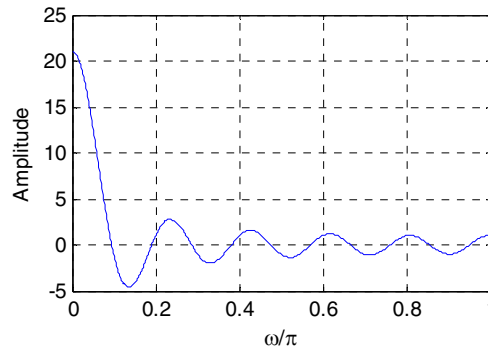


(c) $y_3[n] = \begin{cases} 1 - \frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ $Y_3(e^{j\omega}) = \frac{1}{N} \cdot \frac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$. The plots obtained

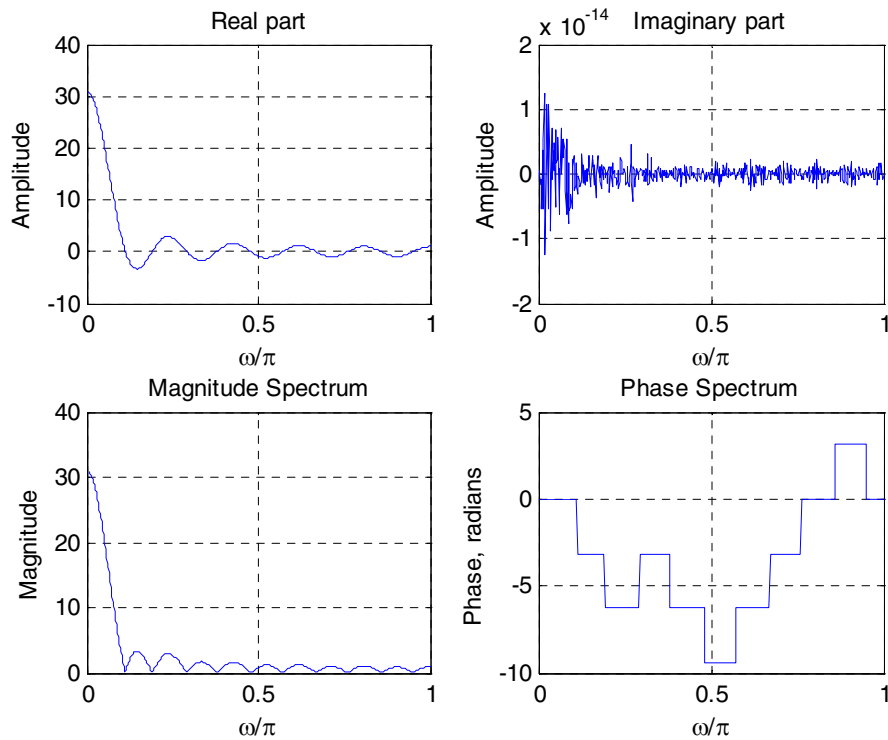
using Program 3_1.m are shown below:



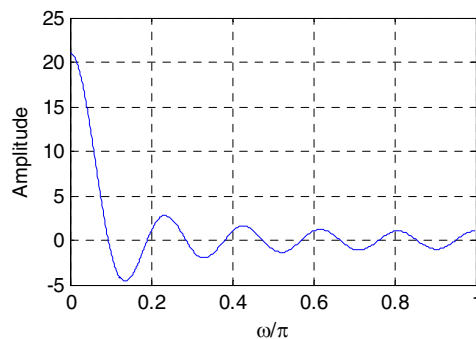
The plot obtained using the function zerophase is shown below:



$$(d) \quad y_4[n] = \begin{cases} N + 1 - \frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases} \quad Y_4(e^{j\omega}) = N \cdot \frac{\sin\left(\omega\left[N + \frac{1}{2}\right]\right)}{\sin(\omega/2)} + \frac{1}{N} \cdot \frac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$$



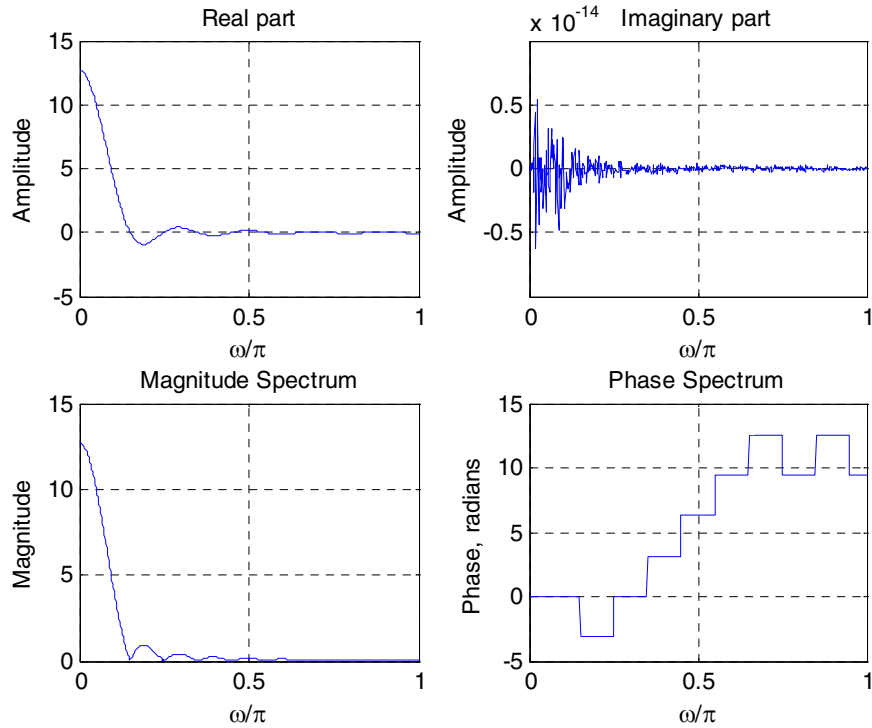
The plot obtained using the function zerophase is shown below:



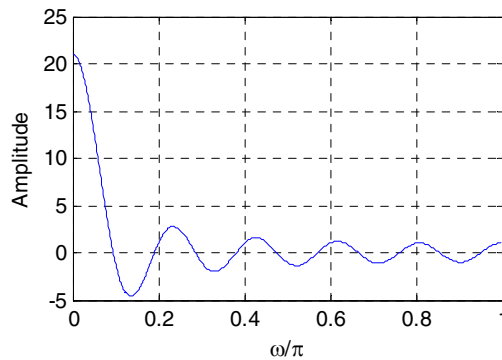
(e) $y_5[n] = \begin{cases} \cos(\pi n/2N), & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$

$$Y_5(e^{j\omega}) = \frac{1}{2} \cdot \frac{\sin\left((\omega - \frac{\pi}{2N})(N + \frac{1}{2})\right)}{\sin\left((\omega - \frac{\pi}{2N})/2\right)} + \frac{1}{2} \cdot \frac{\sin\left((\omega + \frac{\pi}{2N})(N + \frac{1}{2})\right)}{\sin\left((\omega + \frac{\pi}{2N})/2\right)}. \quad \text{The plots obtained using}$$

Program 3_1.m are shown below:

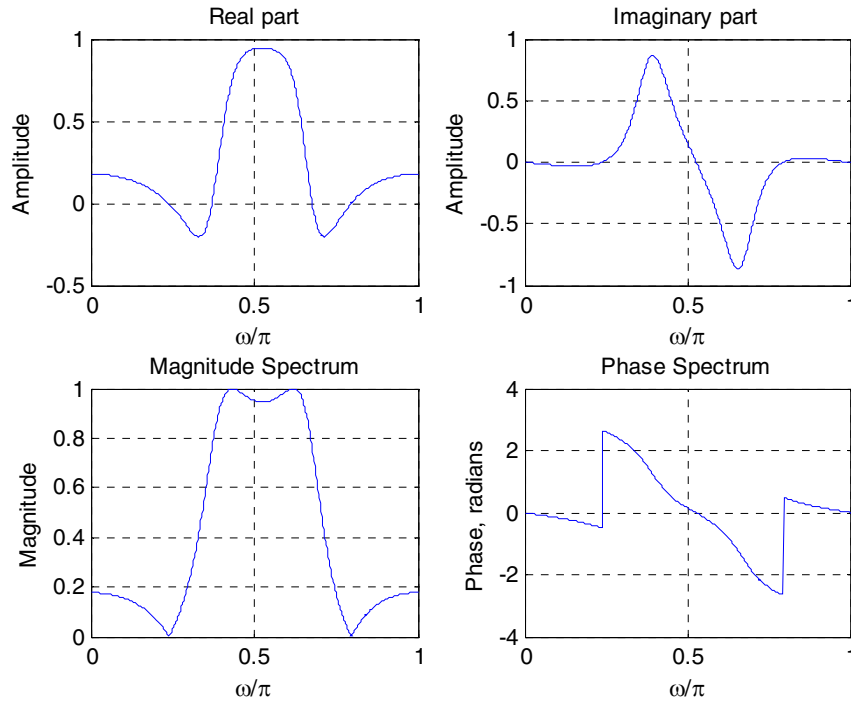


The plot obtained using the function `zerophase` is shown below:



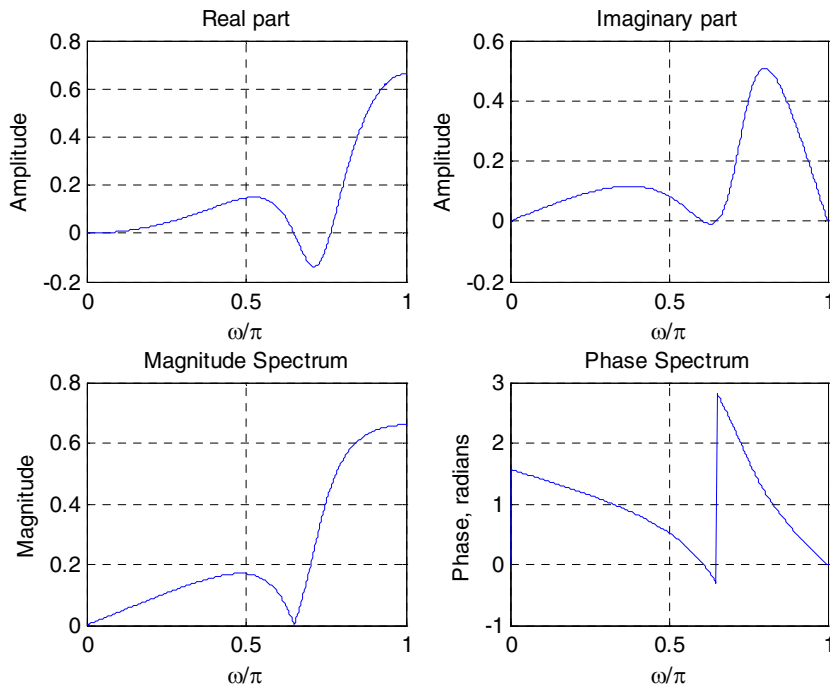
M3.3 (a) $X(e^{j\omega}) = \frac{0.2418(1 + 0.139e^{-j\omega} - 0.3519e^{-j2\omega} + 0.139e^{-j3\omega} + e^{-j4\omega})}{1 + 0.2386e^{-j\omega} + 0.8258e^{-j2\omega} + 0.1393e^{-j3\omega} + 0.4153e^{-j4\omega}}$

The plots obtained using Program 3_1.m are shown below:



$$(b) X(e^{j\omega}) = \frac{0.1397(1 - 0.0911e^{-j\omega} + 0.0911e^{-j2\omega} - e^{-j3\omega})}{1 + 1.1454e^{-j\omega} + 0.7275e^{-j2\omega} + 0.1205e^{-j3\omega}}$$

The plots obtained using Program 3_1.m are shown below:



M3.4 % Property 1

```

N = 8; % Number of samples in sequence
gamma = 0.5; k = 0:N-1;
x = exp(-j*gamma*k); y = exp(-j*gamma*fliplr(k));
% r = x[-n] then y = r[n-(N-1)]
% so if X1(exp(jw)) is DTFT of x[-n], then
% X1(exp(jw)) = R(exp(jw)) = exp(jw(N-1))Y(exp(jw))
[Y,w] = freqz(y,1,512);
X1 = exp(j*w*(N-1)).*Y;
m = 0:511; w = -pi*m/512;
X = freqz(x,1,w);
% Verify X = X1

% Property 2
k = 0:N-1; y = exp(j*gamma*fliplr(k));
[Y,w] = freqz(y,1,512);
X1 = exp(j*w*(N-1)).*Y;
[X,w] = freqz(x,1,512);
% Verify X1 = conj(X)

% Property 3
y = real(x);
[Y3,w] = freqz(y,1,512);
m = 0:511; w0 = -pi*m/512;
X1 = freqz(x,1,w0);
[X,w] = freqz(x,1,512);
% Verify Y3 = 0.5*(X+conj(X1))

% Property 4
y = j*imag(x); [Y4,w] = freqz(y,1,512);
% Verify Y4 = 0.5*(X-conj(X1))

% Property 5
k = 0:N-1; y = exp(-j*gamma*fliplr(k));
xcs = 0.5*[zeros(1,N-1) x] + 0.5*[conj(y) zeros(1,N-1)];
xacs = 0.5*[zeros(1,N-1) x] - 0.5*[conj(y) zeros(1,N-1)];
[Y5,w] = freqz(xcs,1,512);
[Y6,w] = freqz(xacs,1,512);
Y5 = Y5.*exp(j*w*(N-1));
Y6 = Y6.*exp(j*w*(N-1));
% Verify Y5 = real(X) and Y6 = j*imag(X)

```

M3.5

```

N = 8; % Number of samples in sequence
gamma = 0.5; k = 0:N-1;
x = exp(gamma*k); y = exp(gamma*fliplr(k));
xev = 0.5*([zeros(1,N-1) x] + [y zeros(1,N-1)]);
xod = 0.5*([zeros(1,N-1) x] - [y zeros(1,N-1)]);
[X,w] = freqz(x,1,512);
[Xev,w] = freqz(xev,1,512);
[Xod,w] = freqz(xod,1,512);
Xev = exp(j*w*(N-1)).*Xev;

```

```

Xod = exp(j*w*(N-1)).*Xod;
% Verify real(X) = Xev, and j*imag(X) = Xod

M3.5 N = input('The length of the sequence = ');
k = 0:N-1; gamma = -0.5;
g = exp(gamma*k);
% g is an exponential sequence
h = sin(2*pi*k/(N/2));
% h is a sinusoidal sequence with period = N/2
[G,w] = freqz(g,1,512); [H,w] = freqz(h,1,512);

% Property 1
alpha = 0.5; beta = 0.25;
y = alpha*g+beta*h; [Y,w] =
freqz(y,1,512);
% Plot Y and alpha*G+beta*H to verify that they are equal

% Property 2
n0 = 12; % Sequence shifted by 12 samples
y2 = [zeros(1,n0) g];
[Y2,w] = freqz(y2,1,512);
G0 = exp(-j*w*n0).*G;
% Plot G0 and Y2 to verify they are equal

% Property 3
w0 = pi/2; % the value of omega0 = pi/2
r = 256; % the value of omega0 in terms of number of
samples
k = 0:N-1; y3 = g.*exp(j*w0*k);
[Y3,w] = freqz(y3,1,512);
k = 0:511;
w = -w0+pi*k/512; % creating G(exp(w-w0))
G1 = freqz(g,1,w);
% Compare G1 and Y3

% Property 4
k = 0:N-1; y4 = k.*g;
[Y4,w] = freqz(y4,1,512);
% To compute derivative we need sample at pi
y0 = ((-1).^k).*g;
G2 = [G(2:512)' sum(y0)]';
delG = (G2-G)*512/pi;
% Compare Y4, delG

% Property 5
y5 = conv(g,h);
[Y5,w] = freqz(y5,1,512);
% Compare Y5 and G.*H

% Property 6

```

```

y6 = g.*h;
[Y6,w] = freqz(y6,1,512,'whole');
[G0,w] = freqz(g,1,512,'whole');
[H0,w] = freqz(h,1,512,'whole');
% Evaluate the sample value at w = pi/2
% and verify with Y6 at pi/2
H1 = [fliplr(H0(1:129)') fliplr(H0(130:512)')]' ;
val = 1/(512)*sum(G0.*H1);
% Compare val with Y6(129) i.e., sample at pi/2
% Can extend this to other points similarly

% Parsevals theorem
val1 = sum(g.*conj(h)); val2 = sum(G0.*conj(H0))/512;
% Compare val1 with val2

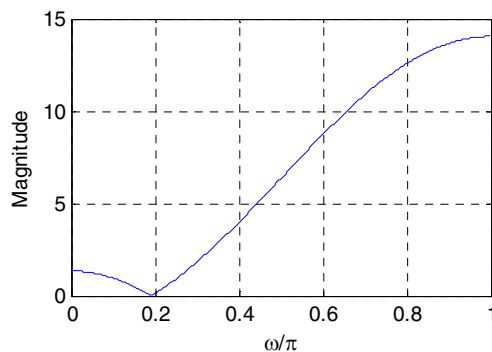
```

M3.7 The DTFT of $nh[n]$ is $j \frac{dH(e^{j\omega})}{d\omega}$. Hence, the group delay $\tau_g(\omega)$ can be computed at a set of N discrete frequency points $\omega_k = 2\pi k / N$, $0 \leq k \leq N - 1$, as follows:

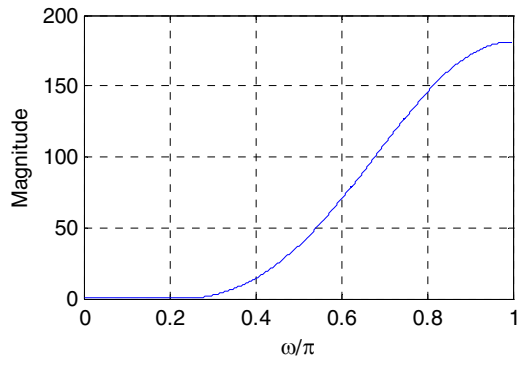
$$\tau_g(\omega_k) = \text{Re} \left(\frac{DFT\{nh[n]\}}{DFT\{h[n]\}} \right),$$

where all DFTs are N -points in length with N greater than or equal to the length of $\{h[n]\}$.

M3.8 `h = [3.8461 -6.3487 3.8461];`
`[H,w] = freqz(h,1,512);`
`plot(w/pi,abs(H)); grid`
`xlabel('\omega/\pi');` `ylabel('Magnitude');`



M3.9 `h = [-13.4866 45.228 -63.8089 45.228 -13.4866];`
`[H,w] = freqz(h,1,512);`
`plot(w/pi,abs(H)); grid`
`xlabel('\omega/\pi');` `ylabel('Magnitude');`



Chapter 4

4.1 Let $\phi(t)$ be an arbitrary continuous-time function with a CTFT $\Phi(j\Omega)$, where

$\Phi(j\Omega) = \int_{-\infty}^{\infty} \phi(t) e^{-j\Omega t} dt$. Let $\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT)$ denote the periodic continuous-time function with a period T obtained by a periodic extension of $\phi(t)$. Note that $\tilde{\phi}_T(t)$ is also given by the convolution of $\phi(t)$ with the periodic impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t + nT), \text{ i.e., } \tilde{\phi}_T(t) = \int_{-\infty}^{\infty} \phi(\tau) p(t - \tau) d\tau.$$

The CTFT of $\tilde{\phi}_T(t)$ is then given by $\mathcal{F}\{\tilde{\phi}_T(t)\} = \Phi(j\Omega) \cdot \mathcal{F}\{p(t)\}$

$$= \Phi(j\Omega) \cdot \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(j(\Omega - n\Omega_T)) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \Phi(jn\Omega_T) \delta(j(\Omega - n\Omega_T)), \quad (4-1)$$

where $\Omega_T = \frac{2\pi}{T}$.

Now, the Fourier series expansion of $\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT)$ is given by

$$\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\Omega_T t}. \text{ A CTFT of both sides of this equation is then}$$

$$\mathcal{F}\{\tilde{\phi}_T(t)\} = \sum_{n=-\infty}^{\infty} a_n \cdot 2\pi \delta(j(\Omega - n\Omega_T)). \quad (4-2)$$

Comparing Eqs. (4-1) and (4-2) we arrive at $a_n = \frac{1}{T} \Phi(jn\Omega_T)$. Substituting this expression in the Fourier expansion of $\tilde{\phi}_T(t)$ we therefore arrive at the Poisson's sum

$$\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \Phi(jn\Omega_T) e^{jn\Omega_T t}.$$

4.2 Consider the continuous-time signal $g_a(t) = \sin(\Omega_m t)$ which is bandlimited to Ω_m . If we sample $g_a(t)$ at a rate $\Omega_T = 2\Omega_m$ starting at $t = 0$, then all its samples are zero. Hence, $g_a(t)$ cannot be recovered from its samples obtained by sampling it at the Nyquist rate $\Omega_T = 2\Omega_m$. As a result, $g_a(t) = \sin(\Omega_m t)$ must be sampled at a rate $\Omega_T > 2\Omega_m$ to recover it fully from its samples.

4.3 (a) Now, the CTFT of $y_1(t)$ is given by $Y_1(j\Omega) = \frac{1}{2\pi} G_a(j\Omega) \circledast G_a(j\Omega)$ where $G_a(j\Omega)$ denotes the CTFT of $g_a(t)$ and \circledast denotes the frequency-domain convolution. The highest frequency present in $y_1(t)$ is therefore twice that of $g_a(t)$ and hence, the Nyquist frequency of $y_1(t)$ is $2\Omega_m$.

(b) The CTFT of $y_2(t)$ is given by $Y_2(j\Omega) = \int_{-\infty}^{\infty} g_a\left(\frac{t}{3}\right)e^{-j\Omega t} dt$
 $= 3 \int_{-\infty}^{\infty} g_a(\tau)e^{-j3\Omega\tau} d\tau = 3G_a(j3\Omega)$. The highest frequency present in $y_2(t)$ is therefore one-third of that of $g_a(t)$ and hence, the Nyquist frequency of $y_2(t)$ is $\Omega_m/3$.

(c) The CTFT of $y_3(t)$ is given by $Y_3(j\Omega) = \int_{-\infty}^{\infty} g_a(3t)e^{-j\Omega t} dt$
 $= \frac{1}{3} \int_{-\infty}^{\infty} g_a(\tau)e^{-j\Omega\tau/3} d\tau = \frac{1}{3}G_a(j\frac{\Omega}{3})$. The highest frequency present in $y_3(t)$ is therefore three times of that of $g_a(t)$ and hence, the Nyquist frequency of $y_3(t)$ is $3\Omega_m$.

(d) The CTFT of $y_4(t)$ is given by

$$Y_4(j\Omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g_a(t-\tau)g_a(\tau)d\tau \right] e^{-j\Omega t} dt = \int_{-\infty}^{\infty} g_a(\tau) \left[\int_{-\infty}^{\infty} g_a(t-\tau)e^{-j\Omega t} dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} g_a(\tau)e^{-j\Omega\tau} G_a(j\Omega) d\tau = G_a(j\Omega) \int_{-\infty}^{\infty} g_a(\tau)e^{-j\Omega\tau} d\tau = G_a(j\Omega)G_a(j\Omega)$$
. The highest frequency present in $y_4(t)$ is therefore the same as that of $g_a(t)$ and hence, the Nyquist frequency of $y_4(t)$ is Ω_m .

(e) Now $g_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_a(j\Omega)e^{j\Omega t} d\Omega$. Differentiating both sides of this equation we get $\frac{dg_a(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\Omega G_a(j\Omega)e^{j\Omega t} d\Omega$. Hence, it follows that the CTFT of $y_5(t) = \frac{dg_a(t)}{dt}$ is simply $j\Omega G_a(j\Omega)$. The highest frequency present in $y_5(t)$ is therefore the same as that of $g_a(t)$ and hence, the Nyquist frequency of $y_5(t)$ is Ω_m .

4.4 By Parseval's relation, the total energy of $g_a(t)$ is given by

$$\mathcal{E}_{g_a}(t) = \int_{-\infty}^{\infty} |g_a(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_a(j\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{-\Omega_m}^{\Omega_m} |G_a(j\Omega)|^2 d\Omega$$
. Likewise, the

$$\text{total energy of } g[n] \text{ is given by } \mathcal{E}_{g[n]} = \sum_{-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left| \frac{1}{T} G_a(j\Omega) \right|^2 d(\Omega T) = \frac{1}{2\pi T} \int_{-\pi/T}^{\pi/T} |G_a(j\Omega)|^2 d\Omega = \frac{1}{2\pi T} \int_{-\Omega_m}^{\Omega_m} |G_a(j\Omega)|^2 d\Omega \\
&= \frac{1}{T} E_{g_a(t)}.
\end{aligned}$$

4.5 Sampling period $T = \frac{2.5}{5000} = \text{sec}$. Hence, the sampling frequency is

$$F_T = \frac{1}{T} = 2000 \text{ Hz.}$$

Therefore, the highest frequency component that could be present in the continuous-time signal has a frequency $\frac{20000}{2} = 10000 \text{ Hz}$.

4.6 Since the continuous-time signal $x_a(t)$ is being sampled at 2 kHz rate, the sampled version of its i -th sinusoidal component with a frequency F_i will generate discrete-time sinusoidal signals with frequencies $F_i \pm 2000n$, $-\infty < n < \infty$. Hence, the frequencies F_{im} generated in the sampled version associated with the sinusoidal components present in are as follows:

$$F_1 = 300 \text{ Hz} \Rightarrow F_{1m} = 300, 1700, 2300, \dots \text{ Hz}$$

$$F_2 = 500 \text{ Hz} \Rightarrow F_{2m} = 500, 1500, 2500, \dots \text{ Hz}$$

$$F_3 = 1200 \text{ Hz} \Rightarrow F_{3m} = 1200, 800, 3200, \dots \text{ Hz}$$

$$F_4 = 2150 \text{ Hz} \Rightarrow F_{4m} = 2150, 150, 4150, \dots \text{ Hz}$$

$$F_5 = 3500 \text{ Hz} \Rightarrow F_{5m} = 3500, 1500, 5500, 500, 7500, \dots \text{ Hz}$$

After filtering by a lowpass filter with a cutoff at 900 Hz, the frequencies of the sinusoidal components in $y_a(t)$ are 150, 300, 500, 800 Hz.

4.7 One possible set of values for the frequencies present in $y_a(t)$ are: $F_1 = 350 \text{ Hz}$, $F_2 = 575 \text{ Hz}$, $F_3 = 815 \text{ Hz}$, and $F_4 = 9650 \text{ Hz}$. Another possible set of values for the frequencies present in $y_a(t)$ are: $F_1 = 350 \text{ Hz}$, $F_2 = 575 \text{ Hz}$, $F_3 = 815 \text{ Hz}$, and $F_4 = 10575 \text{ Hz}$. Hence, the solution is not unique.

4.8 $t = nT = \frac{n}{50}$. Therefore,

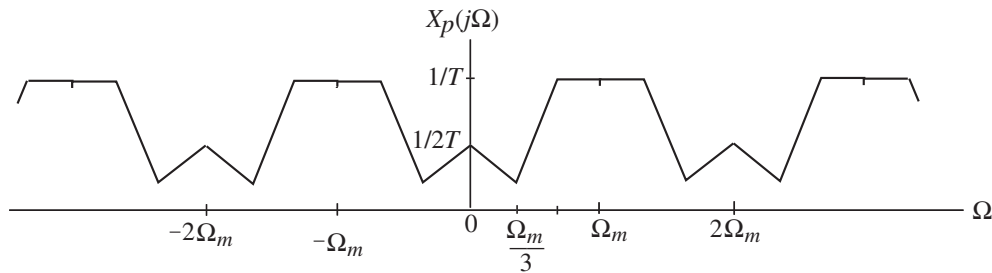
$$\begin{aligned}
x[n] &= 4 \sin\left(\frac{20\pi n}{50}\right) - 5 \cos\left(\frac{24\pi n}{50}\right) + 3 \sin\left(\frac{120\pi n}{50}\right) + 2 \cos\left(\frac{176\pi n}{50}\right) \\
&= 4 \sin\left(\frac{2\pi n}{5}\right) - 5 \cos\left(\frac{12\pi n}{25}\right) + 3 \sin\left(\frac{(10+2)\pi n}{5}\right) + 2 \cos\left(\frac{(100-12)\pi n}{25}\right) \\
&= 4 \sin\left(\frac{2\pi n}{5}\right) - 5 \cos\left(\frac{12\pi n}{25}\right) + 3 \sin\left(\frac{2\pi n}{5}\right) + 2 \cos\left(\frac{12\pi n}{25}\right).
\end{aligned}$$

4.9 Both channels are being sampled at 45 kHz. Therefore, there are a total of $2 \times 45000 = 90000$ samples/sec. Each sample is quantized using 12 bits. Hence, the total bit rate of the two channels after sampling and digitization is 108 kpbs.

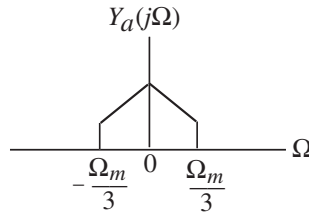
4.10 $h_r(t) = \frac{\sin(\Omega_c t)}{\Omega_T t/2}$. Therefore, $h_r(nT) = \frac{\sin(\Omega_c nT)}{\Omega_T nT/2}$. Since $T = 2\pi/\Omega_T$, we have

$$h_r(nT) = \frac{\sin\left(\frac{2\pi\Omega_c n}{\Omega_T}\right)}{\pi n}. \text{ For } \Omega_c = \Omega_T/2, \text{ we thus have } h_r(nT) = \frac{\sin(\pi n)}{\pi n} = \delta[n].$$

4.11 The spectrum of the sampled signal is as shown below:



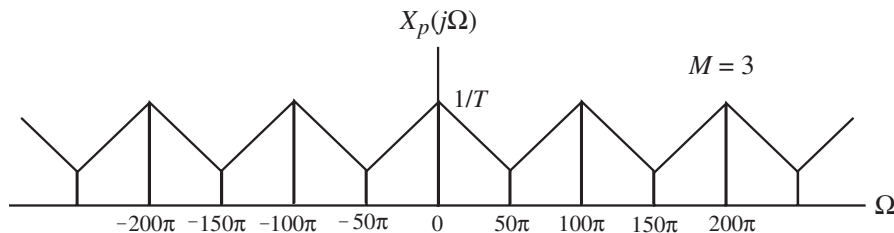
Now, $T = \frac{2\pi}{2\Omega_m} = \frac{\pi}{\Omega_m}$. As a result, $\omega_c = \frac{\Omega_m \pi}{3\Omega_m} = \frac{\pi}{3}$. Hence after lowpass filtering the spectrum of the output continuous-time signal $y_a(t)$ will be as shown below:

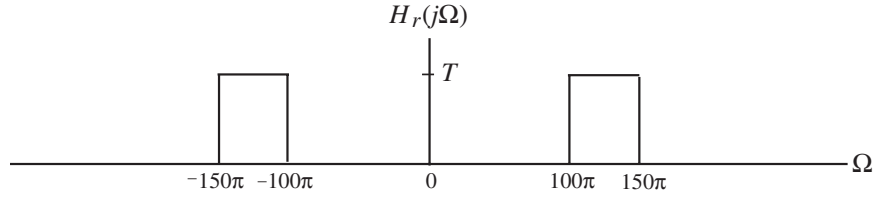


4.12 (a) $\Omega_1 = 100\pi$, $\Omega_2 = 150\pi$. Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 50\pi$. Note $\Delta\Omega$ is an integer multiple of Ω_2 . Hence, we choose the sampling angular frequency as

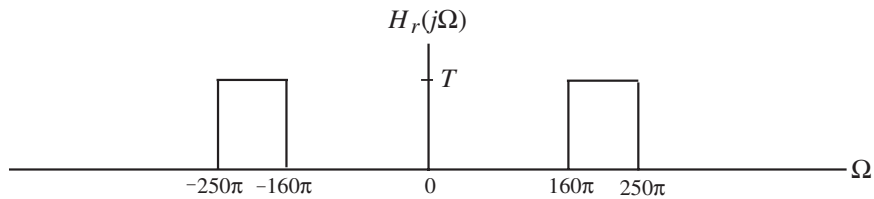
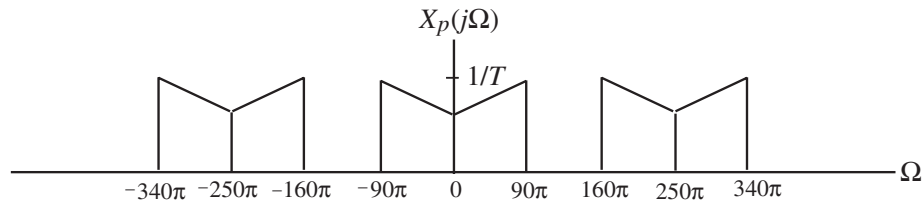
$$\Omega_T = 2\Delta\Omega = 100\pi = \frac{2 \times 150\pi}{M}, \text{ which is satisfied for } M = 3. \text{ The sampling}$$

frequency is therefore 50 Hz. The CTFT $X_p(j\Omega)$ of the sampled sequence and the frequency response $H_r(j\Omega)$ of the desired reconstruction filter are shown below.

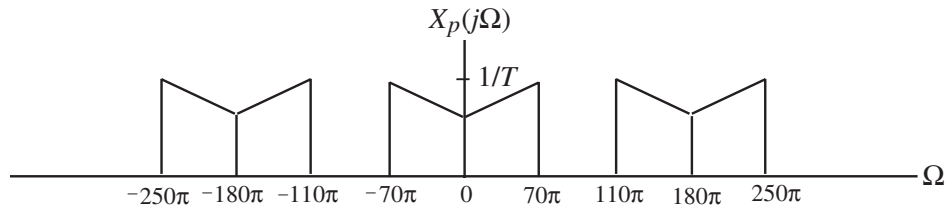


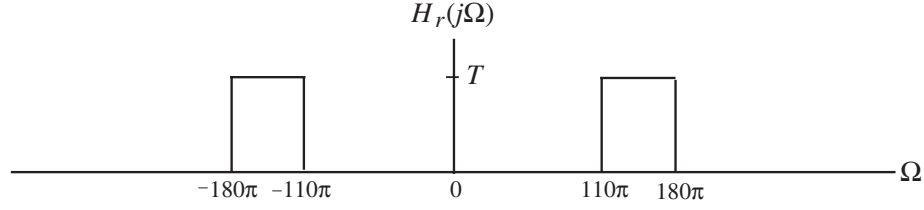


(b) $\Omega_1 = 160\pi$, $\Omega_2 = 250\pi$. Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 90\pi$. Note $\Delta\Omega$ is not an integer multiple of Ω_2 . Hence, we extend the bandwidth to the left by assuming the lowest frequency to be Ω_0 and choose the sampling angular frequency as $\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_0) = \frac{2 \times 250\pi}{M}$, which is satisfied for $\Omega_0 = 125\pi$ and $M = 2$. The sampling frequency is therefore 125 Hz. The CTFT $X_p(j\Omega)$ of the sampled sequence and the frequency response $H_r(j\Omega)$ of the desired reconstruction filter are shown below.



(c) $\Omega_1 = 110\pi$, $\Omega_2 = 180\pi$. Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 70\pi$. Note $\Delta\Omega$ is not an integer multiple of Ω_2 . Hence, we extend the bandwidth to the left by assuming the lowest frequency to be Ω_0 and choose the sampling angular frequency as $\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_0) = \frac{2 \times 180\pi}{M}$, which is satisfied for $\Omega_0 = 90\pi$ and $M = 2$. The sampling frequency is therefore 90 Hz. The CTFT $X_p(j\Omega)$ of the sampled sequence and the frequency response $H_r(j\Omega)$ of the desired reconstruction filter are shown below.





4.13 $\alpha_p = -20 \log_{10}(1 - \delta_p)$ dB and $\alpha_s = -20 \log_{10} \delta_s$ dB. Therefore,

$$\delta_p = 1 - 10^{-\alpha_p/20} \text{ and } \delta_s = 10^{-\alpha_s/20}.$$

(a) $\alpha_p = 0.21$ dB and $\alpha_s = 52$ dB. Hence, $\delta_p = 0.0239$ and $\delta_s = 0.025$.

(b) $\alpha_p = 0.03$ dB and $\alpha_s = 69$ dB. Hence, $\delta_p = 0.0034$ and $\delta_s = 0.00355$.

(c) $\alpha_p = 0.33$ dB and $\alpha_s = 57$ dB. Hence, $\delta_p = 0.0373$ and $\delta_s = 0.0014$.

4.14 $H_a(s) = \frac{a}{s+a}$. Thus, $H_a(j\Omega) = \frac{a}{j\Omega+a}$, and hence,

$$|H_a(j\Omega)|^2 = H_a(j\Omega)H_a(-j\Omega) = \frac{a}{j\Omega+a} \cdot \frac{a}{-j\Omega+a} = \frac{a^2}{\Omega^2+a^2}.$$

As Ω increases from 0 to ∞ , it can be seen that the square-magnitude function $|H_a(j\Omega)|^2$ and hence,

the magnitude function $|H_a(j\Omega)| = \frac{a}{\sqrt{\Omega^2+a^2}}$ decreases monotonically from

$|H_a(j0)| = 1$ to $|H_a(j\infty)| = 0$. Let Ω_c denote the 3-dB cutoff frequency. Then

$$|H_a(j\Omega_c)|^2 = \frac{a^2}{\Omega_c^2+a^2} = \frac{1}{2}, \text{ which implies } \Omega_c = a.$$

4.15 $G_a(s) = \frac{s}{s+a}$. Thus, $G_a(j\Omega) = \frac{j\Omega}{j\Omega+a}$, and hence,

$$|G_a(j\Omega)|^2 = G_a(j\Omega)G_a(-j\Omega) = \frac{j\Omega}{j\Omega+a} \cdot \frac{-j\Omega}{-j\Omega+a} = \frac{\Omega^2}{\Omega^2+a^2}.$$

As Ω increases from 0 to ∞ , it can be seen that the square-magnitude function $|G_a(j\Omega)|^2$ and hence,

the magnitude function $|G_a(j\Omega)| = \frac{\Omega}{\sqrt{\Omega^2+a^2}}$ increases monotonically from

$|G_a(j0)| = 0$ to $|G_a(j\infty)| = 1$. Let Ω_c denote the 3-dB cutoff frequency. Then

$$|G_a(j\Omega_c)|^2 = \frac{\Omega_c^2}{\Omega_c^2+a^2} = \frac{1}{2}, \text{ which implies } \Omega_c = a.$$

4.16 $H_a(s) = \frac{a}{s+a} = \frac{1}{2} \left(1 - \frac{s-a}{s+a} \right) = \frac{1}{2} (A_1(s) - A_2(s))$ and $G_a(s) = \frac{s}{s+a}$
 $= \frac{1}{2} \left(1 + \frac{s-a}{s+a} \right) = \frac{1}{2} (A_1(s) + A_2(s))$, where $A_1(s) = 1$ and $A_2(s) = \frac{s-a}{s+a}$. Now,

$|A_1(j\Omega)| = 1$ and $|A_2(j\Omega)|^2 = \frac{j\Omega - a}{j\Omega + a} \cdot \frac{-j\Omega - a}{-j\Omega + a} = \frac{j\Omega - a}{j\Omega + a} \cdot \frac{j\Omega + a}{j\Omega - a} = 1$ for all values of Ω , $A_1(s)$ and $A_2(s)$ are allpass functions.

4.17 $H_a(s) = \frac{bs}{s^2 + bs + \Omega_o^2}$. Thus, $H_a(j\Omega) = \frac{jb\Omega}{jb\Omega + \Omega_o^2 - \Omega^2}$ and hence,

$$|H_a(j\Omega)|^2 = H_a(j\Omega)H_a(-j\Omega) = \frac{jb\Omega}{jb\Omega + \Omega_o^2 - \Omega^2} \cdot \frac{-jb\Omega}{-jb\Omega + \Omega_o^2 - \Omega^2}$$

$$= \frac{b^2\Omega^2}{(\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2}$$

At $\Omega = 0$, $|H_a(j\Omega)| = 0$, at $\Omega = \infty$, $|H_a(j\Omega)| = 0$, and at $\Omega = \Omega_o$, $|H_a(j\Omega)|$ has the maximum value of 1. Now,

$$\frac{d|H_a(j\Omega)|^2}{d\Omega} = \frac{2b^2\Omega(\Omega_o^2 - \Omega^2)(\Omega_o^2 + \Omega^2)}{((\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2)^2}$$

It therefore follows that in the

frequency range $0 \leq \Omega < \Omega_o$, $\frac{d|H_a(j\Omega)|^2}{d\Omega} > 0$, and in the frequency range

$\Omega_o < \Omega < \infty$, $\frac{d|H_a(j\Omega)|^2}{d\Omega} < 0$. Hence, in the frequency range $0 \leq \Omega < \Omega_o$,

$|H_a(j\Omega)|^2$ is a monotonically increasing function of Ω and in the frequency range $\Omega_o < \Omega < \infty$, $|H_a(j\Omega)|^2$ is a monotonically decreasing function of Ω . Or in other words, $H_a(s)$ has a bandpass magnitude response. The 3-dB cutoff frequencies

are given by the solution of $\frac{b^2\Omega_c^2}{(\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2} = \frac{1}{2}$, or,

$(\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2 = 2b^2\Omega_c^2$, i.e., $\Omega_c^4 - (b^2 + 2\Omega_o^2)\Omega_c^2 + \Omega_o^4 = 0$. Substituting $x = \Omega_c^2$ in the last equation we get $x^2 - (b^2 + 2\Omega_o^2)x + \Omega_o^4 = 0$. Let $x_1 = \Omega_1^2$ and $x_2 = \Omega_2^2$ be the two roots of this quadratic equation. Then, $x_1x_2 = \Omega_1^2\Omega_2^2 = \Omega_o^4$ and $x_1 + x_2 = \Omega_1^2 + \Omega_2^2 = b^2 + 2\Omega_o^2$. Therefore, $\Omega_1\Omega_2 = \Omega_o^2$. From the last two equations we get $\Omega_1^2 + \Omega_2^2 - 2\Omega_1\Omega_2 = (\Omega_2 - \Omega_1)^2 = b^2 + 2\Omega_o^2 - 2\Omega_o^2 = b^2$. Hence, $\Omega_2 - \Omega_1 = b$.

4.18 $G_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + bs + \Omega_o^2}$. Thus, $G_a(j\Omega) = \frac{\Omega_o^2 - \Omega^2}{jb\Omega + \Omega_o^2 - \Omega^2}$ and hence,

$$|G_a(j\Omega)|^2 = G_a(j\Omega)G_a(-j\Omega) = \frac{\Omega_o^2 - \Omega^2}{jb\Omega + \Omega_o^2 - \Omega^2} \cdot \frac{\Omega_o^2 - \Omega^2}{-jb\Omega + \Omega_o^2 - \Omega^2}$$

$$= \frac{(\Omega_o^2 - \Omega^2)^2}{(\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2}. \text{ Note } |G_a(j0)| = |G_a(j\infty)| = 1 \text{ and } |G_a(j\Omega_o)| = 0.$$

Now, $\frac{d|G_a(j\Omega)|^2}{d\Omega} = \frac{2b^2\Omega^3(\Omega^2 - \Omega_o^2)}{((\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2)^2}$. It therefore follows that in the

frequency range $0 \leq \Omega < \Omega_o$, $\frac{d|G_a(j\Omega)|^2}{d\Omega} < 0$, and in the frequency range

$\Omega_o < \Omega < \infty$, $\frac{d|G_a(j\Omega)|^2}{d\Omega} > 0$. Hence, in the frequency range $0 \leq \Omega < \Omega_o$,

$|G_a(j\Omega)|^2$ is a monotonically decreasing function of Ω and in the frequency range $\Omega_o < \Omega < \infty$, $|G_a(j\Omega)|^2$ is a monotonically increasing function of Ω . Or in other words, $G_a(s)$ has a bandstop magnitude response.

The 3-dB cutoff frequencies are given by the solution of $\frac{(\Omega_o^2 - \Omega_c^2)^2}{(\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2} = \frac{1}{2}$,

or, $2(\Omega_o^2 - \Omega_c^2)^2 = (\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2$, i.e., $\Omega_c^4 - (b^2 + 2\Omega_o^2)\Omega_c^2 + \Omega_o^4 = 0$. This last equation is exactly the same as in solution of Problem 4.18 from which we get $\Omega_1\Omega_2 = \Omega_o^2$ and $\Omega_2 - \Omega_1 = b$.

4.19 $H_a(s) = \frac{bs}{s^2 + bs + \Omega_o^2} = \frac{1}{2} \left(1 - \frac{s^2 - bs + \Omega_o^2}{s^2 + bs + \Omega_o^2} \right) = \frac{1}{2} (A_1(s) - A_2(s))$ and

$$G_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + bs + \Omega_o^2} = \frac{1}{2} \left(1 + \frac{s^2 - bs + \Omega_o^2}{s^2 + bs + \Omega_o^2} \right) = \frac{1}{2} (A_1(s) + A_2(s)), \text{ where}$$

$$A_1(s) = 1 \text{ and } A_2(s) = \frac{s^2 - bs + \Omega_o^2}{s^2 + bs + \Omega_o^2}. \text{ Now, } |A_1(j\Omega)| = 1 \text{ and } |A_2(j\Omega)|^2$$

$$= A_2(j\Omega)A_2(-j\Omega) = \frac{-\Omega^2 - jb\Omega + \Omega_o^2}{-\Omega^2 + jb\Omega + \Omega_o^2} \cdot \frac{-\Omega^2 + jb\Omega + \Omega_o^2}{-\Omega^2 - jb\Omega + \Omega_o^2} = 1, \text{ for all values of}$$

Ω , $A_1(s)$ and $A_2(s)$ are allpass functions.

4.20 (a) Let $A_i(s) = \frac{s + \lambda_i^*}{s - \lambda_i}$. Since the pole of $A_i(s)$ is strictly in the left-half s -plane

and hence, $A_i(s)$ is causal and stable. Now $|A_i(j\Omega)|^2 = A_i(j\Omega)A_i^*(j\Omega)$

$$= \frac{j\Omega + \lambda_i^*}{j\Omega - \lambda_i} \cdot \frac{-j\Omega + \lambda_i}{-j\Omega - \lambda_i^*} = \frac{j\Omega + \lambda_i^*}{j\Omega - \lambda_i} \cdot \frac{j\Omega - \lambda_i}{j\Omega + \lambda_i^*} = 1. \text{ Hence, } A_i(s) \text{ is an allpass}$$

function. Since, $A(s) = \prod_{i=1}^N A_i(s)$, it is a product of causal, stable allpass functions, and as a result, is also a causal, stable allpass function.

(b) $|A_i(s)|^2 = A_i(s)A_i^*(s) = \frac{s + \lambda_i^*}{s - \lambda_i} \cdot \frac{s^* + \lambda_i}{s^* - \lambda_i^*} = \frac{|s|^2 + |\lambda_i|^2 + 2 \operatorname{Re}\{s\lambda_i\}}{|s|^2 + |\lambda_i|^2 - 2 \operatorname{Re}\{s^*\lambda_i\}}$. Let

$$s = \sigma + j\Omega \text{ and } \lambda_i = a_i + jb_i. \text{ Then } |A_i(s)|^2 = \frac{(|s|^2 + |\lambda_i|^2 - 2\Omega b_i) + 2\sigma a_i}{(|s|^2 + |\lambda_i|^2 - 2\Omega b_i) - 2\sigma a_i}.$$

Since $a_i < 0$, it follows from the above that $|A_i(s)|^2 < 1$ for $\sigma > 0$, $|A_i(s)|^2 = 1$ for $\sigma = 0$, and $|A_i(s)|^2 > 1$ for $\sigma < 0$.

4.21 $|H_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$. Define $D(\Omega) = \frac{1}{|H_a(j\Omega)|^2} = 1 + (\Omega/\Omega_c)^{2N}$. It

follows then $\frac{d^k D(\Omega)}{d\Omega^k} = 2N(2N-1)\cdots(2N-k+1) \frac{\Omega^{2N-k}}{\Omega_c^{2N}}$. Therefore,

$$\left. \frac{d^k D(\Omega)}{d\Omega^k} \right|_{\Omega=0} = 0 \text{ for } k = 1, 2, \dots, N-1, \text{ or, equivalently, } \left. \frac{d^k |H_a(j\Omega)|}{d\Omega^k} \right|_{\Omega=0} = 0$$

for $k = 1, 2, \dots, N-1$.

4.22 $10 \log_{10} \left(\frac{1}{1 + \varepsilon^2} \right) = -0.25$. Therefore, $\varepsilon^2 = 10^{0.025} - 1 = 0.0593$. Next, from

$$10 \log_{10} \left(\frac{1}{A^2} \right) = -25, \text{ we get } A^2 = 10^{2.5} = 316.2278. \text{ Now, } \frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{6}{1.5} = 4$$

$$\text{and } \frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\varepsilon^2}} = \sqrt{\frac{315.2278}{0.0593}} = 72.9381. \text{ Hence, } N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.0943.$$

We choose $N = 4$ as the filter order.

To verify using MATLAB, we use the code fragment

```
[N, Wn] = buttord(2*pi*1500, 2*pi*6000, 0.25, 25, 's');
```


which yields $N = 4$ and $Wn = 18365.51286$.

4.23 The poles are given by $p_\ell = e^{j\pi(5+2\ell)/12}$, $1 \leq \ell \leq 6$. Hence,

$$\begin{aligned} p_1 &= e^{j(7\pi/12)} = -0.2588 + j0.9659, & p_2 &= e^{j(9\pi/12)} = -0.7071 + j0.7071, \\ p_3 &= e^{j(11\pi/12)} = -0.9659 + j0.2588, & p_4 &= e^{j(13\pi/12)} = p_3^* = -0.9659 - j0.2588, \\ p_5 &= e^{j(15\pi/12)} = p_2^* = -0.7071 - j0.7071, & p_6 &= e^{j(17\pi/12)} = p_1^* = -0.2588 - j0.9659. \end{aligned}$$

The poles can also be determined in MATLAB using the statement `[z,p,k]=buttap(6)` which yields

```
p =
-0.2588 + 0.9659i
-0.2588 - 0.9659i
-0.7071 + 0.7071i
-0.7071 - 0.7071i
-0.9659 + 0.2588i
-0.9659 - 0.2588i
```

4.24 From Eq. (4.41) of text, $T_N(\Omega) = 2\Omega T_{N-1}(\Omega) - T_{N-2}(\Omega)$, where $T_N(\Omega)$ is defined in Eq. (4.40).

Case 1: $|\Omega| \leq 1$. Making use of Eq. (4.40) in Eq. (4.41) we get

$$\begin{aligned} T_N(\Omega) &= 2\Omega \cos((N-1) \cdot \cos^{-1} \Omega) - \cos((N-2) \cdot \cos^{-1} \Omega) \\ &= 2\Omega \cos(N \cos^{-1} \Omega - \cos^{-1} \Omega) - \cos(N \cos^{-1} \Omega - 2 \cos^{-1} \Omega) \\ &= 2\Omega \left[\cos(N \cos^{-1} \Omega) \cos(\cos^{-1} \Omega) + \sin(N \cos^{-1} \Omega) \sin(\cos^{-1} \Omega) \right] \\ &\quad - \left[\cos(N \cos^{-1} \Omega) \cos(2 \cos^{-1} \Omega) + \sin(N \cos^{-1} \Omega) \sin(2 \cos^{-1} \Omega) \right] \\ &= 2\Omega \cos(N \cos^{-1} \Omega) \cos(\cos^{-1} \Omega) - \cos(N \cos^{-1} \Omega) \cos(2 \cos^{-1} \Omega) \\ &= 2\Omega^2 \cos(N \cos^{-1} \Omega) - \cos(N \cos^{-1} \Omega) \left[2 \cos^2(\cos^{-1} \Omega) - 1 \right] \\ &= \cos(N \cos^{-1} \Omega) \left[2\Omega^2 - 2\Omega^2 + 1 \right] = \cos(N \cos^{-1} \Omega). \end{aligned}$$

Case 2: $|\Omega| > 1$. Making use of Eq. (4.40) in Eq. (4.41) we get

$$T_N(\Omega) = 2\Omega \cosh((N-1) \cdot \cosh^{-1} \Omega) - \cosh((N-2) \cdot \cosh^{-1} \Omega)$$

Using the trigonometric identities

$$\begin{aligned} \cosh(A-B) &= \cosh(A) \cosh(B) - \sinh(A) \sinh(B), & \sinh(2A) &= 2 \sinh(A) \cosh(A), \text{ and} \\ \cosh(2A) &= 2 \cosh^2(A) - 1, \text{ and following a similar algebra as in Case 1, we can show} \\ T_N(\Omega) &= \cosh(N \cosh^{-1} \Omega). \end{aligned}$$

4.25 From the solution of Problem 4.22, we have $\frac{1}{k} = 4$ and $\frac{1}{k_1} = 72.9381$. Hence,

$$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 2.4151. \text{ We choose the filter order as } N = 3.$$

The filter order obtained using the MATLAB statement

`[N,Wn]=cheblord(2*pi*1500,2*pi*6000,0.25, 25, 's')` results in $N=3$.

4.26 $10 \log_{10} \left(\frac{1}{1+\varepsilon^2} \right) = -0.25$, which yields $\varepsilon = 0.2434$. $10 \log_{10} \left(\frac{1}{A^2} \right) = -25$, which

yields $A^2 = 316.2278$. Now, $k = \frac{\Omega_p}{\Omega_s} = \frac{1500}{6000} = 0.25$ and $k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}} = \frac{0.2434}{\sqrt{315.2278}} =$

$= 0.0137$. Substituting the value of k in Eq. (4.55a) we get $k' = 0.9682$. Then from Eq. (4.55b) we get $\rho_0 = 0.004$. Substituting the value ρ_0 in Eq. (4.55c) we get $\rho = 0.004$.

Finally, from Eq. (4.54) we arrive at $N = 2.0591$. We choose the next higher integer as the filter order $N = 3$.

The filter order obtained using the MATLAB statement

`[N,Wn]=ellipord(2*pi*1500,2*pi*6000,0.25, 25, 's')` results in $N=3$.

4.27 $B_N(s) = (2N - 1)B_{N-1}(s) + s^2 B_{N-2}(s)$, where $B_1(s) = s + 1$ and $B_2(s) = s^2 + 3s + 3$.

(a) Thus, $B_3(s) = 5B_2(s) + s^2 B_1(s) = 5(s^2 + 3s + 3) + s^2(s + 1) = s^3 + 6s^2 + 15s + 15$,

$$B_4(s) = 7B_3(s) + s^2 B_2(s) = 7(s^3 + 6s^2 + 15s + 15) + s^2(s^2 + 3s + 3) \\ = s^4 + 10s^3 + 45s^2 + 105s + 105.$$

(b) $B_5(s) = 9B_4(s) + s^2 B_3(s) = 9(s^4 + 10s^3 + 45s^2 + 105s + 105) + s^2(s^3 + 6s^2 + 15s + 15) \\ = s^5 + 15s^4 + 105s^3 + 420s^2 + 945s + 945.$

4.28 $\Omega_p = 2\pi \times 0.24$ and $\hat{\Omega}_p = 2\pi \times 3$. The mapping is thus $s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} = \frac{4\pi^2 \times 0.72}{\hat{s}}$.

Denote $K = 4\pi^2 \times 0.72 = 28.4245$. Hence, the desired highpass transfer function is given

$$\text{by } H_{HP}(\hat{s}) = H_{LP}(s) \Big|_{s=K/\hat{s}} = \frac{10}{\left(\frac{K}{\hat{s}}\right)^3 + 4.309\left(\frac{K}{\hat{s}}\right)^2 + 9.2835\left(\frac{K}{\hat{s}}\right) + 10} \\ = \frac{10\hat{s}^3}{K^3 + 4.309K^2\hat{s} + 9.2835K\hat{s}^2 + 10\hat{s}^3} = \frac{10\hat{s}^3}{10\hat{s}^3 + 263.8785\hat{s}^2 + 3481.5\hat{s} + 22966} \\ = \frac{\hat{s}^3}{\hat{s}^3 + 26.38785\hat{s}^2 + 348.15\hat{s} + 2296.6}.$$

4.29 $\Omega_p = 2\pi \times 0.9$ and $\hat{\Omega}_p = 2\pi \times 3$. The mapping is thus $s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} = \frac{4\pi^2 \times 2.7}{\hat{s}}$. Denote

$K = 4\pi^2 \times 2.7 = 106.5917$. Hence, the desired lowpass transfer function is given by

$$\begin{aligned}
H_{LP}(s) &= H_{LP}(s)|_{s=K/\hat{s}} = \frac{\left(\frac{K}{\hat{s}}\right)^3}{\left(\frac{K}{\hat{s}}\right)^3 + 9.238\left(\frac{K}{\hat{s}}\right)^2 + 40.087\left(\frac{K}{\hat{s}}\right) + 100} \\
&= \frac{K^3}{K^3 + 9.238K^2\hat{s} + 40.087K\hat{s}^2 + 100\hat{s}^3} \\
&= \frac{12110.735}{\hat{s}^3 + 42.729\hat{s}^2 + 1049.602\hat{s} + 12110.735}.
\end{aligned}$$

4.30 $\Omega_p = 2\pi \times 0.25 = 0.5\pi$, $\hat{\Omega}_o = 2\pi \times 3 = 6\pi$, $\hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi(0.5) = \pi$. The mapping is thus

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} = 0.5\pi \left(\frac{\hat{s}^2 + 36\pi^2}{\pi\hat{s}} \right) = \frac{\hat{s}^2 + 36\pi^2}{2\hat{s}}.$$

$$\begin{aligned}
H_{BP}(\hat{s}) &= H_{LP}(s)|_{s=(\hat{s}^2 + 36\pi^2)/2\hat{s}} = \frac{0.01 \left[\left(\frac{\hat{s}^2 + 36\pi^2}{2\hat{s}} \right)^2 + 367.93 \right]}{\left(\frac{\hat{s}^2 + 36\pi^2}{2\hat{s}} \right)^2 + 2.269 \left(\frac{\hat{s}^2 + 36\pi^2}{2\hat{s}} \right) + 3.895} \\
&= \frac{0.01(\hat{s}^4 + 2182.33\hat{s}^2 + 126242.18)}{\hat{s}^4 + 4.538\hat{s}^3 + 726.19\hat{s}^2 + 1612.38\hat{s} + 126242.18}.
\end{aligned}$$

4.31 $\hat{\Omega}_p = 2\pi \times 6.5 \times 10^3$ and $\hat{\Omega}_s = 2\pi \times 1.5 \times 10^3$.

$$10 \log_{10} \left(\frac{1}{1+\varepsilon^2} \right) = -0.5, \text{ and hence, } \varepsilon^2 = 10^{0.05} - 1 = 0.122.$$

$$10 \log_{10} \left(\frac{1}{A^2} \right) = -40, \text{ and hence, } A^2 = 10^4. \text{ Therefore, } \frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\varepsilon^2}} = 286.2632.$$

Set $\Omega_p = 1$. Then $\Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = \frac{6.5}{1.5} = \frac{13}{3}$. Thus, $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{13}{3}$. The order of the

prototype lowpass filter is thus given by $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.8579$. As a result, we

choose the filter order as $N = 4$.

The order of the prototype lowpass filter obtained using the MATLAB statement `[N,Wn]=buttord(1,13/3,0.5,40,'s')` results in $N=4$.

The order of the desired highpass filter is also 4.

4.32 $\hat{F}_{p1} = 20 \times 10^3$, $\hat{F}_{p2} = 45 \times 10^3$, $\hat{F}_{s1} = 10 \times 10^3$, and $\hat{F}_{s2} = 50 \times 10^3$. Thus,

$\hat{F}_{p1}\hat{F}_{p2} = 9 \times 10^8$ and $\hat{F}_{s1}\hat{F}_{s2} = 7.5 \times 10^8$. Since $\hat{F}_{p1}\hat{F}_{p2} > \hat{F}_{s1}\hat{F}_{s2}$, we can either increase left stopband edge \hat{F}_{s1} or decrease the left passband edge \hat{F}_{p1} to make $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2}$. We choose to increase \hat{F}_{s1} to a new value given by $\hat{F}_{s1} = 18 \times 10^3$, in which case $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2} = \hat{F}_o^2 = 9 \times 10^8$. The center angular frequency of the bandpass filter is therefore $\hat{\Omega}_o = 2\pi \times 30 \times 10^3$. The passband width is $B_w = \hat{\Omega}_{p1} - \hat{\Omega}_{p2} = 2\pi \times 25 \times 10^3$.

To determine the bandedges of the prototype lowpass filter we set $\Omega_p = 1$ and thus,

$$\Omega_s = \Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}_{s1}^2}{\hat{\Omega}_{s1} B_w} = \frac{30^2 - 18^2}{18 \times 25} = 1.28.$$

Now, $k = \frac{\Omega_p}{\Omega_s} = \frac{1}{1.28} = 0.78125$. Hence, $k' = \sqrt{1 - k^2} = 0.62421826$.

Next, $10 \log_{10}\left(\frac{1}{1 + \varepsilon^2}\right) = -0.25$ or equivalently, $\log_{10}(1 + \varepsilon^2) = 0.025$ which yields

$$\varepsilon^2 = 10^{0.025} - 1 = 0.059253725 \text{ or } \varepsilon = 0.243421. \text{ Likewise, } 10 \log_{10}\left(\frac{1}{A^2}\right) = -50$$

or, equivalently, $\log_{10}(A^2) = 5$ which yields $A^2 = 10^5 = 100000$. Therefore,

$$k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}} = 7.69768 \times 10^{-4}, \rho_0 = \frac{1 - \sqrt{k'}}{2(1 + \sqrt{k'})} = 0.058635856. \text{ As a result,}$$

$$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13} = 0.058637246. \text{ Hence,}$$

$$N = \frac{2 \log_{10}(4/k_1)}{\log_{10}(1/\rho)} = 6.0328. \text{ We choose } N = 7 \text{ as the order of the prototype lowpass}$$

filter.

Note that the order can also estimated using the specifications of the bandpass filter. To this end, the statement to use is

`[N,Wn]=ellipord([20 45],[15 50],0.25,50,'s')` which also yields $N=7$ as the order of the prototype lowpass filter. The order of the desired bandpass filter is therefore $7 \times 2 = 14$.

4.33 $\hat{F}_{p1} = 10 \times 10^6$, $\hat{F}_{p2} = 70 \times 10^6$, $\hat{F}_{s1} = 20 \times 10^6$, and $\hat{F}_{s2} = 45 \times 10^6$. Thus,

$\hat{F}_{p1}\hat{F}_{p2} = 70 \times 10^{13}$ and $\hat{F}_{s1}\hat{F}_{s2} = 90 \times 10^{13}$. Since $\hat{F}_{p1}\hat{F}_{p2} < \hat{F}_{s1}\hat{F}_{s2}$, we can either

increase left passband edge \hat{F}_{p1} or decrease the left stopband edge \hat{F}_{s1} to make

$\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2}$. We choose to increase \hat{F}_{p1} to a new value given by

$$\hat{F}_{p1}\hat{F}_{p2} = \frac{\hat{F}_{s1}\hat{F}_{s2}}{\hat{F}_{p2}} = 12.8571 \times 10^6, \text{ in which case } \hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2} = F_o^2 = 700 \times 10^{12}.$$

The width of the stopband is $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 2\pi \times 25 \times 10^6$ and the center angular frequency of the stopband is $\Omega_o^2 = 4\pi^2 \times 700 \times 10^{12}$.

To determine the bandedges of the prototype lowpass filter we set $\Omega_s = 1$ resulting in its passband edge $\Omega_p = \Omega_p \frac{\hat{\Omega}_{p1} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}_{p1}^2} = 0.4375$.

Now, $10 \log_{10} \left(\frac{1}{1 + \varepsilon^2} \right) = -0.5$ or equivalently, $\log_{10}(1 + \varepsilon^2) = 0.05$ which yields

$$\varepsilon^2 = 10^{0.05} - 1 = 0.1220184543 \text{ or } \varepsilon = 0.349114. \text{ Likewise, } 10 \log_{10} \left(\frac{1}{A^2} \right) = -30$$

or, equivalently, $\log_{10}(A^2) = 3$ which yields $A^2 = 10^3 = 1000$. Therefore,

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{1}{0.4375} = 2.2857 \text{ and } \frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\varepsilon} = \frac{\sqrt{999}}{0.349114} = 90.4836236.$$

Substituting the values of $\frac{1}{k}$ and $\frac{1}{k_1}$ in Eq. (4.43) we get

$$N = \frac{\cosh^{-1}(90.4836236)}{\cosh^{-1}(2.2857)} = 3.5408. \text{ We therefore choose } N = 4 \text{ as the order of the}$$

prototype lowpass filter. The order of the desired bandstop filter is thus 8.

Using the statement `[N,Wn]=cheb1ord(0.4375,1,0.5,30,'s')` we get $N=4$.

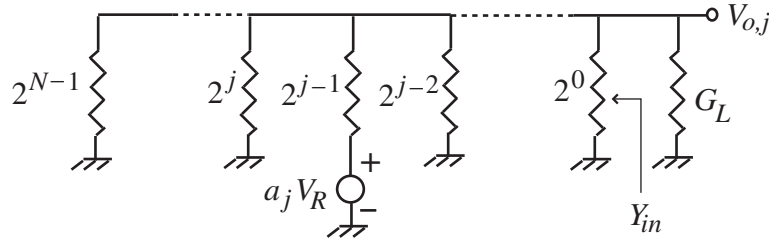
Note that the order can also be estimated using the specifications of the bandstop filter. To this end, the statement to use is

`[N,Wn]=cheb1ord([10 70],[20 45],0.5,30,'s')` which also yields $N=4$ as the order of the prototype lowpass filter.

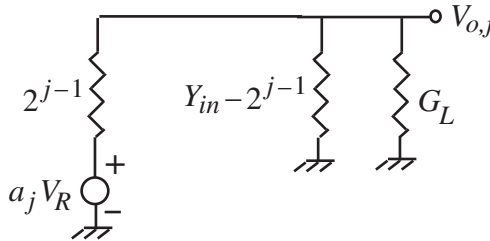
4.34 From Eq. (4.71), the difference in dB in the attenuation levels at Ω_p and Ω_s is given by $20N \log_{10}(\Omega_p / \Omega_s)$. Hence, for $\Omega_o = 2\Omega_p$, the attenuation difference in dB is equal to $20N \log_{10} 2 = 6.0206N$. Likewise, for $\Omega_o = 3\Omega_p$, the attenuation difference in dB is equal to $20N \log_{10} 3 = 9.5424N$. Finally, for $\Omega_o = 4\Omega_p$, the attenuation difference in dB is equal to $20N \log_{10} 4 = 12.0412N$.

4.35 The equivalent representation of the D/A converter of Figure 4.48 reduces to the circuit shown below if j -th bit is ON and the remaining bits are OFF, i.e., $a_j = 1$ and

$$a_k = 0, k \neq j.$$



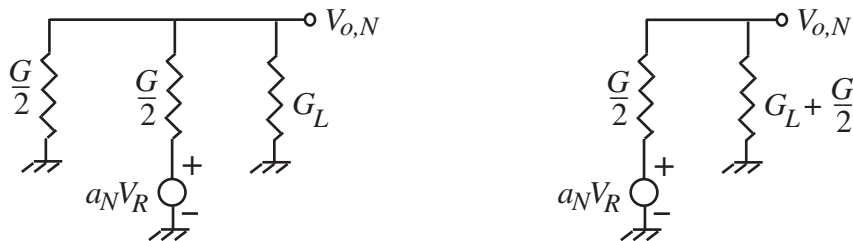
In the above circuit, Y_{in} is the total conductance seen by the load conductance G_L which is given by $Y_{in} = \sum_{i=0}^{N-1} 2^i = 2^N - 1$. The above circuit can be redrawn as indicated below:



Using the voltage-divider relation we then get $V_{o,j} = \frac{2^{j-1}}{Y_{in} + G_L} \cdot a_j V_R$. Using the superposition theorem, the general expression for the output voltage V_o is thus given by

$$V_o = \sum_{j=1}^N \frac{2^{j-1}}{Y_{in} + G_L} \cdot a_j V_R = \sum_{j=1}^N 2^{j-1} a_j \left(\frac{R_L}{1 + (2^N - 1)R_L} \right) V_R.$$

4.36 The equivalent representation of the D/A converter of Figure 4.49 reduces to the circuit shown below on the left if N -th bit is ON and the remaining bits are OFF, i.e., $a_N = 1$ and $a_k = 0, k \neq N$,

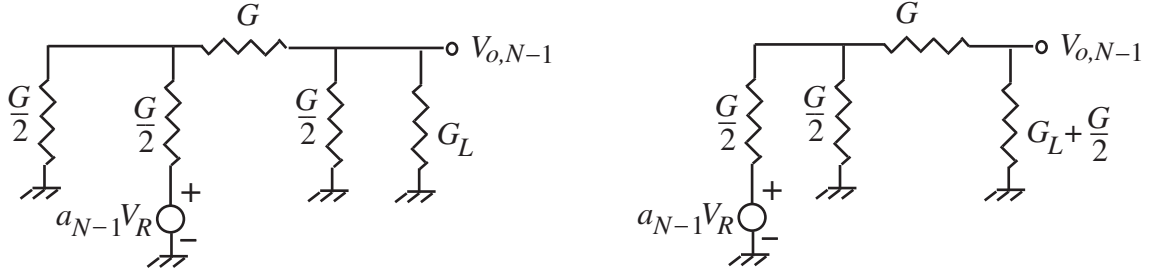


which simplifies to the circuit shown above on the right.

Using the voltage-divider relation we then get

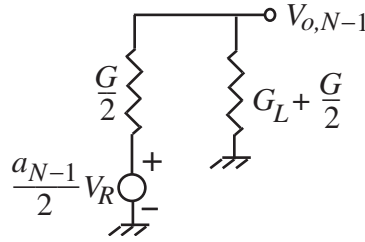
$$V_{o,N} = \frac{\frac{G}{2}}{\frac{G}{2} + G_L + \frac{G}{2}} \cdot a_N V_R = \frac{R_L}{2(R + R_L)} \cdot a_N V_R.$$

The equivalent representation of the D/A converter of Figure 4.49 reduces to the circuit shown below on the left if $(N - 1)$ -th bit is ON and the remaining bits are OFF, i.e., $a_{N-1} = 1$ and $a_k = 0, k \neq N - 1$,



which simplifies to the circuit shown above on the right.

Its Thevenin equivalent circuit is indicated below:



from which we readily obtain

$$V_{o,N-1} = \frac{\frac{G}{2}}{G + G_L} \cdot \frac{a_{N-1}}{2} V_R = \frac{R_L}{2(R_L + R)} \cdot \frac{a_{N-1}}{2} V_R.$$

Following the same procedure we can show that if the ℓ -th bit is ON and the remaining bits are OFF, i.e., $a_\ell = 1$, and $a_k = 0$, $k \neq \ell$, then

$$V_{o,\ell} = \frac{R_L}{2(R_L + R)} \cdot \frac{a_\ell}{2^{N-\ell}} V_R.$$

Hence, in general we have

$$V_o = \sum_{\ell=1}^N \frac{R_L}{2(R_L + R)} \cdot \frac{a_\ell}{2^{N-\ell}} V_R.$$

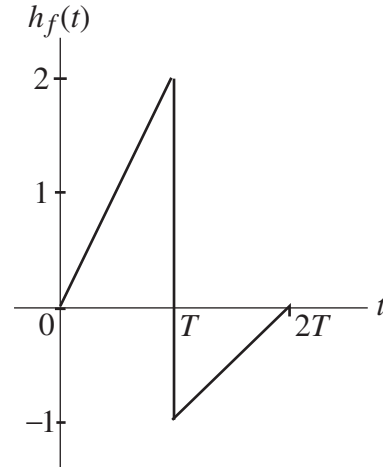
4.37 From the input-output relation of the first-order hold, we get the expression for the impulse response as $h_f(t) = \delta(nT) + \frac{\delta(nT) - \delta(nT - T)}{T}(t - nT)$, $nT \leq t < (n+1)T$. In the range

$0 \leq t < T$, the impulse response is given by $h_f(t) = \delta(0) + \frac{\delta(0) - \delta(-T)}{T}t = 1 + \frac{t}{T}$.

Likewise, in the range $T \leq t < 2T$, the impulse response is given by

$h_f(t) = \delta(T) + \frac{\delta(T) - \delta(0)}{T}(t - T) = 1 - \frac{t}{T}$. Outside these two ranges, $h_f(t) = 0$. Hence we have

$$h_f(t) = \begin{cases} 1 + \frac{t}{T}, & 0 \leq t < T, \\ 1 - \frac{t}{T}, & T \leq t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$



Using the step function we can write

$$\begin{aligned} h_f(t) &= \left(1 + \frac{t}{T}\right) [\mu(t) - \mu(t - T)] + \left(1 - \frac{t}{T}\right) [\mu(t - T) - \mu(t - 2T)] \\ &= \mu(t) + \frac{t}{T} \mu(t) - \frac{2(t - T)}{T} \mu(t - T) - 2\mu(t - T) - \mu(t - 2T) + \frac{(t - 2T)}{T} \mu(t - 2T) + 2\mu(t - 2T). \end{aligned}$$

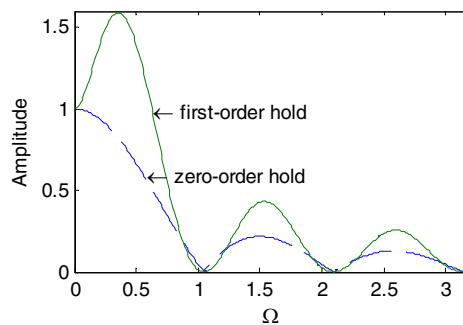
Taking the Laplace transform of the above equation we arrive at the transfer function

$$H_f(s) = \frac{1}{s} + \frac{1}{Ts^2} - \frac{2}{T} \cdot \frac{e^{-sT}}{s^2} - 2 \frac{e^{-sT}}{s} - \frac{e^{-2sT}}{s} + \frac{1}{T} \cdot \frac{e^{-2sT}}{s^2} + 2 \frac{e^{-2sT}}{s} = \left(\frac{1 + sT}{T}\right) \left(\frac{1 - e^{-sT}}{s}\right)^2.$$

Hence, the frequency response is given by

$$H_f(j\Omega) = \left(\frac{1 + j\Omega T}{T}\right) \left(\frac{1 - e^{-j\Omega T}}{j\Omega}\right)^2 = T \sqrt{1 + \Omega^2 T^2} \left(\frac{2 \sin(\Omega T / 2)}{\Omega T / 2}\right)^2 e^{-j\Omega T} e^{j \tan^{-1} \Omega T}. \quad A$$

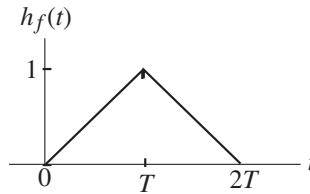
plot of the magnitude responses of the zero-order hold and the first-order hold is shown below:



4.37 From the input-output relation of the linear interpolator, we get the expression for the impulse response as $h_f(t) = \delta(nT - T) + \frac{\delta(nT) - \delta(nT - T)}{T} (t - nT)$, $nT \leq t < (n + 1)T$. In the range $0 \leq t < T$, the impulse response is given by $h_f(t) = \delta(-T) + \frac{\delta(0) - \delta(-T)}{T} t$. Likewise,

in the range $T \leq t < 2T$, the impulse response is given by $h_f(t) = \delta(0) + \frac{\delta(T) - \delta(0)}{T}(t - T)$.

Outside these two ranges, $h_f(t) = 0$. Hence we have $h_f(t) = \begin{cases} \frac{t}{T}, & 0 \leq t < T, \\ 2 - \frac{t}{T}, & T \leq t < 2T, \\ 0, & \text{otherwise.} \end{cases}$



Using the step function we can write

$$h_f(t) = \frac{t}{T}[\mu(t) - \mu(t - T)] + \left(2 - \frac{t}{T}\right)[\mu(t - T) - \mu(t - 2T)]$$

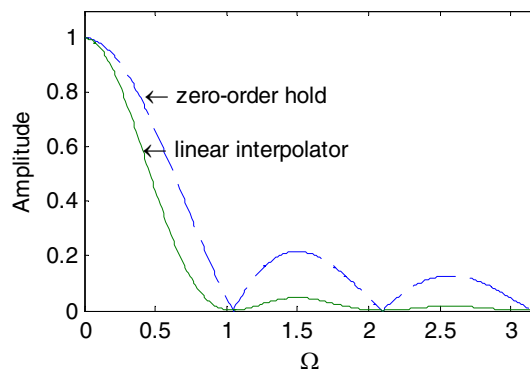
$$= \frac{t}{T}\mu(t) - \frac{2(t - T)}{T}\mu(t - T) + \frac{(t - 2T)}{T}\mu(t - 2T).$$

Taking the Laplace transform of the above equation we arrive at the transfer function

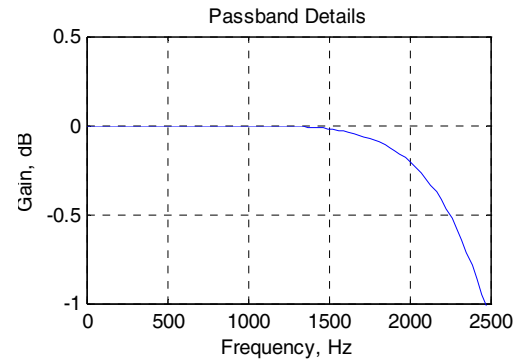
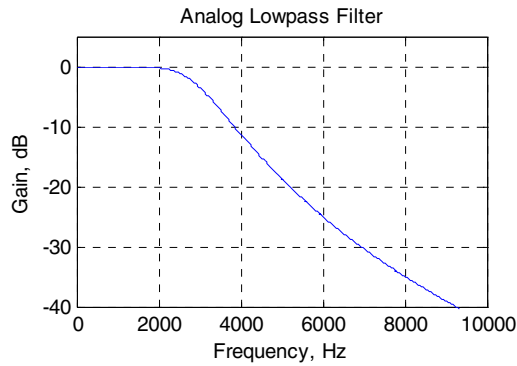
$$H_f(s) = \frac{1}{s^2 T} - \frac{2e^{-sT}}{s^2 T} + \frac{e^{-2sT}}{s^2 T} = T \left(\frac{1 - e^{-sT}}{sT} \right)^2. \text{ Hence, the frequency response is given}$$

by $H_f(j\Omega) = T \left(\frac{1 - e^{-j\Omega T}}{j\Omega T} \right)^2 = T \left(\frac{\sin(\Omega T / 2)}{\Omega T / 2} \right)^2 e^{-j\Omega T}$. A plot of the magnitude

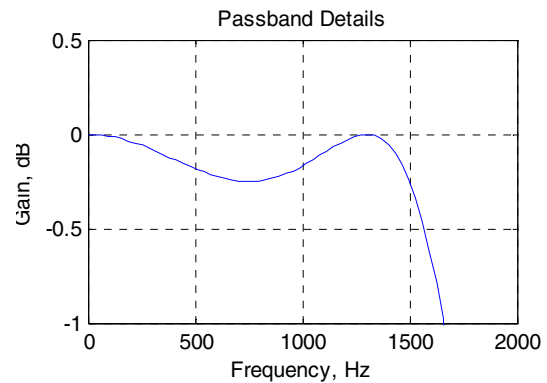
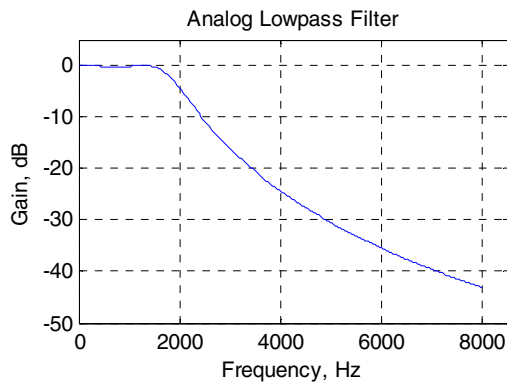
responses of the ideal filter, zero-order hold and the first-order hold is shown below:



M4.1 We use $N = 4$ and $\omega_n = 18365.512865$ computed in Problem 4.22 and use $\omega = 0:2*\pi:2*\pi*10000$; to evaluate the frequency points. The gain plot obtained using Program 4_2 is shown below.



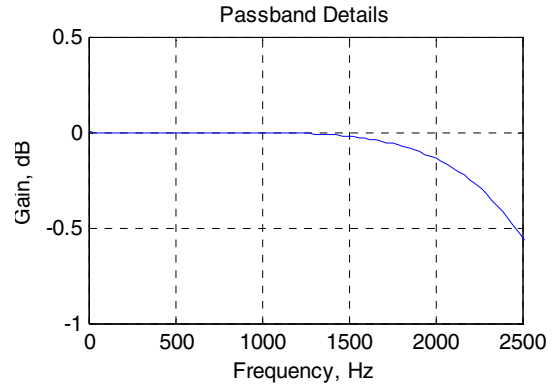
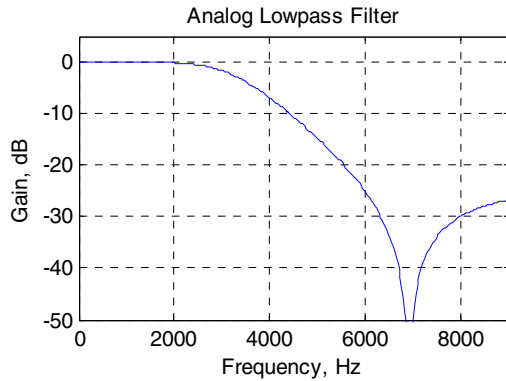
M4.2 We use $N = 3$ computed in Problem 4.23 and $F_p = 2 * \pi * 1500$ and $R_p = 0.25$ and use $\omega = 0:2 * \pi:2 * \pi * 10000$; to evaluate the frequency points. The gain plot obtained using Program 4_3 is shown below.



M4.3 We replace the statement

```
Fp = input('Passband edge frequency in Hz = '); with
Fs = input('Stopband edge frequency in Hz = '); replace
Rp = input('Passband ripple in dB = ');
with Rs = input('Minimum stopband attenuation in dB = '); and
replace [num,den] = cheby1(N,Rp,Fp,'s'); with [num,den] =
cheby2(N,Rs,Fs,'s');
```

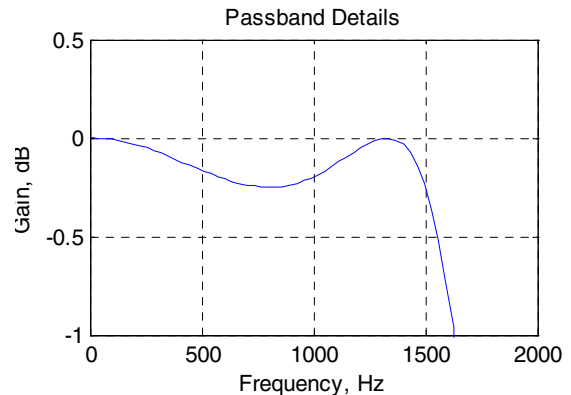
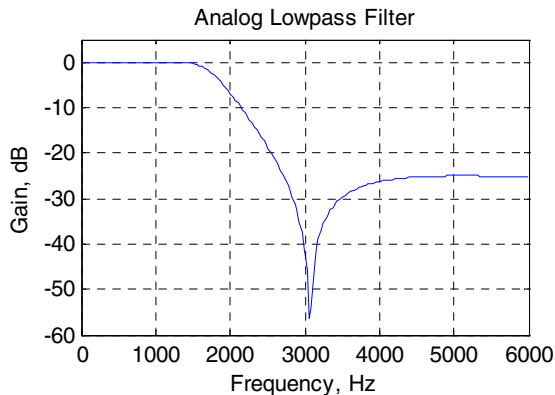
to modify Program 4_3. Next, we run the modified program using $N = 3$ and $R_s = 25$, and $F_s = 2 * \pi * 6000$. The gain response plot generated by the modified program is shown below.



The numerator and the denominator coefficients of the 3rd order Type 2 Chebyshev lowpass filter can be obtained by typing num and den in the command window:

$$H_{LP}(s) = \frac{10138.1864 s^2 + 4.8663294 \times 10^{10}}{s^3 + 7030.255525 s^2 + 2.4198332254 \times 10^7 s + 4.8663294 \times 10^{10}}$$

M4.4 We use $N = 3$ and $W_n = 9424.777960769379$ computed in Problem 4.26 in Program 4_4 and use $\omega = [0: 200: 12000*\pi]$; to evaluate the frequency points. The gain plot generated by running this program is shown below:

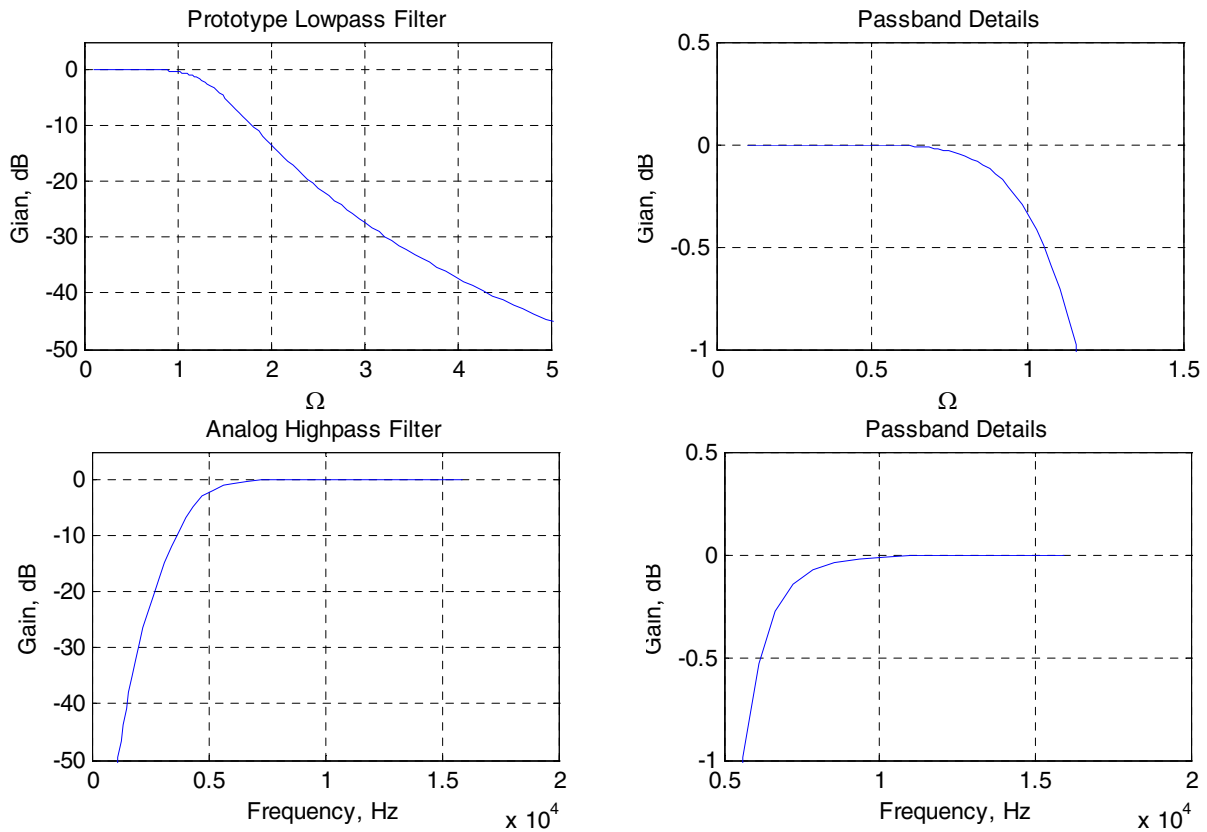


M4.5 The MATLAB program used is as given below:

```
[N,Wn]=buttord(1,13/3,0.5, 40, 's');
[B,A] = butter(N,Wn,'s');
[num,den]=lp2hp(B,A,2*pi*6500);
figure(1)
[h,w]=freqs(B,A);gain = 20*log10(abs(h));
plot(w,gain);grid
xlabel('\Omega');ylabel('Gain, dB');
title('Analog Lowpass Filter');
figure(2)
[h,w]=freqs(num,den);gain = 20*log10(abs(h));
plot(w/(2*pi),gain);grid
xlabel('Frequency, Hz');ylabel('Gain, dB');
title('Analog Highpass Filter');
```

$$H_{LP}(s) = \frac{3.5262257}{s^4 + 3.58086432s^3 + 6.41129464s^2 + 6.72423556s + 3.5262257},$$

$$H_{HP}(s) = \frac{s^4}{s^4 + 7.7880s^3 + 3.0326485 \times 10^9 s^2 + 6.91763168 \times 10^{13} s + 7.8897418 \times 10^{17}}.$$



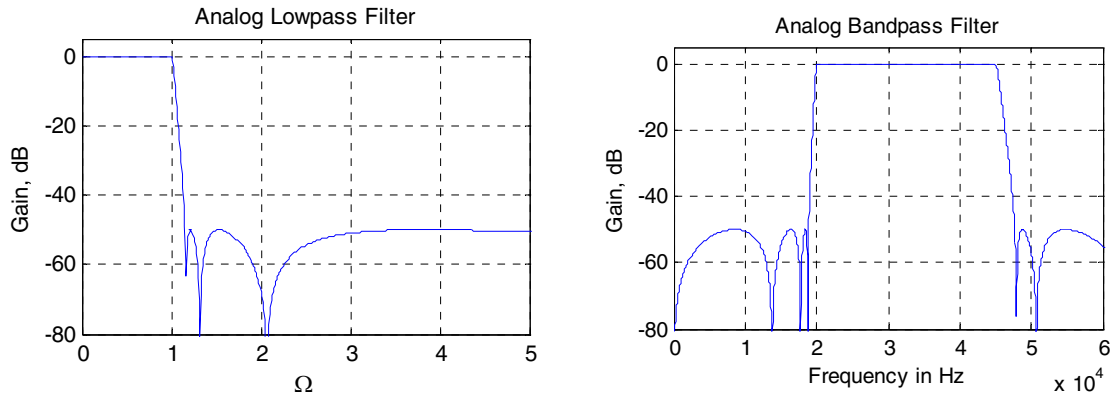
M4.6 The MATLAB program used is given below:

```
[N,Wn] = ellipord(1,1.28,0.25,50,'s');
[B,A] = ellip(N,0.25,50,Wn,'s');
[num,den] = lp2bp(B,A,2*pi*30e3,2*pi*25e3);
```

```
figure(1)
omega = [0:0.01:10];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot(omega,gain); grid; axis([0 5 -80 5]);
xlabel('\Omega'); ylabel('Gain, dB');
title('Analog Lowpass Filter');
```

```
figure(2)
omega = [0:200:100e3*2*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
```

```
plot(omega/(2*pi),gain); grid; axis([0 60e3 -80 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB');
title('Analog Bandpass Filter');
```



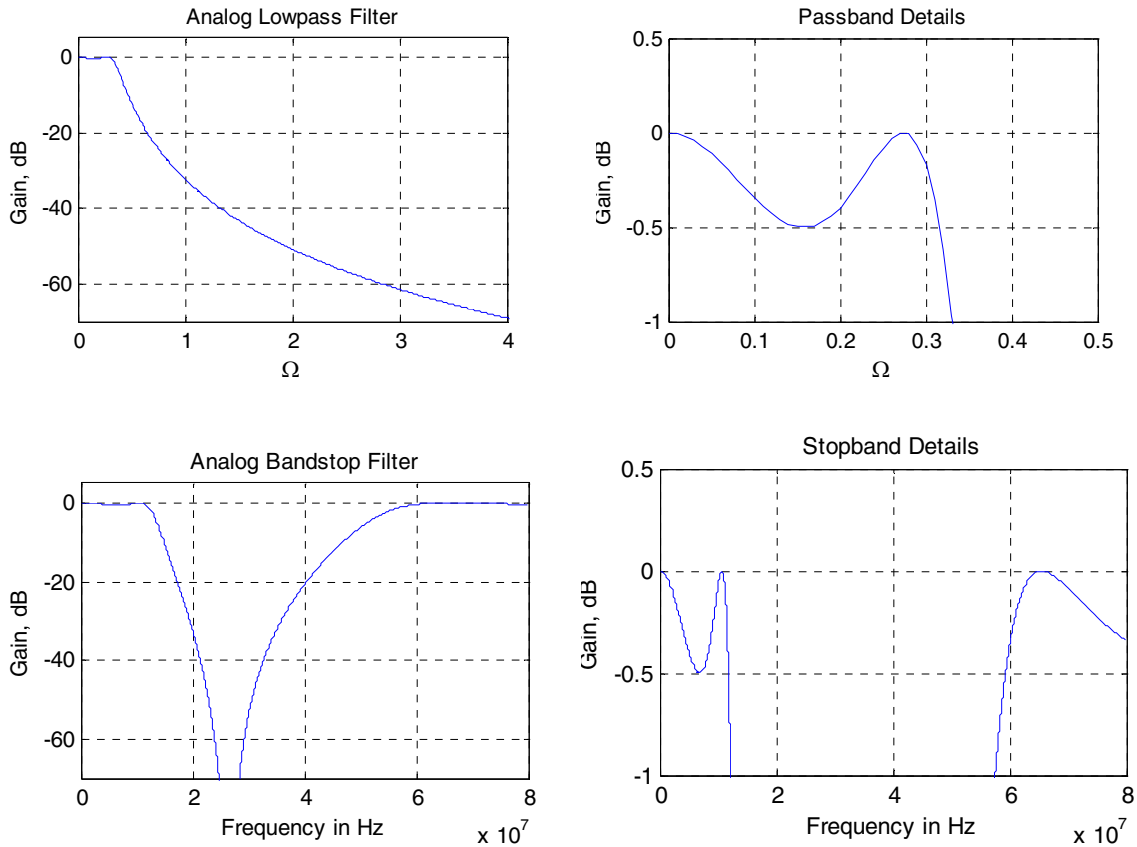
$$H_{LP}(s) = \frac{0.0185s^6 + 0.1364s^4 + 0.2887s^2 + 0.186835}{s^7 + 1.36426s^6 + 2.9795s^5 + 2.7545s^4 + 2.70025s^3 + 1.56215s^2 + 0.7275s + 0.1868}$$

The numerator and denominator coefficients of can be obtained by typing num and den in the Command Window.

M4.7 The MATLAB program used is given below:

```
[N,Wn] = cheblord(0.3157894, 1, 0.5, 30, 's');
[B,A] = cheby1(N,0.5, Wn, 's');
[num,den] = lp2bs(B,A,2*pi*sqrt(700)*10^6, 2*pi*15e6);
figure(1)
omega = [0:0.01:10];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot(omega, gain); grid; axis([0 4 -70 5]);
xlabel('\Omega'); ylabel('Gain, dB');
title('Analog Lowpass Filter');
figure(2)
omega = [0:10000:160e6*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
plot(omega/(2*pi), gain); grid; axis([0 80e6 -70 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB');
title('Analog Bandstop Filter');
```

$$H_{LP}(s) = \frac{0.02253823}{s^3 + 0.3956566s^2 + 0.1530643s + 0.02253823}$$



M4.8 The MATLAB program to generate the plots of Figure 4.56 is given below:

```
% Droop Compensation
w = 0:pi/100:pi;
h1 = freqz([-1/16 9/8 -1/16],1,w);
h2 = freqz(9, [8 1], w);
w1 = 0;
for n = 1:101;
    h3(n) = sin(w1/2)/(w1/2);
    w1 = w1 + pi/100;
end
m1 = 20*log10(abs(h1));
m2 = 20*log10(abs(h2));
m3 = 20*log10(abs(h3));
plot(w/pi,m3,'-',w/pi,m1+m2,'--',w/pi,m2+m3,'-.');grid
xlabel('Normalized frequency');ylabel('Gain, dB');
```

Chapter 5

5.1 Let $\tilde{y}[n] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r]$. Then $\tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n+kN-r]$. Since $\tilde{h}[n]$ is periodic in n with a period N , $\tilde{h}[n+kN-r] = \tilde{h}[n-r]$. Therefore $\tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r] = \tilde{y}[n]$, hence $\tilde{y}[n]$ is also periodic in n with a period N .

5.2 (a) $\tilde{y}[0] = \sum_{r=0}^4 \tilde{x}[r]\tilde{h}[-r] = \tilde{x}[0]\tilde{h}[0] + \tilde{x}[1]\tilde{h}[4] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] + \tilde{x}[4]\tilde{h}[1] = -13$,

$$\tilde{y}[1] = \sum_{r=0}^4 \tilde{x}[r]\tilde{h}[1-r] = \tilde{x}[0]\tilde{h}[1] + \tilde{x}[1]\tilde{h}[0] + \tilde{x}[2]\tilde{h}[4] + \tilde{x}[3]\tilde{h}[3] + \tilde{x}[4]\tilde{h}[2] = -13,$$

$$\tilde{y}[2] = \sum_{r=0}^4 \tilde{x}[r]\tilde{h}[2-r] = \tilde{x}[0]\tilde{h}[2] + \tilde{x}[1]\tilde{h}[1] + \tilde{x}[2]\tilde{h}[0] + \tilde{x}[3]\tilde{h}[4] + \tilde{x}[4]\tilde{h}[3] = -13,$$

$$\tilde{y}[3] = \sum_{r=0}^4 \tilde{x}[r]\tilde{h}[3-r] = \tilde{x}[0]\tilde{h}[3] + \tilde{x}[1]\tilde{h}[2] + \tilde{x}[2]\tilde{h}[1] + \tilde{x}[3]\tilde{h}[0] + \tilde{x}[4]\tilde{h}[4] = -13,$$

$$\tilde{y}[4] = \sum_{r=0}^4 \tilde{x}[r]\tilde{h}[4-r] = \tilde{x}[0]\tilde{h}[4] + \tilde{x}[1]\tilde{h}[3] + \tilde{x}[2]\tilde{h}[2] + \tilde{x}[3]\tilde{h}[1] + \tilde{x}[4]\tilde{h}[0] = -13.$$

Therefore, $\tilde{y}[n] = \{-13, -13, -13, -13, -13\}$, $0 \leq n \leq 4$.

(b) $\tilde{y}[n] = \{1, 1, 1, 1, 1\}$, $0 \leq n \leq 4$.

5.3 Since $\tilde{\psi}_k[n+rN] = \tilde{\psi}_k[n]$, hence all the terms which are not in the range $0, 1, \dots, N-1$, can be accumulated to $\tilde{\psi}_k[n]$, where $0 \leq k \leq N-1$. Hence, in this case the Fourier series representation involves only N complex exponential sequences. Let

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}, \text{ then}$$

$$\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi rn/N} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi(k-r)n/N} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=0}^{N-1} e^{j2\pi(k-r)n/N} \right).$$

Now, from Eq. (5.11), the inner summation is equal to N if $k=r$, otherwise it is equal

to 0. Thus, $\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi rn/N} = \tilde{X}[r]$. Next, we observe $\tilde{X}[k+\ell N]$

$$= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi(k+\ell N)n/N} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} e^{-j2\pi \ell n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} = \tilde{X}[k].$$

5.4 (a) $\tilde{x}_1[n] = \cos\left(\frac{\pi n}{4}\right) = \frac{1}{2} \left\{ e^{j\pi n/4} + e^{-j\pi n/4} \right\}$ The period of $\tilde{x}_1[n]$ is $N = 8$.

$$\begin{aligned} \tilde{X}_1[k] &= \frac{1}{2} \left\{ \sum_{n=0}^7 e^{j2\pi n/8} e^{-j2\pi kn/8} + \sum_{n=0}^7 e^{-j2\pi n/8} e^{-j2\pi kn/8} \right\} \\ &= \frac{1}{2} \left\{ \sum_{n=0}^7 e^{-j2\pi n(k-1)/8} + \sum_{n=0}^7 e^{-j2\pi n(k+1)/8} \right\}. \text{ Now, from Eqn. (5.11) we observe} \\ \sum_{n=0}^7 e^{-j2\pi n(k-1)/8} &= \begin{cases} 8, & \text{for } k=1, \\ 0, & \text{otherwise,} \end{cases} \text{ and } \sum_{n=0}^7 e^{-j2\pi n(k+1)/8} = \begin{cases} 8, & \text{for } k=7, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, $\tilde{X}_1[k] = \begin{cases} 4, & \text{for } k=1,7, \\ 0, & \text{otherwise.} \end{cases}$

(b) $\tilde{x}_2[n] = \sin\left(\frac{\pi n}{3}\right) + 3\cos\left(\frac{\pi n}{4}\right) = \frac{1}{2j} \{ e^{j\pi n/3} - e^{-j\pi n/3} \} + \frac{3}{2} \{ e^{j\pi n/4} + e^{-j\pi n/4} \}$. The period of $\sin\left(\frac{\pi n}{3}\right)$ is 6 and the period of $\cos\left(\frac{\pi n}{4}\right)$ is 8. Hence, the period of $\tilde{x}_2[n]$ is the GCM of (6,8) and is 24.

$$\begin{aligned} \tilde{X}_2[k] &= \frac{1}{2j} \left\{ \sum_{n=0}^{23} e^{j8\pi n/24} e^{-j2\pi kn/24} - \sum_{n=0}^{23} e^{-j8\pi n/24} e^{-j2\pi kn/24} \right\} \\ &+ \frac{3}{2} \left\{ \sum_{n=0}^{23} e^{j6\pi n/24} e^{-j2\pi kn/24} - \sum_{n=0}^{23} e^{-j6\pi n/24} e^{-j2\pi kn/24} \right\} \\ &= \frac{1}{2j} \left\{ \sum_{n=0}^{23} e^{-j2\pi n(k-3)/24} - \sum_{n=0}^{23} e^{-j2\pi n(k+3)/24} \right\} \\ &+ \frac{3}{2} \left\{ \sum_{n=0}^{23} e^{-j2\pi n(k-4)/24} - \sum_{n=0}^{23} e^{-j2\pi n(k+4)/24} \right\}. \text{ Hence } \tilde{X}_2[k] = \begin{cases} -j12, & k=3, \\ j12, & k=21, \\ 36, & k=4,20, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

5.5 Let $\tilde{P}[k]$ denote the coefficients of the Fourier series representation of $\tilde{p}[n]$. Since $\tilde{p}[n]$ is periodic with a period N , then from Eq. (5.185b), we have

$$\tilde{P}[k] = \sum_{n=0}^{N-1} \tilde{p}[n] e^{-j2\pi kn/N} = 1. \text{ Hence, from Eq. (5.185a) we get}$$

$$\tilde{p}[n] = \frac{1}{N} \sum_{\ell=0}^{N-1} \tilde{P}[\ell] e^{j2\pi \ell n/N} = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{j2\pi \ell n/N}.$$

5.6 $\tilde{X}[k] = X(e^{j\omega}) \Big|_{\omega=2\pi k/N} = X(e^{j2\pi k/N}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi kn/N}, -\infty < k < \infty.$

Now, $\tilde{X}[k + \ell N] = X(e^{j2\pi(k+\ell N)/N}) = X(e^{j2\pi k/N} e^{j2\pi \ell}) = X(e^{j2\pi k/N}) = \tilde{X}[k].$

Likewise, $\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N} = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi \ell n/N} \right) e^{j2\pi kn/N}$

$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{\infty} x[\ell] e^{j2\pi(k-\ell)n/N}$. Let $\ell = n + rN$. Then

$\tilde{x}[n] = \frac{1}{N} \sum_{r=-\infty}^{\infty} x[n + rN] \left(\sum_{k=0}^{N-1} e^{-j2\pi kr} \right)$. But $\sum_{k=0}^{N-1} e^{-j2\pi kr} = N$. Hence,

$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$.

5.7 (a) $\tilde{G}[k] = \sum_{n=0}^{N-1} \tilde{g}[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} \tilde{x}[n] \tilde{y}[n] e^{-j2\pi kn/N}$. Now,

$\tilde{x}[n] = \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}[r] e^{-j2\pi rn/N}$. Therefore,

$\tilde{G}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} \tilde{X}[r] \tilde{y}[n] e^{-j2\pi(k-r)n/N} = \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}[r] \sum_{n=0}^{N-1} \tilde{y}[n] e^{-j2\pi(k-r)n/N}$

$= \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}[r] \tilde{Y}[k-r]$.

(b) $\tilde{h}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \tilde{Y}[k] e^{j2\pi kn/N} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{Y}[k] e^{j2\pi k(n-r)/N}$

$= \sum_{r=0}^{N-1} \tilde{x}[r] \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{Y}[k] e^{j2\pi k(n-r)/N} \right) = \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{y}[n-r]$.

5.8 (a) $x_a[n] = \sin(2\pi n/N) = \frac{1}{2j} (e^{j2\pi n/N} - e^{-j2\pi n/N})$. Therefore,

$X_a[k] = \frac{1}{2j} \sum_{n=0}^{N-1} e^{j2\pi n/N} e^{-j2\pi kn/N} - \frac{1}{2j} \sum_{n=0}^{N-1} e^{-j2\pi n/N} e^{-j2\pi kn/N}$

$= \frac{1}{2j} \sum_{n=0}^{N-1} e^{-j2\pi(k-1)n/N} - \frac{1}{2j} \sum_{n=0}^{N-1} e^{-j2\pi(k+1)n/N}$. From Eq. (5.11), the first sum is

equal to N when $k=1$ and 0 otherwise. Likewise, from Eq. (5.11), the second sum is equal to N when $k=N-1$ and 0 otherwise. Therefore,

$$X_a[k] = \begin{cases} N/2j, & k=1, \\ -N/2j, & k=N-1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $x_b[n] = \cos^2\left(\frac{2\pi n}{N}\right) = \frac{1}{2} + \frac{1}{2} \cos\left(\frac{4\pi n}{N}\right)$. Now the N -point DFT of $\frac{1}{2}$ is $\frac{N}{2}$ for $k=0$ and 0 otherwise. From Example 5.2, the N -point DFT of $\cos\left(\frac{4\pi n}{N}\right)$ is $\frac{N}{2}$ for $k=2$ and $k=N-2$ and 0 otherwise. Therefore,

$$X_b[k] = \begin{cases} N/2, & k = 0, \\ N/4, & k = 2, N-2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) $x_c[n] = \cos^3\left(\frac{2\pi n}{N}\right) = \frac{1}{4}\cos\left(\frac{6\pi n}{N}\right) + \frac{3}{4}\cos\left(\frac{2\pi n}{N}\right)$. From Example 5.2, the N -point DFT of $\cos\left(\frac{6\pi n}{N}\right)$ is $\frac{N}{2}$ for $k = 3$ and $k = N-3$ and 0 otherwise. Likewise, from

Example 5.2, the N -point DFT of $\cos\left(\frac{2\pi n}{N}\right)$ is $\frac{N}{2}$ for $k = 1$ and $k = N-1$ and 0

otherwise. Therefore, $X_c[k] = \begin{cases} N/8, & k = 3, N-3, \\ 3N/8, & k = 1, N-1, \\ 0, & \text{otherwise.} \end{cases}$

5.9 (a) $Y_a[k] = \sum_{n=0}^{N-1} \alpha^n W_N^{kn} = \sum_{n=0}^{N-1} (\alpha W_N^k)^n = \frac{1 - \alpha W_N^{kN}}{1 - \alpha W_N^k} = \frac{1 - \alpha}{1 - \alpha W_N^k}$.

(b) $Y_b[k] = 2 \sum_{k \text{ even}} W_N^{kn} - 3 \sum_{k \text{ odd}} W_N^{kn}$. Assume first N is even, i.e., $N = 2L$. Then

$$Y_b[k] = 2 \sum_{r=0}^{L-1} W_{2L}^{k2r} - 3 \sum_{r=0}^{L-1} W_{2L}^{k(2r+1)} = 2 \sum_{r=0}^{L-1} W_L^{kr} - 3W_{2L}^k \sum_{r=0}^{L-1} W_L^{kr} = (2 - 3W_N^k) \left(\frac{1 - W_L^{kL}}{1 - W_L^k} \right) = 0.$$

Next, assume N is odd, i.e., $N = 2L + 1$. Then $Y_b[k] = 2 \sum_{r=0}^L W_{2L}^{k2r} - 3 \sum_{r=0}^{L-1} W_{2L}^{k(2r+1)}$

$$= 2 \sum_{r=0}^L W_L^{kr} - 3W_{2L}^k \sum_{r=0}^{L-1} W_L^{kr} = 2 \left(\frac{1 - W_L^{k(L+1)}}{1 - W_L^k} \right) - 3W_{2L}^k \left(\frac{1 - W_L^{kL}}{1 - W_L^k} \right) = 2 \left(\frac{1 - W_L^k}{1 - W_L^k} \right) = 2.$$

5.10 $x[n] = \cos(\omega_o n) = \frac{1}{2}(e^{j\omega_o n} + e^{-j\omega_o n})$, $0 \leq n \leq N-1$. Therefore,

$$\begin{aligned} X[k] &= \frac{1}{2} \sum_{n=0}^{N-1} e^{j\omega_o n} e^{-j2\pi kn/N} + \frac{1}{2} \sum_{n=0}^{N-1} e^{-j\omega_o n} e^{-j2\pi kn/N} \\ &= \frac{1}{2} \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi k}{N} - \omega_o\right)n} + \frac{1}{2} \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi k}{N} + \omega_o\right)n} \\ &= \frac{1}{2} \cdot \frac{1 - e^{-j\left(\frac{2\pi k}{N} - \omega_o\right)N}}{1 - e^{-j\left(\frac{2\pi k}{N} - \omega_o\right)}} + \frac{1}{2} \cdot \frac{1 - e^{-j\left(\frac{2\pi k}{N} + \omega_o\right)N}}{1 - e^{-j\left(\frac{2\pi k}{N} + \omega_o\right)}} \end{aligned}$$

$$= \frac{1}{2} e^{-j\left(\frac{2\pi k}{N} - \omega_o\right)\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left(\pi k - \frac{\omega_o N}{2}\right)}{\sin\left(\frac{\pi k}{N} - \frac{\omega_o}{2}\right)} + \frac{1}{2} e^{-j\left(\frac{2\pi k}{N} + \omega_o\right)\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left(\pi k + \frac{\omega_o N}{2}\right)}{\sin\left(\frac{\pi k}{N} + \frac{\omega_o}{2}\right)}.$$

5.11
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$$

$$= \sum_{r=0}^{(N/2)-1} x_0[r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x_1[r] W_{N/2}^{rk}$$

$$= X_0[\langle k \rangle_{N/2}] + W_N^k X_1[\langle k \rangle_{N/2}], \quad 0 \leq k \leq N-1.$$

5.12
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{n=0}^{(N/2)-1} x[n] W_N^{nk} + \sum_{n=N/2}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{n=0}^{(N/2)-1} x[n] W_N^{nk} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x\left[\frac{N}{2} + n\right] W_N^{nk}$$

$$= \sum_{n=0}^{(N/2)-1} \left(x[n] + (-1)^k x\left[\frac{N}{2} + n\right] \right) W_N^{nk}. \quad \text{For } k = 2\ell, \text{ we get}$$

$$X[2\ell] = \sum_{n=0}^{(N/2)-1} \left(x[n] + x\left[\frac{N}{2} + n\right] \right) W_N^{2n\ell} = \sum_{n=0}^{(N/2)-1} \left(x[n] + x\left[\frac{N}{2} + n\right] \right) W_{N/2}^{n\ell} = X_0[\ell]$$

and for $k = 2\ell + 1$ we get
$$X[2\ell + 1] = \sum_{n=0}^{(N/2)-1} \left(x[n] - x\left[\frac{N}{2} + n\right] \right) W_N^{(2\ell+1)n}$$

$$= \sum_{n=0}^{(N/2)-1} \left(x[n] + x\left[\frac{N}{2} + n\right] \right) W_N^n \cdot W_{N/2}^{n\ell} = X_1[\ell] \quad \text{where } 0 \leq \ell \leq \frac{N}{2} - 1.$$

5.13
$$g[n] = \frac{1}{2}(x[2n] + x[2n+1]), \quad h[n] = \frac{1}{2}(x[2n] - x[2n+1]), \quad 0 \leq n \leq \frac{N}{2} - 1.$$
 Solving for $x[2n]$ and $x[2n+1]$, we get $x[2n] = g[n] + h[n]$ and $x[2n+1] = g[n] - h[n]$. Therefore,

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$$

$$= \sum_{r=0}^{(N/2)-1} (g[r] + h[r]) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} (g[r] - h[r]) W_{N/2}^{rk}$$

$$= (1 + W_N^k) \sum_{r=0}^{(N/2)-1} g[r] W_{N/2}^{rk} + (1 - W_N^k) \sum_{r=0}^{(N/2)-1} h[r] W_{N/2}^{rk}$$

$$= (1 + W_N^k)G[\langle k \rangle_{N/2}] + (1 - W_N^k)H[\langle k \rangle_{N/2}], \quad 0 \leq k \leq N-1.$$

5.14 $g[n] = a_1x[2n] + a_2x[2n+1]$, $h[n] = a_3x[2n] - a_4x[2n+1]$, $0 \leq n \leq \frac{N}{2} - 1$, with $a_1a_4 \neq a_2a_3$. Solving for $x[2n]$ and $x[2n+1]$, we get $x[2n] = \frac{a_4g[n] - a_2h[n]}{a_1a_4 - a_2a_3}$ and

$$x[2n+1] = \frac{-a_3g[n] + a_1h[n]}{a_1a_4 - a_2a_3}. \quad \text{Therefore,}$$

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r]W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk} \\ &= \sum_{r=0}^{(N/2)-1} \left(\frac{a_4g[r] - a_2h[r]}{a_1a_4 - a_2a_3} \right) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} \left(\frac{-a_3g[r] + a_1h[r]}{a_1a_4 - a_2a_3} \right) W_{N/2}^{rk} \\ &= \frac{1}{a_1a_4 - a_2a_3} \left((a_4 - a_3W_N^k)G[\langle k \rangle_{N/2}] + (-a_2 + a_1W_N^k)H[\langle k \rangle_{N/2}] \right), \quad 0 \leq n \leq N-1. \end{aligned}$$

5.15 (a) $G[k] = \sum_{n=0}^{N-1} x[n]W_{2N}^{nk}$. For k even, i.e., $k = 2\ell$, $G[2\ell] = \sum_{n=0}^{N-1} x[n]W_{2N}^{2\ell n}$

$$= \sum_{n=0}^{N-1} x[n]_{2N}^{\ell n} = X[\ell], \quad 0 \leq \ell \leq N-1.$$

(b) $H[k] = \sum_{n=N}^{2N-1} x[n-N]W_{2N}^{nk}$. Let $m = n - N$ or $n = m + N$. Then

$$H[k] = \sum_{m=0}^{N-1} x[m]W_{2N}^{(m+N)k} = (-1)^k \sum_{m=0}^{N-1} x[m]W_{2N}^{mk}. \quad \text{For } k \text{ even, i.e., } k = 2\ell,$$

$$H[2\ell] = \sum_{n=0}^{N-1} x[n]W_{2N}^{2\ell n} = \sum_{n=0}^{N-1} x[n]W_N^{\ell n} = X[\ell], \quad 0 \leq \ell \leq N-1.$$

5.16 $Y[k] = \sum_{n=0}^{2N-1} y[n]W_{2N}^{nk} = \sum_{n=0}^{2N-1} (g[n] + h[n])W_{2N}^{nk} = G[k] + H[k]$. For k even, i.e.,

$$k = 2\ell, \quad Y[2\ell] = G[2\ell] + H[2\ell] = 2X[\ell], \quad 0 \leq \ell \leq N-1.$$

For k odd, i.e., $k = 2\ell + 1$, $G[2\ell + 1] = \sum_{n=0}^{N-1} x[n]W_{2N}^{(2\ell+1)n} = \sum_{n=0}^{N-1} x[n]W_{2N}^n W_N^{\ell n}$ and

$$H[2\ell + 1] = - \sum_{n=0}^{N-1} x[n]W_{2N}^{(2\ell+1)n} = - \sum_{n=0}^{N-1} x[n]W_{2N}^n W_N^{\ell n} = -G[2\ell + 1], \quad 0 \leq \ell \leq N-1.$$

Hence, for $k = 2\ell + 1$, $Y[2\ell + 1] = G[2\ell + 1] - G[2\ell + 1] = 0$, $0 \leq \ell \leq N-1$.

5.17 $Y[k] = \sum_{n=0}^{MN-1} y[n]W_{MN}^{nk} = \sum_{n=0}^{N-1} x[n]W_{MN}^{nk}$. Thus,

$$Y[kM] = \sum_{n=0}^{N-1} x[n]W_{MN}^{nkM} = \sum_{n=0}^{N-1} x[n]W_N^{nk} = X[k].$$

5.18

5.19 (a) Now, $X[N/2] = \sum_{n=0}^{N-1} x[n]W_N^{nN/2} = \sum_{n=0}^{N-1} (-1)^n x[n]$. Hence if $x[n] = x[N-1-n]$

and N is even, then $X[N/2] = \sum_{n=0}^{N-1} (-1)^n x[n] = 0$.

(b) $X[0] = \sum_{n=0}^{N-1} x[n]$. Hence if $x[n] = -x[N-1-n]$, then $X[0] = 0$.

(c) $X[2\ell] = \sum_{n=0}^{N-1} x[n]W_N^{2n\ell} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{2n\ell} + \sum_{n=(N/2)-1}^{N-1} x[n]W_N^{2n\ell}$
 $= \sum_{n=0}^{(N/2)-1} x[n]W_N^{2n\ell} + \sum_{n=0}^{(N/2)-1} x[n + \frac{N}{2}]W_N^{2n\ell} = \sum_{n=0}^{M-1} (x[n] + x[n + M])W_{2M}^{2n\ell}$. Hence if $x[n] = -x[n + M]$, then $X[2\ell] = 0$ for $0 \leq \ell \leq M-1$.

5.20 $X[2m] = \sum_{n=0}^{N-1} x[n]W_N^{2mn} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{2mn} + \sum_{n=N/2}^{N-1} x[n]W_N^{2mn}$
 $= \sum_{n=0}^{(N/2)-1} x[n]W_N^{2mn} + \sum_{n=0}^{N/2-1} x[n + \frac{N}{2}]W_N^{2mn}W_N^{mN} = \sum_{n=0}^{(N/2)-1} (x[n] + x[n + \frac{N}{2}])W_N^{2mn} = 0$,
 $0 \leq m \leq \frac{N}{2} - 1$. This implies $x[n] + x[n + \frac{N}{2}] = 0$.

5.21 (a) Using the circular convolution property of the DFT given in Table 5.3, we get $\text{DFT}\{x[\langle n - m_1 \rangle_N]\} = W_N^{km_1} X[k]$ and $\text{DFT}\{x[\langle n - m_2 \rangle_N]\} = W_N^{km_2} X[k]$.

Hence, $W[k] = \text{DFT}\{w[n]\} = W_N^{km_1} X[k] + W_N^{km_2} X[k] = (W_N^{km_1} + W_N^{km_2})X[k]$.

(b) $g[n] = \frac{1}{2}(x[n] + (-1)^n x[n]) = \frac{1}{2}(x[n] + W_N^{-(N/2)n} x[n])$ Using the circular convolution property of the DFT given in Table 5.3, we get

$$G[k] = \text{DFT}\{g[n]\} = \frac{1}{2} \left\{ X[k] + X[\langle k - \frac{N}{2} \rangle_N] \right\}$$

(c) Using the circular convolution property of the DFT given in Table 5.3, we get $Y[k] = \text{DFT}\{y[n]\} = X[k] \cdot X[k] = X^2[k]$.

5.22 (a) $\text{DFT}\left\{x\left\langle n - \frac{N}{2} \right\rangle_N\right\} = W_N^{k(N/2)} X[k] = (-1)^k X[k]$. Hence,

$$U[k] = \text{DFT}\{u[n]\} = \text{DFT}\left\{x[n] - x\left\langle n - \frac{N}{2} \right\rangle_N\right\} = X[k] - (-1)^k X[k] = \begin{cases} 2X[k], & \text{for } k \text{ odd,} \\ 0, & \text{for } k \text{ even.} \end{cases}$$

$$\text{(b)} \quad V[k] = \text{DFT}\{v[n]\} = \text{DFT}\left\{x[n] - x\left[n - \frac{N}{2}\right]\right\} = X[k] + X[k] = 2X[k].$$

(c) $y[n] = (-1)^n x[n] = W_N^{(N/2)n} x[n]$. Hence, using the circular frequency-shifting property of the DFT given in Table 5.3, we get

$$Y[k] = \text{DFT}\{y[n]\} = \text{DFT}\left\{W_N^{(N/2)n} x[n]\right\} = X\left[\left\langle k - \frac{N}{2} \right\rangle_N\right].$$

5.23 (a) From the circular frequency-shifting property of the DFT given in Table 5.3, we get $\text{IDFT}\{X[\langle k - m_1 \rangle_N]\} = W_N^{-m_1 n} x[n]$ and $\text{IDFT}\{X[\langle k - m_2 \rangle_N]\} = W_N^{-m_2 n} x[n]$. Hence,

$$\begin{aligned} w[n] &= \text{IDFT}\{W[k]\} = \text{IDFT}\{\alpha X[\langle k - m_1 \rangle_N] + \beta X[\langle k - m_2 \rangle_N]\} \\ &= \alpha W_N^{-m_1 n} x[n] + \beta W_N^{-m_2 n} x[n] = (\alpha W_N^{-m_1 n} + \beta W_N^{-m_2 n}) x[n]. \end{aligned}$$

(b) $G[k] = \frac{1}{2}(X[k] + (-1)^k X[k]) = \frac{1}{2}(X[k] + W_N^{-(N/2)k} X[k])$ Using the circular time-shifting property of the DFT given in Table 5.3, we get

$$g[n] = \text{IDFT}\{G[k]\} = \frac{1}{2}\left(x[n] + x\left\langle n - \frac{N}{2} \right\rangle_N\right).$$

(c) Using the modulation property of the DFT given in Table 5.3, we get

$$y[n] = \text{IDFT}\{Y[k]\} = N \cdot x[n] \cdot x[n] = N \cdot x^2[n].$$

5.24 (a) $X[2m] = \sum_{n=0}^{N-1} x[n] W_N^{2mn} = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2mn} + \sum_{n=N/2}^{N-1} x[n] W_N^{2mn}$

$$= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2mn} + \sum_{n=0}^{N/2-1} x\left[n + \frac{N}{2}\right] W_N^{2m\left(n + \frac{N}{2}\right)}$$

$$= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2mn} + \sum_{n=0}^{N/2-1} x\left[n + \frac{N}{2}\right] W_N^{2mn} W_N^{mn}$$

$$= \sum_{n=0}^{(N/2)-1} \left(x[n] + x\left[n + \frac{N}{2}\right]\right) W_N^{2mn} = \sum_{n=0}^{(N/2)-1} (x[n] - x[n]) W_N^{2mn} = 0, \quad 0 \leq m \leq \frac{N}{2} - 1.$$

$$\text{(b)} \quad X[4\ell] = \sum_{n=0}^{N-1} x[n] W_N^{4\ell n}$$

$$= \sum_{n=0}^{(N/4)-1} x[n] W_N^{4\ell n} + \sum_{n=N/4}^{(N/2)-1} x[n] W_N^{4\ell n} + \sum_{n=N/2}^{(3N/4)-1} x[n] W_N^{4\ell n} + \sum_{n=3N/4}^{N-1} x[n] W_N^{4\ell n}$$

$$\begin{aligned}
&= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n]W_N^{4\ell n} + x\left[n + \frac{N}{4}\right]W_N^{4\ell\left(n + \frac{N}{4}\right)} + x\left[n + \frac{N}{2}\right]W_N^{4\ell\left(n + \frac{N}{2}\right)} + x\left[n + \frac{3N}{4}\right]W_N^{4\ell\left(n + \frac{3N}{4}\right)} \right) \\
&= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n] + x\left[n + \frac{N}{4}\right]W_N^{\ell n} + x\left[n + \frac{N}{2}\right]W_N^{2\ell n} + x\left[n + \frac{3N}{4}\right]W_N^{3\ell n} \right) W_N^{4\ell n} \\
&= \sum_{n=0}^{\frac{N}{4}-1} (x[n] - X[n] + x[n] - x[n])W_N^{4\ell n} = 0 \quad \text{as } W_N^{\ell N} = W_N^{2\ell N} = W_N^{3\ell N} = 1.
\end{aligned}$$

5.25 (a) $X[N-k] = \sum_{n=0}^{N-1} x[n]W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n]W_N^{-kn} + X^*[k].$

(b) $X[0] = \sum_{n=0}^{N-1} x[n]W_N^0 \sum_{n=0}^{N-1} x[n]$ which is real.

(c) $X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} x[n]W_N^{(N/2)n} = \sum_{n=0}^{N-1} (-1)^n x[n]$ which is real.

5.26 $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}.$

(a) $X^*[k] = \sum_{n=0}^{N-1} x^*[n]W_N^{-nk}.$ Replacing n by $N-n$ in the summation we obtain

$$X^*[k] = \sum_{n=0}^{N-1} x^*[N-n]W_N^{-(N-n)k} = \sum_{n=0}^{N-1} x^*[N-n]W_N^{nk}.$$
 Thus,

$$\text{DFT}\{x^*[N-n]\} = \text{DFT}\{x^*[\langle -n \rangle_N]\} = X^*[k].$$

(b) $\text{Re}\{x[n]\} = \frac{1}{2}\{x[n] + x^*[n]\}.$ Taking the DFT of both sides and using the results of Part (a) we get $\text{DFT}\{\text{Re}\{x[n]\}\} = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}.$

(c) $j \text{Im}\{x[n]\} = \frac{1}{2}\{x[n] - x^*[n]\}.$ Thus, $\text{DFT}\{j \text{Im}\{x[n]\}\} = \frac{1}{2}\{X[k] - X^*[\langle -k \rangle_N]\}.$

(d) $x_{cs}[n] = \frac{1}{2}\{x[n] + x^*[\langle -n \rangle_N]\}.$ Using the linearity property and results of Part (b) we get $\text{DFT}\{x_{cs}[n]\} = \frac{1}{2}\{X[k] + X^*[k]\} = \text{Re}\{X[k]\}.$

(e) $x_{ca}[n] = \frac{1}{2}\{x[n] - x^*[\langle -n \rangle_N]\}.$ Using the linearity property and results of Part (b) we get $\text{DFT}\{x_{ca}[n]\} = \frac{1}{2}\{X[k] - X^*[k]\} = j \text{Im}\{X[k]\}.$

5.27 Since for a real sequence, $x[n] = x^*[n]$, taking the DFT of both sides we get $X[k] = X^*[\langle -k \rangle_N]$. This implies

$$\operatorname{Re}\{X[k]\} + j \operatorname{Im}\{X[k]\} = \operatorname{Re}\{X[\langle -k \rangle_N]\} - j \operatorname{Im}\{X[\langle -k \rangle_N]\}.$$

Comparing real and imaginary parts we get $\operatorname{Re}\{X[k]\} = \operatorname{Re}\{X[\langle -k \rangle_N]\}$ and $\operatorname{Im}\{X[k]\} = -\operatorname{Im}\{X[\langle -k \rangle_N]\}$.

$$\begin{aligned} \text{Also, } |X[k]| &= \sqrt{(\operatorname{Re}\{X[k]\})^2 + (\operatorname{Im}\{X[k]\})^2} \\ &= \sqrt{(\operatorname{Re}\{X[\langle -k \rangle_N]\})^2 + (\operatorname{Im}\{X[\langle -k \rangle_N]\})^2} = |X[\langle -k \rangle_N]| \text{ and} \\ \arg\{X[k]\} &= \tan^{-1}\left(\frac{\operatorname{Im}\{X[k]\}}{\operatorname{Re}\{X[k]\}}\right) = \tan^{-1}\left(\frac{-\operatorname{Im}\{X[\langle -k \rangle_N]\}}{\operatorname{Re}\{X[\langle -k \rangle_N]\}}\right) = -\arg\{X[\langle -k \rangle_N]\}. \end{aligned}$$

5.28 (a) $x_1[\langle -n \rangle_9] = \{4 \ 3 \ -5 \ 1 \ -2 \ -2 \ 1 \ -5 \ 3\} = x_1[n]$. Thus, $x_1[n]$ is a circular even sequence and hence, it has a real-valued 9-point DFT.

(b) $x_2[\langle -n \rangle_9] = \{0 \ -5 \ -1 \ -4 \ 3 \ -3 \ 4 \ 1 \ 5\} = -x_2[n]$. Thus, $x_2[n]$ is a circular odd sequence and hence, it has an imaginary-valued 9-point DFT.

(c) $x_3[\langle -n \rangle_9] = \{0 \ -5 \ -1 \ -4 \ 3 \ -3 \ 4 \ 2 \ -5\}$ which is neither equal to $x_3[n]$ nor equal to $-x_3[n]$. Thus, $x_3[n]$ has a complex-valued 9-point DFT.

(d) $x_4[\langle -n \rangle_9] = \{-5 \ 5 \ -2 \ 2 \ 4 \ 4 \ 2 \ -2 \ 5\} = x_4[n]$. Thus, $x_4[n]$ is a circular even sequence and hence, it has a real-valued 9-point DFT.

5.29 (a) $h[n] = g[\langle n-5 \rangle_8]$. Hence, $H[k] = W_8^{-5k} G[k] = e^{j10\pi k/8} G[k] = e^{j5\pi k/4} G[k]$
 $= \{2.6 + j4.1, e^{j5\pi/4}(3 - j2.7), e^{j5\pi/2}(-4.2 + j1.4), e^{j15\pi/4}(3.5 - j2.6),$
 $e^{j5\pi}(0.5), e^{j25\pi/4}(1.3 + j4.4), e^{j15\pi/2}(2.4 - j1.6), e^{j35\pi/4}(-3 + j1.6)\}.$

(b) $H[k] = G[\langle k+3 \rangle_8]$. Hence, $h[n] = W_8^3 g[n] = e^{-j6\pi n/8} g[n] = e^{-j3\pi n/4} g[n]$
 $= \{-0.1 - j0.7, e^{-j3\pi/4}(1.3 + j), e^{-j3\pi/2}(2 + j0.7), e^{-j9\pi/4}(1.1 + j2.2),$
 $e^{-j3\pi}(-0.8 + j0.2), e^{-j15\pi/4}(3.4 - j0.1), e^{-j9\pi/2}(-1.2 + j3.1), e^{-j21\pi/4}(j1.5)\}.$

5.30 (a) $y[n] = \alpha g[n] + \beta h[n]$. Therefore,

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{nk} = \alpha \sum_{n=0}^{N-1} g[n] W_N^{nk} + \beta \sum_{n=0}^{N-1} h[n] W_N^{nk} = \alpha G[k] + \beta H[k].$$

(b) $x[n] = g[\langle n - n_o \rangle_N]$. Therefore, $X[k] = \sum_{n=0}^{N-1} g[\langle n - n_o \rangle_N] W_N^{nk}$

$$\begin{aligned}
&= \sum_{n=0}^{n_o-1} g[N+n-n_o] W_N^{nk} + \sum_{n=n_o}^{N-1} g[n-n_o] W_N^{nk} \\
&= \sum_{n=N-n_o}^{N-1} g[n] W_N^{(n+n_o-N)k} + \sum_{n=0}^{N-n_o-1} g[n_o] W_N^{(n+n_o)k} = W_N^{n_o k} \sum_{n=0}^{N-1} g[n] W_N^{nk} = W_N^{n_o k} G[k].
\end{aligned}$$

(c) $u[n] = W_N^{-k_o n} g[n]$. Therefore, $U[k] = \sum_{n=0}^{N-1} u[n] W_N^{kn} = \sum_{n=0}^{N-1} g[n] W_N^{(k-k_o)n}$

$$= \begin{cases} \sum_{n=0}^{N-1} g[n] W_N^{(k-k_o)n}, & \text{if } k \geq k_o, \\ \sum_{n=0}^{N-1} g[n] W_N^{(N+k-k_o)n}, & \text{if } k < k_o. \end{cases}$$

Thus, $U[k] = \begin{cases} G[k-k_o], & \text{if } k \geq k_o, \\ G[N+k-k_o], & \text{if } k < k_o, \end{cases} = G[\langle k-k_o \rangle_N]$.

(d) $h[n] = G[n]$. Therefore, $H[k] = \sum_{n=0}^{N-1} h[n] W_N^{nk} = \sum_{n=0}^{N-1} G[n] W_N^{nk}$

$$= \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} g[r] W_N^{nr} W_N^{kr} = \sum_{r=0}^{N-1} g[r] \sum_{n=0}^{N-1} W_N^{(k+r)n}.$$

The second sum is nonzero only if $k+r=0$ or else if $r=N-k$ and $k \neq 0$. Hence,

$$H[k] = \begin{cases} Ng[0], & \text{if } k=0, \\ Ng[N-k], & \text{if } k>0, \end{cases} = Ng[\langle -k \rangle_N].$$

(e) $u[n] = \sum_{m=0}^{N-1} g[m] h[\langle -n-m \rangle_N]$. Therefore, $U[k] = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} g[m] h[\langle -n-m \rangle_N] W_N^{nk}$

$$= \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} h[\langle -n-m \rangle_N] W_N^{nk} = \sum_{m=0}^{N-1} g[m] H[k] W_N^{mk} = H[k] G[k].$$

5.31 $\sum_{n=0}^{N-1} g[n] h^*[n] = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} G[k] W_N^{-nk} h^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k] \sum_{n=0}^{N-1} h^*[n] W_N^{-nk}$

$$= \frac{1}{N} \sum_{k=0}^{N-1} G[k] H^*[k].$$

5.32 DFT{ $r_{xy}[\ell]$ } = $\mathcal{R}_{xy}[\ell] = \sum_{\ell=0}^{N-1} r_{xy}[\ell] W_N^{\ell k} = \sum_{\ell=0}^{N-1} \left(\sum_{n=0}^{N-1} x[n] y[\langle \ell+n \rangle_N] \right) W_N^{\ell k}$

$$= \sum_{n=0}^{N-1} x[n] \left(\sum_{\ell=0}^{N-1} y[\langle \ell+n \rangle_N] W_N^{\ell k} \right) = \sum_{n=0}^{N-1} x[n] \left(\sum_{\ell=0}^{N-1} y[m] W_N^{\ell(m-n)} \right)$$

$$= \sum_{n=0}^{N-1} x[n] W_N^{-nk} \left(\sum_{\ell=0}^{N-1} y[\ell] W_N^{\ell m} \right) = X^*[k] Y[k].$$

5.33 Note $X[k]$ is the MN -point DFT of the sequence $x_e[n]$ obtained from $x[n]$ by appending it with $M(N-1)$ zeros. Thus, the length- MN sequence $y[n]$ is given by

$$y[n] = \sum_{\ell=0}^{M-1} x_e[\langle n - N\ell \rangle_{MN}], \quad 0 \leq n \leq MN-1. \quad \text{Taking the } MN\text{-DFT of both sides we}$$

$$\text{get } Y[k] = \left(\sum_{\ell=0}^{M-1} W_{MN}^{Nk\ell} \right) X[k] = \left(\sum_{\ell=0}^{M-1} W_N^{k\ell} \right) X[k].$$

5.34 (a) $X[0] = \sum_{n=0}^9 x[n] = 30.$

(b) $X[5] = \sum_{n=0}^9 (-1)^n x[n] = 0.$

(c) $\sum_{k=0}^9 X[k] = 10 \cdot x[0] = -30.$

(d) The inverse DFT of $e^{-j2\pi k/5} X[k]$ is $x[\langle n-2 \rangle_{10}]$. Thus, $\sum_{k=0}^9 e^{-j2\pi k/5} X[k]$
 $= 10 \cdot x[\langle 0-2 \rangle_{10}] = 10 \cdot x[8] = -100.$

(e) From Parseval's relation, $\sum_{k=0}^9 |X[k]|^2 = 10 \cdot \sum_{n=0}^9 |x[n]|^2 = 38600.$

5.35 $X[7] = X^*[\langle -7 \rangle_{12}] = X^*[5] = 2 - j$, $X[8] = X^*[\langle -8 \rangle_{12}] = X^*[4] = -3 - j2$,
 $X[9] = X^*[\langle -9 \rangle_{12}] = X^*[3] = 6 - j3$, $X[10] = X^*[\langle -10 \rangle_{12}] = X^*[2] = 1 + j12$,
 $X[11] = X^*[\langle -11 \rangle_{12}] = X^*[1] = 8 + j2.$

(a) $x[0] = \frac{1}{12} \sum_{k=0}^{11} X[k] = 4.5,$

(b) $x[6] = \frac{1}{12} \sum_{k=0}^{11} (-1)^k X[k] = -0.8333,$

(c) $\sum_{n=0}^{11} x[n] = X[0] = 11,$

(d) Let $g[n] = e^{j2\pi n/3} x[n] = W_{12}^{-4n} x[n]$. Then, $\text{DFT}\{W_{12}^{-4n} x[n]\} = X[\langle k-4 \rangle_{12}].$

Thus, $\sum_{n=0}^{11} g[n] = \sum_{n=0}^{11} e^{j2\pi n/3} x[n] = X[\langle 0-4 \rangle_{12}] = X[8] = -3 - j2,$

(e) From Parseval's relation, $\sum_{n=0}^{11} |x[n]|^2 = \frac{1}{12} \sum_{k=0}^{11} |X[k]|^2 = 74.8333.$

5.36 Now, $y_C[n] = \sum_{\ell=0}^6 g[\ell]h[\langle n-\ell \rangle_7]$. Hence,

$$\begin{aligned} y_C[0] &= g[0]h[0] + g[1]h[6] + g[2]h[5] + g[3]h[4] + g[4]h[3] + g[5]h[2] + g[6]h[1], \\ y_C[1] &= g[0]h[1] + g[1]h[0] + g[2]h[6] + g[3]h[5] + g[4]h[4] + g[5]h[3] + g[6]h[2], \\ y_C[2] &= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[6] + g[4]h[5] + g[5]h[4] + g[6]h[3], \\ y_C[3] &= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] + g[4]h[6] + g[5]h[5] + g[6]h[4], \\ y_C[4] &= g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] + g[4]h[0] + g[5]h[6] + g[6]h[5], \\ y_C[5] &= g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0] + g[6]h[6], \\ y_C[6] &= g[0]h[6] + g[1]h[5] + g[2]h[4] + g[3]h[3] + g[4]h[2] + g[5]h[1] + g[6]h[0]. \end{aligned}$$

Likewise, $y_L[n] = \sum_{\ell=0}^6 g[\ell]h[n-\ell]$. Hence,

$$\begin{aligned} y_L[0] &= g[0]h[0], \\ y_L[1] &= g[0]h[1] + g[1]h[0], \\ y_L[2] &= g[0]h[2] + g[1]h[1] + g[2]h[0], \\ y_L[3] &= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0], \\ y_L[4] &= g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] + g[4]h[0], \\ y_L[5] &= g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0], \\ y_L[6] &= g[0]h[6] + g[1]h[5] + g[2]h[4] + g[3]h[3] + g[4]h[2] + g[5]h[1] + g[6]h[0], \\ y_L[7] &= g[1]h[6] + g[2]h[5] + g[3]h[4] + g[4]h[3] + g[5]h[2] + g[6]h[1], \\ y_L[8] &= g[2]h[6] + g[3]h[5] + g[4]h[4] + g[5]h[3] + g[6]h[2], \\ y_L[9] &= g[3]h[6] + g[4]h[5] + g[5]h[4] + g[6]h[3], \\ y_L[10] &= g[4]h[6] + g[5]h[5] + g[6]h[4], \\ y_L[11] &= g[5]h[6] + g[6]h[5], \\ y_L[12] &= g[6]h[6]. \end{aligned}$$

Comparing $y_C[n]$ with $y_L[n]$ we observe that

$$\begin{aligned} y_C[0] &= y_L[0] + y_L[7], \\ y_C[1] &= y_L[1] + y_L[8], \\ y_C[2] &= y_L[2] + y_L[9], \\ y_C[3] &= y_L[3] + y_L[10], \\ y_C[4] &= y_L[4] + y_L[11], \\ y_C[5] &= y_L[5] + y_L[12], \\ y_C[6] &= y_L[6]. \end{aligned}$$

5.37 Since $x[n]$ is a length-9 real sequence, its DFT satisfies $X[k] = X^*[\langle -k \rangle_9]$. Therefore, $X[1] = X^*[\langle -1 \rangle_9] = X^*[8] = -7.7 + j3.2$,

$$\begin{aligned}
X[3] &= X^*[\langle -3 \rangle_9] = X^*[6] = 8.6 + j9.6, \\
X[5] &= X^*[\langle -5 \rangle_9] = X^*[4] = -3.5 - j5.3, \\
X[7] &= X^*[\langle -7 \rangle_9] = X^*[2] = 1.2 + j4.1.
\end{aligned}$$

5.38 $X[1] = X^*[\langle -1 \rangle_9] = X^*[8] = 4.5 - j1.6,$
 $X[4] = X^*[\langle -4 \rangle_9] = X^*[5] = -3.1 - j8.2,$
 $X[6] = X^*[\langle -6 \rangle_9] = X^*[3] = -7.2 + 4.1,$
 $X[7] = X^*[\langle -7 \rangle_9] = X^*[2] = 1.2 + j2.3.$

5.39 Since the DFT $X[k]$ is real-valued, $x[n]$ is a circularly even sequence, i.e.,
 $x[n] = x[\langle -n \rangle_{12}]$. Therefore,
 $x[1] = x[\langle -1 \rangle_{12}] = x[11] = -2,$
 $x[4] = x[\langle -4 \rangle_{12}] = x[8] = 9.3,$
 $x[7] = x[\langle -7 \rangle_{12}] = x[5] = 4.1,$
 $x[9] = x[\langle -9 \rangle_{12}] = x[3] = -3.25,$
 $x[10] = x[\langle -10 \rangle_{12}] = x[2] = 0.7.$

5.40 Since the DFT $X[k]$ is imaginary-valued, $x[n]$ is a circularly odd sequence, i.e.,
 $x[n] = -x[\langle -n \rangle_{12}]$. Therefore,
 $x[7] = -x[\langle -7 \rangle_{12}] = -x[5] = 9.3,$
 $x[8] = -x[\langle -8 \rangle_{12}] = -x[4] = -2.87,$
 $x[9] = -x[\langle -9 \rangle_{12}] = -x[3] = -4.1,$
 $x[10] = -x[\langle -10 \rangle_{12}] = -x[2] = 3.25,$
 $x[11] = -x[\langle -11 \rangle_{12}] = -x[1] = -0.7.$

5.41 $X[k] = X^*[\langle -k \rangle_{174}] = X^*[174 - k].$
 $X[9] = X^*[174 - 9] = X^*[165] = -3.4 + j5.9 \Rightarrow X[165] = -3.4 - j5.9.$
 $X[51] = X^*[174 - 51] = X^*[123] = 5 - j1.6 \Rightarrow X[123] = 5 + j1.6.$
 $X[113] = X^*[174 - 113] = X^*[61] = 8.7 - j4.9 \Rightarrow X[61] = 8.7 + j4.9.$
 $X[162] = X^*[174 - 162] = X^*[12] = 7.1 - j2.4 \Rightarrow X[12] = 7.1 + j2.4.$
 $X[k_1] = 7.1 + j2.4, X[k_2] = 8.7 + j4.9, X[k_3] = 5 + j1.6, X[k_4] = -3.4 - j5.9.$

(a) Comparing these 4 DFT samples with the DFT samples given above we conclude
 $k_1 = 12, k_2 = 61, k_3 = 123, k_4 = 165.$

(b) dc value of $\{x[n]\} = X[0] = 11.$

(c)
$$\begin{aligned}
x[n] &= \frac{1}{174} \sum_{k=0}^{173} X[k] W_{174}^{-kn} = \frac{1}{174} (X[0] + 2 \operatorname{Re}\{X[9] W_{174}^{-9n}\} + 2 \operatorname{Re}\{X[51] W_{174}^{-51n}\} \\
&\quad + X[87] W_{174}^{-87n} + 2 \operatorname{Re}\{X[113] W_{174}^{-113n}\} + 2 \operatorname{Re}\{X[162] W_{174}^{-162n}\})
\end{aligned}$$

$$(d) \sum_{n=0}^{173} |x[n]|^2 = \frac{1}{174} \sum_{k=0}^{173} |X[k]|^2 = 86.0279.$$

$$5.42 \quad X[k] = X^*[\langle -k \rangle_{126}] = X^*[126 - k].$$

$$X[0] = 12.8 + j\alpha.$$

$$X[13] = X^*[126 - 13] = X^*[113] = -3.7 + j2.2 \Rightarrow X[113] = -3.7 - j2.2.$$

$$X[51] = X^*[126 - 51] = X^*[75] = -j1.7 \Rightarrow X[75] = j1.7.$$

$$X[13] = X^*[126 - 13] = X^*[113] = -3.7 + j2.2 \Rightarrow X[113] = -3.7 - j2.2.$$

$$X[k_1] = X^*[126 - k_1] = 9.1 - j5.4 \Rightarrow X[126 - k_1] = 9.1 + j5.4.$$

$$X[k_2] = X^*[126 - k_2] = 6.3 + j2.3 \Rightarrow X[126 - k_2] = 6.3 - j2.3.$$

$$X[k_3] = X^*[126 - k_3] = \gamma + j1.7 \Rightarrow X[126 - k_3] = \gamma - j1.7.$$

$$X[k_4] = X^*[126 - k_4] = -3.7 - j2.2 \Rightarrow X[126 - k_4] = -3.7 + j2.2.$$

(a) (b) Since $x[n]$ is a real-valued sequence of length 126, $X[0]$ and $X[63]$ must be real. Thus, $\alpha = 0$ and $\beta = 0$. As $X[126 - k_1]$ and $X[108]$ have the same imaginary part, $\varepsilon = 9.1$ and $k_1 = 126 - 108 = 18$. Likewise, as $X[k_2]$ and $X[79]$ have the same real part, $\delta = -2.3$ and $k_2 = 126 - 79 = 47$. Similarly, as $X[126 - k_3]$ and $X[51]$ have the same imaginary part, $\gamma = 0$ and $k_3 = 126 - 51 = 75$. Finally, as $X[k_4] = X[113]$, $k_4 = 113$.

(c) dc value of $x[n]$ is $X[0] = 12.8$.

$$(d) x[n] = \frac{1}{126} \sum_{k=0}^{125} X[k] W_{126}^{-kn} = \frac{1}{126} \left(X[0] + 2 \operatorname{Re}\{X[13] W_{126}^{-26\pi n / 126}\} \right. \\ \left. + 2 \operatorname{Re}\{X[18] W_{126}^{-36\pi n / 126}\} + 2 \operatorname{Re}\{X[47] W_{126}^{-94\pi n / 126}\} + X[63] W_{126}^{-63\pi n} \right. \\ \left. + 2 \operatorname{Re}\{X[75] W_{126}^{-150\pi n / 126}\} + 2 \operatorname{Re}\{X[113] W_{126}^{-226\pi n / 126}\} \right).$$

$$(e) \sum_{n=0}^{125} |x[n]|^2 = \frac{1}{126} \sum_{k=0}^{125} |X[k]|^2 = 5.767.$$

$$5.43 \quad Y[k] = W_3^{-2k} X[k] = W_9^{-6k} X[k]. \text{ Therefore, } y[n] = x[\langle n - 6 \rangle_9]. \text{ Thus,} \\ y[0] = x[3] = 4, y[1] = x[4] = -3, y[2] = x[5] = 5, y[3] = x[6] = -2, y[4] = x[7] = -2, \\ y[5] = x[8] = 4, y[6] = x[0] = 3, y[7] = x[1] = 5, y[8] = x[2] = 1.$$

$$5.44 \quad H[k] = H^*[\langle -k \rangle_9] = X^*[9 - k]. \text{ Hence, } H[5] = H^*[4] = -6.876 - j11.4883, \\ H[6] = H^*[3] = -j8.6603, H[7] = H^*[2] = 6.0346 + j1.957, \\ H[8] = H^*[1] = 6.8414 + j6.0572.$$

Now $g[n] = e^{j2\pi n/3}h[n] = e^{j6\pi n/9}h[n] = W_9^{-6n}h[n]$. Therefore,
 $G[k] = H[\langle k-6 \rangle_9]$, $0 \leq k \leq 8$. Thus, $G[0] = H[\langle -6 \rangle_9] = H[3] = j8.6603$,
 $G[1] = H[\langle 1-6 \rangle_9] = H[4] = -6.876 - j11.4883$,
 $G[2] = H[\langle 2-6 \rangle_9] = H[5] = H^*[4] = -6.876 + j11.4883$,
 $G[3] = H[\langle 3-6 \rangle_9] = H[6] = H^*[3] = -j8.6603$,
 $G[4] = H[\langle 4-6 \rangle_9] = H[7] = H^*[2] = 6.0346 + j1.957$,
 $G[5] = H[\langle 5-6 \rangle_9] = H[8] = H^*[1] = 6.8414 + j.0572$,
 $G[6] = H[\langle 6-6 \rangle_9] = H[0] = 15$,
 $G[7] = H[\langle 7-6 \rangle_9] = H[1] = 6.8414 - j6.0572$,
 $G[8] = H[\langle 8-6 \rangle_9] = H[2] = 6.0346 - j1.957$.

5.45 (a) $y_L[0] = g[0]h[0] = -4$,

$$y_L[1] = g[0]h[1] + g[1]h[0] = 10,$$

$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] = -6,$$

$$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] = 8,$$

$$y_L[4] = g[1]h[3] + g[2]h[2] = 7,$$

$$y_L[5] = g[2]h[3] = -3.$$

(b) $y_C[0] = g_e[0]h[0] + g_e[1]h[3] + g_e[2]h[2] + g_e[3]h[1]$
 $= g[0]h[0] + g[1]h[3] + g[2]h[2] = 3$,

$$y_C[1] = g_e[0]h[1] + g_e[1]h[0] + g_e[2]h[3] + g_e[3]h[2]$$

 $= g[0]h[1] + g[1]h[0] + g[2]h[3] = 7$,

$$y_C[2] = g_e[0]h[2] + g_e[1]h[1] + g_e[2]h[0] + g_e[3]h[3]$$

 $= g[0]h[2] + g[1]h[1] + g[2]h[0] = -6$,

$$y_C[3] = g_e[0]h[3] + g_e[1]h[2] + g_e[2]h[1] + g_e[3]h[0]$$

 $= g[0]h[3] + g[1]h[2] + g[2]h[1] = 8$.

(c)
$$\begin{bmatrix} G_e[0] \\ G_e[1] \\ G_e[2] \\ G_e[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1+j \\ 6 \\ -1-j \end{bmatrix},$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4-j5 \\ -3 \\ -4+j5 \end{bmatrix}.$$

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G_e[0]H[0] \\ G_e[1]H[1] \\ G_e[2]H[2] \\ G_e[3]H[3] \end{bmatrix} = \begin{bmatrix} 12 \\ 9+j \\ -18 \\ 9-j \end{bmatrix}. \text{ Therefore}$$

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 12 \\ 9+j \\ -18 \\ 9-j \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -6 \\ 8 \end{bmatrix}.$$

(d) $g_e[n] = [2, -1, 3, 0, 0, 0]$, $h_e[n] = [-2, 4, 2, -1, 0, 0]$

$$y_C[0] = g_e[0]h_e[0] + g_e[1]h_e[5] + g_e[2]h_e[4] + g_e[3]h_e[3] + g_e[4]h_e[2] + g_e[5]h_e[1]$$

$$= g[0]h[0] = -4 = y_L[0],$$

$$y_C[1] = g_e[0]h_e[1] + g_e[1]h_e[0] + g_e[2]h_e[5] + g_e[3]h_e[6] + g_e[4]h_e[3] + g_e[5]h_e[2]$$

$$= g[0]h[1] + g[1]h[0] = 10 = y_L[1],$$

$$y_C[2] = g_e[0]h_e[2] + g_e[1]h_e[1] + g_e[2]h_e[0] + g_e[3]h_e[5] + g_e[4]h_e[4] + g_e[5]h_e[3]$$

$$= g[0]h[2] + g[1]h[1] + g[2]h[0] = -6 = y_L[2],$$

$$y_C[3] = g_e[0]h_e[3] + g_e[1]h_e[2] + g_e[2]h_e[1] + g_e[3]h_e[0] + g_e[4]h_e[5] + g_e[5]h_e[4]$$

$$= g[0]h[3] + g[1]h[2] + g[2]h[1] = 8 = y_L[3],$$

$$y_C[4] = g_e[0]h_e[4] + g_e[1]h_e[3] + g_e[2]h_e[2] + g_e[3]h_e[1] + g_e[4]h_e[0] + g_e[5]h_e[5]$$

$$= g[1]h[3] + g[2]h[2] = 7 = y_L[4],$$

$$y_C[5] = g_e[0]h_e[5] + g_e[1]h_e[4] + g_e[2]h_e[3] + g_e[3]h_e[2] + g_e[4]h_e[1] + g_e[5]h_e[0]$$

$$= g[2]h[3] = -3 = y_L[5].$$

5.46 We need to show $g[n] \circledast h[n] = h[n] \circledast g[n]$. Let

$$x[n] = g[n] \circledast h[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N] \text{ and}$$

$$y[n] = h[n] \circledast g[n] = \sum_{m=0}^{N-1} h[m]g[\langle n-m \rangle_N] = \sum_{m=0}^n h[m]g[n-m] + \sum_{m=n+1}^{N-1} h[m]g[N+n-m]$$

$$= \sum_{m=0}^n h[n-m]g[m] + \sum_{m=n+1}^{N-1} h[N+n-m]g[m] = \sum_{m=0}^{N-1} h[\langle n-m \rangle_N]g[m] = x[n].$$

5.47 (a) $y[n] = x_1[n] \circledast x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[\langle n-m \rangle_N]$. Thus,

$$\sum_{n=0}^{N-1} y[n] = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x[\langle n-m \rangle_N] = \left(\sum_{n=0}^{N-1} x_1[n] \right) \left(\sum_{n=0}^{N-1} x_2[n] \right).$$

(b)

$$\sum_{n=0}^{N-1} (-1)^n y[n] = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x[\langle n-m \rangle_N] (-1)^n$$

$$= \left(\sum_{m=0}^{N-1} x_1[m] \right) \left(\sum_{n=0}^{m-1} x_2[N+m-n] (-1)^n + \sum_{n=m}^{N-1} x_2[m-n] (-1)^n \right). \text{ Replacing } n \text{ by}$$

$N + n - m$ in the first sum on the right-hand side and by $n - m$ in the second sum on the right-hand side we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} (-1)^n y[n] &= \left(\sum_{m=0}^{N-1} x_1[m] \right) \left(\sum_{n=0}^{m-1} x_2[n] (-1)^{n-N+m} + \sum_{n=m}^{N-1} x_2[n] (-1)^{n+m} \right) \\ &= \left(\sum_{n=0}^{N-1} (-1)^n x_1[n] \right) \left(\sum_{n=0}^{N-1} (-1)^n x_2[n] \right). \end{aligned}$$

5.48 $y[n] = x[3n]$, $0 \leq n \leq \frac{N}{3} - 1$. Therefore, $Y[k] = \sum_{n=0}^{\frac{N}{3}-1} y[n] W_{N/3}^{nk} = \sum_{n=0}^{\frac{N}{3}-1} x[4n] W_{N/3}^{nk}$.

Now, $x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_N^{-3mn} = \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_{N/3}^{-mn}$. Hence,

$$Y[k] = \frac{1}{N} \sum_{n=0}^{\frac{N}{3}-1} \sum_{m=0}^{N-1} X[m] W_{N/3}^{-mn} W_{N/3}^{nk} = \frac{1}{N} \sum_{m=0}^{N-1} X[m] \sum_{n=0}^{\frac{N}{3}-1} W_{N/3}^{(k-m)n}. \text{ Since}$$

$$\sum_{n=0}^{\frac{N}{3}-1} W_{N/3}^{(k-m)n} = \begin{cases} \frac{N}{3}, & m = k, k + \frac{N}{3}, k + \frac{2N}{3}, k + N, \\ 0, & \text{elsewhere.} \end{cases} \text{ Thus,}$$

$$Y[k] = \frac{1}{3} \left(X[k] + X[k + \frac{N}{3}] + X[k + \frac{2N}{3}] + X[k + N] \right).$$

5.49 $v[n] = x[n] + jy[n]$. Hence, $X[k] = \frac{1}{2} \{V[k] + V^*[\langle -k \rangle_8]\}$ is the 8-point DFT of $x[n]$

and $Y[k] = \frac{1}{2j} \{V[k] - V^*[\langle -k \rangle_8]\}$ is the 8-point DFT of $y[n]$.

$$V[k] = [3 + j7, -2 + j6, 1 - j5, 4 - j9, 5 + j2, 3 - j2, j4, -3 - j8].$$

$$V^*[\langle -k \rangle_8] = [3 + j7, -3 + j8, -j4, 3 + j2, 5 - j2, 4 + j9, 1 + j5, -2 - j6].$$

Therefore,

$$X[k] = \left[3 + j7, -\frac{5}{2} + j7, \frac{1}{2} + j\frac{9}{2}, \frac{7}{2} - j\frac{7}{2}, 5, \frac{7}{2} + j\frac{7}{2}, \frac{1}{2} - j\frac{9}{2}, -\frac{5}{2} - j7 \right],$$

$$Y[k] = \left[0, -1 - j\frac{1}{2}, -\frac{1}{2} - j\frac{1}{2}, -\frac{11}{2} - j\frac{1}{2}, 2, -\frac{11}{2} + j\frac{1}{2}, -\frac{1}{2} + j\frac{1}{2}, -1 + j\frac{1}{2} \right].$$

The IDFT of $V[k]$ obtained using MATLAB is given by

Columns 1 through 4

$$\begin{array}{cccc} 1.3750 & -0.6250i & -0.8044 & + 1.7223i & -1.7500 & + 1.2500i \\ -0.9331 & -0.1187i & & & & \end{array}$$

Columns 5 through 8

$$\begin{array}{cccc} 0.8750 & + 2.6250i & 2.5544 & - 0.2223i & 3.5000 & + 1.2500i \\ -1.8169 & + 1.1187i & & & & \end{array}$$

The same result is obtained by computing the IDFT $x[n]$ of $X[k]$ and the IDFT $y[n]$ of $Y[k]$ using MATLAB and then forming $x[n] + jy[n]$.

5.50 $v[n] = g[n] + jh[n] = [2 - j2, -1 + j4, 3 + j2, -j]$. Therefore,

$$\begin{bmatrix} V[0] \\ V[1] \\ V[2] \\ V[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 - j2 \\ -1 + j4 \\ 3 + j2 \\ -j \end{bmatrix} = \begin{bmatrix} 4 + j3 \\ 4 - j3 \\ 6 - j3 \\ -6 - j5 \end{bmatrix}, \text{ i.e.,}$$

$$V[k] = [4 + j3, 4 - j3, 6 - j3, -6 - j5].$$

Thus, $V^*[\langle -k \rangle_4] = [4 + j3, -6 + j5, 6 + j3, 4 + j3]$. Therefore,

$$G[k] = \frac{1}{2}(V[k] + V^*[\langle -k \rangle_4]) = [4, -1 + j, 6, -1 - j] \text{ and}$$

$$H[k] = \frac{1}{2j}(V[k] - V^*[\langle -k \rangle_4]) = [3, -4 - j5, -3, -4 + j5].$$

5.51 Let $p[n] = \text{IDFT}\{P[k]\}$. Thus,

$$\begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -5 \\ -2 + j5 \\ 4 \\ -2 - j5 \end{bmatrix} = \begin{bmatrix} -1.25 \\ -4.75 \\ 0.75 \\ 0.25 \end{bmatrix}$$

$$\begin{bmatrix} d[0] \\ d[1] \\ d[2] \\ d[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 3 \\ 4 + j \\ -7 \\ 4 - j \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}. \text{ Therefore,}$$

$$X(e^{j\omega}) = \frac{-1.25 - 4.75e^{-j\omega} + 0.75e^{-j2\omega} + 0.25e^{-j3\omega}}{1 + 2e^{-j\omega} - 3e^{-j2\omega} + 3e^{-j3\omega}}.$$

5.52 Let $p[n] = \text{IDFT}\{P[k]\}$. Thus,

$$\begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 8 \\ -5 - j6 \\ -3 \\ -5 + j6 \end{bmatrix} = \begin{bmatrix} -1.25 \\ 5.75 \\ 3.75 \\ -0.25 \end{bmatrix}$$

$$\begin{bmatrix} d[0] \\ d[1] \\ d[2] \\ d[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 3 \\ 4 + j \\ -7 \\ 4 - j \end{bmatrix} = \begin{bmatrix} 2.75 \\ -3.75 \\ -3.25 \\ -1.75 \end{bmatrix}. \text{ Therefore,}$$

$$X(e^{j\omega}) = \frac{-1.25 + 54.75e^{-j\omega} + 3.75e^{-j2\omega} - 0.25e^{-j3\omega}}{2.75 - 3.75e^{-j\omega} - 3.25e^{-j2\omega} - 1.75e^{-j3\omega}}.$$

5.53 $X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$ and $\hat{X}[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$. Now,

$$\begin{aligned}\hat{x}[n] &= \frac{1}{M} \sum_{k=0}^{M-1} \hat{X}[k] W_M^{-nk} = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{m=0}^{N-1} x[m] e^{-j2\pi km/M} W_M^{-nk} \\ &= \frac{1}{M} \sum_{m=0}^{N-1} x[m] \sum_{k=0}^{M-1} e^{-j2\pi k(m-n)/M} = \sum_{r=-\infty}^{\infty} x[n+rM].\end{aligned}$$

Thus, $\hat{x}[n]$ is obtained by shifting $x[n]$ in multiples of M and adding the shifted copies. Since the new sequence is obtained by shifting in multiples of M , hence, to recover the original sequence take any consecutive N samples in the range $0 \leq n \leq N-1$ for any value of r . This would be true only if the shifted copies of $x[n]$ did not overlap with each other, that is, if and only if $M \geq N$.

5.54 (a) $X(e^{j\omega}) = \sum_{n=0}^8 x[n] e^{-j\omega n}$. Therefore, $X_1[k] = \sum_{n=0}^8 x[n] e^{-j2\pi kn/12}$. Hence,

$$\begin{aligned}x_1[n] &= \frac{1}{12} \sum_{k=0}^{11} X[k] e^{j2\pi kn/12} = \frac{1}{12} \sum_{k=0}^{11} \left(\sum_{m=0}^8 x[m] e^{j2\pi km/12} \right) e^{j2\pi kn/12} \\ &= \frac{1}{12} \sum_{m=0}^8 x[m] \sum_{k=0}^{11} e^{j2\pi k(n-m)/12} = \sum_{r=-\infty}^{\infty} x[n+12r] \text{ using the results of Problem 5.53.}\end{aligned}$$

Since $M = 12$ and $N = 9$, $M > N$, and hence, $x[n]$ is recoverable from $x_1[n]$. In fact $x_1[n] = \{1, -2, 3, -4, 5, -4, 3, -2, 1, 0, 0, 0\}$, $0 \leq n \leq 11$, and $x[n]$ is given by the first 9 samples of $x_1[n]$.

(b) Here, $X_2[k] = \sum_{n=0}^8 x[n] e^{-j2\pi kn/8}$. Hence,

$$\begin{aligned}x_1[n] &= \frac{1}{8} \sum_{k=0}^7 X[k] e^{j2\pi kn/8} = \frac{1}{8} \sum_{k=0}^7 \left(\sum_{m=0}^8 x[m] e^{j2\pi km/8} \right) e^{j2\pi kn/8} \\ &= \frac{1}{8} \sum_{m=0}^8 x[m] \sum_{k=0}^7 e^{j2\pi k(n-m)/8} = \sum_{r=-\infty}^{\infty} x[n+8r] \text{ using the results of Problem 5.53.}\end{aligned}$$

Since $M = 8$ and $N = 9$, $M < N$, and hence, $x[n]$ is not recoverable from $x_1[n]$. In fact $x_2[n] = [2, -2, 3, -4, 5, -4, 3, -2, 2, -1, 1]$, $0 \leq n \leq 10$.

5.55 $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$. Let $n = N - m$. Then,

$$x[m] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-k(N-m)} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{km} = \frac{1}{N} \mathcal{F}\{X[k]\}. \text{ Therefore,}$$

$$x[N-n] = \frac{1}{N} \mathcal{F}\{X[k]\} = \frac{1}{N} \mathcal{F}\{\mathcal{F}\{x[n]\}\}, \text{ or, } \mathcal{F}\{\mathcal{F}\{x[n]\}\} = N \cdot x[N-n]. \text{ Hence,}$$

$$\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{x[n]\}\}\}\} = N^2 \cdot x[N-n].$$

$$5.56 \quad y[n] = x[n] \circledast h[n] = \sum_{k=0}^{100} x[k]h[n-k] = \sum_{k=0}^{100} h[k]x[n-k] = \sum_{k=17}^{36} h[k]x[n-k].$$

$$u[n] = x[n] \circledcirc h[n] = \sum_{k=0}^{50} h[k]x[\langle n-k \rangle_{51}] = \sum_{k=17}^{36} h[k]x[\langle n-k \rangle_{51}].$$

Now, for $n \geq 36$, $x[\langle n-k \rangle_{51}] = x[n-k]$. Thus, $y[n] = u[n]$ for $36 \leq n \leq 50$.

5.57 (a) Overlap-add method: Since the impulse response is of length and the DFT size to be used is 128, hence, the number of data samples required for each convolution will be $128 - 109 = 19$. Thus the total number of DFTs required for the length-1300 data sequence is $\left\lceil \frac{1300}{19} \right\rceil = 69$. Also, the DFT of the impulse response needs to be computed once. Hence, the total number of DFTs used are $= 69 + 1 = 70$. The total number of IDFTs used are $= 69$.

(b) Overlap-save method: In this case, since the first $110 - 1 = 109$ points are lost, we need to pad the data sequence with 109 zeros for a total length of 1409. Again, each convolution will result in $128 - 109 = 19$ correct values. Thus the total number of DFTs required for the data are $\left\lceil \frac{1409}{19} \right\rceil = 75$. Again, 1 DFT is required for the impulse response. The total number of DFTs used are $75 + 1 = 76$. The total number of IDFTs used are $= 75$.

$$5.58 \quad (a) \quad y[n] = \begin{cases} x[n/L], & n = 0, L, 2L, \dots, (N-1)L, \\ 0, & \text{elsewhere.} \end{cases}$$

$$Y[k] = \sum_{n=0}^{NL-1} y[n]W_{NL}^{nk} = \sum_{n=0}^{N-1} x[n]W_{NL}^{nLk} = \sum_{n=0}^{N-1} x[n]W_N^{nk}. \quad \text{For } k \geq N, \text{ let } k = k_o + rN$$

$$\text{where } k_o = \langle k \rangle_N. \text{ Then, } Y[k] = Y[k_o + rN] = \sum_{n=0}^{N-1} x[n]W_N^{n(k_o + rN)} = \sum_{n=0}^{N-1} x[n]W_N^{nk_o} \\ = X[k_o] = X[\langle k \rangle_N].$$

(b) Since $Y[k] = X[\langle k \rangle_5]$ for $0 \leq k \leq 20$, a sketch of $Y[k]$ is thus as shown below.

5.59 $x_0[n] = x[2n+1] + x[2n]$, $x_1[n] = x[2n+1] - x[2n]$, $y_0[n] = y[2n+1] + y[2n]$, and $y_1[n] = y[2n+1] - y[2n]$, $0 \leq n \leq \frac{N}{2} - 1$. Since $x[n]$ and $y[n]$ are real, symmetric sequences, it follows that $x_0[n]$ and $y_0[n]$ are real, symmetric sequences, and $x_1[n]$ and $y_1[n]$ are real, antisymmetric sequences. Now, consider the $\frac{N}{2}$ -length sequence $u[n] = x_0[n] + y_1[n] + j(x_1[n] + y_0[n])$. Its conjugate sequence is given by $u^*[n] = x_0[n] + y_1[n] - j(x_1[n] + y_0[n])$. Next, we observe that $u[\langle -n \rangle_{N/2}] = x_0[\langle -n \rangle_{N/2}] + y_1[\langle -n \rangle_{N/2}] + j(x_1[\langle -n \rangle_{N/2}] + y_0[\langle -n \rangle_{N/2}])$

$= x_0[n] - y_1[n] + j(-x_1[n] + y_0[n])$. Its conjugate sequence is given by $u^*[\langle -n \rangle_{N/2}] = x_0[n] - y_1[n] - j(-x_1[n] + y_0[n])$.

By adding the last 4 sequences we get

$$4x_0[n] = u[n] + u^*[n] + u[\langle -n \rangle_{N/2}] + u^*[\langle -n \rangle_{N/2}].$$

From Table 5.3, if $U[k] = \text{DFT}\{u[n]\}$, then $U^*[\langle -k \rangle_{N/2}] = \text{DFT}\{u^*[n]\}$,

$U^*[k] = \text{DFT}\{u^*[\langle -n \rangle_{N/2}]\}$, and $U[\langle -k \rangle_{N/2}] = \text{DFT}\{u[\langle -n \rangle_{N/2}]\}$. Thus,

$$X_0[k] = \text{DFT}\{x_0[n]\} = \frac{1}{4}(U[k] + U^*[\langle -k \rangle_{N/2}] + U[\langle -k \rangle_{N/2}] + U^*[k]).$$
 Similarly,

$$j4x_1[n] = u[n] - u^*[n] - u[\langle -n \rangle_{N/2}] + u^*[\langle -n \rangle_{N/2}].$$
 Hence,

$$X_1[k] = \text{DFT}\{x_1[n]\} = \frac{1}{j4}(U[k] - U^*[\langle -k \rangle_{N/2}] - U[\langle -k \rangle_{N/2}] + U^*[k]).$$
 Likewise,

$$4y_1[n] = u[n] - u[\langle -n \rangle_{N/2}] + u^*[n] - u^*[\langle -n \rangle_{N/2}].$$
 Thus,

$$Y_1[k] = \text{DFT}\{y_1[n]\} = \frac{1}{4}(U[k] - U[\langle -k \rangle_{N/2}] + U^*[\langle -k \rangle_{N/2}] - U^*[k]).$$
 Finally,

$$j4y_0[n] = u[n] + u[\langle -n \rangle_{N/2}] - u^*[n] - u^*[\langle -n \rangle_{N/2}].$$
 Hence,

$$Y_0[k] = \text{DFT}\{y_0[n]\} = \frac{1}{j4}(U[k] + U[\langle -k \rangle_{N/2}] - U^*[\langle -k \rangle_{N/2}] - U^*[k]).$$

5.60 $X_{GDFT}[k, a, b] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi(n+a)(k+b)}{N}}$.

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi(n+a)(k+b)}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} x[r] e^{-j \frac{2\pi(r+a)k+b}{N}} e^{j \frac{2\pi(n+a)k+b}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} x[r] e^{j \frac{2\pi(n+a-r-a)(k+b)}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} x[r] e^{j \frac{2\pi(n-r)(k+b)}{N}}$$

$$= \frac{1}{N} \sum_{r=0}^{N-1} x[r] \sum_{k=0}^{N-1} e^{j \frac{2\pi(n-r)(k+b)}{N}} = \frac{1}{N} \cdot x[n] \cdot N = x[n],$$
 as from Eq. (5.23)

$$\sum_{k=0}^{N-1} e^{j \frac{2\pi(n-r)(k+b)}{N}} = \begin{cases} N, & \text{if } n = r, \\ 0, & \text{otherwise.} \end{cases}$$

5.61 (a) $x[n] = \alpha g[n] + \beta h[n]$. Thus,

$$\begin{aligned} X_{\text{DCT}}[k] &= \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) = \sum_{n=0}^{N-1} (\alpha g[n] + \beta h[n]) \cos\left(\frac{\pi k(2n+1)}{2N}\right) \\ &= \sum_{n=0}^{N-1} \alpha g[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) + \sum_{n=0}^{N-1} \beta h[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) = \alpha G_{\text{DCT}}[k] + \beta H_{\text{DCT}}[k]. \end{aligned}$$

$$(b) G_{\text{DCT}}[k] = \sum_{n=0}^{N-1} g[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) \Rightarrow G_{\text{DCT}}^*[k] = \sum_{n=0}^{N-1} g^*[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right).$$

Therefore, $g^*[n] \stackrel{\text{DCT}}{\leftrightarrow} G_{\text{DCT}}^*[k]$.

$$(c) \text{ Note that } \sum_{n=0}^{N-1} \cos\left(\frac{\pi k(2n+1)}{2N}\right) \cos\left(\frac{\pi m(2n+1)}{2N}\right) = \begin{cases} N, & \text{if } k = m = 0, \\ N/2, & \text{if } k = m \text{ and } k \neq 0, \\ 0, & \text{otherwise.} \end{cases} \text{ Now,}$$

$$g[n]g^*[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \alpha[k]\alpha[m] G_{\text{DCT}}[k] G_{\text{DCT}}^*[m] \cos\left(\frac{\pi(2n+1)k}{2N}\right) \cos\left(\frac{\pi(2n+1)m}{2N}\right).$$

Thus,

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \alpha[k]\alpha[m] G_{\text{DCT}}[k] G_{\text{DCT}}^*[m] \cos\left(\frac{\pi(2n+1)k}{2N}\right) \cos\left(\frac{\pi(2n+1)m}{2N}\right).$$

Using the orthogonality property mentioned earlier we get

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{2N} \sum_{k=0}^{N-1} \alpha[k] |G_{\text{DCT}}[k]|^2.$$

$$5.62 (a) \mathbf{H}_N = \begin{bmatrix} 13 & 13 & 13 & 13 \\ 17 & 7 & -7 & -17 \\ 13 & -13 & -13 & 13 \\ 7 & -17 & 17 & -7 \end{bmatrix}. \text{ The matrix } \mathbf{H}_N \text{ is orthogonal if } \mathbf{H}_N \mathbf{H}_N^T = c\mathbf{I}$$

where \mathbf{I} is the 4×4 identity matrix and c is a constant. Now,

$$\mathbf{H}_N \mathbf{H}_N^T = \begin{bmatrix} 13 & 13 & 13 & 13 \\ 17 & 7 & -7 & -17 \\ 13 & -13 & -13 & 13 \\ 7 & -17 & 17 & -7 \end{bmatrix} \begin{bmatrix} 13 & 17 & 13 & 7 \\ 13 & 7 & -13 & -17 \\ 13 & -7 & -13 & 17 \\ 13 & -17 & 13 & -7 \end{bmatrix} = \begin{bmatrix} 676 & 0 & 0 & 0 \\ 0 & 676 & 0 & 0 \\ 0 & 0 & 676 & 0 \\ 0 & 0 & 0 & 676 \end{bmatrix}.$$

Hence, the matrix is orthogonal and all its rows have the same \mathcal{L}_2 -norm.

$$(b) \mathbf{G}_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}. \text{ Next, we observe}$$

$$\mathbf{G}_N \mathbf{G}_N^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -1 & -1 & -2 \\ 1 & -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \text{ which shows that}$$

the rows of \mathbf{G}_N are orthogonal but do not have the same \mathcal{L}_2 -norms.

5.63 (a) $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Now, $\frac{1}{2} \mathbf{H}_2^t \mathbf{H}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$.

$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}$. Thus,

$$\frac{1}{4} \mathbf{H}_4^t \mathbf{H}_4 = \frac{1}{4} \begin{bmatrix} \mathbf{H}_2^t & \mathbf{H}_2^t \\ \mathbf{H}_2^t & -\mathbf{H}_2^t \end{bmatrix} \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4\mathbf{I}_2 & 4\mathbf{I}_2 \\ 4\mathbf{I}_2 & 4\mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_4.$$

$$\frac{1}{N} \mathbf{H}_N^t \mathbf{H}_N = \frac{1}{N} \begin{bmatrix} \mathbf{H}_{N/2}^t & \mathbf{H}_{N/2}^t \\ \mathbf{H}_{N/2}^t & -\mathbf{H}_{N/2}^t \end{bmatrix} \begin{bmatrix} \mathbf{H}_{N/2} & \mathbf{H}_{N/2} \\ \mathbf{H}_{N/2} & -\mathbf{H}_{N/2} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \mathbf{M}_{N/2} & \mathbf{M}_{N/2} \\ \mathbf{M}_{N/2} & \mathbf{M}_{N/2} \end{bmatrix} = \mathbf{I}_N.$$

(b) From Eq. (5.172), $\sum_{n=0}^{N-1} |x[n]|^2 = \mathbf{x}^t \mathbf{x} = \left(\frac{1}{N} \mathbf{H}_N^t \mathbf{X}_{Haar} \right)^t \left(\frac{1}{N} \mathbf{H}_N^t \mathbf{X}_{Haar} \right)$

$$= \frac{1}{N^2} \mathbf{X}_{Haar}^t \mathbf{H}_N \mathbf{H}_N^t \mathbf{X}_{Haar} = \frac{1}{N} \mathbf{X}_{Haar}^t \mathbf{X}_{Haar} = \frac{1}{N} \sum_{k=0}^{N-1} |X_{Haar}[k]|^2 \text{ as from Eq.}$$

(5.171) $\mathbf{H}_N \mathbf{H}_N^t + N$.

5.64 $X_{DHT}[k] = \sum_{n=0}^{N-1} x[n] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right)$. Now,

$$X_{DHT}[k] \left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right)$$

$$= \sum_{n=0}^{N-1} x[n] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right). \text{ Therefore,}$$

$$\sum_{k=0}^{N-1} X_{DHT}[k] \left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right)$$

$$= \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right).$$

It can be shown that $\sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) \cdot \cos\left(\frac{2\pi mk}{N}\right) = \begin{cases} N, & \text{if } m = n = 0, \\ N/2, & \text{if } m = n \neq 0, \\ N/2, & \text{if } m = N - n, \\ 0, & \text{otherwise,} \end{cases}$

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi nk}{N}\right) \cdot \sin\left(\frac{2\pi mk}{N}\right) = \begin{cases} N, & \text{if } m = n = 0, \\ N/2, & \text{if } m = n \neq 0, \\ -N/2, & \text{if } m = N - n, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi nk}{N}\right) \cdot \cos\left(\frac{2\pi mk}{N}\right) = \sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) \cdot \sin\left(\frac{2\pi mk}{N}\right) = 0.$$

Hence, $x[m] = \frac{1}{N} \sum_{n=0}^{N-1} X_{DHT}[n] \left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right)$.

5.65 (a) $y[n] = x[\langle n - n_o \rangle_N] = \begin{cases} x[n - n_o + N], & 0 \leq n \leq n_o - 1, \\ x[n - n_o], & n_o \leq n \leq N - 1. \end{cases}$

$$\begin{aligned} Y_{\text{DHT}}[k] &= \sum_{n=0}^{N-1} y[n] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \\ &= \sum_{n=0}^{n_o-1} x[n - n_o + N] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) + \sum_{n=n_o}^{N-1} x[n - n_o] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right). \end{aligned}$$

Replacing $n - n_o + N$ by n in the first sum and $n - n_o$ by n in the second sum we get

$$\begin{aligned} Y_{\text{DHT}}[k] &= \sum_{n=N-n_o}^{N-1} x[n] \left(\cos\left(\frac{2\pi(n+n_o)k}{N}\right) + \sin\left(\frac{2\pi(n+n_o)k}{N}\right) \right) \\ &\quad + \sum_{n=0}^{n_o-1} x[n] \left(\cos\left(\frac{2\pi(n+n_o)k}{N}\right) + \sin\left(\frac{2\pi(n+n_o)k}{N}\right) \right) \\ &= \sum_{n=0}^{N-1} x[n] \left(\cos\left(\frac{2\pi(n+n_o)k}{N}\right) + \sin\left(\frac{2\pi(n+n_o)k}{N}\right) \right) \\ &= \cos\left(\frac{2\pi n_o k}{N}\right) \sum_{n=0}^{N-1} x[n] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \\ &\quad + \sin\left(\frac{2\pi n_o k}{N}\right) \sum_{n=0}^{N-1} x[n] \left(\cos\left(\frac{2\pi nk}{N}\right) - \sin\left(\frac{2\pi nk}{N}\right) \right) \\ &= \cos\left(\frac{2\pi n_o k}{N}\right) X_{\text{DHT}}[k] + \sin\left(\frac{2\pi n_o k}{N}\right) X_{\text{DHT}}[-k]. \end{aligned}$$

(b) The N -point DHT of $x[\langle -n \rangle_N]$ is $X_{\text{DHT}}[-k]$.

(c)
$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} X_{\text{DHT}}[k] X_{\text{DHT}}[\ell] \times \left(\sum_{n=0}^{N-1} \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \left(\cos\left(\frac{2\pi n\ell}{N}\right) + \sin\left(\frac{2\pi n\ell}{N}\right) \right) \right).$$

Using the orthogonality property, we observe that the above product is equal to N if

$k = \ell$ and is equal to zero if $k \neq \ell$. Hence,
$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N} \sum_{k=0}^{N-1} (X_{\text{DHT}}[k])^2.$$

5.66 $\cos\left(\frac{2\pi nk}{N}\right) = \frac{1}{2} (W_N^{-nk} + W_N^{nk})$ and $\sin\left(\frac{2\pi nk}{N}\right) = \frac{1}{2j} (W_N^{-nk} - W_N^{nk})$. Therefore,

$$\begin{aligned} X_{\text{DHT}}[k] &= \frac{1}{2} \sum_{n=0}^{N-1} x[n] (W_N^{-nk} + W_N^{nk} - j W_N^{-nk} + j W_N^{nk}) \\ &= \frac{1}{2} (X[N-k] + X[k] - j X[N-k] + j X[k]). \end{aligned}$$

5.67 $y[n] = x[n] \circledast g[n]$. Thus, $Y_{\text{DHT}}[k] = \sum_{n=0}^{N-1} y[n] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right)$
 $= \sum_{r=0}^{N-1} x[r] \sum_{n=0}^{N-1} g[\langle n-r \rangle_N] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right).$

From the results of Problem 5.65 we have

$$\begin{aligned} Y_{\text{DHT}}[k] &= \sum_{\ell=0}^{N-1} x[\ell] \left(G_{\text{DHT}}[k] \cos\left(\frac{2\pi \ell k}{N}\right) + G_{\text{DHT}}[\langle -k \rangle_N] \sin\left(\frac{2\pi \ell k}{N}\right) \right) \\ &= G_{\text{DHT}}[k] \sum_{\ell=0}^{N-1} x[\ell] \cos\left(\frac{2\pi \ell k}{N}\right) + G_{\text{DHT}}[\langle -k \rangle_N] \sum_{\ell=0}^{N-1} x[\ell] \sin\left(\frac{2\pi \ell k}{N}\right) \\ &= \frac{1}{2} G_{\text{DHT}}[k] (X_{\text{DHT}}[k] + X_{\text{DHT}}[\langle -k \rangle_N]) + \frac{1}{2} G_{\text{DHT}}[\langle -k \rangle_N] (X_{\text{DHT}}[k] - X_{\text{DHT}}[\langle -k \rangle_N]) \\ &= \frac{1}{2} X_{\text{DHT}}[k] (G_{\text{DHT}}[k] + G_{\text{DHT}}[\langle -k \rangle_N]) + \frac{1}{2} X_{\text{DHT}}[\langle -k \rangle_N] (G_{\text{DHT}}[k] - G_{\text{DHT}}[\langle -k \rangle_N]). \end{aligned}$$

5.68 (a) $y[n] = \sum_{k=0}^{N-1} (\beta_1 W_N^{-nk} + \beta_2 W_N^{nk}) X_{\text{DCFT}}[k]$
 $= \sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} (\alpha_1 \beta_1 W_N^{(m-n)k} + \alpha_2 \beta_2 W_N^{-(m+n)k} + \alpha_1 \beta_2 W_N^{(m+n)k} + \alpha_1 \beta_2 W_N^{-(m-n)k})$
 $= \sum_{m=0}^{N-1} x[m] N ((\alpha_1 \beta_1 + \alpha_2 \beta_2) \delta[m-n] + (\alpha_2 \beta_1 + \alpha_1 \beta_2) (\delta[m+n] + \delta[m+n-N]))$
 $= N [(\alpha_1 \beta_1 + \alpha_2 \beta_2) x[n] + (\alpha_2 \beta_1 + \alpha_1 \beta_2) (x[n] \delta[n] + x[N-n] (1 - \delta[n]))]$

If we require

$$y[n] = x[n], \quad 0 \leq n \leq N-1,$$

then the following conditions must be satisfied:

$$\alpha_2 \beta_1 + \alpha_1 \beta_2 = 0, \quad (5-1)$$

and

$$N(\alpha_1 \beta_1 + \alpha_2 \beta_2) = 1. \quad (5-2)$$

(b) Let $\alpha_1^2 - \alpha_2^2 \neq 0$. Solving for β_1 and β_2 in Eqs. (5-1) and (5-2) we arrive at

$$\beta_1 = \frac{\alpha_1}{N(\alpha_1^2 - \alpha_2^2)}, \quad \beta_2 = \frac{-\alpha_2}{N(\alpha_1^2 - \alpha_2^2)}.$$

Then, the inverse DCFT is given by

$$x[n] = \frac{1}{N(\alpha_1^2 - \alpha_2^2)} \sum_{k=0}^{N-1} (\alpha_1 W_N^{-nk} - \alpha_2 W_N^{nk}) X_{\text{DCFT}}[k], \quad 0 \leq n \leq N-1.$$

(c) If $\alpha_1 = \alpha_2^* = \alpha_{\text{re}} + j\alpha_{\text{im}}$ with $\alpha_{\text{re}} \neq 0$ and $\alpha_{\text{im}} \neq 0$, the expression for $X_{\text{DCFT}}[k]$ reduces to

$$X_{\text{DCFT}}[k] = \sum_{n=0}^{N-1} \left(2\alpha_{\text{re}} \cos\left(\frac{2\pi nk}{N}\right) + 2\alpha_{\text{im}} \sin\left(\frac{2\pi nk}{N}\right) \right) x[n]$$

which is real. The inverse DCFT is then given by

$$x[n] = \frac{1}{4N\alpha_1\alpha_2} \sum_{k=0}^{N-1} \left(2\alpha_{\text{im}} \cos\left(\frac{2\pi nk}{N}\right) + 2\alpha_{\text{re}} \sin\left(\frac{2\pi nk}{N}\right) \right) X_{\text{DCFT}}[k].$$

(d) It can be easily shown that the discrete Hartley transform (DHT) of Eq. (5.192) is a special case of the real DCFT with $\alpha_{\text{re}} = \alpha_{\text{im}} = \frac{1}{2N}$.

5.69 (a) $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, and

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

(b) From the structure of \mathbf{H}_2 , \mathbf{H}_4 , and \mathbf{H}_8 , it can be seen that $\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}$,

and $\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix}$.

(c) $\mathbf{X}_{HT} = \mathbf{H}_N \mathbf{x}$. Therefore, $\mathbf{x} = \mathbf{H}_N^{-1} \mathbf{X}_{HT} = N \cdot \mathbf{H}_N^T \mathbf{X}_{HT} = N \cdot \mathbf{H}_N^* \mathbf{X}_{HT}$. Hence,

$$x[n] = \sum_{k=0}^{N-1} \mathbf{X}_{HT}[k] (-1)^{\sum_{i=0}^{\ell-1} b_i(n) b_i(k)}, \quad \text{where } b_i(r) \text{ is the } i\text{-th bit in the binary representation of } r.$$

M5.1 (a)

```
N = input('The value of N = ');
k = -N:N;
y = ones(1, 2*N+1);
w = 0:2*pi/255:2*pi;
Y = freqz(y, 1, w);
Ydft = fft(y);
n = 0:1:2*N;
plot(w/pi, abs(Y), n*2/(2*N+1), abs(Ydft), 'o');
xlabel('\omega/\pi'), ylabel('Amplitude');
```

(b) Replace the statement $k = -N:N$; with $k = 0:N$; $y = \text{ones}(1, 2*N+1)$; with $y = \text{ones}(1, N+1)$; the statement $n = 0:1:2*N$; with $n = 0:N$; and the statement $\text{plot}(w/\pi, \text{abs}(Y), n*2/(2*N+1), \text{abs}(Y_{\text{dft}}), 'o')$; with $\text{plot}(w/\pi, \text{abs}(Y), n*2/(N+1), \text{abs}(Y_{\text{dft}}), 'o')$; , in the program of Part (a).

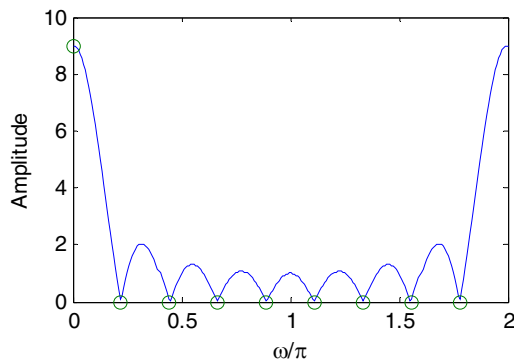
(c) Add the statement $y = y - \text{abs}(k)/N$; below the statement $y = \text{ones}([1, 2*N+1])$; in the program of Part (a).

(d) Replace the statement $y = \text{ones}(1, 2*N+1)$; with $y = N + \text{ones}(1, 2*N+1) - \text{abs}(k)$; in the program of Part (a).

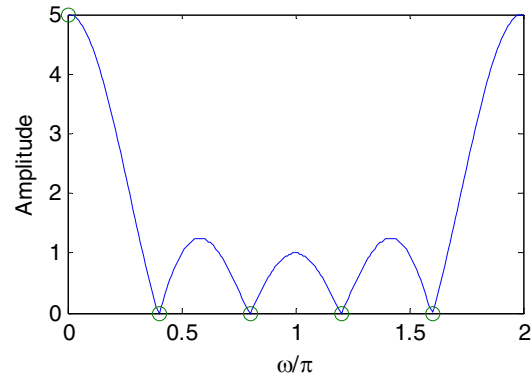
(e) Replace the statement $y = \text{ones}(1, 2*N+1)$; with $y = \cos(\pi*k/(2*N))$; in the program of Part (a).

The plots generated for $N = 4$ are shown below where the circles denote the DFT samples.

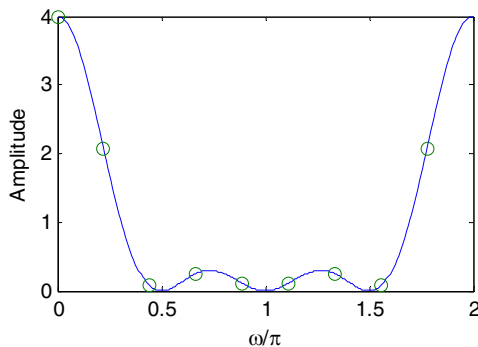
(a)



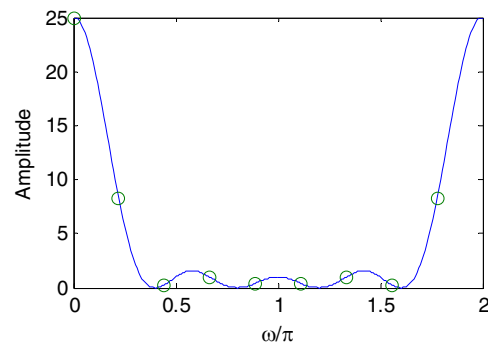
(b)



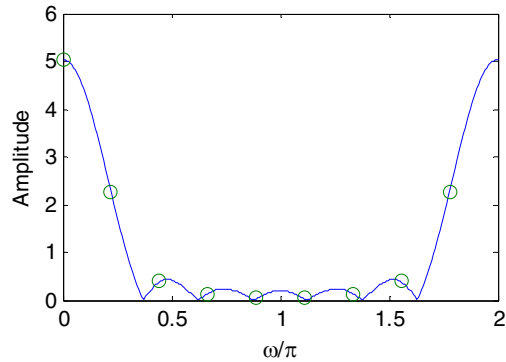
(c)



(d)



(e)



M5.2 The code fragments to be used are as follows:

```
Y = fft(g).*fft(h);
y = ifft(Y);
```

(a) $y[n] = g[n] \otimes h[n]$. The output generated using the above code fragments is

```
Y =
    -6     9   -16    20    -4    45
```

(b) $w[n] = x[n] \otimes v[n]$. The output generated using the above code fragments is

```
w =
Columns 1 through 3
11.0000 +25.0000i   -9.0000 +48.0000i    3.0000 +17.0000i

Columns 4 through 5
29.0000 + 0.0000i   -10.0000 +12.0000i
```

(c) $u[n] = x[n] \otimes y[n]$. The output generated using the above code fragments is

```
u =
   -23.0000   -69.0000    35.0000   105.0000    73.0000
```

```
M5.3 N = 8; % sequence length
gamma = 0.5;
k = 0:N-1;
x = exp(-gamma*k);
X = fft(x);

% Property 1
X1 = fft(conj(x));
G1 = conj([X(1) X(N:-1:2)]);
% Verify X1 = G1

% Property 2
x2 = conj([x(1) x(N:-1:2)]);
X2 = fft(x2);
```

```

% Verify X2 = conj(X)

% Property 3
x3 = real(x);
X3 = fft(x3);
G3 = 0.5*(X+conj([X(1) X(N:-1:2)]));
% Verify X3 = G3

% Property 4
x4 = j*imag(x);
X4 = fft(x4);
G4 = 0.5*(X-conj([X(1) X(N:-1:2)]));
% Verify X4 = G4

% Property 5
x5 = 0.5*(x+conj([x(1) x(N:-1:2)]));
X5 = fft(x5);
% Verify X5 = real(X)

% Property 6
x6 = 0.5*(x-conj([x(1) x(N:-1:2)]));
X6 = fft(x6);
% Verify X6 = j*imag(X)

```

```

M5.4 N = 8;
k = 0:N-1;
gamma = 0.5;
x = exp(-gamma*k);
X = fft(x);

% Property 1
xpe = 0.5*(x+[x(1) x(N:-1:2)]);
xpo = 0.5*(x-[x(1) x(N:-1:2)]);
Xpe = fft(xpe);
Xpo = fft(xpo);
% Verify Xpe = real(X) and Xpo = j*imag(X)

% Property 2
X2 = [X(1) X(N:-1:2)];
% Verify X = conj(X2);
% real(X) = real(X2) and imag(X) = -imag(X2)
% abs(X) = abs(X2) and angle(X) = -angle(X2)

```

```

M5.5 N = 8; % N is length of the sequence(s)
gamma = 0.5;
k = 0:N-1;
g = exp(-gamma*k); h = cos(pi*k/N);
G = fft(g); H=fft(h);

% Property 1

```

```

alpha=0.5; beta=0.25;
x1 = alpha*g+beta*h;
X1 = fft(x1);
% Verify X1=alpha*G+beta*H

% Property 2
n0 = N/2; % n0 is the amount of shift
x2 = [g(n0+1:N) g(1:n0)];
X2 = fft(x2);
% Verify X2(k)= exp(-j*k*n0)G(k)

% Property 3
k0 = N/2;
x3 = exp(-j*2*pi*k0*k/N).*g;
X3 = fft(x3);
G3 = [G(k0+1:N) G(1:k0)];
% Verify X3=G3

% Property 4
x4 = G;
X4 = fft(G);
G4 = N*[g(1) g(8:-1:2)]; % This forms N*(g mod(-k))
% Verify X4 = G4;

% Property 5
% To calculate circular convolution between
% g and h use eqn (3.67)
h1 = [h(1) h(N:-1:2)];
T = toeplitz(h',h1);
x5 = T*g';
X5 = fft(x5');
% Verify X5 = G.*H

% Property 6
x6 = g.*h;
X6 = fft(x6);
H1 = [H(1) H(N:-1:2)];
T = toeplitz(H.', H1); % .' is the nonconjugate transpose
G6 = (1/N)*T*G.';
% Verify G6 = X6.'

M5.6 g = input('Type in first sequence = ');
h = input('Type in second sequence = ');
x = g +i*h;
XF = fft(x);
XFconj = conj(XF);
N = length(g);
YF = zeros(N-1,1)';
for k = 1:N-1;
    YF(k) = XFconj(mod(-k,N)+1);

```

```

end
YF1 = [XFconj(1) YF];
GF = (XF + YF1)/2;
HF = (XF - YF1)/2;

```

M5.7

```

x = [-3 5 45 -15 -9 -19 -8 21 -10 23];
XF = fft(x);
k = 0:9; YF = exp(-i*2*pi*k/5).*XF;
output = [XF(1) XF(6) sum(XF) sum(YF)];;
disp(output)
disp(sum(abs(XF).*abs(XF)))

```

The output data generated by this program is

```

30      0      -30-0.0000i  -100-0.0000i
38600

```

M5.8

```

X = [11 8-i*2 1-i*12 6+i*3 -3+i*2 2+i 15];
k = 8:12; XF(k)=conj(X(mod(-k+2,12)));
XF = [X XF(8:12)];
x = ifft(XF);
n = 0:11; y = exp(i*2*pi*n/3).*x;
output = [x(1) x(7) sum(x) sum(y)];
disp(output)
disp(x.*x)

```

The output data generated by this program is

```

4.5000   -0.8333    11.0000   -3.0000-2.0000i
74.8333

```

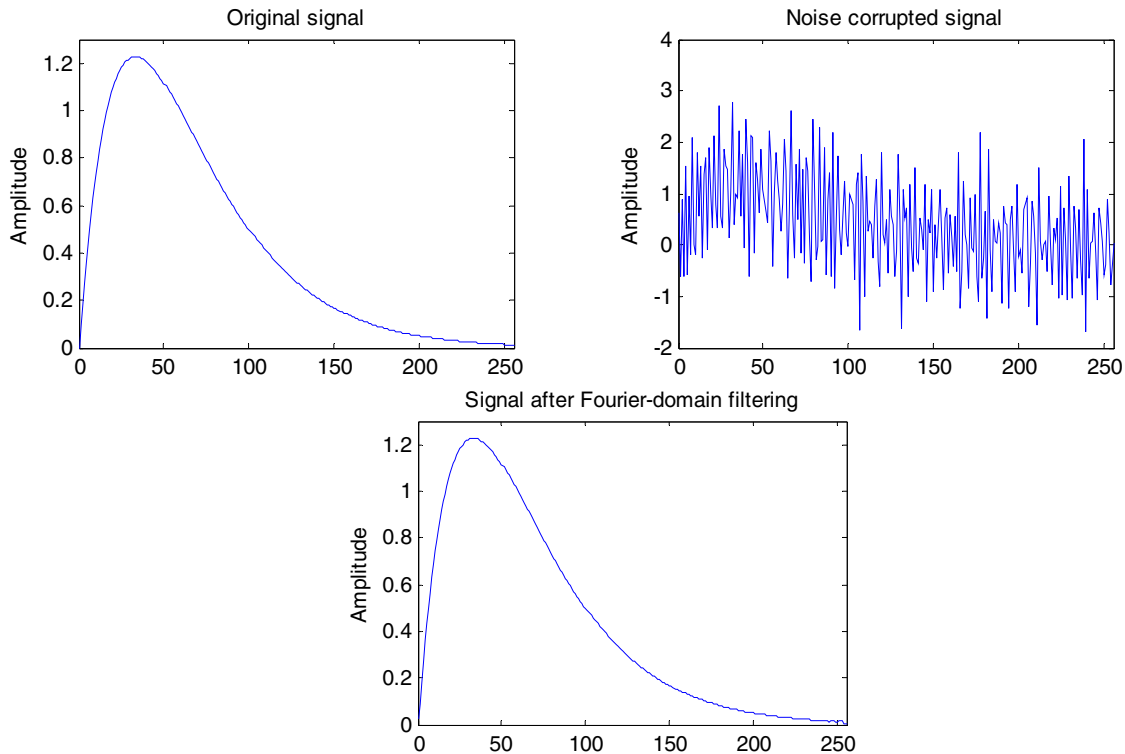
M5.9

```

n=0:255;
x = 0.1*n.*exp(-0.03*n);
plot(n,x);axis([0 255 0 1.3]);
xlabel('n');ylabel('Amplitude');
title('Original signal');
pause
z = [zeros(1,50) ones(1,156) zeros(1,50)];
y = 4*rand(1,256)-1;
YF = z.*fft(y);
yinv = ifft(YF);
s = x + yinv;
plot(n,s);axis([0 255 -2 4]);
xlabel('n');ylabel('Amplitude');
title('Noise corrupted signal');
pause
zc = [ones(1,50) zeros(1,156) ones(1,50)];
SF = zc.*fft(s);
xr = ifft(SF);
plot(n,xr);axis([0 255 0 1.3]);
xlabel('n');ylabel('Amplitude');

```

```
title('Signal after Fourier-domain filtering');
```



```
M5.10 function y = overlapsave(x,h)
X = length(x);      %Length of longer sequence
M = length(h);      %length of shorter sequence
flops(0);
if (M > X)          %Error condition
    disp('error');
end
%clear all
temp = ceil(log2(M)); %Find length of circular
convolution
N = 2^temp;         %zero padding the shorter sequence
if(N > M)
    for i = M+1:N
        h(i) = 0;
    end
end
end
m = ceil((-N/(N-M+1)));
while (m*(N-M+1) <= X)
    if((N+m*(N-M+1)) <= X & ((m*(N-M+1)) > 0))
        for n = 1:N
            x1(n) = x(n+m*(N-M+1));
        end
    end
end
if((m*(N-M+1)) <= 0 & ((N+m*(N-M+1)) >= 0))
%underflow adjustment
```

```

        for n = 1:N
            x1(n) = 0;
        end
        for n = m*(N-M+1):N+m*(N-M+1)
            if(n > 0)
                x1(n-m*(N-M+1)) = x(n);
            end
        end
    end
end
if((N+m*(N-M+1)) > X) %overflow adjustment
    for n = 1:N
        x1(n) = 0;
    end
    for n = 1:(X-m*(N-M+1))
        x1(n) =x (m*(N-M+1)+n);
    end
end
end

w1 = circonv1(h,x1); %circular convolution using DFT
for i = 1:M-1
    y1(i) = 0;
end
for i = M:N
    y1(i) = w1(i);
end
for j = M:N
    if((j+m*(N-M+1)) < (X+M))
        if((j+m*(N-M+1)) > 0)
            y0(j+m*(N-M+1)) = y1(j);
        end
    end
end
end
m = m+1;
end
%disp('Convolution using Overlap Save:');
y = real(y0);

function y = circonv1(x1,x2)
L1 = length(x1); L2 = length(x2);
if L1 ~= L2,
    error('Sequences of unequal lengths'),
end
X1 = fft(x1);
X2 = fft(x2);
X_RES = X1.*X2;
y = ifft(X_RES);

```

The MATLAB program for performing convolution using the overlap-save method is

```

h = [1 1 1]/3;
R = 50;

```

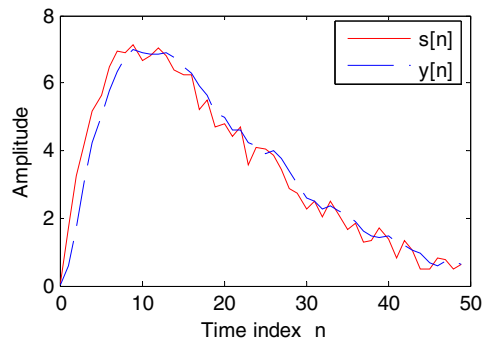


```

d = rand(1,R) - 0.5;
m = 0:1:R-1;
s = 2*m.*(0.9.^m);
x = s + d;
%x = [x x x x x x x];
y = overlapsave(x,h);
k = 0:R-1;
plot(k,x,'r-',k,y(1:R),'b--');
xlabel('Time index n');ylabel('Amplitude');
legend('r-', 's[n]', 'b--', 'y[n]');

```

The output plot generated by the above program is shown below:



Chapter 6

6.1 $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n}$. Therefore, $\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x[n]z^{-n}$

$$= \lim_{z \rightarrow \infty} x[0] + \lim_{z \rightarrow \infty} \sum_{n=1}^{\infty} x[n]z^{-n} = x[0].$$

6.2 (a) $Z\delta\{[n]\} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = \delta[0] = 1$, which converges everywhere in the z -plane.

(b) $x[n] = \alpha^n \mu[n]$. From Table 6.1,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|. \text{ Let } g[n] = nx[n]. \text{ Then,}$$

$$Z\{g[n]\} = G(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}. \text{ Now, } \frac{dX(z)}{dz} = - \sum_{n=-\infty}^{\infty} ng[n]z^{-n-1}. \text{ Hence,}$$

$$z \frac{dX(z)}{dz} = - \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -G(z), \text{ or, } G(z) = -z \frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \quad |z| > |\alpha|.$$

(c) $x[n] = r^n \sin(\omega_o n) \mu[n] = \frac{r^n}{2j} (e^{j\omega_o n} - e^{-j\omega_o n}) \mu[n]$. Using the results of Example

6.1 and the linearity property of the z -transform we get

$$\begin{aligned} Z\{r^n \sin(\omega_o n) \mu[n]\} &= \frac{1}{2j} \left(\frac{1}{1 - r e^{j\omega_o} z^{-1}} \right) - \frac{1}{2j} \left(\frac{1}{1 - r e^{-j\omega_o} z^{-1}} \right) \\ &= \frac{\frac{r}{2j} (e^{j\omega_o} - e^{-j\omega_o}) z^{-1}}{1 - r(e^{j\omega_o} + e^{-j\omega_o}) z^{-1} + r^2 z^{-2}} = \frac{r \sin(\omega_o) z^{-1}}{1 - 2r \cos(\omega_o) z^{-1} + r^2 z^{-2}}, \quad \forall |z| > |r|. \end{aligned}$$

6.3 (a) $x_1[n] = \alpha^n \mu[n-2]$. Note, $x_1[n]$ is a right-sided sequence. Hence, the ROC of its

z -transform is exterior to a circle. Therefore, $X_1(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n-2] z^{-n} = \sum_{n=2}^{\infty} \alpha^n z^{-n}$

Simplifying we get $X_1(z) = \frac{1}{1 - \alpha z^{-1}} - 1 - \alpha z^{-1} = \frac{\alpha^2 z^{-2}}{1 - \alpha z^{-1}}$ whose ROC is given by

$$|z| > |\alpha|.$$

(b) $x_2[n] = -\alpha^n \mu[-n-3]$. Note, $x_2[n]$ is a left-sided sequence. Hence, the ROC of its z -transform is interior to a circle. Therefore,

$$X_2(z) = - \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n-3] z^{-n} = - \sum_{n=-\infty}^{-3} \alpha^n z^{-n} = - \sum_{m=3}^{\infty} \alpha^{-m} z^m = - \sum_{m=3}^{\infty} (z/\alpha)^m$$

Simplifying we get $X_2(z) = \frac{(z/\alpha)^3}{1-(z/\alpha)}$ whose ROC is given by $|z| < |\alpha|$.

(c) $x_3[n] = \alpha^n \mu[n+4]$. Note, $x_3[n]$ is a right-sided sequence. Hence, the ROC of its z -transform is exterior to a circle. Therefore, $X_3(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n+4] z^{-n} = \sum_{n=-4}^{\infty} \alpha^n z^{-n}$

Simplifying we get $X_3(z) = \frac{(\alpha/z)^4}{1-(\alpha/z)}$ whose ROC is given by $|z| > |\alpha|$.

(d) $x_4[n] = \alpha^n \mu[-n]$. Note, $x_4[n]$ is a left-sided sequence. Hence, the ROC of its z -transform is interior to a circle. Therefore, $X_4(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n] z^{-n} = \sum_{n=-\infty}^0 \alpha^n z^{-n}$
 $= \sum_{m=0}^{\infty} \alpha^{-m} z^m = \frac{1}{1-(z/\alpha)}, |z/\alpha| < 1$. Therefore the ROC of $X_4(z)$ is given by $|z| < |\alpha|$.

6.4 $Z\{(0.4)^n \mu[n]\} = \frac{1}{1-0.4z^{-1}}, |z| > 0.4$; $Z\{(-0.6)^n \mu[n]\} = \frac{1}{1+0.6z^{-1}}, |z| > 0.6$;

$Z\{(0.4)^n \mu[-n-1]\} = -\frac{1}{1-0.4z^{-1}}, |z| < 0.4$;

$Z\{(-0.6)^n \mu[-n-1]\} = -\frac{1}{1+0.6z^{-1}}, |z| < 0.6$;

(a) $Z\{x_1[n]\} = \frac{1}{1-0.4z^{-1}} + \frac{1}{1+0.6z^{-1}} = \frac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}, |z| > 0.6$.

(b) $Z\{x_2[n]\} = \frac{1}{1-0.4z^{-1}} + \frac{1}{1+0.6z^{-1}} = \frac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}, 0.4 < |z| < 0.6$.

(c) $Z\{x_3[n]\} = \frac{1}{1-0.4z^{-1}} + \frac{1}{1+0.6z^{-1}} = \frac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}, |z| < 0.4$.

(d) $Z\{x_4[n]\} = \frac{1}{1-0.4z^{-1}} + \frac{1}{1+0.6z^{-1}} = \frac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$. Since the ROC of

the first term is $|z| < 0.4$ and that of the second term is $|z| > 0.6$. Hence, the z -transform of $x_4[n]$ does not converge.

6.5 (a) The ROC of $Z\{x_1[n]\}$ is $|z| > 0.3$, the ROC of $Z\{x_2[n]\}$ is $|z| > 0.7$, the ROC of $Z\{x_3[n]\}$ is $|z| > 0.4$, and the ROC of $Z\{x_4[n]\}$ is $|z| < 0.4$.

(b) (i) The ROC of $Z\{y_1[n]\}$ is $|z| > 0.7$,

(ii) The ROC of $Z\{y_2[n]\}$ is $|z| > 0.4$,

(iii) The ROC of $Z\{y_3[n]\}$ is $0.3 < |z| < 0.4$,

(iv) The ROC of $Z\{y_4[n]\}$ is $|z| > 0.7$,

(v) The ROC of $Z\{x_2[n]\}$ is $|z| > 0.7$, whereas, the ROC of $Z\{x_4[n]\}$ is $|z| < 0.4$.

Hence, the z -transform of $y_5[n]$ does not converge.

(vi) The ROC of $Z\{x_3[n]\}$ is $|z| > 0.4$, whereas, the ROC of $Z\{x_4[n]\}$ is $|z| < 0.4$.

Hence, the z -transform of $y_6[n]$ does not converge.

6.6 $v[n] = \alpha^{|n|} = \alpha^n \mu[n] + \alpha^{-n} \mu[-n-1]$. Now, $Z\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha z^{-1}}$, $|z| > |\alpha|$. (See Table 6.1) and $Z\{\alpha^{-n} \mu[-n-1]\} = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} = \sum_{m=1}^{\infty} \alpha^m z^m = \sum_{m=0}^{\infty} \alpha^m z^m - 1 = \frac{1}{1-\alpha z} - 1 = \frac{\alpha z}{1-\alpha z}$, $|\alpha z| < 1$. Therefore, $\{v[n]\} = V(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha}{z^{-1}-\alpha} = \frac{(1-\alpha^2)z^{-1}}{(1-\alpha z^{-1})(z^{-1}-\alpha)}$ with its ROC given by $|\alpha| < |z| < 1/|\alpha|$.

6.7 (a) $x_1[n] = \alpha^n \mu[n+2] + \beta^n \mu[n+2]$ with $|\beta| > |\alpha|$. Note that $x_1[n]$ is a right-sided sequence. Hence, the ROC of its z -transform is exterior to a circle. Now,

$$Z\{\alpha^n \mu[n+2]\} = \sum_{n=-2}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + (\alpha/z)^{-1} + (\alpha/z)^{-2} = \frac{1}{1-(\alpha/z)} + (\alpha/z)^{-1} + (\alpha/z)^{-2} = \frac{(\alpha/z)^{-2}}{1-(\alpha/z)}$$
 with its ROC given by $|z| > |\alpha|$. Likewise,

$$Z\{\beta^n \mu[n+2]\} = \frac{(\beta/z)^{-2}}{1-(\beta/z)}$$
 with its ROC given by $|z| > |\beta|$. Hence,

$$Z\{x_1[n]\} = X_1(z) = \frac{(\alpha/z)^{-2}}{1-(\alpha/z)} + \frac{(\beta/z)^{-2}}{1-(\beta/z)} = \frac{(\alpha^{-2} + \beta^{-2}) - (\alpha\beta^{-2} + \alpha^{-2}\beta)z^{-1}}{z^{-2}(1-\alpha z^{-1})(1-\beta z^{-1})}$$
 with its ROC given by $|z| > |\beta|$.

(b) $x_2[n] = \alpha^n \mu[-n-2] + \beta^n \mu[n-1]$ with $|\beta| > |\alpha|$. Note that $x_2[n]$ is a two-sided sequence. Now,

$$Z\{\alpha^n \mu[-n-2]\} = \sum_{n=-\infty}^{-2} \alpha^n z^{-n} = \sum_{m=2}^{\infty} \alpha^{-m} z^m = \sum_{m=0}^{\infty} (z/\alpha)^m - 1 - (z/\alpha) - (z/\alpha)^2 = \frac{(z/\alpha)^3}{1-(z/\alpha)}$$
 with its ROC given by $|z| < |\alpha|$. Likewise,

$Z\{\beta^n \mu[n-1]\} = \sum_{n=1}^{\infty} \beta^n z^{-n} = \sum_{n=0}^{\infty} \beta^n z^{-n} - 1 = \frac{1}{1-\beta z^{-1}} - 1 = \frac{\beta z^{-1}}{1-\beta z^{-1}}$ with its ROC given by $|z| > |\beta|$. Since the two ROCs do not intersect, $Z\{x_2[n]\}$ does not converge.

(c) $x_3[n] = \alpha^n \mu[n+1] + \beta^n \mu[-n-2]$ with $|\beta| > |\alpha|$. Note that $x_3[n]$ is a two-sided sequence. Now,

$$\begin{aligned} Z\{\beta^n \mu[-n-2]\} &= \sum_{n=-\infty}^{-2} \beta^n z^{-n} = \sum_{m=2}^{\infty} \beta^{-m} z^m = \sum_{m=0}^{\infty} (z/\beta)^m - 1 - (z/\beta) - (z/\beta)^2 \\ &= \frac{(z/\beta)^3}{1-(z/\beta)} \text{ with its ROC given by } |z| < |\beta|. \text{ Likewise,} \end{aligned}$$

$$Z\{\alpha^n \mu[n-1]\} = \sum_{n=1}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} - 1 = \frac{1}{1-\alpha z^{-1}} - 1 = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} \text{ with its ROC given by } |z| > |\alpha|.$$

6.8 The denominator factor $(z^2 + 0.3z - 0.18) = (z + 0.6)(z - 0.3)$ has poles at $z = -0.6$ and at $z = 0.3$, and the factor $(z^2 - 2z + 4)$ has poles with a magnitude 2. Hence, the four ROCs are defined by the regions: $\mathcal{R}_1: 0 < |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: 0.6 < |z| < 2$, and $\mathcal{R}_4: |z| > 2$. The inverse z -transform associated with the ROC \mathcal{R}_1 is a left-sided sequence, the inverse z -transforms associated with the ROCs \mathcal{R}_2 and \mathcal{R}_3 are two-sided sequences, and the inverse z -transform associated with the ROC \mathcal{R}_4 is a right-sided sequence.

6.9 $X(z) = Z\{x[n]\}$ with an ROC given by \mathcal{R}_x . Using the conjugation property of the z -transform given in Table 6.2, we observe that $Z\{x^*[n]\} = X^*(z^*)$ whose ROC is given by \mathcal{R}_x . Now, $\text{Re}\{x[n]\} = \frac{1}{2}(x[n] + x^*[n])$. Hence, $Z\{\text{Re } x[n]\} = \frac{1}{2}(X(z) + X^*(z))$ whose ROC is also \mathcal{R}_x . Likewise, $\text{Im}\{x[n]\} = \frac{1}{2j}(x[n] - x^*[n])$. Thus, $Z\{\text{Im } x[n]\} = \frac{1}{2j}(X(z) - X^*(z))$ whose ROC is again \mathcal{R}_x .

6.10 $\{x[n]\} = \{2, 3, -1, 0, -4, 3, 1, 2, 4\}, -2 \leq n \leq 6$. Then, $\tilde{X}[k] = X(z)|_{z=e^{j\pi k/3}} = X(z)|_{z=e^{j2\pi k/6}} = X(e^{j\omega})|_{\omega=2\pi k/6}$. Note that $\tilde{X}[k]$ is a periodic sequence with a period 6. Hence, from Eq. (5.49), the inverse of the discrete Fourier series $\tilde{X}[k]$ is given by $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+6r] = x[n-6] + x[n] + x[n+6]$ for $0 \leq n \leq 5$. Let $y[n] = x[n-6] + x[n] + x[n+6], -2 \leq n \leq 6$. Now,

$\{x[n-6]\} = \{0, 0, 0, 0, 0, 0, 2, 3, -1\}$ and
 $\{x[n+6]\} = \{1, 2, 4, 0, 0, 0, 0, 0, 0\}$. Therefore,
 $\{y[n]\} = \{3, 5, 3, 0, -4, 3, 3, 5, 3\}, -2 \leq n \leq 6$. Hence,
 $\{\tilde{x}[n]\} = \{3, 0, -4, 3, 3, 5\}, 0 \leq n \leq 5$.

6.11 $\{x[n]\} = \{4, 2, -1, 5, -3, 1, -2, 4, 2\}, -6 \leq n \leq 2$. Then,

$\tilde{X}[k] = X(z)|_{z=e^{j\pi k/3}} = X(z)|_{z=e^{j2\pi k/6}} = X(e^{j\omega})|_{\omega=2\pi k/6}$. Note that $\tilde{X}[k]$ is a periodic sequence with a period 6. Hence, from Eq. (5.49), the inverse of the discrete Fourier series $\tilde{X}[k]$ is given by $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+6r] = x[n-6] + x[n] + x[n+6]$ for $0 \leq n \leq 5$. Let $y[n] = x[n-6] + x[n] + x[n+6], -6 \leq n \leq 5$. Now,
 $\{x[n-6]\} = \{0, 0, 0, 0, 0, 0, 4, 2, -1, 5, -3, 1\}, -6 \leq n \leq 5$,
 $\{x[n+6]\} = \{-2, 4, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, -6 \leq n \leq 5$.
Therefore, $\{y[n]\} = \{2, 6, 1, 5, -3, 1, 2, 6, 5, -3, 1\}, -6 \leq n \leq 5$.
Hence, $\{\tilde{x}[n]\} = \{2, 6, 1, 5, -3, 1\}, 0 \leq n \leq 5$.

6.12 $X(z) = \sum_{n=0}^{11} x[n]z^{-n}$. $X_0[k] = X(z)|_{z=e^{j2\pi k/9}} = \sum_{n=0}^{11} x[n]e^{-j2\pi kn/9}, 0 \leq k \leq 8$.

Therefore, $x_0[n] = \frac{1}{9} \sum_{k=0}^8 X_0[k]e^{j2\pi kn/9} = \frac{1}{9} \sum_{k=0}^8 \left(\sum_{r=0}^{11} x[r]e^{-j2\pi kr/9} \right) e^{j2\pi kn/9}$
 $= \frac{1}{9} \sum_{k=0}^8 \sum_{r=0}^{11} x[r]e^{j2\pi k(n-r)/9} = \frac{1}{9} \sum_{r=0}^{11} x[r] \sum_{k=0}^8 e^{j2\pi k(n-r)/9} = \frac{1}{9} \sum_{r=0}^{11} x[r] \sum_{k=0}^8 W_9^{-(r-n)k}$.

From Eq. (5.11), $\frac{1}{9} \sum_{k=0}^8 W_9^{-(r-n)k} = \begin{cases} 1, & \text{for } r-n = 9i, \\ 0, & \text{otherwise.} \end{cases}$ Hence,

$x_0[n] = \begin{cases} x[0] + x[9], & \text{for } n = 0 \\ x[n], & \text{otherwise,} \end{cases}$ i.e.,

$\{x_0[n]\} = \{20, 5, 45, -15, -9, -19, -8, 21, -10\}, 0 \leq n \leq 8$.

6.13 (a) $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. Hence, $X(z^3) = \sum_{n=-\infty}^{\infty} x[n]z^{-3n} = \sum_{\substack{r=-\infty \\ m=3r}}^{\infty} x[m/3]z^{-m}$. Define

a new sequence $g[m] = \begin{cases} x[m/3], & m = 0, \pm 3, \pm 6, \dots, \\ 0, & \text{otherwise.} \end{cases}$ We can then express

$X(z^3) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$. Thus, the inverse z -transform of $X(z^3)$ is given by $g[n]$.

$$\text{For } x[n] = (-0.5)^n \mu[n], \quad g[n] = \begin{cases} (-0.5)^{n/3}, & n = 0, 3, 6, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $Y(z) = (1 + z^{-1})X(z^3) = X(z^3) + z^{-1}X(z^3)$. Therefore,

$$y[n] = Z^{-1}\{Y(z)\} = Z^{-1}X(z^3) + Z^{-1}z^{-1}X(z^3) = g[n] + g[n-1], \text{ where } g[n] =$$

$$Z^{-1}X(z^3). \text{ From Part (a), } g[n] = \begin{cases} (-0.5)^{n/3}, & n = 0, 3, 6, \dots, \\ 0, & \text{otherwise.} \end{cases} \text{ Hence,}$$

$$g[n-1] = \begin{cases} (-0.5)^{(n-1)/3}, & n = 1, 4, 7, \dots, \\ 0, & \text{otherwise.} \end{cases} \text{ Therefore,}$$

$$y[n] = \begin{cases} (-0.5)^{n/3}, & n = 0, 3, 6, \dots, \\ (-0.5)^{(n-1)/3}, & n = 1, 4, 7, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

6.14 (a) $X_a(z) = Z\{\mu[n] - \mu[n-5]\} = \frac{1}{1-z^{-1}} - \frac{z^{-5}}{1-z^{-1}} = \frac{1-z^{-5}}{1-z^{-1}}$

$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$. Since has all poles at the origin, the ROC is the entire z -plane except the point $z = 0$, and hence includes the unit circle. On the unit circle,

$$X_a(z)|_{z=e^{j\omega}} = X_a(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}}.$$

(b) $x_b[n] = \alpha^n \mu[n] - \alpha^n \mu[n-8]$, $|\alpha| < 1$. From Table 6.1,

$$X_b(z) = \frac{1}{1-\alpha z^{-1}} - \frac{z^{-8}}{1-\alpha z^{-1}} = \frac{1-z^{-8}}{1-\alpha z^{-1}}. \text{ The ROC is exterior to the circle at}$$

$|z| = |\alpha| < 1$. Hence, the ROC includes the unit circle. On the unit circle,

$$X_b(z)|_{z=e^{j\omega}} = X_b(e^{j\omega}) = \frac{1 - e^{j8\omega}}{1 - \alpha e^{j\omega}}.$$

(c) $x_c[n] = (n+1)\alpha^n \mu[n] = n\alpha^n \mu[n] + \alpha^n \mu[n]$, $|\alpha| < 1$. From Table 6.1,

$$X_c(z) = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z^{-1}} = \frac{1+\alpha z^{-1}}{1-\alpha z^{-1}}. \text{ The ROC is exterior to the circle at}$$

$|z| = |\alpha| < 1$. Hence, the ROC includes the unit circle. On the unit circle,

$$X_c(e^{j\omega}) = \frac{1 + \alpha e^{j\omega}}{1 - \alpha e^{j\omega}}.$$

6.15 (a) $Y_1(z) = \sum_{n=-N}^N z^{-n} = z^N \left(\sum_{n=0}^{2N} z^{-n} \right) = \frac{1-z^{-(2N+1)}}{z^{-N}(1-z^{-1})}$. $Y_1(z)$ has N poles at

$z=0$ and N poles at $z=\infty$. Hence, the ROC is the entire z -plane excluding the points $z=0$ and $z=\infty$, and includes the unit circle. On the unit circle,

$$Y_1(z)|_{z=e^{j\omega}} = Y_1(e^{j\omega}) = \frac{1-e^{j(2N+1)\omega}}{e^{-jN\omega}(1-e^{-j\omega})} = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin(\omega/2)}.$$

(b) $Y_2(z) = \sum_{n=0}^N z^{-n} = \frac{1-z^{-(N+1)}}{1-z^{-1}}$. $Y_2(z)$ has N poles at $z=0$. Hence, the ROC

is the entire z -plane excluding the point $z=0$. On the unit circle,

$$Y_2(z)|_{z=e^{j\omega}} = Y_2(e^{j\omega}) = \frac{1-e^{j(N+1)\omega}}{1-e^{-j\omega}} = e^{-N\omega/2} \frac{\sin\left(\frac{N+1}{2}\omega\right)}{\sin(\omega/2)}.$$

(c) $y_3[n] = \begin{cases} 1-\frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Now, $y_3[n] = y_0[n] \otimes y_0[n]$ where

$$y_0[n] = \begin{cases} 1, & -\frac{N}{2} \leq n \leq \frac{N}{2}, \\ 0, & \text{otherwise.} \end{cases} \text{ Therefore, } Y_3(z) = Y_0^2(z) = \frac{(1-z^{-(N+1)})^2}{z^{-N}(1-z^{-1})^2}.$$

$Y_3(z)$ has $\frac{N}{2}$ poles at $z=0$ and $\frac{N}{2}$ poles at $z=\infty$. Hence, the ROC is the entire z -plane excluding the points $z=0$ and $z=\infty$, and includes the unit circle. On the unit

circle, $Y_3(e^{j\omega}) = Y_0^2(e^{j\omega}) = \frac{\sin^2\left(\omega\left(\frac{N+1}{2}\right)\right)}{\sin^2(\omega/2)}$.

(d) $y_4[n] = \begin{cases} N+1-|n|, & -N \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases} = y_1[n] + N \cdot y_3[n]$, where $y_1[n]$ is the

sequence of Part (a) and $y_3[n]$ is the sequence of Part (c). Therefore,

$$Y_4(z) = Y_1(z) + N \cdot Y_3(z) = \frac{1-z^{-(2N+1)}}{z^{-N}(1-z^{-1})} + N \frac{(1-z^{-(N+1)})^2}{z^{-N}(1-z^{-1})^2}.$$

Since the ROCs of both $Y_1(z)$ and $Y_3(z)$ include the unit circle, the ROC of $Y_4(z)$ also includes the unit

circle. On the unit circle, $Y_4(e^{j\omega}) = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin(\omega/2)} + \frac{\sin^2\left(\omega\left(\frac{N+1}{2}\right)\right)}{\sin^2(\omega/2)}$.

(e) $y_f[n] = \begin{cases} \cos(\pi n / 2N), & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$ Therefore,

$$\begin{aligned}
Y_f(z) &= \frac{1}{2} \sum_{n=-N}^N e^{-j(\pi n/2N)} z^{-n} + \frac{1}{2} \sum_{n=-N}^N e^{j(\pi n/2N)} z^{-n} \\
&= \frac{e^{j(\pi/2)} z^N}{2} \left(\frac{1 - e^{-j(2N+1)(\pi/2N)} z^{-(2N+1)}}{1 - e^{-j(\pi/2N)} z^{-1}} \right) \\
&\quad + \frac{e^{-j(\pi/2)} z^N}{2} \left(\frac{1 - e^{j(2N+1)(\pi/2N)} z^{-(2N+1)}}{1 - e^{j(\pi/2N)} z^{-1}} \right).
\end{aligned}$$

$Y_f(z)$ has N poles at $z=0$ and N poles at $z=\infty$. Hence, the ROC is the entire z -plane excluding the points $z=0$ and $z=\infty$, and includes the unit circle. On the unit

circle, $Y_f(e^{j\omega}) = \frac{1}{2} \frac{\sin\left(\left(\omega - \frac{\pi}{2N}\right)\left(N + \frac{1}{2}\right)\right)}{\sin\left(\left(\omega - \frac{\pi}{2N}\right)/2\right)} + \frac{1}{2} \frac{\sin\left(\left(\omega + \frac{\pi}{2N}\right)\left(N + \frac{1}{2}\right)\right)}{\sin\left(\left(\omega + \frac{\pi}{2N}\right)/2\right)}$.

6.16 $X(z) = -4z^3 + 5z^2 + z - 2 - 3z^{-1} + 2z^{-3}$, $Y(z) = 6z - 3 - z^{-1} + 8z^{-3} + 7z^{-4} - 2z^{-5}$,
 $W(z) = 3z^{-2} + 2z^{-3} + 2z^{-4} - z^{-5} - 2z^{-7} + 5z^{-8}$.

(a) $U(z) = X(z)Y(z)$

$$\begin{aligned}
&= (-4z^3 + 5z^2 + z - 2 - 3z^{-1} + 2z^{-3})(6z - 3 - z^{-1} + 8z^{-3} + 7z^{-4} - 2z^{-5}) \\
&= -24z^4 + 42z^3 - 5z^2 - 20z - 45 + 23z^{-1} + 66z^{-2} - 25z^{-3} - 42z^{-4} - 17z^{-5} \\
&\quad + 22z^{-6} + 14z^{-7} - 4z^{-8}. \text{ Hence,}
\end{aligned}$$

$$\{u[n]\} = \{-24, 42, -5, -20, -45, 23, 66, -25, -42, -17, 22, 14, -4\}, -4 \leq n \leq 8.$$

(b) $V(z) = X(z)W(z)$

$$\begin{aligned}
&= (-4z^3 + 5z^2 + z - 2 - 3z^{-1} + 2z^{-3})(3z^{-2} + 2z^{-3} + 2z^{-4} - z^{-5} - 2z^{-7} + 5z^{-8}) \\
&= -12z + 7 + 5z^{-1} + 10z^{-2} - 16z^{-3} - 3z^{-4} - 28z^{-5} + 30z^{-6} + 13z^{-7} - 6z^{-8} \\
&\quad - 15z^{-4} - 4z^{-10} + 10z^{-11}. \text{ Hence,}
\end{aligned}$$

$$\{v[n]\} = \{-12, 7, 5, 10, -16, -3, -28, 30, 13, -6, -15, -4, 10\}, -1 \leq n \leq 11.$$

(c) $G(z) = W(z)Y(z)$

$$\begin{aligned}
&= (3z^{-2} + 2z^{-3} + 2z^{-4} - z^{-5} - 2z^{-7} + 5z^{-8})(6z - 3 - z^{-1} + 8z^{-3} + 7z^{-4} - 2z^{-5}) \\
&= 18z^{-1} + 3z^{-2} + 3z^{-3} - 14z^{-4} + 25z^{-5} + 26z^{-6} + 60z^{-7} - 11z^{-8} - 16z^{-9} - 14z^{-10} \\
&\quad + 26z^{-11} + 39z^{-12} - 10z^{-13}. \text{ Hence,}
\end{aligned}$$

$$\{g[n]\} = \{18, 3, 3, -14, 25, 26, 60, -11, -16, -14, 26, 39, -10\}, 1 \leq n \leq 13.$$

6.17 $Y_L(z) = \sum_{n=0}^{2N-1} \left(\sum_{m=0}^{N-1} x[m]h[n-m] \right) z^{-n}$ and $Y_C(z) = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[\langle n-m \rangle_N] \right) z^{-n}$.

Now, $Y_L(z)$ can be rewritten as

$$\begin{aligned} Y_L(z) &= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[n-m] \right) z^{-n} + \sum_{n=N}^{2N-1} \left(\sum_{m=0}^{N-1} x[m]h[n-m] \right) z^{-n} \\ &= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[n-m] \right) z^{-n} + \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[k-m-N] \right) z^{-(k-N)}. \text{ Therefore,} \\ \langle Y_L(z) \rangle_{(z^{-N} - 1)} &= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[n-m] \right) z^{-n} + \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[\langle k-m \rangle_N] \right) z^{-k} \\ &= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]h[\langle n-m \rangle_N] \right) z^{-n} = Y_C(z). \end{aligned}$$

6.18 $G(z) = 2 - z^{-1} + 3z^{-2}$, $H(z) = -2 + 4z^{-1} + 2z^{-2} - z^{-3}$. Now,

$$\begin{aligned} Y_L(z) &= G(z)H(z) = (2 - z^{-1} + 3z^{-2})(-2 + 4z^{-1} + 2z^{-2} - z^{-3}) \\ &= -4 + 10z^{-1} - 6z^{-2} + 8z^{-3} + 7z^{-4} - 3z^{-5}. \text{ Therefore,} \\ y_L[n] &= \{-4 \quad 6 \quad 2 \quad 12 \quad 5 \quad -3\}, 0 \leq n \leq 5. \end{aligned}$$

Using the MATLAB statement `y = conv([2 -1 3], [-2 4 2 -1]);` we obtain

$$Y = \begin{matrix} -4 & 10 & -6 & 8 & 7 & -3 \end{matrix}$$

which is seen to be the same result as given above.

$$\begin{aligned} Y_C(z) &= \langle Y_C(z) \rangle_{(z^{-4} - 1)} = \langle -4 + 10z^{-1} - 6z^{-2} + 8z^{-3} + 7z^{-4} - 3z^{-5} \rangle_{(z^{-4} - 1)} \\ &= -4 + 10z^{-1} - 6z^{-2} + 8z^{-3} + 7 - 3z^{-1} = 3 + 7z^{-1} - 6z^{-2} + 8z^{-3}. \text{ Therefore,} \\ y_C[n] &= \{1 \quad 3 \quad 2 \quad 12\}, 0 \leq n \leq 3. \end{aligned}$$

Using the MATLAB statement `y = circonv([2 -1 3 0], [-2 4 2 -1]);` we obtain

$$Y = \begin{matrix} 3 & 7 & -6 & 8 \end{matrix}$$

which is seen to be the same result as given above.

6.19 $G(z) = \frac{P(z)}{D(z)} = \frac{P(z)}{(1 - \lambda_\ell z^{-1})R(z)}$. The residue ρ_ℓ of $G(z)$ at the pole is given by

$$\rho_\ell = \left. \frac{P(z)}{R(z)} \right|_{z=\lambda_\ell}. \text{ Now,}$$

$$D'(z) = \frac{dD(z)}{dz^{-1}} = \frac{d[(1 - \lambda_\ell z^{-1})R(z)]}{dz^{-1}} = -\lambda_\ell R(z) + (1 - \lambda_\ell z^{-1}) \frac{dR(z)}{dz^{-1}}. \text{ Hence,}$$

$$D'(z)|_{z=\lambda_\ell} = -\lambda_\ell R(z)|_{z=\lambda_\ell}. \text{ Therefore, } \rho_\ell = -\lambda_\ell \left. \frac{P(z)}{D'(z)} \right|_{z=\lambda_\ell}.$$

$$6.20 \text{ (a)} \quad X_a(z) = \frac{3z}{(z+0.6)(z-0.3)} = \frac{3z^{-1}}{(1+0.6z^{-1})(1-0.3z^{-1})} = \frac{\rho_1}{1+0.6z^{-1}} + \frac{\rho_2}{1-0.3z^{-1}},$$

$$\text{where } \rho_1 = \frac{3}{z-0.3} \Big|_{z=-0.6} = \frac{3}{-0.9} = -\frac{10}{3}, \quad \rho_2 = \frac{3}{z+0.6} \Big|_{z=0.3} = \frac{3}{0.9} = \frac{10}{3}.$$

$$\text{Therefore, } X_a(z) = -\frac{10/3}{1+0.6z^{-1}} + \frac{10/3}{1-0.3z^{-1}}.$$

There are three ROCs - $\mathcal{R}_1: |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: |z| > 0.6$.

The inverse z -transform associated with the ROC \mathcal{R}_1 is a left-sided sequence:

$$Z^{-1}\{X_a(z)\} = x_a[n] = \frac{10}{3} \left((-0.6)^n - (0.3)^n \right) \mu[-n-1].$$

The inverse z -transform associated with the ROC \mathcal{R}_2 is a two-sided sequence:

$$Z^{-1}\{X_a(z)\} = x_a[n] = -\frac{10}{3} (-0.6)^n \mu[-n-1] + \frac{10}{3} (0.3)^n \mu[n].$$

The inverse z -transform associated with the ROC \mathcal{R}_3 is a right-sided sequence:

$$Z^{-1}\{X_a(z)\} = x_a[n] = \frac{10}{3} \left(-(-0.6)^n + (0.3)^n \right) \mu[n].$$

$$6.20 \text{ (b)} \quad X_b(z) = \frac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{(1+0.6z^{-1})(1-0.3z^{-1})^2} = K + \frac{\rho_1}{1+0.6z^{-1}} + \frac{\gamma_1}{1-0.3z^{-1}} + \frac{\gamma_2}{(1-0.3z^{-1})^2}.$$

$$K = X_b(0) = 0, \quad \rho_1 = \frac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{(1-0.3z^{-1})^2} \Big|_{z=-0.6} = 2.7279,$$

$$\gamma_2 = \frac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{1+0.6z^{-1}} \Big|_{z=0.3} = 0.6190,$$

$$\gamma_1 = \frac{1}{-0.3} \cdot \frac{d}{dz^{-1}} \left(\frac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{1+0.6z^{-1}} \right) \Big|_{z=0.3} = -0.3469. \text{ Hence,}$$

$$X_b(z) = \frac{2.7279}{1+0.6z^{-1}} - \frac{0.3469}{1-0.3z^{-1}} + \frac{0.6190}{(1-0.3z^{-1})^2}.$$

There are three ROCs - $\mathcal{R}_1: |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: |z| > 0.6$.

The inverse z -transform associated with the ROC \mathcal{R}_1 is a left-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(-0.6)^n \mu[-n-1] \\ + (-0.3469 + 0.6190(n+1))(0.3)^n \mu[-n-1].$$

The inverse z -transform associated with the ROC \mathcal{R}_2 is a two-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(-0.6)^n \mu[-n-1] \\ + (-0.3469 + 0.6190(n+1))(0.3)^n \mu[n].$$

The inverse z -transform associated with the ROC \mathcal{R}_3 is a right-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(-0.6)^n \mu[n] + (-0.3469 + 0.6190(n+1))(0.3)^n \mu[n].$$

6.21 $G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \dots + p_M z^{-M}}{1 + d_1 z^{-1} + \dots + d_N z^{-N}}$. Thus, $G(\infty) = \frac{P(\infty)}{D(\infty)}$. Now, a partial-

fraction expansion of $G(z)$ in z^{-1} is given by $G(z) = \sum_{\ell=1}^N \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}}$, from which we obtain

$$G(\infty) = \sum_{\ell=1}^N \rho_\ell = \frac{p_0}{d_0}.$$

6.22 $H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}$, $|z| > r > 0$. By using partial-fraction expansion we write

$$H(z) = \frac{1}{(e^{j\theta} - e^{-j\theta})} \left(\frac{e^{j\theta}}{1 - r e^{j\theta} z^{-1}} - \frac{e^{-j\theta}}{1 - r e^{-j\theta} z^{-1}} \right) = \frac{1}{2 \sin \theta} \left(\frac{e^{j\theta}}{1 - r e^{j\theta} z^{-1}} - \frac{e^{-j\theta}}{1 - r e^{-j\theta} z^{-1}} \right).$$

Thus, $h[n] = \frac{1}{j 2 \sin \theta} (r^n e^{j\theta} e^{jn\theta} \mu[n] - r^n e^{-j\theta} e^{-jn\theta} \mu[n]) = \frac{r^n}{\sin \theta} \left(\frac{e^{j\theta(n+1)} - e^{-j\theta(n+1)}}{2j} \right) \mu[n]$

$$= \frac{r^n \sin((n+1)\theta)}{\sin \theta} \mu[n].$$

6.23 (a) $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = \sum_{m=1}^{\infty} \alpha^{-m} z^m = \sum_{m=1}^{\infty} (z/\alpha)^m = \sum_{m=0}^{\infty} (z/\alpha)^m - 1$

$$= \frac{z/\alpha}{1 - (z/\alpha)} = \frac{1}{1 - \alpha z^{-1}}, |z| < |\alpha|.$$

(b) Using the differentiation property, we obtain from Part (a),

$$Z\{nx[n]\} = -z \frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, |z| < |\alpha|.$$

Therefore, $Z\{y[n]\} = Z\{nx[n] + x[n]\}$

$$= \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} + \frac{1}{1 - \alpha z^{-1}} = \frac{1}{(1 - \alpha z^{-1})^2}, |z| < |\alpha|.$$

6.24 (a) Expanding $X_1(z)$ in a power series we get $X_1(z) = \sum_{n=0}^{\infty} z^{-3n}$, $|z| > 1$. Thus,

$$x_1[n] = \begin{cases} 1, & \text{if } n = 3k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Alternately, using partial-fraction expansion we get

$$X_1(z) = \frac{1}{1 - z^{-3}} = \frac{\frac{1}{3}}{1 - z^{-1}} + \frac{\frac{1}{3}}{1 + (\frac{1}{2} + j\frac{\sqrt{3}}{2})z^{-1}} + \frac{\frac{1}{3}}{1 + (\frac{1}{2} - j\frac{\sqrt{3}}{2})z^{-1}}.$$

Therefore,

$$\begin{aligned}
x_1[n] &= \frac{1}{3}\mu[n] + \frac{1}{3}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\mu[n] + \frac{1}{3}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\mu[n] \\
&= \frac{1}{3}\mu[n] + \frac{1}{3}e^{-j2\pi n/3}\mu[n] + \frac{1}{3}e^{j2\pi n/3}\mu[n] = \frac{1}{3}\mu[n] + \frac{2}{3}\cos(2\pi n/3)\mu[n]. \text{ Thus,} \\
x_1[n] &= \begin{cases} 1, & \text{if } n = 3k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) Expanding $X_2(z)$ in a power series we get $X_2(z) = \sum_{n=0}^{\infty} z^{-4n}$, $|z| > 1$. Thus,

$$x_2[n] = \begin{cases} 1, & \text{if } n = 4k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases} \text{ Alternately, using partial-fraction expansion we get}$$

$$X_2(z) = \frac{\frac{1}{4}}{1-z^{-1}} + \frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{1}{4}}{1+(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}} + \frac{\frac{1}{4}}{1+(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}. \text{ Thus,}$$

$$\begin{aligned}
x_2[n] &= \frac{1}{4}\mu[n] + \frac{1}{4}(-1)^n\mu[n] + \frac{1}{4}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\mu[n] + \frac{1}{4}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\mu[n] \\
&= \frac{1}{4}\mu[n] + \frac{1}{4}(-1)^n\mu[n] + \frac{1}{4}e^{-j2\pi n/3}\mu[n] + \frac{1}{4}e^{j2\pi n/3}\mu[n] \\
&= \frac{1}{4}\mu[n] + \frac{1}{4}(-1)^n\mu[n] + \frac{1}{2}\cos(2\pi n/3)\mu[n]. \text{ Thus, } x_2[n] = \begin{cases} 1, & \text{if } n = 4k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

6.25 (a) $X_1(z) = \log(1 - \alpha z^{-1})$, $|z| > |\alpha|$. Expanding $X_1(z)$ in a power series we get

$$X_1(z) = -\alpha z^{-1} - \frac{\alpha^2 z^{-2}}{2} - \frac{\alpha^3 z^{-3}}{3} - \dots = -\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}. \text{ Therefore,}$$

$$x_1[n] = -\frac{\alpha^n}{n} \mu[n-1].$$

(b) $X_2(z) = \log\left(\frac{\alpha - z^{-1}}{\alpha}\right) = \log\left(1 - (\alpha z)^{-1}\right)$, $|z| < |\alpha|$. Expanding $X_2(z)$ in a power series

$$\text{we get } X_2(z) = -(\alpha z)^{-1} - \frac{(\alpha z)^{-2}}{2} - \frac{(\alpha z)^{-3}}{3} - \dots = -\sum_{n=1}^{\infty} \frac{(\alpha z)^{-n}}{n}. \text{ Therefore,}$$

$$x_2[n] = -\frac{\alpha^{-n}}{n} \mu[n-1].$$

(c) $X_3(z) = \log\left(\frac{1}{1 - \alpha z^{-1}}\right)$, $|z| > |\alpha|$. Expanding $X_3(z)$ in a power series we get

$$X_3(z) = \alpha z^{-1} + \frac{\alpha^2 z^{-2}}{2} + \frac{\alpha^3 z^{-3}}{3} \dots = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}. \text{ Therefore, } x_3[n] = \frac{\alpha^n}{n} \mu[n-1].$$

(d) $X_4(z) = \log\left(\frac{\alpha}{\alpha - z^{-1}}\right) = -\log\left(1 - (\alpha z)^{-1}\right)$, $|z| < |\alpha|$. Expanding $X_4(z)$ in a power series we get $X_4(z) = (\alpha z)^{-1} + \frac{(\alpha z)^{-2}}{2} + \frac{(\alpha z)^{-3}}{3} - \dots = \sum_{n=1}^{\infty} \frac{(\alpha z)^{-n}}{n}$. Therefore,

$$x_4[n] = \frac{\alpha^{-n}}{n} \mu[n-1].$$

6.26 $H(z) = \frac{z^{-1} + 1.7z^{-2}}{(1 - 0.3z^{-1})(1 + 0.5z^{-1})} = k + \frac{\rho_1}{1 - 0.3z^{-1}} + \frac{\rho_2}{1 + 0.5z^{-1}}$, where $k = H(0) = -\frac{34}{3}$,

$$\rho_1 = \left. \frac{z^{-1} + 1.7z^{-2}}{1 + 0.5z^{-1}} \right|_{z=0.3} = \frac{25}{3}, \quad \rho_2 = \left. \frac{z^{-1} + 1.7z^{-2}}{1 - 0.3z^{-1}} \right|_{z=-0.5} = 3.$$

The statement `[r, p, k]=residuez([0 1 1.7], conv([1 -0.3], [1 0.6]));` yields

```
r =
3.0000
8.3333
```

```
p =
-0.5000
0.3000
```

```
k =
-11.3333
```

Thus, $H(z) = -\frac{34}{3} + \frac{25/3}{1 - 0.3z^{-1}} + \frac{3}{1 + 0.5z^{-1}}$. Hence, its inverse z -transform is given by

$$h[n] = -\frac{34}{3} \delta[n] + \frac{25}{3} (0.3)^n \mu[n] + 3(-0.5)^n \mu[n].$$

6.27 $G(z) = Z\{g[n]\} = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ with a ROC given by \mathcal{R}_g and $H(z) = Z\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

with a ROC given by \mathcal{R}_h .

(a) $G^*(z) = \sum_{n=-\infty}^{\infty} g^*[n](z^*)^{-n}$ and $G^*(z^*) = \sum_{n=-\infty}^{\infty} g^*[n]z^{-n}$. Therefore,

$$Z\{g^*[n]\} = G^*(z^*) \text{ with a ROC given by } \mathcal{R}_g.$$

(b) Replace n by $-m$ in the sum defining $G(z)$. Then

$$\sum_{m=-\infty}^{\infty} g[-m]z^m = \sum_{m=-\infty}^{\infty} g[-m](1/z)^{-m} = G(1/z). \text{ Thus, } Z\{g[-n]\} = G(1/z). \text{ Since } z \text{ has}$$

been replaced by $1/z$, the ROC of is given by $1/\mathcal{R}_g$.

(c) Let $y[n] = \alpha g[n] + \beta h[n]$. Then

$Y(z) = Z\{\alpha g[n] + \beta h[n]\} = \alpha Z\{g[n]\} + \beta Z\{h[n]\} = \alpha G(z) + \beta H(z)$. In this case $Y(z)$ will converge wherever both $G(z)$ and $H(z)$ converge. Hence the ROC of $Y(z)$ is $\mathcal{R}_g \cap \mathcal{R}_h$.

(d) $y[n] = g[n - n_o]$. Hence,

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n - n_o]z^{-n} = \sum_{m=-\infty}^{\infty} y[m]z^{-(m+n_o)}$$

$$= z^{-n_o} \sum_{m=-\infty}^{\infty} g[m]z^{-m} = z^{-n_o} G(z).$$

In this case, the ROC of $Y(z)$ is same as that of $G(z)$ except for the possible addition or elimination of the point $z = 0$ or $z = \infty$ (due to the factor of z^{-n_o}).

(e) $y[n] = \alpha^n g[n]$. Hence, $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^n g[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n](z/\alpha)^{-n}$
 $= G(z/\alpha)$. The ROC of $Y(z)$ is $|\alpha| \mathcal{R}_g$.

(f) $y[n] = n g[n]$. Hence, $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} n g[n]z^{-n}$. Now,

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}. \text{ Thus, } \frac{dG(z)}{dz} = - \sum_{n=-\infty}^{\infty} n g[n]z^{-n-1}.$$

Hence, $-z \frac{dG(z)}{dz} = \sum_{n=-\infty}^{\infty} n g[n]z^{-n-1}$. Thus, $Y(z) = Z\{n g[n]\} = - \frac{dG(z)}{dz}$. In this case, the ROC of $Y(z)$ is same as that of $G(z)$ except possibly the point $z = 0$ or $z = \infty$.

6.28 From Eq. (6.111), for $N = 3$, we get

$$\mathbf{D}_3 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{bmatrix}. \text{ The determinant of } \mathbf{D}_3 \text{ is given by}$$

$$\det(\mathbf{D}_3) = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 0 & z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} \\ 0 & z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} \end{vmatrix} = \begin{vmatrix} z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} \\ z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} \end{vmatrix}$$

$$= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} \\ 1 & z_2^{-1} + z_0^{-1} \end{vmatrix} = (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_2^{-1} - z_1^{-1})$$

$$= \prod_{2 \geq k \geq \ell \geq 0} (z_k^{-1} - z_\ell^{-1}).$$

From Eq. (6.111), for $N = 4$, we get

$$\mathbf{D}_4 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{bmatrix}. \quad \text{The determinant of } \mathbf{D}_4 \text{ is given by}$$

$$\begin{aligned} \det(\mathbf{D}_4) &= \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 0 & z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} & z_1^{-3} - z_0^{-3} \\ 0 & z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} & z_2^{-3} - z_0^{-3} \\ 0 & z_3^{-1} - z_0^{-1} & z_3^{-2} - z_0^{-2} & z_3^{-3} - z_0^{-3} \end{vmatrix} \\ &= \begin{vmatrix} z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} & z_1^{-3} - z_0^{-3} \\ z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} & z_2^{-3} - z_0^{-3} \\ z_3^{-1} - z_0^{-1} & z_3^{-2} - z_0^{-2} & z_3^{-3} - z_0^{-3} \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} & z_1^{-2} + z_1^{-1}z_0^{-1} + z_0^{-2} \\ 1 & z_2^{-1} + z_0^{-1} & z_2^{-2} + z_2^{-1}z_0^{-1} + z_0^{-2} \\ 1 & z_3^{-1} + z_0^{-1} & z_3^{-2} + z_3^{-1}z_0^{-1} + z_0^{-2} \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} & z_1^{-2} + z_1^{-1}z_0^{-1} + z_0^{-2} \\ 0 & z_2^{-1} - z_0^{-1} & (z_2^{-1} - z_0^{-1})(z_2^{-1} + z_1^{-1} + z_0^{-1}) \\ 0 & z_3^{-1} - z_0^{-1} & (z_3^{-1} - z_0^{-1})(z_3^{-1} + z_1^{-1} + z_0^{-1}) \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} z_2^{-1} - z_1^{-1} & (z_2^{-1} - z_1^{-1})(z_2^{-1} + z_1^{-1} + z_0^{-1}) \\ z_3^{-1} - z_1^{-1} & (z_3^{-1} - z_1^{-1})(z_3^{-1} + z_1^{-1} + z_0^{-1}) \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1})(z_2^{-1} - z_1^{-1})(z_3^{-1} - z_1^{-1})(z_3^{-1} - z_2^{-1}) \\ &= \prod_{\substack{3 \geq k \geq \ell \geq 0}} (z_k^{-1} - z_\ell^{-1}). \quad \text{Hence, in the general case, } \det(\mathbf{D}_N) = \prod_{N-1 \geq k \geq \ell \geq 0} (z_k^{-1} - z_\ell^{-1}). \quad \text{It follows} \end{aligned}$$

from this expression that the determinant $\det(\mathbf{D}_N)$ is non-zero, i.e., \mathbf{D}_N is non-singular, if the sampling points z_k are distinct.

6.29 $X(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}$. Thus,

$$X_{\text{NDFT}}[0] = X(z_0) = 1 - 2\left(-\frac{1}{2}\right)^{-1} + 3\left(-\frac{1}{2}\right)^{-2} - 4\left(-\frac{1}{2}\right)^{-3} = 1 + 4 + 3 \cdot 4 + 4 \cdot 8 = 49,$$

$$X_{\text{NDFT}}[1] = X(z_1) = 1 - 2 + 3 - 4 = -2,$$

$$X_{\text{NDFT}}[2] = X(z_2) = 1 - 2\left(\frac{1}{2}\right)^{-1} + 3\left(\frac{1}{2}\right)^{-2} - 4\left(\frac{1}{2}\right)^{-3} = 1 - 4 + 3 \cdot 4 - 4 \cdot 8 = -23,$$

$$X_{\text{NDFT}}[3] = X(z_3) = 1 - 2\left(\frac{1}{3}\right)^{-1} + 3\left(\frac{1}{3}\right)^{-2} - 4\left(\frac{1}{3}\right)^{-3} = 1 - 2 \cdot 3 + 3 \cdot 9 - 4 \cdot 27 = -86.$$

$$I_0(z) = (1 - z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1}) = 1 - \frac{11}{6}z^{-1} + z^{-2} - \frac{1}{6}z^{-3} \Rightarrow I_0(-\frac{1}{2}) = 10,$$

$$I_1(z) = (1 + \frac{1}{2}z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1}) = 1 - \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{12}z^{-3} \Rightarrow I_1(1) = \frac{1}{2},$$

$$I_2(z) = (1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{3}z^{-1}) = 1 - \frac{5}{6}z^{-1} - \frac{1}{3}z^{-2} + \frac{1}{6}z^{-3} \Rightarrow I_2(\frac{1}{2}) = -\frac{2}{3},$$

$$I_3(z) = (1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{2}z^{-1}) = 1 - z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} \Rightarrow I_3(\frac{1}{3}) = \frac{5}{2}.$$

$$\begin{aligned} \text{Therefore, } X(z) &= \frac{49}{10}I_0(z) + \frac{-2}{1/2}I_1(z) + \frac{-23}{-2/3}I_2(z) + \frac{-86}{5/2}I_3(z) \\ &= 4.9I_0(z) - 4I_1(z) + 34.5I_2(z) - 34.4I_3(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}. \end{aligned}$$

6.30 (a) $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. Let $\hat{X}(z) = \log(X(z)) \Rightarrow X(z) = e^{\hat{X}(z)}$. Thus,

$$X(e^{j\omega}) = e^{\hat{X}(e^{j\omega})}.$$

(b) $\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega}))e^{j\omega n} d\omega$. If $x[n]$ is real, then $X(e^{j\omega}) = X^*(e^{-j\omega})$.

Therefore, $\log(X(e^{j\omega})) = \log(X^*(e^{-j\omega}))$.

$$\begin{aligned} \hat{x}^*[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X^*(e^{j\omega}))e^{-j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{-j\omega}))e^{-j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega}))e^{j\omega n} d\omega = \hat{x}[n]. \end{aligned}$$

$$\begin{aligned} \text{(c) } \hat{x}_{ev}[n] &= \frac{\hat{x}[n] + \hat{x}[-n]}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \left(\frac{e^{j\omega n} + e^{-j\omega n}}{2} \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \cos(\omega n) d\omega. \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \hat{x}_{od}[n] &= \frac{\hat{x}[n] - \hat{x}[-n]}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2} \right) d\omega \\ &= \frac{j}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \sin(\omega n) d\omega. \end{aligned}$$

6.31 $x[n] = a\delta[n] + b\delta[n-1]$. Thus, $X(z) = Z\{x[n]\} = a + bz^{-1}$. Also,

$$\hat{X}(z) = \log(a + bz^{-1}) = \log(a) + \log(1 + b/az^{-1}) = \log(a) + \sum_{n=-\infty}^{\infty} (-1)^{n-1} \frac{(b/a)^n}{n} z^{-n}.$$

$$\text{Therefore, } \hat{x}[n] = \begin{cases} \log(a), & \text{if } n = 0, \\ (-1)^{n-1} \frac{(b/a)^n}{n}, & \text{for } n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \text{6.32 (a) } \hat{X}(z) &= \log(K) + \sum_{k=1}^{N_\alpha} \log(1 - \alpha_k z^{-1}) + \sum_{k=1}^{N_\gamma} \log(1 - \gamma_k z) - \sum_{k=1}^{N_\beta} \log(1 - \beta_k z^{-1}) - \sum_{k=1}^{N_\delta} \log(1 - \delta_k z) \\ &= \log(K) - \sum_{k=1}^{N_\alpha} \sum_{n=1}^{\infty} \frac{\alpha_k^n}{n} z^{-n} - \sum_{k=1}^{N_\gamma} \sum_{n=1}^{\infty} \frac{\gamma_k^n}{n} z^n - \sum_{k=1}^{N_\beta} \sum_{n=1}^{\infty} \frac{\beta_k^n}{n} z^{-n} - \sum_{k=1}^{N_\delta} \sum_{n=1}^{\infty} \frac{\delta_k^n}{n} z^n. \end{aligned}$$

$$\text{Thus, } \hat{x}[n] = \begin{cases} \log(K), & n = 0, \\ \sum_{k=1}^{N_\beta} \sum_{n=1}^{\infty} \frac{\beta_k^n}{n} - \sum_{k=1}^{N_\alpha} \sum_{n=1}^{\infty} \frac{\alpha_k^n}{n}, & n > 0, \\ \sum_{k=1}^{N_\gamma} \sum_{n=1}^{\infty} \frac{\gamma_k^{-n}}{n} - \sum_{k=1}^{N_\delta} \sum_{n=1}^{\infty} \frac{\delta_k^{-n}}{n}, & n < 0. \end{cases}$$

(b) $|\hat{x}[n]| < N \frac{|r|^n}{|n|}$ as $n \rightarrow \infty$, where r is the maximum value of $\alpha_k, \beta_k, \gamma_k$, and δ_k for all values of k , and N is a constant. Thus, $\hat{x}[n]$ is a decaying bounded sequence as $n \rightarrow \infty$.

(c) From Part (a) if $\alpha_k = \beta_k = 0$, then $\hat{x}[n] = 0$ for all $n > 0$, and is thus anti-causal.

(d) If $\gamma_k = \delta_k = 0$, then $\hat{x}[n] = 0$ for all $n < 0$, and is thus causal.

6.33 If $X(z)$ has no poles and zeros on the unit circle, then from Part (b) of Problem 6.32, $\gamma_k = \delta_k = 0$, then $\hat{x}[n] = 0$ for all $n < 0$.

$$\hat{X}(z) = \log(X(z)). \text{ Therefore, } \frac{d\hat{X}(z)}{dz} = \frac{1}{X(z)} \frac{dX(z)}{dz}. \text{ Thus, } z \frac{dX(z)}{dz} = zX(z) \frac{d\hat{X}(z)}{dz}.$$

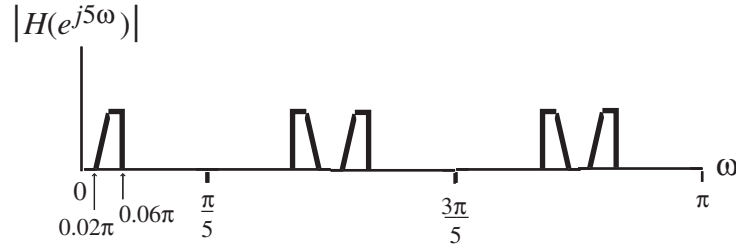
Taking the inverse z -transform we get $nx[n] = \sum_{k=0}^n k \hat{x}[k]x[n-k]$, $n \neq 0$. Or,

$$x[n] = \sum_{k=0}^{n-1} \frac{k}{n} \hat{x}[k]x[n-k] + \hat{x}[n]x[0]. \text{ Hence, } \hat{x}[n] = \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left(\frac{k}{n}\right) \frac{\hat{x}[k]x[n-k]}{x[0]}, n \neq 0.$$

For $n = 0$, $\hat{x}[0] = \hat{X}(z)|_{z=\infty} = X(z)|_{z=\infty} = \log(x[0])$. Thus,

$$\hat{x}[n] = \begin{cases} 0, & n < 0, \\ \log(x[0]), & n = 0, \\ \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left(\frac{k}{n}\right) \frac{\hat{x}[k]x[n-k]}{x[0]}, & n > 0. \end{cases}$$

6.34



6.35 Given the real part of a real, stable transfer function

$$H_{re}(e^{j\omega}) = \frac{\sum_{i=0}^N a_i \cos(i\omega)}{\sum_{i=0}^N b_i \cos(i\omega)} = \frac{A(e^{j\omega})}{B(e^{j\omega})}, \quad (6-1)$$

the problem is to determine the transfer function $H(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^N p_i z^{-i}}{\sum_{i=0}^N p_i z^{-i}}$.

$$\begin{aligned} \text{(a)} \quad H_{re}(e^{j\omega}) &= \frac{1}{2}[H(e^{j\omega}) + H^*(e^{j\omega})] = \frac{1}{2}[H(e^{j\omega}) + H(e^{-j\omega})] \\ &= \frac{1}{2}[H(z) + H(z^{-1})] \Big|_{z=e^{j\omega}}. \end{aligned}$$

Substituting $H(z) = P(z)/D(z)$ in the above we get

$$H_{re}(e^{j\omega}) = \frac{1}{2} \frac{P(z)D(z^{-1}) + P(z^{-1})D(z)}{D(z)D(z^{-1})} \Big|_{z=e^{j\omega}}, \quad (6-2)$$

which is Eq. (6.117).

$$\text{(b)} \quad \text{Comparing Eqs. (6-1) and (6-2) we get } B(e^{j\omega}) = D(z)D(z^{-1}) \Big|_{z=e^{j\omega}}, \quad (6-3)$$

$$A(e^{j\omega}) = \frac{1}{2} [P(z)D(z^{-1}) + P(z^{-1})D(z)] \Big|_{z=e^{j\omega}}. \quad (6-4)$$

$$\text{Now, } D(z) \text{ is of the form } D(z) = Kz^{-N} \prod_{i=1}^N (z - z_i), \quad (6-5)$$

where the z_i 's are the roots of $B(z) = B(e^{j\omega}) \Big|_{e^{j\omega}=z}$ inside the unit circle and K is a scalar

constant. Putting $\omega = 0$ in Eq. (6-3) we get $B(1) = [D(1)]^2$, or $\sqrt{B(1)} = K \prod_{i=1}^N (1 - z_i)$.

Hence,

$$K = \sqrt{B(1)} / \prod_{i=1}^N (1 - z_i). \quad (6-6)$$

(c) By analytic continuation, Eq. (6-4) yields $A(z) = \frac{1}{2}[P(z)D(z^{-1}) + P(z^{-1})D(z)]$. (6-7)

Substituting $A(z) = \frac{1}{2} \sum_{i=0}^N a_i(z^i + z^{-i})$ and the polynomial forms of $P(z)$ and $D(z)$, we get

$\sum_{i=0}^N a_i(z^i + z^{-i}) = \left(\sum_{i=0}^N p_i z^{-i} \right) \left(\sum_{i=0}^N d_i z^i \right) + \left(\sum_{i=0}^N p_i z^i \right) \left(\sum_{i=0}^N d_i z^{-i} \right)$ and equating the coefficients of $(z^i + z^{-i})$ on both sides, we arrive at a set of N equations which can be solved for the numerator coefficients p_i of $H(z)$.

For the given example, i.e., $H_{re}(e) = \frac{1 + \cos(\omega) + \cos(2\omega)}{17 - 8 + \cos(2\omega)}$, we observe

$$A(z) = 1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2}). \quad (6-8)$$

Also, $B(z) = 17 - 4(z^2 + z^{-2})$, which has roots at $z = \pm \frac{1}{2}$ and $z = \pm 2$. Hence,

$$D(z) = Kz^{-2}(z - \frac{1}{2})(z + \frac{1}{2}) = K(z^2 - \frac{1}{4})z^{-2}. \quad (6-9)$$

Also, from Eq. (6-6) we have $K = \sqrt{17 - 8} / (1 - \frac{1}{4}) = 4$, so that $D(z) = 4 - z^{-2}$.

Substituting Eqs. (6-8), (6-9) and $P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2}$ in Eq. (6-7) we get

$$1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2}) = \left[(p_0 + p_1 z^{-1} + p_2 z^{-2})(4 - z^{-2}) + (p_0 + p_1 z + p_2 z^2)(4 - z^{-2}) \right]$$

Equating the coefficients of $(z^i + z^{-i})/2, 0 \leq i \leq 2$, on both sides we get

$4p_0 - p_2 = 1, 3p_1 = 1, 4p_2 - p_0 = 1$. Solving these equations we then arrive at

$$p_0 = p_1 = p_2 = 1/3. \text{ Therefore, } H(z) = \frac{1 + z^{-1} + z^{-2}}{3(4 - z^{-2})}.$$

6.36 $A(z) = \frac{d_M + d_{M-1}z^{-1} + \dots + d_1z^{-M+1} + z^{-M}}{1 + d_1z^{-1} + \dots + d_{M-1}z^{-M+1} + d_Mz^{-M}} \Rightarrow A(1) = 1 \text{ and } A(-1) = -1 \text{ if } M \text{ is odd.}$

In which case, $G(1) = H(1)$ and $G(-1) = H(-1)$. If is even, then $G(1) = H(1)$ and $G(-1) = H(1)$.

6.37 $H(z) = H_1(z)H_2(z) + H_3(z) = (1.2 + 3.3z^{-1} + 0.7z^{-2})(-4.1 - 2.5z^{-1} + 0.9z^{-2}) + 2.3 + 4.3z^{-1} + 0.8z^{-2} = -2.62 - 12.23z^{-1} - 9.24z^{-2} + 1.22z^{-3} + 0.63z^{-4}.$

6.38 (a) $(1 - 0.1z^{-1} + 0.14z^{-2} + 0.49z^{-3})Y(z) = (5 + 9.5z^{-1} + 1.4z^{-2} - 24z^{-3})X(z) \Rightarrow$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5 + 9.5z^{-1} + 1.4z^{-2} - 24z^{-3}}{1 - 0.1z^{-1} + 0.14z^{-2} + 0.49z^{-3}}. \text{ Using Program 6_1.m we factorize } H(z)$$

and develop is pole-zero plot shown below:

Numerator factors

```
1.0000000000000000    3.1000000000000000    4.0000000000000000
1.0000000000000000    -1.2000000000000000    0
```

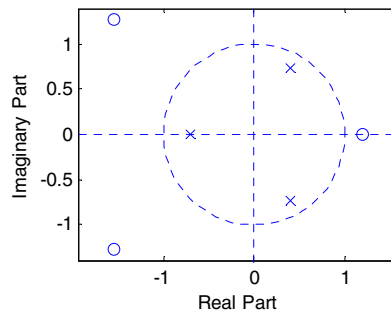
Denominator factors

```
1.0000000000000000    -0.8000000000000000    0.7000000000000000
1.0000000000000000    0.7000000000000000    0
```

Gain constant

5

The factored form of $H(z)$ is thus
$$H(z) = \frac{5(1 + 3.1z^{-1} + 4z^{-2})(1 - 1.2z^{-1})}{(1 - 0.81z^{-1} + 0.7z^{-2})(1 + 0.7z^{-1})}$$



As all poles are inside the unit circle, $H(z)$ is BIBO stable.

(b) $(1 - 0.5z^{-1} + 0.1z^{-2} + 0.3z^{-3} - 0.0936z^{-4})Y(z)$
 $= (5 + 16.5z^{-1} + 14.7z^{-2} - 22.4z^{-3} - 33.6z^{-4})X(z) \Rightarrow$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5 + 16.5z^{-1} + 14.7z^{-2} - 22.4z^{-3} - 33.6z^{-4}}{1 - 0.5z^{-1} + 0.1z^{-2} + 0.3z^{-3} - 0.0936z^{-4}}. \text{ Using Program 6_1.m we}$$

factorize $H(z)$ and develop is pole-zero plot shown below:

Numerator factors

```
1.0000000000000000    -1.2000000000000000    0
1.0000000000000000    3.1000000000000001    4.0000000000000001
1.0000000000000000    1.3999999999999999    0
```

Denominator factors

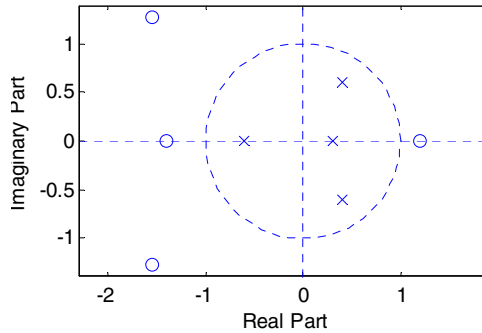
```
1.0000000000000000    0.59950918226500    0
1.0000000000000000    -0.80131282790906    0.52021728677142
1.0000000000000000    -0.29819635435594    0
```

Gain constant
5

The factored form of $H(z)$ is thus

$$H(z) = \frac{5(1 - 1.2z^{-1})(1 + 3.1z^{-1} + 4z^{-2})(1 + 1.4z^{-1})}{(1 + 0.5995z^{-1})(1 - 0.801313z^{-1} + 0.520217z^{-2})(1 - 0.2982z^{-1})}$$

As all poles are inside the unit circle, $H(z)$ is BIBO stable.



6.39 A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields

$$H(z) = -\frac{1.21212}{1 - 0.4z^{-1}} + \frac{2.21212(1 - 0.81781z^{-1})}{1 + 0.5z^{-1} + 0.3z^{-2}}. \quad \text{Comparing the denominator of the}$$

quadratic factor with $1 - 2r \cos(\omega_o)z^{-1} + r^2z^{-2}$ we get $r = \sqrt{0.3} = 0.54772$ and

$$\cos(\omega_o) = -\frac{0.5}{2\sqrt{0.3}}, \text{ or } \omega_o = 2.04478. \text{ Hence, from Table 6.1 we have}$$

$$h[n] = -1.21212(0.4)^n \mu[n] + (\sqrt{0.3})^n \cos(2.04478n) \mu[n].$$

6.40 (a) A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields

$$H(z) = -2 + \frac{5}{1 + 0.6z^{-1}} - \frac{2}{1 - 0.3z^{-1}}. \quad \text{Hence, from Table 6.1 we have}$$

$$h[n] = -2\delta[n] + 5(-0.6)^n \mu[n] - 2(0.3)^n \mu[n].$$

(b) $x[n] = 2.1(0.4)^n \mu[n] + 0.3(-0.3)^n \mu[n]$. Its z -transform is thus given by

$$X(z) = \frac{2.1}{1 - 0.4z^{-1}} + \frac{0.3}{1 + 0.3z^{-1}} = \frac{2.4 + 0.51z^{-1}}{(1 - 0.4z^{-1})(1 + 0.3z^{-1})}, \quad |z| > 0.4. \quad \text{The } z\text{-transform of}$$

$$\text{the output is then given by } Y(z) = \left[\frac{2.4 + 0.51z^{-1}}{(1 - 0.4z^{-1})(1 + 0.3z^{-1})} \right] \cdot \left[\frac{1 - 3.3z^{-1} + 0.36z^{-2}}{1 + 0.3z^{-1} - 0.18z^{-2}} \right].$$

A partial-fraction expansion of $Y(z)$ in z^{-1} using the M-file `residuez` yields

$Y(z) = \frac{9.3}{1+0.6z^{-1}} - \frac{16.8}{1-0.4z^{-1}} + \frac{12.3}{1-0.3z^{-1}} - \frac{2.4}{1+0.3z^{-1}}, |z| > 0.6$. Hence, from Table 6.1 we have $y[n] = \left(9.3(-0.6)^n - 16.8(0.4)^n + 12.3(0.3)^n - 2.4(-0.3)^n\right) \mu[n]$.

6.41 (a) $H(z) = Z\{h[n]\} = \frac{1}{1+0.4z^{-1}}, |z| > 0.4$, $X(z) = Z\{x[n]\} = \frac{1}{1-0.2z^{-1}}, |z| > 0.4$. Thus, $Y(z) = H(z)X(z) = \frac{1}{(1+0.4z^{-1})(1-0.2z^{-1})}, |z| > 0.4$. A partial-fraction expansion of using the M-file `residuez` yields $Y(z) = \frac{2/3}{1+0.4z^{-1}} + \frac{1/3}{1-0.2z^{-1}}$. Hence, from Table 6.1 $y[n] = \frac{2}{3}(-0.4)^n \mu[n] + \frac{1}{3}(0.2)^n \mu[n]$.

(b) $H(z) = Z\{h[n]\} = \frac{1}{1+0.2z^{-1}}, |z| > 0.2$, $X(z) = Z\{x[n]\} = \frac{1}{1+0.2z^{-1}}, |z| > 0.4$. Thus, $Y(z) = H(z)X(z) = \frac{1}{(1+0.2z^{-1})^2}, |z| > 0.2$. Hence, from Table 6.1, $y[n] = (n+1)(-0.2)^n \mu[n]$.

6.42 $Y(z) = Z\{y[n]\} = \frac{2}{1+0.3z^{-1}}, |z| > 0.3$, $X(z) = Z\{x[n]\} = \frac{4}{1-0.6z^{-1}}, |z| > 0.2$. Thus, $H(z) = \frac{Y(z)}{X(z)} = \frac{0.5(1-0.6z^{-1})}{1+0.3z^{-1}}, |z| > 0.3$. A partial-fraction expansion of using the M-file `residuez` yields $H(z) = -1 + \frac{1.5}{1+0.3z^{-1}}$. Hence, from Table 6.1, $h[n] = -\delta[n] + 1.5(-0.3)^n \mu[n]$.

6.43 (a) Taking the $-$ transform of both sides of the difference equation we get $Y(z) = 0.2z^{-1}Y(z) + 0.08z^{-2}Y(z) + 2X(z)$. Hence, $H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-0.2z^{-1}-0.08z^{-2}}$.

(b) A partial-fraction expansion of using the M-file `residuez` yields $H(z) = \frac{4/3}{1-0.4z^{-1}} + \frac{2/3}{1+0.2z^{-1}}$. Hence, from Table 6.1, $h[n] = \frac{4}{3}(0.4)^n \mu[n] + \frac{2}{3}(-0.2)^n \mu[n]$.

(c) Now $S(z) = Z\{s[n]\} = H(z) \cdot Z\{\mu[n]\} = \frac{2}{(1-0.2z^{-1}-0.08z^{-2})(1-z^{-1})}$.

A partial-fraction expansion of using the M-file `residuez` yields

$$S(z) = \frac{2.7778}{1-z^{-1}} - \frac{0.8889}{1-0.4z^{-1}} + \frac{0.1111}{1+0.2z^{-1}}. \text{ Hence, from Table 6.1,}$$

$$s[n] = 2.7778\mu[n] - 0.8889(0.4)^n \mu[n] + 0.1111(-0.2)^n \mu[n].$$

6.44 $H(z) = \frac{1-z^{-2}}{1-(1+\alpha)\cos(\omega_c)z^{-1} + \alpha z^{-2}}$. Thus,

$$H(e^{j\omega}) = \frac{1-e^{j2\omega}}{1-(1+\alpha)\cos(\omega_c)e^{-j\omega} + \alpha e^{-j2\omega}}.$$

$$|H(e^{j\omega})|^2 = \frac{2(1-\cos 2\omega)}{1+(\cos \omega_c)^2(1+\alpha)^2 + \alpha^2 + 2\alpha \cos 2\omega - 2\cos \omega_c(1+\alpha)^2 \cos \omega}$$

$$= \frac{4\sin^2 \omega}{(1+\alpha)^2(\cos \omega - \cos \omega_c) + (1-\alpha)^2 \sin^2 \omega}. \text{ Note that } |H(e^{j\omega})|^2 \text{ is maximum when}$$

$$\cos \omega = \cos \omega_c, \text{ i.e., } \omega = \omega_c. \text{ Then } |H(e^{j\omega_c})|^2 = \frac{4\sin^2 \omega_c}{(1-\alpha)^2 \sin^2 \omega_c} = \frac{4}{(1-\alpha)^2}, \text{ and hence,}$$

$$|H(e^{j\omega_c})| = 2/(1-\alpha).$$

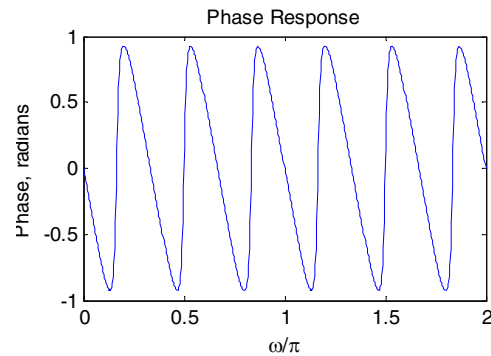
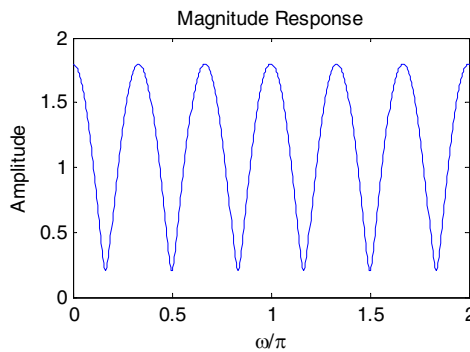
6.45 $H(z) = 1 - \alpha z^{-R} \Rightarrow H(e^{j\omega}) = 1 - \alpha e^{-j\omega R}$. Then, $|H(e^{j\omega})| = \sqrt{1 + \alpha^2 - 2\alpha \cos(\omega R)}$.

$|H(e^{j\omega})|$ is maximum when $\cos(\omega R) = -1$ and is minimum when $\cos(\omega R) = 1$. The

maximum value of $|H(e^{j\omega})|$ is $1 + |\alpha|$, and the minimum value is $1 - |\alpha|$. $|H(e^{j\omega})|$ has R peaks and R dips in the range $0 \leq \omega < 2\pi$.

The peaks are located at $\omega = \omega_k = \frac{2\pi k}{R}$ and the dips are located at $\omega = \omega_k = \frac{(2k+1)\pi}{R}$,

$0 \leq k \leq R-1$.



6.46 $G(e^{j\omega}) = (H(e^{j\omega}))^3 = (1 - \alpha e^{j\omega R})^3$.

6.47 $G(e^{j\omega}) = \sum_{n=0}^{M-1} \alpha^n e^{-j\omega n} = \frac{1 - \alpha^M e^{-j\omega M}}{1 - \alpha e^{-j\omega}}$. Note that $G(e^{j\omega}) = H(e^{j\omega})$ for

$\alpha^n = \frac{1}{M}$, $0 \leq n \leq M-1$. Now, $G(e^{j0}) = \frac{1 - \alpha^M}{1 - \alpha}$. Hence, to make the dc value of the magnitude response equal to unity, the impulse response should be multiplied by a constant $K = \left| \frac{1 - \alpha}{1 - \alpha^M} \right|$.

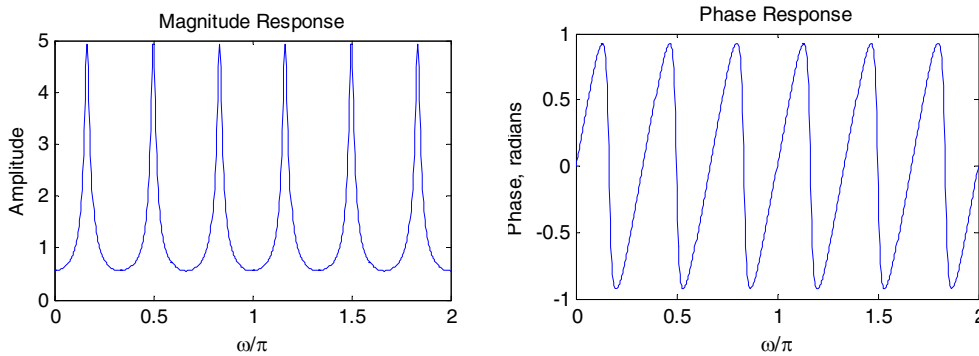
6.48 $Y(e^{j\omega}) = X(e^{j\omega}) + \alpha e^{-j\omega R} Y(e^{j\omega})$. Hence, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \alpha e^{-j\omega R}}$. Maximum

value of $|H(e^{j\omega})|$ is $\frac{1}{1 - |\alpha|}$ and the minimum value is $\frac{1}{1 + |\alpha|}$. There are R peaks and dips in the range $0 \leq \omega \leq 2\pi$. The locations of the peaks and dips are given by

$1 - \alpha e^{-j\omega R} = 1 \pm |\alpha|$ or, $e^{-j\omega R} = \pm \frac{|\alpha|}{\alpha}$. The locations of the peaks are given by

$\omega = \omega_k = \frac{2\pi k}{R}$ and the locations of the dips are given by $\omega = \omega_k = \frac{(2\pi + 1)k}{R}$,

$0 \leq k \leq R-1$. Plots of the magnitude and phase responses of $H(e^{j\omega})$ for $\alpha = 0.8$ and $R = 6$ are shown below:



6.49 $A(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = \frac{(b_0 e^{j\omega} + b_2 e^{-j\omega}) + b_1}{(e^{j\omega} + a_2 e^{-j\omega}) + a_1}$
 $= \frac{b_1 + (b_0 + b_2) \cos \omega + j(b_0 - b_2) \sin \omega}{a_1 + (1 + a_2) \cos \omega + j(1 - a_2) \sin \omega}$. Therefore,

$|A(e^{j\omega})|^2 = \frac{[b_1 + (b_0 + b_2)]^2 \cos^2 \omega + (b_0 - b_2)^2 \sin^2 \omega}{[a_1 + (1 + a_2)]^2 \cos^2 \omega + (1 - a_2)^2 \sin^2 \omega} = 1$. Hence, at $\omega = 0$, we have

$b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$, and at $\omega = \pi/2$, we have $b_0 - b_2 = \pm(1 - a_2)$.

Solution #1: Consider $b_0 - b_2 = 1 - a_2$. Choose $b_0 = 1$, $-b_2 = 1 - a_2$, and $b_2 = a_2$.

Substituting these values in $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$, we get $b_1 = a_1$. In this case,

$$A(e^{j\omega}) = \frac{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = 1, \text{ a trivial solution.}$$

Solution #2: Consider $b_0 - b_2 = a_2 - 1$. Choose $b_0 = a_2$ and $b_2 = 1$. Substituting these values in $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$, we get $b_1 = a_1$. In this case,

$$A(e^{j\omega}) = \frac{a_2 + a_1 e^{-j\omega} + e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}.$$

6.50 From Eq. (2.20), the input-output relation of a factor-of-2 up-sampler is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \dots \\ 0, & \text{otherwise.} \end{cases}$$

The DTFT of $x_u[n]$ is therefore given by

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\omega m} = X(e^{j2\omega}) \text{ where } X(e^{j\omega})$$

is the DTFT of $x[n]$.

6.51 $H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}} = \frac{1}{1 - 0.5 \cos \omega + j0.5 \sin \omega}$. Thus,

$$H(e^{\pm j\pi/4}) = \frac{1}{1 - 0.5 \cos(\pm\pi/4) + j0.5 \sin(\pm\pi/4)} = \frac{1}{0.6464 \pm j0.3536} = 0.6512 \mp j1.1907.$$

Therefore, $|H(e^{\pm j\pi/4})| = 1.3572$ and $\arg\{H(e^{\pm j\pi/4})\} = \theta(\pm j\pi/4) = \mp 1.0703$.

Now, for an input $x[n] = \sin(\omega_o n) \mu[n]$, the steady-state output is given by

$$y[n] = |H(e^{j\omega_o})| \sin(\omega_o n + \theta(\omega_o)) \text{ which for } \omega_o = \pi/4 \text{ reduces to}$$

$$y[n] = |H(e^{j\pi/4})| \sin\left(\frac{\pi}{4} n + \theta(\pi/4)\right) = 1.3572 \sin\left(\frac{\pi}{4} n - 1.0703\right).$$

6.52 To guarantee the stability of $G(z)$, the transformation $z \rightarrow F(z)$ should be such that the unit circle remains inside the ROC after the mapping. If the points inside the unit circle after the mapping remains inside the unit circle, $G(z)$ will be causal and stable. On the other hand, if the points inside the unit circle after the mapping move outside the unit circle, $G(z)$ will be stable but anti-causal. For example, the mapping $z \rightarrow -z$ will ensure that $G(z)$ will be causal and stable, whereas, the mapping $z \rightarrow z^{-1}$ will result in a $G(z)$ that is stable, but anti-causal.

6.53

6.54 $\sum_{n=0}^K |h[n]|^2 = 0.95 \sum_{n=0}^{\infty} |h[n]|^2$. Since $H(z) = 1/(1 - \beta z^{-1})$, $h[n] = (\beta)^n \mu[n]$. Thus,

$$\frac{1 - |\beta|^{2K}}{1 - |\beta|^2} = \frac{0.95}{1 - |\beta|^2}. \text{ Solving this equation for we get } K = 0.5 \frac{\log(0.05)}{\log(|\alpha|)}.$$

6.55 Let the output of the predictor of Figure P6.3(a) be denoted by $E(z)$. Then analysis of this structure yields $E(z) = P(z)[U(z) + E(z)]$ and $U(z) = X(z) - E(z)$. From the first equation we have $E(z) = \frac{P(z)}{1 - P(z)}U(z)$ which when substituted in the second equation yields

$$H(z) = \frac{U(z)}{X(z)} = 1 - P(z).$$

Analyzing Figure P6.3(b) we get $Y(z) = V(z) + P(z)Y(z)$ which leads to

$$G(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 - P(z)}, \text{ which is seen to be the inverse of } H(z).$$

For $P(z) = h_1 z^{-1}$, $H(z) = 1 - h_1 z^{-1}$ and $G(z) = \frac{1}{1 - h_1 z^{-1}}$. Similarly, for

$$P(z) = h_1 z^{-1} + h_2 z^{-2}, H(z) = 1 - h_1 z^{-1} - h_2 z^{-2} \text{ and } G(z) = \frac{1}{1 - h_1 z^{-1} - h_2 z^{-2}}.$$

6.56 $Y(z) = [H_0(z)F_0(z) - H_0(-z)F_0(-z)]X(z)$. Since the output is a delayed replica of the input, we must have $H_0(z)F_0(z) - H_0(-z)F_0(-z) = z^{-r}$. But, $H_0(z) = 1 + \alpha z^{-1}$. Hence, $(1 + \alpha z^{-1})F_0(z) - (1 - \alpha z^{-1})F_0(-z) = z^{-r}$. Let $F_0(z) = a_0 + a_1 z^{-1}$. This implies, $2(a_0 \alpha + a_1)z^{-1} = z^{-r}$.

The solution is therefore, $r = 1$ and $2(a_0 \alpha + a_1) = 1$. One possible solution is thus $a_0 = 1/2$ and $a_1 = 1/4$. Hence, $F_0(z) = 0.25(1 + z^{-1})$.

6.57 $H_1(z) = Z\{h_1[n]\} = 1.2 + \frac{0.5}{1 + 0.5z^{-1}} - \frac{0.6}{1 - 0.2z^{-1}} = \frac{1.1 - 0.04z^{-1} - 0.12z^{-2}}{1 + 0.3z^{-1} - 0.1z^{-2}}$. The transfer function of the inverse transform is thus

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1 + 0.3z^{-1} - 0.1z^{-2}}{1.1 - 0.04z^{-1} - 0.12z^{-2}} = \frac{(1 + 0.5z^{-1})(1 - 0.2z^{-1})}{1.1(1 - 0.349z^{-1})(1 + 0.3126z^{-1})}.$$

As both poles are inside the unit circle, $H_2(z)$ is stable and causal with an ROC $|z| > 0.349$. A partial-fraction expansion is obtained using the M-file `residuez` is

$$H_2(z) = \frac{1}{1.2} + \frac{0.498}{1 - 0.34897z^{-1}} - \frac{0.42224464}{1 + 0.31261z^{-1}}. \text{ Hence,}$$

$$h_2[n] = \frac{1}{1.2} \delta[n] + 0.498(0.34897)^n \mu[n] - 0.42224464(-0.31261)^n \mu[n].$$

6.58 Now, $H(z)|_{z=e^{j\omega}} = H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$. Denote $H'(z) = \frac{dH(z)}{dz}$.

From the above we get $\ln H(z)|_{z=e^{j\omega}} = \ln |H(e^{j\omega})| + j\theta(\omega)$. Therefore,

$$\left. \frac{H'(z)}{H(z)} \cdot \frac{dz}{d\omega} \right|_{z=e^{j\omega}} = \frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|} + j \frac{d\theta(\omega)}{d\omega} = \frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|} - j\tau_g(\omega). \quad (6-A)$$

$$\text{Hence, } \tau_g(\omega) = -z \left. \frac{H'(z)}{H(z)} \right|_{z=e^{j\omega}} - j \frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|}. \quad (6-B)$$

Replacing by in Eq. (6-A) we arrive at

$$\tau_g(\omega) = -z^{-1} \left. \frac{H'(z)}{H(z)} \right|_{z=e^{j\omega}} + j \frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|}. \quad (6-C)$$

Adding Eqs. (6-B) and (6-C), and making use of the notation $T(z) = z \frac{H'(z)}{H(z)}$ we finally get

$$\tau_g(\omega) = \left. \frac{T(z) + T(z^{-1})}{2} \right|_{z=e^{j\omega}}.$$

M6.1 (a) The output data generated by Program 6_1 is as follows:

Numerator factors

| | | |
|--------------------|---------------------|--------------------|
| 1.0000000000000000 | -2.1000000000000000 | 5.0000000000000000 |
| 1.0000000000000000 | -0.4000000000000000 | 0.9000000000000000 |

Denominator factors

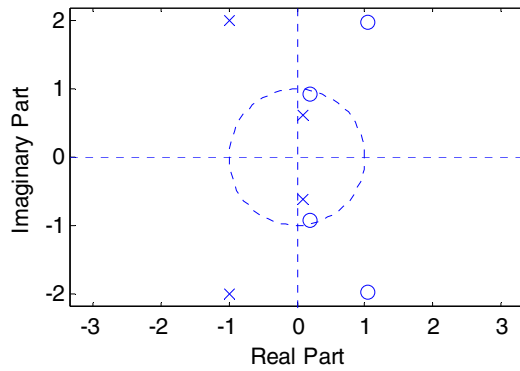
| | | |
|--------------------|---------------------|--------------------|
| 1.0000000000000000 | 2.0000000000000000 | 4.9999999999999999 |
| 1.0000000000000000 | -0.2000000000000000 | 0.4000000000000001 |

Gain constant

0.5000000000000000

$$\text{Hence, } G_1(z) = \frac{0.5(1 - 2.1z^{-1} + 5z^{-2})(1 + 2z^{-1} + 0.9z^{-2})}{(1 + 2z^{-1} + 5z^{-2})(1 - 0.2z^{-1} + 0.4z^{-2})}.$$

The pole-zero plot of $G_1(z)$ is given below:



There are 3 ROCs associated with $G_1(z)$: $\mathcal{R}_1 : |z| < \sqrt{0.4}$, $\mathcal{R}_2 : \sqrt{0.4} < |z| < \sqrt{5}$, and $\mathcal{R}_3 : |z| > \sqrt{5}$. The inverse z -transform corresponding to the ROC \mathcal{R}_1 is a left-sided sequence, the inverse z -transform corresponding to the ROC \mathcal{R}_2 is a two-sided sequence, and the inverse z -transform corresponding to the ROC \mathcal{R}_3 is a right-sided sequence.

(b) The output data generated by Program 6_1 is as follows:

Numerator factors

```
1.0000000000000000    1.2000000000000000    3.9999999999999999
1.0000000000000000    -0.5000000000000000    0.9000000000000001
```

Denominator factors

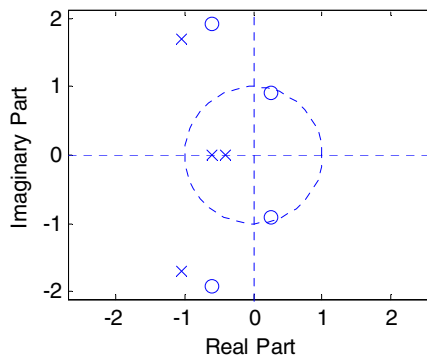
```
1.0000000000000000    2.1000000000000000    4.0000000000000001
1.0000000000000000    0.6000000000000003    0
1.0000000000000000    0.3999999999999997    0
```

Gain constant

1

$$\text{Hence, } G_2(z) = \frac{(1 + 1.2z^{-1} + 4z^{-2})(1 - 0.5z^{-1} + 0.9z^{-2})}{(1 + 2.1z^{-1} + 4z^{-2})(1 + 0.6z^{-1})(1 + 0.4z^{-1})}$$

The pole-zero plot of $G_2(z)$ is given below:



There are 4 ROCs associated with $G_2(z)$: $\mathcal{R}_1 : |z| < 0.4$, $\mathcal{R}_2 : 0.4 < |z| < 0.6$,

$\mathcal{R}_3 : 0.6 < |z| < 2$, and $\mathcal{R}_4 : |z| > 2$. The inverse z -transform corresponding to the ROC \mathcal{R}_1 is a left-sided sequence, the inverse z -transform corresponding to the ROC \mathcal{R}_2 is a two-sided sequence, the inverse z -transform corresponding to the ROC \mathcal{R}_3 is a two-sided sequence, and the inverse z -transform corresponding to the ROC \mathcal{R}_4 is a right-sided sequence.

M6.2 (a) The output data generated by Program 6_3 is as follows:

Residues

-3.33333333333333 3.33333333333333

Poles

-0.600000000000000 0.300000000000000

Constants

0

Hence, the partial-fraction expansion of $X_a(z)$ is given by

$$X_a(z) = -\frac{10/3}{1+0.6z^{-1}} + \frac{10/3}{1-0.3z^{-1}}. \text{ The } z\text{-transform has poles at } z = -0.6 \text{ and at}$$

$z = 0.3$. Thus, it is associated with ROCs as given in the solution of Problem 6.20 which also shows their corresponding inverse z -transform.

(b) The output data generated by Program 6_3 is as follows:

Residues

Columns 1 through 2

2.33333333333333 -3.66666666666667 + 0.00000008829151i

Column 3

4.33333333333333 - 0.00000008829151i

Poles

Columns 1 through 2

-0.600000000000000 0.300000000000000 - 0.00000000722385i

Column 3

0.300000000000000 + 0.00000000722385i

Constants

Hence, the partial-fraction expansion of $X_b(z)$ is given by

$$X_b(z) = \frac{2.3333}{1+0.6z^{-1}} - \frac{3.66666}{1-0.3z^{-1}} + \frac{4.33333}{(1-0.3z^{-1})^2}. \text{ The } z\text{-transform has two poles}$$

at $z = -0.6$ and one at $z = 0.3$. Thus, it is associated with ROCs as given in the solution of Problem 6.20 which also shows their corresponding inverse z -transform.

M6.3 (a) $X_1(z) = 3 - \frac{4}{5+z^{-1}} - \frac{7}{6+z^{-1}} = 3 - \frac{4/5}{1+(1/5)z^{-1}} - \frac{7/6}{1+(1/6)z^{-1}}.$

The output data generated by Program 6_4 is as follows:

Numerator polynomial coefficients
 1.03333333333333 0.73333333333333 0.10000000000000

Denominator polynomial coefficients
 1.00000000000000 0.36666666666667 0.03333333333333

$$\text{Hence, } X_1(z) = \frac{31 + 22z^{-1} + 3z^{-2}}{30 + 11z^{-1} + z^{-2}}.$$

$$\begin{aligned} \text{(b) } X_2(z) &= -2.5 + \frac{3}{1 + 0.4z^{-1}} - \frac{1.4 + z^{-1}}{1 + 0.6z^{-2}} \\ &= -2.5 + \frac{3}{1 + 0.4z^{-1}} - \frac{0.7 - j6454972243679}{1 - j0.774596669z^{-1}} - \frac{0.7 + j6454972243679}{1 + j0.774596669z^{-1}}. \end{aligned}$$

The output data generated by Program 6_4 is as follows:

Numerator polynomial coefficients
 -0.9000 -2.5600 -0.1000 -0.6000

Denominator polynomial coefficients
 1.0000 0.4000 0.6000 0.2400

$$\text{Hence, } X_b(z) = -\frac{0.9 + 2.56z^{-1} + 0.1z^{-2} + 0.6z^{-3}}{1.0 + 0.4z^{-1} + 0.6z^{-2} + 0.24z^{-3}}.$$

$$\begin{aligned} \text{(c) } X_3(z) &= \frac{5}{1 + 0.64z^{-1}} + \frac{6}{4 + 2z^{-1}} + \frac{-4}{(4 + 2z^{-1})^2} \\ &= \frac{5}{1 + 0.64z^{-1}} + \frac{1.5}{1 + 0.5z^{-1}} + \frac{-0.25}{(1 + 0.5z^{-1})^2}. \end{aligned}$$

The output data generated by Program 6_4 is as follows:

Numerator polynomial coefficients
 6.2500 6.5500 1.7300 0

Denominator polynomial coefficients
 1.0000 1.6400 0.8900 0.1600

$$\text{Hence, } X_3(z) = \frac{6.25 + 6.55z^{-1} + 1.73z^{-2}}{1 + 1.64z^{-1} + 0.89z^{-2} + 0.16z^{-3}}.$$

$$\begin{aligned} \text{(d) } X_4(z) &= -5 + \frac{2}{4 + 3z^{-1}} + \frac{z^{-1}}{4 + 3z^{-1} + 0.9z^{-2}} \\ &= -5 + \frac{0.5}{1 + 0.75z^{-1}} - \frac{j0.4303}{1 + (0.3750 - j0.2905)z^{-1}} + \frac{j0.4303}{1 + (0.3750 + j0.2905)z^{-1}}. \end{aligned}$$

The output data generated by Program 6_4 is as follows:

Numerator polynomial coefficients

-4.5000 -6.8750 -3.6375 -0.8438

Denominator polynomial coefficients
 1.0000 1.5000 0.7875 0.1688

$$\text{Hence, } X_4(z) = -\frac{4.5 + 6.875z^{-1} + 3.6375z^{-2} + 0.84375z^{-3}}{1 + 1.5z^{-1} + 0.7875z^{-2} + 0.16875z^{-3}}.$$

M6.4 (a) The inverse z -transform of $X_a(z)$ from its partial-fraction expansion form is thus

$$x_a[n] = 3\delta[n] - \frac{4}{5}(-1/5)^n \mu[n] - \frac{7}{6}(-1/6)^n \mu[n].$$

The first 10 samples of $x_a[n]$ obtained by evaluating this expression in MATLAB are given by

Columns 1 through 4

1.0333333333 0.35444444444 -0.0644074074 0.0118012345

Columns 5 through 8

-0.0021802057 0.0004060343 -0.0000762057 0.0000144076

Columns 9 through 10

-0.0000027426 0.0000005254

The first 10 samples of the inverse z -transform of the rational form of $X_a(z)$ obtained using the M-file `impz` are identical to the samples given above.

(b) The inverse z -transform of $X_b(z)$ from its partial-fraction expansion form is thus

$$x_b[n] = -2.5\delta[n] + 3(-0.4)^n \mu[n] - (0.7 - j0.6454972243679)(j0.774596669241)^n \mu[n] - (0.7 + j0.6454972243679)(-j0.774596669241)^n \mu[n].$$

The first 10 samples of $x_b[n]$ obtained by evaluating this expression in MATLAB are given by

Columns 1 through 4

-0.9000000000 -2.2000000000 1.3200000000 0.4080000000

Columns 5 through 8

-0.4272000000 -0.3907200000 0.3146880000 0.2110848000

Columns 9 through 10

-0.179473920 -0.130386432

The first 10 samples of the inverse z -transform of the rational form of $X_b(z)$ obtained using the M-file `impz` are identical to the samples given above.

(c) The inverse z -transform of $X_c(z)$ from its partial-fraction expansion form is thus

$$x_c[n] = 5(-0.64)^n \mu[n] + 1.5(-0.5)^n \mu[n] - 0.25(n+1)(-0.5)^n \mu[n].$$

The first 10 samples of $x_c[n]$ obtained by evaluating this expression in MATLAB are given by


```
Columns 1 through 5
6.250000000    -3.700000000    2.235500000    -1.373220000
0.854485800
```

```
Columns 6 through 10
-0.536870912    0.3396911337    -0.215996076    0.137807801
-0.0881188675
```

The first 10 samples of the inverse z–transform of the rational form of $X_c(z)$ obtained using the M-file `impz` are identical to the samples given above.

(d) The inverse Z –transform of $X_d(z)$ from its partial-fraction expansion form is thus

$$x_d[n] = -5 + 0.5(-0.75)^n \mu[n] - j0.430331483(-0.375 + j0.29047375)^n \mu[n] \\ + j0.430331483(-0.375 - j0.29047375)^n \mu[n].$$

The first 10 samples of $x_d[n]$ obtained by evaluating this expression in MATLAB are given by

```
Columns 1 through 5
4.500000000000000    -0.12499999991919    0.09374999993940    -
0.12656249997773    0.13710937500069
```

```
Columns 6 through 10
-0.12181640625721    0.09610839844318    -0.07136938476820
0.05192523193410    -0.03790274963350
```

The first 10 samples of the inverse z–transform of the rational form of $X_d(z)$ obtained using the M-file `impz` are identical to the samples given above.

M6.5 To verify using MATLAB that $H_2(z) = \frac{1 + 0.3z^{-1} - 0.1z^{-2}}{1.1 - 0.04z^{-1} - 0.12z^{-2}}$ is the inverse of

$$H_1(z) = \frac{1.1 - 0.04z^{-1} - 0.12z^{-2}}{1 + 0.3z^{-1} - 0.1z^{-2}},$$

we determine the first 20 samples of $h_1[n]$ and $h_2[n]$,

and then form the convolution of these two sequences using the M-file `conv`. The first samples of the convolution result are as follows:

```
Columns 1 through 9
1.0000    0    0.0000    0.0000    0.0000    0.0000    0.0000    -0.0000
```

```
Columns 10 through 18
0.0000    -0.0000    -0.0000    -0.0000    -0.0000    0.0000    -0.0000    0.0000    0.0000
```

```
Columns 19 through 21
-0.0000    -0.0000    -0.0000
```

M6.6 `% As an example try a sequence x = 0:24;`
`% calculate the actual uniform dft`

```

% and then use these uniform samples
% with this ndft program to get the
% original sequence back
% [X,w] = freqz(x,1,25,'whole');
% use freq = X and points = exp(j*w)

freq = input('The sample values = ');
points = input('Frequencies at which samples are taken = ');
L = 1;
len = length(points);
val = zeros(size(1,len));
L = poly(points);
for k = 1:len
    if (freq(k) ~= 0)
        xx = [1 -points(k)];
        [yy, rr] = deconv(L,xx);
        F(k,:) = yy;
        Down = polyval(yy,points(k))*(points(k))*(points(k)^(-len+1));
        F(k,:) = freq(k)/down*yy;
        val = val+F(k,:);
    end
end
coeff = val;

```

Chapter 7

7.1 Impulse response of the moving average filter is: $h[n] = \begin{cases} \frac{1}{M}, & 0 \leq n \leq M-1, \\ 0, & \text{otherwise.} \end{cases}$

Its frequency response is: $H(e^{j\omega}) = \frac{1}{M} \left(\frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}} \right) = \frac{1}{M} \cdot \frac{\sin(M\omega/2)}{\sin(\omega/2)} e^{-jM\omega/2}$.

Now, for a BR transfer function, $|H(e^{j\omega})| \leq 1, \forall \omega$. For the moving-average filter,

$|H(e^{j\omega})| = \frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right|$. We shall show by induction that

$\frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| \leq 1, \forall \omega$. Now, for

$M = 2, \left| H(e^{j\omega}) \right| = \frac{1}{2} \left| \frac{\sin(\omega)}{\sin(\omega/2)} \right| = \frac{1}{2} \left| \frac{2 \sin(\omega/2) \cos(\omega/2)}{\sin(\omega/2)} \right| = \cos(\omega/2) \leq 1, \forall \omega$.

We assume next that $\frac{1}{m} \left| \frac{\sin(m\omega/2)}{\sin(\omega/2)} \right| \leq 1$ for $1 \leq m \leq M-1$. We can express

$\frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| = \frac{1}{M} \left| \frac{\sin((M-1)\omega/2) \cos(\omega/2) + \cos((M-1)\omega/2) \sin(\omega/2)}{\sin(\omega/2)} \right|$
 $\leq \frac{1}{M} \left[\left| \frac{\sin((M-1)\omega/2)}{\sin(\omega/2)} \right| |\cos(\omega/2)| + |\cos((M-1)\omega/2)| \right]$. Now, $|\cos(m\omega/2)| \leq 1$ for

all values of m . Hence, $\frac{1}{M} \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| \leq \frac{1}{M} [M-1+1] \leq 1$.

7.2 $A(z) = \frac{1-d_1^*z}{z-d_1}$. $|A(z)|^2 = A(z)A^*(z) = \frac{(1-d_1^*z)(1-d_1z^*)}{(z-d_1)(z^*-d_1^*)}$. Therefore,

$$1 - |A(z)|^2 = \frac{(z-d_1)(z^*-d_1^*) - (1-d_1^*z)(1-d_1z^*)}{(z-d_1)(z^*-d_1^*)}$$

$$= \frac{|z|^2 + |d_1|^2 - d_1z^* - d_1^*z - 1 - |d_1|^2|z|^2 + d_1z^* + d_1^*z}{(z-d_1)(z^*-d_1^*)} = \frac{(|z|^2 - 1)(1 - |d_1|^2)}{|z-d_1|^2}. \text{ Hence,}$$

$1 - |A_1(z)|^2 \begin{cases} > 0, & \text{if } |z| > 1, \\ = 0, & \text{if } |z| = 1, \\ < 0, & \text{if } |z| < 1. \end{cases}$ Thus, $|A_1(z)|^2 \begin{cases} < 1, & \text{if } |z| > 1, \\ = 1, & \text{if } |z| = 1, \\ > 1, & \text{if } |z| < 1. \end{cases}$ Thus, Eq. (7.20) holds

for any first-order allpass function. A higher-order allpass function can be factored into a product of first-order allpass functions. Since, Eq. (7.20) holds true each of these factors individually, hence, it also holds true for the product.

7.3 An m -th order stable, real allpass transfer function can be expressed as a product

of first-order allpass transfer functions of the form $A_i(z) = \frac{1-d_i^*z}{z-d_i}$. If d_i is

complex, then $A(z)$ has another factor of the form $A_i'(z) = \frac{1-d_i z}{z-d_i^*}$. Now,

$$A_i(e^{j\omega}) = \frac{1-d_i^*e^{j\omega}}{e^{j\omega}-d_i} = e^{-j\omega} \frac{(1-d_i^*e^{j\omega})(1-d_i^*e^{j\omega})}{(1-d_i e^{-j\omega})(1-d_i^*e^{j\omega})}. \text{ Let } d_i = |d_i|e^{j\theta} = \alpha e^{j\theta}.$$

Then, $A_i(e^{j\omega}) = e^{-j\omega} \frac{(1-\alpha e^{-j\theta} e^{j\omega})^2}{[1+\alpha^2-2\alpha \cos(\theta-\omega)]}$. Therefore,

$$\arg\{A_i(e^{j\omega})\} = -\omega + \arg\{(1-\alpha e^{-j\theta} e^{j\omega})^2\} = -\omega + 2 \arg\{1-\alpha e^{-j\theta} e^{j\omega}\}$$

$$= -\omega + 2 \tan^{-1} \left[\frac{\alpha \sin(\theta-\omega)}{1-\alpha \cos(\theta-\omega)} \right]. \text{ We can show similarly,}$$

$$\arg\{A_i'(e^{j\omega})\} = -\omega + 2 \tan^{-1} \left[\frac{\alpha \sin(\theta+\omega)}{1-\alpha \cos(\theta+\omega)} \right].$$

If $d_i = \alpha$ is real, then $\arg\{A_i(e^{j\omega})\} = -\omega + 2 \tan^{-1} \left[\frac{\alpha \sin \omega}{1-\alpha \cos \omega} \right]$. Now for real d_i ,

$$\arg\{A_i(e^{j0})\} - \arg\{A_i(e^{j\pi})\} = \{-0 + 2 \tan^{-1}(-0)\} - \{-\pi + 2 \tan^{-1}(-0)\} = \pi. \text{ For}$$

complex d_i , $\arg\{A_i(e^{j0})\} + \arg\{A_i'(e^{j0})\} - \arg\{A_i(e^{j\pi})\} - \arg\{A_i'(e^{j\pi})\} =$

$$= -0 + 2 \tan^{-1} \left[\frac{\alpha \sin \theta}{1-\alpha \cos \theta} \right] - 0 + 2 \tan^{-1} \left[\frac{-\alpha \sin \theta}{1-\alpha \cos \theta} \right]$$

$$+ \pi - 2 \tan^{-1} \left[\frac{\alpha \sin \theta}{1+\alpha \cos \theta} \right] + \pi - 2 \tan^{-1} \left[\frac{\alpha \sin \theta}{1+\alpha \cos \theta} \right] = 2\pi. \text{ Now,}$$

$\tau(\omega) = -\frac{d}{d\omega} (\arg\{A(e^{j\omega})\})$. Therefore,

$$\int_0^\pi \tau(\omega) d\omega = -\int_0^\pi d[\arg\{A(e^{j\omega})\}] = \arg\{A(e^{j0})\} - \arg\{A(e^{j\pi})\}. \text{ Since}$$

$$\arg\{A(e^{j\omega})\} = \sum_{i=1}^m \arg\{A_i(e^{j\omega})\}, \text{ it follows then } \int_0^\pi \tau(\omega) d\omega = m\pi.$$

7.4 $h[n] = a_1\delta[n-2] - a_2\delta[n-1] - a_3\delta[n] + a_4\delta[n+1] - a_5\delta[n+2]$. Therefore,

$$H(e^{j\omega}) = a_1e^{j2\omega} - a_2e^{j\omega} - a_3 + a_4e^{-j\omega} - a_5e^{-2\omega}$$

$$= a_1(\cos 2\omega + j \sin 2\omega) - a_1(\cos \omega + j \sin \omega) - a_3 + a_4(\cos \omega - j \sin \omega)$$

$$- a_3 + a_4(\cos \omega - j \sin \omega) - a_5(\cos 2\omega - j \sin 2\omega)$$

$$= (a_1 - a_5) \cos 2\omega - (a_2 - a_4) \cos \omega - a_3 + j((a_1 + a_5) \sin 2\omega - (a_2 + a_4) \sin \omega). \text{ To}$$

have a zero-phase frequency response, the imaginary part of must be equal to zero for all values of ω . Hence, for a zero-phase response, $a_1 = -a_5$ and $a_2 = -a_4$.

7.5 $v[n] = x[-n] \otimes h[n]$, and $u[n] = v[-n] = x[n] \otimes h[-n]$. Hence,

$y[n] = (h[n] + h[-n]) \otimes x[n]$. Therefore, $G(e^{j\omega}) = H(e^{j\omega}) + H^*(e^{j\omega})$. Thus, the equivalent frequency response is real and has zero phase.

7.6 Now, $A_M(z) = \pm \frac{z^{-M} D_M^*(1/z^*)}{D_M(z)}$, where

$$D_M(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \dots + d_M z^{-M}.$$

$$|A_M(z)|^2 = A_M(z) A_M^*(1/z^*) = \frac{D_M^*(1/z^*)}{D_M(z)} \cdot \frac{D_M(z)}{D_M^*(1/z^*)} = 1.$$

7.7 Consider the first-order factor $1 + az^{-1}$. Its square-magnitude function is given by $(1 + az^{-1})(1 + az) \Big|_{z=e^{j\omega}} = (1 + a^2) + 2a \cos \omega$. We thus rewrite the given square-magnitude function as

$$\left| H(e^{j\omega}) \right|^2 = H(z) H(z^{-1}) \Big|_{z=e^{j\omega}} = \frac{9[1.0625 + 0.25(z + z^{-1})][1.49 - 0.7(z + z^{-1})]}{[1.36 + 0.6(z + z^{-1})][1.64 + 0.8(z + z^{-1})]} \Big|_{z=e^{j\omega}}.$$

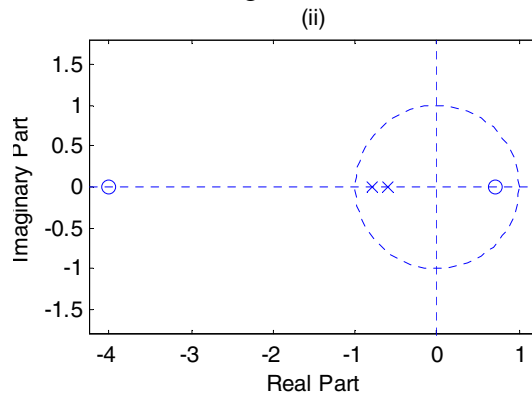
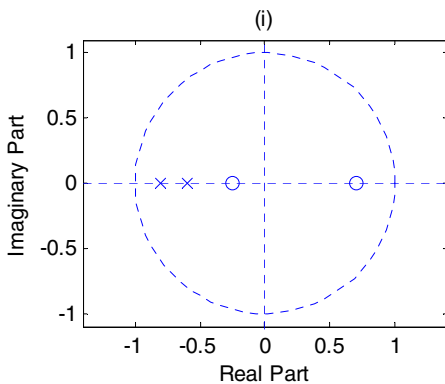
$$\text{Therefore, } H(z) H(z^{-1}) = \frac{9(1 + 0.25z)(1 + 0.25z^{-1})(1 - 0.7z)(1 - 0.7z^{-1})}{(1 + 0.6z)(1 + 0.6z^{-1})(1 + 0.8z)(1 + 0.8z^{-1})}.$$

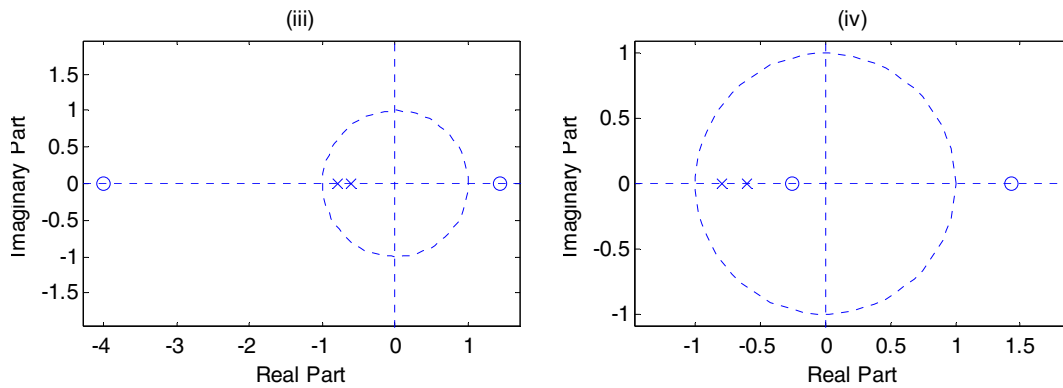
As can be seen, there are 4 possible causal, stable transfer functions:

$$\text{(i) } H(z) = \frac{3(1 + 0.25z^{-1})(1 - 0.7z^{-1})}{(1 + 0.6z^{-1})(1 + 0.8z^{-1})}, \quad \text{(ii) } H(z) = \frac{3(1 + 0.25z)(1 - 0.7z)}{(1 + 0.6z)(1 + 0.8z)},$$

$$\text{(iii) } H(z) = \frac{3(1 + 0.25z)(1 - 0.7z)}{(1 + 0.6z^{-1})(1 + 0.8z^{-1})}, \quad \text{(iv) } H(z) = \frac{3(1 + 0.25z^{-1})(1 - 0.7z^{-1})}{(1 + 0.6z)(1 + 0.8z)}.$$

The zero locations of the four FIR transfer functions are given below:





The zeros of the transfer function of a linear-phase FIR filter exhibits mirror-image symmetry with respect to the unit circle. The zeros of the transfer function of a minimum-phase FIR filter are all inside the unit circle and that of a maximum-phase FIR filter are outside the unit circle.

(a) The transfer functions of (iii) and (iv) have linear-phase as their zeros exhibit mirror-image symmetry.

(b) The transfer function of (i) is minimum-phase as its zeros are inside the unit circle.

(c) The transfer function of (ii) is maximum-phase as its zeros are outside the unit circle.

7.9 $G(z) = 6(1 + 1.7z^{-1} - 2z^{-2})(1 - 0.5z^{-1}) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1})$
 $= 6 + 7.2z^{-1} - 17.1z^{-2} + 6z^{-3}$.

(a) Other transfer functions having the same magnitude responses are:

(i) $G_2(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \left(\frac{2.5 + z^{-1}}{1 + 2.5z^{-1}} \right)$
 $= 6(2.5 + z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) = 15 - 13.5z^{-1} - 1.8z^{-2} + 2.4z^{-3}$.

(ii) $G_3(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \left(\frac{-0.8 + z^{-1}}{1 - 0.8z^{-1}} \right)$
 $= 6(1 + 2.5z^{-1})(-0.8 + z^{-1})(1 - 0.5z^{-1}) = -4.8 - 3.6z^{-1} + 18z^{-2} - 7.56z^{-3}$.

(iii) $G_4(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \left(\frac{-0.5 + z^{-1}}{1 - 0.5z^{-1}} \right)$
 $= 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(-0.5 + z^{-1}) = -3 + 0.9z^{-1} + 16.28z^{-2} - 124z^{-3}$.

(iv) $G_5(z) = 6(1 + 2.5z^{-1})(1 - 0.8z^{-1})(1 - 0.5z^{-1}) \left(\frac{2.5 + z^{-1}}{1 + 2.5z^{-1}} \right) \left(\frac{-0.8 + z^{-1}}{1 - 0.8z^{-1}} \right)$
 $= 6(2.5 + z^{-1})(-0.8 + z^{-1})(1 - 0.5z^{-1}) = -12 + 16.2z^{-1} + 0.9z^{-2} - 3z^{-3}$.

$$\begin{aligned} \text{(v)} \quad G_6(z) &= 6(1+2.5z^{-1})(1-0.8z^{-1})(1-0.5z^{-1}) \left(\frac{-0.8+z^{-1}}{1-0.8z^{-1}} \right) \left(\frac{-0.5+z^{-1}}{1-0.5z^{-1}} \right) \\ &= 6(1+2.5z^{-1})(-0.8+z^{-1})(-0.5+z^{-1}) = 2.4 - 1.8z^{-1} - 13.5z^{-2} + 15z^{-3}. \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad G_7(z) &= 6(1+2.5z^{-1})(1-0.8z^{-1})(1-0.5z^{-1}) \left(\frac{2.5+z^{-1}}{1+2.5z^{-1}} \right) \left(\frac{-0.5+z^{-1}}{1-0.5z^{-1}} \right) \\ &= 6(2.5+z^{-1})(1-0.8z^{-1})(-0.5+z^{-1}) = -7.5 + 18z^{-1} - 3.6z^{-2} - 4.8z^{-3}. \end{aligned}$$

$$\begin{aligned} \text{(vii)} \quad G_8(z) &= 6(1+2.5z^{-1})(1-0.8z^{-1})(1-0.5z^{-1}) \left(\frac{2.5+z^{-1}}{1+2.5z^{-1}} \right) \left(\frac{-0.8+z^{-1}}{1-0.8z^{-1}} \right) \left(\frac{-0.5+z^{-1}}{1-0.5z^{-1}} \right) \\ &= 6(1+2.5z^{-1})(-0.8+z^{-1})(-0.5+z^{-1}) = -6 - 17.1z^{-1} + 7.2z^{-2} + 6z^{-3}. \end{aligned}$$

(b) The minimum phase filter is $G_2(z)$, as all its zeros inside the unit circle. Likewise, the maximum phase filter is $G_6(z)$, as all its zeros outside the unit circle.

(c) The partial energy of impulse responses of each of the above transfer functions for different values of k are given by:

| | $k=0$ | $k=1$ | $k=2$ | $k=3$ |
|----------|-------|--------|--------|--------|
| $G_1(z)$ | 36 | 87.84 | 380.25 | 416.25 |
| $G_2(z)$ | 225 | 407.25 | 410.49 | 416.25 |
| $G_3(z)$ | 23.04 | 36 | 360 | 416.25 |
| $G_4(z)$ | 9 | 9.81 | 272.25 | 416.25 |
| $G_5(z)$ | 144 | 406.44 | 407.25 | 416.25 |
| $G_6(z)$ | 5.76 | 9 | 191.25 | 416.25 |
| $G_7(z)$ | 56.25 | 380.25 | 393.25 | 416.25 |
| $G_8(z)$ | 36.00 | 328.41 | 380.25 | 416.25 |

The partial energy remains the same for values of $k \geq 3$. From the above table it

can be seen that $\sum_{m=0}^n |g_k[m]|^2 \leq \sum_{m=0}^n |g_2[m]|^2$, and

$\sum_{m=0}^{\infty} |g_k[m]|^2 = \sum_{m=0}^{\infty} |g_2[m]|^2 = 416.25$, $1 \leq k \leq 8$, where $G_2(z)$ is the minimum

phase transfer function. Likewise, $\sum_{m=0}^n |g_k[m]|^2 \geq \sum_{m=0}^n |g_6[m]|^2$, $1 \leq k \leq 8$, where

$G_6(z)$ is the maximum phase transfer function.

$$\mathbf{7.9} \quad H_1(z) = 1 - 0.5z^{-1} + 0.8z^{-2} - 0.4z^{-3} + 0.25z^{-4} - 0.125z^{-5} + 0.2z^{-6} - 0.1z^{-7}$$

$$\begin{aligned}
&= (1 - 0.8z^{-2})(1 + z^{-1} + 0.5z^{-2})(1 - z^{-1} + 0.5z^{-2})(1 - 0.5z^{-1}). \\
H_2(z) &= 0.5 + 0.25z^{-1} + 0.4z^{-2} - 0.425z^{-3} + 0.75z^{-4} - 0.75z^{-5} + 0.6z^{-6} - 0.2z^{-7} \\
&= (1 - 0.8z^{-2})(0.5 + z^{-1} + z^{-2})(1 - z^{-1} + 0.5z^{-2})(1 - 0.5z^{-1}). \\
H_3(z) &= -0.25 + 0.25z^{-1} + 0.175z^{-2} + 0.7z^{-3} - 0.45z^{-4} + 0.9z^{-5} - 0.6z^{-6} + 0.4z^{-7} \\
&= (1 - 0.8z^{-2})(0.5 + z^{-1} + z^{-2})(1 - z^{-1} + 0.5z^{-2})(-0.5 + z^{-1}). \\
H_4(z) &= -0.5 + z^{-1} - 0.4z^{-2} + 0.8z^{-3} - 0.125z^{-4} + 0.25z^{-5} - 0.1z^{-6} + 0.2z^{-7} \\
&= (1 - 0.8z^{-2})(1 + z^{-1} + 0.5z^{-2})(1 - z^{-1} + 0.5z^{-2})(-0.5 + z^{-1}). \\
H_5(z) &= -0.1 + 0.2z^{-1} - 0.125z^{-2} + 0.25z^{-3} - 0.4z^{-4} + 0.8z^{-5} - 0.5z^{-6} + z^{-7} \\
&= (-0.8 + z^{-2})(0.5 + z^{-1} + z^{-2})(0.5 - z^{-1} + z^{-2})(-0.5 + z^{-1}) = z^{-7}H_1(z^{-1}).
\end{aligned}$$

Each factor of $H_1(z)$ has roots inside the unit circle. Hence, $H_1(z)$ is a minimum-phase transfer function. Since $H_5(z)$ is a mirror-image of $H_1(z)$, it has all zeros outside the unit circle and is thus a maximum-phase transfer function.

There are 11 other length-7 FIR filters having the same magnitude responses as that of the above filters.

$$\begin{aligned}
\mathbf{7.11} \quad H(e^{j\omega}) &= a_1 + a_2e^{-j\omega} + a_3e^{-j2\omega} + a_4e^{-j3\omega} + a_5e^{-j4\omega} + a_6e^{-j5\omega} \\
&= e^{-j5\omega/2} \left[(a_1e^{j5\omega/2} + a_6e^{-j5\omega/2}) + (a_2e^{j3\omega/2} + a_5e^{-j3\omega/2}) + (a_3e^{j\omega/2} + a_4e^{-j\omega/2}) \right] \\
&= e^{-j\frac{5\omega}{2}} \left[2(a_1 + a_6)\cos\left(\frac{5\omega}{2}\right) + j(a_1 - a_6)\sin\left(\frac{5\omega}{2}\right) + \right. \\
&\quad \left. + (a_2 + a_5)\cos\left(\frac{3\omega}{2}\right) + j(a_2 - a_5)\sin\left(\frac{3\omega}{2}\right) + (a_3 + a_4)\cos\left(\frac{\omega}{2}\right) + j(a_3 - a_4)\sin\left(\frac{\omega}{2}\right) \right]
\end{aligned}$$

It follows from the above that $H(e^{j\omega})$ will have linear-phase, i.e., constant group delay, if the imaginary parts inside the square brackets are zero. Hence, for a constant group delay we must have $a_1 = a_6$, $a_2 = a_5$, and $a_3 = a_4$.

7.12 The frequency response of the LTI discrete-time system is given by

$$\begin{aligned}
H(e^{j\omega}) &= a_1e^{jk\omega} - a_2e^{j(k-1)\omega} + a_2e^{j(k-3)\omega} - a_1e^{j(k-4)\omega} \\
&= e^{-j(2-k)\omega} [a_1(e^{j2\omega} - e^{-j2\omega}) + a_2(e^{j\omega} - e^{-j\omega})] \\
&= -je^{-j(2-k)\omega} 2[a_1 \sin(2\omega) + a_2 \sin \omega] = -e^{-j(2-k-\frac{\pi}{2})\omega} 2[a_1 \sin(2\omega) + a_2 \sin \omega].
\end{aligned}$$

So, for $k = 2 - \frac{\pi}{2}$, the system will have a frequency response that is a real function of ω .

7.13 $H_1(e^{j\omega}) = \alpha + \beta e^{-j\omega}$ and $H_2(e^{j\omega}) = \frac{1}{1 + \gamma e^{-j\omega}}$. Thus,

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \frac{\alpha + \beta e^{-j\omega}}{1 - \gamma e^{-j\omega}}.$$

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H(e^{-j\omega}) = \frac{\alpha + \beta e^{-j\omega}}{1 - \gamma e^{-j\omega}} \cdot \frac{\alpha + \beta e^{j\omega}}{1 - \gamma e^{j\omega}} = \frac{\alpha^2 + \beta^2 + 2\alpha\beta \cos \omega}{1 + \gamma^2 - 2\gamma \cos \omega} = K^2$$

if $\alpha^2 + \beta^2 = K^2(1 + \gamma^2)$ and $\alpha\beta = -K^2\gamma$, i.e., $(\alpha + \beta)^2 = K^2(1 - \gamma)^2$.

7.14 $Y(e^{j\omega}) = \left|X(e^{j\omega})\right|^\alpha e^{j \arg X(e^{j\omega})}$. Hence, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left|X(e^{j\omega})\right|^{(\alpha-1)}$. Since

$H(e^{j\omega})$ is a real function of ω , it has zero phase.

7.15 $H(e^{j\omega}) = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega} = e^{-j\omega}(h[0](e^{j\omega} + e^{-j\omega}) + h[1])$
 $= e^{-j\omega}(2h[0]\cos \omega + h[1])$. Thus, we require $\left|H(e^{j0.3})\right| = 2h[0]\cos(0.3) + h[1] = 1$

and $\left|H(e^{j0.6})\right| = 2h[0]\cos(0.6) + h[1] = 0$. Solving these two equations we get
 $h[0] = 3.8461$ and $h[1] = -6.3487$.

7.16 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} - h[1]e^{-j3\omega} - h[0]e^{-j4\omega}$
 $= e^{-j2\omega}(h[0](e^{j2\omega} - e^{-j2\omega}) + h[1](e^{j\omega} - e^{-j\omega}))$
 $= j2e^{-j2\omega}(h[0]\sin(2\omega) + h[1]\sin(\omega))$. Therefore,

$$\left|H(e^{j\omega})\right| = 2(h[0]\sin(2\omega) + h[1]\sin(\omega)). \text{ Thus,}$$

$$\left|H(e^{j0.3})\right| = 2(h[0]\sin(0.6) + h[1]\sin(0.3)) = 0.3 \text{ and}$$

$$\left|H(e^{j0.6})\right| = 2(h[0]\sin(1.2) + h[1]\sin(0.6)) = 0.8. \text{ Solving these two equations we get}$$

$$h[1] = -0.9873 \text{ and } h[0] = 2.5686.$$

(b) $H(e^{j\omega}) = j2e^{-j2\omega}(2.5686\sin(2\omega) - 0.9873\sin(\omega))$. The plot of the magnitude and phase responses are shown below.

7.17 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} - h[1]e^{-j2\omega} - h[0]e^{-j3\omega}$
 $= e^{-j3\omega/2}(h[0](e^{j3\omega/2} - e^{-j3\omega/2}) + h[1](e^{j\omega/2} - e^{-j\omega/2}))$
 $= j2e^{-j3\omega/2}(h[0]\sin(3\omega/2) + h[1]\sin(\omega/2))$. Therefore,

$$\left|H(e^{j\omega})\right| = 2(h[0]\sin(3\omega/2) + h[1]\sin(\omega/2)). \text{ Thus,}$$

$$\left|H(e^{j0.25})\right| = 2(h[0]\sin(0.375) + h[1]\sin(0.125)) = 0.2 \quad \text{and}$$

$$\left|H(e^{j0.8})\right| = 2(h[0]\sin(1.2) + h[1]\sin(0.4)) = 0.8. \quad \text{Solving these two equations we get}$$

$$h[1] = 6.2573 \quad \text{and} \quad h[0] = -1.8569.$$

$$(b) \quad H(e^{j\omega}) = j2e^{-j3\omega/2}(-1.8569\sin(3\omega/2) + 6.2573\sin(\omega/2)).$$

7.18 (a) $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[1]e^{-j3\omega} + h[0]e^{-j4\omega}$
 $= e^{-j2\omega}(h[0](e^{j2\omega} + e^{-j2\omega}) + h[1](e^{j\omega} + e^{-j\omega}) + h[2])$
 $= e^{-j2\omega}(2h[0]\cos(2\omega) + 2h[1]\cos\omega + h[2]).$ Therefore,

$$\left|H(e^{j\omega})\right| = 2h[0]\cos(2\omega) + 2h[1]\cos\omega + h[2]. \quad \text{Thus,}$$

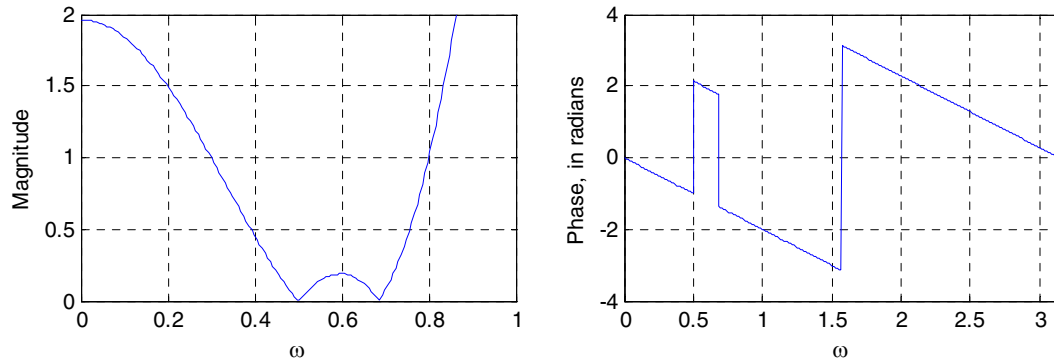
$$\left|H(e^{j0.3})\right| = 2h[0]\cos(0.6) + 2h[1]\cos(0.3) + h[2] = 1,$$

$$\left|H(e^{j0.5})\right| = 2h[0]\cos(1.0) + 2h[1]\cos(0.5) + h[2] = 0, \quad \text{and}$$

$$\left|H(e^{j0.8})\right| = 2h[0]\cos(1.6) + 2h[1]\cos(0.8) + h[2] = 1. \quad \text{Solving these three equations}$$

we get $h[0] = 17.7761$, $h[1] = -58.7339$, and $h[2] = 83.8786$.

(b) $H(e^{j\omega}) = e^{-j2\omega}(35.5522\cos(2\omega) - 117.4677\cos\omega + 83.8786)$. The plot of the magnitude and phase responses are shown below.



7.19 $H(e^{j0}) = 13$, $H(e^{j3\pi/4}) = -3 - j4$, and $H(e^{j\pi}) = -3$. Using the symmetry property of the DTFT of a real sequence, we observe

$$H(e^{j\pi/4}) = H^*(e^{j3\pi/4}) = -3 + j4. \quad \text{Thus, the 4-point DFT } H[k] \text{ of the length-4}$$

sequence $\{h[n]\}$ is given by $\{H[k]\} = \{13, -3 + j4, -3, -3 - j4\}$. Its 4-point

$$\text{inverse DFT is thus given by } \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 13 \\ -3 + j4 \\ -3 \\ -3 - j4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}.$$

Hence, $H(z) = 1 + 2z^{-1} + 4z^{-2} + 6z^{-3}$.

7.20 Now, for a real, anti-symmetric sequence $\{h[n]\}$ of even length, $H(e^{j0}) = 0$.

Using the symmetry property of the DTFT of a real sequence, we observe also

$H(e^{j\pi/4}) = H^*(e^{j3\pi/4}) = -5 + j5$. Hence, the 4-point DFT of the length-4

sequence $\{h[n]\}$ is given by $\{H[k]\} = \{0, -5 + j5, 20, -5 - j5\}$. Its 4-point

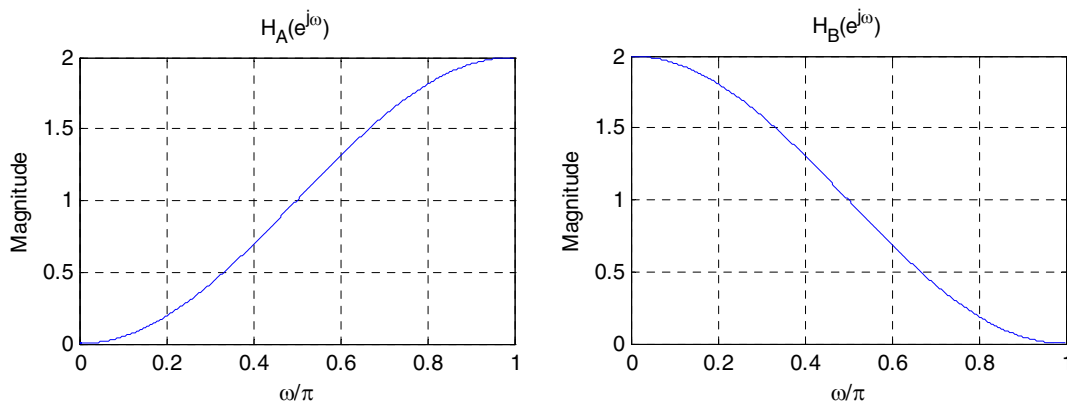
$$\text{inverse DFT is thus given by } \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 0 \\ -5 + j5 \\ 20 \\ -5 - j5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -7.5 \\ -2.5 \\ 7.5 \end{bmatrix}.$$

Hence, $H(z) = 2.5 - 7.5z^{-1} - 2.5z^{-2} + 7.5z^{-3}$.

7.21 (a) $H_A(e^{j\omega}) = 0.5 - e^{-j\omega} + 0.5e^{-j2\omega} = e^{-j\omega}(\cos\omega - 1)$ and

$H_B(e^{j\omega}) = 0.5 + e^{-j\omega} + 0.5e^{-j2\omega} = e^{-j\omega}(\cos\omega + 1)$. Hence,

$|H_A(e^{j\omega})| = |\cos\omega - 1|$ and $|H_B(e^{j\omega})| = |\cos\omega + 1|$. Plots of these two magnitude functions is given below:



As can be seen from the above plots, $h_A[n]$ is a highpass filter and $h_B[n]$ is a lowpass filter.

(b) $h_C[n] = (-1)^n h_A[n] = h_B[n]$. Hence, $H_C(e^{j\omega}) = H_A(e^{j(\omega+\pi)}) = H_B(e^{j\omega})$.

$H_C(e^{j\omega})$ is a shifted version of $H_A(e^{j\omega})$ shifted by π radians.

7.22 $h[n] = h[N - n]$. Thus,

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} h[N-n]z^{-n} = \sum_{k=0}^{N-1} h[k]z^{-(N-k)} = z^{-N} H(z^{-1}).$$

As $H(z)$ has zeros in a mirror-image symmetry in the z -plane, $G(z) = 1/H(z)$ will have poles outside the unit circle making it unstable.

7.23 (a) Since $H(z)$ is a minimum-phase FIR transfer function, any other FIR transfer function $G(z)$ with the same magnitude response as that of $H(z)$ can be expressed as $G(z) = H(z)A(z)$ where $A(z)$ is an allpass function. Now, $g[0] = \lim_{z \rightarrow \infty} G(z)$.

$$\begin{aligned} \text{Thus, } |g[0]| &= \left| \lim_{z \rightarrow \infty} G(z) \right| = \left| \lim_{z \rightarrow \infty} H(z)A(z) \right| = \left| \lim_{z \rightarrow \infty} H(z) \right| \cdot \left| \lim_{z \rightarrow \infty} A(z) \right| \\ &\leq \left| \lim_{z \rightarrow \infty} H(z) \right|, \text{ as } \left| \lim_{z \rightarrow \infty} A(z) \right| < 1 \text{ (see Eq. (7.20)). Hence, } |g[0]| \leq |h[0]|. \end{aligned}$$

(b) If λ_1 is a zero of $H(z)$, then $|\lambda_1| < 1$, since $H(z)$ is a minimum-phase causal stable transfer function with all zeros inside the unit circle. Let

$H(z) = B(z)(1 - \lambda_1 z^{-1})$. It follows that is also a minimum-phase causal transfer function.

Now, consider the transfer function $F(z) = B(z)(\lambda_1^* - z^{-1}) = H(z) \left(\frac{\lambda_1^* - z^{-1}}{1 - \lambda_1 z^{-1}} \right)$. If

$h[n], b[n]$, and $f[n]$ denote, respectively, the inverse z -transform of $H(z), B(z)$, and $F(z)$, then we get

$$h[n] = \begin{cases} b[0], & n = 0, \\ b[n] - \lambda_1 b[n-1], & n \geq 1, \end{cases} \text{ and } f[n] = \begin{cases} \lambda_1^* b[0], & n = 0, \\ \lambda_1^* b[n] - b[n-1], & n \geq 1. \end{cases}$$

Consider

$$\varepsilon = \sum_{n=0}^m |h[n]|^2 - \sum_{n=0}^m |f[n]|^2 = |b[0]|^2 - |\lambda_1^*|^2 |b[0]|^2 + \sum_{n=1}^m |h[n]|^2 - \sum_{n=1}^m |f[n]|^2.$$

$$|h[n]|^2 = |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n], \text{ and}$$

$$|f[n]|^2 = |\lambda_1|^2 |b[n]|^2 + |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n].$$

$$\begin{aligned} \text{Hence, } \varepsilon &= |b[0]|^2 - |\lambda_1|^2 |b[0]|^2 + \sum_{n=1}^m \left(|b[n]|^2 - |\lambda_1|^2 |b[n-1]|^2 \right) \\ &\quad - \sum_{n=1}^m \left(|\lambda_1|^2 |b[n]|^2 - |b[n-1]|^2 \right) = (1 - |\lambda_1|^2) |b[m]|^2. \end{aligned}$$

Since $|\lambda_1| < 1$, $\varepsilon > 0$, i.e., $\sum_{n=0}^m |h[n]|^2 > \sum_{n=0}^m |f[n]|^2$. Hence, $\sum_{n=0}^m |h[n]|^2 \geq \sum_{n=0}^m |g[n]|^2$.

7.24 (a) $H_1(z) = \frac{z^3 + 3z^2 + 2z + 7}{(2z+3)(z^2 + 0.5z + 0.8)} = \frac{(z+3.0867)(z^2 - 0.0867z + 2.2678)}{(2z+3)(z^2 + 0.5z + 0.8)}$ has a pole at

$z = -\frac{3}{2}$, which is outside the unit circle. Hence, $H_1(z)$ is not a stable transfer function. To arrive at a stable transfer function with an identical magnitude response, we multiply $H_1(z)$ with the allpass function $\frac{2z+3}{2+3z}$ resulting in

$$\frac{z^3 + 3z^2 + 2z + 7}{(2z+3)(z^2 + 0.5z + 0.8)} \cdot \frac{2z+3}{2+3z} = \frac{z^3 + 3z^2 + 2z + 7}{(2+3z)(z^2 + 0.5z + 0.8)}.$$

(b) $H_2(z) = \frac{4z^3 - 2z^2 + 5z - 6}{(1.5z^2 + 3z - 5)(z^2 - 0.3z + 0.7)} = \frac{4(z^2 + 0.4181 + 1.6338)(z - 0.9181)}{1.5(z + 3.0817)(z - 1.0817)(z^2 - 0.3z + 0.7)}$ has

two poles outside the unit circle at $z = -3.0817$ and $z = 1.0817$. Hence, $H_2(z)$ is not a stable transfer function. To arrive at a stable transfer function with an identical magnitude response, we multiply $H_2(z)$ with the allpass function

$$\frac{z^2 - 0.3z + 0.7}{0.7z^2 - 0.3z + 1} \text{ resulting in } \frac{4z^3 - 2z^2 + 5z - 6}{(1.5z^2 + 3z - 5)(z^2 - 0.3z + 0.7)} \cdot \frac{z^2 - 0.3z + 0.7}{0.7z^2 - 0.3z + 1}$$

$$= \frac{4z^3 - 2z^2 + 5z - 6}{(5z^2 + 3z - 1.5)(z^2 - 0.3z + 0.7)}.$$

7.25 The transfer function of the simplest notch filter is given by

$H(z) = (1 - e^{j\omega_o} z^{-1})(1 - e^{-j\omega_o} z^{-1}) = 1 - 2 \cos(\omega_o)z^{-1} + z^{-2}$. In the steady-state, the output for an input $x[n] = \cos(\omega_o n)$ is given by $y[n] = H(e^{j\omega_o}) \cos(\omega_o n + \theta(\omega_o))$ where $\theta(\omega_o) = \arg[H(e^{j\omega_o})]$.

(a) Comparing $H(z) = 1 - z^{-1} + z^{-2}$ with $H(z)$ as given above we conclude

$$\cos(\omega_o) = \frac{1}{2}, \text{ or } \omega_o = \frac{\pi}{3}. \text{ Here, } H_1(e^{j\omega_o}) = 1 - e^{-j\omega_o} + e^{-j2\omega_o}$$

$$= 1 - e^{-j\pi/3} + e^{-j2\pi/3} = 1 - (0.5 + j0.866) + (-0.5 + j0.866) = 0.$$

Hence, $y[n] = H_1(e^{j\omega_o}) \cos(\omega_o n + \theta(\omega_o)) = 0$.

(b) Comparing $H_2(z) = 1 - 0.8z^{-1} + z^{-2}$ with $H(z)$ as given above we conclude

$$\cos(\omega_o) = 0.4 \text{ or } \omega_o = 0.369\pi. \text{ Here,}$$

$$H_2(e^{j\omega_o}) = 1 - e^{-j\omega_o} + e^{-j2\omega_o} = 1 - (0.32 + j0.7332) + (-0.68 + j0.7332) = 0.$$

Hence, $y[n] = H_2(e^{j\omega_o}) \cos(\omega_o n + \theta(\omega_o)) = 0$.

(c) Comparing $H_3(z) = 1 - 1.6z^{-1} + z^{-2}$ with $H(z)$ as given above we conclude $\cos(\omega_o) = 0.8$ or $\omega_o = 0.2048\pi$. Here,

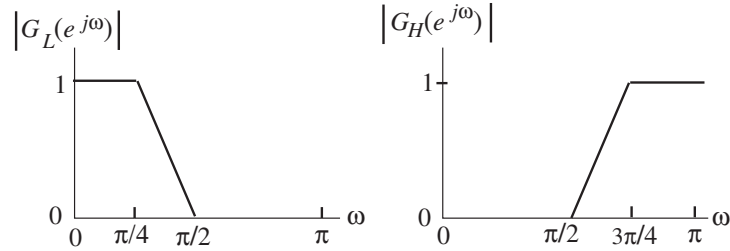
$$H_2(e^{j\omega_o}) = 1 - e^{-j\omega_o} + e^{-j2\omega_o} = 1 - (1.2891 + j0.9599) + (0.2801 + j0.9599) = 0.$$

Hence, $y[n] = H_3(e^{j\omega_o}) \cos(\omega_o n + \theta(\omega_o)) = 0$.

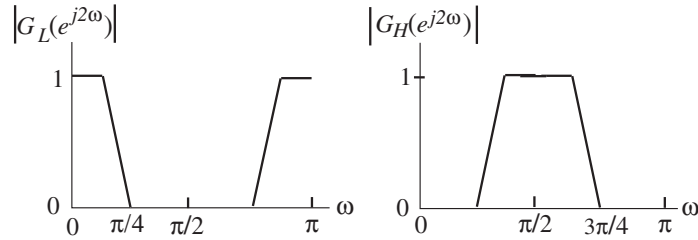
7.26 From the figure, $H_0(z) = \frac{Y_0(z)}{X(z)} = G_L(z)G_H(z^2)$, $H_1(z) = \frac{Y_1(z)}{X(z)} = G_H(z)G_H(z^2)$,

$$H_2(z) = \frac{Y_2(z)}{X(z)} = G_H(z)G_L(z^2), \quad H_3(z) = \frac{Y_3(z)}{X(z)} = G_L(z)G_L(z^2).$$

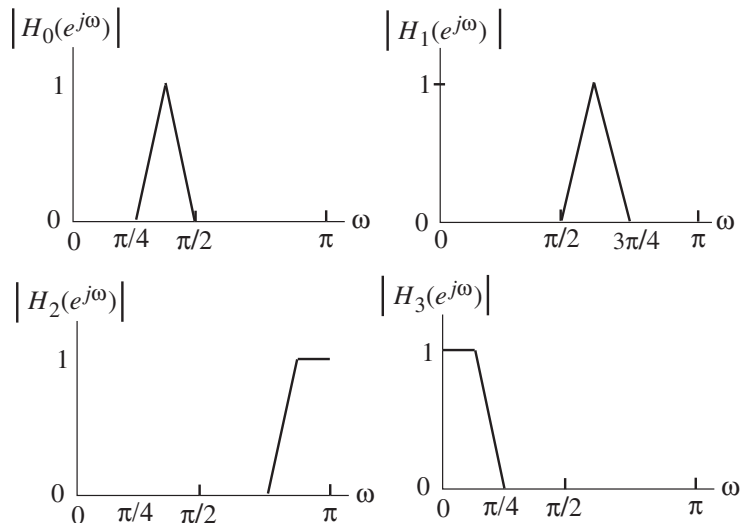
For the magnitude responses of $G_L(z)$ and $G_H(z)$ shown below



the magnitude responses of $G_L(z^2)$ and $G_H(z^2)$ are as shown below



Hence, the magnitude responses of the cascaded filters $H_i(z)$, $0 \leq i \leq 3$, are as indicated below:



7.27 $G(e^{j\omega}) = H_{LP}(e^{j(\pi-\omega)})$. Since, $H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega < \omega_c, \\ 0, & \omega_c \leq \omega < \pi, \end{cases}$

$G(e^{j\omega}) = \begin{cases} 0, & 0 \leq \omega < \pi - \omega_c, \\ 1, & \pi - \omega_c \leq \omega < \pi. \end{cases}$ Hence, $G(e^{j\omega})$ is a highpass filter with a cutoff

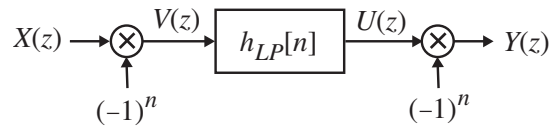
frequency given by $\omega_o = \pi - \omega_c$. Also, since $G(z) = H_{LP}(-z)$, we have

$$g[n] = (-1)^n h_{LP}[n].$$

7.28 $G(z) = H_{LP}(e^{j\omega_o} z) + H_{LP}(e^{-j\omega_o} z)$. Hence, $g[n] = h_{LP}[n]e^{-j\omega_o n} + h_{LP}[n]e^{j\omega_o n} = 2 h_{LP}[n] \cos(\omega_o n)$. Thus, $G(z)$ is a real coefficient bandpass filter with a center frequency at ω_o and a passband width of $2\omega_p$.

7.29 $F(z) = H_{LP}(e^{j\omega_o} z) + H_{LP}(e^{-j\omega_o} z) + H_{LP}(z)$. Hence, $f[n] = (1 + 2 \cos(\omega_o n))h_{LP}[n]$. Thus, $F(z)$ is a real coefficient bandstop filter with a center frequency at ω_o and a stopband width of $(\pi - 3\omega_p)/2$.

7.30

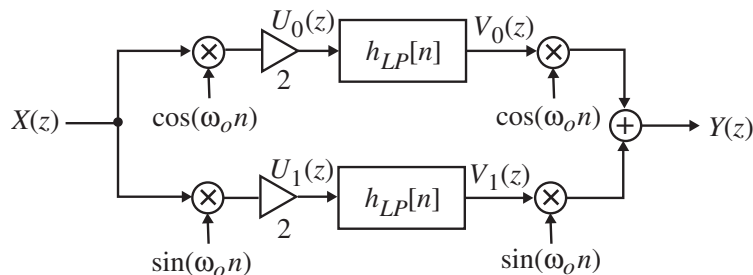


From the above figure, we get $V(z) = X(-z)$, $U(z) = H_{LP}(z)X(-z)$, and

$Y(z) = U(-z) = H_{LP}(-z)X(z)$. Hence, $H_{eq}(z) = \frac{Y(z)}{X(z)} = H_{LP}(-z)$, which is the

highpass filter of Problem 7.27.

7.31



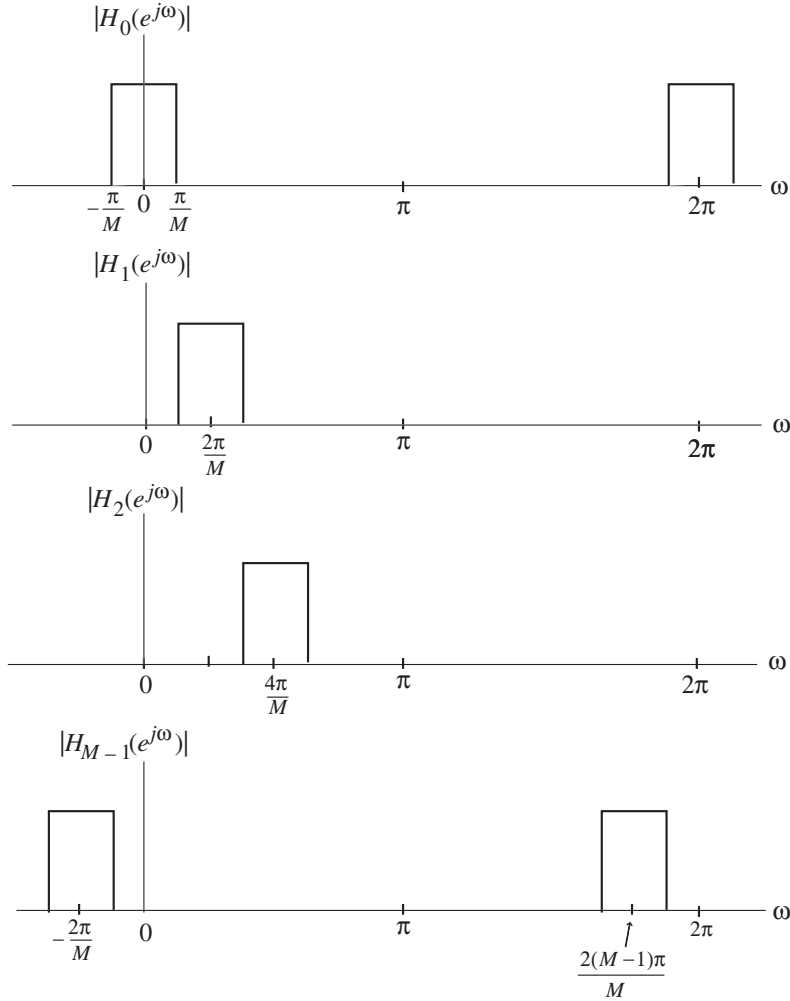
From the above figure, we have $u_0[n] = 2x[n]\cos(\omega_o n) = x[n]e^{j\omega_o n} + x[n]e^{-j\omega_o n}$

or, $U_0(e^{j\omega}) = X(e^{j(\omega+\omega_o)}) + X(e^{j(\omega-\omega_o)})$. Likewise,

$U_1(e^{j\omega}) = jX(e^{j(\omega+\omega_o)}) - jX(e^{j(\omega-\omega_o)})$. We also have

$V_0(z) = H_{LP}(z)X(z e^{j\omega_o}) + H_{LP}(z)X(z e^{-j\omega_o})$, and
 $V_1(z) = j H_{LP}(z)X(z e^{j\omega_o}) - j H_{LP}(z)X(z e^{-j\omega_o})$. Therefore,
 $Y(z) = \frac{1}{2} \left(V_0(z e^{j\omega_o}) + V_0(z e^{-j\omega_o}) \right) + j \frac{1}{2} \left(V_1(z e^{j\omega_o}) - V_1(z e^{-j\omega_o}) \right)$ which after
simplification yields
 $Y(z) = \frac{1}{4} \left\{ H_{LP}(z e^{j\omega_o})X(z e^{j2\omega_o}) + H_{LP}(z e^{j\omega_o})X(z) + H_{LP}(z e^{-j\omega_o})X(z) \right.$
 $\left. + H_{LP}(z e^{-j\omega_o})X(z e^{-j2\omega_o}) \right\} - \frac{1}{4} \left\{ H_{LP}(z e^{j\omega_o})X(z e^{j2\omega_o}) - H_{LP}(z e^{j\omega_o})X(z) \right.$
 $\left. + H_{LP}(z e^{-j\omega_o})X(z) + H_{LP}(z e^{-j\omega_o})X(z e^{-j2\omega_o}) \right\}$. Hence,
 $Y(z) = \frac{1}{2} \left\{ H_{LP}(z e^{j\omega_o}) + H_{LP}(z e^{-j\omega_o}) \right\} X(z)$. Therefore,
 $H_{eq}(z) = \frac{1}{2} \left\{ H_{LP}(z e^{j\omega_o}) + H_{LP}(z e^{-j\omega_o}) \right\}$ Thus, the structure of Figure P7.4
implements the bandpass filter of Problem 7.28.

7.32



The output of the k -th filter is a bandpass signal occupying a bandwidth of $2\pi/M$ and centered at $\omega = k\pi/M$. In general, the k -th filter $H_k(z)$ has a complex impulse response generating a complex output sequence. To realize a real coefficient bandpass filter, one can combine the outputs of $H_k(z)$ and $H_{M-k}(z)$.

7.33 $H_0(z) = \frac{1}{2}(1 + z^{-1})$. Thus, $|H_0(e^{j\omega})| = \cos(\omega/2)$. Now, $G(z) = (H_0(z))^M$.

Hence, $|G(e^{j\omega})|^2 = |H_0(e^{j\omega})|^{2M} = (\cos(\omega/2))^{2M}$. The 3-dB cutoff frequency ω_c of $G(z)$ is thus given by $(\cos(\omega_c/2))^{2M} = \frac{1}{2}$. Hence, $\omega_c = 2 \cos^{-1}(2^{-1/2M})$.

7.34 $H_1(z) = \frac{1}{2}(1 - z^{-1})$. Thus, $|H_1(e^{j\omega})|^2 = \sin^2(\omega/2)$. Let, $F(z) = (H_0(z))^M$. Then, $|F(e^{j\omega})|^2 = (\sin(\omega/2))^{2M}$. The 3-dB cutoff frequency ω_c of $F(z)$ is thus given

by $(\sin(\omega_c/2))^{2M} = \frac{1}{2}$, which yields, $\omega_c = 2 \sin^{-1}(2^{-1/2M})$.

7.35 $H_{LP}(z) = \frac{1-\alpha}{2} \left(\frac{1+z^{-1}}{1-\alpha z^{-1}} \right)$. Note that $H_{LP}(z)$ is stable if $|\alpha| < 1$. Now,

$$\begin{aligned} \alpha &= \frac{1 - \sin(\omega_c)}{\cos(\omega_c)} = \frac{\cos^2(\omega_c/2) + \sin^2(\omega_c/2) - 2\sin(\omega_c/2)\cos(\omega_c/2)}{\cos^2(\omega_c/2) - \sin^2(\omega_c/2)} \\ &= \frac{\cos(\omega_c/2) - \sin(\omega_c/2)}{\cos(\omega_c/2) + \sin(\omega_c/2)} = \frac{1 - \tan(\omega_c/2)}{1 + \tan(\omega_c/2)}. \end{aligned} \quad (7-a)$$

If $0 \leq \omega < \pi$, then $\tan(\omega_c/2) \geq 0$. Hence, $|\alpha| < 1$.

7.36 From Eq. (7-a), $\alpha = \frac{1 - \tan(\omega_c/2)}{1 + \tan(\omega_c/2)}$. Hence, $\tan(\omega_c/2) = \frac{1-\alpha}{1+\alpha}$.

7.37 (a) From Eq.(7.73b), we get $\alpha = \frac{1 - \sin(0.6)}{\cos(0.6)} = 0.5275$. Substituting this value of α

in Eq.(7.71), we arrive at $H_{LP}(z) = 0.2363 \left(\frac{1+z^{-1}}{1-0.5275z^{-1}} \right)$.

(b) From Eq.(7.73b), we get $\alpha = \frac{1 - \sin(0.45)}{\cos(0.45)} = 0.0787$. Substituting this value of

α in Eq.(7.71), we arrive at $H_{LP}(z) = 0.4607 \left(\frac{1+z^{-1}}{1-0.0787z^{-1}} \right)$.

7.38 $H_{HP}(z) = \frac{1+\alpha}{2} \left(\frac{1-z^{-1}}{1-\alpha z^{-1}} \right)$. Thus, $H_{HP}(e^{j\omega}) = \frac{1+\alpha}{2} \left(\frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}} \right)$. Thus,

$$\left| H_{HP}(e^{j\omega}) \right|^2 = \left(\frac{1+\alpha}{2} \right)^2 \left(\frac{2-2\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)} \right). \text{ At -dB cutoff frequency } \omega_c,$$

$$\left| H_{HP}(e^{j\omega_c}) \right|^2 = \frac{1}{2} \text{ which yields } \cos(\omega_c) = \frac{2\alpha}{1+\alpha^2}.$$

7.39 (a) From Eq.(7.73b), we get $\alpha = \frac{1 - \sin(0.6)}{\cos(0.6)} = 0.5275$. Substituting this value of

α in Eq.(7.74), we arrive at $H_{HP}(z) = 0.7637 \left(\frac{1+z^{-1}}{1-0.5275z^{-1}} \right)$.

(b) From Eq.(7.73b), we get $\alpha = \frac{1 - \sin(0.45)}{\cos(0.45)} = 0.0787$. Substituting this value of

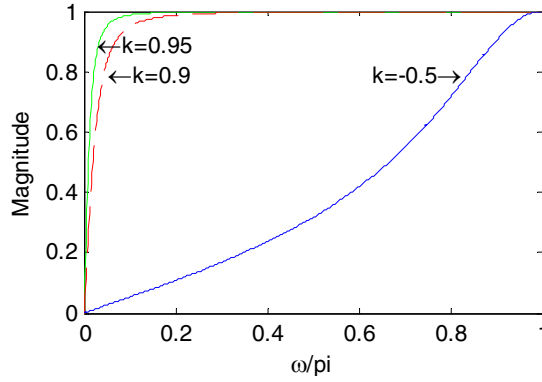
α in Eq.(7.74), we arrive at $H_{HP}(z) = 0.5394 \left(\frac{1+z^{-1}}{1-0.0787z^{-1}} \right)$.

7.40 $H(z) = \frac{1-z^{-1}}{1-kz^{-1}}$. Hence, $|H(e^{j\omega})|^2 = \frac{(1-\cos\omega)^2 + \sin^2\omega}{(1-k\cos\omega)^2 + k^2\sin^2\omega} = \frac{2-2\cos\omega}{1+k^2-2k\cos\omega}$.

Now, $|H(e^{j\pi})|^2 = \frac{4}{(1+k)^2}$. Thus, the scaled transfer function is given by

$H(z) = \frac{1+k}{2} \left(\frac{1-z^{-1}}{1-kz^{-1}} \right)$. A plot of the magnitude responses of the scaled transfer

function for $k = 0.95, 0.9$, and -0.5 are given below:



7.41 $H_{BP}(z) = \frac{1-\alpha}{2} \left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}} \right)$. Thus,

$H_{BP}(e^{j\omega}) = \frac{1-\alpha}{2} \left(\frac{1-e^{-j2\omega}}{1-\beta(1+\alpha)e^{-j\omega} + \alpha e^{-j2\omega}} \right)$. Hence,

$|H_{BP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2}{2} \cdot \frac{2(1-2\cos(2\omega))}{1+\beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)}$
 $= \frac{(1-\alpha)^2 \sin^2\omega}{(1+\alpha)^2(\cos\omega - \beta)^2 + (1-\alpha)^2 \sin^2\omega}$. At the center frequency ω_o ,

$|H_{BP}(e^{j\omega_o})|^2 = 1$. Hence, $(\cos\omega_o - \beta)^2 = 0$ or $\cos\omega_o = \beta$.

At the 3-dB bandedges ω_1 and ω_2 , $|H_{BP}(e^{j\omega_i})|^2 = \frac{1}{2}$, $i = 1, 2$. This implies

$(1+\alpha)^2(\cos\omega_i - \beta)^2 = (1-\alpha)^2 \sin^2\omega_i$, (7-c)

or, $\sin\omega_i = \pm \left(\frac{1+\alpha}{1-\alpha} \right) (\cos\omega_i - \beta)$, $i = 1, 2$. Since, $\omega_1 < \omega_o < \omega_2$, $\sin\omega_1$ must have positive sign and $\sin\omega_2$ must have negative sign, because otherwise, $\sin\omega_2 < 0$

for ω_2 in $(0, \pi)$. Now, Eq. (7-c) can be rewritten as

$$2(1 + \alpha^2) \cos^2 \omega_i - 2\beta(1 + \alpha)^2 \cos \omega_i + \beta^2(1 + \alpha)^2 - (1 - \alpha^2) = 0. \text{ Hence,}$$

$$\cos \omega_1 + \cos \omega_2 = \beta \frac{(1 + \alpha)^2}{1 + \alpha^2}, \text{ and } (\cos \omega_1)(\cos \omega_2) = \frac{\beta^2(1 + \alpha)^2 - (1 - \alpha^2)}{2(1 + \alpha^2)}.$$

$$\begin{aligned} \text{Denote } \omega_{3dB} &= \omega_2 - \omega_1. \text{ Then } \cos \omega_{3dB} = \cos \omega_2 \cos \omega_1 + \sin \omega_2 \sin \omega_1 \\ &= \cos \omega_2 \cos \omega_1 - \left(\frac{1 + \alpha}{1 - \alpha} \right)^2 \left(\cos \omega_2 \cos \omega_1 + \beta^2 (\cos \omega_2 + \cos \omega_1) \right) = \frac{2\alpha}{1 + \alpha^2}. \end{aligned}$$

7.42 (a) Using Eq.(7.76), we get $\beta = \cos(0.55\pi) = -0.1564$. Next, from Eq.(7.78), we get $\frac{2\alpha}{1 + \alpha^2} = \cos(0.25\pi) = 0.7071$, or, equivalently, $0.7071\alpha^2 - 2\alpha + 0.7071 = 0$.

Solution of this quadratic equation yields $\alpha = 2.4142$ and $\alpha = 0.4142$.

Substituting $\alpha = 2.4142$ and $\beta = -0.1564$. in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as

$D(z) = 1 + 0.5340z^{-1} + 2.4142z^{-2}$. Comparing with Eq. (7.136) we note $d_1 = 0.5340$ and $d_2 = 2.4142$. Since the condition of Eq. (7.139) is not satisfied, the corresponding $H_{BP}(z)$ is unstable.

Substituting $\alpha = 0.4142$ and $\beta = -0.1564$. in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as

$D(z) = 1 + 0.2212z^{-1} + 0.4142z^{-2}$. Comparing with Eq. (7.136) we note $d_1 = 0.2212$ and $d_2 = 0.4142$. Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding $H_{BP}(z)$ is a stable transfer function. Hence, the desired

transfer function is
$$H_{BP}(z) = \frac{0.2929(1 - z^{-2})}{1 + 0.2212z^{-1} + 0.4142z^{-2}}.$$

(b) Using Eq.(7.76), we get $\beta = \cos(0.3\pi) = 0.5878$. Next, from Eq.(7.78), we get $\frac{2\alpha}{1 + \alpha^2} = \cos(0.3\pi) = 0.5878$, or, equivalently, $0.5878\alpha^2 - 2\alpha + 0.5878 = 0$.

Solution of this quadratic equation yields $\alpha = 3.0766$ and $\alpha = 0.3249$.

Substituting $\alpha = 3.0766$ and $\beta = 0.5878$. in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as $D(z) = 1 + 2.3968z^{-1} + 0.788z^{-2}$.

Comparing with Eq. (7.136) we note $d_1 = 2.3968$ and $d_2 = 0.788$. Since the condition of Eq. (7.141) is not satisfied, the corresponding $H_{BP}(z)$ is unstable.

Substituting $\alpha = 0.3249$ and $\beta = 0.5878$. in Eq.(7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as $D(z) = 1 + 0.7788z^{-1} + 0.3249z^{-2}$.

Comparing with Eq. (7.136) we note $d_1 = 0.2212$ and $d_2 = 0.4142$. Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding $H_{BP}(z)$ is a stable transfer function. Hence, the desired transfer function is

$$H_{BP}(z) = \frac{0.3376(1 - z^{-2})}{1 + 0.7788z^{-1} + 0.3249z^{-2}}.$$

7.43 $H_{BS}(z) = \frac{1 + \alpha}{2} \left(\frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \right)$. Thus,

$$\begin{aligned} |H_{BS}(e^{j\omega})|^2 &= \left(\frac{1 + \alpha}{2} \right)^2 \cdot \frac{2 + 4\beta^2 - 8\beta \cos(\omega) + 2 \cos(2\omega)}{1 + \beta^2(1 + \alpha)^2 + \alpha^2 + 2\alpha \cos(2\omega) - 2\beta(1 + \alpha)^2 \cos(\omega)} \\ &= \frac{(1 + \alpha)^2 (\cos \omega - \beta)^2}{(1 + \alpha)^2 (\cos \omega - \beta)^2 + (1 - \alpha)^2 \sin^2 \omega}. \end{aligned}$$

At the center frequency ω_o ,

$$|H_{BS}(e^{j\omega_o})|^2 = 0. \text{ Hence, } (\cos \omega_o - \beta)^2 = 0 \text{ or } \cos \omega_o = \beta. \text{ At the 3-dB}$$

bandedges ω_1 and ω_2 , $|H_{BP}(e^{j\omega_i})|^2 = \frac{1}{2}$, $i = 1, 2$. This leads to Eq. (7-c) given in the solution of Problem 7.41. Hence, as in the solution of Problem 7.41,

$$\omega_{3dB} = \frac{2\alpha}{1 + \alpha^2}.$$

7.44 (a) Using Eq.(7.76), we get $\beta = \cos(0.35\pi) = 0.454$. Next, from Eq.(7.78), we get

$$\frac{2\alpha}{1 + \alpha^2} = \cos(0.2\pi) = 0.809, \text{ or, equivalently, } 0.809\alpha^2 - 2\alpha + 0.809 = 0. \text{ Solution}$$

of this quadratic equation yields $\alpha = 1.9627$ and $\alpha = 0.3249$. Substituting $\alpha = 1.9627$ and $\beta = 0.454$ in Eq.(7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as $D(z) = 1 - 1.3451z^{-1} + 1.9627z^{-2}$. Since the condition of Eq. (7.139) is not satisfied, the corresponding $H_{BS}(z)$ is unstable.

Substituting $\alpha = 0.3249$ and $\beta = 0.454$ in Eq.(7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as $D(z) = 1 + 0.6015z^{-1} + 0.3249z^{-2}$. Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding $H_{BS}(z)$ is a stable transfer function. Hence, the desired transfer function is

$$H_{BS}(z) = \frac{0.6624(1 - 0.908z^{-1} + z^{-2})}{1 + 0.6015z^{-1} + 0.3249z^{-2}}.$$

(b) Using Eq.(7.76), we get $\beta = \cos(0.6\pi) = -0.309$. Next, from Eq.(7.78), we get

$$\frac{2\alpha}{1 + \alpha^2} = \cos(0.15\pi) = 0.891, \text{ or, equivalently, } 0.891\alpha^2 - 2\alpha + 0.891 = 0. \text{ Solution}$$

of this quadratic equation yields $\alpha = 1.6319$ and $\alpha = 0.6128$.

Substituting $\alpha = 1.6319$ and $\beta = -0.309$ in Eq.(7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as $D(z) = 1 + 2.345z^{-1} + 1.6319z^{-2}$. Since the condition of Eq. (7.139) is not satisfied, the corresponding $H_{BS}(z)$ is unstable.

Substituting $\alpha = 0.6128$ and $\beta = -0.309$ in Eq.(7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as $D(z) = 1 + 0.4974z^{-1} + 0.6128z^{-2}$. Since the conditions of Eqs. (7.139) and (7.141) are satisfied, the corresponding $H_{BS}(z)$ is a stable transfer function. Hence, the desired transfer

$$\text{function is } H_{BS}(z) = \frac{0.564(1 + 0.618z^{-1} + z^{-2})}{1 + 0.4984z^{-1} + 0.6128z^{-2}}.$$

7.45 $\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)} = 2^{-1/K}$. Let $C = 2^{(K-1)/K}$. Simplifying the first equation

$$\text{we then get } \alpha^2(\cos\omega_c + 1 - C) - 2\alpha(1 + \cos\omega_c - C\cos\omega_c) + 1 + \cos\omega_c - C = 0.$$

Solving the quadratic equation for α we obtain

$$\alpha = \frac{2(1 + (1 - C)\cos\omega_c) \pm \sqrt{4(1 + (1 - C)\cos\omega_c)^2 - 4(1 + \cos\omega_c - C)^2}}{2(1 + \cos\omega_c - C)}$$

$$= \frac{(1 + (1 - C)\cos\omega_c) \pm \sqrt{(2 + 2\cos\omega_c - C - C\cos\omega_c)(C(1 - \cos\omega_c))}}{1 + \cos\omega_c - C}$$

$$= \frac{1 + (1 - C)\cos\omega_c \pm \sin\omega_c\sqrt{2C - C^2}}{1 + \cos\omega_c - C}. \text{ For stability we require } |\alpha| < 1, \text{ hence the}$$

$$\text{desired solution is } \alpha = \frac{1 + (1 - C)\cos\omega_c - \sin\omega_c\sqrt{2C - C^2}}{1 + \cos\omega_c - C}.$$

7.46 $H_{HP}(z) = \frac{1+\alpha}{2} \left(\frac{1-z^{-1}}{1-\alpha z^{-1}} \right)$. $|H_{HP}(e^{j\omega})|^2 = \left(\frac{1+\alpha}{2} \right)^2 \left| \frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}} \right|^2$.

$$|H_{HP}(e^{j\omega})|^{2K} = \left(\frac{1+\alpha}{2} \right)^{2K} \left| \frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}} \right|^{2K} = \left(\frac{1+\alpha}{2} \right)^{2K} \frac{2^K(1-\cos\omega)^K}{(1+\alpha^2-2\alpha\cos\omega)^K}. \text{ At}$$

$$\text{the -dB cutoff frequency } \omega_c, |H_{HP}(e^{j\omega_c})|^{2K} = \frac{1}{2}. \text{ Let } C = 2^{(K-1)/K}.$$

Simplifying the above equation we get

$$\alpha^2(1 - \cos\omega_c - C) - 2\alpha(1 - \cos\omega_c + C\cos\omega_c) + 1 - \cos\omega_c - C = 0. \text{ Solving the quadratic equation for } \alpha \text{ we obtain}$$

$$\alpha = \frac{-2(1 - \cos \omega_c + C \cos \omega_c) \pm 2\sqrt{(1 - \cos \omega_c + C \cos \omega_c)^2 - (1 - \cos \omega_c - C)^2}}{2(1 - \cos \omega_c - C)}.$$

For stability we require $|\alpha| < 1$, hence the desired solution is

$$\alpha = \frac{\sin \omega_c \sqrt{2C - C^2} - (1 - \cos \omega_c + C \cos \omega_c)}{1 - \cos \omega_c - C}.$$

7.47 (a) Analyzing Figure P7.6(a) we get $Y(z) = \left[\frac{1}{2}(1 + \mathcal{A}_1(z)) + \frac{K}{2}(1 - \mathcal{A}_1(z)) \right] X(z)$.

$$\text{Hence, } H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1+K}{2} \right) + \left(\frac{1-K}{2} \right) \mathcal{A}_1(z).$$

(b) Analyzing Figure P7.6(b) we get $Y(z) = \left(\frac{1+K}{2} \right) X(z) + \left(\frac{1-K}{2} \right) \mathcal{A}_1(z) X(z)$.

$$\text{Hence, } H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1+K}{2} \right) + \left(\frac{1-K}{2} \right) \mathcal{A}_1(z).$$

7.48 (a) Analyzing Figure P7.6(a) we get $Y(z) = X(z) + \frac{K}{2}(1 + \mathcal{A}_1(z))X(z)$.

$$\text{Therefore, } H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1+K}{2} \right) + \left(\frac{K}{2} \right) \mathcal{A}_1(z).$$

(b) Analyzing Figure P7.6(b) we get $Y(z) = X(z) + \frac{K}{2}(1 - \mathcal{A}_1(z))X(z)$. Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1+K}{2} \right) - \left(\frac{K}{2} \right) \mathcal{A}_1(z).$$

7.49 $H(e^{j\omega}) = \begin{cases} 1, & \omega_{p1} \leq |\omega| \leq \omega_{p2}, \\ 0, & 0 \leq |\omega| \leq \omega_{s1}, \\ 0, & \omega_{s2} \leq |\omega| < \pi. \end{cases}$ $G(e^{j\omega}) = H(e^{j(\pi-\omega)})$, This implies that the

frequency response of $H(-z)$ is a shifted version of the frequency response of $H(z)$, shifted by π radians. Therefore,

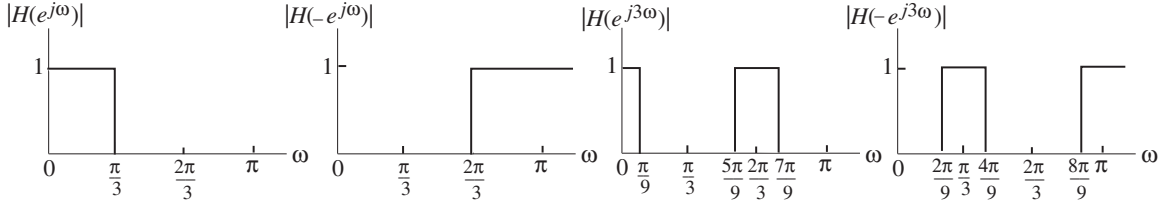
$$G(e^{j\omega}) = H(e^{j(\pi-\omega)}) = \begin{cases} 1, & \pi - \omega_{p2} \leq |\omega| \leq \pi - \omega_{p1}, \\ 0, & 0 \leq |\omega| \leq \pi - \omega_{s2}, \\ 0, & \pi - \omega_{s1} \leq |\omega| < \pi. \end{cases} \quad \text{Hence, } H(-z) \text{ is also a}$$

bandpass filter with passband edges at $\pi - \omega_{p2}$ and $\pi - \omega_{p1}$, and stopband edges at $\pi - \omega_{s2}$ and $\pi - \omega_{s1}$ with $\pi - \omega_{s2} < \pi - \omega_{p1} < \pi - \omega_{p2} < \pi - \omega_{s1}$.

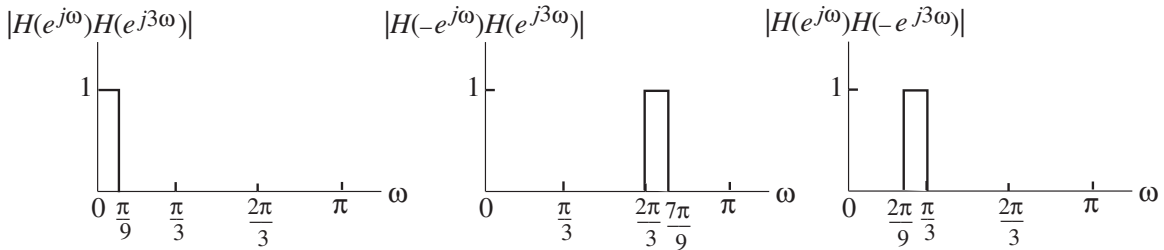
7.50 $H_{LP}(z) = \frac{1-\alpha}{2} \left(\frac{1+z^{-1}}{1-\alpha z^{-1}} \right)$, $G_{HP}(z) = \frac{1-\alpha}{2} \left(\frac{1-z^{-1}}{1+\alpha z^{-1}} \right)$. Let $\beta = -\alpha$. Then,

$$G_{HP}(z) = \frac{1+\beta}{2} \left(\frac{1-z^{-1}}{1-\beta z^{-1}} \right). \text{ Therefore, } \omega_c = \cos^{-1}(\beta) = \cos^{-1}(-\alpha).$$

7.51 The magnitude responses of $H(z), H(-z), H(z^3)$, and $H(-z^3)$ are shown below:



The magnitude responses of $H(z)H(z^3), H(-z)H(z^3)$, and $H(z)H(-z^3)$ are shown below:



7.52 From Eq. (7.49) we observe that the amplitude response $\check{H}(\omega)$ of a Type 1 FIR transfer function is a function of $\cos(\omega n)$. Thus, $\check{H}(\omega + 2\pi k)$ will be a function of $\cos((\omega + 2\pi k)n) = \cos(\omega n + 2\pi kn) = \cos(\omega n)\cos(2\pi kn) - \sin(\omega n)\sin(2\pi kn) = \cos(\omega n)$. Hence, $\check{H}(\omega)$ is a periodic function in ω with a period 2π .

Likewise, from Eq. (7.53) we observe that the amplitude response $\check{H}(\omega)$ of a Type 3 FIR transfer function is a function of $\sin(\omega n)$. Thus, $\check{H}(\omega + 2\pi k)$ will be a function of $\sin((\omega + 2\pi k)n) = \sin(\omega n + 2\pi kn) = \sin(\omega n)\cos(2\pi kn) + \cos(\omega n)\sin(2\pi kn) = \sin(\omega n)$. Hence, $\check{H}(\omega)$ is a periodic function in ω with a period 2π .

Next, from Eq. (7.52) we observe that the amplitude response $\check{H}(\omega)$ of a Type 2 FIR transfer function is a function of $\cos(\omega(n - \frac{1}{2}))$. Thus, $\check{H}(\omega + 4\pi k)$ will be a function of $\cos((\omega + 4\pi k)(n - \frac{1}{2})) = \cos(\omega(n - \frac{1}{2}) + 4\pi k(n - \frac{1}{2})) = \cos(\omega(n - \frac{1}{2}))\cos(4\pi k(n - \frac{1}{2}))$

$-\sin\left(\omega(n - \frac{1}{2})\right)\sin\left(4\pi k(n - \frac{1}{2})\right) = \cos\left(\omega(n - \frac{1}{2})\right)$ as $\cos\left(4\pi k(n - \frac{1}{2})\right) = 1$ and $\sin\left(4\pi k(n - \frac{1}{2})\right) = 0$. Hence, $\check{H}(\omega)$ is a periodic function in ω with a period 4π .

Finally, from Eq. (7.54) we observe that the amplitude response $\check{H}(\omega)$ of a Type 4 FIR transfer function is a function of $\sin\left(\omega(n - \frac{1}{2})\right)$. Thus, $\check{H}(\omega + 4\pi k)$ will be a function of

$$\begin{aligned} \sin\left((\omega + 4\pi k)(n - \frac{1}{2})\right) &= \sin\left(\omega(n - \frac{1}{2}) + 4\pi k(n - \frac{1}{2})\right) = \sin\left(\omega(n - \frac{1}{2})\right)\cos\left(4\pi k(n - \frac{1}{2})\right) \\ &+ \cos\left(\omega(n - \frac{1}{2})\right)\sin\left(4\pi k(n - \frac{1}{2})\right) = \sin\left(\omega(n - \frac{1}{2})\right) \text{ as } \cos\left(4\pi k(n - \frac{1}{2})\right) = 1 \text{ and } \\ &\sin\left(4\pi k(n - \frac{1}{2})\right) = 0. \text{ Hence, } \check{H}(\omega) \text{ is a periodic function in } \omega \text{ with a period } 4\pi. \end{aligned}$$

7.53 The remaining zeros are at: $z_5 = \frac{1}{z_1} = \frac{1}{0.8} = 1.25$; $z_6 = z_2^* = j$; $z_7 = z_3^* = 2 + j2$;

$$z_8 = \frac{1}{z_3} = \frac{1}{2 - j2} = 0.25 + j0.25; \quad z_9 = z_8^* = 0.25 - j0.25; \quad z_{10} = z_4^* = -0.5 - j0.3;$$

$$z_{11} = \frac{1}{z_4} = \frac{1}{-0.5 + j0.3} = -\frac{0.5}{0.34} + j\frac{0.3}{0.34}, \quad z_{12} = z_{11}^* = -\frac{0.5}{0.34} - j\frac{0.3}{0.34}.$$

$$\begin{aligned} H_1(z) &= \prod_{k=1}^{12} (1 - z_k z^{-1}) = 1 - 2.6088z^{-1} + 1.7576z^{-2} + 11.0226z^{-3} + 5.6432z^{-4} \\ &- 24.2166z^{-5} + 9.7711z^{-6} - 24.2166z^{-7} + 5.6432z^{-8} + 11.0226z^{-9} + 1.7576z^{-10} \\ &- 2.6088z^{-11} + z^{-12}. \end{aligned}$$

7.54 The remaining zeros are at: $z_4 = \frac{1}{z_1} = \frac{1}{3.1} = 0.3226$, $z_5 = z_2^* = -2 - j4$,

$$z_6 = \frac{1}{z_2} = \frac{1}{-2 + j4} = -0.1 - j0.2, \quad z_7 = -0.1 - j0.2,$$

$$z_8 = z_3^* = 0.8 - j0.4, \quad z_9 = \frac{1}{z_3} = \frac{1}{0.8 + j0.4} = 1 - j0.5, \quad z_{10} = 1 + j0.5, \quad z_{11} = -1.$$

$$\begin{aligned} H_2(z) &= \prod_{k=1}^{11} (1 - z_k z^{-1}) = 1 - 1.8226z^{-1} + 7.1039z^{-2} - 79.4635z^{-3} + 182.19z^{-4} \\ &- 111.2306z^{-5} - 111.2306z^{-6} + 182.19z^{-7} - 79.4635z^{-8} + 7.1039z^{-9} - 1.8226z^{-10} + z^{-11}. \end{aligned}$$

7.55 The remaining zeros are at: $z_4 = z_1^* = 0.1 + j0.599$,

$$z_5 = \frac{1}{z_1} = \frac{1}{0.1 - j0.599} = 0.2711 + j1.6242, \quad z_6 = z_5^* = 0.2711 - j1.6242,$$

$$z_7 = z_2^* = -0.3 - j0.4, \quad z_8 = \frac{1}{z_2} = -1.2 - j1.6, \quad z_9 = z_8^* = -1.2 + j1.6,$$

$$z_3 = 2, \quad z_{10} = \frac{1}{z_3} = 0.5, \quad z_{11} = 1, \quad z_{12} = -1.$$

$$H_3(z) = 1 - 0.2423z^{-1} + 1.0076z^{-2} - 6.5294z^{-3} + 1.3338z^{-4} - 17.2533z^{-5} \\ + 17.2533z^{-7} - 1.3338z^{-8} + 6.5294z^{-9} - 1.0076z^{-10} + 0.2423z^{-11} - z^{-12}.$$

7.56 The remaining zeros are at: $z_4 = 2.2 - j3.4, z_5 = \frac{1}{z_1} = 0.1341 - j0.2073$,

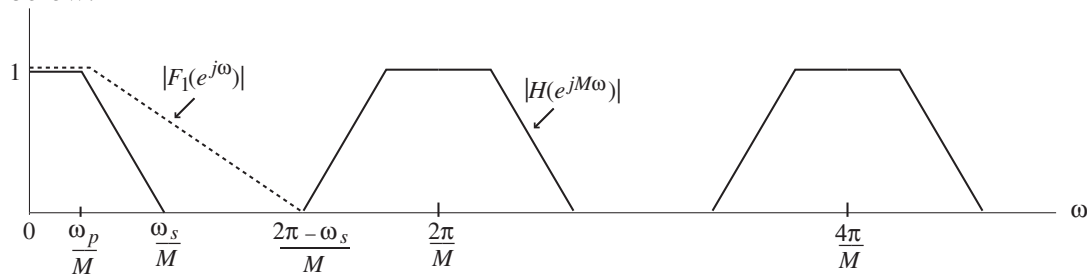
$$z_6 = z_5^* = 0.1341 + j0.2073,$$

$$z_2 = 0.6 + j0.9, \quad z_7 = z_2^* = 0.6 - j0.9, \quad z_8 = \frac{1}{z_2} = 0.5128 - j0.7692,$$

$$z_9 = z_8^* = 0.5128 + j0.7692, \quad z_3 = -0.5, \quad z_{10} = \frac{1}{z_3} = -2, \quad z_{11} = 1.$$

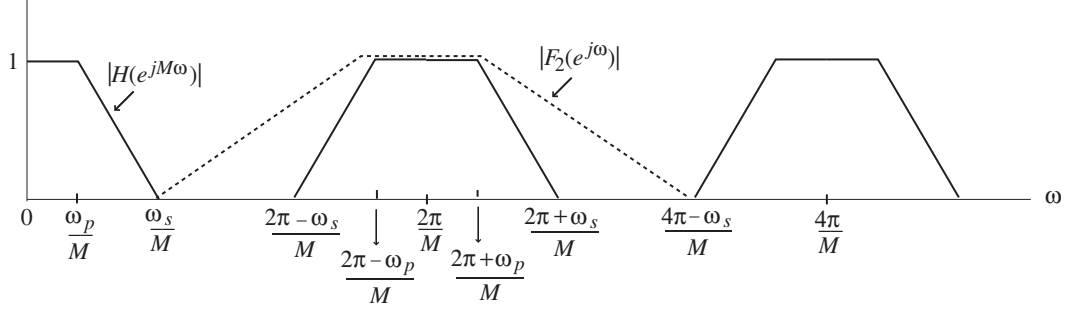
$$H_4(z) = 1 - 5.3939z^{-1} + 19.446z^{-2} - 5.0838z^{-3} - 51.8577z^{-4} + 119.7073z^{-5} \\ - 119.7073z^{-6} + 51.8577z^{-7} + 5.0838z^{-8} - 19.446z^{-9} + 5.3939z^{-10} - z^{-11}.$$

7.57 The magnitude responses of $H(z^M)$ (solid line) and $F_1(z)$ (dashed line) are shown below:



Hence, $G_1(z) = H(z^M)F_1(z)$ is a lowpass filter with a unity passband magnitude, passband edge at ω_p / M and stopband edge at ω_s / M .

The magnitude responses of $H(z^M)$ (solid line) and $F_2(z)$ (dashed line) are shown below:



Hence, $G_2(z) = H(z^M)F_2(z)$ is a bandpass filter with a unity passband magnitude, passband edges at $(2\pi - \omega_p)/M$ and $(2\pi + \omega_p)/M$, and stopband edges at $(2\pi - \omega_s)/M$ and $(2\pi + \omega_s)/M$.

7.58 $H(z) = \sum_{n=0}^N h[n]z^{-n}$ and $H(e^{j\omega}) = \sum_{n=0}^N h[n]e^{-j\omega n}$. The frequency response will

exhibit generalized linear phase if it can be expressed in the form $H(e^{j\omega}) = \check{H}(\omega)e^{-j\alpha\omega}e^{-j\beta}$, where $\check{H}(\omega)$, the amplitude function, is a real function of ω and α and β are constants. We need to examine the case when the order N is even and when N is odd separately. Without any loss of generality, assume first

$N = 5$. Then $H(z) = \sum_{n=0}^5 h[n]z^{-n}$, and

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + h[5]e^{-j5\omega} \\ &= e^{-j5\omega/2} \left(h[0]e^{j5\omega/2} + h[5]e^{-j5\omega/2} + h[1]e^{j3\omega/2} + h[4]e^{-j3\omega/2} + h[2]e^{j\omega/2} \right. \\ &\quad \left. + h[3]e^{-j3\omega/2} \right) = e^{-j5\omega/2} \left[(h[0] + h[5])\cos(5\omega/2) + (h[1] + h[4])\cos(3\omega/2) \right. \\ &\quad \left. + (h[2] + h[3])\cos(\omega/2) \right] + je^{-j5\omega/2} \left[(h[0] - h[5])\sin(5\omega/2) + (h[1] - h[4])\sin(3\omega/2) \right. \\ &\quad \left. + (h[2] - h[3])\sin(\omega/2) \right]. \end{aligned}$$

It follows then that if $h[n] = h[5 - n]$, $0 \leq n \leq 5$, we have $H(e^{j\omega}) = e^{-j5\omega/2}\check{H}(\omega)$, where

$\check{H}(\omega) = 2h[0]\cos(5\omega/2) + 2h[1]\cos(3\omega/2) + 2h[2]\cos(\omega/2)$, which is a real function of ω and as a result, $H(e^{j\omega})$ has generalized phase.

Alternately, if $h[n] = -h[5 - n]$, $0 \leq n \leq 5$, then we have

$$\begin{aligned} H(e^{j\omega}) &= je^{-j5\omega/2}\check{H}(\omega) = e^{-j5\omega/2}e^{j\pi}\check{H}(\omega), \text{ where,} \\ \check{H}(\omega) &= 2h[0]\sin(5\omega/2) + 2h[1]\sin(3\omega/2) + 2h[2]\sin(\omega/2), \text{ which is a real} \\ &\text{function of } \omega \text{ and as a result, } H(e^{j\omega}) \text{ has generalized phase.} \end{aligned}$$

Next, assume $N = 6$. Then $H(z) = \sum_{n=0}^6 h[n]z^{-n}$, and $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + h[5]e^{-j5\omega} + h[6]e^{-j6\omega}$

$$\begin{aligned}
& + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + h[5]e^{-j5\omega} + h[6]e^{-j6\omega} \\
& = e^{-j3\omega} \left[(h[0] + h[6])\cos(3\omega) + (h[1] + h[5])\cos(2\omega) + (h[2] + h[4])\cos(\omega) + h[3] \right] \\
& + je^{-j3\omega} \left[(h[0] - h[6])\sin(3\omega) + (h[1] - h[5])\sin(2\omega) + (h[2] - h[4])\sin(\omega) \right].
\end{aligned}$$

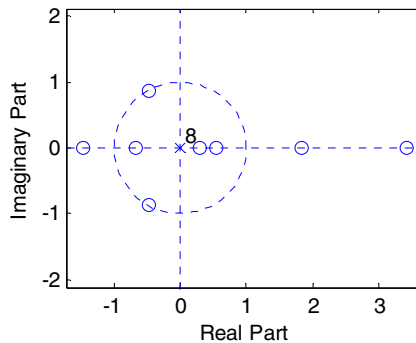
Hence, it follows that if $h[n] = h[6 - n]$, $0 \leq n \leq 6$, then $H(e^{j\omega}) = e^{-j3\omega} \check{H}(\omega)$, where $\check{H}(\omega) = 2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]$, which is a real function of ω and as a result, $H(e^{j\omega})$ has generalized phase.

Alternately, if $h[n] = -h[6 - n]$, $0 \leq n \leq 5$, then we have

$$\begin{aligned}
H(e^{j\omega}) & = je^{-j3\omega} \check{H}(\omega) = e^{-j3\omega} e^{j\pi} \check{H}(\omega), \text{ where,} \\
\check{H}(\omega) & = 2h[0]\sin(3\omega) + 2h[1]\sin(2\omega) + 2h[2]\sin(\omega), \text{ which is a real function of } \omega \text{ and} \\
& \text{as a result, } H(e^{j\omega}) \text{ has generalized phase.}
\end{aligned}$$

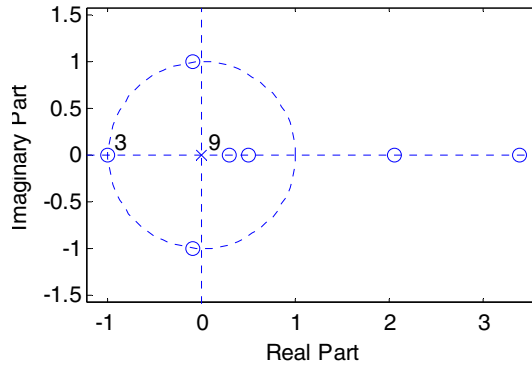
- 7.59** Type 1: $h[n] = \{a \ -b \ -c \ d \ e \ d \ -c \ -b \ a\}$.
Type 2: $h[n] = \{a \ -b \ -c \ d \ e \ e \ d \ -c \ -b \ a\}$.
Type 3: $h[n] = \{a \ -b \ -c \ d \ e \ 0 \ -e \ -d \ c \ b \ -a\}$.
Type 4: $h[n] = \{a \ -b \ -c \ d \ e \ -e \ -d \ c \ b \ -a\}$.

- 7.60 (a)** Type 1: $\{h[n]\} = \{1 \ -3 \ -4 \ 6 \ 8 \ 6 \ -4 \ -3 \ 1\}$. Hence,
 $H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} + 6z^{-5} - 4z^{-6} - 3z^{-7} + z^{-8}$.
The zero plot obtained using the M-file `zplane` is shown below:



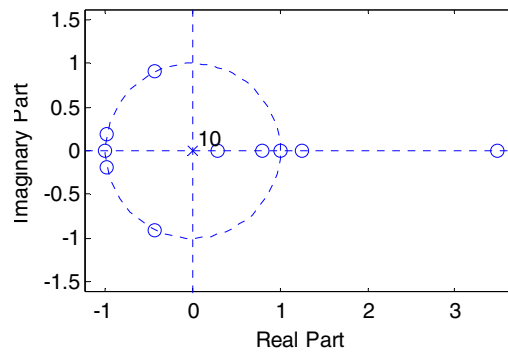
It can be seen from the above that a complex-conjugate zero pair on the unit circle appear singly and real zeros appear in mirror-image symmetry. There are no zeros at $z = 1$.

- (b)** Type 2: $\{h[n]\} = \{1 \ -3 \ -4 \ 6 \ 8 \ 8 \ 6 \ -4 \ -3 \ 1\}$. Hence,
 $H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} + 8z^{-5} + 6z^{-6} - 4z^{-7} - 3z^{-8} + z^{-9}$.
The zero plot obtained using the M-file `zplane` is shown below:



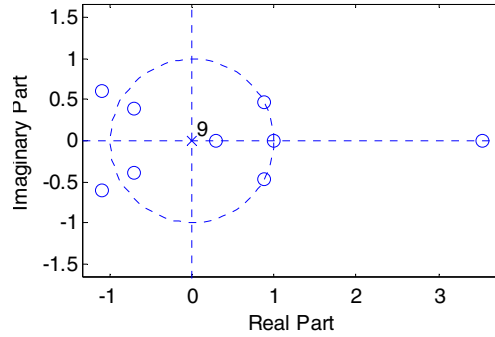
From the above zero plot it can be seen that complex-conjugate zero pairs on the unit circle appear singly and real zeros appear in mirror-image symmetry. There are 3 zeros at $z = -1$.

(c) Type 3: $\{h[n]\} = \{1 \ -3 \ -4 \ 6 \ 8 \ 0 \ -8 \ -6 \ 4 \ 3 \ -1\}$. Hence,
 $H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} - 8z^{-6} - 6z^{-7} + 4z^{-8} + 3z^{-9} - z^{-10}$.
 The zero plot obtained using the M-file `zplane` is shown below:



From the above zero plot it can be seen that complex-conjugate zero pairs on the unit circle appear singly and real zeros appear in mirror-image symmetry. There is one zero at $z = -1$, and one zero at $z = 1$.

(d) Type 4: $\{h[n]\} = \{1 \ -3 \ -4 \ 6 \ 8 \ -8 \ -6 \ 4 \ 3 \ -1\}$. Hence,
 $H(z) = 1 - 3z^{-1} - 4z^{-2} + 6z^{-3} + 8z^{-4} + 8z^{-5} + 6z^{-6} - 4z^{-7} - 3z^{-8} + z^{-9}$.
 The zero plot obtained using the M-file `zplane` is shown below:



From the above zero plot it can be seen that complex-conjugate zeros appear in mirror-image symmetry, a complex-conjugate zero pair on the unit circle appear singly, and real zeros appear in mirror-image symmetry. There is one zero at $z = 1$.

7.61 $H_1(z)$ is of Type 1 and hence, it has a symmetric impulse response of odd length $2N + 1$. Let α be the constant term of $H_1(z)$. Then, the coefficient of the highest power of z^{-1} of $H_1(z)$ is also α .

$H_2(z)$ is of Type 2 and hence, it has a symmetric impulse response of even length $2M$. Let β be the constant term of $H_2(z)$. Then, the coefficient of the highest power of z^{-1} of $H_2(z)$ is also β .

$H_3(z)$ is of Type 3 and hence, it has an anti-symmetric impulse response of odd length $2R + 1$. Let γ be the constant term of $H_3(z)$. Then, the coefficient of the highest power of z^{-1} of $H_3(z)$ is $-\gamma$.

$H_4(z)$ is of Type 4 and hence, it has an anti-symmetric impulse response of even length $2K$. Let δ be the constant term of $H_4(z)$. Then, the coefficient of the highest power of z^{-1} of $H_4(z)$ is $-\delta$.

(a) The length of $H_1(z)H_1(z)$ is $(2N + 1) + (2N + 1) - 1 = 4N + 1$ which is odd.

The constant term of $H_1(z)H_1(z)$ is α^2 and the coefficient of the highest power of z^{-1} of $H_1(z)H_1(z)$ is also α^2 . Hence, $H_1(z)H_1(z)$ is of Type 1.

(b) The length of $H_1(z)H_2(z)$ is $(2N + 1) + (2M) - 1 = 2(N + M)$ which is even.

The constant term of $H_1(z)H_2(z)$ is $\alpha\beta$ and the coefficient of the highest power of z^{-1} of $H_1(z)H_2(z)$ is also $\alpha\beta$. Hence, $H_1(z)H_2(z)$ is of Type 2.

(c) The length of $H_1(z)H_3(z)$ is $(2N + 1) + (2R + 1) - 1 = 2(N + R) + 1$ which is odd.

The constant term of $H_1(z)H_3(z)$ is $\alpha\gamma$ and the coefficient of the highest power of z^{-1} of $H_1(z)H_3(z)$ is $-\alpha\gamma$. Hence, $H_1(z)H_3(z)$ is of Type 3.

(d) The length of $H_1(z)H_4(z)$ is $(2N + 1) + (2K) - 1 = 2(N + K)$ which is even. The constant term of $H_1(z)H_4(z)$ is $\alpha\delta$ and the coefficient of the highest power of z^{-1} of $H_1(z)H_4(z)$ is $-\alpha\delta$. Hence, $H_1(z)H_4(z)$ is of Type 4.

(e) The length of $H_2(z)H_2(z)$ is $(2M) + (2M) - 1 = 4M - 1$ which is odd. The constant term of $H_2(z)H_2(z)$ is β^2 and the coefficient of the highest power of z^{-1} of $H_2(z)H_2(z)$ is also β^2 . Hence, $H_2(z)H_2(z)$ is of Type 1.

(f) The length of $H_3(z)H_3(z)$ is $(2R + 1) + (2R + 1) - 1 = 4R + 1$ which is odd. The constant term of $H_3(z)H_3(z)$ is γ^2 and the coefficient of the highest power of z^{-1} of $H_3(z)H_3(z)$ is also γ^2 . Hence, $H_3(z)H_3(z)$ is of Type 1.

(g) The length of $H_4(z)H_4(z)$ is $(2K) + (2K) - 1 = 4K - 1$ which is odd. The constant term of $H_4(z)H_4(z)$ is δ^2 and the coefficient of the highest power of z^{-1} of $H_4(z)H_4(z)$ is also δ^2 . Hence, $H_4(z)H_4(z)$ is of Type 1.

(h) The length of $H_2(z)H_3(z)$ is $(2M) + (2R + 1) - 1 = 2(M + R)$ which is even. The constant term of $H_2(z)H_3(z)$ is $\beta\gamma$ and the coefficient of the highest power of z^{-1} of $H_2(z)H_3(z)$ is $-\beta\gamma$. Hence, $H_2(z)H_3(z)$ is of Type 4.

(i) The length of $H_3(z)H_4(z)$ is $(2R + 1) + (2K) - 1 = 2(R + K)$ which is even. The constant term of $H_3(z)H_4(z)$ is $\gamma\delta$ and the coefficient of the highest power of z^{-1} of $H_3(z)H_4(z)$ is also $\gamma\delta$. Hence, $H_3(z)H_4(z)$ is of Type 2.

7.62 (a) $F_1(z) = 2.1 - 3.5z^{-1} + 4.2z^{-2} = 2.1(1 - 1.667z^{-1} + 2z^{-2})$. $F_1(z)$ has complex conjugate zeros at $z = 0.8333 \pm j1.1426$. To generate a linear-phase transfer function $H(z)$, we need to multiply $F_1(z)$ with the factor $F_2(z)$ which has complex-conjugate zeros situated in the z -plane with a mirror-image symmetry with respect to the zeros of $F_1(z)$. Hence, $F_2(z) = 2 - 1.667z^{-1} + z^{-2}$, resulting in $H(z) = F_1(z)F_2(z) = 2.1(2 - 5z^{-1} + 7.7778z^{-2} - 5z^{-3} + 2z^{-4}) = 4.2 - 10.5z^{-1} + 16.3333z^{-2} - 10.5z^{-3} + 4.2z^{-4}$.

(b) $F_1(z) = 1.4 + 5.2z^{-1} - 2.2z^{-2} + 3.3z^{-3} = 1.4(1 + 3.7143z^{-1} - 1.5714z^{-2} + 2.3571z^{-3})$. $F_1(z)$ has complex conjugate zeros

at $z = 0.2524 \pm j0.7035$, and a real zero at -4.2192 . To generate a linear-phase transfer function $H(z)$, we need to multiply $F_1(z)$ with the factor $F_2(z)$ which has situated in the z -plane with a mirror-image symmetry with respect to the zeros of $F_1(z)$. Hence, $F_2(z) = 2.3571 - 1.5714z^{-1} + 3.7143z^{-2} + z^{-3}$, resulting in $H(z) = F_1(z)F_2(z) = 1.4(2.3571 + 7.1837z^{-1} - 5.8265z^{-2} + 22.8214z^{-3} - 5.8265z^{-4} + 7.1837z^{-5} + 2.3571z^{-6}) = 3.3 + 10.0571z^{-1} - 8.1671z^{-2} + 31.95z^{-3} - 8.1571z^{-4} + 10.0571z^{-5} + 3.3z^{-6}$.

7.63 We rewrite the polynomial $H(z)$ in the form $f(z) = Kz^{-N} \prod_{i=1}^N (z - \lambda_i)$. Its

logarithmic differential is given by $\frac{f'(z)}{f(z)} = -\frac{N}{z} + \sum_{i=1}^N \frac{1}{z - \lambda_i}$ or

$$z \frac{f'(z)}{f(z)} = -N + \sum_{i=1}^N \frac{z}{z - \lambda_i} = -N + \sum_{i=1}^N \frac{1}{1 - (\lambda_i / z)}$$

For any $|z| > \max_i |\lambda_i|$ we can expand the above in a Taylor series as

$$z \frac{f'(z)}{f(z)} = -N + \sum_{i=1}^N \left[1 + (\lambda_i / z) + (\lambda_i / z)^2 + (\lambda_i / z)^3 + \dots \right], \text{ or}$$

$$z \frac{f'(z)}{f(z)} = -N + N + (S_1 / z) + (S_2 / z)^2 + (S_3 / z)^3 + \dots \quad (1)$$

where $S_m = \sum_{i=1}^N \lambda_i^m$.

Now set $f(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \dots + h[N]z^{-N}$. So that

$$z \frac{f'(z)}{f(z)} = -\frac{h[1]z^{-1} + 2h[2]z^{-2} + \dots + Nh[N]z^{-N}}{h[0] + h[1]z^{-1} + h[2]z^{-2} + \dots + h[N]z^{-N}} \quad (2)$$

Equations (1) and (2) pertain to the same quantity hence identically the right hand sides are the same. Thus the following convolution holds true

$$\{h[0], h[1], h[2], \dots, h[N]\} \circledast \{0, S_1, S_2, S_3, \dots\} = -\{h[1], 2h[2], \dots, Nh[n]\}$$

Hence the Newton Identities.

7.64 The root moments of S_m of $H(z) = K \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) \prod_{i=1}^{n_2} (1 - \beta_i z^{-1})$ are defined as

$S_m = \sum_{i=1}^{n_1} \alpha_i^m + \sum_{i=1}^{n_2} \beta_i^m$. For real $H(z)$, the complex roots occur in conjugate pairs and hence, their corresponding powers are also in this form, thereby making their sum entirely real.

(a) If $H(z)$ is minimum-phase, then $\beta_i = 0$ for all i , and $|\alpha_i| < 1$. Therefore as $m \rightarrow \infty$, S_m will decrease exponentially.

(b) Write $\ln H(z) = \ln K + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1})$ and then expand

$\ln H(z)$ in a Laurent series. The second summation can be re-expressed as

$$\sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1}) = \sum_{i=1}^{n_2} \ln(-\beta_i z^{-1}) \left(1 - \frac{z}{\beta_i}\right). \text{ Hence,}$$

$$\ln H(z) = \ln K + \sum_{i=1}^{n_2} \ln(-\beta_i) - n_2 \ln z + \sum_{i=1}^{n_1} (1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} (1 - \frac{z}{\beta_i}). \text{ Now, } \beta_i \text{ appear}$$

in complex conjugate pairs, hence, $\ln K + \sum_{i=1}^{n_2} \ln(-\beta_i)$ can be written as $\ln K_1$ where K_1 is real.

$$\text{However, } \ln(1 - \alpha_i z^{-1}) = - \left[\alpha_i z^{-1} + \frac{(\alpha_i z^{-1})^2}{2} + \frac{(\alpha_i z^{-1})^3}{3} + \dots \right] \text{ and}$$

$$\ln \left(1 - \frac{z}{\beta_i}\right) = - \left[\frac{z}{\beta_i} + \frac{(z/\beta_i)^2}{2} + \frac{(z/\beta_i)^3}{3} + \dots \right]. \text{ These two Taylor series expansions are}$$

valid since $|\alpha_i z^{-1}| < 1$ and $|z/\beta_i| < 1$ on $|z| = 1$. Thus,

$$\ln H(z) = \ln K_1 - \sum_{m=1}^{\infty} \left(\frac{S_m^{N_1}}{m} z^{-m} + \frac{S_{-m}^{N_2}}{m} z^m \right) \text{ where } S_m^{N_1} = \sum_{i=1}^{n_1} \alpha_i^m \text{ and } S_{-m}^{N_2} = \sum_{i=1}^{n_2} \beta_i^{-m}.$$

On $|z| = 1$ we have $H(z)|_{z=e^{j\omega}} = \check{H}(\omega) e^{j\theta(\omega)}$ and therefore,

$$\ln \check{H}(\omega) + j\theta(\omega) = \ln K_1 - \left(\sum_{m=1}^{\infty} \frac{S_m^{N_1}}{m} e^{-jm\omega} + \sum_{m=1}^{\infty} \frac{S_{-m}^{N_2}}{m} e^{jm\omega} \right) + (-jn_2\omega).$$

On equating real and imaginary parts of the equation we arrive at

$$\ln \check{H}(\omega) = \ln K_1 - \sum_{m=1}^{\infty} \left(\frac{S_m^{N_1} + S_{-m}^{N_2}}{m} \right) \cos(m\omega),$$

$$\theta(\omega) = -n_2\omega + \sum_{m=1}^{\infty} \left(\frac{S_m^{N_1} - S_{-m}^{N_2}}{m} \right) \sin(m\omega).$$

(c) From the expression for the phase as given above, it can be seen that the second term contributes non-linear components to the phase, whilst the only term which is linear is the first ($-n_2\omega$). Thus to have linear phase we must have (1) $n_2 \neq 0$ and (2) $S_m^{N_1} = S_{-m}^{N_2}$. The second condition means that the zeros of $H(z)$ outside the unit circle must be the same in number as those inside the unit circle and the zeros outside must be at locations of the zeros inside the unit circle.

7.65 (a) $A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$. Thus, $A_1(e^{j\omega}) = \frac{d_1 + e^{-j\omega}}{1 + d_1 e^{-j\omega}} = \frac{d_1 e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} + d_1 e^{-j\omega/2}}$

$$= \frac{\alpha e^{j\beta}}{\alpha e^{-j\beta}} = e^{j2\beta}, \text{ where } \alpha e^{j\beta} = d_1 e^{j\omega/2} + e^{-j\omega/2}$$

$= (d_1 + 1)\cos(\omega/2) + j(d_1 - 1)\sin(\omega/2)$. Therefore, phase is given by

$$\theta(\omega) = 2\beta = -2 \tan^{-1} \left(\frac{1-d_1}{1+d_1} \tan(\omega/2) \right).$$

Now, for small values of x , $\tan(x) \cong x$ and

$\tan^{-1}(x) \cong x$. Hence, the approximate expression for the phase at low frequencies

is given by $\theta(\omega) \cong -2 \left(\frac{1-d_1}{1+d_1} \right) \frac{\omega}{2} = - \left(\frac{1-d_1}{1+d_1} \right) \omega$. Therefore, the approximate

expression for the phase delay is given by $\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = \delta \cong \frac{1-d_1}{1+d_1}$ samples.

(b) For $\delta = 0.5$ samples, $d_1 = \frac{1-\delta}{1+\delta} = \frac{0.5}{1.5} = \frac{1}{3}$. Then, $A_1(z) = \frac{\frac{1}{3} + z^{-1}}{1 + \frac{1}{3}z^{-1}}$. Thus, the

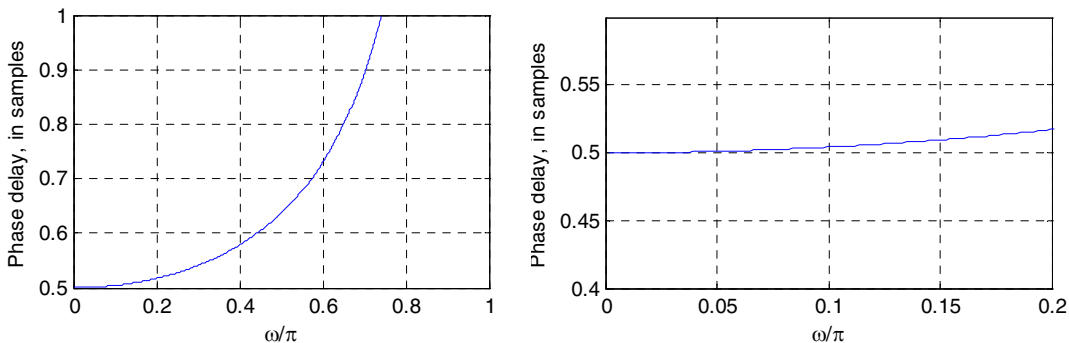
exact phase delay is given by $\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = \frac{2}{\omega} \left(\frac{1-d_1}{1+d_1} \tan(\omega/2) \right) = \frac{2}{\omega} (0.5 \tan(\omega/2))$.

For a sampling rate of 20 kHz, the normalized angular frequency equivalent to 1

kHz is $\omega_o = \frac{10^3}{20 \times 10^3} = \frac{1}{20} = 0.05$. The exact phase delay at ω_o is thus

$$\tau_p(\omega_o) = \frac{2}{\omega_o} (0.5 \tan(\omega_o/2)) = \frac{2}{0.05} (0.5 \tan(0.025)) = 0.500078 \text{ samples, which is}$$

seen to be very close to the desired phase delay of 0.5 samples.



$$\begin{aligned}
7.66 \quad A_2(e^{j\omega}) &= \frac{d_2 + d_1 e^{-j\omega} + e^{-j2\omega}}{1 + d_1 e^{-j\omega} + d_2 e^{-j2\omega}} = \frac{d_2 e^{j\omega} + d_1 + e^{-j\omega}}{e^{j\omega} + d_1 + d_2 e^{-j\omega}} \\
&= \frac{(d_1 + d_2 \cos \omega + \cos \omega) + j(d_2 \sin \omega + \sin \omega)}{(d_1 + d_2 \cos \omega + \cos \omega) + j(\sin \omega - d_2 \sin \omega)} = \frac{(d_1 + (d_2 + 1) \cos \omega) + j(d_2 - 1) \sin \omega}{(d_1 + (d_2 + 1) \cos \omega) - j(d_2 - 1) \sin \omega}.
\end{aligned}$$

Therefore, $\theta(\omega) = 2 \tan^{-1} \left(\frac{(d_2 - 1) \sin \omega}{d_1 + (d_2 + 1) \cos \omega} \right)$. Now,

$$\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = -\frac{2}{\omega} \tan^{-1} \left(\frac{(d_2 - 1) \sin \omega}{d_1 + (d_2 + 1) \cos \omega} \right). \text{ For } \omega \cong 0, \sin \omega = \omega \text{ and}$$

$$\cos \omega = 1. \text{ Then, } \tau_p(\omega) = -\frac{2}{\omega} \tan^{-1} \left(\frac{(d_2 - 1)\omega}{d_1 + (d_2 + 1)} \right). \text{ Also, for } x \cong 0, \tan^{-1} x \cong x.$$

$$\text{Hence, } \tau_p(\omega) = -\frac{2}{\omega} \frac{(d_2 - 1)\omega}{d_1 + (d_2 + 1)} = -\frac{2(d_2 - 1)}{d_1 + d_2 + 1}. \text{ Now, substituting } d_1 = 2 \left(\frac{2 - \delta}{1 + \delta} \right) \text{ and}$$

$$d_2 = \frac{(2 - \delta)(1 - \delta)}{(2 + \delta)(1 + \delta)}, \text{ we can easily show that } -\frac{2(d_2 - 1)}{d_1 + d_2 + 1} = \delta.$$

7.67 Since $G(z)$ is non-minimum phase but causal, it will have some zeros outside the unit circle. Let $z = \alpha$ be one such zero. We can then write $G(z) = P(z)(1 - \alpha z^{-1})$

$$= P(z)(-\alpha^* + z^{-1}) \left(\frac{1 - \alpha z^{-1}}{(-\alpha^* + z^{-1})} \right). \text{ Note that } \left(\frac{1 - \alpha z^{-1}}{(-\alpha^* + z^{-1})} \right) \text{ is a stable first-order}$$

allpass function. If we carry out this operation for all zeros of $G(z)$ that are outside the unit circle, we can write $G(z) = H(z)A(z)$ where $H(z)$ will have all zeros inside the unit circle and will thus be a minimum phase function and will be a product of stable first-order allpass functions, and hence an allpass function.

$$7.68 \quad H(z) = \frac{(3z - 2.1)(z^2 + 2.5z + 5)}{(z - 0.65)(z + 0.48)}. \text{ In order to correct for magnitude distortion we}$$

require the transfer function $G(z)$ to satisfy the following property

$$\left| G(e^{j\omega}) \right| = \frac{1}{\left| H(e^{j\omega}) \right|}. \text{ Hence, one possible solution is}$$

$$G_d(z) = \frac{1}{H(z)} = \frac{(z - 0.65)(z + 0.48)}{(3z - 2.1)(z^2 + 2.5z + 5)}. \text{ Note that the coefficients of the pole}$$

factor $(z^2 + 2.5z + 5)$ in the denominator of $G_d(z)$ do not satisfy the condition of Eq. (7.139) and hence, has roots outside the unit circle making $G_d(z)$ unstable.

To develop a stable transfer function with magnitude response same as $G_d(z)$, we

$$\text{multiply it with the stable allpass function } \frac{z^2 + 2.5z + 5}{5z^2 + 2.5z + 1} \text{ resulting in the transfer}$$

function $G(z) = \frac{(z - 0.65)(z + 0.48)}{(3z - 2.1)(5z^2 + 2.5z + 1)}$ which is the desired stable solution

satisfying the condition $\left|G(e^{j\omega})\right| \left|H(e^{j\omega})\right| = 1$.

7.69 (a) $G(z) = H(z)A(z)$, where $A(z)$ is an allpass function. Then, $g[0] = \lim_{z \rightarrow \infty} G(z)$.

Hence,

$|g[0]| = \left| \lim_{z \rightarrow \infty} G(z) \right| = \left| \lim_{z \rightarrow \infty} H(z)A(z) \right| = \left| \lim_{z \rightarrow \infty} H(z) \right| \left| \lim_{z \rightarrow \infty} A(z) \right| \leq \left| \lim_{z \rightarrow \infty} H(z) \right|$
 $\leq \left| \lim_{z \rightarrow \infty} H(z) \right|$ because $\left| \lim_{z \rightarrow \infty} A(z) \right| < 1$ because of Property 2 of stable allpass function (see Eq. (7.20)). Hence, $|g[0]| \leq |h[0]|$.

(b) If λ_1 is a zero of $H(z)$, then $|\lambda_1| < 1$, since $H(z)$, is a minimum-phase causal stable transfer function. As $H(z)$ has all zeros inside the unit circle, we can write $H(z) = B(z)(1 - \lambda_1 z^{-1})$. It follows that $B(z)$ is also a minimum-phase causal transfer function.

Now consider the transfer function $F(z) = B(z)(\lambda_1^* - z^{-1}) = H(z) \left(\frac{\lambda_1^* - z^{-1}}{1 - \lambda_1 z^{-1}} \right)$. If

$h[n], b[n]$, and $f[n]$ denote, respectively, the inverse z -transforms of $H(z), B(z)$, and $F(z)$, then we get $h[n] = \begin{cases} b[0], & n = 0, \\ b[n] - \lambda_1 b[n-1], & n \geq 1, \end{cases}$ and

$f[n] = \begin{cases} \lambda_1^* b[0], & n = 0, \\ \lambda_1^* b[n] - b[n-1], & n \geq 1. \end{cases}$

Consider $\varepsilon = \sum_{n=0}^m |h[n]|^2 - \sum_{n=0}^m |f[n]|^2 = |b[0]|^2 - |\lambda_1^*|^2 |b[0]|^2 + \sum_{n=1}^m |h[n]|^2 - \sum_{n=1}^m |f[n]|^2$.

Now, $|h[n]|^2 = |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n]$, and

$|f[n]|^2 = |\lambda_1|^2 |b[n]|^2 + |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n]$. Hence,

$\varepsilon = |b[0]|^2 - |\lambda_1|^2 |b[0]|^2 + \sum_{n=1}^m \left(|b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 \right) - \sum_{n=1}^m \left(|\lambda_1|^2 |b[n]|^2 - |b[n-1]|^2 \right)$

$= (1 - |\lambda_1|^2) |b[m]|^2$. Since $|\lambda_1| < 1, \varepsilon > 0$, i.e., $\sum_{n=0}^m |h[n]|^2 > \sum_{n=0}^m |f[n]|^2$. Hence,

$\sum_{n=0}^m |h[n]|^2 \geq \sum_{n=0}^m |g[n]|^2$.

7.70 $H(z) = \frac{(2z+3)(4z-1)}{(z+0.4)(z-0.6)}$ has a zero at $z = \frac{3}{2}$ which is outside the unit circle and is thus a non-minimum phase transfer function. To develop a minimum phase transfer function $G(z)$ such that $|G(e^{j\omega})| = |H(e^{j\omega})|$, we multiply with an allpass function $\frac{3z+2}{2z+3}$ and arrive at $G(z) = \frac{(3z+2)(4z-1)}{(z+0.4)(z-0.6)}$ which is minimum phase.

The first 5 impulse response of $H(z)$ are

$$\{h[n]\} = \{8 \quad 11.6 \quad 1.24 \quad 3.032 \quad 0.904\}, \quad 0 \leq n \leq 4, \text{ and}$$

The first 5 impulse response of $G(z)$ are

$$\{g[n]\} = \{12 \quad 7.4 \quad 2.36 \quad 2.248 \quad 1.016\}, \quad 0 \leq n \leq 4.$$

| m | 0 | 1 | 2 | 3 | 4 |
|-------------------------|-----|--------|----------|----------|----------|
| $\sum_{n=0}^m h[n] ^2$ | 64 | 198.56 | 200.0976 | 209.2906 | 210.1078 |
| $\sum_{n=0}^m g[n] ^2$ | 144 | 198.76 | 204.3296 | 209.3831 | 210.4154 |

It follows from the above that $\sum_{n=0}^m |g[n]|^2 > \sum_{n=0}^m |h[n]|^2$ for $m \geq 1$.

7.71 See Example 7.14.

(a) $H_{BS}(z) = \frac{1}{4}(1+z^{-2})^2$. Thus, $H_{BP}(z) = z^{-2} - \frac{1}{4}(1+z^{-2})^2 = -\frac{1}{4}(1-z^{-2})^2$.

(b) $H_{BS}(z) = \frac{1}{16}(1+z^{-2})(-1+6z^{-2}-z^{-4})$. Thus,
 $H_{BP}(z) = z^{-4} - \frac{1}{16}(1+z^{-2})(-1+6z^{-2}-z^{-4}) = \frac{1}{16}(1-4z^{-2}+6z^{-4}-4z^{-6}+z^{-8})$
 $= \frac{1}{16}(1-z^{-2})^4$.

(c) $H_{BS}(z) = \frac{1}{32}(1+z^{-2})^2(-3+14z^{-2}-3z^{-4})$. Thus,
 $H_{BP}(z) = z^{-4} - \frac{1}{32}(1+z^{-2})^2(-3+14z^{-2}-3z^{-4}) = \frac{1}{32}(3-8z^{-2}+10z^{-4}-8z^{-6}+3z^{-8})$.

7.72 $H_0(z) = A_0(z) + A_1(z)$, and $H_1(z) = A_0(z) - A_1(z)$, where $A_0(z)$ and $A_1(z)$ are allpass functions of orders M and N , respectively. Hence, the orders of $H_0(z)$ and $H_1(z)$ are $M+N$. Now, we can write $A_0(z) = \frac{z^{-M}D_0(z^{-1})}{D_0(z)}$ and

$A_1(z) = \frac{z^{-N} D_1(z^{-1})}{D_1(z)}$. Then,

$$H_0(z) = \frac{P(z)}{D(z)} = \frac{z^{-M} D_0(z^{-1}) D_1(z) + z^{-N} D_0(z) D_1(z^{-1})}{D_0(z) D_1(z)} \text{ and}$$

$$H_1(z) = \frac{Q(z)}{D(z)} = \frac{z^{-M} D_0(z^{-1}) D_1(z) - z^{-N} D_0(z) D_1(z^{-1})}{D_0(z) D_1(z)}. \text{ Since } P(z) \text{ is of degree}$$

$$M + N \text{ and } z^{-(M+N)} P(z^{-1}) = z^{-(M+N)} \left[z^M D_0(z) D_1(z^{-1}) + z^N D_0(z^{-1}) D_1(z) \right] \\ = z^{-M} D_0(z^{-1}) D_1(z) + z^{-N} D_0(z) D_1(z^{-1}) = P(z). \text{ Hence, } P(z) \text{ is symmetric.}$$

Similarly, one can show that $Q(z)$ is anti-symmetric.

7.73 $H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)]$ and $H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)]$. Thus,

$$H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = \frac{1}{4}[A_0(z) + A_1(z)][A_0(z^{-1}) + A_1(z^{-1})] \\ + \frac{1}{4}[A_0(z) - A_1(z)][A_0(z^{-1}) - A_1(z^{-1})] \\ = \frac{1}{4}[A_0(z)A_0(z^{-1}) + A_0(z)A_1(z^{-1}) + A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] \\ + \frac{1}{4}[A_0(z)A_0(z^{-1}) - A_0(z)A_1(z^{-1}) - A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] \\ = \frac{1}{2}[A_0(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] = 1. \text{ Thus, } \left| H_0(e^{j\omega}) \right|^2 + \left| H_1(e^{j\omega}) \right|^2 = 1$$

implying that $H_0(z)$ and $H_1(z)$ form a power-complementary pair.

7.74 $\left| H_0(e^{j\omega}) \right|^2 = \frac{1}{4} \left\{ A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_0^*(e^{j\omega}) + A_0(e^{j\omega})A_1^*(e^{j\omega}) \right. \\ \left. + A_1(e^{j\omega})A_1^*(e^{j\omega}) \right\}$. Since $A_0(z)$ and $A_1(z)$ are allpass functions,

$$A_0(e^{j\omega}) = e^{j\varphi_0(\omega)} \text{ and } A_1(e^{j\omega}) = e^{j\varphi_1(\omega)}. \text{ Therefore,}$$

$$\left| H_0(e^{j\omega}) \right|^2 = \frac{1}{4} \left\{ 2 + e^{j(\varphi_0(\omega) - \varphi_1(\omega))} + e^{-j(\varphi_0(\omega) - \varphi_1(\omega))} \right\} \leq 1 \text{ as maximum values of } \\ e^{j(\varphi_0(\omega) - \varphi_1(\omega))} \text{ and } e^{-j(\varphi_0(\omega) - \varphi_1(\omega))} \text{ are 1. } H_0(z) \text{ is stable since } A_0(z) \text{ and } A_1(z) \\ \text{ are stable allpass functions. Hence, } H_0(z) \text{ is BR.}$$

7.75 $H(z) = \frac{1}{M} \sum_{k=0}^{M-1} A_k(z)$. Thus, $H(z)H(z^{-1}) = \frac{1}{M^2} \sum_{k=0}^{M-1} \sum_{r=0}^{M-1} A_k(z)A_r(z^{-1})$. Hence,

$$\left| H(e^{j\omega}) \right|^2 = \frac{1}{M^2} \sum_{r=0}^{M-1} \sum_{k=0}^{M-1} e^{j(\varphi_k(\omega) - \varphi_r(\omega))} \leq 1. \text{ Also, } H(z) \text{ is stable since}$$

$A_i(z)$, $0 \leq i \leq M-1$, are stable allpass functions. Hence, $H(z)$ is BR.

7.76 $H_{BP}(z) = \frac{1}{2}[1 - A(z)]$ and $H_{BS}(z) = \frac{1}{2}[1 + A(z)]$ where

$$A(z) = \frac{\alpha - \beta(1 + \alpha)z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}$$

is an allpass function. Note

$H_{BP}(z) + H_{BS}(z) = A(z)$ and from the solution of Problem 7.73,

$|H_{BP}(e^{j\omega})|^2 + |H_{BS}(e^{j\omega})|^2 = 1$. Hence, $H_{BP}(z)$ and $H_{BS}(z)$ are doubly-complementary pair.

7.77 $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K$. On the unit circle, this reduces to

$$H(e^{j\omega})H(e^{-j\omega}) + H(-e^{j\omega})H(-e^{-j\omega}) = K, \text{ or, equivalently,}$$

$|H(e^{j\omega})|^2 + |H(-e^{j\omega})|^2 = K$, as $H(z)$ is a real-coefficient transfer function. Now,
 $|H(-e^{j\omega})|^2 = |H(e^{j(\pi+\omega)})|^2$. Hence, for $\omega = \pi/2$, the power-symmetric condition
reduces to $|H(e^{j\pi/2})|^2 + |H(e^{j(\pi+\pi/2)})|^2 = K$. Since $H(z)$ is a real-coefficient

transfer function, $|H(e^{j\omega})|^2$ is an even function of ω , and thus,

$$|H(e^{j\pi/2})|^2 = |H(e^{j(2\pi-\pi/2)})|^2. \text{ As a result, } 2|H(e^{j\pi/2})|^2 = K, \text{ from which we}$$

obtain $10 \log_{10} 2 + 20 \log_{10} |H(e^{j\pi/2})| = 10 \log_{10} K$, or,

$$20 \log_{10} |H(e^{j\pi/2})| = 10 \log_{10} K - 3 \text{ dB.}$$

7.78 $H(z) = A_0(z^2) + z^{-1}A_1(z^2)$. Therefore,

$$\begin{aligned} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) &= [A_0(z^2) + z^{-1}A_1(z^2)][A_0(z^{-2}) + zA_1(z^{-2})] \\ &+ [A_0((-z)^2) - z^{-1}A_1((-z)^2)][A_0((-z)^{-2}) - zA_1((-z)^{-2})] \\ &= A_0(z^2)A_0(z^{-2}) + z^{-1}A_1(z^2)A_0(z^{-2}) + zA_0(z^2)A_1(z^{-2}) + A_1(z^2)A_1(z^{-2}) \\ &+ A_0(z^2)A_0(z^{-2}) - z^{-1}A_1(z^2)A_0(z^{-2}) - zA_0(z^2)A_1(z^{-2}) + A_1(z^2)A_1(z^{-2}) = 4, \text{ as} \\ A_0(z^2)A_0(z^{-2}) &= A_1(z^2)A_1(z^{-2}) = 1. \end{aligned}$$

$$7.79 \quad H(z) = \frac{-0.1 + 0.5z^{-1} + 0.05z^{-2} + 0.05z^{-3} + 0.5z^{-4} - 0.1z^{-5}}{1 + 0.1z^{-2} - 0.2z^{-4}} = \frac{1}{2}[A(z^2) + z^{-1}],$$

where $A(z) = \frac{-0.2 + 0.1z^{-1} + z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$ is a stable allpass function. Thus,

$$\begin{aligned} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) &= \frac{1}{4}[A(z^2) + z^{-1}][A(z^{-2}) + z] \\ &+ \frac{1}{4}[A(z^2) - z^{-1}][A(z^{-2}) - z] = 1. \end{aligned}$$

$$7.80 \quad (a) \quad H_a(z) = 1 - 2z^{-1} + 4.5z^{-2} + 6z^{-3} + z^{-4} + 0.5z^{-5}. \text{ Thus, } H_a(z)H_a(z^{-1}) \\ = (1 - 2z^{-1} + 4.5z^{-2} + 6z^{-3} + z^{-4} + 0.5z^{-5})(1 - 2z + 4.5z^2 + 6z^3 + z^4 + 0.5z^5) \\ = 0.5z^5 + 6.25z^3 + 22.5z + 62.5 + 22.5z^{-1} + 6.25z^{-3} + 0.5z^{-5}.$$

$$\begin{aligned} \text{Next, we compute } H_a(-z)H_a(-z^{-1}) &= \\ &= (1 + 2z^{-1} + 4.5z^{-2} - 6z^{-3} + z^{-4} - 0.5z^{-5})(1 + 2z + 4.5z^2 - 6z^3 + z^4 - 0.5z^5) \\ &= -0.5z^5 - 6.25z^3 - 22.5z + 62.5 - 22.5z^{-1} - 6.25z^{-3} - 0.5z^{-5}. \end{aligned}$$

Hence, $H_a(z)H_a(z^{-1}) + H_a(-z)H_a(-z^{-1}) = 125$.

$$(b) \quad H_b(z) = 1 + \frac{1}{2}z^{-1} + \frac{15}{4}z^{-2} - z^{-4} + 2z^{-5}. \text{ Thus, } H_b(z)H_b(z^{-1}) \\ = \left(1 + \frac{1}{2}z^{-1} + \frac{15}{4}z^{-2} - z^{-4} + 2z^{-5}\right) \left(1 + \frac{1}{2}z + \frac{15}{4}z^2 - z^4 + 2z^5\right) \\ = 2z^5 + 7z^3 + 0.375z + 20.3125 + 0.375z^{-1} + 7z^{-3} + 2z^{-5}.$$

$$\begin{aligned} \text{Next, we compute } H_b(-z)H_b(-z^{-1}) &= \\ &= (1 - 0.5z^{-1} + 3.75z^{-2} - z^{-4} - 2z^{-5})(1 - 0.5z + 3.75z^2 - z^4 - 2z^5) \\ &= -2z^5 - 7z^3 - 0.375z + 20.3125 - 0.375z^{-1} - 7z^{-3} - 2z^{-5}. \end{aligned}$$

Hence, $H_b(z)H_b(z^{-1}) + H_b(-z)H_b(-z^{-1}) = 40.625$.

$$7.81 \quad H(z)H(z^{-1}) = a^2(1 + bz^{-1})(1 + bz) = a^2bz + a^2(1 + b^2) + a^2bz^{-1} = cz + d + cz^{-1}.$$

Thus, $c = a^2b$ and $d = a^2(1 + b^2)$. Now, $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$

$cz + d + cz^{-1} - cz + d - cz^{-1} = 2d$. Therefore, $2d = 2a^2(1 + b^2) = 1$. This

condition is satisfied by $a = \frac{1}{\sqrt{2(1+b^2)}}$. For $b = 1$, $a = \frac{1}{2}$. Other solutions include

$$b = -1, a = \frac{1}{2}, \text{ and } b = 10, a = \frac{1}{\sqrt{10}}.$$

Since $H(z)$ is a first-order causal FIR transfer function, $G(z) = -z^{-1}H(-z^{-1})$ is also a first-order causal FIR transfer function. Now,

$H(z)H(z^{-1}) + G(z)G(z^{-1}) = H(z)H(z^{-1}) + [-z^{-1}H(-z^{-1})][-zH(-z)]$
 $= H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1$. Hence, $H(z)$ and $G(z)$ are power complementary.

7.82 $H(z)H(z^{-1}) = (cz + d + cz^{-1})[d_2z^2 + d_1(1 + d_2)z + (1 + d_1^2 + d_2^2) + d_1(1 + d_2)z^{-1} + d_2z^{-2}]$

Thus, $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2[cd_1(1 + d_2)z^2 + dd_2z^2 + 2cd_1(1 + d_2) + d(1 + d_1^2 + d_2^2) + dd_2z^{-2} + cd_1(1 + d_2)z^{-2}] = 1$. Hence, we require

$dd_2 + cd_1(1 + d_2) = 0$ and $2cd_1(1 + d_2) + d(1 + d_1^2 + d_2^2) = 1$. Solving these two

equations we arrive at $c = \frac{d_2}{d_2(1 + d_2)(2d_2 - 1 - d_1^2 - d_2^2)}$, and

$$d = -\frac{1}{2d_2 - 1 - d_1^2 - d_2^2}. \text{ For } d_1 = d_2 = 1, \text{ we get } c = -\frac{1}{2} \text{ and } d = 1.$$

Since $H(z)$ is a third-order causal FIR transfer function, $G(z) = -z^{-3}H(-z^{-1})$ is also a third-order causal FIR transfer function. Now,

$H(z)H(z^{-1}) + G(z)G(z^{-1}) = H(z)H(z^{-1}) + [-z^{-3}H(-z^{-1})][-z^3H(-z)]$
 $= H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1$. Hence, $H(z)$ and $G(z)$ are power complementary.

7.83 $H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)]$ and $H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)]$, where $A_0(z)$ and $A_1(z)$

are stable allpass transfer functions. From these two equations we obtain

$H_0(z) + H_1(z) = A_0(z)$, and $H_0(z) - H_1(z) = A_1(z)$. Moreover, we have

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1. \text{ Choose } G_0(z) = H_0^2(z) \text{ and } G_1(z) = -H_1^2(z).$$

$$\text{Hence, } \left|G_0(e^{j\omega})\right| + \left|G_1(e^{j\omega})\right| = \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1.$$

7.84 (a) $H_1(z) = \frac{1}{4}(1 + 3z^{-1})$. Therefore, $H_1(z)H_1(z^{-1}) = \frac{1}{16}(1 + 3z^{-1})(1 + 3z) \Rightarrow$

$$\left|H_1(e^{j\omega})\right|^2 = \frac{10 + 6\cos\omega}{16}. \text{ Thus, } \frac{d}{d\omega}\left(\left|H_1(e^{j\omega})\right|^2\right) = -\frac{3}{8}\sin\omega < 0 \text{ for } 0 \leq \omega < \pi.$$

Thus, $\left|H_1(e^{j\omega})\right|$ is a monotonically decreasing function of ω . The maximum value of $\left|H_1(e^{j\omega})\right| = 1$ is at $\omega = 0$, and the minimum value is at $\omega = \pi$. Hence, $H_1(z)$ is BR.

(b) $H_2(z) = \frac{1}{2.2}(1 - 1.2z^{-1})$. Therefore, $H_2(z)H_2(z^{-1}) = \frac{1}{4.84}(1 - 1.2z^{-1})(1 - 1.2z) \Rightarrow$

$$\left|H_2(e^{j\omega})\right|^2 = \frac{2.44 - 2.4 \cos \omega}{4.84}. \text{ Thus, } \frac{d}{d\omega} \left(\left|H_2(e^{j\omega})\right|^2 \right) = \frac{2.4}{4.84} \sin \omega > 0 \text{ for}$$

$0 \leq \omega < \pi$. Thus, $\left|H_2(e^{j\omega})\right|$ is a monotonically increasing function of ω . The maximum value of $\left|H_2(e^{j\omega})\right| = 1$ is at $\omega = \pi$, and the minimum value is at $\omega = 0$. Hence, $H_2(z)$ is BR.

$$(c) \quad H_3(z) = \frac{(1 + \alpha z^{-1})(1 - \beta z^{-1})}{(1 + \alpha)(1 + \beta)} = G_1(z)G_2(z), \text{ where } G_1(z) = \frac{1 + \alpha z^{-1}}{1 + \alpha} \text{ and}$$

$$G_2(z) = \frac{1 - \beta z^{-1}}{1 + \beta}. \text{ Now, } \left|G_1(e^{j\omega})\right|^2 = \frac{1 + \alpha^2 + 2\alpha \cos \omega}{(1 + \alpha)^2}. \text{ Thus,}$$

$$\frac{d}{d\omega} \left(\left|G_1(e^{j\omega})\right|^2 \right) = -\frac{2\alpha \sin \omega}{(1 + \alpha)^2} < 0 \text{ for } 0 \leq \omega < \pi \text{ as } \alpha > 0. \text{ As a result, } \left|G_1(e^{j\omega})\right|$$

is a monotonically decreasing function of ω . The maximum value of $\left|G_1(e^{j\omega})\right| = 1$ is at $\omega = 0$, and the minimum value is at $\omega = \pi$. Hence, $G_1(z)$ is BR.

$$\text{Likewise, } \left|G_2(e^{j\omega})\right|^2 = \frac{1 + \beta^2 - 2\beta \cos \omega}{(1 + \beta)^2}. \text{ Thus, } \frac{d}{d\omega} \left(\left|G_2(e^{j\omega})\right|^2 \right) = \frac{2\beta \sin \omega}{(1 + \beta)^2} > 0$$

for $0 \leq \omega < \pi$ as $\beta > 0$. Thus, $\left|H_2(e^{j\omega})\right|$ is a monotonically increasing function of ω . The maximum value of $\left|G_2(e^{j\omega})\right| = 1$ is at $\omega = \pi$, and the minimum value is at $\omega = 0$. Hence, $G_2(z)$ is BR. Therefore, $H_3(z) = G_1(z)G_2(z)$ is also BR.

$$(d) \quad H_4(z) = \frac{(1 - 0.3z^{-1})(1 + 0.2z^{-1})(1 - 0.5z^{-1})}{2.34} = \left(\frac{1 - 0.3z^{-1}}{1.3} \right) \left(\frac{1 + 0.2z^{-1}}{1.2} \right) \left(\frac{1 - 0.5z^{-1}}{1.5} \right).$$

Since each individual factor on the right-hand side is BR, $H_4(z)$ is BR.

$$7.85 \quad (a) \quad H_1(z) = \frac{2.6 + 2.6z^{-1}}{4.2 + z^{-1}} = \frac{1}{2} \left(1 + \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \right) = \frac{1}{2} (A_0(z) + A_1(z)), \text{ where}$$

$$A_0(z) = 1 \text{ and } A_1(z) = \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \text{ are stable allpass transfer functions. Therefore,}$$

$H_1(z)$ is BR (See solution of Problem 7.74).

$$(b) H_2(z) = \frac{1.6 - 1.6z^{-1}}{4.2 + z^{-1}} = \frac{1}{2} \left(1 - \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \right) = \frac{1}{2} (A_0(z) - A_1(z)), \text{ where}$$

$$A_0(z) = 1 \text{ and } A_1(z) = \frac{1 + 4.2z^{-1}}{4.2 + z^{-1}} \text{ are stable allpass transfer functions. Therefore,}$$

$H_2(z)$ is BR (See solution of Problem 7.74).

$$(c) H_3(z) = \frac{0.1(1 - z^{-2})}{1 + 0.4z^{-1} + 0.8z^{-2}} = \frac{1}{2} \left(1 - \frac{0.8 + 0.4z^{-1} + z^{-2}}{1 + 0.4z^{-1} + 0.8z^{-2}} \right)$$

$$= \frac{1}{2} (A_0(z) - A_1(z)), \text{ where } A_0(z) = 1 \text{ and } A_1(z) = \frac{0.8 + 0.4z^{-1} + z^{-2}}{1 + 0.4z^{-1} + 0.8z^{-2}} \text{ are stable}$$

allpass transfer functions. Therefore, $H_3(z)$ is BR (See solution of Problem 7.74).

$$(d) H_4(z) = \frac{4.5 + 2z^{-1} + 4.5z^{-2}}{5 + 2z^{-1} + 4z^{-2}} = \frac{1}{2} \left(1 + \frac{4 + 2z^{-1} + 5z^{-2}}{5 + 2z^{-1} + 4z^{-2}} \right) = \frac{1}{2} (A_0(z) + A_1(z)),$$

$$\text{where } A_0(z) = 1 \text{ and } A_1(z) = \frac{4 + 2z^{-1} + 5z^{-2}}{5 + 2z^{-1} + 4z^{-2}} \text{ are stable allpass transfer functions.}$$

Therefore, $H_4(z)$ is BR (See solution of Problem 7.74).

7.86 Since $A_1(z)$ and $A_2(z)$ are LBR, $|A_1(e^{j\omega})| = 1$ and $|A_2(e^{j\omega})| = 1$. Thus,

$$A_1(e^{j\omega}) = e^{j\phi_1(\omega)} \text{ and } A_2(e^{j\omega}) = e^{j\phi_2(\omega)}. \text{ Now, } A_1\left(\frac{1}{A_2(e^{j\omega})}\right) = A_1\left(e^{-j\phi_2(\omega)}\right)$$

$$\text{Thus, } |A_1\left(e^{-j\phi_2(\omega)}\right)| = 1. \text{ Thus, } A_1\left(\frac{1}{A_2(z)}\right) \text{ is LBR.}$$

$$\mathbf{7.87} \quad F(z) = z \left(\frac{G(z) + \alpha}{1 + \alpha G(z)} \right). \text{ Thus, } F(e^{j\omega}) = e^{j\omega} \left(\frac{G(e^{j\omega}) + \alpha}{1 + \alpha G(e^{j\omega})} \right) = e^{j\omega} \left(\frac{e^{j\phi(\omega)} + \alpha}{1 + \alpha e^{j\phi(\omega)}} \right)$$

$$\text{since } G(z) \text{ is LBR. Therefore, } |F(e^{j\omega})|^2 = \left| \frac{e^{j\phi(\omega)} + \alpha}{1 + \alpha e^{j\phi(\omega)}} \right|^2$$

$$= \frac{(\cos(\phi(\omega)) + \alpha)^2 + (\sin(\phi(\omega)))^2}{(1 + \alpha \cos(\phi(\omega)))^2 + (\alpha \sin(\phi(\omega)))^2} = \frac{1 + 2\alpha \cos(\phi(\omega)) + \alpha^2}{1 + 2\alpha \cos(\phi(\omega)) + \alpha^2} = 1. \text{ Let } z = \lambda \text{ be a pole}$$

$$\text{of } F(z). \text{ Then } G(z)|_{z=\lambda} = \left. \frac{F(z) - \alpha z}{z - \alpha F(z)} \right|_{z=\lambda} = -\frac{1}{\alpha}, \text{ or, } |G(\lambda)| = |1/\alpha|. \text{ If } |\alpha| < 1, \text{ then}$$

$|G(\lambda)| > 1$, which is satisfied by the LBR $G(z)$ if $|\lambda| < 1$. Hence, $F(z)$ is LBR. The order of $F(z)$ is same as that of $G(z)$.

$G(z)$ can be realized in the form of a two-pair constrained by the transfer function $F(z)$ as shown below:



To this end, we express $G(z)$ in terms of $F(z)$ arriving at

$$G(z) = \frac{-\alpha + z^{-1}F(z)}{1 - \alpha z^{-1}F(z)} = \frac{C + DF(z)}{A + BF(z)}, \text{ where } A, B, C, \text{ and } D \text{ are the chain}$$

parameters of the two-pair. Comparing the above two expressions we get $A = 1$, $B = -\alpha z^{-1}$, $C = -\alpha$, and $D = z^{-1}$. The corresponding transfer parameters are given by $t_{11} = -\alpha$, $t_{21} = 1$, $t_{12} = (1 - \alpha^2)z^{-1}$ and $t_{22} = \alpha z^{-1}$.

7.88 Let $F(z) = G\left(\frac{1}{A(z)}\right)$. Now, $A(z)$ being LBR, $A(e^{j\omega}) = e^{j\phi(\omega)}$. Thus,

$$F(e^{j\omega}) = G\left(\frac{1}{A(e^{j\omega})}\right) = G(e^{-j\phi(\omega)}). \text{ Since } G(z) \text{ is a BR function, } |G(e^{j\omega})| \leq 1.$$

$$\text{Hence, } |F(e^{j\omega})| = \left| G\left(\frac{1}{A(e^{j\omega})}\right) \right| \leq 1.$$

Let $z = \xi$ be a pole of $F(z)$. Hence, $F(z)$ will be a BR function if $|\xi| < 1$. Let $z = \lambda$ be a pole of $G(z)$. Then this pole is mapped to the location $z = \xi$ of $F(z)$ by the relation $\frac{1}{A(z)}\Big|_{z=\xi} = \lambda$, or $A(\xi) = \frac{1}{\lambda}$. Hence, $|A(\xi)| = \frac{1}{|\lambda|} > 1$ because of Eq.

(7.20). This implies, $|\lambda| < 1$. Thus, $G(z)$ is a BR function.

7.89 (a) $H(z) = \frac{2.6(1+z^{-1})}{4.2+z^{-1}}$, $G(z) = \frac{1.6(1-z^{-1})}{4.2+z^{-1}}$. Now,

$$H(z) + G(z) = \frac{2.6(1+z^{-1})}{4.2+z^{-1}} + \frac{1.6(1-z^{-1})}{4.2+z^{-1}} = \frac{4.2+z^{-1}}{4.2+z^{-1}} = 1. \text{ Next,}$$

$$\begin{aligned} H(z)H(z^{-1}) + G(z)G(z^{-1}) &= \frac{2.6(1+z^{-1})}{4.2+z^{-1}} \cdot \frac{2.6(1+z)}{4.2+z} + \frac{1.6(1-z^{-1})}{4.2+z^{-1}} \cdot \frac{1.6(1-z)}{4.2+z} \\ &= \frac{(6.76z + 13.52 + 6.76z^{-1}) + (-2.56z + 5.12 - 2.56z^{-1})}{4.2z^{-1} + 18.64 + 4.2z^{-1}} = \frac{4.2z^{-1} + 18.64 + 4.2z^{-1}}{4.2z^{-1} + 18.64 + 4.2z^{-1}} = 1. \end{aligned}$$

Thus, $\left|H(e^{j\omega})\right|^2 + \left|G(e^{j\omega})\right|^2 = 1$. Hence, $H(z)$ and $G(z)$ are both allpass-complementary and power complementary. As a result, they are doubly complementary.

$$(b) \quad H(z) = \frac{0.1(1-z^{-2})}{1+0.4z^{-1}+0.8z^{-2}}, \quad G(z) = \frac{0.9+0.4z^{-1}+0.9z^{-2}}{1+0.4z^{-1}+0.8z^{-2}}. \quad \text{Now,}$$

$$H(z) + G(z) = \frac{0.1(1-z^{-2})}{1+0.4z^{-1}+0.8z^{-2}} + \frac{0.9+0.4z^{-1}+0.9z^{-2}}{1+0.4z^{-1}+0.8z^{-2}} = \frac{0.8+0.4z^{-1}+z^{-2}}{1+0.4z^{-1}+0.8z^{-2}}$$

implying that and are allpass complementary. Next, $H(z)H(z^{-1}) + G(z)G(z^{-1})$

$$\begin{aligned} &= \frac{0.1(1-z^{-2})}{1+0.4z^{-1}+0.8z^{-2}} \cdot \frac{0.1(1-z^2)}{1+0.4z+0.8z^2} + \frac{0.9+0.4z^{-1}+0.9z^{-2}}{1+0.4z^{-1}+0.8z^{-2}} \cdot \frac{0.9+0.4z+0.9z^2}{1+0.4z+0.8z^2} \\ &= \frac{(-0.01z^2+0.02-0.01z^{-2})+(0.81z^2+0.72z+1.78+0.72z^{-1}+0.81z^{-2})}{0.8z^2+0.72z+1.8+0.72z^{-1}+0.8z^{-2}} \\ &= \frac{0.8z^2+0.72z+1.8+0.72z^{-1}+0.8z^{-2}}{0.8z^2+0.72z+1.8+0.72z^{-1}+0.8z^{-2}} = 1. \quad \text{Hence } H(z) \text{ and } G(z) \text{ are also} \end{aligned}$$

power complementary. As a result, they are doubly complementary.

$$7.90 \quad (a) \quad H_a(z) = \frac{2(2+z^{-1}+2z^{-2})}{5+2z^{-1}+3z^{-2}} = \frac{1}{2}[1+A(z)] \quad \text{where } A(z) = \frac{3+2z^{-1}+5z^{-2}}{5+2z^{-1}+3z^{-2}} \text{ is an}$$

allpass function. Hence, the power complementary transfer function of $H_a(z)$ is

$$\text{given by } G_a(z) = \frac{1}{2}[1-A(z)] = \frac{1}{2}\left[1 - \frac{3+2z^{-1}+5z^{-2}}{5+2z^{-1}+3z^{-2}}\right] = \frac{1-z^{-2}}{5+2z^{-1}+3z^{-2}}.$$

$$(b) \quad H_b(z) = \frac{3+7.5z^{-1}+7.5z^{-2}+3z^{-3}}{8+8z^{-1}+4z^{-2}+z^{-3}} = \frac{3+7.5z^{-1}+7.5z^{-2}+3z^{-3}}{2(1+0.5z^{-1})(4+2z^{-1}+z^{-2})}$$

$$= \frac{1}{2}[A_0(z) + A_1(z)] \quad \text{where } A_0(z) = \frac{1+2z^{-1}+4z^{-2}}{4+2z^{-1}+z^{-2}} \text{ and } A_1(z) = \frac{0.5+z^{-1}}{1+0.5z^{-1}} \text{ are}$$

allpass functions. Hence, the power complementary transfer function of $H_b(z)$ is

$$\text{given by } G_b(z) = \frac{1}{2}[A_0(z) - A_1(z)] = \frac{-1-2.5z^{-1}+2.5z^{-2}+z^{-3}}{8+8z^{-1}+4z^{-2}+z^{-3}}.$$

7.91 From Eq. (7.126) we have $X_1 = AY_2 + BX_2$, $Y_1 = CY_2 + DX_2$. From the first equation, $Y_2 = \frac{1}{A}X_1 - \frac{B}{A}X_2$. Substituting this in the second equation we get

$Y_1 = C\left(\frac{1}{A}X_1 - \frac{B}{A}X_2\right) + DX_2 = \frac{C}{A}X_1 + \frac{AD-BC}{A}X_2$. Comparing the last two equations with Eq. (7.123) we arrive at $t_{11} = \frac{C}{A}, t_{12} = \frac{AD-BC}{A}, t_{21} = \frac{1}{A}, t_{22} = -\frac{B}{A}$.

From Eq. (7.123) we have $Y_1 = t_{11}X_1 + t_{12}X_2, Y_2 = t_{21}X_1 + t_{22}X_2$. From the second equation we get $X_1 = -\frac{t_{22}}{t_{21}}X_2 + \frac{1}{t_{21}}Y_2$. Substituting this in the first equation we get

$$Y_1 = t_{11}\left(-\frac{t_{22}}{t_{21}}X_2 + \frac{1}{t_{21}}Y_2\right) + t_{12}X_2 = \frac{t_{11}}{t_{21}}Y_2 + \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}X_2.$$

Comparing the last two equations with Eq. (7.126) we arrive at $A = \frac{1}{t_{21}}, B = -\frac{t_{22}}{t_{21}},$

$$C = \frac{t_{11}}{t_{21}}, D = \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}.$$

7.92 From Eq. (7.128a) we note $t_{12} = \frac{AD-BC}{A}$ and $t_{21} = \frac{1}{A}$. Hence, $t_{12} = t_{21}$ imply $AD-BC = 1$.

7.93 $\begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} k_1 & (1-k_1^2)z^{-1} \\ 1 & -k_1z^{-1} \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}, \begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} = \begin{bmatrix} k_2 & (1-k_2^2)z^{-1} \\ 1 & -k_2z^{-1} \end{bmatrix} \begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix}$, where $\begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix} = \begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix}$. The transfer matrices of the two two-pairs are given by

$$\tau_1 = \begin{bmatrix} k_1 & (1-k_1^2)z^{-1} \\ 1 & -k_1z^{-1} \end{bmatrix} \text{ and } \tau_2 = \begin{bmatrix} k_2 & (1-k_2^2)z^{-1} \\ 1 & -k_2z^{-1} \end{bmatrix}.$$

The corresponding chain matrices are obtained using Eq. (7.128b) and are given by $\Gamma_1 = \begin{bmatrix} 1 & k_1z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$ and

$$\Gamma_2 = \begin{bmatrix} 1 & k_2z^{-1} \\ k_2 & z^{-1} \end{bmatrix}.$$

Therefore, the chain matrix of the Γ -cascade is given by $\Gamma_1\Gamma_2 = \begin{bmatrix} 1 & k_1z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_2z^{-1} \\ k_2 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1+k_1k_2z^{-1} & k_2z^{-1}+k_1z^{-1} \\ k_1+k_2z^{-1} & k_1k_2z^{-1}+z^{-2} \end{bmatrix}$. Next using Eq.

(7.128a) we arrive at the transfer matrix of the Γ -cascade as

$$\tau = \begin{bmatrix} \frac{k_1 + k_2 z^{-1}}{1 + k_1 k_2 z^{-1}} & \frac{z^{-2}(1 - k_1^2)(1 - k_2^2)}{1 + k_1 k_2 z^{-1}} \\ \frac{1}{1 + k_1 k_2 z^{-1}} & \frac{z^{-1}(k_2 + k_1 z^{-1})}{1 + k_1 k_2 z^{-1}} \end{bmatrix}$$

7.94 $\begin{bmatrix} X_1' \\ Y_1' \end{bmatrix} = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} Y_2' \\ X_2' \end{bmatrix} \begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix} = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} Y_2'' \\ X_2'' \end{bmatrix}$, where $\begin{bmatrix} Y_2' \\ X_2' \end{bmatrix} = \begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix}$. The

chain matrices of the two two-pairs are given by $\Gamma_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$ and

$$\Gamma_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}. \text{ The corresponding transfer matrices are obtained using Eq.}$$

(7.128a) and are given by $\tau_1 = \begin{bmatrix} k_1 & (1 - k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}$ and $\tau_2 = \begin{bmatrix} k_2 & (1 - k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}$. The

transfer matrix τ of the τ -cascade is therefore given by

$$\begin{aligned} \tau_2 \tau_1 &= \begin{bmatrix} k_2 & (1 - k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix} \begin{bmatrix} k_1 & (1 - k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix} \\ &= \begin{bmatrix} k_1 k_2 + z^{-1}(1 - k_2^2) & z^{-1} k_2 (1 - k_1^2) - z^{-2} k_1 (1 - k_2^2) \\ k_1 - k_2 z^{-1} & z^{-1}(1 - k_1^2) + z^{-1} k_2 (1 - k_1^2)z^{-2} \end{bmatrix}. \end{aligned}$$

Using Eq. (7.128b) we thus

arrive at the chain matrix of the τ -cascade as

$$\Gamma = \begin{bmatrix} \frac{1}{k_1 - k_2 z^{-1}} & \frac{-(k_1 k_2 z^{-1} + 1 - k_1^2)z^{-1}}{k_1 - k_2 z^{-1}} \\ \frac{k_1 k_2 + z^{-1}(1 - k_2^2)}{k_1 - k_2 z^{-1}} & \frac{z^{-2}}{k_1 - k_2 z^{-1}} \end{bmatrix}$$

7.95 (a) Analyzing Figure P7.10(a) we obtain $Y_2 = X_1 - k_m z^{-1} X_2$ and

$$Y_1 = k_m Y_2 + z^{-1} X_2 = k_m (X_1 - k_m z^{-1} X_2) z^{-1} X_2 = k_m X_1 + (1 - k_m^2) z^{-1} X_2.$$

Hence, the transfer matrix is given by $\tau = \begin{bmatrix} k_m & (1 - k_m^2)z^{-1} \\ 1 & -k_m z^{-1} \end{bmatrix}$. Using Eq. (7.128b)

we then arrive at the chain matrix $\Gamma = \begin{bmatrix} 1 & k_m z^{-1} \\ k_m & z^{-1} \end{bmatrix}$.

(b) Labeling V_1 the output variable of the top left adder connected to input X_1 we then analyze Figure P7.10(b) and obtain $V_1 = k_m(X_1 - z^{-1}X_2)$, $Y_1 = V_1 + X_1 = (1 + k_m)X_1 - k_m z^{-1}X_2$, and $Y_2 = V_1 + z^{-1}X_2 = k_m X_1 + (1 - k_m)z^{-1}X_2$. Hence, the transfer matrix of the two-pair is given by $\tau = \begin{bmatrix} 1 + k_m & -k_m z^{-1} \\ k_m & (1 - k_m)z^{-1} \end{bmatrix}$. Using

Eq. (7.128b) we then arrive at the chain matrix $\Gamma = \begin{bmatrix} \frac{1}{k_m} & -\frac{(1 - k_m)z^{-1}}{k_m} \\ \frac{1 - k_m}{k_m} & -\frac{1}{k_m} \end{bmatrix}$.

7.96 Solving $G(z) = \frac{H(z) - k_m}{z^{-1}[1 - k_m H(z)]}$ for $H(z)$ we get $H(z) = \frac{k_m + z^{-1}G(z)}{1 + k_m z^{-1}G(z)}$.

For the constrained two-pair $H(z) = \frac{C + D \cdot G(z)}{A + B \cdot G(z)}$. Comparing the last two

equations we thus get $C = k_m, D = z^{-1}, A = 1, B = k_m z^{-1}$. Substituting these values of the chain parameters in Eq. (7.128a) we get

$$t_{11} = \frac{C}{A} = k_m, t_{12} = \frac{AD - BC}{A} = z^{-1}(1 - k_m^2), t_{21} = \frac{1}{A} = 1, t_{22} = -\frac{B}{A} = -k_m z^{-1}.$$

7.97 (a) $H(z) = \frac{\alpha + z^{-1}G(z)}{1 + \alpha z^{-1}G(z)}$. For the constrained two-pair $H(z) = \frac{C + D \cdot G(z)}{A + B \cdot G(z)}$.

Comparing the last two equations we thus get $C = \alpha, D = z^{-1}, A = 1, B = \alpha z^{-1}$.

(b)

7.98 From the results of Problem 7.95, Part (a), we observe that the chain matrix of the

i -th lattice two-pair is given by $\Gamma_i = \begin{bmatrix} 1 & k_i z^{-1} \\ k_i & z^{-1} \end{bmatrix}, i = 1, 2, 3$. Thus, the chain matrix

of the cascade of the three lattice two-pairs is given by

$$\begin{aligned} \Gamma_{\text{cascade}} &= \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_3 z^{-1} \\ k_3 & z^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 + k_2 k_3 z^{-1} + k_1 z^{-1}(k_2 + k_3 z^{-1}) & k_3 z^{-1} + k_2 z^{-1} + k_1 z^{-1}(k_2 k_3 + z^{-2}) \\ k_1(1 + k_2 k_3 z^{-1}) + z^{-1}(k_2 + k_3 z^{-1}) & k_1(k_3 z^{-1} + k_2 z^{-1}) + z^{-1}(k_2 k_3 + z^{-2}) \end{bmatrix}. \end{aligned}$$

From Eq. (7.135a) we obtain $A_3(z) = \frac{C + D}{A + B}$

$$= \frac{1 + (k_2k_3 + k_1k_2 + k_3)z^{-1} + (k_1k_3 + k_2 + k_1k_2k_3)z^{-2} + k_1z^{-3}}{k_1 + (k_1k_3 + k_2 + k_1k_2k_3)z^{-1} + (k_2k_3 + k_1k_2 + k_3)z^{-2} + z^{-3}}$$
 which is seen to be an allpass function.

7.99 Let

$$D(z) = 1 + d_1z^{-1} + d_2z^{-2} = (1 - \lambda_1z^{-1})(1 - \lambda_2z^{-1}) = 1 - (\lambda_1 + \lambda_2)z^{-1} + \lambda_1\lambda_2z^{-2}.$$

Thus, $d_2 = \lambda_1\lambda_2$ and $d_1 = -(\lambda_1 + \lambda_2)$. For stability, $|\lambda_i| < 1, i = 1, 2$. As a result,

$$|d_2| = |\lambda_1\lambda_2| < 1.$$

Case 1: Complex poles: $d_2 > 0$. In this case, $\lambda_2 = \lambda_1^*$. Now,

$$\lambda_1, \lambda_2 = \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2}. \text{ Hence, } \lambda_1 \text{ and } \lambda_2 \text{ will be complex, if } d_1^2 < 4d_2. \text{ In}$$

this case, $\lambda_1 = -\frac{d_1}{2} + \frac{j}{2}\sqrt{4d_2 - d_1^2}$. Thus, $|\lambda_1|^2 = \frac{1}{4}(d_1^2 + 4d_2 - d_1^2) = d_2 < 1$.

Consequently, if the poles are complex and $d_2 < 1$, then they are inside the unit circle.

Case 2: Real poles. In this case we get $-1 < \lambda_i < 1, i = 1, 2$. Since, $|\lambda_i| < 1$, it follows

$$\text{then } |d_1| < |\lambda_1| + |\lambda_2| < 2. \text{ Now, } -1 < \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < 1, \text{ or}$$

$\pm \sqrt{d_1^2 - 4d_2} < 2 + d_1$. It is not possible to satisfy the inequality on the right hand side with a minus sign in front of the square root as it would imply then $d_1 < -2$.

Therefore,

$$\sqrt{d_1^2 - 4d_2} < 2 + d_1, \text{ or } d_1^2 - 4d_2 < 4 + d_1^2 + 4d_1, \text{ or } -d_1 < 1 + d_2. \quad (7-x)$$

Similarly, $\frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < -1$, or $\pm \sqrt{d_1^2 - 4d_2} > -2 + d_1$. Again it is not possible to satisfy the inequality on the right hand side with a plus sign in front of the square root as it would imply then $d_1 > 2$. Therefore, $-\sqrt{d_1^2 - 4d_2} > -2 + d_1$,

or $\sqrt{d_1^2 - 4d_2} < 2 - d_1$, or $d_1^2 - 4d_2 < 4 + d_1^2 - 4d_1$, or equivalently,

$$d_1 < 1 + d_2. \quad (7-y)$$

Combining Eqs. (7-x) and (7-y) we get $|d_1| < 1 + d_2$.

7.100 (a) $D_a(z) = 4(1 + 0.75z^{-1} + 0.25z^{-2}) \Rightarrow d_1 = 0.75, d_2 = 0.5$. Since

$|d_2| = 0.5 < 1$ and $1 + d_2 = 1.5, |d_1| = 0.75 < 1 + d_2$. Hence, both roots of $D_a(z)$ are inside the unit circle.

(b) $D_b(z) = 2(1 + 0.5z^{-1} + 0.5z^{-2}) \Rightarrow d_1 = 0.5, d_2 = 0.5$. Since $|d_2| = 0.5 < 1$ and $1 + d_2 = 1.5, |d_1| = 0.5 < 1 + d_2$. $1 + d_2 = 1.5, |d_1| = 0.75 < 1 + d_2$. Hence, both roots of $D_b(z)$ are inside the unit circle.

(c) $D_c(z) = 3(1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}) \Rightarrow d_1 = \frac{4}{3}, d_2 = -\frac{4}{3}$. Since $|d_2| = \frac{4}{3} > 1$, at least one root of $D_c(z)$ is outside the unit circle.

(d) $D_d(z) = 3(1 - \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}) \Rightarrow d_1 = -\frac{1}{6}, d_2 = -\frac{1}{3}$. Since $|d_2| = \frac{1}{3} < 1$ and $1 + d_2 = \frac{2}{3}, |d_1| = \frac{1}{3} < 1 + d_2$. Hence, both roots of $D_d(z)$ are inside the unit circle.

7.101 (a) $A_3(z) = \frac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}$. Note $|k_3| = 0.25 < 1$. Using Eq.

(7.148) we arrive at $A_2(z) = \frac{\frac{1}{3} + \frac{2}{3}z^{-1} + z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{3}z^{-2}}$. Here, $|k_2| = \frac{1}{3} < 1$. Continuing this

process we get $A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}$. Finally, $|k_1| = 0.5 < 1$. Since $|k_i| < 1, i = 1, 2, 3$,

$H_a(z)$ is stable.

(b) $A_3(z) = \frac{-\frac{1}{3} - \frac{2}{3}z^{-1} + \frac{2}{3}z^{-2} + z^{-3}}{1 + \frac{2}{3}z^{-1} - \frac{2}{3}z^{-2} - \frac{1}{3}z^{-3}}$. Note $|k_3| = \frac{1}{3} < 1$. Using Eq. (7.148) we

arrive at $A_2(z) = \frac{-0.5 + 0.5z^{-1} + z^{-2}}{1 + 0.5z^{-1} - 0.5z^{-2}}$. Here, $|k_2| = 0.5 < 1$. Continuing this

process we get $A_1(z) = \frac{1 + z^{-1}}{1 + z^{-1}}$. Finally, $|k_1| = 1$. Since $|k_1|$ is not less than 1,

$H_b(z)$ is not stable.

(c) $A_4(z) = \frac{-\frac{1}{6} + \frac{1}{2}z^{-1} + \frac{2}{3}z^{-2} + \frac{2}{3}z^{-3} + z^{-4}}{1 + \frac{2}{3}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{2}z^{-3} - \frac{1}{6}z^{-4}}$. Note $|k_4| = \frac{1}{6} < 1$. Using Eq. (7.148)

we get $A_3(z) = \frac{0.6286 + 0.8z^{-1} + 0.7714z^{-2} + z^{-3}}{1 + 0.7714z^{-1} + 0.8z^{-2} + 0.6286z^{-3}}$. Note $|k_3| = 0.6286 < 1$.

Using Eq. (7.148) we next get $A_2(z) = \frac{0.5209 + 0.444z^{-1} + z^{-2}}{1 + 0.444z^{-1} + 0.5209z^{-2}}$. Here,

$|k_2| = 0.209 < 1$. Continuing this process we get $A_1(z) = \frac{0.2919 + z^{-1}}{1 + 0.2919z^{-1}}$. Finally,

$|k_1| = 0.2919 < 1$. Since $|k_i| < 1, i = 1, 2, 3, 4$, $H_c(z)$ is stable.

(d) $A_4(z) = \frac{0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.8\frac{2}{3}z^{-3} + z^{-4}}{1 + 0.8z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}}$. Note $|k_4| = 0.2 < 1$. Using

Eq. (7.148) we get $A_3(z) = \frac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}$. Note $|k_3| = 0.25 < 1$.

Using Eq. (7.148) we next get $A_2(z) = \frac{\frac{1}{3} + \frac{2}{3}z^{-1} + z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{3}z^{-2}}$. Here, $|k_2| = \frac{1}{3} < 1$.

Continuing this process we get $A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}$. Finally, $|k_1| = 0.5 < 1$. Since

$|k_i| < 1, i = 1, 2, 3, 4$, $H_d(z)$ is stable.

(e) $A_5(z) = \frac{0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.5z^{-3} + 0.7z^{-4} + z^{-5}}{1 + 0.7z^{-1} + 0.5z^{-2} + 0.3z^{-3} + 0.2z^{-4} + 0.1z^{-5}}$. Note $|k_5| = 0.1 < 1$.

Using Eq. (7.148) we get

$$A_4(z) = \frac{0.1313 + 0.2525z^{-1} + 0.4747z^{-2} + 0.6869z^{-3} + z^{-4}}{1 + 0.6869z^{-1} + 0.4747z^{-2} + 0.2525z^{-3} + 0.1313z^{-4}}. \text{ Note}$$

$|k_4| = 0.1313 < 1$. Using Eq. (7.148) we next get

$$A_3(z) = \frac{0.1652 + 0.4196z^{-1} + 0.6652z^{-2} + z^{-3}}{1 + 0.6652z^{-1} + 0.4196z^{-2} + 0.1652z^{-3}}. \text{ Note } |k_3| = 0.1652 < 1. \text{ Using}$$

Eq. (7.148) we next get $A_2(z) = \frac{0.3185 + 0.6126z^{-1} + z^{-2}}{1 + 0.6126z^{-1} + 0.3185z^{-2}}$. Here,

$$|k_2| = 0.3185 < 1. \text{ Continuing this process we get } A_1(z) = \frac{0.4646 + z^{-1}}{1 + 0.4646z^{-1}}. \text{ Finally,}$$

$|k_1| = 0.4646 < 1$. Since $|k_i| < 1, i = 1, 2, 3, 4, 5$, $H_e(z)$ is stable.

7.102 (a) $A_5(z) = \frac{0.1 + 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3} + 0.8z^{-4} + z^{-5}}{1 + 0.8z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4} + 0.1z^{-5}}$. Note

$|k_5| = 0.1 < 1$. Using Eq. (7.148) we get

$$A_4(z) = \frac{0.1212 + 0.3434z^{-1} + 0.5657z^{-2} + 0.7879z^{-3} + z^{-4}}{1 + 0.7879z^{-1} + 0.5657z^{-2} + 0.3434z^{-3} + 0.1212z^{-4}}. \text{ Note}$$

$|k_4| = 0.1211 < 1$. Using Eq. (7.148) we next get

$$A_3(z) = \frac{0.2516 + 0.5045z^{-1} + 0.7574z^{-2} + z^{-3}}{1 + 0.7574z^{-1} + 0.5045z^{-2} + 0.2516z^{-3}}. \text{ Note } |k_3| = 0.2516 < 1. \text{ Using}$$

Eq. (7.148) we next get $A_2(z) = \frac{0.3351 + 0.6730z^{-1} + z^{-2}}{1 + 0.6730z^{-1} + 0.3351z^{-2}}.$ Here,

$$|k_2| = 0.3351 < 1. \text{ Continuing this process we get } A_1(z) = \frac{0.5041 + z^{-1}}{1 + 0.5041z^{-1}}. \text{ Finally,}$$

$|k_1| = 0.5041 < 1.$ Since $|k_i| < 1, i = 1, 2, 3, 4, 5,$ $D_a(z)$ has all roots inside the unit circle..

(b) $A_5(z) = \frac{-0.25 + 0.5z^{-1} + 0.625z^{-2} + 0.75z^{-3} + 0.875z^{-4} + z^{-5}}{1 + 0.875z^{-1} + 0.75z^{-2} + 0.625z^{-3} + 0.5z^{-4} - 0.25z^{-5}}.$ Note

$|k_5| = 0.25 < 1.$ Using Eq. (7.148) we get

$$A_4(z) = \frac{0.7667 + 0.8667z^{-1} + 0.9667z^{-2} + 1.0667z^{-3} + z^{-4}}{1 + 1.0667z^{-1} + 0.9667z^{-2} + 0.8667z^{-3} + 0.7667z^{-4}}. \text{ Note}$$

$|k_4| = 0.7667 < 1.$ Using Eq. (7.148) we next get

$$A_3(z) = \frac{0.1186 + 0.5472z^{-1} + 0.9757z^{-2} + z^{-3}}{1 + 0.9757z^{-1} + 0.5472z^{-2} + 0.1186z^{-3}}. \text{ Note } |k_3| = 0.1186 < 1. \text{ Using}$$

Eq. (7.148) we next get $A_2(z) = \frac{0.4376 + 0.9238z^{-1} + z^{-2}}{1 + 0.9238z^{-1} + 0.4376z^{-2}}.$ Here,

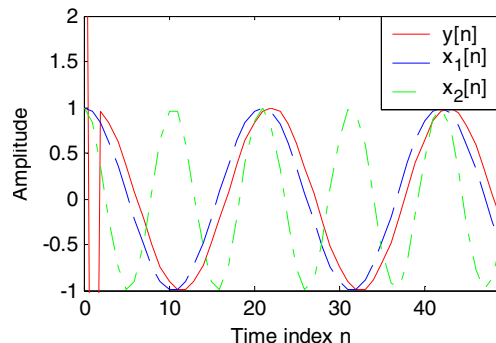
$$|k_2| = 0.4376 < 1. \text{ Continuing this process we get } A_1(z) = \frac{0.6426 + z^{-1}}{1 + 0.6426z^{-1}}. \text{ Finally,}$$

$|k_1| = 0.6426 < 1.$ Since $|k_i| < 1, i = 1, 2, 3, 4, 5,$ $D_b(z)$ has all roots inside the unit circle..

M7.1 The MATLAB code fragments used to simulate the FIR filter are

```
b = [3.8461 -6.3487 3.8461]; zi = [0 0];
n=0:49; x1 = cos(0.3*n); x2=cos(0.6*n);
y = filter(b,1,x1+x2, zi);
```

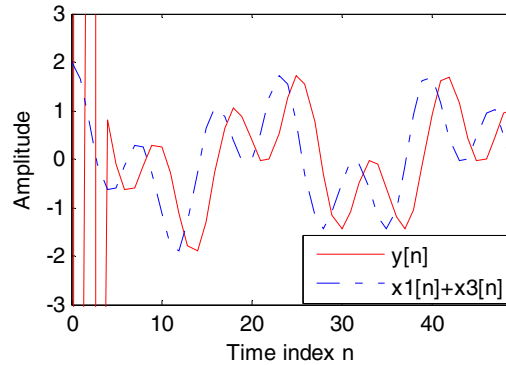
The plot generated by the above program is shown below:



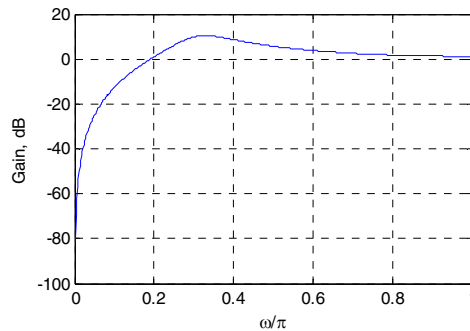
M7.2 The MATLAB code fragments used to simulate the FIR filter are:

```
b = [17.7761 -58.7339 83.8786 -58.7339 17.7761];
n = 0:49; x1 = cos(0.3*n); x2=cos(0.5*n); x3=cos(0.8*n);
y = filter(b,1,x1+x2+x3);
```

The plot generated by the above program is shown below:



M7.3 The gain response of $H(z)$ is shown below:



M7.4 (a) The MATLAB code fragments used to evaluate $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ is shown below:

```
n = [1 -2 3.5]; n = [n fliplr(n)];
d = [8 0 -2 0 -1 0];
k = 0:5; n1 = (-1).^k.*n;
a = conv(n, fliplr(n)) + conv(n1, fliplr(n1));
b = conv(d, fliplr(d));
```

The numerator and denominator of are $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ given by the vectors a and b :

```
a =
0 -8 0 -14 0 69 0 -14 0 -8 0
b =
0 -8 0 -14 0 69 0 -14 0 -8 0
```

Hence $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1$ verifying the power-complementary property of $H(z)$ and $H(-z)$.

(b) The MATLAB code fragments used to evaluate $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ is shown below:

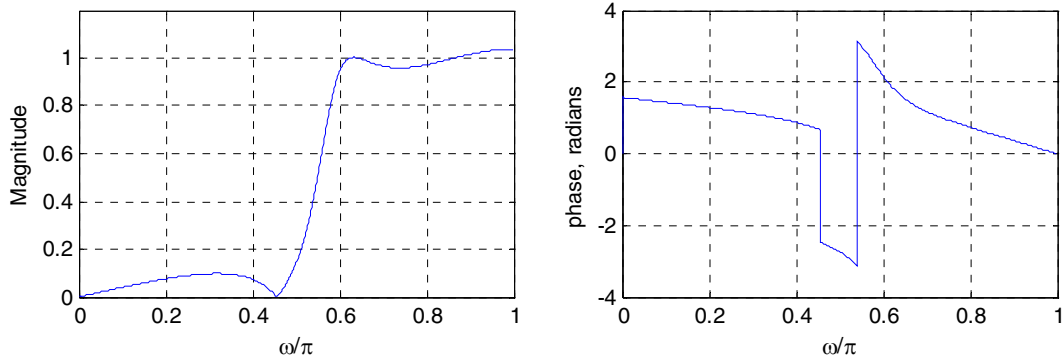
```
n = [1 1.5 5.25 7.25]; n = [n fliplr(n)];
d = [12 0 13 0 4.5 0 0.5 0];
k = 0:7; n1 = (-1).^k.*n;
a = conv(n, fliplr(n)) + conv(n1, fliplr(n1));
b = conv(d, fliplr(d));
```

The numerator and denominator of $H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ are given by the vectors a and b :

```
a =
Columns 1 through 10
0    6.0000    0    60.5000    0    216.7500
0   333.5000    0   216.7500
Columns 11 through 15
0    60.5000    0    6.0000    0
b =
Columns 1 through 10
0    6.0000    0    60.5000    0    216.7500
0   333.5000    0   216.7500
Columns 11 through 15
0    60.5000    0    6.0000    0
```

Hence $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1$ verifying the power-complementary property of $H(z)$ and $H(-z)$.

M7.5 The magnitude and phase responses of $H(z)$ are shown below:

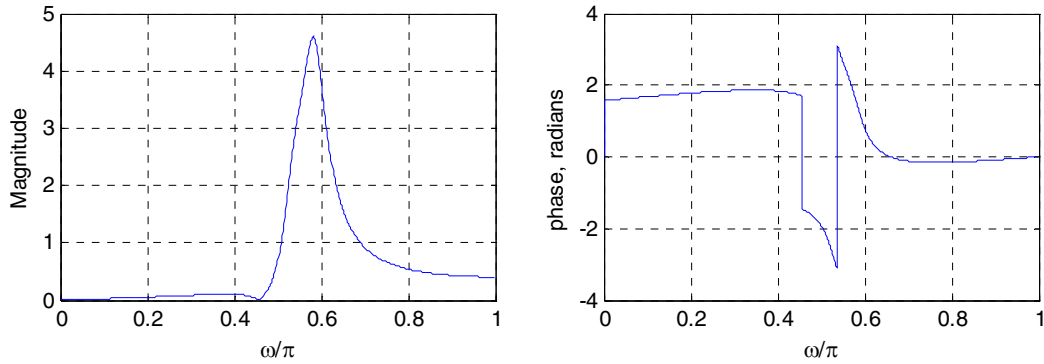


From the magnitude response plot given above it can be seen that represents a highpass filter. The difference equation representation of $H(z)$ is given by

$$y[n] + 0.7074y[n-1] + 0.7976y[n-2] + 0.2004y[n-3]$$

$$= 0.2031x[n] - 0.2588x[n-1] + 0.2588x[n-2] - 0.2031x[n-3].$$

M7.6 The magnitude and phase responses of $H(z)$ are shown below:

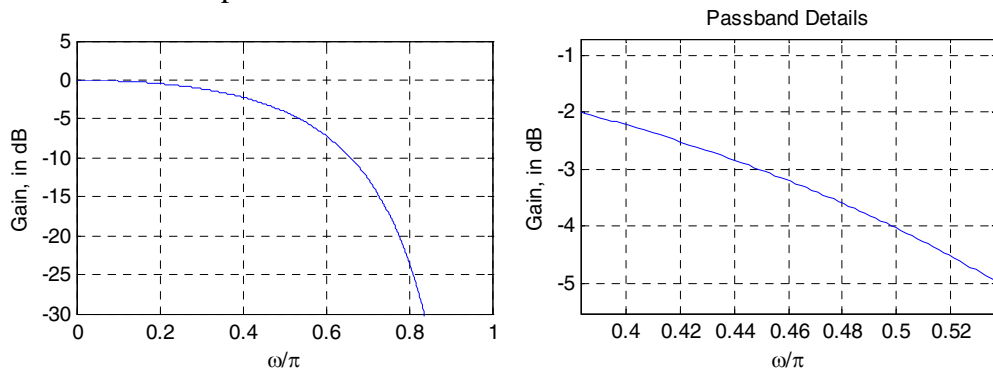


From the magnitude response plot given above it can be seen that represents a bandpass filter. The difference equation representation of $H(z)$ is given by

$$y[n] + 0.6402y[n-1] + 1.7497y[n-2] + 0.5354y[n-3] + 0.7015y[n-4]$$

$$= 0.2031x[n] - 0.2588x[n-1] + 0.2588x[n-2] - 0.2031x[n-3].$$

M7.7 Here $K = 5$. Using Eq. (7.85) we obtain first $C = 1.7411$. Then using Eq. (7.84) we obtain $\alpha = -0.3779$. From Eq. (7.71) we get the transfer function of the lowpass filter as $H_{LP}(z) = \frac{0.6889(1+z^{-1})}{1-0.3779z^{-1}}$. A plot of the gain response of a cascade of 5 lowpass filters is shown below:



M7.8 From the solution of Problem 7.46 we observe

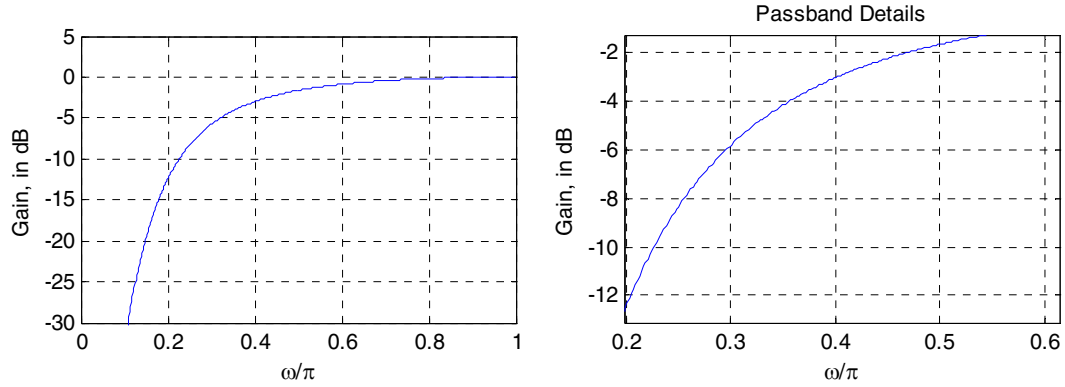
$$\alpha = \frac{\sin \omega_c \sqrt{2C - C^2} - (1 - \cos \omega_c + C \cos \omega_c)}{1 - \cos \omega_c - C}, \text{ where } C = 2^{(K-1)/K}.$$

Substituting $K = 6$ in the second equation we get $C = 1.7818$. Next, substituting this value of C and $\omega_c = 0.4\pi$ in the first equation we arrive at $\alpha = 0.5946$.

From Eq. (7.74) we get the transfer function of the lowpass filter as

$$H_{HP}(z) = \frac{0.7973(1-z^{-1})}{1-0.5946z^{-1}}. \text{ A plot of the gain response of a cascade of 6}$$

highpass filters is shown below:

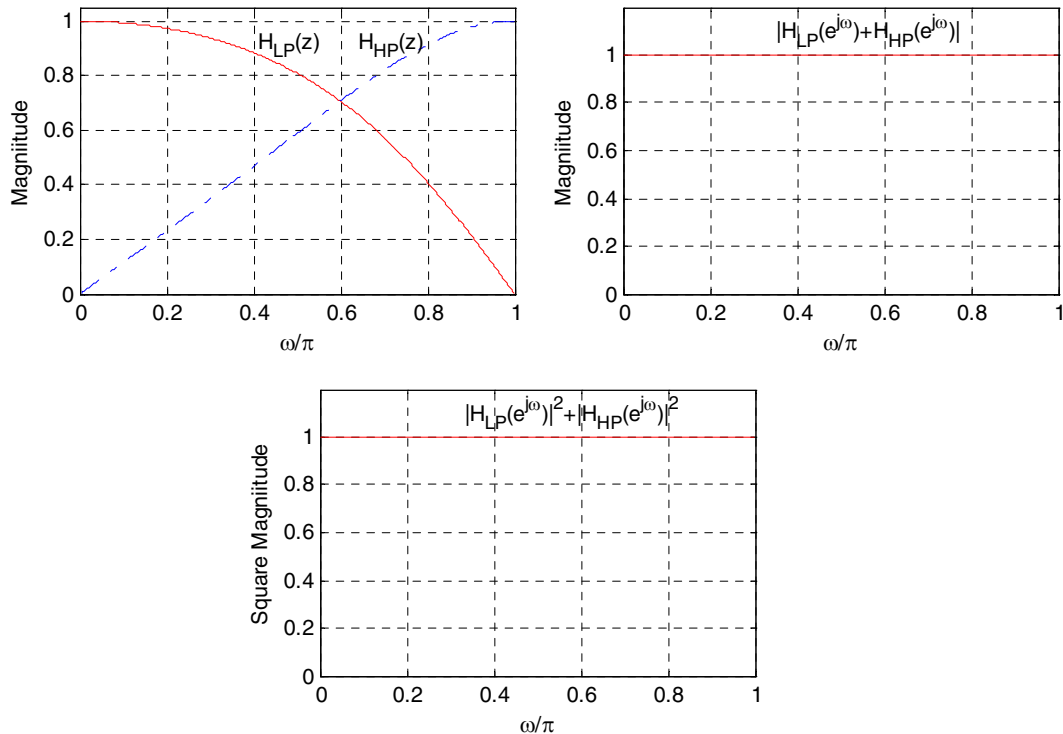


M7.9 Using Eq. (7.73b) we obtain $\alpha = -0.1584$. Sunstituting this value of α in Eqs.

(7.71) and (7.74) we get $H_{LP}(z) = \frac{0.5792(1+z^{-1})}{1-0.1584z^{-1}}$ and $H_{HP}(z) = \frac{0.4208(1-z^{-1})}{1-0.1584z^{-1}}$.

Plots of the magnitude responses of $H_{LP}(z)$ and $H_{HP}(z)$ along with the plot of the magnitude response of $H_{LP}(z) + H_{HP}(z)$ and plot of

$|H_{LP}(e^{j\omega})|^2 + |H_{HP}(e^{j\omega})|^2$ are shown below verifying the doubly complementary property of $H_{LP}(z)$ and $H_{HP}(z)$.



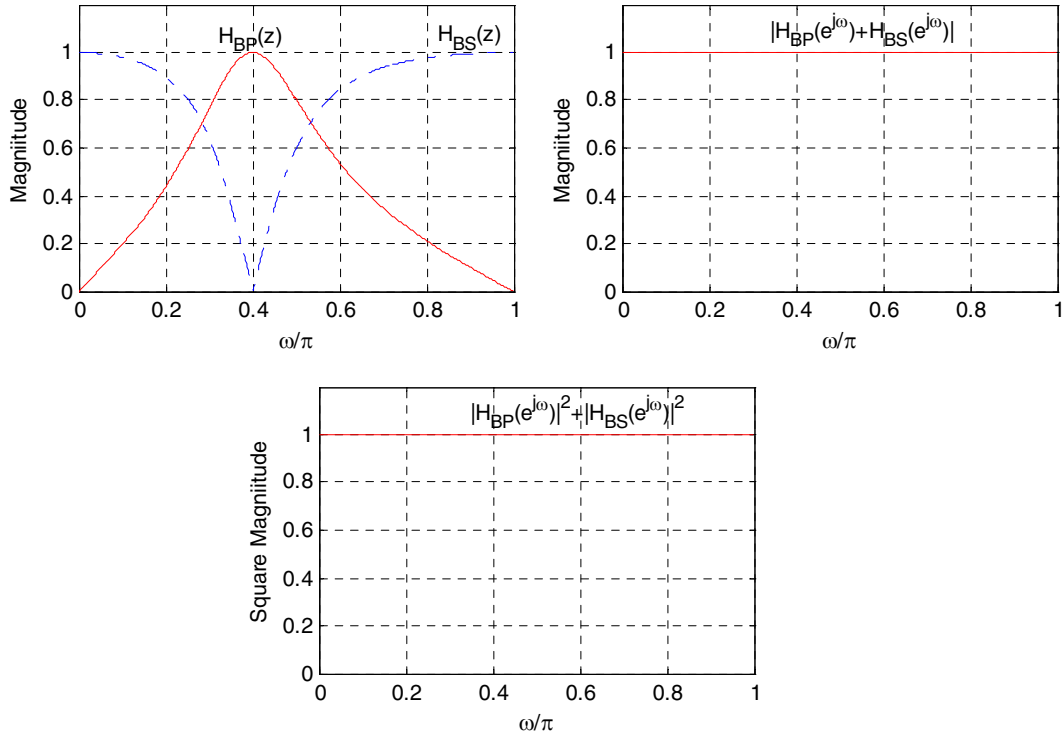
M7.10 From Eq. (7.78) we arrive at the quadratic equation $\alpha^2 - 2.8284\alpha + 1 = 0$ whose solution yields $\alpha = 0.4142$ and $\alpha = 2.4142$. A stable bandpass and bandstop transfer function requires $|\alpha| < 1$. Hence we choose $\alpha = 0.4142$. Next, from Eq.

(7.76) we get $\beta = 0.3090$. Substituting these two parameters in Eqs. (7.77) and

(7.80) we obtain $H_{BP}(z) = \frac{0.2929(1 - z^{-2})}{1 - 0.4370z^{-1} + 0.4142z^{-2}}$ and

$$H_{BS}(z) = \frac{0.7071(1 - 0.6180z^{-1} + z^{-2})}{1 - 0.4370z^{-1} + 0.4142z^{-2}}.$$

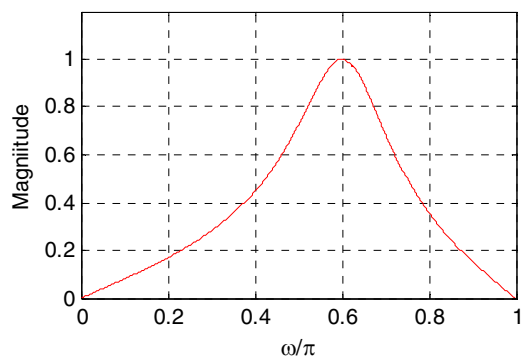
Plots of the magnitude responses of $H_{BP}(z)$ and $H_{BS}(z)$ along with the plot of the magnitude response of $H_{BP}(z) + H_{BS}(z)$ and plot of $|H_{BP}(e^{j\omega})|^2 + |H_{BS}(e^{j\omega})|^2$ are shown below verifying the doubly complementary property of $H_{BP}(z)$ and $H_{BS}(z)$.



M7.11 From Eq. (7.78) we arrive at the quadratic equation $\alpha^2 - 2.4721\alpha + 1 = 0$ whose solution yields $\alpha = 0.5095$ and $\alpha = 1.9626$. A stable bandpass transfer function requires $|\alpha| < 1$. Hence we choose $\alpha = 0.5095$. Next, from Eq. (7.76) we get $\beta = -0.3090$. Substituting these two parameters in Eqs. (7.77) we obtain

$$H_{BP}(z) = \frac{0.2453(1 - z^{-2})}{1 + 0.4665z^{-1} + 0.5095z^{-2}}.$$

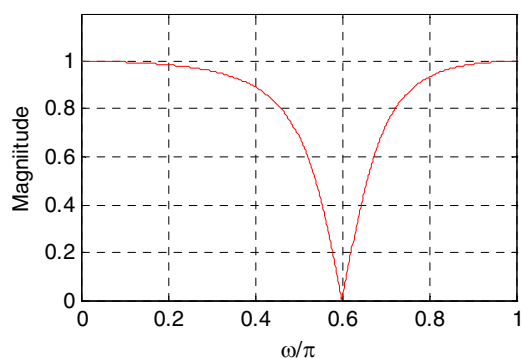
A plot of the magnitude response of $H_{BP}(z)$ is shown below:



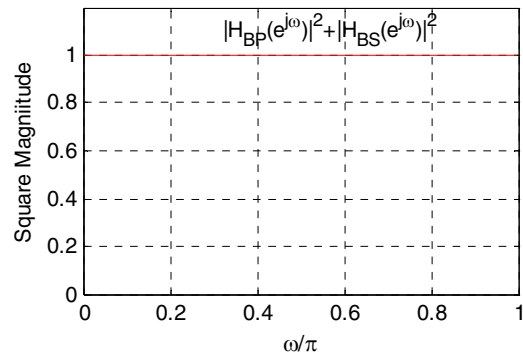
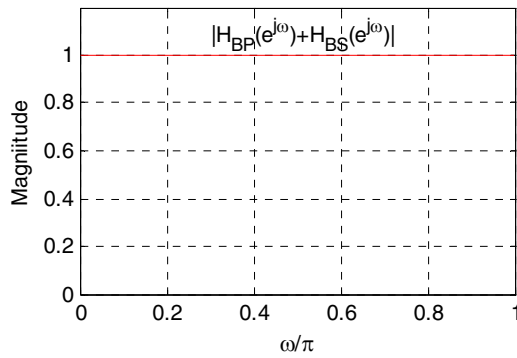
M7.12 From Eq. (7.78) we arrive at the quadratic equation $\alpha^2 - 2.4721\alpha + 1 = 0$ whose solution yields $\alpha = 0.5095$ and $\alpha = 1.9626$. A stable bandstop transfer function requires $|\alpha| < 1$. Hence we choose $\alpha = 0.5095$. Next, from Eq. (7.80) we get $\beta = -0.3090$. Substituting these two parameters in Eqs. (7.77) we obtain

$$H_{BS}(z) = \frac{0.7548(1 + 0.6180z^{-1} + z^{-2})}{1 + 0.4665z^{-1} + 0.5095z^{-2}}. \text{ A plot of the magnitude response of}$$

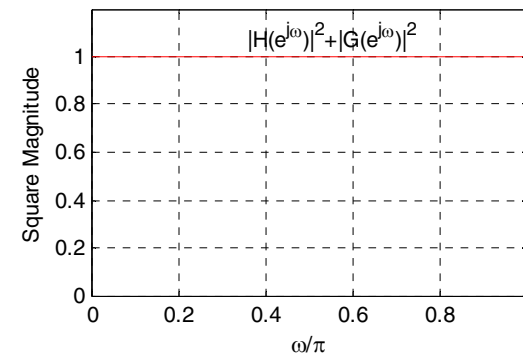
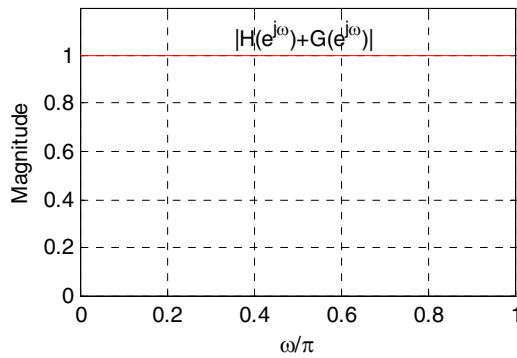
$H_{BS}(z)$ is shown below:



M7.13 A plot of the magnitude response of $H_{BP}(z) + H_{BS}(z)$ and plot of $\left|H_{BP}(e^{j\omega})\right|^2 + \left|H_{BS}(e^{j\omega})\right|^2$ are shown below verifying the doubly complementary property of $H_{BP}(z)$ and $H_{BS}(z)$.

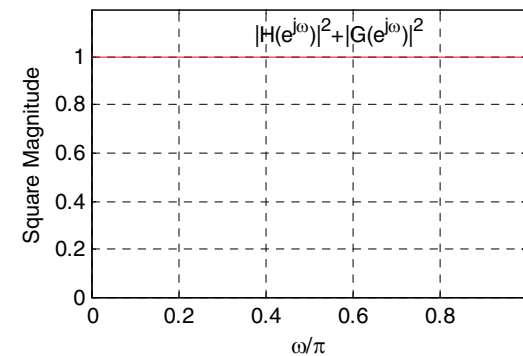
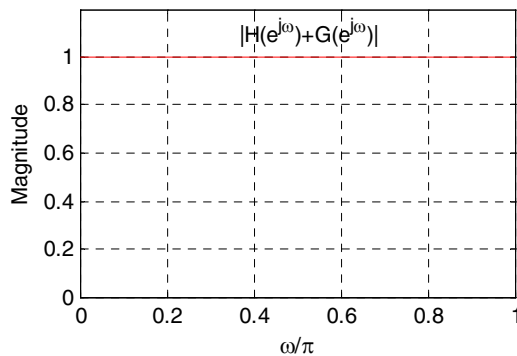


M7.14 (a)



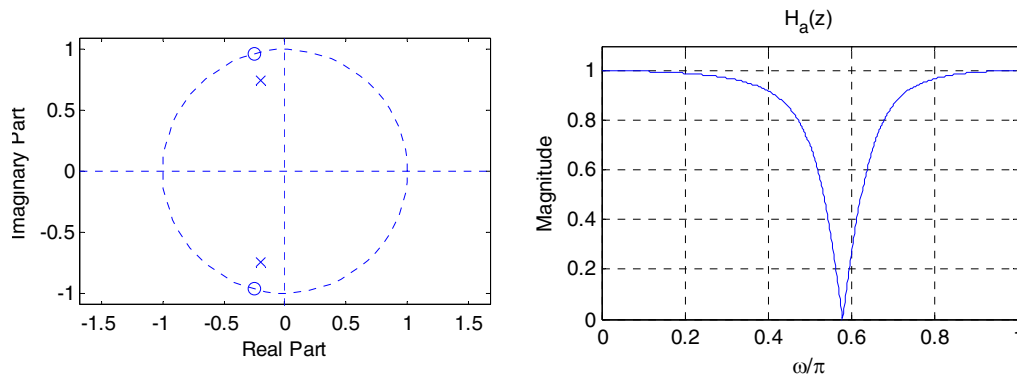
A plot of the magnitude response of $H(z) + G(z)$ and plot of $|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2$ are shown above verifying the doubly complementary property of $H(z)$ and $G(z)$.

(b)



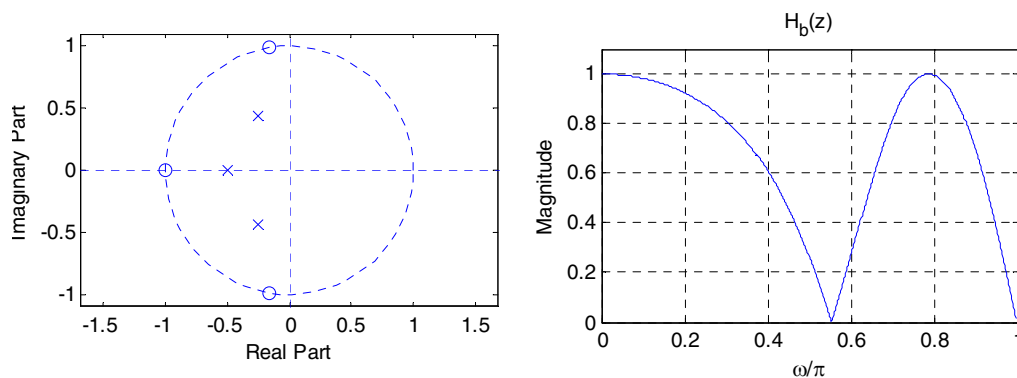
A plot of the magnitude response of $H(z) + G(z)$ and plot of $|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2$ are shown above verifying the doubly complementary property of $H(z)$ and $G(z)$.

M7.15 (a) The pole-zero plot obtained using the M-file `zplane` is shown below:



It can be seen from the above pole-zero plot that the two poles of $H_a(z)$ are inside the unit circle and hence $H_a(z)$ is stable. From the magnitude response plot given above, we observe that $|H_a(e^{j\omega})| \leq 1$ and hence, $H_a(z)$ is a BR function.

(b) The pole-zero plot obtained using the M-file `zplane` is shown below:

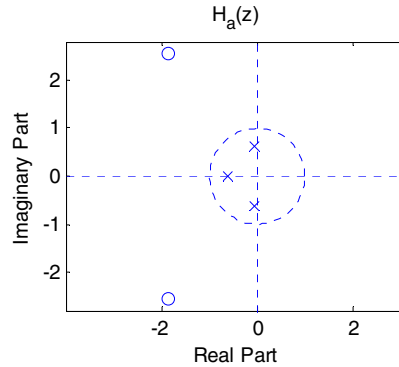


It can be seen from the above pole-zero plot that the three poles of $H_b(z)$ are inside the unit circle and hence $H_b(z)$ is stable. From the magnitude response plot given above, we observe that $|H_a(e^{j\omega})| \leq 1$ and hence, $H_a(z)$ is a BR function.

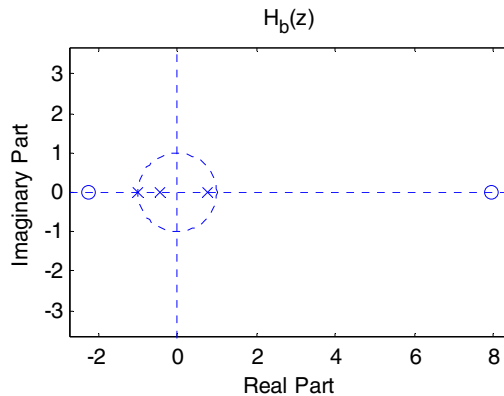
M7.16 (a) The power-complementary transfer function $G_a(z) = Q(z)/D(z)$ to the transfer function $H_a(z) = P(z)/D(z)$ satisfy the relation $G_a(z)G_a(z^{-1}) = 1 - H_a(z)H_a(z^{-1})$, or equivalently the relation $Q(z)Q(z^{-1}) = D(z)D(z^{-1}) - P(z)P(z^{-1})$. Here, $D(z) = 1 + 0.4z^{-1} + 0.6z^{-2}$ and $P(z) = 0.8 + 0.4z^{-1} + 0.8z^{-2}$.

(b)

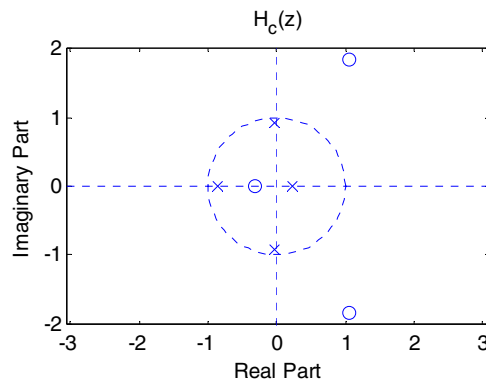
M7.17 (a) The pole-zero plot of $H_a(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that all 3 poles of $H_a(z)$ are inside the unit circle, and hence, $H_a(z)$ is a stable transfer function.



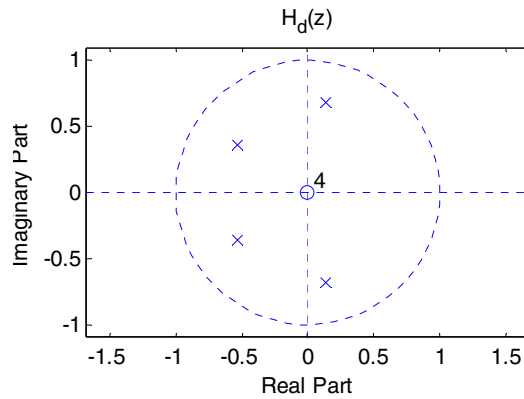
(b) The pole-zero plot of $H_b(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that one pole of $H_b(z)$ is on the unit circle, and hence, $H_b(z)$ is an unstable transfer function.



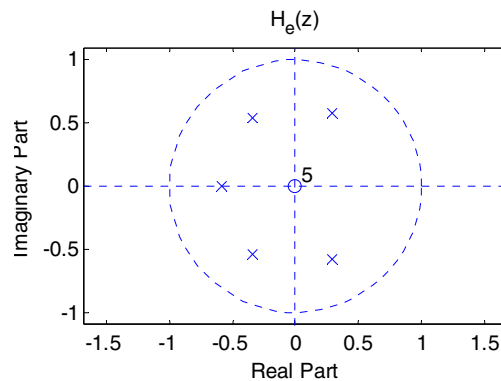
(c) The pole-zero plot of $H_c(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that all 4 poles of $H_c(z)$ are inside the unit circle, and hence, $H_c(z)$ is a stable transfer function.



(d) The pole-zero plot of $H_d(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that all 4 poles of $H_d(z)$ are inside the unit circle, and hence, $H_d(z)$ is a stable transfer function.



(e) The pole-zero plot of $H_e(z)$ obtained using the M-file `zplane` is shown below. From this plot it can be seen that all 5 poles of $H_e(z)$ are inside the unit circle, and hence, $H_e(z)$ is a stable transfer function.



M7.18 (a) The output data generated by running Program 7_2 are:

```
The stability test parameters are
0.2500    0.3333    0.5000
stable =
    1
```

Hence, $H_a(z)$ is a stable transfer function.

(b) The output data generated by running Program 7_2 are:

```
The stability test parameters are
-0.3333   -0.5000    1.0000
stable =
    0
```

Hence, $H_b(z)$ is an unstable transfer function.

(c) The output data generated by running Program 7_2 are:

```
The stability test parameters are
-0.1667    0.6286    0.5209    0.2919
stable =
      1
```

Hence, $H_c(z)$ is a stable transfer function.

(d) The output data generated by running Program 7_2 are:

```
The stability test parameters are
0.2000    0.2500    0.3333    0.5000
stable =
      1
```

Hence, $H_d(z)$ is a stable transfer function.

(e) The output data generated by running Program 7_2 are:

```
The stability test parameters are
0.1000    0.1313    0.1652    0.3185    0.4646
stable =
      1
```

Hence, $H_e(z)$ is a stable transfer function.

M7.19 (a) The output data generated by running Program 7_2 are:

```
The stability test parameters are
0.1000    0.1212    0.2516    0.3351    0.5041
stable =
      1
```

Hence, all roots of $D_a(z)$ are inside the unit circle.

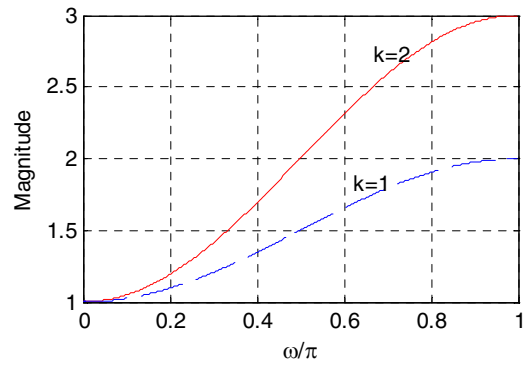
(b) The output data generated by running Program 7_2 are:

(c)

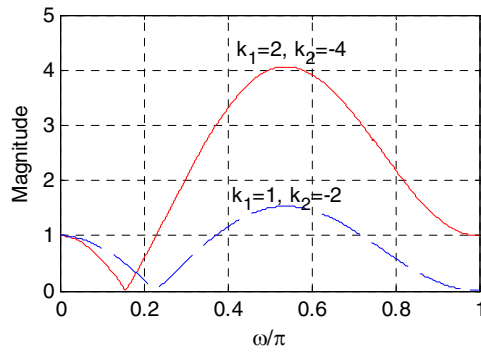
```
The stability test parameters are
-0.2500    0.7667    0.1186    0.4376    0.6426
stable =
      1
```

Hence, all roots of $D_b(z)$ are inside the unit circle.

M7.20 $H(z) = -\frac{k}{4} + \left(1 + \frac{k}{4}\right)z^{-1} - \frac{k}{4}z^{-2}$.

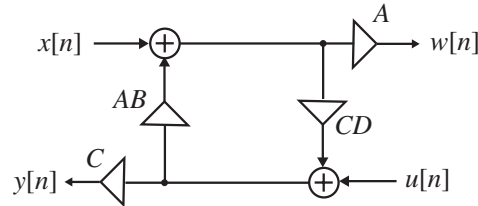


M7.21 $H(z) = -\frac{k_2}{4} + \frac{k_1}{4}z^{-1} + \left(1 - \frac{k_1}{4} + \frac{k_2}{4}\right)z^{-2} + \frac{k_1}{4}z^{-3} - \frac{k_2}{4}z^{-4}$.

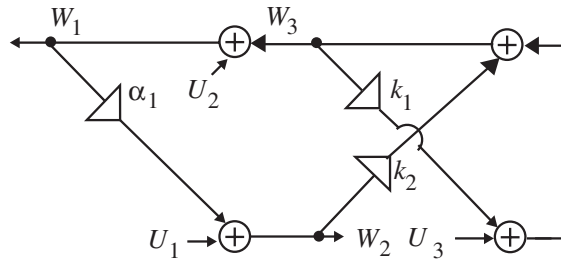


Chapter 8

- 8.1** Analyzing Figure P8.1, we get $w[n] = A(X[n] + CDu[n])$ and $y[n] = C(ABx[n] + u[n])$. A direct implementation of these two equations leads to the structure shown below which has no delay-free loop.



- 8.2 (a)** Figure P8.2(a) with input and output nodes removed and delays removed is as shown below. There are two delay-free loops in the reduced structure shown below. Analysis of this structure yields

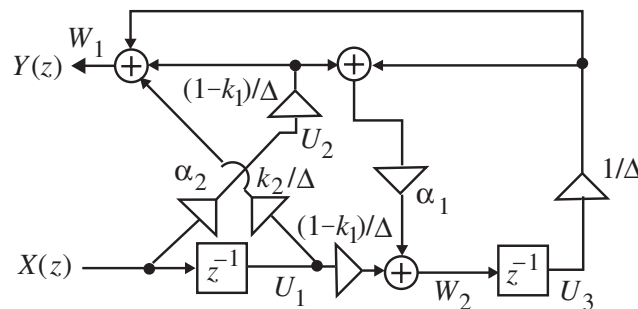


- (1): $W_1 = W_3 + U_2$, (2): $W_2 = \alpha_1 W_1 + U_1$, (3): $W_3 = k_1 W_3 + k_2 W_2 + U_2$.
 From Eq. (3) we get (4): $(1 - k_1)W_3 - k_2 W_2 = U_2$, and from Eq. (1) we get $W_3 = W_1 - U_2$. Substituting the last equation in Eq. (4) we get $(1 - k_1)W_1 - k_2 W_2 = (1 - k_1)U_2 + U_3$ and from Eq. (2) we have $-\alpha_1 W_1 + W_2 = U_1$.

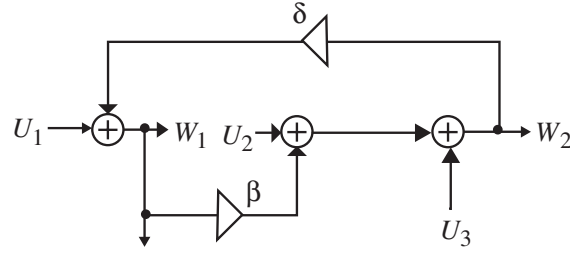
Solving the last two equations we arrive at $W_1 = \frac{k_2}{\Delta} U_1 + \frac{1 - k_1}{\Delta} U_2 + \frac{1}{\Delta} U_3$ and

$W_2 = \frac{1 - k_1}{\Delta} U_1 + \frac{\alpha_1(1 - k_1)}{\Delta} U_2 + \frac{\alpha_1}{\Delta} U_3$. A realization based on the above two

equations after insertion of the input and output nodes and the delays is shown below which has no delay-free loops.



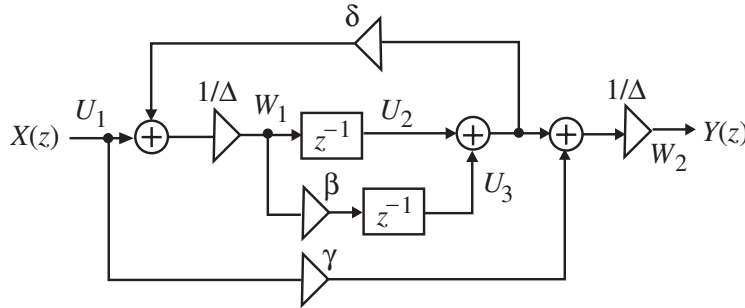
(b) Figure P8.2(b) with input and output nodes removed and delays removed is as shown below. There is one delay-free loop in the reduced structure shown below. Analysis of this structure yields



(1): $W_1 = U_1 + \delta W_2$, and (2): $W_2 = U_2 + \beta W_1 + U_3$. Substituting Eq. (1) in Eq. (2)

we get after some algebra (3): $W_2 = \frac{\beta}{\Delta} U_1 + \frac{1}{\Delta} U_2 + \frac{1}{\Delta} U_3$, where $\Delta = 1 - \beta\delta$.

Substituting Eq. (3) in Eq. (1) we get (5): $W_1 = U_1 + \frac{\beta\delta}{\Delta} U_1 + \frac{\delta}{\Delta} U_2 + \frac{\delta}{\Delta} U_3$. A realization based on Eqs. (3) and (5) after insertion of the input and output nodes and the delays is shown below which has no delay-free loop.



8.3 Analysis yields $Y(z) = G(z)(X(z) - C(z)Y(z))$. Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{G(z)}{1 + G(z)C(z)} = \frac{2}{(1 + 2K) + 3z^{-1}}. \text{ The overall transfer function has}$$

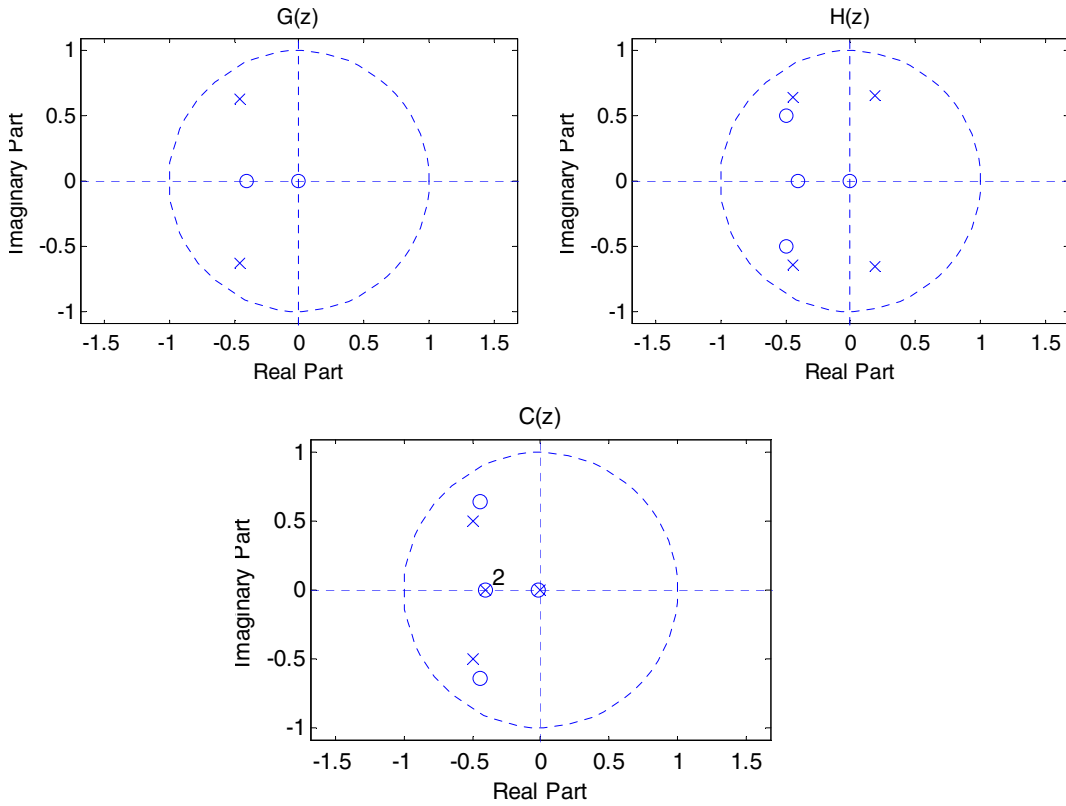
a pole at $z = -\frac{3}{1 + 2K}$. Thus, the system is stable if $\left| \frac{3}{1 + 2K} \right| < 1$, or, $K > 1$ or $K < -2$.

8.4 From the results of Problem 8.3, we have $H(z) = \frac{G(z)}{1 + G(z)C(z)}$ which can be

solved yielding $C(z) = \frac{G(z) - H(z)}{G(z)H(z)}$. Substituting the expressions for $G(z)$ and $H(z)$ in this expression we get

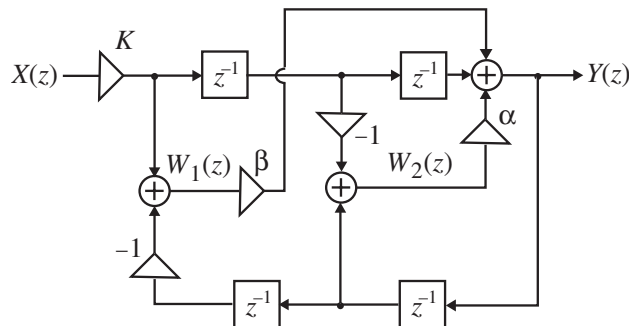
$$C(z) = -\frac{z^{-1}(7.0 + 9.1z^{-1} + 6.87z^{-2} + 1.84z^{-3} + 0.04z^{-4})}{0.04(1 + 1.8z^{-1} + 1.46z^{-2} + 0.56z^{-3} + 0.08z^{-4})}$$

$G(z), C(z),$ and $H(z)$ obtained using `zplane` are shown below:



As can be seen from the above pole-zero plots, $G(z), C(z),$ and $H(z)$ are stable transfer functions.

8.5 The structure with internal variables is shown below. Analysis of this structure yields



$$W_1(z) = K X(z) - z^{-2}Y(z), \quad W_2(z) = -Kz^{-1}X(z) + z^{-1}Y(z), \quad \text{and}$$

$Y(z) = \beta W_1(z) + \alpha W_2(z) + Kz^{-2}X(z)$. Substituting the first two equations into the third, and rearranging we get $(1 - \alpha z^{-1} + \beta z^{-2})Y(z) = K(\beta - \alpha z^{-1} + z^{-2})X(z)$.

Hence, the transfer function is given by $H(z) = \frac{K(\beta - \alpha z^{-1} + z^{-2})}{1 - \alpha z^{-1} + \beta z^{-2}}$.

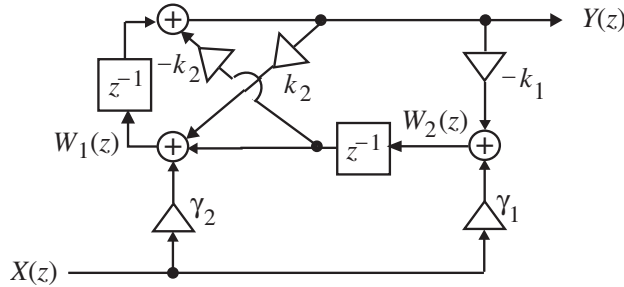
(a) Since the structure employs 2 delays and has a second order transfer function, it is a canonic structure.

(b) and (c) We form

$$H(z)H(z^{-1}) = K^2 \left(\frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}} \right) \left(\frac{\beta z^{-2} - \alpha z^{-1} + 1}{z^{-2} - \alpha z^{-1} + \beta} \right) = K^2. \text{ Therefore,}$$

$$|H(e^{j\omega})| = |K| \text{ for all values of } \omega. \text{ Hence, } |H(e^{j\omega})| = 1 \text{ if } |K| = 1.$$

8.6 The structure with internal variables is shown below. Analysis of this structure yields



$$W_1(z) = \gamma_2 X(z) + k_2 Y(z) + z^{-1} W_2(z), \quad W_2(z) = \gamma_1 X(z) - k_1 Y(z), \text{ and}$$

$Y(z) = z^{-1} W_1(z) - k_2 z^{-1} W_2(z)$. Substituting the second equation in the first and rearranging we get $W_1(z) = (\gamma_2 + \gamma_1 z^{-1}) X(z) + (k_2 - k_1 z^{-1}) Y(z)$. Substituting this equation and the second equation in the third equation we get after some rearrangement $[1 - (k_1 + 1)k_2 z^{-1} + k_1 z^{-2}] Y(z) = [(\gamma_2 - k_2 \gamma_1) z^{-1} + \gamma_1 z^{-2}] X(z)$.

$$\text{Hence, the transfer function is given by } H(z) = \frac{Y(z)}{X(z)} = \frac{(\gamma_2 - k_2 \gamma_1) z^{-1} + \gamma_1 z^{-2}}{1 - (k_1 + 1)k_2 z^{-1} + k_1 z^{-2}}.$$

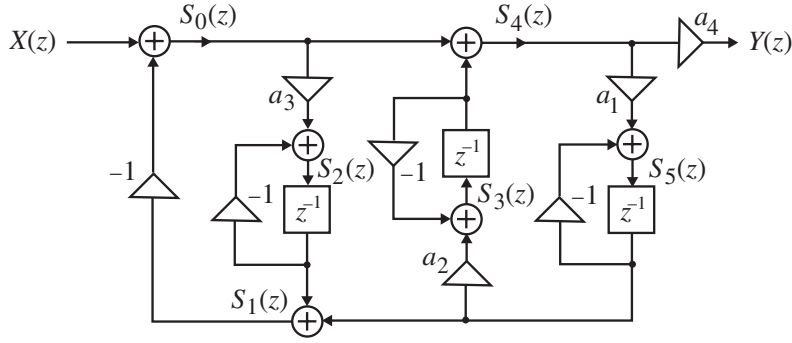
For stability, we must have $|k_1| < 1$ and $|(1 + k_1)k_2| < 1 + k_1$, or $|k_2| < 1$.

8.7 The structure with internal variables is shown on next page. Analysis of this structure yields

$$S_0(z) = X(z) - S_1(z), \quad S_1(z) = z^{-1} S_2(z) + z^{-1} S_5(z), \quad S_2(z) = a_3 S_0(z) - z^{-1} S_2(z),$$

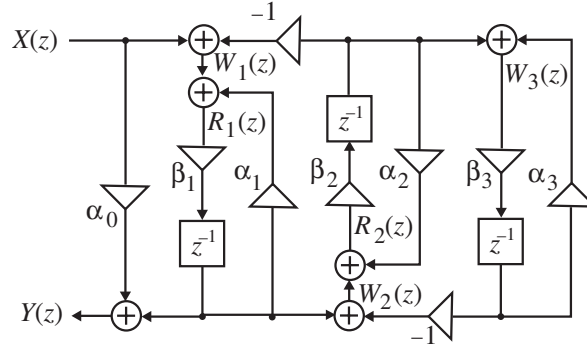
$$S_3(z) = a_3 S_5(z) - z^{-1} S_3(z), \quad S_4(z) = S_0(z) + z^{-1} S_3(z), \quad S_5(z) = a_1 S_4(z) - z^{-1} S_5(z),$$

and $Y(z) = a_4 S_4(z)$. Eliminating $S_0(z), S_1(z), S_2(z), S_3(z), S_4(z)$ and $S_5(z)$ from these equations we get after some algebra



$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_4(1+z^{-1})^3}{1 + (3+a_1+a_3)z^{-1} + (3+2a_1+2a_3-a_1a_2)z^{-2} + (1+a_1+a_3-a_1a_2-a_1a_2a_3)z^{-3}}$$

8.8 The structure with internal variables is shown below. Analysis of this structure yields



$$Y(z) = \alpha_0 X(z) + \beta_1 z^{-1} R_1(z), \quad W_1(z) = X(z) - \beta_2 z^{-1} R_2(z),$$

$$R_1(z) = W_1(z) + \alpha_1 \beta_1 z^{-1} R_1(z), \quad W_2(z) = \beta_1 z^{-1} R_1(z) - \beta_3 z^{-1} W_3(z),$$

$$R_2(z) = W_2(z) + \alpha_2 \beta_2 z^{-1} R_2(z), \quad W_3(z) = \beta_2 z^{-1} R_2(z) + \alpha_3 \beta_3 z^{-1} W_3(z).$$

From the third equation we get $W_1(z) = (1 - \alpha_1 \beta_1 z^{-1}) R_1(z)$. From the sixth equation

we get $W_3(z) = \frac{\beta_2 z^{-1} R_2(z)}{\alpha_3 \beta_3 z^{-1}}$. From the fifth equation we get $R_2(z) = \frac{W_2(z)}{1 - \alpha_2 \beta_2 z^{-1}}$.

Rewriting the fourth equation we get $W_2(z) + \beta_3 z^{-1} W_3(z) = \beta_1 z^{-1} R_1(z)$, in which we substitute the expressions for $W_3(z)$ and $W_1(z)$ resulting in after some algebra

$$R_2(z) = \frac{\beta_1 z^{-1} (1 - \alpha_3 \beta_3 z^{-1})}{\beta_2 \beta_3 z^{-1} + (1 - \alpha_2 \beta_2 z^{-1})(1 - \alpha_3 \beta_3 z^{-1})} R_1(z).$$

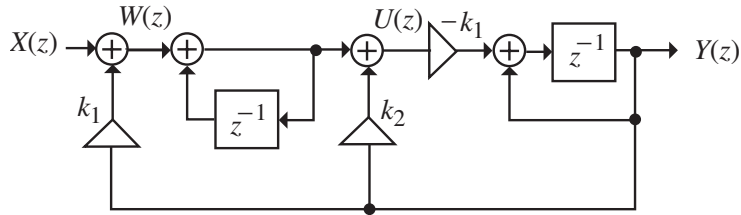
Substituting this expression for $R_2(z)$ and the expression for $W_1(z)$ in the second equation we arrive at

$$R_1(z) = \frac{\beta_2\beta_3z^{-2} + (1 - \alpha_2\beta_2z^{-1})(1 - \alpha_3\beta_3z^{-1})}{\beta_2\beta_3z^{-2}(1 - \alpha_1\beta_1z^{-1}) + (1 - \alpha_1\beta_1z^{-1})(1 - \alpha_2\beta_2z^{-1})(1 - \alpha_3\beta_3z^{-1})} X(z).$$

Substituting the above in the first equation we finally arrive at

$$H(z) = \frac{Y(z)}{X(z)} = \alpha_0 + \frac{\beta_1\beta_2\beta_3z^{-3} + \beta_1z^{-1}(1 - \alpha_2\beta_2z^{-1})(1 - \alpha_3\beta_3z^{-1})}{(1 - \alpha_1\beta_1z^{-1})(1 - \alpha_2\beta_2z^{-1})(1 - \alpha_3\beta_3z^{-1}) + \beta_2\beta_3z^{-2}(1 - \alpha_1\beta_1z^{-1})}.$$

8.9 The structure with internal variables is shown below. Analysis of this structure yields



$$(1): W(z) = X(z) + k_1Y(z), \quad (2): U(z) = \frac{1}{1 - z^{-1}}W(z) + k_2Y(z), \quad \text{and}$$

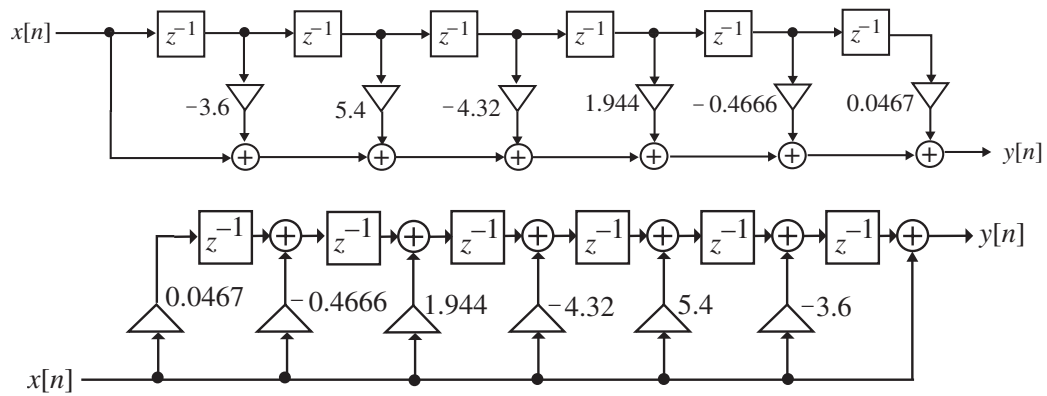
$$(3): Y(z) = \frac{-k_1}{1 - z^{-1}}U(z). \quad \text{Substituting Eq. (2) in Eq. (3) we get}$$

$$(4): Y(z) = \frac{-k_1}{1 - z^{-1}} \left(\frac{1}{1 - z^{-1}}W(z) + k_2Y(z) \right) = \frac{-k_1}{(1 - z^{-1})^2}W(z) - \frac{k_1k_2}{1 - z^{-1}}Y(z).$$

Substituting Eq. (1) in Eq. (4) we then get

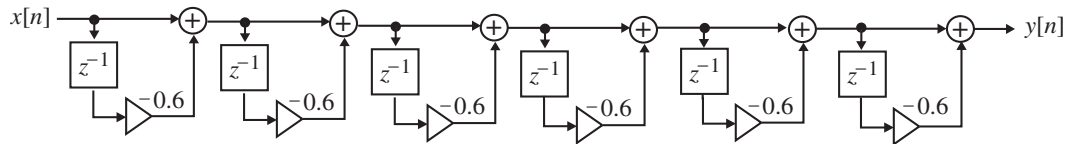
$$\begin{aligned} Y(z) &= -\frac{k_1}{(1 - z^{-1})^2} [X(z) + k_1Y(z)] - \frac{k_1k_2}{1 - z^{-1}}Y(z) \\ &= -\frac{k_1}{(1 - z^{-1})^2} X(z) - \frac{k_1}{1 - z^{-1}} \left[\frac{k_1 + k_2 - k_2z^{-1}}{1 - z^{-1}} \right] Y(z), \quad \text{or,} \\ \left[1 + \frac{k_1(k_1 + k_2 - k_2z^{-1})}{(1 - z^{-1})^2} \right] Y(z) &= -\frac{k_1}{(1 - z^{-1})^2} X(z). \quad \text{Hence,} \\ H(z) = \frac{Y(z)}{X(z)} &= -\frac{k_1}{[1 + k_1(k_1 + k_2)] - (2 + k_1k_2)z^{-1} + z^{-2}}. \end{aligned}$$

8.10 (a) A direct form realization of $H(z) = 1 - 3.6z^{-1} + 5.4z^{-2} - 4.32z^{-3} + 1.944z^{-4} - 0.4666z^{-5} + 0.0467z^{-6}$ and its transformed structure are shown on next page:



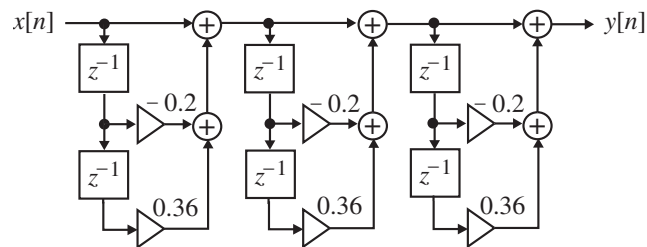
(b) A cascade realization of

$H(z) = (1 - 0.6z^{-1})(1 - 0.6z^{-1})(1 - 0.6z^{-1})(1 - 0.6z^{-1})(1 - 0.6z^{-1})(1 - 0.6z^{-1})$ is shown below:



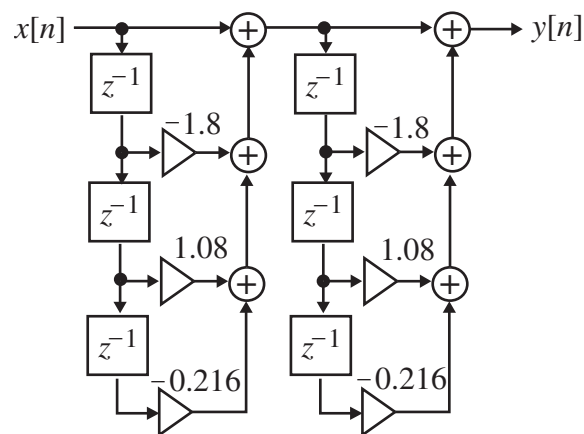
(c) A cascade realization of

$H(z) = (1 - 1.2z^{-1} + 0.36z^{-2})(1 - 1.2z^{-1} + 0.36z^{-2})(1 - 1.2z^{-1} + 0.36z^{-2})$ is shown below:



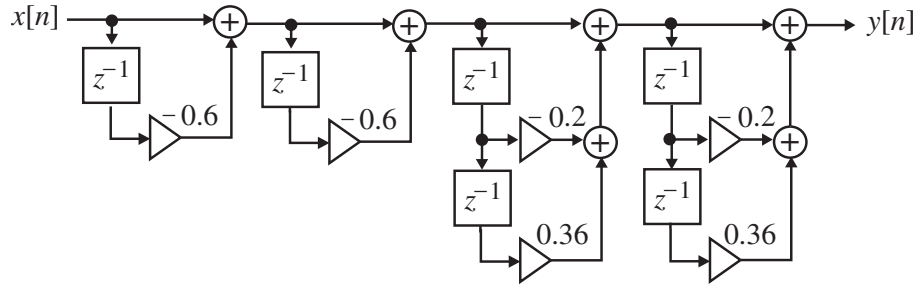
(d) A cascade realization of

$H(z) = (1 - 1.8z^{-1} + 1.08z^{-2} - 0.216z^{-3})(1 - 1.8z^{-1} + 1.08z^{-2} - 0.216z^{-3})$ is shown below:



(e) A cascade realization of

$H(z) = (1 - 0.6z^{-1})(1 - 0.6z^{-1})(1 - 1.2z^{-1} + 0.36z^{-2})(1 - 1.2z^{-1} + 0.36z^{-2})$ is shown below:



All realizations given above require 6 multipliers and 6 two-input adders.

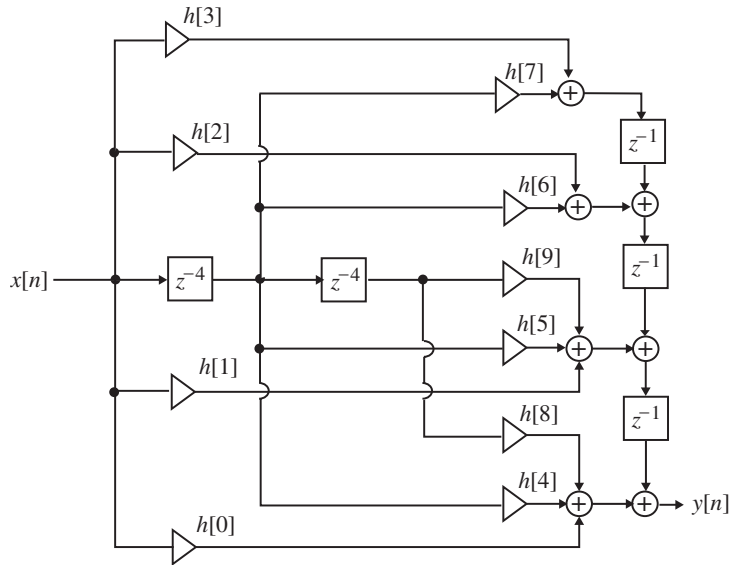
8.11 (a) A four branch polyphase decomposition is given by

$H(z) = E_0(z^4) + E_1(z^4)z^{-1} + E_2(z^4)z^{-2} + E_3(z^4)z^{-3}$, where

$E_0(z) = h[0] + h[4]z^{-1} + h[8]z^{-2}$, $E_1(z) = h[1] + h[5]z^{-1} + h[9]z^{-2}$,

$E_2(z) = h[2] + h[6]z^{-1}$, $E_3(z) = h[3] + h[7]z^{-1}$.

(b) A canonic four-branch polyphase realization of the above transfer function is shown below:



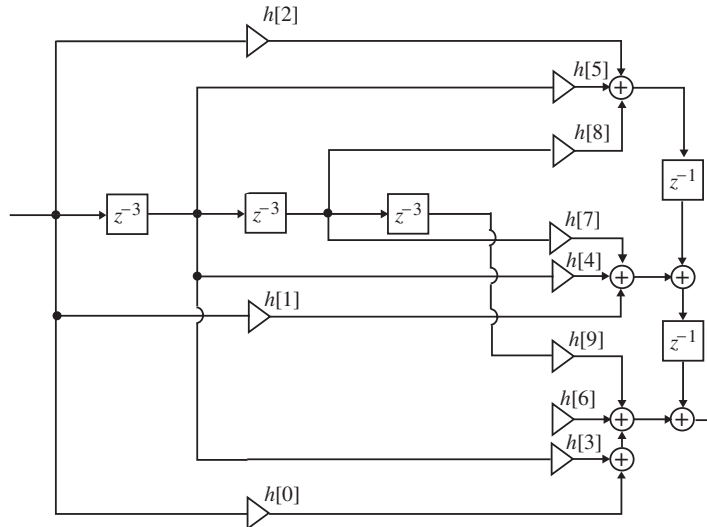
8.12 (a) A three branch polyphase decomposition is given by

$H(z) = E_0(z^3) + E_1(z^3)z^{-1} + E_2(z^3)z^{-2}$, where

$E_0(z) = h[0] + h[3]z^{-1} + h[6]z^{-2} + h[9]z^{-3}$, $E_1(z) = h[1] + h[4]z^{-1} + h[7]z^{-2}$,

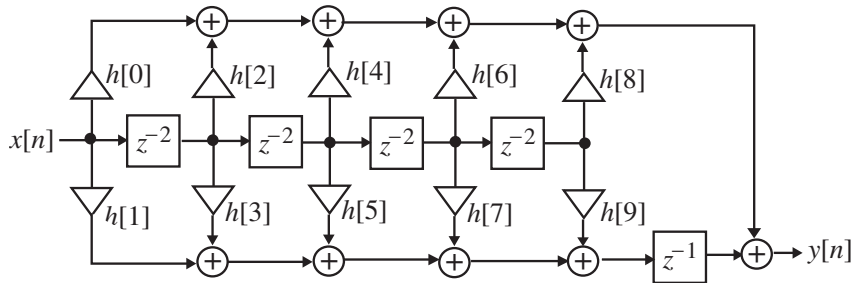
$E_2(z) = h[2] + h[5]z^{-1} + h[8]z^{-2}$.

(b) A canonic three branch polyphase realization of the above transfer function is shown below:

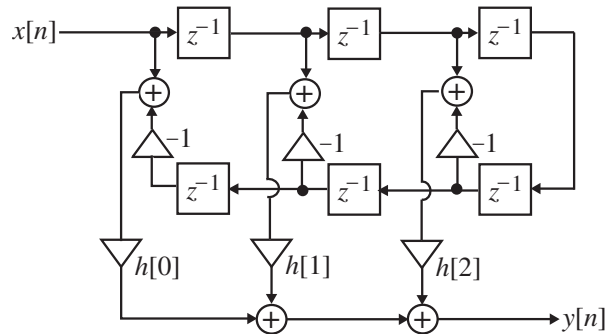


8.13 (a) A two branch polyphase decomposition is given by $H(z) = E_0(z^2) + E_1(z^2)z^{-1}$, where $E_0(z) = h[0] + h[2]z^{-1} + h[4]z^{-2} + h[6]z^{-3} + h[8]z^{-4}$, $E_1(z) = h[1] + h[3]z^{-1} + h[5]z^{-2} + h[7]z^{-3} + h[9]z^{-4}$.

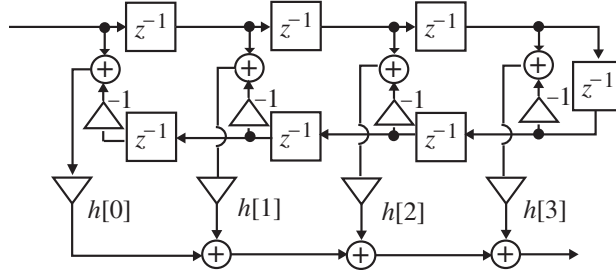
(b) A canonic two branch polyphase realization of the above transfer function is shown below:



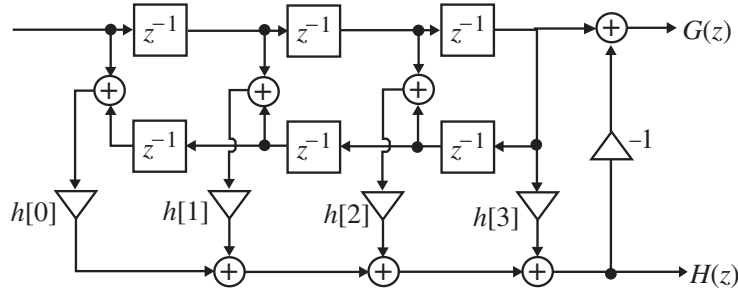
8.14



8.15



8.16 $G(z) = z^{-N/2} - H(z)$. A canonic realization of both $G(z)$ and $H(z)$ is shown below for $N = 6$.



8.17 Without any loss of generality, assume $M = 5$ which means $N = 11$. In this case, the transfer function is given by $H(z)$

$$= z^{-5} [h[5] + h[4](z + z^{-1}) + h[3](z^2 + z^{-2}) + h[2](z^3 + z^{-3}) + h[1](z^4 + z^{-4}) + h[0](z^5 + z^{-5})]$$

Now, the recursion relation for the Chebyshev polynomial is given by

$$T_r(x) = 2xT_{r-1}(x) - T_{r-2}(x), \quad r \geq 2 \quad \text{with } T_0(x) = 1 \quad \text{and } T_1(x) = x. \quad \text{Hence,}$$

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1,$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x,$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1,$$

$$T_5(x) = 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x.$$

We can thus rewrite the expression inside the square brackets given above as

$$h[5] + 2h[4]T_1\left(\frac{z+z^{-1}}{2}\right) + 2h[3]T_2\left(\frac{z+z^{-1}}{2}\right) + 2h[2]T_3\left(\frac{z+z^{-1}}{2}\right) + 2h[1]T_4\left(\frac{z+z^{-1}}{2}\right) + 2h[0]T_5\left(\frac{z+z^{-1}}{2}\right)$$

$$= h[5] + 2h[4]\left(\frac{z+z^{-1}}{2}\right) + 2h[3]\left[2\left(\frac{z+z^{-1}}{2}\right)^2 - 1\right] + 2h[2]\left[4\left(\frac{z+z^{-1}}{2}\right)^3 - 3\left(\frac{z+z^{-1}}{2}\right)\right]$$

$$+ 2h[1]\left[8\left(\frac{z+z^{-1}}{2}\right)^4 - 8\left(\frac{z+z^{-1}}{2}\right)^2 + 1\right] + 2h[0]\left[16\left(\frac{z+z^{-1}}{2}\right)^5 - 20\left(\frac{z+z^{-1}}{2}\right)^3 + 5\left(\frac{z+z^{-1}}{2}\right)\right]$$

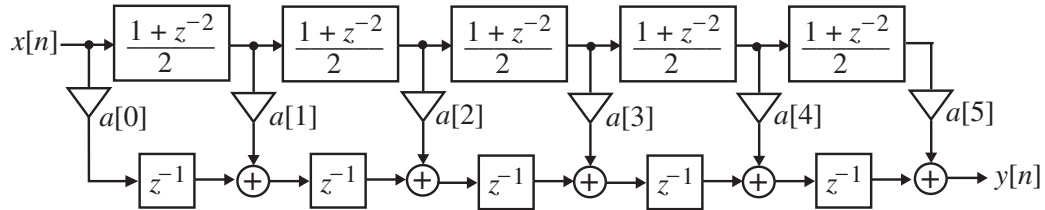
$$= \sum_{n=0}^5 a[n] \left(\frac{z+z^{-1}}{2}\right)^n, \quad \text{where } a[0] = h[5] - 2h[3] + 2h[1], \quad a[1] = 2h[4] - 6h[2] + 10h[0],$$

$$a[2] = 4h[3] - 16h[1], \quad a[3] = 8h[2] - 40h[1], \quad a[4] = 16h[1], \quad \text{and } a[5] = 32h[0]. \quad \text{A}$$

realization of

$$H(z) = z^{-5} \sum_{n=0}^5 a[n] \left(\frac{z+z^{-1}}{2} \right)^n = a[0]z^{-5} + a[1] \left(\frac{1+z^{-2}}{2} \right) z^{-4} + a[2] \left(\frac{1+z^{-2}}{2} \right)^2 z^{-3} + a[3] \left(\frac{1+z^{-2}}{2} \right)^3 z^{-2} + a[4] \left(\frac{1+z^{-2}}{2} \right)^4 z^{-1} + a[5] \left(\frac{1+z^{-2}}{2} \right)^5$$

is shown below:



8.18 Consider $H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)}{D_1(z)} \cdot \frac{P_2(z)}{D_2(z)} \cdot \frac{P_3(z)}{D_3(z)}$. Assume all zeros of $P(z)$ and $D(z)$

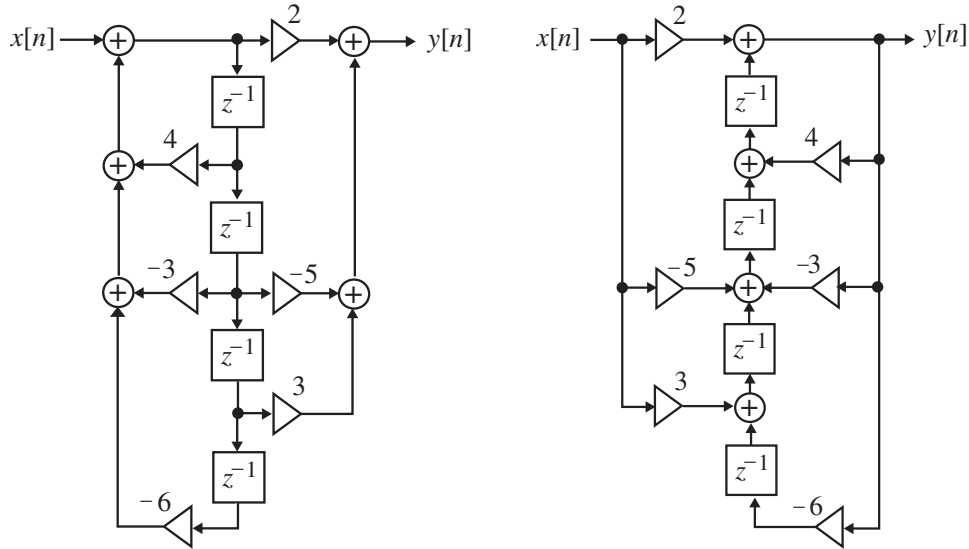
are complex. Note that the numerator of the first stage can be one of the 3 factors, $P_1(z)$, $P_2(z)$, and $P_3(z)$. Likewise, the numerator of the second stage can be one of the remaining 2 factors, and the numerator of the third stage is the remaining factor. Similarly, that the denominator of the first stage can be one of the 3 factors, $D_1(z)$, $D_2(z)$, and $D_3(z)$. Likewise, the denominator of the second stage can be one of the remaining 2 factors, and the denominator of the third stage is the remaining factor. Hence, there are $(3!)^2 = 36$ different types of cascade realizations.

If the zeros of $P(z)$ and $D(z)$ are all real, then $P(z)$ has 6 real zeros and $D(z)$ has 6 real zeros. In this case, then there are $(6!)^2 = 518400$ different types of cascade realizations.

8.19 $H(z) = \prod_{i=1}^K \frac{P_i(z)}{D_i(z)}$. Here, the numerator of the first stage can be chosen in $\binom{K}{1}$ ways,

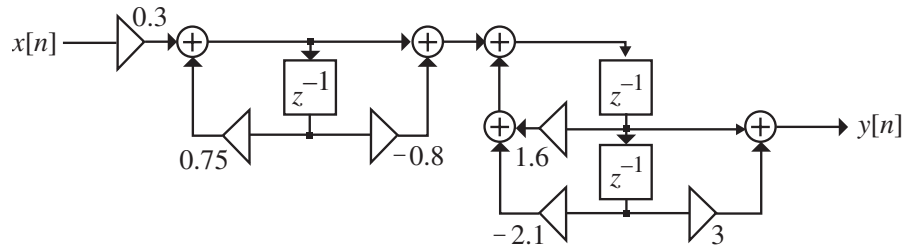
the numerator of the second stage can be chosen in $\binom{K-1}{1}$ ways, and so on, until there is only one possible choice for the numerator of the K -th stage. Likewise, the denominator of the first stage can be chosen in $\binom{K}{1}$ ways, the denominator of the second stage can be chosen in $\binom{K-1}{1}$ ways, and so on, until there is only one possible choice for the denominator of the K -th stage. Hence, the total number of possible cascade realizations are equal to $\binom{K}{1}^2 \binom{K-1}{1}^2 \binom{K-2}{1}^2 \cdots \binom{2}{1}^2 \binom{1}{1}^2 = (K!)^2$.

8.20 A canonic direct form II realization of $H(z) = \frac{3 - 5z^{-2} + 3z^{-3}}{1 - 4z^{-1} + 3z^{-2} + 6z^{-4}}$ is shown below on the left. Its transposed realization is shown below on the right.



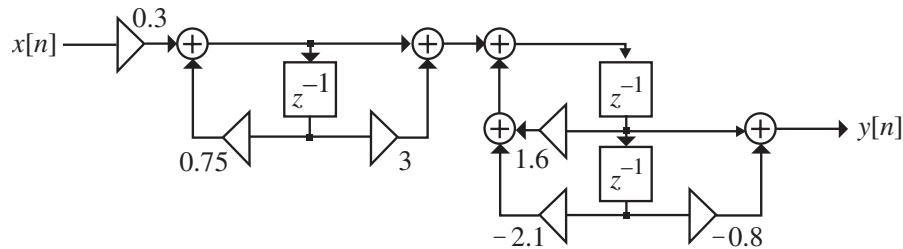
8.21 (a) A cascade canonic realization of $H_1(z)$ based on the decomposition

$$H_1(z) = 0.3 \left(\frac{z^{-1} + 3z^{-2}}{1 - 1.6z^{-1} + 2.1z^{-2}} \right) \left(\frac{1 - 0.8z^{-1}}{1 - 0.75z^{-1}} \right) \text{ is shown below:}$$



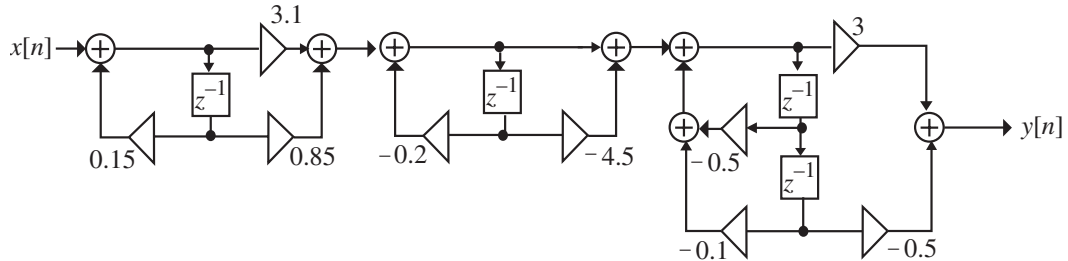
An alternate cascade canonic realization of $H_1(z)$ based on the decomposition

$$H_1(z) = 0.3 \left(\frac{z^{-1} - 0.8z^{-2}}{1 - 1.6z^{-1} + 2.1z^{-2}} \right) \left(\frac{1 + 3z^{-1}}{1 - 0.75z^{-1}} \right) \text{ is shown below:}$$



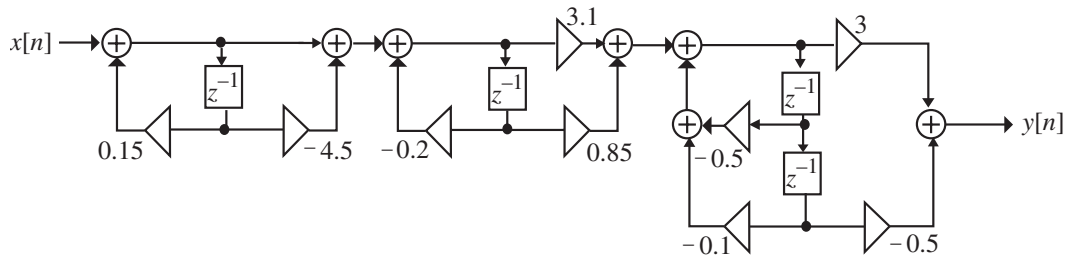
(b) A cascade canonic realization of $H_2(z)$ based on the decomposition

$$H_2(z) = \left(\frac{3.1 + 0.85z^{-1}}{1 - 0.15z^{-1}} \right) \left(\frac{1 - 4.5z^{-1}}{1 + 0.2z^{-1}} \right) \left(\frac{3 - 0.5z^{-2}}{1 + 0.5z^{-1} + 0.1z^{-2}} \right) \text{ is shown below:}$$



An alternate cascade canonic realization of $H_2(z)$ based on the decomposition

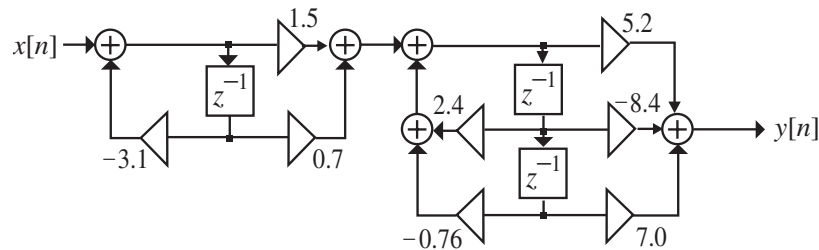
$$H_2(z) = \left(\frac{1 - 4.5z^{-1}}{1 - 0.15z^{-1}} \right) \left(\frac{3.1 + 0.85z^{-1}}{1 + 0.2z^{-1}} \right) \left(\frac{3 - 0.5z^{-2}}{1 + 0.5z^{-1} + 0.1z^{-2}} \right) \text{ is shown below:}$$



(c) The factored form of the denominator of $H_3(z)$ is given by:

$(1 + 3.1z^{-1})(1 - 2.0246z^{-1})(1 - 0.3754z^{-1})$. A cascade canonic realization of $H_3(z)$ based on the decomposition

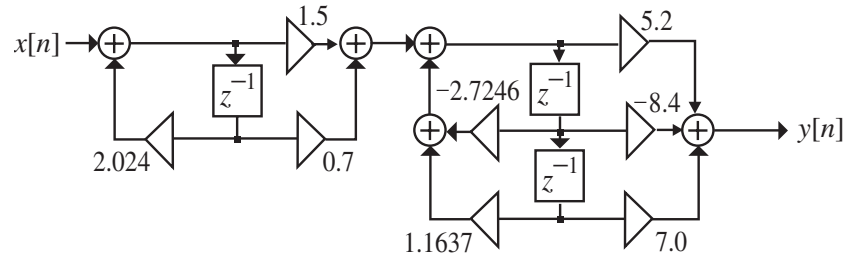
$$H_3(z) = \left(\frac{1.5 + 0.7z^{-1}}{1 + 3.1z^{-1}} \right) \left(\frac{5.2 - 8.4z^{-1} + 7z^{-2}}{(1 - 2.0246z^{-1})(1 - 0.3754z^{-1})} \right) \\ = \left(\frac{1.5 + 0.7z^{-1}}{1 + 3.1z^{-1}} \right) \left(\frac{5.2 - 8.4z^{-1} + 7z^{-2}}{1 - 2.4z^{-1} + 0.76z^{-2}} \right) \text{ is shown below:}$$



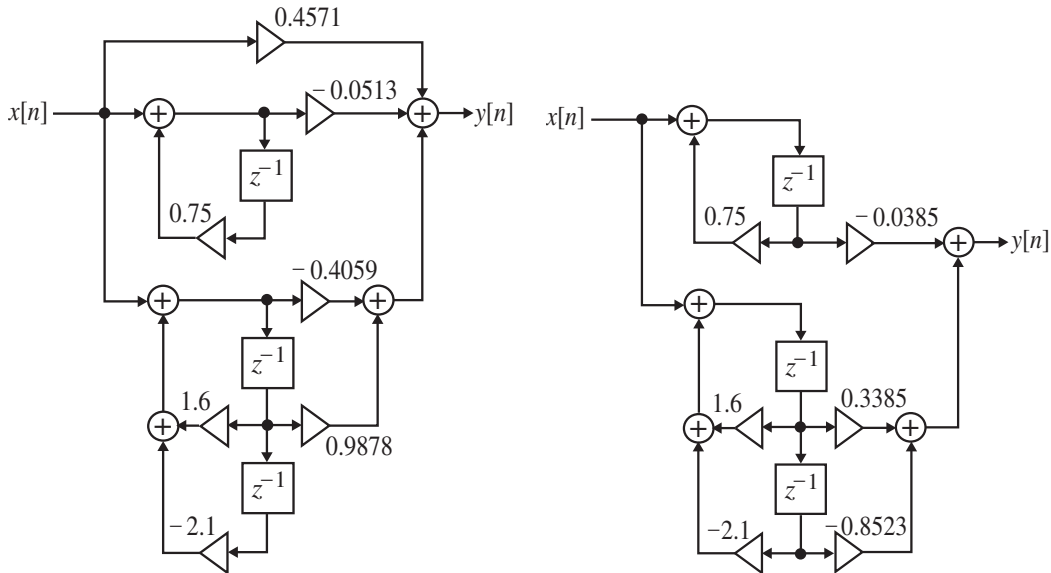
An alternate cascade canonic realization of $H_3(z)$ based on the decomposition

$$H_3(z) = \left(\frac{1.5 + 0.7z^{-1}}{1 - 2.0246z^{-1}} \right) \left(\frac{5.2 - 8.4z^{-1} + 7z^{-2}}{(1 + 3.1z^{-1})(1 - 0.3754z^{-1})} \right)$$

$$= \left(\frac{1.5 + 0.7z^{-1}}{1 - 2.0246z^{-1}} \right) \left(\frac{5.2 - 8.4z^{-1} + 7z^{-2}}{1 + 2.7246z^{-1} - 1.1637z^{-2}} \right) \text{ is shown below:}$$

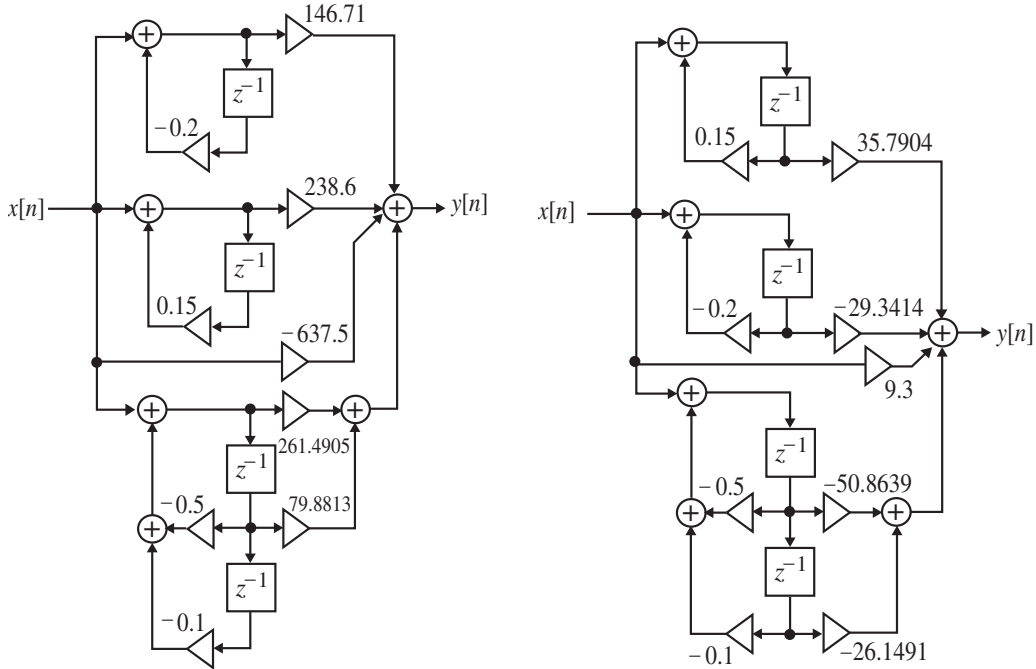


8.22 (a) A partial fraction expansion of $H_1(z)$ in z^{-1} obtained using the M-file `residuez` is given by $H_1(z) = 0.4571 + \frac{-0.0513}{1 - 0.75z^{-1}} + \frac{-0.4059 + 0.9878z^{-1}}{1 - 1.6z^{-1} + 2.1z^{-2}}$. The Parallel Form I realization based on this expansion is shown on the next page on the left side.



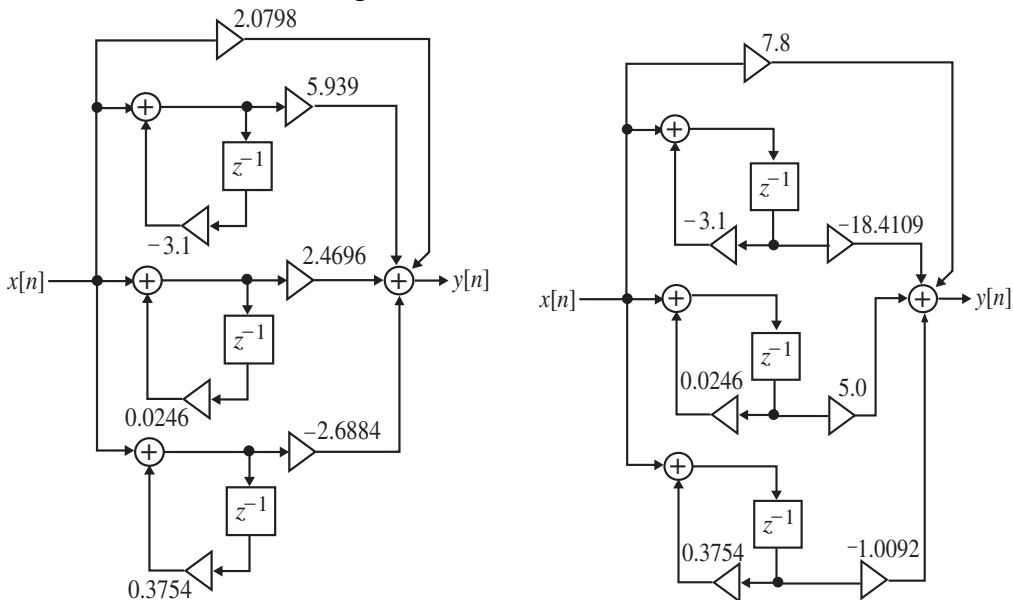
A partial fraction expansion of $H_1(z)$ in z obtained using the M-file `residue` is given by $H_1(z) = \frac{-0.0385z^{-1}}{1 - 0.75z^{-1}} + \frac{0.3385z^{-1} - 0.8523z^{-2}}{1 - 1.6z^{-1} + 2.1z^{-2}}$. The Parallel Form II realization based on this expansion is shown above on the right side.

(b) A partial fraction expansion of $H_2(z)$ in z^{-1} obtained using the M-file `residuez` is given by $H_2(z) = -637.5 + \frac{146.71}{1 + 0.2z^{-1}} + \frac{238.6}{1 - 0.15z^{-1}} + \frac{261.4905 + 79.8813z^{-1}}{1 + 0.5z^{-1} + 0.1z^{-2}}$. The Parallel Form I realization based on this expansion is shown on the next page on the left side.



A partial fraction expansion of $H_2(z)$ in z obtained using `residue` is given by $H_2(z) = 9.3 + \frac{-29.3414z^{-1}}{1+0.2z^{-1}} + \frac{35.7904z^{-1}}{1-0.15z^{-1}} + \frac{-50.8639z^{-1} - 26.1491z^{-2}}{1+0.5z^{-1}+0.1z^{-2}}$. The Parallel Form II realization based on this expansion is shown above on the right side.

(c) A partial fraction expansion of $H_3(z)$ in z^{-1} obtained using `residuez` is given by $H_3(z) = 2.0798 + \frac{5.939}{1+3.1z^{-1}} + \frac{2.4696}{1-0.0246z^{-1}} + \frac{-2.6884}{1-0.3754z^{-1}}$. The Parallel Form I realization based on this expansion is shown below on the left side.

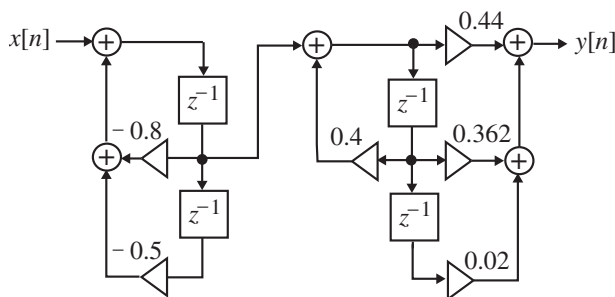


A partial fraction expansion of $H_3(z)$ in z obtained using residue is given by

$H_3(z) = 7.8 + \frac{-18.4109z^{-1}}{1+3.1z^{-1}} + \frac{5.0z^{-1}}{1-0.0246z^{-1}} + \frac{-1.0092z^{-1}}{1-0.3754z^{-1}}$. The Parallel Form II realization based on this expansion is shown on the previous page on the right side.

8.23 A cascade realization of based on the factored form given by

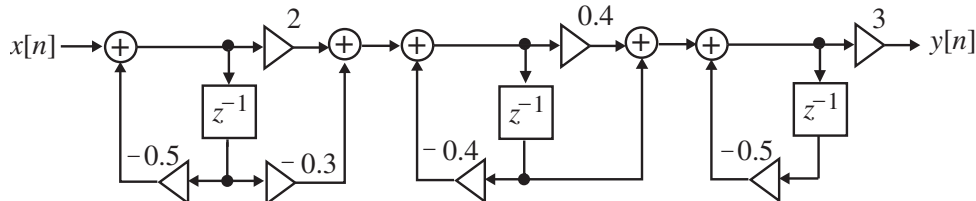
$H(z) = \left(\frac{z^{-1}}{1+0.8z^{-1}+0.5z^{-2}} \right) \left(\frac{0.44+0.362z^{-1}+0.02z^{-2}}{1-0.4z^{-1}} \right)$ is shown below:



8.24 (a) $H(z) = \frac{2-0.3z^{-1}}{1+0.5z^{-1}} \cdot \frac{0.4+z^{-1}}{1+0.4z^{-1}} \cdot \frac{3}{1+0.5z^{-1}} = \frac{2.4+5.64z^{-1}-0.9z^{-2}}{1+1.4z^{-1}+0.65z^{-2}+0.1z^{-3}}$.

(b) $y[n] = 2.4x[n] + 5.64x[n-1] - 0.9x[n-2] - 1.4y[n-1] - 0.65y[n-2] - 0.1y[n-3]$.

(c) A cascade realization of $H(z)$ is shown below:

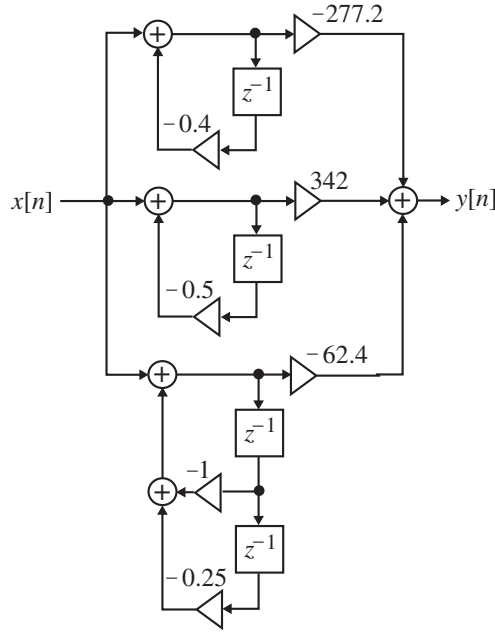


(d) A partial-fraction expansion of $H(z)$ in z^{-1} obtained using residuez is given by

$H(z) = \frac{342}{1+0.5z^{-1}} + \frac{-62.4}{(1+0.5z^{-1})^2} + \frac{-277.2}{1+0.4z^{-1}}$. The Parallel Form I realization based on this expansion is shown on the next page.

(e) The inverse z -transform of the partial-fraction of $H(z)$ given in Part (d) yields

$h[n] = 342(-0.5)^n \mu[n] - 62.4(n+1)(-0.5)^n \mu[n] - 277.2(-0.4)^n \mu[n]$.

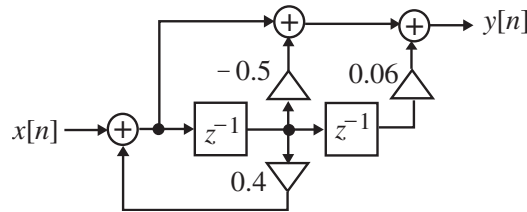


8.25 (a) $Y(z) = \frac{1}{1-0.4z^{-1}} - \frac{0.3z^{-1}}{1-0.4z^{-1}} = \frac{1-0.3z^{-1}}{1-0.4z^{-1}}$, $X(z) = \frac{1}{1-0.2z^{-1}}$. Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-0.3z^{-1})(1-0.2z^{-1})}{1-0.4z^{-1}} = \frac{1-0.5z^{-1}+0.06z^{-2}}{1-0.4z^{-1}}.$$

(b) $y[n] = x[n] - 0.5x[n-1] + 0.06x[n-2] + 0.4y[n-1]$.

(c)



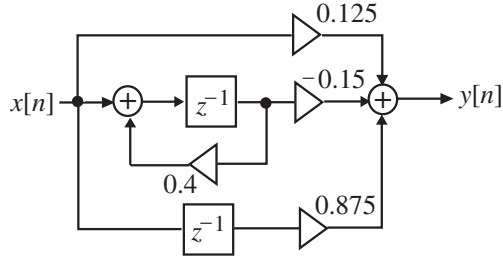
(d) A partial-fraction expansion of $H(z)$ in z^{-1} using `residuez` is given by

$$H(z) = 0.875z^{-1} - 0.15 + \frac{0.125}{1-0.4z^{-1}}$$

whose realization yields the Parallel Form I structure as indicated on the next page.

(e) The inverse z -transform of $H(z)$ yields

$$h[n] = 0.875\delta[n-1] - 0.15\delta[n] + 0.125(0.4)^n \mu[n].$$



(f) $X(z) = \frac{1}{1-0.3z^{-1}} - \frac{0.4z^{-1}}{1-0.3z^{-1}} = \frac{1-0.4z^{-1}}{1-0.3z^{-1}}$. Therefore, $Y(z) = H(z)X(z)$
 $= \left(\frac{(1-0.3z^{-1})(1-0.2z^{-1})}{1-0.4z^{-1}} \right) \left(\frac{1-0.4z^{-1}}{1-0.3z^{-1}} \right) = 1-0.2z^{-1}$, whose inverse z -transform
yields $y[n] = \delta[n] - 0.2\delta[n-1]$.

8.26 Figure P8.12 can be seen to be a Parallel Form II structure. A partial-fraction expansion of $H(z) = \frac{2z^2 - 3.2z - 18.6}{z^2 - 2.6z - 1.2}$ in z obtained using the M-file `residue` is given by $H(z) = 2 + \frac{-3z^{-1}}{1-3z^{-1}} + \frac{5z^{-1}}{1+0.4z^{-1}}$. Comparing the coefficients of the expansion with the corresponding multiplier coefficients in Figure P8.12 we conclude that the multiplier coefficient -5 in the feed forward path should be 5 and the multiplier coefficient -0.5 in the feedback path should be replaced with -0.4 .

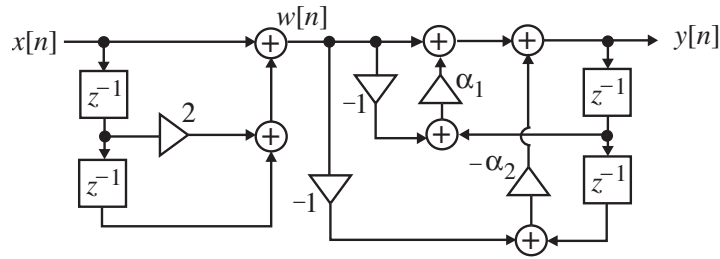
8.27 Figure P8.13 can be seen to be a Parallel Form I structure. A partial-fraction expansion of $H(z) = \frac{4z^2 - 5.6z}{z^2 + 0.2z - 0.08}$ in z^{-1} obtained using the M-file `residuez` is given by $H(z) = \frac{12}{1+0.4z^{-1}} + \frac{-8}{1-0.2z^{-1}} = -8 \left(\frac{-1.5}{1+0.4z^{-1}} + \frac{1}{1-0.2z^{-1}} \right)$. Comparing the coefficients of the expansion with the corresponding multiplier coefficients in Figure P8.12 we conclude that the multiplier coefficient A has a value equal to -8 and the multiplier coefficient B has a value equal to -0.4 .

8.28 (a) The difference equation corresponding to the transfer function $H(z) = \frac{Y(z)}{X(z)}$

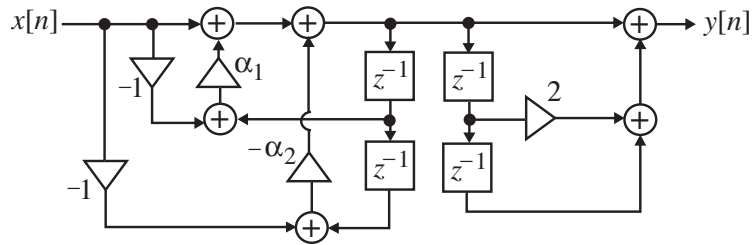
$$= \frac{(1 + \alpha_1 + \alpha_2)(1 + 2z^{-1} + z^{-2})}{1 - \alpha_1 z^{-1} + \alpha_2 z^{-2}}$$
 is given by

$y[n] + \alpha_1 y[n-1] - \alpha_2 y[n-2] = (1 + \alpha_1 + \alpha_2)(x[n] + 2x[n-1] + x[n-2])$, which can be rewritten as $y[n] = (x[n] + 2x[n-1] + x[n-2]) + \alpha_1 (y[n-1] - x[n] - 2x[n-1] - x[n-2]) - \alpha_2 (y[n-2] - x[n] - 2x[n-1] - x[n-2])$. Denoting $w[n] = x[n] + 2x[n-1] + x[n-2]$, the difference equation representation becomes

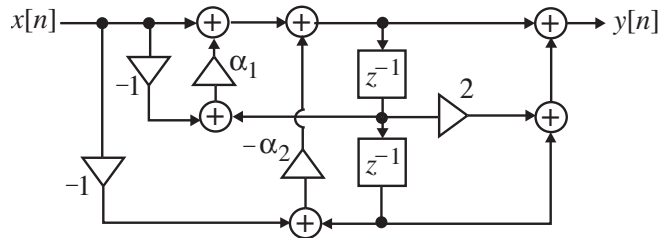
$y[n] = w[n] + \alpha_1(y[n-1] - w[n]) - \alpha_2(y[n-2] - w[n])$. A realization of $H(z)$ based on the last two equations is as indicated below:



An interchange of the two stages leads to an equivalent realization shown below:



Finally, by delay sharing the above structure reduces to the canonic realization shown below:



(b) The difference equation corresponding to the transfer function $H(z) = \frac{Y(z)}{X(z)}$

$$= \frac{(1 - \alpha_2)(1 - z^{-2})}{1 - \alpha_1 z^{-1} + \alpha_2 z^{-2}}$$
 is given by

$$y[n] - \alpha_1 y[n-1] + \alpha_2 y[n-2] = (1 - \alpha_2)(x[n] - x[n-2]) \text{ which can be rewritten as}$$

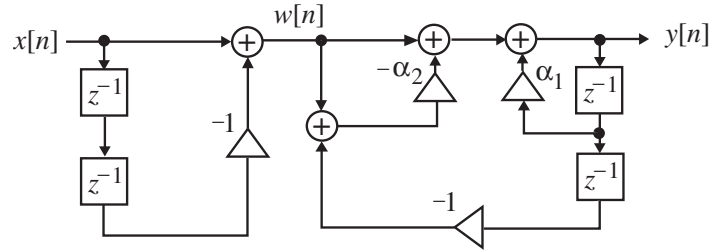
$$y[n] = \alpha_1 y[n-1] - \alpha_2 y[n-2] + x[n] - \alpha_2 x[n] - x[n-2] + \alpha_2 x[n-2]$$

$$= \alpha_1 y[n-1] - \alpha_2 (x[n] - x[n-2] + y[n-2]) + (x[n] - x[n-2]). \text{ Denoting}$$

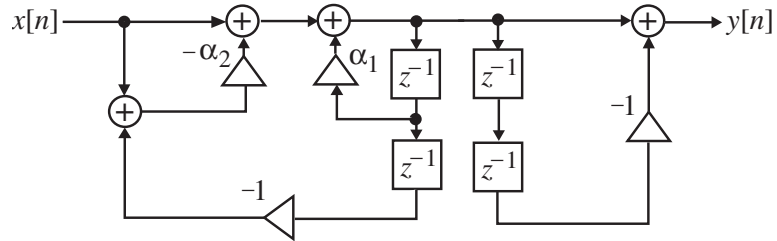
$$w[n] = x[n] - x[n-2], \text{ we can rewrite the last equation as}$$

$$y[n] = \alpha_1 y[n-1] - \alpha_2 (w[n] + y[n-2]) + w[n].$$

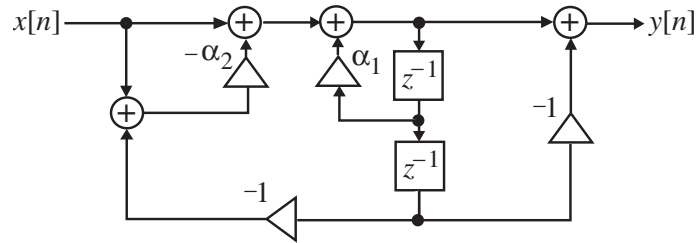
A realization of $H(z)$ based on the last two equations is as shown on next page.



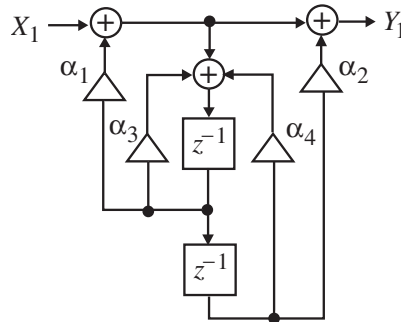
An interchange of the two stages leads to an equivalent realization shown below:



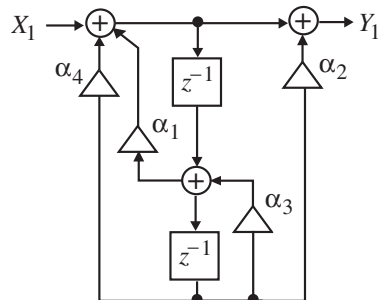
Finally, by delay sharing the above structure reduces to a canonic realization as shown below:



8.29 (a)



(b)



8.30 (a) From the structure of Figure P8.14 it follows that $H_N(z) = \frac{Y_1}{X_1} = \frac{C + D \cdot H_{N-1}(z)}{A + B \cdot H_{N-1}(z)}$,

from which we get $H_{N-1}(z) = \frac{C - A \cdot H_N(z)}{B \cdot H_N(z) - D}$. Substituting the expression for $H_N(z)$ we

$$\text{then arrive at } H_{N-1}(z) = \frac{C \left(1 + \sum_{i=1}^N d_i z^{-i} \right) - A \left(\sum_{i=0}^N p_i \right)}{B \left(\sum_{i=0}^N p_i \right) - D \left(1 + \sum_{i=1}^N d_i z^{-i} \right)}$$

$$= \frac{(C - Ap_0) + (Cd_1 - Ap_1)z^{-1} + \dots + (Cd_{N-1} - Ap_{N-1})z^{-N+1} + (Cd_N - Ap_N)z^{-N}}{(Bp_0 - D) + (Bp_1 - Dd_1)z^{-1} + \dots + (Bp_{N-1} - Dd_{N-1})z^{-N+1} + (Bp_N - Dd_N)z^{-N}}.$$

Substituting the values $A = 1, B = d_N z^{-1}, C = p_0$, and $D = p_N z^{-1}$, we get $H_{N-1}(z)$

$$= \frac{(p_0 - p_0) + (p_0 d_1 - p_1)z^{-1} + \dots + (p_0 d_{N-1} - p_{N-1})z^{-N+1} + (p_0 d_N - p_N)z^{-N}}{(d_N p_0 - p_N)z^{-1} + (d_N p_1 - p_N d_1)z^{-2} + \dots + (d_N p_{N-1} - p_N d_{N-1})z^{-N} + (d_N p_N - p_N d_N)z^{-N-1}}$$

$$= \frac{(p_0 d_1 - p_1) + \dots + (p_0 d_{N-1} - p_{N-1})z^{-N+2} + (p_0 d_N - p_N)z^{-N+1}}{(d_N p_0 - p_N) + (d_N p_1 - p_N d_1)z^{-1} + \dots + (d_N p_{N-1} - p_N d_{N-1})z^{-N+1}}$$

$$= \frac{p'_0 + p'_1 z^{-1} + \dots + p'_{N-2} z^{-N+2} + p'_{N-1} z^{-N+1}}{1 + d'_1 z^{-1} + \dots + d'_{N-2} z^{-N+2} + d'_{N-1} z^{-N+1}} \text{ where}$$

$$p'_k = \frac{p_k d_{k+1} - p_{k+1}}{p_0 d_N - p_N}, 0 \leq k \leq N-1, \text{ and } d'_k = \frac{p_k d_N - p_N d_k}{p_0 d_N - p_N}, 1 \leq k \leq N-1.$$

(b) From the chain parameters, we obtain for the first stage $t_{11} = \frac{C}{A} = p_0$,

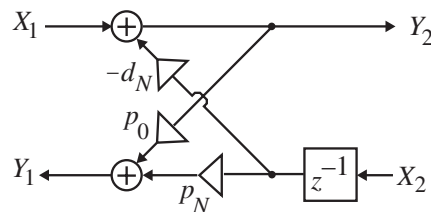
$$t_{12} = \frac{AD - BC}{A} = (p_N - p_0 d_N)z^{-1}, t_{21} = \frac{1}{A} = 1, t_{22} = -\frac{B}{A} = -d_N z^{-1}.$$

The corresponding input-output relations of the two-pair are given by

$$Y_1 = p_0 X_1 + (p_N - p_0 d_N)z^{-1} X_2 = p_0 (X_1 - d_N z^{-1} X_2) + p_N z^{-1} X_2,$$

$$Y_2 = X_1 - d_N z^{-1} X_2. \text{ Substituting the second equation into the first we rewrite it as}$$

$Y_1 = p_0 Y_2 + p_N z^{-1} X_2$. A realization of the two-pair based on the last two equations is therefore as indicated below:

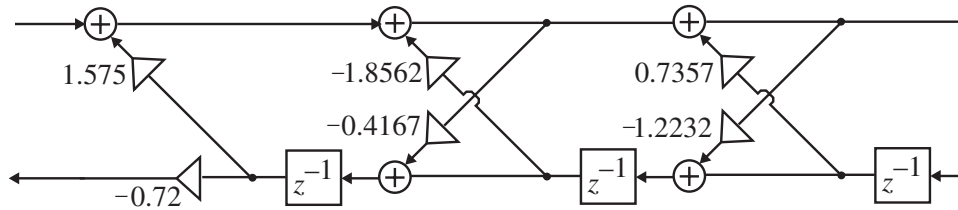


(c) Except for the first stage, all other stages require 2 multipliers. Hence, the total number of multipliers needed to implement an N -th order transfer function $H_N(z)$ is $2N + 1$. The total number of 2-input adders required is $2N$, while the overall realization is canonic requiring N delays.

8.31 (a) From $G_3(z) = H_1(z) = \frac{0.3z^{-1} + 0.66z^{-2} - 0.72z^{-3}}{1 - 2.35z^{-1} + 3.3z^{-2} - 1.575z^{-3}}$, using Eq. (8.132a) and

(8.132b) we get $G_2(z) = \frac{-0.4167 - 0.9167z^{-1} + z^{-2}}{1 - 3.0063z^{-1} + 1.8562z^{-2}}$. Repeating the procedure we get

$G_1(z) = \frac{-1.2232 + z^{-1}}{1 - 0.7357z^{-1}}$. From $G_3(z)$, $G_2(z)$, and $G_1(z)$, we arrive at the cascaded lattice realization shown below:



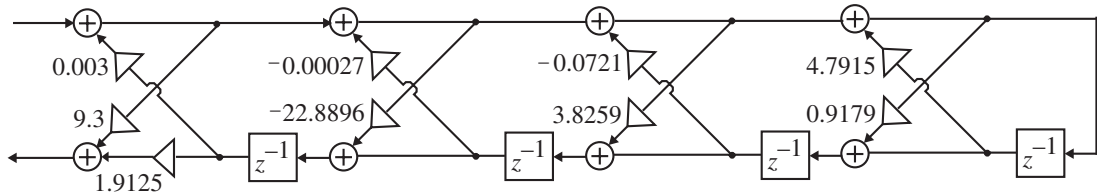
(b) From $G_4(z) = H_2(z) = \frac{9.3 - 39.3z^{-1} - 13.025z^{-2} + 6.55z^{-3} + 1.9125z^{-4}}{1 + 0.55z^{-1} + 0.095z^{-2} - 0.01z^{-3} - 0.003z^{-4}}$, using Eq.

(8.132a) and (8.132b) we get $G_3(z) = \frac{-22.8896 - 7.1679z^{-1} + 3.4235z^{-2} + z^{-3}}{1 + 0.4813z^{-1} + 0.0735z^{-2} + 0.00027z^{-3}}$.

Repeating the procedure twice we get $G_2(z) = \frac{3.8259 + 5.0744z^{-1} + z^{-2}}{1 + 0.4803z^{-1} + 0.0721z^{-2}}$ and

$G_1(z) = \frac{4.4704 + z^{-1}}{1 + 0.1579z^{-1}}$. From $G_4(z)$, $G_3(z)$, $G_2(z)$, and $G_1(z)$, we arrive at the cascaded

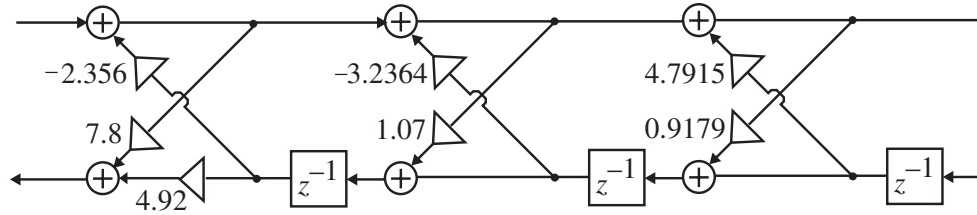
lattice realization shown below:



(c) From $G_3(z) = H_3(z) = \frac{7.8 - 8.96z^{-1} + 4.62z^{-2} + 4.92z^{-3}}{1 + 0.7z^{-1} - 6.68z^{-2} + 2.356z^{-3}}$, using Eq. (8.132a) and

(8.132b) we get $G_2(z) = \frac{1.07 - 4.209z^{-1} + z^{-2}}{1 - 1.8209z^{-1} + 3.2364z^{-2}}$. Repeating the procedure we get

$G_1(z) = \frac{0.9179 + z^{-1}}{1 - 4.7915z^{-1}}$. From $G_3(z), G_2(z)$, and $G_1(z)$, we arrive at the cascaded lattice realization shown below:



8.32 When $H_N(z)$ is an allpass transfer function of the form

$$H_N(z) = A_N(z) = \frac{d_N + d_{N-1}z^{-1} + \dots + d_1z^{-N+1} + z^{-N}}{1 + d_1z^{-1} + \dots + d_{N-1}z^{-N+1} + d_Nz^{-N}},$$

then from Eq. (8.132a), the numerator coefficients of $H_{N-1}(z)$ are given by

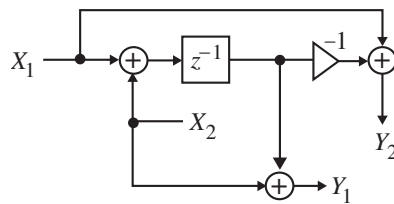
$$p'_k = \frac{p_0 d_{k+1} - p_{k+1}}{p_0 d_N - p_N} = \frac{d_N d_{k+1} - d_{N-k-1}}{d_N^2 - 1},$$

and from Eq. (8.132b) the denominator coefficients of $H_{N-1}(z)$ are given by

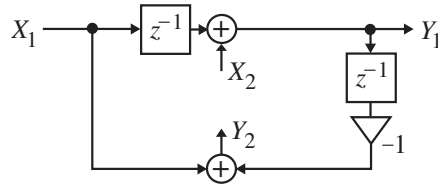
$$d'_{N-k-1} = \frac{p_{N-k-1} d_N - d_{N-k-1}}{p_0 d_N - p_N} = \frac{d_{k+1} d_N - d_{N-k-1}}{d_N^2 - 1} = p'_k,$$

implying $H_{N-1}(z)$ is an allpass transfer function of order $N - 1$. Since here $p_N = 1$ and $p_0 = d_N$, the lattice structure of Problem 8.30 then reduces to the lattice structure employed in the Gray-Markel realization procedure.

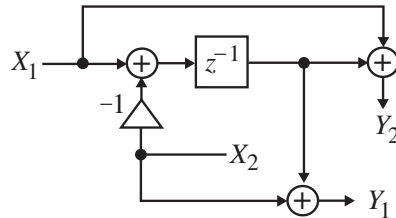
8.33 (a) Consider the realization of Type 1B allpass structure. From its transfer parameters given in Eq. (8.50b) we arrive at $Y_1 = z^{-1}X_1 + (1 + z^{-1})X_2 = z^{-1}(X_1 + X_2) + X_2$, and $Y_2 = (1 - z^{-1})X_1 - z^{-1}X_2 = X_1 - z^{-1}(X_1 + X_2)$. A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 8.24(b).



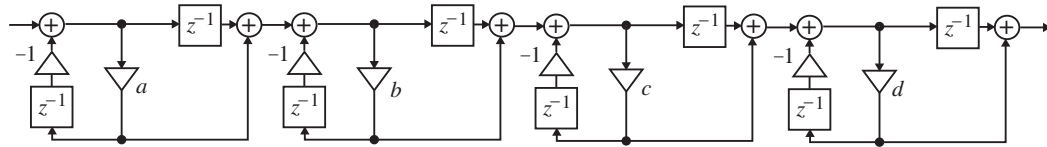
(b) From the transfer parameters of Type 1A_i allpass given in Eq. (8.50c) we obtain $Y_1 = z^{-1}X_1 + X_2$, and $Y_2 = (1 - z^{-2})X_1 - z^{-1}X_2 = X_1 - z^{-1}(z^{-1}X_1 + X_2) = X_1 - z^{-1}Y_1$. A realization of the two-pair based on these two equations is as shown on next page which leads to the structure of Figure 8.24(c).



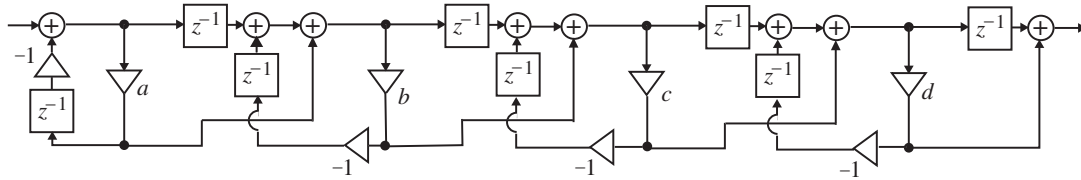
(c) From the transfer parameters of Type 1A₁ allpass given in Eq. (8.50d) we obtain $Y_1 = z^{-1}X_1 + (1 - z^{-1})X_2 = z^{-1}(X_1 - X_2) + X_2$, and $Y_2 = (1 + z^{-1})X_1 - z^{-1}X_2 = X_1 + z^{-1}(X_1 - X_2)$. A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 8.24(d).



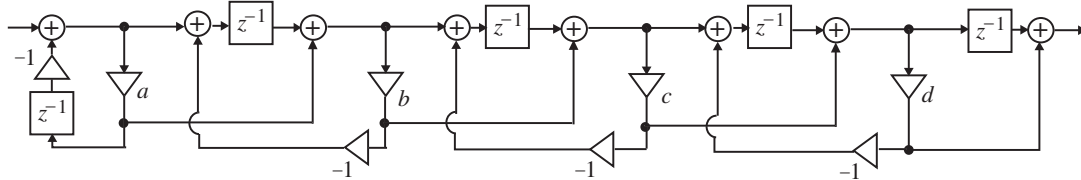
8.34 (a) A cascade connection of 4 Type 1A first-order allpass networks is shown below:



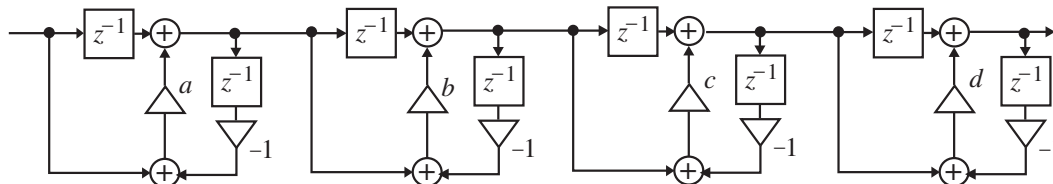
Simple block diagram manipulation of the above structure leads to:



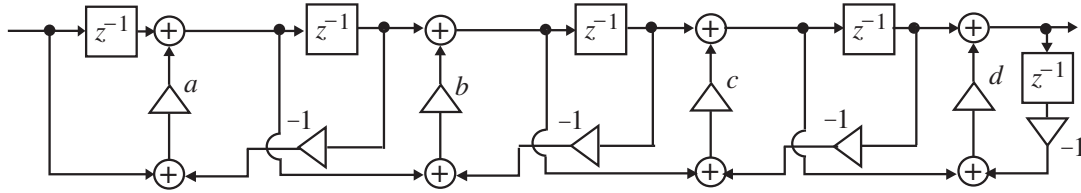
Finally, by delay sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 5 delays compared to 8 delays in the direct realization shown on the previous page.



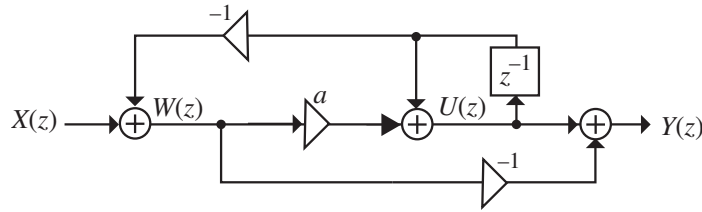
(b) A cascade connection of 4 Type 1A₁ first-order allpass networks is shown below which requires 8 delays.



By delay sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 5 delays.



8.35 The structure of Figure P8.15 with internal variables labeled is shown below:



Its analysis yields (1): $W(z) = X(z) - z^{-1}U(z)$, (2): $U(z) = aW(z) + z^{-1}U(z)$, and (3):

$Y(z) = -W(z) + U(z)$. From Eq. (2) we obtain (4): $U(z) = \frac{a}{1 - z^{-1}}W(z)$. Substituting Eq.

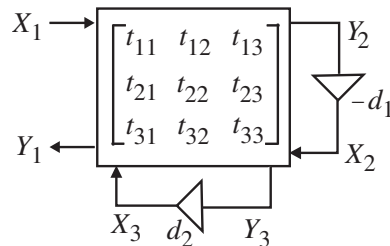
(4) in Eq. (3) we get (5): $Y(z) = -\left(\frac{1 - a - z^{-1}}{1 - z^{-1}}\right)W(z)$. Substituting Eq. (4) in Eq. (1) we get

(6): $X(z) = \left(\frac{1 - a - z^{-1} + az^{-1}}{1 - z^{-1}}\right)W(z)$. Finally, from Eqs. (5) and (6) we arrive at

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-(1-a) + z^{-1}}{1 - (1-a)z^{-1}}.$$

8.36 We realize $A_2(z) = \frac{d_1d_2 + d_1z^{-1} + z^{-2}}{1 + d_1z^{-1} + d_1d_2z^{-2}}$ in the form of a constrained three-pair as

indicated below:



From the above figure, we have
$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \text{ and } X_2 = -d_1Y_2,$$

$X_3 = d_2 Y_3$. From these equations, we get after some algebra $A_2(z) = \frac{Y_1}{X_1} = \frac{N(z)}{D(z)}$, where

$$D(z) = 1 + d_1 t_{22} - d_2 t_{33} + d_1 d_2 (t_{23} t_{32} - t_{22} t_{33}), \text{ and}$$

$$N(z) = t_{11} + d_1 (t_{11} t_{22} - t_{12} t_{21}) - d_2 (t_{11} t_{33} - t_{13} t_{31}) + d_1 d_2 \{t_{21} (t_{12} t_{33} - t_{13} t_{32}) + t_{31} (t_{22} t_{13} - t_{12} t_{23}) + t_{11} (t_{23} t_{32} - t_{22} t_{33})\}$$

Comparing the denominator of the desired allpass transfer function with $D(z)$ we get

$$t_{22} = z^{-1}, t_{33} = 0, t_{23} t_{32} = z^{-2}. \text{ Next, comparing the numerator of the desired allpass}$$

$$\text{transfer function with } N(z) \text{ we get } t_{11} = z^{-2}, t_{12} t_{21} = z^{-1}(z^{-2} - 1), t_{13} t_{31} = 0, \text{ and}$$

$$t_{32} (t_{11} t_{23} - t_{21} t_{13}) + t_{31} (t_{22} t_{13} - t_{12} t_{23}) = 1. \text{ Substituting the appropriate transfer}$$

$$\text{parameters from the previous equations into the last equation we simplify it to}$$

$$t_{13} t_{21} t_{32} + t_{31} t_{12} t_{23} = z^{-4} - 1. \text{ Since } t_{13} t_{31} = 0, \text{ either } t_{13} = 0, \text{ or } t_{31} = 0. \text{ (Both cannot be simultaneously equal to zero, as this will violate the condition } t_{13} t_{21} t_{32} + t_{31} t_{12} t_{23} = z^{-4} - 1.$$

Consider the case $t_{13} = 0$. Then the above equation reduces to $t_{31} t_{12} t_{23} = z^{-4} - 1$. From

$$\text{this equation and } t_{23} t_{32} = z^{-2}, \text{ it follows that } t_{32} = z^{-2}, t_{23} = 1, t_{31} t_{12} = z^{-4} - 1$$

$$= (z^{-1} - 1)(z^{-1} + 1)(z^{-2} + 1). \text{ There are four possible realizable sets of values of } t_{21} \text{ and}$$

$$t_{31} \text{ satisfying the last equation and } t_{12} t_{21} = z^{-1}(z^{-2} - 1). \text{ These lead to 4 different}$$

$$\text{realizable transfer matrices for the three-pair: Type 2A: } \begin{bmatrix} z^{-2} & z^{-2} - 1 & 0 \\ z^{-1} & z^{-1} & 1 \\ z^{-2} + 1 & z^{-2} & 0 \end{bmatrix},$$

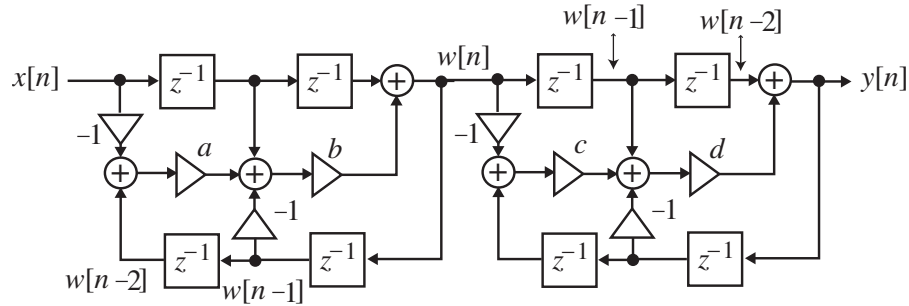
$$\text{Type 2B: } \begin{bmatrix} z^{-2} & z^{-1} + 1 & 0 \\ z^{-1}(z^{-1} - 1) & z^{-1} & 1 \\ (z^{-2} + 1)(z^{-1} - 1) & z^{-2} & 0 \end{bmatrix}, \text{ Type 2C: } \begin{bmatrix} z^{-2} & z^{-1} - 1 & 0 \\ z^{-1}(z^{-1} + 1) & z^{-1} & 1 \\ (z^{-2} + 1)(z^{-1} + 1) & z^{-2} & 0 \end{bmatrix},$$

$$\text{Type 2D: } \begin{bmatrix} z^{-2} & 1 & 0 \\ z^{-1}(z^{-2} - 1) & z^{-1} & 1 \\ z^{-4} - 1 & z^{-2} & 0 \end{bmatrix}. \text{ A realization of each of the above Type 2 allpass}$$

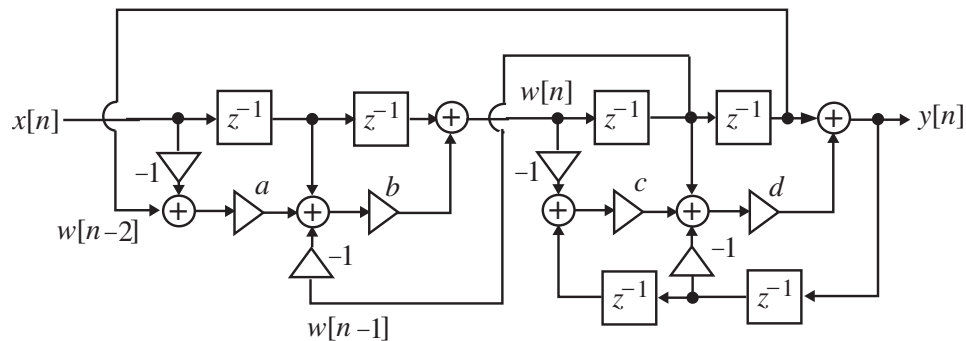
structures is obtained by implementing its respective transfer matrix, and then constraining the Y_2 and X_2 variables through the multiplier d_1 and constraining the Y_3 and X_3 variables through the multiplier $-d_2$ resulting in the four structures shown in Figure 8.25 of the text.

It can be easily shown that the allpass structures obtained for the case $t_{31} = 0$ are precisely the transpose of the structures of Figure 8.25.

8.37 A cascade of two Type 2D second-order allpass networks is shown below which requires 8 delays.



By delay sharing between adjacent allpass sections we arrive at the following equivalent realizations requiring now 6 delays.



The minimum number of delays needed to implement a cascade of Type 2D second-order allpass sections is thus $4 + 2(M - 1) = 2(M + 1)$.

8.38 We realize $A_2(z) = \frac{d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}$ in the form of a constrained three-pair as indicated

in the solution of Problem 8.36. Comparing the numerator and the denominator of the Type 3 allpass transfer function with $N(z)$ and $D(z)$ given in the solution of Problem 8.36 we arrive at

$$t_{11} = z^{-2}, t_{22} = z^{-1}, t_{22} = z^{-2}, t_{33} = z^{-2}, t_{23}t_{32} = z^{-3}, t_{12}t_{212} = z^{-1}(z^{-2} - 1),$$

$$t_{13}t_{31} = z^{-4} - 1, t_{13}t_{21}t_{32} + t_{13}t_{21}t_{23} = z^{-3}(z^{-2} - 1) + z^{-1}(z^{-4} - 1).$$

To solve the last four equations to arrive at a set of realizable set of values for the transfer parameters, $t_{12}, t_{21}, t_{13}, t_{31}, t_{23}$, and t_{32} , we need to pre-select t_{23} and t_{32} satisfying the condition $t_{23}t_{32} = z^{-3}$, and then determine realizable values for the other transfer parameters.

(a) Let $t_{23} = z^{-1}$, $t_{32} = z^{-2}$. There are 4 possible realizable sets of values of t_{12}, t_{21}, t_{13} , and t_{31} satisfying the constraints given above resulting in the following transfer matrices of the three-pair:

$$\begin{aligned} \text{Type 3A: } & \begin{bmatrix} z^{-2} & z^{-2} - 1 & z^{-2} - 1 \\ z^{-1} & z^{-1} & z^{-1} \\ z^{-2} + 1 & z^{-2} & z^{-2} \end{bmatrix}, \text{ Type 3B: } \begin{bmatrix} z^{-2} & z^{-1} + 1 & z^{-1} + 1 \\ z^{-1}(z^{-1} - 1) & z^{-1} & z^{-1} \\ (z^{-1} - 1)(z^{-2} + 1) & z^{-2} & z^{-2} \end{bmatrix}, \\ \text{Type 3C: } & \begin{bmatrix} z^{-2} & z^{-1} - 1 & z^{-1} - 1 \\ z^{-1}(z^{-1} + 1) & z^{-1} & z^{-1} \\ (z^{-1} + 1)(z^{-2} + 1) & z^{-2} & z^{-2} \end{bmatrix}, \text{ Type 3D: } \begin{bmatrix} z^{-2} & 1 & 1 \\ z^{-1}(z^{-2} - 1) & z^{-1} & z^{-1} \\ z^{-4} - 1 & z^{-2} & z^{-2} \end{bmatrix}. \end{aligned}$$

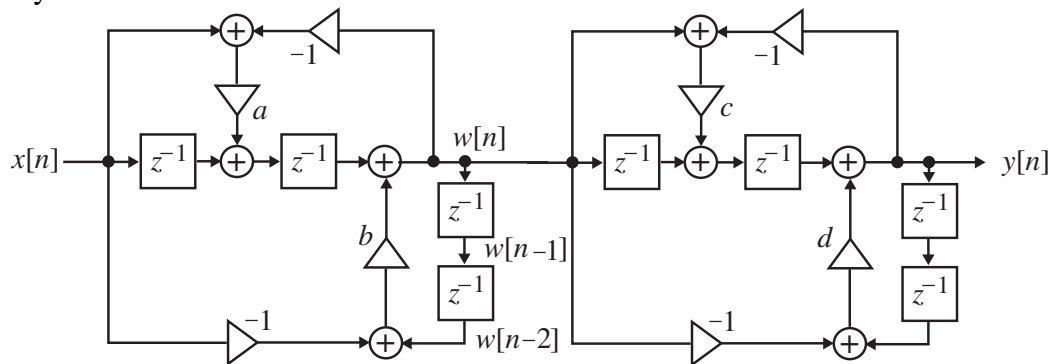
(b) $t_{23} = z^{-2}$, $t_{32} = z^{-1}$. This leads to 4 new realizations which are transpose of the 4 structures developed in Case (a).

(c) $t_{23} = 1$, $t_{32} = z^{-3}$. Here also there are 4 different realizations whose transfer matrices are as given below:

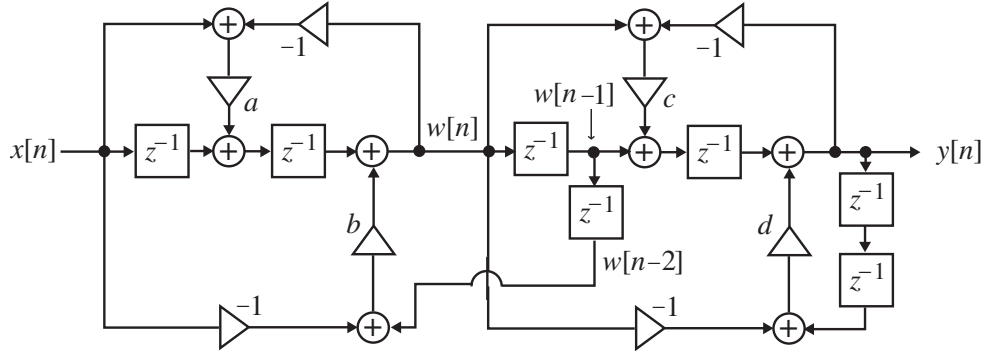
$$\begin{aligned} \text{Type 3E: } & \begin{bmatrix} z^{-2} & z^{-1}(z^{-2} - 1) & z^{-2} - 1 \\ 1 & z^{-1} & z^{-1} \\ z^{-2} + 1 & z^{-3} & z^{-2} \end{bmatrix}, \\ \text{Type 3F: } & \begin{bmatrix} z^{-2} & z^{-1}(z^{-1} + 1) & z^{-1} + 1 \\ z^{-1} - 1 & z^{-1} & 1 \\ (z^{-1} - 1)(z^{-2} + 1) & z^{-3} & z^{-2} \end{bmatrix}, \\ \text{Type 3G: } & \begin{bmatrix} z^{-2} & z^{-1}(z^{-1} - 1) & z^{-1} - 1 \\ z^{-1} + 1 & z^{-1} & 1 \\ (z^{-1} + 1)(z^{-2} + 1) & z^{-3} & z^{-2} \end{bmatrix}, \text{ Type 3H: } \begin{bmatrix} z^{-2} & z^{-1} & 1 \\ z^{-2} - 1 & z^{-1} & 1 \\ z^{-4} - 1 & z^{-3} & z^{-2} \end{bmatrix}. \end{aligned}$$

(d) $t_{23} = z^{-3}$, $t_{32} = 1$. This leads to 4 new realizations which are transpose of the 4 structures developed in Case (b).

8.39 A cascade of two Type 3B second-order allpass networks is shown below which requires 8 delays.

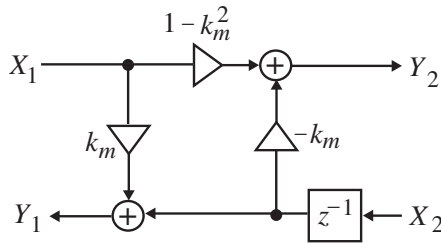


By delay sharing between adjacent allpass sections we arrive at the following equivalent realizations requiring now 7 delays.

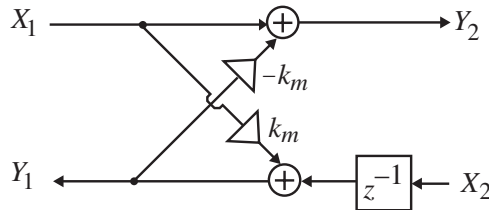


The minimum number of delays needed to implement a cascade of Type 3B second-order allpass sections is thus $4 + 3(M - 1) = 3(M + 1)$.

- 8.40** From the transfer parameters given in Eq. (8.50d) we arrive at the input-output relations of the two-pair as (1): $Y_1 = k_m X_1 + z^{-1} X_2$, and (2): $Y_2 = (1 - k_m^2) X_1 - k_m z^{-1} X_2$. A three-multiplier realization based on these two equations is shown below:

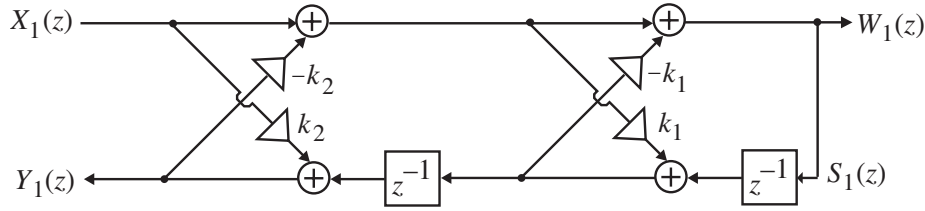


- 8.41** Eq. (2) in the solution of Problem 8.40 can be rewritten as:
 $Y_2 = X_1 - k_m^2 X_1 - k_m z^{-1} X_2 = X_1 - k_m (k_m X_1 + z^{-1} X_2)$. Substituting Eq. (1) of the solution of Problem 8.40 in this equation we then get (3): $Y_2 = X_1 - k_m Y_1$. A realization of the two-pair based on Eq. (3) and Eq. (1) of the solution of Problem 8.40 results in the lattice structure shown below:



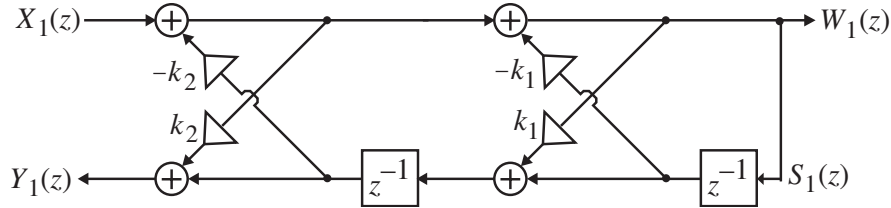
From Eqs. (1) and (2) of the solution of Problem 8.40 we arrive at the transfer parameters of the lattice two-pair given by $t_{11} = k_m$, $t_{12} = z^{-1}$, $t_{21} = 1 - k_m^2$, $t_{22} = -k_m z^{-1}$. The corresponding chain parameters are obtained using Eq. (7.128b)

and are given by $A = \frac{1}{1-k_m^2}$, $B = \frac{k_m z^{-1}}{1-k_m^2}$, $C = \frac{k_m}{1-k_m^2}$, $D = \frac{z^{-1}}{1-k_m^2}$.



Hence, the input-output relation of the all-pole cascade lattice structure given above can be expressed as: $\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \frac{1}{(1-k_2^2)(1-k_1^2)} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} W_1(z) \\ S_1(z) \end{bmatrix}$, from

which we obtain $H_r(z) = \frac{W_1(z)}{X_1(z)} = \frac{(1-k_2^2)(1-k_1^2)}{1+k_1(1+k_2)z^{-1}+k_2z^{-2}}$.



Likewise, from Section 8.8.1, we arrive at input-output relation of the all-pole cascade lattice structure given above as: $\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} W_1(z) \\ S_1(z) \end{bmatrix}$, from

which we obtain $H_s(z) = \frac{W_1(z)}{X_1(z)} = \frac{1}{1+k_1(1+k_2)z^{-1}+k_2z^{-2}}$.

Now a second-order all-pole transfer function can also be expressed in terms of its poles as $H(z) = \frac{1}{1-2r \cos(\omega_o)z^{-1}+r^2z^{-2}}$. Comparing the denominator of with that

of $H_s(z)$ and $H_r(z)$, we observe $k_2 = r^2$ and $k_1(1+k_2) = -2r \cos(\omega_o)$. As $r \cong 1$, $k_1 = -\frac{2r \cos(\omega_o)}{1+r^2} \cong -\cos(\omega_o)$. At the true resonance peak, we have

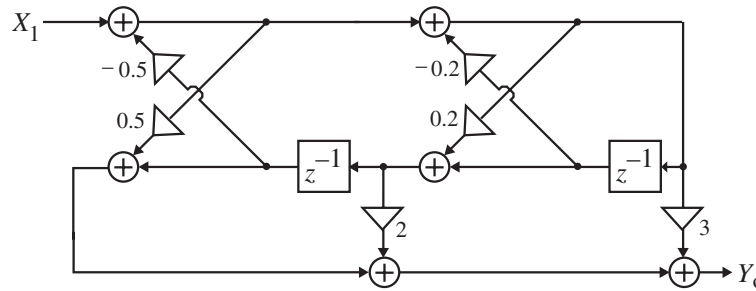
$|H(e^{j\omega_r})| = \frac{1}{(1-r^2)|\sin(\omega_o)|}$. Likewise, at $\omega = \omega_r$, $|H_s(e^{j\omega_r})| = \frac{1}{(1-k_2)\sqrt{1-k_1^2}}$. As $r \rightarrow 1$, $k_2 = r^2 \rightarrow 1$, and $(1-k_1^2)(1-k_2^2) = (1-k_1^2)(1-k_2)(1+k_2) \cong 2(1-k_1^2)(1-k_2)$.

Hence, at $\omega = \omega_r$, $|H_r(e^{j\omega_r})| \cong \frac{2(1-k_1^2)(1-k_2)}{(1-k_2)\sqrt{1-k_1^2}} = 2\sqrt{1-k_1^2} = 2|\sin(\omega_r)|$. This

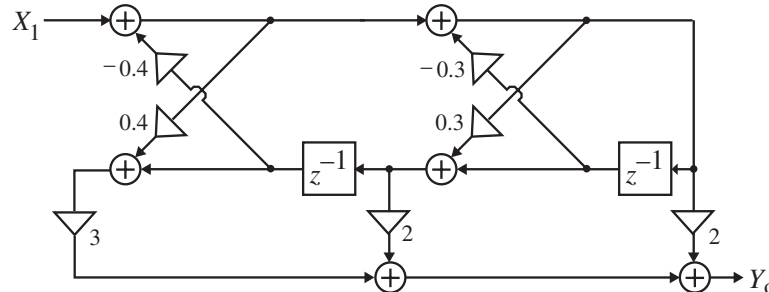
indicates that the gain at the peak resonance is approximately independent of the pole

radius as long as the pole angle is constant and the radius is close to 1.

- 8.42 (a)** $H_1(z) = \frac{3.9 + 2.3z^{-1} + z^{-2}}{1 + 0.3z^{-1} + 0.5z^{-2}}$. Choose $A_2(z) = \frac{0.5 + 0.3z^{-1} + z^{-2}}{1 + 0.3z^{-1} + 0.5z^{-2}}$. Note $k_2 = A_2(\infty) = d_2 = 0.5 < 1$. Using Eq. (8.45) we next determine the coefficient d_1' of the allpass transfer function $A_1(z)$ arriving at $A_1(z) = \frac{0.2 + z^{-1}}{1 + 0.2z^{-1}}$. Here, $k_1 = A_1(\infty) = d_1' = 0.2 < 1$. Therefore, $A_2(z)$ and hence, $H_1(z)$ is stable. The feed-forward coefficients are given by $\alpha_1 = p_2 = 1.0$, $\alpha_2 = p_1 - \alpha_1 d_1' = 2.0$, $\alpha_3 = p_0 - \alpha_1 d_2 - \alpha_2 d_1' = 3.0$. Final realization of $H_1(z)$ is as shown below:



- (b)** $H_2(z) = \frac{2.6 + 0.74z^{-1} + 3z^{-2}}{1 - 0.42z^{-1} + 0.4z^{-2}}$. Choose $A_2(z) = \frac{0.4 - 0.42z^{-1} + z^{-2}}{1 - 0.42z^{-1} + 0.4z^{-2}}$. Note $k_2 = A_2(\infty) = d_2 = 0.4 < 1$. Using Eq. (8.45) we next determine the coefficient d_1' of the allpass transfer function $A_1(z)$ arriving at $A_1(z) = \frac{-0.3 + z^{-1}}{1 - 0.3z^{-1}}$. Here, $|k_1| = |A_1(\infty)| = |d_1'| = 0.3 < 1$. Therefore, $A_2(z)$ and hence, $H_2(z)$ is stable. The feed-forward coefficients are given by $\alpha_1 = p_2 = 3.0$, $\alpha_2 = p_1 - \alpha_1 d_1' = 2.0$, $\alpha_3 = p_0 - \alpha_1 d_2 - \alpha_2 d_1' = 2.0$. Final realization of $H_2(z)$ is as shown below:



- (c)** $H_3(z) = \frac{-3 + 5.192z^{-1} - 3.56z^{-2} + 2z^{-3}}{1 - 0.28z^{-1} + 0.056z^{-2} + 0.4z^{-3}}$. Choose

$$A_3(z) = \frac{0.4 + 0.056z^{-1} - 0.28z^{-2} + z^{-3}}{1 - 0.28z^{-1} + 0.056z^{-2} + 0.4z^{-3}} \quad \text{Note } k_3 = A_3(\infty) = d_3 = 0.4 < 1. \quad \text{Using}$$

Eq. (8.45) we next determine the coefficient d_1' and d_2' of the allpass transfer function

$$A_2(z) \text{ arriving at } A_2(z) = \frac{0.2 - 0.36z^{-1} + z^{-2}}{1 - 0.36z^{-1} + 0.2z^{-2}}. \quad \text{Now } k_2 = A_2(\infty) = d_2 = 0.2 < 1.$$

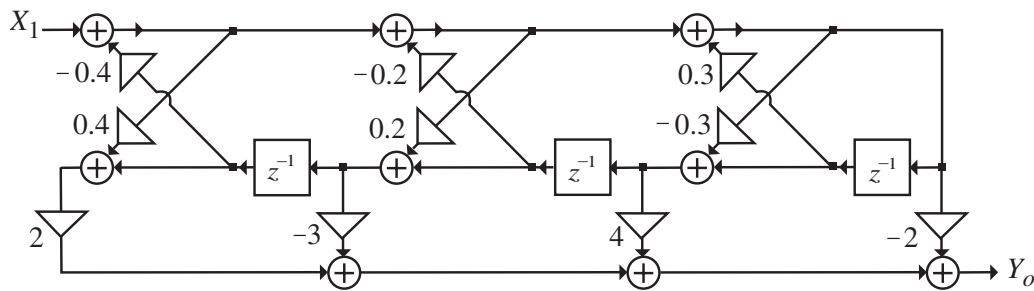
Using Eq. (8.45) we next determine the coefficient d_1'' of the allpass transfer function

$$A_1(z) \text{ arriving at } A_1(z) = \frac{-0.3 + z^{-1}}{1 - 0.3z^{-1}}. \quad \text{Here, } |k_1| = |A_1(\infty)| = |d_1'| = 0.3 < 1. \quad \text{Therefore,}$$

$A_3(z)$ and hence, $H_3(z)$ is stable.

The feed-forward coefficients are given by $\alpha_1 = p_3 = 2.0$, $\alpha_2 = p_2 - \alpha_1 d_1 = -3.0$,

$\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 4.0$, $\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = -2.0$. Final realization of $H_3(z)$ is as shown below:



(d) $H_4(z) = \frac{1.63 + 3.112z^{-1} + 0.84z^{-2} + 2z^{-3}}{1 + 0.42z^{-1} + 0.056z^{-2} - 0.3z^{-3}}. \quad \text{Choose}$

$$A_3(z) = \frac{-0.3 + 0.056z^{-1} + 0.42z^{-2} + z^{-3}}{1 + 0.42z^{-1} + 0.056z^{-2} - 0.3z^{-3}} \quad \text{Note } |k_3| = |A_3(\infty)| = |d_3| = 0.3 < 1. \quad \text{Using}$$

Eq. (8.45) we next determine the coefficients d_1' and d_2' of the allpass transfer function

$$A_2(z) \text{ arriving at } A_2(z) = \frac{0.2 + 0.48z^{-1} + z^{-2}}{1 + 0.48z^{-1} + 0.2z^{-2}}. \quad \text{Now } k_2 = A_2(\infty) = d_2 = 0.2 < 1.$$

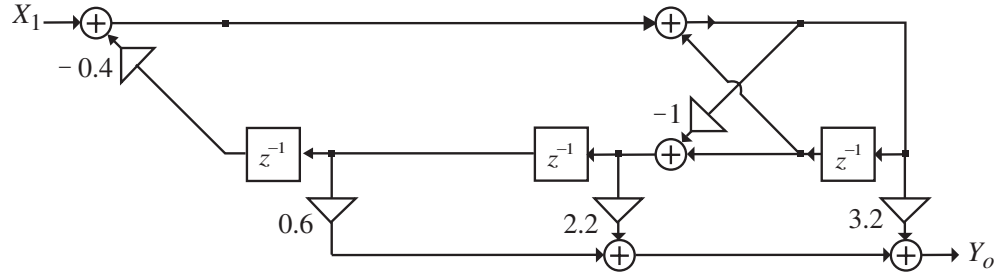
Using Eq. (8.45) we next determine the coefficient d_1'' of the allpass transfer function

$$A_1(z) \text{ arriving at } A_1(z) = \frac{0.4 + z^{-1}}{1 + 0.4z^{-1}}. \quad \text{Here, } k_1 = A_1(\infty) = d_1' = 0.4 < 1. \quad \text{Therefore,}$$

$A_3(z)$ and hence, $H_4(z)$ is stable.

The feed-forward coefficients are given by $\alpha_1 = p_3 = 2.0$, $\alpha_2 = p_2 - \alpha_1 d_1 = 0$,

$\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 3.0$, $\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = 1.0$. Final realization of $H_3(z)$ is as shown on the next page.



(e) $H_5(z) = \frac{2 + 0.6z^{-1} - z^{-2} + 0.3z^{-3}}{1 - 0.1z^{-1} + 0.18z^{-2} + 0.08z^{-3} - 0.08z^{-4}}$. Choose

$A_4(z) = \frac{-0.08 + 0.08z^{-1} + 0.18z^{-2} - 0.1z^{-3} + z^{-4}}{1 - 0.1z^{-1} + 0.18z^{-2} + 0.08z^{-3} - 0.08z^{-4}}$. Note

$|k_4| = |A_4(\infty)| = |d_4| = 0.08 < 1$. Using Eq. (8.45) we next determine the coefficients d'_1 , d'_2 and d'_3 of the allpass transfer function $A_3(z)$ arriving at

$A_3(z) = \frac{0.0725 + 0.195z^{-1} - 0.0942z^{-2} + z^{-3}}{1 - 0.0942z^{-1} + 0.1957z^{-2} + 0.0725z^{-3}}$. Now

$k_3 = A_3(\infty) = d'_3 = 0.0725 < 1$. Using Eq. (8.45) we next determine the coefficients d''_1 and d''_2 of the allpass transfer function $A_2(z)$ arriving at

$A_2(z) = \frac{0.2035 - 0.1090z^{-1} + z^{-2}}{1 - 0.1090z^{-1} + 0.2035z^{-2}}$. Now $k_2 = A_2(\infty) = d''_2 = 0.2035 < 1$. Using Eq.

(8.45) we next determine the coefficient d'''_1 of the allpass transfer function $A_1(z)$

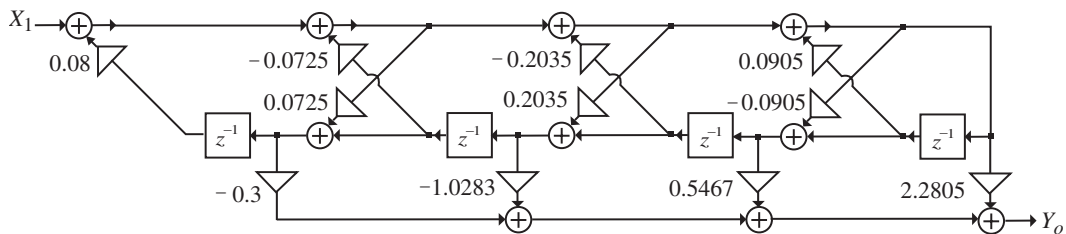
arriving at $A_1(z) = \frac{-0.009058 + z^{-1}}{1 - 0.0905z^{-1}}$. Here, $|k_1| = |A_1(\infty)| = |d'''_1| = 0.0905 < 1$.

Therefore, $A_4(z)$ and hence, $H_5(z)$ is stable.

The feed-forward coefficients are given by $\alpha_1 = p_4 = 0$, $\alpha_2 = p_3 - \alpha_1 d_1 = -0.3$,

$\alpha_3 = p_2 - \alpha_1 d_2 - \alpha_2 d'_1 = -1.0283$, $\alpha_4 = p_1 - \alpha_1 d_3 - \alpha_2 d'_2 - \alpha_3 d''_1 = 0.5467$,

$\alpha_5 = p_0 - \alpha_1 d_4 - \alpha_2 d'_3 - \alpha_3 d''_2 - \alpha_4 d'''_1 = 2.2805$. Final realization of $H_5(z)$ is as shown below:



8.43 (a) $H_1(z) = \frac{0.32z^{-1} + 0.66z^{-2} - 0.72z^{-3}}{1 - 2.35z^{-1} + 3.36z^{-2} - 1.575z^{-3}}$. Choose

$$A_3(z) = \frac{-1.575 + 3.36z^{-1} - 2.35z^{-2} + z^{-3}}{1 - 2.35z^{-1} + 3.36z^{-2} - 1.575z^{-3}}. \text{ Note } |k_3| = |A_3(\infty)| = |d_3| = 1.575 > 1.$$

Therefore, $A_3(z)$ and hence, $H_3(z)$ is unstable. Using Eq. (8.45) we next determine the coefficients d_1' and d_2' of the allpass transfer function $A_2(z)$ arriving at

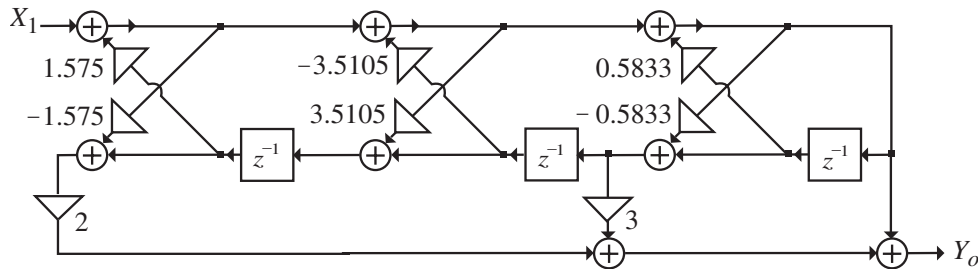
$$A_2(z) = \frac{3.5105 - 2.6308z^{-1} + z^{-2}}{1 - 2.6308z^{-1} + 3.5105z^{-2}}. \text{ Now } k_2 = A_2(\infty) = d_2 = 3.5105 > 1. \text{ Using Eq.}$$

(8.45) we next determine the coefficient d_1'' of the allpass transfer function $A_1(z)$

$$\text{arriving at } A_1(z) = \frac{-0.5833 + z^{-1}}{1 - 0.5833z^{-1}}. \text{ Here, } |k_1| = |A_1(\infty)| = |d_1''| = 0.5833 < 1.$$

The feed-forward coefficients are given by $\alpha_1 = p_3 = 2.0$, $\alpha_2 = p_2 - \alpha_1 d_1 = 0$,

$\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 3.0$, $\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = 1.0$. Final realization of $H_3(z)$ is as shown below:



$$(b) H_2(z) = \frac{9.3 - 39.3z^{-1} - 13.025z^{-2} + 6.552z^{-3} + 1.9125z^{-4}}{1 + 0.55z^{-1} + 0.095z^{-2} - 0.01z^{-3} - 0.003z^{-4}}. \text{ Choose}$$

$$A_4(z) = \frac{-0.003 - 0.01z^{-1} + 0.095z^{-2} + 0.55z^{-3} + z^{-4}}{1 + 0.55z^{-1} + 0.095z^{-2} - 0.01z^{-3} - 0.003z^{-4}}. \text{ Note}$$

$|k_4| = |A_4(\infty)| = |d_4| = 0.003 < 1$. Using Eq. (8.45) we next determine the coefficients d_1' , d_2' and d_3' of the allpass transfer function $A_3(z)$ arriving at

$$A_3(z) = \frac{-0.0084 + 0.0953z^{-1} + 0.55z^{-2} + z^{-3}}{1 + 0.55z^{-1} + 0.0953z^{-2} - 0.0084z^{-3}}. \text{ Now}$$

$|k_3| = |A_3(\infty)| = |d_3'| = 0.0084 < 1$. Using Eq. (8.45) we next determine the coefficients d_1'' and d_2'' of the allpass transfer function $A_2(z)$ arriving at

$$A_2(z) = \frac{0.0999 + 0.5508z^{-1} + z^{-2}}{1 + 0.5508z^{-1} + 0.0999z^{-2}}. \text{ Now } k_2 = A_2(\infty) = d_2 = 0.0999 < 1. \text{ Using Eq.}$$

(8.45) we next determine the coefficient d_1'' of the allpass transfer function $A_1(z)$

$$\text{arriving at } A_1(z) = \frac{0.5088 + z^{-1}}{1 + 0.5088z^{-1}}. \text{ Here, } k_1 = A_1(\infty) = d_1'' = 0.5088 < 1. \text{ As a result,}$$

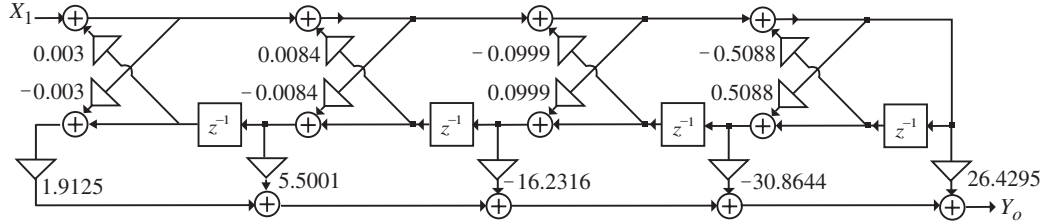
$A_3(z)$, and hence, $H_2(z)$ is stable.

The feed-forward coefficients are given by

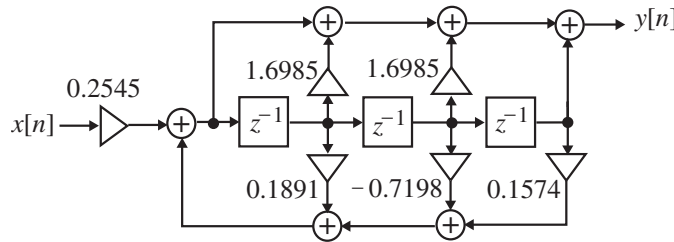
$$\alpha_1 = p_4 = 1.9125, \alpha_2 = p_3 - \alpha_1 d_1 = 5.5001, \alpha_3 = p_2 - \alpha_1 d_2 - \alpha_2 d_1' = -16.2316,$$

$$\alpha_4 = p_1 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = -30.8644,$$

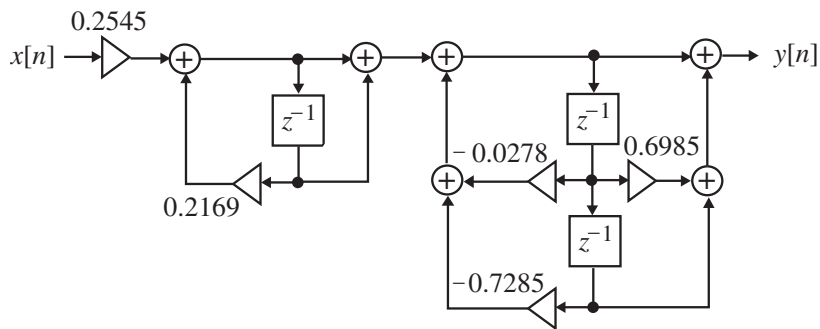
$\alpha_5 = p_0 - \alpha_1 d_4 - \alpha_2 d_3' - \alpha_3 d_2'' - \alpha_4 d_1''' = 26.4295$. Final realization of $H_2(z)$ is as shown below:



- 8.44 (a)** $H(z) = \frac{0.2545(1 + 1.6985z^{-1} + 1.6985z^{-2} + z^{-3})}{1 - 0.1891z^{-1} + 0.7198z^{-2} - 0.1574z^{-3}}$. A direct form II realization of $H(z)$ is shown below:



- (b)** $H(z) = \frac{0.2545(1 + z^{-1})(1 + 0.6985z^{-1} + z^{-2})}{(1 - 0.2169z^{-1})(1 + 0.0278z^{-1} + 0.7258z^{-2})}$. A cascade realization of $H(z)$ is shown below:



- (c)** We first form $A_3(z) = \frac{-0.1574 + 0.71985z^{-1} - 0.1891z^{-2} + z^{-3}}{1 - 0.1891z^{-1} + 0.7198z^{-2} - 0.1574z^{-3}}$. Now

$|k_3| = |A_3(\infty)| = |d_3| = 0.1574 < 1$. Using Eq. (8.45) we next determine the coefficients d_1'' and d_2'' of the allpass transfer function $A_2(z)$ arriving at

$$A_2(z) = \frac{0.7075 - 0.0777z^{-1} + z^{-2}}{1 - 0.0777z^{-1} + 0.7075z^{-2}}. \text{ Now } k_2 = A_2(\infty) = d_2 = 0.7075 < 1. \text{ Using Eq.}$$

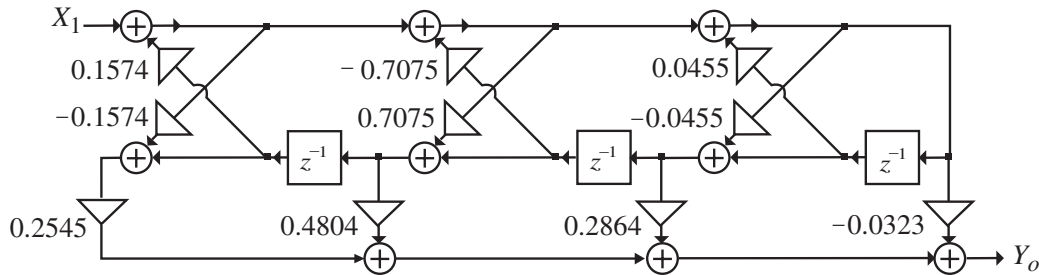
(8.45) we next determine the coefficient d_1'' of the allpass transfer function $A_1(z)$

$$\text{arriving at } A_1(z) = \frac{-0.0455 + z^{-1}}{1 - 0.0455z^{-1}}. \text{ Here, } |k_1| = |A_1(\infty)| = |d_1''| = 0.0455 < 1.$$

The feed-forward coefficients are given by $\alpha_1 = p_3 = 0.2545$, $\alpha_2 = p_2 - \alpha_1 d_1 = 0.4804$,

$$\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 0.2864, \quad \alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = -0.0323. \text{ The}$$

Gray-Markel realization of $H(z)$ is as shown below:

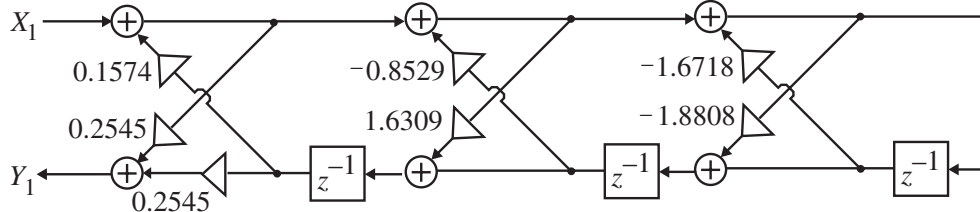


(d) Using Eqs. (8.132a) and (8.132b) we determine the coefficients of $H_2(z)$:

$$H_2(z) = \frac{1.6309 + 0.8456z^{-1} + z^{-2}}{1 + 0.0676z^{-1} + 0.8529z^{-2}}. \text{ Next, using Eqs. (8.132a) and (8.132b) we}$$

determine the coefficients of $H_1(z)$: $H_1(z) = \frac{-1.8808 + z^{-1}}{1 + 1.6718z^{-1}}$. A cascaded lattice

realization of using the method of Problem 8.30 is thus as shown below:



A comparison of the hardware requirements of the four realizations is as follows:

Direct form: No. of multipliers = 7 and no. of 2-input adders = 6,

Cascade form: No. of multipliers = 10 and no. of 2-input adders = 9,

Gray-Markel realization: No. of multipliers = 5 and no. of 2-input adders = 6,

Cascaded lattice realization: No. of multipliers = 6 and no. of 2-input adders = 6.

Hence, the Gray-Markel realization has the smallest hardware requirements.

8.45 (a) A partial-fraction expansion of $G(z)$ is of the form

$$G(z) = d + \sum_{i=1}^{N/2} \frac{v_i}{z - \lambda_i} + \sum_{i=1}^{N/2} \frac{v_i^*}{z - \lambda_i^*}. \text{ If we define } H(z) = \frac{d}{2} + \sum_{i=1}^{N/2} \frac{v_i}{z - \lambda_i}, \text{ then we can}$$

write $G(z) = H(z) + H_*(z)$, where $H_*(z)$ represents the transfer function obtained from $H(z)$ by conjugating its coefficients.

(b) In this case, the partial-fraction expansion of $G(z)$ is of the form

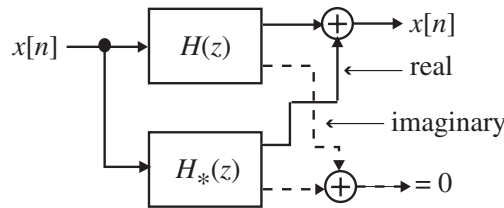
$$G(z) = d + \sum_{i=1}^{N_r} \frac{\rho_i}{z - \zeta_i} + \sum_{i=1}^{N_c/2} \frac{v_i}{z - \lambda_i} + \sum_{i=1}^{N_c/2} \frac{v_i^*}{z - \lambda_i^*},$$

where N_r and N_c are the number of real poles ζ_i 's and complex poles λ_i 's, respectively, with residues ρ_i 's and v_i 's.

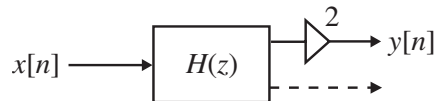
We can thus decompose $G(z)$ as $G(z) = H(z) + H_*(z)$, where

$$H(z) = \frac{d}{2} + \sum_{i=1}^{N_r} \frac{\rho_i/2}{z - \zeta_i} + \sum_{i=1}^{N_c/2} \frac{v_i}{z - \lambda_i}.$$

(c) An implementation of real coefficient $G(z)$ is thus simply a parallel connection of two complex-coefficient filters characterized by transfer functions $H(z)$ and $H_*(z)$ as indicated in the figure below:



However, for a real-valued input $x[n]$, the output of $H(z)$ is the complex conjugate of that of $H_*(z)$. As a result, two times the real part of the output of $H(z)$ is the desired real-valued output sequence $y[n]$ indicating that a single complex-coefficient filter $H(z)$ is sufficient to realize $G(z)$ as indicated below:



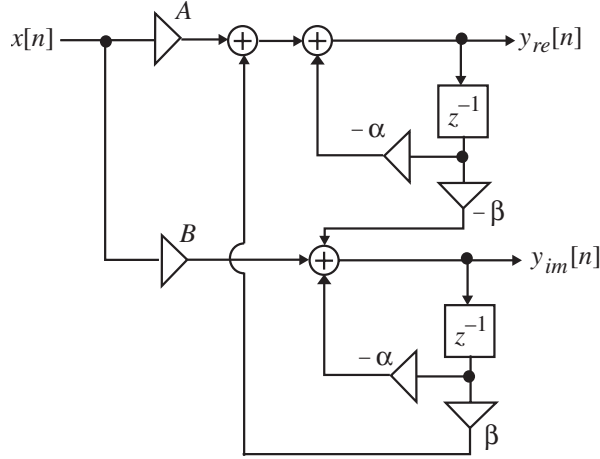
8.46 From $H(z) = \frac{Y(z)}{X(z)} = \frac{A + jB}{1 + (\alpha + j\beta)z^{-1}}$, we arrive at the difference equation

representation $y_{re}[n] + j y_{im}[n] = -(\alpha + j\beta)(y_{re}[n-1] + j y_{im}[n-1]) + Ax[n] + jBx[n]$, which is equivalent to a set of two difference equations involving all real variables and real multiplier coefficients: $y_{re}[n] = -\alpha y_{re}[n-1] + \beta y_{im}[n-1] + Ax[n]$ and

$y_{im}[n] = -\beta y_{re}[n-1] - \alpha y_{im}[n-1] + Bx[n]$. A realization of $H(z)$ based on the last two equations is shown on next page.

To determine the transfer function $Y_{re}(z)/X(z)$, we take the z -transform of the last two difference equations and arrive at $(1 + \alpha z^{-1})Y_{re}(z) - \beta z^{-1}Y_{im}(z) = AX(z)$ and $\beta z^{-1}Y_{re}(z) + (1 + \alpha z^{-1})Y_{im}(z) = BX(z)$. Solving these two equations we get

$\frac{Y_{re}(z)}{X(z)} = \frac{A + (A\alpha + B\beta)z^{-1}}{1 + 2\alpha z^{-1} + (\alpha^2 + \beta^2)z^{-2}}$, which is seen to be a second-order transfer function.



To determine the transfer function $Y_{re}(z)/X(z)$, we take the z -transform of the last two difference equations and arrive at $(1 + \alpha z^{-1})Y_{re}(z) - \beta z^{-1}Y_{im}(z) = AX(z)$ and $\beta z^{-1}Y_{re}(z) + (1 + \alpha z^{-1})Y_{im}(z) = BX(z)$. Solving these two equations we get

$\frac{Y_{re}(z)}{X(z)} = \frac{A + (A\alpha + B\beta)z^{-1}}{1 + 2\alpha z^{-1} + (\alpha^2 + \beta^2)z^{-2}}$, which is seen to be a second-order transfer function.

8.47 An m -th order complex allpass function is given by

$$A_m(z) = \frac{\alpha_m^* + \alpha_{m-1}^* z^{-1} + \dots + \alpha_1^* z^{-m+1} + z^{-m}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \dots + \alpha_{m-1} z^{-m+1} + \alpha_m z^{-m}}. \text{ To generate an } (m-1)\text{-th}$$

order allpass function, we use the recursion $A_{m-1}(z) = \frac{P_{m-1}(z)}{D_{m-1}(z)} = z \left[\frac{A_m(z) - k_m^*}{1 - k_m A_m(z)} \right]$.

Substituting the expression for $A_m(z)$ in the above equation we obtain after some

$$\begin{aligned} \text{algebra } P_{m-1}(z) &= z[\alpha_m^* + \alpha_{m-1}^* z^{-1} + \alpha_{m-2}^* z^{-2} + \dots + \alpha_1^* z^{-m+1} + z^{-m} \\ &\quad - \alpha_m^*(1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \dots + \alpha_{m-1} z^{-m+1} + \alpha_m z^{-m})] \\ &= (\alpha_{m-1}^* - \alpha_m^* \alpha_1) + (\alpha_{m-2}^* - \alpha_m^* \alpha_2) z^{-1} + \dots + (\alpha_1^* - \alpha_m^* \alpha_{m-1}) z^{-m+2} + (1 - |\alpha_m|^2) z^{-m+1}, \end{aligned}$$

$$\text{and } D_{m-1}(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \dots + \alpha_{m-1} z^{-m+1} + \alpha_m z^{-m}$$

$$\begin{aligned} &\quad - \alpha_m(\alpha_m^* + \alpha_{m-1}^* z^{-1} + \dots + \alpha_1^* z^{-m+1} + z^{-m}) \\ &= (1 - |\alpha_m|^2) + (\alpha_1 - \alpha_m \alpha_{m-1}^*) z^{-1} + (\alpha_2 - \alpha_m \alpha_{m-2}^*) z^{-2} + \dots + (\alpha_{m-1} - \alpha_m \alpha_1^*) z^{-m+1}. \end{aligned}$$

Thus, $A_{m-1}(z)$ is a complex allpass function of order $m-1$ and given by

$$A_{m-1}(z) = \frac{\beta_{m-1}^* + \beta_{m-2}^* z^{-1} + \dots + \beta_1^* z^{-m+2} + z^{-m+1}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \dots + \beta_{m-2} z^{-m+2} + \beta_{m-1} z^{-m+1}}, \text{ where}$$

$$\beta_k = \frac{\alpha_k - \alpha_m \alpha_{m-k}^*}{1 - |\alpha_m|^2}, \quad 1 \leq k \leq m-1.$$

To develop a realization of $A_m(z)$ we express $A_m(z)$ in terms of $A_{m-1}(z)$:

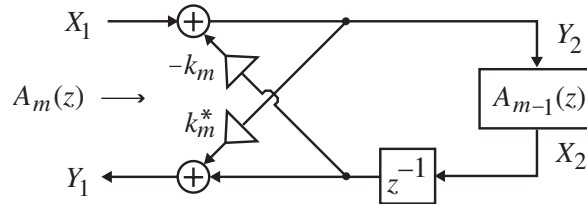
$$A_m(z) = \frac{Y_1}{X_1} = \frac{k_m^* - z^{-1} A_{m-1}(z)}{1 + k_m z^{-1} A_{m-1}(z)}, \text{ and compare it with Eq. (8.47) resulting in the}$$

following expressions for the transfer parameters of the two-pair: $t_{11} = k_m^*$,

$t_{22} = -k_m z^{-1}$, and $t_{12} t_{21} = (1 - k_m k_m^*) z^{-1}$. As in the case of the realization of a real allpass function, there are many possible choices for t_{12} and t_{21} . We choose

$t_{12} = (1 - k_m k_m^*) z^{-1}$, $t_{21} = 1$. The corresponding input-output relations of the two-pair are: $Y_1 = k_m^* X_1 + (1 - k_m^* k_m) z^{-1} X_2 = k_m^* (X_1 - k_m z^{-1} X_2) + z^{-1} X_2$ and

$Y_2 = X_1 - k_m z^{-1} X_2$. A realization of $A_m(z)$ based on the above two-pair relations is indicated below:



$$\mathbf{8.48} \quad \mathbf{(a)} \quad H_1(z) = 4 \left(\frac{1+z^{-1}}{5+3z^{-1}} \right) = \frac{1}{2} \left(\frac{8+8z^{-1}}{5+3z^{-1}} \right) = \frac{1}{2} \left(1 + \frac{3+5z^{-1}}{5+3z^{-1}} \right) = \frac{1}{2} (A_0(z) + A_1(z)),$$

$$\text{where } A_0(z) = 1 \text{ and } A_1(z) = \frac{3+5z^{-1}}{5+3z^{-1}}.$$

$$\mathbf{(b)} \quad H_2(z) = \left(\frac{5-5z^{-1}}{6-4z^{-1}} \right) = \frac{1}{2} \left(\frac{5-5z^{-1}}{3-2z^{-1}} \right) = \frac{1}{2} \left(1 - \frac{-2+3z^{-1}}{3-2z^{-1}} \right) = \frac{1}{2} (A_0(z) - A_1(z)),$$

$$\text{where } A_0(z) = 1 \text{ and } A_1(z) = \frac{-2+3z^{-1}}{3-2z^{-1}}.$$

$$\mathbf{(c)} \quad H_3(z) = \frac{0.5414(1-z^{-1})(1-0.0757z^{-1}+z^{-2})}{(1-0.1768z^{-1})(1-0.004z^{-1}+0.9061z^{-2})} = \frac{1}{2} [A_0(z) - A_1(z)], \text{ where}$$

$$A_0(z) = \frac{0.9061 - 0.004z^{-1} + z^{-2}}{1 - 0.004z^{-1} + 0.9061z^{-2}} \text{ and } A_1(z) = \frac{-0.1768 + z^{-1}}{1 - 0.1768z^{-1}}.$$

$$\mathbf{(d)} \quad H_4(z) = \frac{0.4547(1+z^{-1})(1-0.2859z^{-1}+z^{-2})}{(1+0.0712z^{-1})(1-0.0377z^{-1}+0.8482z^{-2})} = \frac{1}{2} [A_0(z) - A_1(z)], \text{ where}$$

$$A_0(z) = \frac{0.0712 + z^{-1}}{1 + 0.0712z^{-1}} \text{ and } A_1(z) = \frac{0.8482 - 0.0377z^{-1} + z^{-2}}{1 - 0.0377z^{-1} + 0.8482z^{-2}}.$$

8.49 (a) From the given equation we get

$$\begin{aligned} y[2\ell] &= h[0]x[2\ell] + h[1]x[2\ell - 1] + h[2]x[2\ell - 2] + h[3]x[2\ell - 3] + h[4]x[2\ell - 4] + h[5]x[2\ell - 5]. \\ y[2\ell + 1] &= h[0]x[2\ell + 1] + h[1]x[2\ell] + h[2]x[2\ell - 1] + h[3]x[2\ell - 2] + h[4]x[2\ell - 3] + h[5]x[2\ell - 4]. \end{aligned}$$

Rewriting the above equations in matrix form we arrive at

$$\begin{aligned} \begin{bmatrix} y[2\ell] \\ y[2\ell + 1] \end{bmatrix} &= \begin{bmatrix} h[0] & 0 \\ h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[2\ell] \\ x[2\ell + 1] \end{bmatrix} + \begin{bmatrix} h[2] & h[1] \\ h[3] & h[2] \end{bmatrix} \begin{bmatrix} x[2\ell - 2] \\ x[2\ell - 1] \end{bmatrix} \\ &\quad + \begin{bmatrix} h[4] & h[3] \\ h[5] & h[4] \end{bmatrix} \begin{bmatrix} x[2\ell - 4] \\ x[2\ell - 3] \end{bmatrix} + \begin{bmatrix} 0 & h[5] \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x[2\ell] \\ x[2\ell + 1] \end{bmatrix}, \end{aligned}$$

which can be alternately expressed as $\mathbf{Y}_\ell = \mathbf{H}_0\mathbf{X}_\ell + \mathbf{H}_1\mathbf{X}_{\ell-1} + \mathbf{H}_2\mathbf{X}_{\ell-2} + \mathbf{H}_3\mathbf{X}_{\ell-3}$,

$$\begin{aligned} \text{where } \mathbf{Y}_\ell &= \begin{bmatrix} y[2\ell] \\ y[2\ell + 1] \end{bmatrix}, \mathbf{X}_\ell = \begin{bmatrix} x[2\ell] \\ x[2\ell + 1] \end{bmatrix}, \mathbf{H}_0 = \begin{bmatrix} h[0] & 0 \\ h[1] & h[0] \end{bmatrix}, \mathbf{H}_1 = \begin{bmatrix} h[2] & h[1] \\ h[3] & h[2] \end{bmatrix}, \\ \mathbf{H}_2 &= \begin{bmatrix} h[4] & h[3] \\ h[5] & h[4] \end{bmatrix}, \mathbf{H}_3 = \begin{bmatrix} 0 & h[5] \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

(b) Here we have

$$\begin{aligned} y[3\ell] &= h[0]x[3\ell] + h[1]x[3\ell - 1] + h[2]x[3\ell - 2] + h[3]x[3\ell - 3] + h[4]x[3\ell - 4] + h[5]x[3\ell - 5]. \\ y[3\ell + 1] &= h[0]x[3\ell + 1] + h[1]x[3\ell] + h[2]x[3\ell - 1] + h[3]x[3\ell - 2] + h[4]x[3\ell - 3] + h[5]x[3\ell - 4], \\ y[3\ell + 2] &= h[0]x[3\ell + 2] + h[1]x[3\ell + 1] + h[2]x[3\ell] + h[3]x[3\ell - 1] + h[4]x[3\ell - 2] + h[5]x[3\ell - 3]. \end{aligned}$$

Rewriting the above equations in matrix form we arrive at

$$\begin{aligned} \begin{bmatrix} y[3\ell] \\ y[3\ell + 1] \\ y[3\ell + 2] \end{bmatrix} &= \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[3\ell] \\ x[3\ell + 1] \\ x[3\ell + 2] \end{bmatrix} + \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix} \begin{bmatrix} x[3\ell - 3] \\ x[3\ell - 2] \\ x[3\ell - 1] \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & h[2] & h[1] \\ 0 & 0 & h[2] \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[3\ell - 6] \\ x[3\ell - 5] \\ x[3\ell - 4] \end{bmatrix}, \end{aligned}$$

which can be alternately expressed as $\mathbf{Y}_\ell = \mathbf{H}_0\mathbf{X}_\ell + \mathbf{H}_1\mathbf{X}_{\ell-1} + \mathbf{H}_2\mathbf{X}_{\ell-2}$, where

$$\mathbf{Y}_\ell = \begin{bmatrix} y[3\ell] \\ y[3\ell + 1] \\ y[3\ell + 2] \end{bmatrix}, \mathbf{X}_\ell = \begin{bmatrix} x[3\ell] \\ x[3\ell + 1] \\ x[3\ell + 2] \end{bmatrix}, \mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix}, \mathbf{H}_1 = \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix},$$

$$\text{and } \mathbf{H}_2 = \begin{bmatrix} 0 & h[2] & h[1] \\ 0 & 0 & h[2] \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Following a procedure similar to that outlined in Parts (a) and (b) above, we can show that here $\mathbf{Y}_\ell = \mathbf{H}_0 \mathbf{X}_\ell + \mathbf{H}_1 \mathbf{X}_{\ell-1} + \mathbf{H}_2 \mathbf{X}_{\ell-2}$, where

$$\mathbf{Y}_\ell = \begin{bmatrix} y[4\ell] \\ y[4\ell+1] \\ y[4\ell+2] \\ y[4\ell+3] \end{bmatrix}, \quad \mathbf{X}_\ell = \begin{bmatrix} x[4\ell] \\ x[4\ell+1] \\ x[4\ell+2] \\ x[4\ell+3] \end{bmatrix}, \quad \mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} h[4] & h[3] & h[2] & h[1] \\ h[5] & h[4] & h[3] & h[2] \\ 0 & h[5] & h[4] & h[3] \\ 0 & 0 & h[5] & h[4] \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 & h[5] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8.50 (a) $d_0 y[2\ell] + d_1 y[2\ell-1] + d_2 y[2\ell-2] + d_3 y[2\ell-3] + d_4 y[2\ell-4]$
 $= p_0 x[2\ell] + p_1 x[2\ell-1] + p_2 x[2\ell-2] + p_3 x[2\ell-3] + p_4 x[2\ell-4],$
 $d_0 y[2\ell+1] + d_1 y[2\ell] + d_2 y[2\ell-1] + d_3 y[2\ell-2] + d_4 y[2\ell-3]$
 $= p_0 x[2\ell+1] + p_1 x[2\ell] + p_2 x[2\ell-1] + p_3 x[2\ell-2] + p_4 x[2\ell-3].$

Rewriting the above equations in matrix form we arrive at

$$\begin{bmatrix} d_0 & 0 \\ d_1 & d_0 \end{bmatrix} \begin{bmatrix} y[2\ell] \\ y[2\ell+1] \end{bmatrix} + \begin{bmatrix} d_2 & d_1 \\ d_3 & d_2 \end{bmatrix} \begin{bmatrix} y[2\ell-2] \\ y[2\ell-1] \end{bmatrix} + \begin{bmatrix} d_4 & d_3 \\ 0 & d_4 \end{bmatrix} \begin{bmatrix} y[2\ell-4] \\ y[2\ell-3] \end{bmatrix}$$

$$= \begin{bmatrix} p_0 & 0 \\ p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[2\ell] \\ x[2\ell+1] \end{bmatrix} + \begin{bmatrix} p_2 & p_1 \\ p_3 & p_2 \end{bmatrix} \begin{bmatrix} x[2\ell-2] \\ x[2\ell-1] \end{bmatrix} + \begin{bmatrix} p_4 & p_3 \\ 0 & p_4 \end{bmatrix} \begin{bmatrix} x[2\ell-4] \\ x[2\ell-3] \end{bmatrix}.$$

The above matrix equation can be alternately expressed as

$$\mathbf{D}_0 \mathbf{Y}_\ell + \mathbf{D}_1 \mathbf{Y}_{\ell-1} + \mathbf{D}_2 \mathbf{Y}_{\ell-2} = \mathbf{P}_0 \mathbf{X}_\ell + \mathbf{P}_1 \mathbf{X}_{\ell-1} + \mathbf{P}_2 \mathbf{X}_{\ell-2}, \quad \text{where } \mathbf{Y}_\ell = \begin{bmatrix} y[2\ell] \\ y[2\ell+1] \end{bmatrix},$$

$$\mathbf{X}_\ell = \begin{bmatrix} x[2\ell] \\ x[2\ell+1] \end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix} d_0 & 0 \\ d_1 & d_0 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} d_2 & d_1 \\ d_3 & d_2 \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} d_4 & d_3 \\ 0 & d_4 \end{bmatrix},$$

$$\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 \\ p_1 & p_0 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} p_2 & p_1 \\ p_3 & p_2 \end{bmatrix}, \quad \text{and } \mathbf{P}_2 = \begin{bmatrix} p_4 & p_3 \\ 0 & p_4 \end{bmatrix}.$$

(b) $\begin{bmatrix} d_0 & 0 & 0 \\ d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \end{bmatrix} \begin{bmatrix} y[3\ell] \\ y[3\ell+1] \\ y[3\ell+2] \end{bmatrix} + \begin{bmatrix} d_3 & d_2 & d_1 \\ d_4 & d_3 & d_2 \\ 0 & d_4 & d_3 \end{bmatrix} \begin{bmatrix} y[3\ell-3] \\ y[3\ell-2] \\ y[3\ell-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & d_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y[3\ell-6] \\ y[3\ell-5] \\ y[3\ell-4] \end{bmatrix}$

$$= \begin{bmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[3\ell] \\ x[3\ell+1] \\ x[3\ell+2] \end{bmatrix} + \begin{bmatrix} p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \\ 0 & p_4 & p_3 \end{bmatrix} \begin{bmatrix} x[3\ell]-3 \\ x[3\ell-2] \\ x[3\ell-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & p_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[3\ell-6] \\ x[3\ell-5] \\ x[3\ell-4] \end{bmatrix},$$

which can be alternately expressed as

$$\mathbf{D}_0 \mathbf{Y}_\ell + \mathbf{D}_1 \mathbf{Y}_{\ell-1} + \mathbf{D}_2 \mathbf{Y}_{\ell-2} = \mathbf{P}_0 \mathbf{X}_\ell + \mathbf{P}_1 \mathbf{X}_{\ell-1} + \mathbf{P}_2 \mathbf{X}_{\ell-2}, \text{ where } \mathbf{D}_0 = \begin{bmatrix} y[3\ell] \\ y[3\ell+1] \\ y[3\ell+2] \end{bmatrix},$$

$$\mathbf{P}_0 = \begin{bmatrix} x[3\ell] \\ x[3\ell+1] \\ x[3\ell+2] \end{bmatrix}, \mathbf{D}_0 = \begin{bmatrix} d_0 & 0 & 0 \\ d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \end{bmatrix}, \mathbf{D}_1 = \begin{bmatrix} d_3 & d_2 & d_1 \\ d_4 & d_3 & d_2 \\ 0 & d_4 & d_3 \end{bmatrix}, \mathbf{D}_2 = \begin{bmatrix} 0 & 0 & d_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & p_1 & p_0 \end{bmatrix}, \mathbf{P}_1 = \begin{bmatrix} p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \\ 0 & p_4 & p_3 \end{bmatrix}, \text{ and } \mathbf{P}_2 = \begin{bmatrix} 0 & 0 & p_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} & \begin{matrix} \text{(c)} \end{matrix} \begin{bmatrix} d_0 & 0 & 0 & 0 \\ d_1 & d_0 & 0 & 0 \\ d_2 & d_1 & d_0 & 0 \\ d_3 & d_2 & d_1 & d_0 \end{bmatrix} \begin{bmatrix} y[4\ell] \\ y[4\ell+1] \\ y[4\ell+2] \\ y[4\ell+3] \end{bmatrix} + \begin{bmatrix} d_4 & d_3 & d_2 & d_1 \\ 0 & d_4 & d_3 & d_2 \\ 0 & 0 & d_4 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix} \begin{bmatrix} y[4\ell-4] \\ y[4\ell-3] \\ y[4\ell-2] \\ y[4\ell-1] \end{bmatrix} \\ & = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[4\ell] \\ x[4\ell+1] \\ x[4\ell+2] \\ x[4\ell+3] \end{bmatrix} + \begin{bmatrix} p_4 & p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \begin{bmatrix} x[4\ell-4] \\ x[4\ell-3] \\ x[4\ell-2] \\ x[4\ell-1] \end{bmatrix}, \text{ which can be} \end{aligned}$$

alternately written as $\mathbf{D}_0 \mathbf{Y}_\ell + \mathbf{D}_1 \mathbf{Y}_{\ell-1} + \mathbf{D}_2 \mathbf{Y}_{\ell-2} = \mathbf{P}_0 \mathbf{X}_\ell + \mathbf{P}_1 \mathbf{X}_{\ell-1} + \mathbf{P}_2 \mathbf{X}_{\ell-2}$, where

$$\mathbf{Y}_\ell = \begin{bmatrix} y[4\ell] \\ y[4\ell+1] \\ y[4\ell+2] \\ y[4\ell+3] \end{bmatrix}, \mathbf{X}_\ell = \begin{bmatrix} x[4\ell] \\ x[4\ell+1] \\ x[4\ell+2] \\ x[4\ell+3] \end{bmatrix}, \mathbf{D}_0 = \begin{bmatrix} d_0 & 0 & 0 & 0 \\ d_1 & d_0 & 0 & 0 \\ d_2 & d_1 & d_0 & 0 \\ d_3 & d_2 & d_1 & d_0 \end{bmatrix},$$

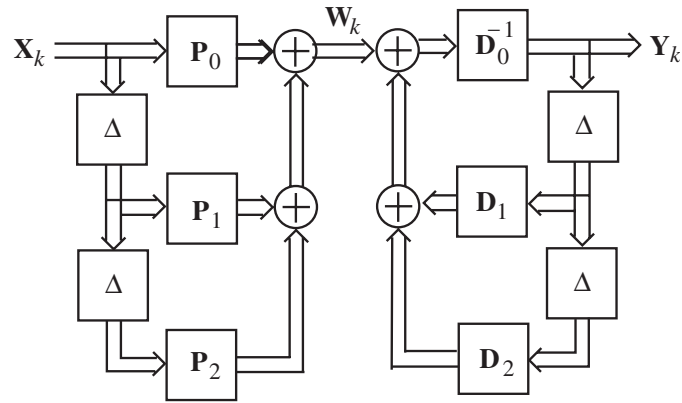
$$\mathbf{D}_1 = \begin{bmatrix} d_4 & d_3 & d_2 & d_1 \\ 0 & d_4 & d_3 & d_2 \\ 0 & 0 & d_4 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}, \mathbf{P}_0 = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \end{bmatrix}, \text{ and } \mathbf{P}_1 = \begin{bmatrix} p_4 & p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix}.$$

8.51 We first rewrite the second-order block difference equation

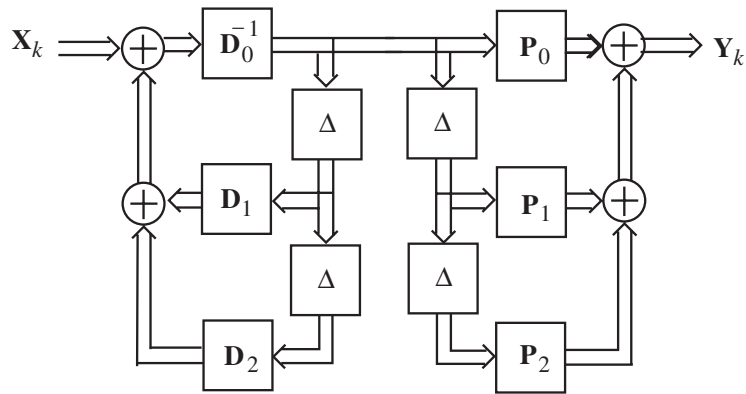
$$\mathbf{D}_0 \mathbf{Y}_\ell + \mathbf{D}_1 \mathbf{Y}_{\ell-1} + \mathbf{D}_2 \mathbf{Y}_{\ell-2} = \mathbf{P}_0 \mathbf{X}_\ell + \mathbf{P}_1 \mathbf{X}_{\ell-1} + \mathbf{P}_2 \mathbf{X}_{\ell-2},$$

as two separate equations: $\mathbf{W}_\ell = \mathbf{P}_0 \mathbf{X}_\ell + \mathbf{P}_1 \mathbf{X}_{\ell-1} + \mathbf{P}_2 \mathbf{X}_{\ell-2}$, and

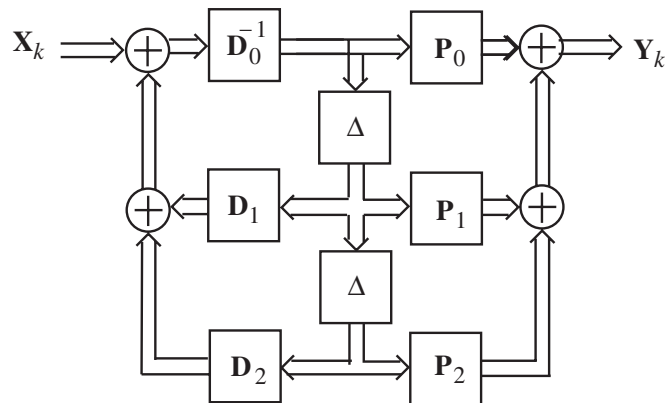
$Y_\ell = -D_0^{-1}D_1Y_{\ell-1} - D_0^{-1}D_2Y_{\ell-2} + D_0^{-1}W_\ell$. As cascade realization of the IIR block digital filter based on the last two equations is thus as shown below:



By interchanging the locations of the two block sections in the above structure we arrive at an equivalent realization as indicated below:

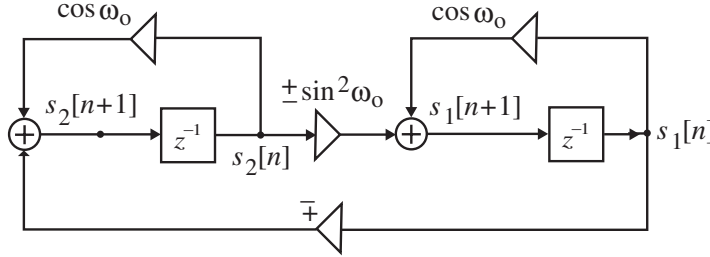


Finally, by delay-sharing the above structure reduces to a canonic realization as shown below:



8.51 By setting $\alpha = \pm\beta \sin \theta$ in Eq. (8.125), the state-space description of the sine-cosine

generator reduces to $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos \omega_o & \pm \sin^2 \omega_o \\ \mp & \cos \omega_o \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$, which leads to the three-multiplier realization shown below:



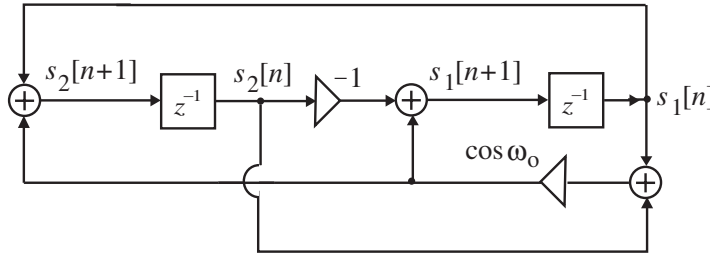
8.53 Let $-\frac{\beta \sin \omega_o}{\alpha} = 1 + \cos \omega_o$. Then $\frac{\alpha}{\beta} \sin \omega_o = \frac{-\sin^2 \omega_o}{\cos \omega_o + 1} = \frac{\cos^2 \omega_o - 1}{\cos \omega_o + 1} = \cos \omega_o - 1$, and

$-\frac{\beta}{\alpha} \sin \omega_o = \cos \omega_o + 1$. Substituting these values in Eq. (8.125) we arrive at

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos \omega_o & \cos \omega_o - 1 \\ \cos \omega_o + 1 & \cos \omega_o \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}. \text{ These equations can be alternately written}$$

as $s_1[n+1] = \cos \omega_o (s_1[n] + s_2[n]) - s_2[n]$, and $s_2[n+1] = \cos \omega_o (s_1[n] + s_2[n]) + s_2[n]$.

A realization based on the last two equations results in a single-multiplier structure as indicated below:



8.54 $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha(C - \cos \omega_o)}{\beta \sin \omega_o} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} C & \frac{\alpha(1 - C \cos \omega_o)}{\beta \sin \omega_o} \\ -\frac{\beta}{\alpha} \sin \omega_o & \cos \omega_o \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$. If

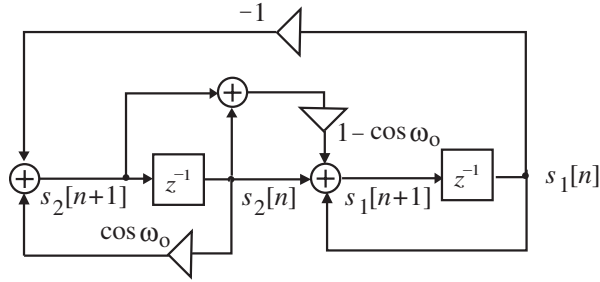
$C = 1$, then

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha(1 - \cos \omega_o)}{\beta \sin \omega_o} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 1 & \frac{\alpha(1 - \cos \omega_o)}{\beta \sin \omega_o} \\ -\frac{\beta}{\alpha} \sin \omega_o & \cos \omega_o \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$$

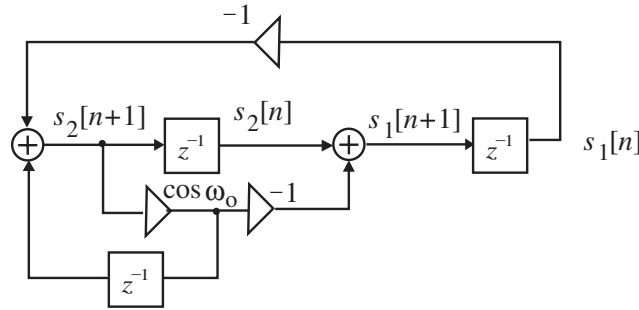
Choose $\alpha = \beta \sin \omega_o$. Then the above equation reduces to

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 - \cos \omega_o \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 1 & 1 - \cos \omega_o \\ -1 & \cos \omega_o \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$$

A two-multiplier realization of the above equation is shown below:



To arrive at a one-multiplier realization we observe that the two equations describing the sine-cosine generator are given by
 $s_1[n+1] = (1 - \cos \omega_o) s_2[n+1] + s_1[n] + (1 - \cos \omega_o) s_2[n]$, and
 $s_2[n+1] = -s_1[n] + \cos \omega_o \cdot s_2[n]$. Substituting the second equation in the first we arrive at an alternate description in the form $s_1[n+1] = -\cos \omega_o \cdot s_2[n+1] + s_2[n]$, and
 $s_2[n+1] = -s_1[n] + \cos \omega_o \cdot s_2[n]$. A realization of these two equations leads to the single-multiplier structure shown below:



8.55 From Figure P8.18(a), the input-output relation of the channel is given

$$\text{by } \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} 1 & H_{12}(z) \\ H_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}. \text{ Likewise, the input-output relation of the}$$

channel separation circuit of Figure P8.18(b) is given by

$$\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & -G_{12}(z) \\ -G_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}. \text{ Hence, the overall system is characterized by}$$

$$\begin{aligned} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} &= \begin{bmatrix} 1 & -G_{12}(z) \\ -G_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & H_{12}(z) \\ H_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} \\ &= \begin{bmatrix} 1 - H_{21}(z)G_{12}(z) & H_{12}(z) - G_{12}(z) \\ H_{21}(z) - G_{21}(z) & 1 - H_{12}(z)G_{21}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}. \end{aligned}$$

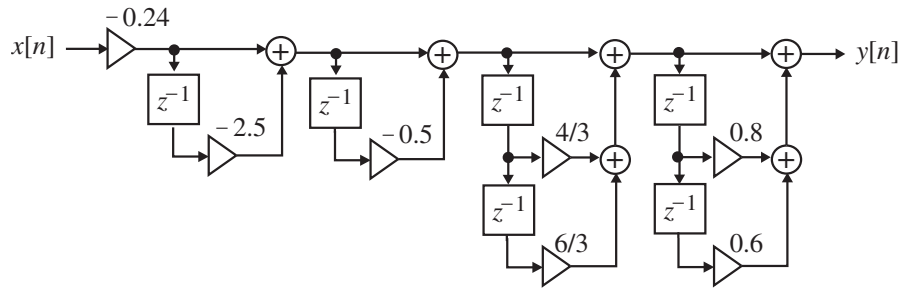
The cross-talk is eliminated if $V_1(z)$ is a function of either $X_1(z)$ or $X_2(z)$, and similarly, if $V_2(z)$ is a function of either $X_1(z)$ or $X_2(z)$. From the above equation it follows that if $H_{12}(z) = G_{12}(z)$, and $H_{21}(z) = G_{21}(z)$, then $V_1(z) = (1 - H_{21}(z)G_{12}(z))X_1(z)$, and $V_2(z) = (1 - H_{12}(z)G_{21}(z))X_2(z)$. Alternately, if $G_{12}(z) = H_{21}^{-1}(z)$, and $G_{21}(z) = H_{12}^{-1}(z)$,

then $V_1(z) = \left(\frac{H_{12}(z)H_{21}(z)-1}{H_{21}(z)} \right) X_2(z)$, and $V_2(z) = \left(\frac{H_{12}(z)H_{21}(z)-1}{H_{12}(z)} \right) X_1(z)$.

M8.1 (a) Using the M-file factor, we factorize $H_1(z)$ resulting in

$$H_1(z) = -0.24(1 - 2.5z^{-1})(1 - 0.4z^{-1})\left(1 + \frac{4}{3}z^{-1} + \frac{6}{3}z^{-2}\right)(1 + 0.8z^{-1} + 0.6z^{-2}).$$

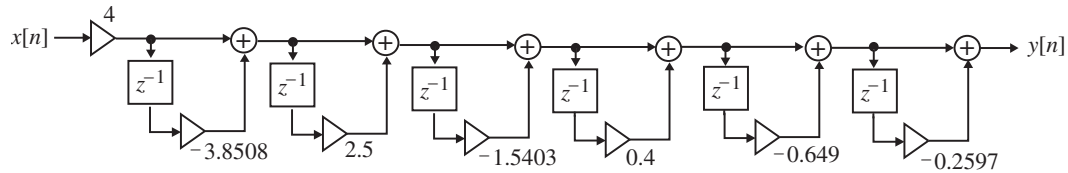
A cascade realization of $H_1(z)$ based on the above factored form is shown below:



(b) Using the M-file factor, we factorize $H_2(z)$ resulting in

$$H_2(z) = 4(1 - 3.8508z^{-1})(1 + 2.5z^{-1})(1 - 1.5403z^{-1})(1 + 0.4z^{-1})(1 - 0.649z^{-1})(1 - 0.2597z^{-1}).$$

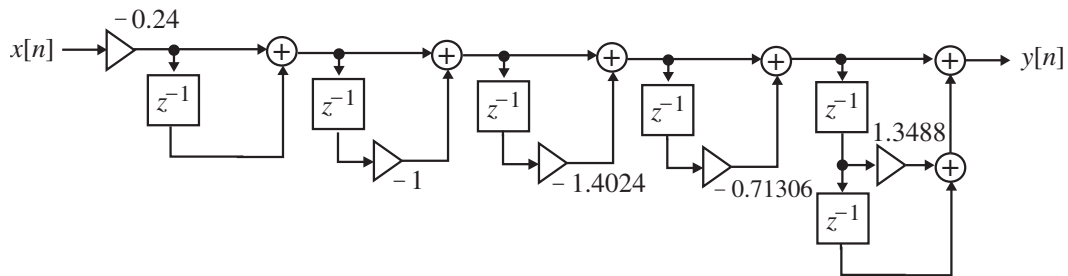
A cascade realization of $H_2(z)$ based on the above factored form is shown below:



(c) Using the M-file factor, we factorize $H_3(z)$ resulting in

$$H_3(z) = -0.24(1 + z^{-1})(1 - z^{-1})(1 - 1.4024z^{-1})(1 - 0.71306z^{-1})(1 + 1.3488z^{-1} + z^{-2}).$$

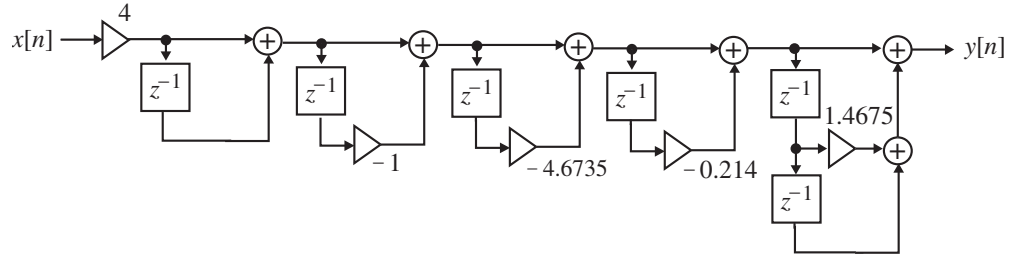
A cascade realization of $H_3(z)$ based on the above factored form is shown below:



(d) Using the M-file factor, we factorize $H_4(z)$ resulting in

$$H_4(z) = 4(1 + z^{-1})(1 - z^{-1})(1 - 4.6735z^{-1})(1 - 0.214z^{-1})(1 + 1.4675z^{-1} + z^{-2}).$$

A cascade realization of $H_4(z)$ based on the above factored form is shown below:



M8.2 (a) Using the M-file `factor`, we factorize the numerator and the denominator of $G(z)$

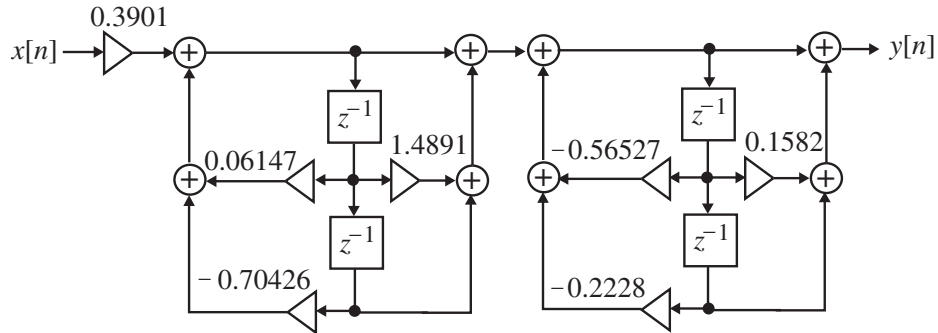
resulting in
$$G(z) = \frac{0.3901(1 + 1.4891z^{-1} + z^{-2})(1 + 0.1582z^{-1} + z^{-2})}{(1 - 0.06147z^{-1} + 0.70426z^{-2})(1 + 0.5653z^{-1} + 0.2228z^{-2})}$$
.

(b) We can write

$$G(z) = 0.3901 \left(\frac{1 + 1.4891z^{-1} + z^{-2}}{1 - 0.06147z^{-1} + 0.70426z^{-2}} \right) \left(\frac{1 + 0.1582z^{-1} + z^{-2}}{1 + 0.5653z^{-1} + 0.2228z^{-2}} \right).$$

A

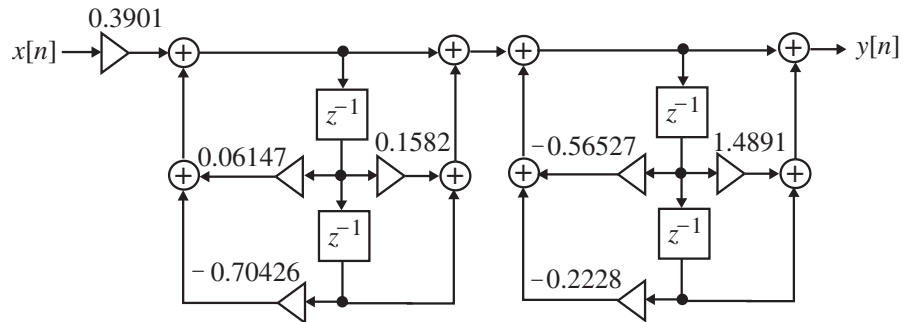
cascade realization of $G(z)$ based on the above decomposition is shown on next page:



Alternately, we can write

$$G(z) = 0.3901 \left(\frac{1 + 0.1582z^{-1} + z^{-2}}{1 - 0.06147z^{-1} + 0.70426z^{-2}} \right) \left(\frac{1 + 1.4891z^{-1} + z^{-2}}{1 + 0.5653z^{-1} + 0.2228z^{-2}} \right).$$

A cascade realization of $G(z)$ based on the above decomposition is shown below:



(c) A partial-fraction expansion of $G(z)$ in z^{-1} obtained using the M-file `residuez` is

given by
$$G(z) = 2.4863 + \frac{-0.3481 - 0.11128z^{-1}}{1 - 0.06147z^{-1} + 0.70425z^{-2}} + \frac{-1.7481 + 0.4094z^{-1}}{1 + 0.5653z^{-1} + 0.2228z^{-2}}.$$

A Parallel Form I realization of $G(z)$ based on the above expansion is shown on the left side top of next page.

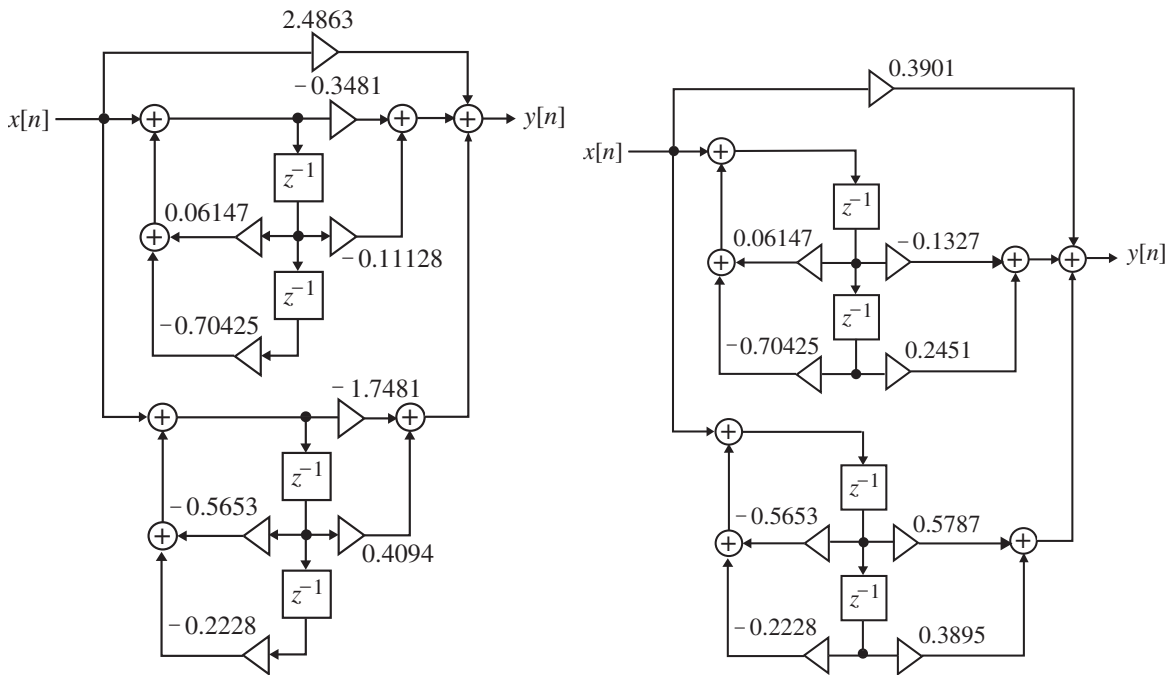
(d) A partial-fraction expansion of $G(z)$ in z obtained using the M-file `residue` is given

$$\text{by } G(z) = 0.3901 + \frac{-0.1327z^{-1} + 0.2451z^{-1}}{1 - 0.06147z^{-1} + 0.70425z^{-2}} + \frac{0.5787z^{-1} + 0.3895z^{-2}}{1 + 0.5653z^{-1} + 0.2228z^{-2}}.$$

Parallel Form II realization of $G(z)$ based on the above expansion is shown on the right side top of next page.

M8.3 (a) Using the M-file `factor`, we factorize the numerator and the denominator of $H(z)$

$$\text{resulting in } H(z) = \frac{0.3549(1 + 0.5958z^{-1} + z^{-2})(1 - 0.03169z^{-1} + z^{-2})}{(1 + 0.6209z^{-1} + 0.9845z^{-2})(1 + 0.6313z^{-1} + 0.5681z^{-2})}.$$



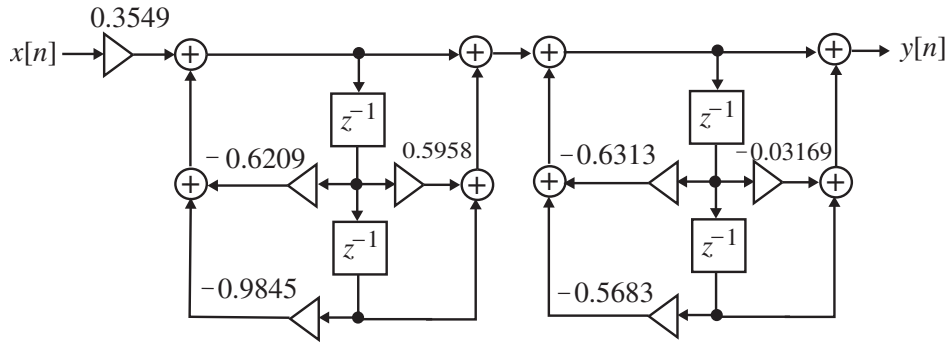
(b) We can write

$$H(z) = 0.3549 \left(\frac{1 + 0.5958z^{-1} + z^{-2}}{1 + 0.6209z^{-1} + 0.9845z^{-2}} \right) \left(\frac{1 - 0.03169z^{-1} + z^{-2}}{1 + 0.6313z^{-1} + 0.5683z^{-2}} \right).$$

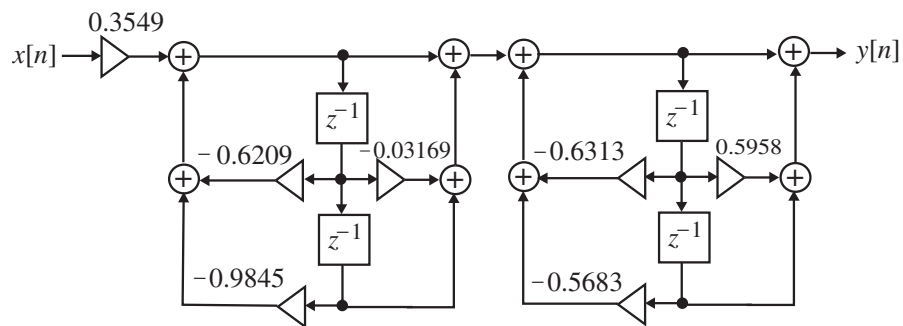
A cascade realization of $H(z)$ based on the above decomposition is shown on top of the next page.

Alternately, we can write

$$H(z) = 0.3549 \left(\frac{1 - 0.03169z^{-1} + z^{-2}}{1 + 0.6209z^{-1} + 0.9845z^{-2}} \right) \left(\frac{1 + 0.5958z^{-1} + z^{-2}}{1 + 0.6313z^{-1} + 0.5683z^{-2}} \right).$$



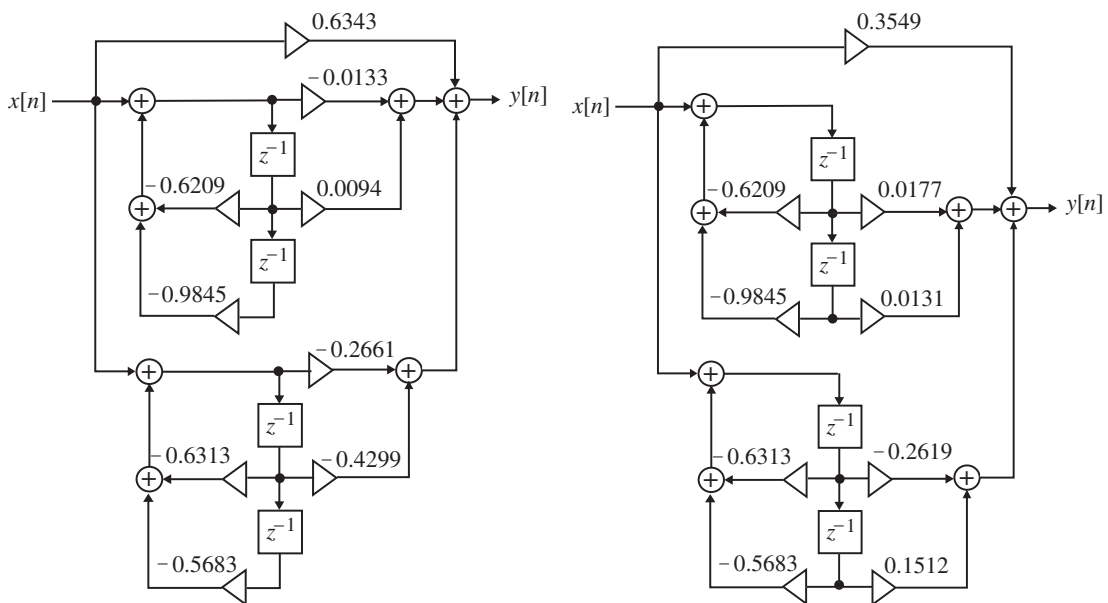
A cascade realization of $H(z)$ based on the decomposition given at the bottom of the previous page is shown below:



(c) A partial-fraction expansion of $G(z)$ in z^{-1} obtained using the M-file `residuez` is given by

$$H(z) = 0.6343 + \frac{-0.0133 + 0.0094z^{-1}}{1 + 0.6209z^{-1} + 0.9845z^{-2}} + \frac{-0.2661 - 0.4299z^{-1}}{1 + 0.6313z^{-1} + 0.5683z^{-2}}.$$

A Parallel Form I realization of $G(z)$ based on the above expansion is shown below on the left side.

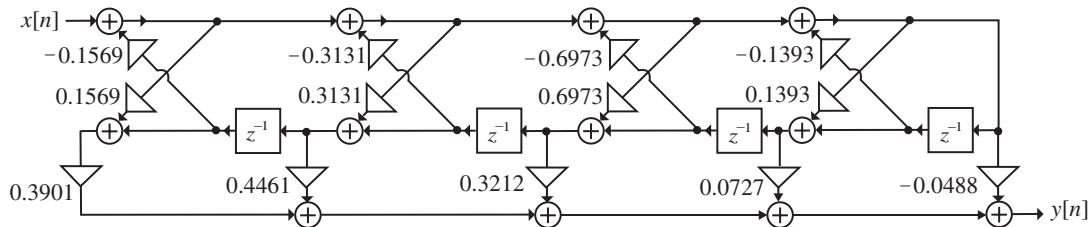


(d) A partial-fraction expansion of $G(z)$ in z obtained using the M-file `residue` is given by $G(z) = 0.3549 + \frac{-0.0177z^{-1} + 0.0131z^{-1}}{1 + 0.6209z^{-1} + 0.9845z^{-2}} + \frac{-0.2619z^{-1} + 0.1512z^{-2}}{1 + 0.6313z^{-1} + 0.5683z^{-2}}$. A Parallel Form II realization of $G(z)$ based on the above expansion is shown at the bottom of the previous page on the right side.

M8.4 Lattice parameters are
 0.1393 0.6973 0.3131 0.1569

Feedforward multipliers are
 0.3901 0.4461 0.3212 0.0727 -0.0488

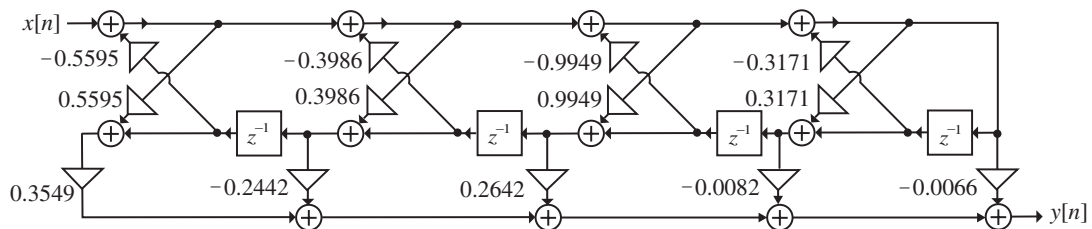
The Gray-Markel tapped cascaded lattice realization of $G(z)$ is shown on next page:



M8.5 Lattice parameters are
 0.3171 0.9949 0.3986 0.5595

Feedforward multipliers are
 0.3549 -0.2442 0.2642 -0.0082 -0.0066

The Gray-Markel tapped cascaded lattice realization of $H(z)$ is shown below:



M8.6 (a) (b) Program 8_7 generated the following error messages for the FIR transfer functions of Parts (a) and (b):

??? Error using ==> signal/private/levdown
 At least one of the reflection coefficients is equal to one.
 The algorithm fails for this case.

(c) (d) Program 8_7 generated the following error messages for the FIR transfer functions of Parts (c) and (d):

Warning: Divide by zero.
 Lattice coefficients are

M8.7 (a) Using the M-file `tf2ca` we arrive at the parallel allpass decomposition of $G(z)$ as:

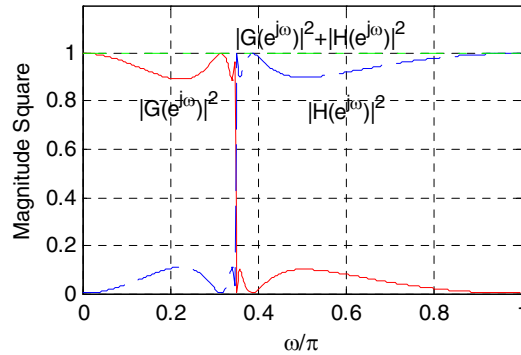
$$G(z) = \frac{1}{2}[A_0(z) + A_1(z)], \text{ where } A_0(z) = \frac{-0.2440 + 1.2092z^{-1} - 1.1436z^{-2} + z^{-3}}{1 - 1.1436z^{-1} + 1.2092z^{-2} - 0.2440z^{-3}} \text{ and}$$

$$A_1(z) = \frac{0.8041 - 0.8171z^{-1} + z^{-2}}{1 - 0.8171z^{-1} + 0.8041z^{-2}}.$$

(b) The power-complementary transfer function is then given by $H(z) = \frac{1}{2}[A_0(z) - A_1(z)]$

$$= \frac{0.524(1 - 3.0011z^{-1} + 4.9948z^{-2} - 4.9948z^{-3} + 3.0011z^{-4} - z^{-5})}{1 - 1.9606z^{-1} + 2.9473z^{-2} - 2.1511z^{-3} + 1.1713z^{-4} - 0.1961z^{-5}}.$$

(c)



M8.8 (a) Using the M-file `tf2ca` we arrive at the parallel allpass decomposition of $G(z)$ as:

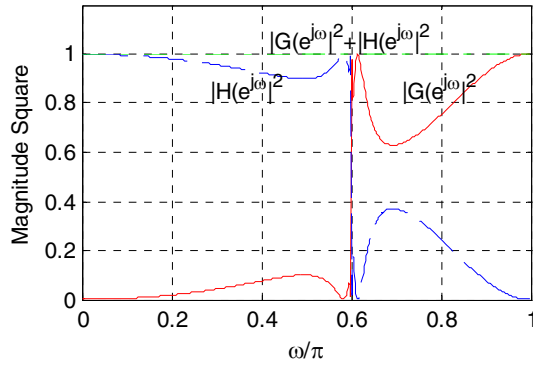
$$G(z) = \frac{1}{2}[A_0(z) - A_1(z)], \text{ where } A_0(z) = \frac{0.9077 + 0.6205z^{-1} + z^{-2}}{1 + 0.6205z^{-1} + 0.9077z^{-2}} \text{ and}$$

$$A_1(z) = \frac{0.3325 + 1.2027z^{-1} + 0.9522z^{-2} + z^{-3}}{1 + 0.9522z^{-1} + 1.2027z^{-2} + 0.3325z^{-3}}.$$

(b) The power-complementary transfer function is then given by $H(z) = \frac{1}{2}[A_0(z) + A_1(z)]$

$$= \frac{(1 + 2.3334z^{-1} + 3.7759z^{-2} + 3.7759z^{-3} + 2.3334z^{-4} + z^{-5})}{1 + 1.5727z^{-1} + 2.712z^{-2} + 1.9431z^{-3} + 1.2979z^{-4} + 0.3018z^{-5}}.$$

(c)



M8.9 The transfer function given cannot be realized as a parallel connection of two real allpass sections as the degree difference between the two allpass transfer functions must be 2.

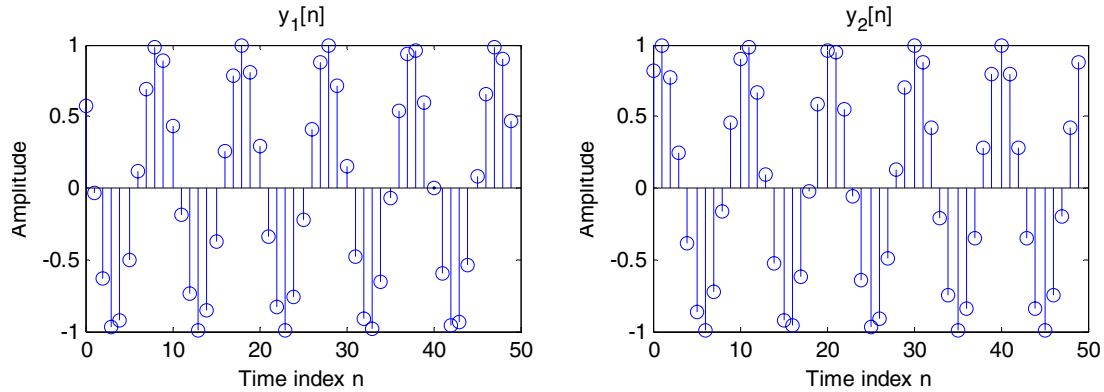
M8.10 The MATLAB program for the simulation of the sine-cosine generator of Problem 8.53 is given below:

```

s10 = 0.1; s20 = 0.1; a = 0.8;
y1 = zeros(1,50); y2 = y1;
for n = 1:50;
    y1(n) = a*(s10 + s20) - s20;
    y2(n) = a*(s10 + s20) + s10;
    s10 = y1(n);
    s20 = y2(n);
end
y1max = max(y1); y2max = max(y2);
k = 1:50;
stem(k-1, y1/y1max);
xlabel('Time index n'); ylabel('Amplitude');
title('y_1[n]');
pause
stem(k-1, y2/y2max);
xlabel('Time index n'); ylabel('Amplitude');
title('y_2[n]');

```

The plots generated by the above program for initial conditions $s1[-1] \quad s2[-1] = 0.1$ are shown below:



The outputs are zero for zero initial conditions. Non-zero initial conditions of equal values appear to have no effects on the outputs. However, unequal initial conditions have effects on the amplitudes and phases of the two output sequences.

M8.11 The MATLAB program for the simulation of the sine-cosine generator of Problem 8.53 is given below:

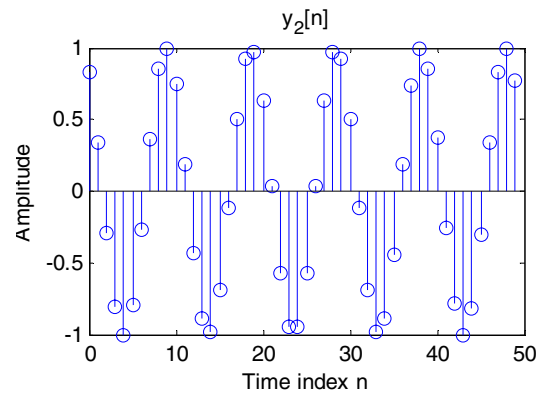
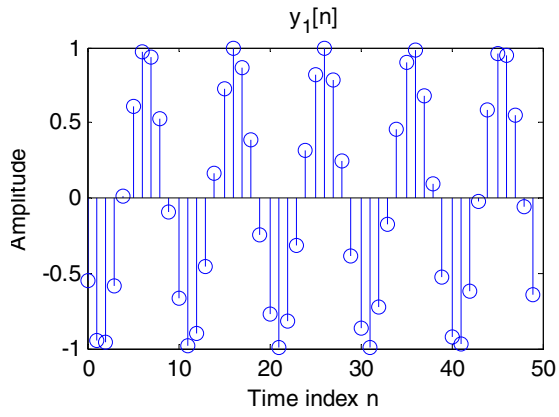
```

s10 = 0.1; s20 = 1; a = 0.8;
y1 = zeros(1,50); y2 = y1;
for n = 1:50;
    y1(n) = -s20 + a*s10;
    y2(n) = -a*y1(n) + s10;
    s10 = y1(n); s20 = y2(n);
end
y1max = max(y1); y2max = max(y2);
k = 1:50;
stem(k-1, y1/y1max);
xlabel('Time index n'); ylabel('Amplitude');
title('y_1[n]');
pause
stem(k-1, y2/y2max);
xlabel('Time index n'); ylabel('Amplitude');
title('y_2[n]');

```

The plots generated by the above program for initial conditions $s1[-1] \ s2[-1] = 0.1$ are shown on top of the next page.:

The outputs are zero for zero initial conditions. Non-zero initial conditions of equal values appear to have no effects on the outputs. However, unequal initial conditions have effects on the amplitudes and phases of the two output sequences.



Chapter 9

9.1 We obtain the solutions by using Eq. (9.3) and Eq. (9.4).

(a) $\delta_p = 1 - 10^{-\alpha_p/20} = 1 - 10^{-0.21/20} = 0.0239$, $\delta_s = 10^{-\alpha_s/20} = 10^{-53/20} = 0.0022$

(b) $\delta_p = 1 - 10^{-\alpha_p/20} = 1 - 10^{-0.17/20} = 0.0194$,
 $\delta_s = 10^{-\alpha_s/20} = 10^{-78/20} = 1.26 \times 10^{-4}$.

9.2 We obtain the solutions by using Eq. (9.3) and Eq. (9.4).

(a) $\alpha_p = -20 \log_{10}(1 - \delta_p) = -20 \log_{10}(1 - 0.02) = 0.1755$ dB,

$\alpha_s = -20 \log_{10}(\delta_s) = -20 \log_{10}(0.03) = 30.458$ dB.

(b) $\alpha_p = -20 \log_{10}(1 - \delta_p) = -20 \log_{10}(1 - 0.055) = 0.4914$ dB,

$\alpha_s = -20 \log_{10}(\delta_s) = -20 \log_{10}(0.033) = 29.630$ dB.

9.3 $G(z) = H^2(z)$, or equivalently, $G(e^{j\omega}) = H^2(e^{j\omega})$. $|G(e^{j\omega})| = |H^2(e^{j\omega})| = |H(e^{j\omega})|^2$.

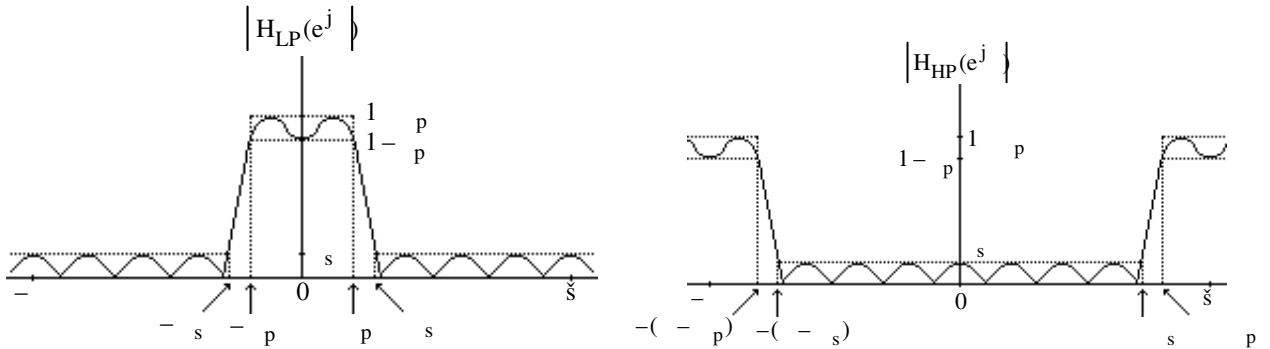
Let δ_p and δ_s denote the passband and stopband ripples of $H(e^{j\omega})$, respectively. Also,

let $\delta_{p,2} = 2\delta_p$, and $\delta_{s,2}$ denote the passband and stopband ripples of $G(e^{j\omega})$,

respectively. Then $\delta_{p,2} = 1 - (1 - \delta_p)^2$, and $\delta_{s,2} = (\delta_s)^2$. For a cascade of M sections,

$\delta_{p,M} = 1 - (1 - \delta_p)^M$, and $\delta_{s,M} = (\delta_s)^M$.

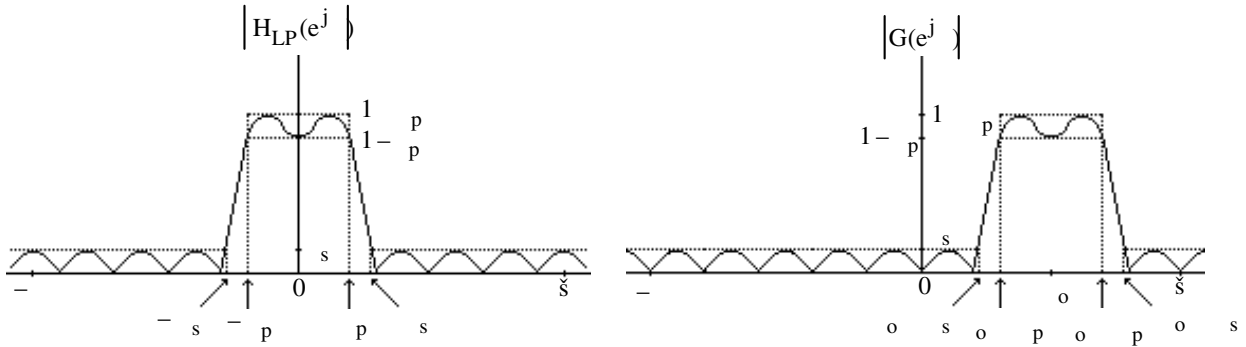
9.4



Therefore, the passband edge and the stopband edge of the highpass filter are given by $\omega_{p,HP} = \pi - \omega_p$, and $\omega_{s,HP} = \pi - \omega_s$, respectively.

9.5 Note that $G(z)$ is a complex bandpass filter with a passband in the range $0 \leq \omega \leq \pi$. Its passband edges are at $\omega_{p,BP} = \omega_o \pm \omega_p$, and stopband edges at $\omega_{s,BP} = \omega_o \pm \omega_s$. A real coefficient bandpass transfer function can be generated according to

$G_{BP}(z) = H_{LP}(e^{j\omega_0} z) + H_{LP}(e^{-j\omega_0} z)$ which will have a passband in the range $0 \leq \omega \leq \pi$ and another passband in the range $-\pi \leq \omega \leq 0$. However because of the overlap of the two spectra a simple formula for the bandedges cannot be derived.



9.6 (a) $h_p(t) = h_a(t) \cdot p(t)$ where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. Thus, $h_p(t) = \sum_{n=-\infty}^{\infty} h_a(t - nT) \delta(t - nT)$.

We also have, $g[n] = h_a(nT)$. Now, $H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{st} dt$ and

$$H_p(s) = \int_{-\infty}^{\infty} h_p(t) e^{st} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} h_a(t - nT) \delta(t - nT) e^{st} dt = \sum_{n=-\infty}^{\infty} h_a(nT) e^{snT}.$$

Comparing the above expression with $G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} h_a(nT) z^{-n}$, we

conclude that $G(z) = H_p(s) \Big|_{s = \frac{1}{T} \ln z}$.

We can also show that a Fourier series expansion of $p(t)$ is given by

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j(2\pi kt/T)}. \quad \text{Therefore,}$$

$$h_p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j(2\pi kt/T)} h_a(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} h_a(t) e^{j(2\pi kt/T)}. \quad \text{Hence,}$$

$$H_p(s) = \sum_{k=-\infty}^{\infty} \frac{1}{T} H_a \left(s + j \frac{2\pi k}{T} \right). \quad \text{As a result, we have}$$

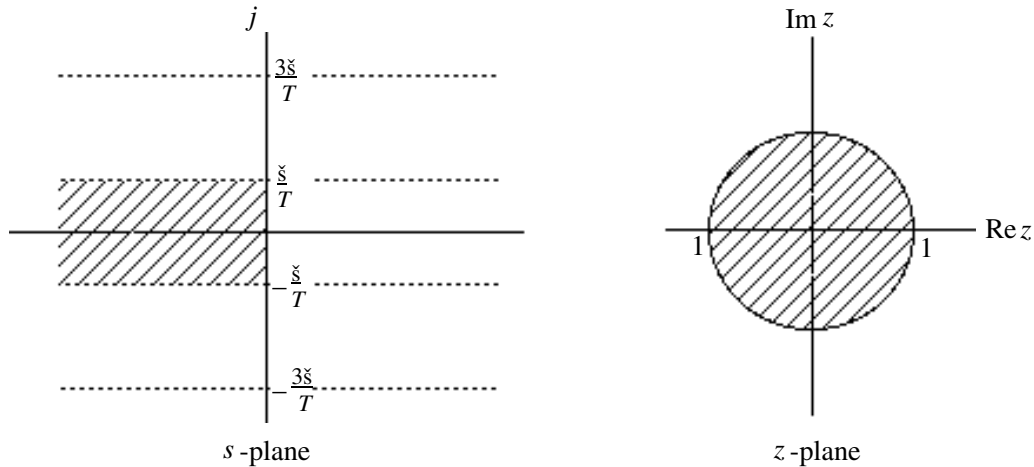
$$G(z) = \sum_{k=-\infty}^{\infty} \frac{1}{T} H_a \left(s + j \frac{2\pi k}{T} \right) \Big|_{s = \frac{1}{T} \ln z}. \quad (7-1)$$

(b) The transformation from the s -plane to z -plane is given by $z = e^{sT}$. If we express $s = \sigma + j\omega$, then we can write $z = re^{j\theta} = e^{\sigma T} e^{j\omega T}$. Therefore,

$|z| < 1$, for $\sigma < 0$,
 $|z| = 1$, for $\sigma = 0$,
 $|z| > 1$, for $\sigma > 0$.

Or in other words, a point in the left-half s -plane is mapped onto a point inside the unit circle in the z -plane, a point in the right-half s -plane is mapped onto a point outside the unit circle in the z -plane, and a point on the j -axis in the s -plane is mapped onto a point on the unit circle in the z -plane. As a result, the mapping has the desirable properties enumerated in Section 9.1.3.

(c) However, all points in the s -plane defined by $s = j\omega = j\frac{2\pi k}{T}$, $k = 0, 1, 2, \dots$, are mapped onto a single point in the z -plane as $z = e^{sT} = e^{j\omega T} = e^{j2\pi k}$. The mapping is illustrated in the figure below



Note that the strip of width $2\pi/T$ in the s -plane for values of s in the range $-\pi/T < \sigma < \pi/T$ is mapped into the entire z -plane, and so are the adjacent strips of width $2\pi/T$. The mapping is many-to-one with infinite number of such strips of width $2\pi/T$.

It follows from the above figure and also from Eq. (7-1) that if the frequency response $H_a(j\omega) = 0$ for $|\omega| > \pi/T$, then $G(e^{j\omega T}) = \frac{1}{T} H_a(j\frac{\omega T}{T})$, for $|\omega| < \pi/T$, and there is no aliasing.

(d) For $z = e^{j\omega T} = e^{j\omega T}$. Or equivalently, $\omega = \frac{\arg z}{T}$.

9.7 Assume $h_a(t)$ is causal. Now, $h_a(t) = \int_0^\infty H_a(s)e^{st} ds$. Hence,

$$g[n] = h_a(nT) = \int_0^\infty H_a(s)e^{snT} ds. \quad \text{Therefore,}$$

$$G(z) = \sum_{n=0}^{\infty} g[n]z^{-n} = \sum_{n=0}^{\infty} \int_0^\infty H_a(s)e^{snT} z^{-n} ds = \int_0^\infty H_a(s) \sum_{n=0}^{\infty} z^{-n} e^{snT} ds = \int_0^\infty \frac{H_a(s)}{1 - e^{sT} z^{-1}} ds.$$

Hence $G(z) = \sum_{\text{all poles of } H_a(s)} \text{Residues} \left[\frac{H_a(s)}{1 - e^{sT} z^{-1}} \right]$.

9.8 $H_a(s) = \frac{A}{s + \alpha}$. The transfer function has a pole at $s = -\alpha$. Now

$$G(z) = \text{Residue}_{\text{at } s=-\alpha} \left[\frac{A}{(s + \alpha)(1 - e^{sT} z^{-1})} \right] = \frac{A}{1 - e^{sT} z^{-1}} \Big|_{s=-\alpha} = \frac{A}{1 - e^{-\alpha T} z^{-1}}.$$

9.9 (a) $H_a(s) = \frac{4(3s + 7)}{(s + 2)(s^2 + 4s + 5)} = \frac{-4}{s + 2} + \frac{2 - 6j}{(s + 2 + j)} + \frac{2 + 6j}{(s + 2 - j)}$

$$= \frac{-4}{s + 2} + \frac{4s - 4}{(s + 2)^2 + 1^2} = \frac{-4}{s + 2} + \frac{4(s + 2)}{(s + 2)^2 + 1^2} + \frac{(-12) \cdot 1^2}{(s + 2)^2 + 1^2}.$$

Using Eqs. (9.58), (9.60), and (9.62), we get

$$G_a(z) = \frac{-4}{1 - e^{-2T} z^{-1}} + \frac{4(1 - z^{-1} e^{-2T} \cos(T))}{1 - 2z^{-1} e^{-2T} \cos(T) + e^{-4T} z^{-2}} + \frac{z^{-1} e^{-2T} \sin(T)}{1 - 2z^{-1} e^{-2T} \cos(T) + e^{-4T} z^{-2}},$$

where $T = 0.3$. Therefore,

$$G_a(z) = \frac{-4}{1 - 0.5488z^{-1}} + \frac{4(1 - 0.5243z^{-1})}{1 - 1.0486z^{-1} + 0.3012z^{-2}} + \frac{0.1622z^{-1}}{1 - 1.0486z^{-1} + 0.3012z^{-2}}$$

$$= \frac{-4}{1 - 0.5488z^{-1}} + \frac{4 - 1.935z^{-1}}{1 - 1.0486z^{-1} + 0.3012z^{-2}}.$$

(b) $H_b(s) = \frac{8s^2 + 37s + 56}{(s^2 + 2s + 10)(s + 4)} = \frac{2}{(s + 4)} + \frac{3 + 0.5j}{(s + 1 + 3j)} + \frac{s - 0.5j}{(s + 1 - 3j)}$

$$= \frac{2}{s + 4} + \frac{6s + 9}{(s + 1)^2 + 9} = \frac{2}{s + 4} + \frac{6(s + 1)}{(s + 1)^2 + 3^2} + \frac{3(3)}{(s + 1)^2 + 3^2}.$$

Using Eqs. (9.58), (9.60), and (9.62), we get

$$G_b(z) = \frac{2}{1 - e^{-4T} z^{-1}} + 6 \frac{1 - z^{-1} e^{-T} \cos(3T)}{1 - 2z^{-1} e^{-T} \cos(3T) + e^{-2T} z^{-2}} + 3 \frac{z^{-1} e^{-T} \sin(3T)}{1 - 2z^{-1} e^{-T} \cos(3T) + e^{-2T} z^{-2}},$$

where $T = 0.3$. Therefore,

$$G_b(z) = \frac{4}{1 - 0.3012z^{-1}} + \frac{6(1 - 0.4605z^{-1})}{1 - 0.9210z^{-1} + 0.5489z^{-2}} + \frac{3(0.5803z^{-1})}{1 - 0.9210z^{-1} + 0.5489z^{-2}}$$

$$= \frac{4}{1-0.3012z^{-1}} + \frac{6-1.0221z^{-1}}{1-0.9210z^{-1}+0.5489z^{-2}}$$

$$\begin{aligned} \text{(c)} \quad H_c(s) &= \frac{s^3+s^2+6s+14}{(s^2+2s+5)(s^2+s+4)} = \frac{4s+1}{s^2+2s+5} + \frac{-3s+2}{s^2+s+4} \\ &= \frac{4(s+2)}{(s+2)^2+1^2} + \frac{-7(1)}{(s+2)^2+1^2} + \frac{-3(s+0.5)}{(s+0.5)^2+(\sqrt{15}/2)^2} + \frac{7/\sqrt{15}(\sqrt{15}/2)}{(s+0.5)^2+(\sqrt{15}/2)^2}. \end{aligned}$$

Using Eqs. (9.58), (9.60), and (9.62), we get

$$\begin{aligned} G_c(z) &= (4) \frac{1-e^{-2T} \cos(T)z^{-1}}{1-2e^{-2T} \cos(T)z^{-1}+e^{-4T}z^{-2}} + (-7) \frac{e^{-2T} \sin(T)z^{-1}}{1-2e^{-2T} \cos(T)z^{-1}+e^{-4T}z^{-2}} \\ &+ (-3) \frac{1-e^{-T/2} \cos(\sqrt{15}T/2)z^{-1}}{1-2e^{-2T} \cos(\sqrt{15}T/2)z^{-1}+e^{-T}z^{-2}} + (7/\sqrt{15}) \frac{e^{-T/2} \sin(\sqrt{15}T/2)z^{-1}}{1-2e^{-2T} \cos(\sqrt{15}T/2)z^{-1}+e^{-T}z^{-2}}, \end{aligned}$$

where $T=0.3$. Hence,

$$\begin{aligned} G_c(z) &= \frac{4(1-0.5243z^{-1})}{1-1.0486z^{-1}+0.3012z^{-2}} - \frac{1.1353z^{-1}}{1-1.0486z^{-1}+0.3012z^{-2}} \\ &+ \frac{-3(1-0.7195z^{-1})}{1-1.4390z^{-1}+0.7408z^{-2}} + \frac{0.8538z^{-1}}{1-1.4390z^{-1}+0.7408z^{-2}}. \end{aligned}$$

$$\text{9.10 (a)} \quad G_a(z) = \frac{3z}{z-e^{-1.5}} + \frac{4z}{z-e^{-1.8}} = \frac{A_1z}{z-e^{-\alpha_1 T}} + \frac{A_2z}{z-e^{-\alpha_2 T}}.$$

Since $T=0.2$, $\alpha_1=7.5$, $\alpha_2=9$, $A_1=3$, $A_2=4$, it follows $H_a(s) = \frac{3}{s+7.5} + \frac{4}{s+9}$.

$$\text{(b)} \quad G_b(z) = \frac{ze^{-1.2} \sin(1.5)}{z^2-2ze^{-1.2} \cos(1.5)+e^{-2.4}} = \frac{ze^{-\beta T} \sin(\lambda T)}{z^2-2ze^{-\beta T} \cos(\lambda T)+e^{-2\beta T}}.$$

Since $T=0.2$, $\lambda=7.5$, $\beta=6$, it follows $H_b(s) = \frac{7.5}{(s+6)^2+7.5^2}$

$$\text{9.11 (a)} \quad H_a(s) = G_a(z) \Big|_{z=4\left(\frac{1+s}{1-s}\right)} = \frac{4\left[4\left(\frac{1+s}{1-s}\right)\right]^2 + 3\left[4\left(\frac{1+s}{1-s}\right)\right] + 4}{10\left[4\left(\frac{1+s}{1-s}\right)\right]^2 + 4\left[4\left(\frac{1+s}{1-s}\right)\right] + 6} = \frac{16(s^2+3s+4)}{75s^2+154s+91}.$$

$$\text{(b)} \quad H_b(s) = G_b(z) \Big|_{z=4\left(\frac{1+s}{1-s}\right)} = \frac{54\left[4\left(\frac{1+s}{1-s}\right)\right]^3 + 62\left[4\left(\frac{1+s}{1-s}\right)\right]^2 + 26\left[4\left(\frac{1+s}{1-s}\right)\right] + 18}{\left(3\left[4\left(\frac{1+s}{1-s}\right)\right] + 1\right)\left(12\left[4\left(\frac{1+s}{1-s}\right)\right]^2 - 4\left[4\left(\frac{1+s}{1-s}\right)\right] + 8\right)}$$

$$= \frac{-(5 + 3s)(425s^2 + 846s + 457)}{4(13s + 11)(27s^2 + 46s + 23)}$$

9.12 For the impulse invariance design:

$$\omega_p = \Omega_p T = 2\pi F_p T = 2\pi(0.45 \times 10^3)(0.3 \times 10^{-3}) = 0.27\pi$$

For the bilinear transformation method:

$$\omega_p = 2 \tan^{-1} \left(\frac{\Omega_p T}{2} \right) = 2 \tan^{-1} (F_p T \pi) = 2 \tan^{-1} (0.45 \times 10^3 \cdot 0.3 \times 10^{-3} \cdot \pi) = 0.2554\pi$$

9.13 For the impulse invariance method: $2\pi F_p = \frac{\omega_p}{T} = \frac{0.56\pi}{0.2 \times 10^{-3}} \Rightarrow F_p = 1.4 \text{ kHz}$

For the bilinear transformation method:

$$F_p = \tan \left(\frac{\omega_p}{2} \right) \cdot \frac{1}{\pi T} = \tan \left(\frac{0.56\pi}{2} \right) \cdot \frac{1}{\pi(0.2 \times 10^{-3})} = 1.923 \text{ kHz.}$$

9.14 The passband and the stopband edges of the analog lowpass filter are assumed to $\Omega_p = 0.25\pi$ and $\Omega_s = 0.55\pi$. The requirements to be satisfied by the analog

lowpass filter are thus $20 \log_{10} |H_a(j0.25\pi)| \geq -0.5 \text{ dB}$ and

$20 \log_{10} |H_a(j0.55\pi)| \leq -15 \text{ dB}$.

From $\alpha_p = 20 \log_{10}(\sqrt{1 + \epsilon^2}) = 0.5$ we obtain $\epsilon^2 = 0.1220184543$. From

$\alpha_s = 20 \log_{10}(A^2) = 15$ we obtain $A^2 = 31.6227766$. From Eq. (4.32), the inverse

discrimination ratio is given by $\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\epsilon} = 15.841979$ and from Eq. (4.31) the inverse

transition ratio is given by $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.2$. Substituting these values in Eq. (4.35) we

obtain $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(15.841979)}{\log_{10}(2.2)} = 3.503885$. We choose $N = 4$.

From Eq. (4.33) we have $\left(\frac{\Omega_p}{\Omega_c} \right)^{2N} = \epsilon^2$. Substituting the values of Ω_p , N , and ϵ^2 we get

$$\Omega_c = 1.3007568(\Omega_p) = 1.021612.$$

Using the statement `[z, p, k] = buttap(4)` we get the poles of the 4-th order Butterworth analog filter with a 3-dB cutoff at 1 rad/s as $p_1 = -0.3827 + j0.9239$,

$p_2 = -0.3827 - j0.9239$, $p_3 = -0.9239 + j0.3827$, and $p_4 = -0.9239 - j0.3827$.

Therefore,

$$H_{an}(s) = \frac{1}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)} = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$$

Next we expand $H_{an}(s)$ in a partial-fraction expansion using the M-file `residue` and arrive at $H_{an}(s) = \frac{-0.9238729s - 0.7071323}{s^2 + 0.7654s + 1} + \frac{0.9238729s + 1.7071323}{s^2 + 1.8478s + 1}$. We next denormalize $H_{an}(s)$ to move the 3-dB cutoff frequency to $\omega_c = 1.021612$ using the M-file

$$\begin{aligned} \text{lp2lp} \text{ resulting in } H_a(s) &= H_{an}\left(\frac{s}{1.021612}\right) \\ &= \frac{-0.943847s - 0.738039}{s^2 + 0.781947948s + 1.0437074244} + \frac{0.943847s + 1.78174665}{s^2 + 1.887749436s + 1.0437074244} \\ &= \frac{-0.943847s - 0.738039}{(s + 0.390974)^2 + (0.9438467)^2} + \frac{0.943847s + 1.78174665}{(s + 0.94387471)^2 + (0.39090656)^2} \end{aligned}$$

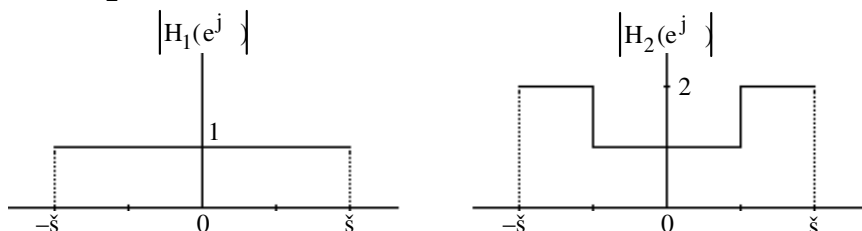
Making use of the M-file `bilinear` we finally arrive at

$$G(z) = \frac{-0.943847z^2 + 0.68178386z}{z^2 - 1.363567724z + 0.4575139} + \frac{0.943847z^2 - 0.25640047z}{z^2 - 0.77823439z + 0.1514122}$$

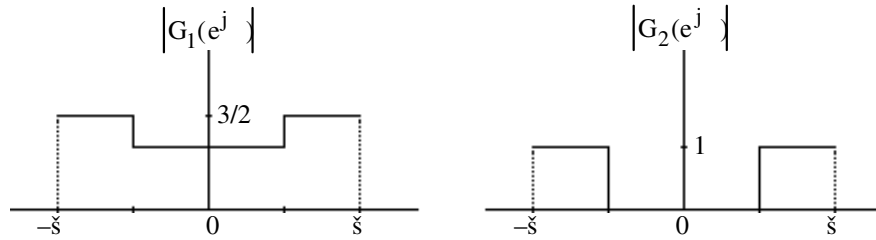
- 9.15** The mapping is given by $s = \frac{1}{T}(1 - z^{-1})$ or equivalently, by $z = \frac{1}{1 - sT}$. For $s = \sigma_o + j\Omega_o$, $z = \frac{1}{1 - \sigma_o T - j\Omega_o T}$. Therefore, $|z|^2 = \frac{1}{(1 - \sigma_o T)^2 + (\Omega_o T)^2}$. Hence, $|z| < 1$ for $\sigma_o < 0$. As a result, a stable $H_a(s)$ results in a stable $H(z)$ after the transformation. However, for $\sigma_o = 0$, $|z|^2 = \frac{1}{1 + (\Omega_o T)^2}$ which is equal to 1 only for $\Omega_o = 0$. Hence, only the point $\Omega_o = 0$ on the $j\Omega$ -axis in the s -plane is mapped onto the point $z = 1$ on the unit circle. Consequently, this mapping is not useful for the design of digital filters via analog filter transformation.

- 9.16** For no aliasing $T \leq \frac{\pi}{\Omega_c}$. Figure below shows the magnitude responses of the digital filters

$H_1(z)$ and $H_2(z)$.



(a) The magnitude responses of the digital filters $G_1(z)$ and $G_2(z)$ are shown below:



(b) As can be seen from the above $G_1(z)$ is a multi-band filter, whereas, $G_2(z)$ is a highpass filter.

9.17 $H_a(s)$ is causal and stable and $|H_a(s)| \leq 1$, $\text{Re}\{s\} < 0$. Now, $G(z) = H_a(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$. Thus,

$G(z)$ is causal and stable. Now,

$$G(e^{j\omega}) = H_a(s)\bigg|_{s=\frac{2}{T}\left(\frac{1-e^{j\omega}}{1+e^{j\omega}}\right)} = H_a(s)\bigg|_{s=j\frac{2}{T}\tan(\omega/2)} = H_a(j\frac{2}{T}\tan(\omega/2)).$$

Therefore, $|G(e^{j\omega})| = |H_a(j\frac{2}{T}\tan(\omega/2))| \leq 1$ for all values of ω . Hence, $G(z)$ is a BR function.

9.18 $G(z) = \frac{1+\alpha}{2} \cdot \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$. For $\beta = \cos \omega_0$, the numerator of $G(z)$ becomes $1-2\cos \omega_0 z^{-1}+z^{-2} = (1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})$ which has roots at $z = e^{\pm j\omega_0}$. The numerator of $G(z^N)$ is then given by $(1-e^{j\omega_0}z^{-N})(1-e^{-j\omega_0}z^{-N})$ whose roots are obtained by solving the equation $z^N = e^{\pm j\omega_0}$, and are given by $z = e^{j(2n \pm \omega_0)/N}$, $0 \leq n \leq N-1$. Hence $G(z^N)$ has N complex conjugate zero-pairs located on the unit circle at angles of $\frac{2n \pm \omega_0}{N}$ radians, $0 \leq n \leq N-1$.

Therefore, there are $2N$ equally spaced zeros on the unit circle starting at $\omega_0/2N$.

9.19 (a) $H(z) = \frac{1}{2}[1+A_4(z)] = \frac{N(z)}{D(z)}$. We can write $A_4(z) = \frac{z^{-2}D_1(z^{-1})}{D_1(z)} \cdot \frac{z^{-2}D_2(z^{-1})}{D_2(z)}$, where

$$D_1(z) = 1 - \beta_1(1 + \alpha_1)z^{-1} + \alpha_1 z^{-2} \text{ and } D_2(z) = 1 - \beta_2(1 + \alpha_2)z^{-1} + \alpha_2 z^{-2}. \text{ Therefore,}$$

$$N(z) = \frac{1}{2} [D_1(z)D_2(z) + z^{-4}D_1(z^{-1})D_2(z^{-1})] \text{ Now,}$$

$z^{-4}N(z^{-1}) = \frac{1}{2} \left[z^{-4}D_1(z^{-1})D_2(z^{-1}) + D_1(z)D_2(z) \right] = N(z)$. Hence, $N(z)$ is a symmetric polynomial. It follows then

$$\begin{aligned} P(z) &= \frac{1}{2} \left[(\alpha_1 - \beta_1(1 + \alpha_1)z^{-1} + z^{-2}) (\alpha_2 - \beta_2(1 + \alpha_2)z^{-1} + z^{-2}) \right. \\ &\quad \left. + (1 - \beta_1(1 + \alpha_1)z^{-1} + \alpha_1z^{-2}) (1 - \beta_2(1 + \alpha_2)z^{-1} + \alpha_2z^{-2}) \right] \\ &= \frac{1 + \alpha_1\alpha_2}{2} \left[1 - \frac{(1 + \alpha_1)(1 + \alpha_2)(\beta_1 + \beta_2)}{1 + \alpha_1\alpha_2} z^{-1} + \frac{2[\alpha_1 + \alpha_2 + \beta_1\beta_2(1 + \alpha_1)(1 + \alpha_2)]}{1 + \alpha_1\alpha_2} z^{-2} \right. \\ &\quad \left. - \frac{(1 + \alpha_1)(1 + \alpha_2)(\beta_1 + \beta_2)}{1 + \alpha_1\alpha_2} z^{-3} + z^{-4} \right] = a(1 + b_1z^{-1} + b_2z^{-2} + b_1z^{-3} + z^{-4}), \end{aligned}$$

$$\text{where} \quad b_1 = -\frac{(1 + \alpha_1)(1 + \alpha_2)(\beta_1 + \beta_2)}{1 + \alpha_1\alpha_2}, \quad (7-a)$$

$$b_2 = \frac{2[\alpha_1 + \alpha_2 + \beta_1\beta_2(1 + \alpha_1)(1 + \alpha_2)]}{1 + \alpha_1\alpha_2}, \quad (7-b)$$

$$(b) \quad a = \frac{1 + \alpha_1\alpha_2}{2}. \quad (7-c)$$

(c) for $z = e^{j\omega}$, we can write $N(e^{j\omega}) = a(1 + b_1e^{-j\omega} + b_2e^{-j2\omega} + b_1e^{-j3\omega} + e^{-j4\omega}) = ae^{-j2\omega}(b_2 + 2b_1 \cos \omega + 2 \cos 2\omega)$. Now, $N(e^{j\omega}) = 0$ for $i = 1, 2$. For $i = 1$, we get

$$b_2 + 2b_1 \cos \omega_1 + 2 \cos 2\omega_1 = 0, \quad (7-d)$$

$$\text{for } i = 2, \text{ we get} \quad b_2 + 2b_1 \cos \omega_2 + 2 \cos 2\omega_2 = 0, \quad (7-e)$$

$$\text{Solving Eqs. (7-d) and (7-e) we get } b_1 = -2(\cos \omega_1 + \cos \omega_2) \quad (7-f)$$

$$\text{and} \quad b_2 = 2(2 \cos \omega_1 \cos \omega_2 + 1). \quad (7-g)$$

$$\text{From Eqs. (7-a) and (7-f) we have } \frac{(1 + \alpha_1)(1 + \alpha_2)(\beta_1 + \beta_2)}{1 + \alpha_1\alpha_2} = 2(\cos \omega_1 + \cos \omega_2), \quad (7-h)$$

and from Eqs. (7-b) and (7-g) we have

$$\frac{2[\alpha_1 + \alpha_2 + \beta_1\beta_2(1 + \alpha_1)(1 + \alpha_2)]}{1 + \alpha_1\alpha_2} = 2(2 \cos \omega_1 \cos \omega_2 + 1). \quad (7-i)$$

Substituting $\alpha_1 = \frac{1 - \tan(B_1/2)}{1 + \tan(B_1/2)}$ and $\alpha_2 = \frac{1 - \tan(B_2/2)}{1 + \tan(B_2/2)}$, and after rearrangement we get

$$\beta_1 + \beta_2 = (\cos \omega_1 + \cos \omega_2)[1 + \tan(B_1/2)\tan(B_2/2)] \stackrel{\Delta}{=} \theta_1, \quad (7-j)$$

and $\beta_1\beta_2 = [1 + \tan(B_1/2)\tan(B_2/2)]\cos\omega_1\cos\omega_2 \triangleq \theta_2$. (7-k)

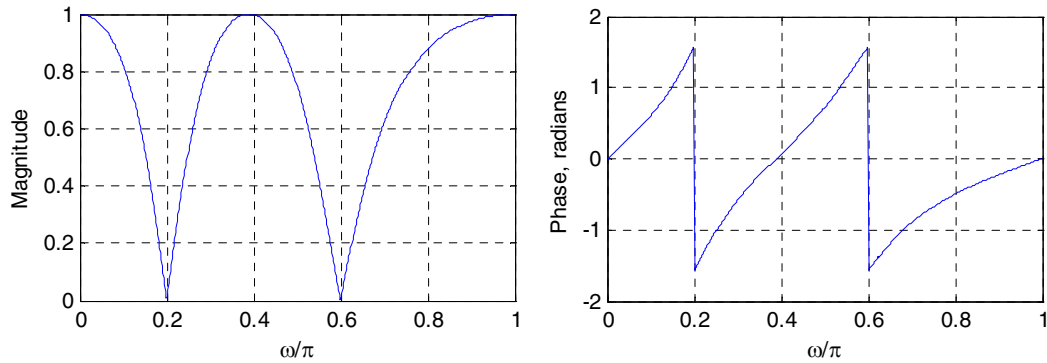
The above two nonlinear equations can be solved yielding

$$\beta_1 = \frac{1 + \sqrt{\frac{2}{1 + 4\beta_2}}}{2} \text{ and } \beta_2 = \frac{2}{1 + \beta_1}.$$

(d) For the double notch filter with the following specifications: $\omega_1 = 0.2\pi$, $\omega_2 = 0.6\pi$, $B_1 = 0.2\pi$, and $B_2 = 0.25\pi$ we get the following values for the parameters of the notch filter transfer function:

$$\alpha_1 = 0.5095, \alpha_2 = 0.4142, \theta_1 = 0.5673, \theta_2 = -0.1491, \beta_1 = 0.7628, \text{ and}$$

$$\beta_2 = -0.1955. H(z) = \frac{1}{2}[1 + A_4(z)]:$$



9.20 A zero (pole) of $H_{LP}(z)$ is given by the factor $(z - z_k)$. After applying the lowpass-to-lowpass transformation, this factor becomes $\frac{\hat{z} - \alpha}{1 - \alpha\hat{z}} - z_k$, and hence the new location of the zero (pole) is given by the roots of the equation

$$\hat{z} - \alpha - z_k + \alpha z_k \hat{z} = (1 + \alpha z_k)\hat{z} - (\alpha + z_k) = 0 \text{ or } \hat{z}_k = \frac{\alpha + z_k}{1 + \alpha z_k}. \text{ For } z_k = -1,$$

$$\hat{z}_k = \frac{\alpha - 1}{1 - \alpha} = -1.$$

9.21 The lowpass-to-bandpass transformation is given by $z = \frac{b + a\hat{z}}{1 + a\hat{z}} \frac{\hat{z}^2}{b\hat{z}^2}$ where $a = \frac{2}{1 + \omega_c}$ and $b = \frac{1}{1 - \omega_c}$. A zero (pole) of $H_{LP}(z)$ is given by the factor $(z - z_k)$. After applying the lowpass-to-bandpass transformation, this factor becomes $\frac{b + a\hat{z}}{1 + a\hat{z}} \frac{\hat{z}^2}{b\hat{z}^2} - z_k$, and hence, the new location of the zero (pole) of the bandpass transfer function is given by the roots of the

equation $(1 - bz_k)z^2 - a(1 - z_k)z - (b - z_k) = 0$, or $z^2 - \frac{a(1 - z_k)}{1 - bz_k}z - \frac{b - z_k}{1 - bz_k} = 0$, whose

solution is given by $\hat{z}_k = \frac{a(1 - z_k)}{2(1 - bz_k)} \pm \sqrt{\left(\frac{a(1 - z_k)}{2(1 - bz_k)}\right)^2 + \frac{b - z_k}{1 - bz_k}}$. For $z_k = 1$, $\hat{z}_k = 1$.

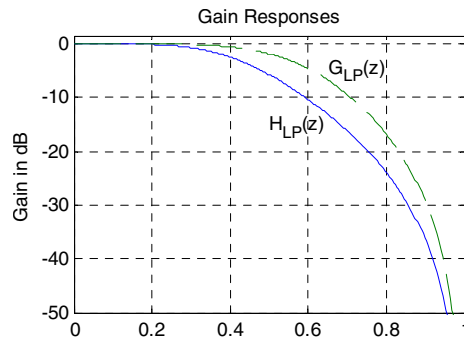
$$9.22 \quad G_{LP}(z) = \frac{0.3404(1 + z^{-1})^2}{1 + 0.1842z^{-1} + 0.1776z^{-2}}, \text{ with } \omega_c = 0.55\pi.$$

$H_{LP}(z)$ for $\hat{\omega}_c = 0.42\pi$

$$\alpha = \frac{\tan\left(\frac{0.55\pi}{2}\right) - \tan\left(\frac{0.42\pi}{2}\right)}{\tan\left(\frac{0.55\pi}{2}\right) + \tan\left(\frac{0.42\pi}{2}\right)} = \frac{\sin\left(\frac{0.55\pi - 0.42\pi}{2}\right)}{\sin\left(\frac{0.55\pi + 0.42\pi}{2}\right)} = 0.2030$$

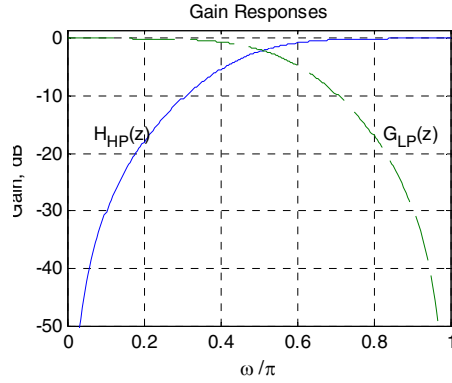
$$H_{LP}(z) = G_{LP}(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}} = \frac{0.3404\left(1 + \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}\right)^2}{1 + 0.1842\left(\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}\right) + 0.1776\left(\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}\right)^2}$$

$$= \frac{0.21623 + 0.43245\hat{z}^{-1} + 0.21623\hat{z}^{-2}}{0.96993 - 0.2863\hat{z}^{-1} + 0.181416\hat{z}^{-2}}.$$



$$9.23 \quad \alpha = \frac{-\cos\left(\frac{0.55\pi + 0.47\pi}{2}\right)}{\cos\left(\frac{0.55\pi - 0.47\pi}{2}\right)} = 0.0317.$$

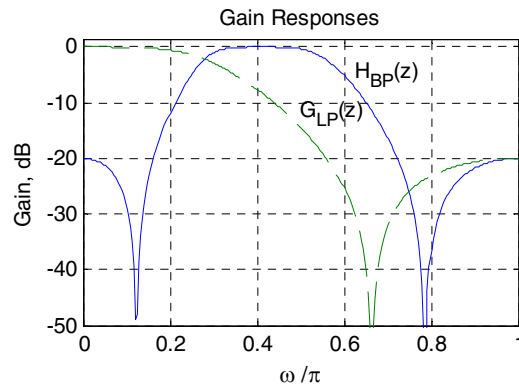
$$H_{HP}(z) = G_{LP}(z)\Big|_{z^{-1} = -\left(\frac{\hat{z}^{-1} + \alpha}{1 + \alpha\hat{z}^{-1}}\right)} = \frac{0.3192 - 0.6383\hat{z}^{-1} + 0.3192\hat{z}^{-2}}{0.9943 - 0.1097\hat{z}^{-1} + 0.1728\hat{z}^{-2}}.$$



9.24 $\omega_c = \hat{\omega}_{c2} - \hat{\omega}_{c1}$, and $\alpha = \cos(\hat{\omega}_c) = 0.2790$

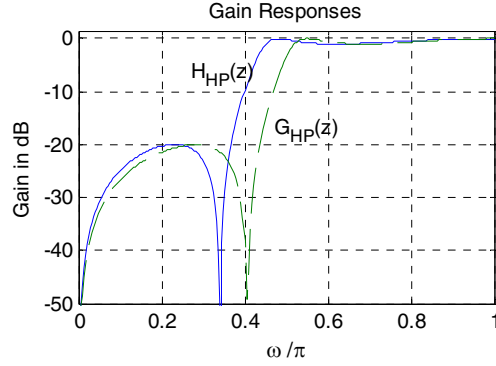
$$z^{-1} = -\hat{z}^{-1} \left(\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}} \right) = \frac{0.2790\hat{z}^{-1} - \hat{z}^{-2}}{1 - 0.2790\hat{z}^{-1}}$$

$$H_{BP}(z) = \frac{0.6918 - 0.1968\hat{z}^{-1} - 0.6234\hat{z}^{-2} - 0.1968\hat{z}^{-3} + 0.6918\hat{z}^{-4}}{3.5588 - 2.6824\hat{z}^{-1} + 3.046\hat{z}^{-2} - 1.2546\hat{z}^{-3} + \hat{z}^{-4}}$$



9.25 $\hat{\omega}_p = 0.45\pi$, and $\omega_p = 0.52\pi$. $\lambda = \frac{\sin\left(\frac{0.52\pi - 0.45\pi}{2}\right)}{\sin\left(\frac{0.52\pi + 0.45\pi}{2}\right)} = 0.1099$.

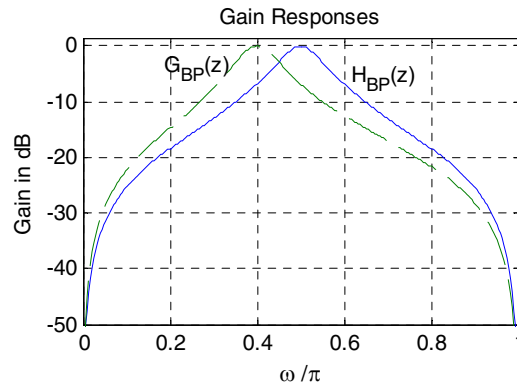
$$H_{HP}(z) = G_{HP}(z) \Big|_{z = \frac{\hat{z}^{-1} - 0.1099}{1 - 0.1099\hat{z}^{-1}}} = \frac{2.873 - 5.622\hat{z}^{-1} + 5.629\hat{z}^{-2} - 2.874\hat{z}^{-3}}{9.7623 - 1.5359\hat{z}^{-1} + 6.711\hat{z}^{-2} + \hat{z}^{-3}}$$



9.26 Eq. (7.79): $H_{BP}(z) = 0.136728736 \left(\frac{1 - z^{-2}}{1 - 0.53353098z^{-1} + 0.726542528z^{-2}} \right)$

$\hat{\omega}_0 = 0.5\pi$, and $\omega_0 = 0.4\pi$. $\lambda = \frac{\sin\left(\frac{0.5\pi - 0.4\pi}{2}\right)}{\sin\left(\frac{0.5\pi + 0.4\pi}{2}\right)} = -0.1584$

$$G_{BP}(z) = H_{BP}(z) \Big|_{z^{-1} = \hat{z}^{-1} + 0.1584} = \frac{0.1333 - 0.1333\hat{z}^{-2}}{1 + 0.1584\hat{z}^{-1}} = \frac{0.1333 - 0.1333\hat{z}^{-2}}{0.933718 + 0.667122\hat{z}^{-2}}$$



9.27 $\omega_0 = 2\pi\left(\frac{80}{400}\right) = 0.4\pi$, $B_\omega = 2\pi\left(\frac{5}{400}\right) = 0.025\pi$. $\alpha = \frac{1 - \tan\left(\frac{B_\omega}{2}\right)}{1 + \tan\left(\frac{B_\omega}{2}\right)} = 0.9244$

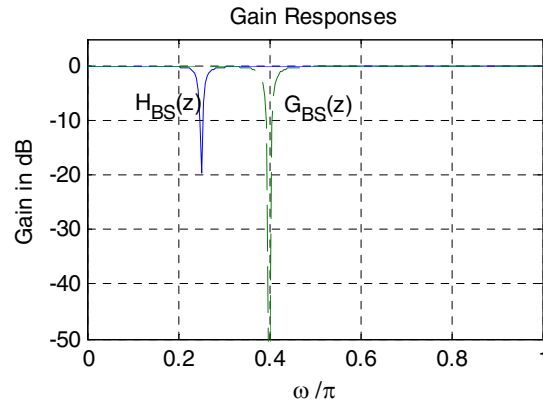
$\beta = \cos(\omega_0) = 0.3090$.

$$G(z) = \frac{1}{2} \frac{(1 + \alpha) - 2\beta(1 + \alpha)z^{-1} + (1 + \alpha)z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} = \frac{0.5(1.9244 - 1.1893z^{-1} + 1.9244z^{-2})}{1 - 0.5946z^{-1} + 0.9244z^{-2}}$$

$$= \frac{0.9622 - 0.5946z^{-1} + 0.9622z^{-2}}{1 - 0.5946z^{-1} + 0.9244z^{-2}}$$

$$\alpha = \frac{\sin\left(\frac{\omega_0 - \hat{\omega}_0}{2}\right)}{\sin\left(\frac{\omega_0 + \hat{\omega}_0}{2}\right)} = 0.2738. \quad \hat{\omega}_0 = \left(\frac{50}{400}\right)2\pi = 0.25\pi.$$

$$H_{BS}(z) = G_{BS}(z) \Big|_{z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha \hat{z}^{-1}}} = \frac{1.0301 - 1.45679\hat{z}^{-1} + 1.0301\hat{z}^{-2}}{1.06017 - 1.45674\hat{z}^{-1} + \hat{z}^{-2}}.$$

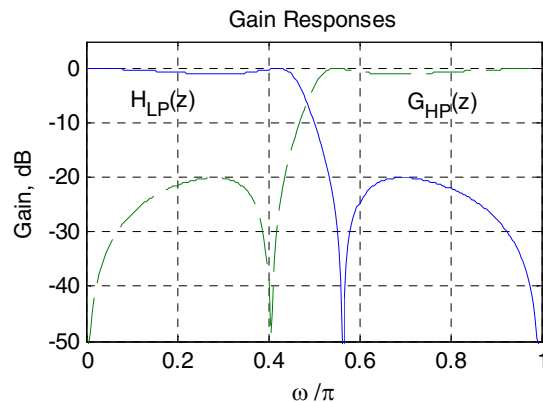


9.28 $\hat{\omega}_p = 0.45\pi$ and $\omega_p = 0.52\pi$.

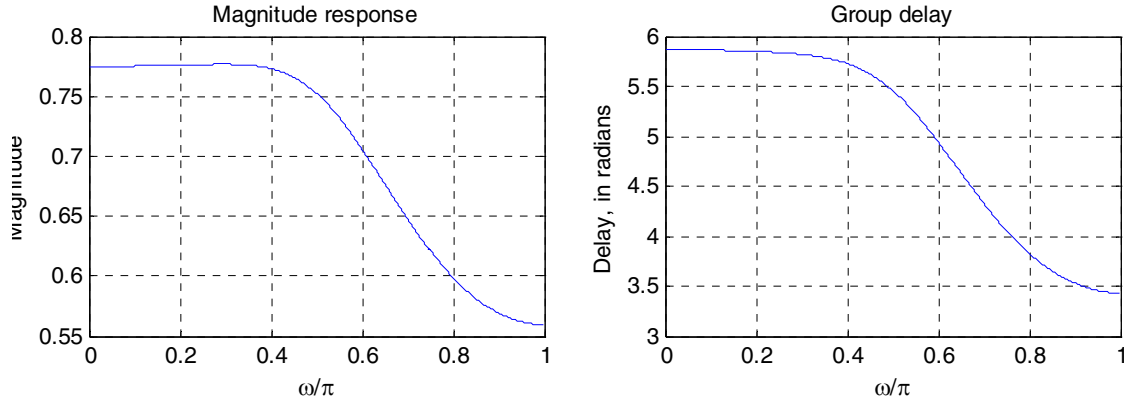
$$G_{HP}(z) = \frac{0.2397(1 - 1.5858z^{-1} + 1.5858z^{-2} - z^{-3})}{1 + 0.3272z^{-1} + 0.7459z^{-2} + 0.179z^{-3}}$$

$$\alpha = \frac{-\cos\left(\frac{\omega_p + \hat{\omega}_p}{2}\right)}{\cos\left(\frac{\omega_p - \hat{\omega}_p}{2}\right)} = -0.0474.$$

$$H_{LP}(z) = G_{HP}(z) \Big|_{z^{-1} = \frac{-\hat{z}^{-1} - \alpha}{1 + \alpha \hat{z}^{-1}}} = \frac{-1.03399 - 1.45578\hat{z}^{-1} - 1.45578\hat{z}^{-2} - 1.03399\hat{z}^{-3}}{-4.72684 + 2.52265\hat{z}^{-1} - 3.7756\hat{z}^{-2} + \hat{z}^{-3}}$$

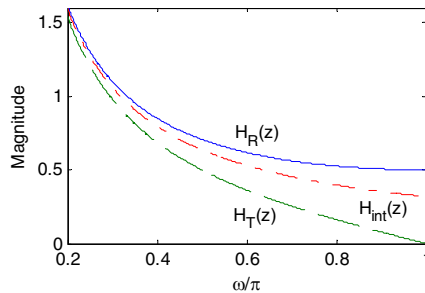


9.29 $D = 6.9231$, hence, $N = 6$. $A(z) = \frac{z^{-6}D(z^{-1})}{D(z)}$.

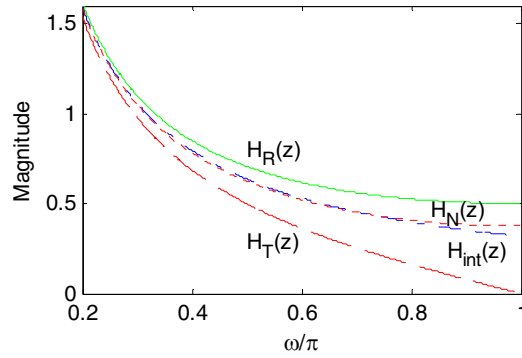


9.30 From Eq. (2.141) $y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)d\tau = y((n-1)T) + T \cdot x((n-1)T)$, which reduces to $y[n] = y[n-1] + T \cdot x[n-1]$. Hence, the corresponding transfer function is given by $H_R(z) = \frac{Tz^{-1}}{1-z^{-1}}$. From Eq. (2.120) $y[n] = y[n-1] + \frac{T}{2}(x[n] + x[n-1])$.

Hence, the corresponding transfer function is given by $H_T(z) = \frac{T}{2} \cdot \frac{1+z^{-1}}{1-z^{-1}}$. From the plot given below it can be seen that the magnitude response of $H_{\text{int}}(z)$ lies between that of $H_R(z)$ and $H_T(z)$.



9.31 $H_N(z) = \frac{3}{4}H_R(z) + \frac{1}{4}H_T(z)$. From the plot given below it can be seen that the magnitude response of $H_N(z)$ lies between that of $H_R(z)$ and $H_T(z)$, and is much closer to that of $H_{\text{int}}(z)$.

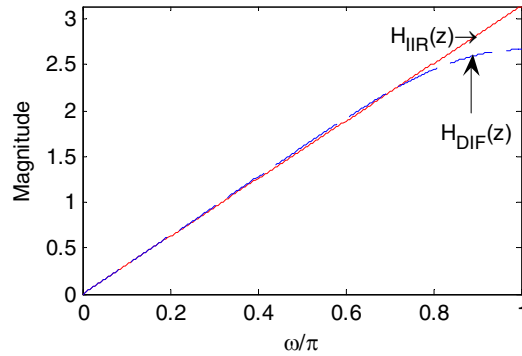


9.32 $H_N(z) = \frac{3}{4} \left(\frac{z^{-1}}{1-z^{-1}} \right) + \frac{1}{8} \left(\frac{1+z^{-1}}{1-z^{-1}} \right) = \frac{1+7z^{-1}}{8(1-z^{-1})}$. Its inverse is given by $H(z) = \frac{8(1-z^{-1})}{1+7z^{-1}}$,

which is unstable as it has a pole at $z = -7$. A stable transfer function with the same magnitude response is obtained by multiplying $H(z)$ with an allpass function $\frac{1+7z^{-1}}{7+z^{-1}}$

resulting in $H_{IIR}(z) = \frac{8(1-z^{-1})}{1+7z^{-1}} \cdot \frac{1+7z^{-1}}{7+z^{-1}} = \frac{8(1-z^{-1})}{7+z^{-1}}$. A plot of the ideal differentiator

$H_{DIF}(z)$ with a frequency response given by Eq. (7.68) and that of the IIR differentiator $H_{IIR}(z)$ is given below. As can be seen the magnitude response of $H_{IIR}(z)$ is very close to that $H_{DIF}(z)$.



M9.1 $F_T = 100$ kHz, $\alpha_p = 0.4$ dB, $F_p = 10$ kHz, $\alpha_s = 50$ dB, & $F_s = 30$ kHz,

$$\omega_p = \frac{2\pi F_p}{F_T} = 0.628, \quad \omega_s = \frac{2\pi F_s}{F_T} = 1.885.$$

Let $T = 2$. $\Omega_p = \tan\left(\frac{\omega_p}{2}\right) = 1.376$, and $\Omega_s = \tan\left(\frac{\omega_s}{2}\right) = 1.376$. Therefore,

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 4.235. \text{ Now, } 20 \log_{10} \left(\frac{1}{\sqrt{1+\varepsilon^2}} \right) = -0.4. \text{ Hence, } \varepsilon^2 = 0.096.$$

From $20 \log_{10} \left(\frac{1}{A} \right) = -50$ we obtain $A^2 = 100,000$. Therefore,

$k_1 = 9.798 \times 10^{-4}$, or $\frac{1}{k_1} = 1020.62$. As a result,

$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 4.80 \rightarrow 5$. Next, solving $\left(\frac{\Omega_s}{\Omega_c} \right)^{10} = A^2 - 1 = 99,999$ we get

$$\Omega_c = \frac{\Omega_s}{99,999^{1/10}} = 0.316\Omega_s = 0.435$$

Using the M-file `buttap`, we determine the normalized analog Butterworth transfer function of 5th order with a 3-dB cutoff frequency at $\Omega_c = 1$, which is:

$$H_{an}(s) = \frac{1}{(s+1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)}$$

Denormalize $H_{an}(s)$ to move Ω_c to 0.316:

$$H_a(s) = H_{an}\left(\frac{s}{0.435}\right) = \frac{1}{(2.30s+1)(5.28s^2 + 1.42s + 1)(5.28s^2 + 3.72s + 1)}$$

$$= \frac{1}{1 + 7.44s + 27.66s^2 + 63.58s^3 + 90.30s^4 + 64.12s^5}$$

$$G(z) = H_a(s) \Big|_{s=\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)}$$

$$= \frac{0.0039 + 0.0197z^{-1} + 0.0394z^{-2} + 0.0394z^{-3} + 0.0197z^{-4} + 0.0039z^{-5}}{1 - 2.3617z^{-1} + 2.6139z^{-2} - 1.5492z^{-3} + 0.4865z^{-4} - 0.0637z^{-5}}$$

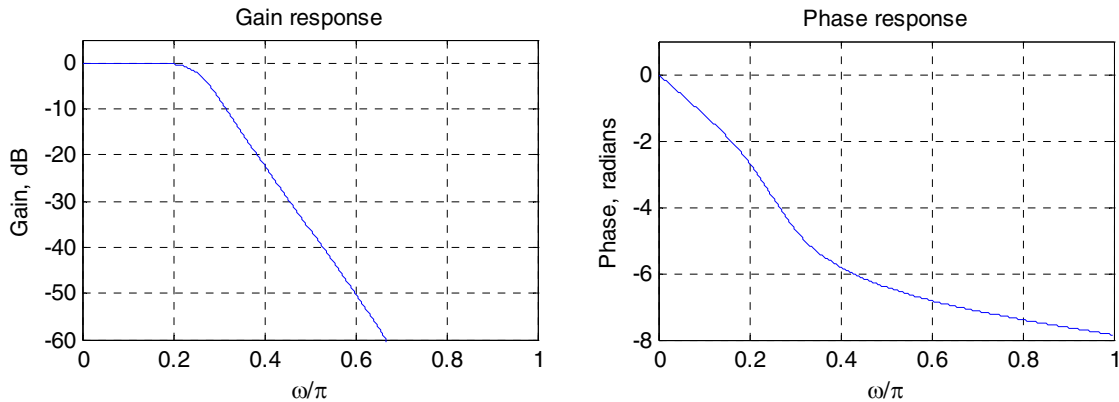
Matlab code is as follows:

```
% Program M9.01
N = 5;
[z, p, k] = buttap(N);
[num, den] = zp2tf(z, p, k);
% s -> s/0.435
den = [64.12 90.30 63.58 27.66 7.44 1];
num = [0 0 0 0 0 1];
% compute z, p, and k
[z, p, k] = tf2zp(num, den);
% perform bilinear transformation with T = 2;
[zd, pd, kd] = bilinear(z, p, k, 1/2);
% get the digital transfer function
[n2, d2] = zp2tf(zd, pd, kd);
% get the frequency response
[h, w] = freqz(n2, d2, 512);
figure(1);
plot(w/pi, 20*log10(abs(h))); grid;
axis([0 1 -60 5]);
xlabel('\omega/\pi'); ylabel('Gain, dB');
```

```

title('Gain response');
figure(2);
plot(w/pi, unwrap(angle(h))); grid;
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, radians');
title('Phase response');

```



M9.2 % Problem M9.02

```

Fp = input('Passband edge frequency in Hz = ');
Fs = input('Stopband edge frequency in Hz = ');
FT = input('Sampling frequency in Hz = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Stopband minimum attenuation in dB = ');
Wp = 2*Fp/FT;
Ws = 2*Fs/FT;
[N, Wn] = buttord(Wp, Ws, Rp, Rs)
[b, a] = butter(N, Wn);
disp('Numerator polynomial');
disp(b)
disp('Denominator polynomial');
disp(a)
[h, w] = freqz(b, a, 512);
plot(w/pi, 20*log10(abs(h))); grid
axis([0 1 -60 5]);
xlabel('\omega/\pi'); ylabel('Magnitude, dB');
pause
plot(w/pi, unwrap(angle(h))); grid
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, radians');

```

M9.3 % Program #M9.03

```

close all;
clear;
clc;

% (a.) Ft = 1Hz;
[B, A] = besself(5, 0.5);

```



```

[BZ,AZ] =impinvar(B,A,1);
[h, w] = freqz(BZ, AZ, 512);
[Gd,W] = grpdelay(BZ,AZ,512);

figure(1);
plot(w/pi, 20*log10(abs(h)));
title('Gain response, , Sampling rate = 1 Hz');
xlabel('\omega/\pi'); ylabel('Gain, in dB');

figure(2);
plot(W/pi, (Gd));
title('Group delay, Sampling rate = 1 Hz');
xlabel('\omega/\pi'); ylabel('Delay, in samples');

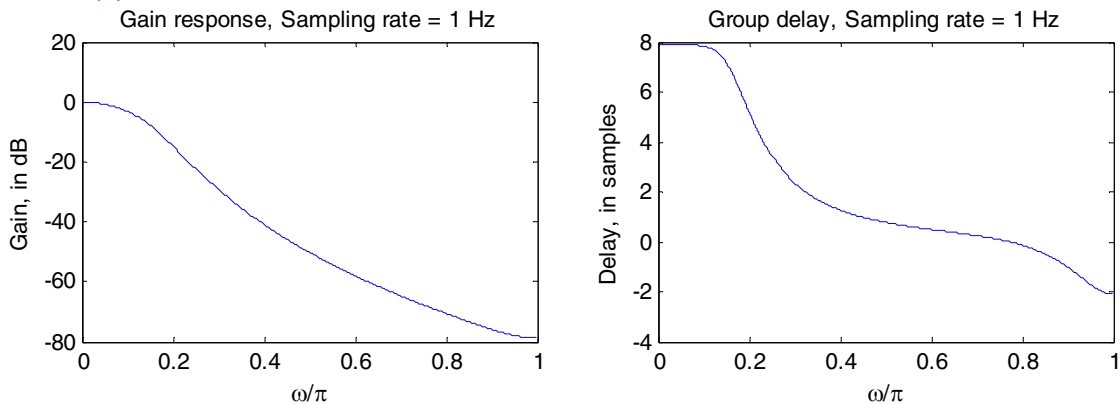
% (b.) Ft = 2Hz;
[B, A] = besself(5, 0.5);
[BZ,AZ] =impinvar(B,A,2);
[h, w] = freqz(BZ, AZ, 512);
[Gd,W] = grpdelay(BZ,AZ,512);

figure(3);
plot(w/pi, 20*log10(abs(h)));
title('Gain response, Sampling rate = 2 Hz');
xlabel('\omega/\pi');ylabel('gain response');

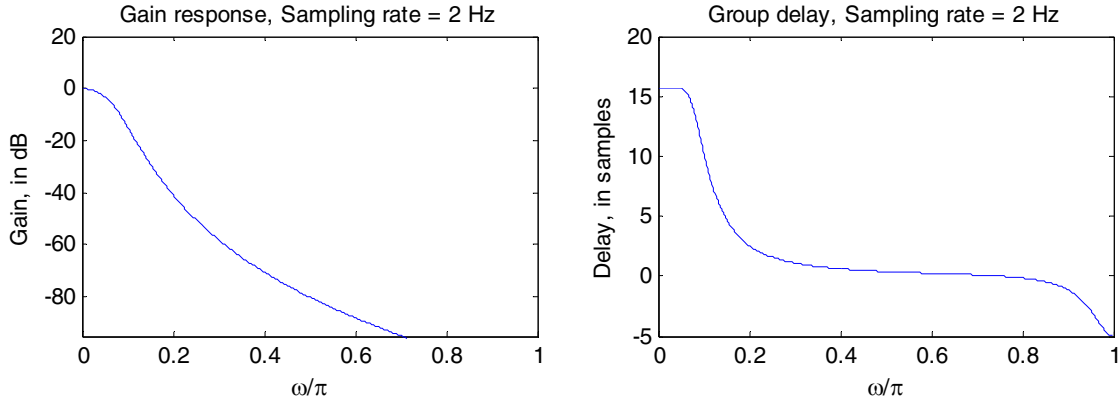
figure(4);
plot(W/pi, (Gd));
title('Group delay, Sampling rate = 2 Hz');
xlabel('\omega/\pi'); ylabel('Delay, in samples');

```

(a) FT = 1 Hz



(b) FT = 2 Hz



M9.4 $\omega_p = 0.628$, $\omega_s = 1.885$.

$$20 \log_{10} |G(e^{0.628j})| \geq -0.4, \quad 20 \log_{10} |G(e^{1.885j})| \leq -50.$$

Impulse invariance method, let $T = 1$ and assume no aliasing. Then

$$20 \log_{10} \left(\frac{1}{\sqrt{1+\varepsilon^2}} \right) = -0.4 \Rightarrow \varepsilon^2 = 0.096,$$

$$20 \log_{10} \left(\frac{1}{A} \right) = -50 \Rightarrow A^2 = 100,000.$$

$$\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\varepsilon^2}} = 1020.62, \quad \frac{1}{k} = \frac{1.885}{0.628} = 3. \text{ Hence,}$$

$$N = \frac{\log_{10} \left(\frac{1}{k_1} \right)}{\log_{10} \left(\frac{1}{k} \right)} = 6.3 \rightarrow 7$$

$$\left(\frac{\Omega_s}{\Omega_c} \right)^{10} = A^2 - 1 = 99,999 \text{ or } \Omega_c = \frac{\Omega_s}{(A^2 - 1)^{1/10}} = 0.596.$$

Using the M-file `butter`, we get the analog Butterworth transfer function $H_a(s)$ of 7th order with a 3-dB cutoff frequency at Ω_c .

$H_a(s)$ is then transformed into a digital transfer function using the M-file `impinvar`, which yields:

$$G(z) = \frac{0.001z^{-2} + 0.0034z^{-3} + 0.0023z^{-4} + 0.0003z^{-5}}{1 - 4.38z^{-1} + 8.54z^{-2} - 9.55z^{-3} + 6.57z^{-4} - 2.77z^{-5} + 0.66z^{-6} - 0.07z^{-7}}.$$

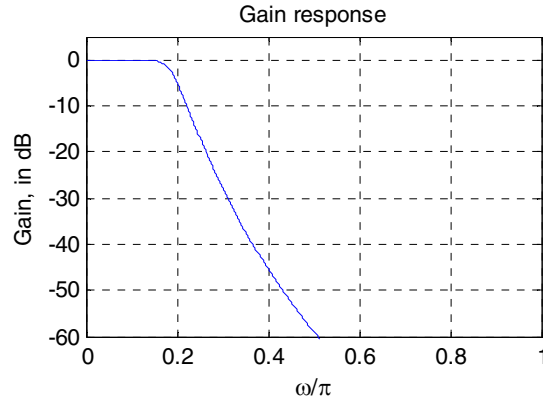
`% Problem M9.04`

```
[B, A] = butter(7, 0.596, 's');
[num, den] =impinvar(B, A, 1);
```

```

% get the frequency response
[h, w] = freqz(num, den, 512);
plot(w/pi, 20*log10(abs(h))); grid;
axis([0 1 -60 5]);
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Gain response');

```



M9.5 $F_T = 100$ kHz, $F_p = 10$ kHz, $F_s = 30$ kHz, $\alpha_p = 0.4$ dB, $\alpha_s = 50$ dB

$$\omega_p = \frac{2\pi F_p}{F_T} = 0.2\pi, \text{ and } \omega_s = \frac{2\pi F_s}{F_T} = 0.6\pi.$$

$$20 \log_{10} |G(e^{j0.628})| \geq -0.4, \quad 20 \log_{10} |G(e^{j1.885})| \leq -50.$$

Impulse invariance method:

Let $T = 1$ and assume no aliasing. In this case, the specifications for $H_a(s)$ are same as that for $G(z)$, i.e.,

$$\Omega_p = 0.628, \quad \Omega_s = 1.885,$$

$$20 \log_{10} |H_a(0.628j)| \geq -0.4, \quad 20 \log_{10} |H_a(1.885j)| \leq -50.$$

$$\text{Now, } 20 \log_{10} \left(\frac{1}{\sqrt{1+\varepsilon^2}} \right) = -0.4 \Rightarrow \varepsilon^2 = 0.096, \text{ and}$$

$$20 \log_{10} \left(\frac{1}{A} \right) = -50 \Rightarrow A^2 = 100,000. \text{ Hence,}$$

$$\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\varepsilon^2}} = 1020.62 \quad \text{and} \quad \frac{1}{k} = \frac{1.885}{0.628} = 3.$$

$$\text{Order of the Type I Chebyshev filter is } N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 4.324 \rightarrow 5$$

Bilinear transformation method:

$$\text{Let } T = 2. \quad \Omega_p = 0.325, \Omega_s = 1.376. \text{ Here, } \frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 4.235, \text{ and}$$

$\varepsilon^2 = 0.096, A^2 = 100,000$. Thus, $\frac{1}{k_1} = 1020.62$.

Order of the Type I Chebyshev filter is $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 3.59 \rightarrow 4$.

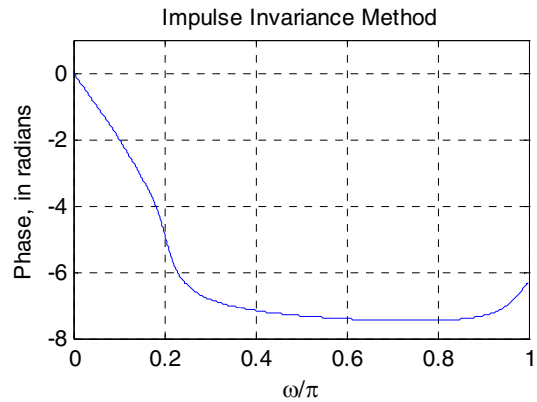
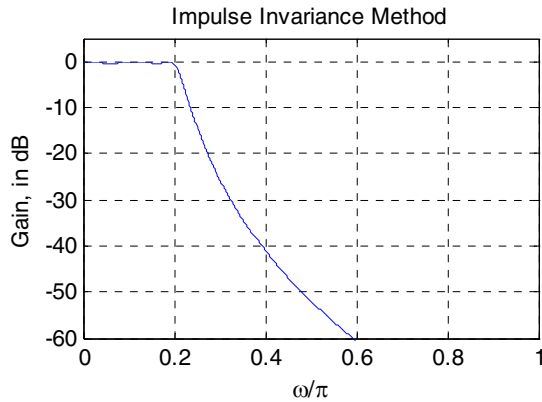
Both designs meet the specifications, while the bilinear transformation method meets with a filter of lower order. MATLAB code is as follows:

```
% Problem #M9.05
% Impulse invariance method
[z, p, k] = cheblap(5, 0.4);
[B, A] = zp2tf(z, p, k);
[BT, AT] = lp2lp(B, A, 0.628);
[num, den] =impinvar(BT, AT, 1);
[h, w] = freqz(num, den, 512);
figure(1);
plot(w/pi, 20*log10(abs(h))); grid;
axis([0 1 -60 5]);
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Impulse Invariance Method');
figure(2);
plot(w/pi, unwrap(angle(h))); grid;
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, in radians');
title('Impulse Invariance Method');

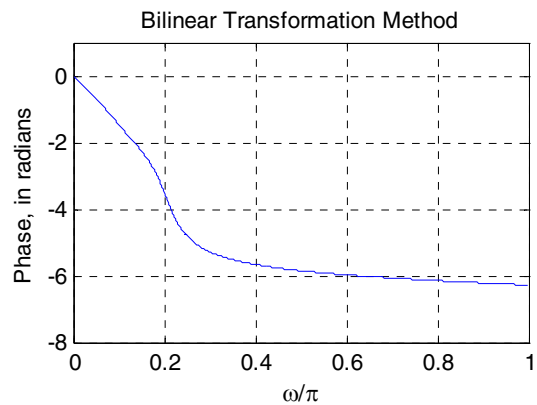
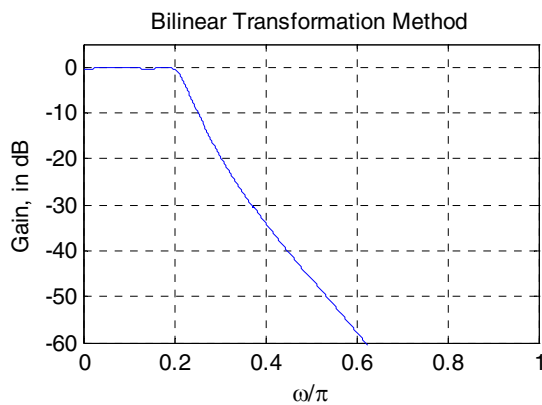
% Bilinear transformation method
[z, p, k] = cheblap(4, 0.4);
[B, A] = zp2tf(z, p, k);
[BT, AT] = lp2lp(B, A, 0.325);
[num, den] = bilinear(BT, AT, 0.5);
[h, w] = freqz(num, den, 512);
figure(3);
plot(w/pi, 20*log10(abs(h))); grid;
axis([0 1 -60 5]);
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Bilinear Transformation Method');

figure(4);
plot(w/pi, unwrap(angle(h))); grid;
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, in radians');
title('Bilinear Transformation Method');
```

Impulse Invariance Method:



Bilinear Transformation Method:



M9.6 % Problem M9.06

```

Wp = input('Normalized passband edge = ');
Ws = input('Normalized stopband edge = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Minimum stopband attenuation in dB = ');
[N, Wn] = cheblord(Wp, Ws, Rp, Rs);
[b, a] = cheby1(N, Rp, Wn);
[h, omega] = freqz(b,a,256);

```

```

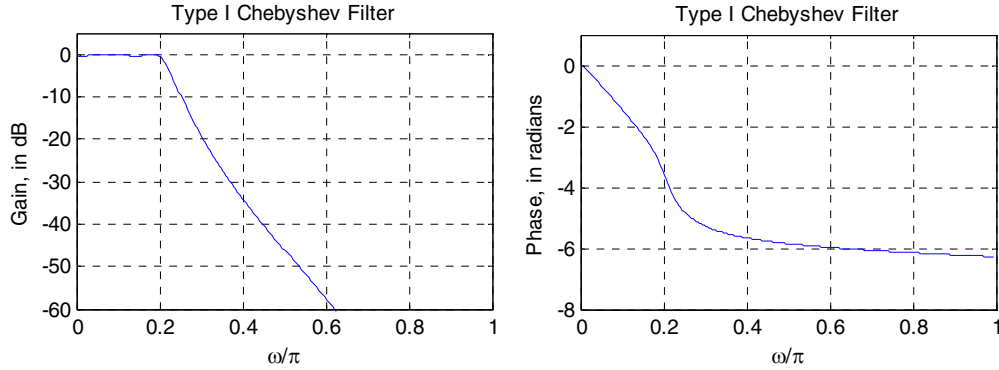
figure(1);
plot(omega/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Type I Chebyshev Filter');
axis([0 1 -60 5]);

```

```

figure(2);
plot(omega/pi, unwrap(angle(h))); grid
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, in radians');
title('Type I Chebyshev Filter');

```

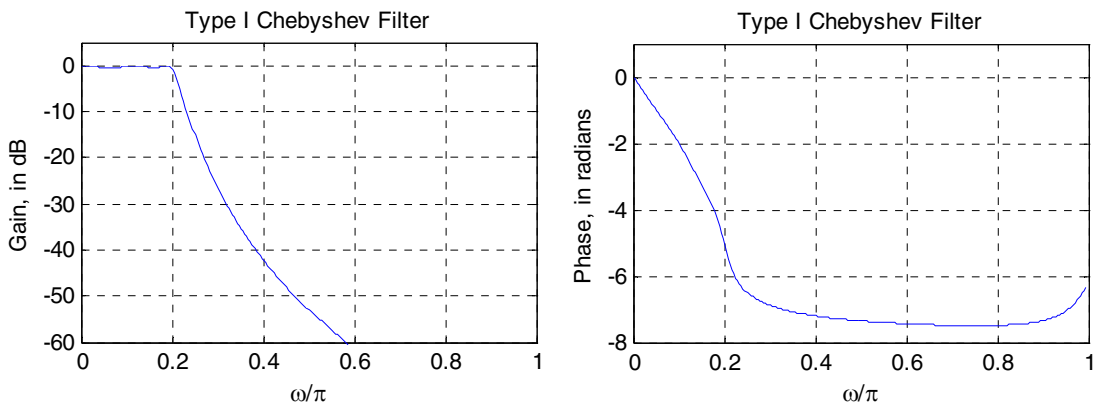


M9.7 % Problem M9.07

```

Wp = input(' Passband edge in radians = ');
Ws = input(' Stopband edge in radians = ');
Rp = input(' Passband ripple in dB = ');
Rs = input(' Stopband minimum attenuation in dB = ');
[N, Wn] = cheblord(Wp, Ws, Rp, Rs, 's');
[B, A] = cheby1(N, 0.5, Wn, 's');
[num, den] =impinvar(B, A, 1);
[h, omega] = freqz(num, den, 256);
figure(1);
plot(omega/pi, 20*log10(abs(h))); grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Type I Chebyshev Filter');
axis([0 1 -60 5]);
figure(2);
plot(omega/pi, unwrap(angle(h))); grid;
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, in radians');
title('Type I Chebyshev Filter');

```



M9.8 Impulse invariance method:

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 3.00, \quad \frac{1}{k_1} = 1020.62. \quad \text{Hence, } N \approx \frac{2 \log_{10}(4/k_1)}{\log_{10}(1/\rho)}, \text{ where}$$

$$k' = \sqrt{1 - k^2}, \rho_0 = \frac{1 - \sqrt{k'}}{2(1 + \sqrt{k'})}$$

$\rho = \rho_0 + 2\rho_0^5 + 15\rho_0^9 + 150\rho_0^{13}$. In our case,

$k' = 0.943$, $\rho_0 = 0.0073$, $\rho = 0.0073$. Hence, $N \approx 3.38 \rightarrow 4$

Bilinear transformation method:

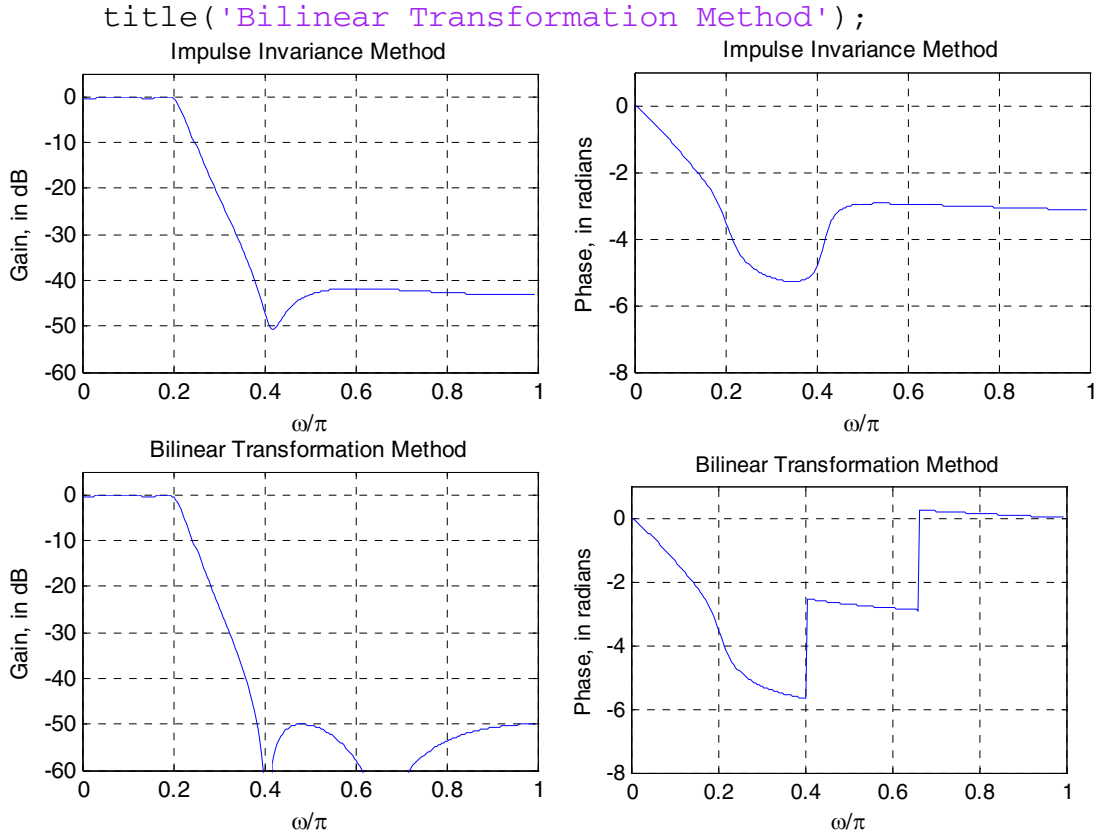
$$\Omega_p = 0.325, \Omega_s = 1.376, \frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 4.235. \quad \frac{1}{k_1} = 1020.62.$$

$k' = 0.972$, $\rho_0 = 0.0036$, $\rho = 0.0036$. Hence, $N \approx 2.96 \rightarrow 3$

We note that the filter design using the impulse invariance method does not meet the specifications due to aliasing. Increasing N from 4 to 6 will meet the specifications.

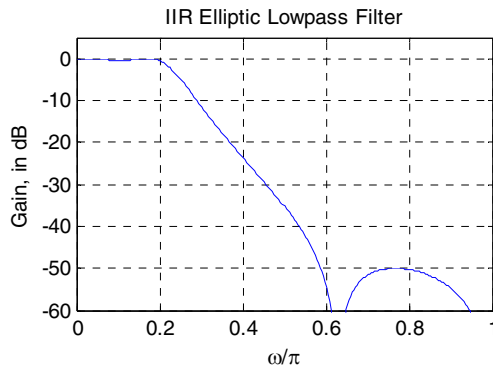
```
% Problem M9_08
% Impulse Invariance Method
[z, p, k] = ellipap(4, 0.4, 50);
[B, A] = zp2tf(z, p, k);
[BT, AT] = lp2lp(B, A, 0.628);
[num, den] =impinvar(BT, AT, 1);
[h, omega] = freqz(num, den, 256);
figure(1);
plot(omega/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Impulse Invariance Method');
axis([0 1 -60 5]);
figure(2);
plot(omega/pi, unwrap(angle(h))); grid
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, in radians');
title('Impulse Invariance Method');

% Bilinear Transformation Method
[z, p, k] = ellipap(4, 0.4, 50);
[B, A] = zp2tf(z, p, k);
[BT, AT] = lp2lp(B, A, 0.325);
[num, den] = bilinear(BT, AT, 0.5);
[h, omega] = freqz(num, den, 256);
figure(3);
plot(omega/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Bilinear Transformation Method');
axis([0 1 -60 5]);
figure(4);
plot(omega/pi, unwrap(angle(h))); grid
axis([0 1 -8 1]);
xlabel('\omega/\pi'); ylabel('Phase, in radians');
```



M9.9 % Program M9_09

```
Wp = input('Normalized passband edge = ');
Ws = input('Normalized stopband edge = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Stopband ripple in dB = ');
[N, Wn] = ellipord(Wp, Ws, Rp, Rs);
[b, a] = ellip(N, Rp, Rs, Wn);
[h, omega] = freqz(b, a, 256);
plot(omega/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('IIR Elliptic Lowpass Filter');
axis([0 1 -60 5]);
```



M9.10 $F_T = 1.5$ MHz, $F_p = 600$ kHz, $F_s = 210$ kHz.

$\alpha_p = 0.4 \text{ dB}, \alpha_s = 45 \text{ dB}$

$$(a) \omega_p = \frac{2\pi F_p}{F_T} = 2.513, \omega_s = \frac{2\pi F_s}{F_F} = 0.880.$$

$$\hat{\Omega}_p = \tan\left(\frac{\omega_p}{2}\right) = 3.076, \hat{\Omega}_s = \tan\left(\frac{\omega_s}{2}\right) = 0.471.$$

Analog highpass specifications are thus:

$$\hat{\Omega}_p = 3.076, \hat{\Omega}_s = 0.471, \alpha_p = 0.4 \text{ dB}, \alpha_s = 45 \text{ dB}$$

$$(b) \Omega_p = 1, \Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = 6.531$$

`% Problem M9.10`

`close all;`

`clear;`

`clc;`

`[N, Wn] = ellipord(1, 6.531, 0.4, 45, 's');`

`[B, A] = ellip(N, 0.4, 45, Wn, 's');`

`[BT, AT] = lp2hp(B, A, 3.076);`

`[num, den] = bilinear(BT, AT, 0.5);`

`[h, omega] = freqz(num, den, 256);`

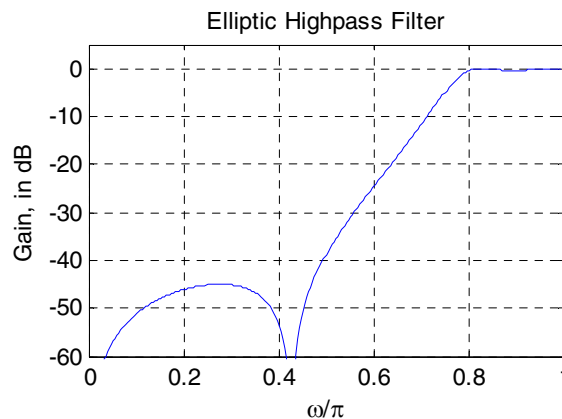
`figure(1);`

`plot(omega/pi, 20*log10(abs(h)));grid;`

`xlabel('\omega/\pi'); ylabel('Gain, in dB');`

`title('Elliptic Highpass Filter');`

`axis([0 1 -60 5]);`



(c) Analog lowpass transfer function coefficient are obtained by displaying B and A:

$$H_{LP}(s) = \frac{0.0556s^2 + 0.8329}{s^3 + 1.3314s^2 + 1.6434s + 0.8329}.$$

Analog highpass transfer function coefficient are obtained by displaying BT and AT:

$$H_{HP}(s) = \frac{s^3 + 0.6315s}{s^3 + 6.0693s^2 + 15.1254s + 34.9438}.$$

Digital highpass transfer function coefficient are obtained by displaying num and den:

$$:G_{HP}(z) = \frac{0.0286 - 0.0415z^{-1} + 0.0415z^{-2} - 0.0286z^{-3}}{1 + 1.9407z^{-1} + 1.5613z^{-2} + 0.4356z^{-3}}.$$

M9.11 $F_T = 9\text{kHz}$, $F_{p1} = 1.2\text{kHz}$, $F_{p2} = 2.2\text{kHz}$, $F_{s1} = 650\text{Hz}$, $F_{s2} = 3\text{kHz}$,

$$\alpha_p = 0.8\text{dB}, \alpha_s = 31\text{dB}$$

(a) $\omega_{p1} = \frac{2\pi F_{p1}}{F_T} = 0.8378$, $\omega_{p2} = \frac{2\pi F_{p2}}{F_T} = 1.536$, $\omega_{s1} = \frac{2\pi F_{s1}}{F_T} = 0.4538$, and

$$\omega_{s2} = \frac{2\pi F_{s2}}{F_T} = 2.094, \text{ with}$$

$$\hat{\Omega}_{p1} = \tan\left(\frac{\omega_{p1}}{2}\right) = 0.445, \hat{\Omega}_{p2} = \tan\left(\frac{\omega_{p2}}{2}\right) = 0.966, \hat{\Omega}_{s1} = \tan\left(\frac{\omega_{s1}}{2}\right) = 0.231,$$

$$\text{and } \hat{\Omega}_{s2} = \tan\left(\frac{\omega_{s2}}{2}\right) = 1.731.$$

$$B_\omega = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 0.521,$$

$$\hat{\Omega}_0^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = 0.430 \neq \hat{\Omega}_{s1}\hat{\Omega}_{s2} = 0.40$$

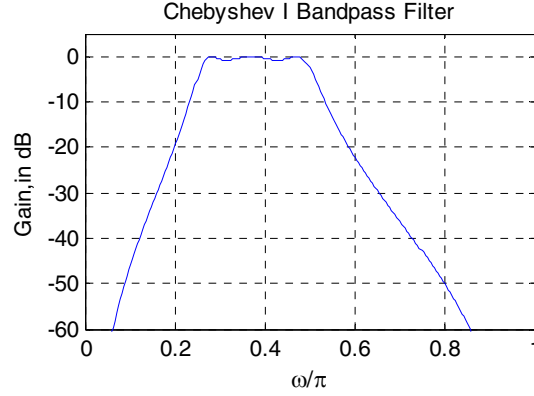
Therefore, we choose to lower the stopband edge to: $\Omega_s = \frac{\hat{\Omega}_0^2 - \hat{\Omega}_{s1}^2}{\hat{\Omega}_{s1} \cdot B_\omega} = 3.13$

(b) The analog lowpass specifications are thus:

$$\Omega_p = 1 \text{ rad}, \Omega_s = 3.13 \text{ rad}, \alpha_p = 0.8 \text{ dB}, \alpha_s = 31 \text{ dB}.$$

`% Problem M9.11`

```
[N, Wn] = cheblord(1, 3.13, 0.8, 31, 's');
[B, A] = cheby1(N, 0.8, Wn, 's');
[BT, AT] = lp2bp(B, A, sqrt(0.43), 0.521);
[num, den] = bilinear(BT, AT, 0.5);
[h, omega] = freqz(num, den, 256);
plot(omega/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Chebyshev I Bandpass Filter');
axis([0 1 -60 5]);
```



(c) Analog lowpass transfer function coefficient are obtained by displaying B and A:

$$H_{LP}(s) = \frac{0.5559}{s^3 + 1.0719s^2 + 1.3244s + 0.5559}.$$

Analog bandpass transfer function coefficient are obtained by displaying BT and AT:

$$H_{BP}(s) = \frac{0.0786s^3}{s^6 + 0.5584s^5 + 1.6495s^4 + 0.5589s^3 + 0.7093s^2 + 0.1033s + 0.0795}.$$

Digital bandpass transfer function coefficient are obtained by displaying den and num:

$$G_{BP}(z) = \frac{0.0169 - 0.0506z^{-2} + 0.0506z^{-4} - 0.0169z^{-6}}{1 - 1.9799z^{-1} + 3.3196z^{-2} - 3.1443z^{-3} + 2.6191z^{-4} + 1.1983z^{-5} + 0.476z^{-6}}.$$

M9.12 $F_T = 8$ kHz, $F_{p1} = 0.9$ kHz, $F_{p2} = 2.1$ kHz, $F_{s1} = 0.6$ kHz, $F_{s2} = 3$ kHz,
 $\alpha_p = 1.5$ dB, $\alpha_s = 30$ dB

(a) $\omega_{p1} = \frac{2\pi F_{p1}}{F_T} = 0.707$, $\omega_{p2} = \frac{2\pi F_{p2}}{F_T} = 1.649$, $\omega_{s1} = \frac{2\pi F_{s1}}{F_T} = 0.471$, and

$\omega_{s2} = \frac{2\pi F_{s2}}{F_T} = 2.356$, with

$\hat{\Omega}_{p1} = \tan\left(\frac{\omega_{p1}}{2}\right) = 0.369$, $\hat{\Omega}_{p2} = \tan\left(\frac{\omega_{p2}}{2}\right) = 1.081$, $\hat{\Omega}_{s1} = \tan\left(\frac{\omega_{s1}}{2}\right) = 0.240$,

and $\hat{\Omega}_{s2} = \tan\left(\frac{\omega_{s2}}{2}\right) = 2.414$.

$B_\omega = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 2.174$

$\hat{\Omega}_0^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = 0.399 \neq \hat{\Omega}_{s1}\hat{\Omega}_{s2} = 0.579$

Therefore, we choose to lower the passband:

$$\hat{\Omega}_{p1} = \frac{\hat{\Omega}_{s2} \hat{\Omega}_{s1}}{\hat{\Omega}_{p2}} = 0.536$$

The analog bandstop filter specifications are thus:

$$\hat{\Omega}_{p1} = 0.536 \text{ rad}, \hat{\Omega}_{p2} = 1.081 \text{ rad}, \hat{\Omega}_{s1} = 0.240 \text{ rad}, \hat{\Omega}_{s2} = 2.414 \text{ rad},$$

$$\alpha_p = 1.5 \text{ dB}, \alpha_s = 30 \text{ dB}.$$

(b) Analog prototype LP filter:

$$\Omega_s = 1 \text{ rad}, \Omega_p = \frac{\hat{\Omega}_{p1} \cdot B_\omega}{\hat{\Omega}_0^2 - \hat{\Omega}_{p1}^2} = 3.995 \text{ rad}, \alpha_p = 1.5 \text{ dB}, \alpha_s = 30 \text{ dB}.$$

`% Problem M9.12`

`close all;`

`clear;`

`clc;`

`[N, Wn] = ellipord(3.995, 1, 1.5, 30, 's');`

`[B, A] = ellip(N, 1.5, 30, Wn, 's');`

`[BT, AT] = lp2bs(B, A, sqrt(0.579), 2.174);`

`[num, den] = bilinear(BT, AT, 0.5);`

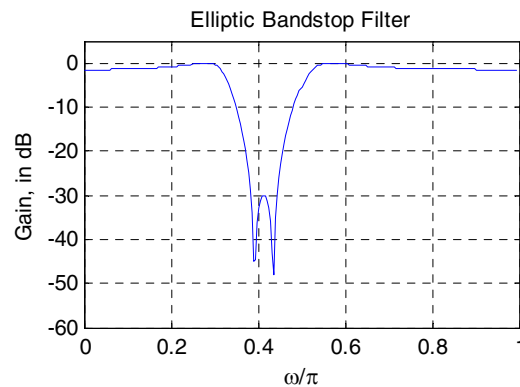
`[h, omega] = freqz(num, den, 256);`

`plot(omega/pi, 20*log10(abs(h)));grid;`

`xlabel('\omega/\pi'); ylabel('Gain, in dB');`

`title('Elliptic Bandstop Filter');`

`axis([0 1 -60 5]);`



(c) Analog lowpass transfer function coefficients are obtained by displaying B and A:

$$H_{LP}(s) = \frac{0.0316s^2 + 12.6706}{s^2 + 3.6122s + 15.059}.$$

Analog bandstop transfer function coefficients are obtained by displaying BT and AT:

$$H_{BS}(s) = \frac{0.8414s^4 + 0.9843s^2 + 0.2821}{s^4 + 0.5215s^3 + 1.4719s^2 + 0.3019s + 0.3352}.$$

Digital bandstop transfer function coefficients are obtained by displaying num and den:

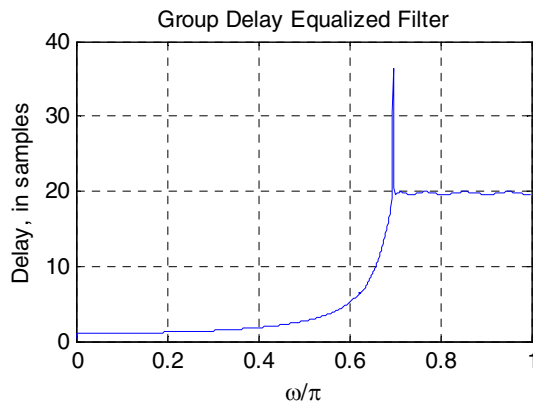
$$G_{BS}(z) = \frac{0.5806 - 0.6163z^{-1} + 1.3145z^{-2} - 0.6163z^{-3} + 0.5806z^{-4}}{1 - 0.8534z^{-1} + 1.3959z^{-2} - 0.6115z^{-3} + 0.5464z^{-4}}.$$

M9.13 % Program M9.13

```
close all;
clear;
clc;
Wp = 0.7; Ws = 0.5;
Rp = 1; Rs = 32;
[N,Wn] = cheblord(Wp,Ws,Rp,Rs);
[b,a] = cheby1(N,Rp,Wn,'high');
[h,omega] = freqz(b,a,256);
plot(omega/pi,20*log10(abs(h)));grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Type I Chebyshev Highpass Filter');

[GdH,w] = grpdelay(b,a,512);
plot(w/pi,GdH); grid
xlabel('\omega/\pi'); ylabel('Delay, in samples');
title('Original Filter Group Delay');

F = 0.7:0.001:1;
g = grpdelay(b,a,F,2); % Equalize the passband
Gd = max(g)-g;
% Design the allpass delay equalizer
[num,den,tau] = iirgrpdelay(2*N, F, [0.7 1], Gd);
[GdA,w] = grpdelay(num,den,512);
plot(w/pi,GdH+GdA); grid
xlabel('\omega/\pi');ylabel('Delay, in samples');
title('Group Delay Equalized Filter');
```



M9.14 % Program M9.14

```
close all;
clear;
```

```

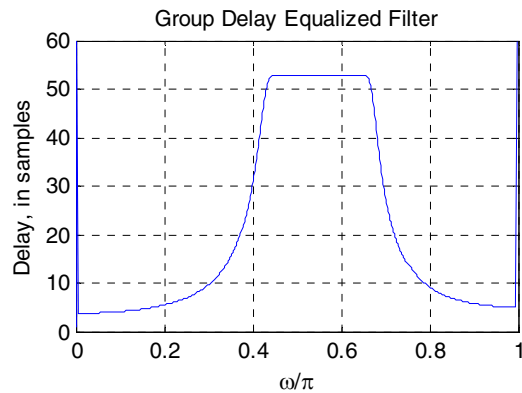
clc;

Wp = [0.45 0.65]; Ws = [0.3 0.75];
Rp = 1; Rs = 40;
[N,Wn] = buttord(Wp, Ws, Rp, Rs);
[b,a] = butter(N,Wn);
[h,omega] = freqz(b,a,256);
gain = 20*log10(abs(h));
plot (omega/pi,gain);grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('IIR Butterworth Bandpass Filter');

[GdH,w] = grpdelay(b,a,512);
plot(w/pi,GdH); grid
xlabel('\omega/\pi'); ylabel('Delay, in samples');
title('Original Filter Group Delay');

F = 0.45:0.001:0.65;
g = grpdelay(b,a,F,2); % Equalize the passband
Gd = max(g)-g;
% Design the allpass delay equalizer
[num,den,tau] = iirgrpdelay(2*N, F, [0.45 0.65], Gd);
[GdA,w] = grpdelay(num,den,512);
plot(w/pi,GdH+GdA); grid
xlabel('\omega/\pi');ylabel('Delay, in samples');
title('Group Delay Equalized Filter');

```



Chapter 10

10.1 To compute the filter orders, we use Kaiser's formula of Eq. (10.3), Bellanger's formula of Eq. (10.4), and Hermann's formula of Eq. (10.5).

Filter #1:

$$\text{Kaiser's formula - } N = \frac{-20 \log_{10}(\sqrt{0.0224 \cdot 0.00012}) - 13}{14.6(0.14375\pi - 0.10625\pi)/2\pi} = 157.097 \approx 158$$

$$\text{Bellanger's formula - } N = \frac{-2 \log_{10}(10 \cdot 0.0224 \cdot 0.00012)}{3(0.14375\pi - 0.10625\pi)/2\pi} - 1 = 162.575 \approx 163$$

Hermann's formula -

$$D_{\infty}(\delta_p, \delta_s) = [0.005309(\log_{10} 0.0224)^2 + 0.07114(\log_{10} 0.0224) - 0.4761] \cdot \log_{10} 0.00012 - [0.00266(\log_{10} 0.0224)^2 + 0.5941(\log_{10} 0.0224) + 0.4278] = 2.8326$$

$$F(\delta_p, \delta_s) = 11.01217 + 0.51244[\log_{10} 0.0224 - \log_{10} 0.00012] = 12.1913$$

$$N = \frac{2.8326 - 12.1913[(0.14375\pi - 0.10625\pi)/2\pi]^2}{(0.14375\pi - 0.10625\pi)/2\pi} = 150.8434 \approx 151$$

Filter #2:

$$\text{Kaiser's formula - } N = \frac{-20 \log_{10}(\sqrt{0.017 \cdot 0.034}) - 13}{14.6(0.2875\pi - 0.2075\pi)/2\pi} = 33.186 \approx 34$$

$$\text{Bellanger's formula - } N = \frac{-2 \log_{10}(10 \cdot 0.017 \cdot 0.034)}{3(0.2875\pi - 0.2075\pi)/2\pi} - 1 = 36.3012 \approx 37$$

Hermann's formula -

$$D_{\infty}(\delta_p, \delta_s) = [0.005309(\log_{10} 0.017)^2 + 0.07114(\log_{10} 0.017) - 0.4761] \cdot \log_{10} 0.034 - [0.00266(\log_{10} 0.017)^2 + 0.5941(\log_{10} 0.017) + 0.4278] = 1.474777$$

$$F(\delta_p, \delta_s) = 11.01217 + 0.51244[\log_{10} 0.017 - \log_{10} 0.034] = 10.85791019$$

$$N = \frac{1.474777 - 10.85791019[(0.2875\pi - 0.2075\pi)/2\pi]^2}{(0.2875\pi - 0.2075\pi)/2\pi} = 36.435 \approx 37$$

Filter #3:

$$\text{Kaiser's formula - } N_k = \frac{-20 \log_{10}(\sqrt{0.0411 \cdot 0.0137}) - 13}{14.6(0.575\pi - 0.345\pi)/2\pi} = 11.6107 \approx 12$$

$$\text{Bellanger's formula - } N = \frac{-2 \log_{10}(10 \cdot 0.0411 \cdot 0.0137)}{3(0.575\pi - 0.345\pi)/2\pi} - 1 = 12.04 \approx 13$$

Hermann's formula -

$$D_{\infty}(\delta_p, \delta_s) = \left[0.005309(\log_{10} 0.0411)^2 + 0.07114(\log_{10} 0.0411) - 0.4761 \right] \cdot \log_{10} 0.0137 - \left[0.00266(\log_{10} 0.0411)^2 + 0.5941(\log_{10} 0.0411) + 0.4278 \right] = 1.4424$$

$$F(\delta_p, \delta_s) = 11.01217 + 0.51244[\log_{10} 0.0411 - \log_{10} 0.0137] = 11.256666$$

$$N = \frac{1.4424 - 11.256666 \left[\frac{(0.575\pi - 0.345\pi)/2\pi \right]^2}{(0.575\pi - 0.345\pi)/2\pi}}{(0.575\pi - 0.345\pi)/2\pi} = 11.248 \approx 12$$

10.2 $N = 75$ and $\omega_s - \omega_p = 0.05\pi$ and we assume $\delta_s = \delta_p$.

(a) Using Kaiser's formula of Eq. (10.3):

$$\delta_s = 10^{\left(\frac{75[14.6 \cdot 0.05\pi/2\pi] + 13}{-20} \right)} = 0.009577 \therefore \alpha_s = 40.375 \text{ dB.}$$

(b) Using Bellanger's formula of Eq. (10.4):

$$\delta_s = \left(0.1 \cdot 10^{\left(\frac{-76.3 \cdot 0.05\pi/2\pi}{2} \right)} \right)^{1/2} = 0.0119 \therefore \alpha_s = 38.5 \text{ dB.}$$

(c) Using Hermann's formula of Eq. (10.5):

$$F = b_1 \therefore D_{\infty}(\delta_s) = N(\omega_s - \omega_p)/2\pi + b_1 \left[(\omega_s - \omega_p)/2\pi \right]^2$$

$$D_{\infty}(\delta_s) = \left[a_1(\log_{10} \delta_s)^2 + a_2(\log_{10} \delta_s) + a_3 \right] (\log_{10} \delta_s) - \left[a_4(\log_{10} \delta_s)^2 + a_5(\log_{10} \delta_s) + a_6 \right]$$

$$D_{\infty}(\delta_s) = a_1(\log_{10} \delta_s)^3 + a_2(\log_{10} \delta_s)^2 + a_3(\log_{10} \delta_s) - a_4(\log_{10} \delta_s)^2 - a_5(\log_{10} \delta_s) - a_6$$

$$D_{\infty}(\delta_s) = a_1(\log_{10} \delta_s)^3 + (a_2 - a_4)(\log_{10} \delta_s)^2 + (a_3 - a_5)(\log_{10} \delta_s) - a_6$$

Let $x = (\log_{10} \delta_s)$, and thus

$$D_{\infty}(\delta_s) = 0.005309x^3 + 0.06848x^2 - 1.0702x - 0.4278 = 1.875697$$

Solving for x gives us three possible solutions:

$$x = -21.3787, x = -1.94654, x = 10.4263$$

The most reasonable solution is the second. Therefore,

$$\delta_s = 0.0113 \therefore \alpha_s = 38.93$$

10.3 $N = 75$ and $\omega_s - \omega_p = 0.05\pi = \Delta\omega$

$$\alpha_s = 2.285(\Delta\omega)N + 8 = 34.9 \text{ dB.}$$

10.4 The ideal L -band digital filter $H_{ML}(z)$ has a frequency response given by

$H_{ML}(e^{j\omega}) = A_k$, for $\omega_{k-1} \leq \omega \leq \omega_k$, $1 \leq k \leq L$, and can be considered as sum of L ideal

bandpass filters with cutoff frequencies at $\omega_{c1}^k = \omega_{k-1}$ and $\omega_{c2}^k = \omega_k$, where $\omega_{c1}^0 = 0$

and $\omega_{c2}^L = \pi$. Now from Eq. (10.47) the impulse response of an ideal bandpass filter is

given by $h_{BP}[n] = \frac{\sin(\omega_{c2}n)}{\pi n} - \frac{\sin(\omega_{c1}n)}{\pi n}$. Therefore ,

$h_{BP}^k[n] = \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n}$. Hence,

$$\begin{aligned} h_{ML}[n] &= \sum_{k=1}^L h_{BP}^k[n] = \sum_{k=1}^L A_k \left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n} \right) \\ &= A_1 \left(\frac{\sin(\omega_1 n)}{\pi n} - \frac{\sin(0n)}{\pi n} \right) + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n} \right) \\ &\quad + A_L \left(\frac{\sin(\omega_L n)}{\pi n} - \frac{\sin(\omega_{L-1}n)}{\pi n} \right) \\ &= A_1 \frac{\sin(\omega_1 n)}{\pi n} + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_{k-1}n)}{\pi n} - A_L \frac{\sin(\omega_{L-1}n)}{\pi n} \\ &= \sum_{k=1}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^L A_k \frac{\sin(\omega_{k-1}n)}{\pi n}. \end{aligned}$$

Since $\omega_L = \pi$, $\sin(\omega_L n) = 0$. We add a term $A_L \frac{\sin(\omega_L n)}{\pi n}$ to the first sum in the above expression and change the index range of the second sum, resulting in

$$h_{ML}[n] = \sum_{k=1}^L A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L-1} A_{k+1} \frac{\sin(\omega_k n)}{\pi n}.$$

Finally, since $A_{L+1} = 0$, we can add a term $A_{L+1} \frac{\sin(\omega_L n)}{\pi n}$ to the second sum. This leads to

$$h_{ML}[n] = \sum_{k=1}^L A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^L A_{k+1} \frac{\sin(\omega_k n)}{\pi n} = \sum_{k=1}^L (A_k - A_{k+1}) \frac{\sin(\omega_k n)}{\pi n}.$$

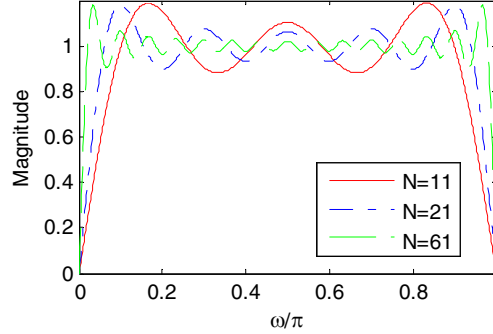
10.5 $H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}$ Therefore,

$$\begin{aligned} h_{HT}[n] &= \frac{1}{2\pi} \int_{-\pi}^0 H_{HT}(e^{j\omega}) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} H_{HT}(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^0 j e^{j\omega n} d\omega - \frac{1}{2\pi} \int_0^{\pi} j e^{j\omega n} d\omega = \frac{2}{2\pi n} (1 - \cos(\pi n)) = \frac{2 \sin^2(\pi n / 2)}{\pi n} \text{ if } n \neq 0. \end{aligned}$$

For $n = 0$, $h_{HT}[0] = \frac{1}{2\pi} \int_{-\pi}^0 j d\omega - \frac{1}{2\pi} \int_0^{\pi} j d\omega = 0$.

$$\text{Hence, } h_{HT}[n] = \begin{cases} 0, & \text{if } n = 0, \\ \frac{2\sin^2(\pi n/2)}{\pi n}, & \text{if } n \neq 0. \end{cases}$$

Since $h_{HT}[n] = -h_{HT}[-n]$, and the length of the truncated impulse response is odd, it is a Type 3 linear-phase FIR filter.



From the frequency response plots given above, we observe the presence of ripples at the bandedges due to the Gibbs phenomenon caused by the truncation of the impulse response.

10.6 $\mathcal{H}\{x[n]\} = \sum_{k=-\infty}^{\infty} h_{HT}[n-k]x[k]$. Hence,

$$\mathcal{F}\{\mathcal{H}\{x[n]\}\} = H_{HT}(e^{j\omega})X(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), & -\pi < \omega < 0, \\ -jX(e^{j\omega}), & 0 < \omega < \pi. \end{cases}$$

(a) Let $y[n] = \mathcal{H}\{\mathcal{H}\{\mathcal{H}\{\mathcal{H}\{x[n]\}\}\}\}$. Hence, $Y(e^{j\omega}) = \begin{cases} j^4 X(e^{j\omega}), & -\pi < \omega < 0, \\ (-j)^4 X(e^{j\omega}), & 0 < \omega < \pi, \end{cases} = X(e^{j\omega})$.

Therefore, $y[n] = x[n]$.

(b) Define $g[n] = \mathcal{H}\{x[n]\}$, and $h^*[n] = x[n]$. Then $\sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = \sum_{\ell=-\infty}^{\infty} g[\ell]h^*[\ell]$.

But from the Parseval's relation in Table 3.4, $\sum_{\ell=-\infty}^{\infty} g[\ell]h^*[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})G(e^{j\omega})d\omega$.

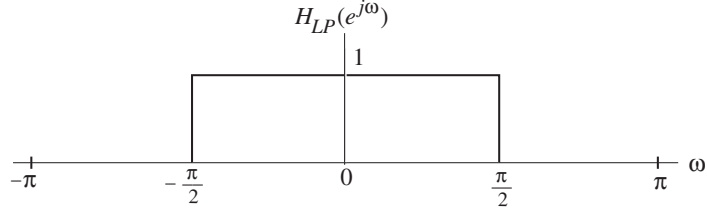
Therefore, $\sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})d\omega$ where

$$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}$$

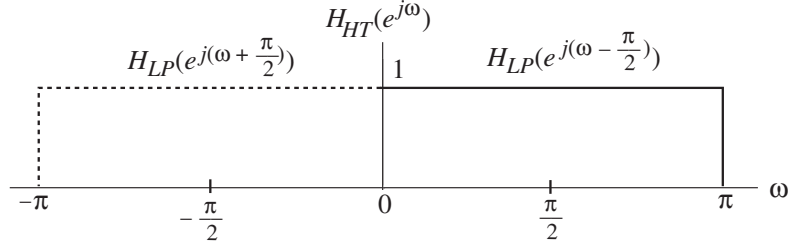
Since the integrand $H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})$ is an odd

function of ω , $\int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})d\omega = 0$. As a result, $\sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = 0$.

10.7 $H_{LP}(z) = \sum_{n=0}^N h_{LP}[n]z^{-n}$. Its frequency response $H_{LP}(e^{j\omega})$ is shown in Figure (a)



(a)



(b)

$$\begin{aligned} H_{HT}(e^{j\omega}) &= H_{LP}(e^{j(\omega + \pi/2)}) + H_{LP}(e^{j(\omega - \pi/2)}) \\ &= \sum_{n=0}^N h_{LP}[n]e^{-jn(\omega + \pi/2)} + \sum_{n=0}^N h_{LP}[n]e^{-jn(\omega - \pi/2)} = \sum_{n=0}^N h_{LP}[n]e^{-jn\omega} (e^{-jn\pi/2} + e^{jn\pi/2}) \\ &= \sum_{n=0}^N 2h_{LP}[n]e^{-jn\omega} \cos\left(\frac{n\pi}{2}\right) \end{aligned}$$

Now, for n odd, $\cos(n\pi/2) = 0$ and hence, we can drop all

odd terms in the above expression. Let $N = 2M$ with N even and let $r = 2n$. Then, we can rewrite the above equation as $H_{HT}(e^{j\omega}) = \sum_{r=0}^M 2h_{LP}[2r]\cos(r\pi)e^{-j2r\omega}$. The

corresponding transfer function of the Hilbert transformer is therefore given by

$$H_{HT}(z) = \sum_{n=0}^M 2h_{LP}[2n]\cos(n\pi)z^{-2n} = \sum_{n=0}^M 2(-1)^n h_{LP}[2n]z^{-2n}.$$

10.8 $H_{DIF}(e^{j\omega}) = j\omega$. Hence,

$$h_{DIF}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega = \frac{j}{2\pi} \int_{-\pi}^{\pi} \omega e^{j\omega n} d\omega = \frac{j}{2\pi} \left(\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right) \Big|_{-\pi}^{\pi}.$$

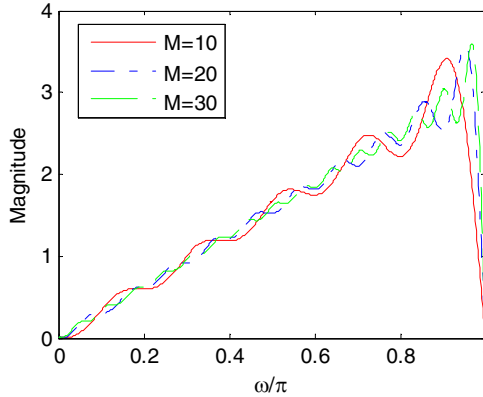
Therefore,

$$h_{DIF}[n] = \frac{\cos(\pi n)}{n} - \frac{\sin(\pi n)}{\pi n^2} = \frac{\cos(\pi n)}{n}, \text{ if } n \neq 0.$$

For $n = 0$, $h_{DIF}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega d\omega = 0$.

Hence, $h_{DIF}[n] = \begin{cases} 0, & n = 0, \\ \frac{\cos(\pi n)}{n}, & |n| > 0. \end{cases}$ Since $h_{DIF}[n] = -h_{DIF}[-n]$, the truncated impulse

response is a Type 3 linear-phase FIR filter. The magnitude responses of the above differentiator for several values of M are given below:



10.9 $N = 2M + 1$. $\hat{h}_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & \text{for } n = M, \\ -\frac{\sin(\omega_c(n-m))}{\pi(n-m)}, & \text{if } n \neq M, 0 \leq n < N, \\ 0, & \text{otherwise.} \end{cases}$

Now, $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = \sum_{n=-\infty}^{\infty} \hat{h}_{HP}[n]z^{-n} + \sum_{n=-\infty}^{\infty} \hat{h}_{LP}[n]z^{-n} = \sum_{n=0}^{N-1} \hat{h}_{HP}[n]z^{-n} + \sum_{n=0}^{N-1} \hat{h}_{LP}[n]z^{-n}$
 $= \sum_{n=0}^{N-1} (\hat{h}_{HP}[n] + \hat{h}_{LP}[n])z^{-n}$.

But $\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \begin{cases} 0, & 0 \leq n \leq N-1, n \neq M, \\ 1, & n=M. \end{cases}$

Hence, $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = z^{-M}$, i.e. the two filters are delay-complementary.

10.10 $H_{LLP}(e^{j\omega}) = \begin{cases} |\omega|, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$ Therefore,

$$\begin{aligned}
h_{LLP}[n] &= \frac{1}{2\pi} \left(- \int_{-\omega_c}^0 \omega e^{j\omega n} d\omega + \int_0^{\omega_c} \omega e^{j\omega n} d\omega \right) \\
&= \frac{1}{2\pi} \left(\left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{-\omega_c}^0 + \left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_0^{\omega_c} \right) \\
&= \frac{1}{2\pi} \left(\frac{\omega_c e^{j\omega_c n} - \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} + e^{-j\omega_c n} - 2}{n^2} \right) = \frac{\omega_c}{\pi n} \sin(\omega_c n) + \frac{\cos(\omega_c n) - 1}{\pi n^2}.
\end{aligned}$$

10.11 $H_{BLDIF}(e^{j\omega}) = \begin{cases} \omega, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$ Hence,

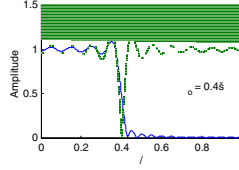
$$\begin{aligned}
h_{BLDIF}[n] &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \omega e^{j\omega n} d\omega = \frac{1}{2\pi} \left(\left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{-\omega_c}^{\omega_c} \right) \\
&= \frac{1}{2\pi} \left(\frac{\omega_c e^{j\omega_c n} + \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n^2} \right) = -j \frac{\omega_c}{\pi n} \cos(\omega_c n) + j \frac{1}{\pi n^2} \sin(\omega_c n).
\end{aligned}$$

10.12 The frequency response of a causal ideal notch filter can thus be expressed as

$$H_{notch}(e^{j\omega}) = \check{H}_{notch}(\omega) e^{j\theta(\omega)} \text{ where } \check{H}_{notch}(\omega) \text{ is the amplitude response which can be expressed as } \check{H}_{notch}(\omega) = \begin{cases} 1, & 0 \leq \omega \leq \omega_o, \\ -1, & \omega_o < \omega < \pi. \end{cases} \text{ It follows then that } \check{H}_{notch}(\omega) \text{ is}$$

related to the amplitude response $\check{H}_{LP}(\omega)$ of the ideal lowpass filter with a cutoff at ω_o through $\check{H}_{notch}(\omega) = \pm[2\check{H}_{LP}(\omega) - 1]$. Hence, the impulse response of the ideal notch filter is given by $h_{notch}[n] = \pm[2h_{LP}[n] - \delta[n]]$, where

$h_{LP}[n] = \frac{\sin(\omega_o n)}{\pi n}$, $-\infty < n < \infty$. The magnitude responses of a length 41 notch filter with a notch frequency at $\omega_o = 0.4\pi$ and its associated length-41 lowpass filter are shown below.



$$10.13 \quad \Phi_R = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_t(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega, \text{ where } H_t(e^{j\omega}) = \sum_{n=-M}^M h_t[n] e^{-j\omega n}.$$

$$\begin{aligned} \text{Using Parseval's relation, we can write } \Phi_R &= \sum_{n=-\infty}^{\infty} |h_t[n] - h_d[n]|^2 \\ &= \sum_{n=-M}^M |h_t[n] - h_d[n]|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]. \end{aligned}$$

$$\begin{aligned} \text{Now, } \Phi_{Haan} &= \sum_{n=-\infty}^{\infty} |h_d[n] \cdot w_{Hann}[n] - h_d[n]|^2 \\ &= \sum_{n=-M}^M \left| h_d[n] \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{2\pi n}{2M+1}\right) \right) - h_d[n] \right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n] \end{aligned}$$

$$\text{Hence, } \Phi_{Excess} = \Phi_R - \Phi_{Haan}$$

$$\begin{aligned} &= \sum_{n=-M}^M |h_d[n] \cdot w_R[n] - h_d[n]|^2 - \sum_{n=-M}^M \left| h_d[n] \cdot \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{2\pi n}{2M+1}\right) \right) - h_d[n] \right|^2 \\ &= - \sum_{n=-M}^M \left| \frac{h_d[n]}{2} \cos\left(\frac{2\pi n}{2M+1}\right) - \frac{h_d[n]}{2} \right|^2 = \frac{1}{2} (1 + 2M) \left| \cos\left(\frac{2\pi M}{2M+1}\right) - 1 \right|^2. \end{aligned}$$

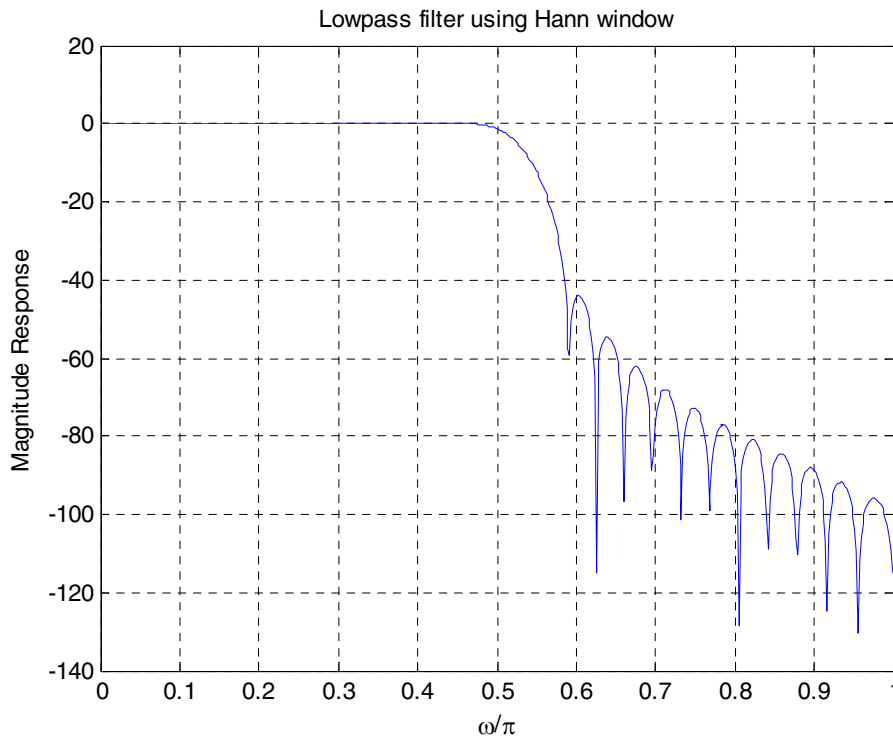
$$10.14 \quad \Phi_R = \sum_{n=-\infty}^{\infty} |h_t[n] - h_d[n]|^2 \text{ and } \Phi_{Hamm} = \sum_{n=-\infty}^{\infty} |h_d[n] \cdot w_{Hamm}[n] - h_d[n]|^2.$$

$$\begin{aligned} \text{Therefore, } \Phi_{Excess} = \Phi_R - \Phi_{Hamm} &= - \sum_{n=-M}^M \left| h_d[n] \left(0.46 \cos\left(\frac{2\pi n}{2M+1}\right) - 0.46 \right) \right|^2 \\ &= - \sum_{n=-M}^M \left| 0.46 h_d[n] \left(\cos\left(\frac{2\pi n}{2M+1}\right) - 1 \right) \right|^2 = 0.46(2M+1) \left| \cos\left(\frac{2\pi M}{2M+1}\right) - 1 \right|^2. \end{aligned}$$

10.15 (a) $\omega_p = 0.47\pi$, $\omega_s = 0.59\pi$, $\delta_p = 0.001$, $\delta_s = 0.007$, $\Delta\omega = 0.12\pi$,
 $\alpha_s = -20 \log_{10} \delta_s = 43.1 \text{ dB}$

From Table 10.2, we see that for fixed-window functions, we can achieve the minimum stopband attenuation by using Hann, Hamming, or Blackman windows. Hann will have the lowest filter length.

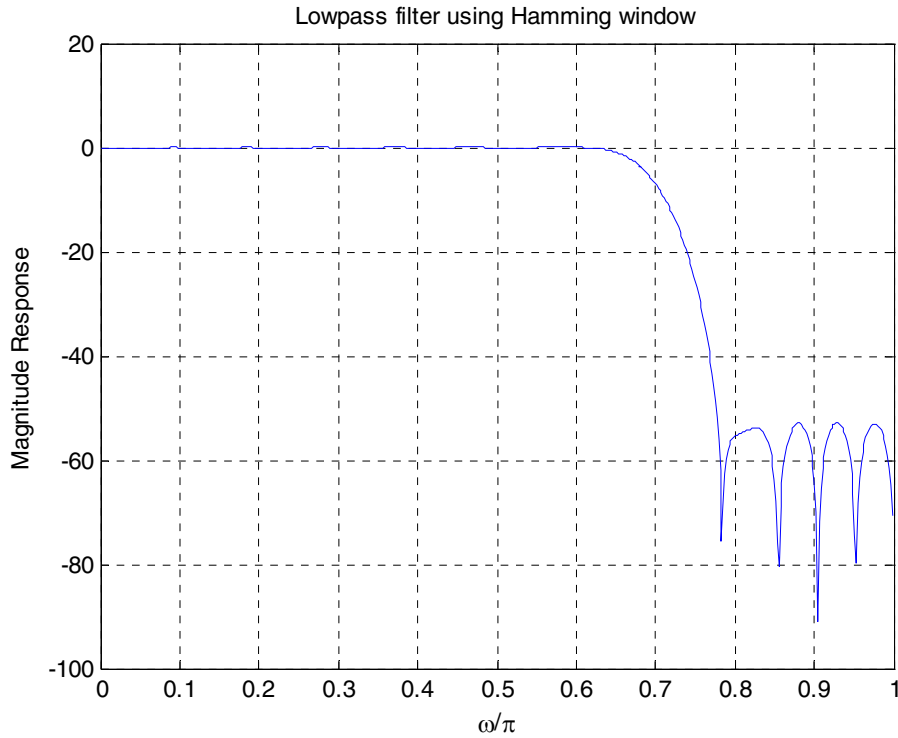
$$N_{Hann} = \lceil 2M + 1 \rceil = 53 \text{ since } M = \frac{3.11\pi}{0.12\pi} = 25.917.$$



(b) $\omega_p = 0.61\pi$, $\omega_s = 0.78\pi$, $\delta_p = 0.001$, $\delta_s = 0.002$, $\Delta\omega = 0.17\pi$,
 $\alpha_s = -20 \log_{10} \delta_s = 54 \text{ dB}$

From Table 10.2, we see that for fixed-window functions, we can achieve the minimum stopband attenuation by using either Hamming, or Blackman windows. Hamming will have the lowest filter length.

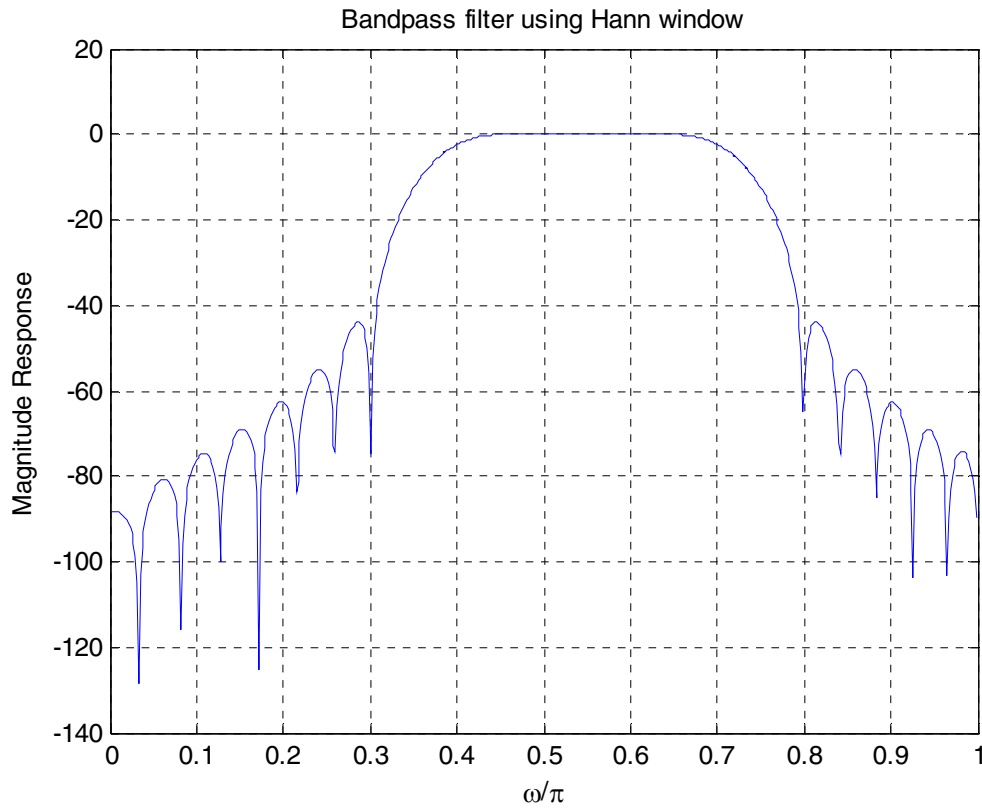
$$N_{Hamm} = \lceil 2M + 1 \rceil = 41 \text{ since } M = \frac{3.32\pi}{0.17\pi} = 19.53.$$



10.16 $\omega_{p1} = 0.45\pi, \omega_{p2} = 0.65\pi, \omega_{s1} = 0.3\pi, \omega_{s2} = 0.8\pi, \Delta\omega_1 = \Delta\omega_2 = 0.15\pi,$
 $\delta_p = 0.01, \delta_{s1} = 0.008, \delta_{s2} = 0.05$
 $\alpha_{s1} = -20 \log_{10} \delta_{s1} = 42 \text{ dB}, \alpha_{s2} = -20 \log_{10} \delta_{s2} = 26 \text{ dB}$

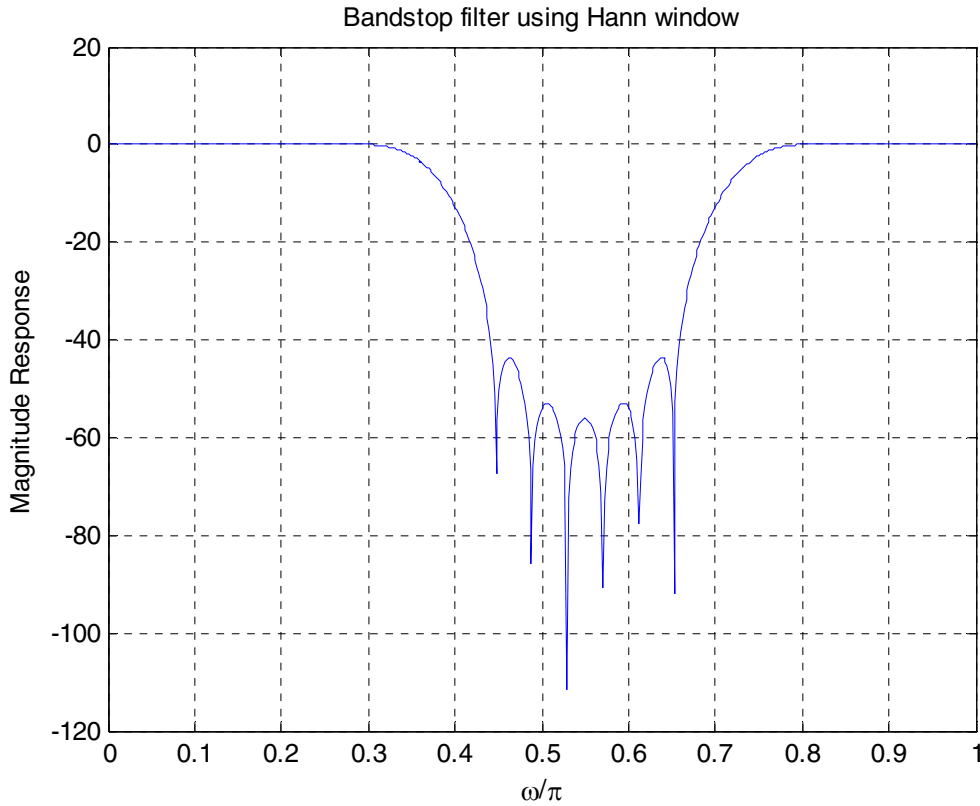
From Table 10.2, we see that the Hann window will have minimum length and meet the minimum stopband attenuation.

$$M = \frac{3.11\pi}{0.15\pi} = 21. \text{ Therefore, } N = 43.$$



10.17 $\omega_{p1} = 0.3\pi, \omega_{p2} = 0.8\pi, \omega_{s1} = 0.45\pi, \omega_{s2} = 0.65\pi, \Delta\omega_1 = \Delta\omega_2 = 0.15\pi,$
 $\delta_{p1} = 0.05, \delta_{p2} = 0.009, \delta_s = 0.02,$
 $\alpha_s = -20 \log_{10} \delta_s = 34 \text{ dB}$

From Table 10.2, we see that the Hann window will have minimum length and meet the minimum stopband attenuation.



10.18 Consider another filter with a frequency response $G(e^{j\omega})$ given by

$$G(e^{j\omega}) = \begin{cases} 0, & 0 \leq \omega \leq \omega_p, \\ \frac{-\pi}{2\Delta\omega} \sin\left(\frac{\pi(\omega - \omega_p)}{\Delta\omega}\right), & \omega_p < \omega \leq \omega_s, \\ \frac{-\pi}{2\Delta\omega} \sin\left(\frac{\pi(\omega + \omega_p)}{\Delta\omega}\right), & -\omega_s \leq \omega \leq -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly $G(e^{j\omega}) = \frac{dH(e^{j\omega})}{d\omega}$. Now,

$$g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega = \frac{-\pi}{8\pi\Delta\omega j} \left(\int_{\omega_p}^{\omega_s} e^{j\left(\frac{\pi(\omega - \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega - \int_{\omega_p}^{\omega_s} e^{-j\left(\frac{\pi(\omega - \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega \right. \\ \left. + \int_{-\omega_s}^{-\omega_p} e^{j\left(\frac{\pi(\omega + \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega - \int_{-\omega_s}^{-\omega_p} e^{-j\left(\frac{\pi(\omega + \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega \right)$$

$$\begin{aligned}
&= \frac{-1}{j8\Delta\omega} \left(e^{-\frac{j\pi\omega_p}{\Delta\omega}} \left[\frac{e^{j\omega_s\left(n+\frac{\pi}{\Delta\omega}\right)} - e^{j\omega_p\left(n+\frac{\pi}{\Delta\omega}\right)}}{j\left(n+\frac{\pi}{\Delta\omega}\right)} \right] - e^{-\frac{j\pi\omega_p}{\Delta\omega}} \left[\frac{e^{j\omega_s\left(n-\frac{\pi}{\Delta\omega}\right)} - e^{j\omega_p\left(n-\frac{\pi}{\Delta\omega}\right)}}{j\left(n-\frac{\pi}{\Delta\omega}\right)} \right] \right. \\
&\quad \left. + e^{\frac{j\pi\omega_p}{\Delta\omega}} \left[\frac{-e^{-j\omega_p\left(n+\frac{\pi}{\Delta\omega}\right)} - e^{-j\omega_s\left(n+\frac{\pi}{\Delta\omega}\right)}}{j\left(n+\frac{\pi}{\Delta\omega}\right)} \right] - e^{\frac{j\pi\omega_p}{\Delta\omega}} \left[\frac{-e^{-j\omega_p\left(n-\frac{\pi}{\Delta\omega}\right)} - e^{-j\omega_s\left(n-\frac{\pi}{\Delta\omega}\right)}}{j\left(n-\frac{\pi}{\Delta\omega}\right)} \right] \right) \\
&= \frac{-1}{4\Delta\omega j} \left(\frac{-\sin(\omega_s n) - \sin(\omega_p n)}{\left(n+\frac{\pi}{\Delta\omega}\right)} - \frac{-\sin(\omega_s n) - \sin(\omega_p n)}{\left(n-\frac{\pi}{\Delta\omega}\right)} \right) = \frac{(\sin(\omega_s n) + \sin(\omega_p n))}{4\Delta\omega j} \left(\frac{-2\pi / \Delta\omega}{n^2 - \frac{\pi^2}{\Delta\omega^2}} \right) \\
&= \frac{\sin(\omega_c n) \cos(\Delta\omega n / 2)}{\Delta\omega j} \left(\frac{1}{\pi \left(1 - (\Delta\omega / \pi)^2 n^2\right)} \right).
\end{aligned}$$

Now, $h[n] = \frac{j}{n} g[n]$. Therefore, $h[n] = \left(\frac{\cos(\Delta\omega n / 2)}{1 - (\Delta\omega / \pi)^2 n^2} \right) \left(\frac{\sin(\omega_c n)}{\pi n} \right)$.

10.19 $w_{GC}[n] = \left[\alpha + \beta \cos\left(\frac{2\pi n}{2M+1}\right) + \gamma \cos\left(\frac{4\pi n}{2M+1}\right) \right] w_R[n]$

$$= \left[\alpha + 2\beta \left(e^{j\left(\frac{2\pi n}{2M+1}\right)} + e^{-j\left(\frac{2\pi n}{2M+1}\right)} \right) + 2\gamma \left(e^{j\left(\frac{4\pi n}{2M+1}\right)} + e^{-j\left(\frac{4\pi n}{2M+1}\right)} \right) \right] w_R[n]$$

Hence, $\Psi_{GC}(e^{j\omega}) = \alpha\Psi_R(e^{j\omega}) + 2\beta\Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)$

$$+ 2\gamma\Psi_R\left(e^{j\left(\omega - \frac{4\pi}{2M+1}\right)}\right) + 2\gamma\Psi_R\left(e^{j\left(\omega + \frac{4\pi}{2M+1}\right)}\right).$$

For the Hann window : $\alpha = 0.5$, $\beta = 0.5$ and $\gamma = 0$. Hence,

$$\Psi_{Hann}(e^{j\omega}) = 0.5\Psi_R(e^{j\omega}) + \Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)$$

$$= 0.5 \frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}.$$

For the Hamming window, $\alpha = 0.54$, $\beta = 0.46$, and $\gamma = 0$. Hence,

$$\begin{aligned} \Psi_{Ham \min g}(e^{j\omega}) &= 0.54\Psi_R(e^{j\omega}) + 0.92\Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 0.92\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right) \\ &= 0.54 \frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + 0.92 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)} + 0.92 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}. \end{aligned} \quad \text{F}$$

or the Blackmann window $\alpha = 0.42$, $\beta = 0.5$ and $\gamma = 0.08$

$$\begin{aligned} \Psi_{Blackman}(e^{j\omega}) &= 0.42\Psi_R(e^{j\omega}) + \Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + \Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right) \\ &\quad + 0.16\Psi_R\left(e^{j\left(\omega - \frac{4\pi}{2M+1}\right)}\right) + 0.16\Psi_R\left(e^{j\left(\omega + \frac{4\pi}{2M+1}\right)}\right) \\ &= 0.42 \frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)} \\ &\quad + 0.16 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)} + 0.16 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)}. \end{aligned}$$

10.20 (a) $H(z) = z^{-D} \approx \sum_{n=0}^N h[n]z^{-n} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \dots + h[N]z^{-N}$

We see that if $\hat{x}_a(t) = \sum_{k=-N_1}^{N_2} P_k(t)x[n+k]$, then $P_k(t) = \prod_{\substack{l=-N_1 \\ l \neq k}}^{N_2} \left(\frac{t-t_l}{t_k-t_l} \right)$ for $-N_1 \leq k \leq N_2$.

Here, we have $\hat{H}(z) = \sum_{n=0}^N h[n]z^{-n}$ and the solution follows if

$$P_k(t) = h[n], N_1 = 0, N_2 = N, k = n, t = D, t_l = k, \text{ and } t_k = n$$

Therefore, we have $h[n] = \prod_{\substack{k=0 \\ k \neq n}}^N \frac{D-k}{n-k}$ for $0 \leq n \leq N$.

(b) $N = 21, D = 90/13, L = 22$. $H(z) \approx \sum_{n=0}^{21} h[n]z^{-n}$, where $h[n] = \prod_{\substack{k=0 \\ k \neq n}}^{21} \frac{90/13 - k}{n - k}$.

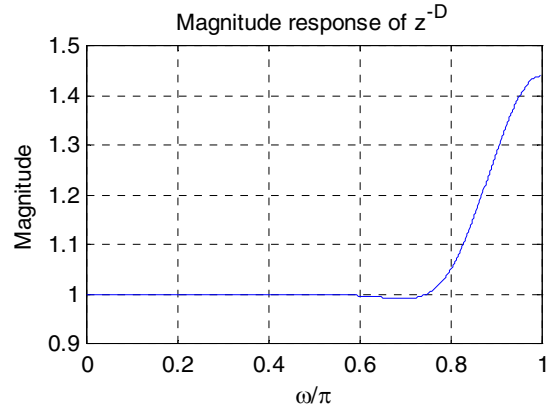
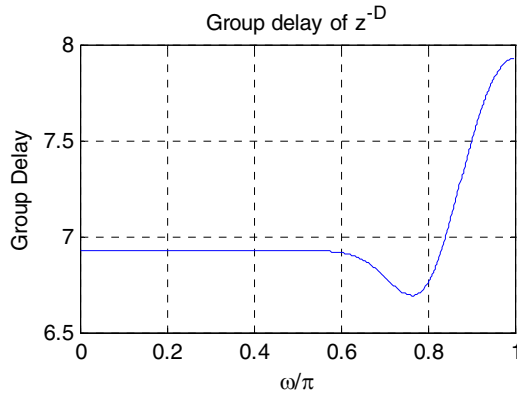
```

% Problem #10.20
D = 90/13;
N = 21;
for n = 0:N,
    for k = 0:N,
        if n ~= k,
            tmp(n+1,k+1) = (D-k)/(n-k);
        else
            tmp(n+1,k+1) = 1;
        end
    end
end
h = prod(tmp');
[Gd,W] = grpdelay(h,1,512);
[H, w] = freqz(h,1,512);

figure(1);
plot(W/pi, Gd);
xlabel('\omega/\pi');
ylabel('Group Delay');
title('Group delay of z^{-D}');
grid;

figure(2);
plot(w/pi, (abs(H)));
xlabel('\omega/\pi');
ylabel('Magnitude');
title('Magnitude response of z^{-D}');
grid;

```



10.21 (a) $x[n] = s[n] + \sum_{k=0}^M A_k \sin(k\omega_o n + \phi_k) = s[n] + r[n]$, where $s[n]$ is the desired signal and

$r[n] = \sum_{k=0}^M A_k \sin(k\omega_o n + \phi_k)$ is the harmonic interference with fundamental

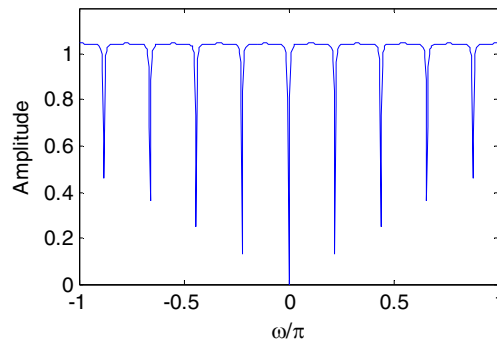
frequency ω_o . Now, $r[n-D] = \sum_{k=0}^M A_k \sin[k\omega_o(n-D) + \phi_k]$

$$= \sum_{k=0}^M A_k \sin(k\omega_o n + \phi_k - 2\pi k) = r[n].$$

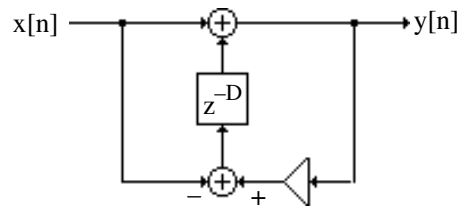
(b) $y[n] = x[n] - x[n-D] = s[n] + r[n] - s[n-D] - r[n-D] = s[n] + r[n] - s[n-D] - r[n]$
 $= s[n] - s[n-D]$. Hence, $y[n]$ does not contain any harmonic disturbances.

(c) $H_c(z) = \frac{1-z^{-D}}{1-\rho^{-D}z^{-D}}$. Thus, $H_c(e^{j\omega}) = \frac{1-e^{-jD\omega}}{1-\rho^{-D}e^{-jD\omega}} = \frac{(1-\cos(D\omega)) + j\sin(D\omega)}{(1-\rho^D \cos(D\omega)) + j\rho^D \sin(D\omega)}$.

Then, $|H_c(e^{j\omega})| = \sqrt{\frac{2(1-\cos(D\omega))}{1-2\rho^D \cos(D\omega) + \rho^{2D}}}$. A plot of $|H_c(e^{j\omega})|$ for $\omega_o = 0.22\pi$ and $\rho = 0.99$ is shown below:



(d)

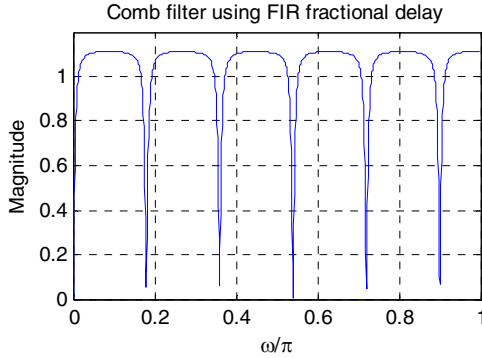


10.22 $H_c(z) = \frac{P(z)}{Q(z)} = \frac{1 - N(z)}{1 - 0.98^D N(z)}$, with $D = 2\pi/0.18\pi$, $N = 18$, and $\rho = 0.98$.

```

% Problem #10.23
close all;
clear;
clc;
D = 2/.18;
N = 18;
rho = 0.98;
for n = 0:N,
    for k = 0:N,
        if n ~= k,
            tmp(n+1,k+1) = (D-k)/(n-k);
        else
            tmp(n+1,k+1) = 1;
        end
    end
end
h = prod(tmp');
[H,w] = freqz(h,1,1024);
Hc = (1-H)./(1-(rho^D)*H);
x = sqrt((2*(1-cos(D*w))./(1-
2*(rho^D)*cos(D*w)+rho^(2*D))));
%plot(w/pi, abs(Hc));
plot(w/pi, x); grid;
xlabel('\omega/\pi');
ylabel('Magnitude');
title('Comb filter using FIR fractional delay');

```

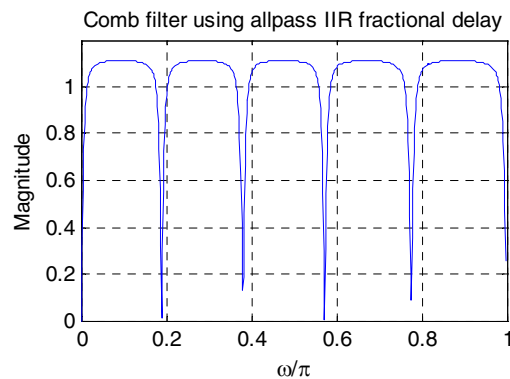


10.23 $H_c(z) = \frac{P(z)}{Q(z)} = \frac{D(z) - z^{-11}D(z^{-1})}{D(z) - 0.98^{90/13}z^{-11}D(z^{-1})}$, with $D = 2\pi/0.18\pi = 11.11$ and $N = 11$.

```

% Problem #10.23
close all;
clear;
clc;
D = 2/0.18;
rho = 0.98;
N = floor(D);
for k = 1:N,
    for n = 0:N,
        p(n+1) = (D-N+n)/(D-N+k+n);
    end
    d(k) = ((-1)^k)*nchoosek(N,k)*prod(p);
end
[H,w] = freqz(fliplr(d)-d, fliplr(d)-(rho^D).*d, 512);
plot(w/pi, abs(H));grid;
xlabel('\omega/\pi');
ylabel('Magnitude');
title('Comb filter using allpass IIR fractional delay');

```



$$10.24 \quad H_{LP}(e^{j\omega}) = \begin{cases} 1, & -\omega_p \leq \omega \leq \omega_p, \\ 1 - \left(\frac{\omega - \omega_p}{\omega_s - \omega_p} \right), & \omega_p < \omega \leq \omega_s, \\ 1 + \left(\frac{\omega + \omega_p}{\omega_s - \omega_p} \right), & -\omega_s < \omega \leq -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\begin{aligned} \text{Now, for } n \neq 0, \quad h_{LP}[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left[\int_{-\omega_p}^{\omega_p} e^{j\omega n} d\omega + \int_{\omega_p}^{\omega_s} \left(1 - \frac{\omega - \omega_p}{\Delta\omega} \right) e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \left(1 + \frac{\omega + \omega_p}{\Delta\omega} \right) e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi} \left[\int_{-\omega_s}^{\omega_s} e^{j\omega n} d\omega - \int_{\omega_p}^{\omega_s} \frac{\omega - \omega_p}{\Delta\omega} e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \frac{\omega + \omega_p}{\Delta\omega} e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi} \left\{ \frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega} \left[\frac{(\omega - \omega_p)e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Bigg|_{\omega_p}^{\omega_s} + \frac{1}{\Delta\omega} \left[\frac{(\omega + \omega_p)e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Bigg|_{-\omega_s}^{-\omega_p} \right\} \\ &= \frac{1}{2\pi} \left\{ \frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega} \left[\frac{\Delta\omega e^{j\omega_s n}}{jn} + \frac{e^{j\omega_s n} - e^{j\omega_p n}}{n^2} + \frac{-\Delta\omega e^{-j\omega_s n}}{jn} - \frac{e^{-j\omega_p n} - e^{-j\omega_s n}}{n^2} \right] \right\} \\ &= \frac{1}{2\pi} \left\{ \frac{2\sin(\omega_s n)}{\pi n} - \frac{2}{\Delta\omega} \left[\frac{\Delta\omega \sin(\omega_s n)}{n} + \frac{\cos(\omega_s n) - \cos(\omega_p n)}{n^2} \right] \right\} \\ &= \frac{1}{\Delta\omega} \left\{ \frac{\cos(\omega_p n)}{\pi n^2} - \frac{\cos(\omega_s n)}{\pi n^2} \right\} = \frac{1}{\Delta\omega} \left\{ \frac{\cos((\omega_c - \Delta\omega/2)n)}{\pi n^2} - \frac{\cos((\omega_c + \Delta\omega/2)n)}{\pi n^2} \right\} \\ &= \frac{2 \sin(\Delta\omega n / 2) \sin(\omega_c n)}{\Delta\omega n \pi n}. \end{aligned}$$

$$\begin{aligned} \text{Next, for } n = 0, \quad h_{LP}[0] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) d\omega = \frac{1}{2\pi} (\text{area under the curve}) \\ &= \frac{1}{2\pi} \frac{2(\omega_s + \omega_p)}{2} = \frac{\omega_c}{\pi}. \end{aligned}$$

$$\text{Hence, } h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \frac{2\sin(\Delta\omega n/2) \sin(\omega_c n)}{\Delta\omega \pi n}, & \text{if } n \neq 0. \end{cases}$$

An alternate approach to solving this problem is as follows. Consider the frequency response

$$G(e^{j\omega}) = \frac{dH_{LP}(e^{j\omega})}{d\omega} = \begin{cases} 0, & -\omega_p \leq \omega \leq \omega_p, \\ -\frac{1}{\Delta\omega}, & \omega_p < \omega < \omega_s, \\ \frac{1}{\Delta\omega}, & -\omega_s < \omega < -\omega_p, \\ 0, & \text{elsewhere.} \end{cases} \quad \text{Its inverse DTFT is given by}$$

$$\begin{aligned} g[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_s}^{-\omega_p} \frac{1}{\Delta\omega} e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{\omega_p}^{\omega_s} \frac{1}{\Delta\omega} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi\Delta\omega} \left(\frac{e^{j\omega n}}{jn} \Big|_{-\omega_s}^{-\omega_p} - \frac{e^{j\omega n}}{jn} \Big|_{\omega_p}^{\omega_s} \right) = \frac{1}{j\pi n \Delta\omega} (\cos(\omega_p n) - \cos(\omega_s n)). \end{aligned}$$

$$\begin{aligned} \text{Thus, } h_{LP}[n] &= \frac{j}{n} g[n] = \frac{1}{\pi n^2 \Delta\omega} (\cos(\omega_p n) - \cos(\omega_s n)) \\ &= \frac{1}{\pi n^2 \Delta\omega} \left[\cos\left(\left(\omega_c - \frac{\Delta\omega}{2}\right)n\right) - \cos\left(\left(\omega_c + \frac{\Delta\omega}{2}\right)n\right) \right] \\ &= \frac{2\sin(\Delta\omega n/2) \sin(\omega_c n)}{\Delta\omega \pi n}, \text{ for } n \neq 0. \end{aligned}$$

$$\text{For } n = 0, h_{LP}[n] = \frac{\omega_c}{\pi}.$$

10.25 Consider the case when the transition region is approximated by a second order spline. In this case the ideal frequency response can be constructed by convolving an ideal, no-transition-band frequency response with a triangular pulse of width $\Delta\omega = \omega_s - \omega_p$, which in turn can be obtained by convolving two rectangular pulses of width $\Delta\omega/2$. In the time domain this implies that the impulse response of a filter with transition band approximated by a second order spline is given by the product of the impulse response of an ideal low pass filter with no transition region and square of the impulse response of a rectangular pulse. Now,

$$H_{LP(ideal)}[n] = \frac{\sin(\omega_c n)}{\pi n} \text{ and } H_{rec}[n] = \frac{\sin(\Delta\omega n / 4)}{\Delta\omega n / 4}. \text{ Hence,}$$

$$H_{LP}[n] = H_{LP(ideal)}[n](H_{rec}[n])^2.$$

Thus for a lowpass filter with a transition region approximated by a second order spline

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left(\frac{\sin(\Delta\omega n / 4)}{\Delta\omega n / 4} \right)^2 \frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

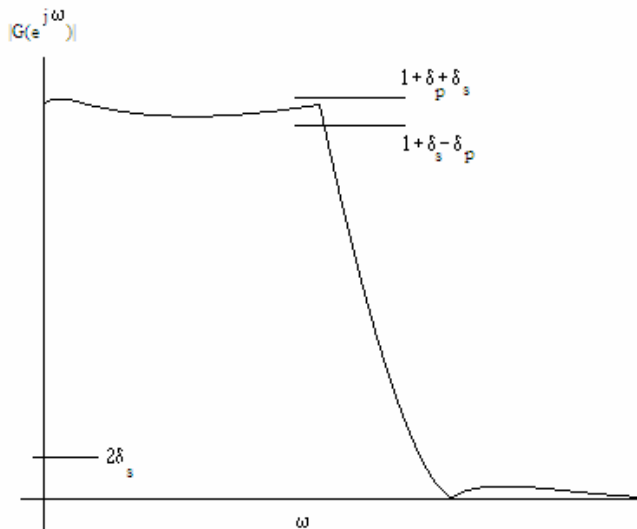
Similarly the frequency response of a lowpass filter with the transition region specified by a P -th order spline can be obtained by convolving in the frequency domain an ideal filter with no transition region with P rectangular pulses of width $\Delta\omega / P$. Hence,

$H_{LP}[n] = H_{LP(ideal)}[n](H_{rec}[n])^P$, where the rectangular pulse is of width $\Delta\omega / P$. Thus

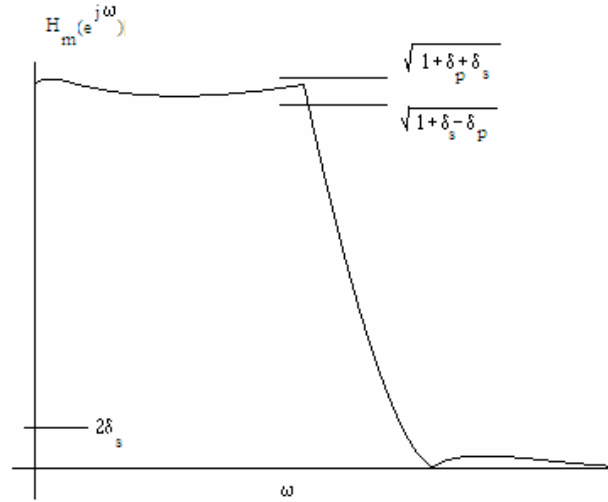
$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left(\frac{\sin(\Delta\omega n / 2P)}{\Delta\omega n / 2P} \right)^P \frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

10.26 From Step 2, we have $G(e^{j\omega}) = \tilde{G}(\omega)e^{-jN\omega} = \delta_s^{(F)}e^{-jN\omega} + \tilde{F}(\omega)e^{-jN\omega}$.

The amplitude response $\tilde{G}(\omega)$ has been obtained by raising the amplitude response $\tilde{F}(\omega)$ by $\delta_s^{(F)}$ and hence, the filter $G(z)$ has double zeros in the stopband.



We may factorize as follows: $G(z) = z^{-N}H_m(z)H_m(z^{-1})$, where $H_m(z)$ is a real-coefficient minimum-phase FIR lowpass filter with half the degree of the original $H(z)$. Since, $\tilde{G}(\omega) \geq 0$, the amplitude response $\tilde{H}_m(\omega)$ of the minimum-phase filter $H_m(z)$ does not oscillate about unity in the passband. Since the original frequency response was raised by $\delta_s^{(F)}$, $\tilde{H}_m(\omega)$ must be normalized by a factor $\sqrt{1 + \delta_s}$. Therefore,



For $\tilde{H}_m(\omega)$, we can see $\delta_s^{(F)} = \sqrt{\frac{2\delta_s}{1 + \delta_s}}$ and $\delta_p^{(F)} = \sqrt{\frac{1 + \delta_p + \delta_s}{1 + \delta_s}} - 1 = \sqrt{1 + \frac{\delta_p}{1 + \delta_s}} - 1$.

10.27 (a) $N = 1$ and hence, $x_a(t) = \alpha_0 + \alpha_1 t$. Without any loss of generality, for $L = 5$, we first fit the data set $\{x[k]\}, -5 \leq k \leq 5$, by the polynomial $x_a(t) = \alpha_0 + \alpha_1 t$ with a minimum mean-square error at $t = -5, -4, \dots, -1, 0, 1, \dots, 5$, and then replace $x[0]$ with a new value $\bar{x}[0] = x(0) = \alpha_0$.

Now, the mean-square error is given by $\varepsilon(\alpha_0, \alpha_1) = \sum_{k=-5}^5 (x[k] - \alpha_0 - \alpha_1 k)^2$. We set

$$\frac{\partial \varepsilon(\alpha_0, \alpha_1)}{\partial \alpha_0} = 0 \text{ and } \frac{\partial \varepsilon(\alpha_0, \alpha_1)}{\partial \alpha_1} = 0 \text{ which yields } 11\alpha_0 + \alpha_1 \sum_{k=-5}^5 k = \sum_{k=-5}^5 x[k], \text{ and}$$

$$\alpha_0 \sum_{k=-5}^5 k + \alpha_1 \sum_{k=-5}^5 k^2 = \sum_{k=-5}^5 kx[k].$$

From the first equation we get $\bar{x}[0] = \alpha_0 = \frac{1}{11} \sum_{k=-5}^5 x[k]$. In the general case we thus

have $\bar{x}[n] = \alpha_0 = \frac{1}{11} \sum_{k=-5}^5 x[k]$ which is a moving average filter of length 11.

(b) $N = 2$, and hence, $x_a(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$. Here, we fit the data set $\{x[k]\}$, $-5 \leq k \leq 5$, by the polynomial $x_a(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$ with a minimum mean-square error at $t = -5, -4, \dots, -1, 0, 1, \dots, 5$, and then replace $x[0]$ with a new value $\bar{x}[0] = x_a(0) = \alpha_0$. The mean-square error is now given by

$$\varepsilon(\alpha_0, \alpha_1, \alpha_2) = \sum_{k=-5}^5 (x[k] - \alpha_0 - \alpha_1 k - \alpha_2 k^2)^2. \quad \text{We set } \frac{\partial \varepsilon(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_0} = 0,$$

$$\frac{\partial \varepsilon(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_1} = 0, \text{ and } \frac{\partial \varepsilon(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_2} = 0, \text{ which yields}$$

$$11\alpha_0 + 110\alpha_2 = \sum_{k=-5}^5 x[k], \quad 110\alpha_1 = \sum_{k=-5}^5 kx[k], \quad 110\alpha_0 + 1958\alpha_2 = \sum_{k=-5}^5 k^2 x[k].$$
 From the

first and the third equations we then get

$$\alpha_0 = \frac{1958 \sum_{k=-5}^5 x[k] - 110 \sum_{k=-5}^5 k^2 x[k]}{(1958 \times 11) - (110)^2} = \frac{1}{429} \sum_{k=-5}^5 (89 - 5k^2) x[k].$$

Hence, here we replace $x[n]$ with a new value $\bar{x}[n] = \alpha_0$ which is a weighted combination of the original data set $\{x[k]\}$, $-5 \leq k \leq 5$:

$$\begin{aligned} \bar{x}[n] &= \frac{1}{429} \sum_{k=-5}^5 (89 - 5k^2) x[n-k] \\ &= \frac{1}{429} (-36x[n+5] + 9x[n+4] + 44x[n+3] + 69x[n+2] + 84x[n+1] + 89x[n] \\ &\quad + 84x[n-1] + 69x[n-2] + 44x[n-3] + 9x[n-4] - 36x[n-5]). \end{aligned}$$

(c) The impulse response of the FIR filter of Part (a) is given by

$$h_1[n] = \frac{1}{11} \{1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\},$$

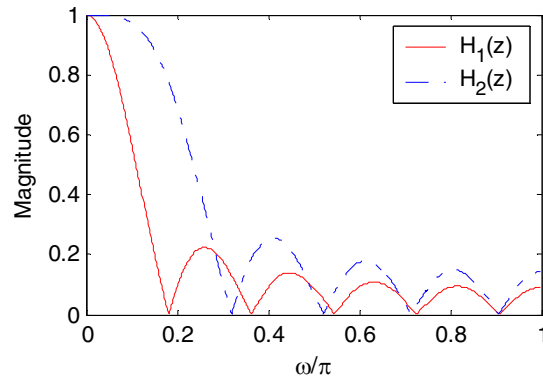
whereas, the impulse response of the FIR filter of Part (b) is given by

$$h_2[n] = \frac{1}{429} \{-36 \ 9 \ 44 \ 69 \ 84 \ 89 \ 84 \ 69 \ 44 \ 9 \ -36\}.$$

The corresponding frequency responses are given by

$$H_1(e^{j\omega}) = \frac{1}{11} \sum_{k=-5}^5 e^{-j\omega k}, \quad \text{and } H_2(e^{j\omega}) = \frac{1}{429} \sum_{k=-5}^5 (89 - 5k^2) e^{-j\omega k}.$$

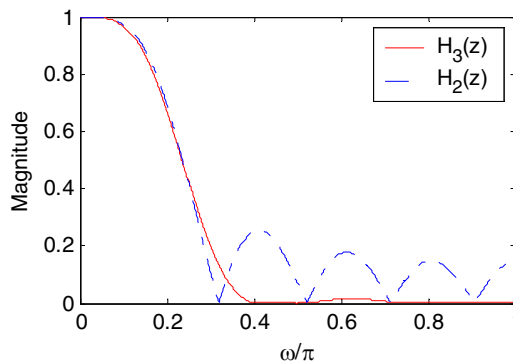
A plot of the magnitude responses of these two filters are shown below from which it can be seen that the filter of Part (b) has a wider passband and thus provides smoothing over a larger frequency range than the filter of Part (a).



10.28
$$y[n] = \frac{1}{320} \left\{ -3x[n-7] - 6x[n-6] - 5x[n-5] + 3x[n-4] + 21x[n-3] + 46x[n-2] \right. \\ \left. + 67x[n-1] + 74x[n] + 67x[n+1] + 46x[n+2] + 21x[n+3] + 3x[n+4] \right. \\ \left. + 5x[n+5] - 6x[n+6] - 3x[n+7] \right\}.$$

Hence,
$$H_3(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{320} \left\{ -3e^{-j7\omega} - 6e^{-j6\omega} - 5e^{-j5\omega} + 3e^{-j4\omega} + 21e^{-j3\omega} \right. \\ \left. 46e^{-j2\omega} + 67e^{-j\omega} + 74 + 67e^{j\omega} + 46e^{j2\omega} + 21e^{j3\omega} + 3e^{j4\omega} - 5e^{j5\omega} - 6e^{j6\omega} - 3e^{j7\omega} \right\} \\ = \frac{1}{160} \left\{ 74 + 67\cos\omega + 46\cos(2\omega) + 21\cos(3\omega) + 3\cos(4\omega) - 5\cos(5\omega) - 6\cos(6\omega) - 3\cos(7\omega) \right\} \\ - 5\cos(5\omega) - 6\cos(6\omega) - 3\cos(7\omega) \left. \right\}.$$

The magnitude response of the above FIR filter $H_3(z)$ is plotted below (solid line) along with that of the FIR filter $H_2(z)$ of Part (b) of Problem 10.27 (dashed line). Note that both filters have roughly the same passband but the Spencer's filter has very large attenuation in the stopband and hence it provides better smoothing than the filter of Part (b).



10.29 (a) $L = 3$. $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Now $P(0) = 0$ is satisfied by the way $P(x)$ has been defined. Also to ensure $P(1) = 1$ we require $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Choose $m = 1$ and $n = 1$. Since $\left. \frac{dP(x)}{dx} \right|_{x=0} = 0$, hence $\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 \Big|_{x=0} = 0$, implying $\alpha_1 = 0$. Also since $\left. \frac{dP(x)}{dx} \right|_{x=1} = 0$, hence $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$. Thus solving the three equations:
 $\alpha_1 + \alpha_2 + \alpha_3 = 1, \alpha_1 = 0$, and $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$
we arrive at $\alpha_1 = 0, \alpha_2 = 3$, and $\alpha_3 = -2$. Therefore, $P(x) = 3x^2 - 2x^3$.

(b) $L = 4$. Hence, $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$. Choose $m = 2$ and $n = 1$ (alternatively one can choose $m = 1$ and $n = 2$ for better stopband performance). Then, $P(1) = 1 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. Also,

$$\left. \frac{dP(x)}{dx} \right|_{x=0} = 0 \Rightarrow \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 \Big|_{x=0} = 0,$$

$$\left. \frac{d^2 P(x)}{dx^2} \right|_{x=0} = 0 \Rightarrow 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 \Big|_{x=0} = 0,$$

$$\left. \frac{dP(x)}{dx} \right|_{x=1} = 0 \Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0.$$

Solving the above four simultaneous equations we get $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 4$, and $\alpha_4 = -3$. Therefore, $P(x) = 4x^3 - 3x^4$.

(c) $L = 5$. Hence $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$. Choose $m = 2$ and $n = 2$. Following a procedure similar to that in parts (a) and (b) we get $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 10, \alpha_4 = -15$, and $\alpha_5 = 6$.

10.30 From Eq. (7.102) we have $\check{H}(\omega) = \sum_{k=1}^M c[k] \sin(\omega k)$. Now

$$\check{H}(\pi - \omega) = \sum_{k=1}^M c[k] \sin((\pi - \omega)k) = - \sum_{k=1}^M c[k] \sin(\omega k) \cos(\pi k) = \sum_{k=1}^M c[k] (-1)^{k+1} \sin(\omega k).$$

Thus, $\check{H}(\omega) = \check{H}(\pi - \omega)$ implies $\sum_{k=1}^M c[k] \sin(\omega k) = \sum_{k=1}^M c[k] (-1)^{k+1} \sin(\omega k)$, or

equivalently, $\sum_{k=1}^M (1 - (-1)^{k+1}) c[k] \sin(\omega k) = 0$, which in turn implies that $c[k] = 0$ for $k = 2, 4, 6, \dots$

But from Eq. (7.103) we have $c[k] = 2h[M - k]$, $1 \leq k \leq M$, or, $h[k] = \frac{1}{2}c[M - k]$. For k even, i.e., $k = 2R$, $h[2R] = \frac{1}{2}c[M - 2R] = 0$ if M is even.

10.31 (a) $H[k] = H(e^{j\omega_k}) = H(e^{j2\pi k/M})$, $0 \leq k \leq M-1$. Thus, $h[n] = \frac{1}{M} \sum_{k=0}^{M-1} H[k] W_M^{-kn}$,

where $W_M = e^{-j2\pi k/M}$.

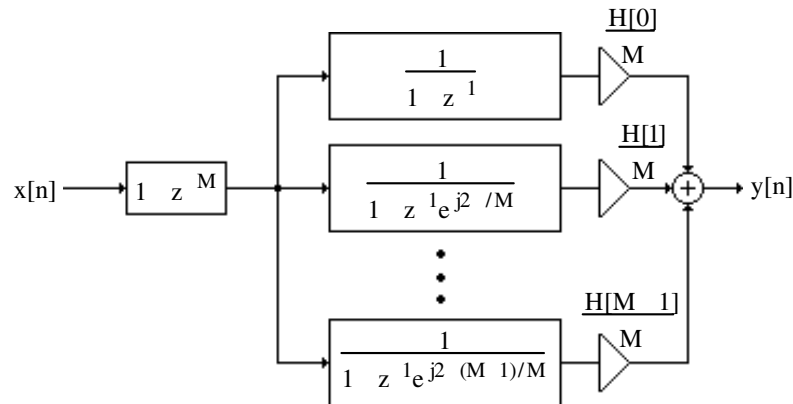
Now, $H(z) = \sum_{n=0}^{M-1} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{M-1} H[k] W_M^{-kn} z^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left(\sum_{n=0}^{M-1} W_M^{-kn} z^{-n} \right)$.

We can write $\sum_{n=0}^{M-1} W_M^{-kn} z^{-n} = \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} - \sum_{n=M}^{\infty} W_M^{-kn} z^{-n}$

$= \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} - W_M^{-kM} z^{-M} \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} = (1 - z^{-M}) \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} = \frac{1 - z^{-M}}{1 - W_M^{-k} z^{-1}}$.

Therefore, $H(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H[k]}{1 - W_M^{-k} z^{-1}}$.

(b)



(c) Note $H(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H[k]}{1 - W_M^{-k} z^{-1}} = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left(\sum_{n=0}^{M-1} W_M^{-kn} z^{-n} \right)$.

On the unit circle the above reduces to $H(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left(\sum_{n=0}^{M-1} W_M^{-kn} e^{-j\omega n} \right)$. For $\omega = j2\pi\ell / M$, we then get from the above $H(e^{j2\pi\ell / M}) = \sum_{k=0}^{M-1} H[k] \left(\frac{1}{M} \sum_{n=0}^{M-1} W_M^{-kn} e^{-2\pi\ell n / M} \right)$

$$= \sum_{k=0}^{M-1} H[k] \left(\frac{1}{M} \sum_{n=0}^{M-1} W_M^{-kn} W_M^{\ell n} \right) = \sum_{k=0}^{M-1} H[k] \left(\frac{1}{M} \sum_{n=0}^{M-1} W_M^{-(k-\ell)n} \right).$$

Using the identity of Eq. (5.23) of text we observe that $\frac{1}{M} \sum_{n=0}^{M-1} W_M^{-(k-\ell)n} = \begin{cases} 1, & \text{if } \ell = k, \\ 0, & \text{otherwise.} \end{cases}$

Hence, $H(e^{j2\pi\ell / M}) = H[\ell]$.

10.32 (a) For Type 1 FIR filter, $H(e^{j\omega}) = e^{-j\frac{\omega(M-1)}{2}} |H(e^{j\omega})|$. Since in the frequency sampling approach we sample the DTFT $H(e^{j\omega})$ at M points given by $\omega = \frac{2\pi k}{M}$, $0 \leq k \leq M-1$, therefore $H[k] = H(e^{j2\pi k / M}) = |H_d(e^{j2\pi k / M})| e^{j2\pi k(M-1) / M}$, $0 \leq k \leq M-1$. Since the filter is of Type 1, $M-1$ is even, thus, $e^{j2\pi k(M-1) / 2} = 1$. Moreover, $h[n]$ being real, $H(e^{j\omega}) = H^*(e^{j\omega})$. Thus, $H(e^{j\omega}) = e^{j\omega(M-1) / 2} |H(e^{j\omega})|$, $\pi \leq \omega < 2\pi$. Hence,

$$H[k] = \begin{cases} |H_d(e^{j2\pi k / M})| e^{-j2\pi k(M-1) / 2M}, & k = 0, 1, 2, \dots, \frac{M-1}{2}, \\ |H_d(e^{j2\pi k / M})| e^{-j2\pi(M-k)(M-1) / 2M}, & k = \frac{M+1}{2}, \frac{M+3}{2}, \dots, M-1. \end{cases}$$

(b) For the Type 2 FIR filter

$$H[k] = \begin{cases} |H_d(e^{j2\pi k / M})| e^{-j2\pi k(M-1) / 2M}, & k = 0, 1, 2, \dots, \frac{M-1}{2}, \\ 0, & k = \frac{M}{2}, \\ |H_d(e^{j2\pi k / M})| e^{-j2\pi(M-k)(M-1) / 2M}, & k = \frac{M}{2} + 1, \dots, M-1. \end{cases}$$

10.33 (a) $\omega_p = 0.55\pi = 1.72788$. The frequency spacing between 2 consecutive DFT samples is given by $\frac{2\pi}{19} = 0.3307$. The desired passband edge is between the frequency samples at $\omega = 2\pi \frac{5}{19}$ and $\omega = 2\pi \frac{6}{19}$. Therefore, the 19-point DFT is given by

$$H[k] = \begin{cases} e^{-j(2\pi/19)9k}, & k = 0, 1, \dots, 5, 13, 14, \dots, 18, \\ 0, & k = 6, \dots, 12. \end{cases}$$

A 19-point IDFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of z^{-1} :

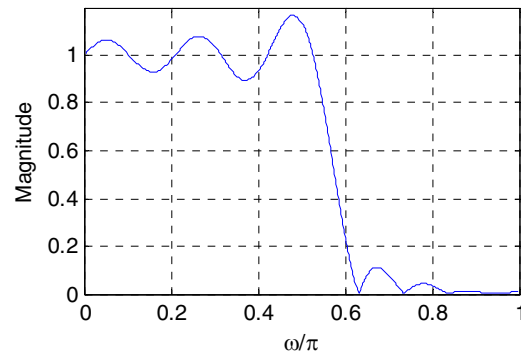
Columns 1 through 10

| | | | | | |
|---------|---------|--------|---------|---------|--------|
| -0.0037 | -0.0022 | 0.0224 | -0.0211 | -0.0231 | 0.0674 |
| -0.0316 | -0.1128 | 0.2888 | 0.6316 | | |

Columns 11 through 19

| | | | | | |
|--------|---------|---------|--------|---------|---------|
| 0.2888 | -0.1128 | -0.0316 | 0.0674 | -0.0231 | -0.0211 |
| 0.0224 | -0.0022 | -0.0037 | | | |

(b)



```
% Problem #10.33
close all;
clear;
clc;

ind = 1;
for k = 0:5,
    H(ind) = exp(-i*2*pi*9*k/19);
    ind = ind + 1;
end

for k = 6:12,
    H(ind) = 0;
    ind = ind + 1;
end

for k = 13:18,
    H(ind) = exp(-i*2*pi*9*k/19);
    ind = ind + 1;
end
h = ifft(H);

figure(1);
stem(real(h));
[FF, w] = freqz(h, 1, 512);
```

```
figure(2);
plot(w/pi, abs(FF)); axis([0 1 0 1.2]);grid;
ylabel('Gain, in dB'); xlabel('\omega/\pi');
```

10.34 (a) $\omega_p = 0.35\pi = 1.09956$. The frequency spacing between 2 consecutive DFT samples is given by $\frac{2\pi}{39} = 0.1611$. The desired passband edge is between the frequency samples at $\omega = 2\pi\frac{6}{39}$ and $\omega = 2\pi\frac{7}{39}$. Therefore, the 39-point DFT is given by

$$H[k] = \begin{cases} e^{-j(2\pi/39)19k}, & k = 0, 1, \dots, 6, 32, 33, \dots, 38, \\ 0, & k = 7, \dots, 31. \end{cases}$$

A 39-point IDFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of z^{-1} :

Columns 1 through 10

| | | | | | |
|--------|--------|---------|---------|---------|---------|
| 0.0006 | 0.0031 | 0.0017 | -0.0054 | -0.0091 | -0.0010 |
| 0.0128 | 0.0146 | -0.0034 | -0.0237 | | |

Columns 11 through 20

| | | | | | |
|---------|--------|--------|--------|---------|---------|
| -0.0192 | 0.0134 | 0.0405 | 0.0227 | -0.0362 | -0.0753 |
| -0.0249 | 0.1222 | 0.2870 | 0.3590 | | |

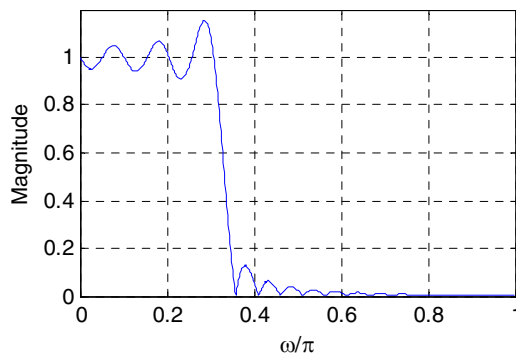
Columns 21 through 30

| | | | | | |
|--------|--------|---------|---------|---------|--------|
| 0.2870 | 0.1222 | -0.0249 | -0.0753 | -0.0362 | 0.0227 |
| 0.0405 | 0.0134 | -0.0192 | -0.0237 | | |

Columns 31 through 39

| | | | | | |
|---------|--------|--------|---------|---------|---------|
| -0.0034 | 0.0146 | 0.0128 | -0.0010 | -0.0091 | -0.0054 |
| 0.0017 | 0.0031 | 0.0006 | | | |

(b)



```
% Problem #10.34
close all;
clear;
clc;
ind = 1;
for k = 0:6,
```

```

H(ind) = exp(-i*2*pi*19*k/39);
ind = ind + 1;
end
for k = 7:31,
H(ind) = 0;
ind = ind + 1;
end
for k = 32:38,
H(ind) = exp(-i*2*pi*19*k/39);
ind = ind + 1;
end

h = ifft(H);
figure(1);
stem(real(h));
[FF, w] = freqz(h, 1, 512);
figure(2);
plot(w/pi, abs(FF)); axis([0 1 0 1.2]); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');

```

10.35 By expressing $\cos(\omega n) = T_n(\cos \omega)$, where $T_n(x)$ is the n -th order Chebyshev polynomial in x , we first rewrite Eq. (10.48) in the form:

$$\tilde{H}(\omega) = \sum_{n=0}^M a[n] \cos(\omega n) = \sum_{n=0}^M \alpha_n \cos^n(\omega).$$

Therefore, we can rewrite Eq. (10.70) repeated below for convenience

$$P(\omega_i) [H(\omega_i) - D(\omega_i)] = (-1)^i \varepsilon, \quad 1 \leq i \leq M+2,$$

in a matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_1) & \cdots & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \cdots & \cos^M(\omega_2) & -1/P(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \cdots & \cos^M(\omega_{M+1}) & (-1)^M / P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) & (-1)^{M+1} / P(\omega_{M+2}) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_M \\ \varepsilon \end{bmatrix} = \begin{bmatrix} D(\omega_1) \\ D(\omega_2) \\ \vdots \\ D(\omega_{M+1}) \\ D(\omega_{M+2}) \end{bmatrix}.$$

Note that the coefficients $\{\alpha_i\}$ are different from the coefficients $\{a[i]\}$ of Eq. (10.70). To

determine the expression of α_i we use Cramer's rule arriving at $\alpha_i = \frac{\Delta_i}{\Delta}$, where

$$\Delta = \det \begin{bmatrix} 1 & \cos(\omega_1) & \cdots & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \cdots & \cos^M(\omega_2) & -1/P(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \cdots & \cos^M(\omega_{M+1}) & (-1)^M / P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) & (-1)^{M+1} / P(\omega_{M+2}) \end{bmatrix}, \text{ and}$$

$$\Delta_\varepsilon = \begin{vmatrix} 1 & \cos(\omega_1) & \cdots & \cos^M(\omega_1) & D(\omega_1) \\ 1 & \cos(\omega_2) & \cdots & \cos^M(\omega_2) & D(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \cdots & \cos^M(\omega_{M+1}) & D(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) & D(\omega_{M+2}) \end{vmatrix} \dots$$

Expanding both determinants using the last column we get $\Delta_\varepsilon = \sum_{i=1}^{M+2} b_i D(\omega_{i+1})$ and

$$\Delta = \sum_{i=1}^{M+2} b_i \frac{(-1)^{i-1}}{P(\omega_i)}, \text{ where}$$

$$b_i = \det \begin{vmatrix} 1 & \cos(\omega_1) & \cos^2(\omega_1) & \cdots & \cos^M(\omega_1) \\ 1 & \cos(\omega_2) & \cos^2(\omega_2) & \cdots & \cos^M(\omega_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\omega_{i-1}) & \cos^2(\omega_{i-1}) & \cdots & \cos^M(\omega_{i-1}) \\ 1 & \cos(\omega_{i+1}) & \cos^2(\omega_{i+1}) & \cdots & \cos^M(\omega_{i+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\omega_{M+2}) & \cos^2(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) \end{vmatrix}.$$

The above matrix is seen to be a Vandermonde matrix and its determinant is given by

$$b_i = \prod_{\substack{k \neq \ell, k > \ell \\ k, \ell \neq i}} (\cos \omega_k - \cos \omega_\ell). \text{ Define } c_i = \frac{b_i}{\prod_{\substack{r=1 \\ r \neq i}}^{M+2} b_r}. \text{ It can be shown by induction that}$$

$$c_i = \prod_{\substack{n=1 \\ n \neq i}}^{M+2} \frac{1}{\cos \omega_i - \cos \omega_n}. \text{ Therefore,}$$

$$\varepsilon = \frac{\sum_{i=1}^{M+2} b_i D(\omega_i)}{\sum_{i=1}^{M+2} b_i \frac{(-1)^i}{P(\omega_i)}} = \frac{c_1 D(\omega_1) + c_2 D(\omega_2) + \cdots + c_{M+2} D(\omega_{M+2})}{\frac{c_1}{P(\omega_1)} - \frac{c_2}{P(\omega_2)} + \cdots + \frac{c_{M+2} (-1)^{M+1}}{P(\omega_{M+2})}}.$$

$$10.36 \quad W(\omega) = \begin{cases} 1 & \omega \in \text{passband,} \\ \frac{\delta_p}{\delta_s} & \omega \in \text{stopband.} \end{cases} \quad W(\omega) = \begin{cases} 1 & 0 \leq \omega \leq 0.45\pi, \\ 4.5 & 0.55\pi \leq \omega \leq \pi. \end{cases}$$

$$10.37 \quad W(\omega) = \begin{cases} 1 & \omega \in \text{passband}, \\ \frac{\delta_p}{\delta_s} & \omega \in \text{stopband}. \end{cases} \quad W(\omega) = \begin{cases} 1 & 0 \leq \omega \leq 0.55\pi, \\ 3.4 & 0.7\pi \leq \omega \leq \pi. \end{cases}$$

$$10.38 \quad W(\omega) = \begin{cases} \frac{\delta_p}{\delta_{s1}} & 0 \leq \omega \leq \omega_{s1}, \\ 1 & \omega_{p1} \leq \omega \leq \omega_{p2}, \\ \frac{\delta_p}{\delta_{s2}} & \omega_{s2} \leq \omega \leq \pi, \end{cases} = \begin{cases} 1.43 & 0 \leq \omega \leq 0.44\pi, \\ 1 & 0.55\pi \leq \omega \leq 0.7\pi, \\ 5 & 0.82\pi \leq \omega \leq \pi. \end{cases}$$

10.39 It follows from Eq. (10.22) that the impulse response of an ideal Hilbert transformer is an antisymmetric sequence. If we truncate it to a finite number of terms between $-M \leq n \leq M$ the impulse response is of length $(2M + 1)$ which is odd. Hence the FIR Hilbert transformer obtained by truncation and satisfying Eq. (10.90) cannot be satisfied by a Type 4 FIR filter.

10.40 (a) $X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}$. Thus,

$$X(\tilde{z}) = X(z) \Big|_{z^{-1} = \frac{-\alpha + \tilde{z}^{-1}}{1 - \alpha\tilde{z}^{-1}}} = \sum_{n=0}^{N-1} x[n] \left(\frac{-\alpha + \tilde{z}^{-1}}{1 - \alpha\tilde{z}^{-1}} \right) = \frac{P(\tilde{z})}{D(\tilde{z})}, \text{ where}$$

$$P(\tilde{z}) = \sum_{n=0}^{N-1} p[n]\tilde{z}^{-1} = \sum_{n=0}^{N-1} x[n](1 - \alpha\tilde{z}^{-1})^{N-1-n} (-\alpha + \tilde{z}^{-1})^n, \text{ and}$$

$$D(\tilde{z}) = \sum_{n=0}^{N-1} d[n]\tilde{z}^{-n} = (1 - \alpha\tilde{z}^{-1})^{N-1}.$$

(b) $\tilde{X}[k] = X(\tilde{z}) \Big|_{\tilde{z} = e^{j2\pi k/N}} = \frac{P(\tilde{z})}{D(\tilde{z})} \Big|_{\tilde{z} = e^{j2\pi k/N}} = \frac{\tilde{P}[k]}{\tilde{D}[k]}$, where $\tilde{P}[k] = P(\tilde{z}) \Big|_{\tilde{z} = e^{j2\pi k/N}}$ is

the N -point DFT of the sequence $p[n]$ and $\tilde{D}[k] = D(\tilde{z}) \Big|_{\tilde{z} = e^{j2\pi k/N}}$ is the N -point DFT of the sequence $d[n]$.

(c) Let $\mathbf{P} = [p[0] \ p[1] \ \cdots \ p[N-1]]^T$ and $\mathbf{X} = [x[0] \ x[1] \ \cdots \ x[N-1]]^T$. Without

any loss of generality, assume $N = 4$ in which case $P(\tilde{z}) = \sum_{n=0}^3 p[n]\tilde{z}^{-n}$

$$\begin{aligned} &= (x[0] - \alpha x[1] + \alpha^2 x[2] - \alpha^3 x[3]) \\ &+ (-3\alpha x[0] + (1 + 2\alpha^2)x[1] - \alpha(2 + \alpha^2)x[2] + 3\alpha^2 x[3])\tilde{z}^{-1} \\ &+ (3\alpha^2 x[0] - \alpha(2 + \alpha^2)x[1] + (1 + 2\alpha^2)x[2] - 3\alpha x[3])\tilde{z}^{-2} \end{aligned}$$

$+ (-\alpha^3 x[0] + \alpha^2 x[1] - \alpha x[2] + \alpha x[3]) \bar{z}^{-3}$. Equating like powers of \bar{z}^{-1} we can write

$\mathbf{P} = \mathbf{Q} \cdot \mathbf{X}$, where $\mathbf{P} = [p[0] \ p[1] \ p[2] \ p[3]]^T$, $\mathbf{X} = [x[0] \ x[1] \ x[2] \ x[3]]^T$ and

$$\mathbf{Q} = \begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 \\ -3\alpha & 1+2\alpha^2 & -\alpha(2+\alpha^2) & 3\alpha^2 \\ 3\alpha^2 & -\alpha(2+\alpha^2) & 1+2\alpha^2 & -3\alpha \\ -\alpha^3 & \alpha^2 & -\alpha^2 & 1 \end{bmatrix}.$$

It can be seen that the elements $q_{r,s}$, $0 \leq r, s \leq 3$, $q_{r,s}$, of the 4 4 matrix \mathbf{Q} can be determined as follows:

(i) The first row is given by $q_{0,s} = (-\alpha)^s$,

(ii) The first column is given by $q_{r,0} = {}^3C_r (-\alpha)^r = \frac{3!}{r!(3-r)!} (-\alpha)^r$, and

(iii) the remaining elements can be obtained using the recurrence relation

$$q_{r,s} = q_{r-1,s-1} - \alpha q_{r,s-1} + \alpha q_{r-1,s}.$$

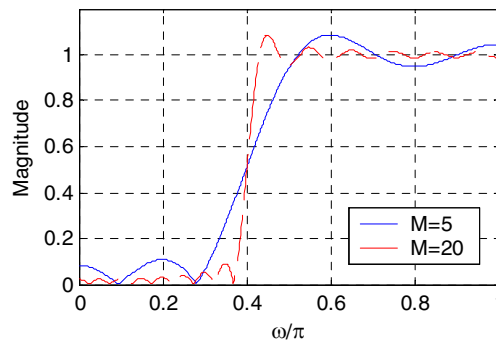
In the general case, we only change the computation of the elements of the first column

using the relation $q_{r,0} = {}^{N-1}C_r (-\alpha)^r = \frac{(N-1)!}{r!(N-1-r)!} (-\alpha)^r$.

M10.1 The impulse response coefficients of the truncated FIR highpass filter with cutoff frequency at 0.4π can be generated using the following MATLAB statements:

```
n = -M:M;
num = -0.4*sinc(0.4*n);
num(M+1) = 0.6;
```

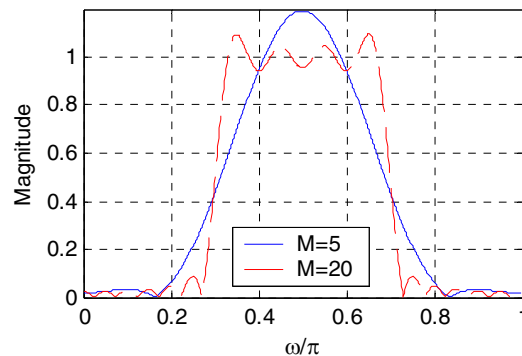
The magnitude responses of the truncated FIR highpass filter for two values of M are shown below:



M10.2 The impulse response coefficients of the truncated FIR bandpass filter with cutoff frequencies at 0.7π and 0.3π can be generated using the following MATLAB statements:

```
n = -M:M;
num = 0.7*sinc(0.7*n) - 0.3*sinc(0.3*n);
```

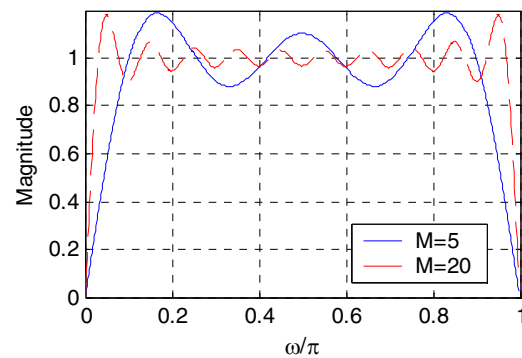
The magnitude responses of the truncated FIR bandpass filter for two values of M are shown below:



M10.3 The impulse response coefficients of the truncated Hilbert transformer can be generated using the following MATLAB statements:

```
n = 1:M;
c = 2*sin(pi*n/2).*sin(pi*n/2); b = c./(pi*n);
num = [-fliplr(b) 0 b];
```

The magnitude responses of the truncated Hilbert transformer for two values of M are shown below:



M10.4

```
% Problem #M10.4
% Cascade of 2 boxcar filters of length 4
K = 2; N = 4; b = firgauss(K,N);
figure(1);
stem(b); xlabel('n'); ylabel('h[n]');
title('Cascade of 2 boxcar filters of length 4');
```



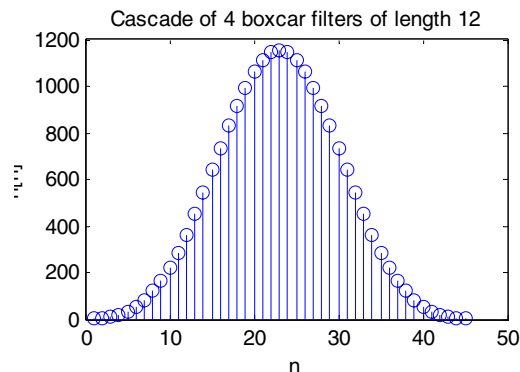
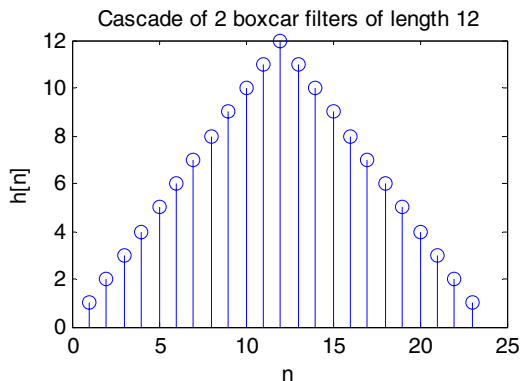
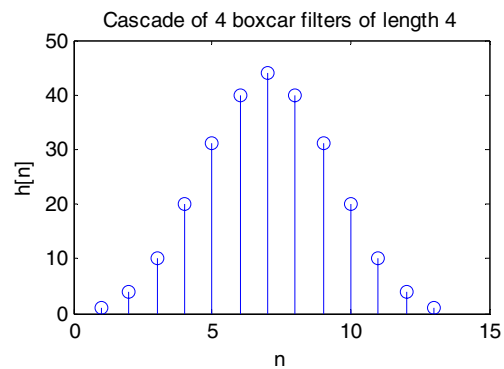
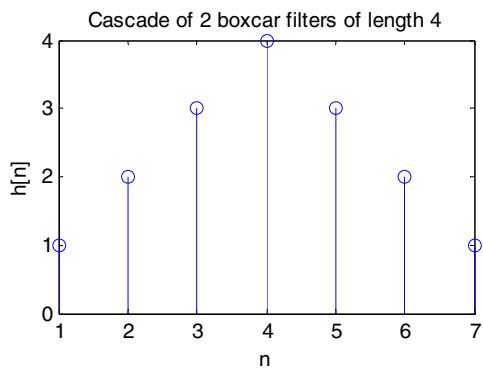
```

% Cascade of 4 boxcar filters of length 4
K = 4;N = 4;b = firgauss(K,N);
figure(2);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 4 boxcar filters of length 4');

% Cascade of 2 boxcar filters of length 12
K = 2;N = 12;b = firgauss(K,N);
figure(3);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 2 boxcar filters of length 12');

% Cascade of 4 boxcar filters of length 12
K = 4;N = 12;b = firgauss(K,N);
figure(4);
stem(b);xlabel('n');ylabel('h[n]');
title('Cascade of 4 boxcar filters of length 12');

```



We can see that by increasing either K or N , the approximation to a Gaussian function gets better. It is noted that increasing the number of boxcar filters K to a number greater than 3 greatly affects the Gaussian shape of the impulse response.

```

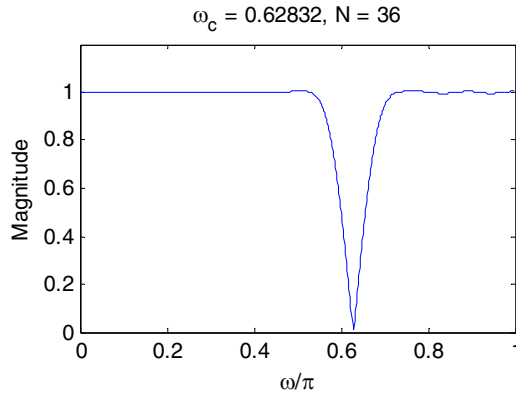
M10.5 % Problem #M10.05
N = 36;fc = 0.2*pi;
M = N/2;
n = -M:1:M;t = fc*n;
lp = fc*sinc(t);

```

```

b = 2*[lp(1:M) (lp(M+1) - 0.5) lp((M+2):N+1)];
bw = b.*hamming(N+1)';
[h2, w] = freqz(bw, 1, 512);
plot(w/pi, abs(h2));axis([0 1 0 1.2]);
xlabel('\omega/\pi');ylabel('Magnitude');
title(['\omega_c = ', num2str(fc), ', N = ', num2str(N)]);

```

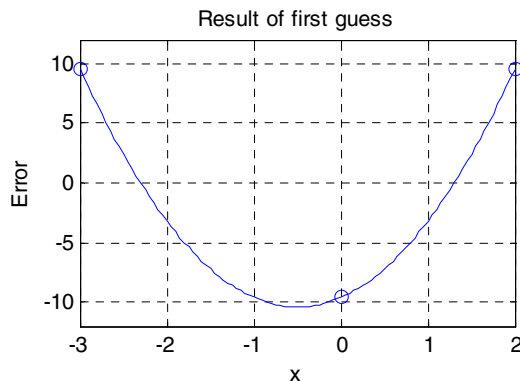


M10.6 $D(x) = 3.2x^2 + 4.05x - 5.5$. Its approximation over the range $-3 \leq x \leq 2$ is given by $A(x) = a_0 + a_1x$. We want to minimize the peak value of the absolute error, i.e.,

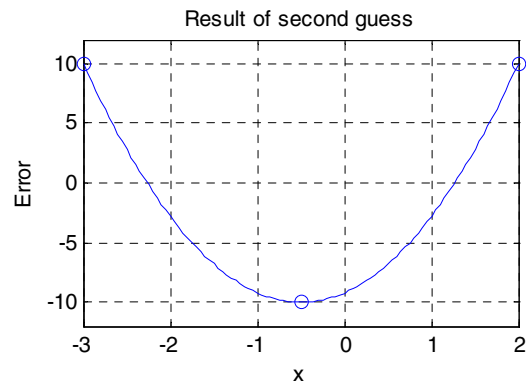
minimize $\max_{-3 \leq x \leq 2} |3.2x^2 + 4.05x - 5.5 - a_0 - a_1x|$. Since there are 3 unknowns, a_0, a_1 , and ε , we need 3 extremal points on x , which we arbitrarily choose as $x_1 = -3$, $x_2 = 0$, and $x_3 = 2$. We then solve the 3 linear equations: $a_0 + a_1x - (-1)^\ell \varepsilon = D(x_\ell)$, $\ell = 1, 2, 3$.

This leads to
$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 11.15 \\ -5.5 \\ 15.4 \end{bmatrix}$$
, whose solution yields $a_0 = 4.1, a_1 = 0.85$,

and $\varepsilon = 9.6$. A plot of the corresponding error $\mathcal{E}_1(x) = 3.2x^2 + 3.2x - 9.6$ is shown below in Figure (c).



(c)



(d)

After looking at $\mathcal{E}_1(x)$, we move the second extremal point x_2 to the location where $\mathcal{E}_1(x)$ is a minimum. The next extremal points are therefore given by $x_1 = -3$, $x_2 = -0.5$, and $x_3 = 2$. The new values of the unknowns are obtained by solving

$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & -0.5 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 11.15 \\ -6.725 \\ 15.4 \end{bmatrix}, \text{ which yields } a_0 = 3.7, a_1 = 0.85, \text{ and } \varepsilon = 10. \text{ A plot}$$

of the corresponding error $\mathcal{E}_2(x) = 3.2x^2 + 3.2x - 9.2$ is shown on the previous page in Figure (d).

```
% Problem #M10.06
x = [-3 0 2];
d = 3.2.*x.^2 + 4.05.*x-5.5;
D = d';
A = [1 -3 1;1 0 -1;1 2 1];
C = inv(A)*D;

y = -3:0.05:2;
E = 3.2.*y.^2 + 4.05.*y - 5.5 - C(1) - C(2).*y;

% Results of first guess
figure(1);
plot(y,E);
axis([-3 2 -12 12]);
xlabel('x');
ylabel('Error');
title('Result of first guess');
hold on;
plot([-3 -3], [E(1) E(1)], 'o');
plot([2 2], [E(end) E(end)], 'o');
plot([0 0], [E(61) E(61)], 'o');
hold off;

x = [-3 -0.5 2];
d = 3.2.*x.^2 + 4.05.*x-5.5;
D = d';
A = [1 -3 1;1 -0.5 -1;1 2 1];
C = inv(A)*D;

y = -3:0.05:2;
E = 3.2.*y.^2 + 4.05.*y - 5.5 - C(1) - C(2).*y;

% Results of second guess
figure(2);
plot(y,E);
axis([-3 2 -12 12]);
xlabel('x');
ylabel('Error');
```

```

title('Result of second guess');
hold on;
plot([-3 -3], [E(1) E(1)], 'o');
plot([2 2], [E(end) E(end)], 'o');
plot([-0.5 -0.5], [E(51) E(51)], 'o');
hold off;

```

M10.7 $D(x) = -5x^3 - 0.2x^2 + 8x + 5.5$. Its approximation over the range $-2 \leq x \leq 2$ is given by

$A(x) = a_0 + a_1x + a_2x^2$. We want to minimize the peak value of the absolute error, i.e.,

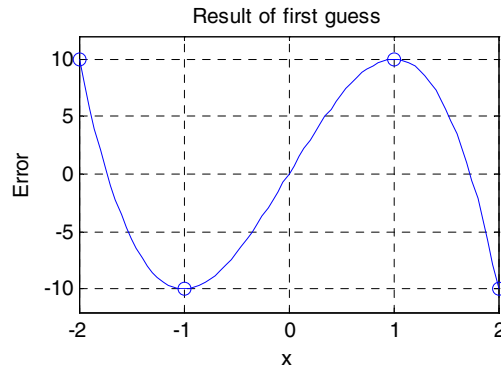
minimize $\max_{-2 \leq x \leq 2} \left| -5x^3 - 0.2x^2 + 8x + 5.5 - a_0 - a_1x - a_2x^2 \right|$. Since there are 4

unknowns, a_0, a_1, a_2 , and ε , we need 4 extremal points on x , which we arbitrarily choose as $x_1 = -2, x_2 = -1, x_3 = 1, \text{ and } x_4 = 2$. We then solve the 4 linear equations:

$a_0 + a_1x + a_2x^2 - (-1)^\ell \varepsilon = D(x_\ell), \ell = 1, 2, 3, 4$. This leads to

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 28.7 \\ 2.3 \\ 8.3 \\ -19.3 \end{bmatrix}, \text{ whose solution yields } a_0 = 5.5, a_1 = -7, a_2 = -0.2,$$

and $\varepsilon = 10$. A plot of the corresponding error $\mathcal{E}_1(x) = -5x^3 + 15x$ is shown below in Figure (e). We observe that these values maximize the error ($\varepsilon = 10$).



```

% Problem #M10.07
x = [-2 -1 1 2];
d = -5.*x.^3 - 0.2.*x.^2 + 8.*x + 5.5;
D = d';
A = [1 -2 4 1; 1 -1 1 -1; 1 1 1 1; 1 2 4 -1];
C = inv(A)*D;

y = -2:0.05:2;
E = -5.*y.^3 - 0.2.*y.^2 + 8.*y + 5.5 - C(1) - C(2).*y -
C(3).*y.^2;

% Results of first guess
figure(1);

```

```

plot(y,E);
axis([-2 2 -12 12]);
xlabel('x');
ylabel('Error');
title('Result of first guess');
hold on;
plot([-2 -2], [E(1) E(1)], 'o');
plot([2 2], [E(end) E(end)], 'o');
plot([-1 -1], [E(21) E(21)], 'o');
plot([1 1], [E(1) E(1)], 'o');
hold off;

```

M10.8 $\omega_p = \frac{8\pi}{18}, \omega_s = \frac{12\pi}{18}, \omega_T = \frac{10\pi}{18}$

```

% Problem #M10.8
wp = 4*(2*pi)/18;
ws = 6*(2*pi)/18;
wc = (wp + ws)/2;
dw = ws - wp;
% Hamming
M = ceil(3.32*pi/dw);N = 2*M+1;n = -M:M;
num = (6/18)*sinc(6*n/18);
wh = hamming(N)';b = num.*wh;

figure(1);
k=0:2*M:stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude');

figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h))); grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Lowpass filter designed using Hamming window');
axis([0 1 -80 10]);

% Hann
M = ceil(3.11*pi/dw);N = 2*M+1;n = -M:M;
num = (6/18)*sinc(6*n/18);
wh = hann(N)';b = num.*wh;

figure(3);
k=0:2*M:stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude');

figure(4);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Gain, in dB');

```

```

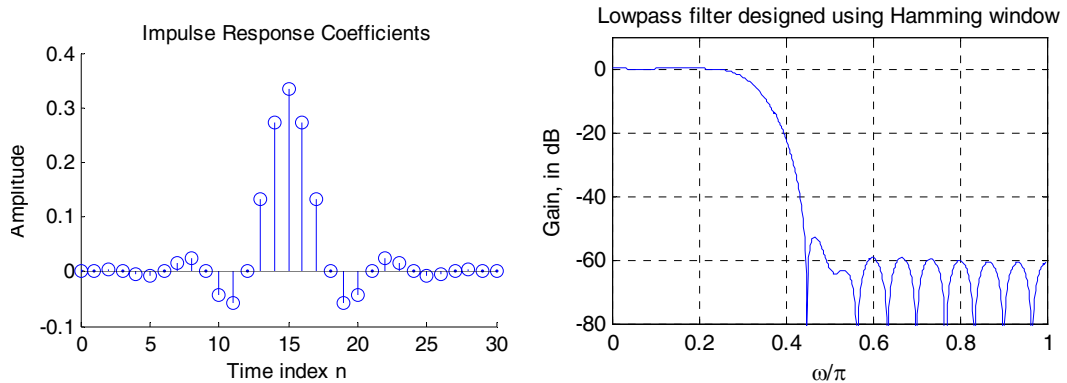
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

% Blackman
M = ceil(5.56*pi/dw);N = 2*M+1;n = -M:M;
num = (6/18)*sinc(6*n/18);
wh = blackman(N)';b = num.*wh;

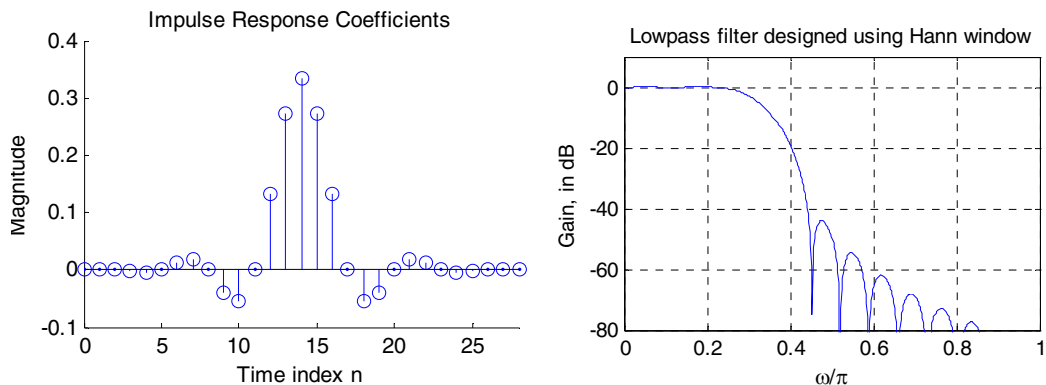
figure(5);
k=0:2*M:stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude');
figure(6);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Gain, in dB');
title('Lowpass filter designed using Blackman window');
axis([0 1 -80 10]);

```

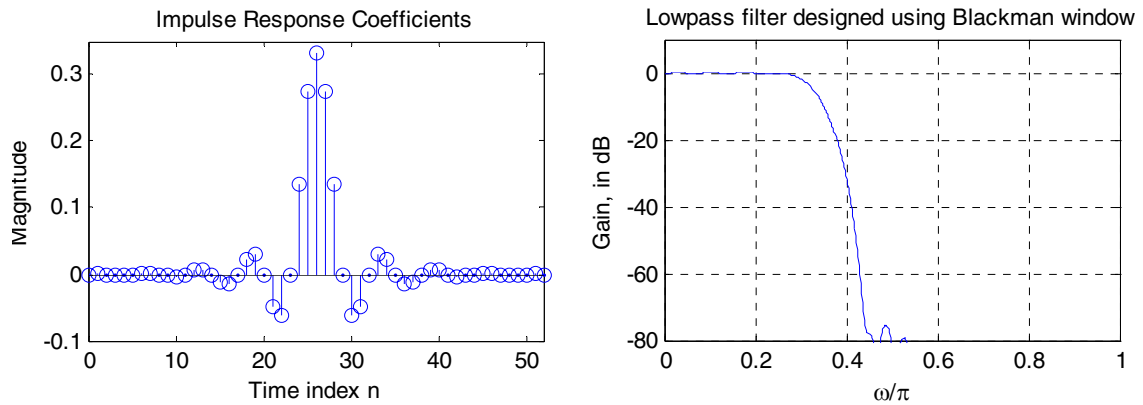
Lowpass filter design using Hamming window: $N = 31$



Lowpass filter design using Hann window: $N = 29$



Lowpass filter design using Blackman window: $N = 53$



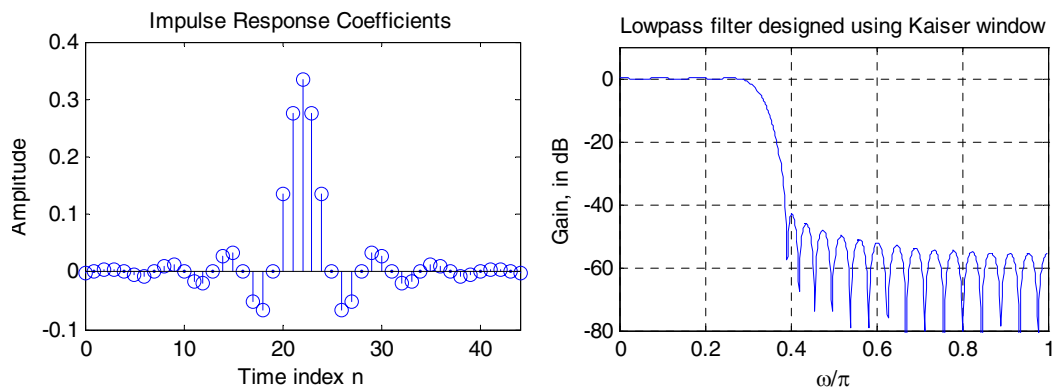
Comments: The Hann window method results in using the lowest filter order. All filters meet the requirements of the specifications.

M10.9 $\alpha_s = 42, \beta = 0.5842(42 - 21)^{0.4} + 0.07886(42 - 21) = 3.631$ using Eq. (10.41).

$$N = \frac{42 - 8}{2.285 \left(\frac{2\pi}{18} \right)} \text{ using Eq. (10.42).}$$

$N = 42.627 \geq 43$ and we choose 44 since N must be even. $M = 22$.

```
% Problem #M10.9
beta = 3.631; N = 44; n = -N/2:N/2;
num = (6/18)*sinc(6*n/18);
wh = kaiser(N+1,beta)'; b = num.*wh;
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n'); ylabel('Amplitude')
figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h))); grid;
xlabel('\omega/\pi'); ylabel('Gain, in dB');
title('Lowpass filter designed using Kaiser window');
axis([0 1 -80 10]);
```



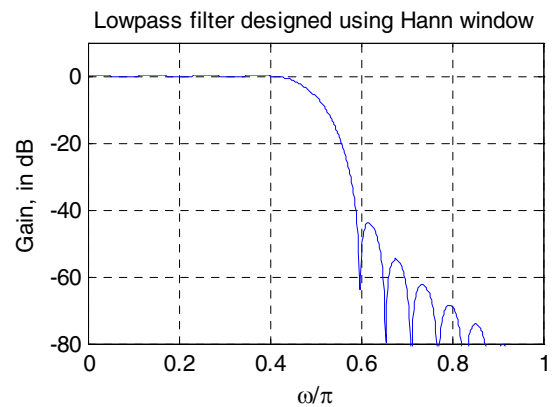
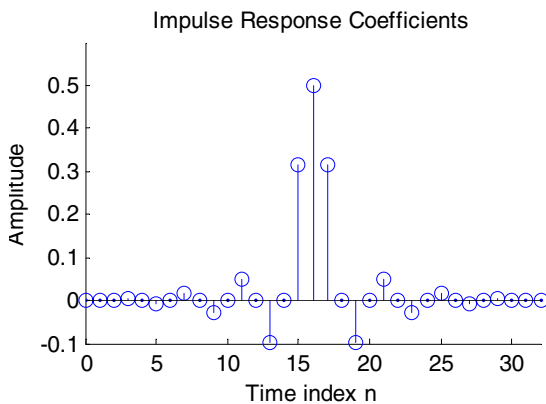
M10.10 $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $\alpha_s = 42$ dB, $\omega_c = 0.5\pi$, $\Delta\omega = 0.2\pi$

We will use the Hann window since it meets the requirements and has the lowest order from Table 10.2.

$$M = \frac{3.11\pi}{\Delta\omega} = 15.55 \rightarrow 16 \Rightarrow N = 32.$$

`% Problem M10.10`

```
n = -16:16;
lp = 0.5*sinc(0.5*n);wh = hanning(33);
b = lp.*wh';
figure(1);
k=0:2*n;stem(k,b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('Amplitude');
figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Gain, in dB');
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);
```



M10.11 We use the same specifications from Problem **M10.10**, but we use the Dolph-Chebyshev

window. $N = \frac{2.056(42) - 16.4}{2.285(0.2\pi)} = 48.7$. We use $N = 50$, which is a much higher order

than in Problem **M10.10**.

`% Problem M10.11`

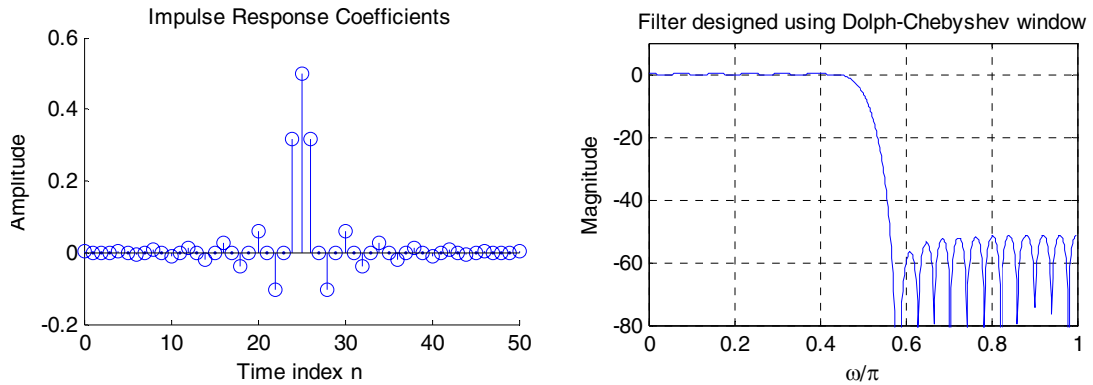
```
n = -25:25;
lp = 0.5*sinc(0.5*n);wh = chebwin(51,42);
b = lp.*wh';
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('Amplitude');
figure(2);
[h, w] = freqz(b,1,512);
```



```

plot(w/pi, 20*log10(abs(h)));grid;
xlabel('\omega/\pi');ylabel('Magnititude');
title('Filter designed using Dolph-Chebyshev window');
axis([0 1 -80 10]);

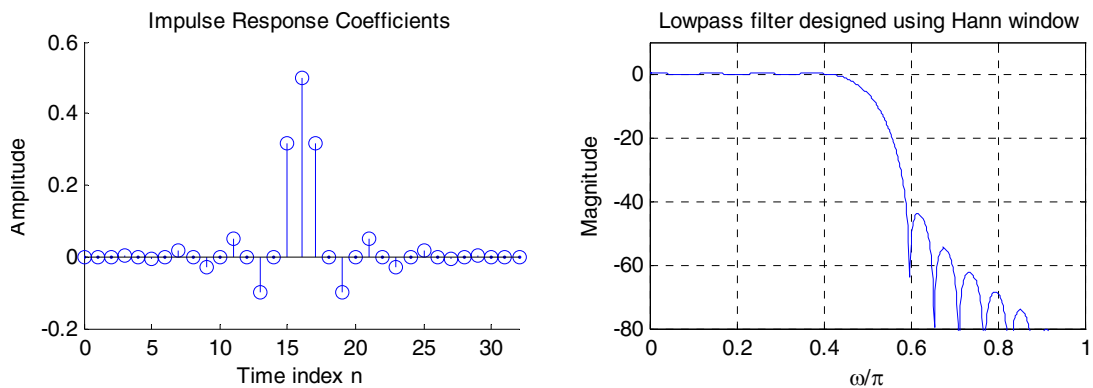
```



```

M10.12 n = -16:16;
b = fir1(32, 0.5, hanning(33));
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('Amplitude');
figure(2);
[h, w] = freqz(b,1,512);
plot(w/pi, 20*log10(abs(h)));
grid;
xlabel('\omega/\pi');ylabel('Magnititude');
title(' Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

```



```

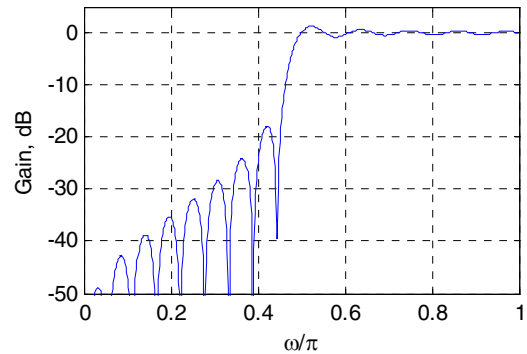
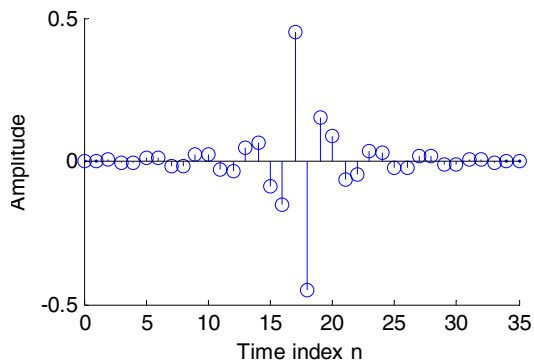
M10.13 N = 35;
for k = 1:N+1,
w(k) = 2*pi*(k-1)/(N+1);
if(w(k) >= 0.45*pi & w(k) <= 1.45*pi) H(k) = 1;
else H(k) = 0;
end
if (w(k) <= pi) phase(k) = i*exp(-i*w(k)*N/2);
else phase(k) = -i*exp(i*(2*pi-w(k))*N/2);

```

```

    end
end
H = H.*phase;
f = ifft(H);
[FF, w] = freqz(f, 1, 512);
k = 0:N;
figure(1);
stem(k, real(f));
xlabel('Time index n');ylabel('Amplitude');
figure(2);
plot(w/pi, 20*log10(abs(FF)));grid
xlabel('\omega/\pi');ylabel('Gain, dB');
axis([0 1 -50 5]);

```

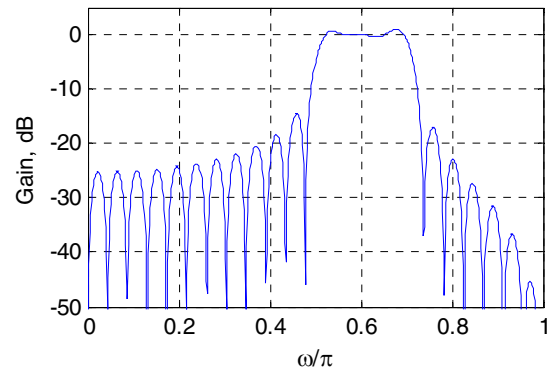
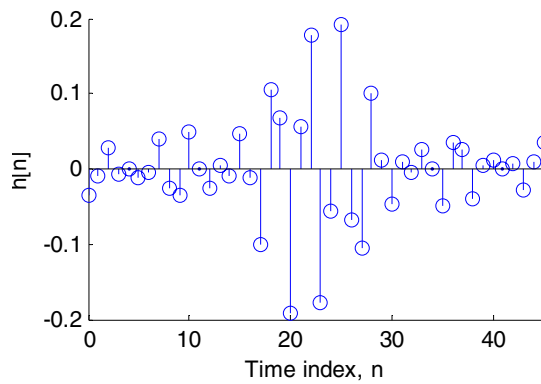


M10.14 $N = 45$;

```

L = N + 1;
for k = 1:L,
    w = 2*pi*(k-1)/L;
    if (w >= 0.5*pi & w <= 0.7*pi) H(k) = i*exp(-i*w*N/2);
    elseif (w >= 1.3*pi & w <= 1.5*pi) H(k) = -
i*exp(i*(2*pi-w)*N/2);
    else H(k) = 0;
    end
end
end
f = ifft(H);
[FF, w] = freqz(f, 1, 512);
k = 0:N;
figure(1);
stem(k, real(f));
xlabel('Time index, n');ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(FF)));grid;
ylabel('Gain, dB');xlabel('\omega/\pi');
axis([0 1 -50 5]);

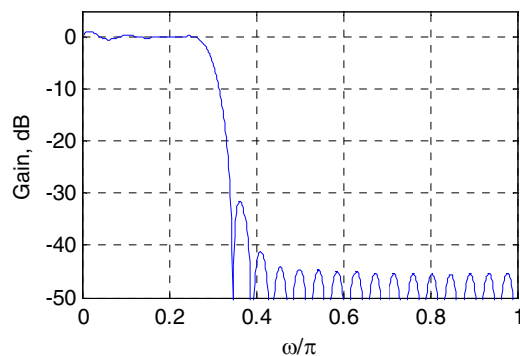
```



```

M10.15 ind = 1;
        for k = 0:6,
            H(ind) = exp(-i*2*pi*19*k/39);
            ind = ind + 1;
        end
        k = 7;
        H(ind) = 0.5*exp(-i*2*pi*19*k/39);
        ind = ind + 1;
        for k = 8:30,
            H(ind) = 0;
            ind = ind + 1;
        end
        k = 31;
        H(ind) = 0.5*exp(-i*2*pi*19*k/39);
        ind = ind + 1;
        for k = 32:38,
            H(ind) = exp(-i*2*pi*19*k/39);
            ind = ind + 1;
        end
        h = ifft(H);
        [FF, w] = freqz(h, 1, 512);
        plot(w/pi, 20*log10(abs(FF)));grid;
        xlabel('\omega/\pi');ylabel('Gain, dB');
        axis([0 1 -50 5]);

```



```

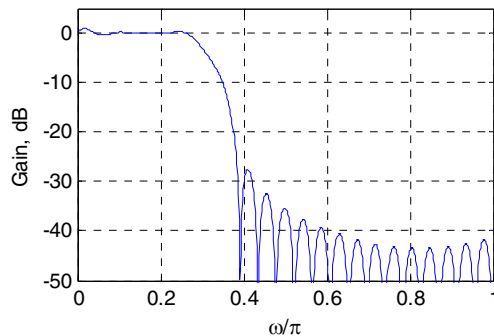
M10.16 ind = 1;

```

```

for k = 0:6,
    H(ind) = exp(-i*2*pi*19*k/39);
    ind = ind + 1;
end
k = 7;
H(ind) = (2/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
k = 8;
H(ind) = (1/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
for k = 9:29,
    H(ind) = 0;
    ind = ind + 1;
end
k = 30;
H(ind) = (1/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
k = 31;
H(ind) = (2/3)*exp(-i*2*pi*19*k/39);
ind = ind + 1;
for k = 32:38,
    H(ind) = exp(-i*2*pi*19*k/39);
    ind = ind + 1;
end
h = ifft(H);
[FF, w] = freqz(h, 1, 512);
plot(w/pi, 20*log10(abs(FF)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
axis([0 1 -50 5]);

```



M10.17 $\omega_p = \frac{8\pi}{18}, \omega_s = \frac{12\pi}{18}, \omega_c = \frac{10\pi}{18}, \Delta\omega = \frac{4\pi}{18}$

```

wp = 4*(2*pi)/18;
ws = 6*(2*pi)/18;
wc = (wp + ws)/2;
dw = ws - wp;
% Hamming
M = ceil(3.32*pi/dw);
N = 2*M;
b = fir1(N, ws/(2*pi));

```

```

[H, w] = freqz(b,1,512);
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Lowpass filter designed using Hamming window');
axis([0 1 -80 10]);

% Hann
M = ceil(3.11*pi/dw);
N = 2*M;
b = fir1(N, ws/(2*pi), hanning(N+1));
[H, w] = freqz(b,1,512);
figure(3);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(4);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Lowpass filter designed using Hann window');
axis([0 1 -80 10]);

% Blackman
M = ceil(5.56*pi/dw);
N = 2*M;
b = fir1(N, ws/(2*pi), blackman(N+1));
[H, w] = freqz(b,1,512);

figure(5);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(6);
plot(w/pi, 20*log10(abs(H)));
grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Lowpass filter designed using Blackman window');
axis([0 1 -80 10]);

% Kaiser
beta = 3.631;
N = 44;
b = fir1(N, ws/(2*pi), kaiser(N+1, beta));
[H, w] = freqz(b,1,512);
figure(7);
stem(b);
title('Impulse Response Coefficients');

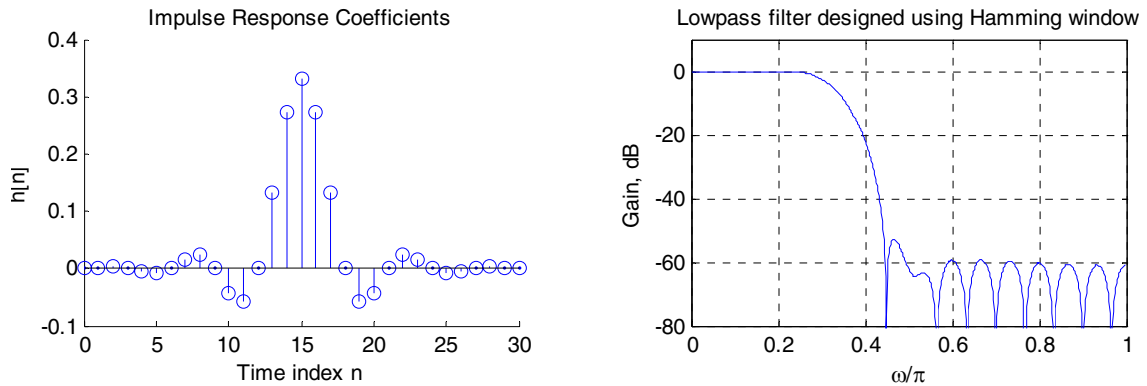
```

```

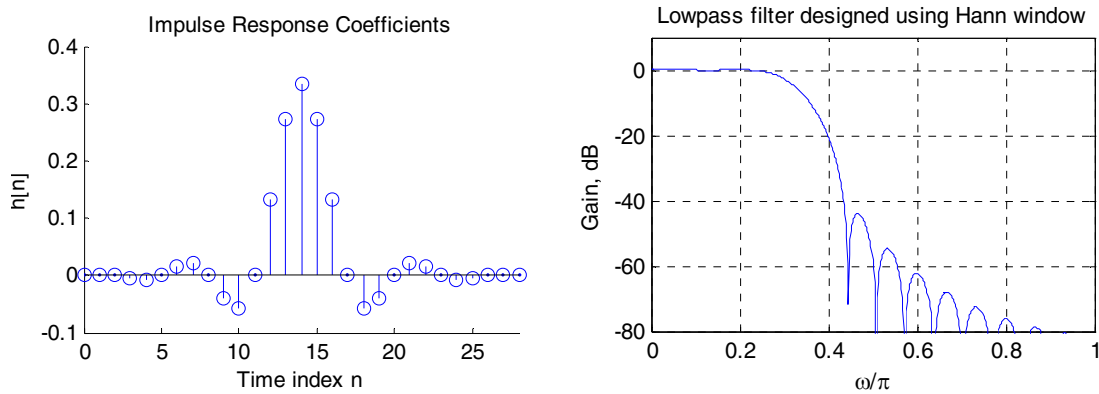
xlabel('Time index n');ylabel('h[n]');
figure(8);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Lowpass filter designed using Kaiser window');
axis([0 1 -80 10]);

```

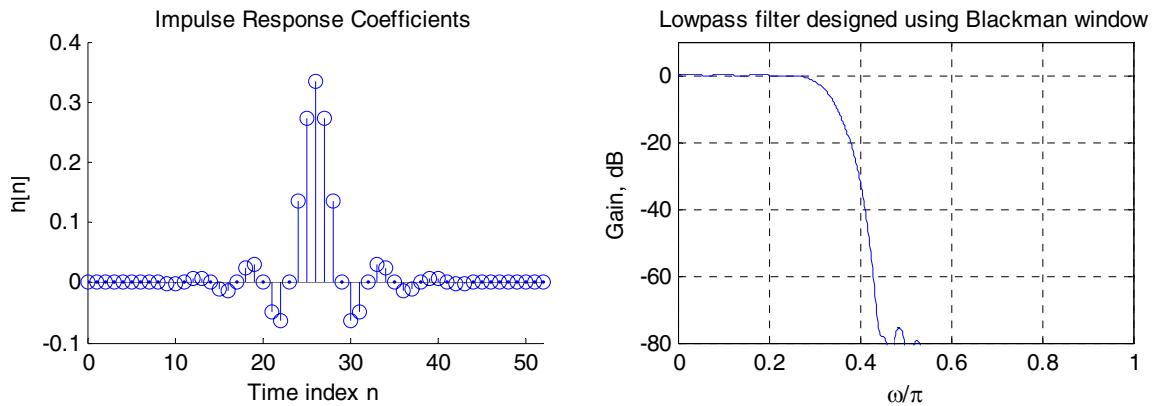
(a) Hamming window using fir1



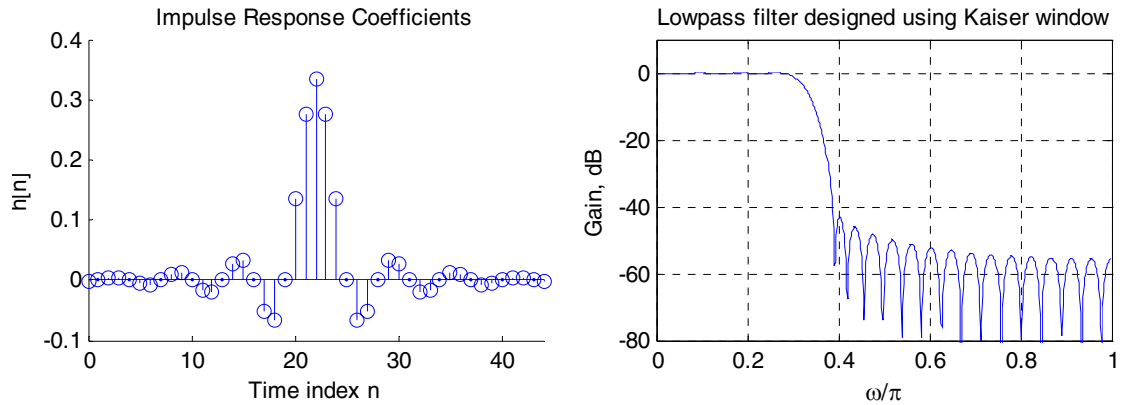
(b) Hann window using fir1



(c) Blackman window using fir1



(d) Kaiser window using fir1



M10.18 $\omega_s = 0.4\pi$, $\omega_p = 0.55\pi$, $\alpha_p = 0.1$ dB, $\alpha_s = 42$ dB, $\Delta\omega = 0.15\pi$, $\omega_c = 0.475\pi$

(a) Hamming: use Eq. (10.33): $M = \frac{3.32\pi}{0.15\pi} = 22.133 \rightarrow 23 \therefore N = 2M = 46$

(b) Hann: $M = \frac{3.11\pi}{0.15\pi} = 20.733 \rightarrow 21 \therefore N = 2M = 42$

(c) Blackman: $M = \frac{5.56\pi}{0.15\pi} = 37.067 \rightarrow 38 \therefore N = 2M = 76$

(d) Kaiser: $\delta_s = 10^{-\alpha_s/20} = 0.00794$

```
% Hamming
N = 46;
b = fir1(N, 0.475, 'high');
[H,w] = freqz(b,1,512);
figure(1);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(2);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Highpass filter designed using Hamming window');
axis([0 1 -80 10]);
```

```
% Hann
N = 42;
b = fir1(N, 0.475, 'high', hanning(N+1));
[H,w] = freqz(b,1,512);
figure(3);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(4);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Highpass filter designed using Hann window');
axis([0 1 -80 10]);
```

```

% Blackman
N = 76;
b = fir1(N, 0.475, 'high', blackman(N+1));
[H,w] = freqz(b,1,512);
figure(5);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');

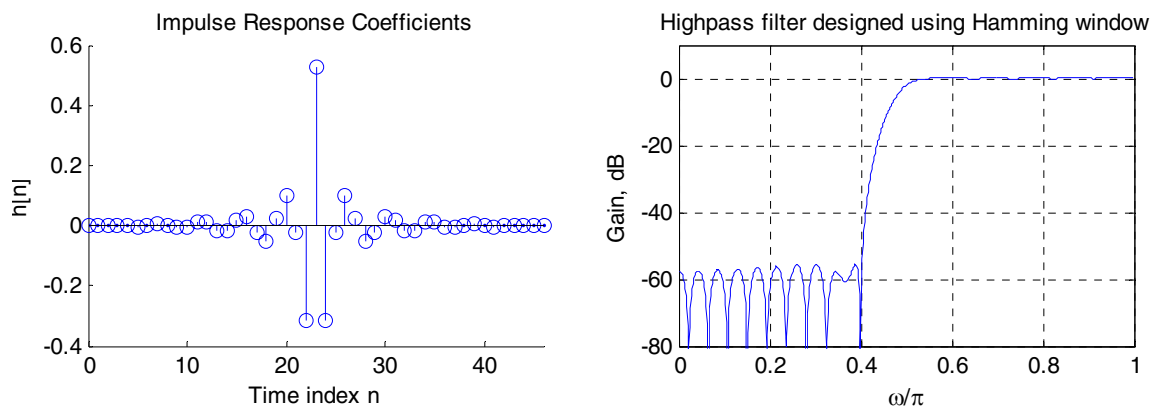
figure(6);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Highpass filter designed using Blackman window');
axis([0 1 -80 10]);

% Kaiser
ds = 0.00794;
[N,Wn,beta,type] = kaiserord([0.4 0.55],[1 0],[ds ds]);
b = fir1(N, 0.475, 'high',kaiser(N+1,beta));
[H,w] = freqz(b,1,512);

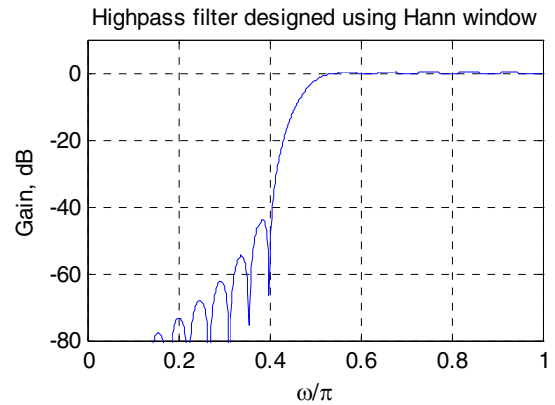
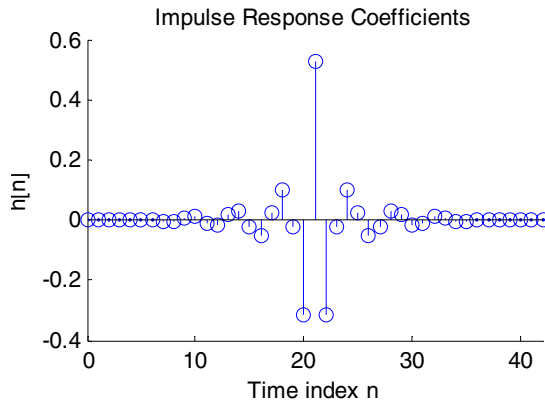
figure(7);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');
figure(8);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Highpass filter designed using Kaiser window');
axis([0 1 -80 10]);

```

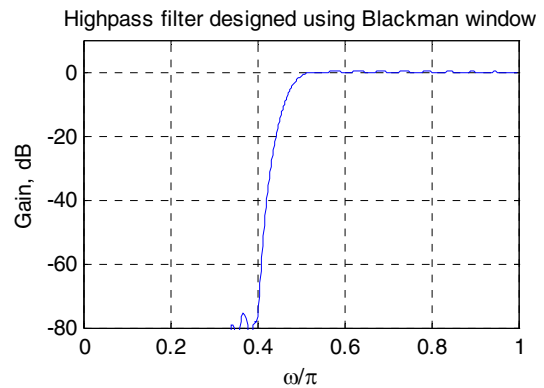
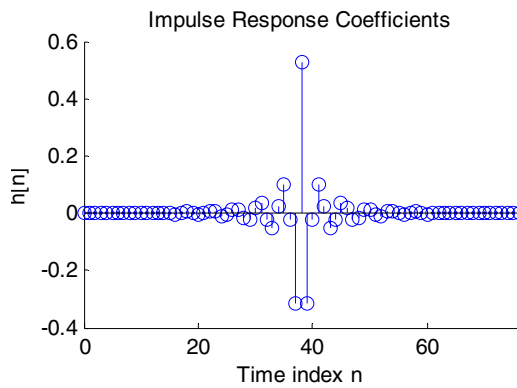
(a) Hamming window using fir1



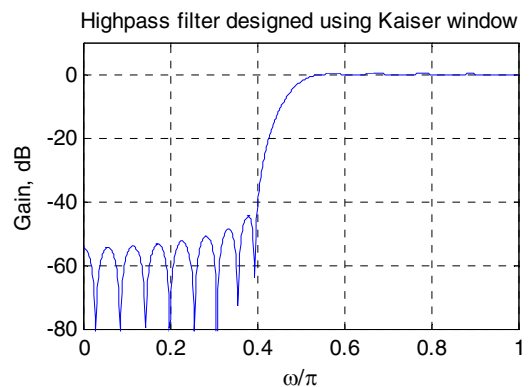
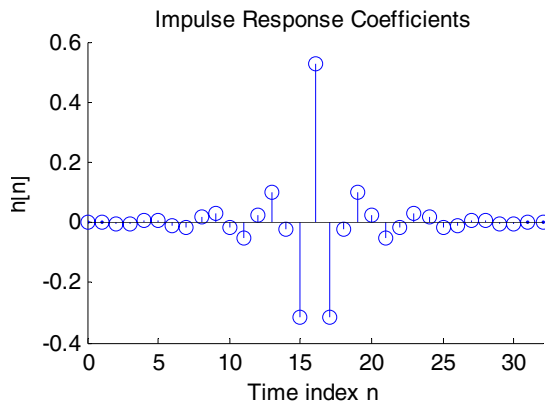
(b) Hann window using fir1



(c) Blackman window using `fir1`



(d) Kaiser window using `fir1`



M10.19 $\omega_{p1} = 0.65\pi$, $\omega_{p2} = 0.85\pi$, $\omega_{s1} = 0.55\pi$, $\omega_{s2} = 0.75\pi$, $\alpha_p = 0.2$ dB, $\alpha_s = 42$ dB

$$\Delta\omega_1 = \omega_{p1} - \omega_{s1} = 0.1\pi, \quad \Delta\omega_2 = \omega_{p2} - \omega_{s2} = 0.1\pi = \Delta\omega$$

(a) Hamming window: $M = \frac{3.32\pi}{0.1\pi} = 33.2 \rightarrow 34 \therefore N = 2M = 68$

(b) Hann: $M = \frac{3.11\pi}{0.1\pi} = 31.1 \rightarrow 32 \therefore N = 2M = 64$

(c) Blackman: $M = \frac{5.56\pi}{0.1\pi} = 55.6 \rightarrow 56 \therefore N = 2M = 112$

(d) Kaiser: $\delta_s = 10^{-\alpha_s/20} = 0.00794, \delta_p = 10^{-\alpha_p/20} = 0.97724$

```
% Problem #M10.19
```

```
% Hamming
```

```
N = 68;
```

```
b = fir1(N, [0.6 0.8]);
```

```
[H, w] = freqz(b,1,512);
```

```
figure(1);
```

```
stem(b);
```

```
title('Impulse Response Coefficients');
```

```
xlabel('Time index n');ylabel('h[n]');
```

```
figure(2);
```

```
plot(w/pi, 20*log10(abs(H)));grid;
```

```
xlabel('\omega/\pi');ylabel('Gain, dB');
```

```
title('Bandpass filter designed using Hamming window');
```

```
axis([0 1 -80 10]);
```

```
% Hann
```

```
N = 64;
```

```
b = fir1(N, [0.6 0.8], hanning(N+1));
```

```
[H, w] = freqz(b,1,512);
```

```
figure(3);
```

```
stem(b);
```

```
title('Impulse Response Coefficients');
```

```
xlabel('Time index n');ylabel('h[n]');
```

```
figure(4);
```

```
plot(w/pi, 20*log10(abs(H)));grid;
```

```
xlabel('\omega/\pi');ylabel('Gain, dB');
```

```
title('Bandpass filter designed using Hann window');
```

```
axis([0 1 -80 10]);
```

```
% Blackman
```

```
N = 112;
```

```
b = fir1(N, [0.6 0.8], blackman(N+1));
```

```
[H, w] = freqz(b,1,512);
```

```
figure(5);
```

```
stem(b);
```

```
title('Impulse Response Coefficients');
```

```
xlabel('Time index n');ylabel('h[n]');
```

```
figure(6);
```

```
plot(w/pi, 20*log10(abs(H)));grid;
```

```
xlabel('\omega/\pi');ylabel('Gain, dB');
```

```
title('Bandpass filter designed using Blackman window');
```

```
axis([0 1 -80 10]);
```

```
% Kaiser
```

```
[N, Wn, beta, type] = kaiserord([0.6 0.8], [1 0], [0.97724  
0.00794]);
```

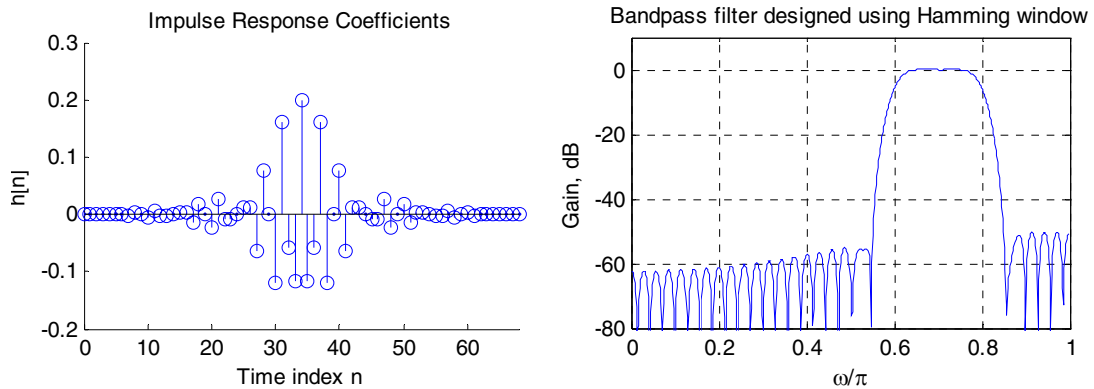
```

b = fir1(2*N, [0.6 0.8], kaiser(2*N+1, beta));
[H, w] = freqz(b,1,512);
figure(7);
stem(b);
title('Impulse Response Coefficients');
xlabel('Time index n');ylabel('h[n]');

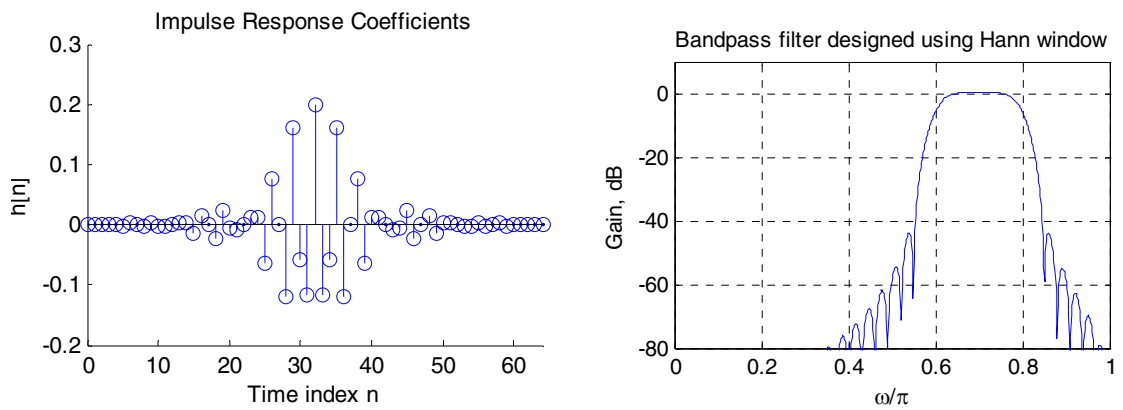
figure(8);
plot(w/pi, 20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Bandpass filter designed using Kaiser window');
axis([0 1 -80 10]);

```

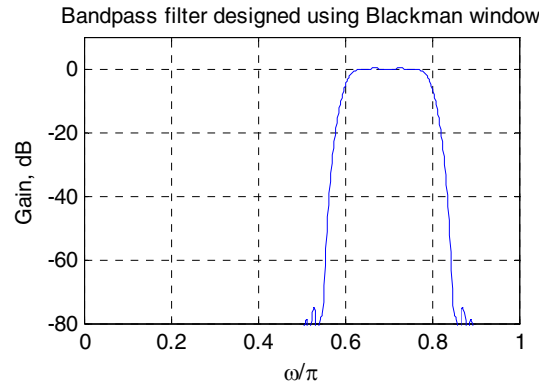
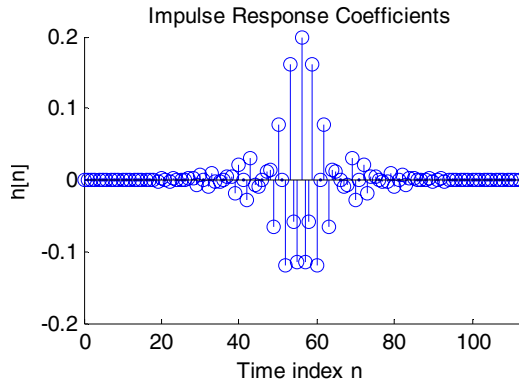
(a) Hamming window using `fir1`



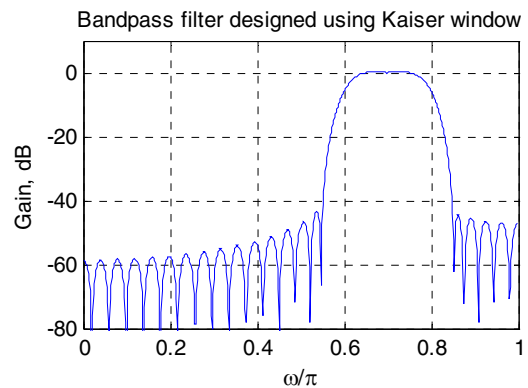
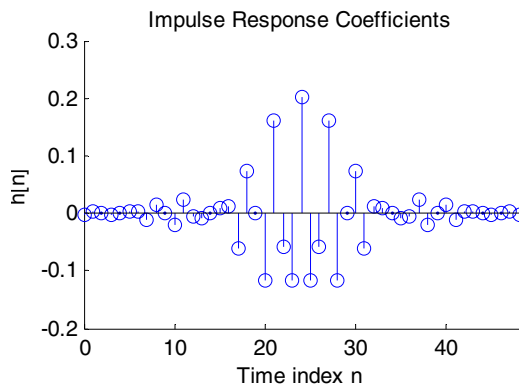
(b) Hann window using `fir1`



(c) Blackman window using `fir1`



(d) Kaiser window using `fir1`



M10.20 $\omega_c = \frac{2\pi \cdot 15}{70} = 0.4286\pi$

`% Problem #M10.20`

`N = 31;`

`d1 = fir1(N-1,0.4286);d2 = fir1(N-1,0.4286,'high');`

`[h1,w] = freqz(d1,1,512);h2 = freqz(d2,1,w);`

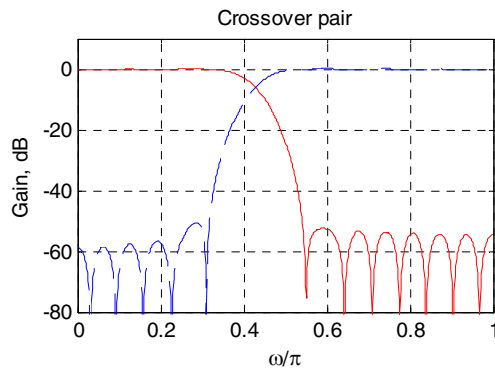
`plot(w/pi,20*log10(abs(h1)),'-r',w/pi,20*log10(abs(h2)),'--b');`

`grid;`

`xlabel('\omega/\pi');ylabel('Gain, dB');`

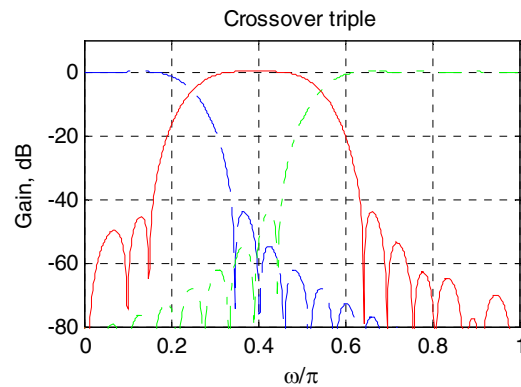
`title('Crossover pair');`

`axis([0 1 -80 10]);`



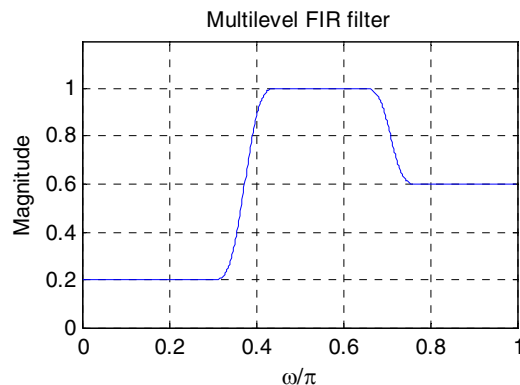
M10.21 $\omega_{c1} = 0.2494, \omega_{c2} = 0.5442$

```
% Problem #M10.21
N = 32;
d1 = fir1(N, 0.2494, hanning(33));
d2 = fir1(N, 0.5442, 'high', hanning(33));
d3 = -d1-d2;
d3(17) = 1-d1(17)-d2(17);
[h1,w] = freqz(d1,1,512);h2 = freqz(d2,1,w);h3 =
freqz(d3,1,w);
g1 = 20*log10(abs(h1)); g2 = 20*log10(abs(h2));
g3 = 20*log10(abs(h3));
plot(w/pi, g1,'--b',w/pi,g2,'-.g',w/pi,g3,'-r');grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Crossover triple');
axis([0 1 -80 10]);
```



M10.22 % Problem #M10.22

```
fpts = [0 0.35 0.4 0.7 0.72 1];
mval = [0.2 0.2 1 1 0.6 0.6];
b = fir2(70, fpts, mval);
[H,w] = freqz(b,1,512);
figure(1);
plot(w/pi,abs(H));grid;
xlabel('\omega/\pi');ylabel('Magnitude');
title('Multilevel FIR filter');
axis([0 1 0 1.2]);
```



M10.23 From Problem 10.36,

$\omega_p = 0.45\pi$, $\omega_s = 0.6\pi$, $\delta_p = 0.2043$, $\delta_s = 0.0454$ and we assume that

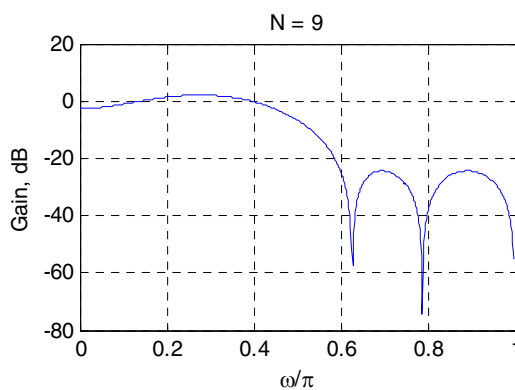
$F_T = 2$. Therefore, $F_p = \frac{0.45\pi \cdot F_T}{2\pi} = 0.45$ and $F_s = 0.6$.

$\alpha_p = -20 \log_{10}(1 - \delta_p) = 1.985$ dB, $\alpha_s = -20 \log_{10}(\delta_s) = 26.858$ dB.

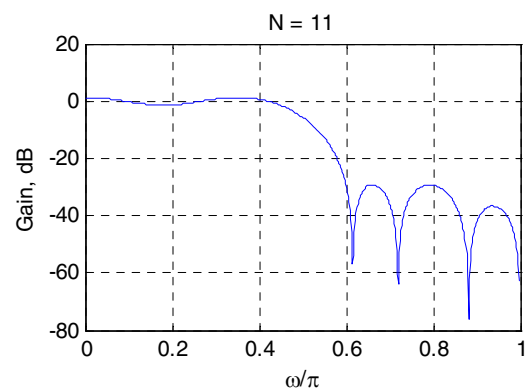
After obtaining the length N using 'remezord', the specifications of the filter were not met. We increased N to 11 to meet the specifications.

```
% Program #M10.23
Ft = 2; Fp = 0.45; Fs = 0.6;
ds = 0.0454; dp = 0.2043;
F = [Fp Fs]; A = [1 0]; DEV = [dp ds];
[N, Fo, Ao, W] = remezord(F, A, DEV, Ft);
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(1);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi'); ylabel('Gain, dB'); title('N = 9');
%axis([0 0.45 -3 3]);
N = 11;
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(2);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi'); ylabel('Gain, dB'); title('N = 11');
```

Using remezord, we get $N = 9$. The corresponding gain response is shown in Figure (e) below:



(e)



(f)

However, specifications are not met in the passband with this filter, so we increase N to 11. The corresponding gain response is shown in Figure (f) above. The specifications are now met.

M10.24 From problem 10.37,

$\omega_p = 0.7\pi, \omega_s = 0.55\pi, \delta_p = 0.03808, \delta_s = 0.0112$ and we assume that

$F_T = 2$. Therefore, $F_p = \frac{0.7\pi \cdot F_T}{2\pi} = 0.7$ and $F_s = 0.55$.

$\alpha_p = -20 \log_{10}(1 - \delta_p) = 0.3372$ dB, $\alpha_s = -20 \log_{10}(\delta_s) = 39.016$ dB.

Using 'remezord', we get an estimate of $N = 20$. However, using this order, the specifications are not met in the stopband, so we need to increase N up to 23 to meet the specifications.

```
% Program #M10.24
```

```
Ft = 2;Fp = 0.7;Fs = 0.55;
```

```
ds = 0.0112;dp = 0.03808;
```

```
F = [Fs Fp];A = [0 1];DEV = [ds dp];
```

```
[N,Fo,Ao,W] = remezord(F,A,DEV,Ft);
```

```
b = remez(N,Fo,Ao,W);
```

```
[H,w] = freqz(b,1,512);
```

```
figure(1);
```

```
plot(w/pi, 20*log10(abs(H)));
```

```
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 20');
```

```
N = 23;
```

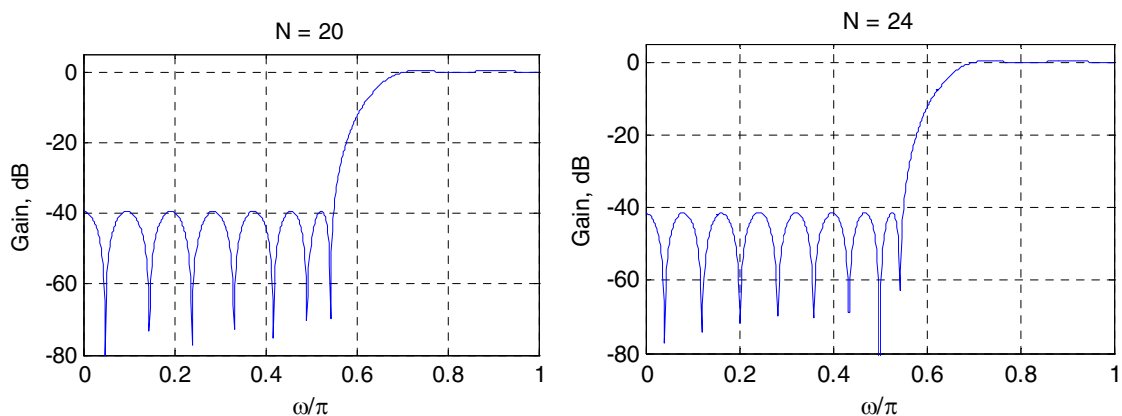
```
b = remez(N,Fo,Ao,W);
```

```
[H,w] = freqz(b,1,512);
```

```
figure(2);
```

```
plot(w/pi,20*log10(abs(H)));
```

```
xlabel('\omega/\pi');ylabel('Gain,dB ');title('N = 23');
```



Note: As odd order symmetric FIR filters must have a gain of zero at the Nyquist frequency. The order has been increased by one by remez.

M10.25 From Problem 10.38,

$\omega_{p1} = 0.55\pi, \omega_{p2} = 0.7\pi, \omega_{s1} = 0.44\pi, \omega_{s2} = 0.82\pi, \delta_p = 0.01, \delta_{s1} = 0.007,$

$\delta_{s2} = 0.002. \alpha_p = -20 \log_{10}(1 - \delta_p) = 0.087$ dB,

$$\alpha_{s1} = -20 \log_{10}(\delta_{s1}) = 43 \text{ dB}, \quad \alpha_{s2} = -20 \log_{10}(\delta_{s2}) = 54 \text{ dB}.$$

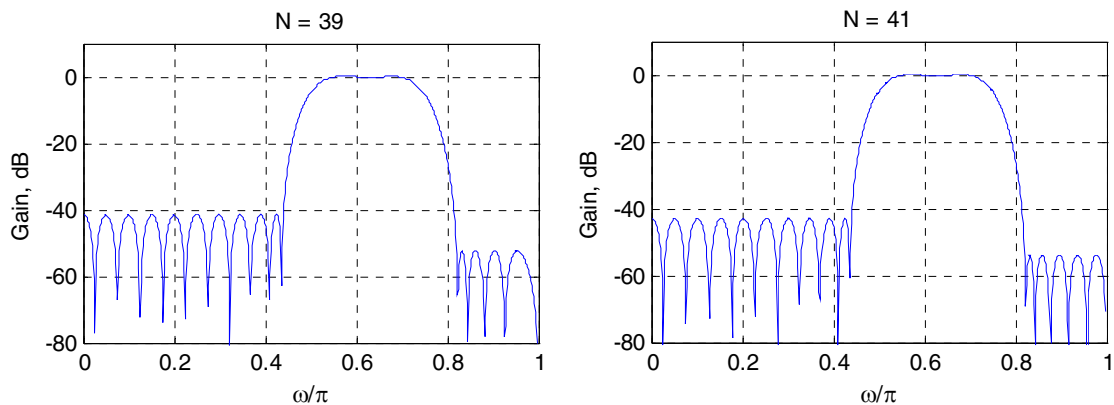
```

% Program #M10.25
Ft = 2;Fp1 = 0.55;Fp2 = 0.7;Fs1 = 0.44;Fs2 = 0.82;
ds1 = 0.007;ds2 = 0.002;dp = 0.01;
F = [Fs1 Fp1 Fp2 Fs2];A = [0 1 0];DEV = [ds1 dp ds2];
[N,Fo,Ao,W] = remezord(F,A,DEV,Ft);
b = remez(N,Fo,Ao,W);
[H, w] = freqz(b, 1, 512);
figure(1);
plot(w/pi,20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 39');
axis([0 1 -80 10]);

N = 41;
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(2);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 41');
axis([0 1 -80 10]);

```

Using remezord, we estimate the filter length to be $N = 39$. However, the minimum stopband attenuation specifications are not met in both stopbands, so we increase N to 41

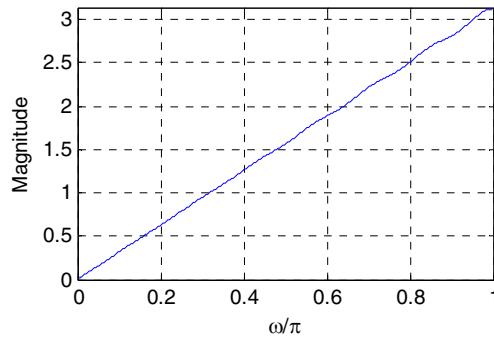


```

M10.26 % Program #M10.26
b = remez(29, [0 1], [0 pi], 'differentiator');
[H, w] = freqz(b, 1, 512);
plot(w/pi,abs(H));grid
xlabel('\omega/\pi');ylabel('Magnitude');
axis([0 1 0 pi]);

```

The magnitude response of the differentiator is given below:

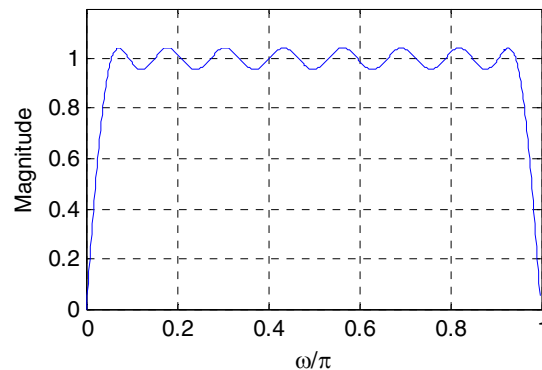


```

M10.27 % Program #M10.27
f = [0.02 0.05 0.07 0.95 0.97 1];
m = [0 0 1 1 0 0];
wt = [1 60 1];
b = remez(30, f, m, wt, 'hilbert');
[H,w] = freqz(b,1,512);
plot(w/pi,abs(H));grid
xlabel('\omega/\pi');ylabel('Magnitude');
axis([0 1 0 1.2]);

```

The magnitude response of the Hilbert transformer is shown below:



M10.28 $\omega_p = 0.35\pi$, $\omega_s = 0.5\pi$, $R_p = 1$ dB, and $R_s = 28$ dB.

```

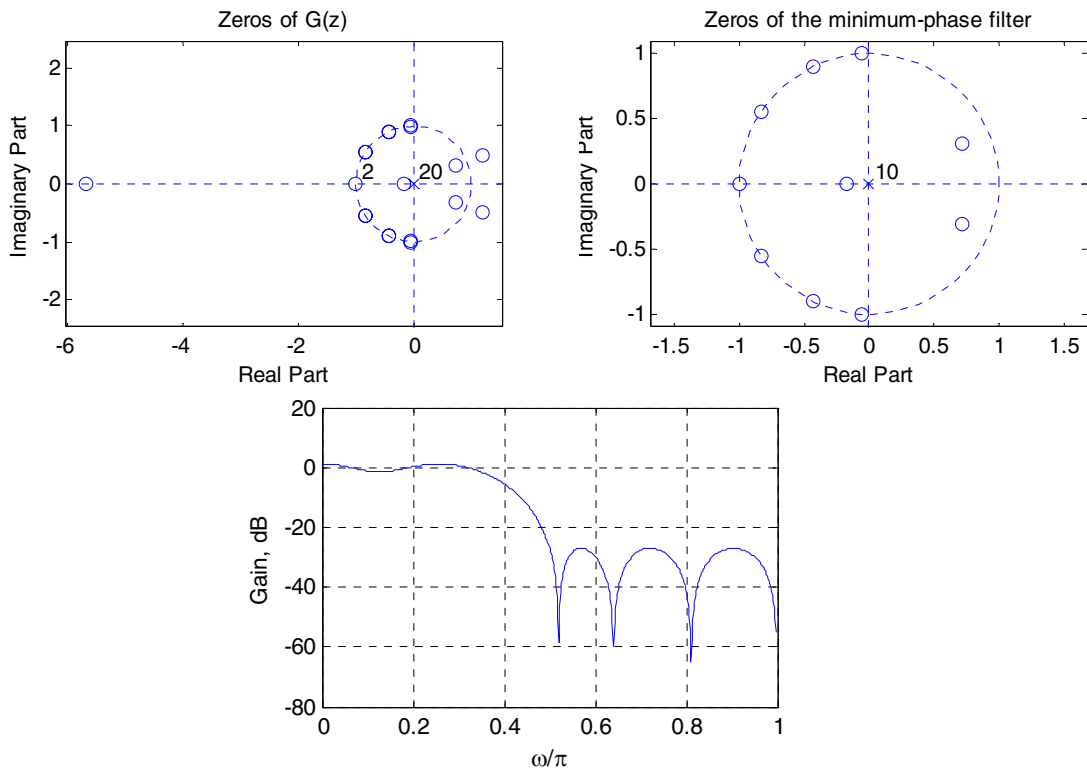
% Program #M10_28
% Design of a minimum-phase lowpass FIR filter
Wp = 0.35; Ws = 0.5; Rp = 1; Rs = 28;
% Desired ripple values of minimum-phase filter
dp = 1 - 10^(-Rp/20); ds = 10^(-Rs/20);
% Compute ripple values of prototype linear-phase filter
Ds = (ds*ds)/(2 - ds*ds);
Dp = (1 + Ds)*((dp + 1)*(dp + 1) - 1);
% Estimate filter order
[N,fpts,mag,wt] = remezord([Wp Ws], [1 0], [Dp Ds]);
% Design the prototype linear-phase filter H(z)
[b,err,res] = remez(N,fpts,mag,wt);
K = N/2;
b1 = b(1:K);

```

```

% Design the linear-phase filter G(z)
lenerr= res.error(length(res.error));
c = [b1 (b(K+1) + lenerr) fliplr(b1)]/(1+Ds);
zplane(c);title('Zeros of G(z)');pause
c1 = c(K+1:N+1);
[y, ssp, iter] = minphase(c1);
zplane(y);title('Zeros of the minimum-phase filter');
pause
[hh,w] = freqz(y,1, 512);
% Plot the gain response of the minimum-phase filter
plot(w/pi, 20*log10(abs(hh)));grid
xlabel('\omega/\pi');ylabel('Gain, dB');

```



M10.29 % Program #M10.29

```

c = [2.4 6.76 26.15 68.43 186.83 326.51 565.53 678.95 805.24
678.95 565.53 326.51 186.83 68.43 26.15 6.76 2.4];
h = firminphase(c)

```

The coefficients of the minimum phase spectral factor are:

```

h = 7.6730 8.5329 18.0722 12.5696 12.6822 4.8388 2.0784 0.5332 0.3128

```

M10.30 % Program #M10.30

```

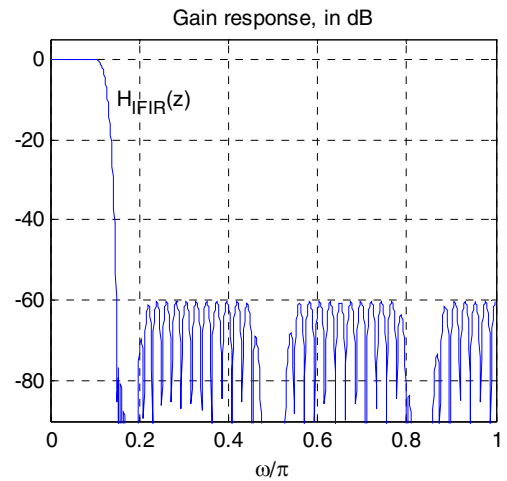
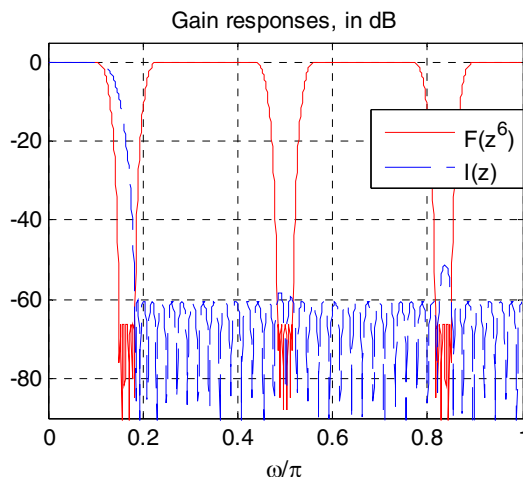
[h,g]=ifir(6,'low',[.1 .15],[.001 .001]);
[hh,w]=freqz(h,1,1024); hg=freqz(g,1,1024);
h = hh.*hg; % Compounded response
Fg = 20*log10(abs(hh)); Ig = 20*log10(abs(hg));
plot(w/pi,Fg,'-r',w/pi,Ig,'--b'); grid;
axis([0 1 -90 5]);

```

```

legend('F(z^6)', 'I(z)');
xlabel('\omega/\pi');title('Gain responses, in dB');
pause;
plot(w/pi,20*log10(abs(h))); grid;
axis([0 1 -90 5]);
xlabel('\omega/\pi');title('Gain response, in dB');
gtext('H_{IFIR}(z)');

```

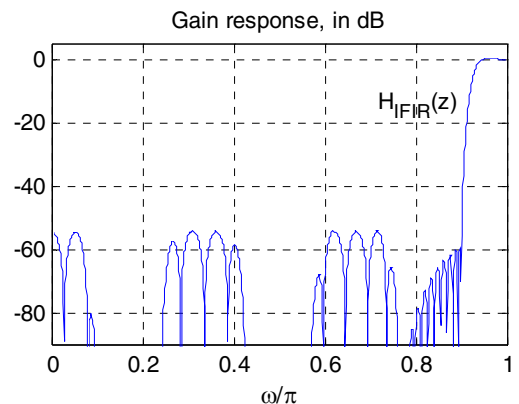
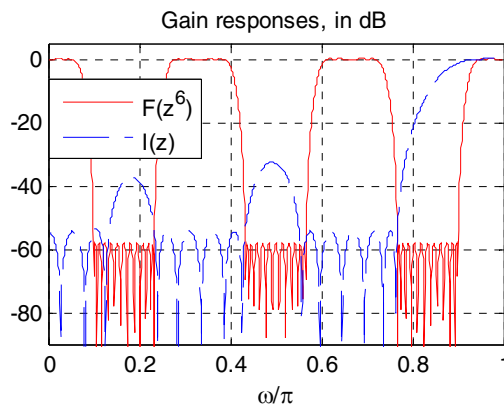


M10.31 % Program #M10.31

```

[h,g]=ifir(6,'high',[.9 .95],[.002 .004]);
[hh,w]=freqz(h,1,1024); hg=freqz(g,1,1024);
h = hh.*hg; % Compounded response
Fg = 20*log10(abs(hh)); Ig = 20*log10(abs(hg));
plot(w/pi,Fg,'-r',w/pi,Ig,'--b'); grid;
axis([0 1 -90 5]);
legend('F(z^6)', 'I(z)');
xlabel('\omega/\pi');title('Gain responses, in dB');
pause;
plot(w/pi,20*log10(abs(h))); grid;
axis([0 1 -90 5]);
xlabel('\omega/\pi');title('Gain response, in dB');
gtext('H_{IFIR}(z)');

```



```

M10.32 % Problem #M10.32
InpN = ceil(2*pi/(0.15*pi));

% Plotting function
% plots the result of using an equalizer of length InpN
%function [N] = plotfunc(InpN);
% Creating filters
Wfilt = ones(1, InpN);
Efilt = remezfunc(InpN, Wfilt);

% Plot running sum filter response
figure(1);
Wfilt = Wfilt/sum(Wfilt);
[hh, w] = freqz(Wfilt, 1, 512);
plot(w/pi, 20*log10(abs(hh)));
axis([0 1 -50 5]);grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Prefilter H(z)');

% Plot equalizer filter response
figure(2);
[hw, w] = freqz(Efilt, 1, 512);
plot(w/pi, 20*log10(abs(hw)));
axis([0 1 -50 5]);grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Equalizer F(z)');

% Plot cascade filter response
figure(3);
Cfilt = conv(Wfilt, Efilt);
[hc, w] = freqz(Cfilt, 1, 512);
plot(w/pi, 20*log10(abs(hc)));
axis([0 1 -80 5]);grid;
xlabel('\omega/\pi');ylabel('Gain, dB');
title('Cascade filter H(z)F(z)');

% Remez function using 1/P(z) as desired amplitude
% and P(z) as weighting
function [N] = remezfunc(Nin, Wfilt);
% Nin : number of tuples in the remez equalizer filter
% Wfilt : the prefilter
a = [0:0.001:0.999]; % The accuracy of the computation
w = a.*pi;wp = 0.05*pi;ws = 0.15*pi;
i = 1;n = 1;
for t = 1:(length(a)/2),

```

```

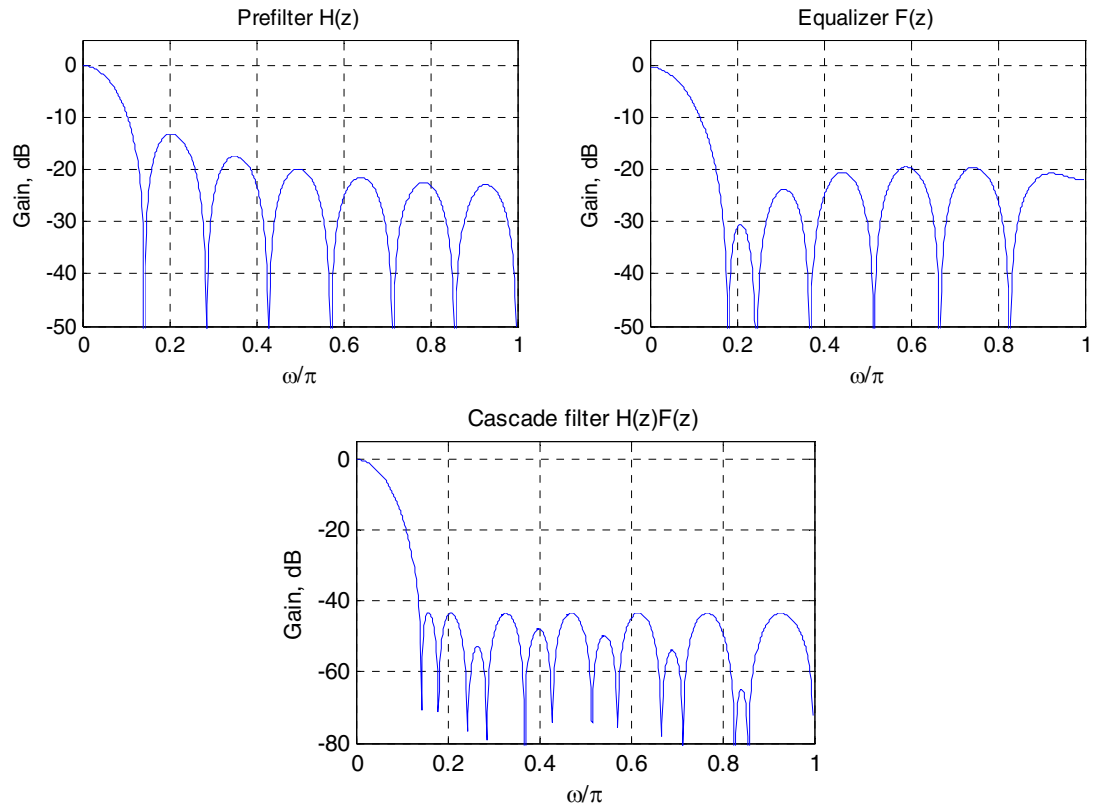
    if w(2*t) < wp
        pas(i) = w(2*t - 1);
        pas(i+1) = w(2*t);
        i = i+2;
    end
    if w(2*t-1) > ws
        sto(n) = w(2*t - 1);
        sto(n+1) = w(2*t);
        n = n+2;
    end
end
w = cat(2, pas, sto);
bi = length(w)/2;
for t1 = 1:bi,
    bw(t1) = (w(2*t1) + w(2*t1-1))/2;
    W(t1) = Weight(bw(t1), Wfilt, ws);
end
W = W/max(W);
for t2 = 1:length(w),
    G(t2) = Hdr(w(t2), Wfilt, wp);
end
G = G/max(G);
N = remez(Nin, w/pi, G, W);

% Weighting function
function[Wout] = Weight(w, Wfilt, ws);
K = 22.8;
L = length(Wfilt);
Wtemp = 0;
Wsum = 0;
for k = 1:L,
    Wtemp = Wfilt(k)*exp((k-1)*i*w);
    Wsum = Wsum + Wtemp;
end
Wout = abs(Wsum);
if w > ws,
    Wout = K*max(Wout);
end

% Desired function
function [Wout] = Hdr(w, Wfilt, ws);
if w <= ws,
    L = length(Wfilt);
    Wtemp = 0;
    Wsum = 0;
    for k = 1:L,
        Wtemp = Wfilt(k)*exp(i*(k-1)*w);
        Wsum = Wsum + Wtemp;
    end
    Wsum = abs(Wsum);
    Wout = 1/Wsum;
end

```

```
else
    Wout = 0;
end
```



Chapter 11

11.1 Analysis yields

$$\begin{aligned} w_1[n] &= \beta_1 \{x[n-1] + \alpha_2 w_2[n-1]\}, w_2[n] = w_1[n] + w_3[n], \\ w_3[n] &= \beta_1 \{\alpha_2 w_2[n-1] + \alpha_2 w_4[n-1]\}, w_4[n] = w_3[n] + w_5[n], \\ w_5[n] &= \beta_3 \{\alpha_2 w_4[n-1] + \alpha_3 w_5[n-1]\}, y[n] = w_1[n] + \alpha_0 x[n]. \end{aligned}$$

In matrix form the above equations can be written as

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 \beta_2 & 0 & \alpha_2 \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \beta_3 & \alpha_3 \beta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} \beta_1 x[n-1] \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

Here the \mathbf{F} matrix is given by:
$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the \mathbf{F} matrix contains nonzero entries above the main diagonal, the above set of equations are not computable.

11.2 A computable set of equations are given by:

$$\begin{aligned} w_1[n] &= \beta_1 \{x[n-1] + \alpha_1 w_2[n-1]\}, w_3[n] = \beta_1 \{\alpha_2 w_2[n-1] + \alpha_2 w_4[n-1]\}, \\ w_5[n] &= \beta_3 \{\alpha_2 w_4[n-1] + \alpha_3 w_5[n-1]\}, w_2[n] = w_1[n] + w_3[n], y[n] = w_1[n] + \alpha_0 x[n], \\ w_4[n] &= w_3[n] + w_5[n]. \end{aligned}$$

In matrix form the above equations can be written as

$$\begin{bmatrix} w_1[n] \\ w_3[n] \\ w_5[n] \\ w_2[n] \\ y[n] \\ w_4[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_3[n-1] \\ w_5[n-1] \\ w_2[n-1] \\ y[n-1] \\ w_4[n-1] \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 \beta_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 \beta_2 & 0 & \alpha_2 \beta_2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \beta_3 & \alpha_3 \beta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_3[n-1] \\ w_5[n-1] \\ w_2[n-1] \\ y[n-1] \\ w_4[n-1] \end{bmatrix} + \begin{bmatrix} \beta_1 x[n-1] \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \\ 0 \end{bmatrix}$$

Here the \mathbf{F} matrix is given by:
$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Since the \mathbf{F} matrix has no nonzero entries above the main diagonal, the new set of equations are computable.

- 11.3** Analysis yields $w_1[n] = \beta_1 \{ \alpha_1 w_2[n-1] + \alpha_2 w_4[n-1] + \alpha_3 w_6[n-1] \}$, $w_2[n] = w_1[n] + x[n]$, $w_3[n] = \beta_2 \{ \alpha_2 w_4[n-1] + \alpha_3 w_6[n-1] \}$, $w_4[n] = w_2[n] + w_3[n]$, $w_5[n] = \beta_3 \{ \alpha_3 w_6[n-1] \}$, $w_6[n] = w_4[n] + w_5[n]$, $y[n] = \alpha_1 w_2[n] + \alpha_2 w_4[n] + \alpha_3 w_6[n] + \alpha_0 x[n]$.

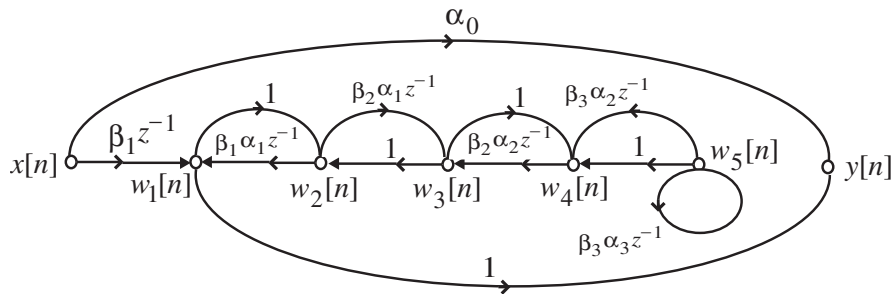
In matrix form the above set of equations are given by

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ w_6[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ w_6[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \beta_1 \alpha_1 & 0 & \beta_1 \alpha_2 & 0 & \beta_1 \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \alpha_2 & 0 & \beta_2 \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_3 \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ w_6[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ x[n] \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

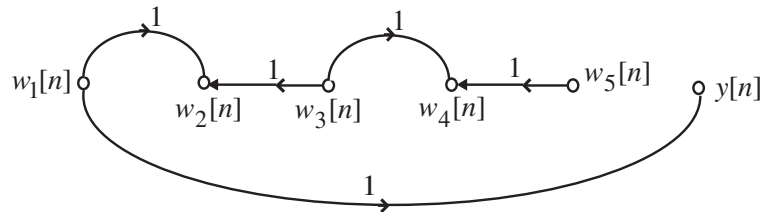
Here the \mathbf{F} matrix is given by: $\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 \end{bmatrix}$.

Since the diagonal of the \mathbf{F} matrix has all zeros, and no nonzero entries above the main diagonal, the new set of equations are computable.

11.4

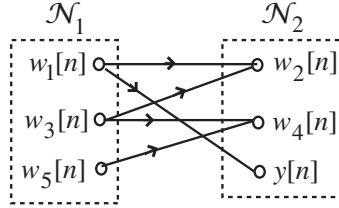


Reduced signal flow graph obtained by removing the branches going out of the input node and the delay branches is:



The node sets of the precedence graph are as follows:

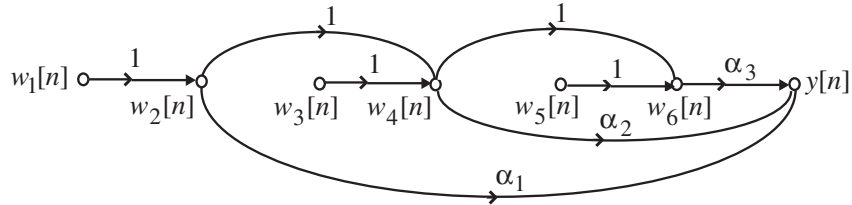
$$\{\mathcal{N}_1\} = \{w_1[n], w_3[n], w_5[n]\}, \quad \{\mathcal{N}_2\} = \{w_2[n], w_4[n], y[n]\}$$



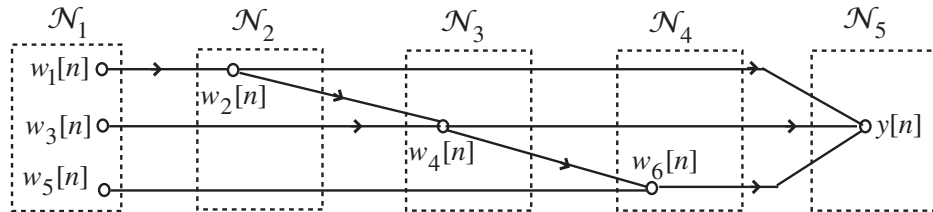
In the above precedence graph, \mathcal{N}_1 contains only outgoing nodes and \mathcal{N}_2 contains only incoming branches. The structure has no delay free loops. A valid computational algorithm is:

$$\begin{aligned} w_1[n] &= \beta_1 \{x[n-1] + \alpha_1 w_2[n-1]\}, & w_3[n] &= \beta_1 \{\alpha_2 w_2[n-1] + \alpha_2 w_4[n-1]\}, \\ w_5[n] &= \beta_3 \{\alpha_2 w_4[n-1] + \alpha_3 w_5[n-1]\}, & w_2[n] &= w_1[n] + w_3[n], & y[n] &= w_1[n] + \alpha_0 x[n], \\ & & w_4[n] &= w_3[n] + w_5[n]. \end{aligned}$$

11.5 Reduced signal flow graph is:



The node sets of the precedence graph are as follows: $\{\mathcal{N}_1\} = \{w_1[n], w_3[n], w_5[n]\}$, $\{\mathcal{N}_2\} = \{w_2[n]\}$, $\{\mathcal{N}_3\} = \{w_4[n]\}$, $\{\mathcal{N}_4\} = \{w_6[n]\}$, $\{\mathcal{N}_5\} = \{y[n]\}$.



A valid computational algorithm is:

$$\begin{aligned} w_1[n] &= \beta_1 \{\alpha_1 w_2[n-1] + \alpha_2 w_4[n-1] + \alpha_3 w_6[n-1]\}, \\ w_3[n] &= \beta_2 \{\alpha_2 w_4[n-1] + \alpha_3 w_6[n-1]\}, & w_5[n] &= \beta_3 \{\alpha_3 w_6[n-1]\}, \\ w_2[n] &= w_1[n] + x[n], & w_4[n] &= w_2[n] + w_3[n], & w_6[n] &= w_4[n] + w_5[n], \\ y[n] &= \alpha_1 w_2[n] + \alpha_2 w_4[n] + \alpha_3 w_6[n] + \alpha_0 x[n]. \end{aligned}$$

11.6 (a) Analysis yields $w_1[n] = w_2[n] - k_1 w_1[n-1]$, $w_2[n] = w_3[n] - k_2 s_2[n-1]$,

$$\begin{aligned} w_3[n] &= x[n] - k_3 s_3[n-1], & s_2[n] &= k_1 w_1[n] + w_1[n-1], & s_3[n] &= k_2 w_2[n] + s_2[n-1], \\ y[n] &= k_3 w_3[n] + s_3[n-1]. \end{aligned}$$

In matrix form we have

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ s_2[n] \\ s_3[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x[n] \\ 0 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ s_2[n] \\ s_3[n] \\ y[n] \end{bmatrix} \\
+ \underbrace{\begin{bmatrix} -k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{G}} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ s_2[n-1] \\ s_3[n-1] \\ y[n-1] \end{bmatrix}.$$

As the diagonal elements of \mathbf{F} -matrix are all zeros, there are no delay-free loops. However, the above set of equations are not computable as there are non-zero elements above the diagonal of \mathbf{F} .

(b) The reduced signal flow-graph representation of Figure P11.3 is shown below:

From the above flow-graph we observe that the set composed of nodes with only outgoing branches is $\mathcal{N}_1 = \{w_3[n]\}$. The set of nodes with incoming branches from \mathcal{N}_1 and outgoing branches is $\mathcal{N}_2 = \{w_2[n]\}$. The set of nodes with incoming branches from \mathcal{N}_1 and \mathcal{N}_2 and outgoing branches is $\mathcal{N}_3 = \{w_1[n]\}$. Finally, the set of nodes with only incoming branches from \mathcal{N}_1 , \mathcal{N}_2 and \mathcal{N}_3 , and no outgoing branches $\mathcal{N}_4 = \{s_2[n], s_3[n], y[n]\}$. Therefore, one possible ordered set of equations that is

computable is given by

$$\begin{bmatrix} w_3[n] \\ w_2[n] \\ w_1[n] \\ s_2[n] \\ s_1[n] \\ y[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ k_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} w_3[n] \\ w_2[n] \\ w_1[n] \\ s_2[n] \\ s_1[n] \\ y[n] \end{bmatrix} \\
+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & -k_2 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{G}} \begin{bmatrix} w_3[n-1] \\ w_2[n-1] \\ w_1[n-1] \\ s_2[n-1] \\ s_1[n-1] \\ y[n-1] \end{bmatrix}.$$

Note: All elements on the diagonal and above diagonal of \mathbf{F} are zeros.

11.7 $H(z) = \frac{p_0 + p_1z^{-1} + p_2z^{-2}}{1 - 3z^{-1} + 1.5z^{-2}}$. From Eq. (11.18) we get,

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 3.2 & 0 & 0 \\ 5.6 & 3.2 & 0 \\ 7.0 & 5.6 & 3.2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 3.2 \\ -4.0 \\ -5.0 \end{bmatrix}. \text{ Hence, } H(z) = \frac{3.2 - 4z^{-1} - 5z^{-2}}{1 - 3z^{-1} + 1.5z^{-2}}.$$

11.8 From Eqn (11.15), $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & & & \\ -2 & 2 & 0 & & & \\ 4 & -2 & 2 & & & \\ \hline 8 & 4 & -2 & & & \\ -12 & 8 & 4 & & & \end{array} \right] \begin{bmatrix} 1 \\ d_1 \\ d_2 \end{bmatrix}$ and from Eqn (11.20),

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\begin{bmatrix} 4 & -2 \\ 8 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -12 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 3.5 \end{bmatrix}. \text{ Finally, from Eqn (11.21),}$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 2 & 0 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.25 \\ 3.5 \end{bmatrix} = \begin{bmatrix} 2 \\ -2.5 \\ 11.5 \end{bmatrix}. \text{ Hence, } H(z) = \frac{2 - 2.5z^{-1} + 11.5z^{-2}}{1 - 0.25z^{-1} + 3.5z^{-2}}.$$

11.9 From Eqn. (11.15) $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & & & & \\ -4 & 2 & 0 & 0 & & & & \\ 8 & -4 & 2 & 0 & & & & \\ -8 & 8 & -4 & 2 & & & & \\ \hline 12 & -8 & 8 & -4 & & & & \\ 16 & 12 & -8 & 8 & & & & \\ -8 & 16 & 12 & -8 & & & & \end{array} \right] \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}$. Next, from Eqn. (11.20),

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -8 & 8 & -4 \\ 12 & -8 & 8 \\ 16 & 12 & -8 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -4.8 \\ -7.4 \end{bmatrix}. \text{ Finally, from Eqn. (11.21),}$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 8 & -4 & 2 & 0 \\ -8 & 8 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \\ -4.8 \\ -7.4 \end{bmatrix} = \begin{bmatrix} 2.0 \\ -3.2 \\ -3.2 \\ -0.4 \end{bmatrix}. \text{ Hence,}$$

$$H(z) = \frac{2 - 3.2z^{-1} - 3.2z^{-2} - 0.4z^{-3}}{1 + 0.4z^{-1} - 4.8z^{-2} - 7.4z^{-3}}.$$

11.10
$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 4 & -4 & 2 & 0 \\ -6 & -6 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.6 \\ 0.2 \\ 1.8 \end{bmatrix} = \begin{bmatrix} 2 \\ -5.2 \\ 6.8 \\ -5.6 \end{bmatrix}.$$
 Hence, $P(z) = 2 - 5.2z^{-1} + 6.8z^{-2} - 5.6z^{-3}$.

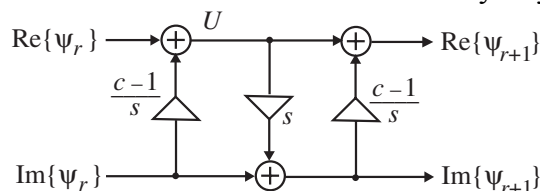
11.11
$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 4 & -2 & 2 & 0 & 0 \\ 6 & 4 & -2 & 2 & 0 \\ -8 & 6 & 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2}{3} \\ 0 \\ -2 \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{2}{3} \\ 2\frac{2}{3} \\ 4\frac{2}{3} \\ -3\frac{1}{3} \end{bmatrix}.$$
 $\{p_i\} = \{2, -\frac{2}{3}, 2\frac{2}{3}, 4\frac{2}{3}, -3\frac{1}{3}\}$

11.12 The k -th sample of an N -point DFT is given by $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}$. Thus, the computation of $X[k]$ requires N complex multiplications and $N-1$ complex additions. Now, each complex multiplication, in turn, requires 4 real multiplications and 2 real additions. Likewise, each complex addition requires 2 real additions. As a result, the N complex multiplications needed to compute $X[k]$ require a total of $4N$ real multiplications and a total of $2N-2$ real additions. Therefore, each sample of the N -point DFT involves $4N$ real multiplications and $4N-2$ real additions. Hence, the computation of all DFT samples thus requires $4N^2$ real multiplications and $(4N-2)N$ real additions.

11.13 Let the two complex numbers be $\alpha = a + jb$ and $\beta = c + jd$. Thus, $\alpha\beta = (a + jb)(c + jd) = (ac - bd) + j(ad + bc)$, which requires 4 real multiplications and 2 real additions. Consider the product $(a + b)(c + d)$, ac , and bd , which require 3 real multiplications and 2 real additions. The imaginary part of $\alpha\beta$ can be formed from $(a + b)(c + d) - ac - bd = ad + bc$, which now requires 2 real additions. Likewise, the real part of $\alpha\beta$ can be formed from $ac - bd$ requiring an additional real additions. Hence, the complex multiplication can be computed using 3 real multiplications and 5 real additions.

11.14 Recall, $W_N = e^{-j2\pi/N} = c + js$, where $c = \cos(2\pi/N)$ and $s = -\sin(2\pi/N)$. Thus, $c^2 + s^2 = 1$. Now, $\Psi_{r+1} = W_N \cdot \Psi_r = (c + js)(\text{Re}\{\Psi_r\} + j\text{Im}\{\Psi_r\}) = (c \cdot \text{Re}\{\Psi_r\} - s \cdot \text{Im}\{\Psi_r\}) + j(c \cdot \text{Im}\{\Psi_r\} + s \cdot \text{Re}\{\Psi_r\})$. Thus, $\text{Re}\{\Psi_{r+1}\} = c \cdot \text{Re}\{\Psi_r\} - s \cdot \text{Im}\{\Psi_r\}$ and $\text{Im}\{\Psi_{r+1}\} = c \cdot \text{Im}\{\Psi_r\} + s \cdot \text{Re}\{\Psi_r\}$.

Figure P11.4 with internal node label is shown below. Its analysis yields



$$(1): U = \text{Re}\{\Psi_r\} + \frac{c-1}{s} \text{Im}\{\Psi_r\}, \quad (2): \text{Im}\{\Psi_{r+1}\} = \text{Im}\{\Psi_r\} + sU, \text{ and}$$

$$(3): \text{Re}\{\Psi_{r+1}\} = U + \frac{c-1}{s} \text{Im}\{\Psi_{r+1}\}. \quad \text{Substituting Eqs. (1) in Eq. (2) we get}$$

$$(4): \text{Im}\{\Psi_{r+1}\} = \text{Im}\{\Psi_r\} + s \left(\frac{c-1}{s} \text{Im}\{\Psi_r\} + \text{Re}\{\Psi_r\} \right) = c \cdot \text{Im}\{\Psi_r\} + s \cdot \text{Re}\{\Psi_r\}. \quad \text{Next,}$$

substituting Eqs. (1) and (4) in Eq. (3) we get (5):

$$(5): \text{Re}\{\Psi_{r+1}\} = \frac{c-1}{s} (c \cdot \text{Im}\{\Psi_r\} + s \cdot \text{Re}\{\Psi_r\}) + \frac{c-1}{s} \text{Im}\{\Psi_r\} + \text{Re}\{\Psi_r\}$$

$$= \frac{c^2-1}{s} \text{Im}\{\Psi_r\} + c \cdot \text{Re}\{\Psi_r\} = \frac{-s^2}{s} \cdot \text{Im}\{\Psi_r\} + c \cdot \text{Re}\{\Psi_r\} = c \cdot \text{Re}\{\Psi_r\} - s \cdot \text{Im}\{\Psi_r\}.$$

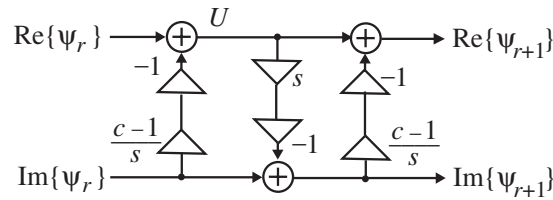
It thus follows that the structure of Figure P11.4 implements the multiplication of a complex signal with the twiddle factor using only 3 real multiplications.

In the case of multiplication by complex twiddle factor $W_N^{-1} = c - js$, we have

$$\Psi_{r+1} = W_N^{-1} \cdot \Psi_r = (c - js)(\text{Re}\{\Psi_r\} + j \text{Im}\{\Psi_r\})$$

$$= (c \cdot \text{Re}\{\Psi_r\} + s \cdot \text{Im}\{\Psi_r\}) + j(c \cdot \text{Im}\{\Psi_r\} - s \cdot \text{Re}\{\Psi_r\}). \quad \text{Thus, here}$$

$\text{Re}\{\Psi_{r+1}\} = c \cdot \text{Re}\{\Psi_r\} + s \cdot \text{Im}\{\Psi_r\}$ and $\text{Im}\{\Psi_{r+1}\} = c \cdot \text{Im}\{\Psi_r\} - s \cdot \text{Re}\{\Psi_r\}$. It follows then that the real and imaginary parts are simply obtained by reversing the sign of s of the real and imaginary parts derived in the case of multiplication by W_N . As a result, the corresponding structure is obtained by cascading an inverter to the real multipliers in Figure P11.4 as indicated below:



11.15 The center frequency bin k : $f_c(k) = \frac{kF_T}{N}$, where $N = \#$ of bins, and F_T is the sampling

frequency. Inverting we have $k(f) = \left\lfloor \frac{fN}{F_T} \right\rfloor$. Therefore, the absolute difference from one of

the given four tones (150 Hz, 375 Hz, 620 Hz, and 850 Hz) to the center of its bin is given by

$dist(N, f) = \left| f - \frac{F_T}{N} \left\lfloor \frac{fN}{F_T} \right\rfloor \right|$. It follows from this equation that the distance goes to zero if

$\frac{F_T}{N} \left\lfloor \frac{fN}{F_T} \right\rfloor = f$ or $\frac{fN}{F_T}$ is an integer.

The total distance is reduced to zero if $\frac{f_i N}{F_T}$ is an integer for $i = 1, \dots, 4$. The minimum value of N for which this true is 500.

However, the total distance can be small, but nonzero, for significantly smaller values of N .

11.16 $H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}}$. Hence, $Y(z) = \frac{X(z)}{1 - W_N^{-k} z^{-1}} = \frac{1 + z^{-N/2}}{1 - W_N^{-k} z^{-1}} = V(z) + z^{-N/2} V(z)$,

where $V(z) = \frac{1}{1 - W_N^{-k} z^{-1}}$. Or, in other words, $y[n] = v[n] + v[n - \frac{N}{2}]$.

Consider $k = 1$: Then $V(z) = \frac{1}{1 - W_N^{-1} z^{-1}}$. This implies,

$$v[n] = W_N^{-n} \mu[n] = \left\{ 1, W_N^{-1}, W_N^{-2}, \dots, W_N^{-N/2}, W_N^{-\frac{(N+1)}{2}}, \dots \right\}$$

$$= \left\{ 1, W_N^{-1}, W_N^{-2}, \dots, -1, -W_N^{-1}, \dots \right\} \text{ since } W_N^{-N/2} = -1, W_N^{-\frac{(N+1)}{2}} = -W_N^{-1}, \text{ and}$$

son on. Thus, $v[n - \frac{N}{2}] = \left\{ 0, 0, 0, \dots, 1, W_N^{-1}, W_N^{-2}, \dots \right\}$ Hence,

$$y[n] = v[n] + v[n - \frac{N}{2}] = \left\{ 1, W_N^{-1}, W_N^{-2}, \dots, W_N^{-\frac{(N-1)}{2}}, 0, 0, 0, \dots \right\}$$

Now, consider $k = \frac{N}{2}$: $V(z) = \frac{1}{1 - W_N^{-N/2} z^{-1}}$. This implies,

$$v[n] = W_N^{-(N/2)n} \mu[n] = \left\{ 1, W_N^{-\frac{N}{2}}, W_N^{-N}, \dots, W_N^{-\frac{N \cdot N}{2}}, W_N^{-\frac{(N+1) \cdot N}{2}}, \dots \right\}$$

Thus, $v[n - \frac{N}{2}] = \left\{ 0, 0, 0, \dots, 0, 1, W_N^{-N/2}, W_N^{-N}, \dots \right\}$

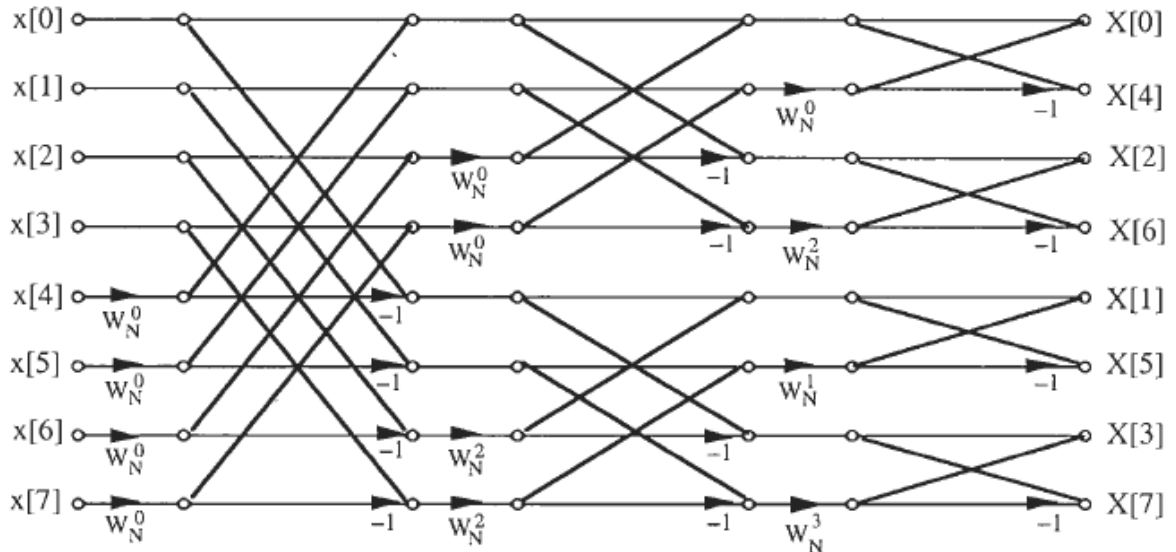
Now, $W_N^{-\frac{N \cdot N}{2}} = (-1)^{N/2}$, $W_N^{-\frac{N}{2}} = -1$, $W_N^{-N} = 1$, $W_N^{-\frac{(N+1) \cdot N}{2}} = (-1)^{N/2} (-1)$, etc. Hence,

$$y[n] = \left\{ 1, -1, 1, -1, \dots, 1 + (-1)^{N/2} \underset{n=N/2}{}, 1 + (-1)^{\frac{N}{2}+1} \underset{n=\frac{N}{2}+1}{}, \dots, \dots \right\}. \text{ Therefore, if } N/2$$

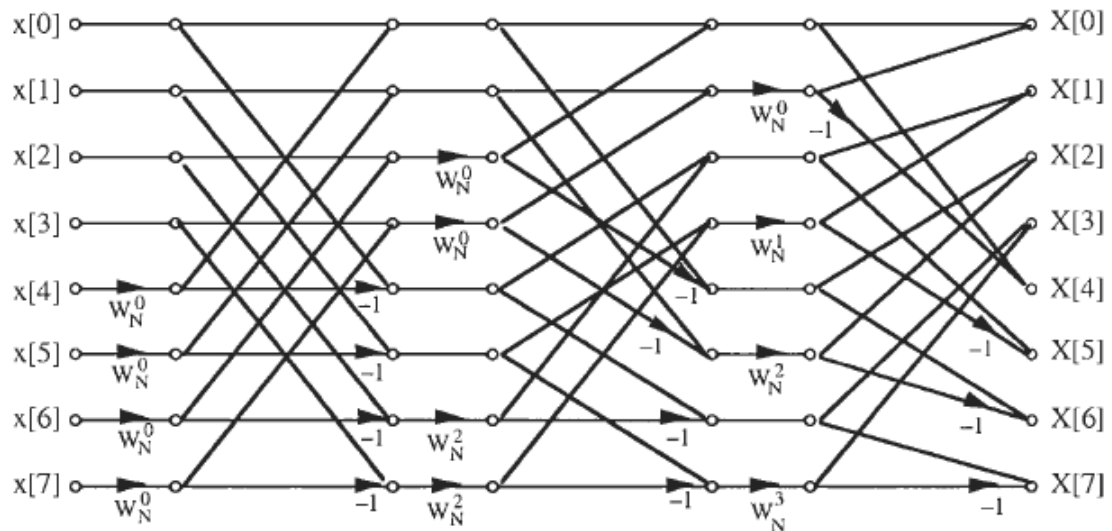
is even, $y[n] = \left\{ 1, -1, 1, -1, \dots, 2, -2, 2, \dots \right\}$, and if $N/2$ is odd,

$$y[n] = \left\{ 1, -1, 1, -1, \dots, \underset{n=N/2}{0}, \underset{n=\frac{N}{2}+1}{0}, 0, \dots \right\}$$

11.17



11.18



11.19

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1]W_N^{nkr_1} + \sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1+1]W_N^{nkr_1+k} + \dots + \sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1+r_1-1]W_N^{nkr_1+(r_1-1)k}$$

$$= \sum_{i=0}^{r_1-1} \left(\sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1+i] W_N^{nkr_1} \right) W_N^{ki}.$$

(N/r₁)-point DFT

Thus, if the (N/r_1) -point DFT has been calculated, we need at the first stage an additional $(r_1 - 1)$ multiplications to compute one sample of the N -point DFT $X[k]$ and, as a result, additional $(r_1 - 1)N$ multiplications are required to compute all N samples of the N -point DFT. Decomposing it further, it follows that additional $(r_2 - 1)N$ multiplications are needed at the second stage, and so on. Therefore, the total number of multiply (add) operations

$$= (r_1 - 1)N + (r_2 - 1)N + \dots + (r_v - 1)N = \left(\sum_{i=1}^v r_i - v \right) N.$$

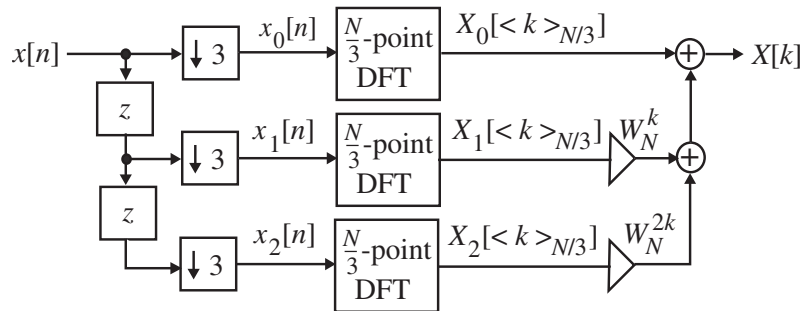
11.20 An examination of the flow-graph of the 8-point DIT FFT algorithm shown in Figure 11.24 reveals that in the first stage all twiddle factors are $W_N^0 = 1$ and in the second stage the twiddle factors are either $W_N^0 = 1$ or $W_N^{N/4} = j$. Hence, there are no twiddle factors with nonunity magnitudes in the first two stages. In all succeeding stages, one of the twiddle factors is $W_N^0 = 1$ and another one is $W_N^{N/4} = j$. The number of twiddle factors with nonzero magnitudes in the i -th stage, $i \geq 3$, is $2^{i-1} - 2$. Hence, the total number of twiddle factors with nonzero magnitudes in an N -point radix-2 FFT algorithm is

$$R(v) = \left(\sum_{i=3}^v 2^{i-1} \right) - 2(v-2), \quad v \geq 3, \quad \text{where } N = 2^v.$$

11.21 Direct computation of M samples of an N -point DFT requires M^2 multiplications, whereas, the Radix-2 FFT algorithm requires $\frac{N}{2} \log_2 N$ multiplications. In order for a N -point radix-2 FFT algorithm to be computationally more efficient than a direct computation of M samples of an N -point DFT, the following inequality must hold: $M > \left\lceil \sqrt{\frac{N}{2} \log_2 N} \right\rceil$.

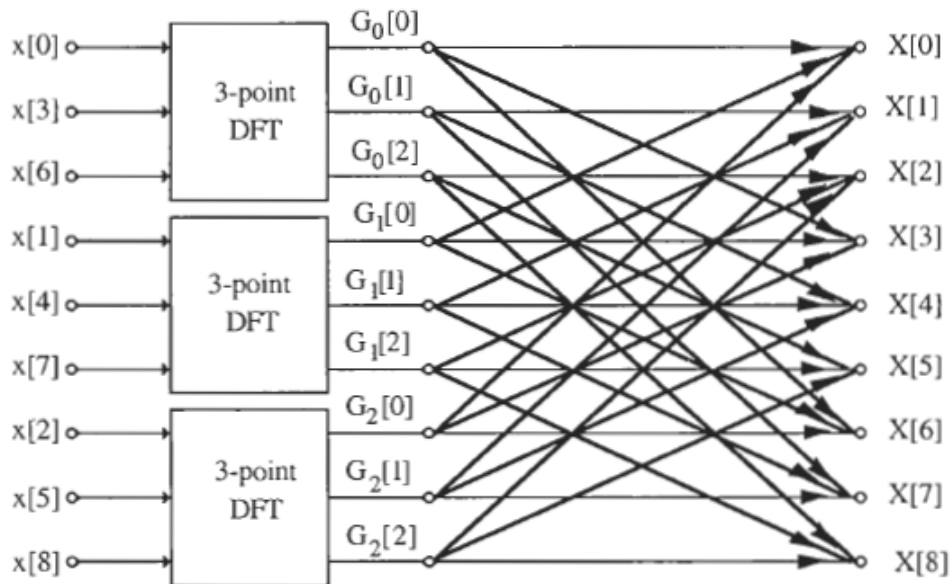
a) $N = 512, M = 49$, b) $N = 1024, M = 72$, c) $N = 2048, M = 107$

11.22 $X(z) = X_0(z^3) + z^{-1}X_1(z^3) + z^{-2}X_2(z^3)$. Thus, the N -point DFT can be expressed as $X[k] = X_0[\langle k \rangle_{N/3}] + W_N^k X_1[\langle k \rangle_{N/3}] + W_N^{2k} X_2[\langle k \rangle_{N/3}]$. Hence, the structural interpretation of the first stage of the radix-3 DFT is as indicated below:

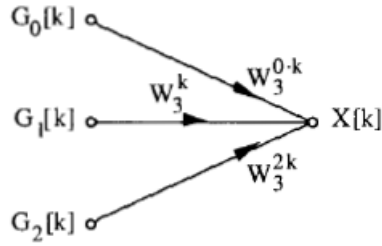


11.23
$$X[k] = \sum_{n=0}^8 x[n]W_9^{nk} = (x[0]W_9^{0 \cdot k} + x[3]W_9^{3k} + x[6]W_9^{6k}) + (x[1]W_9^k + x[4]W_9^{4k} + x[7]W_9^{7k}) + (x[2]W_9^{2k} + x[5]W_9^{5k} + x[8]W_9^{8k})$$

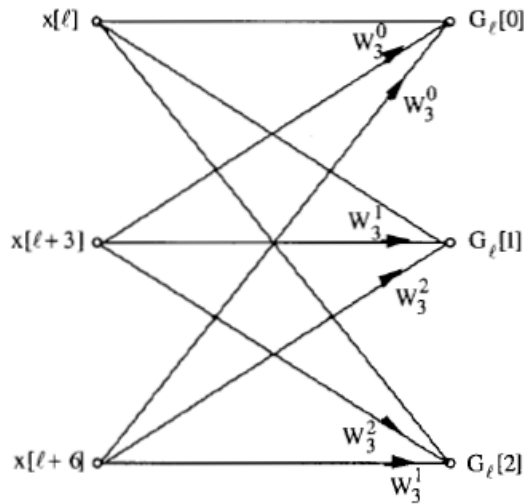
$$= (x[0]W_9^{0 \cdot k} + x[3]W_9^{3k} + x[6]W_9^{6k}) + (x[1]W_9^{0 \cdot k} + x[4]W_9^{3k} + x[7]W_9^{6k})W_9^k + (x[2]W_9^{0 \cdot k} + x[5]W_9^{3k} + x[8]W_9^{6k})W_9^{2k} = G_0[\langle k \rangle_3] + G_1[\langle k \rangle_3]W_9^k + G_2[\langle k \rangle_3]W_9^{2k}$$
, where $G_0[\langle k \rangle_3] = x[0]W_3^{0 \cdot k} + x[3]W_3^k + x[6]W_3^{2k}$, $G_1[\langle k \rangle_3] = x[1]W_3^{0 \cdot k} + x[4]W_3^k + x[7]W_3^{2k}$, $G_2[\langle k \rangle_3] = x[2]W_3^{0 \cdot k} + x[5]W_3^k + x[8]W_3^{2k}$, are 3-point DFTs. A flow-graph representation of this radix-3 DIT FFT computation scheme is shown below,



where the twiddle factors for computing the DFT samples are indicated below for a typical DFT sample:



In the above diagram, the 3-point DFT computation is carried out as indicated below:



11.24
$$X[k] = \sum_{n=0}^{14} x[n]W_{15}^{nk} = (x[0]W_{15}^{0 \cdot k} + x[3]W_{15}^{3k} + x[6]W_{15}^{6k} + x[9]W_{15}^{9k} + x[12]W_{15}^{12k})$$

$$+ (x[1]W_{15}^k + x[4]W_{15}^{4k} + x[7]W_{15}^{7k} + x[10]W_{15}^{10k} + x[13]W_{15}^{13k})$$

$$+ (x[2]W_{15}^{2k} + x[5]W_{15}^{5k} + x[8]W_{15}^{8k} + x[11]W_{15}^{11k} + x[14]W_{15}^{14k})$$

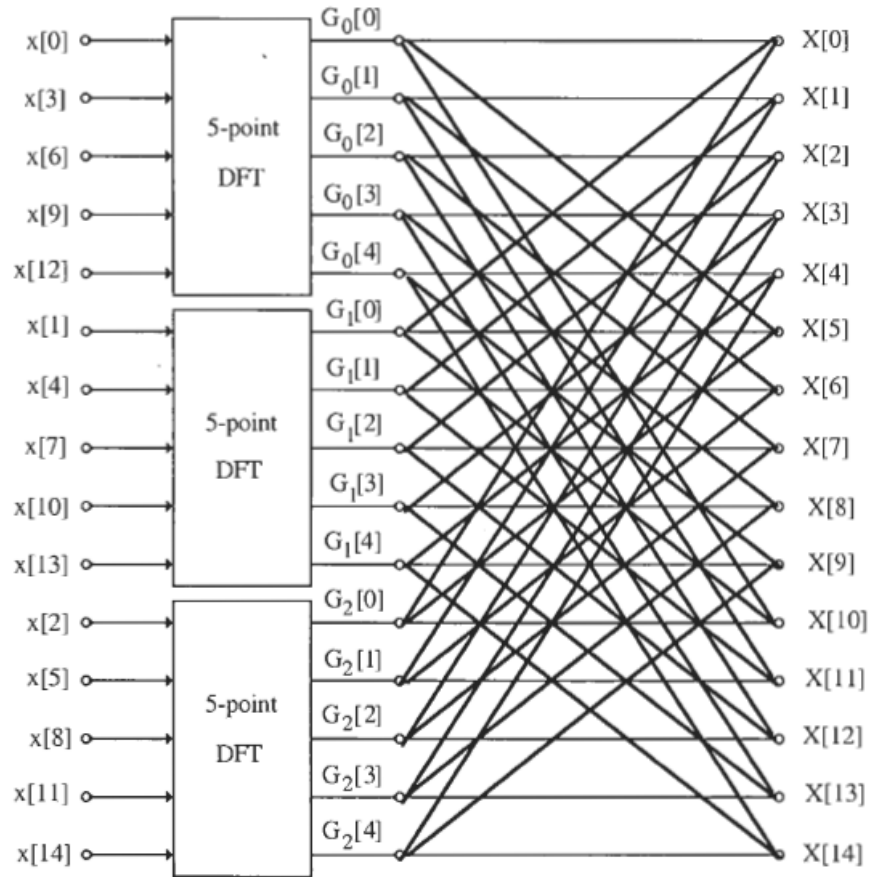
$$= G_0[\langle k \rangle_5] + G_1[\langle k \rangle_5]W_{15}^k + G_2[\langle k \rangle_5]W_{15}^{2k}, \text{ where}$$

$$G_0[\langle k \rangle_5] = x[0]W_5^{0 \cdot k} + x[3]W_5^k + x[6]W_5^{2k} + x[9]W_5^{3k} + x[12]W_5^{4k},$$

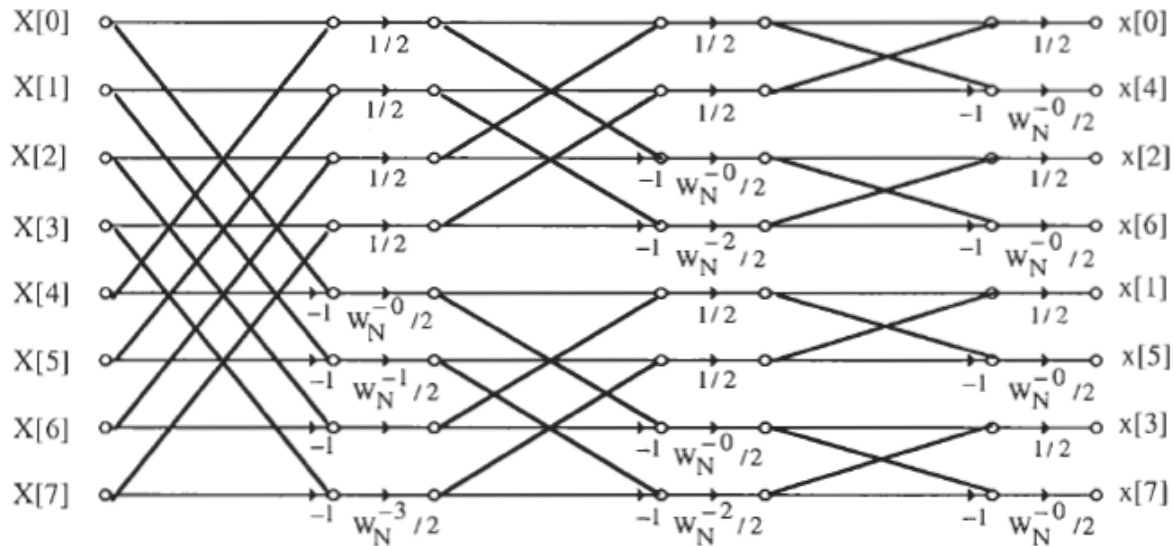
$$G_1[\langle k \rangle_5] = x[1]W_5^{0 \cdot k} + x[4]W_5^k + x[7]W_5^{2k} + x[10]W_5^{3k} + x[13]W_5^{4k}, \text{ and}$$

$$G_2[\langle k \rangle_5] = x[2]W_5^{0 \cdot k} + x[5]W_5^k + x[8]W_5^{2k} + x[11]W_5^{3k} + x[14]W_5^{4k}.$$

A flow-graph representation of this mixed-radix DIT FFT computation scheme is shown below:



11.25



11.26 Now, by definition, $q[n] = \text{Im}\{X[n]\} + j \text{Re}\{X[n]\}$. Its N -point DFT is given by

$$Q[k] = \sum_{n=0}^{N-1} q[n] W_N^{nk}. \text{ Thus,}$$

$$(1): \operatorname{Re}\{Q[k]\} = \sum_{m=0}^{N-1} \left[\operatorname{Im}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) + \operatorname{Re}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) \right],$$

$$(2): \operatorname{Im}\{Q[k]\} = \sum_{m=0}^{N-1} \left[-\operatorname{Im}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) + \operatorname{Re}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) \right].$$

From the definition of the inverse DFT we observe $x[k] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_N^{-mk}$. Hence,

$$(3): \operatorname{Re}\{x[k]\} = \frac{1}{N} \sum_{m=0}^{N-1} \left[\operatorname{Re}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) - \operatorname{Im}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) \right],$$

$$(4): \operatorname{Im}\{x[k]\} = \frac{1}{N} \sum_{m=0}^{N-1} \left[\operatorname{Im}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) + \operatorname{Re}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) \right].$$

Comparing Eqs. (2) and (3) we get $\operatorname{Re}\{x[n]\} = \frac{1}{N} \cdot \operatorname{Im}\{Q[k]\}|_{k=n}$, and comparing Eqs. (1) and (4) we get $\operatorname{Im}\{x[n]\} = \frac{1}{N} \cdot \operatorname{Re}\{Q[k]\}|_{k=n}$.

11.27 $r[n] = X[\langle -n \rangle_N] = \begin{cases} X[0], & \text{if } n = 0, \\ X[N-n], & \text{if } n \neq 0. \end{cases}$ Therefore,

$$\begin{aligned} R[k] &= \sum_{n=0}^{N-1} r[n] W_N^{nk} = r[0] + \sum_{n=1}^{N-1} r[n] W_N^{nk} = X[0] + \sum_{n=1}^{N-1} X[N-n] W_N^{nk} \\ &= X[0] + \sum_{n=1}^{N-1} X[n] W_N^{(N-n)k} = X[0] + \sum_{n=1}^{N-1} X[n] W_N^{-nk} = \sum_{n=0}^{N-1} X[n] W_N^{-nk} = N \cdot x[k]. \text{ Thus,} \\ x[n] &= \frac{1}{N} \cdot R[k]|_{k=n}. \end{aligned}$$

11.28 Let $y[n]$ denote the result of convolving a length- L sequence $x[n]$ with a length- N sequence $h[n]$. The length of $y[n]$ is then $L + N - 1$. Here $L = 16$ and $N = 9$, hence length of $y[n]$ is 24.

Method #1: Direct linear convolution - For a length- L sequence $x[n]$ and a length- N sequence $h[n]$,

$$\# \text{ of real mult.} = 2 \sum_{n=1}^N n + N(L - N - 1) = 2 \sum_{n=1}^9 n + 9(16 - 9 - 1) = 135.$$

Method #2: Linear convolution via circular convolution - Since $y[n]$ is of length 24, to get the correct result we need to pad both sequences with zeros to increase their lengths to 24 before carrying out the circular convolution.

$$\# \text{ of real mult.} = 24 \times 24 = 576.$$

Method #3: Linear convolution via radix-2 FFT - The process involves computing the 16-point FFT $G[k]$ of the length-16 complex sequence $g[n] = x[n] + jh_e[n]$, where $h_e[n]$ is obtained by zero-padding $h[n]$ to length 16. Then, the 16-point DFTs, $X[k]$ and $H_e[k]$, of $x[n]$ and $h_e[n]$, respectively, are recovered from $G[k]$. Finally, the IDFT of the product $Y[k] = X[k] \cdot H_e[k]$ yields $y[n]$.

Now, the first stage of the 16-point radix-2 FFT requires 0 complex multiplications, the second stage requires 0 complex multiplications, the third stage requires 4 complex multiplications, and the last stage requires 6 complex multiplications, resulting in a total of 10 complex multiplications.

of complex mult. to implement $G[k] = 10$

of complex mult. to recover $X[k]$ and $H_e[k]$ from $G[k] = 0$

of complex mult. to form $Y[k] = 16$

of complex mult. to form the IDFT of $Y[k] = 10$

Hence, the total number of complex mult. = 36

A direct implementation of a complex multiplication requires 4 real multiplications resulting in a total of $4 \times 36 = 144$ real multiplications for Method #3. However, if a complex multiply can be implemented using 3 real multiplies (see Problem 11.13), in which case Method #3 requires a total of $3 \times 36 = 108$ real multiplications.

11.29 Let $y[n]$ denote the result of convolving a length- L sequence $x[n]$ with a length- N sequence $h[n]$. The length of $y[n]$ is then $L + N - 1$. Here, $L = 16$ and $N = 10$, hence length of $y[n]$ is 25.

Method #1: Direct linear convolution - For a length- L sequence $x[n]$ and a length- N sequence $h[n]$,

$$\# \text{ of real mult.} = 2 \sum_{n=1}^N n + N(L - N - 1) = 2 \sum_{n=1}^{10} n + 10(16 - 10 - 1) = 160.$$

Method #2: Linear convolution via circular convolution - Since $y[n]$ is of length 24, to get the correct result we need to pad both sequences with zeros to increase their lengths to 24 before carrying out the circular convolution.

of real mult. = $25 \times 25 = 625$.

Method #3: Linear convolution via radix-2 FFT - The process involves computing the 16-point FFT $G[k]$ of the length-16 complex sequence $g[n] = x[n] + jh_e[n]$, where $h_e[n]$ is obtained by zero-padding $h[n]$ to length 16. Then, the 16-point DFTs, $X[k]$ and $H_e[k]$, of $x[n]$ and $h_e[n]$, respectively, are recovered from $G[k]$. Finally, the IDFT of the product $Y[k] = X[k] \cdot H_e[k]$ yields $y[n]$.

Now, the first stage of the 16-point radix-2 FFT requires 0 complex multiplications, the second stage requires 0 complex multiplications, the third stage requires 4 complex

multiplications, and the last stage requires 6 complex multiplications resulting in a total of 10 complex multiplications.

of complex mult. to implement $G[k] = 10$

of complex mult. to recover $X[k]$ and $H_e[k]$ from $G[k] = 0$

of complex mult. to form $Y[k] = 16$

of complex mult. to form the IDFT of $Y[k] = 10$

Hence, the total number of complex mult. = 36

A direct implementation of a complex multiplication requires 4 real multiplications resulting in a total of $4 \times 36 = 144$ real multiplications for Method #3. However, if a complex multiply can be implemented using 3 real multiplies (see Problem 11.13), in which case Method #3 requires a total of $3 \times 36 = 108$ real multiplications.

11.30 (a) Since the impulse response of the filter is of length 72, the transform length N should be greater than 72. If L denotes the number of input samples used for convolution, then $L = N - 71$. So for every L samples of the input sequence, an N -point DFT is computed and multiplied with an N -point DFT of the impulse response sequence $h[n]$ (which needs to be computed only once), and finally an N -point inverse of the product sequence is evaluated. Hence, the total number \mathcal{R}_M of complex multiplications required (assuming N is a power-of-2) is given by $\mathcal{R}_M = \left\lceil \frac{2048}{N-71} \right\rceil (N \log_2 N + N) + \frac{N}{2} \log_2 N$.

It should be noted that in developing the above expression, multiplications due to twiddle factors of values ± 1 and $\pm j$ have not been excluded. The values of \mathcal{R}_M for different values of N are as follows:

1) For $N = 128$, $\mathcal{R}_M = 37312$, 2) For $N = 256$, $\mathcal{R}_M = 28672$, 3) For $N = 512$, $\mathcal{R}_M = 27904$, 4) For $N = 1024$, $\mathcal{R}_M = 38912$.

Hence, $N = 512$ is the appropriate choice for the transform length requiring 27904 complex multiplications or equivalently, $27904 \times 3 = 83712$ real multiplications.

Since the first stage of the FFT calculation process requires only multiplications by ± 1 , the total number of complex multiplications for $N = 128$ is actually

$\mathcal{R}_M = \left\lceil \frac{2048}{N-71} \right\rceil (N \log_2 N + N) + \frac{N}{2} \log_2 N - \frac{N}{2} = 27648$, or equivalently, $27648 \times 3 = 82944$ real multiplications.

(b) For direct convolution, # of real multiplications =

$$2 \sum_{n=1}^N n + N(L - N - 1) = 2 \sum_{n=1}^{72} n + 72(2048 - 72 - 1) = 147456.$$

11.31 (a)

$$\mathbf{V}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & W_8^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^7 \end{bmatrix}, \quad \mathbf{V}_4 = \begin{bmatrix} 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^2 \end{bmatrix},$$

$$\mathbf{V}_2 = \begin{bmatrix} 1 & W_8^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^4 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As can be seen from the above, multiplication by each matrix $\mathbf{V}_k, k = 1, 2, 3$, requires at most 8 complex multiplications.

(b) The transpose of the matrices given in Part (a) are as follows:

$$\mathbf{V}_8^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ W_8^0 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 & 0 & 0 & 0 & W_8^3 \end{bmatrix}, \quad \mathbf{V}_4^t = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ W_8^0 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & W_8^2 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & W_8^0 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_8^2 & 0 & W_8^2 \end{bmatrix},$$

$$\mathbf{V}_2^t = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ W_8^0 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & W_8^0 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_8^0 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_8^0 & W_8^4 \end{bmatrix}, \quad \mathbf{E}^t = \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to show that the flow-graph representation of $\mathbf{D}_8 = \mathbf{E}^t \mathbf{V}_2^t \mathbf{V}_4^t \mathbf{V}_8^t$ is precisely the 8-point DIF FFT algorithm of Figure 11.28.

11.32 $X[2\ell] = \sum_{n=0}^{N-1} x[n]W_N^{2\ell n} = \sum_{n=0}^{\frac{N}{2}-1} x[n]W_N^{2\ell n} + \sum_{n=N/2}^{N-1} x[n]W_N^{2\ell n}, 0 \leq \ell \leq \frac{N}{2} - 1.$ Replacing n by

$n + \frac{N}{2}$ in the right-most sum we get

$$X[2\ell] = \sum_{n=0}^{\frac{N}{2}-1} x[n]W_N^{2\ell n} + \sum_{n=0}^{\frac{N}{2}-1} x[n + \frac{N}{2}]W_N^{2\ell n} = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + x[n + \frac{N}{2}] \right) W_N^{\ell n}, 0 \leq \ell \leq \frac{N}{2} - 1.$$

Likewise,

$$X[4\ell + 1] = \sum_{n=0}^{\frac{N}{4}-1} x[n]W_N^{(4\ell+1)n} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} x[n]W_N^{(4\ell+1)n} + \sum_{n=\frac{N}{2}}^{\frac{3N}{4}-1} x[n]W_N^{(4\ell+1)n} + \sum_{n=\frac{3N}{4}}^{N-1} x[n]W_N^{(4\ell+1)n},$$

where $0 \leq \ell \leq \frac{N}{4} - 1.$ Replacing n by $n + \frac{N}{4}$ in the second sum, n by $n + \frac{N}{2}$ in the third sum, and n by $n + \frac{3N}{4}$ in the fourth sum, we get

$$\begin{aligned} X[4\ell + 1] &= \sum_{n=0}^{\frac{N}{4}-1} x[n]W_N^{4\ell n} W_N^n + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{4}]W_N^{4\ell n} W_N^n W_N^{\ell N} W_N^{N/4} \\ &\quad + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{2}]W_N^{4\ell n} W_N^n W_N^{2\ell N} W_N^{N/2} + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{3N}{4}]W_N^{4\ell n} W_N^n W_N^{3\ell N} W_N^{3N/4}. \end{aligned}$$

Now, $W_N^{\ell N} = W_N^{2\ell N} = W_N^{3\ell N} = 1, W_N^{N/4} = -j, W_N^{N/2} = -1,$ and $W_N^{3N/4} = +j.$ Therefore,

$$X[4\ell + 1] = \sum_{n=0}^{\frac{N}{4}-1} \left\{ \left(x[n] - x[n + \frac{N}{2}] \right) - j \left(x[n + \frac{N}{4}] - x[n + \frac{3N}{4}] \right) \right\} W_N^n W_N^{\ell n}, 0 \leq \ell \leq \frac{N}{4} - 1.$$

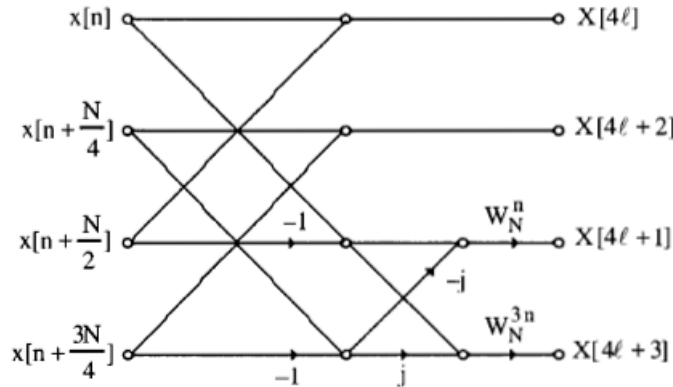
Similarly, $X[4\ell + 3] = \sum_{n=0}^{\frac{N}{4}-1} x[n]W_N^{(4\ell+3)n} + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{4}]W_N^{(4\ell+3)n} W_N^{(4\ell+3)N/4}$

$$+ \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{2}]W_N^{(4\ell+3)n} W_N^{(4\ell+3)N/2} + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{3N}{4}]W_N^{(4\ell+3)n} W_N^{(4\ell+3)3N/4}$$

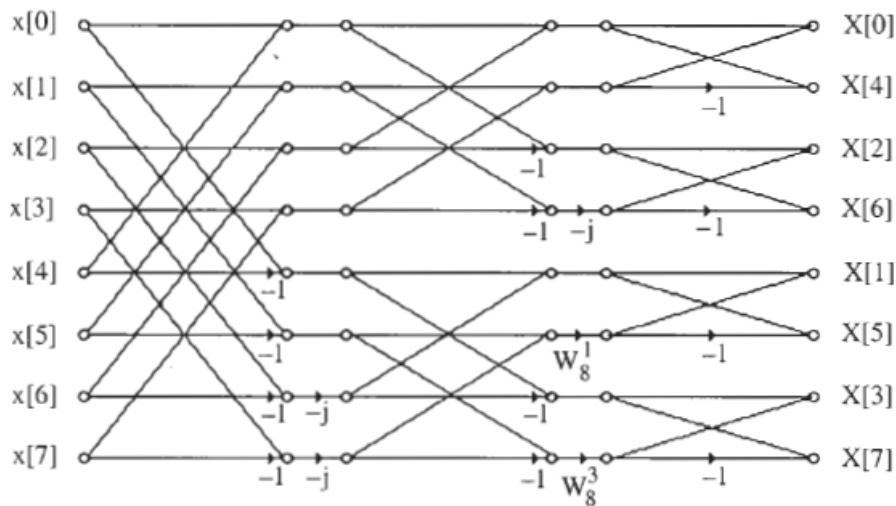
$$= \sum_{n=0}^{\frac{N}{4}-1} x[n]W_N^{4\ell n} W_N^{3n} + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{4}]W_N^{4\ell n} W_N^{3n} W_N^{\ell N} W_N^{3N/4}$$

$$\begin{aligned}
& + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{2}] W_N^{4\ell n} W_N^{3n} W_N^{2\ell N} W_N^{6N/4} + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{4}] W_N^{4\ell n} W_N^{3n} W_N^{3\ell N} W_N^{9N/4} \\
& = \sum_{n=0}^{\frac{N}{4}-1} \left\{ \left(x[n] - x[n + \frac{N}{2}] \right) + j \left(x[n + \frac{N}{4}] - x[n + \frac{3N}{4}] \right) \right\} W_N^{3n} W_N^{\ell n} W_N^{N/4}, \quad 0 \leq \ell \leq \frac{N}{4} - 1.
\end{aligned}$$

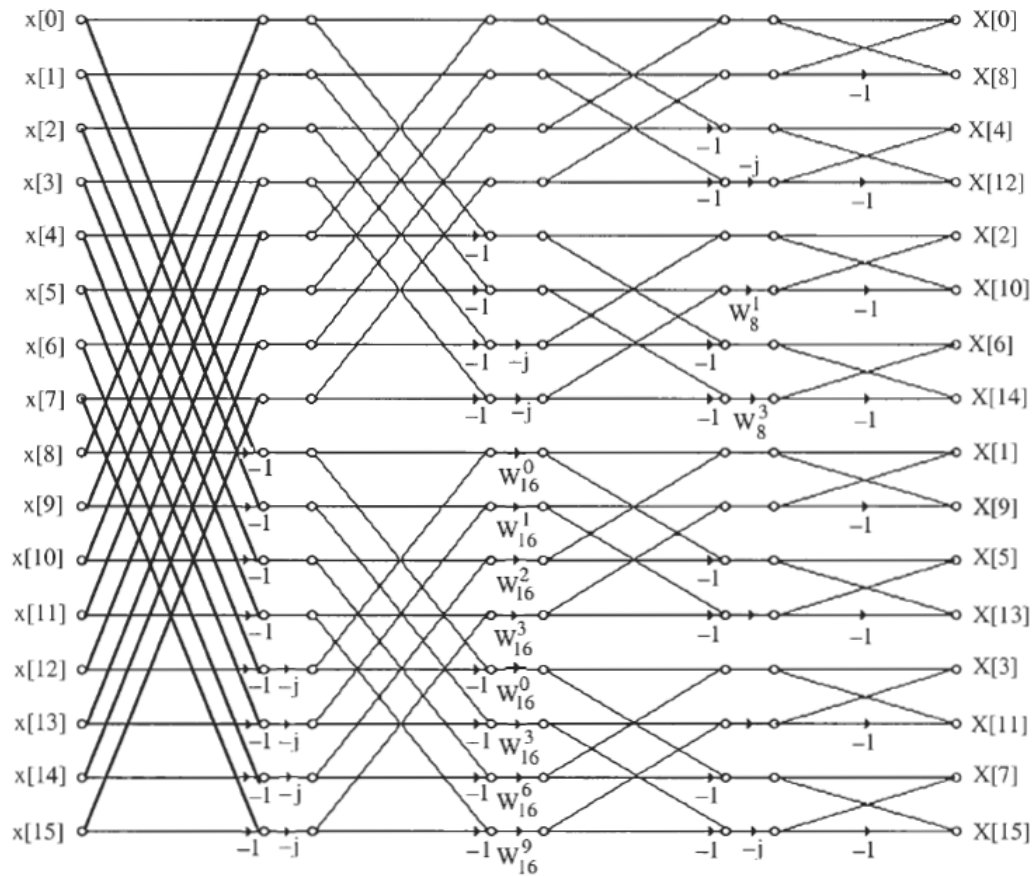
The butterfly here is as shown below which is seen to require two complex multiplications.



11.33 From the flow-graph of the 8-point split-radix FFT algorithm given below it can be seen that the total number of complex multiplications required is 2. On the other hand, the total number of complex multiplications required for a standard DIF FFT algorithm is also 2.



11.34 If multiplications by $\pm j, \pm 1$ are ignored, the flow-graph shown below requires 8 complex multiplications = 24 real multiplications. A radix-2 DIF 16-point FFT algorithm, on the other hand requires 10 complex multiplications = 30 real multiplications.



11.35 (a) $N = 12$. Choose $N_1 = 4$ and $N_2 = 3$. Thus, $n = n_1 + 4n_2$, $\begin{cases} 0 \leq n_1 \leq 3, \\ 0 \leq n_2 \leq 2, \end{cases}$ and

$k = 3k_1 + k_2$, $\begin{cases} 0 \leq k_1 \leq 3, \\ 0 \leq k_2 \leq 2. \end{cases}$ The corresponding index mappings are indicated below:

| | | | | |
|----------------------|--------|--------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 |
| 0 | $x[0]$ | $x[1]$ | $x[2]$ | $x[3]$ |
| 1 | $x[4]$ | $x[5]$ | $x[6]$ | $x[7]$ |
| 2 | $x[8]$ | $x[9]$ | $x[10]$ | $x[11]$ |

| | | | | |
|----------------------|--------|--------|--------|---------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 |
| 0 | $X[0]$ | $X[3]$ | $X[6]$ | $X[9]$ |
| 1 | $X[1]$ | $X[4]$ | $X[7]$ | $X[10]$ |
| 2 | $X[2]$ | $X[5]$ | $X[8]$ | $X[11]$ |

(b) $N = 15$. Choose $N_1 = 3$ and $N_2 = 5$. Thus, $n = n_1 + 3n_2$, $\begin{cases} 0 \leq n_1 \leq 2, \\ 0 \leq n_2 \leq 4, \end{cases}$ and

$k = 5k_1 + k_2$, $\begin{cases} 0 \leq k_1 \leq 2, \\ 0 \leq k_2 \leq 4. \end{cases}$ The corresponding index mappings are indicated below:

| | | | |
|----------------------|---------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 |
| 0 | $x[0]$ | $x[1]$ | $x[2]$ |
| 1 | $x[3]$ | $x[4]$ | $x[5]$ |
| 2 | $x[6]$ | $x[7]$ | $x[8]$ |
| 3 | $x[9]$ | $x[10]$ | $x[11]$ |
| 4 | $x[12]$ | $x[13]$ | $x[14]$ |

| | | | |
|----------------------|--------|--------|---------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 |
| 0 | $X[0]$ | $X[5]$ | $X[10]$ |
| 1 | $X[1]$ | $X[6]$ | $X[11]$ |
| 2 | $X[2]$ | $X[7]$ | $X[12]$ |
| 3 | $X[3]$ | $X[8]$ | $X[13]$ |
| 4 | $X[4]$ | $X[9]$ | $X[14]$ |

(c) $N = 21$. Choose $N_1 = 7$ and $N_2 = 3$. Thus, $n = n_1 + 7n_2$, $\begin{cases} 0 \leq n_1 \leq 6, \\ 0 \leq n_2 \leq 2, \end{cases}$ and

$k = 3k_1 + k_2$, $\begin{cases} 0 \leq k_1 \leq 6, \\ 0 \leq k_2 \leq 2. \end{cases}$ The corresponding index mappings are indicated below:

| | | | | | | | | | | | | | | | |
|----------------------|---------|---------|---------|---------|---------|---------|---------|----------------------|--------|--------|--------|---------|---------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | $x[0]$ | $x[1]$ | $x[2]$ | $x[3]$ | $x[4]$ | $x[5]$ | $x[6]$ | 0 | $X[0]$ | $X[3]$ | $X[6]$ | $X[9]$ | $X[12]$ | $X[15]$ | $X[18]$ |
| 1 | $x[7]$ | $x[8]$ | $x[9]$ | $x[10]$ | $x[11]$ | $x[12]$ | $x[13]$ | 1 | $X[1]$ | $X[4]$ | $X[7]$ | $X[10]$ | $X[13]$ | $X[16]$ | $X[19]$ |
| 2 | $x[14]$ | $x[15]$ | $x[16]$ | $x[17]$ | $x[18]$ | $x[19]$ | $x[20]$ | 2 | $X[2]$ | $X[5]$ | $X[8]$ | $X[11]$ | $X[14]$ | $X[17]$ | $X[20]$ |

(d) $N = 35$. Choose $N_1 = 7$ and $N_2 = 5$. Thus, $n = n_1 + 7n_2$, $\begin{cases} 0 \leq n_1 \leq 6, \\ 0 \leq n_2 \leq 4, \end{cases}$ and

$k = 5k_1 + k_2$, $\begin{cases} 0 \leq k_1 \leq 6, \\ 0 \leq k_2 \leq 4. \end{cases}$ The corresponding index mappings are indicated below:

| | | | | | | | |
|----------------------|---------|---------|---------|---------|---------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | $x[0]$ | $x[1]$ | $x[2]$ | $x[3]$ | $x[4]$ | $x[5]$ | $x[6]$ |
| 1 | $x[7]$ | $x[8]$ | $x[9]$ | $x[10]$ | $x[11]$ | $x[12]$ | $x[13]$ |
| 2 | $x[14]$ | $x[15]$ | $x[16]$ | $x[17]$ | $x[18]$ | $x[19]$ | $x[20]$ |
| 3 | $x[21]$ | $x[22]$ | $x[23]$ | $x[24]$ | $x[25]$ | $x[26]$ | $x[27]$ |
| 4 | $x[28]$ | $x[29]$ | $x[30]$ | $x[31]$ | $x[32]$ | $x[33]$ | $x[34]$ |

| | | | | | | | |
|----------------------|--------|--------|---------|---------|---------|---------|---------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | $X[0]$ | $X[5]$ | $X[10]$ | $X[15]$ | $X[20]$ | $X[25]$ | $X[30]$ |
| 1 | $X[1]$ | $X[6]$ | $X[11]$ | $X[16]$ | $X[21]$ | $X[26]$ | $X[31]$ |
| 2 | $X[2]$ | $X[7]$ | $X[12]$ | $X[17]$ | $X[22]$ | $X[27]$ | $X[32]$ |
| 3 | $X[3]$ | $X[8]$ | $X[13]$ | $X[18]$ | $X[23]$ | $X[28]$ | $X[33]$ |
| 4 | $X[4]$ | $X[9]$ | $X[14]$ | $X[19]$ | $X[24]$ | $X[29]$ | $X[34]$ |

11.36 (a) $N = 12$. Choose $N_1 = 4$ and $N_2 = 3$. $A = 3$, $B = 4$, $C = 3\langle 3^{-1} \rangle_4 = 9$,

$D = 4\langle 4^{-1} \rangle_3 = 4$. Thus, $n = \langle 3n_1 + 4n_2 \rangle_{12}$, $\begin{cases} 0 \leq n_1 \leq 3, \\ 0 \leq n_2 \leq 2, \end{cases}$ $k = \langle 9k_1 + 4k_2 \rangle_{12}$, $\begin{cases} 0 \leq k_1 \leq 3, \\ 0 \leq k_2 \leq 2. \end{cases}$

The corresponding index mappings are indicated below:

| | | | | | | | | | |
|----------------------|--------|---------|---------|--------|----------------------|--------|--------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 | $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 |
| 0 | $x[0]$ | $x[3]$ | $x[6]$ | $x[9]$ | 0 | $X[0]$ | $X[9]$ | $X[6]$ | $X[3]$ |
| 1 | $x[4]$ | $x[7]$ | $x[10]$ | $x[1]$ | 1 | $X[4]$ | $X[1]$ | $X[10]$ | $X[7]$ |
| 2 | $x[8]$ | $x[11]$ | $x[2]$ | $x[5]$ | 2 | $X[8]$ | $X[5]$ | $X[2]$ | $X[11]$ |

(b) $N = 15$. Choose $N_1 = 3$ and $N_2 = 5$. $A = 5$, $B = 3$, $C = 5\langle 5^{-1} \rangle_3 = 10$, $D = 3\langle 3^{-1} \rangle_5 = 6$.

Thus, $n = \langle 5n_1 + 3n_2 \rangle_{15}$, $\begin{cases} 0 \leq n_1 \leq 2, \\ 0 \leq n_2 \leq 4, \end{cases}$ $k = \langle 10k_1 + 6k_2 \rangle_{15}$, $\begin{cases} 0 \leq k_1 \leq 2, \\ 0 \leq k_2 \leq 4. \end{cases}$

The corresponding index mappings are indicated below:

| | | | |
|----------------------|-------|-------|-------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 |
| 0 | x[0] | x[5] | x[10] |
| 1 | x[3] | x[8] | x[13] |
| 2 | x[6] | x[11] | x[1] |
| 3 | x[9] | x[14] | x[4] |
| 4 | x[12] | x[2] | x[7] |

| | | | |
|----------------------|-------|-------|-------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 |
| 0 | X[0] | X[10] | X[5] |
| 1 | X[6] | X[1] | X[11] |
| 2 | X[12] | X[7] | X[2] |
| 3 | X[3] | X[13] | X[8] |
| 4 | X[9] | X[4] | X[14] |

(c) $N = 21$. Choose $N_1 = 7$ and $N_2 = 3$. $A = 3$, $B = 7$, $C = 3\langle 3^{-1} \rangle_4 = 3 \times 5 = 15$,

$D = 7\langle 7^{-1} \rangle_3 = 7 \times 1 = 7$. Thus, $n = \langle 3n_1 + 7n_2 \rangle_{21}$, $\begin{cases} 0 \leq n_1 \leq 6, \\ 0 \leq n_2 \leq 2, \end{cases}$

$k = \langle 15k_1 + 7k_2 \rangle_{21}$, $\begin{cases} 0 \leq k_1 \leq 6, \\ 0 \leq k_2 \leq 2. \end{cases}$ The corresponding index mappings are indicated

below:

| | | | | | | | |
|----------------------|-------|-------|-------|-------|-------|-------|-------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | x[0] | x[3] | x[6] | x[9] | x[12] | x[15] | x[18] |
| 1 | x[7] | x[10] | x[13] | x[16] | x[19] | x[1] | x[4] |
| 2 | x[14] | x[17] | x[20] | x[2] | x[5] | x[8] | x[11] |

| | | | | | | | |
|----------------------|-------|-------|-------|-------|-------|-------|-------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | X[0] | X[15] | X[9] | X[3] | X[18] | X[12] | X[6] |
| 1 | X[7] | X[1] | X[16] | X[10] | X[4] | X[19] | X[13] |
| 2 | X[14] | X[8] | X[2] | X[17] | X[11] | X[5] | X[20] |

(d) $N = 35$. Choose $N_1 = 7$ and $N_2 = 5$. $A = 5$, $B = 7$, $C = 5\langle 5^{-1} \rangle_7 = 5 \times 3 = 15$,

$D = 7\langle 7^{-1} \rangle_5 = 7 \times 3 = 21$. Thus, $n = \langle 5n_1 + 7n_2 \rangle_{35}$, $\begin{cases} 0 \leq n_1 \leq 6, \\ 0 \leq n_2 \leq 4, \end{cases}$

$k = \langle 15k_1 + 21k_2 \rangle_{35}$, $\begin{cases} 0 \leq k_1 \leq 6, \\ 0 \leq k_2 \leq 4. \end{cases}$ The corresponding index mappings are indicated

below:

| | | | | | | | |
|----------------------|-------|-------|-------|-------|-------|-------|-------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | x[0] | x[5] | x[10] | x[15] | x[20] | x[25] | x[30] |
| 1 | x[7] | x[12] | x[17] | x[22] | x[27] | x[32] | x[2] |
| 2 | x[14] | x[19] | x[24] | x[29] | x[34] | x[4] | x[9] |
| 3 | x[21] | x[26] | x[31] | x[1] | x[6] | x[11] | x[16] |
| 4 | x[28] | x[33] | x[3] | x[8] | x[13] | x[18] | x[23] |

| | | | | | | | |
|----------------------|-------|-------|-------|-------|-------|-------|-------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | X[0] | X[15] | X[30] | X[10] | X[25] | X[5] | X[20] |
| 1 | X[21] | X[1] | X[16] | X[31] | X[11] | X[26] | X[6] |
| 2 | X[7] | X[22] | X[2] | X[17] | X[32] | X[12] | X[27] |
| 3 | X[28] | X[8] | X[23] | X[3] | X[18] | X[33] | X[13] |
| 4 | X[14] | X[29] | X[9] | X[24] | X[4] | X[19] | X[34] |

11.37 $N = 12$. Choose $N_1 = 4$ and $N_2 = 3$. $A = 3$, $B = 4$, $C = 3\langle 3^{-1} \rangle_4 = 9$, $D = 4\langle 4^{-1} \rangle_3 = 4$.

$$\text{Thus, } n = \langle 3n_1 + 4n_2 \rangle_{12}, \begin{cases} 0 \leq n_1 \leq 3, \\ 0 \leq n_2 \leq 2, \end{cases} \quad k = \langle 9k_1 + 4k_2 \rangle_{12}, \begin{cases} 0 \leq k_1 \leq 3, \\ 0 \leq k_2 \leq 2. \end{cases}$$

The corresponding index mappings are indicated below:

| | | | | |
|----------------------|--------|---------|---------|--------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 |
| 0 | $x[0]$ | $x[3]$ | $x[6]$ | $x[9]$ |
| 1 | $x[4]$ | $x[7]$ | $x[10]$ | $x[1]$ |
| 2 | $x[8]$ | $x[11]$ | $x[2]$ | $x[5]$ |

| | | | | |
|----------------------|--------|--------|---------|---------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 |
| 0 | $X[0]$ | $X[9]$ | $X[6]$ | $X[3]$ |
| 1 | $X[4]$ | $X[1]$ | $X[10]$ | $X[7]$ |
| 2 | $X[8]$ | $X[5]$ | $X[2]$ | $X[11]$ |

$$\text{Alternately, } k = \langle 9n_1 + 4n_2 \rangle_{12}, \begin{cases} 0 \leq n_1 \leq 3, \\ 0 \leq n_2 \leq 2, \end{cases} \quad k = \langle 3k_1 + 4k_2 \rangle_{12}, \begin{cases} 0 \leq k_1 \leq 3, \\ 0 \leq k_2 \leq 2. \end{cases}$$

The corresponding index mappings are indicated below:

| | | | | |
|----------------------|--------|--------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 |
| 0 | $x[0]$ | $x[9]$ | $x[6]$ | $x[3]$ |
| 1 | $x[4]$ | $x[1]$ | $x[10]$ | $x[7]$ |
| 2 | $x[8]$ | $x[5]$ | $x[2]$ | $x[11]$ |

| | | | | |
|----------------------|--------|---------|---------|--------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 |
| 0 | $X[0]$ | $X[3]$ | $X[6]$ | $X[9]$ |
| 1 | $X[4]$ | $X[7]$ | $X[10]$ | $X[1]$ |
| 2 | $X[8]$ | $X[11]$ | $X[2]$ | $X[5]$ |

Hence, $X[2k] = Y[2k]$, and $X[2k+1] = Y[\langle 6 + (2k+1) \rangle_{12}]$, $k = 0, 1, \dots, 5$.

11.38 (a) $N = 6$. Choose $N_1 = 2$ and $N_2 = 3$. $A = 3$, $B = 2$, $C = 3\langle 3^{-1} \rangle_2 = 3$, $D = 2\langle 2^{-1} \rangle_3 = 4$.

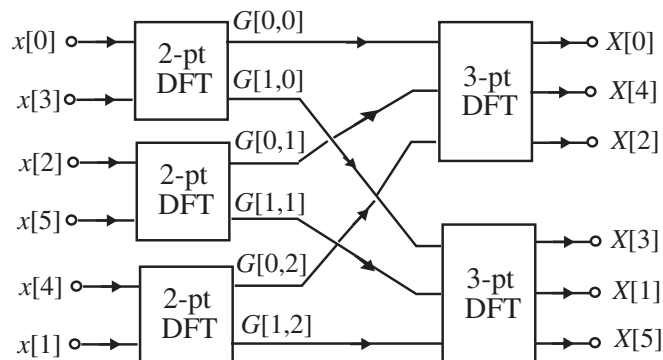
$$\text{Thus, } n = \langle 3n_1 + 2n_2 \rangle_6, \begin{cases} 0 \leq n_1 \leq 1, \\ 0 \leq n_2 \leq 2, \end{cases} \quad k = \langle 3k_1 + 4k_2 \rangle_6, \begin{cases} 0 \leq k_1 \leq 1, \\ 0 \leq k_2 \leq 2. \end{cases}$$

The corresponding index mappings are indicated below:

| | | |
|----------------------|--------|--------|
| $n_2 \backslash n_1$ | 0 | 1 |
| 0 | $x[0]$ | $x[3]$ |
| 1 | $x[2]$ | $x[5]$ |
| 2 | $x[4]$ | $x[1]$ |

| | | |
|----------------------|----------|----------|
| $k_2 \backslash n_1$ | 0 | 1 |
| 0 | $G[0,0]$ | $G[1,0]$ |
| 1 | $G[0,1]$ | $G[1,1]$ |
| 2 | $G[0,2]$ | $G[1,2]$ |

| | | |
|----------------------|--------|--------|
| $k_2 \backslash k_1$ | 0 | 1 |
| 0 | $X[0]$ | $X[3]$ |
| 1 | $X[4]$ | $X[1]$ |
| 2 | $X[2]$ | $X[5]$ |



(b) $N = 10$. Choose $N_1 = 2$ and $N_2 = 5$. $A = 5$, $B = 2$, $C = 5\langle 5^{-1} \rangle_2 = 5$, $D = 2\langle 2^{-1} \rangle_5 = 6$.

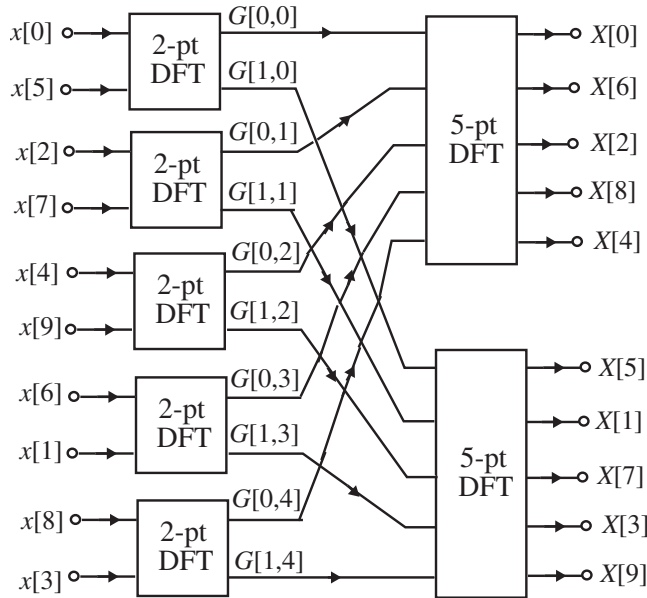
Thus, $n = \langle 5n_1 + 2n_2 \rangle_{10}$, $\begin{cases} 0 \leq n_1 \leq 4, \\ 0 \leq n_2 \leq 2, \end{cases}$ $k = \langle 5k_1 + 6k_2 \rangle_{10}$, $\begin{cases} 0 \leq k_1 \leq 4, \\ 0 \leq k_2 \leq 2. \end{cases}$

The corresponding index mappings are indicated below:

| | | |
|----------------------|-----------|---|
| $n_2 \backslash n_1$ | 0 | 1 |
| 0 | x[0] x[5] | |
| 1 | x[2] x[7] | |
| 2 | x[4] x[9] | |
| 3 | x[6] x[1] | |
| 4 | x[8] x[3] | |

| | | |
|----------------------|---------------|---|
| $k_2 \backslash n_1$ | 0 | 1 |
| 0 | G[0,0] G[1,0] | |
| 1 | G[0,1] G[1,1] | |
| 2 | G[0,2] G[1,2] | |
| 3 | G[0,3] G[1,3] | |
| 4 | G[0,4] G[1,4] | |

| | | |
|----------------------|-----------|---|
| $k_2 \backslash k_1$ | 0 | 1 |
| 0 | X[0] X[5] | |
| 1 | X[6] X[1] | |
| 2 | X[2] X[7] | |
| 3 | X[8] X[3] | |
| 4 | X[4] X[9] | |



(c) $N = 12$. Choose $N_1 = 3$ and $N_2 = 4$. $A = 4$, $B = 3$, $C = 4\langle 4^{-1} \rangle_3 = 4$, $D = 3\langle 3^{-1} \rangle_4 = 3$.

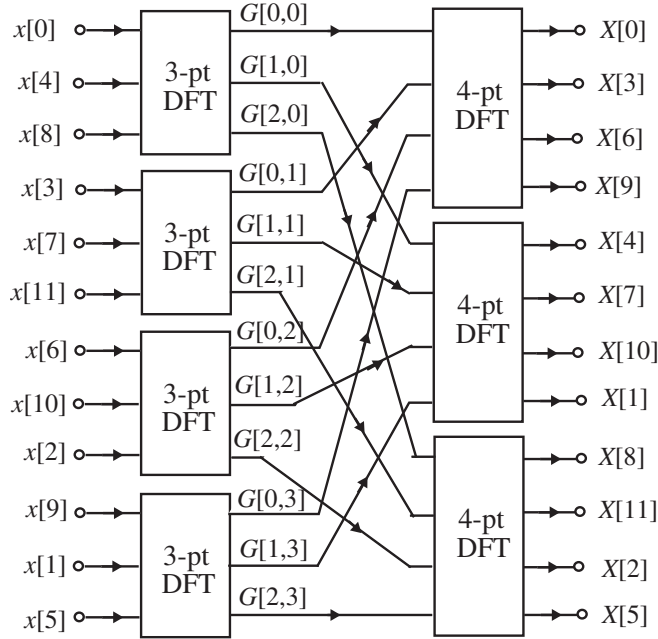
Thus, $n = \langle 4n_1 + 3n_2 \rangle_{12}$, $\begin{cases} 0 \leq n_1 \leq 2, \\ 0 \leq n_2 \leq 3, \end{cases}$ $k = \langle 4k_1 + 3k_2 \rangle_{12}$, $\begin{cases} 0 \leq k_1 \leq 2, \\ 0 \leq k_2 \leq 3. \end{cases}$

The corresponding index mappings are indicated below:

| | | | |
|----------------------|-----------------|---|---|
| $n_2 \backslash n_1$ | 0 | 1 | 2 |
| 0 | x[0] x[4] x[8] | | |
| 1 | x[3] x[7] x[11] | | |
| 2 | x[6] x[10] x[2] | | |
| 3 | x[9] x[1] x[5] | | |

| | | | |
|----------------------|----------------------|---|---|
| $k_2 \backslash k_1$ | 0 | 1 | 2 |
| 0 | G[0,0] G[1,0] G[2,0] | | |
| 1 | G[0,1] G[1,1] G[2,1] | | |
| 2 | G[0,2] G[1,2] G[2,2] | | |
| 3 | G[0,3] G[1,3] G[2,3] | | |

| | | | |
|----------------------|-----------------|---|---|
| $k_2 \backslash k_1$ | 0 | 1 | 2 |
| 0 | X[0] X[4] X[8] | | |
| 1 | X[3] X[7] X[11] | | |
| 2 | X[6] X[10] X[2] | | |
| 3 | X[9] X[1] X[5] | | |



(d) $N = 15$. Choose $N_1 = 5$ and $N_2 = 3$. $A = 3$, $B = 5$, $C = 3\langle 3^{-1} \rangle_5 = 10$,

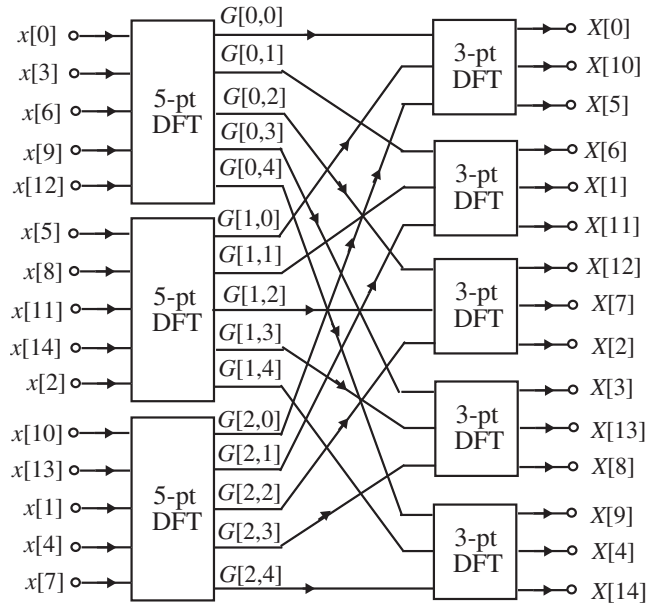
$D = 5\langle 5^{-1} \rangle_3 = 6$. Thus, $n = \langle 5n_1 + 3n_2 \rangle_{10}$, $\begin{cases} 0 \leq n_1 \leq 4, \\ 0 \leq n_2 \leq 2, \end{cases}$ $k = \langle 10k_1 + 6k_2 \rangle_{15}$, $\begin{cases} 0 \leq k_1 \leq 4, \\ 0 \leq k_2 \leq 2. \end{cases}$

The corresponding index mappings are indicated below:

| | | | | | |
|----------------------|---------|---------|---------|---------|---------|
| $n_2 \backslash n_1$ | 0 | 1 | 2 | 3 | 4 |
| 0 | $x[0]$ | $x[3]$ | $x[6]$ | $x[9]$ | $x[12]$ |
| 1 | $x[5]$ | $x[8]$ | $x[11]$ | $x[14]$ | $x[2]$ |
| 2 | $x[10]$ | $x[13]$ | $x[1]$ | $x[4]$ | $x[7]$ |

| | | | | |
|----------------------|----------|----------|----------|----------|
| $k_2 \backslash n_1$ | 0 | 1 | 2 | 3 |
| 0 | $G[0,0]$ | $G[1,0]$ | $G[2,0]$ | $G[3,0]$ |
| 1 | $G[0,1]$ | $G[1,1]$ | $G[2,1]$ | $G[3,1]$ |
| 2 | $G[0,2]$ | $G[1,2]$ | $G[2,2]$ | $G[3,2]$ |

| | | | | | |
|----------------------|---------|---------|---------|---------|---------|
| $k_2 \backslash k_1$ | 0 | 1 | 2 | 3 | 4 |
| 0 | $X[0]$ | $X[6]$ | $X[12]$ | $X[3]$ | $X[9]$ |
| 1 | $X[10]$ | $X[1]$ | $X[7]$ | $X[13]$ | $X[4]$ |
| 2 | $X[5]$ | $X[11]$ | $X[2]$ | $X[8]$ | $X[14]$ |



11.39 Note that $1536 = 256 \times 6$. Now an N -point DFT, with N divisible by 6, can be computed

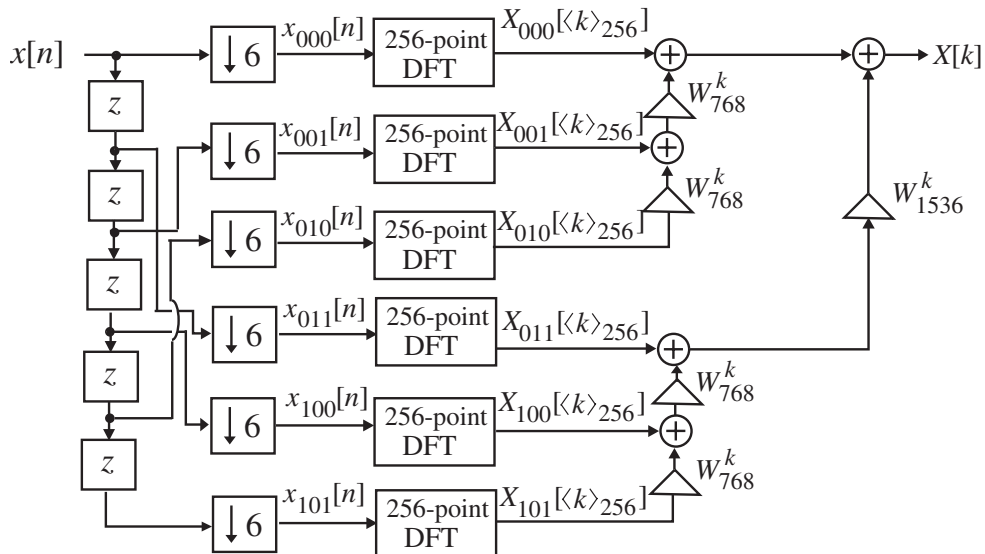
as follows:
$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk} = X_0[\langle k \rangle_{N/6}] + W_N^k X_1[\langle k \rangle_{N/6}] + W_N^{2k} X_2[\langle k \rangle_{N/6}]$$

$+ W_N^{3k} X_3[\langle k \rangle_{N/6}] + W_N^{4k} X_4[\langle k \rangle_{N/6}] + W_N^{5k} X_5[\langle k \rangle_{N/6}]$, where

$$X_\ell[\langle k \rangle_{N/6}] = \sum_{r=0}^{\frac{N}{6}-1} x[6r + \ell]W_{N/6}^{rk}, \quad 0 \leq \ell \leq 5.$$
 For $N = 1536$, we thus get

$$X[k] = X_0[\langle k \rangle_{256}] + W_{1536}^k X_1[\langle k \rangle_{256}] + W_{1536}^{2k} X_2[\langle k \rangle_{256}] + W_{1536}^{3k} X_3[\langle k \rangle_{256}]$$

$+ W_{1536}^{4k} X_4[\langle k \rangle_{256}] + W_{1536}^{5k} X_5[\langle k \rangle_{256}]$, where $X_\ell[\langle k \rangle_{256}] = \sum_{r=0}^{511} x[6r + \ell]W_{256}^{rk}, \quad 0 \leq \ell \leq 5.$



Now an N -point FFT algorithm requires $\frac{N}{2} \log_2 N$ complex multiplications and $N \log_2 N$ complex additions. Hence, an $\frac{N}{6}$ -point FFT algorithm requires $\frac{N}{12} \log_2(N/6)$ complex multiplications and $\frac{N}{6} \log_2(N/6)$ complex additions. In addition, we need $5N$ complex multiplications and $5N$ complex additions to compute the N -point DFT $X[k]$. Hence, for $N = 1536$, the evaluation of $X[k]$ using six 256-point FFT modules requires $\frac{N}{12} \log_2(N/6) + 5N = 128 \times \log_2(256) + 5 \times 1536 = 8704$ complex multiplications and $\frac{N}{6} \log_2(N/6) + 5N = 256 \times \log_2(256) + 5 \times 1536 = 9728$ complex additions. It should be noted that a direct computation of the 3072-point DFT would require 6431296 complex multiplications and 2357760 complex additions.

11.40 (a) # of zero-valued samples to be added is $512 - 498 = 14$.

(b) Direct computation of a 512-point DFT of a length-498 sequence requires $(498)^2 = 248004$ complex multiplications and $497 \times 498 = 247506$ complex additions.

(c) A 512-point Cooley-Tukey type FFT algorithm requires $256 \times \log_2(512) = 2304$ complex multiplications and $512 \times \log_2(512) = 46308$ complex additions.

11.41 $z_\ell = \alpha^\ell$. Hence, $A_o V_o^{-\ell} e^{j\theta_o} e^{-j\ell\phi_o} = \alpha^\ell$. Since α is real, we have $A_o = 1$, $V_o = 1/\alpha$, $\theta_o = 0$, $\phi_o = 0$.

11.42 (a) $Y(z) = H(z)X(z)$ or $y[0] + y[1]z^{-1} + y[2]z^{-2} = (h[0] + h[1]z^{-1})(x[0] + x[1]z^{-1})$. Now,
 $Y(z_0) = Y(-1) = y[0] - y[1] + y[2] = H(-1)X(-1) = (h[0] - h[1])(x[0] - x[1])$,
 $Y(z_1) = Y(\infty) = y[0] = H(\infty)X(\infty) = h[0]x[0]$,
 $Y(z_2) = Y(1) = y[0] + y[1] + y[2] = H(1)X(1) = \{h[0] + h[1]\}(x[0] + x[1])$.

From Eqs. (6.114) and (6.115), we arrive at

$$Y(z) = \frac{I_0(z)}{I_0(z_0)} Y(z_0) + \frac{I_1(z)}{I_1(z_1)} Y(z_1) + \frac{I_2(z)}{I_2(z_2)} Y(z_2), \text{ where } I_0(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}),$$

$$I_1(z) = (1 - z_0 z^{-1})(1 - z_2 z^{-1}), \quad I_2(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1}). \text{ Therefore,}$$

$$\frac{I_0(z)}{I_0(z_0)} = \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - z_1 z_0^{-1})(1 - z_2 z_0^{-1})} = -\frac{1}{2} z^{-1} (1 - z^{-1}),$$

$$\frac{I_1(z)}{I_1(z_1)} = \frac{(1 - z_0 z^{-1})(1 - z_2 z^{-1})}{(1 - z_0 z_1^{-1})(1 - z_2 z_1^{-1})} = (1 - z^{-2}), \text{ and}$$

$$\frac{I_2(z)}{I_2(z_0)} = \frac{(1 - z_0 z^{-1})(1 - z_1 z^{-1})}{(1 - z_0 z_2^{-1})(1 - z_1 z_2^{-1})} = \frac{1}{2} z^{-1} (1 + z^{-1}). \text{ Hence,}$$

$$\begin{aligned}
Y(z) &= -\frac{1}{2}z^{-1}(1-z^{-1})Y(z_0) + (1-z^{-2})Y(z_1) + \frac{1}{2}z^{-1}(1+z^{-1})Y(z_2) \\
&= Y(z_1) + \left(-\frac{1}{2}Y(z_0) + \frac{1}{2}Y(z_2)\right)z^{-1} + \left(\frac{1}{2}Y(z_0) - Y(z_1) + \frac{1}{2}Y(z_2)\right)z^{-2} \\
&= h[0]x[0] + \left(-\frac{1}{2}(h[0] - h[1])(x[0] - x[1]) + \frac{1}{2}(h[0] + h[1])(x[0] + x[1])\right)z^{-1} \\
&\quad + \left(\frac{1}{2}(h[0] - h[1])(x[0] - x[1]) - h[0]x[0] + \frac{1}{2}(h[0] + h[1])(x[0] + x[1])\right)z^{-2} \\
&= h[0]x[0] + (h[0]x[1] + h[1]x[0])z^{-1} + h[1]x[1]z^{-2}.
\end{aligned}$$

Ignoring the multiplications by $\frac{1}{2}$, computation of the coefficients of $Y(z)$ require the values of $Y(z_0)$, $Y(z_1)$, and $Y(z_2)$, which can be evaluated using only 3 multiplications.

(b) $Y(z) = H(z)X(z)$ or

$$y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + y[4]z^{-4} = (h[0] + h[1]z^{-1} + h[2]z^{-2})(x[0] + x[1]z^{-1} + x[2]z^{-2})$$

Now,

$$Y(z_0) = Y(-\frac{1}{2}) = (h[0] - 2h[1] + 4h[2])(x[0] - 2x[1] + 4x[2]),$$

$$Y(z_1) = Y(-1) = (h[0] - h[1] + h[2])(x[0] - x[1] + x[2]), \quad Y(z_2) = Y(\infty) = h[0]x[0],$$

$$Y(z_3) = Y(1) = (h[0] + h[1] + h[2])(x[0] + x[1] + x[2]),$$

$$Y(z_4) = Y(\frac{1}{2}) = (h[0] + 2h[1] + 4h[2])(x[0] + 2x[1] + 4x[2]).$$

From Eqs. (6.114) and (6.115), we arrive at

$$Y(z) = \frac{I_0(z)}{I_0(z_0)}Y(z_0) + \frac{I_1(z)}{I_1(z_1)}Y(z_1) + \frac{I_2(z)}{I_2(z_2)}Y(z_2) + \frac{I_3(z)}{I_3(z_3)}Y(z_3) + \frac{I_4(z)}{I_4(z_4)}Y(z_4), \text{ where}$$

$$I_0(z) = (1 - z_1z^{-1})(1 - z_2z^{-1})(1 - z_3z^{-1})(1 - z_4z^{-1}),$$

$$I_1(z) = (1 - z_0z^{-1})(1 - z_2z^{-1})(1 - z_3z^{-1})(1 - z_4z^{-1}),$$

$$I_2(z) = (1 - z_0z^{-1})(1 - z_1z^{-1})(1 - z_3z^{-1})(1 - z_4z^{-1}),$$

$$I_3(z) = (1 - z_0z^{-1})(1 - z_1z^{-1})(1 - z_2z^{-1})(1 - z_4z^{-1}),$$

$$I_4(z) = (1 - z_0z^{-1})(1 - z_1z^{-1})(1 - z_2z^{-1})(1 - z_3z^{-1}). \text{ Therefore,}$$

$$\frac{I_0(z)}{I_0(z_0)} = \frac{(1 - z_1z^{-1})(1 - z_2z^{-1})(1 - z_3z^{-1})(1 - z_4z^{-1})}{(1 - z_1z_0^{-1})(1 - z_2z_0^{-1})(1 - z_3z_0^{-1})(1 - z_4z_0^{-1})} = \frac{1}{12}z^{-1}(1 - \frac{1}{2}z^{-1})(1 - z^{-2}),$$

$$\frac{I_1(z)}{I_1(z_1)} = \frac{(1 - z_0z^{-1})(1 - z_2z^{-1})(1 - z_3z^{-1})(1 - z_4z^{-1})}{(1 - z_0z_1^{-1})(1 - z_2z_1^{-1})(1 - z_3z_1^{-1})(1 - z_4z_1^{-1})} = -\frac{2}{3}z^{-1}(1 - z^{-1})(1 - \frac{1}{4}z^{-2}),$$

$$\frac{I_2(z)}{I_2(z_2)} = \frac{(1 - z_0z^{-1})(1 - z_1z^{-1})(1 - z_3z^{-1})(1 - z_4z^{-1})}{(1 - z_0z_2^{-1})(1 - z_1z_2^{-1})(1 - z_3z_2^{-1})(1 - z_4z_2^{-1})} = (1 - z^{-2})(1 - \frac{1}{4}z^{-2}),$$

$$\frac{I_3(z)}{I_3(z_3)} = \frac{(1 - z_0z^{-1})(1 - z_1z^{-1})(1 - z_2z^{-1})(1 - z_4z^{-1})}{(1 - z_0z_3^{-1})(1 - z_1z_3^{-1})(1 - z_2z_3^{-1})(1 - z_4z_3^{-1})} = \frac{2}{3}z^{-1}(1 + z^{-1})(1 - \frac{1}{4}z^{-2}),$$

$$\begin{aligned} \frac{I_4(z)}{I_4(z_4)} &= \frac{(1-z_0z^{-1})(1-z_1z^{-1})(1-z_2z^{-1})(1-z_3z^{-1})}{(1-z_0z_4^{-1})(1-z_1z_4^{-1})(1-z_2z_4^{-1})(1-z_3z_4^{-1})} = -\frac{1}{12}z^{-1}\left(1+\frac{1}{2}z^{-1}\right)(1-z^{-2}). \text{ Hence,} \\ Y(z) &= \frac{1}{12}(z^{-1} - \frac{1}{2}z^{-2} - z^{-3} + \frac{1}{4}z^{-4})Y(z_0) - \frac{2}{3}(z^{-1} - z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{4}z^{-4})Y(z_1) \\ &\quad + (1 - \frac{5}{4}z^{-2} + \frac{1}{4}z^{-4})Y(z_2) + \frac{2}{3}(z^{-1} + z^{-2} - \frac{1}{4}z^{-3} - \frac{1}{4}z^{-4})Y(z_3) \\ &\quad - \frac{1}{12}(z^{-1} + \frac{1}{2}z^{-1} - z^{-3} - \frac{1}{2}z^{-4})Y(z_4) \\ &= Y(z_2) + \left(\frac{1}{12}Y(z_0) - \frac{2}{3}Y(z_1) + \frac{2}{3}Y(z_3) - \frac{1}{12}Y(z_4)\right)z^{-1} \\ &\quad + \left(-\frac{1}{24}Y(z_0) + \frac{2}{3}Y(z_1) - \frac{5}{4}Y(z_2) + \frac{2}{3}Y(z_3) - \frac{1}{24}Y(z_4)\right)z^{-2} \\ &\quad + \left(-\frac{1}{12}Y(z_0) + \frac{1}{6}Y(z_1) - \frac{1}{6}Y(z_3) + \frac{1}{12}Y(z_4)\right)z^{-3} \\ &\quad + \left(\frac{1}{24}Y(z_0) - \frac{1}{6}Y(z_1) + \frac{1}{4}Y(z_2) - \frac{1}{6}Y(z_3) + \frac{1}{24}Y(z_4)\right)z^{-4}. \end{aligned}$$

Substituting the expressions for $Y(z_0), Y(z_1), Y(z_2), Y(z_3)$, and $Y(z_4)$, in the above equation, we then arrive at the expressions for the coefficients $\{y[n]\}$ in terms of the coefficients $\{h[n]\}$ and $\{x[n]\}$. Thus, $y[0] = Y(z_2) = h[0]x[0]$,

$$y[1] = \frac{1}{12}Y(z_0) - \frac{2}{3}Y(z_1) + \frac{2}{3}Y(z_3) - \frac{1}{12}Y(z_4) = h[0]x[1] + h[1]x[0],$$

$$y[2] = -\frac{1}{24}Y(z_0) + \frac{2}{3}Y(z_1) - \frac{5}{4}Y(z_2) + \frac{2}{3}Y(z_3) - \frac{1}{24}Y(z_4) = h[0]x[2] + h[1]x[1] + h[2]x[0],$$

$$y[3] = -\frac{1}{12}Y(z_0) + \frac{1}{6}Y(z_1) - \frac{1}{6}Y(z_3) + \frac{1}{12}Y(z_4) = h[1]x[2] + h[2]x[1],$$

$$y[4] = \frac{1}{24}Y(z_0) - \frac{1}{6}Y(z_1) + \frac{1}{4}Y(z_2) - \frac{1}{6}Y(z_3) + \frac{1}{24}Y(z_4) = h[2]x[2].$$

Hence, ignoring the multiplications by $\frac{1}{12}, \frac{2}{3}, \frac{5}{4}, \frac{1}{4}, \frac{1}{6}$, and $\frac{1}{24}$, computation of the coefficients of $Y(z)$ require the values of $Y(z_0), Y(z_1), Y(z_2), Y(z_3)$, and $Y(z_4)$, which can be evaluated using only 5 multiplications.

11.43 $Y(z) = H(z)X(z)$ or $y[0] + y[1]z^{-1} + y[2]z^{-2} = (h[0] + h[1]z^{-1})(x[0] + x[1]z^{-1})$
 $= h[0]x[0] + (h[0]x[1] + h[1]x[0])z^{-1} + h[2]x[2]z^{-2}$. Hence, $y[0] = h[0]x[0]$,
 $y[1] = h[0]x[1] + h[1]x[0]$, $y[2] = h[1]x[1]$. Now,
 $(h[0] + h[1])(x[0] + x[1]) - h[0]x[0] - h[1]x[1] = h[0]x[1] + h[1]x[0] = y[1]$. As a result,
evaluation of $H(z)X(z)$ requires the computation of 3 products, $h[0]x[0]$, $h[1]x[1]$, and
 $(h[0] + h[1])(x[0] + x[1])$. In addition, it requires 4 additions, $h[0] + h[1]$, $x[0] + x[1]$, and
 $(h[0] + h[1])(x[0] + x[1]) - h[0]x[0] - h[1]x[1]$.

11.43 Let the two length- N sequences be denoted by $\{h[n]\}$ and $\{x[n]\}$. Denote the sequence

generated by the linear convolution of $h[n]$ and $x[n]$ as $y[n]$. Let $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$ and $X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}$. Rewrite $H(z)$ and $X(z)$ in the form $H(z) = H_0(z) + z^{-N/2}H_1(z)$, and $X(z) = X_0(z) + z^{-N/2}X_1(z)$, where $H_0(z) = \sum_{n=0}^{(N/2)-1} h[n]z^{-n}$, $H_1(z) = \sum_{n=0}^{(N/2)-1} h[n + \frac{N}{2}]z^{-n}$, $X_0(z) = \sum_{n=0}^{(N/2)-1} x[n]z^{-n}$, $X_1(z) = \sum_{n=0}^{(N/2)-1} x[n + \frac{N}{2}]z^{-n}$. Therefore, we can write

$$Y(z) = (H_0(z) + z^{-N/2}H_1(z))(X_0(z) + z^{-N/2}X_1(z))$$

$$= H_0(z)X_0(z) + z^{-N/2}(H_0(z)X_1(z) + H_1(z)X_0(z)) + z^{-N}H_1(z)X_1(z)$$

$$= Y_0(z) + z^{-N/2}Y_1(z) + z^{-N}Y_2(z), \text{ where } Y_0(z) = H_0(z)X_0(z),$$

$$Y_1(z) = H_0(z)X_1(z) + H_1(z)X_0(z), \text{ and } Y_2(z) = H_1(z)X_1(z).$$

Note that $Y_0(z)$ and $Y_1(z)$ are products of two polynomials of degree $\frac{N}{2}$ each, and hence, require $\left(\frac{N}{2}\right)^2$ multiplications each. Now, we can write $Y_1(z) = (H_0(z) + H_1(z))(X_0(z) + X_1(z)) - Y_0(z) - Y_2(z)$. Since, $(H_0(z) + H_1(z))(X_0(z) + X_1(z))$ is a product of two polynomials of degree $\frac{N}{2}$ each, it can be computed using $\left(\frac{N}{2}\right)^2$ multiplications. As a result, $Y(z) = H(z)X(z)$ can be computed using $3\left(\frac{N}{2}\right)^2$ multiplications instead of N^2 multiplications. If N is a power-of-2, $\frac{N}{2}$ is even, and the same procedure can be applied to compute $Y_0(z), Y_1(z)$, and $Y_2(z)$, reducing further the number of multiplications. This process can be continued until the sequences being convolved are of length 1 each.

Let $\mathcal{R}(N)$ denote the total number of multiplications required to compute the linear convolution of two N -length sequences. Then, in the method outlined above, we have $\mathcal{R}(N) = 3 \cdot \mathcal{R}(N/2)$ with $\mathcal{R}(1) = 1$. A solution of this equation is given by $\mathcal{R}(N) = 3^{\log_2 N}$.

11.45 The dynamic range of a signed B -bit integer η is given by $-(2^{(B-1)} - 1) \leq \eta < (2^{(B-1)} - 1)$ which for $B = 32$ is given by $-(2^{31} - 1) \leq \eta < (2^{31} - 1)$.

(a) For $E = 6$ and $M = 25$, the value of a 32-bit floating-point number is given by $\eta = (-1)^s 2^{E-31}(M)$. Hence, the value of the largest number is $\approx 2^{32}$, and the value of the smallest number is $\approx -2^{32}$. The dynamic range is therefore $\approx 2 \times 2^{32}$.

(b) For $E = 7$ and $M = 24$, the value of a 32-bit floating-point number is given by $\eta = (-1)^s 2^{E-63}(M)$. Hence, the value of the largest number is $\approx 2^{64}$, and the value of the

smallest number is $\approx -2^{64}$. The dynamic range is therefore $\approx 2 \times 2^{64}$.

(c) For $E = 8$ and $M = 23$, the value of a 32-bit floating-point number is given by $\eta = (-1)^s 2^{E-127} (M)$. Hence, the value of the largest number is $\approx 2^{128}$, and the value of the smallest number is $\approx -2^{128}$. The dynamic range is therefore $\approx 2 \times 2^{128}$.

Hence, the dynamic range in a floating-point representation is much larger than that in a fixed-point representation with the same wordlength.

11.46 A 32-bit floating-point number in the IEEE Format has $E = 8$ and $M = 23$. Also, the exponent E is coded in a biased form as $E - 127$ with certain conventions for special cases such as $E = 0, 255$, and $M = 0$ (See page 637 of text).

Now, a positive 32-bit floating-point number represented in the “normalized” form have an exponent in the range $0 < E < 255$, and is of the form $\eta = (-1)^s 2^{E-127} (1_\Delta M)$. Hence, the smallest positive number that can be represented will have $E = 1$, and $M = \underbrace{0 \ 0 \ \dots \ 0 \ 0}_{22 \text{ bits}}$,

and has therefore a value given by $2^{-126} \cong 1.18 \times 10^{-38}$. For the largest positive number, $E = 254$, and $M = \underbrace{1 \ 1 \ \dots \ 1 \ 1}_{22 \text{ bits}}$. Thus, here $\eta = (-1)^0 2^{127} (1_\Delta \underbrace{1 \ 1 \ \dots \ 1 \ 1}_{22 \text{ bits}}) \approx 2^{127} \times 2$

$\cong 3.4 \times 10^{38}$.

Note: For representing numbers less than 2^{-126} , IEEE format uses the “de-normalized” form where $E = 0$, and $\eta = (-1)^s 2^{-126} (0_\Delta M)$. In this case, the smallest positive number that can be represented is given by $\eta = (-1)^0 2^{126} (0_\Delta \underbrace{0 \ 0 \ \dots \ 0 \ 1}_{22 \text{ bits}}) \approx 2^{-149}$

$\cong 1.4013 \times 10^{-45}$.

11.47 For a two’s-complement binary fraction $s_\Delta a_{-1} a_{-2} \dots a_{-b}$, the decimal equivalent for

$s = 0$ is simply $\sum_{i=1}^b a_{-i} 2^{-i}$. For $s = 1$, the decimal equivalent is given by

$$-\left(\sum_{i=1}^b (1 - a_{-i}) 2^{-i} + 2^{-b} \right) = -\sum_{i=1}^b 2^{-i} + \sum_{i=1}^b a_{-i} 2^{-i} - 2^{-b} = -(1 - 2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i} - 2^{-b}$$

$= -1 + \sum_{i=1}^b a_{-i} 2^{-i}$. Hence, the decimal equivalent of $s_\Delta a_{-1} a_{-2} \dots a_{-b}$ is given by

$$-s + \sum_{i=1}^b a_{-i} 2^{-i}.$$

11.48 For a ones’-complement binary fraction $s_\Delta a_{-1} a_{-2} \dots a_{-b}$, the decimal equivalent for

$s = 0$ is simply $\sum_{i=1}^b a_{-i} 2^{-i}$. For $s = 1$, the decimal equivalent is given by

$$-\left(\sum_{i=1}^b (1 - a_{-i}) 2^{-i}\right) = -\sum_{i=1}^b 2^{-i} + \sum_{i=1}^b a_{-i} 2^{-i} = -(1 - 2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i}$$

$$= -1 + \sum_{i=1}^b a_{-i} 2^{-i}. \text{ Hence, the decimal equivalent of } s_{\Delta} a_{-1} a_{-2} \dots a_{-b} \text{ is given by}$$

$$-s(1 - 2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i}.$$

11.49 $\eta_1 = 0.78125_{10} = 0_{\Delta} 11001$, $\eta_2 = 0.68750_{10} = 0_{\Delta} 10110$, $\eta_3 = 0.53125_{10} = 1_{\Delta} 01111$.

$$\eta_1 + \eta_2 = 0_{\Delta} 11001 + 0_{\Delta} 10110 = 1_{\Delta} 01111.$$

Dropping thee overflow bit and adding $(\eta_1 + \eta_2)$ to η_3 :

$$(\eta_1 + \eta_2) + \eta_3 = 0_{\Delta} 01111 + 1_{\Delta} 01111 = 0_{\Delta} 11110 = 0.9375_{10},$$

where we have dropped the carry bit in the MSB location. Note that the final sum is correct inspite of the overflow.

11.49 The transformation $\cos \omega = \alpha + \beta \cos \hat{\omega}$ is equivalent to $\frac{e^{j\omega} + e^{-j\omega}}{2} = \alpha + \beta \left(\frac{e^{j\hat{\omega}} + e^{-j\hat{\omega}}}{2} \right)$,

which by analytic continuation can be expressed as $\frac{z + z^{-1}}{2} = \alpha + \beta \left(\frac{\hat{z} + \hat{z}^{-1}}{2} \right)$. Now, $H(z)$ let

be a Type 1 linear-phase FIR transfer function of degree $2M$. As indicated in Eq. (8.128),

$H(z)$ can be expressed as $H(z) = z^{-M} \sum_{n=0}^M a[n] \left(\frac{z + z^{-1}}{2} \right)^n$ with a frequency response

given by $H(e^{j\omega}) = e^{-jM\omega} \sum_{n=0}^M a[n] (\cos \omega)^n$, with $\check{H}(\omega) = \sum_{n=0}^M a[n] (\cos \omega)^n$ denoting the amplitude function or the zero-phase frequency response. The amplitude function or the zero-phase frequency response of the transformed filter obtained by applying the mapping $\cos \omega = \alpha + \beta \cos \hat{\omega}$ is therefore given by $\check{H}(\hat{\omega}) = \sum_{n=0}^M a[n] (\alpha + \beta \cos \hat{\omega})^n$. Or, equivalently, the transfer function of the transformed filter is given by

$H(\hat{z}) = z^{-M} \sum_{n=0}^M a[n] \left(\alpha + \beta \frac{\hat{z} + \hat{z}^{-1}}{2} \right)^n$ A convenient way to realize $H(\hat{z})$ is to consider the

realization of the parent transfer function $H(z)$ in the form of a Taylor structure as outlined in Problem 8.17 which is obtained by expressing $H(z)$ in the form

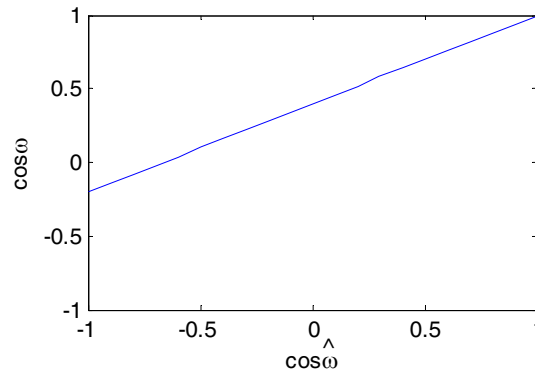
$H(z) = \sum_{n=0}^M a[n] z^{-M+n} \left(\frac{1+z^{-2}}{2} \right)^n$. Similarly, the transformed filter can be realized by

replacing each block $\frac{1+z^{-2}}{2}$ in the Taylor structure by the block $\alpha\hat{z}^{-1} + \beta\frac{1+\hat{z}^{-2}}{2}$.

Now, for a lowpass-to-lowpass transformation we can impose the condition

$\tilde{H}(\hat{\omega})|_{\hat{\omega}=0} = \tilde{H}(\omega)|_{\omega=0}$. This condition is met if $\alpha + \beta = 1$ and $0 \leq \alpha < 1$. In this case, the

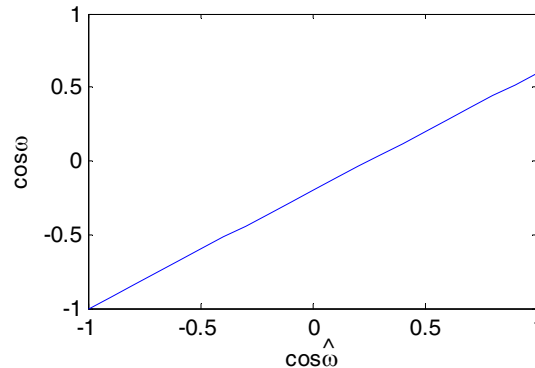
transformation reduces to $\cos \omega = \alpha + (1 - \alpha) \cos \hat{\omega}$. From the plot of the mapping given below, it follows that as α is varied between 0 and 1, $\hat{\omega}_c < \omega_c$.



On the other hand if $\hat{\omega}_c > \omega_c$ is desired along with a lowpass-to-lowpass transformation, we

can impose the condition $\tilde{H}(\hat{\omega})|_{\hat{\omega}=\pi} = \tilde{H}(\omega)|_{\omega=\pi}$. This condition is met if $\beta = 1 + \alpha$ and

$-1 < \alpha \leq 1$. The corresponding transformation is now given by $\cos \omega = \alpha + (1 + \alpha) \cos \hat{\omega}$. From the plot of the mapping given below, it follows that as α is varied between 0 and 1, $\hat{\omega}_c > \omega_c$.



$$\mathbf{11.51 \text{ (a)}} \quad X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}. \quad X(\tilde{z}) = X(z)|_{z^{-1} = \frac{-\alpha + \tilde{z}^{-1}}{1 - \alpha\tilde{z}^{-1}}} = \sum_{n=0}^{N-1} x[n] \left(\frac{-\alpha + \tilde{z}^{-1}}{1 - \alpha\tilde{z}^{-1}} \right)^n = \frac{P(\tilde{z})}{D(\tilde{z})},$$

$$\text{where } P(\tilde{z}) = \sum_{n=0}^{N-1} p[n]\tilde{z}^{-n} = \sum_{n=0}^{N-1} x[n](1 - \alpha\tilde{z}^{-1})^{N-1-n} (-\alpha + \tilde{z}^{-1})^n, \text{ and}$$

$$D(\tilde{z}) = \sum_{n=0}^{N-1} d[n]\tilde{z}^{-n} = (1 - \alpha\tilde{z}^{-1})^{N-1}.$$

(b) $\tilde{X}[k] = X(\tilde{z})|_{\tilde{z}=e^{j2\pi k/N}} = \frac{P(\tilde{z})}{D(\tilde{z})}|_{\tilde{z}=e^{j2\pi k/N}} = \frac{\tilde{P}[k]}{\tilde{D}[k]}$, where $\tilde{P}[k] = P(\tilde{z})|_{\tilde{z}=e^{j2\pi k/N}}$ is the N -point DFT of the sequence $p[n]$ and $\tilde{D}[k] = D(\tilde{z})|_{\tilde{z}=e^{j2\pi k/N}}$ is the N -point DFT of the sequence $d[n]$.

(c) Let $\mathbf{P} = [p[0] \ p[1] \ \dots \ p[N-1]]^t$ and $\mathbf{X} = [x[0] \ x[1] \ \dots \ x[N-1]]^t$. Without any loss of generality, assume $N = 4$ in which case

$$P(\tilde{z}) = \sum_{n=0}^3 p[n]\tilde{z}^{-n} = (x[0] - \alpha x[1] + \alpha^2 x[2] - \alpha^3 x[3]) \\ + (-3\alpha x[0] + (1 + 2\alpha^2)x[1] - \alpha(2 + \alpha^2)x[2] + 3\alpha^2 x[3])\tilde{z}^{-1} \\ + (3\alpha^2 x[0] - \alpha(2 + \alpha^2)x[1] - (1 + \alpha^2)x[2] - 3\alpha x[3])\tilde{z}^{-2} \\ + (-\alpha^3 x[0] + \alpha^2 x[1] - \alpha x[2] + \alpha x[3])\tilde{z}^{-3}. \text{ Equating like powers of } \tilde{z}^{-1} \text{ we can write}$$

$$\mathbf{P} = \mathbf{Q} \cdot \mathbf{X} \text{ or } \begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 \\ -3\alpha & 1 + 2\alpha^2 & -\alpha(2 + \alpha^2) & 3\alpha^2 \\ 3\alpha^2 & -\alpha(2 + \alpha^2) & 1 + 2\alpha^2 & -3\alpha \\ -\alpha^3 & \alpha^2 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}. \text{ It can be shown}$$

that the elements $q_{r,s}$, $0 \leq r, s \leq 3$, of the 4×4 matrix \mathbf{Q} can be determined as follows:

- (i) The first row is given by $q_{0,s} = (-\alpha)^s$,
- (ii) The first column is given by $q_{r,0} = {}^3C_r (-\alpha)^r = \frac{3!}{r!(3-r)!} (-\alpha)^r$, and
- (iii) The remaining elements can be obtained using the recurrence relation $q_{r,s} = q_{r-1,s-1} - \alpha q_{r,s-1} + \alpha q_{r-1,s}$.

In the general case, we only change the computation of the elements of the first column using the relation $q_{r,0} = {}^{N-1}C_r (-\alpha)^r = \frac{(N-1)!}{r!(N-1-r)!} (-\alpha)^r$.

11.52 The highpass transfer function can be expressed as $H(z) = \frac{1}{2}[A_0(z) - A_1(z)]$, where

$$A_0(z) = \frac{0.3038 + z^{-1}}{1 + 0.3038z^{-1}} \text{ and } A_1(z) = \frac{0.639 + 0.4012z^{-1} + z^{-2}}{1 + 0.4012z^{-1} + 0.639z^{-2}}. \text{ The tunable highpass}$$

transfer function is obtained by applying the lowpass-to-lowpass transformation of Eq.

$$(11.115) \text{ where the tuning parameter is given by } \alpha = \frac{\sin[(0.6\pi - \hat{\omega}_p)/2]}{\sin[(0.6\pi + \hat{\omega}_p)/2]}.$$

The tunable highpass transfer function is then given by $H(z, \alpha) = \frac{1}{2}[\hat{A}_0(z) - \hat{A}_1(z)]$, where from Eqn.

$$(11.119) \hat{A}_0(z) = \frac{[0.3038 - 0.9077\alpha] + z^{-1}}{1 + [0.3038 - 0.9077\alpha]z^{-1}} \text{ and from Eqn. (11.123)}$$

$$\hat{A}_1(z) = \frac{[0.639 - 0.1448\alpha] + [0.4012 + 3.439\alpha]z^{-1} + z^{-2}}{1 + [0.4012 + 3.439\alpha]z^{-1} + [0.639 - 0.1448\alpha]z^{-2}}$$

11.53 The lowpass transfer function can be expressed as $H(z) = \frac{1}{2}[A_0(z) + A_1(z)]$ where

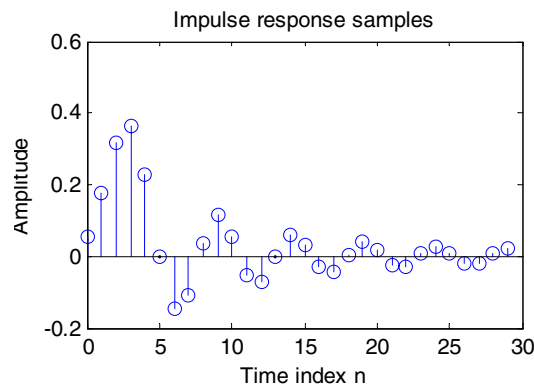
$$A_0(z) = \frac{-0.1584 + z^{-1}}{1 - 0.1584z^{-1}} \text{ and } A_1(z) = \frac{0.3554 - 0.4191z^{-1} + z^{-2}}{1 - 0.4191z^{-1} + 0.3554z^{-2}}. \text{ The tunable bandpass}$$

transfer function is obtained by applying the lowpass-to-bandpass transformation of Eq. (11.124) where the tuning parameter is given by $\beta = \cos \omega_o$ with ω_o denoting the center frequency of the bandpass filter. The tunable bandpass transfer function is then given by

$$H(z, \alpha) = \frac{1}{2}[\hat{A}_0(z) + \hat{A}_1(z)] \text{ where } \hat{A}_0(z) = \frac{-0.1584 - z^{-1} \left(\frac{\beta + z^{-1}}{1 + \beta z^{-1}} \right)}{1 + 0.1584z^{-1} \left(\frac{\beta + z^{-1}}{1 + \beta z^{-1}} \right)} \text{ and}$$

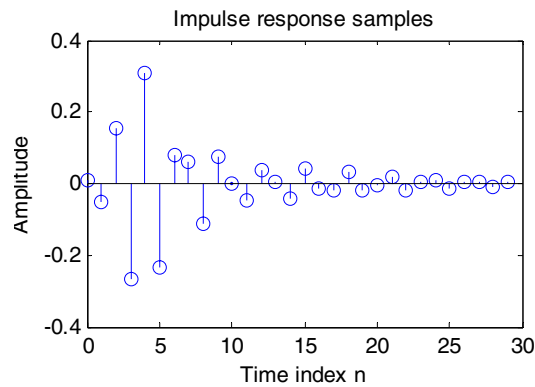
$$\hat{A}_1(z) = \frac{0.3554 + 0.4191z^{-1} \left(\frac{\beta + z^{-1}}{1 + \beta z^{-1}} \right) + z^{-2} \left(\frac{\beta + z^{-1}}{1 + \beta z^{-1}} \right)^2}{1 + 0.4191z^{-1} \left(\frac{\beta + z^{-1}}{1 + \beta z^{-1}} \right) + 0.3554z^{-2} \left(\frac{\beta + z^{-1}}{1 + \beta z^{-1}} \right)^2}$$

M11.1 (a) Numerator coefficients =
 [0.0528 0.0797 0.1295 0.1295 0.0797 0.0528]
 Denominator coefficients =
 [1.0000 -1.8107 2.4947 -1.8801 0.9537 -0.2336]

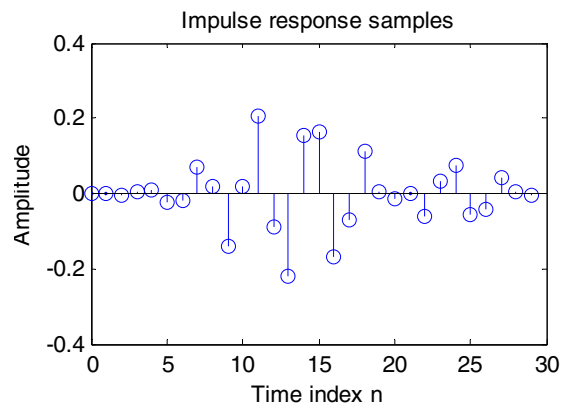


(b) Numerator coefficients =
 [0.0084 -0.0335 0.0502 -0.0335 0.0084]
 Denominator coefficients =

[1.0000 2.3741 2.7057 1.5917 0.4103]



(c) Numerator coefficients = [0.0003 0 -0.0019 0 0.0057 0
 -0.0095 0 0.0095 0 -0.0057 0 0.0019 0 -0.0003]
 Denominator coefficients = [1.0000 1.7451 4.9282 6.1195
 9.8134 9.2245 10.4323 7.5154 6.4091 3.4595
 2.2601 0.8470 0.4167 0.0856 0.0299]

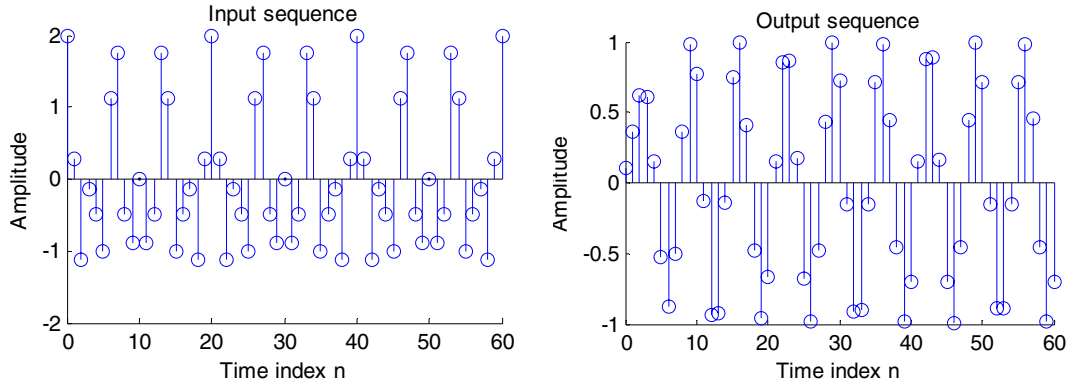


M11.2 The modified MATLAB program is given below:

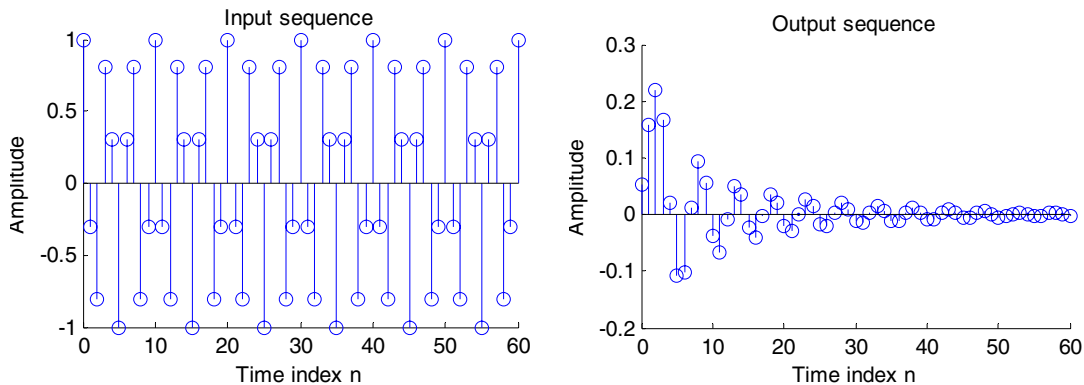
```
n = 0:60;
w = input('Normalized angular frequency vector = ');
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
x1 = cos(w(1)*pi*n); x2 = cos(w(2)*pi*n);
x = x1+x2;
% Generate the output sequence by filtering the input
y = filter(num,den,x);
% Plot the input and the output sequences
figure(1)
stem(n,x);
xlabel('Time index n'); ylabel('Amplitude');
title('Input sequence');
figure(2)
stem(n,y);
xlabel('Time index n'); ylabel('Amplitude');
```

```
title('Output sequence');
```

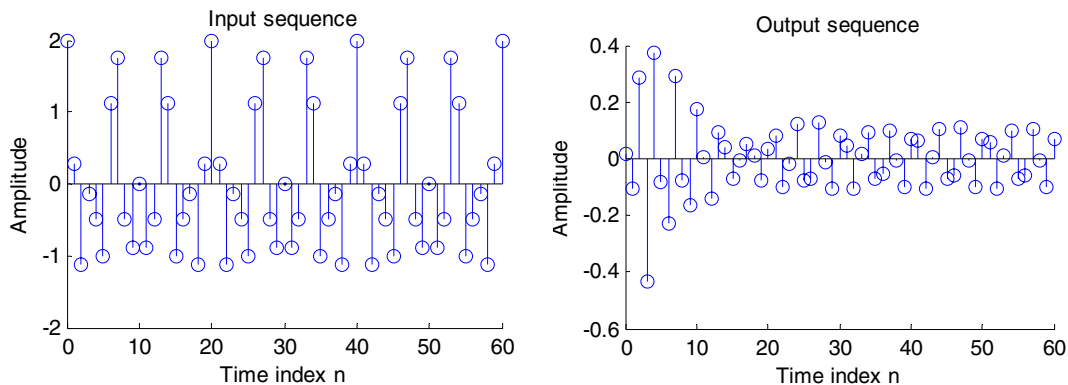
The plots generated by the above program for the filter of Example 9.14 for an input composed of a sum of two sinusoidal sequences of angular frequencies, 0.3π and 0.6π , are given below:



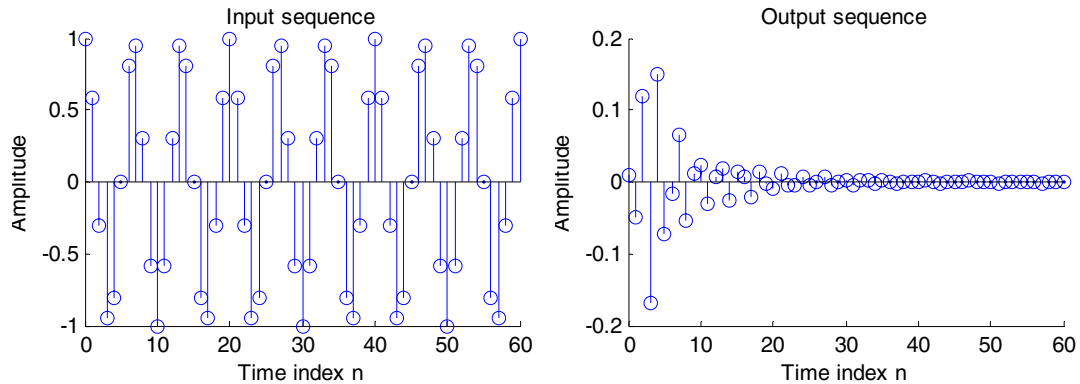
The blocking of the high-frequency signal by the lowpass filter can be demonstrated by replacing the statement `stem(n, x);` with `stem(n, x2);` and the statement `y=filter(num, den, x);` with `y=filter(num, den, x2);`. The plots of the high-frequency input signal and the corresponding output are shown below:



M11.3 The plots generated by the program of Exercise M11.2 for the filter of Example 9.15 for an input composed of a sum of two sinusoidal sequences of angular frequencies, 0.3π and 0.6π , are given below:



The blocking of the low-frequency signal by the highpass filter can be demonstrated by replacing the statement `stem(n, x);` with `stem(n, x1);` and the statement `y=filter(num, den, x);` with `y=filter(num, den, x1);`. The plots of the low-frequency input signal and the corresponding output are shown below:

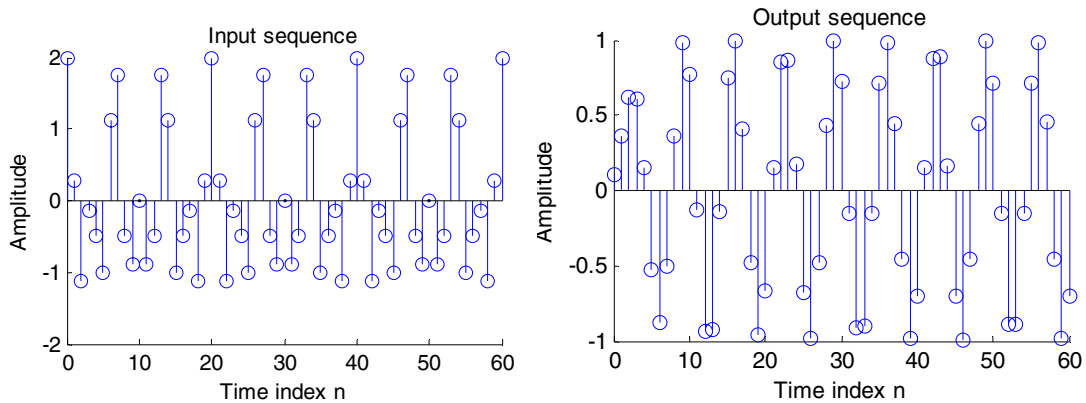


```

M11.4 % The factors for the transfer function of
% the lowpass filter of Example 9.14 are
% num1 = [0.0528 0.0528 0];
% den1 = [1.0000 -0.4909 0];
% num2 = [1.0000 0.6040 1.0000];
% den2 = [1.0000 -0.7624 0.5390];
% num3 = [1.0000 -0.0949 1.0000];
% den3 = [1.0000 -0.5574 0.8828];
N = input('Total number of sections = ');
for k = 1:N;
    num(k, :) = input('Numerator factor = ');
    den(k, :) = input('Denominator factor = ');
end
n = 0:60;
w = input('Normalized angular frequency vector = ');
x1 = cos*w(1)*pi*n; x2 = cos*w(2)*pi*n;
x = x1 + x2;
% Plot the input sequence
figure(1)
stem(n,x);
xlabel('Time index n'); ylabel('Amplitude');
title('Input sequence');
si = [0 0];
for k = 1:N;
    y(k, :) = filter(num(k, :), den(k, :), x, si);
    x = y(k, :);
end
% Plot the output sequence
figure(2)
stem(n,y(N, :));
xlabel('Time index n'); ylabel('Amplitude');
title('Output sequence');

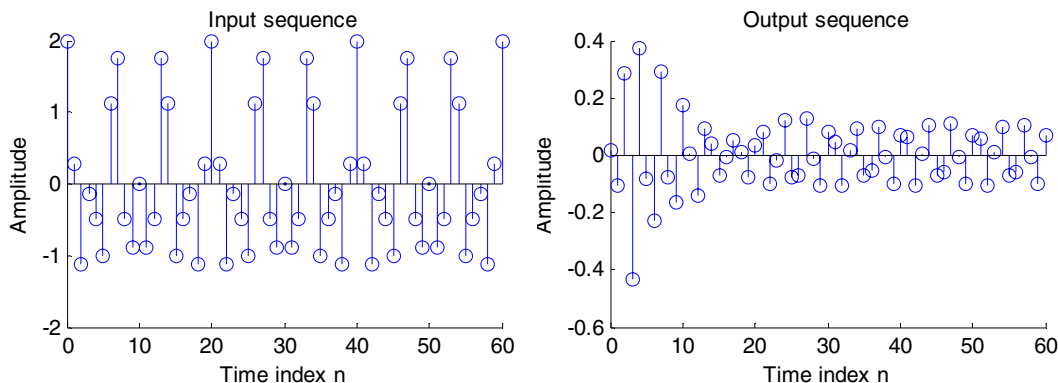
```

The plots generated by the above program for the filter of Example 9.14 for an input composed of a sum of two sinusoidal sequences of angular frequencies, 0.3π and 0.6π , are given below:

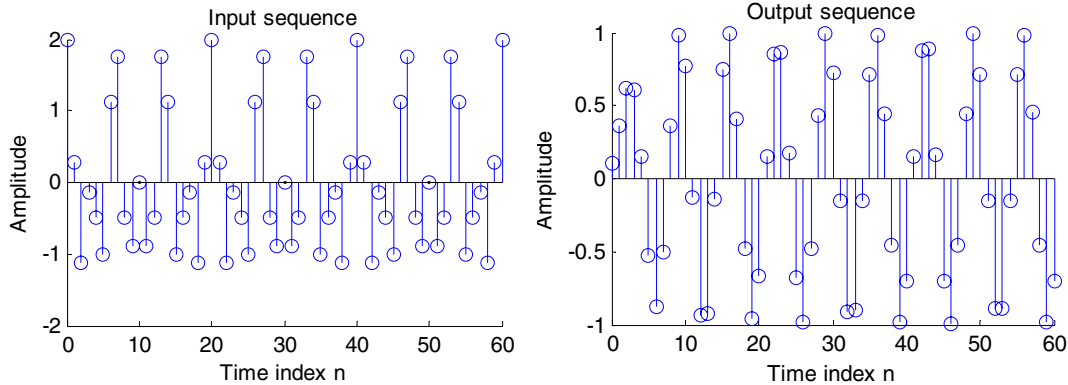


```
M11.5 % The factors for the transfer function of
% the highpass filter of Example 9.15 are
% num1 = [0.0084 -0.0167 0.0084];
% den1 = [1.0000 1.3101 0.5151];
% num2 = [1.0000 -2.0000 1.0000];
% den2 = [1.0000 1.0640 0.7966];
```

The plots generated by the program of Exercise M11.4 for the filter of Example 9.15 for an input composed of a sum of two sinusoidal sequences of angular frequencies, 0.3π and 0.6π , are given below:



M11.6 To apply the function `direct2` to filter a sum of two sinusoidal sequences, we replace the statement `y = filter(num,den,x,si);` in the MATLAB program given in the solution of Exercise M11.2 with the statement `y = direct2(num,den,x,si);`. The plots generated by running the modified program for the data given in this problem are given below:



M11.7 The MATLAB program that can be used to compute all DFT samples using the function `goertzel` and the function `fft` is as follows:

```
clear
N = input('Desired DFT length = ');
x = input('Input sequence = ');
for j = 1:N
    Y(j)=goertzel(x,j);
end
disp('DFT samples computed using goertzel are ');
disp(Y)
disp('DFT samples computed using fft are ');
X = fft(x); disp(X);
```

Results obtained for two input sequences of lengths 8 and 12, respectively, are given below:

```
Desired DFT length = 8
Input sequence = [1 2 3 4 4 3 2 1]

DFT samples computed using goertzel are
Columns 1 through 4
20.0000    -5.8284-2.4142i    -0.0000-0.0000i    -0.1716-0.4142i
Columns 5 through 8
0-0.0000i    -0.1716+0.4142i    0.0000-0.0000i    -5.8284+2.4142i
```

```
DFT samples computed using fft are
Columns 1 through 4
20.0000    -5.8284-2.4142i    0    -0.1716-0.4142i
Columns 5 through 8
0    -0.1716+0.4142i    0    -5.828 +2.4142i
```

```
Desired DFT length = 12
Input sequence = [1 2 4 8 10 12 7 3 -4 5 0 6]

DFT samples computed using goertzel are
Columns 1 through 4
54.0000    -13.0622-21.0885i    1.5000+19.9186i    -4.0000-2.0000i
Columns 5 through 8
```

```

4.5000+2.5981i    -0.9378+10.0885i    -18.0000+0.0000i
-0.9378-10.0885i

Columns 9 through 12
4.5000-2.5981i  -4.0000+2.0000i    1.5000-19.9186i    -
13.0622 +21.0885i

DFT samples computed using fft are
54.0000  -13.0622-21.0885i    1.5000+19.9186i  -4.0000-2.0000i
Columns 5 through 8

4.5000+2.5981i    -0.9378+10.0885i    -18.0000+0.0000i
-0.9378-10.0885i

Columns 9 through 12
4.5000-2.5981i  -4.0000+2.0000i    1.5000-19.9186i    -
13.0622 +21.0885i

```

M11.8 The MATLAB program that can be used to verify the plots of Figure 11.43 is given below:

```

%Program_11_8.m
[z,p,k] = ellip(5,0.5,40,0.4);
a = conv([1 -p(1)], [1 -p(2)]); b = [1 -p(5)];
c = conv([1 -p(3)], [1 -p(4)]);
w = 0:pi/255:pi;
alpha = 0;
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;
an2 = a(3) + (a(3) -1)*a(2)*alpha;
g = b(2) - (1 - b(2)*b(2))*alpha;
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;
cn2 = c(3) + (c(3) -1)*c(2)*alpha;
a = [1 an1 an2]; b = [1 g]; c = [1 cn1 cn2];
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);
h3 = freqz(fliplr(c),c,w);
ha = 0.5*(h1.*h2 + h3); ma = 20*log10(abs(ha));
alpha = 0.1;
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;
an2 = a(3) + (a(3) -1)*a(2)*alpha;
g = b(2) - (1 - b(2)*b(2))*alpha;
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;
cn2 = c(3) + (c(3) -1)*c(2)*alpha;
a = [1 an1 an2]; b = [1 g]; c = [1 cn1 cn2];
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);
h3 = freqz(fliplr(c),c,w);
hb = 0.5*(h1.*h2 + h3); mb = 20*log10(abs(hb));
alpha = -0.25;
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;
an2 = a(3) + (a(3) -1)*a(2)*alpha;
g = b(2) - (1 - b(2)*b(2))*alpha;
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;
cn2 = c(3) + (c(3) -1)*c(2)*alpha;

```

```

a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);
h3 = freqz(fliplr(c),c,w);
hc = 0.5*(h1.*h2 + h3);mc = 20*log10(abs(hc));
plot(w/pi,ma,'r-',w/pi,mb,'b--',w/pi,mc,'g-.');axis([0 1 -80 5]);
xlabel('Normalized frequency');ylabel('Gain, dB');
legend('b--','\alpha = 0.1 ','r-','\alpha = 0 ','g-.','\alpha = -0.25
');

```

M11.9 The MATLAB program that can be used to verify the plots of Figure 11.44 is given below:

```

%Program_11_9.m
w = 0:pi/255:pi;
wc2 = 0.31*pi;
f = [0 0.36 0.46 1];m = [1 1 0 0];
b1 = remez(50, f, m);
h1 = freqz(b1,1,w);
m1 = 20*log10(abs(h1));
n = -25:-1;
c = b1(1:25)./sin(0.41*pi*n);
d = c.*sin(wc2*n);q = (b1(26)*wc2)/(0.4*pi);
b2 = [d q fliplr(d)];
h2 = freqz(b2,1,w);
m2 = 20*log10(abs(h2));
wc3 = 0.51*pi;
d = c.*sin(wc3*n);q = (b1(26)*wc3)/(0.4*pi);
b3 = [d q fliplr(d)];
h3 = freqz(b3,1,w);
m3 = 20*log10(abs(h3));
plot(w/pi,m1,'r-',w/pi,m2,'b--',w/pi,m3,'g-.');
axis([0 1 -80 5]);
xlabel('Normalized frequency');ylabel('Gain, dB');
legend('b--','\omega_c = 0.3\pi','r-','\omega_c = 0.41\pi','g-
.','\omega_c = 0.51\pi')

```

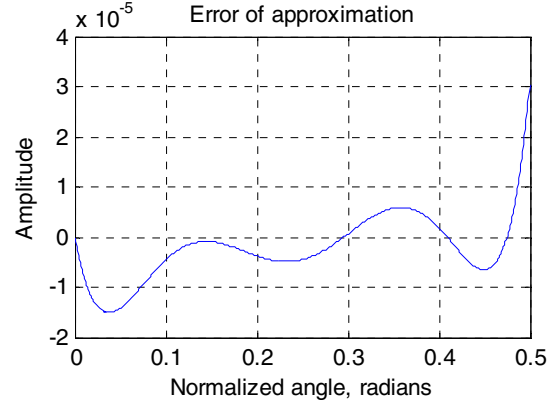
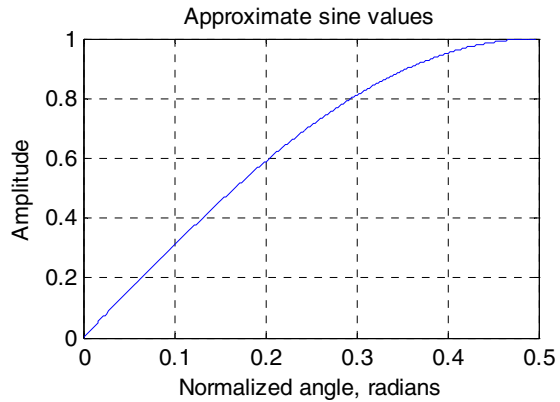
M11.10 The MATLAB program to evaluate Eqs. (11.155) is given below:

```

%Program_11_10.m
x = 0:0.001:0.5;
y = 3.140625*x + 0.0202636*x.^2 - 5.325196*x.^3 +
0.5446778*x.^4 + 1.800293*x.^5;
x1 = pi*x;
z = sin(x1);
plot(x,y);
xlabel('Normalized angle, radians');ylabel('Amplitude');
title('Approximate sine values');grid;axis([0 0.5 0 1]);
pause
plot(x,y-z);
xlabel('Normalized angle, radians');ylabel('Amplitude');
title('Error of approximation');grid;

```

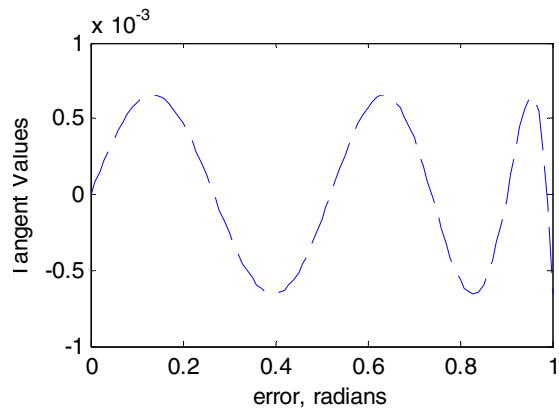
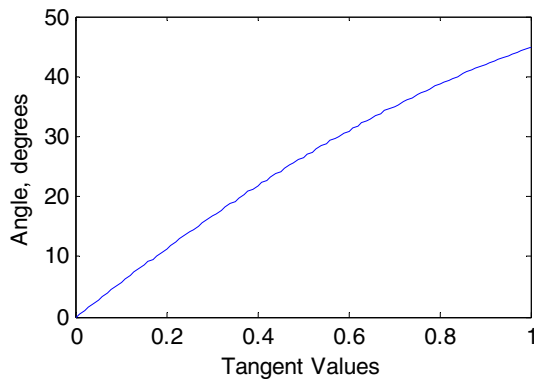
The plots generated by the above program are as indicated below:



M11.11 The MATLAB program to evaluate Eqs. (11.156) and (11.135) is given below:

```
%Program_11_11.m
k = 1;
for x = 0:.01:1
    op1 = 0.318253*x+0.00331*x^2-0.130908*x^3+0.068524*x^4-
0.009159*x^5;
    op2 = 0.999866*x-0.3302995*x^3+0.180141*x^5-
0.085133*x^7+0.0208351*x^9;
    arctan1(k) = op1*180/pi;
    arctan2(k) = 180*op2/pi;
    actual(k) = atan(x)*180/pi;
    k = k+1;
end
subplot(211)
x = 0:.01:1;
plot(x,arctan2);
ylabel('Angle, degrees');xlabel('Tangent Values');
subplot(212)
plot(x,actual-arctan2,'--');
ylabel('Tangent Values');xlabel('error, radians');
```

The plots generated by the above program are as indicated below:



Chapter 12

12.1 Two's Complement Truncation - Assume $x > 0$. The relative error e_t is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{\sum_{i=1}^b a_{-i} 2^{-i} - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M} = \frac{-\sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ Now will}$$

be a minimum if all a_{-i} 's are 1 and will be a maximum if all a_{-i} 's are 0 for $b+1 \leq i \leq \beta$.

Thus, $-\frac{\delta}{M} \leq e_t \leq \frac{0}{M}$. Since, $0.5 \leq M < 1$, hence $-2\delta \leq e_t \leq 0$.

Next, consider $x < 0$. Here, the relative error e_t is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{-1 + \sum_{i=1}^b a_{-i} 2^{-i} + 1 - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ As before,}$$

$-\frac{\delta}{M} \leq e_t \leq \frac{0}{M}$. In this case, $-1 \leq M \leq -0.5$, and hence, $0 \leq e_t \leq 2\delta$.

Ones' Complement Truncation - Assume $x > 0$. The relative error e_t is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{\sum_{i=1}^b a_{-i} 2^{-i} - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M} = \frac{-\sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}.$$

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{\sum_{i=1}^b a_{-i} 2^{-i} - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M} = \frac{-\sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ Now will}$$

be a minimum if all a_{-i} 's are 1 and will be a maximum if all a_{-i} 's are 0 for $b+1 \leq i \leq \beta$.

Thus, $-\frac{\delta}{M} \leq e_t \leq \frac{0}{M}$. Since, $0.5 \leq M \leq 1$, hence $-2\delta \leq e_t \leq 0$.

Next, consider $x < 0$. Here, the relative error e_t is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{-(1 - 2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i} + (1 - 2^{-\beta}) - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M}$$

$$= \frac{(2^{-b} - 2^{-\beta}) + \sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ Now will be a minimum if all } a_{-i} \text{'s are 1 and will be a}$$

maximum if all a_{-i} 's are 0 for $b+1 \leq i \leq \beta$. In this case, $\frac{0}{M} \leq e_t \leq \frac{\delta}{M}$. Since,

$-0.5 \leq M < -1$, hence, $-2\delta \leq e_t \leq 0$.

Sign-Magnitude Truncation - Assume $x > 0$. The relative error e_t is given by

$$e_t = \frac{Q(M) - M}{M} = \frac{-\sum_{i=1}^b a_{-i} 2^{-i} + \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M} = \frac{\sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ Since,}$$

$-0.5 \leq M < -1$, hence $-2\delta \leq e_t < 0$.

Rounding - $\varepsilon = Q(M) - M$. Hence, $-\frac{\delta}{2M} \leq \varepsilon \leq \frac{\delta}{2M}$. Since, $-1 < M \leq -0.5$, hence, $-\delta \leq e_t \leq \delta$.

12.2 The denominator is given by $D(z) = z - \alpha$. Here, the pole is at $z = r e^{j\theta} = \alpha$, where $r = \alpha$ and $\theta = 0$. The components of the pole sensitivities are therefore given by $\frac{\partial r}{\partial \alpha} = 1$ and $\frac{\partial \theta}{\partial \alpha} = 0$.

12.3 The denominator is given by $D(z) = z^2 - \beta(1 + \alpha)z + \alpha = (z - r e^{j\theta})(z - r e^{-j\theta}) = z^2 - 2r \cos \theta z + r^2$. Comparing we get $2r \cos \theta = \beta(1 + \alpha)$ and $r^2 = \alpha$. Taking the partials of both sides of the last two equations we get

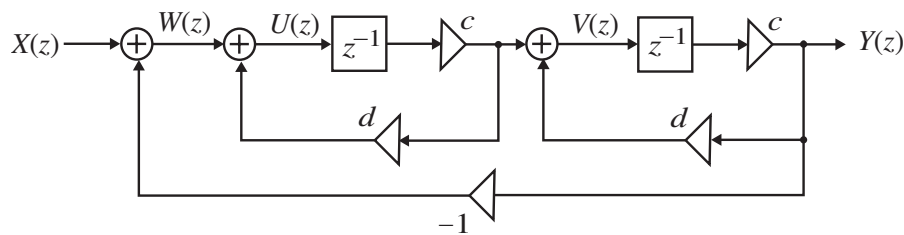
$$\begin{bmatrix} 2r & 0 \\ 2 \cos \theta & -2r \sin \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta & 1 + \alpha \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2r \cos \theta}{1+r^2} & 1+r^2 \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} 2r & 0 \\ 2 \cos \theta & -2r \sin \theta \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \frac{2r \cos \theta}{1+r^2} & 1+r^2 \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2r} & 0 \\ \frac{(1-r^2) \cos \theta}{2r^2(1+r^2) \sin \theta} & -\frac{1+r^2}{2r \sin \theta} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}.$$

Hence, the components of the pole sensitivities are given by

$$\frac{\partial r}{\partial \alpha} = \frac{1}{2r}, \quad \frac{\partial r}{\partial \beta} = 0, \quad \frac{\partial \theta}{\partial \alpha} = \frac{(1-r^2) \cos \theta}{2r^2(1+r^2) \sin \theta}, \quad \text{and} \quad \frac{\partial \theta}{\partial \beta} = -\frac{1+r^2}{2r \sin \theta}.$$

12.4 The digital filter structure of Figure P12.1 with internal variables labeled is shown below. Analysis yields



$$W(z) = X(z) - Y(z), \quad U(z) = W(z) + c d z^{-1} U(z) \Rightarrow U(z) = \frac{W(z)}{1 - c d z^{-1}} = \frac{X(z) - Y(z)}{1 - c d z^{-1}}, \text{ and}$$

$$c z^{-1} V(z) = Y(z) \Rightarrow V(z) = \frac{z Y(z)}{c}. \text{ Also, } V(z) = c z^{-1} U(z) + d Y(z). \text{ Thus,}$$

$$Y(z) \{1 - c d z^{-1} - c d z^{-1} (1 - c d z^{-1}) + c^2 z^{-2}\} = c^2 z^{-2} X(z). \text{ Hence,}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{c^2}{z^2 - 2c d z + c^2(1 + d^2)}.$$

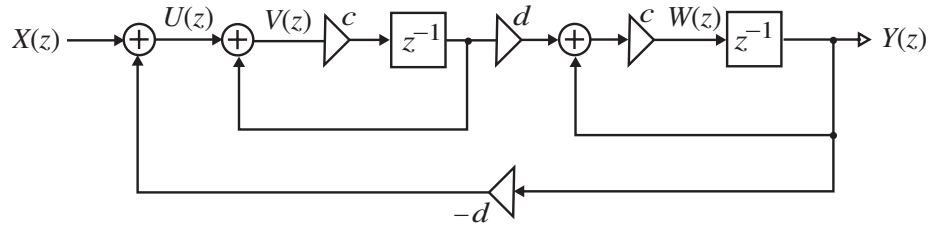
Let $c^2(1+d^2) = r^2$ and $cd = r \cos \theta$. Solving we get $\cos \theta = \frac{d}{\sqrt{1+d^2}}$. Thus,

$$d = \cot \theta \text{ and } c = r \sin \theta. \text{ Hence, } \Delta d = -\operatorname{cosec}^2 \theta (\Delta \theta) \text{ and } \Delta c = \sin \theta (\Delta \theta) + r \cos \theta (\Delta \theta). \text{ Or,}$$

$$\begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix} = \begin{bmatrix} \sin \theta & r \cos \theta \\ 0 & -\operatorname{cosec}^2 \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = -\sin \theta \begin{bmatrix} -\operatorname{cosec}^2 \theta & -r \cos \theta \\ 0 & \sin \theta \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sin \theta} & -r \cos \theta \cdot \cos \theta \\ 0 & -\sin^2 \theta \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix}.$$

The digital filter structure of Figure 12.54 with internal variables labeled is shown below.



Analysis yields $U(z) = X(z) - dY(z)$, $V(z) = U(z) + cz^{-1}V(z) \Rightarrow V(z) = \frac{U(z)}{1 - cz^{-1}}$, and

$$W(z) = cdz^{-1}V(z) + cz^{-1}W(z), \text{ or } W(z) = \frac{cdz^{-1}}{1 - cz^{-1}}V(z) = \frac{cdz^{-1}}{(1 - cz^{-1})^2}U(z). \text{ As a result,}$$

$$Y(z) = cz^{-1}W(z) \text{ or, } \frac{Y(z)}{cz^{-1}} = \frac{cdz^{-1}(X(z) - dY(z))}{(1 - cz^{-1})^2}. \text{ Thus,}$$

$$Y(z)\{(1 - cz^{-1})^2 + c^2dz^{-2}\} = c^2dz^{-2}X(z). \text{ Hence, } H(z) = \frac{Y(z)}{X(z)} = \frac{c^2d}{z^2 - 2cz + c^2(1+d^2)}.$$

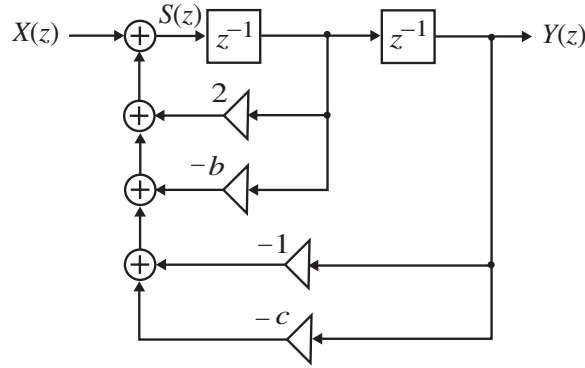
We let $c = r \cos \theta$, and $c^2(1+d^2) = r^2$. Therefore, $1+d^2 = \sec^2 \theta \Rightarrow d = \tan \theta$. Taking partials we then get $\Delta c = \cos \theta (\Delta r) - r \sin \theta (\Delta \theta)$ and $\Delta d = \sec^2 \theta (\Delta \theta)$. Thus,

$$\begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ 0 & \sec^2 \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos \theta} & r \sin \theta \cdot \cos \theta \\ 0 & \cos^2 \theta \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix}.$$

12.5 The digital filter structure of Figure P12.2 with internal variables labeled is shown on the next page. Analysis yields $X(z) + 2z^{-1}S(z) - bz^{-1}S(z) - Y(z) - cY(z) = S(z)$ and $Y(z) = z^{-2}S(z)$.

$$\text{Thus, } H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1 + (b-2)z^{-1} + (1+c)z^{-2}} = \frac{1}{z^2 + (b-2)z + (1+c)}.$$

Let $1+c = r^2$ and $2-b = 2r \cos \theta$. Taking partials, we get $\Delta c = 2r(\Delta r)$ and $-\Delta b = 2 \cos \theta (\Delta r) - 2r \sin \theta (\Delta \theta)$. Therefore, we have



$$\begin{bmatrix} \Delta c \\ \Delta b \end{bmatrix} = \begin{bmatrix} 2r & 0 \\ 2 \cos \theta & -2r \sin \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} \text{ or, } \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2r} & 0 \\ \frac{1}{2r^2 \tan \theta} & -\frac{1}{2r \sin \theta} \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta b \end{bmatrix}.$$

From Eq. (12.19) the pole sensitivities for Figure 12.11 are given by

$$\begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\frac{1}{r} \sin \theta & \frac{1}{r} \cos \theta \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}.$$

12.6 (a) For direct form implementation $B(z) = (z - z_1)(z - z_2)(z - z_3)$, where

$$z_1 = r_1 e^{j\theta_1}, z_2 = r_2 e^{j\theta_2}, \text{ and } z_3 = r_3 e^{j\theta_3}. \text{ Thus, } B(z) = (z^2 - 2r_1 \cos \theta_1 z + r_1^2)(z - r_3) \\ = (z^2 + 0.5625z + 0.7386)(z - 0.0606). \text{ This implies,}$$

$$2r_1 \cos \theta_1 = -0.5625, r_1^2 = 0.7386, r_3 = 0.0606, \text{ and } \theta_3 = \pi. \text{ Thus, } r_1 = \sqrt{0.7386} = 0.8594$$

$$\text{and } \cos \theta_1 = \frac{-0.5625}{2\sqrt{0.7386}} = -0.3273. \text{ Now,}$$

$$\frac{1}{B(z)} = \frac{1}{(z^2 + 0.5625z + 0.7386)(z - 0.0606)} = \\ \frac{-0.6440 + j0.2711}{z + 0.2813 + j0.8121} + \frac{-0.6440 - j0.2711}{z + 0.2813 - j0.8121} + \frac{1.2811}{z + 0.0606}.$$

$$\mathbf{P}_1 = [\cos \theta_1 \quad r_1 \quad r_1^2 \cos \theta_1] = [-0.3273 \quad 0.8594 \quad -0.2417],$$

$$\mathbf{Q}_1 = [\sin \theta_1 \quad 0 \quad r_1^2 \sin \theta_1] = [-0.9449 \quad 0 \quad 0.6979], R_1 = -0.6440, \text{ and } X_1 = 0.2711.$$

$$\text{Likewise, } \mathbf{P}_3 = [\cos \theta_3 \quad r_3 \quad r_3^2 \cos \theta_3] = [-1 \quad 0.0606 \quad -0.0037],$$

$$\mathbf{Q}_3 = [\sin \theta_3 \quad 0 \quad r_3^2 \sin \theta_3] = [0 \quad 0 \quad 0], R_3 = 1.2881, \text{ and } X_3 = 0.$$

$$\text{Thus, } \Delta r_1 = (-R_1 \mathbf{P}_1 + X_1 \mathbf{Q}_1) \cdot \Delta \mathbf{B} = -0.4669 \Delta b_0 + 0.5535 \Delta b_1 + 0.0335 \Delta b_2,$$

$$\Delta \theta_1 = -\frac{1}{r_1} (X_1 \mathbf{P}_1 + R_1 \mathbf{Q}_1) \cdot \Delta \mathbf{B} = 0.8113 \Delta b_0 - 0.2711 \Delta b_1 - 0.4467 \Delta b_2,$$

$$\Delta r_3 = (-R_3 \mathbf{P}_3 + X_3 \mathbf{Q}_3) \cdot \Delta \mathbf{B} = 1.2881 \Delta b_0 - 0.0781 \Delta b_1 + 0.0048 \Delta b_2, \text{ and}$$

$$\Delta \theta_3 = -\frac{1}{r_3} (X_3 \mathbf{P}_3 + R_3 \mathbf{Q}_3) \cdot \Delta \mathbf{B} = 0.$$

(b) Cascade form: $B(z) = (z^2 + c_1z + c_0)(z + d_0) = B_1(z)B_2(z)$, where
 $B_1(z) = z^2 + c_1z + c_0 = z^2 + 0.5625z + 0.7386 = (z - r_1e^{j\theta_1})(z - r_1e^{-j\theta_1}) = z^2 - 2r_1\cos\theta_1z + r_1^2$
and $B_2(z) = z + d_0 = z - 0.0606 = z - r_3e^{j\theta_3}$. Comparing we get
 $2r_1\cos\theta_1 = 0.5625$, $r_1^2 = 0.7386$, $r_3 = 0.0606$, $\theta_3 = \pi$. Solving the first two equations we get
 $r_1 = \sqrt{0.7386} = 0.8594$ and $\cos\theta_1 = \frac{-0.5625}{2\sqrt{0.7386}} = -0.3273$.

Now, $\frac{1}{B(z_1)} = \frac{-j0.6157}{z + 0.2813 - j0.8121} + \frac{j0.6157}{z + 0.2813 + j0.8121}$. Hence, $R_1 = 0$ and

$$X_1 = -0.6157. \quad \mathbf{P}_1 = [\cos\theta_1 \quad r_1] = [-0.3273 \quad 0.8594], \quad \mathbf{Q}_1 = [-\sin\theta_1 \quad 0] = [-0.9449 \quad 0].$$

Next, $\frac{1}{B_2(z)} = \frac{1}{z + 0.0606}$. Hence, $R_3 = 1$ and $X_3 = 0$. Here, $\mathbf{P}_3 = \cos\theta_3 = -1$, and

$$\mathbf{Q}_3 = -\sin\theta_3 = 0. \quad \text{Thus,}$$

$$\Delta r_1 = (-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1) \cdot [\Delta c_0 \quad \Delta c_1]^t = X_1\mathbf{Q}_1 \cdot [\Delta c_0 \quad \Delta c_1]^t = 0.5818\Delta c_0,$$

$$\Delta\theta_1 = -\frac{1}{r_1}(X_1\mathbf{P}_1 + R_1\mathbf{Q}_1) \cdot [\Delta c_0 \quad \Delta c_1]^t = -\frac{1}{r_1} \cdot X_1\mathbf{P}_1 \cdot [\Delta c_0 \quad \Delta c_1]^t = -0.23458\Delta c_0 + 0.6157\Delta c_1,$$

$$\Delta r_3 = (-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3) \cdot \Delta d_0 = -\Delta d_0, \quad \Delta\theta_3 = -\frac{1}{r_3}(X_3\mathbf{P}_3 + R_3\mathbf{Q}_3) \cdot \Delta d_0 = -\frac{1}{r_3}R_3\mathbf{Q}_3 \cdot \Delta d_0 = 0$$

12.7 In terms of the transfer parameters, the input-output relation of the two-pair of Figure P12.3 is given by $Y_1(z) = t_{11}X_1(z) + t_{12}X_2(z)$ and $Y_2(z) = t_{21}X_1(z) + t_{22}X_2(z)$. The constraining equation of the two-pair is given by $X_2(z) = \alpha Y_2(z)$. From Eq. (7.135b) we get

$$H(z) = \frac{Y_1(z)}{X_1(z)} = t_{11} + \frac{\alpha t_{12}t_{21}}{1 - \alpha t_{22}}. \quad \text{Hence, } \frac{\partial H(z)}{\partial \alpha} = \frac{t_{12}t_{21}(1 - \alpha t_{22}) + t_{22}\alpha t_{12}t_{21}}{(1 - \alpha t_{22})^2} = \frac{t_{12}t_{21}}{(1 - \alpha t_{22})^2}.$$

Substituting $X_2(z) = \alpha Y_2(z)$ in $Y_2(z) = t_{21}X_1(z) + t_{22}X_2(z)$, we get the expression for the scaling transfer function $F_\alpha(z) = \frac{Y_2(z)}{X_1(z)} = \frac{t_{21}}{1 - \alpha t_{22}}$.

Next, from the structure given below we observe that the noise transfer function is given by

$$G_\alpha(z) = \left. \frac{Y_1(z)}{U(z)} \right|_{X_1(z)=0}. \quad \text{From the structure we also observe that } X_2(z) = \alpha Y_2(z) + U(z).$$

Substituting this in the transfer relations we arrive at, with $X_1(z) = 0$,

$$Y_1(z) = t_{12}X_2(z) = t_{12}(\alpha X_2(z) + U(z)) \quad \text{and} \quad Y_2(z) = t_{22}X_2(z) = t_{22}(\alpha Y_2(z) + U(z)).$$

From these two equations we obtain, after some algebra,

$$Y_1(z) = \frac{\alpha t_{12}t_{22}}{1 - \alpha t_{22}}U(z) + t_{12}U(z) = \frac{t_{12}}{1 - \alpha t_{22}}U(z). \quad \text{Hence, the noise transfer function is given by}$$

$$G_\alpha(z) = \left. \frac{Y_1(z)}{U(z)} \right|_{X_1(z)=0} = \frac{t_{12}}{1 - \alpha t_{22}}. \quad \text{Therefore, } \frac{\partial H(z)}{\partial \alpha} = \frac{t_{21}}{1 - \alpha t_{22}} \cdot \frac{t_{12}}{1 - \alpha t_{22}} = F_\alpha(z)G_\alpha(z).$$

12.8 (a) $W_N(\omega) = \left[\frac{1}{2N+1} \left(N + \frac{\sin[(N+1)\omega]}{\sin(\omega)} \right) \right]^{1/2} = \left[\frac{1}{2N+1} \left(1 + 4 \sum_{n=1}^{N/2} \cos^2(\omega n) \right) \right]^{1/2}$. Since the

maximum value of $\cos^2(\omega n)$ is 1 and the minimum value is 0, hence,

$$\max\{W_N(\omega)\} = \left[\frac{1}{2N+1} \left(1 + \frac{4N}{2} \right) \right]^{1/2} = 1 \text{ and } \min\{W_N(\omega)\} = \left[\frac{1}{2N+1} \right]^{1/2} > 0. \text{ Therefore,}$$

$$0 < W_N(\omega) \leq 1.$$

(b) $W_N(0) = \left[\frac{1}{2N+1} \left(1 + 4 \sum_{m=1}^{N/2} 1 \right) \right]^{1/2} = 1. \quad W_N(\pi) = \left[\frac{1}{2N+1} \left(1 + 4 \sum_{m=1}^{N/2} 1 \right) \right]^{1/2} = 1.$

(c) $\lim_{N \rightarrow \infty} W_N(\pi) = \left[\lim_{N \rightarrow \infty} \frac{N}{2N+1} + \lim_{N \rightarrow \infty} \frac{1}{2N+1} \cdot \frac{\sin((N+1)\omega)}{\sin(\omega)} \right]^{1/2}$. Since

$$|\sin((N+1)\omega)| \leq 1, \quad \lim_{N \rightarrow \infty} \frac{1}{2N+1} \cdot \frac{\sin((N+1)\omega)}{\sin(\omega)} = 0. \text{ Hence, } \lim_{N \rightarrow \infty} W_N(\pi) = \frac{1}{\sqrt{2}}.$$

12.9 From Eq. (12.75), $\text{SNR}_{A/D} = 6.0206b + 16.81 - 20 \log_{10}(K)$ dB. For a constant value of K , an increase in b by 1 bit, increases the $\text{SNR}_{A/D}$ by 6.0206 dB and an increase in b by 2 bits, increases the $\text{SNR}_{A/D}$ by 12.0412 dB.

If $K = 4$ and $b = 7$, then $\text{SNR}_{A/D} = 6.0206 \times 7 + 16.81 - 20 \log_{10}(4) = 46.9122$ dB. Therefore, for $K = 4$ and $b = 9$, $\text{SNR}_{A/D} = 46.91 + 12.04 = 58.95$ dB; for $K = 4$ and $b = 11$, $\text{SNR}_{A/D} = 58.95 + 12.04 = 70.99$ dB; for $K = 4$ and $b = 13$, $\text{SNR}_{A/D} = 70.99 + 12.04 = 83.03$ dB; for $K = 4$ and $b = 15$, $\text{SNR}_{A/D} = 83.03 + 12.04 = 95.08$ dB.

If $K = 6$ and $b = 7$, then $\text{SNR}_{A/D} = 6.0206 \times 7 + 16.81 - 20 \log_{10}(6) = 43.3902$ dB. Therefore, for $K = 6$ and $b = 9$, $\text{SNR}_{A/D} = 43.39 + 12.04 = 55.43$ dB; for $K = 6$ and $b = 11$, $\text{SNR}_{A/D} = 55.43 + 12.04 = 67.47$ dB; for $K = 6$ and $b = 13$, $\text{SNR}_{A/D} = 67.47 + 12.04 = 79.51$ dB; for $K = 6$ and $b = 15$, $\text{SNR}_{A/D} = 79.51 + 12.04 = 91.56$ dB.

If $K = 8$ and $b = 7$, then $\text{SNR}_{A/D} = 6.0206 \times 7 + 16.81 - 20 \log_{10}(8) = 40.8922$ dB. Therefore, for $K = 8$ and $b = 9$, $\text{SNR}_{A/D} = 40.89 + 12.04 = 52.93$ dB; for $K = 8$ and $b = 11$, $\text{SNR}_{A/D} = 52.93 + 12.04 = 64.97$ dB; for $K = 8$ and $b = 13$, $\text{SNR}_{A/D} = 64.97 + 12.04 = 77.01$ dB; for $K = 8$ and $b = 15$, $\text{SNR}_{A/D} = 77.01 + 12.04 = 89.05$ dB.

12.10 (a) $H(z) = \sum_{k=1}^N \frac{A_k}{z+a_k} = \sum_{k=1}^N C_k \frac{a_k z+1}{z+a_k} - \sum_{k=1}^N C_k a_k = \sum_{k=1}^N C_k F_k(z) + B$, where $C_k = \frac{A_k}{1-a_k^2}$,

$$F_k(z) = \frac{a_k z+1}{z+a_k}, \text{ and } B = -\sum_{k=1}^N C_k a_k.$$

(b) $\sigma_o^2 = \frac{1}{2\pi j} \oint_C H(z)H(z^{-1})z^{-1} dz = \frac{1}{2\pi j} \oint_C \left[\sum_{k=1}^N C_k F_k(z) + B \right] \left[\sum_{k=1}^N C_k F_k(z^{-1}) + B \right] z^{-1} dz$

$$= \frac{1}{2\pi j} \left[\oint_C B^2 z^{-1} dz + B \sum_{k=1}^N C_k \oint_C F_k(z) z^{-1} dz + B \sum_{\ell=1}^N C_\ell \oint_C F_\ell(z^{-1}) z^{-1} dz \right.$$

$$\left. + \sum_{k=1}^N \sum_{\ell=1}^N C_k C_\ell \oint_C F_k(z) F_\ell(z) z^{-1} dz \right].$$

Now, $\frac{1}{2\pi j} \oint_C B^2 z^{-1} dz = B^2$, $\frac{1}{2\pi j} \oint_C F_k(z) z^{-1} dz = a_k$, $\frac{1}{2\pi j} \oint_C F_\ell(z^{-1}) z^{-1} dz = a_\ell$, and

$$\frac{1}{2\pi j} \oint_C F_k(z) F_\ell(z) z^{-1} dz = \frac{1-a_k^2 - a_\ell^2 + a_k a_\ell}{1-a_k a_\ell}. \text{ Therefore,}$$

$$\sigma_o^2 = B^2 + 2 \sum_{k=1}^N B C_k a_k + \sum_{k=1}^N \sum_{\ell=1}^N \frac{1-a_k^2 - a_\ell^2 + a_k a_\ell}{1-a_k a_\ell}. \text{ Since, } \sum_{k=1}^N C_k a_k = -B, \text{ we have}$$

$$\sigma_o^2 = -B^2 + \sum_{k=1}^N \sum_{\ell=1}^N \frac{1-a_k^2 - a_\ell^2 + a_k a_\ell}{1-a_k a_\ell} C_k C_\ell$$

$$= -\sum_{k=1}^N \sum_{\ell=1}^N C_k C_\ell a_k a_\ell + \sum_{k=1}^N \sum_{\ell=1}^N C_k C_\ell \frac{1-a_k^2 - a_\ell^2 + a_k a_\ell}{1-a_k a_\ell}$$

$$= \sum_{k=1}^N \sum_{\ell=1}^N C_k C_\ell \left[\frac{1-a_k^2 - a_\ell^2 + a_k a_\ell}{1-a_k a_\ell} - a_k a_\ell \right] = \sum_{k=1}^N \sum_{\ell=1}^N \frac{A_k}{1-a_k^2} \cdot \frac{A_\ell}{1-a_\ell^2} \left[\frac{1-a_k^2 - a_\ell^2 + a_k a_\ell}{1-a_k a_\ell} - a_k a_\ell \right]$$

$$= \sum_{k=1}^N \sum_{\ell=1}^N \frac{A_k}{1-a_k^2} \cdot \frac{A_\ell}{1-a_\ell^2} \left[\frac{(1-a_k^2)(1-a_\ell^2)}{1-a_k a_\ell} \right] = \sum_{k=1}^N \sum_{\ell=1}^N \frac{A_k A_\ell}{1-a_k a_\ell}. \quad (\text{A})$$

Further, $\sigma_o^2 = \sum_{k=1}^N \frac{A_k^2}{1-a_k^2} + \sum_{k=1}^{N-1} \sum_{\ell=k+1}^N \frac{A_k A_\ell}{1-a_k a_\ell} + \sum_{\ell=1}^{N-1} \sum_{k=\ell+1}^N \frac{A_k A_\ell}{1-a_k a_\ell}$

$$= \sum_{k=1}^N \frac{A_k^2}{1-a_k^2} + 2 \sum_{k=1}^{N-1} \sum_{\ell=k+1}^N \frac{A_k A_\ell}{1-a_k a_\ell}.$$

(c) See Eq. (A).

12.11(a) $H_1(z) = \frac{(z-2)(z+4)}{(z-0.2)(z-0.6)} = 1 + \frac{-16.1}{z-0.6} + \frac{18.9}{z-0.2}$. Making use of Eq. (12.87) and Table

$$12.4, \text{ we get } \sigma_{1,n}^2 = 1 + \frac{(-16.1)^2}{1-(0.6)^2} + \frac{(18.9)^2}{1-(0.2)^2} + \frac{2 \times (-16.1) \times (18.9)}{1-(0.6) \times (0.2)}$$

$$= 1 + 405.0156 + 372.0937 - 691.5682 = 86.5411.$$

Output of Program 12_4.m is 86.5412.

(b) $H_2(z) = \frac{2(3z-1)(2z^2-4z+1)}{(2z-1)(z+0.5)(z^2+0.8z+0.5)} = \frac{25}{z+0.5} + \frac{-0.2174}{z-0.5} + \frac{-18.7826z-21.2174}{z^2+0.8z+0.5}$.

Making use of Eq. (12.87) and Table 12.4, we get $\sigma_{2,n}^2 = \frac{(25)^2}{1-(0.5)^2} + \frac{(0.2174)^2}{1-(0.5)^2}$

$$+ \frac{2 \times 25 \times (-0.2174)}{1+0.25} + \frac{2 \times 25 \times [-18.7826 + (-21.2174) \times (-0.5)]}{1+0.8 \times (-0.5) + 0.5 \times (-0.5)^2}$$

$$+ \frac{2 \times (-0.2174) \times [-18.7826 + (-21.2174) \times 0.5]}{1+0.8 \times 0.5 + 0.5 \times (0.5)^2}$$

$$+ \frac{[(-18.7826)^2 + (-21.2174)^2] \times (1-0.5)^2 - 2 \times (-18.7826) \times (-21.2174) \times (1-0.5) \times 0.8}{[1-(0.5)^2]^2 + 2 \times (0.5) \times (0.8)^2 - [1+(0.5)^2] \times (0.8)^2}$$

$$= 833.3333 + 0.0630 - 8.6960 - 563.7172 + 8.3799 + 704.1211 = 973.4841.$$

$$\sigma_{2,n}^2 = \frac{(25)^2}{1-(0.5)^2} + \frac{(0.2174)^2}{1-(0.5)^2} + \frac{2(25)(-0.2174)}{1+0.25} + \frac{2(25)(-18.7826 + (-21.2174) \times (-0.5))}{1+0.8(-0.5) + 0.5(-0.5)^2}$$

$$+ \frac{2(-0.2174)(-18.7826 + (-21.2174) \times (0.5))}{1+0.8(0.5) + 0.5(0.5)^2}$$

$$+ \frac{[(-18.7826)^2 + (-21.2174)^2](1-0.5)^2 - 2(-18.7826)(-21.2174)(1-0.5)(0.8)}{(1-0.5^2)^2 + 2(0.5)(0.8)^2 - (1+0.5^2)(0.8)^2}$$

$$= 833.3333 + 0.0630 - 8.6960 - 563.7172 - 8.3799 - 704.1211 = 973.4841$$

Output of Program 12_4.m is 973.4831.

(c) $H_3(z) = \frac{(z+1)^2}{z^2+0.6z+0.4} = 1 + \frac{1.4z+0.6}{z^2+0.6z+0.4}$. Making use of Eq. (12.87) and Table 12.4,

we get $\sigma_{3,n}^2 = 1 + \frac{[(1.4)^2 + (0.6)^2][1-(0.4)^2] - 2 \times 1.4 \times 0.6 \times 0.6(1-0.4)}{[1-(0.4)^2]^2 + 2 \times 0.4 \times (0.6)^2 - [1+(0.4)^2](0.6)^2}$

$$= 1 + 2.3333 = 3.3333.$$

Output of Program 12_4.m is 3.3333.

(d) $H_4(z) = \frac{(z-2.5)(z+1.25)(z+1)}{(z-0.4)(z-0.8)(z+0.5)} = 1 + \frac{-0.825}{z+0.8} + \frac{4.1667}{z+0.5} + \frac{-4.4917}{z-0.4}$.

Making use of Eq. (12.87) and Table 12.4, we get

$$\begin{aligned} \sigma_{4,n}^2 &= 1 + \frac{(0.825)^2}{1-(0.8)^2} + \frac{(4.1667)^2}{1-(0.5)^2} + \frac{(4.4917)^2}{1-(0.4)^2} + \frac{2 \times 0.825 \times 4.1667}{1-0.4} + \frac{2 \times 0.825 \times 4.4917}{1+0.32} \\ &+ \frac{2 \times 4.4917 \times 4.1667}{1+0.25} = 1 + 1.8906 + 23.1485 + 24.0183 - 11.4584 + 5.6146 - 31.1926 \\ &= 13.021. \end{aligned}$$

Output of Program 12_4.m is 13.0208.

12.12 $H(z) = A + \frac{B}{z+\beta} + \frac{C}{z+\gamma}$. $\sigma_n^2 = A^2 + \frac{B^2}{1-\beta^2} + \frac{C^2}{1-\gamma^2} + \frac{2BC}{1-\beta\gamma}$. For $\beta = 0.6, \gamma = -0.8$,

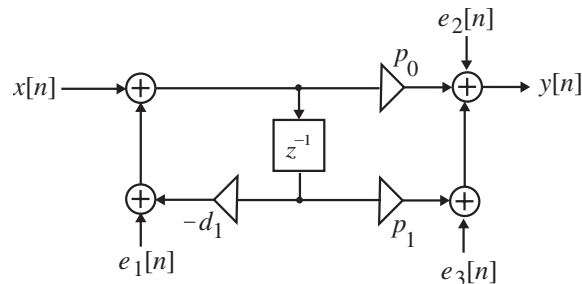
$A = 3, B = -4, C = 2$, we get

$$\sigma_n^2 = 3^2 + \frac{4^2}{1-(0.6)^2} + \frac{2^2}{1-(0.8)^2} + \frac{2 \times (-4) \times 2}{1-(-0.6) \times (0.8)} = 34.3003.$$

Output of Program 12_4.m is 34.3003.

12.13 (a) Quantization of products before addition

Direct Form II - The noise model is shown below.



The noise transfer function from the noise source $e_1[n]$ to the filter output is

$$G_1(z) = \frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1}} = \frac{p_0 z + p_1}{z + d_1} = p_0 + \frac{p_1 - p_0 d_1}{z + d_1}.$$

The corresponding normalized noise variance at the output is $\sigma_1^2 = p_0^2 + \frac{(p_1 - p_0 d_1)^2}{1 - d_1^2}$.

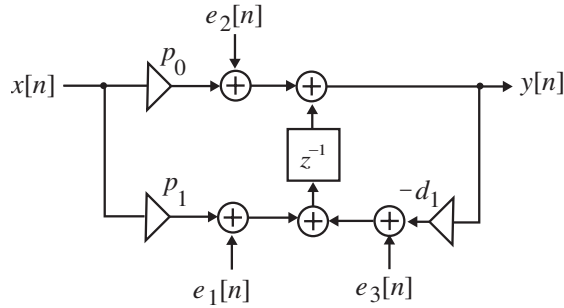
The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$$G_2(z) = G_3(z) = 1. \text{ The corresponding normalized noise variance at the output is}$$

$$\sigma_2^2 = \sigma_3^2 = 1. \text{ Hence the total noise variance at the output is } \sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

$$= 2 + p_0^2 + \frac{(p_1 - p_0 d_1)^2}{1 - d_1^2}.$$

Direct Form II_t - The noise model is shown below.



Analyzing the structure, we have that $Y(z) = \frac{E_1(z)z^{-1} + E_2(z) + E_3(z)z^{-1}}{1 + d_1z^{-1}}$.

The noise transfer function from $e_1[n]$ to the filter output is

$$G_1(z) = \left. \frac{Y(z)}{E_1(z)} \right|_{E_2(z)=E_3(z)=0} = \frac{z^{-1}}{1 + d_1z^{-1}} = \frac{1}{z + d_1}. \text{ The corresponding normalized noise}$$

$$\text{variance at the output is } \sigma_1^2 = \frac{1}{1 - d_1^2}.$$

The noise transfer function from $e_2[n]$ to the filter output is

$$G_2(z) = \left. \frac{Y(z)}{E_2(z)} \right|_{E_1(z)=E_3(z)=0} = \frac{1}{1 + d_1z^{-1}} = \frac{z}{z + d_1} = 1 - \frac{d_1}{z + d_1}. \text{ The corresponding}$$

$$\text{normalized noise variance at the output is } \sigma_2^2 = 1 + \frac{d_1^2}{1 - d_1^2} = \frac{1}{1 - d_1^2}.$$

The noise transfer function from $e_3[n]$ to the filter output is

$$G_3(z) = \left. \frac{Y(z)}{E_3(z)} \right|_{E_1(z)=E_2(z)=0} = \frac{z^{-1}}{1 + d_1z^{-1}} = \frac{1}{z + d_1}. \text{ The corresponding normalized noise}$$

$$\text{variance at the output is } \sigma_3^2 = \frac{1}{1 - d_1^2}.$$

$$\text{Hence the total noise variance at the output is } \sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{3}{1 - d_1^2}.$$

(b) Quantization of products after addition

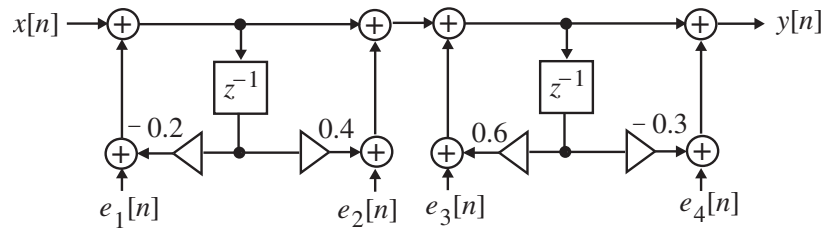
$$\text{Direct Form II} - \sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 2 + p_0^2 + \frac{(p_1 - p_0d_1)^2}{1 - d_1^2}.$$

$$\text{Direct Form II}_t - \sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{3}{1 - d_1^2}.$$

12.14 (a) Quantization of products before addition

Cascade Structure #1: $G(z) = \frac{(1 + 0.4z^{-1})(1 - 0.3z^{-1})}{(1 + 0.2z^{-1})(1 - 0.6z^{-1})}$. The noise model of this structure is as

below



The noise transfer function from the noise source $e_1[n]$ to the filter output is

$$G_1(z) = \frac{(z + 0.4)(z - 0.3)}{(z + 0.2)(z - 0.6)} = 1 + \frac{0.375}{z - 0.6} + \frac{0.125}{z + 0.2}.$$

The corresponding normalized noise variance at the output is

$$\sigma_{1,n}^2 = 1 + \frac{(0.375)^2}{1 - (0.6)^2} + \frac{(0.125)^2}{1 - (-0.2)^2} + \frac{2 \times 0.375 \times 0.125}{1 - 0.6 \times (-0.2)} = 1.3197.$$

Output of Program 12_4.m is 1.3197.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$$G_2(z) = \frac{z - 0.3}{z - 0.6} = 1 + \frac{0.3}{z - 0.6}.$$

The normalized noise variance at the output due to each of these noise sources is

$$\sigma_{2,n}^2 = 1 + \frac{(0.3)^2}{1 - (0.6)^2} = 1.1406.$$

Output of Program 12_4.m is 1.1406.

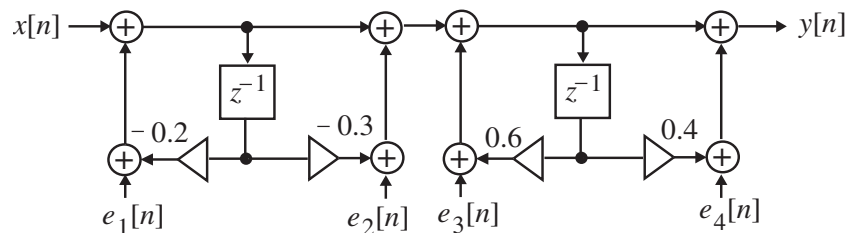
The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 4.6009$.

Cascade Structure #2: $G(z) = \frac{(1 - 0.3z^{-1})(1 + 0.4z^{-1})}{(1 + 0.2z^{-1})(1 - 0.6z^{-1})}$. The noise model of this structure is as

below:



$\sigma_{1,n}^2 = 1.3197$ as in Structure#1.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$$G_2(z) = \frac{z+0.4}{z-0.6} = 1 + \frac{1}{z-0.6}. \text{ Hence, } \sigma_{2,n}^2 = 1 + \frac{(1)^2}{1-(0.6)^2} = 2.5625.$$

Output of Program 12_4.m is 2.5625.

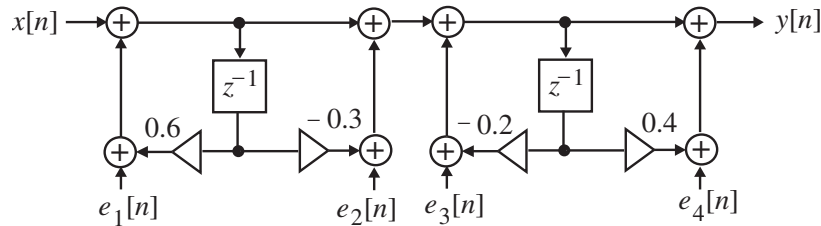
The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 7.4447$.

Cascade Structure #3: $G(z) = \frac{(1-0.3z^{-1})(1+0.4z^{-1})}{(1-0.6z^{-1})(1+0.2z^{-1})}$. The noise model of this structure is as

below:



$\sigma_{1,n}^2 = 1.3197$ as in Structure#1.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$$G_2(z) = \frac{z+0.4}{z+0.2} = 1 + \frac{0.2}{z+0.2}. \text{ Hence, } \sigma_{2,n}^2 = 1 + \frac{(0.2)^2}{1-(0.2)^2} = 1.0417.$$

Output of Program 12_4.m is 1.0417.

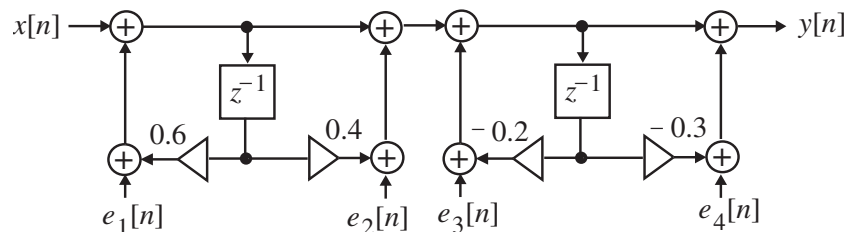
The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 4.4031$.

Cascade Structure #4: $G(z) = \frac{(1+0.4z^{-1})(1-0.3z^{-1})}{(1-0.6z^{-1})(1+0.2z^{-1})}$. The noise model of this structure is as

below:



$\sigma_{1,n}^2 = 1.3197$ as in Structure#1.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$$G_2(z) = \frac{z-0.3}{z+0.2} = 1 + \frac{-0.5}{z+0.2}. \text{ Hence, } \sigma_{2,n}^2 = 1 + \frac{(-0.5)^2}{1-(0.2)^2} = 1.2604.$$

Output of Program 12_4.m is 1.2604.

The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 4.8405$.

Hence, the Cascade Structure #3 has the smallest round off noise variance.

(b) Quantization of products after addition. From the results of Part (a)

Cascade Structure #1: $\sigma_n^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2 + \sigma_{4,n}^2 = 3.4603$.

Cascade Structure #2: $\sigma_n^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2 + \sigma_{4,n}^2 = 4.8822$.

Cascade Structure #3: $\sigma_n^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2 + \sigma_{4,n}^2 = 3.614$.

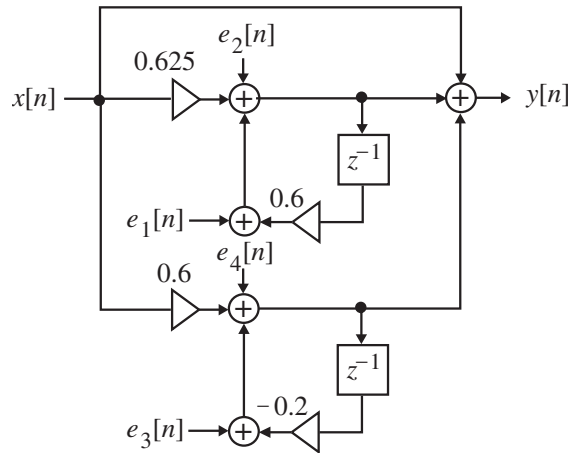
Cascade Structure #4: $\sigma_n^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2 + \sigma_{4,n}^2 = 3.5801$.

Hence, the Cascade Structure #1 has the smallest round off noise variance.

12.15 (a) Quantization of products before addition.

$$\text{Parallel Form \#1: } G(z) = \frac{(1+0.4z^{-1})(1-0.3z^{-1})}{(1+0.2z^{-1})(1-0.6z^{-1})} = 1 + \frac{0.625}{1-0.6z^{-1}} + \frac{-0.625}{1+0.2z^{-1}}.$$

The noise model of this structure is shown below:



The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is

$$G_2(z) = \frac{z}{z-0.6} = 1 + \frac{0.6}{z-0.6}. \text{ The corresponding normalized noise variance at the output is}$$

$$\sigma_{2,n}^2 = 1 + \frac{(0.6)^2}{1-(0.6)^2} = 1.5625.$$

The noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the filter output is

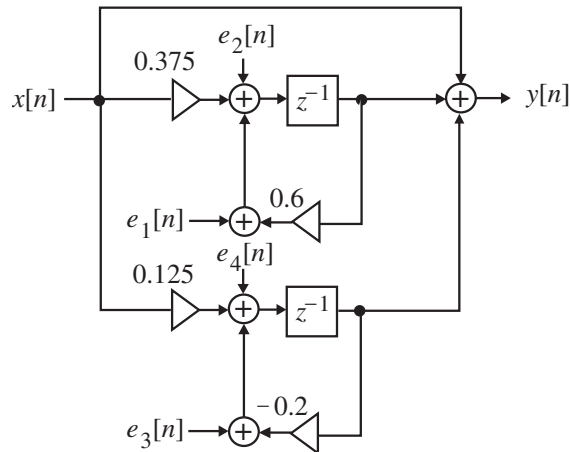
$G_4(z) = \frac{z}{z+0.2} = 1 + \frac{-0.2}{z+0.2}$. The corresponding normalized noise variance at the output is

$$\sigma_{2,n}^2 = 1 + \frac{(-0.2)^2}{1-(0.2)^2} = 1.0417.$$

Hence the total noise variance at the output is $\sigma_n^2 = 2\sigma_{2,n}^2 + 2\sigma_{4,n}^2 = 5.2804$.

Parallel Form #2: $G(z) = 1 + \frac{0.375}{z-0.6} + \frac{0.125}{z+0.2} = 1 + \frac{0.375z^{-1}}{1-0.6z^{-1}} + \frac{0.125z^{-1}}{1+0.2z^{-1}}$.

The noise model of this structure is shown below:



The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is

$G_2(z) = \frac{z^{-1}}{1-0.6z^{-1}} = \frac{1}{z-0.6}$. The corresponding normalized noise variance at the output is

$$\sigma_{2,n}^2 = \frac{(1)^2}{1-(0.6)^2} = 1.5625.$$

The noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the filter output is

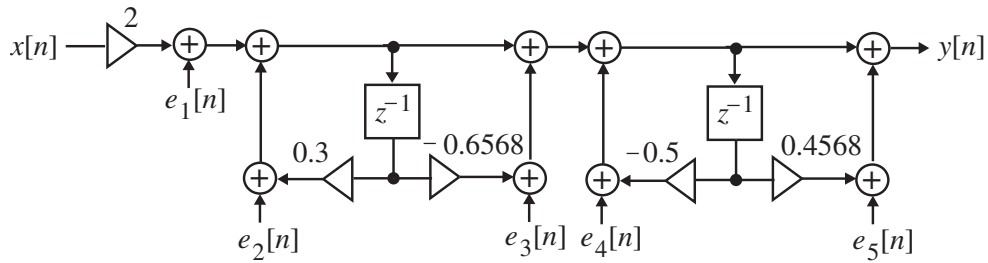
$G_4(z) = \frac{z^{-1}}{1+0.2z^{-1}} = \frac{1}{z+0.2}$. The corresponding normalized noise variance at the output is

$$\sigma_{2,n}^2 = \frac{(1)^2}{1-(0.2)^2} = 1.0417.$$

Hence the total noise variance at the output is $\sigma_n^2 = 2\sigma_{2,n}^2 + 2\sigma_{4,n}^2 = 5.2804$.

Therefore, both parallel forms have identical round off noise variances.

(b) Quantization of products after addition. From the results of Part **(a)**



The noise transfer function from $e_1[n]$ and $e_2[n]$ to filter output is

$$G_1(z) = \frac{1 - 0.2z^{-1} - 0.3z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} = 1 + \frac{-0.0625}{z + 0.5} + \frac{-0.3375}{z - 0.3}$$

The corresponding normalized noise variance at the output is

$$\sigma_{1,n}^2 = (1)^2 + \frac{(-0.0625)^2}{1 - (-0.5)^2} + \frac{(-0.3375)^2}{1 - (0.3)^2} + \frac{(-0.0625) \times (-0.3375)}{1 - 0.3 \times (-0.5)} = 1.1671.$$

Output of Program 12_4.m is 1.1671.

The noise transfer function from $e_3[n]$ and $e_4[n]$ to filter output is

$$G_3(z) = \frac{1 + 0.4568z^{-1}}{1 + 0.5z^{-1}} = \frac{z + 0.4568}{z + 0.5} = 1 + \frac{-0.0432}{z + 0.5}$$

The corresponding normalized noise

$$\text{variance at the output is } \sigma_{3,n}^2 = (1)^2 + \frac{(-0.0432)^2}{1 - (0.5)^2} = 1.0025.$$

Output of Program 12_4.m is 1.0025.

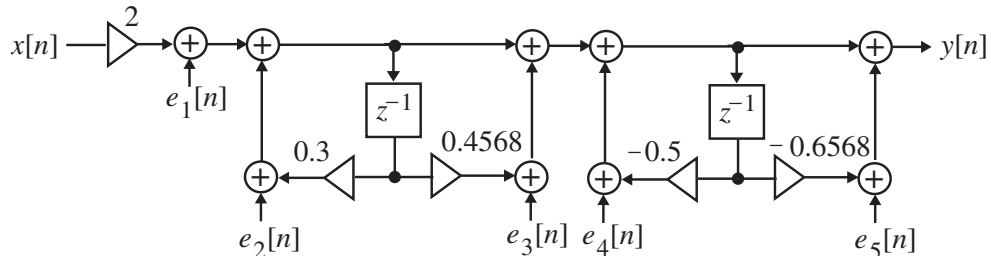
The noise transfer function from the noise source $e_5[n]$ to the filter output is $G_5(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{5,n}^2 = 1$.

Hence the total noise variance at the output is $\sigma_n^2 = 2\sigma_{1,n}^2 + 2\sigma_{3,n}^2 + \sigma_{5,n}^2 = 5.3392$

Cascade Form #2 - $H(z) = 2 \left(\frac{1 + 0.4568z^{-1}}{1 - 0.3z^{-1}} \right) \left(\frac{1 - 0.6568z^{-1}}{1 + 0.5z^{-1}} \right)$. The noise model of this

structure is shown below:



The noise transfer function from $e_1[n]$ and $e_2[n]$ to filter output is same as for the Cascade Structure #1. Hence, the corresponding normalized noise variance at the output is

$$\sigma_{1,n}^2 = 1.1671.$$

The noise transfer function from $e_3[n]$ and $e_4[n]$ to filter output is

$$G_3(z) = \frac{1 - 0.6568z^{-1}}{1 + 0.5z^{-1}} = \frac{z + 0.6568}{z + 0.5} = 1 + \frac{-1.1568}{z + 0.5}$$

The corresponding normalized noise

$$\text{variance at the output is } \sigma_{3,n}^2 = (1)^2 + \frac{(-1.1568)^2}{1 - (0.5)^2} = 2.7842.$$

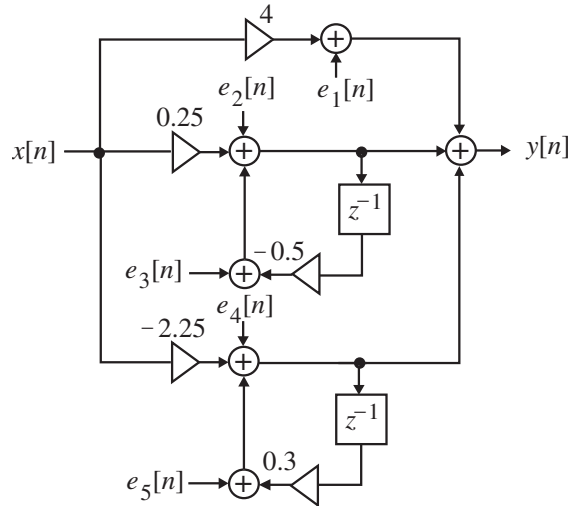
Output of Program 12_4.m is 2.7842.

The noise transfer function from the noise source $e_5[n]$ to the filter output is $G_5(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{5,n}^2 = 1$.

Hence the total noise variance at the output is $\sigma_n^2 = 2\sigma_{1,n}^2 + 2\sigma_{3,n}^2 + \sigma_{5,n}^2 = 6.1184$.

(c) Parallel Form I realization - $H(z) = 4 + \frac{0.25}{1 + 0.5z^{-1}} + \frac{-2.25}{1 - 0.3z^{-1}}$. The noise model of this structure is shown below:



The noise transfer function from $e_2[n]$ and $e_3[n]$ to filter output is

$$G_2(z) = \frac{1}{1 + 0.5z^{-1}} = 1 + \frac{-0.5}{z + 0.5}$$

The corresponding normalized noise variance at the output is

$$\sigma_{2,n}^2 = (1)^2 + \frac{(-0.5)^2}{1 - (0.5)^2} = \frac{4}{3}.$$

The noise transfer function from $e_4[n]$ and $e_5[n]$ to filter output is

$$G_4(z) = \frac{1}{1 - 0.3z^{-1}} = 1 + \frac{0.3}{z - 0.3}$$

The corresponding normalized noise variance at the output is

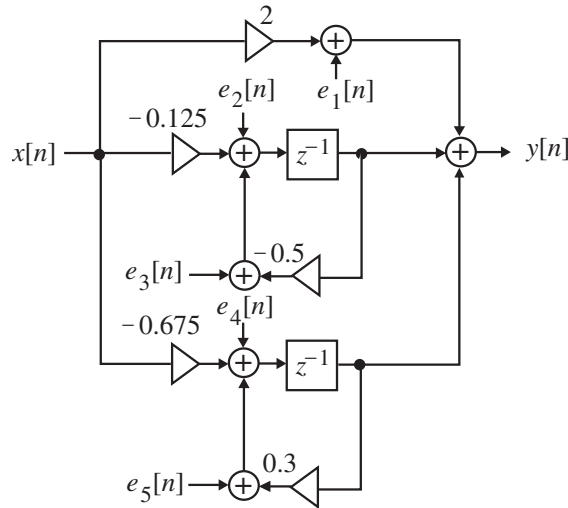
$$\sigma_{4,n}^2 = (1)^2 + \frac{(0.3)^2}{1 - (-0.3)^2} = 1.0989.$$

The noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{1,n}^2 = 1$.

Hence the total noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 5.8645$.

Parallel Form II realization. - $H(z) = 2 + \frac{-0.125}{z+0.5} + \frac{-0.675}{z-0.3}$. The noise model of this structure is shown below:



The noise transfer function from $e_2[n]$ and $e_3[n]$ to filter output is $G_2(z) = \frac{1}{z+0.5}$. The corresponding normalized noise variance at the output is $\sigma_{2,n}^2 = \frac{1}{1-(0.5)^2} = \frac{4}{3}$.

The noise transfer function from $e_4[n]$ and $e_5[n]$ to filter output is $G_4(z) = \frac{1}{z-0.3}$. The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = \frac{1}{1-(-0.3)^2} = 1.0989$.

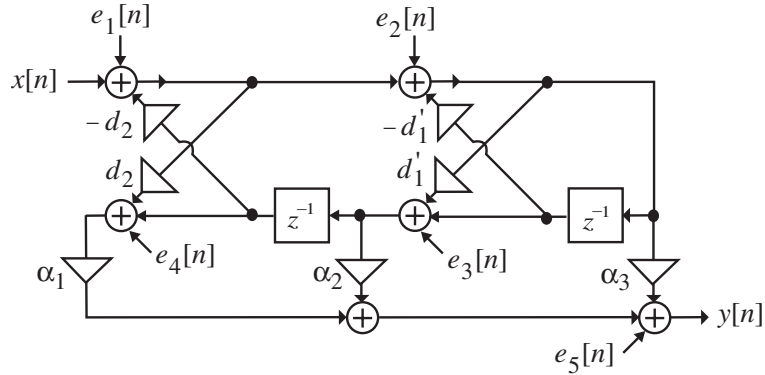
The noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{1,n}^2 = 1$.

Hence the total noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 5.8645$.

12.17 $H(z) = \frac{2 - 0.4z^{-1} - 0.6z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}}$. Using Program 8_5.m, the Gray-Markel realization of $H(z)$ is found to be as shown on top of the next page where $d_2 = 0.2353, d_1' = -0.15, \alpha_1 = -0.6, \alpha_2 = -0.28, \alpha_3 = 1.9759$.

(a) For quantization of products before addition, the noise model is shown on the next page:



Analyzing the above structure, we obtain the noise transfer function for each noise source,

$$e_i[n], 1 \leq i \leq 7: G_1(z) = H(z) = \frac{2 - 0.4z^{-1} - 0.6z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}},$$

$$G_2(z) = \frac{(\alpha_2 d_1' + \alpha_3) + [\alpha_1(1 - d_2^2)d_1' + \alpha_2]z^{-1} + \alpha_1(1 - d_2^2)z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} \\ = \frac{2.0179 - 0.1950z^{-1} - 0.5668z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}},$$

$$G_3(z) = \frac{\alpha_2 + [\alpha_1(1 - d_2^2) - \alpha_2 d_1' d_2 - \alpha_3 d_2]z^{-1} + \alpha_1(1 - d_2^2)d_1' d_2 z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} \\ = \frac{-0.28 - 1.0416z^{-1} + 0.02z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}},$$

$$G_4(z) = \alpha_1 = -0.6, G_5(z) = G_6(z) = G_7(z) = 1.$$

Using Program 12_4.m, we get $\sigma_{1,n}^2 = 4.6683$, $\sigma_{2,n}^2 = 4.4549$, $\sigma_{3,n}^2 = 1.1199$, $\sigma_{4,n}^2 = 0.36$.

Hence, the total normalized output noise variances = $4.6683 + 4.459 + 1.1199 + 0.36 + 3(1) = 13.6072$.

(b) The realization along with the noise model for the quantization of products after addition is the same as in Part **(a)**.

The noise transfer functions from the noise sources $e_2[n]$, $e_3[n]$, and $e_4[n]$, are $G_2(z)$, $G_3(z)$, and $G_4(z)$, respectively and are given in Part **(a)**. Their corresponding noise variances

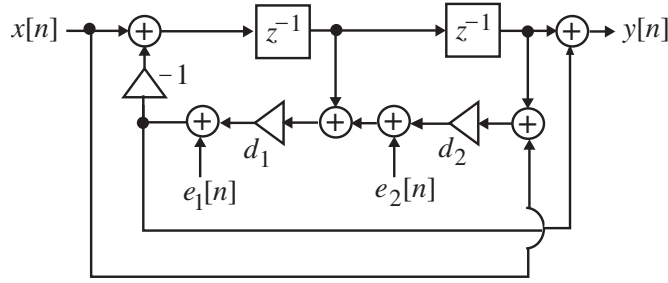
were also derived in Part **(a)** and are given by $\sigma_{2,n}^2 = 4.4549$, $\sigma_{3,n}^2 = 1.1199$, $\sigma_{4,n}^2 = 0.36$.

Hence, the total normalized output noise variances = $4.459 + 1.1199 + 0.365 + 1 = 6.9389$

12.18 The noise model is shown on the next page. The noise transfer function from $e_1[n]$ to the

$$\text{output is } G_1(z) = \frac{1 - z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}} = \frac{z^2 - 1}{z^2 + d_1 z + d_1 d_2} = 1 - \frac{d_1 z + (1 + d_1 d_2)}{z^2 + d_1 z + d_1 d_2}.$$

Its corresponding output noise variance is given by



$$\sigma_{1,n}^2 = 1 + \frac{[d_1^2 + (1 + d_1 d_2)^2](1 - d_1^2 d_2^2) - 2[d_1(1 + d_1 d_2)(1 - d_1 d_2)d_1]}{(1 - d_1^2 d_2^2)^2 + 2d_1^3 d_2 - (1 + d_1^2 d_2^2)d_1^2} = \frac{2}{1 - d_1 d_2}.$$

The noise transfer function from $e_2[n]$ to the output is

$$G_1(z) = \frac{d_1(1 - z^{-2})}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}} = d_1 - \frac{d_1^2 z + (1 + d_1 d_2)d_1}{z^2 + d_1 z + d_1 d_2}. \text{ Its corresponding}$$

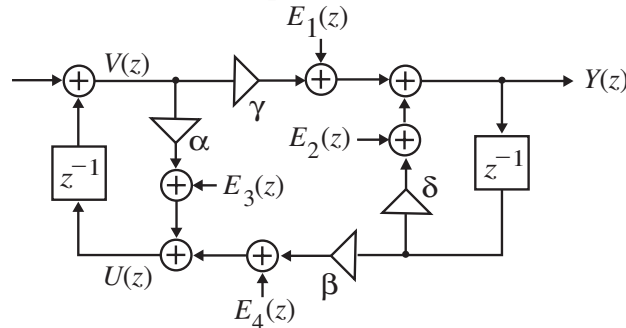
normalized output noise variance is given by

$$\sigma_{2,n}^2 = d_1^2 \sigma_{1,n}^2 = \frac{2d_1^2}{1 - d_1 d_2}.$$

Total output noise variance is therefore $\sigma_{o,n}^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2 = \frac{2(1 + d_1^2)}{1 - d_1 d_2}$.

12.19 (a) Quantization of products before addition

The noise model of the second-order coupled form structure is shown below;



We observe from discussion on Page 675 that $\alpha = \delta = r \cos \theta$, $\beta = -\gamma = r \sin \theta$ and hence, $\alpha + \delta = 2r \cos \theta$, $\alpha \delta - \beta \gamma = r^2$.

Analysis yields $U(z) = \beta z^{-1} Y(z) + \alpha V(z) + E_3(z)$, $V(z) = z^{-1} U(z)$, and

$Y(z) = \gamma V(z) + \delta z^{-1} Y(z) + E_1(z)$. Eliminating the variables $U(z)$ and $V(z)$ from these equations

we arrive at $\frac{1 - (\alpha + \delta)z^{-1} + (\alpha \delta - \beta \gamma)z^{-2}}{1 - \alpha z^{-1}} Y(z) = \frac{\gamma z^{-1}}{1 - \alpha z^{-1}} E_3(z) + E_1(z)$. Hence, the noise

transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the output is given by

$$G_1(z) = \frac{Y(z)}{E_1(z)} \Big|_{E_3(z)=E_4(z)=0} = \frac{1 - \alpha z^{-1}}{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}} = \frac{z^2 - \alpha z}{z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\gamma)}$$

$$= 1 + \frac{\delta z - (\alpha\delta - \beta\gamma)}{z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\gamma)} = 1 + \frac{r \cos(\theta)z - r^2}{z^2 - 2r \cos(\theta)z + r^2},$$

and the noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the output is given by

$$G_3(z) = \frac{Y(z)}{E_3(z)} \Big|_{E_1(z)=E_2(z)=0} = \frac{\gamma z^{-1}}{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}} = \frac{\gamma z}{z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\gamma)}$$

$$= \frac{-r \sin(\theta)z}{z^2 - 2r \cos(\theta)z + r^2}.$$

The output roundoff noise variance due each of the noise sources $e_1[n]$ and $e_2[n]$ is

$$\sigma_{1,n}^2 = 1 + \frac{(r^2 \cos \theta + r^4)(1 - r^4) - 2(-r^3 \cos \theta + r^3 \cos \theta \cdot r^2)(-2r \cos \theta)}{(1 - r^4)^2 + 2r^2(4r^2 \cos^2 \theta) - (1 + r^4)(4r^2 \cos^2 \theta)}$$

$$= 1 + \frac{r^2(1 + r^2)(r^2 + \cos^2 \theta) - 4r^4 \cos^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]},$$

and the output roundoff noise variance due each of the noise sources $e_3[n]$ and $e_4[n]$ is $\sigma_{3,n}^2 = \frac{r^2(1 + r^2) \sin^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}$. Hence, the total

$$\text{roundoff noise variance at the output is } \sigma_n^2 = 2(\sigma_{1,n}^2 + \sigma_{2,n}^2)$$

$$= 2 \left(1 + \frac{r^2(1 + r^2)(r^2 + \cos^2 \theta) - 4r^4 \cos^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]} + \frac{r^2(1 + r^2) \sin^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]} \right).$$

Quantization of products after addition

Here, the total roundoff noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2$

$$= 1 + \frac{r^2(1 + r^2)(r^2 + \cos^2 \theta) - 4r^4 \cos^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]} + \frac{r^2(1 + r^2) \sin^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}.$$

(b) From the solution of Problem 12.4 we observe that the transfer function of the modified

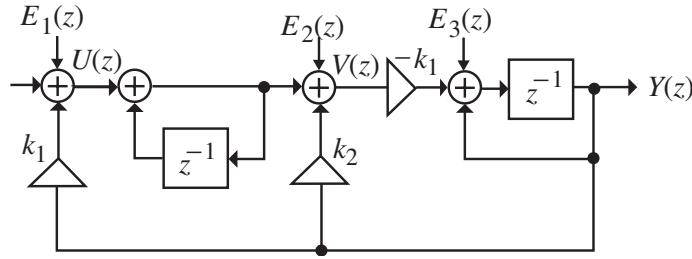
coupled-form structure is $H(z) = \frac{c^2 d}{z^2 - 2cz + c^2(1 + d^2)}$, where $c = r \sin \theta$ and $d = \cot \theta$.

The noise model is shown on top of the next page. Analysis yields

$$Y(z) = \frac{E_4(z) + cz^{-1}E_2(z) + cz^{-1}E_3(z) - cdz^{-1}E_4(z) + c^2z^{-2}E_1(z) - c^2dz^{-2}E_3(z)}{1 - 2cdz^{-1} + c^2(1 + d^2)z^{-2}}.$$

The noise transfer function from $e_1[n]$ to the output is

12.20 The noise model of the Kingsbury structure is shown below:



Analysis yields $U(z) = k_1 Y(z) + E_1(z)$, $V(z) = \frac{1}{1-z^{-1}} U(z) + k_2 Y(z) + E_2(z)$, and

$$Y(z) = [-k_1 V(z) + E_3(z)] \frac{z^{-1}}{1-z^{-1}} = -\frac{k_1 z^{-1}}{1-z^{-1}} V(z) + \frac{z^{-1}}{1-z^{-1}} E_3(z).$$

Eliminating $U(z)$ and $V(z)$ from these equations we arrive at

$$\frac{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}}{(1 - z^{-1})^2} Y(z) = -\frac{k_1 z^{-1}}{(1 - z^{-1})^2} E_1(z) - \frac{k_1 z^{-1}}{1 - z^{-1}} E_2(z) + \frac{z^{-1}}{1 - z^{-1}} E_3(z).$$

Hence, the noise transfer function from $e_1[n]$ to the output is

$$G_1(z) = \left. \frac{Y(z)}{E_1(z)} \right|_{E_2(z)=E_3(z)=0} = \frac{-k_1 z^{-1}}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}} = \frac{-k_1 z}{z^2 - (2r \cos \theta)z + r^2}.$$

The corresponding normalized output roundoff noise variance is thus given by

$$\begin{aligned} \sigma_{1,n}^2 &= \frac{k_1^2 (1 - r^4)}{(1 - r^4)^2 + 2r^2 (-2r \cos \theta)^2 - (1 + r^4)(-2r \cos \theta)^2} = \frac{k_1^2 (1 - r^2)(1 + r^2)}{(1 - r^2)^2 (1 + r^2)^2 - 4r^2 \cos^2 \theta (1 - r^2)^2} \\ &= \frac{k_1^2 (1 + r^2)}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}. \end{aligned}$$

The noise transfer function from $e_2[n]$ to the output is

$$G_2(z) = \left. \frac{Y(z)}{E_2(z)} \right|_{E_1(z)=E_3(z)=0} = \frac{-k_1 z^{-1} (1 - z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}} = \frac{-k_1 z + k_1}{z^2 - (2r \cos \theta)z + r^2}.$$

Its corresponding normalized output roundoff noise variance is thus given by

$$\sigma_{2,n}^2 = \frac{2k_1^2 (1 + r^2)}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}.$$

Finally, the noise transfer function from $e_3[n]$ to the output is

$$G_3(z) = \left. \frac{Y(z)}{E_3(z)} \right|_{E_1(z)=E_2(z)=0} = \frac{z^{-1} (1 - z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}} = \frac{z - 1}{z^2 - (2r \cos \theta)z + r^2}.$$

Its corresponding normalized output roundoff noise variance is thus given by

$$\sigma_{3,n}^2 = \frac{2(1 + r^2)}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}.$$

Hence, the total normalized noise variance due to

product roundoff before (after) summation is given by

$$\sigma_{o,n}^2 = \sigma_{1,n}^2 + \sigma_{2,n}^2 + \sigma_{3,n}^2 = \frac{(3k_1^2 + 2)(1 + r^2)}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}.$$

12.21 Let $e[n]$ be a noise source due to product roundoff inside $G(z)$ with an associated noise transfer function $G_e(z)$ to the output of $G(z)$. Then the normalized noise power at the output of the cascade structure due to $e[n]$ is given by

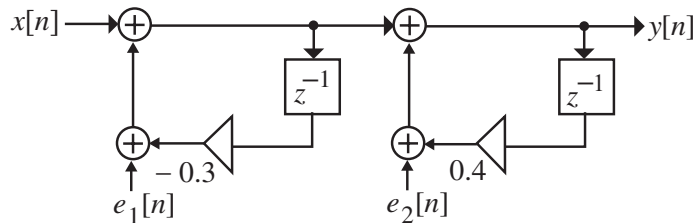
$$\sigma_{e,n}^2 = \frac{1}{2\pi j} \oint_C G_e(z) A_2(z) G_e(z^{-1}) A_2(z^{-1}) z^{-1} dz = \frac{1}{2\pi j} \oint_C G_e(z) G_e(z^{-1}) z^{-1} dz \text{ since}$$

$A_2(z) A_2(z^{-1}) = 1$. Assuming $G(z)$ is realized with the lowest product roundoff noise power $\sigma_{g,n}^2$, the normalized noise power at the output of the cascade due to product roundoff in $G(z)$ is still $\sigma_{g,n}^2$. From the solution of Problem 12.14, we note that the lowest roundoff noise power (assuming quantization before addition) is $\sigma_{g,n}^2 = 4.4031$ for cascade structure #3. Next, from the solution of Problem 12.18, we note that the normalized output noise power due to noise

sources in $A_2(z)$ is $\sigma_{2,n}^2 = \frac{2d_1^2}{1 - d_1 d_2}$. Substituting $d_1 = 0.2$ and $d_1 d_2 = -0.5$, we get

$$\sigma_{2,n}^2 = 0.0533. \text{ Hence, the total output noise variance of the cascade is given by } \sigma_{g,n}^2 + \sigma_{2,n}^2 = 4.4031 + 0.0533 = 4.4564.$$

12.21 The noise model for the structure is as shown below:



The noise transfer function from $e_1[n]$ to the output is $G_1(z) = \frac{1}{(1 + 0.3z^{-1})(1 - 0.4z^{-1})}$
 $= 1 + \frac{0.2286}{z - 0.4} + \frac{-0.1286}{z + 0.3}$. Thus, the normalized noise power at the output due to $e_1[n]$ is

$$\sigma_1^2 = 1 + \frac{(0.2286)^2}{1 - (0.4)^2} + \frac{(-0.1286)^2}{1 - (-0.3)^2} + \frac{2 \times 0.2286 \times (-0.1286)}{1 - (-0.3)(0.4)} = 1.0279.$$

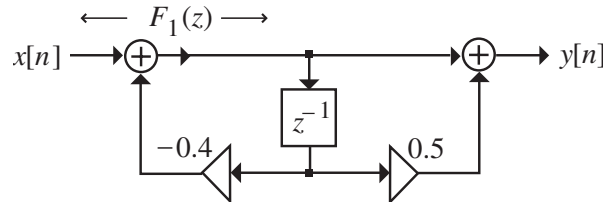
The noise transfer function from $e_2[n]$ to the output is $G_2(z) = \frac{1}{1 - 0.4z^{-1}} = 1 + \frac{0.4}{z - 0.4}$.

Thus, the normalized noise power at the output due to $e_2[n]$ is $\sigma_2^2 = 1 + \frac{(0.4)^2}{1 - (0.4)^2} = 1.1905$.

Therefore, the total normalized noise power at the output = $1.0279 + 1.1905 = 2.2184$.

For a 9-bit signed two's-complement number representation, the quantization level is $\delta = 2^{-8} = 0.0039062$. Hence, the total output noise power due to product roundoff is $\sigma_o^2 = (2.2184) \frac{\delta^2}{12} = 2.8208 \times 10^{-6}$.

12.22 The unscaled structure is shown below:



Now, $F_1(z) = \frac{1}{1+0.4z^{-1}} = 1 + \frac{-0.4}{z+0.4}$. Using Table 12.4 we obtain $\|F_1\|_2^2 = 1 + \frac{(-0.4)^2}{1-(-0.4)^2} =$

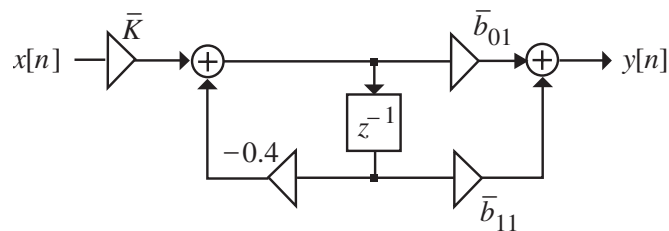
1.1905. Next, $H(z) = \frac{1+0.5z^{-1}}{1-0.4z^{-1}} = 1 + \frac{0.9}{z-0.4}$. Using Table 12.4 we obtain

$\|H\|_2^2 = 1 + \frac{(0.9)^2}{1-(-0.4)^2} = 1.9643$. From Eq. (12.128a), $\|F_1\|_2 = \alpha_1 = 1.0911$, and from Eq.

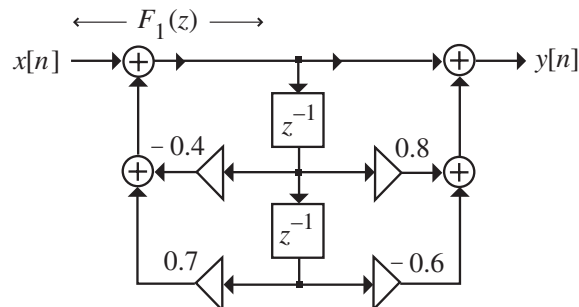
(12.128b), $\|H\|_2 = \alpha_2 = 1.0415$.

The scaled structure is shown below, where $\bar{b}_{01} = \beta_1$ and $\bar{b}_{11} = 0.5\beta_1$. From Eq. (12.129) and (12.132a), $\bar{K} = \beta_0 K = \beta_0 = \frac{1}{\alpha_1} = 0.9165$ and from Eq. (12.132b),

$\beta_1 = \frac{\alpha_1}{\alpha_2} = \frac{1.0911}{1.0415} = 0.7785$. Therefore, $\bar{b}_{01} = \beta_1 = 0.7785$ and $\bar{b}_{11} = 0.5\beta_1 = 0.3893$.



12.24 The unscaled structure is shown below:



Now, $F_1(z) = \frac{1}{1+0.4z^{-1}-0.12z^{-2}} = \frac{z^2}{z^2+0.4z-0.12} = 1 + \frac{-0.45}{z+0.6} + \frac{0.05}{z-0.2}$. Using Table

12.4 we get $\|F_1\|_2^2 = 1 + \frac{(-0.45)^2}{1-(-0.6)^2} + \frac{(0.05)^2}{1-(0.2)^2} + \frac{2 \times (-0.45) \times 0.05}{1-(-0.6) \times 0.2} = 1.2788$. Next,

$H(z) = \frac{1+0.8z^{-1}-0.6z^{-2}}{1+0.4z^{-1}-0.12z^{-2}} = 1 + \frac{0.9}{z+0.6} + \frac{-0.5}{z-0.2}$. Using Table 12.4 we get

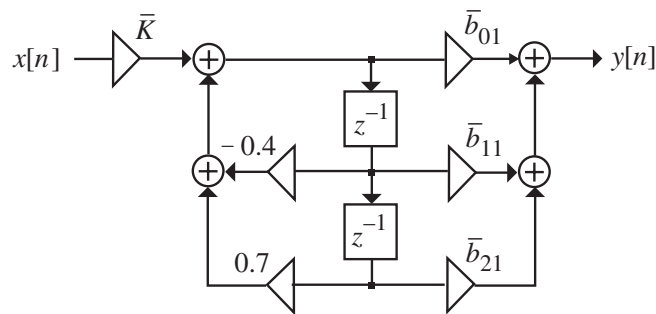
$\|H\|_2^2 = 1 + \frac{(0.9)^2}{1-(-0.6)^2} + \frac{(-0.5)^2}{1-(0.2)^2} + \frac{2 \times 0.9 \times (-0.5)}{1-(-0.6) \times 0.2} = 1.7225$.

From Eq. (12.128a) we get $\|F_1\|_2 = \alpha_1 = 1.1308$ and from Eq. (12.28b) we get

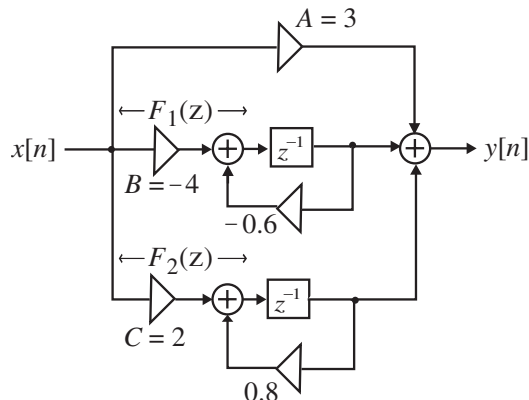
$\|H\|_2 = \alpha_2 = 1.3124$. Hence, from Eq. (12.132a) we have $\beta_0 = \frac{1}{\alpha_1} = 0.8843$ and from Eq.

(12.132b) we have $\beta_1 = \frac{\alpha_1}{\alpha_2} = 0.8616$. The scaled structure is as shown below, where

$\bar{K} = \beta_0 K = \beta_0 = 0.8843$, $\bar{b}_{01} = 1 \times \beta_1 = 0.8616$, $\bar{b}_{11} = 0.8 \times \beta_1 = 0.6893$, and $\bar{b}_{21} = (-0.6) \times \beta_1 = -0.5170$.



12.25 $H(z) = 3 + \frac{-4z^{-1}}{1+0.6z^{-1}} + \frac{2z^{-1}}{1-0.8z^{-1}}$. The unscaled structure is shown below:



Here, $F_1(z) = \frac{-4z^{-1}}{1+0.6z^{-1}} = 1 + \frac{-4.6}{z+0.6}$. Using Table 12.4 we get $\|F_1\|_2^2 = 1 + \frac{(-4.6)^2}{1-(-0.6)^2}$

= 34.0625. Likewise, $F_2(z) = \frac{2z^{-1}}{1-0.8z^{-1}} = 1 + \frac{2.8}{z-0.8}$. Using Table 12.4 we get

$\|F_2\|_2^2 = 1 + \frac{(2.8)^2}{1-(0.8)^2} = 22.7778$. Also, $H(z) = 3 + \frac{-4}{z+0.6} + \frac{2}{z-0.8}$. Using Table 12.4 we get

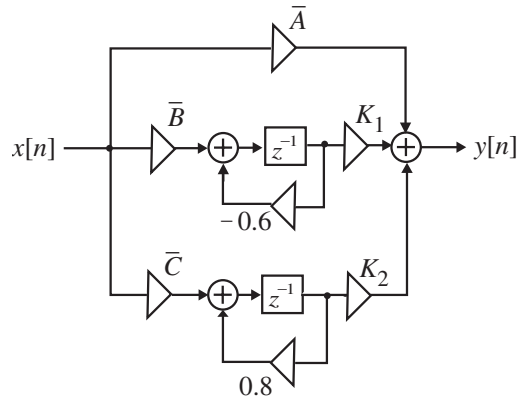
$\|H\|_2^2 = 3^2 + \frac{(-4)^2}{1-(-0.6)^2} + \frac{2^2}{1-(0.8)^2} + \frac{2 \times (-4) \times 2}{1-(-0.6) \times 0.8} = 34.3003$.

Denote $\gamma_0 = \|H\|_2 = 5.8566$, $\gamma_1 = \|F_1\|_2 = 5.8363$, $\gamma_2 = \|F_2\|_2 = 4.7726$.

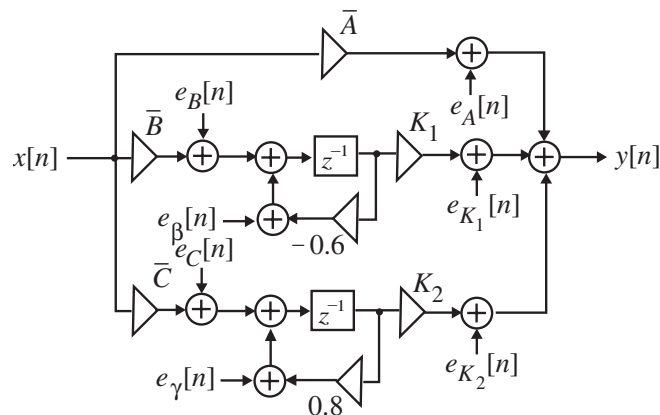
Hence, $\bar{A} = \frac{A}{\gamma_1} = \frac{3}{5.8363} = 0.5140$, $\bar{B} = \frac{B}{\gamma_2} = \frac{-4}{4.7726} = -0.8381$, $\bar{C} = \frac{C}{\gamma_0} = \frac{2}{5.8566} = 0.3415$,

$K_1 = \frac{\gamma_1}{\gamma_0} = 0.9965$, $K_2 = \frac{\gamma_2}{\gamma_0} = 0.8149$.

The scaled structure is shown below:



The noise model is



The noise transfer function from the noise sources $e_B[n]$ and $e_\beta[n]$ is given by:

$G_1(z) = \frac{0.9965z}{z+0.6} = 0.9965 + \frac{-0.5979}{z+0.6}$. Using Table 12.4, we obtain

$$\sigma_1^2 = (0.9965)^2 + \frac{(-0.5979)^2}{1 - (-0.6)^2} = 1.5516.$$

Likewise, the noise transfer function from the noise sources $e_C[n]$ and $e_\gamma[n]$ is given by:

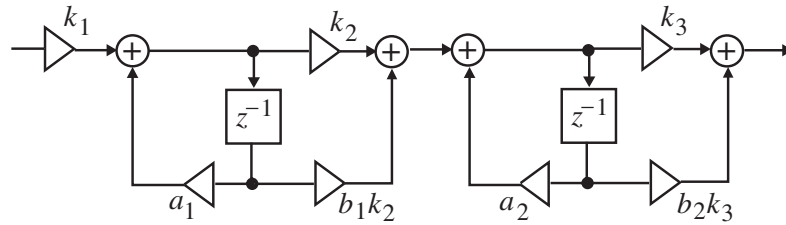
$$G_2(z) = \frac{0.8149z}{z-0.8} = 0.8149 + \frac{1.6519}{z-0.8}. \text{ Using Table 12.4, we obtain}$$

$$\sigma_2^2 = (0.8149)^2 + \frac{(1.6519)^2}{1 - (0.8)^2} = 1.8446.$$

The noise transfer function from the remaining three noise sources is $G_3(z) = 1$. Hence, the normalized output noise variance due to these three noise sources is given by $\sigma_3^2 = 3$.

Therefore, the total normalized output noise variance is $\sigma_0^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 9.7924$.

12.26 The scaled structure is shown below.



(a) Cascade Structure # 1 - $H(z) = \left(\frac{1+0.4z^{-1}}{1+0.2z^{-1}} \right) \left(\frac{1-0.3z^{-1}}{1-0.6z^{-1}} \right)$. Here, $a_1 = -0.2$, $b_1 = 0.4$,

$a_2 = 0.6$, $b_2 = -0.3$. The values of the scaling constants are determined below.

$$F_1(z) = \frac{1}{z+0.2}. \text{ Using Table 12.4 we get } \|F_1\|_2^2 = \frac{1}{1 - (-0.2)^2} = 1.0417. \text{ Hence,}$$

$$\gamma_1 = \|F_1\|_2 = 1.0206. \text{ Likewise,}$$

$$F_2(z) = \left(\frac{1+0.4z^{-1}}{1+0.2z^{-1}} \right) \left(\frac{z^{-1}}{1-0.6z^{-1}} \right) = \frac{z+0.4}{(z+0.2)(z-0.6)} = \frac{1.25}{z-0.6} + \frac{-0.25}{z+0.2}. \text{ Using Table 12.4}$$

$$\text{we get } \|F_2\|_2^2 = \frac{(1.25)^2}{1 - (0.6)^2} + \frac{(-0.25)^2}{1 - (-0.2)^2} + \frac{2 \times 1.25 \times (-0.2)}{1 - 0.6 \times (-0.2)} = 1.9485. \text{ Hence,}$$

$$\gamma_2 = \|F_2\|_2 = 1.3959. \text{ Next, } H(z) = \frac{z^2 + 0.1z - 0.12}{z^2 - 0.4z - 0.12} = 1 + \frac{0.375}{z-0.6} + \frac{0.125}{z+0.2}. \text{ Using Table 12.4}$$

$$\text{we get } \|H\|_2^2 = 1 + \frac{(0.375)^2}{1 - (0.6)^2} + \frac{(0.125)^2}{1 - (-0.2)^2} + \frac{2 \times 0.375 \times 0.125}{1 - 0.6 \times (-0.2)} = 1.3197. \text{ Hence,}$$

$$\gamma_0 = \|H\|_2 = 1.1488.$$

The scaling multipliers are therefore given by $k_1 = \frac{1}{\gamma_1} = 0.9798$, $k_2 = \frac{\gamma_1}{\gamma_2} = 0.7311$, and

$$k_3 = \frac{\gamma_2}{\gamma_0} = 1.2151. \text{ Therefore, } b_1k_2 = 0.2925 \text{ and } b_2k_3 = -0.3645.$$

The noise at the output due to the input multiplier k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.0417$. The noise at the output due to a_1, k_2 , and b_1k_2 have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$$G_2(z) = \frac{1.2151 - 0.3645z^{-1}}{1 - 0.6z^{-1}} = \frac{1.2151z - 0.3645}{z - 0.6} = 1.2151 + \frac{0.3646}{z - 0.6}. \text{ Using Table 12.4 we get}$$

$$\sigma_2^2 = (1.2151)^2 + \frac{(0.3646)^2}{1 - (0.6)^2} = 1.6841.$$

Hence the total noise power (variance) at the output = $2 \times 1.0417 + 3 \times 1.6841 + 2 = 9.1357$. In case the quantization is carried out after addition, then the total noise power at the output = $1.0417 + 1.6841 + 1 = 3.7258$.

(b) Cascade Structure # 2 - $H(z) = \left(\frac{1 - 0.3z^{-1}}{1 + 0.2z^{-1}} \right) \left(\frac{1 + 0.4z^{-1}}{1 - 0.6z^{-1}} \right)$. Here, $a_1 = -0.2$, $b_1 = -0.3$,

$a_2 = 0.6$, $b_2 = 0.4$. The values of the scaling constants are determined below.

$F_1(z) = \frac{1}{z + 0.2}$. From Part **(a)** we have $\|F_1\|_2^2 = 1.0417$ and $\gamma_1 = \|F_1\|_2 = 1.0206$. Likewise,

$$F_2(z) = \left(\frac{1 - 0.3z^{-1}}{1 + 0.2z^{-1}} \right) \left(\frac{z^{-1}}{1 - 0.6z^{-1}} \right) = \frac{z - 0.3}{(z + 0.2)(z - 0.6)} = \frac{0.375}{z - 0.6} + \frac{0.625}{z + 0.2}. \text{ Using Table 12.4}$$

we get $\|F_2\|_2^2 = \frac{(0.375)^2}{1 - (0.6)^2} + \frac{(0.625)^2}{1 - (-0.2)^2} + \frac{2 \times 0.375 \times 0.625}{1 - 0.6 \times (-0.2)} = 1.0452$. Hence,

$\gamma_2 = \|F_2\|_2 = 1.0224$. Next, $H(z) = \frac{z^2 + 0.1z - 0.12}{z^2 - 0.4z - 0.12}$ and from Part **(a)** we have

$$\|H\|_2^2 = 1.3197. \text{ Hence, } \gamma_0 = \|H\|_2 = 1.1488.$$

The scaling multipliers are therefore given by $k_1 = \frac{1}{\gamma_1} = 0.9798$, $k_2 = \frac{\gamma_1}{\gamma_2} = 0.9982$, and

$$k_3 = \frac{\gamma_2}{\gamma_0} = 0.89. \text{ Therefore, } b_1k_2 = -0.2995 \text{ and } b_2k_3 = 0.356.$$

The noise at the output due to the input multiplier k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.0417$. The noise at the output due to a_1, k_2 , and b_1k_2 have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$$G_2(z) = \frac{0.89 + 0.356z^{-1}}{1 - 0.6z^{-1}} = \frac{0.89z + 0.356}{z - 0.6} = 0.89 + \frac{0.89}{z - 0.6}. \text{ Using Table 12.4 we get}$$

$$\sigma_2^2 = (0.89)^2 + \frac{(0.89)^2}{1 - (0.6)^2} = 2.0298.$$

Hence the total noise power (variance) at the output = $2 \times 1.0417 + 3 \times 2.0298 + 2 = 10.1728$. In case the quantization is carried out after addition, then the total noise power at the output = $1.0417 + 2.0298 + 1 = 4.0715$.

(c) Cascade Structure # 3 - $H(z) = \left(\frac{1 - 0.3z^{-1}}{1 - 0.6z^{-1}} \right) \left(\frac{1 + 0.4z^{-1}}{1 + 0.2z^{-1}} \right)$. Here, $a_1 = 0.6, b_1 = -0.3,$

$a_2 = -0.2, b_2 = 0.4$. The values of the scaling constants are determined below.

$$F_1(z) = \frac{1}{z - 0.6}. \text{ Using Table 12.4 we get } \|F_1\|_2^2 = \frac{1}{1 - (0.6)^2} = 1.5625. \text{ Hence,}$$

$$\gamma_1 = \|F_1\|_2 = 1.25. \text{ Likewise, } F_2(z) = \left(\frac{1 - 0.3z^{-1}}{1 - 0.6z^{-1}} \right) \left(\frac{z^{-1}}{1 + 0.2z^{-1}} \right) = \frac{z - 0.3}{(z - 0.6)(z + 0.2)}. \text{ From}$$

Part **(b)** we have $\|F_2\|_2^2 = 1.0452$ and $\gamma_2 = \|F_2\|_2 = 1.0224$. Next, $H(z) = \frac{z^2 + 0.1z - 0.12}{z^2 - 0.4z - 0.12}$

and from Part **(a)** we have $\|H\|_2^2 = 1.3197$. Hence, $\gamma_0 = \|H\|_2 = 1.1488$.

The scaling multipliers are therefore given by $k_1 = \frac{1}{\gamma_1} = 0.8, k_2 = \frac{\gamma_1}{\gamma_2} = 1.2226$, and

$$k_3 = \frac{\gamma_2}{\gamma_0} = 0.89. \text{ Therefore, } b_1k_2 = -0.3668 \text{ and } b_2k_3 = 0.356.$$

The noise at the output due to the input multiplier k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.5625$. The noise at the output due to a_1, k_2 , and b_1k_2 have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$$G_2(z) = \frac{0.89 + 0.356z^{-1}}{1 + 0.2z^{-1}} = \frac{0.89z + 0.356}{z + 0.2} = 0.89 + \frac{0.178}{z + 0.2}. \text{ Using Table 12.4 we get}$$

$$\sigma_2^2 = (0.89)^2 + \frac{(0.178)^2}{1 + (-0.2)^2} = 0.8251.$$

Hence the total noise power (variance) at the output = $2 \times 1.5625 + 3 \times 0.8251 + 2 = 7.6003$. In case the quantization is carried out after addition, then the total noise power at the output = $1.5625 + 0.8251 + 1 = 3.3876$.

(d) Cascade Structure # 4 - $H(z) = \left(\frac{1 + 0.4z^{-1}}{1 - 0.6z^{-1}} \right) \left(\frac{1 - 0.3z^{-1}}{1 + 0.2z^{-1}} \right)$. Here, $a_1 = 0.6, b_1 = 0.4,$

$a_2 = -0.2, b_2 = -0.3$. The values of the scaling constants are determined below.

$F_1(z) = \frac{1}{z-0.6}$. From Part (c) we have $\|F_1\|_2^2 = 1.5625$ and $\gamma_1 = \|F_1\|_2 = 1.25$. Likewise,

$F_2(z) = \left(\frac{1+0.4z^{-1}}{1-0.6z^{-1}} \right) \left(\frac{z^{-1}}{1+0.2z^{-1}} \right) = \frac{z+0.4}{(z-0.6)(z+0.2)}$. From Part (a) we have $\|F_2\|_2^2 = 1.9485$

and $\gamma_2 = \|F_2\|_2 = 1.3959$. Next, $H(z) = \frac{z^2+0.1z-0.12}{z^2-0.4z-0.12}$ and from Part (a) we have

$\|H\|_2^2 = 1.3197$ and $\gamma_0 = \|H\|_2 = 1.1488$.

The scaling multipliers are therefore given by $k_1 = \frac{1}{\gamma_1} = 0.8$, $k_2 = \frac{\gamma_1}{\gamma_2} = 0.8955$, and

$k_3 = \frac{\gamma_2}{\gamma_0} = 1.2151$. Therefore, $b_1k_2 = 0.3582$ and $b_2k_3 = -0.3645$.

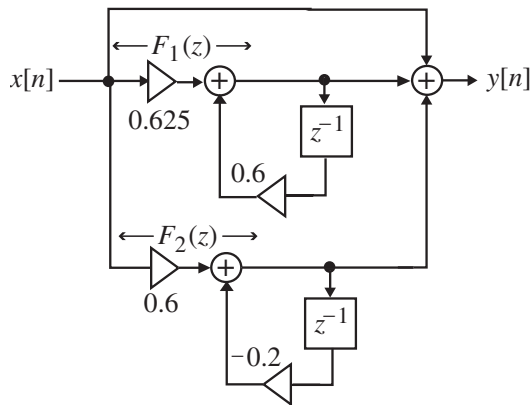
The noise at the output due to the input multiplier k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.5625$. The noise at the output due to a_1, k_2 , and b_1k_2 have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$G_2(z) = \frac{1.2151 - 0.3645z^{-1}}{1 + 0.2z^{-1}} = \frac{1.2151z - 0.3645}{z + 0.2} = 1.2151 + \frac{-0.6075}{z + 0.2}$. Using Table 12.4 we

get $\sigma_2^2 = (0.89)^2 + \frac{(-0.6075)^2}{1 + (-0.2)^2} = 1.8609$.

Hence the total noise power (variance) at the output = $2 \times 1.5625 + 3 \times 1.8609 + 2 = 10.7077$. In case the quantization is carried out after addition, then the total noise power at the output = $1.5625 + 1.8609 + 1 = 5.4234$.

12.27 (a) Parallel Form #1 - $H(z) = \frac{(z+0.4)(z-0.3)}{(z+0.2)(z-0.6)} = 1 + \frac{0.625}{1-0.6z^{-1}} + \frac{0.6}{1+0.2z^{-1}}$. The unscaled structure is shown below:



Here the scaling transfer function $F_1(z)$ for the middle branch is given by $F_1(z) = \frac{0.625z}{z-0.6}$

$= 0.625 + \frac{0.375}{z-0.6}$. From Table 12.4 we get $\|F_1\|_2^2 = (0.625)^2 + \frac{(0.375)^2}{1-(0.6)^2} = 0.6104$. Hence,

$\gamma_1 = \|F_1\|_2 = 0.7813$. Similarly the scaling transfer function $F_2(z)$ for the bottom branch is

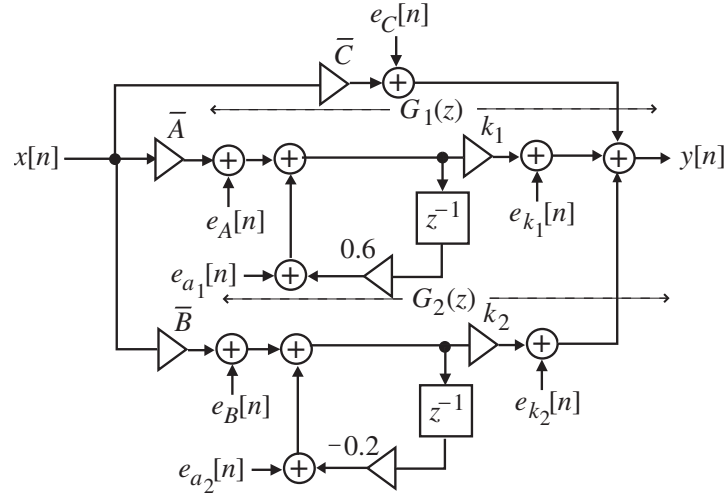
given by $F_2(z) = \frac{0.6z}{z+0.2} = 0.6 + \frac{-0.12}{z+0.2}$. From Table 12.4 we get

$\|F_2\|_2^2 = (0.6)^2 + \frac{(-0.12)^2}{1-(-0.2)^2} = 0.375$. Hence, $\gamma_2 = \|F_2\|_2 = 0.6124$.

Finally, $H(z) = 1 + \frac{0.375}{z-0.6} + \frac{0.125}{z+0.2}$. From Table 12.4 we get

$\|H\|_2^2 = 1 + \frac{(0.375)^2}{1-(0.6)^2} + \frac{(0.125)^2}{1-(-0.2)^2} + \frac{2 \times 0.375 \times 0.125}{1-0.6 \times (-0.2)} = 1.3197$. Hence, $\gamma_0 = \|H\|_2 = 1.1488$.

The scaled structure with noise sources is shown below where



$$\bar{A} = \frac{0.625}{\gamma_1} = 0.7999, \bar{B} = \frac{0.6}{\gamma_2} = 0.9798, \bar{C} = \frac{1}{\gamma_0}, k_1 = \frac{\gamma_1}{\gamma_0} = 0.6801, k_2 = \frac{\gamma_2}{\gamma_0} = 0.5331.$$

The noise transfer function from each of the noise sources $e_C[n]$, $e_{k_1}[n]$, and $e_{k_2}[n]$ is $G_3(z) = 1$. The corresponding output noise variance is $\sigma_3^2 = 1$.

The noise transfer function from each of the noise sources $e_A[n]$ and $e_{a_1}[n]$ is

$$G_1(z) = \frac{k_1}{1-0.6z^{-1}} = \frac{0.6801z}{z-0.6} = 0.6801 + \frac{0.4081}{z-0.6}$$

$$\sigma_1^2 = (0.6801)^2 + \frac{(0.4081)^2}{1-(0.6)^2} = 0.7227.$$

Likewise, the noise transfer function from each of the noise sources $e_B[n]$ and $e_{a_2}[n]$ is

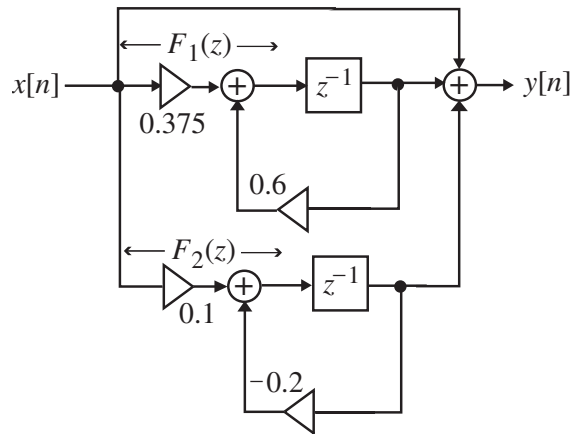
$$G_2(z) = \frac{k_2}{1+0.2z^{-1}} = \frac{0.5331z}{z+0.2} = 0.53301 + \frac{-0.1066}{z+0.2}. \text{ From Table 12.4 we get}$$

$$\sigma_2^2 = (0.5331)^2 + \frac{(-0.1066)^2}{1-(-0.2)^2} = 0.296.$$

Hence the total noise power (variance) at the output, $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + 3\sigma_3^2 = 5.0374$. In case the quantization is carried out after addition, then the total noise power at the output, $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 2.0187$.

(b) Parallel Form #2 - $H(z) = 1 + \frac{0.375}{z-0.6} + \frac{0.125}{z+0.2} = 1 + \frac{0.375z^{-1}}{1-0.6z^{-1}} + \frac{0.125z^{-1}}{1+0.2z^{-1}}$.

The unscaled structure is shown below:



Here the scaling transfer function $F_1(z)$ for the middle branch is given by $F_1(z) = \frac{0.375}{z-0.6}$.

From Table 12.4 we get $\|F_1\|_2^2 = \frac{(0.375)^2}{1-(0.6)^2} = 0.2197$. Hence, $\gamma_1 = \|F_1\|_2 = 0.4687$. Similarly

the scaling transfer function $F_2(z)$ for the bottom branch is given by $F_2(z) = \frac{0.125}{z+0.2}$. From

Table 12.4 we get $\|F_2\|_2^2 = \frac{(0.125)^2}{1-(-0.2)^2} = 0.0163$. Hence, $\gamma_2 = \|F_2\|_2 = 0.1277$.

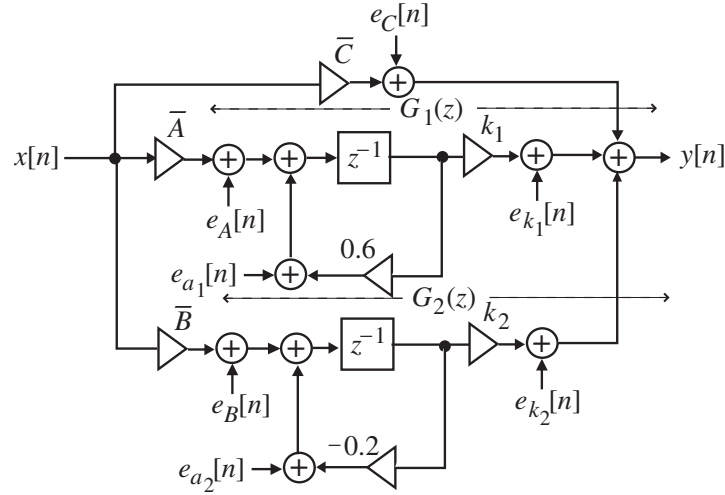
Finally, $H(z) = 1 + \frac{0.375}{z-0.6} + \frac{0.125}{z+0.2}$. From Part (a) we get $\|H\|_2^2 = 1.3197$ and

$$\gamma_0 = \|H\|_2 = 1.1488.$$

The scaled structure with noise sources is shown on the next page where

$$\bar{A} = \frac{0.375}{\gamma_1} = 0.7999, \bar{B} = \frac{0.125}{\gamma_2} = 0.9796, \bar{C} = \frac{1}{\gamma_0} = 0.8705, k_1 = \frac{\gamma_1}{\gamma_0} = 0.4081, k_2 = \frac{\gamma_2}{\gamma_0} = 0.1111.$$

The noise transfer function from each of the noise sources $e_C[n]$, $e_{k_1}[n]$, and $e_{k_2}[n]$ is $G_3(z) = 1$. The corresponding output noise variance is $\sigma_3^2 = 1$.



The noise transfer function from each of the noise sources $e_A[n]$ and $e_{a_1}[n]$ is

$$G_1(z) = \frac{k_1 z^{-1}}{1 - 0.6z^{-1}} = \frac{0.4081}{z - 0.6}. \text{ From Table 12.4 we get } \sigma_1^2 = \frac{(0.4081)^2}{1 - (0.6)^2} = 0.2602.$$

Likewise, the noise transfer function from each of the noise sources $e_B[n]$ and $e_{a_2}[n]$ is

$$G_2(z) = \frac{k_2 z^{-1}}{1 + 0.2z^{-1}} = \frac{0.1111}{z + 0.2}. \text{ From Table 12.4 we get } \sigma_2^2 = \frac{(0.1111)^2}{1 - (-0.2)^2} = 0.0129.$$

Hence the total noise power (variance) at the output, $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + 3\sigma_3^2 = 3.5462$. In case the quantization is carried out after addition, then the total noise power at the output, $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1.2731$.

In either case, Parallel form II structure (after scaling) has the lowest roundoff noise variance.

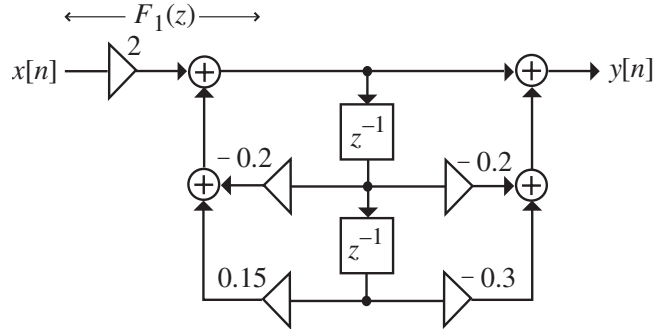
12.28
$$H(z) = \frac{2 - 0.4z^{-1} - 0.6z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} = 2 \left(\frac{z^2 - 0.2z - 0.3}{z^2 + 0.2z - 0.15} \right).$$

(a) Direct Form: The unscaled structure is shown on top of next page.

$$F_1(z) = \frac{2}{1 + 0.2z^{-1} - 0.15z^{-2}} = \frac{2z^2}{z^2 + 0.2z - 0.15} = 2 + \frac{-0.625}{z + 0.5} + \frac{0.225}{z - 0.3}. \text{ From Table 12.4 we}$$

$$\text{get } \|F_1\|_2^2 = 2^2 + \frac{(-0.625)^2}{1 - (-0.5)^2} + \frac{(0.225)^2}{1 - (0.3)^2} + \frac{2 \times (-0.625) \times 0.225}{1 - (-0.5) \times 0.3} = 4.3319. \text{ Hence,}$$

$$\beta_0 = \frac{1}{\|F_1\|_2} = 0.4805. \text{ Next, } H(z) = \frac{2z^2 - 0.4z - 0.6}{z^2 + 0.2z - 0.15} = 2 + \frac{-0.125}{z + 0.5} + \frac{-0.675}{z - 0.3}. \text{ From Table}$$

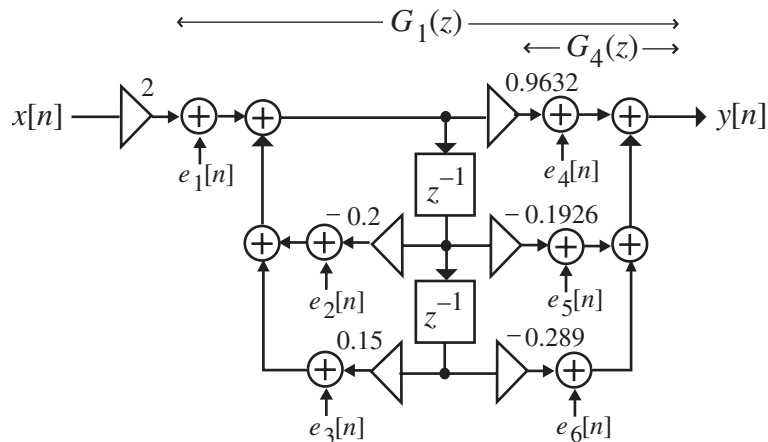


12.4 we get $\|H\|_2^2 = 2^2 + \frac{(-0.125)^2}{1 - (-0.5)^2} + \frac{(-0.675)^2}{1 - (0.3)^2} + \frac{2 \times (-0.125) \times (-0.675)}{1 - (-0.5) \times 0.3} = 4.6683$. Hence,

$$\beta_1 = \frac{\|F_1\|_2}{\|H\|_2} = 0.9632.$$

The scaling multipliers are therefore $\bar{K} = \beta_0 K = 0.4805 \times 2 = 0.961$, $\bar{b}_{01} = \beta_1 = 0.9632$, $\bar{b}_{11} = -0.2 \times \beta_1 = -0.1926$, $\bar{b}_{21} = -0.3 \times \beta_1 = -0.289$.

The scaled structure with the noise sources is shown below:



Quantization before addition: The noise transfer function associated with the noise sources

$$e_1[n], e_2[n], \text{ and } e_3[n] \text{ is given by } G_1(z) = \frac{\bar{b}_{01}z^2 + \bar{b}_{11}z + \bar{b}_{21}}{z^2 + 0.2z - 0.15} = \frac{0.9632z^2 - 0.1926z - 0.289}{z^2 + 0.2z - 0.15}$$

$$= 0.9632 + \frac{-0.0601}{z + 0.5} + \frac{-0.3251}{z - 0.3}. \text{ From Table 12.4 we get}$$

$$\sigma_1^2 = (0.9632)^2 + \frac{(-0.0601)^2}{1 - (-0.5)^2} + \frac{(-0.3251)^2}{1 - (0.3)^2} + \frac{2 \times (-0.0601) \times (-0.3251)}{1 - (-0.5) \times 0.3} = 1.0827. \text{ The noise}$$

transfer function associated with the noise sources $e_4[n], e_5[n], \text{ and } e_6[n]$ is given by

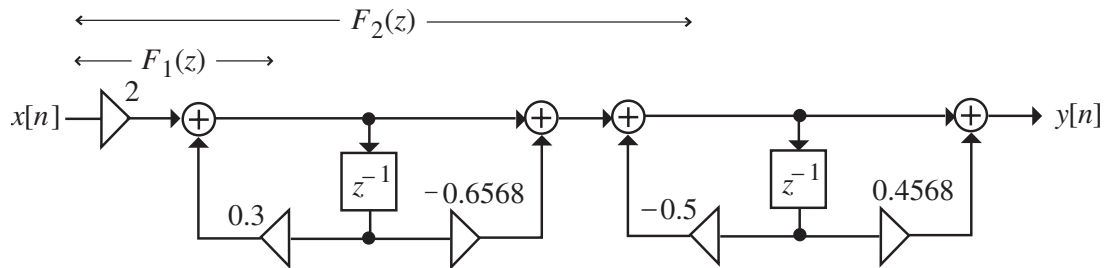
$$G_4(z) = 1. \text{ Hence, } \sigma_4^2 = 1. \text{ Hence, total normalized output noise power is}$$

$$\sigma_o^2 = 3\sigma_1^2 + 3\sigma_4^2 = 3 \times 1.0827 + 3 = 6.2481.$$

Quantization after addition: Here, total normalized output noise power is $\sigma_o^2 = \sigma_1^2 + \sigma_4^2 = 1.0827 + 1 = 2.0827$.

(b) Cascade Structure #1 - $H(z) = 2 \left(\frac{1 - 0.6568z^{-1}}{1 - 0.3z^{-1}} \right) \left(\frac{1 + 0.4568z^{-1}}{1 + 0.5z^{-1}} \right)$.

The unscaled structure is shown below.



$$F_1(z) = \frac{2}{1 - 0.3z^{-1}} = \frac{2z}{z - 0.3} = 2 + \frac{0.6}{z - 0.3}. \text{ From Table 12.4 we get}$$

$$\|F_1\|_2^2 = 2^2 + \frac{(0.6)^2}{1 - (0.3)^2} = 4.3956. \text{ Hence, } \beta_0 = \frac{1}{\|F_1\|_2} = 0.477.$$

$$F_2(z) = \frac{2(1 - 0.6568z^{-1})}{(1 - 0.3z^{-1})(1 + 0.5z^{-1})} = 2 + \frac{-1.446}{z + 0.5} + \frac{-0.2676}{z - 0.3}. \text{ From Table 12.4 we get}$$

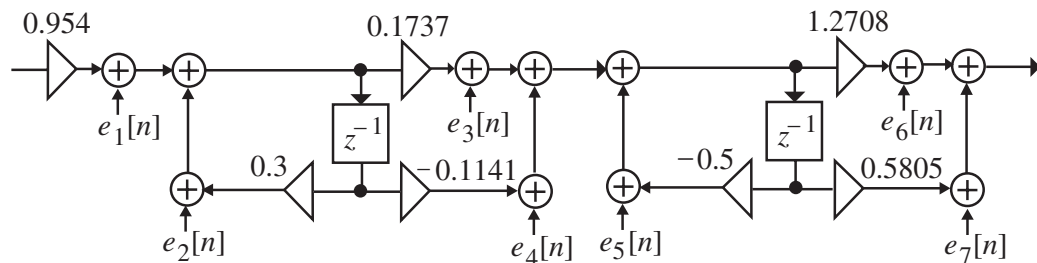
$$\sigma_2^2 = 2^2 + \frac{(-1.446)^2}{1 - (-0.5)^2} + \frac{(-0.2676)^2}{1 - (0.3)^2} + \frac{2 \times (-1.446) \times (-0.2676)}{1 - (-0.5) \times 0.3} = 7.5395. \text{ Hence,}$$

$$\beta_1 = \frac{\|F_1\|_2}{\|F_2\|_2} = 0.1737. \text{ Next, } H(z) = \frac{2 - 0.4z^{-1} - 0.6z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}}. \text{ From Part (a), } \|H\|_2^2 = 4.6683.$$

$$\text{Hence, } \beta_1 = \frac{\|F_2\|_2}{\|H\|_2} = 1.2708.$$

The scaling multipliers are therefore $\bar{K} = \beta_0 K = 0.477 \times 2 = 0.954$, $\bar{b}_{01} = \beta_1 = 0.1737$, $\bar{b}_{11} = -0.6568 \times \beta_1 = -0.1141$, $\bar{b}_{02} = \beta_2 = 1.2708$, $\bar{b}_{12} = 0.4568 \times \beta_2 = 0.5805$.

The scaled structure with the noise sources is shown below:



Quantization before addition - The noise transfer function associated with the noise sources $e_1[n]$ and $e_2[n]$ is

$$G_1(z) = \left(\frac{0.1737 - 0.1141z^{-1}}{1 - 0.3z^{-1}} \right) \left(\frac{1.2708 + 0.5805z^{-1}}{1 + 0.5z^{-1}} \right) = 0.2207 + \frac{-0.0138}{z + 0.5} + \frac{-0.0745}{z - 0.3}. \text{ From}$$

Table 12.4 we get

$$\sigma_1^2 = (0.2207)^2 + \frac{(-0.0138)^2}{1 - (-0.5)^2} + \frac{(-0.0745)^2}{1 - (0.3)^2} + \frac{2 \times (-0.0138) \times (-0.0745)}{1 - (-0.5) \times 0.3} = 0.0569.$$

The noise transfer function associated with the noise sources $e_3[n]$, $e_4[n]$, and $e_5[n]$ is

$$G_3(z) = \frac{1.2708 + 0.5805z^{-1}}{1 + 0.5z^{-1}} = 1.2708 + \frac{-0.0549}{z + 0.5}. \text{ From Table 12.4 we get}$$

$$\sigma_3^2 = (1.2708)^2 + \frac{(-0.0549)^2}{1 - (-0.5)^2} = 1.619. \text{ The noise transfer function associated with the noise}$$

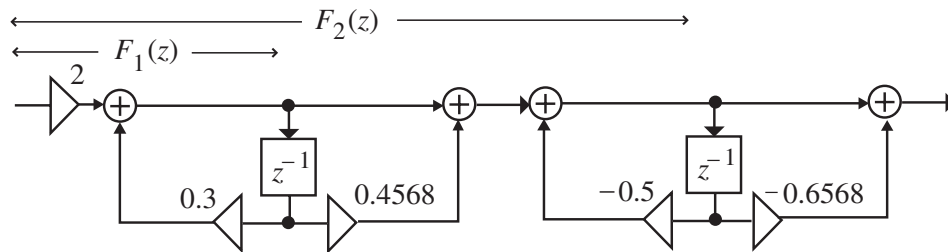
sources $e_6[n]$ and $e_7[n]$ is given by $G_6(z) = 1$. Hence, $\sigma_6^2 = 1$. Hence, total normalized output noise power is $\sigma_o^2 = 2\sigma_1^2 + 3\sigma_3^2 + 2\sigma_6^2 = 2 \times 0.0569 + 3 \times 1.619 + 2 = 6.9708$.

Quantization after addition

Here, total normalized output noise power is $\sigma_o^2 = \sigma_1^2 + \sigma_3^2 + \sigma_6^2 = 0.0569 + 1.619 + 1 = 2.6759$.

(c) Cascade Structure #2 - $H(z) = 2 \left(\frac{1 + 0.4568z^{-1}}{1 - 0.3z^{-1}} \right) \left(\frac{1 - 0.6568z^{-1}}{1 + 0.5z^{-1}} \right)$.

The unscaled structure is shown below:



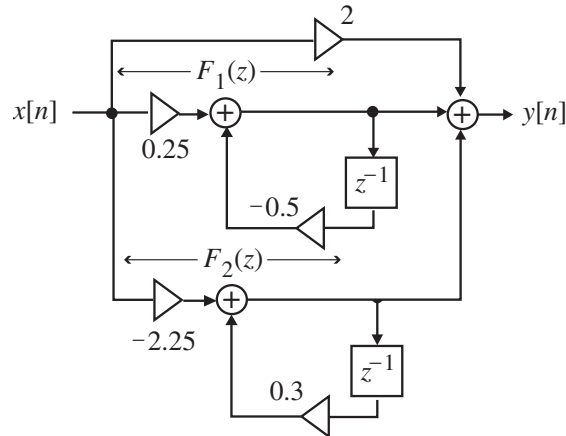
$$F_1(z) = \frac{2}{1 - 0.3z^{-1}} = 2 + \frac{0.6}{z - 0.3}. \text{ Using Table 12.4 we get } \|F_1\|_2^2 = 2^2 + \frac{(0.6)^2}{1 - (0.3)^2} = 4.3956.$$

Hence, $\beta_0 = \frac{1}{\|F_1\|_2} = 0.477$. Likewise,

$$F_2(z) = \frac{2(1 + 0.4568z^{-1})}{(1 - 0.3z^{-1})(1 + 0.5z^{-1})} = 2 + \frac{-0.054}{z + 0.5} + \frac{0.5676}{z - 0.3}. \text{ Using Table 12.4 we get}$$

(d) Parallel Form I Realization - $H(z) = 2 + \frac{0.25}{1 + 0.5z^{-1}} + \frac{-2.25}{1 - 0.3z^{-1}}$.

The unscaled structure is shown below:



$$F_1(z) = \frac{0.25}{1 + 0.5z^{-1}} = 0.25 + \frac{-0.125}{z + 0.5}. \text{ Using Table 12.4 we get}$$

$$\|F_1\|_2^2 = (0.25)^2 + \frac{(-0.125)^2}{1 - (-0.5)^2} = 0.0833. \text{ Hence, } \beta_1 = \frac{1}{\|F_1\|_2} = 3.4641.$$

$$F_2(z) = \frac{-2.25}{1 - 0.3z^{-1}} = -2.25 + \frac{-0.675}{z - 0.3}. \text{ Using Table 12.4 we get}$$

$$\|F_2\|_2^2 = (-2.25)^2 + \frac{(-0.675)^2}{1 - (0.3)^2} = 5.5632. \text{ Hence, } \beta_2 = \frac{1}{\|F_2\|_2} = 0.424.$$

$$H(z) = 2 + \frac{-0.125}{z + 0.5} + \frac{-0.675}{z - 0.3}. \text{ Using Table 12.4 we get}$$

$$\|H\|_2^2 = 2^2 + \frac{(-0.125)^2}{1 - (-0.5)^2} + \frac{(-0.675)^2}{1 - (0.3)^2} + \frac{2 \times (-0.125) \times (-0.675)}{1 - (-0.5) \times 0.3} = 4.6683. \text{ Hence,}$$

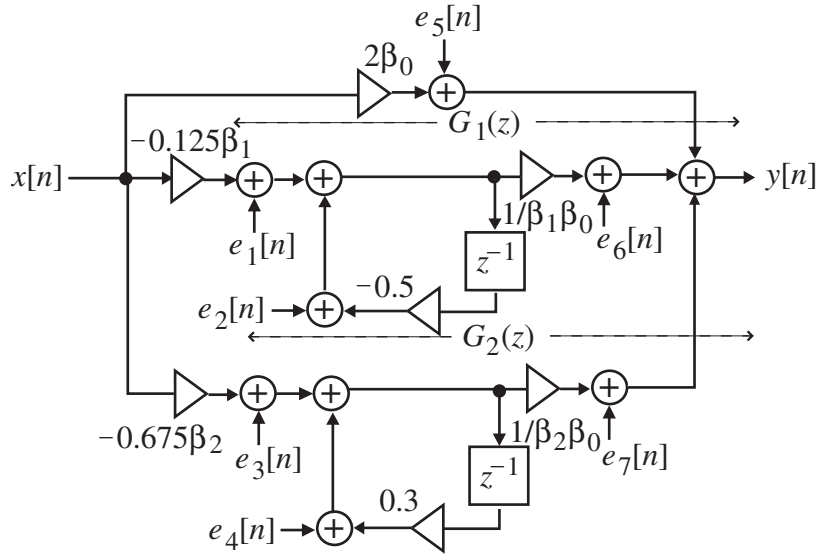
$$\beta_0 = \frac{1}{\|H\|_2} = 0.4628.$$

The scaled structure with the noise sources is shown on top of the next page. The noise transfer function associated with the noise sources $e_1[n]$ and $e_2[n]$ is given by

$$G_1(z) = \frac{1/\beta_0\beta_1}{1 + 0.5z^{-1}} = 0.6238 + \frac{-0.3119}{z + 0.5}. \text{ Using Table 12.4 we obtain,}$$

$$\sigma_1^2 = (0.6238)^2 + \frac{(-0.3119)^2}{1 - (-0.5)^2} = 0.5188. \text{ Likewise, the noise transfer function associated with}$$

the noise sources $e_3[n]$ and $e_4[n]$ is given by



$$G_3(z) = \frac{1/\beta_0\beta_2}{1-0.3z^{-1}} = 0.9162 + \frac{0.2749}{z-0.3}. \text{ Using Table 12.4 we obtain,}$$

$$\sigma_3^2 = (0.9162)^2 + \frac{(0.2749)^2}{1-(0.3)^2} = 0.224.$$

The noise transfer function associated with the remaining three noise sources is $G_5(z) = 1$.

Hence, $\sigma_5^2 = 1$.

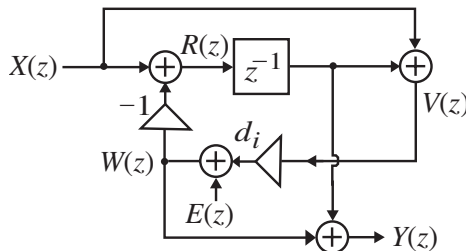
Quantization before addition

Hence, total normalized output noise power is $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_3^2 + 3\sigma_5^2 = 5.8824$.

Quantization after addition

Here, total normalized output noise power is $\sigma_o^2 = \sigma_1^2 + \sigma_3^2 + \sigma_5^2 = 2.4412$.

12.28 (a) The noise model for the allpass structure of Figure P12.9(a) is shown below:



Analysis yields $W(z) = E(z) + d_i V(z)$, $R(z) = X(z) - W(z)$, $V(z) = X(z) + z^{-1}R(z)$, and $Y(z) = W(z) + z^{-1}R(z)$. To determine the noise transfer function $G(z)$ we set $X(z) = 0$ in the

above equations. This leads to $R(z) = -W(z)$, $V(z) = z^{-1}R(z) = -z^{-1}W(z)$, and hence, $W(z) = E(z) - d_i z^{-1}W(z)$ or, $E(z) = (1 + d_i z^{-1})W(z)$. As a result, $Y(z) = W(z) - z^{-1}W(z) = (1 - z^{-1})W(z)$. Consequently, the noise transfer function is given by

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + d_i z^{-1}} = \frac{z - 1}{z + d_i} = 1 + \frac{-(1 + d_i)}{z + d_i}. \text{ Using Table 12.4 we get}$$

$$\sigma_e^2 = 1 + \frac{(1 + d_i)^2}{1 - d_i^2} = \frac{2}{1 - d_i}.$$

(b) Let $G_i(z)$ be the noise transfer function for d_i , $1 \leq i \leq 3$. Then,

$$G_1(z) = \frac{z - 1}{z + d_1} \cdot A_2(z), \quad G_2(z) = \frac{z - 1}{z + d_2}, \quad \text{and} \quad G_3(z) = \frac{z - 1}{z + d_3}.$$

From the results of Part (a), it follows that the noise variances due to d_1, d_2 , and d_3 are given by $\sigma_k^2 = \frac{2}{1 - d_k}$ for $k = 1, 2, 3$. Hence the total noise power at the output of the digital filter structure of Figure P12.9 is given by $\sigma_o^2 = \frac{2}{1 - d_1} + \frac{2}{1 - d_2} + \frac{2}{1 - d_3}$.

12.30 Let the total noise power at the output of $G(z)$ due to product round-off be given by σ_G^2 .

Assuming a total of L multipliers in the realization of $G(z)$ we get

$$\sigma_G^2 = \sum_{\ell=1}^L k_\ell \left(\frac{1}{2\pi} \int_0^{2\pi} |G_\ell(e^{j\omega})|^2 d\omega \right),$$

where $G_\ell(z)$ denotes the noise transfer function due the ℓ -th noise source in $G(z)$. Now if each delay is replaced by two delays, then each of the noise transfer function becomes $G_\ell(z^2)$. Thus, the total noise power at the output due to noise

sources in $G(z^2)$ is given by $\hat{\sigma}_G^2 = \sum_{\ell=1}^L k_\ell \left(\frac{1}{2\pi} \int_0^{2\pi} |G_\ell(e^{j2\omega})|^2 d\omega \right)$. Replacing ω by $\hat{\omega}/2$ in

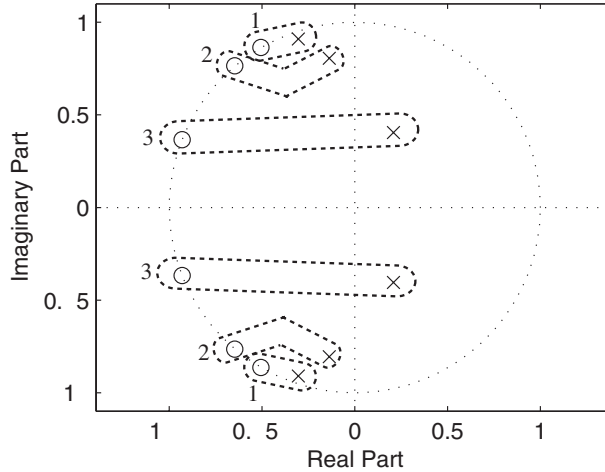
the integral we get $\sum_{\ell=1}^L k_\ell \left(\frac{1}{2\pi} \int_0^{4\pi} |G_\ell(e^{j\hat{\omega}})|^2 \left(\frac{1}{2}\right) d\hat{\omega} \right) = \frac{1}{2} \sum_{\ell=1}^L k_\ell \left(\frac{1}{2\pi} \int_0^{4\pi} |G_\ell(e^{j\hat{\omega}})|^2 d\hat{\omega} \right) = \sigma_G^2$.

Since, $\frac{1}{2\pi} \int_0^{2\pi} |G(e^{j2\omega})|^2 |A(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{j2\omega})|^2 d\omega$, the total noise power at the output of the cascade is still equal to σ_G^2 ;

12.31 For the first factor in the numerator, there are R possible choices of factors. Once this factor has been chosen, there are $R - 1$ choices for the next factor and continuing further we get that the total number of possible ways in which the factors in the numerator can be generated equal to $R(R - 1)(R - 2) \cdots 2 \times 1 = R!$. Similarly, the total number of ways in which the factors in the

denominator can be generated is equal to $R!$. Since the numerator and the denominator are factored independent of each other, hence the total number of possible realizations are $N = (R!)(R!) = (R!)^2$.

12.32 (i) A pole-zero plot of $H_1(z)$ is shown below:



First we pair the poles closest to the unit circle with their nearest zeros resulting in the second-order section

$$H_a(z) = \frac{z^2 + 1.0094z + 1}{z^2 + 0.6081z + 0.9208}$$

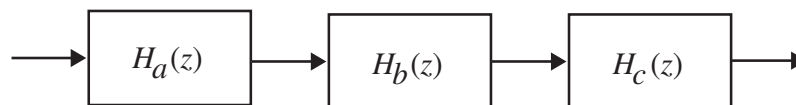
Next, the poles that are closest to the poles of $H_a(z)$ are matched with their nearest zeros resulting in the second-order section

$$H_b(z) = \frac{z^2 + 1.2914z + 1}{z^2 + 0.2749z + 0.6673}$$

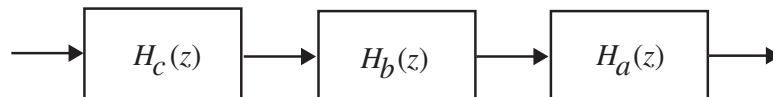
Finally, the remaining poles and zeros are matched yielding the second-order section

$$H_c(z) = \frac{z^2 + 1.8606z + 1}{z^2 - 0.41591z + 0.2066}$$

For ordering the sections to yield the smallest peak output noise due to product round-off under an \mathcal{L}_2 -scaling rule, the sections should be placed from most peaked to least peaked as shown below.

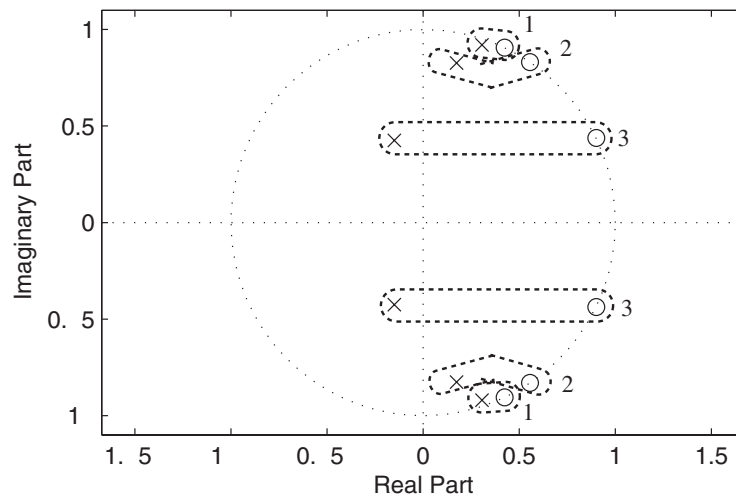


For ordering the sections to yield the smallest peak output noise power due to product round-off under an \mathcal{L}_∞ -scaling rule, the sections should be placed from least peaked to most peaked as shown below.



(ii) A pole-zero plot of $H_2(z)$ is shown below:

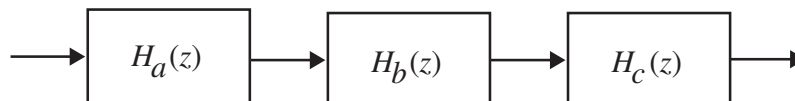
(iii)



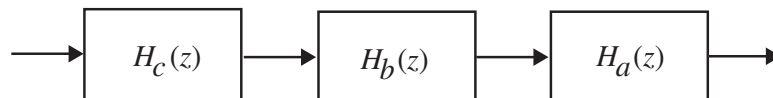
Following the same procedure as in Part (i), we get

$$H_a(z) = \frac{z^2 - 0.8508z + 1}{z^2 - 0.6120z + 0.9208}, H_b(z) = \frac{z^2 - 1.1141z + 1}{z^2 - 0.3474z + 0.7127}, H_c(z) = \frac{z^2 - 1.7988z + 1}{z^2 - 0.2993z + 0.20358}.$$

For ordering the sections to yield the smallest peak output noise due to product round-off under an \mathcal{L}_2 -scaling rule, the sections should be placed from most peaked to least peaked as shown below.



For ordering the sections to yield the smallest peak output noise power due to product round-off under an \mathcal{L}_∞ -scaling rule, the sections should be placed from least peaked to most peaked as shown below.



12.33 $\text{SNR} = \frac{\sigma_x^2}{\sigma_o^2}$. After scaling, σ_x changes to $K\sigma_x$ where K is as given by Eq. (12.151).

Therefore, $\text{SNR} = \frac{\sigma_x^2(1-|\alpha|)^2}{\sigma_o^2}$.

(i) For uniform density function $\sigma_x^2 = \int_{-1}^1 \frac{1}{2}x^2 dx = \frac{1}{3}$. Thus, $\text{SNR} = \frac{(1-|\alpha|)^2}{3\sigma_o^2}$. With

$b = 12$, $\sigma_o^2 = \frac{2^{-24}}{12} = 4.967 \times 10^{-9}$ and $|\alpha| = 0.95$. Hence,

$$\text{SNR}_{dB} = 10 \log_{10} \left(\frac{(1-|\alpha|)^2}{3\sigma_o^2} \right) = 52.24 \text{ dB.}$$

(ii) For Gaussian input with $\sigma_x^2 = \frac{1}{9}$, $\text{SNR} = \frac{(1-|\alpha|)^2}{9\sigma_o^2}$. Again with $b = 12$,

$$\text{SNR}_{dB} = 10 \log_{10} \left(\frac{(1-|\alpha|)^2}{9\sigma_o^2} \right) = 47.97 \text{ dB.}$$

(iii) For a sinusoidal input of known frequency, i.e., $x[n] = \sin(\omega_o n)$. In this case, the average

power = $\sigma_x^2 = \frac{1}{2}$. Hence, $\text{SNR} = \frac{(1-|\alpha|)^2 \sigma_x^2}{\sigma_o^2} = \frac{(1-|\alpha|)^2}{2\sigma_o^2}$. Therefore,

$$\text{SNR}_{dB} = 10 \log_{10} \left(\frac{(1-|\alpha|)^2}{2\sigma_o^2} \right) = 69.91 \text{ dB.}$$

12.34 (a) $H_{LP}(z) = \frac{1}{2}[1 + A_1(z)]$, where $A_1(z) = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$. Hence,

$$|H_{LP}(e^{j\omega})|^2 = \left(\frac{1-\alpha}{2} \right)^2 \frac{2(1+\cos\omega)}{1-2\alpha\cos\omega+\alpha^2}. \text{ Proving } \left. \frac{\partial |H_{LP}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = 0 \text{ is equivalent to}$$

$$\text{proving } \left. \frac{\partial |H_{LP}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = 0. \text{ Now,}$$

$$\frac{\partial |H_{LP}(e^{j\omega})|^2}{\partial \alpha} = -\frac{1-\alpha}{2} \cdot \frac{2(1+\cos\omega)}{1-2\alpha\cos\omega+\alpha^2} - \left(\frac{1-\alpha}{2} \right)^2 \cdot \frac{2(1+\cos\omega)(\alpha-2\cos\omega)}{1-2\alpha\cos\omega+\alpha^2}.$$

$$\text{Thus, } \left. \frac{\partial |H_{LP}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = \frac{-2}{1-\alpha} + \frac{2}{1-\alpha} = 0.$$

(b) $H_{HP}(z) = \frac{1}{2}[1 - A_1(z)]$, where $A_1(z) = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$. Hence,

$$|H_{HP}(e^{j\omega})|^2 = \left(\frac{1+\alpha}{2} \right)^2 \frac{2(1-\cos\omega)}{1-2\alpha\cos\omega+\alpha^2}. \text{ Now,}$$

$$\frac{\partial |H_{HP}(e^{j\omega})|^2}{\partial \alpha} = -\frac{1+\alpha}{2} \cdot \frac{2(1-\cos\omega)}{1-2\alpha\cos\omega+\alpha^2} - \left(\frac{1+\alpha}{2} \right)^2 \cdot \frac{2(1-\cos\omega)(\alpha-2\cos\omega)}{1-2\alpha\cos\omega+\alpha^2}.$$

$$\text{Thus, } \left. \frac{\partial |H_{HP}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = \frac{-2}{1+\alpha} + \frac{2}{1+\alpha} = 0.$$

12.35 (a) $H_{BP}(z) = \frac{1}{2}[1 - A_2(z)]$, where $A_2(z) = \frac{\alpha - \beta(1+\alpha)z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$, with α and β being

$$\text{real. Hence, } |H_{BP}(e^{j\omega})|^2 = \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{2(1-\cos\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right).$$

$$\begin{aligned} \text{Now, } \frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \alpha} &= \left(\frac{-2}{1-\alpha}\right) \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{2(1-\cos\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right) \\ &\quad + \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{2(1-\cos\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right) \times \\ &\quad \left(\frac{2\beta^2(1+\alpha) + 2\alpha + 2\cos(2\omega) - 4\beta(1+\alpha)\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right). \end{aligned}$$

Using the fact $\beta = \cos(\omega_o)$, we get

$$\left. \frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=\omega_o} = \frac{-2}{1-\alpha} - \frac{(2\beta^2 + 2\alpha - 2 - 2\alpha\beta^2)}{1 + \alpha^2 - 2\alpha + 2\alpha\beta^2 - \beta^2 - \alpha^2\beta^2} = \frac{-2}{1-\alpha} + \frac{2}{1-\alpha} = 0.$$

$$\begin{aligned} \text{Similarly, } \frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \beta} &= \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{2(1-\cos\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right) \times \\ &\quad \left(-\frac{2\beta(1+\alpha) - 2(1+\alpha)^2\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right). \end{aligned}$$

$$\text{Again, using the fact } \beta = \cos(\omega_o), \text{ it can be seen that } \left. \frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \beta} \right|_{\omega=\omega_o} = 0.$$

(b) Here, $H_{BS}(z) = \frac{1}{2}[1 + A_2(z)]$, where $A_2(z)$ is as given in Part (a). Hence,

$$|H_{BS}(e^{j\omega})|^2 = \left(\frac{1+\alpha}{2}\right)^2 \left(\frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right). \text{ Thus,}$$

$$\frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \alpha} = \left(\frac{1+\alpha}{2}\right) \left(\frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right)$$

$$-\left(\frac{1+\alpha}{2}\right)^2 \left(\frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right) \times$$

$$\left(\frac{2(1+\alpha)\beta^2 + 2\alpha + 2\cos(2\omega) - 4\beta(1+\alpha)\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right).$$

Substituting $\beta = \cos(\omega_o)$, it can be seen that

$$\left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = \frac{2}{1-\alpha} - \left(\frac{1+\alpha}{2}\right)^2 \left(\frac{2(1+\alpha)(\beta+1)^2}{(1+\alpha)^2(\beta+1)^2} \right) = 0.$$

$$\text{Similarly, } \left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=\pi} = \frac{2}{1-\alpha} - \left(\frac{1+\alpha}{2}\right)^2 \left(\frac{2(1+\alpha)(\beta+1)^2}{(1+\alpha)^2(\beta+1)^2} \right) = 0.$$

$$\text{Now, } \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \beta} = \left(\frac{1+\alpha}{2}\right)^2 \left(\frac{8\beta - 8\beta\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right)$$

$$- \left(\frac{1+\alpha}{2}\right)^2 \left(\frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos\omega}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right) \times$$

$$\left(\frac{(1+\alpha)^2 2(\beta - \cos\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos\omega} \right).$$

Again, substituting $\beta = \cos(\omega_o)$, it can be seen that $\left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \beta} \right|_{\omega=0} = 0$ and

$$\left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \beta} \right|_{\omega=\pi} = 0.$$

12.36 For a BR transfer function $G(z)$ realized in a parallel allpass form, its power-complementary transfer function $H(z)$ is also BR satisfying the condition $|G(e^{j\omega})|^2 = 1 - |H(e^{j\omega})|^2$. Let $\omega = \omega_o$ be a frequency where $|G(e^{j\omega})|$ is a maximum, i.e., $|G(e^{j\omega_o})| = 1$. Then, it follows that $|H(e^{j\omega_o})| = 0$. From the power-complementary condition it follows that

$$2|G(e^{j\omega})| \cdot \frac{\partial |G(e^{j\omega})|}{\partial \omega} = -2|H(e^{j\omega})| \cdot \frac{\partial |H(e^{j\omega})|}{\partial \omega}. \text{ Therefore, at } \omega = \omega_o,$$

$$2|G(e^{j\omega})| \cdot \left. \frac{\partial |G(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = -2|H(e^{j\omega})| \cdot \left. \frac{\partial |H(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o}, \text{ and thus, } \left. \frac{\partial |G(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = 0$$

whether or not $\left. \frac{\partial |H(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = 0$. Hence, low passband sensitivity of $G(z)$ does not necessarily imply low passband sensitivity of $H(z)$.

12.37 Without error feedback – The transfer function of the structure without error feedback is

$$\text{given by } H(z) = \frac{1}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}, \text{ where } r = 1 - \varepsilon. \text{ The}$$

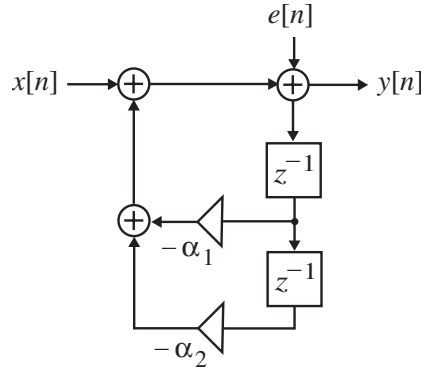
corresponding impulse response $h[n]$ is given by $h[n] = \frac{r^n \sin(n+1)\theta}{\sin \theta} \mu[n]$. To keep the

output $y[n]$ from overflowing we must insert a multiplier of value $\frac{1}{L}$ at the input where

$$L = \sum_{n=0}^{\infty} |h[n]|. \text{ From Eq. (12.164) we get}$$

$$(1): \frac{1}{(1-r)^2(1-2r \cos \theta + r^2)} \leq L^2 \leq \frac{16}{\pi^2(1-r)^2 \sin^2 \theta}.$$

The quantization noise model for $H(z)$ is as shown below:



$$\text{The output noise power is given by } \sigma_n^2 = \sigma_e^2 \sum_{n=0}^{\infty} |h[n]|^2 = \frac{1+r^2}{1-r^2} \cdot \frac{1}{r^4 - 2r^2 \cos \theta + 1} \cdot \sigma_e^2.$$

The output signal power, assuming an input signal of variance σ_x^2 is given by

$$\sigma_y^2 = \frac{\sigma_y^2}{L^2} \sum_{n=0}^{\infty} |h[n]|^2. \text{ Hence, the SNR is given by } \text{SNR} = \frac{S}{N} = \frac{\sigma_y^2}{\sigma_n^2} = \frac{\sigma_x^2}{L^2 \sigma_e^2}. \text{ For a } (b+1)\text{-bit}$$

signed representation, $\sigma_e^2 = \frac{2^{-2b}}{12}$. Hence, here $\text{SNR} = \frac{S}{N} = \frac{\sigma_x^2}{L^2 2^{-2b}}$. Therefore, from the

inequality of Eq. (1) given above, we get

$$\frac{2^{-2b}}{12\sigma_x^2(1-r)^2(1-2r\cos\theta+r^2)} \leq \frac{N}{S} \leq \frac{16 \times 2^{-2b}}{12\sigma_x^2\pi^2(1-r)^2 \sin^2 \theta}.$$

(a) For an WSS uniformly distributed input between $[-1, 1]$, $\sigma_x^2 = \frac{1}{3}$. Hence,

$$\frac{2^{-2b}}{4(1-r)^2(1-2r\cos\theta+r^2)} \leq \frac{N}{S} \leq \frac{4}{\pi^2} \cdot \frac{2^{-2b}}{(1-r)^2 \sin^2 \theta}.$$

If $\varepsilon \rightarrow 0$, and $\theta \rightarrow 0$, then $(1-r)^2 \rightarrow \varepsilon^2$, $\cos(2\theta) = 1 - 2\sin^2 \theta \cong 1 - 2\theta^2$, and $\sin^2 \theta \cong \theta^2$. In this case we have

$$\frac{2^{-2b}}{4\varepsilon^2(\varepsilon^2 + 4\theta^2)} \leq \frac{N}{S} \leq \frac{2^{-2b}}{\pi^2\varepsilon^2\theta^2}.$$

(b) For an input with a Gaussian distribution between $[-1, 1]$, $\sigma_x = \frac{1}{3}$. In this case, we have

$$3 \cdot \frac{2^{-2b}}{4\varepsilon^2(\varepsilon^2 + 4\theta^2)} \leq \frac{N}{S} \leq \frac{3}{\pi^2} \cdot \frac{2^{-2b}}{\varepsilon^2\theta^2}.$$

(c) For a sinusoidal input between $[-1, 1]$ of known frequency ω_o , $\sigma_x^2 = \frac{1}{2}$. The output noise

variance here is therefore $\sigma_n^2 = \sigma_e^2 \sum_{n=0}^{\infty} |h[n]|^2 = \frac{2^{-2b}}{12} \cdot \frac{1+r^2}{1-r^2} \cdot \frac{1}{r^4 - 2r^2 \cos\theta + 1}$. Thus,

$$\frac{N}{S} = \frac{1+r^2}{1-r^2} \cdot \frac{1}{6(r^4 - 2r^2 \cos(2\theta) + 1)} = \frac{2^{-2b}}{24\varepsilon(\varepsilon^2 + \theta^2)} \cong \frac{2^{-2b}}{24\varepsilon\theta^2}.$$

With error feedback

$$G(z) = \frac{1 - 2z^{-1} + z^{-2}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} = \frac{z^2 - 2z + 1}{z^2 - 2r\cos\theta z + 1} = 1 + \frac{2(r\cos\theta - 1)z + 1 - r^2}{z^2 - 2r\cos\theta z + 1}.$$

$$\text{Thus, } \|G\|^2 = 1 + \frac{(4(1-r\cos\theta)^2 + (1-r^2)^2)(1-r^4) + 8r\cos\theta \cdot (1-r^2)^2(r\cos\theta - 1)}{(1-r^2)^2 + 2r^2(4r^2 \cos^2 \theta) - 4(1+r^4)r^2 \cos^2 \theta}.$$

For $r = 1 - \varepsilon$ with $\varepsilon \rightarrow 0$, and $\theta \rightarrow 0$, we get after some manipulation

$$\|G\|^2 = 1 + \frac{\theta^4}{4\varepsilon(\varepsilon^2 + \theta^2)}, \text{ and } \sigma_n^2 = \sigma_e^2 \|G\|^2.$$

Now, remains the same as before since it depends only upon the denominator. Also, the overall transfer function of the structure remains the same as before. The output noise power with

The total output noise power is thus given by $\sigma_o^2 = \|G_1\|_2^2 \sigma_{e1}^2 + \|G_2\|_2^2 \sigma_{e2}^2 + \|G_3\|_2^2 \sigma_{e3}^2$. Hence, the output noise power for a lowpass filter design can be reduced by placing zeros of the noise transfer functions in the passband. For each of the noise transfer functions given above, we can only place one zero at $z = 1$ by choosing $\lambda_i = -1$, $i = 1, 2, 3$, in which case, using the notations $b = -[2 - k_1(k_1 + k_2)]$ and $d = 1 - k_1 k_2$, the noise transfer functions reduce to

$$G_1(z) = \frac{-k_1 z^{-1}(1 - z^{-1})}{1 + b z^{-1} + d z^{-2}} = \frac{-k_1 z + k_1}{z^2 + b z + d},$$

$$G_2(z) = \frac{-k_1 z^{-1}(1 - z^{-1})^2}{1 + b z^{-1} + d z^{-2}} = \frac{-k_1(z^2 + z + 1)}{z(z^2 + b z + d)} = \frac{-k_1 A}{z} + \frac{Cz + D}{z^2 + b z + d},$$

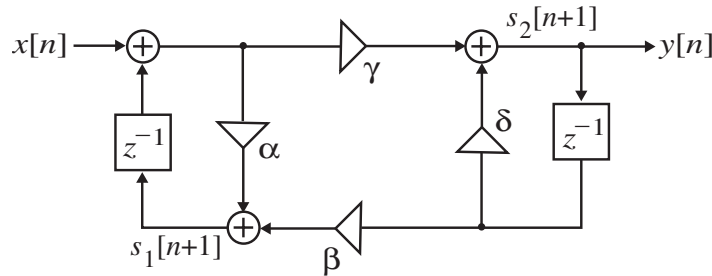
$$G_3(z) = \frac{z^{-1}(1 - z^{-1})^2}{1 + b z^{-1} + d z^{-2}} = \frac{z^2 + z + 1}{z(z^2 + b z + d)} = \frac{A}{z} + \frac{Cz + D}{z^2 + b z + d}, \text{ where } A = \frac{1}{d}, \quad C = 1 - \frac{1}{d},$$

and $D = -\left(2 + \frac{b}{d}\right)$. Using Table 12.4 we then obtain

$$\|G_1\|_2^2 = k_1^2 \cdot \frac{2(1 - d^2) - 2(1 - d)b}{(1 - d^2)^2 + 2db^2 - (1 + d^2)b^2},$$

$$\|G_3\|_2^2 = A^2 + 2BC + \frac{(C^2 + D^2)(1 - d^2) - 2CD(1 - d)b}{(1 - d^2)^2 + 2db^2 - (1 + d^2)b^2}, \text{ and } \|G_2\|_2^2 = k_1^2 \|G_3\|_2^2.$$

12.40 The coupled-form structure with state-variables labeled is shown below:



Its transfer function is given by Eq. (12.19) where $\alpha = \delta = r \cos \theta$ and $\beta = -\gamma = r \sin \theta$.

Analysis of the above structure yields $s_1[n + 1] = \alpha(x[n] + s_1[n]) + \beta s_2[n]$ and

$s_2[n + 1] = \gamma(x[n] + s_1[n]) + \delta s_2[n]$. Rewriting these two equations in matrix form we get

$$\begin{bmatrix} s_1[n + 1] \\ s_2[n + 1] \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} x[n]. \text{ Thus, } \mathbf{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \text{ Therefore,}$$

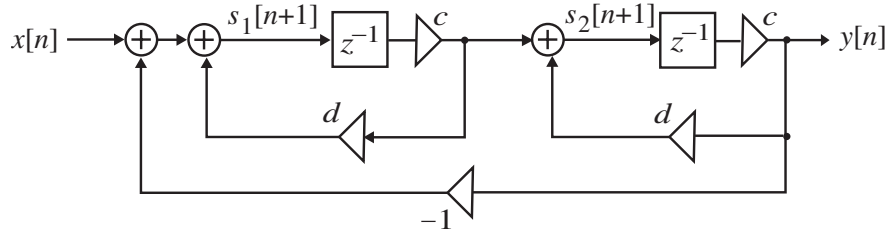
$$\mathbf{A}^t \mathbf{A} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2 + \gamma^2 & \alpha\beta + \gamma\delta \\ \alpha\beta + \gamma\delta & \beta^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}. \text{ Likewise,}$$

$$\mathbf{A} \mathbf{A}^t = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta^2 & \alpha\gamma + \beta\delta \\ \alpha\gamma + \beta\delta & \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}. \text{ Since for stability } r < 1, \mathbf{A} \text{ is a}$$

normal form matrix, and hence the coupled-form structure does not support limit cycles.

12.41 The modified coupled-form structure with state-variables labeled is shown on top of the next page. Its transfer function determined in the solution of Problem 12.4 is given by

$$H(z) = \frac{c^2}{z^2 - 2cdz + c^2(1 + d^2)}. \text{ Now, a second-order transfer function with poles at}$$



$z = re^{\pm j\theta}$ has a denominator given by $z^2 - 2r \cos \theta \cdot z + r^2$. Comparing this denominator with that of $H(z)$, we get $c = r \sin \theta$ and $d = \cot \theta$.

Analyzing the modified coupled-form structure we arrive at the following equations:

$$s_1[n+1] = cd s_1[n] - c s_2[n] + x[n] \quad \text{and} \quad s_2[n+1] = c s_1[n] + cd s_2[n]. \text{ Rewriting these two}$$

equations in matrix form we get $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} cd & -c \\ c & cd \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$. Thus,

$$\mathbf{A} = \begin{bmatrix} cd & -c \\ c & cd \end{bmatrix} = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix}. \text{ Therefore,}$$

$$\mathbf{A}^t \mathbf{A} = \begin{bmatrix} r \cos \theta & r \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}. \text{ Likewise,}$$

$$\mathbf{A} \mathbf{A}^t = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} r \cos \theta & r \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}. \text{ Since for stability } r < 1, \mathbf{A} \text{ is a}$$

normal form matrix, and hence the modified coupled-form structure does not support limit cycles.

12.42 $\delta = \frac{z-1}{\Gamma} \Rightarrow z = \delta \Gamma + 1.$

$$H(z) = \frac{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} = \frac{\beta_0 z^2 + \beta_1 z + \beta_2}{z^2 + \alpha_1 z + \alpha_2}. \text{ Therefore,}$$

$$H(\delta) = H(z) \Big|_{z=\delta\Gamma+1} = \frac{\beta_0 z^2 + \beta_1 z + \beta_2}{z^2 + \alpha_1 z + \alpha_2} \Big|_{z=\delta\Gamma+1} = \frac{\beta_0 (\delta\Gamma + 1)^2 + \beta_1 (\delta\Gamma + 1) + \beta_2}{(\delta\Gamma + 1)^2 + \alpha_1 (\delta\Gamma + 1) + \alpha_2}$$

$$= \frac{\beta_0 (\delta^2 \Gamma^2 + 2\delta\Gamma + 1) + \beta_1 (\delta\Gamma + 1) + \beta_2}{(\delta^2 \Gamma^2 + 2\delta\Gamma + 1) + \alpha_1 (\delta\Gamma + 1) + \alpha_2} = \frac{(\beta_0 \Gamma^2) \delta^2 + (2\beta_0 \Gamma + \beta_1 \Gamma) \delta + (\beta_0 + \beta_1 + \beta_2)}{\Gamma^2 \delta^2 + (2\Gamma + \alpha_1 \Gamma) \delta + (1 + \alpha_1 + \alpha_2)}$$

$$= \frac{(\beta_0 \Gamma^2) + (2\beta_0 \Gamma + \beta_1 \Gamma) \delta^{-1} + (\beta_0 + \beta_1 + \beta_2) \delta^{-2}}{\Gamma^2 + (2\Gamma + \alpha_1 \Gamma) \delta^{-1} + (1 + \alpha_1 + \alpha_2) \delta^{-2}}.$$

12.43 From Section 11.3.1 we know that computation of each DFT sample requires $2N + 4$ real multiplications. Assuming that the quantization noise generated from each multiplier is independent of the noise generated from other multipliers, we get

$$\sigma_r^2 = (2N + 4)\sigma_o^2 = \frac{2^{-2b}(N + 2)}{6}.$$

12.44 $\text{SNR} = \frac{2^{2b}}{N^2}$. Hence, an SNR of 30 dB implies $\frac{2^{2b}}{N^2} = 10^3$, or

$b = \frac{1}{2} \log_2(10^3 \times 512^2) = 12.9829$. We choose $b + 1 = 14$ bits per sample to get an SNR of 30 dB.

12.45 Let $N = 2^v$. Consider the m -th stage. The output sees $4(2^{v-m})$ noise sources from the m -th stage. Each noise source has a variance reduction by a factor of $\left(\frac{1}{4}\right)^{v-m}$ due to multiplication

by $\frac{1}{2}$ at each stage till the output. Hence the total noise variance at the output due to the noises injected in the m -th stage is $4(2^{v-m})(2^{-2(v-m)})\sigma_o^2$. Therefore, the total noise variance at the

$$\begin{aligned} \text{output} = \sigma^2 &= \sum_{m=1}^v 4(2^{v-m})(2^{-2(v-m)})\sigma_o^2 = 4\sigma_o^2 2^{-v} \sum_{m=1}^v 2^m \\ &= 4 \left(\frac{2^{-2b}}{12} \right) 2^{-v} \left(\frac{2(2^v - 1)}{2 - 1} \right) = \frac{2}{3} 2^{-2b} (1 - 2^{-v}) \approx \frac{2}{3} 2^{-2b} \text{ for large } N. \end{aligned}$$

12.46 $\text{SNR} = \frac{2^{2b}}{2N}$. Hence, $b = \frac{1}{2} \log_2(10^3 \times 2 \times 256) = 9.4829$. We choose $b + 1 = 11$ bits per sample to get an SNR of 30 dB.

$\text{SNR} = \frac{2^{2b}}{2N}$. Hence $b = \frac{1}{2} \log_2((10)^{2.5} \times 5 \times 256) = 10.1439$. We choose $b + 1 = 12$ bits per sample to get an SNR of 30dB. CHECK

M12.1 %POLE_PLOT

```
% POLE_PLOT(B) obtains the pole distribution
% plot of a second order transfer function with a denominator
% of the form Z*Z - KZ + L. For stability 0 < L < 1 and
% abs(K) < 1+L so the range of K is (-2,2).
%
% for a B-bit wordlength, 1 bit is reserved for the sign
% as the coefficients are sorted in the sign magnitude form.
% Both L and the K are quantized to B-1 bits.
%
function [] = pole_plot(bits);
point = 'o';
%
```

```

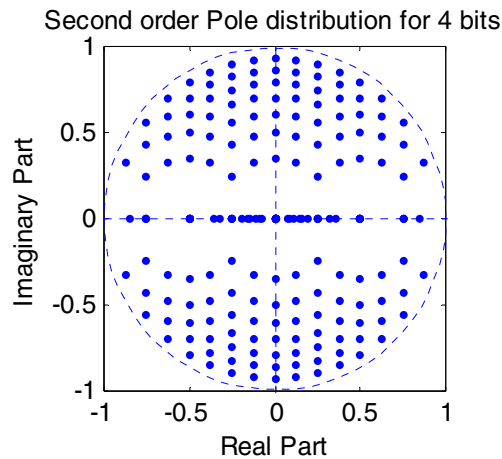
% This part prints the unit circle for reference
% One bit is kept for the sign so effectively for the
% quantization. The remaining number of bits is bits-1
%
bits = bits-1;
%
% sets up the axis
zplane(2,2); % plotting queue of roots using zplane.m with a
% dummy zero and pole
axis([-1 1 -1 1]); % adjusting axis to "hide" dummy zero/pole,
% calling zplane without a zero is not legal in MATLAB release
% 12 and 13
title(['Second order Pole distribution for ',num2str(bits+1),'
bits']);
hold on;
%
% The quantization step.
%
step = power(2,-bits);
for index_1 = 0:1:(power(2,bits)-1) % 0 < index1 < 2^bits-1
    L = index_1*step; % 0 < L < 1
    for index_2 = -(power(2,bits)-1):1:(power(2,bits)-1)
        % -2^bits-1 < index2 < 2^bits-1
        K = 2*index_2*step; % -2 < K < 2
        p = roots([1 -K L]); % finding roots
        if abs(p(1)) < 1 % testing root 1
            if (imag(p(1)) == 0) % real root
                plot(p(1),0,point); % plotting
            else % complex root
                plot(p(1),point); % plotting
            end
        end
        if abs(p(2)) < 1 % testing root 2
            if (imag(p(1)) == 0) % real root
                plot(p(2),0,point); % plotting
            else % complex root
                plot(p(2),point); % plotting
            end
        end
    end
end

% title(['L = ',num2str(L),' K = ',num2str(K)]);
% above 3 lines for debugging/demo purposes
% place break here and step thru to see poles for each L and K
    end
end

title(['Second order Pole distribution for ',num2str(bits+1),'
bits']);
hold off;

```

M12.2 The pole-distribution plot of the second-order direct form structure is obtained by running the MATLAB program given in Exercise M12.1. Running this program using the statement `pole_plot(4)` yields the following plot:



For plotting the pole distribution of the coupled form structure, we use the following MATLAB program:

```
% POLE_PLOT_COUPLED(B) obtains the pole distribution
% plot of a second order transfer function in coupled form
% with the transfer function  $rZ^2 / Z^2 - (a + d)Z + (ad - br)$ 
% For stability  $-1 < a, b, d, r < 1$ 
%
% for a B-bit wordlength, 1 bit is reserved for the sign
% as the coefficients are sorted in the sign magnitude form.
% Both L and the K are quantized to B-1 bits.
%
function [] = pole_plot_coupled(bits);
bits = 4;
point = '.';
%
% This part prints the unit circle for reference
%
% One bit is kept for the sign so effectively for the
% quantization, the remaining number of bits is bits-1
%
bits = bits-1;
%
% sets up the axis
zplane(2,2); % plotting queue of roots using zplane.m with a
% dummy zero and pole
axis([-1 1 -1 1]); % adjusting axis to "hide" dummy zero/pole,
% calling zplane without a zero is not legal in MATLAB release
% 12 and 13
title(['Second-order coupled form pole distribution for ',
num2str(bits+1), ' bits']);
hold on;
%
```



```

% The quantization step.
%
step = power(2,-bits);
for index_1 = -(power(2,bits)-1):1:(power(2,bits)-1)
    a = index_1*step;
    for index_2 = -(power(2,bits)-1):1:(power(2,bits)-1)
        % -2^bits-1 < index2 < 2^bits-1
        b = index_2*step;
        for index_3 = -(power(2,bits)-1):1:(power(2,bits)-1)
            d = index_3*step;
            for index_4 = -(power(2,bits)-1):1:(power(2,bits)-1)
                r = index_4*step;
                p = roots([1 -(a+d) (a*d-b*r)]);
                % finding roots
                if abs(p(1)) < 1 % testing root 1
                    if (imag(p(1)) == 0)
                        % real root
                        plot(p(1),0,point); % plotting
                    else % complex root
                        plot(p(1),point); % plotting
                    end
                end % testing root 2
                if (imag(p(1)) == 0) % real root
                    plot(p(2),0,point); % plotting
                else % complex root
                    plot(p(2),point); % plotting
                end
            end
        end
    end
end
end
% title(['a = ',num2str(a),' b = ',num2str(b),' d =
',num2str(d),' r = ',num2str(r)]);
end
end

title(['Second-order coupled form pole distribution for
',num2str(bits+1),' bits']);
hold off;

```

```

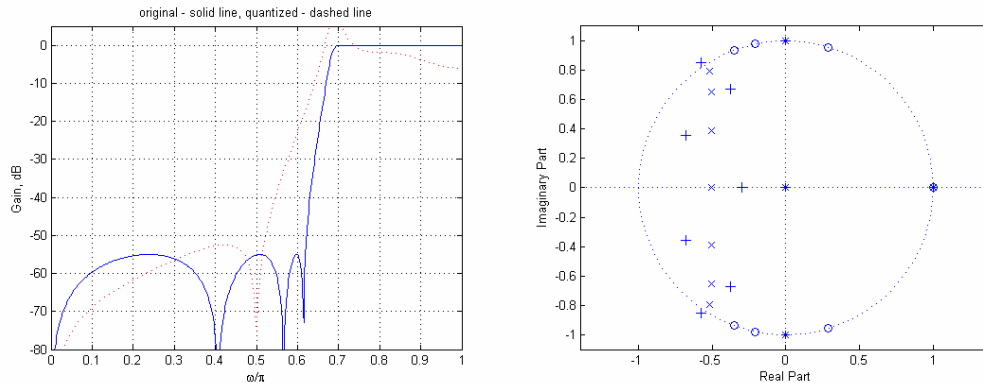
M12.3 % Modified Program 12_1
% Coefficient Quantization Effects on the frequency
% response of a direct form IIR filter
clf;
[b,a] = ellip(7,0.02,55,0.7,'high');
[h,w] = freqz(b,a,512);
g = 20*log10(abs(h));
% Truncate the filter coefficients to 5 bits
bq = a2dT(b,6); aq = a2dT(a,6);
[hq,w] = freqz(bq,aq,512);
gq = 20*log10(abs(hq));

```

```

plot(w/pi,g,'b',w/pi,gq,'r:'); grid; axis([0 1 -80 5]);
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('original - solid line, quantized - dashed line');
pause
% The call to the ZPLANE function below sets the axes
% properties to those corresponding to a pole-zero plot
zplane(b,a);
% The HOLD ON command retains the axes properties as
% they are and does not create new axes for the next plot
hold on;
plotzp(bq,aq)

```



M12.4 MATLAB code fragments are as follows:

```

num = [0.06891875 0.13808186 0.18636107 0.13808186 0.06891875];
den = [1 -1.30613249 1.48301305 -0.77709026 0.2361457];
nvar = filternorm(num,den)^2;
disp('Normalized Output Noise Variance = ');disp(nvar)

```

The computed noise variance is

```

Normalized Output Noise Variance =
0.4026

```

which is seen to be the same as that determined in Example 12.5..

M 12.5 MATLAB code is shown below:

```

N = 7; wn = 0.45; Rp = 0.2; Rs = 50;
[B,A] = ellip(N,Rp,Rs,wn);
zplane(B,A);
z = cplxpair(roots(B)); p = cplxpair(roots(A));
const = B(1)/A(1);
disp('Numerator Factors');
factor = factorize(B);disp(factor)
disp('Denominator Factors');
factor = factorize(A);disp(factor)
disp('Scale factor = ');disp(const)
sos = zp2sos(z,p,const)

```

The above program yields

Numerator Factors

| | | |
|--------|--------|--------|
| 1.0000 | 1.0000 | 0 |
| 1.0000 | 1.0500 | 1.0000 |
| 1.0000 | 0.2577 | 1.0000 |
| 1.0000 | 0.0072 | 1.0000 |

Denominator Factors

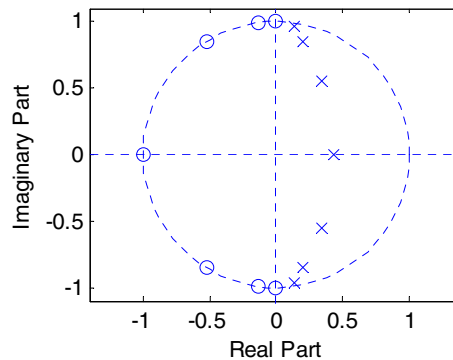
| | | |
|--------|---------|--------|
| 1.0000 | -0.2756 | 0.9379 |
| 1.0000 | -0.3951 | 0.7545 |
| 1.0000 | -0.6694 | 0.4245 |
| 1.0000 | -0.4353 | 0 |

Scale factor =

0.03485569511749

$$H(z) = 0.0349 \left(\frac{1 + 1.05z^{-1} + z^{-2}}{1 - 0.2756z^{-1} + 0.9379z^{-2}} \right) \left(\frac{1 + 0.2577z^{-1} + z^{-2}}{1 - 0.3951 + 0.7545z^{-2}} \right) \times \left(\frac{1 + 0.0072z^{-1} + z^{-2}}{1 - 0.6694 + 0.4245z^{-2}} \right) \left(\frac{1 + z^{-1}}{1 - 0.4353z^{-1}} \right).$$

Note that the ordering has no effect if \mathcal{L}_∞ -scaling is used.



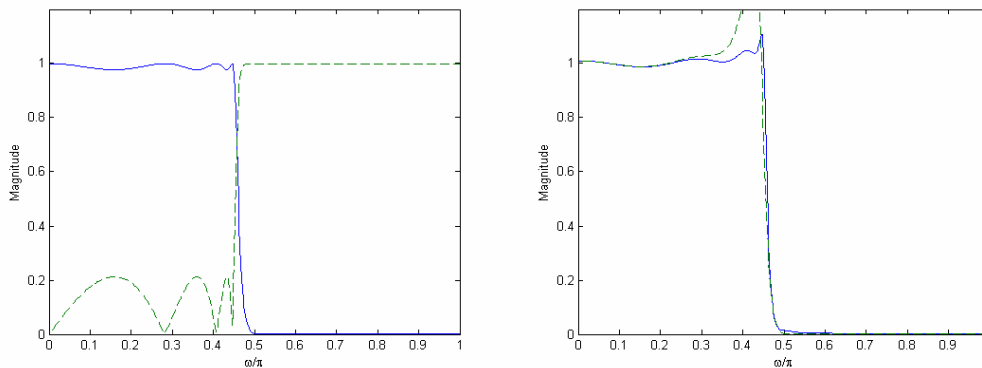
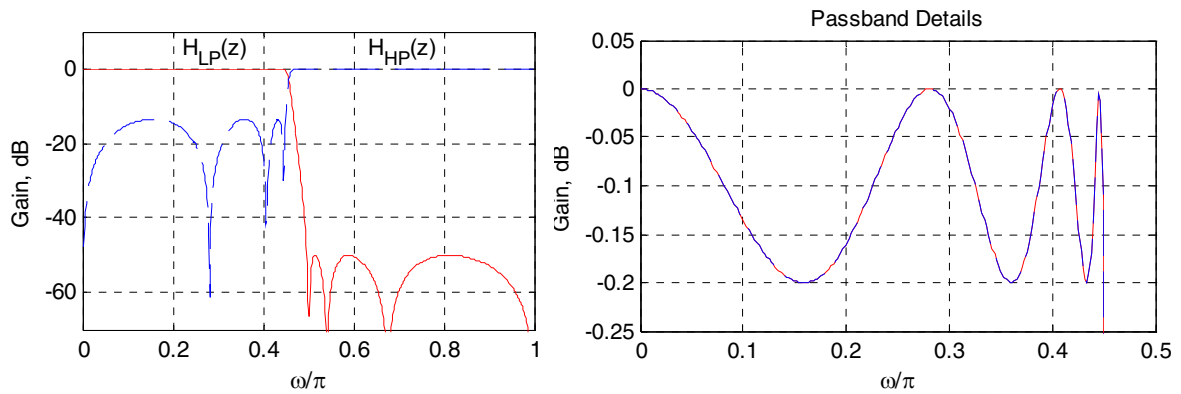
sos =

| | | | | | |
|--------|--------|--------|--------|---------|--------|
| 0.0349 | 0.0349 | 0 | 1.0000 | -0.4353 | 0 |
| 1.0000 | 1.0500 | 1.0000 | 1.0000 | -0.6694 | 0.4245 |
| 1.0000 | 0.2577 | 1.0000 | 1.0000 | -0.3951 | 0.7545 |
| 1.0000 | 0.0072 | 1.0000 | 1.0000 | -0.2756 | 0.9379 |

M 12.6 %Program_M12_6.m

```
[B,A] = ellip(7,0.2,50,0.45);
[d0,d1]=tf2ca(B,A);
num1 = 0.5*(conv(fliplr(d1),d0)+conv(d1,fliplr(d0)));
den = conv(d0,d1);
num2 = 0.5*(-conv(fliplr(d1),d0)+conv(d1,fliplr(d0)));
[h1,w]=freqz(num1,den,512);
[h2,w]=freqz(num2,den,512);
plot(w/pi,20*log10(abs(h1)),'-r',w/pi,20*log10(abs(h2)),'--b');
```

```
xlabel('\omega/\pi');ylabel('Gain, dB');grid
```



M12.7

```
num1 = input('Numerator first factor = ');
num2 = input('Numerator second factor = ');
den1 = input('Denominator first factor = ');
den2 = input('Denominator second factor = ');
```

```
% The scaling functions are
```

```
f1num = 1;
f1den = [den1];
f2num = num1;
f2den = conv(den1,den2);
f3num = conv(num1,num2);
f3den = conv(den1,den2);
x = [1 zeros([1,511])];
```

```
% Impulse Responses
```

```
% Sufficient length for impulse response to have
% decayed to nearly zero
```

```
f1 = filter(f1num,f1den,x);
f2 = filter(f2num,f2den,x);
f3 = filter(f3num,f3den,x);
figure(1),stem(f1(1:50));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_1(z)');
figure(2),stem(f2(1:50));
```

```

ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_2(z)')
figure(3), stem(f3(1:50));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of H(z)')

k1 = sum(f1.*f1);
k2 = sum(f2.*f2);
k3 = sum(f3.*f3);
disp(['L_2 norm of F_1(z) = ' num2str(k1)]);
disp(['L_2 norm of F_2(z) = ' num2str(k2)]);
disp(['L_2 norm of H(z)   = ' num2str(k3)]);

b_0 = 1/sqrt(k1);
b_1 = sqrt(k1)/sqrt(k2);
b_2 = sqrt(k2)/sqrt(k3);
disp(['First scaling factor, b_0 = ' num2str(b_0)]);
disp(['Second scaling factor, b_1 = ' num2str(b_1)]);
disp(['Third scaling factor, b_2 = ' num2str(b_2)]);

% The scaled transfer functions
disp(['Scaled gain = ' num2str(k1)]);
disp('The scaled second order sections are ');
disp([k2*num1 k3*num2 ; den1 den2])

% The noise transfer functions
g1num = conv(num1,num2)*(k2*k3);
g1den = conv(den1,den2)*k3;
g2num = num2;
g2den = den2;
g1 = filter(g1num,g1den,x);
g2 = filter(g2num,g2den,x);
var = sum(f1.*f1)*3+sum(g2.*g2)*5+3;
disp('The normalized noise variance'); disp(var);

```

For the first cascade realization we have

```

L_2 norm of F_1(z) = 4.9808
L_2 norm of F_2(z) = 88.2071
L_2 norm of H(z)   = 64.2444
First scaling factor, b_0 = 0.44807
Second scaling factor, b_1 = 0.23763
Third scaling factor, b_2 = 1.1717
Scaled gain = 4.9808
The scaled second order sections are
    88.2071  -66.9403   88.2071   64.2444   42.1700   64.2444
    1.0000   1.0462   0.8385   1.0000   1.0657   0.4046

The normalized noise variance
35.5077

```

For the second cascade realization we have

L₂ norm of F₁(z) = 2.8179

L₂ norm of F₂(z) = 11.375

L₂ norm of H(z) = 64.2444

First scaling factor, b₀ = 0.59572

Second scaling factor, b₁ = 0.49772

Third scaling factor, b₂ = 0.42078

Scaled gain = 2.8179

The scaled second order sections are

| | | | | | |
|---------|--------|---------|---------|----------|---------|
| 11.3750 | 7.4666 | 11.3750 | 64.2444 | -48.7551 | 64.2444 |
| 1.0000 | 1.0657 | 0.4046 | 1.0000 | 1.0462 | 0.8385 |

The normalized noise variance

106.5131

The L₂ norm of F₁(z) = 4.9808

The L₂ norm of F₂(z) = 11.375

The L₂ norm of H(z) = 64.2444

The first scaling factor, b₀ = 0.44807

The second scaling factor, b₁ = 0.66172

The third scaling factor, b₂ = 0.42078

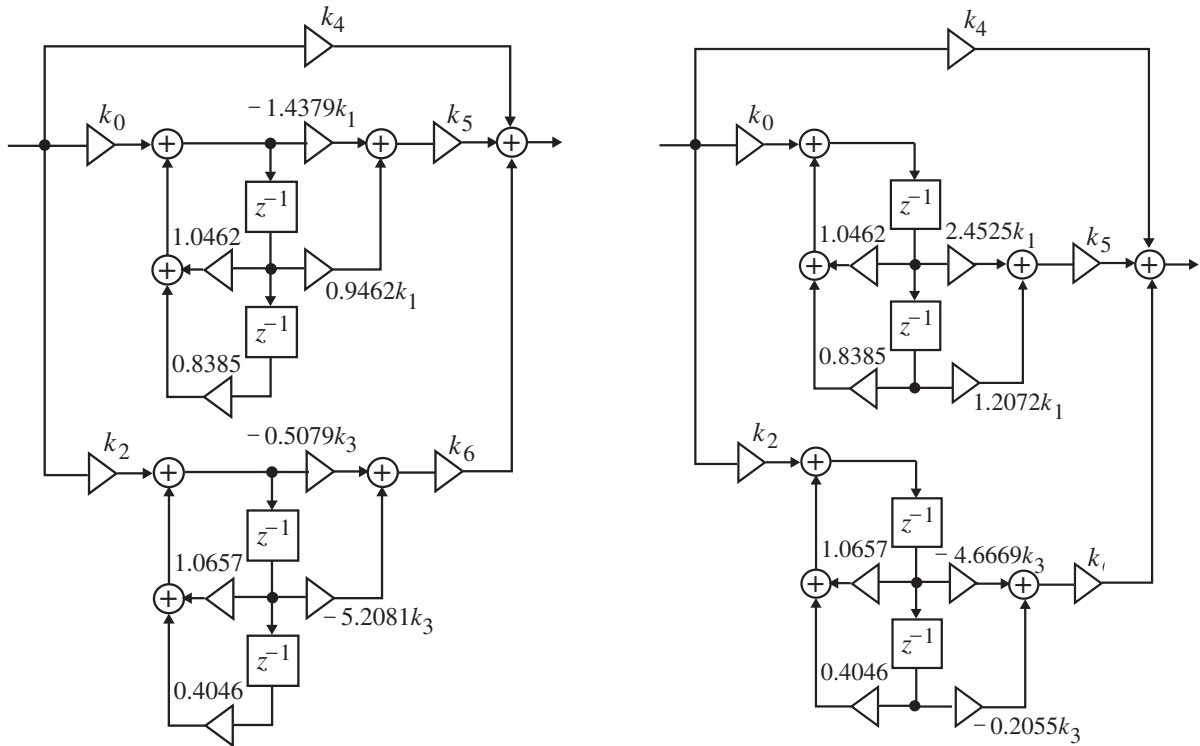
The scaled gain = 4.9808

The scaled second order sections are

| | | | | | |
|---------|--------|---------|---------|----------|---------|
| 11.3750 | 7.4666 | 11.3750 | 64.2444 | -48.7551 | 64.2444 |
| 1.0000 | 1.0462 | 0.8385 | 1.0000 | 1.0657 | 0.4046 |

The normalized noise variance = 98.0695

M12.8 The parallel form I and II structures used for the simulation are shown on top of the next page:



The MATLAB program below can be used to simulate a 4-th order IIR transfer function in both parallel forms.

```

num1 = input('Numerator first factor =');
num2 = input('Numerator Second factor =');
den1 = input('Denominator first factor =');
den2 = input('Denominator second factor =');

% Parallel Form I
num = conv(num1,num2);
den = conv(den1,den2);
[r1,p1,k11] = residuez(num,den);
R1 = [r1(1) r1(2)];
P1 = [p1(1) p1(2)];
R2 = [r1(3) r1(4)];
P2 = [p1(3) p1(4)];

[num11,den11] = residuez(R1,P1,0);
[num12,den12] = residuez(R2,P2,0);

disp('Parallel Form I');
disp('The numerators are');
disp(k11); disp(real(num11)); disp(real(num12));
disp('The denominators are');
disp(real(den11)); disp(real(den12));

imp = [1 zeros([1,2000])];

```

```

f0 = filter([1 0 0],den11,imp);
f1 = filter(num11,den11,imp);
f2 = filter([1 0 0],den12,imp);
f3 = filter(num12,den12,imp);
figure(1),stem(real(f0(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_0(z)')
saveas(gcf,'M12_8_P1_0.tif')
figure(2),stem(real(f1(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_1(z)')
saveas(gcf,'M12_8_P1_1.tif')
figure(3),stem(real(f2(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_2(z)')
saveas(gcf,'M12_8_P1_2.tif')
figure(4),stem(real(f3(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_3(z)')
saveas(gcf,'M12_8_P1_3.tif')

g0 = sum(f0.*conj(f0));
g1 = sum(f1.*conj(f1));
g2 = sum(f2.*conj(f2));
g3 = sum(f3.*conj(f3));
disp(['L_2 norm of F_0(z) = ' num2str(g0)]);
disp(['L_2 norm of F_1(z) = ' num2str(g1)]);
disp(['L_2 norm of F_2(z) = ' num2str(g2)]);
disp(['L_2 norm of F_3(z) = ' num2str(g3)]);

k0 = sqrt(1/g0);
k1 = sqrt(g0/g1);
k2 = sqrt(1/g2);
k3 = sqrt(g2/g3);
f = filter(num,den,imp);
g = sum(f.*conj(f));
k4 = sqrt(1/g);
k5 = k4/(k0*k1);
k6 = k4/(k2*k3);

disp('The scaling constants are');
disp(['k0 = ' num2str(k0)]); disp(['k1 = ' num2str(k1)]);
disp(['k2 = ' num2str(k2)]); disp(['k3 = ' num2str(k3)]);
disp(['k4 = ' num2str(k4)]); disp(['k5 = ' num2str(k5)]);
disp(['k6 = ' num2str(k6)]);

noise = 3*(k5/k0)^2+3*(k6/k2)^2+2*k5^2+2*k6^2+3;
disp(['Product roundoff noise variance = ' num2str(noise)]);

% Parallel From II

```



```

num = conv(num1,num2);
den = conv(den1,den2);
[r1,p1,k11] = residue(num,den);
R1 = [r1(1) r1(2)];
P1 = [p1(1) p1(2)];
R2 =[r1(3) r1(4)];
P2 = [p1(3) p1(4)];

[num11,den11] = residue(R1,P1,0);
[num12,den12] = residue(R2,P2,0);

disp('Parallel Form II');
disp('The numerators are');
disp(k11); disp(real(num11)); disp(real(num12));
disp('The denominators are');
disp(real(den11)); disp(real(den12));

imp = [1 zeros([1,2000])];
f0 = filter([1 0 0],den11,imp);
f1 = filter(num11,den11,imp);
f2 = filter([1 0 0],den12,imp);
f3 = filter(num12,den12,imp);
figure(1),stem(real(f0(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_0(z)')
saveas(gcf,'M12_8_P2_0.tif')
figure(2),stem(real(f1(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_1(z)')
saveas(gcf,'M12_8_P2_1.tif')
figure(3),stem(real(f2(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_2(z)')
saveas(gcf,'M12_8_P2_2.tif')
figure(4),stem(real(f3(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of F_3(z)')
saveas(gcf,'M12_8_P2_3.tif')

g0 = sum(f0.*conj(f0));
g1 = sum(f1.*conj(f1));
g2 = sum(f2.*conj(f2));
g3 = sum(f3.*conj(f3));
disp(['L_2 norm of F_0(z) = ' num2str(g0)]);
disp(['L_2 norm of F_1(z) = ' num2str(g1)]);
disp(['L_2 norm of F_2(z) = ' num2str(g2)]);
disp(['L_2 norm of F_3(z) = ' num2str(g3)]);

k0 = sqrt(1/g0);
k1 = sqrt(g0/g1);
k2 = sqrt(1/g2);

```

```

k3 = sqrt(g2/g3);
f = filter(num,den,imp);
g = sum(f.*conj(f));
k4 = sqrt(1/g);
k5 = k4/(k0*k1);
k6 = k4/(k2*k3);
disp('The scaling constants are');
disp(['k0 = ' num2str(k0)]); disp(['k1 = ' num2str(k1)]);
disp(['k2 = ' num2str(k2)]); disp(['k3 = ' num2str(k3)]);
disp(['k4 = ' num2str(k4)]); disp(['k5 = ' num2str(k5)]);
disp(['k6 = ' num2str(k6)]);

noise = 3*(k5/k0)^2+3*(k6/k2)^2+2*k5^2+2*k6^2+3;
disp(['Product roundoff noise variance = ' num2str(noise)]);

```

Parallel Form I

The numerators are

| | | | |
|---------|---------|--|---|
| 2.9476 | | | |
| -1.4397 | 0.9462 | | 0 |
| -0.5079 | -5.2081 | | 0 |

The denominators are

| | | |
|--------|--------|--------|
| 1.0000 | 1.0462 | 0.8385 |
| 1.0000 | 1.0657 | 0.4046 |

L₂ norm of F₀(z) = 4.9808
L₂ norm of F₁(z) = 22.5061
L₂ norm of F₂(z) = 2.8179
L₂ norm of F₃(z) = 65.8497

The scaling constants are

k0 = 0.44807
k1 = 0.47044
k2 = 0.59572
k3 = 0.20686
k4 = 0.12476
k5 = 0.59188
k6 = 1.0124

Product roundoff noise variance = 19.6501

Parallel Form II

The numerators are

| | | |
|---------|--------|--|
| 1 | | |
| 2.4525 | 1.2072 | |
| -4.6669 | 0.2055 | |

The denominators are

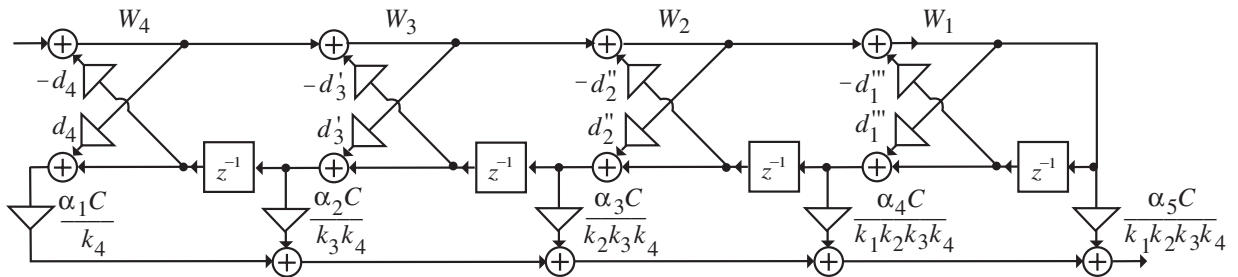
| | | |
|--------|--------|--------|
| 1.0000 | 1.0462 | 0.8385 |
| 1.0000 | 1.0657 | 0.4046 |

L₂ norm of F₀(z) = 4.9808
L₂ norm of F₁(z) = 20.4333

L_2 norm of $F_2(z) = 2.8179$
 L_2 norm of $F_3(z) = 65.5918$
 The scaling constants are
 $k_0 = 0.44807$
 $k_1 = 0.49372$
 $k_2 = 0.59572$
 $k_3 = 0.20727$
 $k_4 = 0.12476$
 $k_5 = 0.56396$
 $k_6 = 1.0104$
 Product roundoff noise variance = 19.0615

Parallel Form II has the lowest output round off noise.

M 12.9 The scaled Gray-Markel cascaded lattice structure used for simulation is shown below:



The MATLAB program that can be used to simulate the Gray-Markel realization of a 4-th order IIR transfer function is as follows:

```

num1 = input('Numerator first factor = ');
num2 = input('Numerator second factor = ');
den1 = input('Denominator first factor = ');
den2 = input('Denominator second factor = ');

num = conv(num1,num2); den = conv(den1,den2);
num = num/den(1); den = den/den(1);

% Gray-Markel realization
[d,alpha] = tf2latc(num,den);
disp('Lattice parameters are'); disp(['d1 = ' num2str(d(1))]);
disp(['d2 = ' num2str(d(2))]); disp(['d3 = ' num2str(d(3))]);
disp(['d4 = ' num2str(d(4))]);

disp('Feed-forward multipliers are');
disp(['alpha1 = ' num2str(alpha(1))]);
disp(['alpha2 = ' num2str(alpha(2))]);
disp(['alpha3 = ' num2str(alpha(3))]);
disp(['alpha4 = ' num2str(alpha(4))]);

imp = [1 zeros([1,499])];
qold1 = 0;
for k = 1:500
    w1 = imp(k)-d(1)*qold1;
  
```

```

        y1(k) = w1;
        qnew1 = w1;
        qold1 = qnew1;
end
k1 = sqrt(1/(sum(y1.*conj(y1))));
figure(1),stem(real(y1(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of W1')
saveas(gcf,'M12_9_1.tif')

imp = [1 zeros([1,499])];
qold1 = 0;qold2 = 0;
for k = 1:500
    w2 = imp(k)-d(2)*qold2*1/k1;
    w1 = k1*w2-d(1)*qold1;
    y2(k)=w2;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qold1 = qnew1;qold2 = qnew2;
end
k2 = sqrt(1/(sum(y2.*conj(y2))));
figure(2),stem(real(y2(1:50)));
ylabel('Amplitude'); xlabel('Samples');
title('Impulse Response of W2')
saveas(gcf,'M12_9_2.tif')

qold1 = 0;qold2 = 0;qold3 = 0;
for k = 1:500
    w3 = imp(k)-d(3)*qold3*1/k2;
    w2 = k2*w3-d(2)*qold2*1/k1;
    w1 = k1*w2-d(1)*qold1;
    y3(k) = w3;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qnew3 = w2*d(2)+qold2*1/k1;
    qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;
end
k3 = sqrt(1/sum(y3.*conj(y3)));
figure(3),stem(real(y3(1:50))); ylabel('Amplitude');
xlabel('Samples'); title('Impulse Response of W3')
saveas(gcf,'M12_9_3.tif')

qold1 = 0;qold2 = 0;qold3 = 0;qold4 = 0;
for k = 1:500
    w4 = imp(k)-d(4)*qold4/k3;
    w3 = k3*w4-d(3)*qold3/k2;
    w2 = k2*w3-d(2)*qold2/k1;
    w1 = k1*w2-d(1)*qold1;
    y4(k) = w4;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qnew3 = w2*d(2)+qold2*1/k1;
    qnew4 = w3*d(3)+qold3*1/k2;
    qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;qold4 = qnew4;
end

```

```

k4 = sqrt(1/sum(y4.*conj(y4)));
figure(4),stem(real(y3(1:50))); ylabel('Amplitude');
xlabel('Samples'); title('Impulse Response of W4')
saveas(gcf,'M12_9_4.tif')

disp('The scaling parameters are');
disp(['k1 = ' num2str(k1)]); disp(['k2 = ' num2str(k2)]);
disp(['k3 = ' num2str(k3)]); disp(['k4 = ' num2str(k4)]);
alpha(5) = alpha(5)/(k1*k2*k3*k4);
alpha(4) = alpha(4)/(k1*k2*k3*k4);
alpha(3) = alpha(3)/(k2*k3*k4);
alpha(2) = alpha(2)/(k3*k4);
alpha(1) = alpha(1)/k4;

%%% Computation of noise variance %%%
imp = [1 zeros([1,499])];
for k = 1:500
    w4 = imp(k)-d(4)*qold4/k3;
    w3 = k3*w4-d(3)*qold3/k2;
    w2 = k2*w3-d(2)*qold2/k1;
    w1 = k1*w2-d(1)*qold1;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qnew3 = w2*d(2)+qold2*1/k1;
    qnew4 = w3*d(3)+qold3*1/k2;
    y11 = w4*d(4)+qold4/k3;
    y0(k) = alpha(1)*y11+alpha(2)*qnew4+alpha(3)*qnew3 +
alpha(4)*qnew2+alpha(5)*qnew1;
    qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;qold4 = qnew4;
end
nv = sum(y0.*conj(y0));
disp(['Product roundoff noise variance = ' num2str(nv)]);

```

The output data generated by this program are as follows:

Lattice parameters are

```

d1 = 0.19149
d2 = 0.75953
d3 = 0.44349
d4 = 0.27506

```

Feed-forward multipliers are

```

alpha1 = -0.9001
alpha2 = -1.7069
alpha3 = 4.6151
alpha4 = -3.5028
alpha5 = 1.8177

```

The scaling parameters are

```

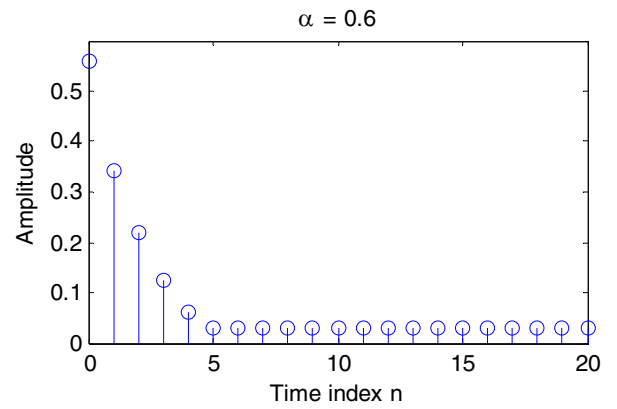
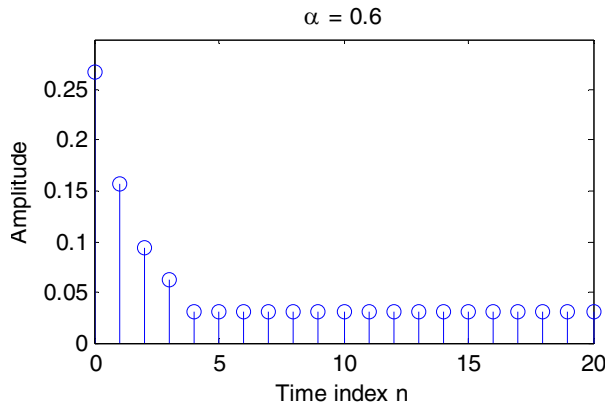
k1 = 0.98149
k2 = 0.65047
k3 = 0.89628
k4 = 0.96143

```

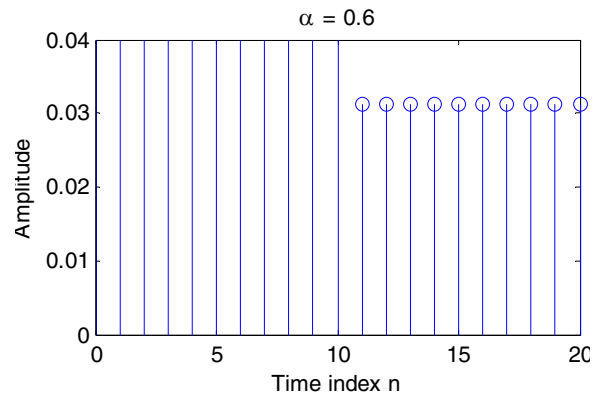
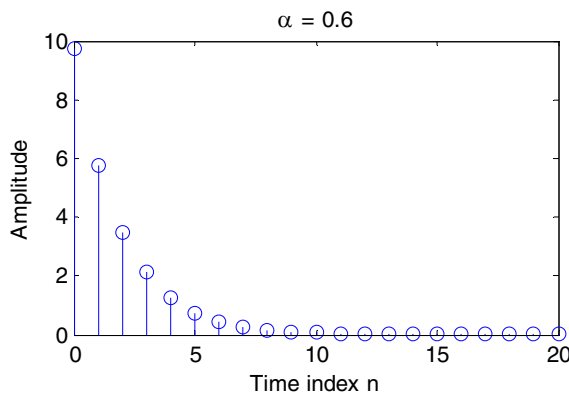
The product roundoff noise variance = 81.1431

M12.10 (a) $y[-1] = 0.3, x[0] = 0.08$

(b) $y[-1] = 0.9, x[0] = 0.03$

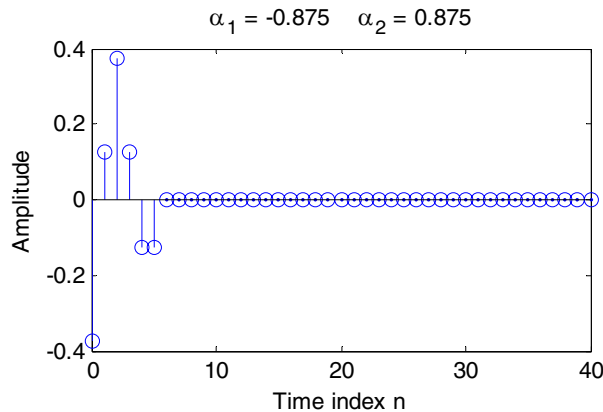


(c) $y[-1] = 8, x[0] = 5$



In all three cases, the condition of Eq. (12.187) is satisfied and hence, the structure exhibits zero input granular limit cycles.

M 12.11 Plot of the output samples generated by the modified program is shown below:



As can be seen from the above plot, output goes to zero values and hence the structure of Figure 12.49 does not support overflow limit cycles if sign-magnitude truncation is used to truncate the sum of products of Eq. (12.195).

Chapter 13

13.1 Up-sampler – Let $x_1[n]$ and $x_2[n]$ be the inputs to a factor-of- L up-sampler with corresponding outputs given by $y_1[n]$ and $y_2[n]$, respectively:

$$y_1[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad y_2[n] = \begin{cases} x_2[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x_3[n] = \alpha x_1[n] + \beta x_2[n]$ be the input to the up-sampler. Then, the corresponding

$$\text{output } y_3[n] \text{ is given by } y_3[n] = \begin{cases} x_3[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \alpha x_1[n/L] + \beta x_2[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \alpha x_1[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases} + \begin{cases} \beta x_2[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \alpha y_1[n] + \beta y_2[n]. \text{ Hence, the up-sampler is a linear system.}$$

Down-sampler - Let $x_1[n]$ and $x_2[n]$ be the inputs to a factor-of- M down-sampler with corresponding outputs given by $y_1[n]$ and $y_2[n]$, respectively:

$$y_1[n] = x_1[Mn] \text{ and } y_2[n] = x_2[Mn]. \text{ Let } x_3[n] = \alpha x_1[n] + \beta x_2[n] \text{ be the input to the}$$

up-sampler. Then, the corresponding output $y_3[n]$ is given by $y_3[n] = x_3[Mn]$

$$= \alpha x_1[Mn] + \beta x_2[Mn] = \alpha y_1[n] + \beta y_2[n]. \text{ Hence, the down-sampler is a linear system.}$$

13.2 Up-Sampler – For inputs $x_1[n]$ and $x_2[n]$, the outputs of the factor-of- L up-sampler are, respectively given by

$$x_{u1}[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad x_{u2}[n] = \begin{cases} x_2[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

Let $x_2[n] = x_1[n - n_o]$, where n_o is an integer. Then, $x_2[n/L] = x_1[(n/L) - n_o]$. Hence,

$$x_{u2}[n] = \begin{cases} x_1[(n - n_o)/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise.} \end{cases} \quad \text{But,}$$

$$x_{u1}[n] = \begin{cases} x_1[(n - n_o)/L], & n - n_o = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Since $x_{u2}[n] \neq x_{u1}[n - n_o]$, the up-sampler is a time-varying system.

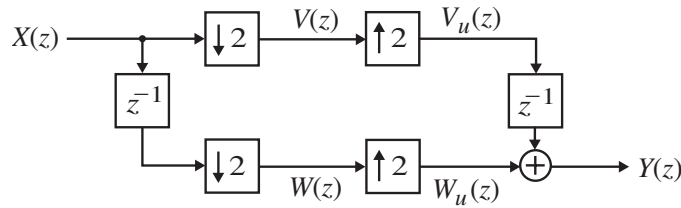
Down-Sampler – For inputs $x_1[n]$ and $x_2[n]$, the outputs of the factor-of- M down-sampler are, respectively given by $y_1[n] = x_1[Mn]$ and $y_2[n] = x_2[Mn]$. Let

$$x_2[n] = x_1[n - n_o], \text{ where } n_o \text{ is an integer. Then, } y_2[n] = x_2[Mn] = x_1[Mn - n_o].$$

However, $y_1[n - n_o] = x_1[M(n - n_o)] = x_1[Mn - Mn_o]$.

Since $y_2[n] \neq y_1[n - n_o]$, the down-sampler is a time-varying system.

13.3



Analysis yields $V(z) = \frac{1}{2} X(z^{1/2}) + \frac{1}{2} X(-z^{1/2})$,

$$W(z) = \frac{z^{-1/2}}{2} X(z^{1/2}) - \frac{z^{-1/2}}{2} X(-z^{1/2}), \quad V_u(z) = \frac{1}{2} X(z) + \frac{1}{2} X(-z),$$

$W_u(z) = \frac{z^{-1}}{2} X(z) - \frac{z^{-1}}{2} X(-z)$. Hence, $Y(z) = z^{-1}V_u(z) + W_u(z) = z^{-1}X(z)$, or in other words $y[n] = x[n-1]$.

13.4 $c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn} = \frac{1}{M} \left(\frac{1 - W_M^{nM}}{1 - W_M^n} \right)$. Hence, if $n \neq rM$, $c[n] = \frac{1}{M} \left(\frac{1-1}{1 - W_M^n} \right) = 0$. On

the other hand, if $n = rM$, then $c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn} = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{krM} = \frac{1}{M} \sum_{k=0}^{M-1} 1 = \frac{M}{M} = 1$.

13.5 (a) For $M = 6$ and $L = 5$,

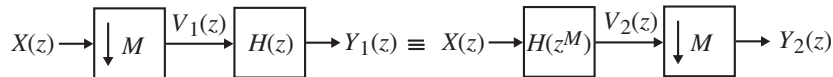
$$\{W_M^{-k}\} = \{W_6^{-k}\} = \{W_6^0, W_6^{-1}, W_6^{-2}, W_6^{-3}, W_6^{-4}, W_6^{-5}\}, \text{ and}$$

$$\{W_M^{-Lk}\} = \{W_6^{-5k}\} = \{W_6^0, W_6^{-5}, W_6^{-10}, W_6^{-15}, W_6^{-20}, W_6^{-25}\}$$

$$= \{W_6^0, W_6^{-5}, W_6^{-4}, W_6^{-3}, W_6^{-2}, W_6^{-1}\} = \{W_6^{-k}\}.$$

(b) For $\{W_M^{-k}\}$ to have same set of values for $0 \leq k \leq M-1$, as $\{W_M^{-kL}\}$, each should have unique values for each k . Therefore $W_M^{-kL} \neq W_M^{-nL}$ for all $k, n \in [0, M-1]$, or $(k-n)L \neq rM$ for any positive integer r , which implies that L and M should be relatively prime.

13.6



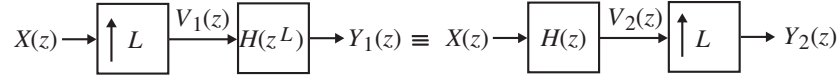
For the figure on the left-hand side we have $V_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^k)$, and

$$Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z) X(z^{1/M} W_M^k).$$

For the figure on the right-hand side we have

$$V_2(z) = H(z^M)X(z), Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(zW_M^{kM})X(z^{1/M}W_M^k) = \frac{1}{M} \sum_{k=0}^{M-1} H(z)X(z^{1/M}W_M^k).$$

Hence, $Y_1(z) = Y_2(z)$.



For the figure on the left-hand side we have $V_1(z) = X(z^L)$, $Y_1(z) = H(z^L)X(z^L)$. For the figure on the right-hand side we have $V_2(z) = H(z)X(z)$, $Y_2(z) = H(z^L)X(z^L)$. Hence, $Y_1(z) = Y_2(z)$.

13.7 (a) The system of Figure P13.1 with internal variables labeled is shown below:



Analysis yields $V(z) = X(z^L)$, $U(z) = G(z)V(z)$, and $Y(z) = \frac{1}{L} \sum_{k=0}^{L-1} U(z^{1/L}W_L^{-k})$.

Substituting the first equation in the second equation we get $U(z) = G(z)X(z^L)$.

Substituting this equation in the expression for $Y(z)$ in the above we get

$$Y(z) = \frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L} \cdot W_L^{-k})X(z \cdot W_L^{-kL}) = \frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L} \cdot W_L^{-k})X(z), \text{ since } W_L^{-kL} = 1.$$

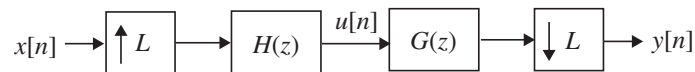
Therefore, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L}W_L^{-k})$. Hence, Figure P13.1 is a LTI system.

(b) It follows from the last equation given above, if $\frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L}W_L^{-k}) = 1$, then

$H(z) = 1$, i.e., $Y(z) = X(z)$, or $y[n] = x[n]$. Or, in other words, the system of Figure

P13.1 is an identity system for $\frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L}W_L^{-k}) = 1$.

13.8 Consider the multirate structure shown below. Analysis yields

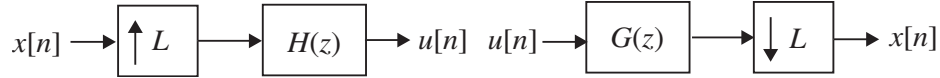


$$U(z) = H(z)X(z^L), \text{ and } Y(z) = \frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L}W_L^k)U(z^{1/L}W_L^k). \text{ Substituting the first}$$

equation into the second we get $Y(z) = \left[\frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L} W_L^k) H(z^{1/L} W_L^k) \right] X(z)$. Hence,

if $\frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L} W_L^k) H(z^{1/L} W_L^k) = 1$, we have $Y(z) = X(z)$, or in other words, the

above multirate structure is an identity system. We break the system as shown below into two parts with the transfer functions and satisfying the relation

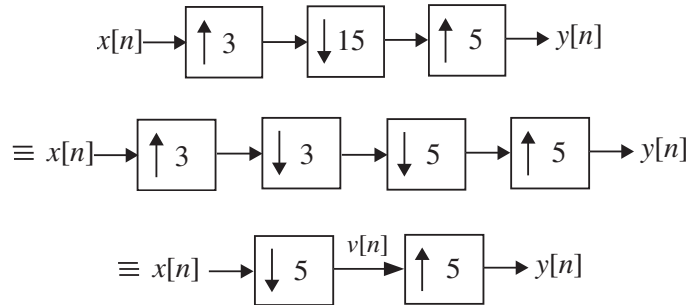


$\frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L} W_L^k) H(z^{1/L} W_L^k) = 1$. If we now place the second system in front of the

first we arrive at the system shown in Figure P13.2 which is an identity system

provided $\frac{1}{L} \sum_{k=0}^{L-1} G(z^{1/L} W_L^k) H(z^{1/L} W_L^k) = 1$.

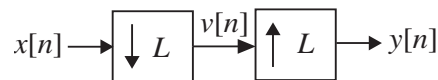
13.9 Making use of the multirate identities we simplify the structure of Figure P13.3 as indicated below:



Analysis of the last structure yields, $v[n] = x[5n]$, and

$$y[n] = \begin{cases} v[n/5], & n = 0, \pm 5, \pm 10, \dots, \\ 0, & \text{otherwise,} \end{cases} \text{ or, } y[n] = \begin{cases} x[n], & n = 0, \pm 5, \pm 10, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

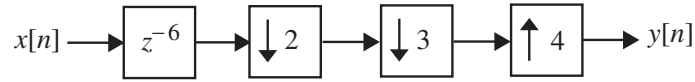
13.10



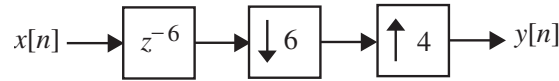
Analysis yields $v[n] = x[Ln]$, and $y[n] = \begin{cases} v[n/L], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise,} \end{cases}$ or,

$$y[n] = \begin{cases} x[n], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

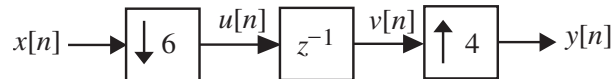
13.11 Since 3 and 4 are relatively prime, we can interchange the positions of the factor-of-3 down-sampler and the factor-of-4 up-sampler as indicated below:



which simplifies to the structure shown below:



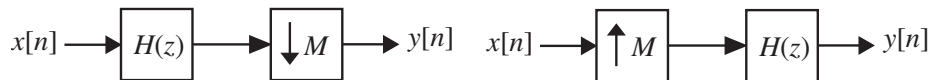
Using the Noble identity of Figure 13.14(a) we redraw the above structure as indicated below:



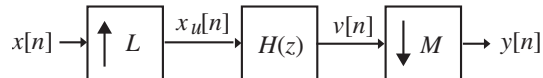
Analysis yields $u[n] = x[6n]$, $v[n] = u[n-1] = x[6n-1]$, and

$$y[n] = \begin{cases} v[n/4], & n = 0, \pm 4, \pm 8, \dots, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} x[(3n/2) - 1], & n = 0, \pm 4, \pm 8, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

13.12 As outlined in Section 8.2, the transpose of a digital filter structure is obtained by reversing all paths, replacing the pick-off node with an adder and vice-versa, and interchanging the input and the output nodes. Moreover, in a multirate structure, the transpose of a factor-of- M down-sampler is a factor-of- M up-sampler and vice-versa. Applying these operations to the factor-of- M decimator shown below on the left-hand side, we arrive at a factor-of- M up-sampler shown below on the right-hand side.



13.13 (a) To prove Eq. (13.20), consider the fractional-rate sampling rate converter of Figure 13.16(b) with internal variables labeled as shown below:



Analysis yields $x_u[Ln] = x[n]$, $n = 0, \pm 1, \pm 2, \dots$ and $v[n] = \sum_{\ell=-\infty}^{\infty} h[n-\ell]x_u[\ell]$.

Substituting the first equation in the second we get $v[n] = \sum_{m=-\infty}^{\infty} h[n-Lm]x[m]$. Finally,

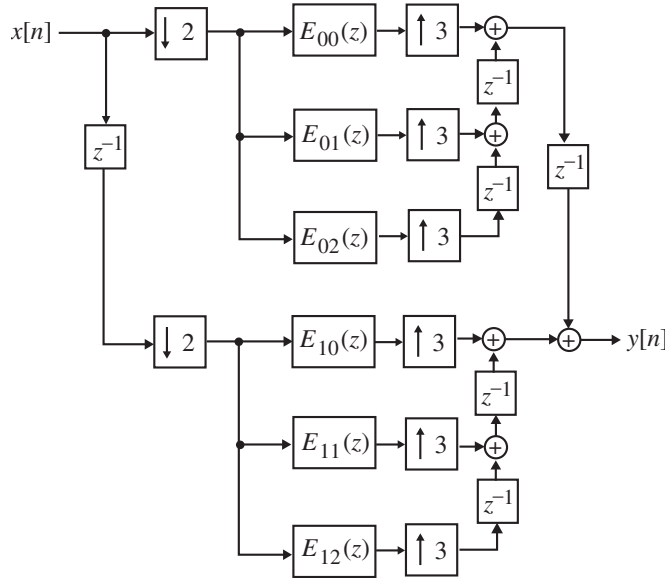
$$y[n] = v[Mn] = \sum_{m=-\infty}^{\infty} h[Mn-Lm]x[m].$$

(b) Next, to prove Eq. (13.21), we make use of the z-domain relations of the down-sampler and the up-sampler. From Eq. (13.17) we have

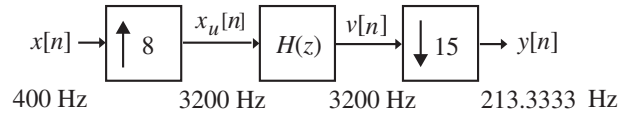
$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z^{1/M} W_M^{-k}) X_u(z^{1/M} W_M^{-k}). \text{ But } X_u(z) = X(z^L). \text{ Hence,}$$

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z^{1/M} W_M^{-k}) X(z^{L/M} W_M^{-Lk}).$$

13.14



13.15

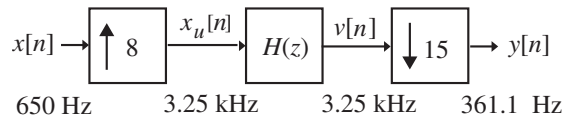


(a) $F_T = 400$ Hz, $L = 8$, $M = 15$. Now, the sampling rate of $x_u[n]$ and $v[n]$ is $400L = 400 \times 8 = 3.2$ kHz. Hence, the sampling rate of $y[n]$ is $3200/M = 3200/15 = 213.33$ Hz.

(b) The normalized stopband edge angular frequency of $H(z)$ (for no aliasing)

$$\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right) = \frac{\pi}{M} = \frac{\pi}{15}. \text{ Hence, the stopband edge frequency is } F_s = \frac{1}{30} \text{ Hz.}$$

13.16

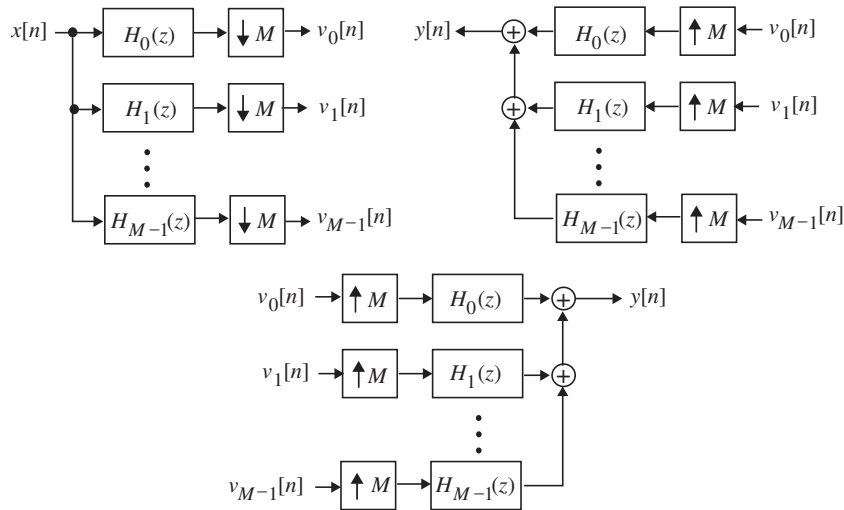


(a) $F_T = 650$ Hz, $L = 5$, $M = 9$. Now, the sampling rate of $x_u[n]$ and $v[n]$ is $650L = 650 \times 5 = 3.25$ kHz. Hence, the sampling rate of $y[n]$ is $3250/M = 3250/9 = 361.1$ Hz.

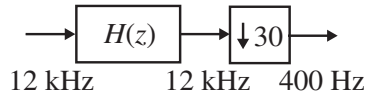
(b) The normalized stopband edge angular frequency of $H(z)$ (for no aliasing)

$$\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right) = \frac{\pi}{M} = \frac{\pi}{9}. \text{ Hence, the stopband edge frequency is } \frac{\pi}{9} \text{ Hz.}$$

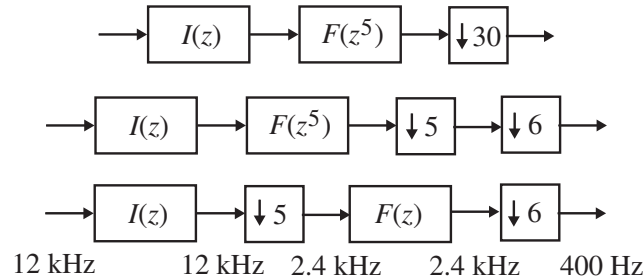
13.17 Applying the transpose operation to the M-channel analysis filter bank shown below on the left-hand side, we arrive at the M-channel synthesis filter bank shown below on the right-hand side.



13.18 Specifications for $H(z)$ are: $F_p = 180$ Hz, $F_s = 200$ Hz, $\delta_p = 0.002$, $\delta_s = 0.001$.



$$H(z) = I(z)F(z^5)$$



Specifications for $F(z)$ are: $F_p = 900$ Hz, $F_s = 1000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.001$.

Here $\Delta f = \frac{100}{12000}$. Using Eq. (10.3) we arrive at the order of $F(z)$ given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.001} - 13}{14.6 \left(\frac{100}{12000} \right)} = 386.3 \Rightarrow N_F = 387$$

Specifications for $I(z)$ are: $F_p = 180$ Hz, $F_s = 2200$ Hz, $\delta_p = 0.001$, $\delta_s = 0.001$.

Here $\Delta f = \frac{2020}{12000}$. Using Eq. (10.3) we arrive at the order of $I(z)$ given by

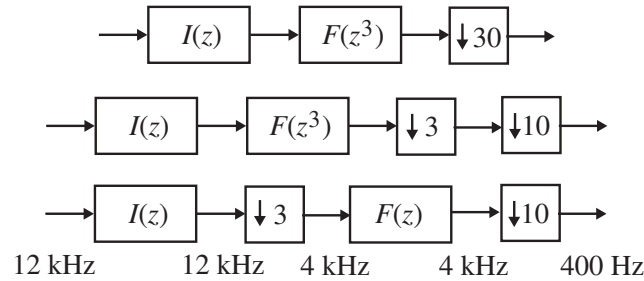
$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.001} - 13}{14.6 \left(\frac{2020}{12000} \right)} = 19.12 \Rightarrow N_I = 207$$

Hence, $\mathcal{R}_{M,F} = (387 + 1) \frac{2400}{6} = 155,200$ mps, $\mathcal{R}_{M,I} = (20 + 1) \frac{12000}{5} = 50,400$ mps.

The above realization requires a total of 205,600 mps. As a result, the computational complexity is slightly higher than in Example 13.10.

13.19 Specifications for $H(z)$ are: $F_p = 180$ Hz, $F_s = 200$ Hz, $\delta_p = 0.002$, $\delta_s = 0.001$.

$$H(z) = I(z)F(z^3)$$



Specifications for $F(z)$ are: $F_p = 540$ Hz, $F_s = 600$ Hz, $\delta_p = 0.001$, $\delta_s = 0.001$.

Here $\Delta f = \frac{60}{12000}$. Using Eq. (10.3) we arrive at the order of $F(z)$ given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.001} - 13}{14.6 \left(\frac{60}{12000} \right)} = 643.8 \Rightarrow N_F = 644$$

Specifications for $I(z)$ are: $F_p = 180$ Hz, $F_s = 3800$ Hz, $\delta_p = 0.001$, $\delta_s = 0.001$.

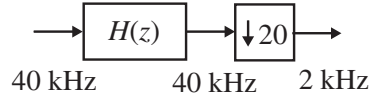
Here $\Delta f = \frac{3260}{12000}$. Using Eq. (10.3) we arrive at the order of $I(z)$ given by

$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.001} - 13}{14.6 \left(\frac{3260}{12000} \right)} = 11.85 \Rightarrow N_I = 12$$

Hence, $\mathcal{R}_{M,F} = (644 + 1)\frac{4000}{10} = 262,000$ mps, $\mathcal{R}_{M,I} = (12 + 1)\frac{12000}{3} = 52,000$ mps.

The above realization requires a total of 314,000 mps. As a result, the computational complexity is higher than in Example 13.10 and that in Problem 13.18.

13.20 (a) The desired down-sampling factor is $M = \frac{40000}{2,000} = 20$. The general structure of the desired decimator is thus as shown below:



Now, the normalized stopband edge angular frequency of a factor-of-20 decimator is $\omega_s = \frac{\pi}{20}$. Hence, the desired stopband edge frequency in this case is

$F_s = \frac{20000}{20} = 1000$ Hz. The specifications of the decimation filter $H(z)$ is thus as

follows: $F_p = 800$ Hz, $F_s = 1000$ Hz, $\delta_p = 0.002$, $\delta_s = 0.002$. Here $\Delta f = \frac{200}{40000}$.

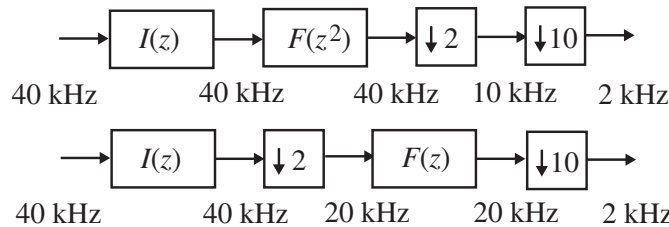
Using Eq. (10.3) we arrive at the order of $H(z)$ given by

$$N_H = \frac{-20 \log_{10} \sqrt{0.002 \times 0.002} - 13}{14.6 \left(\frac{200}{40000} \right)} = 561.4 \Rightarrow 562. \text{ Therefore, the computational}$$

complexity is given by $\mathcal{R}_{M,H} = (562 + 1)\frac{40000}{20} = 1126,000$ mps.

(b) For a two-stage realization of the decimator, there are 4 possible realizations of the decimation filter:

Realization #1 – $H(z) = F(z^2)I(z)$



Specifications for $F(z)$ and $I(z)$ are as follows:

$F(z)$: $F_p = 1600$ Hz, $F_s = 2000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus, $\Delta f = \frac{400}{40000}$.

$I(z)$: $F_p = 800$ Hz, $F_s = 19,000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus, $\Delta f = \frac{18200}{40000}$.

Orders of $F(z)$ and $I(z)$ are given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{400}{40000} \right)} = 301.29 \Rightarrow 302$$

$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{18200}{40000} \right)} = 6.62 \Rightarrow 7$$

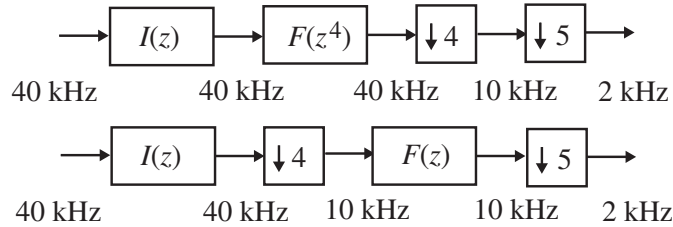
Computational complexities of the two sections are:

$$\mathcal{R}_{M,F} = (302 + 1) \frac{20000}{10} = 606,000 \text{ mps and } \mathcal{R}_{M,I} = (7 + 1) \frac{40000}{2} = 160,000 \text{ mps.}$$

Hence, the total computational complexity of the two-stage realization is

$$\mathcal{R}_M = \mathcal{R}_{M,F} + \mathcal{R}_{M,I} = 766,000 \text{ mps.}$$

Realization #2 – $H(z) = F(z^4)I(z)$



Specifications for $F(z)$ and $I(z)$ are as follows:

$$F(z): F_p = 3200 \text{ Hz, } F_s = 4000 \text{ Hz, } \delta_p = 0.001, \delta_s = 0.002, \text{ and thus, } \Delta f = \frac{800}{40000}.$$

$$I(z): F_p = 800 \text{ Hz, } F_s = 9,000 \text{ Hz, } \delta_p = 0.001, \delta_s = 0.002, \text{ and thus, } \Delta f = \frac{8200}{40000}.$$

Orders of $F(z)$ and $I(z)$ are given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{8000}{40000} \right)} = 150.65 \Rightarrow 151$$

$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{8200}{40000} \right)} = 14.69 \Rightarrow 15$$

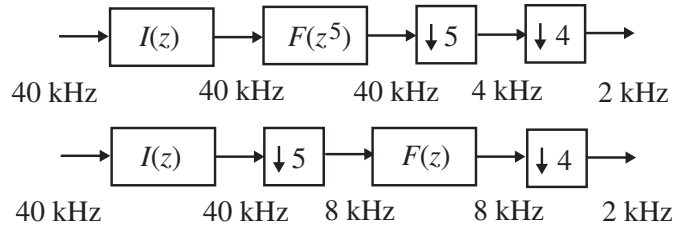
Computational complexities of the two sections are:

$$\mathcal{R}_{M,F} = (151 + 1) \frac{10000}{5} = 304,000 \text{ mps and } \mathcal{R}_{M,I} = (15 + 1) \frac{40000}{4} = 160,000 \text{ mps.}$$

Hence, the total computational complexity of the two-stage realization is

$$\mathcal{R}_M = \mathcal{R}_{M,F} + \mathcal{R}_{M,I} = 464,000 \text{ mps.}$$

Realization #3 – $H(z) = F(z^5)I(z)$



Specifications for $F(z)$ and $I(z)$ are as follows:

$F(z)$: $F_p = 4000$ Hz, $F_s = 5000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus,

$$\Delta f = \frac{1000}{40000}.$$

$I(z)$: $F_p = 800$ Hz, $F_s = 7,000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus, $\Delta f = \frac{6200}{40000}$.

Orders of $F(z)$ and $I(z)$ are given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{1000}{40000} \right)} = 120.52 \Rightarrow 121$$

$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{6200}{40000} \right)} = 19.44 \Rightarrow 20$$

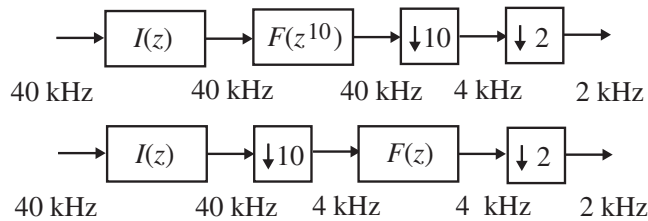
Computational complexities of the two sections are:

$$\mathcal{R}_{M,F} = (121 + 1) \frac{8000}{4} = 244,000 \text{ mps and } \mathcal{R}_{M,I} = (20 + 1) \frac{40000}{5} = 168,000 \text{ mps.}$$

Hence, the total computational complexity of the two-stage realization is

$$\mathcal{R}_M = \mathcal{R}_{M,F} + \mathcal{R}_{M,I} = 412,000 \text{ mps.}$$

Realization #4 – $H(z) = F(z^{10})I(z)$



Specifications for $F(z)$ and $I(z)$ are as follows:

$F(z)$: $F_p = 8000$ Hz, $F_s = 10,000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus, $\Delta f = \frac{2000}{40000}$.

$I(z)$: $F_p = 800$ Hz, $F_s = 3,000$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus, $\Delta f = \frac{2200}{40000}$.

Orders of $F(z)$ and $I(z)$ are given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{2000}{40000} \right)} = 60.26 \Rightarrow 61$$

$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{2200}{40000} \right)} = 54.78 \Rightarrow 55$$

Computational complexities of the two sections are:

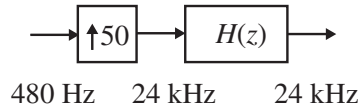
$$\mathcal{R}_{M,F} = (61 + 1) \frac{4000}{2} = 124,000 \text{ mps and } \mathcal{R}_{M,I} = (55 + 1) \frac{40000}{10} = 224,000 \text{ mps.}$$

Hence, the total computational complexity of the two-stage realization is

$$\mathcal{R}_M = \mathcal{R}_{M,F} + \mathcal{R}_{M,I} = 348,000 \text{ mps.}$$

Hence, the optimum two-stage design with the lowest computational complexity is the Realization #4.

- 13.21 (a)** The desired up-sampling factor is $L = \frac{24000}{480} = 50$. The general structure of the desired interpolator is thus as shown below:



Now, the normalized stopband edge angular frequency of a factor-of-50 interpolator is

$$\omega_s = \frac{\pi}{50}. \text{ Hence, the desired stopband edge frequency in this case is}$$

$$F_s = \frac{12000}{50} = 240 \text{ Hz. The specifications of the decimation filter } H(z) \text{ is thus as}$$

$$\text{follows: } F_p = 190 \text{ Hz, } F_s = 240 \text{ Hz, } \delta_p = 0.002, \delta_s = 0.002. \text{ Here } \Delta f = \frac{50}{24000}.$$

Using Eq. (10.3) we arrive at the order of $H(z)$ given by

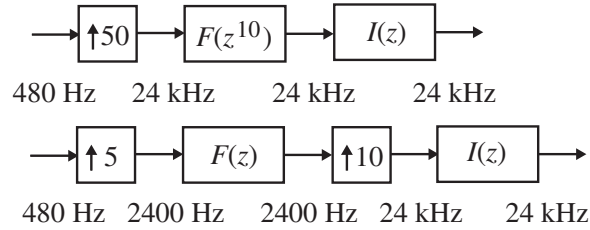
$$N_H = \frac{-20 \log_{10} \sqrt{0.002 \times 0.002} - 13}{14.6 \left(\frac{50}{24000} \right)} = 1347.27 \Rightarrow 1348.$$

$$\text{Computational complexity is thus } \mathcal{R}_{M,H} = (1348 + 1) \frac{24000}{50} = 647,520 \text{ mps.}$$

(b) $H(z) = F(z^{10})I(z)$.

Specifications for $F(z)$ and $I(z)$ are as follows:

$$F(z): F_p = 1900 \text{ Hz, } F_s = 2400 \text{ Hz, } \delta_p = 0.001, \delta_s = 0.002, \text{ and thus, } \Delta f = \frac{50}{24000}.$$



$I(z)$: $F_p = 190$ Hz, $F_s = 2160$ Hz, $\delta_p = 0.001$, $\delta_s = 0.002$, and thus, $\Delta f = \frac{1970}{24000}$.

Orders of $F(z)$ and $I(z)$ are given by

$$N_F = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{500}{24000} \right)} = 144.62 \Rightarrow 145$$

$$N_I = \frac{-20 \log_{10} \sqrt{0.001 \times 0.002} - 13}{14.6 \left(\frac{1970}{24000} \right)} = 36.71 \Rightarrow 37$$

Computational complexities of the two sections are:

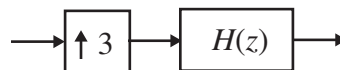
$$\mathcal{R}_{M,F} = (145 + 1) \frac{2400}{5} = 70,080 \text{ mps} \text{ and } \mathcal{R}_{M,I} = (37 + 1) \frac{24000}{10} = 91,200 \text{ mps.}$$

Hence, the total computational complexity of the two-stage realization is

$$\mathcal{R}_M = \mathcal{R}_{M,F} + \mathcal{R}_{M,I} = 161,280 \text{ mps.}$$

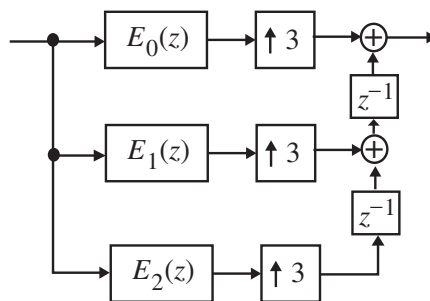
Therefore, the complexity in a two-stage design with $H(z) = F(z^{10})I(z)$. is approximately 25% of that of the single-stage design.

13.22 A computationally efficient realization of a factor-of-3 interpolator as shown below



is obtained from the 3-band polyphase decomposition of $H(z)$ given by

$H(z) = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$. The general form of the polyphase representation of the interpolator is as shown below:



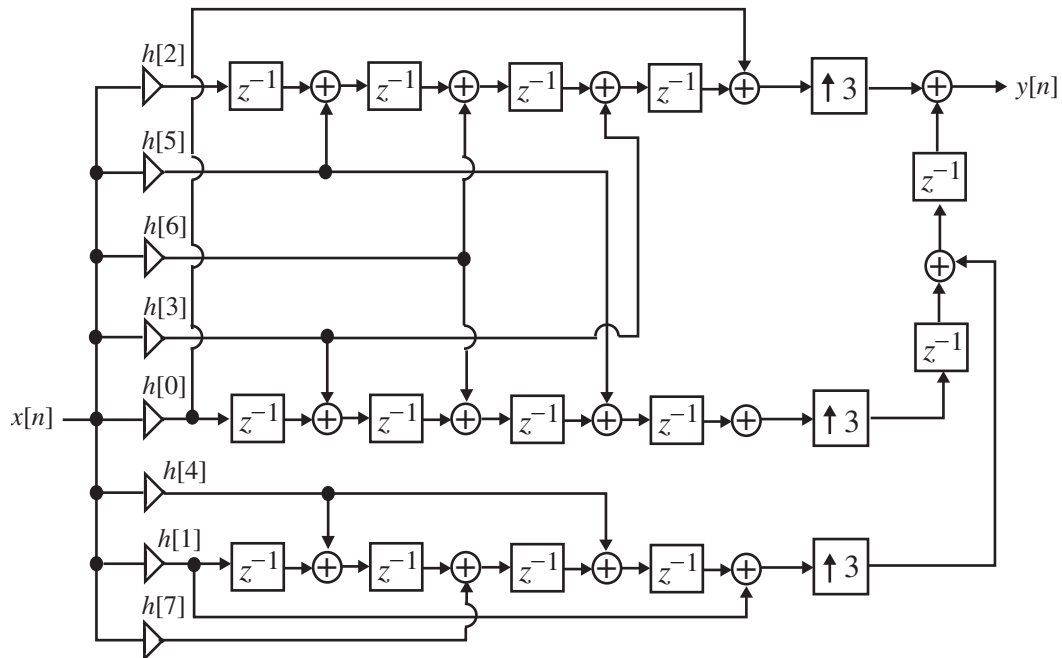
Since $H(z)$ is a length-15 linear-phase transfer function, $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[6]z^{-8} + h[5]z^{-9} + h[4]z^{-10} + h[3]z^{-11} + h[2]z^{-12} + h[1]z^{-13} + h[0]z^{-14}$, the transfer functions of the sub-filters are as follows:

$$E_0(z) = h[0] + h[3]z^{-1} + h[6]z^{-2} + h[5]z^{-3} + h[2]z^{-4},$$

$$E_1(z) = h[1] + h[4]z^{-1} + h[7]z^{-2} + h[4]z^{-3} + h[1]z^{-4},$$

$$E_2(z) = h[2] + h[5]z^{-1} + h[6]z^{-2} + h[3]z^{-3} + h[0]z^{-4}.$$

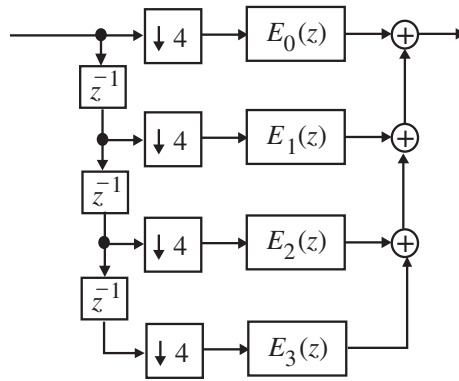
A computationally efficient realization of the factor-of-3 interpolator is obtained by sharing common multipliers as shown below:



13.23 A computationally efficient realization of a factor-of-4 decimator as shown below



is obtained from the 4-band polyphase decomposition of $H(z)$ given by $H(z) = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3) + z^{-3}E_3(z^3)$. The general form of the polyphase representation of the interpolator is as shown below:



Since $H(z)$ is a length-16 linear-phase transfer function, $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[7]z^{-8} + h[6]z^{-9} + h[5]z^{-10} + h[4]z^{-11} + h[3]z^{-12} + h[2]z^{-13} + h[1]z^{-14} + h[0]z^{-15}$, the transfer functions of the sub-filters are as follows:

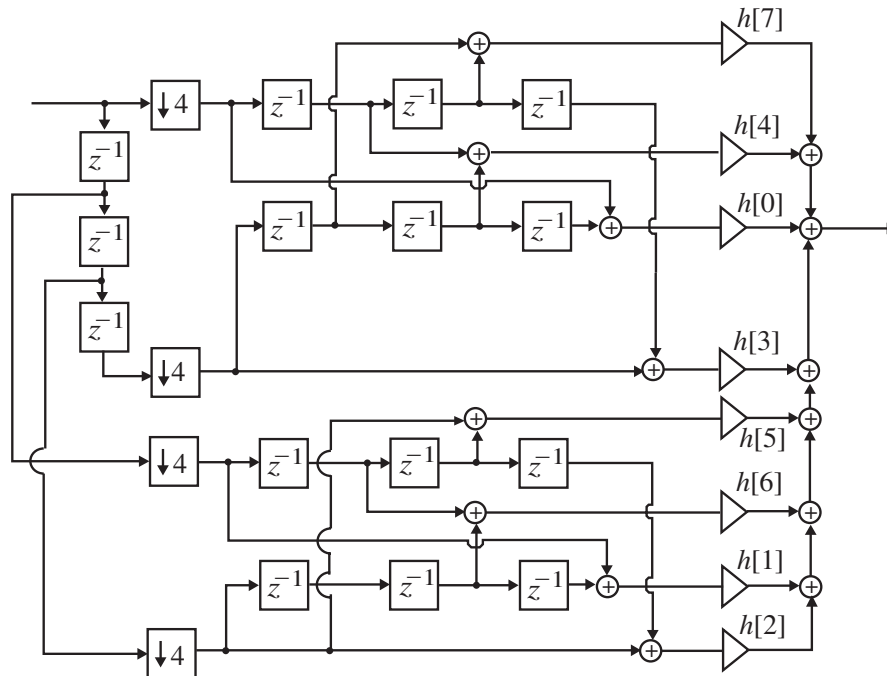
$$E_0(z) = h[0] + h[4]z^{-1} + h[7]z^{-2} + h[3]z^{-3},$$

$$E_1(z) = h[1] + h[5]z^{-1} + h[6]z^{-2} + h[2]z^{-3},$$

$$E_2(z) = h[2] + h[6]z^{-1} + h[5]z^{-2} + h[1]z^{-3}.$$

$$E_3(z) = h[3] + h[7]z^{-1} + h[4]z^{-2} + h[0]z^{-3}.$$

A computationally efficient realization of the factor-of-4 decimator is obtained by sharing common multipliers as shown below



13.24
$$H(z) = \sum_{i=0}^{N-1} z^{-i} = (1 + z^{-1}) + (z^{-2} + z^{-3}) + \dots + (z^{-(N-2)} + z^{-(N-1)})$$

$$= (1 + z^{-1})(1 + z^{-2} + z^{-2} + \dots + z^{-(N-2)}) = (1 + z^{-1})G(z^2), \text{ where } G(z) = \sum_{i=0}^{(N/2)-1} z^{-i}.$$

Using a similar technique we can show that $G(z) = (1 + z^{-1}) \left(\sum_{i=0}^{(N/4)-1} z^{-2i} \right)$. Therefore we

$$\text{can write } H(z) = (1 + z^{-1})(1 + z^{-2}) \left(\sum_{i=0}^{(N/4)-1} z^{-4i} \right) = (1 + z^{-1})(1 + z^{-2})F(z^4), \text{ where}$$

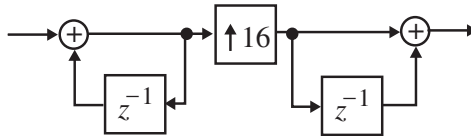
$$F(z) = \sum_{i=0}^{(N/4)-1} z^{-i}. \text{ Continuing this decomposition process further we arrive at}$$

$$H(z) = (1 + z^{-1})(1 + z^{-2}) \dots (1 + z^{-2^{K-1}}), \text{ where } N = 2^K.$$

The transfer function of a box-car decimation filter of length-16 can be expressed as:

$$H(z) = \sum_{i=0}^{15} z^{-i} = \frac{1 - z^{-16}}{1 - z^{-1}}.$$

As a result, a computationally efficient realization of a factor-of-16 decimator using a length-16 boxcar decimation filter is as shown below:



13.25 Let $u[n]$ denote the output of the factor-of- L interpolator. Then

$$\mathcal{E} = \frac{\sum_{n=-\infty}^{\infty} (u[n] - u[n-1])^2}{\sum_{n=-\infty}^{\infty} u^2[n]}, \quad (13-1)$$

and

$$C = \frac{\sum_{n=-\infty}^{\infty} u[n]u[n-1]}{\sum_{n=-\infty}^{\infty} u^2[n]}. \quad (13-2)$$

Substituting Eq. (13-2) in Eq. (13-1) we get $\mathcal{E} = 2(1 - C)$. Hence, as $C \rightarrow 1$, i.e., as the signal $u[n]$ becomes highly correlated, $\mathcal{E} \rightarrow 0$.

Now, by Parseval's relation, $\sum_{n=-\infty}^{\infty} u[n]v[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(e^{j\omega})V^*(e^{j\omega})d\omega$, where $U(e^{j\omega})$

and $V(e^{j\omega})$ are the DTFTs of $u[n]$ and $v[n]$, respectively. If we let $v[n] = u[n-1]$ in the numerator of Eq. (13-1), and $v[n] = u[n]$ in the denominator of Eq. (13-1), then we can write

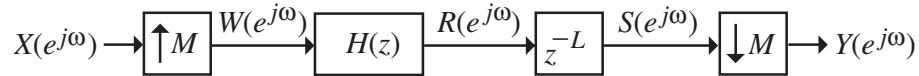
$$E = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} |U(e^{j\omega})|^2 e^{j\omega} d\omega}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |U(e^{j\omega})|^2 d\omega} = \frac{\int_{-\pi}^{\pi} |U(e^{j\omega})|^2 \cos(\omega) d\omega}{\int_{-\pi}^{\pi} |U(e^{j\omega})|^2 d\omega},$$

assuming $u[n]$ to be a real sequence. If $x[n]$ is assumed to be a broadband signal with a flat magnitude spectrum, i.e., $|X(e^{j\omega})| = 1$ for $0 \leq |\omega| \leq \pi$, then the magnitude spectrum of is bandlimited to the range $0 \leq |\omega| \leq \pi/L$, i.e.,

$$|U(e^{j\omega})| = \begin{cases} 1, & 0 \leq |\omega| < \pi/L, \\ 0, & \text{otherwise.} \end{cases} \quad \text{Therefore, } C = \frac{\int_0^{\pi/L} \cos(\omega) d\omega}{\int_0^{\pi/L} d\omega} = \frac{\sin(\pi/L)}{(\pi/L)}. \quad \text{Hence, as}$$

$L \rightarrow \infty, C \rightarrow 1.$

13.26



Analysis of the above structure yields $W(e^{j\omega}) = X(e^{j\omega M}),$

$R(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega M}),$ $S(e^{j\omega}) = e^{-j\omega L} R(e^{j\omega}) = e^{-j\omega L} H(e^{j\omega})X(e^{j\omega M}),$

$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} S(e^{-j\omega k/M} e^{j\omega/M}).$ If the filter $H(e^{j\omega})$ is assumed to be close to

an ideal lowpass filter with a cutoff at ω/M , we can assume that all images of

$X(e^{j\omega})$ are suppressed leaving only the term in the expression for $Y(e^{j\omega})$. Hence, we

can write $Y(e^{j\omega}) = \frac{1}{M} S(e^{j\omega/M}) = \frac{1}{M} H(e^{j\omega/M}) e^{-j\omega L/M} X(e^{j\omega}).$ Since $H(z)$ is a

Type 1 FIR filter with exact linear phase and a delay of $(N-1)/2 = KM$ samples and a magnitude response equal to M in the passband, we have

$Y(e^{j\omega}) = e^{-j\omega K} e^{-j\omega L/M} X(e^{j\omega}).$ Thus, the structure of Figure P13.7 Is

approximately an allpass filter with a fixed delay of K samples and a variable noninteger delay of L/M samples.

13.27 An ideal M -th band lowpass filter $H(z)$ is characterized by a frequency response

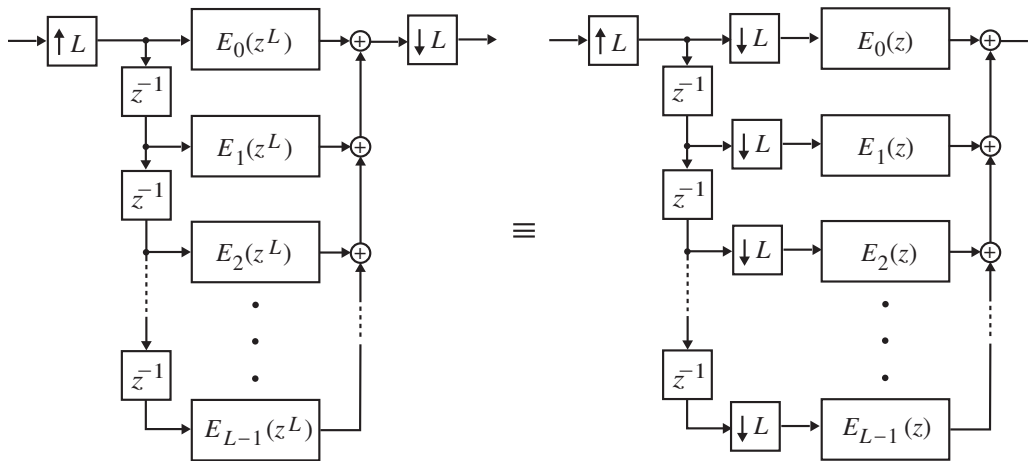
$H(e^{j\omega}) = \begin{cases} 1, & -\pi/M \leq \omega \leq \pi/M, \\ 0, & \text{otherwise.} \end{cases}$ The transfer function $H(z)$ can be expressed in

an M -branch polyphase form as $H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M)$. From the above we observe

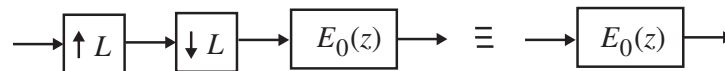
$$\sum_{r=0}^{M-1} H(zW_M^r) = M H_0(z^M). \text{ Therefore, } H_0(e^{j\omega M}) = \frac{1}{M} \sum_{r=0}^{M-1} H(e^{j(\omega-2\pi r/M)}) = \frac{1}{M}.$$

Or in other words, $H_0(z^M)$ is an allpass function.

13.28 An equivalent realization of the structure of Figure P13.38 obtained by realizing the filter in a Type 1 polyphase form is shown below on the left. By moving the down-sampler through the system and invoking the cascade equivalence #1 of Figure 13.14 we arrive at the structure shown below on the right.



The structure on the right hand side reduces to the one shown below on the left from which we arrive at the simplified equivalent structure shown below on the right.



13.29 (a) $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$. Let $H_0(z) = \sum_{i=0}^{(N/2)-1} (h[2i] + h[2i+1])z^{-2i}$ and

$$H_1(z) = \sum_{i=0}^{(N/2)-1} (h[2i] - h[2i+1])z^{-2i}. \text{ Then}$$

$$(1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2) = \sum_{i=0}^{(N/2)-1} h[2i]z^{-2i} + \sum_{i=0}^{(N/2)-1} h[2i+1]z^{-2i} + 1 = H(z).$$

(b) $H(z) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2)$

$= (H_0(z^2) + H_1(z^2)) + z^{-1}(H_0(z^2) - H_1(z^2)) = E_0(z^2) + z^{-1}E_1(z^2)$. Therefore, $E_0(z) = H_0(z) + H_1(z)$ and $E_1(z) = H_0(z) - H_1(z)$.

(c) Now,
$$H(z) = \begin{bmatrix} 1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} H_0(z^2) \\ H_1(z^2) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} \end{bmatrix} \begin{bmatrix} H_0(z^2) + H_1(z^2) \\ H_0(z^2) - H_1(z^2) \end{bmatrix}$$

$$= (H_0(z^2) + H_1(z^2)) + z^{-1}(H_0(z^2) - H_1(z^2)) = (1 + z^{-1})H_0(z^2) + (1 - z^{-1})H_1(z^2).$$

(d) If $L = 2$, i.e., $N = 2^2$, then we can express

$$H_0(z^2) = (1 + z^{-2})H_{00}(z^4) + (1 - z^{-2})H_{01}(z^4) \text{ and}$$

$$H_1(z^2) = (1 + z^{-2})H_{10}(z^4) + (1 - z^{-2})H_{11}(z^4). \text{ Substituting these expressions in}$$

$$H(z) = (1 + z^{-1})H_0(z^2) + (1 - z^{-1})H_1(z^2) \text{ we get}$$

$$\begin{aligned} H(z) &= (1 + z^{-1}) \left[(1 + z^{-2})H_{00}(z^4) + (1 - z^{-2})H_{01}(z^4) \right] \\ &\quad + (1 - z^{-1}) \left[(1 + z^{-2})H_{10}(z^4) + (1 - z^{-2})H_{11}(z^4) \right] \\ &= (1 + z^{-1})(1 + z^{-2})H_{00}(z^4) + (1 + z^{-1})(1 - z^{-2})H_{01}(z^4) \\ &\quad + (1 - z^{-1})(1 + z^{-2})H_{10}(z^4) + (1 - z^{-1})(1 - z^{-2})H_{11}(z^4) \\ &= \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} H_{00}(z^4) \\ H_{01}(z^4) \\ H_{10}(z^4) \\ H_{11}(z^4) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \mathbf{R}_4 \begin{bmatrix} \hat{H}_0(z^4) \\ \hat{H}_1(z^4) \\ \hat{H}_2(z^4) \\ \hat{H}_3(z^4) \end{bmatrix}. \end{aligned}$$

Continuing this process it is easy to establish that for we have

$$H(z) = \begin{bmatrix} 1 & z^{-1} & \dots & z^{-(L-1)} \end{bmatrix} \mathbf{R}_L \begin{bmatrix} \hat{H}_0(z^L) \\ \hat{H}_1(z^L) \\ \vdots \\ \hat{H}_{L-1}(z^L) \end{bmatrix}.$$

13.30 Now
$$H(z) = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \mathbf{R}_4 \begin{bmatrix} \hat{H}_0(z^4) \\ \hat{H}_1(z^4) \\ \hat{H}_2(z^4) \\ \hat{H}_3(z^4) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \begin{bmatrix} E_0(z^4) \\ E_1(z^4) \\ E_2(z^4) \\ E_3(z^4) \end{bmatrix}.$$

Therefore,
$$\begin{bmatrix} \hat{H}_0(z) \\ \hat{H}_1(z) \\ \hat{H}_2(z) \\ \hat{H}_3(z) \end{bmatrix} = \mathbf{R}_4^{-1} \begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \\ E_3(z) \end{bmatrix} = \frac{1}{4} \mathbf{R}_4 \begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \\ E_3(z) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \\ E_3(z) \end{bmatrix}.$$

A length-16 Type 1 linear-phase FIR transfer function is of the form

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + h_5 z^{-5} + h_6 z^{-6} + h_7 z^{-7} + h_7 z^{-8} + h_6 z^{-9}$$

$$+h_5z^{-10} + h_4z^{-11} + h_3z^{-12} + h_2z^{-13} + h_1z^{-14} + h_0z^{-15}.$$

Hence, $E_0(z) = h_0 + h_4z^{-1} + h_7z^{-2} + h_3z^{-3}$, $E_1(z) = h_1 + h_5z^{-1} + h_6z^{-2} + h_2z^{-3}$,

$E_2(z) = h_2 + h_6z^{-1} + h_5z^{-2} + h_1z^{-3}$, $E_3(z) = h_3 + h_7z^{-1} + h_4z^{-2} + h_0z^{-3}$. Thus,

$\hat{H}_0(z) = g_0 + g_1z^{-1} + g_1z^{-2} + g_0z^{-3}$, $\hat{H}_1(z) = g_2 + g_3z^{-1} - g_3z^{-2} - g_2z^{-3}$,

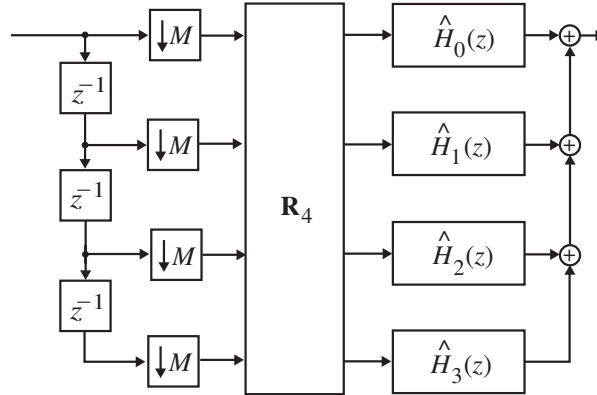
$\hat{H}_2(z) = g_4 + g_5z^{-1} - g_5z^{-2} - g_4z^{-3}$, $\hat{H}_3(z) = g_6 + g_7z^{-1} + g_7z^{-2} + g_6z^{-3}$, where

$$g_0 = \frac{1}{4}(h_0 + h_1 + h_2 + h_3), g_1 = \frac{1}{4}(h_4 + h_5 + h_6 + h_7), g_2 = \frac{1}{4}(h_0 - h_1 + h_2 - h_3),$$

$$g_3 = \frac{1}{4}(h_4 - h_5 + h_6 - h_7), g_4 = \frac{1}{4}(h_0 + h_1 - h_2 - h_3), g_5 = \frac{1}{4}(h_4 + h_5 - h_6 - h_7),$$

$$g_6 = \frac{1}{4}(h_0 - h_1 - h_2 + h_3), g_7 = \frac{1}{4}(h_4 - h_5 - h_6 + h_7). \text{ Note that } \hat{H}_0(z) \text{ and } \hat{H}_3(z)$$

are Type 1 linear-phase FIR transfer functions, whereas, $\hat{H}_1(z)$ and $\hat{H}_2(z)$ are Type 2 linear-phase FIR transfer functions. A computationally efficient realization of a factor-of-4 decimator using a four-band structural subband decomposition of the decimation filter $H(z)$ is shown below:



Because of the symmetry or anti-symmetry in the impulse responses of the subband filters, each subband filter can be realized using only 2 multipliers. Hence, the final realization uses only 8 multipliers. Note also that by delay-sharing, the total number of delays in implementing the four subband filters can be reduced to 3.

13.31 $y[n] = P_{-2}(\alpha)x[n-2] + P_{-1}(\alpha)x[n-1] + P_0(\alpha)x[n] + P_1(\alpha)x[n+1] + P_2(\alpha)x[n+2]$

where $P_{-2}(\alpha) = \frac{(\alpha+1)\alpha(\alpha-1)(\alpha-2)}{(-2+1)(-2)(-2-1)(-2-2)} = \frac{1}{24}(\alpha^4 - 2\alpha^3 - \alpha^2 + 2\alpha),$

$$P_{-1}(\alpha) = \frac{(\alpha+2)\alpha(\alpha-1)(\alpha-2)}{(-1+2)(-1)(-1-1)(-1-2)} = -\frac{1}{6}(\alpha^4 - \alpha^3 - 4\alpha^2 + 4\alpha),$$

$$P_0(\alpha) = \frac{(\alpha+2)(\alpha+1)(\alpha-1)(\alpha-2)}{(0+2)(0+1)(0-1)(0-2)} = \frac{1}{4}(\alpha^4 - 5\alpha^2 + 4),$$

$$P_1(\alpha) = \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-2)}{(1+2)(1+1)(1-0)(1-2)} = -\frac{1}{6}(\alpha^4 + \alpha^3 - 4\alpha^2 - 4\alpha),$$

$$P_2(\alpha) = \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{(2+2)(2+1)(2-0)(2-1)} = \frac{1}{24}(\alpha^4 + 2\alpha^3 - \alpha^2 - 2\alpha).$$

We consider the computation of $y[n]$, $y[n+1]$, $y[n+2]$, $y[n+3]$ using 5 input samples: $x[n-2]$ through $x[n+2]$.

For $\alpha_0 = 0$, $P_{-2}(\alpha_0) = 0$, $P_{-1}(\alpha_0) = 0$, $P_0(\alpha_0) = 1$, $P_1(\alpha_0) = 0$, and $P_2(\alpha_0) = 0$. For $\alpha_1 = 5/4$, $P_{-2}(\alpha_1) = -0.022$, $P_{-1}(\alpha_1) = 0.127$, $P_0(\alpha_1) = -0.3428$, $P_1(\alpha_1) = 1.1426$, and $P_2(\alpha_1) = 0.0952$.

For $\alpha_2 = 10/4$, $P_{-2}(\alpha_2) = 0.2734$, $P_{-1}(\alpha_2) = -1.4062$, $P_0(\alpha_2) = 2.9531$, $P_1(\alpha_2) = -3.2812$, and $P_2(\alpha_2) = 2.4609$.

For $\alpha_3 = 15/4$, $P_{-2}(\alpha_3) = 3.5718$, $P_{-1}(\alpha_3) = -17.2949$, $P_0(\alpha_3) = 32.86$, $P_1(\alpha_3) = -29.873$, and $P_2(\alpha_3) = 11.7358$.

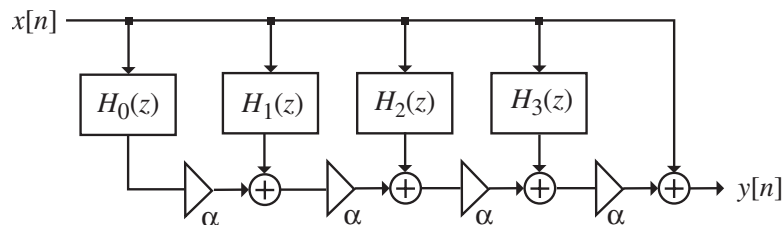
The block filter implementation is thus given by

$$\begin{bmatrix} y[n] \\ y[n+1] \\ y[n+2] \\ y[n+3] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -0.022 & 0.127 & -0.3428 & 1.1426 & 0.0952 \\ 0.2734 & -1.4062 & 2.9531 & -3.2812 & 2.4609 \\ 3.5718 & -17.2949 & 32.86 & -29.873 & 11.7358 \end{bmatrix} \begin{bmatrix} x[n-2] \\ x[n-1] \\ x[n] \\ x[n+1] \\ x[n+2] \end{bmatrix}. \quad \text{Another}$$

implementation is given by

$$\begin{aligned} y[n] = & \alpha^4 \left(\frac{1}{24} x[n-2] - \frac{1}{6} x[n-1] + \frac{1}{4} x[n] - \frac{1}{6} x[n+1] + \frac{1}{24} x[n+2] \right) \\ & + \alpha^3 \left(-\frac{1}{12} x[n-2] + \frac{1}{6} x[n-1] - \frac{1}{6} x[n+1] + \frac{1}{12} x[n+2] \right) \\ & + \alpha^2 \left(-\frac{1}{24} x[n-2] + \frac{4}{6} x[n-1] - \frac{5}{4} x[n] + \frac{4}{6} x[n+1] - \frac{1}{24} x[n+2] \right) \\ & + \alpha \left(\frac{1}{12} x[n-2] - \frac{4}{6} x[n-1] + \frac{4}{6} x[n+1] - \frac{1}{12} x[n+2] \right) + x[n]. \end{aligned}$$

The Farrow structure implementation of the interpolator is shown below:



where $H_0(z) = \frac{1}{24}z^{-2} - \frac{1}{6}z^{-1} + \frac{1}{4} - \frac{1}{6}z + \frac{1}{24}z^2$, $H_1(z) = -\frac{1}{12}z^{-2} + \frac{1}{6}z^{-1} + \frac{1}{6}z - \frac{1}{24}z^2$,

$$H_2(z) = -\frac{1}{24}z^{-2} + \frac{4}{6}z^{-1} - \frac{5}{4} + \frac{4}{6}z - \frac{1}{24}z^2, \text{ and } H_3(z) = \frac{1}{12}z^{-2} - \frac{4}{6}z^{-1} + \frac{4}{6}z - \frac{1}{12}z^2.$$

13.32 From Eq. (13.75), we have $a_0 = -a_4 \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ t_4 & t_1 & t_2 & t_3 \\ t_4^2 & t_1^2 & t_2^2 & t_3^2 \\ t_4^3 & t_1^3 & t_2^3 & t_3^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ t_0 & t_1 & t_2 & t_3 \\ t_0^2 & t_1^2 & t_2^2 & t_3^2 \\ t_0^3 & t_1^3 & t_2^3 & t_3^3 \end{vmatrix}}$. Both the numerator and the

denominator are determinants of Vandermonde matrices and have a nonzero value if $t_i \neq t_j, i \neq j$. From the solution of Problem 6.28, we get

$$a_0 = a_4 \frac{(t_1-t_4)(t_2-t_4)(t_3-t_4)(t_2-t_1)(t_3-t_1)(t_3-t_2)}{(t_1-t_0)(t_2-t_0)(t_3-t_0)(t_2-t_1)(t_3-t_1)(t_3-t_2)} = a_4 \frac{(t_1-t_4)(t_2-t_4)(t_3-t_4)}{(t_1-t_0)(t_2-t_0)(t_3-t_0)}, \text{ or}$$

$$a_0 = -a_4 \frac{(t_4-t_1)(t_4-t_2)(t_4-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}.$$

13.33 From Eq. (13.74) we get $a_1 = -a_4 \frac{(t_4-t_0)(t_2-t_0)(t_3-t_0)(t_2-t_4)(t_3-t_4)(t_3-t_2)}{(t_1-t_0)(t_2-t_0)(t_3-t_0)(t_2-t_1)(t_3-t_1)(t_3-t_2)}$

$$= -a_4 \frac{(t_4-t_0)(t_2-t_4)(t_3-t_4)}{(t_1-t_0)(t_2-t_1)(t_3-t_1)}.$$
 Substituting the value of a_4 given by Eq. (13.77) in

the above we arrive at $a_1 = \frac{1}{(t_1-t_0)(t_1-t_2)(t_1-t_3)(t_1-t_4)}$. Likewise, from Eq. (13.74)

we get $a_2 = a_4 \frac{(t_1-t_0)(t_4-t_0)(t_3-t_0)(t_4-t_1)(t_3-t_1)(t_3-t_4)}{(t_1-t_0)(t_2-t_0)(t_3-t_0)(t_2-t_1)(t_3-t_1)(t_3-t_2)}$

$$= a_4 \frac{(t_4-t_0)(t_4-t_1)(t_3-t_4)}{(t_2-t_0)(t_2-t_1)(t_3-t_2)}.$$
 Substituting the value of a_4 given by Eq. (13.77) in the

above we arrive at $a_0 = \frac{1}{(t_2-t_0)(t_2-t_1)(t_2-t_3)(t_2-t_4)}$. Finally, from Eq. (13.74) we

get $a_3 = -a_4 \frac{(t_1-t_0)(t_2-t_0)(t_4-t_0)(t_2-t_1)(t_4-t_1)(t_4-t_2)}{(t_1-t_0)(t_2-t_0)(t_3-t_0)(t_2-t_1)(t_3-t_1)(t_3-t_2)}$

$$= -a_4 \frac{(t_4-t_0)(t_4-t_1)(t_4-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)}.$$
 Substituting the value of a_4 given by Eq. (13.77) in the

above we arrive at $a_3 = \frac{1}{(t_3-t_0)(t_3-t_1)(t_3-t_2)(t_3-t_4)}$.

13.34 $t_i = i, m \leq i \leq m+4$.

$$\begin{aligned}
a_m &= \frac{1}{(m-m-1)(m-m-2)(m-m-3)(m-m-4)} = \frac{1}{24}, \\
a_{m+1} &= \frac{1}{(m+1-m)(m+1-m-2)(m+1-m-3)(m+1-m-4)} = -\frac{1}{6}, \\
a_{m+2} &= \frac{1}{(m+2-m)(m+2-m-1)(m+2-m-3)(m+2-m-4)} = \frac{1}{4}, \\
a_{m+3} &= \frac{1}{(m+3-m)(m+3-m-1)(m+3-m-2)(m+3-m-4)} = -\frac{1}{6}, \\
a_{m+4} &= \frac{1}{(m+4-m)(m+4-m-1)(m+4-m-2)(m+4-m-3)} = \frac{1}{24}.
\end{aligned}$$

From Eq. (13.69), we have the expressions for $B_m^{(3)}(t)$ given by

$$B_m^{(3)}(t) = \begin{cases} 0, & t < m, \\ \frac{1}{24}(t-m)^3, & m \leq t < m+1, \\ \frac{1}{24}(t-m)^3 - \frac{1}{6}(t-m-1)^3, & m+1 \leq t < m+2, \\ \frac{1}{24}(t-m)^3 - \frac{1}{6}(t-m-1)^3 + \frac{1}{4}(t-m-2)^3, & m+2 \leq t < m+3, \\ \frac{1}{24}(t-m)^3 - \frac{1}{6}(t-m-1)^3 + \frac{1}{4}(t-m-2)^3 - \frac{1}{6}(t-m-3)^3, & m+3 \leq t < m+4, \\ 0, & t \geq m+4. \end{cases}$$

The normalized 3rd order B-spline is then given by $\beta_m^{(3)}(t) = (m+4-m)B_m^{(3)}(t)$

$$= 4B_m^{(3)}(t) = \begin{cases} 0, & t < m, \\ \frac{1}{6}(t-m)^3, & m \leq t < m+1, \\ \frac{1}{6}(t-m)^3 - \frac{2}{3}(t-m-1)^3, & m+1 \leq t < m+2, \\ \frac{1}{6}(t-m)^3 - \frac{2}{3}(t-m-1)^3 + (t-m-2)^3, & m+2 \leq t < m+3, \\ \frac{1}{6}(t-m)^3 - \frac{2}{3}(t-m-1)^3 + (t-m-2)^3 - \frac{2}{3}(t-m-3)^3, & m+3 \leq t < m+4, \\ 0, & t \geq m+4. \end{cases}$$

Substituting $t = 1 + \alpha$ and evaluating $\beta_m^{(3)}(t)$ for $m = -1, 0, 1, 2$, we have

$$\begin{aligned}
\beta_{-1}^{(3)}(\alpha) &= -\frac{\alpha^3}{6} + \frac{\alpha^2}{2} - \frac{\alpha}{2} + \frac{1}{6}, & \beta_0^{(3)}(\alpha) &= \frac{\alpha^3}{2} + \alpha^2 + \frac{2}{3}, \\
\beta_1^{(3)}(\alpha) &= -\frac{\alpha^3}{2} + \frac{\alpha^2}{2} + \frac{\alpha}{2} + \frac{1}{6}, & \beta_2^{(3)}(\alpha) &= \frac{\alpha^3}{6}.
\end{aligned}$$

Substituting back in Eq. (13.91), we have $y[n] = \sum_{k=-1}^2 \beta_k^{(3)}(\alpha)x[n+k]$

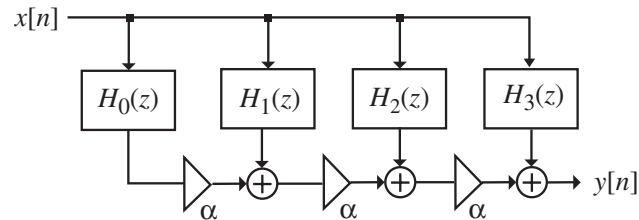
$$\begin{aligned}
&= \left(-\frac{\alpha^3}{6} + \frac{\alpha^2}{2} - \frac{\alpha}{2} + \frac{1}{6} \right) x[n-1] + \left(\frac{\alpha^3}{2} + \alpha^2 + \frac{2}{3} \right) x[n] + \left(-\frac{\alpha^3}{2} + \frac{\alpha^2}{2} + \frac{\alpha}{2} + \frac{1}{6} \right) x[n+1] \\
&+ \left(\frac{\alpha^3}{6} \right) x[n+2] = \left(\frac{1}{6} x[n-1] + \frac{2}{3} x[n] + \frac{1}{6} x[n+1] \right) + \left(-\frac{1}{2} x[n-1] + \frac{1}{2} x[n+1] \right) \alpha \\
&+ \left(\frac{1}{2} x[n-1] - x[n] + \frac{1}{2} x[n+1] \right) \alpha^2 + \left(-\frac{1}{6} x[n-1] + \frac{1}{2} x[n] - \frac{1}{2} x[n+1] + \frac{1}{6} x[n+2] \right) \alpha^3.
\end{aligned}$$

In the z -domain, the input-output relation is thus given by

$$Y(z) = H_3(z) + H_2(z)\alpha + H_1(z)\alpha^2 + H_0(z)\alpha^3, \text{ where}$$

$$H_0(z) = -\frac{1}{6}z^{-1} + \frac{1}{2} - \frac{1}{2}z + \frac{1}{6}z^2, \quad H_1(z) = \frac{1}{2}z^{-1} - 1 + \frac{1}{2}z, \quad H_2(z) = -\frac{1}{2}z^{-1} + \frac{1}{2}z,$$

$H_3(z) = \frac{1}{6}z^{-1} + \frac{2}{3} + \frac{1}{6}z$. The corresponding Farrow structure is shown on top of the next page:



13.35 For the factor-of-4/3 interpolator design, if we use cubic B-spline with uniformly spaced knots at the problem reduces exactly to the design given in the solution of Problem **13.34**.

13.36 From Eq. (13.94) with $k = 0$ we have $H(z) = \alpha + \sum_{i=1}^{L-1} z^{-i} E_i(z^L)$. Substituting the expression for on the left-hand side Eq. (13.97) we get **INCOMPLETE**

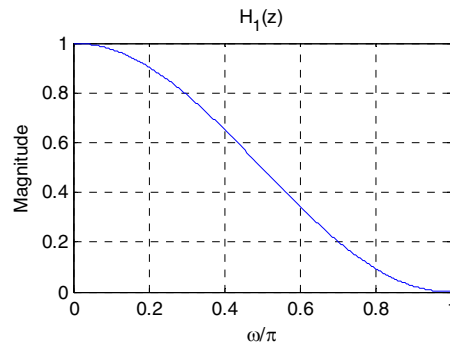
13.37 For a half-band zero-phase lowpass filter, the transfer function is of the form

$$H(z) = h[0] + z^{-1} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} h[2n] z^{-2n}, \text{ where } h[0] = \frac{1}{2}. \text{ If the half-band filter has a zero at}$$

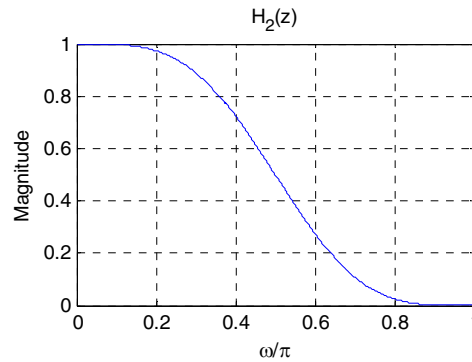
$$z = -1, \text{ then } H(-1) = h[0] - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} h[2n] = 0, \text{ or } h[0] = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} h[2n].$$

13.38 From Eq. (13.99), a zero-phase half-band filter $H(z)$ satisfies the condition $H(z) + H(-z) = \text{a constant}$.

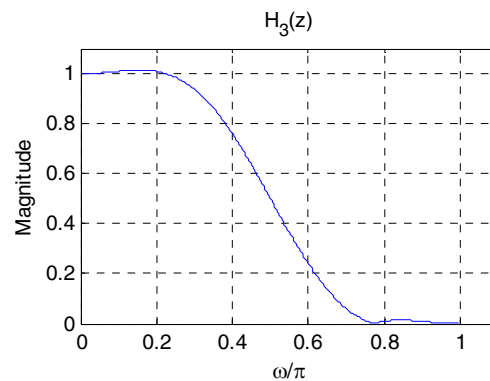
(a) The zero-phase equivalent here is given by $H_1(z) = z + 2 + z^{-1}$. Hence,
 $H_1(z) + H_1(-z) = z + 2 + z^{-1} - z + 2 - z^{-1} = 4$. A plot of the scaled magnitude response of $H_1(z)$ is given below:



(b) The zero-phase equivalent here is given by $H_2(z) = -z^3 + 9z + 16 + 9z^{-1} - z^{-3}$. Hence,
 $H_2(z) + H_2(-z) = -z^3 + 9z + 16 + 9z^{-1} - z^{-3} + z^3 - 9z + 16 - 9z^{-1} + z^{-3} = 32$.
A plot of the scaled magnitude response of $H_2(z)$ is given below:

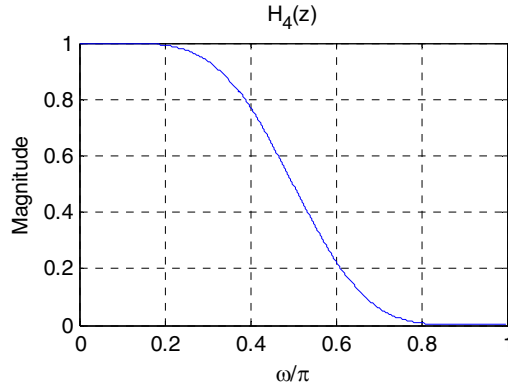


(c) The zero-phase equivalent here is given by
 $H_3(z) = -3z^3 + 19z + 326 + 19z^{-1} - 3z^{-3}$. Hence,
 $H_3(z) + H_3(-z) = -3z^3 + 19z + 326 + 19z^{-1} - 3z^{-3} + 3z^3 - 19z + 326 - 19z^{-1} + 3z^{-3} = 64$.
A plot of the scaled magnitude response of $H_3(z)$ is given below:



(d) The zero-phase equivalent here is given by

$H_4(z) = 3z^5 - 25z^3 + 150z + 256 + 150z^{-1} - 25z^{-3} + 3z^{-5}$. Hence, $H_4(z) + H_4(-z) = 3z^5 - 25z^3 + 150z + 256 + 150z^{-1} - 25z^{-3} + 3z^{-5} - 3z^5 + 25z^3 - 150z + 256 - 150z^{-1} + 25z^{-3} - 3z^{-5} = 512$. A plot of the scaled magnitude response of $H_4(z)$ is given below:



13.39 (a) A function $F(\omega)$ of has p -th order zero at a frequency $\omega = \omega_i$ if

$$\left. \frac{d^p F(\omega)}{d\omega^p} \right|_{\omega=\omega_i} = 0. \text{ The function } H(e^{j\omega}) \text{ has } p \text{ zeros at } \cos \omega = 1, \text{ i.e., at } \omega = \pi.$$

Hence, $\left. \frac{d^p H(e^{j\omega})}{d\omega^p} \right|_{\omega=\pi} = 0$. Moreover, the order of the highest power of

$(1 - \cos \omega)/2$ is $p - 1$. As a result, $\left. \frac{d^{p-1} H(e^{j\omega})}{d\omega^{p-1}} \right|_{\omega=0} = 0$.

(b) Now, $\frac{1 + \cos \omega}{2} = e^{j\omega} \left(\frac{1 + e^{-j\omega}}{2} \right)^2$ and $\frac{1 - \cos \omega}{2} = -e^{j\omega} \left(\frac{1 - e^{-j\omega}}{2} \right)^2$.

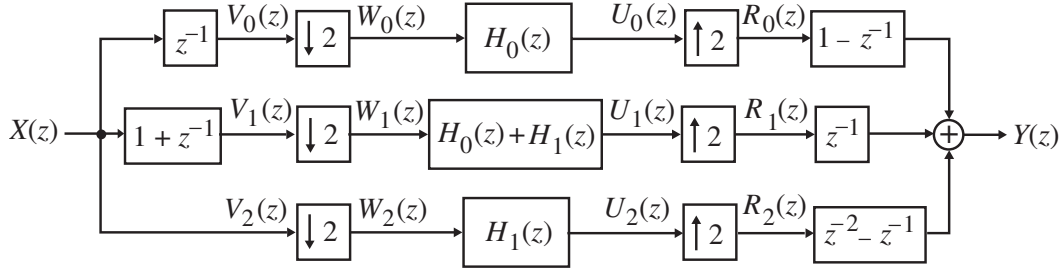
Substituting these expressions in Eq. (13.120) we arrive at

$$H(e^{j\omega}) = e^{jp\omega} \left(\frac{1 + e^{-j\omega}}{2} \right)^p \sum_{\ell=0}^{p-1} \binom{p-1+\ell}{\ell} (-e^{-j\omega})^\ell \left(\frac{1 - e^{-j\omega}}{2} \right)^\ell. \text{ Replacing with}$$

$e^{j\omega}$ in z the above we get the expression for the zero-phase transfer function as

$$H(z) = z^p \left(\frac{1 + z^{-1}}{2} \right)^p \sum_{\ell=0}^{\ell-1} \binom{p-1+\ell}{\ell} (-z)^\ell \left(\frac{1 - z^{-1}}{2} \right)^\ell.$$

13.40 (a)



Analysis of the above structure yields $\mathbf{V}(z) = \begin{bmatrix} V_0(z) \\ V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1} \\ 1 + z^{-1} \\ 1 \end{bmatrix} X(z)$,

$$\mathbf{W}(z) = \begin{bmatrix} W_0(z) \\ W_1(z) \\ W_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1/2} \\ 1 + z^{-1/2} \\ 1 \end{bmatrix} X(z^{1/2}) + \begin{bmatrix} -z^{-1/2} \\ 1 - z^{-1/2} \\ 1 \end{bmatrix} X(z^{-1/2}),$$

$$\mathbf{U}(z) = \begin{bmatrix} U_0(z) \\ U_1(z) \\ U_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1/2} H_0(z) \\ (1 + z^{-1/2})(H_0(z) + H_1(z)) \\ H_1(z) \end{bmatrix} X(z^{1/2}) + \begin{bmatrix} -z^{-1/2} H_0(z) \\ (1 - z^{-1/2})(H_0(z) + H_1(z)) \\ H_1(z) \end{bmatrix} X(z^{-1/2}),$$

$$\mathbf{R}(z) = \begin{bmatrix} R_0(z) \\ R_1(z) \\ R_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1} H_0(z^2) \\ (1 + z^{-1})(H_0(z^2) + H_1(z^2)) \\ H_1(z^2) \end{bmatrix} X(z) + \begin{bmatrix} -z^{-1} H_0(z) \\ (1 - z^{-1})(H_0(z^2) + H_1(z^2)) \\ H_1(z^2) \end{bmatrix} X(-z).$$

$$\begin{aligned} Y(z) &= (1 - z^{-1})R_0(z) + z^{-1}R_1(z) + (z^{-2} - z^{-1})R_2(z) \\ &= \left[(1 - z^{-1})z^{-1}H_0(z^2) + (1 + z^{-1})z^{-1}(H_0(z^2) + H_1(z^2)) + (z^{-2} - z^{-1})H_1(z^2) \right] X(z) \\ &\quad + \left[-(1 - z^{-1})z^{-1}H_0(z) + (1 - z^{-1})z^{-1}(H_0(z^2) + H_1(z^2)) + (z^{-2} - z^{-1})H_1(z^2) \right] X(-z) \\ &= \left[2z^{-1}H_0(z^2) + 2z^{-2}H_1(z^2) \right] X(z) = 2z^{-1} [H_0(z^2) + z^{-1}H_1(z^2)] X(z). \text{ Hence,} \\ T(z) &= 2z^{-1} [H_0(z^2) + z^{-1}H_1(z^2)]. \end{aligned}$$

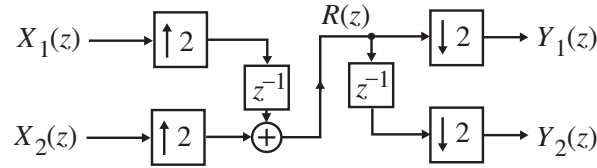
(b) $T(z) = 2z^{-1} \left[\frac{1}{2} H(z) + \frac{1}{2} H(-z) + \frac{1}{2} H(z) - \frac{1}{2} H(-z) \right] = 2z^{-1} H(z)$.

(c) Length of $H_0(z) = K$ and length of $H_1(z) = K$.

(d) The total computational complexity of the above structure is $3K \frac{F_T}{2}$

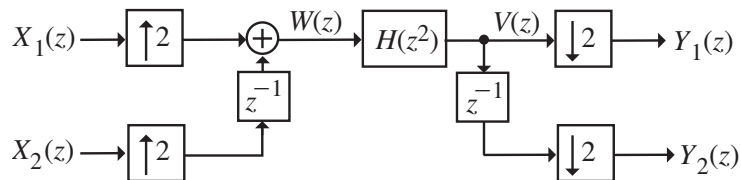
multiplications per second, where is the sampling frequency in Hz. On the other hand, a direct implementation of $H(z)$ requires $2KF_T$ multiplications per second.

13.41



Analysis yields $R(z) = z^{-1}X_1(z^2) + X_2(z^2)$, $Y_1(z) = R(z^{1/2}) + R(-z^{1/2})$
 $= [z^{-1/2}X_1(z) + X_2(z)] + [-z^{-1/2}X_1(z) + X_2(z)] = 2X_2(z)$,
 $Y_2(z) = z^{-1/2}R(z^{1/2}) - z^{-1/2}R(-z^{1/2})$
 $= [z^{-1}X_1(z) + z^{-1/2}X_2(z)] + [z^{-1}X_1(z) - z^{-1/2}X_2(z)] = 2z^{-1}X_1(z)$. Thus, the output $y_1[n]$ is a scaled replica of the input $x_2[n]$ while the output $y_2[n]$ is a scaled replica of the delayed input $x_1[n-1]$.

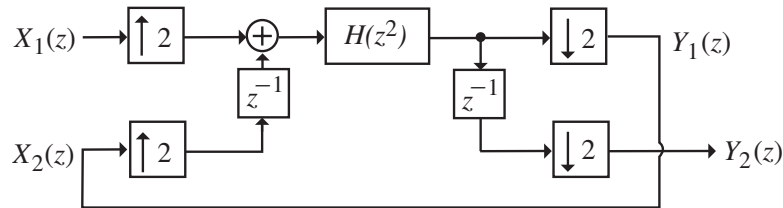
13.42



Analysis yields $W(z) = X_1(z^2) + z^{-1}X_2(z^2)$,
 $V(z) = H(z^2)X_1(z^2) + z^{-1}H(z^2)X_2(z^2)$,
 $Y_1(z) = \frac{1}{2}(V(z^{1/2}) + V(-z^{1/2})) = H(z)X_1(z)$,
 $Y_2(z) = \frac{1}{2}(z^{-1/2}V(z^{1/2}) - z^{-1/2}V(-z^{1/2})) = z^{-1}H(z)X_2(z)$.

Therefore, $\frac{Y_1(z)}{X_1(z)} = H(z)$, and $\frac{Y_2(z)}{X_2(z)} = z^{-1}H(z)$. Hence, the system is time-invariant.

13.43

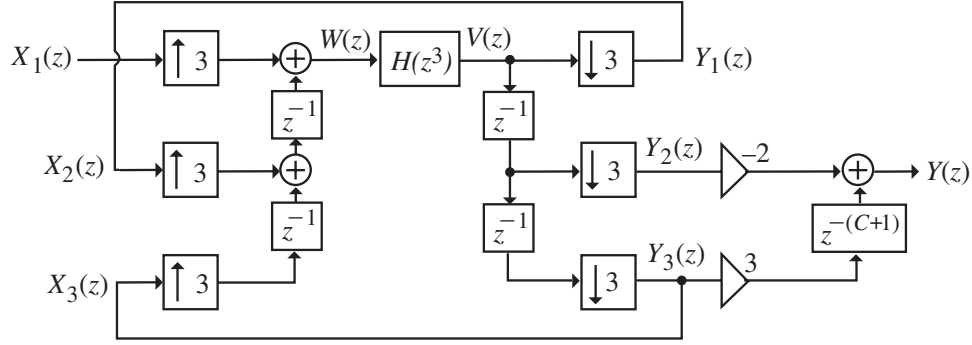


From the solution of Problem 13.42, $\frac{Y_1(z)}{X_1(z)} = H(z)$, and $\frac{Y_2(z)}{X_2(z)} = z^{-1}H(z)$. Here now,

$X_2(z) = Y_1(z)$, and hence, $Y_2(z) = z^{-1}H(z)X_2(z) = z^{-1}H(z)Y_1(z) = z^{-1}H^2(z)X_1(z)$.

Thus, $\frac{Y_2(z)}{X_1(z)} = z^{-1}H^2(z)$. Hence, the system is time-invariant.

13.44



Analysis yields $W(z) = X_1(z^3) + z^{-1}Y_1(z^3) + z^{-2}X_3(z^3)$,

$V(z) = H(z^3)W(z) = H(z^3)X_1(z^3) + z^{-1}H(z^3)X_2(z^3) + z^{-2}H(z^3)X_3(z^3)$,

$Y_1(z) = \frac{1}{3} \left[V(z^{1/3}) + V(z^{1/3}e^{-j2\pi/3}) + V(z^{1/3}e^{-j4\pi/3}) \right]$

$= \frac{1}{3} \left[H(z)X_1(z) + H(ze^{-j2\pi})X_1(ze^{-j2\pi}) + H(ze^{-j4\pi})X_1(ze^{-j4\pi}) \right]$

$+ \frac{1}{3} \left[z^{-1/3}H(z)X_2(z) + z^{-1/3}e^{-j2\pi/3}H(ze^{-j2\pi})X_2(ze^{-j2\pi}) + z^{-1/3}e^{-j4\pi/3}H(ze^{-j4\pi})X_2(ze^{-j4\pi}) \right]$

$+ \frac{1}{3} \left[z^{-2/3}H(z)X_3(z) + z^{-2/3}e^{-j4\pi/3}H(ze^{-j4\pi})X_3(ze^{-j4\pi}) + z^{-2/3}e^{-j8\pi/3}H(ze^{-j8\pi})X_3(ze^{-j8\pi}) \right]$

$= \frac{1}{3} \left[H(z)X_1(z) + H(z)X_1(z) + H(z)X_1(z) \right]$

$+ \frac{1}{3} \left[z^{-1/3}H(z)X_2(z) + z^{-1/3}e^{-j2\pi/3}H(z)X_2(z) + z^{-1/3}e^{-j4\pi/3}H(z)X_2(z) \right]$

$+ \frac{1}{3} \left[z^{-2/3}H(z)X_3(z) + z^{-2/3}e^{-j4\pi/3}H(z)X_3(z) + z^{-2/3}e^{-j8\pi/3}H(z)X_3(z) \right]$

$= H(z)X_1(z)$,

$Y_2(z) = \frac{1}{3} \left[z^{-1/3}V(z^{1/3}) + z^{-1/3}e^{j2\pi/3}V(z^{1/3}e^{-j2\pi/3}) + z^{-1/3}e^{j4\pi/3}V(z^{1/3}e^{-j4\pi/3}) \right]$

$= z^{-1}H(z)X_3(z)$,

$Y_3(z) = \frac{1}{3} \left[z^{-1/3}V(z^{1/3}) + z^{-1/3}e^{j4\pi/3}V(z^{1/3}e^{-j2\pi/3}) + z^{-1/3}e^{j8\pi/3}V(z^{1/3}e^{-j4\pi/3}) \right]$

$= z^{-2}H(z)X_2(z)$.

Now, $X_3(z) = Y_3(z)$ and $X_2(z) = Y_1(z)$. Hence,

$Y_2(z) = z^{-1}H(z)X_3(z) = z^{-1}H(z)Y_3(z) = z^{-1}H(z) \left[z^{-1}H(z)X_2(z) \right] = z^{-2}H^2(z)Y_1(z)$

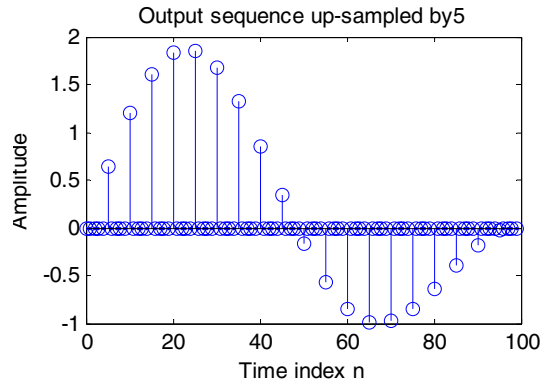
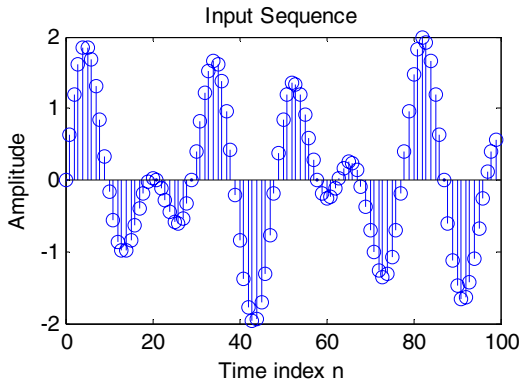
$= z^{-2}H^3(z)X_1(z)$. Therefore, $Y(z) = -2Y_2(z) + 3z^{-(C+1)}Y_3(z)$

$= -2z^{-2}H^3(z)X_1(z) + 3z^{-(C+1)}z^{-1}H^2(z)X_1(z) = z^{-2} \left[3H^2(z) - 2H^3(z) \right] X_1(z)$ for

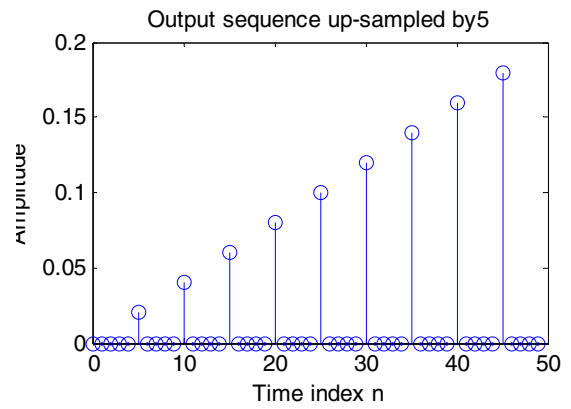
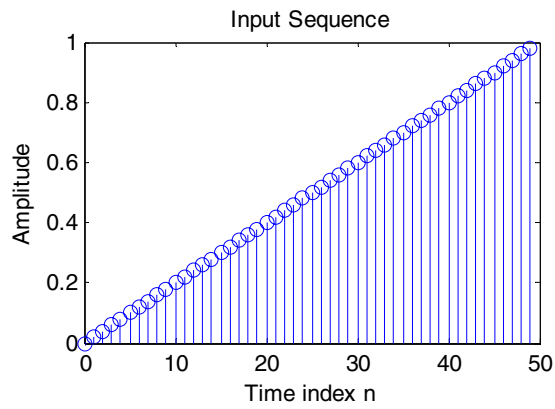
$C = 0$. Thus, the transfer function of the system of Figure P13.12 is

$G(z) = z^{-2} \left[3H^2(z) - 2H^3(z) \right]$.

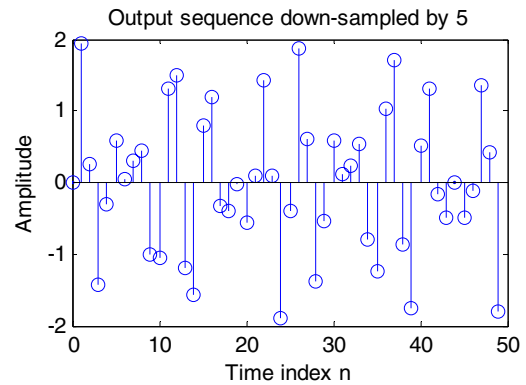
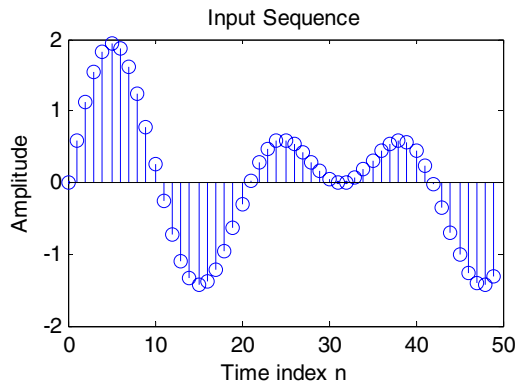
M13.1 (a) (i)



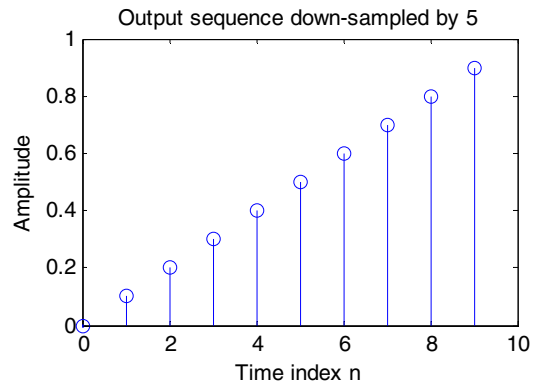
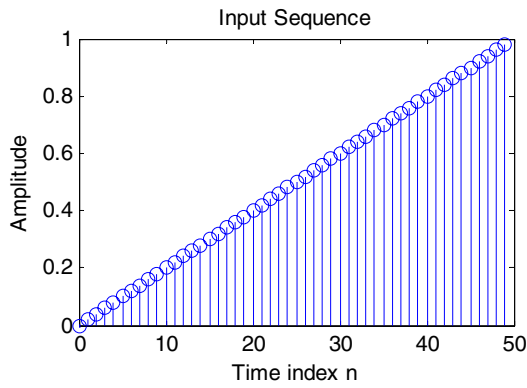
(ii)



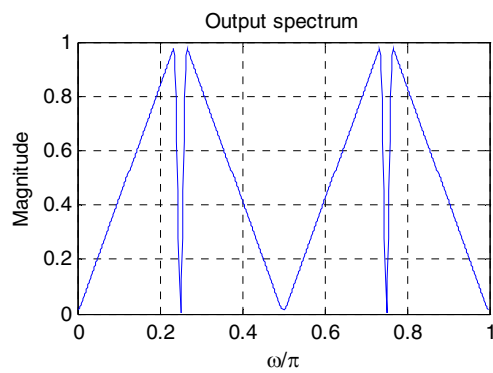
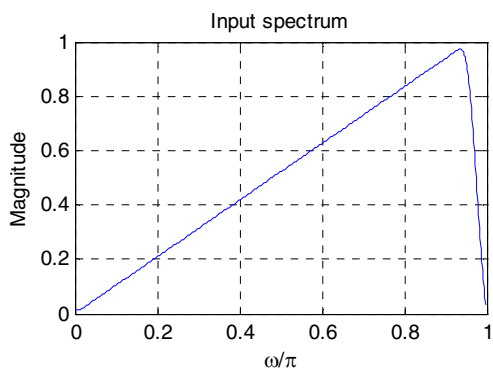
M13.2 (a) (i)



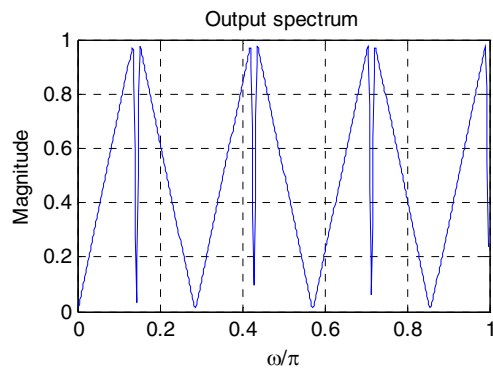
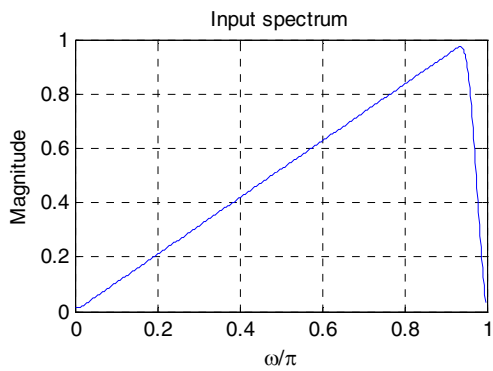
(ii)



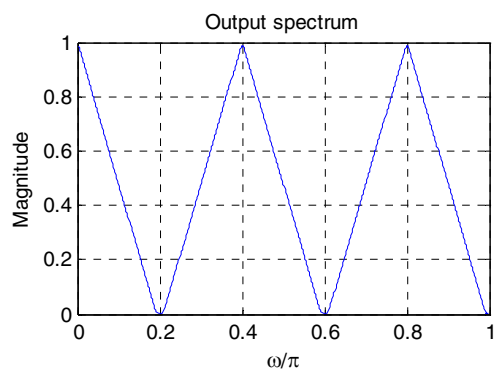
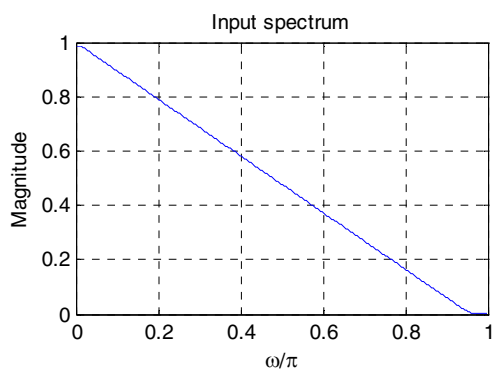
M13.3 (a)



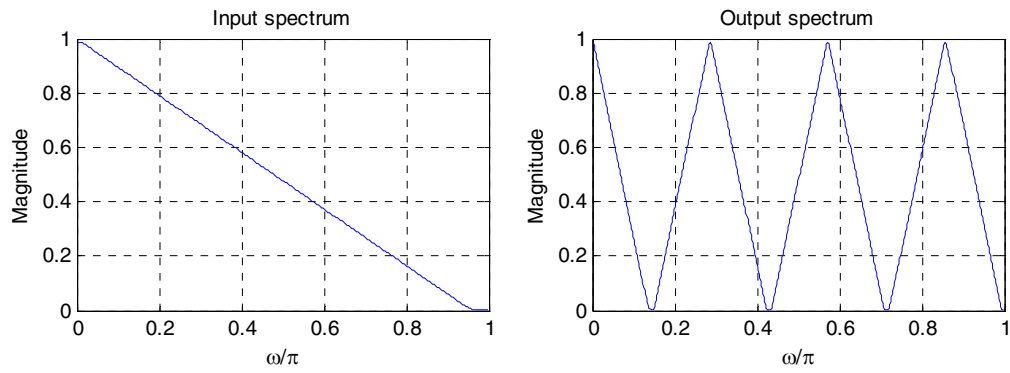
(b)



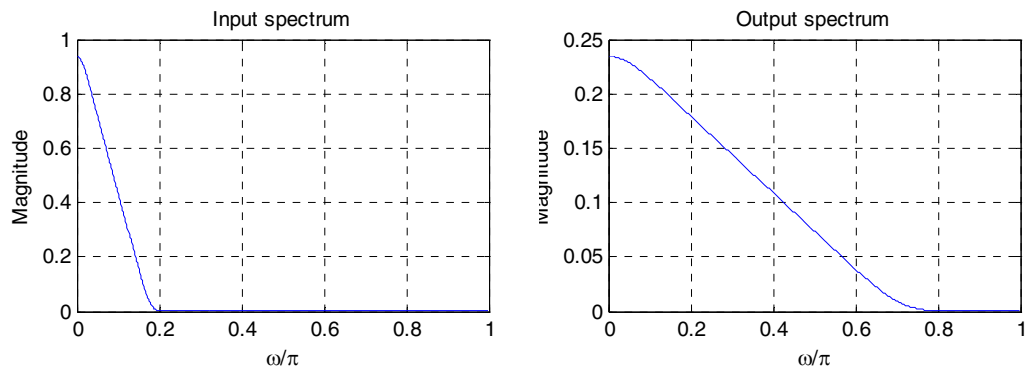
M13.4 (a)



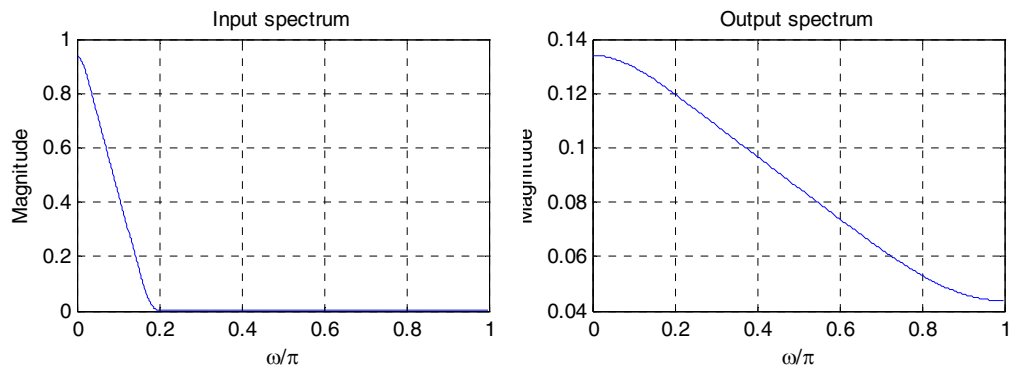
(b)



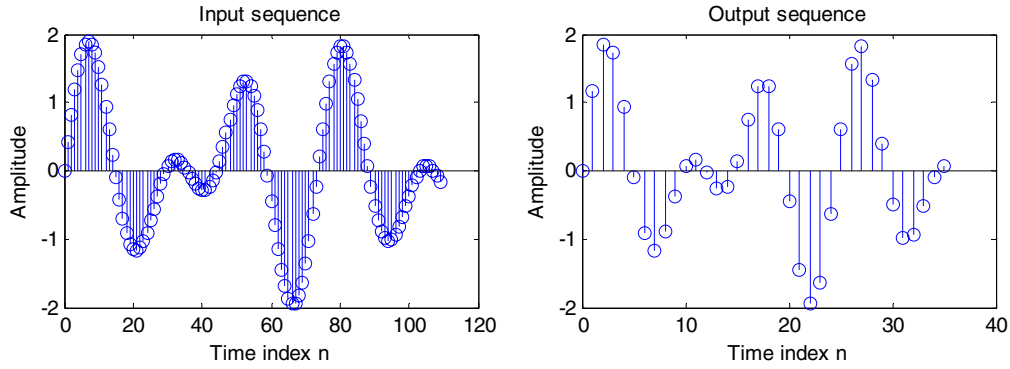
M13.5 (a)



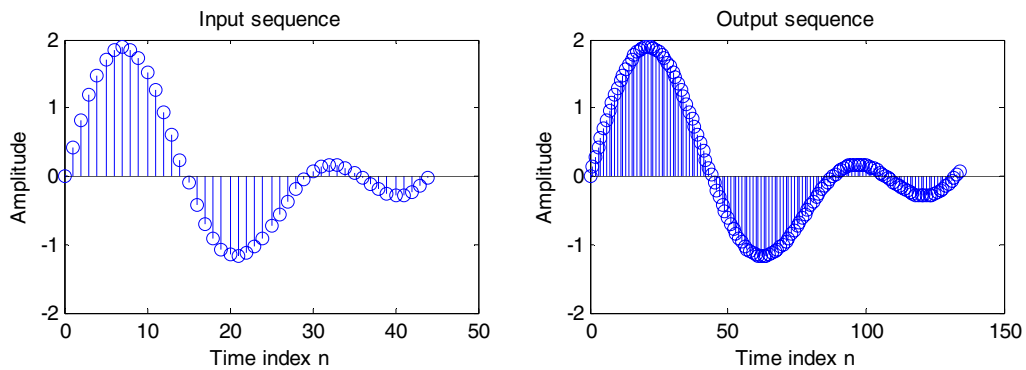
(b)



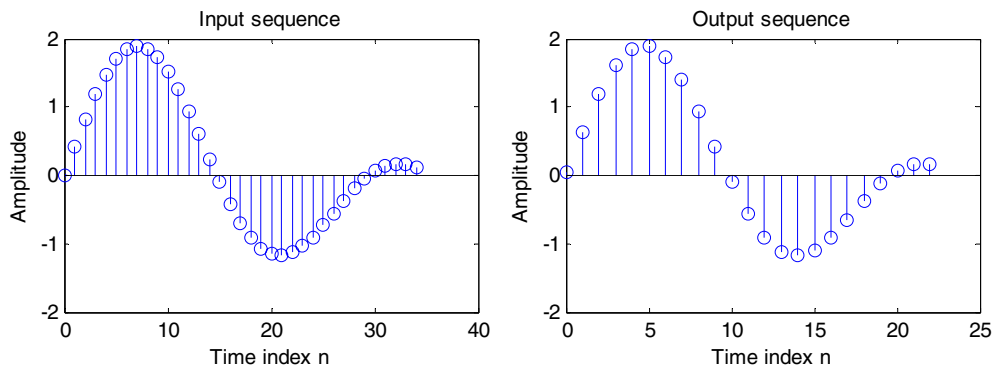
M13.6 (a)



M13.7 (a)



M13.8 (a)



M13.9 Using Program 13_9.m we arrive at the transfer function of the desired elliptic half-

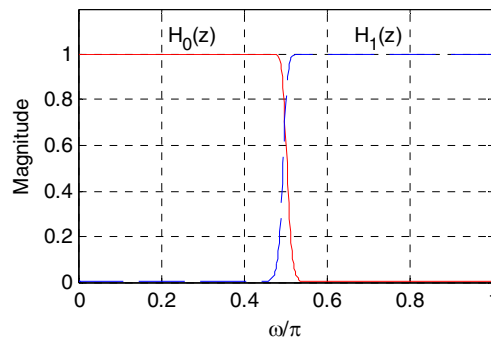
band lowpass filter: $H_0(z) = \frac{1}{2} [\mathcal{A}_0(z^2) + z^{-1} \mathcal{A}_1(z^2)]$ where

$$\mathcal{A}_0(z^2) = \frac{0.0192 + 0.3903z^{-2} + 1.2456z^{-4} + z^{-6}}{1 + 1.2456z^{-2} + 0.3903z^{-4} + 0.0192z^{-6}} \text{ and}$$

$$\mathcal{A}_1(z^2) = \frac{0.1206 + 0.8884z^{-2} + 1.7442z^{-4} + z^{-6}}{1 + 1.7442z^{-2} + 0.8884z^{-4} + 0.1206z^{-6}}. \text{ The power-complementary half-}$$

band highpass transfer function is given by $H_0(z) = \frac{1}{2} [\mathcal{A}_0(z^2) - z^{-1} \mathcal{A}_1(z^2)]$ A plot

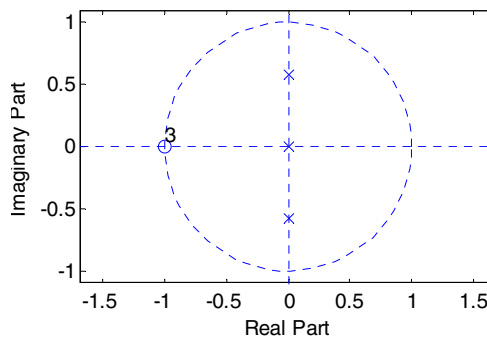
of the magnitude responses of the above half-band lowpass and highpass filters is shown below:



M13.10 (a) A digital lowpass half-band filter can be designed by applying a bilinear transformation to an analog lowpass Butterworth transfer function with a 3-dB cutoff frequency at 1 rad/sec. The 3-dB cutoff frequency of the digital lowpass Butterworth half-band filter is therefore at $\omega_c = 2 \tan^{-1}(1)/\pi = 0.5$.

To design a 3rd-order digital lowpass Butterworth half-band filter we use the MATLAB statement `[num,den] = butter(3,0.5)`; which yields

$$H(z) = \frac{0.1667(1+z^{-1})^3}{1+0.3333z^{-2}}$$
. As can be seen from the pole-zero plot of given below all poles are on the imaginary axis:



Using the MATLAB statement

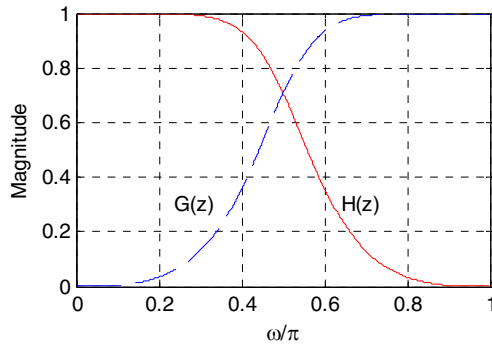
$$[d1,d2] = tf2ca([1 \ 3 \ 3 \ 1]/6, [1 \ 0 \ 1/3 \ 0]);$$

we arrive at the parallel allpass decomposition of $H(z)$ as

$$H(z) = \frac{1}{2}[A_0(z^2) + z^{-1}A_1(z^2)], \text{ where } A_0(z^2) = \frac{\frac{1}{3} + z^{-2}}{1 + \frac{1}{3}z^{-2}} \text{ and } A_1(z^2) = 1. \text{ Hence,}$$

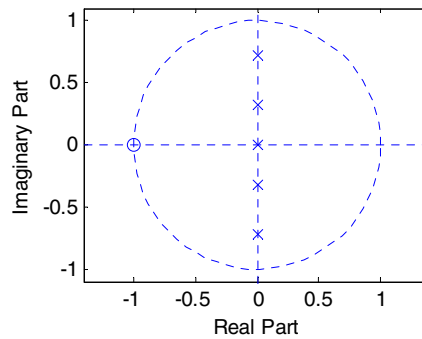
the power-complementary highpass transfer function is given by

$$G(z) = \frac{1}{2}[A_0(z^2) - z^{-1}A_1(z^2)] = \frac{0.1667(1-3z^{-1}+3z^{-2}-z^{-3})}{1+0.3333z^{-2}}.$$



(b) To design a 5th-order digital lowpass Butterworth half-band filter we use the MATLAB statement `[num, den] = butter(5, 0.5)`; which yields

$$H(z) = \frac{0.0528(1+z^{-1})^5}{1+0.6334z^{-2}+0.0557z^{-4}}$$
. As can be seen from the pole-zero plot of given below all poles are on the imaginary axis:



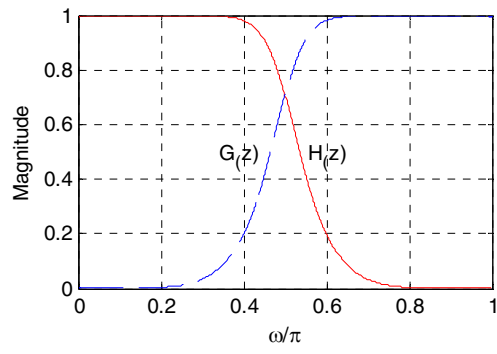
Using the MATLAB statement

`[d1, d2] = tf2ca(0.0528*[1 5 10 10 5 1], [1 0 0.6334 0 0.0557]);`
we arrive at the parallel allpass decomposition of $H(z)$ as

$$H(z) = \frac{1}{2}[A_0(z^2) + z^{-1}A_1(z^2)], \text{ where } A_0(z^2) = \frac{0.1056 + z^{-2}}{1 + 0.1056z^{-2}} \text{ and}$$

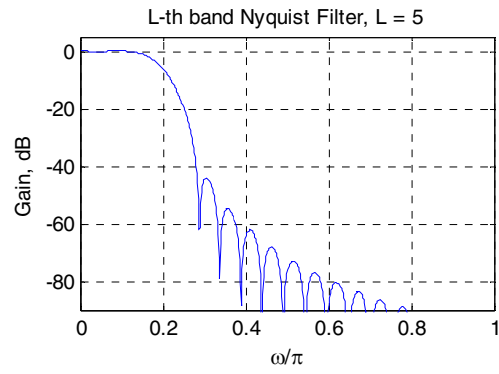
$$A_1(z^2) = \frac{0.5279 + z^{-2}}{1 + 0.5279z^{-2}}. \text{ Hence, the power-complementary highpass transfer function}$$

$$\text{is given by } G(z) = \frac{1}{2}[A_0(z^2) - z^{-1}A_1(z^2)] = \frac{0.0528(1-z^{-1})^5}{1+0.6334z^{-2}+0.0557z^{-4}}.$$



M13.11

M13.12



Chapter 14

14.1 (a) 2-band polyphase decomposition – Using $P(z) = 1 - cz^{-1}$, we express

$$H(z) = \frac{a + bz^{-1}}{1 + cz^{-1}} \text{ as}$$

$$\begin{aligned} H(z) &= \frac{(a + bz^{-1})(1 - cz^{-1})}{(1 + cz^{-1})(1 - cz^{-1})} = \frac{a + (b - ac)z^{-1} - bcz^{-2}}{1 - c^2z^{-2}} \\ &= \left(\frac{a - bcz^{-2}}{1 - c^2z^{-2}} \right) + z^{-1} \left(\frac{b - ac}{1 - c^2z^{-2}} \right). \text{ Hence, } E_0(z^2) = \frac{a - bcz^{-2}}{1 - c^2z^{-2}} \text{ and} \\ E_1(z^2) &= \frac{b - ac}{1 - c^2z^{-2}}. \end{aligned}$$

(b) 3-band polyphase decomposition – Using $P(z) = 1 - cz^{-1} + c^2z^{-2}$, we rewrite $H(z)$ as

$$\begin{aligned} H(z) &= \frac{(a + bz^{-1})(1 - cz^{-1} + c^2z^{-2})}{(1 + cz^{-1})(1 - cz^{-1} + c^2z^{-2})} = \frac{a + (b - ac)z^{-1} + c(ac - b)z^{-2} + bc^2z^{-3}}{1 - c^3z^{-3}} \\ &= \left(\frac{a + bc^2z^{-3}}{1 - c^3z^{-3}} \right) + z^{-1} \left(\frac{b - ac}{1 - c^3z^{-3}} \right) + z^{-2} \left(\frac{c(ac - b)}{1 - c^3z^{-3}} \right). \text{ Hence, } E_0(z^3) = \frac{a + bc^2z^{-3}}{1 - c^3z^{-3}}, \\ E_1(z^3) &= \frac{b - ac}{1 - c^3z^{-3}}, \quad E_2(z^3) = \frac{c(ac - b)}{1 - c^3z^{-3}}. \end{aligned}$$

14.2 (a) $H_a(z) = \frac{a + bz^{-1}}{1 + cz^{-1}}$. $H_a(zW_2^1) = H_a(-z) = \frac{a - bz^{-1}}{1 - cz^{-1}}$. Thus,

$$\begin{aligned} E_0(z^2) &= \frac{1}{2}[H_a(z) + H_a(-z)] = \frac{1}{2} \left(\frac{a + bz^{-1}}{1 + cz^{-1}} + \frac{a - bz^{-1}}{1 - cz^{-1}} \right) = \frac{a - bcz^{-2}}{1 - c^2z^{-2}} \text{ and} \\ z^{-1}E_1(z^2) &= \frac{1}{2}[H_a(z) - H_a(-z)] = \frac{1}{2} \left(\frac{a + bz^{-1}}{1 + cz^{-1}} - \frac{a - bz^{-1}}{1 - cz^{-1}} \right) = \frac{(b - ac)z^{-1}}{1 - c^2z^{-2}}. \end{aligned}$$

Hence, a 2-band polyphase decomposition of $H_a(z)$ is given by

$$H_a(z) = \left(\frac{a - bcz^{-2}}{1 - c^2z^{-2}} \right) + z^{-1} \left(\frac{b - ac}{1 - c^2z^{-2}} \right).$$

(b) $H_b(z) = \frac{3 - 4z^{-1} + 2.1z^{-2}}{1 - 0.8z^{-1} + 0.7z^{-2}}$. $H_b(zW_2^1) = H_b(-z) = \frac{3 + 4z^{-1} + 2.1z^{-2}}{1 + 0.8z^{-1} + 0.7z^{-2}}$.

Thus,

$$E_0(z^2) = \frac{1}{2}[H_b(z) + H_b(-z)] = \frac{1}{2} \left(\frac{3 - 4z^{-1} + 2.1z^{-2}}{1 - 0.8z^{-1} + 0.7z^{-2}} + \frac{3 + 4z^{-1} + 2.1z^{-2}}{1 + 0.8z^{-1} + 0.7z^{-2}} \right)$$

$$= \frac{3 + z^{-2} + 1.47z^{-4}}{1 + 0.76z^{-2} + 0.49z^{-4}}.$$

$$z^{-1}E_1(z^2) = \frac{1}{2}[H_b(z) - H_b(-z)] = \frac{1}{2} \left(\frac{3 - 4z^{-1} + 2.1z^{-2}}{1 - 0.8z^{-1} + 0.7z^{-2}} - \frac{3 + 4z^{-1} + 2.1z^{-2}}{1 + 0.8z^{-1} + 0.7z^{-2}} \right)$$

$$= \frac{-1.6z^{-1} - 1.12z^{-3}}{1 + 0.76z^{-2} + 0.49z^{-4}}. \text{ Hence, } E_1(z^2) = \frac{-1.6 - 1.12z^{-2}}{1 + 0.76z^{-2} + 0.49z^{-4}}.$$

Hence, a 2-band polyphase decomposition of $H_b(z)$ is given by

$$H_b(z) = \left(\frac{3 + z^{-2} + 1.47z^{-4}}{1 + 0.76z^{-2} + 0.49z^{-4}} \right) + z^{-1} \left(\frac{-1.6 - 1.12z^{-2}}{1 + 0.76z^{-2} + 0.49z^{-4}} \right).$$

$$(c) H_c(z) = \frac{4 + 2.5z^{-1} - 3.5z^{-2} + 2z^{-3}}{1 - 0.4z^{-1} + 0.78z^{-2} + 0.18z^{-3}}, H_c(-z) = \frac{4 - 2.5z^{-1} - 3.5z^{-2} - 2z^{-3}}{1 + 0.4z^{-1} + 0.78z^{-2} - 0.18z^{-3}}.$$

$$E_0(z^2) = \frac{1}{2}[H_c(z) + H_c(-z)]$$

$$= \frac{1}{2} \left(\frac{4 + 2.5z^{-1} - 3.5z^{-2} + 2z^{-3}}{1 - 0.4z^{-1} + 0.78z^{-2} + 0.18z^{-3}} + \frac{4 - 2.5z^{-1} - 3.5z^{-2} - 2z^{-3}}{1 + 0.4z^{-1} + 0.78z^{-2} - 0.18z^{-3}} \right)$$

$$= \frac{4 + 0.62z^{-2} - 2.38z^{-4} - 0.36z^{-6}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}}.$$

$$z^{-1}E_1(z^2) = \frac{1}{2}[H_c(z) - H_c(-z)] =$$

$$= \frac{1}{2} \left(\frac{4 + 2.5z^{-1} - 3.5z^{-2} + 2z^{-3}}{1 - 0.4z^{-1} + 0.78z^{-2} + 0.18z^{-3}} - \frac{4 - 2.5z^{-1} - 3.5z^{-2} - 2z^{-3}}{1 + 0.4z^{-1} + 0.78z^{-2} - 0.18z^{-3}} \right)$$

$$= \frac{4.1z^{-1} + 1.83z^{-3} + 2.19z^{-5}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}}. \text{ Thus,}$$

$$E_1(z^2) = \frac{4.1 + 1.83z^{-2} + 2.19z^{-4}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}}.$$

Hence, a 2-band polyphase decomposition of $H_c(z)$ is given by

$$H_c(z) = \left(\frac{4 + 0.62z^{-2} - 2.38z^{-4} - 0.36z^{-6}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}} \right) + z^{-1} \left(\frac{4.1 + 1.83z^{-2} + 2.19z^{-4}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}} \right).$$

14.3 (a) $H_1(z) = \frac{3 - 4z^{-1}}{1 - 0.5z^{-1}} = \sum_{k=0}^2 z^{-k} E_k(z^3)$. We can write,

$$\begin{bmatrix} H_1(z) \\ H_1(W_3^1 z) \\ H_1(W_3^2 z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix} \begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix}. \text{ Thus,}$$

$$\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix}^{-1} \begin{bmatrix} H_1(z) \\ H_1(W_3^1 z) \\ H_1(W_3^2 z) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix} \begin{bmatrix} H_1(z) \\ H_1(W_3^1 z) \\ H_1(W_3^2 z) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} E_0(z^3) &= \frac{1}{3}[H_1(z) + H_1(zW_3^1) + H_1(zW_3^2)] \\ &= \frac{1}{3} \left(\frac{3-4z^{-1}}{1-0.5z^{-1}} + \frac{3-4e^{j2\pi/3}z^{-1}}{1-0.5e^{j2\pi/3}z^{-1}} + \frac{3-4e^{j4\pi/3}z^{-1}}{1-0.5e^{j4\pi/3}z^{-1}} \right) = \frac{3-z^{-3}}{1-(0.5)^3z^{-3}}, \end{aligned}$$

$$\begin{aligned} z^{-1}E_1(z^3) &= \frac{1}{3}[H_1(z) + W_3^1 H_1(zW_3^1) + W_3^2 H_1(zW_3^2)] \\ &= \frac{1}{3} \left(\frac{3-4z^{-1}}{1-0.5z^{-1}} + e^{j2\pi/3} \frac{3-4e^{j2\pi/3}z^{-1}}{1-0.5e^{j2\pi/3}z^{-1}} + e^{j4\pi/3} \frac{3-4e^{j4\pi/3}z^{-1}}{1-0.5e^{j4\pi/3}z^{-1}} \right) \\ &= z^{-1} \left(\frac{-2.5}{1-(0.5)^3z^{-3}} \right), \end{aligned}$$

$$\begin{aligned} z^{-2}E_2(z^3) &= \frac{1}{3}[H_1(z) + W_3^2 H_1(zW_3^1) + W_3^1 H_1(zW_3^2)] \\ &= \frac{1}{3} \left(\frac{3-4z^{-1}}{1-0.5z^{-1}} + e^{j4\pi/3} \frac{3-4e^{j2\pi/3}z^{-1}}{1-0.5e^{j2\pi/3}z^{-1}} + e^{j2\pi/3} \frac{3-4e^{j4\pi/3}z^{-1}}{1-0.5e^{j4\pi/3}z^{-1}} \right) \\ &= z^{-2} \left(\frac{-1.25}{1-(0.5)^3z^{-3}} \right). \end{aligned}$$

$$\text{Hence, } E_0(z) = \frac{3-z^{-1}}{1-0.125z^{-1}}, \quad E_1(z) = \frac{-2.5}{1-0.125z^{-1}}, \quad E_2(z) = \frac{-1.25}{1-0.125z^{-1}}.$$

(b) $H_2(z) = \frac{4+2.1z^{-1}-3.4z^{-2}}{1-0.8z^{-1}+0.6z^{-2}} = \sum_{k=0}^2 z^{-k} E_k(z^3)$. We can write,

$$\begin{bmatrix} H_2(z) \\ H_2(W_3^1 z) \\ H_2(W_3^2 z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix} \begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix}. \text{ Thus,}$$

$$\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix}^{-1} \begin{bmatrix} H_2(z) \\ H_2(W_3^1 z) \\ H_2(W_3^2 z) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix} \begin{bmatrix} H_2(z) \\ H_2(W_3^1 z) \\ H_2(W_3^2 z) \end{bmatrix}.$$

$$\text{Therefore, } E_0(z^3) = \frac{1}{3}[H_2(z) + H_2(zW_3^1) + H_2(zW_3^2)]$$

$$= \frac{1}{3} \left(\frac{4 + 2.1z^{-1} - 3.4z^{-2}}{1 - 0.8z^{-1} + 0.6z^{-2}} + \frac{4 + 2.1e^{j2\pi/3}z^{-1} - 3.4e^{j4\pi/3}z^{-2}}{1 - 0.8e^{j2\pi/3}z^{-1} + 0.6e^{j4\pi/3}z^{-2}} + \frac{4 + 2.1e^{j4\pi/3}z^{-1} - 3.4e^{j2\pi/3}z^{-2}}{1 - 0.8e^{j4\pi/3}z^{-1} + 0.6e^{j2\pi/3}z^{-2}} \right)$$

$$= \frac{4 - 0.716z^{-3} - 1.224z^{-6}}{1 + 0.928z^{-3} + 0.216z^{-6}},$$

$$z^{-1}E_1(z^3) = \frac{1}{3} [H_2(z) + W_3^1 H_2(zW_3^1) + W_3^2 H_2(zW_3^2)] = \frac{1}{3} \left(\frac{4 + 2.1z^{-1} - 3.4z^{-2}}{1 - 0.8z^{-1} + 0.6z^{-2}} \right.$$

$$\left. + e^{j2\pi/3} \frac{4 + 2.1e^{j2\pi/3}z^{-1} - 3.4e^{j4\pi/3}z^{-2}}{1 - 0.8e^{j2\pi/3}z^{-1} + 0.6e^{j4\pi/3}z^{-2}} + e^{j4\pi/3} \frac{4 + 2.1e^{j4\pi/3}z^{-1} - 3.4e^{j2\pi/3}z^{-2}}{1 - 0.8e^{j4\pi/3}z^{-1} + 0.6e^{j2\pi/3}z^{-2}} \right)$$

$$= \frac{-1.56z^{-3} - 0.876z^{-6}}{1 + 0.928z^{-3} + 0.216z^{-6}},$$

$$z^{-1}E_2(z^3) = \frac{1}{3} [H_2(z) + W_3^2 H_2(zW_3^1) + W_3^1 H_2(zW_3^2)] = \frac{1}{3} \left(\frac{4 + 2.1z^{-1} - 3.4z^{-2}}{1 - 0.8z^{-1} + 0.6z^{-2}} \right.$$

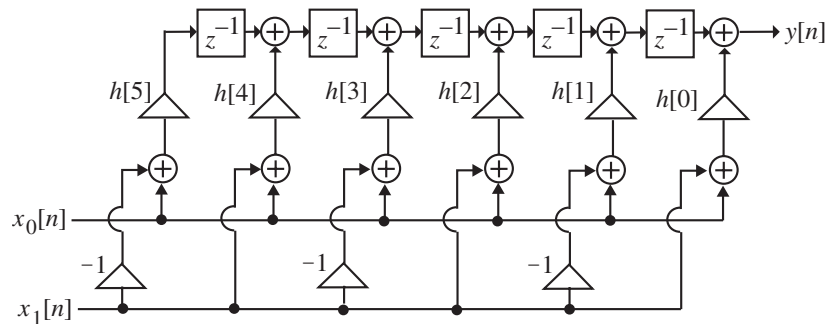
$$\left. + e^{j4\pi/3} \frac{4 + 2.1e^{j2\pi/3}z^{-1} - 3.4e^{j4\pi/3}z^{-2}}{1 - 0.8e^{j2\pi/3}z^{-1} + 0.6e^{j4\pi/3}z^{-2}} + e^{j2\pi/3} \frac{4 + 2.1e^{j4\pi/3}z^{-1} - 3.4e^{j2\pi/3}z^{-2}}{1 - 0.8e^{j4\pi/3}z^{-1} + 0.6e^{j2\pi/3}z^{-2}} \right)$$

$$= \frac{5.3z^{-3} + 2.312z^{-6}}{1 + 0.928z^{-3} + 0.216z^{-6}}.$$

Hence, $E_0(z) = \frac{4 - 0.716z^{-1} - 1.224z^{-2}}{1 + 0.928z^{-1} + 0.216z^{-2}}$, $E_1(z) = \frac{-1.56z^{-1} - 0.876z^{-2}}{1 + 0.928z^{-1} + 0.216z^{-2}}$,

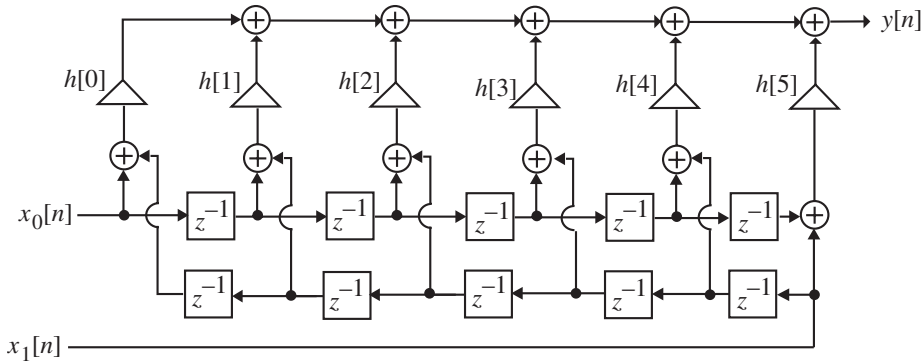
$$E_2(z) = \frac{5.3z^{-1} + 2.312z^{-2}}{1 + 0.928z^{-1} + 0.216z^{-2}}.$$

- 14.4** $H_0(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5}$, and $H_1(z) = H_0(-z) = h[0] - h[1]z^{-1} + h[2]z^{-2} - h[3]z^{-3} + h[4]z^{-4} - h[5]z^{-5}$. A realization of $H_0(z)$ and $H_1(z)$ in the form of Figure P14.1 using 5 delays and 6 multipliers is shown below:

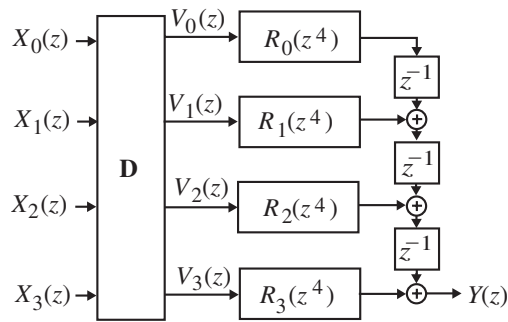


- 14.5** $H_0(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5}$, and $H_1(z) = z^{-5}H_0(z^{-1}) = h[5] + h[4]z^{-1} + h[3]z^{-2} + h[2]z^{-3} + h[1]z^{-4} + h[0]z^{-5}$.

$H_1(z) = H_0(-z) = h[0] - h[1]z^{-1} + h[2]z^{-2} - h[3]z^{-3} + h[4]z^{-4} - h[5]z^{-5}$. A realization of $H_0(z)$ and $H_1(z)$ in the form of Figure P14.1 using 6 multipliers is shown below:



14.6 (a) The structure of Figure P14.2(a) with internal variables labeled is shown below:



Analyzing the above structure we arrive at

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ V_2(z) \\ V_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} X_0(z) \\ X_1(z) \\ X_2(z) \\ X_3(z) \end{bmatrix},$$

and $Y(z) = R_3(z^4)V_3(z) + z^{-1}R_2(z^4)V_2(z) + z^{-2}R_1(z^4)V_1(z) + z^{-3}R_0(z^4)V_0(z)$.

The first equation leads to

$$V_0(z) = X_0(z) + X_1(z) + X_2(z) + X_3(z), \quad V_1(z) = X_0(z) - jX_1(z) - X_2(z) + jX_3(z),$$

$$V_2(z) = X_0(z) - X_1(z) + X_2(z) - X_3(z), \quad V_3(z) = X_0(z) + jX_1(z) - X_2(z) - jX_3(z).$$

Substituting these 4 equations in the equation for $Y(z)$ and solving for the transfer functions $G_i(z) = Y(z)/X_i(z)$, $0 \leq i \leq 3$, we arrive at

$$G_0(z) = R_3(z^4) + z^{-1}R_2(z^4) + z^{-2}R_1(z^4) + z^{-3}R_0(z^4),$$

$$G_1(z) = jR_3(z^4) - z^{-1}R_2(z^4) - jz^{-2}R_1(z^4) + z^{-3}R_0(z^4),$$

$$G_2(z) = -R_3(z^4) + z^{-1}R_2(z^4) - z^{-2}R_1(z^4) + z^{-3}R_0(z^4),$$

$$G_3(z) = -jR_3(z^4) - z^{-1}R_2(z^4) + jz^{-2}R_1(z^4) + z^{-3}R_0(z^4).$$

Substituting the expressions for $R_i(z)$, $0 \leq i \leq 3$, we finally arrive at

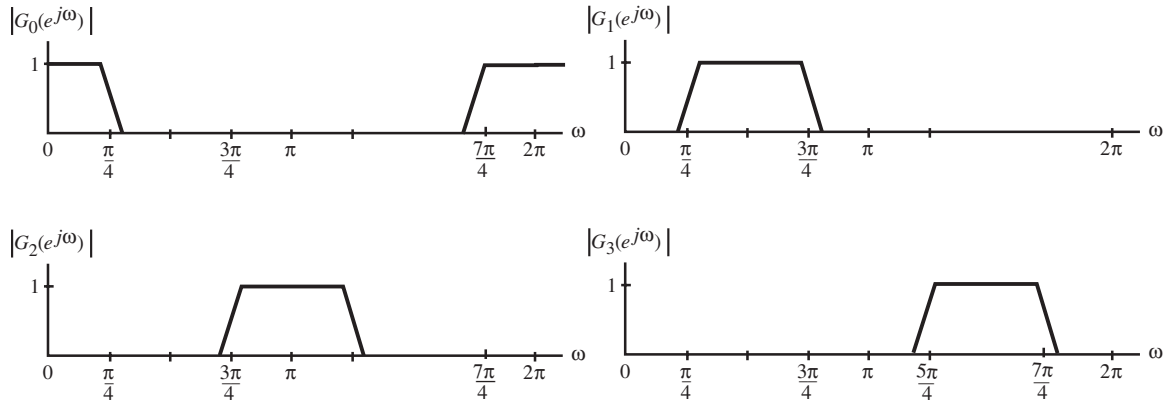
$$G_0(z) = 2 + z^{-1} + 4z^{-2} + z^{-3} - 1.5z^{-4} + 0.3z^{-5} - 0.9z^{-6} + 3.7z^{-7} + 3.1z^{-8} - 0.8z^{-9} + 2.3z^{-10} + 1.7z^{-11},$$

$$G_1(z) = j2 - z^{-1} - j4z^{-2} + z^{-3} - j1.5z^{-4} - 0.3z^{-5} + j0.9z^{-6} + 3.7z^{-7} + j3.1z^{-8} + 0.8z^{-9} - j2.3z^{-10} + 1.7z^{-11},$$

$$G_2(z) = -2 + z^{-1} - 4z^{-2} + z^{-3} + 1.5z^{-4} + 0.3z^{-5} + 0.9z^{-6} + 3.7z^{-7} - 3.1z^{-8} - 0.8z^{-9} - 2.3z^{-10} + 1.7z^{-11},$$

$$G_3(z) = -j2 - z^{-1} + j4z^{-2} + z^{-3} + j1.5z^{-4} - 0.3z^{-5} - j0.9z^{-6} + 3.7z^{-7} - j3.1z^{-8} + 0.8z^{-9} + j2.3z^{-10} + 1.7z^{-11}.$$

(b) The magnitude responses of all 4 analysis filters are shown below:



14.7 $Y(z) = [H_0(z)G_0(z) + H_1(z)G_1(z)]X(z)$. Now, $H_0(z) = \frac{1+z^{-1}}{2}$ and $H_1(z) = \frac{1-z^{-1}}{2}$.

Choose $G_0(z) = \frac{1+z^{-1}}{2}$ and $G_1(z) = -\frac{1-z^{-1}}{2}$. Then,

$$Y(z) = \left(\frac{1}{4}(1+z^{-1})^2 - \frac{1}{4}(1-z^{-1})^2 \right) X(z) = \frac{1}{4}(1+2z^{-1}+z^{-2} - 1+2z^{-1}-z^{-2})X(z)$$

$= z^{-1}X(z)$. Or in other words, $y[n] = x[n-1]$ indicating that the structure of Figure P14.3 is a perfect reconstruction system with the above choices of filters.

14.8 (a) Since $H_0(z)$ and $H_1(z)$ are power-complementary,

$$H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1. \text{ Now,}$$

$$Y(z) = (H_0(z)G_0(z) + H_1(z)G_1(z))X(z)$$

$$= z^{-N} (H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}))X(z) = z^{-N}X(z). \text{ Or in other words,}$$

$y[n] = x[n-N]$ indicating that the structure of Figure P14.3 is a perfect reconstruction system with the above choices of filters.

(b) If $H_0(z)$ and $H_1(z)$ are causal FIR transfer functions of order N each, $H_0(z)$ and $H_1(z)$ are polynomials in z^{-1} . As a result, $H_0(z^{-1})$ and $H_1(z^{-1})$ are polynomials in z with the highest power being z^N . Hence, $z^{-N}H_0(z^{-1})$ and $z^{-N}H_1(z^{-1})$ are polynomials in z^{-1} , making the synthesis filters $G_0(z)$ and $G_1(z)$ causal FIR transfer functions of order N each.

(c) From Figure P14.3, for perfect reconstruction we require $H_0(z)G_0(z) + H_1(z)G_1(z) = z^{-N}$. From Part (a) we note that the perfect reconstruction condition is satisfied with $G_0(z) = z^{-N}H_0(z^{-1})$ and $G_1(z) = z^{-N}H_1(z^{-1})$, if $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$, i.e., if $H_0(z)$ and $H_1(z)$ are power-complementary. The last condition is satisfied if and only if $H_0(z) = \frac{z^{-N/2}}{2}(z^{-n_o} + z^{n_o})$ and $H_1(z) = \frac{z^{-N/2}}{2}(z^{-n_o} - z^{n_o})$. As a result, $G_0(z)$ and $G_1(z)$ are also of the form $G_0(z) = \frac{z^{-N/2}}{2}(z^{-n_o} + z^{n_o})$ and $G_1(z) = \frac{z^{-N/2}}{2}(z^{-n_o} - z^{n_o})$.

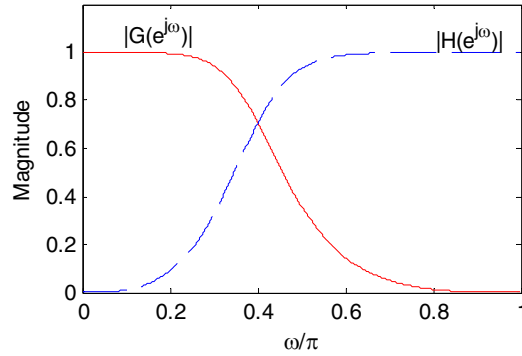
14.9 (a)
$$G(z) = \frac{0.0985(1+z^{-1})^3}{(1-0.1584z^{-1})(1-0.4189z^{-1}+0.3554z^{-2})}$$

$$= \frac{1}{2} \left[\frac{0.3554 - 0.4189z^{-1} + z^{-2}}{1 - 0.4189z^{-1} + 0.3554z^{-2}} + \frac{-0.1584 + z^{-1}}{1 - 0.1584z^{-1}} \right].$$
 Hence,
$$\mathcal{A}_0(z) = \frac{0.3554 - 0.4189z^{-1} + z^{-2}}{1 - 0.4189z^{-1} + 0.3554z^{-2}} \text{ and } \mathcal{A}_1(z) = \frac{-0.1584 + z^{-1}}{1 - 0.1584z^{-1}}.$$

(b) $G(z)$ can be realized with only 3 multipliers as a parallel connection of two allpass filters in the form of Figure 8.43, where the allpass filter $\mathcal{A}_0(z)$ is realized using any one of the Type 2 or Type 3 second-order allpass structures requiring 2 multipliers and the allpass filter $\mathcal{A}_1(z)$ is realized using any one of the Type 1 first-order allpass structures requiring 1 multiplier.

(c)
$$H(z) = \frac{1}{2} [\mathcal{A}_0(z) - \mathcal{A}_1(z)] = \frac{0.2564(1-z^{-1})^3}{(1-0.1584z^{-1})(1-0.4189z^{-1}+0.3554z^{-2})}.$$

(d) A plot of the magnitude responses of $G(z)$ and $H(z)$ is shown on top of the next page:



14.10 (a)
$$G(z) = \frac{0.1868(1 + 1.0902z^{-1} + 1.0902z^{-2} + z^{-3})}{(1 - 0.3628z^{-1})(1 - 0.5111z^{-1} + 0.7363z^{-2})}$$

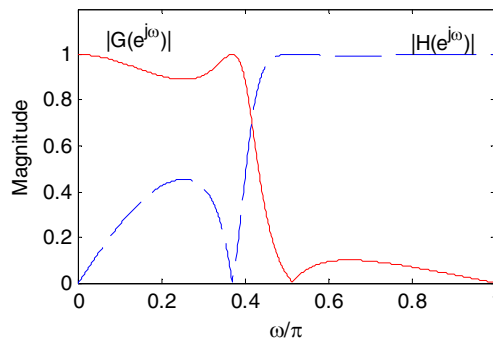
$$= \frac{1}{2} \left[\frac{0.7363 - 0.5111z^{-1} + z^{-2}}{1 - 0.5111z^{-1} + 0.7363z^{-2}} + \frac{-0.3628 + z^{-1}}{1 - 0.3628z^{-1}} \right]. \text{ Hence,}$$

$$\mathcal{A}_0(z) = \frac{0.7363 - 0.5111z^{-1} + z^{-2}}{1 - 0.5111z^{-1} + 0.7363z^{-2}} \text{ and } \mathcal{A}_1(z) = \frac{-0.3628 + z^{-1}}{1 - 0.3628z^{-1}}.$$

(b) $G(z)$ can be realized with only 3 multipliers as a parallel connection of two allpass filters in the form of Figure 8.43, where the allpass filter $\mathcal{A}_0(z)$ is realized using any one of the Type 2 or Type 3 second-order allpass structures requiring 2 multipliers and the allpass filter $\mathcal{A}_1(z)$ is realized using any one of the Type 1 first-order allpass structures requiring 1 multiplier.

(c)
$$H(z) = \frac{1}{2} [\mathcal{A}_0(z) - \mathcal{A}_1(z)] = \frac{0.5495(1 - 1.7866z^{-1} - 1.7866z^{-2} + z^{-3})}{(1 - 0.3628z^{-1})(1 - 0.5111z^{-1} + 0.7363z^{-2})}.$$

(d) A plot of the magnitude responses of $G(z)$ and $H(z)$ is shown below:



14.11 An N -th order, with N odd, elliptic lowpass transfer function $H(z)$ satisfies the condition

$$H(z)H(z^{-1}) = \frac{1}{1 + \varepsilon^2 R_N(z)R_N(z^{-1})}, \quad (\text{A})$$

where $R_N(z)$ is a rational function of the form

$$R_N(z) = \left(\frac{1-z^{-1}}{1-z^{-1}} \right) \prod_{\ell=0}^{N-1} \frac{(1-z^{-1}e^{j\phi_\ell})(1-z^{-1}e^{-j\phi_\ell})}{(1-z^{-1}e^{j\xi_\ell})(1-z^{-1}e^{-j\xi_\ell})}. \quad (\text{B})$$

In the above equation, the frequencies ξ_ℓ are the transmission zeros at which $H(e^{j\omega})$ is equal to 0, i.e., $H(e^{j\xi_\ell}) = 0$, and the frequencies ϕ_ℓ are the reflection zeros at which $|H(e^{j\omega})|$ is equal to the maximum value of 1, i.e., $|H(e^{j\phi_\ell})| = 1$.

From Eq. (B) it follows that

$$R_N(z) = -R_N(z^{-1}). \quad (\text{C})$$

Now, as $H(z)$ satisfies the power-symmetric condition, we have

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1, \text{ i.e.,}$$

$$H(z)H(z^{-1}) = 1 - H(-z)H(-z^{-1}). \quad (\text{D})$$

From the above equation it follows that the transmission zeros of $H(-z)$ are at frequencies $\pi - \phi_\ell$ and its reflection zeros are at $\pi - \xi_\ell$. As a result, $\phi_\ell + \xi_\ell = \pi$.

Hence, it follows that $R_N(-z) = 1/R_N(z)$, or equivalently,

$$R_N(z)R_N(-z) = 1. \quad (\text{E})$$

From Eq. (A) we have $H(-z)H(-z^{-1}) = \frac{1}{1 + \varepsilon^2 R_N(-z)R_N(-z^{-1})}$, which when

substituted in Eq. (D) yields

$$H(z)H(z^{-1}) = 1 - \frac{1}{1 + \varepsilon^2 R_N(-z)R_N(-z^{-1})} = \frac{1}{1 + \frac{1}{\varepsilon^2 R_N(-z)R_N(-z^{-1})}}. \text{ In view of}$$

Eq. (E), the above equation reduces to

$$H(z)H(z^{-1}) = \frac{1}{1 + \frac{1}{\varepsilon^2} R_N(z)R_N(z^{-1})}. \quad (\text{F})$$

Comparing Eqs. (A) and (F) we thus conclude that $\varepsilon^2 = 1$, and hence,

$$H(z)H(z^{-1}) = \frac{1}{1 + R_N(z)R_N(z^{-1})}. \text{ Now at a pole } z = \lambda \text{ of } H(z),$$

$R_N(\lambda)R_N(\lambda^{-1}) = -1$. From this relation, and Eqs. (C) and (E), it follows then that

$$\left| \frac{R_N(\lambda)}{R_N(-\lambda)} \right| = 1. \text{ Also, for a lowpass power-symmetric transfer function, } \xi_\ell > \pi/2$$

and $\phi_\ell < \pi/2$. Consequently, the poles of the rational function $R_N(z)/R_N(-z)$ must lie on the left half of the z -plane.

Since, the magnitude of the rational function $R_N(z)/R_N(-z)$ on the imaginary axis is 1, using maximum-modulus theorem it can be shown that

$|R_N(z)/R_N(-z)| < 1$ for $\text{Re } z > 0$. In a similar manner by replacing z with $-z$, it can be shown that $|R_N(-z)/R_N(z)| < 1$ for $\text{Re } z > 0$, or, equivalently, $|R_N(z)/R_N(-z)| > 1$ for $\text{Re } z < 0$. Thus,

$$\left| \frac{R_N(z)}{R_N(-z)} \right| \begin{cases} < 1, & \text{Re } z > 0, \\ = 1, & \text{Re } z = 0, \\ > 1, & \text{Re } z < 0, \end{cases}$$

or, in other words, all poles of $H(z)$ lie on the imaginary axis of the z -plane.

14.12 Now the magnitude-square function of an N -th order analog lowpass Butterworth

function $G_a(s)$ is given by $|G_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$, where Ω_c is the 3-dB

cutoff angular frequency. For $\Omega_c = 1$, then $|G_a(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$. The

corresponding transfer function of an N -th order analog highpass Butterworth function is simply $G_a(1/s)$, whose the magnitude-square function is given by

$$\left| G_a\left(\frac{1}{j\Omega}\right) \right|^2 = \frac{\Omega^{2N}}{1 + \Omega^{2N}}. \text{ As a result,}$$

$$|G_a(j\Omega)|^2 + \left| G_a\left(\frac{1}{j\Omega}\right) \right|^2 = \frac{1}{1 + \Omega^{2N}} + \frac{\Omega^{2N}}{1 + \Omega^{2N}} = 1. \text{ Now the bilinear transformation}$$

maps the analog angular frequency Ω to the digital angular frequency ω through the relation $e^{j\omega} = \frac{1 - j\Omega}{1 + j\Omega}$. As $-e^{j\omega} = \frac{1 - (1/j\Omega)}{1 + (1/j\Omega)}$, the analog angular frequency $1/\Omega$ is

mapped to the digital angular frequency $\pi + \omega$. Hence, the relation

$$|G_a(j\Omega)|^2 + \left| G_a\left(\frac{1}{j\Omega}\right) \right|^2 = 1, \text{ becomes } |H_0(e^{j\omega})|^2 + |H_0(e^{j(\pi+\omega)})|^2 =$$

$$|H_0(e^{j\omega})|^2 + |H_0(-e^{j\omega})|^2 = |H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1, \text{ where } H_0(e^{j\omega}) \text{ is the}$$

frequency response of the digital lowpass filter $H_0(z)$ obtained by applying the bilinear transformation to $G_a(s)$ and $H_1(e^{j\omega})$ is the frequency response of the digital highpass filter $H_0(z)$ obtained by applying the bilinear transformation to $G_a(1/s)$. Note that a

digital transfer function satisfying the condition $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$, is called power-symmetric.

Moreover, from the relation it follows that the analog 3-dB cutoff angular frequency $\Omega_c = 1$ is mapped into the digital 3-dB cutoff angular frequency $\omega_c = \pi/2$. Hence, $H_0(z)$ is a digital half-band lowpass filter.

Let $H_0(z) = \frac{P_0(z)}{D_0(z)}$, where $P_0(z)$ and $D_0(z)$ are polynomials in z^{-1} . Hence,

$$H_0(e^{j\omega}) = \frac{P_0(e^{j\omega})}{D_0(e^{j\omega})} \text{ and } H_0(-e^{j\omega}) = \frac{P_0(-e^{j\omega})}{D_0(-e^{j\omega})}. \text{ Now,}$$

$$\begin{aligned} |H_0(-e^{j\omega})|^2 &= \frac{P_0(-e^{j\omega})P_0^*(-e^{j\omega})}{D_0(-e^{j\omega})D_0^*(-e^{j\omega})} = 1 - |H_0(e^{j\omega})|^2 = 1 - \frac{P_0(e^{j\omega})P_0^*(e^{j\omega})}{D_0(e^{j\omega})D_0^*(e^{j\omega})} \\ &= \frac{D_0(e^{j\omega})D_0^*(e^{j\omega}) - P_0(e^{j\omega})P_0^*(e^{j\omega})}{D_0(e^{j\omega})D_0^*(e^{j\omega})}. \end{aligned}$$

Note that there are no common factors between

$P_0(e^{j\omega})$ and $D_0(e^{j\omega})$, and between $P_0^*(e^{j\omega})$ and $D_0^*(e^{j\omega})$. As a result, there are no common factors between $P_0(-e^{j\omega})P_0^*(-e^{j\omega})$ and $D_0(-e^{j\omega})D_0^*(-e^{j\omega})$. This implies then $D_0(e^{j\omega})D_0^*(e^{j\omega}) = D_0(-e^{j\omega})D_0^*(-e^{j\omega})$. Consequently, $D_0(e^{j\omega}) = D_0(-e^{j\omega})$, or $D_0(e^{j\omega}) = d_0(e^{j2\omega})$. Hence, $D_0(z) = d_0(z^2)$. Since $H_0(z) = \frac{P_0(z)}{D_0(z)} = \frac{P_0(z)}{d_0(z^2)}$, it

follows then $H_1(z) = \frac{P_0(-z)}{d_0(z^2)}$. We have shown earlier that $H_0(z)$ and $H_1(z)$ are power-

complementary. Also, $P_0(z)$ is a symmetric polynomial of odd order and $P_1(z)$ is an anti-symmetric polynomial of odd order. As a result, we can express

$H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)]$ and $H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)]$, where $A_0(z)$ and $A_1(z)$ are stable allpass functions for stable $H_0(z)$ and $H_1(z)$. But, $H_1(z) = H_0(-z)$. Hence,

$H_0(z) = \frac{1}{2}[A_0(-z) - A_1(-z)]$. It therefore follows that $A_0(z) = A_0(-z) = \mathcal{A}_0(z^2)$, and

$A_1(z) = -A_1(-z) = z^{-1}\mathcal{A}_1(z^2)$. Thus, $H_0(z) = \frac{1}{2}[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)]$.

14.13 From Eq. (14.18), $Y(z) = T(z)X(z) + A(z)X(-z)$. Let $Z^{-1}\{T(z)\} = t[n]$, and $Z^{-1}\{A(z)\} = a[n]$. Then, an inverse z -transform of Eq. (14.18) yields

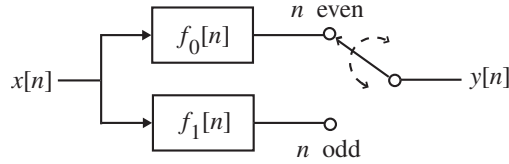
$$y[n] = \sum_{\ell=-\infty}^{\infty} t[\ell]x[n-\ell] + \sum_{\ell=-\infty}^{\infty} a[\ell](-1)^{n-\ell}x[n-\ell] = \sum_{\ell=-\infty}^{\infty} (t[\ell] + (-1)^{n-\ell}a[\ell])x[n-\ell].$$

Define $f_0[n] = t[n] + (-1)^{-n}a[n]$ and $f_1[n] = t[n] - (-1)^{-n}a[n]$. Then we can write

$$y[n] = \begin{cases} \sum_{\ell=-\infty}^{\infty} f_0[\ell]x[n-\ell], & \text{for } n \text{ even,} \\ \sum_{\ell=-\infty}^{\infty} f_1[\ell]x[n-\ell], & \text{for } n \text{ odd.} \end{cases}$$

The corresponding equivalent

realization of the 2-channel QMF bank is therefore as indicated below:



As can be seen from the above representation, the QMF bank is, in general, a time-varying system with a period 2. Note that $A(z) = 0$, if then it becomes a linear, time-invariant system.

14.14 $G_a(s) = \frac{1}{(s+1)(s^2+s+1)}$. The digital transfer function $H_0(z)$ obtained by a bilinear transformation is given by

$$\begin{aligned} H_0(z) &= G_a(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{(z+1)^3}{(z-1+z+1)[(z-1)^2+(z^2-1)+(z+1)^2]} = \frac{(z+1)^3}{2z(3z^2+1)} \\ &= \frac{(1+z^{-1})^3}{6+2z^{-2}} = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{6+2z^{-2}} = \frac{1}{2} \left(\frac{1+3z^{-2}}{3+z^{-2}} + z^{-1} \right) = \frac{1}{2} [\mathcal{A}_0(z^2) + z^{-1} \mathcal{A}_1(z^2)], \end{aligned}$$

where $\mathcal{A}_0(z) = \frac{1+3z^{-1}}{3+z^{-1}}$ and $\mathcal{A}_1(z) = 1$.

The corresponding power-complementary transfer function is given by

$$H_1(z) = \frac{1}{2} [\mathcal{A}_0(z^2) - z^{-1} \mathcal{A}_1(z^2)] = \frac{1}{2} \left(\frac{1+3z^{-2}}{3+z^{-2}} - z^{-1} \right) = \frac{(1-z^{-1})^3}{6+2z^{-2}}.$$

A realization of the analysis part of the QMF bank is as shown in Figure 14.11 where the first-order allpass transfer function $\mathcal{A}_0(z)$ is realized using any one of the single-multiplier structure of Figure 8.24 and the zero-th order allpass transfer function $\mathcal{A}_1(z)$ is replaced with a direct connection between the input and the output.

14.15 $G_a(s) = \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1}$. The digital transfer function $H_0(z)$ obtained by a bilinear transformation is given by

$$H_0(z) = \frac{(1+z^{-1})^5}{18.9443 + 12z^{-2} + 1.0557z^{-4}} = \frac{0.0528(1+z^{-1})^5}{1 + 0.6334z^{-2} + 0.0557z^{-4}}$$

$$\begin{aligned}
&= \frac{0.0528 + 0.5279z^{-2} + 0.2639z^{-4}}{1 + 0.6334z^{-2} + 0.0557z^{-4}} + \frac{0.2639z^{-1} + 0.5279z^{-3} + 0.0528z^{-5}}{1 + 0.6334z^{-2} + 0.0557z^{-4}} \\
&= \frac{0.0528(1 + 9.4704z^{-2})}{1 + 0.1056z^{-2}} + z^{-1} \frac{0.2639(1 + 1.8948z^{-2})}{1 + 0.5278z^{-2}} \\
&= \frac{1}{2} [\mathcal{A}_0(z^2) + z^{-1} \mathcal{A}_1(z^2)], \text{ where } \mathcal{A}_0(z) = \frac{0.1056 + z^{-1}}{1 + 0.1056z^{-1}} \text{ and} \\
&\mathcal{A}_1(z) = \frac{0.5278 + z^{-1}}{1 + 0.5278z^{-1}}.
\end{aligned}$$

Its power-complementary transfer function is given by

$$H_1(z) = \frac{1}{2} [\mathcal{A}_0(z^2) - z^{-1} \mathcal{A}_1(z^2)] = \frac{1}{2} \left[\frac{0.1056 + z^{-2}}{1 + 0.1056z^{-2}} - z^{-1} \frac{0.5278 + z^{-2}}{1 + 0.5278z^{-2}} \right].$$

In the realization of a magnitude-preserving QMF bank as shown in Figure 14.11, the realization of the allpass filters $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ require 1 multiplier each, and hence, the realization of the analysis (and the synthesis) filter bank requires a total of 2 multipliers.

14.16 (a) Total number of multipliers required is $4(2N - 1)$. Hence, the total number of multiplications per second is equal to $4(2N - 1)F_T = 4(2N - 1)/T$, where $F_T = 1/T$ is the sampling frequency in Hz.

(b) In Figure 14.11, $H_0(z) = \frac{1}{2} [\mathcal{A}_0(z^2) + z^{-1} \mathcal{A}_1(z^2)]$ and

$H_1(z) = \frac{1}{2} [\mathcal{A}_0(z^2) - z^{-1} \mathcal{A}_1(z^2)]$. If the order of $\mathcal{A}_0(z)$ is K and the order of $\mathcal{A}_1(z)$ is L , then the order of $H_0(z)$ is $2K + 2L + 1 = N$. Hence, $K + L = (N - 1)/2$. The total number of multipliers needed to implement is K , while the total number of multipliers needed to implement is L . Hence, the total number of multipliers required to implement the QMF bank of Figure 14.11 is $2(K + L) = N - 1$. However, the multipliers here are operating t half of the sampling rate of the input $x[n]$. As a result, the total number of multiplications per second in this case is $(N - 1)F_T / 2 = (N - 1)/2T$.

14.17 The 4 filters of the 2-channel QMF bank are: $H_0(z) = 4z^{-2}$, $H_1(z) = z^{-1}$, $G_0(z) = 0.25z^{-1}$, $G_1(z) = z^{-1}$. Substituting these transfer functions in Eq. (14.21) we get $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 4z^{-2} \times 0.25z^{-1} - z^{-1} \times z^{-2} = z^{-3} - z^{-3} = 0$, implying aliasing cancellation condition holds. Next, substituting the transfer functions in Eq. (14.27) we get

$H_0(z)G_0(z) + H_1(z)G_1(z) = 4z^{-2} \times 0.25z^{-1} + z^{-1} \times z^{-2} = 2z^{-3}$. Hence, it is a perfect reconstruction QMF bank.

14.18 The aliasing cancellation condition is satisfied if the synthesis filters satisfy Eq. (14.37). Hence, we choose $H_1(z) = G_0(-z) = d - ez^{-1} + fz^{-2} - gz^{-3} + hz^{-4}$, and $G_1(z) = -H_0(-z) = -(a - bz^{-1} + cz^{-2}) = -a - bz^{-1} - cz^{-2}$.

14.19 The 4 filters satisfy the aliasing cancellation condition of Eq. (14.21) and the perfect reconstruction condition of Eq. (14.27). Interchanging $H_0(z)$ and $G_0(z)$, and interchanging $H_1(z)$ and $G_1(z)$ in Eq. (14.21) we arrive at the new aliasing term $A(z) = G_0(-z)H_0(z) + G_1(-z)H_1(z)$. Hence,

$A(-z) = G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0$, implying that the aliasing condition is still satisfied after interchanging the analysis and synthesis filters. It also follows from Eq. (14.27), an interchange of the analysis and synthesis filters does not change the expression for the distortion transfer function and as a result the perfect reconstruction condition is still satisfied.

14.20 From Eq. (14.73), it can be seen that the normalized product filter $P(z)$ is of odd length $2L + 1$. Let the degrees of $H_0(z)$ and $G_0(z)$ be N and K , respectively. Hence, it follows from Eq. (14.76), $N + K = 2L$. Therefore, either N and K are both even or both odd. As a result, $H_0(z)$ and $G_0(z)$ cannot be of even and odd lengths, respectively.

Moreover, $P(z)$ is a symmetric polynomial. If $H_0(z)$ is a symmetric polynomial and $G_0(z)$ is an antisymmetric polynomial, their product cannot be a symmetric polynomial.

14.21 $P(z) = (1 + z^{-1})^3(1 + z)^3(az^2 + bz + c + bz^{-1} + az^{-2}) = az^5 + (6a + b)z^4 + (15a + 6b + c)z^3 + (20a + 16b + 6c)z^2 + (16a + 26b + 15c)z + (12a + 30b + 20c) + (16a + 26b + 15c)z^{-1} + (20a + 16b + 6c)z^{-2} + (15a + 6b + c)z^{-3} + (6a + b)z^{-4} + az^{-5}$.

Since the even powers of $P(z)$ must be zeros and the coefficient of z^0 be equal to 1, we must have $6a + b = 0$, $20a + 16b + 6c = 0$, and $12a + 30b + 20c = 1$. Solving these 3 equations we arrive at $a = 0.01171875$, $b = -0.0703125$, $c = 0.1484375$.

Hence, $P(z) = (1 + z^{-1})^3(1 + z)^3(0.1171875z^2 - 0.0703125z + 0.1484375 - 0.0703125z^{-1} + 0.1171875z^{-2}) = 0.1171875z^5(1 + z^{-1})^6(1 - 5.4255z^{-1} + 9.4438z^{-2}) \times (1 - 0.5745z^{-1} + 0.1059z^{-2})$.

One possible factorization of $P(z)$ is given by

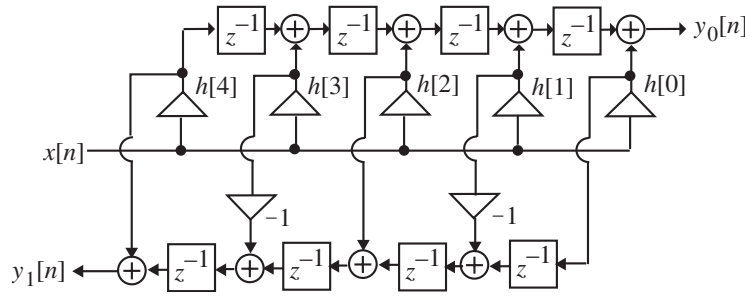
$H_0(z) = 0.01171875(1+z^{-1})(1-5.4255z^{-1}+9.4438z^{-2})(1-0.5745z^{-1}+0.1059z^{-2})$,
and $G_0(z) = (1+z^{-1})^5$. Thus, the highpass analysis filter is given by
 $H_1(z) = G_0(-z) = (1-z^{-1})^5$.

14.22 Without any loss of generality, let $N = 4$. Let

$H_0(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}$. Then,

$H_1(z) = z^{-4}H_0(-z^{-1}) = z^{-4}(h[0] - h[1]z + h[2]z^2 - h[3]z^3 + h[4]z^4)$

$= h[4] - h[3]z^{-1} + h[2]z^{-2} - h[1]z^{-3} + h[0]z^{-4}$. A realization of the analysis filter bank using $N + 1 = 5$ multipliers and $2N = 8$ two-input adders is shown on top of the next page:



14.23 For an orthogonal filter bank, the two analysis filters $H_0(z) = \sum_{n=0}^N h_0[n]z^{-n}$

and $H_1(z) = \sum_{n=0}^N h_1[n]z^{-n}$ are causal FIR filters satisfy the power

complementary property, i.e., $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = \gamma$, where $\gamma > 0$.

We assume $h_0[0] \neq 0$ and $h_1[0] \neq 0$. It follows from Section 7.3 that a linear-phase FIR filter has either a symmetric or an anti-symmetric impulse response. As a result, if $H_0(z)$ and $H_1(z)$ are linear-phase transfer functions, then they are of the form $H_0(z) = e^{-j\beta_0} z^{-N} H_0(z^{-1})$ and $H_1(z) = e^{-j\beta_1} z^{-N} H_1(z^{-1})$.

Substituting the expressions for $H_0(z^{-1})$ and $H_1(z^{-1})$ in the power-

complementary condition we arrive at $e^{j\beta_0} z^N H_0^2(z) + e^{j\beta_1} z^N H_1^2(z) = \gamma$, which can be rewritten as

$[e^{j\beta_0/2} H_0(z) + je^{j\beta_1/2} H_1(z)][e^{j\beta_0/2} H_0(z) - je^{j\beta_1/2} H_1(z)] = \gamma z^{-N}$. Both quantities inside the square brackets are causal FIR transfer functions. Hence, we have

$$e^{j\beta_0/2} H_0(z) + je^{j\beta_1/2} H_1(z) = \alpha_0 z^{-N_0}, \quad (\text{G})$$

and

$$e^{j\beta_0/2} H_0(z) - je^{j\beta_1/2} H_1(z) = \alpha_1 z^{-N_1}, \quad (\text{H})$$

where $\alpha_0\alpha_1 = \gamma$ and $N_0 + N_1 = N$. Adding Eq. (H) from Eq. (G) we get

$$H_0(z) = e^{-j\beta_0/2}(\alpha_0 z^{-N_0} + \alpha_1 z^{-N_1}) \text{ and}$$

$H_1(z) = -je^{-j\beta_1/2}(\alpha_0 z^{-N_0} - \alpha_1 z^{-N_1})$. Hence, as the analysis filters have transfer functions that are weighted sum of two delays, they can not be used to design filters with any practical frequency response specifications.

14.24 From Eq. (14.90) we have $H_0(z) = -z^{-3}H_1(-z^{-1}) = -z^{-3}(a - bz + cz^2 - dz^3)$
 $= d - cz^{-1} + bz^{-2} - az^{-3}$. Next, from Eq. (14.92) we get
 $G_0(z) = z^{-3}H_0(z^{-1}) = z^{-3}(d - cz + bz^2 - az^3) = -a + bz^{-1} - cz^{-2} + dz^{-3}$ and
 $G_1(z) = z^{-3}H_1(z^{-1}) = z^{-3}(a + bz + cz^2 + dz^3) = d + cz^{-1} + bz^{-2} + az^{-3}$.

14.25 Substituting the transfer functions in Eq. (14.21) we get
 $H_0(-z)G_0(z) + H_1(-z)G_1(z) = (3 - 4z^{-1})(-0.5 + z^{-1}) + (1 - 2z^{-1})(1.5 - 2z^{-1})$
 $= (-1.5 + 5z^{-1} - 4z^{-2}) + (1.5 - 5z^{-1} + 4z^{-2}) = 0$ implying that the aliasing
cancellation condition is satisfied. Next, substituting the transfer functions in Eq.
(14.27) we get

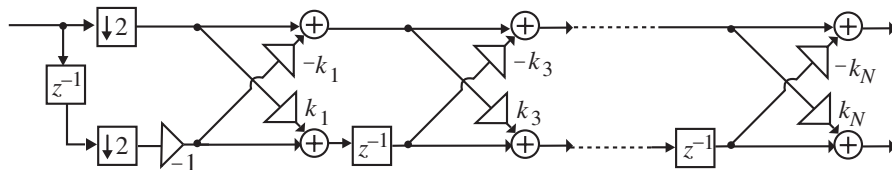
$$H_0(z)G_0(z) + H_1(z)G_1(z) = (3 + 4z^{-1})(-0.5 + z^{-1}) + (1 + 2z^{-1})(1.5 - 2z^{-1})$$

$$= (-1.5 + z^{-1} + 4z^{-2}) + (1.5 + z^{-1} - 4z^{-2}) = 2z^{-1}$$

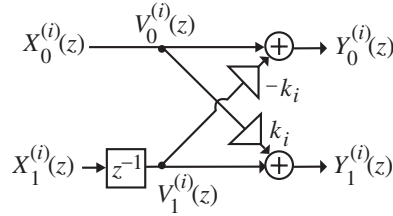
implying that the perfect reconstruction condition is satisfied.

14.26 $H_0(z) = a + bz^{-1} + cz^{-2} + dz^{-3} + ez^{-4} + fz^{-5}$. From Eq. (14.90) we get
 $H_1(z) = z^{-5}H_0(-z^{-1}) = z^{-5}(a - bz + cz^2 - dz^3 + ez^4 - fz^5)$
 $= -f + ez^{-1} - dz^{-2} + cz^{-3} - bz^{-4} + az^{-5}$. From Eq. (14.92), we arrive at
 $G_0(z) = z^{-5}H_0(z^{-1}) = f + ez^{-1} + dz^{-2} + cz^{-3} + bz^{-4} + az^{-5}$ and
 $G_1(z) = z^{-5}H_1(z^{-1}) = a - bz^{-1} + cz^{-2} - dz^{-3} + ez^{-4} - fz^{-5}$.

14.27 To develop the realization of the synthesis filter bank we redraw Figure P14.4 as shown below:



The i -th stage of the above analysis filter bank is of the form shown below:

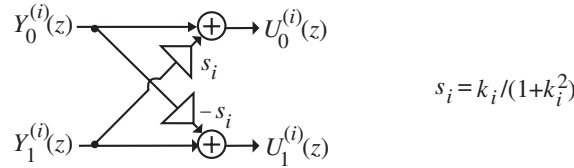


The input-output relation of the lattice part of the above figure is given by

$$\begin{bmatrix} Y_0^{(i)}(z) \\ Y_1^{(i)}(z) \end{bmatrix} = \begin{bmatrix} 1 & k_i \\ -k_i & 1 \end{bmatrix} \begin{bmatrix} V_0^{(i)}(z) \\ V_1^{(i)}(z) \end{bmatrix} \text{ which can be solved for the input variables}$$

leading to
$$\begin{bmatrix} U_0^{(i)}(z) \\ U_1^{(i)}(z) \end{bmatrix} = \begin{bmatrix} 1 & k_i \\ -k_i & 1 \end{bmatrix}^{-1} \begin{bmatrix} Y_0^{(i)}(z) \\ Y_1^{(i)}(z) \end{bmatrix} = \frac{1}{1+k_i^2} \begin{bmatrix} 1 & -k_i \\ k_i & 1 \end{bmatrix} \begin{bmatrix} Y_0^{(i)}(z) \\ Y_1^{(i)}(z) \end{bmatrix}, \text{ a}$$

realization of which is indicated below:



where $s_i = k_i / (1 + k_i^2)$. A cascade of the above two lattice structures thus has an input-output relation given by

$$\begin{bmatrix} U_0^{(i)}(z) \\ U_1^{(i)}(z) \end{bmatrix} = \frac{1}{1+k_i^2} \begin{bmatrix} 1 & -k_i \\ k_i & 1 \end{bmatrix} \begin{bmatrix} 1 & k_i \\ -k_i & 1 \end{bmatrix} \begin{bmatrix} V_0^{(i)}(z) \\ V_1^{(i)}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_0^{(i)}(z) \\ V_1^{(i)}(z) \end{bmatrix}.$$

Likewise, the input-output relation of the delay part of the i -th stage of the analysis

filter bank is given by
$$\begin{bmatrix} V_0^{(i)}(z) \\ V_1^{(i)}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} X_0^{(i)}(z) \\ X_1^{(i)}(z) \end{bmatrix}$$
 which can be solved for the

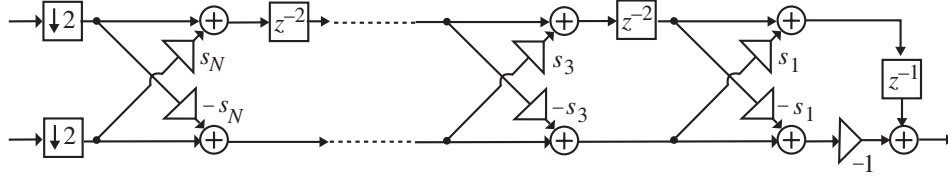
input variables leading to
$$\begin{bmatrix} W_0^{(i)}(z) \\ W_1^{(i)}(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_0^{(i)}(z) \\ V_1^{(i)}(z) \end{bmatrix},$$
 a realization of which is

indicated below:

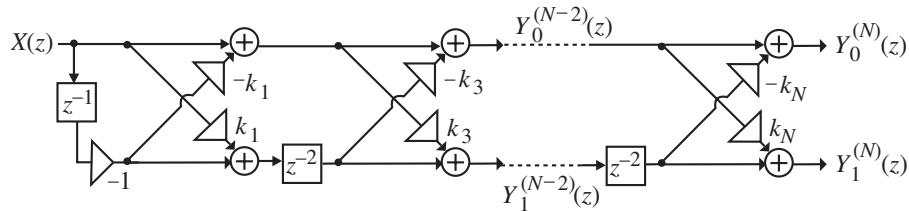
$$V_0^{(i)}(z) \longrightarrow \boxed{z^{-1}} \longrightarrow W_0^{(i)}(z)$$

$$V_1^{(i)}(z) \longrightarrow W_1^{(i)}(z)$$

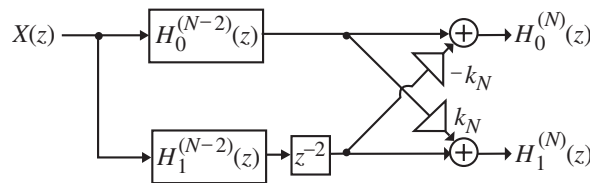
Hence, the cascaded-lattice realization of the synthesis filter bank of the two-channel orthogonal filter bank is as shown below where $s_i = k_i / (1 + k_i^2)$.



14.28 Figure P14.4 with the down-samplers removed and internal variables labeled is shown below:



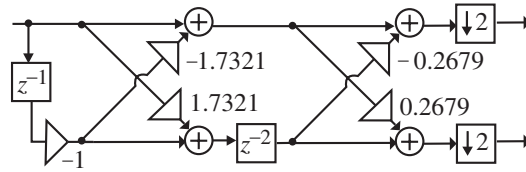
Denoting the transfer functions $H_0^{(r)}(z) = Y_0^{(r)}(z)/X(z)$ and $H_1^{(r)}(z) = Y_1^{(r)}(z)/X(z)$, we arrive at an equivalent representation of the above analysis filter bank as indicated below:



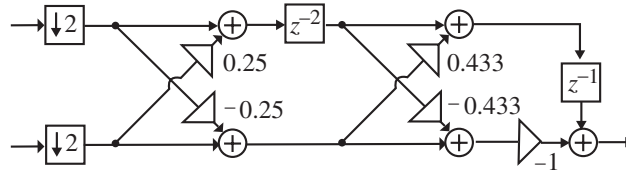
Analysis yields $H_0^{(N)}(z) = H_0^{(N-2)}(z) + k_N z^{-2} H_1^{(N-2)}(z)$ and $H_1^{(N)}(z) = -k_N H_0^{(N-2)}(z) + z^{-2} H_1^{(N-2)}(z)$. Solving these two equations, we get $(1 + k_N^2) H_0^{(N-2)}(z) = H_0^{(N)}(z) - k_N H_1^{(N)}(z)$ and $(1 + k_N^2) z^{-2} H_1^{(N-2)}(z) = k_N H_0^{(N)}(z) + H_1^{(N)}(z)$. Choose k_N such that the highest power of z^{-1} in $H_0^{(N)}(z) - k_N H_1^{(N)}(z)$ is eliminated. By construction, the next highest power of z^{-1} also is removed reducing the order of $H_0^{(N-2)}(z)$ to $N - 2$. Moreover, the coefficients of z^0 and z^{-1} in $k_N H_0^{(N)}(z) + H_1^{(N)}(z)$ are also zero for the above choice of k_N , resulting in a causal $H_1^{(N-2)}(z)$ of order $N - 2$. Continuing this process, we arrive at a cascaded lattice realization of the analysis filter bank of an orthogonal QMF bank.

From Eq. (14.87) we have $H_0^{(3)}(z) = 0.3415 + 0.5915z^{-1} + 0.1585z^{-2} - 0.0915z^{-3}$ and from Eq. (14.89a) we have $H_1^{(3)}(z) = -0.0915 - 0.1585z^{-1} + 0.5915z^{-2} - 0.3415z^{-3}$. We now form $(1 + k_3^2) H_0^{(1)}(z) = H_0^{(3)}(z) - k_3 H_1^{(3)}(z) = (0.3415 + 0.0915k_3) + (0.5915 + 0.1585k_3)z^{-1} + (0.1585 - 0.5915k_3)z^{-2} + (-0.0915 + 0.3415k_3)z^{-3}$.

Choose $k_3 = 0.0915/0.3415 = 0.2679$. Then $1.0718H_0^{(1)}(z) = H_0^{(3)}(z) - k_3H_1^{(3)}(z) = 0.366 + 0.634z^{-1}$, implying, $H_0^{(1)}(z) = 0.3415 + 0.5915z^{-1} = 0.3415(1 + 1.7321z^{-1})$. We next form $(1 + k_3^2)z^{-2}H_1^{(1)}(z) = [k_3H_0^{(3)}(z) + H_1^{(3)}(z)]$ or, $1.0718z^{-2}H_1^{(1)}(z) = [0.2679H_0^{(3)}(z) + H_1^{(3)}(z)] = 0.634 - 0.366z^{-1}$, implying, $H_1^{(1)}(z) = 0.5915 - 0.3415z^{-1} = 0.3415(1.7321 - z^{-1})$. Hence, $k_1 = 1.7321$. A cascaded lattice realization of the analysis filter bank is as shown below:



From the solution of Problem 14.27 we arrive at the structure of the synthesis filter bank shown on the next page to ensure the realization of the orthogonal perfect reconstruction QMF bank.



14.29 (a) $H_0(z) = 1 + 3z^{-1} + 14z^{-2} + 22z^{-3} - 12z^{-4} + 4z^{-5}$. Hence,

$$H_1(z) = -z^{-5}H_0(-z^{-1}) = 4 + 12z^{-1} + 22z^{-2} - 14z^{-3} + 3z^{-4} - z^{-5}.$$

(b) To develop the cascade lattice realization of the analysis filter bank, we rewrite the analysis transfer functions as $H_0^{(5)}(z) = H_0(z)$ and $H_1^{(5)}(z) = H_1(z)$. Next, we determine $H_0^{(3)}(z)$ using the relation $(1 + k_5^2)H_0^{(3)}(z) = H_0^{(5)}(z) - k_5H_1^{(5)}(z)$ and choose k_5 so that the coefficient of z^{-5} is eliminated. Now,

$$(1 + k_5^2)H_0^{(3)}(z) = (1 - 4k_5) + (3 - 12k_5)z^{-1} + (14 - 22k_5)z^{-2} + (22 + 14k_5)z^{-3} + (-12 - 3k_5)z^{-4} + (4 + k_5)z^{-5}.$$

We choose $k_5 = -4$. Then $(1 + k_5^2)H_0^{(3)}(z) = 17H_0^{(3)}(z) = 17 + 51z^{-1} + 102z^{-2} - 34z^{-3}$. Note that as expected the above choice of k_5 also cancels the coefficient of z^{-4} . We thus have,

$$H_0^{(3)}(z) = 1 + 3z^{-1} + 6z^{-2} - 2z^{-3}.$$

Next we determine $H_1^{(3)}(z)$ using the relation

$$(1 + k_5^2)z^{-2}H_1^{(3)}(z) = k_5H_0^{(5)}(z) + H_1^{(5)}(z)$$

which leads to

$$17z^{-2}H_1^{(3)}(z) = -34z^{-2} - 102z^{-3} + 51z^{-4} - 17z^{-5}.$$

It should be noted that here,

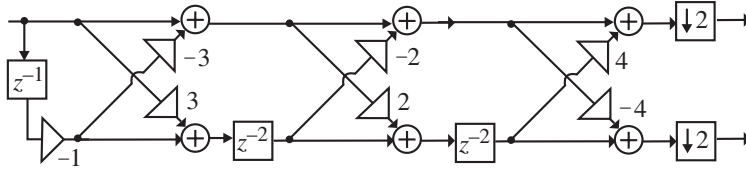
as expected, the choice of k_5 has resulted in zero-valued coefficients of z^0 and z^{-1} . Hence, $H_1^{(3)}(z) = -2 - 6z^{-1} + 3z^{-2} - z^{-3}$. It can be verified that the same expression for $H_1^{(3)}(z)$ is obtained from $H_1^{(3)}(z) = -z^{-3}H_0^{(3)}(-z^{-1})$.

Next, we determine $H_0^{(1)}(z)$ using the relation

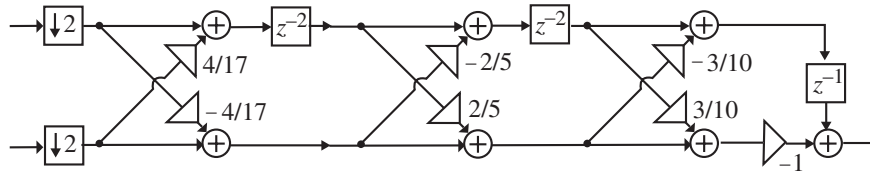
$(1 + k_3^2)H_0^{(1)}(z) = H_0^{(3)}(z) - k_3H_1^{(3)}(z)$ and choose $k_3 = 2$ so that the coefficient of z^{-3} is eliminated. This results in $H_0^{(1)}(z) = 1 + 3z^{-1}$ and $H_1^{(1)}(z) = 3 - z^{-1}$.

Analysis of the structure of Figure P14.4 yields $H_0^{(1)}(z) = 1 + k_1z^{-1}$ and

$H_1^{(1)}(z) = k_1 - z^{-1}$. Hence, $k_1 = 3$. The final realization of the analysis filter bank is shown below:



From the solution of Problem 14.27 we arrive at the structure of the synthesis filter bank shown below to ensure the realization of a paraunitary perfect reconstruction QMF bank.



14.30 (a) $H_0(z) = 0.5 - z^{-1} + 10.5z^{-2} - 13.5z^{-3} - 5z^{-4} - 2.5z^{-5}$. Hence,
 $H_1(z) = -z^{-5}H_0(-z^{-1}) = -2.5 + 5z^{-1} - 13.5z^{-2} - 10.5z^{-3} - z^{-4} - 0.5z^{-5}$.

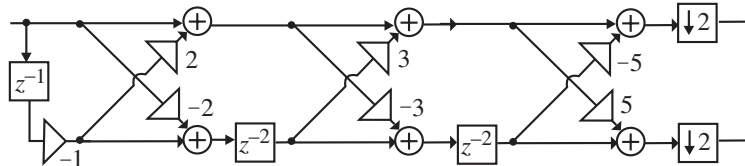
(b) To develop the cascade lattice realization of the analysis filter bank, we rewrite the analysis transfer functions as $H_0^{(5)}(z) = H_0(z)$ and $H_1^{(5)}(z) = H_1(z)$. Next, we determine $H_0^{(3)}(z)$ using the relation $(1 + k_5^2)H_0^{(3)}(z) = H_0^{(5)}(z) - k_5H_1^{(5)}(z)$

and choose k_5 so that the coefficient of z^{-5} is eliminated. Now,

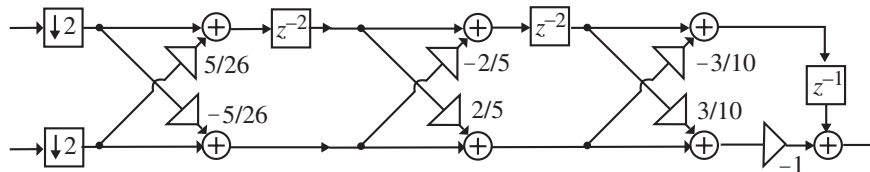
$$(1 + k_5^2)H_0^{(3)}(z) = (0.5 + 2.5k_5) + (-1 - 5k_5)z^{-1} + (10.5 + 13.5k_5)z^{-2} + (-13.5 + 10.5k_5)z^{-3} + (-5 + k_5)z^{-4} + (-2.5 + 0.5k_5)z^{-5}.$$

We choose $k_5 = 5$. Then $(1 + k_5^2)H_0^{(3)}(z) = 26H_0^{(3)}(z) = 13 - 26z^{-1} + 78z^{-2} + 39z^{-3}$. Note that as expected the above choice of k_5 also cancels the coefficient of z^{-4} . We thus have,

$H_0^{(3)}(z) = 0.5 - z^{-1} + 3z^{-2} + 1.5z^{-3}$. Next we determine $H_1^{(3)}(z)$ using the relation $(1 + k_5^2)z^{-2}H_1^{(3)}(z) = k_5H_0^{(5)}(z) + H_1^{(5)}(z)$ which leads to $26z^{-2}H_1^{(3)}(z) = 39z^{-2} - 78z^{-3} - 26z^{-4} - 13z^{-5}$. It should be noted that here, as expected, the choice of k_5 has resulted in zero-valued coefficients of z^0 and z^{-1} . Hence, $H_1^{(3)}(z) = 1.5 - 3z^{-1} - z^{-2} - 0.5z^{-3}$. It can be verified that the same expression for $H_1^{(3)}(z)$ is obtained from $H_1^{(3)}(z) = -z^{-3}H_0^{(3)}(-z^{-1})$. Next, we determine $H_0^{(1)}(z)$ using the relation $(1 + k_3^2)H_0^{(1)}(z) = H_0^{(3)}(z) - k_3H_1^{(3)}(z)$ and choose $k_3 = -3$ so that the coefficient of z^{-3} is eliminated. This results in $H_0^{(1)}(z) = 0.5 - z^{-1} = 0.5(1 - 2z^{-1})$ and $H_1^{(1)}(z) = -1 - 0.5z^{-1} = 0.5(-2 - z^{-1})$. Analysis of the structure of Figure P14.4 yields $H_0^{(1)}(z) = 1 + k_1z^{-1}$ and $H_1^{(1)}(z) = k_1 - z^{-1}$. Hence, $k_1 = -2$. The final realization of the analysis filter bank is shown on top of next page:



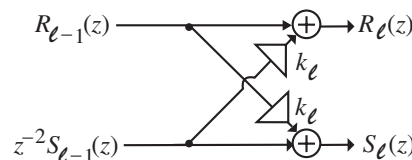
From the solution of Problem 14.27 we arrive at the structure of the synthesis filter bank shown below to ensure the realization of a paraunitary perfect reconstruction QMF bank.



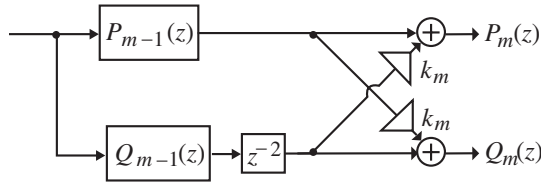
14.31 (a) The input-output relation of the ℓ -th two-pair is given by

$$\begin{bmatrix} R_\ell(z) \\ S_\ell(z) \end{bmatrix} = \begin{bmatrix} 1 & k_\ell \\ k_\ell & 1 \end{bmatrix} \begin{bmatrix} R_{\ell-1}(z) \\ z^{-2}S_{\ell-1}(z) \end{bmatrix}, \text{ from which we obtain}$$

$R_\ell(z) = R_{\ell-1}(z) + k_\ell[z^{-2}S_{\ell-1}(z)]$ and $S_\ell(z) = k_\ell R_{\ell-1}(z) + [z^{-2}S_{\ell-1}(z)]$. The realization of the two-pair is thus as shown below:



(b) The portion of Figure P14.5 upto the m -th stage is then as shown below:



Analysis of the above structure yields $P_m(z) = P_{m-1}(z) + k_m z^{-2} Q_{m-1}(z)$ and $Q_m(z) = k_m P_{m-1}(z) + z^{-2} Q_{m-1}(z)$.

(c) Analysis of the structure of Figure P14.5 yields $H_0(z) = P_M(z) + Q_M(z) = P_M(z) + z^{-(2M+1)} P_M(z^{-1})$. Hence, $H_0(z^{-1}) = P_M(z^{-1}) + z^{(2M+1)} P_M(z)$. Therefore, $z^{-(2M+1)} H_0(z^{-1}) = z^{-(2M+1)} P_M(z^{-1}) + P_M(z) = H_0(z)$. Since the order of $H_0(z)$ is $2M+1$, $H_0(z)$ is a Type 2 linear-phase FIR transfer function.

Similarly, analysis of the structure of Figure P14.5 yields $H_1(z) = P_M(z) - Q_M(z) = P_M(z) - z^{-(2M+1)} P_M(z^{-1})$. Hence, $H_1(z^{-1}) = P_M(z^{-1}) - z^{(2M+1)} P_M(z)$. Therefore, $z^{-(2M+1)} H_1(z^{-1}) = z^{-(2M+1)} P_M(z^{-1}) - P_M(z) = -H_1(z)$. Since the order of $H_1(z)$ is $2M+1$, $H_1(z)$ is a Type 4 linear-phase FIR transfer function.

(c) It follows from Figure P14.5 that

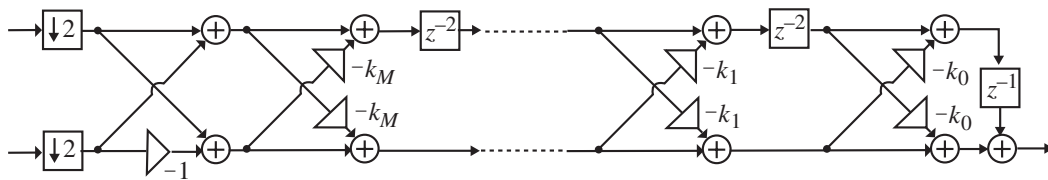
$$\mathbf{E}(z) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & k_M \\ k_M & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \cdots \begin{bmatrix} 1 & k_1 \\ k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_0 \\ k_0 & 1 \end{bmatrix}$$

$$= \mathbf{D}_2 \Gamma(z) \mathbf{T}_M \cdots \mathbf{T}_1 \Gamma(z) \mathbf{T}_0, \text{ where } \mathbf{D}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } \Gamma(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}. \text{ We}$$

determine the synthesis bank for perfect reconstruction by choosing

$$\mathbf{R}(z) = \alpha z^{-M} \mathbf{E}^{-1}(z) = \alpha z^{-M} \mathbf{D}_2^{-1} \Gamma^{-1}(z) \mathbf{T}_M^{-1} \cdots \mathbf{T}_1^{-1} \Gamma^{-1}(z) \mathbf{T}_0^{-1}.$$

The final realization of the synthesis filter bank is shown below which satisfies the perfect reconstruction condition within a scale factor.



14.32 From Figure P14.5 we observe that $H_0(z) = P_M(z) + Q_M(z)$ and $H_1(z) = P_M(z) - Q_M(z)$. Solving these equations we get

$$P_M(z) = \frac{1}{2} [H_0(z) + H_1(z)] \text{ and } Q_M(z) = \frac{1}{2} [H_0(z) - H_1(z)].$$

solution of Problem 14.31, Part (b), we have $P_m(z) = P_{m-1}(z) + k_m z^{-2} Q_{m-1}(z)$ and $Q_m(z) = k_m P_{m-1}(z) + z^{-2} Q_{m-1}(z)$. Solving these two equations we get

$$P_{m-1}(z) = \frac{1}{1-k_m^2} [P_m(z) - k_m Q_m(z)] \text{ and } Q_{m-1}(z) = \frac{1}{(1-k_m^2)z^{-2}} [Q_m(z) - k_m P_m(z)].$$

To synthesize the analysis filter bank, choose k_m so that the coefficient of the highest power of $P_{m-1}(z)$ in z^{-1} is equal to 0. It can be shown that this choice of also guarantees that the coefficient of the second highest power of is also 0. Hence, the order of $P_{m-1}(z)$ is 2 less that of $P_m(z)$.

(a) Here $M = 2$. Hence, $P_2(z) = \frac{1}{2}[H_0(z) + H_1(z)] = 1 + 2z^{-1} + 12z^{-2} + 15z^{-3} + 4z^{-4} + z^{-5}$ and $Q_2(z) = \frac{1}{2}[H_0(z) - H_1(z)] = 2 + 4z^{-1} + 15z^{-2} + 12z^{-3} + 2z^{-4} + z^{-5}$.

We form

$$P_2(z) - k_2 Q_2(z) = (1 - 2k_2) + (2 - 4k_2)z^{-1} + (12 - 15k_2)z^{-2} + (15 - 12k_2)z^{-3} + (4 - 2k_2)z^{-4} + (2 - k_2)z^{-5}.$$

Choose $k_2 = 2$. Then

$$P_1(z) = \frac{1}{1-k_2^2} [P_2(z) - k_2 Q_2(z)] = 1 + 2z^{-1} + 6z^{-2} + 3z^{-3}.$$

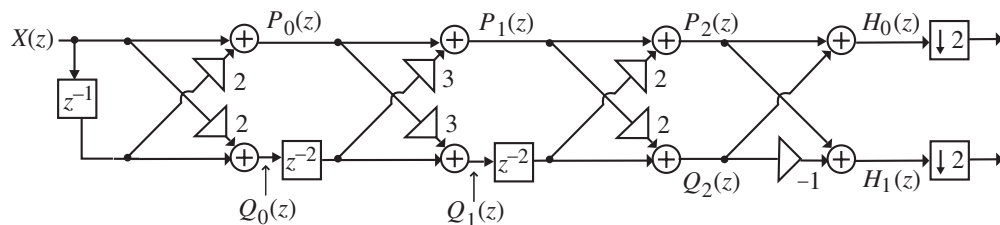
Likewise, $Q_1(z) = \frac{1}{(1-k_2^2)z^{-2}} [Q_2(z) - k_2 P_2(z)] = 3 + 6z^{-1} + 2z^{-2} + z^{-3}$. Next, we form

$$P_1(z) - k_1 Q_1(z) = (1 - 3k_1) + (2 - 6k_1)z^{-1} + (6 - 2k_1)z^{-2} + (3 - k_1)z^{-3}.$$

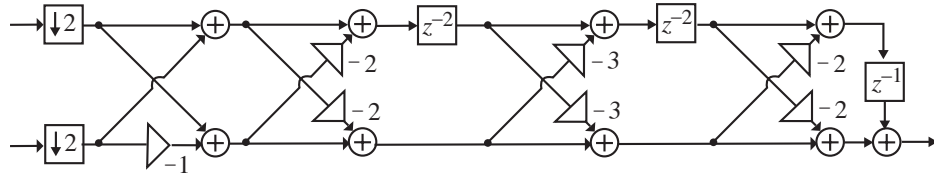
Choose $k_1 = 3$. Then, $P_0(z) = \frac{1}{1-k_1^2} [P_1(z) - k_1 Q_1(z)] = 1 + 2z^{-1}$. Likewise,

$$Q_0(z) = \frac{1}{(1-k_1^2)z^{-2}} [Q_1(z) - k_1 P_1(z)] = 2 + z^{-1}.$$

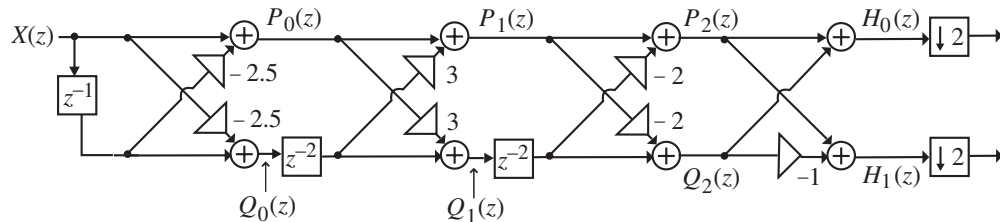
From Figure P14.5 we have $P_0(z) = 1 + k_0 z^{-1}$ and $Q_0(z) = k_0 + z^{-1}$. Hence, it follows that $k_0 = 2$. As a result, the analysis filter bank is as shown below:



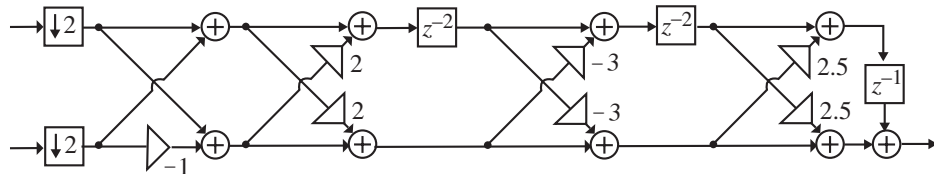
The corresponding synthesis filter bank for a perfect reconstruction filter bank is as shown below:



(b) Here $M = 2$. Hence, $P_2(z) = \frac{1}{2}[H_0(z) + H_1(z)] = 1 - 2.5z^{-1} - 13.5z^{-2} + 18z^{-3} + 5z^{-4} - 2z^{-5}$ and $Q_2(z) = \frac{1}{2}[H_0(z) - H_1(z)] = -2 + 5z^{-1} + 18z^{-2} - 13.5z^{-3} - 2.5z^{-4} + z^{-5}$. We form $P_2(z) - k_2Q_2(z) = (1 + 2k_2) - (2.5 + 5k_2)z^{-1} - (13.5 + 18k_2)z^{-2} + (18 + 13.5k_2)z^{-3} + (5 + 2.5k_2)z^{-4} - (2 + k_2)z^{-5}$. Choose $k_2 = -2$. Then $P_1(z) = \frac{1}{1 - k_2^2}[P_2(z) - k_2Q_2(z)] = 1 - 2.5z^{-1} - 7.5z^{-2} + 3z^{-3}$. Likewise, $Q_1(z) = \frac{1}{(1 - k_2^2)z^{-2}}[Q_2(z) - k_2P_2(z)] = 3 - 7.5z^{-1} - 2.5z^{-2} + z^{-3}$. Next, we form $P_1(z) - k_1Q_1(z) = (1 - 3k_1) - (2.5 - 7.5k_1)z^{-1} - (7.5 - 2.5k_1)z^{-2} + (3 - k_1)z^{-3}$. Choose $k_1 = 3$. Then, $P_0(z) = \frac{1}{1 - k_1^2}[P_1(z) - k_1Q_1(z)] = 1 - 2.5z^{-1}$. Likewise, $Q_0(z) = \frac{1}{(1 - k_1^2)z^{-2}}[Q_1(z) - k_1P_1(z)] = -2.5 + z^{-1}$. Hence, $k_0 = -2.5$. As a result, the analysis filter bank is as shown below:

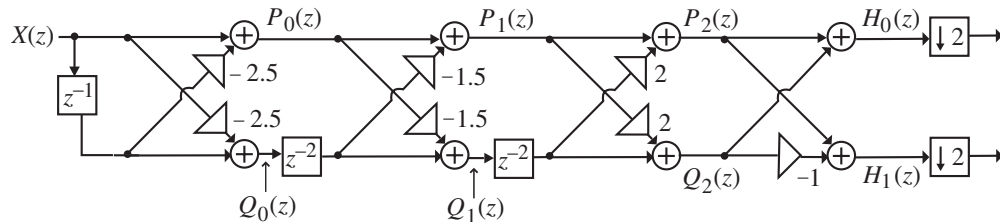


The corresponding synthesis filter bank for a perfect reconstruction filter bank is as shown below:

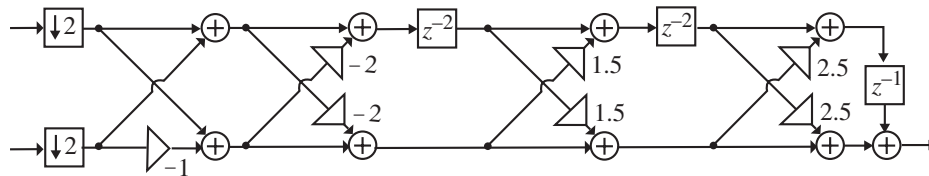


(c) Here $M = 2$. Hence, $P_2(z) = \frac{1}{2}[H_0(z) + H_1(z)] = 1 - 2.5z^{-1} + 0.75z^{-2} + 6z^{-3}$

$-5z^{-4} + 2z^{-5}$ and $Q_2(z) = \frac{1}{2}[H_0(z) - H_1(z)]$
 $= 2 - 5z^{-1} + 6z^{-2} + 0.75z^{-3} - 2.5z^{-4} + z^{-5}$. We form
 $P_2(z) - k_2Q_2(z) = (1 - 2k_2) - (2.5 - 5k_2)z^{-1} - (0.75 - 6k_2)z^{-2} + (6 - 0.75k_2)z^{-3}$
 $- (5 - 2.5k_2)z^{-4} + (2 - k_2)z^{-5}$. Choose $k_2 = 2$. Then $P_1(z) = \frac{1}{1 - k_2^2}[P_2(z) - k_2Q_2(z)]$
 $= 1 - 2.5z^{-1} + 3.75z^{-2} - 1.5z^{-3}$. Likewise,
 $Q_1(z) = \frac{1}{(1 - k_2^2)z^{-2}}[Q_2(z) - k_2P_2(z)] = -1.5 + 3.75z^{-1} - 2.5z^{-2} + z^{-3}$. Next, we form
 $P_1(z) - k_1Q_1(z) = (1 + 1.5k_1) - (2.5 - 3.75k_1)z^{-1} + (3.75 + 2.5k_1)z^{-2} - (1.5 + k_1)z^{-3}$.
 Choose $k_1 = -1.5$. Then, $P_0(z) = \frac{1}{1 - k_1^2}[P_1(z) - k_1Q_1(z)] = 1 - 2.5z^{-1}$. Likewise,
 $Q_0(z) = \frac{1}{(1 - k_1^2)z^{-2}}[Q_1(z) - k_1P_1(z)] = -2.5 + z^{-1}$. Hence, $k_0 = -2.5$. As a result, the
 analysis filter bank is as shown below:



The corresponding synthesis filter bank for a perfect reconstruction filter bank is as shown below:



14.33 To show the system of Figure 14.18 is, in general, periodic with a period L , we need to show that if $\hat{X}_1(z)$ is the output for an input $X_1(z)$, and $\hat{X}_2(z)$ is the output for an input $X_2(z)$, then if $X_2(z) = z^{-L}X_1(z)$, i.e., $x_2[n] = x_1[n - L]$, then the corresponding output satisfies $\hat{X}_2(z) = z^{-L}\hat{X}_1(z)$, i.e., $\hat{x}_2[n] = \hat{x}_1[n - L]$. Now,

$$\hat{X}(z) = \mathbf{g}^T(z)\mathbf{H}^{(m)}(z) \begin{bmatrix} X(z) \\ X(zW) \\ \vdots \\ X(zW^{L-1}) \end{bmatrix}, \text{ where } \mathbf{g}(z) \text{ is the vector composed of the synthesis}$$

bank filters given by Eq. (14.98) and $\mathbf{H}^{(m)}(z)$ is the analysis filter bank modulation

matrix given Eq. (14.102). Now, $\hat{X}_2(z) = \mathbf{g}^T(z)\mathbf{H}^{(m)}(z) \begin{bmatrix} X_2(z) \\ X_2(zW) \\ \vdots \\ X_2(zW^{L-1}) \end{bmatrix}$

$$= \mathbf{g}^T(z)\mathbf{H}^{(m)}(z) \begin{bmatrix} z^{-L}X_1(z) \\ z^{-L}X_1(zW) \\ \vdots \\ z^{-L}X_1(zW^{L-1}) \end{bmatrix} = z^{-L} \mathbf{g}^T(z)\mathbf{H}^{(m)}(z) \begin{bmatrix} X_1(z) \\ X_1(zW) \\ \vdots \\ X_1(zW^{L-1}) \end{bmatrix} = z^{-L} \hat{X}_1(z).$$

As a result, the structure of Figure 14.18 is, in general, a time-varying system with a period of L .

14.34 $H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2)$, $H_1(z) = E_{10}(z^2) + z^{-1}E_{11}(z^2)$. Thus,

$$\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix}.$$

(a) $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$. Hence, $H_1(z) = H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2)$.

Thus, $E_{00}(z) = E_0(z)$, $E_{01}(z) = E_1(z)$, $E_{10}(z) = E_0(z)$, and $E_{11}(z) = -E_1(z)$.

Therefore,
$$\mathbf{E}(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix}.$$

(b) Two cases need to be considered – N odd and N even. Consider first the N odd case. For simplicity let $H_0(z) = a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}$. Then,

$$H_1(z) = z^{-3}H_0(z^{-1}) = z^{-3}(a_0 + a_1z + a_2z^2 + a_3z^3) = a_3 + a_2z^{-1} + a_1z^{-2} + a_0z^{-3}.$$

Thus, $E_{00}(z) = E_0(z) = a_0 + a_2z^{-1}$, $E_{01}(z) = E_1(z) = a_1 + a_3z^{-1}$, and

$$E_{10}(z) = a_3 + a_1z^{-1}, E_{11}(z) = a_2 + a_0z^{-1} \Rightarrow E_{10}(z) = z^{-1}E_1(z^{-1}) \text{ and}$$

$$E_{11}(z) = z^{-1}E_0(z^{-1}).$$

Next, consider the N even case. Assume $H_0(z) = a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + a_4z^{-4}$.

Then,

$$H_1(z) = z^{-4}H_0(z^{-1}) = z^{-4}(a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4)$$

$$= a_4 + a_3z^{-1} + a_2z^{-2} + a_1z^{-3} + a_0z^{-4}. \text{ Thus,}$$

$$E_{00}(z) = E_0(z) = a_0 + a_2z^{-1} + a_4z^{-2}, E_{01}(z) = E_1(z) = a_1 + a_3z^{-1}, \text{ and}$$

$$E_{10}(z) = a_4 + a_2z^{-1} + a_0z^{-2}, E_{11}(z) = a_3 + a_1z^{-1} \Rightarrow E_{10}(z) = z^{-2}E_0(z^{-1}) \text{ and}$$

$$E_{11}(z) = z^{-1}E_1(z^{-1}).$$

In the general case, for N odd with $N = 2K + 1$, $E_{11}(z) = z^{-K}E_0(z^{-1})$ and

$$E_{10}(z) = z^{-K}E_1(z^{-1}) \Rightarrow \mathbf{E}(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ z^{-(N-1)/2}E_1(z^{-1}) & z^{-(N-1)/2}E_0(z^{-1}) \end{bmatrix}.$$

Likewise, for N even with $N = 2K$, $E_{10}(z) = z^{-K}E_0(z^{-1})$ and

$$E_{11}(z) = z^{-(K-1)}E_1(z^{-1}) \Rightarrow \mathbf{E}(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ z^{-N/2}E_0(z^{-1}) & z^{-(N-2)/2}E_1(z^{-1}) \end{bmatrix}.$$

14.35 (a) $H_0(z) = \frac{1}{4}[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)]$ and $H_1(z) = \frac{1}{4}[\mathcal{A}_0(z^2) - z^{-1}\mathcal{A}_1(z^2)]$.

Hence, $\mathbf{E}(z) = \begin{bmatrix} \mathcal{A}_0(z) & \mathcal{A}_1(z) \\ \mathcal{A}_0(z) & -\mathcal{A}_1(z) \end{bmatrix}$, and $\mathbf{R}(z) = \begin{bmatrix} \mathcal{A}_1(z) & \mathcal{A}_1(z) \\ \mathcal{A}_0(z) & -\mathcal{A}_0(z) \end{bmatrix}$.

(b) To prove $\mathbf{E}(z)$ is lossless, we need to show $\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega}) = c^2\mathbf{I}$ for some constant c : $\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega}) =$

$$\begin{bmatrix} \mathcal{A}_0^*(e^{j\omega}) & \mathcal{A}_0^*(e^{j\omega}) \\ \mathcal{A}_1^*(e^{j\omega}) & -\mathcal{A}_1^*(e^{j\omega}) \end{bmatrix} \begin{bmatrix} \mathcal{A}_0(e^{j\omega}) & \mathcal{A}_1(e^{j\omega}) \\ \mathcal{A}_0(e^{j\omega}) & -\mathcal{A}_1(e^{j\omega}) \end{bmatrix}$$

$$= \begin{bmatrix} |\mathcal{A}_0(e^{j\omega})|^2 + |\mathcal{A}_0(e^{j\omega})|^2 & 0 \\ 0 & |\mathcal{A}_1(e^{j\omega})|^2 + |\mathcal{A}_1(e^{j\omega})|^2 \end{bmatrix} = 2\mathbf{I}. \text{ Hence, } \mathbf{E}(z) \text{ is}$$

lossless.

(c) $\mathbf{R}(z)\mathbf{E}(z) = \frac{1}{4} \begin{bmatrix} 2\mathcal{A}_1(z)\mathcal{A}_0(z) & 0 \\ 0 & 2\mathcal{A}_1(z)\mathcal{A}_0(z) \end{bmatrix} = \frac{\mathcal{A}_1(z)\mathcal{A}_0(z)}{2}\mathbf{I}$. Therefore,

$$\mathbf{R}(z) = \frac{\mathcal{A}_1(z)\mathcal{A}_0(z)}{2}\mathbf{E}^{-1}(z).$$

(d) As in part (c), $\mathbf{R}(z)\mathbf{E}(z) = \frac{\mathcal{A}_1(z)\mathcal{A}_0(z)}{2}\mathbf{I}$.

14.36 $\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} = \begin{bmatrix} -2z^{-2} + 8z^{-1} - 6 & 2 - z^{-1} \\ -3 + 2z^{-1} & 1 \end{bmatrix}$. Hence,

$$H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2) = -6 + 2z^{-1} + 8z^{-2} - z^{-3} - 2z^{-4} \text{ and}$$

$$H_1(z) = E_{10}(z^2) + z^{-1}E_{11}(z^2) = -3 + z^{-1} + 2z^{-2}.$$

Now, $\det(\mathbf{E}(z)) = (-2z^{-2} + 8z^{-1} - 6) - (2 - z^{-1})(-3 + 2z^{-1}) = z^{-1}$. From Eq. (14.133) we have

$$\mathbf{R}(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} = cz^{-K}\mathbf{E}^{-1}(z) = \frac{cz^{-K}}{z^{-1}} \begin{bmatrix} -2z^{-2} + 8z^{-1} - 6 & -2 + z^{-1} \\ 3 - 2z^{-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2z^{-2} + 8z^{-1} - 6 & -2 + z^{-1} \\ 3 - 2z^{-1} & 1 \end{bmatrix} \text{ for } K = 1 \text{ and } c = 1. \text{ Therefore,}$$

$$G_0(z) = R_{00}(z^2) + z^{-1}R_{01}(z^2) = -6 - 2z^{-1} + 8z^{-2} + z^{-3} - 2z^{-4} \text{ and}$$

$$G_1(z) = R_{10}(z^2) + z^{-1}R_{11}(z^2) = 3 + z^{-1} - 2z^{-2}.$$

14.37 $\mathbf{P} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 2 \end{bmatrix}$. From Eqs. (14.121) and (14.122a) to (14.122c), we thus get

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}, \text{ resulting in the analysis filters}$$

$$H_0(z) = 2 - 2z^{-1} + z^{-2}, H_1(z) = 3 + z^{-1} - 2z^{-2}, H_2(z) = 2 + 4z^{-1} + 2z^{-2}.$$

$$\text{Now, } 4\mathbf{P}^{-1} = \begin{bmatrix} 0.8 & 0.64 & 0.24 \\ -0.8 & 0.16 & 0.56 \\ 0.8 & -0.96 & 0.64 \end{bmatrix}. \text{ Using Eqs. (14.124) and (14.125a) to (14.125c),}$$

$$\text{we arrive at } [G_0(z) \ G_1(z) \ G_2(z)] = [z^{-2} \ z^{-1} \ 1] \begin{bmatrix} 0.8 & 0.64 & 0.24 \\ -0.8 & 0.16 & 0.56 \\ 0.8 & -0.96 & 0.64 \end{bmatrix}, \text{ leading to}$$

$$\text{the synthesis filters } G_0(z) = 0.8 - 0.8z^{-1} + 0.8z^{-2}, G_1(z) = -0.96 + 0.16z^{-1} + 0.64z^{-2}, \\ G_2(z) = 0.64 + 0.56z^{-1} + 0.24z^{-2}.$$

14.38 $\mathbf{R}(z) = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$. For $y[n] = 4x[n-3]$ to hold, we require

$$\mathbf{R}(z)\mathbf{E}(z) = 4\mathbf{I}. \text{ Hence, } \mathbf{E}(z) = 4[\mathbf{R}(z)]^{-1} = 4 \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}^{-1} \\ = \begin{bmatrix} -1.4118 & 2.5882 & -1.8824 & 0.2353 \\ 0.2353 & 0.2353 & 1.6471 & -0.7059 \\ 2.5882 & -1.4118 & 2.1176 & 0.2353 \\ 1.8824 & -2.1176 & 1.1765 & 2.3529 \end{bmatrix}.$$

14.39 $[H_0(z) \ H_1(z) \ H_2(z)] = [z^{-2} \ z^{-1} \ 1] \begin{bmatrix} 1 & 2 & 2 \\ 3 & -3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$. Therefore,

$$H_0(z) = 2 + 3z^{-1} + z^{-2}, H_1(z) = 1 - 3z^{-1} + 2z^{-2}, H_2(z) = 2 + z^{-1} + 2z^{-2}. \text{ The}$$

corresponding Type 1 polyphase component matrix is thus given by $\mathbf{E}(z) = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$.

For perfect reconstruction we require the Type 2 polyphase component matrix to be given

$$\text{by } \mathbf{R}(z) = [\mathbf{E}(z)]^{-1} = \begin{bmatrix} 8/3 & 5/3 & -3 \\ -2/3 & -2/3 & 1 \\ -8/3 & 4/3 & 3 \end{bmatrix}. \text{ Hence, the synthesis filters are obtained from}$$

$$[G_0(z) \ G_1(z) \ G_2(z)] = [z^{-2} \ z^{-1} \ 1] \begin{bmatrix} 8/3 & 5/3 & -3 \\ -2/3 & -2/3 & 1 \\ -8/3 & 4/3 & 3 \end{bmatrix} \text{ resulting in}$$

$$G_0(z) = -\frac{8}{3} - \frac{2}{3}z^{-1} + \frac{8}{3}z^{-2}, \quad G_1(z) = \frac{4}{3} - \frac{2}{3}z^{-1} + \frac{5}{3}z^{-2}, \quad \text{and} \quad G_2(z) = 3 + z^{-1} + 3z^{-2}.$$

14.40 $\mathbf{E}(z) = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 1 & -1 & -2 \\ 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & 2 \end{bmatrix}$. For perfect reconstruction we require

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \frac{1}{3} \begin{bmatrix} 1 & 0.5 & 0 & -0.5 \\ -1 & 0.25 & 0 & 1.25 \\ -0.4 & -0.95 & 1.2 & 0.05 \\ 1.2 & -0.15 & -0.6 & -0.15 \end{bmatrix}. \text{ Thus,}$$

$$[G_0(z) \ G_1(z) \ G_2(z) \ G_3(z)] = \frac{1}{3} [z^{-3} \ z^{-2} \ z^{-1} \ 1] \begin{bmatrix} 1 & 0.5 & 0 & -0.5 \\ -1 & 0.25 & 0 & 1.25 \\ -0.4 & -0.95 & 1.2 & 0.05 \\ 1.2 & -0.15 & -0.6 & -0.15 \end{bmatrix}.$$

Hence,

$$G_0(z) = \frac{1}{3}(1.2 - 0.4z^{-1} - z^{-2} + z^{-3}), \quad G_1(z) = \frac{1}{3}(-0.15 - 0.95z^{-1} + 0.25z^{-2} + 0.5z^{-3}),$$

$$G_2(z) = \frac{1}{3}(-0.6 + 1.2z^{-1}), \quad G_3(z) = \frac{1}{3}(-0.15 + 0.05z^{-1} + 1.25z^{-2} - 0.5z^{-3}).$$

14.41 $G_0(z) = -2 + 3z^{-1} + 2z^{-2}$, $G_1(z) = 3 + z^{-1} - 2z^{-2}$, $G_2(z) = 2 - z^{-1} + z^{-2}$.

Rewriting these equations in an matrix form we get

$$[G_0(z) \ G_1(z) \ G_2(z)] = [z^{-2} \ z^{-1} \ 1] \begin{bmatrix} 2 & -2 & 2 \\ 3 & 1 & -1 \\ -2 & 3 & 1 \end{bmatrix}. \text{ This implies,}$$

$$\mathbf{R}(z) = \begin{bmatrix} 2 & -2 & 2 \\ 3 & 1 & -1 \\ -2 & 3 & 1 \end{bmatrix}. \text{ For perfect reconstruction we require}$$

$$\mathbf{E}(z) = \mathbf{R}^{-1}(z) = \begin{bmatrix} 0.125 & 0.25 & 0 \\ -0.0313 & 0.1875 & 0.25 \\ 0.3438 & -0.0625 & 0.25 \end{bmatrix}. \text{ Hence,}$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 0.125 & 0.25 & 0 \\ -0.0313 & 0.1875 & 0.25 \\ 0.3438 & -0.0625 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix} \text{ implying}$$

$$H_0(z) = 0.125 + 0.25z^{-1}, \quad H_1(z) = -0.0313 + 0.1875z^{-1} + 0.25z^{-2},$$

$$H_2(z) = 0.3438 - 0.0625z^{-1} + 0.25z^{-2}.$$

14.42 If the filter bank is alias-free, then

$$\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{L-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{L-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{L-1}) & H_1(zW^{L-1}) & \cdots & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{The above}$$

L equations are

$$(1): H_0(z)G_0(z) + H_1(z)G_1(z) + \dots + H_{L-1}(z)G_{L-1}(z) = T(z),$$

$$(2): H_0(zW^r)G_0(z) + H_1(zW^r)G_1(z) + \dots + H_{L-1}(zW^r)G_{L-1}(z) = 0, 1 \leq r \leq L-1.$$

Replacing z by zW^{L-r} in the above equation we get

$$H_0(zW^L)G_0(zW^{L-r}) + H_1(zW^L)G_1(zW^{L-r}) + \dots + H_{L-1}(zW^L)G_{L-1}(zW^{L-r}) = 0,$$

or equivalently,

$$(3): H_0(z)G_0(zW^{L-r}) + H_1(z)G_1(zW^{L-r}) + \dots + H_{L-1}(z)G_{L-1}(zW^{L-r}) = 0,$$

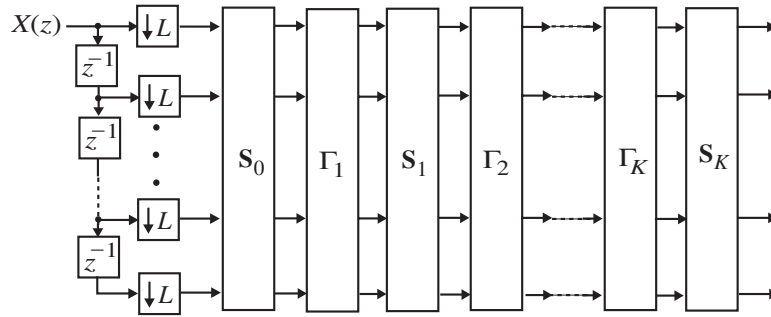
$0 \leq r \leq L-1$. Rewriting Eqs. (1) and (3) in matrix form we get

$$\begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{L-1}(z) \\ G_0(zW) & G_1(zW) & \cdots & G_{L-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ G_0(zW^{L-1}) & G_1(zW^{L-1}) & \cdots & G_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, if an L -channel filter bank is alias-free with a given set of analysis and synthesis filters, then the filter bank is still alias-free if the analysis and synthesis filters are interchanged.

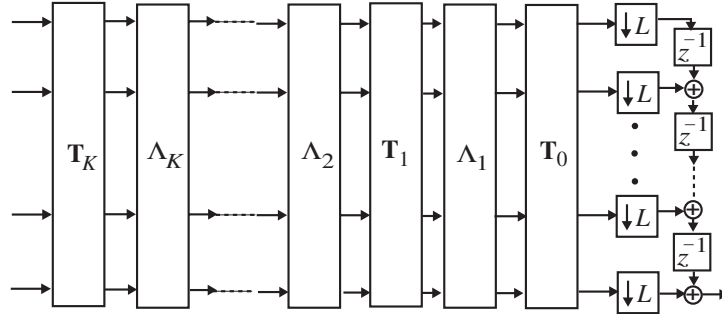
14.43 An equivalent representation of the structure of Figure P14.6 is shown below

$$\text{where } \Gamma_m = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{-1} \end{bmatrix}.$$



The corresponding synthesis bank for designing an L -channel perfect reconstruction bank is thus as shown below where $\mathbf{T}_m = \mathbf{S}_m^{-1}$ and

$$\mathbf{\Lambda}_m = z^{-2} \mathbf{\Gamma}_m^{-1} = \begin{bmatrix} z^{-1} & 0 & \cdots & 0 \\ 0 & z^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$



14.44
$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -11-3z^{-1} & 7+3z^{-1} \\ -1 & 3+2z^{-1} & -2-2z^{-1} \\ 0 & -z^{-1} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}. \text{ Hence, } H_0(z) = 5 - 11z^{-1} + 4z^{-2} + 3z^{-3},$$

$$H_1(z) = -1 + 3z^{-1} - 2z^{-3}, \quad H_2(z) = -z^{-2} + z^{-3}.$$

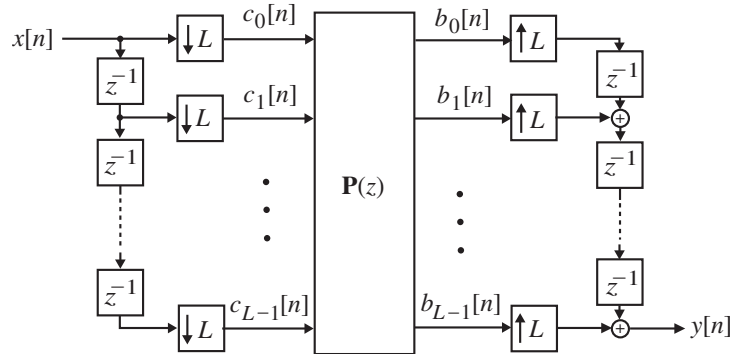
$$[G_0(z) \quad G_1(z) \quad G_2(z)] = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 3z^{-1} + 1 & 8z^{-1} + 3 & 7z^{-1} + 4 \\ z^{-1} + 2 & 3z^{-1} + 6 & 3z^{-1} + 8 \\ 1 & 3 & 4 \end{bmatrix}. \text{ Hence,}$$

$$G_0(z) = 1 + 2z^{-1} + 2z^{-2} + 3z^{-3}, \quad G_1(z) = 3 + 6z^{-1} + 6z^{-2} + 8z^{-3},$$

$$G_2(z) = 4 + 8z^{-1} + 7z^{-2} + 7z^{-3}.$$

14.45 Figure 14.20 with internal variables labeled is shown below:



From the above figure it follows that we can express the z -transforms of $\{c_\ell[n]\}$ as

$$C_\ell(z) = \frac{1}{L} \sum_{k=0}^{L-1} (z^{1/L} W^k)^{-\ell} X((z^{1/L} W^k)), \quad 0 \leq \ell \leq L-1, \quad \text{where } W = e^{-j2\pi/L}.$$

Likewise, the z -transforms of $\{b_s[n]\}$ can be expressed as

$$B_s(z) = \sum_{\ell=0}^{L-1} P_{s,\ell}(z) C_\ell(z), \quad 0 \leq s \leq L-1, \quad \text{where } P_{s,\ell}(z) \text{ denotes the } z\text{-transforms the}$$

(s, ℓ) -th element of $\mathbf{P}(z)$. Finally, the z -transform of the output $y[n]$ is given by

$$\begin{aligned} Y(z) &= \sum_{s=0}^{L-1} z^{-(L-1-s)} B_s(z^L) = \sum_{s=0}^{L-1} z^{-(L-1-s)} \sum_{\ell=0}^{L-1} P_{s,\ell}(z^L) C_\ell(z^L) \\ &= \frac{1}{L} \sum_{s=0}^{L-1} z^{-(L-1-s)} \sum_{\ell=0}^{L-1} P_{s,\ell}(z^L) \sum_{k=0}^{L-1} (z W^k)^{-\ell} X(z W^k) \\ &= \frac{1}{L} \sum_{k=0}^{L-1} X(z W^k) \sum_{\ell=0}^{L-1} W^{-k\ell} \sum_{s=0}^{L-1} z^{-(L-1-s)} z^{-\ell} P_{s,\ell}(z^L). \end{aligned}$$

In the above expression, terms of the form $X(z W^k)$, $k \neq 0$, represent the contribution coming from aliasing. Hence, the expression for $Y(z)$ is free from these aliasing terms for any arbitrary input $x[n]$ if and only if

$$\sum_{\ell=0}^{L-1} W^{-k\ell} \sum_{s=0}^{L-1} z^{-(L-1-s)} z^{-\ell} P_{s,\ell}(z^L) = 0, \quad k \neq 0.$$

The above expression can be written in a matrix form as $\mathbf{D}^\dagger \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

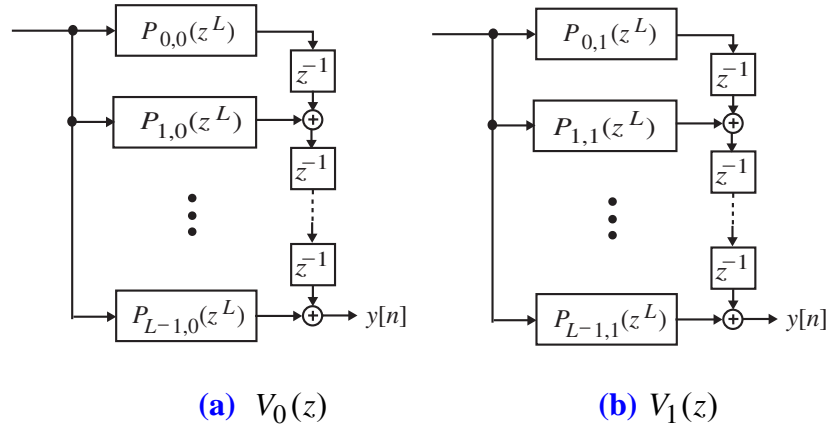
where \mathbf{D} is the DFT matrix, $T(z)$ is transfer function of the alias-free system, and

$$V_\ell(z) = \sum_{s=0}^{L-1} z^{-\ell} z^{-(L-1-s)} P_{s,\ell}(z^L). \quad \text{Since } \mathbf{D}\mathbf{D}^\dagger = \mathbf{L}\mathbf{I}, \text{ the above matrix equation can}$$

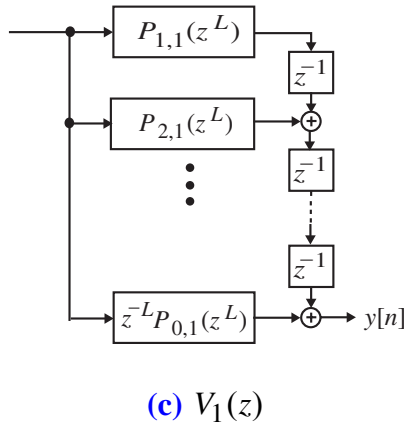
be alternately written as $\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \mathbf{D} \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. This implies that

$V_\ell(z) = V(z)$, $0 \leq \ell \leq L-1$, as the first column of \mathbf{D} has all elements equal to 1. As a result, the L -channel QMF bank is alias-free if and only if $V_\ell(z)$ is the same for all ℓ .

The two figures below show the polyphase realizations of $V_0(z)$ and $V_1(z)$:



The realization of $V_1(z)$ can be redrawn as indicated below:



Because of the constraint $V_0(z) = V_1(z)$, the polyphase components in Figures (a) and (c) should be the same. From these two figures it follows that the first column of $\mathbf{P}(z)$ is an upwards-shifted version of the second column, with the topmost element appearing with a z^{-1} attached. This type of relation holds for the k -th column and the $(k+1)$ -th column of $\mathbf{P}(z)$. As a result, $\mathbf{P}(z)$ is a pseudo-circulant matrix of the form of Eq. (14.155).

14.46 $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$, $H_1(z) = E_0(z^2) - z^{-1}E_1(z^2)$. Hence,

$$\mathbf{E}(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix}. \text{ Likewise,}$$

$$G_0(z) = E_0(z^2) + z^{-1}E_1(z^2) = z^{-1}R_{00}(z^2) + R_{10}(z^2),$$

$$G_1(z) = -E_0(z^2) + z^{-1}E_1(z^2) = z^{-1}R_{01}(z^2) + R_{11}(z^2). \text{ Hence,}$$

$$\mathbf{R}(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} = \begin{bmatrix} E_1(z) & E_1(z) \\ E_0(z) & -E_0(z) \end{bmatrix}. \text{ Therefore,}$$

$$\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z) = \begin{bmatrix} E_1(z) & E_1(z) \\ E_0(z) & -E_0(z) \end{bmatrix} \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix} = \begin{bmatrix} 2E_0(z)E_1(z) & 0 \\ 0 & 2E_0(z)E_1(z) \end{bmatrix}.$$

Here, $P_0(z) = 2E_0(z)E_1(z)$ and $P_1(z) = 0$. As a result, $\mathbf{P}(z)$ is pseudo-circulant.

14.47 (a) $H_0(z) = 1 - 2z^{-1} + 4.5z^{-2} + 6z^{-3} + z^{-4} + 0.5z^{-5}$,

$$H_0(z^{-1}) = 0.5z^5 + z^4 + 6z^3 + 4.5z^2 - 2z + 1. \text{ Therefore,}$$

$$H_0(z)H_0(z^{-1}) = 0.5z^5 + 6.25z^3 + 22.5z + 62.5 + 22.5z^{-1} + 6.25z^{-3} + 0.5z^{-5}. \text{ Thus,}$$

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 125. \text{ Hence, } H_0(z) \text{ is a power-symmetric function.}$$

The highpass analysis filter is given by

$$H_1(z) = z^{-5}H_0(-z^{-1}) = -0.5 + z^{-1} - 6z^{-2} + 4.5z^{-3} + 2z^{-4} + z^{-5}. \text{ The two synthesis}$$

filters are time-reversed versions of the analysis filters as per Eq. (14.92):

$$G_0(z) = 2z^{-5}H_0(z^{-1}) = 1 + 2z^{-1} + 12z^{-2} + 9z^{-3} - 4z^{-4} + 2z^{-5} \text{ and}$$

$$G_1(z) = 2z^{-5}H_1(z^{-1}) = 2 + 4z^{-1} + 9z^{-2} - 12z^{-3} + 2z^{-4} - z^{-5}.$$

(b) $H_0(z) = 1 + \frac{1}{2}z^{-1} + \frac{15}{4}z^{-2} - z^{-4} + 2z^{-5}$, $H_0(z^{-1}) = 2z^5 - z^4 + \frac{15}{4}z^2 + \frac{1}{2}z + 1$.

Therefore,

$$H_0(z)H_0(z^{-1}) = 2z^5 + 10.75z^3 - 3.375z + 20.3125 - 3.375z^{-1} + 10.75z^{-3} + 2z^{-5}. \text{ Thus,}$$

$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 40.625$. Hence, $H_0(z)$ is a power-symmetric function. The highpass analysis filter is given by

$$H_1(z) = z^{-5}H_0(-z^{-1}) = -2 - z^{-1} + \frac{15}{4}z^{-3} - \frac{1}{2}z^{-4} + z^{-5}. \text{ The two synthesis filters are}$$

time-reversed versions of the analysis filters as per Eq. (14.92):

$$G_0(z) = 2z^{-5}H_0(z^{-1}) = 4 - 2z^{-1} + \frac{15}{2}z^{-3} + z^{-4} + 2z^{-5} \text{ and}$$

$$G_1(z) = 2z^{-5}H_1(z^{-1}) = 2 - z^{-1} + \frac{15}{2}z^{-3} - 2z^{-4} - 4z^{-5}.$$

14.48 $H_0(z) = 1 + az^{-1} + z^{-2}$ and $H_1(z) = 1 + az^{-1} + bz^{-2} + az^{-3} + z^{-4}$. The corresponding

synthesis filters are given by $G_0(z) = H_1(-z) = 1 - az^{-1} + bz^{-2} - az^{-3} + z^{-4}$, and

$$G_1(z) = -H_0(-z) = -1 + az^{-1} - z^{-2}.$$

To show that the filter bank is alias-free and satisfies the perfect reconstruction property we

need to show that $\begin{bmatrix} H_0(z) & H_0(-z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} cz^{-K} \\ 0 \end{bmatrix}$, where $c \neq 0$. Now,

$$H_0(z)G_0(z) + H_1(z)G_1(z) = (1 + az^{-1} + z^{-2})(1 - az^{-1} + z^{-2} - az^{-3} + z^{-4})$$

$$+ (-1 + az^{-1} - z^{-2})(1 + az^{-1} + z^{-2} + az^{-3} + z^{-4}) = 2a(b - 2)z^{-3}, \text{ and}$$

$$H_0(-z)G_0(z) + H_1(z)G_1(-z) = H_0(-z)H_1(-z) - H_0(-z)H_1(-z) = 0. \text{ Thus, if } a \neq 0 \text{ and}$$

$b \neq 2$, the above filter bank is alias-free and also satisfies the perfect reconstruction property.

14.49 If $H_0(z)$ is required to have 2 zeros at $z = -1$, then it is of the form

$H_0(z) = (1 + z^{-1})^2 C(z)$, where $C(z)$ is a first-order polynomial. Now,

$P(z) = H_0(z)H_0(z^{-1}) = (1 + z^{-1})^2(1 + z)^2 R(z)$, where $R(z)$ is a zero-phase polynomial of the form $R(z) = az + b + az^{-1}$. For perfect reconstruction we require, $P(z) + P(-z) = 2$, i.e.,

$(1 + z^{-1})^2(1 + z)^2(az + b + az^{-1}) + (1 - z^{-1})^2(1 - z)^2(-az + b - az^{-1}) = 2$. Since the above equation must hold for all z , we observe that at $z = 1$, we get

$2a + b = \frac{1}{8}$. Likewise, at $z = j$, we get $b = \frac{1}{4}$. Hence, $a = -\frac{1}{16}$. Therefore,

$P(z) = (1 + z^{-1})^2(1 + z)^2(-\frac{1}{16}z + \frac{1}{4} - \frac{1}{16}z^{-1})$. The analysis filter $H_0(z)$ is

obtained by a spectral factorization of $P(z)$. Three choices of spectral factorization resulting in linear-phase analysis filters are given in Section 14.3.3.

14.50

14.51 (a) An orthogonal perfect reconstruction filter bank is obtained by choosing the minimum-phase spectral factor of $P(z)$ as the lowpass analysis filter $H_0(z)$ and then determining the lowpass synthesis filter $G_0(z)$ using the relation

$G_0(z) = z^{-5}H_0(z^{-1})$. The minimum-phase spectral factor of $P(z)$ is given by

$H_0(z) = 0.33267(1 + z^{-1})^3(1 - 0.5745z^{-1} + 0.1059z^{-2})$. Hence,

$G_0(z) = z^{-5}H_0(z^{-1}) = 0.33267(1 + z^{-1})^3(0.1059 - 0.5745z^{-1} + z^{-2})$.

(b) A perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of even length is given by

$H_0(z) = (1 + z^{-1})(a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4})$ and $G_0(z) = (1 + z^{-1})^5$.

Another possible design of a perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of even length is given by

$H_0(z) = (1 + z^{-1})^3(a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4})$ and $G_0(z) = (1 + z^{-1})^3$.

(c) A perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of odd length is given by

$H_0(z) = (a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4})$ and $G_0(z) = (1 + z^{-1})^6$. Another

possible design of a perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of odd length is given by

$H_0(z) = (1 + z^{-1})^2(a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4})$ and $G_0(z) = (1 + z^{-1})^4$.

14.52

14.53

14.54 The polyphase matrices for the structure of Figure P14.7(a) are given by

$$\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} \text{ and } \mathbf{R}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix}. \text{ Therefore,}$$

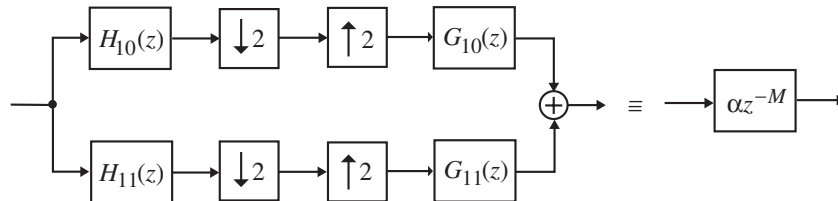
$$\mathbf{R}(z)\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Hence, the structure of Figure P14.7(a) is a perfect reconstruction QMF bank.}$$

14.55 The polyphase matrices for the structure of Figure P14.7(b) are given by

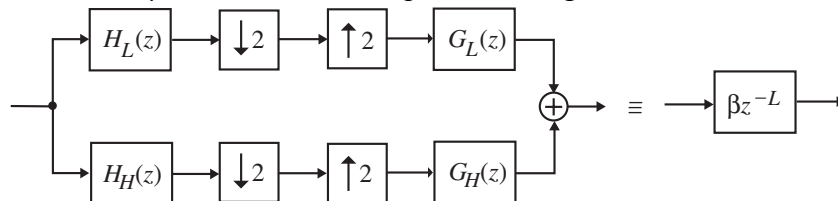
$$\mathbf{E}(z) = \begin{bmatrix} 1 & Q(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} \text{ and } \mathbf{R}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -Q(z) \\ 0 & 1 \end{bmatrix}. \text{ Therefore,}$$

$$\begin{aligned} \mathbf{R}(z)\mathbf{E}(z) &= \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -Q(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & Q(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ Hence, the structure of Figure P14.7(b) is a perfect reconstruction QMF bank.} \end{aligned}$$

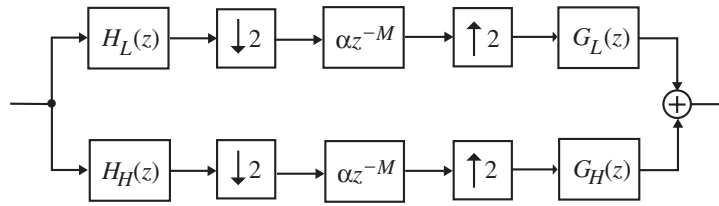
14.56 If the 2-channel QMF banks in the middle of the structure of Figure 14.24 are of perfect reconstruction type, then each of these two filter banks have a distortion transfer function of the form αz^{-M} , where M is a positive integer.



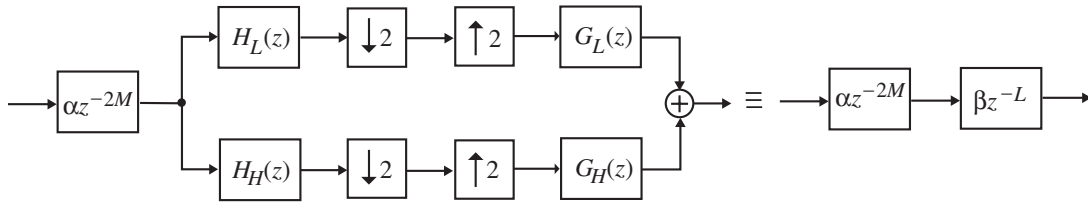
Likewise, the 2-channel analysis filter bank on the left with the 2-channel synthesis filter bank on the right form a perfect reconstruction QMF bank with a distortion transfer function βz^{-L} , where L is a positive integer:



Hence, an equivalent representation of Figure 14.24 is as indicated below:

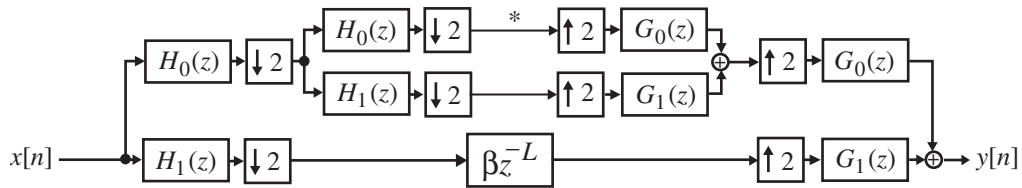


which reduces to

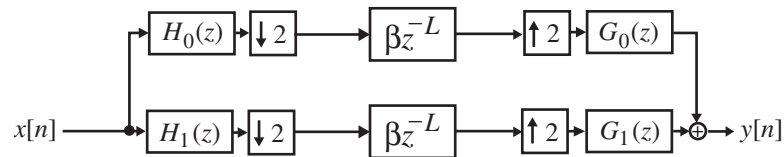


Thus, the overall structure is also of perfect reconstruction type with have a distortion transfer function given by $\alpha\beta z^{-(2M+L)}$.

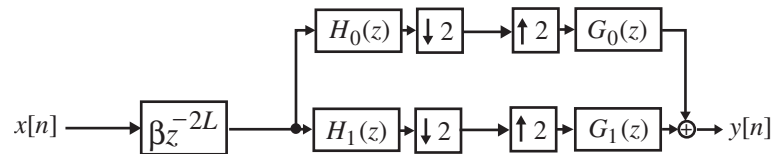
14.57 We analyze the 3-channel filter bank of Figure 14.27(b). If the 2-channel QMF bank of Figure 14.27(a) is of perfect reconstruction type with a distortion transfer function βz^{-L} , the structure of Figure 14.27(b) should be implemented as indicated below to ensure perfect reconstruction:



An equivalent representation of the above structure is as shown below:



which reduces to



verifying the perfect reconstruction property.

In a similar manner, the perfect reconstruction property of Figure 14.27(c) can be proved.

M14.1 The specifications of the corresponding zero-phase half-band filter are as follows: stopband edge $\omega_s = 0.65\pi$ and a minimum stopband attenuation of

$\alpha_s = 2 \times 30 + 6.02 = 66.02$ dB. The desired stopband ripple is therefore

$\delta_s = 10^{-\alpha_s/20} = 10^{-3.301} = 0.0005$. The passband edge of the half-band filter is at $\omega_p = \pi - 0.6\pi = 0.4\pi$. Using the function `remezord` we then estimate the order

of $F(z)$ and using the function `remez` we next design $Q(z)$. To this end the code fragments used are

```
[N, fpts, mag, wt] = remezord([0.4 0.6], [1 0], [0.0005 0.005]);
```

The order of $F(z)$ is found to be 30 which is of the form $4K + 2 = N$ for $K = 4$. The order of $H_0(z)$ is therefore 15. The filter $Q(z)$ is designed using the statement

```
[q, err] = remez(N, fpts, mag, wt);
```

To determine the coefficients $\{f[n]\}$ of the filter $F(z)$ we add `err` to the central coefficient `q[16]`. Next using the statement `h0 = firminphase(f)`; we determine the minimum-phase spectral factor of $F(z)$ which are the coefficients of the lowpass analysis filter $H_0(z)$:

```
Columns 1 through 7
  0.2818    0.5076    0.3582   -0.0386   -0.1749
 0.0083    0.1047
Columns 8 through 14
-0.0089   -0.0663    0.0122    0.0418   -0.0134   -
 0.0319    0.0335
Columns 15 through 16
-0.0121    0.0008
```

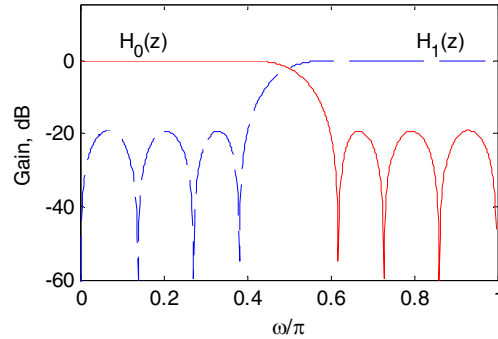
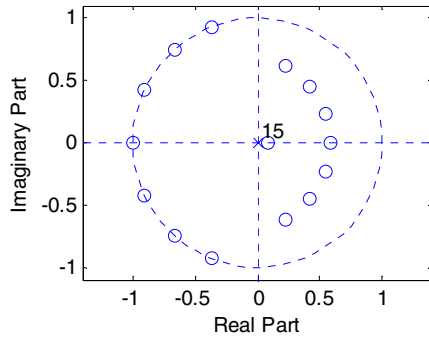
The highpass analysis filter $H_1(z)$ is obtained using the code fragments

```
k = 0:15;
h1 = ((-1).^k) .* h0;
```

The synthesis filters $G_0(z)$ and $G_1(z)$ can be found from the analysis filters using the code fragments

```
G0 = fliplr(h0);
G1 = fliplr(h1);
```

Plots of the roots of and the gain responses of the two analysis filters are shown below:



M14.2 The following MATLAB programs can be used to design the analysis filters corresponding to a two-channel QMF paraunitary lattice filter bank. (The program uses the function `fminunc` from the MATLAB Optimization Toolbox, which in turn uses the functions `cubici2`, `cubici3`, `optint`, `searchq`, and `updhess`.)

```

% Program for the design of a two-channel QMF
% lattice filter bank.
Len = input('The length of the filter = ');
if (mod(Len,2) ~= 0)
    sprintf('Length has to be an even number')
    Len = Len+1;
end
ord = Len/2-1;
ws = 0.55*pi;
kinit = [1;zeros([ord,1])];
% set the parameters for the optimization routine
options = optimset('MaxIter', 2500, 'Display', 'off');
kfin = fminunc('filtopt',kinit,options,Len,ws);
e00old = 1;
e01old = kfin(1);
e10old = -kfin(1);
e11old = 1;
for k = 2:length(kfin)
    e00new = [e00old 0]-kfin(k)*[0 e01old];
    e01new = kfin(k)*[e00old 0]+[0 e01old];
    e10new = [e10old 0]-kfin(k)*[0 e11old];
    e11new = kfin(k)*[e10old 0]+[0 e11old];
    e00old = e00new;
    e01old = e01new;
    e10old = e10new;
    e11old = e11new;
end
E1 = [e00old;e01old];
h0 = E1(:);
scale_factor = abs(sum(h0));
h0 = h0/scale_factor;
E2 = [e10old;e11old];
h1 = E2(:);

```

```

h1 = h1/scale_factor;
[H0,W] = freqz(h0,1,1024);
[H1,W] = freqz(h1,1,1024);
plot(W/pi,20*log10(abs(H0)), W/pi, 20*log10(abs(H1)));
grid on
title('Gain response of the analysis filters');
xlabel('\omega/\pi'); ylabel('Gain, dB');

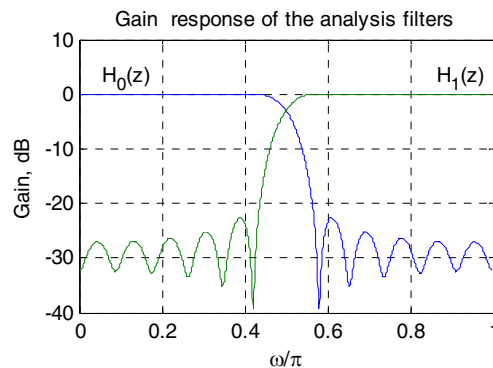
```

```

function val = filtopt(kval,Len,ws)
e00old = 1;
e01old = kval(1);
e10old = -kval(1);
e11old = 1;
for k = 2:length(kval)
    e00new = [e00old 0]-kval(k)*[0 e01old];
    e01new = kval(k)*[e00old 0]+[0 e01old];
    e10new = [e10old 0]-kval(k)*[0 e11old];
    e11new = kval(k)*[e10old 0]+[0 e11old];
    e00old = e00new;
    e01old = e01new;
    e10old = e10new;
    e11old = e11new;
end
E1 = [e00old;e01old];
h0 = E1(:);
[H0,W] = freqz(h0,1,1024);
val = 0;
for k = 1:length(W)
    if (W(k) > ws)
        val = val+abs(H0(k))^2;
    end
end
end

```

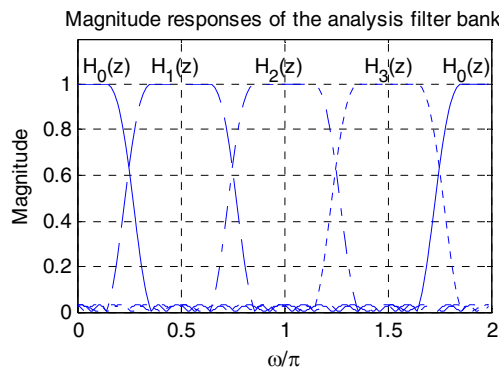
Due to the non-linear nature of the function to be optimized, different values of k_{init} should be used to optimize the analysis filter's gain response. The gain responses of the two analysis filters is as shown below. From the gain responses, the minimum stopband attenuation of the analysis filters is observed to be about 23 dB.



M14.3 The MATLAB program used to generate the prototype lowpass filter and the analysis filters of the 4-channel uniform DFT filter bank is given below:

```
L = 19; f = [0 0.15 0.35 1]; m = [1 1 0 0]; w = [10 1];
N = 4; WN = exp(-2*pi*j/N);
plottag = ['- ' ; '--' ; '-.' ; ':' ];
h = zeros(N,L);
n = 0:L-1;
h(1,:) = remez(L-1, f, m, w);
for i = 1:N-1
    h(i+1,:) = h(1,:).*(WN.^(-i*n));
end;
clf;
for i = 1:N
    [H,w] = freqz(h(i,:), 1, 256, 'whole');
    plot(w/pi, abs(H), plottag(i,:));
    hold on;
end;
grid on;
hold off;
xlabel('\omega/\pi'); ylabel('Magnitude');
title('Magnitude responses of the analysis filter bank');
```

The plots generated by the above program is given below:



M14.4 The first 8 impulse response coefficients of Johnston's 16A lowpass filter $H_L(z)$ are given by

0.001050167, -0.005054526, -0.002589756, 0.0276414, -0.009666376, -0.09039223, 0.09779817, 0.4810284

The remaining 8 coefficients are given by flipping the coefficients left to right, From Eq. (14.98), the highpass filter in the tree-structured 3-channel filter bank is given by

$H_2(z) = H_H(z) = z^{-15}H_L(z^{-1})$. The two remaining filters are given by

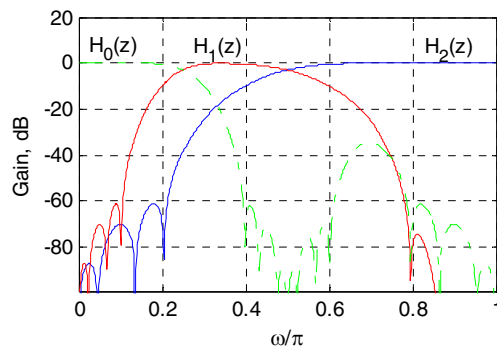
$H_0(z) = H_L(z)H_L(z^2)$ and $H_1(z) = H_L(z)H_H(z^2)$. The MATLAB program used to generate the gain plots of the 3 analysis filters is given by:

```

G1 = [0.10501670e-2 -0.50545260e-2 -0.25897560e-2
0.27641400e-1 -0.96663760e-2 -0.90392230e-1 0.97798170e-1
0.48102840];
G = [G1 fliplr(G1)];
n = 0:15;
H0 = (-1).^n.*G;
Hsqar = zeros(1,31); Gsqar = zeros(1,31);
Hsqar(1:2:31) = H0; Gsqar(1:2:31) = G;
H1 = conv(Hsqar,G); H2 = conv(Gsqar,G);
[h0,w0] = freqz(H0,[1]); [h1,w1] = freqz(H1,[1]); [h2,w2] =
freqz(H2,[1]);
g0 = 20*log10(abs(h0));g1 = 20*log10(abs(h1));
g2 = 20*log10(abs(h2));
plot(w0/pi,g0,'b-',w1/pi,g1,'r-',w2/pi,g2,'g-.');
axis([0 1 -120 20]);
grid on;
xlabel('\omega/\pi');ylabel('Gain, dB');

```

The plots generated are given below:



M14.5