TABLE OF CONTENTS

SECTION 6 - FREQUENTLY USED DISCRETE DISTRIBUTIONS

SECTION 7 - FREQUENTLY USED CONTINUOUS DISTRIBUTIONS

SECTION 8 - JOINT, MARGINAL, AND CONDITIONAL DISTRIBUTIONS

SECTION 9 - TRANSFORMATIONS OF RANDOM VARIABLES

SECTION 10 - RISK MANAGEMENT CONCEPTS

INTRODUCTORY COMMENTS

This study guide is designed to help in the preparation for the Society of Actuaries Exam P-Casualty Actuarial Society Exam 1. The study manual is divided into two main parts. The first part consists of a summary of notes and illustrative examples related to the material described in the exam catalog as well as a series of problem sets and detailed solutions related to each topic. Many of the examples and problems in the problem sets are taken from actual exams (and from the sample question list posted on the SOA website).

The second part of the study manual consists of eight practice exams, with detailed solutions, which are designed to cover the range of material that will be found on the exam. The questions on these practice exams are not from old Society exams and may be somewhat more challenging, on average, than questions from previous actual exams. Between the section of notes and the section with practice exams I have included the normal distribution table provided with the exam.

I have attempted to be thorough in the coverage of the topics upon which the exam is based. I have been, perhaps, more thorough than necessary on a couple of topics, particularly order statistics in Section 9 of the notes and some risk management topics in Section 10 of the notes.

Section 0 of the notes provides a brief review of a few important topics in calculus and algebra. This manual will be most effective, however, for those who have had courses in college calculus at least to the sophomore level and courses in probability to the sophomore or junior level.

If you are taking the Exam P for the first time, be aware that a most crucial aspect of the exam is the limited time given to take the exam (3 hours). It is important to be able to work very quickly and accurately. Continual drill on important concepts and formulas by working through many problems will be helpful. It is also very important to be disciplined enough while taking the exam so that an inordinate amount of time is not spent on any one question. If the formulas and reasoning that will be needed to solve a particular question are not clear within 2 or 3 minutes of starting the question, it should be abandoned (and returned to later if time permits). Using the exams in the second part of this study manual and simulating exam conditions will also help give you a feeling for the actual exam experience.

If you have any comments, criticisms or compliments regarding this study guide, please contact the publisher, ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you bring them to my attention. Any errors that are found will be posted in an errata file at the ACTEX website, www.actexmadriver.com .

It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

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NOTES, EXAMPLES

AND PROBLEM SETS

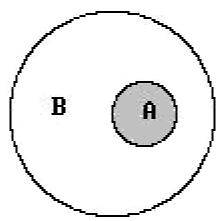
SECTION 0 - REVIEW OF ALGEBRA AND CALCULUS

In this introductory section, a few important concepts that are preliminary to probability topics will be reviewed. The concepts reviewed are set theory, graphing an inequality in two dimensions, properties of functions, differentiation, integration and geometric series. Students with a strong background in calculus who are familiar with these concepts can skip this section.

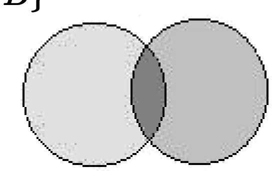
SET THEORY

A **set** is a collection of **elements**. The phrase **"$\boldsymbol{x}$ is an element of $\boldsymbol{A}$"** is denoted by $\boldsymbol{x \in A}$, and **"$\boldsymbol{x}$ is not an element of $\boldsymbol{A}$"** is denoted by $\boldsymbol{x \notin A}$.

Subset of a set: $\boldsymbol{A \subset B}$ means that each element of the set A is an element of the set B.
B may contain elements which are not in A, but A is totally contained within B. For instance, if A is the set of all odd, positive integers, and B is the set of all positive integers, then $A = \{1\,,\,3\,,\,5\,,\,\ldots\}$ and $B = \{1\,,\,2\,,\,3\,,\,\ldots\}$. For these two sets it is easy to see that $A \subset B$, since any member of A (any odd positive integer) is a member of B (is a positive integer).
The Venn diagram below illustrates A as a subset of B.



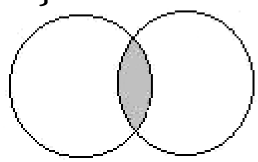
Union of sets: $\boldsymbol{A \cup B}$ is the set of all elements in either A or B (or both).
$\boldsymbol{A \cup B = \{x \,|\, x \in A}$ **or** $\boldsymbol{x \in B\}}$



$A \cup B$

If A is the set of all positive even integers $(A = \{2\,,\,4\,,\,6\,,\,8\,,10\,,\,12\,,\,\ldots\})$ and B is the set of all positive integers which are multiples of 3 $(B = \{3\,,\,6\,,\,9\,,\,12\,,\,\ldots\})$, then $A \cup B = \{2\,,\,3\,,\,4\,,\,6\,,\,8\,,\,9\,,\,10\,,\,12\,,\,\ldots\}$ is the set of positive integers which are either multiples of 2 or are multiples of 3 (or both).

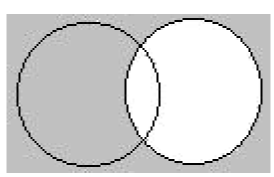
Intersection of sets: $\boldsymbol{A \cap B}$ is the set of all elements that are in both A and B.
$\boldsymbol{A \cap B = \{x \,|\, x \in A}$ **and** $\boldsymbol{x \in B\}}$



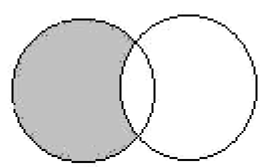
$A \cap B$

If A is the set of all positive even integers and B is the set of all positive integers which are a multiple of 3, then $A \cap B = \{6\ ,\ 12\ ,\ \ldots\}$ is the set of positive integers which are a multiple of 6. The elements of $A \cap B$ must satisfy the properties of **both** A and B. In this example, that means an element of $A \cap B$ must be a multiple of 2 and must also be a multiple of 3, and therefore must be a multiple of 6.

The complement of the set $\boldsymbol{B}$: The complement of B consists of all elements **not in $\boldsymbol{B}$**, and is denoted $\boldsymbol{B'}$, $\boldsymbol{\overline{B}}$ or $\boldsymbol{\sim B}$. $\boldsymbol{B' = \{x \,|\, x \notin B\}}$. When referring to the complement of a set, it is usually understood that there is some "full set", and the complement of B consists of the elements of the full set which are not in B. For instance, if B is the set of all positive even integers, and if the "full set" is the set of all positive integers, then B' consists of all positive odd integers. The set difference of "set A minus B" is $\boldsymbol{A - B = \{x \,|\, x \in A \ and \ x \notin B\}}$ and consists of all elements that are in A but not in B. Note that $\boldsymbol{A - B = A \cap B'}$. $A - B$ can also be described as the set that results when the intersection $A \cap B$ is removed from A.



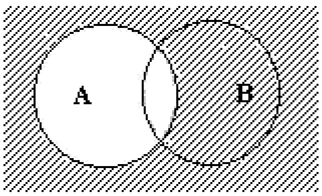
$B' = \overline{B}$



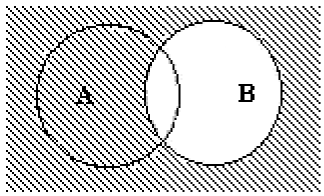
$A - B = A \cap B'$

Example 0-1: Verify the following set relationships (DeMorgan's Laws):
(i) $(A \cup B)' = A' \cap B'$ (the complement of the union of A and B is the intersection of the complements of A and B)
(ii) $(A \cap B)' = A' \cup B'$ (the complement of the intersection of A and B is the union of the complements of A and B)

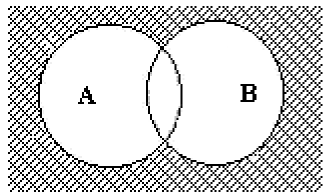
Solution: (i) Since the union of A and B consists of all points in either A **or** B, any point not in $A \cup B$ is in neither A nor B, and therefore must be in both the complement of A **and** the complement of B; this is the intersection of A' and B'. The reverse implication holds in a similar way; if a point is in the intersection of A' and B' then it is not in A **and** it is not in B, so it is not in $A \cup B$, and therefore it is in $(A \cup B)'$. Therefore, $(A \cup B)'$ and $A' \cap B'$ consist of the same collection of points, they are the same set.



A'

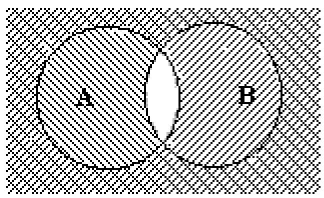


B'



$(A \cup B)' = A' \cap B'$

(ii) The solution is very similar to (i).



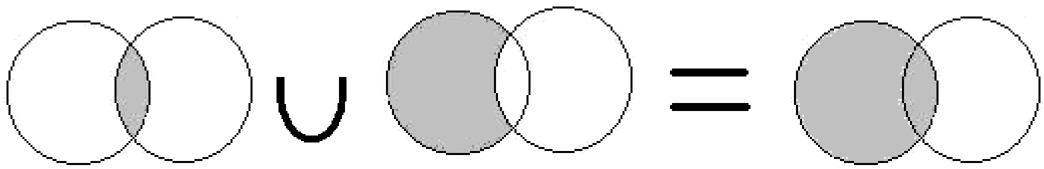
$(A \cap B)' = A' \cup B'$ □

Empty set: The **empty set** is the set that contains no elements, and is denoted $\emptyset$. It is also referred to as the **null set**. Sets A and B are called **disjoint sets** if $A \cap B = \emptyset$.

Relationships involving sets:

1. $A \cup B = B \cup A\,;\ A \cap B = B \cap A\,;\ A \cup A = A\,;\ A \cap A = A$
2. $A \cup \phi = A\,;\ A \cap \phi = \phi\,;\ A - \phi = A$
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$ (this can be seen
   from the Venn diagram in the paragraph above describing subset)
6. For any sets A and B, $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$
7. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
8. For any set A, $\phi \subset A$ (the empty set is a subset of any other set A)

An important rule (that follows from point 4 above) is the following.
For any two sets A and B, we have $A=(A\cap B)\cup(A\cap B')$ **.**



$A\cap B$ $\qquad$ $A\cap B'$ $\qquad$ A

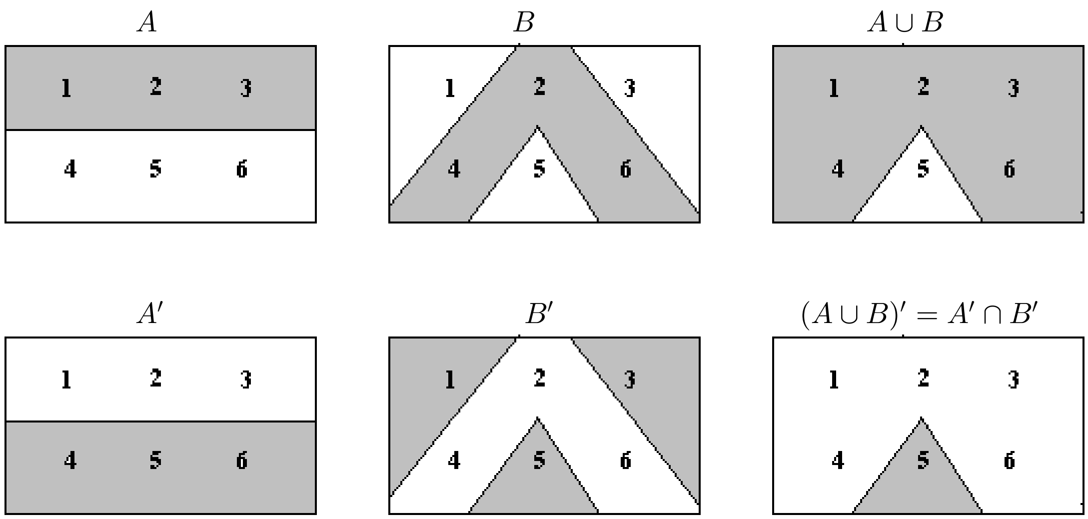
Related to this is the property that if a finite set is made up of the union of disjoint sets, then the number of elements in the union is the sum of the numbers in each of the component sets. For a finite set S, we define $n(S)$ to be the number of elements in S.
Two useful relationships for counting elements in a set are
$n(A)=n(A\cap B)+n(A\cap B')$ **(true since $A\cap B$ and $A\cap B'$ are disjoint), and**
$n(A\cup B)=n(A)+n(B)-n(A\cap B)$ **(cancels the double counting of $A\cap B$).**
This rule can be extended to three sets,

$$\begin{aligned} n(A\cup B\cup C) = {} & n(A)+n(B)+n(C) \\ & -n(A\cap B)-n(A\cap C)-n(B\cap C) \\ & +n(A\cap B\cap C)\,. \end{aligned}$$

The main application of set algebra is in a probability context in which we use set algebra to describe events and combinations of events (this appears in the next section of this study guide). An understanding of set algebra and Venn diagram representations can be quite helpful in describing and finding event probabilities.

Example 0-2: Suppose that the "total set" S consists of the possible outcomes that can occur when tossing a six-faced die. Then $S=\{1,2,3,4,5,6\}$. We define the following subsets of S:
$A=\{1,2,3\}$ (a number less than 4 is tossed) ,
$B=\{2,4,6\}$ (an even number is tossed) ,
$C=\{4\}$ (a 4 is tossed) .
Then $A\cup B=\{1,2,3,4,6\}$; $A\cap B=\{2\}$;
A and C are disjoint since $A\cap C=\emptyset$; $C\subset B$;
$A'=\{4,5,6\}$ (complement of A); $B'=\{1,3,5\}$; $A\cup B=\{1,2,3,4,6\}$;
and $(A\cup B)'=\{5\}=A'\cap B'$ (this illustrates one of DeMorgan's Laws).
This is illustrated in the following Venn diagrams with sets identified by shaded regions.

Example 0-2 continued:



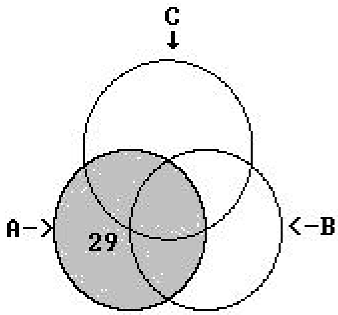
□

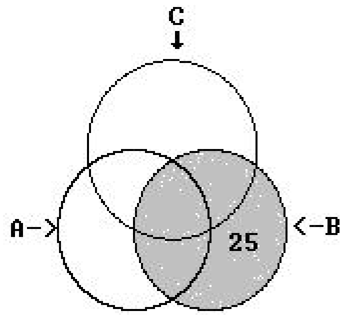
Venn diagrams can sometimes be useful when analyzing the combinations of intersections and unions of sets and the numbers of elements in various. The following examples illustrates this.

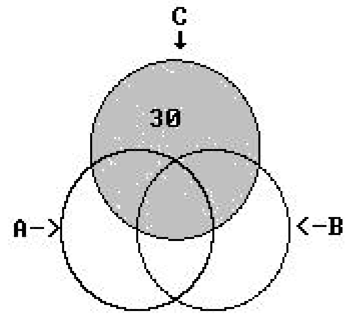
Example 0-3: A heart disease researcher has gathered data on 40,000 people who have suffered heart attacks. The researcher identifies three variables associated with heart attack victims:
A - smoker , B - heavy drinker , C - sedentary lifestyle .
The following data on the 40,000 victims has been gathered:
29,000 were smokers ; 25,000 were heavy drinkers ; 30,000 had a sedentary lifestyle ;
22,000 were both smokers and heavy drinkers ;
24,000 were both smokers and had a sedentary lifestyle ;
20,000 were both heavy drinkers and had a sedentary lifestyle ; and
20,000 were smokers, and heavy drinkers and had a sedentary lifestyle.
Determine how many victims were:
(i) neither smokers, nor heavy drinkers, nor had a sedentary lifestyle;
(ii) smokers but not heavy drinkers;
(iii) smokers but not heavy drinkers and did not have a sedentary lifestyle?
(iv) either smokers or heavy drinkers (or both) but did not have a sedentary lifestyle?

Solution: It is convenient to represent the data in Venn diagram form. For a subset S, $n(S)$ denotes the number of elements in that set (in thousands). The given information can be summarized in Venn diagram form as follows:

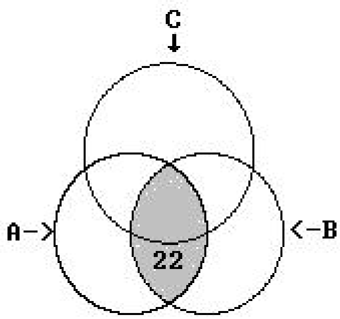
Example 0-3 continued:

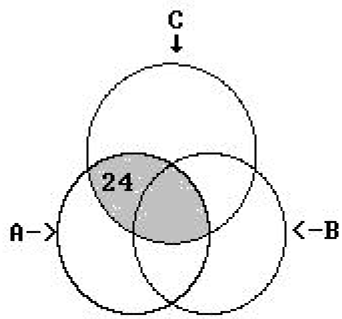


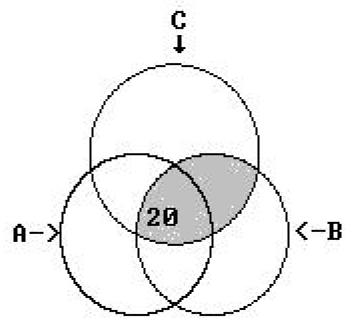




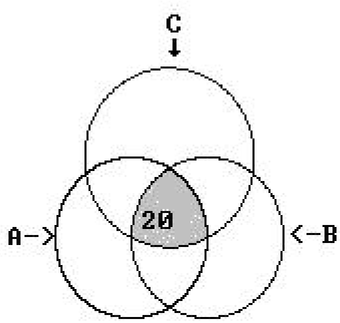
$n(A) = 29{,}000$ (smoker) $n(B) = 25,000$ (heavy drinker) $n(C) = 30,000$ (sedentary lifestyle)





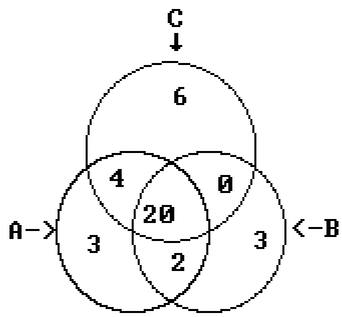


$n(A \cap B) = 22,000$ (smoker and heavy drinker) $n(A \cap C) = 24,000$ (smoker and sedentary lifestyle) $n(B \cap C) = 20,000$ (heavy drinker and sedentary lifestyle)



$n(A \cap B \cap C) = 20,000$ (smoker and heavy drinker and sedentary lifestyle)

Example 0-3 continued:
Working from the inside outward in the Venn diagrams, we can identify the number within each minimal subset of all of the intersections:



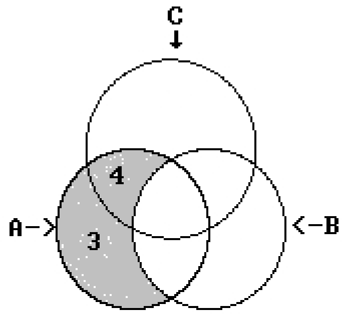
A typical calculation to fill in this diagram is as follows. We are given $n(A \cap B \cap C) = 20,000$ and $n(A \cap B) = 22,000$; we use the relationship
$22,000 = n(A \cap B) = n(A \cap B \cap C) + n(A \cap B \cap C') = 20,000 + n(A \cap B \cap C')$
to get $n(A \cap B \cap C') = 2,000$ (this shows that the 22,000 victims in $A \cap B$ who are both smokers and heavy drinker can be subdivided into those who also have a sedentary lifestyle $n(A \cap B \cap C) = 20,000$, and those who do not have a sedentary lifestyle, $n(A \cap B \cap C')$, the other $= 2,000$). Other entries are found in a similar way. From the diagram we can gain additional insight into other combinations of subsets. For instance, 6,000 of the victims have a sedentary lifestyle, but are neither smokers nor heavy drinkers; this is the entry "6", which in set notation is $n(A' \cap B' \cap C) = 6,000$. Also, the number of victims who were both heavy drinkers and had a sedentary lifestyle but were not smokers is 0.

We can now find the requested numbers.
(i) The number of victims who had at least one of the three specified conditions is $n(A \cup B \cup C)$, which, from the diagram can be calculated from the disjoint components:
$n(A \cup B \cup C) = 20,000 + 2,000 + 4,000 + 0 + 3,000 + 3,000 + 6,000 = 38,000$.
The "total set" in this example is the set of all 40,000 victims. Therefore, there were 2,000 heart attack victims who had none of the three specified conditions; this is the complement of $n(A \cup B \cup C)$. Algebraically, we have used the extension of one of DeMorgan's laws to the case of three sets, "none of A or B or C" $= (A \cup B \cup C)' = A' \cap B' \cap C' =$ "not A" and "not B" and "not C" .

Example 0-3 continued:

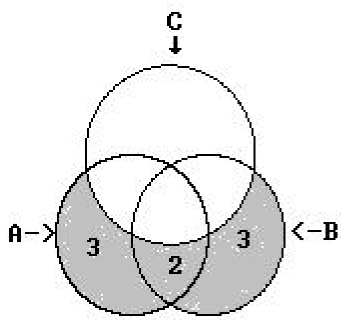
(ii) The number of victims who were smokers but not heavy drinkers is

$n(A \cap B') = 3,000 + 4,000$. This can be seen from the following Venn diagram



(iii) The number of victims who were smokers but not heavy drinkers and did not have a sedentary lifestyle is $n(A \cap B' \cap C') = 3,000$ (part of the group in (ii)).

(iv) The number of victims who were either smokers or heavy drinkers (or both) but did not have a sedentary lifestyle is $n[(A \cup B) \cap C']$. This is illustrated in the following Venn diagram.



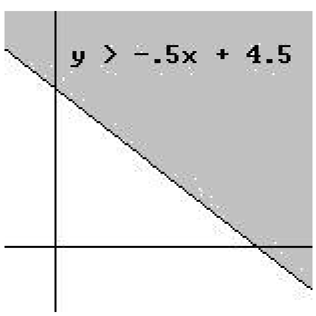
$n[(A \cup B) \cap C'] = 3,000 + 2,000 + 3,000 = 8,000$. □

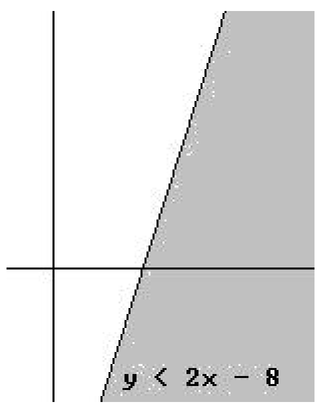
GRAPHING AN INEQUALITY IN TWO DIMENSIONS

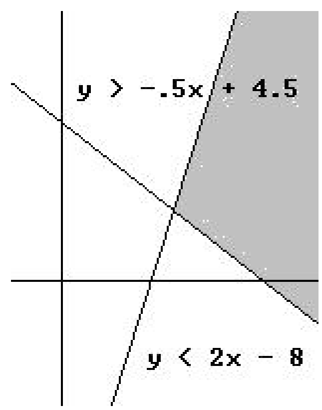
The joint distribution of a pair of random variables X and Y is sometimes defined over a two dimensional region which is described in terms of linear inequalities involving x and y. The region represented by the inequality $y > ax + b$ is the region above the line $y = ax + b$ (and $y < ax + b$ is the region below the line).

Example 0-4: Using the lines $y = -\frac{1}{2}x + \frac{9}{2}$ and $y = 2x - 8$, find the region in the x-y plane that satisfies both of the inequalities $y > -\frac{1}{2}x + \frac{9}{2}$ and $y < 2x - 8$.

Solution: We graph each of the straight lines, and then determine which side of the line is represented by the inequality. The first graph below is the graph of the line $y = -\frac{1}{2}x + \frac{9}{2}$, along with the shaded region, which is the region $y > -\frac{1}{2}x + \frac{9}{2}$, consisting of all points "above" that line. The second graph below is the graph of the line $y = 2x - 8$, along with the shaded region, which is the region $y < 2x - 8$, consisting of all points "below" that line. The third graph is the intersection (first region and second region) of the two regions.





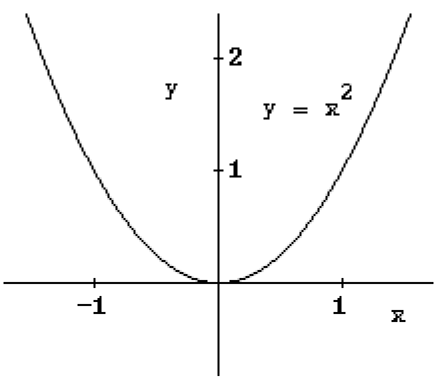


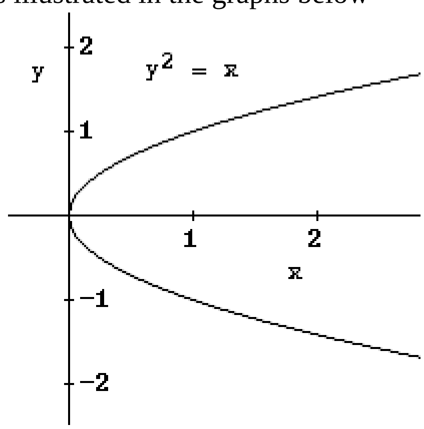
□

PROPERTIES OF FUNCTIONS

Definition of a function f: A function $f(x)$ is defined on a subset (or the entire set) of real numbers. For each x, the function defines a number $f(x)$. The **domain** of the function f is the set of x-values for which the function is defined. The **range of f** is the set of all $f(x)$ values that can occur for x's in the domain. Functions can be defined in a more general way, but we will be concerned only with real valued functions of real numbers. Any relationship between two real variables (say x and y) can be represented by its graph in the (x, y)-plane. If the function $y = f(x)$ is graphed, then for any x in the domain of f, the vertical line at x will intersect the graph of the function at exactly one point; this can also be described by saying that for each value of x there is (at most) one related value of y.

Example 0-5: (i) $y = x^2$ defines a function since for each x there is exactly one value x^2. The domain of the function is all real numbers (each real number has a square). The range of the function is all real numbers ≥ 0 , since for any real x, the square is $x^2 \geq 0$.
(ii) $y^2 = x$ does not define a function since if $x > 0$, there are two values of y for which $y^2 = x$. These two values are $\pm\sqrt{x}$. This is illustrated in the graphs below



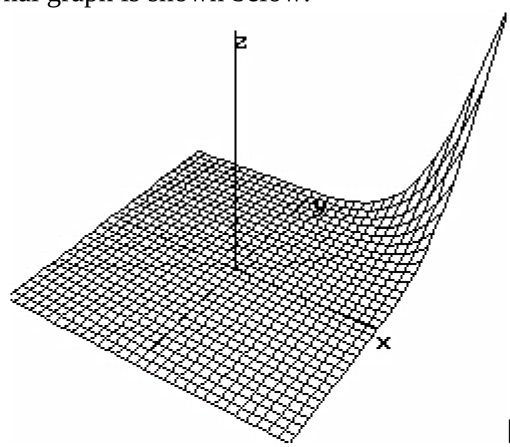


□

Functions defined piecewise: A function that is defined in different ways on separate intervals is called a **piecewise defined function**. The absolute value function is an example of a piecewise defined function: $|x| = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$.

Multivariate function: A function of more than one variable is called a multivariate function.

Example 0-6: $z = f(x,y) = e^{x+y}$ is a function of two variables, the domain is the entire 2-dimensional plane (the set $\{(x,y)|$ x,y are both real numbers$\}$) , and the range is the set of strictly positive real numbers. The function could be graphed in 3-dimensional x-y-z space. The domain would be the (horizontal) x-y plane, and the range would be the (vertical) z-dimension.
The 3-dimensional graph is shown below.

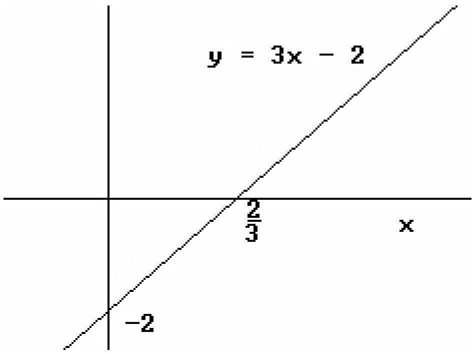


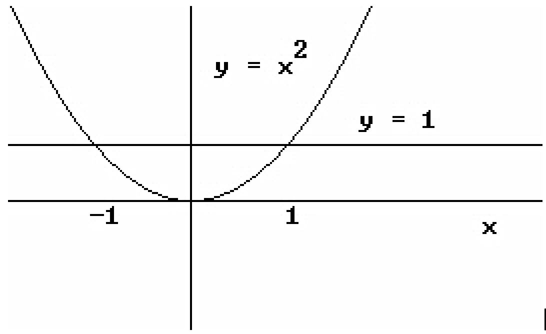
□

The concept of the inverse of a function is important when formulating the distribution of a transformed random variable. A preliminary concept related to the inverse of a function is that of a one-to-one function.

One-to-one function: The function f is called a one-to-one if the equation $f(x) = y$ has at most one solution x for each y (or equivalently, different x-values result in different $f(x)$ values). If a graph is drawn of a one-to-one function, any horizontal line crosses the graph in at most one place.

Example 0-7: The function $f(x) = 3x - 2$ is one-to-one, since for each value of y, the relation $y = 3x - 2$ has exactly one solution for x in terms of y; $x = \frac{y+2}{3}$. The function $g(x) = x^2$ with the whole set of real numbers as its domain is not one-to-one, since for each $y > 0$, there are two solutions for x in terms of y for the relation $y = x^2$ (those two solutions are $x = \sqrt{y}$ and $x = -\sqrt{y}$; note that if we restrict the domain of $g(x) = x^2$ to the positive real numbers, it becomes a one-to-one function). The graphs are below.





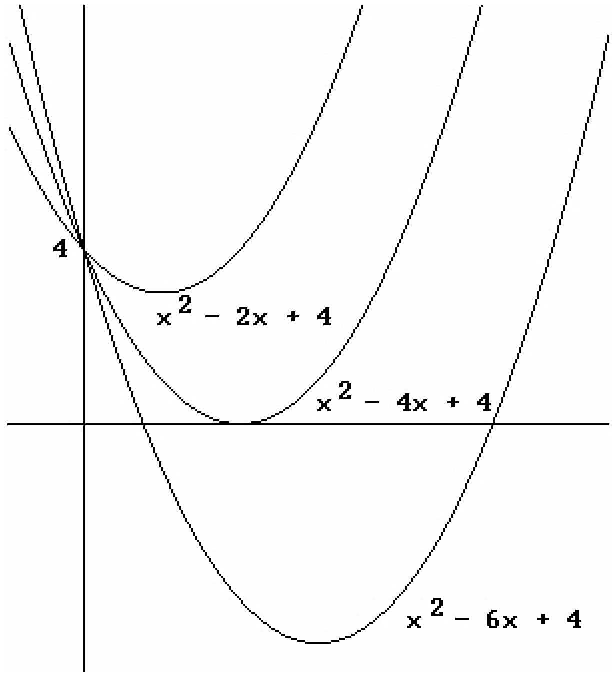
□

Inverse of function f: The inverse of the function f is denoted f^{-1}. The inverse exists only if f is one-to-one, in which case, $f^{-1}(y) = x$ is the (unique) value of x which satisfies $f(x) = y$ (finding the inverse of $y = f(x)$ means that we solve for x in terms of y, $x = f^{-1}(y)$). For instance, for the function $y = 2x^3 = f(x)$, if $x = 1$ then $y = f(1) = 2(1^3) = 2$, so that $1 = f^{-1}(2) = (2/2)^{1/3}$. For the example just considered, the inverse function applied to $y = 2$ is the value of x for which $f(x) = 2$, or equivalently, $2x^3 = 2$, from which we get $x = 1$.

Example 0-8: (i) The inverse of the function $y = 5x - 1 = f(x)$ is the function $x = \frac{y+1}{5} = f^{-1}(y)$ (we solve for x in terms of y).
(ii) Given the function $y = x^2 = f(x)$, solving for x in terms of y results in $x = \pm\sqrt{y}$, so there are two possible values of x for each value of y; this function does not have an inverse. However, if the function is defined to be $y = x^2 = f(x)$ **for $x \geq 0$ only**, then $x = +\sqrt{y} = f^{-1}(y)$ would be the inverse function, since f is one-to-one on its domain which consists of non-negative numbers. □

Quadratic functions and equations: A quadratic function is of the form $p(x) = ax^2 + bx + c$. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $r_1, r_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. The quadratic equation has
(i) distinct real roots if $b^2 - 4ac > 0$,
(ii) distinct complex roots if $b^2 - 4ac < 0$, and
(iii) equal real roots if $b^2 - 4ac = 0$.

Example 0-9: The quadratic equation $x^2 - 6x + 4 = 0$ has two distinct real solutions: $x = 3 \pm \sqrt{5}$. The quadratic equation $x^2 - 4x + 4 = 0$ has both roots equal: $x = 2$. The quadratic equation $x^2 - 2x + 4 = 0$ has two distinct complex roots: $x = 1 \pm i\sqrt{3}$.



□

Exponential and logarithmic functions: Exponential functions are of the form $f(x) = b^x$, where $b > 0,\ b \neq 1$, and the inverse of this function is denoted $log_b(y)$.
Thus $y = b^x \Leftrightarrow log_b(y) = x$. The log function with base e is the **natural logarithm**, $log_e(y) = ln\, y$ (also written $log\, y$). Some important properties of these functions are:

$b^0 = 1$	$log_b(1) = 0$
$domain(f) = \mathbb{R} = range(f^{-1})$	$range(f) = (0, +\infty) = domain(f^{-1})$
$b^{log_b(y)} = y$ for $y > 0$	$log_b(b^x) = x$ for all x
$b^x = e^{x \cdot ln\, b}$	$log_b(y) = \frac{ln\, y}{ln\, b}$
$(b^x)^y = b^{xy}$	$log_b(y^k) = k \cdot log_b(y)$
$b^x b^y = b^{x+y}$	$log_b(yz) = log_b(y) + log_b(z)$
$b^x / b^y = b^{x-y}$	$log_b(y/z) = log_b(y) - log_b(z)$

For the function e^x, we have $e^{ln\, y} = y$ for an $y > 0$, and for the natural log function, we have $ln\, e^x = x$ for any real number x.

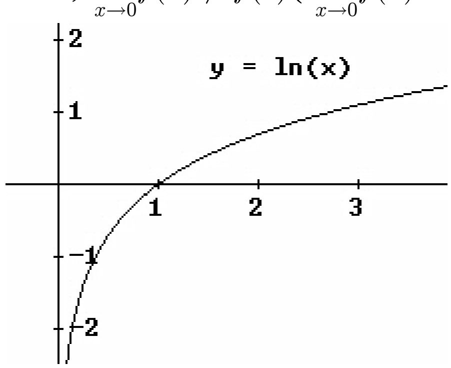
LIMITS AND CONTINUITY

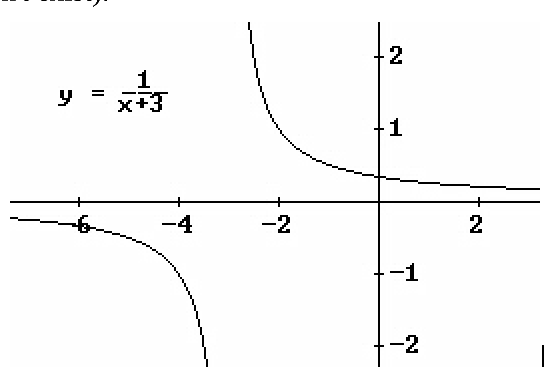
Intuitive definition of limit: The expression $\lim_{x \to c} f(x) = L$ means that as x gets close to (approaches) the number c, the value of $f(x)$ gets close to L.

Example 0-10: $\lim_{x \to 1}(x+3) = 4$, $\lim_{x \to +\infty} e^{-x} = 0$ and $\lim_{x \to 1} \frac{x^2+2x-3}{x-1} = \lim_{x \to 1}(x+3) = 4$ (for this last limit, note that $\frac{x^2+2x-3}{x-1} = \frac{(x+3)(x-1)}{x-1} = x+3$ if $x \neq 1$, but in taking this limit we are only concerned with what happens "near" $x = 1$, that fact that $\frac{x^2+2x-3}{x-1} = \frac{0}{0}$ at $x = 1$ does not mean that the limit does not exist; it means that the function does not exist at the point $x = 1$).□

Continuity: **The function $\boldsymbol{f}$ is continuous at the point $\boldsymbol{x = c}$** if there is no "break" or "hole" in the graph of $y = f(x)$, or equivalently, if $\boldsymbol{\lim_{x \to c} f(x) = f(c)}$. In Example 0-10 above, the third function is not continuous at $x = 1$ because $f(1) = \frac{0}{0}$ is not defined. Another reason for a discontinuity in $f(x)$ occurring at $x = c$ is that the limit of $f(x)$ is different from the left than it is from the right.

Example 0-11: (i) If $f(x) = ln\, x$ and $c = 0$ then f is discontinuous at $c = 0$ since the function $ln\, x$ is not defined at the point $x = 0$ (this would also be the case for the function $f(x) = \frac{1}{x+3}$ and $c = -3$).
(ii) If $f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$, then $f(x)$ is discontinuous at $x = 0$ since even though $f(0)$ is defined, $\lim_{x \to 0} f(x) \neq f(0)$ ($\lim_{x \to 0} f(x)$ doesn't exist).





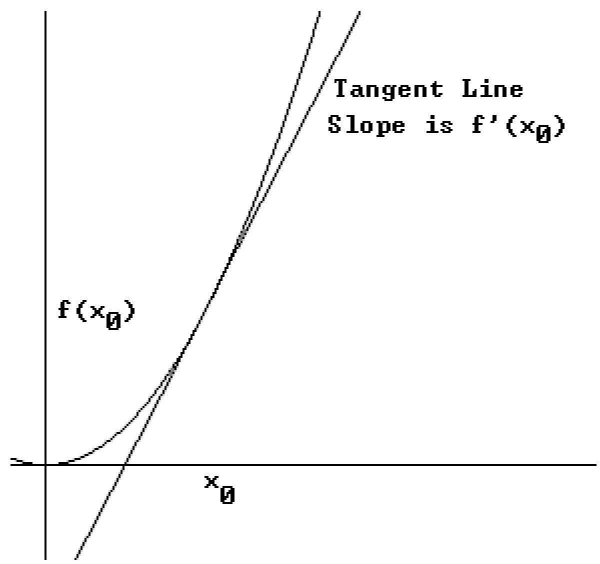
□

DIFFERENTATION

Geometric interpretation of derivative: The derivative of the function $f(x)$ at the point $x = x_0$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$. The derivative of $f(x)$ at $x = x_0$ is denoted $f'(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$.

This is also referred to as the derivative of f with respect to x at the point $x = x_0$.
The algebraic definition of $f'(x_0)$ is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} .$$



The second derivative of f at x_0 is the derivative of $f'(x)$ at the point x_0. It is denoted $f''(x_0)$ or $f^{(2)}(x_0)$ or $\left.\frac{d^2 f}{dx^2}\right|_{x=x_0}$. The n-th order derivative of f at x_0 (n repeated applications of differentiation) is denoted $f^{(n)}(x_0) = \left.\frac{d^n f}{dx^n}\right|_{x=x_0}$.

The derivative as a rate of change: Perhaps the most important interpretation of the derivative $f'(x_0)$ is as the **"instantaneous" rate at which the function is increasing or decreasing as x increases (if $f' > 0$, the graph of $y = f(x)$ is rising, with the tangent line to the graph having positive slope, and if $f' < 0$, the graph of $y = f(x)$ is falling)**, and if $f'(x_0) = 0$ then the tangent line at that point is horizontal (has slope 0). This interpretation is the one most commonly used when analyzing physical, economic or financial processes.

The following is a summary of some important differentiation rules.

Rules of differentiation:	$f(x)$	$f'(x)$
	c (a constant)	0
Power rule -	cx^n ($n \in \mathbb{R}$)	cnx^{n-1}
	$g(x) + h(x)$	$g'(x) + h'(x)$
Product rule -	$g(x) \cdot h(x)$	$g'(x) \cdot h(x) + g(x) \cdot h'(x)$
	$u(x)v(x)w(x)$	$u'vw + uv'w + uvw'$
Quotient rule -	$\frac{g(x)}{h(x)}$	$\frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$
Chain rule -	$g(h(x))$	$g'(h(x)) \cdot h'(x)$
	$e^{g(x)}$	$g'(x) \cdot e^{g(x)}$
	$ln(g(x))$	$\frac{g'(x)}{g(x)}$
	a^x $(a > 0)$	$a^x \, ln\, a$
	e^x	e^x
	$ln\, x$	$\frac{1}{x}$
	$log_b x$	$\frac{1}{x\, lnb}$
	$sin\, x$	$cos\, x$
	$cos\, x$	$-\, sin\, x$

Example 0-12: What is the derivative of $f(x) = 4x(x^2+1)^3$?
Solution: We apply the product rule and chain rule: $f(x) = g(x) \cdot h(x)$,
where $g(x) = 4x$, $h(x) = (x^2+1)^3$, $g'(x) = 4$, $h'(x) = 3(x^2+1)^2 \cdot 2x$.
$f'(x) = 4x \cdot 3(x^2+1)^2 \cdot 2x + 4(x^2+1)^3 = 4(x^2+1)^2(7x^2+1)$.
Notice that $h(x) = (x^2+1)^3 = [w(x)]^3 = h(w(x))$, where $h(w) = w^3$ and $w(x) = x^2+1$.
The chain rule tells us that $h'(x) = h'(w) \cdot w'(x) = 3w^2 \cdot (2x) = 3(x^2+1)^2 \cdot (2x)$. □

L'Hospital's rules for calculating limits: A limit of the form $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ is said to be in indeterminate form if both the numerator and denominator go to 0, or if both the numerator and denominator go to $\pm\infty$. L'Hospital's rules are:

1. **IF** $\begin{cases} \text{(i) } \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f'(c) \text{ exists, } \textbf{and} \\ \text{(iii) } g'(c) \text{ exists and is } \neq 0 \end{cases}$ **THEN** $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$

2. **IF** $\begin{cases} \text{(i) } \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f \text{ and } g \text{ are differentiable near } c, \textbf{ and} \\ \text{(iii) } \lim\limits_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists} \end{cases}$ **THEN** $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$

In 1 or 2, the conditions $\lim\limits_{x \to c} f(x) = 0$ and $\lim\limits_{x \to c} g(x) = 0$ can be replaced by the conditions $\lim\limits_{x \to c} f(x) = \pm\infty$ and $\lim\limits_{x \to c} g(x) = \pm\infty$, and the point c can be replaced by $\pm\infty$ with the conclusions remaining valid.

Example 0-13: Find $\lim\limits_{x \to 2} \frac{3^{x/2}-3}{3^x-9}$.

Solution: The limits in both the numerator and denominator are 0, so we apply l'Hospital's rule. $\frac{d}{dx} 3^x = 3^x \, ln\, 3$, and $\frac{d}{dx} 3^{x/2} = 3^{x/2} \cdot \frac{1}{2} \, ln\, 3$, so that $\lim\limits_{x \to 2} \frac{3^{x/2}-3}{3^x-9} = \lim\limits_{x \to 2} \frac{3^{x/2} \cdot \frac{1}{2} \, ln\, 3}{3^x \, ln\, 3} = \frac{1}{6}$. This limit can also be found by factoring the denominator into $3^x - 9 = (3^{x/2} - 3)(3^{x/2} + 3)$, and then canceling out the factor $3^{x/2} - 3$ in the numerator and denominator. □

Differentiation of functions of several variables - partial differentiation:

Given the function $f(x, y)$, a function of two variables, the partial derivative of f with respect to x at the point (x_0, y_0) is found by differentiating f with respect to x and regarding the variable y as constant - then substitute in the values $x = x_0$ and $y = y_0$. The partial derivative of f with respect to x is usually denoted $\frac{\partial f}{\partial x}$. The partial derivative with respect to y is defined in a similar way: "Higher order" partial derivatives can be defined - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$; and "mixed partial" derivatives can be defined (the order of partial differentiation does not usually matter) - $\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \, \partial x}$.

Example 0-14: If $f(x, y) = x^y$ for $x, y > 0$ then find $\left.\frac{\partial f}{\partial x}\right|_{(4,\frac{1}{2})}$ and $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(4,\frac{1}{2})}$.

Solution: $\frac{\partial f}{\partial x} = \left. y x^{y-1} \right|_{(4,\frac{1}{2})} = (\frac{1}{2})(4)^{-1/2} = \frac{1}{4}$, and

$\frac{\partial f}{\partial y} = x^y \, (ln\, x)$ and $\frac{\partial^2 f}{\partial y^2} = \left. x^y \, (ln\, x)^2 \right|_{(4,\frac{1}{2})} = 4^{1/2} \, (ln\, 4)^2 = 2 \, (ln\, 4)^2$. □

INTEGRATION

Geometric interpretation of the "definite integral" - the area under the curve:

Given a function $f(x)$ on the interval $[a,b]$, the definite integral of $f(x)$ over the interval is denoted $\int_a^b f(x)\,dx$, and is equal to the "signed" area between the graph of the function and the x-axis from $x=a$ to $x=b$. Signed area is positive when $f(x)>0$ and is negative when $f(x)<0$. What is meant by signed area here is the area from the interval(s) where $f(x)$ is positive minus the area from the intervals where $f(x)$ is negative.

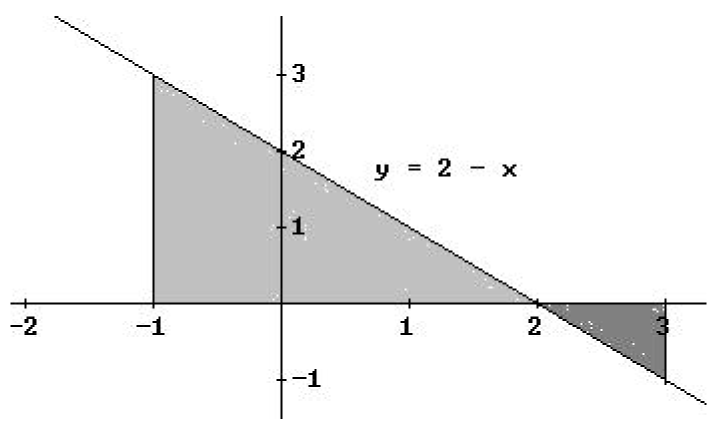
Integration is related to the antiderivative of a function. Given a function $f(x)$, an antiderivative of $f(x)$ is any function $F(x)$ which satisfies the relationship $F'(x)=f(x)$. According to the Fundamental Theorem of Calculus, the definite integral for $f(x)$ can be found by first finding $F(x)$, an antiderivative of $f(x)$. The basic relationships relating integration and differentiation are:

(i) If $F'(x)=f(x)$ for $a\le x\le b$, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

(ii) If $G(x)=\int_a^x g(t)\,dt$, then $G'(x)=g(x)$

Example 0-15: Find the definite integral of the function $f(x)=2-x$ on the interval $[-1,3]$.

Solution: The graph of the function is given below. It is clear that $f(x)>0$ for $x<2$, and $f(x)<0$ for $x>2$. An antiderivative for $f(x)$ is $F(x)=2x-\frac{x^2}{2}$. The definite integral will be $\int_{-1}^3 (2-x)\,dx = F(3)-F(-1) = (6-\frac{3^2}{2})-(-2-\frac{(-1)^2}{2}) = 4$. Note that the area between the graph and the x-axis from $x=-1$ to $x=2$ is $\frac{1}{2}(3)(3)=\frac{9}{2}$, and the signed area between the graph and the x-axis from $x=2$ to $x=3$ is $-\frac{1}{2}(1)(1)=-\frac{1}{2}$. The total signed area is $\frac{9}{2}-\frac{1}{2}=4$.



□

Antiderivatives of some frequently used functions:

$f(x)$	$\int f(x)dx$ (antiderivative)
$g(x)+h(x)$	$\int g(x)dx + \int h(x)dx + c$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
$\frac{1}{x}$	$ln\,x + c$
e^x	$e^x + c$
$a^x\ (a>0)$	$\frac{a^x}{ln\,a} + c$
xe^{ax}	$\frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + c$.
$sin\,x$	$-cos\,x + c$
$cos\,x$	$sin\,x + c$

Integration of f on $[a,b]$ when f is not defined at a or b, or when a or b is $\pm\infty$:

Integration over an infinite interval (an "improper integral") is defined by taking limits:
$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$, with a similar definition applying to $\int_{-\infty}^b f(x)\,dx$,
and $\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to+\infty}\int_{-a}^a f(x)\,dx$.

If f is not defined at $x=a$ (also called an improper integral), or if f is discontinuous at $x=a$,
then $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$.

A similar definition applies if f is not defined at $x=b$, or if f is discontinuous at $x=b$.

If $f(x)$ has a discontinuity at the point $x=c$ in the interior of $[a,b]$, then
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example 0-16:

(a) $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+}\int_c^1 x^{-1/2}dx = \lim_{c\to 0^+}\left[2x^{1/2}\Big|_{x=c}^{x=1}\right] = \lim_{c\to 0^+}[2-2\sqrt{c}] = 2$,

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{c\to\infty}\int_1^c x^{-1/2}dx = \lim_{c\to\infty}\left[2x^{1/2}\Big|_{x=1}^{x=c}\right] = \lim_{c\to\infty}[2\sqrt{c}-2] = +\infty$.

(b) $\int_1^{+\infty}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_{x=1}^{x=b}\right] = \lim_{b\to\infty}[-\frac{1}{b}-(-1)] = 1$

(c) $\int_{-\infty}^1 \frac{1}{x^2}\,dx$. Note that $\frac{1}{x^2}$ has a discontinuity at $x=0$, so that
$\int_{-\infty}^1 \frac{1}{x^2}\,dx = \int_{-\infty}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$. The second integral is
$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}[-1+\frac{1}{a}] = +\infty$, thus, the second improper integral does not exist (when $\lim_{\to}\int$ is infinite or does not exist, the integral is said to "diverge"). □

A few other useful integration rules are:

(i) for integer $n \geq 0$ and real number $c > 0$ $\int_0^\infty x^n \, e^{-cx} \, dx = \frac{n!}{c^{n+1}}$.

(ii) if $G(x) = \int_a^{h(x)} f(u) \, du$, then $G'(x) = f[h(x)] \cdot h'(x)$,

(iii) if $G(x) = \int_x^b f(u) \, du$, then $G'(x) = -f(x)$,

(iv) if $G(x) = \int_{g(x)}^b f(u) \, du$, then $G'(x) = -f[g(x)] \cdot g'(x)$,

(v) if $G(x) = \int_{g(x)}^{h(x)} f(u) \, du$, then $G'(x) = f[h(x)] \cdot h'(x) - f[g(x)] \cdot g'(x)$.

Double integral: Given a continuous function of two variables, $f(x,y)$ on the rectangular region bounded by $x = a$, $x = b$, $y = c$ and $y = d$, it is possible to define the definite integral of f over the region. It can be expressed in one of two equivalent ways:

$$\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy$$

The interpretation of the first expression is $\int_a^b [\int_c^d f(x,y) \, dy] \, dx$, in which the "inside integral" is $\int_c^d f(x,y) \, dy$, and it is calculated assuming that the value of x is constant (it is an integral with respect to the variable y). When this definite "inside integral" has been calculated, it will be a function of x alone, which can then be integrated with respect to x from $x = a$ to $x = b$. The second equivalent expression has a similar interpretation; $\int_a^b f(x,y) \, dx$ is calculated assuming that y is constant; this results in a function of y alone which is then integrated with respect to y from $y = c$ to $y = d$. Double integration arises in the context of finding probabilities for a joint distribution of continuous random variables.

Example 0-17: Find $\int_0^1 \int_1^2 \frac{x^2}{y} \, dy \, dx$.

Solution: First we assume that x is constant and find $\int_1^2 \frac{x^2}{y} \, dy = x^2(ln\, y)\Big|_{y=1}^{y=2} = x^2(ln\, 2)$.

Then we find $\int_0^1 [x^2(ln\, 2)] \, dx = (ln\, 2) \cdot \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{ln\, 2}{3}$.

We can also write the integral as $\int_1^2 \int_0^1 \frac{x^2}{y} \, dx \, dy$, and first find

$\int_0^1 \frac{x^2}{y} \, dx = \frac{x^3}{3y}\Big|_{x=0}^{x=1} = \frac{1}{3y}$. Then, $\int_1^2 \frac{1}{3y} \, dy = \frac{1}{3}(ln\, y)\Big|_{y=1}^{y=2} = \frac{1}{3}(ln\, 2)$. $\square$

For double integration over the rectangular two-dimensional region $a \leq x \leq b$, $c \leq y \leq d$, as the expression $\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy$ indicates, it is possible to calculate the double integral by integrating with respect to the variables in either order (y first and x second for the integral on the left, and x first and y second for the integral on the right of the " $=$ " sign).

Formulations of probabilities and expectations for continuous joint distributions sometimes involve integrals over a non-rectangular two-dimensional region. It will still be possible to arrange the integral for integration in either order ($dy\,dx$ or $dx\,dy$), but care must be taken in setting up the limits of integration. If the limits of integration are properly specified, then the double integral will be the same whichever order of integration is used. Note also that in some situations, it may be more efficient to formulate the integration in one order than in the other.

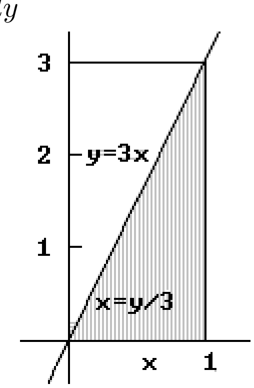
Example 0-18: Which of the following integrals is equal to $\int_0^1\int_0^{3x} f(x,y)\,dy\,dx$ for every function for which the integral exists?

A) $\int_0^3\int_0^{y/3} f(x,y)\,dx\,dy$ B) $\int_0^1\int_{3x}^3 f(x,y)\,dx\,dy$ C) $\int_0^3\int_{3y}^1 f(x,y)\,dx\,dy$

D) $\int_0^1\int_0^{x/3} f(x,y)\,dx\,dy$ E) $\int_0^3\int_{y/3}^1 f(x,y)\,dx\,dy$

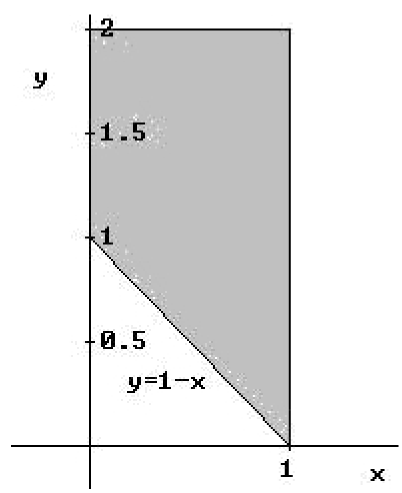
Solution: The graph at the right illustrates the region of integration. The region is $0 \le x \le 1$, $0 \le y \le 3x$. Writing $y = 3x$ as $x = \frac{y}{3}$, we see that the inequalities translate into $0 \le y \le 3$, and $\frac{y}{3} \le x \le 1$.

Answer: E □



Example 0-19: The function $f(x,y)$ is to be integrated over the two-dimensional region defined by the following constraints: $0 \le x \le 1$ and $1-x \le y \le 2$. Formulate the double integration in the $dy\,dx$ order and then in the $dx\,dy$ order.

Solution: The graph at the right illustrates the region of integration. The region is $0 \le x \le 1$, $1-x \le y \le 2$. The integral can be formulated in the $dy\,dx$ order as $\int_0^1\int_{1-x}^2 f(x.y)\,dy\,dx$; for each x, the integral in the vertical direction starts on the line $y = 1-x$ and continues to the upper boundary $y = 2$. To use the $dx\,dy$ order, we must split the integral into two double integrals; $\int_0^1\int_{1-y}^1 f(x,y)\,dx\,dy$ to cover the triangular area below $y = 1$, and $\int_1^2\int_0^1 f(x,y)\,dx\,dy$ to cover the square area above $y = 1$. □



There are a few integration techniques that are useful to know. The integrations that arise on Exam P are usually straightforward, but knowing a few additional techniques of integration are sometimes useful in simplifying an integral in an efficient way.

The Method of Substitution: Substitution is a basic technique of integration that is used to rewrite the integral in a standard form for which the antiderivative is well known. In general, to find $\int f(x)\,dx$ we may make the substitution $u = g(x)$ for an "appropriate" function $g(x)$. We then define the "differential" du to be $du = g'(x)\,dx$, and we try to rewrite $\int f(x)\,dx$ as an integral with respect to the variable u.

For example, to find $\int (x^3-1)^{4/3}x^2dx$, we let $u = x^3-1$, so that $du = 3x^2dx$, or equivalently, $\frac{1}{3}\cdot du = x^2dx$; then the integral can be written as $\int u^{4/3}\cdot\frac{1}{3}\,du$, which has antiderivative $\int u^{4/3}\cdot\frac{1}{3}\,du = \frac{1}{3}\cdot\int u^{4/3}du = \frac{1}{3}\cdot\frac{u^{7/3}}{7/3} = \frac{1}{7}u^{7/3}(+c)$.

We can then write the antiderivative in terms of the original variable x -
$\int (x^3-1)^{4/3}x^2dx = \frac{1}{7}u^{7/3} = \frac{1}{7}(x^3-1)^{7/3}$.

The main point to note in applying the substitution technique is that the choice of $u = g(x)$ should result in an antiderivative which is easier to find than was the original antiderivative.

Example 0-20: Find $\int_0^1 x\sqrt{1-x^2}dx$.

Solution: Let $u = 1-x^2$ Then $du = -2x\,dx$, so that $-\frac{1}{2}\cdot du = x\,dx$, and the antiderivative can be written as $\int u^{1/2}\cdot(-\frac{1}{2})du = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}$. The definite integral is then $\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_{x=0}^{x=1} = -0-(-\frac{1}{3}) = \frac{1}{3}$.

Note that once the appropriate substitution has been made, the definite integral may be calculated in terms of the variable u: $u(0) = 1$ and $u(1) = 0$ -

$\int_0^1 x\sqrt{1-x^2}dx = \int_{u(0)=1}^{u(1)=0} u^{1/2}\cdot(-\frac{1}{2})du = -\frac{1}{3}u^{3/2}\Big|_{u=1}^{u=0} = -0-(-\frac{1}{3}) = \frac{1}{3}$. □

Integration by parts:

This technique of integration is based upon the product rule
$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$. This can be rewritten as
$f(x) \cdot g'(x) = \frac{d}{dx}[f(x) \cdot g(x)] - f'(x) \cdot g(x)$, which means that the antiderivative of
$f(x) \cdot g'(x)$ can be written as $\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx$.
This technique is useful if $f'(x) \cdot g(x)$ has an easier antiderivative to find than $f(x) \cdot g'(x)$.
Given an integral, it may not be immediately apparent how to define $f(x)$ and $g(x)$ so that the integration by parts technique applies and results in a simplification. It may be necessary to apply integration by parts more than once to simplify an integral.

Example 0-21: Find $\int xe^{ax}\, dx$, where a is a constant.
Solution: If we define $f(x) = x$ and $g(x) = \frac{e^{ax}}{a}$, then $g'(x) = e^{ax}$, and
$\int xe^{ax}\, dx = \int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$.
Since $f'(x) = 1$, it follows that $\int f'(x)g(x)\, dx = \int \frac{e^{ax}}{a} dx = \frac{e^{ax}}{a^2}$, and therefore
$\boldsymbol{\int xe^{ax}\, dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + c}$ **.**
An alternative to integration by parts is the following approach:
$\boldsymbol{\frac{d}{da}\int e^{ax}\, dx = \int xe^{ax}\, dx}$ and $\boldsymbol{\frac{d}{da}\int e^{ax}\, dx = \frac{d}{da}\ \frac{e^{ax}}{a} = \frac{axe^{ax}-e^{ax}}{a^2}}$
so it follows that $\boldsymbol{\int xe^{ax}\, dx = \frac{axe^{ax}-e^{ax}}{a^2} = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}}$ **.**
This integral has appeared a number of times on the exam, usually with $a < 0$ (it is valid for any $a \neq 0$) and it is important to be familiar with it. □

NOTE: An extension of Example 0-21 shows that
for integer $n \geq 0$ and $c > 0$ $\boldsymbol{\int_0^\infty x^n\, e^{-cx}\, dx = \frac{n!}{c^{n+1}}}$.
This is another useful identity for the exam.

GEOMETRIC AND ARITHMETIC PROGRESSIONS

Geometric progression: $a,\ ar,\ ar^2,\ ar^3,\ \ldots,$ sum of the first n terms is
$a+ar+ar^2+\cdots+ar^{n-1}=a[1+r+r^2+\cdots+r^{n-1}]=a\cdot\frac{r^n-1}{r-1}=a\cdot\frac{1-r^n}{1-r}$,
if $-1<r<1$ then the series can be summed to ∞, $a+ar+ar^2+\cdots=\frac{a}{1-r}$

Arithmetic progression: $a,\ a+d,\ a+2d,\ a+3d,\ \ldots,$
sum of the first n terms of the series is $na+d\cdot\frac{n(n-1)}{2}$,
a special case is the sum of the first n integers - $1+2+\cdots+n=\frac{n(n+1)}{2}$

Example 0-22: A product sold 10,000 units last week, but sales are forecast to decrease 2% per week if no advertising campaign is implemented. If an advertising campaign is implemented immediately, the sales will decrease by 1% of the previous week's sales but there will be 200 new sales for the week (starting with this week). Under this model, calculate the number of sales for the 10-th week, 100-th week and 1000-th week of the advertising campaign (last week is week 0, this week is week 1 of the campaign).

Solution: Week 1 sales: $(.99)(10,000)+200$,
Week 2 sales: $(.99)[(.99)(10,000)+200]+200=(.99)^2(10,000)+(200)[1+.99]$
Week 3 sales: $(.99)[(.99)^2(10,000)+(200)[1+.99]]+200$
$=(.99)^3(10,000)+(200)[1+.99+.99^2]$
$\vdots$
Week 10 sales: $(.99)^{10}(10,000)+(200)[1+.99+.99^2+\cdots+.99^9]$
$=(.99)^{10}(10,000)+(200)[\frac{1-.99^{10}}{1-.99}]=10,956.2.$
Week 100 sales: $(.99)^{100}(10,000)+(200)[\frac{1-.99^{100}}{1-.99}]=16,339.7.$
Week 1000 sales: $(.99)^{1000}(10,000)+(200)[\frac{1-.99^{1000}}{1-.99}]=19,999.6.$ □

PROBLEM SET 0

Review of Algebra and Calculus

1. The manufacturer of a certain product is conducting a market survey. The manufacturer started a major marketing campaign at the end of last year and is trying to determine the effect of that campaign on consumer use of the product. 15,000 individuals are surveyed. The following information is obtained.
- 4,500 used the product last year,
- 7,500 used the product this year, and
- 4,000 used the product both last year and this year.

Of those surveyed, determine:
(i) the number who used the product either last year or this year, or both years,
(ii) the number that did not use the product either last year or this year,
(iii) the number who used the product last year but not this year, and
(iv) the number who used the product this year but not last year.

2. A group of 5000 undergraduate college students were surveyed regarding the following characteristics:
- participate in extracurricular activities,
- have a double major, and
- have a part-time job .

The following data was obtained.
2600 participated in extracurricular activities , 1200 had a double major,
2500 had a part time-job ,
400 both participated in extracurricular activities and had a double major ,
1000 both participated in extracurricular activities and had a part-time job ,
300 both had a double major and had a part-time job ,
200 participated in extracurricular activities and had a double major and had a part-time job.
Determine each of the following.
(i) The number who had a double major but did not participate in extra-curricular activities and did not have a part-time job.
(ii) The number who had a double major and either participated in extra-curricular activities or had a part-time job , but not both.
(iii) The number who neither participated in extracurricular activities nor had a part-time job.

3. A group of 1000 patients each diagnosed with a certain disease is being analyzed with regard to the disease symptoms present. The symptoms are labeled A, B and C, and each patient has at least one symptom. The following information has also been determined:
- 900 have either symptom A or B (or both),
- 900 have either symptom A or C (or both),
- 800 have either symptom B or C (or both),
- 650 have symptom A,
- 500 have symptom B,
- 550 have symptom C.

Determine each of the following.

(i) The number who had both symptoms A and B.

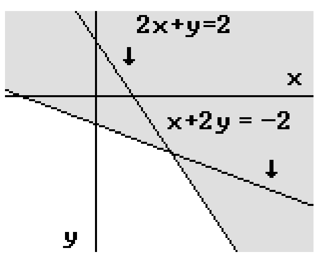
(ii) The number who had either symptom A or B (or both) but not C.

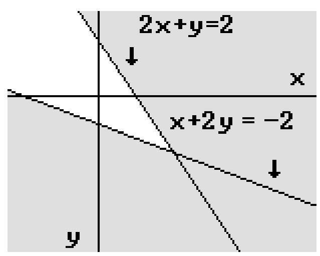
(iii) The number who had all three symptoms.

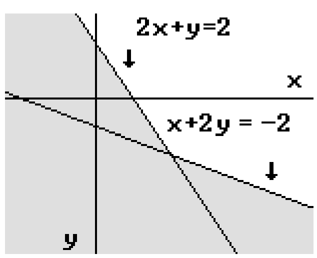
4. $\lim\limits_{N\to\infty} \frac{5^N}{N!} =$

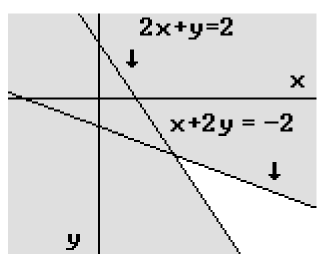
A) 0 B) $\frac{1}{2}$ C) $5\,ln\,5$ D) $+\infty$ E) None of A,B,C,D

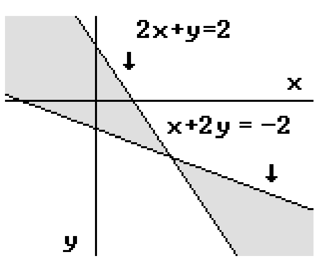
5. Which region of the plane represents the solution set to the inequalities
$\{(x,y) : 2x + y > 2\} \cup \{(x,y) : x + 2y < -2\}$?

A) 

B) 

C) 

D) 

E) 

6. For what real values of k are the roots of $kx^2 + 3x - 2 = 0$ not real numbers?
A) $k < -\frac{9}{8}$ B) $k < \frac{9}{8}$ C) $k > -\frac{9}{8}$ D) $k < -\frac{8}{9}$ E) $k < \frac{8}{9}$

7. If $f(x) = f^{-1}(x)$ then $f(x)$ may equal:
A) 2^x B) $\frac{1}{2^x}$ C) 1 D) $\frac{1}{x}$ E) $\frac{1}{\sqrt{x}}$

8. Let $f(x) = x^4 e^x$. Determine the nth derivative of f at $x = 0$.
A) 0 B) 1 C) $n(n-1)(n-2)$ D) $n(n-1)(n-2)(n-3)$ E) $n!$

9. A model for world population assumes a population of 6 billion at reference time 0, with population increasing to a limiting population of 30 billion. The model assumes that the rate of population growth at time $t \geq 0$ is $\frac{Ae^t}{(.02A+e^t)^2}$ billion per year, where t is regarded as a continuous variable. According to this model, at what time will the population reach 10 billion (nearest .1)?
A) .3 B) .4 C) .5 D) .6 E) .6

10. Calculate the area of the closed region in the xy-plane bounded by $y = x - 5$ and $y^2 = 2x + 5$.
A) 8 B) $\frac{74}{3}$ C) $\frac{98}{3}$ D) $\frac{122}{3}$ E) $\frac{128}{3}$

11. Let $F(x) = \int_0^{x^{1/3}} \sqrt{1+t^4}\, dt$. $F'(0) =$
A) 0 B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) 1 E) Does not exist

12. Let f be a continuous function on $\mathbb{R}^2$ and let $I = \int_0^2 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\, dy\, dx$.
Which of the following expressions is equal to I with the order of integration reversed?
A) $\int_{-2}^{2} \int_{y^2}^{2} f(x,y)\, dx\, dy$ D) $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{2} f(x,y)\, dx\, dy$
B) $\int_{-2}^{2} \int_{y}^{\sqrt{2}} f(x,y)\, dx\, dy$ E) $\int_0^2 \int_2^{y^2} f(x,y)\, dx\, dy$
C) $\int_0^2 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx\, dy$

13. Calculate $\int_0^1 \int_x^1 \frac{1}{1+y^2}\, dy\, dx$.

A) $1 - ln\,2$ B) $\frac{1}{2}\,ln\,2$ C) $\frac{\pi}{4}$ D) $\frac{1}{2} + ln\,2$ E) $\frac{\pi}{2} - \frac{1}{2}\,ln\,2$

Question 14 and 15 relate to the following information. Smith begins a new job at a salary of 100,000 . Smith expects to receive a 5% raise every year until he retires.

14. Suppose that Smith works for 35 years. Determine the total salary earned over Smith's career (nearest million).

A) 5 B) 6 C) 7 D) 8 E) 9

15. At the end of each year, Smith's employer deposits 3% of Smith's salary (for the year just finished) into a fund earning 4% per year compounded each year. Find the value of the fund just after the final deposit at the end of Smith's 35th year of employment (nearest 10,000)

A) 450,000 B) 460,000 C) 470,000 D) 480,000 E) 490,000

PROBLEM SET 0 SOLUTIONS

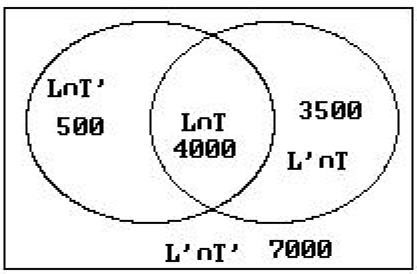
1. The "total set" is the set of all those who were surveyed and consists of 15,000 individuals. We define the following sets:

L - those who used the product last year,

T - those who used the product this year.

We are given $n(L) = 4,500$, $n(T) = 7,500$ and $n(L \cap T) = 4,000$.

This can be represented in Venn diagram form as follows (not to scale):



(i) The number who used the product either last year or this year, or both years is $n(L \cup T) = 500 + 4000 + 3500 = 8000$.

(ii) The number that did not use the product either last year or this year is $n[(L \cup T)'] = n(L' \cap T') = n(\text{total set}) - n(L \cup T) = 15,000 - 8,000 = 7,000$.

(iii) The number who used the product last year but not this year is $n(L \cap T') = 500$.

(iv) The number who used the product this year but not last year is $n(L' \cap T) = 3500$.

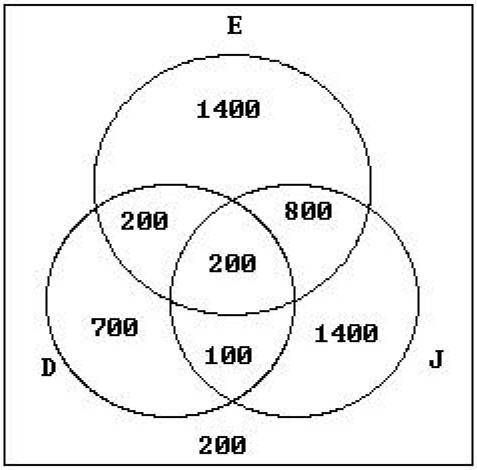
2. Following the same method applied in Example 1-4 of the notes of this study material, we get the Venn diagram entries below for the numbers in the various combinations.

$E =$ participated in extracurricular activities ,

$D =$ had a double major , and

$J =$ had a part-time job.

An example illustrating the calculation of one of the entries is the following. Since there are 1000 who both participated in extra-curricular activities and had a part-time job, and since 200 were in all three groups, the number who both participated in extra-curricular activities and had a part-time job but didn't have a double major is $1000 - 200 = 800$ (this would be $n(E \cap D' \cap J)$).

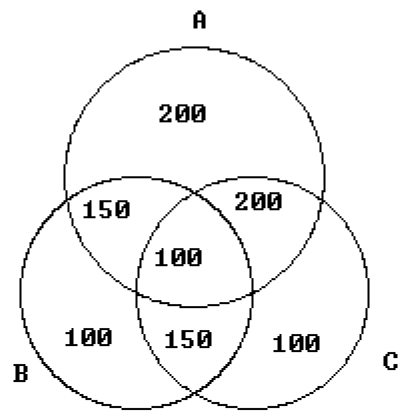


(i) The number who had a double major but did not participate in extra-curricular activities and did not have a part-time job is $n(D \cap E' \cap J') = 700$.

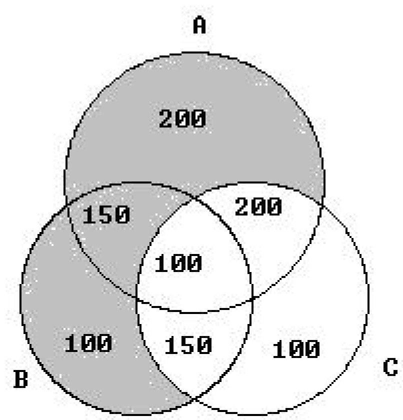
(ii) The number who had a double major and either participated in extra-curricular activities or had a part-time job , but not both is
$n(D \cap E \cap J') + n(D \cap E' \cap J) = 200 + 100 = 300$.

(iii) The number who neither participated in extracurricular activities nor had a part-time job is $n(E' \cap J') = n[(E \cup J)'] = 700 + 200 = 900$ (those in $(E \cup J)'$ with double major and those in $(E \cup J)'$ without double major).

3. This is quite similar to the previous problem, but the information is based on different combinations of sets. We are given
$n(A) = 650$, $n(B) = 500$, $n(C) = 550$, $n(A \cup B) = 900$, $n(A \cup C) = 900$,
$n(B \cup C) = 800$ and $n(A \cup B \cup C) = 1000$.
(i) We are asked for $n(A \cap B)$. We can use the relationship
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ to get $n(A \cap B) = 650 + 500 - 900 = 250$.
In order to solve (ii) and (iii) we can use the given information to fill in the numbers for the component subsets. We have $n(A \cap C) = n(A) + n(C) - n(A \cup C) = 300$,
$n(B \cap C) = 250$. We then use
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
so that $1000 = 650 + 500 + 550 - 250 - 300 - 250 + n(A \cap B \cap C)$, from which we get $n(A \cap B \cap C) = 100$. We can express the numbers in the component sets in the following Venn diagram.



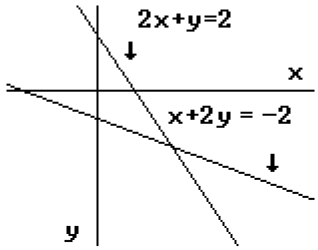
(ii) The number who had either symptom A or B (or both) but not C is $n[(A \cup B) \cap C'] = 200 + 150 + 100 = 450$. This represented in the following Venn diagram.



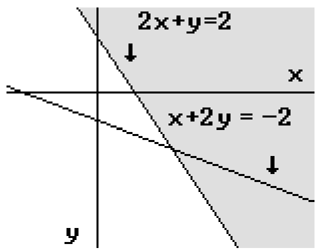
(iii) The number who had all three symptoms is $n(A \cap B \cap C)$, which we have already identified as 100.

4. $\lim_{N\to\infty} \frac{5^N}{N!} = \frac{5\cdot5\cdot5\cdot5\cdot5\cdot5\cdot5\cdots}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdots} \leq \frac{5^5}{5!} \cdot \left(\frac{5}{6}\right)^{\infty} = 0$. Answer: A

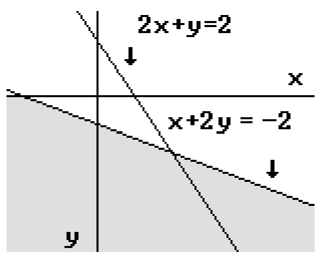
5. The graphs of the lines are illustrated in the following graph.



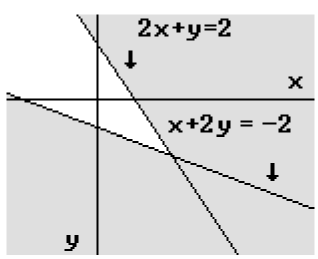
The inequality $2x+y>2$ is represented by the region "above" the line $2x+y=2$:



The inequality $x+2y<-2$ is represented by the region "below" the line $x+2y=-2$:



The union of the two regions is:



Answer: B

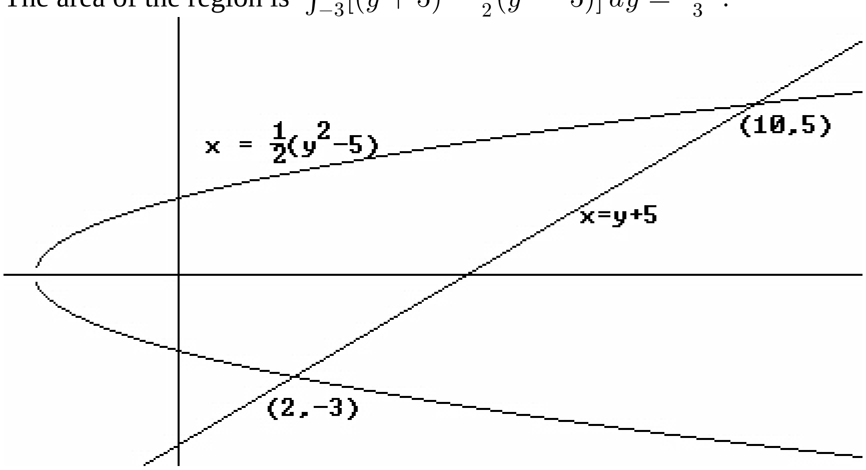
6. The quadratic equation ax^2+bx+c has no real roots if $b^2-4ac<0$.
Thus, $9+8k<0 \rightarrow k<-\frac{9}{8}$. Answer: A

7. $1/(1/x) = x$. Answer: D

8. 1st derivative - $e^x(x^4 + 4x^3)$; 2nd derivative - $e^x(x^4 + 8x^3 + 12x^2)$
3rd derivative - $e^x(x^4 + 12x^3 + 36x^2 + 24x)$
4th derivative - $e^x(x^4 + 16x^3 + 72x^2 + 96x + 24)$
It might be possible to determine a general expression for the nth derivative in terms of n then substitute $x = 0$. However, note that the 1st, 2nd and 3rd derivatives $(n = 1, 2, 3)$ must be 0 if $x = 0$, but the 4th derivative is not 0 at $x = 0$. This eliminates answers A, B, C and E.

9. We define $F(t)$ to be the population at time t. Then $F(0) = 6$, $\lim_{t\to\infty} F(t) = 30$,
and $F'(t) = \frac{Ae^t}{(.02A+e^t)^2}$. Then (using the substitution $u = .02A + e^t$), we have
$F(s) - F(0) = \int_0^s F'(t)\,dt = \int_0^s \frac{Ae^t}{(.02A+e^t)^2}\,dt = -\frac{A}{.02A+e^t}\Big|_{t=0}^{t=s} = \frac{A}{.02A+1} - \frac{A}{.02A+e^s}$,
so that $F(s) = 6 + \frac{A}{.02A+1} - \frac{A}{.02A+e^s}$.
Then $\lim_{s\to\infty} F(s) = 6 + \frac{A}{.02A+1} = 30 \rightarrow \frac{A}{.02A+1} = 24 \rightarrow A = 46.15$.
Therefore, $F(s) = 30 - \frac{46.15}{.923+e^s}$. In order to have $F(t) = 10$, we have
$30 - \frac{46.15}{.923+e^s} = 10 \rightarrow s = .325$. Answer: A

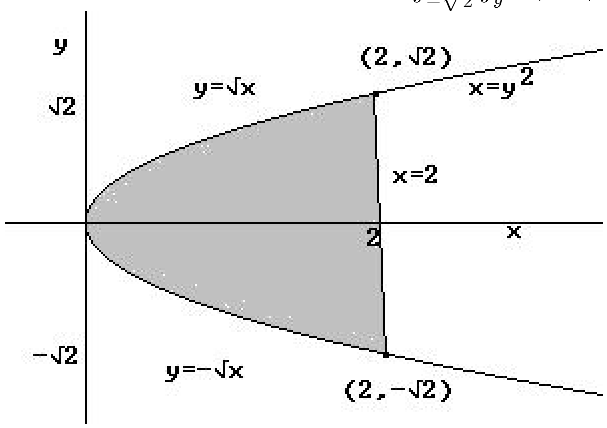
10. The line and the parabola intersect at y-values that are the solutions of
$y + 5 = \frac{1}{2}(y^2 - 5)$, so that $y = -3\ (x = 2)\ ,\ 5\ (x = 10)$.
The graph below indicates the closed region bounded by the line and the parabola.
The area of the region is $\int_{-3}^{5}[(y+5) - \frac{1}{2}(y^2 - 5)]\,dy = \frac{128}{3}$.



Answer: E

11. $F(x) = f(g(x))$, where $g(x) = x^{1/3}$ and $f(z) = \int_0^z \sqrt{1+t^4}\,dt$. Applying the Chain Rule results in $F'(x) = f'(g(x)) \cdot g'(x) = f'(x^{1/3}) \cdot \frac{1}{3}x^{-2/3} = \sqrt{1+(x^{1/3})^4} \cdot \frac{1}{3}x^{-2/3}$. At $x = 0$, this becomes $\frac{1}{0^+}$. Answer: E

12. The region of integration is illustrated in the graph below. For each x between 0 and 2, we have $-\sqrt{x} \le y \le \sqrt{x}$, or equivalently, for $-\sqrt{2} \le y \le \sqrt{2}$, we have $y^2 \le x \le 2$. The integral becomes $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{2} f(x,y)\,dx\,dy$.



Answer: D

13. $\int_0^1 \int_x^1 \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \int_0^y \frac{1}{1+y^2}\,dx\,dy = \int_0^1 \frac{y}{1+y^2}\,dy = \frac{1}{2}\,ln(1+y^2)\Big|_0^1 = \frac{1}{2}\,ln\,2$

Note that if we try to solve the integral directly as written, we get

$\int_0^1 \int_x^1 \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \Big[arctan(y)\Big|_x^1\Big]dx = \int_0^1 [\frac{\pi}{4} - arctan(x)]\,dx$

which is a more difficult integral to determine. Answer: B

14. Total salary $= 100,000[1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{34}] = 100,000 \cdot \frac{1.05^{35}-1}{1.05-1}$ $= 9,032,031$. Note that salary in 35th year has grown for 34 years since the first year of employment. Answer: E

15. Value at end of 35 years of deposits is

$100,000(.03)(1.04)^{34} + 100,000(.03)(1.05)(1.04)^{33} + 100,000(.03)(1.05)^2(1.04)^{32} +$
$\cdots + 100,000(.03)(1.05)^{33}(1.04) + 100,000(.03)(1.05)^{34}$
$= 100,000(.03)[(1.04)^{34} + (1.05)(1.04)^{33} + (1.05)^2(1.04)^{32}$
$+ \cdots + (1.05)^{33}(1.04) + (1.05)^{34}]$

(this represents accumulation of 1st yr. deposit, 2nd year deposit, 3rd year deposit, . . . , 34th year deposit, and 35th year deposit) . If we factor out $(1.04)^{34}$ the sum becomes

$100,000(.03)(1.04)^{34}[1 + \frac{1.05}{1.04} + (\frac{1.05}{1.04})^2 + \cdots + (\frac{1.05}{1.04})^{33} + (\frac{1.05}{1.04})^{34}]$
$= 100,000(.03)(1.04)^{34}[\frac{(\frac{1.05}{1.04})^{35}-1}{\frac{1.05}{1.04}-1}] = 470,978$. Answer: C

SECTION 1 - BASIC PROBABILITY CONCEPTS

PROBABILITY SPACES AND EVENTS

Sample point and sample space: A sample point is the simple outcome of a random experiment. The probability space (also called sample space) is the collection of all possible sample points related to a specified experiment. When the experiment is performed, one of the sample points will be the outcome. The probability space is the "full set" of possible outcomes of the experiment.

Mutually exclusive outcomes: Outcomes are mutually exclusive if they cannot occur simultaneously. They are also referred to as **disjoint** outcomes.

Exhaustive outcomes: Outcomes are exhaustive if they combine to be the entire probability space, or equivalently, if at least one of the outcomes must occur whenever the experiment is performed.

Event: Any collection of sample points, or any subset of the probability space is referred to as an event. We say that "event A has occurred" if the experimental outcome was one of the sample points in A.

Union of events $\boldsymbol{A}$ and $\boldsymbol{B}$: $A \cup B$ denotes the union of events A and B, and consists of all sample points that are in either A or B.

Union of events $\boldsymbol{A_1, A_2, ..., A_n}$: $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ denotes the union of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are in at least one of the A_i's. This definition can be extended to the union of infinitely many events.

Intersection of events $\boldsymbol{A_1, A_2, ..., A_n}$: $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$ denotes the intersection of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are simultaneously in all of the A_i's. ($A \cap B$ is also denoted $A \cdot B$ or AB).

Mutually exclusive events $A_1, A_2, \ldots, A_n$: Two events are mutually exclusive if they have no sample points in common, or equivalently, if they have **empty intersection**. Events $A_1, A_2, \ldots, A_n$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$, where $\emptyset$ denotes the empty set with no sample points. Mutually exclusive events cannot occur simultaneously.

Exhaustive events $B_1, B_2, \ldots, B_n$: If $B_1 \cup B_2 \cup \cdots \cup B_n = S$, the entire probability space, then the events $B_1, B_2, \ldots, B_n$ are referred to as exhaustive events.

Complement of event A: The complement of event A consists of all sample points in the probability space that are **not in A**. The complement is denoted $\overline{A},\ \sim A,\ A'$ or A^c and is equal to $\{x : x \notin A\}$. When the underlying random experiment is performed, to say that the complement of A has occurred is the same as saying that A has not occurred.

Subevent (or subset) A of event B: If event B contains all the sample points in event A, then A is a subevent of B, denoted $A \subset B$. The occurrence of event A implies that event B has occurred.

Partition of event A: Events $C_1, C_2, \ldots, C_n$ form a partition of event A if $A = \bigcup_{i=1}^{n} C_i$ and the C_i's are mutually exclusive.

DeMorgan's Laws:

(i) $(A \cup B)' = A' \cap B'$, to say that $A \cup B$ has not occurred is to say that A has not occurred <u>and</u> B has not occurred ; this rule generalizes to any number of events;

$$\left(\bigcup_{i=1}^{n} A_i\right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'$$

(ii) $(A \cap B)' = A' \cup B'$, to say that $A \cap B$ has not occurred is to say that either A has not occurred <u>or</u> B has not occurred (or both have not occurred) ; this rule generalizes to any number of events, $\left(\bigcap_{i=1}^{n} A_i\right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'$

Indicator function for event A: The function $I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is the indicator function for event A, where x denotes a sample point. $I_A(x)$ is 1 if event A has occurred.

Basic set theory was reviewed in Section 0 of these notes. The Venn diagrams presented there apply in the sample space and event context presented here.

Example 1-1: Suppose that an "experiment" consists of tossing a six-faced die. The **probability space** of outcomes consists of the set $\{1,2,3,4,5,6\}$, each number being a **sample point** representing the number of spots that can turn up when the die is tossed. The outcomes 1 and 2 (or more formally, $\{1\}$ and $\{2\}$) are an example of **mutually exclusive** outcomes, since they cannot occur simultaneously on one toss of the die. The collection of all the outcomes (sample points) 1 to 6 are **exhaustive** for the experiment of tossing a die since one of those outcomes must occur. The collection $\{2,4,6\}$ represents the **event** of tossing an even number when tossing a die. We define the following events.

$A = \{1,2,3\} =$ "a number less than 4 is tossed" ,

$B = \{2,4,6\} =$ "an even number is tossed" ,

$C = \{4\} =$ "a 4 is tossed" ,

$D = \{2\} =$ "a 2 is tossed" .

Then (i) $A \cup B = \{1,2,3,4,6\}$,

(ii) $A \cap B = \{2\}$,

(iii) A and C are mutually exclusive since $A \cap C = \emptyset$ (when the die is tossed it is not possible for both A and C to occur),

(iv) $D \subset B$,

(v) $A' = \{4,5,6\}$ (complement of A) ,

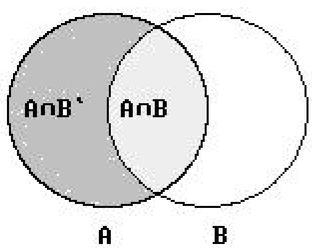
(vi) $B' = \{1,3,5\}$,

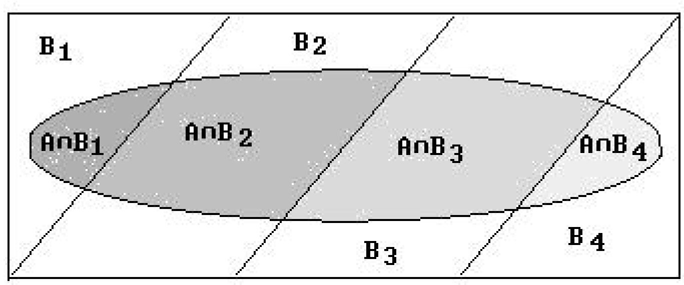
(vii) $A \cup B = \{1,2,3,4,6\}$, so that $(A \cup B)' = \{5\} = A' \cap B'$ (this illustrates one of DeMorgan's Laws). □

Some rules concerning operations on events:

(i) $A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$ and
$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$ for any events
$A, B_1, B_2, ..., B_n$

(ii) If $B_1, B_2, ..., B_n$ are exhaustive events ($\bigcup_{i=1}^{n} B_i = S$, the entire probability space),
then for any event A,
$A = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$.
If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive events, then they form a **partition of the probability space**. For example, the events $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$ and $B_3 = \{5, 6\}$ form a partition of the probability space for the outcomes of tossing a single die. The general idea of a partition is illustrated in the diagram below. As a special case of a partition, if B is any event, then B and B' form a partition of the probability space. We then get the following identity for any two events A and B:
$\boldsymbol{A = A \cap (B \cup B') = (A \cap B) \cup (A \cap B')}$; note also that $A \cap B$ and $A \cap B'$ form a partition of event A.





(iii) For any event A, $A \cup A' = S$, the entire probability space, and $A \cap A' = \emptyset$

(iv) $A \cap B' = \{x : x \in A \text{ and } x \notin B\}$ is sometimes denoted $A - B$, and consists of all sample points that are in event A but not in event B

(v) If $A \subset B$ then $A \cup B = B$ and $A \cap B = A$.

PROBABILITY

Probability function for a discrete probability space: A discrete probability space (or sample space) is a set of a finite or countable infinite number of sample points. $P[a_i]$ or p_i denotes the probability that sample point (or outcome) a_i occurs. There is some underlying "random experiment" whose outcome will be one of the a_i's in the probability space. Each time the experiment is performed, one of the a_i's will occur. The probability function P must satisfy the following two conditions:

(i) $0 \leq P[a_i] \leq 1$ for each a_i in the sample space, and

(ii) $P[a_1] + P[a_2] + \cdots = \sum_{\text{all } i} P[a_i] = 1$ (total probability for a probability space is always 1).

This definition applies to both finite and infinite probability spaces.

Tossing an ordinary die is an example of an experiment with a finite probability space $\{1, 2, 3, 4, 5, 6\}$. An example of an experiment with an infinite probability space is the tossing of a coin until the first head appears. The toss number of the first head could be any positive integer, 1, or 2, or 3, ... The probability space for this experiment is the infinite set of positive integers $\{1, 2, 3, ...\}$ since the first head could occur on any toss starting with the first toss. The notion of discrete random variable covered later is closely related to the notion of discrete probability space and probability function.

Uniform probability function: If a probability space has a finite number of sample points, say k points, $a_1, a_2, ..., a_k$, then the probability function is said to be **uniform** if each sample point has the same probability of occurring ; $P[a_i] = \frac{1}{k}$ for each $i = 1, 2, ..., k$. Tossing a fair die would be an example of this, with $k = 6$.

Probability of event A: An event A consists of a subset of sample points in the probability space. In the case of a discrete probability space, the probability of A is
$P[A] = \sum_{a_i \in A} P[a_i]$, the sum of $P[a_i]$ over all sample points in event A.

Example 1-2: In tossing a "fair" die, it is assumed that each of the six faces has the same chance of $\frac{1}{6}$ of turning up. If this is true, then the probability function $P(j) = \frac{1}{6}$ for $j = 1, 2, 3, 4, 5, 6$ is a uniform probability function on the sample space $\{1, 2, 3, 4, 5, 6\}$.
The event "an even number is tossed" is $A = \{2, 4, 6\}$, and has probability
$P[A] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$. □

Continuous probability space: An experiment can result in an outcome which can be any real number in some interval. For example, imagine a simple game in which a pointer is spun randomly and ends up pointing at some spot on a circle. The angle from the vertical (measured clockwise) is between 0 and 2π. The probability space is the interval $(0, 2\pi]$, the set of possible angles that can occur. We regard this as a continuous probability space. In the case of a continuous probability space (an interval), we describe probability by assigning probability to subintervals rather than individual points. If the spin is "fair", so that all points on the circle are equally likely to occur, then intuition suggests that the probability assigned to an interval would be the fraction that the interval is of the full circle. For instance, the probability that the pointer ends up between "3 O'clock" and "9 O'clock" (between $\pi/2$ or 90° and $3\pi/2$ or 270° from the vertical) would be .5, since that subinterval is one-half of the full circle. The notion of a continuous random variable, covered later in this study guide, is related to a continuous probability space.

Some rules concerning probability:

(i) $\boldsymbol{P[S] = 1}$ **if** $\boldsymbol{S}$ **is the entire probability space** (when the underlying experiment is performed, some outcome must occur with probability 1; for instance $S = \{1, 2, 3, 4, 5, 6\}$ for the die toss example).

(ii) $\boldsymbol{P[\emptyset] = 0}$ (the probability of no face turning up when we toss a die is 0).

(iii) If events $\boldsymbol{A_1, A_2, \ldots, A_n}$ are mutually exclusive (also called disjoint) then

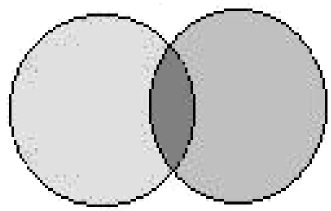
$$\boldsymbol{P[\bigcup_{i=1}^{n} A_i] = P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = \sum_{i=1}^{n} P[A_i]}\ .$$

This extends to infinitely many mutually exclusive events. This rule is similar to the rule discussed in Section 0 of this study guide, where it was noted that the number of elements in the union of mutually disjoint sets is the sum of the numbers of elements in each set. When we have mutually exclusive events and we add the event probabilities, there is no double counting.

(iv) For any event A, $\boldsymbol{0 \le P[A] \le 1}$.

(v) If $A \subset B$ then $P[A] \le P[B]$.

(vi) For any events A, B and C, $\boldsymbol{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$.
This relationship can be explained as follows. We can formulate $A \cup B$ as the union of three mutually exclusive events as follows: $A \cup B = (A \cap B') \cup (A \cap B) \cup (B \cap A')$.
This is expressed in the following Venn diagram.



$A \cap B'$ $A \cap B$ $B \cap A'$

Since these are mutually exclusive events, it follows that
$P[A \cup B] = P[A \cap B'] + P[A \cap B] + P[B \cap A']$.
From the Venn diagram we see that $A = (A \cap B') \cup (A \cap B)$, so that
$\boldsymbol{P[A] = P[A \cap B'] + P[A \cap B]}$, and we also see that $P[B \cap A'] = P[B] - P[A \cap B]$.
It then follows that
$P[A \cup B] = (P[A \cap B'] + P[A \cap B]) + P[B \cap A'] = P[A] + P[B] - P[A \cap B]$.

We subtract $P[A \cap B]$ because $P[A] + P[B]$ counts $P[A \cap B]$ twice. $P[A \cup B]$ is the probability that at least one of the two events A, B occurs. This was reviewed in Section 0, where a similar rule was described for counting the number of elements in $A \cup B$.

For the union of three sets we have
$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$$

(vii) For any event A, $\boldsymbol{P[A'] = 1 - P[A]}$.

(viii) For any events A and B, $\boldsymbol{P[A] = P[A \cap B] + P[A \cap B']}$
(this was mentioned in (vi), it is illustrated in the Venn diagram above).

(ix) For exhaustive events $B_1, B_2, ..., B_n$, $P[\bigcup_{i=1}^{n} B_i] = 1$.
If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive, they form a **partition** of the entire probability space, and for any event A,
$$\boldsymbol{P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n] = \sum_{i=1}^{n} P[A \cap B_i]} \,.$$

(x) If P is a uniform probability function on a probability space with k points, and if event A consists of m of those points, then $P[A] = \frac{m}{k}$.

(xi) The words "percentage" and "proportion" are used as alternatives to "probability". As an example, if we are told that the percentage or proportion of a group of people that are of a certain type is 20%, this is generally interpreted to mean that a randomly chosen person from the group has a 20% probability of being of that type. This is the "long-run frequency" interpretation of probability. As another example, suppose that we are tossing a fair die. In the long-run frequency interpretation of probability, to say that the probability of tossing a 1 is $\frac{1}{6}$ is the same as saying that if we repeatedly toss the die, the proportion of tosses that are 1's will approach $\frac{1}{6}$.

(xii) for any events $A_1, A_2, ..., A_n$, $P[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} P[A_i]$, with equality holding if and only if the events are mutually exclusive

Example 1-3: Suppose that $P[A \cap B] = .2$, $P[A] = .6$, and $P[B] = .5$.
Find $P[A' \cup B']$, $P[A' \cap B']$, $P[A' \cap B]$ and $P[A' \cup B]$.
Solution: Using probability rules we get the following.

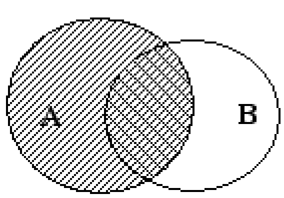
$$P[A' \cup B'] = P[(A \cap B)'] = 1 - P[A \cap B] = .8 \text{ .}$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = .6 + .5 - .2 = .9$$
$$\rightarrow P[A' \cap B'] = P[(A \cup B)'] = 1 - P[A \cup B] = 1 - .9 = .1 \text{ .}$$

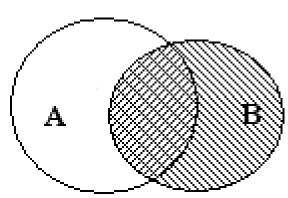
$$P[B] = P[B \cap A] + P[B \cap A'] \rightarrow P[A' \cap B] = .5 - .2 = .3 \text{ .}$$

$$P[A' \cup B] = P[A'] + P[B] - P[A' \cap B] = .4 + .5 - .3 = .6 \text{ .}$$

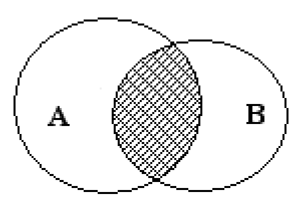
The following Venn diagrams illustrate the various combinations of intersections, unions and complements of the events A and B.



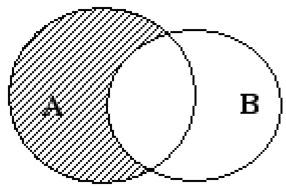
$P[A] = .6$



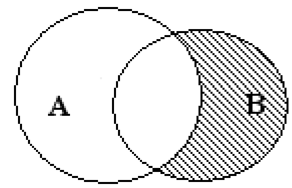
$P[B] = .5$



$P[A \cap B] = .2$

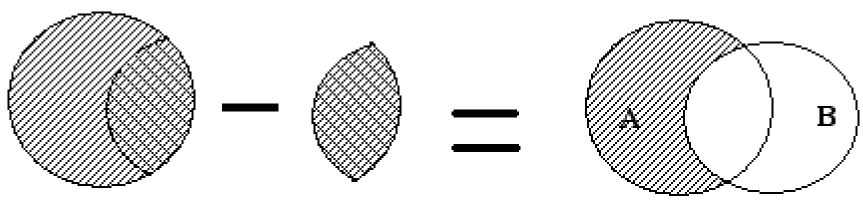


$P[A \cap B'] = .4$

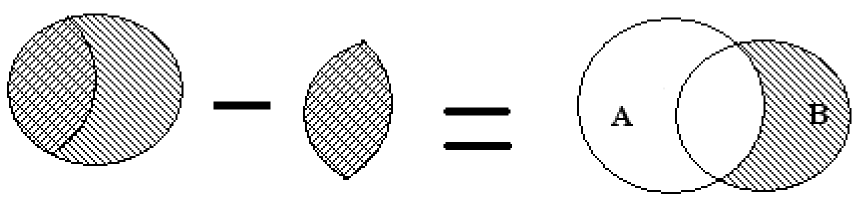


$P[A' \cap B] = .3$

From the following Venn diagrams we see that $P[A \cap B'] = P[A] - P[A \cap B] = .6 - .2 = .4$ and $P[A' \cap B] = P[B] - P[A \cap B] = .5 - .2 = .3$.

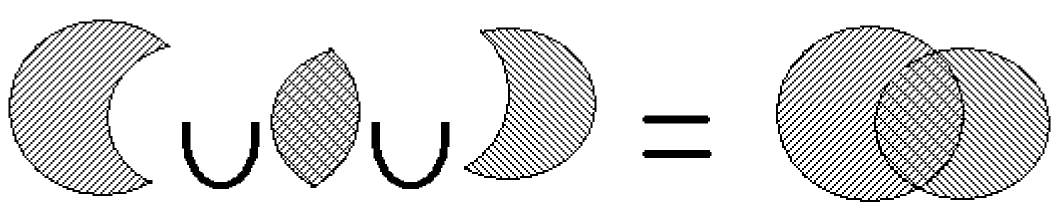


$$P[A] \quad - \quad P[A \cap B] \quad = \quad P[A \cap B']$$
$$.6 \quad - \quad .2 \quad = \quad .4$$



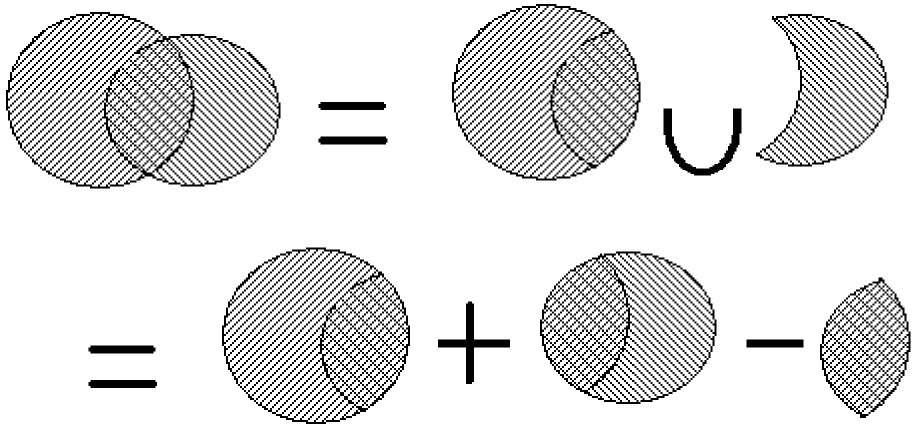
$$P[B] \quad - \quad P[A \cap B] \quad = \quad P[A' \cap B]$$
$$.5 \quad - \quad .2 \quad = \quad .3$$

The following Venn diagrams shows how to find $P[A \cup B]$.



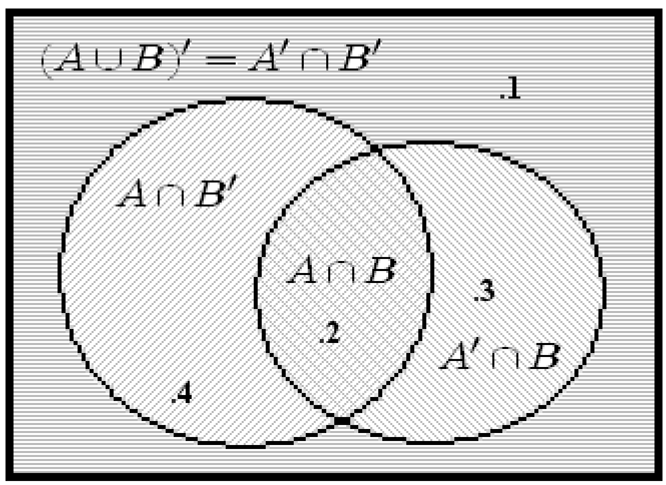
$$P[A \cap B'] \quad + \quad P[A \cap B] \quad + \quad P[A' \cap B] \quad = \quad P[A \cup B]$$
$$.4 \quad + \quad .2 \quad + \quad .3 \quad = \quad .9\,.$$

The relationship $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ is explained in the following Venn

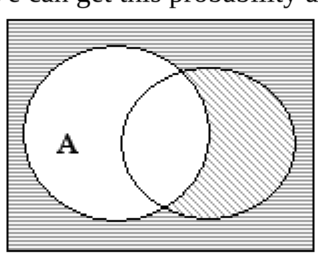


$$P[A \cup B] \quad = \quad P[A] \quad + \quad P[B] \quad - \quad P[A \cap B]$$
$$= \quad .6 + .5 - .2 = .9 \ .$$

The components of the events and their probabilities are summarized in the following diagram.



We can represent a variety of events in Venn diagram form and find their probabilities from the component events described in the previous diagram. For instance, the complement of A is the combined shaded region in the following Venn diagram, and the probability is $P[A'] = .3 + .1 = .4$. We can get this probability also from $P[A'] = 1 - P[A] = 1 - .6 = .4$.



Another efficient way of summarizing the probability information for events A and B is in the form of a table.

	$P[A] = .6$ (given)	$P[A']$
$P[B] = .5$ (given)	$P[A \cap B] = .2$ (given)	$P[A' \cap B]$
$P[B']$	$P[A \cap B']$	$P[A' \cap B']$

Complementary event probabilities can be found from $P[A'] = 1 - P[A] = .4$ and $P[B'] = 1 - P[B] = .5$. Also, across each row or along each column, the "intersection probabilities" add up to the single event probability at the end or top:
$P[B] = P[A \cap B] + P[A' \cap B] \rightarrow .5 = .2 + P[A' \cap B] \rightarrow P[A' \cap B] = .3$,
$P[A] = P[A \cap B] + P[A \cap B'] \rightarrow .6 = .2 + P[A \cap B'] \rightarrow P[A \cap B'] = .4$, and
$P[A'] = P[A' \cap B] + P[A' \cap B'] \rightarrow .4 = .3 + P[A' \cap B'] \rightarrow P[A' \cap B'] = .1$ or
$P[B'] = P[A \cap B'] + P[A' \cap B'] \rightarrow .5 = .4 + P[A' \cap B'] \rightarrow P[A' \cap B'] = .1$.
These calculations can be summarized in the next table.

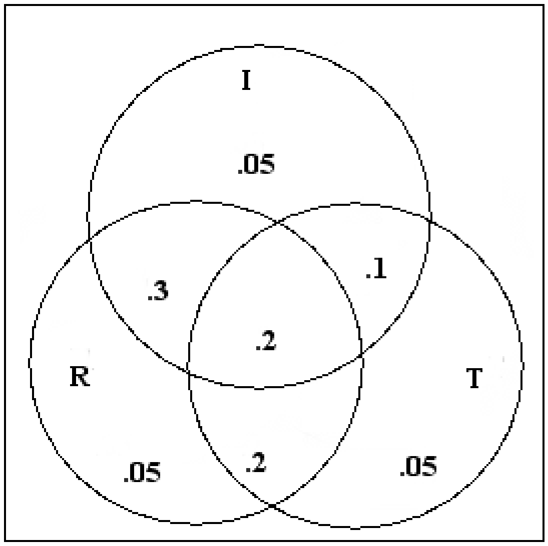
		$P[A] = .6$ (given)	$\Rightarrow$	$P[A'] = 1 - P[A] = .4$
		$\Uparrow$		$\Uparrow$
$P[B] = .5$	$\Leftarrow$	$P[A \cap B] = .2$	$+$	$P[A' \cap B] = \mathbf{.3}$
		$+$		$+$
$P[B'] = \mathbf{.5}$	$\Leftarrow$	$P[A \cap B'] = \mathbf{.4}$	$+$	$P[A' \cap B'] = \mathbf{.1}$

$\square$

Example 1-4: A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have electric irons (I), 55% have electric toasters (T), 50% have (IR), 40% have (RT), 30% have (IT), and 20% have all three. Find the probability that a household has at least one of these appliances.

Solution: $P[R \cup I \cup T] = P[R] + P[I] + P[T]$
$- P[R \cap I] - P[R \cap T] - P[I \cap T] + P[R \cap I \cap T]$
$= .75 + .65 + .55 - .5 - .4 - .3 + .2 = .95$.

The following diagram deconstructs the three events.

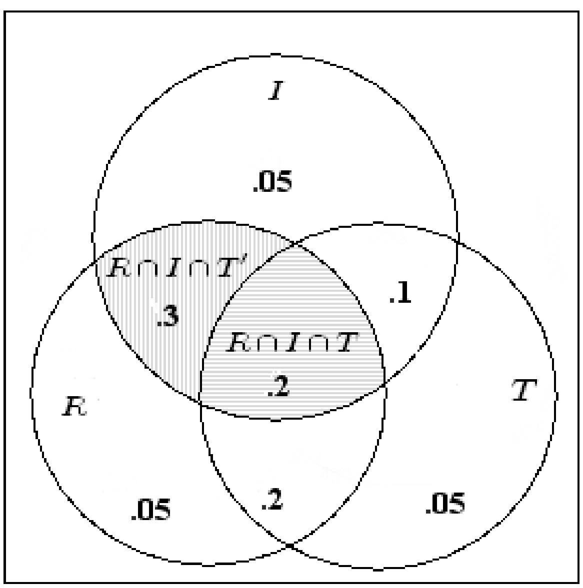


The entries in this diagram were calculated from the "inside out". For instance, since $P(R \cap I) = .5$ (given) , and since $P(R \cap I \cap T) = .2$ (also given) , it follows that $P(R \cap I \cap T') = .3$, since

$.5 = P(R \cap I) = P(R \cap I \cap T) + P(R \cap I \cap T') = .2 + P(R \cap I \cap T')$

(this uses the rule $P(A) = P(A \cap B) + P(A \cap B')$, where $A = R \cap I$ and $B = T$).

This is illustrated in the following diagram.



The value ".05" that is inside the diagram for event R refers to $P(R \cap I' \cap T')$ (the proportion who have radios but neither irons nor toasters). This can be found in the following way.

First we find $P(R \cap I')$:

$.75 = P(R) = P(R \cap I) + P(R \cap I') = .5 + P(R \cap I') \rightarrow P(R \cap I') = .25.$

$P(R \cap I')$ is the proportion with radios but not irons; this is the ".05" inside R combined with the ".2" in the lower triangle inside $R \cap T$. Then we find $P(R \cap I' \cap T)$:

$.4 = P(R \cap T) = P(R \cap I \cap T) + P(R \cap I' \cap T)$

$= .2 + P(R \cap I' \cap T) \rightarrow P(R \cap I' \cap T) = .2$.

Finally we find $P(R \cap I' \cap T')$:

$.25 = P(R \cap I') = P(R \cap I' \cap T) + P(R \cap I' \cap T')$

$= .2 + P(R \cap I' \cap T') \rightarrow P(R \cap I' \cap T') = .05$.

The other probabilities in the diagram can be found in a similar way. Notice that $P(R \cup I \cup T)$ is the sum of the probabilities of all the disjoint pieces inside the three events,

$P(R \cup I \cup T) = .05 + .05 + .05 + .1 + .2 + .3 + .2 = .95$.

We can also use the rule

$P(R \cup I \cup T) = P(R) + P(I) + P(T) - P(R \cap I) - P(R \cap T) - P(I \cap T) + P(R \cap I \cap T)$

$= .75 + .65 + .55 - .5 - .4. - 3 + .2 = .95$.

Either way, this implies that 5% of the households have none of the three appliances. □

It is possible that information is given in terms of numbers of units in each category rather than proportion of probability of each category that was given in Example 1-4.

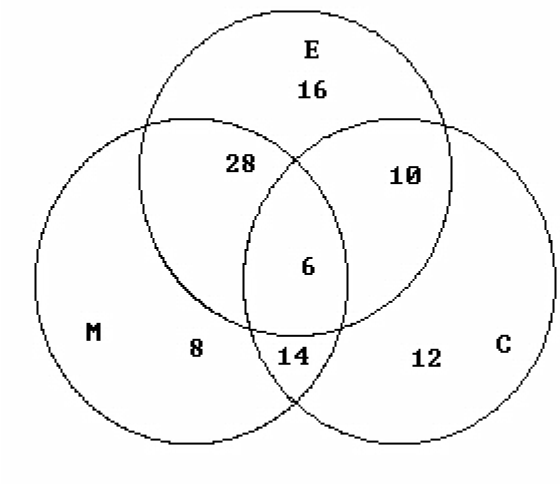
Example 1-5: In a survey of 120 students, the following data was obtained.
60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects.
Find the number of students who took
(i) none of the subjects,
(ii) Math, but not English or Chemistry,
(iii) English and Math but not Chemistry.

Solution:

Since $E \cap M$ has 34 and $E \cap M \cap C$ has 6 , it follows that $E \cap M \cap C'$ has 28 .
The other entries are calculated in the same way (very much like the previous example).
(i) The total number of students taking any of the three subjects is $E \cup M \cup C$, and is $16 + 28 + 6 + 10 + 8 + 14 + 12 = 94$. The remaining 26 (out of 120) students are not taking any of the three subjects (this could be described as the set $E' \cap M' \cap C'$).
(ii) $M \cap E' \cap C'$ has 8 students .
(iii) $E \cap M \cap C'$ has 28 students .

Example 1-5 continued

The following diagram illustrates how the numbers of students can be deconstructed.



□

PROBLEM SET 1

Basic Probability Concepts

1. A survey of 1000 people determines that 80% like walking and 60% like biking, and all like at least one of the two activities. What is the probability that a randomly chosen person in this survey likes biking but not walking?

A) 0 B) .1 C) .2 D) .3 E) .4

2. A survey of 1000 Canadian sports fans who indicated they were either hockey fans or lacrosse fans or both, had the following result.

- 800 indicated that they were hockey fans
- 600 indicated that they were lacrosse fans

Based on the sample, find the probability that a Canadian sports fan is not a hockey fan given that she/he is a lacrosse fan.

A) $\frac{1}{5}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) 1

3. (SOA) Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

A) 0.26 B) 0.38 C) 0.40 D) 0.48 E) 0.62

4. (SOA) An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44 Calculate the number of blue balls in the second urn.

A) 4 B) 20 C) 24 D) 44 E) 64

5. (SOA) An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{5}{12}$, respectively. Determine the probability that a randomly chosen employee will choose no supplementary coverage.

A) 0 B) $\frac{47}{144}$ C) $\frac{1}{2}$ D) $\frac{97}{144}$ E) $\frac{7}{9}$

6. (SOA) An auto insurance company has 10,000 policyholders.
Each policyholder is classified as
(i) young or old;
(ii) male or female; and
(iii) married or single.
Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

A) 280 B) 423 C) 486 D) 880 E) 896

7. (SOA) An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is $\frac{1}{3}$. What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A?

A) 0.280 B) 0.311 C) 0.467 D) 0.484 E) 0.700

8. (SOA) The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35% . Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

A) 0.05 B) 0.12 C) 0.18 D) 0.25 E) 0.35

9. (SOA) You are given $P[A \cup B] = 0.7$ and $P[A \cup B'] = 0.9$. Determine $P[A]$.

A) 0.2 B) 0.3 C) 0.4 D) 0.6 E) 0.8

10. (SOA) A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gymnastics

(ii) 29% watched baseball

(iii) 19% watched soccer

(iv) 14% watched gymnastics and baseball

(v) 12% watched baseball and soccer

(vi) 10% watched gymnastics and soccer

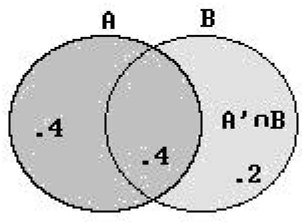
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

A) 24 B) 36 C) 41 D) 52 E) 60

PROBLEM SET 1 SOLUTIONS

1. Let $A =$ "like walking" and $B =$ "like biking" . We use the interpretation that "percentage" and "proportion" are taken to mean "probability".
We are given $P(A) = .8$, $P(B) = .6$ and $P(A \cup B) = 1$.
From the diagram below we can see that since $A \cup B = A \cup (B \cap A')$ we have
$P(A \cup B) = P(A) + P(A' \cap B) \rightarrow P(A' \cap B) = .2$ is the proportion of people who like biking but (and) not walking . In a similar way we get $P(A \cap B') = .4$.



An algebraic approach is the following. Using the rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get $1 = .8 + .6 - P(A \cap B) \rightarrow P(A \cap B) = .4$. Then, using the rule $P(B) = P(B \cap A) + P(B \cap A')$, we get $P(B \cap A') = .6 - .4 = .2$. Answer: C

2. From the given information, 400 of those surveyed are both hockey and lacrosse fans, 200 are lacrosse fans and not hockey fans, and 400 are hockey fans an not lacrosse fans.
This is true because there are 1000 fans in the survey, but a combined total of 800 + 600 = 1400 sports preferences, so that 400 must be fans of both. Of the 600 lacrosse fans, 400 are also hockey fans, so 200 are not hockey fans The probability that a Canadian sports fans is not a hockey fan given that she/he is a lacrosse fan is $\frac{200}{600} = \frac{1}{3}$. Answer: C

3. C - chiropractor visit ; T - therapist visit.
We are given $P(C \cap T) = .22$, $P(C' \cap T') = .12$, $P(C) = P(T) + .14$.
$.88 = 1 - P(C' \cap T') = P(C \cup T) = P(C) + P(T) - P(C \cap T)$
$= P(T) + .14 + P(T) - .22 \rightarrow P(T) = .48$. Answer: D

4. Suppose there are B blue balls in urn II.

$P[\text{both balls are same color}] = P[\text{both blue} \cup \text{both red}] = P[\text{both blue}] + P[\text{both red}]$

(the last equality is true since the events "both blue" and "both red" are disjoint).

$P[\text{both blue}] = P[\text{blue from urn I} \cap \text{blue from urn II}]$

$= P[\text{blue from urn I}] \cdot P[\text{blue from urn II}]$ (choices from the two urns are independent)

$= (\frac{6}{10})(\frac{B}{16+B})$,

$P[\text{both red}] = P[\text{red from urn I} \cap \text{red from urn II}]$

$= P[\text{red from urn I}] \cdot P[\text{red from urn II}] = (\frac{4}{10})(\frac{16}{16+B})$,

We are given $(\frac{6}{10})(\frac{B}{16+B}) + (\frac{4}{10})(\frac{16}{16+B}) = .44 \rightarrow \frac{6B+64}{10(16+B)} = .44 \rightarrow B = 4$.

Answer: A

5. Since someone who chooses coverage must choose exactly two supplementary coverages, in order for someone to choose coverage A, they must choose either A-and-B or A-and-C. Thus, the proportion of $\frac{1}{4}$ of individuals that choose A is

$P[A \cap B] + P[A \cap C] = \frac{1}{4}$ (where this refers to the probability that someone chosen at random in the company chooses coverage A). In a similar way we get

$P[B \cap A] + P[B \cap C] = \frac{1}{3}$ and $P[C \cap A] + P[C \cap B] = \frac{5}{12}$.

Then, $(P[A \cap B] + P[A \cap C]) + (P[B \cap A] + P[B \cap C]) + (P[C \cap A] + P[C \cap B])$

$= 2(P[A \cap B] + P[A \cap C] + P[B \cap C]) = \frac{1}{4} + \frac{1}{3} + \frac{5}{12} = 1$.

It follows that $P[A \cap B] + P[A \cap C] + P[B \cap C] = \frac{1}{2}$.

This is the probability that a randomly chosen individual chooses some form of coverage, since if someone who chooses coverage chooses exactly two of A,B, and C. Therefore, the probability that a randomly chosen individual does not choose any coverage is the probability of the complementary event, which is also $\frac{1}{2}$. Answer: C

6. We identify the following subsets of the set of 10,000 policyholders:

$Y =$ young, with size 3000 (so that $Y' =$ old has size 7000) ,

$M =$ male, with size 4600 (so that $M' =$ female has size 5400), and

$C =$ married, with size 7000 (so that $C' =$ single has size 3000).

We are also given that $Y \cap M$ has size 1320 , $M \cap C$ has size 3010 ,

$Y \cap C$ has size 1400 , and $Y \cap M \cap C$ has size 600 .

We wish to find the size of the subset $Y \cap M' \cap C'$.

We use the following rules of set theory:

6. continued

(i) if two finite sets are disjoint (have no elements in common, also referred to as empty intersection), then the total number of elements in the union of the two sets is the sum of the numbers of elements in each of the sets;

(ii) for any sets A and B, $A = (A \cap B) \cup (A \cap B')$, and $A \cap B$ and $A \cap \overline{B}$ are disjoint.

Applying rule (ii), we have $Y = (Y \cap M) \cup (Y \cap M')$. Applying rule (i), it follows that the size of $Y \cap M'$ must be $3000 - 1320 = 1680$.

We now apply rule (ii) to $Y \cap C$ to get $Y \cap C = (Y \cap C \cap M) \cup (Y \cap C \cap M')$.

Applying rule (i), it follows that $Y \cap C \cap M'$ has size $1400 - 600 = 800$.

Now applying rule (ii) to $Y \cap M'$ we get $Y \cap M' = (Y \cap M' \cap C) \cup (Y \cap M' \cap C')$.

Applying rule (i), it follows that $Y \cap M' \cap C'$ has size $1680 - 800 = 880$.

Within the "Young" category, which we are told is 3000, we can summarize the calculations in the following table. This is a more straightforward solution.

	Married 1400 (given)	Single $1600 = 3000 - 1400$
Male 1320 (given)	600 (given)	$720 = 1320 - 600$
Female $1680 =$ $3000 - 1320$	$800 = 1400 - 600$	$880 = 1600 - 720$

Answer: D

7. We are given

$P[A \cap B' \cap C'] = P[A' \cap B \cap C'] = P[A' \cap B' \cap C] = .1$

(having exactly one risk factor means not having either of the other two).

We are also given

$P[A \cap B \cap C'] = P[A \cap B' \cap C] = P[A' \cap B \cap C] = .12.$

And we are given

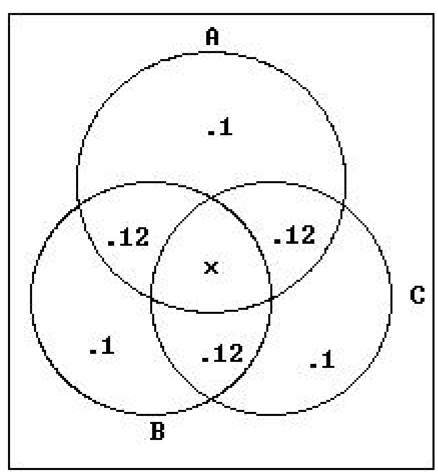
$P[A \cap B \cap C | A \cap B] = \frac{1}{3}$.

We are asked to find $P[A' \cap B' \cap C' | A']$.

From $P[A \cap B \cap C | A \cap B] = \frac{1}{3}$ we get $\frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{1}{3}$, and then

$P[A \cap B \cap C] = \frac{1}{3} \cdot P[A \cap B]$.

The following Venn diagram illustrates the situation:



7. continued

We see that $P[A\cap B\cap C]=x$ and $P[A\cap B]=x+.12$, so that

$x=\frac{1}{3}\cdot(x+.12)\to x=\ P[A\cap B\cap C]=.06$.

Alternatively, we can use the rule $P[D]=P[D\cap E]+P[D\cap E']$ to get

$P[A\cap B]=P[A\cap B\cap C]+P[A\cap B\cap C']=P[A\cap B\cap C]+.12$.

Then, $P[A\cap B]=P[A\cap B\cap C]+.12=\frac{1}{3}\cdot P[A\cap B]+.12\to P[A\cap B]=.18$

and $P[A\cap B\cap C]=\frac{1}{3}\cdot(.18)=.06$.

We can also see from the diagram that $P[A\cap B']=.1+.12=.22$.

Alternatively, we can use the rule above again to get

$P[A\cap B']=P[A\cap B'\cap C]+P[A\cap B'\cap C']=.12+.1=.22$.

Then, $P[A]=P[A\cap B]+P[A\cap B']=.18+.22=.4$, and $P[A']=1-P[A]=.6$.

We are asked to find $P[A'\cap B'\cap C'|A']=\frac{P[A'\cap B'\cap C']}{P[A']}=\frac{P[A'\cap B'\cap C']}{.6}$, so we must find

$P[A'\cap B'\cap C']$. From the Venn diagram, we see that

$P[A'\cap B'\cap C']=1-(.1+.1+.1+.12+.12+.12+.06)=.28$.

Finally, $P[A'\cap B'\cap C'|A']=\frac{P[A'\cap B'\cap C']}{P[A']}=\frac{P[A'\cap B'\cap C']}{.6}=\frac{.28}{.6}=.467$. Answer: C

8. We identify events as follows:

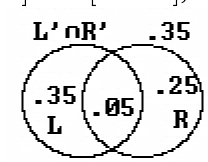
L : lab work needed

R : referral to a specialist needed

We are given $P[L'\cap R']=.35$, $P[R]=.3$, $P[L]=.4$. It follows that

$P[L\cup R]=1-P[L'\cap R']=.65$, and then since

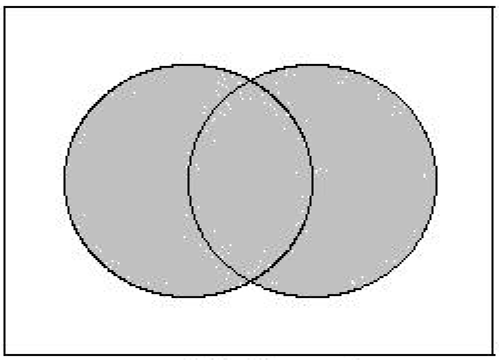
$P[L\cup R]=P[L]+P[R]-P[L\cap R]$, we get $P[L\cap R]=.3+.4-.65=.05$.



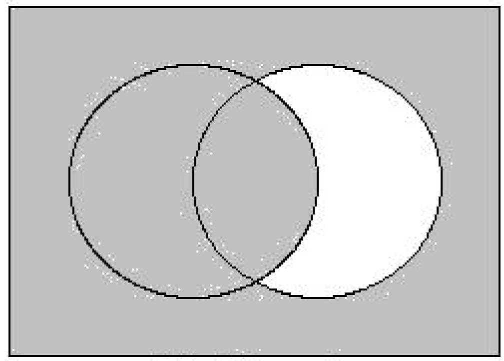
These calculations can be summarized in the following table

	L , .4 given	L' , .6 $.6=1-.4$	
R , .3 given	$L\cap R$ $.05=.4-.35$	$L'\cap R$ $.25=.3-.05$	
R' , .7 $.7=1-.3$	$L\cap R'$ $.35=.7-.35$	$L'\cap R'$, .35 given	Answer: A

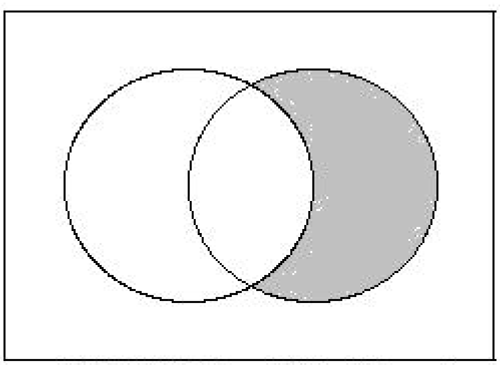
9. $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, $P[A \cup B'] = P[A] + P[B'] - P[A \cap B']$.
We use the relationship $P[A] = P[A \cap B] + P[A \cap B']$. Then
$P[A \cup B] + P[A \cup B'] = P[A] + P[B] - P[A \cap B] + P[A] + P[B'] - P[A \cap B']$
$= 2P[A] + 1 - P[A] = P[A] + 1$ (since $P[B] + P[B'] = 1$) .
Therefore, $.7 + .9 = P[A] + 1$ so that $P[A] = .6$.
An alternative solution is based on the following Venn diagrams.



P[A∪B] = .7



P[A∪B'] = .9



P[(A∪B')'] = P[A'∩B] = .1

In the third diagram, the shaded area is the complement of that in the second diagram (using De Morgan's Law, we have $(A \cup B')' = A' \cap B'' = A' \cap B$) . Then it can be seen from diagrams 1 and 3 that $A = (A \cup B) - (A' \cap B)$, so that
$P[A] = P[A \cup B] - P[A' \cap B] = .7 - .1 = .6$. Answer: D

10. We identify the following events:

G - watched gymnastics , B - watched baseball , S - watched soccer .

We wish to find $P[G' \cap B' \cap S']$. By DeMorgan's rules we have

$P[G' \cap B' \cap S'] = 1 - P[G \cup B \cup S]$.

We use the relationship

$$P[G \cup B \cup S] = P[G] + P[B] + P[S] \\ - \left(P[G \cap B] + P[G \cap S] + P[B \cap S]\right) + P[G \cap B \cap S] .$$

We are given $P[G] = .28$, $P[B] = .29$, $P[S] = .19$,

$P[G \cap B] = .14$, $P[G \cap S] = .10$, $P[B \cap S] = .12$, $P[G \cap B \cap S] = .08$.

Then $P[G \cup B \cup S] = .48$ and $P[G' \cap B' \cap S'] = 1 - .48 = .52$. Answer: D

SECTION 2 - CONDITIONAL PROBABILITY AND INDEPENDENCE

Conditional probability of event B given event A:

If $P[A] > 0$, then $\boldsymbol{P[B|A] = \frac{P[B\cap A]}{P[A]}}$ is defined to be the conditional probability that event B occurs given that event A has occurred. Events A and B may be related so that if we know that event A has occurred, the **conditional probability of event B occurring given that event A has occurred** might not be the same as the unconditional probability of event B occurring if we had no knowledge about the occurrence of event A. For instance, if a fair 6-sided die is tossed and if we know that the outcome is even, then the conditional probability is 0 of tossing a 3 given that the toss is even. If we did not know that the toss was even, if we had no knowledge of the nature of the toss, then tossing a 3 would have an unconditional probability of $\frac{1}{6}$, the same as all other possible tosses that could occur.

When we condition on event A, we are assuming that event A has occurred so that A becomes the new probability space, and all conditional events must take place within event A (the new probability space). Dividing by $P[A]$ scales all probabilities so that A is the entire probability space, and $P[A|A] = 1$. To say that event B has occurred given that event A has occurred means that both B and A ($B \cap A$) have occurred within the probability space A. This explains the numerator $P(B \cap A)$ in the definition of the conditional probability $P[B|A]$.

Rewriting $P[B|A] = \frac{P[B\cap A]}{P[A]}$, the equation that defines conditional probability, results in $\boldsymbol{P[B \cap A] = P[B|A] \cdot P[A]}$, which is referred to as the **multiplication rule**.

Example 2-1: Suppose that a fair six-sided die is tossed. The probability space is $S = \{1, 2, 3, 4, 5, 6\}$. We define the following events:
$A =$ "the number tossed is even" $= \{2, 4, 6\}$, $B =$ "the number tossed is ≤ 3" $= \{1, 2, 3\}$,
$C =$ "the number tossed is a 1 or a 2" $= \{1, 2\}$,
$D =$ "the number tossed doesn't start with the letters 'f' or 't'" $= \{1, 6\}$.

The conditional probability of B given A is
$P[B|A] = \frac{P[\{1,2,3\}\cap\{2,4,6\}]}{P[\{2,4,6\}]} = \frac{P[\{2\}]}{P[\{2,4,6\}]} = \frac{1/6}{1/2} = \frac{1}{3}$. The interpretation of this conditional probability is that if we know that event A has occurred, then the toss must be 2, 4 or 6. Since the original 6 possible tosses of a die were equally likely, if we are given the additional information

Example 2-1 continued

that the toss is 2, 4 or 6, it seems reasonable that each of those is equally likely, each with a probability of $\frac{1}{3}$. Then within the reduced probability space A, the (conditional) probability that event B occurs is the probability, in the reduced space, of tossing a 2; this is $\frac{1}{3}$.

For events B and C defined above, the conditional probability of B given C is $P[B|C] = 1$. To say that C has occurred means that the toss is 1 or 2. It is then guaranteed that event B has occurred (the toss is a 1, 2 or 3), since $C \subset B$.

The conditional probability of A given C is $P[A|C] = \frac{1}{2}$. □

Example 2-2: If $P[A] = \frac{1}{6}$ and $P[B] = \frac{5}{12}$, and $P[A|B] + P[B|A] = \frac{7}{10}$, find $P[A \cap B]$.

Solution: $P[B|A] = \frac{P[A \cap B]}{P[A]} = 6P[A \cap B]$ and $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{12}{5} P[A \cap B]$

$\rightarrow (6 + \frac{12}{5}) \cdot P[A \cap B] = \frac{7}{10} \rightarrow P[A \cap B] = \frac{1}{12}$. □

IMPORTANT NOTE: The following manipulation of event probabilities arises from time to time: $\boldsymbol{P[B] = P[B|A] \cdot P(A) + P[B|A'] \cdot P(A')}$.

This relationship is a version of the **Law of Total Probability.** This relationship is valid since for any events A and B, we have $P[B] = P[B \cap A] + P[B \cap A']$. We then use the relationships $P[B \cap A] = P[B|A] \cdot P(A)$ and $P[B \cap A'] = P[B|A'] \cdot P(A')$. If we know the conditional probabilities for event B given some other event A and if we also know the conditional probability of B given the complement A', and if we are given the (unconditional) probability of event A, then we can find the (unconditional) probability of event B. An application of this concept occurs when an experiment has two (or more) steps. The following example illustrates this idea.

Example 2-3: Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. An Urn is chosen at random, and a ball is randomly selected from that Urn. Find the probability that the ball chosen is white.

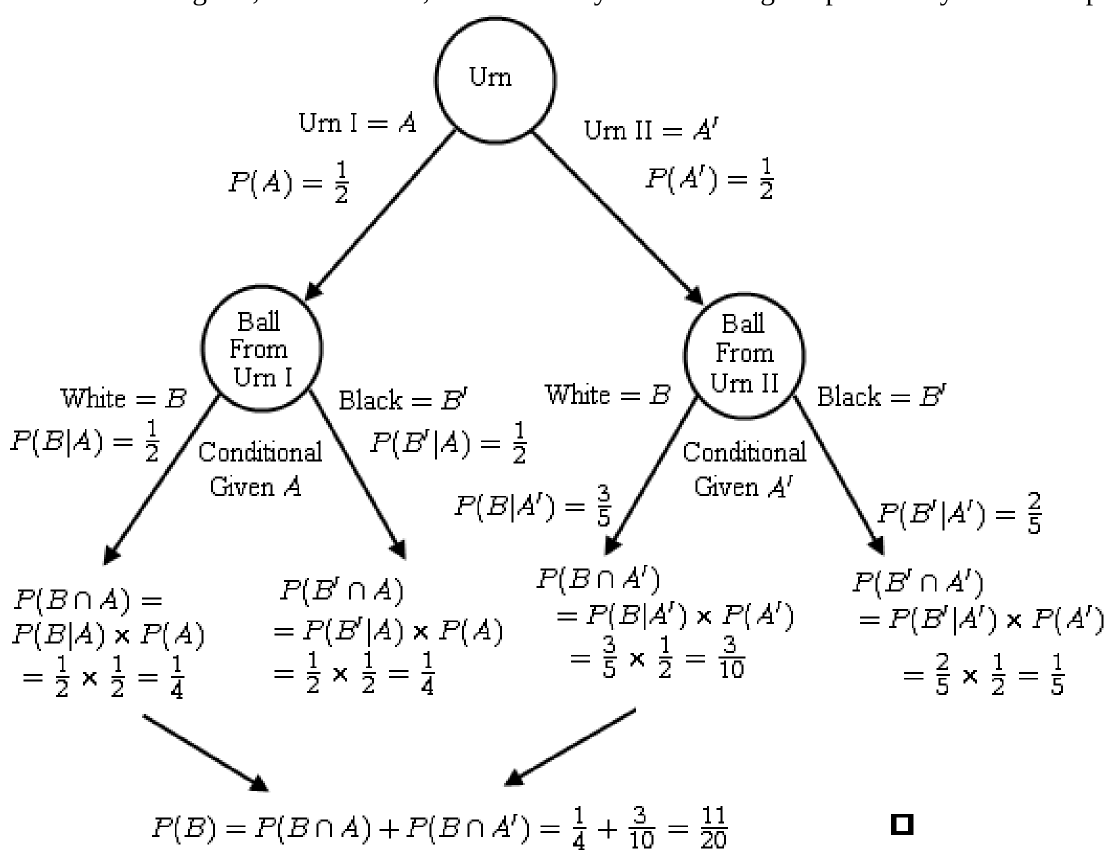
Solution: Let A be the event that Urn I is chosen and A' is the event that Urn II is chosen. The implicit assumption is that both Urns are equally likely to be chosen (this is the meaning of "an Urn is chosen at random"). Therefore, $P[A] = \frac{1}{2}$ and $P[A'] = \frac{1}{2}$. Let B be the event that the ball chosen is white. If we know that Urn I was chosen, then there is $\frac{1}{2}$ probability of choosing a white ball (2 white out of 4 balls, it is assumed that each ball has the same chance of being chosen); this can be described as $P[B|A] = \frac{1}{2}$. In a similar way, if Urn II is chosen, then $P[B|A'] = \frac{3}{5}$ (3 white out of 5 balls). We can now apply the relationship described prior to this example. $P[B \cap A] = P[B|A] \cdot P[A] = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$, and
$P[B \cap A'] = P[B|A'] \cdot P[A'] = (\frac{3}{5})(\frac{1}{2}) = \frac{3}{10}$. Finally,
$P[B] = P[B \cap A] + P[B \cap A'] = \frac{1}{4} + \frac{3}{10} = \frac{11}{20}$.

The order of calculations (1-2-3) can be summarized in the following table

	A	A'
B	**1.** $P(B \cap A) = P[B|A] \cdot P[A]$	**2.** $P(B \cap A') = P[B|A'] \cdot P[A']$

3. $P(B) = P(B \cap A) + P(B \cap A')$

An event tree diagram, shown below, is another way of illustrating the probability relationships.



□

IMPORTANT NOTE: An exam question may state that "an item is to chosen at random" from a collection of items". Unless there is an indication otherwise, this is interpreted to mean that each item has the same chance of being chosen. Also, if we are told that a fair coin is tossed randomly, then we interpret that to mean that the head and tail each have the probability of .5 occurring. Of course, if we are told that the coin is "loaded" so that the probability of tossing a head is 2/3 and tail is 1/3, then random toss means the head and tail will occur with those stated probabilities.

Bayes' rule and Bayes' Theorem (basic form):

For any events A and B with $P[B] > 0$, $\boldsymbol{P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B|A] \cdot P[A]}{P[B]}}$.

The usual way that this is applied is in the case that we are given the values of

$P[A]$, $P[B|A]$ and $P[B|A']$ (from $P[A]$ we get $P[A'] = 1 - P[A]$),

and we are asked to find $P[A|B]$ (in other words, we are asked to "turn around" the conditioning of the events A and B). We can summarize this process by calculating the quantities in the following table in the order indicated numerically (1-2-3-4) (other entries in the table are not necessary in this calculation, but might be needed in related calculations).

	A , $P(A)$ given	A' , $P(A')$ given
B	$P[B\|A]$ given **1.** $P[B \cap A] = P[B\|A] \cdot P[A]$	$P[B\|A']$ given **2.** $P[B \cap A'] = P[B\|A'] \cdot P[A']$
	$\Downarrow$	$\Downarrow$

3. $P[B] = P[B \cap A] + P[B \cap A']$

Also, we can find

B'	$P[B'\|A] = 1 - P[B\|A]$ $P[B' \cap A] = P[B'\|A] \cdot P[A]$	$P[B'\|A'] = 1 - P[B\|A']$ $P[A' \cap B'] = P[B'\|A'] \cdot P[A']$

and $P[B'] = P[B' \cap A] + P[B' \cap A']$

(but we could have found $P[B']$ from $P[B'] = 1 - P[B]$, once $P[B]$ was found).

Step **4**: $P[A|B] = \frac{P[A \cap B]}{P[B]}$.

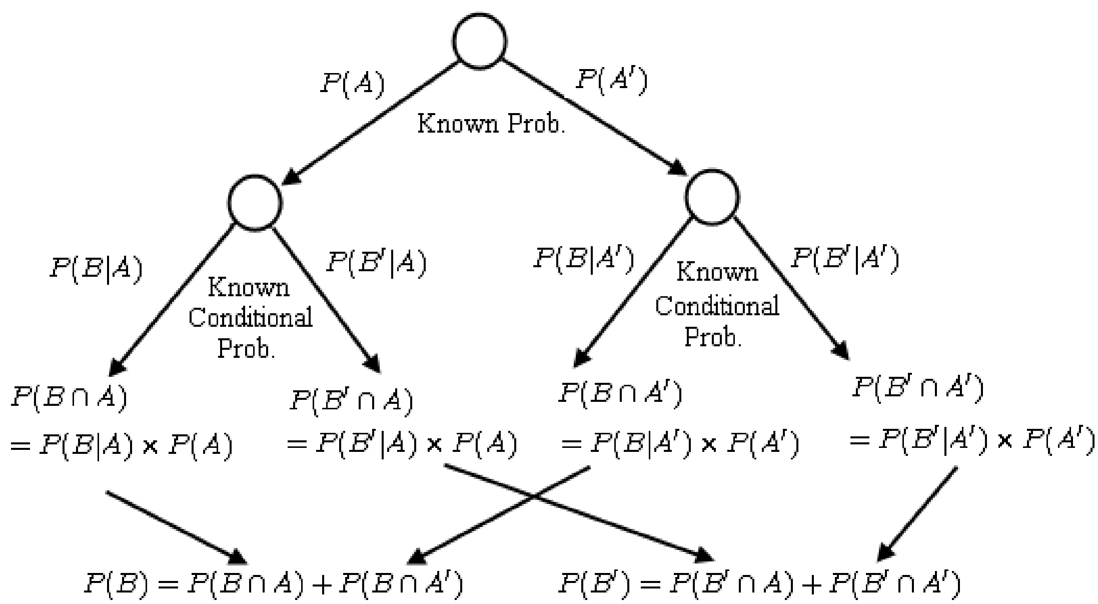
This can also be summarized in the following sequence of calculations.

$$\begin{array}{lcl} P[A]\ ,\ P[B|A]\ ,\ \text{given} & & P[A'] = 1 - P[A]\ ,\ P[B|A']\ ,\ \text{given} \\ \Downarrow & & \Downarrow \\ P[B \cap A] & & P[B \cap A'] \\ = P[B|A] \cdot P[A] & & = P[B|A'] \cdot P[A'] \\ & \Downarrow & \\ & P[B] = P[B \cap A] + P[B \cap A'] & \end{array}$$

Algebraically, we have done the following calculation:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B]}{P[B \cap A] + P[B \cap A']} = \frac{P[B|A] \cdot P[A]}{P[B|A] \cdot P[A] + P[B|A'] \cdot P[A']}\ ,$$

where all the factors in the final expression were originally known. Note that the numerator is one of the components of the denominator. The following event tree is similar to the one in Example 2-3, illustrating the probability relationships.



Note that at the bottom of the event tree, $P(B')$ is also equal to $1 - P(B)$.

Exam questions that involve conditional probability and make use of Bayes rule (and its extended form reviewed below) occur frequently. The key starting point is identifying and labeling unconditional events and conditional events and their probabilities in an efficient way.

Example 2-4: Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. One ball is chosen at random from Urn I and transferred to Urn II, and then a ball is chosen at random from Urn II. The ball chosen from Urn II is observed to be white. Find the probability that the ball transferred from Urn I to Urn II was white.

Solution: Let A denote the event that the ball transferred from Urn I to Urn II was white and let B denote the event that the ball chosen from Urn II is white. We are asked to find $P[A|B]$. From the simple nature of the situation (and the usual assumption of uniformity in such a situation, meaning that all balls are equally likely to be chosen from Urn I in the first step), we have $P[A]=\frac{1}{2}$ (2 of the 4 balls in Urn I are white), and $P[A']=\frac{1}{2}$.

If the ball transferred is white, then Urn II has 4 white and 2 black balls, and the probability of choosing a white ball out of Urn II is $\frac{2}{3}$; this is $P[B|A]=\frac{2}{3}$.

If the ball transferred is black, then Urn II has 3 white and 3 black balls, and the probability of choosing a white ball out of Urn II is $\frac{1}{2}$; this is $P[B|A']=\frac{1}{2}$.

All of the information needed has been identified. From the table described above, we do the calculations in the following order:

1. $P[B\cap A]=P[B|A]\cdot P[A]=(\frac{2}{3})(\frac{1}{2})=\frac{1}{3}$
2. $P[B\cap A']=P[B|A']\cdot P[A']=(\frac{1}{2})(\frac{1}{2})=\frac{1}{4}$
3. $P[B]=P[B\cap A]+P[B\cap A']=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$
4. $P[A|B]=\frac{P[A\cap B]}{P[B]}=\frac{1/3}{7/12}=\frac{4}{7}$. □

Example 2-5: Identical twins come from the same egg and hence are of the same sex. Fraternal twins have a 50-50 chance of being the same sex. Among twins, the probability of a fraternal set is p and an identical set is $q=1-p$. If the next set of twins are of the same sex, what is the probability that they are identical?

Solution: Let B be the event "the next set of twins are of the same sex", and let A be the event "the next sets of twins are identical". We are given $P[B|A]=1$, $P[B|A']=.5$ $P[A]=q$, $P[A']=p=1-q$. Then $P[A|B]=\frac{P[B\cap A]}{P[B]}$ is the probability we are asked to find. But $P[B\cap A]=P[B|A]\cdot P[A]=q$, and $P[B\cap A']=P[B|A']\cdot P[A']=.5p$. Thus, $P[B]=P[B\cap A]+P[B\cap A']=q+.5p=q+.5(1-q)=.5(1+q)$, and $P[A|B]=\frac{q}{.5(1+q)}$.

Example 2-5 continued

This can be summarized in the following table

	$A =$ identical, $P[A] = q$	$A' =$ not identical $P[A'] = 1 - q$
$B =$ same sex	$P[B\vert A] = 1$ (given) , $P[B \cap A]$ $= P[B\vert A] \cdot P[A] = q$	$P[B\vert A'] = .5$ (given) $P[B \cap A']$ $= P[B\vert A'] \cdot P[A'] = .5(1-q)$

$$\Downarrow$$

$$P[B] = P[B \cap A] + P[B \cap A'] = q + .5(1-q) = .5(1+q) \text{ .}$$

Then, $P[A|B] = \frac{P[B \cap A]}{P[B]} = \frac{q}{.5(1+q)}$.

The event tree shown on page 63 can be applied to this example. □

Bayes' rule and Bayes' Theorem (extended form):

If $A_1, A_2, ..., A_n$ form a partition of the entire probability space S, then

$$P[A_j|B] = \frac{P[B \cap A_j]}{P[B]} = \frac{P[B \cap A_j]}{\sum_{i=1}^{n} P[B \cap A_i]} = \frac{P[B|A_j] \cdot P[A_j]}{\sum_{i=1}^{n} P[B|A_i] \cdot P[A_i]} \quad \text{for each } j = 1, 2, ..., n.$$

For example, if the A's form a partition of $n = 3$ events, we have

$$P[A_1|B] = \frac{P[A_1 \cap B]}{P[B]} = \frac{P[B|A_1] \cdot P[A_1]}{P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3]}$$

$$= \frac{P[B|A_1] \cdot P[A_1]}{P[B|A_1] \cdot P[A_1] + P[B|A_2] \cdot P[A_2] + P[B|A_3] \cdot P[A_3]}$$

The relationship in the denominator, $P[B] = \sum_{i=1}^{n} P[B|A_i] \cdot P[A_i]$ is the general Law of Total Probability. The values of $P[A_j]$ are called prior probabilities, and the value of $P[A_j|B]$ is called a posterior probability. The basic form of Bayes' rule is just the case in which the partition consists of two events, A and A'. The main application of Bayes' rule occurs in the situation in which the $P[A_i]$ probabilities are known and the $P[B|A_h]$ probabilities are known, and we are asked to find $P[A_j|B]$ for one of the j's. The series of calculations can be summarized in a table as in the basic form of Bayes' rule. This is illustrated in the following example.

Example 2-6: Three dice have the following probabilities of throwing a "six": p, q, r, respectively. One of the dice is chosen at random and thrown (each is equally likely to be chosen). A "six" appeared. What is the probability that the die chosen was the first one?

Solution: The event " a 6 is thrown" is denoted by B and A_1, A_2 and A_3 denote the events that die 1, die 2 and die 3 was chosen.

$P[A_1|B] = \frac{P[A_1 \cap B]}{P[B]} = \frac{P[B|A_1] \cdot P[A_1]}{P[B]} = \frac{p \cdot \frac{1}{3}}{P[B]}$.

But $P[B] = P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3]$

$= P[B|A_1] \cdot P[A_1] + P[B|A_2] \cdot P[A_2] + P[B|A_3] \cdot P[A_3]$

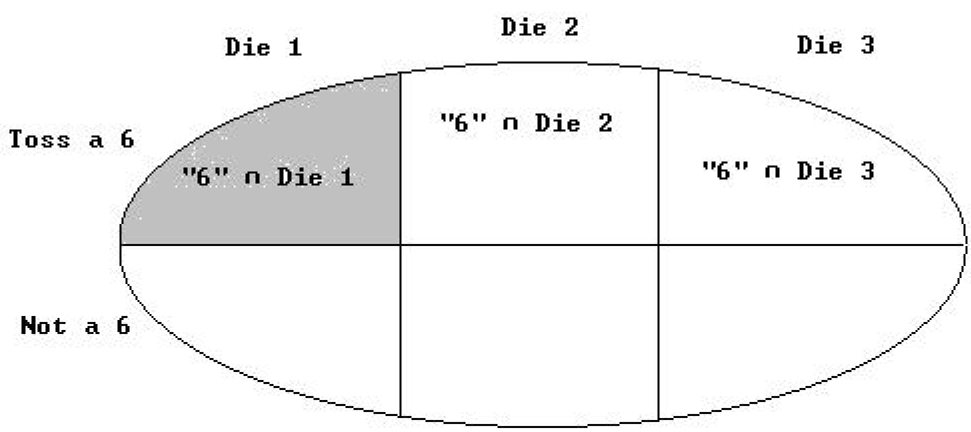
$= p \cdot \frac{1}{3} + q \cdot \frac{1}{3} + r \cdot \frac{1}{3} = \frac{p+q+r}{3} \Rightarrow P[A_1|B] = \frac{p \cdot \frac{1}{3}}{P[B]} = \frac{p \cdot \frac{1}{3}}{(p+q+r) \cdot \frac{1}{3}} = \frac{p}{p+q+r}$.

These calculations can be summarized in the following table.

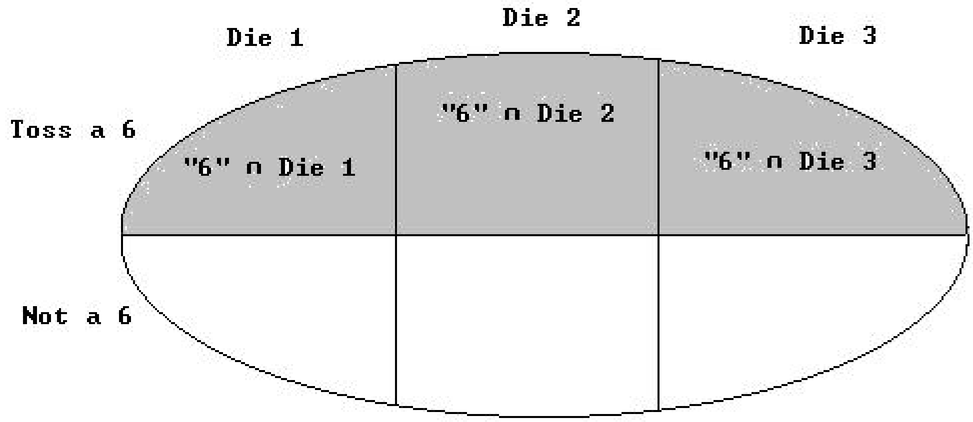
	Die 1 , $P(A_1) = \frac{1}{3}$ (given)	Die 2 , $P(A_2) = \frac{1}{3}$ (given)	Die 3 , $P(A_3) = \frac{1}{3}$ (given)
Toss "6", B	$P[B\|A_1] = p$ (given) $P[B \cap A_1]$ $= P[B\|A_1] \cdot P[A_1]$ $= p \cdot \frac{1}{3}$	$P[B\|A_2] = q$ (given) $P[B \cap A_2]$ $= P[B\|A_2] \cdot P[A_2]$ $= q \cdot \frac{1}{3}$	$P[B\|A_3] = r$ (given) $P[B \cap A_3]$ $= P[B\|A_3] \cdot P[A_3]$ $= r \cdot \frac{1}{3}$

$P[B] = p \cdot \frac{1}{3} + q \cdot \frac{1}{3} + r \cdot \frac{1}{3} = \frac{1}{3}(p + q + r)$.

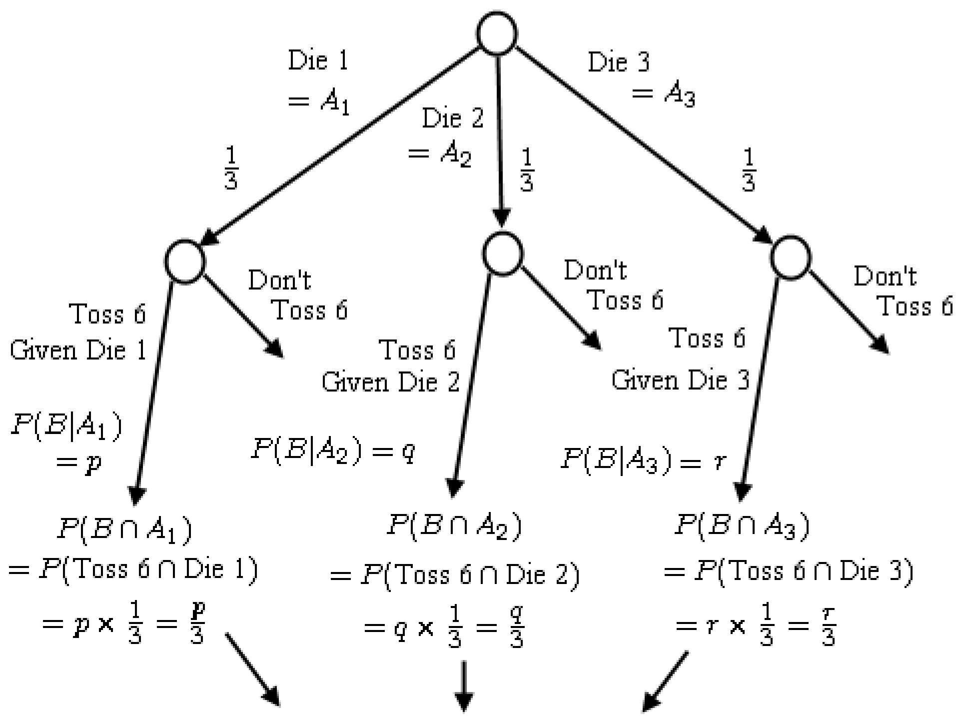
In terms of Venn diagrams, the conditional probability is the ratio of the shaded area probability in the first diagram to the shaded area probability in the second diagram.



Example 2-6 continued



The event tree representing the probabilities has three branches at the top node to represent the three die types that can be chosen in the first step of the process.



$$\begin{aligned} P(\text{Toss } 6) &= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) \\ &= P(\text{Toss } 6 \cap \text{Die } 1) + P(\text{Toss } 6 \cap \text{Die } 2) + P(\text{Toss } 6 \cap \text{Die } 3) \\ &= \frac{p}{3} + \frac{q}{3} + \frac{r}{3} = \frac{p+q+r}{3} \end{aligned}$$

□

In Example 2-6 there is a certain symmetry to the situation and general reasoning can provide a shortened solution. In the conditional probability $P[\text{die }1|\text{"6"}] = \frac{P[(\text{die }1)\cap(\text{"6"})]}{P[\text{"6"}]}$, we can think of the denominator as the combination of the three possible ways a "6" can occur, $p+q+r$, and we can think of the numerator as the "6" occurring from die 1, with probability p. Then the conditional probability is the fraction $\frac{p}{p+q+r}$. The symmetry involved here is in the assumption that each die was equally likely to be chosen, so there is a $\frac{1}{3}$ chance of any one die being chosen. This factor of $\frac{1}{3}$ cancels in the numerator and denominator of $\frac{p\cdot\frac{1}{3}}{(p+q+r)\cdot\frac{1}{3}}$. If we had not had this symmetry, we would have to apply different "weights" to the three dice.

Another example of this sort of symmetry is a variation on Example 2-3 above. Suppose that Urn I has 2 white and 3 black balls and Urn II has 4 white and 1 black balls. An Urn is chosen at random and a ball is chosen. The reader should verify using the usual conditional probability rules that the probability of choosing a white is $\frac{6}{10}$. This can also be seen by noting that if we consider the 10 balls together, 6 of them are white, so that the chance of picking a white out of the 10 is $\frac{6}{10}$. This worked because of two aspects of symmetry, equal chance for picking each Urn, and same number of balls in each Urn.

Independent events A and B: If events A and B satisfy the relationship $P[A\cap B] = P[A]\cdot P[B]$, then the events are said to be independent or stochastically independent or statistically independent. The independence of (non-empty) events A and B is equivalent to $P[A|B] = P[A]$, and also is equivalent to $P[B|A] = P[B]$.

Example 2-1 continued: A fair six-sided die is tossed.
$A =$ "the number tossed is even" $= \{2,4,6\}$, $B =$ "the number tossed is ≤ 3" $= \{1,2,3\}$,
$C =$ "the number tossed is a 1 or a 2" $= \{1,2\}$,
$D =$ "the number tossed doesn't start with the letters 'f' or 't'" $= \{1,6\}$.
We have the following probabilities: $P[A] = \frac{1}{2},\ P[B] = \frac{1}{2},\ P[C] = \frac{1}{3},\ P[D] = \frac{1}{3}$.
Events A and B are not independent since $\frac{1}{6} = P[A\cap B] \neq P[A]\cdot P[B] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. We also see that A and B are not independent because $P[B|A] = \frac{1}{3} \neq \frac{1}{2} = P[B]$.
Also, B and C are not independent, since $P[B\cap C] = \frac{1}{3} \neq \frac{1}{2}\cdot\frac{1}{3} = P[B]\cdot P[C]$ (also since $P[B|C] = 1 \neq \frac{1}{2} = P[B]$). Events A and C are independent, since
$P[A\cap C] = \frac{1}{6} = \frac{1}{2}\cdot\frac{1}{3} = P[A]\cdot P[C]$ (alternatively,
$P[A|C] = \frac{1}{2} = P[A]$, so that A and C are independent).
The reader should check that both A and B are independent of D. □

Mutually independent events: Events $A_1, A_2, ..., A_n$ are said to be mutually independent if the following relationships are satisfied. For any two events, say A_i and A_j, we have $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$. For any three events, Say A_i, A_j, A_k , we have] $P(A_i \cap A_j \cap A_k) = P(A_i) \cdot P(A_j) \cdot P(A_k)$. This must be true for any four events, any five events, etc.

Some rules concerning conditional probability and independence are:

(i) $P[A \cap B] = P[B|A] \cdot P[A] = P[A|B] \cdot P[B]$ for any events A and B

(ii) If $P[A_1 \cap A_2 \cap \cdots \cap A_{n-1}] > 0$, then
$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1] \cdot P[A_2|A_1] \cdot P[A_3|A_1 \cap A_2] \cdots P[A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}]$

(iii) $\boldsymbol{P[A'|B] = 1 - P[A|B]}$

(iv) $P[A \cup B|C] = P[A|C] + P[B|C] - P[A \cap B|C]$

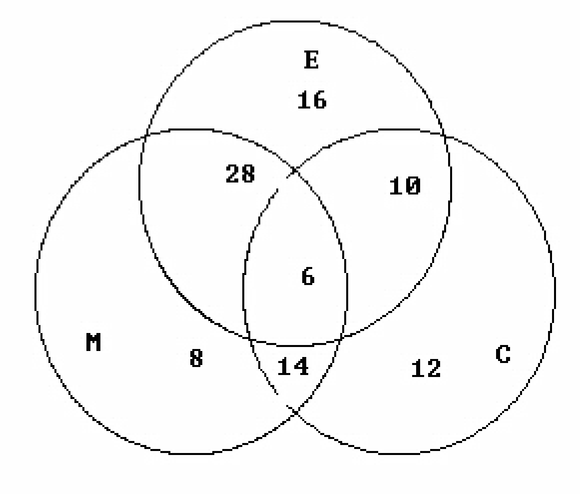
(v) if $A \subset B$ then $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]}{P[B]}$, and $P[B|A] = 1$; properties (iv) and (v) are the same properties satisfied by unconditional events

(vi) if A and B are independent events then A' and B are independent events, A and B' are independent events, and A' and B' are independent events

(vii) since $P[\emptyset] = P[\emptyset \cap A] = 0 = P[\emptyset] \cdot P[A]$ for any event A, it follows that $\emptyset$ is independent of any event A

Example 2-7: Suppose that events A and B are independent. Find the probability, in terms of $P[A]$ and $P[B]$, that exactly one of the events A and B occurs.
Solution: $P[\text{exactly one of } A \text{ and } B] = P[(A \cap B') \cup (A' \cap B)]$.
Since $A \cap B'$ and $B \cap A'$ are mutually exclusive, it follows that
$P[\text{exactly one of } A \text{ and } B] = P[A \cap B'] + P[A' \cap B]$.
Since A and B are independent, it follows that A and B' are also independent, as are B and A'.
Then $P[(A \cap B') \cup (A' \cap B)] = P[A] \cdot P[B'] + P[B] \cdot P[A']$
$= P[A](1 - P[B]) + P[B](1 - P[A]) = P[A] + P[B] - 2P[A] \cdot P[B]$ □

Example 2-8: In a survey of 94 students, the following data was obtained.
60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects.
Find the following proportions.
(i) Of those who took Math, the proportion who took neither English nor Chemistry,
(ii) Of those who took English or Math, the proportion who also took Chemistry.
Solution: The following diagram illustrates how the numbers of students can be deconstructed. We calculate proportion of the numbers in the various subsets.

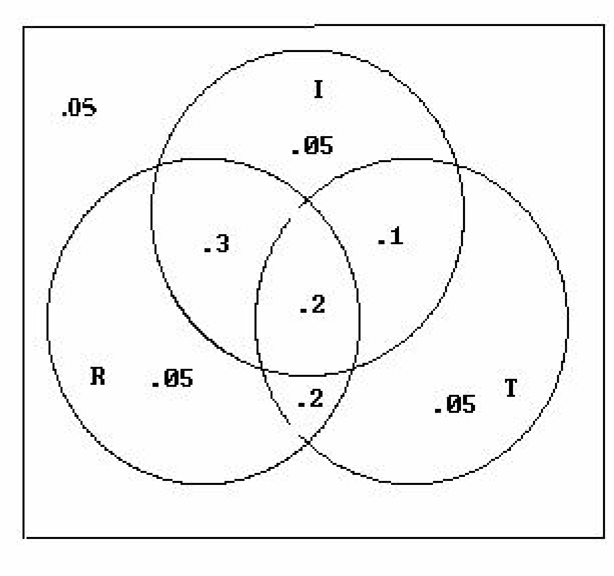


(i) 56 students took Math, and 8 of them took neither English nor Chemistry.
$P(E' \cap C'|M) = \frac{P(E' \cap C' \cap M)}{P(M)} = \frac{8}{56} = \frac{1}{7}$.

(ii) 82 ($= 8 + 14 + 6 + 28 + 16 + 10$ in $E \cup M$) students took English or Math (or both), and 30 of them ($= 14 + 6 + 10$ in $(E \cup M) \cap C$) also took Chemistry .
$P(C|E \cup M) = \frac{P[C \cap (E \cup M)]}{P(E \cup M)} = \frac{30}{82} = \frac{15}{41}$. □

Example 2-9: A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have irons (I), 55% have electric toasters (T), 50% have (IR), 40% have (RT), 30% have (IT), and 20% have all three.
Find the following proportions.
(i) Of those households that have a toaster, find the proportion that also have a radio.
(ii) Of those households that have a toaster but no iron, find the proportion that have a radio.

Solution: This is a continuation of Example 1-3 given earlier in the study guide.
The diagram below deconstructs the three events.



(i) This is $P[R|T]$. The language "of those households that have a toaster" means, "given that the household has a toaster", so we are being asked for a conditional probability.
Then, $P[R|T] = \frac{P[R \cap T]}{P[T]} = \frac{.4}{.55} = \frac{8}{11}$.
(ii) This is $P[R|T \cap I'] = \frac{P[R \cap T \cap I']}{P[T \cap I']} = \frac{.2}{.25} = \frac{4}{5}$. □

Example 2-8 presents a "population" of 94 individuals, each with some combination of various properties (took English, took Math, took Chemistry). We found conditional probabilities involving the various properties by calculating proportions in the following way
$P(A|B) = \frac{number\,of\,individuals\,satisfying\,both\,properties\,A\,and\,B}{number\,of\,indiviudals\,satisfying\,property\,B}$.
We could approach Example 2-9 in a similar way by creating a "model population" with the appropriate attributes. Since we are given percentages of households with various properties, we can imagine a model population of 100 households, in which 75 have radios (R, 75%), 65 have irons (I), 55 have electric toasters (T), 50 have (IR), 40 have (RT), 30 have (IT), and 20 have all three. The diagram in the solution could modified by changing the decimals to numbers out of 100 - so .2 becomes 10, etc. Then so solve (i), since 55 have toasters and 40 have both a radio and a toaster, the proportion of those who have toasters that also have a radio is $\frac{44}{50}$.

Creating a model population is sometimes an efficient way of solving a problem involving conditional probabilities, particularly when applying Bayes rule. The following example illustrates this.

Example 2-10 (SOA): A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

Solution: 13. We identify the following events:

D : a person has the disease , TP : a person tests positive for the disease

We are given $P(D) = .01$, $P(D') = .99$, $P(TP|D) = .95$, $P(TP'|D) = .05$, $P(TP|D') = .005$, $P(TP'|D') = .995$.

We wish to find $P(D|TP)$.

We first solve the problem using rules of conditional probability.

We have $P(D|TP) = \frac{P(D \cap TP)}{P(TP)}$.

We also have, $P(D \cap TP) = P(TP|D) \cdot P(D) = (.95)(.01) = .0095$, and

$P(TP) = P(D \cap TP) + P(D' \cap TP)$

$= P(TP|D) \cdot P(D) + P(TP|D') \cdot P(D') = (.95)(.01) + (.005)(.99) = .01445$.

Then, $P(D|TP) = \frac{P(D \cap TP)}{P(TP)} = \frac{.0095}{.01445} = .657$.

We can also solve this problem with the model population approach. We imagine a model population of 100,000 individuals. In this population, the number with disease is #$(D) = 1000$ (.01 of the population) , the number without disease is #$(D') = 99,000$ (.99 of the population). Since $P(TP|D) = .95$, it follows that 95% of those with the disease will test positive, so the number who have the disease and test positive is #$(TP \cap D) = .95 \times 1000 = 950$ (this just reflects the fact that $P(TP \cap D) = P(TP|D) \times P(D) = .95 \times .01 = .0095$, so that $.0095 \times 100,000 = 950$ in the population have the disease and test positive. In the same way, we find #$(TP \cap D') = .005 \times 99,000 = 495$ is the number who do not have disease but test positive. Therefore, the total number who test positive is

#$(TP) =$ #$(TP \cap D) +$ #$(TP \cap D') = 950 + 495 = 1445$.

The probability that a person has the disease given that the test indicates the presence of the disease is the proportion that have the disease and test positive as a fraction of all those who test positive, $P(D|TP) = \frac{\#(TP \cap D)}{\#(D)} = \frac{950}{1445} = .657$.

Example 2-10 continued

The following table summarizes the calculations in the conditional probability approach.

	$P(D) = .01$, given	$P(D') = .99$ $= 1 - .01$
TP	$P(TP\|D) = .95$, given $P(TP \cap D)$ $= P(TP\|D) \cdot P(D)$ $= (.95)(.01) = .0095$	$P(TP\|D') = .005$, given $P(TP \cap D')$ $= P(TP\|D') \cdot P(D')$ $= (.005)(.99) = .00495$
TP'	$P(TP'\|D) = 1 - P(TP\|D)$ $= .05$, $P(TP' \cap D)$ $= P(TP'\|D) \cdot P(D)$ $= (.05)(.01) = .0005$	$P(TP'\|D') = 1 - P(TP\|D')$ $= .995$, $P(TP' \cap D')$ $= P(TP'\|D') \cdot P(D')$ $= (.995)(.99) = .98505$

$P(TP) = P(TP \cap D) + P(TP \cap D') = .0095 + .00495 = .01445$.

$P(D|TP) = \frac{P(D \cap TP)}{P(TP)} = \frac{.0095}{.01445} = .657$. Answer: B

PROBLEM SET 2

Conditional Probability and Independence

1. Let A, B, C and D be events such that $B = A'$, $C \cap D = \emptyset$, and
$P[A] = \frac{1}{4}$, $P[B] = \frac{3}{4}$, $P[C|A] = \frac{1}{2}$, $P[C|B] = \frac{3}{4}$, $P[D|A] = \frac{1}{4}$, $P[D|B] = \frac{1}{8}$
Calculate $P[C \cup D]$.
A) $\frac{5}{32}$ B) $\frac{1}{4}$ C) $\frac{27}{32}$ D) $\frac{3}{4}$ E) 1

2. You are given that $P[A] = .5$ and $P[A \cup B] = .7$.
Actuary 1 assumes that A and B are independent and calculates $P[B]$ based on that assumption.
Actuary 2 assumes that A and B mutually exclusive and calculates $P[B]$ based on that assumption. Find the absolute difference between the two calculations.
A) 0 B) .05 C) .10 D) .15 E) .20

3. (SOA) An actuary studying the insurance preferences of automobile owners makes the following conclusions:
(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15 .
What is the probability that an automobile owner purchases neither collision nor disability coverage?
A) 0.18 B) 0.33 C) 0.48 D) 0.67 E) 0.82

4. Two bowls each contain 5 black and 5 white balls. A ball is chosen at random from bowl 1 and put into bowl 2. A ball is then chosen at random from bowl 2 and put into bowl 1. Find the probability that bowl 1 still has 5 black and 5 white balls.
A) $\frac{2}{3}$ B) $\frac{3}{5}$ C) $\frac{6}{11}$ D) $\frac{1}{2}$ E) $\frac{6}{13}$

5. (SOA) An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.

(ii) 70% of the customers insure more than one car.

(iii) 20% of the customers insure a sports car.

(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

A) 0.13 B) 0.21 C) 0.24 D) 0.25 E) 0.30

6. (SOA) An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims. Calculate the probability that a claim submitted to the insurance company includes operating room charges.

A) 0.10 B) 0.20 C) 0.25 D) 0.40 E) 0.80

7. Let A, B and C be events such that $P[A|C] = .05$ and $P[B|C] = .05$. Which of the following statements must be true?

A) $P[A \cap B|C] = (.05)^2$ B) $P[A' \cap B'|C] \geq .90$ C) $P[A \cup B|C] \leq .05$

D) $P[A \cup B|C'] \geq 1 - (.05)^2$ E) $P[A \cup B|C'] \geq .10$

8. A system has two components placed in series so that the system fails if either of the two components fails. The second component is twice as likely to fail as the first. If the two components operate independently, and if the probability that the entire system fails is .28, find the probability that the first component fails.

A) $\frac{.28}{3}$ B) $.10$ C) $\frac{.56}{3}$ D) $.20$ E) $\sqrt{.14}$

9. A ball is drawn at random from a box containing 10 balls numbered sequentially from 1 to 10. Let X be the number of the ball selected, let R be the event that X is an even number, let S be the event that $X \geq 6$, and let T be the event that $X \leq 4$. Which of the pairs (R,S), (R,T), and (S,T) are independent?

A) (R,S) only B) (R,T) only C) (S,T) only

D) (R,S) and (R,T) only E) (R,S), (R,T) and (S,T)

10. (SOA) A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

A) 0.20 B) 0.25 C) 0.35 D) 0.42 E) 0.57

11. If E_1, E_2 and E_3 are events such that $P[E_1|E_2] = P[E_2|E_3] = P[E_3|E_1] = p$,
$P[E_1 \cap E_2] = P[E_1 \cap E_3] = P[E_2 \cap E_3] = r$, and $P[E_1 \cap E_2 \cap E_3] = s$,
find the probability that at least one of the three events occurs.

A) $1 - \frac{r^3}{p^3}$ B) $\frac{3p}{r} - r + s$ C) $\frac{3r}{p} - 3r + s$ D) $\frac{3p}{r} - 6r + s$ E) $\frac{3r}{p} - r + s$

12. (SOA) A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease. Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

A) 0.115 B) 0.173 C) 0.224 D) 0.327 E) 0.514

13. In a T-maze, a laboratory rat is given the choice of going to the left and getting food or going to the right and receiving a mild electric shock. Assume that before any conditioning (in trial number 1) rats are equally likely to go the left or to the right. After having received food on a particular trial, the probability of going to the left and right become .6 and .4, respectively on the following trial. However, after receiving a shock on a particular trial, the probabilities of going to the left and right on the next trial are .8 and .2, respectively. What is the probability that the animal will turn left on trial number 2?

A) .1 B) .3 C) .5 D) .7 E) .9

14. In the game show "Let's Make a Deal", a contestant is presented with 3 doors. There is a prize behind one of the doors, and the host of the show knows which one. When the contestant makes a choice of door, at least one of the other doors will not have a prize, and the host will open a door (one not chosen by the contestant) with no prize. The contestant is given the option to change his choice after the host shows the door without a prize. If the contestant switches doors, what is the probability that he gets the door with the prize?

A) 0 B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

15. (SOA) A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.

(ii) 22% have low blood pressure.

(iii) 15% have an irregular heartbeat.

(iv) Of those with an irregular heartbeat, one-third have high blood pressure.

(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

A) 2% B) 5% C) 8% D) 9% E) 20%

16. (SOA) An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policy-holder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred?

A) 0.0001 B) 0.0010 C) 0.0071 D) 0.0141 E) 0.2817

17. (SOA) The probability that a randomly chosen male has a circulation problem is 0.25 . Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker?

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{2}{5}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

18. (SOA) A study of automobile accidents produced the following data:

Model year	Proportion of all vehicles	Probability of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997 .

A) 0.22 B) 0.30 C) 0.33 D) 0.45 E) 0.50

19. (SOA) An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16-20	0.06	0.08
21-30	0.03	0.15
31-65	0.02	0.49
66-99	0.04	0.28

A randomly selected driver that the company insures has an accident.
Calculate the probability that the driver was age 16-20.

A) 0.13 B) 0.16 C) 0.19 D) 0.23 E) 0.40

20. (SOA) Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;
(ii) 30% of the emergency room patients were serious;
(iii) the rest of the emergency room patients were stable;
(iv) 40% of the critical patients died
(v) 10% of the serious patients died; and
(vi) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

A) 0.06 B) 0.29 C) 0.30 D) 0.39 E) 0.64

21. Let A, B and C be mutually independent events such that $P[A] = .5$, $P[B] = .6$ and $P[C] = .1$. Calculate $P[A' \cup B' \cup C]$.

A) $.69$ B) $.71$ C) $.73$ D) $.98$ E) 1.00

22. (SOA) An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

A) 20 B) 29 C) 41 D) 53 E) 70

23. (SOA) An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of
driver

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	.15
Young Adult	16%	.08
Midlife	45%	.04
Senior	31%	.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

A) 0.06 B) 0.16 C) 0.19 D) 0.22 E) 0.25

24. Urn 1 contains 5 red and 5 blue balls. Urn 2 contains 4 red and 6 blue balls, and Urn 3 contains 3 red balls. A ball is chosen at random from Urn 1 and placed in Urn 2. Then a ball is chosen at random from Urn 2 and placed in Urn 3. Finally, a ball is chosen at random from Urn 3. Find the probabilities that all three balls chosen are red.

A) $\frac{5}{11}$ B) $\frac{5}{12}$ C) $\frac{5}{21}$ D) $\frac{5}{22}$ E) $\frac{5}{33}$

PROBLEM SET 2 SOLUTIONS

1. Since C and D have empty intersection, $P[C \cup D] = P[C] + P[D]$.
Also, since A and B are "exhaustive" events (since they are complementary events, their union is the entire sample space, with a combined probability of
$P[A \cup B] = P[A] + P[B] = 1$).
We use the rule $P[C] = P[C \cap A] + P[C \cap A']$, and the rule $P[C|A] = \frac{P[A \cap C]}{P[A]}$ to get
$P[C] = P[C|A] \cdot P[A] + P[C|A'] \cdot P[A'] = \frac{1}{2} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{11}{16}$ and
$P[D] = P[D|A] \cdot P[A] + P[D|A'] \cdot P[A'] = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{3}{4} = \frac{5}{32}$.
Then, $P[C \cup D] = P[C] + P[D] = \frac{27}{32}$. Answer: C.

2. Actuary 1: Since A and B are independent, so are A' and B'.
$P[A' \cap B'] = 1 - P[A \cup B] = .3$.
But $.3 = P[A' \cap B'] = P[A'] \cdot P[B'] = (.5)P[B'] \rightarrow P[B'] = .6 \rightarrow P[B] = .4$.
Actuary 2: $.7 = P[A \cup B] = P[A] + P[B] = .5 + P[B] \rightarrow P[B] = .2$.
Absolute difference is $|.4 - .2| = .2$. Answer: E

3. We identify the following events:
$D =$ an automobile owner purchases disability coverage, and
$C =$ an automobile owner purchases collision coverage.
We are given that
(i) $P[C] = 2P[D]$, (ii) C and D are independent, and (iii) $P[C \cap D] = .15$.
From (ii) it follows that $P[C \cap D] = P[C] \cdot P[D]$, and therefore,
$.15 = 2P[D] \cdot P[D] = 2(P[D])^2$, from which we get $P[D] = \sqrt{.075} = .27386$.
Then, $P[C] = 2P[D] = .54772$, $P[D'] = 1 - P[D] = .72614$, and
$P[C'] = 1 - P[C] = .45228$.
Since C and D are independent, so are C' and D', and therefore, the probability that an automobile owner purchases neither disability coverage nor collision coverage is
$P[C' \cap D'] = P[C'] \cdot P[D'] = .328$. Answer: B

4. Let C be the event that bowl 1 has 5 black balls after the exchange.
Let B_1 be the event that the ball chosen from bowl 1 is black, and
let B_2 be the event that the ball chosen from bowl 2 is black.
Event C is the disjoint union of $B_1 \cap B_2$ and $B_1' \cap B_2'$ (black-black or white-white picks), so that $P[C] = P[B_1 \cap B_2] + P[B_1' \cap B_2']$.
The black-black combination has probability $(\frac{6}{11})(\frac{1}{2})$,
since there is a $\frac{5}{10}$ chance of picking black from bowl 1, and then (with 6 black in bowl 2, which now has 11 balls) $\frac{6}{11}$ is the probability of picking black from bowl 2. This is
$P[B_1 \cap B_2] = P[B_2|B_1] \cdot P[B_1] = (\frac{6}{11})(\frac{1}{2})$.
In a similar way, the white-white combination has probability $(\frac{6}{11})(\frac{1}{2})$.
Then $P[C] = (\frac{6}{11})(\frac{1}{2}) + (\frac{6}{11})(\frac{1}{2}) = \frac{6}{11}$. Answer: C

5. We identify the following events:
A - the policyholder insures exactly one car (so that A' is the event that the policyholder insures more than one car), and
S - the policyholder insures a sports car.
We are given $P[A'] = .7$ (from which it follows that $P[A] = .3$) , and $P[S] = .2$ (and $P[S'] = .8$). We are also given the conditional probability $P[S|A'] = .15$;
"of those customers who insure more than one car", means that we are looking at a conditional event given A' .
We are asked to find $P[A \cap S']$.
We create the following probability table, with the numerals in parentheses indicating the order in which calculations are performed.

	A , .3	A' , .7
S , .2	**(2)** $P[S \cap A]$ $= P[S] - P[S \cap A']$ $= .2 - .105 = .095$	**(1)** $P[S \cap A'] = P[S\|A'] \cdot P[A']$ $= (.15)(.7) = .105$
S' , .8	**(3)** $P[A \cap S']$ $= P[A] - P[A \cap S]$ $= .3 - .095 = .205$	

We can solve this problem with a model population of 1000 individuals with auto insurance. $\#A = 300$ (since 70% insure more than one car), and $\#S = 200$. From $P[S|A'] = .15$ we get $\#S \cap A' = .15 \times \#A' = .15 \times 700 = 105$. Then $\#S \cap A = \#S - \#S \cap A' = 200 - 105 = 95$, and $\#S' \cap A = \#A - \#S \cap A = 300 - 95 = 205$ is the number that insure exactly one car and the car is not a sports car. Therefore $P[S' \cap A] = .205$. Answer: B

6. We define the following events.

E - the claim includes emergency room charges ,

O - the claim includes operating room charges.

We are given $P[E \cup O] = .85$, $P[E'] = .25$ and E and O are independent.

We are asked to find $P[O]$.

We use the probability rule $P[E \cup O] = P[E] + P[O] - P[E \cap O]$.

Since E and O are independent, we have $P[E \cap O] = P[E] \cdot P[O] = (.75)P[O]$

(since $P[E] = 1 - P[E'] = 1 - .25 = .75$).

Therefore, $.85 = P[E \cup O] = .75 + P[O] - .75P[O]$.

Solving for $P[O]$ results in $P[O] = .40$. Answer: D

7. $P[A' \cap B'|C] = P[(A \cup B)'|C] = 1 - P[A \cup B|C] \geq .9$,

since $P[A \cup B|C] \leq P[A|C] + P[B|C] = .1$. Answer: B

8. $.28 = P[C_1 \cup C_2] = P[C_1] + P[C_2] - P[C_1 \cap C_2] = P[C_1] + 2P[C_1] - 2(P[C_1])^2$

Solving the quadratic equation results in $P[C_1] = .1$ (or 1.4, but we disregard this solution since $P[C_1]$ must be ≤ 1). Alternatively, each of the five answers can be substituted into the expression above for $P[C_1]$ to see which one satisfies the equation. Answer: B

9. $P[R] = .5$, $P[S] = .5$, $P[T] = .4$.

$P[R \cap S] = P[6, 8, 10] = .3 \neq (.5)(.5) = P[R] \cdot P[S]$ → R, S are not independent

$P[R \cap T] = P[2, 4] = .2 = (.5)(.4) = P[R] \cdot P[T]$ → R, T are independent

$P[S \cap T] = P[\emptyset] = 0 \neq (.5)(.4) = P[S] \cdot P[T]$ → S, T are not independent. Answer: B

10. We identify the following events

N - non-smoker , L - light smoker , H - heavy smoker ,

D - dies during the 5-year study .

We are given $P[N] = .50$, $P[L] = .30$, $P[H] = .20$.

We are also told that $P[D|L] = 2P[D|N] = \frac{1}{2}P[D|H]$

(the probability that a light smoker dies during the 5-year study period is $P[D|L]$;

it is the conditional probability of dying during the period given that the individual is a light smoker). We wish to find the conditional probability $P[H|D]$.

We will find this probability from the basic definition of conditional probability,

$P[H|D] = \frac{P[H \cap D]}{P[D]}$. These probabilities can be found from the following probability table.

The numerals indicate the order in which the calculations are made.

We are not given specific values for $P[D|L]$, $P[D|N]$, or $P[D|H]$, so will let $P[D|N] = k$,

and then $P[D|L] = 2k$ and $P[D|H] = 4k$.

	N , .5	L , .3	H , .2
D	**(1)** $P[D \cap N]$ $= P[D|N] \cdot P[N]$ $= (k)(.5) = .5k$	**(2)** $P[D \cap L]$ $= P[D|L] \cdot P[L]$ $= (2k)(.3) = .6k$	**(3)** $P[D \cap H]$ $= P[D|H] \cdot P[H]$ $= (4k)(.2) = .8k$

(4) $P[D] = P[D \cap N] + P[D \cap L] + P[D \cap H] = .5k + .6k + .8k = 1.9k$.

(5) $P[H|D] = \frac{P[H \cap D]}{P[D]} = \frac{.8k}{1.9k} = .42$. Answer: D

11. $P[E_1|E_2] = \frac{P[E_1 \cap E_2]}{P[E_2]} = p \rightarrow P[E_2] = \frac{r}{p}$, and similarly $P[E_3] = P[E_1] = \frac{r}{p}$.

Then, $P[E_1 \cup E_2 \cup E_3]$

$= P[E_1] + P[E_2] + P[E_3] - (P[E_1 \cap E_2] + P[E_1 \cap E_3] + P[E_2 \cap E_3])$

$+ P[E_1 \cap E_2 \cap E_3] = 3(\frac{r}{p}) - 3r + s$. Answer: C

12. In this group of 937 man, we regard proportions of people with certain conditions to be probabilities. We are given the population of 937 men. We identify the following conditions:
DH - died from causes related to heart disease , and
PH - had a parent with heart disease.
We are given $\#PH = 312$, so if follows that $\#PH' = 937 - 312 = 625$.
We are also given $\#DH = 210$ and $\#DH \cap PH = 102$.
It follows that $\#DH \cap (PH') = \#DH - \#DH \cap PH = 210 - 102 = 108$.
Then the probability of dying due to heart disease given that neither parent suffered from heart disease is the proportion $\frac{\#DH \cap (PH')}{\#PH'} = \frac{108}{625}$.

The solution in terms of conditional probability rules is as follows. From the given information, we have
$P[DH] = \frac{210}{937}$ (proportion who died from causes related to heart disease)
$P[PH] = \frac{312}{937}$ (proportion who have parent with heart disease)
$P[DH|PH] = \frac{102}{312}$ (prop. who died from heart disease given that a parent has heart disease).
We are asked to find $P[DH|PH']$ (PH' is the complement of event PH, so that PH' is the event that neither parent had heart disease). Using event algebra, we have
$P[DH|PH] = \frac{P[DH \cap PH]}{P[PH]} \Rightarrow P[DH \cap PH] = P[DH|PH] \cdot P[PH] = (\frac{102}{312})(\frac{312}{937}) = \frac{102}{937}$.
We now use the rule $P[A] = P[A \cap B] + P[A \cap \overline{B}]$.
Then $P[DH] = P[DH \cap PH] + P[DH \cap PH'] \rightarrow \frac{210}{937} = \frac{102}{937} + P[DH \cap PH']$
$\Rightarrow P[DH \cap PH'] = \frac{108}{937}$.
Finally, $P[DH|PH'] = \frac{P[DH \cap PH']}{P[PH']} = \frac{108/937}{1 - P[PH]} = \frac{108/937}{1 - \frac{312}{937}} = \frac{108}{625} = .1728$.

These calculations can be summarized in the following table.

	DH , 210 given	DH' , 727 $= 937 - 210$
PH , 312 given	$DH \cap PH = 102$ given	$DH' \cap PH = 210$ $= 312 - 102$
PH' , 625 $= 937 - 312$	$DH \cap PH' = 108$ $= 210 - 102$	$DH' \cap PH' = 517$ $= 727 - 210$ or $= 625 - 108$

$P[DH|PH'] = \frac{P[DH \cap PH']}{P[PH']} = \frac{\#[DH \cap PH']}{\#[PH']} = \frac{108}{625} = .1728$.

In this example, probability of an event is regarded as the proportion of a group that experiences that event. Answer: B

13. $L1 =$ turn left on trial 1, $R1 =$ turn right on trial 1, $L2 =$ turn left on trial 2 .
We are given that $P[L1] = P[R1] = .5$.
$P[L2] = P[L2 \cap L1] + P[L2 \cap R1]$ since $L1, R1$ form a partition .
$P[L2|L1] = .6$ (if the rat turns left on trial 1 then it gets food and has a .6 chance of turning left on trial 2). Then $P[L2 \cap L1] = P[L2|L1] \cdot P[L1] = (.6)(.5) = .3$.
In a similar way, $P[L2 \cap R1] = P[L2|R1] \cdot P[R1] = (.8)(.5) = .4$.
Then, $P[L2] = .3 + .4 = .7$.
In a model population of 10 rats, $\#L1 = \#R1 = 5$, and $\#L2 \cap L1 = .6 \times 5 = 3$ and $\#L2 \cap R1 = .8 \times 5 = 4$. Then the number turning left on trial 2 will be $\#L2 = \#L2 \cap L1 + \#L2 \cap R1 = 3 + 4 = 7$, so the probability of a rat turning left on trial 2 is $7/10 = .7$ Answer: D

14. We define the events $A =$ prize door is chosen after contestant switches doors , $B =$ prize door is initial one chosen by contestant . Then $P[B] = \frac{1}{3}$, since each door is equally likely to hold the prize initially. To find $P[A]$ we use the Law of Total Probability.
$P[A] = P[A|B] \cdot P[B] + P[A|B'] \cdot P[B'] = (0)(\frac{1}{3}) + (1)(\frac{2}{3}) = \frac{2}{3}$.
If the prize door is initially chosen, then after switching, the door chosen is not the prize door, so that $P[A|B] = 0$. If the prize door is not initially chosen, then since the host shows the other non-prize door, after switching the contestant definitely has the prize door, so that $P[A|B'] = 1$.
Answer: E

15. This question can be put into the context of probability event algebra. First we identify events: $H =$ high blood pressure , $L =$ low blood pressure , $N =$ normal blood pressure , $R =$ regular heartbeat , $I = R' =$ irregular heartbeat

We are told that 14% of patients have high blood pressure, which can be represented as $P[H] = .14$, and similarly $P[L] = .22$, and therefore $P[N] = 1 - P[H] - P[L] = .64$.
We are given $P[I] = .15$, so that $P[R] = 1 - P[I] = .85$.
We are told that "of those with an irregular heartbeat, one-third have high blood pressure". This is the conditional probability that given I (irregular heartbeat) the probability of H (high blood pressure) is $P[H|I] = \frac{1}{3}$. Similarly, we are given $P[I|N] = \frac{1}{8}$.

We are asked to find the portion of patients who have both a regular heartbeat and low blood pressure; this is $P[R \cap L]$. Since every patient is exactly one of H, L or N, we have
$P[R \cap L] + P[R \cap H] + P[R \cap N] = P[R] = .85$, so that
$P[R \cap L] = .85 - P[R \cap H] - P[R \cap N]$.

15. continued

From the conditional probabilities we have

$\frac{1}{3} = P[H|I] = \frac{P[H \cap I]}{P[I]} = \frac{P[H \cap I]}{.15} \rightarrow P[H \cap I] = .05$, and

$\frac{1}{8} = P[I|N] = \frac{P[I \cap N]}{P[N]} = \frac{P[I \cap N]}{.64} \rightarrow P[I \cap N] = .08$.

Then, since all patients are exactly one of I and R, we have

$P[H \cap I] + P[H \cap R] = P[H] = .14 \rightarrow P[H \cap R] = .14 - .05 = .09$, and

$P[I \cap N] + P[R \cap N] = P[N] = .64 \rightarrow P[R \cap N] = .64 - .08 = .56$.

Finally, $P[R \cap L] = .85 - P[R \cap H] - P[R \cap N] = .85 - .09 - .56 = .20$.

These calculations can be summarized in the following table.

	H , .14 given $\Downarrow$	N , .64 $= 1 - .14 - .22$ $\Downarrow$	L , .22 given
I , .15 given	$P(H \mid I) = \frac{1}{3}$ given $P(H \cap I)$ $= P(H \mid I) \cdot P(I)$ $= (\frac{1}{3})(.15) = .05$	$P(I \mid N) = \frac{1}{8}$ given $P(N \cap I)$ $= P(I \mid N) \cdot P(N)$ $= (\frac{1}{8})(.64) = .08 \Rightarrow$	 $P(L \cap I)$ $= P(I) - P(H \cap I) - P(N \cap I)$ $= .15 - .05 - .08 = .02$ $\Downarrow$
R , .85 $= 1 - .15$			$P(R \cap L)$ $= P(L) - P(L \cap I)$ $= .22 - .02 = .2$

Note that the entries $P(R \cap H)$ and $P(R \cap N)$ can also be calculated from this table.

The model population solution is as follows. Suppose that the model population has 2400 individuals. Then we have the following

$\#H = .14 \times 2400 = 336$, $\#L = 528$, $\#N = 1536$, $\#I = 360$, $\#R = 2040$.

Since one-third of those with an irregular heartbeat have high blood pressure, we get $\#I \cap H = 120$, and since one-eighth of those with normal blood pressure have an irregular heartbeat we get $\#N \cap I = 192$. We wish to find $\#R \cap L$.

From $\#I = \#I \cap H + \#I \cap L + \#I \cap N$, we get $360 = 120 + \#I \cap L + 192$, so that $\#I \cap L = 48$. Then from $\#L = \#I \cap L + \#R \cap L$ we get $528 = 48 + \#R \cap L$, so that $\#R \cap L = 480$. Finally, the probability of having a regular heartbeat and low blood pressure is the proportion of the population with those properties, which is $\frac{480}{2400} = .2$. Answer: E

16. This is a typical exercise involving conditional probability. We first label the events, and then identify the probabilities.

S - standard policy P - preferred policy

U- ultra-preferred policy D- death occurs in the next year.

We are given $P[S] = .50$, $P[P] = .40$, $P[U] = .10$,
$P[D|S] = .01$, $P[D|P] = .005$, $P[D|U] = .001$.
We are asked to find $P[U|D]$.

The model population solution is as follows. Suppose there is a model population of 10,000 insured lives. Then $\#S = 5000$, $\#P = 4000$ and $\#U = 1000$.
From $P[D|S] = .01$ we get $\#D \cap S = .01 \times 5000 = 50$, and we also get
$\#D \cap P = .005 \times 4000 = 20$ and $\#D \cap U = .001 \times 1000 = 1$.
Then $\#D = 50 + 20 + 1 = 71$, and $P[U|D]$ is the proportion who are ultra-preferred as a proportion of all who died. This is $\frac{1}{71} = .0141$.

The conditional probability approach to solving the problem is as follows.
The basic formulation for conditional probability is $P[U|D] = \frac{P[U \cap D]}{P[D]}$.
We use the following relationships:
$P[A \cap B] = P[A|B] \cdot P[B]$, and
$P[A] = P[A \cap C_1] + P[A \cap C_2] + \cdots + P[A \cap C_n]$, for a partition $C_1, C_2..., C_n$.

In this problem, events S , P and U form a partition of all policyholders.
Using the relationships we get
$P[U \cap D] = P[D|U] \cdot P[U] = (.001)(.1) = .0001$, and
$P[D] = P[D \cap S] + P[D \cap P] + P[D \cap U]$
$= P[D|S] \cdot P[S] + P[D|P] \cdot P[P] + P[D|U] \cdot P[U]$
$= (.01)(.5) + (.005)(.4) + (.001)(.1) = .0071$.

Then, $P[U|D] = \frac{P[U \cap D]}{P[D]} = \frac{P[D|U] \cdot P[U]}{P[D|S] \cdot P[S] + P[D|P] \cdot P[P] + P[D|U] \cdot P[U]}$
$= \frac{(.001)(.1)}{(.01)(.5)+(.005)(.4)+(.001)(.1)} = \frac{.0001}{.0071} = .0141$.
Notice that the numerator is one of the factors of the denominator. This will always be the case when we are "reversing" conditional probabilities such as has been done here; we are to find $P[U|D]$ from being given information about $P[D|U]$, $P[D|S]$, $P[D|P]$, etc.

16. continued

From the calculations already made it is easy to find the probability that the deceased policyholder was preferred;

$P[P|D] = \frac{P[P\cap D]}{P[D]} = \frac{P[D|P]\cdot P[P]}{P[D|S]\cdot P[S]+P[D|P]\cdot P[P]+P[D|U]\cdot P[U]}$
$= \frac{(.005)(.4)}{(.01)(.5)+(.005)(.4)+(.001)(.1)} = \frac{.0020}{.0071} = .2817$.

And $P[S|D]$ is $\frac{(.01)(.5)}{(.01)(.5)+(.005)(.4)+(.001)(.1)} = \frac{.0050}{.0071} = .7042$.

The calculations can be summarized in the following table.

	S , .5 given	P , .4 given	U , .1 given
D	$P(D\|S) = .01$ given	$P(D\|P) = .005$ given	$P(D\|U) = .001$ given
	$P(D\cap S)$ $= P(D\|S)\cdot P(S)$ $= (.01)(.5) = .005$	$P(D\cap P)$ $= P(D\|P)\cdot P(P)$ $= (.005)(.4) = .002$	$P(D\cap U)$ $= P(D\|U)\cdot P(U)$ $= (.001)(.1) = .0001$

$P(D) = P[D\cap S] + P[D\cap P] + P[D\cap U] = .005 + .002 + .0001 = .0071$.

$P[U|D] = \frac{P[U\cap D]}{P[D]} = \frac{.0001}{.0071} = .0141$. Answer: D

17. We identify the following events:

C - a randomly chosen male has a circulation problem ,

S - a randomly chosen male is a smoker.

We are given the following probabilities:

$P[C] = .25$, $P[S|C] = 2P[S|C']$.

From the rule $P[A\cap B] = P[A|B]\cdot P[B]$, we get

$P[S\cap C] = P[S|C]\cdot P[C] = (.25)P[S|C]$, and

$P[S\cap C'] = P[S|C']\cdot P[C'] = P[S|C']\cdot(1-P[C]) = (.75)(\frac{1}{2})P[S|C]$,

so that $P[S] = P[S\cap C] + P[S\cap C'] = (.25)P[S|C] + (.75)(\frac{1}{2})P[S|C] = .625P[S|C]$.

We are asked to find $P[C|S]$. This is $P[C|S] = \frac{P[C\cap S]}{P[S]} = \frac{(.25)P[S|C]}{.625P[S|C]} = .4$.

Note that the way in which information was provided allowed us to formulate various probabilities in terms of $P[S|C]$ (but we do not have enough to find $P[S|C]$). Answer: C

18. We identify events as follows:
97: the model year is 1997 , 98: the model year is 1998 , 99: the model year is 1999
OO : other, the model year is not 1997, 1998 or 1999
A : the car is involved in an accident

We are given $P[97] = .16$, $P[98] = .18$, $P[99] = .20$, $P[00] = .46$,
$P[A|97] = .05$, $P[A|98] = .02$, $P[A|99] = .03$, $P[A|\text{other}] = .04$.

The model population solution is as follows. Suppose there are 10,000 automobiles in the study.
Then $\#97 = 1600$, $\#98 = 1800$, $\#99 = 2000$, $\#OO = 4600$.
From $P[A|97] = .05$ we get $\#A \cap 97 = .05 \times 1600 = 80$, and in a similar way we get
$\#A \cap 98 = .02 \times 1800 = 36$, $\#A \cap 99 = .03 \times 2000 = 60$
and $\#A \cap OO = .04 \times 4600 = 184$.

We are given that an automobile from one of 97, 98 or 99 was involved in an accident, and we wish to find the probability that it was a 97 model. This is the conditional probability
$P[97|A \cap (97 \cup 98 \cup 99)]$. This will be the proportion
$\frac{\#A\cap 97}{\#A\cap 97+\#A\cap 98+\#A\cap 99} = \frac{80}{80+36+60} = \frac{80}{176} = .4545$.

The conditional probability apporach to solve the problem is as follows.
We use the conditional probability rule $P[C|D] = \frac{P[C\cap D]}{P[D]}$, so that
$P[97|A \cap (97 \cup 98 \cup 99)] = \frac{P[97\cap[A\cap(97\cup 98\cup 99)]]}{P[A\cap(97\cup 98\cup 99)]}$.

From set algebra, we have $97 \cap [A \cap (97 \cup 98 \cup 99)] = 97 \cap A$, and
$A \cap (97 \cup 98 \cup 99) = (A \cap 97) \cup (A \cap 98) \cup (A \cap 99)$.

Since the events 97, 98 and 99 are disjoint, we get
$P[A \cap (97 \cup 98 \cup 99)] = P[(A \cap 97) \cup (A \cap 98) \cup (A \cap 99)]$
$= P[A \cap 97] + P[A \cap 98] + P[A \cap 99]$.

From conditional probability rules we have
$P[A \cap 97] = P[A|97] \cdot P[97] = (.05)(.16) = .008$, and similarly
$P[A \cap 98] = (.02)(.18) = .0036$, and $P[A \cap 99] = (.03)(.20) = .006$.
Then, $P[A \cap (97 \cup 98 \cup 99)] = .008 + .0036 + .006 = .0176$.

Therefore, the probability we are trying to find is
$P[97|A \cap (97 \cup 98 \cup 99)] = \frac{P[97\cap[A\cap(97\cup 98\cup 99)]]}{P[A\cap(97\cup 98\cup 99)]}$
$= \frac{P[97\cap A]}{P[A\cap(97\cup 98\cup 99)]} = \frac{.008}{.0176} = .4545$.

18. continued

These calculations can be summarized in the following table.

	97 , $.16$ given	98 , $.18$ given	99 , $.20$ given	Other , $.46$ given
A	$P(A\|97)$ $=.05$ given	$P(A\|98)$ $=.02$ given	$P(A\|99)$ $=.03$ given	$P(A\|\text{Other})$ $=.04$ given
	$P(A\cap 97)$ $P[A\|97]\cdot P[97]$ $=(.05)(.16)$ $=.008$	$P(A\cap 98)$ $P[A\|98]\cdot P[98]$ $=(.02)(.18)$ $=.0036$	$P(A\cap 99)$ $P[A\|99]\cdot P[99]$ $=(.03)(.20)$ $=.006$	$P(A\cap \text{Other})$ $P[A\|\text{O}]\cdot P[\text{O}]$ $=(.04)(.46)$ $=.0184$

Then, $P[97|A\cap(97\cup 98\cup 99)] = \frac{.008}{.008+.0036+.006} = .4545$.

Note that the denominator is the sum of the first three of the intersection probabilities, since the condition is that the auto was 97, 98 or 99. If the question had asked for the probability that the model year was 97 given that an accident occurred (without restricting to 97, 98, 99) then the probability would be $\frac{.008}{.008+.0036+.006+.0184}$; we would include all model years in the denominator. If the question had asked for the probability that the model year was 97 given that an accident occurred and the automobile was from one of the model years 97 or 98, then the probability would be $\frac{.008}{.008+.0036}$; we would include only the 97 and 98 model years.

Answer: D

19. We identify the following events:

A - the driver has an accident ,

T (teen) - age of driver is 16-20 , Y (young) - age of driver is 21-30 ,

M (middle age) - age of driver is 31-65 , S (senior) - age of driver is 66-99 .

The final column in the table lists the probabilities of T, Y, M and S , and the middle column gives the conditional probability of A given driver age. The table can be interpreted as

Age	Probability of Accident	Portion of Insured Drivers
16-20	$P[A\|T]=.06$	$P[T]=.08$
21-30	$P[A\|Y]=.03$	$P[Y]=.15$
31-65	$P[A\|M]=.02$	$P[M]=.49$
66-99	$P[A\|S]=.04$	$P[S]=.28$

We are asked to find $P[T|A]$.

19. continued

We construct the following probability table, with numerals in parentheses indicating the order of the calculations.

	$T, .08$	$Y, .15$	$M, .49$	$S, .28$
A	**(1)** $P[A \cap T]$ $= P[A\|T] \cdot P[T]$ $= (.06)(.08)$ $= .0048$	**(2)** $P[A \cap Y]$ $= P[A\|Y] \cdot P[Y]$ $= (.03)(.15)$ $= .0045$	**(3)** $P[A \cap M]$ $= P[A\|M] \cdot P[M]$ $= (.02)(.49)$ $= .0098$	**(4)** $P[A \cap S]$ $= P[A\|S] \cdot P[S]$ $= (.04)(.28)$ $= .0112$

(5) $P[A] = P[A \cap T] + P[A \cap Y] + P[A \cap M] + P[A \cap S] = .0303$

(6) $P[T|A] = \frac{P[A \cap T]}{P[A]} = \frac{.0048}{.0303} = .158$. Answer: B

20. We label the following events:

C - critical , S - serious , T -stable , D - died , $\overline{D}$ - survived.

The following information is given

$P(C) = .1$, $P(S) = .3$, $P(T) = .6 = 1 - P(C) - P(S)$,

$P(D|C) = .4$, $P(D|S) = .1$, $P(D|T) = .01$.

We are asked to find $P(S|D)$. This can be done by using the following table of probabilities.

The rules being used here is $P(A \cap B) = P(A|B) \cdot P(B)$,

and $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$ if $B_1, B_2, \ldots, B_n$ form a partition of the probability space. In this case, C, S, T form a partition since all patients are exactly one of these three conditions.

	C	S	T
D	$P(D \cap C)$ $= P(D\|C) \cdot P(C)$ $= (.4)(.1) = .04$	$P(D \cap S)$ $= P(D\|S) \cdot P(S)$ $= (.1)(.3) = .03$	$P(D \cap T)$ $= P(D\|T) \cdot P(T)$ $= (.01)(.6) = .006$

$\rightarrow P(D) = P(D \cap C) + P(D \cap S) + P(D \cap T) = .04 + .03 + .006 = .076$

	C	S	T
D'	$P(D' \cap C)$ $= P(D'\|C) \cdot P(C)$ $= (.6)(.1) = .06$	$P(D' \cap S)$ $= P(D'\|S) \cdot P(S)$ $= (.9)(.3) = .27$	$P(D' \cap T)$ $= P(D'\|T) \cdot P(T)$ $= (.99)(.6) = .594$

$\rightarrow P(D') = P(D' \cap C) + P(D' \cap S) + P(D' \cap T) = .06 + .27 + .594 = .924$

It was not necessary to do the calculations for D', since $P(D') = 1 - P(D) = 1 - .076 = .924$.

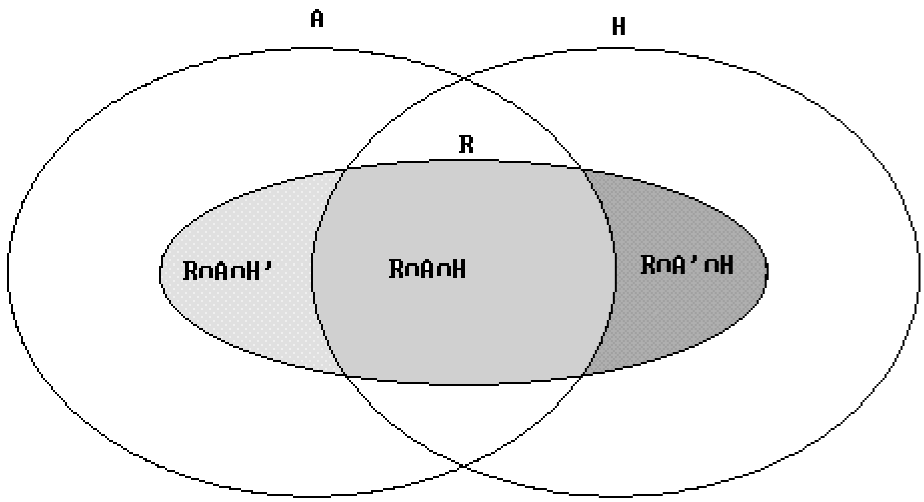
The probability in question is $P(S|D') = \frac{P(S \cap D')}{P(D')} = \frac{.27}{.924} = .292$. Answer: B

21. $P[A' \cup B' \cup C]$
$= P[A'] + P[B'] + P[C] - (P[A' \cap B'] + P[A' \cap C] + P[B' \cap C]) + P[A' \cap B' \cap C]$
$= .5 + .4 + .1 - [(.5)(.4) + (.5)(.1) + (.4)(.1)] + (.5)(.4)(.1) = .73$.
If events X and Y are independent, then so are X' and Y, X and Y', and X' and Y' .
Alternatively using DeMorgan's Law, we have
$P[A' \cup B' \cup C] = 1 - P[(A' \cup B' \cup C)'] = 1 - P[A'' \cap B'' \cap C'] = 1 - P[A \cap B \cap C']$
$= 1 - P[A] \cdot P[B] \cdot P[C'] = 1 - (.5)(.6)(.9) = .73$. Answer: C

22. We define the following events
R - renew at least one policy next year
A - has an auto policy , H - has a homeowner policy
A policyholder with an auto policy only can be described by the event $A \cap H'$, and
a policyholder with a homeowner policy only can be described by the event $A' \cap H$.
We are given $P[R|A \cap H'] = .4$, $P[R|A' \cap H] = .6$ and $P[R|A \cap H] = .8$.
We are also given $P[A] = .65$, $P[H] = .5$ and $P[A \cap H] = .15$.
We are asked to find $P[R]$.

We use the rule
$P[R] = P[R \cap A \cap H] + P[R \cap A' \cap H] + P[R \cap A \cap H'] + P[R \cap A' \cap H']$.

Since renewal can only occur if there is at least one policy, it follows that $P[R \cap A' \cap H'] = 0$;
in other words, of there is no auto policy (event A') and there is no homeowner policy (event H'),
then there can be no renewal. An alternative way of saying the same thing is that R is a subset
(subevent) of $A \cup H$.
(Note also that $P[A \cup H] = P[A] + P[H] - P[A \cap H] = .65 + .5 - .15 = 1$, so this also show
that R must be a subevent of $A \cup H$, and it also shows that
$P[A' \cap H'] = 1 - P[A \cup H] = 1 - 1 = 0$ so that $A' \cap H' = \phi$).
This can be illustrated in the following diagram.



We find $P[R \cap A \cap H]$, $P[R \cap A' \cap H]$ and $P[R \cap A \cap H']$ by using the rule
$P[C \cap D] = P[C|D] \cdot P[D]$:
$P[R \cap A \cap H] = P[R|A \cap H] \cdot P[A \cap H] = (.8)(.15) = .12$,
$P[R \cap A' \cap H] = P[R|A' \cap H] \cdot P[A' \cap H] = (.6)P[A' \cap H]$,
$P[R \cap A \cap H'] = P[R|A \cap H'] \cdot P[A \cap H'] = (.4)P[A \cap H']$.

In order to complete the calculations we must find $P[A' \cap H]$ and $P[A \cap H']$.
From the diagram above, or using the probability rule, we have
$P[A] = P[A \cap H] + P[A \cap H'] \rightarrow .65 = .15 + P[A \cap H'] \rightarrow P[A \cap H'] = .5$, and
$P[H] = P[A \cap H] + P[A' \cap H] \rightarrow .5 = .15 + P[A' \cap H] \rightarrow P[A' \cap H] = .35$.
Then $P[R \cap A' \cap H] = (.6)(.35) = .21$ and $P[R \cap A \cap H'] = (.4)(.5) = .2$.
Finally, $P[R] = .12 + .21 + .2 = .53$. 53% of policyholders will renew. Answer: D

23. We are given $P(\text{teen}) = .08$, $P(\text{young adult}) = .16$, $P(\text{midlife}) = .45$ and $P(\text{senior}) = .31$. We are also given the conditional probabilities
$P(\text{at least one collision}|\text{teen}) = .15$, $P(\text{at least one collision}|\text{young adult}) = .08$,
$P(\text{at least one collision}|\text{midlife}) = .04$, $P(\text{at least one collision}|\text{senior}) = .05$.
We wish to find $P(\text{young adult}|\text{at least one collision})$.
Using the definition of conditional probability, we have
$P(\text{young adult}|\text{at least one collision}) = \frac{P(\text{young adult} \cap \text{at least one collision})}{P(\text{at least one collision})}$.
We use the rule $P(A \cap B) = P(A|B) \cdot P(B)$, to get
$P(\text{young adult} \cap \text{at least one collision}) = P(\text{at least one collision} \cap \text{young adult})$
$= P(\text{at least one collision}|\text{young adult}) \cdot P(\text{young adult}) = (.08)(.16) = .0128$.

24. continued

We also have

$P(\text{at least one collision}) = P(\text{at least one collision} \cap \text{teen})$
$+ P(\text{at least one collision} \cap \text{young adult}) + P(\text{at least one collision} \cap \text{midlife})$
$+ P(\text{at least one collision} \cap \text{senior})$
$= P(\text{at least one collision}|\text{teen}) \cdot P(\text{young teen})$
$+ P(\text{at least one collision}|\text{young adult}) \cdot P(\text{young adult})$
$+ P(\text{at least one collision}|\text{midlife}) \cdot P(\text{midlife})$
$+ P(\text{at least one collision}|\text{senior}) \cdot P(\text{senior})$
$= (.15)(.08) + (.08)(.16) + (.04)(.45) + (.05)(.31) = .0583$.

Then $P(\text{young adult}|\text{at least one collision}) = \frac{.0128}{.0583} = .2196$.

These calculations can be summarized in the following table.

	T , .08 given	Y , .16 given	M , .45 given	S , .31 given
At least one collision	$P(C\|T)$ $= .15$ given	$P(C\|Y)$ $= .08$ given	$P(C\|M)$ $= .04$ given	$P(C\|S)$ $= .05$ given
	$P(C \cap T)$ $= (.15)(.08)$ $= .012$	$P(C \cap Y)$ $= (.08)(.16)$ $= .0128$	$P(C \cap M)$ $= (.04)(.45)$ $= .018$	$P(C \cap S)$ $= (.05)(.31)$ $= .0165$

$P(\text{at least one Collision}) = P(C) = P(C \cap T) + P(C \cap Y) + P(C \cap M) + P(C \cap S)$
$= .012 + .0128 + .018 + .0165 = .0593$.

$P(\text{young adult}|\text{at least one collision}) = P(Y|C) = \frac{P(Y \cap C)}{P(C)} = \frac{.0128}{.0583} = .2196$. Answer: D

24. R_1 , R_2 and R_3 denote the events that the 1st, 2nd and 3rd ball chosen is red, respectively.

$P(R_3 \cap R_2 \cap R_1) = P(R_3|R_2 \cap R_1) \cdot P(R_2 \cap R_1)$
$= P(R_3|R_2 \cap R_1) \cdot P(R_2|R_1) \cdot P(R_1) = 1 \cdot \frac{5}{11} \cdot \frac{5}{10} = \frac{5}{22}$. Answer: D

SECTION 3 - COMBINATORIAL PRINCIPLES, PERMUTATIONS AND COMBINATIONS

Factorial notation: $n!$ denotes the quantity $n(n-1)(n-2)\cdots 2\cdot 1$;
$0!$ is defined to be equal to 1.

Permutations:

(a) Given n distinct objects, the number of different ways in which the objects may be **ordered** (or **permuted**) is $n!$. For example, the set of 3 letters $\{a, b, c\}$ can be ordered in the following $3! = 6$ ways: abc, acb, bac, bca, cab, cba.

We say that we are choosing an ordered subset of size k **without replacement** from a collection of n objects if after the first object is chosen, the next object is chosen from the remaining $n-1$, the next after that from the remaining $n-2$, etc. The number of ways of doing this is $\frac{n!}{(n-k)!} = n\cdot(n-1)\cdot\cdots(n-k+1)$, and is denoted ${}_nP_k$ or $P_{n,k}$ or $P(n,k)$.
Using the set $\{a, b, c\}$ again, the number of ways of choosing an ordered subset of size 2 is $\frac{3!}{(3-2)!} = \frac{6}{1} = 6$ - ab, ac, ba, bc, ca, cb.

(b) Given n objects, of which n_1 are of Type 1, n_2 are of Type 2, . . . , and n_t are of Type t ($t \geq 1$ is an integer), and $n = n_1 + n_2 + \cdots + n_t$, the number of ways of ordering all n objects (where objects of the same Type are indistinguishable) is
$\frac{n!}{n_1!\cdot n_2!\cdots n_t!}$, which is sometimes denoted $\binom{n}{n_1\ n_2\ \cdots\ n_t}$.
For example, the set $\{a, a, b, b, c\}$ has 5 objects, 2 are a's (Type 1), 2 are b's (Type 2) and 1 is c (Type 3). According to the formula above, there should be $\frac{5!}{2!\cdot 2!\cdot 1!} = 30$ distinct ways of ordering the 5 objects. These are
$aabbc$, $aabcb$, $aacbb$, $ababc$, $abacb$, $abbac$, $abbca$, $abcab$, $abcba$, $acabb$, $acbab$, $acbba$, $bbaac$, $bbaca$, $bbcaa$, $babac$, $babca$, $baabc$, $baacb$, $bacba$, $bacab$, $bcbaa$, $bcaba$, $bcaab$, $caabb$, $cabab$, $cabba$, $cbaab$, $cbaba$, $cbbaa$

Combinations:

(a) Given n distinct objects, the number of ways of choosing a subset of size $k \leq n$ without replacement and without regard to the order in which the objects are chosen is $\frac{n!}{k! \cdot (n-k)!}$, which is usually denoted $\binom{n}{k}$ (or ${}_nC_k$, $C_{n,k}$ or $C(n,k)$) and is read "n choose k". $\binom{n}{k}$ is also called a **binomial coefficient** (and can be defined for any real number n and non-negative integer k). Note that if n is an integer and k is a non-negative integer, then $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$, and $\binom{n}{0} = \binom{n}{n} = 1$, and
$\binom{n}{1} = \binom{n}{n-1} = n$, and $\binom{n}{k} = \binom{n}{n-k}$.
Using the set $\{a, b, c\}$ again, the number of ways of choosing a subset of size 2 without replacement is $\binom{3}{2} = \frac{3!}{2! \cdot (3-2)!} = 3$; the subsets are $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. When considering <u>combinations</u>, the order of the elements in the set is irrelevant, so $\{a, b\}$ is considered the same <u>combination</u> as $\{b, a\}$. When conisdering <u>permutations</u>, the order is important, so $\{a, b\}$ is a different permutation from $\{b, a\}$.

The name "binomial coefficient" arises from the fact that these factors appear as coefficients in a "binomial expansion". For instance,
$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$= \binom{4}{0}x^4y^0 + \binom{4}{1}x^{4-1}y^1 + \binom{4}{2}x^{4-2}y^2 + \binom{4}{3}x^{4-3}y^3 + \binom{4}{4}x^{4-4}y^4$.
A general form of this expansion is found in the binomial theorem.

Binomial Theorem: In the power series expansion of $(1+t)^N$, the coefficient of t^k is $\binom{N}{k}$, so that $(1+t)^N = \sum_{k=0}^{\infty} \binom{N}{k} \cdot t^k = 1 + Nt + \frac{N(N-1)}{2}t^2 + \frac{N(N-1)(N-2)}{6}t^3 + \cdots$
If N is an integer, then the summation stops at $k = N$ and the series is valid for any real number t, but if N is not an integer, then the series is valid if $|t| < 1$.

(b) Given n objects, of which n_1 are of Type 1, n_2 are of Type 2, . . . , and n_t are of Type t ($t \geq 1$ is an integer), and $n = n_1 + n_2 + \cdots + n_t$, the number of ways of choosing a subset of size $k \leq n$ (without replacement) with k_1 objects of Type 1, k_2 objects of Type 2,..., and k_t objects of Type t, where $k = k_1 + k_2 + \cdots + k_t$ is $\binom{n_1}{k_1} \cdot \binom{n_2}{k_2} \cdots \binom{n_t}{k_t}$. A general form of the relationship is found in the multinomial theorem (on the next page).

Multinomial Theorem: In the power series expansion of $(t_1 + t_2 + \cdots + t_s)^N$ where N is a positive integer, the coefficient of $t_1^{k_1} \cdot t_2^{k_2} \cdots t_s^{k_s}$ (where $k_1 + k_2 + \cdots + k_s = N$) is $\binom{N}{k_1\ k_2\ \cdots\ k_s} = \frac{N!}{k_1! \cdot k_2! \cdots k_s!}$. For example, in the expansion of $(1 + x + y)^4$, the coefficient of xy^2 is the coefficient of $1^1 x^1 y^2$, which is $\binom{4}{1\ 1\ 2} = \frac{4!}{1! \cdot 1! \cdot 2!} = 12$.

Important Note

In questions involving coin flips or dice tossing, it is understood, unless indicated otherwise successive flips or tosses are independent of one another. Also, in making a random selection of an object from a collection of n objects, it is understood, unless otherwise indicated, that each object has the same chance of being chosen, which is $\frac{1}{n}$. In questions that arise involving choosing k objects at random from a total of n objects, or in constructing a random permutation of a collection of objects, it is understood that each of the possible choices or permutations is equally likely to occur. For instance, if a purse contains one quarter, one dime, one nickel and one penny, and two coins are chosen, there are $\binom{4}{2} = 6$ possible ways of choosing two coins without regard to order of choosing; these are Q-D , Q-N , Q-P , D-N , D-P , N-P (the choice Q-D is regarded as the same as D-Q, etc.). It would be understood that each of the 6 possible ways are equally likely, and each has (uniform) probability of $\frac{1}{6}$ of occurring; the probability space would consist of the 6 possible pairs of coins, and each sample point would have probability $\frac{1}{6}$. Then, the probability of a particular event occurring would be $\frac{j}{6}$, where j is the number of sample points in the event. If A is the event "one of the coins is either a quarter or a dime", then $P[A] = \frac{5}{6}$, since event A consists of the 5 of the sample points $\{$Q-D , Q-N , Q-P , D-N , D-P $\}$.

Example 3-1: An ordinary die and a die whose faces have $2,\ 3,\ 4,\ 6,\ 7,\ 9,$ dots are tossed independently of one another, and the total number of dots on the two dice is recorded as N. Find the probability that $N \geq 10$.

Solution: It is assumed that for each die, each face has a $\frac{1}{6}$ probability of turning up. If the number of dots turning up on die 1 and die 2 are d_1 and d_2, respectively, then the tosses that result in $N = d_1 + d_2 \geq 10$ are $(1,9)$, $(2,9)$, $(3,7)$, $(3,9)$, $(4,6)$, $(4,7)$, $(4,9)$, $(5,6)$, $(5,7)$, $(5,9)$, $(6,4)$, $(6,6)$, $(6,7)$, $(6,9)$, 14 combinations out of a total of $6 \times 6 = 36$ combinations that can possibly occur. Since each of the 36 (d_1, d_2) combinations is equally likely, the probability is $\frac{14}{36}$. □

Example 3-2: Three nickels, one dime and two quarters are in a purse. In picking three coins at one time (without replacement), what is the probability of getting a total of at least 35 cents?

Solution: In order to get at least 35 cents, at least one quarter must be chosen. The possible choices are 1Q $+$ any 2 of the non-quarters , or 2Q $+$ any 1 of the non-quarters.

The total number of ways of choosing three coins from the six coins is $\binom{6}{3} = 20$.

If we label the two quarters as Q_1 and Q_2 , then the number of ways of choosing the three coins so that only Q_1 (and not Q_2) is in the choice is $\binom{4}{2} = 6$ (this is the number of ways of choosing the other two coins from the three nickels and one dime), and therefore, the number of choices that contain only Q_2 (and not Q_1) is also 6.

The number of ways of choosing the three coins so that both Q_1 and Q_2 are in the choice is 4 (this is the number of ways of choosing the other coin from the three nickels and one dime). Thus, the total number of choices for which at least one of the three coins chosen is a quarter is 16. The probability in question is $\frac{16}{20}$.

An alternative approach is to find the number of three coin choices that do not contain any quarters is $\binom{4}{3} = 4$ (the number of ways of choosing the three coins from the 4 non-quarters), so that number of choices that contain at least one quarter is $20 - 4 = 16$. $\square$

Example 3-3: A and B draw coins in turn without replacement from a bag containing 3 dimes and 4 nickels. A draws first. It is known that A drew the first dime. Find the probability that A drew it on the first draw.

Solution: $P[A \text{ draws dime on first draw} | A \text{ draws first dime}] = \frac{P[A \text{ draws dime on first draw}]}{P[A \text{ draws first dime}]}$

$P[A \text{ draws dime on first draw}] = \frac{3}{7}$. Since there only 3 dimes, in order for A to draw the first dime, this must happen on A's first, second or third draw. Thus,

$P[A \text{ draws first dime}] = P[A \text{ draws dime on first draw}]$

$\quad + P[A \text{ draws first dime on second draw}] + P[A \text{ draws first dime on third draw}]$.

$P[A \text{ draws dime on second draw}] = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} = \frac{6}{35}$, since A's first draw is one of the four non-dimes, and B's first draw is one of the three remaining non-dimes after A's draw, and A's second draw is one of the three dimes of the five remaining coins. In a similar way,

$P[A \text{ draws first dime on third draw}] = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot 1 = \frac{1}{35}$.

Then, $P[A \text{ draws first dime}] = \frac{3}{7} + \frac{6}{35} + \frac{1}{35} = \frac{22}{35}$, and

$P[A \text{ draws dime on first draw} | A \text{ draws first dime}] = \frac{3/7}{22/35} = \frac{15}{22}$. $\square$

Example 3-4: Three people, X, Y and Z, in order, roll an ordinary die. The first one to roll an even number wins. The game continues until someone rolls an even number. Find the probability that X will win.

Solution: Since X rolls first, fourth, seventh, etc. until the game ends, the probability that X will win is the probability that in throwing a die, the first even number will occur on the 1st, or 4th, or 7th, or . . . throw. The probability that the first even number occurs on the n-th throw is $(\frac{1}{2})^{n-1}(\frac{1}{2}) = \frac{1}{2^n}$. This is true since it requires $n-1$ odd throws followed by an even throw. Assuming independence of successive throws, with $A_i =$ "throw i is even", the probability that the first even throw occurs on throw n is

$$P[A_1' \cap A_2' \cap \cdots \cap A_{n-1}' \cap A_n] = P[A_1'] \cdot P[A_2'] \cdots P[A_{n-1}'] \cdot P[A_n]$$
$$= (\tfrac{1}{2})^{n-1}(\tfrac{1}{2}) = \tfrac{1}{2^n} .$$

Thus, P[first even throw is on 1st, or 4th, or 7th, or ...]

$$= \tfrac{1}{2} + \tfrac{1}{2^4} + \tfrac{1}{2^7} + \cdots = \tfrac{1}{2}(1 + \tfrac{1}{8} + \tfrac{1}{8^2} + \cdots) = \tfrac{4}{7} . \qquad \square$$

Example 3-5: Urn I contains 7 red and 3 black balls, and Urn II contains 4 red and 5 black balls. After a randomly selected ball is transferred from Urn I to Urn II, 2 balls are randomly drawn from Urn II without replacement. Find the probability that both balls drawn from Urn II are red.

Solution: Define the following events:

R_1: the ball transferred from Urn I to Urn II is red

B_1: the ball transferred from Urn I to Urn II is black

R_2: two red balls are selected from Urn II after the transfer from Urn I to Urn II .

Since R_1 and B_1 are mutually exclusive,

$$P[R_2] = P[R_2 \cap (R_1 \cup B_1)] = P[R_2 \cap R_1] + P[R_2 \cap B_1]$$
$$= P[R_2|R_1] \cdot P[R_1] + P[R_2|B_1] \cdot P[B_1] = \frac{\binom{5}{2}}{\binom{10}{2}} \cdot \frac{7}{10} + \frac{\binom{4}{2}}{\binom{10}{2}} \cdot \frac{3}{10} = \frac{44}{225} . \ \square$$

Example 3-6: A calculator has a random number generator button which, when pressed, displays a random digit $0, 1, ..., 9$. The button is pressed four times. Assuming that the numbers generated are independent of one another, find the probability of obtaining one "0", one "5", and two "9"'s in any order.

Solution: There are $10^4 = 10,000$ four-digit orderings that can arise, from 0-0-0-0 to 9-9-9-9 . From the notes above on permutations, if we have four digits, with one "0", one "5" and two "9"'s, the number of orderings is $\frac{4!}{1!\cdot 1!\cdot 2!} = 12$.

The probability in question is then $\frac{12}{10,000}$. $\square$

Example 3-7: In Canada's national 6-49 lottery, a ticket has 6 numbers each from 1 to 49, with no repeats. Find the probability of matching all 6 numbers if the numbers are all randomly chosen. The ticket cost is $2. If you match exactly 3 of the 6 numbers chosen, you win $10. Find the probability of winning $10.

Solution: There are $\binom{49}{6} = \frac{49!}{6! \cdot 43!} = \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 13,983,816$ possible combinations of 6 numbers from 1 to 49 (we are choosing 6 numbers from 1 to 49 without replacement), so the probability of matching all 6 number is $\frac{1}{13,983,816} = .0000000715112$ (about 1 in 14 million).

Suppose you have bought a lottery ticket. There are $\binom{6}{3} = 20$ ways of picking 3 numbers from the 6 numbers on your ticket. Suppose we look at one of those subsets of 3 numbers from your ticket. In order for the winning ticket number to match exactly those 3 of your 6 numbers, the other 3 winning ticket numbers must come from the 43 numbers between 1 and 49 that are not numbers on your ticket. There are $\binom{43}{3} = \frac{43 \times 42 \times 41}{3 \times 2 \times 1} = 12,341$ ways of doing that, and since there are 20 subsets of 3 numbers on your ticket, there are $20 \times 12,341 = 246,820$ ways in which the winning ticket numbers match exactly 3 of your ticket numbers. Since there are a total of 13,983,816 ways of picking 6 out of 49 numbers, your chance of matching exactly 3 of the winning numbers is $\frac{246,820}{13,983,816} = .01765$ (about $\frac{1}{57}$). So you have about a one in 57 chance of turning $2 into $10. □

Example 3-8: In a poker hand of 5 cards from an ordinary deck of 52 cards, a "full house" is a hand that consist of 3 of one rank and 2 of another rank (such as 3 kings and 2 5's). If 5 cards are dealt at random from an ordinary deck, find the probability of getting a full house.

Solution: There are $\binom{52}{5} = 2,598,960$ possible hands that can be dealt from the 52 cards. There are 13 ranks from deuce (2) to ace, and there are $\binom{13}{2} = 78$ pairs of ranks. For each pair of ranks, there are $\binom{4}{3} \times \binom{4}{2} = 24$ combinations consisting of 3 cards of the first rank and 2 cards of the second rank, and there are 24 combinations consisting of 2 cards of the first rank and 3 cards of the second rank, for a total of 48 possible full house hands based on those two ranks. Since there are 78 pairs of ranks, there are $78 \times 48 = 3744$ distinct poker hands that are a full house. The probability of being dealt a full house is $\frac{3744}{2,598,960} = .00144058$ (a little better chance than 1 in 700). □

PROBLEM SET 3

Combinatorial Principles

1. A class contains 8 boys and 7 girls. The teacher selects 3 of the children at random and without replacement. Calculate the probability that number of boys selected exceeds the number of girls selected.

A) $\frac{512}{3375}$ B) $\frac{28}{65}$ C) $\frac{8}{15}$ D) $\frac{1856}{3375}$ E) $\frac{36}{65}$

2. There are 97 men and 3 women in an organization. A committee of 5 people is chosen at random, and one these 5 is randomly designated as chairperson. What is the probability that the committee includes all 3 women and has one of the women as chairperson?

A) $\frac{3(4!\,97!)}{2(100!)}$ B) $\frac{5!\,97!}{2(100!)}$ C) $\frac{3(5!\,97!)}{2(100!)}$ D) $\frac{3!\,5!\,97!}{100!}$ E) $\frac{3^3\,97^2}{100^5}$

3. A box contains 4 red balls and 6 white balls. A sample of size 3 is drawn without replacement from the box. What is the probability of obtaining 1 red ball and 2 white balls, given that at least 2 of the balls in the sample are white?

A) $\frac{1}{2}$ B) $\frac{2}{3}$ C) $\frac{3}{4}$ D) $\frac{9}{11}$ E) $\frac{54}{55}$

4. When sent a questionnaire, 50% of the recipients respond immediately. Of those who do not respond immediately, 40% respond when sent a follow-up letter. If the questionnaire is sent to 4 persons and a follow-up letter is sent to any of the 4 who do not respond immediately, what is the probability that at least 3 never respond?

A) $(.3)^4 + 4(.3)^3(.7)$ B) $4(.3)^3(.7)$ C) $(.1)^4 + 4(.1)^3(.9)$

D) $.4(.3)(.7)^3 + (.7)^4$ E) $(.9)^4 + 4(.9)^3(.1)$

5. A box contains 35 gems, of which 10 are real diamonds and 25 are fake diamonds. Gems are randomly taken out of the box, one at a time without replacement. What is the probability that exactly 2 fakes are selected before the second real diamond is selected?

A) $\frac{225}{5236}$ B) $\frac{675}{5236}$ C) $\frac{\binom{25}{2}\binom{10}{2}}{\binom{35}{4}}$ D) $\binom{3}{2}(\frac{10}{35})^2(\frac{25}{35})^2$ E) $\binom{4}{2}(\frac{10}{35})^2(\frac{25}{35})^2$

6. (SOA) An insurance company determines that N the number of claims received in a week, is a random variable with $P[N=n] = \frac{1}{2^{n+1}}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly seven claims will be received during a given two-week period.

A) $\dfrac{1}{256}$ B) $\dfrac{1}{128}$ C) $\dfrac{7}{512}$ D) $\dfrac{1}{64}$ E) $\dfrac{1}{32}$

7. Three boxes are numbered 1, 2 and 3. For $k = 1, 2, 3$, box k contains k blue marbles and $5 - k$ red marbles. In a two-step experiment, a box is selected and 2 marbles are drawn from it without replacement. If the probability of selecting box k is proportional to k, what is the probability that the two marbles drawn have different colors?

A) $\frac{17}{60}$ B) $\frac{34}{75}$ C) $\frac{1}{2}$ D) $\frac{8}{15}$ E) $\frac{17}{30}$

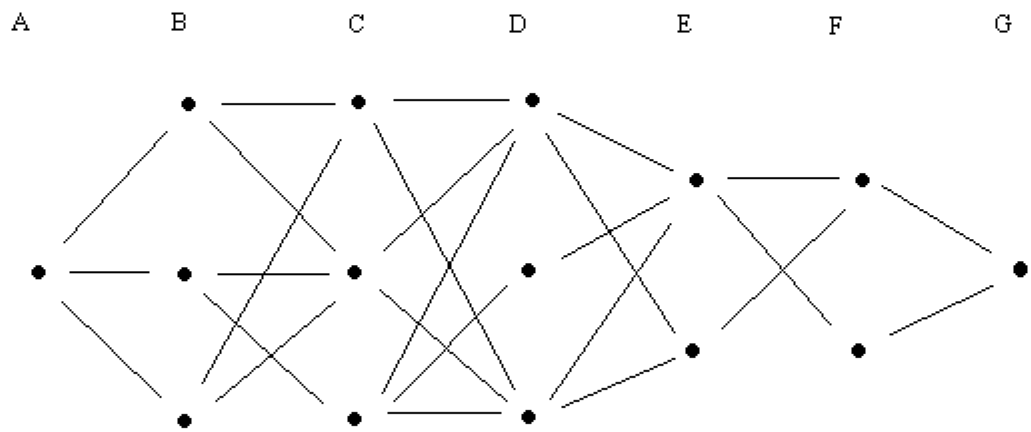
8. In Canada's national 6-49 lottery, a ticket has 6 numbers each from 1 to 49, with no repeats. Find the probability of matching exactly 4 of the 6 winning numbers if the winning numbers are all randomly chosen.

A) .00095 B) .00097 C) .00099 D) .00101 E) .00103

9. A number X is chosen at random from the series 2,5,8, ... and another number Y is chosen at random from the series 3,7,11, ... Each series has 100 terms. Find $P[X = Y]$.

A) $.0025$ B) $.0023$ C) $.0030$ D) $.0021$ E) $.0033$

10. In the following diagram, A, B,... refer to successive states through which a traveler must pass in order to get from A to G, moving from left to right. A path consists of a sequence of line segments from one state to the next. A path must always move to the next state until reaching state G. Determine the number of possible paths from A to G.



A) 30 B) 32 C) 34 D) 36 E) 38

11. A store has 80 modems in its inventory, 30 coming from Source A and the remainder from Source B. Of the modems from Source A, 20% are defective. Of the modems from Source B, 8% are defective. Calculate the probability that exactly two out of a random sample of five modems from the store's inventory are defective.

A) 0.010 B) 0.078 C) 0.102 D) 0.105 E) 0.125

PROBLEM SET 3 SOLUTIONS

1. There are $\binom{15}{3} = \frac{15!}{3!\cdot 12!} = \frac{15\times14\times13}{3\times2\times1} = 455$ ways of selecting 3 children from a group of 15 without replacement. The number of boys selected exceeds the number of girls selected if either (i) 3 boys and 0 girls are selected, or (ii) 2 boys and 1 girl are selected .
There are $\binom{8}{3}\cdot\binom{7}{0} = \frac{8!}{3!\times5!}\cdot\frac{7!}{0!\times7!} = 56$ ways in which selection (i) can occur , and there are $\binom{8}{2}\cdot\binom{7}{1} = \frac{8!}{2!\times6!}\cdot\frac{7!}{1!\times6!} = 196$ ways in which selection (ii) can occur.
The probability of either (i) or (ii) occurring is $\frac{56+196}{455} = \frac{36}{65}$. Answer: E

2. Let A be the event that the committee has a woman as chairperson, and let B be the event that the committee includes all 3 women. Then, $P[A\cap B] = P[A|B]\cdot P[B]$.
The conditional probability $P[A|B]$ is equal to $\frac{3}{5}$ since the chairperson is chosen at random from the 5 committee members, and, given B, 3 of the committee members are women. There are $\binom{100}{5}$ ways of choosing a 5-member committee from the group of 100. Out of all 5-member committees, there are $\binom{97}{2}$ committees that include all 3 women (i.e., 2 men are chosen from the 97 men). Thus,
$P[B] = \frac{\binom{97}{2}}{\binom{100}{5}} = (\frac{97!}{2!\,95!})/(\frac{100!}{5!\,95!}) = \frac{5!\,97!}{2!\,100!}$,
and $P[A\cap B] = \frac{5!\,97!}{2!\,100!}\cdot\frac{3}{5} = \frac{3\cdot4!\,97!}{2!\,100!}$. Answer: A

3. $P[R, 2W|\text{at least } 2W] = \frac{P[R,2W]}{P[\text{at least } 2W]} = \frac{\binom{4}{1}\binom{6}{2}}{\binom{4}{1}\binom{6}{2}+\binom{4}{0}\binom{6}{3}} = \frac{3}{4}$. Answer: C

4. The probability that an individual will not respond to either the questionnaire or the follow-up letter is $(.5)(.6) = .3$. The probability that all 4 will not respond to either the questionnaire or the follow-up letter is $(.3)^4$.
$P[\text{3 don't respond}] = P[\text{1 response on 1st round, no additional responses on 2nd round}]$
$+ P[\text{no responses on 1st round, 1 response on 2nd round}]$
$= 4[(.5)^4(.6)^3] + 4[(.5)^4(.6)^3(.4)] = 4(.3)^3(.7)$. Then,
$P[\text{at least 3 don't respond}] = (.3)^4 + 4(.3)^3(.7)$. Answer: A

5. Exactly 2 fakes must be picked in the first 3 picks and the second real diamond must occur on the 4th pick. The possible ways in which this may occur are (F-fake, R-real)
$FFRR$ (prob. $\frac{25\times24\times10\times9}{35\times34\times33\times32}$) , $FRFR$ (prob. $\frac{25\times10\times24\times9}{35\times34\times33\times32}$) , $RFFR$ (prob. $\frac{10\times25\times24\times9}{35\times34\times33\times32}$) .
The overall probability is $3\cdot\frac{25\times24\times10\times9}{35\times34\times33\times32}=\frac{675}{5236}$. Answer: B

6. The following combinations result in a total of 7 claims in a 2 week period:

Week 1 , Prob.	Week 2 - Prob.	Combined Probability
0 , $\frac{1}{2}$	7 , $\frac{1}{2^8}$	$\frac{1}{2}\cdot\frac{1}{2^8}=\frac{1}{2^9}$
1 , $\frac{1}{2^2}$	6 , $\frac{1}{2^7}$	$\frac{1}{2^2}\cdot\frac{1}{2^7}=\frac{1}{2^9}$
	$\vdots$	
7 , $\frac{1}{2^8}$	1 , $\frac{1}{2}$	$\frac{1}{2^8}\cdot\frac{1}{2}=\frac{1}{2^9}$

The total probability of exactly 7 claims in a two week period is $8\cdot\frac{1}{2^9}=\frac{1}{64}$.
Answer: D

7. If the probability of selecting box 1 is p, then $p+2p+3p=1\rightarrow p=\frac{1}{6}$.
Then the probability in question is
$P[\text{2 different colors}|\text{box 1 selected}]\cdot P[\text{box 1 selected}]$
$+P[\text{2 different colors}|\text{box 2 selected}]\cdot P[\text{box 2 selected}]$
$+P[\text{2 different colors}|\text{box 3 selected}]\cdot P[\text{box 3 selected}]$
$=\frac{1\cdot4}{\binom{5}{2}}\cdot\frac{1}{6}+\frac{2\cdot3}{\binom{5}{2}}\cdot\frac{2}{6}+\frac{3\cdot2}{\binom{5}{2}}\cdot\frac{3}{6}=\frac{34}{10\times6}=\frac{17}{30}$. Answer: E

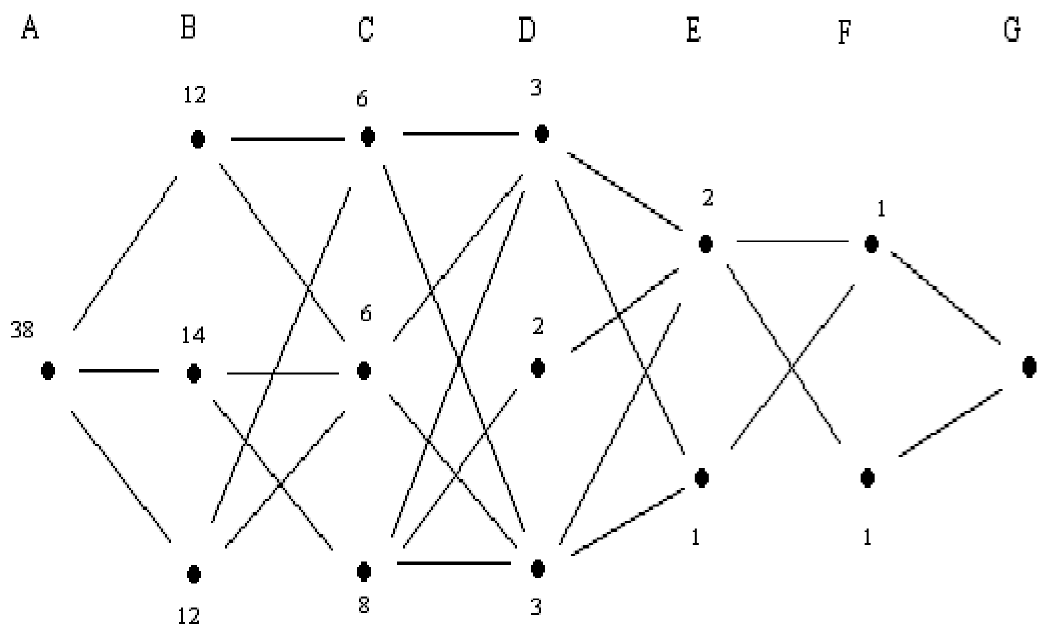
8. Suppose you have bought a lottery ticket. There are $\binom{6}{4}=15$ ways of picking 4 numbers from the 6 numbers on your ticket. Suppose we look at one of those subsets of 4 numbers from your ticket. In order for the winning ticket number to match exactly those 4 of your 6 numbers, the other 2 winning ticket numbers must come from the 43 numbers between 1 and 49 that are not numbers on your ticket. There are $\binom{43}{2}=\frac{43\times42}{2\times1}=903$ ways of doing that, and since there are 15 subsets of 4 numbers on your ticket, there are $15\times903=13,545$ ways in which the winning ticket numbers match exactly 3 of your ticket numbers. Since there are a total of 13,983,816 ways of picking 6 out of 49 numbers, your chance of matching exactly 4 of the winning numbers is $\frac{13{,}545}{13{,}983{,}816}=.00096862$. Answer: B

9. There are $100^2 = 10,000$ equally likely possible choices of (X, Y). Of these choices, the pairs that equal X and Y are $(11,11)$-1 , $(23,23)$-2 , $(35,35)$-3 , $\ldots$, $(299,299)$-25 (they are of the form $(12k-1, 12k-1)$. The probability is $\frac{25}{10{,}000}$. Answer: A

10. This problem can be solved by a "backward induction" on the diagram. At each node we find the number of paths from that node to state G. We first apply backward induction to the two nodes in state F. At the upper node there is 1 path to G and at the lower node there is 1 path to G.

Then we look at the notes in state E and look at the next segments that can be taken. We see that there are $1+1=2$ possible paths from the upper node at F to G and 1 possible path from the lower node.

We continue in this way at state D. From the top node of state D there are $2+1=3$ paths to state G, from the middle node of state D there are paths, and from the lower node there are 3 paths. Continuing in this way back to state A, there will be a total of 38 paths from state A. The diagram below indicates the number of paths to state G from each node.



11. The probability is $\dfrac{\text{number of ways of choosing 2 defective and 3 non-defective}}{\text{number of ways of choosing 5 modems}}$.

There are a total of $.2 \times 30 + .08 \times 50 = 10$ defective modems in total.

The number of ways of choosing 5 modems at random from the 80 modems is $\binom{80}{5}$.

The number of ways of choosing 2 defective and 3 non-defective is $\binom{10}{2} \times \binom{70}{3}$,

since there are 10 defective and 70 non-defective. the probability is $\frac{\binom{80}{5}}{\binom{10}{2}\times\binom{70}{3}} = .102$.

Answer: C

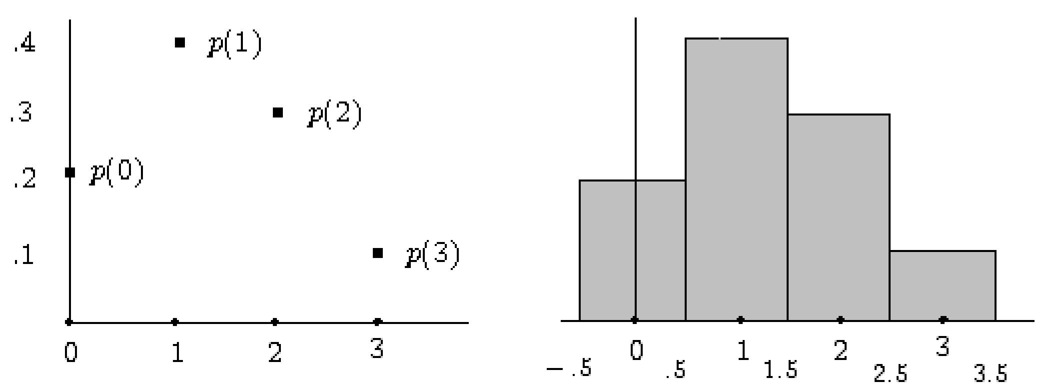
SECTION 4 - RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

Random variable X: The formal definition of a random variable is that it is a function on a probability space S. This function assigns a real number $X(s)$ to each sample point $s \in S$. The less formal, but more typical way to describe a random variable is to describe the possible values that can occur and the probabilities of those values occurring. It is usually implicitly understood that there is some underlying random experiment whose outcome determines the value of X. For example, suppose that a gamble based on the outcome of the toss of a die pays \$10 if an even number is tossed, and pays \$20 if an odd number is tossed. If the die is a fair die, then there is probability of $\frac{1}{2}$ of tossing an even number and the same probability of $\frac{1}{2}$ of tossing an odd number. If the gamble had been described in terms of the flip of a fair coin with a payoff of \$10 if a head is flipped and a payoff of \$20 if a tail is flipped, then the probabilities of \$10 and \$20 are still each $\frac{1}{2}$. The crucial components of the description of this random variable are the possible outcomes (\$10 and \$20) and their probabilities (both $\frac{1}{2}$), and the actual experiment (even-or-odd die toss, or head-or-tail coin flip) leading to the outcome is not particularly significant, except that it tells us the probabilities of the possible outcomes. It would be possible to define this random variable without any reference to die toss or coin flip. We would say that the random variable X takes on either the value 10 or the value 20 and the probability is $\frac{1}{2}$ for each of these outcomes. That completely describes the random variable.

Discrete random variable: The random variable X is discrete and is said to have a **discrete distribution** if it can take on values only from a finite or countable infinite sequence (usually the integers or some subset of the integers). As an example, consider the following two random variables related to successive tosses of a coin:
$X = 1$ if the first head occurs on an even-numbered toss, $X = 0$ if the first head occurs on an odd-numbered toss;
$Y = n$, where n is the number of the toss on which the first head occurs.
Both X and Y are discrete random variables, where X can take on only the values 0 or 1, and Y can take on any positive integer value. The "probability space" or set of possible outcomes for X is $\{0, 1\}$, and the probability space for Y is $\{1, 2, 3, 4, \ldots\}$.

Probability function of a discrete random variable: The probability function (pf) of a discrete random variable is usually denoted $p(x)$, $f(x)$, $f_X(x)$ or p_x, and is equal to the probability that the value x occurs. This probability is sometimes denoted $P[X = x]$.
The probability function must satisfy
(i) $\boldsymbol{0 \leq p(x) \leq 1}$ for all x , and (ii) $\boldsymbol{\sum_x p(x) = 1}$.
Given a set A of real numbers (possible outcomes of X), the probability that X is one of the values in A is $\boldsymbol{P[X \in A] = \sum_{x \in A} p(x) = P[A]}$.

Probability plot and histogram: The probability function of a discrete random variable can be described in a probability plot or in a histogram. Suppose that X has the probability function $p(0) = .2$, $p(1) = .4$, $p(2) = .3$ and $p(3) = .1$ (note that the required conditions (i) and (ii) listed above are satisfied for this random variable X). The graph below on the left is the probability plot, and the graph at the right is the histogram for this distribution. For an integer valued random variable, a histogram is a bar graph. For each integer k, the base of the bar is from $k - \frac{1}{2}$ to $k + \frac{1}{2}$, and the height of the bar is the probability $p(k)$ at the point $X = k$. Histograms are also used to graph distributions that are described in interval form.

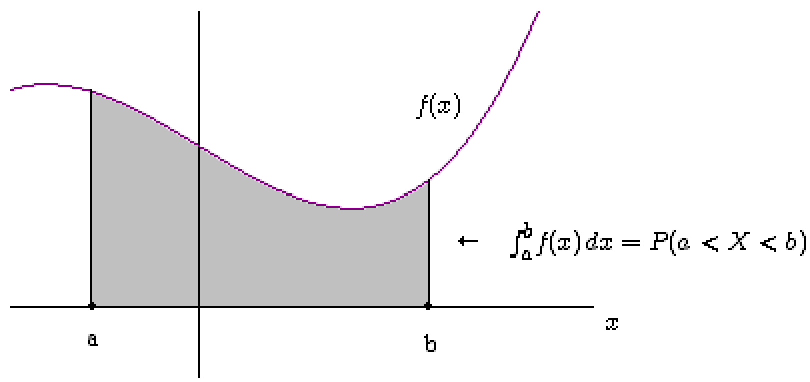


We can find various probabilities for this random variable X. For example
$P[X \text{ is odd}] = P[X = 1, 3] = P[X = 1] + P[X = 3] = .4 + .1 = .5$ and
$P[X \leq 2] = P[X = 0, 1, 2] = P[X = 0] + P[X = 1] + P[X = 2] = .9$.
We can find conditional probabilities also. For example,
$P[X \geq 1 | X \leq 2] = \frac{P[(X \geq 1) \cap (X \leq 2)]}{P[X \leq 2]} = \frac{P[X=1,2]}{P[X=0,1,2]} = \frac{.7}{.9} = \frac{7}{9}$.

The probability at a point of a discrete random variable is sometimes called a **probability mass**. X above has a probability mass of .2 at $X = 0$, etc.

Continuous random variable: A continuous random variable usually can assume numerical values from an interval of real numbers, or perhaps the whole set of real numbers. The probability space for the random variable is this interval. As an example, the length of time between successive streetcar arrivals at a particular (in service) streetcar stop could be regarded as a continuous random variable (assuming that time measurement can be made perfectly accurate).

Probability density function: A continuous random variable X has a **probability density function** (**pdf**) usually denoted $f(x)$ or $f_X(x)$, which is a continuous function except possibly at a finite number of points. Probabilities related to X are found by integrating the density function over an interval. The probability that X is in the interval (a,b) is
$\boldsymbol{P[X \in (a,b)] = P[a < X < b]}$ **, which is defined to be equal to** $\boldsymbol{\int_a^b f(x)\,dx}$ (probability on an interval for a continuous random variable is the area under the density curve on that interval) . Note that for a continuous random variable $P[X=c]=0$ for any individual point c, since $P[X=c]=\int_c^c f(x)\,dx = 0$. For a continuous random variable there can only be probability over an interval, not at a single point.



Note that for a continuous random variable X, the following are all equal:

$P(a < X < b)$, $P(a < X \le b)$, $P(a \le X < b)$, $P(a \le X \le b)$.

This is true since the probability at a single point is 0, so it doesn't matter whether or not we include the endpoints a and b or not.

For a discrete random variable, probabilities are calculated as the sum of probabilities at individual points, so is does matter whether not an endpoint of an interval is included. For instance, for a fair die toss for which X denotes the outcome of the toss,, $P(X \le 3) = \frac{3}{6}$, but $P(X < 3) = \frac{2}{6}$.

The pdf $\boldsymbol{f(x)}$ must satisfy (i) $\boldsymbol{f(x) \ge 0}$ for all $\boldsymbol{x}$, and (ii) $\boldsymbol{\int_{-\infty}^{\infty} f(x)\,dx = 1}$.

Condition (ii) can be restated by saying that the integral of $f(x)$ over the probability space must be 1. Often, the region of non-zero density (the probability space of X) is a finite interval, and $f(x) = 0$ outside that interval. If $f(x)$ is continuous except at a finite number of points, then probabilities are defined and calculated as if $f(x)$ was continuous everywhere (the discontinuities are ignored).

Example 4-1: Suppose that X has density function $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ elsewhere} \end{cases}$.

(i) Show that f satisfies the requirements for being a density function.

(ii) Find $P[.2 < X < .5]$.

(iii) Find $P[.2 < X < .5|X > .25]$.

Solution: (i) f satisfies the requirements for a density function, since $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 2x\,dx = 1$.

(ii) $P[.2 < X < .5] = \int_{.2}^{.5} 2x\,dx = x^2\Big|_{.2}^{.5} = .21$. Note that this is equal to $P(.2 \leq X \leq .5)$.

(iii) $P[.2 < X < .5|X > .25] = \frac{P[(.2<X<.5)\cap(X>.25)]}{P[X>.25]}$

$$= \frac{P[.25<X<.5]}{P[X>.25]} = \frac{\int_{.25}^{.5} 2x\,dx}{\int_{.25}^{1} 2x\,dx} = \frac{.1875}{.9375} = .2 \,. \quad \square$$

Example 4-2: Y has the pdf $f(y) = \frac{20{,}000}{(100+y)^3}$ for $y > 0$.

(i) Show that f satisfies the requirements for being a density function.

(ii) Find $P(Y > t)$ if $t > 0$.

(iii) Find $P(Y > t + y|Y > t)$ if $t > 0$.

Solution: (i) $\int_0^{\infty} \frac{20{,}000}{(100+y)^3}\,dy = \frac{20{,}000(100+y)^{-2}}{-2}\Big|_{y=0}^{y=\infty} = -0 + \frac{20{,}000}{2(100^2)} = 1$,

and $f(y) \geq 0$ for all y.

(ii) $P(Y > t) = \int_t^{\infty} \frac{20{,}000}{(100+y)^3}\,dy = \frac{20{,}000(100+y)^{-2}}{-2}\Big|_{y=t}^{y=\infty} = -0 + \frac{20{,}000}{2(100+t)^2} = (\frac{100}{100+t})^2$

(iii) $P(Y > t + y|Y > t) = \frac{P(Y>t+y)}{P(Y>t)} = \frac{10{,}000}{(100+t+y)^2} \Big/ \frac{10{,}000}{(100+t)^2} = (\frac{100+t}{100+t+y})^2$. $\square$

Mixed distribution: A random variable may have some points with non-zero probability mass combined with a continuous pdf on one or more intervals. Such a random variable is said to have a **mixed distribution**. The probability space is a combination of the set of discrete points of probability for the discrete part of the random variable along with the intervals of density for the continuous part. The sum of the probabilities at the discrete points of probability plus the integral of the density function on the continuous region for X is the total probability for X, and this must be 1. For example, suppose that X has probability of $.5$ at $X = 0$, and X is a continuous random variable on the interval $(0, 1)$ with density function $f(x) = x$ for $0 < x < 1$, and X has no density or probability elsewhere. This satisfies the requirements for a random variable since the total probability over the probability space is
$P[X = 0] + \int_0^1 f(x)\,dx = .5 + \int_0^1 x\,dx = .5 + .5 = 1$.
Then, $P[0 < X < .5] = \int_0^{.5} x\,dx = .125,$ and
$P[0 \leq X < .5] = P[X = 0] + P[0 < X < .5] = .5 + .125 = .625$
(since $X = 0$ is a discrete point of probability, we must include that probability in any interval that includes $X = 0$).

Cumulative distribution function (and survival function): Given a random variable X, the cumulative distribution function of X (also called the **distribution function**, or **cdf**) is $\boldsymbol{F(x) = P[X \leq x]}$ (also denoted $F_X(x)$). $F(x)$ is the cumulative probability to the left of (and including) the point x. The **survival function** is the complement of the distribution function, $\boldsymbol{S(x) = 1 - F(x) = P[X > x]}$. The event $X > x$ is referred to as a "tail" (or right tail) of the distribution.

For a discrete random variable with probability function $p(x)$, $\boldsymbol{F(x) = \sum_{w \leq x} p(w)}$, and in this case $F(x)$ is a "step function", it has a jump (or step increase) at each point with non-zero probability, while remaining constant until the next jump.

If X has a continuous distribution with density function $f(x)$, then $\boldsymbol{F(x) = \int_{-\infty}^{x} f(t)\,dt}$ and $F(x)$ is a continuous, differentiable, non-decreasing function such that
$\boldsymbol{\frac{d}{dx} F(x) = F'(x) = -S'(x) = f(x)}$. If X has a mixed distribution, then $F(x)$ is continuous except at the points of non-zero probability mass, where $F(x)$ will have a jump.

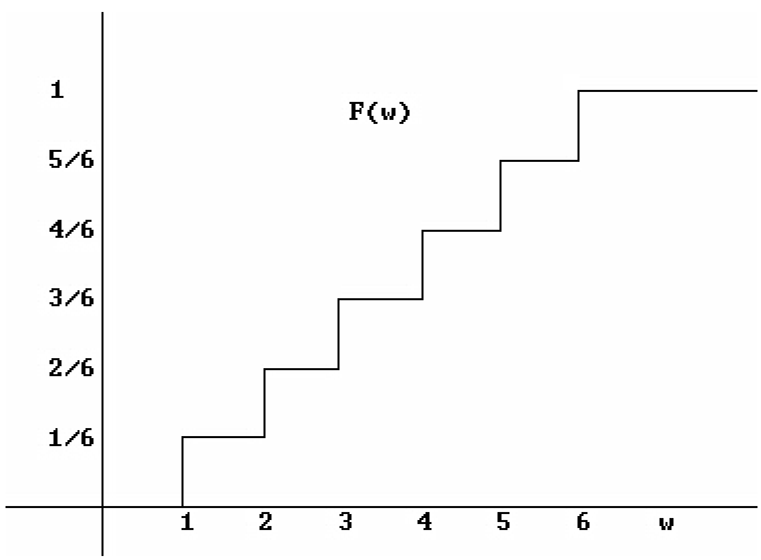
For any cdf $\boldsymbol{P[a < X \leq b] = F(b) - F(a)}$, $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$.

Examples of probability, density and distribution functions:

Example 4-3: Discrete Random Variable on a Finite Number of Points (finite support)
$W =$ number turning up when tossing one fair die, so W has probability function
$p(w) = p_W(w) = P[W = w] = \frac{1}{6}$ for $w = 1, 2, 3, 4, 5, 6$.

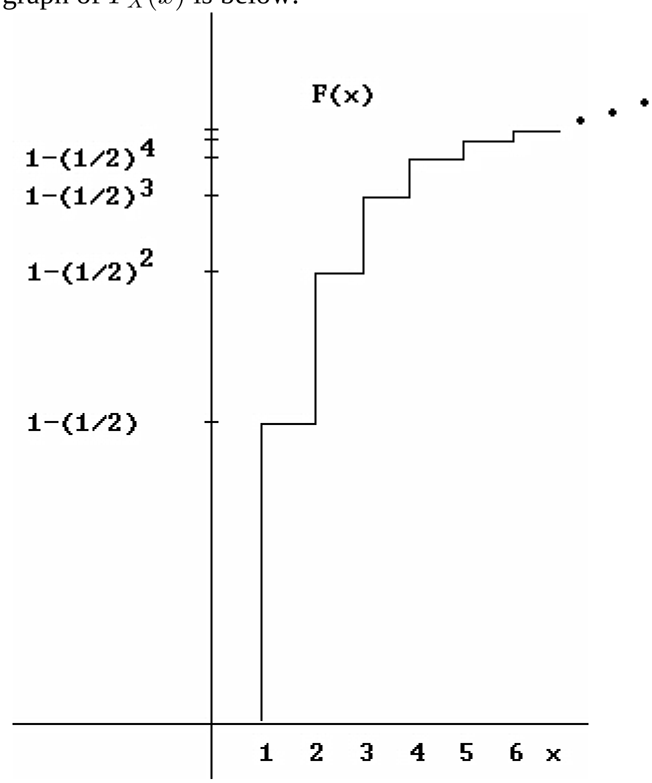
$$F_W(w) = P[W \leq w] = \begin{cases} 0 \text{ if } & w < 1 \\ 1/6 \text{ if } & 1 \leq w < 2 \\ 2/6 \text{ if } & 2 \leq w < 3 \\ 3/6 \text{ if } & 3 \leq w < 4 \\ 4/6 \text{ if } & 4 \leq w < 5 \\ 5/6 \text{ if } & 5 \leq w < 6 \\ 1 \text{ if } & w \geq 6 \end{cases}$$

The graph of the cdf (cumulative distribution function) is a step-function that increases at each point of probability by the amount of probability at that point (all 6 points have probability $\frac{1}{6}$ in this example). Since the support of W is finite (the "support" is the region of non-zero probability; for W that is the set of integers from 1 to 6), $F_W(w)$ reaches 1 at the largest point $W = 6$ (and stays at 1 for all $w \geq 6$).



□

Example 4-4: Discrete Random Variable on an Infinite Number of Points (infinite support)
$X =$ number of successive independent tosses of a fair coin until the first head turns up.
X can be any integer ≥ 1, and the probability function of X is $p_X(x) = \frac{1}{2^x}$, since
$P[\text{first head on toss } x] = P[(\text{toss } 1,T) \cap (\text{toss } 2,T) \cap \cdots \cap (\text{toss } x-1,T) \cap (\text{toss } x,H)]$
$= P[\text{toss } 1,T] \cdot P[\text{toss } 2,T] \cdots P[\text{toss } x-1,T] \cdot P[\text{toss } x,H] = (\frac{1}{2})(\frac{1}{2})\cdots(\frac{1}{2})(\frac{1}{2}) = \frac{1}{2^x}$.
The cdf is $F_X(x) = P[X \leq x] = P[X=1] + P[X=2] + \cdots + P[X=x] = \sum_{k=1}^{x} \frac{1}{2^k} = 1 - \frac{1}{2^x}$
for $x = 1, 2, 3, \ldots$. The graph of this cdf is a step-function that increases at each point of probability by the amount of probability at that point. Since the support of X is infinite (the support in this case is the set of integers ≥ 1) $F_X(x)$ never reaches 1, but approaches 1 as a limit as $x \to \infty$. The graph of $F_X(x)$ is below.



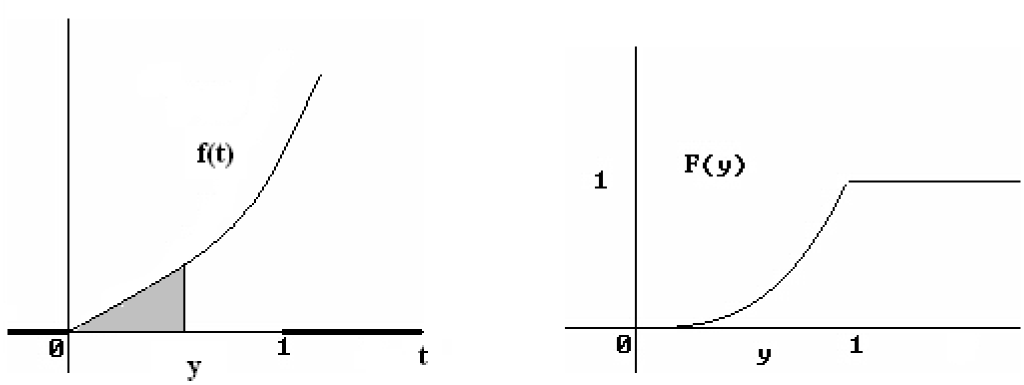
The probability that the first head occurs on an even numbered toss is
$P(X \text{ is even}) = P(X = 2, 4, 6, \ldots) = P(X=2) + P(X=4) + P(X=6) + \cdots$
$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1}{2^2} \times [1 + \frac{1}{2^2} + (\frac{1}{2^2})^2 + \cdots] = \frac{1}{2^2} \Big/ [1 - \frac{1}{2^2}] = \frac{1}{3} .$$
The probability that the first head occurs on, or after the k-th toss ($k = 1, 2, \ldots$) is
$P(X \geq k) = P(X=k) + P(X=k+1) + P(X=k+2) + \cdots$
$= \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots = \frac{1}{2^k} \times [1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots] = \frac{1}{2^k} \times 2 = \frac{1}{2^{k-1}}$,
for $k = 1, 2, 3, \ldots$ □

The typical behavior of the cdf $F(x)$ is to tend to increase toward 1 as x increases. Depending on the nature of the random variable, $F(x)$ may actually reach 1 at some point, as in Example 4-3, or $F(x)$ might approach 1 as a limit, as in Example 4-4. For a continuous random variable, $F(x)$ has similar increasing behavior, but will be increasing continuously rather than in the series of steps we have seen for a discrete random variable. $F(x)$ will never decrease, but it may remain "flat" for a while, as can be seen in the previous two examples.

Example 4-5: Continuous Random Variable on a Finite Interval

Y is a continuous random variable on the interval $(0,1)$ with density function $f_Y(y) = \begin{cases} 3y^2 \text{ for } 0<y<1 \\ \text{0, elsewhere} \end{cases}$. The cdf is $F_Y(y) = \int_{-\infty}^{y} f(t)\,dt = \int_0^y 3t^2\,dt = y^3$ if $y \le 1$.

Then $F_Y(y) = \int_0^y f_Y(t)\,dt = \begin{cases} 0 \text{ if } y<0 \\ y^3 \text{ if } 0 \le y < 1 \\ 1 \text{ if } y \ge 1 \end{cases}$. The graphs of $f_Y(y)$ and $F_Y(y)$ are as follows. The heavy line in the graph of $f_Y(y)$ indicates that the density is 0 outside the interval $(0,1)$. Note that the cdf increases continuously, reaching 1 at the right end of the interval for the probability space.



Some other probabilities are $P(Y \le \frac{1}{2}) = F(\frac{1}{2}) = \frac{1}{8}$, and

for $0 \le y \le t \le 1$, $P(Y \le y | Y \le t) = \frac{P(Y \le y \cap Y \le t)}{P(Y \le t)} = \frac{P(Y \le y)}{P(Y \le t)} = \frac{y^3}{t^3} = (\frac{y}{t})^3$. □

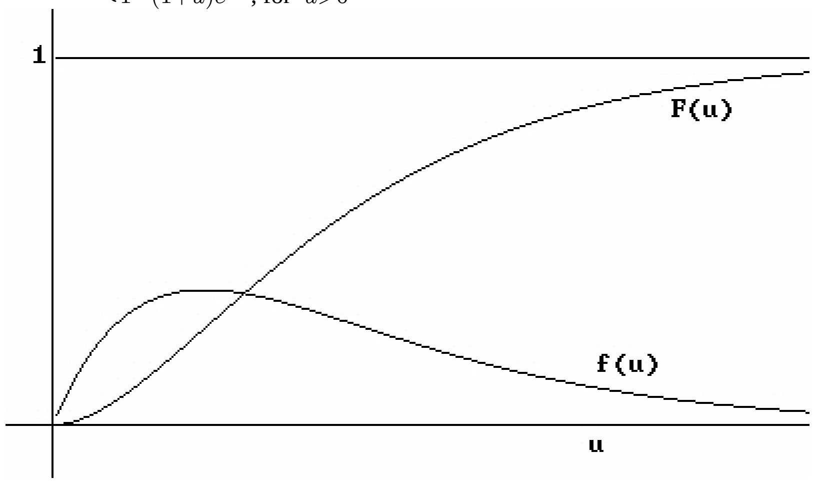
Example 4-6: Continuous Random Variable on an Infinite Interval

U is a continuous random variable on the interval $(0,\infty)$ with density function.

$f_U(u) = \begin{cases} ue^{-u} \text{ for } u>0 \\ 0 \text{ for } u\le 0 \end{cases}$.

The cdf is $F_U(u) = \int_{-\infty}^{u} f(t)\,dt = \int_0^u te^{-t}\,dt = -te^{-t} - e^{-t}\Big|_{t=0}^{t=u} = 1-(1+u)e^{-u}$ for $u>0$

Then $F_U(u) = \begin{cases} 0 \text{ for } u\le 0 \\ 1-(1+u)e^{-u}, \text{ for } u>0 \end{cases}$. $F(u)$ increases, approaching a limit of 1 as $u\to\infty$.



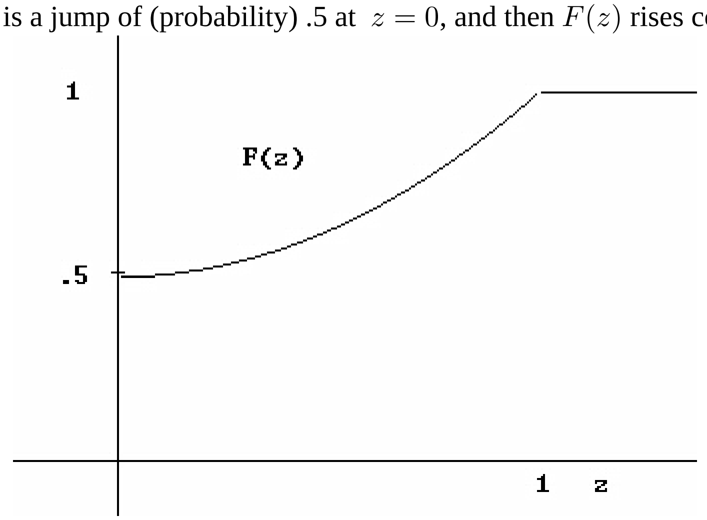
□

Example 4-7: Mixed Random Variable

Z has a mixed distribution on the interval $[0,1)$. Z has probability of .5 at $Z=0$, and Z has density function $f_Z(z) = z$ for $0<z<1$, and Z has no density or probability elsewhere.

The cdf of Z is $F_Z(z) = \begin{cases} 0 \text{ if } z<0 \\ .5 \text{ if } z=0 \\ .5+\frac{1}{2}z^2 \text{ if } 0<z<1 \\ 1 \text{ if } z\ge 1 \end{cases}$.

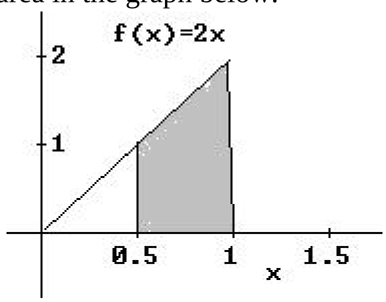
Note that there is a jump of (probability) .5 at $z=0$, and then $F(z)$ rises continuously on $(0,1)$.



□

Some results and formulas relating to this section:

(i) For a continuous random variable X,
$P[a < X < b] = P[a \leq X < b] = P[a < X \leq b] = P[a \leq X \leq b] = \int_a^b f_X(x)\,dx$,
so that when calculating the probability for a continuous random variable on an interval, it is irrelevant whether or not the endpoints are included. For the density function
$f_X(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ otherwise} \end{cases}$, we have $P[.5 < X \leq 1] = \int_{.5}^1 2x\,dx = x^2\Big|_{x=.5}^{x=1} = 1 - (.5)^2 = .75$.
This is illustrated in the shaded area in the graph below.



Also, for a continuous random variable, $P[X = a] = 0$, the probability at a single point is 0. Non-zero probabilities only exist over an interval, not at a single point.

(ii) For a continuous random variable, the hazard rate or failure rate is
$h(x) = \frac{f(x)}{1-F(x)} = -\frac{d}{dx}\, ln[1 - F(x)]$.

(iii) If X has a mixed distribution, then $P[X = t]$ will be non-zero for some value(s) of t, and $P[a < X < b]$ will not always be equal to $P[a \leq X \leq b]$ (they will not be equal if X has a non-zero probability mass at either a or b).

(iv) $f(x)$ may be defined **piecewise**, meaning that $f(x)$ is defined by a different algebraic formula on different intervals. Example 4-13 below illustrates this.

(v) Independence of random variables: A more technical definition of independence of random variables will be given in a later section of these notes. One of the important consequences of random variables X and Y being independent is that
$P[(a < X \leq b) \cap (c < Y \leq d)] = P[a < X \leq b] \times P[c < Y \leq d]$.
In general, what we mean by saying that random variables X and Y are independent is that if A is any event involving only X (such as $a < X \leq b$), and B is any event involving only Y, then A and B are independent events.

(vi) Conditional distribution of X given event A: Suppose that $f_X(x)$ is the density function or probability function of X, and suppose that A is an event. The conditional pdf or pf of "X given A" is $f_{X|A}(x|A) = \begin{cases} \frac{f(x)}{P(A)} & \text{if } x \text{ is an outcome in event } A \\ 0 & \text{if } x \text{ is not an outcome in event } A \end{cases}$.

For example, suppose that $f_X(x) = \begin{cases} 2x & \text{for } 0<x<1 \\ 0, & \text{otherwise} \end{cases}$,

and suppose that A is the event that $X \leq \frac{1}{2}$.

Then $P(A) = P(X \leq \frac{1}{2}) = \frac{1}{4}$, and for $0 < x \leq \frac{1}{2}$, $f_{X|A}(x|X \leq \frac{1}{2}) = \frac{2x}{1/4} = 8x$, and for $x > \frac{1}{2}$, $f_{X|A}(x|X \leq \frac{1}{2}) = 0$ (if we are given that $X \leq \frac{1}{2}$, then it is not possible for $x > \frac{1}{2}$, so the conditional density is 0 if $x > \frac{1}{2}$).

The conditional density must satisfy the same requirements as any probability density, it must integrate to 1 over its probability space. This is true for the example just presented, since $\int_0^{1/2} f_{X|A}(x|X \leq \frac{1}{2})\, dx = \int_0^{1/2} 8x\, dx = 1.$

Example 4-8: A die is loaded in such a way that the probability of the face with j dots turning up is proportional to j for $j = 1, 2, 3, 4, 5, 6$. What is the probability, in one roll of the die, that an even number of dots will turn up?

Solution: Let X denote the random variable representing the number of dots that appears when the die is rolled once. Then, $P[X = k] = R \cdot k$ for $k = 1, 2, 3, 4, 5, 6$, where R is the proportional constant. Since the sum of all of the probabilities of points that can occur must be 1, it follows that $R \cdot [1 + 2 + 3 + 4 + 5 + 6] = 1$, so that $R = \frac{1}{21}$.

Then, $P[\text{even number of dots turns up}] = P[2] + P[4] + P[6] = \frac{2+4+6}{21} = \frac{4}{7}$. □

Example 4-9: An ordinary single die is tossed repeatedly and independently until the first even number turns up. The random variable X is defined to be the number of the toss on which the first even number turns up. Find the probability that X is an even number.

Solution: X is a discrete random variable that can take on an integer value of 1 or more. The probability function for X is $p(x) = P[X = x] = (\frac{1}{2})^x$ (this is the probability of $x - 1$ successive odd tosses followed by an even toss, we are using the independence of successive tosses). Then,

$P[X \text{ is even}] = P[2] + P[4] + P[6] + \cdots = (\frac{1}{2})^2 + (\frac{1}{2})^4 + (\frac{1}{2})^6 + \cdots = \frac{(\frac{1}{2})^2}{1-(\frac{1}{2})^2} = \frac{1}{3}.$ □

Example 4-10: Suppose that the continuous random variable X has density function $f(x) = 3 - 48x^2$ for $-.25 \leq x \leq .25$ (and $f(x) = 0$ elsewhere). Find $P[\frac{1}{8} \leq X \leq \frac{5}{16}]$.

Solution: $P[.125 \leq X \leq .3125] = P[.125 \leq X \leq .25]$, since there is no density for X at points greater than .25. The probability is $\int_{.125}^{.25}(3 - 48x^2)\,dx = \frac{5}{32}$. □

Example 4-11: Suppose that the continuous random variable X has the cumulative distribution function $F(x) = \frac{1}{1+e^{-x}}$ for $-\infty < x < \infty$. Find X's density function.

Solution: The density function for a continuous random variable is the first derivative of the distribution function. The density function of X is $f(x) = F'(x) = \frac{e^{-x}}{(1+e^{-x})^2}$. □

Example 4-12: X is a random variable for which $P[X \leq x] = 1 - e^{-x}$ for $x \geq 1$, and $P[X \leq x] = 0$ for $x < 1$. Which of the following statements is true?

A) $P[X = 2] = 1 - e^{-2}$ and $P[X = 1] = 1 - e^{-1}$

B) $P[X = 2] = 1 - e^{-2}$ and $P[X \leq 1] = 1 - e^{-1}$

C) $P[X = 2] = 1 - e^{-2}$ and $P[X < 1] = 1 - e^{-1}$

D) $P[X < 2] = 1 - e^{-2}$ and $P[X < 1] = 1 - e^{-1}$

E) $P[X < 2] = 1 - e^{-2}$ and $P[X = 1] = 1 - e^{-1}$

Solution: Since $P[X \leq x] = 1 - e^{-x}$ for $x \geq 1$, it follows that $P[X \leq 1] = 1 - e^{-1}$.
But $P[X \leq x] = 0$ if $x < 1$, and thus $P[X < 1] = 0$, so that $P[X = 1] = 1 - e^{-1}$ (since $P[X \leq 1] = P[X < 1] + P[X = 1]$). This eliminates answers C and D. Since the distribution function for X is continuous (and differentiable) for $x > 1$, it follows that $P[X = x] = 0$ for $x > 1$. This eliminates answers A, B and C. This is an example of a random variable X with a mixed distribution, a point of probability at $X = 1$, with $P(X = 1) = 1 - e^{-1}$, and a continuous distribution for $X > 1$ with pdf $f(x) = e^{-x}$ for $x > 1$.

Answer: E □

Example 4-13: A continuous random variable X has the density function

$f(x) = \begin{cases} 2x & 0 < x < \frac{1}{2} \\ \frac{4-2x}{3} & \frac{1}{2} \leq x < 2 \\ 0, & \text{elsewhere} \end{cases}$. Find $P[.25 < X \leq 1.25]$.

Solution: $P[.25 < X \leq 1.25] = \int_{.25}^{1.25} f(x)\,dx = \int_{.25}^{.5} 2x\,dx + \int_{.5}^{1.25} \frac{4-2x}{3}\,dx = \frac{3}{4}$.

Note that since X is a continuous random variable, the probability $P[.25 \leq X < 1.25]$ would be the same as $P[.25 < X \leq 1.25]$. This is an example of a density function defined piecewise. The only consequence of this is that in finding a probability for an interval that contains the point $\frac{1}{2}$, we must set up two integrals, one integral ending at right hand limit $\frac{1}{2}$, and the other integral starting at left hand limit $\frac{1}{2}$.

Also, note that if the density function was defined to be $g(x) = \begin{cases} 2x & 0 < x < \frac{1}{2}, \\ 0 & x = 1/2 \\ \frac{4-2x}{3} & \frac{1}{2} < x \leq 2 \end{cases}$

(0 density at $x = 1/2$), then all probabilities are unchanged (since the two density functions f and g differ at only one point, probability calculations, which are based on integrals of the density function over an interval, are the same for both f and g). □

Example 4-14: The density function for the continuous random variable U is

$f_U(u) = \begin{cases} e^{-u} \text{ for } u>0 \\ 0, \text{ for } u \leq 0 \end{cases}$. Find the probability $P[U \leq 2|U > 1]$.

Solution: $P[U \leq 2|U > 1] = \frac{P[(U \leq 2) \cap (U>1)]}{P[U>1]} = \frac{P[1<U\leq 2]}{P[U>1]}$.

$P[1 < U \leq 2] = \int_1^2 e^{-u}\,du = e^{-1} - e^{-2}$, $P[U > 1] = \int_1^\infty e^{-u}\,du = e^{-1}$.

$P[U \leq 2|U > 1] = \frac{e^{-1}-e^{-2}}{e^{-1}} = 1 - e^{-1}$. □

Example 4-15: An ordinary single die is tossed repeatedly until the first even number turns up. The random variable X is defined to be the number of the toss on which the first even number turns up. We define the following two events: $A = X$ is even , $B = X$ is a multiple of 3 . Determine whether or not events A and B are independent.

Solution: This is the same distribution as in Example 4-9. X is a discrete random variable that can take on an integer value of 1 or more. The probability function for X is

$$p(x) = P[X = x] = (\tfrac{1}{2})^x \ .$$

Then, $P[A] = P[X = 2 \text{ or } 4 \text{ or } 6 \ldots] = (\frac{1}{2})^2 + (\frac{1}{2})^4 + (\frac{1}{2})^6 + \cdots = (\frac{1}{2})^2 \cdot [\frac{1}{1-(\frac{1}{2})^2}] = \frac{1}{3}$,

and $P[B] = P[X = 3 \text{ or } 6 \text{ or } 9 \ldots] = (\frac{1}{2})^3 + (\frac{1}{2})^6 + (\frac{1}{2})^9 + \cdots = (\frac{1}{2})^3 \cdot [\frac{1}{1-(\frac{1}{2})^3}] = \frac{1}{7}$.

$A \cap B = X$ is a multiple of 6 (multiple of 2 and of 3).

Example 4-15 continued

Then $P[A \cap B] = P[X = 6 \text{ or } 12 \text{ or } 18 \ldots]$

$= (\frac{1}{2})^6 + (\frac{1}{2})^{12} + (\frac{1}{2})^{18} + \cdots = (\frac{1}{2})^6 \cdot [\frac{1}{1-(\frac{1}{2})^6}] = \frac{1}{63}$.

We note that $P[A \cap B] = \frac{1}{63} \neq \frac{1}{3} \cdot \frac{1}{7} = P[A] \cdot P[B]$, and therefore, A and B are not independent. □

Example 4-16: A random sample of 4 independent random variables X_1, X_2, X_3, X_4 is obtained. Each of the X_i's has a density function of the form $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ elsewhere} \end{cases}$.

We define the following two random variables:

$Y = max\{X_1, X_2, X_3, X_4\}$ and $Z = min\{X_1, X_2, X_3, X_4\}$.

Find the density functions of Y and Z.

Solution: For Y we first find the distribution function.

$P[Y \leq y] = P[max\{X_1, X_2, X_3, X_4\} \leq y]$

$= P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y) \cap (X_4 \leq y)]$

$= P[X_1 \leq y] \cdot P[X_2 \leq y] \cdot P[X_3 \leq y] \cdot P[X_4 \leq y] = (y^2)(y^2)(y^2)(y^2) = y^8$, $0 < y < 1$.

(We use the cdf of X, $P[X \leq y] = \int_0^y 2x\,dx = y^2$.)

Thus, $F_Y(y) = P[Y \leq y] = y^8 \rightarrow f_Y(y) = F_Y'(y) = 8y^7$, $0 < y < 1$.

For Z we find the survival function (complement of the distribution function).

$P[Z > z] = P[min\{X_1, X_2, X_3, X_4\} > z]$

$= P[(X_1 > z) \cap (X_2 > z) \cap (X_3 > z) \cap (X_4 > z)]$

$= P[X_1 > z] \cdot P[X_2 > z] \cdot P[X_3 > z] \cdot P[X_4 > z] = (1 - z^2)^4$, $0 < z < 1$.

Then $F_Z(z) = P[Z \leq z] = 1 - P[Z > z] = 1 - (1 - z^2)^4$, $0 < z < 1$, and

$f_Z(z) = F_Z'(z) = 4(1 - z^2)^3(2z) = 8z(1 - z^2)^3$, $0 < z < 1$.

Y and Z are examples of order statistics on a collection of independent random variables. A little later we will consider order statistics in more detail. □

Example 4-17: Example 4-4 considers the random variable
$X=$ number of successive independent tosses of a fair coin until the first head turns up.
X can be any integer ≥ 1, and the probability function of X is $p_X(x)=P[X=x]=\frac{1}{2^x}$, for $x=1,2,3,\ldots$

(a) Find the probability function of the conditional distribution of X given that the first head occurs on an odd numbered toss. Find the probability that the first head occurs within the first 3 tosses given that the first head occurs on an odd numbered toss.

(b) Find the probability function of the conditional distribution of X given that the first head occurs within the first 5 tosses. Find the probability that the first head occurs within the first 3 tosses given that the first head occurs within the first 5 tosses.

Solution: (a) A is the event that the first head occurs on an odd numbered toss.

$$P(A)=P[X=1]+P[X=3]+\cdots=\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}\cdots$$
$$=\frac{1}{2}\cdot[1+\frac{1}{2^2}+(\frac{1}{2^2})^2+(\frac{1}{2^2})^3+\cdots]=\frac{1}{2}\cdot\frac{1}{1-(\frac{1}{2})^2}=\frac{2}{3}\ .$$

Then $p_{X|A}(x|X \text{ is odd})=\frac{p_X(x)}{P(A)}=\frac{(\frac{1}{2})^x}{\frac{2}{3}}=\frac{3}{2}\cdot\frac{1}{2^x}$ if x is odd,

and $p_{X|A}(x|X \text{ is odd})=0$ if x is even. Then

$$P[X\leq 3|X \text{ is odd}]=p_{X|A}(1|X \text{ is odd})+p_{X|A}(2|X \text{ is odd})+p_{X|A}(3|X \text{ is odd})$$
$$=\frac{3}{2}\cdot\frac{1}{2}+0+\frac{3}{2}\cdot\frac{1}{2^3}=.9375\ .$$

Note that we can also find $P[X\leq 3|X \text{ is odd}]$ using the definition of conditional probability;

$$P[X\leq 3|X \text{ is odd}]=\frac{P[X\leq 3\cap X \text{ is odd}]}{P[X \text{ is odd}]}=\frac{P[X=1]+P[X=3]}{2/3}=\frac{3}{2}\cdot\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{2^3}\ .$$

(b) B is the event that the first head occurs within the first 5 tosses.

$$P(B)=P[X\leq 5]=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}=\frac{31}{32}\ .$$

$p_{X|B}(x|X\leq 5)=\frac{(\frac{1}{2})^x}{\frac{31}{32}}$ if $x=1,2,3,4,5$, and $p_{X|B}(x|X\leq 5)=0$ if $x>5$.

$$P[X\leq 3|X\leq 5]=p_{X|B}(1|X\leq 5)+p_{X|B}(2|X\leq 5)+p_{X|B}(3|X\leq 5)$$
$$=\frac{32}{31}\cdot\frac{1}{2}+\frac{32}{31}\cdot\frac{1}{2^2}+\frac{32}{31}\cdot\frac{1}{2^3}=.903\ .$$

Alternatively,

$$P[X\leq 3|X\leq 5]=\frac{P[X\leq 3\cap X\leq 5]}{P[X\leq 5]}=\frac{P[X=1]+P[X=2]+P[X=3]}{P[X\leq 5]}=\frac{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}{\frac{31}{32}}=.903\ .\ \square$$

Example 4-18: Bob just read a news report that suggested that one-quarter of all cars on the road are imports, and the rest are domestic. Bob decides to test this suggestion by watching the cars go by his house. Bob assumes that each successive car that goes by has a $\frac{1}{4}$ chance of being an import and a $\frac{3}{4}$ chance of being domestic. Bob knows cars, and he can tell the difference between imports and domestic cars. If Bob's assumption is correct, find the probability that Bob will see at least 2 imports pass his house before the 3rd domestic car passes his house.

Solution: As soon as the 4th car passes his house, Bob will know whether or not at least 2 imports passed before the third domestic. If 2, 3 or 4 of the first 4 cars are imports, then the 2nd import passed his house before the 3rd domestic. If 0 or 1 of the first 4 cars are imports then the 3rd domestic passed his house before the 2nd import.

The probability of 2 of the first 4 cars being imports is the probability of any one of the following 6 successions of 4 cars occurring:
$IIDD$, $IDID$, $IDDI$, $DIID$, $DIDI$, $DDII$.
Each one of those has a chance of $(\frac{1}{4})^2(\frac{3}{4})^2$ occurring, for a total probability of $6 \times \frac{9}{256} = \frac{27}{128}$.

The probability of 3 of the first 4 cars being imports is the probability of any one of the following 4 successions of 4 cars occurring: $IIID$, $IIDI$, $IDII$, $DIII$.
Each one of those has a chance of $(\frac{1}{4})^3(\frac{3}{4})$ occurring, for a total probability of $4 \times \frac{3}{256} = \frac{3}{64}$.

The probability that all 4 of the first 4 cars being imports is $(\frac{1}{4})^4 = \frac{1}{256}$.

Therefore, the overall total probability of at least 2 imports passing Bob's house before the 3rd domestic car passes his house is $\frac{27}{128} + \frac{3}{64} + \frac{1}{256} = \frac{67}{256}$. □

PROBLEM SET 4

Random Variables and Probability Distributions

1. Let X be a discrete random variable with probability function
$P[X=x]=\frac{2}{3^x}$ for $x=1,2,3,\ldots$ What is the probability that X is even?
A) $\frac{1}{4}$ B) $\frac{2}{7}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

2. (SOA) In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0$, $p_{n+1}=\frac{1}{5}p_n$, where p_n represents the probability that the policyholder files n claims during the period. Under this assumption, what is the probability that a policyholder files more than one claim during the period?
A) 0.04 B) 0.16 C) 0.20 D) 0.80 E) 0.96

3. Let X be a continuous random variable with density function
$f(x)=\begin{cases} 6x(1-x) & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$. Calculate $P[\,|X-\frac{1}{2}|>\frac{1}{4}]$.
A) .0521 B) .1563 C) .3125 D) .5000 E) .8000

4. Let X be a random variable with distribution function

$$F(x)=\begin{cases} 0 & \text{for } x<0 \\ \frac{x}{8} & \text{for } 0\leq x<1 \\ \frac{1}{4}+\frac{x}{8} & \text{for } 1\leq x<2 \\ \frac{3}{4}+\frac{x}{12} & \text{for } 2\leq x<3 \\ 1 & \text{for } x\geq 3 \end{cases}$$

. Calculate $P[1\leq X\leq 2]$.

A) $\frac{1}{8}$ B) $\frac{3}{8}$ C) $\frac{7}{16}$ D) $\frac{13}{24}$ E) $\frac{19}{24}$

5. (SOA) In a small metropolitan area, annual losses due to storm, fire, and theft are independently distributed random variables. The pdf's are:

	Storm	Fire	Theft
$f(x)$	e^{-x}	$\frac{2e^{-2x/3}}{3}$	$\frac{5e^{-5x/12}}{12}$

Determine the probability that the maximum of these losses exceeds 3 .
A) 0.002 B) 0.050 C) 0.159 D) 0.287 E) 0.414

6. Let $X_1,\ X_2$ and X_3 be three independent, identically distributed random variables each with density function $f(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$. Let $Y = max\{X_1, X_2, X_3\}$. Find $P[Y > \frac{1}{2}]$.

A) $\frac{1}{64}$ B) $\frac{37}{64}$ C) $\frac{343}{512}$ D) $\frac{7}{8}$ E) $\frac{511}{512}$

7. Let the distribution function of X for $x > 0$ be $F(x) = 1 - \sum_{k=0}^{3} \frac{x^k e^{-x}}{k!}$.

What is the density function of X for $x > 0$?

A) e^{-x} B) $\frac{x^2 e^{-x}}{2}$ C) $\frac{x^3 e^{-x}}{6}$ D) $\frac{x^3 e^{-x}}{6} - e^{-x}$ E) $\frac{x^3 e^{-x}}{6} + e^{-x}$

8. Let X have the density function $f(x) = \frac{3x^2}{\theta^3}$ for $0 < x < \theta$, and $f(x) = 0$, otherwise. If $P[X > 1] = \frac{7}{8}$, find the value of θ.

A) $\frac{1}{2}$ B) $(\frac{7}{8})^{1/3}$ C) $(\frac{8}{7})^{1/3}$ D) $2^{1/3}$ E) 2

9. (SOA) A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by $V = 100,000Y$ where Y is a random variable with density function $f(y) = \begin{cases} k(1-y)^4 & \text{for } 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases}$ where k is a constant. What is the conditional probability that V exceeds 40,000, given that V exceeds 10,000?

A) 0.08 B) 0.13 C) 0.17 D) 0.20 E) 0.51

10. (SOA) An insurance company insures a large number of homes. The insured value, X of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2?

A) 0.578 B) 0.684 C) 0.704 D) 0.829 E) 0.875

11. (SOA) Two life insurance policies, each with a death benefit of 10,000 and a one-time premium of 500, are sold to a couple, one for each person. The policies will expire at the end of the tenth year. The probability that only the wife will survive at least ten years is 0.025, the probability that only the husband will survive at least ten years is 0.01, and the probability that both of them will survive at least ten years is 0.96. What is the expected excess of premiums over claims, given that the husband survives at least ten years?

A) 350 B) 385 C) 397 D) 870 E) 897

12. X_1 and X_2 are two independent random variables, but they have the same density function $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0\text{, elsewhere} \end{cases}$. Find the probability that the maximum of X_1 and X_2 is at least .5.

A) .92 B) .94 C) .96 D) .98 E) 1.00

13. For two random variables, the "distance" between two distributions is defined to be the maximum, $\underset{all\ x}{max} |F_1(x) - F_2(x)|$ over the range for which F_1 and F_2 are defined, where $F(x)$ is the cumulative distribution function. Find the distance between the following two distributions:

(i) uniform on the interval $[0,1]$,

(ii) pdf is $f(x) = \frac{1}{(x+1)^2}$ for $0 < x < \infty$.

A) 0 B) $\frac{1}{4}$ C) $\frac{1}{2}$ D) $\frac{3}{4}$ E) 1

14. A family health insurance policy pays the total of the first three claims in a year. If there is one claim during the year, the amount claimed is uniformly distributed between 100 and 500. If there are two claims in the year, the total amount claimed is uniformly distributed between 200 and 1000, and if there are three claims in the year, the total amount claimed is uniformly distributed between 500 and 2000. The probabilities of 0, 1, 2 and 3 claims in the year are $.5$, $.3$, $.1$, $.1$ respectively. Find the probability that the insurer pays at least 500 in total claims for the year.

A) .10 B) .12 C) .14 D) .16 E) .18

15. (SOA) The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20-x) & \text{for } 0 < x < 20 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16?

A) $\frac{1}{25}$ B) $\frac{1}{9}$ C) $\frac{1}{8}$ D) $\frac{1}{3}$ E) $\frac{3}{7}$

16. (SOA) The lifetime of a machine part has a continuous distribution on the interval $(0\,,\,40)$ with probability density function f, where $f(x)$ is proportional to $(10+x)^{-2}$.
Calculate the probability that the lifetime of the machine part is less than 6.

A) 0.04 B) 0.15 C) 0.47 D) 0.53 E) 0.94

17. X is a continuous random variable with density function $f(x) = ce^{-x}$, $x > 1$.
Find $P[X < 3|X > 2]$.

A) $1 - e^{-1}$ B) e^{-1} C) $1 - e^{-2}$ D) $e^{-1} - e^{-2}$ E) $e^{-2} - e^{-3}$

PROBLEM SET 4 SOLUTIONS

1. $P[X \text{ is even}] = P[X=2] + P[X=4] + P[X=6] + \cdots$
$= \frac{2}{3} \cdot [\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \cdots] = \frac{2}{3^2} \cdot \frac{1}{1-\frac{1}{3^2}} = \frac{1}{4}$. Answer: A

2. A requirement for a valid distribution is $\sum_{k=0}^{\infty} p_k = 1$.
Since $p_n = \frac{1}{5}p_{n-1} = \frac{1}{5} \cdot \frac{1}{5}p_{n-2} = \frac{1}{5} \cdot \frac{1}{5} \cdots \frac{1}{5}p_0 = (\frac{1}{5})^n p_0$, it follows that
$1 = \sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} (\frac{1}{5})^k p_0 = p_0 \cdot \frac{1}{1-\frac{1}{5}}$ (infinite geometric series) so that $p_0 = \frac{4}{5}$
and $p_k = (\frac{1}{5})^k (\frac{4}{5})$. Then,
$P[N>1] = 1 - P[N=0 \text{ or } 1] = 1 - p_0 - p_1 = 1 - \frac{4}{5} - (\frac{1}{5})(\frac{4}{5}) = \frac{1}{25}$. Answer: A

3. $P[\left|X - \frac{1}{2}\right| \le \frac{1}{4}] = P[-\frac{1}{4} \le X - \frac{1}{2} \le \frac{1}{4}] = P[\frac{1}{4} \le X \le \frac{3}{4}] = \int_{1/4}^{3/4} 6x(1-x)\,dx$
$= .6875 \rightarrow P[\left|X - \frac{1}{2}\right| > \frac{1}{4}] = 1 - P[\left|X - \frac{1}{2}\right| \le \frac{1}{4}] = .3125$. Answer: C

4. $P[1 \le X \le 2] = P[X \le 2] - P[X < 1] = F(2) - \lim_{x \to 1^-} F(x) = \frac{11}{12} - \frac{1}{8} = \frac{19}{24}$.
Answer: E

5. $P[max\{S,F,T\} > 3] = 1 - P[max\{S,F,T,\} \le 3]$.
$P[max\{S,F,T,\} \le 3] = P[(S \le 3) \cap (F \le 3) \cap (T \le 3)]$
$= P[S \le 3] \cdot P[F \le 3] \cdot P[T \le 3]$
$= (1 - e^{-3/1})(1 - e^{-3/1.5})(1 - e^{-3/2.4}) = .586$.
$P[max\{S,F,T\} > 3] = 1 - .586 = .414$. Answer: E

6. $P[Y > \frac{1}{2}] = 1 - P[Y \le \frac{1}{2}] = 1 - P[(X_1 \le \frac{1}{2}) \cap (X_2 \le \frac{1}{2}) \cap (X_3 \le \frac{1}{2})]$
$= 1 - (P[X \le \frac{1}{2}])^3 = 1 - [\int_0^{1/2} 3x^2 dx]^3 = 1 - (\frac{1}{8})^3 = \frac{511}{512}$. Answer: E

7. $f(x) = F'(x) = -\sum_{k=0}^{3} \frac{kx^{k-1}e^{-x} - x^k e^{-x}}{k!} = e^{-x} \cdot \sum_{k=0}^{3} \left[\frac{x^k - kx^{k-1}}{k!}\right]$
$= e^{-x} \cdot [1 + \frac{x-1}{1} + \frac{x^2-2x}{2} + \frac{x^3-3x^2}{6}] = \frac{e^{-x}x^3}{6}$. Answer: C

8. Since $f(x) = 0$ if $x > \theta$, and since $P[X > 1] = \frac{7}{8}$, we must conclude that $\theta > 1$.
Then, $P[X > 1] = \int_1^\theta f(x)\,dx = \int_1^\theta \frac{3x^2}{\theta^3}\,dx = 1 - \frac{1}{\theta^3} = \frac{7}{8}$, or equivalently, $\theta = 2$.
Answer: E

9. In order for $f(y)$ to be a properly defined density function it must be true that
$\int_0^1 f(y)\,dy = \int_0^1 k(1-y)^4\,dy = 1 \rightarrow k \cdot (\frac{1}{5}) = 1 \rightarrow k = 5$.
We wish to find the conditional probability $P[100,000Y > 40,000|100,000Y > 10,000]$.
For events A and B, the definition of the conditional probability $P[A|B]$
is $P[A|B] = \frac{P[A \cap B]}{P[B]}$. With $A = 100,000Y > 40,000$ and
$B = 100,000Y > 10,000$, we have $A \cap B = A$, and,
$P[100,000Y > 40,000|100,000Y > 10,000] = \frac{P[100{,}000Y > 40{,}000]}{P[100{,}000Y > 10{,}000]}$.
From the density function for Y we have
$P[100,000Y > 40,000] = P[Y > .4] = \int_{.4}^1 f(y)\,dy = \int_{.4}^1 5(1-y)^4\,dy = (.6)^5$, and
$P[100,000Y > 10,000] = P[Y > .1] = \int_{.1}^1 f(y)\,dy = \int_{.1}^1 5(1-y)^4\,dy = (.9)^5$.
The conditional probability in question is $\frac{(.6)^5}{(.9)^5} = .132$. Answer: B

10. We are asked to find a conditional probability $P[X < 2|X \geq 1.5]$.
The definition of conditional probability is $P[A|B] = \frac{P[A \cap B]}{P[B]}$.
Then, $P[X < 2|X \geq 1.5] = \frac{P[1.5 \leq X < 2]}{P[X \geq 1.5]}$.
From the given density function of X we get
$P[X \geq 1.5] = \int_{1.5}^\infty 3x^{-4}\,dx = \frac{1}{(1.5)^3} = .29630$ and
$P[1.5 \leq X < 2] = \int_{1.5}^2 3x^{-4}\,dx = \frac{1}{(1.5)^3} - \frac{1}{(2)^3} = .17130$.
Then, $P[X < 2|X \geq 1.5] = \frac{P[1.5 \leq X < 2]}{P[X \geq 1.5]} = \frac{.17130}{.29630} = .578$. Answer: A

11. W is the event that the wife will survive at least 10 years, and H is the event that the husband will survive at least 10 years. We are given $P[W \cap H'] = .025$, $P[W' \cap H] = .01$, and $P[W \cap H] = .96$. Given that the husband survives at least 10 years, the probability that the wife survives at least 10 years is $P[W|H] = \frac{P[W \cap H]}{P[H]} = \frac{.96}{P[H]}$, and the probability that the wife does not survive at least 10 years is $P[\bar{W}|H] = \frac{P[\bar{W} \cap H]}{P[H]} = \frac{.01}{P[H]}$.
We can find $P[H] = P[W \cap H] + P[\bar{W} \cap H] = .96 + .01 = .97$, or use the table

		W		W'
$P[H]$	$\Leftarrow$	$P[W \cap H] = .96$	$+$	$P[W' \cap H] = .01$
$= .97$		given		given

Given that the husband survives 10 years, the claim will either be 0 if the wife survives 10 years, and 10,000 if the wife does not survive 10 years.
The expected amount of claim given that the husband survives 10 years is
$(0) \cdot P[W|H] + (10,0000) \cdot P[\bar{W}|H] = 10,000 \cdot \frac{1}{97} = 103.09$.
The total premium is 1,000 (for the two insurance policies), so that the excess premium over expected claim is $1000 - 103.09 = 897$. Answer: E

12. $P[max\{X_1, X_2\} \geq .5] = 1 - P[(X_1 < .5) \cap (X_2 < .5)]$
$= 1 - P[X_1 < .5] \cdot P[X_2 < .5]$.
$P[X_1 < .5] = \int_0^{.5} f(x)\, dx = \int_0^{.5} 2x\, dx = .25$, and $P[X_2 < .5] = .25$ also.
Then $P[max\{X_1, X_2\} \geq .5] = 1 - (.25)(.25) = .9375$. Answer: B

13. $F_1(x) = x$ for $0 \leq x \leq 1$, and $F_1(x) = 1$ for $x > 1$.
$F_2(x) = \int_0^x \frac{1}{(t+1)^2}\, dt = 1 - \frac{1}{x+1}$ for $x > 0$.
For $0 \leq x \leq 1$, $|F_1(x) - F_2(x)| = |x + \frac{1}{x+1} - 1|$, which is maximized at either $x = 0$, 1 or a critical point; critical points occur where $1 - \frac{1}{(x+1)^2} = 0$, or $x = 0$.
$|F_1(0) - F_2(0)| = 0$, $|F_1(1) - F_2(1)| = \frac{1}{2}$.
For $x > 1$, $|F_1(x) - F_2(x)| = |1 + \frac{1}{x+1} - 1| = \frac{1}{x+1}$, which decreases.
The distance between the two distributions is $\frac{1}{2}$. Answer: C

14. $T =$ total claims for the year, $N =$ number of claims for the year .

$P[T \geq 500] = \sum_{k=0}^{3} P[(T \geq 500) \cap (N = k)]$

$= P[(T \geq 500) \cap (N = 2)] + P[(T \geq 500) \cap (N = 3)]$

(this is true since if there are 0 or 1 claim, then total must be $\leq$ 500).

$P[(T \geq 500) \cap (N = 2)] = P[T \geq 500 | N = 2] \cdot P[N = 2] = \frac{1000-500}{1000-200} \cdot (.1) = .0625$, and

$P[(T \geq 500) \cap (N = 3)] = P[T \geq 500 | N = 3] \cdot P[N = 3] = 1 \cdot (.1) = .1$.

Then $P[T \geq 500] = .0625 + .1 = .1625$. Answer: D

15. We are asked to find the conditional probability

$P[X > 16 | X > 8] = \frac{P[X>16]}{P[X>8]}$.

$P[X > 8] = \int_8^{20} .005(20 - x)\, dx = .36$,

$P[X > 16] = \int_{16}^{20} .005(20 - x)\, dx = .04$.

$P[X > 16 | X > 8] = \frac{.04}{.36} = \frac{1}{9}$. Answer: B

16. $f(x) = c(10 + x)^{-2}$, $0 < x < 40$.

The total probability must be 1, so that $\int_0^{40} c(10 + x)^{-2}\, dx = c[\frac{1}{10} - \frac{1}{50}] = 1$.

Therefore, $c = 12.5$ and $f(x) = 12.5(10 + x)^{-2}$.

Then, $P[X < 6] = \int_0^6 f(x)\, dx = \int_0^6 12.5(10 + x)^{-2}\, dx = -12.5(10 + x)^{-1}\Big|_{x=0}^{x=6} = .46875$.

Answer: C

17. $P[X < 3 | X > 2] = \frac{P[2<X<3]}{P[X>2]}$.

$P[X > 2] = \int_2^{\infty} ce^{-x}\, dx = ce^{-2}$, $P[2 < X < 3] = \int_2^3 ce^{-x}\, dx = c(e^{-2} - e^{-3})$,

$P[X < 3 | X > 2] = \frac{c(e^{-2}-e^{-3})}{ce^{-2}} = 1 - e^{-1}$.

Note that we can find c, from $1 = \int_1^{\infty} f(x)\, dx = \int_1^{\infty} ce^{-x}\, dx = ce^{-1} \rightarrow c = e$;

but this is not necessary for this exercise. Answer: A

SECTION 5 - EXPECTATION AND OTHER DISTRIBUTION PARAMETERS

Expected value of a random variable:

A random variable is a numerical outcome from an experiment or from a random procedure. If it is possible to repeat the experiment many times, the numerical outcomes will fluctuate from one experiment to the next because of the variability inherent in the behavior of a random variable. Although successive numerical outcomes of the random variable will fluctuate, as more and more random outcomes are observed, the numerical average of those outcomes will tend to stabilize. For instance, if we repeatedly toss a fair die, each successive outcome will be an integer from 1 to 6, but as the number of successive tosses n gets large, if we calculate the average outcome of the n tosses, it will tend toward a constant limit. This is the average value, or **expected value**, or the **mean** of the random variable.

For a random variable X, the expected value (also called the **expectation**) is denoted $\boldsymbol{E[X]}$, or $\boldsymbol{\mu_X}$ or $\boldsymbol{\mu}$. The mean is interpreted as the "average" of the random outcomes.

Mean of a Discrete Random Variable

For a discrete random variable, the expected value of X is $\boldsymbol{\sum x \cdot p(x) = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + \cdots}$, where the sum is taken over all points x at which X has non-zero probability. For instance, if X is the result of one toss of a fair die, then $E[X] = (1) \cdot (\frac{1}{6}) + (2) \cdot (\frac{1}{6}) + \cdots + (6) \cdot (\frac{1}{6}) = \frac{7}{2}$. The meaning of this value is that if the die is tossed many times, then the "long-run" average of the numbers turning up is $\frac{7}{2}$.

We can see this from another point of view. Suppose that we toss the die n times. Then, "on average", we expect that there will be $\frac{n}{6}$ tosses that are 1, and the same for 2, 3, 4, 5, and 6. Therefore, the average outcome would be the total of all the tosses, divided by n, which is $\frac{(\frac{n}{6})(1)+(\frac{n}{6})(2)+(\frac{n}{6})(3)+(\frac{n}{6})(4)+(\frac{n}{6})(5)+(\frac{n}{6})(6)}{n} = (1) \cdot (\frac{1}{6}) + (2) \cdot (\frac{1}{6}) + \cdots + (6) \cdot (\frac{1}{6}) = \frac{7}{2}$, which is the same as the formal definition of the mean.

Note that the mean of a random variable X is not necessarily one of the possible outcomes for X ($\frac{7}{2}$ is not a possible outcome when tossing a die).

Mean of a Continuous Random Variable

For a continuous random variable, the expected value is $\int_{-\infty}^{\infty} \boldsymbol{x \cdot f(x)\,dx}$.

Although this integral is written with lower limit $-\infty$ and upper limit ∞, the interval of integration is the interval of non-zero-density for X. For instance, if $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ elsewhere} \end{cases}$,

then $E[X] = \int_0^1 x \cdot (2x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$. Note that even though this random variable is defined on the interval $(0,1)$, the mean is not the midpoint of that interval. This could have been anticipated since the density is higher for x values near 1 than it is for x values near 0. The distribution is weighted more heavily toward 1 than 0. This can also be seen, for instance, by noting that $P[0 < X < \frac{1}{2}] = \frac{1}{4}$, and $P[\frac{1}{2} < X < 1] = \frac{3}{4}$.

The expected value is the "average" over the range of values that X can be, in the sense of the average being the "weighted center" (not necessarily the "geographic center") of the distribution. In the case of the die toss example above, all outcomes were equally likely, so it is not surprising that the mean was in the middle of the possible outcomes. In the continuous example in the previous paragraph, the mean of $\frac{2}{3}$ reflected the higher density for the values close to 1.

Expectation of $\boldsymbol{h(x)}$: If h is a function, then $\boldsymbol{E[h(X)]}$ is equal to $\sum_x \boldsymbol{h(x) \cdot p(x)}$ if X is a discrete random variable, and it is equal to $\int_{-\infty}^{\infty} \boldsymbol{h(x) \cdot f(x)\,dx}$ if X is a continuous random variable. Suppose that $h(x) = \sqrt{x}$. Then for the die toss example,

$E[h(X)] = E[\sqrt{X}] = (\sqrt{1}) \cdot (\frac{1}{6}) + (\sqrt{2}) \cdot (\frac{1}{6}) + \cdots + (\sqrt{6}) \cdot (\frac{1}{6}) = 1.805$.

For the continuous random variable with density function $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ elsewhere} \end{cases}$,

we have $E[h(X)] = E[\sqrt{X}] = \int_0^1 \sqrt{x} \cdot (2x)\,dx = \frac{4}{5}$.

Example 5-1: Let X equal the number of tosses of a fair die until the first "1" appears. Find $E[X]$.

Solution: X is a discrete random variable that can take on an integer value ≥ 1. The probability that the first 1 appears on the x-th toss is $p(x) = (\frac{5}{6})^{x-1}(\frac{1}{6})$ for $x \geq 1$ ($x-1$ tosses that are not 1 followed by a 1). This is the probability function of X. Then

$E[X] = \sum_{k=1}^{\infty} k \cdot f(k) = \sum_{k=1}^{\infty} k \cdot (\frac{5}{6})^{k-1}(\frac{1}{6}) = (\frac{1}{6})[1 + 2(\frac{5}{6}) + 3(\frac{5}{6})^2 + \cdots]$.

We use the general increasing geometric series relation $1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$,

so that $E[X] = (\frac{1}{6}) \cdot \frac{1}{(1-\frac{5}{6})^2} = 6$. □

Example 5-1 used an identity involving an infinite increasing geometric series. It is worthwhile knowing this identity. There are a number of ways to derive the infinite increasing geometric series formula used in Example 5-1. For instance, from the equation
$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$, if we differentiate both sides of the equation, we get
$1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$.

Example 5-2: A fair die is tossed until the first 1 appears. Let x equal the number of tosses required, $x = 1, 2, 3, ...$ You are to receive $(.5)^x$ dollars if the first 1 appears on the x-th toss. What is the expected amount that you will receive?

Solution: This is the same distribution as in Example 5-1 above, with the probability that the first 1 appears on the x-th toss being $(\frac{5}{6})^{x-1}(\frac{1}{6})$ for $x \geq 1$ ($x-1$ tosses that are not 1, followed by a 1), and the amount received in that case is $h(x) = (.5)^x$. Then, the expected amount received is $E[h(X)] = E[(.5)^X] = \sum_{k=1}^{\infty}(.5)^k(\frac{5}{6})^{k-1}(\frac{1}{6}) = (\frac{1}{12})[1 + (\frac{5}{12}) + (\frac{5}{12})^2 + \cdots] = \frac{1}{7}$. □

Moments of a random variable: If $n \geq 1$ is an integer, the **n-th moment of X** is $\boldsymbol{E[X^n]}$. If the mean of X is μ, then the **n-th central moment of X (about the mean μ)** is $E[(X-\mu)^n]$.

Example 5-3: You are given that $\theta > 0$ is a constant, and the density function of X is $f(x) = \theta e^{-x\theta}$, for $x > 0$, and 0 elsewhere. Find the n-th moment of X, where n is a non-negative integer (assuming that $\theta > 0$).

Solution: The n-th moment of X is $E[X^n] = \int_0^{\infty} x^n \cdot \theta e^{-x\theta} dx$. Applying integration by parts, this can be written as
$\int_0^{\infty} x^n \, d(-e^{-x\theta}) = -x^n e^{-x\theta}\Big|_{x=0}^{x=\infty} - \int_0^{\infty} -nx^{n-1}e^{-x\theta}\,dx = \int_0^{\infty} nx^{n-1}e^{-x\theta}dx$.
Repeatedly applying integration by parts results in $E[X^n] = \frac{n!}{\theta^n}$.
An Alternative to integration by parts is the method mentioned on page 23.
It is worthwhile noting the general form of the integral that appears in this example; if $k \geq 0$ is an integer and $a > 0$, then by repeated applications of integration by parts, we have $\int_0^{\infty} t^k e^{-at} dt = \frac{k!}{a^{k+1}}$.
In this example $\int_0^{\infty} x^n \theta e^{-x\theta} dx = \theta \int_0^{\infty} x^n e^{-x\theta} dx = \theta \cdot \frac{n!}{\theta^{n+1}} = \frac{n!}{\theta^n}$. □

Symmetric Distribution: If X is a continuous random variable with pdf $f(x)$, and if c is a point for which $f(c+t) = f(c-t)$ for all $t > 0$, then X is said to have a symmetric distribution about the point $x = c$. For such a distribution, the mean will be the point of symmetry, $E[X] = c$. This will be shown in more detail later in the notes, and we will review a couple of specific symmetric distributions.

Variance of X:

Let us go back to the die toss example again. We saw that the mean of X, the outcome of the fair die toss, was $\frac{7}{2}$. Each time the die is tossed, the actual outcome is from 1 to 6, so there will be some "deviation" or distance in the outcome from the mean of $\frac{7}{2}$. Sometimes the deviation will be $\frac{1}{2}$ (for tosses that are 3 or 4), sometimes the deviation will be $\frac{3}{2}$ and sometimes it will be $\frac{5}{2}$. Now suppose that we consider a modification to this die toss example in which we use a modified die which has three sides each with 1 on them, and 3 sides each with 6 on them. The random variable Y that represents the outcome of the modified die has the following probability distribution: $P[Y=1] = \frac{1}{2}$, $P[Y=6] = \frac{1}{2}$. The mean of Y is $\frac{7}{2}$, but each time the modified die is tossed, the deviation of the outcome from the mean is always $\frac{5}{2}$. We see that both the original die and the modified die have the same mean, but the modified die tends to have outcomes that have larger deviation from the mean.

In probability theory, there is a generally accepted way of measuring the derivation from the mean that occurs in a random variable. This is called the **variance** of the random variable. The variance of X is denoted $\boldsymbol{Var[X]}$, $\boldsymbol{V[X]}$, $\boldsymbol{\sigma_X^2}$ or $\boldsymbol{\sigma^2}$. The variance is defined as follows:
$\boldsymbol{Var[X] = E[(X-\mu_X)^2]}$ (the variance is the 2nd central moment of X about its mean).

It is possible to show that $\boldsymbol{Var[X] = E[X^2] - (E[X])^2 = E[X^2] - \mu_X^2}$.
This is usually the most efficient way to calculate variance.

The variance is a measure of the "dispersion" of X about the mean. A large variance indicates significant levels of probability or density for points far from $E[X]$. The variance is always ≥ 0. The variance of X is equal to 0 only if X has a discrete distribution with a single point and probability 1 at that point (not random at all).

For the original standard die toss example, we have
$E[X^2] = (1^2) \cdot (\frac{1}{6}) + (2^2) \cdot (\frac{1}{6}) + \cdots + (6^2) \cdot (\frac{1}{6}) = \frac{91}{6}$, and
$Var[X] = E[X^2] - (E[X])^2 = \frac{91}{6} - (\frac{7}{2})^2 = \frac{35}{12}$.

For the continuous random variable with density function $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ elsewhere} \end{cases}$, we have
$E[X^2] = \int_0^1 x^2 \cdot (2x)\,dx = \frac{1}{2}$, and $Var[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$.

Standard deviation of X: The standard deviation of the random variable X is the square root of the variance, and is denoted $\sigma_X = \sqrt{Var[X]}$. The **coefficient of variation** of X is $\frac{\sigma_X}{\mu_X}$.

Example 5-4: A continuous random variable X has density function

$$f_X(x) = \begin{cases} 1-|x| \text{ if } |x|<1 \\ 0, \text{ elsewhere} \end{cases} .$$

The continuous random variable W has density function

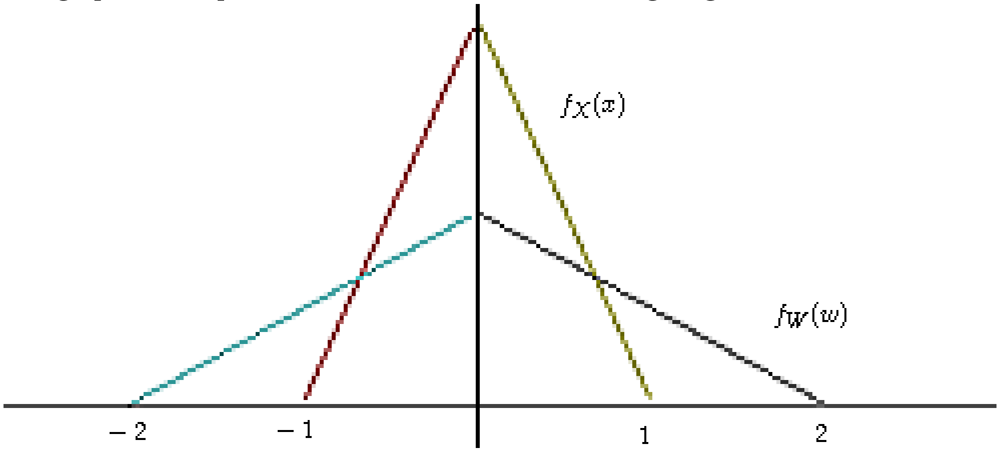
$$f_W(w) = \begin{cases} .5-.25|w| \text{ if } |w|<2 \\ 0, \text{ elsewhere} \end{cases} .$$

Find the mean and variance of X and W.

Solution: The density of X is symmetric about 0 (since $|-x| = |x|$, it follows that $f_X(x) = f_X(-x)$), so that $E[X] = 0$.
This can be verified directly:
$E[X] = \int_{-1}^1 x(1-|x|)\,dx = \int_{-1}^0 x(1+x)\,dx + \int_0^1 x(1-x)\,dx = -\frac{1}{6} + \frac{1}{6} = 0.$
Then, $Var[X] = E[X^2] - (E[X])^2 = E[X^2] = \int_{-1}^1 x^2(1-|x|)\,dx$
$= \int_{-1}^0 x^2(1+x)\,dx + \int_0^1 x^2(1-x)\,dx = \frac{1}{6}$.
The pdf of W is also symmetric about 0 for the same reason, and has mean $E[W] = 0$, or
$E[W] = \int_{-2}^2 x(.5-.25|x|)\,dx = \int_{-2}^0 x(.5+.25x)\,dx + \int_0^2 x(.5-.25x)\,dx = -\frac{1}{3} + \frac{1}{3} = 0.$
Then $Var[W] = E[W^2] - (E[W])^2 = E[W^2] = \int_{-2}^2 x^2(.5-.25|x|)\,dx$
$= \int_{-2}^0 x^2(.5+.25x)\,dx + \int_0^2 x^2(.5-.25x)\,dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.
The graphs of the pdfs of X and W are in the diagram on the following page. We see that the pdf of W is more widely dispersed about its mean than the pdf of X is, and so we would anticipate a larger variance for W by comparing the graphs. Comparison of pdf's to determine relative size of variance might not always be as straightforward as it is in
this example.

Example 5-4 continued

The graphs of the pdfs of X and W are in the following diagram:



□

Moment generating function of random variable X: The moment generating function of X (mgf) is denoted $M_X(t)$, $m_X(t)$, $M(t)$ or $m(t)$, and it is defined to be $\boldsymbol{M_X(t) = E[e^{tX}]}$.

If X is a discrete random variable then $\boldsymbol{M_X(t) = \Sigma e^{tx} p(x)}$.

If X is a continuous random variable then $\boldsymbol{M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx}$.

Some important properties that moment generating functions satisfy are

(i) It is always true that $M_X(0) = 1$. For instance, in the continuous case,
$M_X(0) = \int_{-\infty}^{\infty} e^{0 \cdot x} f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$.

(ii) The moments of X can be found from the successive derivatives of $M_X(t)$.
$\boldsymbol{M_X'(0) = E[X]}$, $\boldsymbol{M_X''(0) = E[X^2]}$, $M_X^{(n)}(0) = E[X^n]$, $\frac{d^2}{dt^2} ln[M_X(t)]\Big|_{t=0} = Var[X]$.

For instance, in the continuous case,
$M_X'(t) = \frac{d}{dt} M_X(t) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} \frac{d}{dt} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} x\, e^{tx} f(x)\, dx$, so that
$M_X'(0) = \int_{-\infty}^{\infty} x\, e^{0 \cdot x} f(x)\, dx = \int_{-\infty}^{\infty} x\, f(x)\, dx = E[X]$.

(iii) The moment generating function of X might not exist for all real numbers, but usually exists on some interval of real numbers.

Example 5-5: The pdf of X is $f(x) = 5e^{-5x}$ for $x > 0$. Find the moment generating function of X and use it to find the first and second moments of X, and the variance of X.

Solution: $M_X(t) = \int_0^\infty e^{tx} \cdot 5e^{-5x}\,dx = 5\int_0^\infty e^{-(5-t)x}\,dx = \frac{5}{5-t}$

(we have used the integration rule $\int_0^\infty e^{-at}\,dt = \frac{1}{a}$ if $a > 0$).

This integration is valid if $5 - t > 0$, or equivalently, if $t < 5$.

Then $\frac{d}{dt} M_X(t)\Big|_{t=0} = M_X'(0) = \frac{5}{(5-0)^2} = \frac{1}{5} = E[X]$,

and $\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = M_X''(0) = \frac{2\times 5}{(5-0)^3} = \frac{2}{25} = E[X^2]$.

The variance of X is $Var[X] = E[X^2] - (E[X])^2 = \frac{1}{25}$. □

Example 5-6: Find the moment generating function for each of these two random variables.

(i) $X =$ outcome of a die toss, $p(x) = P[X = x] = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$.

(ii) X is a continuous random variable with density function $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0\text{, elsewhere} \end{cases}$.

Solution: (i) $M_X(t) = E[e^{tX}] = \sum_{x=1}^{6} e^{tx} \cdot p(x)$

$= e^t \cdot \frac{1}{6} + e^{2t} \cdot \frac{1}{6} + e^{3t} \cdot \frac{1}{6} + e^{4t} \cdot \frac{1}{6} + e^{5t} \cdot \frac{1}{6} + e^{6t} \cdot \frac{1}{6} = \frac{1}{6} e^t (\frac{e^{6t}-1}{e^t-1})$.

Note that $M_X'(t) = e^t \cdot \frac{1}{6} + 2e^{2t} \cdot \frac{1}{6} + 3e^{3t} \cdot \frac{1}{6} + 4e^{4t} \cdot \frac{1}{6} + 5e^{5t} \cdot \frac{1}{6} + 6e^{6t} \cdot \frac{1}{6}$

and $\frac{d}{dt} M_X(t)\Big|_{t=0} = M_X'(0) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2} = E[X]$.

Also

$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = M_X''(0) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6} = E[X^2]$.

The variance of X is $E[X^2] - (E[X])^2 = \frac{91}{6} - (\frac{7}{2})^2 = \frac{35}{12}$.

(ii) $M_X(t) = \int_0^1 e^{tx} \cdot f(x)\,dx = \int_0^1 e^{tx} \cdot 2x\,dx = 2(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2})\Big|_{x=0}^{x=1} = 2(\frac{e^t}{t} - \frac{e^t-1}{t^2}) = M_X(t)$.

Then $M_X'(t) = 2(\frac{te^t-e^t}{t^2} - \frac{t^2e^t-2te^t+2t}{t^4}) = 2(\frac{t^2e^t-2te^t+2e^t-2}{t^3})$.

Note that $M_X'(0)$ is found as a limit, $\lim_{t\to 0} 2(\frac{t^2e^t-2te^t+2e^t-2}{t^3}) = \frac{2}{3} = E[X]$ (by l'Hospital's rule).

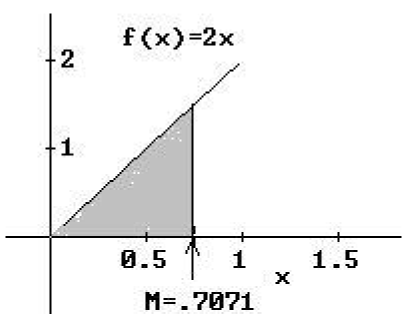
The antiderivative of $e^{tx} \cdot 2x$ was found by integration by parts. A useful point to note is the general antiderivative $\boldsymbol{\int xe^{cx}\,dx = \frac{xe^{cx}}{c} - \frac{e^{cx}}{c^2}}$, if c is a constant. This antiderivative has come up from time to time on previous exams. It is much more straightforward to find $E[X]$ and $E[X^2]$ directly as $\int_0^1 x \cdot f(x)\,dx = \int_0^1 x \cdot 2x\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$ and $\int_0^1 x^2 \cdot f(x)\,dx$ for the random variable in part (ii). □

Example 5-7: The moment generating function of X is given as $\frac{\alpha}{\alpha-t}$ for $t<\alpha$, where $\alpha>0$. Find $Var[X]$.

Solution: $Var[X]=E[X^2]-(E[X])^2$. $E[X]=M_X'(0)=\frac{\alpha}{(\alpha-t)^2}\Big|_{t=0}=\frac{1}{\alpha}$,
and $E[X^2]=M_X''(0)=\frac{2\alpha}{(\alpha-t)^3}\Big|_{t=0}=\frac{2}{\alpha^2}\rightarrow Var[X]=\frac{2}{\alpha^2}-(\frac{1}{\alpha})^2=\frac{1}{\alpha^2}$.
Alternatively, $ln\,M_X(t)=ln(\frac{\alpha}{\alpha-t})=ln\,\alpha-ln(\alpha-t)\rightarrow\frac{d}{dt}\,ln[M_X(t)]=\frac{1}{\alpha-t}$
and $\frac{d^2}{dt^2}\,ln[M_X(t)]=\frac{1}{(\alpha-t)^2}$ so that $Var[X]=\frac{d^2}{dt^2}\,ln[M_X(t)]\Big|_{t=0}=\frac{1}{\alpha^2}$.
This is like Example 5-5 above, with 5 replaced by α. □

Percentiles of a distribution: If $0<p<1$, then the $100p$-th percentile of the distribution of X is the number c_p which satisfies both of the following inequalities:
$P[X\le c_p]\ge p$ and $P[X\ge c_p]\ge 1-p$. For a continuous random variable, it is sufficient to find the c_p for which $P[X\le c_p]=p$. If $p=.5$, **the 50-th percentile of a distribution is referred to as the median of the distribution**; it is the point M for which $P[X\le M]=.5$. The median M is the 50% probability point, half of the distribution probability is to the left of M and half is to the right. If X has a symmetric distribution about the point $x=c$, then the mean and the median of X will be equal to c.

For the continuous random variable with density function $f(x)=\begin{cases}2x \text{ for } 0<x<1\\ 0, \text{ elsewhere}\end{cases}$,
the median is M, where $\int_0^M 2x\,dx=M^2=.5$, so that $M=\sqrt{.5}=.7071$. This is illustrated in the graph below. The shaded area below has probability .5, and is to the left of $M=.7071$, the median.



Example 5-8: The continuous random variable X has pdf $f(x) = \frac{1}{2} \cdot e^{-|x|}$ for $-\infty < x < \infty$. Find the 87.5-th percentile of the distribution.

Solution: The 87.5-th percentile is the number b for which

$.875 = P[X \leq b] = \int_{-\infty}^{b} f(x)\, dx = \int_{-\infty}^{b} \frac{1}{2} \cdot e^{-|x|}\, dx$.

Note that this distribution is symmetric about 0, since $f(-x) = f(x)$, so the mean and median are both 0. Thus, $b > 0$, and so

$\int_{-\infty}^{b} \frac{1}{2} \cdot e^{-|x|}\, dx = \int_{-\infty}^{0} \frac{1}{2} \cdot e^{-|x|}\, dx + \int_{0}^{b} \frac{1}{2} \cdot e^{-|x|}\, dx = .5 + \int_{0}^{b} \frac{1}{2} \cdot e^{-x}\, dx$

$= .5 + \frac{1}{2}(1 - e^{-b}) = .875 \Rightarrow b = -ln(.25) = ln\, 4$. □

The mode of a distribution: The mode is any point m at which the probability or density function $f(x)$ is maximized. The mode of the distribution in the graph above is 1, since the maximum value of $f(x)$ occurs at $x = 1$.

The skewness of a distribution: If the mean of random variable X is μ and the variance is σ^2 then the skewness is defined to be $E[(X - \mu)^3]/\sigma^3$. If skewness is positive, the distribution is said to be skewed to the right, and if skewness is negative it is skewed to the left.

Some results and formulas relating to distribution moments:

(i) The mean of a random variable X might not exist, it might be $+\infty$ or $-\infty$, and the variance of X might be $+\infty$. For example, the continuous random variable X with pdf $f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \geq 1 \\ 0, & \text{otherwise} \end{cases}$ has expected value $\int_{1}^{\infty} x \cdot \frac{1}{x^2}\, dx = +\infty$.

(ii) For any constants $a_1,\ a_2$ and b and functions h_1 and h_2,

$\boldsymbol{E[a_1 h_1(X) + a_2 h_2(X) + b] = a_1 E[h_1(X)] + a_2 E[h_2(X)] + b.}$

As a special case, $\boldsymbol{E[aX + b] = aE[X] + b}$.

(iii) If X is a random variable defined on the interval $[a, \infty)$ ($f(x) = 0$ for $x < a$), then $E[X] = a + \int_{a}^{\infty} [1 - F(x)]\, dx$, and if X is defined on the interval $[a, b]$, where $b < \infty$, then $E[X] = a + \int_{a}^{b} [1 - F(x)]\, dx$. This relationship is valid for any random variable, discrete, continuous or with a mixed distribution. As a special, case, if X is a non-negative random variable (defined on $[0, \infty)$ or $(0, \infty)$) then

$E[X] = \int_{0}^{\infty} [1 - F(x)]\, dx$

(iv) Jensen's inequality: If h is a function and X is a random variable such that $\frac{d^2}{dx^2}\,h(x) = h''(x) \geq 0$ at all points x with non-zero density or probability for X, then $E[h(X)] \geq h(E[X])$, and if $h'' > 0$ then $E[h(X)] > h(E[X])$. The inequality reverses if $h'' \leq 0$. For example, if $h(x) = x^2$, then $h''(x) = 2 \geq 0$ for any x, so that $E[X^2] \geq (E[X])^2$ (this is also true since $Var[X] = E[X^2] - (E[X])^2 \geq 0$ for any random variable X). As another example, if X is a positive random variable (i.e., X has non-zero density or probability only for $x \geq 0$), and $h(x) = \sqrt{x}$, then $h''(x) = \frac{-1}{4x^{3/2}} < 0$ for $x > 0$, and it follows from Jensen's inequality that $E[\sqrt{X}] < \sqrt{E[X]}$.

(v) If a and b are constants, then $Var[aX + b] = a^2 Var[X]$.

(vi) Chebyshev's inequality: If X is a random variable with mean μ_X and standard deviation σ_X, then for any real number $r > 0$, $P[|X - \mu_X| > r\sigma_X] \leq \frac{1}{r^2}$.

(vii) Suppose that for the random variable X, the moment generating function $M_X(t)$ exists in an interval containing the point $t = 0$. Then $\boldsymbol{\frac{d^n}{dt^n}\, M_X(t)\Big|_{t=0} = M_X^{(n)}(0) = E[X^n]}$, the n-th moment of X, and
$\frac{d}{dt}\, ln[M_X(t)]\Big|_{t=0} = \frac{M_X'(0)}{M_X(0)} = E[X]$, and $\frac{d^2}{dt^2}\, ln[M_X(t)]\Big|_{t=0} = Var[X]$.

The Taylor series expansion of $M_X(t)$ expanded about the point $t = 0$ is
$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, E[X^k] = 1 + t \cdot E[X] + \frac{t^2}{2} \cdot E[X^2] + \frac{t^3}{6} \cdot E[X^3] + \cdots$
Therefore, if we are given a moment generating function and we are able to formulate the Taylor series expansion about the point $t = 0$, we can identify the successive moments of X.

If X has a discrete distribution with probability space $\{x_1, x_2, x_3, ...\}$ and probability function $P(X = x_k) = p_k$, then the moment generating function is
$M_X(t) = e^{tx_1} \cdot p_1 + e^{tx_2} \cdot p_2 + e^{tx_3} \cdot p_3 + \cdots$.
Conversely, if we are given a moment generating function in this form (a sum of exponential factors), then we can identify the points of probability and their probabilities. This is illustrated in Example 15-17 below.

If X_1 and X_2 are random variables, and $M_{X_1}(t) = M_{X_2}(t)$ for all values of t in an interval containing $t = 0$, then X_1 and X_2 have identical probability distributions.

(viii) The median (50-th percentile) and other percentiles of a distribution are not always unique. For example, if X is the discrete random variable with probability function $f(x) = .25$ for $x = 1, 2, 3, 4$, then the median of X would be any point from 2 to 3, but the usual convention is to set the median to be the midpoint between the two "middle" values of X, $M = 2.5$.

(ix) The distribution of the random variable X is said to be **symmetric about the point c** if $f(c+t) = f(c-t)$ for any value of t. It follows that the expected value of X and the median of X is c. Also, for a symmetric distribution, any odd-order central moments about the mean are 0, this means that $E[(X-\mu)^k] = 0$ if k is an odd integer ≥ 1 .

To see that the mean is c

$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{c} x \cdot f(x)\, dx + \int_{c}^{\infty} x \cdot f(x)\, dx$.

If we apply the change of variable $t = x - c$ to the each integral on the right, the first becomes $\int_{-\infty}^{0} (c+t) \cdot f(c+t)\, dt = c\int_{-\infty}^{0} f(c+t)\, dt + \int_{-\infty}^{0} t \cdot f(c+t)\, dt$, and the second becomes $\int_{0}^{\infty} (c+t) \cdot f(c+t)\, dt = c\int_{0}^{\infty} f(c+t)\, dt + \int_{0}^{\infty} t \cdot f(c+t)\, dt$.

Then, $\int_{-\infty}^{0} f(c+t)\, dt + \int_{0}^{\infty} f(c+t)\, dt = \int_{-\infty}^{\infty} f(c+t)\, dt = 1$ (this can be seen if we change the variable to $u = c+t$, and the integral becomes $\int_{-\infty}^{\infty} f(u)\, du = 1$ since f is a pdf). Also, $\int_{-\infty}^{0} t \cdot f(c+t)\, dt + \int_{0}^{\infty} t \cdot f(c+t)\, dt = 0$ (this can be seen if we change the variable in the first integral to $u = -t$, so the first integral becomes $-\int_{0}^{\infty} u \cdot f(c+u)\, du$, which is the negative of the second integral.)

(x) If $E[X] = \mu$, $Var[X] = \sigma^2$ and $Z = \frac{X-\mu}{\sigma}$, then $E[Z] = 0$ and $Var[Z] = 1$.

(xi) A "mixture" of distributions: Given any finite collection of random variables, $X_1, X_2, \ldots, X_k$ with density or probability functions, say $f_1(x), f_2(x), \ldots f_k(x)$, where k is a non-negative integer, and given a set of "weights", $\alpha_1, \alpha_2, \ldots, \alpha_k$, where $0 \leq \alpha_i \leq 1$ for each i and $\sum_{i=1}^{k} \alpha_i = 1$, it is possible to construct a new density function:

$f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_k f_k(x)$, which is a "weighted average" of the original density functions. It then follows that the resulting distribution X, whose density/probability function is f, has moments and moment generating function which are weighted averages of the original distribution moments and moment generating functions:

$E[X^n] = \alpha_1 E[X_1^n] + \alpha_2 E[X_2^n] + \cdots + \alpha_k E[X_k^n]$ and

$M_X(t) = \alpha_1 M_{X_1}(t) + \alpha_2 M_{X_2}(t) + \cdots + \alpha_k M_{X_k}(t)$.

Mixtures of distributions will be considered again later in the study guide.

Example 5-9: The skewness of a random variable is defined to be $\frac{E[(X-\mu)^3]}{\sigma^3}$, where $\mu = E[X]$ and $\sigma^2 = Var[X]$. Find the skewness of the random variable X with pdf $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0\text{, elsewhere} \end{cases}$.

Solution: $\mu = E[X] = \int_0^1 x \cdot (2x)\,dx = \frac{2}{3}$, $E[X^2] = \int_0^1 x^2 \cdot (2x)\,dx = \frac{1}{2}$.

$\sigma^2 = Var[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$.

$E[(X-\mu)^3] = E[(X-\frac{2}{3})^3] = \int_0^1 (x-\frac{2}{3})^3 \cdot (2x)\,dx$

$= \int_0^1 (x^3 - 2x^2 + \frac{4}{3}x - \frac{8}{27})(2x)\,dx = -\frac{1}{135}$.

Then $\frac{E[(X-\mu)^3]}{\sigma^3} = \frac{-1/135}{(1/18)^{3/2}} = -.566$.

Since the skewness is negative, this random variable is said to be skewed to the left (of its mean). □

Example 5-10: In Neverland there is a presidential election every year. The president must put his investments into a blind trust that earns compound interest at rate 10% (compounded annually). Neverland has no term limits for elected officials. The current president was just elected and has a blind trust worth $1,000,000 right now. The current president is very popular and he assesses his chance of being re-elected to be .75 each year from now on. The next election is one year from now, and elections will continue every year. Assuming the re-election probability stays the same year after year, find the expected value of the blind trust when the current president first loses an election in the future.

Solution: Let X denote the number of years until the current president is not re-elected. The distribution of X is

X :	1	2	3	$\ldots$	n	$\ldots$
$p(x)$:	$.25$	$(.75)(.25)$	$(.75)^2(.25)$	$\ldots$	$(.75)^{n-1}(.25)\ldots$	
Blind Trust	$1.1M$	$(1.1)^2M$	$(1.1)^3M$		$(1.1)^nM$	

The expected value of the blind trust at the time the current president first loses an election is

$E[(1.1)^X] = (1.1)(.25) + (1.1)^2(.75)(.25) + (1.1)^3(.75)^2(.25) + \cdots$

$= (1.1)(.25)[1 + (1.1)(.75) + ((1.1)(.75))^2 + \cdots] = (1.1)(.25)[\frac{1}{1-(1.1)(.75)}] = 1.57$

(million). Note that there is implicit assumption that there is no upper limit on how long the current president can survive. □

Example 5-11: Smith finds the carnival game "over-under-seven" irresistible. The game involves the random toss of two fair dice. If a player bets 1 on "over" and the total on the dice is over 7, then the player wins 1 (otherwise he loses the 1 he bet). If he bets 1 on "under" and the total on the dice is under 7 then he wins 1 (otherwise he loses). If he bets 1 on "seven" and the total on the dice is 7 then he wins 4 (otherwise he loses 1). Smith devises the following strategy. His first bet is 1 on "under". If he wins, he walks away with his net gain of 1. If he loses, he doubles his bet to 2 on "under". If he wins, he walks away with his net gain of 1. If he loses, he doubles his bet to 4 and bets on "under", etc. Smith walks away as soon as he wins. Find his expected gain, and the number of bets it will take to win, on average.

Solution: Smith's net gain is 1 as soon as he first wins. Therefore, whenever he wins, his net gain is 1, so his expected gain is 1. The probability of winning when betting "under" is $\frac{15}{36} = \frac{5}{12}$, since there are 15 winning dice combinations out of a total of $6 \times 6 = 36$ dice combinations. 1-1 , 1-2 , 1-3 , 1-4 , 1-5 , 2-1 , 2-2 , 2-3 , 2-4 , 3-1 , 3-2 , 3-3 , 4-1 , 4-2 , 5-1 are winners for "under". Let X be the bet number of his first win. The probability function for X is

X :	1	2	3	...	n
$p(x)$	$\frac{5}{12}$	$(\frac{7}{12})(\frac{5}{12})$	$(\frac{7}{12})^2(\frac{5}{12})$	...	$(\frac{7}{12})^{n-1}(\frac{5}{12})$

Then $E[X] = \sum_{n=1}^{\infty} n \cdot (\frac{7}{12})^{n-1}(\frac{5}{12}) = (\frac{5}{12}) \cdot [1 + 2(\frac{7}{12}) + 3(\frac{7}{12})^2 + \cdots]$

$= (\frac{5}{12}) \cdot [\frac{1}{(1-\frac{7}{12})^2}] = \frac{12}{5} = 2.4$.

We have used the identity $1 + 2a + 3a^2 + \cdots = \frac{1}{(1-a)^2}$ for $|a| < 1$. □

Example 5-12: The pdf of X_1 is $f_1(x) = e^{-x}$ for $x > 0$ and the pdf for X_2 is $f_2(x) = 2e^{-2x}$ for $x > 0$. We define a new random variable X with pdf $f(x) = .5e^{-x} + e^{-2x}$ for $x > 0$. Find the mean and variance of X.

Solution: $f(x) = (.5)(e^{-x}) + (.5)(2e^{-2x})$, which shows that X is a mixture of X_1 and X_2 with mixing weights .5 for X_1 and .5 for X_2 . The first and second moments of X_1 and X_2 are

$E[X_1] = \int_0^{\infty} x \cdot e^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_{x=0}^{x=\infty} = 1$,

$E[X_1^2] = \int_0^{\infty} x^2 \cdot e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_{x=0}^{x=\infty} = 2$.

$E[X_2] = \int_0^{\infty} x \cdot 2e^{-2x}\,dx = -xe^{-2x} - \frac{1}{2}e^{-x}\Big|_{x=0}^{x=\infty} = \frac{1}{2}$,

$E[X_2^2] = \int_0^{\infty} x^2 \cdot 2e^{-2x}\,dx = -x^2e^{-2x} - xe^{-2x} - \frac{1}{2}e^{-2x}\Big|_{x=0}^{x=\infty} = \frac{1}{2}$.

Then $E[X] = \int_0^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot [(.5)f_1(x) + (.5)f_2(x)]\,dx$

$= (.5)E[X_1] + (.5)E[X_2] = (.5)(1) + (.5)(\frac{1}{2}) = .75$,

and similarly, $E[X^2] = (.5)E[X_1^2] + (.5)E[X_2^2] = (.5)(2) + (.5)(\frac{1}{2}) = 1.25$.

Then, $Var[X] = E[X^2] - (E[X])^2 = 1.25 - (.75)^2 = .6875$.

Note that $Var[X]$ is <u>not</u> $.5Var[X_1] + .5Var[X_2]$. □

Example 5-13: The pdf of X is $f_X(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ 0, \text{ otherwise} \end{cases}$.

Find the mean and cdf of the conditional distribution of X given that $X \leq \frac{1}{2}$.

Solution: If A is the event that $X \leq \frac{1}{2}$, then $P(A) = P[X \leq \frac{1}{2}] = \frac{1}{4}$,

and $f_{X|A}(x|X \leq \frac{1}{2}) = \frac{2x}{1/4} = 8x$, for $0 < x \leq \frac{1}{2}$, and $f_{X|A}(x|X \leq \frac{1}{2}) = 0$, otherwise.

$E[X|X \leq \frac{1}{2}] = \int_0^{1/2} x \cdot f_{X|A}(x|X \leq \frac{1}{2})\, dx = \int_0^{1/2} x \cdot 8x\, dx = \frac{1}{3}$.

The cdf of the conditional distribution is

$F_{X|A}(t) = P[X \leq t|X \leq \frac{1}{2}] = \int_0^t f_{X|A}(x)\, dx = \int_0^t 8x\, dx = 4t^2$ for $0 \leq t \leq \frac{1}{2}$,

and $F_{X|A}(t) = 1$ for $t > \frac{1}{2}$.

Note also that we can find the cdf of the conditional distribution from

$F_{X|A}(t) = P[X \leq t|X \leq \frac{1}{2}] = \frac{P[X \leq t \cap X \leq \frac{1}{2}]}{P[X \leq \frac{1}{2}]} = \frac{t^2}{1/4}$ for $t \leq \frac{1}{2}$. □

Example 5-14: The continuous random variable X has pdf $f(x) = 1$ for $0 < x < 1$.

X_1, X_2 and X_3 are independent random variables, all with the same distribution as X.

$Y = max\{X_1, X_2, X_3\}$ and $Z = min\{X_1, X_2, X_3\}$. Find $E[Y - Z]$.

Solution: When dealing with the maximum or minimum of a collection of random variables, it is usually most efficient to work with the cdf F or the survival function $S = 1 - F$.

The cdf of Y is $F_Y(y) = P(Y \leq y) = P(max\{X_1, X_2, X_3\} \leq y)$.

In order for the inequality $max\{X_1, X_2, X_3\} \leq y$ to be true, it must be true that each of $X_1, X_2,$ and X_3 are $\leq y$. Therefore, $P(max\{X_1, X_2, X_3\} \leq y) = P(X_1 \leq y \cap X_2 \leq y \cap X_3 \leq y)$.

Since X_1, X_2 and X_3 are independent random variables, this is equal to

$P(X_1 \leq y) \cdot P(X_2 \leq y) \cdot P(X_3 \leq y) = [F_X(y)]^3$.

From the definition of the distribution of X, we have $F_X(x) = P(X \leq x) = \int_0^x 1\, dt = x$.

It follows that $F_Y(y) = y^3$. Since X is defined on the interval $0 < x < 1$, the same is true for Y. We can find $E[Y]$ by first finding $f_Y(y) = F_Y'(y) = 3y^2$, and then

$E[Y] = \int_0^1 y \cdot f_Y(y)\, dy = \int_0^1 y \cdot 3y^2\, dy = \int_0^1 3y^3\, dy = \frac{3}{4}$.

Since $Y \geq 0$ (Y is a non-negative random variable), we can also formulate the mean of Y as $E[Y] = \int_0^\infty [1 - F_Y(y)]\, dy$. We note that $F_Y(y) = y^3$ for $y \geq 1$, and therefore,

$E[Y] = \int_0^1 (1 - y^3)\, dy = \frac{3}{4}$.

The survival function of Z is $S_Z(z) = P(Z > z)] = P(min\{X_1, X_2, X_3\} > z)$.

In order for the inequality $min\{X_1, X_2, X_3\} > z$ to be true, it must be true that each of $X_1, X_2,$ and X_3 are $> z$. Therefore, $P(min\{X_1, X_2, X_3\} > z) = P(X_1 > z \cap X_2 > z \cap X_3 > z)$.

Example 15-14 continued

Since X_1, X_2 and X_3 are independent random variables, $S_Z(z)$ is equal to

$S_Z(z) = P(X_1 > z) \cdot P(X_2 > z) \cdot P(X_3 > z) = [S_X(z)]^3 = [1 - F_X(z)]^3 = [1 - z]^3$.

Then $f_Z(z) = F'_Z(z) = -S'_Z(z) = 3(1-z)^2$ (using the chain rule), and Z is defined on the interval $0 < z < 1$.

Then, $E[Z] = \int_0^1 z \cdot f_Z(z)\,dz = \int_0^1 z \cdot 3(1-z)^2\,dz = \frac{1}{4}$.

Finally, $E[Y - Z] = E[Y] - E[Z] = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

Alternatively, since $F_Z(z) = 1$ for $z \geq 1$, we have

$E[Z] = \int_0^\infty [1 - F_Z(z)]\,dz = \int_0^1 [1 - F_Z(z)]\,dx = \int_0^1 S_Z(z)\,dz = \int_0^1 (1-z)^3\,dz = \frac{1}{4}$. □

Example 5-15: Smith and Jones both love to gamble. Smith finds a casino that offers a game of chance in which a fair coin is tossed until the first head appears. If the first head appears on toss number X $(X = 1, 2, 3, ...)$, the game pays out $\$3000(1 - \frac{1}{2^X})$. The casino charges \$2100 to play the game. Jones realizes that, on average, the first head will occur on the 2nd toss, so he estimates that the on average, the payout on a game will be $3000(1 - \frac{1}{2^2}) = 2250$, and thinks this is a good game to play. Smith, who is a deeper thinker than Jones, uses Jensen's inequality to get an idea of what the expected payout will be. Smith then calculates the exact expected payout. Describe Smith's conclusions:

Solution: The amount paid out in the game is $h(x) = 3000(1 - \frac{1}{2^x})$.

Smith notices that $h'(x) = 3000(\frac{1}{2^x})(ln\,2)$ and $h''(x) = -3000(\frac{1}{2^x})(ln\,2)^2 < 0$.

It follows from Jensen's inequality that $E[h(X)] < h(E[X])$.

$E[h(X)] = E[3000(1 - \frac{1}{2^X})]$ is the expected payout, and

$h(E[X]) = 3000(1 - \frac{1}{2^{E[X]}}) = 3000(1 - \frac{1}{2^2}) = 2250$

(we can find $E[X]$ in a way similar to the method applied in Example 5-1 above, where we found the expected toss number of the first "1" when a fair die is repeatedly tossed; for the toss of the fair coin, the expected toss number of the first head is $E[X] = 2$).

Jensen's inequality shows that the expected payout will be less than 2250.

The exact expected payout is $E[3000\ (1 - \frac{1}{2^X})] = 3000(1 - E[\frac{1}{2^X}])$, so we find

$E[\frac{1}{2^X}] = \frac{1}{2} \cdot P(X = 1) + \frac{1}{2^2} \cdot P(X = 2) + \frac{1}{2^3} \cdot P(X = 3) + \cdots$

$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2^2} \cdot \frac{1}{2^2} + \frac{1}{2^3} \cdot \frac{1}{2^3} + \cdots = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}$.

Then $E[3000\ (1 - \frac{1}{2^X})] = 3000(1 - \frac{1}{3}) = 2000$ is the expected payout. □

Example 5-16: For each of the following random variables, use Chebyshev's inequality to get an upper bound on the probabilities $P[\,|X-E[X]|\,>\sqrt{Var[X]}\,]$ and $P[\,|X-E[X]|\,>2\sqrt{Var[X]}\,]$, and also calculate the exact probabilities.

(a) X is the result of a fair die toss, 1,2,3,4,5,6 , each with probability $\frac{1}{6}$.

(b) X is continuous with pdf $f_X(x)=1$ for $0<x<1$.

(c) X is continuous with pdf $f_X(x)=e^{-x}$ for $0<x$.

Solution: According to Chebyshev's inequality, $P[\,|X-E[X]|\,>r\sqrt{Var[X]}\,]\le\frac{1}{r^2}$.

Therefore, with $r=1$, using Chebyshev's inequality, we have

$P[\,|X-E[X]|\,>\sqrt{Var[X]}\,]\le 1$ (since any probability must be ≤ 1, this will always be true).

With $r=2$, using Chebyshev's inequality, we have $P[\,|X-E[X]|\,>2\sqrt{Var[X]}\,]\le\frac{1}{4}$.

(a) $E[X]=\frac{7}{2}$, $Var[X]=E[X^2]-(E[X])^2=\frac{35}{12}$.

Exact probabilities are

$P[\,|X-E[X]|\,>\sqrt{Var[X]}\,]=P[\,|X-\frac{7}{2}|\,>\sqrt{\frac{35}{12}}\,]=1-P[\,|X-\frac{7}{2}|\,\le\sqrt{\frac{35}{12}}\,]$

$=1-P[-\sqrt{\frac{35}{12}}<X-\frac{7}{2}<\sqrt{\frac{35}{12}}]=1-P[1.79<X<5.2]$

$=1-P[X=2,3,4,5]=\frac{2}{6}\le 1$, and

$P[\,|X-E[X]|\,>2\sqrt{Var[X]}\,]=P[\,|X-\frac{7}{2}|\,>2\sqrt{\frac{35}{12}}\,]=1-P[\,|X-\frac{7}{2}|\,\le 2\sqrt{\frac{35}{12}}\,]$

$=1-P[-2\sqrt{\frac{35}{12}}<X-\frac{7}{2}<2\sqrt{\frac{35}{12}}]=1-P[.08<X<6.9]$

$=1-P[X=1,2,3,4,5,6]=0\le\frac{1}{4}$.

(b) $E[X]=\int_0^1 x\,dx=\frac{1}{2}$ and $E[X^2]=\int_0^1 x^2\,dx=\frac{1}{3}$, so $Var[X]=\frac{1}{3}-(\frac{1}{2})^2=\frac{1}{12}$.

Exact probabilities are

$P[\,|X-E[X]|\,>\sqrt{Var[X]}\,]=P[\,|X-\frac{1}{2}|\,>\sqrt{\frac{1}{12}}\,]=1-P[\,|X-\frac{1}{2}|\,\le\sqrt{\frac{1}{12}}\,]$

$=1-P[-\sqrt{\frac{1}{12}}<X-\frac{1}{2}<\sqrt{\frac{1}{12}}]=1-P[\frac{1}{2}-\sqrt{\frac{1}{12}}<X<\frac{1}{2}+\sqrt{\frac{1}{12}}]$

$=1-2\sqrt{\frac{1}{12}}=.423\le 1$

and

$P[\,|X-E[X]|\,>2\sqrt{Var[X]}\,]=P[\,|X-\frac{1}{2}|\,>2\sqrt{\frac{1}{12}}\,]=1-P[\,|X-\frac{1}{2}|\,\le 2\sqrt{\frac{1}{12}}\,]$

$=1-P[-2\sqrt{\frac{1}{12}}<X-\frac{1}{2}<2\sqrt{\frac{1}{12}}]=1-P[\frac{1}{2}-2\sqrt{\frac{1}{12}}<X<\frac{1}{2}+2\sqrt{\frac{1}{12}}]$

$=1-P[-.077<X<1.077]=0\le\frac{1}{4}$.

Example 5-16 continued

(c) $E[X]=\int_0^\infty x\cdot e^{-x}\,dx=1$ and $E[X^2]=\int_0^\infty x^2\cdot e^{-x}\,dx=2$ (we use the general rule $\int_0^\infty x^k\cdot e^{-cx}\,dx=\frac{k!}{c^{k+1}}$ if k is an integer and $c>0$).

Then $Var[X]=E[X^2]-(E[X])^2=1$.

Exact probabilities are

$P[\,|X-E[X]|\,>\sqrt{Var[X]}\,]=P[\,|X-1|\,>1\,]=1-P[\,|X-1|\,\le 1\,]$

$=1-P[-1<X-1<1]=1-P[0<X<2]=1-\int_0^2 e^{-x}\,dx$

$=1-(1-e^{-2})=e^{-2}=.135\le 1$

and

$P[\,|X-E[X]|\,>2\sqrt{Var[X]}\,]=P[\,|X-1|\,>2\,]=1-P[\,|X-1|\,\le 2\,]$

$=1-P[-2<X-1<2]=1-P[-1<X<3]=1-P[0<X<3]$

$=1-\int_0^3 e^{-x}\,dx=1-(1-e^{-3})=e^{-3}=.050\le\frac{1}{4}$. □

Example 5-17: Each of the following is a moment generating function for a discrete non-negative integer-valued random variable. For each random variable, find the mean, variance and the probability $P(X\le 2)$.

(a) $\frac{1}{6}(1+2e^t+3e^{2t})$,

(b) e^{2e^t-2} ,

Solution: (a) $M'(t)=\frac{1}{6}(2e^t+6e^{2t})\rightarrow M'(0)=\frac{4}{3}=E[X]$,

$M''(t)=\frac{1}{6}(2e^t+12e^{2t})\rightarrow M''(0)=\frac{7}{3}=E[X^2]\rightarrow Var[X]=\frac{7}{3}-(\frac{4}{3})^2=\frac{5}{9}$.

$M(t)=p_0+p_1e^t+p_2e^{2t}+p_3e^{3t}+\cdots=\frac{1}{6}(1+2e^t+3e^{2t})=\frac{1}{6}+\frac{1}{3}e^t+\frac{1}{2}e^{2t}$

$\rightarrow p_0=\frac{1}{6}$, $p_1=\frac{1}{3}$, $p_2=\frac{1}{2}$, and $P(X\le 2)=1$.

(b) $ln[M(t)]=2e^t-2\rightarrow\frac{d}{dt}\,ln[M(t)]=2e^t\rightarrow\frac{d}{dt}\,ln[M(t)]\Big|_{t=0}=2=E[X]$.

$\frac{d^2}{dt^2}\,ln[M(t)]=2e^t\rightarrow\frac{d}{dt^2}\,ln[M(t)]\Big|_{t=0}=2=Var[X]$.

$M(t)=e^{-2}e^{2e^t}$. We use the Taylor expansion $e^y=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots$, so that

$e^{2e^t}=1+2e^t+\frac{(2e^t)^2}{2!}+\frac{(2e^t)^3}{3!}+\cdots$, and then

$M(t)=e^{-2}\cdot[1+2e^t+\frac{(2e^t)^2}{2!}+\frac{(2e^t)^3}{3!}+\cdots]=e^{-2}+2e^{-2}e^t+2e^{-2}e^{2t}+\frac{4}{3}e^{-2}e^{3t}+\cdots$.

Therefore, $p_0=e^{-2}$, $p_1=2e^{-2}$, $p_2=2e^{-2}$, $p_3=\frac{4}{3}e^{-2}$, . . . ,

and $P(X\le 2)=e^{-2}+2e^{-2}+2e^{-2}=5e^{-2}$. □

PROBLEM SET 5

Expectation and Other Distribution Parameters

1. If X is a random variable with density function $f(x) = \begin{cases} 1.4e^{-2x}+.9e^{-3x} & \text{for } x>0 \\ 0, & \text{elsewhere} \end{cases}$,

then $E[X] =$

A) $\frac{9}{20}$ B) $\frac{5}{6}$ C) 1 D) $\frac{230}{126}$ E) $\frac{23}{10}$

2. Let X be a continuous random variable with density function

$f(x) = \begin{cases} \frac{1}{30}x(1+3x) & \text{for } 1<x<3 \\ 0, & \text{otherwise} \end{cases}$. Find $E[\frac{1}{X}]$.

A) $\frac{1}{12}$ B) $\frac{7}{15}$ C) $\frac{45}{103}$ D) $\frac{11}{20}$ E) $\frac{14}{15}$

3. (SOA) Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & \text{for } -2 \le x \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X.

A) $\frac{1}{5}$ B) $\frac{3}{5}$ C) 1 D) $\frac{28}{15}$ E) $\frac{12}{5}$

4. (SOA) An insurer's annual weather related loss, X, is a random variable with density function

$$f(x) = \begin{cases} \frac{2.5(200)^{2.5}}{x^{3.5}} & \text{for } x \ge 200 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30th and 70th percentiles of X.

A) 35 B) 93 C) 124 D) 131 E) 298

5. Let X be a continuous random variable with density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x>0 \\ 0, \text{otherwise} \end{cases}$

If the median of this distribution is $\frac{1}{3}$, then $\lambda =$

A) $\frac{1}{3}\,ln\,\frac{1}{2}$ B) $\frac{1}{3}\,ln\,2$ C) $2\,ln\,\frac{3}{2}$ D) $3\,ln\,2$ E) 3

6. Let X be a continuous random variable with density function

$f(x) = \begin{cases} \frac{1}{9}x(4-x) & \text{for } 0<x<3 \\ 0, \text{otherwise} \end{cases}$. What is the mode of X?

A) $\frac{4}{9}$ B) 1 C) $\frac{3}{2}$ D) $\frac{7}{4}$ E) 2

7. (SOA) A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260 . If a tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive), what will be the variance of the annual cost of maintaining and repairing a car?

A) 208 B) 260 C) 270 D) 312 E) 374

8. A system made up of 7 components with independent, identically distributed lifetimes will operate until any of 1 of the system's components fails. If the lifetime X of each component has density function $f(x) = \begin{cases} \frac{3}{x^4} & \text{for } 1<x \\ 0, \text{otherwise} \end{cases}$,

what is the expected lifetime until failure of the system?

A) 1.02 B) 1.03 C) 1.04 D) 1.05 E) 1.06

9. (SOA) A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim Size	20	30	40	50	60	70	80
Probability	.15	.10	.05	.20	.10	.10	.30

What percentage of the claims are within one standard deviation of the mean claim size?

A) 45% B) 55% C) 68% D) 85% E) 100%

10. (SOA) An actuary determines that the claim size for a certain class of accidents is a random variable, X, with moment generating function $M_X(t) = \frac{1}{(1-2500t)^4}$.

Determine the standard deviation of the claim size for this class of accidents.

A) 1,340 B) 5,000 C) 8,660 D) 10,000 E) 11,180

11. Let X be a random variable with mean 0 and variance 4. Calculate the largest possible value of $P[|X| \geq 8]$, according to Chebyshev's inequality.

A) $\frac{1}{16}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$ E) $\frac{15}{16}$

12. If the moment generating function for the random variable X is $M_X(t) = \frac{1}{1+t}$,

find the third moment of X about the point $x = 2$.

A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) $\frac{3}{2}$ D) -38 E) $-\frac{19}{3}$

13. (SOA) A company insures homes in three cities, J, K, and L . Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent. The moment generating functions for the loss distributions of the cities are:

$$M_J(t) = (1-2t)^{-3} \quad , \quad M_K(t) = (1-2t)^{-2.5} \quad , \quad M_L(t) = (1-2t)^{-4.5}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

A) 1,320 B) 2,082 C) 5,760 D) 8,000 E) 10,560

14. (SOA) Let X_1 , X_2 , X_3 be a random sample from a discrete distribution with probability function

$$p(x) = \begin{cases} 1/3 & \text{for } x = 0 \\ 2/3 & \text{for } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the moment generating function, $M(t)$, of $Y = X_1X_2X_3$.

A) $\frac{19}{27} + \frac{8}{27}e^t$ B) $1 + 2e^t$ C) $(\frac{1}{3} + \frac{2}{3}e^t)^3$ D) $\frac{1}{27} + \frac{8}{27}e^{3t}$ E) $\frac{1}{3} + \frac{2}{3}e^{3t}$

15. Two balls are dropped in such a way that each ball is equally likely to fall into any one of four holes. Both balls may fall into the same hole. Let X denote the number of unoccupied holes at the end of the experiment. What is the moment generating function of X?

A) $\frac{7}{4} - \frac{1}{2}t$ if $t = 2$ or 3, 0 otherwise B) $\frac{9}{4}t + \frac{21}{8}t^2$ C) $\frac{1}{4}(3e^{2t} + e^{3t})$

D) $\frac{1}{4}(e^{2t} + e^{3t})$ E) $\frac{1}{4}(e^{3t/4} + 3e^{t/4})$

16. A lottery is designed so that the winning number is a randomly chosen 4-digit number (0000 to 9999). The prize is designed as follows: if your ticket matches the last 2 digits (in order) of the winning number, you win \$50 , match last 3 digits (in order) and win \$500 (but not the \$50 for matching the last 2), match all 4 digits in order and win \$5000 (but not \$500 or \$50). The cost to buy a lottery ticket is \$2. Find the ticket holder's expected net gain.

A) -1.40 B) $-.60$ C) 0 D) $.60$ E) 1.40

17. (SOA) An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$, where $0 < x < \infty$. Determine the company's expected monthly claims.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) 1 E) 3

18. The cumulative distribution function for a loss random variable X is

$F(x) = \begin{cases} 0, \text{ for } x<0 \\ 1-\frac{1}{2}e^{-x}, \text{ for } x \geq 0 \end{cases}$. Find the moment generating function of X as a function of t.

A) $\frac{1}{1-t}$, $t < 1$ B) $\frac{1}{2-2t}$, $t < 1$ C) $\frac{2-t}{2-2t}$, $t < 1$

D) $\frac{1}{2t} + \frac{1}{2(1+t)}$, $t < 1$ E) Undefined

19. Suppose that the random variable X has moment generating function $M(t) = \frac{e^{at}}{1-bt^2}$ for $-1 < t < 1$. It is found that the mean and variance of X are 3 and 2 respectively. Find $a + b$.

A) 0 B) 1 C) 2 D) 3 E) 4

20. An actuary uses the following distribution for the random variable T the time until death for a new born baby : $f(t) = \frac{t}{5000}$ for $0 < t < 100$. At the time of birth an insurance policy is set up to pay an amount of $(1.1)^t$ at time t if death occurs at that instant. Find the expected payout on this insurance policy. (nearest 100).

A. 2000 B. 2200 C. 2400 D. 2600 E. 2800

21. A life insurer has created a special one year term insurance policy for a pair of business people who travel to high risk locations. The insurance policy pays nothing if neither die in the year, \$100,000 if exactly one of the two die, and \$$K > 0$ if both die. The insurer determines that there is a probability of .1 that at least one of the two will die during the year and a probability of .08 that exactly one of the two will die during the year. You are told that the standard deviation of the payout is \$74,000. Find the expected payout for the year on this policy.

A. 18,000 B. 21,000 C. 24,000 D. 27,000 E. 30,000

22. The board of directors of a corporation wishes to purchase "headhunter insurance" to cover the cost of replacing up to 3 of the corporations high-ranking executives, should they leave during the next year to take another job. The board wants the insurance policy to pay \$1,000,000 $\times K^2$, where $K = 0, 1, 2$ or 3 is the number of the three executives that leave within the next year. An actuary analyzes the past experience of the corporation's retention of executives at that level, and estimates the following probabilities for the number who will leave:

$P[K = 0] = .8$, $P[K = 1] = .1$, $P[K = 2] = P[K = 3] = .05$.

Find the expected payment the insurer will make for the year on this policy.

A. 250,000 B. 500,000 C. 750,000 D. 1,000,000 E. 2,000,000

23. (SOA) A random variable has the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x^2 - 2x + 2}{2} & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

Calculate the variance of X

A) $\frac{7}{72}$ B) $\frac{1}{8}$ C) $\frac{5}{36}$ D) $\frac{4}{32}$ E) $\frac{23}{12}$

24. Smith is offered the following gamble: he is to choose a coin at random from a large collection of coins and toss it randomly. $\frac{3}{4}$ of the coins in the collection are loaded towards a head (LH) and $\frac{1}{4}$ are loaded towards a tail. If a coin is loaded towards a head, then when the coin is tossed randomly, there is a $\frac{3}{4}$ probability that a head will turn up and a $\frac{1}{4}$ probability that a tail will turn up. Similarly, if the coin is loaded towards tails, then there is a $\frac{3}{4}$ chance of tossing a tail on any given toss. If Smith tosses a head, he loses $100, and if he tosses a tail, he wins $200. Smith is allowed to obtain "sample information" about the gamble. When he chooses the coin at random, he is allowed to toss it once before deciding to accept the gamble with that same coin. Suppose Smith tosses a head on the sample toss. Find Smith's expected gain/loss on the gamble if it is accepted.

A) loss of 20 B) loss of 10 C) loss of 0 D) gain of 10 E) gain of 20

25. The loss amount, X, for a medical insurance policy has cumulative distribution function

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right), & \text{for } 0 \le x \le 3 \\ 1, & \text{for } x > 3 \end{cases}$$

. Calculate the mode of the distribution.

A) 2/3 B) 1 C) 3/2 D) 2 E) 3

26. Smith is offered the following gamble: he is to choose a coin at random from a large collection of coins and toss it randomly. The proportion of the coins in the collection that are loaded towards a head is p. If a coin is loaded towards a head, then when the coin is tossed randomly, there is a $\frac{3}{4}$ probability that a head will turn up and a $\frac{1}{4}$ probability that a tail will turn up. Similarly, if the coin is loaded towards tails, then there is a $\frac{3}{4}$ probability that a tail will turn up and a $\frac{1}{4}$ probability that a head will turn up. If Smith tosses a head, he loses $100, and if he tosses a tail, he wins $200. Find the proportion p for which Smith's expected gain is 0 when taking the gamble.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{5}{6}$

27. The random variable X has density function $f(x) = ce^{-|x|}$ for $-\infty < x < \infty$.
Find the variance of X.

A) 0 B) .5 C) 1.0 D) 1.5 E) 2.0

PROBLEM SET 5 SOLUTIONS

1. $E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} (1.4xe^{-2x} + .9xe^{-3x})\,dx$
$= (-.7xe^{-2x} - .35e^{-2x} - .3xe^{-3x} - .1e^{-3x})\Big|_{x=0}^{x=\infty} = .45$.

The integrals were found by integration by parts. Note that we could also have used $\int_0^{\infty} x^k e^{-ax} dx = \frac{k!}{a^{k+1}}$ if k is an integer ≥ 0, and $a > 0$. Answer: A

2. $E[\frac{1}{X}] = \int_1^3 \frac{1}{x} \cdot \frac{1}{30} x(1+3x)\,dx = \frac{7}{15}$. Answer: B

3. $E[X] = \int_{-2}^{4} x \cdot f(x)\,dx = \int_{-2}^{4} x \cdot \frac{|x|}{10}\,dx$.

For $x < 0$, $|x| = -x$ and for $x > 0$, $|x| = x$.

Then, $E[X] = \int_{-2}^{0} x \cdot (-\frac{x}{10})\,dx + \int_0^4 x \cdot (\frac{x}{10})\,dx = (.1)(-\int_{-2}^{0} x^2\,dx + \int_0^4 x^2\,dx)$
$= (.1)\big[-\frac{0^3-(-2)^3}{3} + \frac{4^3-0^3}{3}\big] = \frac{28}{15}$. Answer: D

4. The cdf of X is $F(y) = \int_{200}^{y} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx$.

The 30-th percentile of X, say d, is the point for which $P[X \leq c] = .3$.

Therefore, $.3 = \int_{200}^{c} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx = -\frac{(200)^{2.5}}{x^{2.5}}\Big|_{x=200}^{x=c} = 1 - (\frac{200}{c})^{2.5} = .3$.

Solving for c results in $c = 230.7$.

The 70-th percentile of X, say d, is the point for which $P[X \leq d] = .7$.

Therefore, $.7 = \int_{200}^{d} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx = -\frac{(200)^{2.5}}{x^{2.5}}\Big|_{x=200}^{x=d} = 1 - (\frac{200}{d})^{2.5} = .7$.

Solving for d results in $d = 323.7$. Then $d - c = 93$. Answer: B

5. $\int_0^{1/3} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda/3} = \frac{1}{2} \rightarrow \lambda = 3\,ln\,2$. Answer: D

6. The mode is the point at which $f(x)$ is maximized. $f'(x) = -\frac{1}{9}x + \frac{1}{9}(4-x) = \frac{4}{9} - \frac{2}{9}x$.

Setting $f'(x) = 0$ results in $x = 2$.

Since $f''(2) = -\frac{2}{9} < 0$, that point is a relative maximum. Answer: E

7. Let X denote the annual cost. We are given that $Var[X] = 260$.
If annual cost increases by 20% to $1.2X$, the variance is
$Var[1.2X] = (1.2)^2 \cdot Var[X] = (1.44)(260) = 374.4$. Answer: E

8. Let T be the time until failure for the system. In order for the system to not fail by time $t > 0$, it must be the case that none of the components have failed by time t. For a given component, with time until failure of W, $P[W > t] = \int_t^\infty \frac{3}{x^4}\,dx = \frac{1}{t^3}$. Thus,
$P[T > t] = P[(W_1 > t) \cap (W_2 > t) \cap \cdots \cap (W_7 > t)]$
$= P[W_1 > t] \cdot P[W_2 > t] \cdots P[W_7 > t] = \frac{1}{t^{21}}$ (because of independence of the W_i's).
The cumulative distribution function for T is
$F_T(t) = P[T \le t] = 1 - P[T > t] = 1 - \frac{1}{t^{21}}$, so the density function for T is $f_T(t) = \frac{21}{t^{22}}$.
The expected value of T is then $E[T] = \int_1^\infty t \cdot \frac{21}{t^{22}}\,dt = \frac{21}{20} = 1.05$.
Alternatively, once the cdf of T is known, since the region of density for T is $t > 1$, the expected value of T is $E[T] = 1 + \int_1^\infty [1 - F_T(t)]\,dt = 1 + \int_1^\infty \frac{1}{t^{21}}\,dt = 1 + \frac{1}{20}$.
Answer: D

9. Mean claim size $= E[X] = (20)(.15) + (30)(.1) + \cdots + (80)(.3) = 55$;
$E[X^2] = (20)^2(.15) + (30)^2(.1) + \cdots + (80)^2(.3) = 3500$.
$Var[X] = E[X^2] - (E[X])^2 = 3500 - 55^2 = 475$.
Standard deviation $= \sqrt{Var[X]} = \sqrt{475} = 21.79$.
The claim sizes within one standard deviation of the mean claim size of 55 are those claim sizes between $55 - 21.79 = 33.21$ and $55 + 21.79 = 76.79$; those claim sizes are 40, 50, 60 and 70. The total probability of those claim sizes is $.05 + .2 + .1 + .1 = .45$.
Answer: A

10. The standard deviation is $\sqrt{Var[X]}$, and $Var[X] = E[X^2] - (E[X])^2$.
The moment generating function can be used to find the moments of X,
$E[X^k] = M^{(k)}(0)$ (k-th derivative evaluated at 0).
$M(t) = (1 - 2500t)^{-4} \to M'(t) = (-4)(1 - 2500t)^{-5}(-2500)$
$\to E[X] = M'(0) = 10,000$,
$M''(t) = (-5)(-4)(1 - 2500t)^{-6}(-2500)^2$
$\to E[X^2] = M''(0) = 125 \times 10^6$.

10. continued

$Var[X] = 125 \times 10^6 - (10,000)^2 = 25,000,000 \to \sqrt{Var[X]} = 5,000$.

Alternatively, it is also true that $Var[X] = \frac{d^2}{dt^2} ln[M(t)]\Big|_{t=0}$. In this case,

$\frac{d^2}{dt^2} ln[M(t)] = \frac{d^2}{dt^2} [-4ln(1-2500t)] = (-4)(-2500)^2(-1)(1-2500t)^{-2}$,

and when $t = 0$ this becomes $Var[X] = 25,000,000$, as before.

Another alternative solution would be to notice that $M(t) = \frac{1}{(1-ct)^\alpha}$ is the moment generating function for a gamma distribution with mean αc , and variance αc^2 . In this case, $\alpha = 4$ and $c = 2500$, so that the variance is $4(2500)^2 = 25,000,000$. Answer: B

11. $P[\,|X-\mu| > r \cdot \sigma] \leq \frac{1}{r^2}$. In this case $\mu = 0$ and $\sigma^2 = 4$, so that $r = 4$,

and $P[|X| > 8] = P[|X| > 4\sigma] \leq \frac{1}{16}$. Answer: A

12. $E[(X-2)^3] = E[X^3] - E[6X^2] + E[12X] - E[8]$

$= M_X^{(3)}(0) - 6M_X^{(2)}(0) + 12M'_X(0) - 8$.

$M'_X(t) = -\frac{1}{(1+t)^2} \to M'_X(0) = -1$, $M_X^{(2)}(t) = \frac{2}{(1+t)^3} \to M_X^{(2)}(0) = 2$,

$M_X^{(3)}(t) = -\frac{6}{(1+t)^4} \to M_X^{(3)}(0) = -6$. Then, $E[(X-2)^3] = -38$. Answer: D

13. $X = J + K + L$. Because of independence of J, K and L, we have

$M_X(t) = M_J(t) \cdot M_K(t) \cdot M_L(t) = (1-2t)^{-3} \cdot (1-2t)^{-2.5} \cdot (1-2t)^{-4.5} = (1-2t)^{-10}$.

We use the property of the moment generating function that states

$\frac{d^n}{dt^n} M_Z(t)\Big|_{t=0} = E[Z^n]$. Then,

$\frac{d^3}{dt^3} M_X(t) = (-10)(-2)(-11)(-2)(-12)(-2)(1-2t)^{-13}$

and evaluated at $t = 0$ this is $10,560$.

Alternatively, we can write

$(J+K+L)^3 = J^3 + K^3 + L^3 + 3J^2L + \cdots$, and find each expectation separately (much too tedious and too much work). Answer: E

14. Since each X is either 0 or 1, it follows that

$Y = X_1X_2X_3 = 1$ only if $X_1 = X_2 = X_3 = 1$, and $Y = 0$ otherwise.

Since X_1, X_2, X_3 form a random sample, they are mutually independent. Therefore,

$P[Y=1] = P[(X_1=1) \cap (X_2=1) \cap (X_3=1)]$

$= P[(X_1=1)] \cdot P[(X_2=1)] \cdot P[(X_3=1)] = (\frac{2}{3})(\frac{2}{3})(\frac{2}{3}) = \frac{8}{27}$,

and $P[Y=0] = 1 - P[Y=1] = \frac{19}{27}$.

The moment generating function of Y is

$M(t) = E[e^{tY}] = e^{t\cdot 0} \cdot P[Y=0] + e^{t\cdot 1} \cdot P[Y=1]$

$= 1 \cdot (\frac{19}{27}) + e^t \cdot (\frac{8}{27}) = \frac{19}{27} + \frac{8}{27}e^t$. Answer: A

15. Let A denote the event that the two balls fall into separate holes, and B denote the event that the two balls fall into the same hole. Event A is equivalent to $X=2$ holes being left unoccupied, and event B is equivalent to $X=3$ holes being left unoccupied.

When the balls are dropped, both balls have the same chance of dropping into any of the four holes. Therefore, each of the 16 possible outcomes (i.e., ball 1 in hole 1, 2, 3 or 4 and ball 2 in hole 1, 2, 3 or 4) is equally likely to occur. Four of these outcomes result in the two balls dropping into the same hole (event B) and the other twelve outcomes result in the balls dropping into separate holes (event A). Therefore,

$P[A] = P[X=2] = \frac{12}{16} = \frac{3}{4}$, and $P[B] = P[X=3] = \frac{4}{16} = \frac{1}{4}$.

The moment generating function of a discrete random variable X is

$M(t) = E[e^{tX}] = \sum e^{tx} \cdot P[X=x]$. The moment generating function for the random variable X described in this problem is

$M(t) = e^{2t} \cdot P[X=2] + e^{3t} \cdot P[X=3] = e^{2t} \cdot \frac{3}{4} + e^{3t} \cdot \frac{1}{4} = \frac{1}{4}(3e^{2t} + e^{3t})$.

Answer: C

16. $P[\text{match only last 2}] = \frac{90}{10{,}000}$, since the 100's can be any one of 9 wrong digits and the 1000's can be any one of 10.

$P[\text{match only last 3}] = \frac{9}{10{,}000}$, since the 1000's can be any one of 9 wrong digits.

$P[\text{match all 4}] = \frac{1}{10{,}000}$.

Expected gain from ticket is

$(50)(\frac{90}{10{,}000}) + (500)(\frac{9}{10{,}000}) + (5000)(\frac{1}{10{,}000}) - 2 = -.60$. Answer: B

17. The pdf is $c(1+x)^{-4}$ for $0<\infty$. Therefore, $1=\int_0^{\infty} c(1+x)^{-4}\,dx=\frac{c}{3}$,
from which we get $c=3$, and $f(x)=3(1+x)^{-4}$.
$E[X]=\int_0^{\infty} x\cdot f(x)\,dx=\int_0^{\infty} x\cdot 3(1+x)^{-4}\,dx$. This can be found using integration by parts:
$\int_0^{\infty} x\cdot 3(1+x)^{-4}\,dx=\int_0^{\infty} x\cdot d[-(1+x)^{-3}]$
$=-x(1+x)^{-3}\Big|_{x=0}^{x=\infty}-\int_0^{\infty}-(1+x)^{-3}\,dx=0+\int_0^{\infty}(1+x)^{-3}\,dx=\frac{1}{2}$.
Alternatively, the cumulative distribution function of X is
$F(x)=\int_0^x f(t)\,dt=\int_0^x 3(1+t)^{-4}\,dt=1-(1+x)^{-3}$, and we use the rule for non-negative random variables $E[X]=\int_0^{\infty}[1-F(x)]\,dx=\int_0^{\infty}(1+x)^{-3}\,dx=\frac{1}{2}$. Answer: C

18. Since $F(x)=0$ if $x<0$ but $F(0)=1-\frac{1}{2}$, it follows that $P[X=0]=\frac{1}{2}$.
Since $F(x)$ is differentiable for $x>0$, it follows that the density function of X for $x>0$ is $f(x)=F'(x)=\frac{1}{2}e^{-x}$. The moment generating function of X is then
$M_X(t)=E[e^{tX}]=e^{t\cdot 0}\cdot P[X=0]+\int_0^{\infty} e^{tx}\cdot\frac{1}{2}e^{-x}dx=\frac{1}{2}+\frac{-1}{2(t-1)}=\frac{2-t}{2-2t}$,
for $t<1$ (the improper integral converges only if $t<1$). Answer: C

19. $\frac{d}{dt}\,ln\,M(t)\Big|_{t=0}=\mu$ (mean) , $\frac{d^2}{dt^2}\,ln\,M(t)\Big|_{t=0}=\sigma^2$ (variance)
$\rightarrow\ \frac{d}{dt}\,ln\,\frac{e^{at}}{1-bt^2}=\frac{d}{dt}[at-ln(1-bt^2)]=a+\frac{2bt}{1-bt^2}$, substitute $t=0\rightarrow a=3$,
and $\frac{d^2}{dt^2}\,ln\,\frac{e^{at}}{1-bt^2}=\frac{d^2}{dt^2}[at-ln(1-bt^2)]=\frac{2b+2b^2t^2}{(1-bt^2)^2}$,
substitute $t=0\rightarrow 2b=2\ \Rightarrow\ b=1\rightarrow a+b=4$. Answer: E

20. The expected payout is $\int_0^{100}\frac{t}{5000}\,(1.1)^t\,dt=\frac{1}{5000}\left(\frac{t(1.1)^t}{ln\,1.1}-\frac{(1.1)^t}{[ln\,1.1]^2}\right)\Big|_{t=0}^{t=100}=2588$
Answer: D

21. The expected payout is $100,000(.08)+K(.02)=8000+.02K$ (since there is a .9 chance that neither dies and a .08 chance that exactly 1 dies, there must be a probability of .02 that both die). The variance is $74,000^2=100,000^2(.08)+K^2(.02)-(8000+.02K)^2$
$=736,000,000-320K+.0196.K^2\rightarrow K=500,000$ (or $-483,673$, we discard the negative root). The expected payout is $100,000(.08)+500,000(.02)=18,000$.
Answer: A

22. The expected payment is
$(.8)(0)+(.1)(1,000,000)+(.05)(4,000,000+9,000,000)=750,000$.
Answer: C

23. From the definition of $F(x)$ we see that $F(1)=\frac{1}{2}$. This indicates that X has a point of probability at $X=1$ with $P[X=1]=\frac{1}{2}$. For $1<x<2$, the density function for X is $f(x)=F'(x)=x-1$. We formulate the variance of X as $Var[X]=E[X^2]-(E[X])^2$.
$E[X]=(1)\cdot P[X=1]+\int_1^2 x\cdot f(x)\,dx=(1)(\frac{1}{2})+\int_1^2 x(x-1)\,dx=\frac{1}{2}+\frac{7}{3}-\frac{3}{2}=\frac{4}{3}$.
$E[X^2]=(1^2)\cdot P[X=1]+\int_1^2 x^2\cdot f(x)\,dx=(1)(\frac{1}{2})+\int_1^2 x^2(x-1)\,dx=\frac{1}{2}+\frac{15}{4}-\frac{7}{3}=\frac{23}{12}$.
$Var[X]=\frac{23}{12}-(\frac{4}{3})^2=\frac{5}{36}$. Answer: C

24. We identify the following events:
H- toss a head ; T- toss a tail ; LH- coin is loaded toward heads
LT- coin is loaded toward tails .
We are given $P[LH]=\frac{3}{4}$, $P[LT]=\frac{1}{4}$,
$P[H|LH]=\frac{3}{4}$, $P[T|LH]=\frac{1}{4}$, $P[H|LT]=\frac{1}{4}$, $P[T|LT]=\frac{3}{4}$.
We must find the conditional probabilities $P[\text{2nd flip } H|\text{1st flip } H]$.
Then his expected gain is
$(-100)P[\text{2nd flip } H|\text{1st flip } H]+(200)(1-P[\text{2nd flip } H|\text{1st flip } H])$.
$P[\text{2nd flip } H|\text{1st flip } H]=\frac{P[\text{2nd flip } H\cap\text{1st flip } H]}{P[\text{1st flip } H]}$.
To find both the numerator and denominator we use the rules
$P(A)=P(A\cap B)+P(A\cap\overline{B})$ and $P(A\cap B)=P(A|B)\cdot P(B)$.
The denominator is
$P[1H]=P[1H\cap LH]+P[1H\cap LT]$
$=P[1H|LH]\cdot P[LH]+P[1H|LT]\cdot P[LT]=(\frac{3}{4})(\frac{3}{4})+(\frac{1}{4})(\frac{1}{4})=\frac{5}{8}$.
The numerator is
$P[2H\cap 1H]=P[2H\cap 1H|LH]\cdot P[LH]+P[2H\cap 1H|LT]\cdot P[LT]$
$=(\frac{3}{4})^2(\frac{3}{4})+(\frac{1}{4})^2(\frac{1}{4})=\frac{7}{16}$
(if the coin is LH then each of the two flips has probability $\frac{3}{4}$ of being head, and if the coin is LT then each of the two flips has probability $\frac{1}{4}$ of being tail).
Then $P[\text{2nd flip } H|\text{1st flip } H]=\frac{P[\text{2nd flip } H\cap\text{1st flip } H]}{P[\text{1st flip } H]}=\frac{7/16}{5/8}=.7$.
Then, the expected gain is
$(-100)P[\text{2nd flip } H|\text{1st flip } H]+(200)[1-P[\text{2nd flip } H|\text{1st flip } H]]$.
$=(-100)(.7)+(200)(.3)=-10$. Answer: B

25. The mode of a distribution is the point x at which the density function $f(x)$ is maximized. From the distribution function $F(x)$, we can find the density function $f(x) = F'(x) = \frac{1}{9}(4x - x^2)$ for $0 \le x \le 3$. We now find where the maximum of $f(x)$ occurs on the interval $[0, 3]$. The critical points of $f(x)$ occur where $f'(x) = 0$:
$f'(x) = \frac{1}{9}(4 - 2x) = 0 \to x = 2$. To find the maximum of $f(x)$ on the interval, we calculate $f(x)$ at the critical points and at the endpoints of the interval:
$f(0) = 0$, $f(2) = \frac{4}{9}$, $f(3) = \frac{1}{3}$. The mode is at $x = 2$. Answer: D

26. $P[\text{toss head}] = P[\text{toss head} \cap \text{loaded head}] + P[\text{toss head} \cap \text{loaded tail}]$
$= P[\text{toss head}|\text{loaded head}] \cdot P[\text{loaded head}] + P[\text{toss head}|\text{loaded tail}] \cdot P[\text{loaded tail}]$
$= (\frac{3}{4})(p) + (\frac{1}{4})(1 - p) = \frac{1}{4} + \frac{1}{2}p$. Then, $P[\text{toss tail}] = 1 - P[\text{toss head}] = \frac{3}{4} - \frac{1}{2}p$.
Smith's expected gain is
$(-100) \cdot P[\text{toss head}] + (200) \cdot P[\text{toss tail}]$
$= (-100) \cdot (\frac{1}{4} + \frac{1}{2}p) + (200)(\frac{3}{4} - \frac{1}{2}p) = 125 - 150p$.
In order for Smith's expected gain to be 0 we must have $125 - 150p = 0 \to p = \frac{5}{6}$.
Answer: E

27. In order for $f(x)$ to be a properly defined density function, it must satisfy $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
The integral is $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ce^{-|x|}\,dx = c \cdot [\int_{-\infty}^{0} e^x\,dx + \int_0^{\infty} e^{-x}\,dx] = c \cdot [1 + 1] = 1$,
from which it follows that $c = \frac{1}{2}$.
Then, $Var[X] = E[X^2] - (E[X])^2$.
$E[X] = \int_{-\infty}^{\infty} \frac{1}{2}xe^{-|x|}\,dx = \frac{1}{2} \cdot [\int_{-\infty}^{0} xe^x\,dx + \int_0^{\infty} xe^{-x}\,dx]$.
We use the integration by parts rule $\int xe^{ax}\,dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$.
Then $\int_{-\infty}^{0} xe^x\,dx = xe^x - e^x \Big|_{x=-\infty}^{x=0} = 0 - 1 - (0 - 0) = -1$,
and $\int_0^{\infty} xe^{-x}\,dx = -xe^{-x} - e^{-x} \Big|_{x=0}^{x=\infty} = 0 - 0 - (0 - 1) = 1$.
Therefore, $E[X] = -1 + 1 = 0$.
To find $E[X^2]$ we use integration by parts. $\int x^2e^{ax}\,dx = \frac{x^2e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot 2x\,dx$.
$E[X^2] = \frac{1}{2}[\int_{-\infty}^{0} x^2e^x\,dx + \int_0^{\infty} x^2e^{-x}\,dx]$.
Then $\int_{-\infty}^{0} x^2e^x\,dx = x^2e^x \Big|_{x=-\infty}^{x=0} - \int_{-\infty}^{0} e^x \cdot 2x\,dx = 0 - 0 - 2(-1) = 2$,
and $\int_0^{\infty} x^2e^{-x}\,dx = -x^2e^{-x} \Big|_{x=0}^{x=\infty} + \int_0^{\infty} e^{-x} \cdot 2x\,dx = -0 + 0 + 2(1) = 2$.
Therefore, $E[X^2] = \frac{1}{2}[2 + 2] = 2$, and $Var[X] = 2 - 0 = 2$. Answer: E

SECTION 6 - FREQUENTLY USED DISCRETE DISTRIBUTIONS

Uniform distribution on N points (where $N \geq 1$ is an integer):

The probability function is $\boldsymbol{p(x) = \frac{1}{N}}$ **for** $\boldsymbol{x = 1, 2, \ldots, N}$, and $p(x) = 0$ otherwise.
Since each x has the same probability of occurring, it seems reasonable that the mean (the average) is the midpoint of the set of successive integers. The average is midway between the smallest and largest possible value of X, $\boldsymbol{E[X] = \frac{N+1}{2}}$. Another way of seeing this, is by using the rule for summing consecutive integers, $1 + 2 + \cdots + N = \frac{N(N+1)}{2}$. It then follows that $E[X] = \sum_{x=1}^{N} x \cdot p(x) = 1 \cdot \frac{1}{N} + 2 \cdot \frac{1}{N} + \cdots + N \cdot \frac{1}{N}$

$$= [1 + 2 + \cdots + N] \cdot \frac{1}{N} = \frac{N(N+1)}{2} \cdot \frac{1}{N} = \frac{N+1}{2} .$$

The variance $\boldsymbol{Var[X] = \frac{N^2-1}{12}}$, and the moment generating function is
$M_X(t) = \sum_{j=1}^{N} \frac{e^{jt}}{N} = \frac{e^t(e^{Nt}-1)}{N(e^t-1)}$ for any real t. The outcome of tossing a fair die is an example of the discrete uniform distribution with $N = 6$. It is unlikely that the moment generating function for the discrete uniform distribution will come up in an Exam P question.

Example 6-1: Suppose that X is a discrete random variable that is uniformly distributed on the even integers $x = 0, 2, 4, \ldots, 22$, so that the probability function of X is $p(x) = \frac{1}{12}$ for each even integer x from 0 to 22. Find $E[X]$ and $Var[X]$.

Solution: If we consider the transformation $Y = \frac{X+2}{2}$, then the random variable Y is distributed on the points $Y = 1, 2, \ldots, 12$, with probability function $p_Y(y) = \frac{1}{12}$ for each integer y from 1 to 12. Thus, Y has the discrete uniform distribution described above with $N = 12$, and $E[Y] = \frac{12+1}{2} = \frac{13}{2}$ and $Var[Y] = \frac{12^2-1}{12} = \frac{143}{12}$.
But we want the mean and variance of X. Since $Y = \frac{X+2}{2}$, we rewrite this as $X = 2Y - 2$ and use rules for expectation and variance to get $E[X] = 2 \cdot E[Y] - 2 = 11$, and
$Var[X] = 4 \cdot Var[Y] = \frac{143}{3}$. □

Binomial distribution with parameters n and p
(integer $n \geq 1$ and $0 \leq p \leq 1$)

Suppose that a single trial of an experiment results in either success with probability p, or failure with probability $1-p=q$. If n independent trials of the experiment are performed, and X is the number of successes that occur, then X is an integer between 0 and n. X is said to have a binomial distribution with parameters n and p (sometimes denoted $X \sim B(n,p)$).

$\boldsymbol{p(x) = \binom{n}{x} p^x (1-p)^{n-x}}$ **for** $\boldsymbol{x = 0, 1, 2, \ldots, n}$**,** where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

$p(x)$ is the probability that there will be exactly x successes in the n trials of the experiment. The average number of successes in the n trials is

$\boldsymbol{E[X] = np}$, **the mean of the binomial distribution**,

and the variance is

$\boldsymbol{Var[X] = np(1-p)}$**, the variance of the binomial distribution** .

The moment generating function is $M_X(t) = (1-p+pe^t)^n$.

Note that since $Var[X] = E[X^2] - (E[X])^2$, it follows that the second moment of X is $E[X^2] = np(1-p) + (np)^2$ for the binomial distribution.

In the special case of $n=1$ (a single trial), the distribution is referred to as a **Bernoulli distribution**. If $X \sim B(n,p)$, then X is the sum of n independent Bernoulli random variables each with distribution $B(1,p)$.

For example, if $n=3$, $p=\frac{1}{2}$, the binomial random variable has the following distribution:

X :	0	1	2	3
$p(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This would describe the distribution of the number of heads occurring in three tosses of a fair coin. The probabilities are found as follows:

$p(0) = P[X=0] = \binom{3}{0} \frac{1}{2}^0 (1-\frac{1}{2})^{3-0} = \frac{1}{8}$, $p(1) = P[X=1] = \binom{3}{1} \frac{1}{2}^1 (1-\frac{1}{2})^{3-1} = \frac{3}{8}$, etc.

The mean is $np = 3(\frac{1}{2}) = \frac{3}{2}$ and the variance is $np(1-p) = 3(\frac{1}{2})(\frac{1}{2}) = \frac{3}{4}$.

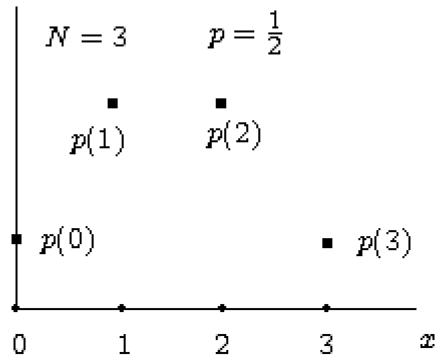
As another example, if $n=3$, $p=.2$, the distribution is

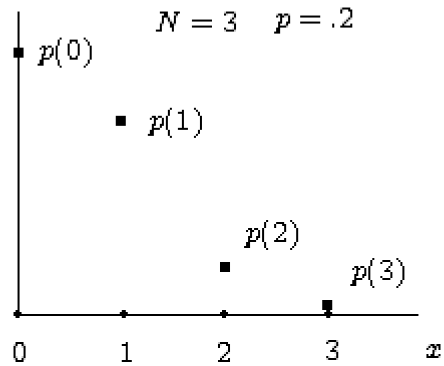
X :	0	1	2	3
$p(x)$:	.512	.384	.096	.008

As an example of a probability calculation, $P[X=2] = \binom{3}{2} (.2)^2 (1-.2)^{3-2} = .096$.

The mean is $np = 3(.2) = .6$ and the variance is $np(1-p) = 3(.2)(.8) = .48$.

The graphs of the probability functions of these two binomial distributions are:.





For the case of $n = 3$, we can illustrate fairly easily why the binomial probability function is $p(x) = \binom{3}{x} p^x (1-p)^{3-x}$ for $x = 0, 1, 2, 3$. We will use the notation S and F to denote success and failure of a particular trial of the underlying experiment. In order to have $X = 0$ successes in 3 trials, the trials must be FFF. The probability of each F is $1 - p$, so the probability of 3 F's in a row (because of independence of successive trials) is
$(1-p)(1-p)(1-p) = (1-p)^3 = \binom{3}{0} p^0 (1-p)^3$.

In order to have $X = 1$ success, that success must occur on either the 1st, 2nd or 3rd trial. Therefore, the result of the 3 trials must be either SFF, FSF, or FFS. The probability of any one of those three sequences is $p(1-p)(1-p) = p(1-p)^2$. The combined probability of all three sequences is $3p(1-p)^2 = \binom{3}{1} p^1 (1-p)^2$. Similar reasoning explains the other probabilities for this binomial distribution.

Example 6-2: Smith and Jones each write the same multiple choice test. The test has 5 questions, and each question has 5 answers (exactly one of which is right). Smith and Jones are not very well prepared for the test and they answer the questions randomly.
(i) Find the probability that they both get the same number of answers correct.
(ii) Find the probability that their papers are identical, assuming that they have answered independently of one another.

Solution: (i) Let X be the number of answers that Smith gets correct. Then X has a binomial distribution with $n = 5$, $p = .2$, and the probability function is $P[X = k] = \binom{5}{k} (.2)^k (.8)^{5-k}$.

$X:$	0	1	2	3	4	5
$p(x):$	.32768	.4096	.2048	.0512	.0064	.00032

Example 6-2 continued

The number of answers Jones gets correct, say Y, has the same distribution. Then,

$P[X=Y]=P[(X=0)\cap(Y=0)]+P[(X=1)\cap(Y=1)]+\cdots+P[(X=5)\cap(Y=5)]$

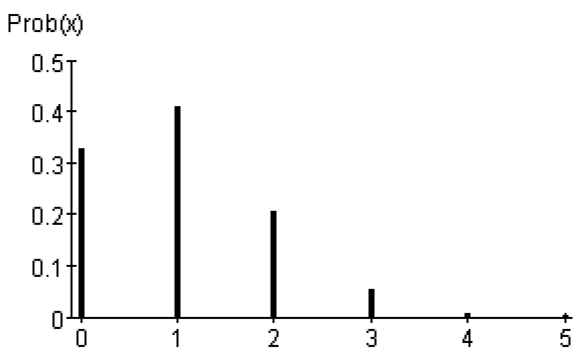
$=P[X=0]\cdot P[Y=0]+P[X=1]\cdot P[Y=1]+\cdots+P[X=5]\cdot P[Y=5]$

(this follows from independence of X and Y), which is equal to

$(.32768)^2+(.4096)^2+(.2048)^2+(.0512)^2+(.0064)^2+(.00032)^2=.3198$.

(ii) For a particular question, the probability that Jones picks (at random) the same answer as Smith is .2. Since all 5 questions are answered independently, the probability of both papers being identical is $(.2)^5=.00032$. □

The following is a probability plot for the binomial distribution in Example 6-2.



Example 6-3: If X is the number of "6"'s that turn up when 72 ordinary dice are independently thrown, find the expected value of X^2.

Solution: X has a binomial distribution with $n=72$ and $p=\frac{1}{6}$. Then

$E[X]=np=12$, and $Var[X]=np(1-p)=10$. But $Var[X]=E[X^2]-(E[X])^2$,

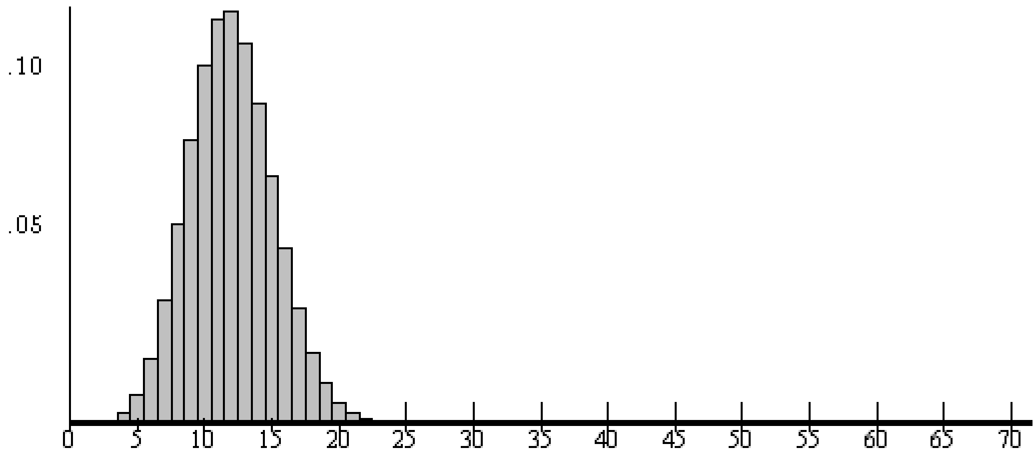
so that $E[X^2]=Var[X]+(E[X])^2=10+12^2=154$.

Note that the probability function of X is $P[X=k]=\binom{72}{k}(\frac{1}{6})^k(\frac{5}{6})^{72-k}$ for $k=0,1,2,...,72$.

Below is a histogram for the distribution of X. The probabilities are very small for X-values above 20 or so. For instance, $P[X=25]=.00010195$, $P[X=30]=3.5\times10^{-7}$ and $P[X=2]=.0002035$. The mode of the distribution occurs at $X=12$, with a probability of $P[X=12]=.12525$. $X=12$ is also the mean of the distribution. As n gets larger in a binomial distribution, the histogram takes on more of a bell shape. Later on we will see the normal approximation applied to a distribution.

Example 6-3 continued

The normal distribution is a continuous random variable with a bell-shaped density that can sometimes be used to approximate other distributions such as the binomial.



□

Poisson distribution with parameter $\lambda\,(\lambda > 0)$

This distribution is defined for all integers $0, 1, 2, \ldots$

The probability function is $\boldsymbol{p(x) = \frac{e^{-\lambda}\lambda^x}{x!}}$ **for** $\boldsymbol{x = 0, 1, 2, 3, \ldots,}$

λ may be referred to as the Poisson parameter for the distribution.

The mean and variance are equal to the Poisson parameter, $\boldsymbol{E[X] = Var[X] = \lambda}$,

and the moment generating function is $M_X(t) = e^{\lambda(e^t-1)}$.

The Poisson distribution is often used as a model for counting the number of events of a certain type that occur in a certain period of time. Suppose that X represents the number of customers arriving for service at a bank in a one hour period, and that a model for X is the Poisson distribution with parameter λ. Under some reasonable assumptions (such as independence of the numbers arriving in different time intervals) it is possible to show that the number arriving in any time period also has a Poisson distribution with the appropriate parameter that is "scaled" from λ. Suppose that $\lambda = 40$, meaning that X, the number of bank customers arriving in one hour, has a mean of 40. If Y represents the number of customers arriving in 2 hours, then Y has a Poisson distribution with a parameter of 80. For any time interval of length t, the number of customers arriving in that time interval has a Poisson distribution with parameter (mean) $\lambda t = 40t$. For instance, the number of customers arriving during a 15-minute period $(t = \frac{1}{4}$ hour) will have a Poisson distribution with parameter (mean) $40 \times \frac{1}{4} = 10$.

As an example, for $\lambda = .5$, the following is a partial description of the Poisson distribution:

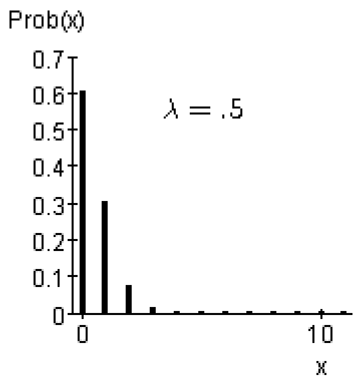
$X:$	0	1	2	3	4	5
$p(x):$	$.6065$	$.3033$	$.0758$	$.0126$	$.0016$	$.0002$ . .

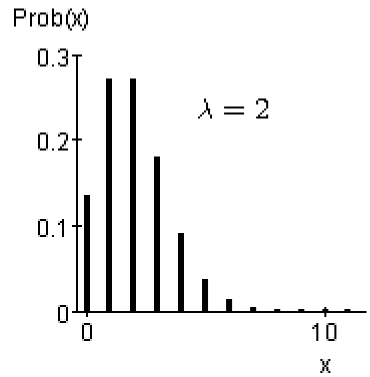
An example of the calculation of these values is $P[X=3] = p(3) = \frac{e^{-.5}(.5)^3}{3!} = .012636$.

For $\lambda = 2$, the following is a partial description of the Poisson distribution:

$X:$	0	1	2	3	4	$5\ldots$
$p(x):$	$.1353$	$.2707$	$.2707$	$.1804$	$.0902$	$.0361$ . .

Histograms of these two Poisson distributions are as follows:





Example 6-4: The number of home runs in a baseball game is assumed to have a Poisson distribution with a mean of 3. As a promotion, a company pledges to donate \$10,000 to charity for each home run hit up to a maximum of 3. Find the expected amount that the company will donate. Another company pledges to donate \$C for each home run over 3 hit during the game, and C is chosen so that the second company's expected donation is the same as the first. Find C.

Solution: Let X be the number of home runs hit in the game and let Y be the first company's donation, and let Z be the second company's donation. The probability function for X is $P[X=n] = \frac{e^{-3}3^n}{n!}$. The distribution of X is

$X:$	0	1	2	3	4	5 ...
$p(x):$	$.0498$	$.1494$	$.2240$	$.2240$	$.1680$	$.1008$
$Y:$	0	$10,000$	$20,000$	$30,000$	$30,000$	$30,000$
$Z:$	0	0	0	0	C	$2C$

$$E[Y] = (0)(.0498) + (10,000)(.1494) + (20,000)(.2240) + (30,000)(.2240 + .1680 + \cdots)$$
$$= (10,000)(.1494) + (20,000)(.2240) + (30,000)(1 - p(0) - p(1) - p(2))$$
$$= (10,000)(.1494) + (20,000)(.2240) + (30,000)(1 - .0498 - .1494 - .2240) = 23,278 \text{ .}$$

To find $E[Z]$ we look at X as the sum of two new random variables U and W..

$X:$	0	1	2	3	4	5	6	...	x	...
$U:$	0	1	2	3	3	3	3	...	3	...
$W:$	0	0	0	0	1	2	3	...	$x-3$	...

Example 6-4 continued

We see that $X = U + W$, and therefore, $3 = E[X] = E[U] + E[W]$. We can find $E[U]$ in the same way we found $E[Y]$ above,

$E[U] = 0 \times p(0) + 1 \times p(1) + 2 \times p(2) + 3 \times [1 - p(0) - p(1) - p(2)]$

$= .1494 + 2(.2240) + 3(1 - .0498 - .1494 - .2240) = 2.3278$ (reduced by a factor of 10,000 from Y). It follows that $E[W] = E[X] - E]U] = 3 - 2.3278 = .6722$. Then, since $Z = CW$, we get $E[Z] = .6722C$. In order for this to be 23,278 we require $C = \frac{23{,}278}{.6722} = 34{,}630$. □

Example 6-5: Assume that the number of hits, X, per baseball game, has a Poisson distribution. If the probability of a no-hit game is $\frac{1}{10{,}000}$, find the probability of having 4 or more hits in a particular game.

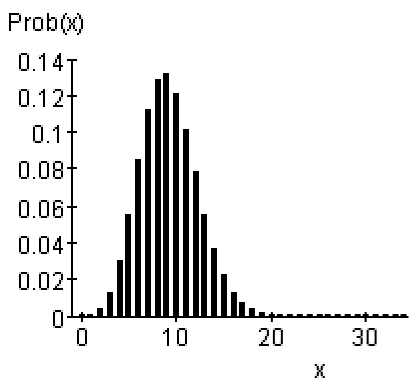
Solution: $P[X = 0] = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-\lambda} = \frac{1}{10{,}000} \rightarrow \lambda = ln\, 10{,}000$.

$P[X \geq 4] = 1 - (P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3])$

$= 1 - (\frac{e^{-\lambda} \cdot \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} + \frac{e^{-\lambda} \cdot \lambda^2}{2!} + \frac{e^{-\lambda} \cdot \lambda^3}{3!})$

$= 1 - (\frac{1}{10{,}000} + \frac{ln 10{,}000}{10{,}000} + \frac{(ln 10{,}000)^2}{2(10{,}000)} + \frac{(ln 10{,}000)^3}{6(10{,}000)}) = .9817.$ □

The mean in Example 6-5 is $\lambda = ln(10{,}000) = 9.21$. The histogram for this Poisson distribution is given below. As λ increases, the histogram becomes more bell-shaped.



Note that for the Poisson distribution with mean λ, we have the following relationship between successive probabilities: $P(X = n + 1) = P(X = n) \cdot \frac{\lambda}{n+1}$.

For Exam P it is very important to be familiar with the binomial and Poisson distributions. The following distributions have all come up on the exam at one time or another as well, but not as often as the binomial and Poisson distributions.

Geometric distribution with parameter p ($0 \leq p \leq 1$)

Suppose that a single trial of an experiment results in either success with probability p, or failure with probability $1-p=q$. The experiment is performed with successive independent trials until the first success occurs. If X represents the number of failures until the first success, then X is a discrete random variable that can be $0,1,2,3,\ldots$. X is said to have a **geometric distribution with parameter p. The probability function for X is**
$p(x)=(1-p)^x p$ for $x=0,1,2,3,\ldots$.
The **mean and variance of X** are $\boldsymbol{E[X]=\frac{1-p}{p}}=\frac{q}{p}$, $\boldsymbol{Var[X]=\frac{1-p}{p^2}}=\frac{q}{p^2}$.
The moment generating function is $M_X(t)=\frac{p}{1-(1-p)e^t}$.
The geometric distribution has the lack of memory property, $P[X=n+k|X\geq n]=P[X=k]$.
An equivalent description of the geometric distribution uses the parameter β, where $\beta=\frac{1-p}{p}$.

Another version of a geometric distribution is the random variable Y, the number of the experiment on which the first success occurs. Y is related to X just defined: $Y=X+1$ and $P[Y=y]=P[X=y-1]=(1-p)^{y-1}p$ for $y=1,2,3,\ldots$ An example of this was seen with tossing a coin until a head turns up. Y is the toss number of the first head, X is the number of tails until the first head: $E[Y]=E[X]+1=\frac{1}{p}$, $Var[Y]=Var[X]=\frac{1-p}{p^2}$.

Example 6-6: In tossing a fair die repeatedly (and independently on successive tosses), find the probability of getting the first "1" on the t-th toss. Find the expected number of tosses before the first "1" is tossed.

Solution: The probability that the first "1" occurs on the first toss is $\frac{1}{6}$. The probability that the first "1" occurs on the second toss is $(\frac{5}{6})(\frac{1}{6})$ (a toss other than "1" followed by a "1").
The probability that the first "1" occurs on the t-th toss is $(\frac{5}{6})^{t-1}(\frac{1}{6})$ ($t-1$ rolls other than "1" followed by a "1"). If we define tossing a "1" as a success, then the number of failures until the first success has a geometric distribution described as X above, with $p=\frac{1}{6}$:
$P[X=0]=\frac{1}{6}$, $P[X=1]=(\frac{5}{6})(\frac{1}{6})$, $\ldots$, $P[X=t]=(\frac{5}{6})^{t-1}(\frac{1}{6})$. The mean of X is $E[X]=\frac{1}{p}-1=6-1=5$. We expect 5 "failures" until the first success.
If the question had asked us to find the expected toss number of the first "1", then this would be the version of the geometric distribution defined as Y above, with $p=\frac{1}{6}$. Then Y is the toss number at which the first "1" occurs. The mean of Y is $E[Y]=\frac{1}{p}=6$. This is reasonable, since there are six possible outcomes from the die toss, we would expect 6 tosses would be needed on average to get the first "1". □

Negative binomial distribution with parameters r and p ($r > 0$ and $0 < p \leq 1$)

The probability function is

$$p(x) = \binom{r+x-1}{x} p^r (1-p)^x = \binom{r+x-1}{r-1} p^r (1-p)^x \text{ for } x = 0, 1, 2, 3, ...,$$

The mean and variance and moment generating function are

$$E[X] = \frac{r(1-p)}{p} \; , \; Var[X] = \frac{r(1-p)}{p^2} \; , \; M_X(t) = \left[\frac{p}{1-(1-p)e^t}\right]^r .$$

If r is an integer, then the negative binomial random variable X can be interpreted as follows. Suppose that an experiment ends in either failure or success, and the probability of success for a particular trial of the experiment is p. Suppose further that the experiment is performed repeatedly (independent trials) until the r-th success occurs. If X is the number of failures until the r-th success occurs, then X has a negative binomial distribution with parameters r and p. The distribution is defined even if r is not an integer. Note that $r + x$ is the total number of trials until the r-th success. If r is not an integer then $\binom{r+x-1}{r-1} = \frac{(r+x-1)(r+x-2)\cdots(r+1)(r)}{x!}$.

The notation q is sometimes used to represent $1-p$.

The geometric distribution is a special case of the negative binomial with $r = 1$.

We can get some insight into the algebraic form of the negative binomial probability function by considering an example. Suppose that a fair die is tossed until the 3rd "1" turns up. We will define "success" to be a "1" turning up on a die toss, and "failure" will be the result that a "1" didn't turn up on the toss. Suppose that we wish to find the probability that there are (exactly) 2 failures before the 3rd success. In order for there to be 2 failures before the 3rd success, the 3rd success must be on the 5th toss, and of the previous 4 tosses there must be 2 failures and 2 successes. The number of non-"1"'s on the first 4 trials has a binomial distribution, with $n = 4$ and probability $\frac{5}{6}$ (we are assuming that the die is fair). Therefore, the probability of 2 failures on the first 4 trials is $\binom{4}{2}(\frac{1}{6})^2(\frac{5}{6})^2$. The probability of 2 failures in the first 4 trials followed by a success on the 5th trial is (because of independence) $\binom{4}{2}(\frac{5}{6})^2(\frac{1}{6})^2 \times \frac{1}{6} = \binom{4}{2}(\frac{5}{6})^2(\frac{1}{6})^3$. This is the negative binomial probability function $P[X = 2]$ for $r = 3$, $p = \frac{1}{6}$.

More generally, in order to have x failures before the r-th success, we must have the rth success occur on trial number $r + x$ (x failures and $r-1$ successes in the first $x + r - 1$ trials, followed by a success on the r-th trial). Therefore, we must have x failures in the first $x + r - 1$ trials, followed by a success on trial number $x + r$. The number of failures in the first $x + r - 1$ trials has a binomial distribution, and the probability of x failures in the first $x + r - 1$ trials is $\binom{x+r-1}{x}(1-p)^x p^{r-1}$. The probability of a success on the r-th trial is p, so the total probability of x failures before the r-th success is $\binom{x+r-1}{x}(1-p)^x p^{r-1} \times p = \binom{x+r-1}{x}(1-p)^x p^r$.

Keep in mind the binomial coefficient relationship $\binom{n}{k} = \binom{n}{n-k}$, which might be useful in some circumstances.

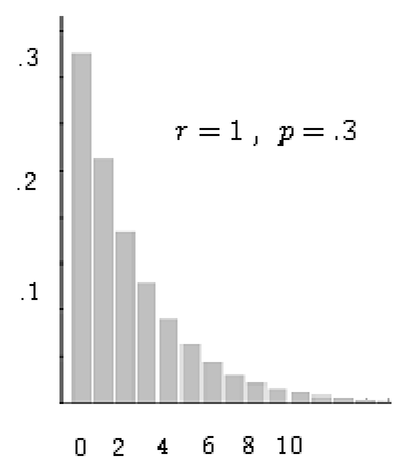
Example 6-7: In tossing a fair die repeatedly (and independently on successive tosses), find the probability of getting the third "1" on the t-th toss.

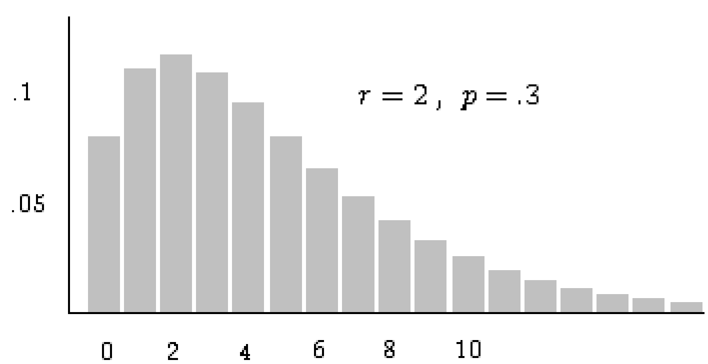
Solution: The negative binomial random variable X with parameters $r = 3$ and $p = \frac{1}{6}$ is the number of failures (failure means tossing 2,3,4,5 or 6) until the 3rd success. The probability that the 3rd success (3rd "1") occurs on the t-th toss is the same as the probability of $x = t - 3$ failures before the 3rd success. Thus, if X is the number of failures until the 3rd success, X has a negative binomial distribution with $r = 3$ and $p = \frac{1}{6}$. Then, the probability that $X = t - 3$ is

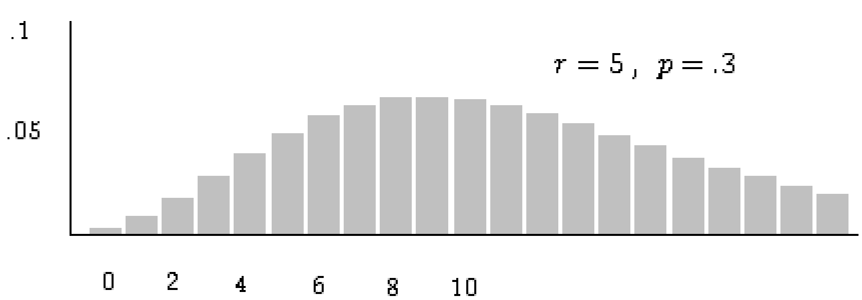
$P[X = t - 3] = p(t - 3) = \binom{r+x-1}{x} p^r (1-p)^x = \binom{3+t-3-1}{t-3} (\frac{1}{6})^3 (\frac{5}{6})^{t-3}$

$= \binom{t-1}{t-3} (\frac{1}{6})^3 (\frac{5}{6})^{t-3} = \frac{(t-1)!}{(t-3)! \cdot 2!} (\frac{1}{6})^3 (\frac{5}{6})^{t-3} = \frac{(t-1)(t-2)}{2} (\frac{1}{6})^3 (\frac{5}{6})^{t-3}$. $\square$

As mentioned above, the geometric distribution is a special case of the negative binomial with $r = 1$. The graphs below show the histogram for negative binomial distributions with $p = .3$ and (i) $r = 1$ (geometric) , (ii) $r = 2$, and (iii) $r = 5$.







Hypergeometric distribution with integer parameters M, K and n ($M > 0,\ 0 \le K \le M$ and $1 \le n \le M$):

In a group of M objects, suppose that K are of Type I and $M - K$ are of Type II.
If a subset of n objects is randomly chosen without replacement from the group of M objects, let X denote the number that are of Type I in the subset of size n. X is said to have a hypergeometric distribution. X is a non-negative integer that satisfies
$X \le n$, $X \le K$, $0 \le X$ and $n - (M - K) \le X$.
The probability function for X is $p(x) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}}$.
x can't be larger than n or K, so $x \le min[n, K]$, and since there are $M - K$ Type II objects in total, x must be at least $n - (M - K)$.
The probability function is explained as follows. There are $\binom{M}{n}$ ways of choosing the subset of n objects from the entire group of M objects. The number of choices that result in x objects of Type I and $n - x$ objects of Type II is $\binom{K}{x}\binom{M-K}{n-x}$.
The mean and variance of X are $E[X] = \frac{nK}{M}$, $Var[X] = \frac{nK(M-K)(M-n)}{M^2 \cdot (M-1)}$.

Example 6-8: An urn contains 6 blue and 4 red balls. 6 balls are chosen at random and without replacement from the urn. If X is the number of red balls chosen, find the standard deviation of X.

Solution: This is a hypergeometric distribution with $M = 10$, $K = 4$ and $n = 6$.
The probability function of X is $f(x) = \frac{\binom{4}{x}\binom{6}{6-x}}{\binom{10}{6}}$, for $x = 0, 1, 2, 3, 4$.
The variance is $Var[X] = \frac{nK(M-K)(M-n)}{M^2 \cdot (M-1)} = .64$. Standard deviation is $\sqrt{.64} = .8$. □

The hypergeometric distribution has rarely come up on the exams that have been publicly released.

Multinomial distribution with parameters n, p_1, $p_2, \ldots, p_k$ (where n is a positive integer and $0 \leq p_i \leq 1$ for all $i = 1, 2, ..., k$ and $p_1 + p_2 + \cdots + p_k = 1$)

Suppose that an experiment has k possible outcomes, with probabilities $p_1,\ p_2, ..., p_k$ respectively. Each time the experiment is performed, it results in one of the possible outcomes. If the experiment is performed n successive times (independently), let X_i denote the number of experiments that resulted in outcome i, so that $X_1 + X_2 + \cdots + X_k = n$.
The multinomial distribution probability function is
$P[X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k] = p(x_1, x_2, ..., x_k) = \frac{n!}{x_1! \cdot x_2! \cdots x_k!} \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_k^{x_k}$.

For each i from $i = 1$ to $i = k$, X_i is a random variable with a mean and variance similar to the binomial mean and variance: $E[X_i] = np_i$, $Var[X_i] = np_i(1 - p_i)$.
Also, for i and j between 1 and n, X_i and X_j are related. A little later on in this study guide we will review joint distributions and relationships between random variables. One measure of the relationship between random variables is the covariance. For the multinomial distribution, the covariance between X_i and X_j is $Cov[X_i\ ,\ X_j] = -np_ip_j$.

For example, the toss of a fair die results in one of $k = 6$ outcomes, with probabilities $p_i = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$. If the die is tossed n times, then with $X_i =$ # of tosses resulting in face "i" turning up, the distribution of $X_1, X_2, ..., X_6$ is a multinomial distribution. In $n = 10$ tosses of the die, the probability that there are exactly 2-"1"'s , 1-"2" , 0-"3"'s , 3-"4"'s , 1-"5" and 3-"6"'s is $\frac{10!}{2! \cdot 1! \cdot 0! \cdot 3! \cdot 1! \cdot 3!} \cdot (\frac{1}{6})^2(\frac{1}{6})^1(\frac{1}{6})^0(\frac{1}{6})^3(\frac{1}{6})^1(\frac{1}{6})^3$.

Recursive relationship for the binomial, Poisson and negative binomial:

The probability function for each these three distributions satisfies the following recursive relationship $\frac{p_k}{p_{k-1}} = a + \frac{b}{k}$ for $k = 1, 2, 3, \ldots$.
Poisson with parameter λ: $\frac{p_k}{p_{k-1}} = \frac{e^{-\lambda}\lambda^k/k!}{e^{-\lambda}\lambda^{k-1}/(k-1)!} = \frac{\lambda}{k} \rightarrow a = 0\ ,\ b = \lambda$.
Binomial with parameters n and p: $a = -\frac{p}{1-p}\ ,\ b = \frac{(n+1)p}{1-p}$.
Negative binomial with parameters r and p: $a = 1 - p\ ,\ b = (r-1)(1-p)$.
For instance, for a Poisson distribution with $\lambda = 2$, we have $a = 0$, $b = 2$, and
$p_k = (a + \frac{b}{k}) \cdot p_{k-1} = \frac{2}{k} \cdot \frac{e^{-2} \cdot 2^{k-1}}{(k-1)!} = \frac{e^{-2} \cdot 2^k}{k!}$.

In the released exams, the binomial, Poisson and geometric have come up regularly. The negative binomial arises occasionally, and the, multinomial and hypergeometric have rarely occurred.

SUMMARY OF DISCRETE DISTRIBUTIONS

Distribution	Parameters	Prob. Fn., $p(x)$	Mean, $E[X]$	Variance, $Var[X]$	MGF, $M_X(t)$
Uniform	$N > 0$, integer	$\frac{1}{N}$, $x = 1, 2, ..., N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\frac{e^t(e^{Nt}-1)}{N(e^t-1)}$
Binomial	$n > 0$ integer, $0 < p < 1$	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, ..., n$	np	$np(1-p)$	$(1 - p + pe^t)^n$
Poisson	$\lambda > 0$	$\frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, ...$	λ	λ	$e^{\lambda(e^t-1)}$
Geometric	$0 < p < 1$	$(1-p)^x \cdot p$, $x = 0, 1, 2, ...$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$
Negative Binomial	$r > 0$, $0 < p < 1$	$\binom{r+x-1}{x} p^r (1-p)^x$, $x = 0, 1, 2, ...$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{p}{1-(1-p)e^t}\right]^r$
Hypergeometric	$M > 0$, $0 \leq K \leq M$, $1 \leq n \leq M$, integers	$\frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}}$ $x \leq min[n, K]$	$\frac{nK}{M}$	$\frac{nK(M-K)(M-n)}{M^2(M-1)}$	
Multinomial	n, $p_1, p_2, ..., p_k$ $0 < p_i < 1$	$\frac{n!}{x_1! \cdot x_2! \cdots x_k!} \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_k^{x_k}$ $x_1 + x_2 + \cdots + x_k = n$	$E[X_i] = np_i$	$Var[X_i] = np_i(1-p_i)$	

PROBLEM SET 6

Frequently Used Discrete Distributions

1. X has a discrete uniform distribution on the integers $0, 1, 2, \ldots, n$ and Y has a discrete uniform distribution on the integers $1, 2, 3, \ldots, n$. Find $Var[X] - Var[Y]$.

A) $\frac{2n+1}{12}$ B) $\frac{1}{12}$ C) 0 D) $-\frac{1}{12}$ E) $-\frac{2n+1}{12}$

2. The probability that a particular machine breaks down in any day is .20 and is independent of the breakdowns on any other day. The machine can break down only once per day. Calculate the probability that the machine breaks down two or more times in ten days.

A) $.1075$ B) $.0400$ C) $.2684$ D) $.6242$ E) $.9596$

3. (SOA) A company prices its hurricane insurance using the following assumptions:
(i) In any calendar year, there can be at most one hurricane.
(ii) In any calendar year, the probability of a hurricane is 0.05 .
(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.
Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

A) 0.06 B) 0.19 C) 0.38 D) 0.62 E) 0.92

4. (SOA) A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

A) 0.096 B) 0.192 C) 0.235 D) 0.376 E) 0.469

5. (SOA) A hospital receives $1/5$ of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials. For Company X's shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective. What is the probability that this shipment came from Company X?

A) 0.10 B) 0.14 C) 0.37 D) 0.63 E) 0.86

6. (SOA) A company establishes a fund of 120 from which it wants to pay an amount, C, to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year, independent of any other employee. Determine the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.

A) 24 B) 30 C) 40 D) 60 E) 120

7. Let X be a Poisson random variable with $E[X] = ln\,2$. Calculate $E[cos(\pi X)]$.

A) 0 B) $\frac{1}{4}$ C) $\frac{1}{2}$ D) 1 E) $2\,ln\,2$

8. (SOA) The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is considered to be a continuous random variable with pdf ce^{-cx}. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year. What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

A) 0.15 B) 0.34 C) 0.43 D) 0.57 E) 0.66

9. (SOA) An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

A) $\frac{1}{\sqrt{3}}$ B) 1 C) $\sqrt{2}$ D) 2 E) 4

10. (SOA) An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4. What is the expected benefit under this policy?

A) 2234 B) 2400 C) 2500 D) 2667 E) 2694

11. (SOA) A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists. Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay $100(ticket\,cost + 50\,penalty)$ to the tourist. What is the expected revenue of the tour operator?

A) 935 B) 950 C) 967 D) 976 E) 985

12. A fair die is tossed until a 2 is obtained. If X is the number of trials required to obtain the first 2, what is the smallest value of x for which $P[X \leq x] \geq \frac{1}{2}$?

A) 2 B) 3 C) 4 D) 5 E) 6

13. (SOA) A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool. This month, the insurance company writes 4 new policies for adults earning their first driver's license. What is the probability that these 4 will contain at least two more high-risk drivers than low-risk drivers?

A) 0.006 B) 0.012 C) 0.018 D) 0.049 E) 0.073

14. A box contains 10 white and 15 black marbles. Let X denote the number of white marbles in a selection of 10 marbles selected at random and without replacement. Find $\frac{Var[X]}{E[X]}$.

A) $\frac{1}{8}$ B) $\frac{3}{16}$ C) $\frac{2}{8}$ D) $\frac{5}{16}$ E) $\frac{3}{8}$

15. A multiple choice test has 10 questions, and each question has 5 answer choices (exactly one of which is correct). A student taking the test guesses randomly on all questions. Find the probability that the student will actually get at least as many correct answers as she would expect to get with the random guessing approach.

A) .624 B) .591 C) .430 D) .322 E) .302

16. An analysis of auto accidents shows that one in four accidents results in an insurance claim. In a series of independent accidents, find the probability that the first accident resulting in an insurance claim is one of the first 3 accidents.

A) .50 B) .52 C) .54 D) .56 E) .58

17. An insurer has 5 independent one-year term life insurance policies. The face amount on each policy is 100,000. The probability of a claim occurring in the year for any given policy is .2. Find the probability the insurer will have to pay more than the total expected claim for the year.

A) .06 B) .11 C) .16 D) .21 E) .26

18. The number of claims per year from a particular auto insurance policy has a Poisson distribution with a mean of 1, and probability function p_k. Based on a number of years of experience, the insurer decides to change the distribution, so that the new probability of 0 claims is $p_0^* = .5$, and the new probabilities p_k^* for $k \geq 1$ are proportional to the old (Poisson) probabilities according to the relationship $p_k^* = c \cdot p_k$ for $k \geq 1$. Find the mean of the new claim number distribution.

A) .79 B) .63 C) .5 E) .37 E) .21

19. The probability generating function of a discrete non-negative integer valued random variable N is a function of the real variable t: $P(t) = \sum_{k=0}^{\infty} t^k \cdot P[N = k] = E[t^N]$.

Which of the following is the correct expression for the probability generating function of the Poisson random variable with mean 2?

A) e^{-2t} B) e^{1-2t} C) e^{2t} D) e^{2t-1} E) $e^{2(t-1)}$

20. An insurer issues two independent policies to individuals of the same age.
The insurer models the distribution of the completed number of years until death for each individual, and uses the geometric distribution $P[N=k]=(.99)^k(.01)$, where $k=0,1,2,...$ and N is the completed number of years until death for each individual.
Find the probability that the two individuals die in the same year.
A) .001 B) .003 C) .005 D) .007 E) .009

21. An insurer uses the Poisson distribution with mean 4 as the model for the number of warranty claims per month on a particular product. Each warranty claim results in a payment of 1 by the insurer. Find the probability that the total payment by the insurer in a given month is less than one standard deviation above the average monthly payment.
A) .9 B) .8 C) .7 D) .6 E) .5

22. As part of the underwriting process for insurance, each prospective policyholder is tested for high blood pressure. Let X represent the number of tests completed when the first person with high blood pressure is found. The expected value of X is 12.5.
Calculate the probability that the sixth person tested is the first one with high blood pressure.
A) 0.000 B) 0.053 C) 0.080 D) 0.316 E) 0.394

23. Let X be a random variable with moment generating function

$$M(t)=\left(\frac{2+e^t}{3}\right)^9 , \quad -\infty<t<\infty \ .$$

Calculate the variance of X.
A) 2 B) 3 C) 8 D) 9 E) 11

24. According to the house statistician, a casino estimates that it has a 51% chance of winning on any given hand of blackjack. The casino also assumes that blackjack hands are independent of one another. The casino randomly monitors its blackjack dealers, and as soon as a dealer is found to lose 5 hands in a row, the casino stops the game at that dealer's table and checks the deck of cards that the dealer is using. The casino has just started monitoring a dealer. What is the chance that the game will be stopped at the table sometime within the next 8 hands of blackjack?
A) .07 B) .09 C) .11 D) .13 E) .15

25. (SOA) A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is $\frac{3}{5}$. The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months. Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs.

A) 0.01 B) 0.12 C) 0.23 D) 0.29 E) 0.41

26. (SOA) Each time a hurricane arrives, a new home has a 0.4 probability of experiencing damage. The occurrences of damage in different hurricanes are independent. Calculate the mode of the number of hurricanes it takes for the home to experience damage from two hurricanes.

A) 2 B) 3 C) 4 D) 5 E) 6

27. For a certain discrete random variable on the non-negative integers, the probability function satisfies the relationships

$P(0) = P(1)$ and $P(k+1) = \frac{1}{k} \cdot P(k)$ for $k = 1, 2, 3, ...$ Find $P(0)$.

A) $ln\, e$ B) $e - 1$ C) $(e+1)^{-1}$ D) e^{-1} E) $(e-1)^{-1}$

PROBLEM SET 6 SOLUTIONS

1. $Var[X] = \frac{(n+1)^2-1}{12}$ and $Var[Y] = \frac{n^2-1}{12}$ so that
$Var[X] - Var[Y] = \frac{2n+1}{12}$. Answer: A

2. Since the breakdowns from one day to another are independent, the number of breakdowns (successes) in 10 days, X, has a binomial distribution with $n = 10$ and $p = .2$.
$P[X \geq 2] = 1 - (P[X = 0] + P[X = 1])$
$= 1 - \binom{10}{0}(.2)^0(.8)^{10} - \binom{10}{1}(.2)^1(.8)^9 = .6242$. Answer: D

3. The number of hurricanes N in 20 years will have a binomial distribution with $n = 20$ (years) and $p = .05$ (chance of hurricane in any one year).
$P[N < 3] = P[N = 0] + P[N = 1] + P[N = 2]$
$= \binom{20}{0}(.05)^0(.95)^{20} + \binom{20}{1}(.05)^1(.95)^{19} + \binom{20}{2}(.05)^2(.95)^{18} = .9245$.
Note that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Answer: E

4. For each group, the number who complete the study is a binomial random variable with $n = 10$ trials (10 people in the group, each is a "trial") and probability $p = .8$ of any individual in the group completing the study (probability of success for one trial).
The probability of at least 9 completing the study in group 1 is $P[N = 9] + P[N = 10]$.
This is $\binom{10}{9}(.8)^9(.2)^1 + \binom{10}{10}(.8)^{10}(.2)^0 = (10)(.13422)(.2) + (1)(.10737) = .376$.
The probability that less than 9 complete the study in group 1 is $1 - .376 = .624$.
The same is true for group 2.

We are asked to find the probability that at least 9 participants complete the study in one of the two groups, but not in both of the groups. Since the two groups are independent, this will be
P[at least 9 complete study in group 1] $\cdot$ P[less than 9 complete study in group 2]
$+ P$[less than 9 complete study in group 1] $\cdot$ P[at least 9 complete study in group 2]
$= (.376)(.624) + (.624)(.376) = .469$. Answer: E

5. Event X - vaccine is from company X ,
$\overline{X}$ - vaccine is from a company other than X. $P(X) = \frac{1}{5}$, $P(\overline{X}) = \frac{4}{5}$.
In a sample of 30 vials from Company X, the number of ineffective vials has a binomial distribution with $n = 30$ trials (vials) and probability $p = .10$ of any particular one being ineffective. In a sample of 30 vials from a Company other than X, the number of ineffective vials has a binomial distribution with $n = 30$ trials (vials) and probability $p = .02$ of any particular one being ineffective.
Therefore, $P[\text{1 ineffective}|\text{shipment is from Company } X] = \binom{30}{1}(.1)^1(.9)^{29} = .1413$,
and
$P[\text{1 ineffective}|\text{shipment is from Company other than } X] = \binom{30}{1}(.02)^1(.98)^{29} = .3340$.
We wish to find $P[\text{shipment is from Company } X|\text{1 ineffective}]$
$= \frac{P[(\text{shipment is from Company } X)\cap(\text{1 ineffective})]}{P[\text{1 ineffective}]}$.
$P[(\text{shipment is from Company } X) \cap (\text{1 ineffective})]$
$= P[\text{1 ineffective}|\text{shipment is from Company } X] \cdot P[\text{shipment is from Company } X]$
$= (.1413)(\frac{1}{5})$.
$P[\text{1 ineffective}] = P[(\text{1 ineffective}) \cap X] + P[(\text{1 ineffective}) \cap \overline{X}]$
$= P[(\text{1 ineffective})|X] \cdot P[X] + P[(\text{1 ineffective})|\overline{X}] \cdot P[\overline{X}] = (.1413)(\frac{1}{5}) + (.3340)(\frac{4}{5})$.
Then, $P[\text{shipment is from Company } X|\text{1 ineffective}] = \frac{(.1413)(\frac{1}{5})}{(.1413)(\frac{1}{5})+(.3340)(\frac{4}{5})} = .096$.

This can be described by the following probability table.

	Company X	Other Companies
	$P[X] = .2$, given	$P[X'] = .8$
1 Ineff	$P[\text{1 Ineff}|X] = .1413$	$P[\text{1 Ineff}|\text{other}] = .3340$
	given (calc. for binomial dist.)	given (calc. for binomial dist.)
	$\Downarrow$	$\Downarrow$
	$P[\text{1 Ineff} \cap X]$	$P[\text{1 Ineff} \cap \text{Other}]$
	$= (.1413)(.2)$	$= (.3340)(.8)$

$\Downarrow$

$P[\text{I ineff}] = (.1413)(.2) + (.3340)(.8)$

$\Downarrow$

$P[X|\text{1 ineff}] = \frac{(.1413)(.2)}{(.1413)(.2)+(.3340)(.8)} = .096$. Answer: A

6. N is the number of people who achieve high performance. N has a binomial distribution with $n = 20$ and $p = .02$. We wish to find the largest C for which
$P[NC > 120] < .01$. From the binomial distribution, we have
$P[N = 0] = \binom{20}{0}(.02)^0(.98)^{20} = .6676$,
$P[N = 1] = \binom{20}{1}(.02)^1(.98)^{19} = .2725$,
$P[N = 2] = \binom{20}{2}(.02)^2(.98)^{28} = .0528$.
Therefore, $P[N > 1] = .0599$, $P[N > 2] = .0071$.
If $\frac{120}{C} \geq 2$ then $P[NC > 120] = P[N > \frac{120}{C}] \leq P[N > 2] = .0071 < .01$,
but if $\frac{120}{C} < 2$ then $P[NC > 120] = P[N > \frac{120}{C}] \leq P[N \geq 2] = .0599 > .01$.
In order to satisfy $P[NC > 120] < .01$ we must have $\frac{120}{C} \geq 2$, or equivalently,
$C \leq 60$.

An alternative approach to this problem is to look at each possible value of C in the answers, and find the probability $P[NC > 120]$ for each. Starting with the largest possible value, we get for answer E, $P[120N > 120] = P[N > 1] = 1 - P[N = 0] - P[N = 1]$
$= 1 - .6676 - .2725 = .0599 > .01$, so $C = 120$ does not satisfy the probability requirement.
Then for answer D, we get
$P[60N > 120] = P[N > 2] = 1 - P[N = 0] - P[N = 1] - P[N = 2]$
$= 1 - .6676 - .2725 - .0528 = .0071 < .01$, so that $C = 60$ does satisfy the requirement.
Answer: D

7. Since X has a Poisson distribution, it can take on the non-negative integer values $0, 1, 2, ...$
With $E[X] = ln(2)$, the probability function of X is $P[X = x] = \frac{e^{-ln(2)}[ln(2)]^x}{x!} = \frac{1}{2} \cdot \frac{[ln(2)]^x}{x!}$
The transformed random variable $cos(\pi X)$ can take on the values
$cos(0) = 1,\ cos(\pi) = -1,\ cos(2\pi) = 1,\ -1\,,\ 1\,,\ -1\,,\ \ldots$
Then
$E[cos(\pi X)] = \sum_{x=0}^{\infty} cos(\pi x) \cdot \frac{1}{2} \cdot \frac{[ln(2)]^x}{x!} = \frac{1}{2} \cdot \sum_{x=0}^{\infty} (-1)^x \cdot \frac{[ln(2)]^x}{x!} = \frac{1}{2} \cdot \sum_{x=0}^{\infty} \frac{[-ln(2)]^x}{x!}$
$= \frac{1}{2} \cdot e^{-ln(2)} = \frac{1}{4}$ (we use the identity, $\sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a$) . Answer: B

8. We interpret proportion as probability. The statement "30% of high-risk drivers will be involved in an accident in the first 50 days of the year" is interpreted as
$P[$a high-risk driver is involved in an accident in the first 50 days of the year$] = .3$.
This can be written as $P[T \leq 50] = .3$, where T is the time, in days, until an accident occurs for a high-risk driver. We are told that T has an exponential distribution. Suppose that the mean of T is μ. Then the density function of T is $f(t) = \frac{1}{\mu}e^{-t/\mu}$, and $P[T \leq 50] = 1 - e^{-50/\mu}$.
From $.3 = P[T \leq 50] = 1 - e^{-50/\mu}$, we get $e^{-50/\mu} = .7$, and then $\mu = -\frac{50}{ln\,.7}$.
The proportion of high-risk drivers that are expected to have an accident in the first 80 days of the year is interpreted as a probability,
$P[T \leq 80] = 1 - e^{-80/\mu} = 1 - e^{-80/(-50/ln\,.7)} = 1 - e^{1.6\,ln\,.7} = 1 - (.7)^{1.6} = .43$.
Answer: C

9. N has a Poisson distribution with mean λ . $P[N = 2] = e^{-\lambda} \cdot \frac{\lambda^2}{2!}$ and
$P[N = 4] = e^{-\lambda} \cdot \frac{\lambda^4}{4!}$. We are told that $P[N = 2] = 3P[N = 4]$, so that
$e^{-\lambda} \cdot \frac{\lambda^2}{2} = 3e^{-\lambda} \cdot \frac{\lambda^4}{24} \Rightarrow \lambda = 2$.
The variance of the Poisson distribution is equal to the mean, $\lambda = 2$. Answer: D

10. The probability that the device fails in year $n = 1, 2, 3, \ldots$,
is $(.6)^{n-1}(.4)$ ($n - 1$ years of non-failure followed year of failure).
This is a version of the geometric distribution.

Year of Failure	Prob.	Amount Paid
1	.4	4000
2	$(.6)(.4) = .24$	3000
3	$(.6)^2(.4) = .144$	2000
4	$(.6)^3(.4) = .0864$	1000
≥ 5		0

The expected amount paid is
$(4000)(.4) + (3000)(.24) + (2000)(.144) + (1000)(.0864) = 2694.4$.
Answer: E

11. The tour operator collects 21 fares, $21 \times 50 = 1050$. Let N denote the number of ticket holders who show up. The tour operator does not have to make a refund if $N \leq 20$. If $N = 21$, the tour operator must pay 100. The number of ticket holders that show up has a binomial distribution based on $n = 21$ (ticket holders) and $p = .98$ (probability of any particular ticket holder showing up). Then $P[N = 21] = \binom{21}{21}(.98)^{21}(.02)^0 = .65426$.
The expected amount the tour operator must pay in refund and penalty is
$(0) \cdot P[N \leq 20] + (100) \cdot P[N = 21] = (100)(.65426) = 65.43$.
The expected revenue (after refund and penalty) is $1050 - 65.43 = 984.57$.
Answer: E

12. $P[X = k] = (\frac{5}{6})^{k-1}(\frac{1}{6})$.
$P[X \leq x] = \sum_{k=1}^{x} P[X = k] = \frac{1}{6} \cdot \left[\frac{1-(\frac{5}{6})^x}{1/6}\right] = 1 - (\frac{5}{6})^x \geq \frac{1}{2} \rightarrow (\frac{5}{6})^x \leq \frac{1}{2} \rightarrow x \geq 4.$
Note that $X - 1$ has a geometric distribution with $p = \frac{1}{6}$. Answer: C

13. This problem involves the multinomial distribution. The multinomial distribution with parameters $n,\ p_1,\ p_2, \ldots,\ p_k$ (where n is a positive integer and $0 \leq p_i \leq 1$ for all $i = 1, 2, ..., k$ and $p_1 + p_2 + \cdots p_k = 1)$ is defined in the following way.
Suppose that an experiment has k possible outcomes, with probabilities $p_1,\ p_2, ..., p_k$ respectively. If the experiment is performed n successive times (independently), let X_i denote the number of experiments that resulted in outcome i, so that
$X_1 + X_2 + \cdots + X_k = n$. The multinomial probability function is
$f(x_1, x_2, ..., x_k) = \frac{n!}{x_1! \cdot x_2! \cdots x_k!} \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_k^{x_k}$.
In this problem, the "experiment" outcome is the type of driver, which has three outcomes. These are "low-risk", with probability $p_1 = .5$, "moderate-risk", with probability $p_2 = .3$, and "high-risk", with probability $p_3 = .2$. The "experiment" (choosing a driver) is performed $n = 4$ times. We want to find the probability that $X_3 \geq X_1 + 2$ (number of high-risk drivers at least two more than the number of low-risk drivers). We can look at the outcomes that result in this event:

X_1 , number of low-risk drivers	X_2 , number of moderate-risk rivers	X_3 , number of high-risk drivers
0	0	4
0	1	3
0	2	2
1	0	3

13. continued

These are the only outcomes that result in this event.

$f(0,0,4) = \frac{4!}{0!\cdot 0!\cdot 4!} \cdot (.5)^0 \cdot (.3)^0 \cdot (.2)^4 = .0016$,

$f(0,1,3) = \frac{4!}{0!\cdot 1!\cdot 3!} \cdot (.5)^0 \cdot (.3)^1 \cdot (.2)^3 = .0096$,

$f(0,2,2) = \frac{4!}{0!\cdot 2!\cdot 2!} \cdot (.5)^0 \cdot (.3)^2 \cdot (.2)^2 = .0216$,

$f(1,0,3) = \frac{4!}{1!\cdot 0!\cdot 3!} \cdot (.5)^1 \cdot (.3)^0 \cdot (.2)^3 = .0160$.

The total probability of this event is then $.0016 + .0096 + .0216 + .0160 = .0488$.

Answer: D

14. X has a hypergeometric distribution with $M = 25$ marbles, $K = 10$ white marbles, and $n = 10$ marbles chosen. Then $E[X] = \frac{nK}{M} = \frac{(10)(10)}{25} = 4$, and $Var[X] = \frac{nK(M-K)(M-n)}{M^2(M-1)} = \frac{3}{2} \Rightarrow \frac{Var[X]}{E[X]} = \frac{(M-K)(M-n)}{M(M-1)} = \frac{3}{8}$. Answer: E

15. The probability of a correct guess is .2 on any particular question. The number of correct guesses forms a binomial distribution based on $n = 10$ trials (10 questions), with a probability of $p = .2$ of success (correct answer) on each trial. The expected number of correct guesses is $np = 10(.2) = 2$. The probability of getting at least 2 correct is $P[X \geq 2]$. The binomial distribution probability is $P[X = k] = \binom{n}{k}(.2)^k(.8)^{10-k}$. Then,

$$P[X \geq 2] = 1 - (P[X=0] + P[X=1]) = 1 - \binom{10}{0}(.2)^0(.8)^{10} - \binom{10}{1}(.2)^1(.8)^9$$

$= .624$. Answer: A

16. Out of 3 independent accidents, the number that result in a claim has a binomial distribution with $n = 3$ and $p = \frac{1}{4}$. The probability that none of the 3 accidents result in a claim is $\binom{3}{0}(\frac{1}{4})^0(\frac{3}{4})^3 = .422$. The probability that there is a least one claim in the 3 accidents is $1 - .422 = .578$.

Alternatively, the probability that the first accident resulting in a claim is the k-th accident is $(.75)^{k-1}(.25)$ (geometric distribution). Thus, the probability is $(.75)^0(.25) + (.75)^1(.25) + (.75)^2(.25) = .578$. Answer: E

17. The expected claim from any one policy is $(100,000)(.2) = 20,000$, so the overall expected claim from all 5 policies is 100,000 . The total claim for the year will be more than 100,000 if there are 2 or more claims. This probability is

$P[N \geq 2] = 1 - P[N = 0] - P[N = 1]$, where N is the number of claims.

N has a binomial distribution with $n = 5$, $p = .2$.

$P[N = 0] + P[N = 1] = \binom{5}{0}(.2)^0(.8)^5 + \binom{5}{1}(.2)^1(.8)^4 = .73728$

$\Rightarrow \; P[N \geq 2] = 1 - .73728 = .26272$. Answer: E

18. $1 = p_0^* + \sum_{k=1}^{\infty} p_k^*$, so that since $p_0^* = .5$, we must have $\sum_{k=1}^{\infty} p_k^* = c\sum_{k=1}^{\infty} p_k = .5$.

However, $1 = p_0 + \sum_{k=1}^{\infty} p_k = e^{-1} + \sum_{k=1}^{\infty} p_k \to \sum_{k=1}^{\infty} p_k = 1 - e^{-1} = .6321$,

so that $c = \frac{.5}{.6321} = .7910$. Then, the new expectation is

$\sum_{k=0}^{\infty} kp_k^* = \sum_{k=1}^{\infty} kp_k^* = \sum_{k=1}^{\infty} kcp_k = c\sum_{k=1}^{\infty} kp_k = c\sum_{k=0}^{\infty} kp_k$

$= c \times \text{old expectation} = c \cdot 1 = .7910$. Answer: A

19. $P(t) = \sum_{k=0}^{\infty} t^k \cdot P[N = k] = E[t^N] = \sum_{k=0}^{\infty} t^k \cdot e^{-2} \cdot \frac{2^k}{k!}$

$= \sum_{k=0}^{\infty} e^{-2} \cdot \frac{(2t)^k}{k!} = e^{-2} \cdot \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} = e^{-2}e^{2t} = e^{2(t-1)}$. Answer: E

20. $P[\text{death in same year}] = \sum_{k=0}^{\infty} P[\text{both die after } k \text{ complete years}]$

$= \sum_{k=0}^{\infty} P[\text{person 1 dies after } k \text{ complete years}] \cdot P[\text{person 2 dies after } k \text{ complete years}]$

$= \sum_{k=0}^{\infty} (.99)^k(.01)(.99)^k(.01) = (.0001)\sum_{k=0}^{\infty} [(.99)^2]^k = \frac{.0001}{1-(.99)^2} = .005$. Answer: C

21. Average monthly payment is 4, variance is 4 (variance of Poisson is equal to mean).

Probability that total payment is less than $4 + 2 = 6$ is

$P[N \leq 5] = e^{-4}[1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}] = .785$. Answer: B

22. This problem makes use of the geometric distribution. The experiment being performed is the blood pressure test on an individual. We define "success" of the experiment to mean that the individual has high blood pressure. We denote the probability of a success occurring in a particular trial by p. Since X is the number of persons tested until the first person with high blood pressure is found, it is like the version of the geometric distribution described as Y earlier in the study guide, where Y is the trial number of the first success (the trial number of the first success is 1, or 2, or 3, ...). The mean of this form of the geometric distribution is $\frac{1}{p}$, so that $\frac{1}{p} = 12.5$ and therefore $p = .08$. The probability that the first success occurs on the 6th trial (first case of high blood pressure is the 6th individual) is $(1-p)^5 p$, since there will be 5 failures and then the first success. This probability is $(.92)^5(.08) = .0527$. Answer: B

23. One of the applications of the moment generating function $M_X(t)$ for the random variable X is to calculate the moments of X - for an integer $k \geq 1$,
$E[X^k] = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}$. Therefore, $Var[X] = E[X^2] - (E[X])^2 = M_X^{(2)}(0) - [M_X'(0)]^2$.
In this problem, $M_X'(t) = 9\left(\frac{2+e^t}{3}\right)^8 (\frac{e^t}{3})$, so that $M_X'(0) = 3$, and
$M_X^{(2)}(t) = 72\left(\frac{2+e^t}{3}\right)^7 (\frac{e^t}{3})^2 + 9\left(\frac{2+e^t}{3}\right)^8 (\frac{e^t}{3})$, so that $M_X^{(2)}(0) = 11$,
and then, $Var[X] = 11 - 3^2 = 2$.
Alternatively, $Var[X] = \frac{d^2}{dt^2}[ln\, M_X(t)]\Big|_{t=0}$. In this problem, $ln\, M_X(t) = 9\, ln(2+e^t) - 9\, ln\, 3$, so that
$\frac{d}{dt}[ln\, M_X(t)] = \frac{9e^t}{2+e^t}$, and $\frac{d^2}{dt^2}[ln\, M_X(t)] = \frac{(2+e^t)(9e^t)-(9e^t)(e^t)}{(2+e^t)^2}$, and then
$Var[X] = \frac{(3)(9)-(9)(1)}{(3)^2} = 2$.

A much faster solution is based on the following fact. The moment generating function of the binomial random variable with parameters n (number of trials) and p (probability of success) is $(1-p+pe^t)^n$. In this case, the mgf corresponds to the binomial distribution with $n = 9$ and $p = \frac{1}{3}$, and therefore the variance is $np(1-p) = 9(\frac{1}{3})(\frac{2}{3}) = 2$. Answer: A

24. The game will be stopped only under the following circumstances:
L L L L L , W L L L L L , L W L L L L L , W W L L L L L ,
L L W L L L L L , W L W L L L L L , L W W L L L L L , W W W L L L L L ,
where W refers to win and L refers to loss. The sum of the probabilities is
$(.49)^5 + (.51)(.49)^5 + (.49)(.51)(.49)^5 + (.51)^2(.49)^5 + (.49)^2(.51)(.49)^5$
$+ (.51)(.49)(.51)(.49)^5 + (.49)(.51)^2(.49)^5 + (.51)^3(49)^5 = .0715$. Answer: A

25. We define the random variable X to be the number of months in which no accidents have occurred when the fourth month of accidents has occurred. We wish to find $P[X \geq 4]$. This can be written as $1 - P[X = 0, 1, 2$ or $3]$.
$P[X = 0] = P$[first 4 months all have accidents] $= (\frac{3}{5})^4 = .1296$.
$P[X = 1] = P$[1 of the first 4 months has no accidents and 3 have accidents and 5th month has accidents] $= \binom{4}{1}(\frac{2}{5})^1(\frac{3}{5})^3(\frac{3}{5}) = 4(\frac{2}{5})(\frac{3}{5})^4 = .2074$.
$P[X = 2] = P$[2 of the first 5 months has no accidents and 3 have accidents and 6th month has accidents] $= \binom{5}{2}(\frac{2}{5})^2(\frac{3}{5})^3(\frac{3}{5}) = 10(\frac{2}{5})^2(\frac{3}{5})^4 = .2074$.
$P[X = 3] = P$[3 of the first 6 months has no accidents and 3 have accidents and 7th month has accidents] $= \binom{6}{3}(\frac{2}{5})^3(\frac{3}{5})^3(\frac{3}{5}) = 20(\frac{2}{5})^3(\frac{3}{5})^4 = .1659$.
$P[X \geq 4] = 1 - .1296 - .2074 - .2074 - .1659 = .29$.
X has a negative binomial distribution with $r = 4$ and $p = \frac{3}{5}$. Answer: D

26. If "failure" refers to a hurricane that results in no damage and "success" refers a hurricane that causes damage, then the distribution of X the number of failures until the 2nd success has a negative binomial distribution with probability function
$p(X = x) = \binom{r+x-1}{r-1} p^r (1-p)^x$ for $x = 0, 1, 2, 3, ...,$ where $r = 2$ and $p = .4$
We can also describe this in terms of total number of hurricanes $n = r + x = 2 + x$,
so that $P(N = n) = P(X = n - 2) = \binom{n-1}{2-1} p^2 (1-p)^{n-2} = \binom{n-1}{2-1} (.4)^2 (.6)^{n-2}$
for $n = 2 + x = 2, 3, ...$. We wish find n that maximizes this probability.
$P(N = 2) = .16$, $P(N = 3) = 2 \times .16 \times .6 = .192$, $P(N = 4) = .1728$.
The probabilities continue to decrease, because for $n \geq 3$, we have $\frac{n}{n-1} \times .6 < 1$.
Answer: B

27. $P(2) = P(1) = P(0)$, $P(3) = \frac{1}{2} \cdot P(2) = \frac{1}{2!} \cdot P(0)$, . . .
$P(k) = \frac{1}{(k-1)!} \cdot P(0)$. The probability function must satisfy the requirement
$\sum_{i=0}^{\infty} P(i) = 1$ so that $P(0) + \sum_{i=1}^{\infty} \frac{1}{(i-1)!} \cdot P(0) = P(0)(1 + e) = 1$
(this uses the series expansion for e^x at $x = 1$). Then, $P(0) = \frac{1}{e+1}$. Answer: C

SECTION 7 - FREQUENTLY USED CONTINUOUS DISTRIBUTIONS

Note that for a continuous random variable X, the following probabilities are the same:
$P[a < X < b]$, $P[a < X \le b]$, $P[a \le X < b]$, $P[a \le X \le b]$.

Uniform distribution on the interval (a, b) (where $-\infty < a < b < \infty$):

The density function is constant, $\boldsymbol{f(x) = \frac{1}{b-a}}$ **for** $\boldsymbol{a < x < b}$, and $f(x) = 0$ otherwise.

The distribution function is $F(x) = \int_a^x f(x)\,dx = \frac{x-a}{b-a}$, so $\boldsymbol{F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}}$.

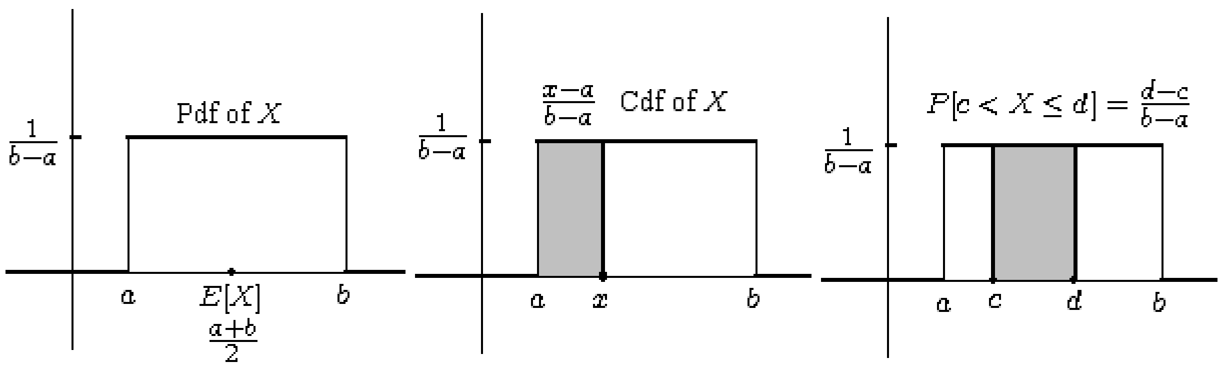
The mean and variance are $\boldsymbol{E[X] = \frac{a+b}{2}}$ and $\boldsymbol{Var[X] = \frac{(b-a)^2}{12}}$.

The moment generating function is $M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)\cdot t}$ for any real t .

The n-th moment of X is $E[X^n] = \frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}$. The median is $\frac{a+b}{2}$, the same as the mean.

This is a symmetric distribution about the mean; the mean is the midpoint of the interval (a, b).

The probability of the subinterval (c, d) of (a, b) is $P[c < X \le d] = \frac{d-c}{b-a}$.

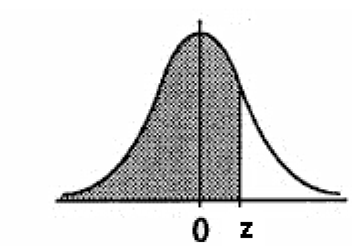


Example 7-1: Suppose that X has a uniform distribution on the interval $(0, a)$, where $a > 0$. Find $P[X > X^2]$.

Solution: If $a \le 1$, then $X > X^2$ is always true for $0 < X < a$, so that $P[X > X^2] = 1$.
If $a > 1$, then $X > X^2$ only if $X < 1$, which has probability
$P[X < 1] = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{a}\,dx = \frac{1}{a}$. Thus, $P[X > X^2] = min[1, \frac{1}{a}]$. □

The Normal Distribution

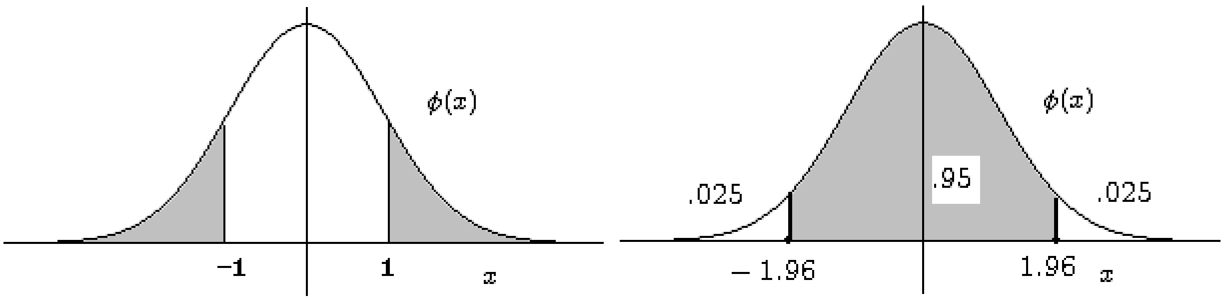
The **standard normal** distribution, $Z \sim N(0,1)$, has a mean of 0 and variance of 1. A table of probabilities for the standard normal distribution is provided on the exam. The density function is $\phi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}$ for $-\infty < z < \infty$. $E[Z] = 0$, $Var[Z] = 1$.
The moment generating function is $M_Z(t) = exp\left[\frac{t^2}{2}\right]$.
The density function has the following bell-shaped graph. The shaded area is the distribution function $P[Z \le z]$, which is denoted $\Phi(z)$. The graph and an excerpt from the standard normal distribution table are given on the following page.



z	0.00	0.01	0.02	0.03
0.0	0.5000	0.5040	0.5080	0.5120
0.1	0.5398	0.5438	0.5478	0.5517
0.2	0.5793	0.5832	0.5871	0.5910
0.3	0.6179	0.6217	0.6255	0.6293
0.4	0.6554	0.6591	0.6628	0.6664
0.5	0.6915	0.6950	0.6985	0.7019
0.6	0.7257	0.7291	0.7324	0.7357
0.7	0.7580	0.7611	0.7642	0.7673
0.8	0.7881	0.7910	0.7939	0.7967
0.9	0.8159	0.8186	0.8212	0.8238

A normal distribution table is provided at the exam. The full table can be found just before the practice exam section later in this study guide. The entries in the table are probabilities of the form $\Phi(z) = P[Z \le z]$. The 95-th percentile of Z is 1.645 (sometimes denoted $z_{.05}$) since $\Phi(1.645) = .950$ (the shaded region to the left of $z = 1.645$ in the graph above).

We use the symmetry of the standard normal distribution to find $\Phi(z)$ for negative values of z. For instance, $\Phi(-1) = P[Z \le -1] = P[Z \ge 1] = 1 - \Phi(1)$ since the two regions have the same area (probability). This is illustrated in the left graph below. The two outside areas are equal, the left area is $\Phi(-1)$ and the right area is $1 - \Phi(1)$. Notice also in the right graph below that $P[-1.96 \le Z \le 1.96] = .95$, since $P[Z > 1.96] = 1 - \Phi(1.96) = .025$, and this area is deleted from both ends of the curve.

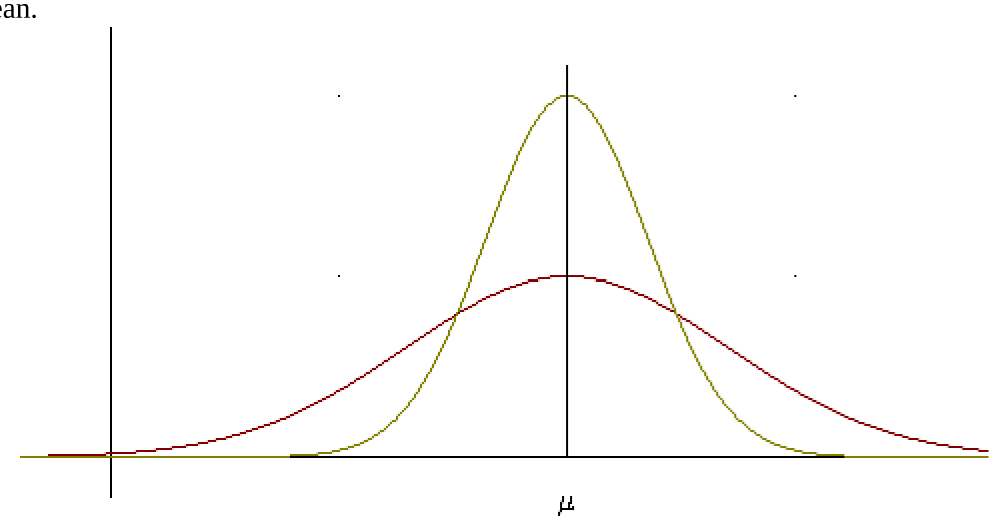


The general form of the normal distribution has mean μ and variance σ^2. This is a continuous distribution with a "bell-shaped" density function similar to that of the standard normal, but symmetric around the mean μ. The pdf is $f(x) = \frac{1}{\sigma\cdot\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/2\sigma^2}$ for $-\infty < x < \infty$.

and the mean, variance and moment generating function of X are

$$\boldsymbol{E[X] = \mu\,,\ Var[X] = \sigma^2\,,\ M_X(t) = exp\Big[\mu t + \frac{\sigma^2 t^2}{2}\Big].}$$

Note also that for the normal distribution, **mean = median = mode =** $\boldsymbol{\mu}$. μ is the "center" of the distribution, and the variance σ^2 is a measure of how widely dispersed the distribution is. The graph shows the density functions of two normal distributions with a common mean μ. The distribution with the "flatter" graph has the larger variance, and is more widely dispersed around the mean.



Given any normal random variable $X \sim N(\mu, \sigma^2)$, it is possible to find $P[r < X < s]$ by first "standardizing" . This means that we define the random variable Z as follows: $Z = \frac{X-\mu}{\sigma}$.

Then $P[r < X < s] = P[\frac{r-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{s-\mu}{\sigma}] = \Phi(\frac{s-\mu}{\sigma}) - \Phi(\frac{r-\mu}{\sigma})$.

For example, suppose that X has a normal distribution with mean 1 and variance 4. Then $P[X \leq 2.5] = P[\frac{X-1}{\sqrt{4}} \leq \frac{2.5-1}{\sqrt{4}}] = P[Z \leq .75] = \Phi(.75) = .7734$.

We have found $\Phi(.75)$ from the standard normal table.

The 95-th percentile of X can be found as follows. Let us denote the 95-th percentile of X by c. Then $P[X \leq c] = .95 \Rightarrow P[\frac{X-1}{\sqrt{4}} \leq \frac{c-1}{\sqrt{4}}] = \Phi(\frac{c-1}{\sqrt{4}}) = .95 \Rightarrow \frac{c-1}{\sqrt{4}} = 1.645 \Rightarrow c = 4.29.$ We have used the value 1.645, which is the 95-th percentile of the standard normal.

Example 7-2: If for a certain normal random variable X, $P[X < 500] = .5$ and $P[X > 650] = .0227$, find the standard deviation of X.

Solution: The normal distribution is symmetric about its mean, with $P[X < \mu] = .5$ for any normal random variable. Thus, for this normal X we have $\mu = 500$. Then, $P[X > 650] = .0227 = P\left[\frac{X-500}{\sigma} > \frac{150}{\sigma}\right]$. From the standard normal table, we see that $\Phi(2.00) = .9773$. Since $\frac{X-500}{\sigma}$ has a standard normal distribution, it follows from the table for the standard normal distribution that $\frac{150}{\sigma} = 2.00$ and $\sigma = 75$. □

Approximating a distribution using a normal distribution

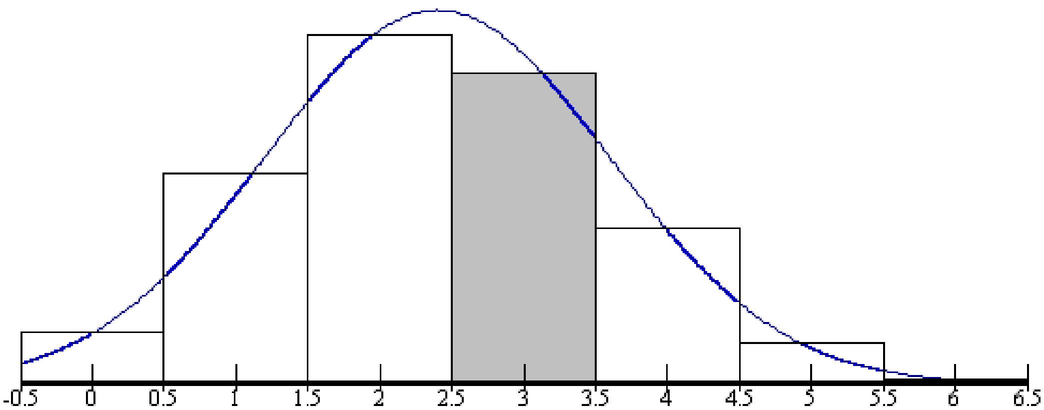
Given a random variable X with mean μ and variance σ^2, probabilities related to the distribution of X are sometimes approximated by assuming the distribution of X is approximately $N(\mu, \sigma^2)$. The SOA/CAS probability exam regularly has questions involving the normal approximation. It has sometimes been the case that a question asks for the approximate probability for some interval. This will almost always mean that the normal approximation should be applied, even if it is not specifically mentioned. Later in the study guide we will see the justification for using the normal approximation for a sum of random variables. It is in this context that approximate probabilities have come up on the exam.

Integer correction for the normal approximation to an integer-valued random variable:

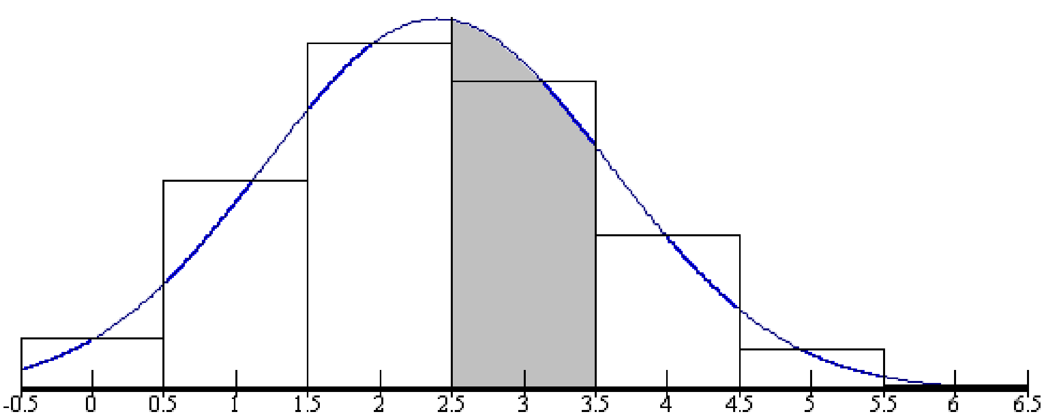
The normal distribution is continuous, but it can be used to approximate a discrete integer- valued distribution. In such a case (if instructed to do so) we can apply the following procedure.

If X is discrete and integer-valued then an **"integer correction"** may be applied in the following way. If n and m are integers, the probability $P[n \leq X \leq m]$ is approximated by using a normal random variable Y with the same mean and variance as X, and then finding the probability $P[n - \frac{1}{2} \leq Y \leq m + \frac{1}{2}]$. We extend the interval $[n, m]$ to $[n - \frac{1}{2}, m + \frac{1}{2}]$. The reasoning behind this can be seen from the following graphs, in which a normal density function is superimposed over the histogram of an integer-valued random variable.

The integer-valued random variable X in the following graphs happens to be a binomial with $N = 6$ and $p = .4$, so the mean and variance are $(6)(.4) = 2.4$ and $(6)(.4)(.6) = 1.44$. The way in which the normal approximation is applied, is to use the normal distribution with the same mean ($\mu = 2.4$) and variance ($\sigma^2 = 1.44$) as the original distribution. The density function in the graphs is of that normal distribution. In the first graph, the shaded region is the actual binomial probability that the outcome of the binomial distribution is 3.
This actual probability is $\binom{6}{3}(.4)^3(.6)^{6-3} = .27648$.

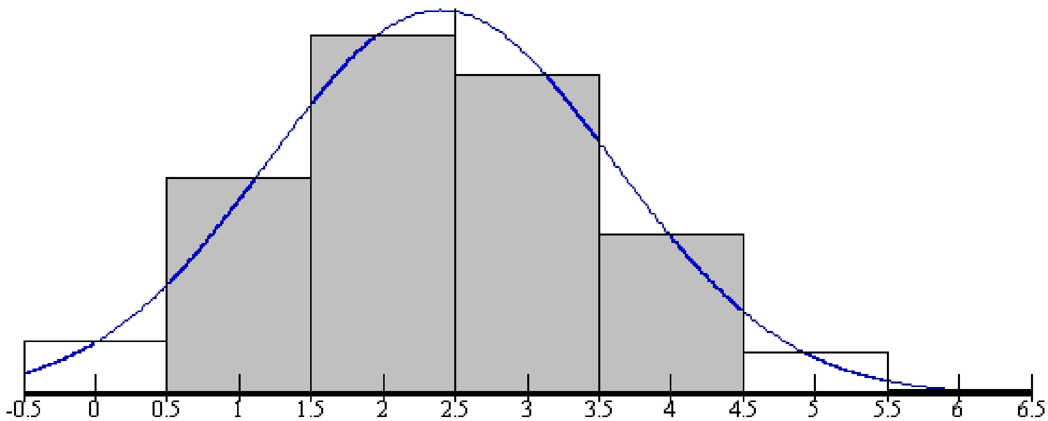


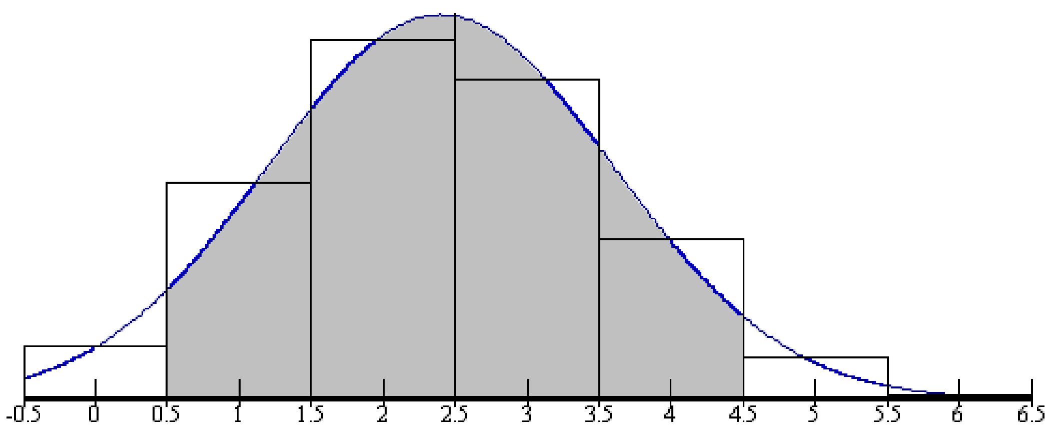
Since the normal distribution Y is a continuous distribution, in order to calculate a probability using the normal distribution, we must integrate the density over an interval. In order to use the normal distribution to approximate the probability that the binomial outcome is 3, we integrate over an interval of length 1 centered at $x = 3$. This is the integral from 2.5 to 3.5 of the normal density, so the normal approximation probability is $P[2.5 < Y < 3.5]$. This is the shaded region in the next graph.



For a normal distribution with mean 2.4 and variance 1.44, this probability is:
$P[2.5 < Y < 3.5] = P[\frac{2.5-2.4}{\sqrt{1.44}} < \frac{Y-2.4}{\sqrt{1.44}} < \frac{3.5-2.4}{\sqrt{1.44}}] = P[.0833 < Z < .9167]$
$= \Phi(.92) - \Phi(.08) = .8186 - .5319 = .29$. The exact binomial probability is .27648 .

In general, the normal approximation for an integer value $X = k$ is the normal distribution (Y) probability on the interval from $k - \frac{1}{2}$ to $k + \frac{1}{2}$ ($P[k - \frac{1}{2} < Y < k + \frac{1}{2}]$). For the probability of several successive integer values, we have a series of intervals. For instance, to find the probability that $1 \leq X \leq 4$, we would approximate the probability at $X = 1$, $X = 2$, $X = 3$ and $X = 4$ and add them up. This corresponds to finding the normal probability for Y on the intervals from .5 to 1.5 , from 1.5 to 2.5, from 2.5 to 3.5 and from 3.5 to 4.5. When we combine these, we get the probability from .5 to 4.5 , $P[.5 < Y < 4.5]$. This is illustrated in the following graphs. The shaded region of the first graph is the actual binomial probability $P[1 \leq X \leq 4]$. The shaded region of the second graph is the normal approximation probability $P[.5 < Y < 4.5]$.





Note that if we were asked to approximate the probability $P[1 < X < 4]$, then this would be $P[X = 2 \text{ or } 3]$, which we would approximate it as $P[1.5 < Y < 3.5]$. If we were asked to approximate $P[X \leq 5]$, we would approximate it as $P[Y < 5.5]$.

There is a little bit of a vague area regarding the use of the integer correction on the exam and it may be worthwhile to calculate probabilities both with and without the integer correction. If the probability corresponding to the integer correction is one of the possible answers, it should be the correct answer (unless there is an indication that the integer correction should not be used).

We will consider sums of independent random variables in more detail later in this study guide, but one rule to make a note of now is the following. If X_1 and X_2 are independent normal random variables with means μ_1 and μ_2, and variances σ_1^2 and σ_2^2, respectively, then $W = X_1 + X_2$ is also a normal random variable, and has mean $\mu_1 + \mu_2$, and variance $\sigma_1^2 + \sigma_2^2$.

Example 7-3: Suppose that a multiple choice exam has 40 questions, each with 5 possible answers. A well prepared student feels that he has a probability of $.5$ of getting any particular question correct, with independence from one question to another. Apply the normal approximation to X, the number of correct answers out of 40, to determine the probability of getting at least 25 correct. Find the probability with the integer correction, and then without the correction.

Solution: The number of questions answered correctly, say X, has a binomial distribution with mean $(40)(.5) = 20$ and variance $(40)(.5)(.5) = 10$. Applying the normal approximation Y to X, with integer correction to find the probability of answering at least 25 correct, we get

$P[X \geq 25] = P[Y \geq 24.5] = P[\frac{Y-20}{\sqrt{10}} \geq \frac{24.5-20}{\sqrt{10}}] = P[Z \geq 1.42] = 1 - \Phi(1.42) = .078$.

Without the integer correction, the probability is

$P[Y \geq 25] = P[\frac{Y-20}{\sqrt{10}} \geq \frac{25-20}{\sqrt{10}}] = P[Z \geq 1.58] = 1 - \Phi(1.58) = .057$.

There is a noticeable difference between the two approaches. If X has a much larger standard deviation, then the difference is not so noticeable. □

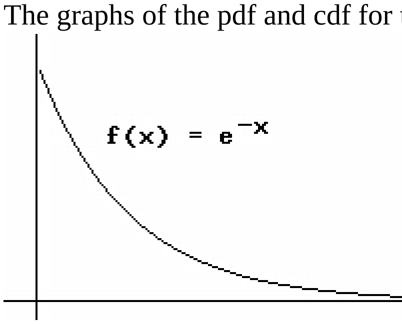
Exponential distribution with mean $\frac{1}{\lambda} > 0$

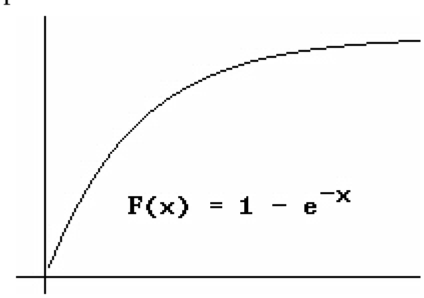
The density function is $\boldsymbol{f(x) = \lambda e^{-\lambda x}}$ **for** $\boldsymbol{x > 0}$**,** and $f(x) = 0$ otherwise.
The distribution function is $\boldsymbol{F(x) = 1 - e^{-\lambda x}}$ **for** $\boldsymbol{x \geq 0}$, and
the survival function is $\boldsymbol{S(x) = 1 - F(x) = P[X > x] = e^{-\lambda x}}$.
The mean is $\boldsymbol{E[X] = \frac{1}{\lambda}}$, the variance is $\boldsymbol{Var[X] = \frac{1}{\lambda^2}}$, and
the moment generating function is $M_X(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$
The k-th moment is $E[X^k] = \int_0^\infty x^k \cdot \lambda e^{-\lambda x} dx = \frac{k!}{\lambda^k}$, $k = 1, 2, 3, ...$

An alternative, but equivalent way to describe the exponential distribution is with the density function $f(x) = \frac{1}{\theta} e^{-x/\theta}$, using the parameter θ. Then $\theta = \frac{1}{\lambda}$ (the mean of X) is the relationship linking the parameters in these two descriptions of the exponential distribution. Some probability textbooks use one definition and some use the other. This makes it somewhat ambiguous if we are told that X has an exponential distribution with "parameter 3". Does this mean $\lambda = 3$, or does it mean $\theta = 3$? It is more precise to be told that X has an exponential distribution with **a mean of** 3. Then we know that $\frac{1}{\lambda} = 3$ based on the first definition, and $\theta = 3$ based on the second. Either way, the density function is $\frac{1}{3} e^{-x/3}$.

The exponential distribution is often used as a model for the time until some specific event occurs, say the time until the next earthquake at a certain location.

The graphs of the pdf and cdf for the exponential distribution with mean 1 are





Example 7-4: The random variable T has an exponential distribution such that $P[T \leq 2] = 2 \cdot P[T > 4]$. Find $Var[T]$.
Solution: Suppose that T has mean $\frac{1}{\lambda}$. Then $P[T \leq 2] = 1 - e^{-2\lambda} = 2P[T > 4] = 2e^{-4\lambda}$
$\Rightarrow 2x^2 + x - 1 = 0$, where $x = e^{-2\lambda}$. Solving the quadratic equation results in $x = \frac{1}{2}, -1$. We ignore the negative root, so that $e^{-2\lambda} = \frac{1}{2}$, and $\lambda = \frac{1}{2} ln\, 2$.
Then, $Var[T] = \frac{1}{\lambda^2} = \frac{4}{(ln\, 2)^2}$. □

Example 7-5: The initial cost of a machine is 3. The lifetime of the machine has an exponential distribution with a mean of 3 years. The manufacturer is considering offering a warranty and considers two types of warranties. Warranty 1 pays 3 if the machine fails in the first year, 2 if the machine fails in the second year, and 1 if the machine fails in the third year, with no payment if the machine fails after 3 years. Warranty 2 pays $3e^{-t}$ if the machine fails at time t years (with no limit on the time of failure). Find the expected warranty payment under each of the two warranties.

Solution: The pdf of T is $f(t) = \frac{1}{3}e^{-t/3}$, and the cdf is $F(t) = 1 - e^{-t/3}$.

Let X be the amount paid by warranty 1. The distribution of X is

$X:$	3	2	1	0
$p(x):$	$P[0 < T \leq 1]$	$P[1 < T \leq 2]$	$P[2 < T \leq 3]$	$P[T > 3]$
	$1 - e^{-\frac{1}{3}} = .2835$	$e^{-\frac{1}{3}} - e^{-\frac{2}{3}} = .2031$	$e^{-\frac{2}{3}} - e^{-1} = .1455$	$e^{-1} = .3679$

$E[X] = (3)(.2835) + (2)(.2031) + (1)(.1455) = 1.40$.

Let Y be the amount paid by warranty 2. Then $Y = 3e^{-T}$.

$E[Y] = \int_0^{\infty} 3e^{-t} \cdot \frac{1}{3}e^{-t/3}\,dt = \int_0^{\infty} e^{-4t/3}\,dt = \frac{3}{4}$. □

There are a few additional properties satisfied by the exponential distribution that are worth noting.

(i) Lack of memory property: for $x, y > 0$,

$P[X > x+y | X > x] = \frac{P\{X>x+y \cap X>x]}{P[X>x]} = \frac{P\{X>x+y]}{P[X>x]} = \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = e^{-\lambda y} = P[X > y]$.

We can interpret this as follows. Suppose that X represents the time, measured from now, in weeks until the next insurance claim filed by a company, and suppose also that X has an exponential distribution with mean $\frac{1}{\lambda}$. Suppose that 5 weeks have passed without an insurance claim, and we want to know the distribution of the time until the next insurance claim as measured from our new time origin, which is 5 weeks after the previous time origin. According to the lack of memory property, the fact that there have been no claims in the past 5 weeks is irrelevant, and measuring time starting from our new time origin, the time until the next claim is exponential with the same mean $\frac{1}{\lambda}$. In fact, no matter how many claims have occurred in the past 5 weeks, as measured from now, the time until the next claim has an exponential distribution with mean $\frac{1}{\lambda}$; the distribution has "forgotten" what has happened prior to now and the "clock" measuring time until the next claim is restarted now..

(ii) Link between the exponential distribution and Poisson distribution:
Suppose that X has an exponential distribution with mean $\frac{1}{\lambda}$ and we regard X as the time between successive occurrences of some type of event (say the event is the arrival of a new insurance claim at an insurance office), where time is measured in some appropriate units (seconds, minutes, hours or days, etc.). Now, we imagine that we choose some starting time (say labeled as $t = 0$), and from now we start recording times between successive events. For instance, the first claim may arrive in 2 days, then the next claim arrives 3 days after that, etc. Let N represent the number of events (claims) that have occurred when one unit of time has elapsed. Then N will be a random variable related to the times of the occurring events. It can be shown that the distribution of N is Poisson with a mean of λ.

(iii) The minimum of a collection of independent exponential random variables:
Suppose that independent random variables $Y_1, Y_2, ..., Y_n$ each have exponential distributions with means $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$, respectively. Let $Y = min\{Y_1, Y_2, \ldots, Y_n\}$. Then Y has an exponential distribution with mean $\frac{1}{\lambda_1+\lambda_2+\cdots+\lambda_n}$. An interpretation of this relationship is as follows. Suppose that an insurer has two types of insurance policies, basic coverage and extended coverage. Suppose insurance policies are independent of one another and that the time X_B until a claim from a basic policy is exponential with a mean of 4 weeks, and the time X_E until a claim from an extended policy is exponential with a mean of 2 weeks. The time until the next claim of any type is $X = min\{X_B, X_E\}$. X will be exponential with mean $\frac{1}{\frac{1}{4}+\frac{1}{2}} = \frac{4}{3}$ weeks. Another way of interpreting this is that the average number of claims per week for basic policies is $\frac{1}{4}$ (one every 4 weeks) and the average number of claims per week for extended policies is $\frac{1}{2}$, so the average number of claims per week for the two policy types combined is $\frac{3}{4}$.
We can descibe the average of $\frac{3}{4}$ claims per weeks an average of $\frac{4}{3}$ weeks between claims.

Example 7-6: Verify algebraically the validity of properties (i) and (iii) of the exponential distribution described above.

Solution: (i) Suppose that X has an exponential distribution with parameter λ. Then
$P[X > x + y|X > x] = \frac{P[(X>x+y)\cap(X>x)]}{P[X>x]} = \frac{P[X>x+y]}{P[X>x]} = \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = e^{-\lambda y}$,
and $P[X > y] = e^{-\lambda y}$.

Example 7-6 continued

(iii) Suppose that independent random variables $Y_1, Y_2, ..., Y_n$ have exponential distributions with means $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$ respectively. Let $Y = min\{Y_1, Y_2, \ldots, Y_n\}$. Then,

$P[Y > y] = P[Y_i > y$ for all $i = 1, 2, ..., n] = P[(Y_1 > y) \cap (Y_2 > y) \cap \cdots \cap (Y_n > y)]$

$= P[Y_1 > y] \cdot P[Y_2 > y] \cdots P[Y_n > y]$ (because of independence of the Y_i's)

$= (e^{-\lambda_1 y})(e^{-\lambda_2 y}) \cdots (e^{-\lambda_n y}) = e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)y}$. The cdf of Y is then

$F_Y(y) = P[Y \leq y] = 1 - P[Y > y] = 1 - e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)y}$ and the pdf of Y is

$f_Y(y) = F'_Y(y) = (\lambda_1 + \lambda_2 + \cdots + \lambda_n) e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)y}$, which is the pdf of an exponential distribution with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. □

Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$

The pdf is $f(x) = \frac{\beta^\alpha \cdot x^{\alpha-1} \cdot e^{-\beta x}}{\Gamma(\alpha)}$ for $x > 0$, and $f(x) = 0$ otherwise.

$\Gamma(\alpha)$ is the **gamma function**, which is defined for $\alpha > 0$ to be $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \cdot e^{-y} dy$.

If n is a positive integer it can be shown that $\Gamma(n) = (n-1)!$.

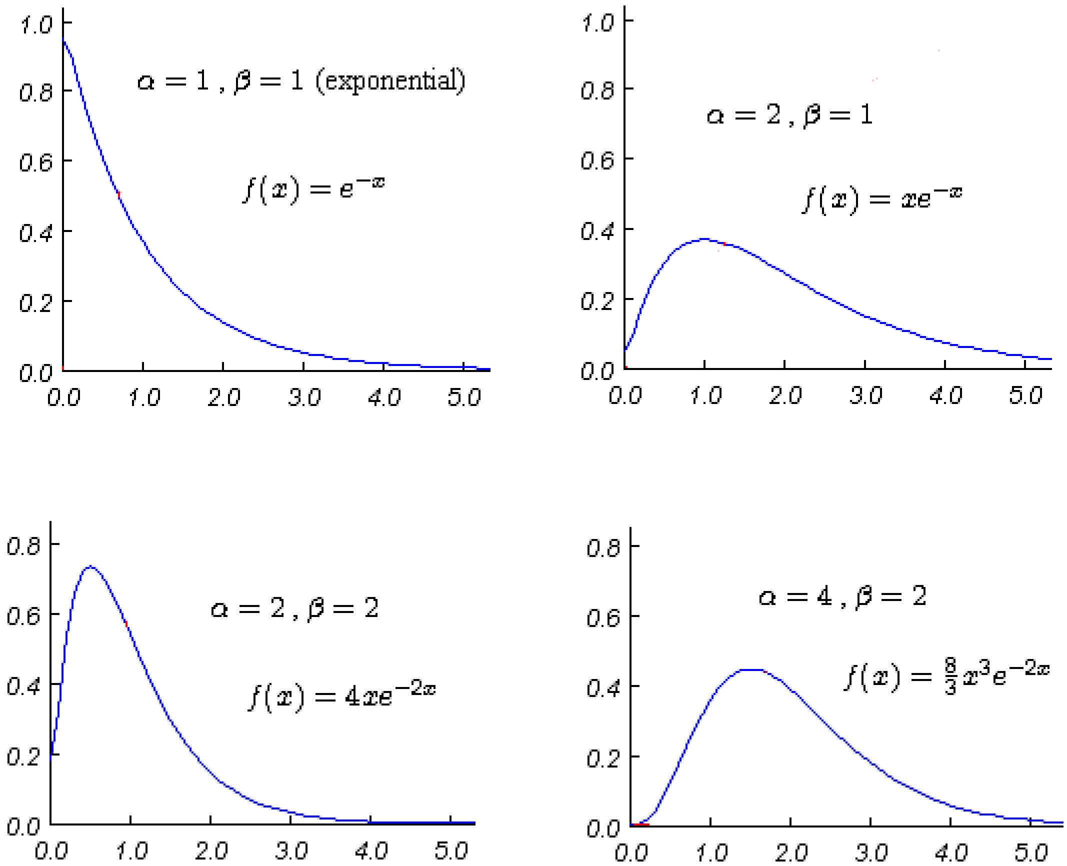
The mean, variance and moment generating function of X are $E[X] = \frac{\alpha}{\beta}$, $Var[X] = \frac{\alpha}{\beta^2}$, and $M_X(t) = (\frac{\beta}{\beta - t})^\alpha$ for $t < \beta$.

If this distribution was to show up on Exam P, it would likely have the parameter α as an integer n, in which case the density would be $f(x) = \frac{\beta^\alpha \cdot x^{n-1} \cdot e^{-\beta x}}{(n-1)!}$.

An alternative, but equivalent parametrization of X uses $\theta = \frac{1}{\beta}$ and the same α, so the pdf, mean, and variance are written as $f(x) = \frac{x^{\alpha-1} \cdot e^{-x/\theta}}{\theta^\alpha \cdot \Gamma(\alpha)}$, $E[X] = \alpha\theta$ and $Var[X] = \alpha\theta^2$.

Note that the exponential distribution with mean $\frac{1}{\lambda}$ is a special case of the gamma distribution with $\alpha = 1$ and $\beta = \lambda$. The graphs below illustrate the density functions of a few gamma distributions with various combinations of parameters α and β.

The cdf of the gamma distribution can be complicated. If α is an integer then it is possible to find the cdf by integration by parts. But this would tend to be quite tedious unless α is 1 or 2.



Gamma distribution density functions

As α gets larger, the pdf becomes more weighted to the right and is more spread out. As β gets larger, the pdf is more weighted to the left and becomes more peaked.

For Exam P it is very important to be familiar with the uniform, normal and exponential distributions.

Example 7-7: An insurer will pay 80% of the loss incurred on a loss of amount X. The loss random variable X has pdf $f(x) = \frac{3{,}000{,}000}{x^4}$ for $x > 100$, and $f(x) = 0$ for $x \leq 100$.
Find the standard deviation of the amount paid by the insurer.

Solution: If Y is the amount paid by the insurer, then $Y = .8X$, so that
$Var[Y] = (.8)^2 Var[X]$, and $\sqrt{Var[Y]} = (.8)\sqrt{Var[X]}$.
$Var[X] = E[X^2] - (E[X])^2$, where $E[X] = \int_{100}^{\infty} x \cdot \frac{3{,}000{,}000}{x^4}\, dx = 150$ and
$E[X^2] = \int_{100}^{\infty} x^2 \cdot \frac{3{,}000{,}000}{x^4}\, dx = 30,000$. Then $Var[X] = 30,000 - (150)^2 = 7,500$,
and $\sqrt{Var[Y]} = (.8)\sqrt{7,500} = 69.3$. $\square$

SUMMARY OF CONTINUOUS DISTRIBUTIONS

Distribution	Parameters	PDF, $p(x)$	Mean, $E[X]$	Variance, $Var[X]$	MGF, $M_X(t)$
Uniform	$a < b$	$\frac{1}{b-a},\ a < x < b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)\cdot t}$
Normal	μ (any number), $\sigma^2 > 0$	$\frac{1}{\sigma\cdot\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/2\sigma^2}$ $-\infty < x < \infty$	μ	σ^2	$exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$
Exponential	$\frac{1}{\lambda} = \theta > 0$	$\lambda e^{-\lambda x},\ x > 0$ $F(x) = 1 - e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
Gamma	$\alpha > 0$, $\beta > 0$	$\frac{\beta^\alpha \cdot x^{\alpha-1}\cdot e^{-\beta x}}{\Gamma(\alpha)}$, $x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$(\frac{\beta}{\beta-t})^\alpha$

PROBLEM SET 7

Frequently Used Continuous Distributions

1. Let X be a random variable with a continuous uniform distribution on the interval $(1, a)$ where $a > 1$. If $E[X] = 6 \cdot Var[X]$, then $a =$

A) 2 B) 3 C) $3\sqrt{2}$ D) 7 E) 8

2. A large wooden floor is laid with strips 2 inches wide and with negligible space between strips. A uniform circular disk of diameter 2.25 inches is dropped at random on the floor. What is the probability that the disk touches three of the wooden strips?

A) $\frac{1}{\sqrt{\pi}}$ B) $\frac{1}{\pi}$ C) $\frac{1}{4}$ D) $\frac{1}{8}$ E) $\frac{1}{\pi^2}$

3. If X has a continuous uniform distribution on the interval from 0 to 10, then what is $P[X + \frac{10}{X} > 7]$?

A) $\frac{3}{10}$ B) $\frac{31}{70}$ C) $\frac{1}{2}$ D) $\frac{39}{70}$ E) $\frac{7}{10}$

Problems 4 and 5 relate to the following information. Three individuals are running a one kilometer race. The completion time for each individual is a random variable. X_i is the completion time, in minutes, for person i.

X_1 : uniform distribution on the interval $[2.9\,,\ 3.1]$

X_2 : uniform distribution on the interval $[2.7\,,\ 3.1]$

X_3 : uniform distribution on the interval $[2.9\,,\ 3.3]$

The three completion times are independent of one another.

4. Find the probability that the earliest completion time is less than 3 minutes.

A) .89 B) .91 C) .94 D) .96 E) .98

5. Find the probability that the latest completion time is less than 3 minutes (nearest .01).

A) .03 B) .06 C) .09 D) .12 E) .15

6. A student received a grade of 80 in a math final where the mean grade was 72 and the standard deviation was s. In the statistics final, he received a 90, where the mean grade was 80 and the standard deviation was 15. If the standardized scores (i.e., the scores adjusted to a mean of 0 and standard deviation of 1) were the same in each case, then $s =$

A) 10 B) 12 C) 16 D) 18 E) 20

7. If X has a standard normal distribution and $Y = e^X$, what is the k-th moment of Y?

A) 0 B) 1 C) $e^{k/2}$ D) $e^{k^2/2}$

E) 1 if $k = 2m - 1$ and $e^{(2m-1)(2m-3)\cdots 3\cdot 1}$ if $k = 2m$

8. (SOA) For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000. For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000. Assume that the total claim amounts of the two companies are independent. What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

A) 0.180 B) 0.185 C) 0.217 D) 0.223 E) 0.240

9. (SOA) The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively. What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

A) $\frac{1}{18}\left(1-e^{-2/3}-e^{-1/2}+e^{-7/6}\right)$ B) $\frac{1}{18}e^{-7/6}$ C) $1 - e^{-2/3} - e^{-1/2} + e^{-7/6}$

D) $1 - e^{-2/3} - e^{-1/2} + e^{-1/3}$ E) $1 - \frac{1}{3}e^{-2/3} - \frac{1}{6}e^{-1/2} + \frac{1}{18}e^{-7/6}$

10. Let X be a continuous random variable with density function

$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $-\infty < x < \infty$. Calculate $E[X|X \geq 0]$.

A) 0 B) $\frac{1}{\sqrt{2\pi}}$ C) $\frac{1}{2}$ D) $\sqrt{\frac{2}{\pi}}$ E) 1

11. (SOA) Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation 0.0056h . The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation 0.0044h . Assuming the two measurements are independent random variables, what is the probability that their average value is within 0.005h of the height of the tower?

A) 0.38 B) 0.47 C) 0.68 D) 0.84 E) 0.90

12. A new car battery is sold for \$100 with a 3-year limited warranty. If the battery fails at time t ($0 < t < 3$), the battery manufacturer will refund $\$100(1-\frac{t}{3})$. After analyzing battery performance, the battery manufacturer uses the (continuous) uniform distribution on the interval $(0, n)$ as the model for time until failure for the battery (n in years). The battery manufacturer determines that the expected cost of the warranty is \$10. Find n.

A) 3 B) 5 C) 10 D) 15 E) 30

13. (SOA) The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

A) 6,321 B) 7,358 C) 7,869 D) 10,256 E) 12,642

14. (SOA) A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000 ?

A) 3858 B) 4449 C) 5382 D) 5644 E) 7235

15. (SOA) The time to failure of a component in an electronic device has an exponential distribution with a median of four hours. Calculate the probability that the component will work without failing for at least five hours.

A) 0.07 B) 0.29 C) 0.38 D) 0.42 E) 0.57

16. An insurer uses the exponential distribution with mean μ as the model for the total annual claim occurring from a particular insurance policy in the current one year period. The insurer assumes an inflation factor of 10% for the one year period following the current one year period. Using the insurer's assumption, find the coefficient of variation ($\frac{standard\,deviation}{expected\,value}$) for the annual claim paid on the policy for the one year period following the current one year period.

A) 1.21 B) 1.1 C) 1 D) $\frac{1}{1.1}$ E) $\frac{1}{1.21}$

17. Average loss size per policy on a portfolio of policies is 100. Actuary 1 assumes that the distribution of loss size has an exponential distribution with a mean of 100, and Actuary 2 assumes that the distribution of loss size has a pdf of $f_2(x) = \frac{2\theta^2}{(x+\theta)^3}$, $x > 0$. If m_1 and m_2 represent the median loss sizes for the two distributions, find $\frac{m_1}{m_2}$.

A) .6 B) 1.0 C) 1.3 D) 1.7 E) 2.0

18. The time until the occurrence of a major hurricane is exponentially distributed. It is found that it is 1.5 times as likely that a major hurricane will occur in the next ten years as it is that the next major hurricane will occur in the next five years. Find the expected time until the next major hurricane.

A) 5 B) $5\,ln\,2$ C) $\frac{5}{ln\,2}$ D) $10\,ln\,2$ E) $\frac{10}{ln\,2}$

PROBLEM SET 7 SOLUTIONS

1. $E[X] = \frac{1+a}{2}$ and $Var[X] = \frac{(a-1)^2}{12}$, so that
$\frac{a+1}{2} = 6 \cdot \frac{(a-1)^2}{12} \Rightarrow a^2 - 3a = 0 \Rightarrow a = 0, 3 \Rightarrow a = 3$ (since $a > 0$). Answer: B

2. Let us focus on the left-most point p on the disk. Consider two adjacent strips on the floor. Let the interval $[0, 2]$ represent the distance as we move across the left strip from left to right. If p is between 0 and 1.75, then the disk lies within the two strips.
If p is between 1.75 and 2, the disk will lie on 3 strips (the first two and the next one to the right). Since any point between 0 and 2 is equally likely as the left most point p on the disk (i.e. uniformly distributed between 0 and 2) it follows that the probability that the disk will touch three strips is $\frac{.25}{2} = \frac{1}{8}$. Answer: D

3. Since the density function for X is $f(x) = \frac{1}{10}$ for $0 < x < 10$, we can regard X as being positive. Then
$P[X + \frac{10}{X} > 7] = P[X^2 - 7X + 10 > 0] = P[(X-5)(X-2) > 0]$
$= P[X > 5] + P[X < 2]$ (since $(t-5)(t-2) > 0$ if either both $t-5, t-2 > 0$ or both $t-5, t-2 < 0$) $= \frac{5}{10} + \frac{2}{10} = \frac{7}{10}$. Answer: E

4. $f_{X_1}(t) = \frac{1}{.2} = 5$ for $2.9 \le t \le 3.1$, $F_{X_1}(t) = P[X_1 \le t] = 5(t-2.9)$ for $2.9 \le t \le 3.1$.
$f_{X_2}(t) = \frac{1}{.4} = 2.5$ for $2.7 \le t \le 3.1$, $F_{X_2}(t) = P[X_2 \le t] = 2.5(t-2.7)$ for $2.7 \le t \le 3.1$.
$f_{X_3}(t) = \frac{1}{.4} = 2.5$ for $2.9 \le t \le 3.3$, $F_{X_3}(t) = P[X_3 \le t] = 2.5(t-2.9)$ for $2.9 \le t \le 3.3$.
$P[min(X_1, X_2, X_3) < 3] = 1 - P[min(X_1, X_2, X_3) \ge 3]$
$= 1 - P[(X_1 \ge 3) \cap (X_2 \ge 3) \cap (X_3 \ge 3)]$
$= 1 - [1 - F_{X_1}(3)] \cdot [1 - F_{X_2}(3)] \cdot [1 - F_{X_3}(3)]$
$= 1 - [1 - 5(3-2.9)] \cdot [1 - 2.5(3-2.7)] \cdot [1 - 2.5(3-2.9)] = .90625$. Answer: B

5. $P[max(X_1, X_2, X_3) < 3] = P[(X_1 < 3) \cap (X_2 < 3) \cap (X_3 < 3)]$
$= F_{X_1}(3) \cdot F_{X_2}(3) \cdot F_{X_3}(3) = [5(3-2.9)] \cdot [2.5(3-2.7)] \cdot [2.5(3-2.9)] = .09375$.
Answer: C

6. The standardized statistics score is $\frac{90-80}{15} = \frac{2}{3}$. The standardized math score is $\frac{80-72}{s} = \frac{8}{s} = \frac{2}{3} \rightarrow s = 12$. Answer: B

7. The k-th moment of Y is $E[Y^k] = E[e^{kX}] = M_X(k) = e^{k^2/2}$ (since $\mu = 0$ and $\sigma^2 = 1$).
Answer: D

8. We denote by X_A and X_B the total claim amount for the coming year for Company A and B, respectively. We are asked to find $P[X_B > X_A]$. X_A is a mixture of two parts.
There is a discrete part,
P[Company A has no claims in the coming year] $= P[X_A = 0] = .6$,
and a continuous part
P[Company A has some claims in the coming year]
$= P[X_A$ has a normal distribution with mean 10,000 and standard deviation 2,000$] = .4$.
X_B is similar. There is a discrete part
P[Company B has no claims in the coming year] $= P[X_B = 0] = .7$,
and a continuous part
P[Company B has some claims in the coming year]
$= P[X_B$ has a normal distribution with mean 9,000 and standard deviation 2,000$] = .3$.
Therefore, $X_A = \begin{cases} 0 & \text{prob. .6} \\ Y_A\text{, normal, mean 10,000, std. dev. 2000} & \text{prob. .4} \end{cases}$ and
$X_B = \begin{cases} 0 & \text{prob. .7} \\ Y_B\text{, normal, mean 9,000, std. dev. 2000} & \text{prob. .3} \end{cases}$.

We use the following probability rule:
$P[C] = P[C|D_1] \cdot P[D_1] + P[C|D_2] \cdot P[D_2] + \cdots + P[C|D_n] \cdot P[D_n]$,
for any event C and any partition of events $D_1, D_2, ..., D_n$.
In this case, the event C is $X_B > X_A$, and the partition has 4 events,
D_1 : Company A has no claims and Company B has no claims,
D_2 : Company A has no claims and Company B has some claims,
D_3 : Company A has some claims and Company B has no claims,
D_4 : Company A has some claims and Company B has some claims.
The companies have independent claim amounts, so we can use the following rule for independent events: $P[U \cap V] = P[U] \cdot P[V]$.

8. continued

The probabilities of the partition events are

$P[D_1] = P$[(no claims for company A) $\cap$ (no claims for company B)]

$= P$[no claims for company A] $\cdot$ P[no claims for company B] $= (.6)(.7) = .42$.

$P[D_2] = (6.)(.3) = .18$, $P[D_3] = (.4)(.7) = .28$, $P[D_4] = (.4)(.3) = .12$.

Using the partition rule above, we have

$P[X_B > X_A] = P[X_B > X_A|D_1] \cdot P[D_1] + P[X_B > X_A|D_2] \cdot P[D_2]$

$+ P[X_B > X_A|D_3] \cdot P[D_3] + P[X_B > X_A|D_4] \cdot P[D_4]$.

$P[X_B > X_A|D_1] = 0$, since in this case $X_A = X_B = 0$.

$P[X_B > X_A|D_2] = P[Y_B > 0] = P[\frac{Y_B - 9{,}000}{2{,}000} > \frac{0-9{,}000}{2{,}000}]$

$= P[Z > -4.5] = \Phi(4.5) = 1$. In this case $X_A = 0$ and $X_B = Y_B$ has a normal distribution, and we can standardize the probability; Z has a standard normal distribution (mean 0, standard deviation 1).

$P[X_B > X_A|D_3] = P[Y_A < 0] = P[\frac{Y_A - 10{,}000}{2{,}000} < \frac{0-10{,}000}{2{,}000}]$

$= P[Z < -5] = \Phi(-5) = 1 - \Phi(5) = 0$. In this case, $X_B = 0$ and $X_A = Y_A$ has a normal distribution, and we can standardize the probability.

$P[X_B > X_A|D_4] = P[Y_B > Y_A] = P[Y_B - Y_A > 0]$. Since claims from the two companies are independent, Y_A and Y_B are independent. The sum or difference of normal random variables is normal, and the mean is the sum or difference of the means. The mean of $Y_B - Y_A$ is $9,000 - 10,000 = -1,000$. Since Y_A and Y_B are independent,

$Var[Y_B - Y_A] = 2000^2 + 2000^2 = 8,000,000$.

Then standardizing $Y_B - Y_A$, we get

$P[X_B > X_A|D_4] = P[Y_B > Y_A] = P[Y_B - Y_A > 0]$

$= P[\frac{Y_B - Y_A - (-1{,}000)}{\sqrt{8{,}000{,}000}} > \frac{0-(-1{,}000)}{\sqrt{8{,}000{,}000}}] = P[Z > .3536] = 1 - \Phi(.35) = .363$

(from the normal distribution table, we get $\Phi(.35) = (.5)\Phi(.3) + (.5)\Phi(.4) = .637$).

Finally,

$P[X_B > X_A] = (0)(.42) + (1)(.18) + (0)(.28) + (.363)(.12) = .223$.

8. continued

This solution can be summarized using some "general intuitive reasoning" as follows. The only way that X_B can be greater than X_A is if Company B has some claims. If Company A has no claims (prob. .6) and Company B has some claims (prob. .3), then the probability that Company B's claim amount will exceed Company A's claim is 1. If Company A has some claims (prob. .4) and Company B has some claims (prob. .3), then the probability that Company B's claims exceed Company A's claims is .363 (as outlined above). The overall probability that Company B's claims exceed Company A's claims is $(.6)(.3)(1) + (.4)(.3)(.363) = .223$.

Answer: D

9. Let T_g be the time until a claim from the good driver. Then the pdf of T_g is $f_g(t) = \frac{1}{6}e^{-t/6}$ (exponential with a mean of 6). Let T_b be the time until a claim from the bad driver. Then the pdf of T_b is $f_b(t) = \frac{1}{3}e^{-t/3}$ (exponential with a mean of 3).

Let A be the event that the first claim from a good driver will be filed within 3 years.

$P[A] = P[T_g < 3] = \int_0^3 \frac{1}{6}e^{-t/6}dt = 1 - e^{-1/2}$.

Let B be the event that the first claim from a bad driver will be filed within 2 years.

$P[B] = P[T_b < 2] = \int_0^2 \frac{1}{3}e^{-t/3}dt = 1 - e^{-2/3}$.

The probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years is $P[A \cap B]$. Since T_g and T_b are independent, so are the events A and B. Therefore,

$P[A \cap B] = P[A] \cdot P[B] = (1 - e^{-1/2})(1 - e^{-2/3}) = 1 - e^{-1/2} - e^{-2/3} + e^{-7/6}$.

Answer: C

10. X has a $N(0,1)$ distribution, so that the density function of the conditional distribution is $f(x \,|\, X > 0) = \frac{f(x)}{P[X>0]} = \frac{f(x)}{1/2} = 2f(x)$. The conditional expectation is

$\int_0^\infty x \cdot f(x \,|\, X > 0)\, dx = \int_0^\infty 2x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = -\frac{2}{\sqrt{2\pi}} e^{-x^2/2} \Big|_{x=0}^{x=\infty} = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$

Answer: D

11. $E(E_1) = E(E_2) = 0$, $Var(E_1) = (.0056h)^2$, $Var(E_2) = (.0044h)^2$.
$E_1 + E_2$ is normal with $E(E_1 + E_2) = 0$, and since E_1 and E_2 are independent,
$Var(E_1 + E_2) = Var(E_1) + Var(E_2) = (.0056h)^2 + (.0044h)^2 = (.00712h)^2$.
The average of E_1 and E_2 is $A = \frac{1}{2}(E_1 + E_2)$, which is also normal with mean 0 and variance $Var(A) = \frac{1}{4}Var(E_1 + E_2) = (.00356h)^2$.
The average height is within $.005h$ of the height of the tower if the absolute error in the average is less than $.005h$. We wish to find $P(|A| < .005h) = P(-.005h < A < .005h)$.
We standardize A to get
$P(-.005h < A < .005h) = P(\frac{-.005h-E(A)}{\sqrt{Var(A)}} < \frac{A-E(A)}{\sqrt{Var(A)}} < \frac{.005h-E(A)}{\sqrt{Var(A)}})$
$= P(\frac{-.005h-0}{.00356h} < Z < \frac{.005h-0}{.00356h}) = P(-1.4 < Z < 1.4)$,
where Z has a standard normal distribution. From the standard normal table,
$P(Z < 1.4) = .9192$, so that $P(Z < -1.4) = P(Z > 1.4) = .0808$,
and therefore, $P(-1.4 < Z < 1.4) = .9192 - .0808 = .8384$. Answer: D

12. The expected payout on the warranty is $\int_0^3 100(1 - \frac{t}{3}) \cdot \frac{1}{n}\, dt = \frac{300}{2n} = 10$
$\rightarrow n = 15$. Answer: D

13. The exponential distribution with a mean of 2 has density function
$f(x) = \frac{1}{2}e^{-x/2}$, for $x > 0$, and distribution function
$P[X \le x] = F(x) = 1 - e^{-x/2}$, for $x > 0$.
The probability that a printer will fail in the first year is
$P[X \le 1] = F(1) = 1 - e^{-1/2} = .39347$, so that the expected number of failures in the first year out of 100 printers is 39.347 .
The probability that a printer will fail in the second year is
$P[1 < X \le 2] = F(2) - F(1) = e^{-1/2} - e^{-2/2} = .23865$, so that the expected number of failures in the first year out of 100 printers is 23.865 .
The expected amount the manufacturer will pay in refunds is
$(200)(39.347) + (100)(23.865) = 10,256$. Answer: D

14. The density function for the time of failure T is $f(t) = .1e^{-.1t}$ (exponential with mean 10).
The amount paid is $P(t) = \begin{cases} x & 0 < t \le 1 \\ .5x & 1 < t \le 3 \\ 0 & t > 3 \end{cases}$. The expected amount paid is
$E[P(T)] = \int_0^\infty P(t) \cdot f(t)\, dx = \int_0^1 .1xe^{-.1t}\, dt + \int_1^3 .05xe^{-.1t}\, dt$
$= x[1 - e^{-.1}] + .5x[e^{-.1} - e^{-.3}] = .17717x$.
In order for this to be 1000, we must have $.17717x = 1000 \rightarrow x = 5644$. Answer: D

15. The exponential time until failure random variable T has density function of the form $f(t) = \lambda e^{-\lambda t}$, and had distribution function $F(t) = P[T \le t] = 1 - e^{-\lambda t}$ for $t > 0$.
The median of the distribution is the time point m that satisfies the relationship $F(m) = .5$; in other words, m is the time point for which there is a 50% probability of failure by time m. We are given that $m = 4$, and therefore $F(4) = 1 - e^{-4\lambda} = .5$, from which it follows that $e^{-4\lambda} = .5$. We are asked to find $P[T > 5] = 1 - F(5) = e^{-5\lambda}$.
Using the relationship $e^{-5\lambda} = (e^{-4\lambda})^{1.25}$, we get $P[T > 5] = e^{-5\lambda} = (.5)^{1.25} = .420$.
Notice that we could solve for λ from the equation $e^{-4\lambda} = .5$, but it is not necessary.
Answer: D

16. The coefficient of variation of a random variable X is $\frac{\sqrt{Var[X]}}{E[X]}$.
If X denotes the claim amount for the current one year period, then $E[X] = \mu$, $Var[X] = \mu^2$.
The claim amount for the one year period following the current one year period is $1.1X$, with mean $E[1.1X] = (1.1)E[X] = 1.1\mu$, and variance
$Var[1.1X] = (1.1)^2 Var[X] = (1.1)^2 \mu^2$.
The coefficient of variation in the following period is
$\frac{\sqrt{Var[1.1X]}}{E[1.1X]} = \frac{\sqrt{(1.1)^2 Var[X]}}{(1.1)E[X]} = \frac{\sqrt{Var[X]}}{E[X]} = \frac{\sqrt{\mu^2}}{\mu} = 1$. Answer: C

17. The cdf for distribution 1 is $F_1(x) = 1 - e^{-x/100}$. The median m_1 must satisfy
$.5 = F_1(m_1) = 1 - e^{-m_1/100} \rightarrow m_1 = 69.3$.
The cdf for distribution 2 is $F_2(x) = \int_0^x f_2(t)\,dt = \int_0^x \frac{2\theta^2}{(t+\theta)^3}\,dt = 1 - \frac{\theta^2}{(x+\theta)^2}$.
The mean of distribution 2 is $E[X_2] = \int_0^\infty [1 - F_2(x)]\,dx = \int_0^\infty \frac{\theta^2}{(x+\theta)^2}\,dx = \theta = 100$.
Therefore, the cdf of distribution 2 is $F_2(x) = 1 - \frac{100^2}{(x+100)^2}$, and the median m_2 satisfies
$.5 = F_2(m_2) = 1 - \frac{100^2}{(m_2+100)^2} \Rightarrow m_2 = 41.4$. Then $\frac{m_1}{m_2} = \frac{69.3}{41.4} = 1.67$.
Answer: D

18. $T =$ time, in years, until next major hurricane, is exponentially distributed with mean μ. The density function of T is $f(t) = \frac{1}{\mu} e^{-t/\mu}$, and cumulative distribution function is
$F(t) = P[T \le t] = 1 - e^{-t/\mu}$. We are given $P[T \le 10] = 1.5P[T \le 5]$, so that
$1 - e^{-10/\mu} = (1.5)[1 - e^{-5/\mu}]$. This can be written as $e^{-10/\mu} - 1.5e^{-5/\mu} + .5 = 0$.
With $x = e^{-5/\mu}$, this becomes the quadratic equation $x^2 - 1.5x + .5 = 0$. with roots
$x = 1$, $.5$. Therefore, $e^{-5/\mu}$ is either 1 or .5 . It is not possible to have $e^{-5/\mu} = 1$,
since that would require $\mu = \infty$. Therefore, $e^{-5/\mu} = .5$, so that $\mu = \frac{-5}{ln\,\frac{1}{2}} = \frac{5}{ln\,2}$.
Answer: C

SECTION 8 - JOINT, MARGINAL, AND CONDITIONAL DISTRIBUTIONS

Joint distribution of random variables X and Y

A random variable X is a numerical outcome that results from some random experiment, such as the number that turns up when tossing a die. It is possible that an experiment may result in two or more numerical outcomes. A simple example would be the numbers that turn up when tossing two dice. X could be the number that turns up on the first die and Y could be the number on the second die. Another example could be the following experiment. A coin is tossed and if the outcome is head then toss one die, and if the outcome is tails then toss two dice. We could set $X = 1$ for a head and $X = 2$ for a tail and $Y =$ total on the dice thrown. In both of the examples just described, we have a pair of random variables X and Y, that result from the experiment. X and Y might be unrelated or independent of one another (as in the example of the toss of two independent dice), or they might be related to each other (as in the coin-dice example).

We describe the probability distribution of two or more random variables together as a **joint distribution**. As in the case of a single discrete random variable, we still describe probabilities for each possible pair of outcomes for a pair of discrete random variables. In the case of a pair of random variables X and Y, there would be probabilities of the form $P[(X = x) \cap (Y = y)]$ for each pair (x, y) of possible outcomes. For a pair of continuous random variables X and Y, there would be a density function to describe density over a two dimensional region.

A joint distribution of two random variables has a probability function or probability density function $f(x, y)$ that is a function of two variables (sometimes denoted $f_{X,Y}(x, y)$). It is defined over a two-dimensional region. For joint distributions of continuous random variables X and Y, the region of probability (the probability space) is usually a rectangle or triangle in the x-y plane.

If X and Y are discrete random variables, then $f(x, y) = P[(X = x) \cap (Y = y)]$ is the joint probability function, and it must satisfy
(i) $0 \leq f(x, y) \leq 1$ and (ii) $\sum_x \sum_y f(x, y) = 1$.

If X and Y are continuous random variables, then $f(x, y)$ must satisfy
(i) $f(x, y) \geq 0$ and (ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = 1$.

In the two dice example described above, if the two dice are tossed independently of one another then $f(x,y) = P[(X=x) \cap (Y=y)] = P[X=x] \times P[Y=y] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ for each pair with $x = 1,2,3,4,5,6$ and $y = 1,2,3,4,5,6$. The coin-die toss example above is more complicated because the number of dice tossed depends on whether the toss is head or tails. If the coin toss is a head then $X = 1$ and $Y = 1,2,3,4,5,6$ so
$f(1,y) = P[(X=1) \cap (Y=y)] = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ for $y = 1,2,3,4,5,6$.
If the coin toss is tail then $X = 2$ and $Y = 2,3,\ldots,12$ with
$f(2,2) = P[(X=2) \cap (Y=2)] = \frac{1}{2} \times \frac{1}{36} = \frac{1}{72}$,
$f(2,3) = P[(X=2) \cap (Y=3)] = \frac{1}{2} \times \frac{2}{36} = \frac{1}{36}$, etc.

It is possible to have a joint distribution in which one variable is discrete and one is continuous, or either has a mixed distribution. The joint distribution of two random variables can be extended to a joint distribution of any number of random variables.

If A is a subset of two-dimensional space, then $P[(X,Y) \in A]$ is the summation (discrete case) or double integral (continuous case) of $f(x,y)$ over the region A.

Example 8-1: X and Y are discrete random variables which are jointly distributed with the probability function $f(x,y)$ defined in the following table:

		X		
		-1	0	1
	1	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$
Y	0	$\frac{1}{9}$	0	$\frac{1}{6}$
	-1	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$

From this table we see, for example, that $P[X=0, Y=-1] = f(0,-1) = \frac{1}{9}$.

Find (i) $P[X+Y=1]$, (ii) $P[X=0]$ and (iii) $P[X<Y]$.

Solution: (i) We identify the (x,y)-points for which $X+Y=1$, and the probability is the sum of $f(x,y)$ over those points. The only x,y combinations that sum to 1 are the points $(0,1)$ and $(1,0)$. Therefore, $P[X+Y=1] = f(0,1) + f(1,0) = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$.

(ii) We identify the (x,y)-points for which $X=0$. These are $(0,-1)$ and $(0,1)$ (we omit $(0,0)$ since there is no probability at that point). $P[X=0] = f(0,-1) + f(0,1) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$

(iii) The (x,y)-points satisfying $X<Y$ are $(-1,0)$, $(-1,1)$ and $(0,1)$.
Then $P[X<Y] = f(-1,0) + f(-1,1) + f(0,1) = \frac{1}{9} + \frac{1}{18} + \frac{1}{9} = \frac{5}{18}$. □

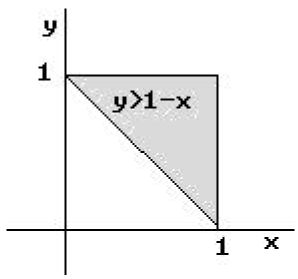
Example 8-2: Suppose that $f(x,y) = K(x^2+y^2)$ is the density function for the joint distribution of the continuous random variables X and Y defined over the unit square bounded by the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, find K. Find $P[X+Y \geq 1]$.

Solution: In order for $f(x,y)$ to be a properly defined joint density, the (double) integral of the density function over the region of density must be 1, so that

$1 = \int_0^1 \int_0^1 K(x^2+y^2)\,dy\,dx = K \cdot \frac{2}{3} \;\;\Rightarrow\;\; K = \frac{3}{2}$

$\Rightarrow\;\; f(x,y) = \frac{3}{2}(x^2+y^2)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

In order to find the probability $P[X+Y \geq 1]$, we identify the two dimensional region representing $X+Y \geq 1$. This is generally found by drawing the boundary line for the inequality, which is $x+y=1$ (or $y = 1-x$) in this case, and then determining which side of the line is represented in the inequality. We can see that $x+y \geq 1$ is equivalent to $y \geq 1-x$. This is the shaded region in the graph below.



The probability $P[X+Y \geq 1]$ is found by integrating the joint density over the two-dimensional region. It is possible to represent two-variable integrals in either order of integration. In some cases one order of integration is more convenient than the other. In this case there is not much advantage of one direction of integration over the other.

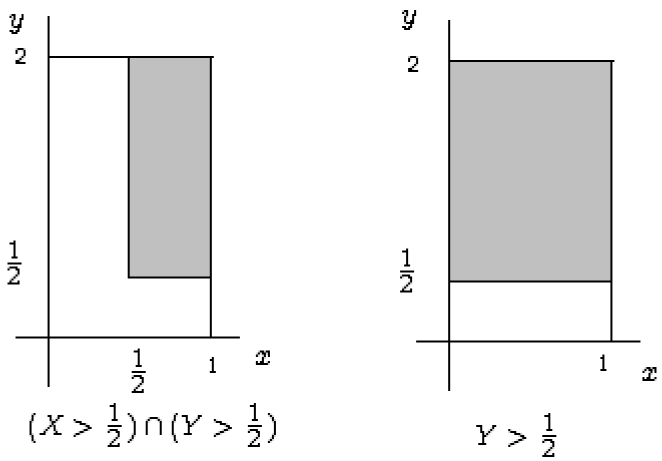
$P[X+Y \geq 1] = \int_0^1 \int_{1-x}^1 \frac{3}{2}(x^2+y^2)\,dy\,dx = \int_0^1 \frac{1}{2}(3x^2y + y^3\Big|_{y=1-x}^{y=1})\,dx$

$= \int_0^1 \frac{1}{2}(3x^2 + 1 - 3x^2(1-x) - (1-x)^3)\,dx = \frac{3}{4}$.

Reversing the order of integration, we have $x \geq 1-y$, so that

$P[X+Y \geq 1] = \int_0^1 \int_{1-y}^1 \frac{3}{2}(x^2+y^2)\,dx\,dy = \frac{3}{4}$. □

Example 8-3: Continuous random variables X and Y have a joint distribution with density function $f(x,y) = x^2 + \frac{xy}{3}$ for $0 < x < 1$ and $0 < y < 2$.
Find the conditional probability $P[X > \frac{1}{2} | Y > \frac{1}{2}]$.
Solution: We use the usual definition $P[A|B] = \frac{P[A \cap B]}{P[B]}$.
$P[X > \frac{1}{2} | Y > \frac{1}{2}] = \frac{P[(X > \frac{1}{2}) \cap (Y > \frac{1}{2})]}{P[Y > \frac{1}{2}]}$.
These regions are described in the following diagram



$P[(X > \frac{1}{2}) \cap (Y > \frac{1}{2})] = \int_{1/2}^{1} \int_{1/2}^{2} [x^2 + \frac{xy}{3}] \, dy \, dx = \frac{43}{64}$.
$P[Y > \frac{1}{2}] = \int_{1/2}^{2} [\int_{0}^{1} f(x,y) \, dx] \, dy = \int_{1/2}^{2} \int_{0}^{1} [x^2 + \frac{xy}{3}] \, dx \, dy = \frac{13}{16}$
$\rightarrow P[X > \frac{1}{2} | Y > \frac{1}{2}] = \frac{43/64}{13/16} = \frac{43}{52}$. □

Cumulative distribution function of a joint distribution: If random variables X and Y have a joint distribution, then the cumulative distribution function is
$F(x,y) = P[(X \le x) \cap (Y \le y)]$.
In the continuous case, $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t) \, dt \, ds$,
and in the discrete case, $F(x,y) = \sum_{s=-\infty}^{x} \sum_{t=-\infty}^{y} f(s,t)$.
In the continuous case, $\frac{\partial^2}{\partial x \, \partial y} F(x,y) = f(x,y)$.

Example 8-4: The cumulative distribution function for the joint distribution of the continuous random variables X and Y is $F(x,y) = (.2)(3x^3 y + 2x^2 y^2)$, for $0 \le x \le 1$ and $0 \le y \le 1$.
Find $f(\frac{1}{2}, \frac{1}{2})$.
Solution: $f(x,y) = \frac{\partial^2}{\partial x \, \partial y} F(x,y) = (.2)(9x^2 + 8xy) \rightarrow f(\frac{1}{2}, \frac{1}{2}) = \frac{17}{20}$. □

Expectation of a function of jointly distributed random variables

If $h(x,y)$ is a function of two variables, and X and Y are jointly distributed random variables, then the **expected value of $h(X,Y)$** is defined to be

$\boldsymbol{E[h(X,Y)] = \sum_x \sum_y h(x,y) \cdot f(x,y)}$ in the discrete case, and

$\boldsymbol{E[h(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\, dy\, dx}$ in the continuous case.

Example 8-5: X and Y are discrete random variables which are jointly distributed with the following probability function $f(x,y)$ (from Example 8-1):

		X		
		-1	0	1
	1	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$
Y	0	$\frac{1}{9}$	0	$\frac{1}{6}$
	-1	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$

Find $E[X \cdot Y]$.

Solution: $E[XY] = \sum_x \sum_y xy \cdot f(x,y) = (-1)(1)(\frac{1}{18}) + (-1)(0)(\frac{1}{9}) + (-1)(-1)(\frac{1}{6})$

$+ (0)(1)(\frac{1}{9}) + (0)(0)(0) + (0)(-1)(\frac{1}{9})$

$+ (1)(1)(\frac{1}{6}) + (1)(0)(\frac{1}{6}) + (1)(-1)(\frac{1}{9}) \ = \frac{1}{6}$. □

Example 8-6: Suppose that $f(x,y) = \frac{3}{2}(x^2+y^2)$ is the density function for the joint distribution of the continuous random variables X and Y defined over the unit square defined on the region $0 \le x \le 1$ and $0 \le y \le 1$. Find $E[X^2+Y^2]$.

Solution:

$E[X^2+Y^2] = \int_0^1 \int_0^1 (x^2+y^2) \cdot f(x,y)\, dy\, dx = \int_0^1 \int_0^1 (x^2+y^2)(\frac{3}{2})(x^2+y^2)\, dy\, dx$

$= \int_0^1 (1.5x^4 + x^2 + .3)\, dx = \frac{14}{15}$. □

Marginal distribution of X found from a joint distribution of X and Y

If X and Y have a joint distribution with joint density or probability function $f(x,y)$, then the **marginal distribution of X** has a probability function or density function denoted $f_X(x)$, which is equal to $\boldsymbol{f_X(x) = \sum_y f(x,y)}$ in the discrete case, and is equal to $\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy}$ in the continuous case. The density function for the marginal distribution of Y is found in a similar way, $f_Y(y)$ is equal to either $f_Y(y) = \sum_x f(x,y)$ or $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

For instance, $f_X(1) = \sum_y f(1,y)$ in the discrete case. What we are doing is "adding up" the probability for all points whose x-value is 1 to get the overall probability that X is 1. The marginal distribution of X describes the random behavior of X as a single random variable.

Care must be taken when the probability space is triangular or some other non-rectangular shape. In this case one must be careful to set the limits of integration properly when finding a marginal density. This is illustrated in Example 8-9 below.

If the cumulative distribution function of the joint distribution of X and Y is $F(x,y)$, then the cdf for the marginal distributions of X and Y are

$F_X(x) = \lim_{y\to\infty} F(x,y)$ and $F_Y(y) = \lim_{x\to\infty} F(x,y)$.

This concept of marginal distribution can be extended to define the marginal distribution of any one (or subcollection) variable in a multivariate distribution. Marginal probability functions and marginal density functions must satisfy all the requirements of probability and density functions. A marginal probability function must sum to 1 over all points of probability and a marginal density function must integrate to 1.

Example 8-7: Find the marginal distributions of X and Y for the joint distribution in Example 8-1 .

Solution: The joint distribution was given as

			X	
		-1	0	1
	1	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$
Y	0	$\frac{1}{9}$	0	$\frac{1}{6}$
	-1	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$

To find the marginal probability function for X, we first note that X can be -1 , 0 or 1.
We wish to find $f_X(-1) = P[X = -1]$, $f_X(0)$ and $f_X(1)$.

Example 8-7 continued

As noted above, to find $f_X(x)$ we sum over the other variable Y:

$f_X(-1) = \sum_{\text{all } y} f(-1, y) = f(-1, -1) + f(-1, 0) + f(-1, 1) = \frac{1}{6} + \frac{1}{9} + \frac{1}{18} = \frac{1}{3}$,

and in a similar way we get $f_X(0) = \frac{1}{9} + 0 + \frac{1}{9} = \frac{2}{9}$ and $f_X(1) = \frac{1}{9} + \frac{1}{6} + \frac{1}{6} = \frac{4}{9}$.

In Example 8-1 we saw that $P[X = 0] = \frac{2}{9}$. What we were finding was the marginal probability $f_X(0)$. Note also that $\sum_{\text{all } x} f_X(x) = f_X(-1) + f_X(0) + f_X(1) = \frac{1}{3} + \frac{2}{9} + \frac{4}{9} = 1$.

This verifies that $f_X(x)$ satisfies the requirements of a probability function.

The marginal probability function of Y is found in the same way, except that sum over x (across each row in the table above).

$f_Y(-1) = \frac{1}{6} + \frac{1}{9} + \frac{1}{9} = \frac{7}{18}$, $f_Y(0) = \frac{1}{9} + 0 + \frac{1}{6} = \frac{5}{18}$ and $f_Y(1) = \frac{1}{18} + \frac{1}{9} + \frac{1}{6} = \frac{1}{3}$. □

Example 8-8: Find the marginal distributions of X and Y for the joint distribution in Example 8-2 .

Solution: The joint density function is $f(x, y) = \frac{3}{2}(x^2 + y^2)$ for $0 \le x \le 1$ and $0 \le y \le 1$.

The marginal density function of X is found by integrating out the other variable y.

$f_X(x) = \int_{\text{all } y} f(x, y)\, dy = \int_0^1 f(x, y)\, dy = \int_0^1 \frac{3}{2}(x^2 + y^2)\, dy = \frac{3}{2}x^2 + \frac{1}{2}$ for $0 \le x \le 1$.

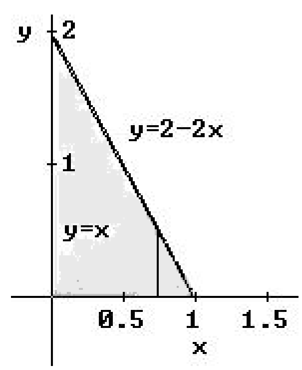
We can verify that this is a proper density function by checking that $\int_0^1 f_X(x)\, dx = 1$.

In a similar way, $f_Y(y) = \frac{3}{2}y^2 + \frac{1}{2}$ for $0 \le y \le 1$. □

Example 8-9: Continuous random variables X and Y have a joint distribution with density function $f(x, y) = \frac{3(2-2x-y)}{2}$ in the region bounded by $y = 0$, $x = 0$ and $y = 2 - 2x$. Find the density function for the marginal distribution of X for $0 < x < 1$.

Find $P[X > \frac{1}{2}]$ and find $P[X > Y]$.

Solution: The region of joint density is illustrated in the graph at the right. Note that X must be in the interval $(0, 1)$ and Y must be in the interval $(0, 2)$ and the joint probability space is triangular. Since $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$, we note that given a value of x in $(0, 1)$, the possible values of y (with non-zero density for $f(x, y)$) must satisfy $0 < y < 2 - 2x$, so that



$f_X(x) = \int_0^{2-2x} f(x, y)\, dy$

$= \int_0^{2-2x} \frac{3(2-2x-y)}{2}\, dy = 3(1 - x)^2$.

Example 8-9 continued

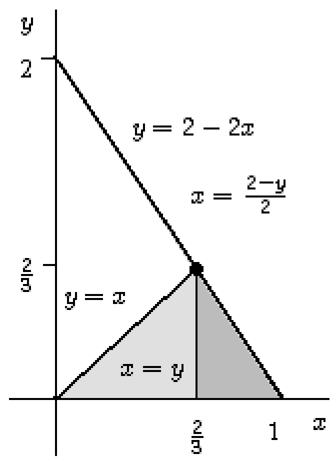
Once we have the marginal density function for X, we can find $P[X > \frac{1}{2}]$.

$P[X > \frac{1}{2}] = \int_{1/2}^{1} f_X(x)\,dx = \int_{1/2}^{1} 3(1-x)^2\,dx = \frac{1}{8}$.

Note that we could find $P[X > \frac{1}{2}]$ by identifying the two-dimensional region and integrating $f(x,y)$. This would come out to be $\int_{1/2}^{1}\int_{0}^{2-2x} f(x,y)\,dy\,dx = \int_{1/2}^{1} f_X(x)\,dx$,

which is the same as finding the marginal density first. We could also have reversed the order of integration in x and y, so that $P[X > \frac{1}{2}] = \int_0^1\int_{1/2}^{(2-y)/2} f(x,y)\,dx\,dy$. This involves a little more algebra.

The region for which $X > Y$ is identified in the graph below. The line $y = x$ intersects with the line $y = 2 - 2x$ at the point $(\frac{2}{3}, \frac{2}{3})$. The region we are looking for is $y < x$, and lies below the line $y = x$.



The probability can be expressed as a double integral. If we set the order of integration with dx on the outside then we must integrate in two pieces, first from 0 to $\frac{2}{3}$ beneath the line $y = x$, and then from $\frac{2}{3}$ to 1 beneath the line $y = 2 - 2x$.

$P[X > Y] = \int_0^{2/3}\int_0^{x} \frac{3(2-2x-y)}{2}\,dy\,dx + \int_{2/3}^{1}\int_0^{2-2x} \frac{3(2-2x-y)}{2}\,dy\,dx = \frac{8}{27} + \frac{1}{27} = \frac{1}{3}$.

The integral can be found in the $dx\,dy$ integration order. In that case,

$P[X > Y] = \int_0^{2/3}\int_y^{(2-y)/2} \frac{3(2-2x-y)}{2}\,dx\,dy = \frac{1}{3}$.

In this case it is more efficient to express the integral with dy on the outside.

There is one other note on this example. The probability space was originally described as "the region bounded by $y = 0$, $x = 0$ and $y = 2 - 2x$". We might also see this region defined in the following way: $0 < x < \frac{2-y}{2}$ and $y > 0$. The reader can check that this is the same region. □

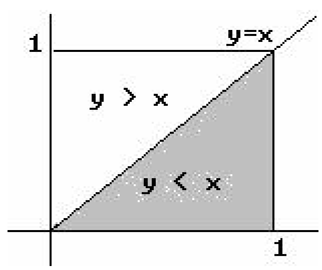
Independence of random variables X and Y

Random variables X and Y with density functions $f_X(x)$ and $f_Y(y)$ are said to be independent (or stochastically independent) if the probability space is rectangular ($a \leq x \leq b$, $c \leq y \leq d$, where the endpoints can be infinite) and if the joint density function is of the form $\boldsymbol{f(x,y) = f_X(x) \cdot f_Y(y)}$. Independence of X and Y is also equivalent to the factorization of the cumulative distribution function $\boldsymbol{F(x,y) = F_X(x) \cdot F_Y(y)}$ **for all** $\boldsymbol{(x,y)}$.

For the discrete joint distribution in Example 8-1 we can see that X and Y are not independent, because, for instance, $f(-1,-1) = \frac{1}{6} \neq \frac{1}{3} \cdot \frac{7}{18} = f_X(-1) \cdot f_Y(-1)$. For the continuous joint distribution of Example 8-8, we see that
$f(x,y) = \frac{3}{2}(x^2+y^2) \neq (\frac{3}{2}x^2 + \frac{1}{2})(\frac{3}{2}y^2 + \frac{1}{2}) = f_X(x) \cdot f_Y(y)$, so X and Y are not independent.

Example 8-10: Suppose that X and Y are independent continuous random variables with the following density functions: $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(y) = 2y$ for $0 < y < 1$.
Find $P[Y < X]$.

Solution: Since X and Y are independent, the density function of the joint distribution of X and Y is $f(x,y) = f_X(x) \cdot f_Y(y) = 2y$, and is defined on the rectangle created by the intervals for X and Y, which, in this case, is the unit square. The graph below illustrates the region for the probability in question. $P[Y < X] = \int_0^1 \int_0^x 2y\,dy\,dx = \frac{1}{3}$.



□

Conditional distribution of Y given $X = x$

The way in which a conditional distribution is defined follows the basic definition of conditional probability, $P[A|B] = \frac{P[A\cap B]}{P[B]}$. In fact, given a discrete joint distribution, this is exactly how a conditional distribution is defined. Example 8-1 described a discrete joint distribution of X and Y, and then Example 8-7 showed how to formulate the marginal distributions of X and Y. We now wish to formulate a conditional distribution. For instance, for the joint distribution of Example 8-1, suppose we wish to describe the conditional distribution of X given $Y = 1$. What we are trying to describe are conditional probabilities of the form $P[X = x|Y = 1]$.

We find these conditional probabilities in the usual way that conditional probability is defined.
$P[X = -1|Y = 1] = \frac{P[(X=-1)\cap(Y=1)]}{P[Y=1]}$.
The denominator is the marginal probability that $Y = 1$, $f_Y(1) = \frac{1}{3}$. The numerator is the joint probability $f(-1,1) = \frac{1}{18}$, which is found in the joint probability table. Then,
$P[X = -1|Y = 1] = \frac{f(-1,1)}{f_Y(1)} = \frac{1/18}{1/3} = \frac{1}{6}$. We would denote this conditional probability $f_{X|Y}(-1|Y = 1)$. In a similar way, we can get $f_{X|Y}(0|Y = 1) = \frac{f(0,1)}{f_Y(1)} = \frac{1/9}{1/3} = \frac{1}{3}$, and $f_{X|Y}(1|Y = 1) = \frac{f(1,1)}{f_Y(1)} = \frac{1/6}{1/3} = \frac{1}{2}$. This completely describes the conditional distribution of X given $Y = 1$. As with any discrete distribution, probabilities for a conditional distribution must add to 1, and this is the case for this conditional distribution, since
$f_{X|Y}(-1|Y = 1) + f_{X|Y}(0|Y = 1) + f_{X|Y}(1|Y = 1) = \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$.

A conditional distribution satisfies all the same properties of any distribution. We can find a conditional mean, a conditional variance, etc. For instance, the conditional mean of X given $Y = 1$ in the example we have just been considering is

$$\begin{aligned}E[X|Y = 1] &= \sum_{all\, x} x \cdot f_{X|Y}(x|Y = 1)\\ &= (-1)f_{X|Y}(-1|Y = 1) + (0)f_{X|Y}(0|Y = 1) + (1)f_{X|Y}(1|Y = 1)\\ &= (-1)(\tfrac{1}{6}) + (0)(\tfrac{1}{3}) + (1)(\tfrac{1}{2}) = \tfrac{1}{3}.\end{aligned}$$

We can find the second conditional moment in a similar way,
$E[X^2|Y = 1] = \sum_{all\, x} x^2 \cdot f_{X|Y}(x|Y = 1)$. Then the conditional variance would be
$Var[X|Y = -1] = E[X^2|Y = -1] - (E[X|Y = -1])^2$.

The expression for conditional probability that was used above in the discrete case was $\boldsymbol{f_{X|Y}(x|Y=y)=\frac{f(x,y)}{f_Y(y)}}$. This can be applied to find a conditional distribution of Y given $X=x$ also, so that we define $f_{Y|X}(y|X=x)=\frac{f(x,y)}{f_X(x)}$.

We also apply this same algebraic form to define the conditional density in the continuous case, with $f(x,y)$ being the joint density and $f_X(x)$ being the marginal density. In the continuous case, the conditional mean of Y given $X=x$ would be

$$\boldsymbol{E[Y|X=x]=\int y\cdot f_{Y|X}(y|X=x)\,dy}\ ,$$

where the integral is taken over the appropriate interval for the conditional distribution of Y given $X=x$. The conditional density/probability is also written as $f_{Y|X}(y|x)$, or $f(y|x)$.

If X and Y are independent random variables, then

$$f_{Y|X}(y|X=x)=\frac{f(x,y)}{f_X(x)}=\frac{f_X(x)\cdot f_Y(y)}{f_X(x)}=f_Y(y)$$

and in a similar way we have $f_{X|Y}(x|Y=y)=f_X(x)$, which indicates that the density of Y does not depend on X and vice-versa.

The conditional density function must satisfy the usual requirement of a density function, $\int_{-\infty}^{\infty} f_{Y|X}(y|x)\,dy=1$. Note also that if $f_X(x)$, the marginal density of X is known, and if $f_{Y|X}(y|X=x)$, the conditional density of Y given $X=x$, is also known, then the joint density of X and Y can be formulated as

$$\boldsymbol{f(x,y)=f_{Y|X}(y|X=x)\cdot f_X(x)}\ .$$

Example 8-11: Find the conditional distribution of Y given $X=-1$ for the joint distribution of Example 8-1. Find the conditional expectation of Y given $X=-1$.

Solution: The marginal probability function for X was found in Example 8-7, where it was found that $f_X(-1)=\frac{1}{3}$. The conditional probability function of Y given $X=-1$ is $f_{Y|X}(y|X=-1)=\frac{f(-1,y)}{f_X(-1)}=\frac{f(-1,y)}{1/3}$. Then,

$f_{Y|X}(-1|X=-1)=\frac{f(-1,-1)}{1/3}=\frac{1/6}{1/3}=\frac{1}{2}$, $f_{Y|X}(0|X=-1)=\frac{f(-1,0)}{1/3}=\frac{1/9}{1/3}=\frac{1}{3}$,

and $f_{Y|X}(1|X=-1)=\frac{f(-1,1)}{1/3}=\frac{1/18}{1/3}=\frac{1}{6}$.

$E[Y|X=-1]=\sum_{\text{all } y} y\cdot f_{Y|X}(y|X=-1)=(-1)(\frac{1}{2})+(0)(\frac{1}{3})+(1)(\frac{1}{6})=-\frac{1}{3}$. □

Example 8-12: Find the conditional density and conditional expectation and conditional variance of X given $Y = .3$ for the joint distribution of Example 8-2.

Solution: $f_{X|Y}(x|Y = .3) = \frac{f(x,.3)}{f_Y(.3)} = \frac{\frac{3}{2}(x^2+.3^2)}{\frac{3}{2}.3^2+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+.09)}{.635}$.

The conditional expectation is

$E[X|Y = .3] = \int_0^1 x \cdot f_{X|Y}(x|Y = .3)\,dx = \int_0^1 x \cdot \frac{\frac{3}{2}(x^2+.09)}{.635}\,dx = .697$.

The conditional second moment of X given $Y = .3$ is

$E[X^2|Y = .3] = \int_0^1 x^2 \cdot f_{X|Y}(x|Y = .3)\,dx = \int_0^1 x^2 \cdot \frac{\frac{3}{2}(x^2+.09)}{.635}\,dx = .543$.

The conditional variance is

$Var[X|Y = .3] = E[X^2|Y = .3] - (E[X|Y = .3])^2 = .543 - (.697)^2 = .057$. □

Example 8-13: Continuous random variables X and Y have a joint distribution with density function $f(x, y) = \frac{\pi}{2}(sin\,\frac{\pi}{2}y)e^{-x}$ for $0 < x < \infty$ and $0 < y < 1$. Find $P[X > 1|Y = \frac{1}{2}]$.

Solution: $P[X > 1|Y = \frac{1}{2}] = \frac{P[(X>1)\cap(Y=\frac{1}{2})]}{f_Y(\frac{1}{2})}$.

$P[(X > 1) \cap (Y = \frac{1}{2})] = \int_1^\infty f(x, \frac{1}{2})\,dx = \int_1^\infty \frac{\pi}{2}(sin\,\frac{\pi}{4})e^{-x}dx = \frac{\pi\sqrt{2}}{4} \cdot e^{-1}$.

$f_Y(\frac{1}{2}) = \int_0^\infty f(x, \frac{1}{2})\,dx = \int_0^\infty \frac{\pi}{2}(sin\,\frac{\pi}{4})e^{-x}dx = \frac{\pi\sqrt{2}}{4} \quad \Rightarrow \quad P[X > 1|Y = \frac{1}{2}] = e^{-1}$.

Note that $f_X(x) = \int_0^1 f(x, y)\,dy = \int_0^1 \frac{\pi}{2}(sin\,\frac{\pi}{2}y)e^{-x}\,dy = e^{-x}$, and

$f_Y(y) = \int_0^\infty f(x, y)\,dx = \int_0^\infty \frac{\pi}{2}(sin\,\frac{\pi}{2}y)e^{-x}\,dx = \frac{\pi}{2}(sin\,\frac{\pi}{2}y)$.

Then we see that X and Y are independent.

This follows from $f(x, y) = (e^{-x})(\frac{\pi}{2} \cdot sin\,\frac{\pi}{2}y) = f_X(x) \cdot f_Y(y)$.

From independence it follows that $P[X > 1|Y = \frac{1}{2}] = P[X > 1] = e^{-1}$ (same as the result above).

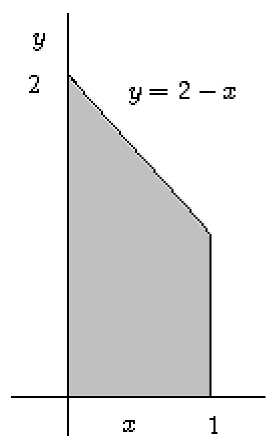
As a final comment on this example, we could have noticed at the start that since $f(x, y)$ can be factored into separate functions in x and y, we might anticipate that X and Y are independent. Since the joint distribution is defined on a rectangular area, it follows that X and Y must be independent if the pdf factors into a function of x only multiplied by a function of y only. Furthermore, the factor involving x is e^{-x}. Since this integrates to 1 over the range for X, it must be the marginal density of X, $f_X(x) = e^{-x}$. Then, because of independence, we have $P[X > 1|Y = \frac{1}{2}] = P[X > 1]$, and from the marginal density, this is $\int_1^\infty e^{-x}\,dx = e^{-1}$. □

Example 8-14: X is a continuous random variable with density function $f_X(x) = x + \frac{1}{2}$ for $0 < x < 1$. X is also jointly distributed with the continuous random variable Y, and the conditional density function of Y given $X = x$ is
$f_{Y|X}(y|X = x) = \frac{x+y}{x+\frac{1}{2}}$ for $0 < x < 1$ and $0 < y < 1$. Find $f_Y(y)$ for $0 < y < 1$.
Solution: $f(x, y) = f(y|x) \cdot f_X(x) = \frac{x+y}{x+\frac{1}{2}} \cdot (x + \frac{1}{2}) = x + y$.
Then, $f_Y(y) = \int_0^1 f(x, y)\, dx = y + \frac{1}{2}$. □

As Example 8-14 shows, we can construct the joint density $f(x, y)$ from knowing the conditional density $f_{Y|X}(y|x)$ and the marginal density $f_X(x)$ using the relationship $f(x, y) = f(y|x) \cdot f_X(x)$. When doing this, care must be taken to ensure that proper two-dimensional region is being formulated for the joint distribution. The following example illustrates this point.

Example 8-15: A company will experience a loss X that is uniformly distributed between 0 and 1. The company pays a bonus to its employees that is uniformly distributed on the interval $(0, 2 - X)$, which depends on the amount of the loss that occurred. Find the expected amount of the bonus paid.

Solution: X has marginal pdf $f_X(x) = 1$ for $0 < x < 1$. Let Y be the bonus paid. The conditional distribution of Y given $X = x$ is a uniform distribution on the interval $(0, 2 - x)$, with conditional density $f_{Y|X}(y|x) = \frac{1}{2-x}$. The joint density of X and Y is
$f(x, y) = f_{Y|X}(y|x) \cdot f_X(x) = 1 \cdot \frac{1}{2-x} = \frac{1}{2-x}$
and the two-dimensional region of probability is the region $0 < y < 2 - x$ and $0 < x < 1$. This region is the shaded area in the graph at the right.



The expected value of Y can be found as $E[Y] = \int_0^1 \int_0^{2-x} y \cdot \frac{1}{2-x}\, dy\, dx = \frac{3}{4}$.
We could have found the marginal density on Y, and then have found $E[Y]$, but that is more awkward because for $0 < y \leq 1$ we have $f_Y(y) = \int_0^1 \frac{1}{2-x}\, dx = ln\, 2$, and for $1 < y \leq 2$ we have $f_Y(y) = \int_0^{2-y} \frac{1}{2-x}\, dx = ln\, 2 - ln\, y$. □

Example 8-16: Suppose that X has a continuous distribution with pdf $f_X(x) = 2x$ on the interval $(0,1)$, and $f_X(x) = 0$ elsewhere. Suppose that Y is a continuous random variable such that the conditional distribution of Y given $X = x$ is uniform on the interval $(0,x)$. Find the mean and variance of Y.

Solution: We find the unconditional (marginal) distribution of Y. We are given $f_X(x) = 2x$ for $0 < x < 1$, and $f_{Y|X}(y|X = x) = \frac{1}{x}$ for $0 < y < x$.
Then, $f(x,y) = f(y|x) \cdot f_X(x) = \frac{1}{x} \cdot 2x = 2$ for $0 < y < x < 1$.
The unconditional (marginal) distribution of Y has pdf.
$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 2\,dx = 2(1-y)$ for $0 < y < 1$ (and $f_Y(y)$ is 0 elsewhere). Then $E[Y] = \int_0^1 y \cdot 2(1-y)\,dy = \frac{1}{3}$, $E[Y^2] = \int_0^1 y^2 \cdot 2(1-y)\,dy = \frac{1}{6}$,
and $Var[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{6} - (\frac{1}{3})^2 = \frac{1}{18}$. □

Covariance between random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$: If random variables X and Y are jointly distributed with joint density/probability function $f(x,y)$, X and Y might not be independent of one another. We have seen some examples in which X and Y are independent, and some in which they are not. There are a couple of numerical measures that describe, in some sense, the dependence that exists between X and Y. Covariance is one such measure.

The covariance between X and Y is

$$\boldsymbol{Cov[X,Y] = E[(X - E[X])(Y - E[Y])]}$$
$$\boldsymbol{= E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X] \cdot E[Y]}.$$

A positive covariance between X and Y is an indication that "large" values of X (values of X that are bigger than $E[X]$) tend to occur paired with "large" values of Y, and the same for "small" values of X and Y. Negative covariance indicates the opposite relationship. Covariance near or at 0 indicates that the "size" of X value is not related to the "size" of the Y values to which they are paired.

Note that $Cov[X,X] = Var[X]$. One important application of the covariance is in finding the variance of the sum of X and Y. Suppose that a, b and c are constants. Then it can be shown that

$$\boldsymbol{Var[aX + bY + c] = a^2 Var[X] + b^2 Var[Y] + 2ab Cov[X,Y]}\ .$$

Coefficient of correlation between random variables X and Y:

The coefficient of correlation between random variables X and Y is
$\boldsymbol{\rho(X,Y) = \rho_{X,Y} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}}$, where σ_X and σ_Y are the standard deviations of X and Y respectively. Note that $-1 \leq \rho_{X,Y} \leq 1$ always.

Example 8-17: Find $Cov[X,Y]$ for the jointly distributed discrete random variables in Example 8-1 above.

Solution: $Cov[X,Y] = E[XY] - E[X] \cdot E[Y]$. In Example 8-5 it was found that $E[XY] = \frac{1}{6}$.
The marginal probability function for X is $P[X=1] = \frac{1}{6} + \frac{1}{6} + \frac{1}{9} = \frac{4}{9}$,
$P[X=0] = \frac{2}{9}$ and $P[X=-1] = \frac{1}{3}$, and the mean of X is
$E[X] = (1)(\frac{4}{9}) + (0)(\frac{2}{9}) + (-1)(\frac{1}{3}) = \frac{1}{9}$.
In a similar way, the probability function of Y is found to be $P[Y=1] = \frac{1}{3}$, $P[Y=0] = \frac{5}{18}$, and $P[Y=-1] = \frac{7}{18}$, with a mean of $E[Y] = -\frac{1}{18}$.
Then, $Cov[X,Y] = \frac{1}{6} - (\frac{1}{9})(-\frac{1}{18}) = \frac{14}{81}$. □

Example 8-18: The coefficient of correlation between random variables X and Y is $\frac{1}{3}$, and $\sigma_X^2 = a$, $\sigma_Y^2 = 4a$. The random variable Z is defined to be $Z = 3X - 4Y$, and it is found that $\sigma_Z^2 = 114$. Find a.

Solution: $\sigma_Z^2 = Var[Z] = 9Var[X] + 16Var[Y] + 2 \cdot (3)(-4)\,Cov[X,Y]$.
Since $Cov[X,Y] = \rho[X,Y] \cdot \sigma_X \cdot \sigma_Y = \frac{1}{3} \cdot \sqrt{a} \cdot \sqrt{4a} = \frac{2a}{3}$, it follows that
$114 = \sigma_Z^2 = 9a + 16(4a) - 24(\frac{2a}{3}) = 57a \rightarrow a = 2$. □

Moment generating function of a joint distribution: Given jointly distributed random variables X and Y, the moment generating function of the joint distribution is $M_{X,Y}(t_1,t_2)=E[e^{t_1X+t_2Y}]$. This definition can be extended to the joint distribution of any number of random variables. It can be shown that $E[X^nY^m]$ is equal to the multiple partial derivative evaluated at 0, $E[X^nY^m]=\frac{\partial^{n+m}}{\partial^n t_1\,\partial^m t_2}\,M_{X,Y}(t_1,t_2)\Big|_{t_1=t_2=0}$.

Example 8-19: Suppose that X and Y are random variables whose joint distribution has moment generating function $M(t_1,t_2)=(\frac{1}{4}e^{t_1}+\frac{3}{8}e^{t_2}+\frac{3}{8})^{10}$, for all real t_1 and t_2.
Find the covariance between X and Y.

Solution: $Cov[X,Y]=E[XY]-E[X]\cdot E[Y]$.

$E[XY]=\frac{\partial^2}{\partial t_1\,\partial t_2}\,M_{X,Y}(t_1,t_2)\Big|_{t_1=t_2=0}$

$=(10)(9)(\frac{1}{4}e^{t_1}+\frac{3}{8}e^{t_2}+\frac{3}{8})^8(\frac{1}{4}e^{t_1})(\frac{3}{8}e^{t_2})\Big|_{t_1=t_2=0}=\frac{135}{16}$,

$E[X]=\frac{\partial}{\partial t_1}\,M_{X,Y}(t_1,t_2)\Big|_{t_1=t_2=0}=(10)(\frac{1}{4}e^{t_1}+\frac{3}{8}e^{t_2}+\frac{3}{8})^9(\frac{1}{4}e^{t_1})\Big|_{t_1=t_2=0}=\frac{5}{2}$,

$E[Y]=\frac{\partial}{\partial t_2}\,M_{X,Y}(t_1,t_2)\Big|_{t_1=t_2=0}=(10)(\frac{1}{4}e^{t_1}+\frac{3}{8}e^{t_2}+\frac{3}{8})^9(\frac{3}{8}e^{t_2})\Big|_{t_1=t_2=0}=\frac{15}{4}$,

$\Rightarrow\quad Cov[X,Y]=\frac{135}{16}-(\frac{5}{2})(\frac{15}{4})=-\frac{15}{16}$. $\square$

The Bivariate normal distribution:

Suppose that X and Y are normal random variables with means and variances $E[X]=\mu_X$, $Var[X]=\sigma_X^2$, $E[Y]=\mu_Y$, $Var[Y]=\sigma_Y^2$, and with correlation coefficient ρ_{XY}. X and Y are said to have a bivariate normal distribution. The conditional mean and variance of Y given $X=x$ are

$$\boldsymbol{E[Y|X=x]=\mu_Y+\rho_{XY}\cdot\frac{\sigma_Y}{\sigma_X}\cdot(x-\mu_X)=\mu_Y+\frac{Cov(X,Y)}{\sigma_X^2}\cdot(x-\mu_X)},$$

and $\boldsymbol{Var[Y|X=x]=\sigma_Y^2\cdot(1-\rho_{XY}^2)}$. Similar relationships apply for the conditional distribution of X given $Y=y$.

If X and Y are independent, then $\rho_{XY}=0$, and vice-versa.

The pdf of the bivariate normal distribution is

$$f(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\cdot exp\Big[-\frac{1}{2(1-\rho^2)}\cdot\Big[\big(\frac{x-\mu_X}{\sigma_X}\big)^2+\big(\frac{y-\mu_Y}{\sigma_Y}\big)^2-2\rho\big(\frac{x-\mu_X}{\sigma_X}\big)\big(\frac{y-\mu_Y}{\sigma_Y}\big)\Big]\Big].$$

The bivariate normal has occurred infrequently on Exam P.

Some results and formulas related to this section are

(i) If X and Y are independent, then for any functions g and h,
$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$, and in particular, $E[X \cdot Y] = E[X] \cdot E[Y]$.

(ii) The density/probability function of jointly distributed variables X and Y can be written in the form $f(x,y) = f_{Y|X}(y|X=x) \cdot f_X(x) = f_{X|Y}(x|Y=y) \cdot f_Y(y)$

(iii) $Cov[X,Y] = E[X \cdot Y] - \mu_X \cdot \mu_Y = E[XY] - E[X] \cdot E[Y]$.
$Cov[X,Y] = Cov[Y,X]$. If X and Y are independent, then $E[X \cdot Y] = E[X] \cdot E[Y]$ and $Cov[X,Y] = 0$. For constants $a,\ b,\ c,\ d,\ e,\ f$ and random variables $X,\ Y,\ Z$ and W,
$Cov[aX + bY + c\ ,\ dZ + eW + f]$
$= adCov[X,Z] + aeCov[X,W] + bdCov[Y,Z] + beCov[Y,W]$

(iv) $Var[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$
$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$
$= E[X^2] + E[2XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2$
$= Var[X] + Var[Y] + 2 \cdot Cov[X,Y]$
If X and Y are independent, then $Var[X+Y] = Var[X] + Var[Y]$.
For any X, Y, $Var[aX + bY + c] = a^2 Var[X] + b^2 Var[Y] + 2ab \cdot Cov[X,Y]$

(v) If X and Y have a joint distribution which is uniform (constant density) on the two dimensional region R (usually R will be a triangle, rectangle or circle in the (x,y) plane) then the pdf of the joint distribution is $\frac{1}{Area\,of\,R}$ inside the region R (and the pdf is 0 outside). The probability of any event A (represented by a subset of R) is the proportion $\frac{Area\,of\,A}{Area\,of\,R}$. Also the conditional distribution of Y given $X = x$ has a uniform distribution on the line segment (or segments) defined by the intersection of the region R with the line $X = x$.
The marginal distribution of Y might or might not be uniform

(vi) $E[h_1(X,Y) + h_2(X,Y)] = E[h_1(X,Y)] + E[h_2(X,Y)]$, and in particular,
$E[X+Y] = E[X] + E[Y]$ and $E[\sum X_i] = \sum E[X_i]$

(vii) $\lim_{x \to -\infty} F(x,y) = \lim_{y \to -\infty} F(x,y) = 0$

(viii) $P[(x_1 < X \le x_2) \cap (y_1 < Y \le y_2)] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$

(ix) $P[(X \le x) \cup (Y \le y)] = F_X(x) + F_Y(y) - F(x, y) \le 1$

(x) For any jointly distributed random variables X and Y, $-1 \le \rho_{XY} \le 1$

(xi) $M_{X,Y}(t_1, 0) = E[e^{t_1 X}] = M_X(t_1)$ and $M_{X,Y}(0, t_2) = E[e^{t_2 Y}] = M_Y(t_2)$

(xii) $\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2)\Big|_{t_1=t_2=0} = E[X]$, $\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=t_2=0} = E[Y]$
$\frac{\partial^{r+s}}{\partial^r t_1\, \partial^s t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=t_2=0} = E[X^r \cdot Y^s]$

(xiii) If $M(t_1, t_2) = M(t_1, 0) \cdot M(0, t_2)$ for t_1 and t_2 in a region about $(0, 0)$, then X and Y are independent.

(xiv) If $Y = aX + b$ then $M_Y(t) = e^{bt} M_X(at)$.

(xv) If X and Y are jointly distributed, then for any y, $E[X|Y = y]$ depends on y, say $E[X|Y = y] = h(y)$. It can then be shown that $E[h(Y)] = E[X]$; this is more usually written in the form $\boldsymbol{E[\, E[X|Y]\,] = E[X]}$.
It can also be shown that $\boldsymbol{Var[X] = E[\, Var[X|Y]\,] + Var[\, E[X|Y]\,]}$.
This relationship has come up on several questions on recent exams.

Example 8-20: For random variables X, Y and Z, you are given that $Var[X] = Var[Y] = 1$, $Var[Z] = 2$, $Cov(X, Y) = -1$, $Cov(X, Z) = 0$, $Cov(Y, Z) = 1$.
Find the covariance between $X + 2Y$ and $Y + 2Z$.

Solution:

$Cov(X + 2Y, Y + 2Z) = Cov(X, Y) + Cov(X, 2Z) + Cov(2Y, Y) + Cov(2Y, 2Z)$
$= -1 + 2Cov(X, Z) + 2Cov(Y, Y) + 4Cov(Y, Z)$
$= -1 + 2(0) + 2Var[Y] + 4(1) = -1 + 2(1) + 4 = 5$. □

Example 8-21: Random variables X and Y are jointly distributed on the region $0 \le y \le x$, $y \le 1$, $x \le 2$. The joint distribution has a constant density over the entire region.

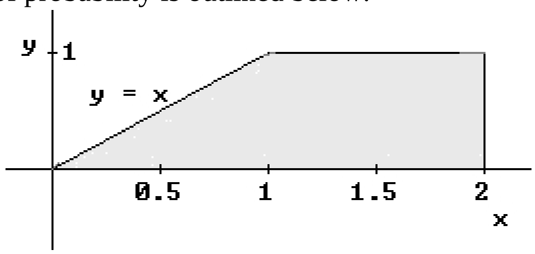
(i) Find the joint density.

(ii) Find the marginal densities of X and Y.

(iii) Find the conditional densities of X given $Y = y$, and Y given $X = x$.

(iv) Find the probabilities $P[X > 1]$, $P[Y \le \frac{1}{2}]$, and $P[X + Y \le 1]$.

Solution: The region of probability is outlined below.



(i) Since we are told that the joint density is constant on the region of probability that constant must be equal to $\frac{1}{\text{area of region of probability}}$. The area of the region of probability is 1.5 , so that $f(x,y) = \frac{1}{1.5} = \frac{2}{3}$ for (x,y) points in the region of probability.

(ii) The marginal density function of X is $f_X(x) = \int_0^1 f(x,y)\,dy$.
If $0 \le x \le 1$, the probability region is $0 \le y \le x$, so that $f_X(x) = \int_0^x \frac{2}{3}\,dy = \frac{2x}{3}$.
If $1 \le x \le 2$, the probability region is $0 \le y \le 1$, so that $f_X(x) = \int_0^1 \frac{2}{3}\,dy = \frac{2}{3}$.

The marginal density of Y is $f_Y(y) = \int_y^2 \frac{2}{3}\,dx = \frac{2}{3}(2-y)$ for $0 \le y \le 1$.

(iii) The conditional density of X given $Y = y$ is $f_{X|Y}(x|Y=y) = \frac{f(x,y)}{f_Y(y)} = \frac{2/3}{2(2-y)/3} = \frac{1}{2-y}$.
Note that the interval of probability for the conditional distribution of X given $Y = y$ is $y \le x \le 2$. The conditional density of X given $Y = y$ is uniform on that interval.

The conditional density of Y given $X = x$ is $f_{Y|X}(y|X=x) = \frac{f(x,y)}{f_X(x)}$.
For $0 \le x \le 1$ this is $\frac{2/3}{2x/3} = \frac{1}{x}$, a uniform distribution for $0 \le y \le x$.
For $1 \le x \le 2$ this is $\frac{2/3}{2/3} = 1$, a uniform distribution for $0 \le y \le 1$.
Note that in all cases the conditional distributions are uniform. This will always be the case if the joint distribution is uniform on its probability space.

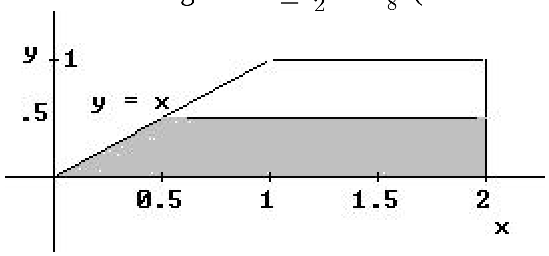
Example 8-21 continued

(iv) $P[X > 1]$ can be found in two ways. Since we have the marginal density for X, we can use it to find $P[X > 1] = \int_1^2 f_X(x)\,dx = \int_1^2 \frac{2}{3}\,dx = \frac{2}{3}$.

Alternatively, since the joint distribution is uniform, the probability of any event A is the proportion $\frac{Area\,of\,A}{Area\,of\,R}$. The area of the full probability space R was already found to be 1.5 . The area of the region $X > 1$ is 1. Therefore, $P[X > 1] = \frac{1}{1.5} = \frac{2}{3}$.

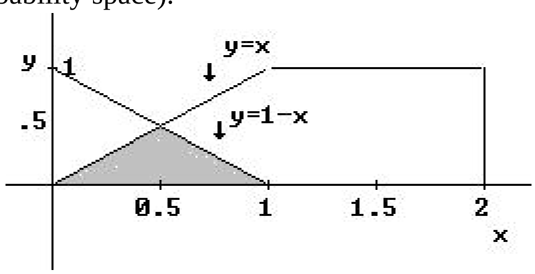
The same comment applies to $P[Y \leq \frac{1}{2}]$. From the marginal density of Y we have $P[Y \leq \frac{1}{2}] = \int_0^{1/2} \frac{2}{3}(2-y)\,dy = \frac{7}{12}$.

Alternatively, the area of the region $Y \leq \frac{1}{2}$ is $\frac{7}{8}$ (outlined in the diagram below).



The probability $P[Y \leq \frac{1}{2}]$ is the proportion $\frac{7/8}{3/2} = \frac{7}{12}$.

The region $X + Y \leq 1$ is outlined in the diagram below (the region below $y = 1 - x$ within the original probability space).



The area of the region $X + Y \leq 1$ is $\frac{1}{4}$, so the probability is $P[X + Y \leq 1] = \frac{1/4}{3/2} = \frac{1}{6}$.

We could also find the probability by integrating the joint density over the appropriate region, $P[X + Y \leq 1] = \int_0^{.5}\int_y^{1-y} \frac{2}{3}\,dx\,dy = \int_0^{.5} \frac{2}{3}(1-2y)\,dy = \frac{1}{6}$. □

Example 8-22: Suppose that W is the 3-point discrete uniform random variable $\{1,2,3\}$, with $P[W=1]=P[W=2]=P[W=3]=\frac{1}{3}$, and suppose that the conditional distribution of Y given $W=w$ is exponential with mean w.

Find the (unconditional) mean and (unconditional) variance of Y.

Solution: The conditional pdf of Y given $W=w$ is $f_{Y|W}(y|W=w)=\frac{1}{w}e^{-y/w}$.

Also, we are given $E[Y|W=w]=w$, so that $E[Y|W]=\begin{cases} 1 & \text{if } W=1\text{, prob. } \frac{1}{3} \\ 2 & \text{if } W=2\text{, prob. } \frac{1}{3} \\ 3 & \text{if } W=3\text{, prob. } \frac{1}{3} \end{cases}$.

We see that $\boldsymbol{E[Y|W]}$ **is a 3-point random variable.**

Let us use the notation $Z=E[Y|W]$ for this 3-point random variable.

Then $E[Z]=E[\,E[Y|W]\,]=(1)(\frac{1}{3})+(2)(\frac{1}{3})+(3)(\frac{1}{3})=2$.

This is $E[Y]$ according to the rule cited above..

Also, as a random variable, $Z=E[Y|W]$ has a variance:

$Var[Z]=Var[\,E[Y|W]\,]=E[Z^2]-(E[Z])^2=E[(E[Y|W])^2]-(E[\,E[Y|W]\,])^2$

$=[(1^2)(\frac{1}{3})+(2^2)(\frac{1}{3})+(3^2)(\frac{1}{3})]-2^2=\frac{2}{3}$.

The variance of an exponential random variable is the square of the mean, so since the conditional distribution of Y given W is exponential with mean w, the conditional variance of Y given $W=w$ is $Var[Y|W=w]=w^2$. As with the conditional mean of $Y|W$, **the conditional variance of $\boldsymbol{Y|W}$ is a random variable dependent on the outcome of $\boldsymbol{W}$.**

$Var[Y|W]=\begin{cases} Var[Y|W=1]=1 & \text{if } W=1\text{, prob. } \frac{1}{3} \\ Var[Y|W=2]=4 & \text{if } W=2\text{, prob. } \frac{1}{3} \\ Var[Y|W=3]=9 & \text{if } W=3\text{, prob. } \frac{1}{3} \end{cases}$.

$Var[Y|W]$ is a three-point random variable. Let us use the notation $U=Var[Y|W]$.

Then $E[U]=E[\,Var[Y|W]\,]$

$=(Var[Y|w=1])(\frac{1}{3})+(Var[Y|w=2])(\frac{1}{3})+(Var[Y|w=3])(\frac{1}{3})$

$=(1)(\frac{1}{3})+(4)(\frac{1}{3})+(9)(\frac{1}{3})=\frac{14}{3}$.

Using the variance rule cited above in (xv) above, we have

$Var[Y]=E[\,Var[Y|W]\,]+Var[\,E[Y|W]\,]=E[U]+Var[Z]=\frac{14}{3}+\frac{2}{3}=\frac{16}{3}$. □

PROBLEM SET 8

Joint, Marginal and Conditional Distributions

1. A wheel is spun with the numbers 1, 2 and 3 appearing with equal probability of $\frac{1}{3}$ each. If the number 1 appears, the player gets a score of 1.0; if the number 2 appears, the player gets a score of 2.0; if the number 3 appears, the player gets a score of X, where X is a normal random variable with mean 3 and standard deviation 1. IF W represents the player's score on 1 spin of the wheel, then what is $P[W \leq 1.5]$?

A) .13 B) .33 C) .36 D) .40 E) .64

2. Let X and Y be discrete loss random variables with joint probability function

$$f(x,y) = \begin{cases} \frac{y}{24x} \text{ for } x=1,2,4\,;\ y=2,4,8\,;\ x\leq y \\ 0, \text{otherwise} \end{cases}.$$

An insurance policy pays the full amount of loss X and half of loss Y. Find the probability that the total paid by the insurer is no more than 5.

A) $\frac{1}{8}$ B) $\frac{7}{24}$ C) $\frac{3}{8}$ D) $\frac{5}{8}$ E) $\frac{17}{24}$

3. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} .75x \text{ for } 0<x<2 \text{ and } 0<y\leq 2-x \\ 0, \text{otherwise} \end{cases}.$$

What is $P[X > 1]$?

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{3}{4}$

4. Let X and Y be independent random variables with $\mu_X = 1$, $\mu_Y = -1$, $\sigma_X^2 = \frac{1}{2}$, and $\sigma_Y^2 = 2$. Calculate $E[(X+1)^2(Y-1)^2]$.

A) 1 B) $\frac{9}{2}$ C) 16 D) 17 E) 27

5. (SOA) Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2, $f(t_1, t_2)$, is constant over the region $0 < t_1 < 6$, $0 < t_2 < 6$, $t_1 + t_2 < 10$, and zero otherwise. Determine $E[T_1 + T_2]$, the expected time between a car accident and payment of the claim.

A) 4.9 B) 5.0 C) 5.7 D) 6.0 E) 6.7

6. (SOA) A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:
$P(X=0, Y=0) = 0.800$, $P(X=1, Y=0) = 0.050$,
$P(X=0, Y=1) = 0.025$, $P(X=1, Y=1) = 0.125$.
Calculate $Var(Y|X=1)$.
A) 0.13 B) 0.15 C) 0.20 D) 0.51 E) 0.71

7. (SOA) A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold.
$P(X=0, Y=0) = \frac{1}{6}$, $P(X=1, Y=0) = \frac{1}{12}$, $P(X=1, Y=1) = \frac{1}{6}$,
$P(X=2, Y=0) = \frac{1}{12}$, $P(X=2, Y=1) = \frac{1}{3}$, $P(X=2, Y=2) = \frac{1}{6}$.
What is the variance of X ?
A) 0.47 B) 0.58 C) 0.83 D) 1.42 E) 2.58

8. (SOA) Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f(x,y) = \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)} \qquad x > 1, \; y > 1.$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3.
A) $\frac{1}{9}$ B) $\frac{2}{9}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) $\frac{8}{9}$

9. (SOA) An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X, has a marginal density function of 1 for $0 < x < 1$. Given $X = x$, the size of the payment for damage to the other driver's car, Y, has conditional density of 1 for $x < y < x+1$. If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.5?
A) $\frac{3}{8}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) $\frac{7}{8}$ E) $\frac{15}{16}$

10. (SOA) The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f(x,y) = \begin{cases} \frac{6}{125{,}000}(50-x-y) & \text{for } 0 < x < 50-y < 50 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that both components are still functioning 20 months from now?

A) $\frac{6}{125{,}000}\int_0^{20}\int_0^{20}(50-x-y)\,dy\,dx$

B) $\frac{6}{125{,}000}\int_{20}^{30}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

C) $\frac{6}{125{,}000}\int_{20}^{30}\int_{20}^{50-x-y}(50-x-y)\,dy\,dx$

D) $\frac{6}{125{,}000}\int_{20}^{50}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

E) $\frac{6}{125{,}000}\int_{20}^{50}\int_{20}^{50-x-y}(50-x-y)\,dy\,dx$

11. (SOA) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{8}{3}xy & \text{for } 0 \le x \le 1 \text{ , } x \le y \le 2x \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the covariance of X and Y.

A) 0.04 B) 0.25 C) 0.67 D) 0.80 E) 1.24

12. (SOA) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 15y & \text{for } x^2 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

Let g be the marginal density function of Y. Which of the following represents g?

A) $g(y) = \begin{cases} 15y & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

B) $g(y) = \begin{cases} \frac{15y^2}{2} & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$

C) $g(y) = \begin{cases} \frac{15y^2}{2} & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

D) $g(y) = \begin{cases} 15y^{3/2}(1-y^{1/2}) & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$

E) $g(y) = \begin{cases} 15y^{3/2}(1-y^{1/2}) & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

13. (SOA) An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{2x+2-y}{4} & \text{for } 0<x<1 \text{ and } 0<y<2 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that the total loss is at least 1 ?

A) 0.33 B) 0.38 C) 0.41 D) 0.71 E) 0.75

14. (SOA) Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval (0, 12). Given $X = x$, Y is uniformly distributed on the interval $(0, x)$. Determine $Cov(X, Y)$ according to this model.

A) 0 B) 4 C) 6 D) 12 E) 24

15. (SOA) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 24xy & \text{for } 0<x<1 \text{ and } 0 \le y \le 1-x \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $P[Y < X | X = \frac{1}{3}]$.

A) $\frac{1}{27}$ B) $\frac{2}{27}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{4}{9}$

16. (SOA) A joint density function is given by

$$f(x,y) = \begin{cases} kx & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise,} \end{cases}$$ where k is a constant.

What is Cov(X, Y)?

A) $-\frac{1}{6}$ B) 0 C) $\frac{1}{9}$ D) $\frac{1}{6}$ E) $\frac{2}{3}$

17. (SOA) An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

		Annual number of tornadoes in county Q			
		0	1	2	3
Annual number	0	0.12	0.06	0.05	0.02
of tornadoes	1	0.13	0.15	0.12	0.03
in county P	2	0.05	0.15	0.10	0.02

Calculate the conditional variance of the annual number of tornadoes in county Q, given that there are no tornadoes in county P .

A) 0.51 B) 0.84 C) 0.88 D) 0.99 E) 1.76

18. (SOA) A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0<s<1$ and $0<t<1$. What is the probability that the device fails during the first half hour of operation?

A) $\int_0^{0.5}\int_0^{0.5} f(s,t)\,ds\,dt$ B) $\int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$ C) $\int_{0.5}^{1}\int_{0.5}^{1} f(s,t)\,ds\,dt$

D) $\int_0^{0.5}\int_0^{1} f(s,t)\,ds\,dt + \int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$ E) $\int_0^{0.5}\int_{0.5}^{1} f(s,t)\,ds\,dt + \int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$

19. (SOA) A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy. Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f(x,y)=2(x+y)$ on the region where the density is positive. Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

A) 0.010 B) 0.013 C) 0.108 D) 0.417 E) 0.500

20. (SOA) The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } \ 0<x<1,\ x<y<x+1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X=x$?

A) $\frac{1}{12}$ B) $\frac{7}{6}$ C) $x+\frac{1}{2}$ D) $x^2-\frac{1}{6}$ E) $x^2+x+\frac{1}{3}$

21. Let X and Y be continuous random variables having a bivariate normal distribution with means μ_X and μ_Y, common variance σ^2, and correlation coefficient ρ_{XY}. Let F_X and F_Y be the cumulative distribution functions of X and Y respectively. Determine which of the following is a necessary and sufficient condition for $F_X(t) \geq F_Y(t)$ for all t.

A) $\mu_X \geq \mu_Y$ B) $\mu_X \leq \mu_Y$ C) $\mu_X \geq \rho_{XY}\mu_Y$ D) $\mu_X \leq \rho_{XY}\mu_Y$ E) $\rho_{XY} \geq 0$

22. If the joint probability density function of X_1 , X_2 is $f(x_1, x_2) = 1$, for $0 < x_1 < 1$ and $0 < x_2 < 1$, and 0 otherwise, then the moment generating function $M(t_1, t_2)$, $t_1, t_2 \neq 0$ of the joint distribution is

A) $\frac{e^{t_1}-1}{t_1}$ B) $\frac{(e^{t_1}-1)(e^{t_2}-1)}{t_1 t_2}$ C) $\frac{(e^{t_1}+1)(e^{t_2}+1)}{t_1 t_2}$ D) $\frac{1}{t_1 t_2}$ E) $e^{t_1+t_2} - 1$

23. The moment generating function for the joint distribution of random variables X and Y is $M_{X,Y}(t_1, t_2) = \frac{1}{3(1-t_2)} + \frac{2}{3}e^{t_1} \cdot \frac{2}{(2-t_2)}$, for $t_2 < 1$. Find $Var[X]$.

A) $\frac{1}{18}$ B) $\frac{1}{9}$ C) $\frac{1}{6}$ D) $\frac{2}{9}$ E) $\frac{1}{3}$

24. (SOA) A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x, y) = \begin{cases} 6[1-(x+y)] & \text{for } x > 0,\ y > 0,\ x + y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2.

A) 0.360 B) 0.480 C) 0.488 D) 0.512 E) 0.520

25. The distribution of Smith's future lifetime is X, an exponential random variable with mean α, and the distribution of Brown's future lifetime is Y, an exponential random variable with mean β. Smith and Brown have future lifetimes that are independent of one another. Find the probability that Smith outlives Brown.

A) $\frac{\alpha}{\alpha+\beta}$ B) $\frac{\beta}{\alpha+\beta}$ C) $\frac{\alpha-\beta}{\alpha}$ D) $\frac{\beta-\alpha}{\beta}$ E) $\frac{\alpha}{\beta}$

26. X and Y are continuous losses with joint distribution

$$f(x,y) = \begin{cases} \frac{3}{4}(2-x-y) & \text{for } 0<x<2,\ 0<y<2, \text{ and } x+y<2 \\ 0, & \text{otherwise} \end{cases}.$$

An insurance policy pays the total $X+Y$.

Find the expected amount the policy will pay.

A) 0 B) .5 C) 1 D) 1.5 E) 2

27. A pair of loss random variables X and Y have joint density function

$$f(x,y) = \begin{cases} 6xy+3x^2 & \text{for } 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}.$$ Find the probability that the loss Y is no more than .5 .

A) .015625 B) .03125 C) .0625 D) .125 E) .25

28. The joint density function of two random losses X and Y is

$$f(x,y) = \begin{cases} x+y, & \text{for } 0<x<1 \text{ and } 0<y<1 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the probability that loss X is less than double the loss Y.

A) $\frac{7}{32}$ B) $\frac{1}{4}$ C) $\frac{3}{4}$ D) $\frac{19}{24}$ E) $\frac{7}{8}$

29. (SOA) A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails. Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

$$f(x,y) = \begin{cases} 6e^{-x}e^{-2y} & \text{for } 0<x<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time at which the device fails?

A) 0.33 B) 0.50 C) 0.67 D) 0.83 E) 1.50

30. (SOA) Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident. X and Y have joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{64}(10-xy^2) & \text{for } 2 \le x \le 10 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected age of an insured automobile involved in an accident.

A) 4.9 B) 5.2 C) 5.8 D) 6.0 E) 6.4

31. A health insurance policy for a family of three covers up to two claims per person during a year. The joint probability function for the number of claims by the three family members is $f(x,y,z) = \frac{6-x-y-z}{81}$, where x,y,z can each be $0,1$ or 2, and X,Y and Z are the number of claims for person 1, 2 and 3 in the family. Find the probability that the total number of claims for the family in the year is 2 given that person 1 has no claims for the year.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{6}$ D) $\frac{1}{9}$ E) $\frac{1}{12}$

32. (SOA) Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \le t_1 \le t_2 \le L$ where L is a positive constant. Determine the expected value of the sum of the squares of T_1 and T_2 .

A) $\frac{L^2}{3}$ B) $\frac{L^2}{2}$ C) $\frac{2L^2}{3}$ D) $\frac{3L^2}{4}$ E) L^2

33. (SOA) Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200. The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further. Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200. Determine the probability that the company considers the two bids further.

A) 0.10 B) 0.19 C) 0.20 D) 0.41 E) 0.60

34. The distribution of loss due to fire damage to a warehouse is:

Amount of Loss	Probability
0	0.900
500	0.060
1,000	0.030
10,000	0.008
50,000	0.001
100,000	0.001

Given that a loss is greater than zero, calculate the expected amount of the loss.

A) 290 B) 322 C) 1,704 D) 2,900 E) 32,222

35. (SOA) A family buys two policies from the same insurance company. Losses under the two policies are independent and have continuous uniform distributions on the interval from 0 to 10. One policy has a deductible of 1 and the other has a deductible of 2. The family experiences exactly one loss under each policy. Calculate the probability that the total benefit paid to the family does not exceed 5.

A) 0.13 B) 0.25 C) 0.30 D) 0.32 E) 0.42

36. Let X and Y be discrete random variables with joint probability function

$$p(x,y) = \begin{cases} \frac{2x+y}{12} & \text{for } (x,y) = (0,1)\,,\ (0,2)\,,\ (1,2)\,,\ (1,3) \\ 0 & \text{otherwise} \end{cases} .$$

Determine the marginal probability function for X.

A) $p(x) = \begin{cases} \frac{1}{6} & \text{for } x = 0 \\ \frac{5}{6} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$ B) $p(x) = \begin{cases} \frac{1}{4} & \text{for } x = 0 \\ \frac{3}{4} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$

C) $p(x) = \begin{cases} \frac{1}{3} & \text{for } x = 0 \\ \frac{2}{3} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$ D) $p(x) = \begin{cases} \frac{2}{9} & \text{for } x = 1 \\ \frac{3}{9} & \text{for } x = 2 \\ \frac{4}{9} & \text{for } x = 3 \\ 0 & \text{otherwise} \end{cases}$

E) $p(x) = \begin{cases} \frac{y}{12} & \text{for } x = 0 \\ \frac{2+y}{12} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$

37. (SOA) An insurance policy pays a total medical benefit consisting of two parts for each claim. Let X represent the part of the benefit that is paid to the surgeon, and let Y represent the part that is paid to the hospital. The variance of X is 5000, the variance of Y is 10,000, and the variance of the total benefit, $X + Y$, is 17,000. Due to increasing medical costs, the company that issues the policy decides to increase X by a flat amount of 100 per claim and to increase Y by 10% per claim. Calculate the variance of the total benefit after these revisions have been made.

A) 18,200 B) 18,800 C) 19,300 D) 19,520 E) 20,670

E) The correct answer is not given by A, B, C, or D

38. (SOA) Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X) = 5,\ E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$. and $Var(X+Y) = 8$. Let $C_1 = X+Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the application of that surcharge. Calculate $Cov(C_1, C_2)$.

A) 8.80 B) 9.60 C) 9.76 D) 11.52 E) 12.32

39. In reviewing some data on smoking (X, number of packages of cigarettes smoked per year), income (Y, in thousands per year) and health (Z, number of visits to the family physician per year) for a sample of males, it is found that

$E[X] = 10$, $Var[X] = 25$, $E[Y] = 50$, $Var[Y] = 100$, $E[Z] = 6$, $Var[Z] = 4$,

and $Cov(X,Y) = -10$, $Cov(X,Z) = 2.5$ (covariances).

Dr. N.A. Ively, a young statistician, attempts to describe the variable Z in terms of X and Y by the relation $Z = X + cY$, where c is a constant to be determined . Dr. Ively's methodology for determining c is to find the value of c for which $Cov(X,Z)$ remains equal to 2.5 when Z is replaced by $X + cY$. What value of c does Dr. Ively find?

A) 2.00 B) 2.25 C) 2.50 D) -2.00 E) -2.25

40. (SOA) An insurance policy is written to cover a loss X where X has density function

$$f(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

The time (in hours) to process a claim of size x, where $0 \le x \le 2$, is uniformly distributed on the interval from x to $2x$. Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.

A) 0.17 B) 0.25 C) 0.32 D) 0.58 E) 0.83

41. (SOA) An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days. What is the probability that the next claim will be a Deluxe Policy claim?

A) 0.172 B) 0.223 C) 0.400 D) 0.487 E) 0.500

42. (SOA) The joint probability density for X and Y is

$$f(x,y)=\begin{cases}2e^{-(x+2y)} & \text{for } x>0\,,\ y>0\\ 0 & \text{otherwise.}\end{cases}$$

Calculate the variance of Y given that $X>3$ and $Y>3$.

A) 0.25 B) 0.50 C) 1.00 D) 3.25 E) 3.50

43. (SOA) The definition of Y, given X, is uniform on the interval $[0,X]$. The marginal density of X is

$$f(x)=\begin{cases}2x & \text{for } 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

Determine the conditional density of X, given $Y=y$, where positive.

A) 1 B) 2 C) $2x$ D) $\frac{1}{y}$ E) $\frac{1}{1-y}$

44. (SOA) A man purchases a life insurance policy on his 40th birthday, The policy will pay 5000 only if he dies before his 50th birthday and will pay 0 otherwise. The length of lifetime, in years, of a male born the same year as the insured has the cumulative distribution function

$$F(t)=\begin{cases}0 & \text{for } t\le 0\\ 1-e^{(1-1.1^t)/1000} & \text{otherwise}\end{cases}.$$

Calculate the expected payment to the man under this policy.

A) 333 B) 348 C) 421 D) 549 E) 574

45. (SOA) The number of workplace injuries, N, occurring in a factory on any given day is Poisson distributed with mean λ. The parameter λ is a random variable that is determined my the level of activity in the factory, and is uniformly distributed on the interval $[0,3]$.

Calculate $Var[N]$

A) λ B) 2λ C) 0.75 D) 1.50 E) 2.25

46. (SOA) A fair die is rolled repeatedly. Let X be the number of rolls neede to obtain a 5 and Y the number of rolls needed to obtain a 6. Calculate $E(X|Y=2)$.

A) 5.0 B) 5.2 C) 6.0 D) 6.6 E) 6.8

47. Let X and Y be identically distributed independent random variables such that the moment generating function of $X+Y$ is
$M(t) = 0.09e^{-2t} + 0.24e^{-t} + 0.34 + 0.24e^{t} + 0.09e^{2t}$ for $-\infty < t < \infty$.
Calculate $P[X \leq 0]$.
A) .33   B) .34   C) .50   D) .67   E) .70

48. New dental and medical plan options will be offered to state employees next year. An actuary uses the following density function to model the joint distribution of the proportion X of state employees who will choose Dental Option 1 and the proportion Y who will choose Medical Option 1 under the new plan options:

$$f(x,y) = \begin{cases} 0.50, & \text{for } 0 < x < 0.5 \text{ and } 0 < y < 0.5 \\ 1.25, & \text{for } 0 < x < 0.5 \text{ and } 0.5 < y < 1 \\ 1.50, & \text{for } 0.5 < x < 1 \text{ and } 0 < y < 0.5 \\ 0.75, & \text{for } 0.5 < x < 1 \text{ and } 0.5 < y < 1 \end{cases}$$

Calculate $Var(Y|X = 0.75)$.
A) 0.00   B) 0.061   C) 0.076   D) 0.083   E) 0.141

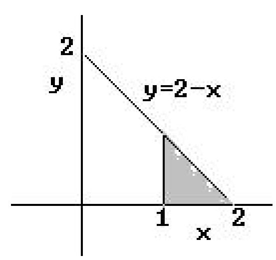
49. The joint density function for the pair of random variables X and Y is
$f(x,y) = \frac{1}{2}e^{-x} \cdot sin\, y$, $0 < x < \infty$, $0 < y < \pi$.
Find $P[(X < 1) \cap (Y < \frac{\pi}{2})]$.
A) $\frac{1-e^{-1}}{2}$ B) $\frac{e-1}{2}$ C) $\frac{2}{e-1}$ D) $\frac{2}{1-e^{-1}}$ E) $\frac{e}{\pi}$

PROBLEM SET 8 SOLUTIONS

1. Let N denote the number that appears on the wheel, so that
$P[N=1]=P[N=2]=P[N=3]=\frac{1}{3}$. Then, conditioning over N,
$P[W\leq 1.5]=P[W\leq 1.5|N=1]\cdot P[N=1]+]P[W\leq 1.5|N=2]\cdot P[N=2]$
$\quad +P[W\leq 1.5|N=3]\cdot P[N=3]$.
If $N=1$ then $W=1$, so that $P[W\leq 1.5|N=1]=1$, and
if $N=2$ then $W=2$, so that $P[W\leq 1.5|N=2]=0$.
If $N=3$ then $W\sim N(3,1)$ so that
$P[W\leq 1.5|N=3]=P[\frac{W-3}{1}\leq\frac{1.5-3}{1}|N=3]=P[Z\leq -1.5]=.07$
(Z has a standard normal distribution - the probability is found from the table).
Then, $P[W\leq 1.5]=1\cdot\frac{1}{3}+0\cdot\frac{1}{3}+(.07)\cdot\frac{1}{3}=.357$. Answer: C

2. This discrete distribution has the following 8 points and probabilities:
$(1,2)$, $\frac{1}{12}$; $(1,4)$, $\frac{1}{6}$; $(1,8)$, $\frac{1}{3}$; $(2,2)$, $\frac{1}{24}$; $(2,4)$, $\frac{1}{12}$; $(2,8)$, $\frac{1}{6}$;
$(4,4)$, $\frac{1}{24}$; $(4,8)$, $\frac{1}{12}$. The event $X+\frac{Y}{2}\leq 5$ occurs at the points
$(1,2)$, $(1,4)$, $(1,8)$, $(2,2)$ and $(2,4)$. The total probability of this event occurring is
$\frac{1}{12}+\frac{1}{6}+\frac{1}{3}+\frac{1}{24}+\frac{1}{12}=\frac{17}{24}$. Answer: E

3. $P[X>1]$
$=\int_1^2\int_0^{2-x}\frac{3}{4}x\,dy\,dx$
$=\int_1^2\frac{3}{4}x(2-x)\,dx$
$=\int_1^2\frac{3}{4}(2x-x^2)\,dx=\frac{1}{2}$.

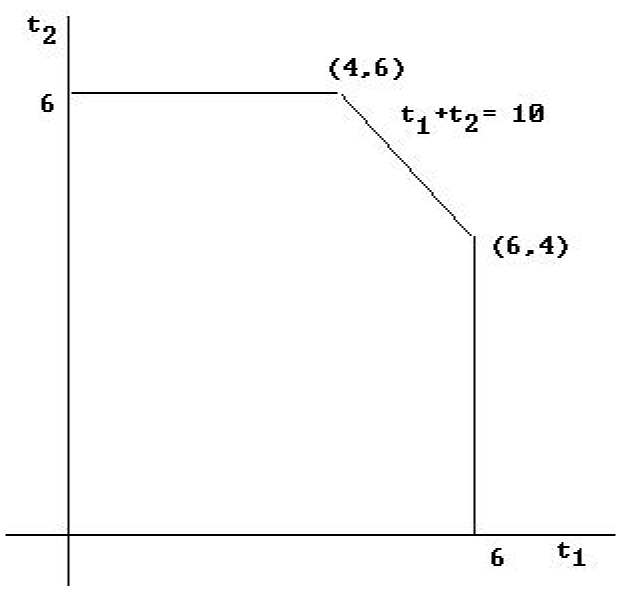


Answer: D

4. It follows from the independence of X and Y that
$E[(X+1)^2(Y-1)^2]=E[(X+1)^2]\cdot E[(Y-1)^2]$.
$E[(X+1)^2]=E[X^2+2X+1]=E[X^2]+2E[X]+1$, and since
$\sigma_X^2=E[X^2]-(E[X])^2$, we have $E[X^2]=\sigma_X^2+(E[X])^2=\frac{3}{2}$, and then
$E[(X+1)^2]=E[X^2+2X+1]=\frac{3}{2}+2(1)+1=\frac{9}{2}$.
In a similar way, $E[Y^2]=\sigma_Y^2+(E[Y])^2=3$, and

4. continued
$E[(Y-1)^2] = E[Y^2 - 2Y + 1] = 3 - 2(-1) + 1 = 6$,
so that $E[(X+1)^2(Y-1)^2] = E[(X+1)^2] \cdot E[(Y-1)^2] = \frac{9}{2} \cdot 6 = 27$.
Note that we could also find $E[(X+1)^2]$ in the following way:
$(X+1)^2 = X^2 + 2X + 1 = X^2 - 2X + 1 + 4X = (X-1)^2 + 4X$, and then
$E[(X+1)^2] = E[(X-1)^2] + 4E[X] = \sigma_X^2 + 4\mu_X = \frac{9}{2}$ (since
$Var[X] = E[(X-\mu_X)^2]$, and $\mu_X = 1$). $E[(Y-1)^2]$ can be found in a similar way.
Answer: E

5. Since the joint density is a constant, say c, over the probability region, and since the total probability in the region must be 1, it follows that $c \times$ (Region Area) $= 1$,
so that $c = \frac{1}{Region\ Area}$. The area of the region is the area of the 6 by 6 square minus the area of the upper right triangle. This is $36 - \frac{1}{2} \times 2 \times 2 = 34$, so that $c = \frac{1}{34}$.
Then, $E[T_1 + T_2] = \int_0^4 \int_0^6 (t_1 + t_2)(\frac{1}{34})\, dt_2\, dt_1 + \int_4^6 \int_0^{10-t_1} (t_1 + t_2)(\frac{1}{34})\, dt_2\, dt_1$.
$\int_0^6 (t_1 + t_2)(\frac{1}{34})\, dt_2 = (6t_1 + 18)(\frac{1}{34}) \Rightarrow \int_0^4 (6t_1 + 18)(\frac{1}{34})\, dt_1 = (48 + 72)(\frac{1}{34})$.
$\int_0^{10-t_1} (t_1 + t_2)(\frac{1}{34})\, dt_2 = [(10 - t_1)t_1 + \frac{(10-t_1)^2}{2}](\frac{1}{34}) = (50 - \frac{1}{2}t_1^2)(\frac{1}{34})$
$\rightarrow \int_4^6 (50 - \frac{1}{2}t_1^2)(\frac{1}{34})\, dt_1 = (\frac{224}{3})(\frac{1}{34})$.
Then, $E[T_1 + T_2] = (48 + 72)(\frac{1}{34}) + (\frac{224}{3})(\frac{1}{34}) = 5.73$.



Answer: C

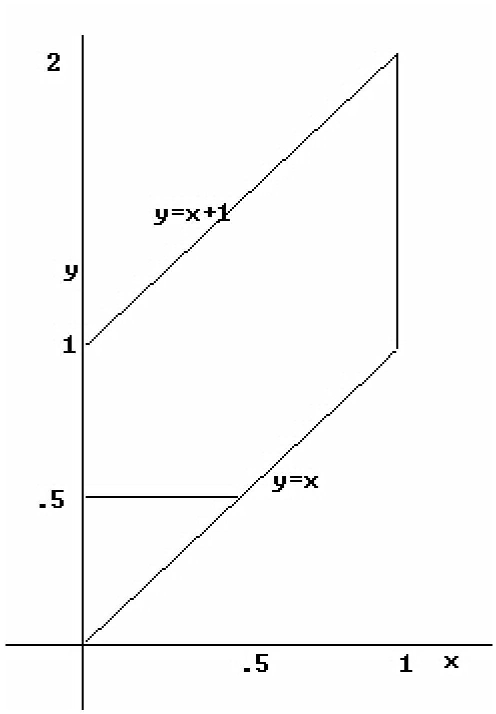
6. We first find the conditional distribution of Y given $X = 1$.
$P[Y=0|X=1] = \frac{P[X=1,Y=0]}{P[X=1]} = \frac{.05}{P[X=1]}$, $P[Y=1|X=1] = \frac{.125}{P[X=1]}$;
this requires $P[X=1] = P[X=1,Y=0] + P[X=1,Y=1] = .05 + .125 = .175$.
The conditional distribution of Y given $X = 1$ is
$P[Y=0|X=1] = \frac{P[X=1,Y=0]}{P[X=1]} = \frac{.05}{.175} = \frac{2}{7}$, $P[Y=1|X=1] = \frac{.125}{.175} = \frac{5}{7}$.
The conditional variance is $Var[Y|X=1] = E[Y^2|X=1] - (E[Y|X=1])^2$, where
$E[Y^2|X=1] = (0^2)(\frac{2}{7}) + (1^2)(\frac{5}{7}) = \frac{5}{7}$, $E[Y|X=1] = (0)(\frac{2}{7}) + (1)(\frac{5}{7}) = \frac{5}{7}$.
Then $Var[Y|X=1] = \frac{5}{7} - (\frac{5}{7})^2 = .204$. Answer: C

7. The marginal distribution of X is found by summing probabilities over the other variable Y.
$P[X=0] = \sum_{y=0}^{2} P[X=0, Y=y] = \frac{1}{6} + 0 + 0 = \frac{1}{6}$,
$P[X=1] = \sum_{y=0}^{2} P[X=1, Y=y] = \frac{1}{12} + \frac{1}{6} + 0 = \frac{1}{4}$,
$P[X=2] = \sum_{y=0}^{2} P[X=2, Y=y] = \frac{1}{12} + \frac{1}{3} + \frac{1}{6} = \frac{7}{12}$.
$E[X] = \sum_{x=0}^{2} x \cdot P[X=x] = (0)(\frac{1}{6}) + (1)(\frac{1}{4}) + (2)(\frac{7}{12}) = \frac{17}{12}$,
$E[X^2] = \sum_{x=0}^{2} x^2 \cdot P[X=x] = (0^2)(\frac{1}{6}) + (1^2)(\frac{1}{4}) + (2^2)(\frac{7}{12}) = \frac{31}{12}$,
$Var[X] = E[X^2] - (E[X])^2 = \frac{31}{12} - (\frac{17}{12})^2 = .576$. Answer: B

8. We wish to find $P[1 < Y < 3|X=2] = \int_1^3 f(y|X=2)\, dy = \int_1^3 \frac{f(2,y)}{f_X(2)}\, dy$.
$f(2,y) = \frac{2}{2^2(2-1)} \cdot y^{-[(2)(2)-1]/(2-1)} = \frac{1}{2} y^{-3}$,
$f_X(2) = \int_1^\infty f(2,y)\, dy = \int_1^\infty \frac{1}{2} y^{-3}\, dy = \frac{1}{4} \Rightarrow \frac{f(2,y)}{f_X(2)} = 2y^{-3}$.
$P[1 < Y < 3|X=2] = \int_1^3 2y^{-3}\, dy = -y^{-2}\Big|_{y=1}^{y=3} = -\frac{1}{9} - (-1) = \frac{8}{9}$.
Answer: E

9. We are given that the marginal distribution of X is uniform on the interval $(0,1)$, so that $f_X(x) = 1$ for $0 < x < 1$. We are also given that the conditional distribution of Y given $X = x$ is uniform on the interval $x < y < x+1$, so that $f_{Y|X}(y|X=x) = 1$ for $x < y < x+1$. The density function for the joint distribution of X and Y is $f_{X,Y}(x,y) = f_{Y|X}(y|X=x) \cdot f_X(x) = 1$ on the region $0 < x < y < x+1 < 2$; this region of probability is the parallelogram in the graph below .

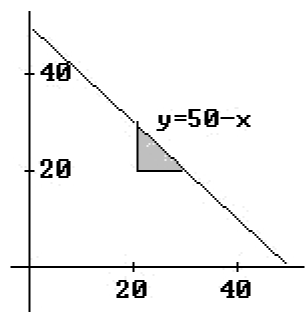
The event "damage to the other driver's car will be greater than .5" when an accident occurs is the event "$Y > .5$" . This is the upper region of the parallelogram below. The probability is the double integral of the joint distribution density on that region. We can also find the probability by first finding the probability of the lower triangular region and then taking the complement.



$P[Y > .5] = 1 - P[Y \leq .5]$.

$P[Y \leq .5] = \int_0^{.5}\int_x^{.5} 1\,dy\,dx = \int_0^{.5}(.5-x)\,dx = \frac{1}{8} \quad \Rightarrow \quad P[Y > .5] = \frac{7}{8}$. Answer: D

10. The region of joint density is the region in the first quadrant below the line $y = 50 - x$ (with horizontal intercept 50 and vertical intercept 50). If X and Y denote the failure time of the two components, then the event that both components are still functioning 20 months from now has probability $P[(X > 20) \cap (Y > 20)]$. This is the shaded region in the graph at the right. From the graph it can be seen that the region of probability for this event is the triangular region bounded on the left by $x = 20$ and on the right by $x = 30$, and bounded below by $y = 20$, and bounded above by $y = 50 - x$.
The probability of the event is $\int_{20}^{30}\int_{20}^{50-x} f(x,y)\,dy\,dx$.



Answer: B

11. $Cov(X,Y) = E[XY] - E[X] \cdot E[Y]$.
We use the expression $E[X] = \int\int x \cdot f(x,y)\,dy\,dx$ to find $E[X]$. Since the region of probability is defined with $x \leq y \leq 2x$, we apply double integration in the $dy\,dx$ order. It would be possible to reverse the order, but that would not make the solution any more efficient. $E[Y]$ and $E[XY]$ are found in a similar way.

$$E[X] = \int_0^1\int_x^{2x} x \cdot \tfrac{8}{3} \cdot xy\,dy\,dx = \tfrac{8}{3} \cdot \int_0^1\int_x^{2x} x^2y\,dy\,dx$$
$$= \tfrac{8}{3} \cdot \int_0^1 [\tfrac{x^2y^2}{2}\Big|_{y=x}^{y=2x}]\,dx = \tfrac{8}{3} \cdot \int_0^1 [\tfrac{3x^4}{2}]\,dx = \tfrac{8}{3} \cdot \tfrac{3}{2} \cdot \tfrac{1}{5} = \tfrac{4}{5} \ .$$
$$E[Y] = \int_0^1\int_x^{2x} y \cdot \tfrac{8}{3} \cdot xy\,dy\,dx = \tfrac{8}{3} \cdot \int_0^1\int_x^{2x} xy^2\,dy\,dx$$
$$= \tfrac{8}{3} \cdot \int_0^1 [\tfrac{xy^3}{3}\Big|_{y=x}^{y=2x}]\,dx = \tfrac{8}{3} \cdot \int_0^1 [\tfrac{7x^4}{3}]\,dx = \tfrac{8}{3} \cdot \tfrac{7}{3} \cdot \tfrac{1}{5} = \tfrac{56}{45} \ .$$
$$E[XY] = \int_0^1\int_x^{2x} xy \cdot \tfrac{8}{3} \cdot xy\,dy\,dx = \tfrac{8}{3} \cdot \int_0^1\int_x^{2x} x^2y^2\,dy\,dx$$
$$= \tfrac{8}{3} \cdot \int_0^1 [\tfrac{x^2y^3}{3}\Big|_{y=x}^{y=2x}]\,dx = \tfrac{8}{3} \cdot \int_0^1 [\tfrac{7x^5}{3}]\,dx = \tfrac{8}{3} \cdot \tfrac{7}{3} \cdot \tfrac{1}{6} = \tfrac{28}{27} \ .$$

Then $Cov(X,Y) = \frac{28}{27} - (\frac{4}{5})(\frac{56}{45}) = .041$. Answer: A

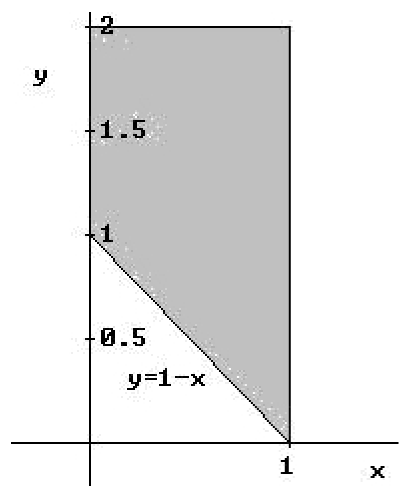
12. The range $x^2 \le y \le x$ is only valid for $0 \le x \le 1$. This is true since $x^2 > x$ for $x > 1$ and $x^2 > 0 > x$ for $x < 0$. Therefore, the range for y is $0 \le x^2 \le y \le x \le 1$, so that $0 \le y \le 1$. Also, the inequality $x^2 \le y$ is equivalent to $x \le \sqrt{y}$, so that $x^2 \le y \le x$ is equivalent to $y \le x \le \sqrt{y}$. The marginal density function of Y is found by integrating the joint density over the range for the other variable x;

$g(y) = \int_y^{\sqrt{y}} 15y\,dx = 15yx\Big|_{x=y}^{x=\sqrt{y}} = 15y(\sqrt{y} - y) = 15(y^{3/2} - y^2)$ for $0 \le y \le 1$.

Note that it is true that for any particular x we have $x^2 \le y \le x$. However, since x can be any number from 0 to 1, y can also be any number from 0 to 1. Answer: E

13. We are asked to find $P[X + Y \ge 1]$. The joint distribution of X and Y is defined on the rectangle $0 < X < 1$, $0 < Y < 2$. The region representing the probability in question is the region on and above the line $x + y = 1$ that is inside the rectangle. The probability is the double integral of the joint density function, integrated over the region of probability. This can be expressed as

$\int_0^1 \int_{1-x}^2 f(x,y)\,dy\,dx = \int_0^1 \int_{1-x}^2 \frac{2x+2-y}{4}\,dy\,dx$, which becomes $\int_0^1 \frac{5x^2+6x+1}{8}\,dx = \frac{17}{24} = .71$.



Answer: D

14. $f_{Y|X}(y|X = x) = \frac{1}{x}$ for $0 < y < x$, $f_X(x) = \frac{1}{12}$ for $0 < x < 12$.

$f_{X,Y}(x,y) = f_{Y|X}(y|X = x) \cdot f_X(x) = \frac{1}{12x}$ for $0 < y < x < 12$.

Then $E[XY] = \int_0^{12}\int_0^x xy \cdot \frac{1}{12x}\,dy\,dx = \int_0^{12}\int_0^x \frac{y}{12}\,dy\,dx = 24$.

We are given that X has a uniform distribution on $(0,12)$, and therefore, $E[X] = 6$.

Also, $E[Y] = \int_0^{12}\int_0^x y \cdot \frac{1}{12x}\,dy\,dx = \int_0^{12} \frac{x}{24}dx = 3$ (note that if we had used the reverse order of integration then we would have

$E[Y] = \int_0^{12}\int_y^{12} y \cdot \frac{1}{12x}\,dx\,dy = \int_0^{12} \frac{y}{12}\,ln(\frac{12}{y})\,dy$, which would require integration by parts).

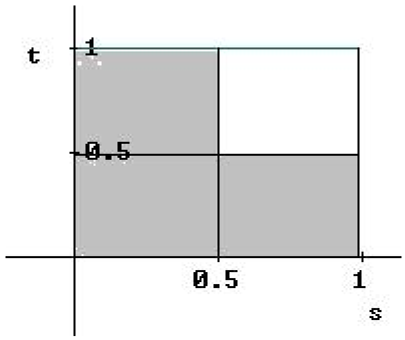
Finally, $Cov(X,Y) = 24 - (6)(3) = 6$. Answer: C

15. If we find the conditional density function $f_{Y|X}(y|X=\frac{1}{3})$, then
$P[Y<X|X=\frac{1}{3}]=P[Y<\frac{1}{3}|X=\frac{1}{3}]=\int_0^{1/3} f_{Y|X}(y|X=\frac{1}{3})\,dy$.
The conditional density is $f_{Y|X}(y|X=\frac{1}{3})=\frac{f(\frac{1}{3},y)}{f_X(\frac{1}{3})}$.
The joint density is $f(\frac{1}{3},y)=24(\frac{1}{3})y=8y$, $0<y<1-\frac{1}{3}$,
and the marginal density of X at $X=\frac{1}{3}$ is $f_X(\frac{1}{3})=\int_0^{2/3} 24(\frac{1}{3})y\,dy=\frac{16}{9}$.
The conditional density is $f_{Y|X}(y|X=\frac{1}{3})=\frac{8y}{\frac{16}{9}}=\frac{9y}{2}$.
The conditional probability is $P[Y<X|X=\frac{1}{3}]=\int_0^{1/3}\frac{9y}{2}\,dy=\frac{1}{4}$. Answer: C

16. We first find the value of the constant k that makes $f(x,y)$ a properly defined density function for the joint distribution. The requirement that must be met is that the double integral of $f(x,y)$ over the x-y region of density must be 1. The x-y region of density is the square $0<x<1$, $0<y<1$. Thus,
$\int_0^1\int_0^1 kx\,dy\,dx=k\int_0^1 x\,dx=k(\frac{1}{2})=1$, from which we get $k=2$ and $f(x,y)=2x$.
We use following definition of $Cov(X,Y)$: $Cov(X,Y)=E[XY]-E[X]\cdot E[Y]$.
$E[XY]=\int_0^1\int_0^1(xy)f(x,y)\,dy\,dx=\int_0^1\int_0^1(xy)(2x)\,dy\,dx=\frac{1}{3}$,
$E[X]=\int_0^1\int_0^1(x)f(x,y)\,dy\,dx=\int_0^1\int_0^1(x)(2x)\,dy\,dx=\frac{2}{3}$, and
$E[Y]=\int_0^1\int_0^1(y)f(x,y)\,dy\,dx=\int_0^1\int_0^1(y)(2x)\,dy\,dx=\frac{1}{2}$.
Then, $Cov(X,Y)=\frac{1}{3}-\frac{2}{3}\cdot\frac{1}{2}=0$.
There is another way that this covariance of 0 could have been found.
The density function of the marginal distribution of X is $f_X(x)=\int_0^1 2x\,dy=2x$ for $0<x<1$, and the density function of the marginal distribution of Y is $f_Y(y)=\int_0^1 2x\,dx=1$ for $0<y<1$. We can then see that $f(x,y)=2x=(2x)(1)=f_X(x)\cdot f_Y(y)$, which indicates that X and Y are independent. If two random variables are independent, then they have covariance of 0. Answer: B

17. The distribution of the number of tornadoes in county Q given there are none in county P is
$P[Q=n|P=0]=\frac{P[(Q=n)\cap(P=0)]}{P[P=0]}$, for $n=0,1,2,3$.
The denominator is $P[P=0]=P[(Q=0)\cap(P=0)]+P[(Q=1)\cap(P=0)]$
$+P[(Q=2)\cap(P=0)]+P[(Q=3)\cap(P=0)]=.12+.06+.05+.02=.25$.
Then, $P[Q=0|P=0]=\frac{P[(Q=0)\cap(P=0)]}{P[P=0]}=\frac{.12}{.25}=.48$, $P[Q=1|P=0]=.24$,
$P[Q=2|P=0]=.20$, $P[Q=3|P=0]=.08$.
Then, $E[Q|P=0]=(0)(.48)+(1)(.24)+(2)(.2)+(3)(.08)=.88$,
$E[Q^2|P=0]=(0^2)(.48)+(1^2)(.24)+(2^2)(.2)+(3^2)(.08)=1.76$, and
$Var[Q|P=0]=E[Q^2|P=0]-(E[Q|P=0])^2=1.76-(.88)^2=.9856$. Answer: D

18. Suppose that the times of failure of the two devices are X and Y.
We wish to find the probability that at least one failure occurs by time .5 .
This is $P[(X < .5) \cup (Y < .5)]$. The region of probability is the shaded in the graph below. The probability is the double integral over region E (let t be the horizontal, s vertical).
The first integral corresponds to the square $.5 < s < 1$, $0 < t < .5$, and the second integral corresponds to the rectangle $0 < s < .5$, $0 < t < 1$.



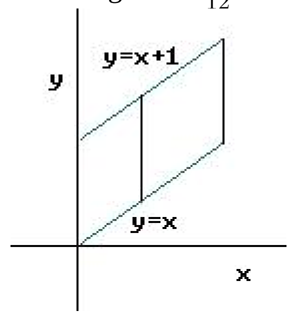
Answer: E

19. We must have $Y \le X$ since no more than proportion X buy the supplementary policy. The region of positive joint density is the triangular region $0 \le y \le x \le 1$
(below the line $y = x$, inside the unit square). We wish to find the conditional probability $P[Y < .05|X = .1]$. The conditional density for Y given $X = .1$ is
$f_{Y|X}(y|X = .1) = \frac{f(.1,y)}{f_X(.1)}$, where $f_X(.1)$ is the density function of the marginal distribution of X at .1 . In general, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ gives the density of the marginal distribution of X from a joint distribution. In this case, based on where the joint density is non-zero, we have
$f_X(.1) = \int_0^{.1} f(.1,y)\,dy = \int_0^{.1} 2(.1+y)\,dy = .03$.
Then, $f_{Y|X}(y|X = .1) = \frac{f(.1,y)}{f_X(.1)} = \frac{2(.1+y)}{.03}$, $y < .1$ (since $Y < X$), and
$P[Y < .05|X = .1] = \int_0^{.05} \frac{2(.1+y)}{.03}\,dy = .4167$. Answer: D

20. To find the conditional variance of Y given that $X = x$, we must find the density function of the conditional distribution of Y given $X = x$. This is
$f_{Y|X}(y|X = x) = \frac{f(x,y)}{f_X(x)}$. We must find the density function of the marginal distribution of X. This is found by integrating the joint distribution density with respect to y over the appropriate region: $f_X(x) = \int_x^{x+1} 2x\,dy = 2x$ for $0 < x < 1$.
Then, $f_{Y|X}(y|X = x) = \frac{2x}{2x} = 1$ and this conditional density is valid for $x < y < x + 1$.

20. continued

Therefore, the conditional distribution of Y given $X = x$ is uniform on the interval $x < y < x + 1$ (a uniform distribution has a constant density). The variance of the continuous uniform distribution on an interval of length 1 is $\frac{1}{12}$.



Answer: A

21. $F_X(t) = P[X \le t] = P[\frac{X-\mu_X}{\sigma} \le \frac{t-\mu_X}{\sigma}] = P[W \le \frac{t-\mu_X}{\sigma}]$, where $W \sim N(0,1)$

$F_Y(t) = P[Y \le t] = P[\frac{Y-\mu_Y}{\sigma} \le \frac{t-\mu_Y}{\sigma}] = P[V \le \frac{t-\mu_Y}{\sigma}]$, where $V \sim N(0,1)$.

$F_X(t) \ge F_Y(t)$ is equivalent to $\frac{t-\mu_X}{\sigma} \ge \frac{t-\mu_Y}{\sigma}$, which is equivalent to $\mu_X \le \mu_Y$.

Note that the fact the X and Y have a bivariate distribution with correlation coefficient ρ_{XY} is irrelevant - we are comparing probabilities of the marginal distributions of X and Y (however, we do use the fact that X and Y have common variance σ^2). Answer: B

22. The moment generating function of X_1 and X_2 is

$M(t_1, t_2) = E[e^{t_1X_1+t_2X_2}] = \int_0^1\int_0^1 e^{t_1x_1+t_2x_2}dx_1\,dx_2 = \frac{(e^{t_1}-1)(e^{t_2}-1)}{t_1t_2}$. Answer: B

23. The moment generating function for X is $M_X(t_1) = M(t_1, 0) = \frac{1}{3} + \frac{2}{3}e^{t_1}$.

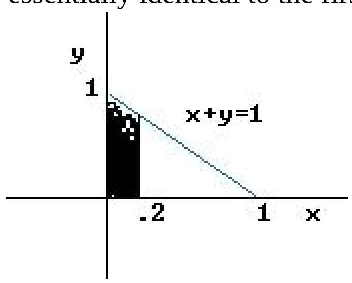
Then, $E[X] = M'_X(0) = \frac{2}{3}$, and $E[X^2] = M''_X(0) = \frac{2}{3}$, so that

$Var[X] = \frac{2}{3} - (\frac{2}{3})^2 = \frac{2}{9}$. Answer: D

24. We wish to find $P[X < .2]$. The region of density for the joint distribution is below the line $x + y = 1$. The region of probability for the event in question is shaded below. The probability is found by integrating the joint density over that region.
$P[X < .2] = \int_0^{.2}\int_0^{1-x} 6[1-(x+y)]\,dy\,dx = \int_0^{.2} 3(1-x)^2\,dx = .488$.

It would also be possible to solve this problem by first finding the marginal distribution of X, and then find $P[X < .2]$. The density function of the marginal distribution of X is
found by integrating the joint density in the "y-direction" over the appropriate range.
Since $x + y < 1$ is equivalent to $y < 1 - x$, the appropriate range for integration over y is from $y = 0$ to $y = 1 - x$. Therefore,
$f_X(x) = \int_0^{1-x} f(x,y)\,dy = \int_0^{1-x} 6[1-(x+y)]\,dy = 3(1-x)^2$.
This is exactly the "inside integral" in the double integration above.
Then, $P[X < .2] = \int_0^{.2} 3(1-x)^2\,dx = .488$, as before.

This second approach is essentially identical to the first approach.



Answer: C

25. $P[Y < X] = \int_0^{\infty}\int_y^{\infty} f_X(x)f_Y(y)\,dx\,dy$ (since X and Y are independent, the joint density function of X and Y is the product of the two separate density functions).
The density function of X is $\frac{1}{\alpha}e^{-x/\alpha}$, and of Y is $\frac{1}{\beta}e^{-x/\beta}$, so that
$P[Y < X] = \int_0^{\infty}\int_y^{\infty} \frac{1}{\alpha}e^{-x/\alpha}\frac{1}{\beta}e^{-y/\beta}dx\,dy = \int_0^{\infty}\frac{1}{\beta}e^{-y/\beta}e^{-y/\alpha}\,dy = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{\alpha}{\alpha+\beta}$.

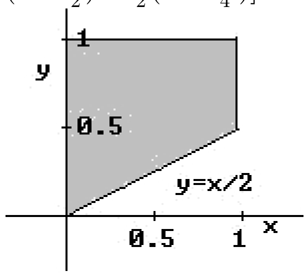
Answer: A

26. The marginal density function of X is $f_X(x) = \int_0^{2-x}\frac{3}{4}(2-x-y)\,dy = \frac{3}{8}(2-x)^2$,
and then $E[X] = \int_0^2 x\cdot\frac{3}{8}(2-x)^2dx = .5$. In a similar way, the marginal density function of Y is $f_Y(y) = \int_0^{2-y}\frac{3}{4}(2-x-y)\,dx = \frac{3}{8}(2-y)^2$,
and then $E[Y] = \int_0^2 y\cdot\frac{3}{8}(2-y)^2dy = .5$ (this could be anticipated from the symmetry of x and y in the joint density function). Then, $E[X+Y] = 1$. Alternatively, we can find
$E[X+Y] = \int_0^2\int_0^{2-x}(x+y)\cdot\frac{3}{4}(2-x-y)\,dy\,dx = 1.$ Answer: C

27. The marginal density function of Y is $f_Y(y) = \int_0^y (6xy + 3x^2)dx = 4y^3$, and then $P[Y < .5] = \int_0^{.5} 4y^3\,dy = (.5)^4$. Answer: C

28. The shaded region in the graph below corresponds to the event that $X < 2Y$. The probability is $P[X < 2Y] = P[\frac{X}{2} < Y] = \int_0^1 \int_{x/2}^1 (x+y)\,dy\,dx$

$= \int_0^1 [x(1-\frac{x}{2}) + \frac{1}{2}(1-\frac{x^2}{4})]dx = \frac{19}{24}$.



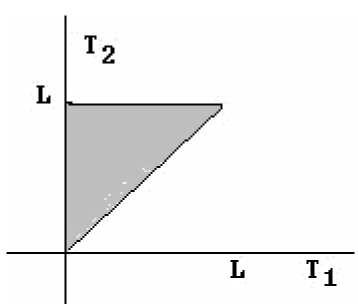
Answer: D

29. The device fails when the second circuit fails, which is at time Y. We wish to find $E[Y]$. This is $E[Y] = \int_0^\infty \int_x^\infty y \cdot 6e^{-x}e^{-2y}\,dy\,dx = \int_0^\infty 6e^{-x}(\int_x^\infty y \cdot e^{-2y}\,dy)dx$.
We find $\int_x^\infty y \cdot e^{-2y}\,dy$ by integration by parts :
$\int_x^\infty y \cdot e^{-2y}\,dy = \int_x^\infty y\,d(-\frac{1}{2}e^{-2y}) = -\frac{1}{2}ye^{-2y}\Big|_{y=x}^{y=\infty} - \int_x^\infty (-\frac{1}{2}e^{-2y})dy$
$= -0 + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x}$.
Then, $E[Y] = \int_0^\infty 6e^{-x}(\frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x})\,dx = \int_0^\infty (3xe^{-3x} + \frac{3}{2}e^{-3x})\,dx$.
From integration by parts, we get $\int_0^\infty 3xe^{-3x}\,dx = \int_0^\infty x\,d(-e^{-3x}) = -\frac{1}{3}xe^{-3x}\Big|_{x=0}^{x=\infty} = \frac{1}{3}$.
Also, $\int_0^\infty \frac{3}{2}e^{-3x}\,dx = \frac{1}{2}$, so that $E[Y] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.
Alternatively, we can find the pdf of the marginal distribution of Y first:
$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 6e^{-x}e^{-2y}\,dx = 6(1-e^{-y})e^{-2y} = 6(e^{-2y} - e^{-3y})$, for $0 < y < \infty$.
Then, $E[Y] = \int_0^\infty y \cdot 6(e^{-2y} - e^{-3y})\,dy$. After integration by parts, this becomes $\frac{5}{6}$.
Answer: D

30. $E[X] = \int_2^{10}\int_0^1 x \cdot \frac{1}{64}(10 - xy^2)\,dy\,dx$.
The "inside" integral is
$\int_0^1 x \cdot \frac{1}{64}(10 - xy^2)\,dy = \frac{1}{64} \cdot \int_0^1 (10x - x^2y^2)\,dy = \frac{1}{64} \cdot [10x - \frac{x^2}{3}]$.
The complete integral is
$\int_2^{10} \frac{1}{64} \cdot [10x - \frac{x^2}{3}]\,dx = \frac{1}{64} \cdot [(5x^2 - \frac{x^3}{9})\Big|_{x=2}^{x=10}] = 5.8$.
Note that we could have found $f_X(x)$, the marginal density function of X first and the have found $E[X]$. This would be done as follows:
$f_X(x) = \int_0^1 \frac{1}{64}(10 - xy^2)\,dy = \frac{1}{64}(10 - \frac{x}{3})$, and then
$E[X] = \int_2^{10} x \cdot f_X(x)\,dx = \int_2^{10} x \cdot \frac{1}{64}(10 - \frac{x}{2})\,dx = \frac{1}{64} \cdot \int_2^{10}(10x - \frac{x^2}{3})\,dx$
$= \frac{1}{64} \cdot (5x^2 - \frac{x^3}{9})\Big|_{x=2}^{x=10}] = 5.8$ (as in the first approach). Answer: C

31. $P[X + Y + Z = 2|X = 0] = \frac{P[(Y+Z=2)\cap(X=0)]}{P[X=0]}$.
$P[X = 0] = \sum_{y=0}^{2}\sum_{z=0}^{2} f(0, y, z) = \frac{1}{81}[(6 - 0) + (6 - 1) + (6 - 2)$
$+ (6 - 1) + (6 - 2) + (6 - 3) + (6 - 2) + (6 - 3) + (6 - 4)] = \frac{36}{81}$.
$P[(Y + Z = 2) \cap (X = 0)] = f(0, 0, 2) + f(0, 1, 1) + f(0, 2, 0) = \frac{4}{81} + \frac{4}{81} + \frac{4}{81}$.
$P[X + Y + Z = 2|X = 0] = \frac{12/81}{36/81} = \frac{1}{3}$. Answer: B

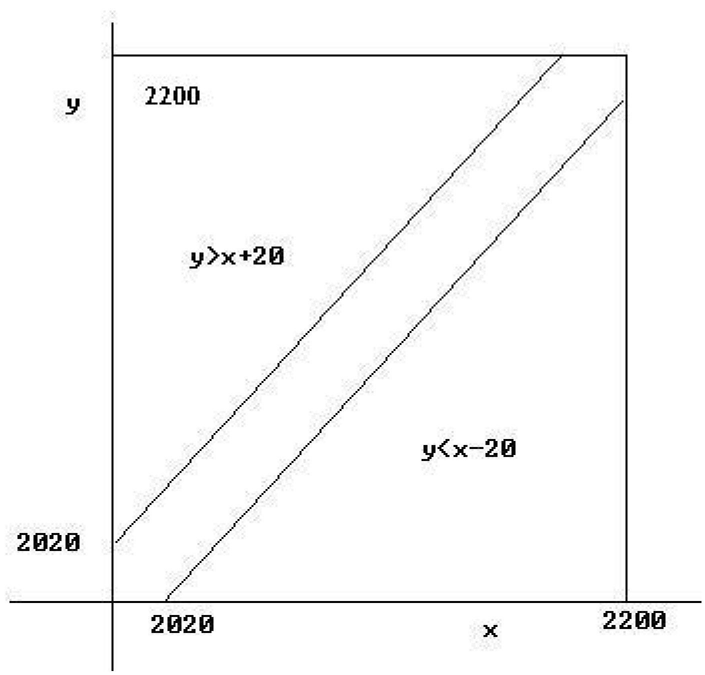
32. The graph at the right indicates the region of non-zero density for the joint distribution of T_1 and T_2. The expected value is $\int_0^L\int_0^{t_2}(t_1^2 + t_2^2)\,f(t_1, t_2)\,dt_1\,dt_2$. We are told that the joint distribution is uniform over the triangular region, and therefore the joint density function $f(t_1, t_2)$ is constant over the region and numerically equal to $\frac{1}{\text{area of region}} = \frac{2}{L^2}$ (half of the $L \times L$ square). The expected value is $\int_0^L\int_0^{t_2}(t_1^2 + t_2^2) \cdot \frac{2}{L^2}\,dt_1\,dt_2 = \frac{2}{L^2} \cdot \int_0^L \frac{4t_2^3}{3}\,dt_2 = \frac{2L^2}{3}$.



Answer: C

33. The company considers the two bids further if the two bids are within 20 of one another. If we let X be the amount of the first bid and Y the amount of the second bid, then the (x, y) region for which the company will consider the bids further satisfies
$x - 20 < y < x + 20$. This is the complement of the union of the two regions
$(y \leq x - 20) \cup (y \geq x + 20)$. We are told that both X and Y are uniformly distributed between 2000 and 2200, so that $f_X(x) = \frac{1}{200} = .005$ for $2000 \leq x \leq 2200$, and
$f_Y(y) = \frac{1}{200} = .005$ for $2000 \leq y \leq 2200$. Since X and Y are independent, the density function of the joint distribution is $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) = (.005)^2$ for
$2000 \leq x \leq 2200$ and $2000 \leq y \leq 2200$. The probability of a two-dimensional region is the double integral of the joint density over the region. Since the joint density is constant, the probability is the region of the area multiplied by that constant.

The region $y \leq x - 20$ (and inside the square) has probability
$\frac{1}{2} \times 180 \times 180 \times (.005)^2 = .405$ (since the triangular region below the lower line has base 180 and height 180). The region $y \geq x + 20$ also has probability $\frac{1}{2} \times 180 \times 180 \times (.005)^2 = .405$ (same size triangle). Therefore,
$P[\text{the two bids are within 20 of one another}] = P[X - 20 < Y < X + 20]$
$= 1 - .405 - .405 = .19$.



Answer: B

34. Conditional expectations are usually found as follows.
$E[W|W>a] = \frac{\int_a^\infty w\, f_W(w)\, dw}{P[W>a]}$ (with summation used in the discrete case), with a similar formulation for $E[W|W<a]$ or $E[W|a<W<b]$. In the specific case that W is a non-negative random variable ($W \geq 0$), we have $E[W|W>0] = \frac{\Sigma w\, f(w)}{P[W>0]}$ (in the discrete case), and notice that the numerator is $E[W]$ (unconditional expectation), so that $E[W|W>0] = \frac{E[W]}{P[W>0]}$. In this case,
$E[L] = (0)(.9) + (500)(.06) + \cdots + (100,000)(.001) = 290$, and $P[L>0] = .1$,so that the conditional expectation becomes $E[L|L>0] = \frac{290}{.1} = 2900$.

The problem can be solved in an alternative way. We first determine the conditional distribution of L given that $L>0$ (where L denotes that amount of the loss). L has a discrete distribution, and the probability function of L given that $L>0$ is found from the following relationship.
For $x>0$, $P[L=x|L>0] = \frac{P[L=x]}{P[L>0]} = \frac{P[L=x]}{.1} = 10P[L=x]$.
The conditional distribution of L given $L>0$ is

x	$P[L=x\|L>0]$
500	.600
$1,000$	.300
$10,000$	.080
$50,000$	.010
$100,000$	.010

The expected amount of the loss given that the loss is greater than 0 is the expectation of this conditional distribution of L given that $L>0$. This expectation is
$E[L|L>0] = (500)(.6) + (1,000)(.3) + (10,000)(.08)$
$+ (50,000)(.01) + (100,000)(.01) = 2,900$. Answer: D

35. Let X and Y denote the two loss amounts (<u>not</u> payment amounts).
We consider the following combinations of X and Y that result in the total benefit <u>payment</u> not exceeding 5.

Case 1: $0 < X \leq 1$ (so loss X results in no payment) and $0 < Y \leq 7$ (so that loss Y results in a maximum payment of 5 after applying the deductible of 2).

Case 2: $1 < X \leq 6$ (so loss X results in a maximum payment of 5 after the deductible of 1 is applied) and $0 < Y \leq 2$ (so loss Y results in no payment).

Case 3: $1 < X \leq 6$ and $2 \leq Y \leq 7$ and $(X-1)+(Y-2) \leq 5$ ($X-1$ is paid for loss X and $Y-2$ is paid for loss Y). The last condition is equivalent to $X+Y \leq 8$. The probability that the total benefit paid does not exceed 5 is the sum of the probabilities for Cases 1, 2 and 3.

$P[\text{Case 1}] = P[(0 < X \leq 1) \cap (0 < Y \leq 7)]$

$\qquad = P[0 < X \leq 1] \cdot P[0 < Y \leq 7] = (\frac{1}{10})(\frac{7}{10}) = .07$

(we have used the independence of X and Y to find the probability of the intersection)

$P[\text{Case 2}] = P[(1 < X \leq 6) \cap (0 < Y \leq 2)]$

$\qquad = P[1 < X \leq 6] \cdot P[0 < Y \leq 2] = (\frac{5}{10})(\frac{2}{10}) = .10$.

$P[\text{Case 3}] = \int_1^6 \int_2^{8-x} f(x,y)\,dy\,dx = \int_1^6 \int_2^{8-x} f_X(x) \cdot f_Y(y)\,dy\,dx$

$\qquad = \int_1^6 \int_2^{8-x} (\frac{1}{10})(\frac{1}{10})\,dy\,dx = \int_1^6 \int_2^{8-x} (.01)\,dy\,dx$

$\qquad = (.01)\int_1^6 [8-x-2]\,dx = (.01)(6x - \frac{x^2}{2}\Big|_{x=1}^{x=6}) = .125$

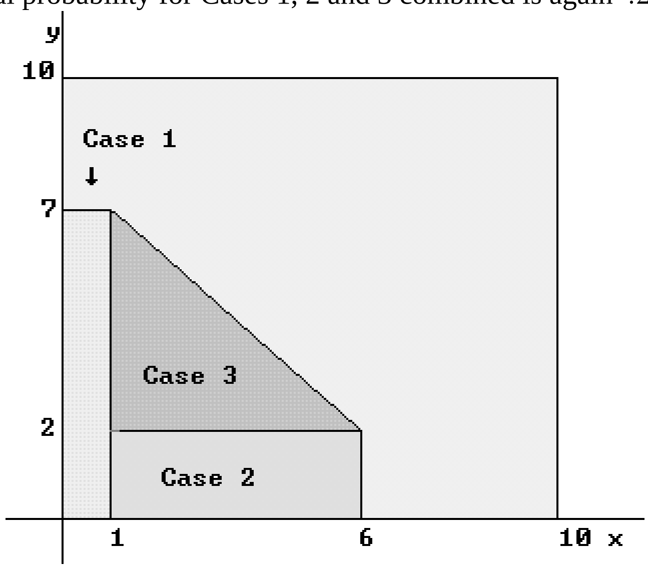
(note that $f(x,y) = f_X(x) \cdot f_Y(y)$ because X and Y are independent).

The total probability is $.07 + .10 + .125 = .295$.

Once we have identified Cases 1, 2 and 3, this problem could be approached from a graphical point of view. Since X and Y are independent and uniform, the joint distribution of X and Y is uniform on the square $0 < x < 10$, $0 < y < 10$, with joint density $(.1)(.1) = .01$.
Since the joint distribution is uniform, the probability of any event involving X and Y is equal to the constant density (.01 in this case) multiplied by the area of the region representing the event.

The three regions for Cases 1, 2 and 3 are indicated in the graph below.

The 10×10 square is the full region for the joint distribution.
The rectangular area for Case 1 is $1 \times 7 = 7$ for a probability of $7 \times .01 = .07$.
The rectangular area for Case 2 is $5 \times 2 = 10$ for a probability of $10 \times .01 = .10$.
The triangular are for Case 3 is $\frac{1}{2} \times 5 \times 5 = 12.5$ for a probability of $12.5 \times .01 = .125$.
The total probability for Cases 1, 2 and 3 combined is again $.295$.



Answer: C

36. $p_X(x) = \sum_y p(x,y)$. $p_X(0) = p(0,1) + p(0,2) = \frac{1}{12} + \frac{2}{12} = \frac{1}{4}$,
$p_X(1) = p(1,2) + p(1,3) = \frac{4}{12} + \frac{5}{12} = \frac{3}{4}$, and $p_X(x) = 0$, otherwise. Answer: B

37. The new amount paid to the surgeon is $X' = X + 100$, and the new amount of hospital charges is $Y' = 1.1Y$. We wish to find
$Var[X' + Y'] = Var[X'] + Var[Y'] + 2Cov(X', Y')$.
$Var[X'] = Var[X + 100] = Var[X] = 5,000$, and
$Var[Y'] = Var[1.1Y] = (1.1^2)Var[Y] = (1.21)(10,000) = 12,100$.
$Cov(X', Y') = Cov(X + 100, 1.1Y) = 1.1Cov(X, Y)$.
We have used the covariance rule $Cov(aU + b, cW + d) = acCov(U, W)$.
We still must know $Cov(X, Y)$ to complete the problem.
We are given $Var[X + Y] = 17,000$, and we use the relationship
$17,000 = Var[X + Y] = Var[X] + Var[Y] + 2Cov(X, Y)$
$= 5,000 + 10,000 + 2Cov(X, Y) \rightarrow Cov(X, Y) = 1,000$.
Then $Cov(X', Y') = 1.1Cov(X, Y) = 1,100$.
Finally, $Var[X' + Y'] = Var[X'] + Var[Y'] + 2Cov(X', Y')$
$= 5,000 + 12,100 + 2(1,100) = 19,300$. Answer: C

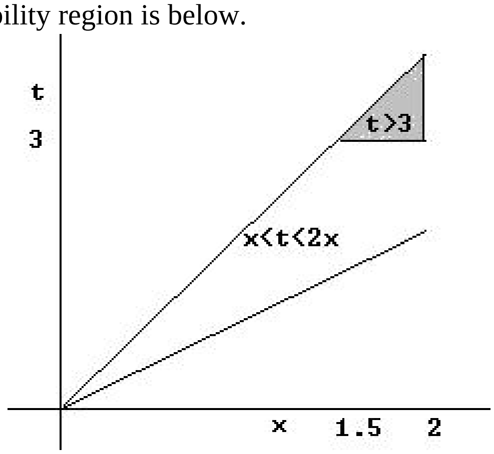
38. $C_1 = X + Y$, $C_2 = X + 1.2Y$.
We use the following rules: $Cov(U, U) = Var(U)$,
$Cov(aU + bV + c, dS + eT + f)$
$= adCov(U, S) + aeCov(U, T) + bdCov(V, S) + beCov(V, T)$,
and $Cov(U, V) = Cov(V, U)$.
Then, $Cov(C_1, C_2) = Cov(X + Y, X + 1.2Y)$
$= Cov(X, X) + 1.2Cov(X, Y) + Cov(Y, X) + 1.2Cov(Y, Y)$
$= Var(X) + 2.2Cov(X, Y) + 1.2Var(Y)$.
From the given information, we have $Var(X) = E(X^2) - [E(X)]^2 = 2.4$,
$Var(Y) = E(Y^2) - [E(Y)]^2 = 2.4$. Also,
$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
$\rightarrow 8 = 2.4 + 2.4 + 2Cov(X, Y) \rightarrow 1.6$.
Then, $Cov(C_1, C_2) = 2.4 + 2.2(1.6) + 1.2(2.4) = 8.8$.
Answer: A

39. $Cov(X, X + cY) = Cov(X, X) + cCov(X, Y) = Var[X] + cCov(X, Y)$
$= 25 - 10c$. This is set equal to $Cov(X, Z) = 2.5$, so that
$25 - 10c = 2.5 \rightarrow c = 2.25$. Answer: B

40. Distribution of T given claim amount $X = x$ is uniform on interval $(x, 2x)$ and has pdf $f_{T|X}(t|X = x) = \frac{1}{x}$ for $x < t < 2x$. The pdf of X is $f_X(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$.
The density function of the joint distribution between T and X is
$f_{X,T}(x, t) = f_{T|X}(t|X = x) \cdot f_X(x) = (\frac{1}{x})(\frac{3}{8}x^2) = \frac{3}{8}x$ for $0 \leq x < t < 2x \leq 4$
(since $x \leq 2$, it follows that $2x \leq 4$).
The event $T > 3$ is illustrated in the graph below. In order to have $T > 3$, it must be true that $x \geq 1.5$, since if $x < 1.5$ then $t < 2x < 3$. Thus, the region of probability for the event $T > 3$ is $1.5 \leq x \leq 2$, and $3 < t < 2x$. The probability is
$P[T > 3] = \int_{1.5}^{2}\int_{3}^{2x} f_{X,T}(x, t)\, dt\, dx = \int_{1.5}^{2}\int_{3}^{2x} \frac{3}{8}x\, dt\, dx = \int_{1.5}^{2} \frac{3}{8}x(2x - 3)\, dx = \frac{11}{64}$.
Alternatively, we can express the conditional probability $P[T > 3|X = x]$ as

$$P[T > 3|X = x] = \begin{cases} 0 & \text{if } x \leq 1.5 \text{ (since then } 2x \leq 3) \\ \frac{2x-3}{x} & \text{if } 1.5 < x \leq 2 \end{cases} .$$

Then, $P[T > 3] = \int_{1.5}^{2} P[T > 3|X = x] \cdot f_X(x)\, dx = \int_{1.5}^{2} \frac{2x-3}{x} \cdot \frac{3x^2}{8} dx = \frac{11}{64}$.
The graph of the probability region is below.



Answer: A

41. $T_B \sim$ exponential mean 2 , pdf $f_B(s) = \frac{1}{2}e^{-s/2}$.
$T_D \sim$ exponential mean 3 , $f_{\mathrm{D}}(\mathrm{t}) = \frac{1}{3}e^{-t/3}$.
Since T_B and T_D are independent, the joint density is
$f_{B,D}(s,t) = f_B(s) \cdot f_D(t) = (\frac{1}{2}e^{-s/2})(\frac{1}{3}e^{-t/3})$, and
$P[T_D < T_B] = \int_0^\infty \int_t^\infty (\frac{1}{2}e^{-s/2})(\frac{1}{3}e^{-t/3})\, ds\, dt$
$= \int_0^\infty \frac{1}{3}e^{-5t/6}\, dt = .4$.
A more general reasoning approach to the solution is the following.
In the next 6 days we expect 3 Basic claims (one every 2 days) and 2 Deluxe claims
(one every 3 days). Of the next 5 claims, there is a $\frac{2}{5} = .4$ chance that it is from a Deluxe policy on average. Answer: C

42. We first note that X and Y are independent. This is true because the joint density can be factored in a function of x alone multiplied by a function of y alone. Another way to verify independence is to note that the marginal density of X is
$f_X(x) = \int_0^\infty f(x,y)\, dy = \int_0^\infty 2e^{-(x+2y)}\, dy = e^{-x}$ for $x > 0$,
and the marginal density of Y is
$f_Y(y) = \int_0^\infty f(x,y)\, dx = \int_0^\infty 2e^{-(x+2y)}\, dx = 2e^{-2y}$ for $y > 0$.
Then, since $f(x,y) = f_X(x) \cdot f_Y(y)$ and since the region of density is rectangular (the entire first quadrant), it follows that X and Y are independent. Since they are independent, the variance of Y does not depend on X, so the conditional variance of Y given X is the same as the variance of Y, and the conditional variance of Y given that $X > 3$ and $Y > 3$ is the same as the conditional variance of Y given that $Y > 3$. The conditional density of Y given that $Y > 3$ is
$f(y|Y > 3) = \frac{f_Y(y)}{P(Y>3)}$ for $y > 3$. $P(Y > 3) = \int_3^\infty 2e^{-2y}\, dy = e^{-8}$, so
$f(y|Y > 3) = \frac{2e^{-2y}}{e^{-8}}$ for $y > 3$. If we make the change of variable $z = y - 3$
then this becomes $f(z|Z > 0) = \frac{2e^{-2(z-3)}}{e^{-8}} = 2e^{-2z}$ for $z > 0$.
This is an exponential density with a mean of $\frac{1}{2}$. The variance of an exponential distribution is the square of the mean, which is $\frac{1}{2^2} = \frac{1}{4}$.
Another point to note, once we have determined that the marginal distribution of Y is exponential with mean 0.5, because of the "lack-of-memory" property of the exponential distribution, the conditional distribution of Y given $Y > a$ is still exponential with mean 0.5, for any $a > 0$.
Answer: A.

43. The conditional density of Y given X is $f_{Y|X}(y|x) = \frac{1}{x}$ on the interval $[0,x]$ (uniform). The region of non-zero joint density of X and Y is on the region $0 < y < x < 1$, since Y has non-zero density only on the interval $[0,x]$. The joint density of X and Y on that region is $f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \frac{1}{x} \cdot 2x = 2$ for $0 < x < y < 1$. We can summarize the joint density function as $f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 < y < x < 1 \\ 0 & \text{if } 0 < x < y < 1 \end{cases}$

(note that we can ignore the region $y = x$, since it has area 0 in the two-dimensional region of probability). This is a (joint) uniform distribution on the triangular region $0 < x < y < 1$. The marginal density of Y is $f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx = \int_0^y 0\,dx + \int_y^1 2\,dx = 2(1-y)$,

and this is defined on the region $0 < y < 1$. This is true because $f(x,y) = 0$ for $0 < y < x < 1$ and $f(x,y) = 2$ for $0 < x < y < 1$.

Then the conditional density of X given $Y = y$ is

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$ for $y < x < 1$, and 0 otherwise.

It is also true in general that i the joint distribution of X and Y is uniform (has constant density) on a region, then the conditional density of Y given X, or of X given Y will be uniform on the appropriate region of non-zero density. Once we have determined that the joint density of X and Y is 2 (constant, and therefore uniform) on the region $0 < y < x < 1$, we know that the conditional density of X given Y will be constant on the interval of definition. The interval of definition for X given Y is $y < x < 1$, which has length $1 - y$. Therefore, the conditional density of X given Y is the constant $\frac{1}{1-y}$ (the density of a uniform is $\frac{1}{\text{interval length}}$)

Answer: E

44. The key point to note in this problem is that we are given the cdf of survival for someone born in the same year as the insured. This is not the cdf of survival for a 40-year old, it is the cdf of survival for a newborn. If we define X to be the time until death for the 40-year old, then the distribution of X is the conditional distribution of T (time from birth until death) given that $T > 40$ (given survival from birth to age 40). The expected payment is

$5000 \times P(X < 10) = 5000 \times P(T < 50|T > 40) = 5000 \times \frac{P(40<T<50)}{P(T>40)}$

$= 5000 \times \frac{F_T(50) - F_T(40)}{1 - F_T(40)} = 5000 \times .0696 = 348$. Answer: B

45. We use the conditioning formula for variance.
$Var[N] = E[\,Var[N|\lambda]\,] + Var[\,E[N|\lambda]\,]$.
Since the distribution of workplace accidents is Poisson with mean λ, we have $E[N|\lambda] = \lambda$ and $Var[N|\lambda] = \lambda$. Then, since the distribution of λ is uniform on the interval $[0,3]$, we have $E[\,Var[N|\lambda]\,] = E[\lambda] = 1.5$, and $Var[\,E[N|\lambda]\,] = Var[\lambda] = \frac{3^2}{12} = .75$ (the variance of a uniform random variable is the square of the interval length divided by 12). Then $Var[N] = 1.5 + .75 = 2.25$. Answer: E

46. Since $Y = 2$, a 5 can be rolled on the first roll or the third or later roll, and also there is no 6 on the first roll. Given that there is no 6 on the first roll, the probability of a 5 on the first roll is $\frac{1}{5}$. The number of rolls until a 5 appears has a geometric distribution with probability function $P(X = n) = (\frac{1}{6})(\frac{5}{6})^{n-1}$, with mean $\frac{1}{1/6} = 6$. If the first 5 does not appear on the first roll, then it will appear on the third or later roll, so the expected number of rolls needed to roll a 5 given that a 5 did not occur on the first two rolls is $2 + 6 = 8$. The overall expected number of rolls until a 5 given that the first 6 is on the second roll is $(1)(.2) + (8)(.8) = 6.6$.
Answer: D

47. If W is a discrete random variable with probability function $P(W = k) = p_k$, then the moment generating function of W is $M_W(t) = \Sigma p_k e^{-kt}$. Also, the moment generating function of the sum of independent random variables is the product of the separate moment generating functions. Then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$. Since X and Y are identically distributed, we have $M_X(t) = M_Y(t)$, so $.09e^{-2t} + .24e^{-t} + .34 + .24e^t + .09e^{2t} = [M_X(t)]^2$.
Algebraically, we see that $M_X(t) = .3e^{-t} + .4 + .3e^t$, so that he distribution of X (and Y) is $P(X = \,-1) + .3$, $P(X = 0) = .4$, $P(X = 1) = .3$. Then $P(X \leq 0) = .7$.
Answer: E

48. The marginal density for X at 0,75 is
$f_X(0.75) = \int_0^1 f(0.75, y)\, dy = \int_0^{0.5} 1.5\, dy \, + \, \int_{0.5}^1 0.75\, dy = 1.125$.
The conditional density of Y given $X = 0.75$ is then
$$f_{Y|X}(y|x = .75) = \frac{f(.75,y)}{f_X(.75)} = \begin{cases} \frac{1.5}{1.125} = \frac{4}{3} & \text{for } 0 < y < .5 \\ \frac{.75}{1.125} = \frac{2}{3} & \text{for } .5 < y < 1 \end{cases}$$
Then, $E(Y|X = .75) = \int_0^{.5} \frac{4}{3} y\, dy + \int_{.5}^1 \frac{2}{3} y\, dy = \, = .417$, and
$E(Y^2|X = .75) = \int_0^{.5} \frac{4}{3} y^2\, dy + \int_{.5}^1 \frac{2}{3} y^2\, dy = .25$.
$Var(Y|X = .75) = .25 - .417^2 = .076$. Answer: C

49. The joint density can be written as $f(x, y) = e^{-x} \cdot (\frac{1}{2} sin\, y) = g(x) \cdot h(y)$.
Since the joint density is defined on a rectangular region and since it factors into the form $g(x) \cdot h(y)$, it follows that the marginal distributions of X and Y are independent. Therefore
$P[(X < 1) \cap (Y < \frac{\pi}{2})] = P[X < 1] \cdot P[Y < \frac{\pi}{2}] = (\int_0^1 e^{-x}\, dx)(\int_0^{\pi/2} \frac{1}{2} sin\, y\, dy)$
$= (1 - e^{-1})(\frac{1}{2})$. Answer: A

SECTION 9 - FUNCTIONS AND TRANSFORMATIONS OF RANDOM VARIABLES

Distribution of a transformation of a continuous random variable X

Suppose that X is a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$, and suppose that $u(x)$ is a one-to-one function (usually u is either strictly increasing, such as $u(x) = e^x$, $\sqrt{x}$ or $\ln x$, or u is strictly decreasing, such as $u(x) = e^{-x}$ or $\frac{1}{x}$). As a one-to-one function, u has an inverse function v, so that $v(u(x)) = x$. The random variable $Y = u(X)$ is referred to as a **transformation of X**. The pdf of Y can be found in one of two ways (they are actually equivalent)

(i) $f_Y(y) = f_X(v(y)) \cdot |v'(y)|$,

(ii) if u is a strictly increasing function, then

$F_Y(y) = P[Y \le y] = P[u(X) \le y] = P[X \le v(y)] = F_X(v(y))$, and $f_Y(y) = F_Y'(y)$.

Distribution of a transformation of a discrete random variable X

Suppose that X is a discrete random variable with probability function $f(x)$. If $u(x)$ is a function of x, and Y is a random variable defined by the equation $Y = u(X)$, then Y is a discrete random variable with probability function $g(y) = \sum_{y=u(x)} f(x)$. Given a value of y, find all values of x for which $y = u(x)$ (say $u(x_1) = u(x_2) = \cdots = u(x_t) = y$), and then $g(y)$ is the sum of those $f(x_i)$ probabilities.

If X and Y are independent random variables, and u and v are functions, then the random variables $u(X)$ and $v(Y)$ are independent.

Example 9-1: The random variable X has an exponential distribution with a mean of 1. The random variable Y is defined to be $Y = 2\ln X$. Find $f_Y(y)$, the pdf of Y.

Solution: $F_Y(y) = P[Y \le y] = P[2\ln X \le y] = P[X \le e^{y/2}]$.

We can now use the cdf of X, $F_X(t) = 1 - e^{-t}$, so that

$F_Y(y) = P[X \le e^{y/2}] = F_X(e^{y/2}) = 1 - e^{-e^{y/2}}$.

Then $f_Y(y) = F_Y'(y) = \frac{d}{dy}\,(1 - e^{-e^{y/2}}) = \frac{1}{2}e^{y/2} \cdot e^{-e^{y/2}}$.

Alternatively, $Y = 2\ln X$. We see that $y = 2\ln x$ is a strictly increasing function of x with inverse function $x = v(y) = e^{y/2}$ and $X = e^{Y/2}$. It follows that

$f_Y(y) = f_X(v(y)) \cdot |v'(y)| = f_X(e^{y/2}) \cdot \left|\frac{d}{dy}\,e^{y/2}\right| = e^{-e^{y/2}} \cdot \frac{1}{2}e^{y/2}$. □

Transformation of jointly distributed random variables X and Y

Suppose that the random variables X and Y are jointly distributed with joint density function $f(x,y)$. Suppose also that u and v are functions of the variables x and y. Then $U = u(X,Y)$ and $V = v(X,Y)$ are also random variables with a joint distribution. We wish to find the joint density function of U and V, say $g(u,v)$. This is a two-variable version of the transformation procedure outlined on the previous page. In the one variable case we required that the transformation had an inverse. There is a similar requirement in the two variable case. We must be able to find inverse functions, $h(u,v)$ and $k(u,v)$ such that $x = h(u(x,y), v(x,y))$, and $y = k(u(x,y), v(x,y))$. The joint density of U and V is then
$g(u,v) = f(h(u,v), k(u,v)) \cdot |\frac{\partial h}{\partial u} \cdot \frac{\partial k}{\partial v} - \frac{\partial h}{\partial v} \cdot \frac{\partial k}{\partial u}|$.
The factor $|\frac{\partial h}{\partial u} \cdot \frac{\partial k}{\partial v} - \frac{\partial h}{\partial v} \cdot \frac{\partial k}{\partial u}|$ is referred to as the Jacobian of the transformation.

This procedure sometimes arises in the context of being given a joint distribution between X and Y, and being asked to find the pdf of some function $U = u(X,Y)$. In this case, we try to find a second function $v(X,Y)$ that will simplify the process of finding the joint distribution of U and V. Then, after we have found the joint distribution of U and V, we can find the marginal distribution of U.

Example 9-2: Suppose that X and Y are independent exponential random variables, each with mean 1. Suppose that $U = \frac{Y}{X}$ and $V = X$. Find the joint distribution of U and V and the marginal distribution of U.

Solution: $U = u(X,Y) = \frac{Y}{X}$ and $V = v(X,Y) = X$, so that $u(x,y) = \frac{y}{x}$ and $v(x,y) = x$.
We can invert these transformations in the following way. $x = v = h(u,v)$, and
$y = \frac{y}{x} \cdot x = u \cdot v = k(u,v)$. Since X and Y are independent, the joint density of X and Y is $f(x,y) = f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)}$. According to the two-variable transformation method outlined above, the joint density of U and V is
$g(u,v) = f(h(u,v), k(u,v)) \cdot |\frac{\partial h}{\partial u} \cdot \frac{\partial k}{\partial v} - \frac{\partial h}{\partial v} \cdot \frac{\partial k}{\partial u}| = f(v, uv) \cdot |0 \cdot u - 1 \cdot v|$
$= e^{-(v+uv)} \cdot v = ve^{-v(u+1)}$ (since $\frac{\partial h}{\partial u} = 0$, $\frac{\partial k}{\partial v} = u$, $\frac{\partial h}{\partial v} = 1$ and $\frac{\partial k}{\partial u} = v$).
We also note that X and Y are defined on the region $x > 0$ and $y > 0$, so U and V are defined on the region $u > 0$ and $v > 0$. The marginal density of U is
$g_U(u) = \int_0^\infty g(u,v)\, dv = \int_0^\infty ve^{-v(u+1)}\, dv = -\frac{ve^{-v(u+1)}}{(u+1)} - \frac{e^{-v(u+1)}}{(u+1)^2} \Big|_{v=0}^{v=\infty} = \frac{1}{(u+1)^2}$.
We used the integration by parts rule $\int ve^{-av}\, dv = -\frac{ve^{-av}}{a} - \frac{e^{-av}}{a^2}$, with $a = u+1$,
and the fact that $\lim_{v\to\infty} \frac{ve^{-v(u+1)}}{(u+1)} = 0$. □

The distribution of a sum of random variables:

(i) If X_1 and X_2 are random variables, and $Y = X_1 + X_2$, then
$\boldsymbol{E[Y] = E[X_1] + E[X_2]}$ and $\boldsymbol{Var[Y] = Var[X_1] + Var[X_2] + 2Cov[X_1, X_2]}$

(ii) If X_1 and X_2 are discrete non-negative integer-valued random variables with joint probability function $f(x_1, x_2)$, then for an integer $k \geq 0$,
$P[X_1 + X_2 = k] = \sum_{x_1=0}^{k} f(x_1, k - x_1)$ (this considers all combinations of X_1 and X_2 whose sum is k). If X_1 and X_2 are independent with probability functions $f_1(x_1)$ and $f_2(x_2)$, respectively, then $P[X_1 + X_2 = k] = \sum_{x_1=0}^{k} f_1(x_1) \cdot f_2(k - x_1)$ (this is the **convolution method** of finding the distribution of the sum of independent discrete random variables).

(iii) If X_1 and X_2 are continuous random variables with joint density function $f(x_1, x_2)$ then the density function of $Y = X_1 + X_2$ is $f_Y(y) = \int_{-\infty}^{\infty} f(x_1, y - x_1)\, dx_1$.
If X_1 and X_2 are independent continuous random variables with density functions $f_1(x_1)$ and $f_2(x_2)$, then the density function of $Y = X_1 + X_2$ is $f_Y(y) = \int_{-\infty}^{\infty} f_1(x_1) \cdot f_2(y - x_1)\, dx_1$.
If $X_1 \geq 0$ and $X_2 \geq 0$, then $f_Y(y) = \int_0^y f(x_1, y - x_1)\, dx_1$.
This is the continuous version of the convolution method.

(iv) If $X_1, X_2, ..., X_n$ are random variables, and the random variable Y is defined to be $Y = \sum_{i=1}^{n} X_i$, then $E[Y] = \sum_{i=1}^{n} E[X_i]$ and $Var[Y] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} Cov[X_i, X_j]$.
If $\boldsymbol{X_1, X_2, ..., X_n}$ are mutually independent random variables, then
$\boldsymbol{Var[Y] = \sum_{i=1}^{n} Var[X_i]}$ **and** $\boldsymbol{M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)}$

(v) If $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ are random variables and $a_1, a_2, ..., a_n,\ b,\ c_1, c_2, ..., c_m$ and d are constants, then $Cov[\sum_{i=1}^{n} a_i X_i + b\ ,\ \sum_{j=1}^{m} c_j Y_j + d] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i c_j Cov[X_i, Y_j]$

(vi) The Central Limit Theorem: Suppose that X is a random variable with mean μ and standard deviation σ and suppose that $X_1, X_2, ..., X_n$ are n independent random variables with the same distribution as X. Let $Y_n = X_1 + X_2 + \cdots + X_n$. Then $E[Y_n] = n\mu$ and $Var[Y_n] = n\sigma^2$, and as n increases, the distribution of Y_n approaches a normal distribution $N(n\mu, n\sigma^2)$. This is a justification for using the normal distribution as an approximation to the distribution of a sum of random variables. **When an exam question asks for a probability involving a sum of a large number of independent random variables, it is usually asking for the normal approximation to be applied.** As mentioned earlier in Section 7 of these notes, when applying the normal approximation to an integer random variable, we may be asked to use the integer correction.

(vii) Sums of certain distributions: Suppose that $X_1, X_2, ..., X_k$ are independent random variables and $Y = \sum_{i=1}^{k} X_i$

distribution of X_i	distribution of Y
Bernoulli $B(1,p)$	binomial $B(k,p)$
binomial $B(n_i,p)$	binomial $B(\Sigma n_i, p)$
Poisson λ_i	Poisson $\Sigma\lambda_i$
geometric p	negative binomial $k,\ p$
negative binomial r_i , p	negative binomial Σr_i , p
normal $N(\mu_i, \sigma_i^2)$	$N(\Sigma\mu_i, \Sigma\sigma_i^2)$
exponential with mean μ	gamma with $\alpha = k$, $\beta = 1/\mu$
gamma with α_i , β	gamma with $\Sigma\alpha_i$, β
Chi-square with k_i df	Chi-square with Σk_i df

Example 9-3: Suppose that X and Y are independent discrete integer-valued random variables with X uniformly distributed on the integers 1 to 5, and Y having the following probability function: $f_Y(0) = .3$, $f_Y(1) = .5$, $f_Y(3) = .2$. Let $Z = X + Y$. Find $P[Z = 5]$.

Solution: Using the fact that $f_X(x) = .2$ for $x = 1,2,3,4,5$, and the convolution method for independent discrete random variables, we have $f_Z(5) = \sum_{i=1}^{5} f_X(i) f_Y(5-i)$

$= (.2)(0) + (.2)(.2) + (.2)(0) + (.2)(.5) + (.2)(.3) = .20$ □

Example 9-4: X_1 and X_2 are independent exponential random variables each with a mean of 1. Find $P[X_1 + X_2 < 1]$.

Solution: Using the convolution method, the density function of $Y = X_1 + X_2$ is $f_Y(y) = \int_0^y f_{X_1}(t) \cdot f_{X_2}(y-t)\,dt = \int_0^y e^{-t} \cdot e^{-(y-t)} dt = ye^{-y}$, so that

$P[X_1 + X_2 < 1] = P[Y < 1] = \int_0^1 ye^{-y} dy = [-ye^{-y} - e^{-y}]\Big|_{y=0}^{y=1} = 1 - 2e^{-1}$

(the last integral required integration by parts). □

Example 9-5: Given n independent random variables $X_1, X_2, ..., X_n$ each having the same variance of σ^2, and defining $U = 2X_1 + X_2 + \cdots + X_{n-1}$ and $V = X_2 + X_3 + \cdots + 2X_n$, find the coefficient of correlation between U and V.

Solution: $\rho_{UV} = \frac{Cov[U,V]}{\sigma_U \sigma_V}$; $\sigma_U^2 = (4 + 1 + 1 + \cdots + 1)\sigma^2 = (n+2)\sigma^2 = \sigma_V^2$.

Since the X's are independent, if $i \neq j$ then $Cov[X_i, X_j] = 0$. Then, noting that $Cov[W, W] = Var[W]$, we have

$Cov[U, V] = Cov[2X_1, X_2] + Cov[2X_1, X_3] + \cdots + Cov[X_{n-1}, 2X_n]$

$= Var[X_2] + Var[X_3] + \cdots + Var[X_{n-1}] = (n-2)\sigma^2$.

Then, $\rho_{UV} = \frac{(n-2)\sigma^2}{(n+2)\sigma^2} = \frac{n-2}{n+2}$. □

Example 9-6: Independent random variables X, Y and Z are identically distributed. Let $W = X + Y$. The moment generating function of W is $M_W(t) = (.7 + .3e^t)^6$. Find the moment generating function of $V = X + Y + Z$.

Solution: For independent random variables,

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = (.7 + .3e^t)^6$. Since X and Y have identical distributions, they have the same moment generating function. Thus,

$M_X(t) = (.7 + .3e^t)^3$, and then $M_V(t) = M_X(t) \cdot M_Y(t) \cdot M_Z(t) = (.7 + .3e^t)^9$.

Alternatively, note that the moment generating function of the binomial $B(n, p)$ is $(1 - p + pe^t)^n$. Thus, $X + Y$ has a $B(6, .3)$ distribution, and each of X, Y and Z has a $B(3, .3)$ distribution, so that the sum of these independent binomial distributions is $B(9, .3)$, with mgf $(.7 + .3e^t)^9$. □

Example 9-7: The birth weight of males is normally distributed with mean 6 pounds, 10 ounces, standard deviation 1 pound. For females, the mean weight is 7 pounds, 2 ounces with standard deviation 12 ounces. Given two independent male/female births, find the probability that the baby boy outweighs the baby girl.

Solution: Let random variables X and Y denote the boy's weight and girl's weight, respectively. Then, $W = X - Y$ has a normal distribution with mean $6\frac{10}{16} - 7\frac{2}{16} = -\frac{1}{2}$ lb. and variance $\sigma_X^2 + \sigma_Y^2 = 1 + \frac{9}{16} = \frac{25}{16}$.

Then, $P[X > Y] = P[X - Y > 0] = P\Big[\frac{W-(-\frac{1}{2})}{\sqrt{25/16}} > \frac{-(-\frac{1}{2})}{\sqrt{25/16}}\Big] = P[Z > .4]$,

where Z has standard normal distribution (W was standardized). Referring to the standard normal table, this probability is $.34$. □

Example 9-8: If the number of typographical errors per page typed by a certain typist follows a Poisson distribution with a mean of λ, find the probability that the total number of errors in 10 randomly selected pages is 10.

Solution: The 10 randomly selected pages have independent distributions of errors per page. The sum of m independent Poisson random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_m$ has a Poisson distribution with parameter $\Sigma\lambda_i$. Thus, the total number of errors in the 10 randomly selected pages has a Poisson distribution with parameter 10λ. The probability of 10 errors in the 10 pages is $\frac{e^{-10\lambda}(10\lambda)^{10}}{10!}$. □

Example 9-9: Smith estimates his chance of winning a particular hand of blackjack at a casino is .45, his probability of losing is .5, and his probability of breaking even on a hand is .05. He is playing at a \$10 table, which means that on each play, he either wins \$10, loses \$10 or breaks even, with the stated probabilities. Smith plays 100 times. What is the approximate probability that he has won money on the 100 plays of the game in total?

Solution: Suppose that X is the gain on a particular play of the game. Then

$E[X] = (10)(.45) + (-10)(.5) = -.5$ is his expected gain on each play, and

$E[X^2] = (100)(.45) + (100)(.5) = 95 \Rightarrow Var[X] = E[X^2] - (E[X])^2 = 94.75$.

$W = \sum_{i=1}^{100} X_i \to E[W] = -50$, $Var[W] = 9475$.

The use of "approximate" in the context of the sum of a large number of independent random variables (the X's) indicates that we are to apply the normal approximation to find the probability. $P[W > 0] = P[\frac{W-E[W]}{\sqrt{Var[W]}} > \frac{0-E[W]}{\sqrt{Var[W]}}]$. We assume that W has an approximate normal distribution. Then $P[W > 0] = P[\frac{W-E[W]}{\sqrt{Var[W]}} > \frac{0-E[W]}{\sqrt{Var[W]}}] = P[Z > \frac{50}{\sqrt{9475}}]$

$= P[Z > .51] = 1 - P[Z \leq .51] = .305$. □

The distribution of the maximum or minimum of a collection of independent random variables: Suppose that X_1 and X_2 are independent random variables. We define two new random variables related to X_1 and X_2: $U = max\{X_1\ ,\ X_2\}$ and $V = min\{X_1\ ,\ X_2\}$. We wish to find the distributions of U and V. Suppose that we know that the distribution functions of X_1 and X_2 are $F_1(x) = P[X_1 \leq x]$ and $F_2(x) = P[X_2 \leq x]$, respectively. We can formulate the distribution functions of U and V in terms of F_1 and F_2 as follows.

$F_U(u) = P[U \leq u] = P[max\{X_1\ ,\ X_2\} \leq u] = P[(X_1 \leq u) \cap (X_2 \leq u)]$
(if the larger of X_1 and X_2 is $\leq u$, then so is the smaller one, so both are $\leq u$).
Since X_1 and X_2 are independent, we have
$P[(X_1 \leq u) \cap (X_2 \leq u)] = P[X_1 \leq u] \cdot P[X_2 \leq u] = F_1(u) \cdot F_2(u)$.
Therefore, the distribution function of U is $F_U(u) = F_1(u) \cdot F_2(u)$.

$F_V(v) = P[V \leq v] = 1 - P[V > v]$
$= 1 - P[min\{X_1\ ,\ X_2\} > v] = 1 - P[(X_1 > v) \cap (X_2 > v)]$
(if the smaller of X_1 and X_2 is $> v$, then so is the larger one, so both are $> v$).
Since X_1 and X_2 are independent, we have
$P[(X_1 > v) \cap (X_2 > v)] = P[X_1 > v] \cdot P[X_2 > v] = [1 - F_1(v)] \cdot [1 - F_2(v)]$.
Therefore, the distribution function of V is $F_V(v) = 1 - [1 - F_1(v)] \cdot [1 - F_2(v)]$.

Example 9-10: A homeowner is accepting sealed bids from two prospective buyers on their offering price to purchase his home. The homeowner assumes that the two bidders will formulate their bids independently of one another. The homeowner assumes a probability distribution for the bid that will be offered by each of the two bidders. For one of the bidders, the homeowner assumes that the bid will be uniformly distributed between 100,000 and 120,000. For the other bidder, the homeowner assumes that the bid will be uniformly distributed between 90,000 and 140,000. Find the probability that the larger of the two bids is over 110,000.

Solution: Let us denote the two bids as X_1 and X_2, so that X_1 has a uniform distribution on the interval $(100,000\ ,\ 120,000)$, and X_2 has a uniform distribution on the interval $(90,000\ ,\ 140,000)$. The distribution function of X_1 and X_2 are
$F_1(x) = \frac{x-100,000}{20,000}$ for $100,000 < x < 120,000$,
$F_2(x) = \frac{x-90,000}{50,000}$ for $90,000 < x < 140,000$.
The larger of the two bids is $U = max\{X_1\ ,\ X_2\}$. Then
$P[U > 110,000] = 1 - P[U \leq 110,000] = 1 - P[(X_1 \leq 110,000) \cap (X_2 \leq 110,000)]$
$= 1 - F_1(110,000) \cdot F_2(110,000) = 1 - (\frac{110,000-100,000}{20,000})(\frac{110,000-90,000}{50,000})$
$= 1 - (\frac{1}{2})(\frac{2}{5}) = \frac{4}{5}$. □

It is possible to extend the case of the max or min of two random variables to the max or min of any collection of independent random variables. For instance, if $X_1 X_2, ..., X_n$ are independent random variables with cdf's $F_1(x)$, $F_2(x)$, $\ldots$, $F_n(x)$, and if $U = max\{X_1 X_2, ..., X_n\}$,
then the cdf of U is $F_U(u) = P[U \leq u] = P[max\{X_1 X_2, ..., X_n\} \leq u]$
$= P[(X_1 \leq u) \cap (X_2 \leq u) \cap \cdots \cap (X_n \leq u)] = F_1(u) \cdot F_2(u) \cdots F_n(u)$.
If $V = min\{X_1 X_2, ..., X_n\}$, then the cdf of V is
$F_V(v) = P[V \leq v] = 1 - P[V > v] = 1 - P[min\{X_1 X_2, ..., X_n\} > v]$
$= 1 - P[(X_1 > v) \cap (X_2 > v) \cap \cdots \cap (X_n > v)]$
$= 1 - [1 - F_1(v)] \cdot [1 - F_2(v)] \cdots [1 - F_n(v)]$.

Order statistics

For a random variable X, a **random sample of size n** is a collection of n independent X_i's all having the same distribution as X. For instance, if X is the outcome that results from tossing a fair die, and the die is tossed independently 10 times, then the outcomes $X_1, X_2, ..., X_{10}$ form a random sample of size 10. We can think of the X_i's as 10 separate independent random variables (when we actually toss the die, we will have 10 numerical outcomes, but in advance of tossing the die we can still think of the outcomes as random variables). When we toss the die 10 times, we will get values between 1 and 6, and they will occur in a random order. For instance, the outcomes might be 5 , 2 , 4 , 4 , 1 , 5 , 2 , 6 , 3 , 1 .

Suppose in advance of actually tossing the die, we decide that we will summarize the 10 outcomes by placing them in increasing order. So the first actual outcome X_1 might not be the smallest numerical outcome, etc. We define 10 new variables Y_1 , Y_2 , $\ldots$, Y_{10} , so that the Y's are the same collection of numbers as the X's, but they have been put in increasing order.
Y_1 is the smallest of the X's, Y_2 is the next smallest, . . . , Y_{10} is the largest.
In general Y_k is the k-th from the smallest of the X_i's.

We can imagine that we will do this even before the die is actually tossed, so that we can think of the Y's as random variables as well. In fact, Y_1 is just the minimum of the X's,
$Y_1 = min\{X_1, X_2, ..., X_{10}\}$, and Y_{10} is the maximum of the X's (and we also have the Y's that are in between).

We saw in the previous example and comments how to find the distribution of the max and the min of a collection of independent random variables, and that would apply to Y_1 and Y_{10} . The Y_i's that we get in this procedure are called the order statistics of the random sample of X's.
In Example 9-10 we had X_1 and X_2 with different distributions. We are assuming now that although the X_i's are independent, they all have the same distribution (such as the outcome of tossing a die), say with density function $f(x)$ and distribution function $F(x)$.

We now wish to describe the distribution of each of the order statistics $Y_1, Y_2, ..., Y_n$. The density function of Y_k can be described in terms $f(x)$ and $F(x)$, the density function and distribution function of X. For each $k = 1, 2, ..., n$ the pdf of Y_k is

$$\boldsymbol{g_k(t) = \frac{n!}{(k-1)!(n-k)!} [F(t)]^{k-1}[1 - F(t)]^{n-k} f(t)} \text{ .}$$

We will not give the general derivation of this density, but the derivation of the density $g_1(t)$ for Y_1 is not difficult to find. If we consider Y_1(the "first order" statistic of the sample of $X's$), its pdf according to the expression above with $k = 1$ is $g_1(t) = n[1 - F(t)]^{n-1} \cdot f(t)$. We saw on the previous page that the cdf of the minimum of $X_1 X_2, ..., X_n$ (it was called V) was $F_V(v) = 1 - [1 - F_1(v)] \cdot [1 - F_2(v)] \cdots [1 - F_n(v)]$. Since V is the first order statistic, $Y_1 = V$ and $F_{Y_1}(t) = 1 - [1 - F_1(t)] \cdot [1 - F_2(t)] \cdots [1 - F_n(t)] = 1 - [1 - F(t)]^n$
(since each F_i is the cdf of X). Then the pdf of Y_1 is

$$g_1(t) = \frac{d}{dt} F_{Y_1}(t) = \frac{d}{dt} (1 - [1 - F(t)]^n) = n[1 - F(t)]^{n-1} \cdot f(t) \text{ .}$$

Since Y_k is one of the X's, it takes on the same possible values as X, so the probability space for each Y is the same as the probability space for X.

The largest order statistic Y_n is the same as the random variable $U = max\{X_1 X_2, ..., X_n\}$ described on the previous page. The cdf of Y_n is $F_{Y_n}(t) = [F(t)]^n$, and the pdf is

$$f_{Y_n}(t) = \frac{d}{dt} F_{Y_n}(t) = \frac{d}{dt} [F(t)]^n = n [F(t)]^{n-1} f(t) \text{ ,}$$

which can be found from the general form of the pdf of Y_k noted above.

For the other order statistics, Y_2 , Y_3 , $\ldots Y_{n-1}$, the cdf's tend to be more complicated (but we do have the pdf of $g_k(t)$ of Y_k for $k = 1, 2, ..., n$ described above). It is possible to formulate the joint distribution of the order statistics Y_1 , Y_2 , $\ldots Y_n$.

The joint density of $Y_1, Y_2, ..., Y_n$ is $g(y_1, y_2, ..., y_n) = n! f(y_1) f(y_2) \cdots f(y_n)$.

Example 9-11: An airport shuttle service driver is waiting for three passengers to arrive. The passengers will be arriving on three separate flights. The shuttle driver assumes that the times until arrival of the three flights are independent of one another, but each time until arrival has an exponential distribution (as measured from now) with a mean of 1 hour. Find the expected time until the 2nd arriving flight.

Solution: We let $X_1,\ X_2$ and X_3 be the three arrival times. The time until the 2nd arriving flight is Y_2, the second order statistics of the 3 X's. We wish to find the expected value of Y_2 . The pdf of each X is $f(t) = e^{-t}$, and the cdf of each X is $F(t) = 1 - e^{-t}$. The pdf of Y_2 can be found from the general form described earlier: $n = 3$, $k = 2$,

$$g_2(t) = \frac{3!}{(2-1)!(3-2)!}\,[F(t)]^{2-1} \cdot [1 - F(t)]^{3-2} \cdot f(t)$$

$$= 6(1 - e^{-t})(e^{-t})(e^{-t}) = 6(e^{-2t} - e^{-3t})\ ,\ t > 0\,.$$

The expected value of Y_2 is

$$E[Y_2] = \int_0^\infty t\,g_2(t)\,dt = \int_0^\infty t \cdot 6(e^{-2t} - e^{-3t})\,dt = 6[\int_0^\infty te^{-2t}\,dt - \int_0^\infty te^{-3t}\,dt]$$

$$= 6\left[-\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\Big|_{t=0}^{t=\infty} - \left(-\frac{te^{-3t}}{3} - \frac{e^{-3t}}{9}\Big|_{t=0}^{t=\infty}\right)\right] = 6(\tfrac{1}{4} - \tfrac{1}{9}) = \tfrac{5}{6}\ .$$

The reader might recall that near the end of Section 7 of this study guide there was a summary of some properties of the exponential distribution. In particular, it was pointed out that the minimum of a collection of independent exponential random variables is also exponential. In this example, the order statistic is Y_1, the minimum of three independent exponential random variables, each with a mean of 1. According to the comments in Section 7 (and also, using the methods of order statistics developed in this section) the distribution of Y_1 will be exponential with a mean of $\frac{1}{3}$. We can expect the first flight arrival to occur in 20 minutes.

Also, recall the exponential integration formula, for an integer $k \geq 0$, $\int_0^\infty t^k \cdot e^{-ct}\,dt = \frac{k!}{c^{k+1}}$. This can be used to calculate the integrals above. □

Mixtures of Distributions

Suppose that X_1 and X_2 are random variables with density (or probability) functions $f_1(x)$ and $f_2(x)$, and suppose a is a number with $0 < a < 1$. We define a new random variable X by defining a new density function $f(x) = a \cdot f_1(x) + (1-a) \cdot f_2(x)$. This newly defined density function will satisfy the requirements for being a properly defined density function. Furthermore, all moments, probabilities and the moment generating function of the newly defined random variable are of the following "weighted-average" form:

$E[X] = aE[X_1] + (1-a)E[X_2]$, $E[X^2] = aE[X_1^2] + (1-a)E[X_2^2]$,

$F_X(x) = P[X \leq x] = aP[X_1 \leq x] + (1-a)P[X_2 \leq x] = aF_1(x) + (1-a)F_2(x)$,

$M_X(t) = aM_{X_1}(t) + (1-a)M_{X_2}(t)$.

The random variable X is called a mixture of X_1 and X_2, and a and $1-a$ are referred to as mixing weights. As mentioned in Section 5, this notion of mixture can be extended to a mixture of any number of random variables.

One place where this "weighted average" relationship **does not work** is in the formulation of the variance of X. **WE DO NOT USE** $Var[X] = aVar[X_1] + (1-a)Var[X_2]$, **it is incorrect**. We must use the earlier relationship above to get the second and first moments of X, and then $\boldsymbol{Var[X] = E[X^2] - (E[X])^2}$, and we would find $E[X]$ and $E[X^2]$ using the weighted-average approach described above.

Another point to note is the following. It appears that the mixture random variable X is equal to $aX_1 + (1-a)X_2$. **This is incorrect.** X is **<u>not</u>** a sum of random variables. X is totally defined by the definition of the pdf $f(x) = a \cdot f_1(x) + (1-a) \cdot f_2(x)$.

A special case of a mixture occurs when X_1 is the constant 0. This situation can be described in the following way. Suppose there is probability a that a loss does not occur, and probability $1-a$ that a loss does occur, and if the loss does occur, the loss amount is a random variable X_2. The overall loss amount is $X = \begin{cases} 0 & \text{if loss does not occur, prob. } a \\ X_2 & \text{if loss does occur, prob. } 1-a \end{cases}$.

This is a mixture of the constant "random" variable X_1 which is always 0, and the loss random variable X_2, with mixing weight a applied to 0 and mixing weight $1-a$ applied to X_2.

Then the expected value of X will be $a(0) + (1-a)E[X_2] = (1-a)E[X_2]$,

and the second moment of X will be $a(0^2) + (1-a)E[X_2^2] = (1-a)E[X_2^2]$.

Example 9-12: Suppose there are two urns containing balls. Urn I contains 5 red and 5 blue balls and Urn II contains 8 red and 2 blue balls. A die is tossed, and if the number turning up is even then a ball is picked from Urn I, and if the number turning up is odd then a ball is picked from Urn II. X is the number of red balls chosen (0 or 1). We can formulate the distribution of X as a mixture of X_1 and X_2, where random variable X_1 is the number of red balls chosen from Urn I and X_2 is the number of red balls chosen from Urn II. Since each urn is equally likely to be chosen, the mixing weights are $a = .5$, $1 - a = .5$. Then
$P[X = 1] = aP[X_1 = 1] + (1 - a)P[X_2 = 1] = (.5)(.5) + (.5)(.8) = .65$, and
$P[X = 0] = aP[X_1 = 0] + (1 - a)P[X_2 = 0] = (.5)(.5) + (.5)(.2) = .35$ □

Example 9-13: An insurer has three risk classifications for policies: low, medium and high. 25% of the company's policies are low risk, 70% are medium risk and 5% are high risk. An individual policy loss is exponentially distributed with the following mean: low risk has mean 1, medium risk has mean 2 and high risk has mean 5. A policy is chosen from the insurer's portfolio of policies, but the risk class is not known. Find the expected loss that will be experienced by the policy, and find the probability that the policy will experience a loss of at least 1.
Solution: We define three loss random variables. X_1 (low risk) has an exponential distribution with a mean of 1, X_2 (medium risk) has an exponential distribution with a mean of 2, and X_3 (high risk) has an exponential distribution with a mean of 5.

Since there is a 25% chance that the chosen policy is low risk, and a 70% chance that it is medium risk and a 5% chance that it is high risk, the distribution of the loss from the chosen policy is a mixture of X_1, X_2 and X_3 , with mixing weights of .25 applied to X_1, .70 applied to X_2 and .05 applied to X_3 . The pdf of X is
$f(x) = .25f_1(x) + .70f_2(x) + .05f_3(x) = .25 \times e^{-x} + .70 \times \frac{1}{2}e^{-x/2} + .05 \times \frac{1}{5}e^{-x/5}$.

The expected value of X is
$E[X] = .25 \times E[X_1] + .70 \times E[X_2] + .05 \times E[X_3] = (.25)(1) + (.70)(2) + (.05)(5) = 1.90.$

The cdf of X is $F_X(x) = .25F_1(x) + .70F_2(x) + .05F_3(x)$
$= .25 \times (1 - e^{-x}) + .70 \times (1 - e^{-x/2}) + .05 \times (1 - e^{-x/5})$, so
$P[X > 1] = 1 - F_X(1) = 1 - [.25 \times (1 - e^{-1}) + .70 \times (1 - e^{-1/2}) + .05 \times (1 - e^{-1/5})]$
$= 1 - .44 = .56$. □

PROBLEM SET 9

Functions and Transformations of Random Variables

1. (SOA) The profit for a new product is given by $Z = 3X - Y - 5$. X and Y are independent random variables with $Var(X) = 1$ and $Var(Y) = 2$.
What is the variance of Z ?
A) 1 B) 5 C) 7 D) 11 E) 16

2. Let X_1, X_2, X_3 be independent discrete random variables, with the probability function $P[X_i = k] = \binom{n_i}{k} p^k (1-p)^{n_i - k}$ for $k = 0, 1, ..., n_i$ for $i = 1, 2, 3$ and $0 < p < 1$.
Determine the probability function of $S = X_1 + X_2 + X_3$, $P[S = s]$.
A) $\binom{n_1+n_2+n_3}{s} p^s (1-p)^{n_1+n_2+n_3-s}$ B) $\sum_{i=1}^{3} \frac{n_i}{n_1+n_2+n_3} \binom{n_i}{s} p^s (1-p)^{n_i-s}$
C) $\prod_{i=1}^{3} \binom{n_i}{s} p^s (1-p)^{n_i-s}$ D) $\sum_{i=1}^{3} \binom{n_i}{s} p^s (1-p)^{n_i-s}$
E) $\binom{n_1 n_2 n_3}{s} p^s (1-p)^{n_1 n_2 n_3 - s}$

3. (SOA) The time, T, that a manufacturing system is out of operation has cumulative distribution function $F(t) = \begin{cases} 1 - \left(\frac{2}{t}\right)^2 & \text{for } t > 2 \\ 0 & \text{otherwise.} \end{cases}$
The resulting cost to the company is $Y = T^2$.
Determine the density function of Y, for $y > 4$.
A) $\frac{4}{y^2}$ B) $\frac{8}{y^{3/2}}$ C) $\frac{8}{y^3}$ D) $\frac{16}{y}$ E) $\frac{1024}{y^5}$

4. Let X and Y be two independent random variables with moment generating functions
$$M_X(t) = e^{t^2+2t} \quad , \quad M_Y(t) = e^{3t^2+t}$$
Determine the moment generating function of $X + 2Y$.
A) $e^{t^2+2t} + 2e^{3t^2+t}$ B) $e^{t^2+2t} + e^{12t^2+2t}$ C) e^{7t^2+4t} D) $2e^{4t^2+3t}$ E) e^{13t^2+4t}

5. Let X_1 and X_2 be random variables with joint moment generating function $M(t_1, t_2) = .3 + .1e^{t_1} + .2e^{t_2} + .4e^{t_1+t_2}$. What is $E[2X_1 - X_2]$?
A) $-.1$ B) $.4$ C) $.8$ D) $.2e + .4e^2$ E) $.3 + .1e^{3t_1} + .2e^{-t_2} + .4e^{3t_1 - t_2}$

6. (SOA) An investment account earns an annual rate R that follows a uniform distribution on the interval $(0.04\,,\,0.08)$. The value of a 10,000 initial investment in this account after one year is given by $V = 10,000e^R$. Determine the cumulative distribution function, $F(v)$, of V for values of v that satisfy $0 < F(v) < 1$.

A) $\frac{10{,}000e^{v/10{,}000} - 10{,}408}{425}$ B) $25e^{v/10{,}000} - 0.04$ C) $\frac{v - 10{,}408}{10{,}833 - 10{,}408}$

D) $\frac{25}{v}$ E) $25\left[ln(\frac{v}{10{,}000}) - 0.04\right]$

7. Let X and Y be discrete random variables with joint probability function $f(x, y)$ given by the following table:

		$x=0$	$x=1$	$x=2$
y	0	0	.40	.20
	1	.20	.20	0

What is the variance of $Y - X$?

A) .16 B) .64 C) 1.04 D) 1.25 E) 1.4

8. Let X_1 and X_2 be two independent observations from a normal distribution with mean and variance 1. If $E[c|X_1 - X_2|] = 1$, then $c =$

A) $\sqrt{\pi}$ B) $\frac{1}{\sqrt{\pi}}$ C) $\frac{\sqrt{2\pi}}{4}$ D) $\frac{2}{\sqrt{\pi}}$ E) $\frac{\sqrt{\pi}}{2}$

9. Let X, Y and Z be independent Poisson Random variables with $E[X] = 3$, $E[Y] = 1$, and $E[Z] = 4$. What is $P[X + Y + Z \leq 1]$?

A) $12e^{-12}$ B) $9e^{-8}$ C) $\frac{13}{12}e^{-1/12}$ D) $9e^{-1/8}$ E) $\frac{9}{8}e^{-1/8}$

10. (SOA) The monthly profit of Company I can be modeled by a continuous random variable with density function f . Company II has a monthly profit that is twice that of Company I. Determine the probability density function of the monthly profit of Company II.

A) $\frac{1}{2}f(\frac{x}{2})$ B) $f(\frac{x}{2})$ C) $2f(\frac{x}{2})$ D) $2f(x)$ E) $2f(2x)$

11. (SOA) An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$ where X is an exponential random variable with mean 1 year. Determine the probability density function $f(y)$, for $y > 0$, of the random variable Y.

A) $10y^{0.8}e^{-y^{-0.2}}$ B) $8y^{-02.}e^{-10y^{0.8}}$ C) $8y^{-0.2}e^{-(0.1y)^{1.25}}$

D) $(0.1y)^{1.25}e^{-0.125(0.1y)^{0.25}}$ E) $0.125(0.1y)^{0.25}e^{-(0.1y)^{1.25}}$

12. Let X, Y and Z have means 1, 2 and 3, respectively, and variances 4, 5 and 9, respectively. The covariance of X and Y is 2, the covariance of X and Z is 3, and the covariance of Y and Z is 1. What are the mean and variance, respectively, of the random variable $3X + 2Y - Z$?

A) 4 and 31 B) 4 and 65 C) 4 and 67 D) 14 and 13 E) 14 and 65

13. (SOA) A device containing two key components fails when, and only when, both components fail. The lifetimes, T_1 and T_2, of these components are independent with common density function $f(t) = e^{-t}$, $t > 0$. The cost, X, of operating the device until failure is $2T_1 + T_2$. Which of the following is the density function of X for $x > 0$?

A) $e^{-x/2} - e^{-x}$ B) $2\left(e^{-x/2} - e^{-x}\right)$ C) $\dfrac{x^2e^{-x}}{2}$ D) $\dfrac{e^{-x/2}}{2}$ E) $\dfrac{e^{-x/3}}{3}$

14. (SOA) A company has two electric generators. The time until failure for each generator follows an exponential distribution with mean 10. The company will begin using the second generator immediately after the first one fails. What is the variance of the total time that the generators produce electricity?

A) 10 B) 20 C) 50 D) 100 E) 200

15. (SOA) A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with a mean of 1. Premiums and claims are independent. Let X denote the ratio of claims to premiums. What is the density function of X?

A) $\dfrac{1}{2x+1}$ B) $\dfrac{2}{(2x+1)^2}$ C) e^{-x} D) $2e^{-2x}$ E) xe^{-x}

16. (SOA) Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes. Let R denote the average rate, in customers per minute, at which the representative responds to inquiries. Which of the following is the density function of the random variable R on the interval $\frac{10}{12} \leq R \leq \frac{10}{8}$?

A) $\frac{12}{5}$ B) $3 - \frac{5}{2r}$ C) $3r - \frac{5\,ln(r)}{2}$ D) $\frac{10}{r^2}$ E) $\frac{5}{2r^2}$

17. (SOA) A charity receives 2025 contributions. Contributions are assumed to be independent and identically distributed with mean 3125 and standard deviation 250. Calculate the approximate 90th percentile for the distribution of the total contributions received.

A) 6,328,000 B) 6,338,000 C) 6,343,000 D) 6,784,000 E) 6,977,000

18. (SOA) An insurance company issues 1250 vision care insurance policies. The number of claims filed by a policyholder under a vision care insurance policy during one year is a Poisson random variable with mean 2 . Assume the numbers of claims filed by distinct policyholders are independent of one another. What is the approximate probability that there is a total of between 2450 and 2600 claims during a one-year period?

A) 0.68 B) 0.82 C) 0.87 D) 0.95 E) 1.00

19. The number of claims received each day by a claims center has a Poisson distribution. On Mondays, the center expects to receive 2 claims but on other days of the week, the claims center expects to receive 1 claim per day. The numbers of claims received on separate days are mutually independent of one another. Find the probability that the claims center receives at least 3 claims in a 5 day week (Monday to Friday).

A) .90 B) .92 C) .94 D) .96 E) .98

20. In analyzing the risk of a catastrophic event, an insurer uses the exponential distribution with mean α as the distribution of the time until the event occurs. The insurer has n independent catastrophe policies of this type. Find the expected time until the insurer will have the first catastrophe claim.

A) $n\alpha$ B) α/n C) α^n D) $\alpha^{1/n}$ E) n/α

21. (SOA) In an analysis of healthcare data, ages have been rounded to the nearest multiple of 5 years. The difference between the true age and the rounded age is assumed to be uniformly distributed on the interval from -2.5 years to 2.5 years. The healthcare data are based on a random sample of 48 people. What is the approximate probability that the mean of the rounded ages is within 0.25 years of the mean of the true ages?

A) 0.14 B) 0.38 C) 0.57 D) 0.77 E) 0.88

22. (SOA) A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

(i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.

(ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25.

(iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.

Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

A) 0.60 B) 0.67 C) 0.75 D) 0.93 E) 0.99

23. An insurer has a portfolio of 1000 independent one-year insurance policies. For any particular policy there is a probability of .01 of a loss occurring within the year. For any particular policy, if a loss occurs, the expected loss is \$2000 with a standard deviation of \$1000. Find the standard deviation of the insurer's aggregate payout for the year (nearest 1000).

A) 6000 B) 7000 C) 8000 D) 9000 E) 10,000

24. (SOA) Claims filed under auto insurance policies follow a normal distribution with mean 19,400 and standard deviation 5,000 . What is the probability that the average of 25 randomly selected claims exceeds 20,000 ?

A) 0.01 B) 0.15 C) 0.27 D) 0.33 E) 0.45

25. (SOA) You are given the following information about N, the annual number of claims for a randomly selected insured:

$$P(N=0) = \frac{1}{2}$$
$$P(N=1) = \frac{1}{3}$$
$$P(N>1) = \frac{1}{6}$$

Let S denote the total annual claim amount for an insured. When $N=1, S$ is exponentially distributed with mean 5 . When $N>1, S$ is exponentially distributed with mean 8.
Determine $P(4<S<8)$.

A) 0.04 B) 0.08 C) 0.12 D) 0.24 E) 0.25

26. (SOA) A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1. A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes. What is the smallest number of bulbs to be purchased so that the succession of light bulbs, produces light for at least 40 months with probability at least 0.9772?

A) 14 B) 16 C) 20 D) 40 E) 55

27. A financial analyst tracking the price of a particular stock uses the uniform distribution between 1 and 2 as the model for the distribution of the stock price P one year from now. A second analyst analyzing the same stock price uses the uniform distribution on the interval from 10 to 100 as the model for the distribution of 10^Q one year from now (Q is the stock price one year from now). Find $m_P - m_Q$, the difference in the median stock price one year from now as estimated by the first and second analyst.

A) $.24$ B) $.12$ C) 0 D) $-.12$ E) $-.24$

28. An actuary is reviewing a study she performed on the size of claims made ten years ago under homeowners insurance policies. In her study, she concluded that the size of claims followed an exponential distribution and that the probability that a claim would be less than $1,000 was 0.250. The actuary feels that the conclusions she reached in her study are still valid today with one exception: every claim made today would be twice the size of a similar claim made ten years ago as a result of inflation. Calculate the probability that the size of a claim made today is less than $1,000.

A) 0.063 B) 0.125 C) 0.134 D) 0.163 E) 0.250

29. An automobile insurance company divides its policyholders into two groups: good drivers and bad drivers. For the good drivers, the amount of an average claim is 1400, with a variance of 40,000. For the bad drivers, the amount of an average claim is 2000, with a variance of 250,000. Sixty percent of the policyholders are classified as good drivers.
Calculate the variance of the amount of a claim for a policyholder.

A) 124,000 B) 145,000 C) 166,000 D) 210,400 E) 235,000

30. An insurance company designates 10% of its customers as high risk and 90% as low risk. The number of claims made by a customer in a calendar year is Poisson distributed with mean θ and is independent of the number of claims made by that customer in the previous calendar year. For high risk customers $\theta = 0.6$, while for low risk customers $\theta = 0.1$. Calculate the probability that a customer of unknown risk profile who made exactly one claim in 1997 will make exactly one claim in 1998.

A) 0.08 B) 0.12 C) 0.16 D) 0.20 E) 0.24

31. (SOA) Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y:

$$\begin{aligned} E(X) &= 50 \\ E(Y) &= 20 \\ Var(X) &= 50 \\ Var(Y) &= 30 \\ Cov(X,Y) &= 10 \end{aligned}$$

One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period. Approximate the value of $P(T < 7100)$.

A) 0.62 B) 0.84 C) 0.87 D) 0.92 E) 0.97

32. For a certain type of insurance policy, the actual loss amount has an exponential distribution with a mean of λ. An insurer will pay 75% of the loss that occurs. Find the moment generating function for the random variable representing the amount paid by the insurer.

A) $\frac{.75}{.75-s\lambda}$ B) $\frac{1}{.75-.75s\lambda}$ C) $\frac{.75}{1-.75s\lambda}$ D) $\frac{1}{1-.75s\lambda}$ E) $\frac{1}{1-1s\lambda}$.

33. (SOA) The total claim amount for a health insurance policy follows a distribution with density function

$$f(x) = \frac{1}{1000} e^{-x/1000} \ , \ x > 0 \ .$$

The premium for the policy is set at 100 over the expected total claim amount. If 100 policies are sold, what is the approximate probability that the insurance company will have claims exceeding the premiums collected?

A) 0.001 B) 0.159 C) 0.333 D) 0.407 E) 0.460

34. A company finds that the time it takes to process a randomly selected insurance claim has a uniform distribution on the interval from 1 to 2 hours. A claims adjuster has developed a new method for processing claims such that if the claim processing time under the current method is t hours, then the claim processing time under his new method is $ln\, t$ hours. Find the density function $f(t)$ for the claim processing time under the new method.

A) $ln\, t$ B) $t\, ln\, t$ C) t D) te^t E) e^t

35. X and Y are random variables with correlation coefficient .75 , and with $E[X] = Var[X] = 1$, and $E[Y] = Var[Y] = 2$. Find $Var[X + 2Y]$.

A) 9 B) $9 + \sqrt{2}$ C) $9 + 2\sqrt{2}$ D) $9 + 3\sqrt{2}$ E) $9 + 4\sqrt{2}$

36. (SOA) Claim amounts for wind damage to insured homes are independent random variables with common density function

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{for } x > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where x is the amount of a claim in thousands. Suppose 3 such claims will be made. What is the expected value of the largest of the three claims?

A) 2025 B) 2700 C) 3232 D) 3375 E) 4500

37. (SOA) A company agrees to accept the highest of four sealed bids on a property. The four bids are regarded as four independent random variables with common cumulative distribution function

$$F(x) = \frac{1}{2}(1+\sin \pi x) \quad \text{for} \quad \frac{3}{2} \le x \le \frac{5}{2}.$$

Which of the following represents the expected value of the accepted bid?

A) $\pi \int_{3/2}^{5/2} x \cos \pi x \, dx$ B) $\frac{1}{16}\int_{3/2}^{5/2} (1+\sin \pi x)^4 \, dx$ C) $\frac{1}{16}\int_{3/2}^{5/2} x(1+\sin \pi x)^4 \, dx$

D) $\frac{1}{4}\pi \int_{3/2}^{5/2} \cos \pi x (1+\sin \pi x)^3 \, dx$ E) $\frac{1}{4}\pi \int_{3/2}^{5/2} x \cos \pi x (1+\sin \pi x)^3 \, dx$

38. (SOA) A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$f(x,y) = \frac{x+y}{27}$ for $0 < x < 3$ and $0 < y < 3$.

Calculate the probability that the device fails during its first hour of operation.

A) 0.04 B) 0.41 C) 0.44 D) 0.59 E) 0.96

39. Let $Y_1, \ldots, Y_n$ be the order statistics of a random sample of size n from a uniform distribution on the interval $(0,2)$. What is $P\left[Y_1 < \frac{1}{2} < Y_n\right]$?

A) $\frac{3^n - 1}{4^n}$ B) $\frac{3^n + 1}{4^n}$ C) $\frac{4^n - 3^n - 1}{4^n}$ D) $\frac{4^n - 3^n + 1}{4^n}$ E) $\frac{4^n + 3^n + 1}{4^n}$

40. The random variables $X_1, X_2, X_3, X_4,$ and X_5 are independent and identically distributed. The random variable $Y = X_1 + X_2 + X_3 + X_4 + X_5$ has moment generating function $M_Y(t) = e^{15e^t - 15}$. Find the variance of X_1.

A) $\sqrt{3}$ B) 3 C) $\sqrt{15}$ D) 15 E) 225

41. (SOA) X and Y are independent random variables with common moment generating function $M(t) = e^{t^2/2}$. Let $W = X + Y$ and $Z = Y - X$.

Determine the joint moment generating function $M(t_1, t_2)$ of W and Z.

A) $e^{2t_1^2 + 2t_2^2}$ B) $e^{(t_1 - t_2)^2}$ C) $e^{(t_1 + t_2)^2}$ D) $e^{2t_1 t_2}$ E) $e^{t_1^2 + t_2^2}$

PROBLEM SET 9 SOLUTIONS

1. We use the probability rule $Var[aX + bY + c] = a^2Var[X] + b^2Var[Y] + 2abCov[X, Y]$, where a, b, c are constants and X and Y are random variables.
Since X and Y are independent, we have $Cov[X, Y] = 0$.
Therefore, $Var[3X - Y - 5] = 3^2Var[X] + (-1)^2Var[Y] = (9)(1) + (1)(2) = 11$.
Answer: D

2. $X_1,\ X_2$ and X_3 are independent binomial random variables, all with the same value of p, and therefore, $S = X_1 + X_2 + X_3$ has a binomial distribution with parameters p and $n = n_1 + n_2 + n_3$. The probability function of S is
$P[S = s] = \binom{n}{s} p^s(1-p)^{n-s} = \binom{n_1 + n_2 + n_3}{s} p^s(1-p)^{n_1+n_2+n_3-s}$. Answer: A

3. The density function for Y is $f_Y(y)$. If we can find $F_Y(y)$, the cumulative distribution function for Y then $f_Y(y) = F'_Y(y)$. We can find $F_Y(y)$ from the relationship between Y and T and from $F_T(t)$ (the cdf of T).
$F_Y(y) = P[Y \leq y] = P[T^2 \leq y] = P[0 < T \leq \sqrt{y}\,]$
(the description of $F_T(t)$ indicates that T is defined for only positive numbers).
Therefore, $F_Y(y) = F_T(\sqrt{y}) = 1 - (\frac{2}{\sqrt{y}})^2 = 1 - \frac{4}{y}$.
The density function for Y is $f_Y(y) = F'_Y(y) = \frac{4}{y^2}$. Answer: A

4. $M_{X+2Y}(t) = E[e^{t(X+2Y)}] = E[e^{tX} \cdot e^{2tY}] = E[e^{tX}] \cdot E[e^{2tY}] = M_X(t) \cdot M_Y(2t)$
$= exp(t^2 + 2t) \cdot exp[3(2t)^2 + 2t] = exp(13t^2 + 4t)$. Note that the equality $E[e^{tX} \cdot e^{2tY}] = E[e^{tX}] \cdot E[e^{2tY}]$ follows from the independence of X and Y. Answer: E

5. $E[X_1] = \frac{\partial}{\partial t_1} M(t_1, t_2)\Big|_{t_1=t_2=0}$ and $E[X_2] = \frac{\partial}{\partial t_2} M(t_1, t_2)\Big|_{t_1=t_2=0}$
$\frac{\partial}{\partial t_1} M(t_1, t_2) = .1e^{t_1} + .4e^{t_1+t_2} \rightarrow E[X_1] = .5$,
$\frac{\partial}{\partial t_2} M(t_1, t_2) = .2e^{t_2} + .4e^{t_1+t_2} \rightarrow E[X_2] = .6$,
$\Rightarrow \quad E[2X_1 - X_2] = 2E[X_1] - E[X_2] = .4$. Answer: B

6. $F(v) = P[V \le v] = P[10{,}000e^R \le v] = P[R \le ln(\frac{v}{10{,}000})]$.
If X has a uniform distribution on the interval (a,b) then if $a < x < b$,
$P[X \le x] = \frac{x-a}{b-a}$. Since R is uniform on $(.04, .08)$ it follows that
$P[R \le ln(\frac{v}{10{,}000})] = \frac{ln(\frac{v}{10{,}000})-.04}{.08-.04} = 25[ln(\frac{v}{10{,}000}) - .04]$ if $.04 < ln(\frac{v}{10{,}000}) < .08$.
Answer: E

7. The distribution of $W = Y - X$ is discrete with possible values $0,\ -1,\ -2,$ and 1.
The probabilities are $f_W(-2) = .2$ (this occurs only if $Y = 0$ and $X = 2$),
$f_W(-1) = .4$ $(Y = 0,\ X = 1)$, $f_W(0) = .2$ $(Y = 1,\ X = 1)$, and
$f_W(1) = .2$ $(Y = 1,\ X = 0)$. Then $E[W] = -.6$, and $E[W^2] = 1.4$, so that
$Var[W] = 1.4 - (-.6)^2 = 1.04$. Answer: C

8. $W = X_1 - X_2$ has a normal distribution with a mean of $1 - 1 = 0$, and a variance of
$1 + 1 = 2$. Then, $E[c|X_1 - X_2|\,] = E[c|W|\,] = c\int_{-\infty}^{\infty}|w| \cdot f_W(w)\,dw$
$= \int_{-\infty}^{0}(-w)f_W(w)\,dw + \int_0^{\infty}w\,f_W(w)\,dw = 2c\int_0^{\infty}w\,f_W(w)\,dw$.
But $f_W(w) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-w^2/4}$ (from the pdf for $N(\mu, \sigma^2)$), so that
$\int_0^{\infty}w\,f_W(w)\,dw = \int_0^{\infty}w \cdot \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-w^2/4}dw = -\frac{1}{\sqrt{\pi}} \cdot e^{-w^2/4}\Big|_{w=0}^{w=\infty} = \frac{1}{\sqrt{\pi}}$.
Thus, $2c \cdot \frac{1}{\sqrt{\pi}} = 1 \rightarrow c = \frac{\sqrt{\pi}}{2}$. Answer: E

9. As the sum of independent Poisson random variables, $W = X + Y + Z$ has a Poisson distribution with parameter $3 + 1 + 4 = 8$, so that
$P[W \le 1] = P[W = 0] + P[W = 1] = e^{-8} + \frac{e^{-8}\cdot 8}{1!} = 9e^{-8}$. Answer: B

10. Let us denote the cumulative distribution function of Company I's monthly profit by
$F(x) = P[X \le x]$, and let us denote Company II's density function and cumulative distribution function of monthly profit by $G(y) = P[Y \le y]$ and $g(y)$, respectively.
Company II's monthly profit is $Y = 2X$. The cumulative distribution function for Company II's monthly profit is $G(y) = P[Y \le y] = P[2X \le y] = P[X \le \frac{y}{2}] = \int_0^{y/2}f(x)\,dx$.
The density function for Company II's profit is then $G'(y) = \frac{d}{dy}\int_0^{y/2}f(x)\,dx = f(\frac{y}{2}) \cdot \frac{1}{2}$
(this uses the differentiation rule $\frac{d}{dy}\int_a^{h(y)}k(s)\,ds = k(h(y)) \cdot h'(y)$;
in this case, $h(y) = \frac{y}{2}$ and $k(s) = f(s)$).
Looking at the cdf of the new random variable is a standard method for determining the density function of a random variable that is defined in terms of or as a transformation of another random variable. Answer: A

11. We first find the distribution function of Y, $F_Y(y)$. Then $f_Y(y) = F'_Y(y)$.
$F_Y(y) = P[Y \leq y] = P[10X^{.8} \leq y] = P[X \leq (.1y)^{1.25}] = 1 - e^{-(.1y)^{1.25}}$.
Then, $f_Y(y) = \frac{d}{dy}[1 - e^{-(.1y)^{1.25}}] = -e^{-(.1y)^{1.25}} \cdot (-1.25(.1y)^{.25}(.1))$
$= .125(.1y)^{.25}e^{-(.1y)^{1.25}}$. Answer: E

12. $E[3X + 2Y - Z] = 3E[X] + 2E[Y] - E[Z] = 3(1) + 2(2) - 1(3) = 4$.
$Var[3X + 2Y - Z] = 9Var[X] + 4Var[Y] + Var[Z]$
$+ 2(6Cov[X,Y] - 3Cov[X,Z] - 2Cov[Y,Z]) = 67$. Answer: C

13. Let $W = 2T_1$. Then the cdf of W is
$F_W(w) = P[W \leq w] = P[2T_1 \leq w] = P[T_1 \leq \frac{w}{2}] = F_{T_1}(\frac{w}{2}) = 1 - e^{-w/2}$.
Then the pdf of W is $f_W(w) = F'_W(w) = \frac{1}{2}e^{-w/2}$.
The density of $Y = 2T_1 + T_2 = W + T_2$ can be found by convolution.
$f_Y(y) = \int_0^y f_W(w) \cdot f_{T_2}(y - w)\, dw = \int_0^y \frac{1}{2}e^{-w/2} \cdot e^{-(y-w)}\, dw = \frac{1}{2}e^{-y}\int_0^y e^{w/2}\, dw$
$= \frac{1}{2}e^{-y}(\frac{e^{y/2}-1}{1/2}) = e^{-y/2} - e^{-y}$, $y > 0$.
Alternatively, the density function of the joint distribution of T_1 and T_2 is
$f_{T_1,T_2}(s,t) = f_{T_1}(s) \cdot f_{T_2}(t) = e^{-s} \cdot e^{-t}$ (by independence). Then with $Y = 2T_1 + T_2$
$P[Y \leq y] = P[2T_1 + T_2 \leq y] = \int_0^{y/2}\int_0^{y-2s} e^{-s} \cdot e^{-t}\, dt\, ds = \int_0^{y/2} e^{-s} \cdot (1 - e^{-(y-2s)})\, ds$
$= \int_0^{y/2} e^{-s} \cdot\, ds - \int_0^{y/2} e^{-(y-s)})\, ds = 1 - e^{-y/2} - e^{-y}(e^{y/2} - 1) = 1 + e^{-y} - 2e^{-y/2}$
$\Rightarrow f_Y(y) = \frac{d}{dy}(1 - e^{-y} - 2e^{-y/2}) = e^{-y/2} - e^{-y}$, $y > 0$. Answer: A

14. Since the second generator starts after the first one fails, the total time that the generators are working is the sum of the two separate working times: $T = T_1 + T_2$.
The length of time the second generator operates is not related to how long the first generated operated, so T_1 and T_2 are independent. Therefore,
$Var[T_1 + T_2] = Var[T_1] + Var[T_2]$
(in general, $Var[X + Y] = Var[X] + Var[Y] + 2Cov[X,Y]$, but if X and Y are independent, then $Cov[X,Y] = 0$).
Each of T_1 and T_2 has an exponential distribution with mean 10. The variance of an exponential random variable is the square of the mean, so that each of T_1 and T_2 has a variance of 100.
Therefore, $Var[T] = 200$. Answer: E

15. The random variable X is defined to be $X = \frac{C}{P}$, where C (claims) has an exponential distribution with mean 1, and P (premiums) has an exponential distribution with mean 2. If we can find $F_X(x)$, the distribution function of X, then the density function is $f_X(x) = \frac{d}{dx} F_X(x)$.
$F_X(x) = P[X \le x] = P[\frac{C}{P} \le x] = P[C \le Px]$.
The 2-dimensional (C, P) region described by this event (with P on the horizontal axis and C on the vertical axis) is all points (p, c) (≥ 0) below the line $c = px$.
The density function of C is $f_C(c) = e^{-c}$ (exponential with mean 1) and the density function of P is $f_P(p) = \frac{1}{2} e^{-p/2}$ (exponential with mean 2). Since P and C are independent, the density function of the joint distribution of P and C is
$f(p, c) = f_P(p) \cdot f_C(c) = \frac{1}{2} e^{-p/2} \cdot e^{-c}$. The probability $P[C \le Px]$ is
$\int_0^\infty \int_0^{px} \frac{1}{2} e^{-p/2} \cdot e^{-c} \, dc \, dp = \int_0^\infty \frac{1}{2} e^{-p/2} \cdot (1 - e^{-px}) \, dp$
$= \int_0^\infty \frac{1}{2} e^{-p/2} \, dp - \int_0^\infty \frac{1}{2} e^{-p(x+\frac{1}{2})} \, dp = 1 - \frac{1}{2x+1}$.
This is $F_X(x)$, so that the density function of X is
$f_X(x) = \frac{d}{dx} [1 - \frac{1}{2x+1}] = \frac{2}{(2x+1)^2}$. Answer: B

16. Since T has a uniform distribution on the interval from 8 to 12, T's distribution function is
$F_T(t) = P[T \le t] = \frac{t-8}{12-8}$, and $P[T \ge t] = \frac{12-t}{12-8}$, for $8 \le t \le 12$.
$R = \frac{10}{T}$; $F_R(r) = P[R \le r] = P[\frac{10}{T} \le r] = P[T \ge \frac{10}{r}] = \frac{12 - \frac{10}{r}}{12-8} = 3 - \frac{2.5}{r}$.
The density function of R is $f_R(r) = F_R'(r) = \frac{d}{dr} F_R(r) = \frac{d}{dr} [3 - \frac{2.5}{r}] = \frac{2.5}{r^2}$.
Alternatively, $f_T(t) = \frac{1}{4}$ for $8 \le t \le 12$. $R = \frac{10}{T} = g(T) \rightarrow T = \frac{10}{R} = h(R)$
$\rightarrow f_R(r) = f_T(h(r)) \cdot |h'(r)| = \frac{1}{4} \cdot |\frac{-10}{r^2}| = \frac{2.5}{r^2}$. Answer: E

17. The standard approximation to the sum (total) of a collection of independent random variables is the normal approximation. The total contribution is $T = C_1 + C_2 + \cdots + C_{2025}$, the sum of the 2025 contributions. C_i is the amount of the i-th contribution, the C_i's are mutually independent, and each has mean $E[C_i] = 3125$ and variance $Var[C_i] = (250)^2$.
The mean and variance of T are $E[T] = \sum_{i=1}^{2025} E[C_i] = (2025)(3125) = 6,328,125$ and
$Var[T] = \sum_{i=1}^{2025} Var[C_i] = (2025)(250^2) = 126,562,500$.
We will denote the 90th percentile of T by p. We find the approximate 90th percentile of T by applying the normal approximation to T. We wish to find p so that $P[T \le p] = .9$.
We standardize the probability: $P[T \le p] = P[\frac{T - 6,328,125}{\sqrt{126,562,500}} \le \frac{p - 6,328,125}{\sqrt{126,562,500}}] = .90$.

17. continued

$\frac{T-6,328,125}{\sqrt{126,562,500}}$ is approximately standard normal (mean 0, variance 1), so that $\frac{p-6,328,125}{\sqrt{126,562,500}}$ is the 90-th percentile of the standard normal distribution. From the table for the standard normal distribution, we see that $\Phi(1.282) = .90$. Therefore we have $\frac{p-6,328,125}{\sqrt{126,562,500}} = 1.282$, from which we get $p = 6,342,547.5$. Answer: C

18. For policyholder i, let X_i be the number of claims filed in the year, $i = 1, 2, ..., 1250$. Each X_i is Poisson with a mean of 2, and therefore has variance of 2 also; $E[X_i] = 2$, $Var[X_i] = 2$. The total number of claims in the year is $T = \sum_{i=1}^{1250} X_i$. Since the X_i's are mutually independent, the distribution of T is approximately normal. The mean of T is $E[T] = E[\sum_{i=1}^{1250} X_i] = \sum_{i=1}^{1250} E[X_i] = (1250)(2) = 2500$, and the variance of T is $Var[T] = Var[\sum_{i=1}^{1250} X_i] = \sum_{i=1}^{1250} Var[X_i] = (1250)(2) = 2500$ (since the X_i's are independent, there are no covariances between X_i's). We wish to find $P[2450 \le T \le 2600]$, by using the normal approximation for T. Applying the normal approximation we get

$P[2450 \le T \le 2600] = P[\frac{2450-E[T]}{\sqrt{Var[T]}} \le \frac{T-E[T]}{\sqrt{Var[T]}} \le \frac{2600-E[T]}{\sqrt{Var[T]}}]$

$= P[\frac{2450-2500}{\sqrt{2500}} \le \frac{T-2500}{\sqrt{2500}} \le \frac{2600-2500}{\sqrt{2500}}] = P[-1 \le Z \le 2]$

$= \Phi(2) - [1 - \Phi(1)] = .9772 - (1 - .8413) = .8185$ (from the normal the table provided with the exam). Answer: B

19. The sum of independent Poisson random variables is also Poisson, so that the number of claims occurring in a 5 day week has a Poisson distribution with 6 claims expected. Then

$P[X \ge 3] = 1 - P[X = 0] - P[X = 1] - P[X = 2]$

$= 1 - e^{-6} - e^{-6} \cdot \frac{6}{1!} - e^{-6} \cdot \frac{6^2}{2!} = 1 - .0620 = .938$. Answer: C

20. Let T_i represent the time until a catastrophe occurs on policy i, and let T represent the time until the first catastrophe occurs. Then

$P[T > t] = P[\text{All } T_i > t] = P[(T_1 > t) \cap (T_2 > t) \cap \cdots \cap (T_n > t)]$

$= P[T_1 > t] \cdot P[T_2 > t] \cdots P[T_n > t]$ (this last equality follows from the independence of the T_i's). From the exponential distribution, we have $P[T_i > t] = e^{-t/\alpha}$, so that $P[T > t] = (e^{-t/\alpha})^n = e^{-tn/\alpha}$, thus T has an exponential distribution with mean α/n.

Answer: B

21. For any given round age X_i, the error E_i is uniform between -2.5 and 2.5.
Therefore, $E[E_i] = 0$ and $Var[E_i] = \frac{25}{12}$ (the variance of the uniform distribution on the interval from a to b has a mean of $\frac{a+b}{2}$ and a variance of $\frac{(b-a)^2}{12}$) .
The total error in 48 independent rounded ages is $\sum_{i=1}^{48} E_i$, which has a mean of 0, and variance $48(\frac{25}{12}) = 100$. The mean of the errors in the 48 rounded ages, $\overline{E} = \frac{1}{48}\sum_{i=1}^{48} E_i$ has expected value 0 and variance $Var[\overline{E}] = (\frac{1}{48})^2(100)$. Using the normal approximation for the distribution of E (since it is the sum of a relatively large number of independent and identically distributed random variables) it follows that $\overline{E}$ has an approximate normal distribution, and then
$P[|\overline{E}| < .25] = P[-.25 < \overline{E} < .25] = P[\frac{-.25-E(\overline{E})}{\sqrt{Var(\overline{E})}} < \frac{\overline{E}-E(\overline{E})}{\sqrt{Var(\overline{E})}} < \frac{.25-E(\overline{E})}{\sqrt{Var(\overline{E})}}]$
$= P[-1.2 < Z < 1.2]$, where Z has a standard normal distribution.
$P[-1.2 < Z < 1.2] = P[Z < 1.2] - P[Z > 1.2] = 2P[Z < 1.2] - 1$
$= 2(.8849) - 1 = .7698$. Answer: D

22. For a given new hire, the number of pensions N that the city will provide at retirement is either 0, 1 or 2, with probabilities $P[N = 0] = .6$ (no longer with the police force) ,
$P[N = 1] = (.4)(.25) = .1$ (stays with police force and is not married) ,
$P[N = 2] = (.4)(.75) = .3$ (stays with the force and is married).
The mean of N is $(1)(.1) + (2)(.3) = .7$, and the variance is
$E[N^2] - (E[N])^2 = [(1^2)(.1) + (2^2)(.3)] - (.7)^2 = .81$.
The number of pensions provided by the city for 100 (independent) new hires is
$T = N_1 + N_2 + \cdots + N_{100}$. We can use the normal approximation for T.
$E[T] = 100E[N] = 70$, $Var[T] = 100Var[N] = 81$.
$P[T \le 90] = P[\frac{T-70}{\sqrt{81}} \le \frac{90-70}{\sqrt{81}}] = P[Z \le 2.22] = \Phi(2.22) = .9867$.
Answer: E

23. Let X denote the amount paid out on a particular policy. Then
$E[X] = E[X|\text{no claim}]P[\text{no claim}] + E[X|\text{claim occurs}]P[\text{claim}]$
$= (0)(.99) + (2000)(.01) = 20$, and
$E[X^2] = E[X^2|\text{no claim}]P[\text{no claim}] + E[X^2|\text{claim occurs}]P[\text{claim}]$
$= (0^2)(.99) + E[X^2|\text{claim occurs}](.01)$.
However, $1000^2 = Var[X|\text{claim occurs}] = E[X^2|\text{claim occurs}] - (E[X|\text{claim occurs}])^2$
$= E[X^2|\text{claim occurs}] - (2000)^2 \Rightarrow E[X^2|\text{claim occurs}] = 5,000,000$,
so that $E[X^2] = (0^2)(.99) + (5,000,000)(.01) = 50,000$,
and then $Var[X] = 50,000 - 20^2 = 49,600$. The variance on 1000 independent policies is 49,600,000, and the standard deviation is $\sqrt{49,600,000} = 7,043$. Answer: B

24. Suppose the 25 random claim amounts are $X_1, X_2, \ldots, X_{25}$, where each X_i has a normal distribution with mean 19,400 and standard deviation 5,000 (variance 25,000,000). Since the claims are randomly chosen, they are independent of one another. The sum of normal random variables is normal and multiplying a normal random variable by a constant results in a normal random variable. Therefore the average of the claims $A = \frac{1}{25}(X_1 + X_2 + \cdots + X_{25})$ has a normal distribution. Using the basic rules for expected value, we get the mean of A,

$E[A] = \frac{1}{25}E[X_1 + X_2 + \cdots + X_{25}] = \frac{1}{25}(E[X_1] + E[X_2] + \cdots + E[X_{25}])$

$= (\frac{1}{25})(25)(19,400) = 19,400$.

Since the X_i's are mutually independent, they have covariances of 0, and we get the variance of A, $Var[A] = Var[\frac{1}{25}(X_1 + X_2 + \cdots + X_{25})] = (\frac{1}{25})^2 Var[X_1 + X_2 + \cdots + X_{25}]$

$= (\frac{1}{25})^2(Var[X_1] + Var[X_2] + \cdots + Var[X_{25}]) = (\frac{1}{25})^2(25)(25,000,000) = 1,000,000$.

Therefore, A has a normal distribution with mean 19,400 and variance 1,000,000 .

Then, $P[A > 20,000] = P[\frac{A-19,400}{\sqrt{1,000,000}} > \frac{20,000-19,400}{\sqrt{1,000,000}}] = P[Z > .6]$, where Z has a standard normal distribution. From the standard normal table distributed with the exam, we have

$P[Z > .6] = 1 - \Phi(.6) = 1 - .7257 = .2743$. Answer: C

25. S is a mixture of three components:

(i) the constant 0, with probability $\frac{1}{2}$,

(ii) exponential distribution X_1 with mean 5, probability $\frac{1}{3}$, and

(iii) exponential distribution X_2 with mean 8, probability $\frac{1}{6}$.

For the exponential distribution with mean μ, the cdf is $F(t) = 1 - e^{-t/\mu}$.

Then, $P[4 < S < 8] = \frac{1}{2}(0) + \frac{1}{3}P[4 < X_1 < 8] + \frac{1}{6}P[4 < X_2 < 8]$

$= \frac{1}{3}[e^{-4/5} - e^{-8/5}] + \frac{1}{6}[e^{-4/8} - e^{-8/8}] = .123$. Answer: C

26. (SOA) Suppose that n bulbs are purchased. Then the total lifetime of the bulbs will be $T_n = X_1 + X_2 + \cdots + X_n$, which has a normal distribution with mean $3n$ and variance n. The probability that total lifetime will be at least 40 is

$P[T_n \geq 40] = P[\frac{T_n - 3n}{\sqrt{n}} \geq \frac{40-3n}{\sqrt{n}}] = 1 - \Phi(\frac{40-3n}{\sqrt{n}})$.

We want this probability to be at least .9772. Trial and error using the possible answers results in

A) $n = 14 \rightarrow 1 - \Phi(\frac{40-42}{\sqrt{14}}) = 1 - \Phi(-.53) = \Phi(.53) = .7107$,

B) $n = 16 \rightarrow 1 - \Phi(\frac{40-48}{\sqrt{16}}) = 1 - \Phi(-2.0) = \Phi(2.0) = .9772$.

The probability is reached with $n = 16$. Answer: B

27. The median of P is the midpoint of the uniform interval, 1.5 ,
The median of Q is m_Q, where $P[Q \leq m_Q] = .5$.
But $P[Q \leq m_Q] = P[10^Q \leq 10^{m_Q}] = \frac{10^{m_Q}-10}{100-10} = .5 \rightarrow m_Q = 1.74$.
Then, $m_P - m_Q = -.24$. Answer: E

28. We assume that the exponential distribution used by the actuary 10 years ago for the claim amount X had a parameter λ . Then, 10 years ago,
$P[X < 1000] = 1 - e^{-1000\lambda} = .25 \rightarrow e^{-1000\lambda} = .75$.
If Y denotes the random variable for a claim amount today, then $Y = 2X$, since every claim made today is twice the size of a similar claim made ten years ago. Then
$P[Y < 1000] = P[2X < 1000] = P[X < 500] = 1 - e^{-500\lambda}$
$= 1 - (e^{-1000\lambda})^{1/2} = 1 - (.75)^{1/2} = .134$. Answer: C

29. C as a mixture of two distributions, based on the two groups. Good drivers are group 1 with claim distribution X_1 and bad drivers are group 2 with claim distribution X_2. The mixing factors are $\alpha_1 = .6$ (60% of drivers are classified as good) and $\alpha_2 = .4$. Then, moments of C are the "weighted" moments of X_1 and X_2 . Thus,
$E[C] = (.6)E[X_1] + (.4)E[X_2] = (.6)(1400) + (.4)(2000) = 1,640$.
Since $Var[X_1] = E[X_1^2] - (E[X_1])^2$, and since we are given $Var[X_1] = 40,000$ and $E[X_1] = 1400$, it follows that $E[X_1^2] = 2,000,000$, and in a similar way we get $E[X_2^2] = 4,250,000$. Then, $E[C^2] = (.6)E[X_1^2] + (.4)E[X_2^2] = 2,900,000$.
Finally, $Var[C] = E[C^2] - (E[C])^2 = 2,9000,000 - (1,640)^2 = 210,400$. Answer: D

30. We wish to find $P[N_2 = 1|N_1 = 1]$. Using the definition of conditional probability, we have $P[N_2 = 1|N_1 = 1] = \frac{P[N_2=1 \cap N_1=1]}{P[N_1=1]}$. We find the numerator and denominator by conditioning on the value of θ (since we don't know θ).

$$
\begin{aligned}
P[N_1 = 1] &= P[N_1 = 1 \cap \theta = .6] + P[N_1 = 1 \cap \theta = .1] \\
&= P[N_1 = 1|\theta = .6] \cdot P[\theta = .6] + P[N_1 = 1|\theta = .1] \cdot P[\theta = .1] \\
&= e^{-.6}(.6) \cdot (.1) + e^{-.1}(.1)(.9) .
\end{aligned}
$$

30. continued

$P[N_1 = 1 \cap N_2 = 1] = P[N_1 = 1 \cap N_2 = 1 \cap \theta = .6] + P[N_1 = 1 \cap N_2 = 1 \cap \theta = .1]$

$= P[N_1 = 1 \cap N_2 = 1|\theta = .6] \cdot P[\theta = .6] + P[N_1 = 1 \cap N_2 = 1|\theta = .1] \cdot P[\theta = .1]$

$= P[N_1 = 1|\theta = .6] \cdot P[N_2 = 1|\theta = .6] \cdot P[\theta = .6]$

$+ P[N_1 = 1|\theta = .1] \cdot P[N_2 = 1|\theta = .1] \cdot P[\theta = .1]$

$= e^{-.6}(.6) \cdot e^{-.6}(.6) \cdot (.1) + e^{-.1}(.1) \cdot e^{-.1}(.1) \cdot (.9)$.

Note that we have used independence of N_1 and N_2 given $\theta = .6$ and also given $\theta = .1$.

Then $P[N_2 = 1|N_1 = 1] = \frac{e^{-.6}(.6)\cdot(.1)+e^{-.1}(.1)(.9)}{e^{-.6}(.6)\cdot e^{-.6}(.6)\cdot(.1)+e^{-.1}(.1)\cdot e^{-.1}(.1)\cdot(.9)e^{-.6}(.6)\cdot(.1)+e^{-.1}(.1)(.9)} = .159$.

Answer: C

31. $T = X_1 + Y_1 + X_2 + Y_2 + \cdots + X_{100} + Y_{100} = T_1 + T_2 + \cdots + T_{100}$.

Since the individuals are randomly selected, they are independent of one another, the T_i's are independent of one another. $E[T_i] = E[X] + E[Y] = 50 + 20 = 70$,

$Var[T_i] = Var[X] + Var[Y] + 2Cov[X, Y] = 50 + 30 + 2(10) = 100$.

$E[T] = 100E[T_i] = 7000$, $Var[T] = 100Var[T_i] = 10,000$.

We apply the normal approximation to T to get

$P[T < 7100] = P[\frac{T-7000}{\sqrt{10{,}000}} < \frac{7100-7000}{\sqrt{10{,}000}}] = P[Z < 1] = \Phi(1) = .8413$. Answer: B

32. If X denotes the actual loss, then the pdf of X is $f_X(t) = \frac{1}{\lambda}e^{-t/\lambda}$, $t > 0$.

The moment generating function is

$M_X(r) = E[e^{rX}] = \int_0^\infty e^{rt} \cdot \frac{1}{\lambda}e^{-t/\lambda}\, dt = \frac{1}{\lambda}\int_0^\infty e^{-(\frac{1}{\lambda}-r)t} dt = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}-r} = \frac{1}{1-\lambda r}$.

The amount paid by the insurer is $Y = .75X$, and the moment generating function of Y is

$M_Y(s) = E[e^{sY}] = E[e^{s(.75X)}] = E[e^{.75sX}] = M_X(.75s) = \frac{1}{1-.75s\lambda}$. Answer: D

33. This is the pdf for an exponential distribution with a mean of 1000. The expected claim per policy is 1000, and the variance is 1000^2. The premium collected is 1100 per policy. For 100 policies, a total of 110,000 is collected in premium. The total claim is

$W = X_1 + X_2 + \cdots + X_{100}$, and $E[W] = 100E[X] = 100(1000) = 100,000$,

and $Var[W] = 100Var[X] = 100 \cdot 1000^2$. The central limit theorem suggests that W has an approximately normal distribution. Thus,

$P[W > 110,000] = P[\frac{W-E[W]}{\sqrt{Var[W]}} > \frac{110{,}000-E[W]}{\sqrt{Var[W]}}] = P[Z > \frac{110{,}000-100{,}000}{\sqrt{100\cdot 1000^2}}]$

$= P[Z > 1] = 1 - .8413 = .1587$ (Z has an approximately standard normal distribution).

Answer: B

34. Current method has pdf $g(t) = 1$ for $1 < t < 2$.
The cdf of this uniform distribution is $G(t) = P[T \leq t] = t - 1$ for $1 < t < 2$, and $G(t) = 1$ for $t \geq 2$.
The new method claim processing time is $U = ln\, T$, where T is uniformly distributed from 1 to 2. The range for U is $0 = ln\, 1 \leq u \leq ln\, 2$.
The cdf of U is
$F(u) = P[U \leq u] = P[ln\, T \leq u] = P[T \leq e^u] = e^u - 1$ for $0 < u < ln\, 2$,
and $f(u) = 0$ for $u \geq ln\, 2$.
The pdf of U if $f(u) = F'(u) = e^u$ for $0 < u < ln\, 2$, and $f(u) = 0$ otherwise .
Answer: E

35. The correlation coefficient between X and Y is equal to $\frac{Cov[X,Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}}$.
Therefore, $.75 = \frac{Cov[X,Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}} = \frac{Cov[X,Y]}{\sqrt{1]}\sqrt{2}} \Rightarrow Cov[X,Y] = .75\sqrt{2}$.
Then, $Var[X + 2Y] = Var[X] + 2^2 Var[Y] + 2(2)Cov[X,Y]$
$= 1 + 8 + 3\sqrt{2} = 9 + 3\sqrt{2}$. Answer: D

36. $X_i =$ amount of claim i, $i = 1, 2, 3$.
Largest claim is $Y = Max\{X_1, X_2, X_3\}$.
The density function of Y is the derivative of the distribution function of Y,
$f_Y(y) = F'_Y(y)$. The distribution function of Y can be found as follows.
$P[Y \leq y] = P[$each of X_1, X_2, X_3 is $\leq y]$
$= P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y)]$
$= P[(X_1 \leq y)] \cdot P[(X_2 \leq y) \cdot P[(X_3 \leq y)]$. y is measured in thousands.
The last is equality follows from the independence of X_1, X_2 and X_3
($P[A \cap B] = P[A] \cdot P[B]$ for independent events).
$P[X \leq y] = \int_1^y \frac{3}{x^4}\, dx = 1 - \frac{1}{y^3}$ (since each X must be ≥ 1, the same is true for Y). Then
$F_Y(y) = P[Y \leq y] = (1 - \frac{1}{y^3})^3$, from which we get
$f_Y(y) = 3(1 - \frac{1}{y^3})^2(\frac{3}{y^4}) = 9[\frac{1}{y^4} - \frac{2}{y^7} + \frac{1}{y^{10}}]$.
Then, $E[Y] = \int_1^\infty y \cdot f_Y(y)\, dy = \int_1^\infty y \cdot\ 9[\frac{1}{y^4} - \frac{2}{y^7} + \frac{1}{y^{10}}]\, dy$
$= 9\int_1^\infty [\frac{1}{y^3} - \frac{2}{y^6} + \frac{1}{y^9}]\, dy = 9[\frac{1}{2} - 2(\frac{1}{5}) + \frac{1}{8}] = 2.025$.
The measurement is in thousands, so the expected value of the largest claim is 2025.
Answer: A

37. Suppose the bids are X_1, X_2, X_3 and X_4. To find the expected value of the highest bid we need to find the distribution of the highest bid. Let us call the highest bid Y. We can find the distribution function of Y as follows. In order for it to be true that $Y \leq y$, it must be true that $X_1 \leq y$ and $X_2 \leq y$ and $X_3 \leq y$ and $X_4 \leq y$. Thus,
$P[Y \leq y] = P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y) \cap (X_4 \leq y)]$.
Since X_1, X_2, X_3 and X_4 are mutually independent (we are given that the bids are independent), it follows that the events $X_1 \leq y$ and $X_2 \leq y$ and $X_3 \leq y$ and $X_4 \leq y$ are mutually independent. Therefore, $P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y) \cap (X_4 \leq y)$
$= P[(X_1 \leq y)] \cdot P[(X_2 \leq y)] \cdot P[(X_3 \leq y)] \cdot P[(X_4 \leq y)]$
$= F(y) \cdot F(y) \cdot F(y) \cdot F(y) = [\frac{1}{2}(1 + sin\,\pi x)]^4 = P[Y \leq y] = F_Y(y)$.
The density function for Y is then $f_Y(y) = F_Y'(y) = 4[\frac{1}{2}(1 + sin\,\pi x)]^3(\frac{1}{2})(\pi\,cos\,\pi x)$.
The range for Y is between $\frac{3}{2}$ and $\frac{5}{2}$ since the maximum bid must be one of the X's, and all X's are in that range. The mean of Y is then
$E[Y] = \int_{3/2}^{5/2} y \cdot f_Y(y)\,dy = \int_{3/2}^{5/2} y \cdot (\frac{1}{4})(1 + sin\,\pi y)]^3)(\pi\,cos\,\pi y)\,dy$. Answer E.

38. The device fails as soon as either component fails. The probability of failure within the first hour is $P[(X \leq 1) \cup (Y \leq 1)]$. There are a couple of ways in which this can be found.
We can use the probability rule
$P[(X \leq 1) \cup (Y \leq 1)] = P[X \leq 1] + P[Y \leq 1] - P[(X \leq 1) \cap (Y \leq 1)]$,
but this will require three separate double integrals (although the first two are equal because of the symmetry of the distribution).

Alternatively, we can use DeMorgan's rule, $P[A \cup B] = 1 - P[A' \cap B']$, so that
$P[(X \leq 1) \cup (Y \leq 1)] = 1 - P[(X > 1) \cap (Y > 1)]$.
Since both X and Y are between 0 and 3, we get
$P[(X > 1) \cap (Y > 1)] = \int_1^3 \int_1^3 (\frac{x+y}{27})\,dy\,dx = \frac{1}{27} \cdot \int_1^3 [(xy + \frac{1}{2}y^2)\Big|_{y=1}^{y=3}]\,dx$
$= \frac{1}{27} \cdot \int_1^3 (2x+4)\,dx = \frac{1}{27} \cdot (x^2 + 4x)\Big|_{x=1}^{x=3} = \frac{16}{27}$.
Then, $P[(X \leq 1) \cup (Y \leq 1)] = 1 - \frac{16}{27} = \frac{11}{27} = .407$. Answer: B

39. $1-P\left[Y_1<\frac{1}{2}<Y_n\right] = P\left[Y_n \le \frac{1}{2}\right]+P\left[Y_1 \ge \frac{1}{2}\right]$.

$P\left[Y_n \le \frac{1}{2}\right] = P\left[\text{all } Y_i\text{'s} \le \frac{1}{2}\right] = \left(P\left[Y \le \frac{1}{2}\right]\right)^n = \frac{1}{4^n}$.

Similarly, $P\left[Y_1 \ge \frac{1}{2}\right] = P\left[\text{all } Y_i\text{'s} \ge \frac{1}{2}\right] = \left(P\left[Y \ge \frac{1}{2}\right]\right)^n = (\frac{3}{4})^n$.

Thus, $P\left[Y_1<\frac{1}{2}<Y_n\right] = 1-\frac{1}{4^n}-\frac{3^n}{4^n} = \frac{4^n-1-3^n}{4^n}$. Answer: C

40. Since the X_i's are independent and all X_i's have the same variance, say $Var[X]$. it follow that $Var[Y]=\sum_{i=1}^{5}Var[X_i]=5Var[X]$, so that $Var[X]=\frac{1}{5}Var[Y]$.

From the mgf of Y we get $E[Y]=M_Y'(0)$ and $E[Y^2]=M_Y''(0)$.

$M_Y'(t)=e^{15e^t-15}\cdot(15e^t)$, $M_Y''(t)=e^{15e^t-15}\cdot(15e^t)^2+e^{15e^t-15}\cdot(15e^t)$

so that $E[Y]=M_Y'(0)=e^0\cdot(15e^0)=15$, and

$E[Y^2]=M_Y''(0)=e^0(15e^0)^2+e^0(15e^0)=240$.

Then $Var[Y]=E[Y^2]-(E[Y])^2=240-(15)^2=15$, and $Var[X]=\frac{1}{5}Var[Y]=3$.

There are two alternative ways to find the variance of Y. The first alternative uses the fact that $Var[Y]=\frac{d^2}{dt^2}(ln\, M_Y(t))\Big|_{t=0}$. In this case, $ln\, M_Y(t)=15e^t-15$,

and $\frac{d^2}{dt^2}(ln\, M_Y(t))=\frac{d^2}{dt^2}(15e^t-15)=15e^t \rightarrow \frac{d^2}{dt^2}(ln\, M_Y(t))\Big|_{t=0}=15e^0=15$.

The second alternative requires making the observation that if Z has a Poisson distribution with mean λ, then the mgf of Z is $M_Z(t)=e^{\lambda(e^t-1)}$. In this case it can be seen that Y has a Poisson distribution with $\lambda=15$. Therefore the variance (and mean) of Y is 15 and the variance of each X is $Var[X]=\frac{1}{5}\times Var[Y]=3$. Answer: B

41. $M(t_1,t_2)=E[e^{t_1W+t_2Z}]=E[e^{t_1(X+Y)+t_2(Y-X)}]=E[e^{(t_1-t_2)X+(t_1+t_2)Y}]$

$=E[e^{(t_1-t_2)X}\cdot e^{(t_1+t_2)Y}]=E[e^{(t_1-t_2)X}]\cdot E[e^{(t_1+t_2)Y}]$

(this equality follows from the independence of X and Y)

$=M_X(t_1-t_2)\cdot M_Y(t_1+t_2)=e^{(t_1-t_2)^2/2}\cdot e^{(t_1+t_2)^2/2}=e^{t_1^2+t_2^2}$.

Answer: E

SECTION 10 - REVIEW OF RISK MANAGEMENT CONCEPTS

LOSS DISTRIBUTIONS AND INSURANCE

Loss and insurance: When someone is subject to the risk of incurring a financial loss, the loss is generally modeled using a random variable or some combination of random variables. The loss is often related to a particular time interval. For example, an individual may own property that might suffer some damage during the following year. Someone who is at risk of a financial loss may choose some form of **insurance** protection to reduce the impact of the loss. An **insurance policy** is a contract between the party that is at risk (the **policyholder**) and an **insurer**. This contract generally calls for the policyholder to pay the insurer some specified amount, the **insurance premium**, and in return, the insurer will reimburse certain **claims** to the policyholder. A claim is all or part of the loss that occurs, depending on the nature of the insurance contract.

Modeling a loss random variable: There are a few ways of modeling a random loss/claim for a particular insurance policy, depending on the nature of the loss. Unless indicated otherwise, we will assume the amount paid to the policyholder as a claim is the amount of the loss that occurs. Once the random variable X representing the loss has been determined, the expected value of the loss, $E[X]$, is referred to as the **pure premium** for the policy. $E[X]$ is also the **expected claim** on the insurer. Note that in general, one of the outcomes of X might be 0, since it may be possible that no loss occurs. The following are the basic models used for describing the loss random variable. For a random variable X a measure of the risk is $\sigma^2 = Var[X]$. The **unitized risk** or **coefficient of variation** for the random variable X is defined to be $\frac{\sqrt{Var[X]}}{E[X]} = \frac{\sigma}{\mu}$.

Many insurance policies do not cover the full amount of the loss that occurs, but only provide partial coverage. There are a few standard types of partial insurance coverage that can be applied to a basic **ground up loss** (full loss) random variable X. These are described starting on the following page.

PARTIAL INSURANCE COVERAGE

(i) Deductible insurance: A deductible insurance specifies a **deductible amount**, say d. If a loss of amount X occurs, the insurer pays nothing if the loss is less than d, and pays the policyholder the amount of the loss in excess of d if the loss is greater than d. The amount paid by the insurer can be described as $\boldsymbol{Y = \begin{cases} 0 \text{ if } X \leq d \\ X-d \text{ if } X>d \end{cases} = Max\{X-d, 0\}}$. This is also denoted $(X-d)_+$. The expected payment made by the insurer when a loss occurs would be $\boldsymbol{\int_d^\infty (x-d)\, f_X(x)\, d}x$ in the continuous case; (this is also called the *expected cost per loss*). With integration by parts, this can be shown to be equal to $\boldsymbol{\int_d^\infty [1-F_X(x)]\, dx}$. This type of policy is also referred to as an ordinary deductible insurance.

Two variations on deductible insurance are the franchise deductible, and the disappearing deductible. These are less likely to appear on the exam.

(a) **Franchise deductible:** A franchise deductible of amount d refers to the situation in which the insurer pays 0 if the loss is below d but pays the full amount of loss if the loss is above d; the amount paid by the insurer can be described as $\begin{cases} 0 \text{ if } X \leq d \\ X \text{ if } X>d \end{cases}$.

(b) **Disappearing deductible:** A disappearing deductible with lower limit d and upper limit d' (where $d < d'$) refers to the situation in which the insurer pays 0 if the loss is below d, the insurer pays the full loss if the loss amount is above d', and the deductible amount reduces linearly from d to 0 as the loss increases from d to d'; the amount paid by the insurer can be described as $\begin{cases} 0 & X \leq d \\ d' \cdot \frac{X-d}{d'-d} & d < X \leq d' \\ X & X > d' \end{cases}$

Example 10-1: A discrete loss random variable X has the following two-point distribution: $P[X=3] = P[X=12] = 0.5$. An ordinary deductible insurance policy is set up for this loss, with deductible d. It is found that the expected claim on the insurer is 3. Find d.

Solution: The claim on the insurer is $Y = \begin{cases} 0 \text{ if } X \leq d \\ X-d \text{ if } X>d \end{cases}$.

We proceed by "trial-and-error". Suppose our initial "guess" is that $d \leq 3$.

Then the claim on the insurer will be either $3-d$ or $12-d$, each with probability 0.5. so that the expected claim on the insurer will be $(3-d)(.5) + (12-d)(.5) = 3$, which implies that $d = 4.5$. This contradicts our "guess" that $d \leq 3$.

This indicates that the guess was wrong. Thus, $d > 3$, so that the claim on the insurer will be 0 (if $X = 3$) or $12 - d$, each with probability 0.5. The expected claim on the insurer will then be $(12-d)(.5) = 3 \rightarrow d = 6$. □

Example 10-2: A loss random variable is uniformly distributed between 0 and 1000. A deductible of 200 is applied before any insurance payment. Find the expected amount paid by the insurer when a loss occurs.

Solution: Expected insurance payment is

$E[(X-200)_+] = \int_{200}^{1000} (x-200)(\frac{1}{1000})\,dx = 320\,.$ □

Example 10-3: A loss random variable is exponentially distributed with a mean of 1000. A deductible of 200 is applied before any insurance payment. Find the expected amount paid by the insurer when a loss occurs.

Solution: Expected insurance payment is

$E[(X-200)_+] = \int_{200}^{\infty} (x-200)(\frac{e^{-x/1000}}{1000})\,dx = \int_{200}^{\infty} [1-F_X(x)]\,dx\,.\,.$

We can calculate this integral three ways.

(a) $\int_{200}^{\infty} (x-200)(\frac{e^{-x/1000}}{1000})\,dx = \int_{200}^{\infty} x(\frac{e^{-x/1000}}{1000})\,dx - 200\int_{200}^{\infty}(\frac{e^{-x/1000}}{1000})\,dx$.

$= -xe^{-x/1000} - 1000e^{-x/1000} + 200e^{-x/1000}\Big|_{x=200}^{\infty}$

$= -0 - 0 + 200e^{-.2} + 800e^{-.2} = 818.73$.

(b) Apply the change of variable $y = x - 200$. The integral becomes $\int_0^{\infty} y \cdot \frac{e^{-y/1000}}{1000}\,dy$ and we can use the general rule $\int_0^{\infty} t^n e^{-ct}\,dt = \frac{n!}{c^{n+1}}$ (when n is an integer ≥ 0 and $c > 0$) to get $\int_0^{\infty} y \cdot \frac{e^{-(y+200)/1000}}{1000}\,dy = \frac{e^{-.2}}{1000} \cdot \int_0^{\infty} ye^{-y/1000}\,dy$

$= (.0008187)(\frac{1}{(1/1000)^2}) = 818.73$.

(c) $\int_{200}^{\infty} [1-F_X(x)]\,dx = \int_{200}^{\infty} e^{-x/1000}\,dx = 1000e^{-.2} = 818.73$. □

(ii) Policy limit: A **policy limit of amount $\boldsymbol{u}$** indicates that the insurer will pay a maximum amount of u when a loss occurs. Therefore, the amount paid by the insurer is $\begin{cases} \boldsymbol{X} \textbf{ if } \boldsymbol{X \le u} \\ u \text{ if } X > u \end{cases}$.

The expected payment made by the insurer per loss would be

$\int_0^u x \cdot f_X(x)\,dx + u \cdot [1 - F_X(u)]$ in the continuous case.

This can be shown to be equal to $\int_0^u [1 - F_X(x)]\,dx$.

NOTE: Suppose that X is the loss random variable. An insurance policy which pays the loss amount in excess of deductible c pays $Y_1 = \begin{cases} 0 & \text{if } X \le c \\ X - c & \text{if } X > c \end{cases}$,

and an insurance policy which pays the loss up to a limit of c pays $Y_2 = \begin{cases} X \text{ if } X \le c \\ c \text{ if } X > c \end{cases}$.

The combined payment of the two policies is $Y_1 + Y_2 = X$, since Y_2 covers any loss up to amount c and Y_1 covers the loss in excess of c. If an insurance policy with an ordinary deductible of c is purchased, then the part of the loss not paid by the insurance policy will be algebraically the same as the amount paid on a policy with policy limit c, and vice versa.

Questions on ordinary deductible and policy limit have come up regularly on the exam.

Example 10-4: A loss random variable is uniformly distributed between 0 and 1000. An insurance policy pays the loss up to a maximum of 200. Find the expected amount paid by the insurer when a loss occurs.

Solution: The expected amount paid by the insurer is

$\int_0^{200} x \cdot (\frac{1}{1000})\,dx + 200 \cdot [1 - F_X(200)] = 20 + (200)(1 - .2) = 180$.

This is the same loss random variable as in Example 10-2. In that example, with a deductible of 200, the expected amount paid by the insurer was 320. As pointed out above in the notes above, the combination of an insurance with a deductible of 200 and an insurance with a policy limit of 200 is the full loss. Therefore, it is no coincidence that the expected amounts paid by the insurer on the combination of the policy in this example and the one in Example 10-2 add up to the overall expected loss of 500. □

It is possible to combine a deductible insurance with a policy limit. If a policy has a deductible of d and a limit of $u - d$ then the claim amount paid by the insurer can be described as

$\begin{cases} 0 \text{ if } X \le d \\ X - d \text{ if } d < X \le u \\ u - d \text{ if } X > u \end{cases}$. Note that "the deductible is applied after the policy limit is applied".

This means that a loss of amount greater than u triggers the maximum insurance payment of amount $u-d$. The expected payment made by the insurer per loss would be
$\int_d^u (x-d)\cdot f_X(x)\,dx + (u-d)\cdot[1-F_X(u)]$ in the continuous case.
This can be shown to be equal to $\int_d^u [1-F_X(x)]\,dx$.

Example 10-5: A loss random variable is uniformly distributed on $(0\,,\,1000)$.
(a) Find the mean and variance of the insurance payment if a deductible of 250 is imposed.
(b) Find the mean and variance of the insurance payment if a policy limit of 500 is imposed.
(c) Find the mean and variance if a policy limit of 250 and a deductible of 250 is imposed.

Solution: Suppose the insurance payment is Y.
(i) $E[Y] = \int_{250}^{1000}(x-250)\cdot\frac{1}{1000}\,dx = \frac{(x-250)^2}{2(1000)}\Big|_{x=250}^{x=1000} = 281.25$,
$E[Y^2] = \int_{250}^{1000}(x-250)^2\cdot\frac{1}{1000}\,dx = \frac{(x-250)^3}{3(1000)}\Big|_{x=250}^{x=1000} = 140,625$,
$Var[Y] = E[Y^2]-(E[Y])^2 = 140,625-(281.25)^2 = 61,523$.
(ii) $E[Y] = \int_0^{500} x\cdot\frac{1}{1000}\,dx + 500[1-F(500)] = 125+500(1-\frac{1}{2}) = 375$,
$E[Y^2] = \int_0^{500} x^2\cdot\frac{1}{1000}\,dx + (500)^2[1-F(500)] = 41,667+125,000 = 166,667$,
$Var[Y] = E[Y^2]-(E[Y])^2 = 26,042$.
(iii) The policy limit is $u-d = u-250 = 250$, so that $u=500$. Then
$E[Y] = \int_{250}^{500}(x-250)\cdot\frac{1}{1000}\,dx + 250[1-F(500)] = 31.25+(250)(\frac{1}{2}) = 156.25$,
$E[Y^2] = \int_{250}^{500}(x-250)^2\cdot\frac{1}{1000}\,dx + (250)^2[1-F(500)] = 5208+31,250 = 36,458$,
$Var[Y] = E[Y^2]-(E[Y])^2 = 12,044$. □

(iii) Proportional insurance - Proportional insurance specifies a fraction α $(0<\alpha<1)$, and if a loss of amount X occurs, the insurer pays the policyholder αX, the specified fraction of the full loss.

Models for describing a loss random variable X:

Case 1 (most likely): The complete description of X is given:

In this case, if X is continuous, the density function $f(x)$ or distribution function $F(x)$ is given. If X is discrete, the probability function (or possibly the distribution function) is given. One typical (and simple) example of the discrete case is a loss random variable of the form

$$X = \begin{cases} K & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{cases}$$

(this might arise in a one- year term life insurance in which the death benefit is K, paid if the policyholder dies within the year, and probability of death within the year is q).
Another example of a discrete loss random variable (with more than two points) is the following example of dental expenses for a family over a one-year period.

Amount of Dental Expense	Probability
\$ 0	.1
200	.1
400	.3
800	.4
1500	.1

In some problems, all that is needed is the mean and variance of X, and sometimes that is the only information about X that is given (rather than the full description of X's distribution).

Case 2 (less likely): The probability q of a non-negative loss is given, and the <u>conditional</u> distribution B of loss amount <u>given that a loss has occurred</u> is given:

The probability of no loss occurring is $1-q$, and the loss amount X is 0 if no loss occurs; thus, $P[X=0]=1-q$. If a loss does occur, the loss amount is the random variable B, so that $X=B$. The random variable B is the loss amount **<u>given that a loss has occurred</u>**, so that B is really the **conditional distribution of the loss amount X given that a loss occurs**. The random variable B might be described in detail, or only the mean and variance of B might be known. Note that if $E[B]$ and $Var[B]$ are given, then $E[B^2]=Var[B]+(E[B])^2$ (this is needed in the formulation of $Var[X]$).

We can formulate X as a "mixture" of two random variables, W_1 and W_2, where $W_1=0$ is the constant random variable (not really random at all), and $W_2=B$, and with mixing weights $\alpha_1=1-q$ and $\alpha_2=q$. Then the first two moments of X are
$E[X]=(1-q)E[W_1]+q\cdot E[W_2]=\boldsymbol{q\cdot E[B]}$, since $E[W_1]=0$, and

$E[X^2] = (1-q)E[W_1^2] + q \cdot E[W_2^2] = \boldsymbol{q \cdot E[B^2]}$. Then,
$\boldsymbol{Var[X] = q \cdot E[B^2] - (q \cdot E[B])^2} = qVar[B] + q(1-q)(E[B])^2$
(note that $Var[X]$ **is not** $q \cdot Var[B]$).
The mixing weight $1-q$ applies to $W_1 = 0$, which means that there is a probability $1-q$ the no loss will occur (loss $= 0$). The mixing weight q (the probability of a loss occurring) is applied to B, the random loss amount when a loss occurs.

For example, the loss due to fire damaging a particular property might be modeled this way. Suppose that $q = .01$ is the probability that fire damage occurs, and given that fire damage occurs, the amount of damage, B, has a uniform distribution between \$$10,000$ and \$$50,000$. **Keep in mind that $\boldsymbol{B}$ is the loss amount given that a loss has occurred, whereas $\boldsymbol{X}$ is the overall loss amount**. Then, $E[B] =$ \$$30,000$ and $E[B^2] = \frac{3{,}100{,}000{,}000}{3}$ (first and second moment of the uniform distribution on $(10,000\,,\,50,000)$. Using the formulas above,
$E[X] = (.01)(30,000) =$ \$300 , and
$Var[X] = q \cdot E[B^2] - (q \cdot E[B])^2 = (.01)(\frac{3{,}100{,}000{,}000}{3}) - (300)^2 = \frac{30{,}730{,}000}{3}$.

Example 10-6: For a one-year dental insurance policy for a family, we consider the following two models for annual claims X:

Model 1:

Amount of Dental Expense (X)	Probability
\$ 0	.1
200	.1
400	.3
800	.4
1500	.1

Model 2: There is a probability of $.1$ that no claim occurs , $P[X = 0] = .1$. If a claim does occur, the claim amount random variable B, has mean $E[B] = 677.78$ and variance $Var[B] = 132,839.51$.

For each loss model, find $E[X]$ and $Var[X]$.

Solution: Model 1: In this case the complete description of X is given (Case 1 mentioned above). $E[X] = 0(.1) + 200(.1) + 400(.3) + 800(.4) + 1500(.1) = 610$,
$E[X^2] = 0^2(.1) + 200^2(.1) + 400^2(.3) + 800^2(.4) + 1500^2(.1) = 533,000$
$Var[X] = 533,000 - 610^2 = 160,900$.

Model 2: In this case, the probability of a claim occurring is given ($q = .9$) along with the mean and variance of the conditional distribution B of claim amount given that a claim occurs (Case 2 mentioned above). $E[X] = q \cdot E[B] = (.9)(677.78) = 610$,
$E[X^2] = q \cdot E[B^2] = (.9)[Var[B] + (E[B])^2] = (.9)[132,839.51 + (677.78)^2] = 533,003$.
$Var[X] = E[X^2] - (E[X])^2 = 160,900$.
Note that it is not a coincidence that the mean and variance of X turned out to be the same for Models 1 and 2. This is true because the mean and variance of B in Model 2 were chosen as the conditional mean and variance of the distribution in Model 1 given that a claim occurs. □

Modeling the aggregate claims in a portfolio of insurance policies,
The Individual Risk Model

The individual risk model assumes that the portfolio consists of a specific number, say n, of insurance policies, with the claim for one period on policy i being the random variable X_i. X_i would be modeled in one of the ways described above for an individual policy loss random variable. Unless mentioned otherwise, it is assumed that the X_i's are mutually independent random variables. Then the aggregate claim is the random variable
$\boldsymbol{S = \sum_{i=1}^{n} X_i}$, with $\boldsymbol{E[S] = \sum_{i=1}^{n} E[X_i]}$ and $\boldsymbol{Var[S] = \sum_{i=1}^{n} Var[X_i]}$.
If $E[X_i] = \mu$ and $Var[X_i] = \sigma^2$ for each $i = 1, 2, ..., n$, then the coefficient of variation of the aggregate claim distribution S is $\frac{\sqrt{Var[S]}}{E[S]} = \frac{\sqrt{nVar[X]}}{nE[X]} = \frac{\sigma}{\mu\sqrt{n}}$, which goes to 0 as $n \to \infty$.

Example 10-7: An insurer has a portfolio of 1000 one-year term life insurance policies just issued to 1000 different (independent) individuals. Each policy will pay $1000 in the event that the policyholder dies within the year. For 500 of the policies, the probability of death is .01 per policyholder, and for the other 500 policies the probability of death is .02 per policyholder. Find the expected value and the standard deviation of the aggregate claim that the insurer will pay.

Solution: The aggregate claim random variable is $S = \sum_{i=1}^{1000} X_i$, where X_i is the claim from policy i. Then $E[S] = \sum_{i=1}^{1000} E[X_i]$ and since the claims are independent, $Var[S] = \sum_{i=1}^{1000} Var[X_i]$.
If Y_1 is one of the 500 policies with death probability .01, then $Y_1 = \begin{cases} 0 & \text{prob. .99} \\ 1000 & \text{prob. 01} \end{cases} \Rightarrow$
$E[Y_1] = 1000(.01) = 10$, $Var[Y_1] = E[Y_1^2] - (E[Y_1])^2 = 9900$.
If Y_2 is one of the 500 policies with death probability .02, then $E[Y_2] = 20,\ Var[Y_2] = 19,600$.
Thus, $E[S] = 500(10) + 500(20) = 15,000$,
$Var[S] = \sum_{i=1}^{1000} Var[X_i] = 500(9900) + 500(19,600) = 14,750,000 \Rightarrow \sqrt{Var[S]} = 3,841.$ □

Example 10-8: Two portfolios of independent insurance policies have the following characteristics:

Portfolio A:

Class	Number in Class per Policy	Probability of Claim	Claim Amount
1	2,000	0.05	1
2	500	0.10	2

Portfolio B:

Class	Number in Class per Policy	Probability of Claim	Claim Amount Distribution Mean	Variance
1	2,000	0.05	1	1
2	500	0.10	2	4

The aggregate claims in the portfolios are denoted by S_A and S_B.
Find $\frac{Var[S_A]}{Var[S_B]}$.

Solution: In this example, Portfolio B information is given in the following form: for policy i, the probability of a claim occurring is given, q_i, and the mean and variance of the conditional distribution of claim amount given a claim occurs is given, $E[B_i] = \mu_i$, $Var[B_i] = \sigma_i^2$.
Note that for each policy in Portfolio A, $Var[B_i] = 0$. When the loss random variable is given in this form, we have for policy i, $E[X_i] = q_i \cdot E[B_i]$, and
$Var[X_i] = q_i(1-q_i)(E[B_i])^2 + q_i \cdot Var[B_i]$, and
for a portfolio of independent policies, $E[S] = \sum E[X_i]$ and $Var[S] = \sum Var[X_i]$.

For Portfolio A, any policy in Class 1 has
$Var[X_i] = q_i(1-q_i)(E[B_i])^2 + q_i \cdot Var[B_i] = (.05)(.95)(1^2) + (.05)(0) = .0475$
and any policy in Class 2 has $Var[X_i] = (.10)(.90)(2^2) + (.10)(0) = .36$, so that
$Var[S_A] = 2000(.0475) + 500(.36) = 275$.

For Portfolio B, any policy in Class 1 has $Var[X_i] = (.05)(.95)(1^2) + (.05)(1) = .0975$
and any policy in Class 2 has $Var[X_i] = (.10)(.90)(2^2) + (.10)(4) = .76$, so that
$Var[S_B] = 2000(.0975) + 500(.76) = 575$.
Then, $\frac{Var[S_A]}{Var[S_B]} = .478$. □

The normal approximation to aggregate claims: For an aggregate claims distribution S, if the mean and variance of S are known ($E[S]$ and $Var[S]$), it is possible to approximate probabilities for S by using the normal distribution. The 95-th percentile of aggregate claims is the number Q for which $P[S \leq Q] = .95$. If S is assumed to have a distribution which is approximately normal, then by standardizing S we have $P[S \leq Q] = P[\frac{S-E[S]}{\sqrt{Var[S]}} \leq \frac{Q-E[S]}{\sqrt{Var[S]}}] = .95$, so that $\frac{Q-E[S]}{\sqrt{Var[S]}}$ is equal to the 95-th percentile of the standard normal distribution (which is found to be 1.645 when referring to the standard normal table), so that Q can be found; $Q = E[S] + 1.645 \cdot \sqrt{Var[S]}$. If the insurer collects total premium of amount Q, then (assuming that it is reasonable to use the approximation) there is a 95% chance (approximately) that aggregate claims will be less than the premium collected, and there is a 5% chance that aggregate claims will exceed the premium. Since S is a sum of many independent individual policy loss random variables, the Central Limit Theorem suggests that the normal approximation to S is not unreasonable. **The normal approximation has come up frequently on past exams.** The normal approximation and the integer correction when applied to integer-valued random variables was discussed earlier in Section 7 of this study guide.

Example 10-9: An insurance company provides insurance to three classes of independent insureds with the following characteristics:

			For Each Insured	
Class	Number in Class	Probability of a Claim	Expected Claim Amount	Variance of Claim Amount
1	500	0.05	5	5
2	1000	0.10	10	10
3	500	0.15	5	5

For each class, the amount of premium collected is $(1+\theta)$(expected claims), where θ is the same for all three classes. Using the normal approximation to aggregate claims, find θ so that the probability that total claims exceed the amount of premium collected is .05 .

Solution: We wish to find $Q = (1+\theta)E[S]$, so that $P[S > Q] = .05$, or equivalently, $P[S \leq Q] = .95$.

Applying the normal approximation and standardizing S, this can be written in the form $P[S \leq Q] = P[\frac{S-E[S]}{\sqrt{Var[S]}} \leq \frac{Q-E[S]}{\sqrt{Var[S]}}] = .95$, so that $\frac{Q-E[S]}{\sqrt{Var[S]}} = 1.645$ (the 95-th percentile of the standard normal distribution). Thus, once $E[S]$ and $Var[S]$ are found, we can find $Q = (1+\theta)E[S]$, and then find θ.

$E[S] = \Sigma E[X_i] = \Sigma q_i \cdot E[B_i] = 500(.05)(5) + 1000(.1)(10) + 500(.15)(5) = 1500$,

Example 10-9 continued
since there are 500 policies in class 1, each with expected claim $(.05)(5)$, and similarly for classes 2 and 3. The policies are independent so that the variance of the sum of all policy claims is the sum of the variances (no covariances when independence is assumed). The variance of a claim for a policy from class 1 is $q(1-q)\cdot(E[B])^2 + q\cdot Var[B] = (.05)(.95)(5^2)+(.05)(5)$, and there are 500 of those policies, and similarly for classes 2 and 3.

$Var[S] = \Sigma Var[X_i] = \Sigma[q_i(1-q_i)\cdot(E[B_i])^2 + q_i \cdot Var[B_i]\,]$

$= 500[(.05)(.95)(5^2)+(.05)(5)] + 1000[(.1)(.9)(10^2)+(.1)(10)]$

$\quad + 500[(.15)(.85)(5^2)+(.15)(5)] = 12,687.5$.

Then, $Q = 1685.29$, and $\theta = .1235$. □

Example 10-10: Suppose that a multiple choice exam has 40 questions, each with 5 possible answers. A well prepared student feels that he has a probability of .5 of getting any particular question correct, with independence from question to question. Apply the normal approximation to X, the number of correct answers out of 40 to determine the probability of getting at least 25 correct. Find the probability with the integer correction, and then without the correction.

Solution: The number of questions answered correctly, say X, has a binomial distribution with mean $(40)(.5) = 20$ and variance $(40)(.5)(.5) = 10$. Applying the normal approximation to X, with integer correction since the binomial distribution is a discrete integer-valued random variable to find the probability of answering at least 25 correct, we get

$P[X \geq 25] = P[X \geq 24.5] = P[\frac{X-20}{\sqrt{10}} \geq \frac{24.5-20}{\sqrt{10}}] = P[Z \geq 1.42] = 1 - \Phi(1.42) = .078$.

Without the integer correction, the probability is

$P[X \geq 25] = P[\frac{X-20}{\sqrt{10}} \geq \frac{25-20}{\sqrt{10}}] = P[Z \geq 1.58] = 1 - \Phi(1.58) = .057$.

There is a noticeable difference between the two approaches. If X has a much larger standard deviation, then the difference is no so noticeable. □

Mixture of Loss Distributions

A portfolio of policies might consist of two or more classes of policyholders, as in the previous example. In the previous example, the number of policies in each class was known. It may be the case that the number of policies in each class is not known but the proportion of policies in each class is known. In such a situation, we might be asked to describe the distribution of the loss for a randomly chosen policy from the portfolio of policies. This will be a mixture of the distributions representing the different classes of policyholders. Mixtures of distributions were considered near the end of Section 9 of the study guide. The following example illustrates this idea.

Example 10-11: The insurer of a portfolio of automobile insurance policies classifies each policy as either high risk, medium risk or low risk. The portfolio consists of 10% high risk policies, 30% medium risk and 60% low risk. The claim means and variances for the three risk classes are

	mean	variance
high risk	10	16
medium risk	4	4
low risk	1	1

A policy is chosen at random from the portfolio. Find the mean and variance of this policy.

Solution: The distribution of the randomly chosen policy is the mixture of the three risk class claim distributions, using the percentages as the mixing factors. If X denotes the claim for the randomly chosen policy, then all moments of X (pdf and cdf also) are the weighted averages of the moments for the component distributions in the mixture.

$E[X] = (.1)(10) + (.3)(4) + (.6)(1) = 2.8$ is the mean.

Since $Var[X] = E[X^2] - (E[X])^2$, we need $E[X^2]$ in order to find the variance of X.

Let X_H denote the claim random variable for a high risk policy. Then

$16 = Var[X_H] = E[X_H^2] - (E[X_H])^2 = E[X_H^2] - (10)^2$, from which we get $E[X_H^2] = 116$.

In a similar way we get $E[X_M^2] = 4 + (4)^2 = 20$ and $E[X_L^2] = 1 + (1)^2 = 2$.

Then $E[X^2] = (.1)(116) + (.3)(20) + (.6)(2) = 18.8$, and $Var[X] = 18.8 - (2.8)^2 = 10.96$.

Note that the variance of X is not the weighted average of the variances of X_H , X_M and X_L

We can use the conditional variance formula $\boldsymbol{Var[X] = E[\,Var[X|Y]\,] + Var[\,E[X|Y]\,]}$ to get the variance of X. If we define Y to be the 3-point random variable $Y = \{H, M, L\}$ with probabilities $P(Y = H) = .1$ (high risk) , $P(Y = M) = .3$ and $P(Y = L) = .6$, then from the information given, we have $E(X|Y = H) = 10$, $Var(X|Y = H) = 16$,

$E(X|Y = M) = 4$, $Var(X|Y = M) = 4$, and $E(X|Y = L) = 1$, $Var(X|Y = L) = 1$.

Then, from the double expectation rule, we have

$E[X] = E[\,E[X|Y]\,] = 10 \times .1 + 4 \times .3 + 1 \times .6 = 2.8$.

Example 10-11 continued

We can think of $Var[X|Y]$ as a 3-point random variable

$$Var(X|Y) = \begin{cases} Var(X|Y=H) = 16 & \text{prob. .1} \\ Var(X|Y=M) = 4 & \text{prob. .3} \\ Var(X|Y=L) = 1 & \text{prob. .6} \end{cases}$$

And then we get the expected value of that 3-point random variable

$E[Var[X|Y]] = 16 \times .1 + 4 \times .3 + 1 \times .6 = 3.4$.

To find $Var[E[X|Y]]$, we think of $E(X|Y)$ as a 3-point random variable,

$$E(X|Y) = \begin{cases} E(X|Y=H) = 10 & \text{prob. .1} \\ E(X|Y=M) = 4 & \text{prob. .3} \\ E(X|Y=L) = 1 & \text{prob. .6} \end{cases} .$$

We then find the variance of this 3-point random variable:

$10^2 \times .1 + 4^2 \times .3 + 1^2 \times .6 - (10 \times .1 + 4 \times .3 + 1 \times .6)^2 = 15.4 - 2.8^2 = 7.56$.

This is $Var[E[X|Y]]$.

Then $Var[X] = E[Var[X|Y]] + Var[E[X|Y]] = 3.4 + 7.56 = 10.96$. □

Loss Distribution Formulated By Conditioning

A loss distribution may be presented in a conditional format in the following way. We may be given the conditional distribution of the loss variable X given some other variable. For instance, in Example 10-11, we were presented with the distribution of claim for three types of policies, high, medium and low risk. What we are given is actually the conditional claim X given risk type. We are given the conditional mean $E(X|\text{risk type})$ and the conditional variance $Var(X|\text{risk type})$. If we denote "risk type" as Y, we can think of a randomly chosen policy as coming from the distribution of Y where $Y = \{H, M, L\}$ with probabilities $P(Y = H) = .1$ (high risk) , $P(Y = M) = .3$ and $P(Y = L) = .6$. Then we can use the double expectation rule and the conditional variance rule to find the overall or total mean claim $E(X)$ and overall claim variance $Var(X)$.

A special case that arises when a loss distribution is formulated by conditioning in this way is the following. Suppose that the number of losses, say N, in specified period of time has a Poisson distribution with mean λ. Suppose that the severity of each loss is a random variable X, and suppose that the number of losses and the amount of each loss are mutually independent. The mean and variance of the aggregate loss S can be found as follows.

$E[S] = E[E[S|N]] = E[N \times E[X]] = E[N] \times E[X] = \lambda \times E[X]$, and

$Var[S] = E[Var[S|N]] + Var[E[S|N]] = E[N \times Var[X]] + Var[N \times E[X]]$

$= E[N] \times Var[X] + Var[N] \times (E[X])^2 = \lambda \times (Var[X] + E[X]^2) = \lambda \times E[X^2]$.

Example 10-12: 40. (SOA) The number of hurricanes that will hit a certain house in the next ten years is Poisson distributed with mean 4. Each hurricane results in a loss that is exponentially distributed with mean 1000. Losses are mutually independent of the number of hurricanes. Calculate the mean and variance of the total loss due to hurricanes hitting the house in the next ten years.

Solution: The number of hurricanes in 10 years is N, which is Poisson with a mean of 4. The size of each hurricane loss X is exponential with mean 1000. The average loss in 10 years is $E[S] = E[N] \times E[X] = 4 \times 1000 = 4000$. The variance of the loss in 10 years is
$Var[S] = E[N] \times Var[X] + Var[N] \times (E[X])^2$
$= 4 \times 1000^2 + 4 \times 1000^2 = 8,000,000.$
Alternatively, since N has a Poisson distribution, we also have
$Var[S] = E[N[\times E[X^2] = 4 \times (2 \times 1000^2) = 8,000,000$. □

PROBLEM SET 10

Loss Distributions and Insurance

1. (SOA) An insurance policy pays an individual 100 per day for up to 3 days of hospitalization and 25 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability function

$$P(X=k)=\begin{cases} \frac{6-k}{15} & \text{for } \ k=1,2,3,4,5 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected payment for hospitalization under this policy.

A) 85 B) 163 C) 168 D) 213 E) 255

2. An insurance company issues insurance contracts to two classes of independent lives, as shown below.

Class	Probability of Death	Benefit Amount	Number in Class
A	0.01	200,000	500
B	0.05	100,000	300

The company wants to collect an amount, in total, equal to the 95-th percentile of the distribution of total claims. The company will collect an amount from each life insured that is proportional to that life's expected claim. That is, the amount for life j with expected claim $E[X_j]$ would be $kE[X_j]$. Calculate k.

A) 1.30 B) 1.32 C) 1.34 D) 1.36 E) 1.38

3. (SOA) An insurance policy reimburses a loss up to a benefit limit of 10 . The policyholder's loss, Y, follows a distribution with density function:

$$f(y)=\begin{cases} \frac{2}{y^3} & \text{for } \ y>1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

A) 1.0 B) 1.3 C) 1.8 D) 1.9 E) 2.0

4. (SOA) A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E[X]$.

A) $2 + \frac{1}{3}e^{-6}$ B) $2 - 2e^{-2/3} + 5e^{-4/3}$ C) 3 D) $2 + 3e^{-2/3}$ E) 5

5. (SOA) The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X has density function

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 0 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Let Y be the age of the machine at the time of replacement. Determine the variance of Y.

A) 1.3 B) 1.4 C) 1.7 D) 2.1 E) 7.5

6. (SOA) An insurance policy is written to cover a loss, X, where X has a uniform distribution on [0, 1000] . At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible?

A) 250 B) 375 C) 500 D) 625 E) 750

7. A derivative investment speculator identifies certain technical financial conditions which, when they arise, allow her to place an investment whose return will be normally distributed with a mean return of \$100,000 and a standard deviation of \$20,000. Experience indicates that the number of times per year these specific financial conditions arise has a Poisson distribution with a mean of 3. Assuming that the financial conditions arise independently of one another and that the speculator places the investment each time it arises, find the probability that the speculator earns less than \$100,000 in a year in total on her investments.

A) .12 B) .18 C) .24 D) .30 E) .36

8. (SOA) An insurance company sells a one-year automobile policy with a deductible of 2 The probability that the insured will incur a loss is 0.05. If there is a loss, the probability of a loss of amount N is $\frac{K}{N}$, for $N = 1, \ldots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur. Determine the net premium for this policy.

A) 0.031 B) 0.066 C) 0.072 D) 0.110 E) 0.150

9. A company's dental plan pays the annual dental expenses above a deductible of \$100 for each of 50 employees. The distribution of annual dental expenses X for an individual employee is

$$X = \begin{cases} 0, \text{ prob. } .1 \\ 100, \text{ prob. } .2 \\ 200, \text{ prob. } .4 \\ 500, \text{ prob. } .2 \\ 1000, \text{ prob. } .1 \end{cases}.$$

Using the normal approximation, find the 95th percentile of the aggregate annual claims distribution that the company pays (nearest \$10).

A) 11,640 B) 12,640 C) 13,640 D) 14,640 E) 15,640

10. A portfolio of independent one-year insurance policies has three classes of policies:

Class	Number in Class	Probability of Claim per Policy	Claim Amount
1	1000	.01	1
2	2000	.02	1
3	500	.04	2

Find the standard deviation of the aggregate one-year claims distribution.

A) 10.0 B) 10.4 C) 10.8 D) 11.2 E) 11.6

11. A loss random variable X has the following (cumulative) distribution function:

$$F(x) = \begin{cases} 0, \text{ if } x < 0 \\ .2 + .3x, \text{ if } 0 \le x < 2 \\ 1, \text{ if } x \ge 2 \end{cases}.$$

An insurer will provide proportional insurance on this loss, covering fraction α of the loss ($0 < \alpha < 1$). The expected claim on the insurer is $.5$. Find α.

A) .25 B) .3 C) .45 D) .5 E) .65

12. If a loss occurs, the amount of loss will be uniform between \$1000 and \$2,000. The probability of the loss occurring is .2. An insurance policy pays the total loss, if a loss occurs. Find the standard deviation of the amount paid by the insurer.

A) 584 B) 614 C) 634 D) 654 E) 674

13. X and Y are random losses with joint density function

$$f(x,y) = \frac{x}{500{,}000} \text{ for } 0 < x < 100 \text{ and } 0 < y < 100 .$$

An insurance policy on the losses pays the total of the two losses to a maximum payment of 100. Find the expected payment the insurer will make on this policy (nearest 1).

A) 90 B) 92 C) 94 D) 96 E) 98

14. The number of claims N that can result from a small group insurance policy is 0, 1 or 2, each with probability $\frac{1}{3}$. Information about the aggregate loss S incurred by the insurer is available in conditional form: $E[S|N=0]=0$, $Var[S|N=0]=0$,
$E[S|N=1]=10$, $Var[S|N=1]=5$, $E[S|N=2]=20$, $Var[S|N=2]=8$.
Find the unconditional variance of the aggregate loss S.

A) 13/3 B) 6.5 C) 13 D) 200/3 E) 71

15. In modeling the behavior of insurance claims, a risk manager uses an exponential distribution with mean μ as the distribution describing the claim size random variable.
The risk manager forecasts that claim sizes will increase next year, with the average claim size increasing by 10% from this year to next. The risk manager plans to continue using the exponential distribution as the model for claim amounts next year. The risk manager calculates the median of the claim size distribution this year, M_0 and for next year, M_1.
Find M_1/M_0 .

A) 1 B) $1 + ln\,1.1$ C) 1.1 D) $e^{.1}$ E) $e^{1.1}$

16. (SOA) An auto insurance company insures an automobile worth 15,000 for one year under a policy with a 1,000 deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} 0.5003e^{-x/2} & \text{for } \ 0 < x < 15 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected claim payment?

A) 320 B) 328 C) 352 D) 380 E) 540

17. An insurer finds that for automobile drivers classified as high risk, the number of accidents in one year has a binomial distribution with $n = 2$ and $p = .02$, and for drivers classified as low risk, the number of accidents in one year has a Bernoulli distribution with $n = 1$ and $p = .01$. The insurer's portfolio is made up of 25% policies on high risk drivers and 75% low risk drivers. Suppose that a driver has had no accidents in the past year.
Find the probability that the same driver will have no accidents in the next year.

A) .980 B) . 983 C) .986 D) .991 E) .994

18. (SOA) A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \frac{2.5(.6)^{2.5}}{x^{3.5}} & \text{for } \ x > 0.6 \\ 0 & \text{otherwise.} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2. What is the mean of the manufacturer's annual losses not paid by the insurance policy?

A) 0.84 B) 0.88 C) 0.93 D) 0.95 E) 1.00

19. Let X and Y be random variables with joint density function

$f(x, y) = 2x$ for $0 < x < 1$ and $0 < y < 1$.

An insurance policy is written to cover the loss $X + Y$. The policy has a deductible of 1.
Calculate the expected payment under the policy.

A) 1/4 B) 1/3 C) 1/2 D) 7/12 E) 5/6

20. (SOA) An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X , the probability that the insurance payment is less than or equal to 0.5 is 0.64. Calculate C.

A) .1 B) .3 C) .4 D) .6 E) .8

21. (SOA) The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250. In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $(0, 1500)$. Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

A) 361 B) 403 C) 433 D) 464 E) 521

22. An insurer models the claim random variable X for a certain insurance policy as follows:

x	$P[X = x]$
0	0.30
50	0.10
200	0.10
500	0.20
$1,000$	0.20
$10,000$	0.10

The insurer wishes to summarize the claim amount distribution with the parameters:

$q =$ probability a non-zero claim occurs ,

$B =$ conditional distribution of claim amount given that a claim occurs.

Find the standard deviation of B (nearest 100).

A) 3200 B) 3300 C) 3400 D) 3500 E) 3600

23. (SOA) A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and 10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5 . What is the expected amount paid to the company under this policy during a one-year period?

A) 2,769 B) 5,000 C) 7,231 D) 8,347 E) 10,578

24. A loss random variable has density function $f(x) = xe^{-x}$ for $x > 0$. An insurance policy on the loss has a deductible of 2. Find the expected insurance payment.

A) .46 B) .50 C) .54 D) .58 E) .62

25. A loss random variable has density function $f(x) = 2 - 2x$ for $0 \leq x \leq 1$.
At what level should a policy limit be set so that the expected insurer payment is one-half of the overall expected loss?

A) .11 B) .16 C) .21 D) .26 E) .31

26. (SOA) A baseball team has scheduled its opening game for April 1. If it rains on April l, the game is postponed and will be played on the next day that it does not rain. The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed. The insurance company determines that the number of consecutive days of rain beginning on April 1 is a Poisson random variable with mean 0.6. What is the standard deviation of the amount the insurance company will have to pay?

A) 668 B) 699 C) 775 D) 817 E) 904

27. (SOA) An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100 . Losses incurred follow an exponential distribution with mean 300 . What is the 95th percentile of actual losses that exceed the deductible?

A) 600 B) 700 C) 800 D) 900 E) 1000

Problems 28 and 29 are based on the following information.
A portfolio of independent insurance policies is divided into three classes:

Class	Number in Class	Probability of Claim per Policy	Claim Amount
1	1000	0.01	1
2	2000	0.02	1
3	500	0.04	2

28. The insurer charges a premium Q that is the 95-th percentile of the aggregate claims distribution based on the normal approximation to the aggregate claims distribution.
Find Q (nearest 5).

A) 90 B) 95 C) 100 D) 105 E) 110

29. The insurer calculates the variance of the aggregate claims random variable. The insurer then changes the assumptions regarding the claims and now supposes that the individual policy claim amounts are also random variables, and that the claim amount listed in the table above is the expected claim amount for each of the policies, and the variance of the claim amount per policy is σ^2. The insurer recalculates the variance of the aggregate claims and finds that it is 67% larger than the initial calculation. Find σ^2.

A) 1.0 B) 1.2 C) 1.4 D) 1.6 E) 1.8

30. An insurer with aggregate claim distribution S charges a premium which includes a relative security loading of θ (i.e., the premium is $Q = (1+\theta)E[S]$). The insurer purchases proportional reinsurance, in which the reinsurer will pay the fraction $\alpha \cdot \mathrm{S}$ of aggregate claims. The reinsurer charges a premium that includes a relative security loading of θ'. After reinsurance, the ceding insurer's resulting effective relative security loading is θ''. Which is the correct expression for α in terms of θ, θ', and θ''?

A) $\frac{\theta-\theta'}{\theta''-\theta'}$ B) $\frac{\theta-\theta''}{\theta'-\theta''}$ C) $\frac{\theta'-\theta}{\theta''-\theta'}$ D) $\frac{\theta'-\theta''}{\theta''-\theta}$ E) $\frac{\theta''-\theta'}{\theta'-\theta}$

31. (SOA) An insurance policy reimburses dental expense, X, up to a maximum benefit of 250 . The probability density function for X is:

$$f(x) = \begin{cases} ce^{-.004x} & \text{for } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant. Calculate the median benefit for this policy.

(A) 161 B) 165 C) 173 D) 182 E) 250

32. For a certain insurance, individual losses in 2000 were uniformly distributed over $(0\,,\,1000)$. A deductible of 100 is applied to each loss. In 2001, individual losses have increased 5%, and are still uniformly distributed. A deductible of 100 is still applied to each loss. Determine the percentage increase in the standard deviation of amount paid.

A) Less than 5.0% B) At least 5.0% but less than 5.5%

C) At least 5.5% but less than 6.0% D) At least 6.0% but less than 6.5%

E) At least 6.5%

33. Losses follow a distribution with density function $f(x) = \frac{1}{1000}e^{-x/1000}$, $0 < x < \infty$. There is a policy deductible of 500 and 10 losses are expected to exceed the deductible each year. Determine the expected number of losses that would exceed the deductible each year if all loss amounts doubled, but the deductible remained at 500.

A) Less than 10 B) At least 10, but less than 12

C) At least 12, but less than 14 D) At least 14, but less than 16

E) At least 16

34. You are given the following:

- the underlying distribution for year 2000 losses is given by $f(x) = e^{-x}$, $x > 0$, where losses are expressed in millions of dollars
- inflation of 5% impacts all claims uniformly from 2000 to 2001
- under a basic limits policy, payments on individual losses are capped at 1 million per year

Find the inflation rate from 2000 to 2001 on expected payments.

A) Less than 1.5% B) At least 1.5%, but less than 2.5%

C) At least 2.5%, but less than 3.5% D) At least 3.5%, but less than 4.5%

E) At least 4.5%

35. An insurer issues a one-year malpractice liability insurance policy to a medical clinic. If a malpractice suit is brought against the clinic, the distribution of the total of legal plus settlement costs to the clinic is assumed to be uniformly distributed between \$100,000 and \$1,000,000 . The number of malpractice suits brought against the clinic in one year, N, is assumed to have the following distribution $P[N=0] = .96$, $P[N=1] = .03$, $P[N=2] = .01$. The insurer charges a premium which is equal to $E[Y] + \sqrt{Var[Y]}$
where Y is the annual total of the clinic's claims. It is assumed that the number of malpractice suits and the costs arising from each suit are mutually independent. Find the premium charged by the insurer (nearest 25,000).

A) 100,000 B) 125,000 C) 150,000 D) 175,000 E) 200,000

36. For a certain insurance, individual losses last year were uniformly distributed over the interval $(0, 1000)$. A deductible of 100 is applied to each loss (the insurer pays the loss in excess of the deductible of 100). This year, individual losses are uniformly distributed over the interval $(0, 1050)$ and a deductible of 100 is still applied. Determine the percentage increase in the expected amount paid by the insurer from last year to this year.

A) 2 B) 4 C) 6 D) 8 E) 10

37. A life insurance company covers 16,000 mutually independent lives for 1-year term life insurance:

Class	Benefit Amount	Number Covered	Probability of Claim
1	1	8000	0.025
2	2	3500	0.025
3	4	4500	0.025

The insurance company's retention limit is 2 units per life (the insurer only covers an individual life up to a payment of 2). Reinsurance is purchased for 0.03 per benefit unit. The ceding insurer collects a premium of $Q = E[S] + 2\sqrt{Var[S]} + R$, where S denotes the distribution of retained claims and R is the cost of reinsurance. Find Q.

A) Less than 900 B) At least 900 but less than 910 C) At least 910 but less than 920
D) At least 920 but less than 930 E) At least 930

38. (SOA) The cumulative distribution function for health care costs experienced by a policyholder is modeled by the function $F(x) = \begin{cases} 1 - e^{-x/100} & \text{for } x > 0 \\ 0 & \text{otherwise,} \end{cases}$

The policy has a deductible of 20. An insurer reimburses the policyholder for 100% of health care costs between 20 and 120 less the deductible. Health care costs above 120 are reimbursed at 50%. Let G be the cumulative distribution function of reimbursements given that the reimbursement is positive. Calculate $G(115)$.

A) 0.683 B) 0.727 C) 0.741 D) 0.757 E) 0.777

39. (SOA) The amount of a claim that a car insurance company pays out follows an exponential distribution. By imposing a deductible of d, the insurance company reduces the expected claim payment by 10%. Calculate the percentage reduction on the variance of the claim payment.

A) 1% B) 5% C) 10% D) 20% E) 25%

40. (SOA) A motorist makes three driving errors, each independently resulting in an accident with probability 0.25. Each accident results in a loss that is exponentially distributed with mean 0.80. Losses are mutually independent and independent of the number of accidents. The motorist's insurer reimburses 70% of each loss due to an accident. Calculate the variance of the total unreimbursed loss the motorist experiences due to accidents resulting from these driving errors.

A) 0.0432 B) 0.0756 C) 0.1782 D) 0.2520 E) 0.4116

PROBLEM SET 10 SOLUTIONS

1. The probability function is

x :	1	2	3	4	5
$P[X=x]$:	$\frac{5}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
Amount paid :	100	200	300	325	350

Expected amount paid

$=(100)(\frac{5}{15})+(200)(\frac{4}{15})+(300)(\frac{3}{15})+(325)(\frac{2}{15})+(350)(\frac{1}{15})=213.3$. Answer: D

2. This is an example of the "individual risk model" for total claims. When there are a large number of policies, it is generally assumed that the distribution of total claims, S, can be approximated by a normal distribution. The total claim random variable is

$S=X_1+X_2+\cdots+X_{800}$ (there are a total of 800 policies),

where X_i denotes the claim from policy i (which may be 0). We wish to find the amount needed, say G, so that $P[S\leq G]=.95$ (G is the 95-th percentile of the distribution of total claims S). If we knew the mean and variance of S, then we can write

$P[S\leq G]=P\left[\frac{S-E[S]}{\sqrt{Var[S]}}\leq\frac{G-E[S]}{\sqrt{Var[S]}}\right]=.95$.

We have "standardized" S, meaning that $\frac{S-E[S]}{\sqrt{Var[S]}}$ has a standard normal distribution,

and, therefore $\frac{G-E[S]}{\sqrt{Var[S]}}$ is the 95-th percentile of the standard normal, which is 1.645.

Thus, $\frac{G-E[S]}{\sqrt{Var[S]}}=1.645$. So, if we know the numerical values of $E[S]$ and $Var[S]$, then we can find G, the 95-th percentile of S (under the normal approximation).

Since $S=X_1+X_2+\cdots+X_{800}$, it follows that

$E[S]=\sum_{i=1}^{800}E[X_i]$, and $Var[S]=\sum_{i=1}^{800}Var[X_i]$ (it is assumed that the X_i's are independent, and therefore the variance of the sum is the sum of the variances with no covariance factors).

In the notes on risk topics earlier in this study material, the following comments were made.

Suppose that the probability of a non-negative loss occurring is specified (usually denoted q, with $1-q=p$ being the probability no loss occurs), and the **conditional distribution of the loss amount given that a loss occurs** is specified, say random variable B. The random variable B might be described in detail, or only the mean and variance of B might be given.

2. continued

In this case, $X = 0$ if no loss occurs (probability p) and $X = B$ if a loss does occur (probability q). It is possible to use a mixture of distributions formulation

$E[X] = q \cdot E[B]$ and $E[X^2] = qE[B^2]$, so that $Var[X] = q \cdot E[B^2] - (q \cdot E[B])^2$.

In this problem there are 500 policies with $q = .01$, $E[B_i] = 200,000$ and $E[B_i^2] = 200,000^2$, and there are 300 policies with $q = .05$, $E[B_i] = 100,000$ and $E[B_i^2] = 100,000^2$.

According to the comments above, for each of the first 500 policies,

$E[X_i] = q \cdot E[B_i] = (.01)(200,000) = 2,000$,

$Var[X_i] = (.01)(200,000)^2 - [(.01)(200,000)]^2 = 396,000,000$,

and for each of the next 300 policies, $E[X_i] = (.05)(100,000) = 5,000$,

$Var[X_i] = (.05)(100,000)^2 - [(.05)(100,000)]^2 = 475,000,000$.

Then, $E[S] = (500)(2,000) + (300)(5,000) = 2,500,000$ and

$Var[S] = (500)(396,000,000) + (300)(475,000,000) = 3.405 \times 10^{11}$.

Then, $G = E[S] + 1.645\sqrt{Var[S]} = 3,460,000$ (nearest 1,000).

The company wishes to collect $kE[X_i]$ from policyholder i, for a total amount collected equal to $kE[S]$. Thus, $k(2,500,000) = 3,460,000$, so that $k = 1.38$. Answer: E

3. Amount paid $= \begin{cases} y & 1 < y \leq 10 \\ 10 & y > 10 \end{cases}$.

Expected amount paid $= \int_1^{10} y \cdot f(y)\, dy + (10)P(Y > 10)$.

$P[Y > 10] = \int_{10}^{\infty} f(y)\, dy = \int_{10}^{\infty} \frac{2}{y^3}\, dy = \frac{1}{100}$.

Expected amount paid $= \int_1^{10} y \cdot \frac{2}{y^3}\, dy + (10)(.01) = 1.8 + .1 = 1.9$. Answer: D

4. The pdf of T is $f(t) = \frac{1}{3}e^{-t/3}$ for $t > 0$.

$X = Max(T, 2) = \begin{cases} 2 & T \leq 2 \\ T & T > 2 \end{cases}$. This random variable has a discrete point,

$P[X = 2] = P[T < 2] = \int_0^2 \frac{1}{3}e^{-t/3}\, dt = 1 - e^{-2/3}$.

$E[X] = (2)(1 - e^{-2/3}) + \int_2^{\infty} t \cdot \frac{1}{3}e^{-t/3}\, dt$.

Integration by parts gives us

$\int_2^{\infty} t \cdot \frac{1}{3}^{-t/3}\, dt = \int_2^{\infty} t \cdot d(-e^{-t/3}) = -te^{-t/3}\Big|_2^{\infty} - \int_2^{\infty} -e^{-t/3} dt$

$= 2e^{-2/3} + 3e^{-2/3} = 5e^{-2/3}$.

Recall that in general, the antiderivative of xe^{ax} is $\frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$ (valid for any $a \neq 0$).

Then, $E[X] = (2)(1 - e^{-2/3}) + 5e^{-2/3} = 2 + 3e^{-2/3}$. Answer: D

5. $Y = Min\{X, 4\}$ (Y is the smaller of X and 4, since the machine will be replaced at time 4 if it is still operating). Y is a combination of a continuous distribution and one discrete point. The density function of Y is the same as that of X for $Y < 4$, and the probability that $Y = 4$ is the probability that $X \geq 4$. $P[X \geq 4] = \int_4^5 \frac{1}{5}\, dx = \frac{1}{5}$.

Therefore, $f_Y(y) = \begin{cases} \frac{1}{5} & Y < 4 \\ \frac{1}{5} & Y = 4 \end{cases}$.

We use the following formulation for variance of Y, $Var[Y] = E[Y^2] - (E[Y])^2$.

$E[Y] = \int_0^4 y \cdot (\frac{1}{5})\, dy + 4(\frac{1}{5}) = \frac{12}{5}$, $E[Y^2] = \int_0^4 y^2 \cdot (\frac{1}{5})\, dy + 4^2(\frac{1}{5}) = \frac{112}{15}$.

$Var[Y] = \frac{112}{15} - (\frac{12}{5})^2 = 1.71$. Answer: C

6. Expected payment with no deductible is 500 (mean of a uniform distribution on interval from 0 to 1000). With deductible d, the amount paid on a loss of amount x is

$\begin{cases} 0 & x \leq d \\ x - d & x > d \end{cases}$, and the expected payment is

$\int_d^{1000} (x-d)(.001)\, dx = (.0005)(1000-d)^2$.

In order for this to be 25% of the expected payment with no deductible, we must have

$(.0005)(1000-d)^2 = (.25)(500) \to d = 500$. Answer: C

7. Let N denote the Poisson random variable, the number of times in one year the financial conditions arise for the investment to be made, and let X denote the aggregate return on the investments for the year. Then

$P[X < 100,000] = P[X < 100,000|N = 0] \cdot P[N = 0]$
$+ P[X < 100,000|N = 1] \cdot P[N = 1] + \cdots$

If $N = 0$, then no investments were made and $P[X < 100,000|N = 0] = 1$, and if $N = 1$, then one investment was made whose return is normal with mean $100,000$ so that $P[X < 100,000|N = 1] = .5$. If $N = 2$, then X is the sum of two independent normal random variables with total mean $200,000$ and variance $2 \cdot 20,000^2 = 28,284^2$, so that

$P[X < 100,000|N = 2] = P[\frac{X-200,000}{28,284} < \frac{100,000-200,000}{28,284}] = P[Z < -3.54]$

where Z has a standard normal distribution - this probability is essentially 0, and so will be $P[X < 100,000|N = 3, 4, ...]$. Since $P[N = 0] = e^{-3} = .0498$ and $P[N = 1] = e^{-3}(3) = .1494$, it follows that

$P[X < 100,000] = (1)(.0498) + (.5)(.1494) = .12$. Answer: A

8. In order for the loss random variable to be properly defined, the probabilities must add to 1: $\frac{K}{1}+\frac{K}{2}+\frac{K}{3}+\frac{K}{4}+\frac{K}{5}=1 \to K=\frac{1}{2.2833}=.4380$.
The net premium for the policy is the expected amount paid by the policy.
The amount paid by the policy can be regarded as a mixture of the outcome 0 (amount paid if no loss occurs) with probability .95, and the outcome Y (amount paid after deductible if a loss occurs) with probability .05. The expected amount paid by the policy is $(0)(.95)+E(Y)\cdot(.05)$.
Y is the amount paid after the deductible is applied, given that a loss has occurred.
Therefore, $P(Y=0)=P(N=1 \text{ or } 2)=\frac{K}{1}+\frac{K}{2}=.657$,
$P(Y=1)=P(N=3)=\frac{K}{3}=.146$, $P(Y=2)=P(N=4)=\frac{K}{4}=.110$,
$P(Y=3)=P(N=5)=.088$. Then $E(Y)=(1)(.146)+(2)(.110)+(3)(.088)=.63$ and the expected amount paid by the policy is $(.05)(.63)=.0315$. Answer: A

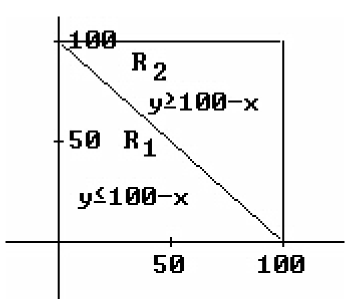
9. For an individual employee, the distribution of the amount the company pays above the deductible of \$100 is $Y=\begin{cases}0, & \text{prob. } .3\\ 100, & \text{prob. } .4\\ 400, & \text{prob. } .2\\ 900, & \text{prob. } .1\end{cases}$, with $E[Y]=210$ and $E[Y^2]=117,000$,
so that $Var[Y]=72,900$. If S is the aggregate amount paid by the company in one year, then $E[S]=50E[Y]=10,500$ and $Var[S]=50Var[Y]=3,645,000$.
The 95-th percentile if S is a, where $P[S\le a]=.95$. This probability can be rewritten as $P[\frac{S-E[S]}{\sqrt{Var[S]}}\le\frac{a-E[S]}{\sqrt{Var[S]}}]=.95$, and then applying the normal approximation to S, $\frac{S-E[S]}{\sqrt{Var[S]}}$ has an approximately standard normal distribution.
Then, $\frac{a-E[S]}{\sqrt{Var[S]}}=\frac{a-10,500}{\sqrt{3,645,000}}$ is equal to the 95-th percentile of the standard normal distribution, which is 1.645 . Thus, $\frac{a-10,500}{\sqrt{3,645,000}}=1.645 \to a=13,641$. Answer: C

10. A policy with probability of claim p and claim amount C has a two-point claim distribution $X=\begin{cases}0, & \text{prob. } 1-p\\ C, & \text{prob. } p\end{cases}$. Then, $E[X]=Cp$, $E[X^2]=C^2p$ and
$Var[X]=C^2p-(Cp)^2=C^2p(1-p)$. Since the policies are mutually independent,
the variance of the aggregate claim S is the sum of the variances of the individual policy claims:
$Var[S]=1000(1^2)(.01)(.99)+2000(1^2)(.02)(.98)+500(2^2)(.04)(.96)=125.9$, and the standard deviation of S is $\sqrt{Var[S]}=\sqrt{125.9}=11.22$. Answer: D

11. The expected loss is $E[X]=\int_0^\infty [1-F(x)]\,dx=\int_0^2 (.8-.3x)\,dx=1$.
The expected claim on the insurer is $.5=E[\alpha X]=\alpha E[X]=\alpha$. Answer: D

12. The insurance payout Y has a mixed distribution - there is a .8 probability that $Y=0$, and the conditional density of Y given that a claim has occurred is
$f_Y(y|\text{claim})=\frac{f_Y(y)}{prob.\,claim\,occurs}=.001$ for $1000\le y\le 2000$, so that
$f_Y(y)=.0002$ for $1000\le y\le 2000$: $f(y)=\begin{cases} .8\,, & \text{if } y=0 \\ .0002, & \text{if } 1000\le y\le 2000\end{cases}$.
Then $E[Y]=(.8)(0)+\int_{1000}^{2000}(.0002)y\,dy=300$, and
$E[Y^2]=(.8)(0^2)+\int_{1000}^{2000}(.0002)y^2\,dy=\frac{1{,}400{,}000}{3}\rightarrow Var[Y]=\frac{1{,}130{,}000}{3}\approx 614^2$.
Alternatively,
$E[Y]=E[Y|\text{no claim occurs}]P[\text{no claim}]+E[Y|\text{claim occurs}]P[\text{claim}]$
$=(0)(.8)+(1500)(.2)=300$, and
$E[Y^2]=E[Y^2|\text{no claim occurs}]P[\text{no claim}]+E[Y^2|\text{claim occurs}]P[\text{claim}]$
$=(0^2)(.8)+(\frac{7{,}000{,}000}{3})(.2)=\frac{1{,}400{,}000}{3}$. Answer: B

13. The maximum payment on the policy occurs when $X+Y\ge 100$. The expected payment is
$\int_{R_1}\int (x+y)\frac{x}{500{,}000}\,dy\,dx \quad +\int_{R_2}\int (100)\frac{x}{500{,}000}\,dy\,dx$
where R_1 and R_2 are the regions in the square $\{(x,y)|0\le x\le 100,\ 0\le y\le 100\}$ represented in the graph below: $R_1=\{(x,y)|x+y\le 100\}$, $R_2=\{(x,y)|x+y\ge 100\}$.
The expected payment is $\int_0^{100}\int_0^{100-x}(x+y)\frac{x}{500{,}000}\,dydx+\int_0^{100}\int_{100-x}^{100}(100)\frac{x}{500{,}000}\,dy\,dx$
$=\frac{1}{500{,}000}\int_0^{100}[x^2(100-x)+\frac{1}{2}x(100-x)^2]\,dx+\int_0^{100}\frac{x^2}{5000}\,dx=25+66.7\approx 92$.



Answer: B

14. We use the conditional variance approach:
$Var[S] = E[Var[S|N]] + Var[E[S|N]]$.

$Var[S|N]$ is a 3-point random variable - $Var[S|N] = \begin{cases} 0, \text{ if } N=0 \text{ , prob. } 1/3 \\ 5, \text{ if } N=1 \text{ , prob. } 1/3 \\ 8, \text{ if } N=2 \text{ , prob. } 1/3 \end{cases}$,

so that $E[Var[S|N]] = (0+5+8)\cdot\frac{1}{3} = \frac{13}{3}$.

$E[S|N]$ is also a 3-point random variable - $E[S|N] = \begin{cases} 0, \text{ if } N=0 \text{ , prob. } 1/3 \\ 10, \text{ if } N=1 \text{ , prob. } 1/3 \\ 20, \text{ if } N=2 \text{ , prob. } 1/3 \end{cases}$, so that

$E[E[S|N]] = (0+10+20)\cdot\frac{1}{3} = 10$, and

$E[(E[S|N])^2] = (0+10^2+20^2)\cdot\frac{1}{3} = \frac{500}{3}$, and then

$Var[E[S|N]] = E[(E[S|N])^2] - (E[E[S|N]])^2 = \frac{500}{3} - 10^2 = \frac{200}{3}$.

Then, $Var[S] = \frac{13}{3} + \frac{200}{3} = \frac{213}{3} = 71$. Answer: E

15. With mean μ the exponential distribution has parameter $1/\mu$, and median M_0 which satisfies $P[X > M_0] = \frac{1}{2} = e^{-M_0/\mu} \rightarrow M_0 = -\mu \ln \frac{1}{2}$.

Next year, with mean 1.1μ, the exponential distribution has parameter $1/1.1\mu$, and median M_1 which satisfies $P[X' > M_1] = \frac{1}{2} = e^{-M_1/1.1\mu} \rightarrow M_1 = -1.1\mu \ln \frac{1}{2}$.

Thus, $M_1/M_0 = 1.1$. Answer: C

16. The amount paid by the insurer is a mixture of 3 components.

There is a .94 probability that the damage and claim is 0.

There is a .04 probability of partial damage, and a .02 probability of total loss on the car.

If there is total loss on the car, the insurer pays 14,000 (15,000 minus the deductible of 1,000). If there is partial damage, then the expected amount paid by the insurer after deductible is $1000\int_1^{15}(x-1)(.5003e^{-x/2})\,dx$. This integral can be simplified by integration by parts:

$u(x) = x-1$, $dv(x) = .5003e^{-x/2}dx \rightarrow v(x) = -1.0006e^{-x/2}$.

$\int_0^{15}(x-1)(.5003e^{-x/2})\,dx = (x-1)(-1.0006e^{-x/2})\Big|_{x=1}^{x=15} - \int_1^{15} -1.0006e^{-x/2}dx$

$= -.00775 + \int_1^{15} 1.0006e^{-x/2}dx = -.00775 + 1.21268 = 1.205$.

Therefore, if there is partial damage, the expected amount paid by the insurer is $1000(1.205) = 1205$. The overall expected amount paid by the insurer is

$(.94)(0) + (.04)(1205) + (.02)(14,000) = 328.20$. Answer: B

17. A_1 - the number of accidents in the first year,
A_2 - the number of accidents in the second year.
We wish to find $P[A_2 = 0|A_1 = 0] = \frac{P[(A_2=0)\cap(A_1=0)]}{P[A_1=0]}$.
We find these probabilities by conditioning over the age of the driver:
$P[A_1 = 0] = P[A_1 = 0|\text{high risk}] \cdot P[\text{high risk}] + P[A_1 = 0|\text{low risk}] \cdot P[\text{low risk}]$
$= (.98)^2(.25) + (.99)(.75) = .9826$, and
$P[(A_1 = 0) \cap (A_2 = 0)] = P[(A_1 = 0) \cap (A_2 = 0)|\text{high risk}] \cdot P[\text{high risk}]$
$+ P[(A_1 = 0) \cap (A_2 = 0)|\text{low risk}] \cdot P[\text{low risk}]$
$= [(.98)^2]^2(.25) + (.99)^2(.75) = .96567 \rightarrow P[A_2 = 0|A_1 = 0] = \frac{.96567}{.9826} = .9828$.
Answer: B

18. The losses Y not paid by the insurance policy can be described in terms of total losses X as follows $Y = \begin{cases} X & \text{if } X \leq 2 \\ 2 & \text{if } X > 2 \end{cases}$. The expected value of Y is

$$E[Y] = \int_{.6}^{2} x \cdot f(x)\,dx + \int_{2}^{\infty} 2 \cdot f(x)\,dx = \int_{.6}^{2} x \cdot \frac{2.5(.6)^{2.5}}{x^{3.5}}\,dx + \int_{2}^{\infty} 2 \cdot \frac{2.5(.6)^{2.5}}{x^{3.5}}\,dx$$

$$= \int_{.6}^{2} \frac{2.5(.6)^{2.5}}{x^{2.5}}\,dx + \frac{5(.6)^{2.5}}{(-2.5)x^{2.5}}\Bigg|_{x=2}^{x=\infty} = \frac{2.5(.6)^{1.5}}{(-1.5)x^{1.5}}\Bigg|_{x=.6}^{x=2} + \left(-0 - \frac{5(.6)^{2.5}}{(-2.5)2^{2.5}} \right)$$

$$= \frac{2.5(.6)^{2.5}}{(-1.5)2^{1.5}} - \frac{2.5(.6)^{2.5}}{(-1.5)(.6)^{1.5}} + .09859 = .934 .$$

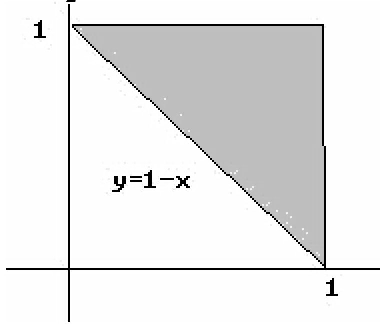
Answer: C

19. Given a function $h(x, y)$, to find the expectation of that function of the two random variables X and Y with joint density function $f(x, y)$, we calculate the integral $\int\int h(x, y) \cdot f(x, y)\,dy\,dx$, where the integral is taken over the two dimensional region of density for the joint distribution. Any region over which $h(x, y) = 0$ can be ignored. In this problem, $h(x, y)$ is the insurer's payment when the random losses are amounts x and y. What it means to say that the policy has a deductible of 1 is that the insurer pays losses <u>in excess</u> of 1. The insurer pays $X + Y - 1$ if $X + Y > 1$ and if $X + Y \leq 1$, the insurer pays 0. The amount paid by the insurer can also be described as $h(x, y) = max\{x + y - 1\,,\, 0\}$.

The bivariate distribution of X and Y has density only on the unit square, and $x + y > 1$ in the shaded upper triangular region of the unit square (above the line $x + y = 1$), so that $h(x, y) = 0$ on the lower triangular region. Therefore, the expectation can found by integrating over the upper triangular region. The upper triangular region corresponds to
$\{(x, y) : 0 \leq x \leq 1\,,\, 1 - x \leq y \leq 1\}$.

19. continued

The expectation is $\int\int h(x,y)\cdot f(x,y)\,dy\,dx = \int_0^1\int_{1-x}^1 (x+y-1)\cdot 2x\,dy\,dx$
$= \int_0^1 [(x-1)\cdot x + \frac{1}{2}(1-(1-x)^2)]\cdot 2x\,dx = \int_0^1 x^3\,dx = \frac{1}{4}$.



Answer: A

20. The insurance payment is $Y = \begin{cases} 0 & X \le c \\ X - C & C < X < 1 \end{cases}$.

The insurance payment is less than .5 if the $X - C < .5$, or equivalently, if $X < C + .5$.
It must be true that $C \le .5$, because if $C > .5$ then $C + .5 > 1$ and then $P[X < C + .5] = 1$ since $P[X < 1] = 1$.
$P[X < C + .5] = \int_0^{c+.5} 2x\,dx = (c+.5)^2$. In order for this to be equal to .64 we must have $(c+.5)^2 = .64 \to c + .5 = .8$ (we ignore the negative square root since $X > 0$) $\to c = .3$.
Answer: B

21. If the repair amount is X, then the insurance pays
$Y = \begin{cases} 0 & X \le 250 \\ X - 250 & X > 250 \end{cases}$.
Then $E[Y] = \int_{250}^{1500} (x-250)\cdot f_X(x)\,dx = \int_{250}^{1500} (x-250)\cdot \frac{1}{1500}\,dx = 520.83$,
and $E[Y^2] = \int_{250}^{1500} (x-250)^2\cdot f_X(x)\,dx = \int_{250}^{1500} (x-250)^2\cdot \frac{1}{1500}\,dx = 434,028$.
Then $Var[Y] = 434,028 - (520.83)^2 = 162,764$, and the standard deviation of Y is $\sqrt{Var[Y]} = 403$. Answer: B

22. The probability of a non-zero claim occurring is $q = 1 - .3 = .7$.
B can take on the values $50,\ 200,\ 500,\ 1000$ or $10,000$, with probabilities
$P[B = 50] = P[X = 50|\text{non-zero claim occurs}] = \frac{P[X=50]}{P[X>0]} = \frac{.1}{q} = \frac{.1}{.7} = \frac{1}{7}$,
and in a similar way, $P[B = 200] = \frac{1}{7}$, $P[B = 500] = \frac{2}{7}$, $P[B = 1000] = \frac{2}{7}$,
$P[B = 10,000] = \frac{1}{7}$. Then,
$Var[B] = E[B^2] - (E[B])^2 = 14,648,983 - (1892.86)^2 = 11,066,000$ (nearest 1000)
and $\sqrt{Var[B]} = 3327$. Answer: B

23. We denote by N the Poisson random variable representing the number of major snowstorms for the year. We are given that $E[N] = 1.5$. The probability function for this Poisson distribution is $p(n) = P[N = n] = \frac{e^{-1.5}(1.5)^n}{n!}$.

The amount paid to the company is $c_0 = 0$ if $N = 0$ or 1, but is $c_n = 10,000n$ if $N = n \geq 2$.

N :	0	1	2	3	4	...
Amt. paid to company, c_N	0	0	$10,000$	$20,000$	$30,000$	...
Loss absorbed by company	0	$10,000$	$10,000$	$10,000$	$10,000$	...
Total loss	0	$10,000$	$20,000$	$30,000$	$40,000$	...

Note that $E[\text{Total loss}] = E[\text{Amt. paid to company}] + E[\text{Loss absorbed by company}]$

Total loss $= 10,000N \Rightarrow E[\text{Total loss}] = E[10,000N] = 10,000E[N] = 15,000$.

$E[\text{Loss absorbed by company}]$

$= 0 \cdot p(0) + 10,000 \cdot [p(1) + p(2) + p(3) + p(4) + \cdots = 10,000[1 - p(0)]$

$= 10,000 \cdot [1 - \frac{e^{-1.5}(1.5)^0}{0!}] = 7,769$.

Therefore, $E[\text{Amt. paid to company}] = 15,000 - 7,769 = 7,231$.

Note that the amount paid to the company is the total loss above a deductible of 10,000 .

Answer: C

24. The expected insurance payment is $\int_2^\infty (x-2) \cdot f(x)\,dx = \int_2^\infty (x-2) \cdot xe^{-x}\,dx$.

With the change of variable $y = x - 2$, this integral becomes

$\int_0^\infty y(y+2)e^{-y-2}\,dy = e^{-2}[\int_0^\infty y^2e^{-y}\,dy + 2\int_0^\infty ye^{-y}] = e^{-2}[\frac{2}{1^3} + 2(\frac{1}{1^2})] = .5413$.

Answer: C

25. The overall expected loss is $\int_0^1 x(2-2x)\,dx = \frac{1}{3}$.

With policy limit u, the expected insurance payment is $\int_0^u x \cdot f(x)\,dx + u \cdot [1 - F(u)]$.

In this case, $F(u) = \int_0^u (2-2x)\,dx = 2u - u^2$.

Therefore, the expected insurance payment is

$\int_0^u x(2-2x)\,dx + u[1 - 2u + u^2] = u^2 - \frac{2u^3}{3} + u - 2u^2 + u^3 = \frac{u^3}{3} - u^2 + u$.

In order for this to be one-half of expected total loss we must have $\frac{u^3}{3} - u^2 + u = \frac{1}{6}$.

We do not solve the cubic equation. We substitute in the possible answers to see which is closest. We see that with $u = .21$ we get $\frac{(.21)^3}{3} - (.21)^2 + .21 = .169$, which is the closest to $\frac{1}{6}$ of all the possible values given in the answers. Answer: C

26. Let N be the number of consecutive days of rain starting on April 1. Then the amount paid by insurance is

N :	0	1	≥ 2
Amt paid by ins.:	0	1000	2000

We are told that N has a Poisson distribution with mean .6, so that $P[N = k] = \frac{e^{-.6}(.6)^k}{k!}$.

We note that $P[N \geq 2] = 1 - P[N = 0 \text{ or } 1] = 1 - e^{-.6} - e^{-.6}(.6) = .1219$.

The first two moments of X are

$E[X] = (0)e^{-.6} + (1000)e^{-.6}(.6) + (2000)(.1219) = 573$ and

$E[X^2] = (0)e^{-.6} + (1000^2)e^{-.6}(.6) + (2000^2)(.1219) = 816,893$.

Then $Var[X] = E[X^2] - (E[X])^2 = 488,461$, and the standard deviation is

$\sqrt{488,461} = 699$. Answer: B

27. The pdf of the exponential distribution with mean 300 is $f(x) = \frac{1}{300}e^{-x/300}\ x > 0$,

and the cdf is $F(x) = 1 - e^{-x/300}$. "Actual losses that exceed the deductible" refers to losses above 100, given that the loss is above 100. The 95-th percentile is c, where

$.95 = \frac{P[100<X\leq c]}{P[X>100]} = \frac{F(c)-F(100)}{1-F(100)} = \frac{(1-e^{-c/300})-(1-e^{-100/300})}{1-(1-e^{-100/300})} = \frac{e^{-1/3}-e^{-c/300}}{e^{-1/3}}$.

Solving for c results in $c = 998.7$ Answer: E

28. Information is given for individual policies in the form

$q_i =$ probability of a non-zero claim from policy i

$\mu_i =$ claim amount from policy i, given that a claim occurs on policy i , $\sigma_i^2 = 0$

Let S denote the aggregate claims. Then the expected claim from policy i is $q_i \cdot \mu_i$

and the variance is $q_i(1 - q_i) \cdot \mu_i^2$.

$E[S] = \sum q_i \cdot \mu_i = 1000(.01)(1) + 2000(.02)(1) + 500(.04)(2) = 90$, and

$$Var[S] = \sum q_i(1 - q_i) \cdot \mu_i^2 = 1000(.01)(.99)(1^2) + 2000(.02)(.98)(1^2)$$

$$+ 500(.04)(.96)(2^2) = 125.9 .$$

The 95-th percentile of S is Q, where $P[S \leq Q] = .95$. Standardizing S results in

$P[\frac{S-E[S]}{\sqrt{Var[S]}} \leq \frac{Q-E[S]}{\sqrt{Var[S]}}] = .95$. Applying the normal approximation results in

$\frac{Q-E[S]}{\sqrt{Var[S]}} = \frac{Q-90}{\sqrt{125.9}} = 1.645 \rightarrow Q = 108.5$ (the 95-th percentile of the standard normal distribution is 1.645). Answer: E

29. With the insurer's initial assumption, information is given for individual policies in the form
$q_i =$ probability of a non-zero claim from policy i ,
$\mu_i =$ claim amount from policy i, given that a claim occurs on policy i , $\sigma_i^2 = 0$
The insurer's revised assumption results in $\sigma_i^2 = \sigma^2$ for all i.
Under the initial assumption, the variance of a claim from policy i is $q_i(1-q_i) \cdot \mu_i^2$,
so that the variance of aggregate claims is
$\sum q_i(1-q_i) \cdot \mu_i^2 = 1000(.01)(.99)(1^2) + 2000(.02)(.98)(1^2) + 500(.04)(.96)(2^2) = 125.9$.

Under the revised assumption, the variance of a claim from policy i is
$q_i(1-q_i) \cdot \mu_i^2 + q_i \cdot \sigma^2$, so that the variance of aggregate claims is
$\sum [q_i(1-q_i) \cdot \mu_i^2 + q_i \cdot \sigma^2] = 1000[(.01)(.99)(1^2) + (.01)\sigma^2]$

$+ 2000[(.02)(.98)(1^2) + (.02)\sigma^2] + 500[(.04)(.96)(2^2) + (.04)\sigma^2] = 125.9 + 70\sigma^2.$
We are given that $125.9 + 70\sigma^2 = 1.67(125.9) \rightarrow \sigma^2 = 1.2$. Answer: B

30. The ceding insurer' initial premium is $(1+\theta) \cdot E[S]$. The expected claim on the reinsurer is $\alpha \cdot E[S]$, so the premium paid by the ceding insurer to the reinsurer is
$(1+\theta') \cdot \alpha \cdot E[S]$. The retained premium for the ceding insurer is
$(1+\theta) \cdot E[S] - (1+\theta') \cdot \alpha \cdot E[S] = [1 + \theta - \alpha(1+\theta')] \cdot E[S]$, and the expected retained claim for the ceding insurer is $(1-\alpha) \cdot E[S]$. Thus, the effective relative security loading for the ceding insurer after reinsurance is θ'', where
$(1+\theta'') \cdot (1-\alpha) \cdot E[S] = [1 + \theta - \alpha(1+\theta')] \cdot E[S]$, from which we can solve for
α in terms of θ, θ' and θ'': $\alpha = \frac{\theta - \theta''}{\theta' - \theta''}$. Answer: B

31. The density function for an exponential distribution with mean $\frac{1}{\lambda}$ is $\lambda e^{-\lambda x}$, $x > 0$.
In order for $f(x)$ to be a proper probability density function, we must have $c = .004$.
Alternatively, $\int_0^\infty f(x)\,dx = 1 \rightarrow \int_0^\infty ce^{-.004x}\,dx = \frac{c}{.004} = 1 \rightarrow c = .004$.
If R is the reimbursed amount and X is the actual expense, then
$R = \begin{cases} X & X \leq 250 \\ 250 & X > 250 \end{cases}$. The median benefit is the amount k for which
$P[R \leq k] = .5$. From the distribution of X, we see that
$P[X \leq r] = \int_0^r f(x)\,dx = 1 - e^{-.004r}$. Solving $P[X \leq r] = .5$
results in $1 - e^{-.004r} = .5 \rightarrow r = 173.29$.
Therefore, the median of R is 173.29 . Answer: C

32. In 2000, with no policy limit and with a deductible of 100, the expected amount paid per loss is $E[Y]$, where $Y = \begin{cases} 0 & X < 100 \\ X - 100 & X \geq 100 \end{cases}$.

The density function of X is $f_X(x) = .001$ for $0 < x < 1000$ (0 otherwise) and the distribution function $F_X(x) = .001x$ for $0 < x < 1000$.

The expectation is

(i) $E[Y] = \int_{100}^{1000} (x-100)(.001)\,dx = 405$ or

(ii) $\int_{100}^{1000} [1 - F_X(x)]\,dx = \int_{100}^{1000} [1 - .001x]\,dx = 405$.

The variance of amount paid is $E[Y^2] - (E[Y])^2$.

$E[Y^2] = \int_{100}^{1000} (x-100)^2 (.001)\,dx = 243,000$, so that the standard deviation of amount paid per loss in 2000 is $\sqrt{243,000 - (405)^2} = 281.0$.

In 2001, the loss W is uniform on $(0\,,\,1050)$. The amount paid by the insurance is

$Z = \begin{cases} 0 & W < 100 \\ W - 100 & W \geq 100 \end{cases}$, and the variance of Z is

$$E[Z^2] - (E[Z])^2 = \int_{100}^{1050} (w-100)^2 (\tfrac{1}{1050})\,dw - [\int_{100}^{1050} (w-100)(\tfrac{1}{1050})\,dx]^2$$
$$= 272,182 - (429.76)^2 = 87,487,$$

and the standard deviation is 295.8. The percentage increase in standard deviation from 2000 to 2001 is $\frac{295.8}{281.0} - 1 = .053$. Answer: B

33. The probability of a given loss exceeding 500 is $e^{-500/1000} = e^{-1/2} = .60653$.

If there are n exposures, then the expected number of losses exceeding the deductible will be $ne^{-1/2} = .60653n$. We are told that this is 10, so that $n = 10e^{1/2}$.

If all loss amounts doubled, the loss distribution will be exponential with mean 2000, so that $F(x) = 1 - e^{-x/2000}$, and the expected number of losses exceeding 500 will be

$n[1 - F_{new}(500)] = 10e^{1/2}e^{-1/4} = 10e^{1/4} = 12.84$. Answer: C

34. In year 2000, $E[\text{payment}] = \int_0^1 [1 - F(x)]\,dx = \int_0^1 e^{-x}dx = 1 - e^{-1} = .6321$ (million).

After 5% inflation, in year 2001,

$P[Y < y] = P[1.05X < y] = P[X < \frac{y}{1.05}] = 1 - e^{-y/1.05}$, exponential with mean 1.05 (million). In year 2001,

$E[\text{payment}] = \int_0^1 [1 - F_Y(x)]\,dx = \int_0^1 e^{-x/1.05}dx = 1.05 - 1.05e^{-1/1.05} = .6449$.

The inflation rate on expected losses is $\frac{.6449}{.6321} = 1.0202$. Answer: B

35. This is a case in which the single claim amount distribution X (severity distribution) is given, and the distribution of the number of claims per year N (frequency distribution) is given.
We can regard the total claim amount as a mixture of 3 distributions:
$Z_1 = 0$ when there are $N = 0$ suits (prob. .96)
$Z_2 = X$ when there is $N = 1$ suit (prob. .03) and
$Z_3 = X_1 + X_2$ when there are $N = 2$ suits (prob. .01) .
Then $E[Y] = (.96)E[Z_1] + (.03)E[Z_2] + (.01)E[Z_3]$
$= (.96)(0) + (.03)(550,000) + (.01)(550,000)(2) = 27,500$.
To find $E[Y^2]$ note that $E[Z_2^2] = E[X^2] = \int_{100,000}^{1,000,000} x^2(\frac{1}{900,000})\,dx = 3.7 \times 10^{11}$.
and $E[Z_3^2] = E[(X_1 + X_2)^2] = E[X_1^2] + 2E[X_1X_2] + E[X_2^2]$,
where by independence $E[X_1X_2] = E[X_1] \cdot E[X_2] = 550,000^2$.
Then $E[Y^2] = (.96)E[Z_1^2] + (.03)E[Z_2^2] + (.01)E[Z_3^2]$
$= (.96)(0) + (.03)(3.7 \times 10^{11})$
$+ (.01)(3.7 \times 10^{11} + 2 \times 550,000^2 + 3.7 \times 10^{11}) = 2.455 \times 10^{10}$.
$Var[Y] = E[Y^2] - (E[Y])^2 = 2.38 \times 10^{10}$.
The premium charged by the insurer is
$E[Y] + \sqrt{Var[Y]} = 27,500 + 154,250 = 181,750$. Answer: D

36. Last year, for a loss of amount X, the amount paid by the insurer was
$Y^* = \begin{cases} 0 \text{ if } X \le 100 \\ X-100 \text{ if } X > 100 \end{cases}$. Last year the pdf of the loss random variable X was
$f_X(x) = \frac{1}{1000}$ (uniform distribution on the interval $(0, 1000)$). The expected payment by the insurer last year was $\int_{100}^{1000}(x - 100) \cdot \frac{1}{1000}\,dx = 405$.
This year, for a loss of amount X, the amount paid by the insurer is still
$Y^* = \begin{cases} 0 \text{ if } X \le 100 \\ X-100 \text{ if } X > 100 \end{cases}$, but this year the pdf of the loss random variable X is
$f_X(x) = \frac{1}{1050}$ (uniform distribution on the interval $(0, 1050)$).
The expected payment by the insurer this year is $\int_{100}^{1050}(x - 100) \cdot \frac{1}{1050}\,dx = 429.76$.
The percentage increase is $100(\frac{429.76}{405} - 1) = 6.11$. Answer: C

37. The ceding insurer will cover all claims from classes 1 and 2, and will cover the first 2 units of claim from any policy in class 3. The ceding insurer purchases 2 units of reinsurance for each of the policies with benefit amount 4, for a total of $4500(2) = 9000$ units reinsured. The cost of the reinsurance is $R = 9000(.03) = 270$. The retained claim distribution S consists of 8000 (Class 1) policies with $q = .025$ and $E[B_i] = 1$ and 8000 policies ($3500 + 4500$, Classes 2 and 3 combined) with $q = .025$ and $E[B_i] = 2$. We are using the notation mentioned earlier, B_i is the conditional claim from policy i given that a claim occurs. Then, X_i is related to B_i through the relationships $E[X_i] = q_i E[B_i]$ and $Var[X_i] = q_i(1-q_i)(E[B_i])^2 + q_i \cdot Var[B_i]$. In this case $Var[B_i] = 0$ for all policies; this is generally assumed for term life insurances. Then,
$E[S] = \sum E[X_i] = 8000(.025)(1) + 8000(.025)(2) = 600$ and
$Var[S] = \sum Var[X_i] = 8000(.025)(.975)(1^2) + 8000(.025)(.975)(2^2) = 975$.
Then, $Q = 600 + 2\sqrt{975} + 270 = 932.45$. Answer: E

38. The distribution of costs is exponential with a mean of 100. From the lack-of-memory property of the exponential distribution, the conditional distribution of costs given cost is greater than 20 is also exponential with mean 100. Reimbursement is 100 if health care cost is 120 and reimbursement is 115 if health care cost is 120+30 (since 50% of the additional 30 is reimbursed). The conditional probability that reimbursement is below 115 given that reimbursement is positive is the probability that an exponential random variable with mean 100 is less than 130 (conditional probability that total cost is less than 150 given that it is at least 120). This is
$1 - e^{-130/100} = 727$. Answer: B

39. Suppose that the mean of the exponential distribution is λ. Then with deductible amount d, the expected payout is $E[(X-d)_+] = \int_d^\infty [1 - F_X(x)]\,dx = \int_d^\infty e^{-x/\lambda}\,dx = \lambda e^{-d/\lambda}$.
Since the expected payout with deductible is 90% of the expected claim payment, it follows that $e^{-d/\lambda} = .9$. This is also $P(X > d)$ for the exponential distribution.
Then $E[(X-d)^2 | X > d] = \int_d^\infty (x-d)^2 \,\frac{e^{-x/\lambda}}{\lambda}\,dx$. If we apply the change of variable $y = x - d$, we get $\int_0^\infty y^2\, e^{-d/\lambda}\, \frac{e^{-y/\lambda}}{\lambda}\,dy = e^{-d/\lambda} \times 2\lambda^2 = .9 \times 2\lambda^2 = 1.8\lambda^2$.
This is because $\int_0^\infty y^2 \,\frac{e^{-y/\lambda}}{\lambda}\,dy = 2\lambda^2$ is the second moment of an exponential distribution and $e^{-d/\lambda}$ was already determined to be .9 .
Then $Var[(X-d)_+] = E[(X-d)^2 | X > d] - (\,E[(X-d)_+])^2 = 1.8\lambda^2 - (.9\lambda)^2 = .99\lambda^2$.
This is $.01\lambda^2$ less (1%) than the variance of X. Answer: A

40. The number of accidents has a binomial distribution with $p = .25$ and $n = 3$, so the mean number of driving errors is $E[N] = 3 \times .25 = .75$ and the variance of the number of driving errors is $Var[N] = 3 \times .25 \times .75 = .5625$.
If the loss is X, then $E[X] = .8$ and $Var[X] = .8^2 = .64$.
The unreimbursed loss for a single accident is $Y = .3X$, with mean $.3 \times .8 = .24$ and variance $.3^2 \times .64 = .0576$. The variance of the total unreimbursed loss S is $Var[E[S|N]] + E[Var[S|N]]$. But $E[S|N] = .24N$ and $Var[S|N] = .0576N$ since there are N independent accidents.
Then $E[Var[S|N]] = E[.0576N] = .0576 \times E[N] = .0576 \times .75 = .0432$,
and $Var[E[S|N]] = Var[.24N] = .24^2 \times Var[N] = .24^2 \times .5625 = .0324$.
Finally, $Var[S] = .0432 + .324 = .0756$. Answer: B

TABLE FOR THE

NORMAL DISTRIBUTION

NORMAL DISTRIBUTION TABLE

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z<z)
The value of z to the first decimal is given in the left column. The second decimal place is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

PRACTICE EXAMS

Note that some of the questions on these practice exams are somewhat more challenging than the typical exam questions.

PRACTICE EXAM 1

1. If E and F are events for which $P[E \cup F] = 1$, then $P[E' \cup F'] =$
A) 0 B) $P[E'] + P[F'] - P[E'] \cdot P[F']$ C) $P[E'] + P[F']$
D) $P[E'] + P[F'] - 1$ E) 1

2. Sixty percent of new drivers have had driver education. During their first year, new drivers without driver education have a probability of .08 of having an accident, but new drivers with driver education have only a .05 probability of an accident. What is the probability a new driver has had driver education, given that the driver has had no accidents the first year?
A) $\frac{5}{6}$ B) $\frac{(.92)(.4)}{(.95)(.6)+(.92)(.4)}$ C) $\frac{(.95)(.4)}{(.95)(.6)+(.92)(.4)}$
D) $\frac{(.95)(.4)}{(.95)(.4)+(.92)(.6)}$ E) $\frac{(.95)(.6)}{(.95)(.6)+(.92)(.4)}$

3. A loss distribution random variable X has a pdf of $f(x) = ae^{-x} + be^{-2x}$ for $x > 0$. If the mean of X is 1, find the probability $P[X < 1]$.
A) .52 B) .63 C) .74 D) .85 E) .96

4. If $f_X(x) = xe^{-x^2/2}$ for $x > 0$, and $Y = ln\,X$, find the density function for Y.
A) $e^{2y-\frac{1}{2}e^{2y}}$ B) $(ln\,y)e^{-(ln\,y)^2/2}$ C) $e^{y-\frac{1}{2}e^{2y}}$ D) $ye^{-y^2/2}$ E) $e^{-\frac{1}{2}e^{2y}}$

5. An insurer estimates that Smith's time until death is uniformly distributed on the interval $[0,5]$ and Jones' time until death is uniform on the interval $[0,10]$. The insurer assumes that the two times of death are independent of one another. Find the probability that Smith is the first of the two to die.
A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

6. If X has a normal distribution with mean 1 and variance 4, then $P[X^2 - 2X \leq 8] = ?$

A) $.13$ B) $.43$ C) $.75$ D) $.86$ E) $.93$

7. The pdf of X is $f(x) = \begin{cases} 2x \text{ for } 0<x<1 \\ \text{0, elsewhere} \end{cases}$. The mean of X is μ.

Find $\frac{E[\,|X-\mu|\,]}{Var[X]}$.

A) $\frac{20}{9}$ B) $\frac{26}{9}$ C) $\frac{32}{9}$ D) $\frac{19}{81}$ E) $\frac{22}{81}$

8. Two players put one dollar into a pot. They decide to throw a pair of dice alternately. The first one who throws a total of 5 on both dice wins the pot. How much should the player who starts add to the pot to make this a fair game?

A) $\frac{9}{17}$ B) $\frac{8}{17}$ C) $\frac{1}{8}$ D) $\frac{2}{9}$ E) $\frac{8}{9}$

9. An analysis of economic data shows that the annual income of a randomly chosen individual from country A has a mean of $18,000 and a standard deviation of $6000,
and the annual income of a randomly chosen individual from country B has a mean of $31,000 and a standard deviation of $8000. 100 individuals are chosen at random from Country A and 100 from Country B. Find the approximate probability that the average annual income from the group chosen from Country B is at least $15,000 larger than the average annual income from the group chosen from Country A (all amounts are in US$).

A) .9972 B) .8413 C) .5000 D) .1587 E) .0228

10. Three individuals are running a one kilometer race. The completion time for each individual is a random variable. X_i is the completion time, in minutes, for person i.

X_1 : uniform distribution on the interval $[2.9\,,\ 3.1]$

X_2 : uniform distribution on the interval $[2.7\,,\ 3.1]$

X_3 : uniform distribution on the interval $[2.9\,,\ 3.3]$

The three completion times are independent of one another.

Find the expected latest completion time (nearest .1).

A) 2.9 B) 3.0 C) 3.1 D) 3.2 E) 3.3

11. The amount of liability claim Y in a motor vehicle accident has a uniform distribution on the interval $(0,1)$, and the amount of property damage in that accident has a uniform distribution on the interval $(0, \sqrt{y})$. Find the density function of X, the amount of property damage in an accident.

A) $2(1-x)$ B) $2x$ C) $2(1-x^{1/4})$ D) $\frac{1}{\sqrt{x}}-1$ E) $\frac{1}{2\sqrt{x}}$

12. A loss random variable X has a uniform distribution on the interval $[0,1000]$.
Find the variance of the insurer payment per loss if there is a deductible of amount 100 and a policy limit (maximum insurance payment) of amount 400 (nearest 1000).

A) 20,000 B) 21,000 C) 22,000 D) 23,000 E) 24,000

13. Let $X_1, ..., X_n$ be independent Poisson random variables with expectations $\lambda_1, ..., \lambda_n$, respectively. $Z = \sum_{i=1}^{n} aX_i$, where a is a constant. Find the moment generating function of Z.

A) $exp\Big(t\sum_{i=1}^{n} a\lambda_i + \frac{1}{2}t^2\sum_{i=1}^{n} a^2\lambda_i\Big)$ B) $exp\Big(\sum_{i=1}^{n} a\lambda_i(\mathrm{e}^t-1)\Big)$

C) $exp\Big(t\sum_{i=1}^{n} a\lambda_i + \frac{1}{2}t^2\sum_{i=1}^{n} a^2\lambda_i^2\Big)$ D) $exp\Big(\sum_{i=1}^{n}\lambda_i(\mathrm{e}^{at}-1)\Big)$ E) $\Big[\prod_{i=1}^{n}\lambda_i\Big][e^{at}-1]^n$

14. Let X be a random variable with mean 3 and variance 2, and let Y be a random variable such that for every x, the conditional distribution of Y given $X=x$ has a mean of x and a variance of x^2. What is the variance of the marginal distribution of Y?

A) 2 B) 4 C) 5 D) 11 E) 13

15. Let X and Y be discrete random variables with joint probabilities given by

	X	1	5
Y	2	$\theta_1+\theta_2$	$\theta_1+2\theta_2$
	4	$\theta_1+2\theta_2$	$\theta_1+\theta_2$

Let the parameters θ_1 and θ_2 satisfy the usual assumption associated with a joint probability distribution and the additional constraints $-.25 \le \theta_1 \le .25$ and $0 \le \theta_2 \le .35$. If X and Y are independent, then $(\theta_1, \theta_2) =$

A) $(0, \frac{1}{6})$ B) $(\frac{1}{4}, 0)$ C) $(-\frac{1}{4}, \frac{1}{3})$ D) $(-\frac{1}{8}, \frac{1}{4})$ E) $(\frac{1}{16}, \frac{1}{8})$

16. A machine has two components. The machine will continue to operate as long as at least one of the two components is working. Measured from when a new machine begins continuous operation, the time (in years) until failure of component 1 is X and the time (in years) until failure of component 2 is Y. The density function of the joint distribution between X and Y is $f(x,y) = x + y$, $0 < x < 1$, $0 < y < 1$. Find the probability that a new machine is still in operation 6 months after it began operating.

A) $\frac{31}{32}$ B) $\frac{15}{16}$ C) $\frac{7}{8}$ D) $\frac{3}{4}$ E) $\frac{1}{2}$

17. For a Poisson random variable X with mean λ it is found that it is twice as likely for X to be less than 3 as it is for X to be greater than or equal to 3. Find λ (nearest .1).

A) 2.0 B) 2.2 C) 2.4 D) 2.6 E) 2.8

18. Let X and Y be discrete random variables with joint probability function

$f(x,y) = \begin{cases} \frac{2^{x+1-y}}{9} \text{ for } x=1,2 \text{ and } y=1,2 \\ 0, \text{otherwise} \end{cases}$. Calculate $E[\frac{X}{Y}]$.

A) $\frac{8}{9}$ B) $\frac{5}{4}$ C) $\frac{4}{3}$ D) $\frac{25}{18}$ E) $\frac{5}{3}$

19. People passing by a city intersection are asked for the month in which they were born. It is assumed that the population is uniformly divided by birth month, so that any randomly passing person has an equally likely chance of being born in any particular month. Find the minimum number of people needed so that the probability that no two people have the same birth month is less than .5 .

A) 2 B) 3 C) 4 D) 5 E) 6

20. Under a group insurance policy, an insurer agrees to pay 100% of the medical bills incurred during the year by employees of a small company, up to a maximum total of one million dollars. The total amount of bills incurred, X, has probability density function

$$f(x) = \begin{cases} \frac{x(4-x)}{9}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$, where x is measured in millions.

Calculate the total amount, in millions of dollars, the insurer would expect to pay under this policy.

A) 0.120 B) 0.301 C) 0.935 D) 2.338 E) 3.495

21. Let X be a random variable with moment generating function $M(t) = \left(\frac{2+e^t}{3}\right)^9$ for $-\infty < x < \infty$. Find the variance of X.

A) 2 B) 3 C) 8 D) 9 E) 11

22. A carnival gambling game involves spinning a wheel and then tossing a coin. The wheel lands on one of three colors, red, white or blue. There is a 1/2 chance that the wheel lands on red, and there is a 3/8 chance of white and a 1/8 chance of blue. A coin of the color indicated by the wheel is then tossed. Red coins have a 50% chance of tossing a head, white coins have a 3/4 chance of tossing a head, and blue coins have a 7/8 chance of tossing a head. If the game player tosses a head, she wins $100, if she does not toss a head she wins 0. Find the cost to play the game so that the carnival wins an average of $1 per play of the game.

A) Less than 62 B) At least 62 but less than 64 C) At least 64 but less than 66
D) At least 66 but less than 68 E) At least 68

23. Fred, Ned and Ted each have season tickets to the Toronto Rock (Lacrosse).
Each one of them might, or might not attend any particular game. The probabilities describing their attendance for any particular game are

$P[$at least one of them attends the game$] = .95$,

$P[$at least two of them attend the game$] = .80$, and

$P[$all three of them attend the game$] = .50$.

Their attendance pattern is also symmetric in the following way

$P(F) = P(N) = P(T)$ and $P(F \cap N) = P(F \cap T) = P(N \cap T)$,

where F, N and T denote the events that Fred, Ned and Ted attended the game, respectively. For a particular game, find the probability that Fred and Ned attended.

A) .15 B) .30 C) .45 D) .60 E) .75

24. An insurer will pay the amount of a loss in excess of a deductible amount α. Suppose that the loss amount has a continuous uniform distribution between 0 and $C > \alpha$. When a loss occurs, let the expected payout on the policy be $f(\alpha)$. Find $f'(\alpha)$.

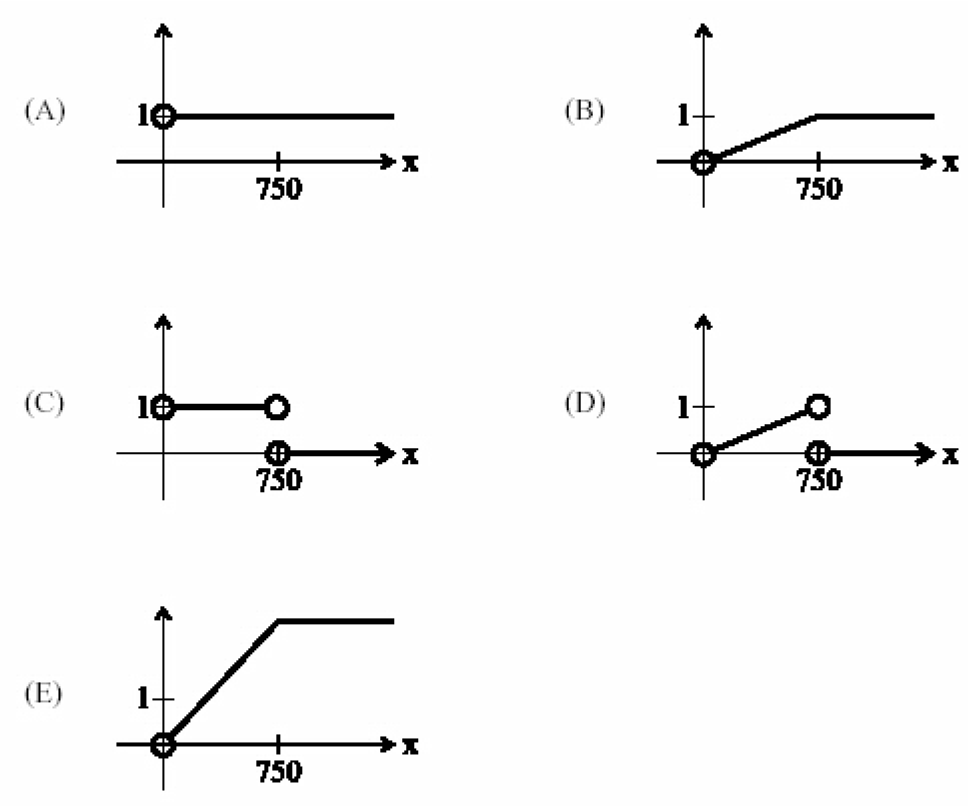
A) $\frac{\alpha}{C}$ B) $-\frac{\alpha}{C}$ C) $\frac{\alpha}{C}+1$ D) $\frac{\alpha}{C}-1$ E) $1-\frac{\alpha}{C}$

25. Coins K and L are weighted so the probabilities of heads are .3 and .1, respectively. Coin K is tossed 5 times and coin L is tossed 10 times. If all the tosses are independent,
what is the probability that coin K will result in heads 3 times and coin L will result in heads 6 times?

A) $\binom{5}{3}(.3)^3(.7)^2+\binom{10}{6}(.1)^3(.9)^2$ B) $\binom{5}{3}(.3)^3(.7)^2\binom{10}{6}(.1)^6(.9)^4$

C) $\binom{15}{9}(.4)^9(.6)^6$ D) $\frac{\binom{5}{3}\binom{10}{6}}{\binom{15}{9}}$ E) $(.6)(.9)$

26. An insurance policy is written that reimburses the policyholder for all losses incurred up to a benefit limit of 750 . Let $f(x)$ be the benefit paid on a loss of x . Which of the following most closely resembles the graph of the derivative of f ?



27. The value, v, of an appliance is based on the number of years since purchase, t, as follows: $v(t) = e^{(7-.2t)}$. If the appliance fails within seven years of purchase, a warranty pays the owner the value of the appliance. After seven years the warranty pays nothing. The time until failure of the appliance has an exponential distribution with a mean of 10. Calculate the expected payment from the warranty.

A) 98.70 B) 109.66 C) 270.43 D) 320.78 E) 352.16

28. A test for a disease correctly diagnoses a diseased person as having the disease with probability .85. The test incorrectly diagnoses someone without the disease as having the disease with a probability of .10. If 1% of the people in a population have the disease, what is the chance that a person from this population who tests positive for the disease actually has the disease?

A) $.0085$ B) $.0791$ C) $.1075$ D) $.1500$ E) $.9000$

29. Let X and Y be discrete random variables with joint probability function $f(x,y)$ given by the following table:

		x			
		2	3	4	5
	0	.05	.05	.15	.05
y	1	.40	0	0	0
	2	.05	.15	.10	0

For this joint distribution, $E[X] = 2.85$ and $E[Y] = 1$. Calculate $Cov[X,Y]$.

A) $-.20$ B) $-.15$ C) $.95$ D) 2.70 E) 2.85

30. One of the questions asked by an insurer on an application to purchase a life insurance policy is whether or not the applicant is a smoker. The insurer knows that the proportion of smokers in the general population is .30, and assumes that this represents the proportion of applicants who are smokers. The insurer has also obtained information regarding the honesty of applicants:

- 40% of applicants that are smokers say that they are non-smokers on their applications,
- none of the applicants who are non-smokers lie on their applications.

What proportion of applicants who say they are non-smokers are actually non-smokers?

A) 0 B) $\frac{6}{41}$ C) $\frac{12}{41}$ D) $\frac{35}{41}$ E) 1

PRACTICE EXAM 1 - SOLUTIONS

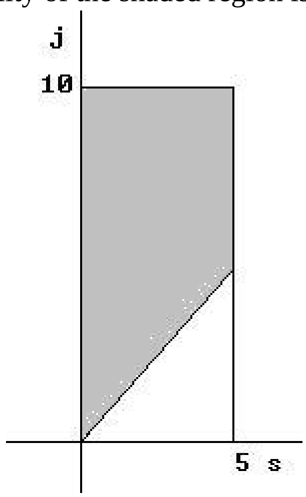
1. $P[E' \cup F'] = P[E'] + P[F'] - P[E' \cap F']$.
But $E' \cap F' = (E \cup F)'$, so that $P[E' \cap F'] = P[(E \cup F)'] = 1 - P[E \cup F] = 1 - 1 = 0$,
so that $P[E' \cup F'] = P[E'] + P[F']$. Answer: C

2. We define the following events:
A - the new driver has had driver education
B - the new driver has had an accident in his first year.
We are to find $P[A|\overline{B}] = \frac{P[A \cap \overline{B}]}{P[\overline{B}]}$, and we are given $P[A] = .6$, $P[B|\overline{A}] = .08$,
and $P[B|A] = .05$. Using rules of probability, $P[\overline{B}|A] = 1 - P[B|A] = .95$,
and hence, $P[A \cap \overline{B}] = P[\overline{B}|A] \cdot P[A] = (.95)(.6)$. Also, $P[\overline{A}] = 1 - P[A] = .4$.
But, $P[\overline{B}|\overline{A}] = 1 - P[B|\overline{A}] = 1 - .08 = .92$, and hence
$P[\overline{A} \cap \overline{B}] = P[\overline{B}|\overline{A}] \cdot P[\overline{A}] = (.92)(.4)$.
Thus, $P[\overline{B}] = P[A \cap \overline{B}] + P[\overline{A} \cap \overline{B}] = (.95)(.6) + (.92)(.4)$, and then
$P[A|\overline{B}] = \frac{P[A \cap \overline{B}]}{P[\overline{B}]} = \frac{(.95)(.6)}{(.95)(.6)+(.92)(.4)}$. Answer: E

3. $f(x) = ae^{-x} + be^{-2x} \rightarrow \int_0^\infty f(x)\,dx = a + \frac{1}{2}b = 1$.
We use the following integral rule for integer $k \geq 0$ and $c > 0$
$\int_0^\infty x^k\, e^{-cx}\,dx = \frac{k!}{c^{k+1}}$, to get
$E[X] = \int_0^\infty x\, f(x)\,dx = a + \frac{1}{4}b = 1$.
Solving the equations results in $a = 1$, $b = 0$. The probability is
$P[X < 1] = \int_0^1 e^{-x}\,dx = 1 - e^{-1} = .632$. Answer: B

4. The function $y = u(x) = ln\, x$ is strictly increasing (and thus, one-to-one) for all $x > 0$, with the inverse function being $x = e^y = v(y)$. Then
$f_Y(y) = f_X(v(y)) \cdot |v'(y)| = f_X(e^y) \cdot e^y = e^y \cdot e^{-(e^y)^2/2} \cdot e^y = e^{2y - \frac{1}{2}e^{2y}}$.
Alternatively, $F_Y(y) = P[Y \leq y] = P[ln\, X \leq y] = P[X \leq e^y]$, and
$F_X(x) = P[X \leq x] = \int_0^x te^{-t^2/2}dt = 1 - e^{-x^2/2}$
$\Rightarrow\ F_Y(y) = P[X \leq e^y] = \int_0^{e^y} te^{-t^2/2}dt = 1 - e^{-(e^y)^2/2}$
$\Rightarrow\ f_Y(y) = F'_Y(y) = e^{2y - \frac{1}{2}e^{2y}}$. Answer: A

5. The joint density of time until death is $f_{S,J}(s,j) = f_S(s) \cdot f_J(j) = \frac{1}{5} \cdot \frac{1}{10} = \frac{1}{50}$.
The rectangle below is the region of density for the joint distribution and the shaded region represents the event $S < J$ (Smith's time of death is less than that of Jones). The probability is $\int_0^5 \int_s^{10} \frac{1}{50}\, dj\, ds = \int_0^5 \frac{10-s}{50}\, ds = .75$. Alternatively, since both S and J are uniformly distributed, and they are independent, it follows that the joint distribution is uniformly distributed on the rectangle and the probability of any region is the fraction of the region of the full rectangle on which the joint distribution is defined. The shaded region can be seen to be .75 of area of the rectangle, and therefore probability of the shaded region is .75.



Answer: E

6. Since $X \sim N(1,4)$, $Z = \frac{X-1}{2}$ has a standard normal distribution. The probability in question can be written as

$P[X^2 - 2x \leq 8] = P[X^2 - 2X + 1 \leq 9] = P[(X-1)^2 \leq 9] = P[-3 \leq X - 1 \leq 3]$
$= P[-1.5 \leq \frac{X-1}{2} \leq 1.5] = P[-1.5 \leq Z \leq 1.5] = \Phi(1.5) - [1 - \Phi(1.5)] = .86$

(from the standard normal table). Answer: D

7. $E[X] = \int_0^1 x \cdot (2x)\, dx = \frac{2}{3}$. $E[X^2] = \int_0^1 x^2 \cdot (2x)\, dx = \frac{1}{2}$.
$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$.
$E[|X - \mu|] = E[|X - \frac{2}{3}|] = \int_0^{2/3} (\frac{2}{3} - x)(2x)\, dx + \int_{2/3}^1 (x - \frac{2}{3})(2x)\, dx = \frac{16}{81}$.
Then $\frac{E[|X-\mu|]}{Var[X]} = \frac{16/81}{1/18} = \frac{32}{9}$. Answer: C

8. Player 1 throws the dice on throws 1, 3, 5, . . . and the probability that player wins on throw $2k+1$ is $(\frac{8}{9})^{2k} \cdot \frac{1}{9}$ for $k = 0, 1, 2, 3, \ldots$ (there is a $\frac{1}{9}$ probability of throwing a total of 5 on any one throw of the pair of dice). The probability that player 1 wins the pot is
$\frac{1}{9} + (\frac{8}{9})^2 \cdot \frac{1}{9} + (\frac{8}{9})^4 \cdot \frac{1}{9} + \cdots = \frac{1}{9} \cdot \frac{1}{1-(\frac{8}{9})^2} = \frac{9}{17}$.
Player 2 throws the dice on throws 2, 4, 6, . . The probability that player 2 wins the pot on throw $2k$ is $(\frac{8}{9})^{2k-1} \cdot \frac{1}{9}$ for $k = 1, 2, 3, \ldots$ and the probability that player 2 wins is
$\frac{8}{9} \cdot \frac{1}{9} + (\frac{8}{9})^3 \cdot \frac{1}{9} + (\frac{8}{9})^5 \cdot \frac{1}{9} + \cdots = \frac{8}{9} \cdot \frac{1}{9} \cdot \frac{1}{1-(\frac{8}{9})^2} = \frac{8}{17} = 1 - \frac{9}{17}$.
If player 1 puts $1+c$ dollars into the pot, then his expected gain is $1 \cdot \frac{9}{17} - (1+c) \cdot \frac{8}{17}$.
and player 2's expected gain is $(1+c) \cdot \frac{8}{17} - 1 \cdot \frac{9}{17}$. In order for the two players to have the same expected gain, we must have $1 \cdot \frac{9}{17} - (1+c) \cdot \frac{8}{17} = 0$, so that $c = \frac{1}{8}$. Answer: C

9. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$ be the average annual income of the 100 individuals from Country A.
Then $E[S] = (\frac{1}{100})\sum_{i=1}^{100} E[X_i] = (\frac{1}{100})(100)(18,000) = 18,000$,
and $Var[S] = (\frac{1}{100})^2\sum_{i=1}^{100} Var[X_i] = (\frac{1}{100})^2(100)(6,000)^2 = 360,000$
(being randomly chosen, the X_i's are independent, and covariances between any pair is 0). In a similar way, let $T = \frac{1}{100}\sum_{i=1}^{100} Y_i$ be the average annual income of the 100 individuals from Country B. Then $E[T] = (\frac{1}{100})\sum_{i=1}^{100} E[Y_i] = (\frac{1}{100})(100)(31,000) = 31,000$,
and $Var[T] = (\frac{1}{100})^2\sum_{i=1}^{100} Var[Y_i] = (\frac{1}{100})^2(100)(8,000)^2 = 640,000$.
Since the sample sizes are each 100, both S and T have distributions which are approximately normal (sample size 30 is usually the number taken in practice to use the normal approximation to the sum or mean of a random sample).

We wish to find $P[T > S + 15,000]$. $W = T - S$ has normal distribution with mean $E[W] = E[T-S] = E[T] - E[S] = 31,000 - 18,000 = 13,000$, and with variance $Var[W] = Var[T-S] = Var[T] + Var[S] = 640,000 + 360,000 = 1,000,000$
(T and S are independent, since the samples are drawn from two different Countries; therefore there is no covariance between T and S). Then,
$P[W > 15,000] = P[\frac{W-13,000}{\sqrt{1,000,000}} > \frac{15,000-13,000}{\sqrt{1,000,000}}] = P[Z > 2]$
$= 1 - \Phi(2) = 1 - .9772 = .0228$ (we transform W to get $Z = \frac{W-E[W]}{\sqrt{Var[W]}}$, which has a standard normal distribution). Answer: E

10. $Y = max(X_1\ ,\ X_2\ ,\ X_3)$. $f_Y(y) = F_Y'(y)$, where

$F_Y(y) = P[Y \leq y] = P[max(X_1\ ,\ X_2\ ,\ X_3) \leq y] = P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y)]$

$= P[X_1 \leq y] \cdot P[X_2 \leq y] \cdot P[X_3 \leq y]$

$$= \begin{cases} 5(y-2.9) \cdot 2.5(y-2.7) \cdot 2.5(y-2.9) = 31.25(y^3 - 8.5y^2 + 24.07y - 22.707) & \text{for } 2.9 \leq y \leq 3.1 \\ 2.5(y-2.9) & \text{for } 3.1 \leq y \leq 3.3 \end{cases}$$

and $F_Y(y) = 0$ for $y \leq 2.9$.

Then, $f_Y(y) = F_Y'(y) = \begin{cases} 31.25(3y^2 - 17y + 24.07) & \text{for } 2.9 \leq y \leq 3.1 \\ 2.5 & \text{for } 3.1 \leq y \leq 3.3 \end{cases}$.

Finally, $E[Y] = \int_{2.9}^{3.1} y \cdot 31.25(3y^2 - 17y + 24.07)\, dy + \int_{3.1}^{3.3} y \cdot 2.5\, dy = \frac{73}{48} + 1.6 = 3.12$.

An alternative solution uses the fact that for a non-negative random variable, $Y \geq 0$, the mean can be expressed in the form $E[Y] = \int_0^\infty [1 - F_Y(y)]\, dy$.

In this case,

$E[Y] = \int_0^{2.9} [1-0]\, dy + \int_{2.9}^{3.1} [1 - 31.25(y^3 - 8.5y^2 + 24.07y - 22.707)]\, dy$

$+ \int_{3.1}^{3.3} [1 - 2.5(y-2.9)]\, dy = 2.9 + \frac{41}{240} + \frac{1}{20} = 3.12$. Answer: C

11. The marginal density of X is $f(x) = \int_0^1 f(x,y)\, dy$.

The joint density of X and Y can be constructed from the conditional density of X given Y and the marginal density of Y, $f(x,y) = f(x|y) \cdot g(y)$. Then

$f(x) = \int_0^1 f(x,y)\, dy = \int_0^1 f(x|y) \cdot g(y)\, dy$.

But $f(x|y) = \frac{1}{\sqrt{y}}$ for $0 < x < \sqrt{y}$, or equivalently, for $x^2 < y < 1$.

Thus, $f(x) = \int_{x^2}^1 \frac{1}{\sqrt{y}} \cdot 1\, dy = 2y^{1/2}\Big|_{y=x^2}^{y=1} = 2(1-x)$. Answer: A

12. $Y =$ amount paid by insurance $= \begin{cases} 0 & X < 100 \\ X - 100 & 100 \leq X < 500 \\ 400 & X \geq 500 \end{cases}$

$E[Y] = \int_{100}^{500} (x-100) \cdot \frac{1}{1000}\, dx + 400[1 - F_X(500)] = 80 + 400(1 - \frac{1}{2}) = 280$.

$E[Y^2] = \int_{100}^{500} (x-100)^2 \cdot \frac{1}{1000}\, dx + 400^2[1 - F_X(500)] = \frac{64{,}000}{3} + 400^2(1 - \frac{1}{2}) = \frac{304{,}000}{3}$.

$Var[Y] = E[Y^2] - (E[Y])^2 = \frac{68{,}800}{3} = 22{,}933$. Answer: D

13. $M_Z(t) = E[e^{tZ}] = E[exp(t\sum_{i=1}^n aX_i) = E[e^{atX_1} \cdot e^{atX_2} \cdots e^{atX_n}]$, and since the X_i's are independent, and since a Poisson random variable Y with mean λ_i has mgf.

$M_Y(r) = e^{\lambda_i(e^r - 1)}$, this becomes

$M_Z(t) = \prod_{i=1}^n E[e^{atX_i}] = \prod_{i=1}^n M_{X_i}(at) = \prod_{i=1}^n e^{\lambda_i(e^{at}-1)} = exp[\sum_{i=1}^n \lambda_i(e^{at} - 1)]$. Answer: D

14. $Var[Y] = Var[E[Y|X]] + E[Var[Y|X]]$.
We are given $E[Y|X=x] = x$ and $Var[Y|X=x] = x^2$, so that
$E[Y|X] = X$ and $Var[Y|X] = X^2$, and then
$Var[Y] = Var[X] + E[X^2]$. We are given $E[X] = 3$, and $Var[X] = 2$,
so that $2 = Var[X] = E[X^2] - (E[X])^2 \rightarrow E[X^2] = 11$, and therefore,
$Var[Y] = 2 + 11 = 13$. Answer: E

15. Since the total probability must be 1, we have $4\theta_1 + 6\theta_2 = 1$.
The marginal distributions of X and Y have
$P[X=1] = P[X=5] = P[Y=2] = P[Y=4] = 2\theta_1 + 3\theta_2 = \frac{1}{2}$. Then, because of independence, $P[X=1, Y=2] = P[X=1] \cdot P[Y=2] = \frac{1}{4} = \theta_1 + \theta_2$.
Solving the two equations in θ_1 and θ_2 ($4\theta_1 + 6\theta_2 = 1$ and $\theta_1 + \theta_2 = \frac{1}{4}$) results in
$\theta_1 = \frac{1}{4}$, $\theta_2 = 0$. Answer: B

16. The new machine is still operating if at least one component is still working. The machine is no longer operating if both components have stopped working.
P[the new machine is still operating at time 6 months ($\frac{1}{2}$-year)]
$= 1 - P$[machine is no longer operating at $\frac{1}{2}$-year]
$= 1 - P[(X < \frac{1}{2}) \cap (Y < \frac{1}{2})] = 1 - \int_0^{1/2}\int_0^{1/2} f(x,y)\,dy\,dx$
$= 1 - \int_0^{1/2}\int_0^{1/2} (x+y)\,dy\,dx = 1 - \int_0^{1/2} (\frac{1}{2}x + \frac{1}{8})\,dx = 1 - \frac{1}{8} = \frac{7}{8}$. Answer: C

17. $P[X<3] = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2}e^{-\lambda} = e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})$.
$P[X \geq 3] = 1 - P[X<3] = 1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})$.
We are given that $P[X<3] = 2P[X \geq 3] \rightarrow e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) = 2[1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})]$
$\rightarrow e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) = \frac{2}{3}$.
It is not possible to solve this equation algebraically, but we can substitute the 5 possible answers for λ to see which is closest. For $\lambda = 2$, we get $e^{-2}(1 + 2 + \frac{2^2}{2}) = .677$, which turns out to be the closest answer. Answer: A

18. $E[\frac{X}{Y}] = \sum_{x=1}^{2}\sum_{y=1}^{2} \frac{X_i}{Y_i} \cdot p(X_i, Y_i) = 1 \cdot \frac{2}{9} + \frac{1}{2} \cdot \frac{1}{9} + 2 \cdot \frac{4}{9} + 1 \cdot \frac{2}{9} = \frac{25}{18}$ Answer: D

19. $A_2 =$ event that second person has different birth month from the first.
$P(A_2) = \frac{11}{12} = .9167$.
$A_3 =$ event that third person has different birth month from first and second.
Then, the probability that all three have different birthdays is
$P[A_3 \cap A_2] = P[A_3|A_2] \cdot P(A_2) = (\frac{10}{12})(\frac{11}{12}) = .7639$.
$A_4 =$ event that fourth person has different birth month from first three.
Then, the probability that all four have different birthdays is
$P[A_4 \cap A_3 \cap A_2] = P[A_4|A_3 \cap A_2] \cdot P[A_3 \cap A_2]$
$= P[A_4|A_3 \cap A_2] \cdot P[A_3|A_2] \cdot P(A_2) = (\frac{9}{12})(\frac{10}{12})(\frac{11}{12}) = .5729$.
$A_5 =$ event that fifth person has different birth month from first four.
Then, the probability that all five have different birthdays is
$P[A_5 \cap A_4 \cap A_3 \cap A_2] = P[A_5|A_4 \cap A_3 \cap A_2] \cdot P[A_4 \cap A_3 \cap A_2]$
$= P[A_5|A_4 \cap A_3 \cap A_2] \cdot P[A_4|A_3 \cap A_2] \cdot P[A_3|A_2] \cdot P(A_2)$
$= (\frac{8}{12})(\frac{9}{12})(\frac{10}{12})(\frac{11}{12}) = .3819$. Answer: D

20. The insurer will pay $L = \begin{cases} x & \text{if } x \le 1 \text{ (million)} \\ 1 & \text{if } x > 1 \text{ (million)} \end{cases}$.
The expected payment by the insurer will be
$E[L] = \int_0^1 x \cdot f(x)\,dx + \int_1^3 1 \cdot f(x)\,dx = \int_0^1 x \cdot \frac{x(4-x)}{9}\,dx + \int_1^3 \frac{x(4-x)}{9}\,dx$
$= \frac{13}{108} + \frac{22}{27} = \frac{101}{108} = .935.$ Answer: C

21. $Var[X] = E[X^2] - (E[X])^2$, and $E[X] = M'(0)$, $E[X^2] = M''(0)$.
$M'(t) = 9(\frac{2+e^t}{3})^8 \cdot \frac{e^t}{3}$, $M''(t) = 9 \cdot 8(\frac{2+e^t}{3})^7 \cdot (\frac{e^t}{3})^2 + 9(\frac{2+e^t}{3})^8 \cdot \frac{e^t}{3}$.
Then, $M'(0) = 3$ and $M''(0) = 8 + 3 = 11$, so that $Var[X] = 11 - 3^2 = 2$.
Alternatively, $Var[X] = \frac{d^2}{dt^2} \ln M(t)\Big|_{t=0}$. In this case,
$\ln M(t) = 9 \ln(\frac{2+e^t}{3}) = 9\,[\ln(2+e^t) - \ln 3]$,
so that $\frac{d}{dt} \ln M(t) = \frac{9e^t}{2+e^t}$, and $\frac{d^2}{dt^2} \ln M(t) = \frac{(2+e^t)(9e^t)-(9e^t)(e^t)}{(2+e^t)^2}$,
and then $\frac{d^2}{dt^2} \ln M(t)\Big|_{t=0} = \frac{(3)(9)-(9)(1)}{(3)^2} = 2$.
A quicker alternative is to recognize that the given MGF is the MGF of a binomial random variable with $n = 9$ and $p = \frac{1}{3}$. In general, the MGF of a binomial random variable with parameters n and p is $M(t) = [pe^t + (1-p)]^n$. Each random variable has its own unique MGF, so $M(t) = (\frac{2+e^t}{3})^9$ must be the MGF for the binomial with $n = 9$ and $p = \frac{1}{3}$.
The variance of X is then $np(1-p) = 9(\frac{1}{3})(\frac{2}{3}) = 2$. Answer: A

22. The amounts won for each coin type are the component distributions:

Red coin: $X_R = \begin{cases} 100 & \text{prob. } \frac{1}{2} \\ 0 & \text{prob. } \frac{1}{2} \end{cases}$, White coin: $X_W = \begin{cases} 100 & \text{prob. } \frac{3}{4} \\ 0 & \text{prob. } \frac{1}{4} \end{cases}$,

Blue coin: $X_B = \begin{cases} 100 & \text{prob. } \frac{7}{8} \\ 0 & \text{prob. } \frac{1}{8} \end{cases}$. $P(\text{Red}) = \frac{1}{2}$, $P(\text{White}) = \frac{3}{8}$, $P(\text{Blue}) = \frac{1}{8}$.

$Y =$ amount won is a mixture of $X_R,\ X_W$ and X_B , with the mixing weights above.
Then $P(Y = 100)$
$= P(X_R = 100) \cdot P(\text{Red}) + P(X_W = 100) \cdot P(\text{White}) + P(X_B = 100) \cdot P(\text{Blue})$
$= (\frac{1}{2})(\frac{1}{2}) + (\frac{3}{4})(\frac{3}{8}) + (\frac{7}{8})(\frac{1}{8}) = \frac{41}{64}$, and $P(Y = 0) = \frac{23}{64}$.
The expected amount won is $100(\frac{41}{64}) = 64.0625$ on a play of the game, so the carnival should charge the player 65.0625 per play. Answer: C

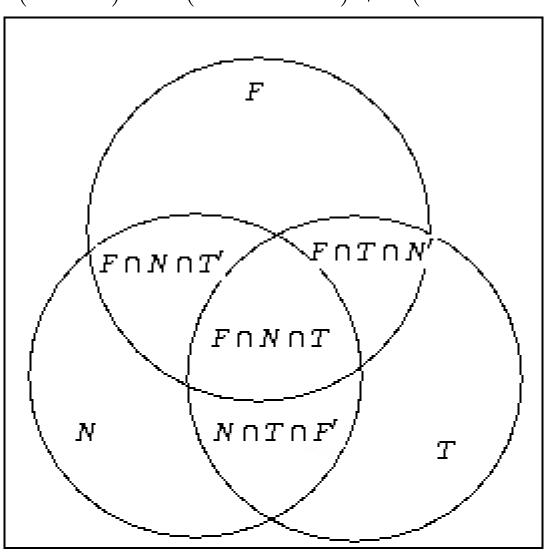
23. From the given probabilities, it follows that

P[exactly two of them attend the game]
$= P$[at least two of them attend the game] $- P$[all three of them attend the game]
$= .80 - .50 = .30$.

We also know that P[exactly two of them attend the game]
$= P(F \cap N \cap T') + P(F \cap N' \cap T) + P(F' \cap N \cap T) = .3$.

From the symmetry of the probabilities, the three on the right hand side of the equation are equal, so that $P(F \cap N \cap T') = P(F \cap N' \cap T) = P(F' \cap N \cap T) = .1$.
They are equal because $P(F \cap N \cap T') = P(F \cap N) - P(F \cap N \cap T)$ and
$P(F \cap T \cap N') = P(F \cap T) - P(F \cap N \cap T)$ and
$P(N \cap T \cap F') = P(N \cap T) - P(F \cap N \cap T)$, which are all equal.

The probability that Fred and Ned attend is then
$P(F \cap N) = P(F \cap N \cap T') + P(F \cap N \cap T) = .1 + .5 = .6$.



Answer: D

24. $f(\alpha)=\int_{\alpha}^{C}(x-\alpha)\cdot\frac{1}{C}\,dx=\frac{(C-\alpha)^2}{2C}\ \rightarrow\ f'(\alpha)=\frac{\alpha}{C}-1$. Answer: D

25. Because of independence, $P[(K=3)\cap(L=6)]=P[K=3]\cdot P[L=6]$
$=\left[\binom{5}{3}(.3)^3(.7)^2\right]\left[\binom{10}{6}(.1)^6(.9)^4\right]$
(K and L both have binomial distributions). Answer: B

26. $f(x)=\begin{cases} x & x\leq 750 \\ 750 & x>750 \end{cases}\ \rightarrow\ f'(x)=\begin{cases} 1 & x\leq 750 \\ 0 & x>750 \end{cases}$.
This is the graph in C. Answer: C

27. The expected value of the warranty is $E[w(m)]=\int_0^{\infty}w(m)\cdot f(m)\,dm$, where $f(m)$ is the density function of the appliance failing at time m. We are given that the failure time has an exponential distribution with a mean of 10. The mean of an exponential distribution with parameter λ is $\frac{1}{\lambda}=10$, so that $\lambda=.1$. and the density function is $f(m)=.1e^{-.1m}$. The expected value of the warranty is $E[w(m)]=\int_0^7 v(m)\cdot .1e^{-.1m}\,dm=\int_0^7 e^{(7-.2m)}\cdot .1e^{-.1m}\,dm$
$=.1e^7\int_0^7 e^{-.3m}dm=.1e^7[\frac{1-e^{-2.1}}{.3}]=320.78$. Answer: D

28. We define the following events: D - a person has the disease ,
TP - a person tests positive for the disease. We are given $P[TP|D]=.85$ and
$P[TP|D']=.10$ and $P[D]=.01$. We wish to find $P[D|TP]$.

With a model population of 10,000, there would be $10,000\times P(D)=100=\#D$ people with the disease and 9,900 without the disease. The number that have the disease and test positive is $\#D\cap TP=\#D\times P[TP|D]=100\times .85=85$ and the number that do not have the disease and test positive is $\#D'\cap TP=\#D'\times P[TP|D']=9,900\times .1=990$. The total number who test positive is $\#D=\#D\cap TP\ +\ \#D'\cap TP=85+990=1075$. The probability that someone who tests positive actually has the disease is the proportion $\frac{\#D\cap TP}{TP}=\frac{85}{1075}=.0791$.

The conditional probability approach to solving the problem is as follows.
Using the formulation for conditional probability we have $P[D|TP]=\frac{P[D\cap TP]}{P[TP]}$.
But $P[D\cap TP]=P[TP|D]\cdot P[D]=(.85)(.01)=.0085$, and
$P[D'\cap TP]=P[TP|D']\cdot P[D']=(.10)(.99)=.099$. Then,
$P[TP]=P[D\cap TP]+P[D'\cap TP]=.1075\rightarrow P[D|TP]=\frac{.0085}{.1075}=.0791$.

28. continued

The following table summarizes the calculations.

$$P[D] = .01 \text{ , given} \quad \Rightarrow \quad P[D'] = 1 - P[D] = .99$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$P[D \cap TP] \qquad\qquad\qquad\qquad P[D' \cap TP]$$

$$= P[TP|D] \cdot P[D] = .0085 \qquad\qquad = P[TP|D'] \cdot P[D'] = .099$$

$$\Downarrow$$

$$P[TP] = P[D \cap TP] + P[D' \cap TP] = .1075$$

$$\Downarrow$$

$$P[D|TP] = \frac{P[D \cap TP]}{P[TP]} = \frac{.0085}{.1075} = .0791 \text{ .}$$

Answer: B

29. $Cov[X, Y] = E[XY] - E[X] \cdot E[Y] = E[XY] - 2.85$.

$$E[XY] = \sum_{x=2}^{5} \sum_{y=0}^{2} xy \cdot f(x, y) = 2 \cdot 0 \cdot (.05) + 2 \cdot 1 \cdot (.40) + \cdots + 5 \cdot 1 \cdot (0) + 5 \cdot 2 \cdot (0)$$

$$= 2.7 - 2.85 = -.15 \text{ .}$$

Answer: B

30. We identify the following events:

S - the applicant is a smoker, NS - the applicant is a non-smoker $= S'$

DS - the applicant declares to be a smoker on the application

DN - the applicant declares to be non-smoker on the application $= DS'$.

The information we are given is $P[S] = .3$, $P[NS] = .7$, $P[DN|S] = .4$, $P[DS|NS] = 0$.

We wish to find $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]}$.

With a model population of 100 there are 30 $= \#S$ smokers and 70 $= \#NS$ non-smokers. The number of smokers who declare that they are non-smokers is $\#DN \cap S = \#S \times P[DN|S] = 30 \times .4 = 12$ and since non-smokers don't lie, the number of non-smokers who declare that they are non-smokers is equal to the number of non-smokers, so $\#DN \cap NS = NS = 70$. The total number of people who declare that they are non-smokers is $\#DN \cap S + \#DN \cap NS = 12 + 70 = 82 = \#DN$.

Then, the proportion of applicants who say they are non-smokers that are actually non-smokers is $\frac{\#DN \cap NS}{DN} = \frac{70}{82} = \frac{35}{41}$.

The conditional probability approach to the solution is on the next page.

30. continued

We calculate $.4 = P[DN|S] = \frac{P[DN \cap S]}{P[S]} = \frac{P[DN \cap S]}{.3} \rightarrow P[DN \cap S] = .12$,

and $0 = P[DS|NS] = \frac{P[DS \cap NS]}{P[NS]} = \frac{P[DS \cap NS]}{.7} \rightarrow P[DS \cap NS] = 0$.

Using the rule $P[A] = P[A \cap B] + P[A \cap B']$, and noting that $DS = DN'$ and $S = NS'$

we have $P[DS \cap S] = P[S] - P[DN \cap S] = .3 - .12 = .18$, and

$P[DN \cap NS] = P[NS] - P[DS \cap NS] = .7 - 0 = .7$, and

$P[DN] = P[DN \cap NS] + P[DN \cap S] = .7 + .12 = .82$.

Then, $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{.7}{.82} = \frac{35}{41}$.

These calculations can be summarized in the order indicated in the following table.

	$P(S)$, .3 given	$\Rightarrow$	**1.** $P(NS) = 1 - P(S) = .7$ $\Downarrow$
6. DS $\Leftarrow$ $P(DS)$ $= P(DS \cap S)$ $+ P(DS \cap NS)$ $= .18 + 0 = .18$ $\Downarrow$	**5.** $P(DS \cap S)$ $= P(S) - P(DN \cap S)$ $= .3 - .12 = .18$ $\Uparrow$		**2.** $P(DS|NS) = 0$, given $P(DS \cap NS)$ $= P(DS|NS) \cdot P(NS)$ $= (0)(.7) = 0$
7. DN $P(DN)$ $= 1 - P(DS)$ $= 1 - .18$ $= .82$	**4.** $P(DN|S) = .4$ given $P(DN \cap S)$ $= P(DN|S) \cdot P(S)$ $= (.4)(.3) = .12$		**3.** $P(DN \cap NS) =$ $= P(NS) - P(DS \cap NS)$ $= .7 - 0 = .7$

Then, **8.** $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{.7}{.82} = \frac{35}{41}$.   Answer: D

PRACTICE EXAM 2

1. Let X have the density function $f(x) = \begin{cases} \frac{2x}{k^2} & \text{for } 0 \le x \le k \\ 0, & \text{otherwise} \end{cases}$.

For what value of k is the variance of X equal to 2?

A) 2 B) 6 C) 9 D) 18 E) 36

2. A life insurer classifies insurance applicants according to the following attributes:

M - the applicant is male

H - the applicant is a homeowner

Out of a large number of applicants the insurer has identified the following information:

40% of applicants are male, 40% of applicants are homeowners and

20% of applicants are female homeowners.

Find the percentage of applicants who are male and do not own a home.

A) 10% B) 20% C) 30% D) 40% E) 50%

3. Two components in an electrical circuit have continuous failure times X and Y. Both components will fail by time 1, but the circuit is designed so that the combined times until failure is also less than 1, so that the joint distribution of failure times satisfies the requirements $0 < x + y < 1$. How many of the following joint density functions are consistent with an expected combined time until failure less than $\frac{1}{2}$ for the two components?

I. $f(x,y) = 2$ II. $f(x,y) = 3(x+y)$ III. $f(x,y) = 6x$ IV. $f(x,y) = 6y$

A) 0 B) 1 C) 2 D) 3 E) 4

4. In a "wheel of fortune" game, the contestant spins a dial and it ends up pointing to a number uniformly distributed between 0 and 1 (continuous). After 10,000 independent spins of the wheel find the approximate probability that the average of the 10,000 spins is less than .499.

A) Less than .34 B) At least .34 but less than .35 C) At least .35 but less than .36

D) At least .36 but less than .37 E) At least .37

5. A continuous random variable U has density function $f_U(u) = \begin{cases} 1-|u| & \text{for } -1<u<1 \\ 0, & \text{otherwise} \end{cases}$.

Which of the pairs of the following events are independent?

I. $-1 < U < 0$ II. $-\frac{1}{2} < U < \frac{1}{2}$ III. $0 < U < 1$.

A) I and II only B) I and III only C) II and III only

D) I and II, II and III only E) No pairs are independent

6. An excess-of-loss insurance policy has a deductible of 1 and pays a maximum amount of 1. The loss random variable being insured by the policy has an exponential distribution with a mean of 1. Find the expected claim paid by the insurer on this policy.

A) $e^{-1} - 2e^{-2}$ B) $e^{-1} - e^{-2}$ C) $2(e^{-1} - e^{-2})$ D) e^{-1} E) $2e^{-2}$

7. If $f(x) = (k+1)x^2$ for $0 < x < 1$, find the moment generating function of X.

A) $\frac{e^t(6+6t+3t^2)}{t^3}$ B) $\frac{e^t(6-6t+3t^2)}{t^3}$ C) $\frac{e^t(6+6t+3t^2)}{t^3} - \frac{6}{t^3}$

D) $\frac{e^t(6+6t+3t^2)}{t^3} + \frac{6}{t^3}$ E) $\frac{e^t(6-6t+3t^2)}{t^3} - \frac{6}{t^3}$

8. Urn 1 contains 5 red and 5 blue balls. Urn 2 contains 4 red and 6 blue balls, and Urn 3 contains 3 red balls. A ball is chosen at random from Urn 1 and placed in Urn 2. Then a ball is chosen at random from Urn 2 and placed in Urn 3. Finally, a ball is chosen at random from Urn 3. Find the probability that the ball chosen from Urn 3 is red.

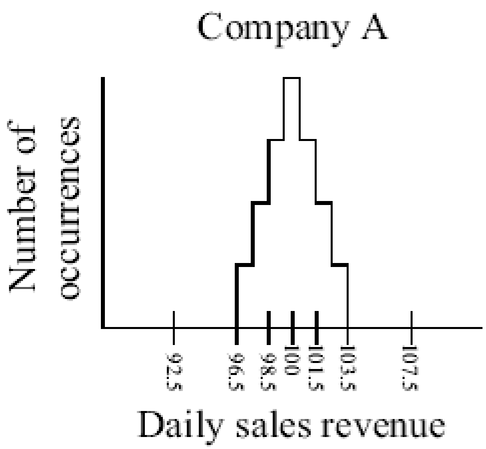
A) $\frac{15}{88}$ B) $\frac{30}{88}$ C) $\frac{45}{88}$ D) $\frac{60}{88}$ E) $\frac{75}{88}$

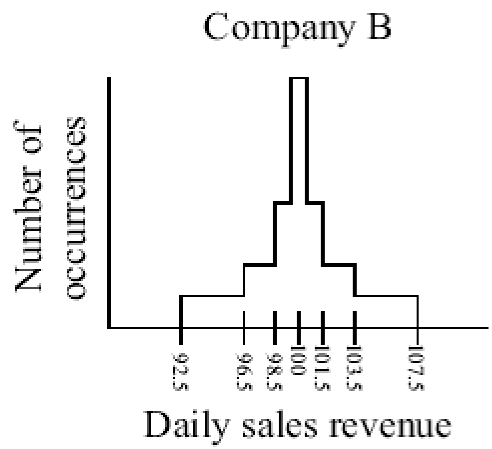
9. The amount of time taken by a machine repair person to repair a particular machine is a random variable with an exponential distribution with a mean of 1 hour. The repair person's employer pays the repair person a bonus of 2 whenever a repair takes less than $\frac{1}{4}$ hours, and a bonus of 1 if the repair takes between $\frac{1}{2}$ and $\frac{1}{4}$ hours. Find the average bonus received per machine repaired.

A) Less than .3 B) At least .3 but less than .4 C) At least .4 but less than .5

D) At least .5 but less than .6 E) At least .6

10. A stock market analyst has recorded the daily sales revenue for two companies over the last year and displayed them in the histograms below.





The analyst noticed that a daily sales revenue above 100 for Company A was always accompanied by a daily sales revenue below 100 for Company B, and vice versa. Let X denote the daily sales revenue for Company A and let Y denote the daily sales revenue for Company B, on some future day. Assuming that for each company the daily sales revenues are independent and identically distributed, which of the following is true?

(A) Var(X) > Var(Y) and Var(X + Y) > Var(X) + Var(Y) .

(B) Var(X) > Var(Y) and Var(X + Y) < Var(X) + Var(Y) .

(C) Var(X) > Var(Y) and Var(X + Y) = Var(X) + Var(Y) .

(D) Var(X) < Var(Y) and Var(X + Y) > Var(X) + Var(Y) .

(E) Var(X) < Var(Y) and Var(X + Y) < Var(X) + Var(Y) .

11. Bob and Doug are both 100-metre sprinters. Bob's sprint time is normally distributed with a mean of 10.00 seconds and Doug's sprint time is also normally distributed, but with a mean of 9.90 seconds. Both have the same standard deviation in sprint time of σ. Assuming that Bob and Doug have independent sprint times, and given that there is .95 chance that Doug beats Bob in any given race, find σ.

A) .040 B) .041 C) .042 D) .043 E) .044

12. Let X and Y be continuous random variables with joint cumulative distribution function $F(x,y) = \frac{1}{250}(20xy - x^2y - xy^2)$ for $0 \le x \le 5$ and $0 \le y \le 5$.

Determine $P[X > 2]$.

A) $\frac{3}{125}$ B) $\frac{11}{50}$ C) $\frac{12}{25}$ D) $\frac{1}{250}(39y - 3y^2)$ E) $\frac{1}{250}(36y - 2y^2)$

13. In a survey of males over the age of 30, it is found 50% are married, 40% smoke, 30% own a home and 60% own a car. It is also found that 30% are non-smoking bachelors, 40% are married car owners, 36% are non-smoking car owners, 25% own both a home and a car and 20% are married and own a home and a car. Which of the following statements is true regarding independence among the attributes of being married, being a smoker, being a car owner and being a home owner?

A) Being single and owning a car are independent

B) Being married and smoking are not independent

C) Being a smoker and owning a car are independent

D) Being a home owner and being a car owner are independent

E) Being married, being a home owner and being a car owner are mutually independent

14. A loss random variable is uniformly distributed on the integers from 0 to 11.
An insurance pays the loss in excess of a deductible of 5.5 . Find the expected amount not covered by the insurance.

A) 2 B) 3 C) 4 D) 5 E) 6

15. An insurer offers an "all or nothing" policy of the following type. If the loss being insured is for an amount of D or more, then the insurance policy pays the full amount, but if the loss is less than D then the policy pays nothing. Assuming that the distribution of the loss has an exponential distribution with a mean of 2, and that $D = 2$, find the expected payout on the policy.

A) $\frac{1}{e}$ B) $\frac{2}{e}$ C) $\frac{4}{e}$ D) e E) $2e$

16. Suppose that X is a random variable with moment generating function
$M(t) = \sum_{j=0}^{\infty} \frac{e^{(tj-1)}}{j!}$. Find $P[X = 2]$.

A) 0 B) $\frac{1}{2e}$ C) $\frac{e}{2}$ D) $\frac{1}{2}$ E) $\sum_{j=0}^{\infty} \frac{e^{2j-1}}{j!}$

17. A supplier of a testing device for a type of component claims that the device is highly reliable, with $P[A|B] = P[A'|B'] = .95$, where
$A =$ device indicates component is faulty, and $B =$ component is faulty .
You plan to use the testing device on a large batch of components of which 5% are faulty.
Find the probability that the component is faulty given that the testing device indicates that the component is faulty .

A) 0 B) .05 C) .15 D) .25 E) .50

18. Customers arrive randomly and independently at a service window, and the time between arrivals has an exponential distribution with a mean of 12 minutes. Let X equal the number of arrivals per hour. What is $P[X = 10]$?

A) $\frac{10e^{-12}}{10!}$ B) $\frac{10^{12}e^{-10}}{10!}$ C) $\frac{12^{10}e^{-10}}{10!}$ D) $\frac{12^{10}e^{-12}}{10!}$ E) $\frac{5^{10}e^{-5}}{10!}$

19. Workplace accidents are categorized in three groups: minor, moderate, and severe. The probability that a given accident is minor is .5, that it is moderate is .4, and that it is severe is .1. Two accidents occur independently in one month. Calculate the probability that neither accident is severe and at most one is moderate.

A) $.25$ B) $.40$ C) $.45$ D) $.56$ E) $.65$

20. Let X be a Poisson random variable with mean λ. If $P[X = 1|X \leq 1] = .8$, what is the value of λ?

A) 4 B) $-\,ln\,2$ C) $.8$ D) $.25$ E) $-\,ln\,.8$

21. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} x+y \text{ for } 0<x<1 \text{ , } 0<y<1 \\ 0, \text{otherwise} \end{cases} .$$

What is the marginal density function for X, where nonzero?

A) $y + \frac{1}{2}$ B) $2x$ C) x D) $\frac{x+x^2}{2}$ E) $x + \frac{1}{2}$

22. Let X_1, X_2, X_3 be uniform random variables on the interval $(0,1)$ with $Cov[X_i, X_j] = \frac{1}{24}$ for $i, j = 1, 2, 3$, $i \neq j$. Calculate the variance of $X_1 + 2X_2 - X_3$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{5}{12}$ D) $\frac{1}{2}$ E) $\frac{11}{12}$

23. The joint density for liability damage X and collision damage Y when a claim occurs is $f(x,y)$, $0 \le X \le 3$, $0 \le Y \le 1$. Which of the following represents the probability that total loss will exceed 1?

A) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$ B) $1 - \int_0^{1-y} \int_0^{1-x} f(x,y)\,dy\,dx$ C) $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$

D) $1 - \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$ E) $\int_0^1 \int_0^{x-1} f(x,y)\,dy\,dx$

24. Medical researchers have identified three separate genes that individuals may or may not be born with. The researchers have found that 25% of the population have gene A, 20% have gene B and 10% have gene C. Furthermore, in any individual, the presence of gene A is independent of the presence of genes B or C, but no people can have both genes B and C. Find the probability that a randomly chosen individual has at least one of the three genes.

A. .450 B. .475 C. .500 D. .525 E. .550

25. An insurance contract reimburses a family's automobile accident losses up to a maximum of two accidents per year. The joint probability distribution for the number of accidents of a three person family (X,Y,Z) is $p(x,y,z) = k(x+2y+z)$, where

$x = 0,1$, $y = 0,1,2$, $z = 0,1,2$, and

x,y,z are the number of accidents incurred by X, Y and Z, respectively.

Determine the expected number of unreimbursed accident losses given that X is not involved in any of the accidents.

A) 5/21 B) 1/3 C) 5/9 D) 4/63 E) 7/9

26. Let X be a continuous random variable with density function $f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$.

Find $E[\,|X - E[X]|\,]$.

A) 0 B) $\frac{2}{9}$ C) $\frac{32}{81}$ D) $\frac{64}{81}$ E) $\frac{4}{3}$

27. Let X and Y be continuous random variables with joint density function

$f(x,y) = \begin{cases} 1 & \text{for } 0 < y < 1 - |x| \text{ and } -1 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$.

What is $Var[X]$?

A) $\frac{1}{18}$ B) $\frac{1}{6}$ C) $\frac{2}{9}$ D) $\frac{11}{18}$ E) $\frac{2}{3}$

28. A pair of fair dice is tossed 2160 times. X is the number of times a total of 2 occurs. Find the approximate probability that X is less than 55 using the continuity correction.

A) .24 B) .26 C) .28 D) .30 E) .32

29. Suppose the remaining lifetimes of a husband and wife are independent and uniformly distributed on the interval [0,40]. An insurance company offers two products to married couples:

One which pays when the husband dies; and

One which pays when both the husband and wife have died.

Calculate the covariance of the two payment times.

A) 0.0 B) 44.4 C) 66.7 D) 200.0 E) 466.7

30. The mortality of a certain type of transistor is such that the probability of its breakdown in the interval $(t,\, t+dt)$ is given by: $ce^{-ct}\,dt,\ t>0,\ c>0$. If 10 of these transistors are taken at random, then the probability that the 10th transistor that breaks down will do so during time $(v,\, v+dv)$ is

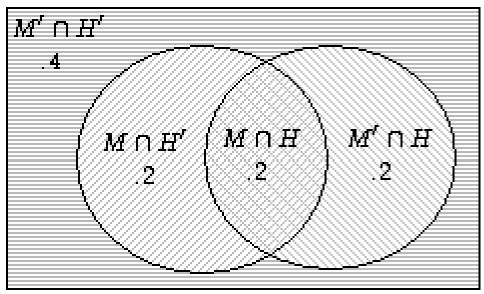
A.) $10c(1-e^{-cv})^9e^{-cv}\,dv$ B) $10ce^{-10cv}\,dv$ C) $10ce^{-9cv}(1-e^{-cv})\,dv$

D) $10c(1-e^{-9cv})e^{-cv}\,dv$ E. $c(1-e^{-cv})^9e^{-cv}\,dv$

PRACTICE EXAM 2 - SOLUTIONS

1. $E[X] = \int_0^k x \cdot \frac{2x}{k^2}\,dx = \frac{2k}{3}$, $E[X^2] = \int_0^k x^2 \cdot \frac{2x}{k^2}\,dx = \frac{k^2}{2}$
$\Rightarrow \; Var[X] = \frac{k^2}{2} - (\frac{2k}{3})^2 = \frac{k^2}{18} = 2 \rightarrow k = 6$. Answer: B

2. $P[M] = .4$, $P[M'] = .6$, $P[H] = .4$, $P[H'] = .6$, $P[M' \cap H] = .2$,
We wish to find $P[M \cap H']$. From probability rules, we have
$.6 = P[H'] = P[M' \cap H'] + P[M \cap H']$, and
$.6 = P[M'] = P[M' \cap H] + P[M' \cap H'] = .2 + P[M' \cap H']$.
Thus, $P[M' \cap H'] = .4$ and then $P[M \cap H'] = .2$. The following diagram identifies the component probabilities.



The calculations above can also be summarized in the following table. The events across the top of the table categorize individuals as male (M) or female (M'), and the events down the left side of the table categorize individuals as homeowners (H) or non-homeowners (H').

	$P(M) = .4$, given		$P(M') = 1 - .4 = .6$
$P(H) = .4$ given	$P(M \cap H)$ $= P(H) - P(M' \cap H) = .4 - .2 = .2$	$\Leftarrow$	$P(M' \cap H) = .2$, given
	$\Downarrow$		
$P(H') = 1 - .4 = .6$	$P(M \cap H') = P(M) - P(M \cap H) = .4 - .2 = .2$		

Answer: B

3. $E[X+Y] = \int_0^1 \int_0^{1-x} (x+y) f(x,y)\,dy\,dx$
I. $\int_0^1 \int_0^{1-x} 2(x+y)\,dy\,dx = \frac{2}{3}$. Not correct.
II. $\int_0^1 \int_0^{1-x} 3(x+y)(x+y)\,dy\,dx = \frac{3}{4}$. Not correct.
III .$\int_0^1 \int_0^{1-x} 6x(x+y)\,dy\,dx = \frac{3}{4}$. Not correct.
IV .$\int_0^1 \int_0^{1-x} 6y(x+y)\,dy\,dx = \frac{3}{4}$. Not correct.
Note that III and IV will have the same outcome by the symmetry of x and y. Answer: A

4. The outcome of a single spin is X, which has a uniform distribution on the interval $(0,1]$. The mean and variance of X are $\frac{1}{2}$ and $\frac{1}{12}$. If $W = \sum_{i=1}^{10,000} X_i$, then W has an approximately normal distribution with mean $10,000(\frac{1}{2}) = 5000$ and variance $10,000(\frac{1}{12}) = 833.33$.
The average of the 10,000 spins is $A = \frac{W}{10,000}$.
$P[A < .499] = P[W < 4990] = P[\frac{W-5000}{\sqrt{833.33}} < \frac{4990-5000}{\sqrt{833.33}}]$
$\doteq \Phi(\frac{4990-5000}{\sqrt{833.33}}) = \Phi(-.346) = 1 - \Phi(.35) = 1 - .637 = .363$. Answer: D

5. $P(I) = \int_{-1}^{0}(1-|u|)\,du = \int_{-1}^{0}(1+u)\,du = \frac{1}{2}$,
$P(II) = \int_{-1/2}^{1/2}(1-|u|)\,du = \int_{-1/2}^{0}(1+u)\,du + \int_{0}^{1/2}(1-u)\,du = \frac{3}{4}$,
$P(III) = \int_{0}^{1}(1-|u|)\,du = \int_{0}^{1}(1-u)\,du = \frac{1}{2}$.
$P(I \cap II) = P(-\frac{1}{2} < U < 0) = \int_{-1/2}^{0}(1+u)\,du = \frac{3}{8} = (\frac{1}{2})(\frac{3}{4}) = P(I) \cdot P(II)$,
which shows that I and II are independent.
$P(I \cap III) = P(\emptyset) = 0 \neq (\frac{1}{2})(\frac{1}{2}) = P(I) \cdot P(III)$, which shows that I and III are not independent.
$P(II \cap III) = \int_{0}^{1/2}(1-|u|)\,du = \frac{3}{8} = (\frac{3}{4})(\frac{1}{2}) = P(II) \cdot P(III)$, which shows that II and III are independent. Answer: D

From the density function for Y we have
$P[100,000Y > 40,000] = P[Y > .4] = \int_{.4}^{1} f(y)\,dy = \int_{.4}^{1} 5(1-y)^4\,dy = (.6)^5$, and
$P[100,000Y > 10,000] = P[Y > .1] = \int_{.1}^{1} f(y)\,dy = \int_{.1}^{1} 5(1-y)^4\,dy = (.9)^5$.
The conditional probability in question is $\frac{(.6)^5}{(.9)^5} = .132$. Answer: B

6. The loss random variable is X, which has density function $f_X(x) = e^{-x}$ for $x > 1$.
The amount paid by the insurance is $Y = \begin{cases} 0 \text{ if } X \leq 1 \\ X-1 \text{ if } 1 < X \leq 2 \\ 1 \text{ if } X > 2 \end{cases}$.
Then $E[Y] = \int_{1}^{2}(x-1)e^{-x}dx + 1 \cdot P[X > 2] = -xe^{-x}\Big|_{x=1}^{x=2} + e^{-2} = e^{-1} - e^{-2}$.
Answer: B

7. Since $\int_0^1 f(x)\,dx = 1$, it follows that $(k+1)\cdot\frac{1}{3} = 1$, so that $k=2$, and $f(x) = 3x^2$. Then, $M_X(t) = E[e^{tX}] = \int_0^1 e^{tx}\cdot 3x^2 dx$. Applying integration by parts, we have

$$\int_0^1 e^{tx}\cdot 3x^2 dx = \int_0^1 3x^2\ d(\tfrac{e^{tx}}{t}) = \frac{3x^2e^{tx}}{t}\Big|_{x=0}^{x=1} - \int_0^1 \frac{6xe^{tx}}{t}\,dx$$
$$= \frac{3e^t}{t} - \int_0^1 \frac{6x}{t}\ d(\tfrac{e^{tx}}{t}) = \frac{3e^t}{t} - \left[\frac{6xe^{tx}}{t^2}\Big|_{x=0}^{x=1} - \int_0^1 \frac{6e^{tx}}{t^2}dx\right]$$
$$= \frac{3e^t}{t} - \frac{6e^t}{t^2} + \frac{6(e^t-1)}{t^3} = \frac{e^t(6-6t+3t^2)}{t^3} - \frac{6}{t^3}.$$ Answer: E

8. $P(R_3) = P(R_3\cap R_2\cap R_1) + P(R_3\cap R_2\cap B_1) + P(R_3\cap B_2\cap R_1) + P(R_3\cap B_2\cap B_1)$.
$P(R_3\cap R_2\cap R_1) = P(R_3|R_2\cap R_1)\cdot P(R_2|R_1)\cdot P(R_1) = 1\cdot\frac{5}{11}\cdot\frac{1}{2} = \frac{5}{22}$,
$P(R_3\cap R_2\cap B_1) = P(R_3|R_2\cap B_1)\cdot P(R_2|B_1)\cdot P(B_1) = 1\cdot\frac{4}{11}\cdot\frac{1}{2} = \frac{2}{11}$,
$P(R_3\cap B_2\cap R_1) = P(R_3|B_2\cap R_1)\cdot P(B_2|R_1)\cdot P(R_1) = \frac{3}{4}\cdot\frac{6}{11}\cdot\frac{1}{2} = \frac{9}{44}$,
$P(R_3\cap B_2\cap B_1) = P(R_3|B_2\cap B_1)\cdot P(B_2|B_1)\cdot P(B_1) = \frac{3}{4}\cdot\frac{7}{11}\cdot\frac{1}{2} = \frac{21}{88}$.
Finally, $P(R_3) = \frac{5}{22} + \frac{2}{11} + \frac{9}{44} + \frac{21}{88} = \frac{75}{88}$. Answer: E

9. Average bonus $= 2\cdot P[T\le\frac{1}{4}] + 1\cdot P[\frac{1}{4} < T \le \frac{1}{2}]$
$= 2[1-e^{-1/4}] + [e^{-1/4} - e^{-1/2}] = .615$. Answer: E

10. The histogram for Company B is more widely dispersed about its mean than the histogram for Company A, and therefore $Var(Y) > Var(X)$. Since daily sales revenue above 100 for Company A is always associated with daily sales revenue below 100 for Company B and vice-versa, the covariance between X and Y is negative. Therefore,
$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) < Var(X) + Var(Y)$. Answer: E

11. $B - D \sim N(.1, 2\sigma^2)$. $P[B > D] = P[B-D > 0] = P[\frac{B-D-.1}{\sigma\sqrt{2}} > \frac{0-.1}{\sigma\sqrt{2}}] = .95$
$\Rightarrow\ \frac{0-.1}{\sigma\sqrt{2}} = -1.645$ (since $P[Z < 1.645] = .95$) $\rightarrow\ \sigma = .043$. Answer: D

12. $F_X(2) = P[X\le 2] = \lim_{y\to\infty} F(2,y) = F(2,5) = \frac{130}{250} = \frac{13}{25}$,
so that $P[X > 2] = 1 - P[X\le 2] = \frac{12}{25}$. Answer: C

13. M - married, S - smoker, C - car owner, H - home owner
We are given, $P[M] = .5$, $P[S] = .4$, $P[C] = .6$, $P[H] = .3$, $P[M' \cap S'] = .3$,
$P[M \cap C] = .4$, $P[S' \cap C] = .36$, $P[H \cap C] = .25$, $P[M \cap C \cap H] = .2$.
Then, $P[M' \cap C] = P[C] - P[M \cap C] = .6 - .4 = .2$,
but $P[M'] \cdot P[C] = (.5)(.6) = .3 \rightarrow M', C$ not independent $\rightarrow$ A is false.
$P[M' \cap S'] = .3$, $P[M'] \cdot P[S'] = (.5)(.6) = .3 \rightarrow M', S'$ are independent
$\rightarrow M, S$ are independent $\rightarrow$ B is false.
$P[S' \cap C] = .36 = (.6)(.6) = P[S'] \cdot P[C] \rightarrow S', C$ are independent
$\rightarrow S, C$ are independent $\rightarrow$ C is true.
We can also check $P[H \cap C] = .25 \neq (.3)(.6) = P[H] \cdot P[C]$
$\rightarrow H, C$ not independent $\rightarrow$ D is false,
$P[M \cap C \cap H] = .2 \neq (.5)(.6)(.3) = P[M] \cdot P[C] \cdot P[H]$
$\rightarrow M, C, H$ are not mutually independent. Answer: C

14. The amount not covered is

Loss	0	1	2	3	4	5	6	7	$\cdots$	11
Amt. Not Covered	0	1	2	3	4	5	5.5	5.5	$\cdots$	5.5
Prob.	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\cdots$	$\frac{1}{12}$

The expected amount not covered by the insurance is
$(\frac{1}{12})[0 + 1 + 2 + 3 + 4 + 5 + 5.5(6)] = 4$. Answer: C

15. The expected payment on the policy will be
$\int_2^\infty x \cdot \frac{1}{2} e^{-x/2}\, dx = -xe^{-x/2} - 2e^{-x/2}\Big|_{x=2}^{x=\infty} = 4e^{-1}$. Answer: C

16. The moment generating function for a non-negative discrete integer-valued random variable X with probability function f is defined to be $M(t) = E[e^{tX}] = \sum_{j=0}^{\infty} e^{tj} \cdot f(j)$.
Since we are given that $M(t) = \sum_{j=0}^{\infty} \frac{e^{(tj-1)}}{j!}$, and it is known that the distribution of a random variable is uniquely determined by its moment generating function (i.e., there is precisely one probability distribution with that specified mgf), it follows that
$f(j) = \frac{e^{-1}}{j!} = \frac{1}{e \cdot j!}$. Since $f(j) = P[X = j]$, it follows that $P[X = 2] = \frac{1}{2e}$.
Answer: B

17. With a model population of 10,000, we have $\#B = 10,000 \times .05 = 500$ faulty components and $\#B' = 9,500$ working components. We also have $\#A \cap B = \#B \times P[A|B] = 475$ devices that are faulty and that test as faulty, and we have $\#A' \cap B' = \#B' \times P[A'|B'] = 9,025$ components that are working and do not test faulty. Therefore, there are $\#A \cap B' = \#B' - \#A' \cap B' = 9,500 - 9,025 = 475$ components that are working and test faulty. The total number of components that test faulty is $\#A = \#A \cap B + \#A \cap B' = 950$. The probability that a component is faulty given that it test faulty is the proportion $\frac{\#A \cap B}{A} = \frac{1}{2}$.

The conditional probability approach to solving the problem is as follows. We can calculate entries in the following table in the order indicated.

	A	A'
B $P[B] = .05$ (given)	$P[A|B] = .95$ (given) **1.** $P[A \cap B] = P[A|B] \cdot P[B] = .0475$	
B' $P[B']$ $= 1 - P[B]$ $= .95$	**3.** $P[A \cap B']$ $= P[B'] - P[A' \cap B']$ $= .0475$ **4.** $P[A] = P[A \cap B] + P[A \cap B'] = .095$	**2.** $P[A' \cap B']$ $= P[A'|B'] \cdot P[B']$ $= .9025$

5. $P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{.0475}{.095} = .5$. Answer: E

18. When the time between successive arrivals has an exponential distribution with mean $\frac{1}{\alpha}$ (units of time), then the number of arrivals per unit time has a Poisson distribution with parameter (mean) α. The time between successive arrivals has an exponential distribution with mean $\frac{1}{5}$ hours (12 minutes). Thus, the number of arrivals per hour has a Poisson distribution with parameter 5, so that $P[X = 10] = \frac{e^{-5}5^{10}}{10!}$. Answer: E

19. A_1 denotes the severity of accident 1 and A_2 denotes the severity of accident 2.
MI denotes the event that an accident is minor,
MO denotes the event that an accident is moderate, and
S denotes the event that an accident is severe.
The probability in question is the probability that either both are minor or exactly one is moderate (and because of independence, $P[A_1 \cap A_2] = P[A_1] \cdot P[A_2]$) :

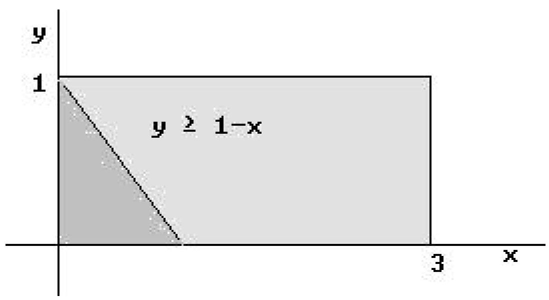
$$P[(A_1 = MI) \cap (A_2 = MI)] + P[(A_1 = MI) \cap (A_2 = MO)]$$
$$+ P[(A_1 = MO) \cap (A_2 = MI) = (.5)(.5) + (.5)(.4) + (.4)(.5) = .65$$ Answer: E

20. $P[X=1|X\leq 1]=\frac{P[X=1]}{P[X=0]+P[X=1]}=\frac{e^{-\lambda}\cdot\lambda^1/1!}{(e^{-\lambda}\cdot\lambda^0/0!)+(e^{-\lambda}\cdot\lambda^1/1!)}=\frac{\lambda}{\lambda+1}=.8$
$\rightarrow \lambda=4$. Answer: A

21. $f_X(x)=\int_0^1(x+y)\,dy=x+\frac{1}{2}$. Answer: E

22. The variance of a uniform random variable on the interval $[a,b]$ is $\frac{(b-a)^2}{12}$.
$Var[X_1+2X_2-X_3]=Var[X_1]+4Var[X_2]+Var[X_3]$
$+2\cdot 2Cov[X_1,X_2]-2Cov[X_1,X_3]-2\cdot 2Cov[X_2,X_3]$
$=\frac{1}{12}+\frac{4}{12}+\frac{1}{12}+\frac{4}{24}-\frac{2}{24}-\frac{4}{24}=\frac{5}{12}$. Answer: C

23. The region of probability is the lightly shaded region below. It is the complement of the darkly shaded region. The probability of the darkly shaded region is $\int_0^1\int_0^{1-y}f(x,y)\,dx\,dy$. Therefore, the probability of the lighter region is $1-\int_0^1\int_0^{1-y}f(x,y)\,dx\,dy$.



Answer: D

24. We are given $P[A]=.25$, $P[B]=.2$, $P[C]=.1$, $P[B\cap C]=0$, and from independence, we have $P[A\cap B]=(.25)(.2)=.05$, $P[A\cap C]=(.25)(.1)=.025$.
Using the probability rule for the union of events, we have
$P[A\cup B\cup C]=P[A]+P[B]+P[C]$
$-P[A\cap B]-P[A\cap C]-P[B\cap C]+P[A\cap B\cap C]$.
Since no people have both genes B and C, it is also true that no one has all three genes.
Thus, $P[A\cup B\cup C]=.25+.2+.1-.05-.025-0+0=.475$. Answer: B

25. In order to be a probability distribution, we must have $\Sigma\Sigma\Sigma p(x,y,z)=1$:
$k[0+1+2+3+4+5+1+2+3+4+5+6+2+3+4+5+6+7]=1$
$\rightarrow k=\frac{1}{63}$. Given that $X=0$, the conditional distribution of Y and Z is
$p(y,z|X=0)=\frac{p(0,y,z)}{p_X(0)}$. $p(0,y,z)=\frac{1}{63}(2y+z)$, and
$p_X(0)=\sum_y\sum_z p(0,y,z)=\frac{1}{63}(0+2+4+1+3+5+2+4+6)=\frac{3}{7}$.

The conditional probabilities for Y, Z are

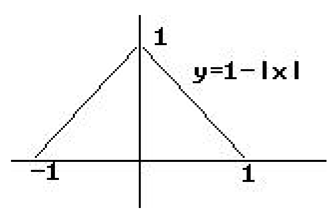
(y,z)	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$
$p(y,z\|X=0)$	0	$\frac{2/63}{3/7}$	$\frac{4/63}{3/7}$	$\frac{1/63}{3/7}$	$\frac{3/63}{3/7}$	$\frac{5/63}{3/7}$	$\frac{2/63}{3/7}$	$\frac{4/63}{3/7}$	$\frac{6/63}{3/7}$
Number of unreimbursed accidents	0	0	0	0	0	1	0	1	2

Expected number of unreimbursed accidents is $1\cdot\frac{5/63}{3/7}+1\cdot\frac{4/63}{3/7}+2\cdot\frac{6/63}{3/7}=\frac{7}{9}$.

Answer: E

26. $E[X]=\int_0^2 x\cdot\frac{x}{2}\,dx=\frac{4}{3}$. Then, $X-E[X]=X-\frac{4}{3}$, which is negative for $0\le x\le\frac{4}{3}$ and is positive for $\frac{4}{3}\le x\le 2$. Thus, $\left|X-E[X]\right|=\frac{4}{3}-X$ if $0\le X\le\frac{4}{3}$ and $\left|X-E[X]\right|=X-\frac{4}{3}$ if $\frac{4}{3}\le x\le 2$. Then,
$E[\,\left|X-E[X]\right|\,]=\int_0^{4/3}(\frac{4}{3}-x)\cdot\frac{x}{2}dx+\int_{4/3}^2(x-\frac{4}{3})\cdot\frac{x}{2}dx=\frac{32}{81}$. Answer: C

27. The marginal distribution of X is



$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,dy$
$\int_0^{1-|x|}1\,dy=1-|x|$ for $-1\le x\le 1$.
$E[X]=\int_{-1}^1 x(1-|x|)\,dx$
$=\int_{-1}^0 x(1+x)\,dx+\int_0^1 x(1-x)\,dx=0$,
$E[X^2]=\int_{-1}^1 x^2(1-|x|)\,dx=\int_{-1}^0 x^2(1+x)\,dx+\int_0^1 x^2(1-x)\,dx$
$=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$. $Var[X]=E[X^2]-(E[X])^2=\frac{1}{6}$. Answer: B

28. The probability of tossing a total of 2 is $p=\frac{1}{36}$ (both dice have to turn up "1"). The number of "2"s occurring in $n=2160$ tosses of the dice has a binomial distribution with mean $np=60$ and variance $np(1-p)=58.33$. Applying the normal approximation with continuity correction, we have
$P[X<55]=P[\frac{X-60}{\sqrt{58.33}}<\frac{54.5-60}{\sqrt{58.33}}]=\Phi(-.72)=1-\Phi(.72)=1-.7642=.2358$.

Answer: A

29. $T =$ time of husband's death , $U =$ time of wife's death , $W = max(T, U)$.
$Cov(T, W) = E[TW] - E[T]E[W]$. $E[T] = 20$, expectation of uniform $[0, 40]$.
$F_W(t) = P[W \leq t] = P[(T \leq t) \cap (U \leq t)] = P[T \leq t] \cdot P[U \leq t]$
$= (\frac{t}{40})(\frac{t}{40}) = \frac{t^2}{1600}$ for $0 \leq t \leq 40$.
The pdf of W is $f_W(t) = F'_W(t) = \frac{t}{800}$, and $E[W] = \int_0^{40} t \cdot \frac{t}{800}\, dt = \frac{80}{3}$.
Alternatively, $E[W] = E[max(T, U)] = \int_0^{40}\int_0^{40} max(t, u)\, (\frac{1}{40})(\frac{1}{40})\, du\, dt$
$= \int_0^{40}\int_0^{t} t \cdot \frac{1}{1600}\, du\, dt + \int_0^{40}\int_t^{40} u \cdot \frac{1}{1600}\, du\, dt = \frac{40}{3} + \frac{40}{3} = \frac{80}{3}$.
Let $h(t, u) = t \cdot max(t, u)$. Then,
$E[TW] = E[h(T, U)] = \int_0^{40}\int_0^{40} t \cdot max(t, u)\, (\frac{1}{40})(\frac{1}{40})\, du\, dt$
$= \int_0^{40}\int_0^{t} t^2 \cdot \frac{1}{1600}\, du\, dt + \int_0^{40}\int_t^{40} tu \cdot \frac{1}{1600}\, du\, dt = 400 + 200 = 600$.
$Cov(T, W) = 600 - (20)(\frac{80}{3}) = \frac{200}{3}$. Answer: C

30. The density function for the time of breakdown t for a particular transistor is ce^{-ct} for $t > 0$. Thus, the cumulative distribution function for the break down time of transistor k is $P[T_k \leq t] = \int_0^t ce^{-cs}\, ds = 1 - e^{-ct}$ for $t > 0$. Let W denote the break down time of the last (10th) transistor. Then the event that the 10th transistor breaks down by time t is equivalent to the event that all transistors break down by time t (if the last one breaks down by time t then all the others have already broken down by that time). Thus,

$$G(t) = P[W \leq t] = P[(T_1 \leq t) \cap (T_2 \leq t) \cap \cdots \cap (T_{10} \leq t)]$$
$$= P[T_1 \leq t] \cdot P[T_2 \leq t] \cdots P[T_{10} \leq t] = (1 - e^{-ct})^{10}.$$

The second last equality is a consequence of the assumption of independence of the T_k's. Thus, the density function of W is $g(t) = G'(t) = 10(1 - e^{-ct})^9 ce^{-ct}$. Answer: A

PRACTICE EXAM 3

1. Let X_1 and X_2 form a random sample from a Poisson distribution. The Poisson distribution has a mean of 1. If $Y = min[X_1, X_2]$, then $P[Y{=}1] =$

A) $\frac{2e-1}{e^2}$ B) $\frac{2e-3}{e^2}$ C) $\frac{e-1}{e}$ D) $\frac{3-e}{e}$ E) $\frac{1}{e}$

2. X and Y are random losses with the following joint density function:
$f(x,y) = \frac{3}{4}x$ for $0 < x < y < 2$, and 0 elsewhere. Find the probability that the total loss $X+Y$ is no greater than 2.

A) $\frac{1}{12}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{1}{2}$

3. An insurer classifies flood hazard based on geographical areas, with hazard categorized as low, medium and high. The probability of a flood occurring in a year in each of the three areas is

Area Hazard	low	medium	high
Prob. of Flood	.001	.02	.25

The insurer's portfolio of policies consists of a large number of policies with 80% low hazard policies, 18% medium hazard policies and 2% high hazard policies. Suppose that a policy had a flood claim during a year. Find the probability that it is a high hazard policy.

A) .50 B) .53 C) .56 D) .59 E) .62

4. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 6x \text{ for } 0<x<y<1 \\ 0, \text{otherwise} \end{cases} .$$

Note that $E[X] = \frac{1}{2}$ and $E[Y] = \frac{3}{4}$. What is $Cov[X,Y]$?

A) $\frac{1}{40}$ B) $\frac{1}{20}$ C) $\frac{1}{10}$ D) $\frac{1}{5}$ E) 1

5. A marketing survey indicates that 60% of the population owns an automobile, 30% owns a house, and 20% owns both an automobile and a house. Calculate the probability that a person chosen at random owns an automobile or a house, but not both.

A) 0.4 B) 0.5 C) 0.6 D) 0.7 E) 0.9

6. The number of injury claims per month is modeled by a random variable N with

$$P[N=n] \;=\; \frac{1}{(n+1)(n+2)}, \quad \text{where } n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

A) $\frac{1}{3}$ B) $\frac{2}{5}$ C) $\frac{1}{2}$ D) $\frac{3}{5}$ E) $\frac{5}{6}$

7. A survey of a large number randomly selected males over the age of 50 shows the following results:

- the proportion found to have diabetes is .02
- the proportion found to have heart disease is .03
- the proportion having neither heart disease nor diabetes is .96 .

Find the proportion that have both diabetes and heart disease.

A) 0 B) .001 C) .006 D) .01 E) .05

8. Customers at Fred's Cafe win a 100 dollar prize if their cash register receipts show a star on each of the five consecutive days Monday,..., Friday in any one week. The cash register is programmed to print stars on a randomly selected 10% of the receipts. If Mark eats at Fred's once each day for four consecutive weeks and the appearance of stars is an independent process, what is the standard deviation of X, where X is the number of dollars won by Mark in the four-week period?

A) .61 B) .62 C) .63 D) .64 E) .65

9. Let (X,Y) have joint density function $f(x,y) = \begin{cases} 2 \text{ for } 0<x<y<1 \\ 0, \text{otherwise} \end{cases}$.

For $0<x<1$, what is $Var[Y|X=x]$?

A) $\frac{1}{18}$ B) $\frac{(1-x)^2}{12}$ C) $\frac{1+x}{2}$ D) $\frac{1}{3}$

E) Cannot be determined from the given information

10. A health insurer finds that health claims for an individual in a one year period are random and depend upon whether or not the individual is a smoker. For a smoker, the expected health claim in a year is $500 with a standard deviation of $200, and for a non-smoker, the expected health claim is $200 with a standard deviation of $100. The insurer estimates that 30% of the population are smokers. The insurer accepts a group health insurance policy with a large number of members in the group. Find the standard deviation for the aggregate claims for a randomly selected member of the group.

A) 184.1 B) 186.8 C) 189.5 D) 192.1 E) 194.7

11. At a certain large university the weights of male students and female students are approximately normally distributed with means and standard deviations of (180,20) and (130,15), respectively. If a male and female are selected at random, what is the probability that the sum of their weights is less than 280?

A) 0.1587 B) 0.1151 C) 0.0548 D) 0.0359 E) 0.0228

12. A loss distribution is uniformly distributed on the interval from 0 to 100 .
Two insurance policies are being considered to cover part of the loss.
Insurance policy 1 insures 80% of the loss.
Insurance policy 2 covers the loss up to a maximum insurance payment of $L < 100$.
Both policies have the same expected payment by the insurer. Find the ratio
$\frac{Var[\text{insurer payment under policy 2}]}{Var[\text{insurer payment under policy 1}]}$ (nearest .1).

A) 1.5 B) 1.2 C) .9 D) .6 E) .3

13. The random variable X has an exponential distribution with mean $\frac{1}{b}$. It is found that $M_X(-b^2) = 0.2$. Find b.

A) 1 B) 2 C) 3 D) 4 E) 5

14. An inspector has been informed that a certain gambling casino uses a "fixed" deck of cards one-quarter of the time in its blackjack games. With a fair deck, the probability of the casino winning a particular hand of blackjack is .52, but with a fixed deck the probability of the casino winning a particular hand is .75 . The inspector visits the casino and plays 3 games of blackjack (from the same deck of cards), losing all of them. Find the conditional probability that the deck was fixed given that the inspector lost all 3 games.

A) 0 B) .25 C) .50 D) .75 E) 1

15. Let X and Y be continuous random variables with joint density function $f(x,y) = \begin{cases} xy \text{ for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$. What is $P[\frac{X}{2} \leq Y \leq X]$?

A) $\frac{3}{32}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{3}{8}$ E) $\frac{3}{4}$

16. The life (in days) of a certain machine has an exponential distribution with a mean of 1 day. The machine comes supplied with one spare. Find the density function (t measure in days) of the combined life of the machine and its spare if the life of the spare has the same distribution as the first machine, but is independent of the first machine.

A) te^{-t} B) $2e^{-t}$ C) e^{-t} D) $(t-1)e^{-t}$ E) $2te^{-t}$

17. An insurer finds that the time until occurrence of a claim from its property insurance division is exponentially distributed with a mean of 1 unit of time, and the time until occurrence of a claim from its life insurance division is exponentially distributed with a mean of 2 units of time. Claims occur independently in the two divisions. Find the expected time until the first claim occurrence, property or life.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{5}{6}$

18. A factory makes three different kinds of bolts: Bolt A, Bolt B and Bolt C. The factory produces millions of each bolt every year, but makes twice as many of Bolt B as it does Bolt A. The number of Bolt C made is twice the total of Bolts A and B combined. Four bolts made by the factory are randomly chosen from all the bolts produced by the factory in a given year. Which of the following is most nearly equal to the probability that the sample will contain two of Bolt B and two of Bolt C?

A) $\frac{8}{243}$ B) $\frac{96}{625}$ C) $\frac{384}{2410}$ D) $\frac{32}{243}$ E) $\frac{1}{6}$

19. Events X, Y and Z satisfy the following relationships:
$X \cap Y' = \phi$, $Y \cap Z' = \phi$, $P(X' \cap Y) = a$, $P(Y' \cap Z) = b$, $P(Z) = c$.
Find $P(X)$ in terms of a, b and c.
A) $a + b + c$ B) $a + b - c$ C) $c + b - a$ D) $c + a - b$ E) $c - b - a$

20. Let Z_1, Z_2, Z_3 be independent random variables each with mean 0 and variance 1, and let $X = 2Z_1 - Z_3$ and $Y = 2Z_2 + Z_3$. What is ρ_{XY} ?
A) -1 B) $-\frac{1}{3}$ C) $-\frac{1}{5}$ D) 0 E) $\frac{3}{5}$

21. Suppose that X has a binomial distribution based on 100 trials with a probability of success of .2 on any given trial. Find the approximate probability $P[15 \leq X \leq 25]$ using the integer correction.
A) .17 B) .34 C) .50 D) .67 E) .83

22. The model for the amount of damage to a particular property during a one-month period is as follows: there is a .99 probability of no damage, there is a .01 probability that damage will occur, and if damage does occur, it is uniformly distributed between 1000 and 2000. An insurance policy pays the amount of damage up to a policy limit of 1500. It is later found that the original model for damage when damage does occur was incorrect, and should have been uniformly distributed between 1000 and 5000. Find the amount by which the insurer's expected payment was understated when comparing the original model with the corrected model.
A) $\frac{11}{16}$ B) $\frac{13}{16}$ C) $\frac{15}{16}$ D) $\frac{17}{16}$ E) $\frac{19}{16}$

23. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the fifth toss?
A) $\frac{8}{81}$ B) $\frac{40}{243}$ C) $\frac{16}{81}$ D) $\frac{80}{243}$ E) $\frac{3}{5}$

24. The repair costs for boats in a marina have the following characteristics:

Boat Type	Number of boats	Probability that repair is needed	Mean of repair cost given a repair	Variance of repair cost given a repair
Power Boats	100	0.3	300	10,000
Sailboats	300	0.1	1000	400,000
Luxury Yachts	50	0.6	5000	2,000,000

At most one repair is required per boat each year. The marina budgets an amount, Y, equal to the aggregate mean repair costs plus the standard deviation of the aggregate repair costs.

Calculate Y.

A) 200,000 B) 210,000 C) 220,000 D) 230,000 E) 240,000

25. A writes to B and does not receive an answer. Assuming that one letter in n is lost in the mail, find the chance that B received the letter. It is to be assumed that B would have answered the letter if he had received it.

A) $\frac{n}{n-1}$ B) $\frac{n-1}{n^2} + \frac{1}{n}$ C) $\frac{n-1}{2n-1}$ D) $\frac{n}{2n-1}$ E) $\frac{n-1}{n^2}$

26. A study is done of people who have been charged by police on a drug-related crime in a large urban area. A conviction must take place in order for there to be a sentence of jail time.

The following information is determined:

(a) 75% are convicted.

(b) 10% of those convicted actually did not commit the crime.

(c) 25% of those not convicted actually did commit the crime.

(d) 2% of those who actually did not commit the crime are jailed.

(e) 20% of those who actually did commit the crime are not jailed.

Find the probability that someone charged with a drug-related crime who is convicted but not sentenced to jail time actually did not commit the crime.

A) .35 B) .40 C) .45 D) .50 E) .55

27. An insurance policy covers losses incurred by Jim and Bob who work at ABC Company. Jim and Bob each have a probability of 40% of incurring a loss during a year, and their losses are independent of one another. Jim is allowed at most one loss per year, and so is Bob. The policy reimburses the full amount of the total losses of Jim and Bob combined up to an annual maximum of 8000. If Jim has a loss, the amount is uniformly distributed on [1000, 5000], and the same is true for Bob. Given that Jim has incurred a loss in excess of 2000, determine the probability that total losses will exceed reimbursements made by the policy.

A) $\frac{1}{20}$ B) $\frac{1}{15}$ C) $\frac{1}{10}$ D) $\frac{1}{8}$ E) $\frac{1}{6}$

28. Let Y be a continuous random variable with cumulative distribution function $F(y) = \begin{cases} 0 \text{ for } y \leq a \\ 1-e^{-\frac{1}{2}(y-a)^2} \text{ otherwise} \end{cases}$, where a is a constant. Find the 75th percentile of Y.

A) $F(.75)$ B) $a - \sqrt{2\,ln\,2}$ C) $a + \sqrt{2\,ln\,2}$ D) $a - 2\sqrt{ln\,2}$ E) $a + 2\sqrt{ln\,2}$

29. Let X be a continuous random variable with density function $f(x) = \begin{cases} 2x^{-2} \text{ for } x \geq 2 \\ 0, \text{otherwise} \end{cases}$.

Determine the density function of $Y = \frac{1}{X-1}$ for $0 < y \leq 1$.

A) $\frac{1}{y^2}$ B) $\frac{2}{(y+1)^2}$ C) $\frac{2}{(y-1)^2}$ D) $2(\frac{y}{y+1})^2$ E) $2(\frac{y+1}{y})^2$

30. X and Y are loss random variables, with X discrete and Y continuous. The joint density function of X and Y is $f(x,y) = \frac{(x+1)e^{-y/2}}{12}$ for $x = 0, 1, 2$ and $0 < y < \infty$.

Find the probability that the total loss, $X + Y$ is less than 2.

A) $\frac{1}{6}(3 - 2e^{-1/2} - e^{-1})$ B) $\frac{1}{6}(3 - e^{-1/2} - 2e^{-1})$ C) $\frac{1}{3}(3 - 2e^{-1/2} - e^{-1})$

D) $\frac{1}{3}(3 - e^{-1/2} - e^{-1})$ E) $\frac{1}{2}(3 - 2e^{-1/2} - e^{-1})$

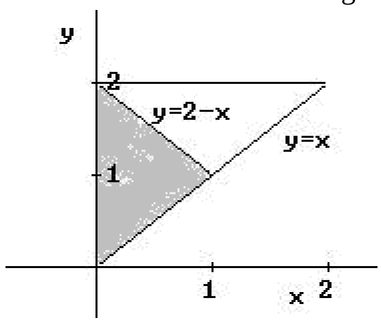
PRACTICE EXAM 3 - SOLUTIONS

1. $P[Y{=}1] = P[(X_1{=}1) \cap (X_2 \geq 1)] + P[(X_2{=}1) \cap (X_1 \geq 2)]$

$$= P[X_1{=}1] \cdot P[X_2 \geq 1] + P[X_2{=}1] \cdot P[X_1 \geq 2]$$
$$= P[X_1{=}1] \cdot \Big(1 - P[X_2{=}0]\Big) + P[X_2{=}1] \cdot \Big(1 - P[X_1 \leq 1]\Big)$$
$$= e^{-1}(1-e^{-1}) + e^{-1}(1-2e^{-1}) = 2e^{-1} - 3e^{-2} = \tfrac{2e-3}{e^2}.$$

Answer: B

2. The event $X + Y \leq 2$ is equivalent to the event $Y \leq 2 - X$. The region of probability is shaded in the graph at the right. The probability is found by integrating the joint density function over the two-dimensional region. $\int_0^1 \int_x^{2-x} \frac{3}{4} x \, dy \, dx = \int_0^1 \frac{3}{4} x(2-2x) \, dx = \frac{1}{4}$.



Answer: C

3. This is a classical Bayesian probability situation. Let C denote the event that a flood claim occurred. We wish to find $P(H|C)$. With a model populatiojn of 10,000 we have $\#L = 8,000$, $\#M = 1800$ and $\#H = 200$. Also, $\#C \cap L = \#L \times P(C \cap L) = 8$, and similarly, $\#C \cap M = 36$ and $\#C \cap H = 50$. The probability that a policy is high hazard given that there was a claim is $\frac{\#C \cap H}{C} = \frac{50}{8+36+50} = \frac{50}{94} = .532$.

The conditional probability approach to solving the problem is as follows.
We can summarize the information in the following table, with the order of calculations indicated.

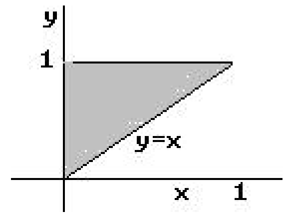
	L , $P(L) = .8$ (given)	M , $P(M) = .18$ (given)	H , $P(H) = .02$ (given)
C	$P(C \mid L) = .001$ (given)	$P(C \mid M) = .02$ (given)	$P(C \mid H) = .25$ (given)
	1. $P(C \cap L)$ $= P(C \mid L) \cdot P(L)$ $= .0008$	**2.** $P(C \cap M)$ $= P(C \mid M) \cdot P(M)$ $= .0036$	**3.** $P(C \cap H)$ $= P(C \mid H) \cdot P(H)$ $= .005$

4. $P(C) = P(C \cap L) + P(C \cap M) + P(C \cap H) = .0094$.

5. $P(H|C) = \frac{P(H \cap C)}{P(C)} = \frac{.005}{.0094} = .532$.

Answer: B

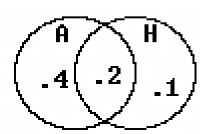
4. $Cov[X,Y]=E[XY]-E[X]\cdot E[Y]$
The region of probability is the triangle above the line $y=x$ in the unit square $0\le x\le 1$, $0\le y\le 1$.
$E[XY]=\int_0^1\int_0^y xy\cdot 6x\,dx\,dy=\frac{2}{5}$
$\rightarrow Cov[X,Y]=\frac{2}{5}-\frac{1}{2}\cdot\frac{3}{4}=\frac{1}{40}$.
Alternatively,
$E[XY]=\int_0^1\int_x^1 xy\cdot 6x\,dy\,dx=\frac{2}{5}$.



Answer: A

5. We identify the events A and H: $A\ =$ a randomly chosen person owns an automobile, $H\ =$ a randomly chosen person owns a house.
We are given $P[A]=.60$, $P[H]=.30$, $P[A\cap H]=.20$.
We wish to find $P[A\cap H']+P[A'\cap H]$
(the event $A\cap H'$ is the event that a randomly chosen person owns an automobile but does not own a house, and the reverse for the event $A'\cap H$).
In a population of 10, 6 would own an automobile, 3 would own a house, and 2 would own at least one is $6+3-2=7$ and the number owning exactly one is $7-2=5$, so the probability of owning exactly one is $\frac{5}{10}=.5$.

The event probabliity approach to solving the problem is as follows.
The diagram at the right indicates the breakdown of the components of the two events. We can see that $P[A'\cap H]=.1$ and $P[A\cap H']=.4$, so the desired probability is .5.



Also, from rules of probability we have $P[A\cap H']=P[A]-P[A\cap H]=.60-.20=.40$, and similarly, $P[A'\cap H]=P[H]-P[A\cap H]=.30-.20=.10$. Therefore, the probability in question is $.40+.10=.50$. An alternative way to consider the problem is to start with the event $A\cup H$, which is the event that a randomly chosen person either owns an automobile, or owns a house (or owns both), and then note that this event is the disjoint union of the event in question and the event $A\cap H$, so that
"either A or H or both" $=$ "exactly one of A or H" $\cup$ "both A and H".

5. continued

It follows that the probability in question is

$P[A$ or H, but not both$] = P[A \cup H] - P[A \cap H]$

$= \Big(P[A] + P[H] - P[A \cap H]\Big) - P[A \cap H] = (.60 + .30 - .20) - .20 = .50$.

The following table also summarizes the calculations in a simple way.

	$P(A) = .6$, given	$\Rightarrow$	$P(A') = 1 - .6 = .4$
$P(H) = .3$ given	$P(A \cap H) = .2$ given	$\Rightarrow$	$P(A' \cap H)$ $= P(H) - P(A \cap H)$ $= .3 - .2 = .1$
$\Downarrow$	$\Downarrow$		
$P(H')$ $= 1 - .3 = .7$	$P(A \cap H')$ $= P(A) - P(A \cap H)$ $= .6 - .2 = .4$		

Then $P(A \cap H') + P(A' \cap H) = .4 + .1 = .5$. Answer: B

6. We are asked to find $P[N \geq 1 | N \leq 4] = \frac{P[1 \leq N \leq 4]}{P[N \leq 4]}$.

$P[1 \leq N \leq 4] = P[N = 1] + P[N = 2] + P[N = 3] + P[N = 4]$

$= \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \frac{1}{(4)(5)} + \frac{1}{(5)(6)} = \frac{1}{3}$.

$P[N \leq 4] = P[N = 0] + P[1 \leq N \leq 4] = \frac{1}{(1)(2)} + \frac{1}{3} = \frac{5}{6}$.

$P[N \geq 1 | N \leq 4] = \frac{1/3}{5/6} = .4$. Answer: B

7. We identify events as follows:

D : randomly chosen individual has diabetes

H : randomly chosen individual has heart disease

We are given $P[D] = .02$, $P[H] = .03$, $P[D' \cap H'] = .96$.

Using rules of probability, we have

$.98 = P[D'] = P[D' \cap H] + P[D' \cap H'] \rightarrow P[D' \cap H] = .98 - .96 = .02$, and

$.03 = P[H] = P[H \cap D] + P[H \cap D'] \rightarrow P[H \cap D] = .03 - .02 = .01$.

These calculations are summarized in the following table.

	$P(D) = .02$, given	$\Rightarrow$	$P(D') = 1 - .02 = .98$
	$\Downarrow$		
$P(H) = .03$ given	$P(H \cap D)$ $= P(D) - P(H' \cap D) = .02 - .01 = .01$		
$\Downarrow$	$\Uparrow$		
$P(H') = .97$ $= 1 - .03$	$P(D \cap H')$ $= P(H') - P(D' \cap H') = .97 - .96 = .01$	$\Leftarrow$	$P(D' \cap H') = .96$, given

Answer: D

8. $P[\text{win in a given week}] = (.1)^5 = p$. Then the number of wins in four weeks, N, has a binomial distribution $B(4,p)$, and $X = 100N$ dollars is the amount won in 4 weeks. Then, $Var[X] = 100^2 Var[N] = 100^2(4)(p)(1-p) \Rightarrow$ the standard deviation of X is $200\sqrt{p(1-p)} = .63$. Answer: C

9. $Var[Y|X=x] = E[Y^2|X=x] - (E[Y|X=x])^2$.
$f_{Y|X}(y|X=x) = \frac{f(x,y)}{f_X(x)}$, where $f_X(x) = \int_x^1 2\,dy = 2(1-x)$.
Thus, $f_{Y|X}(y|X=x) = \frac{1}{1-x}$ so that $E[Y|X=x] = \int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1+x}{2}$
and $E[Y^2|X=x] = \int_x^1 y^2 \cdot \frac{1}{1-x}\,dy = \frac{1+x+x^2}{3}$, and then
$Var[Y|X] = \frac{1+x+x^2}{3} - [\frac{1+x}{2}]^2 = \frac{(1-x)^2}{12}$.
Alternatively, note that given any joint uniform distribution, any related conditional distribution is also uniform. Given $X = x$, Y has a uniform distribution on $(x,1)$ and thus has a variance of $\frac{(1-x)^2}{12}$. Answer: B

10. This is an example of a mixture of distributions. X_1 is the annual claim amount for a smoker and X_2 is the annual claim amount for a non-smoker. We are given
$E[X_1] = 500$, $\sqrt{Var[X_1]} = 200$, $E[X_2] = 200$, $\sqrt{Var[X_1]} = 100$.
We are also given the mixing weights $\alpha_1 = .3$ (proportion of the population that are smokers), and $\alpha_2 = .7$. The distribution of the annual claim amount for a randomly chosen individual from the group is X, which is a mixture of X_1 and X_2 .
$E[X] = (.3)E[X_1] + (.7)E[X_2] = 290$.
$E[X^2] = (.3)E[X_1^2] + (.7)E[X_2^2]$.
We know that $Var[X_1] = 40{,}000 = E[X_1^2] - (E[X_1])^2 = E[X_1^2] - (500)^2$
$\rightarrow E[X_1^2] = 290{,}000$, and
$Var[X_2] = 10{,}000 = E[X_2^2] - (E[X_2])^2 = E[X_2^2] - (200)^2 \rightarrow E[X_2^2] = 50{,}000$.
Then, $E[X^2] = (.3)E[X_1^2] + (.7)E[X_2^2] = 122{,}000$.
Finally, $Var[X] = E[X^2] - (E[X])^2 = 122{,}000 - (290)^2 = 37{,}900$
and the standard deviation is $\sqrt{37{,}500} = 194.7$. Answer: E

11. Let X and Y denote the random variables of the weights of the male and female students respectively. Since the students are chosen at random, X and Y are independent. But then $W = X + Y$ is normal with mean $\mu_W = \mu_X + \mu_Y = 310$ and variance $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 = 625$. Thus, $P[W < 280] = P\left[\frac{W-310}{\sqrt{625}} < \frac{280-310}{\sqrt{625}}\right] = P[Z < -1.2]$, (where Z has a standard normal distribution) $= .1151$. Answer: B

12. Expected payment under policy 2 is

$\int_0^L y(.01)dy + L \cdot P[X > L] = .005L^2 + (L)(\frac{100-L}{100}) = L - .005L^2$.

This is equal to the expected payment under policy 1, which is

$(.8)E[X] = (.8)(50) = 40$. Solving $L - .005L^2 = 40$ results in

$L = 55.28$, 144.72. We discard 144.72 as a limit since it is larger than the maximum loss amount. Thus, $L = 55.28$.

The variance of insurer payment under policy 1 is

$Var[.8X] = .64Var[X] = (.64)(\frac{100^2}{12}) = 533.33$.

Under policy 2,

$E[(\text{insurer payment})^2] = \int_0^{55.28} y^2(.01)dy + (55.28)^2 \cdot P[X > 55.28]$

$= 563.10 + (55.28)^2[\frac{100-55.28}{100}] = 1929.69$, and

$Var[\text{insurer payment}] = 1929.69 - (40)^2 = 329.69$.

$\frac{Var[\text{insurer payment under policy 2}]}{Var[\text{insurer payment under policy 1}]} = \frac{329.69}{533.33} = .618$. Answer: D

13. $M_X(t) = \frac{b}{b-t} \Rightarrow M_X(-b^2) = \frac{b}{b-(-b^2)} = \frac{b}{b+b^2} = \frac{1}{1+b} = .2 \Rightarrow b = 4$.

Answer: D

14. Let $A =$ 'deck is fixed', $X =$ number of games lost out of 3 games.

We wish to find $P[A|X = 3]$.

X has a binomial distribution with $n = 3$ and p depends on whether or not the deck is fixed.

We use the usual Bayesian approach.

$P[A|X = 3] = \frac{P[X=3|A] \cdot P[A]}{P[X=3]} = \frac{P[X=3|A] \cdot P[A]}{P[X=3|A] \cdot P[A] + P[X=3|A'] \cdot P[A']}$.

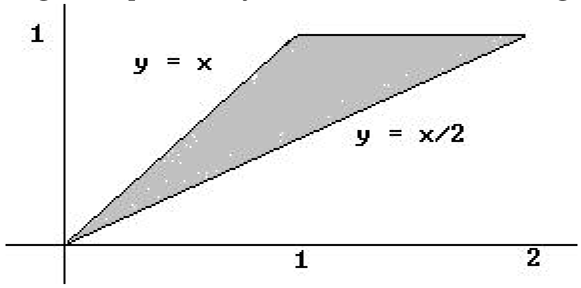
We are given that $P[A] = .25$ (the casino uses a fixed deck one-quarter the time).

Also, if the deck is fixed then $p = .75$, and $P[X = 3|A] = (.75)^3 = .421875$.

If the deck is fair, then $p = .52$, and $P[X = 3|A'] = (.52)^3 = .140608$.

Then $P[A|X = 3] = \frac{(.421875)(.25)}{(.421875)(.25) + (.140608)(.75)} = .50$. Answer: C

15. The region of probability is shown in the shaded figure below



The probability is $\int_0^1\int_{x/2}^{x} xy\,dy\,dx + \int_1^2\int_{x/2}^{1} xy\,dy\,dx = \frac{3}{32} + \frac{9}{32} = \frac{3}{8}$.
Alternatively, the probability is $\int_0^1\int_y^{2y} xy\,dx\,dy = \;= \frac{3}{8}$. Answer: D

16. $T = T_1 + T_2$, where T_i is the random lifetime of machine i (n days). Since T_1 and T_2 are independent, the joint density of $T_1$and T_2 is $f(t_1, t_2) = e^{-t_1}e^{-t_2}$.
Applying the convolution method for the sum of random variables results in
$f_T(t) = \int_0^t f(s, t-s)\,ds = \int_0^t e^{-s}e^{-(t-s)}ds = te^{-t}$. Answer: A

17. $X =$ time until property claim, $Y =$ time until life claim.
$f(x) = e^{-x}$, $g(y) = \frac{1}{2}e^{-y/2}$.
$T =$ time until next claim $= min(X, Y)$.
$P[T > t] = P[X > t] \cdot P[Y > t] = e^{-t} \cdot e^{-t/2} = e^{-3t/2}$.
Pdf of T is $h(t) = \frac{d}{dt}P[T \le t] = -\frac{d}{dt}P[T > t] = -\frac{d}{dt}(e^{-3t/2}) = \frac{3}{2}e^{-3t/2}$.
This is the pdf of an exponential random variable with mean $\frac{2}{3}$. Answer: D

18. Because of the proportions in which the bolts are produced, a randomly selected bolt will have a $\frac{1}{9}$ chance of being of type A, a $\frac{2}{9}$ chance of being of type B, and a $\frac{2}{3}$ chance of being of type C. A random selection of size n from the production of bolts will have a multinomial distribution with parameters n , $p_A = \frac{1}{9}$, $p_B = \frac{2}{9}$ and $p_C = \frac{2}{3}$, with probability function $P[N_A = n_A, N_B = n_b, N_C = n_c] = \frac{n!}{n_A!\,n_B!\,n_C!}(\frac{1}{9})^{n_A}(\frac{2}{9})^{n_B}(\frac{2}{3})^{n_C}$
With $n = 4$, $P[N_A = 0, N_B = 2, N_C = 2] = \frac{4!}{0!\,2!\,2!}(\frac{1}{9})^0(\frac{2}{9})^2(\frac{2}{3})^2 = \frac{32}{243}$ Answer: D

19. $X = (X \cap Y) \cup (X \cap Y') \rightarrow X = X \cap Y \rightarrow$
$P(Y) = P(Y \cap X) + P(Y \cap X') = P(X) + a.$
$Y = (Y \cap Z) \cup (Y \cap Z') \rightarrow Y = Y \cap Z \rightarrow c = P(Z) = P(Z \cap Y) + P(Z \cap Y') = P(Y) + b.$
Then, $P(X) + a + b = c \rightarrow P(X) = c - b - a$. {}It is also true that $X \subset Y \subset Z$, so that
$c = P(Z) = P(X) + P(Y - X) + P(Z - Y)$
$= P(X) + P(Y \cap X') + P(Z \cap Y') = P(X) + a + b$. Answer: E

20. $\rho_{XY} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = \frac{Cov[2Z_1 - Z_3, 2Z_2 + Z_3]}{\sqrt{Var[2Z_1 - Z_3] \cdot Var[2Z_2 + Z_3]}}$.
$Cov[2Z_1 - Z_3, 2Z_2 + Z_3]$
$= 4Cov[Z_1, Z_2] + 2Cov[Z_1, Z_3] - 2Cov[Z_3, Z_2] - Cov[Z_3, Z_3]$
$= 4(0) + 2(0) - 2(0) - Var[Z_3] = -1$ (since $Cov[Z_3, Z_3] = Var[Z_3]$ and independent random variables have covariance of 0).
$Var[2Z_1 - Z_3] = 4Var[Z_1] + Var[Z_3] - 2(2Cov[Z_1, Z_3]) = 5$,
$Var[2Z_2 + Z_3] = 4Var[Z_2] + Var[Z_3] + 2(2Cov[Z_2, Z_3]) = 5$.
Thus, the correlation is $\rho_{XY} = \frac{-1}{\sqrt{(5) \cdot (5)}} = -\frac{1}{5}$. Answer: C

21. The mean and variance of X are $E[X] = 100(.2) = 20$, $Var[X] = 100(.2)(.8) = 16$.
Using the normal approximation with integer correction, we assume that X is approximately normal and find
$P[14.5 \leq X \leq 25.5] = P[\frac{14.5-20}{\sqrt{16}} \leq \frac{X-20}{\sqrt{16}} \leq \frac{25.5-20}{\sqrt{16}}] = P[-1.375 \leq Z \leq 1.375]$,
where Z has a standard normal distribution.
$P[-1.375 \leq Z \leq 1.375] = \Phi(1.375) - \Phi(-1.375) = \Phi(1.375) - [1 - \Phi(1.375)]$
$= 2\Phi(1.375) - 1$.
From the standard normal table we have $\Phi(1.3) = .9032$ and $\Phi(1.4) = .9192$. Using linear interpolation (since 1.375 is $\frac{3}{4}$ of the way from 1.3 to 1.4) we have
$\Phi(1.375) = (.25)\Phi(1.3) + (.75)\Phi(1.4) = .9152$, and then the probability in question is
$2(.9152) - 1 = .8304$. Answer: E

22. When damage occurs, the pdf of the amount of damage is .001 for the uniform distribution on the interval from 1000 to 2000 .

Expected insurer payment $= (.01) \cdot$ Expected payment given that damage occurs

$= (.01)[\int_{1000}^{1500} x(.001)\,dx + 1500 \cdot P(\text{damage exceeds 1500 when damage occurs})]$

$= (.01)[625 + (1500)(\frac{2000-1500}{2000-1000})] = 13.75$. For the corrected model,

Expected insurer payment $= (.01) \cdot$ Expected payment given that damage occurs

$= (.01)[\int_{1000}^{1500} x(.00025)\,dx + 1500 \cdot P(\text{damage exceeds 1500 when damage occurs})]$

$= (.01)[156.25 + (1500)(\frac{5000-1500}{5000-1000})] = 14.6875$.

Increase in expected value is .9375 . Answer: C

23. $P[\text{head}] = \frac{1}{3}$, $P[\text{tail}] = \frac{2}{3}$.

$P[\text{3rd head on 5th toss}] = P[\text{(2 heads in first 4 tosses)} \cap \text{(head on 5th toss)}]$

$= P[\text{2 heads in first 4 tosses}] \cdot P[\text{head on 5th toss}]$

$= \binom{4}{2}(\frac{1}{3})^2(\frac{2}{3})^2(\frac{1}{3}) = \frac{8}{81}$.

Note that the number of tails, X, that are tossed until the 3rd head occurs can also be regarded as negative binomial distribution with $p = \frac{1}{3}$ and $r = 3$, and we are finding $P[X = 2]$.

Answer: A

24. For each boat type we find the mean and the variance of the repair cost. The mean of the aggregate repair costs is the sum of the mean repair costs for the 450 boats, and assuming independence of boat repair costs for all 450 boats, the variance of the aggregate cost is the sum of the variances for the 450 boats.

For each type of boat, the repair cost is a mixture of 0 (if no repair is needed) and X_i (repair cost variable for boat i if a repair is needed). The mean repair cost for boat i is

$E[X_i] \times$ prob. repaid is needed for boat i , and the second moment of the repair cost for boat i is $E[X_i^2] \times$ prob. repaid is needed for boat i (note that $E[X_i^2] = Var[X_i] + (E[X_i])^2$).

The variance of the repaid cost for boat i is the second moment minus the square of the first moment.

Power boats: Mean repair cost for one boat $= 300(.3) = 90$,

second moment of repair cost for one boat $= [10,000 + 300^2](.3) = 30,000$.

Variance of repair cost for one power boat $= 30,000 - 90^2 = 21,900$.

24. continued

Sailboats: Mean repair cost for one boat $= 1000(.1) = 100$,

second moment of repair cost for one boat $= [400,000 + 1000^2](.1) = 140,000$.

Variance of repair cost for one power boat $= 140,000 - 100^2 = 130,000$.

Luxury Yachts: Mean repair cost for one boat $= 5000(.6) = 3000$,

second moment of repair cost for one boat $= [2,000,000 + 5000^2](.6) = 16,200,000$.

Variance of repair cost for one power boat $= 16,200,000 - 3000^2 = 7,200,000$.

The mean of the aggregate repair cost is $100(90) + 300(100) + 50(3000) = 189,000$,

and the variance is $100(21,900) + 300(130,000) + 50(7,200,000) = 401,190,000$.

The amount budgeted by the marina is $189,000 + \sqrt{401,190,000} = 209,030$. Answer: B

25. $P[B \text{ received the letter}|A \text{ did not receive an answer after writing to } B]$

$= \frac{P[(B \text{ received the letter}) \cap (A \text{ did not receive an answer after writing to } B)]}{P[A \text{ did not receive an answer after writing to } B]}$.

But, $P[A \text{ does not receive a reply after writing to } B]$

$= P[(A \text{ does not receive a reply after writing to } B) \cap (B \text{ received } A\text{'s letter})]$

$+ P[(A \text{ does not receive a reply after writing to } B) \cap (B \text{ did not receive } A\text{'s letter})]$.

$P[(A \text{ does not receive a reply after writing to } B) \cap (B \text{ received } A\text{'s letter})]$

$= P[A \text{ does not receive a reply after writing to } B|B \text{ received } A\text{'s letter}]$

$\times P[B \text{ received } A\text{'s letter}] = \frac{1}{n} \cdot \frac{n-1}{n}$, and

25. continued

$P[(A \text{ does not receive a reply after writing to } B) \cap (B \text{ did not receive } A\text{'s letter})]$

$= P[A \text{ does not receive a reply after writing to } B|B \text{ did not receive } A\text{'s letter}]$

$\times P[B \text{ did not receive } A\text{'s letter}] = 1 \cdot \frac{1}{n}$.

Therefore,

$P[A \text{ does not receive a reply after writing to } B] = \frac{n-1}{n} \cdot \frac{1}{n} + \frac{1}{n} = \frac{2n-1}{n^2}$ and

$P[B \text{ received the letter}|A \text{ did not receive an answer after writing to } B]$

$= \frac{P[(B \text{ received the letter}) \cap (A \text{ did not receive an answer after writing to } B)]}{P[A \text{ did not receive an answer after writing to } B]}$

$= \frac{(n-1)/n^2}{(2n-1)/n^2} = \frac{n-1}{2n-1}$. Answer: C

26. Our "probability space" consists of all people who have been charged with a drug-related offence. We define the following events:

C - the person is convicted $\quad$ T - the person is sentenced to jail time

D - the person did actually commit the crime.

Since jail time is sentenced only to those who are convicted, we have

$T \subset C$, so that $P(T \cap C) = P(T)$.

We are also given the following information:

$P(C) = .75$, $P(D'|C) = .10$, $P(D|C') = .25$, $P(T|D') = .02$, $P(T'|D) = .20$

We wish to find $P(D'|C \cap T')$. In the model population approach, $P(D'|C \cap T') = \frac{\#D' \cap C \cap T'}{\#C \cap T'}$.

With a model population of 100,000 we have $\#C = 75,000$ and $\#C' = 25,000$.

From this we get $\#D' \cap C = \#C \times P(D'|C) = 75,000 \times .1 = 7,500$ so that

$\#D \cap C = \#C - \#D' \cap C = 67,500$. We also get $\#D \cap C' = 25,000 \times .25 = 6,250$.

Then, $\#D = \#D \cap C + \#D \cap C' = 67,500 + 6,250 = 73,750$ and $\#D' = 26,250$.

We now get $\#T \cap D' = 26,250 \times .02 = 525$ and $\#T' \cap D = 73,750 \times .2 = 14,750$,

and then $\#T \cap D = \#D - \#T' \cap D = 73,750 - 14,750 = 59,000$ and

$\#T = \#T \cap D' + \#T \cap D = 525 + 59,000 = 59,525$ and $\#T' = 40,475$.

Since $T \subset C$ we have $\#T = \#T \cap C$ and $\#D' \cap C \cap T = \#D' \cap T$ so that

$\#C \cap T' = \#C - \#C \cap T = \#C - \#T = 15,475$

and $\#D' \cap C \cap T = \#D' \cap T = 525$.

Then $\#D' \cap C \cap T' = \#D' \cap C - \#D' \cap C \cap T = 7,500 - 525 = 6,975$.

Finally, $P(D'|C \cap T') = \frac{\#D' \cap C \cap T'}{\#C \cap T'} = \frac{6,975}{15,475} = .4507$.

The conditional probability solution is as follows. From the given information, we get

$P(D' \cap C) = P(D'|C) \cdot P(C) = (.1)(.75) = .075$, and

$P(D \cap C') = P(D|C') \cdot P(C') = (.25)(.25) = .0625$.

Since $P(C) = .75)$ and $P(C') + .25$, we get

$P(D \cap C) = P(C) - P(D' \cap C) = .75 - .075 = .675$.

Then $P(D) = P(D \cap C) + P(D \cap C') = .675 + .0625 = .7375$, and $P(D') = .2625$.

Then, $P(T \cap D') = P(T|D') \cdot P(D') = (.02)(.2625) = .00525$ and

$P(T' \cap D) = P(T'|D) \cdot P(D) = (.20)(.7375) = .1475$, so that

$P(T \cap D) = P(D) - P(T' \cap D) = .7375 - .1475 = .59$ and

$P(T) = P(T \cap D) + P(T \cap D') = .59 + .00525 = .59525$, and $P(T') = .40475$.

26. continued

Then $P(C \cap T') = P(C) - P(C \cap T) = P(C) - P(T) = .75 - .59525 = .15475$, and
$P(D' \cap C \cap T') = P(D' \cap C) - P(D' \cap C \cap T) = .075 - .00525 = .06975$
(note that $P(D' \cap C \cap T) = P(D' \cap T)$ because $T \subset C$).
Finally, $P(D'|C \cap T') = \frac{P(D' \cap C \cap T')}{P(C \cap T')} = \frac{.06975}{.15475} = .4507$. Answer: C

27. Suppose that X_1 is the amount of Jim's loss and X_2 is Bob's loss. Since there is a .6 chance of no loss for an individual, the pdf of loss amount X is $f(x) = \begin{cases} .6 & X_2 = 0 \\ .0001 & 1000 \le X_2 \le 5000 \end{cases}$,
and $f(x) = 0$ otherwise. We wish to find $P[X_1 + X_2 > 8000 | X_1 > 2000]$. This is equal to $\frac{P[(X_1+X_2>8000)\cap(X_1>2000)]}{P[X_1>2000]}$. From the pdf for X we have
$P[X_1 > 2000] = (.0001)(5000 - 2000) = .3$.
If $X_1 \le 3000$, then it is impossible for $X_1 + X_2$ to be > 8000. Then,
$P[(X_1 + X_2 > 8000) \cap (X_1 > 2000)] = \int_{3000}^{5000}\int_{8000-x_1}^{5000} (.0001)^2 \, dx_2 \, dx_1$
$\int_{3000}^{5000} (.0001)^2(x_1 - 3000) \, dx_1 = \frac{(.0001)^2(x_1-3000)^2}{2}\Big|_{x_1=3000}^{x_2=5000} = .02$.
The conditional probability in question is then
$\frac{P[(X_1+X_2>8000)\cap(X_1>2000)]}{P[X_1>2000]} = \frac{.02}{.3} = \frac{1}{15}$. Answer: B

28. Let us denote the 75th percentile of Y by c. Thus, $F(c) = .75$, so that
$1 - e^{-\frac{1}{2}(c-a)^2} = .75$. Solving this equation for c results in $e^{-\frac{1}{2}(c-a)^2} = .25$,
or equivalently, $\frac{1}{2}(c-a)^2 = ln\, 4 \rightarrow c = a + \sqrt{2\, ln\, 4} = a + 2\sqrt{ln\, 2}$. Answer: E

29. The transformation $Y = u(X) = \frac{1}{X-1}$ (for $X \ge 2$) is a decreasing function, and therefore is invertible, $X = u^{-1}(Y) = v(Y) = \frac{1}{Y} + 1$. Then using the standard method for finding the density of a transformed random variable, we have the density function of Y is
$g(y) = f(v(y)) \cdot \left|\frac{d}{dy}\, v(y)\right| = 2(\frac{1}{y} + 1)^{-2} \cdot \left| -\frac{1}{y^2}\right| = \frac{2}{(y+1)^2}$.
Alternatively, $F_Y(y) = P[Y \le y] = P[\frac{1}{X-1} \le y] = P[X \ge 1 + \frac{1}{y}]$
$= \int_{(y+1)/y}^{\infty} 2x^{-2} dx = \frac{2y}{y+1} \rightarrow f_Y(y) = F_Y'(y) = \frac{2}{(y+1)^2}$. Answer: B

30. $P[X + Y < 2] = P[(X = 0) \cap (Y < 2)] + P[(X = 1) \cap (Y < 1)]$
$P[(X = 0) \cap (Y < 2)] = \int_0^2 f(0, y) \, dy = \int_0^2 \frac{1}{12} e^{-y/2} dy = \frac{1}{6}[1 - e^{-1}]$,
$P[(X = 1) \cap (Y < 1)] = \int_0^1 f(1, y) \, dy = \int_0^1 \frac{2}{12} e^{-y/2} dy = \frac{1}{3}[1 - e^{-1/2}]$
$\Rightarrow P[X + Y < 2] = \frac{1}{6}[1 - e^{-1}] + \frac{1}{3}[1 - e^{-1/2}] = \frac{1}{6}[3 - 2e^{-1/2} - e^{-1}]$. Answer: A

PRACTICE EXAM 4

1. Event C is a subevent of $A \cup B$. Which of the following must be true.

I. $A \cup C = B \cup C$

II. $P(C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

III. $A' \cap C \subset B$

A) All but I B) All but II C) All but III D) II only E) III only

2. A dental insurance policy covers three procedures: orthodontics, fillings and extractions. During the life of the policy, the probability that the policyholder needs:

· orthodontic work is 1/2

· orthodontic work or a filling is 2/3

· orthodontic work or an extraction is 3/4

· a filling and an extraction is 1/8

The need for orthodontic work is independent of the need for a filling and is also independent of the need for an extraction. Calculate the probability that the policyholder will need a filling or an extraction during the life of the policy.

A) 7/24 B) 3/8 C) 2/3 D) 17/24 E) 5/6

3. A loss random variable has a continuous uniform distribution between 0 and \$100 . An insurer will insure the loss amount above a deductible c. The variance of the amount that the insurer will pay is 69.75 . Find c.

A) 65 B) 70 C) 75 D) 80 E) 85

4. The joint probability of the three discrete random variables X, Y, Z is $f_{X,Y,Z}(x,y,z) = \frac{xy+xz^2}{24}$ for $x = 1,2$, $y = 1,2$, $z = 0,1$.

How many of the following statements are true?

I. X and Y are independent

II. X and Z are independent

III. Y and Z are independent

A) 0 B) 1 C) 2 D) 3

5. Let A and B be events such that $P[A] = .7$ and $P[B] = .9$.
Calculate the largest possible value of $P[A \cup B] - P[A \cap B]$.
A) $.20$ B) $.34$ C) $.40$ D) $.60$ E) 1.60

6. Bob has a fair die and tosses it until a "1" appears. Doug also has a fair die, and he tosses it until a "1" appears. Joe also has a fair die and he tosses it until a "1" appears. They each stop tossing as soon as a "1" turns up on their die. We define the random variable X to be the total number of tosses that Bob, Doug and Joe made (including the first "1" that each of them tossed). Find $Var[X]$.
A) 50 B) 60 D) 70 D) 80 E) 90

7. Let X and Y be independent continuous random variables with common density function

$$f(t) = \begin{cases} 1 \text{ for } 0<t<1 \\ 0, \text{otherwise} \end{cases}.$$

What is $P[X^2 \geq Y^3]$?
A) $\frac{1}{3}$ B) $\frac{2}{5}$ C) $\frac{3}{5}$ D) $\frac{2}{3}$ E) 1

8. An auto insurer's portfolio of policies is broken into two classes - low risk, which make up 75% of the policies, and high risk, which make up 25% of the policies. The number of claims per year that occur from a policy in the low risk group has a Poisson distribution with a mean of .2, and the number of claims per year that occur from a policy in the high risk group has a Poisson distribution with a mean of 1.5 . A policy is chosen at random from the insurer's portfolio. Find the probability that there will be exactly one claim during the year on that policy.
A) .21 B) .25 C) .29 D) .33 E) .37

9. Every member of an insured group has an annual claim amount distribution that is exponentially distributed. The expected claim amount of a randomly chosen member of the group is $\frac{1}{c}$, where c is uniformly distributed between 1 and 2. Find the probability that a randomly chosen member of the group has annual claim less than 1.
A) Less than .4 B) At least .4 but less than .5 C) At least .5 but less than .6
D) At least .6 but less than .7 E) At least .7

10. X has pdf $f(x) = e^{-x}$ for $x > 0$. If $a > 0$ and A is the event that $X > a$, find $f_{X|A}(x|x > a)$ the density of the conditional distribution of X given that $X > a$.

A) e^{-x} B) $e^{-(x-a)}$ C) e^{-x-a} D) e^{-ax} E) ae^{-ax}

11. A company is planning to begin a production process on a certain day. Government approval is needed in order to begin production, and it is possible that the approval might not be granted until after the planned starting day. For each day that the start of the process is delayed the company will incur a cost of 100,000 . The company has determined that the number of days that the process will be delayed is a random variable with probability function

$p(n) = \frac{1}{(n+1)(n+2)}$ for $n = 0, 1, 2, \ldots$.

The company purchases "delay insurance" which will have the insurer pay the company's cost up to a maximum of 500,000 . Find the expected cost to the insurer.

A) 100,000 B) 115,000 C) 130,000 D) 145,000 E) 160,000

12. A fair coin is tossed. If a head occurs, 1 fair die is rolled; if a tail occurs, 2 fair dice are rolled. If Y is the total on the die or dice, then $P[Y = 6] =$

A) $\frac{1}{9}$ B) $\frac{5}{36}$ C) $\frac{11}{72}$ D) $\frac{1}{6}$ E) $\frac{11}{36}$

13. Let X and Y be continuous random variables with joint density function

$f(x,y) = \begin{cases} 2 \text{ for } 0<x<y<1 \\ 0, \text{ otherwise} \end{cases}$.

Determine the density function of the conditional distribution of Y given $X = x$, where $0 < x < 1$.

A) $\frac{1}{1-x}$ for $x < y < 1$ B) $2(1-x)$ for $x < y < 1$ C) 2 for $x < y < 1$

D) $\frac{1}{y}$ for $x < y < 1$ E) $\frac{1}{1-y}$ for $x < y < 1$

14. A carnival sharpshooter game charges \$25 for 25 shots at a target. If the shooter hits the bullseye fewer than 5 times then he gets no prize. If he hits the bullseye 5 times he gets back \$10. For each additional bullseye over 5 he gets back an additional \$5. The shooter estimates that he has a .2 probability of hitting the bullseye on any given shot. What is the shooter's expected gain if he plays the game (nearest \$1)?

A) -15 B) -10 C) -5 D) 0 E) 5

15. Let X and Y be random losses with joint density function

$f(x,y) = e^{-(x+y)}$ for $x > 0$ and $y > 0$.

An insurance policy is written to reimburse $X + Y$.

Calculate the probability that the reimbursement is less than 1.

A) e^{-2} B) e^{-1} C) $1 - e^{-1}$ D) $1 - 2e^{-1}$ E) $1 - 2e^{-2}$

16. A company has annual losses that can be described by the continuous random variable X, with density function $f(x)$. The company wishes to obtain insurance coverage that covers annual losses above a deductible. The company is trying to choose between deductible amounts d_1 and d_2, where $d_1 < d_2$. With deductible d_1 the expected annual losses that would not be covered by insurance is E_1, and with deductible d_2 the expected annual losses that would not be covered by insurance is E_2 . Which of the following is the correct expression for $E_2 - E_1$?

A) $\int_{d_1}^{d_2} x \cdot f(x)\,dx$

B) $(d_2 - d_1)[F(d_2) - F(d_1)]$

C) $\int_{d_1}^{d_2} x \cdot f(x)\,dx + (d_2 - d_1)[F(d_2) - F(d_1)]$

D) $\int_{d_1}^{d_2} x \cdot f(x)\,dx + (d_2 - d_1)$

E) $\int_{d_1}^{d_2} x \cdot f(x)\,dx + (d_2 - d_1) - [d_2F(d_2) - d_1F(d_1)]$

17. Two components in an electrical circuit have continuous failure times X and Y. Both components will fail by time 1, but the circuit is designed so that the combined times until failure is also less than 1, so that the joint distribution of failure times satisfies the requirements $0 < x + y < 1$. Suppose that the joint density is constant on the probability space. Find the probability that both components will fail by time $\frac{1}{2}$.

A) $\frac{1}{16}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) 1

18. A discrete integer valued random variable has the following probability function:

$P[X = n] = a_n - a_{n+1}$, where the a's are numbers which satisfy the following conditions:

(i) $a_0 = 1$ (ii) $a_0 > a_1 > a_2 > \cdots > a_k > a_{k+1} > \cdots > 0$.

Find the probability $P[X \leq 5 | X > 1]$.

A) $1 - \frac{a_5}{a_2}$ B) $1 - \frac{a_5}{a_1}$ C) $a_1 - a_5$ D) $\frac{a_2}{a_1} - \frac{a_5}{a_2}$ E) $\frac{a_2 - a_6}{a_2}$.

19. In a carnival sharpshooter game the shooter pays \$10 and takes successive shots at a target until he misses. Each time he hits the target he gets back \$3. The game is over as soon as he misses a target. The sharpshooter estimates his probability of hitting the target on any given shot as p. According to this estimate he expects to gain \$2 on the game. Find p.

A) .5 B) .6 C) .7 D) .8 E) .9

20. In a model for hospital room charges X and hospital surgical charges Y for a particular type of hospital admission, the region of probability (after scaling units) is $0 \leq y \leq 2x+1 \leq 3$ and $x \geq 0$. The joint density function of X and Y is $f(x,y) = .3(x+y)$.
Find $E[Y-X]$, the expected excess of surgical charges over room charges for an admission.

A) $-\frac{3}{4}$ B) $-\frac{1}{4}$ C) 0 D) $\frac{1}{4}$ E) $\frac{3}{4}$

21. Auto claim amounts, in thousands, are modeled by a random variable with density function $f(x) = xe^{-x}$ for $x > 0$. The company expects to pay 100 claims if there is no deductible. How many claims does the company expect to pay if the company decides to introduce a deductible of 1000?

A) 26 B) 37 C) 50 D) 63 E) 74

22. Let X have a uniform distribution on the interval $(1,3)$. What is the probability that the sum of 2 independent observations of X is greater than 5?

A) $\frac{1}{18}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

23. Let X_1, X_2 and X_3 be three independent continuous random variables each with density function $f(x) = \begin{cases} \sqrt{2}-x & \text{for } 0<x<\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$.
What is the probability that exactly 2 of the 3 random variables exceeds 1?

A) $\frac{3}{2} - \sqrt{2}$ B) $3 - 2\sqrt{2}$ C) $3(\sqrt{2}-1)(2-\sqrt{2})^2$
D) $(\frac{3}{2} - \sqrt{2})^2(\sqrt{2} - \frac{1}{2})$ E) $3(\frac{3}{2} - \sqrt{2})^2(\sqrt{2} - \frac{1}{2})$

24. An insurer has a portfolio of 1000 independent one-year term insurance policies. For any one policy, there is a probability of .01 that there will be a claim. Use the normal approximation to find the probability that the insurer will experience at least 15 claims using the integer correction.

A) .08 B) .10 C) .12 D) .14 E) .16

25. The number of claims occurring in a period has a Poisson distribution with mean λ. The insurer determines the conditional expectation of expected number of claims in the period given that at least one claim has occurred, say $e(\lambda)$. Find $\lim_{\lambda \to 0} e(\lambda)$.

A) 0 B) e^{-1} C) 1 D) e E) ∞

26. An insurer notices that for a particular class of policies, whenever the claim amount is over 1000, the average amount by which the claim exceeds 1000 is 500. The insurer assumes that the claim amount distribution has a uniform distribution on the interval $[\,0\,,\,c\,]$, where $c > 1000$. Find the value of c that is consistent with the observation of the insurer.

A) 1500 B) 2000 C) 2500 D) 4000 E) 5000

27. Let X and Y be discrete random variables with joint probability function

$$f(x,y) = \begin{cases} \frac{(x+1)(y+2)}{54} & \text{for } x=0,1,2\,;\ y=0,1,2 \\ 0, & \text{otherwise} \end{cases} .$$

What is $E[Y|X=1]$?

A) $\frac{11}{27}$ B) 1 C) $\frac{11}{9}$ D) $\frac{y+2}{9}$ E) $\frac{y^2+2y}{9}$

28. The following probabilities of three events in a sample space are given:

$P[A] = 0.6$, $P[B] = 0.5$, $P[C] = 0.4$,

$P[A \cup B] = 1$, $P[A \cup C] = 0.7$, $P[B \cup C] = 0.7$.

Find $P[A \cap B \cap C | (A \cap B) \cup (A \cap C) \cup (B \cap C)]$.

A) 0.1 B) 0.15 C) 0.2 D) 0.25

E) Cannot be determined from the given information

29. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 1-|x| & \text{for } -1<x<1 \\ 0, & \text{otherwise} \end{cases} .$$

Determine the density function of $Y = X^2$ where nonzero.

A) $\frac{1}{\sqrt{y}} - 1$ B) $2\sqrt{y} - y$ C) $2\sqrt{y}$ D) $1 - \sqrt{y}$ E) $\frac{1}{\sqrt{y}}$

30. The claim amount random variable B has the following distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2,000 & 0 \le x < 1000 \\ .75 & x = 1000 \\ (x + 11,000)/16,000 & 1000 < x < 5000 \\ 1 & x \ge 5000 \end{cases}.$$

What is $E[B] + \sqrt{Var(B)}$?

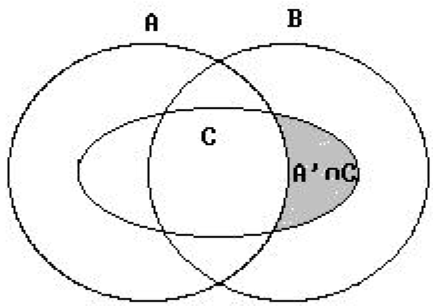
A) 2400 B) 2450 C) 2500 D) 2550 E) 2600

PRACTICE EXAM 4 - SOLUTIONS

1. I. False.

II. $C = (A \cap C) \cup (B \cap C) \rightarrow P(C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$. True.

III. The following diagram explains this situation.



$C = (A \cap C) \cup (A' \cap C)$, and $C = (A \cap C) \cup (B \cap C)$

$\rightarrow (A' \cap C) \subset (A \cap C) \cup (B \cap C)$

$\rightarrow (A' \cap C) = [\,(A' \cap C) \cap (A \cap C)] \cup [(A' \cap C) \cap (B \cap C)]$

$= \emptyset \cup (A' \cap B \cap C) \subset B$. True. Answer: A

2. We identify the events:

O - orthodontic work will be needed during the lifetime of the policy

F - a filling will be needed during the lifetime of the policy

E - an extraction will be needed during the lifetime of the policy

We wish to find $P[F \cup E]$.

We are given: $P[O] = \frac{1}{2}$, $P[O \cup F] = \frac{2}{3}$, $P[O \cup E] = \frac{3}{4}$, $P[F \cap E] = \frac{1}{8}$.

Using rules of probability, we have $P[O \cup F] = P[O] + P[F] - P[O \cap F]$,

and since O and F are independent, we have $P[O \cap F] = P[O] \cdot P[F]$, so that

$P[O \cup F] = \frac{2}{3} = P[O] + P[F] - P[O] \cdot P[F] = \frac{1}{2} + P[F] - \frac{1}{2} \cdot P[F]$,

from which it follows that $P[F] = \frac{1}{3}$.

In a similar way, since O and E are independent,

$P[O \cup E] = \frac{3}{4} = P[O] + P[E] - P[O] \cdot P[E] = \frac{1}{2} + P[E] - \frac{1}{2} \cdot P[E]$,

from which it follows that $P[E] = \frac{1}{2}$.

Now, $P[F \cup E] = P[F] + P[E] - P[F \cap E] = \frac{1}{3} + \frac{1}{2} - \frac{1}{8} = \frac{17}{24}$. Answer: D

3. The insurer pays $Y = \begin{cases} 0, \text{ if } x<c \\ x-c, \text{ if } c<x\leq 100 \end{cases}$ with constant density .01 .
Then $E[Y] = \int_c^{100} \frac{x-c}{100}\,dx = \frac{(100-c)^2}{200}$ and $E[Y^2] = \int_c^{100} \frac{(x-c)^2}{100}\,dx = \frac{(100-c)^3}{300}$
so that $Var[Y] = \frac{(100-c)^3}{300} - [\frac{(100-c)^2}{200}]^2$. Substituting the possible answers, we see that with $c = 70$, the variance of Y is 69.75 . Answer: B

4. The joint probability table is

x	y	z	$f_{X,Y,Z}(x,y,z)$
1	1	0	$1/24$
1	1	1	$2/24$
1	2	0	$2/24$
1	2	1	$3/24$
2	1	0	$2/24$
2	1	1	$4/24$
2	2	0	$4/24$
2	2	1	$6/24$

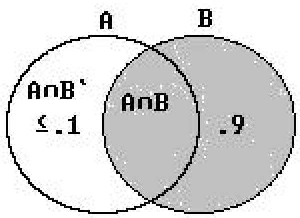
The marginal probability functions are
$f_X(1) = \frac{1+2+2+3}{24} = \frac{1}{3}$, $f_X(2) = \frac{2+4+4+6}{24} = \frac{2}{3}$,
$f_Y(1) = \frac{1+2+2+4}{24} = \frac{3}{8}$, $f_Y(2) = \frac{2+3+4+6}{24} = \frac{5}{8}$,
$f_Z(0) = \frac{1+2+2+4}{24} = \frac{3}{8}$, $f_Z(1) = \frac{2+3+4+6}{24} = \frac{5}{8}$.

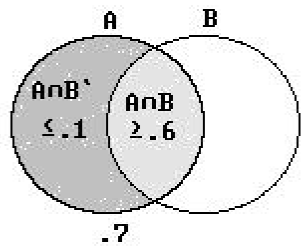
The joint distribution of X and Y is
$f_{X,Y}(1,1) = \frac{1+2}{24} = \frac{1}{8} = (\frac{1}{3})(\frac{3}{8}) = f_X(1)\cdot f_Y(1)$,
$f_{X,Y}(1,2) = \frac{2+3}{24} = \frac{5}{24} = (\frac{1}{3})(\frac{5}{8}) = f_X(1)\cdot f_Y(2)$,
$f_{X,Y}(2,1) = \frac{2+4}{24} = \frac{1}{4} = (\frac{2}{3})(\frac{3}{8}) = f_X(2)\cdot f_Y(1)$,
$f_{X,Y}(2,2) = \frac{4+6}{24} = \frac{5}{12} = (\frac{2}{3})(\frac{5}{8}) = f_X(2)\cdot f_Y(2)$.
Since $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$ for all x,y it follows that X and Y are independent.

The joint distribution of X and Z is
$f_{X,Z}(1,0) = \frac{1}{8} = (\frac{1}{3})(\frac{3}{8}) = f_X(1)\cdot f_Z(0)$,
$f_{X,Z}(1,1) = \frac{5}{24} = (\frac{1}{3})(\frac{5}{8}) = f_X(1)\cdot f_Z(1)$,
$f_{X,Z}(2,0) = \frac{1}{4} = (\frac{2}{3})(\frac{3}{8}) = f_X(2)\cdot f_Z(0)$,
$f_{X,Z}(2,1) = \frac{5}{12} = (\frac{2}{3})(\frac{5}{8}) = f_X(2)\cdot f_Z(1)$.
Since $f_{X,Z}(x,z) = f_X(x)\cdot f_Z(z)$ for all x,z it follows that X and Z are independent.

For the joint distribution of Y and Z we have
$f_{Y,Z}(1,0) = \frac{1}{8} \neq (\frac{3}{8})(\frac{3}{8}) = f_Y(1)\cdot f_Z(0)$.
It follows that Y and Z are not independent. Answer: C

5. $P[A \cup B] - P[A \cap B] = P[A] + P[B] - 2P[A \cap B] = 1.6 - 2P[A \cap B]$.
This will be maximized if $P[A \cap B]$ is minimized.
But $.7 = P[A] = P[A \cap B'] + P[A \cap B]$, and the maximum possible value of $P[A \cap B']$ is .1 (since $P[B]$ is .9, it follows that $P[A \cap B'] \leq P[B'] = .1$), so that the minimum possible value for $P[A \cap B]$ is .6, and then the maximum of $P[A \cup B] - P[A \cap B]$ is $1.6 - 2(.6) = .4$.





Answer: C

6. If we consider Doug first and define "success" to be tossing a "1" and "failure" to be a toss that is not a "1" then the number of tosses Doug makes, say X, before his first 1 is the number of failures before the first success. X has a geometric distribution with $p = \frac{1}{6}$ (p is the probability of success on a single trial). Doug's total number of tosses is $X + 1$, and
$Var(X+1) = Var(X) = \frac{1-p}{p^2} = \frac{1-\frac{1}{6}}{(\frac{1}{6})^2} = 30$.
Bob and Joe have the same distribution for the number of tosses until a first "1", and since they toss independently of Doug and each other, the variance of the total number of tosses is just the sum of the three variances, which is $30 + 30 + 30 = 90$. Answer: E

7. Since both X and Y are between 0 and 1, the event $X^2 > Y^3$ is equivalent to $X > Y^{3/2}$. Since X and Y are independent, their joint density is
$f(x,y) = f_X(x) \cdot f_Y(y) = 1$. Then,
$P[X^2 > Y^3] = \int_0^1 \int_{y^{3/2}}^1 1\, dx\, dy = \int_0^1 (1 - y^{3/2})\, dy = \frac{3}{5}$. Answer: C

8. $P[N=1] = P[N=1|\text{low risk}] \cdot P[\text{low risk}] + P[N=1|\text{high risk}] \cdot P[\text{high risk}]$
$P[N=1|\text{low risk}] = e^{-.2}(.2)$, $P[N=1|\text{high risk}] = e^{-1.5}(1.5)$,
$P[\text{low risk}] = .75$, $P[\text{high risk}] = .25$
$\rightarrow P[N=1] = (.75)e^{-.2}(.2) + (.25)e^{-1.5}(1.5) = .206$. Answer: A

9. $P[X<1|c]=1-e^{-c}$.
$P[X<1]=\int_1^2 P[X<1|c]\cdot f(c)\,dc=\int_1^2(1-e^{-c})\,dc=.767$. Answer: E

10. $P(A)=P[X>a]=\int_a^\infty e^{-x}\,dx=e^{-a}$.
$f_{X|A}(x|X>a)=\frac{f_X(x)}{P(A)}=\frac{e^{-x}}{e^{-a}}=e^{-(x-a)}$, for $x>a$, and $f_{X|A}(x|X>a)=0$ for $x\leq a$.
Answer: B

11. The amount covered by the insurance (in 100,000's) is

Days of delay	0	1	2	3	4	5	6	$\cdots$
Insurer Cost	0	1	2	3	4	5	5	$\cdots$

The expected insurer cost is
$0\cdot p(0)+1\cdot p(1)+2\cdot p(2)+3\cdot p(3)+4\cdot p(4)+5\cdot P[N\geq 5]$
$=0\cdot\frac{1}{2}+1\cdot\frac{1}{6}+2\cdot\frac{1}{12}+3\cdot\frac{1}{20}+4\cdot\frac{1}{30}+5\cdot[1-\frac{1}{2}-\frac{1}{6}-\frac{1}{12}-\frac{1}{20}-\frac{1}{30}]=1.45$.
Answer: D

12. If 1 fair die is rolled, the probability of rolling a 6 is $\frac{1}{6}$, and if 2 fair dice are rolled, the probability of rolling a 6 is $\frac{5}{36}$ (of the 36 possible rolls from a pair of dice, the rolls 1-5, 2-4, 3-3, 4-2 and 5-1 result in a total of 6), Since the coin is fair, the probability of rolling a head or tail is .5. Thus, the probability that $Y=6$ is $(.5)(\frac{1}{6})+(.5)(\frac{5}{36})=\frac{11}{72}$.
Answer: C

13. The region of joint density is the triangular region above the line $y=x$ and below the horizontal line $y=1$ for $0<x<1$. The conditional density of y given $X=x$ is
$f(y\,|\,X=x)=\frac{f(x,y)}{f_X(x)}$, where $f_X(x)$ is the marginal density function of x.
$f_X(x)=\int_{-\infty}^\infty f(x,y)\,dy=\int_x^1 2\,dy=2(1-x)$, so that $f(y\,|\,X=x)=\frac{2}{2(1-x)}=\frac{1}{1-x}$
and the region of density for the conditional distribution of Y given $X=x$ is $x<y<1$.
It is true in general that if a joint distribution is uniform (has constant density in a region) then any conditional (though not necessarily marginal) distribution will be uniform on it restricted region of probability - the conditional distribution of Y given $X=x$ is uniform on the interval $x<y<1$, with constant density $\frac{1}{1-x}$. Answer: A

14. No. of bullseyes:	0	1	2	3	4	5	6	7 ...	25
Prize:	0	0	0	0	0	10	15	20 ...	110
$5X-15$:	-15	-10	-5	0	5	10	15	20 ...	110

Let $X=$ number of bullseyes. X has a binomial distribution with $n=25$, $p=.2$, and $E[X]=5$. $p(x)=\binom{25}{x}(.2)^x(.8)^{25-x}$.

Note that for 5 bullseyes or more the prize is $5X-15$.

We can find the expected prize by first finding $E[5X-15]$ and adjusting for the factors corresponding to $X=0,1,2,3,4$. Therefore,

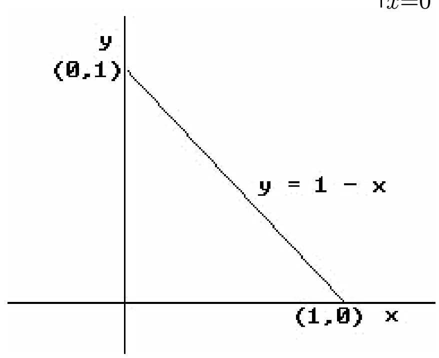
Expected prize

$$= E[5X-15]+15\cdot p(0)+10\cdot p(1)+5\cdot p(2)+0\cdot p(3)-5p(4)$$
$$= 5E[X]-15+15\binom{25}{0}(.2)^0(.8)^{25}+10\binom{25}{1}(.2)(.8)^{24}+5\binom{25}{2}(.2)^2(.8)^{23}$$
$$+(0)\binom{25}{3}(.2)^3(.8)^{22}-5\binom{25}{4}(.2)^4(.8)^{21}=9.71 .$$

The expected gain is $9.71-25=-15.29$. Answer: A

15. The probability in question is found by integrating the joint density function $f(x,y)$ over the two-dimensional region that represents the event. This two-dimensional region is $\{(x,y): x+y<1,\ x>0\ ,\ y>0\}=\{(x,y): y<1-x,\ x>0\ ,\ y>0\}$. This region is represented in the shaded area in the graph below. The probability is

$$\int_0^1\int_0^{1-x} f(x,y)\,dy\,dx=\int_0^1\int_0^{1-x} e^{-(x+y)}\,dy\,dx = \int_0^1 e^{-x}[-e^{-y}\Big|_{y=0}^{y=1-x}]\,dx$$
$$=\int_0^1 e^{-x}[1-e^{x-1}]\,dx=\int_0^1[e^{-x}-e^{-1}]\,dx = [-e^{-x}\Big|_{x=0}^{x=1}]-e^{-1}=1-2e^{-1} .$$

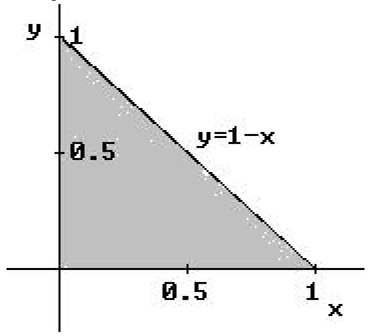


Answer: D

16. $E_1=\int_0^{d_1} x\cdot f(x)\,dx+\int_{d_1}^{\infty} d_1\cdot f(x)\,dx=\int_0^{d_1} x\cdot f(x)\,dx+d_1[1-F(d_1)]$,

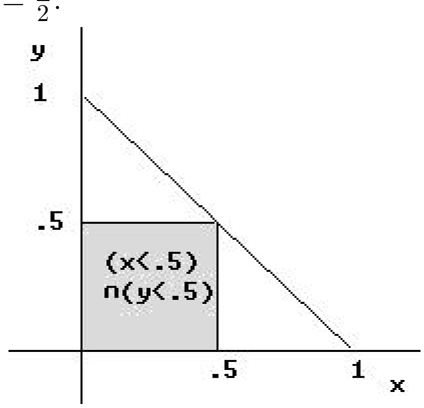
and similarly, $E_2=\int_0^{d_2} x\cdot f(x)\,dx+d_2[1-F(d_2)]$. Answer: E

17. The region of probability is the shaded area below.



The joint density is $f(x,y) = \frac{1}{Area\,of\,probability\,space} = \frac{1}{1/2} = 2$.

Since the joint density is constant, the probability that both components will fail by time $\frac{1}{2}$ is $\frac{area\,of\,region\,A}{total\,area\,of\,probability\,space}$, where A is the region representing the event that both components fail by time $\frac{1}{2}$, $(X < \frac{1}{2}) \cap (Y < \frac{1}{2})$. This area is $\frac{1}{4}$, so that
$P[(X < \frac{1}{2}) \cap (Y < \frac{1}{2})] = \frac{1/4}{1/2} = \frac{1}{2}$.



Alternatively, we can also formulate the probability as
$\int_0^{1/2}\int_0^{1/2} f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^{1/2} 2\,dy\,dx = \frac{1}{2}$. Answer: D

18. $P[X \leq 5 | X > 1] = \frac{P[1<X\leq 5]}{P[X>1]}$.

$P[X > 1] = 1 - P[X = 0,1] = 1 - (1 - a_1) - (a_1 - a_2) = a_2$.

$P[1 < X \leq 5] = P[X = 2,3,4,5]$

$= (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + (a_5 - a_6) = a_2 - a_6$.

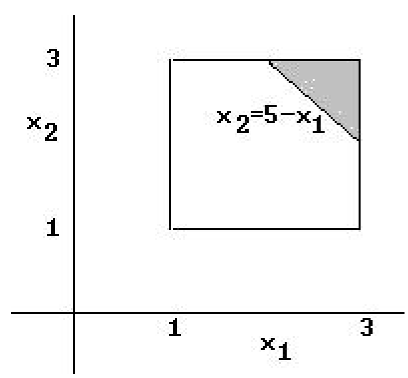
$P[X \leq 5 | X > 1] = \frac{a_2 - a_6}{a_2}$. Answer: E

19. Let X be the number of targets he hits until the first miss. Then the probability function for X is $P[X=k]=p^k(1-p)$. This is the form of the geometric distribution in which we count the number of failures until the first success ("failure" in this context means hitting the target, and "success" means missing the target); however we are using reversing the use of p, so in this case the probability of success is $1-p$. Therefore, the expected value of X is $E[X]=\frac{1-(1-p)}{1-p}=\frac{p}{1-p}$. The expected gain from the game is $3E[X]-10$ (3 dollars for each hit minus the initial cost). To have an expected gain of 2, we have $3(\frac{p}{1-p})-10=2 \rightarrow p=.8$. Answer: D

20. $E[Y-X]=\int_0^1\int_0^{2x+1}(y-x)(.3)(x+y)\,dy\,dx=\int_0^1(.2x^3+.9x^2+.6x+.1)\,dx=\frac{3}{4}$.
Answer: E

21. There are 100 policies (each claim will result in a payment if there is no deductible).
The probability of a claim being above 1000 (one thousand) is $\int_1^\infty xe^{-x}\,dx= -xe^{-x}-e^{-x}\Big|_{x=1}^{\infty}=2e^{-1}=.7358$. Of the 100 policies, the expected number that will have claim amounts over 1000 is $(1000)(.7358)=73.6 \rightarrow 74$ .Answer: E

22. The probability $P[X_1+X_2>5]$ is the integral of the joint density of $X_1$and X_2 over the shaded region at the right.
This region is $2\le x_1\le 3$ and $5-x_1\le x_2\le 3$. The probability is $\int_2^3\int_{5-x_1}^3\frac{1}{2}\cdot\frac{1}{2}dx_2\,dx_1=\frac{1}{8}$.



Answer: B

23. $P[X\le 1]=\int_0^1(\sqrt{2}-x)\,dx=\sqrt{2}-\frac{1}{2}$, $P[X>1]=1-P[X\le 1]=\frac{3}{2}-\sqrt{2}$.
With 3 independent random variables, $X_1,\ X_2$ and X_3, there are 3 ways in which exactly 2 of the X_i's exceed 1 (either X_1,X_2 or X_1,X_3 or X_2,X_3). Each way has probability $(P[X>1])^2\cdot P[X\le 1]=(\frac{3}{2}-\sqrt{2})^2(\sqrt{2}-\frac{1}{2})$ for a total probability of $3\cdot(\frac{3}{2}-\sqrt{2})^2(\sqrt{2}-\frac{1}{2})$. Answer: E

24. The total number of claims follows a binomial distribution with $n = 1000$ trials and $q = .01$ probability of "success" (claim) for each trial. Since the binomial distribution is the sum of independent Bernoulli trials, the normal approximation applies to the total number of claims N. $E[N] = 1000(.01) = 10$ and $Var[N] = 1000(.01)(.99) = 9.9$.
Then $P[N \geq 15] = P[N \geq 14.5] = P[\frac{N-E[N]}{\sqrt{Var[N]}} \geq \frac{14.5-10}{\sqrt{9.9}}] = P[Z \geq 1.43]$
$= 1 - \Phi(1.43) = 1 - .9234 = .0766$ (interpolation between $\Phi(1.4)$ and $\Phi(1.5)$ in the normal tables provided with the exam). Answer: A

25. $E[N|N \geq 1] = \sum_{n=1}^{\infty} n \cdot f(n|N \geq 1) = \sum_{n=1}^{\infty} n \cdot \frac{f(n)}{1-f(0)} = \sum_{n=0}^{\infty} n \cdot \frac{f(n)}{1-f(0)}$
$= \frac{1}{1-f(0)} \cdot \sum_{n=0}^{\infty} n \cdot f(n) = \frac{E[N]}{1-f(0)} = \frac{\lambda}{1-e^{-\lambda}}$.
$\lim_{\lambda \to 0} E[N|N \geq 1] = \lim_{\lambda \to 0} \frac{\lambda}{1-e^{-\lambda}} =$ (by l'Hospital's rule) $\lim_{\lambda \to 0} \frac{1}{e^{-\lambda}} = 1$. Answer: C

26. If the claim amount X is uniform on the interval $[\,0\,,\,c\,]$, then the conditional density $f(x|X > 1000) = \frac{f(x)}{P[X>1000]} = \frac{1/c}{(c-1000)/c} = \frac{1}{c-1000}$, for $1000 < x < c$.
This conditional density is uniform on the interval $[\,0\,,\,c-1000\,]$, and has a mean of $\frac{c-1000}{2}$. In order for this to be 500, we must have $\frac{c-1000}{2} = 500$, so that $c = 2000$.
Answer: B

27. $f_X(1) = P[X = 1] = \sum_{y=-\infty}^{\infty} f(1, y) = f(1, 0) + f(1, 1) + f(1, 2) = \frac{1}{3}$.
Then we have conditional probabilities $P[Y = 0|X = 1] = \frac{f(1,0)}{P[X=1]} = \frac{4/54}{1/3} = \frac{2}{9}$,
and similarly, $P[Y = 1|X = 1] = \frac{1}{3}$ and $P[Y = 2|X = 1] = \frac{4}{9}$.
Then, $E[Y|X = 1] = 0 \cdot \frac{2}{9} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{4}{9} = \frac{11}{9}$. Answer: C

28. $P[A \cap B \cap C | (A \cap B) \cup (A \cap C) \cup (B \cap C)] = \frac{P[A \cap B \cap C]}{P[(A \cap B) \cup (A \cap C) \cup (B \cap C)]}$.

$P[A \cap B] = P[A] + P[B] - P[A \cup B] = .1$, and similarly, $P[A \cap C] = .3$,

$P[B \cap C] = .2$. Since $P[A \cup B] = 1$, it follows that $P[A \cup B \cup C] = 1$. Then,

since $1 = P[A \cup B \cup C]$

$= P[A] + P[B] + P[C] - (P[A \cap B] + P[A \cap C] + P[B \cap C]) + P[A \cap B \cap C]$

it follows that $P[A \cap B \cap C] = .1$. Also,

$P[(A \cap B) \cup (A \cap C) \cup (B \cap C)]$

$= P[A \cap B] + P[A \cap C] + P[B \cap C] - 3P[A \cap B \cap C] + P[A \cap B \cap C] = .4$,

so that $\frac{P[A \cap B \cap C]}{P[(A \cap B) \cup (A \cap C) \cup (B \cap C)]} = .25$. Answer: D

29. For $0 \le y < 1$, $F_Y(y) = P[Y \le y] = P[X^2 \le y] = P[|X| \le \sqrt{y}\,]$

$= \int_{-\sqrt{y}}^{\sqrt{y}} (1 - |x|)\, dx = \int_{-\sqrt{y}}^{0} (1 + x)\, dx + \int_0^{\sqrt{y}} (1 - x)\, dx = 2\sqrt{y} - y$.

Then, $f_Y(y) = F_Y'(y) = \frac{1}{\sqrt{y}} - 1$ for $0 \le y < 1$.

Note that in this case the transformation $u(x) = x^2$ is not one-to-one on the region of probability of X ($-1 < x < 1$), we cannot use the $f_Y(y) = f_X(v(y)) \cdot \left| \frac{d}{dy}\, v(y) \right|$

approach. Answer: A

30. The pdf of B is $f(x) = 0$ for $x < 0$ and $f(x) = 0$ for $x \ge 5000$,

it is $f(x) = .0005$ for $0 \le x \le 1000$, and it is $f(x) = .0000625$ for $1000 < x < 5000$.

There is a point mass of probability with $f(x) = .25$ at $x = 1000$ (B has a mixed distribution).

$E[B] = \int_0^{1000} x \cdot (.0005) dx \; + \; (1000)(.25) + \int_{1000}^{5000} x \cdot (.0000625) dx \; = 1250$,

$E[B^2] = \int_0^{1000} x^2 \cdot (.0005) dx + (1000^2)(.25) + \int_{1000}^{5000} x^2 \cdot (.0000625) dx = 3,000,000$

$Var[B] = E[B^2] - (E[B])^2 = 1,437,500$. $E[B] + \sqrt{Var[B]} = 2449$. Answer: B.

PRACTICE EXAM 5

1. X has pdf $f(x) = \frac{c}{(x+\theta)^{\alpha+1}}$, defined for $x > 0$, where α and θ are both > 0.
Find $F(x)$ for $x > 0$.
A) $(\frac{\theta}{x+\theta})^{\alpha}$ B) $1 - (\frac{\theta}{x+\theta})^{\alpha}$ C) $(\frac{\theta}{x+\theta})^{\alpha+1}$ D) $1 - (\frac{\theta}{x+\theta})^{\alpha+1}$ E) $(\frac{\theta}{x+\theta})^{\alpha-1}$

2. Micro Insurance Company issued insurance policies to 32 independent risks. For each policy, the probability of a claim is 1/6. The benefit amount given that there is a claim has probability density function $f(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$.
Calculate the expected value of total benefits paid.
A) $\frac{16}{9}$ B) $\frac{8}{3}$ C) $\frac{32}{9}$ D) $\frac{16}{3}$ E) $\frac{32}{3}$

3. Let A, B and C be events such that A and B are independent, B and C are mutually exclusive, $P[A] = \frac{1}{4}$, $P[B] = \frac{1}{6}$, and $P[C] = \frac{1}{2}$. Find $P[(A \cap B)' \cup C]$.
A) $\frac{11}{24}$ B) $\frac{3}{4}$ C) $\frac{5}{6}$ D) $\frac{23}{24}$ E) 1

4. If the mean and variance of random variable X are 2 and 8, find the first three terms in the Taylor series expansion of the moment generating function of X about the point $t = 0$.
A) $2t + 2t^2$ B) $1 + 2t + 6t^2$ C) $1 + 2t + 2t^2$ D) $1 + 2t + 4t^2$ E) $1 + 2t + 12t^2$

5. A small commuter plane has 30 seats. The probability that any particular passenger will not show up for a flight is 0.10, independent of other passengers. The airline sells 32 tickets for the flight. Calculate the probability that more passengers show up for the flight than there are seats available.
A) 0.0042 B) 0.0343 C) 0.0382 D) 0.1221 E) 0.1564

6. The moment generating function for the random variable X is $M_X(t) = Ae^t + Be^{2t}$.
You are given that $Var[X] = \frac{2}{9}$ and $A < \frac{1}{2}$. Find $E[X]$.
A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) 1 D) $\frac{4}{3}$ E) $\frac{5}{3}$

7. The exponential distribution with mean 1 is being used as the model for a loss distribution. An actuary attempts to "discretize" the distribution by assigning a probability to $k+\frac{1}{2}$ for $k=0,1,2,\ldots$. The probability assigned to $k+\frac{1}{2}$ is $P[k<X\leq k+1]$, where X is the exponential random variable with mean 1. Find the mean of the discretized distribution.

A) 1.00 B) 1.02 C) 1.04 D) 1.06 E) 1.08

8. A casino manager creates a model for the number of customers who play on a particular gambling machine during a 2-hour period. If $0\leq t\leq 2$ (in hours),
then the probability that k people play on the machine during the time interval from time 0 to time t is $\binom{10}{k}(.5t)^k(1-.5t)^{10-k}$ (binomial). T denotes the time (measured from time 0) at which the first person plays on the machine. Find the pdf of T.

A) $10(1-.5t)^9$ B) $5(1-.5t)^9$ C) $5.5(1-.5t)^{10}$
D) $10(.5t)^{-9}$ E) $5(.5t)^9$

9. A loss distribution this year is exponentially distributed with mean 1000. An insurance policy pays the loss amount up to a maximum of 500. As a result of inflation,
the loss distribution next year will be uniformly distributed between 0 and 1250. The insurer increases the maximum amount of payment to u so that the insurer's expected payment is 25% higher next year. Find u.

A) 525 B) 559 C) 673 D) 707 E) 779

10. Of the following statements regarding the sums of independent random variables, how many are true?

I. The sum of independent Poisson random variables has a Poisson distribution.
II. The sum of independent exponential random variables has an exponential distribution.
III. The sum of independent geometric random variables has a geometric distribution.
IV. The sum of independent normal random variables has a normal distribution.

A) 0 B) 1 C) 2 D) 3 E) 4

11. Among the questions asked in a marketing study were the following:

(i) Are you a member of a group health insurance plan?

(ii) Are you a member of a fitness club?

It was found that 80% of the respondents answered "YES" to at least one of those two questions and 80% answered "NO" to at least one of those two questions. Find the percentage that answered "YES" to exactly one of those two questions.

A) .2 B) .3 C) .4 D) .5 E) .6

12. A manufacturing company ships 10,000 units of a product per shipment. In any given shipment there are a proportion of units that are defective. The company has determined that 25% of all shipments have a defective proportion of .2 and the other 75% of the shipments have a defective proportion of .1. A shipment is selected at random and 10 units of the product are chosen at random from that shipment. Find the probability that at least 2 of the units in that sample are defective (nearest .05).

A) .30 B) .35 C) .40 D) .45 E) .50

13. A population of insured individuals consists of a% low risk, b% medium risk and c% high risk. The number of claims in a year for a low risk individual has a Poisson distribution with a mean of 1 claim, for a medium risk individual the number of claims in a year is Poisson with a mean of 2, and for a high risk individual the number of annual claims is Poisson with a mean of 3. An individual is picked at random from the population and it is found that the mean number of claims for the year is 2.1 and the variance of the number pf claims for the year is 2.59. For a randomly chosen member of the population find the probability of no claims occurring in the year.

A) .088 B) .116 C) .156 D) .198 E) .228

14. An insurer is considering taking over a group of policies. The policies in the group are identically distributed and mutually independent of one another. Each policy in the group has a claim distribution which is exponentially distributed with mean 100. The premium for each policy is 120. The insurer wants a probability of at least 95% that the premium received will be enough to cover the claims. Using the normal approximation, determine the minimum number of policies needed in the group in order for the insurer's criterion to be met.

A) 60 B) 62 C) 64 D) 66 E) 68

15. X has a distribution with pdf $f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$, $x > 0$.
Y has an exponential distribution with mean $\frac{1}{\alpha}$.
Which of the following is the correct transformation linking Y and X?
A) $Y = ln(X+\theta)$ B) $Y = ln(\frac{X}{\theta})$ C) $Y = ln(\frac{X+\theta}{\theta})$
D) $Y = ln(\frac{X+\theta}{X})$ E) $Y = ln(\frac{X}{X+\theta})$

16. Binary digits are transmitted over a communication system. If a 1 is sent, it will be received as a 1 with probability .95 and as a 0 with probability .05. If a 0 is sent, it will be received as a 0 with probability .99 and as a 1 with probability .01. A series of 0's and 1's is sent in random order, with 0's and 1's each being equally likely. If a digit is received as a 1 find the probability that it was sent as a 1.
A) less than .96 B) at least .96 but less than .97 C) at least .97 but less than .98
D) at least .98 but less than .99 E) at least .99

17. The model chosen for a discrete, integer-valued, non-negative random variable N with mean 2 is the binomial distribution with n trials and probability of success p on each trial. Various combinations of n and p are considered, and $P[N=0]$ is calculated. Find $\lim_{n\to\infty} P[N=0]$.
A) e^{-2} B) e^{-1} C) 0 D) $\frac{1}{2}$ E) 1

18. When a fire occurs, the model for fire damage on a particular property is based on the following joint distribution for X (structural damage) and Y (damage to contents):
$f(x,y) = ax + by$, $0 < x < 2$, $0 < y < 1$ (scaled to appropriately sized units). The probability that X is greater than Y is $\frac{5}{6}$. Find the total expected damage if a fire occurs, $E[X+Y]$.
A) $\frac{10}{9}$ B) $\frac{16}{9}$ C) $\frac{22}{9}$ D) $\frac{28}{9}$ E) $\frac{34}{9}$

19. A machine requires the continual operation of two independent devices in order to keep functioning. The machine breaks down as soon as the first device stops operating.
The time until failure of Device 1 is uniformly distributed between time 0 and time 1, and the time until failure of Device 2 has pdf $f(t) = 2t$, $0 < t < 1$. Find the expected time until the machine breaks down.
A) $\frac{1}{12}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{5}{12}$

20. X and Y have a bivariate normal distribution with $E[X]=0$ and $E[Y]=1$.
It is also known that $E[X|Y=9]=8$ and $E[Y|X=9]=2$. Find ρ, the coefficient of correlation between X and Y.
A) $-\frac{1}{2}$ B) $-\frac{1}{3}$ C) 0 D) $\frac{1}{3}$ E) $\frac{1}{2}$

21. Pick the correct relationship:
A) $P[A\cap B]\leq P[A]\cdot P[B]$ for any events A and B
B) $P[A\cup B]\geq P[A]+P[B]$ for any events A and B
C) $P[A\cap B']\geq P[A]-P[B]$ for any events A and B
D) $P[A\cup B|C]=P[A|C]+P[B|C]$ for independent events A and B and any event C
E) $P[A|B]=P[B|A]$ for any events A and B

22. A city lotto is held each week. The lotto ticket costs \$1, and the lotto prize is \$10 and there is a $\frac{1}{30}$ chance of winning the prize. Smith decides to try his luck at the lotto, and decides to buy 1 ticket each week until he wins, at which time he will stop. Find Smith's expected gain for his lotto-ticket enterprise.
A) -20 B) -15 C) -10 D) -5 E) 0

23. When a fire occurs, the model for fire damage on a particular property is based on a joint distribution for X (structural damage) and Y (damage to contents). The marginal distribution of X has density function $f_X(x)=2-2x$ for $0<x<1$. If the amount structural damage is x, then the distribution of damage to contents is uniform on the interval $(0,\frac{x}{2})$. Find the expected amount of damage to contents when a fire occurs.
A) 1 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{6}$ E) $\frac{1}{12}$

24. A loss has a distribution which is uniform between 0 and 1. An insurer issues a policy in this loss which pays the amount of the loss above a deductible of amount d,
where $0<d<1$. The expected claim on the insurer is c, where $0<c<\frac{1}{2}$.
Find the amount of the deductible.
A) $\sqrt{2c}$ B) $1-\sqrt{2c}$ C) $1+\sqrt{2c}$ D) $\frac{1}{\sqrt{2c}}+1$ E) $\frac{1}{\sqrt{2c}}-1$

25. X has an exponential distribution with a mean of 1. Y is defined to be the conditional distribution of $X-2$ given that $X>2$, so for instance, for $c>0$, we have $P[Y>c]=P[X-2>c|X>2]$. What is the distribution of Y?

A) Exponential with mean 1 B) Exponential with mean 2 C) Exponential with mean $\frac{1}{2}$

D) Exponential with mean e E) Exponential with mean e^2

26. A die is being loaded so that the probability of tossing a 1 is p and the probability of tossing a 6 is $\frac{1}{3}-p$. The probabilities of tossing a 2, 3, 4 or 5 are all $\frac{1}{6}$. X is the outcome from tossing the die once. Find the value of p for which the variance of X is maximized.

A) 0 B) $\frac{1}{12}$ C) $\frac{1}{6}$ D) $\frac{1}{4}$ E) $\frac{1}{3}$

27. The daily high temperature in Toronto in January is normally distributed with a mean of -5 degrees Celsius and a standard deviation of 4 degrees Celsius. The daily high temperature in Winnipeg in January is normally distributed with a mean of -10 degrees Celsius and a standard deviation of 8 degrees Celsius. Assuming that daily high temperatures in Toronto and Winnipeg are independent of one another, find the probability that on a given day in January, the high temperatures for that day in Toronto and Winnipeg are within 1 degree Celsius of each other (nearest .025).

A) .025 B) .050 C) .075 D) .100 E) .125

28. The joint distribution of random variables X and Y has pdf

$$f(x,y)=x+y\ ,\ 0<x<1\,,\ 0<y<1\ .$$

The joint distribution of random variables Y and Z has pdf

$$g(y,z)=3(y+\tfrac{1}{2})z^2\ ,\ 0<y<1\,,\ 0<z<1\ .$$

Which of the following could be the pdf of the joint distribution of X and Z?

A) $x+\frac{3}{2}z^2\ ,\ 0<x<1\,,\ 0<z<1$

B) $x+\frac{1}{2}+3z^2\ ,\ 0<x<1\,,\ 0<z<1$

C) $3(x+\frac{1}{2})z^2\ ,\ 0<x<1\,,\ 0<z<1$

D) $x+z\ ,\ 0<x<1\,,\ 0<z<1$

E) $4xz\ ,\ 0<x<1\,,\ 0<z<1$

29. A small company wishes to insure against losses incurred in the case of a strike by the company's employees. An insurer agrees to pay \$100,000 for each strike that occurs within the next year, up to a maximum payment of \$300,000 . The distribution used to model strike behavior is $P[n \text{ strikes within the next year}] = (.8)(.2)^n$, $n \geq 0$.
The small company estimates that it will lose \$150,000 for each strike that occurs.
For the year, find the company's expected loss that is not covered by the insurance.
A) 12,100 B) 12,300 C) 12,500 D) 12,700 E) 12,900

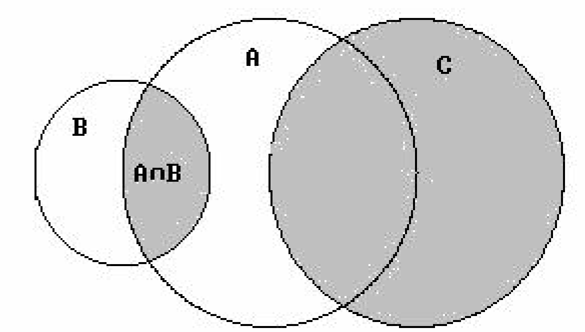
30. X and Y have a joint distribution with pdf $f(x,y) = e^{-(x+y)}$, $x > 0$, $y > 0$.
The random variable U is defined to be equal to $U = e^{-(x+y)}$.
Find the pdf of U, $f_U(u)$.
A) $f_U(u) = 1$ for $0 < u < 1$ B) $f_U(u) = \frac{1}{u^2}$ for $u > 1$
C) $f_U(u) = -ln\, u$ for $0 < u < 1$ D) $f_U(u) = 2u$ for $0 < u < 1$
E) $f_U(u) = e^{-u}$ for $u > 0$

PRACTICE EXAM 5 - SOLUTIONS

1. Since $\int_0^\infty f(x)\,dx = 1$, it follows that $c = 1/\int_0^\infty \frac{1}{(x+\theta)^{\alpha+1}}\,dx = \alpha\theta^\alpha$,
so that $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$. Then $F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{\alpha\theta^\alpha}{(t+\theta)^{\alpha+1}}\,dt = 1 - (\frac{\theta}{x+\theta})^\alpha$.
This is called a Pareto distribution with parameters α and θ. Answer: B

2. In this case, for a particular policy, the probability of a claim occurring is $q = \frac{1}{6}$, and if a claim occurs the (conditional) distribution of the amount of the claim B is defined by the given density function $f(y) = 2(1-y)$ for $0 < y < 1$. Then, the expected claim from any one policy is $E[X] = q \cdot E[B] = (\frac{1}{6}) \cdot \int_0^1 y \cdot 2(1-y)\,dy = (\frac{1}{6}) \cdot (\frac{1}{3}) = \frac{1}{18}$.
The expected value of total benefits paid (claims) is the sum of the individual policy expected values for the 32 policies; this is $(32)(\frac{1}{18}) = \frac{16}{9}$. Answer: A

3. $A \cap B \subset B$ and since B and C are mutually exclusive, $A \cap B$ is disjoint from C .
It then follows that $C \subset (A \cap B)'$, and thus $(A \cap B)' \cup C = (A \cap B)'$.
This can be seen in the diagram below.



Then $P[(A \cap B)' \cup C] = P[(A \cap B)'] = 1 - P[A \cap B]$.
Since A and B are independent, $P[A \cap B] = P[A] \cdot P[B] = \frac{1}{24}$.
Thus, $P[(A \cap B)' \cup C] = \frac{23}{24}$. Answer: D

4. $M(t) = E[e^{tX}] = E[1 + tX + \frac{t^2X^2}{2} + \cdots] = E[1] + t \cdot E[X] + \frac{t^2}{2} \cdot E[X^2] + \cdots$
$E[1] = 1$, $E[X]$ is given as 2, and $Var[X] = E[X^2] - (E[X])^2$ is given to be 8,
so that $E[X^2] = 12$. Then the first 3 terms of the expansion of $M(t)$ are $1 + 2t + 6t^2$.
Answer: B

5. The number of passengers who show up is a random variable N, which has a binomial distribution with $n = 32$, $p = .9$ (each of the 32 ticket holders can be regarded as a trial experiment that can end in success - showing up - prob. .9, or failure - not showing up - prob. .1). In order for more passengers show up for the flight than there are seats available, N must be 31 or 32. Therefore, the probability that more passengers show for the flight than there are seats available is $P[N = 31 \text{ or } 32] = P[N = 31] + P[N = 32]$.
Using the formulation for binomial distribution probabilities we have
$P[N = 31] = \binom{32}{31}(.9)^{31}(.1)^1 = .12209$, and $P[N = 32] = \binom{32}{32}(.9)^{32}(.1)^0 = .03434$. The probability in question is $P[N = 31] + P[N = 32] = .1564$. Answer: E

6. $M_X(t) = Ae^t + Be^{2t}$. For any random variable, $M(0) = 1 \rightarrow A + B = 1$
$\rightarrow M_X(t) = Ae^t + (1 - A)e^{2t}$.
$E[X] = M_X'(0) = A + 2(1 - A) = 2 - A$.
$E[X^2] = M_X''(0) = A + 4(1 - A) = 4 - 3A$.
$Var[X] = E[X^2] - (E[X])^2 = 4 - 3A - (2 - A)^2 = A - A^2 = \frac{2}{9}$
$\rightarrow A^2 - A + \frac{2}{9} = 0 \rightarrow A = \frac{1}{3}$ or $\frac{2}{3}$.
Since we are given that $A < \frac{1}{2}$, it follows that $A = \frac{1}{3}$ and $E[X] = 2 - \frac{1}{3} = \frac{5}{3}$. Answer: E

7. Y is the discretized distribution. $P[Y = k + \frac{1}{2}] = P[k < X \leq k + 1] = e^{-k} - e^{-k-1}$
for $k = 0, 1, 2, \ldots$

$$E[Y] = \sum_{k=0}^{\infty}(k + \tfrac{1}{2})(e^{-k} - e^{-k-1}) = \sum_{k=0}^{\infty}k(e^{-k} - e^{-k-1}) + \tfrac{1}{2}\sum_{k=0}^{\infty}(e^{-k} - e^{-k-1}).$$

$$\sum_{k=0}^{\infty}(e^{-k} - e^{-k-1}) = (e^0 - e^{-1}) + (e^{-1} - e^{-2}) + \cdots = 1.$$

$$\sum_{k=0}^{\infty}ke^{-k} = \frac{e^{-1}}{(1-e^{-1})^2}\ , \ \sum_{k=0}^{\infty}ke^{-k-1} = \frac{e^{-2}}{(1-e^{-1})^2}$$

$$\rightarrow \sum_{k=0}^{\infty}k(e^{-k} - e^{-k-1}) = \frac{e^{-1}}{(1-e^{-1})^2} - \frac{e^{-2}}{(1-e^{-1})^2} = \frac{e^{-1}}{1-e^{-1}}.$$

$E[Y] = \frac{e^{-1}}{1-e^{-1}} + \frac{1}{2} = 1.08$. Answer: E

8. The distribution function of T is

$F_T(t) = P[T \leq t] = 1 - P[T > t] = 1 - P[\text{first play occurs after time } t]$

$= 1 - P[\text{no one plays from time 0 to } t] = 1 - \binom{10}{0}(.5t)^0(1 - .5t)^{10} = 1 - (1 - .5t)^{10}$.

The pdf of T is $f_T(t) = F_T'(t) = 5(1 - .5t)^9$. Answer: B

9. This year's expected insurance payment is

$\int_0^{500} x \cdot (.001e^{-.001x})\, dx + 500 \cdot P[X > 500]$

$= \int_0^{500} (1 - F(x))\, dx = \int_0^{500} e^{-.001x}\, dx = 1000(1 - e^{-.5}) = 393.47$.

Next year's expected insurance payment is

$\int_0^{u} x \cdot (.0008)\, dx + u \cdot P[X > u] = \int_0^{u} (1 - F_1(x))\, dx$

$= \int_0^{u} (1 - .0008x)\, dx = u - .0004u^2$.

We want $u - .0004u^2 = (1.25)(393.47) = 491.84$.

Solving the quadratic equation results in $u = 673$ or 1827 .

We ignore the root $u = 1827$, since it is above the maximum loss.

Answer: C

10. I. True.

II. False. The sum of independent exponentials, each with the same mean, has a gamma distribution.

III. False.

IV. True.

Answer: C

11. $A =$ "YES" to (i) , $B =$ "YES" to (ii)

$P[A \cup B] = .8$, $P[A' \cup B'] = .8 \rightarrow P[A \cap B] = P[(A' \cup B')'] = 1 - .8 = .2$

$\rightarrow P[\text{"YES' to exactly one}] = P[A \cap B'] + P[A' \cap B]$

$= P[A \cup B] - P[A \cap B] = .8 - .2 = .6$. Answer: E

12. N, number of defective in a sample of size 10, has a binomial distribution with probability p of any one being defective.
$P[N \geq 2|p] = 1 - P[N0 \text{ or } 1|p] = 1 - \binom{10}{0}(1-p)^{10} - \binom{10}{1}(1-p)^9 \cdot p$.
For shipment with .2 defective,
$P[N \geq 2|p = .2] = 1 - (.8)^{10} - (10)(.8)^9(.2) = .6242$.
For shipment with .1 defective,
$P[N \geq 2|p = .1] = 1 - (.9)^{10} - (10)(.9)^9(.1) = .2639$.
$P[N \geq 2] = P[N \geq 2|p = .2] \cdot P[p = .2] + P[N \geq 2|p = .1] \cdot P[p = .1]$
$= (.6242)(.25) + (.2639)(.75) = .354$. Answer: B

13. With mixing weight a applied to the Poisson with mean 1 and mixing weight b applied to the Poisson with mean 2, the mean of X is $E(X) = a + 2b + 3(1 - a - b) = 2.1$.
The second moment of X is $E(X^2) = 2a + 6b + 12(1 - a - b) = 2.59 + 2.1^2 = 7.0$.
We get the two equations $2a + b = .9$ and $10a + 6b = 5$.
Solving these two equations results in $a = .2$ and $b = .5$.
Then $P(X = 0) = .2e^{-1} + .5e^{-2} + .3e^{-3} = .156$. Answer: C

14. Claim on policy i is X_i . With n policies, the aggregate claim is $C = \sum_{i=1}^{n} X_i$, and the mean of the aggregate claim is
$E[C] = E[X_1 + X_2 + \cdots + X_n] = nE[X] = 100n$, and the variance is
$Var[C] = Var[X_1 + X_2 + \cdots + X_n] = nVar[X] = 10,000n$ (since the X_i's are independent, the variance of the sum is the sum of the variances, and variance of the exponential distribution is the square of the mean). Total premium collected is $120n$.
Using the normal approximation the probability that total premium exceeds total claims is
$P[C < 120n] = P[\frac{C - E[C]}{Var[C]} < \frac{120n - 100n}{\sqrt{10{,}000n}}]$. $\frac{C - E[C]}{Var[C]}$ is approximately normal, so in order for this probability to be $\geq$ .95, we must have $\frac{120n - 100n}{\sqrt{10{,}000n}} \geq 1.645$ (from the normal table). This is equivalent to $.2\sqrt{n} \geq 1.645$, or equivalently, $n \geq 67.7$. Answer: E

15. If $Y = g(X)$ with inverse function $X = k(Y)$, then $f_Y(y) = f_X(k(y)) \cdot |k'(y)|$.
Also, we can think of X as a function of Y, so that $f_X(x) = f_Y(g(x)) \cdot |g'(x)|$.
But we are given that Y is exponential with mean $\frac{1}{\alpha}$, so that $f_Y(y) = \alpha e^{-\alpha y}$.
Therefore, $f_X(x) = \alpha e^{-\alpha g(x)} \cdot |g'(x)| = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$.
We can try each of the transformations $Y = g(X)$ to see which one satisfies the relationship.
A) $g(x) = ln(x+\theta) \rightarrow g'(x) = \frac{1}{x+\theta}$ and
$\alpha e^{-\alpha g(x)} \cdot |g'(x)| = \alpha e^{-\alpha\, ln(x+\theta)} \cdot |\frac{1}{x+\theta}| = \frac{\alpha}{(x+\theta)^{\alpha+1}}$. Incorrect.
B) $g(x) = ln(\frac{x}{\theta}) \rightarrow g'(x) = \frac{1}{x}$ and
$\alpha e^{-\alpha g(x)} \cdot |g'(x)| = \alpha e^{-\alpha\, ln(x/\theta)} \cdot |\frac{1}{x}| = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$. Incorrect.
C) $g(x) = ln(\frac{x+\theta}{\theta}) \rightarrow g'(x) = \frac{1}{x+\theta}$ and
$\alpha e^{-\alpha g(x)} \cdot |g'(x)| = \alpha e^{-\alpha\, ln(\frac{x+\theta}{\theta})} \cdot |\frac{1}{x+\theta}| = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$. Correct.
Answer: C

16. $A = 1$ was sent , $B = 1$ was received .
$P[A] = P[A'] = .5$. $P[B|A] = .95$, $P[B|A'] = .01$.
$P[B|A] = \frac{P[B\cap A]}{P[A]} \rightarrow P[B \cap A] = (.95)(.5) = .475$,
$P[B|A'] = \frac{P[B\cap A']}{P[A']} \rightarrow P[B \cap A'] = (.01)(.5) = .005$.
$P[B] = P[B \cap A] + P[B \cap A'] = .475 + .005 = .48$.
$P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{.475}{.48} = .9896$. Answer: D

17. $P[N = 0] = (1-p)^n$. Since $E[N] = np = 2$, it follows that $p = \frac{2}{n}$.
Then $\lim\limits_{n\to\infty} P[N = 0] = \lim\limits_{n\to\infty} (1-p)^n = \lim\limits_{n\to\infty} (1 - \frac{2}{n})^n$.
We consider $log[(1 - \frac{2}{n})^n] = n\, log(1 - \frac{2}{n}) = \frac{log(1-\frac{2}{n})}{\frac{1}{n}}$.
Applying l'Hopital's rule, we get

$$\lim_{n\to\infty} log[(1 - \frac{2}{n})^n] = \lim_{n\to\infty} \frac{log(1-\frac{2}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{2/n^2}{1-\frac{2}{n}}}{-\frac{1}{n^2}} = -2 .$$

Therefore, $\lim\limits_{n\to\infty} (1 - \frac{2}{n})^n = e^{-2} = \lim\limits_{n\to\infty} P[N = 0]$. Answer: A

18. In order to be a proper density function, $f(x,y)$ must satisfy the relationship
$\int_0^2 \int_0^1 (ax+by)\,dy\,dx = 1$, so that $2a+b=1$.
From the given probability, we have $\int_0^1 \int_y^2 (ax+by)\,dx\,dy = \frac{5}{6}$,
so that $\int_0^1 [\frac{a}{2}(4-y^2) + b(2y-y^2)]\,dy = \frac{11a}{6} + \frac{2b}{3} = \frac{5}{6} \to 11a + 4b = 5$.
Solving the two equations $2a+b=1$, $11a+4b=5$ results in $a=b=\frac{1}{3}$.
Then, $E[X+Y] = \int_0^2 \int_0^1 (x+y)(\frac{1}{3}x + \frac{1}{3}y)\,dy\,dx = \frac{1}{3}\int_0^2 \int_0^1 (x^2+2xy+y^2)\,dy\,dx$
$= \frac{1}{3}\int_0^2 (x^2 + x + \frac{1}{3})\,dx = \frac{1}{3}(\frac{16}{3}) = \frac{16}{9}$. Answer: B

19. $f_1(s) = 1$, $0 < s < 1$, $f_2(t) = 2t$, $0 < t < 1$.
Time to breakdown is $X = min(S,T)$.
The distribution function of X is
$F_X(x) = P[X \le x] = 1 - P[X > x] = 1 - P[(S > x) \cap (T > x)]$.
Since S and T are independent, we have
$P[(S > x) \cap (T > x)] = P[S > x] \cdot P[T > x]$.
$P[S > x] = \int_x^1 1\,ds = 1 - x$, and $P[T > x] = \int_x^1 2t\,dt = 1 - x^2$.
Therefore, $F_X(x) = 1 - (1-x)(1-x^2) = x + x^2 - x^3$, and the
pdf of X is $f_X(x) = F_X'(x) = 1 + 2x - 3x^2$.
The mean of X is $E[X] = \int_0^1 x \cdot f_X(x)\,dx = \int_0^1 (x + 2x^2 - 3x^3)\,dx = \frac{5}{12}$.
Note that since $X \ge 0$, the mean of X can be formulated as
$E[X] = \int_0^\infty [1 - F_X(x)]\,dx = \int_0^\infty S(x) \cdot T(x)\,dx = \int_0^1 (1-x)(1-x^2)\,dx = \frac{5}{12}$.
Answer: E

20. For a bivariate normal distribution, we have the following.
$E[X|Y=y] = \mu_X + \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot (y - \mu_Y) = 0 + \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot (y-1) \to 8 = \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot 8$.
$E[Y|X=x] = \mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X} \cdot (x - \mu_X) = 1 + \rho \cdot \frac{\sigma_Y}{\sigma_X} \cdot (x-0) \to 2 = 1 + \rho \cdot \frac{\sigma_Y}{\sigma_X} \cdot 9$.
Therefore, $\rho \cdot \frac{\sigma_X}{\sigma_Y} = 1$ and $\rho \cdot \frac{\sigma_Y}{\sigma_X} = \frac{1}{9}$, from which we get
$(\rho \cdot \frac{\sigma_X}{\sigma_Y})(\rho \cdot \frac{\sigma_Y}{\sigma_X}) = \rho^2 = (1)(\frac{1}{9}) = \frac{1}{9}$.
Since $8 = E[X|Y=9] = E[X] + \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot 8 > E[X] = 0$, it follows that $\rho > 0$ (because $\frac{\sigma_X}{\sigma_Y} > 0$ always). Therefore, ρ is the positive square root of $\frac{1}{9}$, which is $\frac{1}{3}$. Answer: D

21. $P[A \cap B'] = P[A] - P[A \cap B] \ge P[A] - P[B]$, since $P[B] \ge P[A \cap B]$.
Answer: C

22. The probability of first win occurring in week n is $(\frac{29}{30})^{n-1}(\frac{1}{30})$.

If week n is Smith's first week then his net gain is $10-n$ dollars (10 dollar prize minus n weeks with cost of 1 per week). Smith's expected net gain is

$\sum_{n=1}^{\infty}(10-n)(\frac{29}{30})^{n-1}(\frac{1}{30}) = 10\sum_{n=1}^{\infty}(\frac{29}{30})^{n-1}(\frac{1}{30}) - \sum_{n=1}^{\infty}n(\frac{29}{30})^{n-1}(\frac{1}{30})$

The week number in which the first win occurs is the form of the geometric distribution on the integers 1 , 2 , 3 , ... with $p=\frac{1}{30}$. The mean is $\frac{1}{p}=30=\sum_{n=1}^{\infty}n(\frac{29}{30})^{n-1}(\frac{1}{30})$.

Also, $\sum_{n=1}^{\infty}(\frac{29}{30})^{n-1}(\frac{1}{30})=1$. Smith's net gain is then $10(1)-30=-20$. Answer: A

23. $f_{Y|X}(y|x)=\frac{2}{x}$ for $0<y<\frac{x}{2}$. The joint density of X and Y is

$f(x,y)=f_{Y|X}(y|x)\cdot f_X(x)=\frac{2}{x}\cdot(2-2x)=\frac{4}{x}-4$, $0<x<1$, $0<y<\frac{x}{2}$.

$E[Y]=\int_0^1\int_0^{x/2}y\cdot(\frac{4}{x}-4)\,dy\,dx=\int_0^1(\frac{x}{2}-\frac{x^2}{2})\,dx=\frac{1}{12}$.

A second approach is somewhat more work. The region of joint density can also be described as $0<y<\frac{1}{2}$, $0<2y<x<1$.

The density function of the marginal distribution of Y is

$f_Y(y)=\int_{2y}^1(\frac{4}{x}-4)\,dx=(4\,ln\,x\,-\,4x)\Big|_{x=2y}^{x=1}=(-4)-(4\,ln\,2y\,-8y)$

$=8y-4\,ln\,2y\,-\,4$ for $0<y<\frac{1}{2}$.

The mean of Y is $E[Y]=\int_0^{1/2}y\cdot(8y-4\,ln\,2y\,-\,4)\,dy$

$=\int_0^{1/2}(8y^2-4y\,ln\,2y\,-4y)\,dy$. This approach requires finding the antiderivative of $y\,ln\,2y$. This can be done by integration by parts.

$\int y\,ln\,2y\;dy=\int ln\,2y\,d(\frac{1}{2}y^2)=\frac{1}{2}y^2\cdot(ln\,2y)-\int(\frac{1}{2}y^2)\,d(ln\,2y)$

$=\frac{1}{2}y^2\cdot(ln\,2y)-\int(\frac{1}{2}y^2)(\frac{1}{y})\,dy=\frac{1}{2}y^2\cdot(ln\,2y)-\frac{y}{4}$. Then,

$E[Y]=\int_0^{1/2}(8y^2-4y\,ln\,2y\,-4y)\,dy=\frac{8}{3}y^3-4(\frac{1}{2}y^2\cdot(ln\,2y)-\frac{y^2}{4})-2y^2\Big|_{y=0}^{y=1/2}$

$=\frac{1}{3}-4(\frac{1}{8}\,ln\,1\,-\frac{1}{16})-\frac{1}{2}=\frac{1}{12}$. Answer: E

24. The expected amount paid by the insurer is

$\int_d^1(x-d)\,f_X(x)\,dx=\int_d^1(x-d)\,dx=\frac{1}{2}(1-d)^2$.

Then, $\frac{1}{2}(1-d)^2=c\rightarrow d=1-\sqrt{2c}$. Answer: B

25. For $c > 0$ we have

$P[Y > c] = P[X - 2 > c|X > 2] = P[X > c + 2|X > 2] = \frac{P[(X>c+2)\cap(X>2)]}{P[X>2]}$.

$P[X > 2] = e^{-2}$ ($= \int_2^\infty e^{-x}\,dx$) and since $c > 0$,

$P[(X > c + 2) \cap (X > 2)] = P[X > c + 2] = e^{-(c+2)}$.

Then, $P[Y > c] = 1 - F_Y(c) = \frac{e^{-c-2}}{e^{-2}} = e^{-c}$, so that $f_Y(c) = F_Y'(c) = e^{-c}$.

Therefore, Y has an exponential distribution with mean 1. Answer: A

26. $E[X] = (1)(p) + (2 + 3 + 4 + 5)(\frac{1}{6}) + (6)(\frac{1}{3} - p) = \frac{13}{3} - 5p$.

$E[X^2] = (1^2)(p) + (2^2 + 3^2 + 4^2 + 5^2)(\frac{1}{6}) + (6^2)(\frac{1}{3} - p) = 21 - 35p$.

$Var[X] = E[X^2] - (E[X])^2 = 21 - 35p - (\frac{13}{3} - 5p)^2 = \frac{20}{9} + \frac{25}{3}p - 25p^2$.

$Var[X]$ is maximized where $\frac{d}{dp}(\frac{20}{9} + \frac{25}{3}p - 25p^2) = \frac{25}{3} - 50p = 0$,

so that $p = \frac{1}{6}$. Answer: C

27. $X =$ Toronto high temp. $N(-5, 16)$, $Y =$ Winnipeg high temp. $N(-10, 64)$.

High temperature difference is $X - Y \sim N(5, 80)$.

P[high temp. diff. $\leq 1] = P[|X - Y| \leq 1] = P[-1 \leq X - Y \leq 1]$

$= P[\frac{-6}{\sqrt{80}} \leq Z \leq \frac{-4}{\sqrt{80}}] = \Phi(-.45) - \Phi(-.67) = [1 - \Phi(.45)] - [1 - \Phi(.67)]$

$= \Phi(.67) - \Phi(.45) = .748 - .673 = .075$. Answer: C

28. Marginal distribution of X has pdf $f_X(x) = \int_0^1 f(x, y)\,dy = x + \frac{1}{2}$, $0 < x < 1$.

Marginal distribution of Y has pdf $f_Y(y) = \int_0^1 f(x, y)\,dx = y + \frac{1}{2}$, $0 < y < 1$,

or $f_Y(y) = \int_0^1 g(y, z)\,dz = y + \frac{1}{2}$, $0 < y < 1$.

Marginal distribution of Z has pdf $f_Z(z) = \int_0^1 g(y, z)\,dy = 3z^2$, $0 < z < 1$.

Only C has marginal distributions for X and Z that are correct. Answer: C

29. The number of strikes in the year has a geometric distribution with mean $\frac{.2}{.8} = .25$. The expected loss to the company in the year (before any insurance coverage) is $150,000(.25) = 37,500$. The expected amount paid by the insurance company in the year is
$0 \cdot P[N=0] + 100,000 \cdot P[N=1] + 200,000 \cdot P[N=2] + 300,000 \cdot P[N \geq 3]$
$= 100,000[(.8)(.2) + 2(.8)(.2)^2 + 3(1 - .8 - (.8)(.2) + (.8)(.2)^2] = 24,800$.
The expected loss to the company during the year not covered by insurance is
$37,500 - 24,800 = 12,700$. Answer: D

30. Using the transformations $U = e^{-(X+Y)} = u(X,Y)$ and $V = X = v(X,Y)$, we have inverse transformations $X = V = h(U,V)$ and $Y = -\ln U - V = k(U,V)$.
Applying the "Jacobian" method to find the joint distribution of transformed random variables U and V, we have
$g(u,v) = f(h(u,v), k(u,v)) \cdot |\frac{\partial h}{\partial u} \cdot \frac{\partial k}{\partial v} - \frac{\partial h}{\partial v} \cdot \frac{\partial k}{\partial u}|$
$= f(v, -\ln u - v) \cdot |0 \cdot (-1) - 1 \cdot (-\frac{1}{u})| = e^{-(v - \ln u - v)} \cdot \frac{1}{u} = 1$.
Since $-\ln U = X + Y > X = V$, the region of joint density is $0 < V < -\ln U$ and $0 < U < 1$. The marginal pdf of U is $f_U(u) = \int_0^{-\ln u} 1\,dv = -\ln u$, on the region $0 < u < 1$.
Answer: C

PRACTICE EXAM 6

1. A survey of the public determines the following about the "Lord of the Rings" trilogy (3 movies).

Have Seen #1	Have Seen #2	Have Seen #3	Percentage of Public
No	No	No	50%
Yes	?	?	35%
?	Yes	?	33%
?	?	Yes	31%
Yes	No	No	8%
Yes	Yes	No	4%
Yes	Yes	Yes	20%

Based on this information, determine the percentage of the public that has seen exactly one of the three "Lord of the Rings" movies.

A) 15 B) 17 C) 19 D) 21 E) 23

2. Suppose that events A and B are independent and suppose that $A \subseteq B$. Which of the following pairs of values is impossible?

A) $P(A) = \frac{1}{3}$ and $P(B) = 1$ B) $P(A) = \frac{1}{2}$ and $P(B) = 1$

C) $P(A) = 0$ and $P(B) = \frac{1}{2}$ D) $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$

E) $P(A) = 1$ and $P(B) = 1$

3. For a particular disease, it is found that 1% of the population will develop the disease and 2% of the population has a family history of having the disease. A genetic test is devised to predict whether or not the individual will develop the disease. For those with a family history of the disease, 20% of the time the genetic test predicts that the individual will develop the disease and for those with no family history of the disease, 1% of the time the genetic test predicts that the individual will develop the disease. The genetic test is not perfect, and individuals are followed to determine whether or not they actually develop the disease. It is found that for those who have a family history of the disease and for whom the genetic test predicts the disease will develop, 80% actually develop the disease. It is also found that for those who have a family history of the disease and for whom the genetic test does not predict the disease will develop, 10% actually develop the disease. Find the probability that someone with a family history of the disease will develop the disease.

A) .20 B) .22 C) .24 D) .26 E) .28

4. You are given the following information:

$$P(A|B) = .4 = P(A'|B') \text{ and } P(A) = .5$$

Find $P(B)$.

A) .4 B) .5 C) .6 D) .7 E) .8

5. If with each new birth, boys and girls are equally likely to be born, find the probability that in a family with three children, exactly one is a girl.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

6. Six digits from 2 , 3 , 4 , 5 , 6 , 7 , 8 are chosen and arranged in a row without replacement to create a 6-digit number. Find the probability that the resulting number is divisible by 2.

A) $\frac{5}{14}$ B) $\frac{3}{7}$ C) $\frac{1}{2}$ D) $\frac{4}{7}$ E) $\frac{9}{14}$

7. A bag contains 3 red balls, 2 white balls and 3 blue balls. Three balls are selected randomly from the bag **with replacement**. Given that no blue ball has been selected, calculate the probability that the number of red balls exceeds the number of white balls chosen.

A) $\frac{3}{8}$ B) $\frac{3}{5}$ C) $\frac{7}{10}$ D) $\frac{81}{512}$ E) $\frac{81}{125}$

8. A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions (each question has the same chance of being selected). To pass, he needs to answer at least two questions correctly. What is the probability that the student will pass the examination if he only knows the answers to 90 questions on the list?

A) Less than .96 B) At least .96 but less than .97 C) At least .97 but less than .98
D) At least .98 but less than .99 E) At least .99

9. A random variable X has a probability mass of 0.2 at $X = 0$ and a probability mass of 0.1 at $X = 1$. For all other values, X has the following density function:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 2x & 1 < x < c \\ 0 & x \geq c \end{cases}$$, where c is a constant.

Find $P(X < 1 | X > .5)$

A) Less than .6 B) At least .6 but less than .7 C) At least .7 but less than .8
D) At least .8 but less than .9 E) At least .9

10. An ordinary fair die is tossed independently until two consecutive tosses result in the same face turning up. X denotes the toss number on which this happens, so $X \geq 2$. Which of the following is $F(x)$, the CDF of X for $x \geq 2$?

A) $1 - (\frac{5}{6})^{x-1}$ B) $1 - (\frac{5}{6})^{x}$ C) $1 - (\frac{1}{6})^{x-1}$ D) $1 - (\frac{1}{6})^{x}$ E) $(\frac{5}{6})^{x}$

11. A discrete random variable X has the following probability function

x :	10	20	30	40	50
$f(x)$:	0.1	0.1	0.4	0.3	0.1

Denote by μ_X and σ_X the mean and the standard deviation of X. Find $P(|X - \mu_X| \leq \sigma_X)$.

A) 1 B) 0.8 C) 0.7 D) 0.5 E) 0.4

12. The continuous random variable X has pdf $f(x) = \frac{(k+1)x^k}{c^{k+1}}$ for $0 < x < c$, where $k > 0$.
The coefficient of variation of a random variable is defined to be $\frac{\sqrt{Variance}}{Mean}$.
Find the coefficient of variation of X.

A) $\frac{1}{\sqrt{(k+1)(k+2)}}$ B) $\frac{1}{\sqrt{(k+2)(k+3)}}$ C) $\frac{1}{\sqrt{(k+1)(k+3)}}$ D) $\frac{1}{k+1}$ E) $\frac{1}{k+3}$

13. X and Y are discrete random variables on the integers $\{0, 1, 2\}$, with moment generating functions $M_X(t)$ and $M_Y(t)$. You are given the following:
$M_X(t) + M_Y(t) = \frac{3}{4} + \frac{3}{4}e^t + \frac{1}{2}e^{2t}$ and $M_X(t) - M_Y(t) = \frac{1}{4} - \frac{1}{4}e^t$.
Find $P(X = 1)$.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

14. X is a continuous random variable for the density function $f(x)=\frac{|x|}{10}$ for $-4\leq x\leq 2$, and $f(x)=0$ otherwise. Find $|E(X)-m|$, where m is the median of X.

A) Less than .2 B) At least .2 but less than .4 C) At least .4 but less than .6

D) At least .6 but less than .8 E) At least .8

15. Smith plays a gambling game in which his probability of winning on any given play of the game is .4. If Smith bets 1 and wins, the amount he wins is 1, and if he loses, then he loses the amount of 1 that he bet. Smith devises the following strategy. If he loses a game, he doubles the amount that he bets on the next play of the game. He continues this strategy of doubling after each loss until he wins for the first time. He stops as soon as he wins for the first time. Smith has a limited amount of money to gamble, say \$$c$. If he loses all \$$c$ he goes broke and stops playing the game. Find the minimum amount c that Smith needs in order for him to have a probability of at least .95 of eventually winning before he goes broke.

A) 7 B) 15 C) 31 D) 63 E) 127

16. A production process for electronic components has a followup inspection procedure. Inspectors assign a rating of high, medium or low to each component inspected. Long run inspection data have yielded the following probabilities for component ratings:

$P(\text{high})=.5$, $P(\text{medium})=.4$, $P(\text{low})=.1$.

Find the probability that in the next batch of 5 components inspected, at least 3 are rated high, and at most 1 is rated low.

A) Less than .10 B) At least .10 but less than .20 C) At least .20 but less than .30

D) At least .30 but less than .40 E) At least .40

17. A teacher in a high school class of 25 students must pick 5 students from the class for a school board math test. The students must be chosen randomly from the class. According to the teacher's assessment, there are 3 exceptional math students in the class and all the rest are average. Find the probability that at least 2 of the exceptional students are chosen for the test.

A) .01 B) .03 C) .05 D) .07 E) .09

18. Smith is a quality control analyst who uses the exponential distribution with a mean of 10 years as the model for the exact time until failure for a particular machine. Smith is really only interested in the integer number of years, say X, until the machine fails, so if failure is within the first year, Smith regards that as 0 (integer) years until failure, and if the machine does not fail during the first year but fails in the second year, the Smith regards that as 1 (integer) year until failure, etc. Smith's colleague Jones, who is also a quality control analyst reviews Smith's model for the random variable X and has two comments:

I. X has a geometric distribution.

II. The mean of X is 10.

Determine which, if any, of the statements made by Jones are true?

A) Neither are true B) Only Statement I is true C) Only Statement II is true

D) Both are true E) None of A, B, C or D is correct

19. X_1, X_2, X_3, X_4 and X_5 are independent normal random variables with $E(X_i) = Var(X_i) = i$ for $i = 1, 2, 3, 4, 5$.

We define Y to be $Y = \frac{1}{5}\sum_{i=1}^{5} X_i$. What is the 50th percentile of $Y - 3$?

A) 0 B) 1.645 C) 3.84 D) 11.07 E) 19.21

20. According to the definition of the beta distribution X on the interval (0,1) with integer parameters $a \geq 1$ and $b \geq 1$, the pdf is $f(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} \cdot x^{a-1}(1-x)^{b-1}$.

Which of the following statements are true?

I. If $a = b$ then $E(X) = \frac{1}{2}$.

II. If $a = b$ then $Var(X) = \frac{1}{8a+1}$.

III. As k increases, $E(X^k)$ increases.

A) I only B) II only C) III only D) All but II E) All but III

21. The joint pdf of X and Y is $f(x, y) = kx^{-3}e^{-y/3}$ for $1 < x < \infty$ and $1 < y < \infty$.

Find $E(X)$ (the mean of the marginal distribution of X).

A) 4 B) 2 C) 1 D) $\frac{1}{2}$ E) $\frac{1}{4}$

22. Smith and Jones are financial analysts who enter a stock picking contest. Smith picks the stock of Company A and Jones picks the stock of Company B. Over the next week, the gain in the price of stock A will be X and the gain in the price of stock B will be Y, where X and Y have the joint density $f(x,y)=\frac{2}{3}(x+2y)$ for $0<x<1$ and $0<y<1$.
If the gain in Smith's stock for the week is greater than the gain in Jones' stock, Jones will pay Smith $1000. How much should Smith pay Jones if the gain in Smith's stock is less than that of Jones' stock in order that Smith's expected return in this contest is 0?
A) 400 B) 800 C) 1200 D) 1600 E) 2000

23. X and Y are normal random variables with means μ_X and μ_Y, standard deviations σ_X and σ_Y, and correlation coefficient ρ. $F_X(t)$ and $F_Y(t)$ denote the cdf's of X and Y respectively.
If $\sigma_X = 2\sigma_Y$, for what values of t is it true that $F_X(t) \geq F_Y(t)$?
A) $t \leq 2\mu_Y - \mu_X$ B) $t \leq 2\mu_X - \mu_Y$ C) $t \leq \rho(\mu_X + \mu_Y)$
D) $t \leq \rho(\mu_X - \mu_Y)$ E) All real numbers t

24. X and Y are continuous random variables with pdf $f(x,y)=2$ for $0 \leq x \leq y \leq 1$, and $f(x,y)=0$ otherwise. Find the conditional expectation of Y given $X=x$.
A) $\frac{1}{2}$ B) $\frac{x}{2}$ C) $\frac{x+1}{2}$ D) $\frac{1-x}{2}$ E) x

25. $X_1, X_2, X_3, ...$ is a sequence of independent random variables, each with mean 0 and variance 1. We define Y_k to be $X_1 + X_2 + \cdots + X_k$.
If $k<j$, what is the coefficient of correlation between Y_k and Y_j ?
A) kj B) $\sqrt{kj}$ B) $\frac{k}{j}$ D) $\sqrt{\frac{k}{j}}$ E) $\frac{j}{k}$

26. X has an exponential distribution with mean 1 and $Y = X^2 - 1$. Find $F_Y(3)$.
A) $\frac{e^{-3}}{3}$ B) $\frac{e^{-2}}{4}$ C) $1-2e^{-2}$ D) $1-e^{-2}$ E) $1-e^{-3}$

27. A financial analyst uses the following model for the daily change in the price of a certain stock: $ln(\frac{X_{k+1}}{X_k})$ has a distribution with a mean of .01 and a variance of .0009 ,
where X_i is the stock closing price for trading day i. Assuming that the daily changes in price are independent from one day to the next, and assuming that the stock price closed at 1 on day 1, use the normal approximation to find the probability that the stock price is at least 4 at the close of trading day 101.

A) .05 B) .1 C) .5 D) .9 E) .95

28. An insurance policy has a deductible of 1 and pays a maximum of 1. The underlying loss random variable being insured by the policy has an exponential distribution with a mean of 1. Find the expected amount paid by the insurer on this policy.

A) $2e^{-2}$ B) e^{-1} C) $e^{-1} - e^{-2}$ D) $e^{-1} - 2e^{-2}$ E) $2(e^{-1} - e^{-2})$

29. X has a uniform distribution on the interval $(0, 2)$, and $Y = max\{X, 1\}$.
Find $Var(Y)$.

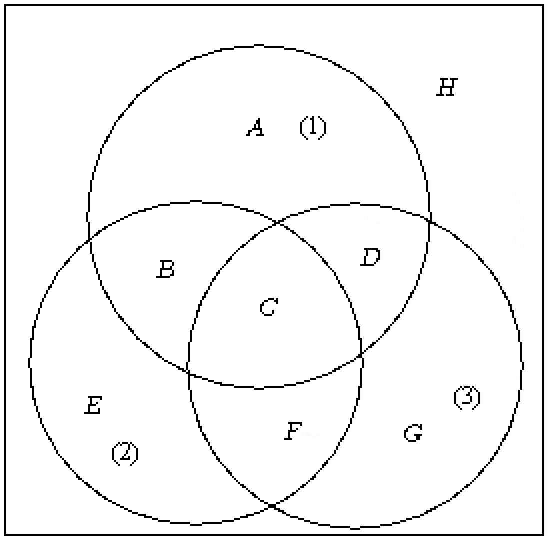
A) .05 B) .10 C) .15 D) .20 E) .25

30. X has a distribution with the following cdf $F(x) = 1 - e^{-(x/\theta)^\tau}$,
where $\tau > 0$ and $\theta > 0$. The random variable Y is defined to be $Y = g(X)$.
Which of the following transformations results in a distribution of Y which is exponential with a mean of 1?

A) $g(x) = x^\tau$ B) $g(x) = x^{1/\tau}$ C) $g(x) = (\frac{x}{\theta})^\tau$ D) $g(x) = (\frac{x}{\theta})^{1/\tau}$ E) $g(x) = \frac{\theta}{x^\tau}$

PRACTICE EXAM 6 - SOLUTIONS

1. We can represent the events in the following diagram:



The top circle, $A \cup B \cup C \cup D$ represents the event of having seen #1 of the movie series, the lower left circle, $E \cup B \cup C \cup F$ represents the event of having seen #2 of the movie series, the lower right circle, $G \cup F \cup C \cup D$ represents the event of having seen #3 of the movie series, and H represents the event of having seen none of the three movies.

From the given information, we know that the percentage for event H is $h = 50$.
The second line of the information table indicates that 35% of the public has seen movie #1 but we don't know about movies #2 and #3 for this group. This is interpreted as the percentage for $A \cup B \cup C \cup D$ is $a + b + c + d = 35$.

Similarly, the percentage for $E \cup B \cup C \cup F$ is $e + b + c + f = 33$, and
the percentage for $G \cup F \cup C \cup D$ is $g + f + c + d = 31$.
The 5th line of the table indicates that 8% have seen movie #1 and not movies #2 or #3.
Therefore, the percentage for event A is $a = 8$.
Event B is the event of having seen both #1 and #2 but not #3 and this has percentage $b = 4$,
and event C is the event of have seen all three, and this has percentage $c = 20$.

1. continued

The event of having seen exactly one of the three movies is the combination $A \cup E \cup G$.

This will be $a+e+g$.

We know that $a+b+c+d+e+f+g+h=100$ percent, since everyone either sees a movie or doesn't. This leads to the following 8 equations:

$h=50$ (1) , $a+b+c+d=35$ (2) , $e+b+c+f=33$ (3) , $g+f+c+d=31$ (4) ,
$a=8$ (5) , $b=4$ (6) , $c=20$ (7) , $a+b+c+d+e+f+g+h=1$ (8) .

From equations (3), (6) and (7) we get $e+f=9$ (9) .

From equations (1), (2) and (8) we get $e+f+g=15$ (10) .

From equations (9) and (10) we get $g=6$ (11) .

From equations (2), (5) and (6) we get $c+d=23$ (12) .

From equations (11), (12) and (4) we get $f=2$ (13) .

From equations (9) and (13) we get $e=7$.

Then $a+e+g=8+7+6=21$ is the percentage that has seen exactly one of the three movies.

Once we have determined the individual values of a,b,c,d,e,f,g,h , we can find the percentage for any combination. For instance, the percentage of people who have seen #1 and #3 but not #2 is $d=3$. Answer: D

2. Since A and B are independent, we must have $P(A \cap B) = P(A) \cdot P(B)$.

Since $A \subseteq B$, it follows that $A \cap B = A$.

Therefore, we must have $P(A) = P(A) \cdot P(B)$.

The only way this can be true is if $P(A)$ is 0 or 1 , or if $P(B)$ is 1.

Only D does not satisfy one of these conditions. Answer: D

3. We denote events as follows:

F - an individual has a family history of the disease

T - the genetic test indicates that an individual will develop the disease

D - an individual will develop the disease

We are given the probabilities $P(D) = .01$ and $P(F) = .02$.

The language "for those with a family history of the disease, 20% of the time the genetic test predicts that the individual will develop the disease" describes the conditional probability

$P($the genetic test indicates that the individual will develop

the disease$|$the individual has a family history of the disease$) = P(T|F) = .20$.

In a similar way $P(T|F') = .01$, $P(D|T \cap F) = .80$ and $P(D|T' \cap F) = .10$.

We are asked to find $P(D|F) = \frac{P(D \cap F)}{P(F)}$.

We are given $P(F) = .02$ so that $P(F') = .98$.

Then, since $.20 = P(T|F) = \frac{P(T \cap F)}{P(F)}$,

we get $P(T \cap F) = P(T|F) \cdot P(F) = (.2)(.02) = .004$.

Then, since $P(F) = P(T \cap F) + P(T' \cap F)$ it follows that $P(T' \cap F) = .02 - .004 = .016$.

Also, $P(D \cap T \cap F) = P(D|T \cap F) \cdot P(T \cap F) = (.80)(.004) = .0032$.

Similarly $P(D \cap T' \cap F) = P(D|T' \cap F) \cdot P(T' \cap F) = (.10)(.016) = .0016$.

Then $P(D \cap F) = P(D \cap T \cap F) + P(D \cap T' \cap F) = .0032 + .0016 = .0048$.

Finally, $P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{.0048}{.02} = .24$. Answer: C

4. $P(A \cap B) = P(A|B) \cdot P(B) = .4 \times P(B)$.

$P(A|B') = 1 - P(A'|B') = .6$ and $P(A \cap B') = P(A|B') \cdot P(B') = .6 \times [1 - P(B)]$.

Then

$.5 = P(A) = P(A \cap B) + P(A \cap B') = .4 \times P(B) + .6 \times [1 - P(B)] = .6 - .2 \times P(B)$.

Solving for $P(B)$ results in $P(B) = .5$. Answer: B

5. There are 8 possible orderings of births in a family of three children:

GGG , GGB , GBG , GBB , BGG , BGB , BBG , BBB .

\# of these orderings result in a family with exactly one girl: GBB , BGB , BBG .

Each of the orderings has a probability of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ of occurring, so the probability of exactly one girl is $\frac{3}{8}$. Answer: C

6. To find the probability that a certain type of combination or arrangement occurs, the probability is usually formulated as $\frac{\text{number of combinations or arrangement of the specific type required}}{\text{total number of all combinations or arrangements}}$.
For these problems, the denominator is the total number of all 6 digit numbers that can be created by choosing 6 digits without replacement from 2 , 3 , 4 , 5 , 6 , 7 , 8. The total number of 6-digit numbers is $7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5,040$ since the first digit can be any one of the 7 integers, the second digit can be any one of the remaining 6 integers, etc.

The number is even if it ends in 2 , 4 , 6 or 8 . For each of these 4 cases, there are $6 \times 5 \times 4 \times 3 \times 2 = 720$ arrangements of the first 5 digits in the number, since the other 5 digits are chosen from the 6 remaining integers. The numerator of the probability is $4 \times 720 = 2880$, and the probability is $\frac{2880}{5040} = \frac{4}{7}$.
An alternative solution is to note that there are 7 possible equally likely final digits for the 6-digit number, and 4 of them make the number even. The probability is $\frac{4}{7}$. Answer: D

7. We wish to find $P(A|B) = \frac{P(A \cap B)}{P(B)}$, where
$A =$ more red than white are chosen, and $B =$ no blue are chosen.
Out of 8 balls in the bag, 5 are not blue, so for each of the three trials, the probability of picking a ball that is not blue is $\frac{5}{8}$; since the ball chosen is always replaced, the probability remains the same on each subsequent choice. Then, $P(B) = \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} = \frac{125}{512}$.
On any given trial, the probability of a red ball being chosen is $\frac{3}{8}$ and the probability of a white ball being chosen is $\frac{2}{8}$.
$A \cap B =$ no blue are chosen and more red than white are chosen.
The following sequences of choices result in $A \cap B$ occurring: RRR , RRW , RWR , WRR.
The probabilities of these sequences is
$P(RRR) = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{27}{512}$, $P(RRW) = \frac{3}{8} \times \frac{3}{8} \times \frac{2}{8} = \frac{18}{512}$,
$P(RWR) = \frac{3}{8} \times \frac{2}{8} \times \frac{3}{8} = \frac{18}{512}$, $P(WRR) = \frac{2}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{18}{512}$.
Then, $P(A \cap B) = \frac{27+18+18+18}{512} = \frac{81}{512}$.
Finally, $P(A|B) = \frac{81/512}{125/512} = \frac{81}{125}$. Answer: E

8. The student will pass if at least two of the three questions chosen come from the 90 questions that the student knows. The number of ways of choosing 3 questions from 100 is
$\binom{100}{3} = \frac{100!}{97!\cdot 3!} = \frac{100\times 99\times 98}{6} = 161,700$.
The student will pass if either
(i) all three questions are chosen from the 90 he knows the answers to, or
(ii) exactly two of the three questions are chosen from the 90 he knows the answers to and the other is chosen from the other 10.
The number of ways of (i) occurring is $\binom{90}{3} = \frac{90!}{87!\cdot 3!} = \frac{90\times 89\times 88}{6} = 117,480$, and the number of ways of (ii) occurring is $\binom{90}{2} \times \binom{10}{1} = \frac{90!}{88!\cdot 2!} \times \frac{10!}{9!\cdot 1!} = \frac{90\times 89}{2} \times 10 = 40,050$.
The probability that the student gets at least 2 of the 3 questions right is $\frac{117,480+40,050}{161,700} = .974$.
Answer: C

9. $P(.5 < X < 1) = \int_{.5}^{1} x\,dx \;=\; .375$.
$P(X < 1|X > .5) = \frac{P(.5<X<1)}{P(X>.5)}$.
Although we have not determined the value of c, we know that
$P(X > .5) = 1 - P(X \leq .5) = 1 - [P(X = 0) + P(0 < X \leq .5)]$
$= 1 - [.2 + \int_0^{.5} x\,dx] \;= 1 - .325 = .675$.
Then $P(X < 1|X > .5) = \frac{P(.5<X<1)}{P(X>.5)} = \frac{.375}{.675} = .556$. Answer: A

10. $F(x) = P(X \leq x)$.
$P(X = 2) = \frac{1}{6}$, since $X = 2$ if the second toss is the same as the first, and there is a $\frac{1}{6}$ chance of that. There is a $\frac{5}{6}$ chance that the 2nd toss is not the same as the first, and then a $\frac{1}{6}$ chance that the 3rd toss is the same as the 2nd, so $P(X = 3) = \frac{5}{6} \times \frac{1}{6}$. There is a $\frac{5}{6}$ chance that the 3rd toss is not the same as the 2nd, and then a $\frac{1}{6}$ chance that the 4th toss is the same as the 3rd, so
$P(X = 4) = (\frac{5}{6})^2 \times \frac{1}{6}$. Continuing in this way, we see that $P(X = n) = (\frac{5}{6})^{n-2} \times \frac{1}{6}$.
Then $F(x) = \sum_{n=2}^{x} P(X = n) = \sum_{n=2}^{x} (\frac{5}{6})^{n-2} \times \frac{1}{6} = [1 + \frac{5}{6} + (\frac{5}{6})^2 + \cdots + (\frac{5}{6})^{x-2}] \times \frac{1}{6}$.
We use the geometric series expression $1 + a + a^2 + \cdots + a^k = \frac{1-a^{k+1}}{1-a}$, to get
$F(x) = \frac{1-(\frac{5}{6})^{x-1}}{1-\frac{5}{6}} \times \frac{1}{6} = 1 - (\frac{5}{6})^{x-1}$. Answer: A

11. $\mu_X = (10\times .1)+(20\times .1)+(30\times .4)+(40\times .3)+(50\times .1) = 32$.
$\sigma_X^2 = E[(X-\mu_X)^2]$
$= (10-32)^2(.1)+(20-32)^2(.1)+(30-32)^2(.4)+(40-32)^2(.3)+(50-32)^2(.1)$
$= 116$
Alternatively, $\sigma_X^2 = E(X^2) - (E[X])^2$.
$E(X^2) = (10^2\times .1)+(20^2\times .1)+(30^2\times .4)+(40^2\times .3)+(50^2\times .1) = 1140$,
so that $\sigma_X^2 = 1140 - 32^2 = 116$, and $\sigma_X = \sqrt{116} = 10.77$.
Then, $P(|X-\mu_X|\le\sigma_X) = P(|X-32|\le 10.77) = P(-10.77\le X-32\le 10.77)$
$= P(21.23\le X\le 42.77) = P(X = 30 \text{ or } 40) = 0.7$. Answer: C

12. $E(X) = \int_0^c x\cdot f(x)\,dx = \int_0^c \frac{(k+1)x^{k+1}}{c^{k+1}}\,dx = \frac{k+1}{k+2}\cdot c$.
$E(X^2) = \int_0^c x^2\cdot f(x)\,dx = \int_0^c \frac{(k+1)x^{k+2}}{c^{k+1}}\,dx = \frac{k+1}{k+3}\cdot c^2$.
$Var(X) = E(X^2)-(E[X])^2 = \frac{k+1}{k+3}\cdot c^2 - (\frac{k+1}{k+2}\cdot c)^2 = [\frac{k+1}{k+3} - (\frac{k+1}{k+2})^2]c^2$
$= \frac{k+1}{(k+3)(k+2)^2}\cdot c^2$.
The coefficient of variation of X is $\sqrt{\frac{k+1}{(k+3)(k+2)^2}\cdot c^2}\Big/(\frac{k+1}{k+2}\cdot c) = \frac{1}{\sqrt{(k+1)(k+3)}}$.
Answer: C

13. We denote the probability function of X by $p_0^X = P(X=0)$, $p_1^X = P(X=1)$, and $p_2^X = P(X=2)$. With similar notation for Y.
Then $M_X(t) = E[e^{tX}] = e^0\cdot p_0^X + e^t\cdot p_1^X + e^{2t}\cdot p_2^X$ and
$M_Y(t) = E[e^{tY}] = e^0\cdot p_0^Y + e^t\cdot p_1^Y + e^{2t}\cdot p_2^Y$.
Then $M_X(t)+M_Y(t) = p_0^X + e^t\cdot p_1^X + e^{2t}\cdot p_2^X + p_0^Y + e^t\cdot p_1^Y + e^{2t}\cdot p_2^Y$
$= p_0^X + p_0^Y + (p_1^X + p_1^Y)e^t + (p_2^X + p_2^Y)e^{2t}$,
and it follows that $p_0^X + p_0^Y = \frac{3}{4}$, $p_1^X + p_1^Y = \frac{3}{4}$ and $p_2^X + p_2^Y = \frac{1}{2}$.
In a similar way, we have
$M_X(t)-M_Y(t) = p_0^X + e^t\cdot p_1^X + e^{2t}\cdot p_2^X - p_0^Y - e^t\cdot p_1^Y - e^{2t}\cdot p_2^Y$
$= p_0^X - p_0^Y + (p_1^X - p_1^Y)e^t + (p_2^X - p_2^Y)e^{2t}$,
and it follows that $p_0^X - p_0^Y = \frac{1}{4}$, $p_1^X - p_1^Y = -\frac{1}{4}$ and $p_2^X - p_2^Y = 0$.
From these equations, we see that $p_1^X + p_1^Y + p_1^X - p_1^Y = 2\,p_1^X = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$,
and therefore $P(X=1) = p_1^X = \frac{1}{4}$. Answer: B

14. $E(X) = \int_{-4}^{2} x\,f(x)\,dx = \int_{-4}^{0} x \cdot (\frac{-x}{10})\,dx + \int_{0}^{2} x \cdot (\frac{x}{10})\,dx = -\frac{28}{15}$.
The median m satisfies the relationship $\int_{-4}^{m} f(x)\,dx = \frac{1}{2}$.
We know that $\int_{-4}^{0} f(x)\,dx = \int_{-4}^{0} (\frac{-x}{10})\,dx = \frac{4}{5}$, so we must have $-4 \leq m \leq 0$.
Then $\int_{-4}^{m} (\frac{-x}{10})\,dx = \frac{16-m^2}{20}$. Setting this equal to $\frac{1}{2}$, we get $m = -\sqrt{6}$ (we take the negative square root because we know that the median is between -4 and 0).
Then $|E(X) - m| = |-\frac{28}{15} - (-\sqrt{6})| = .583$. Answer: C

15. We can approach this problem by trial and error.
If Smith has $c = 1$, then Smith will go broke if he loses the first game (prob. .6), so there is a .4 chance that Smith wins before going broke.
If Smith has $c = 1 + 2 = 3$, then Smith will go broke if he loses the first two games (prob. $.6 \times .6 = .36$), so there is a .64 chance that Smith wins before going broke.
If Smith has $c = 1 + 2 + 4 = 7$, then Smith will go broke if he loses the first three games (prob. $(.6)^3 = .216$), so there is a .784 chance that Smith wins before going broke.
If Smith has $c = 1 + 2 + 4 + 8 = 15$, then Smith will go broke if he loses the first four games (prob. $(.6)^4 = .1296$), so there is a .8704 chance that Smith wins before going broke.
If Smith has $c = 1 + 2 + 4 + 8 + 16 = 31$, then Smith will go broke if he loses the first five games (prob. $(.6)^5 = .0778$), so there is a .9222 chance that Smith wins before going broke.
If Smith has $c = 1 + 2 + 4 + 8 + 16 + 32 = 63$, then Smith will go broke if he loses the first six games (prob. $(.6)^6 = .0467$), so there is a .9533 chance that Smith wins before going broke.
Answer: D

16. We can use the multinomial distribution. For a batch of n components tested, the probability that a are rated high, b are rated medium and c are rated low is $\frac{n!}{a!\,b!\,c!}\,(.5)^a(.4)^b(.1)^c$.
In a batch of 5, in order to have at least 3 high and at most 1 low, the following combinations are possible 5H , 4H and 1M , 4H and 1L , 3H and 1M and 1L .
The probabilities of these combinations are
$P(5H) = \frac{5!}{5!\,0!\,0!}\,(.5)^5(.4)^0(.1)^0 = .03125$,
$P(4H \text{ and } 1M) = \frac{5!}{4!\,1!\,0!}\,(.5)^4(.4)^1(.1)^0 = .125$,
$P(4H \text{ and } 1L) = \frac{5!}{4!\,0!\,1!}\,(.5)^4(.4)^0(.1)^1 = .03125$,
$P(3H \text{ and } 1M \text{ and } 1L) = \frac{5!}{3!\,1!\,1!}\,(.5)^3(.4)^1(.1)^1 = .1$.
$P(3H \text{ and } 2M) = \frac{5!}{3!\,2!\,0!}\,(.5)^3(.4)^2(.1)^0 = .2$.
The total probability is $.03125 + .125 + .03125 + .1 + .2 = .4875$. Answer: E

17. The number of exceptional students chosen, say X, can be described as having a hypergeometric distribution. For $k = 0, 1, 2, 3$ we have $P(X = k) = \frac{\binom{3}{k}\binom{22}{5-k}}{\binom{25}{5}}$

(k of the 3 exceptional students are chosen, and $22 - k$ of the average students are chosen).
Then the probability that at least 2 exceptional students are chosen for the test is
$P(X = 2 \text{ or } 3) = \frac{\binom{3}{2}\binom{22}{3}}{\binom{25}{5}} + \frac{\binom{3}{3}\binom{22}{2}}{\binom{25}{5}} = \frac{(3)(1540)+(1)(231)}{53{,}130} = .0913$. Answer: E

18. We denote by W the exponential distribution with a mean of 10,
so the pdf of W is $f_W(w) = \frac{e^{-w/10}}{10}$.
Then the distribution of X can be found from the distribution of W.
X is a discrete integer-valued random variable ≥ 0.

$X = 0$ if $0 < W \leq 1$ (if failure is in the first year), and the probability is
$P(X = 0) = P(0 < W \leq 1) = \int_0^1 f_W(w)\,dw = \int_0^1 .1e^{-.1w}\,dw = 1 - e^{-.1} = .095163$.

$X = 1$ if $1 < W \leq 2$ (if failure is in the second year), and the probability is
$P(X = 1) = P(1 < W \leq 2) = \int_1^2 .1e^{-.1w}\,dw = e^{-.1} - e^{-.2} = .086107$.

$X = k$ if $k < W \leq k + 1$ (if failure is in the first year), and the probability is
$P(X = k) = P(k < W \leq k + 1) = \int_k^{k+1} .1e^{-.1w}\,dw$
$= e^{-.1k} - e^{-.1(k+1)} = (e^{-.1})^k(1 - e^{-.1})$.

The commonly used definition of the geometric distribution is as follows.
Suppose that $0 < p < 1$ and suppose that Z is an integer-valued random variable ≥ 0
with probability function $P(Z = j) = (1 - p)^k \cdot p$ for $j = 0, 1, 2, \ldots$.
Z is said to have a geometric distribution with parameter p, and the mean of Z is
$E[Z] = \frac{1-p}{p}$ (and the variance of Z is $Var[Z] = \frac{1-p}{p^2}$).

From the probability function of X described above, if we let $p = 1 - e^{-.1}$, then
$1 - p = e^{-.1}$ and $P(X = k) = (e^{-.1})^k(1 - e^{-.1}) = (1 - p)^k \cdot p$.
We see that X has a geometric distribution, with $p = 1 - e^{-.1}$.
The mean of X is $\frac{1-p}{p} = \frac{e^{-.1}}{1-e^{-.1}} = 9.508$.
Jones is correct about the distribution of X being geometric, but is wrong about the mean of X.
Answer: B

19. The sum of normal random variables is normal, and a constant multiple of a normal random variable is normal. Therefore Y is normal and the mean of Y is $\frac{1}{5}\sum_{i=1}^{5} E(X_i) = \frac{1}{5}(1+2+3+4+5) = 3$. Then the mean of $Y-3$ is 0.
The mean and median (50th percentile) of a normal random variable are the same, so that the median of $Y-3$ is equal to 0. Answer: A

20. Since $f(x)$ is a pdf, it follows that $\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}$,
and this is valid for any integers a and b.
Then $E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}\;x^a(1-x)^{b-1}dx$
$= \frac{(a+b-1)!}{(a-1)!(b-1)!}\int_0^1 x^a(1-x)^{b-1}dx = \frac{(a+b-1)!}{(a-1)!(b-1)!}\times\frac{a!(b-1)!}{(a+b)!} = \frac{a}{a+b}$
(this follows by using $a+1$ instead of a for the pdf).
Therefore, if $a=b$ then $E(X) = \frac{1}{2}$, so statement I is true.

$E(X^2) = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}\;x^{a+1}(1-x)^{b-1}dx = \frac{(a+b-1)!}{(a-1)!(b-1)!}\times\frac{(a+1)!(b-1)!}{(a+b+1)!}$
$= \frac{a(a+1)}{(a+b)(a+b+1)}$. If $a=b$ then $E(X^2) = \frac{a(a+1)}{(2a)(2a+1)} = \frac{a+1}{2(2a+1)}$.
Then with $a=b$ we have $Var(X) = \frac{a+1}{2(2a+1)} - (\frac{1}{2})^2 = \frac{a+1}{4a+2} - \frac{1}{4} = \frac{2}{4(4a+2)} = \frac{1}{8a+4}$.
Statement II is false.

$E(X^k) = \int_0^1 x^k\,f(x)\,dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}\;x^{a+k-1}(1-x)^{b-1}dx$.
Since $0 < x < 1$, as k increases x is raised to a higher power, so x^{a+k-1} becomes smaller numerically, and so does the integral. Statement III is false.
Answer: A

21. In order to be a properly defined pdf we must have $\int_1^\infty\int_1^\infty f(x,y)\,dy\,dx = 1$.
Therefore
$\int_1^\infty\int_1^\infty kx^{-3}e^{-y/3}\,dy\,dx = k\int_1^\infty 3e^{-1/3}x^{-3}dx = k\cdot 3e^{-1/3}\cdot\frac{1}{2} = \frac{3}{2}ke^{-1/3} = 1$,
and it follows that $k = \frac{2e^{1/3}}{3}$.
The marginal density of X is
$f_X(x) = \int_1^\infty f(x,y)\,dy = \int_1^\infty \frac{2e^{1/3}}{3}\cdot x^{-3}e^{-y/3}\,dy = \frac{2e^{1/3}}{3}\cdot x^{-3}\cdot 3e^{-1/3} = 2x^{-3}$
for $1 < x < \infty$.
Then $E(X) = \int_1^\infty x\,f_X(x)\,dx = \int_1^\infty x\cdot 2x^{-3}\,dx = \int_1^\infty 2x^{-2}\,dx = 2$.
Answer: B

22. Smith wins 1000 if $X > Y$. The probability of this is $\int_0^1 \int_0^x \frac{2}{3}(x+2y)\,dy\,dx = \frac{4}{9}$.
Therefore, the probability that $X < Y$ is $\frac{5}{9}$.
Suppose that Smith pays Jones \$$C$ if $X < Y$.
Then Smith's expected return is $1000(\frac{4}{9}) - C(\frac{5}{9}) = \frac{4000-5C}{9}$.
In order for this to be 0 we must have $C = 800$. Answer: B

23. Standardizing X and Y, we have $\frac{X-\mu_X}{\sigma_X}$ has a standard normal distribution, as does $\frac{Y-\mu_Y}{\sigma_Y}$.
Then $F_X(t) = P(X \le t) = P(\frac{X-\mu_X}{\sigma_X} \le \frac{t-\mu_X}{\sigma_X}) = \Phi(\frac{t-\mu_X}{\sigma_X})$, and similarly, $F_Y(t) = \Phi(\frac{t-\mu_Y}{\sigma_Y})$.
Then $F_X(t) \ge F_Y(t)$ if $\Phi(\frac{t-\mu_X}{\sigma_X}) \ge \Phi(\frac{t-\mu_Y}{\sigma_Y})$, which occurs if $\frac{t-\mu_X}{\sigma_X} \ge \frac{t-\mu_Y}{\sigma_Y}$.
This inequality can be written as $t - \mu_X \ge (t-\mu_Y) \cdot \frac{\sigma_X}{\sigma_Y} = 2(t-\mu_Y)$
(since we were given that $\sigma_X = 2\sigma_Y$).
The inequality can be rewritten as $t \le 2\mu_Y - \mu_X$. Answer: A

24. The conditional density of Y given $X = x$ is $f(y|x) = \frac{f(x,y)}{f_X(x)}$, where $f_X(x)$ is the marginal density of X. The marginal density of X is
$f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x)$ on the interval $x \le y \le 1$.
The conditional density of Y given $X = x$ is $f(y|x) = \frac{2}{2(1-x)} = \frac{1}{1-x}$ for $x \le y \le 1$.
The conditional mean of Y given $X = x$ is $\int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$.
We also could have noted that $f(y|x) = \frac{1}{1-x}$ is a uniform density on the interval $x \le y \le 1$.
so the mean is the midpoint of the interval, $\frac{x+1}{2}$. Answer: C

25. The coefficient of correlation is $\frac{Cov(Y_k,Y_j)}{\sqrt{Var(Y_k)\times Var(Y_j)}}$.
$Var(Y_k) = Var(\sum_{i=1}^{k} X_i) = \sum_{i=1}^{k} Var(X_i)$ (because of independence) $= 1 + 1 + \cdots + 1 = k$,
and similarly, $Var(Y_j) = j$.
Since $k < j$, it follows that $Y_j = Y_k + X_{k+1} + \cdots + X_j = Y_k + (Y_j - Y_k)$
and therefore $Cov(Y_k, Y_j) = Cov(Y_k, Y_k) + Cov(Y_k, Y_j - Y_k) = Var(Y_k) + 0 = k$.
Since $Y_j - Y_k = X_{k+1} + \cdots + X_j$, we see that $Y_j - Y_k$ is independent of Y_k, and therefore $Cov(Y_k, Y_j - Y_k) = 0$.
The coefficient of correlation is $\frac{k}{\sqrt{kj}} = \sqrt{\frac{k}{j}}$. Answer: D

26. $F_Y(3) = P(Y \le 3) = P(X^2 - 1 \le 3) = P(X^2 \le 4) = P(-2 \le X \le 2) = F_X(2)$.
The pdf of $X\,is\;f(x) = e^{-x}$ since X is exponential with mean 1. Then,
$F_X(2) = \int_0^2 e^{-x}\,dx = 1 - e^{-2}$. Answer: D

27. From the properties of natural log, we have

$ln(\frac{X_2}{X_1}) + ln(\frac{X_3}{X_2}) + \cdots + ln(\frac{X_{101}}{X_{100}}) = ln(\frac{X_{101}}{X_1}) = ln(X_{101})$.

This is the sum of 100 independent rv's each with mean .01 and variance .0009, so $ln(X_{101})$ has a distribution which is approximately normal with mean $100(.01) = 1$ and variance $100(.0009) = .09$. The probability that X_{101} is at least 4 is

$P(X_{101} \geq 4) = P[ln(X_{101}) \geq ln4] = P[\frac{ln(X_{101})-1}{\sqrt{.09}} \geq \frac{ln\,4-1}{\sqrt{.09}}] = 1 - \Phi(1.29) = .10$.

Answer: B

28. The maximum payment will occur if the loss is 2 or more, since the deductible would bring the payment to 1. If Y is the amount paid by the insurance and X is the underlying loss, then

$$Y = \begin{cases} 0 & X \leq 1 \\ X-1 & 1 < X \leq 2 \\ 1 & X > 2 \end{cases} .$$

The pdf of X is e^{-x}, so the expected value of Y is $\int_1^2 (x-1)e^{-x}\,dx + P(X > 2)$.

$\int_1^2 (x-1)e^{-x}\,dx = \int_1^2 xe^{-x}\,dx - \int_1^2 e^{-x}\,dx = (-xe^{-x} - e^{-x}\Big|_{x=1}^{x=2}) - (e^{-1} - e^{-2})$

$= e^{-1} - 2e^{-2}$, and

$P(X > 2) = \int_2^\infty e^{-x}\,dx = e^{-2}$.

Then $E(Y) = e^{-1} - 2e^{-2} + e^{-2} = e^{-1} - e^{-2}$. Answer: C

29. $Y = \begin{cases} 1 & X \leq 1 \\ X & X > 1 \end{cases}$. The pdf of X is $f(x) = \frac{1}{2}$.

$E(Y) = P(X < 1) + \int_1^2 x \cdot \frac{1}{2}\,dx = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$ and

$E(Y^2) = P(X < 1) + \int_1^2 x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2} + \frac{7}{6} = \frac{5}{3}$.

$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{5}{3} - (\frac{5}{4})^2 = .1042$. Answer: B

30. For the exponential random variable Y with mean 1, we have $P(Y > y) = e^{-y}$.

For this distribution, we have $P(X > x) = e^{-(x/\theta)^\tau}$.

With transformation A we have

$P(Y > y) = P(X^\tau > y) = P(X > y^{1/\tau}) = e^{-(y^{1/\tau}/\theta)^\tau} = e^{-y/\theta^\tau} \neq e^{-y}$.

With transformation B we have

$P(Y > y) = P(X^{1/\tau} > y) = P(X > y^\tau) = e^{-(y^\tau/\theta)^\tau} = e^{-y^{\tau^2}/\theta^\tau} \neq e^{-y}$.

With transformation C we have

$P(Y > y) = P((\frac{X}{\theta})^\tau > y) = P(X > \theta y^{1/\tau}) = e^{-(\theta y^{1/\tau}/\theta)^\tau} = e^{-y}$. This is the correct probability for the exponential distribution with mean 1. Answer: C

PRACTICE EXAM 7

1. A study of the relationship between blood pressure and cholesterol level showed the following results for people who took part in the study:
(a) of those who had high blood pressure, 50% had a high cholesterol level, and
(b) of those who had high cholesterol level, 80% had high blood pressure.
Of those in the study who had at least one of the conditions of high blood pressure or high cholesterol level, what is the proportion who had both conditions?
A) $\frac{1}{3}$ B) $\frac{4}{9}$ C) $\frac{5}{9}$ D) $\frac{2}{3}$ E) $\frac{7}{9}$

2. A study of international athletes shows that of the two performance-enhancing steroids Dianabol and Winstrol, 5% of athletes use Dianabol and not Winstrol, 2% use Winstrol and not Dianabol, and 1% use both. A breath test has been developed to test for the presence of the these drugs in an athlete. Past use of the test has resulted in the following information regarding the accuracy of the test. Of the athletes that are using both drugs, the test indicates that 75% are using both drugs, 15% are using Dianabol only and 10% are using Winstrol only. In 80% of the athletes that are using Dianabol but not Winstrol, the test indicates they are using Dianabol but not Winstrol, and for the other 20% the test indicates they are using both drugs. In 60% of the athletes that are using Winstrol but not Dianabol, the test indicates that they are using Winstrol only, and for the other 40% the test indicates they are using both drugs. For all athletes that are using neither Dianabol nor Winstrol, the test always indicates that they are using neither drug.
Of those athletes who test positive for Dianabol but not Winstrol, find the percentage that are using both drugs.
A) 1.2% B) 2.4% C) 3.6% D) 4.8% E) 6.0%

3. The random variable N has the following characteristics:
(i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.
(ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.
Which of the following is a correct expression for Prob$(N = 2)$?
A) $0.125p^2$ B) $0.375 + 0.125p$ C) $0.375 + 0.125p^2$
D) $0.375 - 0.125p^2$ E) $0.375 - 0.125p$

4. An insurance company does a study of claims that arrive at a regional office. The study focuses on the days during which there were at most 2 claims. The study finds that for the days on which there were at most 2 claims, the average number of claims per day is 1.2 . The company models the number of claims per day arriving at that office as a Poisson random variable. Based on this model, find the probability that at most 2 claims arrive at that office on a particular day.

A) .62 B) .64 C) .66 D) .68 E) .70

5. An actuarial trainee working on loss distributions encounters a special distribution. The student reads a discussion of the distribution and sees that the density of X is $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$ on the region $X > \theta$, where α and θ must both be > 0, and the mean is $\frac{\alpha\theta}{\alpha-1}$ if $\alpha > 1$.
The student is analyzing loss data that is assumed to follow such a distribution, but the values of α and θ are not specified, although it is known that $\theta < 200$. The data shows that the average loss for all losses is 180, and the average loss for all losses that are above 200 is 300.
Find the median of the loss distribution.

A) Less than 100 B) At least 100, but less than 120 C) At least 120, but less than 140
D) At least 140, but less than 160 E) At least 160

6. An insurance claims administrator verifies claims for various loss amounts.
For a loss claim of amount x, the amount of time spent by the administrator to verify the claim is uniformly distributed between 0 and $1 + x$ hours. The amount of each claim received by the administrator is uniformly distributed between 1 and 2. Find the average amount of time that an administrator spends on a randomly arriving claim.

A) 1.125 B) 1.250 C) 1.375 D) 1.500 E) 1.625

7. A husband and wife have a health insurance policy. The insurer models annual losses for the husband separately from the wife. X is the annual loss for the husband and Y is the annual loss for the wife. X has a uniform distribution on the interval $(0, 5)$ and Y has a uniform distribution on the interval $(0, 7)$, and X and Y are independent. The insurer applies a deductible of 2 to the combined annual losses, and the insurer pays a maximum of 8 per year. Find the expected annual payment made by the insurer for this policy.

A) 2 B) 3 C) 4 D) 5 E) 6

8. X has a Poisson distribution with a mean of 2.
Y has a geometric distribution on the integers 0,1,2,..., also with mean 2.
X and Y are independent.
Find $P(X=Y)$.
A) $\frac{e^{-2/3}}{3}$ B) $\frac{e^{-1/3}}{3}$ C) $\frac{1}{3}$ D) $\frac{e^{1/3}}{3}$ E) $\frac{e^{2/3}}{3}$

9. X has a uniform distribution on the interval $(0,1)$.
The random variable Y is defined by $Y=X^{-k}$, where $k>0$.
Find the mean of Y, assuming that it is finite.
A) $\frac{1}{k}$ B) $\frac{1}{1-k}$ C) k D) $\frac{k}{1-k}$ E) $\frac{1}{k-1}$

10. The marginal distributions of X and Y are both normal with mean 0, but X has a variance of 1, and Y has a variance of 4.
X and Y have a bivariate normal distribution with the following joint pdf:

$$f(x,y)=\frac{.3125}{\pi}\cdot e^{-.78125(x^2-.6xy+.25y^2)} .$$

Find the coefficient of correlation between $X+Y$ and $X-Y$.
A) Less than $-.6$ B) At least $-.6$, but less than $-.2$ C) At least $-.2$, but less than $.2$
D) At least $.2$, but less than $.6$ E) At least $.6$

11. An insurer is considering insuring two independent risks. The loss for each risk has an exponential distribution with a mean of 1. The insurer is considering issuing two separate insurance policies, one for each risk, each of which has a policy limit (maximum payment) of 2. The insurer is also considering issuing a single policy covering the combined loss on both risks, with a policy limit of 4. We denote by A the expected insurance payment for each of the two se prate policies, and we denote by B the expected insurance payment for the single policy covering the combined loss. Find B/A .
A) 1.4 B) 1.8 C) 2.2 D) 2.6 E) 3.0

12. At the start of a year, Smith is presented with an investment proposal. Smith's payoff from the investment is related to the closing value of an international financial stock index on the last day of the year. If the closing value of the index on the last day of the year is X, Smith's payoff will be $Y = Min\{ Max\{X, 20\} , 50 \}$.
At the start of the year, when Smith is considering this proposal, Smith's model for X is that X has a continuous uniform distribution on the interval $(0, 100)$.
Based on Smith's model, find the expected payoff.
A) Less than 30 B) At least 30, but less than 32 C) At least 32, but less than 34
D) At least 34, but less than 36 E) At least 36

13. In the 2006 World Cup of soccer, according to an online ranking service, Brazil, England and Germany are the three most highly ranked teams to win the tournament. A survey of soccer fans asks the fans to rank from most likely to least likely the chance of each those country's teams winning the world cup. The survey found that 50% of the fans ranked Brazil first, 30% ranked Brazil second, 30% ranked England second, 50% ranked England third, and 20% ranked Brazil first and England second. Of the fans surveyed who ranked England first, find the proportion who ranked Brazil last.
A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

14. In the 2006 World Cup of soccer, according to an online ranking service, Brazil, England and Germany are the three most highly ranked teams to win the tournament. A survey of soccer fans asks the fans to rank from most likely to least likely the chance of each those country's teams winning the world cup. The survey found the following:
- 2/3 of those who ranked Germany first ranked Brazil second ,
- 1/7 of those who didn't rank Germany first ranked Brazil second ,
- 30% of those surveyed ranked Brazil second.

Of those surveyed who ranked Brazil second, find the proportion that ranked Germany third.
A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

15. In the Canadian national lottery called "6-49", a ticket consists of 6 distinct numbers from 1 to 49 chosen by the player. The lottery chooses 6 distinct numbers at random from 1 to 49. If a player's ticket matches at least 3 of the 6 numbers chosen at random, then the player wins a prize. The next lottery is next Wednesday. A lottery player buys the following two tickets for next Wednesday's lottery:

Ticket 1 - 1 , 2 , 3 , 4 , 5 , 6 Ticket 2 - 7 , 8 , 9 , 10 , 11 , 12

Find the player's chance of not matching any of the 6 random numbers chosen on either of her two tickets.

A) Less than .1 B) At least .1, but less than .15 C) At least .15, but less than .2
D) At least .2, but less than .25 E) At least .25

16. A loaded six-sided die has the following probability function:
$P(X=1)=P(X=3)=P(X=5)=\frac{1}{9}$,
$P(X=2)=P(X=4)=P(X=6)=\frac{2}{9}$.
The die is tossed repeatedly until the outcome is 1, 2 or 3.
The first 1, 2 or 3 is the random variable Y. Find the variance of Y.
A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

17. X has a discrete non-negative integer valued distribution with a mean of 5 and a variance of 10. Two new distributions are created from X.
Y has the same probability function as X for $Y=2,3,4,\ldots$, but $P(Y=0)=0$ and
$P(Y=1)=P(X=0)+P(X=1)$.
Z has the same probability function as X for $Z=3,4,\ldots$, but
$P(Z=0)=P(Z=1)=0$ and $P(Z=2)=P(X=0)+P(X=1)+P(X=2)$
You are given that the mean of Y is 5.1 and the mean of Z is 5.3 . Find the variance of Z.
A) 7.0 B) 7.2 C) 7.4 D) 7.6 E) 7.8

18. The time until failure of a machine is modeled as an exponential distribution with a mean of 3 years. A warranty on the machine provides the following schedule of refunds:

- if the machine fails within 1 year, the full purchase price is refunded,
- if the machine fails after 1 year but before 2 years, 3/4 of the purchase price is refunded,
- if the machine fails after 2 years but before 4 years, 1/2 of the purchase price is refunded, and
- if the machine fails after 4 years, 1/4 of the purchase price is refunded.

Find the expected fraction of the purchase price that will be refunded under the warranty.

A) Less than .2 B) At least .2, but less than .4 C) At least .4, but less than .6

D) At least .6, butt less than .8 E) At least .8

19. A loss random variable X is uniformly distributed on the interval $[0\,,\,1000]$.

An insurance policy on the loss pays the following amount:

(i) 0 if the loss is below 200 ,

(ii) one-half of the loss in excess of 200 if the loss is between 200 and 500 , and

(iii) 150 plus one-quarter of the loss in excess of 500 if the loss is at least 500.

Y is the amount paid by the insurer when a loss occurs. Find the coefficient of variation of Y.

A) Less than .2 B) At least .2, but less than .4 C) At least .4, but less than .6

D) At least .6, butt less than .8 E) At least .8

20. A fair 6-sided die with faces numbered 1 to 6 is tossed successively and independently until the total of the faces is at least 14. Find the probability that at least 4 tosses are needed.

A) Less than .2 B) At least .2, but less than .4 C) At least .4, but less than .6

D) At least .6, butt less than .8 E) At least .8

21. A loss random variable has an exponential distribution with mean 800.

If an insurer imposes a policy limit of u on the loss, the insurer will pay a maximum of u when a loss occurs. The expected payment by the insurer with a policy limit of u is A. If instead the insurer imposes a policy limit of $2u$ on the loss, the expected payment by the insurer will be $1.2865A$ when a loss occurs. Find u.

A) 250 B) 500 C) 1000 D) 2000 E) 4000

22. An insurer is insuring 800 independent losses. 400 of the losses each have an exponential distribution with mean 1, and the other 400 losses each have an exponential distribution with mean 2. The insurer applies the normal approximation to find each of the following:
(a) the 95-th percentile of the aggregate of the first 400 losses with mean 1 each, say A ,
(b) the 95-th percentile of the aggregate of the second 400 losses with mean 2 each, say B , and
(c) the 95-th percentile of the aggregate of all 800 losses, say C .
Find $\frac{C}{A+B}$.

A) Less than .2 B) At least .2, but less than .4 C) At least .4, but less than .6
D) At least .6, butt less than .8 E) At least .8

23. A model describes the time until a loss occurs, X, and the size of the loss, Y.
X has pdf $f_X(x) = \frac{1}{x^2}$ for $x > 1$.
The conditional distribution of Y given $X = x$ has pdf $f_{Y|X}(y|X = x) = \frac{1}{x}$ for $x < y < 2x$.
Find pdf of the marginal distribution of Y, $f_Y(y)$.

A) $\begin{cases} \frac{1}{2} - \frac{1}{2y^2} & 1 < y < 2 \\ \frac{3}{2y^2} & y \geq 2 \end{cases}$ B) $\begin{cases} \frac{1}{2} - \frac{1}{3y^3} & 1 < y < 2 \\ \frac{2}{3y^2} & y \geq 2 \end{cases}$ C) $\begin{cases} \frac{1}{2} + \frac{1}{2y^2} & 1 < y < 2 \\ \frac{3}{2y^2} & y \geq 2 \end{cases}$

D) $\begin{cases} \frac{1}{2} + \frac{1}{3y^3} & 1 < y < 2 \\ \frac{3}{2y^2} & y \geq 2 \end{cases}$ E) $\begin{cases} \frac{1}{3} - \frac{1}{4y^3} & 1 < y < 2 \\ \frac{2}{3y^2} & y \geq 2 \end{cases}$

24. X and Y have a bivariate normal distribution, and X and Y each have marginal distributions that are standard normal (mean 0, variance 1).
You are given $P(X > Y + 1) = .2119$. Find $P(X > Y + 2)$.

A) .050 B) .055 C) .060 D) .065 E) .070

25. The Toronto Blue Jays baseball team holds a Children's Hospital Day. The Blue Jays will donate \$100,000 for each home run hit after the 2nd home run in the game. The team's model for the number of home runs hit in the game is Poisson with a mean of 4. Find the expected amount that the Blue Jays will donate.

A) Less than 150,000 B) At least 150,000, but less than 175,000
C) At least 175,000, but less than 200,000 D) At least 200,000, but less than 225,000
E) At least 225,000

26. In the Texas Hold'em poker game. each person is dealt two cards before any betting begins. For an ordinary deck of cards (spades, hearts, diamonds, clubs, 4 aces, 4 kings, etc), find is the probability that a randomly chosen player has a pair in the first two cards received.

A) .0188 B) .0288 C) .0388 D) .0488 E) .0588

27. X has a mean of 2 and a variance of 4. $aX+b$ has a mean of 5 and a variance of 1. What is ab assuming that $a>0$?

A) 1 B) 2 C) 3 D) 4 E) 5

28. X and Y have the following joint distribution:

		X	
		1	2
Y	1	c	$2c$
	2	$c/2$	c

Find $COV(X,Y)$.

A) $-\frac{4}{3}$ B) $-\frac{2}{3}$ C) 0 D) $\frac{2}{3}$ E) $\frac{4}{3}$

29. X and Y are independent continuous random variables, with X uniformly distributed on the interval $[0,\theta]$ and Y uniformly distributed on the interval $[0,2\theta]$. Find $P(Y<3X)$.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{5}{6}$

30. An insurance company is considering insuring a loss. The amount of the loss is uniformly distributed on the interval $[0\,,\,1000]$. The insurer considers two possible insurance policies.

Policy 1 - The insurer applies a deductible of 100 to the loss, and if the loss is above 100, the insurer limits the payment to a maximum payment amount of 500.

Policy 2 - If the loss is above 500 the insurer pays 400. If the loss is below 500, there is no deductible.

Find the ratio $\frac{\text{Expected insurance payment with Policy 1}}{\text{Expected insurance payment with Policy 2}}$.

A) $\frac{4}{5}$ B) $\frac{5}{6}$ C) 1 D) $\frac{6}{5}$ E) $\frac{5}{4}$

PRACTICE EXAM 7 - SOLUTIONS

1. We will use B to denote the event that a randomly chosen person in the study has high blood pressure, and C will denote the event high cholesterol level.
The information given tells us that $P(C|B) = .50$ and $P(C|B) = .80$.
We wish to find $P(B \cap C|B \cup C)$. This is

$$\frac{P[(B\cap C)\cap(B\cup C)]}{P(B\cup C)} = \frac{P[B\cap C)]}{P(B)+P(C)-P(B\cap C)} = \frac{1}{\frac{P(B)+P(C)-P(B\cap C)}{P(B\cap C)}} = \frac{1}{[\frac{1}{P(C|B)}+\frac{1}{P(C|B)}-1]}$$

$$= \frac{1}{\frac{1}{.5}+\frac{1}{.8}-1} = \frac{1}{2.25} = \frac{4}{9} \ .$$

Answer: B

2. We define the following events:
D - the athlete uses Dianabol
W - the athlete uses Winstrol
TD - the test indicates that the athlete uses Dianabol
TW - the test indicates that the athlete uses Winstrol

We are given the following probabilities
$P(D \cap W') = .05$, $P(D' \cap W) = .02$, $P(D \cap W) = .01$,
$P(TD \cap TW|D \cap W) = .75$, $P(TD \cap TW'|D \cap W) = .15$, $P(TD' \cap TW|D \cap W) = .1$,
$P(TD \cap TW|D \cap W') = .2$, $P(TD \cap TW'|D \cap W') = .8$,
$P(TD \cap TW|D' \cap W) = .4$, $P(TD' \cap TW|D' \cap W) = .6$.
We wish to find $P(D \cap W|TD \cap TW') = \frac{P(D\cap W\cap TD\cap TW')}{P(TD\cap TW')}$.
The numerator is $P(D \cap W \cap TD \cap TW') = P(TD \cap TW'|D \cap W) \cdot P(D \cap W)$
$= (.15)(.01) = .0015$.
The denominator is
$P(TD \cap TW') = P(TD \cap TW' \cap D \cap W) + P(TD \cap TW' \cap D' \cap W)$
$+ P(TD \cap TW' \cap D \cap W') + P(TD \cap TW' \cap D' \cap W')$
We have used the rule $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots$, where $B_1, B_2, \ldots$
forms a partition. The partition in this case is $B_1 = D \cap W$, $B_2 = D' \cap W$,
$B_3 = D \cap W'$, $B_4 = D' \cap W'$, since an athlete must be using both, one or neither of the drugs.

2. continued

We have just seen that $P(TD\cap TW'\cap D\cap W)=.0015$.

In a similar way, we have

$P(TD\cap TW'\cap D'\cap W)=P(TD\cap TW'|D'\cap W)\cdot P(D'\cap W)=(0)(.02)=0$, and

$P(TD\cap TW'\cap D\cap W')=P(TD\cap TW'|D\cap W')\cdot P(D\cap W')=(.8)(.05)=.04$, and

$P(TD\cap TW'\cap D'\cap W')=P(TD\cap TW'|D'\cap W')\cdot P(D'\cap W')=(0)(.92)=0$

(note that $P(D'\cap W')=1-P(D\cup W)$

$=1-P(D\cap W')-P(D'\cap W)-P(D\cap W)=.92$.

Then, $P(D\cap W|TD\cap TW')=\frac{.0015}{.0015+0+.04+0}=.036$, 3.6% . Answer: C

3. $P(N=2)=pP(N_1=2)+(1-p)P(N_2=2)=p(.5)^2+(1-p)6(.5)^4=.375-.125p$.
We have used the binomial probabilities $\binom{m}{k}q^k(1-q)^{m-k}$. Answer: E

4. Suppose that the mean number of claims per day arriving at the office is λ.

Let X denote the number of claims arriving in one day.

Then the probability of at most 2 claims in one day is $P(X\leq 2)=e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2e^{-\lambda}}{2}$.

The conditional probability of 0 claims arriving on a day given that there are at most 2 for the day is $P(X=0|X\leq 2)=\frac{P(X=0)}{P(X\leq 2)}=\frac{e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2e^{-\lambda}}{2}}=\frac{1}{1+\lambda+\frac{\lambda^2}{2}}$.

The conditional probability of 1 claim arriving on a day given that there are at most 2 for the day is $P(X=1|X\leq 2)=\frac{P(X=1)}{P(X\leq 2)}=\frac{\lambda e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2e^{-\lambda}}{2}}=\frac{\lambda}{1+\lambda+\frac{\lambda^2}{2}}$.

The conditional probability of 2 claims arriving on a day given that there are at most 2 for the day is $P(X=2|X\leq 2)=\frac{P(X=2)}{P(X\leq 2)}=\frac{\frac{\lambda^2e^{-\lambda}}{2}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2e^{-\lambda}}{2}}=\frac{\frac{\lambda^2}{2}}{1+\lambda+\frac{\lambda^2}{2}}$.

The expected number of claims per day, given that there were at most 2 claims per day is

$(0)\left(\frac{1}{1+\lambda+\frac{\lambda^2}{2}}\right)+(1)\left(\frac{\lambda}{1+\lambda+\frac{\lambda^2}{2}}\right)+(2)\left(\frac{\frac{\lambda^2}{2}}{1+\lambda+\frac{\lambda^2}{2}}\right)=\frac{\lambda+\lambda^2}{1+\lambda+\frac{\lambda^2}{2}}$.

We are told that this is 1.2 .

Therefore $\lambda+\lambda^2=(1.2)(1+\lambda+\frac{\lambda^2}{2})$, which becomes the quadratic equation $.4\lambda^2-.2\lambda-1.2=0$. Solving the equation results in $\lambda=2$ or -1.5 , but we ignore the negative root. The probability of at most 2 claims arriving at the office on a particular day is $P(X\leq 2)=e^{-2}+2e^{-2}+\frac{2^2e^{-2}}{2}=.6767$. Answer: D

5. The distribution function will be $F(y) = \int_\theta^y f(x)\,dx = \int_\theta^y \frac{\alpha\theta^\alpha}{x^{\alpha+1}}\,dx = 1 - \frac{\theta^\alpha}{y^\alpha}$.
The median m occurs where $F(m) = \frac{1}{2}$. If α and θ were known, we could find the median.
The average loss for all losses is $\frac{\alpha\theta}{\alpha-1} = 180$, but both θ and α are not known.
The conditional distribution of loss amount x given that $X > 200$ is
$f(x|X > 200) = \frac{f(x)}{P(X>200)} = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \Big/ \frac{\theta^\alpha}{200^\alpha} = \frac{\alpha 200^\alpha}{x^{\alpha+1}}$.
This random variable has a mean of $\frac{200\alpha}{\alpha-1}$. We are given that this mean is 300,
so $\frac{200\alpha}{\alpha-1} = 300$, and therefore $\alpha = 3$.
Then, from $\frac{\alpha\theta}{\alpha-1} = 180$, we get $\frac{3\theta}{2} = 180$, so that $\theta = 120$.
The median m satisfies the relation $\frac{1}{2} = F(m) = 1 - \frac{\theta^\alpha}{m^\alpha} = 1 - \left(\frac{120}{m}\right)^3$, so that $m = 151.2$.
Answer: D

6. $X =$ amount of loss claim, uniformly distributed on $(1,2)$, so $f_X(x) = 1$ for $1 < x < 2$.
$Y =$ amount of time spent verifying claim.
We are given that the conditional distribution of Y given $X = x$ is uniform on $(0, 1+x)$,
so $f(y|x) = \frac{1}{1+x}$ for $0 < y < 1+x$.

We wish to find $E[Y]$. The joint density of X and Y is
$f(x,y) = f(y|x) \cdot f_X(x) = \frac{1}{1+x}$ for $0 < y < 1+x$ and $1 < x < 2$.
There are a couple of ways to find $E[Y]$:

(i) $E[Y] = \int\int y\, f(x,y)\,dy\,dx$ or $E[Y] = \int\int y\, f(x,y)\,dx\,dy$, with careful setting of the integral limits, or
(ii) $E[Y] = \int y\, f_Y(y)\,dy$, where $f_Y(y)$ is the pdf of the marginal distribution of Y.
$E[Y] = \int_1^2 \int_0^{1+x} y \cdot$
(iii) The double expectation rule, $E[Y] = E[\,E[Y|X]\,]$.

If we apply the first approach for method (i), we get
$E[Y] = \int_1^2 \int_0^{1+x} y \cdot \frac{1}{1+x}\,dy\,dx = \int_1^2 \frac{(1+x)^2}{2(1+x)}\,dy = \int_1^2 \frac{1+x}{2}\,dx = \frac{5}{4}$.

If we apply the second approach for method (i), we must split the double integral into
$E[Y] = \int_0^2 \int_1^2 y \cdot \frac{1}{1+x}\,dx\,dy + \int_2^3 \int_{y-1}^2 y \cdot \frac{1}{1+x}\,dx\,dy$
The first integral becomes $\int_0^2 y\, ln\left(\frac{3}{2}\right)dy = 2\,ln\left(\frac{3}{2}\right)$.
The second integral becomes $\int_2^3 y\,[ln3 - ln\,y]\,dy = \frac{5}{2}\,ln3 - \int_2^3 y\,ln\,y\,dy$.
The integral $\int_2^3 y\,ln\,y\,dy$ is found by integration by parts.

6. continued

Let $\int y\,ln\,y\,dy = A$.

Let $u = y$ and $dv = ln\,y\,dy$, then $v = y\,lny - y$ (antiderivative of $ln\,y$), and then

$A = \int y\,ln\,y\,dy = y(y\,lny - y) - \int (y\,ln\,y - y)\,dy = y^2\,ln\,y - y^2 - A + \frac{y^2}{2}$,

so that $A = \int y\,ln\,y\,dy = \frac{1}{2}y^2\,ln\,y - \frac{y^2}{4}$.

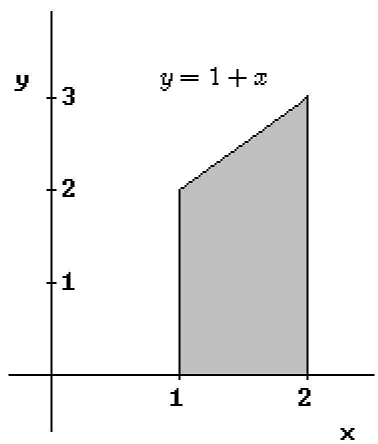
Then $\int_2^3 y\,ln\,y\,dy = \frac{1}{2}y^2\,ln\,y - \frac{y^2}{4}\Big|_2^3 = \frac{9}{2}\,ln3 - \frac{9}{4} - (\frac{4}{2}\,ln\,2 - 1) = \frac{9}{2}\,ln3 - 2\,ln2 - \frac{5}{4}$.

Finally, $E[Y] = 2\,ln(\frac{3}{2}) + \frac{5}{2}\,ln3 - \int_2^3 y\,ln\,y\,dy$

$= 2\,ln\,3 - 2\,ln\,2 + \frac{5}{2}\,ln3 - (\frac{9}{2}\,ln3 - 2\,ln2 - \frac{5}{4}) = \frac{5}{4}$.

The first order of integration for method (i) was clearly the more efficient one.

(ii) This method is equivalent to the second approach in method (i), because we find $f_Y(y)$ from the relationship $f_Y(y) = \int f(x,y)\,dx$. The two-dimensional region of probability for the joint distribution is $1 < x < 2$ and $0 < y < 1 + x$. This is illustrated in the graph below



For $0 < y < 2$, $f_Y(y) = \int_1^2 f(x,y)\,dx = \int_1^2 \frac{1}{1+x}\,dx = ln(\frac{3}{2})$

and for $2 \le x < 3$, $f_Y(y) = \int_{y-1}^2 f(x,y)\,dx = \int_{y-1}^2 \frac{1}{1+x}\,dx = ln\,3 - ln\,y$.

Then $E[Y] = \int_0^2 y\,ln(\frac{3}{2})\,dy + \int_2^3 y\,[ln3 - ln\,y]\,dy$, which is the same as the second part of method (i).

(iii) According to the double expectation rule, for any two random variables U and W, we have $E[U] = E[\,E[U|W]\,]$. Therefore, $E[Y] = E[\,E[Y|X]\,]$.

We are told that the conditional distribution of Y given $X = x$ is uniform on the interval $(0, 1+x)$, so $E[Y|X] = \frac{1+X}{2}$.

Then $E[\,E[Y|X]\,] = E[\frac{1+X}{2}] = \frac{1}{2} + \frac{1}{2}E[X] = \frac{1}{2} + \frac{1}{2}(\frac{3}{2}) = \frac{5}{4}$,

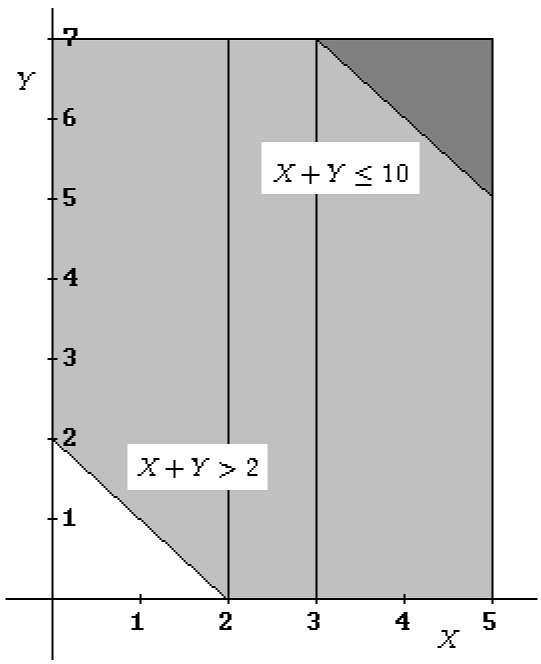
since X is uniform on $(1,2)$ and X has mean $\frac{3}{2}$. Answer: B

7. The joint distribution of X and Y has pdf $f(x,y) = \frac{1}{5} \cdot \frac{1}{7} = \frac{1}{35}$ on the rectangle $0 < x < 5$ and $0 < y < 7$. The insurer pays $X+Y-2$ if the combined loss $X+Y$ is > 2. The maximum payment of 8 is reached if $X+Y-2 \geq 8$, or equivalently, if $X+Y \geq 10$. Therefore, the insurer pays $X+Y-2$ if $2 < X+Y \leq 10$ (the lighter shaded region in the diagram below), and the insurer pays 8 if $X+Y > 10$ (the darker shaded region in the diagram below). The expected amount paid by the insurer is a combination of two integrals:

$\int\int (x+y-2) \cdot \frac{1}{35}\, dy\, dx$, where the integral is taken over the region $2 < x+y \leq 10$ (the lightly shaded region), plus

$\int\int 8 \cdot \frac{1}{35}\, dy\, dx$, where the integral is taken over the region $X+Y > 10$ (the darker region).

The second integral is $\frac{8}{35} \cdot (2) = \frac{16}{35}$, since the area of the darkly shaded triangle is 2 (it is a 2×2 right triangle).



The first integral can be broken into three integrals:

$$\int_0^2 \int_{2-x}^7 (x+y-2) \cdot \frac{1}{35}\, dy\, dx + \int_2^3 \int_0^7 (x+y-2) \cdot \frac{1}{35}\, dy\, dx + \int_3^5 \int_0^{10-x} (x+y-2) \cdot \frac{1}{35}\, dy\, dx$$

$$= \frac{1}{35} \cdot \left[\int_0^2 \frac{(x+5)^2}{2}\, dx + \int_2^3 \frac{7(2x+3)}{2}\, dx + \int_3^5 \frac{60+4x-x^2}{2}\, dx \right]$$

$$= \frac{1}{35} \cdot \left[\frac{109}{3} + 28 + \frac{179}{3} \right] = \frac{124}{35}.$$

The total expected insurance payment is $\frac{16}{35} + \frac{124}{35} = \frac{140}{35} = 4$.   Answer: C

8. The probability function of X is $P(X=k)=\frac{e^{-2}\cdot 2^k}{k!}$.

The general probability function of a geometric distribution on $0,1,2,\ldots$ is of the form $P(Y=k)=p(1-p)^k$ for $k=0,1,2,\ldots$ and the mean is $\frac{1-p}{p}$.

Since the mean is 2, we have $\frac{1-p}{p}=2$, from which we get $p=\frac{1}{3}$,

so the probability function of Y is $P(Y=k)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^k$.

$P(X=Y)=P(X=Y=0)+P(X=Y=1)+\cdots=\sum_{k=0}^{\infty}P(X=Y=k)$.

Since X and Y are independent, we have

$P(X=Y=k)=P(X=k)\cdot P(Y=k)=\frac{e^{-2}\cdot 2^k}{k!}\cdot\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^k=\frac{e^{-2}}{3}\cdot\frac{(4/3)^k}{k!}$.

Then, $P(X=Y)=\sum_{k=0}^{\infty}P(X=Y=k)=\sum_{k=0}^{\infty}\frac{e^{-2}}{3}\cdot\frac{(4/3)^k}{k!}=\frac{e^{-2}}{3}\cdot\sum_{k=0}^{\infty}\frac{(4/3)^k}{k!}$.

The Taylor series expansion for e^x is $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$, so it follows that $\sum_{k=0}^{\infty}\frac{(4/3)^k}{k!}=e^{4/3}$.

Then, $P(X=Y)=\frac{e^{-2}}{3}\cdot e^{4/3}=\frac{e^{-2/3}}{3}$. Answer: A

9. $X=Y^{-1/k}=h(Y)$.

According to the method by which we find the density of a transformed random variable, the pdf of Y is $g(y)=f(h(y))\cdot|h'(y)|$, where f is the pdf of X.

Since X is uniform on $(0,1)$, we know that $f(x)=1$.

Therefore, $g(y)=\left|-\frac{y^{-(k+1)/k}}{k}\right|=\frac{y^{-(k+1)/k}}{k}$. Since $y=x^{-k}$, it follows that $y>1$, since $0<x<1$.

The mean of Y will be $\int_1^{\infty}y\cdot g(y)\,dy=\int_1^{\infty}y\cdot\frac{y^{-(k+1)/k}}{k}\,dy=\int_1^{\infty}\frac{y^{-1/k}}{k}\,dy=\frac{y^{(k-1)/k}}{k-1}\Big|_{y=1}^{y=\infty}$.

This will be ∞ if $k\geq 1$. If $k<1$, then $E[Y]=-\frac{1}{k-1}=\frac{1}{1-k}$. Answer: B

10. If X and Y have a bivariate normal distribution for which

X has mean μ_X and standard deviation σ_X, and

Y has mean μ_Y and standard deviation σ_Y, and

the coefficient of correlation between X and Y is ρ, then

the general bivariate normal joint pdf is

$f(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\cdot exp\left[-\frac{1}{2(1-\rho^2)}\cdot\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right.$.

We are given that $f(x,y)=\frac{.3125}{\pi}\cdot e^{-.78125(x^2-.6xy+.25y^2)}$.

10. continued

From the general form of the joint pdf, we see that $\frac{2\rho}{\sigma_X\sigma_Y} = .6$, so that $\rho = .6$.

The covariance between $X+Y$ and $X-Y$ is

$Cov(X+Y, X-Y) = Cov(X,X) + Cov(X,-Y) + Cov(Y,X) + Cov(Y,-Y)$

$= Var(X) - Cov(X,Y) + Cov(Y,X) - Var(Y) = Var(X) - Var(Y) = 1-4 = -3$.

The coefficient of correlation between $X+Y$ and $X-Y$ is $\frac{Cov(X+Y,X-Y)}{\sqrt{Var(X+Y)\cdot Var(X-Y)}}$.

$Var(X+Y) = Var(X) + Var(Y) + 2\rho\sqrt{Var(X)\cdot Var(Y)} = 1+4+2(.6)\sqrt{(1)(4)} = 7.4$

and

$Var(X-Y) = Var(X) + Var(Y) - 2\rho\sqrt{Var(X)\cdot Var(Y)} = 1+4-2(.6)\sqrt{(1)(4)} = 2.6.$

The coefficient of correlation between $X+Y$ and $X-Y$ is

$\frac{Cov(X+Y,X-Y)}{\sqrt{Var(X+Y)\cdot Var(X-Y)}} = \frac{-3}{\sqrt{(7.4)(2.6)}} = -.684$.   Answer: A

11. Suppose that X is an exponential random variable with mean 1.

The pdf of X is $f_X(x) = e^{-x}$, $x > 0$. The insurance policy for a single risk with policy limit 2 will pay $\begin{cases} x & \text{if } 0 < x \le 2 \\ 2 & \text{if } x > 2 \end{cases}$.

The expected amount paid for one policy is

$A = \int_0^2 x\cdot f_X(x)\,dx + 2\cdot P(X>2) = \int_0^2 x\cdot e^{-x}\,dx + 2\cdot e^{-2}$

$= (-xe^{-x} - e^{-x})\Big|_{x=0}^{x=2} + 2e^{-2} = (-2e^{-2} - e^{-2}) - (0-1) + 2e^{-2} = 1 - e^{-2}$.

Note that, for a non-negative random variable $X \ge 0$, with a policy limit u, the expected insurance payment is $\int_0^u [1 - F_X(x)]\,dx$. In the case of the exponential distribution with mean 1, $F_X(x) = 1 - e^{-x}$, so the expected insurance payment with a policy limit of 2 is $\int_0^2 [1 - (1 - e^{-x})]\,dx = \int_0^2 e^{-x}\,dx = 1 - e^{-2}$.

Suppose that X_1 and X_2 are the independent exponential losses on the two risks. The combined loss is $Y = X_1 + X_2$, and the insurance on the combined losses will apply a limit of 4 to Y. The sum of two independent exponential random variables, each with a mean of 1, is a gamma random variable with pdf $f_Y(y) = ye^{-y}$, $y > 0$. This can be verified a couple of ways.

(i) Convolution:

$f_Y(y) = \int_0^y f_{X_1}(x)\cdot f_{X_2}(y-x)\,dx = \int_0^y e^{-x}\cdot e^{-(y-x)}\,dx = \int_0^y e^{-y}\,dx = ye^{-y}$

(ii) Transformation of random variables:

Since X_1 and X_2 are independent, the joint distribution of X_1 and X_2 has pdf

$f(x_1,x_2) = f_{X_1}(x_1)\cdot f_{X_2}(x_2) = e^{-x_1}\cdot e^{-x_2}$

$U = X_1$, $Y = X_1 + X_2 \to X_1 = U$, $X_2 = Y - U$

$\to$ pdf of U, Y is $g(u,y) = f(u, y-u)\cdot \begin{vmatrix} \frac{\partial}{\partial u}\, u & \frac{\partial}{\partial u}\, y-u \\ \frac{\partial}{\partial y}\, u & \frac{\partial}{\partial y}\, y-u \end{vmatrix} = e^{-u}\cdot e^{-(y-u)}\cdot 1 = e^{-y}$,

and the joint distribution of U and Y is defined on the region $0 < u < y$

(this is true because $u = x_1 < x_1 + x_2 = y$).

The marginal density of Y is $f_Y(y) = \int_0^y g(u,y)\,du = \int_0^y e^{-y}\,dy = ye^{-y}$.

We impose a limit of 4 for the insurance policy on Y, the combination of the two exponential losses. The amount paid by the insurance is $\begin{cases} y & \text{if } 0 < y \le 4 \\ 4 & \text{if } y > 4 \end{cases}$

The expected insurance payment is $\int_0^4 y\cdot f_Y(y)\,dy \, + \, 4\cdot P(Y > 4)$.

$\int_0^4 y\cdot f_Y(y)\,dy = \int_0^4 y\cdot ye^{-y}\,dy = \int_0^4 y^2\cdot e^{-y}\,dy$.

Applying integration by parts, this becomes

$-y^2e^{-y}\Big|_{y=0}^{y=4} - \int_0^4(-e^{-y})(2y)\,dy = -16e^{-4} + 2\int_0^4 ye^{-y}\,dy$

$= -16e^{-4} + 2\cdot[-ye^{-y} - e^{-y}\Big|_{y=0}^{y=4}] = -16e^{-4} + 2[-4e^{-4} - e^{-4} - (0-1)] = 2 - 26e^{-4}$.

$P(Y > 4) = \int_4^\infty f_Y(y)\,dy = \int_4^\infty ye^{-y}\,dy = (-ye^{-y} - e^{-y})\Big|_{y=4}^{y=\infty}$

$= (-0-0) - (-4e^{-4} - e^{-4}) = 5e^{-4}$.

Expected insurance payment of the combined policy is $2 - 26e^{-4} + 4(5e^{-4}) = 2 - 6e^{-4} = B$.

The ratio B/A is $\frac{2-6e^{-4}}{1-e^{-2}} = 2.186$. Answer: C

12. If the index closes below 20, then $Y = Max\{X, 20\} = 20$,

and if the index closes above 50, then $Y = Min\{\,Max\{X,20\}\,,\,50\} \,=\, 50$.

If the index closes between 20 and 50, then

$Y = Min\{\,Max\{X,20\}\,,\,50\} = Min\{X,\,50\} = X$.

Therefore, $Y = \begin{cases} 20 & X \le 20 \\ X & 20 < X \le 50 \\ 50 & X > 50 \end{cases}$.

$E(Y) = \int_0^{20} 20\cdot f_X(x)\,dx + \int_{20}^{50} x\cdot f_X(x)\,dx + \int_{50}^{100} 50\cdot f_X(x)\,dx$.

X has pdf $f_X(x) = \frac{1}{100} = .01$, so

$E(Y) = \int_0^{20} 20(.01)\,dx + \int_{20}^{50} x\cdot(.01)\,dx + \int_{50}^{100} 50(.01)\,dx = 4 + 10.5 + 25 = 39.5$.

Answer: E

13. There are 6 possible rankings that a surveyed fan can choose:

BEG , BGE , EBG , EGB , GEB , GBE

We are given the following:

$P(BEG) + P(BGE) = .5$, $P(EBG) + P(GBE) = .3$, $P(BEG) + P(GEB) = .3$,
$P(BGE) + P(GBE) = .5$, $P(BEG) = .2$.

We wish to find $P(EGB|EGB \cup EBG) = \frac{P(EGB)}{P(EGB)+P(EBG)}$.

Since 80% ranked England either second or third, it follows that 20% ranked England first, so
$P(EGB \cup EBG) = P(EGB) + P(EBG) = .2$.

From the given information, we have $.2 + P(BGE) = .5 \rightarrow P(BGE) = .3$.

Then, $.3 + P(GBE) = .5 \rightarrow P(GBE) = .2$.

Then, $P(EBG) + .2 = .3 \rightarrow P(EBG) = .1$, and then $P(EGB) + .1 = .2 \rightarrow P(EGB) = .1$.

Finally, $\frac{P(EGB)}{P(EGB)+P(EBG)} = \frac{.1}{.1+.1} = \frac{1}{2}$. Answer: C

14. We define the following events:

$B2$ - a surveyed individual ranked Brazil second ,

$G1$ - a surveyed individual ranked Germany first .

We wish to find $P(G3|B2) = \frac{P(B2 \cap G3)}{P(B2)}$.

We are given $P(B2) = .3$,

and we are given the conditional probabilities $P(B2|G1) = \frac{2}{3}$ and $P(B2|G1') = \frac{1}{7}$.

From $P(B2|G1) = \frac{P(B2 \cap G1)}{P(G1)} = \frac{2}{3}$ we get $P(B2 \cap G1) = \frac{2}{3} \cdot P(G1)$, and from
$P(B2|G1') = \frac{1}{7}$ we get $P(B2 \cap G1') = \frac{1}{7} \cdot P(G1') = \frac{1}{7} \cdot [1 - P(G1)]$.

Therefore $.3 = P(B2) = P(B2 \cap G1) + P(B2 \cap G1') = \frac{2}{3} \cdot P(G1) + \frac{1}{7} \cdot P(G1')$
$= \frac{2}{3} \cdot P(G1) + \frac{1}{7} \cdot [1 - P(G1)]$, from which we get $P(G1) = .3$.

Then, $P(B2 \cap G3) = P(B2 \cap G1') = \frac{1}{7} \cdot [1 - P(G1)] = .1$
and $P(G3|B2) = \frac{P(B2 \cap G3)}{P(B2)} = \frac{.1}{.3} = \frac{1}{3}$. Answer: B

15. In order to have no matching number on either ticket, the 6 randomly chosen numbers must come from the 37 other numbers, 13 , 14 , . . . , 49 . The probability in question is the ratio of the number of random ticket draws that result in the event over the total possible number of random ticket draws.

$P(A) = \frac{\binom{37}{6}}{\binom{49}{6}} = \frac{\text{\# randomly chosen tickets that avoid 1,2,...,12}}{\text{total number of possible randomly chosen tickets}} = \frac{37!/(31!\ 6!)}{49!/(43!\ 6!)} = .166248$.

Answer: C

16. Y can be thought of as the conditional distribution of X given that X is not 4, 5 or 6. The probability function of Y is

$P(Y=1)=P(X=1|X\neq 4,5,6)=\frac{P(X=1)}{P(X\neq 4,5,6)}=\frac{1/9}{4/9}=\frac{1}{4}$,

$P(Y=2)=P(X=2|X\neq 4,5,6)=\frac{P(X=2)}{P(X\neq 4,5,6)}=\frac{2/9}{4/9}=\frac{1}{2}$,

$P(Y=3)=P(X=3|X\neq 4,5,6)=\frac{P(X=3)}{P(X\neq 4,5,6)}=\frac{1/9}{4/9}=\frac{1}{4}$.

$E[Y]=(1)\left(\frac{1}{4}\right)+(2)\left(\frac{1}{2}\right)+(3)\left(\frac{1}{4}\right)=2$,

$E[Y^2]=(1^2)\left(\frac{1}{4}\right)+(2^2)\left(\frac{1}{2}\right)+(3^2)\left(\frac{1}{4}\right)=\frac{9}{2}$.

$Var[Y]=E[Y^2]-(E[Y])^2=\frac{9}{2}-2^2=\frac{1}{2}$.   Answer: C

17. We wish to find $Var[Z]=E[Z^2]-(E[Z])^2=E[Z^2]-(5.3)^2$.

Let us denote $P(X=0)=p_0$, $P(X=1)=p_1$, $P(X=2)=p_2$, etc.

Then $P(Y=0)=0$, $P(Y=1)=p_0+p_1$, $P(Y=2)=p_2$, etc.

Then $P(Z=0)=P(Z=1)=0$, $P(Z=2)=p_0+p_1+p_2$,

$5.0=E[X]=p_1+2p_2+3p_3+\cdots$ and

$5.1=E[Y]=(p_0+p_1)+2p_2+3p_3+\cdots$ and

$5.3=E[Z]=2(p_0+p_1+p_2)+3p_3+\cdots$.

Therefore, $.1=E[Y]-E[X]=p_0$ and $.2=E[Z]-E[Y]=p_0+p_1=.1+p_1$,

so that $p_1=.1$.

From $10=Var[X]=E[X^2]-(E[X])^2=E[X^2]-25$, we have

$E[X^2]=35=p_1+4p_2+9p_3+\cdots$

Then, $E[Z^2]=4(p_0+p_1+p_2)+9p_3+\cdots=4p_0+3p_1+E[X^2]$

$=4(.1)+3(.1)+35=35.7$, and

$Var[Z]=35.7-(5.3)^2=7.61$.   Answer: D

18. Suppose that T is the time until failure of the machine. $P(a<T\leq b)=e^{-a/3}-e^{-b/3}$.

The fraction of the purchase price refunded is a random variable X that can be described in the following way:

$$X=\begin{cases} 1 & 0<T\leq 1 \quad \text{prob. } 1-e^{-1/3} \\ 3/4 & 1<T\leq 2 \quad \text{prob. } e^{-1/3}-e^{-2/3} \\ 1/2 & 2<T\leq 3 \quad \text{prob. } e^{-2/3}-e^{-4/3} \\ 1/4 & T>3 \quad \text{prob. } e^{-4/3} \end{cases}$$

Then, $E[X]=1-e^{-1/3}+\frac{3}{4}(e^{-1/3}-e^{-2/3})+\frac{1}{2}(e^{-2/3}-e^{-4/3})+\frac{1}{4}e^{-4/3}$

$=1-\frac{1}{4}e^{-1/3}-\frac{1}{4}e^{-2/3}-\frac{1}{4}e^{-4/3}=.627$.   Answer: D

19. The coefficient of variation of Y is $\frac{\sqrt{Var(Y)}}{E(Y)}$.

$E(Y) = \int_{200}^{500} \frac{1}{2}(x-200)(.001)\,dx + \int_{500}^{1000}[150+\frac{1}{4}(x-500)](.001)\,dx = \frac{45}{2} + \frac{425}{4} = \frac{515}{4}$.

$E(Y^2) = \int_{200}^{500} [\frac{1}{2}(x-200)]^2(.001)\,dx + \int_{500}^{1000}[150+\frac{1}{4}(x-500)]^2(.001)\,dx$

$= 2250 + \frac{139{,}375}{6} = \frac{152{,}875}{6}$.

$Var(Y) = \frac{152{,}875}{6} - (\frac{515}{4})^2 = \frac{427{,}325}{48} = 8902.6$.

The coefficient of variation is $\frac{\sqrt{8902.6}}{128.75} = .733$. Answer: D

20. P[At least 4 tosses are needed] $= 1 - P$[at most 3 tosses are needed] .

It is not possible to reach the total of 14 on 1 or 2 tosses.

There are $6 \times 6 \times 6 = 216$ possible sets of 3 consecutive tosses.

The following sets of 3 consecutive tosses result in a total of at least 14 on the faces that turn up.

(a) Three 6's (6 on each toss) ; 1 set.

(b) Two 6's and 2 to 5 on the other toss ; $4 \times 3 = 12$ sets

(6,6,2 , and 6,2,6 and 2,6,6 , and the same with 3 or 4 or 5 instead of 2).

(c) One 6 and either 5-5, or 4-5, or 4-4, or 3-5; $3+6+\ +3+6 = 18$ sets

(6,5,5 or 5,6,5 or 5,5,6, and 6,4,5 in six arrangements, and 6-3-5 in six arrangements).

(d) No 6's, and either three 5's, or two 5's and a 4; $1+3=4$ sets.

Total of $1+12+18+4 = 35$ sets out of 216 possible sets.

Probability is P[At least 4 tosses are needed] $= 1 - \frac{35}{216} = .838$. Answer: E

21. The exponential distribution with mean θ has pdf $f(t) = \frac{1}{\theta}e^{-t/\theta}$ and cdf $F(x) = 1 - e^{-x/\theta}$.

For a non-negative loss random variable L with cdf $F(y)$, if a policy limit of u is imposed, the expected payment by the insurer when a loss occurs is $\int_0^u [1-F(y)]\,dy$.

For the exponential loss random variable with mean 800 and with limit u, the expected amount paid by the insurer when a loss occurs is $\int_0^u e^{-x/\theta}\,dx = 800[1-e^{-u/800}]$.

If the limit is $2u$, the expected payment by the insurer when a loss occurs is $800[1-e^{-2u/800}]$.

We are given that $800[1-e^{-2u/800}] = 1.2865(800[1-e^{-u/800}])$.

After canceling 800 and factoring the difference of squares

$1 - e^{-2u/800} = (1-e^{-u/800})(1+e^{-u/800})$,

this equation becomes $1 + e^{-u/800} = 1.2865$, so that $u = 1000$. Answer: C

22. The mean and variance of the exponential loss with mean 1 are 1 and 1, and the mean and variance of the exponential distribution with mean 2 are 2 and 4.

(a) $S_a = X_1 + \cdots + X_{400}$. $E[S_a] = 400(1) = 400$, $Var[S_a] = 400(1) = 400$.

$P[S_a \leq A] = P[\frac{S_a - 400}{\sqrt{400}} \leq \frac{A-400}{\sqrt{400}}] = .95 \rightarrow \frac{A-400}{\sqrt{400}} = 1.645 \rightarrow A = 432.9$.

(b) $S_b = Y_1 + \cdots + Y_{400}$. $E[S_b] = 400(2) = 800$, $Var[S_b] = 400(4) = 1600$.

$P[S_b \leq B] = P[\frac{S_b - 800}{\sqrt{1600}} \leq \frac{B-800}{\sqrt{1600}}] = .95 \rightarrow \frac{B-800}{\sqrt{1600}} = 1.645 \rightarrow B = 865.8$.

(c) $S_c = X_1 + \cdots + X_{400} + Y_1 + \cdots + Y_{400}$.

$E[S_c] = 400 + 800 = 1200$, $Var[S_c] = 400 + 1600 = 2000$.

$P[S_c \leq C] = P[\frac{S_c - 1200}{\sqrt{2000}} \leq \frac{C-1200}{\sqrt{2000}}] = .95 \rightarrow \frac{C-1200}{\sqrt{2000}} = 1.645 \rightarrow A = 1273.6$

$\frac{C}{A+B} = \frac{1273.6}{432.9+865.8} = .9807$. Answer: E

23. Since $x < y < 2x$, it follows that $\frac{y}{2} < x < y$.

Also, since $x > 1$ it follows that $x > max\{\frac{y}{2}, 1\}$, and $y > 1$.

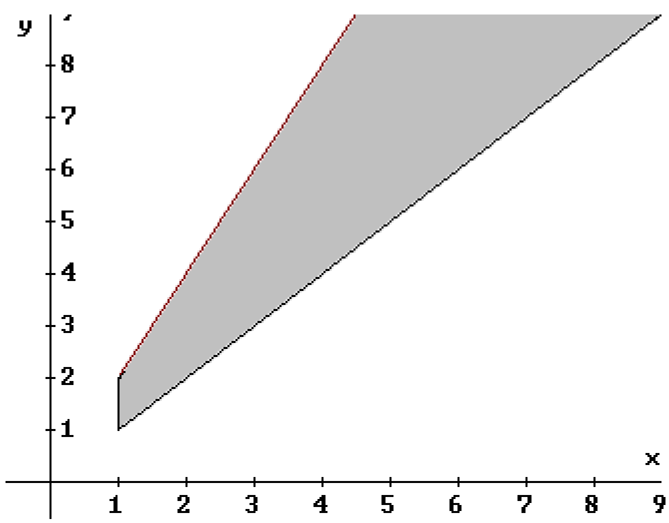
Therefore, if $1 < y \leq 2$, it follows that $x > 1$, and if $y > 2$ then $x > \frac{y}{2}$.

The joint density of X and Y is $f(x,y) = f(y|x) \cdot f_X(x) = \frac{1}{x} \cdot \frac{1}{x^2} = \frac{1}{x^3}$.

If $1 < y < 2$, then this joint pdf is defined for $1 < x < y$,

and if $y \geq 2$, then this joint pdf is defined for $\frac{y}{2} < x < y$.

The shaded region below is the region of joint density.



The pdf of the marginal distribution of Y is $f_Y(y) = \int f(x,y)\, dx$.

For $1 < y < 2$, we get $f_Y(y) = \int_1^y \frac{1}{x^3}\, dx = \frac{1}{2} - \frac{1}{2y^2}$.

For $y \geq 2$, we get $f_Y(y) = \int_{y/2}^y \frac{1}{x^3}\, dx = \frac{4}{2y^2} - \frac{1}{2y^2} = \frac{3}{2y^2}$. Answer: A

24. Suppose that the covariance between X and Y is C. Then $X-Y$ has a normal distribution with mean $1-1=0$ and variance

$Var[X-Y] = Var[X] + Var[Y] - 2\,Cov(X,Y) = 1+1-C = 2-C$.

Then, $P(X-Y>1) = P(\frac{X-Y}{\sqrt{2-C}} > \frac{1}{\sqrt{2-C}}) = .2119$.

$Z = \frac{X-Y}{\sqrt{2-C}}$ has a standard normal distribution, and from the standard normal table,

we get $\frac{1}{\sqrt{2-C}} = .80$.

Then, $P(X > Y+2) = P(X-Y > +2) = P(\frac{X-Y}{\sqrt{2-C}} > \frac{2}{\sqrt{2-C}}) = P(Z > 1.6) = .0548.$

Answer: B

25. N denotes the number of home runs hit in the game. $E[N]=4$ and N has a Poisson distribution. The amount donated X (multiples of 100,000) can be summarized as follows: Define Y to be $Y=N-X$.

N	0	1	2	3	4	5	$\ldots$
X	0	0	0	1	2	3	$\ldots$
Y	0	1	2	2	2	2	$\ldots$

We know that $X+Y=N$ so that $E[X]+E[Y]=E[N]=4$.

But we also can see that Y can only be 0, 1 or 2, and

$P(Y=0)=P(N=0)=e^{-4}$, $P(Y=1)=P(N=1)=4e^{-4}$

and $P(Y=2)=P(N\geq 2)=1-P(N=0,1)=1-5e^{-4}$.

Therefore, $E[X]=4-E[Y]=4-(1)(4e^{-4})-(2)[1-5e^{-4}]=2.11$

and the expected amount paid by the Blue Jays is 211,000 . Answer: D

26. There are 6 possible pairs of aces (Spade-Heart, Spade-Diamond, Spade-Club, Heart-Diamond, Heart-Club, Diamond-Club, and there are 13 possible ranks (ace, king,...), for a total of 78 possible pairs in the first two cards. There are $\binom{52}{2} = \frac{52\cdot 51}{2} = 1326$ possible two-card combinations that can be received in the first two cards. The probability of getting a pair in the first two cards is $\frac{78}{1326} = .0588$. Answer: E

27. $Var(aX+b) = a^2 Var(X) \to 4a^2 = 1 \to a = \frac{1}{2}$.

$E(aX+b) = aE(X)+b \to 2a+b=5 \to b=4 \to ab=2$. Answer: B

28. In order to be a properly defined joint distribution, it must be true that $c + 2c + \frac{c}{2} + c = 1$.
Therefore, $c = \frac{2}{9}$.
Then $E(XY) = (1)(1)(\frac{2}{9}) + (1)(2)(\frac{1}{9}) + (2)(1)(\frac{4}{9}) + (2)(2)(\frac{2}{9}) = \frac{20}{9}$.
The marginal distribution of X has $P(X = 1) = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}$, and $P(X = 2) = \frac{2}{3}$,
so $E(X) = \frac{5}{3}$.
The marginal distribution of Y has $P(Y = 1) = \frac{2}{9} + \frac{4}{9} = \frac{2}{3}$, and $P(Y = 2) = \frac{1}{3}$,
so $E(Y) = \frac{4}{3}$.
The covariance is $COV(X,Y) = E(XY) - E(X)E(Y) = \frac{20}{9} - (\frac{5}{3})(\frac{4}{3}) = 0$.
An alternative solution follows from the observation that X and Y are independent. Once we have determined the marginal distributions of X and Y, we can check to see if
$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for each (x, y) pair. For instance,
$P(X = 1, Y = 1) = c = \frac{2}{9}$ and $P(X = 1) \cdot P(Y = 1) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$. This turns out to be true for all (x, y) pairs. It follows that X and Y are independent, from which it follows that the covariance between X and Y is 0. Answer: C

29. If $X > \frac{2\theta}{3}$, then $2\theta > 3X > Y$, so
$P(Y < 3X) = \int_0^{2\theta/3}\int_0^{3x} \frac{1}{\theta} \cdot \frac{1}{2\theta}\, dy\, dx + \int_{2\theta/3}^{\theta} \frac{1}{\theta}\, dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Answer: D

30. For Policy 1, the maximum payment amount of 500 is reached if the loss is 600 or more because the deductible of 100 is applied first.
Expected insurance payment with Policy 1 is
$\int_{100}^{600} (x - 100) \cdot \frac{1}{1000}\, dx + 500P(X > 600) = 125 + 500\left(\frac{4}{10}\right) = 325$.
Expected insurance payment with Policy 2 is
$\int_0^{500} x \cdot \frac{1}{1000}\, dx + 400P(X > 500) = 125 + 400\left(\frac{5}{10}\right) = 325$. Answer: C

PRACTICE EXAM 8

1. X is a continuous random variable with density function $f(x) = \begin{cases} |x| & \text{for } -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$.

Find $E[\,|X|\,]$.

A) 0 B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) 1 E) $\frac{4}{3}$

2. As part of the underwriting process for insurance, each prospective policyholder is tested for diabetes. Let X represent the number of tests completed when the first person with diabetes pressure is found. The expected value of X is 8. Calculate the probability that the fourth person tested is the first one with diabetes.

A) 0.000 B) 0.050 C) 0.084 D) 0.166 E) 0.394

3. If X has a normal distribution with mean 1 and variance 4, then $P[X^2 - 4X \le 0] =$?

A) Less than .15 B) At least .15 but less than .35

C) At least .35 but less than .55 D) At least .55 but less than .75

E) At least .75

4. Let X and Y be discrete random variables with joint probability function $f(x, y)$ given by the following table:

		x			
		2	3	4	5
y	0	.05	.05	.15	.05
	1	.40	0	0	0
	2	.05	.15	.10	0

Calculate $Cov[X - Y, X + Y]$.

A) Less than -1 B) At least -1 but less than 0

C) At least 0 but less than 1 D) At least 1 but less than 2

E) At least 2

5. Let X and Y be continuous random variables with joint density function
$f(x,y) = \begin{cases} c(y-x) \text{ for } 0<x<y<1 \\ 0, \text{otherwise} \end{cases}$.
What is the mean of the marginal distribution of X?
A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

6. According to NBA playoff statistics, if a team has won 3 games and lost 1 game out of the first 4 games during a "best of 7" playoff series, that team has an 80% chance of winning the series. Statistics also show that if a team has won 3 games and lost 1 game out of the first 4 games and then loses the 5th game, that team has a 65% chance of winning the series. Find the probability that a team that has won 3 games and lost 1 game out of the first 4 games will win the next game.
A) $\frac{2}{7}$ B) $\frac{3}{7}$ C) $\frac{4}{7}$ D) $\frac{5}{7}$ E) $\frac{6}{7}$

7. A statistician for the National Hockey League has created a model for the number of goals scored per 60-minute game by the Ottawa Senators and the Buffalo Sabres. According to the model, the number of goals scored per game by the Senators has a geometric distribution, $X_{OTT} = 0, 1, 2, ...$ with a mean of 3.5. The model also has a similar geometric distribution for the number of goals scored per 60-minute game by the Sabres, X_{BUF}, with a mean of 3.0. Assuming that X_{OTT} and X_{BUF} are independent, find the probability that Buffalo wins the game in 60 minutes by at least 2 goals.
A) .1 B) .2 C) .3 D) .4 E) .5

8. n fair six-sided dice are tossed independently of one another.
Find the probability that the sum is even.
A) $\frac{1}{2} - \frac{(n-1)(n-2)(n-3)}{6n^3}$ B) $\frac{1}{2} - \frac{(n-1)(n-2)}{6n^2}$ C) $\frac{1}{2}$
D) $\frac{1}{2} + \frac{(n-1)(n-2)(n-3)}{6n^3}$ E) $\frac{1}{2} + \frac{(n-1)(n-2)}{6n^2}$

9. A fair coin is tossed 100 times. The tosses are independent of one another. The number of heads tossed is X. It is desired to find the smallest integer value k which satisfies the probability relationship $P(50 - k \leq X \leq 50 + k) \geq .95$.
Find k by applying the normal approximation with integer correction to the distribution of X.
A) 6 B) 7 C) 8 D) 9 E) 10

10. The loss random variable X has an exponential distribution with a mean of $\theta > 0$. An insurance policy pays Y, where $Y = \begin{cases} \frac{X}{2} & \text{if } X \le \theta \\ X & \text{if } X > \theta \end{cases}$

Find $E[Y]$.

A) $\frac{\theta}{2}(1+e^{-1})$ B) $\frac{\theta}{2}(1+2e^{-1})$ C) $\frac{\theta}{2}(1-e^{-1})$ D) $\frac{\theta}{2}(1-2e^{-1})$ E) $\theta(1-e^{-1})$

11. X has a continuous uniform distribution on the interval $[0,1]$ and the conditional distribution of Y given $X=x$ is a continuous uniform distribution on the interval $[x,2]$.

Find $E[Y]$.

A) $\frac{3}{4}$ B) 1 C) $\frac{5}{4}$ D) $\frac{3}{2}$ E) $\frac{7}{4}$

12. A loss random variable X has a continuous uniform distribution on the interval $(0,100)$. An insurance policy on the loss pays the full amount of the loss if the loss is less than or equal to 40. If the loss is above 40 but less than or equal to 80, then the insurance pays 40 plus one-half of the loss in excess of 40. If the loss is above 80, the insurance pays 60. If Y denotes the amount paid by the insurance when a loss occurs, find the variance of Y.

A) $\frac{1020}{3}$ B) $\frac{1040}{3}$ C) $\frac{1060}{3}$ D) $\frac{1080}{3}$ E) $\frac{1110}{3}$

13. A survey of a large number of adult city dwellers identified two characteristics involving personal transportation:

- have a driver's licence
- own a bicycle .

The following information was determined.

- 80% of those surveyed had a driver's licence or owned a bicycle, or both
- $\frac{1}{3}$ of those who had a driver's license also owned a bike
- $\frac{1}{2}$ of those who owned a bike also had a driver's license.

Of those surveyed who didn't own a bike, find the fraction that didn't have a driver's license.

A) $\frac{1}{3}$ B) $\frac{4}{9}$ C) $\frac{5}{9}$ D) $\frac{2}{3}$ E) $\frac{7}{9}$

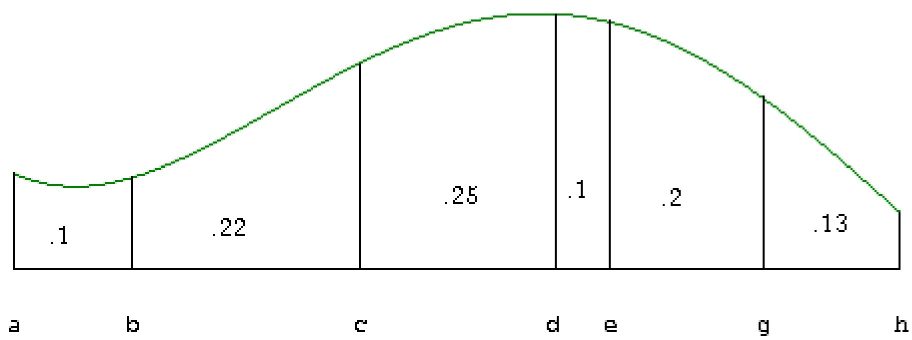
14. A particular large calculus class has two term tests and a final exam.
Students are not allowed to drop the course before the first term test.
Class records for past years show the following:

- 80% of students pass the first test
- 30% of students who fail the first term test drop the course before the second test
- 10% of students who pass the first term test drop the course before the second test
- 90% of students who pass the first term test and take the second test pass the second test
- 80% of students who fail the first term test and take the second test pass the second test
- 50% of students who fail the second term test drop the course before the final exam
- none of students who pass the second term test drop the course before the final exam.

Find the fraction of students who drop the course.

A) Less than $\frac{1}{20}$ B) At least $\frac{1}{20}$ but less than $\frac{1}{10}$ C) At least $\frac{1}{10}$ but less than $\frac{3}{20}$
D) At least $\frac{3}{20}$ but less than $\frac{1}{5}$ E) At least $\frac{1}{5}$

15. The graph below is the pdf of a continuous random variable X on the interval $[a, h]$.
The numerical values represent probabilities for the subintervals.
Find the conditional probability $P[b < X < e \mid (c < X < g) \cap (X < d)]$.



A) Less than .15 B) At least .15 but less than .35 C) At least .35 but less than .55
D) At least .55 but less than .75 E) At least .75

16. An urn has 6 identically shaped balls. 4 of the balls are white and 2 of the balls are blue. A ball is chosen at random from the urn and replaced with a white ball. The procedure is done repeatedly. Find the probability that after the n-th application of this procedure there is exactly one blue ball in the urn.

A) $(\frac{5}{6})^n + (\frac{2}{3})^n$ B) $(\frac{5}{6})^n - (\frac{2}{3})^n$ C) $2[(\frac{5}{6})^n + (\frac{2}{3})^n]$

D) $2[(\frac{5}{6})^n - (\frac{2}{3})^n]$ E) $2(\frac{5}{12})^n$

17. X has a Poisson distribution with a mean of 1, so the probability function for X is

$$P(X = x) = \frac{e^{-1}}{x!} \text{ for } x = 0, 1, 2, ...$$

Y is a new random variable on the non-negative integers. The probability function of Y is related to that of X as follows. A number α is given, with $0 < \alpha < 1$.

$$P(Y = 0) = \alpha \ , \ P(Y = x) = c \cdot P(X = x) \text{ for } x = 1, 2, ...$$

The number c is found so that Y satisfies the requirement for being a random variable $\sum_{x=0}^{\infty} P(Y = x) = 1$.

Find the mean of Y in terms of α and e.

A) $\frac{1-\alpha}{1-e^{-\alpha}}$ B) $\frac{1-\alpha}{e^{\alpha}-1}$ C) $\frac{1-\alpha}{1-e^{-1}}$ D) $\frac{1-\alpha}{e-1}$ E) $\frac{\alpha}{e-1}$

18. The Toronto Maple Leafs have two suppliers for hockey sticks, Crosscheck Lumber, and Sticks R Us. The Leafs get equal numbers of sticks from each supplier, and since the team logo is branded on every stick, after the sticks are delivered, it is not possible to tell what supplier provided any particular stick. The team estimates that on average, 10% of the sticks from Crosscheck lumber are defective and 20% of the sticks from Sticks R Us are defective. A Leaf player examines 10 sticks from a recent shipment from a supplier but doesn't know who the supplier was. The player finds 2 defective sticks out of the 10 sticks. Find the probability that the supplier of those sticks was Crosscheck Lumber.

A) Less than .11 B) At least .11 but less than .22 C) At least .22 but less than .33

D) At least .33 but less than .44 E) At least .44

19. The Winnipeg Rangers hockey team is considering a one-time charitable program of making a donation to the Winnipeg Children's Hospital. The donation will be related to how many goals they score in their next game. The team statistician has determined that the number of goals scored by the Rangers in a game has a Poisson distribution with a mean of 3.
The Rangers are planning donate $K for each goal they score up to a maximum of 3 goals.
Find the value of K that would make the Rangers' expected donations for game to be $5000..
A) Less than 2000 B) At least 2000 but less than 2100
C) At least 2100 but less than 2200 D) At least 2200 but less than 2300 E) At least 2300

20. X has a distribution which is partly continuous and partly discrete.
X has a discrete point of probability at $X = 1$ with probability p, where $0 < p < 1$.
On the interval $(0, 1)$ X has a constant density of $\frac{1-p}{2}$,
and on the interval $(1, 2)$ X has a constant density of $\frac{1-p}{2}$.
Find the variance of X in terms of p
A) $\frac{1-p}{3}$ B) $\frac{2-p}{3}$ C) $\frac{1-p}{2}$ D) $\frac{2-p}{2}$ E) $\frac{1+p}{2}$

21. X has the following pdf: $f(x) = \begin{cases} x - \frac{x^2}{2} & \text{if } 0 < x \leq 1 \\ \frac{x^2}{2} - x + 1 & \text{if } 1 < x < 2 \end{cases}$, and 0 otherwise.
The random variable Y is defined as follows: $Y = X^2$. Find $F_Y(2)$.
A) .33 B) .48 C) .55 D) .67 E) .80

22. You are given the following:
• X_1 has a binomial distribution with a mean of 2 and a variance of 1.
• X_2 has a Poisson distribution with a variance of 2.
• X_1 and X_2 are independent.
• $Y = X_1 + X_2$.
What is $P(Y < 3)$?
A) $\frac{11}{16}e^{-2}$ B) $\frac{15}{16}e^{-2}$ C) $\frac{19}{16}e^{-2}$ D) $\frac{23}{16}e^{-2}$ E) $\frac{27}{16}e^{-2}$

23. X has pdf $f(x) = x$ for $0 < x < 1$.

Also, $P(X = 0) = a$ and $P(X = 1) = b$, and $P(X < 0) = P(X > 1) = 0$.

For what value of a is $Var(X)$ maximized?

A) $0 \leq a < .1$ B) $.1 \leq a < .2$ C) $.2 \leq a < .3$ D) $.3 \leq a < .4$ E) $a \geq .4$

24. You are given the events $A \neq \emptyset$ and $B \neq \emptyset$ satisfy the relationships

(i) $P(A \cap B) > 0$ and

(ii) $P(A|B) = P(B|A)$ (conditional probabilities) .

How many of the following statements always must be true?

I. A and B are independent. II. $P(A) = P(B)$ III. $A = B$

A) None B) 1 C) 2 D) All 3 E) None of A,B,C or D is correct

25. A loss random variable is uniformly distributed on the interval $(0\,,\ 2000)$.

An insurance policy on this loss has an ordinary deductible of 500 for loss amounts up to 1000. If the loss is above 1000, the insurance pays half of the loss amount.

Find the standard deviation of the amount paid by the insurance when a loss occurs.

A) Less than 250 B) At least 250, but less than 300 C) At least 300, but less than 350

D) At least 350, but less than 400 E) At least 400

26. Random variables X and Y have a joint distribution with joint pdf

$$f(x, y) = \frac{2x+y}{12} \text{ for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2$$

Find the conditional probability $P(X + Y \geq 2 | X \leq 1)$.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

27. The pdf of X is $f(x) = ax + b$ on the interval $[0, 2]$ and the pdf is 0 elsewhere.

You are given that the median of X is 1.25. Find the variance of X.

A) Less than .05 B) At least .05 but less than .15 C) At least .15 but less than .25

D) At least .25 but less than .35 E) At least .35

28. In the casino game of roulette, a wheel with 38 equally likely spots is spun, and a ball is dropped at random into one of the 38 spots. The 38 spots are numbers 1 to 36 along with 0 and 00. On a spin of the wheel, a gambler can bet that the ball will drop into a specified spot. If the ball does drop into that spot, the gamble gets back the amount that he bet plus 36 time the amount that he bet. If that spot does not turn up, the gambler loses the amount bet. A gambler can also bet that the outcome of the spin will be even. If the ball drops into an even number spot from 2 to 36, the gambler gets back his bet plus an amount equal to the amount that he bet (the bet is lost if the spot is 0 or 00). On every spin, Gambler 1 always bets 1 that the ball will drop in the spot with the number 1, and Gambler 2 always bets 1 that the ball will drop into an even numbered spot. X_1 denotes the net profit of Gambler 1 after n spins, and X_2 denotes the net profit of Gambler 2 after the n spins. Find $E(X_2 - X_1)$.

A) $-\frac{n}{19}$ B) $-\frac{n}{38}$ C) 0 D) $\frac{n}{38}$ E) $\frac{n}{19}$

29. A loss random variable X has a Poisson distribution with a mean of λ

An insurance policy on the loss has a policy limit of 1.

The expected insurance payment when a loss occurs is .8892 .

Find the expected insurance payment when a loss occurs for a policy on the same loss variable if the policy limit is 2.

A) Less than .35 B) At least .35 but less than .70 C) At least .70 but less than 1.05

D) At least 1.05 but less than 1.4 E) At least 1.4

30. An insurer has two lines of business: auto insurance and home fire insurance.

People with a home fire insurance policy can add flood insurance coverage, but only if the policy already has fire coverage. You are given the following information about the insurer's customers:

- 80% of all customers have an auto insurance policy
- 40% of all customers have a fire insurance policy
- 25% of customers with an auto insurance policy also have a fire insurance policy
- 50% of customers with a fire insurance policy also have flood insurance
- 50% of customers with flood insurance coverage also have auto insurance

Of the insurer's customers that have fire insurance, find the fraction that have neither auto insurance nor flood insurance coverage.

A) .05 B) .10 C) .15 D) .20 E) .25

PRACTICE EXAM 8 - SOLUTIONS

1. $E[\,|X|\,] = \int_{-1}^{1} |x|\, f(x)\, dx = \int_{-1}^{1} |x| \cdot |x|\, dx = \int_{-1}^{1} |x|^2\, dx = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$.
Answer: C

2. This problem makes use of the geometric distribution. The experiment being performed is the diabetes test on an individual. We define "success" of the experiment to mean that the individual has high diabetes. We denote the probability of a success occurring in a particular trial by p. Since X is the number of persons tested until the first person with diabetes is found, it is a version of the geometric distribution, where Y is the trial number of the first success (the trial number of the first success is 1, or 2, or 3, ...).
The probability function if $P(Y = k) = (1-p)^{k-1}p$, $k = 1, 2, 3, ...$
The mean of this form of the geometric distribution is $\frac{1}{p}$, so that
$\frac{1}{p} = 8$ and therefore $p = \frac{1}{8}$. The probability that the first success occurs on the 4th trial (first case of diabetes is the 4th individual) is $(1-p)^3 p$, since there will be 3 failures and then the first success. This probability is $(\frac{7}{8})^3(\frac{1}{8}) = .08374$. Answer: C

3. Since $X \sim N(1, 4)$, $Z = \frac{X-1}{2}$ has a standard normal distribution. The probability in question can be written as
$P[X^2 - 4X \leq 0] = P[X^2 - 4X + 4 \leq 4] = P[(X-2)^2 \leq 4] = P[-2 \leq X - 2 \leq 2]$
$= P[-1 \leq X - 1 \leq 3]$
$= P[-.5 \leq \frac{X-1}{2} \leq 1.5] = P[-.5 \leq Z \leq 1.5] = \Phi(1.5) - [1 - \Phi(.5)]$
$= .9332 - .3085 = .6247$. (from the standard normal table). Answer: D

4. $Cov[X-Y, X+Y] = Cov[X,X] + Cov[X,Y] - Cov[Y,X] - Cov[Y,Y]$
$= Var[X] - Var[Y]$
The marginal distribution of X has probability function
$P(X=2) = .5$, $P(X=3) = .2$, $P(X=4) = .25$, $P(X=5) = .05$.
$E[X] = (2)(.5) + (3)(.2) + (4)(.25) + (5)(.05) = 2.85$.
$E[X^2] = (4)(.5) + (9)(.2) + (16)(.25) + (25)(.05) = 9.05$.
$Var[X] = E[X^2] = (E[X])^2 = 9.05 - 2.85^2 = .9275$.

4. continued
The marginal distribution of Y has probability function
$P(Y=0)=.3\,,\ P(Y=1)=.4\,,\ P(Y=2)=.3\,.$
$E[Y]=(1)(.4)+(2)(.3)=1.0\,.$
$E[Y^2]=(1)(.4)+(4)(.3)=1.6\,.$
$Var[Y]=E[Y^2]=(E[Y])^2=1.6-1^2=.6\,.$
$Cov[X-Y,X+Y]=Var[X]-Var[Y]=.9275-.6=.3275\,.$ Answer: C

5. In order for this to be a properly defined joint pdf, we must have
$\int_0^1\int_x^1 c(y-x)\,dy\,dx\ =\ 1\,.$
$\int_x^1 c(y-x)\,dy=c[\frac{1-x^2}{2}-x(1-x)]=\frac{c(1-x)^2}{2}\,,$ and $\int_0^1 \frac{c(1-x)^2}{2}\,dx=\frac{c}{6}\,.$
Therefore, $c=6$.
$f_X(x)=\int_x^1 6(y-x)\,dy=3(1-x)^2\,,\ 0<x<1$
$E[X]=\int_0^1 3x(1-x)^2\,dx=3\int_0^1[x-2x^2+x^3]\,dx=3[\frac{1}{2}-2(\frac{1}{3})+\frac{1}{4}]=.25\,.$
Answer: B

6. We define the following events and probabilities:
$W=$ team wins the best-of-7 series,
$G=$ team loses game 5,
$T=$ team wins 3 of the first 4 games,
$q=$ probability team wins 5th game given that it has won 3 of the first 4 games.
Our objective is to find $q=P[G'|T]$.
We are given $P[W|T]=.8$ and $P[W|T\cap G]=.65\,.$
$P[W\cap G|T]=\frac{P[W\cap G\cap T]}{P[T]}=\frac{P[W\cap G\cap T]}{P[G\cap T]}\cdot\frac{P[G\cap T]}{P[T]}$
$=P[W|G\cap T]\cdot P[G|T]=(.65)(1-q)\,.$

$.8=P[W|T]=P[W\cap G|T]+P[W\cap G'|T]=(.65)(1-q)+P[W\cap G'|T]\,.$
$P[W\cap G'|T]=\frac{P[W\cap G'\cap T]}{P[T]}=\frac{P[W\cap G'\cap T]}{P[G'\cap T]}\cdot\frac{P[G'\cap T]}{P[T]}$
$=P[W|G'\cap T]\cdot P[G'|T]=q$
(this is true, since $P[W|G'\cap T]=1$, because winning 3 out of the first 4 and then winning the 5th game results in winning the series) . Therefore, $.8=(.65)(1-q)+q\ \rightarrow\ q=\frac{.15}{.35}=\frac{3}{7}\,.$

6. continued

An alternative solution is as follows.

$P[W'|T] = .2$ and $P[W'|T \cap G] = .35$ (these are the complement of the given probabilities $P[W|T] = .8$ and $P[W|T \cap G] = .65$).

First note that $P[W' \cap T] = P[W' \cap G \cap T]$, because in order to win 3 of the first 4 games and lose the series, it must be true that the team loses the 5th (and all subsequent games). Therefore,

$.2 = P[W'|T] = \frac{P[W' \cap T]}{P[T]} = \frac{P[W' \cap G \cap T]}{P[T]} = \frac{P[W' \cap G \cap T]}{P[G \cap T]} \cdot \frac{P[G \cap T]}{P[T]}$

$= P[W'|T \cap G] \cdot P[G|T] = (.35) \cdot P[G|T]$. It follows that $P[G|T] = \frac{.2}{.35} = \frac{4}{7}$,

and then $q = P[G'|T] = 1 - \frac{4}{7} = \frac{3}{7}$. Answer: B

7. The geometric distribution $X = 0, 1, 2, \ldots$ has probability function $P[X = k] = (1-p)^k p$ and has mean $\frac{1-p}{p}$. For the Senators, we have $\frac{1-p_{OTT}}{p_{OTT}} = 3.5$, so that $p_{OTT} = \frac{1}{4.5}$.

For the Sabres, we have $\frac{1-p_{BUF}}{p_{BUF}} = 3$, so that $p_{OTT} = \frac{1}{4}$.

$P[X_{OTT}] = k = (1 - \frac{1}{4.5})^k (\frac{1}{4.5}) = (\frac{7}{9})^k (\frac{2}{9})$ and $P[X_{BUF} = k] = (1 - \frac{1}{4})^k (\frac{1}{4}) = (\frac{3}{4})^k (\frac{1}{4})$.

$P[X_{BUF} \geq n] = \sum_{k=n}^{\infty} (\frac{3}{4})^k (\frac{1}{4}) = (\frac{3}{4})^n$.

$P[X_{BUF} \geq X_{OTT} + 2]$

$= P[X_{BUF} \geq 2 | X_{OTT} = 0] \cdot P[X_{OTT} = 0] + P[X_{BUF} \geq 3 | X_{OTT} = 1] \cdot P[X_{OTT} = 1] + \cdots$

$= \sum_{n=0}^{\infty} P[X_{BUF} \geq n+2 | X_{OTT} = n] \cdot P[X_{OTT} = n]$

$= \sum_{n=0}^{\infty} P[X_{BUF} \geq n+2] \cdot P[X_{OTT} = n]$ (because of independence of X_{BUF} and X_{OTT})

$= \sum_{n=0}^{\infty} (\frac{3}{4})^{n+2} (\frac{7}{9})^n (\frac{2}{9}) = (\frac{3}{4})^2 (\frac{2}{9}) \sum_{n=0}^{\infty} (\frac{3}{4} \cdot \frac{7}{9})^n = \frac{1}{8} \cdot \frac{1}{1 - \frac{7}{12}} = .30$. Answer: C

8. The probability of an even outcome when tossing a single $(n = 1)$ die is $\frac{1}{2}$.

The probabilities for the sum when tossing two dice are

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The probability that the sum is even is $\frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{18}{36} = \frac{1}{2}$.

8. continued

To see that the probability is always $\frac{1}{2}$, suppose that E_{n-1} is the event that sum of the first $n-1$ tosses is even. Then in order for the sum of the n dice to be even, we must have either E_{n-1} occurring and the n-th toss is even, or E'_{n-1} occurring (complement) and the n-th toss is odd. Because of independence of the tosses, we get

$P(\text{sum of } n \text{ tosses is even}) = P(E_{n-1})(\frac{1}{2}) + P(E'_{n-1})(\frac{1}{2}) = (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) = \frac{1}{2}$.

Since $P(E_1) = \frac{1}{2}$, it follows that $P(E_k) = \frac{1}{2}$ for any k. Answer: C

9. X has a binomial distribution with 100 trials and probability $\frac{1}{2}$ of success. The expected number of heads is $100(\frac{1}{2}) = 50$ and the variance of the number of heads is $100(\frac{1}{2})(\frac{1}{2}) = 25$.

Using the normal approximation with integer correction, we want to satisfy the relationship

$P(50 - k - .5 \leq X \leq 50 + k + .5) \geq .95$.

Applying the normal approximation, we have

$P(50 - k - .5 \leq X \leq 50 + k + .5) = P(\frac{-k-.5}{\sqrt{25}} \leq \frac{X-50}{\sqrt{25}} \leq \frac{k+.5}{\sqrt{25}}) = P(-c \leq Z \leq c)$,

where Z is standard normal. In order for this probability to be at least .95, it must be true that $\Phi(c) \geq .975$. This is true because we want to eliminate less than .025 probability from the left and right side of Z.

From the standard normal table, $\Phi(1.96) = .975$, and therefore, we must have $c \geq 1.96$.

Then, $\frac{k+.5}{5} \geq 1.96 \rightarrow k \geq 9.3$. The smallest integer is $k = 10$.

Using the normal approximation, $P(40 \leq X \leq 60) \geq .95$ but $P(41 \leq X \leq 59) < .95$.

Answer: E

10. $E[Y] = \int_0^\theta \frac{x}{2} \cdot \frac{1}{\theta} e^{-x/\theta}\, dx \,+\, \int_\theta^\infty x \cdot \frac{1}{\theta} e^{-x/\theta}\, dx$

$= \frac{1}{2}(-xe^{-x/\theta} - \theta e^{-x/\theta})\Big|_{x=0}^{x=\theta} \,+\, (-xe^{-x/\theta} - \theta e^{-x/\theta})\Big|_{x=\theta}^{x=\infty}$

$= \frac{\theta}{2}(1 - 2e^{-1}) + 2\theta e^{-1} = \frac{\theta}{2}(1 + 2e^{-1})$ Answer: B

11. The pdf of X is $f_X(x) = 1$ for $0 \le x \le 1$, and the conditional pdf of Y given $X = x$ is $f_{Y|X}(y|x) = \frac{1}{2-x}$ for $x \le Y \le 2$.

The joint density of X and Y is $f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \frac{1}{2-x}$ defined on the region $0 \le x \le 1$ and $x \le y \le 2$.

$E[Y] = \int_0^1 \int_x^2 y \cdot \frac{1}{2-x}\, dy\, dx = \int_0^1 \frac{4-x^2}{2(2-x)}\, dx = \int_0^1 \frac{2+x}{2}\, dx = \frac{5}{4}$.

An alternative, solution makes use of the rule $E[Y] = E[\,E[Y|X]\,]$.

Since the conditional distribution of $Y|X = x$ is uniform on the interval from x to 2, it follows that $E[Y|X] = \frac{X+2}{2}$. Then,

$E[Y] = E[\,E[Y|X]\,] = E[\frac{X+2}{2}] = \frac{1}{2}E[X] + 1$.

Since X is uniform on the interval from 0 to 1, $E[X] = \frac{1}{2}$.

Then, $E[Y] = (\frac{1}{2})(\frac{1}{2}) + 1 = \frac{5}{4}$. Answer: C

12. $Y = \begin{cases} X & X \le 40 \\ 40 + \frac{1}{2}(x-40) & 40 < X \le 80 \\ 60 & X > 80 \end{cases}$. $Var[Y] = E[Y^2] - (E[Y])^2$.

$E[Y] = \int_0^{40} x \cdot (.01)\, dx + \int_{40}^{80} (20 + .5x)(.01)\, dx + \int_{80}^{100} 60(.01)\, dx$

$= 8 + 20 + 12 = 40$.

$E[Y^2] = \int_0^{40} x^2 \cdot (.01)\, dx + \int_{40}^{80} (20 + .5x)^2(.01)\, dx + \int_{80}^{100} 60^2(.01)\, dx$

$= \frac{640}{3} + \frac{3040}{3} + 720 = \frac{5840}{3}$.

$Var[Y] = \frac{5840}{3} - 40^2 = \frac{1040}{3}$. Answer: B

13. $A =$ have driver's license $B =$ own a bike

$P(A \cup B) = .8 = P(A) + P(B) - P(A \cap B)$

$P(B|A) = \frac{1}{3} = \frac{P(A \cap B)}{P(A)}$, $P(A|B) = \frac{1}{2} = \frac{P(A \cap B)}{P(B)}$.

$\frac{P(A \cup B)}{P(A \cap B)} = \frac{.8}{P(A \cap B)} = \frac{P(A)+P(B)-P(A \cap B)}{P(A \cap B)} = \frac{1}{1/3} + \frac{1}{1/2} - 1 = 4$,

and it follows that $P(A \cap B) = .2$.

Then $P(A) = 3P(A \cap B) = .6$ and $P(B) = 2P(A \cap B) = .4$.

We wish to find $P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{P[(A \cup B)']}{P(B')} = \frac{1-P(A \cup B)}{1-P(B)} = \frac{1-.8}{1-.4} = \frac{1}{3}$.

Answer: A

14. A student can drop the course after the first test but before the second test.
The fraction of the original group of students that drop the course after the first test but before the second test is
$P[\text{drop after 1st test but before 2nd test}]$
$= P[\text{drop after 1st test but before 2nd test} \cap \text{pass 1st test}]$
$\quad + P[\text{drop after 1st test but before 2nd test} \cap \text{fail 1st test}]$
$= P[\text{drop after 1st test but before 2nd test} \mid \text{pass 1st test}] \cdot P[\text{pass 1st test}]$
$\quad + P[\text{drop after 1st test but before 2nd test} \mid \text{fail 1st test}] \cdot P[\text{fail 1st test}]$
$= (.1)(.8) + (.3)(.2) = .14$

A student can drop the course after the second test but before the final exam.
$P[\text{drop after 2nd test but before final exam}]$
$= P[\text{drop after 2nd test but before final exam} \cap \text{pass 1st test} \cap \text{take 2nd test} \cap \text{pass 2nd test}]$
$\quad + P[\text{drop after 2nd test but before final exam} \cap \text{pass 1st test} \cap \text{take 2nd test} \cap \text{fail 2nd test}]$
$P[\text{drop after 2nd test but before final exam} \cap \text{fail 1st test} \cap \text{take 2nd test} \cap \text{pass 2nd test}]$
$\quad + P[\text{drop after 2nd test but before final exam} \cap \text{fail 1st test} \cap \text{take 2nd test} \cap \text{fail 2nd test}]$

We find these probabilities in the following way:
$P[\text{drop after 2nd test but before final exam} \cap \text{fail 1st test} \cap \text{take 2nd test} \cap \text{fail 2nd test}]$
$= P[\text{drop after 2nd test but before final exam} | \text{fail 1st test} \cap \text{take 2nd test} \cap \text{fail 2nd test}]$
$\times P[\text{fail 2nd test} | \text{take 2nd test} \cap \text{fail 1st test}] \times P[\text{take 2nd test} | \text{fail 1st test}] \times P[\text{fail 1st test}]$
$= (.5)(.2)(.7)(.2) = .014$.
Similarly,
$P[\text{drop after 2nd test but before final exam} \cap \text{pass 1st test} \cap \text{take 2nd test} \cap \text{fail 2nd test}]$
$= (.5)(.1)(.9)(.8) = .036$.

$P[\text{drop after 2nd test but before final exam} \cap \text{fail 1st test} \cap \text{take 2nd test} \cap \text{pass 2nd test}]$ and
$P[\text{drop after 2nd test but before final exam} \cap \text{pass 1st test} \cap \text{take 2nd test} \cap \text{pass 2nd test}]$
are both 0, since anyone who passes the 2nd test does not drop the course.

The probability of dropping the course is $.14 + .014 + .036 = .19$. Answer: D

15. The region $(c < X < g) \cap (X < d)$ is $c < X < d$, so the probability is
$P[b < X < e \mid c < X < d] = \frac{P[(b<X<e) \cap (c<X<d)]}{P[c<X<d]}$.
The region $(b < X < e) \cap (c < X < d)$ is $c < X < d$, so
$P[b < X < e \mid c < X < d] = \frac{P[c<X<d]}{P[c<X<d]} = 1$. Answer: E

16. In order for there to be one blue ball in the urn after the n application, it must be true that a blue ball was chosen exactly once in the n applications of the procedure. The blue ball could have been chosen on the 1st, or 2nd, . . . , or n-th application.

P(blue ball chosen on 1st application and no blue ball chosen in next $n-1$ applications)
$= \frac{1}{3} \cdot (\frac{5}{6})^{n-1}$.
P(1st blue ball chosen on 2nd application and no blue ball chosen in next $n-2$ applications)
$= \frac{2}{3} \cdot \frac{1}{3} \cdot (\frac{5}{6})^{n-2}$.
$\vdots$
P(1st blue ball chosen on k-th application and no blue ball chosen in next $n-k$ applications)
$= (\frac{2}{3})^{k-1} \cdot \frac{1}{3} \cdot (\frac{5}{6})^{n-k}$.
$\vdots$
P(1st blue ball chosen on n-th application) $= (\frac{2}{3})^{n-1} \cdot \frac{1}{3}$.
The probability in question is the sum of these:

$$\sum_{k=1}^{n} (\tfrac{2}{3})^{k-1} \cdot \tfrac{1}{3} \cdot (\tfrac{5}{6})^{n-k} = \tfrac{1}{3} \cdot (\tfrac{2}{3})^{-1} \cdot (\tfrac{5}{6})^{n} \cdot \sum_{k=1}^{n} (\tfrac{2}{3})^{k} (\tfrac{5}{6})^{-k}$$
$$= \tfrac{1}{2} \cdot (\tfrac{5}{6})^{n} \cdot \sum_{k=1}^{n} (\tfrac{2/3}{5/6})^{k} = \tfrac{1}{2} \cdot (\tfrac{5}{6})^{n} \cdot \sum_{k=1}^{n} (.8)^{k} = \tfrac{1}{2} \cdot (\tfrac{5}{6})^{n} \cdot (.8) \cdot \sum_{k=1}^{n} (.8)^{k-1}$$
$$= (.4) \cdot (\tfrac{5}{6})^{n} \cdot \tfrac{1-(.8)^{n}}{1-.8} = 2[(\tfrac{5}{6})^{n} - (\tfrac{2}{3})^{n}]$$. Answer: D

17. Since $\sum_{x=0}^{\infty} P(X=x) = 1$, it follows that $\sum_{x=1}^{\infty} P(X=x) = 1 - P(X=0) = 1 - e^{-1}$.
Then, $\sum_{x=1}^{\infty} P(Y=x) = c \cdot \sum_{x=1}^{\infty} P(X=x) = c(1-e^{-1})$.
But it is also true that $\sum_{x=1}^{\infty} P(Y=x) = 1 - P(Y=0) = 1 - \alpha$.
Therefore, $c(1-e^{-1}) = 1 - \alpha$, so that $c = \frac{1-\alpha}{1-e^{-1}}$.
The mean of Y is

$$E[Y] = \sum_{x=0}^{\infty} x \cdot P(Y=x) = \sum_{x=1}^{\infty} x \cdot P(Y=x) = \sum_{x=1}^{\infty} x \cdot c \cdot P(X=x) = c \cdot \sum_{x=o}^{\infty} x \cdot P(X=x)$$
$$= c \cdot E[X] = c = \frac{1-\alpha}{1-e^{-1}}$$. Answer: C

18. We define the following events:

C - shipment is from Crosscheck Lumber

S - shipment is from Sticks R Us

$2D$ - 2 sticks are defective

We wish to find $P(C|2D)$. This is $\frac{P(C\cap 2D)}{P(2D)}$.

The numerator can be formulated as $P(2D|C)\cdot P(C)$.

We are given that $P(C)=.5$. For a shipment from Crosscheck Lumber, the number of sticks that are defective in a batch of 10 sticks has a binomial distribution with $n=10$ and $p=.1$ (prob. of a particular stick being defective).

Therefore, $P(2D|C)=\binom{10}{2}(.1)^2(.9)^8=.193710$.

The numerator is $P(C\cap 2D)=(.193710)(.5)=.096855$.

The denominator can be formulated as $P(2D)=P(C\cap 2D)+P(S\cap 2D)$ since the shipment must be either C or S. We find $P(S\cap 2D)$ in the same way as $P(C\cap 2D)$. $P(S\cap 2D)=P(2D|S)\cdot P(S)=\binom{10}{2}(.2)^2(.8)^8\cdot(.5)=.150995$.

Then, $P(C|2D)=\frac{P(C\cap 2D)}{P(2D)}=\frac{P(C\cap 2D)}{P(C\cap 2D)+P(S\cap 2D)}=\frac{.096855}{.096855+.150995}=.39$.

Answer: D

19. The donation is $\begin{cases} 0 & \text{Prob. } e^{-3} \\ K & \text{Prob. } 3e^{-3} \\ 2K & \text{Prob. } \frac{9e^{-3}}{2} \\ 3K & \text{Prob. } 1-(e^{-3}+3e^{-3}+\frac{9e^{-3}}{2}) \end{cases}$

The expected donation is

$K\cdot 3e^{-3}+2K\cdot\frac{9e^{-3}}{2}+3K\cdot[\,1-(e^{-3}+3e^{-3}+\frac{9e^{-3}}{2})]=2.328K$.

Setting this equal to 5000 results in $K=2148$. Answer: C

20. Since X has a symmetric distribution about the point $X=1$, it follows that $E[X]=1$. The second moment of X is

$E[X^2]=\int_0^1 x^2\cdot\frac{1-p}{2}\,dx+1^2\cdot p+\int_1^2 x^2\cdot\frac{1-p}{2}\,dx$

$=\frac{1}{3}\cdot\frac{1-p}{2}+p+\frac{7}{3}\cdot\frac{1-p}{2}=\frac{4}{3}-\frac{p}{3}$.

The variance of X is $Var[X]=E[X^2]-(E[X])^2=\frac{4}{3}-\frac{p}{3}-1=\frac{1-p}{3}$. Answer: A

21. The cdf of X is $F_X(t) = \int_0^t f(x)\,dx = \begin{cases} \frac{t^2}{2} - \frac{t^3}{6} & \text{if } 0 < t \le 1 \\ \frac{1}{3} + \frac{t^3-1}{6} - \frac{t^2-1}{2} + t - 1 & \text{if } 1 < x < 2 \end{cases}$

The cdf of Y is $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{2})$.

Since $1 < \sqrt{2} < 2$, we get

$F_Y(2) = F_X(\sqrt{2}) = \frac{1}{3} + \frac{(\sqrt{2})^3-1}{6} - \frac{(\sqrt{2})^2-1}{2} + \sqrt{2} - 1 = .55$. Answer: C

22. X_1 is binomial with $np = 2$ and $np(1-p) = 1$.

It follows that $1 - p = \frac{1}{2}$, and $p = \frac{1}{2}$, and $n = 4$.

The probability function of X_1 is $P(X_1 = k) = \binom{4}{k}(\frac{1}{2})^k(\frac{1}{2})^{4-k} = \binom{4}{k}(\frac{1}{2})^4$.

The probability function of X_2 is $P(X_2 = j) = \frac{2^j e^{-2}}{j!}$.

$P(Y < 3) = P(Y = 0) + P(Y = 1) + P(Y = 2)$.

$P(Y = 0) = P(X_1 = 0 \cap X_2 = 0) = P(X_1 = 0) \times P(X_2 = 0)$

$= \binom{4}{0}(\frac{1}{2})^4 \times \frac{2^0 e^{-2}}{0!} = \frac{1}{16}e^{-2}$.

$P(Y = 1) = P(X_1 = 0 \cap X_2 = 1) + P(X_1 = 1 \cap X_2 = 0)$

$= P(X_1 = 0) \times P(X_2 = 1) + P(X_1 = 1) \times P(X_2 = 0)$

$= \binom{4}{0}(\frac{1}{2})^4 \times \frac{2^1 e^{-2}}{1!} + \binom{4}{1}(\frac{1}{2})^4 \times \frac{2^0 e^{-2}}{0!} = \frac{2}{16}e^{-2} + \frac{4}{16}e^{-2} = \frac{6}{16}e^{-2}$.

$P(Y = 2) = P(X_1 = 0 \cap X_2 = 2) + P(X_1 = 1 \cap X_2 = 1) + P(X_1 = 2 \cap X_2 = 0)$

$= P(X_1 = 0) \times P(X_2 = 2) + P(X_1 = 1) \times P(X_2 = 1) + P(X_1 = 2) \times P(X_2 = 0)$

$= \binom{4}{0}(\frac{1}{2})^4 \times \frac{2^2 e^{-2}}{2!} + \binom{4}{1}(\frac{1}{2})^4 \times \frac{2^1 e^{-2}}{1!} + \binom{4}{2}(\frac{1}{2})^4 \times \frac{2^0 e^{-2}}{0!}$

$= \frac{2}{16}e^{-2} + \frac{8}{16}e^{-2} + \frac{6}{16}e^{-2} = e^{-2}$.

Then, $P(Y < 3) = \frac{1}{16}e^{-2} + \frac{6}{16}e^{-2} + e^{-2} = \frac{23}{16}e^{-2}$. Answer: D

23. In order to be a properly defined random variable, we must have

$P(X = 0) + P(0 < X < 1) + P(X = 1) = 1$, so that

$a + \int_0^1 x\,dx + b = a + \frac{1}{2} + b = 1$. Therefore, $a + b = \frac{1}{2}$.

$Var(X) = E(X^2) - [E(X)]^2$.

$E(X) = 0 \times a + \int_0^1 x \times x\,dx + 1 \times b = \frac{1}{3} + b$, and

$E(X^2) = 0 \times a^2 + \int_0^1 x^2 \times x\,dx + 1^2 \times b = \frac{1}{4} + b$.

Then, $Var(X) = \frac{1}{4} + b - (\frac{1}{3} + b)^2 = \frac{5}{36} + \frac{b}{3} - b^2$.

$Var(X)$ will be maximized if $\frac{d}{db}\left[\frac{5}{36} + \frac{b}{3} - b^2\right] = \frac{1}{3} - 2b = 0$.

This occurs at $b = \frac{1}{6}$. Then $a = \frac{1}{2} - b = \frac{1}{3}$. Answer: D

24. $P(A|B) = \frac{P(A \cap B)}{P(A)}$ and $P(B|A) = \frac{P(A \cap B)}{P(B)}$.
Since $P(A \cap B) > 0$ it follows that $P(A) = P(B)$, so II is true.
If $A = \{1,2,3\}$ and $B = \{3,4,5\}$ when tossing a fair die then the conditions are satisfied, but I is false since $P(A \cap B) = \frac{1}{6} \neq P(A) \times P(B)$, and III is false.
Answer: B

25. The amount paid by the insurance is Y, where $Y = \begin{cases} 0 & \text{if } X \leq 500 \\ X - 500 & \text{if } 500 < X \leq 1000 \\ \frac{X}{2} & \text{if } 1000 < X < 2000 \end{cases}$.

$Var(Y) = E(Y^2) - [E(Y)]^2$.

$E(Y) = \int_{500}^{1000} (x - 500) \times \frac{1}{2000}\, dx + \int_{1000}^{2000} \frac{x}{2} \times \frac{1}{2000}\, dx = \frac{125}{2} + 375 = \frac{875}{2}$.

$E(Y^2) = \int_{500}^{1000} (x - 500)^2 \times \frac{1}{2000}\, dx + \int_{1000}^{2000} (\frac{x}{2})^2 \times \frac{1}{2000}\, dx = \frac{62{,}500}{3} + \frac{875{,}000}{3} = \frac{937{,}500}{3}$.

$Var(Y) = \frac{937{,}500}{3} - (\frac{875}{2})^2 = 121{,}093.75$.

Standard deviation of Y is $\sqrt{Var(Y)} = \sqrt{121{,}093.75} = 348$ .Answer: C

26. $P(X + Y \geq 2 | X \leq 1) = \frac{P(X+Y \geq 2 \cap X \leq 1)}{P(X \leq 1)}$.

$P(X \leq 1) = \int_0^1 \int_0^2 \frac{2x+y}{12}\, dy\, dx = \frac{1}{3}$.

$P(X + Y \geq 2 \cap X \leq 1) = \int_0^1 \int_{2-x}^2 \frac{2x+y}{12}\, dy\, dx = \int_0^1 \frac{3x^2+4x}{24}\, dx = \frac{1}{8}$.

$P(X + Y \geq 2 | X \leq 1) = \frac{1/8}{1/3} = \frac{3}{8}$. Answer: C

27. Since $f(x)$ is a pdf, we know that $\int_0^2 f(x)\, dx = 2a + 2b = 1$.
$F(x) = \int_0^t f(t)\, dt = \frac{at^2}{2} + bt$, so $F(\frac{5}{4}) = \frac{25a}{32} + \frac{5b}{4} = \frac{1}{2}$.
Solving these two equations results in $a = \frac{4}{15}$, $b = \frac{7}{30}$.
The mean of X is $E(X) = \int_0^2 x(\frac{4x}{15} + \frac{7}{30})\, dx = \frac{53}{45}$
and the second moment of X is $E(X^2) = \int_0^2 x^2(\frac{4x}{15} + \frac{7}{30})\, dx = \frac{76}{45}$.
The variance of X is $E(X^2) - [E(X)]^2 = \frac{76}{45} - \left(\frac{53}{45}\right)^2 = \frac{611}{45^2} = .302$. Answer: D

28. Let Y denote the net profit of Gambler 1 for one spin.
Then Y is either -1 with probability $\frac{37}{38}$ or Y is 36 with probability $\frac{1}{38}$.
Then $E(Y) = -1 \times \frac{37}{38} + 36 \times \frac{1}{38} = -\frac{1}{38}$.
The net profit after n spins for Gambler 1 is $X_1 = Y_1 + Y_2 + \cdots + Y_n$,
and the expected profit is
$E(X_1) = E(Y_1) + E(Y_2) + \cdots + E(Y_n) = -\frac{1}{38} \times n = -\frac{n}{38}$.

Let Z denote the net profit of Gambler 2 for one spin.
Then Z is either -1 with probability $\frac{20}{38}$ or Z is 1 with probability $\frac{18}{38}$.
Then $E(Z) = -1 \times \frac{20}{38} + 1 \times \frac{18}{38} = -\frac{2}{38}$.
The net profit after n spins for Gambler 2 is $X_2 = Z_1 + Z_2 + \cdots + Z_n$,
and the expected profit is
$E(X_2) = E(Z_1) + E(Z_2) + \cdots + E(Z_n) = -\frac{2}{38} \times n = -\frac{2n}{38}$.

Then $E(X_2 - X_1) = E(X_2) - E(X_1) = -\frac{2n}{38} - (-\frac{n}{38}) = -\frac{n}{38}$
Answer: B

29. The expected insurance payment is $P(X \geq 1) = .8892$.
Therefore $P(X=0) = .1108 = e^{-\lambda}$, and it follows that $\lambda = -ln(.1108) = 2.200$.
The expected insurance payment on a policy with a deductible of 2 is
$1 \cdot P(X=1) + 2 \cdot P(X \geq 2)$
$= P(X=1) + 2 \cdot [1 - P(X = 0 \text{ or } 1)]$
$= \frac{e^{-\lambda} \cdot \lambda}{1!} + 2 \cdot [1 - e^{-\lambda} - \frac{e^{-\lambda} \cdot \lambda}{1!}]$
$= (.1108)(2.2) + 2 \cdot [1 - .1108 - (.1108)(2.2)] = 1.53$. Answer: E

30. We use the following notation:
A - customer has an auto policy
F - customer has a fire insurance policy
L - customer has flood insurance coverage

We are given: $P(A) = .8$, $P(F) = .4$, $P(F|A) = .25$, $P(L|F) = .5$, $P(A|L) = .5$
We also know that $L \cap F = L$, so from $.5 = P(L|F) = \frac{P(L \cap F)}{P(F)} = \frac{P(L)}{P(F)} = \frac{P(L)}{.4}$
we get $P(L) = .2$.

We wish to find $P(A' \cap L' | F) = \frac{P(A' \cap L' \cap F)}{P(F)}$.

30. continued

$P(A' \cap L' \cap F) = P(F) - P[(A \cup L) \cap F] = P(F) - P[(A \cap F) \cup (L \cap F)]$
$= P(F) - P[(A \cap F) \cup L] = P(F) - [P(A \cap F) + P(L) - P(A \cap F \cap L)]$
$= P(F) - [P(A \cap F) + P(L) - P(A \cap L)]$

From the given information we have $.25 = P(F|A) = \frac{P(F \cap A)}{P(A)} = \frac{P(F \cap A)}{.8}$,
so that $P(F \cap A) = .2$, and $.5 = P(A|L) = \frac{P(A \cap L)}{P(L)} = \frac{P(A \cap L)}{.2}$ so that $P(A \cap L) = .1$.

Then, $P(A' \cap L'|F) = \frac{P(A' \cap L' \cap F)}{P(F)} = \frac{P(F) - [P(A \cap F) + P(L) - P(A \cap L)]}{P(F)} = \frac{.4 - [.2 + .2 - .1]}{.4} = .25$.

Answer: E