CALCULUS

6th Edition

Single & Multivariable

INSTRUCTOR SOLUTIONS MANUAL

Hughes-Hallett Gleason McCallum et al.

CHAPTER ONE

Solutions for Section 1.1

Exercises

1. Since t represents the number of years since 1970, we see that $f(35)$ represents the population of the city in 2005. In 2005, the city's population was 12 million.

2. Since $T = f(P)$, we see that $f(200)$ is the value of T when $P = 200$; that is, the thickness of pelican eggs when the concentration of PCBs is 200 ppm.

3. If there are no workers, there is no productivity, so the graph goes through the origin. At first, as the number of workers increases, productivity also increases. As a result, the curve goes up initially. At a certain point the curve reaches its highest level, after which it goes downward; in other words, as the number of workers increases beyond that point, productivity decreases. This might, for example, be due either to the inefficiency inherent in large organizations or simply to workers getting in each other's way as too many are crammed on the same line. Many other reasons are possible.

4. The slope is $(1-0)/(1-0) = 1$. So the equation of the line is $y = x$.

5. The slope is $(3-2)/(2-0) = 1/2$. So the equation of the line is $y = (1/2)x + 2$.

6. Using the points $(-2, 1)$ and $(2, 3)$, we have

$$\text{Slope } = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}.$$

Now we know that $y = (1/2)x + b$. Using the point $(-2, 1)$, we have $1 = -2/2 + b$, which yields $b = 2$. Thus, the equation of the line is $y = (1/2)x + 2$.

7. Slope = $\dfrac{6-0}{2-(-1)} = 2$ so the equation is $y - 6 = 2(x-2)$ or $y = 2x + 2$.

8. Rewriting the equation as $y = -\frac{5}{2}x + 4$ shows that the slope is $-\frac{5}{2}$ and the vertical intercept is 4.

9. Rewriting the equation as

$$y = -\frac{12}{7}x + \frac{2}{7}$$

shows that the line has slope $-12/7$ and vertical intercept $2/7$.

10. Rewriting the equation of the line as

$$-y = \frac{-2}{4}x - 2$$
$$y = \frac{1}{2}x + 2,$$

we see the line has slope $1/2$ and vertical intercept 2.

11. Rewriting the equation of the line as

$$y = \frac{12}{6}x - \frac{4}{6}$$
$$y = 2x - \frac{2}{3},$$

we see that the line has slope 2 and vertical intercept $-2/3$.

12. (a) is (V), because slope is positive, vertical intercept is negative
(b) is (IV), because slope is negative, vertical intercept is positive
(c) is (I), because slope is 0, vertical intercept is positive
(d) is (VI), because slope and vertical intercept are both negative
(e) is (II), because slope and vertical intercept are both positive
(f) is (III), because slope is positive, vertical intercept is 0

13. (a) is (V), because slope is negative, vertical intercept is 0
(b) is (VI), because slope and vertical intercept are both positive
(c) is (I), because slope is negative, vertical intercept is positive
(d) is (IV), because slope is positive, vertical intercept is negative
(e) is (III), because slope and vertical intercept are both negative
(f) is (II), because slope is positive, vertical intercept is 0

14. The intercepts appear to be $(0, 3)$ and $(7.5, 0)$, giving

$$\text{Slope } = \frac{-3}{7.5} = -\frac{6}{15} = -\frac{2}{5}.$$

The y-intercept is at $(0, 3)$, so a possible equation for the line is

$$y = -\frac{2}{5}x + 3.$$

(Answers may vary.)

15. $y - c = m(x - a)$

16. Given that the function is linear, choose any two points, for example (5.2, 27.8) and (5.3, 29.2). Then

$$\text{Slope} = \frac{29.2 - 27.8}{5.3 - 5.2} = \frac{1.4}{0.1} = 14.$$

Using the point-slope formula, with the point $(5.2, 27.8)$, we get the equation

$$y - 27.8 = 14(x - 5.2)$$

which is equivalent to

$$y = 14x - 45.$$

17. $y = 5x - 3$. Since the slope of this line is 5, we want a line with slope $-\frac{1}{5}$ passing through the point $(2, 1)$. The equation is $(y - 1) = -\frac{1}{5}(x - 2)$, or $y = -\frac{1}{5}x + \frac{7}{5}$.

18. The line $y + 4x = 7$ has slope -4. Therefore the parallel line has slope -4 and equation $y - 5 = -4(x - 1)$ or $y = -4x + 9$. The perpendicular line has slope $\frac{-1}{(-4)} = \frac{1}{4}$ and equation $y - 5 = \frac{1}{4}(x - 1)$ or $y = 0.25x + 4.75$.

19. The line parallel to $y = mx + c$ also has slope m, so its equation is

$$y = m(x - a) + b.$$

The line perpendicular to $y = mx + c$ has slope $-1/m$, so its equation will be

$$y = -\frac{1}{m}(x - a) + b.$$

20. Since the function goes from $x = 0$ to $x = 4$ and between $y = 0$ and $y = 2$, the domain is $0 \le x \le 4$ and the range is $0 \le y \le 2$.

21. Since x goes from 1 to 5 and y goes from 1 to 6, the domain is $1 \le x \le 5$ and the range is $1 \le y \le 6$.

22. Since the function goes from $x = -2$ to $x = 2$ and from $y = -2$ to $y = 2$, the domain is $-2 \le x \le 2$ and the range is $-2 \le y \le 2$.

23. Since the function goes from $x = 0$ to $x = 5$ and between $y = 0$ and $y = 4$, the domain is $0 \le x \le 5$ and the range is $0 \le y \le 4$.

24. The domain is all numbers. The range is all numbers ≥ 2, since $x^2 \ge 0$ for all x.

25. The domain is all x-values, as the denominator is never zero. The range is $0 < y \le \frac{1}{2}$.

26. The value of $f(t)$ is real provided $t^2 - 16 \ge 0$ or $t^2 \ge 16$. This occurs when either $t \ge 4$, or $t \le -4$. Solving $f(t) = 3$, we have

$$\begin{aligned} \sqrt{t^2 - 16} &= 3 \\ t^2 - 16 &= 9 \\ t^2 &= 25 \end{aligned}$$

so

$$t = \pm 5.$$

27. We have $V = kr^3$. You may know that $V = \frac{4}{3}\pi r^3$.

28. If distance is d, then $v = \frac{d}{t}$.

29. For some constant k, we have $S = kh^2$.

30. We know that E is proportional to v^3, so $E = kv^3$, for some constant k.

31. We know that N is proportional to $1/l^2$, so

$$N = \frac{k}{l^2}, \quad \text{for some constant } k.$$

Problems

32. The year 1983 was 25 years before 2008 so 1983 corresponds to $t = 25$. Thus, an expression that represents the statement is:

$$f(25) = 7.019$$

33. The year 2008 was 0 years before 2008 so 2008 corresponds to $t = 0$. Thus, an expression that represents the statement is:

$$f(0) \text{ meters.}$$

34. The year 1965 was $2008 - 1865 = 143$ years before 2008 so 1965 corresponds to $t = 143$. Similarly, we see that the year 1911 corresponds to $t = 97$. Thus, an expression that represents the statement is:

$$f(143) = f(97)$$

35. Since $t = 1$ means one year before 2008, then $t = 1$ corresponds to the year 2007. Similarly, $t = 0$ corresponds to the year 2008. Thus, $f(1)$ and $f(0)$ are the average annual sea level values, in meters, in 2007 and 2008, respectively. Because 1 millimeter is the same as 0.001 meters, an expression that represents the statement is:

$$f(0) = f(1) + 0.001.$$

Note that there are other possible equivalent expressions, such as: $f(1) - f(0) = 0.001$.

36. (a) Each date, t, has a unique daily snowfall, S, associated with it. So snowfall is a function of date.
(b) On December 12, the snowfall was approximately 5 inches.
(c) On December 11, the snowfall was above 10 inches.
(d) Looking at the graph we see that the largest increase in the snowfall was between December 10 to December 11.

37. (a) When the car is 5 years old, it is worth \$6000.
(b) Since the value of the car decreases as the car gets older, this is a decreasing function. A possible graph is in Figure 1.1:

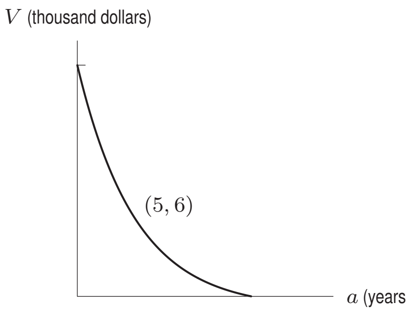


Figure 1.1

(c) The vertical intercept is the value of V when $a = 0$, or the value of the car when it is new. The horizontal intercept is the value of a when $V = 0$, or the age of the car when it is worth nothing.

38. (a) The story in (a) matches Graph (IV), in which the person forgot her books and had to return home.
(b) The story in (b) matches Graph (II), the flat tire story. Note the long period of time during which the distance from home did not change (the horizontal part).
(c) The story in (c) matches Graph (III), in which the person started calmly but sped up later.

The first graph (I) does not match any of the given stories. In this picture, the person keeps going away from home, but his speed decreases as time passes. So a story for this might be: *I started walking to school at a good pace, but since I stayed up all night studying calculus, I got more and more tired the farther I walked.*

39. (a) $f(30) = 10$ means that the value of f at $t = 30$ was 10. In other words, the temperature at time $t = 30$ minutes was 10°C. So, 30 minutes after the object was placed outside, it had cooled to 10 °C.
(b) The intercept a measures the value of $f(t)$ when $t = 0$. In other words, when the object was initially put outside, it had a temperature of a°C. The intercept b measures the value of t when $f(t) = 0$. In other words, at time b the object's temperature is 0 °C.

40. (a) The height of the rock decreases as time passes, so the graph falls as you move from left to right. One possibility is shown in Figure 1.2.

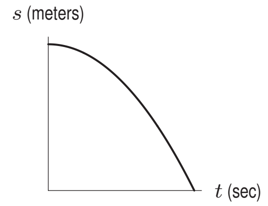


Figure 1.2

(b) The statement $f(7) = 12$ tells us that 7 seconds after the rock is dropped, it is 12 meters above the ground.
(c) The vertical intercept is the value of s when $t = 0$; that is, the height from which the rock is dropped. The horizontal intercept is the value of t when $s = 0$; that is, the time it takes for the rock to hit the ground.

41. (a) We find the slope m and intercept b in the linear equation $C = b + mw$. To find the slope m, we use

$$m = \frac{\Delta C}{\Delta w} = \frac{12.32 - 8}{68 - 32} = 0.12 \text{ dollars per gallon.}$$

We substitute to find b:

$$\begin{aligned} C &= b + mw \\ 8 &= b + (0.12)(32) \\ b &= 4.16 \text{ dollars.} \end{aligned}$$

The linear formula is $C = 4.16 + 0.12w$.
(b) The slope is 0.12 dollars per gallon. Each additional gallon of waste collected costs 12 cents.
(c) The intercept is \$4.16. The flat monthly fee to subscribe to the waste collection service is \$4.16. This is the amount charged even if there is no waste.

42. We are looking for a linear function $y = f(x)$ that, given a time x in years, gives a value y in dollars for the value of the refrigerator. We know that when $x = 0$, that is, when the refrigerator is new, $y = 950$, and when $x = 7$, the refrigerator is worthless, so $y = 0$. Thus $(0, 950)$ and $(7, 0)$ are on the line that we are looking for. The slope is then given by

$$m = \frac{950}{-7}$$

It is negative, indicating that the value decreases as time passes. Having found the slope, we can take the point $(7, 0)$ and use the point-slope formula:

$$y - y_1 = m(x - x_1).$$

So,

$$\begin{aligned} y - 0 &= -\frac{950}{7}(x - 7) \\ y &= -\frac{950}{7}x + 950. \end{aligned}$$

43. (a) The first company's price for a day's rental with m miles on it is $C_1(m) = 40 + 0.15m$. Its competitor's price for a day's rental with m miles on it is $C_2(m) = 50 + 0.10m$.

(b) See Figure 1.3.

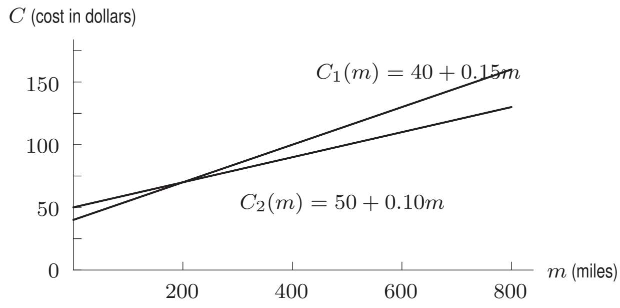


Figure 1.3

(c) To find which company is cheaper, we need to determine where the two lines intersect. We let $C_1 = C_2$, and thus

$$\begin{aligned} 40 + 0.15m &= 50 + 0.10m \\ 0.05m &= 10 \\ m &= 200. \end{aligned}$$

If you are going more than 200 miles a day, the competitor is cheaper. If you are going less than 200 miles a day, the first company is cheaper.

44. (a) Charge per cubic foot = $\dfrac{\Delta\$}{\Delta \text{ cu. ft.}} = \dfrac{55 - 40}{1600 - 1000} = \$0.025/\text{cu. ft.}$
Alternatively, if we let c = cost, w = cubic feet of water, b = fixed charge, and m = cost/cubic feet, we obtain $c = b + mw$. Substituting the information given in the problem, we have

$$\begin{aligned} 40 &= b + 1000m \\ 55 &= b + 1600m. \end{aligned}$$

Subtracting the first equation from the second yields $15 = 600m$, so $m = 0.025$.

(b) The equation is $c = b + 0.025w$, so $40 = b + 0.025(1000)$, which yields $b = 15$. Thus the equation is $c = 15 + 0.025w$.

(c) We need to solve the equation $100 = 15 + 0.025w$, which yields $w = 3400$. It costs \$100 to use 3400 cubic feet of water.

45. See Figure 1.4.

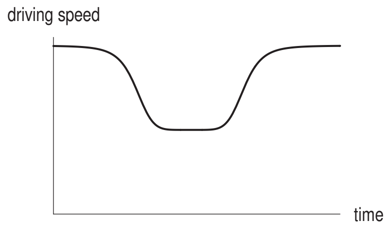


Figure 1.4

46. See Figure 1.5.

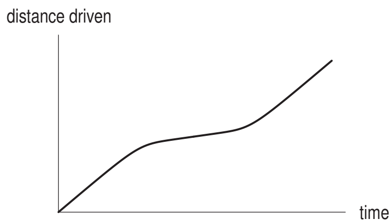


Figure 1.5

47. See Figure 1.6.

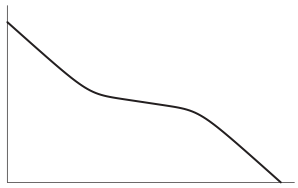


Figure 1.6

48. See Figure 1.7.

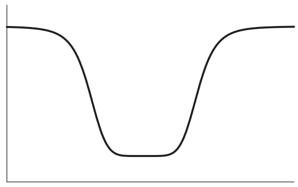


Figure 1.7

49. (a) (i) $f(1985) = 13$
(ii) $f(1990) = 99$

(b) The average yearly increase is the rate of change.

$$\text{Yearly increase} = \frac{f(1990) - f(1985)}{1990 - 1985} = \frac{99 - 13}{5} = 17.2 \text{ billionaires per year.}$$

(c) Since we assume the rate of increase remains constant, we use a linear function with slope 17.2 billionaires per year. The equation is

$$f(t) = b + 17.2t$$

where $f(1985) = 13$, so

$$13 = b + 17.2(1985)$$
$$b = -34{,}129.$$

Thus, $f(t) = 17.2t - 34{,}129$.

50. (a) The largest time interval was 2008–2009 since the percentage growth rate increased from -11.7 to 7.3 from 2008 to 2009. This means the US consumption of biofuels grew relatively more from 2008 to 2009 than from 2007 to 2008. (Note that the percentage growth rate was a decreasing function of time over 2005–2007.)

(b) The largest time interval was 2005–2007 since the percentage growth rates were positive for each of these three consecutive years. This means that the amount of biofuels consumed in the US steadily increased during the three year span from 2005 to 2007, then decreased in 2008.

51. (a) The largest time interval was 2005–2007 since the percentage growth rate decreased from -1.9 in 2005 to -45.4 in 2007. This means that from 2005 to 2007 the US consumption of hydroelectric power shrunk relatively more with each successive year.

(b) The largest time interval was 2004–2007 since the percentage growth rates were negative for each of these four consecutive years. This means that the amount of hydroelectric power consumed by the US industrial sector steadily decreased during the four year span from 2004 to 2007, then increased in 2008.

52. (a) The largest time interval was 2004–2006 since the percentage growth rate increased from -5.7 in 2004 to 9.7 in 2006. This means that from 2004 to 2006 the US price per watt of a solar panel grew relatively more with each successive year.

(b) The largest time interval was 2005–2006 since the percentage growth rates were positive for each of these two consecutive years. This means that the US price per watt of a solar panel steadily increased during the two year span from 2005 to 2006, then decreased in 2007.

53. (a) Since 2008 corresponds to $t = 0$, the average annual sea level in Aberdeen in 2008 was 7.094 meters.

(b) Looking at the table, we see that the average annual sea level was 7.019 fifty years before 2008, or in the year 1958. Similar reasoning shows that the average sea level was 6.957 meters 125 years before 2008, or in 1883.

(c) Because 125 years before 2008 the year was 1883, we see that the sea level value corresponding to the year 1883 is 6.957 (this is the sea level value corresponding to $t = 125$). Similar reasoning yields the table:

Year	1883	1908	1933	1958	1983	2008
S	6.957	6.938	6.965	6.992	7.019	7.094

54. (a) We find the slope m and intercept b in the linear equation $S = b + mt$. To find the slope m, we use

$$m = \frac{\Delta S}{\Delta t} = \frac{66 - 113}{50 - 0} = -0.94.$$

When $t = 0$, we have $S = 113$, so the intercept b is 113. The linear formula is

$$S = 113 - 0.94t.$$

(b) We use the formula $S = 113 - 0.94t$. When $S = 20$, we have $20 = 113 - 0.94t$ and so $t = 98.9$. If this linear model were correct, the average male sperm count would drop below the fertility level during the year 2038.

55. (a) This could be a linear function because w increases by 5 as h increases by 1.

(b) We find the slope m and the intercept b in the linear equation $w = b + mh$. We first find the slope m using the first two points in the table. Since we want w to be a function of h, we take

$$m = \frac{\Delta w}{\Delta h} = \frac{171 - 166}{69 - 68} = 5.$$

Substituting the first point and the slope $m = 5$ into the linear equation $w = b + mh$, we have $166 = b + (5)(68)$, so $b = -174$. The linear function is

$$w = 5h - 174.$$

The slope, $m = 5$, is in units of pounds per inch.

(c) We find the slope and intercept in the linear function $h = b + mw$ using $m = \Delta h/\Delta w$ to obtain the linear function

$$h = 0.2w + 34.8.$$

Alternatively, we could solve the linear equation found in part (b) for h. The slope, $m = 0.2$, has units inches per pound.

56. We will let

$$\begin{aligned} T &= \text{amount of fuel for take-off,} \\ L &= \text{amount of fuel for landing,} \\ P &= \text{amount of fuel per mile in the air,} \\ m &= \text{the length of the trip in miles.} \end{aligned}$$

Then Q, the total amount of fuel needed, is given by

$$Q(m) = T + L + Pm.$$

57. (a) The variable costs for x acres are $\$200x$, or $0.2x$ thousand dollars. The total cost, C (again in thousands of dollars), of planting x acres is:

$$C = f(x) = 10 + 0.2x.$$

This is a linear function. See Figure 1.8. Since $C = f(x)$ increases with x, f is an increasing function of x. Look at the values of C shown in the table; you will see that each time x increases by 1, C increases by 0.2. Because C increases at a constant rate as x increases, the graph of C against x is a line.

(b) See Figure 1.8 and Table 1.1.

Table 1.1
Cost of planting seed

x	C
0	10
2	10.4
3	10.6
4	10.8
5	11
6	11.2

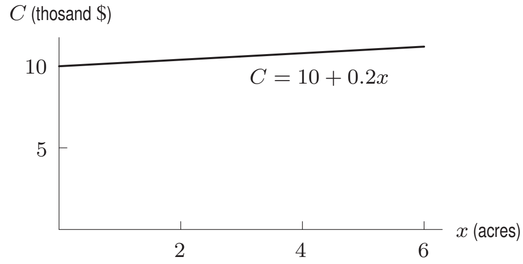


Figure 1.8

(c) The vertical intercept of 10 corresponds to the fixed costs. For $C = f(x) = 10 + 0.2x$, the intercept on the vertical axis is 10 because $C = f(0) = 10 + 0.2(0) = 10$. Since 10 is the value of C when $x = 0$, we recognize it as the initial outlay for equipment, or the fixed cost.

The slope 0.2 corresponds to the variable costs. The slope is telling us that for every additional acre planted, the costs go up by 0.2 thousand dollars. The rate at which the cost is increasing is 0.2 thousand dollars per acre. Thus the variable costs are represented by the slope of the line $f(x) = 10 + 0.2x$.

58. See Figure 1.9.

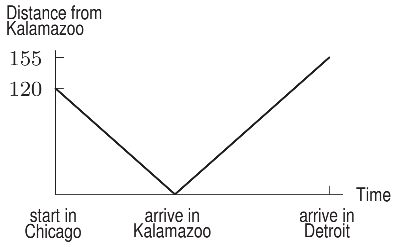


Figure 1.9

59. (a) The line given by $(0, 2)$ and $(1, 1)$ has slope $m = \frac{2-1}{-1} = -1$ and y-intercept 2, so its equation is

$$y = -x + 2.$$

The points of intersection of this line with the parabola $y = x^2$ are given by

$$\begin{aligned} x^2 &= -x + 2 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0. \end{aligned}$$

The solution $x = 1$ corresponds to the point we are already given, so the other solution, $x = -2$, gives the x-coordinate of C. When we substitute back into either equation to get y, we get the coordinates for C, $(-2, 4)$.

(b) The line given by $(0, b)$ and $(1, 1)$ has slope $m = \frac{b-1}{-1} = 1 - b$, and y-intercept at $(0, b)$, so we can write the equation for the line as we did in part (a):

$$y = (1 - b)x + b.$$

We then solve for the points of intersection with $y = x^2$ the same way:

$$\begin{aligned} x^2 &= (1 - b)x + b \\ x^2 - (1 - b)x - b &= 0 \\ x^2 + (b - 1)x - b &= 0 \\ (x + b)(x - 1) &= 0 \end{aligned}$$

Again, we have the solution at the given point $(1, 1)$, and a new solution at $x = -b$, corresponding to the other point of intersection C. Substituting back into either equation, we can find the y-coordinate for C is b^2, and thus C is given by $(-b, b^2)$. This result agrees with the particular case of part (a) where $b = 2$.

60. Looking at the given data, it seems that Galileo's hypothesis was incorrect. The first table suggests that velocity is not a linear function of distance, since the increases in velocity for each foot of distance are themselves getting smaller. Moreover, the second table suggests that velocity is instead proportional to *time*, since for each second of time, the velocity increases by 32 ft/sec.

Strengthen Your Understanding

61. The line $y = 0.5 - 3x$ has a negative slope and is therefore a decreasing function.

62. If y is directly proportional to x we have $y = kx$. Adding the constant 1 to give $y = 2x + 1$ means that y is not proportional to x.

63. One possible answer is $f(x) = 2x + 3$.

64. One possible answer is $q = \dfrac{8}{p^{1/3}}$.

65. False. A line can be put through any two points in the plane. However, if the line is vertical, it is not the graph of a function.

66. True. Suppose we start at $x = x_1$ and increase x by 1 unit to $x_1 + 1$. If $y = b + mx$, the corresponding values of y are $b + mx_1$ and $b + m(x_1 + 1)$. Thus y increases by

$$\Delta y = b + m(x_1 + 1) - (b + mx_1) = m.$$

67. False. For example, let $y = x + 1$. Then the points $(1, 2)$ and $(2, 3)$ are on the line. However the ratios

$$\frac{2}{1} = 2 \qquad \text{and} \qquad \frac{3}{2} = 1.5$$

are different. The ratio y/x is constant for linear functions of the form $y = mx$, but not in general. (Other examples are possible.)

68. False. For example, if $y = 4x + 1$ (so $m = 4$) and $x = 1$, then $y = 5$. Increasing x by 2 units gives 3, so $y = 4(3) + 1 = 13$. Thus, y has increased by 8 units, not $4 + 2 = 6$. (Other examples are possible.)

69. (b) and (c). For $g(x) = \sqrt{x}$, the domain and range are all nonnegative numbers, and for $h(x) = x^3$, the domain and range are all real numbers.

Solutions for Section 1.2

Exercises

1. The graph shows a concave up function.

2. The graph shows a concave down function.

3. This graph is neither concave up or down.

4. The graph is concave up.

5. Initial quantity $= 5$; growth rate $= 0.07 = 7\%$.

6. Initial quantity $= 7.7$; growth rate $= -0.08 = -8\%$ (decay).

7. Initial quantity $= 3.2$; growth rate $= 0.03 = 3\%$ (continuous).

8. Initial quantity $= 15$; growth rate $= -0.06 = -6\%$ (continuous decay).

9. Since $e^{0.25t} = \left(e^{0.25}\right)^t \approx (1.2840)^t$, we have $P = 15(1.2840)^t$. This is exponential growth since 0.25 is positive. We can also see that this is growth because $1.2840 > 1$.

10. Since $e^{-0.5t} = (e^{-0.5})^t \approx (0.6065)^t$, we have $P = 2(0.6065)^t$. This is exponential decay since -0.5 is negative. We can also see that this is decay because $0.6065 < 1$.

11. $P = P_0(e^{0.2})^t = P_0(1.2214)^t$. Exponential growth because $0.2 > 0$ or $1.2214 > 1$.

12. $P = 7(e^{-\pi})^t = 7(0.0432)^t$. Exponential decay because $-\pi < 0$ or $0.0432 < 1$.

13. (a) Let $Q = Q_0 a^t$. Then $Q_0 a^5 = 75.94$ and $Q_0 a^7 = 170.86$. So

$$\frac{Q_0 a^7}{Q_0 a^5} = \frac{170.86}{75.94} = 2.25 = a^2.$$

So $a = 1.5$.

(b) Since $a = 1.5$, the growth rate is $r = 0.5 = 50\%$.

14. (a) Let $Q = Q_0 a^t$. Then $Q_0 a^{0.02} = 25.02$ and $Q_0 a^{0.05} = 25.06$. So

$$\frac{Q_0 a^{0.05}}{Q_0 a^{0.02}} = \frac{25.06}{25.02} = 1.001 = a^{0.03}.$$

So

$$a = (1.001)^{\frac{100}{3}} = 1.05.$$

(b) Since $a = 1.05$, the growth rate is $r = 0.05 = 5\%$.

15. (a) The function is linear with initial population of 1000 and slope of 50, so $P = 1000 + 50t$.

(b) This function is exponential with initial population of 1000 and growth rate of 5%, so $P = 1000(1.05)^t$.

16. (a) This is a linear function with slope -2 grams per day and intercept 30 grams. The function is $Q = 30 - 2t$, and the graph is shown in Figure 1.10.

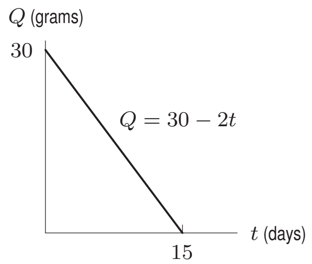


Figure 1.10

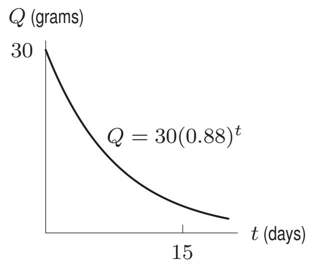


Figure 1.11

(b) Since the quantity is decreasing by a constant percent change, this is an exponential function with base $1 - 0.12 = 0.88$. The function is $Q = 30(0.88)^t$, and the graph is shown in Figure 1.11.

17. The function is increasing and concave up between D and E, and between H and I. It is increasing and concave down between A and B, and between E and F. It is decreasing and concave up between C and D, and between G and H. Finally, it is decreasing and concave down between B and C, and between F and G

18. (a) It was decreasing from March 2 to March 5 and increasing from March 5 to March 9.

(b) From March 5 to 8, the average temperature increased, but the rate of increase went down, from 12° between March 5 and 6 to 4° between March 6 and 7 to 2° between March 7 and 8.

From March 7 to 9, the average temperature increased, and the rate of increase went up, from 2° between March 7 and 8 to 9° between March 8 and 9.

Problems

19. (a) A linear function must change by exactly the same amount whenever x changes by some fixed quantity. While $h(x)$ decreases by 3 whenever x increases by 1, $f(x)$ and $g(x)$ fail this test, since both change by different amounts between $x = -2$ and $x = -1$ and between $x = -1$ and $x = 0$. So the only possible linear function is $h(x)$, so it will be given by a formula of the type: $h(x) = mx + b$. As noted, $m = -3$. Since the y-intercept of h is 31, the formula for $h(x)$ is $h(x) = 31 - 3x$.

(b) An exponential function must grow by exactly the same factor whenever x changes by some fixed quantity. Here, $g(x)$ increases by a factor of 1.5 whenever x increases by 1. Since the y-intercept of $g(x)$ is 36, $g(x)$ has the formula $g(x) = 36(1.5)^x$. The other two functions are not exponential; $h(x)$ is not because it is a linear function, and $f(x)$ is not because it both increases and decreases.

20. Table A and Table B could represent linear functions of x. Table A could represent the constant linear function $y = 2.2$ because all y values are the same. Table B could represent a linear function of x with slope equal to $11/4$. This is because x values that differ by 4 have corresponding y values that differ by 11, and x values that differ by 8 have corresponding y values that differ by 22. In Table C, y decreases and then increases as x increases, so the table cannot represent a linear function. Table D does not show a constant rate of change, so it cannot represent a linear function.

21. Table D is the only table that could represent an exponential function of x. This is because, in Table D, the ratio of y values is the same for all equally spaced x values. Thus, the y values in the table have a constant percent rate of decrease:

$$\frac{9}{18} = \frac{4.5}{9} = \frac{2.25}{4.5} = 0.5.$$

Table A represents a constant function of x, so it cannot represent an exponential function. In Table B, the ratio between y values corresponding to equally spaced x values is not the same. In Table C, y decreases and then increases as x increases. So neither Table B nor Table C can represent exponential functions.

22. (a) Let P represent the population of the world, and let t represent the number of years since 2010. Then we have $P = 6.91(1.011)^t$.

(b) According to this formula, the population of the world in the year 2020 (at $t = 10$) will be $P = 6.9(1.011)^{10} = 7.71$ billion people.

(c) The graph is shown in Figure 1.12. The population of the world has doubled when $P = 13.82$; we see on the graph that this occurs at approximately $t = 63.4$. Under these assumptions, the doubling time of the world's population is about 63.4 years.

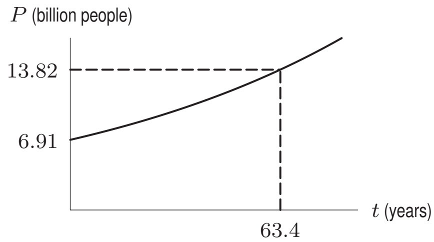
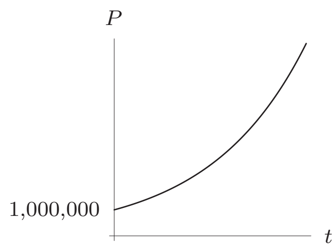


Figure 1.12

23. (a) We have $P_0 = 1$ million, and $k = 0.02$, so $P = (1{,}000{,}000)(e^{0.02t})$.

(b)



24. The doubling time t depends only on the growth rate; it is the solution to

$$2 = (1.02)^t,$$

since 1.02^t represents the factor by which the population has grown after time t. Trial and error shows that $(1.02)^{35} \approx 1.9999$ and $(1.02)^{36} \approx 2.0399$, so that the doubling time is about 35 years.

25. (a) We have

$$\text{Reduced size } = (0.80) \cdot \text{ Original size}$$

or

$$\text{Original size} = \frac{1}{(0.80)} \text{ Reduced size } = (1.25) \text{ Reduced size,}$$

so the copy must be enlarged by a factor of 1.25, which means it is enlarged to 125% of the reduced size.

(b) If a page is copied n times, then

$$\text{New size } = (0.80)^n \cdot \text{ Original.}$$

We want to solve for n so that

$$(0.80)^n = 0.15.$$

By trial and error, we find $(0.80)^8 = 0.168$ and $(0.80)^9 = 0.134$. So the page needs to be copied 9 times.

26. (a) See Figure 1.13.

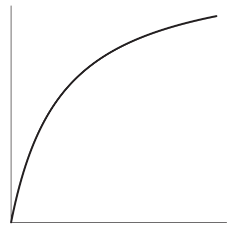


Figure 1.13

(b) "The rate at which new people try it" is the rate of change of the total number of people who have tried the product. Thus, the statement of the problem is telling you that the graph is concave down—the slope is positive but decreasing, as the graph shows.

27. (a) Advertising is generally cheaper in bulk; spending more money will give better and better marginal results initially, (Spending \$5,000 could give you a big newspaper ad reaching 200,000 people; spending \$100,000 could give you a series of TV spots reaching 50,000,000 people.) See Figure 1.14.

(b) The temperature of a hot object decreases at a rate proportional to the difference between its temperature and the temperature of the air around it. Thus, the temperature of a very hot object decreases more quickly than a cooler object. The graph is decreasing and concave up. See Figure 1.15 (We are assuming that the coffee is all at the same temperature.)

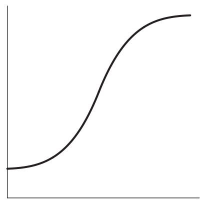


Figure 1.14

Figure 1.15

28. (a) This is the graph of a linear function, which increases at a constant rate, and thus corresponds to $k(t)$, which increases by 0.3 over each interval of 1.

(b) This graph is concave down, so it corresponds to a function whose increases are getting smaller, as is the case with $h(t)$, whose increases are 10, 9, 8, 7, and 6.

(c) This graph is concave up, so it corresponds to a function whose increases are getting bigger, as is the case with $g(t)$, whose increases are 1, 2, 3, 4, and 5.

29. (a) This is a linear function, corresponding to $g(x)$, whose rate of decrease is constant, 0.6.

(b) This graph is concave down, so it corresponds to a function whose rate of decrease is increasing, like $h(x)$. (The rates are -0.2, -0.3, -0.4, -0.5, -0.6.)

(c) This graph is concave up, so it corresponds to a function whose rate of decrease is decreasing, like $f(x)$. (The rates are -10, -9, -8, -7, -6.)

30. Since we are told that the rate of decay is *continuous*, we use the function $Q(t) = Q_0 e^{rt}$ to model the decay, where $Q(t)$ is the amount of strontium-90 which remains at time t, and Q_0 is the original amount. Then

$$Q(t) = Q_0 e^{-0.0247t}.$$

So after 100 years,

$$Q(100) = Q_0 e^{-0.0247 \cdot 100}$$

and

$$\frac{Q(100)}{Q_0} = e^{-2.47} \approx 0.0846$$

so about 8.46% of the strontium-90 remains.

31. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(0, 3)$ and $(2, 12)$ are on the graph, so

$$3 = y_0 a^0 = y_0$$

and

$$12 = y_0 \cdot a^2 = 3 \cdot a^2, \quad \text{giving} \quad a = \pm 2.$$

Since $a > 0$, our equation is $y = 3(2^x)$.

32. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(-1, 8)$ and $(1, 2)$ are on the graph, so

$$8 = y_0 a^{-1} \quad \text{and} \quad 2 = y_0 a^1$$

Therefore $\dfrac{8}{2} = \dfrac{y_0 a^{-1}}{y_0 a} = \dfrac{1}{a^2}$, giving $a = \frac{1}{2}$, and so $2 = y_0 a^1 = y_0 \cdot \frac{1}{2}$, so $y_0 = 4$.
Hence $y = 4\left(\frac{1}{2}\right)^x = 4(2^{-x})$.

33. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(1, 6)$ and $(2, 18)$ are on the graph, so

$$6 = y_0 a^1 \quad \text{and} \quad 18 = y_0 a^2$$

Therefore $a = \frac{y_0 a^2}{y_0 a} = \dfrac{18}{6} = 3$, and so $6 = y_0 a = y_0 \cdot 3$; thus, $y_0 = 2$. Hence $y = 2(3^x)$.

34. The difference, D, between the horizontal asymptote and the graph appears to decrease exponentially, so we look for an equation of the form

$$D = D_0 a^x$$

where $D_0 = 4 =$ difference when $x = 0$. Since $D = 4 - y$, we have

$$4 - y = 4a^x \quad \text{or} \quad y = 4 - 4a^x = 4(1 - a^x)$$

The point $(1, 2)$ is on the graph, so $2 = 4(1 - a^1)$, giving $a = \frac{1}{2}$.
Therefore $y = 4(1 - (\frac{1}{2})^x) = 4(1 - 2^{-x})$.

35. Since f is linear, its slope is a constant

$$m = \frac{20 - 10}{2 - 0} = 5.$$

Thus f increases 5 units for unit increase in x, so

$$f(1) = 15, \quad f(3) = 25, \quad f(4) = 30.$$

Since g is exponential, its growth factor is constant. Writing $g(x) = ab^x$, we have $g(0) = a = 10$, so

$$g(x) = 10 \cdot b^x.$$

Since $g(2) = 10 \cdot b^2 = 20$, we have $b^2 = 2$ and since $b > 0$, we have

$$b = \sqrt{2}.$$

Thus g increases by a factor of $\sqrt{2}$ for unit increase in x, so

$$g(1) = 10\sqrt{2}, \quad g(3) = 10(\sqrt{2})^3 = 20\sqrt{2}, \quad g(4) = 10(\sqrt{2})^4 = 40.$$

Notice that the value of $g(x)$ doubles between $x = 0$ and $x = 2$ (from $g(0) = 10$ to $g(2) = 20$), so the doubling time of $g(x)$ is 2. Thus, $g(x)$ doubles again between $x = 2$ and $x = 4$, confirming that $g(4) = 40$.

36. We see that $\frac{1.09}{1.06} \approx 1.03$, and therefore $h(s) = c(1.03)^s$; c must be 1. Similarly $\frac{2.42}{2.20} = 1.1$, and so $f(s) = a(1.1)^s$; $a = 2$. Lastly, $\frac{3.65}{3.47} \approx 1.05$, so $g(s) = b(1.05)^s$; $b \approx 3$.

37. (a) Because the population is growing exponentially, the time it takes to double is the same, regardless of the population levels we are considering. For example, the population is 20,000 at time 3.7, and 40,000 at time 6.0. This represents a doubling of the population in a span of $6.0 - 3.7 = 2.3$ years.

How long does it take the population to double a second time, from 40,000 to 80,000? Looking at the graph once again, we see that the population reaches 80,000 at time $t = 8.3$. This second doubling has taken $8.3 - 6.0 = 2.3$ years, the same amount of time as the first doubling.

Further comparison of any two populations on this graph that differ by a factor of two will show that the time that separates them is 2.3 years. Similarly, during any 2.3 year period, the population will double. Thus, the doubling time is 2.3 years.

(b) Suppose $P = P_0 a^t$ doubles from time t to time $t + d$. We now have $P_0 a^{t+d} = 2P_0 a^t$, so $P_0 a^t a^d = 2P_0 a^t$. Thus, canceling P_0 and a^t, d must be the number such that $a^d = 2$, no matter what t is.

38. (a) After 50 years, the amount of money is

$$P = 2P_0.$$

After 100 years, the amount of money is

$$P = 2(2P_0) = 4P_0.$$

After 150 years, the amount of money is

$$P = 2(4P_0) = 8P_0.$$

(b) The amount of money in the account doubles every 50 years. Thus in t years, the balance doubles $t/50$ times, so

$$P = P_0 2^{t/50}.$$

39. (a) Since $162.5 = 325/2$, there are 162.5 mg remaining after H hours.
Since $81.25 = 162.5/2$, there are 81.25 mg remaining H hours after there were 162.5 mg, so $2H$ hours after there were 325 mg.
Since $40.625 = 81.25/2$, there are 41.625 mg remaining H hours after there were 81.25 mg, so $3H$ hours after there were 325 mg.

(b) Each additional H hours, the quantity is halved. Thus in t hours, the quantity was halved t/H times, so

$$A = 325\left(\frac{1}{2}\right)^{t/H}.$$

40. (a) The quantity of radium decays exponentially, so we know $Q = Q_0 a^t$. When $t = 1620$, we have $Q = Q_0/2$ so

$$\frac{Q_0}{2} = Q_0 a^{1620}.$$

Thus, canceling Q_0, we have

$$a^{1620} = \frac{1}{2}$$
$$a = \left(\frac{1}{2}\right)^{1/1620}.$$

Thus the formula is $Q = Q_0\left(\left(\frac{1}{2}\right)^{1/1620}\right)^t = Q_0\left(\frac{1}{2}\right)^{t/1620}$.

(b) After 500 years,

$$\text{Fraction remaining} = \frac{1}{Q_0}\cdot Q_0\left(\frac{1}{2}\right)^{500/1620} = 0.80740.$$

so 80.740% is left.

41. Let Q_0 be the initial quantity absorbed in 1960. Then the quantity, Q, of strontium-90 left after t years is

$$Q = Q_0\left(\frac{1}{2}\right)^{t/29}.$$

Since $2010 - 1960 = 50$ years, in 2010

$$\text{Fraction remaining} = \frac{1}{Q_0}\cdot Q_0\left(\frac{1}{2}\right)^{50/29} = \left(\frac{1}{2}\right)^{50/29} = 0.30268 = 30.268\%.$$

42. Direct calculation reveals that each 1000 foot increase in altitude results in a longer takeoff roll by a factor of about 1.096. Since the value of d when $h = 0$ (sea level) is $d = 670$, we are led to the formula

$$d = 670(1.096)^{h/1000},$$

where d is the takeoff roll, in feet, and h is the airport's elevation, in feet.

Alternatively, we can write

$$d = d_0 a^h,$$

where d_0 is the sea level value of d, $d_0 = 670$. In addition, when $h = 1000$, $d = 734$, so

$$734 = 670a^{1000}.$$

Solving for a gives

$$a = \left(\frac{734}{670}\right)^{1/1000} = 1.00009124,$$

so

$$d = 670(1.00009124)^h.$$

43. (a) Since the annual growth factor from 2005 to 2006 was $1 + 1.866 = 2.866$ and $91(1 + 1.866) = 260.806$, the US consumed approximately 261 million gallons of biodiesel in 2006. Since the annual growth factor from 2006 to 2007 was $1 + 0.372 = 1.372$ and $261(1 + 0.372) = 358.092$, the US consumed about 358 million gallons of biodiesel in 2007.

(b) Completing the table of annual consumption of biodiesel and plotting the data gives Figure 1.16.

Year	2005	2006	2007	2008	2009
Consumption of biodiesel (mn gal)	91	261	358	316	339

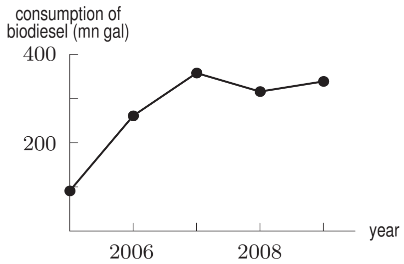


Figure 1.16

44. (a) False, because the annual percent growth is not constant over this interval.

(b) The US consumption of biodiesel more than doubled in 2005 and more than doubled again in 2006. This is because the annual percent growth was larger than 100% for both of these years.

(c) The US consumption of biodiesel more than tripled in 2005, since the annual percent growth in 2005 was over 200%

45. (a) Since the annual growth factor from 2006 to 2007 was $1 - 0.454 = 0.546$ and $29(1 - 0.454) = 15.834$, the US consumed approximately 16 trillion BTUs of hydroelectric power in 2007. Since the annual growth factor from 2005 to 2006 was $1 - 0.10 = 0.90$ and $\frac{29}{(1 - 0.10)} = 32.222$, the US consumed about 32 trillion BTUs of hydroelectric power in 2005.

(b) Completing the table of annual consumption of hydroelectric power and plotting the data gives Figure 1.17.

Year	2004	2005	2006	2007	2008	2009
Consumption of hydro. power (trillion BTU)	33	32	29	16	17	19

(c) The largest decrease in the US consumption of hydroelectric power occurred in 2007. In this year, the US consumption of hydroelectric power dropped by about 13 trillion BTUs to 16 trillion BTUs, down from 29 trillion BTUs in 2006.

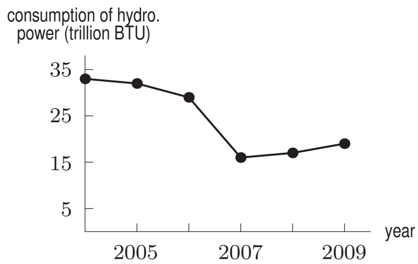


Figure 1.17

46. (a) From the figure we can read-off the approximate percent growth for each year over the previous year:

Year	2005	2006	2007	2008	2009
% growth over previous yr	25	50	30	60	29

Since the annual growth factor from 2006 to 2007 was $1 + 0.30 = 1.30$ and

$$\frac{341}{(1+0.30)} = 262.31,$$

the US consumed approximately 262 trillion BTUs of wind power energy in 2006. Since the annual growth factor from 2007 to 2008 was $1 + 0.60 = 1.60$ and $341(1 + 0.60) = 545.6$, the US consumed about 546 trillion BTUs of wind power energy in 2008.

(b) Completing the table of annual consumption of wind power and plotting the data gives Figure 1.18.

Year	2005	2006	2007	2008	2009
Consumption of wind power (trillion BTU)	175	262	341	546	704

(c) The largest increase in the US consumption of wind power energy occurred in 2008. In this year the US consumption of wind power energy rose by about 205 trillion BTUs to 546 trillion BTUs, up from 341 trillion BTUs in 2007.

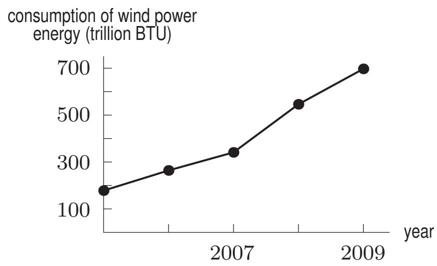


Figure 1.18

47. (a) The US consumption of wind power energy increased by at least 40% in 2006 and in 2008, relative to the previous year. In 2006 consumption increased by just under 50% over consumption in 2005, and in 2008 consumption increased by about 60% over consumption in 2007. Consumption did not decrease during the time period shown because all the annual percent growth values are positive, indicating a steady increase in the US consumption of wind power energy between 2005 and 2009.

(b) Yes. From 2006 to 2007 consumption increased by about 30%, which means $x(1+0.30)$ units of wind power energy were consumed in 2007 if x had been consumed in 2006. Similarly,

$$(x(1 + 0.30))(1 + 0.60)$$

units of wind power energy were consumed in 2008 if x had been consumed in 2006 (because consumption increased by about 60% from 2007 to 2008). Since

$$(x(1 + 0.30))(1 + 0.60) = x(2.08) = x(1 + 1.08),$$

the percent growth in wind power consumption was about 108%, or just over 100%, in 2008 relative to consumption in 2006.

Strengthen Your Understanding

48. The function $y = e^{-0.25x}$ is decreasing but its graph is concave up.

49. The graph of $y = 2x$ is a straight line and is neither concave up or concave down.

50. One possible answer is $q = 2.2(0.97)^t$.

51. One possible answer is $f(x) = 2(1.1)^x$.

52. One possibility is $y = e^{-x} - 5$.

53. False. The y-intercept is $y = 2 + 3e^{-0} = 5$.

54. True, since, as $t \to \infty$, we know $e^{-4t} \to 0$, so $y = 5 - 3e^{-4t} \to 5$.

55. False. Suppose $y = 5^x$. Then increasing x by 1 increases y by a factor of 5. However increasing x by 2 increases y by a factor of 25, not 10, since

$$y = 5^{x+2} = 5^x \cdot 5^2 = 25 \cdot 5^x.$$

(Other examples are possible.)

56. True. Suppose $y = Ab^x$ and we start at the point (x_1, y_1), so $y_1 = Ab^{x_1}$. Then increasing x_1 by 1 gives $x_1 + 1$, so the new y-value, y_2, is given by

$$y_2 = Ab^{x_1+1} = Ab^{x_1}b = (Ab^{x_1})b,$$

so

$$y_2 = by_1.$$

Thus, y has increased by a factor of b, so $b = 3$, and the function is $y = A3^x$.

However, if x_1 is increased by 2, giving $x_1 + 2$, then the new y-value, y_3, is given by

$$y_3 = A3^{x_1+2} = A3^{x_1}3^2 = 9A3^{x_1} = 9y_1.$$

Thus, y has increased by a factor of 9.

57. True. For example, $f(x) = (0.5)^x$ is an exponential function which decreases. (Other examples are possible.)

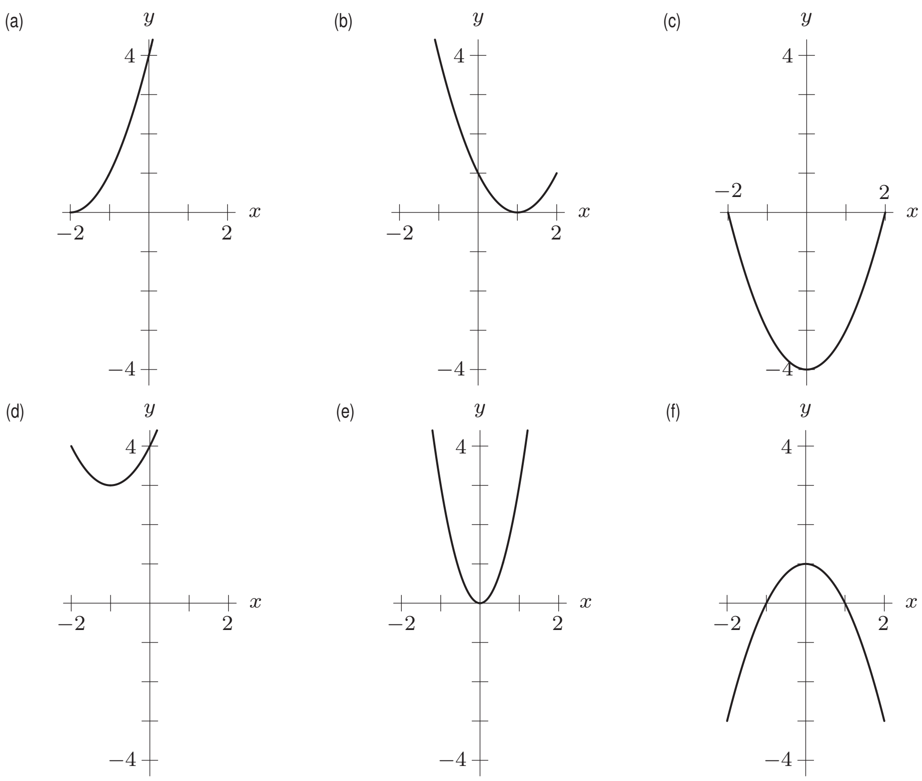
58. True. If $b > 1$, then $ab^x \to 0$ as $x \to -\infty$. If $0 < b < 1$, then $ab^x \to 0$ as $x \to \infty$. In either case, the function $y = a + ab^x$ has $y = a$ as the horizontal asymptote.

59. True, since $e^{-kt} \to 0$ as $t \to \infty$, so $y \to 20$ as $t \to \infty$.

Solutions for Section 1.3

Exercises

1.



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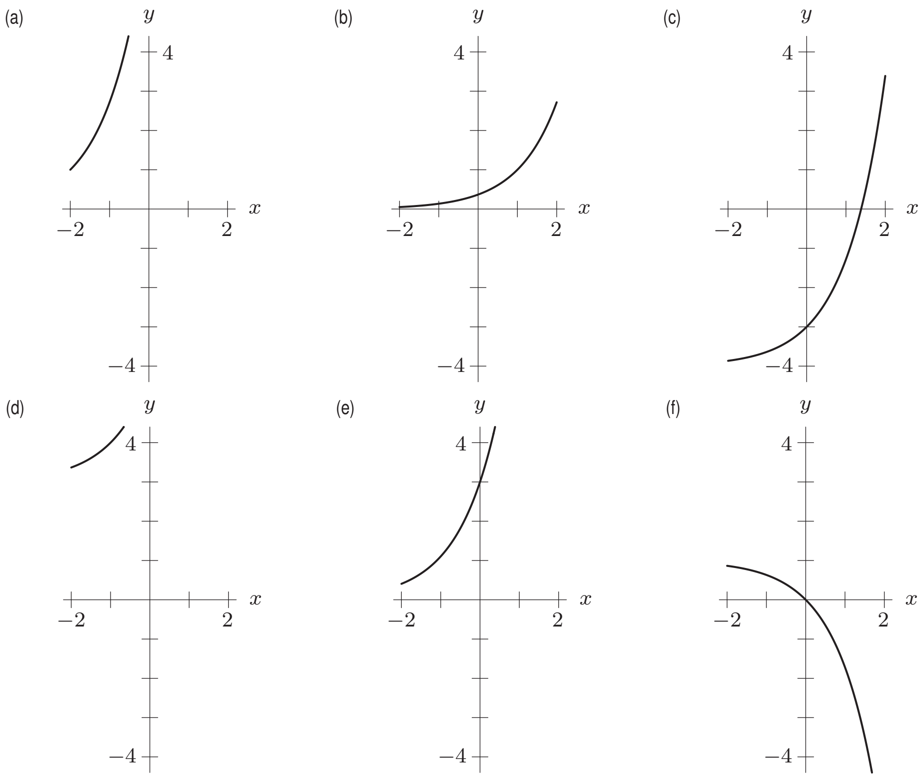

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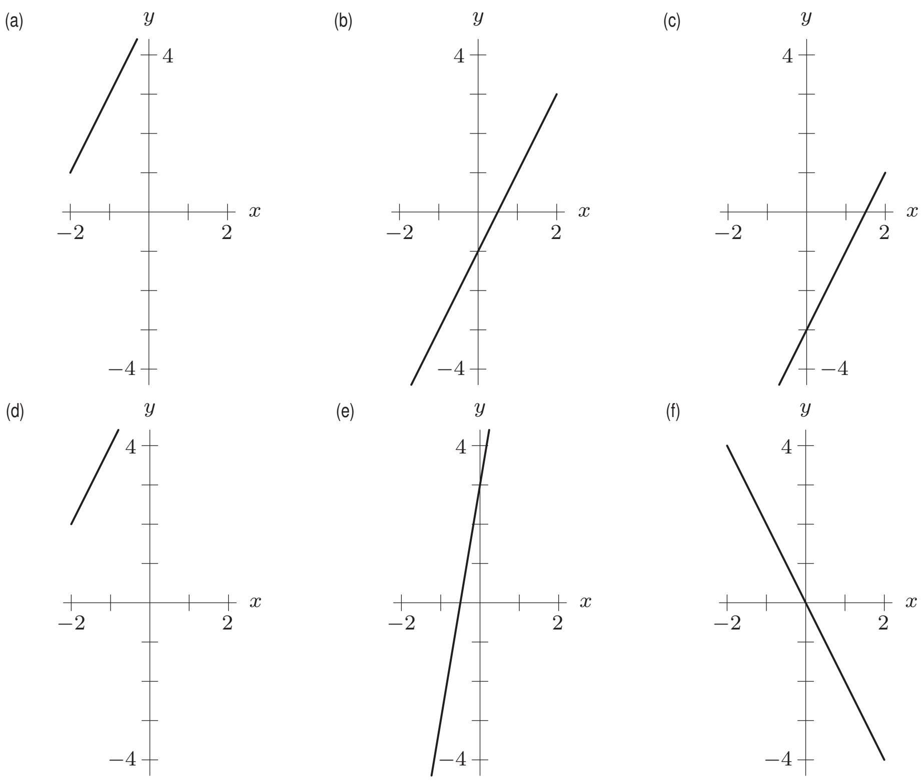


4. This graph is the graph of $m(t)$ shifted upward by two units. See Figure 1.19.

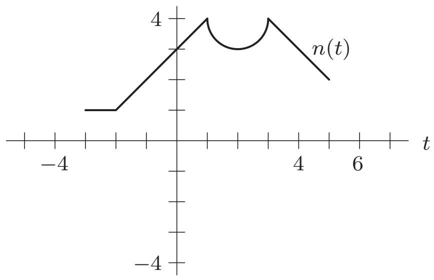


Figure 1.19

5. This graph is the graph of $m(t)$ shifted to the right by one unit. See Figure 1.20.

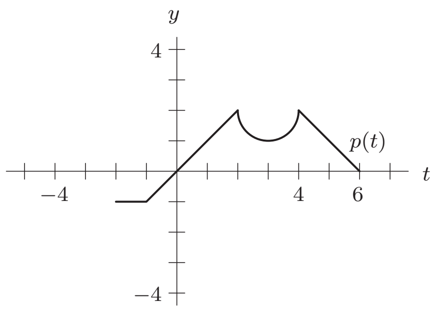


Figure 1.20

6. This graph is the graph of $m(t)$ shifted to the left by 1.5 units. See Figure 1.21.

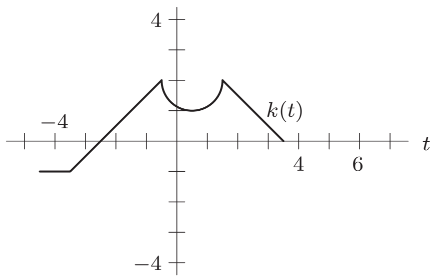


Figure 1.21

7. This graph is the graph of $m(t)$ shifted to the right by 0.5 units and downward by 2.5 units. See Figure 1.22.

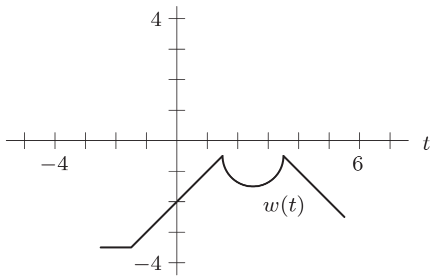


Figure 1.22

8. (a) $f(g(1)) = f(1+1) = f(2) = 2^2 = 4$
(b) $g(f(1)) = g(1^2) = g(1) = 1+1 = 2$
(c) $f(g(x)) = f(x+1) = (x+1)^2$
(d) $g(f(x)) = g(x^2) = x^2+1$
(e) $f(t)g(t) = t^2(t+1)$

9. (a) $f(g(1)) = f(1^2) = f(1) = \sqrt{1+4} = \sqrt{5}$
(b) $g(f(1)) = g(\sqrt{1+4}) = g(\sqrt{5}) = (\sqrt{5})^2 = 5$
(c) $f(g(x)) = f(x^2) = \sqrt{x^2+4}$
(d) $g(f(x)) = g(\sqrt{x+4}) = (\sqrt{x+4})^2 = x+4$
(e) $f(t)g(t) = (\sqrt{t+4})t^2 = t^2\sqrt{t+4}$

10. (a) $f(g(1)) = f(1^2) = f(1) = e^1 = e$
(b) $g(f(1)) = g(e^1) = g(e) = e^2$
(c) $f(g(x)) = f(x^2) = e^{x^2}$
(d) $g(f(x)) = g(e^x) = (e^x)^2 = e^{2x}$
(e) $f(t)g(t) = e^t t^2$

11. (a) $f(g(1)) = f(3 \cdot 1 + 4) = f(7) = \frac{1}{7}$
(b) $g(f(1)) = g(1/1) = g(1) = 7$
(c) $f(g(x)) = f(3x+4) = \frac{1}{3x+4}$
(d) $g(f(x)) = g\left(\frac{1}{x}\right) = 3\left(\frac{1}{x}\right) + 4 = \frac{3}{x} + 4$
(e) $f(t)g(t) = \frac{1}{t}(3t+4) = 3 + \frac{4}{t}$

12. (a) $g(2+h) = (2+h)^2 + 2(2+h) + 3 = 4 + 4h + h^2 + 4 + 2h + 3 = h^2 + 6h + 11$.
(b) $g(2) = 2^2 + 2(2) + 3 = 4 + 4 + 3 = 11$, which agrees with what we get by substituting $h = 0$ into (a).
(c) $g(2+h) - g(2) = (h^2 + 6h + 11) - (11) = h^2 + 6h$.

13. (a) $f(t+1) = (t+1)^2 + 1 = t^2 + 2t + 1 + 1 = t^2 + 2t + 2$.
(b) $f(t^2+1) = (t^2+1)^2 + 1 = t^4 + 2t^2 + 1 + 1 = t^4 + 2t^2 + 2$.
(c) $f(2) = 2^2 + 1 = 5$.
(d) $2f(t) = 2(t^2+1) = 2t^2 + 2$.
(e) $(f(t))^2 + 1 = \left(t^2+1\right)^2 + 1 = t^4 + 2t^2 + 1 + 1 = t^4 + 2t^2 + 2$.

14. $m(z+1) - m(z) = (z+1)^2 - z^2 = 2z + 1$.

15. $m(z+h) - m(z) = (z+h)^2 - z^2 = 2zh + h^2$.

16. $m(z) - m(z-h) = z^2 - (z-h)^2 = 2zh - h^2$.

17. $m(z+h) - m(z-h) = (z+h)^2 - (z-h)^2 = z^2 + 2hz + h^2 - (z^2 - 2hz + h^2) = 4hz$.

18. (a) $f(25)$ is q corresponding to $p = 25$, or, in other words, the number of items sold when the price is 25.
(b) $f^{-1}(30)$ is p corresponding to $q = 30$, or the price at which 30 units will be sold.

19. (a) $f(10{,}000)$ represents the value of C corresponding to $A = 10{,}000$, or in other words the cost of building a 10,000 square-foot store.
(b) $f^{-1}(20{,}000)$ represents the value of A corresponding to $C = 20{,}000$, or the area in square feet of a store which would cost $20,000 to build.

20. $f^{-1}(75)$ is the length of the column of mercury in the thermometer when the temperature is 75°F.

21. (a) The equation is $y = 2x^2 + 1$. Note that its graph is narrower than the graph of $y = x^2$ which appears in gray. See Figure 1.23.

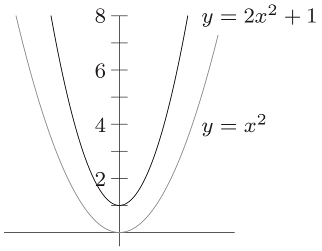


Figure 1.23

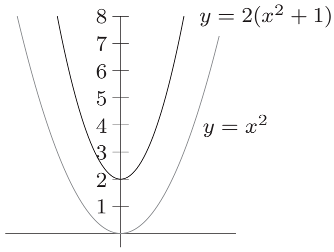


Figure 1.24

(b) $y = 2(x^2 + 1)$ moves the graph up one unit and *then* stretches it by a factor of two. See Figure 1.24.
(c) No, the graphs are not the same. Since $2(x^2 + 1) = (2x^2 + 1) + 1$, the second graph is always one unit higher than the first.

22. Figure 1.25 shows the appropriate graphs. Note that asymptotes are shown as dashed lines and x- or y-intercepts are shown as filled circles.

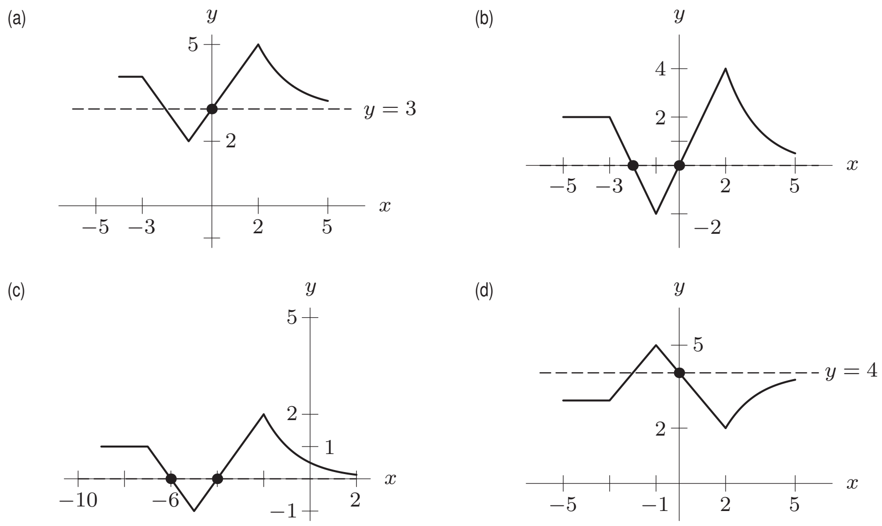


Figure 1.25

23. The function is not invertible since there are many horizontal lines which hit the function twice.

24. The function is not invertible since there are horizontal lines which hit the function more than once.

25. Since a horizontal line cuts the graph of $f(x) = x^2 + 3x + 2$ two times, f is not invertible. See Figure 1.26.

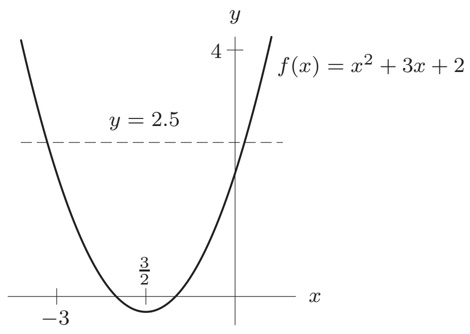


Figure 1.26

26. Since a horizontal line cuts the graph of $f(x) = x^3 - 5x + 10$ three times, f is not invertible. See Figure 1.27.

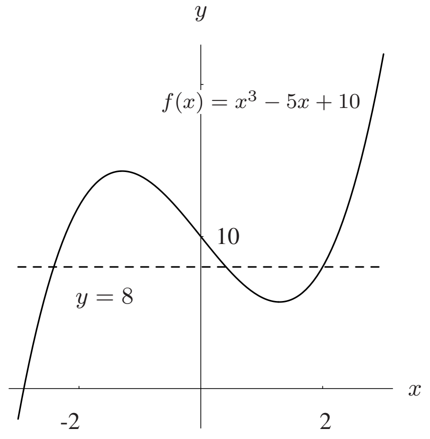


Figure 1.27

27. Since any horizontal line cuts the graph once, f is invertible. See Figure 1.28.

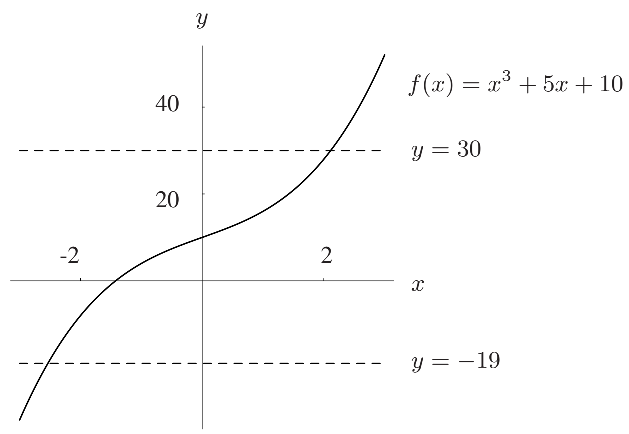


Figure 1.28

28.

$$f(-x) = (-x)^6 + (-x)^3 + 1 = x^6 - x^3 + 1.$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.

29.

$$f(-x) = (-x)^3 + (-x)^2 + (-x) = -x^3 + x^2 - x.$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.

30. Since

$$f(-x) = (-x)^4 - (-x)^2 + 3 = x^4 - x^2 + 3 = f(x),$$

we see f is even

31. Since

$$f(-x) = (-x)^3 + 1 = -x^3 + 1,$$

we see $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd

32. Since

$$f(-x) = 2(-x) = -2x = -f(x),$$

we see f is odd.

33. Since

$$f(-x) = e^{(-x)^2-1} = e^{x^2-1} = f(x),$$

we see f is even.

34. Since

$$f(-x) = (-x)((-x)^2 - 1) = -x(x^2 - 1) = -f(x),$$

we see f is odd

35. Since

$$f(-x) = e^{-x} + x,$$

we see $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd

Problems

36. $f(x) = x^3, \quad g(x) = x + 1.$

37. $f(x) = x + 1, \quad g(x) = x^3.$

38. $f(x) = \sqrt{x}, \quad g(x) = x^2 + 4$

39. $f(x) = e^x, \quad g(x) = 2x$

40. This looks like a shift of the graph $y = -x^2$. The graph is shifted to the left 1 unit and up 3 units, so a possible formula is $y = -(x + 1)^2 + 3$.

41. This looks like a shift of the graph $y = x^3$. The graph is shifted to the right 2 units and down 1 unit, so a possible formula is $y = (x - 2)^3 - 1$.

42. (a) We find $f^{-1}(2)$ by finding the x value corresponding to $f(x) = 2$. Looking at the graph, we see that $f^{-1}(2) = -1$.
(b) We construct the graph of $f^{-1}(x)$ by reflecting the graph of $f(x)$ over the line $y = x$. The graphs of $f^{-1}(x)$ and $f(x)$ are shown together in Figure 1.29.

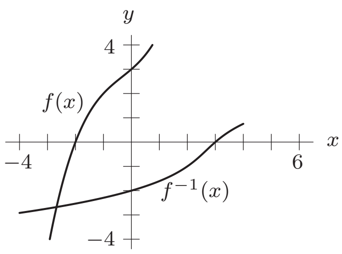


Figure 1.29

43. Values of f^{-1} are as follows

x	3	−7	19	4	178	2	1
$f^{-1}(x)$	1	2	3	4	5	6	7

The domain of f^{-1} is the set consisting of the integers $\{3, -7, 19, 4, 178, 2, 1\}$.

44. f is an increasing function since the amount of fuel used increases as flight time increases. Therefore f is invertible.

45. Not invertible. Given a certain number of customers, say $f(t) = 1500$, there could be many times, t, during the day at which that many people were in the store. So we don't know which time instant is the right one.

46. Probably not invertible. Since your calculus class probably has less than 363 students, there will be at least two days in the year, say a and b, with $f(a) = f(b) = 0$. Hence we don't know what to choose for $f^{-1}(0)$.

47. Not invertible, since it costs the same to mail a 50-gram letter as it does to mail a 51-gram letter.

48. The volume of the balloon t minutes after inflation began is: $g(f(t))$ ft^3.

49. The volume of the balloon if its radius were twice as big is: $g(2r)$ ft^3.

50. The time elapsed is: $f^{-1}(30)$ min.

51. The time elapsed is: $f^{-1}(g^{-1}(10{,}000))$ min.

52. We have $v(10) = 65$ but the graph of u only enables us to evaluate $u(x)$ for $0 \le x \le 50$. There is not enough information to evaluate $u(v(10))$.

53. We have approximately $v(40) = 15$ and $u(15) = 18$ so $u(v(40)) = 18$.

54. We have approximately $u(10) = 13$ and $v(13) = 60$ so $v(u(10)) = 60$.

55. We have $u(40) = 60$ but the graph of v only enables us to evaluate $v(x)$ for $0 \le x \le 50$. There is not enough information to evaluate $v(u(40))$.

56. (a) Yes, f is invertible, since f is increasing everywhere.
(b) The number $f^{-1}(400)$ is the year in which 400 million motor vehicles were registered in the world. From the picture, we see that $f^{-1}(400)$ is around 1979.
(c) Since the graph of f^{-1} is the reflection of the graph of f over the line $y = x$, we get Figure 1.30.

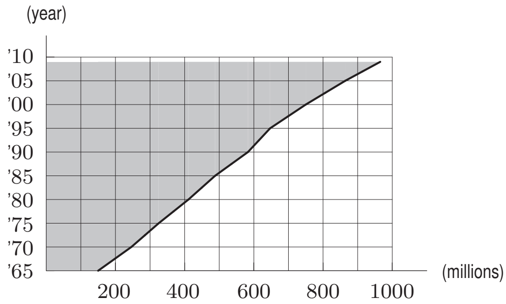


Figure 1.30: Graph of f^{-1}

57. $f(g(1)) = f(2) \approx 0.4$.

58. $g(f(2)) \approx g(0.4) \approx 1.1$.

59. $f(f(1)) \approx f(-0.4) \approx -0.9$.

60. Computing $f(g(x))$ as in Problem 57, we get Table 1.2. From it we graph $f(g(x))$ in Figure 1.31.

Table 1.2

x	$g(x)$	$f(g(x))$
-3	0.6	-0.5
-2.5	-1.1	-1.3
-2	-1.9	-1.2
-1.5	-1.9	-1.2
-1	-1.4	-1.3
-0.5	-0.5	-1
0	0.5	-0.6
0.5	1.4	-0.2
1	2	0.4
1.5	2.2	0.5
2	1.6	0
2.5	0.1	-0.7
3	-2.5	0.1

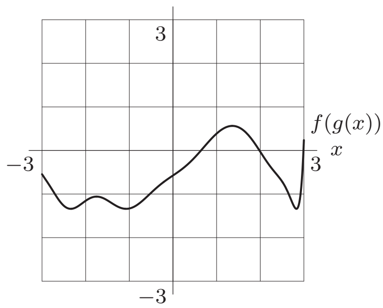


Figure 1.31

61. Using the same way to compute $g(f(x))$ as in Problem 58, we get Table 1.3. Then we can plot the graph of $g(f(x))$ in Figure 1.32.

Table 1.3

x	$f(x)$	$g(f(x))$
-3	3	-2.6
-2.5	0.1	0.8
-2	-1	-1.4
-1.5	-1.3	-1.8
-1	-1.2	-1.7
-0.5	-1	-1.4
0	-0.8	-1
0.5	-0.6	-0.6
1	-0.4	-0.3
1.5	-0.1	0.3
2	0.3	1.1
2.5	0.9	2
3	1.6	2.2

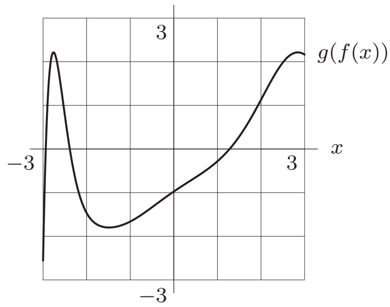


Figure 1.32

62. Using the same way to compute $f(f(x))$ as in Problem 59, we get Table 1.4. Then we can plot the graph of $f(f(x))$ in Figure 1.33.

Table 1.4

x	$f(x)$	$f(f(x))$
-3	3	1.6
-2.5	0.1	-0.7
-2	-1	-1.2
-1.5	-1.3	-1.3
-1	-1.2	-1.3
-0.5	-1	-1.2
0	-0.8	-1.1
0.5	-0.6	-1
1	-0.4	-0.9
1.5	-0.1	-0.8
2	0.3	-0.6
2.5	0.9	-0.4
3	1.6	0

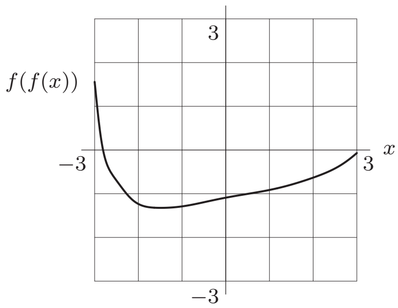


Figure 1.33

63. (a) The graph shows that $f(15)$ is approximately 48. So, the place to find find 15 million-year-old rock is about 48 meters below the Atlantic sea floor.

(b) Since f is increasing (not decreasing, since the depth axis is reversed!), f is invertible. To confirm, notice that the graph of f is cut by a horizontal line at most once.

(c) Look at where the horizontal line through 120 intersects the graph of f and read downward: $f^{-1}(120)$ is about 35. In practical terms, this means that at a depth of 120 meters down, the rock is 35 million years old.

(d) First, we standardize the graph of f so that time and depth are increasing from left to right and bottom to top. Points (t, d) on the graph of f correspond to points (d, t) on the graph of f^{-1}. We can graph f^{-1} by taking points from the original graph of f, reversing their coordinates, and connecting them. This amounts to interchanging the t and d axes, thereby reflecting the graph of f about the line bisecting the 90° angle at the origin. Figure 1.34 is the graph of f^{-1}. (Note that we cannot find the graph of f^{-1} by flipping the graph of f about the line $t = d$ in because t and d have different scales in this instance.)

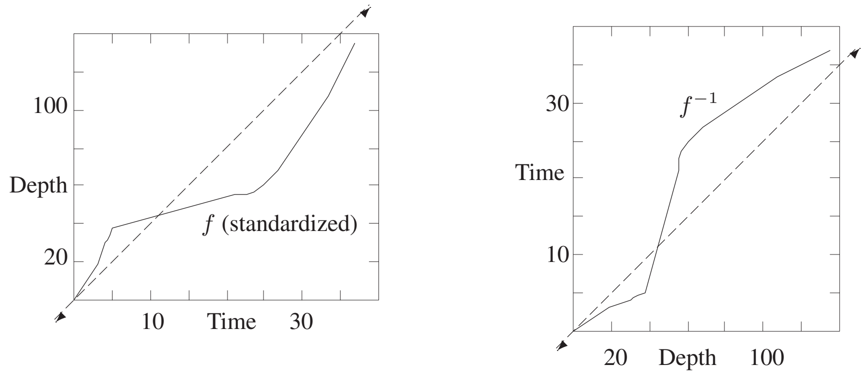


Figure 1.34: Graph of f, reflected to give that of f^{-1}

64. The tree has $B = y - 1$ branches on average and each branch has $n = 2B^2 - B = 2(y-1)^2 - (y-1)$ leaves on average. Therefore

$$\text{Average number of leaves } = Bn = (y-1)(2(y-1)^2 - (y-1)) = 2(y-1)^3 - (y-1)^2.$$

65. The volume, V, of the balloon is $V = \frac{4}{3}\pi r^3$. When $t = 3$, the radius is 10 cm. The volume is then

$$V = \frac{4}{3}\pi(10^3) = \frac{4000\pi}{3} \text{ cm}^3.$$

66. (a) The function f tells us C in terms of q. To get its inverse, we want q in terms of C, which we find by solving for q:

$$\begin{aligned} C &= 100 + 2q, \\ C - 100 &= 2q, \\ q &= \frac{C - 100}{2} = f^{-1}(C). \end{aligned}$$

(b) The inverse function tells us the number of articles that can be produced for a given cost.

67. Since $Q = S - Se^{-kt}$, the graph of Q is the reflection of y about the t-axis moved up by S units.

68.

x	$f(x)$	$g(x)$	$h(x)$
-3	0	0	0
-2	2	2	-2
-1	2	2	-2
0	0	0	0
1	2	-2	-2
2	2	-2	-2
3	0	0	0

Strengthen Your Understanding

69. The graph of $f(x) = -(x+1)^3$ is the graph of $g(x) = -x^3$ shifted left by 1 unit.

70. Since $f(g(x)) = 3(-3x - 5) + 5 = -9x - 10$, we see that f and g are not inverse functions.

71. While $y = 1/x$ is sometimes referred to as the *multiplicative* inverse of x, the inverse of f is $f^{-1}(x) = x$.

72. One possible answer is $g(x) = 3 + x$. (There are many answers.)

73. One possibility is $f(x) = x^2 + 2$.

74. Let $f(x) = 3x$, then $f^{-1}(x) = x/3$. Then for $x > 0$, we have $f(x) > f^{-1}(x)$.

75. We have

$$g(x) = f(x+2)$$

because the graph of g is obtained by moving the graph of f to the left by 2 units. We also have

$$g(x) = f(x) + 3$$

because the graph of g is obtained by moving the graph of f up by 3 units. Thus, we have $f(x+2) = f(x)+3$. The graph of f climbs 3 units whenever x increases by 2. The simplest choice for f is a linear function of slope $3/2$, for example $f(x) = 1.5x$, so $g(x) = 1.5x + 3$.

76. True. The graph of $y = 10^x$ is moved horizontally by h units if we replace x by $x - h$ for some number h. Writing $100 = 10^2$, we have $f(x) = 100(10^x) = 10^2 \cdot 10^x = 10^{x+2}$. The graph of $f(x) = 10^{x+2}$ is the graph of $g(x) = 10^x$ shifted two units to the left.

77. True. If f is increasing then its reflection about the line $y = x$ is also increasing. An example is shown in Figure 1.35. The statement is true.

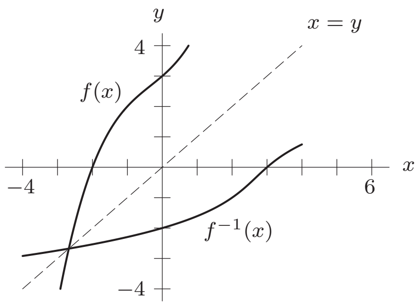


Figure 1.35

78. True. If $f(x)$ is even, we have $f(x) = f(-x)$ for all x. For example, $f(-2) = f(2)$. This means that the graph of $f(x)$ intersects the horizontal line $y = f(2)$ at two points, $x = 2$ and $x = -2$. Thus, f has no inverse function.

79. False. For example, $f(x) = x$ and $g(x) = x^3$ are both odd. Their inverses are $f^{-1}(x) = x$ and $g^{-1}(x) = x^{1/3}$.

80. False. For $x < 0$, as x increases, x^2 decreases, so e^{-x^2} increases.

81. True. We have $g(-x) = g(x)$ since g is even, and therefore $f(g(-x)) = f(g(x))$.

82. False. A counterexample is given by $f(x) = x^2$ and $g(x) = x + 1$. The function $f(g(x)) = (x+1)^2$ is not even because $f(g(1)) = 4$ and $f(g(-1)) = 0 \neq 4$.

83. True. The constant function $f(x) = 0$ is the only function that is both even and odd. This follows, since if f is both even and odd, then, for all x, $f(-x) = f(x)$ (if f is even) and $f(-x) = -f(x)$ (if f is odd). Thus, for all x, $f(x) = -f(x)$ i.e. $f(x) = 0$, for all x. So $f(x) = 0$ is both even and odd and is the only such function.

84. Let $f(x) = x$ and $g(x) = -2x$. Then $f(x) + g(x) = -x$, which is decreasing. Note f is increasing since it has positive slope, and g is decreasing since it has negative slope.

85. This is impossible. If $a < b$, then $f(a) < f(b)$, since f is increasing, and $g(a) > g(b)$, since g is decreasing, so $-g(a) < -g(b)$. Therefore, if $a < b$, then $f(a) - g(a) < f(b) - g(b)$, which means that $f(x) + g(x)$ is increasing.

86. Let $f(x) = e^x$ and let $g(x) = e^{-2x}$. Note f is increasing since it is an exponential growth function, and g is decreasing since it is an exponential decay function. Then $f(x)g(x) = e^{-x}$, which is decreasing.

87. This is impossible. As x increases, $g(x)$ decreases. As $g(x)$ decreases, so does $f(g(x))$ because f is increasing (an increasing function increases as its variable increases, so it decreases as its variable decreases).

Solutions for Section 1.4

Exercises

1. Using the identity $e^{\ln x} = x$, we have $e^{\ln(1/2)} = \frac{1}{2}$.

2. Using the identity $10^{\log x} = x$, we have

$$10^{\log(AB)} = AB$$

3. Using the identity $e^{\ln x} = x$, we have $5A^2$.

4. Using the identity $\ln(e^x) = x$, we have 2AB.

5. Using the rules for ln, we have

$$\begin{aligned} \ln\left(\frac{1}{e}\right) + \ln AB &= \ln 1 - \ln e + \ln A + \ln B \\ &= 0 - 1 + \ln A + \ln B \\ &= -1 + \ln A + \ln B. \end{aligned}$$

6. Using the rules for ln, we have $2A + 3e\ln B$.

7. Taking logs of both sides

$$\begin{aligned} \log 3^x &= x\log 3 = \log 11 \\ x &= \frac{\log 11}{\log 3} = 2.2. \end{aligned}$$

8. Taking logs of both sides

$$\begin{aligned} \log 17^x &= \log 2 \\ x\log 17 &= \log 2 \\ x &= \frac{\log 2}{\log 17} \approx 0.24. \end{aligned}$$

9. Isolating the exponential term

$$\begin{aligned} 20 &= 50(1.04)^x \\ \frac{20}{50} &= (1.04)^x. \end{aligned}$$

Taking logs of both sides

$$\begin{aligned} \log\frac{2}{5} &= \log(1.04)^x \\ \log\frac{2}{5} &= x\log(1.04) \\ x &= \frac{\log(2/5)}{\log(1.04)} = -23.4. \end{aligned}$$

10.

$$\begin{aligned} \frac{4}{7} &= \frac{5^x}{3^x} \\ \frac{4}{7} &= \left(\frac{5}{3}\right)^x \end{aligned}$$

Taking logs of both sides

$$\begin{aligned} \log\left(\frac{4}{7}\right) &= x\log\left(\frac{5}{3}\right) \\ x &= \frac{\log(4/7)}{\log(5/3)} \approx -1.1. \end{aligned}$$

11. To solve for x, we first divide both sides by 5 and then take the natural logarithm of both sides.

$$\begin{aligned}\frac{7}{5} &= e^{0.2x}\\ \ln(7/5) &= 0.2x\\ x &= \frac{\ln(7/5)}{0.2} \approx 1.68.\end{aligned}$$

12. $$\begin{aligned}\ln(2^x) &= \ln(e^{x+1})\\ x\ln 2 &= (x+1)\ln e\\ x\ln 2 &= x+1\\ 0.693x &= x+1\\ x &= \frac{1}{0.693-1} \approx -3.26\end{aligned}$$

13. To solve for x, we first divide both sides by 600 and then take the natural logarithm of both sides.

$$\begin{aligned}\frac{50}{600} &= e^{-0.4x}\\ \ln(50/600) &= -0.4x\\ x &= \frac{\ln(50/600)}{-0.4} \approx 6.212.\end{aligned}$$

14. $$\begin{aligned}\ln(2e^{3x}) &= \ln(4e^{5x})\\ \ln 2 + \ln(e^{3x}) &= \ln 4 + \ln(e^{5x})\\ 0.693 + 3x &= 1.386 + 5x\\ x &= -0.347\end{aligned}$$

15. Using the rules for ln, we get

$$\begin{aligned}\ln 7^{x+2} &= \ln e^{17x}\\ (x+2)\ln 7 &= 17x\\ x(\ln 7 - 17) &= -2\ln 7\\ x &= \frac{-2\ln 7}{\ln 7 - 17} \approx 0.26.\end{aligned}$$

16. $$\begin{aligned}\ln(10^{x+3}) &= \ln(5e^{7-x})\\ (x+3)\ln 10 &= \ln 5 + (7-x)\ln e\\ 2.303(x+3) &= 1.609 + (7-x)\\ 3.303x &= 1.609 + 7 - 2.303(3)\\ x &= 0.515\end{aligned}$$

17. Using the rules for ln, we have

$$\begin{aligned}2x - 1 &= x^2\\ x^2 - 2x + 1 &= 0\\ (x-1)^2 &= 0\\ x &= 1.\end{aligned}$$

18. $$\begin{aligned}4e^{2x-3} &= e+5\\ \ln 4 + \ln(e^{2x-3}) &= \ln(e+5)\\ 1.3863 + 2x - 3 &= 2.0436\\ x &= 1.839.\end{aligned}$$

19. $t = \dfrac{\log a}{\log b}$.

20. $t = \dfrac{\log\left(\frac{P}{P_0}\right)}{\log a} = \dfrac{\log P - \log P_0}{\log a}$.

21. Taking logs of both sides yields

$$nt = \frac{\log\left(\frac{Q}{Q_0}\right)}{\log a}.$$

Hence

$$t = \frac{\log\left(\frac{Q}{Q_0}\right)}{n \log a} = \frac{\log Q - \log Q_0}{n \log a}.$$

22. Collecting similar terms yields

$$\left(\frac{a}{b}\right)^t = \frac{Q_0}{P_0}.$$

Hence

$$t = \frac{\log\left(\frac{Q_0}{P_0}\right)}{\log\left(\frac{a}{b}\right)}.$$

23. $t = \ln \dfrac{a}{b}$.

24. $\ln \dfrac{P}{P_0} = kt$, so $t = \dfrac{\ln \frac{P}{P_0}}{k}$.

25. Since we want $(1.5)^t = e^{kt} = (e^k)^t$, so $1.5 = e^k$, and $k = \ln 1.5 = 0.4055$. Thus, $P = 15e^{0.4055t}$. Since 0.4055 is positive, this is exponential growth.

26. We want $1.7^t = e^{kt}$ so $1.7 = e^k$ and $k = \ln 1.7 = 0.5306$. Thus $P = 10e^{0.5306t}$.

27. We want $0.9^t = e^{kt}$ so $0.9 = e^k$ and $k = \ln 0.9 = -0.1054$. Thus $P = 174e^{-0.1054t}$.

28. Since we want $(0.55)^t = e^{kt} = (e^k)^t$, so $0.55 = e^k$, and $k = \ln 0.55 = -0.5978$. Thus $P = 4e^{-0.5978t}$. Since -0.5978 is negative, this represents exponential decay.

29. If $p(t) = (1.04)^t$, then, for p^{-1} the inverse of p, we should have

$$\begin{aligned}
(1.04)^{p^{-1}(t)} &= t, \\
p^{-1}(t)\log(1.04) &= \log t, \\
p^{-1}(t) &= \frac{\log t}{\log(1.04)} \approx 58.708 \log t.
\end{aligned}$$

30. Since f is increasing, f has an inverse. To find the inverse of $f(t) = 50e^{0.1t}$, we replace t with $f^{-1}(t)$, and, since $f(f^{-1}(t)) = t$, we have

$$t = 50e^{0.1f^{-1}(t)}.$$

We then solve for $f^{-1}(t)$:

$$\begin{aligned}
t &= 50e^{0.1f^{-1}(t)} \\
\frac{t}{50} &= e^{0.1f^{-1}(t)} \\
\ln\left(\frac{t}{50}\right) &= 0.1f^{-1}(t) \\
f^{-1}(t) &= \frac{1}{0.1}\ln\left(\frac{t}{50}\right) = 10\ln\left(\frac{t}{50}\right).
\end{aligned}$$

31. Using $f(f^{-1}(t)) = t$, we see

$$f(f^{-1}(t)) = 1 + \ln f^{-1}(t) = t.$$

So

$$\begin{aligned}
\ln f^{-1}(t) &= t - 1 \\
f^{-1}(t) &= e^{t-1}.
\end{aligned}$$

Problems

32. The population has increased by a factor of $48{,}000{,}000/40{,}000{,}000 = 1.2$ in 10 years. Thus we have the formula

$$P = 40{,}000{,}000(1.2)^{t/10},$$

and $t/10$ gives the number of 10-year periods that have passed since 2000.

In 2000, $t/10 = 0$, so we have $P = 40{,}000{,}000$.
In 2010, $t/10 = 1$, so $P = 40{,}000{,}000(1.2) = 48{,}000{,}000$.
In 2020, $t/10 = 2$, so $P = 40{,}000{,}000(1.2)^2 = 57{,}600{,}000$.
To find the doubling time, solve $80{,}000{,}000 = 40{,}000{,}000(1.2)^{t/10}$, to get $t = 38.02$ years.

33. In ten years, the substance has decayed to 40% of its original mass. In another ten years, it will decay by an additional factor of 40%, so the amount remaining after 20 years will be $100 \cdot 40\% \cdot 40\% = 16$ kg.

34. We can solve for the growth rate k of the bacteria using the formula $P = P_0 e^{kt}$:

$$\begin{aligned} 1500 &= 500e^{k(2)} \\ k &= \frac{\ln(1500/500)}{2}. \end{aligned}$$

Knowing the growth rate, we can find the population P at time $t = 6$:

$$\begin{aligned} P &= 500e^{(\frac{\ln 3}{2})6} \\ &\approx 13{,}500 \text{ bacteria.} \end{aligned}$$

35. (a) Assuming the US population grows exponentially, we have population $P(t) = 281.4e^{kt}$ at time t years after 2000. Using the 2010 population, we have

$$\begin{aligned} 308.7 &= 281.4e^{10k} \\ k &= \frac{\ln(308.7/281.4)}{10} = 0.00926. \end{aligned}$$

We want to find the time t in which

$$\begin{aligned} 350 &= 281.4e^{0.00926t} \\ t &= \frac{\ln(350/281.4)}{0.00926} = 23.56 \text{ years.} \end{aligned}$$

This model predicts the population to go over 350 million 23.56 years after 2000, in the year 2023.

(b) Evaluate $P = 281.4e^{0.00926t}$ for $t = 20$ to find $P = 338.65$ million people.

36. If C_0 is the concentration of NO_2 on the road, then the concentration x meters from the road is

$$C = C_0 e^{-0.0254x}.$$

We want to find the value of x making $C = C_0/2$, that is,

$$C_0 e^{-0.0254x} = \frac{C_0}{2}.$$

Dividing by C_0 and then taking natural logs yields

$$\ln\left(e^{-0.254x}\right) = -0.0254x = \ln\left(\frac{1}{2}\right) = -0.6931,$$

so

$$x = 27 \text{ meters.}$$

At 27 meters from the road the concentration of NO_2 in the air is half the concentration on the road.

37. (a) Since the percent increase in deaths during a year is constant for constant increase in pollution, the number of deaths per year is an exponential function of the quantity of pollution. If Q_0 is the number of deaths per year without pollution, then the number of deaths per year, Q, when the quantity of pollution is x micrograms per cu meter of air is

$$Q = Q_0(1.0033)^x.$$

(b) We want to find the value of x making $Q = 2Q_0$, that is,

$$Q_0(1.0033)^x = 2Q_0.$$

Dividing by Q_0 and then taking natural logs yields

$$\ln\left((1.0033)^x\right) = x\ln 1.0033 = \ln 2,$$

so

$$x = \frac{\ln 2}{\ln 1.0033} = 210.391.$$

When there are 210.391 micrograms of pollutants per cu meter of air, respiratory deaths per year are double what they would be in the absence of air pollution.

38. (a) Since there are 4 years between 2004 and 2008 we let t be the number of years since 2004 and get:

$$450{,}327 = 211{,}800e^{r4}.$$

Solving for r, we get

$$\begin{aligned} \frac{450{,}327}{211{,}800} &= e^{r4} \\ \ln\left(\frac{450{,}327}{211{,}800}\right) &= 4r \\ r &= 0.188583 \end{aligned}$$

Substituting $t = 1, 2, 3$ into

$$211{,}800\, e^{(0.188583)t},$$

we find the three remaining table values:

Year	2004	2005	2006	2007	2008
Number of E85 vehicles	211,800	255,756	308,835	372,930	450,327

(b) If N is the number of E85-powered vehicles in 2003, then

$$211{,}800 = Ne^{0.188583}$$

or

$$N = \frac{211{,}800}{e^{0.188583}} = 175{,}398 \text{ vehicles.}$$

(c) From the table, we can see that the number of E85 vehicles slightly more than doubled from 2004 to 2008, so the percent growth between these years should be slightly over 100%:

$$\begin{matrix}\text{Percent growth from} \\ \text{2004 to 2008}\end{matrix} = 100\left(\frac{450{,}327}{211{,}800} - 1\right) = 1.12619 = 112.619\%.$$

39. (a) The initial dose is 10 mg.
(b) Since $0.82 = 1 - 0.18$, the decay rate is 0.18, so 18% leaves the body each hour.
(c) When $t = 6$, we have $A = 10(0.82)^6 = 3.04$. The amount in the body after 6 hours is 3.04 mg.
(d) We want to find the value of t when $A = 1$. Using logarithms:

$$\begin{aligned} 1 &= 10(0.82)^t \\ 0.1 &= (0.82)^t \\ \ln(0.1) &= t\ln(0.82) \\ t &= 11.60 \text{ hours.} \end{aligned}$$

After 11.60 hours, the amount is 1 mg.

40. (a) Since the initial amount of caffeine is 100 mg and the exponential decay rate is -0.17, we have $A = 100e^{-0.17t}$.
(b) See Figure 1.36. We estimate the half-life by estimating t when the caffeine is reduced by half (so $A = 50$); this occurs at approximately $t = 4$ hours.

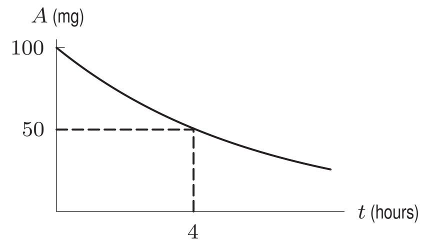


Figure 1.36

(c) We want to find the value of t when $A = 50$:

$$\begin{aligned} 50 &= 100e^{-0.17t} \\ 0.5 &= e^{-0.17t} \\ \ln 0.5 &= -0.17t \\ t &= 4.077. \end{aligned}$$

The half-life of caffeine is about 4.077 hours. This agrees with what we saw in Figure 1.36.

41. Since $y(0) = Ce^0 = C$ we have that $C = 2$. Similarly, substituting $x = 1$ gives $y(1) = 2e^{\alpha}$ so

$$2e^{\alpha} = 1.$$

Rearranging gives $e^{\alpha} = 1/2$. Taking logarithms we get $\alpha = \ln(1/2) = -\ln 2 = -0.693$. Finally,

$$y(2) = 2e^{2(-\ln 2)} = 2e^{-2\ln 2} = \frac{1}{2}.$$

42. The function e^x has a vertical intercept of 1, so must be A. The function $\ln x$ has an x-intercept of 1, so must be D. The graphs of x^2 and $x^{1/2}$ go through the origin. The graph of $x^{1/2}$ is concave down so it corresponds to graph C and the graph of x^2 is concave up so it corresponds to graph B.

43. (a) $B(t) = B_0 e^{0.067t}$
(b) $P(t) = P_0 e^{0.033t}$
(c) If the initial price is \$50, then

$$\begin{aligned} B(t) &= 50e^{0.067t} \\ P(t) &= 50e^{0.033t}. \end{aligned}$$

We want the value of t such that

$$\begin{aligned} B(t) &= 2P(t) \\ 50e^{0.067t} &= 2 \cdot 50e^{0.033t} \\ \frac{e^{0.067t}}{e^{0.033t}} &= e^{0.034t} = 2 \\ t = \frac{\ln 2}{0.034} &= 20.387 \text{ years}. \end{aligned}$$

Thus, when $t = 20.387$ the price of the textbook was predicted to be double what it would have been had the price risen by inflation only. This occurred in the year 2000.

44. (a) We assume $f(t) = Ae^{-kt}$, where A is the initial population, so $A = 100{,}000$. When $t = 110$, there were 3200 tigers, so

$$3200 = 100{,}000e^{-k \cdot 110}$$

Solving for k gives

$$\begin{aligned} e^{-k \cdot 110} &= \frac{3200}{100{,}000} = 0.0132 \\ k = -\frac{1}{110} \ln(0.0132) &= 0.0313 = 3.13\% \end{aligned}$$

so

$$f(t) = 100{,}000e^{-0.0313t}.$$

(b) In 2000, the predicted number of tigers was

$$f(100) = 100{,}000e^{-0.0313(100)} = 4372.$$

In 2010, we know the number of tigers was 3200. The predicted percent reduction is

$$\frac{3200 - 4372}{4372} = -0.268 = -26.8\%.$$

Thus the actual decrease is larger than the predicted decrease.

45. The population of China, C, in billions, is given by

$$C = 1.34(1.004)^t$$

where t is time measured from 2011, and the population of India, I, in billions, is given by

$$I = 1.19(1.0137)^t.$$

The two populations will be equal when $C = I$, thus, we must solve the equation:

$$1.34(1.004)^t = 1.19(1.0137)^t$$

for t, which leads to

$$\frac{1.34}{1.19} = \frac{(1.0137)^t}{(1.0004)^t} = \left(\frac{1.0137}{1.004}\right)^t.$$

Taking logs on both sides, we get

$$t \log \frac{1.0137}{1.004} = \log \frac{1.34}{1.19},$$

so

$$t = \frac{\log(1.34/1.19)}{\log(1.0137/1.004)} = 12.35 \text{ years.}$$

This model predicts the population of India will exceed that of China in 2023.

46. Let A represent the revenue (in billions of dollars) at Apple t years since 2005. Since $A = 3.68$ when $t = 0$ and we want the continuous growth rate, we write $A = 3.68e^{kt}$. We use the information from 2010, that $A = 15.68$ when $t = 5$, to find k:

$$\begin{aligned} 15.68 &= 3.68e^{k\cdot 5} \\ 4.26 &= e^{5k} \\ \ln(4.26) &= 5k \\ k &= 0.2899. \end{aligned}$$

We have $A = 3.68e^{0.2899t}$, which represents a continuous growth rate of 28.99% per year.

47. Let $P(t)$ be the world population in billions t years after 2010.

(a) Assuming exponential growth, we have

$$P(t) = 6.9e^{kt}.$$

In 2050, we have $t = 40$ and we expect the population then to be 9 billion, so

$$9 = 6.9e^{k\cdot 40}.$$

Solving for k, we have

$$e^{k\cdot 40} = \frac{9}{6.9}$$

$$k = \frac{1}{40}\ln\left(\frac{9}{6.9}\right) = 0.00664 = 0.664\% \text{ per year.}$$

(b) The "Day of 7 Billion" should occur when

$$7 = 6.9e^{0.00664t}.$$

Solving for t gives

$$e^{0.00664t} = \frac{7}{6.9}$$

$$t = \frac{\ln(7/6.9)}{0.00664} = 2.167 \text{ years.}$$

So the "Day of 7 Billion" should be 2.167 years after the end of 2010. This is 2 years and $0.167 \cdot 365 = 61$ days; so 61 days into 2013. That is, March 2, 2013.

48. If r was the average yearly inflation rate, in decimals, then $\frac{1}{4}(1+r)^3 = 2{,}400{,}000$, so $r = 211.53$, i.e. $r = 21{,}153\%$.

49. To find a half-life, we want to find at what t value $Q = \frac{1}{2}Q_0$. Plugging this into the equation of the decay of plutonium-240, we have

$$\begin{aligned}\frac{1}{2} &= e^{-0.00011t}\\ t &= \frac{\ln(1/2)}{-0.00011} \approx 6{,}301 \text{ years.}\end{aligned}$$

The only difference in the case of plutonium-242 is that the constant -0.00011 in the exponent is now -0.0000018. Thus, following the same procedure, the solution for t is

$$t = \frac{\ln(1/2)}{-0.0000018} \approx 385{,}081 \text{ years.}$$

50. Given the doubling time of 5 hours, we can solve for the bacteria's growth rate;

$$\begin{aligned}2P_0 &= P_0e^{k5}\\ k &= \frac{\ln 2}{5}.\end{aligned}$$

So the growth of the bacteria population is given by:

$$P = P_0e^{\ln(2)t/5}.$$

We want to find t such that

$$3P_0 = P_0e^{\ln(2)t/5}.$$

Therefore we cancel P_0 and apply ln. We get

$$t = \frac{5\ln(3)}{\ln(2)} = 7.925 \text{ hours.}$$

51. (a) The pressure P at 6194 meters is given in terms of the pressure P_0 at sea level to be

$$\begin{aligned}P &= P_0e^{-0.00012h}\\ &= P_0e^{(-0.00012)6194}\\ &= P_0e^{-0.74328}\\ &\approx 0.4756P_0 \quad \text{or about 47.6\% of sea level pressure.}\end{aligned}$$

(b) At $h = 12{,}000$ meters, we have

$$\begin{aligned}P &= P_0e^{-0.00012h}\\ &= P_0e^{(-0.00012)12{,}000}\\ &= P_0e^{-1.44}\\ &\approx 0.2369P_0 \quad \text{or about 23.7\% of sea level pressure.}\end{aligned}$$

52. We know that the y-intercept of the line is at (0,1), so we need one other point to determine the equation of the line. We observe that it intersects the graph of $f(x) = 10^x$ at the point $x = \log 2$. The y-coordinate of this point is then

$$y = 10^x = 10^{\log 2} = 2,$$

so $(\log 2, 2)$ is the point of intersection. We can now find the slope of the line:

$$m = \frac{2-1}{\log 2 - 0} = \frac{1}{\log 2}.$$

Plugging this into the point-slope formula for a line, we have

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 1 &= \frac{1}{\log 2}(x - 0)\\ y &= \frac{1}{\log 2}x + 1 \approx 3.3219x + 1.\end{aligned}$$

53. If t is time in decades, then the number of vehicles, V, in millions, is given by

$$V = 246(1.155)^t.$$

For time t in decades, the number of people, P, in millions, is given by

$$P = 308.7(1.097)^t.$$

There is an average of one vehicle per person when $\frac{V}{P} = 1$, or $V = P$. Thus, we solve for t in the equation:

$$246(1.155)^t = 308.7(1.097)^t,$$

which leads to

$$\left(\frac{1.155}{1.097}\right)^t = \frac{(1.155)^t}{(1.097)^t} = \frac{308.7}{246}$$

Taking logs on both sides, we get

$$t \log \frac{1.155}{1.097} = \log \frac{308.7}{246},$$

so

$$t = \frac{\log(308.7/246)}{\log(1.155/1.097)} = 4.41 \text{ decades.}$$

This model predicts one vehicle per person in 2054

54. We assume exponential decay and solve for k using the half-life:

$$e^{-k(5730)} = 0.5 \quad \text{so} \quad k = 1.21 \cdot 10^{-4}.$$

Now find t, the age of the painting:

$$e^{-1.21 \cdot 10^{-4} t} = 0.995, \quad \text{so} \quad t = \frac{\ln 0.995}{-1.21 \cdot 10^{-4}} = 41.43 \text{ years.}$$

Since Vermeer died in 1675, the painting is a fake.

55. Yes, $\ln(\ln(x))$ means take the ln of the value of the function $\ln x$. On the other hand, $\ln^2(x)$ means take the function $\ln x$ and square it. For example, consider each of these functions evaluated at e. Since $\ln e = 1$, $\ln^2 e = 1^2 = 1$, but $\ln(\ln(e)) = \ln(1) = 0$. See the graphs in Figure 1.37. (Note that $\ln(\ln(x))$ is only defined for $x > 1$.)

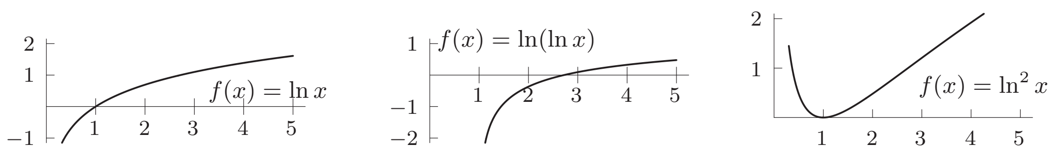


Figure 1.37

56. (a) The y-intercept of $h(x) = \ln(x + a)$ is $h(0) = \ln a$. Thus increasing a increases the y-intercept.
(b) The x-intercept of $h(x) = \ln(x+a)$ is where $h(x) = 0$. Since this occurs where $x+a = 1$, or $x = 1-a$, increasing a moves the x-intercept to the left.

57. The vertical asymptote is where $x + a = 0$, or $x = -a$. Thus increasing a moves the vertical asymptote to the left.

58. (a) The y-intercept of $g(x) = \ln(ax + 2)$ is $g(0) = \ln 2$. Thus increasing a does not effect the y-intercept.
(b) The x-intercept of $g(x) = \ln(ax + 2)$ is where $g(x) = 0$. Since this occurs where $ax + 2 = 1$, or $x = -1/a$, increasing a moves the x-intercept toward the origin. (The intercept is to the left of the origin if $a > 0$ and to the right if $a < 0$.)

59. The vertical asymptote is where $x + 2 = 0$, or $x = -2$, so increasing a does not effect the vertical asymptote.

60. The vertical asymptote is where $ax + 2 = 0$, or $x = -2/a$. Thus increasing a moves the vertical asymptote toward the origin. (The asymptote is to the left of the origin for $a > 0$ and to the right of the origin for $a < 0$.)

Strengthen Your Understanding

61. The function $-\log|x|$ is even, since $|-x| = |x|$, which means $-\log|-x| = -\log|x|$.

62. We have

$$\ln(100x) = \ln(100) + \ln x.$$

In general, $\ln(100x) \neq 100 \cdot \ln x$.

63. One possibility is $f(x) = -x$, because $\ln(-x)$ is only defined if $-x > 0$.

64. One possibility is $f(x) = \ln(x-3)$.

65. True, as seen from the graph.

66. False, since $\log(x-1) = 0$ if $x - 1 = 1$, so $x = 2$.

67. False. The inverse function is $y = 10^x$.

68. False, since $ax + b = 0$ if $x = -b/a$. Thus $y = \ln(ax+b)$ has a vertical asymptote at $x = -b/a$.

Solutions for Section 1.5

Exercises

1. See Figure 1.38.

$$\sin\left(\frac{3\pi}{2}\right) = -1 \quad \text{is negative.}$$
$$\cos\left(\frac{3\pi}{2}\right) = 0$$
$$\tan\left(\frac{3\pi}{2}\right) \quad \text{is undefined.}$$

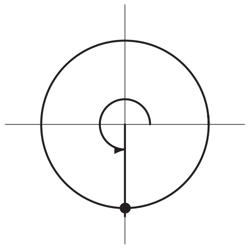


Figure 1.38

2. See Figure 1.39.

$$\sin(2\pi) = 0$$
$$\cos(2\pi) = 1 \text{ is positive.}$$
$$\tan(2\pi) = 0$$

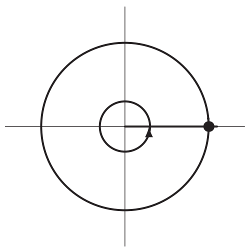


Figure 1.39

3. See Figure 1.40.

$$
\begin{aligned}
\sin\frac{\pi}{4} &\quad \text{is positive} \\
\cos\frac{\pi}{4} &\quad \text{is positive} \\
\tan\frac{\pi}{4} &\quad \text{is positive}
\end{aligned}
$$

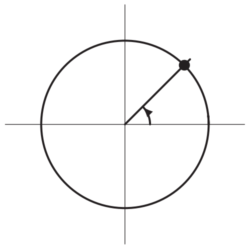


Figure 1.40

4. See Figure 1.41.

$$
\begin{aligned}
\sin 3\pi &= 0 \\
\cos 3\pi &= -1 \text{ is negative} \\
\tan 3\pi &= 0
\end{aligned}
$$

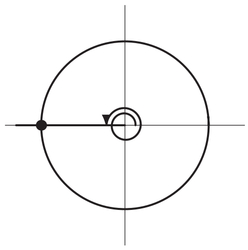


Figure 1.41

5. See Figure 1.42.

$$
\begin{aligned}
\sin\left(\frac{\pi}{6}\right) &\quad \text{is positive.} \\
\cos\left(\frac{\pi}{6}\right) &\quad \text{is positive.} \\
\tan\left(\frac{\pi}{6}\right) &\quad \text{is positive.}
\end{aligned}
$$

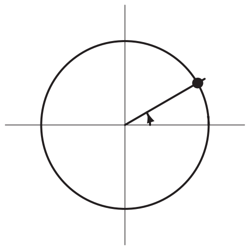


Figure 1.42

6. See Figure 1.43.

$$\begin{aligned} \sin\frac{4\pi}{3} &\quad \text{is negative} \\ \cos\frac{4\pi}{3} &\quad \text{is negative} \\ \tan\frac{4\pi}{3} &\quad \text{is positive} \end{aligned}$$

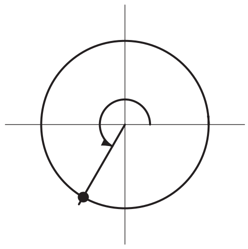


Figure 1.43

7. See Figure 1.44.

$$\begin{aligned} \sin\left(\frac{-4\pi}{3}\right) &\quad \text{is positive.} \\ \cos\left(\frac{-4\pi}{3}\right) &\quad \text{is negative.} \\ \tan\left(\frac{-4\pi}{3}\right) &\quad \text{is negative.} \end{aligned}$$

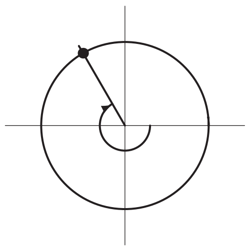


Figure 1.44

8. $4 \text{ radians} \cdot \dfrac{180^\circ}{\pi \text{ radians}} = \left(\dfrac{720}{\pi}\right)^\circ \approx 240^\circ$. See Figure 1.45.

$$\begin{aligned} \sin 4 &\quad \text{is negative} \\ \cos 4 &\quad \text{is negative} \\ \tan 4 &\quad \text{is positive.} \end{aligned}$$

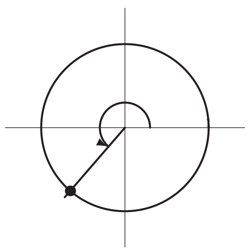


Figure 1.45

9. -1 radian $\cdot \frac{180°}{\pi \text{ radians}} = -\left(\frac{180°}{\pi}\right) \approx -60°$. See Figure 1.46.

$$\begin{array}{ll} \sin(-1) & \text{is negative} \\ \cos(-1) & \text{is positive} \\ \tan(-1) & \text{is negative.} \end{array}$$

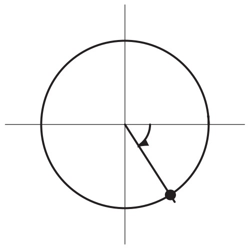


Figure 1.46

10. The period is $2\pi/3$, because when t varies from 0 to $2\pi/3$, the quantity $3t$ varies from 0 to 2π. The amplitude is 7, since the value of the function oscillates between -7 and 7.

11. The period is $2\pi/(1/4) = 8\pi$, because when u varies from 0 to 8π, the quantity $u/4$ varies from 0 to 2π. The amplitude is 3, since the function oscillates between 2 and 8.

12. The period is $2\pi/2 = \pi$, because as x varies from $-\pi/2$ to $\pi/2$, the quantity $2x + \pi$ varies from 0 to 2π. The amplitude is 4, since the function oscillates between 4 and 12.

13. The period is $2\pi/\pi = 2$, since when t increases from 0 to 2, the value of πt increases from 0 to 2π. The amplitude is 0.1, since the function oscillates between 1.9 and 2.1.

14. This graph is a sine curve with period 8π and amplitude 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right)$.

15. This graph is a cosine curve with period 6π and amplitude 5, so it is given by $f(x) = 5\cos\left(\frac{x}{3}\right)$.

16. This graph is an inverted sine curve with amplitude 4 and period π, so it is given by $f(x) = -4\sin(2x)$.

17. This graph is an inverted cosine curve with amplitude 8 and period 20π, so it is given by $f(x) = -8\cos\left(\frac{x}{10}\right)$.

18. This graph has period 6, amplitude 5 and no vertical or horizontal shift, so it is given by

$$f(x) = 5\sin\left(\frac{2\pi}{6}x\right) = 5\sin\left(\frac{\pi}{3}x\right).$$

19. The graph is a cosine curve with period $2\pi/5$ and amplitude 2, so it is given by $f(x) = 2\cos(5x)$.

20. The graph is an inverted sine curve with amplitude 1 and period 2π, shifted up by 2, so it is given by $f(x) = 2 - \sin x$.

21. This can be represented by a sine function of amplitude 3 and period 18. Thus,

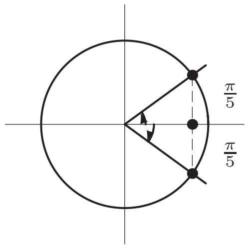
$$f(x) = 3\sin\left(\frac{\pi}{9}x\right).$$

22. This graph is the same as in Problem 14 but shifted up by 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right) + 2$.

23. This graph has period 8, amplitude 3, and a vertical shift of 3 with no horizontal shift. It is given by

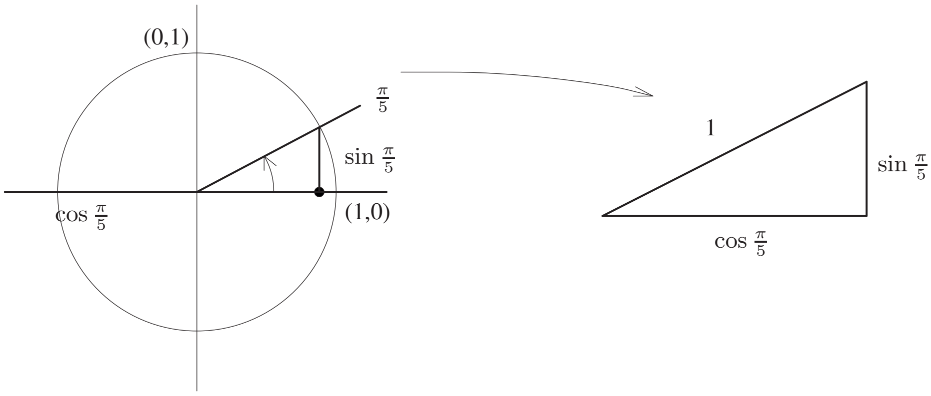
$$f(x) = 3 + 3\sin\left(\frac{2\pi}{8}x\right) = 3 + 3\sin\left(\frac{\pi}{4}x\right).$$

24.



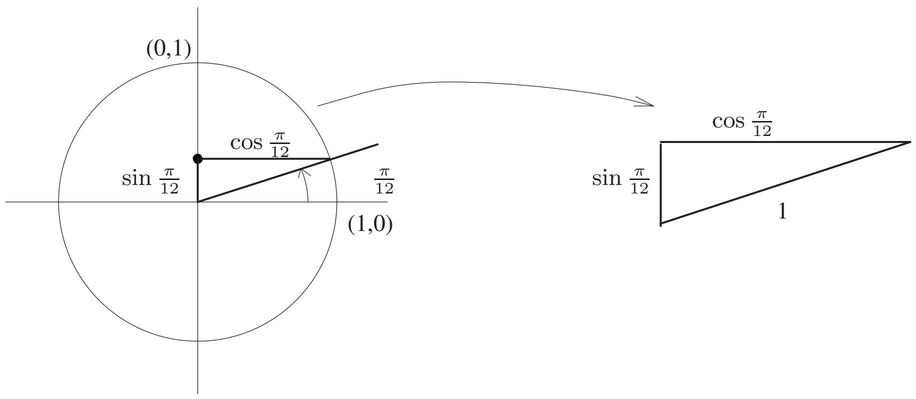
$$\cos\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5} \quad \text{(by picture)}$$
$$= 0.809.$$

25.



By the Pythagorean Theorem, $(\cos\frac{\pi}{5})^2 + (\sin\frac{\pi}{5})^2 = 1^2$;
so $(\sin\frac{\pi}{5})^2 = 1 - (\cos\frac{\pi}{5})^2$, and $\sin\frac{\pi}{5} = \sqrt{1-(\cos\frac{\pi}{5})^2} = \sqrt{1-(0.809)^2} \approx 0.588$.
We take the positive square root since by the picture we know that $\sin\frac{\pi}{5}$ is positive.

26.



By the Pythagorean Theorem, $(\cos\frac{\pi}{12})^2+(\sin\frac{\pi}{12})^2 = 1^2$; so $(\cos\frac{\pi}{12})^2 = 1-(\sin\frac{\pi}{12})^2$ and $\cos\frac{\pi}{12} = \sqrt{1-(\sin\frac{\pi}{12})^2} = \sqrt{1-(0.259)^2} \approx 0.966$. We take the positive square root since by the picture we know that $\cos\frac{\pi}{12}$ is positive.

27. We first divide by 5 and then use inverse sine:

$$\frac{2}{5} = \sin(3x)$$
$$\sin^{-1}(2/5) = 3x$$
$$x = \frac{\sin^{-1}(2/5)}{3} \approx 0.1372.$$

There are infinitely many other possible solutions since the sine is periodic.

28. We first isolate $\cos(2x+1)$ and then use inverse cosine:

$$\begin{aligned}1 &= 8\cos(2x+1)-3\\ 4 &= 8\cos(2x+1)\\ 0.5 &= \cos(2x+1)\\ \cos^{-1}(0.5) &= 2x+1\\ x &= \frac{\cos^{-1}(0.5)-1}{2} \approx 0.0236.\end{aligned}$$

There are infinitely many other possible solutions since the cosine is periodic.

29. We first isolate $\tan(5x)$ and then use inverse tangent:

$$\begin{aligned}8 &= 4\tan(5x)\\ 2 &= \tan(5x)\\ \tan^{-1}2 &= 5x\\ x &= \frac{\tan^{-1}2}{5} = 0.221.\end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

30. We first isolate $(2x+1)$ and then use inverse tangent:

$$\begin{aligned}1 &= 8\tan(2x+1)-3\\ 4 &= 8\tan(2x+1)\\ 0.5 &= \tan(2x+1)\\ \arctan(0.5) &= 2x+1\\ x &= \frac{\arctan(0.5)-1}{2} = -0.268.\end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

31. We first isolate $\sin(5x)$ and then use inverse sine:

$$\begin{aligned}8 &= 4\sin(5x)\\ 2 &= \sin(5x).\end{aligned}$$

But this equation has no solution since $-1 \le \sin(5x) \le 1$.

Problems

32. (a) $h(t) = 2\cos(t-\pi/2)$
(b) $f(t) = 2\cos t$
(c) $g(t) = 2\cos(t+\pi/2)$

33. $\sin x^2$ is by convention $\sin(x^2)$, which means you square the x first and then take the sine.
$\sin^2 x = (\sin x)^2$ means find $\sin x$ and then square it.
$\sin(\sin x)$ means find $\sin x$ and then take the sine of that.
Expressing each as a composition: If $f(x) = \sin x$ and $g(x) = x^2$, then
$\sin x^2 = f(g(x))$
$\sin^2 x = g(f(x))$
$\sin(\sin x) = f(f(x))$.

34. Suppose P is at the point $(3\pi/2, -1)$ and Q is at the point $(5\pi/2, 1)$. Then

$$\text{Slope } = \frac{1-(-1)}{5\pi/2 - 3\pi/2} = \frac{2}{\pi}.$$

If P had been picked to the right of Q, the slope would have been $-2/\pi$.

35. (a) See Figure 1.47.

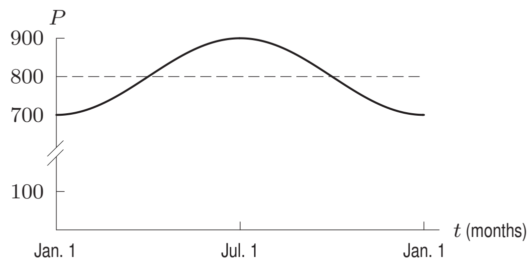


Figure 1.47

(b) Average value of population $= \frac{700+900}{2} = 800$, amplitude $= \frac{900-700}{2} = 100$, and period $= 12$ months, so $B = 2\pi/12 = \pi/6$. Since the population is at its minimum when $t = 0$, we use a negative cosine:

$$P = 800 - 100\cos\left(\frac{\pi t}{6}\right).$$

36. We use a cosine of the form

$$H = A\cos(Bt) + C$$

and choose B so that the period is 24 hours, so $2\pi/B = 24$ giving $B = \pi/12$.

The temperature oscillates around an average value of 60° F, so $C = 60$. The amplitude of the oscillation is 20° F. To arrange that the temperature be at its lowest when $t = 0$, we take A negative, so $A = -20$. Thus

$$A = 60 - 20\cos\left(\frac{\pi}{12}t\right).$$

37. (a) $f(t) = -0.5 + \sin t, \quad g(t) = 1.5 + \sin t, \quad h(t) = -1.5 + \sin t, \quad k(t) = 0.5 + \sin t.$

(b) The values of $g(t)$ are one more than the values of $k(t)$, so $g(t) = 1 + k(t)$. This happens because $g(t) = 1.5 + \sin t = 1 + 0.5 + \sin t = 1 + k(t)$.

(c) Since $-1 \le \sin t \le 1$, adding 1.5 everywhere we get $0.5 \le 1.5 + \sin t \le 2.5$ and since $1.5 + \sin t = g(t)$, we get $0.5 \le g(t) \le 2.5$. Similarly, $-2.5 \le -1.5 + \sin t = h(t) \le -0.5$.

38. Depth $= 7 + 1.5\sin\left(\frac{\pi}{3}t\right)$

39. (a) Beginning at time $t = 0$, the voltage will have oscillated through a complete cycle when $\cos(120\pi t) = \cos(2\pi)$, hence when $t = \frac{1}{60}$ second. The period is $\frac{1}{60}$ second.

(b) V_0 represents the amplitude of the oscillation.

(c) See Figure 1.48.

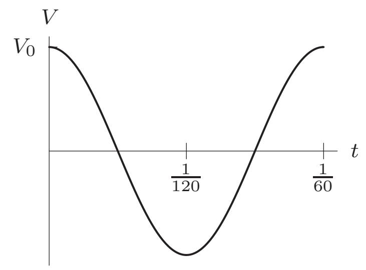


Figure 1.48

40. (a) When the time is t hours after 6 am, the solar panel outputs $f(t) = P(\theta(t))$ watts. So,

$$f(t) = 10\sin\left(\frac{\pi}{14}t\right)$$

where $0 \le t \le 14$ is the number of hours after 6 am.

(b) The graph of $f(t)$ is in Figure 1.49:

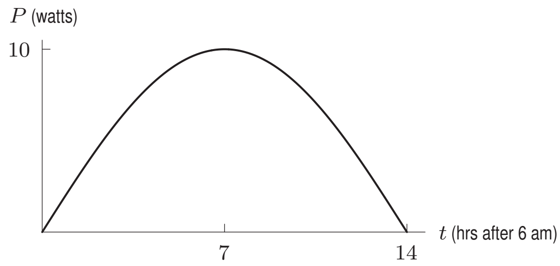


Figure 1.49

(c) The power output is greatest when $\sin(\pi t/14) = 1$. Since $0 \leq \pi t/14 \leq \pi$, the only point in the domain of f at which $\sin(\pi t/14) = 1$ is when $\pi t/14 = \pi/2$. Therefore, the power output is greatest when $t = 7$, that is, at 1 pm. The output at this time will be $f(7) = 10$ watts.

(d) On a typical winter day, there are 9 hours of sun instead of the 14 hours of sun. So, if t is the number of hours since 8 am, the angle between a solar panel and the sun is

$$\phi = \frac{14}{9}\theta = \frac{\pi}{9}t \quad \text{where } 0 \leq t \leq 9.$$

The solar panel outputs $g(t) = P(\phi(t))$ watts:

$$g(t) = 10\sin\left(\frac{\pi}{9}t\right)$$

where $0 \leq t \leq 9$ is the number of hours after 8 am.

41. The function R has period of π, so its graph is as shown in Figure 1.50. The maximum value of the range is v_0^2/g and occurs when $\theta = \pi/4$.

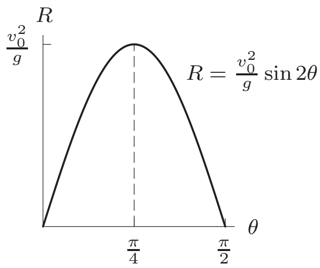


Figure 1.50

42. Over the one-year period, the average value is about 75° and the amplitude of the variation is about $\frac{90-60}{2} = 15°$. The function assumes its minimum value right at the beginning of the year, so we want a negative cosine function. Thus, for t in years, we have the function

$$f(t) = 75 - 15\cos\left(\frac{2\pi}{12}t\right).$$

(Many other answers are possible, depending on how you read the chart.)

43. (a) $D =$ the average depth of the water.
(b) $A =$ the amplitude $= 15/2 = 7.5$.
(c) Period = 12.4 hours. Thus $(B)(12.4) = 2\pi$ so $B = 2\pi/12.4 \approx 0.507$.
(d) C is the time of a high tide.

44. Using the fact that 1 revolution $= 2\pi$ radians and 1 minute = 60 seconds, we have

$$\begin{aligned} 200\frac{\text{rev}}{\text{min}} &= (200)\cdot 2\pi\frac{\text{rad}}{\text{min}} = 200\cdot 2\pi\frac{1}{60}\frac{\text{rad}}{\text{sec}} \\ &\approx \frac{(200)(6.283)}{60} \\ &\approx 20.94 \text{ radians per second.} \end{aligned}$$

Similarly, 500 rpm is equivalent to 52.36 radians per second.

45. 200 revolutions per minute is $\frac{1}{200}$ minutes per revolution, so the period is $\frac{1}{200}$ minutes, or 0.3 seconds.

46. The earth makes one revolution around the sun in one year, so its period is one year.

47. The moon makes one revolution around the earth in about 27.3 days, so its period is 27.3 days $\approx$ one month.

48. (a) The period of the tides is $2\pi/0.5 = 4\pi = 12.566$ hours.

(b) The boat is afloat provided the water is deeper than 2.5 meters, so we need

$$d(t) = 5 + 4.6\sin(0.5t) > 2.5.$$

Figure 1.51 is a graph of $d(t)$, with time t in hours since midnight, $0 \le t \le 24$. The boat leaves at $t = 12$ (midday). To find the latest time the boat can return, we need to solve the equation $d(t) = 5 + 4.6\sin(0.5t) = 2.5$.

A quick way to estimate the solution is to trace along the line $y = 2.5$ in Figure 1.51 until we get to the first point of intersection to the right of $t = 12$. The value we want is about $t = 20$. Thus the water remains deep enough until about 8 pm.

To find t analytically, we solve

$$\begin{aligned} 5 + 4.6\sin(0.5t) &= 2.5 \\ \sin(0.5t) &= -\frac{2.5}{4.6} = -0.5435 \\ t &= \frac{1}{0.5}\arcsin(-0.5435) = -1.149. \end{aligned}$$

This is the value of t immediately to the left of the vertical axis. The water is also 2.5 meters deep one period later at $t = -1.149 + 12.566 = 11.417$. This is shortly before the boat leaves, while the water is rising. We want the next time the time the water is this depth.

The water was at its deepest (that is, $d(t)$ was a maximum) when $t = 12.566/4 = 3.142$. From the figure, the time between when the water was 2.5 meters and when it was deepest was $3.142 + 1.149 = 4.291$ hours. Thus, the value of t that we want is

$$t = 11.417 + 2 \cdot 4.291 = 19.999.$$

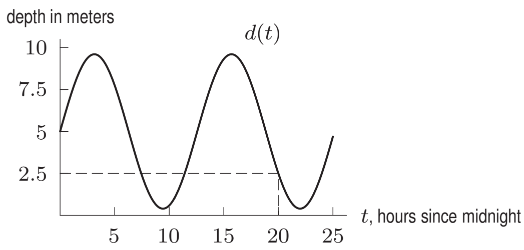


Figure 1.51

49. Since b is a positive constant, f is a vertical shift of $\sin t$ where the midline lies above the t-axis. So f matches Graph C.

Function g is the sum of $\sin t$ plus a linear function $at + b$. We suspect then that the graph of g might periodically oscillate about a line $at + b$, just like the graph of $\sin t$ oscillates about its midline. When adding $at + b$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, at + b)$ that lies both on the graph of g, and on the line $at + b$. See Figure 1.52. So g matches Graph B.

Function h is the sum of $\sin t$ plus an increasing exponential function $e^{ct} + d$. We suspect then that the graph of g might periodically oscillate about the graph of $e^{ct} + d$, just like the graph of $\sin t$ oscillates about its midline. When adding $e^{ct} + d$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, e^{ct} + d)$ that lies both on the graph of h, and on the graph of $e^{ct} + d$. See Figure 1.52. So h matches Graph A. Note that the oscillations on the graph of h may not be visible for all t values.

Function r is the sum of $\sin t$ plus an decreasing exponential function $-e^{ct} + b$. We suspect then that the graph of r might periodically oscillate about the graph of $-e^{ct} + b$, just like the graph of $\sin t$ oscillates about its midline. When adding $-e^{ct} + b$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, -e^{ct} + b)$ that lies both on the graph of r, and on the graph of $-e^{ct} + b$. See Figure 1.52. So r matches Graph D. Note that the oscillations on the graph of r may not be visible for all t values.

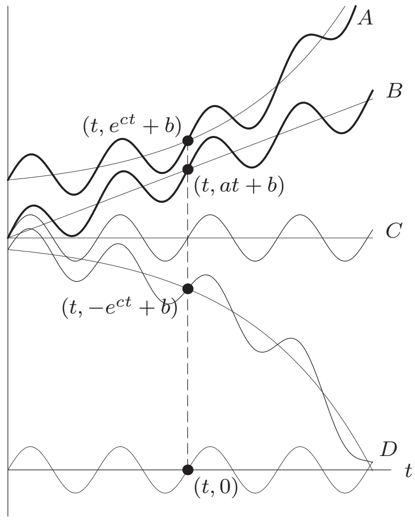


Figure 1.52

50. (a) The monthly mean CO_2 increased about 10 ppm between December 2005 and December 2010. This is because the black curve shows that the December 2005 monthly mean was about 381 ppm, while the December 2010 monthly mean was about 391 ppm. The difference between these two values, $391 - 381 = 10$, gives the overall increase.

(b) The average rate of increase is given by

$$\text{Average monthly increase of monthly mean} = \frac{391 - 381}{60 - 0} = \frac{1}{6} \text{ ppm/month.}$$

This tells us that the slope of a linear equation approximating the black curve is $1/6$. Since the vertical intercept is about 381, a possible equation for the approximately linear black curve is

$$y = \frac{1}{6}t + 381,$$

where t is measured in months since December 2005.

(c) The period of the seasonal CO_2 variation is about 12 months since this is approximately the time it takes for the function given by the blue curve to complete a full cycle. The amplitude is about 3.5 since, looking at the blue curve, the average distance between consecutive maximum and minimum values is about 7 ppm. So a possible sinusoidal function for the seasonal CO_2 cycle is

$$y = 3.5 \sin\left(\frac{\pi}{6}t\right).$$

(d) Taking $f(t) = 3.5 \sin\left(\frac{\pi}{6}t\right)$ and $g(t) = \frac{1}{6}t + 381$, we have

$$h(t) = 3.5 \sin\left(\frac{\pi}{6}t\right) + \frac{1}{6}t + 381.$$

See Figure 1.53.

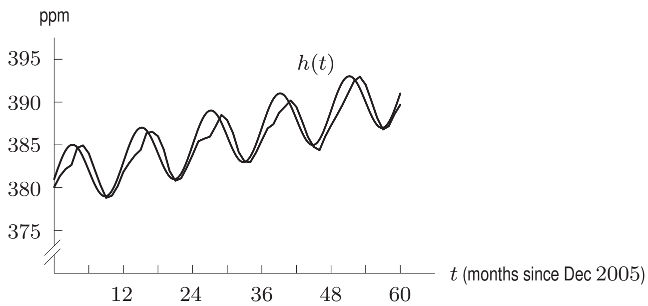


Figure 1.53

51. (a) The period is 2π.

(b) After π, the values of $\cos 2\theta$ repeat, but the values of $2\sin\theta$ do not (in fact, they repeat but flipped over the x-axis). After another π, that is after a total of 2π, the values of $\cos 2\theta$ repeat *again*, and now the values of $2\sin\theta$ repeat also, so the function $2\sin\theta + 3\cos 2\theta$ repeats at that point.

52. Figure 1.54 shows that the cross-sectional area is one rectangle of area hw and two triangles. Each triangle has height h and base x, where

$$\frac{h}{x} = \tan\theta \quad \text{so} \quad x = \frac{h}{\tan\theta}.$$

$$\text{Area of triangle} = \frac{1}{2}xh = \frac{h^2}{2\tan\theta}$$

$$\begin{aligned}\text{Total area} &= \text{Area of rectangle} + 2(\text{Area of triangle})\\ &= hw + 2\cdot\frac{h^2}{2\tan\theta} = hw + \frac{h^2}{\tan\theta}.\end{aligned}$$

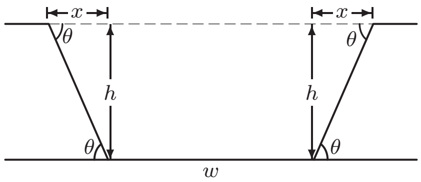


Figure 1.54

53. (a) Two solutions: 0.4 and 2.7. See Figure 1.55.

(b) $\arcsin(0.4)$ is the first solution approximated above; the second is an approximation to $\pi - \arcsin(0.4)$.

(c) By symmetry, there are two solutions: -0.4 and -2.7.

(d) $-0.4 \approx -\arcsin(0.4)$ and $-2.7 \approx -(\pi - \arcsin(0.4)) = \arcsin(0.4) - \pi$.

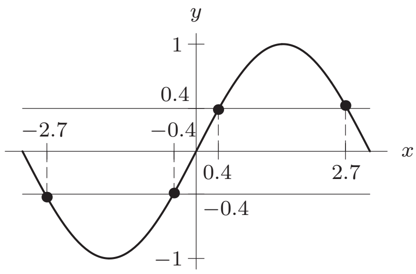


Figure 1.55

54. The ramp in Figure 1.56 rises 1 ft over a horizontal distance of x ft.

(a) For a 1 ft rise over 12 ft, the angle in radians is $\theta = \arctan(1/12) = 0.0831$. To find the angle in degrees, multiply by $180/\pi$. Hence

$$\theta = \frac{180}{\pi}\arctan\frac{1}{12} = 4.76^\circ.$$

(b) We have

$$\theta = \frac{180}{\pi}\arctan\frac{1}{8} = 7.13^\circ.$$

(c) We have

$$\theta = \frac{180}{\pi}\arctan\frac{1}{20} = 2.86^\circ.$$

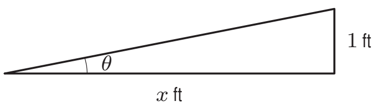


Figure 1.56

55. (a) A table of values for $g(x)$ is:

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$\arccos x$	3.14	2.50	2.21	1.98	1.77	1.57	1.37	1.16	0.93	0.64	0

(b) See Figure 1.57.

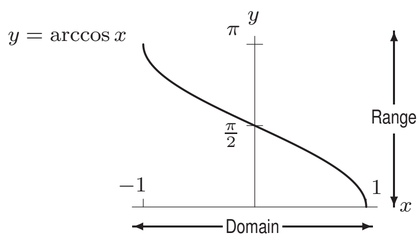


Figure 1.57

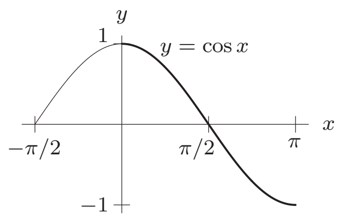


Figure 1.58

(c) The domain of arccos is $-1 \leq x \leq 1$, because its inverse, cosine, takes all values from -1 to 1. The domains of arccos and arcsin are the same because their inverses, sine and cosine, have the same range.
(d) Figure 1.57 shows that the range of $y = \arccos x$ is $0 \leq \theta \leq \pi$.
(e) The range of an inverse function is the domain of the original function. The arcsine is the inverse function to the piece of the sine having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence, the range of the arcsine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. But the piece of the cosine having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ does not have an inverse, because there are horizontal lines that intersect its graph twice. Instead, we define arccosine to be the inverse of the piece of cosine having domain $[0, \pi]$, so the range of arccosine is $[0, \pi]$, which is different from the range of arcsine. See Figure 1.58.

Strengthen Your Understanding

56. Increasing the value of B decreases the period. For example, $f(x) = \sin x$ has period 2π, whereas $g(x) = \sin(2x)$ has period π.

57. The maximum value of $A\sin(Bx)$ is A, so the maximum value of $y = A\sin(Bx) + C$ is $y = A + C$.

58. For $B > 0$, the period of $y = \sin(Bx)$ is $2\pi/B$. Thus, we want

$$\frac{2\pi}{B} = 23 \quad \text{so} \quad B = \frac{2\pi}{23}.$$

The function is $f(x) = \sin(2\pi x/23)$

59. The midline is $y = (1200 + 2000)/2 = 1600$ and the amplitude is $y = (2000 - 1200)/2 = 400$, so a possible function is

$$f(x) = 400(\cos x) + 1600.$$

60. False, since $\cos\theta$ is decreasing and $\sin\theta$ is increasing.

61. False. The period is $2\pi/(0.05\pi) = 40$

62. True. The period is $2\pi/(200\pi) = 1/100$ seconds. Thus, the function executes 100 cycles in 1 second.

63. False. If $\theta = \pi/2, 3\pi/2, 5\pi/2\ldots$, then $\theta - \pi/2 = 0, \pi, 2\pi\ldots$, and the tangent is defined (it is zero) at these values.

64. False: When $x < 0$, we have $\sin|x| = \sin(-x) = -\sin x \neq \sin x$.

65. False: When $\pi < x < 2\pi$, we have $\sin|x| = \sin x < 0$ but $|\sin x| > 0$.

66. False: When $\pi/2 < x < 3\pi/2$, we have $\cos|x| = \cos x < 0$ but $|\cos x| > 0$.

67. True: Since $\cos(-x) = \cos x$, $\cos|x| = \cos x$.

68. False. For example, $\sin(0) \neq \sin((2\pi)^2)$, since $\sin(0) = 0$ but $\sin((2\pi)^2) = 0.98$.

69. True. Since $\sin(\theta + 2\pi) = \sin\theta$ for all θ, we have $g(\theta + 2\pi) = e^{\sin(\theta+2\pi)} = e^{\sin\theta} = g(\theta)$ for all θ.

70. False. A counterexample is given by $f(x) = \sin x$, which has period 2π, and $g(x) = x^2$. The graph of $f(g(x)) = \sin(x^2)$ in Figure 1.59 is not periodic with period 2π.

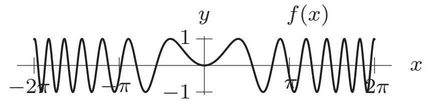


Figure 1.59

71. True. If $g(x)$ has period k, then $g(x + k) = g(x)$. Thus we have

$$f(g(x + k)) = f(g(x))$$

which shows that $f(g(x))$ is periodic with period k.

72. True, since $|\sin(-x)| = |-\sin x| = \sin x$.

Solutions for Section 1.6

Exercises

1. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to -\infty$.

2. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to 0$.

3. Since $f(x)$ is an even power function with a negative leading coefficient, it follows that $f(x) \to -\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

4. Since $f(x)$ is an odd power function with a positive leading coefficient, it follows that $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

5. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of $5x^4$. Thus, as $x \to \pm\infty$, we see that $f(x) \to +\infty$.

6. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of $-5x^3$. Thus, as $x \to +\infty$, we see that $f(x) \to -\infty$ and as $x \to -\infty$, we see that $f(x) \to +\infty$.

7. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in the numerator and denominator. Thus, as $x \to \pm\infty$, we see that $f(x)$ behaves like $\dfrac{3x^2}{x^2} = 3$. We have $f(x) \to 3$ as $x \to \pm\infty$.

8. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in the numerator and denominator. Thus, as $x \to \pm\infty$, we see that $f(x)$ behaves like $\dfrac{-3x^3}{2x^3} = -3/2$. We have $f(x) \to -3/2$ as $x \to \pm\infty$.

9. As $x \to \pm\infty$, we see that $3x^{-4}$ gets closer and closer to 0, so $f(x) \to 0$ as $x \to \pm\infty$.

10. As $x \to +\infty$, we have $f(x) \to +\infty$. As $x \to -\infty$, we have $f(x) \to 0$.

11. The power function with the higher power dominates as $x \to \infty$, so $0.2x^5$ is larger.

12. An exponential growth function always dominates a power function as $x \to \infty$, so $10e^{0.1x}$ is larger.

13. An exponential growth function always dominates a power function as $x \to \infty$, so 1.05^x is larger.

14. The lower-power terms in a polynomial become insignificant as $x \to \infty$, so we are comparing $2x^4$ to the leading term $10x^3$. In comparing two power functions, the higher power dominates as $x \to \infty$, so $2x^4$ is larger.

15. The lower-power terms in a polynomial become insignificant as $x \to \infty$, so we are comparing the leading term $20x^4$ to the leading term $3x^5$. In comparing two power functions, the higher power dominates as $x \to \infty$, so the polynomial with leading term $3x^5$ is larger. As $x \to \infty$, we see that $25 - 40x^2 + x^3 + 3x^5$ is larger.

16. A power function with positive exponent dominates a log function, so as $x \to \infty$, we see that $\sqrt{x}$ is larger.

17. (I) **(a)** Minimum degree is 3 because graph turns around twice.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(II) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.
(III) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(IV) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(V) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.

18. (a) From the x-intercepts, we know the equation has the form

$$y = k(x+2)(x-1)(x-5).$$

Since $y = 2$ when $x = 0$,

$$\begin{aligned} 2 &= k(2)(-1)(-5) = k \cdot 10 \\ k &= \frac{1}{5}. \end{aligned}$$

Thus we have

$$y = \frac{1}{5}(x+2)(x-1)(x-5).$$

19. (a) Because our cubic has a root at 2 and a double root at -2, it has the form

$$y = k(x+2)(x+2)(x-2).$$

Since $y = 4$ when $x = 0$,

$$\begin{aligned} 4 &= k(2)(2)(-2) = -8k, \\ k &= -\frac{1}{2}. \end{aligned}$$

Thus our equation is

$$y = -\frac{1}{2}(x+2)^2(x-2).$$

20. $f(x) = k(x+3)(x-1)(x-4) = k(x^3 - 2x^2 - 11x + 12)$, where $k < 0$. ($k \approx -\frac{1}{6}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

21. $f(x) = kx(x+3)(x-4) = k(x^3 - x^2 - 12x)$, where $k < 0$. ($k \approx -\frac{2}{9}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

22. $f(x) = k(x+2)(x-1)(x-3)(x-5) = k(x^4 - 7x^3 + 5x^2 + 31x - 30)$, where $k > 0$. ($k \approx \frac{1}{15}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

23. $f(x) = k(x+2)(x-2)^2(x-5) = k(x^4 - 7x^3 + 6x^2 + 28x - 40)$, where $k < 0$. ($k \approx -\frac{1}{15}$ if the scales are equal; otherwise one can't tell how large k is.)

24. There are only two functions, h and p, which can be put in the form $y = Cb^x$, where C and b are constants:

$$p(x) = \frac{a^3 b^x}{c} = (a^3/c)b^x, \quad \text{where } C = a^3/c \text{ since } a, c \text{ are constants.}$$

$$h(x) = \frac{-1}{5^{x-2}} = -5^{-(x-2)} = -5^{-x+2} = (-25)5^{-x}.$$

Thus, h and p are the only exponential functions.

25. There is only one function, r, who can be put in the form $y = Ax^2 + Bx + C$:

$$r(x) = -x + b - \sqrt{cx^4} = -\sqrt{c}\,x^2 - x + b, \quad \text{where } A = -\sqrt{c}, \text{ since } c \text{ is a constant.}$$

Thus, r is the only quadratic function.

26. There is only one function, q, which is linear. Function q is a constant linear function whose vertical intercept is the constant ab^2/c, since a, b and c are constants.

Problems

27. Consider the end behavior of the graph; that is, as $x \to +\infty$ and $x \to -\infty$. The ends of a degree 5 polynomial are in Quadrants I and III if the leading coefficient is positive or in Quadrants II and IV if the leading coefficient is negative. Thus, there must be at least one root. Since the degree is 5, there can be no more than 5 roots. Thus, there may be 1, 2, 3, 4, or 5 roots. Graphs showing these five possibilities are shown in Figure 1.60.

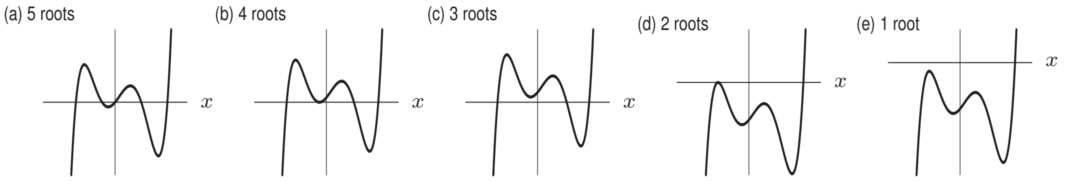


Figure 1.60

28. $g(x) = 2x^2$, $h(x) = x^2 + k$ for any $k > 0$. Notice that the graph is symmetric about the y-axis and $\lim_{x\to\infty} f(x) = 2$.

29. The graphs of both these functions will resemble that of x^3 on a large enough window. One way to tackle the problem is to graph them both (along with x^3 if you like) in successively larger windows until the graphs come together. In Figure 1.61, f, g and x^3 are graphed in four windows. In the largest of the four windows the graphs are indistinguishable, as required. Answers may vary.

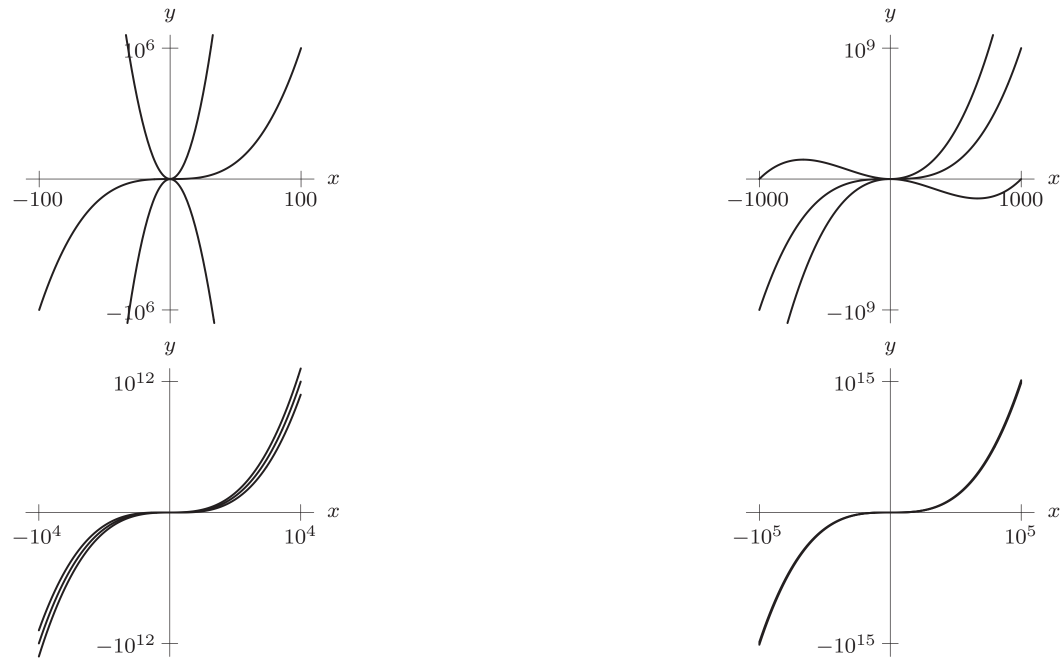


Figure 1.61

30. (a) A polynomial has the same end behavior as its leading term, so this polynomial behaves as $-5x^4$ globally. Thus we have:
$f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.
(b) Polynomials behave globally as their leading term, so this rational function behaves globally as $(3x^2)/(2x^2)$, or $3/2$. Thus we have:
$f(x) \to 3/2$ as $x \to -\infty$, and $f(x) \to 3/2$ as $x \to +\infty$.
(c) We see from a graph of $y = e^x$ that
$f(x) \to 0$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.

31. Substituting $w = 65$ and $h = 160$, we have

(a)
$$s = 0.01(65^{0.25})(160^{0.75}) = 1.3 \text{ m}^2.$$

(b) We substitute $s = 1.5$ and $h = 180$ and solve for w:
$$1.5 = 0.01w^{0.25}(180^{0.75}).$$
We have
$$w^{0.25} = \frac{1.5}{0.01(180^{0.75})} = 3.05.$$
Since $w^{0.25} = w^{1/4}$, we take the fourth power of both sides, giving
$$w = 86.8 \text{ kg}.$$

(c) We substitute $w = 70$ and solve for h in terms of s:
$$s = 0.01(70^{0.25})h^{0.75},$$
so
$$h^{0.75} = \frac{s}{0.01(70^{0.25})}.$$
Since $h^{0.75} = h^{3/4}$, we take the $4/3$ power of each side, giving
$$h = \left(\frac{s}{0.01(70^{0.25})}\right)^{4/3} = \frac{s^{4/3}}{(0.01^{4/3})(70^{1/3})}$$
so
$$h = 112.6s^{4/3}.$$

32. Let $D(v)$ be the stopping distance required by an Alpha Romeo as a function of its velocity. The assumption that stopping distance is proportional to the square of velocity is equivalent to the equation
$$D(v) = kv^2$$
where k is a constant of proportionality. To determine the value of k, we use the fact that $D(70) = 177$.
$$D(70) = k(70)^2 = 177.$$
Thus,
$$k = \frac{177}{70^2} \approx 0.0361.$$
It follows that
$$D(35) = \left(\frac{177}{70^2}\right)(35)^2 = \frac{177}{4} = 44.25 \text{ ft}$$
and
$$D(140) = \left(\frac{177}{70^2}\right)(140)^2 = 708 \text{ ft}.$$
Thus, at half the speed it requires one fourth the distance, whereas at twice the speed it requires four times the distance, as we would expect from the equation. (We could in fact have figured it out that way, without solving for k explicitly.)

33. (a) Since the rate R varies directly with the fourth power of the radius r, we have the formula
$$R = kr^4$$
where k is a constant.

(b) Given $R = 400$ for $r = 3$, we can determine the constant k.

$$400 = k(3)^4$$
$$400 = k(81)$$
$$k = \frac{400}{81} \approx 4.938.$$

So the formula is

$$R = 4.938r^4$$

(c) Evaluating the formula above at $r = 5$ yields

$$R = 4.928(5)^4 = 3086.42\frac{\text{cm}^3}{\text{sec}}.$$

34. Let us represent the height by h. Since the volume is V, we have

$$x^2h = V.$$

Solving for h gives

$$h = \frac{V}{x^2}.$$

The graph is in Figure 1.62. We are assuming V is a positive constant.

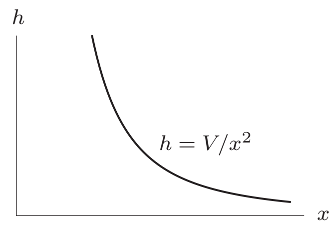


Figure 1.62

35. (a) Let the height of the can be h. Then

$$V = \pi r^2 h.$$

The surface area consists of the area of the ends (each is πr^2) and the curved sides (area $2\pi rh$), so

$$S = 2\pi r^2 + 2\pi rh.$$

Solving for h from the formula for V, we have

$$h = \frac{V}{\pi r^2}.$$

Substituting into the formula for S, we get

$$S = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

(b) For large r, the $2V/r$ term becomes negligible, meaning $S \approx 2\pi r^2$, and thus $S \to \infty$ as $r \to \infty$.
(c) The graph is in Figure 1.63.

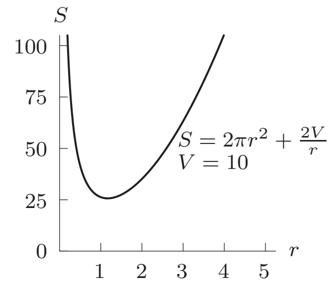


Figure 1.63

36. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that

$$f(x) \to \frac{5x}{2x} = \frac{5}{2}.$$

There is a horizontal asymptote at $y = 5/2$.
To find the vertical asymptotes, we set the denominator equal to zero. When $2x + 3 = 0$, we have $x = -3/2$ so there is a vertical asymptote at $x = -3/2$.

37. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that

$$f(x) \to \frac{x^2}{x^2} = 1.$$

There is a horizontal asymptote at $y = 1$.
To find the vertical asymptotes, we set the denominator equal to zero. When $x^2 - 4 = 0$, we have $x = \pm 2$ so there are vertical asymptotes at $x = -2$ and at $x = 2$.

38. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that

$$f(x) \to \frac{5x^3}{x^3} = 5.$$

There is a horizontal asymptote at $y = 5$.
To find the vertical asymptotes, we set the denominator equal to zero. When $x^3 - 27 = 0$, we have $x = 3$ so there is a vertical asymptote at $x = 3$.

39. (a) The object starts at $t = 0$, when $s = v_0(0) - g(0)^2/2 = 0$. Thus it starts on the ground, with zero height.

(b) The object hits the ground when $s = 0$. This is satisfied at $t = 0$, before it has left the ground, and at some later time t that we must solve for.

$$0 = v_0 t - gt^2/2 = t\,(v_0 - gt/2)$$

Thus $s = 0$ when $t = 0$ and when $v_0 - gt/2 = 0$, i.e., when $t = 2v_0/g$. The starting time is $t = 0$, so it must hit the ground at time $t = 2v_0/g$.

(c) The object reaches its maximum height halfway between when it is released and when it hits the ground, or at

$$t = (2v_0/g)/2 = v_0/g.$$

(d) Since we know the time at which the object reaches its maximum height, to find the height it actually reaches we just use the given formula, which tells us s at any given t. Substituting $t = v_0/g$,

$$\begin{aligned} s &= v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0^2}{g^2}\right) = \frac{v_0^2}{g} - \frac{v_0^2}{2g} \\ &= \frac{2v_0^2 - v_0^2}{2g} = \frac{v_0^2}{2g}. \end{aligned}$$

40. The pomegranate is at ground level when $f(t) = -16t^2 + 64t = -16t(t-4) = 0$, so when $t = 0$ or $t = 4$. At time $t = 0$ it is thrown, so it must hit the ground at $t = 4$ seconds. The symmetry of its path with respect to time may convince you that it reaches its maximum height after 2 seconds. Alternatively, we can think of the graph of $f(t) = -16t^2 + 64t = -16(t-2)^2 + 64$, which is a downward parabola with vertex (i.e., highest point) at $(2, 64)$. The maximum height is $f(2) = 64$ feet.

41. (a) (i) If $(1, 1)$ is on the graph, we know that

$$1 = a(1)^2 + b(1) + c = a + b + c.$$

(ii) If $(1, 1)$ is the vertex, then the axis of symmetry is $x = 1$, so

$$-\frac{b}{2a} = 1,$$

and thus

$$a = -\frac{b}{2}, \text{ so } b = -2a.$$

But to be the vertex, $(1, 1)$ must also be on the graph, so we know that $a + b + c = 1$. Substituting $b = -2a$, we get $-a + c = 1$, which we can rewrite as $a = c - 1$, or $c = 1 + a$.

(iii) For $(0, 6)$ to be on the graph, we must have $f(0) = 6$. But $f(0) = a(0^2) + b(0) + c = c$, so $c = 6$.

(b) To satisfy all the conditions, we must first, from (a)(iii), have $c = 6$. From (a)(ii), $a = c - 1$ so $a = 5$. Also from (a)(ii), $b = -2a$, so $b = -10$. Thus the completed equation is

$$y = f(x) = 5x^2 - 10x + 6,$$

which satisfies all the given conditions.

42. The function is a cubic polynomial with positive leading coefficient. Since the figure given in the text shows that the function turns around once, we know that the function has the shape shown in Figure 1.64. The function is below the x-axis for $x = 5$ in the given graph, and we know that it goes to $+\infty$ as $x \to +\infty$ because the leading coefficient is positive. Therefore, there are exactly three zeros. Two zeros are shown, and occur at approximately $x = -1$ and $x = 3$. The third zero must be to the right of $x = 10$ and so occurs for some $x > 10$.

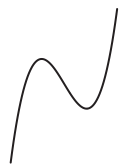


Figure 1.64

43. We use the fact that at a constant speed, Time = Distance/Speed. Thus,

$$\begin{aligned}\text{Total time} &= \text{Time running} + \text{Time walking}\\ &= \frac{3}{x} + \frac{6}{x-2}.\end{aligned}$$

Horizontal asymptote: x-axis.
Vertical asymptote: $x = 0$ and $x = 2$.

44. (a) II and III because in both cases, the numerator and denominator each have x^2 as the highest power, with coefficient $= 1$. Therefore,

$$y \to \frac{x^2}{x^2} = 1 \quad \text{as } x \to \pm\infty.$$

(b) I, since

$$y \to \frac{x}{x^2} = 0 \quad \text{as } x \to \pm\infty.$$

(c) II and III, since replacing x by $-x$ leaves the graph of the function unchanged.
(d) None
(e) III, since the denominator is zero and $f(x)$ tends to $\pm\infty$ when $x = \pm 1$.

45. $h(t)$ cannot be of the form ct^2 or kt^3 since $h(0.0) = 2.04$. Therefore $h(t)$ must be the exponential, and we see that the ratio of successive values of h is approximately 1.5. Therefore $h(t) = 2.04(1.5)^t$. If $g(t) = ct^2$, then $c = 3$ since $g(1.0) = 3.00$. However, $g(2.0) = 24.00 \neq 3 \cdot 2^2$. Therefore $g(t) = kt^3$, and using $g(1.0) = 3.00$, we obtain $g(t) = 3t^3$. Thus $f(t) = ct^2$, and since $f(2.0) = 4.40$, we have $f(t) = 1.1t^2$.

46. The graphs are shown in Figure 1.65.

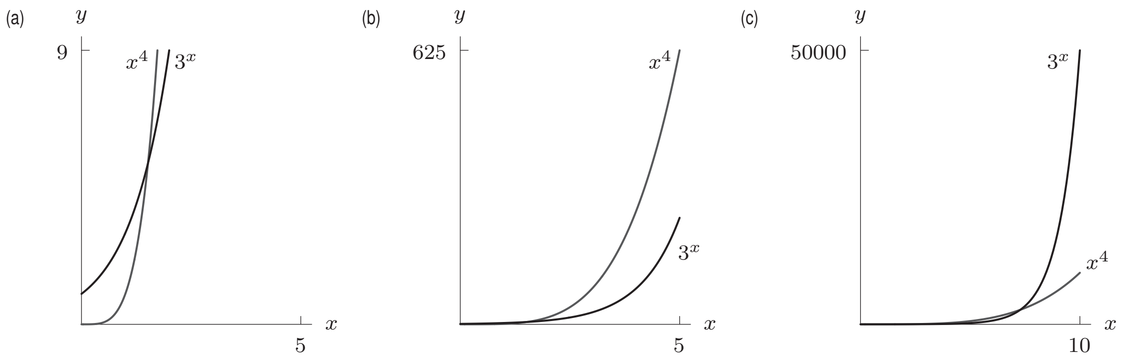


Figure 1.65

47. (a) $R(P) = kP(L - P)$, where k is a positive constant.
(b) A possible graph is in Figure 1.66.

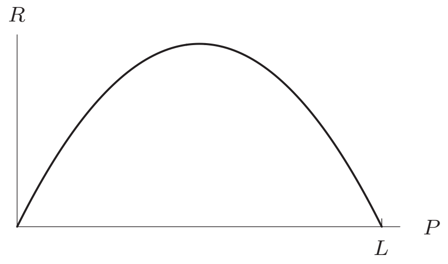


Figure 1.66

48. Since the parabola opens upward, we must have $a > 0$. To determine a relationship between x and y at the point of intersection P, we eliminate a from the parabola and circle equations. Since $y = x^2/a$, we have $a = x^2/y$. Putting this into the circle equation gives $x^2 + y^2 = 2x^4/y^2$. Rewrite this as

$$\begin{aligned} x^2y^2 + y^4 &= 2x^4 \\ y^4 + x^2y^2 - 2x^4 &= 0 \\ (y^2 + 2x^2)(y^2 - x^2) &= 0. \end{aligned}$$

This means $x^2 = y^2$ (since y^2 cannot equal $-2x^2$). Thus $x = y$ since P is in the first quadrant. So P moves out along the line $y = x$ through the origin.

49. (a) The graph is shown in Figure 1.67. The graph represented by the exact formula has a vertical asymptote where the denominator is undefined. This happens when

$$1 - \frac{v^2}{c^2} = 0, \text{ or at } v^2 = c^2.$$

Since $v > 0$, the graph of the exact formula has a vertical asymptote where

$$v = c = 3 \cdot 10^8 \text{m/sec.}$$

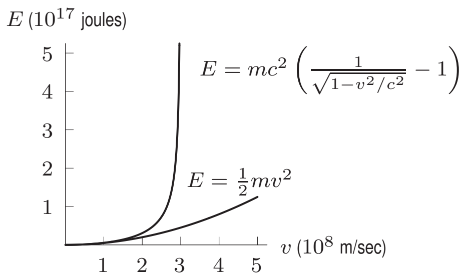


Figure 1.67

(b) The first formula does not give a good approximation to the exact formula when the graphs are not close together. This happens for $v > 1.5 \cdot 10^8$ m/sec. For $v < 1.5 \cdot 10^8$ m/sec, the graphs look close together. However, the vertical scale we are using is so large and the graphs are so close to the v-axis that a more careful analysis should be made. We should zoom in and redraw the graph.

Strengthen Your Understanding

50. The graph of a polynomial of degree 5 cuts the horizontal axis at most five times, but it could be fewer. For example, $f(x) = x^5$ cuts the x-axis only once.

51. The rational function $f(x) = (x^3 + 1)/x$ has no horizontal asymptotes. To see this, observe that

$$y = f(x) = \frac{x^3 + 1}{x} \approx \frac{x^3}{x} = x^2$$

for large x. Thus, $y \to \infty$ as $x \to \pm\infty$.

52. One possibility is $p(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$.

53. A possible function is

$$f(x) = \frac{3x}{x - 10}.$$

54. One possibility is

$$f(x) = \frac{1}{x^2 + 1}.$$

55. Let $f(x) = \dfrac{1}{x + 7\pi}$. Other answers are possible.

56. Let $f(x) = \dfrac{1}{(x-1)(x-2)(x-3)\cdots(x-16)(x-17)}$. This function has an asymptote corresponding to every factor in the denominator. Other answers are possible.

57. The function $f(x) = \dfrac{x-1}{x-2}$ has $y = 1$ as the horizontal asymptote and $x = 2$ as the vertical asymptote. These lines cross at the point $(2, 1)$. Other answers are possible.

58. False. The polynomial $f(x) = x^2 + 1$, with degree 2, has no real zeros.

59. True. If the degree of the polynomial, $p(x)$, is n, then the leading term is $a_n x^n$ with $a_n \neq 0$.

If n is odd and a_n is positive, $p(x)$ tends toward ∞ as $x \to \infty$ and $p(x)$ tends toward $-\infty$ as $x \to -\infty$. Since the graph of $p(x)$ has no breaks in it, the graph must cross the x-axis at least once.

If n is odd and a_n is negative, a similar argument applies, with the signs reversed, but leading to the same conclusion.

60. (a), (c), (d), (e), (b). Notice that $f(x)$ and $h(x)$ are decreasing functions, with $f(x)$ being negative. Power functions grow slower than exponential growth functions, so $k(x)$ is next. Now order the remaining exponential functions, where functions with larger bases grow faster.

Solutions for Section 1.7

Exercises

1. Yes, because $x - 2$ is not zero on this interval.

2. No, because $x - 2 = 0$ at $x = 2$.

3. Yes, because $2x - 5$ is positive for $3 \leq x \leq 4$.

4. Yes, because the denominator is never zero.

5. Yes, because $2x + x^{2/3}$ is defined for all x.

6. No, because $2x + x^{-1}$ is undefined at $x = 0$.

7. No, because $\cos(\pi/2) = 0$.

8. No, because $\sin 0 = 0$.

9. No, because $e^x - 1 = 0$ at $x = 0$.

10. Yes, because $\cos\theta$ is not zero on this interval.

11. We have that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

12. We have that $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

13. We have that $f(0) = -1 < 0$ and $f(1) = 1 - \cos 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

14. Since f is not continuous at $x = 0$, we consider instead the smaller interval $[0.01, 1]$. We have that $f(0.01) = 2^{0.01} - 100 < 0$ and $f(1) = 2 - 1/1 = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0.01, 1]$, and hence in $[0, 1]$, such that $f(c) = k = 0$.

15. (a) At $x = 1$, on the line $y = x$, we have $y = 1$. At $x = 1$, on the parabola $y = x^2$, we have $y = 1$. Thus, $f(x)$ is continuous. See Figure 1.68.

(b) At $x = 3$, on the line $y = x$, we have $y = 3$. At $x = 3$, on the parabola $y = x^2$, we have $y = 9$. Thus, $g(x)$ is not continuous. See Figure 1.69.

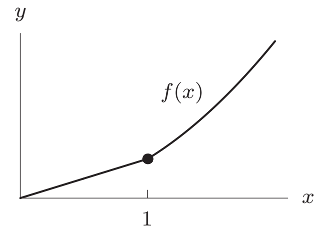


Figure 1.68

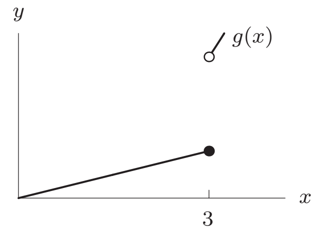


Figure 1.69

Problems

16. (a) Even if the car stops to refuel, the amount of fuel in the tank changes smoothly, so the fuel in the tank is a continuous function; the quantity of fuel cannot suddenly change from one value to another.

(b) Whenever a student joins or leaves the class the number jumps up or down immediately by 1 so this is not a continuous function, unless the enrollment does not change at all.

(c) Whenever the oldest person dies the value of the function jumps down to the age of the next oldest person, so this is not a continuous function.

17. Two possible graphs are shown in Figures 1.70 and 1.71.

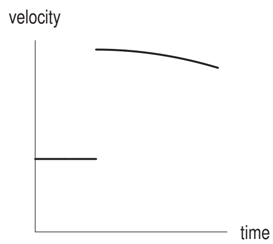


Figure 1.70: Velocity of the car

Figure 1.71: Distance

The distance moved by the car is continuous. (Figure 1.71 has no breaks in it.) In actual fact, the velocity of the car is also continuous; however, in this case, it is well-approximated by the function in Figure 1.70, which is not continuous on any interval containing the moment of impact.

18. The voltage $f(t)$ is graphed in Figure 1.72.

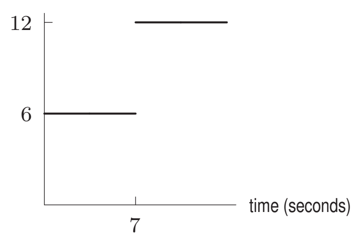


Figure 1.72: Voltage change from 6V to 12V

Using formulas, the voltage, $f(t)$, is represented by

$$f(t) = \begin{cases} 6, & 0 < t \le 7 \\ 12, & 7 < t \end{cases}$$

Although a real physical voltage is continuous, the voltage in this circuit is well-approximated by the function $f(t)$, which is not continuous on any interval around 7 seconds.

19. The value of y on the line $y = kx$ at $x = 3$ is $y = 3k$. To make $f(x)$ continuous, we need

$$3k = 5 \quad \text{so} \quad k = \frac{5}{3}.$$

See Figure 1.73.

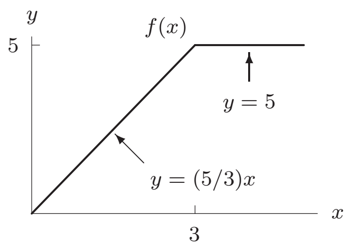


Figure 1.73

20. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $3x^2$ takes the value $3(2^2) = 12$ at $x = 2$, we choose k so that the graph of kx goes through the point $(2, 12)$. Thus $k = 6$.

21. If the graphs of $y = t + k$ and $y = kt$ meet at $t = 5$, we have

$$\begin{aligned} 5 + k &= 5k \\ k &= 5/4. \end{aligned}$$

See Figure 1.74.

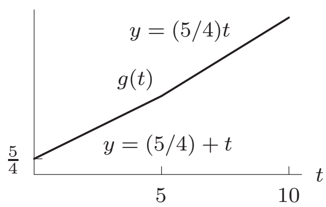


Figure 1.74

22. At $x = \pi$, the curve $y = k \cos x$ has $y = k \cos \pi = -k$. At $x = \pi$, the line $y = 12 - x$ has $y = 12 - \pi$. If $h(x)$ is continuous, we need

$$\begin{aligned} -k &= 12 - \pi \\ k &= \pi - 12. \end{aligned}$$

23. (a) See Figure 1.75.
(b) For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.
Since $(x-2)^2 + 3$ takes on the value $(2-2)^2 + 3 = 3$ at $x = 2$, we choose k so that $kx = 3$ at $x = 2$, so $2k = 3$ and $k = 3/2$.
(c) See Figure 1.76.

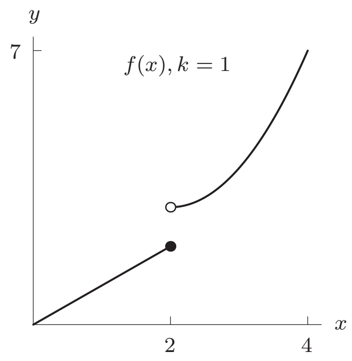


Figure 1.75

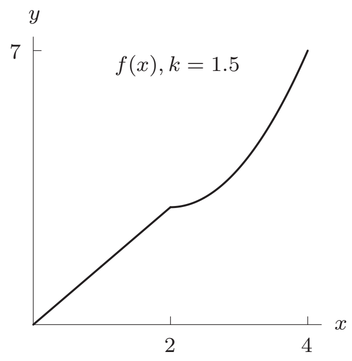


Figure 1.76

24. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.
Since $x + 3$ takes the value $1 + 3 = 4$ at $x = 1$, we choose k so that the graph of kx goes through the point $(1, 4)$. Thus $k = 4$.

25. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.
Since kx takes the value $k \cdot 1 = k$ at $x = 1$, we choose k so that the graph of $2kx + 3$ goes through the point $(1, k)$. This gives

$$2k \cdot 1 + 3 = k$$
$$k = -3.$$

26. For any value of k, the function is continuous at every point except $x = \pi$. We choose k to make the function continuous at $x = \pi$.
Since $k \sin x$ takes the value $k \sin \pi = 0$ at $x = \pi$, we cannot choose k so that the graph of $x + 4$ goes through the point $(\pi, 0)$. Thus, this function is discontinuous for all values of k.

27. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.
Since $x + 1$ takes the value $2 + 1 = 3$ at $x = 2$, we choose k so that the graph of e^{kx} goes through the point $(2, 3)$. This gives

$$e^{2k} = 3$$
$$2k = \ln 3$$
$$k = \frac{\ln 3}{2}$$

28. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.
Since $\sin(kx)$ takes the value $\sin k$ at $x = 1$, we choose k so that the graph of $0.5x$ goes through the point $(1, \sin k)$. This gives

$$\sin k = 0.5$$
$$k = \sin^{-1} 0.5 = \frac{\pi}{6}.$$

Other solutions are possible.

29. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $\ln(kx + 1)$ takes the value $\ln(2k + 1)$ at $x = 2$, we choose k so that the graph of $x + 4$ goes through the point $(2, \ln(2k + 1))$. This gives

$$\begin{aligned} \ln(2k+1) &= 2 + 4 = 6 \\ 2k + 1 &= e^6 \\ k &= \frac{e^6 - 1}{2}. \end{aligned}$$

30. (a) The initial value is when $t = 0$, and we see that $P(0) = e^{k \cdot 0} = e^0 = 1000$.
(b) Since the function is continuous, at $t = 12$, we have $e^{kt} = 100$ and we solve for k:

$$\begin{aligned} e^{12k} &= 100 \\ 12k &= \ln 100 \\ k &= \frac{\ln 100}{12} = 0.384. \end{aligned}$$

(c) The population is increasing exponentially for 12 months and then becoming constant.

31. For $x > 0$, we have $|x| = x$, so $f(x) = 1$. For $x < 0$, we have $|x| = -x$, so $f(x) = -1$. Thus, the function is given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 1 \end{cases},$$

so f is not continuous on any interval containing $x = 0$.

32. The graph of g suggests that g is not continuous on any interval containing $\theta = 0$, since $g(0) = 1/2$.

33. The drug first increases linearly for half a second, at the end of which time there is 0.6 ml in the body. Thus, for $0 \le t \le 0.5$, the function is linear with slope $0.6/0.5 = 1.2$:

$$Q = 1.2t \quad \text{for} \quad 0 \le t \le 0.5.$$

At $t = 0.5$, we have $Q = 0.6$. For $t > 0.5$, the quantity decays exponentially at a continuous rate of 0.002, so Q has the form

$$Q = Ae^{-0.002t} \qquad 0.5 < t.$$

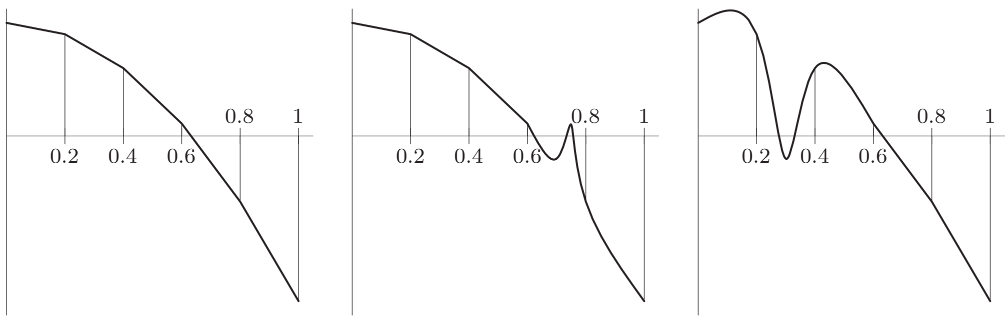
We choose A so that $Q = 0.6$ when $t = 0.5$:

$$\begin{aligned} 0.6 &= Ae^{-0.002(0.5)} = Ae^{-0.001} \\ A &= 0.6e^{0.001}. \end{aligned}$$

Thus

$$Q = \begin{cases} 1.2t & 0 \le t \le 0.5 \\ 0.6e^{0.001}e^{-.002t} & 0.5 < t. \end{cases}$$

34.



35. Since polynomials are continuous, and since $p(5) < 0$ and $p(10) > 0$ and $p(12) < 0$, there are two zeros, one between $x = 5$ and $x = 10$, and another between $x = 10$ and $x = 12$. Thus, $p(x)$ is a cubic with at least two zeros.

If $p(x)$ has only two zeros, one would be a double zero (corresponding to a repeated factor). However, since a polynomial does not change sign at a repeated zero, $p(x)$ cannot have a double zero and have the signs it does.

Thus, $p(x)$ has three zeros. The third zero can be greater than 12 or less than 5. See Figures 1.77 and 1.78.

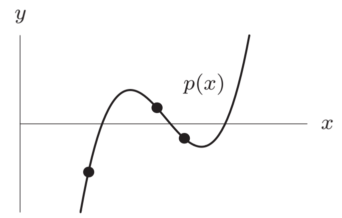


Figure 1.77

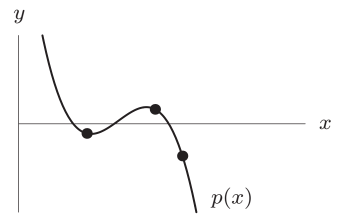


Figure 1.78

36. (a) The graphs of $y = e^x$ and $y = 4 - x^2$ cross twice in Figure 1.79. This tells us that the equation $e^x = 4 - x^2$ has two solutions.

Since $y = e^x$ increases for all x and $y = 4 - x^2$ increases for $x < 0$ and decreases for $x > 0$, these are only the two crossing points.

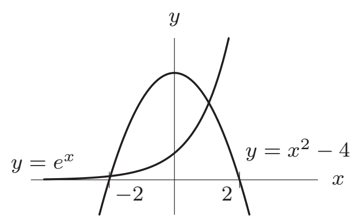


Figure 1.79

(b) Values of $f(x)$ are in Table 1.5. One solution is between $x = -2$ and $x = -1$; the second solution is between $x = 1$ and $x = 2$.

Table 1.5

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	12.0	5.0	0.1	-2.6	-3	-0.3	7.4	25.1	66.6

37. (a) Figure 1.80 shows a possible graph of $f(x)$, yours may be different.

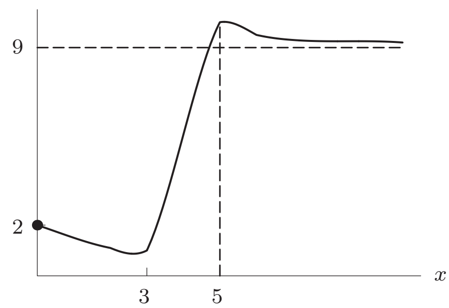


Figure 1.80

(b) In order for f to approach the horizontal asymptote at 9 from above it is necessary that f eventually become concave up. It is therefore not possible for f to be concave down for all $x > 6$.

38. (a) Since $f(x)$ is not continuous at $x = 1$, it does not satisfy the conditions of the Intermediate Value Theorem.

(b) We see that $f(0) = e^0 = 1$ and $f(2) = 4 + (2-1)^2 = 5$. Since e^x is increasing between $x = 0$ and $x = 1$, and since $4 + (x-1)^2$ is increasing between $x = 1$ and $x = 2$, any value of k between $e^1 = e$ and $4 + (1-1)^2 = 4$, such as $k = 3$, is a value such that $f(x) = k$ has no solution.

Strengthen Your Understanding

39. The Intermediate Value theorem only makes this guarantee for a *continuous* function, not for any function.

40. The Intermediate Value Theorem guarantees that for at least one value of x between 0 and 2, we have $f(x) = 5$, but it does not tell us which value(s) of x give $f(x) = 5$.

41. We want a function which has a value at every point but where the graph has a break at $x = 15$. One possibility is

$$f(x) = \begin{cases} 1 & x \geq 15 \\ -1 & x < 15 \end{cases}$$

42. One example is $f(x) = 1/x$, which is not continuous at $x = 0$. The Intermediate Value Theorem does not apply on an interval that contains a point where a function is not continuous.

43. Let $f(x) = \begin{cases} 1 & x \leq 2 \\ x & x > 2 \end{cases}$. Then $f(x)$ is continuous at every point in $[0, 3]$ except at $x = 2$. Other answers are possible.

44. Let $f(x) = \begin{cases} x & x \leq 3 \\ 2x & x > 3 \end{cases}$. Then $f(x)$ is increasing for all x but $f(x)$ is not continuous at $x = 3$. Other answers are possible.

45. False. For example, let $f(x) = \begin{cases} 1 & x \leq 3 \\ 2 & x > 3 \end{cases}$, then $f(x)$ is defined at $x = 3$ but it is not continuous at $x = 3$. (Other examples are possible.)

46. False. A counterexample is graphed in Figure 1.81, in which $f(5) < 0$.

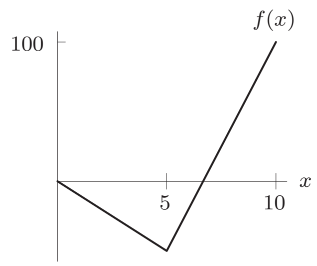


Figure 1.81

47. False. A counterexample is graphed in Figure 1.82.

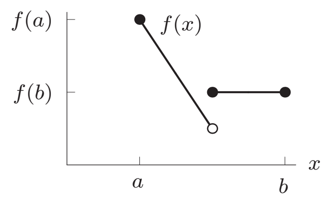


Figure 1.82

Solutions for Section 1.8

Exercises

1. (a) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 3, so the limit appears to be about 3.
(b) As x approaches 0 from either side, the values of $f(x)$ get closer and closer to 7. (Recall that to find a limit, we are interested in what happens to the function near x but not at x.) The limit appears to be about 7.
(c) As x approaches 2 from either side, the values of $f(x)$ get closer and closer to 3 on one side of $x = 2$ and get closer and closer to 2 on the other side of $x = 2$. Thus the limit does not exist.
(d) As x approaches 4 from either side, the values of $f(x)$ get closer and closer to 8. (Again, recall that we don't care what happens right at $x = 4$.) The limit appears to be about 8.

2. (a) $\lim_{x\to 1^-} f(x) = 1$.
(b) $\lim_{x\to 1^+} f(x)$ does not exist.
(c) $\lim_{x\to 1} f(x)$ does not exist.
(d) $\lim_{x\to 2^-} f(x) = 1$.
(e) $\lim_{x\to 2^+} f(x) = 1$.
(f) $\lim_{x\to 2} f(x) = 1$.

3. From the graphs of f and g, we estimate $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^-} g(x) = 5$,
$\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1^+} g(x) = 1$.
(a) $\lim_{x\to 1^-} (f(x) + g(x)) = 3 + 5 = 8$
(b) $\lim_{x\to 1^+} (f(x) + 2g(x)) = \lim_{x\to 1^+} f(x) + 2 \lim_{x\to 1^+} g(x) = 4 + 2(1) = 6$
(c) $\lim_{x\to 1^-} (f(x)g(x)) = (\lim_{x\to 1^-} f(x))(\lim_{x\to 1^-} g(x)) = (3)(5) = 15$
(d) $\lim_{x\to 1^+} (f(x)/g(x)) = \left(\lim_{x\to 1^+} f(x)\right) / \left(\lim_{x\to 1^+} g(x)\right) = 4/1 = 4$

4. We see that $f(x)$ goes to $-\infty$ on both ends, so one possible graph is shown in Figure 1.83. Other answers are possible.

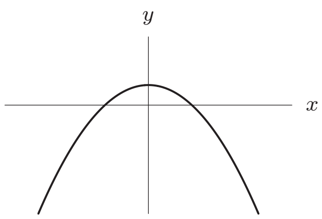


Figure 1.83

5. We see that $f(x)$ goes to $+\infty$ on the left and to $-\infty$ on the right. One possible graph is shown in Figure 1.84. Other answers are possible.

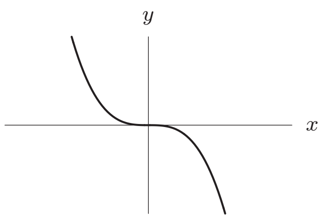


Figure 1.84

6. We see that $f(x)$ goes to $+\infty$ on the left and approaches a y-value of 1 on the right. One possible graph is shown in Figure 1.85. Other answers are possible.

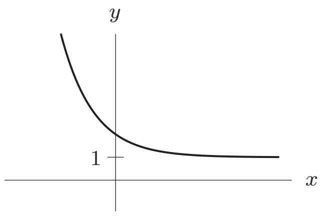


Figure 1.85

7. We see that $f(x)$ approaches a y-value of 3 on the left and goes to $-\infty$ on the right. One possible graph is shown in Figure 1.86. Other answers are possible.

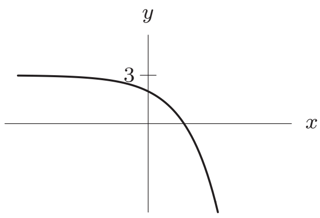


Figure 1.86

8. We see that $f(x)$ goes to $+\infty$ on the right and that it also passes through the point $(-1, 2)$. (Notice that this must be a point on the graph since the instructions require that $f(x)$ be defined and continuous.) One possible graph is shown in Figure 1.87. Other answers are possible.

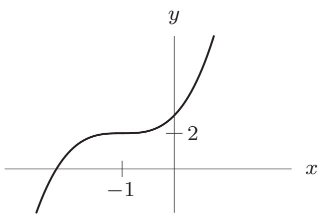


Figure 1.87

9. We see that $f(x)$ goes to $+\infty$ on the left and that it also passes through the point $(3, 5)$. (Notice that this must be a point on the graph since the instructions require that $f(x)$ be defined and continuous.) One possible graph is shown in Figure 1.88. Other answers are possible.

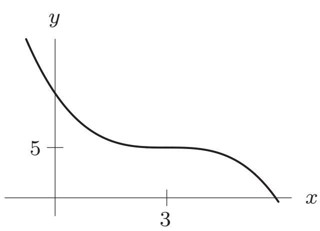


Figure 1.88

10. We see that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$.

11. As $x \to \pm\infty$, we know that $f(x)$ behaves like its leading term $-2x^3$. Thus, we have $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$.

12. As $x \to \pm\infty$, we know that $f(x)$ behaves like its leading term x^5. Thus, we have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$.

13. As $x \to \pm\infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{3x^3}{5x^3} = \frac{3}{5},$$

we have $\lim_{x\to-\infty} f(x) = 3/5$ and $\lim_{x\to+\infty} f(x) = 3/5$.

14. As $x \to \pm\infty$, we know that x^{-3} gets closer and closer to zero, so we have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = 0$.

15. We have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$.

16. The break in the graph at $x = 0$ suggests that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. See Figure 1.89.

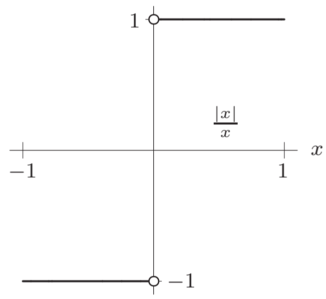


Figure 1.89

17. For $-1 \le x \le 1,\ -1 \le y \le 1$, the graph of $y = x\ln|x|$ is in Figure 1.90. The graph suggests that

$$\lim_{x\to 0} x\ln|x| = 0.$$

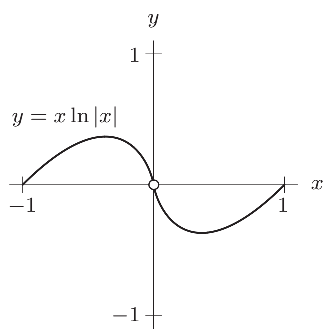


Figure 1.90

18. Since $\lim_{x\to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x\to 0^-} \frac{|x|}{x} = -1$, we say that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. In addition $f(x)$ is not defined at 0. Therefore, $f(x)$ is not continuous on any interval containing 0.

19. For $-0.5 \le \theta \le 0.5$, $0 \le y \le 3$, the graph of $y = \dfrac{\sin(2\theta)}{\theta}$ is shown in Figure 1.91. Therefore, $\lim_{\theta\to 0} \dfrac{\sin(2\theta)}{\theta} = 2$.

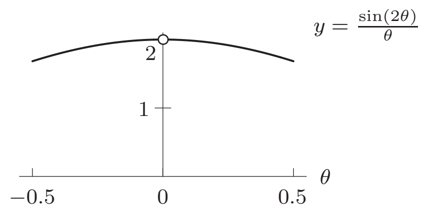


Figure 1.91

20. For $-1 \le \theta \le 1$, $-1 \le y \le 1$, the graph of $y = \dfrac{\cos\theta - 1}{\theta}$ is shown in Figure 1.92. Therefore, $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta} = 0$.

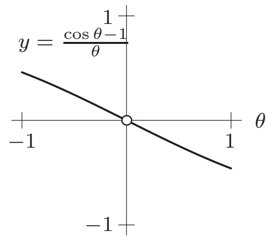


Figure 1.92

21. For $-90^\circ \le \theta \le 90^\circ$, $0 \le y \le 0.02$, the graph of $y = \dfrac{\sin\theta}{\theta}$ is shown in Figure 1.93. Therefore, by tracing along the curve, we see that in degrees, $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 0.01745\ldots$.

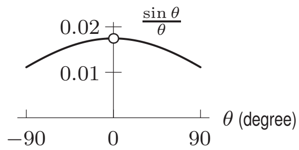


Figure 1.93

22. For $-0.5 \le \theta \le 0.5$, $0 \le y \le 0.5$, the graph of $y = \dfrac{\theta}{\tan(3\theta)}$ is shown in Figure 1.94. Therefore, by tracing along the curve, we see that $\lim_{\theta\to 0} \dfrac{\theta}{\tan(3\theta)} = 0.3333\ldots$.

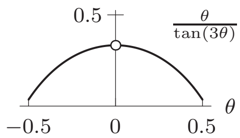


Figure 1.94

23. A graph of $y = \dfrac{e^h - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 3$ appears to indicate $y \to 1$ as $h \to 0$. Therefore, we estimate that $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

24. A graph of $y = \dfrac{e^{5h}-1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 6$ appears to indicate $y \to 5$ as $h \to 0$. Therefore, we estimate that $\lim_{h\to 0} \dfrac{e^{5h}-1}{h} = 5$.

25. A graph of $y = \dfrac{2^h-1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 1$ appears to indicate $y \to 0.7$ as $h \to 0$. By zooming in on the graph, we can estimate the limit more accurately. Therefore, we estimate that

$$\lim_{h\to 0} \frac{2^h-1}{h} = 0.693.$$

26. A graph of $y = \dfrac{3^h-1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 1.5$ appears to indicate $y \to 1.1$ as $h \to 0$. By zooming in on the graph, we can estimate the limit more accurately. Therefore, we estimate that

$$\lim_{h\to 0} \frac{3^h-1}{h} = 1.098.$$

27. A graph of $y = \dfrac{\cos(3h)-1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $-1 \le y \le 1$ appears to indicate $y \to 0$ as $h \to 0$. Therefore, we estimate that

$$\lim_{h\to 0} \frac{\cos(3h)-1}{h} = 0.$$

28. A graph of $y = \dfrac{\sin(3h)}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 4$ appears to indicate $y \to 3$ as $h \to 0$. Therefore, we estimate that

$$\lim_{h\to 0} \frac{\sin(3h)}{h} = 3.$$

29. $f(x) = \dfrac{|x-4|}{x-4} = \begin{cases} \dfrac{x-4}{x-4} & x > 4 \\ -\dfrac{x-4}{x-4} & x < 4 \end{cases} = \begin{cases} 1 & x > 4 \\ -1 & x < 4 \end{cases}$

Figure 1.95 confirms that $\lim_{x\to 4^+} f(x) = 1$, $\lim_{x\to 4^-} f(x) = -1$ so $\lim_{x\to 4} f(x)$ does not exist.

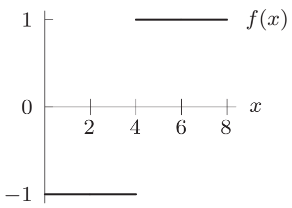


Figure 1.95

30. $f(x) = \dfrac{|x-2|}{x} = \begin{cases} \dfrac{x-2}{x}, & x > 2 \\ -\dfrac{x-2}{x}, & x < 2 \end{cases}$

Figure 1.96 confirms that $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 0$.

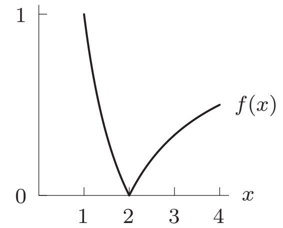


Figure 1.96

31. $f(x) = \begin{cases} x^2 - 2 & 0 < x < 3 \\ 2 & x = 3 \\ 2x+1 & 3 < x \end{cases}$

Figure 1.97 confirms that $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-2) = 7$ and that $\lim_{x\to 3+} f(x) = \lim_{x\to 3+} (2x+1) = 7$, so $\lim_{x\to 3} f(x) = 7$. Note, however, that $f(x)$ is not continuous at $x = 3$ since $f(3) = 2$.

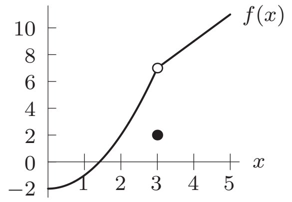


Figure 1.97

32. The graph in Figure 1.98 suggests that

$$\text{if } -0.05 < \theta < 0.05, \quad \text{then} \quad 0.999 < (\sin\theta)/\theta < 1.001.$$

Thus, if θ is within 0.05 of 0, we see that $(\sin\theta)/\theta$ is within 0.001 of 1.

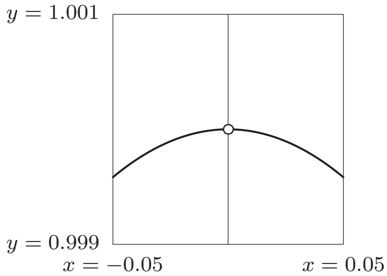


Figure 1.98: Graph of $(\sin\theta)/\theta$ with $-0.05 < \theta < 0.05$

33. The statement

$$\lim_{h\to a} g(h) = K$$

means that we can make the value of $g(h)$ as close to K as we want by choosing h sufficiently close to, but not equal to, a.

In symbols, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|g(h) - K| < \epsilon \quad \text{for all } 0 < |h-a| < \delta.$$

Problems

34. At $x = 0$, the function is not defined. In addition, $\lim_{x\to 0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = 0$.

35. Since $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$, we see that $\lim_{x\to 0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = 0$

36. Since $x/x = 1$ for $x \neq 0$, this function $f(x) = 1$ for all x. Thus, $f(x)$ is continuous for all x.

37. Since $2x/x = 2$ for $x \neq 0$, we have $\lim_{x\to 0} f(x) = 2$, so

$$\lim_{x\to 0} f(x) \neq f(0) = 3.$$

Thus, $f(x)$ is not continuous at $x = 0$.

38. The answer (see the graph in Figure 1.99) appears to be about 2.7; if we zoom in further, it appears to be about 2.72, which is close to the value of $e \approx 2.71828$.

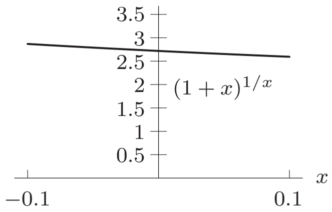


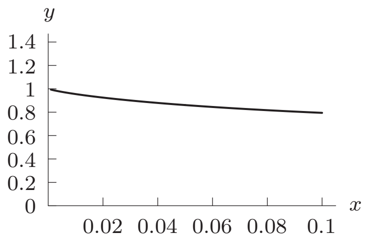
Figure 1.99

39. We use values of h approaching, but not equal to, zero. If we let $h = 0.01, 0.001, 0.0001, 0.00001$, we calculate the values 2.7048, 2.7169, 2.7181, and 2.7183. If we let $h = -0.01\ -0.001, -0.0001, -0.00001$, we get values 2.7320, 2.7196, 2.7184, and 2.7183. These numbers suggest that the limit is the number $e = 2.71828\ldots$. However, these calculations cannot tell us that the limit is exactly e; for that a proof is needed.

40. When $x = 0.1$, we find $xe^{1/x} \approx 2203$. When $x = 0.01$, we find $xe^{1/x} \approx 3 \times 10^{41}$. When $x = 0.001$, the value of $xe^{1/x}$ is too big for a calculator to compute. This suggests that $\lim_{x\to 0^+} xe^{1/x}$ does not exist (and in fact it does not).

41. If $x > 1$ and x approaches 1, then $p(x) = 55$. If $x < 1$ and x approaches 1, then $p(x) = 34$. There is not a single number that $p(x)$ approaches as x approaches 1, so we say that $\lim_{x\to 1} p(x)$ does not exist.

42. The limit appears to be 1; a graph and table of values is shown below.



x	x^x
0.1	0.7943
0.01	0.9550
0.001	0.9931
0.0001	0.9990
0.00001	0.9999

43. The only change is that, instead of considering all x near c, we only consider x near to and greater than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c < x < c + \delta$." Thus, we define

$$\lim_{x\to c^+} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c < x < c + \delta$, then $|f(x) - L| < \epsilon$.

44. The only change is that, instead of considering all x near c, we only consider x near to and less than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c - \delta < x < c$." Thus, we define

$$\lim_{x\to c^-} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.

45. Instead of being "sufficiently close to c," we want x to be "sufficiently large." Using N to measure how large x must be, we define

$$\lim_{x\to\infty} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $N > 0$ (sufficiently large) such that if $x > N$, then $|f(x) - L| < \epsilon$.

46. From Table 1.6, it appears the limit is 1. This is confirmed by Figure 1.100. An appropriate window is $-0.0033 < x < 0.0033$, $0.99 < y < 1.01$.

Table 1.6

x	$f(x)$	x	$f(x)$
0.1	1.3	−0.0001	0.9997
0.01	1.03	−0.001	0.997
0.001	1.003	−0.01	0.97
0.0001	1.0003	−0.1	0.7

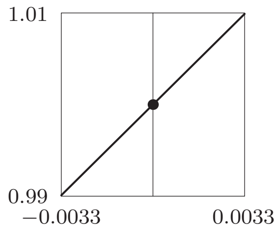


Figure 1.100

47. From Table 1.7, it appears the limit is −1. This is confirmed by Figure 1.101. An appropriate window is $-0.099 < x < 0.099$, $-1.01 < y < -0.99$.

Table 1.7

x	$f(x)$	x	$f(x)$
0.1	−0.99	−0.0001	−0.99999999
0.01	−0.9999	−0.001	−0.999999
0.001	−0.999999	−0.01	−0.9999
0.0001	−0.99999999	−0.1	−0.99

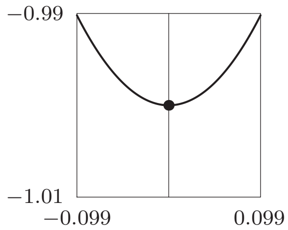


Figure 1.101

48. From Table 1.8, it appears the limit is 0. This is confirmed by Figure 1.102. An appropriate window is $-0.005 < x < 0.005$, $-0.01 < y < 0.01$.

Table 1.8

x	$f(x)$	x	$f(x)$
0.1	0.1987	−0.0001	−0.0002
0.01	0.0200	−0.001	−0.0020
0.001	0.0020	−0.01	−0.0200
0.0001	0.0002	−0.1	−0.1987

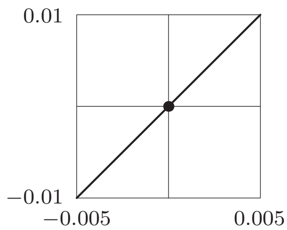


Figure 1.102

49. From Table 1.9, it appears the limit is 0. This is confirmed by Figure 1.103. An appropriate window is $-0.0033 < x < 0.0033$, $-0.01 < y < 0.01$.

Table 1.9

x	$f(x)$	x	$f(x)$
0.1	0.2955	−0.0001	−0.0003
0.01	0.0300	−0.001	−0.0030
0.001	0.0030	−0.01	−0.0300
0.0001	0.0003	−0.1	−0.2955

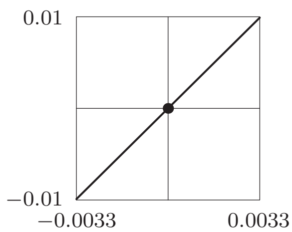


Figure 1.103

50. From Table 1.10, it appears the limit is 2. This is confirmed by Figure 1.104. An appropriate window is $-0.0865 < x < 0.0865$, $1.99 < y < 2.01$.

Table 1.10

x	$f(x)$
0.1	1.9867
0.01	1.9999
0.001	2.0000
0.0001	2.0000

x	$f(x)$
−0.0001	2.0000
−0.001	2.0000
−0.01	1.9999
−0.1	1.9867

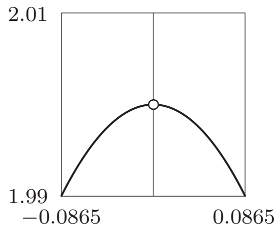


Figure 1.104

51. From Table 1.11, it appears the limit is 3. This is confirmed by Figure 1.105. An appropriate window is $-0.047 < x < 0.047$, $2.99 < y < 3.01$.

Table 1.11

x	$f(x)$
0.1	2.9552
0.01	2.9996
0.001	3.0000
0.0001	3.0000

x	$f(x)$
−0.0001	3.0000
−0.001	3.0000
−0.01	2.9996
−0.1	2.9552

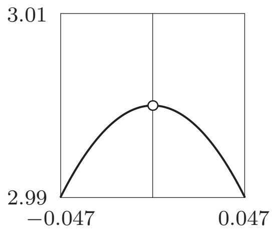


Figure 1.105

52. From Table 1.12, it appears the limit is 1. This is confirmed by Figure 1.106. An appropriate window is $-0.0198 < x < 0.0198$, $0.99 < y < 1.01$.

Table 1.12

x	$f(x)$
0.1	1.0517
0.01	1.0050
0.001	1.0005
0.0001	1.0001

x	$f(x)$
−0.0001	1.0000
−0.001	0.9995
−0.01	0.9950
−0.1	0.9516

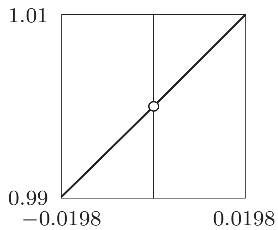


Figure 1.106

53. From Table 1.13, it appears the limit is 2. This is confirmed by Figure 1.107. An appropriate window is $-0.0049 < x < 0.0049$, $1.99 < y < 2.01$.

Table 1.13

x	$f(x)$
0.1	2.2140
0.01	2.0201
0.001	2.0020
0.0001	2.0002

x	$f(x)$
−0.0001	1.9998
−0.001	1.9980
−0.01	1.9801
−0.1	1.8127

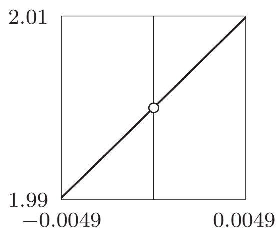


Figure 1.107

54. Divide numerator and denominator by x:

$$f(x) = \frac{x+3}{2-x} = \frac{1+3/x}{2/x-1},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+3/x}{2/x-1} = \frac{\lim_{x\to\infty}(1+3/x)}{\lim_{x\to\infty}(2/x-1)} = \frac{1}{-1} = -1.$$

55. Divide numerator and denominator by x:

$$f(x) = \frac{\pi+3x}{\pi x-3} = \frac{(\pi+3x)/x}{(\pi x-3)/x},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\pi/x+3}{\pi-3/x} = \frac{\lim_{x\to\infty}(\pi/x+3)}{\lim_{x\to\infty}(\pi-3/x)} = \frac{3}{\pi}.$$

56. Divide numerator and denominator by x^2:

$$f(x) = \frac{x-5}{5+2x^2} = \frac{(1/x)-(5/x^2)}{(5/x^2)+2},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(1/x)-(5/x^2)}{(5/x^2)+2} = \frac{\lim_{x\to\infty}((1/x)-(5/x^2))}{\lim_{x\to\infty}((5/x^2)+2)} = \frac{0}{2} = 0.$$

57. Divide numerator and denominator by x^2, giving

$$f(x) = \frac{x^2+2x-1}{3+3x^2} = \frac{1+2/x-1/x^2}{3/x^2+3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+2/x-1/x^2}{3/x^2+3} = \frac{\lim_{x\to\infty}(1+2/x-1/x^2)}{\lim_{x\to\infty}(3/x^2+3)} = \frac{1}{3}.$$

58. Divide numerator and denominator by x, giving

$$f(x) = \frac{x^2+4}{x+3} = \frac{x+4/x}{1+3/x},$$

so

$$\lim_{x\to\infty} f(x) = +\infty.$$

59. Divide numerator and denominator by x^3, giving

$$f(x) = \frac{2x^3-16x^2}{4x^2+3x^3} = \frac{2-16/x}{4/x+3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2-16/x}{4/x+3} = \frac{\lim_{x\to\infty}(2-16/x)}{\lim_{x\to\infty}(4/x+3)} = \frac{2}{3}.$$

60. Divide numerator and denominator by x^5, giving

$$f(x) = \frac{x^4+3x}{x^4+2x^5} = \frac{1/x+3/x^4}{1/x+2},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(1/x+3/x^4)}{\lim_{x\to\infty}(1/x+2)} = \frac{0}{2} = 0.$$

61. Divide numerator and denominator by e^x, giving

$$f(x) = \frac{3e^x + 2}{2e^x + 3} = \frac{3 + 2e^{-x}}{2 + 3e^{-x}},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(3 + 2e^{-x})}{\lim_{x\to\infty}(2 + 3e^{-x})} = \frac{3}{2}.$$

62. Since $\lim_{x\to\infty} 2^{-x} = \lim_{x\to\infty} 3^{-x} = 0$, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2^{-x} + 5}{3^{-x} + 7} = \frac{\lim_{x\to\infty}(2^{-x} + 5)}{\lim_{x\to\infty}(3^{-x} + 7)} = \frac{5}{7}.$$

63. $f(x) = \dfrac{2e^{-x} + 3}{3e^{-x} + 2}$, so $\lim\limits_{x\to\infty} f(x) = \dfrac{\lim_{x\to\infty}(2e^{-x} + 3)}{\lim_{x\to\infty}(3e^{-x} + 2)} = \dfrac{3}{2}$.

64. Because the denominator equals 0 when $x = 4$, so must the numerator. This means $k^2 = 16$ and the choices for k are 4 or -4.

65. Because the denominator equals 0 when $x = 1$, so must the numerator. So $1 - k + 4 = 0$. The only possible value of k is 5.

66. Because the denominator equals 0 when $x = -2$, so must the numerator. So $4 - 8 + k = 0$ and the only possible value of k is 4.

67. Division of numerator and denominator by x^2 yields

$$\frac{x^2 + 3x + 5}{4x + 1 + x^k} = \frac{1 + 3/x + 5/x^2}{4/x + 1/x^2 + x^{k-2}}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 2$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 2$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 2$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 2$.

68. For the numerator, $\lim\limits_{x\to-\infty}\left(e^{2x} - 5\right) = -5$. If $k > 0$, $\lim\limits_{x\to-\infty}\left(e^{kx} + 3\right) = 3$, so the quotient has a limit of $-5/3$. If $k = 0$, $\lim\limits_{x\to-\infty}\left(e^{kx} + 3\right) = 4$, so the quotient has limit of $-5/4$. If $k < 0$, the limit of the quotient is given by $\lim\limits_{x\to-\infty}(e^{2x} - 5)/(e^{kx} + 3) = 0$.

69. Division of numerator and denominator by x^3 yields

$$\frac{x^3 - 6}{x^k + 3} = \frac{1 - 6/x^3}{x^{k-3} + 3/x^3}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 3$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 3$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 3$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 3$.

70. We divide both the numerator and denominator by 3^{2x}, giving

$$\lim_{x\to\infty} \frac{3^{kx} + 6}{3^{2x} + 4} = \frac{3^{(k-2)x} + 6/3^{2x}}{1 + 4/3^{2x}}.$$

In the denominator, $\lim\limits_{x\to\infty} 1 + 4/3^{2x} = 1$. In the numerator, if $k < 2$, we have $\lim\limits_{x\to\infty} 3^{(k-2)x} + 6/3^{2x} = 0$, so the quotient has a limit of 0. If $k = 2$, we have $\lim\limits_{x\to\infty} 3^{(k-2)x} + 6/3^{2x} = 1$, so the quotient has a limit of 1. If $k > 2$, we have $\lim\limits_{x\to\infty} 3^{(k-2)x} + 6/3^{2x} = \infty$, so the quotient has a limit of ∞.

71. In the denominator, we have $\lim\limits_{x\to-\infty} 3^{2x} + 4 = 4$. In the numerator, if $k < 0$, we have $\lim\limits_{x\to-\infty} 3^{kx} + 6 = \infty$, so the quotient has a limit of ∞. If $k = 0$, we have $\lim\limits_{x\to-\infty} 3^{kx} + 6 = 7$, so the quotient has a limit of $7/4$. If $k > 0$, we have $\lim\limits_{x\to-\infty} 3^{kx} + 6 = 6$, so the quotient has a limit of $6/4$.

72. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.2$, $\delta \le 0.1$.
For $\epsilon = 0.1$, $\delta \le 0.05$.
For $\epsilon = 0.02$, $\delta \le 0.01$.
For $\epsilon = 0.01$, $\delta \le 0.005$.
For $\epsilon = 0.002$, $\delta \le 0.001$.
For $\epsilon = 0.001$, $\delta \le 0.0005$.

73. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.46$.
For $\epsilon = 0.01$, $\delta \le 0.21$.
For $\epsilon = 0.001$, $\delta < 0.1$. Thus, we can take $\delta \le 0.09$.

74. The results of Problem 72 suggest that we can choose $\delta = \epsilon/2$. For any $\epsilon > 0$, we want to find the δ such that

$$|f(x) - 3| = |-2x + 3 - 3| = |2x| < \epsilon.$$

Then if $|x| < \delta = \epsilon/2$, it follows that $|f(x) - 3| = |2x| < \epsilon$. So $\lim_{x\to 0}(-2x + 3) = 3$.

75. (a) Since $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$ the sequence of x-values

$$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$$

works. These x-values $\to 0$ and are zeroes of $f(x)$.

(b) Since $\sin(n\pi/2) = 1$ for $n = 1, 5, 9 \ldots$ the sequence of x-values

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$$

works.

(c) Since $\sin(n\pi)/2 = -1$ for $n = 3, 7, 11, \ldots$ the sequence of x-values

$$\frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi} \ldots$$

works.

(d) Any two of these sequences of x-values show that if the limit were to exist, then it would have to have two (different) values: 0 and 1, or 0 and -1, or 1 and -1. Hence, the limit can not exist.

76. From Table 1.14, it appears the limit is 0. This is confirmed by Figure 1.108. An appropriate window is $-0.015 < x < 0.015$, $-0.01 < y < 0.01$.

Table 1.14

x	$f(x)$
0.1	0.0666
0.01	0.0067
0.001	0.0007
0.0001	0.0001

x	$f(x)$
-0.0001	-0.0001
-0.001	-0.0007
-0.01	-0.0067
-0.1	-0.0666

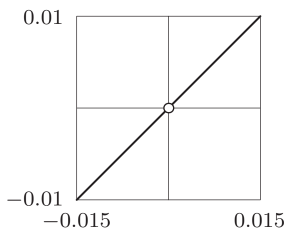


Figure 1.108

77. From Table 1.15, it appears the limit is 0. This is confirmed by Figure 1.109. An appropriate window is $-0.0029 < x < 0.0029$, $-0.01 < y < 0.01$.

Table 1.15

x	$f(x)$
0.1	0.3365
0.01	0.0337
0.001	0.0034
0.0001	0.0004

x	$f(x)$
-0.0001	-0.0004
-0.001	-0.0034
-0.01	-0.0337
-0.1	-0.3365

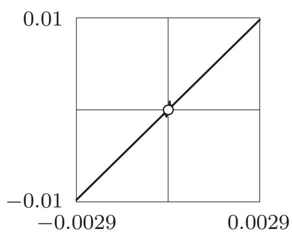


Figure 1.109

78. (a) If $b = 0$, then the property says $\lim_{x\to c} 0 = 0$, which is easy to see is true.

(b) If $|f(x) - L| < \frac{\epsilon}{|b|}$, then multiplying by $|b|$ gives

$$|b||f(x) - L| < \epsilon.$$

Since

$$|b||f(x) - L| = |b(f(x) - L)| = |bf(x) - bL|,$$

we have

$$|bf(x) - bL| < \epsilon.$$

(c) Suppose that $\lim_{x\to c} f(x) = L$. We want to show that $\lim_{x\to c} bf(x) = bL$. If we are to have

$$|bf(x) - bL| < \epsilon,$$

then we will need

$$|f(x) - L| < \frac{\epsilon}{|b|}.$$

We choose δ small enough that

$$|x - c| < \delta \quad \text{implies} \quad |f(x) - L| < \frac{\epsilon}{|b|}.$$

By part (b), this ensures that

$$|bf(x) - bL| < \epsilon,$$

as we wanted.

79. Suppose $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$. Then we need to show that

$$\lim_{x\to c}(f(x) + g(x)) = L_1 + L_2.$$

Let $\epsilon > 0$ be given. We need to show that we can choose $\delta > 0$ so that whenever $|x - c| < \delta$, we will have $|(f(x) + g(x)) - (L_1 + L_2)| < \epsilon$. First choose $\delta_1 > 0$ so that $|x - c| < \delta_1$ implies $|f(x) - L_1| < \frac{\epsilon}{2}$; we can do this since $\lim_{x\to c} f(x) = L_1$. Similarly, choose $\delta_2 > 0$ so that $|x - c| < \delta_2$ implies $|g(x) - L_2| < \frac{\epsilon}{2}$. Now, set δ equal to the smaller of δ_1 and δ_2. Thus $|x - c| < \delta$ will make both $|x - c| < \delta_1$ and $|x - c| < \delta_2$. Then, for $|x - c| < \delta$, we have

$$\begin{aligned}|f(x) + g(x) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)|\\ &\le |(f(x) - L_1)| + |(g(x) - L_2)|\\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This proves $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$, which is the result we wanted to prove.

80. (a) We need to show that for any given $\epsilon > 0$, there is a $\delta > 0$ so that $|x - c| < \delta$ implies $|f(x)g(x)| < \epsilon$. If $\epsilon > 0$ is given, choose δ_1 so that when $|x - c| < \delta_1$, we have $|f(x)| < \sqrt{\epsilon}$. This can be done since $\lim_{x\to 0} f(x) = 0$. Similarly, choose δ_2 so that when $|x - c| < \delta_2$, we have $|g(x)| < \sqrt{\epsilon}$. Then, if we take δ to be the smaller of δ_1 and δ_2, we'll have that $|x - c| < \delta$ implies both $|f(x)| < \sqrt{\epsilon}$ and $|g(x)| < \sqrt{\epsilon}$. So when $|x - c| < \delta$, we have $|f(x)g(x)| = |f(x)|\,|g(x)| < \sqrt{\epsilon} \cdot \sqrt{\epsilon} = \epsilon$. Thus $\lim_{x\to c} f(x)g(x) = 0$.

(b) $(f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2$
$= f(x)g(x) - L_1 g(x) - L_2 f(x) + L_1 L_2 + L_1 g(x) + L_2 f(x) - L_1 L_2 = f(x)g(x).$

(c) $\lim_{x\to c}(f(x) - L_1) = \lim_{x\to c} f(x) - \lim_{x\to c} L_1 = L_1 - L_1 = 0$, using the second limit property. Similarly, $\lim_{x\to c}(g(x) - L_2) = 0$.

(d) Since $\lim_{x\to c}(f(x) - L_1) = \lim_{x\to c}(g(x) - L_2) = 0$, we have that $\lim_{x\to c}(f(x) - L_1)(g(x) - L_2) = 0$ by part (a).

(e) From part (b), we have

$$\begin{aligned}\lim_{x\to c} f(x)g(x) &= \lim_{x\to c}((f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2)\\ &= \lim_{x\to c}(f(x) - L_1)(g(x) - L_2) + \lim_{x\to c} L_1 g(x) + \lim_{x\to c} L_2 f(x) + \lim_{x\to c}(-L_1 L_2)\\ &\quad \text{(using limit property 2)}\\ &= 0 + L_1 \lim_{x\to c} g(x) + L_2 \lim_{x\to c} f(x) - L_1 L_2\\ &\quad \text{(using limit property 1 and part (d))}\\ &= L_1 L_2 + L_2 L_1 - L_1 L_2 = L_1 L_2.\end{aligned}$$

81. We will show $f(x) = x$ is continuous at $x = c$. Since $f(c) = c$, we need to show that

$$\lim_{x\to c} f(x) = c$$

that is, since $f(x) = x$, we need to show

$$\lim_{x\to c} x = c.$$

Pick any $\epsilon > 0$, then take $\delta = \epsilon$. Thus,

$$|f(x) - c| = |x - c| < \epsilon \quad \text{for all} \quad |x - c| < \delta = \epsilon.$$

82. Since $f(x) = x$ is continuous, Theorem 1.3 on page 64 shows that products of the form $f(x) \cdot f(x) = x^2$ and $f(x) \cdot x^2 = x^3$, etc., are continuous. By a similar argument, x^n is continuous for any $n > 0$.

83. If c is in the interval, we know $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then,

$$\begin{aligned}\lim_{x\to c} (f(x) + g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad \text{by limit property 2}\\ &= f(c) + g(c), \quad \text{so } f + g \text{ is continuous at } x = c.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to c} (f(x)g(x)) &= \lim_{x\to c} f(x) \lim_{x\to c} g(x) \quad \text{by limit property 3}\\ &= f(c)g(c) \quad \text{so } fg \text{ is continuous at } x = c.\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{by limit property 4}\\ &= \frac{f(c)}{g(c)}, \quad \text{so } \frac{f}{g} \text{ is continuous at } x = c.\end{aligned}$$

Strengthen Your Understanding

84. Though $P(x)$ and $Q(x)$ are both continuous for all x, it is possible for $Q(x)$ to be equal to zero for some x. For any such value of x, where $Q(x) = 0$, we see that $P(x)/Q(x)$ is undefined, and thus not continuous. For example,

$$\frac{P(x)}{Q(x)} = \frac{x}{x-1}$$

is not defined or continuous at $x = 1$.

85. The left- and right-hand limits are not the same:

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = -1,$$

but

$$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = 1.$$

Since the left- and right-hand limits are not the same, the limit does not exist and thus is not equal to 1.

86. For f to be continuous at $x = c$, we need $\lim_{x\to c} f(x)$ to exist and to be equal to $f(c)$.

87. One possibility is

$$f(x) = \frac{(x+3)(x-1)}{x-1}.$$

We have $\lim_{x\to 1} f(x) = 4$ but $f(1)$ is undefined.

88. One example is

$$f(x) = \frac{2|x|}{x}.$$

89. True, by Property 3 of limits in Theorem 1.2, since $\lim_{x\to 3} x = 3$.

90. False. If $\lim_{x\to 3} g(x)$ does not exist, then $\lim_{x\to 3} f(x)g(x)$ may not even exist. For example, let $f(x) = 2x + 1$ and define g by:

$$g(x) = \begin{cases} 1/(x-3) & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

Then $\lim_{x\to 3} f(x) = 7$ and $g(3) = 4$, but $\lim_{x\to 3} f(x)g(x) \neq 28$, since $\lim_{x\to 3}(2x+1)/(x-3)$ does not exist.

91. True, by Property 2 of limits in Theorem 1.2.

92. True, by Properties 2 and 3 of limits in Theorem 1.2.

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(f(x) + g(x) + (-1)f(x)) = \lim_{x\to 3}(f(x) + g(x)) + (-1)\lim_{x\to 3} f(x) = 12 + (-1)7 = 5.$$

93. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 2.99 \\ 1000 & \text{if } x = 2.99. \end{cases}$$

Then $f(2.9) = 2(2.9) + 1 = 6.8$, whereas $f(2.99) = 1000$.

94. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 3.01 \\ -1000 & \text{if } x = 3.01. \end{cases}$$

Then $f(3.1) = 2(3.1) + 1 = 7.2$, whereas $f(3.01) = -1000$.

95. True. Suppose instead that $\lim_{x\to 3} g(x)$ does not exist but $\lim_{x\to 3}(f(x)g(x))$ did exist. Since $\lim_{x\to 3} f(x)$ exists and is not zero, then $\lim_{x\to 3}((f(x)g(x))/f(x))$ exists, by Property 4 of limits in Theorem 1.2. Furthermore, $f(x) \neq 0$ for all x in some interval about 3, so $(f(x)g(x))/f(x) = g(x)$ for all x in that interval. Thus $\lim_{x\to 3} g(x)$ exists. This contradicts our assumption that $\lim_{x\to 3} g(x)$ does not exist.

96. False. For some functions we need to pick smaller values of δ. For example, if $f(x) = x^{1/3} + 2$ and $c = 0$ and $L = 2$, then $f(x)$ is within 10^{-3} of 2 if $|x^{1/3}| < 10^{-3}$. This only happens if x is within $(10^{-3})^3 = 10^{-9}$ of 0. If $x = 10^{-3}$ then $x^{1/3} = (10^{-3})^{1/3} = 10^{-1}$, which is too large.

97. False. The definition of a limit guarantees that, for any positive ϵ, there is a δ. This statement, which guarantees an ϵ for a specific $\delta = 10^{-3}$, is not equivalent to $\lim_{x\to c} f(x) = L$. For example, consider a function with a vertical asymptote within 10^{-3} of 0, such as $c = 0$, $L = 0$, $f(x) = x/(x - 10^{-4})$.

98. True. This is equivalent to the definition of a limit.

99. False. Although x may be far from c, the value of $f(x)$ could be close to L. For example, suppose $f(x) = L$, the constant function.

100. False. The definition of the limit says that if x is within δ of c, then $f(x)$ is within ϵ of L, not the other way round.

101. (a) This statement follows: if we interchange the roles of f and g in the original statement, we get this statement.
(b) This statement is true, but it does not follow directly from the original statement, which says nothing about the case $g(a) = 0$.
(c) This follows, since if $g(a) \neq 0$ the original statement would imply f/g is continuous at $x = a$, but we are told it is not.
(d) This does not follow. Given that f is continuous at $x = a$ and $g(a) \neq 0$, then the original statement says g continuous implies f/g continuous, not the other way around. In fact, statement (d) is not true: if $f(x) = 0$ for all x, then g could be any discontinuous, non-zero function, and f/g would be zero, and therefore continuous. Thus the conditions of the statement would be satisfied, but not the conclusion.

Solutions for Chapter 1 Review

Exercises

1. The line of slope m through the point (x_0, y_0) has equation

$$y - y_0 = m(x - x_0),$$

so the line we want is

$$\begin{aligned} y - 0 &= 2(x - 5) \\ y &= 2x - 10. \end{aligned}$$

2. We want a function of the form $y = a(x-h)^2 + k$, with $a < 0$ because the parabola opens downward. Since (h, k) is the vertex, we must take $h = 2$, $k = 5$, but we can take any negative value of a. Figure 1.110 shows the graph with $a = -1$, namely $y = -(x-2)^2 + 5$.

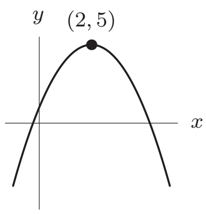


Figure 1.110: Graph of $y = -(x-2)^2 + 5$

3. A parabola with x-intercepts ± 1 has an equation of the form

$$y = k(x-1)(x+1).$$

Substituting the point $x = 0, y = 3$ gives

$$3 = k(-1)(1) \quad \text{so} \quad k = -3.$$

Thus, the equation we want is

$$\begin{aligned} y &= -3(x-1)(x+1) \\ y &= -3x^2 + 3. \end{aligned}$$

4. The equation of the whole circle is

$$x^2 + y^2 = (\sqrt{2})^2,$$

so the bottom half is

$$y = -\sqrt{2 - x^2}.$$

5. A circle with center (h, k) and radius r has equation $(x-h)^2 + (y-k)^2 = r^2$. Thus $h = -1$, $k = 2$, and $r = 3$, giving

$$(x+1)^2 + (y-2)^2 = 9.$$

Solving for y, and taking the positive square root gives the top half, so

$$\begin{aligned} (y-2)^2 &= 9 - (x+1)^2 \\ y &= 2 + \sqrt{9 - (x+1)^2}. \end{aligned}$$

See Figure 1.111.

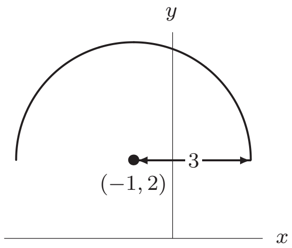


Figure 1.111: Graph of $y = 2 + \sqrt{9 - (x+1)^2}$

6. A cubic polynomial of the form $y = a(x-1)(x-5)(x-7)$ has the correct intercepts for any value of $a \neq 0$. Figure 1.112 shows the graph with $a = 1$, namely $y = (x-1)(x-5)(x-7)$.

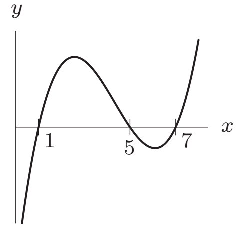


Figure 1.112: Graph of $y = (x-1)(x-5)(x-7)$

7. Since the vertical asymptote is $x = 2$, we have $b = -2$. The fact that the horizontal asymptote is $y = -5$ gives $a = -5$. So

$$y = \frac{-5x}{x-2}.$$

8. The amplitude of this function is 5, and its period is 2π, so $y = 5\cos x$.

9. See Figure 1.113.

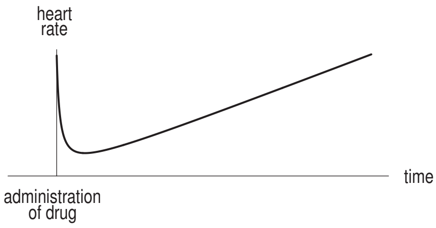


Figure 1.113

10. Factoring gives

$$g(x) = \frac{(2-x)(2+x)}{x(x+1)}.$$

The values of x which make $g(x)$ undefined are $x = 0$ and $x = -1$, when the denominator is 0. So the domain is all $x \neq 0, -1$. Solving $g(x) = 0$ means one of the numerator's factors is 0, so $x = \pm 2$.

11. (a) The domain of f is the set of values of x for which the function is defined. Since the function is defined by the graph and the graph goes from $x = 0$ to $x = 7$, the domain of f is $[0, 7]$.

(b) The range of f is the set of values of y attainable over the domain. Looking at the graph, we can see that y gets as high as 5 and as low as -2, so the range is $[-2, 5]$.

(c) Only at $x = 5$ does $f(x) = 0$. So 5 is the only zero of $f(x)$.

(d) Looking at the graph, we can see that $f(x)$ is decreasing on $(1, 7)$.

(e) The graph indicates that $f(x)$ is concave up at $x = 6$.

(f) The value $f(4)$ is the y-value that corresponds to $x = 4$. From the graph, we can see that $f(4)$ is approximately 1.

(g) This function is not invertible, since it fails the horizontal-line test. A horizontal line at $y = 3$ would cut the graph of $f(x)$ in two places, instead of the required one.

12. (a) $f(n) + g(n) = (3n^2 - 2) + (n + 1) = 3n^2 + n - 1$.
(b) $f(n)g(n) = (3n^2 - 2)(n + 1) = 3n^3 + 3n^2 - 2n - 2$.
(c) The domain of $f(n)/g(n)$ is defined everywhere where $g(n) \neq 0$, i.e. for all $n \neq -1$.
(d) $f(g(n)) = 3(n + 1)^2 - 2 = 3n^2 + 6n + 1$.
(e) $g(f(n)) = (3n^2 - 2) + 1 = 3n^2 - 1$.

13. (a) Since $m = f(A)$, we see that $f(100)$ represents the value of m when $A = 100$. Thus $f(100)$ is the minimum annual gross income needed (in thousands) to take out a 30-year mortgage loan of \$100,000 at an interest rate of 6%.
(b) Since $m = f(A)$, we have $A = f^{-1}(m)$. We see that $f^{-1}(75)$ represents the value of A when $m = 75$, or the size of a mortgage loan that could be obtained on an income of \$75,000.

14. Taking logs of both sides yields $t \log 5 = \log 7$, so $t = \dfrac{\log 7}{\log 5} = 1.209$.

15. $t = \dfrac{\log 2}{\log 1.02} \approx 35.003$.

16. Collecting similar factors yields $\left(\frac{3}{2}\right)^t = \frac{5}{7}$, so

$$t = \frac{\log\left(\frac{5}{7}\right)}{\log\left(\frac{3}{2}\right)} = -0.830.$$

17. Collecting similar factors yields $\left(\frac{1.04}{1.03}\right)^t = \frac{12.01}{5.02}$. Solving for t yields

$$t = \frac{\log\left(\frac{12.01}{5.02}\right)}{\log\left(\frac{1.04}{1.03}\right)} = 90.283.$$

18. We want $2^t = e^{kt}$ so $2 = e^k$ and $k = \ln 2 = 0.693$. Thus $P = P_0 e^{0.693t}$.

19. We want $0.2^t = e^{kt}$ so $0.2 = e^k$ and $k = \ln 0.2 = -1.6094$. Thus $P = 5.23e^{-1.6094t}$.

20. $f(x) = \ln x, \quad g(x) = x^3$. (Another possibility: $f(x) = 3x, \quad g(x) = \ln x$.)

21. $f(x) = x^3, \quad g(x) = \ln x$.

22. The amplitude is 5. The period is 6π. See Figure 1.114.

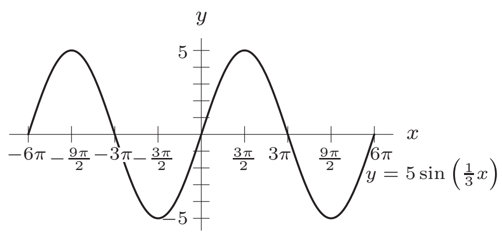


Figure 1.114

23. The amplitude is 2. The period is $2\pi/5$. See Figure 1.115.

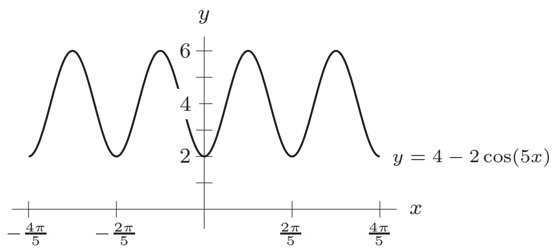
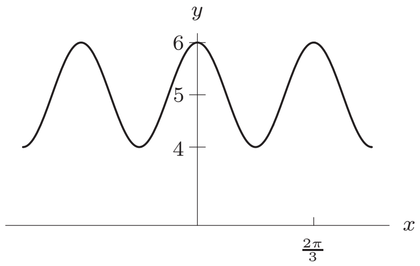


Figure 1.115

24. (a) We determine the amplitude of y by looking at the coefficient of the cosine term. Here, the coefficient is 1, so the amplitude of y is 1. Note that the constant term does not affect the amplitude.
(b) We know that the cosine function $\cos x$ repeats itself at $x = 2\pi$, so the function $\cos(3x)$ must repeat itself when $3x = 2\pi$, or at $x = 2\pi/3$. So the period of y is $2\pi/3$. Here as well the constant term has no effect.
(c) The graph of y is shown in the figure below.



25. (a) Since $f(x)$ is an odd polynomial with a positive leading coefficient, it follows that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.
(b) Since $f(x)$ is an even polynomial with negative leading coefficient, it follows that $f(x) \to -\infty$ as $x \to \pm\infty$.
(c) As $x \to \pm\infty$, $x^4 \to \infty$, so $x^{-4} = 1/x^4 \to 0$.
(d) As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in its numerator and denominator. So as $x \to \pm\infty$, $f(x) \to 6$.

26. Exponential growth dominates power growth as $x \to \infty$, so $10 \cdot 2^x$ is larger.

27. As $x \to \infty$, $0.25x^{1/2}$ is larger than $25{,}000x^{-3}$.

28. This is a line with slope $-3/7$ and y-intercept 3, so a possible formula is

$$y = -\frac{3}{7}x + 3.$$

29. Starting with the general exponential equation $y = Ae^{kx}$, we first find that for $(0, 1)$ to be on the graph, we must have $A = 1$. Then to make $(3, 4)$ lie on the graph, we require

$$\begin{aligned} 4 &= e^{3k} \\ \ln 4 &= 3k \\ k &= \frac{\ln 4}{3} \approx 0.4621. \end{aligned}$$

Thus the equation is

$$y = e^{0.4621x}.$$

Alternatively, we can use the form $y = a^x$, in which case we find $y = (1.5874)^x$.

30. This looks like an exponential function. The y-intercept is 3 and we use the form $y = 3e^{kt}$. We substitute the point $(5, 9)$ to solve for k:

$$\begin{aligned} 9 &= 3e^{k5} \\ 3 &= e^{5k} \\ \ln 3 &= 5k \\ k &= 0.2197. \end{aligned}$$

A possible formula is

$$y = 3e^{0.2197t}.$$

Alternatively, we can use the form $y = 3a^t$, in which case we find $y = 3(1.2457)^t$.

31. $y = -kx(x+5) = -k(x^2+5x)$, where $k > 0$ is any constant.

32. Since this function has a y-intercept at $(0,2)$, we expect it to have the form $y = 2e^{kx}$. Again, we find k by forcing the other point to lie on the graph:

$$\begin{aligned} 1 &= 2e^{2k} \\ \frac{1}{2} &= e^{2k} \\ \ln\left(\frac{1}{2}\right) &= 2k \\ k &= \frac{\ln(\frac{1}{2})}{2} \approx -0.34657. \end{aligned}$$

This value is negative, which makes sense since the graph shows exponential decay. The final equation, then, is

$$y = 2e^{-0.34657x}.$$

Alternatively, we can use the form $y = 2a^x$, in which case we find $y = 2(0.707)^x$.

33. $z = 1 - \cos\theta$

34. $y = k(x+2)(x+1)(x-1) = k(x^3+2x^2-x-2)$, where $k > 0$ is any constant.

35. $x = ky(y-4) = k(y^2-4y)$, where $k > 0$ is any constant.

36. $y = 5\sin\left(\frac{\pi t}{20}\right)$

37. This looks like a fourth degree polynomial with roots at -5 and -1 and a double root at 3. The leading coefficient is negative, and so a possible formula is

$$y = -(x+5)(x+1)(x-3)^2.$$

38. This looks like a rational function. There are vertical asymptotes at $x = -2$ and $x = 2$ and so one possibility for the denominator is $x^2 - 4$. There is a horizontal asymptote at $y = 3$ and so the numerator might be $3x^2$. In addition, $y(0) = 0$ which is the case with the numerator of $3x^2$. A possible formula is

$$y = \frac{3x^2}{x^2-4}.$$

39. There are many solutions for a graph like this one. The simplest is $y = 1 - e^{-x}$, which gives the graph of $y = e^x$, flipped over the x-axis and moved up by 1. The resulting graph passes through the origin and approaches $y = 1$ as an upper bound, the two features of the given graph.

40. The graph is a sine curve which has been shifted up by 2, so $f(x) = (\sin x) + 2$.

41. This graph has period 5, amplitude 1 and no vertical shift or horizontal shift from $\sin x$, so it is given by

$$f(x) = \sin\left(\frac{2\pi}{5}x\right).$$

42. Since the denominator, x^2+1, is continuous and never zero, $g(x)$ is continuous on $[-1,1]$.

43. Since

$$h(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)},$$

we see that $h(x)$ is not defined at $x = -1$ or at $x = 1$, so $h(x)$ is not continuous on $[-1,1]$.

44. (a) $\lim_{x\to 0} f(x) = 1$.
(b) $\lim_{x\to 1} f(x)$ does not exist.
(c) $\lim_{x\to 2} f(x) = 1$.
(d) $\lim_{x\to 3^-} f(x) = 0$.

45. $f(x) = \dfrac{x^3|2x-6|}{x-3} = \begin{cases} \dfrac{x^3(2x-6)}{x-3} = 2x^3, & x > 3 \\ \dfrac{x^3(-2x+6)}{x-3} = -2x^3, & x < 3 \end{cases}$

Figure 1.116 confirms that $\lim_{x\to 3^+} f(x) = 54$ while $\lim_{x\to 3^-} f(x) = -54$; thus $\lim_{x\to 3} f(x)$ does not exist.

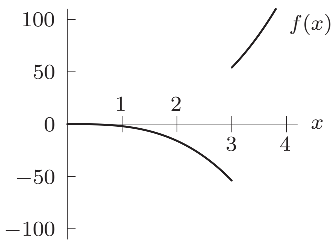


Figure 1.116

46. $f(x) = \begin{cases} e^x & -1 < x < 0 \\ 1 & x = 0 \\ \cos x & 0 < x < 1 \end{cases}$

Figure 1.117 confirms that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$, and that $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \cos x = \cos 0 = 1$, so $\lim_{x\to 0} f(x) = 1$.

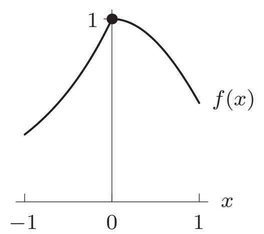


Figure 1.117

Problems

47. (a) More fertilizer increases the yield until about 40 lbs.; then it is too much and ruins crops, lowering yield.
(b) The vertical intercept is at $Y = 200$. If there is no fertilizer, then the yield is 200 bushels.
(c) The horizontal intercept is at $a = 80$. If you use 80 lbs. of fertilizer, then you will grow no apples at all.
(d) The range is the set of values of Y attainable over the domain $0 \le a \le 80$. Looking at the graph, we can see that Y goes as high as 550 and as low as 0. So the range is $0 \le Y \le 550$.
(e) Looking at the graph, we can see that Y is decreasing at $a = 60$.
(f) Looking at the graph, we can see that Y is concave down everywhere, so it is certainly concave down at $a = 40$.

48. (a) Given the two points $(0, 32)$ and $(100, 212)$, and assuming the graph in Figure 1.118 is a line,

$$\text{Slope} = \frac{212-32}{100} = \frac{180}{100} = 1.8.$$

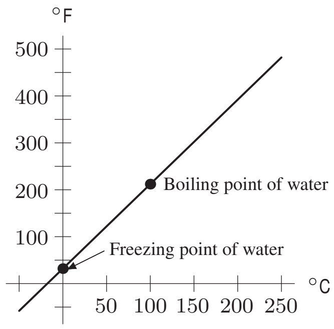


Figure 1.118

(b) The °F-intercept is $(0, 32)$, so

$$°\text{F} = 1.8(°\text{C}) + 32.$$

(c) If the temperature is 20°Celsius, then

$$°\text{F} = 1.8(20) + 32 = 68°\text{F}.$$

(d) If °F = °C, then

$$\begin{aligned} °\text{C} &= 1.8°\text{C} + 32 \\ -32 &= 0.8°\text{C} \\ °\text{C} &= -40° = °\text{F}. \end{aligned}$$

49. (a) We have the following functions.

(i) Since a change in p of \$5 results in a decrease in q of 2, the slope of $q = D(p)$ is $-2/5$ items per dollar. So

$$q = b - \frac{2}{5}p.$$

Now we know that when $p = 550$ we have $q = 100$, so

$$\begin{aligned} 100 &= b - \frac{2}{5} \cdot 550 \\ 100 &= b - 220 \\ b &= 320. \end{aligned}$$

Thus a formula is

$$q = 320 - \frac{2}{5}p.$$

(ii) We can solve $q = 320 - \frac{2}{5}p$ for p in terms of q:

$$\begin{aligned} 5q &= 1600 - 2p \\ 2p &= 1600 - 5q \\ p &= 800 - \frac{5}{2}q. \end{aligned}$$

The slope of this function is $-5/2$ dollars per item, as we would expect.

(b) A graph of $p = 800 - \dfrac{5}{2}q$ is given in Figure 1.119.

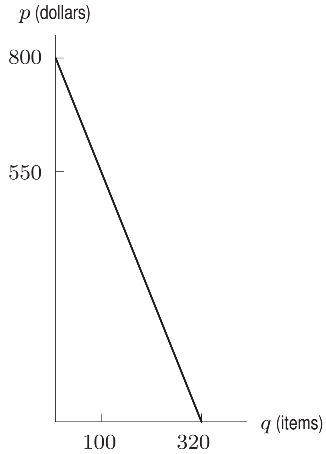


Figure 1.119

50. See Figure 1.120.

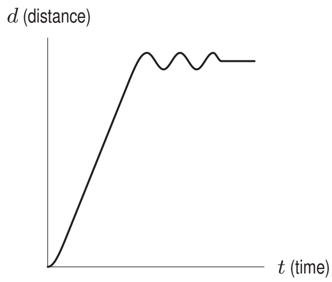


Figure 1.120

51. (a) (i) If the atoms are moved farther apart, then $r > a$ so, from the graph, F is negative, indicating an attractive force, which pulls the atoms back together.

(ii) If the atoms are moved closer together, then $r < a$ so, from the graph, F is positive, indicating an attractive force, which pushes the atoms apart again.

(b) At $r = a$, the force is zero. The answer to part (a)(i) tells us that if the atoms are pulled apart slightly, so $r > a$, the force tends to pull them back together; the answer to part (a)(ii) tells us that if the atoms are pushed together, so $r < a$, the force tends to push them back apart. Thus, $r = a$ is a stable equilibrium.

52. If the pressure at sea level is P_0, the pressure P at altitude h is given by

$$P = P_0\left(1 - \frac{0.4}{100}\right)^{h/100},$$

since we want the pressure to be multiplied by a factor of $(1 - 0.4/100) = 0.996$ for each 100 feet we go up to make it decrease by 0.4% over that interval. At Mexico City $h = 7340$, so the pressure is

$$P = P_0(0.996)^{7340/100} \approx 0.745P_0.$$

So the pressure is reduced from P_0 to approximately $0.745P_0$, a decrease of 25.5%.

53. Assuming the population of Ukraine is declining exponentially, we have population $P(t) = 45.7e^{kt}$ at time t years after 2009. Using the 2010 population, we have

$$\begin{aligned} 45.42 &= 45.7e^{-k\cdot 1} \\ k &= -\ln\left(\frac{45.42}{45.7}\right) = 0.0061. \end{aligned}$$

We want to find the time t at which

$$45 = 45.7e^{-0.0061t}$$
$$t = -\frac{\ln(45/45.7)}{0.0061} = 2.53 \text{ years.}$$

This model predicts the population to go below 45 million 2.53 years after 2009, in the year 2011.

54. (a) We compound the daily inflation rate 30 times to get the desired monthly rate r:

$$\left(1+\frac{r}{100}\right)^1 = \left(1+\frac{0.67}{100}\right)^{30} = 1.2218.$$

Solving for r, we get $r = 22.18$, so the inflation rate for April was 22.18%.

(b) We compound the daily inflation rate 365 times to get a yearly rate R for 2006:

$$\left(1+\frac{R}{100}\right)^1 = \left(1+\frac{0.67}{100}\right)^{365} = 11.4426.$$

Solving for R, we get $R = 1044.26$, so the yearly rate was 1044.26% during 2006. We could have obtained approximately the same result by compounding the monthly rate 12 times. Computing the annual rate from the monthly gives a lower result, because 12 months of 30 days each is only 360 days.

55. (a) The US consumption of hydroelectric power increased by at least 10% in 2009 and decreased by at least 10% in 2006 and in 2007, relative to each corresponding previous year. In 2009 consumption increased by 11% over consumption in 2008. In 2006 consumption decreased by 10% over consumption in 2005, and in 2007 consumption decreased by about 45% over consumption in 2006.

(b) False. In 2009 hydroelectric power consumption increased only by 11% over consumption in 2008.

(c) True. From 2006 to 2007 consumption decreased by 45.4%, which means $x(1 - 0.454)$ units of hydroelectric power were consumed in 2007 if x had been consumed in 2006. Similarly,

$$(x(1-0.454))(1+0.051)$$

units of hydroelectric power were consumed in 2008 if x had been consumed in 2006, and

$$(x(1-0.454)(1+0.051))(1+0.11)$$

units of hydroelectric power were consumed in 2009 if x had been consumed in 2006. Since

$$x(1-0.454)(1+0.051)(1+0.11) = x(0.637) = x(1-0.363),$$

the percent growth in hydroelectric power consumption was -36.3%, in 2009 relative to consumption in 2006. This amounts to about 36% decrease in hydroelectric power consumption from 2006 to 2009.

56. (a) For each 2.2 pounds of weight the object has, it has 1 kilogram of mass, so the conversion formula is

$$k = f(p) = \frac{1}{2.2}p.$$

(b) The inverse function is

$$p = 2.2k,$$

and it gives the weight of an object in pounds as a function of its mass in kilograms.

57. Since $f(x)$ is a parabola that opens upward, we have $f(x) = ax^2 + bx + c$ with $a > 0$. Since $g(x)$ is a line with negative slope, $g(x) = b + mx$, with slope $m < 0$. Therefore

$$g(f(x)) = b + m(ax^2 + bx + c) = max^2 + mbx + mc + b.$$

The coefficient of x^2 is ma, which is negative. Thus, the graph is a parabola opening downward.

58. (a) is $g(x)$ since it is linear. (b) is $f(x)$ since it has decreasing slope; the slope starts out about 1 and then decreases to about $\frac{1}{10}$. (c) is $h(x)$ since it has increasing slope; the slope starts out about $\frac{1}{10}$ and then increases to about 1.

59. Given the doubling time of 2 hours, $200 = 100e^{k(2)}$, we can solve for the growth rate k using the equation:

$$2P_0 = P_0e^{2k}$$
$$\ln 2 = 2k$$
$$k = \frac{\ln 2}{2}.$$

Using the growth rate, we wish to solve for the time t in the formula

$$P = 100e^{\frac{\ln 2}{2}t}$$

where $P = 3{,}200$, so

$$\begin{aligned} 3{,}200 &= 100e^{\frac{\ln 2}{2}t} \\ t &= 10 \text{ hours}. \end{aligned}$$

60. (a) The y-intercept of $f(x) = a\ln(x+2)$ is $f(0) = a\ln 2$. Since $\ln 2$ is positive, increasing a increases the y-intercept.
(b) The x-intercept of $f(x) = a\ln(x+2)$ is where $f(x) = 0$. Since this occurs where $x+2 = 1$, so $x = -1$, increasing a does not affect the x-intercept.

61. Since the factor by which the prices have increased after time t is given by $(1.05)^t$, the time after which the prices have doubled solves

$$\begin{aligned} 2 &= (1.05)^t \\ \log 2 &= \log(1.05^t) = t\log(1.05) \\ t &= \frac{\log 2}{\log 1.05} \approx 14.21 \text{ years}. \end{aligned}$$

62. Using the exponential decay equation $P = P_0e^{-kt}$, we can solve for the substance's decay constant k:

$$\begin{aligned} (P_0 - 0.3P_0) &= P_0e^{-20k} \\ k &= \frac{\ln(0.7)}{-20}. \end{aligned}$$

Knowing this decay constant, we can solve for the half-life t using the formula

$$\begin{aligned} 0.5P_0 &= P_0e^{\ln(0.7)t/20} \\ t &= \frac{20\ln(0.5)}{\ln(0.7)} \approx 38.87 \text{ hours}. \end{aligned}$$

63. (a) We know the decay follows the equation

$$P = P_0e^{-kt},$$

and that 10% of the pollution is removed after 5 hours (meaning that 90% is left). Therefore,

$$\begin{aligned} 0.90P_0 &= P_0e^{-5k} \\ k &= -\frac{1}{5}\ln(0.90). \end{aligned}$$

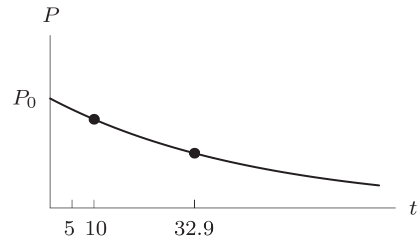
Thus, after 10 hours:

$$\begin{aligned} P &= P_0e^{-10((-0.2)\ln 0.90)} \\ P &= P_0(0.9)^2 = 0.81P_0 \end{aligned}$$

so 81% of the original amount is left.
(b) We want to solve for the time when $P = 0.50P_0$:

$$\begin{aligned} 0.50P_0 &= P_0e^{t((0.2)\ln 0.90)} \\ 0.50 &= e^{\ln(0.90^{0.2t})} \\ 0.50 &= 0.90^{0.2t} \\ t &= \frac{5\ln(0.50)}{\ln(0.90)} \approx 32.9 \text{ hours}. \end{aligned}$$

(c)



(d) When highly polluted air is filtered, there is more pollutant per liter of air to remove. If a fixed amount of air is cleaned every day, there is a higher amount of pollutant removed earlier in the process.

64. Since the amount of strontium-90 remaining halves every 29 years, we can solve for the decay constant;

$$0.5P_0 = P_0e^{-29k}$$
$$k = \frac{\ln(1/2)}{-29}.$$

Knowing this, we can look for the time t in which $P = 0.10P_0$, or

$$0.10P_0 = P_0e^{\ln(0.5)t/29}$$
$$t = \frac{29\ln(0.10)}{\ln(0.5)} = 96.336 \text{ years.}$$

65. One hour.

66. (a) V_0 represents the maximum voltage.
(b) The period is $2\pi/(120\pi) = 1/60$ second.
(c) Since each oscillation takes $1/60$ second, in 1 second there are 60 complete oscillations.

67. The US voltage has a maximum value of 156 volts and has a period of $1/60$ of a second, so it executes 60 cycles a second.
The European voltage has a higher maximum of 339 volts, and a slightly longer period of $1/50$ seconds, so it oscillates at 50 cycles per second.

68. (a) The amplitude of the sine curve is $|A|$. Thus, increasing $|A|$ stretches the curve vertically. See Figure 1.121.
(b) The period of the wave is $2\pi/|B|$. Thus, increasing $|B|$ makes the curve oscillate more rapidly—in other words, the function executes one complete oscillation in a smaller interval. See Figure 1.122.

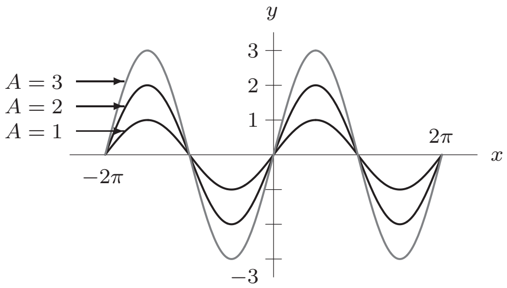


Figure 1.121

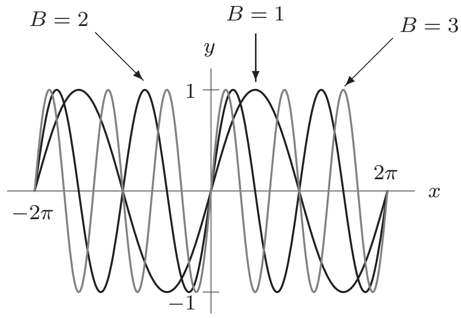


Figure 1.122

69. (a) (i) The water that has flowed out of the pipe in 1 second is a cylinder of radius r and length 3 cm. Its volume is

$$V = \pi r^2(3) = 3\pi r^2.$$

(ii) If the rate of flow is k cm/sec instead of 3 cm/sec, the volume is given by

$$V = \pi r^2(k) = \pi r^2 k.$$

(b) (i) The graph of V as a function of r is a quadratic. See Figure 1.123.

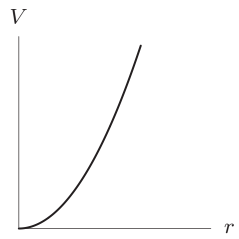


Figure 1.123

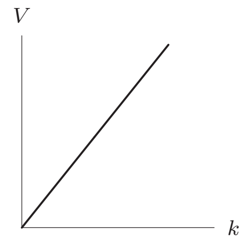


Figure 1.124

(ii) The graph of V as a function of k is a line. See Figure 1.124.

70. Looking at g, we see that the ratio of the values is:

$$\frac{3.12}{3.74} \approx \frac{3.74}{4.49} \approx \frac{4.49}{5.39} \approx \frac{5.39}{6.47} \approx \frac{6.47}{7.76} \approx 0.83.$$

Thus g is an exponential function, and so f and k are the power functions. Each is of the form ax^2 or ax^3, and since $k(1.0) = 9.01$ we see that for k, the constant coefficient is 9.01. Trial and error gives

$$k(x) = 9.01x^2,$$

since $k(2.2) = 43.61 \approx 9.01(4.84) = 9.01(2.2)^2$. Thus $f(x) = ax^3$ and we find a by noting that $f(9) = 7.29 = a(9^3)$ so

$$a = \frac{7.29}{9^3} = 0.01$$

and $f(x) = 0.01x^3$.

71. (a) See Figure 1.125.

(b) The graph is made of straight line segments, rising from the x-axis at the origin to height a at $x = 1$, b at $x = 2$, and c at $x = 3$ and then returning to the x-axis at $x = 4$. See Figure 1.126.

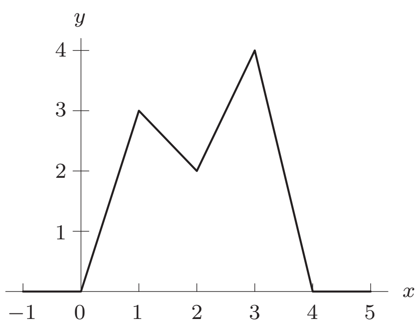


Figure 1.125

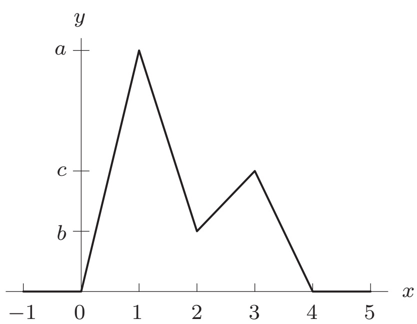


Figure 1.126

72. (a) Reading the graph of θ against t shows that $\theta \approx 5.2$ when $t = 1.5$. Since the coordinates of P are $x = 5\cos\theta$, $y = 5\sin\theta$, when $t = 1.5$ the coordinates are

$$(x, y) \approx (5\cos 5.2, 5\sin 5.2) = (2.3, -4.4).$$

(b) As t increases from 0 to 5, the angle θ increases from 0 to about 6.3 and then decreases to 0 again. Since $6.3 \approx 2\pi$, this means that P starts on the x-axis at the point $(5, 0)$, moves counterclockwise the whole way around the circle (at which time $\theta \approx 2\pi$), and then moves back clockwise to its starting point.

73. (a) III
(b) IV
(c) I
(d) II

74. The functions $y(x) = \sin x$ and $z_k(x) = ke^{-x}$ for $k = 1, 2, 4, 6, 8, 10$ are shown in Figure 1.127. The values of $f(k)$ for $k = 1, 2, 4, 6, 8, 10$ are given in Table 1.16. These values can be obtained using either tracing or a numerical root finder on a calculator or computer.

From Figure 1.127 it is clear that the smallest solution of $\sin x = ke^{-x}$ for $k = 1, 2, 4, 6$ occurs on the first period of the sine curve. For small changes in k, there are correspondingly small changes in the intersection point. For $k = 8$ and $k = 10$, the solution jumps to the second period because $\sin x < 0$ between π and 2π, but ke^{-x} is uniformly positive. Somewhere in the interval $6 \le k \le 8$, $f(k)$ has a discontinuity.

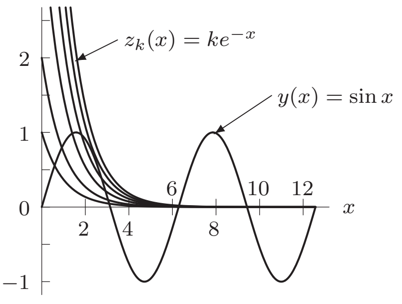


Figure 1.127

Table 1.16

k	$f(k)$
1	0.588
2	0.921
4	1.401
6	1.824
8	6.298
10	6.302

75. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.1$
For $\epsilon = 0.05$, $\delta \le 0.05$.
For $\epsilon = 0.0007$, $\delta \le 0.00007$.

76. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.45$.
For $\epsilon = 0.001$, $\delta \le 0.0447$.
For $\epsilon = 0.00001$, $\delta \le 0.00447$.

77. For any values of k, the function is continuous on any interval that does not contain $x = 2$.

Since $5x^3 - 10x^2 = 5x^2(x-2)$, we can cancel $(x-2)$ provided $x \ne 2$, giving

$$f(x) = \frac{5x^3 - 10x^2}{x-2} = 5x^2 \qquad x \ne 2.$$

Thus, if we pick $k = 5(2)^2 = 20$, the function is continuous.

78. At $x = 0$, the curve $y = k\cos x$ has $y = k\cos 0 = k$. At $x = 0$, the curve $y = e^x - k$ has $y = e^0 - k = 1 - k$. If $j(x)$ is continuous, we need

$$k = 1 - k, \quad \text{so} \quad k = \frac{1}{2}.$$

CAS Challenge Problems

79. (a) A CAS gives $f(x) = (x-a)(x+a)(x+b)(x-c)$.
(b) The graph of $f(x)$ crosses the x-axis at $x = a$, $x = -a$, $x = -b$, $x = c$; it crosses the y-axis at a^2bc. Since the coefficient of x^4 (namely 1) is positive, the graph of f looks like that shown in Figure 1.128.

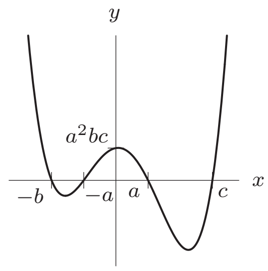


Figure 1.128: Graph of $f(x) = (x-a)(x+a)(x+b)(x-c)$

80. (a) A CAS gives $f(x) = -(x-1)^2(x-3)^3$.

(b) For large $|x|$, the graph of $f(x)$ looks like the graph of $y = -x^5$, so $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. The answer to part (a) shows that f has a double root at $x = 1$, so near $x = 1$, the graph of f looks like a parabola touching the x-axis at $x = 1$. Similarly, f has a triple root at $x = 3$. Near $x = 3$, the graph of f looks like the graph of $y = x^3$, flipped over the x-axis and shifted to the right by 3, so that the "seat" is at $x = 3$. See Figure 1.129.

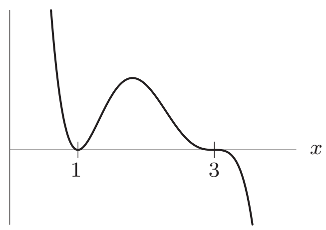


Figure 1.129: Graph of $f(x) = -(x-1)^2(x-3)^3$

81. (a) As $x \to \infty$, the term e^{6x} dominates and tends to ∞. Thus, $f(x) \to \infty$ as $x \to \infty$.
As $x \to -\infty$, the terms of the form e^{kx}, where $k = 6, 5, 4, 3, 2, 1$, all tend to zero. Thus, $f(x) \to 16$ as $x \to -\infty$.

(b) A CAS gives

$$f(x) = (e^x + 1)(e^{2x} - 2)(e^x - 2)(e^{2x} + 2e^x + 4).$$

Since e^x is always positive, the factors $(e^x + 1)$ and $(e^{2x} + 2e^x + 4)$ are never zero. The other factors each lead to a zero, so there are two zeros.

(c) The zeros are given by

$$e^{2x} = 2 \quad \text{so} \quad x = \frac{\ln 2}{2}$$
$$e^x = 2 \quad \text{so} \quad x = \ln 2.$$

Thus, one zero is twice the size of the other.

82. (a) Since $f(x) = x^2 - x$,

$$f(f(x)) = (f(x))^2 - f(x) = (x^2 - x)^2 - (x^2 - x) = x - 2\,x^3 + x^4.$$

Using the CAS to define the function $f(x)$, and then asking it to expand $f(f(f(x)))$, we get

$$f(f(f(x))) = -x + x^2 + 2\,x^3 - 5\,x^4 + 2\,x^5 + 4\,x^6 - 4\,x^7 + x^8.$$

(b) The degree of $f(f(x))$ (that is, f composed with itself 2 times) is $4 = 2^2$. The degree of $f(f(f(x)))$ (that is, f composed with itself 3 times), is $8 = 2^3$. Each time you substitute f into itself, the degree is multiplied by 2, because you are substituting in a degree 2 polynomial. So we expect the degree of $f(f(f(f(f(f(x))))))$ (that is, f composed with itself 6 times) to be $64 = 2^6$.

83. (a) A CAS or division gives

$$f(x) = \frac{x^3 - 30}{x - 3} = x^2 + 3x + 9 - \frac{3}{x-3},$$

so $p(x) = x^2 + 3x + 9$, and $r(x) = -3$, and $q(x) = x - 3$.

(b) The vertical asymptote is $x = 3$. Near $x = 3$, the values of $p(x)$ are much smaller than the values of $r(x)/q(x)$. Thus

$$f(x) \approx \frac{-3}{x-3} \qquad \text{for } x \text{ near 3.}$$

(c) For large x, the values of $p(x)$ are much larger than the value of $r(x)/q(x)$. Thus

$$f(x) \approx x^2 + 3x + 9 \qquad \text{as } x \to \infty, x \to -\infty.$$

(d) Figure 1.130 shows $f(x)$ and $y = -3/(x-3)$ for x near 3. Figure 1.131 shows $f(x)$ and $y = x^2 + 3x + 9$ for $-20 \le x \le 20$. Note that in each case the graphs of f and the approximating function are close.

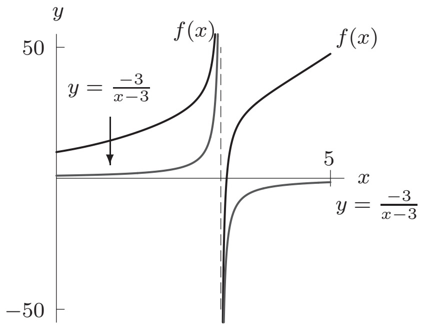


Figure 1.130: Close-up view of $f(x)$ and $y = -3/(x-3)$

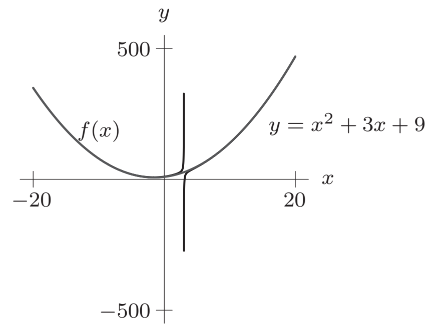


Figure 1.131: Far-away view of $f(x)$ and $y = x^2 + 3x + 9$

84. Using the trigonometric expansion capabilities of your CAS, you get something like

$$\sin(5x) = 5\cos^4(x)\,\sin(x) - 10\cos^2(x)\sin^3(x) + \sin^5(x).$$

Answers may vary. To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\sin(5x) = 5\,\sin(x)\left(1 - \sin^2(x)\right)^2 - 10\sin^3(x)\,\left(1 - \sin^2(x)\right) + \sin^5(x).$$

Finally, using the CAS to simplify,

$$\sin(5x) = 5\,\sin(x) - 20\sin^3(x) + 16\sin^5(x).$$

85. Using the trigonometric expansion capabilities of your computer algebra system, you get something like

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\sin^2(x) + \sin^4(x).$$

Answers may vary.

(a) To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\,\left(1 - \cos^2(x)\right) + \left(1 - \cos^2(x)\right)^2.$$

Finally, using the CAS to simplify,

$$\cos(4x) = 1 - 8\cos^2(x) + 8\cos^4(x).$$

(b) This time we use $\sin^2(x) = 1 - \cos^2(x)$ to get rid of powers of sine. We get

$$\cos(4x) = \left(1 - \sin^2(x)\right)^2 - 6\sin^2(x)\,\left(1 - \sin^2(x)\right) + \sin^4(x) = 1 - 8\sin^2(x) + 8\sin^4(x).$$

PROJECTS FOR CHAPTER ONE

1. Notice that whenever x increases by 0.5, $f(x)$ increases by 1, indicating that $f(x)$ is linear. By inspection, we see that $f(x) = 2x$.

Similarly, $g(x)$ decreases by 1 each time x increases by 0.5. We know, therefore, that $g(x)$ is a linear function with slope $\frac{-1}{0.5} = -2$. The y-intercept is 10, so $g(x) = 10 - 2x$.

$h(x)$ is an even function which is always positive. Comparing the values of x and $h(x)$, it appears that $h(x) = x^2$.

$F(x)$ is an odd function that seems to vary between -1 and 1. We guess that $F(x) = \sin x$ and check with a calculator.

$G(x)$ is also an odd function that varies between -1 and 1. Notice that $G(x) = F(2x)$, and thus $G(x) = \sin 2x$.

Notice also that $H(x)$ is exactly 2 more than $F(x)$ for all x, so $H(x) = 2 + \sin x$.

2. (a) Begin by finding a table of correspondences between the mathematicians' and meteorologists' angles.

θ_{met} (in degrees)	0	45	90	135	180	225	270	315
θ_{math} (in degrees)	270	225	180	135	90	45	0	315

The table is linear for $0 \leq \theta_{\text{met}} \leq 270$, with θ_{math} decreasing by 45 every time θ_{met} increases by 45, giving slope $\Delta\theta_{\text{met}}/\Delta\theta_{\text{math}} = 45/(-45) = -1$.

The interval $270 < \theta_{\text{met}} < 360$ needs a closer look. We have the following more detailed table for that interval:

θ_{met}	280	290	300	310	320	330	340	350
θ_{math}	350	340	330	320	310	300	290	280

Again the table is linear, this time with θ_{math} decreasing by 10 every time θ_{met} increases by 10, again giving slope -1. The graph of θ_{math} against θ_{met} contains two straight line sections, both of slope -1. See Figure 1.132.

(b) See Figure 1.132.

$$\theta_{\text{math}} = \begin{cases} 270 - \theta_{\text{met}} & \text{if } 0 \leq \theta_{\text{met}} \leq 270 \\ 630 - \theta_{\text{met}} & \text{if } 270 < \theta_{\text{met}} < 360 \end{cases}.$$

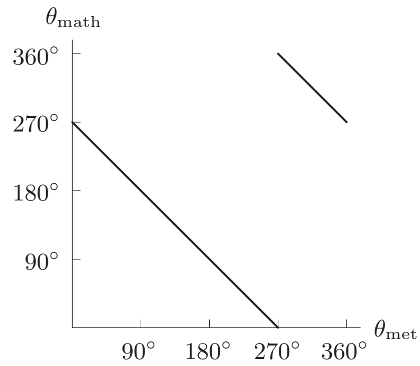


Figure 1.132

CHAPTER TWO

Solutions for Section 2.1

Exercises

1. For t between 2 and 5, we have

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{400-135}{5-2} = \frac{265}{3} \text{ km/hr.}$$

The average velocity on this part of the trip was $265/3$ km/hr.

2. The average velocity over a time period is the change in position divided by the change in time. Since the function $x(t)$ gives the position of the particle, we find the values of $x(0) = -2$ and $x(4) = -6$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta x(t)}{\Delta t} = \frac{x(4)-x(0)}{4-0} = \frac{-6-(-2)}{4} = -1 \text{ meters/sec.}$$

3. The average velocity over a time period is the change in position divided by the change in time. Since the function $x(t)$ gives the position of the particle, we find the values of $x(2) = 14$ and $x(8) = -4$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta x(t)}{\Delta t} = \frac{x(8)-x(2)}{8-2} = \frac{-4-14}{6} = -3 \text{ angstroms/sec.}$$

4. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we read off the graph that $s(0) = 1$ and $s(3) = 4$. Thus,

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3)-s(0)}{3-0} = \frac{4-1}{3} = 1 \text{ meter/sec.}$$

5. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we read off the graph that $s(1) = 2$ and $s(3) = 6$. Thus,

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3)-s(1)}{3-1} = \frac{6-2}{2} = 2 \text{ meters/sec.}$$

6. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(2) = e^2 - 1 = 6.389$ and $s(4) = e^4 - 1 = 53.598$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(4)-s(2)}{4-2} = \frac{53.598-6.389}{2} = 23.605\ \mu\text{m/sec.}$$

7. The average velocity over a time period is the change in the distance divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(\pi/3) = 4 + 3\sqrt{3}/2$ and $s(7\pi/3) = 4 + 3\sqrt{3}/2$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(7\pi/3)-s(\pi/3)}{7\pi/3-\pi/3} = \frac{4+3\sqrt{3}/2-(4+3\sqrt{3}/2)}{2\pi} = 0 \text{ cm/sec.}$$

Though the particle moves, its average velocity is zero, since it is at the same position at $t = \pi/3$ and $t = 7\pi/3$.

8. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{3.63 - 3}{0.1} = 6.3 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{3.0603 - 3}{0.01} = 6.03 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{3.006003 - 3}{0.001} = 6.003 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 6, so we estimate the instantaneous velocity at $t = 1$ to be 6 m/sec.

9. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 0$ and $t = 0.1$. We have

$$\text{Average velocity} = \frac{f(0.1) - f(0)}{0.1 - 0} = \frac{0.004 - 0}{0.1} = 0.04 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(0.01) - f(0)}{0.01 - 0} = \frac{0.000004}{0.01} = 0.0004 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(0.001) - f(0)}{1.001 - 1} = \frac{4 \times 10^{-9} - 0}{0.001} = 4 \times 10^{-6} \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 0$ the average velocity appears to be getting closer and closer to 0, so we estimate the instantaneous velocity at $t = 0$ to be 0 m/sec.

Looking at a graph of $s = f(t)$ we see that a line tangent to the graph at $t = 0$ is horizontal, confirming our result.

10. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{0.808496 - 0.909297}{0.1} = -1.00801 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{0.900793 - 0.909297}{0.01} = -0.8504 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{0.908463 - 0.909297}{0.001} = -0.834 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to -0.83, so we estimate the instantaneous velocity at $t = 1$ to be -0.83 m/sec. In this case, more estimates with smaller values of h would be very helpful in making a better estimate.

11. See Figure 2.1.

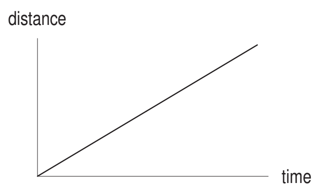


Figure 2.1

12. See Figure 2.2.

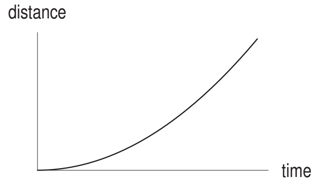


Figure 2.2

13. See Figure 2.3.

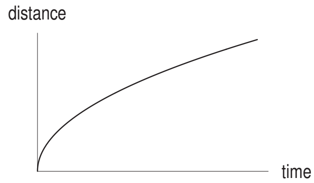


Figure 2.3

Problems

14. Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{(3+0.1)^3-27}{0.1} = 27.91$$
$$\frac{(3+0.01)^3-27}{0.01} = 27.09$$
$$\frac{(3+0.001)^3-27}{0.001} = 27.009.$$

These calculations suggest that $\lim_{h\to 0} \frac{(3+h)^3-27}{h} = 27.$

15. Using radians,

h	$(\cos h - 1)/h$
0.01	-0.005
0.001	-0.0005
0.0001	-0.00005

These values suggest that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$

16. Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{7^{0.1}-1}{0.1} = 2.148$$
$$\frac{7^{0.01}-1}{0.01} = 1.965$$
$$\frac{7^{0.001}-1}{0.001} = 1.948$$
$$\frac{7^{0.0001}-1}{0.0001} = 1.946.$$

This suggests that $\lim_{h\to 0} \frac{7^h-1}{h} \approx 1.9.$

17. Using $h = 0.1, 0.01, 0.001$, we see

h	$(e^{1+h} - e)/h$
0.01	2.7319
0.001	2.7196
0.0001	2.7184

These values suggest that $\lim_{h\to 0} \dfrac{e^{1+h} - e}{h} = 2.7$. In fact, this limit is e.

18.

Slope	-3	-1	0	$1/2$	1	2
Point	F	C	E	A	B	D

19. The slope is positive at A and D; negative at C and F. The slope is most positive at A; most negative at F.

20. $0 <$ slope at $C <$ slope at $B <$ slope of $AB < 1 <$ slope at A. (Note that the line $y = x$, has slope 1.)

21. Since $f(t)$ is concave down between $t = 1$ and $t = 3$, the average velocity between the two times should be less than the instantaneous velocity at $t = 1$ but greater than the instantaneous velocity at time $t = 3$, so $D < A < C$. For analogous reasons, $F < B < E$. Finally, note that f is decreasing at $t = 5$ so $E < 0$, but increasing at $t = 0$, so $D > 0$. Therefore, the ordering from smallest to greatest of the given quantities is

$$F < B < E < 0 < D < A < C.$$

22.

$$\begin{pmatrix}\text{Average velocity} \\ 0 < t < 0.2\end{pmatrix} = \frac{s(0.2) - s(0)}{0.2 - 0} = \frac{0.5}{0.2} = 2.5 \text{ ft/sec.}$$

$$\begin{pmatrix}\text{Average velocity} \\ 0.2 < t < 0.4\end{pmatrix} = \frac{s(0.4) - s(0.2)}{0.4 - 0.2} = \frac{1.3}{0.2} = 6.5 \text{ ft/sec.}$$

A reasonable estimate of the velocity at $t = 0.2$ is the average: $\frac{1}{2}(6.5 + 2.5) = 4.5$ ft/sec.

23. One possibility is shown in Figure 2.4.

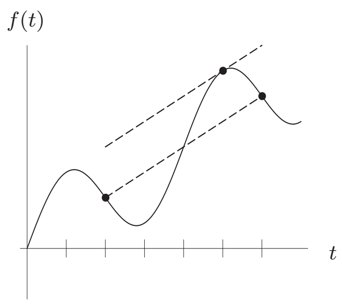


Figure 2.4

24. (a) When $t = 0$, the ball is on the bridge and its height is $f(0) = 36$, so the bridge is 36 feet above the ground.

(b) After 1 second, the ball's height is $f(1) = -16 + 50 + 36 = 70$ feet, so it traveled $70 - 36 = 34$ feet in 1 second, and its average velocity was 34 ft/sec.

(c) At $t = 1.001$, the ball's height is $f(1.001) = 70.017984$ feet, and its velocity about $\frac{70.017984-70}{1.001-1} = 17.984 \approx 18$ ft/sec.

(d) We complete the square:

$$\begin{aligned} f(t) &= -16t^2 + 50t + 36 \\ &= -16\left(t^2 - \frac{25}{8}t\right) + 36 \\ &= -16\left(t^2 - \frac{25}{8}t + \frac{625}{256}\right) + 36 + 16\left(\frac{625}{256}\right) \\ &= -16(t - \tfrac{25}{16})^2 + \tfrac{1201}{16} \end{aligned}$$

so the graph of f is a downward parabola with vertex at the point $(25/16, 1201/16) = (1.6, 75.1)$. We see from Figure 2.5 that the ball reaches a maximum height of about 75 feet. The velocity of the ball is zero when it is at the peak, since the tangent is horizontal there.

(e) The ball reaches its maximum height when $t = \frac{25}{16} = 1.6$.

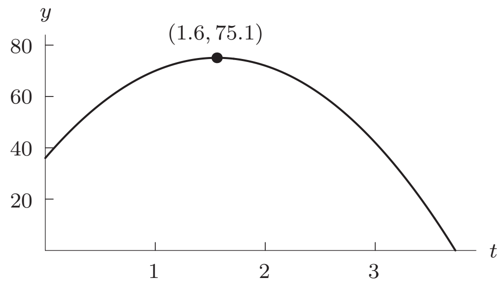


Figure 2.5

25. $\lim_{h\to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h\to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h\to 0}(4 + h) = 4$

26. $\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim_{h\to 0} \frac{h(3 + 3h + h^2)}{h} = \lim_{h\to 0} 3 + 3h + h^2 = 3.$

27. $\lim_{h\to 0} \frac{3(2+h)^2 - 12}{h} = \lim_{h\to 0} \frac{12 + 12h + 3h^2 - 12}{h} = \lim_{h\to 0} \frac{h(12 + 3h)}{h} = \lim_{h\to 0} 12 + 3h = 12.$

28. $\lim_{h\to 0} \frac{(3+h)^2 - (3-h)^2}{2h} = \lim_{h\to 0} \frac{9 + 6h + h^2 - 9 + 6h - h^2}{2h} = \lim_{h\to 0} \frac{12h}{2h} = \lim_{h\to 0} 6 = 6.$

Strengthen Your Understanding

29. Speed is the magnitude of velocity, so it is always positive or zero; velocity has both magnitude and direction.

30. We expand and simplify first

$$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{(4 + 4h + h^2) - 4}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0}(4 + h) = 4.$$

31. Since the tangent line to the curve at $t = 4$ is almost horizontal, the instantaneous velocity is almost zero. At $t = 2$ the slope of the tangent line, and hence the instantaneous velocity, is relatively large and positive.

32. $f(t) = t^2$. The slope of the graph of $y = f(t)$ is negative for $t < 0$ and positive for $t > 0$.
Many other answers are possible.

33. One possibility is the position function $s(t) = t^2$. Any function that is symmetric about the line $t = 0$ works.
For $s(t) = t^2$, the slope of a tangent line (representing the velocity) is negative at $t = -1$ and positive at $t = 1$, and that the magnitude of the slopes (the speeds) are the same.

34. False. For example, the car could slow down or even stop at one minute after 2 pm, and then speed back up to 60 mph at one minute before 3 pm. In this case the car would travel only a few miles during the hour, much less than 50 miles.

35. False. Its average velocity for the time between 2 pm and 4 pm is 40 mph, but the car could change its speed a lot during that time period. For example, the car might be motionless for an hour then go 80 mph for the second hour. In that case the velocity at 2 pm would be 0 mph.

36. True. During a short enough time interval the car can not change its velocity very much, and so it velocity will be nearly constant. It will be nearly equal to the average velocity over the interval.

37. True. The instantaneous velocity is a limit of the average velocities. The limit of a constant equals that constant.

38. True. By definition, Average velocity = Distance traveled/Time.

39. False. Instantaneous velocity equals a *limit* of difference quotients.

Solutions for Section 2.2

Exercises

1. The derivative, $f'(2)$, is the rate of change of x^3 at $x = 2$. Notice that each time x changes by 0.001 in the table, the value of x^3 changes by 0.012. Therefore, we estimate

$$f'(2) = \begin{array}{c}\text{Rate of change}\\ \text{of } f \text{ at } x = 2\end{array} \approx \frac{0.012}{0.001} = 12.$$

The function values in the table look exactly linear because they have been rounded. For example, the exact value of x^3 when $x = 2.001$ is 8.012006001, not 8.012. Thus, the table can tell us only that the derivative is approximately 12. Example 5 on page 95 shows how to compute the derivative of $f(x)$ exactly.

2. With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{f(1.01) - f(1)}{0.01} = 3.0301 \quad \text{and} \quad \frac{f(0.99) - f(1)}{-0.01} = 2.9701.$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{f(1.001) - f(1)}{0.001} = 3.003001 \quad \text{and} \quad \frac{f(0.999) - f(1)}{-0.001} = 2.997001.$$

The values of these difference quotients suggest that the limit is about 3.0. We say

$$f'(1) = \begin{array}{c}\text{Instantaneous rate of change of } f(x) = x^3\\ \text{with respect to } x \text{ at } x = 1\end{array} \approx 3.0.$$

3. (a) Using the formula for the average rate of change gives

$$\begin{array}{c}\text{Average rate of change}\\ \text{of revenue for } 1 \le q \le 2\end{array} = \frac{R(2) - R(1)}{1} = \frac{160 - 90}{1} = 70 \text{ dollars/kg}.$$

$$\begin{array}{c}\text{Average rate of change}\\ \text{of revenue for } 2 \le q \le 3\end{array} = \frac{R(3) - R(2)}{1} = \frac{210 - 160}{1} = 50 \text{ dollars/kg}.$$

So we see that the average rate decreases as the quantity sold in kilograms increases.

(b) With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{R(2.01) - R(2)}{0.01} = 59.9 \text{ dollars/kg} \quad \text{and} \quad \frac{R(1.99) - R(2)}{-0.01} = 60.1 \text{ dollars/kg}.$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{R(2.001) - R(2)}{0.001} = 59.99 \text{ dollars/kg} \quad \text{and} \quad \frac{R(1.999) - R(2)}{-0.001} = 60.01 \text{ dollars/kg}.$$

The values of these difference quotients suggest that the instantaneous rate of change is about 60 dollars/kg. To confirm that the value is exactly 60, that is, that $R'(2) = 60$, we would need to take the limit as $h \to 0$.

4. (a) Using a calculator we obtain the values found in the table below:

x	1	1.5	2	2.5	3
e^x	2.72	4.48	7.39	12.18	20.09

(b) The average rate of change of $f(x) = e^x$ between $x = 1$ and $x = 3$ is

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1} = \frac{e^3 - e}{3 - 1} \approx \frac{20.09 - 2.72}{2} = 8.69.$$

(c) First we find the average rates of change of $f(x) = e^x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$:

$$\text{Average rate of change} = \frac{f(2) - f(1.5)}{2 - 1.5} = \frac{e^2 - e^{1.5}}{2 - 1.5} \approx \frac{7.39 - 4.48}{0.5} = 5.82$$

$$\text{Average rate of change} = \frac{f(2.5) - f(2)}{2.5 - 2} = \frac{e^{2.5} - e^2}{2.5 - 2} \approx \frac{12.18 - 7.39}{0.5} = 9.58.$$

Now we approximate the instantaneous rate of change at $x = 2$ by averaging these two rates:

$$\text{Instantaneous rate of change} \approx \frac{5.82 + 9.58}{2} = 7.7.$$

5. (a)

Table 2.1

x	1	1.5	2	2.5	3
$\log x$	0	0.18	0.30	0.40	0.48

(b) The average rate of change of $f(x) = \log x$ between $x = 1$ and $x = 3$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\log 3 - \log 1}{3 - 1} \approx \frac{0.48 - 0}{2} = 0.24$$

(c) First we find the average rates of change of $f(x) = \log x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$.

$$\frac{\log 2 - \log 1.5}{2 - 1.5} = \frac{0.30 - 0.18}{0.5} \approx 0.24$$

$$\frac{\log 2.5 - \log 2}{2.5 - 2} = \frac{0.40 - 0.30}{0.5} \approx 0.20$$

Now we approximate the instantaneous rate of change at $x = 2$ by finding the average of the above rates, i.e.

$$\begin{pmatrix} \text{the instantaneous rate of change} \\ \text{of } f(x) = \log x \text{ at } x = 2 \end{pmatrix} \approx \frac{0.24 + 0.20}{2} = 0.22.$$

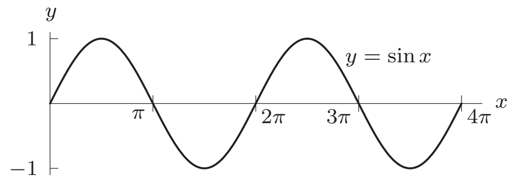
6. In Table 2.2, each x increase of 0.001 leads to an increase in $f(x)$ by about 0.031, so

$$f'(3) \approx \frac{0.031}{0.001} = 31.$$

Table 2.2

x	2.998	2.999	3.000	3.001	3.002
$x^3 + 4x$	38.938	38.969	39.000	39.031	39.062

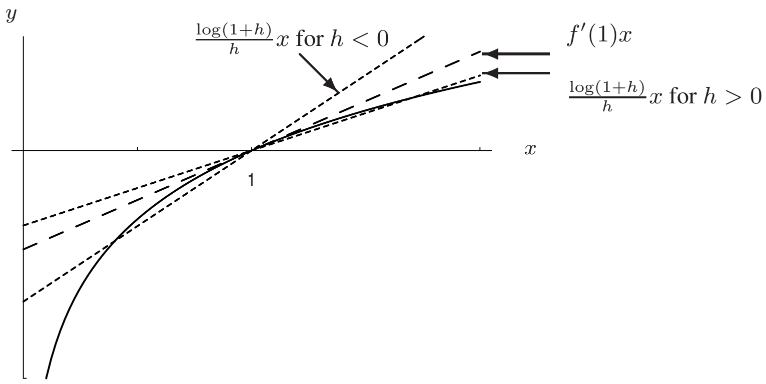
7.



Since $\sin x$ is decreasing for values near $x = 3\pi$, its derivative at $x = 3\pi$ is negative.

8. $f'(1) = \lim_{h\to 0} \dfrac{\log(1+h) - \log 1}{h} = \lim_{h\to 0} \dfrac{\log(1+h)}{h}$

Evaluating $\frac{\log(1+h)}{h}$ for $h = 0.01, 0.001$, and 0.0001, we get $0.43214, 0.43408, 0.43427$, so $f'(1) \approx 0.43427$. The corresponding secant lines are getting steeper, because the graph of $\log x$ is concave down. We thus expect the limit to be more than 0.43427 . If we consider negative values of h, the estimates are too large. We can also see this from the graph below:



9. We estimate $f'(2)$ using the average rate of change formula on a small interval around 2. We use the interval $x = 2$ to $x = 2.001$. (Any small interval around 2 gives a reasonable answer.) We have

$$f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{3^{2.001} - 3^2}{2.001 - 2} = \frac{9.00989 - 9}{0.001} = 9.89.$$

10. (a) The average rate of change from $x = a$ to $x = b$ is the slope of the line between the points on the curve with $x = a$ and $x = b$. Since the curve is concave down, the line from $x = 1$ to $x = 3$ has a greater slope than the line from $x = 3$ to $x = 5$, and so the average rate of change between $x = 1$ and $x = 3$ is greater than that between $x = 3$ and $x = 5$.

(b) Since f is increasing, $f(5)$ is the greater.

(c) As in part (a), f is concave down and f' is decreasing throughout so $f'(1)$ is the greater.

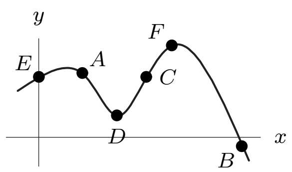
11. Since $f'(x) = 0$ where the graph is horizontal, $f'(x) = 0$ at $x = d$. The derivative is positive at points b and c, but the graph is steeper at $x = c$. Thus $f'(x) = 0.5$ at $x = b$ and $f'(x) = 2$ at $x = c$. Finally, the derivative is negative at points a and e but the graph is steeper at $x = e$. Thus, $f'(x) = -0.5$ at $x = a$ and $f'(x) = -2$ at $x = e$. See Table 2.3.

Thus, we have $f'(d) = 0, f'(b) = 0.5, f'(c) = 2, f'(a) = -0.5, f'(e) = -2$.

Table 2.3

x	$f'(x)$
d	0
b	0.5
c	2
a	−0.5
e	−2

12. One possible choice of points is shown below.



Problems

13. The statements $f(100) = 35$ and $f'(100) = 3$ tell us that at $x = 100$, the value of the function is 35 and the function is increasing at a rate of 3 units for a unit increase in x. Since we increase x by 2 units in going from 100 to 102, the value of the function goes up by approximately $2 \cdot 3 = 6$ units, so

$$f(102) \approx 35 + 2 \cdot 3 = 35 + 6 = 41.$$

14. The answers to parts (a)–(d) are shown in Figure 2.6.

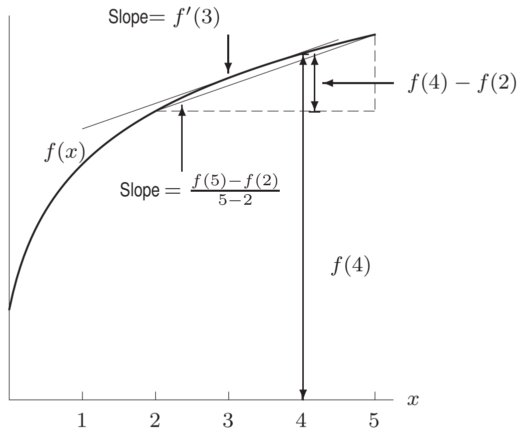


Figure 2.6

15. (a) Since f is increasing, $f(4) > f(3)$.

(b) From Figure 2.7, it appears that $f(2) - f(1) > f(3) - f(2)$.

(c) The quantity $\dfrac{f(2) - f(1)}{2 - 1}$ represents the slope of the secant line connecting the points on the graph at $x = 1$ and $x = 2$. This is greater than the slope of the secant line connecting the points at $x = 1$ and $x = 3$ which is $\dfrac{f(3) - f(1)}{3 - 1}$.

(d) The function is steeper at $x = 1$ than at $x = 4$ so $f'(1) > f'(4)$.

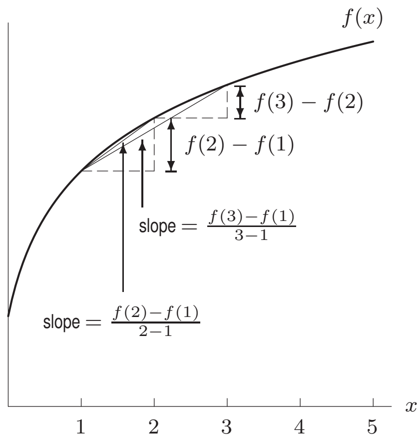


Figure 2.7

16. Figure 2.8 shows the quantities in which we are interested.

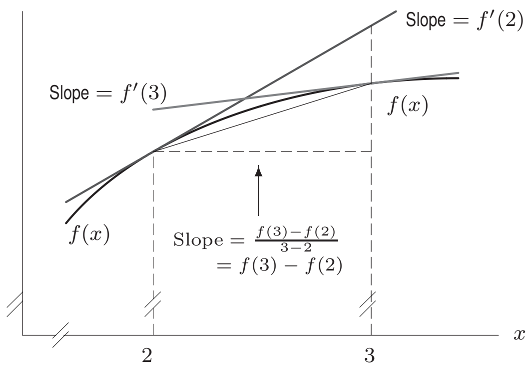


Figure 2.8

The quantities $f'(2)$, $f'(3)$ and $f(3) - f(2)$ have the following interpretations:

- $f'(2)$ = slope of the tangent line at $x = 2$
- $f'(3)$ = slope of the tangent line at $x = 3$
- $f(3) - f(2) = \frac{f(3)-f(2)}{3-2}$ = slope of the secant line from $f(2)$ to $f(3)$.

From Figure 2.8, it is clear that $0 < f(3) - f(2) < f'(2)$. By extending the secant line past the point $(3, f(3))$, we can see that it lies above the tangent line at $x = 3$.

Thus

$$0 < f'(3) < f(3) - f(2) < f'(2).$$

17. The coordinates of A are $(4, 25)$. See Figure 2.9. The coordinates of B and C are obtained using the slope of the tangent line. Since $f'(4) = 1.5$, the slope is 1.5

From A to B, $\Delta x = 0.2$, so $\Delta y = 1.5(0.2) = 0.3$. Thus, at C we have $y = 25 + 0.3 = 25.3$. The coordinates of B are $(4.2, 25.3)$.

From A to C, $\Delta x = -0.1$, so $\Delta y = 1.5(-0.1) = -0.15$. Thus, at C we have $y = 25 - 0.15 = 24.85$. The coordinates of C are $(3.9, 24.85)$.

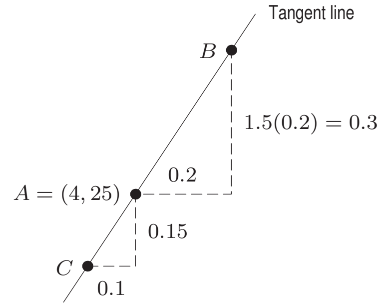


Figure 2.9

18. (a) Since the point $B = (2, 5)$ is on the graph of g, we have $g(2) = 5$.
(b) The slope of the tangent line touching the graph at $x = 2$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{5 - 5.02}{2 - 1.95} = \frac{-0.02}{0.05} = -0.4.$$

Thus, $g'(2) = -0.4$.

19. See Figure 2.10.

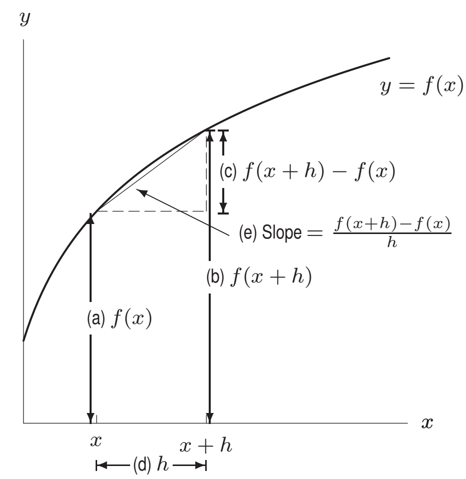


Figure 2.10

20. See Figure 2.11.

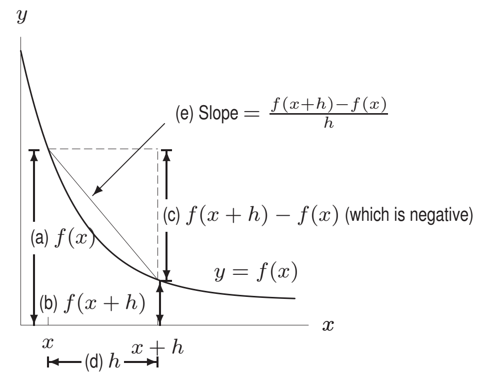


Figure 2.11

21. (a) For the line from A to B,

$$\text{Slope} = \frac{f(b) - f(a)}{b - a}.$$

(b) The tangent line at point C appears to be parallel to the line from A to B. Assuming this to be the case, the lines have the same slope.

(c) There is only one other point, labeled D in Figure 2.12, at which the tangent line is parallel to the line joining A and B.

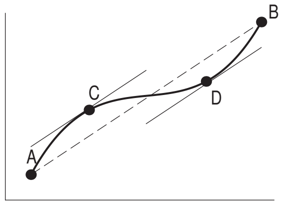


Figure 2.12

22. (a) Figure 2.13 shows the graph of an even function. We see that since f is symmetric about the y-axis, the tangent line at $x = -10$ is just the tangent line at $x = 10$ flipped about the y-axis, so the slope of one tangent is the negative of that of the other. Therefore, $f'(-10) = -f'(10) = -6$.

(b) From part (a) we can see that if f is even, then for any x, we have $f'(-x) = -f'(x)$. Thus $f'(-0) = -f'(0)$, so $f'(0) = 0$.

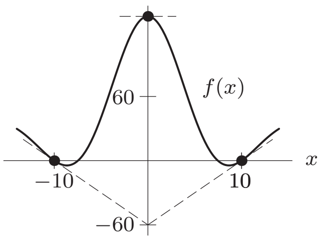


Figure 2.13

23. Figure 2.14 shows the graph of an odd function. We see that since g is symmetric about the origin, its tangent line at $x = -4$ is just the tangent line at $x = 4$ flipped about the origin, so they have the same slope. Thus, $g'(-4) = 5$.

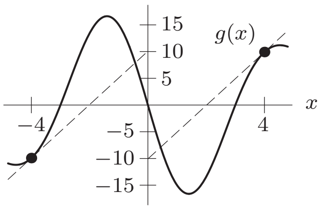


Figure 2.14

24. (a)

$$f'(0) = \lim_{h\to 0} \frac{\overbrace{\sin h}^{h \text{ in degrees}} - \overbrace{\sin 0}^{0}}{h} = \frac{\sin h}{h}.$$

To four decimal places,

$$\frac{\sin 0.2}{0.2} \approx \frac{\sin 0.1}{0.1} \approx \frac{\sin 0.01}{0.01} \approx \frac{\sin 0.001}{0.001} \approx 0.01745$$

so $f'(0) \approx 0.01745$.

(b) Consider the ratio $\frac{\sin h}{h}$. As we approach 0, the numerator, $\sin h$, will be much smaller in magnitude if h is in degrees than it would be if h were in radians. For example, if $h = 1^\circ$ radian, $\sin h = 0.8415$, but if $h = 1$ degree, $\sin h = 0.01745$. Thus, since the numerator is smaller for h measured in degrees while the denominator is the same, we expect the ratio $\frac{\sin h}{h}$ to be smaller.

25. We find the derivative using a difference quotient:

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2+3+h-(3^2+3)}{h}$$
$$= \lim_{h\to 0} \frac{9+6h+h^2+3+h-9-3}{h} = \lim_{h\to 0} \frac{7h+h^2}{h} = \lim_{h\to 0}(7+h) = 7.$$

Thus at $x = 3$, the slope of the tangent line is 7. Since $f(3) = 3^2 + 3 = 12$, the line goes through the point $(3, 12)$, and therefore its equation is

$$y - 12 = 7(x-3) \quad \text{or} \quad y = 7x - 9.$$

The graph is in Figure 2.15.

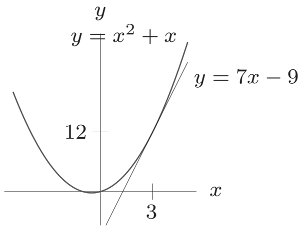


Figure 2.15

26. Using a difference quotient with $h = 0.001$, say, we find

$$f'(1) \approx \frac{1.001\ln(1.001) - 1\ln(1)}{1.001 - 1} = 1.0005$$
$$f'(2) \approx \frac{2.001\ln(2.001) - 2\ln(2)}{2.001 - 2} = 1.6934$$

The fact that f' is larger at $x = 2$ than at $x = 1$ suggests that f is concave up between $x = 1$ and $x = 2$.

27. We want $f'(2)$. The exact answer is

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^{2+h}-4}{h},$$

but we can approximate this. If $h = 0.001$, then

$$\frac{(2.001)^{2.001}-4}{0.001} \approx 6.779$$

and if $h = 0.0001$ then

$$\frac{(2.0001)^{2.0001}-4}{0.0001} \approx 6.773,$$

so $f'(2) \approx 6.77$.

28. Notice that we can't get all the information we want just from the graph of f for $0 \le x \le 2$, shown on the left in Figure 2.16. Looking at this graph, it looks as if the slope at $x = 0$ is 0. But if we zoom in on the graph near $x = 0$, we get the graph of f for $0 \le x \le 0.05$, shown on the right in Figure 2.16. We see that f does dip down quite a bit between $x = 0$ and $x \approx 0.11$. In fact, it now looks like $f'(0)$ is around -1. Note that since $f(x)$ is undefined for $x < 0$, this derivative only makes sense as we approach zero from the right.

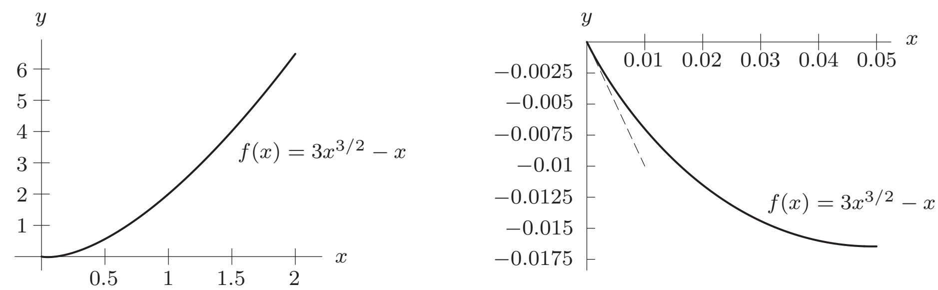


Figure 2.16

We zoom in on the graph of f near $x = 1$ to get a more accurate picture from which to estimate $f'(1)$. A graph of f for $0.7 \leq x \leq 1.3$ is shown in Figure 2.17. [Keep in mind that the axes shown in this graph don't cross at the origin!] Here we see that $f'(1) \approx 3.5$.

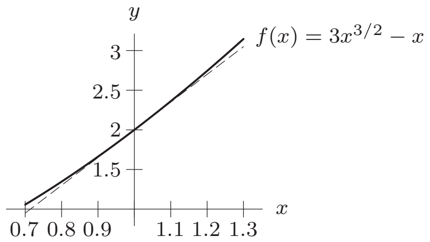


Figure 2.17

29.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(\cos(1+h)) - \ln(\cos 1)}{h}$$

For $h = 0.001$, the difference quotient $= -1.55912$; for $h = 0.0001$, the difference quotient $= -1.55758$. The instantaneous rate of change of f therefore appears to be about -1.558 at $x = 1$.
At $x = \frac{\pi}{4}$, if we try $h = 0.0001$, then

$$\text{difference quotient} = \frac{\ln[\cos(\frac{\pi}{4} + 0.0001)] - \ln(\cos\frac{\pi}{4})}{0.0001} \approx -1.0001.$$

The instantaneous rate of change of f appears to be about -1 at $x = \frac{\pi}{4}$.

30. The quantity $f(0)$ represents the population on October 17, 2006, so $f(0) = 300$ million.

The quantity $f'(0)$ represents the rate of change of the population (in millions per year). Since

$$\frac{1 \text{ person}}{11 \text{ seconds}} = \frac{1/10^6 \text{ million people}}{11/(60 \cdot 60 \cdot 24 \cdot 365) \text{ years}} = 2.867 \text{ million people/year},$$

so we have $f'(0) = 2.867$.

31. We want to approximate $P'(0)$ and $P'(7)$. Since for small h

$$P'(0) \approx \frac{P(h) - P(0)}{h},$$

if we take $h = 0.01$, we get

$$\begin{aligned} P'(0) &\approx \frac{1.267(1.007)^{0.01} - 1.267}{0.01} = 0.00884 \text{ billion/year} \\ &= 8.84 \text{ million people/year in 2000}, \\ P'(7) &\approx \frac{1.267(1.007)^{7.01} - 1.267(1.007)^{7}}{0.01} = 0.00928 \text{ billion/year} \\ &= 9.28 \text{ million people/year in 2007} \end{aligned}$$

32. (a) From Figure 2.18, it appears that the slopes of the tangent lines to the two graphs are the same at each x. For $x = 0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ at 0 are

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h)-0}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \frac{1}{2}h \\ &= 0, \end{aligned}$$

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} \\ &= \lim_{h\to 0} \frac{g(h)-g(0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}h^2+3-3}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \frac{1}{2}h \\ &= 0. \end{aligned}$$

For $x = 2$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2-\frac{1}{2}(2)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2)-2}{h} \\ &= \lim_{h\to 0} \frac{2+2h+\frac{1}{2}h^2-2}{h} \\ &= \lim_{h\to 0} \frac{2h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(2+\frac{1}{2}h\right) \\ &= 2, \end{aligned}$$

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2+3-(\frac{1}{2}(2)^2+3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2-\frac{1}{2}(2)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2)-2}{h} \\ &= \lim_{h\to 0} \frac{2+2h+\frac{1}{2}(h^2)-2}{h} \\ &= \lim_{h\to 0} \frac{2h+\frac{1}{2}(h^2)}{h} \\ &= \lim_{h\to 0} \left(2+\frac{1}{2}h\right) \\ &= 2. \end{aligned}$$

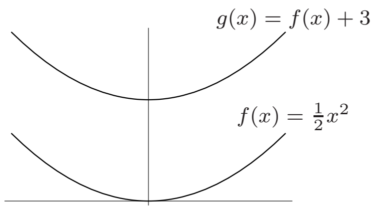


Figure 2.18

For $x = x_0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2+2x_0h+h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0 + \frac{1}{2}h\right) \\ &= x_0, \end{aligned}$$

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 + 3 - (\frac{1}{2}(x_0)^2+3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}(x_0)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2+2x_0h+h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0 + \frac{1}{2}h\right) \\ &= x_0. \end{aligned}$$

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)+C-(f(x)+C)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= f'(x). \end{aligned}$$

33. As h gets smaller, round-off error becomes important. When $h = 10^{-12}$, the quantity $2^h - 1$ is so close to 0 that the calculator rounds off the difference to 0, making the difference quotient 0. The same thing will happen when $h = 10^{-20}$.

34. (a) Table 2.4 shows that near $x = 1$, every time the value of x increases by 0.001, the value of x^2 increases by approximately 0.002. This suggests that

$$f'(1) \approx \frac{0.002}{0.001} = 2.$$

Table 2.4 *Values of $f(x) = x^2$ near $x = 1$*

x	x^2	Difference in successive x^2 values
0.998	0.996004	
		0.001997
0.999	0.998001	
		0.001999
1.000	1.000000	
		0.002001
1.001	1.002001	
		0.002003
1.002	1.004004	
↑ x increments of 0.001		↑ All approximately 0.002

(b) The derivative is the limit of the difference quotient, so we look at

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}.$$

Using the formula for f, we have

$$f'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{(1+2h+h^2)-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h}.$$

Since the limit only examines values of h close to, but not equal to zero, we can cancel h in the expression $(2h + h^2)/h$. We get

$$f'(1) = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0}(2+h).$$

This limit is 2, so $f'(1) = 2$. At $x = 1$ the rate of change of x^2 is 2.

(c) Since the derivative is the rate of change, $f'(1) = 2$ means that for small changes in x near $x = 1$, the change in $f(x) = x^2$ is about twice as big as the change in x. As an example, if x changes from 1 to 1.1, a net change of 0.1, then $f(x)$ changes by about 0.2. Figure 2.19 shows this geometrically. Near $x = 1$ the function is approximately linear with slope of 2.

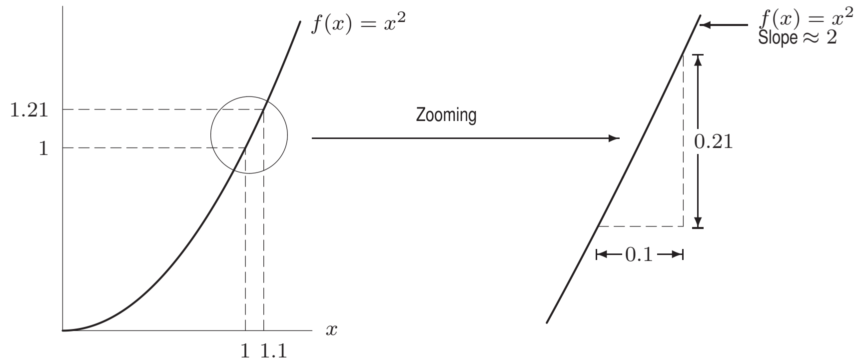


Figure 2.19: Graph of $f(x) = x^2$ near $x = 1$ has slope ≈ 2

35. $\lim_{h\to 0} \frac{(-3+h)^2 - 9}{h} = \lim_{h\to 0} \frac{9 - 6h + h^2 - 9}{h} = \lim_{h\to 0} \frac{h(-6+h)}{h} = \lim_{h\to 0} -6 + h = -6.$

36. $\lim_{h\to 0} \frac{(2-h)^3 - 8}{h} = \lim_{h\to 0} \frac{8 - 12h + 6h^2 - h^3 - 8}{h} = \lim_{h\to 0} \frac{h(-12 + 6h - h^2)}{h} = \lim_{h\to 0} -12 + 6h - h^2 = -12.$

37. $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{1+h} - 1\right) = \lim_{h\to 0} \frac{1-(1+h)}{(1+h)h} = \lim_{h\to 0} \frac{-1}{1+h} = -1$

38. $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(1+h)^2} - 1\right) = \lim_{h\to 0} \frac{1-(1+2h+h^2)}{h(1+h)^2} = \lim_{h\to 0} \frac{-2-h}{(1+h)^2} = -2$

39. $\sqrt{4+h} - 2 = \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{\sqrt{4+h}+2} = \frac{4+h-4}{\sqrt{4+h}+2} = \frac{h}{\sqrt{4+h}+2}.$

Therefore $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$

40. $\frac{1}{\sqrt{4+h}} - \frac{1}{2} = \frac{2-\sqrt{4+h}}{2\sqrt{4+h}} = \frac{(2-\sqrt{4+h})(2+\sqrt{4+h})}{2\sqrt{4+h}(2+\sqrt{4+h})} = \frac{4-(4+h)}{2\sqrt{4+h}(2+\sqrt{4+h})}.$

Therefore $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{4+h}} - \frac{1}{2}\right) = \lim_{h\to 0} \frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} = -\frac{1}{16}$

41. Using the definition of the derivative, we have

$$\begin{aligned} f'(10) &= \lim_{h\to 0} \frac{f(10+h) - f(10)}{h} \\ &= \lim_{h\to 0} \frac{5(10+h)^2 - 5(10)^2}{h} \\ &= \lim_{h\to 0} \frac{500 + 100h + 5h^2 - 500}{h} \\ &= \lim_{h\to 0} \frac{100h + 5h^2}{h} \\ &= \lim_{h\to 0} \frac{h(100+5h)}{h} \\ &= \lim_{h\to 0} 100 + 5h \\ &= 100. \end{aligned}$$

42. Using the definition of the derivative, we have

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} \\
&= \lim_{h\to 0} \frac{(-2+h)^3-(-2)^3}{h} \\
&= \lim_{h\to 0} \frac{(-8+12h-6h^2+h^3)-(-8)}{h} \\
&= \lim_{h\to 0} \frac{12h-6h^2+h^3}{h} \\
&= \lim_{h\to 0} \frac{h(12-6h+h^2)}{h} \\
&= \lim_{h\to 0} (12-6h+h^2),
\end{aligned}$$

which goes to 12 as $h \to 0$. So $f'(-2) = 12$.

43. Using the definition of the derivative

$$\begin{aligned}
g'(-1) &= \lim_{h\to 0} \frac{g(-1+h)-g(-1)}{h} \\
&= \lim_{h\to 0} \frac{((-1+h)^2+(-1+h))-((-1)^2+(-1))}{h} \\
&= \lim_{h\to 0} \frac{(1-2h+h^2-1+h)-(0)}{h} \\
&= \lim_{h\to 0} \frac{-h+h^2}{h} = \lim_{h\to 0}(-1+h) = -1.
\end{aligned}$$

44.

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{((1+h)^3+5)-(1^3+5)}{h} \\
&= \lim_{h\to 0} \frac{1+3h+3h^2+h^3+5-1-5}{h} = \lim_{h\to 0} \frac{3h+3h^2+h^3}{h} \\
&= \lim_{h\to 0}(3+3h+h^2) = 3.
\end{aligned}$$

45.

$$\begin{aligned}
g'(2) &= \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{2-(2+h)}{h(2+h)2} = \lim_{h\to 0} \frac{-h}{h(2+h)2} \\
&= \lim_{h\to 0} \frac{-1}{(2+h)2} = -\frac{1}{4}
\end{aligned}$$

46.

$$\begin{aligned}
g'(2) &= \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{(2+h)^2}-\frac{1}{2^2}}{h} \\
&= \lim_{h\to 0} \frac{2^2-(2+h)^2}{2^2(2+h)^2h} = \lim_{h\to 0} \frac{4-4-4h-h^2}{4h(2+h)^2} \\
&= \lim_{h\to 0} \frac{-4h-h^2}{4h(2+h)^2} = \lim_{h\to 0} \frac{-4-h}{4(2+h)^2} \\
&= \frac{-4}{4(2)^2} = -\frac{1}{4}.
\end{aligned}$$

47. As we saw in the answer to Problem 41, the slope of the tangent line to $f(x) = 5x^2$ at $x = 10$ is 100. When $x = 10$, $f(x) = 500$ so $(10, 500)$ is a point on the tangent line. Thus $y = 100(x - 10) + 500 = 100x - 500$.

48. As we saw in the answer to Problem 42, the slope of the tangent line to $f(x) = x^3$ at $x = -2$ is 12. When $x = -2$, $f(x) = -8$ so we know the point $(-2, -8)$ is on the tangent line. Thus the equation of the tangent line is $y = 12(x + 2) - 8 = 12x + 16$.

49. We know that the slope of the tangent line to $f(x) = x$ when $x = 20$ is 1. When $x = 20$, $f(x) = 20$ so $(20, 20)$ is on the tangent line. Thus the equation of the tangent line is $y = 1(x - 20) + 20 = x$.

50. First find the derivative of $f(x) = 1/x^2$ at $x = 1$.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1^2}}{h} \\ &= \lim_{h\to 0} \frac{1^2 - (1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1 - (1 + 2h + h^2)}{h(1+h)^2} \\ &= \lim_{h\to 0} \frac{-2h - h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2 - h}{(1+h)^2} = -2 \end{aligned}$$

Thus the tangent line has a slope of -2 and goes through the point $(1, 1)$, and so its equation is

$$y - 1 = -2(x - 1) \quad \text{or} \quad y = -2x + 3.$$

Strengthen Your Understanding

51. The graph of $f(x) = \log x$ is increasing, so $f'(0.5) > 0$.

52. The derivative of a function at a point is the slope of the tangent line, not the tangent line itself.

53. $f(x) = e^x$.

Many other answers are possible.

54. A linear function is of the form $f(x) = ax + b$. The derivative of this function is the slope of the line $y = ax + b$, so $f'(x) = a$, so $a = 2$. One such function is $f(x) = 2x + 1$.

55. True. The derivative of a function is the limit of difference quotients. A few difference quotients can be computed from the table, but the limit can not be computed from the table.

56. True. The derivative $f'(10)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point where $x = 10$. When you zoom in on $y = f(x)$ close enough it is not possible to see the difference between the tangent line and the graph of f on the calculator screen. The line you see on the calculator is a little piece of the tangent line, so its slope is the derivative $f'(10)$.

57. True. This is seen graphically. The derivative $f'(a)$ is the slope of the line tangent to the graph of f at the point P where $x = a$. The difference quotient $(f(b) - f(a))/(b - a)$ is the slope of the secant line with endpoints on the graph of f at the points where $x = a$ and $x = b$. The tangent and secant lines cross at the point P. The secant line goes above the tangent line for $x > a$ because f is concave up, and so the secant line has higher slope.

58. (a). This is best observed graphically.

Solutions for Section 2.3

Exercises

1. (a) We use the interval to the right of $x = 2$ to estimate the derivative. (Alternately, we could use the interval to the left of 2, or we could use both and average the results.) We have

$$f'(2) \approx \frac{f(4) - f(2)}{4 - 2} = \frac{24 - 18}{4 - 2} = \frac{6}{2} = 3.$$

We estimate $f'(2) \approx 3$.

(b) We know that $f'(x)$ is positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing, so it appears that $f'(x)$ is positive for $0 < x < 4$ and is negative for $4 < x < 12$.

2. For $x = 0$, 5, 10, and 15, we use the interval to the right to estimate the derivative. For $x = 20$, we use the interval to the left. For $x = 0$, we have

$$f'(0) \approx \frac{f(5) - f(0)}{5 - 0} = \frac{70 - 100}{5 - 0} = \frac{-30}{5} = -6.$$

Similarly, we find the other estimates in Table 2.5.

Table 2.5

x	0	5	10	15	20
$f'(x)$	-6	-3	-1.8	-1.2	-1.2

3. The graph is that of the line $y = -2x + 2$. The slope, and hence the derivative, is -2. See Figure 2.20.

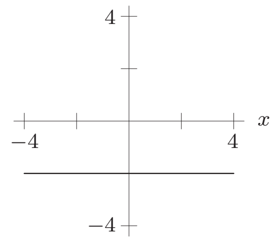


Figure 2.20

4. See Figure 2.21.

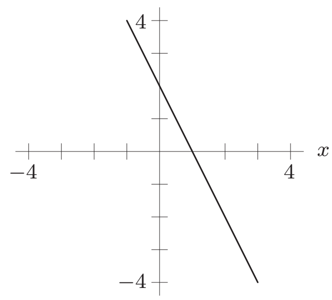


Figure 2.21

5. See Figure 2.22.

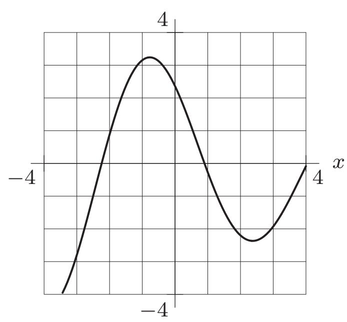


Figure 2.22

6. See Figure 2.23.

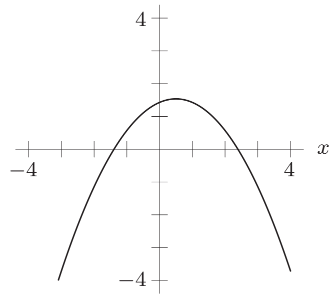


Figure 2.23

7. The slope of this curve is approximately -1 at $x = -4$ and at $x = 4$, approximately 0 at $x = -2.5$ and $x = 1.5$, and approximately 1 at $x = 0$. See Figure 2.24.

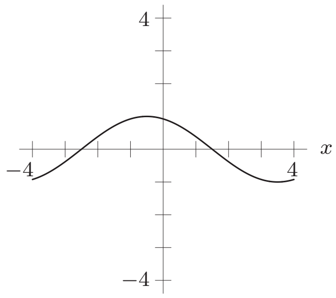


Figure 2.24

8. See Figure 2.25.

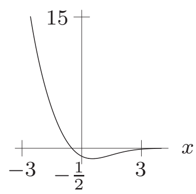


Figure 2.25

9. See Figure 2.26.

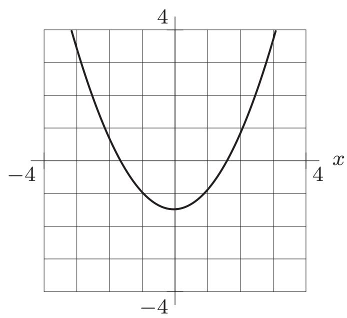


Figure 2.26

10. See Figure 2.27.

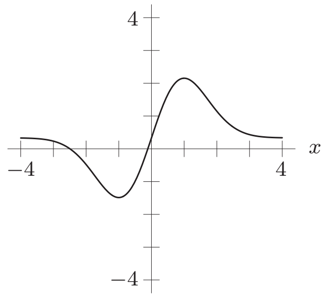


Figure 2.27

11. See Figure 2.28.

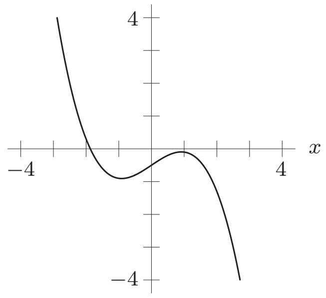


Figure 2.28

12. See Figure 2.29.

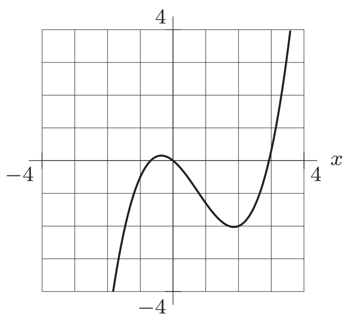


Figure 2.29

13. See Figures 2.30 and 2.31.

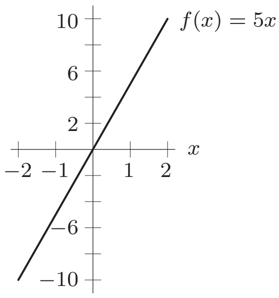


Figure 2.30

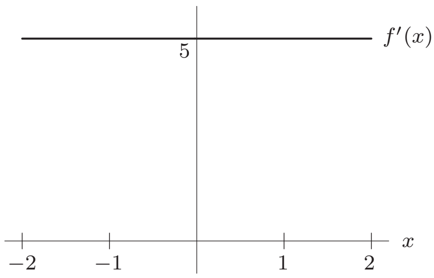


Figure 2.31

14. See Figures 2.32 and 2.33.

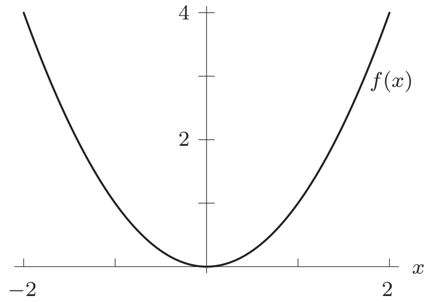


Figure 2.32

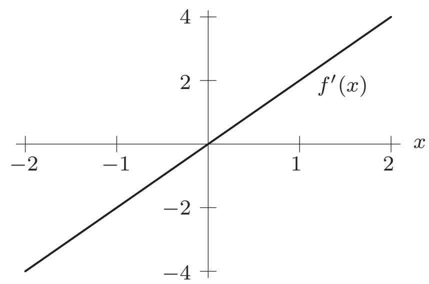


Figure 2.33

15. See Figures 2.34 and 2.35.

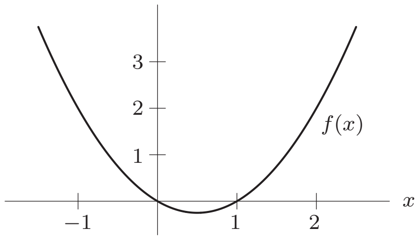


Figure 2.34

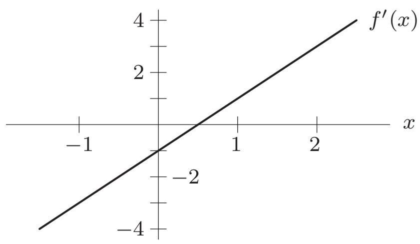


Figure 2.35

16. The graph of $f(x)$ and its derivative look the same, as in Figures 2.36 and 2.37.

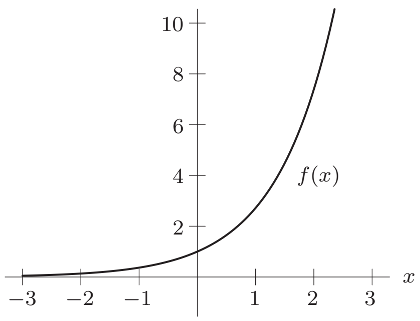


Figure 2.36

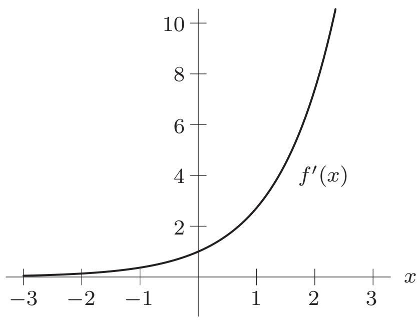


Figure 2.37

17. See Figures 2.38 and 2.39.

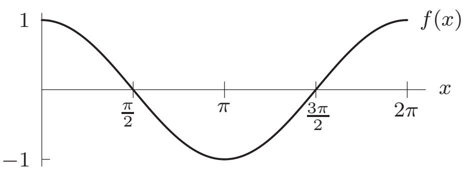


Figure 2.38

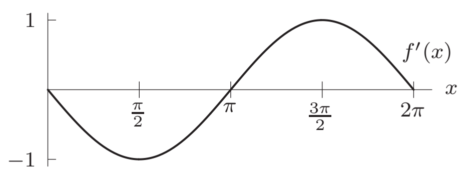


Figure 2.39

18. See Figures 2.40 and 2.41.

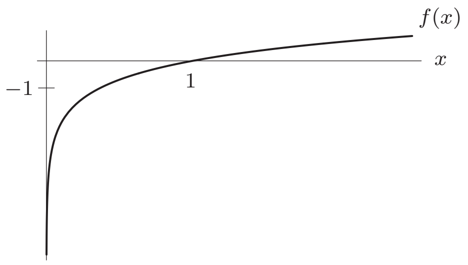


Figure 2.40

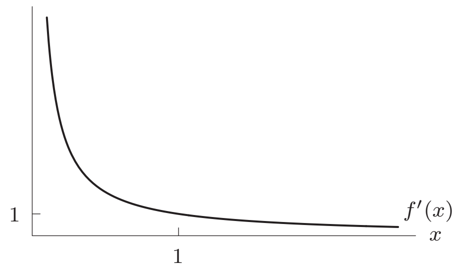


Figure 2.41

19. Since $1/x = x^{-1}$, using the power rule gives

$$k'(x) = (-1)x^{-2} = -\frac{1}{x^2}.$$

Using the definition of the derivative, we have

$$\begin{aligned} k'(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{x-(x+h)}{h(x+h)x} \\ &= \lim_{h\to 0}\frac{-h}{h(x+h)x} = \lim_{h\to 0}\frac{-1}{(x+h)x} = -\frac{1}{x^2}. \end{aligned}$$

20. Since $1/x^2 = x^{-2}$, using the power rule gives

$$l'(x) = -2x^{-3} = -\frac{2}{x^3}.$$

Using the definition of the derivative, we have

$$\begin{aligned} l'(x) &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0}\frac{x^2-(x+h)^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2} = \lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^2x^2} = -\frac{2}{x^3}. \end{aligned}$$

21. Using the definition of the derivative,

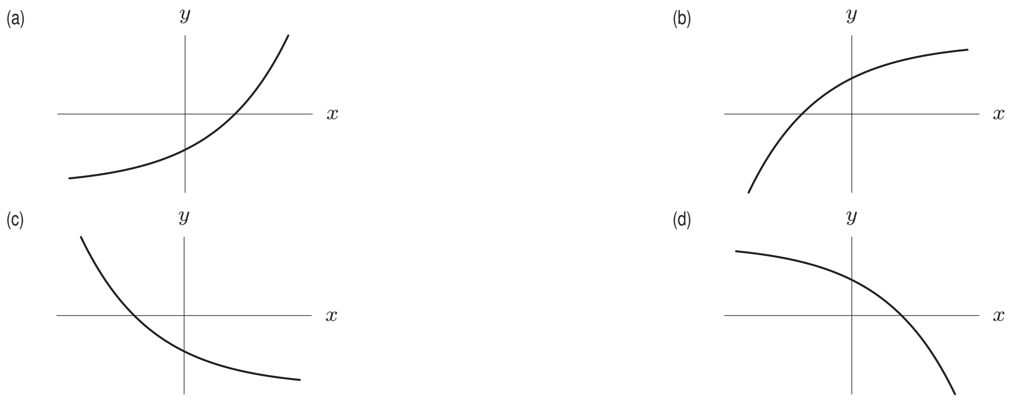
$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2-3-(2x^2-3)}{h} \\ &= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-3-2x^2+3}{h} = \lim_{h\to 0}\frac{4xh+2h^2}{h} \\ &= \lim_{h\to 0}(4x+2h) = 4x. \end{aligned}$$

22. Using the definition of the derivative, we have

$$\begin{aligned} m'(x) &= \lim_{h\to 0}\frac{m(x+h)-m(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h+1}-\frac{1}{x+1}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x+1-x-h-1}{(x+1)(x+h+1)}\right) = \lim_{h\to 0}\frac{-h}{h(x+1)(x+h+1)} \\ &= \lim_{h\to 0}\frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)^2}. \end{aligned}$$

Problems

23.



24. Since $f'(x) > 0$ for $x < -1$, $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x > -1$, $f(x)$ is decreasing on this interval.
Since $f'(x) = 0$ at $x = -1$, the tangent to $f(x)$ is horizontal at $x = -1$.
One possible shape for $y = f(x)$ is shown in Figure 2.42.

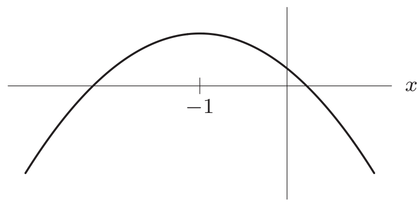


Figure 2.42

25.

x	$\ln x$
0.998	−0.0020
0.999	−0.0010
1.000	0.0000
1.001	0.0010
1.002	0.0020

x	$\ln x$
1.998	0.6921
1.999	0.6926
2.000	0.6931
2.001	0.6936
2.002	0.6941

x	$\ln x$
4.998	1.6090
4.999	1.6092
5.000	1.6094
5.001	1.6096
5.002	1.6098

x	$\ln x$
9.998	2.3024
9.999	2.3025
10.000	2.3026
10.001	2.3027
10.002	2.3028

At $x = 1$, the values of $\ln x$ are increasing by 0.001 for each increase in x of 0.001, so the derivative appears to be 1. At $x = 2$, the increase is 0.0005 for each increase of 0.001, so the derivative appears to be 0.5. At $x = 5$, $\ln x$ increases by 0.0002 for each increase of 0.001 in x, so the derivative appears to be 0.2. And at $x = 10$, the increase is 0.0001 over intervals of 0.001, so the derivative appears to be 0.1. These values suggest an inverse relationship between x and $f'(x)$, namely $f'(x) = \frac{1}{x}$.

26. We know that $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$. For this problem, we'll take the average of the values obtained for $h = 1$ and $h = -1$; that's the average of $f(x+1) - f(x)$ and $f(x) - f(x-1)$ which equals $\dfrac{f(x+1) - f(x-1)}{2}$. Thus,
$f'(0) \approx f(1) - f(0) = 13 - 18 = -5$.
$f'(1) \approx (f(2) - f(0))/2 = (10 - 18)/2 = -4$.
$f'(2) \approx (f(3) - f(1))/2 = (9 - 13)/2 = -2$.
$f'(3) \approx (f(4) - f(2))/2 = (9 - 10)/2 = -0.5$.
$f'(4) \approx (f(5) - f(3))/2 = (11 - 9)/2 = 1$.
$f'(5) \approx (f(6) - f(4))/2 = (15 - 9)/2 = 3$.
$f'(6) \approx (f(7) - f(5))/2 = (21 - 11)/2 = 5$.

$f'(7) \approx (f(8) - f(6))/2 = (30 - 15)/2 = 7.5.$
$f'(8) \approx f(8) - f(7) = 30 - 21 = 9.$
The rate of change of $f(x)$ is positive for $4 \leq x \leq 8$, negative for $0 \leq x \leq 3$. The rate of change is greatest at about $x = 8$.

27. The value of $g(x)$ is increasing at a decreasing rate for $2.7 < x < 4.2$ and increasing at an increasing rate for $x > 4.2$.

$$\frac{\Delta y}{\Delta x} = \frac{7.4 - 6.0}{5.2 - 4.7} = 2.8 \qquad \text{between } x = 4.7 \text{ and } x = 5.2$$
$$\frac{\Delta y}{\Delta x} = \frac{9.0 - 7.4}{5.7 - 5.2} = 3.2 \qquad \text{between } x = 5.2 \text{ and } x = 5.7$$

Thus $g'(x)$ should be close to 3 near $x = 5.2$.

28. (a) x_3 (b) x_4 (c) x_5 (d) x_3

29. This is a line with slope 1, so the derivative is the constant function $f'(x) = 1$. The graph is the horizontal line $y = 1$. See Figure 2.43.

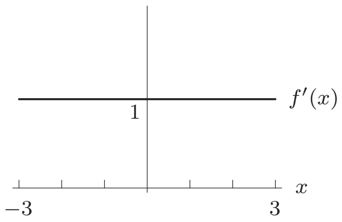


Figure 2.43

30. This is a line with slope -2, so the derivative is the constant function $f'(x) = -2$. The graph is a horizontal line at $y = -2$. See Figure 2.44.

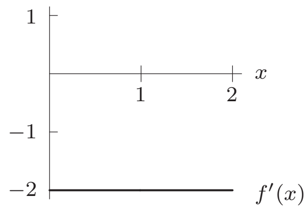


Figure 2.44

31. See Figure 2.45.

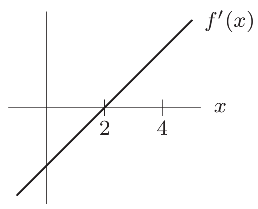


Figure 2.45

32. See Figure 2.46.

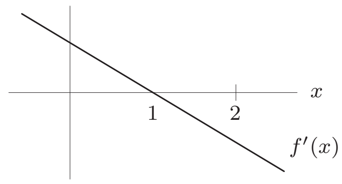


Figure 2.46

33. See Figure 2.47.

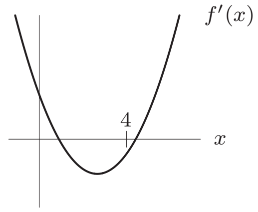


Figure 2.47

34. See Figure 2.48.

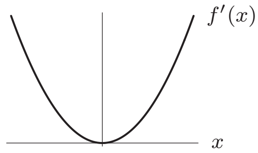


Figure 2.48

35. See Figure 2.49.

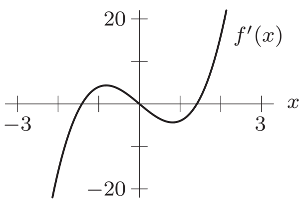


Figure 2.49

36. One possible graph is shown in Figure 2.50. Notice that as x gets large, the graph of $f(x)$ gets more and more horizontal. Thus, as x gets large, $f'(x)$ gets closer and closer to 0.

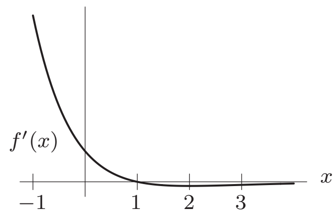


Figure 2.50

37. See Figure 2.51.

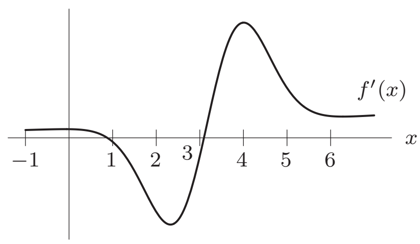


Figure 2.51

38. See Figure 2.52.

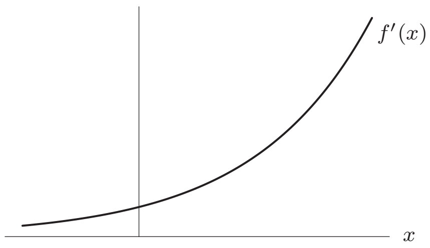


Figure 2.52

39. See Figure 2.53.

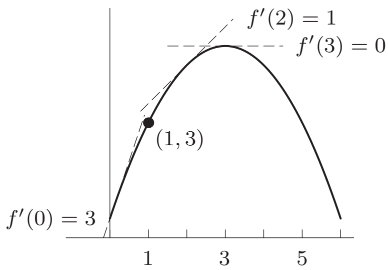
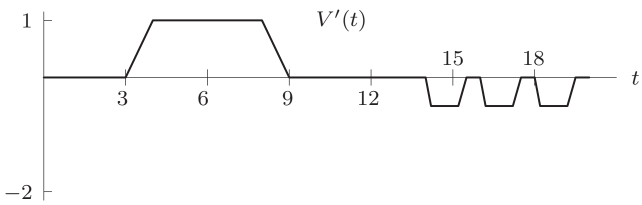


Figure 2.53

40. (a) Graph II
(b) Graph I
(c) Graph III

41. (a) $t = 3$
(b) $t = 9$
(c) $t = 14$
(d)



42. The derivative is zero whenever the graph of the original function is horizontal. Since the current is proportional to the derivative of the voltage, segments where the current is zero alternate with positive segments where the voltage is increasing and negative segments where the voltage is decreasing. See Figure 2.54. Note that the derivative does not exist where the graph has a corner.

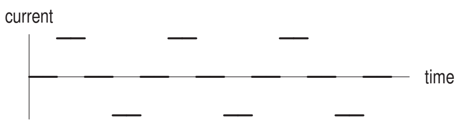


Figure 2.54

43. (a) The function f is increasing where f' is positive, so for $x_1 < x < x_3$.
(b) The function f is decreasing where f' is negative, so for $0 < x < x_1$ or $x_3 < x < x_5$.

44. On intervals where $f' = 0$, f is not changing at all, and is therefore constant. On the small interval where $f' > 0$, f is increasing; at the point where f' hits the top of its spike, f is increasing quite sharply. So f should be constant for a while, have a sudden increase, and then be constant again. A possible graph for f is shown in Figure 2.55.

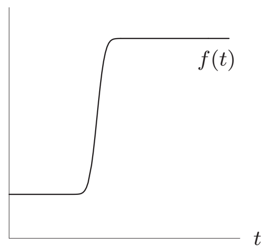
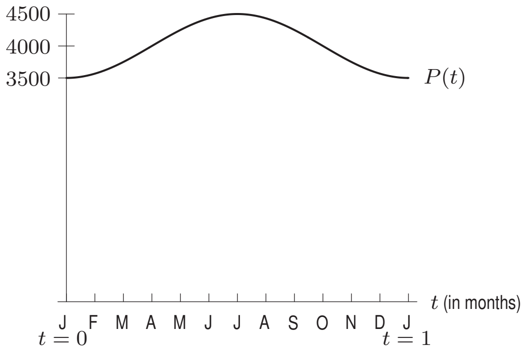


Figure 2.55: Step function

45. (a) The population varies periodically with a period of 1 year. See below.



(b) The population is at a maximum on July 1^{st}. At this time $\sin(2\pi t - \frac{\pi}{2}) = 1$, so the actual maximum population is $4000 + 500(1) = 4500$. Similarly, the population is at a minimum on January 1^{st}. At this time, $\sin(2\pi t - \frac{\pi}{2}) = -1$, so the minimum population is $4000 + 500(-1) = 3500$.
(c) The rate of change is most positive about April 1^{st} and most negative around October 1^{st}.
(d) Since the population is at its maximum around July 1^{st}, its rate of change is about 0 then.

46. The derivative of the accumulated federal debt with respect to time is shown in Figure 2.56. The derivative represents the rate of change of the federal debt with respect to time and is measured in trillions of dollars per year.

Figure 2.56

47. From the given information we know that f is increasing for values of x less than -2, is decreasing between $x = -2$ and $x = 2$, and is constant for $x > 2$. Figure 2.57 shows a possible graph—yours may be different.

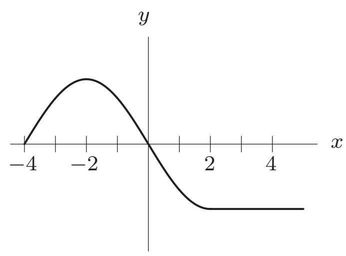


Figure 2.57

48. Since $f'(x) > 0$ for $1 < x < 3$, we see that $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x < 1$ and for $x > 3$, we see that $f(x)$ is decreasing on these intervals.
Since $f'(x) = 0$ for $x = 1$ and $x = 3$, the tangent to $f(x)$ will be horizontal at these x's.
One of many possible shapes of $y = f(x)$ is shown in Figure 2.58.

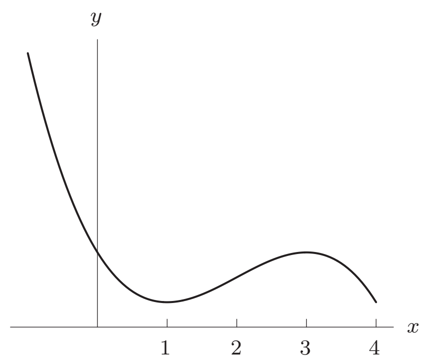


Figure 2.58

49. If $\lim_{x\to\infty} f(x) = 50$ and $f'(x)$ is positive for all x, then $f(x)$ increases to 50, but never rises above it. A possible graph of $f(x)$ is shown in Figure 2.59. If $\lim_{x\to\infty} f'(x)$ exists, it must be zero, since f looks more and more like a horizontal line. If $f'(x)$ approached another positive value c, then f would look more and more like a line with positive slope c, which would eventually go above $y = 50$.

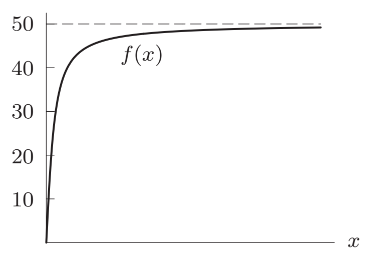
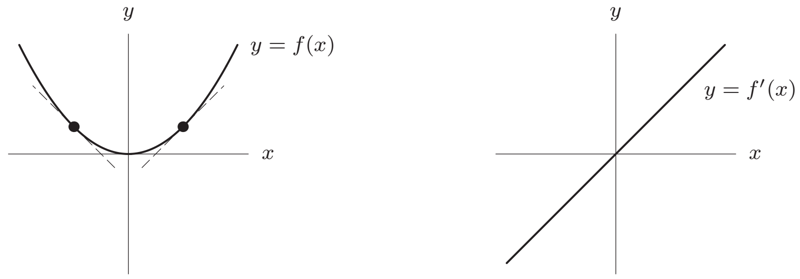


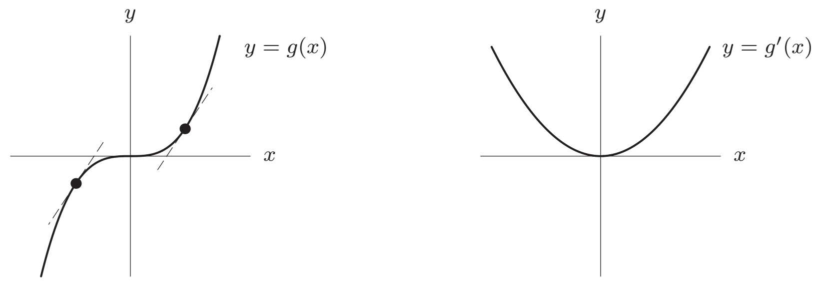
Figure 2.59

50. If $f(x)$ is even, its graph is symmetric about the y-axis. So the tangent line to f at $x = x_0$ is the same as that at $x = -x_0$ reflected about the y-axis.



So the slopes of these two tangent lines are opposite in sign, so $f'(x_0) = -f'(-x_0)$, and f' is odd.

51. If $g(x)$ is odd, its graph remains the same if you rotate it 180° about the origin. So the tangent line to g at $x = x_0$ is the tangent line to g at $x = -x_0$, rotated 180°.



But the slope of a line stays constant if you rotate it 180°. So $g'(x_0) = g'(-x_0)$; g' is even.

Strengthen Your Understanding

52. Since $f(x) = \cos x$ is decreasing on some intervals, its derivative $f'(x)$ is negative on those intervals, and the graph of $f'(x)$ is below the x-axis where $\cos x$ is decreasing.

53. In order for $f'(x)$ to be greater than zero, the slope of $f(x)$ has to be greater than zero. For example, $f(x) = e^{-x}$ is positive for all x but since the graph is decreasing everywhere, $f(x)$ has negative derivative for all x.

54. Two different functions can have the same rate of change. For example, $f(x) = 1, g(x) = 2$ both are constant, so $f'(x) = g'(x) = 0$ but $f(x) \neq g(x)$.

55. $f(t) = t(1-t)$. We have $f(t) = t - t^2$, so $f'(t) = 1 - 2t$ so the velocity is positive for $0 < t < 0.5$ and negative for $0.5 < t < 1$.

Many other answers are possible.

56. Every linear function is of the form $f(x) = b + mx$ and has derivative $f'(x) = m$. One family of functions with the same derivative is $f(x) = b + 2x$.

57. True. The graph of a linear function $f(x) = mx + b$ is a straight line with the same slope m at every point. Thus $f'(x) = m$ for all x.

58. True. Shifting a graph vertically does not change the shape of the graph and so it does not change the slopes of the tangent lines to the graph.

59. False. If $f'(x)$ is increasing then $f(x)$ is concave up. However, $f(x)$ may be either increasing or decreasing. For example, the exponential decay function $f(x) = e^{-x}$ is decreasing but $f'(x)$ is increasing because the graph of f is concave up.

60. False. A counterexample is given by $f(x) = 5$ and $g(x) = 10$, two different functions with the same derivatives: $f'(x) = g'(x) = 0$.

Solutions for Section 2.4

Exercises

1. (a) The statement $f(200) = 1300$ means that it costs \$1300 to produce 200 gallons of the chemical.

(b) The statement $f'(200) = 6$ means that when the number of gallons produced is 200, costs are increasing at a rate of \$6 per gallon. In other words, it costs about \$6 to produce the next (the 201^{st}) gallon of the chemical.

2. (a) The statement $f(5) = 18$ means that when 5 milliliters of catalyst are present, the reaction will take 18 minutes. Thus, the units for 5 are ml while the units for 18 are minutes.

(b) As in part (a), 5 is measured in ml. Since f' tells how fast T changes per unit a, we have f' measured in minutes/ml. If the amount of catalyst increases by 1 ml (from 5 to 6 ml), the reaction time decreases by about 3 minutes.

3. (Note that we are considering the average temperature of the yam, since its temperature is different at different points inside it.)

(a) It is positive, because the temperature of the yam increases the longer it sits in the oven.

(b) The units of $f'(20)$ are °F/min. The statement $f'(20) = 2$ means that at time $t = 20$ minutes, the temperature T would increase by approximately $2°$F if the yam is in the oven an additional minute.

4. (a) As the cup of coffee cools, the temperature decreases, so $f'(t)$ is negative.

(b) Since $f'(t) = dH/dt$, the units are degrees Celsius per minute. The quantity $f'(20)$ represents the rate at which the coffee is cooling, in degrees per minute, 20 minutes after the cup is put on the counter.

5. (a) The function f takes quarts of ice cream to cost in dollars, so 200 is the amount of ice cream, in quarts, and \$600 is the corresponding cost, in dollars. It costs \$600 to produce 200 quarts of ice cream.

(b) Here, 200 is in quarts, but the 2 is in dollars/quart. After producing 200 quarts of ice cream, the cost to produce one additional quart is about \$2.

6. (a) If the price is \$150, then 2000 items will be sold.

(b) If the price goes up from \$150 by \$1 per item, about 25 fewer items will be sold. Equivalently, if the price is decreased from \$150 by \$1 per item, about 25 more items will be sold.

7. Units of $C'(r)$ are dollars/percent. Approximately, $C'(r)$ means the additional amount needed to pay off the loan when the interest rate is increased by 1%. The sign of $C'(r)$ is positive, because increasing the interest rate will increase the amount it costs to pay off a loan.

8. The units of $f'(x)$ are feet/mile. The derivative, $f'(x)$, represents the rate of change of elevation with distance from the source, so if the river is flowing downhill everywhere, the elevation is always decreasing and $f'(x)$ is always negative. (In fact, there may be some stretches where the elevation is more or less constant, so $f'(x) = 0$.)

9. Units of $P'(t)$ are dollars/year. The practical meaning of $P'(t)$ is the rate at which the monthly payments change as the duration of the mortgage increases. Approximately, $P'(t)$ represents the change in the monthly payment if the duration is increased by one year. $P'(t)$ is negative because increasing the duration of a mortgage decreases the monthly payments.

10. Since B is measured in dollars and t is measured in years, dB/dt is measured in dollars per year. We can interpret dB as the extra money added to your balance in dt years. Therefore dB/dt represents how fast your balance is growing, in units of dollars/year.

11. (a) This means that investing the \$1000 at 5% would yield \$1649 after 10 years.

(b) Writing $g'(r)$ as dB/dr, we see that the units of dB/dr are dollars per percent (interest). We can interpret dB as the extra money earned if interest rate is increased by dr percent. Therefore $g'(5) = \frac{dB}{dr}|_{r=5} \approx 165$ means that the balance, at 5% interest, would increase by about \$165 if the interest rate were increased by 1%. In other words, $g(6) \approx g(5) + 165 = 1649 + 165 = 1814$.

12. (a) The units of lapse rate are the same as for the derivative dT/dz, namely (units of T)/(units of z) $=$ $°C$/km.

(b) Since the lapse rate is 6.5, the derivative of T with respect to z is $dT/dz = -6.5°C$/km. The air temperature drops about $6.5°$ for every kilometer you go up.

Problems

13. (a) Since $W = f(c)$ where W is weight in pounds and c is the number of Calories consumed per day:

$f(1800) = 155$	means that	consuming 1800 Calories per day results in a weight of 155 pounds.
$f'(2000) = 0$	means that	consuming 2000 Calories per day causes neither weight gain nor loss.
$f^{-1}(162) = 2200$	means that	a weight of 162 pounds is caused by a consumption of 2200 Calories per day.

(b) The units of dW/dc are pounds/(Calories/day).

14. (a) Let $f(t)$ be the volume, in cubic km, of the Greenland Ice Sheet t years since 2011 (Alternatively, in year t). We are given information about $f'(t)$, which has unit km^3 per year.
(b) If t is in years since 2011, we know $f'(0)$ is between -224 and -82 cubic km/year. (Alternatively, $f'(2011)$ is between -224 and -82.)

15. The graph is increasing for $0 < t < 15$ and is decreasing for $15 < t < 30$. One possible graph is shown in Figure 2.60. The units on the horizontal axis are years and the units on the vertical axis are people.

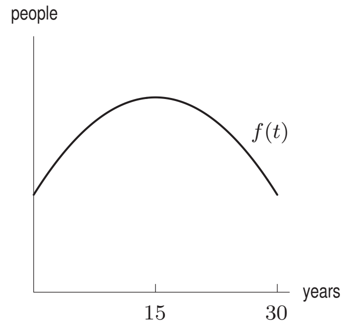


Figure 2.60

The derivative is positive for $0 < t < 15$ and negative for $15 < t < 30$. Two possible graphs are shown in Figure 2.61. The units on the horizontal axes are years and the units on the vertical axes are people per year.

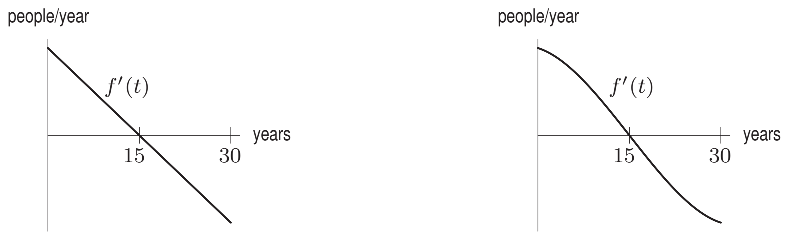


Figure 2.61

16. Since $f(t) = 1.34(1.004)^t$, we have

$$f(9) = 1.34(1.004)^9 = 1.389.$$

To estimate $f'(9)$, we use a small interval around 9:

$$f'(9) \approx \frac{f(9.001) - f(9)}{9.001 - 9} = \frac{1.34(1.004)^{9.001} - 1.34(1.004)^9}{0.001} = 0.0055.$$

We see that $f(9) = 1.389$ billion people and $f'(9) = 0.0055$ billion (that is, 5.5 million) people per year. Since $t = 9$ in 2020, this model predicts that the population of China will be about 1,389,000,000 people in 2009 and growing at a rate of about 5,500,000 people per year at that time.

17. $f(10) = 240{,}000$ means that if the commodity costs \$10, then 240,000 units of it will be sold. $f'(10) = -29{,}000$ means that if the commodity costs \$10 now, each \$1 increase in price will cause a decline in sales of 29,000 units.

18. Let p be the rating points earned by the CBS Evening News, let R be the revenue earned in millions of dollars, and let $R = f(p)$. When $p = 4.3$,

$$\text{Rate of change of revenue} \approx \frac{\$5.5 \text{ million}}{0.1 \text{ point}} = 55 \text{ million dollars/point.}$$

Thus

$$f'(4.3) \approx 55.$$

19. (a) The units of P are millions of people, the units of t are years, so the units of $f'(t)$ are millions of people per year. Therefore the statement $f'(6) = 2$ tells us that at $t = 6$ (that is, in 1986), the population of Mexico was increasing at 2 million people per year.
(b) The statement $f^{-1}(95.5) = 16$ tells us that the year when the population was 95.5 million was $t = 16$ (that is, in 1996).
(c) The units of $(f^{-1})'(P)$ are years per million of population. The statement $(f^{-1})'(95.5) = 0.46$ tells us that when the population was 95.5 million, it took about 0.46 years for the population to increase by 1 million.

20. (a) When $t = 10$, that is, at 10 am, 3.1 cm of rain has fallen.
(b) We are told that when 5 cm of rain has fallen, 16 hours have passed ($t = 16$); that is, 5 cm of rain has fallen by 4 pm.
(c) The rate at which rain is falling is 0.4 cm/hr at $t = 10$, that is, at 10 am.
(d) The units of $(f^{-1})'(5)$ are hours/cm. Thus, we are being told that when 5 cm of rain has fallen, rain is falling at a rate such that it will take 2 additional hours for another centimeter to fall.

21. (a) The depth of the water is 3 feet at time $t = 5$ hours.
(b) The depth of the water is increasing at 0.7 feet/hour at time $t = 5$ hours.
(c) When the depth of the water is 5 feet, the time is $t = 7$ hours.
(d) Since 5 is the depth in feet and $h^{-1}(5)$ is time in hours, the units of $(h^{-1})'$ are hours/feet. Thus, $(h^{-1})'(5) = 1.2$ tells us that when the water depth is 5 feet, the rate of change of time with depth is 1.2 hours per foot. In other words, when the depth is 5 feet, water is entering at a rate such that it takes 1.2 hours to add an extra foot of water.

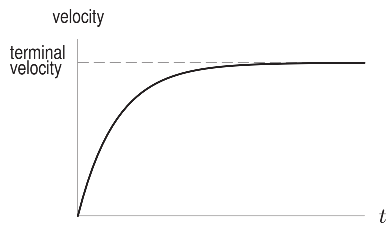
22. (a) The pressure in dynes/cm^2 at a depth of 100 meters.
(b) The depth of water in meters giving a pressure of $1.2 \cdot 10^6$ dynes/cm^2.
(c) The pressure at a depth of h meters plus a pressure of 20 dynes/cm^2.
(d) The pressure at a depth of 20 meters below the diver.
(e) The rate of increase of pressure with respect to depth, at 100 meters, in units of dynes/cm^2 per meter. Approximately, $p'(100)$ represents the increase in pressure in going from 100 meters to 101 meters.
(f) The depth, in meters, at which the rate of change of pressure with respect to depth is 100,000 dynes/cm^2 per meter.

23. The units of $g'(t)$ are inches/year. The quantity $g'(10)$ represents how fast Amelia Earhart was growing at age 10, so we expect $g'(10) > 0$. The quantity $g'(30)$ represents how fast she was growing at age 30, so we expect $g'(30) = 0$ because she was probably not growing taller at that age.

24. Units of $g'(55)$ are mpg/mph. The statement $g'(55) = -0.54$ means that at 55 miles per hour the fuel efficiency (in miles per gallon, or mpg) of the car decreases at a rate of approximately one half mpg as the velocity increases by one mph.

25. Units of dP/dt are barrels/year. dP/dt is the change in quantity of petroleum per change in time (a year). This is negative. We could estimate it by finding the amount of petroleum used worldwide over a short period of time.

26. (a)



(b) The graph should be concave down because air resistance decreases your acceleration as you speed up, and so the slope of the graph of velocity is decreasing.
(c) The slope represents the acceleration due to gravity.

27. (a) The derivative, dW/dt, measures the rate of change of water in the bathtub in gallons per minute.
(b) (i) The interval $t_0 < t < t_p$ represents the time before the plug is pulled. At that time, the rate of change of W is 0 since the amount of water in the tub is not changing.
(ii) Since dW/dt represents the rate at which the amount of water in the tub is changing, after the plug is pulled and water is leaving the tub, the sign of dW/dt is negative.
(iii) Once all the water has drained from the tub, the amount of water in the tub is not changing, so $dW/dt = 0$.

28. (a) The company hopes that increased advertising always brings in more customers instead of turning them away. Therefore, it hopes $f'(a)$ is always positive.
(b) If $f'(100) = 2$, it means that if the advertising budget is \$100,000, each extra dollar spent on advertising will bring in about \$2 worth of sales. If $f'(100) = 0.5$, each dollar above \$100 thousand spent on advertising will bring in about \$0.50 worth of sales.

(c) If $f'(100) = 2$, then as we saw in part (b), spending slightly more than \$100,000 will increase revenue by an amount greater than the additional expense, and thus more should be spent on advertising. If $f'(100) = 0.5$, then the increase in revenue is less than the additional expense, hence too much is being spent on advertising. The optimum amount to spend is an amount that makes $f'(a) = 1$. At this point, the increases in advertising expenditures just pay for themselves. If $f'(a) < 1$, too much is being spent; if $f'(a) > 1$, more should be spent.

29. (a) The derivative has units of people/second, so we find the rate of births, deaths, and migrations per second and combine them.

$$\begin{aligned}\text{Birth rate } &= \frac{1}{8} \text{ people per second}\\ \text{Death rate } &= \frac{1}{13} \text{ people per second}\\ \text{Migration rate } &= \frac{1}{27} \text{ people per second}\end{aligned}$$

Thus

$$f'(0) = \text{ Rate of change of population } = \frac{1}{8} - \frac{1}{13} + \frac{1}{27} = 0.0851 \text{ people/second.}$$

In other words, the population is increasing at 0.0851 people per second.

(b) From the answer to part (a), we see that it took $1/0.0851 = 11.75 \approx 12$ seconds to add one person.

30. Since $O'(2000) = -1,25$, we know the ODGI is decreasing at 1.25 units per year. To reduce the ODGI from 95 to 0 will take $95/1.25 = 76$ years. Thus the ozone hole is predicted to recover by 2076.

31. Since

$$\frac{P(67) - P(66)}{67 - 66} \approx P'(66),$$

we may think of $P'(66)$ as an estimate of $P(67) - P(66)$, and the latter is the number of people between 66 and 67 inches tall. Alternatively, since

$$\frac{P(66.5) - P(65.5)}{66.5 - 65.5} \text{ is a better estimate of } P'(66),$$

we may regard $P'(66)$ as an estimate of the number of people of height between 65.5 and 66.5 inches. The units for $P'(x)$ are people per inch. Since there are about 300 million people in the US, we guess that there are about 250 million full-grown persons in the US whose heights are distributed between 60 inches (5 ft) and 75 inches (6 ft 3 in). There are probably quite a few people of height 66 inches—between one and two times what we would expect from an even, or uniform, distribution— because 66 inches is nearly average. An even distribution would yield

$$P'(66) = \frac{250 \text{ million}}{15 \text{ ins}} \approx 17 \text{ million people per inch,}$$

so we expect $P'(66)$ to be between 17 and 34 million people per inch.

The value of $P'(x)$ is never negative because $P(x)$ is never decreasing. To see this, let's look at an example involving a particular value of x, say $x = 70$. The value $P(70)$ represents the number of people whose height is less than or equal to 70 inches, and $P(71)$ represents the number of people whose height is less than or equal to 71 inches. Since everyone shorter than 70 inches is also shorter than 71 inches, $P(70) \leq P(71)$. In general, $P(x)$ is 0 for small x, and increases as x increases, and is eventually constant (for large enough x).

32. (a) The units of compliance are units of volume per units of pressure, or liters per centimeter of water.

(b) The increase in volume for a 5 cm reduction in pressure is largest between 10 and 15 cm. Thus, the compliance appears maximum between 10 and 15 cm of pressure reduction. The derivative is given by the slope, so

$$\text{Compliance } \approx \frac{0.70 - 0.49}{15 - 10} = 0.042 \text{ liters per centimeter.}$$

(c) When the lung is nearly full, it cannot expand much more to accommodate more air.

33. Solving for $dp/d\delta$, we get

$$\frac{dp}{d\delta} = \left(\frac{p}{\delta + (p/c^2)}\right)\gamma.$$

(a) For $\delta \approx 10$ g/cm^3, we have $\log \delta \approx 1$, so, from Figure 2.38 in the text, we have $\gamma \approx 2.6$ and $\log p \approx 13$.

Thus $p \approx 10^{13}$, so $p/c^2 \approx 10^{13}/(9 \cdot 10^{20}) \approx 10^{-8}$, and

$$\frac{dp}{d\delta} \approx \frac{10^{13}}{10 + 10^{-8}} 2.6 \approx 2.6 \cdot 10^{12}.$$

The derivative can be interpreted as the ratio between a change in pressure and the corresponding change in density. The fact that it is so large says that a very large change in pressure brings about a very small change in density. This says that cold iron is not a very compressible material.

(b) For $\delta \approx 10^6$, we have $\log \delta \approx 6$, so, from Figure 2.38 in the text, $\gamma \approx 1.5$ and $\log p \approx 23$.
Thus $p \approx 10^{23}$, so $p/c^2 \approx 10^{23}/(9 \cdot 10^{20}) \approx 10^2$, and

$$\frac{dp}{d\delta} \approx \frac{10^{23}}{10^6 + 10^2} 1.5 \approx 1.5 \cdot 10^{17}.$$

This tells us that the matter in a white dwarf is even less compressible than cold iron.

Strengthen Your Understanding

34. Since we are not given the units of either t or s we cannot conclude that the units of the derivative are meters/second.

35. Since air is leaking from the balloon, the radius of the balloon must be decreasing, so $r'(t) < 0$.

36. Since T has units of minutes, its derivative with respect to P will have units of minutes/page.

37. Let $T(P)$ be the time, in years, to repay a loan of P dollars, then the derivative dT/dP is given in years/dollar.
There are many other possible answers.

38. Let $m = f(t)$ be the total distance, in miles, driven in a car, t days since it was purchased. Then the derivative dm/dt is given in miles/day.
There are many other possible answers.

39. True. The two sides of the equation are different frequently used notations for the very same quantity, the derivative of f at the point a.

40. True. The derivatives $f'(t)$ and $g'(t)$ measure the same thing, the rate of chemical production at the same time t, but they measure it in different units. The units of $f'(t)$ are grams per minute, and the units of $g'(t)$ are kilograms per minute. To convert from kg/min to g/min, multiply by 1000.

41. False. The derivatives $f'(t)$ and $g'(t)$ measure different things because they measure the rate of chemical production at different times. There is no conversion possible from one to the other.

42. (b) and (e) (b), (e)

43. (b) and (d) are equivalent, with (d) containing the most information. Notice that (a) and (c) are wrong.

Solutions for Section 2.5

Exercises

1. (a) Increasing, concave up
(b) Decreasing, concave down

2. (a) Since the graph is below the x-axis at $x = 2$, the value of $f(2)$ is negative.
(b) Since $f(x)$ is decreasing at $x = 2$, the value of $f'(2)$ is negative.
(c) Since $f(x)$ is concave up at $x = 2$, the value of $f''(2)$ is positive.

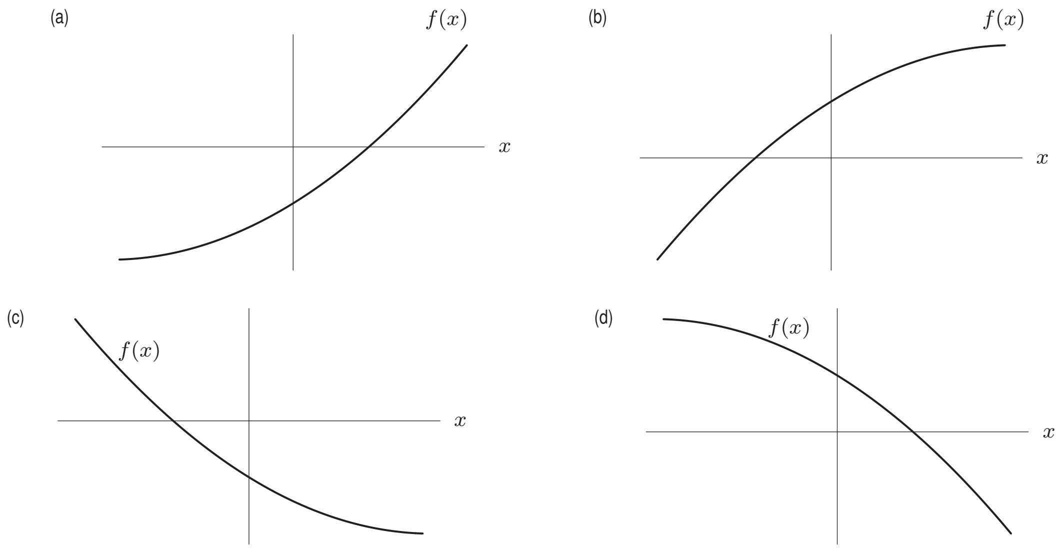
3. At B both dy/dx and d^2y/dx^2 could be positive because y is increasing and the graph is concave up there. At all the other points one or both of the derivatives could not be positive.

4. The two points at which $f' = 0$ are A and B. Since f' is nonzero at C and D and f'' is nonzero at all four points, we get the completed Table 2.6:

Table 2.6

Point	f	f'	f''
A	$-$	0	$+$
B	$+$	0	$-$
C	$+$	$-$	$-$
D	$-$	$+$	$+$

5.



6. The graph must be everywhere decreasing and concave up on some intervals and concave down on other intervals. One possibility is shown in Figure 2.62.

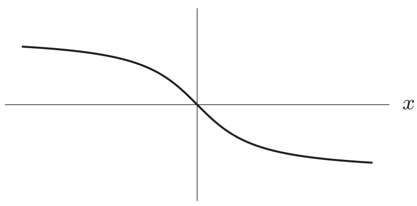


Figure 2.62

7. Since velocity is positive and acceleration is negative, we have $f' > 0$ and $f'' < 0$, and so the graph is increasing and concave down. See Figure 2.63.

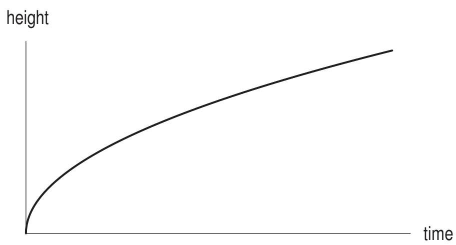


Figure 2.63

8. $f'(x) = 0$
$f''(x) = 0$

9. $f'(x) < 0$
$f''(x) = 0$

10. $f'(x) > 0$
$f''(x) > 0$

11. $f'(x) < 0$
$f''(x) > 0$

12. $f'(x) > 0$
$f''(x) < 0$

13. $f'(x) < 0$
$f''(x) < 0$

14. The velocity is the derivative of the distance, that is, $v(t) = s'(t)$. Therefore, we have

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h\to 0} \frac{(5(t+h)^2 + 3) - (5t^2 + 3)}{h} \\ &= \lim_{h\to 0} \frac{10th + 5h^2}{h} \\ &= \lim_{h\to 0} \frac{h(10t + 5h)}{h} = \lim_{h\to 0} (10t + 5h) = 10t \end{aligned}$$

The acceleration is the derivative of velocity, so $a(t) = v'(t)$:

$$\begin{aligned} a(t) &= \lim_{h\to 0} \frac{10(t+h) - 10t}{h} \\ &= \lim_{h\to 0} \frac{10h}{h} = 10. \end{aligned}$$

Problems

15. (a) The derivative, $f'(t)$, appears to be positive since the number of cars is increasing. The second derivative, $f''(t)$, appears to be negative during the period 1975–1990 because the rate of change is increasing. For example, between 1975 and 1980, the rate of change is $(121.6 - 106.7)/5 = 2.98$ million cars per year, while between 1985 and 1990, the rate of change is 1.16 million cars per year.

(b) The derivative, $f'(t)$, appears to be negative between 1990 and 1995 since the number of cars is decreasing, but increasing between 1995 and 2000. The second derivative, $f''(t)$, appears to be positive during the period 1990–2000 because the rate of change is increasing. For example, between 1990 and 1995, the rate of change is $(128.4 - 133.7)/5 = -1.06$ million cars per year, while between 1995 and 2000, the rate of change is 1.04 million cars per year.

(c) To estimate $f'(2000)$ we consider the interval 2000–2005

$$f'(2005) \approx \frac{f(2005) - f(2000)}{2005 - 2000} \approx \frac{136.6 - 133.6}{5} = \frac{3}{5} = 0.6.$$

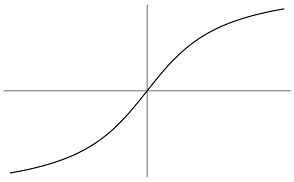
We estimate that $f'(2005) \approx 0.6$ million cars per year. The number of passenger cars in the US was increasing at a rate of about 600,000 cars per year in 2005.

16. To measure the average acceleration over an interval, we calculate the average rate of change of velocity over the interval. The units of acceleration are ft/sec per second, or (ft/sec)/sec, written ft/sec^2.

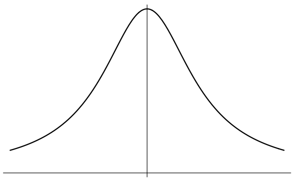
$$\begin{array}{c}\text{Average acceleration}\\ \text{for } 0 \le t \le 1\end{array} = \frac{\text{Change in velocity}}{\text{Time}} = \frac{v(1) - v(0)}{1} = \frac{30 - 0}{1} = 30 \text{ ft/sec}^2$$

$$\begin{array}{c}\text{Average acceleration}\\ \text{for } 1 \le t \le 2\end{array} = \frac{52 - 30}{2 - 1} = 22 \text{ ft/sec}^2$$

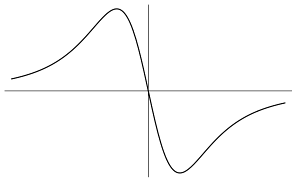
17. (a)



(b)



(c)



18. Since the graph of this function is a line, the second derivative (of any linear function) is 0. See Figure 2.64.

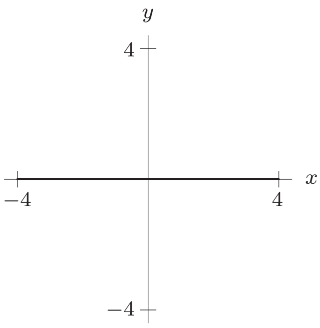


Figure 2.64

19. See Figure 2.65.

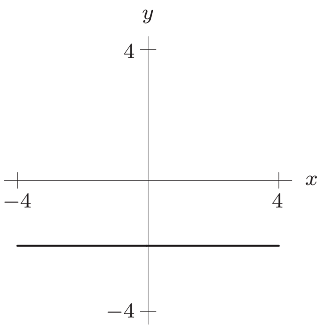


Figure 2.65

20. See Figure 2.66.

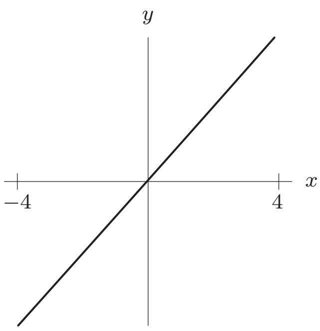


Figure 2.66

21. See Figure 2.67.

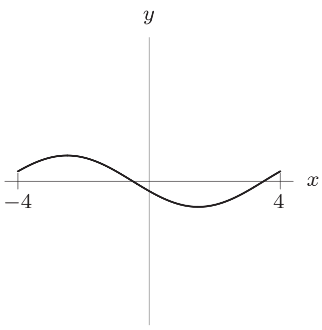


Figure 2.67

22. See Figure 2.68.

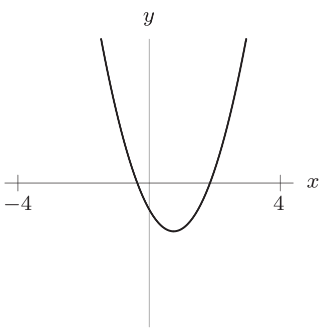


Figure 2.68

23. See Figure 2.69.

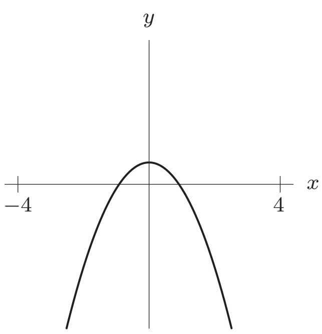
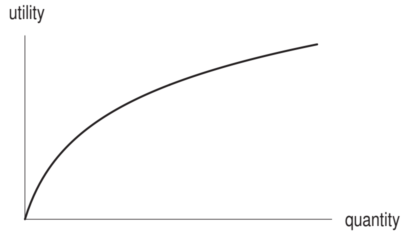


Figure 2.69

24. (a) $dP/dt > 0$ and $d^2P/dt^2 > 0$.

(b) $dP/dt < 0$ and $d^2P/dt^2 > 0$ (but dP/dt is close to zero).

25. (a)



(b) As a function of quantity, utility is increasing but at a decreasing rate; the graph is increasing but concave down. So the derivative of utility is positive, but the second derivative of utility is negative.

26. (a) Let $N(t)$ be the number of people below the poverty line. See Figure 2.70.

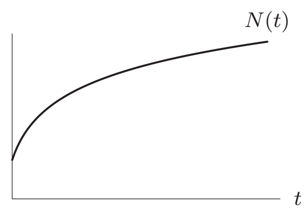


Figure 2.70

(b) dN/dt is positive, since people are still slipping below the poverty line. d^2N/dt^2 is negative, since the rate at which people are slipping below the poverty line, dN/dt, is decreasing.

27. (a) The EPA will say that the rate of discharge is still rising. The industry will say that the rate of discharge is increasing less quickly, and may soon level off or even start to fall.

(b) The EPA will say that the rate at which pollutants are being discharged is leveling off, but not to zero—so pollutants will continue to be dumped in the lake. The industry will say that the rate of discharge has decreased significantly.

28. (a) At x_4 and x_5, because the graph is below the x-axis there.
(b) At x_3 and x_4, because the graph is sloping down there.
(c) At x_3 and x_4, because the graph is sloping down there. This is the same condition as part (b).
(d) At x_2 and x_3, because the graph is bending downward there.
(e) At x_1, x_2, and x_5, because the graph is sloping upward there.
(f) At x_1, x_4, and x_5, because the graph is bending upward there.

29. (a) At t_3, t_4, and t_5, because the graph is above the t-axis there.
(b) At t_2 and t_3, because the graph is sloping up there.
(c) At t_1, t_2, and t_5, because the graph is concave up there
(d) At t_1, t_4, and t_5, because the graph is sloping down there.
(e) At t_3 and t_4, because the graph is concave down there.

30. Since f' is everywhere positive, f is everywhere increasing. Hence the greatest value of f is at x_6 and the least value of f is at x_1. Directly from the graph, we see that f' is greatest at x_3 and least at x_2. Since f'' gives the slope of the graph of f', f'' is greatest where f' is rising most rapidly, namely at x_6, and f'' is least where f' is falling most rapidly, namely at x_1.

31. To the right of $x = 5$, the function starts by increasing, since $f'(5) = 2 > 0$ (though f may subsequently decrease) and is concave down, so its graph looks like the graph shown in Figure 2.71. Also, the tangent line to the curve at $x = 5$ has slope 2 and lies above the curve for $x > 5$. If we follow the tangent line until $x = 7$, we reach a height of 24. Therefore, $f(7)$ must be smaller than 24, meaning 22 is the only possible value for $f(7)$ from among the choices given.

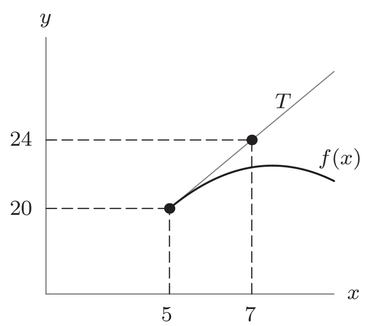


Figure 2.71

32. (a) From the information given, $C(1994) = 3200$ ppt and $C(2010) = 2750$ ppt.
(b) Since the change has been approximately linear, the rate of change is constant:

$$C'(1994) = C'(2010) = \frac{2750 - 3200}{2010 - 1994} = -28.125 \text{ ppt per year}.$$

(c) The slope is -28.125 and $C(1994) = 3200$.
If t is the year, we have

$$C(t) = 3200 - 28.125(t - 1994).$$

(d) We solve

$$\begin{aligned} 1850 &= 3200 - 28.125(t - 1994) \\ 28.125(t - 1994) &= 3200 - 1850 \\ t &= 1994 + \frac{3200 - 1850}{28.125} = 2042. \end{aligned}$$

The CFC level in the atmosphere above the US is predicted to return to the original level in 2042.
(e) Since $C''(t) > 0$, the graph bends upward, so the answer to part (d) is too early. The CFCs are expected to reach their original level later than 2042.

Strengthen Your Understanding

33. A linear function is neither concave up nor concave down.

34. When the acceleration of a car is zero, the car is not speeding up or slowing down. This happens whenever the velocity is constant. The car does not have to be stationary for this to happen.

35. One possibility is $f(x) = b + ax$, $a \neq 0$.

36. One possibility is $f(x) = x^2$. We have $f'(x) = 2x$, which is zero at $x = 0$ but $f''(x) = 2$.
There are many other possible answers.

37. True. The second derivative $f''(x)$ is the derivative of $f'(x)$. Thus the derivative of $f'(x)$ is positive, and so $f'(x)$ is increasing.

38. True. Instantaneous acceleration is a derivative, and all derivatives are limits of difference quotients. More precisely, instantaneous acceleration $a(t)$ is the derivative of the velocity $v(t)$, so

$$a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}.$$

39. True.

40. True; $f(x) = x^3$ is increasing over any interval.

41. False; $f(x) = x^2$ is monotonic on intervals which do not contain the origin (unless the origin is an endpoint).

Solutions for Section 2.6

Exercises

1. (a) Function f is not continuous at $x = 1$.
(b) Function f appears not differentiable at $x = 1, 2, 3$.

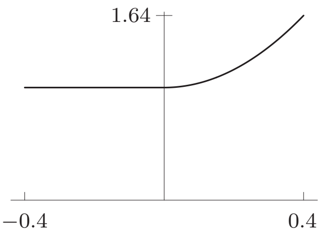
2. (a) Function g appears continuous at all x-values shown.
(b) Function g appears not differentiable at $x = 2, 4$. At $x = 2$, the curve is vertical, so the derivative does not exist. At $x = 4$, the graph has a corner, so the derivative does not exist.

3. No, there are sharp turning points.

4. Yes.

Problems

5. Yes, f is differentiable at $x = 0$, since its graph does not have a "corner" at $x = 0$. See below.



Another way to see this is by computing:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{(h+|h|)^2}{h} = \lim_{h\to 0}\frac{h^2+2h|h|+|h|^2}{h}.$$

Since $|h|^2 = h^2$, we have:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{2h^2+2h|h|}{h} = \lim_{h\to 0} 2(h+|h|) = 0.$$

So f is differentiable at 0 and $f'(0) = 0$.

6. As we can see in Figure 2.72, f oscillates infinitely often between the x-axis and the line $y = 2x$ near the origin. This means a line from $(0, 0)$ to a point $(h, f(h))$ on the graph of f alternates between slope 0 (when $f(h) = 0$) and slope 2 (when $f(h) = 2h$) infinitely often as h tends to zero. Therefore, there is no limit of the slope of this line as h tends to zero, and thus there is no derivative at the origin. Another way to see this is by noting that

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin(\frac{1}{h})+h}{h} = \lim_{h\to 0}\left(\sin\left(\frac{1}{h}\right)+1\right)$$

does not exist, since $\sin(\frac{1}{h})$ does not have a limit as h tends to zero. Thus, f is not differentiable at $x = 0$.

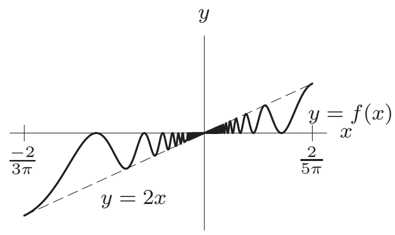


Figure 2.72

7. We can see from Figure 2.73 that the graph of f oscillates infinitely often between the curves $y = x^2$ and $y = -x^2$ near the origin. Thus the slope of the line from $(0, 0)$ to $(h, f(h))$ oscillates between h (when $f(h) = h^2$ and $\frac{f(h)-0}{h-0} = h$) and $-h$ (when $f(h) = -h^2$ and $\frac{f(h)-0}{h-0} = -h$) as h tends to zero. So, the limit of the slope as h tends to zero is 0, which is the derivative of f at the origin. Another way to see this is to observe that

$$\begin{aligned}\lim_{h\to 0}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0}\left(\frac{h^2\sin(\frac{1}{h})}{h}\right)\\ &= \lim_{h\to 0} h\sin(\frac{1}{h})\\ &= 0,\end{aligned}$$

since $\lim_{h\to 0} h = 0$ and $-1 \le \sin(\frac{1}{h}) \le 1$ for any h. Thus f is differentiable at $x = 0$, and $f'(0) = 0$.

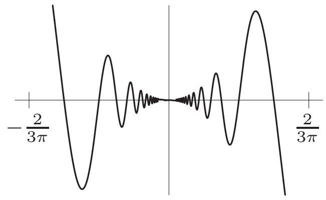


Figure 2.73

8. (a) The graph is concave up everywhere, except at $x = 2$ where the derivative is undefined. This is the case if the graph has a corner at $x = 2$. One possible graph is shown in Figure 2.74.
(b) The graph is concave up for $x < 2$ and concave down for $x > 2$, and the derivative is undefined at $x = 2$. This is the case if the graph is vertical at $x = 2$. One possible graph is shown in Figure 2.75.

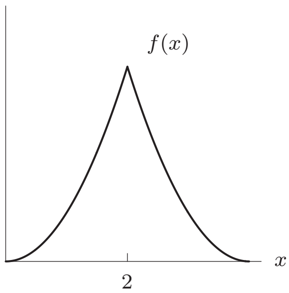


Figure 2.74

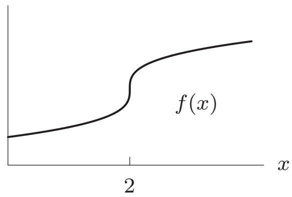


Figure 2.75

9. We want to look at

$$\lim_{h\to 0} \frac{(h^2 + 0.0001)^{1/2} - (0.0001)^{1/2}}{h}.$$

As $h \to 0$ from positive or negative numbers, the difference quotient approaches 0. (Try evaluating it for $h = 0.001$, 0.0001, etc.) So it appears there is a derivative at $x = 0$ and that this derivative is zero. How can this be if f has a corner at $x = 0$?

The answer lies in the fact that what appears to be a corner is in fact smooth—when you zoom in, the graph of f looks like a straight line with slope 0! See Figure 2.76.

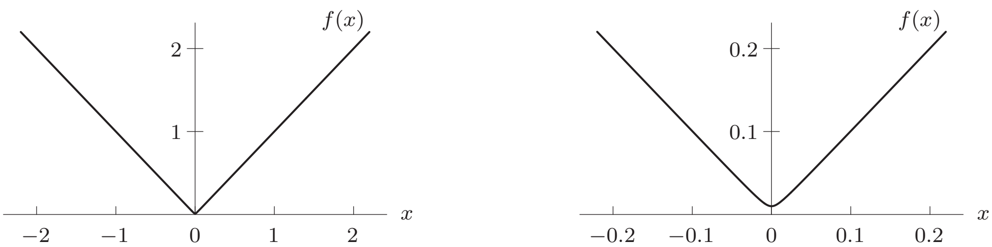


Figure 2.76: Close-ups of $f(x) = (x^2 + 0.0001)^{1/2}$ showing differentiability at $x = 0$

10. (a)

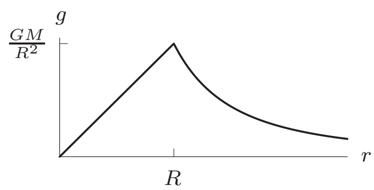


Figure 2.77

(b) The graph certainly looks continuous. The only point in question is $r = R$. Using the second formula with $r = R$ gives

$$g = \frac{GM}{R^2}.$$

Then, using the first formula with r approaching R from below, we see that as we get close to the surface of the earth

$$g \approx \frac{GMR}{R^3} = \frac{GM}{R^2}.$$

Since we get the same value for g from both formulas, g is continuous.

(c) For $r < R$, the graph of g is a line with a positive slope of $= \frac{GM}{R^3}$. For $r > R$, the graph of g looks like $1/x^2$, and so has a negative slope. Therefore the graph has a "corner" at $r = R$ and so is not differentiable there.

11. (a) The graph of Q against t does not have a break at $t = 0$, so Q appears to be continuous at $t = 0$. See Figure 2.78.

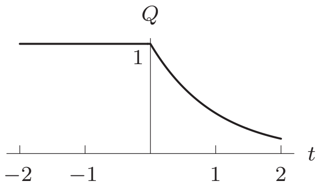


Figure 2.78

(b) The slope dQ/dt is zero for $t < 0$, and negative for all $t > 0$. At $t = 0$, there appears to be a corner, which does not disappear as you zoom in, suggesting that I is defined for all times t except $t = 0$.

12. (a) Notice that B is a linear function of r for $r \leq r_0$ and a reciprocal for $r > r_0$. The constant B_0 is the value of B at $r = r_0$ and the maximum value of B. See Figure 2.79.

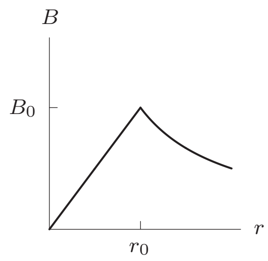


Figure 2.79

(b) B is continuous at $r = r_0$ because there is no break in the graph there. Using the formula for B, we have

$$\lim_{r \to r_0^-} B = \frac{r_0}{r_0} B_0 = B_0 \quad \text{and} \quad \lim_{r \to r_0^+} B = \frac{r_0}{r_0} B_0 = B_0.$$

(c) The function B is not differentiable at $r = r_0$ because the graph has a corner there. The slope is positive for $r < r_0$ and the slope is negative for $r > r_0$.

13. (a) Since

$$\lim_{r \to r_0^-} E = kr_0$$

and

$$\lim_{r \to r_0^+} E = \frac{kr_0^2}{r_0} = kr_0$$

and

$$E(r_0) = kr_0,$$

we see that E is continuous at r_0.

(b) The function E is not differentiable at $r = r_0$ because the graph has a corner there. The slope is positive for $r < r_0$ and the slope is negative for $r > r_0$.
(c) See Figure 2.80.

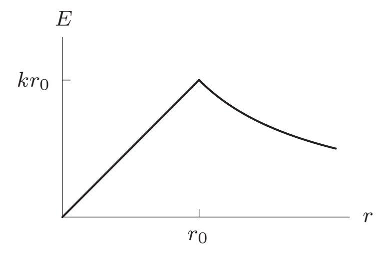


Figure 2.80

14. (a) The graph of $g(r)$ does not have a break or jump at $r = 2$, and so $g(r)$ is continuous there. See Figure 2.81. This is confirmed by the fact that

$$g(2) = 1 + \cos(\pi 2/2) = 1 + (-1) = 0$$

so the value of $g(r)$ as you approach $r = 2$ from the left is the same as the value when you approach $r = 2$ from the right.

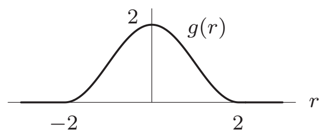


Figure 2.81

(b) The graph of $g(r)$ does not have a corner at $r = 2$, even after zooming in, so $g(r)$ appears to be differentiable at $r = 0$. This is confirmed by the fact that $\cos(\pi r/2)$ is at the bottom of a trough at $r = 2$, and so its slope is 0 there. Thus the slope to the left of $r = 2$ is the same as the slope to the right of $r = 2$.

15. (a) The graph of ϕ does not have a break at $y = 0$, and so ϕ appears to be continuous there. See figure Figure 2.82.

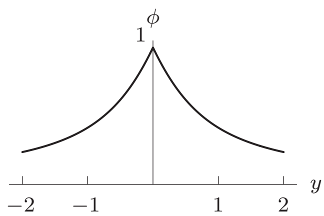


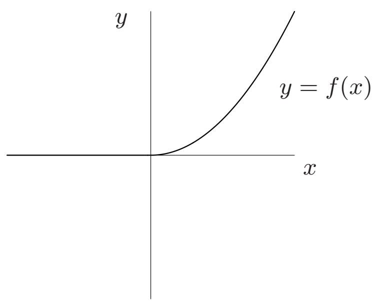
Figure 2.82

(b) The graph of ϕ has a corner at $y = 0$ which does not disappear as you zoom in. Therefore ϕ appears not be differentiable at $y = 0$.

16. (a) The graph of

$$f(x) = \begin{cases} 0 & \text{if } x < 0. \\ x^2 & \text{if } x \geq 0. \end{cases}$$

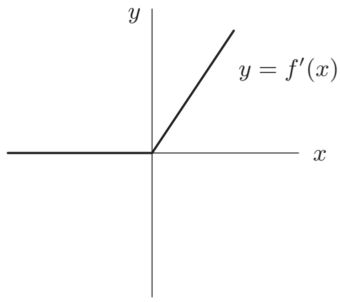
is shown to the right. The graph is continuous and has no vertical segments or corners, so $f(x)$ is differentiable everywhere.



By Example 4 on page 94,

$$f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

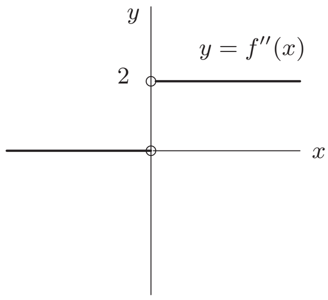
So its graph is shown to the right.



(b) The graph of the derivative has a corner at $x = 0$ so $f'(x)$ is not differentiable at $x = 0$. The graph of

$$f''(x) = \begin{cases} 0 \text{ if } x < 0 \\ 2 \text{ if } x > 0 \end{cases}$$

looks like:



The second derivative is not defined at $x = 0$. So it is certainly neither differentiable nor continuous at $x = 0$.

Strengthen Your Understanding

17. There are several ways in which a function can fail to be differentiable at a point, one of which is because the graph has a sharp corner at the point. Other cases are when the function is not continuous at a point or if the graph has a vertical tangent line.

18. The converse of this statement is true. However, a function can be continuous and not differentiable at a point; for example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

19. $f(x) = |x - 2|$. This is continuous but not differentiable at $x = 2$.

20. $f(x) = \sqrt{x}$, $x \geq 0$. This is invertible but $f'(x) = 1/(2\sqrt{x})$, which is not defined at $x = 0$.

21. Let

$$f(x) = \frac{x^2 - 1}{x^2 - 4}.$$

Since $x^2 - 1 = (x - 1)(x + 1)$, this function has zeros at $x = \pm 1$. However, at $x = \pm 2$, the denominator $x^2 - 4 = 0$, so $f(x)$ is undefined and not differentiable.

22. True. Let $f(x) = |x - 3|$. Then $f(x)$ is continuous for all x but not differentiable at $x = 3$ because its graph has a corner there. Other answers are possible.

23. True. If a function is differentiable at a point, then it is continuous at that point. For example, $f(x) = x^2$ is both differentiable and continuous on any interval. However, *one* example does not establish the truth of this statement; it merely illustrates the statement.

24. False. Being continuous does not imply differentiability. For example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

25. True. If a function were differentiable, then it would be continuous. For example,
$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ is neither differentiable nor continuous at $x = 0$. However, *one* example does not establish the truth of this statement; it merely illustrates the statement.

26. False. For example, $f(x) = |x|$ is not differentiable at $x = 0$, but it is continuous at $x = 0$.

27. (a) This is not a counterexample, since it does not satisfy the conditions of the statement, and therefore does not have the potential to contradict the statement.

(b) This contradicts the statement, because it satisfies its conditions but not its conclusion. Hence it is a counterexample. Notice that this counterexample could not actually exist, since the statement is true.

(c) This is an example illustrating the statement; it is not a counterexample.

(d) This is not a counterexample, for the same reason as in part (a).

Solutions for Chapter 2 Review

Exercises

1. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(3) = 72$ and $s(10) = 144$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(10) - s(3)}{10 - 3} = \frac{144 - 72}{7} = \frac{72}{7} = 10.286 \text{ cm/sec.}$$

2. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = 12 \cdot 3 - 3^2 = 27$ and $s(1) = 12 \cdot 1 - 1^2 = 11$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{27 - 11}{2} = 8 \text{ mm/sec.}$$

3. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = \ln 3$ and $s(1) = \ln 1$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{\ln 3 - \ln 1}{2} = \frac{\ln 3}{2} = 0.549 \text{ mm/sec.}$$

4. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = 11$ and $s(1) = 3$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{11 - 3}{2} = 4 \text{ mm/sec.}$$

5. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = 4$ and $s(1) = 4$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{4 - 4}{2} = 0 \text{ mm/sec.}$$

Though the particle moves, its average velocity over the interval is zero, since it is at the same position at $t = 1$ and $t = 3$.

6. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values on the graph of $s(3) = 2$ and $s(1) = 3$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{2 - 3}{2} = -\frac{1}{2} \text{ mm/sec.}$$

7. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values on the graph of $s(3) = 2$ and $s(1) = 2$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{2 - 2}{2} = 0 \text{ mm/sec.}$$

Though the particle moves, its average velocity over the interval is zero, since it is at the same position at $t = 1$ and $t = 3$.

8. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{7.84 - 7}{0.1} = 8.4 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{7.0804 - 7}{0.01} = 8.04 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{7.008004 - 7}{0.001} = 8.004 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 8, so we estimate the instantaneous velocity at $t = 1$ to be 8 m/sec.

9. See Figure 2.83.

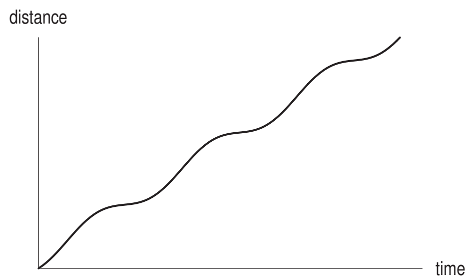


Figure 2.83

10. See Figure 2.84.

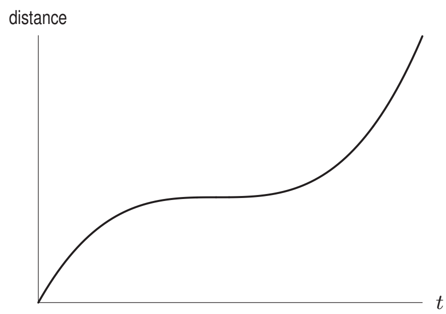


Figure 2.84

11. (a) Figure 2.85 shows a graph of $f(x) = x \sin x$.

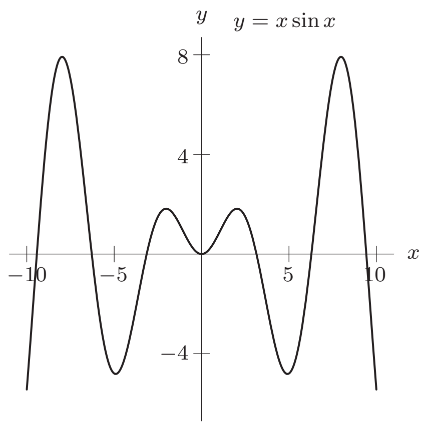


Figure 2.85

(b) Seven, since $x \sin x = 0$ at $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi$.
(c) From the graph, we see $x \sin x$ is increasing at $x = 1$, decreasing at $x = 4$.
(d) We calculate both average rates of change

$$\frac{f(2) - f(0)}{(2-0)} = \frac{2\sin 2 - 0}{2} = \sin 2 \approx 0.91$$
$$\frac{f(8) - f(6)}{(8-6)} = \frac{8\sin 8 - 6\sin 6}{2} \approx 4.80.$$

So the average rate of change over $6 \leq x \leq 8$ is greater.
(e) From the graph, we see the slope is greater at $x = -9$.

12. (a) Using the difference quotient

$$f'(0.6) \approx \frac{f(0.8) - f(0.4)}{0.8 - 0.4} = \frac{0.5}{0.4} = 1.25.$$

Substituting $x = 0.6$, we have $y = 3.9$, so the tangent line is $y - 3.9 = 1.25(x - 0.6)$, that is $y = 1.25x + 3.15$.
(b) The equation from part (a) gives

$$f(0.7) \approx 1.25(0.7) + 3.15 = 4.025$$
$$f(1.2) \approx 1.25(1.2) + 3.15 = 4.65$$
$$f(1.4) \approx 1.25(1.4) + 3.15 = 4.9$$

The estimate for $f(0.7)$ is likely to be reliable as 0.7 is close to 0.6 (and $f(0.8) = 4$, which is not too far off). The estimate for $f(1.2)$ is less reliable as 1.2 is outside the given data (from 0 to 1.0). The estimate for $f(1.4)$ less reliable still.

13. See Figure 2.86.

Figure 2.86

14. See Figure 2.87.

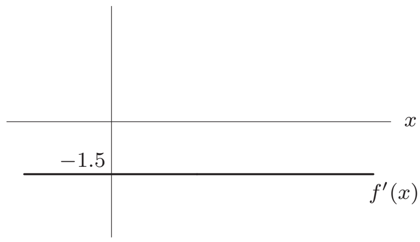


Figure 2.87

15. See Figure 2.88.

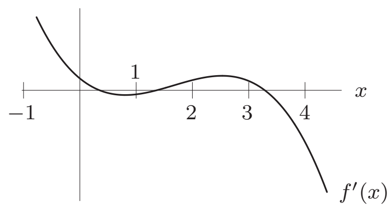


Figure 2.88

16. See Figure 2.89.

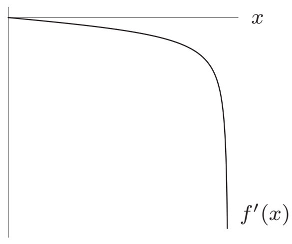


Figure 2.89

17. See Figure 2.90.

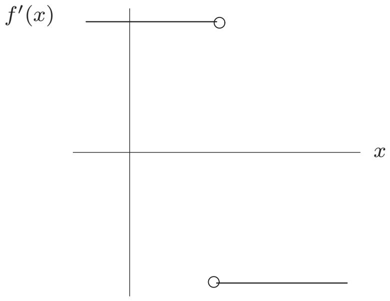


Figure 2.90

18. See Figure 2.91.

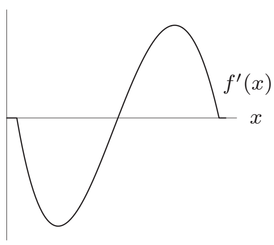


Figure 2.91

19. See Figure 2.92.

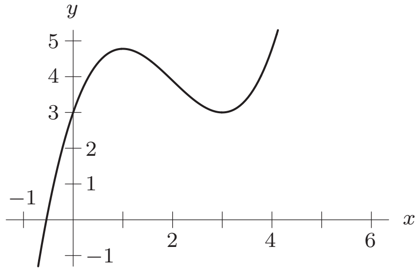


Figure 2.92

20. See Figure 2.93.

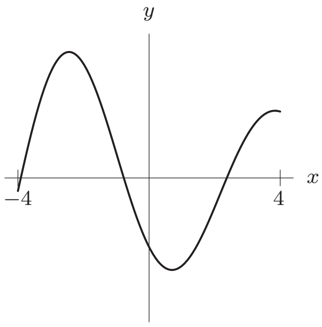


Figure 2.93

21. See Figure 2.94.

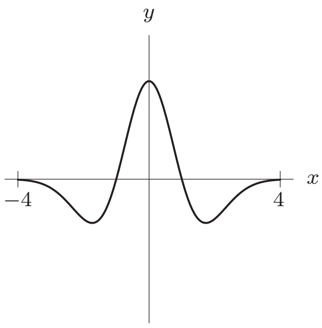


Figure 2.94

22. Using the definition of the derivative

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{5(x+h)^2+x+h-(5x^2+x)}{h} \\
&= \lim_{h\to 0} \frac{5(x^2+2xh+h^2)+x+h-5x^2-x}{h} \\
&= \lim_{h\to 0} \frac{10xh+5h^2+h}{h} \\
&= \lim_{h\to 0} (10x+5h+1) = 10x+1
\end{aligned}$$

23. Using the definition of the derivative, we have

$$\begin{aligned}
n'(x) &= \lim_{h\to 0} \frac{n(x+h)-n(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\left(\frac{1}{x+h}+1\right)-\left(\frac{1}{x}+1\right)\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) \\
&= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.
\end{aligned}$$

24. We need to look at the difference quotient and take the limit as h approaches zero. The difference quotient is

$$\frac{f(3+h)-f(3)}{h} = \frac{[(3+h)^2+1]-10}{h} = \frac{9+6h+h^2+1-10}{h} = \frac{6h+h^2}{h} = \frac{h(6+h)}{h}.$$

Since $h \neq 0$, we can divide by h in the last expression to get $6+h$. Now the limit as h goes to 0 of $6+h$ is 6, so

$$f'(3) = \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6.$$

So at $x = 3$, the slope of the tangent line is 6. Since $f(3) = 3^2 + 1 = 10$, the tangent line passes through $(3, 10)$, so its equation is

$$y - 10 = 6(x-3), \quad \text{or} \quad y = 6x - 8.$$

25. By joining consecutive points we get a line whose slope is the average rate of change. The steeper this line, the greater the average rate of change. See Figure 2.95.

(a) (i) C and D. Steepest slope.
(ii) B and C. Slope closest to 0.

(b) A and B, and C and D. The two slopes are closest to each other.

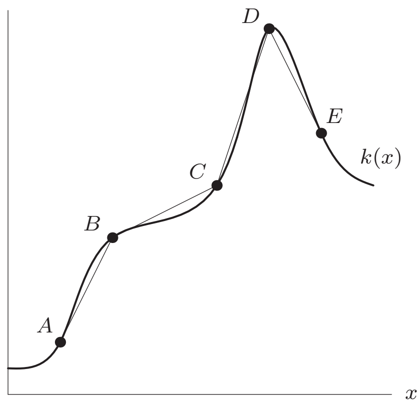


Figure 2.95

26. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(3(x+h)-1)-(3x-1)}{h} \\ &= \lim_{h\to 0} \frac{3x+3h-1-3x+1}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3. \end{aligned}$$

27. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(5(x+h)^2)-(5x^2)}{h} \\ &= \lim_{h\to 0} \frac{5(x^2+2xh+h^2)-5x^2}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{5x^2+10xh+5h^2-5x^2}{h} \\
&= \lim_{h\to 0} \frac{10xh+5h^2}{h} \\
&= \lim_{h\to 0} (10x+5h) \\
&= 10x.
\end{aligned}
$$

28. Using the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^2+4)-(x^2+4)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2+4-x^2-4}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h} \\
&= \lim_{h\to 0} (2x+h) \\
&= 2x.
\end{aligned}
$$

29. Using the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(3(x+h)^2-7)-(3x^2-7)}{h} \\
&= \lim_{h\to 0} \frac{(3(x^2+2xh+h^2)-7)-(3x^2-7)}{h} \\
&= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-7-3x^2+7}{h} \\
&= \lim_{h\to 0} \frac{6xh+3h^2}{h} \\
&= \lim_{h\to 0} (6x+3h) \\
&= 6x.
\end{aligned}
$$

30. Using the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0} (3x^2+3xh+h^2) \\
&= 3x^2.
\end{aligned}
$$

31. $\displaystyle \lim_{h\to 0} \frac{(a+h)^2-a^2}{h} = \lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} = \lim_{h\to 0} (2a+h) = 2a$

32. $\displaystyle \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{a+h}-\frac{1}{a}\right) = \lim_{h\to 0} \frac{a-(a+h)}{(a+h)ah} = \lim_{h\to 0} \frac{-1}{(a+h)a} = \frac{-1}{a^2}$

33. $\displaystyle \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(a+h)^2}-\frac{1}{a^2}\right) = \lim_{h\to 0} \frac{a^2-(a^2+2ah+h^2)}{(a+h)^2a^2h} = \lim_{h\to 0} \frac{(-2a-h)}{(a+h)^2a^2} = \frac{-2}{a^3}$

34. $\sqrt{a+h}-\sqrt{a} = \dfrac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{\sqrt{a+h}+\sqrt{a}} = \dfrac{a+h-a}{\sqrt{a+h}+\sqrt{a}} = \dfrac{h}{\sqrt{a+h}+\sqrt{a}}.$

Therefore $\lim\limits_{h\to 0}\dfrac{\sqrt{a+h}-\sqrt{a}}{h} = \lim\limits_{h\to 0}\dfrac{1}{\sqrt{a+h}+\sqrt{a}} = \dfrac{1}{2\sqrt{a}}$

35. We combine terms in the numerator and multiply top and bottom by $\sqrt{a}+\sqrt{a+h}$.

$$\frac{1}{\sqrt{a+h}}-\frac{1}{\sqrt{a}} = \frac{\sqrt{a}-\sqrt{a+h}}{\sqrt{a+h}\sqrt{a}} = \frac{(\sqrt{a}-\sqrt{a+h})(\sqrt{a}+\sqrt{a+h})}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}$$
$$= \frac{a-(a+h)}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}$$

Therefore $\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{1}{\sqrt{a+h}}-\dfrac{1}{\sqrt{a}}\right) = \lim\limits_{h\to 0}\dfrac{-1}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})} = \dfrac{-1}{2(\sqrt{a})^3}$

Problems

36. The function is everywhere increasing and concave up. One possible graph is shown in Figure 2.96.

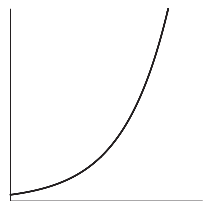


Figure 2.96

37. First note that the line $y = t$ has slope 1. From the graph, we see that

$$0 < \text{Slope at } C < \text{Slope at } B < \text{Slope between } A \text{ and } B < 1 < \text{Slope at } A.$$

Since instantaneous velocity is represented by the slope at a point and average velocity is represented by the slope between two points, we have

$$0 < \text{ Inst. vel. at } C < \text{ Inst. vel. at } B < \text{ Av. vel. between } A \text{ and } B < 1 < \text{ Inst. vel. at } A.$$

38. (a) The only graph in which the slope is 1 for all x is Graph (III).
(b) The only graph in which the slope is positive for all x is Graph (III).
(c) Graphs where the slope is 1 at $x = 2$ are Graphs (III) and (IV).
(d) Graphs where the slope is 2 at $x = 1$ are Graphs (II) and (IV).

39. (a) Velocity is zero at points A, C, F, and H.
(b) These are points where the acceleration is zero, at which the particle switches from speeding up to slowing down or vice versa.

40. (a) The derivative, $f'(t)$, appears to be positive between 2003–2005 and 2006–2007, since the number of cars increased in these intervals. The derivative, $f'(t)$, appears to be negative from 2005–2006, since the number of cars decreased then.
(b) We use the average rate of change formula on the interval 2005 to 2007 to estimate $f'(2006)$:

$$f'(2006) \approx \frac{135.9-136.6}{2007-2005} = \frac{-0.7}{2} = -0.35.$$

We see that $f'(2006) \approx -0.35$ million cars per year. The number of passenger cars in the US was decreasing at a rate of about 0.35 million, or 350,000, cars per year in 2006.

41. (a) If $f'(t) > 0$, the depth of the water is increasing. If $f'(t) < 0$, the depth of the water is decreasing.
(b) The depth of the water is increasing at 20 cm/min when $t = 30$ minutes.
(c) We use 1 meter $= 100$ cm, 1 hour $= 60$ min. At time $t = 30$ minutes

$$\text{Rate of change of depth} = 20\frac{\text{cm}}{\text{min}} = 20\frac{\text{cm}}{\text{min}}\cdot\frac{60 \text{ min}}{1 \text{ hr}}\cdot\frac{1 \text{ m}}{100 \text{ cm}} = 12 \text{ meters/hour}.$$

42. Since $f(t) = 45.7e^{-0.0061t}$, we have
$$f(6) = 45.7e^{-0.0061\cdot 6} = 44.058.$$
To estimate $f'(6)$, we use a small interval around 6:
$$f'(6) \approx \frac{f(6.001) - f(6)}{6.001 - 6} = \frac{45.7e^{-0.0061\cdot 6.001} - 45.7e^{-0.0061\cdot 6}}{0.001} = -0.269.$$
We see that $f(6) = 44.058$ million people and $f'(6) = -0.269$ million (that is, $-269{,}000$) people per year. Since $t = 6$ in 2015, this model predicts that the population of Ukraine will be about $44{,}058{,}000$ people in 2015 and declining at a rate of about 269,000 people per year at that time.

43. (a) The units of $R'(3)$ are thousands of dollars per (dollar per gallon).
The derivative $R'(3)$ tells us the rate of change of revenue with price. That is, $R'(3)$ gives approximately how much the revenue changes if the gas price increases by \$1 per gallon from \$3 per gallon.

(b) The units of $R^{-1}(5)$ are dollars per gallon. Thus, the units of $(R^{-1})'(5)$ are dollars/gallon per thousand dollars.
The derivative $(R^{-1})'(5)$ tells us the rate of change of price with revenue. That is, $(R^{-1})'(5)$ gives approximately how much the price of gas changes if the revenue increases by \$1000 from \$5000 to \$6000.

44. (a) A possible example is $f(x) = 1/|x-2|$ as $\lim_{x\to 2} 1/|x-2| = \infty$.

(b) A possible example is $f(x) = -1/(x-2)^2$ as $\lim_{x\to 2} -1/(x-2)^2 = -\infty$.

45. For $x < -2$, f is increasing and concave up. For $-2 < x < 1$, f is increasing and concave down. At $x = 1$, f has a maximum. For $x > 1$, f is decreasing and concave down. One such possible f is in Figure 2.97.

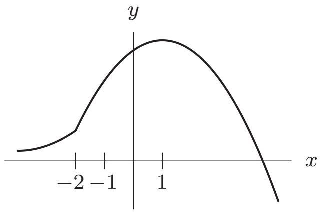


Figure 2.97

46. Since $f(2) = 3$ and $f'(2) = 1$, near $x = 2$ the graph looks like the segment shown in Figure 2.98.

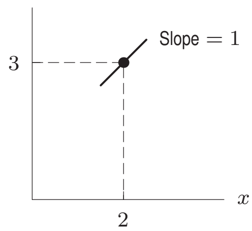


Figure 2.98

(a) If $f(x)$ is even, then the graph of $f(x)$ near $x = 2$ and $x = -2$ looks like Figure 2.99. Thus $f(-2) = 3$ and $f'(-2) = -1$.

(b) If $f(x)$ is odd, then the graph of $f(x)$ near $x = 2$ and $x = -2$ looks like Figure 2.100. Thus $f(-2) = -3$ and $f'(-2) = 1$.

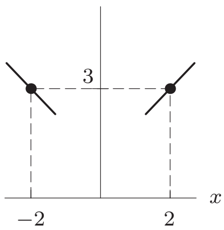


Figure 2.99: For f even

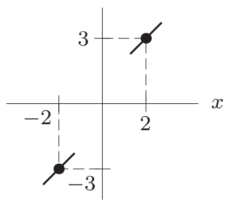


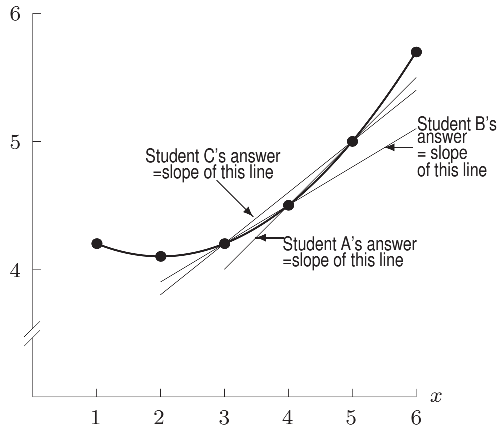
Figure 2.100: For f odd

47. The slopes of the lines drawn through successive pairs of points are negative but increasing, suggesting that $f''(x) > 0$ for $1 \le x \le 3.3$ and that the graph of $f(x)$ is concave up.

48. Using the approximation $\Delta y \approx f'(x)\Delta x$ with $\Delta x = 2$, we have $\Delta y \approx f'(20) \cdot 2 = 6 \cdot 2$, so

$$f(22) \approx f(20) + f'(20) \cdot 2 = 345 + 6 \cdot 2 = 357.$$

49. (a)

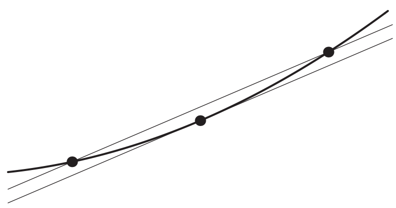


(b) The slope of f appears to be somewhere between student A's answer and student B's, so student C's answer, halfway in between, is probably the most accurate.

(c) Student A's estimate is $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, while student B's estimate is $f'(x) \approx \frac{f(x)-f(x-h)}{h}$. Student C's estimate is the average of these two, or

$$f'(x) \approx \frac{1}{2}\left[\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right] = \frac{f(x+h)-f(x-h)}{2h}.$$

This estimate is the slope of the chord connecting $(x-h, f(x-h))$ to $(x+h, f(x+h))$. Thus, we estimate that the tangent to a curve is nearly parallel to a chord connecting points h units to the right and left, as shown below.



50. (a) Since the point $A = (7, 3)$ is on the graph of f, we have $f(7) = 3$.

(b) The slope of the tangent line touching the curve at $x = 7$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{3.8 - 3}{7.2 - 7} = \frac{0.8}{0.2} = 4.$$

Thus, $f'(7) = 4$.

51. At point A, we are told that $x = 1$ and $f(1) = 3$. Since $A = (x_2, y_2)$, we have $x_2 = 1$ and $y_2 = 3$. Since $h = 0.1$, we know $x_1 = 1 - 0.1 = 0.9$ and $x_3 = 1 + 0.1 = 1.1$.

Now consider Figure 2.101. Since $f'(1) = 2$, the slope of the tangent line AD is 2. Since $AB = 0.1$,

$$\frac{\text{Rise}}{\text{Run}} = \frac{BD}{0.1} = 2,$$

so $BD = 2(0.1) = 0.2$. Therefore $y_1 = 3 - 0.2 = 2.8$ and $y_3 = 3 + 0.2 = 3.2$.

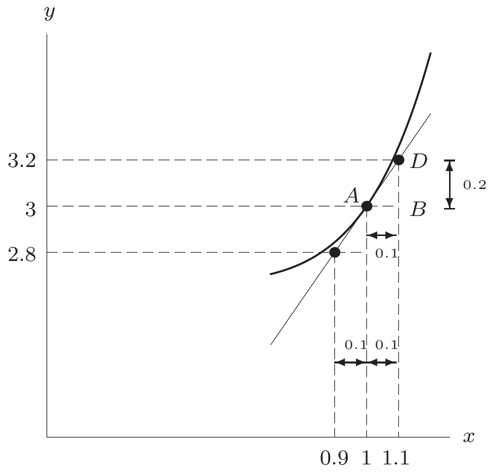


Figure 2.101

52. A possible graph of $y = f(x)$ is shown in Figure 2.102.

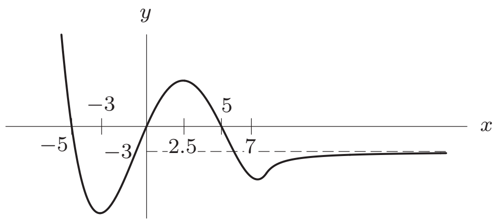


Figure 2.102

53. (a) Negative.
(b) $dw/dt = 0$ for t bigger than some t_0 (the time when the fire stops burning).
(c) $|dw/dt|$ increases, so dw/dt decreases since it is negative.

54. (a) The yam is cooling off so T is decreasing and $f'(t)$ is negative.
(b) Since $f(t)$ is measured in degrees Fahrenheit and t is measured in minutes, df/dt must be measured in units of $^\circ\text{F/min}$.

55. (a) The statement $f(140) = 120$ means that a patient weighing 140 pounds should receive a dose of 120 mg of the painkiller. The statement $f'(140) = 3$ tells us that if the weight of a patient increases by one pound (from 140 pounds), the dose should be increased by about 3 mg.
(b) Since the dose for a weight of 140 lbs is 120 mg and at this weight the dose goes up by about 3 mg for one pound, a 145 lb patient should get about an additional $3(5) = 15$ mg. Thus, for a 145 lb patient, the correct dose is approximately

$$f(145) \approx 120 + 3(5) = 135 \text{ mg}.$$

56. Suppose $p(t)$ is the average price level at time t. Then, if $t_0 =$ April 1991,
"Prices are still rising" means $p'(t_0) > 0$.
"Prices rising less fast than they were" means $p''(t_0) < 0$.
"Prices rising not as much less fast as everybody had hoped" means $H < p''(t_0)$, where H is the rate of change in rate of change of prices that people had hoped for.

57. The rate of change of the US population is $P'(t)$, so

$$P'(t) = 0.8\% \cdot \text{ Current population } = 0.008P(t).$$

58. (a) See Figure 2.103.

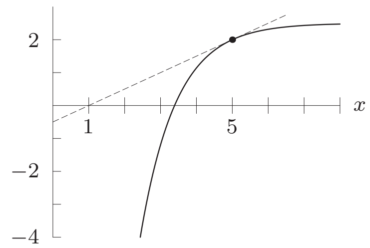


Figure 2.103

(b) Exactly one. There can't be more than one zero because f is increasing everywhere. There does have to be one zero because f stays below its tangent line (dotted line in above graph), and therefore f must cross the x-axis.

(c) The equation of the (dotted) tangent line is $y = \frac{1}{2}x - \frac{1}{2}$, and so it crosses the x-axis at $x = 1$. Therefore the zero of f must be between $x = 1$ and $x = 5$.

(d) $\lim_{x\to-\infty} f(x) = -\infty$, because f is increasing and concave down. Thus, as $x \to -\infty$, $f(x)$ decreases, at a faster and faster rate.

(e) Yes.

(f) No. The slope is decreasing since f is concave down, so $f'(1) > f'(5)$, i.e. $f'(1) > \frac{1}{2}$.

59. (a) $f'(0.6) \approx \dfrac{f(0.8) - f(0.6)}{0.8 - 0.6} = \dfrac{4.0 - 3.9}{0.2} = 0.5.$ $\quad f'(0.5) \approx \dfrac{f(0.6) - f(0.4)}{0.6 - 0.4} = \dfrac{0.4}{0.2} = 2.$

(b) Using the values of f' from part (a), we get $f''(0.6) \approx \dfrac{f'(0.6) - f'(0.5)}{0.6 - 0.5} = \dfrac{0.5 - 2}{0.1} = \dfrac{-1.5}{0.1} = -15.$

(c) The maximum value of f is probably near $x = 0.8$. The minimum value of f is probably near $x = 0.3$.

60. (a) Slope of tangent line $= \lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h}$. Using $h = 0.001$, $\frac{\sqrt{4.001}-\sqrt{4}}{0.001} = 0.249984$. Hence the slope of the tangent line is about 0.25.

(b)

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 0.25(x - 4) \\ y - 2 &= 0.25x - 1 \\ y &= 0.25x + 1 \end{aligned}$$

(c) $f(x) = kx^2$
If $(4, 2)$ is on the graph of f, then $f(4) = 2$, so $k \cdot 4^2 = 2$. Thus $k = \frac{1}{8}$, and $f(x) = \frac{1}{8}x^2$.

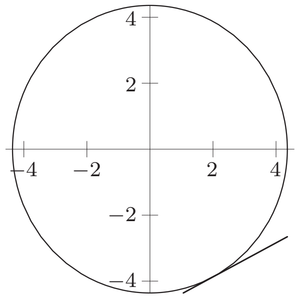
(d) To find where the graph of f crosses then line $y = 0.25x + 1$, we solve:

$$\begin{aligned} \frac{1}{8}x^2 &= 0.25x + 1 \\ x^2 &= 2x + 8 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \\ x = 4 &\text{ or } x = -2 \\ f(-2) &= \frac{1}{8}(4) = 0.5 \end{aligned}$$

Therefore, $(-2, 0.5)$ is the other point of intersection. (Of course, $(4, 2)$ is a point of intersection; we know that from the start.)

61. (a) The slope of the tangent line at $(0, \sqrt{19})$ is zero: it is horizontal.
The slope of the tangent line at $(\sqrt{19}, 0)$ is undefined: it is vertical.

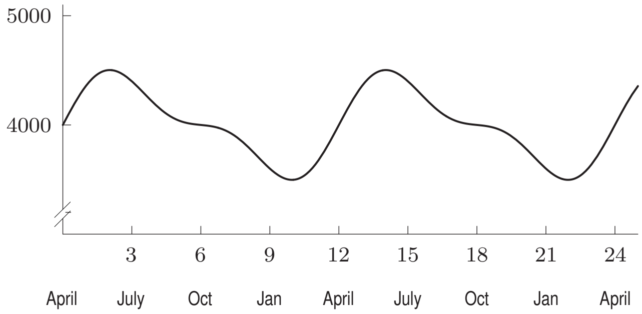
(b) The slope appears to be about $\frac{1}{2}$. (Note that when x is 2, y is about -4, but when x is 4, y is approximately -3.)



(c) Using symmetry we can determine: Slope at $(-2, \sqrt{15})$: about $\frac{1}{2}$. Slope at $(-2, -\sqrt{15})$: about $-\frac{1}{2}$. Slope at $(2, \sqrt{15})$: about $-\frac{1}{2}$.

62. (a) IV, (b) III, (c) II, (d) I, (e) IV, (f) II

63. (a) The population varies periodically with a period of 12 months (i.e. one year).



(b) The herd is largest about June 1^{st} when there are about 4500 deer.
(c) The herd is smallest about February 1^{st} when there are about 3500 deer.
(d) The herd grows the fastest about April 1^{st}. The herd shrinks the fastest about July 15 and again about December 15.
(e) It grows the fastest about April 1^{st} when the rate of growth is about 400 deer/month, i.e about 13 new fawns per day.

64. (a) The graph looks straight because the graph shows only a small part of the curve magnified greatly.
(b) The month is March: We see that about the 21^{st} of the month there are twelve hours of daylight and hence twelve hours of night. This phenomenon (the length of the day equaling the length of the night) occurs at the equinox, midway between winter and summer. Since the length of the days is increasing, and Madrid is in the northern hemisphere, we are looking at March, not September.
(c) The slope of the curve is found from the graph to be about 0.04 (the rise is about 0.8 hours in 20 days or 0.04 hours/day). This means that the amount of daylight is increasing by about 0.04 hours (about $2\frac{1}{2}$ minutes) per calendar day, or that each day is $2\frac{1}{2}$ minutes longer than its predecessor.

65. (a) A possible graph is shown in Figure 2.104. At first, the yam heats up very quickly, since the difference in temperature between it and its surroundings is so large. As time goes by, the yam gets hotter and hotter, its rate of temperature increase slows down, and its temperature approaches the temperature of the oven as an asymptote. The graph is thus concave down. (We are considering the average temperature of the yam, since the temperature in its center and on its surface will vary in different ways.)

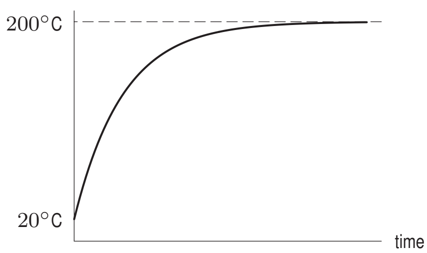


Figure 2.104

(b) If the rate of temperature increase were to remain 2°/min, in ten minutes the yam's temperature would increase $20°$, from $120°$ to $140°$. Since we know the graph is not linear, but concave down, the actual temperature is between $120°$ and $140°$.

(c) In 30 minutes, we know the yam increases in temperature by $45°$ at an average rate of $45/30 = 1.5°$/min. Since the graph is concave down, the temperature at $t = 40$ is therefore between $120 + 1.5(10) = 135°$ and $140°$.

(d) If the temperature increases at 2°/minute, it reaches $150°$ after 15 minutes, at $t = 45$. If the temperature increases at $1.5°$/minute, it reaches $150°$ after 20 minutes, at $t = 50$. So t is between 45 and 50 mins.

66. (a) We construct the difference quotient using $\text{erf}(0)$ and each of the other given values:

$$\begin{aligned} \text{erf}'(0) &\approx \frac{\text{erf}(1) - \text{erf}(0)}{1-0} = 0.84270079 \\ \text{erf}'(0) &\approx \frac{\text{erf}(0.1) - \text{erf}(0)}{0.1-0} = 1.1246292 \\ \text{erf}'(0) &\approx \frac{\text{erf}(0.01) - \text{erf}(0)}{0.01-0} = 1.128342. \end{aligned}$$

Based on these estimates, the best estimate is $\text{erf}'(0) \approx 1.12$; the subsequent digits have not yet stabilized.

(b) Using $\text{erf}(0.001)$, we have

$$\text{erf}'(0) \approx \frac{\text{erf}(0.001) - \text{erf}(0)}{0.001-0} = 1.12838$$

and so the best estimate is now 1.1283.

67. (a)

Table 2.7

x	$\frac{\sinh(x+0.001)-\sinh(x)}{0.001}$	$\frac{\sinh(x+0.0001)-\sinh(x)}{0.0001}$	so $f'(0) \approx$	$\cosh(x)$
0	1.00000	1.00000	1.00000	1.00000
0.3	1.04549	1.04535	1.04535	1.04534
0.7	1.25555	1.25521	1.25521	1.25517
1	1.54367	1.54314	1.54314	1.54308

(b) It seems that they are approximately the same, i.e. the derivative of $\sinh(x) = \cosh(x)$ for x = 0, 0.3, 0.7, and 1.

68. (a) Since the sea level is rising, we know that $a'(t) > 0$ and $m'(t) > 0$. Since the rate is accelerating, we know that $a''(t) > 0$ and $m''(t) > 0$.

(b) The rate of change of sea level for the mid-Atlantic states is between 2 and 4, we know $2 < a'(t) < 4$. (Possibly also $a'(t) = 2$ or $a'(t) = 4$.)
Similarly, $2 < m'(t) < 10$. (Possibly also $m'(t) = 2$ or $m'(t) = 10$.)

(c) (i) If $a'(t) = 2$, then sea level rise $= 2 \cdot 100 = 200$ mm.
If $a'(t) = 4$, then sea level rise $= 4 \cdot 100 = 400$ mm.
So sea level rise is between 200 mm and 400 mm.

(ii) The shortest amount of time for the sea level in the Gulf of Mexico to rise 1 meter occurs when the rate is largest, 10 mm per year. Since 1 meter $=$ 1000 mm,
shortest time to rise 1 meter $= 1000/10 = 100$ years.

CAS Challenge Problems

69. The CAS says the derivative is zero. This can be explained by the fact that $f(x) = \sin^2 x + \cos^2 x = 1$, so $f'(x)$ is the derivative of the constant function 1. The derivative of a constant function is zero.

70. (a) The CAS gives $f'(x) = 2\cos^2 x - 2\sin^2 x$. Form of answers may vary.

(b) Using the double angle formulas for sine and cosine, we have

$$\begin{aligned} f(x) &= 2\sin x \cos x = \sin(2x) \\ f'(x) &= 2\cos^2 x - 2\sin^2 x = 2(\cos^2 x - \sin^2 x) = 2\cos(2x). \end{aligned}$$

Thus we get

$$\frac{d}{dx}\sin(2x) = 2\cos(2x).$$

71. (a) The first derivative is $g'(x) = -2axe^{-ax^2}$, so the second derivative is

$$g''(x) = \frac{d^2}{dx^2}e^{-ax^2} = \frac{-2\,a}{e^{ax^2}} + \frac{4a^2x^2}{e^{ax^2}}.$$

Form of answers may vary.

(b) Both graphs get narrow as a gets larger; the graph of g'' is below the x-axis along the interval where g is concave down, and is above the x-axis where g is concave up. See Figure 2.105.

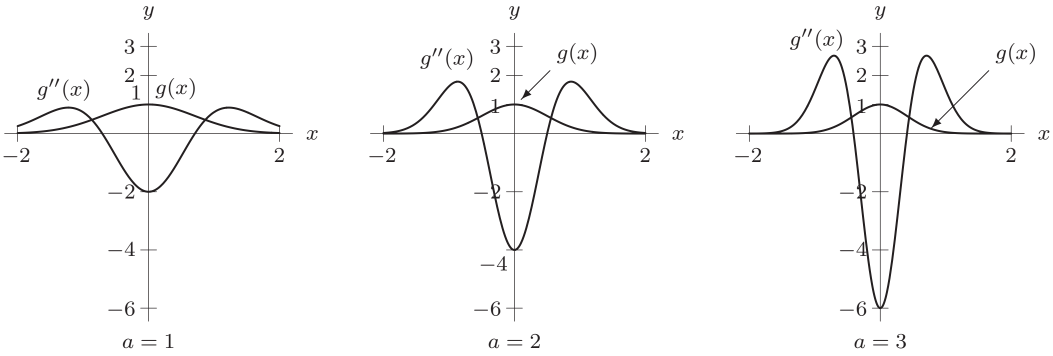


Figure 2.105

(c) The second derivative of a function is positive when the graph of the function is concave up and negative when it is concave down.

72. (a) The CAS gives the same derivative, $1/x$, in all three cases.

(b) From the properties of logarithms, $g(x) = \ln(2x) = \ln 2 + \ln x = f(x) + \ln 2$. So the graph of g is the same shape as the graph of f, only shifted up by $\ln 2$. So the graphs have the same slope everywhere, and therefore the two functions have the same derivative. By the same reasoning, $h(x) = f(x) + \ln 3$, so h and f have the same derivative as well.

73. (a) The computer algebra system gives

$$\frac{d}{dx}(x^2+1)^2 = 4x(x^2+1)$$
$$\frac{d}{dx}(x^2+1)^3 = 6x(x^2+1)^2$$
$$\frac{d}{dx}(x^2+1)^4 = 8x(x^2+1)^3$$

(b) The pattern suggests that

$$\frac{d}{dx}(x^2+1)^n = 2nx(x^2+1)^{n-1}.$$

Taking the derivative of $(x^2+1)^n$ with a CAS confirms this.

74. (a) Using a CAS, we find

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}(\sin x\cos x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1.$$

(b) The product of the derivatives of $\sin x$ and $\cos x$ is $\cos x(-\sin x) = -\cos x\sin x$. On the other hand, the derivative of the product is $\cos^2 x - \sin^2 x$, which is not the same. So no, the derivative of a product is not always equal to the product of the derivatives.

PROJECTS FOR CHAPTER TWO

1. (a) $S(0) = 12$ since the days are always 12 hours long at the equator.

(b) Since $S(0) = 12$ from part (a) and the formula gives $S(0) = a$, we have $a = 12$. Since $S(x)$ must be continuous at $x = x_0$, and the formula gives $S(x_0) = a + b\arcsin(1) = 12 + b\left(\frac{\pi}{2}\right)$ and also $S(x_0) = 24$, we must have $12 + b\left(\frac{\pi}{2}\right) = 24$ so $b\left(\frac{\pi}{2}\right) = 12$ and $b = \frac{24}{\pi} \approx 7.64$.

(c) $S(32°13') \approx 14.12$ and $S(46°4') \approx 15.58$.

(d)

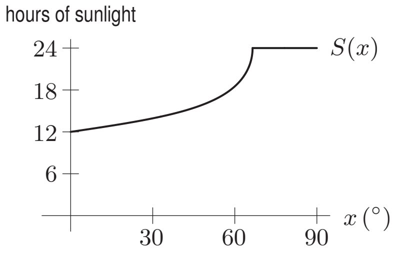


Figure 2.106

(e) The graph in Figure 2.106 appears to have a corner at $x_0 = 66°30'$. We compare the slope to the right of x_0 and to the left of x_0. To the right of S_0, the function is constant, so $S'(x) = 0$ for $x > 66°30'$.

We estimate the slope immediately to the left of x_0. We want to calculate the following:

$$\lim_{h\to 0^-} \frac{S(x_0+h) - S(x_0)}{h}.$$

We approximate it by taking $x_0 = 66.5$ and $h = -0.1,\ -0.01,\ -0.001$:

$$\frac{S(66.49) - S(66.5)}{-0.1} \approx \frac{22.3633 - 24}{-0.1} = 16.38,$$

$$\frac{S(66.499) - S(66.5)}{-0.01} \approx \frac{23.4826 - 24}{-0.01} = 51.83,$$

$$\frac{S(66.4999) - S(66.5)}{-0.001} \approx \frac{23.8370 - 24}{-0.001} = 163.9.$$

These approximations suggest that, for $x_0 = 66.5$,

$$\lim_{h\to 0^-} \frac{S(x_0+h) - S(x_0)}{h} \quad \text{does not exist.}$$

This evidence suggests that $S(x)$ is not differentiable at x_0. A proof requires the techniques found in Chapter 3.

2. (a) (i) Estimating derivatives using difference quotients (but other answers are possible):

$$P'(1900) \approx \frac{P(1910) - P(1900)}{10} = \frac{92.0 - 76.0}{10} = 1.6 \text{ million people per year}$$

$$P'(1945) \approx \frac{P(1950) - P(1940)}{10} = \frac{150.7 - 131.7}{10} = 1.9 \text{ million people per year}$$

$$P'(2000) \approx \frac{P(2000) - P(1990)}{10} = \frac{281.4 - 248.7}{10} = 3.27 \text{ million people per year}$$

(ii) The population growth rate was at its greatest at some time between 1950 and 1960.

(iii) $P'(1950) \approx \dfrac{P(1960) - P(1950)}{10} = \dfrac{179.0 - 150.7}{10} = 2.83$ million people per year,
so $P(1956) \approx P(1950) + P'(1950)(1956 - 1950) = 150.7 + 2.83(6) \approx 167.7$ million people.

(iv) If the growth rate between 2000 and 2010 was the same as the growth rate from 1990 to 2000, then the total population should be about 314 million people in 2010.

(b) (i) $f^{-1}(100)$ is the point in time when the population of the US was 100 million people (somewhere between 1910 and 1920).

(ii) The derivative of $f^{-1}(P)$ at $P = 100$ represents the ratio of change in time to change in population, and its units are years per million people. In other words, this derivative represents about how long it took for the population to increase by 1 million, when the population was 100 million.

(iii) Since the population increased by $105.7 - 92.0 = 13.7$ million people in 10 years, the average rate of increase is 1.37 million people per year. If the rate is fairly constant in that period, the amount of time it would take for an increase of 8 million people (100 million $-$ 92.0 million) would be

$$\frac{8 \text{ million people}}{1.37 \text{ million people/year}} \approx 5.8 \text{ years} \approx 6 \text{ years}$$

Adding this to our starting point of 1910, we estimate that the population of the US reached 100 million around 1916, i.e. $f^{-1}(100) \approx 1916$.

(iv) Since it took 10 years between 1910 and 1920 for the population to increase by $105.7 - 92.0 = 13.7$ million people, the derivative of $f^{-1}(P)$ at $P = 100$ is approximately

$$\frac{10 \text{ years}}{13.7 \text{ million people}} = 0.73 \text{ years/million people}$$

(c) (i) Clearly the population of the US at any instant is an integer that varies up and down every few seconds as a child is born, a person dies, or a new immigrant arrives. So $f(t)$ has "jumps;" it is not a smooth function. But these jumps are small relative to the values of f, so f appears smooth unless we zoom in very closely on its graph (to within a few seconds).

Major land acquisitions such as the Louisiana Purchase caused larger jumps in the population, but since the census is taken only every ten years and the territories acquired were rather sparsely populated, we cannot see these jumps in the census data.

(ii) We can regard rate of change of the population for a particular time t as representing an estimate of how much the population will increase during the year after time t.

(iii) Many economic indicators are treated as smooth, such as the Gross National Product, the Dow Jones Industrial Average, volumes of trading, and the price of commodities like gold. But these figures only change in increments, not continuously.

CHAPTER THREE

Solutions for Section 3.1

Exercises

1. The derivative, $f'(x)$, is defined as
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
If $f(x) = 7$, then
$$f'(x) = \lim_{h\to 0} \frac{7-7}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

2. The definition of the derivative says that
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
Therefore,
$$f'(x) = \lim_{h\to 0} \frac{[17(x+h)+11] - [17x+11]}{h} = \lim_{h\to 0} \frac{17h}{h} = 17.$$

3. The x is in the exponent and we haven't learned how to handle that yet.

4. $y' = 3x^2$. (power rule)

5. $y' = \pi x^{\pi-1}$. (power rule)

6. $y' = 12x^{11}$.

7. $y' = 11x^{10}$.

8. $y' = 11x^{-12}$.

9. $y' = -12x^{-13}$.

10. $y' = 3.2x^{2.2}$.

11. $y' = -\frac{3}{4}x^{-7/4}$.

12. $y' = \frac{4}{3}x^{1/3}$.

13. $y' = \frac{3}{4}x^{-1/4}$.

14. $\dfrac{dy}{dx} = 2x + 5$.

15. $f'(t) = 3t^2 - 6t + 8$.

16. $f'(x) = -4x^{-5}$.

17. Since $g(t) = \dfrac{1}{t^5} = t^{-5}$, we have $g'(t) = -5t^{-6}$.

18. Since $f(z) = -\dfrac{1}{z^{6.1}} = -z^{-6.1}$, we have $f'(z) = -(-6.1)z^{-7.1} = 6.1z^{-7.1}$.

19. Since $y = \dfrac{1}{r^{7/2}} = r^{-7/2}$, we have $\dfrac{dy}{dx} = -\dfrac{7}{2}r^{-9/2}$.

20. Since $y = \sqrt{x} = x^{1/2}$, we have $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2}$.

21. $f'(x) = \frac{1}{4}x^{-3/4}$.

22. Since $h(\theta) = \dfrac{1}{\sqrt[3]{\theta}} = \theta^{-1/3}$, we have $h'(\theta) = -\dfrac{1}{3}\theta^{-4/3}$.

23. Since $f(x) = \sqrt{\dfrac{1}{x^3}} = \dfrac{1}{x^{3/2}} = x^{-3/2}$, we have $f'(x) = -\dfrac{3}{2}x^{-5/2}$.

24. $h(x) = ax \cdot \ln e = ax$, so $h'(x) = a$.

25. $y' = 6x^{1/2} - \frac{5}{2}x^{-1/2}$.

26. $f'(t) = 6t - 4$.

27. $y' = 17 + 12x^{-1/2}$.

28. $y' = 2z - \frac{1}{2z^2}$.

29. The power rule gives $f'(x) = 20x^3 - \dfrac{2}{x^3}$.

30. $h'(w) = 6w^{-4} + \dfrac{3}{2}w^{-1/2}$

31. $y' = -12x^3 - 12x^2 - 6$.

32. $y' = 15t^4 - \frac{5}{2}t^{-1/2} - \frac{7}{t^2}$.

33. $y' = 6t - \frac{6}{t^{3/2}} + \frac{2}{t^3}$.

34. Since $y = \sqrt{x}(x+1) = x^{1/2}x + x^{1/2} \cdot 1 = x^{3/2} + x^{1/2}$, we have $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2} + \dfrac{1}{2}x^{-1/2}$.

35. Since $y = t^{3/2}(2 + \sqrt{t}) = 2t^{3/2} + t^{3/2}t^{1/2} = 2t^{3/2} + t^2$, we have $\frac{dy}{dx} = 3t^{1/2} + 2t$.

36. Since $h(t) = \dfrac{3}{t} + \dfrac{4}{t^2} = 3t^{-1} + 4t^{-2}$, we have $h'(t) = -3t^{-2} - 8t^{-3}$.

37. Since $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2}) = \theta\theta^{-1/2} - \theta\theta^{-2} = \theta^{1/2} - \theta^{-1}$, we have $h'(\theta) = \dfrac{1}{2}\theta^{-1/2} + \theta^{-2}$.

38. $y = x + \frac{1}{x}$, so $y' = 1 - \frac{1}{x^2}$.

39. $f(z) = \dfrac{z}{3} + \dfrac{1}{3}z^{-1} = \dfrac{1}{3}\left(z + z^{-1}\right)$, so $f'(z) = \dfrac{1}{3}\left(1 - z^{-2}\right) = \dfrac{1}{3}\left(\dfrac{z^2-1}{z^2}\right)$.

40. $g'(x) = \dfrac{d}{dx}\left(x^{\frac{1}{2}} + x^{-1} + x^{-\frac{3}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}} - x^{-2} - \dfrac{3}{2}x^{-\frac{5}{2}}$.

41. $y = \frac{\theta}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta}} = \sqrt{\theta} - \frac{1}{\sqrt{\theta}}$
$y' = \frac{1}{2\sqrt{\theta}} + \frac{1}{2\theta^{3/2}}$.

42. Since $g(t) = \dfrac{\sqrt{t}(1+t)}{t^2} = \dfrac{t^{1/2} \cdot 1 + t^{1/2}t}{t^2} = \dfrac{t^{1/2}}{t^2} + \dfrac{t^{3/2}}{t^2} = t^{-3/2} + t^{-1/2}$, we have $g'(t) = -\dfrac{3}{2}t^{-5/2} - \dfrac{1}{2}t^{-3/2}$.

43. $j'(x) = \dfrac{3x^2}{a} + \dfrac{2ax}{b} - c$

44. Since $f(x) = \dfrac{ax+b}{x} = \dfrac{ax}{x} + \dfrac{b}{x} = a + bx^{-1}$, we have $f'(x) = -bx^{-2}$.

45. Since $h(x) = \dfrac{ax+b}{c} = \dfrac{a}{c}x + \dfrac{b}{c}$, we have $h'(x) = \dfrac{a}{c}$.

46. Since $4/3$, π, and b are all constants, we have

$$\frac{dV}{dr} = \frac{4}{3}\pi(2r)b = \frac{8}{3}\pi rb.$$

47. Since w is a constant times q, we have $dw/dq = 3ab^2$.

48. Since a, b, and c are all constants, we have

$$\frac{dy}{dx} = a(2x) + b(1) + 0 = 2ax + b.$$

49. Since a and b are constants, we have

$$\frac{dP}{dt} = 0 + b\frac{1}{2}t^{-1/2} = \frac{b}{2\sqrt{t}}.$$

Problems

50. So far, we can only take the derivative of powers of x and the sums of constant multiples of powers of x. Since we cannot write $\sqrt{x+3}$ in this form, we cannot yet take its derivative.

51. The x is in the exponent and we have not learned how to handle that yet.

52. $g'(x) = \pi x^{(\pi-1)} + \pi x^{-(\pi+1)}$, by the power and sum rules.

53. $y' = 6x$. (power rule and sum rule)

54. We cannot write $\frac{1}{3x^2+4}$ as the sum of powers of x multiplied by constants.

55. $y' = -2/3z^3$. (power rule and sum rule)

56.

$$\begin{aligned} y' &= 3x^2 - 18x - 16 \\ 5 &= 3x^2 - 18x - 16 \\ 0 &= 3x^2 - 18x - 21 \\ 0 &= x^2 - 6x - 7 \\ 0 &= (x+1)(x-7) \\ x &= -1 \text{ or } x = 7. \end{aligned}$$

When $x = -1$, $y = 7$; when $x = 7$, $y = -209$.
Thus, the two points are $(-1, 7)$ and $(7, -209)$.

57. Differentiating gives

$$f'(x) = 6x^2 - 4x \quad \text{so} \quad f'(1) = 6 - 4 = 2.$$

Thus the equation of the tangent line is $(y - 1) = 2(x - 1)$ or $y = 2x - 1$.

58. (a) We have $f(2) = 8$, so a point on the tangent line is $(2, 8)$. Since $f'(x) = 3x^2$, the slope of the tangent is given by

$$m = f'(2) = 3(2)^2 = 12.$$

Thus, the equation is

$$y - 8 = 12(x - 2) \quad \text{or} \quad y = 12x - 16.$$

(b) See Figure 3.1. The tangent line lies below the function $f(x) = x^3$, so estimates made using the tangent line are underestimates.

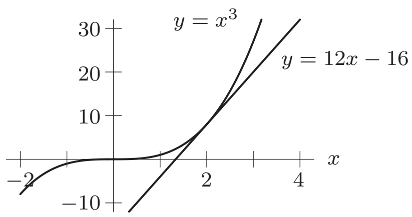


Figure 3.1

59. To calculate the equation of the tangent line to $y = f(x) = x^2 + 3x - 5$ at $x = 2$, we need to find the y-coordinate and the slope at $x = 2$. The y-coordinate is

$$y = f(2) = 2^2 + 3(2) - 5 = 5,$$

so a point on the line is $(2, 5)$. The slope is found using the derivative: $f'(x) = 2x + 3$. At the point $x = 2$, we have

$$\text{Slope} = f'(2) = 2(2) + 3 = 7.$$

The equation of the line is

$$\begin{aligned} y - 5 &= 7(x - 2) \\ y &= 7x - 9. \end{aligned}$$

When we graph the function and the line together, the line $y = 7x - 9$ appears to lie tangent to the curve $y = x^2 + 3x - 5$ at the point $x = 2$ as we expect.

60. The slope of the tangent line is the value of the first derivative at $x = 2$. Differentiating gives

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^3}{2} - \frac{4}{3x}\right) &= \frac{d}{dx}\left(\frac{1}{2}x^3 - \frac{4}{3}x^{-1}\right)\\ &= \frac{1}{2}\cdot 3x^2 - \frac{4}{3}(-1)x^{-2}\\ &= \frac{3}{2}x^2 + \frac{4}{3x^2}.\end{aligned}$$

For $x = 2$,

$$f'(2) = \frac{3}{2}(2)^2 + \frac{4}{3(2)^2} = 6 + \frac{1}{3} = 6.333$$

and

$$f(2) = \frac{2^3}{2} - \frac{4}{3(2)} = 4 - \frac{2}{3} = 3.333.$$

To find the y-intercept for the tangent line equation at the point $(2, 3.333)$, we substitute in the general equation, $y = b + mx$, and solve for b.

$$\begin{aligned}3.333 &= b + 6.333(2)\\ -9.333 &= b.\end{aligned}$$

The tangent line has the equation

$$y = -9.333 + 6.333x.$$

61. The slopes of the tangent lines to $y = x^2 - 2x + 4$ are given by $y' = 2x - 2$. A line through the origin has equation $y = mx$. So, at the tangent point, $x^2 - 2x + 4 = mx$ where $m = y' = 2x - 2$.

$$\begin{aligned}x^2 - 2x + 4 &= (2x - 2)x\\ x^2 - 2x + 4 &= 2x^2 - 2x\\ -x^2 + 4 &= 0\\ -(x + 2)(x - 2) &= 0\\ x &= 2, -2.\end{aligned}$$

Thus, the points of tangency are $(2, 4)$ and $(-2, 12)$. The lines through these points and the origin are $y = 2x$ and $y = -6x$, respectively. Graphically, this can be seen in Figure 3.2.

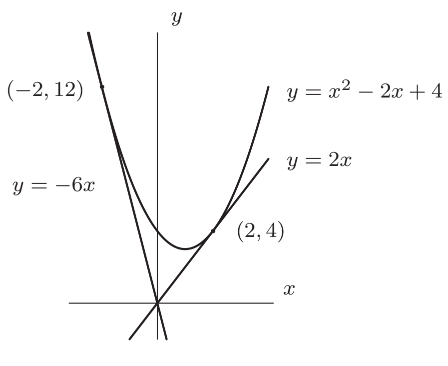


Figure 3.2

62. Decreasing means $f'(x) < 0$:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

so $f'(x) < 0$ when $x < 3$ and $x \neq 0$. Concave up means $f''(x) > 0$:

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

so $f''(x) > 0$ when

$$\begin{aligned}12x(x - 2) &> 0\\ x < 0 \quad \text{or} \quad x &> 2.\end{aligned}$$

So, both conditions hold for $x < 0$ or $2 < x < 3$.

63. The graph increases when $dy/dx > 0$:

$$\frac{dy}{dx} = 5x^4 - 5 > 0$$
$$5(x^4 - 1) > 0 \quad \text{so} \quad x^4 > 1 \quad \text{so} \quad x > 1 \text{ or } x < -1.$$

The graph is concave up when $d^2y/dx^2 > 0$:

$$\frac{d^2y}{dx^2} = 20x^3 > 0 \quad \text{so} \quad x > 0.$$

We need values of x where $\{x > 1 \text{ or } x < -1\}$ AND $\{x > 0\}$, which implies $x > 1$. Thus, both conditions hold for all values of x larger than 1.

64.

$$\begin{aligned} f'(x) = 12x^2 + 12x - 23 &\geq 1 \\ 12x^2 + 12x - 24 &\geq 0 \\ 12(x^2 + x - 2) &\geq 0 \\ 12(x + 2)(x - 1) &\geq 0. \end{aligned}$$

Hence $x \geq 1$ or $x \leq -2$.

65.

$$\begin{aligned} f'(x) &= 3(2x - 5) + 2(3x + 8) = 12x + 1 \\ f''(x) &= 12. \end{aligned}$$

66. (a) We have $p(x) = x^2 - x$. We see that $p'(x) = 2x - 1 < 0$ when $x < \frac{1}{2}$. So p is decreasing when $x < \frac{1}{2}$.

(b) We have $p(x) = x^{1/2} - x$, so

$$\begin{aligned} p'(x) = \frac{1}{2}x^{-1/2} - 1 &< 0 \\ \frac{1}{2}x^{-1/2} &< 1 \\ x^{-1/2} &< 2 \\ x^{1/2} &> \frac{1}{2} \\ x &> \frac{1}{4}. \end{aligned}$$

Thus $p(x)$ is decreasing when $x > \frac{1}{4}$.

(c) We have $p(x) = x^{-1} - x$, so

$$\begin{aligned} p'(x) = -1x^{-2} - 1 &< 0 \\ -x^{-2} &< 1 \\ x^{-2} &> -1, \end{aligned}$$

which is always true where x^{-2} is defined since $x^{-2} = 1/x^2$ is always positive. Thus $p(x)$ is decreasing for $x < 0$ and for $x > 0$.

67. Since W is proportional to r^3, we have $W = kr^3$ for some constant k. Thus, $dW/dr = k(3r^2) = 3kr^2$. Thus, dW/dr is proportional to r^2.

68. Since $f(t) = 700 - 3t^2$, we have $f(5) = 700 - 3(25) = 625$ cm. Since $f'(t) = -6t$, we have $f'(5) = -30$ cm/year. In the year 2010, the sand dune will be 625 cm high and eroding at a rate of 30 centimeters per year.

69. (a) Velocity $v(t) = \frac{dy}{dt} = \frac{d}{dt}(1250 - 16t^2) = -32t$.
Since $t \geq 0$, the ball's velocity is negative. This is reasonable, since its height y is decreasing.

(b) Acceleration $a(t) = \frac{dv}{dt} = \frac{d}{dt}(-32t) = -32$.
So its acceleration is the negative constant -32.

(c) The ball hits the ground when its height $y = 0$. This gives

$$1250 - 16t^2 = 0$$
$$t = \pm 8.84 \text{ seconds}$$

We discard $t = -8.84$ because time t is nonnegative. So the ball hits the ground 8.84 seconds after its release, at which time its velocity is

$$v(8.84) = -32(8.84) = -282.88 \text{ feet/sec} = -192.84 \text{ mph}.$$

70. (a) The average velocity between $t = 0$ and $t = 2$ is given by

$$\text{Average velocity} = \frac{f(2) - f(0)}{2 - 0} = \frac{-4.9(2^2) + 25(2) + 3 - 3}{2 - 0} = \frac{33.4 - 3}{2} = 15.2 \text{ m/sec}.$$

(b) Since $f'(t) = -9.8t + 25$, we have

$$\text{Instantaneous velocity} = f'(2) = -9.8(2) + 25 = 5.4 \text{ m/sec}.$$

(c) Acceleration is given $f''(t) = -9.8$. The acceleration at $t = 2$ (and all other times) is the acceleration due to gravity, which is -9.8 m/sec^2.

(d) We can use a graph of height against time to estimate the maximum height of the tomato. See Figure 3.3. Alternately, we can find the answer analytically. The maximum height occurs when the velocity is zero and $v(t) = -9.8t + 25 = 0$ when $t = 2.6$ sec. At this time the tomato is at a height of $f(2.6) = 34.9$. The maximum height is 34.9 meters.

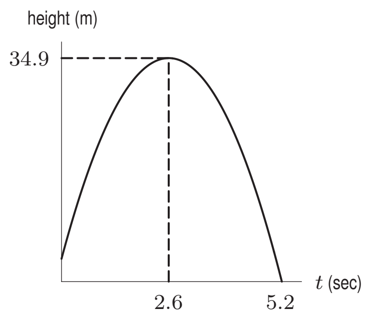


Figure 3.3

(e) We see in Figure 3.3 that the tomato hits ground at about $t = 5.2$ seconds. Alternately, we can find the answer analytically. The tomato hits the ground when

$$f(t) = -4.9t^2 + 25t + 3 = 0.$$

We solve for t using the quadratic formula:

$$t = \frac{-25 \pm \sqrt{(25)^2 - 4(-4.9)(3)}}{2(-4.9)}$$
$$t = \frac{-25 \pm \sqrt{683.8}}{-9.8}$$
$$t = -0.12 \quad \text{and} \quad t = 5.2.$$

We use the positive values, so the tomato hits the ground at $t = 5.2$ seconds.

71. Recall that $v = dx/dt$. We want to find the acceleration, dv/dt, when $x = 2$. Differentiating the expression for v with respect to t using the chain rule and substituting for v gives

$$\frac{dv}{dt} = \frac{d}{dx}(x^2 + 3x - 2) \cdot \frac{dx}{dt} = (2x + 3)v = (2x + 3)(x^2 + 3x - 2).$$

Substituting $x = 2$ gives

$$\text{Acceleration} = \left.\frac{dv}{dt}\right|_{x=2} = (2(2) + 3)(2^2 + 3 \cdot 2 - 2) = 56 \text{ cm/sec}^2.$$

72. (a) Given $h(t) = f(t) - g(t)$, this means $h'(t) = f'(t) - g'(t)$, so

$$\begin{aligned} h(0) &= f(0) - g(0) &&= 2000 - 1500 = 500 \\ h'(0) &= f'(0) - g'(0) = 11 - 13.5 &&= -2.5. \end{aligned}$$

This tells us that initially there are 500 more acre-feet of water in the first reservoir than the second, but that this difference drops at an initial rate of 2.5 acre-feet per day.

(b) Assume h' is constant, this means $h'(t) = -2.5$. It also means that h is a linear function:

$$\begin{aligned} h(t) &= \underbrace{\text{Starting value}}_{h(0)} + \underbrace{\text{Rate of change}}_{h'(0)} \times \underbrace{\text{Time}}_{t} \\ &= 500 - 2.5t. \end{aligned}$$

To find zeros of h, we write:

$$\begin{aligned} h(t) &= 0 \\ 500 - 2.5t &= 0 \\ t &= 200. \end{aligned}$$

This tells us the difference in water level will be zero—that is, the water levels will be equal—after 200 days.

73. (a) We have

$$\frac{dg}{dr} = GM\frac{d}{dr}\left(\frac{1}{r^2}\right) = GM\frac{d}{dr}\left(r^{-2}\right) = GM(-2)r^{-3} = -\frac{2GM}{r^3}.$$

(b) The derivative dg/dr is the rate of change of acceleration due to the pull of gravity with respect to distance. The further away from the center of the earth, the weaker the pull of gravity is. So g is decreasing and therefore its derivative, dg/dr, is negative.

(c) By part (a),

$$\left.\frac{dg}{dr}\right|_{r=6400} = -\left.\frac{2GM}{r^3}\right|_{r=6400} = -\frac{2(6.67 \times 10^{-20})(6 \times 10^{24})}{(6400)^3} \approx -3.05 \times 10^{-6}.$$

(d) Since the magnitude of dg/dr is small, the value of g is not changing much near $r = 6400$. It is reasonable to assume that g is a constant near the surface of the earth.

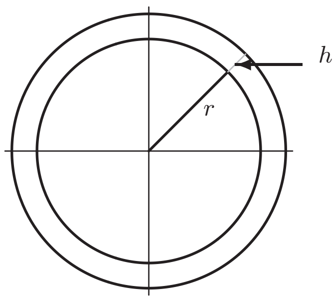
74. (a) $T = 2\pi\sqrt{\dfrac{l}{g}} = \dfrac{2\pi}{\sqrt{g}}\left(l^{\frac{1}{2}}\right)$, so $\dfrac{dT}{dl} = \dfrac{2\pi}{\sqrt{g}}\left(\dfrac{1}{2}l^{-\frac{1}{2}}\right) = \dfrac{\pi}{\sqrt{gl}}$.

(b) Since $\dfrac{dT}{dl}$ is positive, the period T increases as the length l increases.

75. (a) $A = \pi r^2$
$\frac{dA}{dr} = 2\pi r$.

(b) This is the formula for the circumference of a circle.

(c) $A'(r) \approx \frac{A(r+h)-A(r)}{h}$ for small h. When $h > 0$, the numerator of the difference quotient denotes the area of the region contained between the inner circle (radius r) and the outer circle (radius $r + h$). See figure below. As h approaches 0, this area can be approximated by the product of the circumference of the inner circle and the "width" of the region, i.e., h. Dividing this by the denominator, h, we get A' = the circumference of the circle with radius r.



We can also think about the derivative of A as the rate of change of area for a small change in radius. If the radius increases by a tiny amount, the area will increase by a thin ring whose area is simply the circumference at that radius times the small amount. To get the rate of change, we divide by the small amount and obtain the circumference.

76. $V = \frac{4}{3}\pi r^3$. Differentiating gives $\frac{dV}{dr} = 4\pi r^2 =$ surface area of a sphere.

The difference quotient $\frac{V(r+h)-V(r)}{h}$ is the volume between two spheres divided by the change in radius. Furthermore, when h is very small, the difference between volumes, $V(r+h) - V(r)$, is like a coating of paint of depth h applied to the surface of the sphere. The volume of the paint is about $h \cdot$ (Surface Area) for small h: dividing by h gives back the surface area.

Thinking about the derivative as the rate of change of the function for a small change in the variable gives another way of seeing the result. If you increase the radius of a sphere a small amount, the volume increases by a very thin layer whose volume is the surface area at that radius multiplied by that small amount.

77. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. This means $f'(1) = n \cdot 1^{n-1} = n \cdot 1 = n$, because any power of 1 equals 1.

78. Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(2) = (an)2^{n-1} = 3$, and $f'(4) = (an)4^{n-1} = 24$. Therefore,

$$\frac{f'(4)}{f'(2)} = \frac{24}{3}$$
$$\frac{(an)4^{n-1}}{(an)2^{n-1}} = \left(\frac{4}{2}\right)^{n-1} = 8$$
$$2^{n-1} = 8, \text{ and thus } n = 4.$$

Substituting $n = 4$ into the expression for $f'(2)$, we get $3 = a(4)(8)$, or $a = 3/32$.

79. Yes. To see why, we substitute $y = x^n$ into the equation $13x\dfrac{dy}{dx} = y$. We first calculate $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^n) = nx^{n-1}$. The differential equation becomes

$$13x(nx^{n-1}) = x^n$$

But $13x(nx^{n-1}) = 13n(x \cdot x^{n-1}) = 13nx^n$, so we have

$$13n(x^n) = x^n$$

This equality must hold for all x, so we get $13n = 1$, so $n = 1/13$. Thus, $y = x^{1/13}$ is a solution.

80. (a)

$$\begin{aligned}
\frac{d(x^{-1})}{dx} &= \lim_{h\to 0} \frac{(x+h)^{-1} - x^{-1}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right] \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2} = -1x^{-2}.
\end{aligned}$$

$$\begin{aligned}
\frac{d(x^{-3})}{dx} &= \lim_{h\to 0} \frac{(x+h)^{-3} - x^{-3}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^3} - \frac{1}{x^3}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^3 - (x+h)^3}{x^3(x+h)^3}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^3 - (x^3 + 3hx^2 + 3h^2x + h^3)}{x^3(x+h)^3}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{-3hx^2 - 3xh^2 - h^3}{x^3(x+h)^3}\right] \\
&= \lim_{h\to 0} \frac{-3x^2 - 3xh - h^2}{x^3(x+h)^3} \\
&= \frac{-3x^2}{x^6} = -3x^{-4}.
\end{aligned}$$

(b) For clarity, let $n = -k$, where k is a positive integer. So $x^n = x^{-k}$.

$$\frac{d(x^{-k})}{dx} = \lim_{h\to 0} \frac{(x+h)^{-k} - x^{-k}}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^k} - \frac{1}{x^k}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^k - (x+h)^k}{x^k(x+h)^k}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^k - x^k - khx^{k-1} - \overbrace{\ldots - h^k}^{\text{terms involving } h^2 \text{ and higher powers of } h}}{x^k(x+h)^k}\right] \\
&= \frac{-kx^{k-1}}{x^k(x)^k} = \frac{-k}{x^{k+1}} = -kx^{-(k+1)} = -kx^{-k-1}.
\end{aligned}
$$

81. (a) We see that $f(1) = a \cdot 1 = a$, and the graph of $x^2 + 3$ goes through the point $(1, 4)$, so $f(x)$ is continuous when $a = 4$.

(b) No, $f(x)$ does not have a derivative at $(1, 4)$. There is a corner there. We can see this without a graph by noticing that

$$\frac{d}{dx}(x^2 + 3) = 2x$$

has a value of 2 at $x = 1$, but

$$\frac{d}{dx}(4x) = 4$$

has a value of 4 at $x = 1$.

82. Since $ax + b$ and $x^2 + 3$ are each differentiable for all x, the function is differentiable on $0 < x < 2$ except at $x = 1$. For continuity, we must have $a \cdot 1 + b = 1^2 + 3$.

For differentiability, the derivatives at $x = 1$ of the two pieces must be the same. Since

$$\frac{d}{dx}(ax + b) = a \quad \text{and} \quad \frac{d}{dx}(x^2 + 3) = 2x,$$

at $x = 1$, we have $a = 2 \cdot 1 = 2$. We can then solve for b:

$$2 \cdot 1 + b = 4, \text{ so } b = 2.$$

Strengthen Your Understanding

83. Since the derivative of a constant is zero, $\frac{d}{dx}(x^2 + a) = 2x$ for any constant a.

84. The function can be written as $f(x) = x^{-2}$ so the power rule gives $f'(x) = -2x^{-3} = -2/x^3$.

85. One possible example is $f(x) = x^2$ and $g(x) = 3x$. More generally, $f(x) = x^2 + c$ and $g(x) = 3x + k$ work for any c and k.

86. Any function of the form $g(x) = x^2 + c$, where c is a positive constant works. One possibility is $g(x) = x^2 + 1$.

87. If $f(x) = 3x^2$, we have $f'(x) = 6x$ and $f''(x) = 6$. Other answers are possible.

88. True. Since $d(x^n)/dx = nx^{n-1}$, the derivative of a power function is a power function, so the derivative of a polynomial is a polynomial.

89. False, since

$$\frac{d}{dx}\left(\frac{\pi}{x^2}\right) = \frac{d}{dx}\left(\pi x^{-2}\right) = -2\pi x^{-3} = \frac{-2\pi}{x^3}.$$

90. True. The slope of $f(x) + g(x)$ at $x = 2$ is the sum of the derivatives, $f'(2) + g'(2) = 3.1 + 7.3 = 10.4$.

91. True. Since $f''(x) > 0$ and $g''(x) > 0$ for all x, we have $f''(x) + g''(x) > 0$ for all x, which means that $f(x) + g(x)$ is concave up.

92. False. Let $f(x) = 2x^2$ and $g(x) = x^2$. Then $f(x) - g(x) = x^2$, which is concave up for all x.

Solutions for Section 3.2

Exercises

1. $f'(x) = 2e^x + 2x$.
2. $y' = 10t + 4e^t$.
3. Using the chain rule gives $f'(x) = 5\ln(a)a^{5x}$.
4. $f'(x) = 12e^x + (\ln 11)11^x$.
5. $y' = 10x + (\ln 2)2^x$.
6. $f'(x) = (\ln 2)2^x + 2(\ln 3)3^x$.
7. $\dfrac{dy}{dx} = 4(\ln 10)10^x - 3x^2$.
8. $z' = (\ln 4)e^x$.
9. $\dfrac{dy}{dx} = \dfrac{1}{3}(\ln 3)3^x - \dfrac{33}{2}(x^{-\frac{3}{2}})$.
10. Since $y = 2^x + \dfrac{2}{x^3} = 2^x + 2x^{-3}$, we have $\dfrac{dy}{dx} = (\ln 2)2^x - 6x^{-4}$.
11. $z' = (\ln 4)^2 4^x$.
12. $f'(t) = (\ln(\ln 3))(\ln 3)^t$.
13. $\dfrac{dy}{dx} = 5 \cdot 5^t \ln 5 + 6 \cdot 6^t \ln 6$
14. $h'(z) = (\ln(\ln 2))(\ln 2)^z$.
15. $f'(x) = ex^{e-1}$.
16. $\dfrac{dy}{dx} = \pi^x \ln \pi$
17. $f'(x) = (\ln \pi)\pi^x$.
18. This is the sum of an exponential function and a power function, so $f'(x) = \ln(\pi)\pi^x + \pi x^{\pi-1}$.
19. Since e and k are constants, e^k is constant, so we have $f'(x) = (\ln k)k^x$.
20. $f(x) = e^{1+x} = e^1 \cdot e^x$. Then, since e^1 is just a constant,
    $f'(x) = e \cdot e^x = e^{1+x}$.
21. $f(t) = e^t \cdot e^2$. Then, since e^2 is just a constant, $f'(t) = \frac{d}{dt}(e^t e^2) = e^2 \frac{d}{dt} e^t = e^2 e^t = e^{t+2}$.
22. $f'(\theta) = ke^{k\theta}$
23. $y'(x) = a^x \ln a + ax^{a-1}$
24. $f'(x) = \pi^2 x^{(\pi^2-1)} + (\pi^2)^x \ln(\pi^2)$
25. $g'(x) = \dfrac{d}{dx}(2x - x^{-1/3} + 3^x - e) = 2 + \dfrac{1}{3x^{\frac{4}{3}}} + 3^x \ln 3$.
26. $f'(x) = 6x(e^x - 4) + (3x^2 + \pi)e^x = 6xe^x - 24x + 3x^2 e^x + \pi e^x$.

Problems

27. $y' = 2x + (\ln 2)2^x$.
28. $y' = \frac{1}{2}x^{-\frac{1}{2}} - \ln \frac{1}{2}(\frac{1}{2})^x = \frac{1}{2\sqrt{x}} + \ln 2(\frac{1}{2})^x$.
29. We can take the derivative of the sum $x^2 + 2^x$, but not the product.
30. $f(s) = 5^s e^s = (5e)^s$, so $f'(s) = \ln(5e) \cdot (5e)^s = (1 + \ln 5)5^s e^s$.
31. Since $y = e^5 e^x$, $y' = e^5 e^x = e^{x+5}$.
32. $y = e^{5x} = (e^5)^x$, so $y' = \ln(e^5) \cdot (e^5)^x = 5e^{5x}$.
33. The exponent is x^2, and we haven't learned what to do about that yet.
34. $f'(z) = (\ln \sqrt{4})(\sqrt{4})^z = (\ln 2)2^z$.

35. We can't use our rules if the exponent is $\sqrt{\theta}$.

36. This is the composition of two functions each of which we can take the derivative of, but we don't know how to take the derivative of the composition.

37. Once again, this is a product of two functions, 2^x and $\frac{1}{x}$, each of which we can take the derivative of; but we don't know how to take the derivative of the product.

38. The derivative is

$$P'(t) = 300(\ln 1.044)(1.044)^t$$

so

$$P'(5) = 300(\ln 1.044)(1.044)^5 = 16.021.$$

The value

$$P'(5) = 16.021$$

means that when $t = 5$, the population is increasing by approximately 16 animals per year.

39. Since $P = 1 \cdot (1.05)^t$, $\frac{dP}{dt} = \ln(1.05)1.05^t$. When $t = 10$,

$$\frac{dP}{dt} = (\ln 1.05)(1.05)^{10} \approx \$0.07947/\text{year} \approx 7.95¢/\text{year}.$$

40. (a) Substituting $t = 4$ gives $V(4) = 25(0.85)^4 = 25(0.522) = 13.050$. Thus the value of the car after 4 years is \$13,050.

(b) We have a function of the form $f(t) = Ca^t$. We know that such functions have a derivative of the form $(C \ln a) \cdot a^t$. Thus, $V'(t) = (25 \ln 0.85) \cdot (0.85)^t = -4.063(0.85)^t$. The units are the change in value (in thousands of dollars) with respect to time (in years), or thousands of dollars/year.

(c) Substituting $t = 4$ gives $V'(4) = -4.063(0.85)^4 = -4.063(0.522) = -2.121$. This means that at the end of the fourth year, the value of the car is decreasing by \$2121 per year.

(d) The function $V(t)$ is positive and decreasing, so that the value of the automobile is positive and decreasing. The function $V'(t)$ is negative, and its magnitude is decreasing, meaning the value of the automobile is always dropping, but the yearly loss of value decreases as time goes on. The graphs of $V(t)$ and $V'(t)$ confirm that the value of the car decreases with time. What they do not take into account are the *costs* associated with owning the vehicle. At some time, t, it is likely that the yearly costs of owning the vehicle will outweigh its value. At that time, it may no longer be worthwhile to keep the car.

41. For t in years since 2009, the population of Mexico is given by the formula

$$M = 111(1 + 0.0113)^t = 111(1.0113)^t \text{ million}$$

and that of the US by

$$U = 307(1 + 0.00975)^t = 307(1.00975)^t \text{ million},$$

The rate of change of each population, in people/year is given by

$$\left.\frac{dM}{dt}\right|_{t=0} = 111\left.\frac{d}{dt}(1.0113)^t\right|_{t=0} = \left.111(1.0113)^t \ln(1.0113)\right|_{t=0} = 1.247 \text{ million people per year}$$

$$\text{and} \quad \left.\frac{dU}{dt}\right|_{t=0} = 307\left.\frac{d}{dt}(1.00975)^t\right|_{t=0} = \left.307(1.00975)^t \ln(1.00975)\right|_{t=0} = 2.979 \text{ million people per year.}$$

Since $\left.\frac{dU}{dt}\right|_{t=0} > \left.\frac{dM}{dt}\right|_{t=0}$, the population of the US was growing faster in 2009.

42. Differentiating gives

$$\text{Rate of change of price} = \frac{dV}{dt} = 75(1.35)^t \ln 1.35 \approx 22.5(1.35)^t \text{ dollar/yr.}$$

43. (a) $f(x) = 1 - e^x$ crosses the x-axis where $0 = 1 - e^x$, which happens when $e^x = 1$, so $x = 0$. Since $f'(x) = -e^x$, $f'(0) = -e^0 = -1$.

(b) $y = -x$

(c) The negative of the reciprocal of -1 is 1, so the equation of the normal line is $y = x$.

44. Since $y = 2^x$, $y' = (\ln 2)2^x$. At $(0, 1)$, the tangent line has slope $\ln 2$ so its equation is $y = (\ln 2)x + 1$. At c, $y = 0$, so $0 = (\ln 2)c + 1$, thus $c = -\frac{1}{\ln 2}$.

45.

$$g(x) = ax^2 + bx + c \qquad f(x) = e^x$$
$$g'(x) = 2ax + b \qquad f'(x) = e^x$$
$$g''(x) = 2a \qquad f''(x) = e^x$$

So, using $g''(0) = f''(0)$, etc., we have $2a = 1$, $b = 1$, and $c = 1$, and thus $g(x) = \frac{1}{2}x^2 + x + 1$, as shown in Figure 3.4.

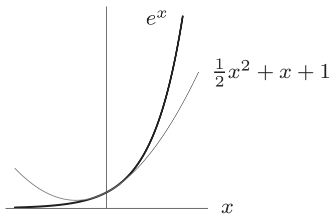


Figure 3.4

The two functions do look very much alike near $x = 0$. They both increase for large values of x, but e^x increases much more quickly. For very negative values of x, the quadratic goes to ∞ whereas the exponential goes to 0. By choosing a function whose first few derivatives agreed with the exponential when $x = 0$, we got a function which looks like the exponential for x-values near 0.

46. The first and second derivatives of e^x are e^x. Thus, the graph of $y = e^x$ is concave up. The tangent line at $x = 0$ has slope $e^0 = 1$ and equation $y = x + 1$. A graph that is always concave up is always above any of its tangent lines. Thus $e^x \geq x + 1$ for all x, as shown in Figure 3.5.

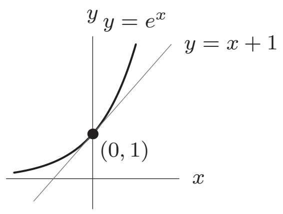


Figure 3.5

47. For $x = 0$, we have $y = a^0 = 1$ and $y = 1 + 0 = 1$, so both curves go through the point $(0, 1)$ for all values of a.

Differentiating gives

$$\left.\frac{d(a^x)}{dx}\right|_{x=0} = \left. a^x \ln a\right|_{x=0} = a^0 \ln a = \ln a$$
$$\left.\frac{d(1+x)}{dx}\right|_{x=0} = 1.$$

The graphs are tangent at $x = 0$ if

$$\ln a = 1 \qquad \text{so} \qquad a = e.$$

48. We are interested in when the derivative $\frac{d(a^x)}{dx}$ is positive and when it is negative. The quantity a^x is always positive. However $\ln a > 0$ for $a > 1$ and $\ln a < 0$ for $0 < a < 1$. Thus the function a^x is increasing for $a > 1$ and decreasing for $a < 1$.

Strengthen Your Understanding

49. The function is an exponential function so the power rule does not apply. The derivative of f is $f'(x) = (\ln 2)2^x$.

50. Since π and e are constants, f is a constant function, so $f'(x) = 0$.

51. The derivative of $f(x) = a^x$ is $f'(x) = (\ln a)a^x$ which is negative when $\ln a < 0$. Since $\ln a < 0$ when $0 < a < 1$, we see that 0.5^x is such a function. There are many other possible examples.

52. A possibility is $f(x) = e^x$. Then $f'(x) = e^x$, $f''(x) = e^x$, and $f'''(x) = e^x$, so $f'''(x) = f(x)$.

53. False. If $f(x) = \ln x$, then $f'(x) = 1/x$, which is decreasing for $x > 0$.

54. False, since $f'(x) = f(x) = e^x$ for all x.

55. False. If $f(x) = |x|$, then $f(x)$ is not differentiable at $x = 0$ and $f'(x)$ does not exist at $x = 0$.

Solutions for Section 3.3

Exercises

1. By the product rule, $f'(x) = 2x(x^3 + 5) + x^2(3x^2) = 2x^4 + 3x^4 + 10x = 5x^4 + 10x$. Alternatively, $f'(x) = (x^5 + 5x^2)' = 5x^4 + 10x$. The two answers should, and do, match.

2. Using the product rule,

$$f'(x) = (\ln 2)2^x 3^x + (\ln 3)2^x 3^x = (\ln 2 + \ln 3)(2^x \cdot 3^x) = \ln(2\cdot 3)(2\cdot 3)^x = (\ln 6)6^x$$

or, since $2^x \cdot 3^x = (2\cdot 3)^x = 6^x$,

$$f'(x) = (6^x)' = (\ln 6)(6^x).$$

The two answers should, and do, match.

3. $f'(x) = x \cdot e^x + e^x \cdot 1 = e^x(x + 1)$.

4. $y' = 2^x + x(\ln 2)2^x = 2^x(1 + x\ln 2)$.

5. $y' = \frac{1}{2\sqrt{x}}2^x + \sqrt{x}(\ln 2)2^x$.

6. $\dfrac{dy}{dt} = 2te^t + (t^2 + 3)e^t = e^t(t^2 + 2t + 3)$.

7. $f'(x) = (x^2 - x^{\frac{1}{2}}) \cdot 3^x(\ln 3) + 3^x\left(2x - \dfrac{1}{2}x^{-\frac{1}{2}}\right) = 3^x\left[(\ln 3)(x^2 - x^{\frac{1}{2}}) + \left(2x - \dfrac{1}{2\sqrt{x}}\right)\right]$.

8. $y' = (3t^2 - 14t)e^t + (t^3 - 7t^2 + 1)e^t = (t^3 - 4t^2 - 14t + 1)e^t$.

9. $f'(x) = \dfrac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \dfrac{e^x(1 - x)}{(e^x)^2} = \dfrac{1 - x}{e^x}$.

10. $g'(x) = \dfrac{50xe^x - 25x^2e^x}{e^{2x}} = \dfrac{50x - 25x^2}{e^x}$.

11. $\dfrac{dy}{dx} = \dfrac{1 \cdot 2^t - (t + 1)(\ln 2)2^t}{(2^t)^2} = \dfrac{2^t(1 - (t + 1)\ln 2)}{(2^t)^2} = \dfrac{1 - (t + 1)\ln 2}{2^t}$

12. $g'(w) = \dfrac{3.2w^{2.2}(5^w) - (\ln 5)(w^{3.2})5^w}{5^{2w}} = \dfrac{3.2w^{2.2} - w^{3.2}(\ln 5)}{5^w}$.

13. $q'(r) = \dfrac{3(5r + 2) - 3r(5)}{(5r + 2)^2} = \dfrac{15r + 6 - 15r}{(5r + 2)^2} = \dfrac{6}{(5r + 2)^2}$

14. $g'(t) = \dfrac{(t + 4) - (t - 4)}{(t + 4)^2} = \dfrac{8}{(t + 4)^2}$.

15. $\dfrac{dz}{dt} = \dfrac{3(5t + 2) - (3t + 1)5}{(5t + 2)^2} = \dfrac{15t + 6 - 15t - 5}{(5t + 2)^2} = \dfrac{1}{(5t + 2)^2}$.

16. $z' = \dfrac{(2t + 5)(t + 3) - (t^2 + 5t + 2)}{(t + 3)^2} = \dfrac{t^2 + 6t + 13}{(t + 3)^2}$.

17. $f'(t) = \dfrac{d}{dt}\left(2te^t - \dfrac{1}{\sqrt{t}}\right) = 2e^t + 2te^t + \dfrac{1}{2t^{3/2}}$.

18. Divide and then differentiate

$$f(x) = x + \frac{3}{x}$$
$$f'(x) = 1 - \frac{3}{x^2}.$$

19. $w = y^2 - 6y + 7.\quad w' = 2y - 6, y \neq 0.$

20. $g'(t) = -4(3+\sqrt{t})^{-2}\left(\frac{1}{2}t^{-1/2}\right) = \dfrac{-2}{\sqrt{t}(3+\sqrt{t})^2}$

21. $\dfrac{d}{dz}\left(\dfrac{z^2+1}{\sqrt{z}}\right) = \dfrac{d}{dz}(z^{\frac{3}{2}} + z^{-\frac{1}{2}}) = \dfrac{3}{2}z^{\frac{1}{2}} - \dfrac{1}{2}z^{-\frac{3}{2}} = \dfrac{\sqrt{z}}{2}(3 - z^{-2}).$

22.

$$\frac{dw}{dz} = \frac{(-3)(5+3z) - (5-3z)(3)}{(5+3z)^2}$$
$$= \frac{-15 - 9z - 15 + 9z}{(5+3z)^2} = \frac{-30}{(5+3z)^2}$$

23. $h'(r) = \dfrac{d}{dr}\left(\dfrac{r^2}{2r+1}\right) = \dfrac{(2r)(2r+1) - 2r^2}{(2r+1)^2} = \dfrac{2r(r+1)}{(2r+1)^2}.$

24. Notice that you can cancel a z out of the numerator and denominator to get

$$f(z) = \frac{3z}{5z+7}, \qquad z \neq 0$$

Then

$$f'(z) = \frac{(5z+7)3 - 3z(5)}{(5z+7)^2}$$
$$= \frac{15z + 21 - 15z}{(5z+7)^2}$$
$$= \frac{21}{(5z+7)^2}, z \neq 0.$$

[If you used the quotient rule correctly without canceling the z out first, your answer should simplify to this one, but it is usually a good idea to simplify as much as possible before differentiating.]

25. $w'(x) = \dfrac{17e^x(2^x) - (\ln 2)(17e^x)2^x}{2^{2x}} = \dfrac{17e^x(2^x)(1-\ln 2)}{2^{2x}} = \dfrac{17e^x(1-\ln 2)}{2^x}.$

26. $h'(p) = \dfrac{2p(3+2p^2) - 4p(1+p^2)}{(3+2p^2)^2} = \dfrac{6p + 4p^3 - 4p - 4p^3}{(3+2p^2)^2} = \dfrac{2p}{(3+2p^2)^2}.$

27. Either notice that $f(x) = \dfrac{x^2+3x+2}{x+1}$ can be written as $f(x) = \dfrac{(x+2)(x+1)}{x+1}$ which reduces to $f(x) = x+2$, giving $f'(x) = 1$, or use the quotient rule which gives

$$f'(x) = \frac{(x+1)(2x+3) - (x^2+3x+2)}{(x+1)^2}$$
$$= \frac{2x^2 + 5x + 3 - x^2 - 3x - 2}{(x+1)^2}$$
$$= \frac{x^2 + 2x + 1}{(x+1)^2}$$
$$= \frac{(x+1)^2}{(x+1)^2}$$
$$= 1.$$

28. We use the quotient rule. We have

$$f'(x) = \frac{(cx+k)(a) - (ax+b)(c)}{(cx+k)^2} = \frac{acx + ak - acx - bc}{(cx+k)^2} = \frac{ak - bc}{(cx+k)^2}.$$

29. Using the product and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= 3(x^2+5)^2(2x)(3x^3-2)^2 + (x^2+5)^3[2(3x^3-2)(9x^2)] \\ &= 3(2x)(x^2+5)^2(3x^3-2)[(3x^3-2) + (x^2+5)(3x)] \\ &= 6x(x^2+5)^2(3x^3-2)[6x^3+15x-2].\end{aligned}$$

30. $f'(x) = \frac{d}{dx}(2-4x-3x^2)(6x^e - 3\pi) = (-4-6x)(6x^e - 3\pi) + (2-4x-3x^2)(6ex^{e-1})$.

Problems

31. Using the product rule, we know that $h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is -1 so we have $g'(x) = -1$ on this interval.

(a) We have $h'(1) = f'(1)\cdot g(1) + f(1)\cdot g'(1) = 2\cdot 3 + 2(-1) = 6 - 2 = 4$.
(b) Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, we cannot use the product rule (and, in this case, $h'(2)$ does not exist).
(c) We have $h'(3) = f'(3)\cdot g(3) + f(3)\cdot g'(3) = (-2)1 + 2(-1) = -2 - 2 = -4$.

32. Using the quotient rule, we know that $k'(x) = (f'(x)\cdot g(x) - f(x)\cdot g'(x))/(g(x))^2$, when f and g are differentiable. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is -1 so we have $g'(x) = -1$ on this interval.

(a) We have
$$k'(1) = \frac{f'(1)\cdot g(1) - f(1)\cdot g'(1)}{(g(1))^2} = \frac{2\cdot 3 - 2(-1)}{3^2} = \frac{6+2}{9} = \frac{8}{9}.$$
(b) Since $f(x)$ has a corner at $x = 2$, we know that the quotient rule does not apply. We know that $f'(2)$ does not exist, and k is not differentiable at $x = 2$.
(c) We have
$$k'(3) = \frac{f'(3)\cdot g(3) - f(3)\cdot g'(3)}{(g(3))^2} = \frac{(-2)1 - 2(-1)}{1^2} = \frac{-2+2}{1} = 0.$$

33. Using the quotient rule, we know that $j'(x) = (g'(x)\cdot f(x) - g(x)\cdot f'(x))/(f(x))^2$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is -1 so we have $g'(x) = -1$ on this interval.

(a) We have
$$j'(1) = \frac{g'(1)\cdot f(1) - g(1)\cdot f'(1)}{(f(1))^2} = \frac{(-1)2 - 3\cdot 2}{2^2} = \frac{-2-6}{4} = \frac{-8}{4} = -2.$$
(b) Since $f(x)$ has a corner at $x = 2$, so the quotient rule does not apply. We know that $f'(2)$ does not exist, so $j'(2)$ does not exist.
(c) We have
$$j'(3) = \frac{g'(3)\cdot f(3) - g(3)\cdot f'(3)}{(f(3))^2} = \frac{(-1)2 - 1(-2)}{2^2} = \frac{-2+2}{4} = 0.$$

34. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the product rule,
$$h'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) \approx (0.5)2 + (-0.4)1 = 0.6.$$

35. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the quotient rule, to one decimal place
$$k'(1) = \frac{f'(1) \cdot g(1) - f(1) \cdot g'(1)}{(g(1))^2} \approx \frac{(0.5)2 - (-0.4)1}{2^2} = 0.4.$$

36. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the product rule, to one decimal place
$$h'(2) = f'(2) \cdot g(2) + f(2) \cdot g'(2) \approx 1.1(1.6) + 0.3(-0.5) = 1.6.$$

37. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the quotient rule, to one decimal place
$$k'(2) = \frac{f'(2) \cdot g(2) - f(2) \cdot g'(2)}{(g(2))^2} \approx \frac{1.1(1.6) - 0.3(-0.5)}{(1.6)^2} = 0.7.$$

38. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the quotient rule, to one decimal place
$$l'(1) = \frac{g'(1) \cdot f(1) - g(1) \cdot f'(1)}{(f(1))^2} \approx \frac{1(-0.4) - 2(0.5)}{(-0.4)^2} = -8.8.$$

39. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the quotient rule, to one decimal place
$$l'(2) = \frac{g'(2) \cdot f(2) - g(2) \cdot f'(2)}{(f(2))^2} \approx \frac{(-0.5)0.3 - 1.6(1.1)}{(0.3)^2} = -21.2.$$

40.
$$\begin{aligned} f(t) &= \frac{1}{e^t} \\ f'(t) &= \frac{e^t \cdot 0 - e^t \cdot 1}{(e^t)^2} \\ &= \frac{-1}{e^t} = -e^{-t}. \end{aligned}$$

41. $f(x) = e^x \cdot e^x$
$f'(x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$.

42.
$$\begin{aligned} f(x) &= e^x e^{2x} \\ f'(x) &= e^x (e^{2x})' + (e^x)' e^{2x} \\ &= 2e^x e^{2x} + e^x e^{2x} \text{ (from Problem 41)} \\ &= 3e^{3x}. \end{aligned}$$

43. We have
$$\begin{aligned} f'(x) &= e^x + xe^x \\ f''(x) &= e^x + e^x + xe^x = (2 + x)e^x. \end{aligned}$$
Since $f(x)$ is concave up when $f''(x) > 0$, we see that $f(x)$ is concave up when $x > -2$.

44. Using the quotient rule, we have

$$\begin{aligned} g'(x) &= \frac{0-1(2x)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2} \\ g''(x) &= \frac{-2(x^2+1)^2 + 2x(4x^3+4x)}{(x^2+1)^4} \\ &= \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} \\ &= \frac{-2(x^2+1)+8x^2}{(x^2+1)^3} \\ &= \frac{2(3x^2-1)}{(x^2+1)^3}. \end{aligned}$$

Since $(x^2+1)^3 > 0$ for all x, we have $g''(x) < 0$ if $(3x^2-1) < 0$, or when

$$\begin{aligned} 3x^2 &< 1 \\ -\frac{1}{\sqrt{3}} < x &< \frac{1}{\sqrt{3}}. \end{aligned}$$

45. Since $f(0) = -5/1 = -5$, the tangent line passes through the point $(0,-5)$, so its vertical intercept is -5. To find the slope of the tangent line, we find the derivative of $f(x)$ using the quotient rule:

$$f'(x) = \frac{(x+1)\cdot 2 - (2x-5)\cdot 1}{(x+1)^2} = \frac{7}{(x+1)^2}.$$

At $x=0$, the slope of the tangent line is $m = f'(0) = 7$. The equation of the tangent line is $y = 7x-5$.

46. This is the same function we were asked to differentiate in Problem 24, so we know that, if $x \neq 0$,

$$f'(x) = \frac{21}{(5x+7)^2}.$$

So at $x = 1$,

$$\begin{aligned} y = f(1) &= \frac{3}{12} = \frac{1}{4}, \\ y' &= \frac{21}{144} = \frac{7}{48}. \end{aligned}$$

So,

$$\begin{aligned} y - \frac{1}{4} &= \frac{7}{48}(x-1). \\ y &= \frac{7}{48}x + \frac{5}{48}. \end{aligned}$$

47. (a) Although the answer you would get by using the quotient rule is equivalent, the answer looks simpler in this case if you just use the product rule:

$$\begin{aligned} \frac{d}{dx}\left(\frac{e^x}{x}\right) &= \frac{d}{dx}\left(e^x \cdot \frac{1}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2} \\ \frac{d}{dx}\left(\frac{e^x}{x^2}\right) &= \frac{d}{dx}\left(e^x \cdot \frac{1}{x^2}\right) = \frac{e^x}{x^2} - \frac{2e^x}{x^3} \\ \frac{d}{dx}\left(\frac{e^x}{x^3}\right) &= \frac{d}{dx}\left(e^x \cdot \frac{1}{x^3}\right) = \frac{e^x}{x^3} - \frac{3e^x}{x^4}. \end{aligned}$$

(b) $\dfrac{d}{dx}\dfrac{e^x}{x^n} = \dfrac{e^x}{x^n} - \dfrac{ne^x}{x^{n+1}}.$

48. By the product rule, we have

$$\frac{d}{dx}(x^2 f(x)) = \frac{d}{dx}(x^2)\cdot f(x) + x^2 \cdot f'(x) = 2xf(x) + x^2 f'(x).$$

49. By the product rule, we have

$$\begin{aligned}\frac{d}{dx}(4^x(f(x)+g(x))) &= \frac{d}{dx}(4^x)\cdot(f(x)+g(x))+4^x\cdot\frac{d}{dx}(f(x)+g(x))\\ &= (\ln 4\cdot 4^x)(f(x)+g(x))+4^x(f'(x)+g'(x))\\ &= 4^x(\ln 4\cdot f(x)+\ln 4\cdot g(x)+f'(x)+g'(x)).\end{aligned}$$

50. By the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)+1}\right)=\frac{f'(x)(g(x)+1)-f(x)\cdot\frac{d}{dx}(g(x)+1)}{(g(x)+1)^2}=\frac{f'(x)g(x)+f'(x)-f(x)g'(x)}{(g(x)+1)^2}.$$

51. By the quotient rule, we have

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)g(x)}{h(x)}\right) &= \frac{\frac{d}{dx}(f(x)g(x))\cdot h(x)-(f(x)g(x))\cdot h'(x)}{(h(x))^2}\\ &= \frac{(f'(x)g(x)+f(x)g'(x))\cdot h(x)-f(x)g(x)h'(x)}{(h(x))^2}\\ &= \frac{f'(x)g(x)h(x)+f(x)g'(x)h(x)-f(x)g(x)h'(x)}{(h(x))^2}.\end{aligned}$$

52. (a) We have $h'(2)=f'(2)+g'(2)=5-2=3$.
(b) We have $h'(2)=f'(2)g(2)+f(2)g'(2)=5(4)+3(-2)=14$.
(c) We have $h'(2)=\dfrac{f'(2)g(2)-f(2)g'(2)}{(g(2))^2}=\dfrac{5(4)-3(-2)}{4^2}=\dfrac{26}{16}=\dfrac{13}{8}$.

53. (a) $G'(z)=F'(z)H(z)+H'(z)F(z)$, so
$G'(3)=F'(3)H(3)+H'(3)F(3)=4\cdot 1+3\cdot 5=19$.
(b) $G'(w)=\dfrac{F'(w)H(w)-H'(w)F(w)}{[H(w)]^2}$, so $G'(3)=\dfrac{4(1)-3(5)}{1^2}=-11$.

54. Use the product rule and rewrite the expression as

$$f'(x)g(x)+f(x)g'(x)-g(x)+4f'(x).$$

At $x=3$, we have

$$f'(3)g(3)+f(3)g'(3)-g(3)+4f'(3).$$

Using the given values,

$$\frac{1}{2}\cdot 12+6\cdot\frac{4}{3}-12+4\cdot\frac{1}{2}=6+8-12+2=4.$$

55. $f'(x)=10x^9e^x+x^{10}e^x$ is of the form $g'h+h'g$, where

$$g(x)=x^{10},\ g'(x)=10x^9$$

and

$$h(x)=e^x,\ h'(x)=e^x.$$

Therefore, using the product rule, let $f=g\cdot h$, with $g(x)=x^{10}$ and $h(x)=e^x$. Thus

$$f(x)=x^{10}e^x.$$

56. (a) $f(140)=15{,}000$ says that 15,000 skateboards are sold when the cost is \$140 per board.
$f'(140)=-100$ means that if the price is increased from \$140, roughly speaking, every dollar of increase will decrease the total sales by 100 boards.

(b) $\dfrac{dR}{dp} = \dfrac{d}{dp}(p \cdot q) = \dfrac{d}{dp}\left(p \cdot f(p)\right) = f(p) + pf'(p)$.
So,

$$\begin{aligned}\left.\frac{dR}{dp}\right|_{p=140} &= f(140) + 140f'(140) \\ &= 15{,}000 + 140(-100) = 1000.\end{aligned}$$

(c) From (b) we see that $\left.\dfrac{dR}{dp}\right|_{p=140} = 1000 > 0$. This means that the revenue will increase by about \$1000 if the price is raised by \$1.

57. We want dR/dr_1. Solving for R:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_2 + r_1}{r_1 r_2}, \text{ which gives } R = \frac{r_1 r_2}{r_2 + r_1}.$$

So, thinking of r_2 as a constant and using the quotient rule,

$$\frac{dR}{dr_1} = \frac{r_2(r_2 + r_1) - r_1 r_2(1)}{(r_2 + r_1)^2} = \frac{r_2^2}{(r_1 + r_2)^2}.$$

58. (a) If the museum sells the painting and invests the proceeds $P(t)$ at time t, then t years have elapsed since 2000, and the time span up to 2020 is $20 - t$. This is how long the proceeds $P(t)$ are earning interest in the bank. Each year the money is in the bank it earns 5% interest, which means the amount in the bank is multiplied by a factor of 1.05. So, at the end of $(20 - t)$ years, the balance is given by

$$B(t) = P(t)(1 + 0.05)^{20-t} = P(t)(1.05)^{20-t}.$$

(b)

$$B(t) = P(t)(1.05)^{20}(1.05)^{-t} = (1.05)^{20}\frac{P(t)}{(1.05)^t}.$$

(c) By the quotient rule,

$$B'(t) = (1.05)^{20}\left[\frac{P'(t)(1.05)^t - P(t)(1.05)^t \ln 1.05}{(1.05)^{2t}}\right].$$

So,

$$\begin{aligned}B'(10) &= (1.05)^{20}\left[\frac{5000(1.05)^{10} - 150{,}000(1.05)^{10}\ln 1.05}{(1.05)^{20}}\right] \\ &= (1.05)^{10}(5000 - 150{,}000 \ln 1.05) \\ &\approx -3776.63.\end{aligned}$$

59. Note first that $f(v)$ is in $\frac{\text{liters}}{\text{km}}$, and v is in $\frac{\text{km}}{\text{hour}}$.

(a) $g(v) = \frac{1}{f(v)}$. (This is in $\frac{\text{km}}{\text{liter}}$.) Differentiating gives

$$g'(v) = \frac{-f'(v)}{(f(v))^2}.$$

So,

$$\begin{aligned}g(80) &= \frac{1}{0.05} = 20\tfrac{\text{km}}{\text{liter}}. \\ g'(80) &= \frac{-0.0005}{(0.05)^2} = -\frac{1}{5}\tfrac{\text{km}}{\text{liter}} \text{ for each } 1\tfrac{\text{km}}{\text{hr}} \text{ increase in speed.}\end{aligned}$$

(b) $h(v) = v \cdot f(v)$. (This is in $\frac{\text{km}}{\text{hour}} \cdot \frac{\text{liters}}{\text{km}} = \frac{\text{liters}}{\text{hour}}$.) Differentiating gives

$$h'(v) = f(v) + v \cdot f'(v),$$

so

$$\begin{aligned}h(80) &= 80(0.05) = 4\tfrac{\text{liters}}{\text{hr}}. \\ h'(80) &= 0.05 + 80(0.0005) = 0.09\tfrac{\text{liters}}{\text{hr}} \text{ for each } 1\tfrac{\text{km}}{\text{hr}} \text{ increase in speed.}\end{aligned}$$

(c) Part (a) tells us that at 80 km/hr, the car can go 20 km on 1 liter. Since the first derivative evaluated at this velocity is negative, this implies that as velocity increases, fuel efficiency decreases, i.e., at higher velocities the car will not go as far on 1 liter of gas. Part (b) tells us that at 80 km/hr, the car uses 4 liters in an hour. Since the first derivative evaluated at this velocity is positive, this means that at higher velocities, the car will use more gas per hour.

60. Assume for $g(x) \neq f(x)$, $g'(x) = g(x)$ and $g(0) = 1$. Then for

$$h(x) = \frac{g(x)}{e^x}$$

$$h'(x) = \frac{g'(x)e^x - g(x)e^x}{(e^x)^2} = \frac{e^x(g'(x) - g(x))}{(e^x)^2} = \frac{g'(x) - g(x)}{e^x}.$$

But, since $g(x) = g'(x)$, $h'(x) = 0$, so $h(x)$ is constant. Thus, the ratio of $g(x)$ to e^x is constant. Since $\frac{g(0)}{e^0} = \frac{1}{1} = 1$, $\frac{g(x)}{e^x}$ must equal 1 for all x. Thus $g(x) = e^x = f(x)$ for all x, so f and g are the same function.

61. (a) $f'(x) = (x-2) + (x-1)$.

(b) Think of f as the product of two factors, with the first as $(x-1)(x-2)$. (The reason for this is that we have already differentiated $(x-1)(x-2)$).

$$f(x) = [(x-1)(x-2)](x-3).$$

Now $f'(x) = [(x-1)(x-2)]'(x-3) + [(x-1)(x-2)](x-3)'$
Using the result of a):

$$\begin{aligned} f'(x) &= [(x-2)+(x-1)](x-3) + [(x-1)(x-2)] \cdot 1 \\ &= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2). \end{aligned}$$

(c) Because we have already differentiated $(x-1)(x-2)(x-3)$, rewrite f as the product of two factors, the first being $(x-1)(x-2)(x-3)$:

$$f(x) = [(x-1)(x-2)(x-3)](x-4)$$

$$\text{Now } f'(x) = [(x-1)(x-2)(x-3)]'(x-4) + [(x-1)(x-2)(x-3)](x-4)'.$$

$$\begin{aligned} f'(x) &= [(x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)](x-4) \\ &\quad + [(x-1)(x-2)(x-3)] \cdot 1 \\ &= (x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) \\ &\quad + (x-1)(x-2)(x-4) + (x-1)(x-2)(x-3). \end{aligned}$$

From the solutions above, we can observe that when f is a product, its derivative is obtained by differentiating each factor in turn (leaving the other factors alone), and adding the results.

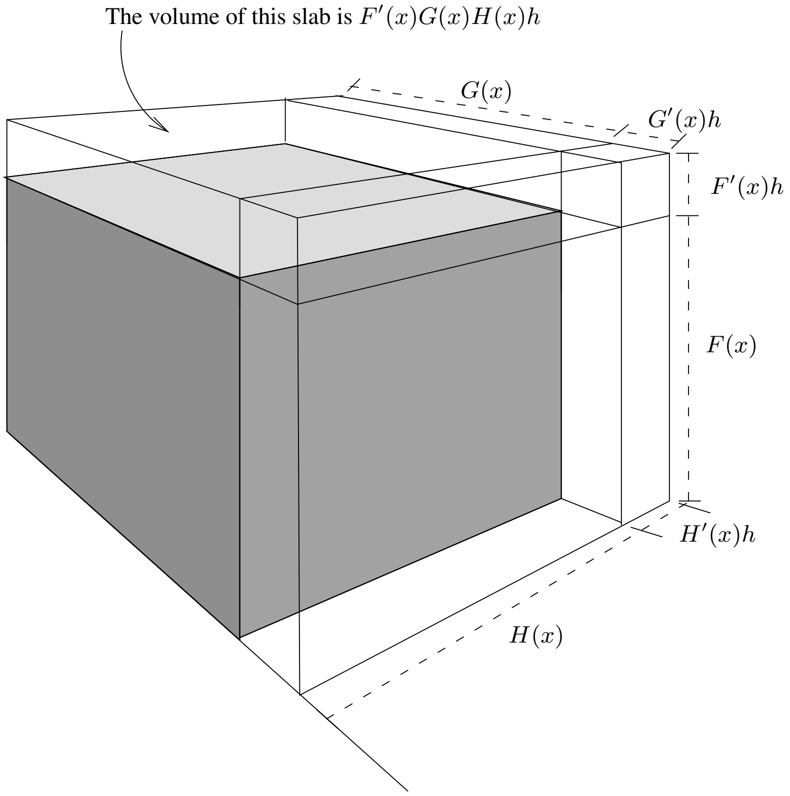
62. From the answer to Problem 61, we find that

$$\begin{aligned} f'(x) &= (x-r_1)(x-r_2)\cdots(x-r_{n-1}) \cdot 1 \\ &\quad + (x-r_1)(x-r_2)\cdots(x-r_{n-2}) \cdot 1 \cdot (x-r_n) \\ &\quad + (x-r_1)(x-r_2)\cdots(x-r_{n-3}) \cdot 1 \cdot (x-r_{n-1})(x-r_n) \\ &\quad + \cdots + 1 \cdot (x-r_2)(x-r_3)\cdots(x-r_n) \\ &= f(x)\left(\frac{1}{x-r_1} + \frac{1}{x-r_2} + \cdots + \frac{1}{x-r_n}\right). \end{aligned}$$

63. (a) We can approximate $\frac{d}{dx}[F(x)G(x)H(x)]$ using the large rectangular solids by which our original cube is increased:

Volume of whole − volume of original solid = change in volume.

$F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x)$ = change in volume.



As in the book, we will ignore the smaller regions which are added (the long, thin rectangular boxes and the small cube in the corner.) This can be justified by recognizing that as $h \to 0$, these volumes will shrink much faster than the volumes of the big slabs and will therefore be insignificant. (Note that these smaller regions have an h^2 or h^3 in the formulas of their volumes.) Then we can approximate the change in volume above by:

$$\begin{aligned} F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x) \approx & F'(x)G(x)H(x)h \quad \text{(top slab)} \\ & + F(x)G'(x)H(x)h \quad \text{(front slab)} \\ & + F(x)G(x)H'(x)h \quad \text{(other slab).} \end{aligned}$$

Dividing by h gives

$$\begin{aligned} & \frac{F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x)}{h} \\ & \approx F'(x)G(x)H(x) + F(x)G'(x)H(x) + F(x)G(x)H'(x). \end{aligned}$$

Letting $h \to 0$

$$(FGH)' = F'GH + FG'H + FGH'.$$

(b) Verifying,

$$\begin{aligned} \frac{d}{dx}[(F(x) \cdot G(x)) \cdot H(x)] &= (F \cdot G)'(H) + (F \cdot G)(H)' \\ &= [F'G + FG']H + FGH' \\ &= F'GH + FG'H + FGH' \end{aligned}$$

as before.

(c) From the answer to (b), we observe that the derivative of a product is obtained by differentiating each factor in turn (leaving the other factors alone), and adding the results. So, in general,

$$(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 \cdots f_{n-1} f_n'.$$

64. (a) Since $x = a$ is a double zero of a polynomial $P(x)$, we can write $P(x) = (x-a)^2Q(x)$, so $P(a) = 0$. Using the product rule, we have

$$P'(x) = 2(x-a)Q(x) + (x-a)^2Q'(x).$$

Substituting in $x = a$, we see $P'(a) = 0$ also.

(b) Since $P(a) = 0$, we know $x = a$ is a zero of P, so that $x - a$ is a factor of P and we can write

$$P(x) = (x-a)Q(x),$$

where Q is some polynomial. Differentiating this expression for P using the product rule, we get

$$P'(x) = Q(x) + (x-a)Q'(x).$$

Since we are told that $P'(a) = 0$, we have

$$P'(a) = Q(a) + (a-a)Q'(a) = 0$$

and so $Q(a) = 0$. Therefore $x = a$ is a zero of Q, so again we can write

$$Q(x) = (x-a)R(x),$$

where R is some other polynomial. As a result,

$$P(x) = (x-a)Q(x) = (x-a)^2R(x),$$

so that $x = a$ is a double zero of P.

65. Using the product rule

$$\begin{aligned}\frac{d^2}{dx^2}\left(f(x)g(x)\right) &= \frac{d}{dx}\left(\frac{d}{dx}\left(f(x)g(x)\right)\right) = \frac{d}{dx}\left(f'(x)g(x) + f(x)g'(x)\right)\\ &= f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)\\ &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).\end{aligned}$$

Strengthen Your Understanding

66. The derivative of x^2e^x is not the product of the derivatives of the factors. Applying the product rule gives

$$\frac{df}{dx} = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) = (2x + x^2)e^x.$$

67. The terms in the numerator of $f'(x)$ are transposed. Applying the quotient rule with $u(x) = x$ and $v(x) = x + 1$ we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\cdot v - u\cdot\frac{dv}{dx}}{v^2}$$

so

$$f'(x) = \frac{(x+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x+1)}{(x+1)^2}.$$

68. Rewrite $f(x)$ as $f(x) = e^x(x+1)$ then the product rule gives $f'(x) = e^x(x+1) + e^x\cdot 1 = (x+2)e^x$.

69. One possibility is $f(x) = e^x\sin x$. More complicated examples include $g(x) = 10^x\sin(3x)$.

70. Functions like $f(x) = x^2$ can be differentiated using the power rule. However, we can also view it as $f(x) = x\cdot x$ and apply the product rule.

71. False. The product rule gives

$$(fg)' = fg' + f'g.$$

Differentiating this and using the product rule again, we get

$$(fg)'' = f'g' + fg'' + f'g' + f''g = fg'' + 2f'g' + f''g.$$

Thus, the right hand side is not equal to $fg'' + f''g$ in general.

72. True; looking at the statement from the other direction, if both $f(x)$ and $g(x)$ are differentiable at $x = 1$, then so is their quotient, $f(x)/g(x)$, as long as it is defined there, which requires that $g(1) \neq 0$. So the only way in which $f(x)/g(x)$ can be defined but not differentiable at $x = 1$ is if either $f(x)$ or $g(x)$, or both, is not differentiable there.

73. False. Let $f(x) = x^2$ and $g(x) = x^2 - 1$. Let $h(x) = f(x)g(x)$. Then $h''(x) = 12x^2 - 2$. Since $h''(0) < 0$, clearly h is not concave up for all x.

74. (a) This is not a counterexample. Although the product rule says that $(fg)' = f'g + fg'$, that does not rule out the possibility that also $(fg)' = f'g'$. In fact, if f and g are both constant functions, then both $f'g + fg'$ and $f'g'$ are zero, so they are equal to each other.

(b) This is not a counterexample. In fact, it agrees with the product rule:

$$\frac{d}{dx}(xf(x)) = \left(\frac{d}{dx}(x)\right) f(x) + x\frac{d}{dx}f(x) = f(x) + xf'(x) = xf'(x) + f(x).$$

(c) This is not a counterexample. Although the product rule says that

$$\frac{d}{dx}(f(x)^2) = \frac{d}{dx}f(x) \cdot f(x) = f'(x)f(x) + f(x)f'(x) = 2f(x)f'(x),$$

it could be true that $f'(x) = 1$, so that the derivative is also just $2f(x)$. In fact, $f(x) = x$ is an example where this happens.

(d) This would be a counterexample. If $f'(a) = g'(a) = 0$, then

$$\frac{d}{dx}(f(x)g(x))\Big|_{x=a} = f'(a)g(a) + f(a)g'(a) = 0.$$

So fg cannot have positive slope at $x = a$. Of course such a counterexample could not exist, since the product rule is true.

Solutions for Section 3.4

Exercises

1. $f'(x) = 99(x+1)^{98} \cdot 1 = 99(x+1)^{98}$.

2. $w' = 100(t^3+1)^{99}(3t^2) = 300t^2(t^3+1)^{99}$.

3. $g'(x) = \frac{d}{dx}\left((4x^2+1)^7\right) = 7(4x^2+1)^6 \frac{d}{dx}(4x^2+1) = 7(4x^2+1)^6 \cdot 8x = 56x(4x^2+1)^6$.

4. $f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$.

5. $\frac{dy}{dx} = \frac{d}{dx}(\sqrt{e^x+1}) = \frac{d}{dx}(e^x+1)^{1/2} = \frac{1}{2}(e^x+1)^{-1/2}\frac{d}{dx}(e^x+1) = \frac{e^x}{2\sqrt{e^x+1}}$.

6. $w' = 100(\sqrt{t}+1)^{99}\left(\frac{1}{2\sqrt{t}}\right) = \frac{50}{\sqrt{t}}(\sqrt{t}+1)^{99}$.

7. $h'(w) = 5(w^4 - 2w)^4(4w^3 - 2)$

8. $s'(t) = 3(3t^2+4t+1)^2 \cdot (6t+4) = 3(6t+4)(3t^2+4t+1)^2$.

9. We can write $w(r) = (r^4+1)^{1/2}$, so

$$w'(r) = \frac{1}{2}(r^4+1)^{-1/2}(4r^3) = \frac{2r^3}{\sqrt{r^4+1}}.$$

10. $k'(x) = 4(x^3+e^x)^3(3x^2+e^x)$.

11. $f'(x) = 2e^{2x}[x^2+5^x] + e^{2x}[2x+(\ln 5)5^x] = e^{2x}[2x^2+2x+(\ln 5+2)5^x]$.

12. $y' = \frac{3}{2}e^{\frac{3}{2}w}$.

13. $g(x) = \pi e^{\pi x}$.

14. $\frac{dB}{dt} = 15e^{0.20t} \cdot 0.20 = 3e^{0.20t}$.

15. $\frac{dw}{dx} = 100e^{-x^2} \cdot (-2x) = -200xe^{-x^2}$.

16. $f(\theta) = (2^{-1})^\theta = (\frac{1}{2})^\theta$ so $f'(\theta) = (\ln \frac{1}{2})2^{-\theta}$.

17. $y' = (\ln \pi)\pi^{(x+2)}$.

18. $g'(x) = 2(\ln 3)3^{(2x+7)}$.

19. $f'(t) = 1 \cdot e^{5-2t} + te^{5-2t}(-2) = e^{5-2t}(1-2t)$.

20. $p'(t) = 4e^{4t+2}$.

21. Using the product rule gives $v'(t) = 2te^{-ct} - ce^{-ct}t^2 = (2t - ct^2)e^{-ct}$.

22. $\frac{d}{dt}e^{(1+3t)^2} = e^{(1+3t)^2}\frac{d}{dt}(1+3t)^2 = e^{(1+3t)^2} \cdot 2(1+3t) \cdot 3 = 6(1+3t)e^{(1+3t)^2}$.

23. $w' = \frac{1}{2\sqrt{s}}e^{\sqrt{s}}$.

24. $y' = -4e^{-4t}$.

25. $y' = \frac{3s^2}{2\sqrt{s^3+1}}$.

26. $y' = 1 \cdot e^{-t^2} + te^{-t^2}(-2t)$

27. $f'(z) = \frac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}$.

28. $z'(x) = \frac{(\ln 2)2^x}{3\sqrt[3]{(2^x+5)^2}}$.

29. $z' = 5 \cdot \ln 2 \cdot 2^{5t-3}$.

30. $w' = \frac{3}{2}\sqrt{x^2 \cdot 5^x}[2x(5^x) + (\ln 5)(x^2)(5^x)] = \frac{3}{2}x^2\sqrt{5^{3x}}(2 + x\ln 5)$.

31. $f(y) = \left[10^{(5-y)}\right]^{\frac{1}{2}} = 10^{\frac{5}{2}-\frac{1}{2}y}$

$f'(y) = (\ln 10)\left(10^{\frac{5}{2}-\frac{1}{2}y}\right)\left(-\frac{1}{2}\right) = -\frac{1}{2}(\ln 10)(10^{\frac{5}{2}-\frac{1}{2}y})$.

32. We can write this as $f(z) = \sqrt{z}e^{-z}$, in which case it is the same as problem 27. So $f'(z) = \frac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}$.

33. $y' = \frac{\frac{2^z}{2\sqrt{z}} - (\sqrt{z})(\ln 2)(2^z)}{2^{2z}} = \frac{1 - 2z\ln 2}{2^{z+1}\sqrt{z}}$.

34. $y' = 2\left(\frac{x^2+2}{3}\right)\left(\frac{2x}{3}\right) = \frac{4}{9}x\left(x^2+2\right)$

35. We can write $h(x) = \left(\frac{x^2+9}{x+3}\right)^{1/2}$, so

$$h'(x) = \frac{1}{2}\left(\frac{x^2+9}{x+3}\right)^{-1/2}\left[\frac{2x(x+3) - (x^2+9)}{(x+3)^2}\right] = \frac{1}{2}\sqrt{\frac{x+3}{x^2+9}}\left[\frac{x^2+6x-9}{(x+3)^2}\right].$$

36.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2} \\ &= \frac{(e^x+e^{-x})^2 - (e^x-e^{-x})^2}{(e^x+e^{-x})^2} = \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2} \\ &= \frac{4}{(e^x+e^{-x})^2}\end{aligned}$$

37. $y' = \frac{-(3e^{3x}+2x)}{(e^{3x}+x^2)^2}$.

38. $h'(z) = \frac{-8b^4z}{(a+z^2)^5}$

39. $f'(x) = -\frac{1}{2}(x^3+1)^{-\frac{3}{2}} \cdot (3x^2) = -1.5x^2(x^3+1)^{-1.5}$.

40. $f'(z) = -2(e^z+1)^{-3} \cdot e^z = \dfrac{-2e^z}{(e^z+1)^3}$.

41. $w' = (2t+3)(1-e^{-2t}) + (t^2+3t)(2e^{-2t})$.

42. $h'(x) = (\ln 2)(3e^{3x})2^{e^{3x}} = 3e^{3x}2^{e^{3x}} \ln 2$.

43. $f'(x) = 6(e^{5x})(5) + (e^{-x^2})(-2x) = 30e^{5x} - 2xe^{-x^2}$.

44. $f'(x) = e^{-(x-1)^2} \cdot (-2)(x-1)$.

45.

$$\begin{aligned} f'(w) &= (e^{w^2})(10w) + (5w^2+3)(e^{w^2})(2w) \\ &= 2we^{w^2}(5+5w^2+3) \\ &= 2we^{w^2}(5w^2+8). \end{aligned}$$

46. The power and chain rules give

$$f'(\theta) = -1(e^\theta + e^{-\theta})^{-2} \cdot \frac{d}{d\theta}(e^\theta + e^{-\theta}) = -(e^\theta + e^{-\theta})^{-2}(e^\theta + e^{-\theta}(-1)) = -\left(\frac{e^\theta - e^{-\theta}}{(e^\theta + e^{-\theta})^2}\right).$$

47. We write $y = (e^{-3t^2}+5)^{1/2}$, so

$$\begin{aligned} \frac{dy}{dt} = \frac{1}{2}(e^{-3t^2}+5)^{-1/2} \cdot \frac{d}{dt}(e^{-3t^2}+5) &= \frac{1}{2}(e^{-3t^2}+5)^{-1/2} \cdot e^{-3t^2} \cdot \frac{d}{dt}(-3t^2) \\ &= \frac{1}{2}(e^{-3t^2}+5)^{-1/2} \cdot e^{-3t^2} \cdot (-6t) = -\frac{3te^{-3t^2}}{\sqrt{e^{-3t^2}+5}}. \end{aligned}$$

48. Using the product and chain rules, we have

$$\begin{aligned} \frac{dz}{dt} &= 9(te^{3t}+e^{5t})^8 \cdot \frac{d}{dt}(te^{3t}+e^{5t}) = 9(te^{3t}+e^{5t})^8(1 \cdot e^{3t} + t \cdot e^{3t} \cdot 3 + e^{5t} \cdot 5) \\ &= 9(te^{3t}+e^{5t})^8(e^{3t} + 3te^{3t} + 5e^{5t}). \end{aligned}$$

49. $f'(y) = e^{e^{(y^2)}}\left[(e^{y^2})(2y)\right] = 2ye^{[e^{(y^2)}+y^2]}$.

50. $f'(t) = 2(e^{-2e^{2t}})(-2e^{2t})2 = -8(e^{-2e^{2t}+2t})$.

51. Since a and b are constants, we have $f'(x) = 3(ax^2+b)^2(2ax) = 6ax(ax^2+b)^2$.

52. Since a and b are constants, we have $f'(t) = ae^{bt}(b) = abe^{bt}$.

53. We use the product rule. We have

$$f'(x) = (ax)(e^{-bx}(-b)) + (a)(e^{-bx}) = -abxe^{-bx} + ae^{-bx}.$$

54. Using the product and chain rules, we have

$$\begin{aligned} g'(\alpha) &= e^{\alpha e^{-2\alpha}} \cdot \frac{d}{dx}(\alpha e^{-2\alpha}) = e^{\alpha e^{-2\alpha}}(1 \cdot e^{-2\alpha} + \alpha e^{-2\alpha}(-2)) \\ &= e^{\alpha e^{-2\alpha}}(e^{-2\alpha} - 2\alpha e^{-2\alpha}) \\ &= (1-2\alpha)e^{-2\alpha}e^{\alpha e^{-2\alpha}}. \end{aligned}$$

55. We have

$$\begin{aligned} \frac{dy}{dx} &= \left(ae^{-be^{-cx}}\right)' \\ &= \left(-be^{-cx}\right)'\left(ae^{-be^{-cx}}\right) \quad \text{chain rule} \end{aligned}$$

$$= -ab\left(-ce^{-cx}\right)e^{-be^{-cx}}$$
$$= abce^{-cx}e^{-be^{-cx}}.$$

This can also be written $\frac{dy}{dx} = abce^{-be^{-cx}-cx}$.

56. One approach is to expand this at the outset:

$$\begin{aligned} y &= \left(e^x - e^{-x}\right)^2 \\ &= \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right) \\ &= \underbrace{(e^x)^2}_{e^{2x}} - \underbrace{e^x e^{-x}}_{1} - \underbrace{e^{-x}e^x}_{1} + \underbrace{\left(e^{-x}\right)^2}_{e^{-2x}} \\ &= e^{2x} + e^{-2x} - 2, \\ \text{so} \quad \frac{dy}{dx} &= \left(e^{2x} + e^{-2x} - 2\right)' \\ &= 2e^{2x} - 2e^{-2x}. \end{aligned}$$

Another approach is to use the Chain Rule:

$$\begin{aligned} \frac{dy}{dx} &= \left(\left(e^x - e^{-x}\right)^2\right)' \\ &= 2\left(e^x - e^{-x}\right)\cdot\left(e^x - e^{-x}\right)' \\ &= 2\left(e^x - e^{-x}\right)\cdot\left(e^x + e^{-x}\right) \\ &= 2\left((e^x)^2 + e^x e^{-x} - e^{-x}e^x - \left(e^{-x}\right)^2\right) \\ &= 2\left(e^{2x} - e^{-2x}\right), \end{aligned}$$

which is equivalent to our first answer.

Problems

57. When f and g are differentiable, the chain rule gives $h'(x) = f'(g(x)) \cdot g'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is -1 so we have $g'(x) = -1$ on this interval.

(a) We have $h'(1) = f'(g(1)) \cdot g'(1) = (f'(3))(-1) = (-2)(-1) = 2$.
(b) Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, the chain rule does not apply.
(c) We have $h'(3) = f'(g(3)) \cdot g'(3) = (f'(1))(-1) = 2(-1) = -2$.

58. When f and g are differentiable, the chain rule gives $u'(x) = g'(f(x)) \cdot f'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is -1 so we have $g'(x) = -1$ on this interval.

(a) We have $u'(1) = g'(f(1)) \cdot f'(1) = (g'(2))2 = (-1)2 = -2$.
(b) Since $f(x)$ has a corner at $x = 2$, the chain rule is not applicable.
(c) We have $u'(3) = g'(f(3)) \cdot f'(3) = (g'(2))(-2) = (-1)(-2) = 2$.

59. When f and g are differentiable, the chain rule gives $v'(x) = f'(f(x)) \cdot f'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

(a) To use the chain rule, we need $f'(f(1)) = f'(2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, the chain rule does not apply.
(b) Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, the chain rule does not apply.

(c) To use the chain rule, we need $f'(f(3)) = f'(2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, the chain rule does not apply.

60. When f and g are differentiable, the chain rule gives $w'(x) = g'(g(x)) \cdot g'(x)$. We use slope to compute the derivatives. Since $g(x)$ is linear on the interval shown, with slope equal to -1, we have $g'(x) = -1$ on this interval.

(a) We have $w'(1) = g'(g(1)) \cdot g'(1) = (g'(3))(-1) = (-1)(-1) = 1$.
(b) We have $w'(2) = g'(g(2)) \cdot g'(2) = (g'(2))(-1) = (-1)(-1) = 1$.
(c) We have $w'(3) = g'(g(3)) \cdot g'(3) = (g'(1))(-1) = (-1)(-1) = 1$.

61. The chain rule gives

$$\left.\frac{d}{dx}f(g(x))\right|_{x=30} = f'(g(30))g'(30) = f'(55)g'(30) = (1)(\frac{1}{2}) = \frac{1}{2}.$$

62. The chain rule gives

$$\left.\frac{d}{dx}f(g(x))\right|_{x=70} = f'(g(70))g'(70) = f'(60)g'(70) = (1)(0) = 0.$$

63. The chain rule gives

$$\left.\frac{d}{dx}g(f(x))\right|_{x=30} = g'(f(30))f'(30) = g'(20)f'(30) = (1/2)(-2) = -1.$$

64. The chain rule gives

$$\left.\frac{d}{dx}g(f(x))\right|_{x=70} = g'(f(70))f'(70) = g'(30)f'(70) = (1)(\frac{1}{2}) = \frac{1}{2}.$$

65. We have $f(2) = (2-1)^3 = 1$, so $(2, 1)$ is a point on the tangent line. Since $f'(x) = 3(x-1)^2$, the slope of the tangent line is

$$m = f'(2) = 3(2-1)^2 = 3.$$

The equation of the line is

$$y - 1 = 3(x - 2) \quad \text{or} \quad y = 3x - 5.$$

66.

$$f(x) = 6e^{5x} + e^{-x^2} \qquad f'(x) = 30e^{5x} - 2xe^{-x^2}$$
$$f(1) = 6e^5 + e^{-1} \qquad f'(1) = 30e^5 - 2(1)e^{-1}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (6e^5 + e^{-1}) &= (30e^5 - 2e^{-1})(x - 1) \\ y - (6e^5 + e^{-1}) &= (30e^5 - 2e^{-1})x - (30e^5 - 2e^{-1}) \\ y &= (30e^5 - 2e^{-1})x - 30e^5 + 2e^{-1} + 6e^5 + e^{-1} \\ &\approx 4451.66x - 3560.81. \end{aligned}$$

67. To calculate the equation of the tangent line, we need to find the function value and the slope at $t = 2$. The function value is

$$f(2) = 100e^{-0.3(2)} = 54.8812,$$

so a point on the line is $(2, 54.8812)$.

The slope is found using the derivative: $f'(t) = 100e^{-0.3t}(-0.3)$. At the point $t = 2$, we have

$$\text{Slope} = f'(2) = 100e^{-0.3(2)}(-0.3) = -16.4643.$$

The equation of the line is

$$\begin{aligned} y - 54.8812 &= -16.4643(t - 2) \\ y &= -16.464t + 87.810. \end{aligned}$$

68. The graph is concave down when $f''(x) < 0$.

$$\begin{aligned} f'(x) &= e^{-x^2}(-2x) \\ f''(x) &= \left[e^{-x^2}(-2x)\right](-2x) + e^{-x^2}(-2) \\ &= \frac{4x^2}{e^{x^2}} - \frac{2}{e^{x^2}} \\ &= \frac{4x^2-2}{e^{x^2}} < 0 \end{aligned}$$

The graph is concave down when $4x^2 < 2$. This occurs when $x^2 < \frac{1}{2}$, or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

69. We rewrite $e^{-x} = 1/e^x$ so that we can use the quotient rule, then

$$\begin{aligned} f(x) &= \frac{x}{e^x}, \\ f'(x) &= \frac{1\cdot e^x - x\cdot e^x}{(e^x)^2} = \frac{(1-x)e^x}{(e^x)^2} = \frac{1-x}{e^x}, \\ f''(x) &= \frac{-1\cdot e^x - (1-x)e^x}{(e^x)^2} = \frac{-e^x - e^x + xe^x}{(e^x)^2} = \frac{(-2+x)e^x}{(e^x)^2} = \frac{x-2}{e^x}. \end{aligned}$$

Since $e^{-x} > 0$, for all x, we have $f''(x) < 0$ if $x - 2 < 0$, that is, $x < 2$.

70.

$$\begin{aligned} f'(x) &= [10(2x+1)^9(2)][(3x-1)^7] + [(2x+1)^{10}][7(3x-1)^6(3)] \\ &= (2x+1)^9(3x-1)^6[20(3x-1) + 21(2x+1)] \\ &= [(2x+1)^9(3x-1)^6](102x+1) \end{aligned}$$

$$\begin{aligned} f''(x) = \; & [9(2x+1)^8(2)(3x-1)^6 + (2x+1)^9(6)(3x-1)^5(3)](102x+1) \\ & + (2x+1)^9(3x-1)^6(102). \end{aligned}$$

71. (a) $P(12) = 10e^{0.6(12)} = 10e^{7.2} \approx 13{,}394$ fish. There are 13,394 fish in the area after 12 months.

(b) We differentiate to find $P'(t)$, and then substitute in to find $P'(12)$:

$$\begin{aligned} P'(t) &= 10(e^{0.6t})(0.6) = 6e^{0.6t} \\ P'(12) &= 6e^{0.6(12)} \approx 8037 \text{ fish/month.} \end{aligned}$$

The population is growing at a rate of approximately 8037 fish per month.

72. (a) With μ and σ constant, differentiating $m(t) = e^{\mu t + \sigma^2 t^2/2}$ with respect to t gives

$$m'(t) = e^{ut+\sigma^2 t^2/2} \cdot \left(\mu + \frac{2\sigma^2 t}{2}\right) = e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t).$$

Thus,

$$\text{Mean } = m'(0) = e^0(\mu + 0) = \mu.$$

(b) Differentiating $m'(t) = e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)$, we have

$$m''(t) = e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)^2 + e^{\mu t + \sigma^2 t^2/2}\sigma^2.$$

Thus

$$\text{Variance } = m''(0) - (m'(0))^2 = e^0\mu^2 + e^0\sigma^2 - \mu^2 = \sigma^2.$$

73. (a) If

$$p(x) = k(2x),$$

then

$$p'(x) = k'(2x)\cdot 2.$$

When $x = \frac{1}{2}$,

$$p'\left(\frac{1}{2}\right) = k'\left(2\cdot\frac{1}{2}\right)(2) = 2\cdot 2 = 4.$$

(b) If

$$q(x) = k(x+1),$$

then

$$q'(x) = k'(x+1)\cdot 1.$$

When $x = 0$,

$$q'(0) = k'(0+1)(1) = 2\cdot 1 = 2.$$

(c) If

$$r(x) = k\left(\frac{1}{4}x\right),$$

then

$$r'(x) = k'\left(\frac{1}{4}x\right)\cdot\frac{1}{4}.$$

When $x = 4$,

$$r'(4) = k'\left(\frac{1}{4}4\right)\frac{1}{4} = 2\cdot\frac{1}{4} = \frac{1}{2}.$$

74. Yes. To see why, simply plug $x = \sqrt[3]{2t+5}$ into the expression $3x^2\dfrac{dx}{dt}$ and evaluate it. To do this, first we calculate $\dfrac{dx}{dt}$. By the chain rule,

$$\frac{dx}{dt} = \frac{d}{dt}(2t+5)^{\frac{1}{3}} = \frac{2}{3}(2t+5)^{-\frac{2}{3}} = \frac{2}{3}[(2t+5)^{\frac{1}{3}}]^{-2}.$$

But since $x = (2t+5)^{\frac{1}{3}}$, we have (by substitution)

$$\frac{dx}{dt} = \frac{2}{3}x^{-2}.$$

It follows that $3x^2\dfrac{dx}{dt} = 3x^2\left(\dfrac{2}{3}x^{-2}\right) = 2.$

75. We see that $m'(x)$ is nearly of the form $f'(g(x))\cdot g'(x)$ where

$$f(g) = e^g \quad\text{and}\quad g(x) = x^6,$$

but $g'(x)$ is off by a multiple of 6. Therefore, using the chain rule, let

$$m(x) = \frac{f(g(x))}{6} = \frac{e^{(x^6)}}{6}.$$

76. (a) $H(x) = F(G(x))$
$H(4) = F(G(4)) = F(2) = 1$

(b) $H(x) = F(G(x))$
$H'(x) = F'(G(x))\cdot G'(x)$
$H'(4) = F'(G(4))\cdot G'(4) = F'(2)\cdot 6 = 5\cdot 6 = 30$

(c) $H(x) = G(F(x))$
$H(4) = G(F(4)) = G(3) = 4$

(d) $H(x) = G(F(x))$
$H'(x) = G'(F(x))\cdot F'(x)$
$H'(4) = G'(F(4))\cdot F'(4) = G'(3)\cdot 7 = 8\cdot 7 = 56$

(e) $H(x) = \frac{F(x)}{G(x)}$
$H'(x) = \frac{G(x)\cdot F'(x) - F(x)\cdot G'(x)}{[G(x)]^2}$
$H'(4) = \frac{G(4)\cdot F'(4) - F(4)\cdot G'(4)}{[G(4)]^2} = \frac{2\cdot 7 - 3\cdot 6}{2^2} = \frac{14-18}{4} = \frac{-4}{4} = -1$

77. (a) Differentiating $g(x) = \sqrt{f(x)} = (f(x))^{1/2}$, we have

$$g'(x) = \frac{1}{2}(f(x))^{-1/2}\cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

$$g'(1) = \frac{f'(1)}{2\sqrt{f(1)}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}.$$

(b) Differentiating $h(x) = f(\sqrt{x})$, we have

$$h'(x) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$
$$h'(1) = f'(\sqrt{1}) \cdot \frac{1}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{3}{2}.$$

78. (a) Using the chain rule, $h'(x) = e^{f(x)} \cdot f'(x)$. Since $e^{f(x)} \neq 0$ for all values of x, the derivative $h'(x)$ is zero when $f'(x) = 0$ and only then.

From the graph we see that $f'(x) = 0$ occurs when $x \approx -2.5$ and $x \approx 1.5$.

(b) Taking the derivative of p, we have $p'(x) = f'(e^x) \cdot e^x$. Since $e^x \neq 0$ for any value of x, the derivative $p'(x)$ is zero only when $f'(e^x) = 0$.

From the graph we see that $f'(y) = 0$ for $y \approx 1.5$ and $y \approx -2.5$. Thus, $f'(e^x)$ equals zero when $e^x = 1.5$ or $e^x = -2.5$. Since e^x never equals -2.5, the only solution to $f'(e^x) = 0$ is $x = \ln(1.5)$.

79. We have $h(0) = f(g(0)) = f(d) = d$. From the chain rule, $h'(0) = f'(g(0))g'(0)$. From the graph of g, we see that $g'(0) = 0$, so $h'(0) = f'(g(0)) \cdot 0 = 0$.

80. We have $h(-c) = f(g(-c)) = f(-b) = 0$. From the chain rule,

$$h'(-c) = f'(g(-c))g'(-c).$$

Since g is increasing at $x = -c$, we know that $g'(-c) > 0$. We have

$$f'(g(-c)) = f'(-b),$$

and since f is decreasing at $x = -b$, we have $f'(g(-c)) < 0$. Thus,

$$h'(-c) = \underbrace{f'(g(-c))}_{-} \cdot \underbrace{g'(-c)}_{+} < 0,$$

so h is decreasing at $x = -c$.

81. We have

$$h'(a) = f'(g(a))g'(a).$$

From the graph of g, we see that g is decreasing at $x = a$, so $g'(a) < 0$. We have

$$f'(g(a)) = f'(b),$$

and from the graph of f, we see that f is increasing at $x = b$, so $f'(b) > 0$. Thus,

$$h'(a) = \underbrace{f'(g(a))}_{+} \cdot \underbrace{g'(a)}_{-} < 0,$$

so h is decreasing at $x = a$.

82. We have $h(d) = f(g(d)) = f(-d) = d$ so $h(d)$ is positive. From the chain rule,

$$h'(d) = f'(g(d))g'(d).$$

We have

$$f'(g(d)) = f'(-d).$$

From the graph of f, we see that $f'(-d) < 0$, and from the graph of g, we see that $g'(d) < 0$. This means the sign of $h'(d)$ is the product of two negative numbers, so $h'(d) > 0$.

83. On the interval $-d < x < -b$, we see that the value of $g(x)$ increases from $-d$ to 0. On the interval $-d < x < 0$, the value of $f(x)$ decreases from d to $-d$. Thus, the value of $h(x) = f(g(x))$ decreases on the interval $-d < x < -b$ from

$$h(-d) = f(g(-d)) = f(-d) = d \quad \text{to} \quad h(-b) = f(g(-b)) = f(0) - d.$$

Confirming this using derivatives and the chain rule, we see

$$h'(x) = f'(g(x)) \cdot g'(x),$$

and since $g'(x)$ is negative on $-d < x < -b$ and $f'(g(x))$ is positive on this interval, the value of $h(x)$ is decreasing.

84. We have $f(0) = 6.91$ and $f(10) = 6.91e^{0.011(10)} = 7.713$. The derivative of $f(t)$ is

$$f'(t) = 6.91e^{0.011t} \cdot 0.011 = 0.0760e^{0.011t},$$

and so $f'(0) = 0.0760$ and $f'(10) = 0.0848$.

These values tell us that in 2010 (at $t = 0$), the population of the world was 6.91 billion people and the population was growing at a rate of 0.0760 billion people per year. In the year 2020 (at $t = 10$), this model predicts that the population of the world will be 7.71 billion people and growing at a rate of 0.0848 billion people per year.

85. Since the population is 308.75 million on April 1, 2010 and growing exponentially, $P = 308.75e^{kt}$, where P is the population in millions and t is time in years since the census in 2010. Then

$$\frac{dP}{dt} = 308.75ke^{kt},$$

so, since the population is growing at 2.85 million/year on April 1, 2010,

$$\left.\frac{dP}{dt}\right|_{t=0} = 308.75ke^{k\cdot 0} = 308.75k = 2.85$$

$$k = \frac{2.85}{308.75} = 0.00923,$$

so $P = 308.75e^{0.00923t}$.

86. (a) The function $f(t)$ is linear; $g(t)$ is exponential; $h(t)$ is quadratic (polynomial of degree 2.)

(b) In 2010, we have $t = 60$. For $f(t)$, the rate of change is

$$f'(t) = 1.3,$$

$$f'(60) = 1.3 \text{ ppm per year.}$$

For $g(t)$, the rate of change is

$$g'(t) = 304(0.0038)e^{0.0038t} = 1.1552e^{0.0038t},$$

$$g'(60) = 1.1552e^{0.0038(60)} = 1.451 \text{ ppm per year .}$$

For $h(t)$, the rate of change is

$$h'(t) = 0.0135 \cdot 2t + 0.5133 = 0.027t + 0.5135,$$

$$h'(60) = 0.027(60) + 0.5133 = 2.133 \text{ ppm per year .}$$

(c) Since $1.3 < 1.451 < 2.133$, we see
Linear prediction < Exponential < Quadratic.

(d) The linear growth rate remains constant and will always be the smallest. The other two growth rates increase with time, and eventually the exponential growth rate will overtake the quadratic growth rate.
For example, if $t = 1000$, we have

$$f'(1000) = 1.3$$

$$g'(1000) = 1.1552e^{0.0038(1000)} = 51.639$$

$$h'(1000) = 0.027(1000) + 0.5133 = 27.513.$$

Thus, at $t = 1000$, the exponential growth rate is largest.

87. We have $f(5) = 5.1e^{0.043(5)} = 6.3$ billion dollars. Since $f'(t) = 5.1e^{0.043t} \cdot 0.043$, we have

$$f'(5) = 5.1e^{0.043(5)}(0.043) = 0.272 \text{ billion dollars per year.}$$

In 2013, net sales at Hershey are predicted to be 6.3 billion dollars and to be increasing at a rate of 0.272 billion dollars per year.

88. Since we're given that the instantaneous rate of change of T at $t = 30$ is 2, we want to choose a and b so that the derivative of T agrees with this value. Differentiating, $T'(t) = ab \cdot e^{-bt}$. Then we have

$$2 = T'(30) = abe^{-30b} \text{ or } e^{-30b} = \frac{2}{ab}.$$

We also know that at $t = 30$, $T = 120$, so

$$120 = T(30) = 200 - ae^{-30b} \text{ or } e^{-30b} = \frac{80}{a}.$$

Thus

$$\frac{80}{a} = e^{-30b} = \frac{2}{ab},$$

so

$$b = \frac{1}{40} = 0.025 \quad \text{and} \quad a = 169.36.$$

89. (a) $\dfrac{dB}{dt} = P\left(1+\dfrac{r}{100}\right)^t \ln\left(1+\dfrac{r}{100}\right)$. The expression $\dfrac{dB}{dt}$ tells us how fast the amount of money in the bank is changing with respect to time for fixed initial investment P and interest rate r.

(b) $\dfrac{dB}{dr} = Pt\left(1+\dfrac{r}{100}\right)^{t-1}\dfrac{1}{100}$. The expression $\dfrac{dB}{dr}$ indicates how fast the amount of money changes with respect to the interest rate r , assuming fixed initial investment P and time t.

90. (a) We see from the formula that \$1000 was deposited initially, and that the money is earning interest at 8% compounded continuously.

(b) We have $f(10) = 1000e^{0.08(10)} = 2225.54$ dollars. Since $f'(t) = 1000e^{0.08t} \cdot 0.08$, we have

$$f'(10) = 1000e^{0.08(10)}(0.08) = 178.04 \text{ dollars per year.}$$

Ten years after the money was deposited, the balance is \$2225.54 and is growing at a rate of \$178.04 per year.

91. (a)

$$\begin{aligned}\frac{dm}{dv} &= \frac{d}{dv}\left[m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]\\ &= m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)\\ &= \frac{m_0 v}{c^2}\frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)^3}}.\end{aligned}$$

(b) $\dfrac{dm}{dv}$ represents the rate of change of mass with respect to the speed v.

92. (a) For $t < 0$, $I = \dfrac{dQ}{dt} = 0$.

For $t > 0$, $I = \dfrac{dQ}{dt} = -\dfrac{Q_0}{RC}e^{-t/RC}$.

(b) For $t > 0$, $t \to 0$ (that is, as $t \to 0^+$),

$$I = -\frac{Q_0}{RC}e^{-t/RC} \to -\frac{Q_0}{RC}.$$

Since $I = 0$ just to the left of $t = 0$ and $I = -Q_0/RC$ just to the right of $t = 0$, it is not possible to define I at $t = 0$.

(c) Q is not differentiable at $t = 0$ because there is no tangent line at $t = 0$.

93. Let f have a zero of multiplicity m at $x = a$ so that

$$f(x) = (x-a)^m h(x), \quad h(a) \neq 0.$$

Differentiating this expression gives

$$f'(x) = (x-a)^m h'(x) + m(x-a)^{(m-1)}h(x)$$

and both terms in the sum are zero when $x = a$ so $f'(a) = 0$. Taking another derivative gives

$$f''(x) = (x-a)^m h''(x) + 2m(x-a)^{(m-1)} h'(x) + m(m-1)(x-a)^{(m-2)} h(x).$$

Again, each term in the sum contains a factor of $(x-a)$ to some positive power, so at $x = a$ this will evaluate to 0. Differentiating repeatedly, all derivatives will have positive integer powers of $(x-a)$ until the m^{th} and will therefore vanish. However,

$$f^{(m)}(a) = m!h(a) \neq 0.$$

94. Using the chain and product rule:

$$\begin{aligned}\frac{d^2}{dx^2}\left(f(g(x))\right) &= \frac{d}{dx}\left(\frac{d}{dx}\left(f(g(x))\right)\right) = \frac{d}{dx}\left(f'(g(x))\cdot g'(x)\right)\\ &= f''(g(x))\cdot g'(x)\cdot g'(x) + f'(g(x))\cdot g''(x)\\ &= f''(g(x))\cdot\left(g'(x)\right)^2 + f'(g(x))\cdot g''(x).\end{aligned}$$

95. Using the chain and product rules:

$$\begin{aligned}\frac{d^2}{dx^2}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(\frac{d}{dx}\left(f(x)(g(x))^{-1}\right)\right) = \frac{d}{dx}\left(f'(x)(g(x))^{-1} - f(x)(g(x))^{-2}g'(x)\right)\\ &= f''(x)(g(x))^{-1} - f'(x)(g(x))^{-2}g'(x) - f'(x)(g(x))^{-2}g'(x)\\ &\qquad + 2f(x)(g(x))^{-3}(g'(x))^2 - f(x)(g(x))^{-2}g''(x)\\ &= f''(x)(g(x))^{-1} - 2f'(x)(g(x))^{-2}g'(x) + 2f(x)(g(x))^{-3}(g'(x))^2 - f(x)(g(x))^{-2}g''(x).\end{aligned}$$

Strengthen Your Understanding

96. The derivative of the inside function, e^x, is missing.
Let $z = h(x) = e^x + 2$, so $g(z) = z^5$ and $g'(z) = 5z^4 \cdot h'(x)$. Taking the derivative of h, we have $h'(x) = e^x$, so

$$g'(x) = 5(e^x+2)^4 e^x = 5e^x(e^x+2)^4.$$

97. To calculate the derivative of e^{x^2} we need to apply the chain rule. Take $z = g(x) = x^2$ as the inside function and $f(z) = e^z$ as the outside function. Since $g'(x) = 2x$ and $f'(z) = e^z$, the chain rule gives

$$w'(x) = e^z \cdot 2x = 2xe^{x^2}.$$

98. Two possibilities are

$$f(x) = \sin(e^x) \qquad \text{and} \qquad g(x) = e^{\sin x}.$$

More complicated examples include

$$f(x) = \sin(3\cdot 10^x) \qquad \text{and} \qquad g(x) = 5\cdot 7^{3\sin 2x}.$$

99. One possibility is $f(x) = (x^2+1)^2$, which can be differentiated using the chain rule with x^2+1 as the inside function and x^2 as the outside function:

$$f'(x) = 2(x^2+1)^1 \cdot 2x.$$

In addition, we can expand the function into a polynomial: $f(x) = x^4 + 2x^2 + 1$, and now differentiate term-by-term:

$$f'(x) = 4x^3 + 4x.$$

100. False; for example, if both $f(x)$ and $g(x)$ are constant functions, such as $f(x) = 6$, $g(x) = 10$, then $(fg)'(x) = 0$, and $f'(x) = 0$ and $g'(x) = 0$.

101. False; for example, if both f and g are constant functions, then the derivative of $f(g(x))$ is zero, as is the derivative of $f(x)$. Another example is $f(x) = 5x + 7$ and $g(x) = x + 2$.

102. False. Let $f(x) = e^{-x}$ and $g(x) = x^2$. Let $h(x) = f(g(x)) = e^{-x^2}$. Then $h'(x) = -2xe^{-x^2}$ and $h''(x) = (-2 + 4x^2)e^{-x^2}$. Since $h''(0) < 0$, clearly h is not concave up for all x.

Solutions for Section 3.5

Exercises

1.

Table 3.1

x	$\cos x$	Difference Quotient	$-\sin x$
0	1.0	-0.0005	0.0
0.1	0.995	-0.10033	-0.099833
0.2	0.98007	-0.19916	-0.19867
0.3	0.95534	-0.296	-0.29552
0.4	0.92106	-0.38988	-0.38942
0.5	0.87758	-0.47986	-0.47943
0.6	0.82534	-0.56506	-0.56464

2. $r'(\theta) = \cos\theta - \sin\theta$.

3. $s'(\theta) = -\sin\theta\sin\theta + \cos\theta\cos\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta$.

4. $z' = -4\sin(4\theta)$.

5. $f'(x) = \cos(3x)\cdot 3 = 3\cos(3x)$.

6. $\dfrac{dy}{dt} = 5\cos(3t)\cdot 3 = 15\cos(3t)$.

7. $\dfrac{dP}{dt} = 4(-\sin(2t))\cdot 2 = -8\sin(2t)$.

8. $\dfrac{d}{dx}\sin(2-3x) = \cos(2-3x)\dfrac{d}{dx}(2-3x) = -3\cos(2-3x)$.

9. Using the chain rule gives $R'(x) = 3\pi\sin(\pi x)$.

10. $g'(\theta) = 2\sin(2\theta)\cos(2\theta)\cdot 2 - \pi = 4\sin(2\theta)\cos(2\theta) - \pi$

11. $g'(t) = 3(2+\sin(\pi t))^2\cdot(\cos(\pi t)\cdot\pi) = 3\pi\cos(\pi t)(2+\sin(\pi t))^2$.

12. $f'(x) = (2x)(\cos x) + x^2(-\sin x) = 2x\cos x - x^2\sin x$.

13. $w' = e^t\cos(e^t)$.

14. $f'(x) = (e^{\cos x})(-\sin x) = -\sin x e^{\cos x}$.

15. $f'(y) = (\cos y)e^{\sin y}$.

16. $z' = e^{\cos\theta} - \theta(\sin\theta)e^{\cos\theta}$.

17. Using the chain rule gives $R'(\theta) = 3\cos(3\theta)e^{\sin(3\theta)}$.

18. $g'(\theta) = \dfrac{\cos(\tan\theta)}{\cos^2\theta}$

19. $w'(x) = \dfrac{2x}{\cos^2(x^2)}$

20.

$$\begin{aligned} f(x) &= (1-\cos x)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(1-\cos x)^{-\frac{1}{2}}(-(-\sin x)) \\ &= \frac{\sin x}{2\sqrt{1-\cos x}}. \end{aligned}$$

21. $f'(x) = \frac{1}{2}(3+\sin(8x))^{-\frac{1}{2}} \cdot (\cos(8x)\cdot 8) = 4\cos(8x)(3+\sin(8x))^{-0.5}$.

22. $f'(x) = [-\sin(\sin x)](\cos x)$.

23. $f'(x) = \dfrac{\cos x}{\cos^2(\sin x)}$.

24. $k'(x) = \frac{3}{2}\sqrt{\sin(2x)}(2\cos(2x)) = 3\cos(2x)\sqrt{\sin(2x)}$.

25. $f'(x) = 2\cdot[\sin(3x)] + 2x[\cos(3x)]\cdot 3 = 2\sin(3x) + 6x\cos(3x)$

26. $y' = e^{\theta}\sin(2\theta) + 2e^{\theta}\cos(2\theta)$.

27. $f'(x) = (e^{-2x})(-2)(\sin x) + (e^{-2x})(\cos x) = -2\sin x(e^{-2x}) + (e^{-2x})(\cos x) = e^{-2x}[\cos x - 2\sin x]$.

28. $z' = \dfrac{\cos t}{2\sqrt{\sin t}}$.

29. $y' = 5\sin^4\theta\cos\theta$.

30. $g'(z) = \dfrac{e^z}{\cos^2(e^z)}$.

31. $z' = \dfrac{-3e^{-3\theta}}{\cos^2(e^{-3\theta})}$.

32. $w' = (-\cos\theta)e^{-\sin\theta}$.

33. $\dfrac{dQ}{dx} = -\sin(e^{2x})\cdot(e^{2x}\cdot 2) = -2e^{2x}\sin(e^{2x})$.

34. $h'(t) = 1\cdot(\cos t) + t(-\sin t) + \frac{1}{\cos^2 t} = \cos t - t\sin t + \frac{1}{\cos^2 t}$.

35. $f'(\alpha) = -\sin\alpha + 3\cos\alpha$

36. $k'(\alpha) = (5\sin^4\alpha\cos\alpha)\cos^3\alpha + \sin^5\alpha(3\cos^2\alpha(-\sin\alpha)) = 5\sin^4\alpha\cos^4\alpha - 3\sin^6\alpha\cos^2\alpha$

37. $f'(\theta) = 3\theta^2\cos\theta - \theta^3\sin\theta$.

38. $y' = -2\cos w\sin w - \sin(w^2)(2w) = -2(\cos w\sin w + w\sin(w^2))$

39. $y' = \cos(\cos x + \sin x)(\cos x - \sin x)$

40. $y' = 2\cos(2x)\sin(3x) + 3\sin(2x)\cos(3x)$.

41. We use the quotient rule. We have

$$\frac{dP}{dt} = \frac{(-\sin t)\cdot t^3 - (3t^2)\cdot\cos t}{(t^3)^2} = \frac{-t^3\sin t - 3t^2\cos t}{t^6} = \frac{-t\sin t - 3\cos t}{t^4}.$$

42. $t'(\theta) = \dfrac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta} = -\dfrac{(\sin^2\theta + \cos^2\theta)}{\sin^2\theta} = -\dfrac{1}{\sin^2\theta}$.

43. Using the power and quotient rules gives

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\frac{1-\sin x}{1-\cos x}\right)^{-1/2}\left[\frac{-\cos x(1-\cos x) - (1-\sin x)\sin x}{(1-\cos x)^2}\right] \\ &= \frac{1}{2}\sqrt{\frac{1-\cos x}{1-\sin x}}\left[\frac{-\cos x(1-\cos x) - (1-\sin x)\sin x}{(1-\cos x)^2}\right] \\ &= \frac{1}{2}\sqrt{\frac{1-\cos x}{1-\sin x}}\left[\frac{1-\cos x-\sin x}{(1-\cos x)^2}\right]. \end{aligned}$$

44. $\dfrac{d}{dy}\left(\dfrac{y}{\cos y + a}\right) = \dfrac{\cos y + a - y(-\sin y)}{(\cos y + a)^2} = \dfrac{\cos y + a + y\sin y}{(\cos y + a)^2}$.

45. The quotient rule gives $G'(x) = \dfrac{2\sin x\cos x(\cos^2 x + 1) + 2\sin x\cos x(\sin^2 x + 1)}{(\cos^2 x + 1)^2}$

or, using $\sin^2 x + \cos^2 x = 1$,

$G'(x) = \dfrac{6\sin x\cos x}{(\cos^2 x + 1)^2}$.

46. $\dfrac{dy}{dt} = a\cos(bt)\cdot b = ab\cos(bt)$.

47. $\dfrac{dP}{dt} = a(-\sin(bt + c) \cdot b) = -ab\sin(bt + c).$

Problems

48. We begin by taking the derivative of $y = \sin(x^4)$ and evaluating at $x = 10$:

$$\frac{dy}{dx} = \cos(x^4) \cdot 4x^3.$$

Evaluating $\cos(10{,}000)$ on a calculator (in radians) we see $\cos(10{,}000) < 0$, so we know that $dy/dx < 0$, and therefore the function is decreasing.
Next, we take the second derivative and evaluate it at $x = 10$, giving $\sin(10{,}000) < 0$:

$$\frac{d^2y}{dx^2} = \underbrace{\cos(x^4) \cdot (12x^2)}_{\text{negative}} + \underbrace{4x^3 \cdot (-\sin(x^4))(4x^3)}_{\substack{\text{positive, but much} \\ \text{larger in magnitude}}}.$$

From this we can see that $d^2y/dx^2 > 0$, thus the graph is concave up.

49. To calculate the equation of the tangent line, we need to find the y-coordinate and the slope at $t = \pi$. The y-coordinate is

$$y = f(\pi) = 3\sin(2\pi) + 5 = 5,$$

so a point on the line is $(\pi, 5)$.
The slope is found using the derivative: $f'(t) = 3\cos(2t) \cdot 2 = 6\cos(2t)$. At the point $t = \pi$, we have

$$\text{Slope} = f'(\pi) = 6\cos(2\pi) = 6.$$

The equation of the line is

$$\begin{aligned} y - 5 &= 6(t - \pi) \\ y &= 6t - 13.850. \end{aligned}$$

50. The pattern in the table below allows us to generalize and say that the $(4n)^{\text{th}}$ derivative of $\cos x$ is $\cos x$, i.e.,

$$\frac{d^4y}{dx^4} = \frac{d^8y}{dx^8} = \cdots = \frac{d^{4n}y}{dx^{4n}} = \cos x.$$

Thus we can say that $d^{48}y/dx^{48} = \cos x$. From there we differentiate twice more to obtain $d^{50}y/dx^{50} = -\cos x$.

n	1	2	3	4	$\cdots$	48	49	50
n^{th} derivative	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$		$\cos x$	$-\sin x$	$-\cos x$

51. Differentiating with respect to t using the chain rule and substituting for dx/dt gives

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}(x\sin x) \cdot \frac{dx}{dt} = (\sin x + x\cos x)x\sin x.$$

52. We see that $q'(x)$ is of the form

$$\frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2},$$

with $f(x) = e^x$ and $g(x) = \sin x$. Therefore, using the quotient rule, let

$$q(x) = \frac{f(x)}{g(x)} = \frac{e^x}{\sin x}.$$

53. Since $F'(x)$ is of the form $\sin u$, we can make an initial guess that

$$F(x) = \cos(4x),$$

then

$$F'(x) = -4\sin(4x)$$

so we're off by a factor of -4. To fix this problem, we modify our guess by a factor of -4, so the next try is

$$F(x) = -(1/4)\cos(4x),$$

which has

$$F'(x) = \sin(4x).$$

54. (a) We have

$$f'(x) = 2(\sin x)^1(\cos x) + 2(\cos x)^1(-\sin x) = 2\sin x\cos x - 2\sin x\cos x = 0.$$

(b) Since $\sin^2 x + \cos^2 x = 1$, we see that $f(x) = 1$ so $f'(x) = 0$.

55. We have

$$\begin{aligned} h'(x) &= \left(f\left(x^2\right)\right)' \\ &= f'\left(x^2\right)\left(x^2\right)' && \text{Chain rule} \\ &= \sin\left(\left(x^2\right)^2\right)\cdot 2x && \text{since } f'(x) = \sin\left(x^2\right) \\ &= 2x\sin\left(x^4\right) \\ \text{so}\quad h''(x) &= \left(2x\sin\left(x^4\right)\right)' \\ &= (2x)'\sin\left(x^4\right) + 2x\left(\sin\left(x^4\right)\right)' && \text{Product rule} \\ &= 2\sin\left(x^4\right) + 2x\cos\left(x^4\right)\left(x^4\right)' \\ &= 2\sin\left(x^4\right) + 2x\cos\left(x^4\right)4x^3 \\ &= 2\sin\left(x^4\right) + 8x^4\cos\left(x^4\right). \end{aligned}$$

56. (a) Differentiating gives

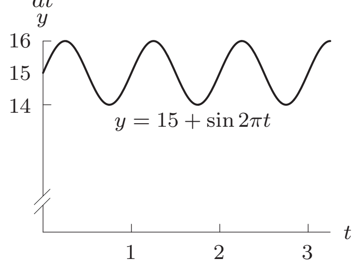
$$\frac{dy}{dt} = -\frac{4.9\pi}{6}\sin\left(\frac{\pi}{6}t\right).$$

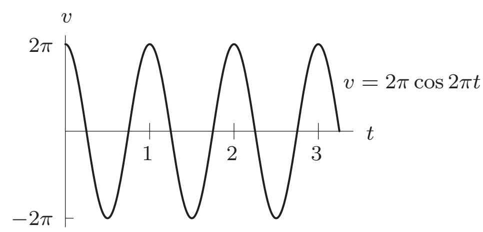
The derivative represents the rate of change of the depth of the water in feet/hour.

(b) The derivative, dy/dt, is zero where the tangent line to the curve y is horizontal. This occurs when $dy/dt = \sin(\frac{\pi}{6}t) = 0$, or at $t = 6$, 12, 18 and 24 (6 am, noon, 6 pm, and midnight). When $dy/dt = 0$, the depth of the water is no longer changing. Therefore, it has either just finished rising or just finished falling, and we know that the harbor's level is at a maximum or a minimum.

57. (a) $v(t) = \dfrac{dy}{dt} = \dfrac{d}{dt}(15 + \sin(2\pi t)) = 2\pi\cos(2\pi t)$.

(b)





58. (a) Differentiating, we find

$$\begin{aligned} \text{Rate of change of voltage with time} = \frac{dV}{dt} &= -120\pi\cdot 156\sin(120\pi t) \\ &= -18720\pi\sin(120\pi t)\ \text{volts per second.} \end{aligned}$$

(b) The rate of change of voltage with time is zero when $\sin(120\pi t) = 0$. This occurs when $120\pi t$ equals any multiple of π. For example, $\sin(120\pi t) = 0$ when $120\pi t = \pi$, or at $t = 1/120$ seconds. Since there are an infinite number of multiples of π, there are many times when the rate of change dV/dt is zero.

(c) The maximum value of the rate of change is $18720\pi = 58810.6$ volts/sec.

59. (a) When $\sqrt{\frac{k}{m}}t = \frac{\pi}{2}$ the spring is farthest from the equilibrium position. This occurs at time $t = \frac{\pi}{2}\sqrt{\frac{m}{k}}$

$v = A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)$, so the maximum velocity occurs when $t = 0$

$a = -A\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right)$, so the maximum acceleration occurs when $\sqrt{\frac{k}{m}}t = \frac{3\pi}{2}$, which is at time $t = \frac{3\pi}{2}\sqrt{\frac{m}{k}}$

(b) $T = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}}$

(c) $\dfrac{dT}{dm} = \dfrac{2\pi}{\sqrt{k}} \cdot \dfrac{1}{2}m^{-\frac{1}{2}} = \dfrac{\pi}{\sqrt{km}}$

Since $\dfrac{dT}{dm} > 0$, an increase in the mass causes the period to increase.

60. (a) The population varies periodically with a period of 1 year. See Figure 3.6.

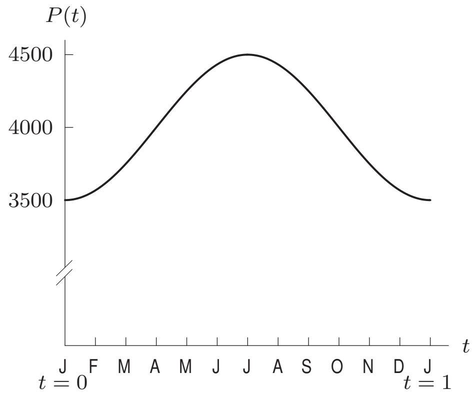


Figure 3.6

(b) The population is at a maximum on July 1st. At this time $\sin(2\pi t - \frac{\pi}{2}) = 1$, so the actual maximum population is $4000 + 500(1) = 4500$. Similarly, the population is at a minimum on January 1st. At this time, $\sin(2\pi t - \frac{\pi}{2}) = -1$, so the minimum population is $4000 + 500(-1) = 3500$.

(c) The rate of change is most positive about April 1st and most negative around October 1st.

(d) Since the population is at its maximum around July 1st, its rate of change is about 0 then.

61. (a) The function $d(t)$ is increasing at a constant rate for the period $0 \le t \le 2$, when the derivative of $d(t)$ is k.

(b) The functions $d(t)$ must be continuous, since the depth of water cannot shift suddenly and instantly (even the fastest change takes some amount of time), so we know that

$$2k = 50 + \sin(0.2), \quad \text{so} \quad k = 25.099.$$

This means that the derivative for $0 < t < 2$ is 25.099, whereas the derivative for $t > 2$ is $0.1\cos(0.1t)$. In other words

$$d(t) = \begin{cases} 25.099t & 0 \le t \le 2 \\ 50 + \sin(0.1t) & t > 2, \end{cases}$$

so

$$d'(t) = \begin{cases} 25.099 & 0 \le t < 2 \\ 0.1\cos(0.1t) & t > 2, \end{cases}$$

At $t = 2$, the derivative is undefined, since $0.1\cos(0.1 \cdot 2) \ne 25.099$.

62. (a) Using triangle OPD in Figure 3.7, we see

$$\frac{OD}{a} = \cos\theta \quad \text{so} \quad OD = a\cos\theta$$
$$\frac{PD}{a} = \sin\theta \quad \text{so} \quad PD = a\sin\theta.$$

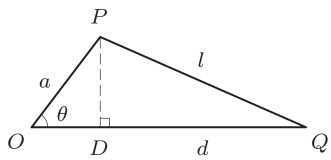


Figure 3.7

Using triangle PQD, we have

$$(PD)^2 + d^2 = l^2$$

so

$$a^2 \sin^2\theta + d^2 = l^2, \quad d = \sqrt{l^2 - a^2\sin^2\theta}.$$

Thus,

$$\begin{aligned} x &= OD + DQ \\ &= a\cos\theta + \sqrt{l^2 - a^2\sin^2\theta}. \end{aligned}$$

(b) Differentiating, regarding a and l as constants,

$$\begin{aligned} \frac{dx}{d\theta} &= -a\sin\theta + \frac{1}{2}\frac{(-2a^2\sin\theta\cos\theta)}{\sqrt{l^2 - a^2\sin^2\theta}} \\ &= -a\sin\theta - \frac{a^2\sin\theta\cos\theta}{\sqrt{l^2 - a^2\sin^2\theta}}. \end{aligned}$$

We want to find dx/dt. Using the chain rule and the fact that $d\theta/dt = 2$, we have

$$\frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = 2\frac{dx}{d\theta}.$$

(i) Substituting $\theta = \pi/2$, we have

$$\begin{aligned} \frac{dx}{dt} = 2\frac{dx}{d\theta}\bigg|_{\theta=\pi/2} &= -2a\sin\left(\frac{\pi}{2}\right) - 2\frac{a^2\sin(\frac{\pi}{2})\cos(\frac{\pi}{2})}{\sqrt{l^2 - a^2\sin^2(\frac{\pi}{2})}} \\ &= -2a \text{ cm/sec.} \end{aligned}$$

(ii) Substituting $\theta = \pi/4$, we have

$$\begin{aligned} \frac{dx}{dt} = 2\frac{dx}{d\theta}\bigg|_{\theta=\pi/4} &= -2a\sin\left(\frac{\pi}{4}\right) - 2\frac{a^2\sin(\frac{\pi}{4})\cos(\frac{\pi}{4})}{\sqrt{l^2 - a^2\sin^2(\frac{\pi}{4})}} \\ &= -a\sqrt{2} - \frac{a^2}{\sqrt{l^2 - a^2/2}} \text{ cm/sec.} \end{aligned}$$

63. The tangent lines to $f(x) = \sin x$ have slope $\frac{d}{dx}(\sin x) = \cos x$. The tangent line at $x = 0$ has slope $f'(0) = \cos 0 = 1$ and goes through the point $(0, 0)$. Consequently, its equation is $y = g(x) = x$. The approximate value of $\sin(\pi/6)$ given by this equation is $g(\pi/6) = \pi/6 \approx 0.524$.

Similarly, the tangent line at $x = \frac{\pi}{3}$ has slope

$$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

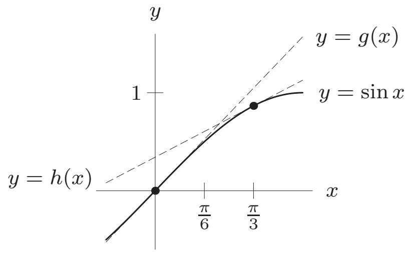
and goes through the point $(\pi/3, \sqrt{3}/2)$. Consequently, its equation is

$$y = h(x) = \frac{1}{2}x + \frac{3\sqrt{3} - \pi}{6}.$$

The approximate value of $\sin(\pi/6)$ given by this equation is then

$$h\left(\frac{\pi}{6}\right) = \frac{6\sqrt{3} - \pi}{12} \approx 0.604.$$

The actual value of $\sin(\pi/6)$ is $\frac{1}{2}$, so the approximation from 0 is better than that from $\pi/3$. This is because the slope of the function changes less between $x = 0$ and $x = \pi/6$ than it does between $x = \pi/6$ and $x = \pi/3$. This is illustrated by the following figure.



64. If the graphs of $y = \sin x$ and $y = ke^{-x}$ are tangent, then the y-values and the derivatives, $\frac{dy}{dx} = \cos x$ and $\frac{dy}{dx} = -ke^{-x}$, are equal at that point, so

$$\sin x = ke^{-x} \qquad \text{and} \qquad \cos x = -ke^{-x}.$$

Thus $\sin x = -\cos x$ so $\tan x = -1$. The smallest x-value is $x = 3\pi/4$, which leads to the smallest k value

$$k = \frac{\sin(3\pi/4)}{e^{-3\pi/4}} = 7.46.$$

When $x = \frac{3\pi}{4}$, we have $y = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$ so the point is $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

65. For $f(x)$ and $g(x)$ to be tangent at $x = 0$ we must have $f(0) = g(0)$ and $f'(0) = g'(0)$.

From $f(0) = g(0)$ we have

$$(1 + R)\cos(-wt) = T\cos(-wt)$$

and thus

$$1 + R = T.$$

Since

$$f'(x) = -k_1 \sin(k_1 x - wt) + k_1 R \sin(-k_1 x - wt)$$
$$g'(x) = -k_2 T \sin(k_2 x - wt)$$

the condition $f'(0) = g'(0)$ gives

$$-k_1 \sin(-wt) + k_1 R \sin(-wt) = -k_2 T \sin(-wt)$$

and thus

$$-k_1 + k_1 R = -k_2 T.$$

Substituting $T = 1 + R$ and solving for R yields

$$-k_1 + k_1 R = -k_2(1 + R)$$
$$(k_1 + k_2)R = k_1 - k_2$$
$$R = \frac{k_1 - k_2}{k_1 + k_2}.$$

Finally

$$T = 1 + R = 1 + \frac{k_1 - k_2}{k_1 + k_2} = \frac{2k_1}{k_1 + k_2}.$$

66. (a) If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{(\sin x \cos h + \sin h \cos x) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}. \end{aligned}$$

(b) $\frac{\cos h-1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$, as $h \to 0$. Thus, $f'(x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$.

(c) Similarly,

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\ &= -\sin x. \end{aligned}$$

67. (a) Sector OAQ is a sector of a circle with radius $\frac{1}{\cos\theta}$ and angle $\Delta\theta$. Thus its area is the left side of the inequality. Similarly, the area of Sector OBR is the right side of the equality. The area of the triangle OQR is $\frac{1}{2}\Delta\tan\theta$ since it is a triangle with base $\Delta\tan\theta$ (the segment QR) and height 1 (if you turn it sideways, it is easier to see this). Thus, using the given fact about areas (which is also clear from looking at the picture), we have

$$\frac{\Delta\theta}{2\pi} \cdot \pi \left(\frac{1}{\cos\theta}\right)^2 \le \frac{1}{2} \cdot \Delta(\tan\theta) \le \frac{\Delta\theta}{2\pi} \cdot \pi \left(\frac{1}{\cos(\theta+\Delta\theta)}\right)^2.$$

(b) Dividing the inequality through by $\frac{\Delta\theta}{2}$ and canceling the π's gives:

$$\left(\frac{1}{\cos\theta}\right)^2 \le \frac{\Delta\tan\theta}{\Delta\theta} \le \left(\frac{1}{\cos(\theta+\Delta\theta)}\right)^2$$

Then as $\Delta\theta \to 0$, the right and left sides both tend toward $\left(\frac{1}{\cos\theta}\right)^2$ while the middle (which is the difference quotient for tangent) tends to $(\tan\theta)'$. Thus, the derivative of tangent is "squeezed" between two values heading toward the same thing and must, itself, also tend to that value. Therefore, $(\tan\theta)' = \left(\frac{1}{\cos\theta}\right)^2$.

(c) Take the identity $\sin^2\theta + \cos^2\theta = 1$ and divide through by $\cos^2\theta$ to get $(\tan\theta)^2 + 1 = (\frac{1}{\cos\theta})^2$. Differentiating with respect to θ yields:

$$\begin{aligned} 2(\tan\theta)\cdot(\tan\theta)' &= 2\left(\frac{1}{\cos\theta}\right)\cdot\left(\frac{1}{\cos\theta}\right)' \\ 2\left(\frac{\sin\theta}{\cos\theta}\right)\cdot\left(\frac{1}{\cos\theta}\right)^2 &= 2\left(\frac{1}{\cos\theta}\right)\cdot(-1)\left(\frac{1}{\cos\theta}\right)^2(\cos\theta)' \\ 2\frac{\sin\theta}{\cos^3\theta} &= (-1)2\frac{1}{\cos^3\theta}(\cos\theta)' \\ -\sin\theta &= (\cos\theta)'. \end{aligned}$$

(d)

$$\begin{aligned} \frac{d}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) &= \frac{d}{d\theta}(1) \\ 2\sin\theta\cdot(\sin\theta)' + 2\cos\theta\cdot(\cos\theta)' &= 0 \\ 2\sin\theta\cdot(\sin\theta)' + 2\cos\theta\cdot(-\sin\theta) &= 0 \\ (\sin\theta)' - \cos\theta &= 0 \\ (\sin\theta)' &= \cos\theta. \end{aligned}$$

Strengthen Your Understanding

68. The function $\sin(\cos x)$ is a composition of two functions and requires the chain rule to differentiate. The correct derivative computation is

$$\frac{d}{dx}\sin(\cos x) = \cos(\cos x)\cdot(-\sin x).$$

69. The function f is not a product; it is a composition. We need to use the chain rule. Let $z = g(x) = \sin x$, so $f(z) = \sin z$. Then, using the chain rule, $f'(x) = \cos z \cdot g'(x) = \cos(\sin x)\cos x$.

70. One possibility is $f(x) = \cos(x^2)$ whose derivative is $f'(x) = -\sin(x^2) \cdot 2x = -2x\sin(x^2)$.

71. The function $f(x) = \sin x$ satisfies this condition because

$$f'(x) = \frac{d}{dx}\sin x = \cos x$$
$$f''(x) = \frac{d}{dx}\cos x = -\sin x = -f(x).$$

There are many other possibilities, including $f(x) = \cos x$.

72. True, since $\cos\theta$ and therefore $\cos^2\theta$ are periodic, and

$$\frac{d}{d\theta}(\tan\theta) = \frac{1}{\cos^2\theta}.$$

73. True. If $f(x)$ is periodic with period c, then $f(x+c) = f(x)$ for all x. By the definition of the derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

and

$$f'(x+c) = \lim_{h\to 0}\frac{f(x+c+h) - f(x+c)}{h}.$$

Since f is periodic, for any $h \neq 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x+c+h) - f(x+c)}{h}.$$

Taking the limit as $h \to 0$, we get that $f'(x) = f'(x+c)$, so f' is periodic with the same period as $f(x)$.

74. False; the fourth derivative of $\cos t + C$, where C is any constant, is indeed $\cos t$. But any function of the form $\cos t + p(t)$, where $p(t)$ is a polynomial of degree less than or equal to 3, also has its fourth derivative equal to $\cos t$. So $\cos t + t^2$ will work.

Solutions for Section 3.6

Exercises

1. $f'(t) = \dfrac{2t}{t^2+1}$.

2. $f'(x) = \dfrac{-1}{1-x} = \dfrac{1}{x-1}$.

3. $f'(x) = \dfrac{1}{5x^2+3}\cdot(10x) = \dfrac{10x}{5x^2+3}$.

4. $f'(x) = 4x + 3\dfrac{1}{x} = 4x + \dfrac{3}{x}$.

5. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(x+1)^2}}$.

6. $f'(x) = \dfrac{1}{1+(3x)^2}\cdot 3 = \dfrac{3}{1+9x^2}$.

7. $\dfrac{dP}{dt} = 3\dfrac{1}{x^2+5x+3}\cdot(2x+5) = \dfrac{6x+15}{x^2+5x+3}$.

8. $\dfrac{dQ}{dx} = a\dfrac{1}{bx+c}\cdot b = \dfrac{ab}{bx+c}$.

9. Since $\ln(e^{2x}) = 2x$, the derivative $f'(x) = 2$.

10. Since $e^{\ln(e^{2x^2+3})} = e^{2x^2+3}$, the derivative $f'(x) = 4xe^{2x^2+3}$.

11. $f'(x) = \dfrac{1}{1-e^{-x}} \cdot (-e^{-x})(-1) = \dfrac{e^{-x}}{1-e^{-x}}.$

12. $f'(\alpha) = \dfrac{1}{\sin\alpha} \cdot \cos\alpha = \dfrac{\cos\alpha}{\sin\alpha}.$

13. $f'(x) = \dfrac{1}{e^x+1} \cdot e^x.$

14. $\dfrac{dy}{dx} = \ln x + x\left(\dfrac{1}{x}\right) - 1 = \ln x$

15. $j'(x) = \dfrac{ae^{ax}}{(e^{ax}+b)}$

16. We use the product rule. We have

$$\frac{dy}{dx} = 3x^2 \cdot \ln x + x^3 \cdot \frac{1}{x} = 3x^2 \ln x + x^2.$$

17. Using the product and chain rules gives $h'(w) = 3w^2\ln(10w) + w^3\dfrac{10}{10w} = 3w^2\ln(10w) + w^2.$

18. $f'(x) = \dfrac{1}{e^{7x}} \cdot (e^{7x})7 = 7.$

(Note also that $\ln(e^{7x}) = 7x$ implies $f'(x) = 7$.)

19. Note that $f(x) = e^{\ln x} \cdot e^1 = x \cdot e = ex$. So $f'(x) = e$. (Remember, e is just a constant.) You might also use the chain rule to get:

$f'(x) = e^{(\ln x)+1} \cdot \frac{1}{x}$.

[Are the two answers the same? Of course they are, since

$$e^{(\ln x)+1}\left(\frac{1}{x}\right) = e^{\ln x} \cdot e\left(\frac{1}{x}\right) = xe\left(\frac{1}{x}\right) = e.]$$

20. $f'(\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$

21. $f(t) = \ln t$ (because $\ln e^x = x$ *or* because $e^{\ln t} = t$), so $f'(t) = \frac{1}{t}$.

22. $f'(y) = \dfrac{2y}{\sqrt{1-y^4}}.$

23. $s'(x) = \dfrac{d}{dx}(\arctan(2-x)) = \dfrac{-1}{1+(2-x)^2}.$

24. $g(\alpha) = \alpha$, so $g'(\alpha) = 1$.

25. $g'(t) = e^{\arctan(3t^2)}\left(\dfrac{1}{1+(3t^2)^2}\right)(6t) = e^{\arctan(3t^2)}\left(\dfrac{6t}{1+9t^4}\right).$

26. $g'(t) = \dfrac{-\sin(\ln t)}{t}.$

27. $h'(z) = (\ln 2)z^{(\ln 2 - 1)}.$

28. $h'(w) = \arcsin w + \dfrac{w}{\sqrt{1-w^2}}.$

29. Note that $f(x) = kx$ so, $f'(x) = k$.

30. Using the chain rule gives $r'(t) = \dfrac{2}{\sqrt{1-4t^2}}.$

31. $j'(x) = -\sin\left(\sin^{-1} x\right) \cdot \left[\dfrac{1}{\sqrt{1-x^2}}\right] = -\dfrac{x}{\sqrt{1-x^2}}$

32. $f'(x) = -\sin(\arctan 3x)\left(\dfrac{1}{1+(3x)^2}\right)(3) = \dfrac{-3\sin(\arctan 3x)}{1+9x^2}.$

33. $f'(z) = -1(\ln z)^{-2} \cdot \dfrac{1}{z} = \dfrac{-1}{z(\ln z)^2}.$

34. Using the quotient rule gives

$$g'(t) = \frac{\left(\frac{k}{kt}+1\right)(\ln(kt)-t) - (\ln(kt)+t)\left(\frac{k}{kt}-1\right)}{(\ln(kt)-t)^2}$$

$$g'(t) = \frac{\left(\frac{1}{t}+1\right)(\ln(kt)-t) - (\ln(kt)+t)\left(\frac{1}{t}-1\right)}{(\ln(kt)-t)^2}$$
$$g'(t) = \frac{\ln(kt)/t - 1 + \ln(kt) - t - \ln(kt)/t - 1 + \ln(kt) + t}{(\ln(kt)-t)^2}$$
$$g'(t) = \frac{2\ln(kt)-2}{(\ln(kt)-t)^2}.$$

35. $f'(w) = 3w^{-1/2} - 2w^{-3} + 5\frac{1}{w} = \frac{3}{\sqrt{w}} - \frac{2}{w^3} + \frac{5}{w}.$

36. $\frac{dy}{dx} = 2(\ln x + \ln 2) + 2x\left(\frac{1}{x}\right) - 2 = 2(\ln x + \ln 2) = 2\ln(2x)$

37. Using the chain rule gives $f'(x) = \frac{\cos x - \sin x}{\sin x + \cos x}.$

38. $f'(t) = \frac{1}{\ln t} \cdot \frac{1}{t} = \frac{1}{t\ln t}$

39. Using the chain rule gives

$$\begin{aligned} T'(u) &= \left[\frac{1}{1+\left(\frac{u}{1+u}\right)^2}\right]\left[\frac{(1+u)-u}{(1+u)^2}\right] \\ &= \frac{(1+u)^2}{(1+u)^2+u^2}\left[\frac{1}{(1+u)^2}\right] \\ &= \frac{1}{1+2u+2u^2}. \end{aligned}$$

40. Since $\ln\left[\left(\frac{1-\cos t}{1+\cos t}\right)^4\right] = 4\ln\left[\left(\frac{1-\cos t}{1+\cos t}\right)\right]$ we have

$$\begin{aligned} a'(t) &= 4\left(\frac{1+\cos t}{1-\cos t}\right)\left[\frac{\sin t(1+\cos t) + \sin t(1-\cos t)}{(1+\cos t)^2}\right] \\ &= \left[\frac{1+\cos t}{1-\cos t}\right]\left[\frac{8\sin t}{(1+\cos t)^2}\right] \\ &= \frac{8\sin t}{1-\cos^2 t} \\ &= \frac{8}{\sin t}. \end{aligned}$$

41. $f'(x) = -\sin(\arcsin(x+1))\left(\frac{1}{\sqrt{1-(x+1)^2}}\right) = \frac{-(x+1)}{\sqrt{1-(x+1)^2}}.$

Problems

42. (a) We have
$$f'(x) = \frac{1}{3x}\cdot 3 = \frac{3}{3x} = \frac{1}{x}.$$
(b) Using properties of logs, we have $f(x) = \ln 3 + \ln x$.
(c) Since $\ln 3$ is a constant, the derivative of $f(x) = \ln 3 + \ln x$ is
$$f'(x) = 0 + \frac{1}{x} = \frac{1}{x}.$$
Yes, the answer is the same as that obtained in part (a), as it should be.

43. Differentiating

$$f'(x) = \frac{1}{x^2+1}\cdot 2x = 2x(x^2+1)^{-1}$$

$$\begin{aligned} f''(x) &= 2(x^2+1)^{-1} - 2x(x^2+1)^{-2}\cdot 2x \\ &= \frac{2}{(x^2+1)} - \frac{4x^2}{(x^2+1)^2} = \frac{2x^2+2}{(x^2+1)^2} - \frac{4x^2}{(x^2+1)^2} \\ &= \frac{2(1-x^2)}{(x^2+1)^2}. \end{aligned}$$

Since $(x^2+1)^2 > 0$ for all x, we see that $f''(0) > 0$ for $1-x^2 > 0$ or $x^2 < 1$. That is, $\ln(x^2+1)$ is concave up on the interval $-1 < x < 1$.

44. Let

$$g(x) = \arcsin x$$

so

$$\sin[g(x)] = x.$$

Differentiating,

$$\begin{aligned} \cos[g(x)]\cdot g'(x) &= 1 \\ g'(x) &= \frac{1}{\cos[g(x)]} \end{aligned}$$

Using the fact that $\sin^2\theta + \cos^2\theta = 1$, and $\cos[g(x)] \geq 0$, since $-\frac{\pi}{2} \leq g(x) \leq \frac{\pi}{2}$, we get

$$\cos[g(x)] = \sqrt{1-(\sin[g(x)])^2}.$$

Therefore,

$$g'(x) = \frac{1}{\sqrt{1-(\sin[g(x)])^2}}$$

Since $\sin[g(x)] = x$, we have

$$g'(x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1.$$

45. Let

$$g(x) = \log x.$$

Then

$$10^{g(x)} = x.$$

Differentiating,

$$\begin{aligned} (\ln 10)[10^{g(x)}]g'(x) &= 1 \\ g'(x) &= \frac{1}{(\ln 10)[10^{g(x)}]} \\ g'(x) &= \frac{1}{(\ln 10)x}. \end{aligned}$$

46. $\text{pH} = 2 = -\log x$ means $\log x = -2$ so $x = 10^{-2}$. Rate of change of pH with hydrogen ion concentration is

$$\frac{d}{dx}\text{pH} = -\frac{d}{dx}(\log x) = \frac{-1}{x(\ln 10)} = -\frac{1}{(10^{-2})\ln 10} = -43.4$$

47. We have $g(5000) = 20\ln(0.001(5000)) = 32.189$ years. Since

$$g'(F) = 20\cdot\frac{1}{0.001F}\cdot 0.001 = \frac{20}{F},$$

we have

$$g'(5000) = \frac{20}{5000} = 0.004$$

years per dollar. It takes about 32.189 years for the investment to grow to \$5000, and it takes about 0.004 years (or about 1.5 days) for the investment to grow by one more dollar.

48. The marginal revenue, MR, is obtained by differentiating the total revenue function, R. We use the chain rule so

$$MR = \frac{dR}{dq} = \frac{1}{1+1000q^2} \cdot \frac{d}{dq}\left(1000q^2\right) = \frac{1}{1+1000q^2} \cdot 2000q.$$

When $q = 10$,

$$\text{Marginal revenue} = \frac{2000 \cdot 10}{1 + 1000 \cdot 10^2} = \$0.2\text{/unit}.$$

49. (a) For $y = \ln x$, we have $y' = 1/x$, so the slope of the tangent line is $f'(1) = 1/1 = 1$. The equation of the tangent line is $y - 0 = 1(x-1)$, so, on the tangent line, $y = g(x) = x - 1$.

(b) Using a value on the tangent line to approximate $\ln(1,1)$, we have

$$\ln(1.1) \approx g(1.1) = 1.1 - 1 = 0.1.$$

Similarly, $\ln(2)$ is approximated by

$$\ln(2) \approx g(2) = 2 - 1 = 1.$$

(c) From Figure 3.8, we see that $f(1.1)$ and $f(2)$ are below $g(x) = x - 1$. Similarly, $f(0.9)$ and $f(0.5)$ are also below $g(x)$. This is true for any approximation of this function by a tangent line since f is concave down ($f''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$). Thus, for a given x-value, the y-value given by the function is always below the value given by the tangent line.

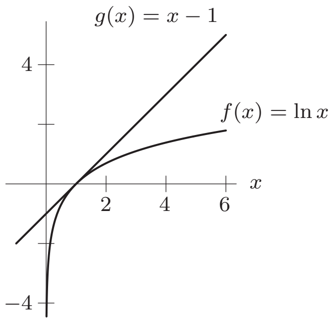


Figure 3.8

50. (a) Let $g(x) = ax^2 + bx + c$ be our quadratic and $f(x) = \ln x$. For the best approximation, we want to find a quadratic with the same value as $\ln x$ at $x = 1$ and the same first and second derivatives as $\ln x$ at $x = 1$. $g'(x) = 2ax + b$, $g''(x) = 2a$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$.

$$\begin{aligned} g(1) &= a(1)^2 + b(1) + c \quad f(1) = 0 \\ g'(1) &= 2a(1) + b \quad f'(1) = 1 \\ g''(1) &= 2a \quad f''(1) = -1 \end{aligned}$$

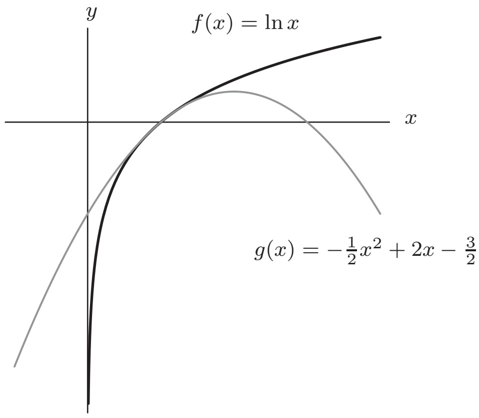
Thus, we obtain the equations

$$\begin{aligned} a + b + c &= 0 \\ 2a + b &= 1 \\ 2a &= -1 \end{aligned}$$

We find $a = -\frac{1}{2}$, $b = 2$ and $c = -\frac{3}{2}$. Thus our approximation is:

$$g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$$

(b) From the graph below, we notice that around $x = 1$, the value of $f(x) = \ln x$ and the value of $g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$ are very close.



(c) $g(1.1) = 0.095 \quad g(2) = 0.5$
Compare with $f(1.1) = 0.0953$, $f(2) = 0.693$.

51. (a)

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \cdot (-\frac{1}{x^2}) \\ &= \frac{1}{1+x^2} + \left(-\frac{1}{x^2+1}\right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0 \end{aligned}$$

(b) f is a constant function. Checking at a few values of x,

Table 3.2

x	$\arctan x$	$\arctan x^{-1}$	$f(x) = \arctan x + \arctan x^{-1}$
1	0.785392	0.7853982	1.5707963
2	1.1071487	0.4636476	1.5707963
3	1.2490458	0.3217506	1.5707963

52. The closer you look at the function, the more it begins to look like a line with slope equal to the derivative of the function at $x = 0$. Hence, functions whose derivatives at $x = 0$ are equal will look the same there.

The following functions look like the line $y = x$ since, in all cases, $y' = 1$ at $x = 0$.

$y = x$ $\quad y' = 1$
$y = \sin x$ $\quad y' = \cos x$
$y = \tan x$ $\quad y' = \frac{1}{\cos^2 x}$
$y = \ln(x+1)$ $\quad y' = \frac{1}{x+1}$

The following functions look like the line $y = 0$ since, in all cases, $y' = 0$ at $x = 0$.

$y = x^2$ $\quad y' = 2x$
$y = x \sin x$ $\quad y' = x\cos x + \sin x$
$y = x^3$ $\quad y' = 3x^2$
$y = \frac{1}{2}\ln(x^2+1)$ $\quad y' = 2x \cdot \frac{1}{2} \cdot \frac{1}{x^2+1} = \frac{x}{x^2+1}$
$y = 1 - \cos x$ $\quad y' = \sin x$

The following functions look like the line $x = 0$ since, in all cases, as $x \to 0^+$, the slope $y' \to \infty$.

$y = \sqrt{x}$ $\quad y' = \frac{1}{2\sqrt{x}}$
$y = \sqrt{\frac{x}{x+1}}$ $\quad y' = \frac{(x+1)-x}{(x+1)^2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{x}{x+1}}} = \frac{1}{2(x+1)^2} \cdot \sqrt{\frac{x+1}{x}}$
$y = \sqrt{2x - x^2}$ $\quad y' = (2-2x)\frac{1}{2} \cdot \frac{1}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$

53. Since the chain rule gives $h'(x) = n'(m(x))m'(x) = -2$ we must find values a and x such that $a = m(x)$ and $n'(a)m'(x) = -2$.

Calculating slopes from the graph of n gives

$$n'(a) = \begin{cases} 1 & \text{if} \quad 0 < a < 50 \\ 1/2 & \text{if} \quad 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of m gives

$$m'(x) = \begin{cases} -2 & \text{if} \quad 0 < x < 50 \\ 2 & \text{if} \quad 50 < x < 100. \end{cases}$$

The only values of the derivative n' are 1 and $1/2$ and the only values of the derivative m' are 2 and -2. In order to have $n'(a)m'(x) = -2$ we must therefore have $n'(a) = 1$ and $m'(x) = -2$. Thus $0 < a < 50$ and $0 < x < 50$.

Now $a = m(x)$ and from the graph of m we see that $0 < m(x) < 50$ for $25 < x < 75$.

The two conditions on x we have found are both satisfied when $25 < x < 50$. Thus $h'(x) = -2$ for all x in the interval $25 < x < 50$. The question asks for just one of these x values, for example $x = 40$.

54. Since the chain rule gives $h'(x) = n'(m(x))m'(x) = 2$ we must find values a and x such that $a = m(x)$ and $n'(a)m'(x) = 2$.

Calculating slopes from the graph of n gives

$$n'(a) = \begin{cases} 1 & \text{if} \quad 0 < a < 50 \\ 1/2 & \text{if} \quad 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of m gives

$$m'(x) = \begin{cases} -2 & \text{if} \quad 0 < x < 50 \\ 2 & \text{if} \quad 50 < x < 100. \end{cases}$$

The only values of the derivative n' are 1 and $1/2$ and the only values of the derivative m' are 2 and -2. In order to have $n'(a)m'(x) = 2$ we must therefore have $n'(a) = 1$ and $m'(x) = 2$. Thus $0 < a < 50$ and $50 < x < 100$.

Now $a = m(x)$ and from the graph of m we see that $0 < m(x) < 50$ for $25 < x < 75$.

The two conditions on x we have found are both satisfied when $50 < x < 75$. Thus $h'(x) = 2$ for all x in the interval $50 < x < 75$. The question asks for just one of these x values, for example $x = 60$.

55. Since the chain rule gives $h'(x) = n'(m(x))m'(x) = 1$ we must find values a and x such that $a = m(x)$ and $n'(a)m'(x) = 1$.

Calculating slopes from the graph of n gives

$$n'(a) = \begin{cases} 1 & \text{if} \quad 0 < a < 50 \\ 1/2 & \text{if} \quad 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of m gives

$$m'(x) = \begin{cases} -2 & \text{if} \quad 0 < x < 50 \\ 2 & \text{if} \quad 50 < x < 100. \end{cases}$$

The only values of the derivative n' are 1 and $1/2$ and the only values of the derivative m' are 2 and -2. In order to have $n'(a)m'(x) = 1$ we must therefore have $n'(a) = 1/2$ and $m'(x) = 2$. Thus $50 < a < 100$ and $50 < x < 100$.

Now $a = m(x)$ and from the graph of m we see that $50 < m(x) < 100$ for $0 < x < 25$ or $75 < x < 100$.

The two conditions on x we have found are both satisfied when $75 < x < 100$. Thus $h'(x) = 1$ for all x in the interval $75 < x < 100$. The question asks for just one of these x values, for example $x = 80$.

56. Since the chain rule gives $h'(x) = n'(m(x))m'(x) = -1$ we must find values a and x such that $a = m(x)$ and $n'(a)m'(x) = -1$.

Calculating slopes from the graph of n gives

$$n'(a) = \begin{cases} 1 & \text{if} \quad 0 < a < 50 \\ 1/2 & \text{if} \quad 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of m gives

$$m'(x) = \begin{cases} -2 & \text{if} \quad 0 < x < 50 \\ 2 & \text{if} \quad 50 < x < 100. \end{cases}$$

The only values of the derivative n' are 1 and $1/2$ and the only values of the derivative m' are 2 and -2. In order to have $n'(a)m'(x) = -1$ we must therefore have $n'(a) = 1/2$ and $m'(x) = -2$. Thus $50 < a < 100$ and $0 < x < 50$.

Now $a = m(x)$ and from the graph of m we see that $50 < m(x) < 100$ for $0 < x < 25$ or $75 < x < 100$.

The two conditions on x we have found are both satisfied when $0 < x < 25$. Thus $h'(x) = -1$ for all x in the interval $0 < x < 25$. The question asks for just one of these x values, for example $x = 10$.

57. We have

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))}.$$

From the graph of $f(x)$ we see that $f^{-1}(5) = 13$. From the graph of $f'(x)$ we see that $f'(13) = 0.36$. Thus $(f^{-1})'(5) = 1/0.36 = 2.8$.

58. We have

$$(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))}.$$

From the graph of $f(x)$ we see that $f^{-1}(10) = 23$. From the graph of $f'(x)$ we see that $f'(23) = 0.62$. Thus $(f^{-1})'(10) = 1/0.62 = 1.6$.

59. We have

$$(f^{-1})'(15) = \frac{1}{f'(f^{-1}(15))}.$$

From the graph of $f(x)$ we see that $f^{-1}(15) = 30$. From the graph of $f'(x)$ we see that $f'(30) = 0.73$. Thus $(f^{-1})'(15) = 1/0.73 = 1.4$.

60. Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent } = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.$$

Thus, $f'(2) = 3$.

By the chain rule

$$h'(2) = 3(f(2))^2 \cdot f'(2) = 3 \cdot 5^2 \cdot 3 = 225.$$

61. Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent } = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.$$

Thus, $f'(2) = 3$.

By the chain rule

$$k'(2) = -(f(2))^{-2} \cdot f'(2) = -5^{-2} \cdot 3 = -0.12.$$

62. Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent } = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.$$

Thus, $f'(2) = 3$. Since g is the inverse function of f and $f(2) = 5$, we know $f^{-1}(5) = 2$, so $g(5) = 2$.

Differentiating, we have

$$g'(2) = \frac{1}{f'(g(5))} = \frac{1}{f'(2)} = \frac{1}{3}.$$

63. (a) Since $f(x) = x^3$, we have $f'(x) = 3x^2$. Thus, $f'(2) = 3(2)^2 = 12$.

(b) To find $f^{-1}(x)$, we switch xs and ys and solve for y.

Since $y = x^3$, we get $x = y^3$.

Solving for y gives $y = \sqrt[3]{x}$.

Thus, $f^{-1}(x) = \sqrt[3]{x}$.

(c) To find $(f^{-1})'(x)$, we differentiate. Since $f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$, we get

$$(f^{-1})'(x) = \frac{1}{3}x^{-2/3}.$$

Thus,

$$(f^{-1})'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}.$$

(d) The point $(2, 8)$ is on the graph of f. Thus the point $(8, 2)$ is on the graph of f^{-1}, so $f^{-1}(8) = 2$. Therefore,

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(2)} = \frac{1}{12}.$$

64. (a) Since $f(x) = 2x^5 + 3x^3 + x$, we differentiate to get $f'(x) = 10x^4 + 9x^2 + 1$.
(b) Because $f'(x)$ is always positive, we know that $f(x)$ is increasing everywhere. Thus, $f(x)$ is a one-to-one function and is invertible.
(c) To find $f(1)$, substitute 1 for x into $f(x)$. We get $f(1) = 2(1)^5 + 3(1)^3 + 1 = 2 + 3 + 1 = 6$.
(d) To find $f'(1)$, substitute 1 for x into $f'(x)$. We get $f'(1) = 10(1)^4 + 9(1)^2 + 1 = 20$.
(e) Since $f(1) = 6$, we have $f^{-1}(6) = 1$, so

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(1)} = \frac{1}{20}.$$

65. To find $(f^{-1})'(3)$, we first look in the table to find that $3 = f(9)$, so $f^{-1}(3) = 9$. Thus,

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(9)} = \frac{1}{5}.$$

66. (a) The statement $f(2) = 4023$ tells us that when the price is \$2 per gallon, 4023 gallons of gas are sold.
(b) Since $f(2) = 4023$, we have $f^{-1}(4023) = 2$. Thus, 4023 gallons are sold when the price is \$2 per gallon.
(c) The statement $f'(2) = -1250$ tells us that if the price increases from \$2 per gallon, the sales decrease at a rate of 1250 gallons per \$1 increase in price.
(d) The units of $(f^{-1})'(4023)$ are dollars per gallon. We have

$$(f^{-1})'(4023) = \frac{1}{f'(f^{-1}(4023))} = \frac{1}{f'(2)} = -\frac{1}{1250} = -0.0008.$$

Thus, when 4023 gallons are already sold, sales decrease at the rate of one gallon per price increase of 0.0008 dollars. In others words, an additional gallon is sold if the price drops by 0.0008 dollars.

67. (a) Knowing $f(2005) = 296$ tells us that the US population was 296 million in the year 2005.
(b) Since $f(2005) = 296$, we have $f^{-1}(296) = 2005$. This tells us that the year in which the US population was 296 million was 2005.
(c) Knowing $f'(2005) = 2.65$ tells us that in the year 2005, the US population was growing at a rate of 2.65 million people per year.
(d) Using parts (b) and (c), we have

$$(f^{-1})'(296) = \frac{1}{f'(f^{-1}(296))} = \frac{1}{f'(2005)} = \frac{1}{2.65} = 0.377.$$

The units of the derivative of f^{-1} are years per million people (the reciprocal of the units of f'). The statement $(f^{-1})'(296) = 0.377$ tells us that when the US population was 296 million, it took 0.377 of a year (between 4 and 5 months) for the population to increase by another million.

68. Each grid mark on the horizontal axis represents 5 years and each grid mark on the vertical axis represents 100 million vehicles.

(a) When $t = 20$, the year is 1985. Reading from the graph, we find that in 1985

$$f(20) \approx 500 \text{ million vehicles.}$$

This tells us that 20 years after 1965, in 1985, there were 500 million registered vehicles.

(b) Drawing a tangent line to the curve at $t = 20$, we have

$$\text{Slope } = f'(20) \approx \frac{175}{10} = 17.5 \text{ million vehicles/year.}$$

Thus, 20 years after 1965, in 1985, the number of registered vehicles was increasing at 17.5 million vehicles per year.

(c) From the graph or part (a)

$$f^{-1}(500) = 20 \text{ years.}$$

Thus, there were 500 million cars registered when $t = 20$, that is, in 1985.

(d) We have

$$(f^{-1})'(500) = \frac{1}{f'(f^{-1}(500))} = \frac{1}{f'(20)} = \frac{1}{17.5} = 0.0571 \text{ years/million vehicles.}$$

Thus, when 500 million vehicles were already registered, it took 0.0571 year, or about 21 days, for another million to be registered.

69. We have $(f^{-1})'(8) = 1/f'(f^{-1}(8))$. From the graph we see $f^{-1}(8) = 4$. Thus $(f^{-1})'(8) = \dfrac{1}{f'(4)} = \dfrac{1}{3.0}$.

70. Since $f(20) = 10$, we have $f^{-1}(10) = 20$, so $(f^{-1})'(10) = \dfrac{1}{f'(f^{-1}(10))} = \dfrac{1}{f'(20)}$. Therefore $(f^{-1})'(10)f'(20) = 1$.

Option (b) is wrong.

71. All three values equal 1.

(a) We have $f^{-1}(A) = a$, so $(f^{-1})'(A) = \frac{1}{f'(f^{-1}(A))} = \frac{1}{f'(a)}$. Thus $f'(a)(f^{-1})'(A) = 1$.
(b) We have $f^{-1}(B) = b$, so $(f^{-1})'(B) = \frac{1}{f'(f^{-1}(B))} = \frac{1}{f'(b)}$. Thus $f'(b)(f^{-1})'(B) = 1$.
(c) We have $f^{-1}(C) = c$, so $(f^{-1})'(C) = \frac{1}{f'(f^{-1}(C))} = \frac{1}{f'(c)}$. Thus $f'(c)(f^{-1})'(C) = 1$.

72. A continuous invertible function $f(x)$ cannot be increasing on one interval and decreasing on another because it would fail the horizontal line test. The same is true of the inverse function $f^{-1}(x)$. Either $f^{-1}(x)$ is increasing and $(f^{-1})'(x) \geq 0$ for all x, or $f^{-1}(x)$ is decreasing and $(f^{-1})'(x) \leq 0$ for all x. We can not have both $(f^{-1})'(10) = 8$ and $(f^{-1})'(20) = -6$.

73. (a) The definition of the derivative of $\ln(1 + x)$ at $x = 0$ is

$$\lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h)}{h} = \left.\frac{1}{1+x}\right|_{x=0} = 1.$$

(b) The rules of logarithms give

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \frac{1}{h} \ln(1+h) = \lim_{h\to 0} \ln(1+h)^{1/h} = 1.$$

Thus, taking e to both sides and using the fact that $e^{\ln A} = A$, we have

$$\begin{aligned} e^{\lim_{h\to 0} \ln(1+h)^{1/h}} &= \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = e^1 \\ \lim_{h\to 0} (1+h)^{1/h} &= e. \end{aligned}$$

This limit is sometimes used as the definition of e.

(c) Let $n = 1/h$. Then as $h \to 0^+$, we have $n \to \infty$. Since

$$\lim_{h\to 0^+} (1+h)^{1/h} = \lim_{h\to 0} (1+h)^{1/h} = e,$$

we have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

This limit is also sometimes used as the definition of e.

Strengthen Your Understanding

74. To calculate the derivative of $\ln(1+x^4)$ we need to apply the chain rule. Take $z = g(x) = 1 + x^4$ as the inside function and $f(z) = \ln z$ as the outside function. Since $g'(x) = 4x^3$ and $f'(z) = 1/z$, the chain rule gives

$$\frac{dw}{dx} = \frac{1}{z} \cdot 4x^3 = \frac{4x^3}{1+x^4}.$$

75. The function f is not a product. We need the chain rule, so $f'(x) = \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln x}$.

76. The formula for $(f^{-1})'(2)$ is wrong. We need $f'(f^{-1}(2))$, and we are not given $f^{-1}(2)$.

77. Any exponential function of the form $c \cdot a^x$ with $c \neq 0$, $a > 0$ and $a \neq e$ is a constant multiple of its derivative but it is not equal to its derivative: $\frac{d}{dx}(c \cdot a^x) = (\ln a)(c \cdot a^x)$. One example is $f(x) = 2^x$.

78. Since

$$\frac{d}{dx} \ln(x) = \frac{1}{x},$$

any function of the form $y = c \ln x$ for some constant c will have derivative c/x. A particular example is $y = 2 \ln x$.

79. $f(x) = \ln x$ is a possible answer since

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx} f(cx) = \frac{d}{dx} \ln(cx) = \frac{c}{cx} = \frac{1}{x}$$

by the chain rule.

80. For the statement to be true, we need $f'(x) = 1$, so $f(x) = x$ is a function to try. Then $f^{-1}(x) = x$, so

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(x)} = \frac{1}{1} = 1.$$

81. False. Since

$$\frac{d}{dx} \ln(x^2) = \frac{1}{x^2} \cdot 2x = \frac{2}{x} \quad \text{and} \quad \frac{d^2}{dx^2} \ln(x^2) = \frac{d}{dx}\left(\frac{2}{x}\right) = -\frac{2}{x^2},$$

we see that the second derivative of $\ln(x^2)$ is negative for $x > 0$. Thus, the graph is concave down.

82. False; For example, the inverse function of $f(x) = x^3$ is $x^{1/3}$, and the derivative of $x^{1/3}$ is $(1/3)x^{-2/3}$, which is not $1/f'(x) = 1/(3x^2)$.

Solutions for Section 3.7

Exercises

1. We differentiate implicitly both sides of the equation with respect to x.

$$2x + 2y\frac{dy}{dx} = 0\,,$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

2. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} 2x + 3y^2\frac{dy}{dx} &= 0 \\ 3y^2\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{3y^2}. \end{aligned}$$

3. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} 2x + \left(y + x\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} &= y^2 + x(2y)\frac{dy}{dx}\ , \\ x\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 2xy\frac{dy}{dx} &= y^2 - y - 2x\ , \\ \frac{dy}{dx} &= \frac{y^2 - y - 2x}{x - 3y^2 - 2xy}. \end{aligned}$$

4. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} 2x + 2y\frac{dy}{dx} + 3 - 5\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} - 5\frac{dy}{dx} &= -2x - 3 \\ (2y - 5)\frac{dy}{dx} &= -2x - 3 \\ \frac{dy}{dx} &= \frac{-2x - 3}{2y - 5}. \end{aligned}$$

5. Implicit differentiation gives

$$1 \cdot y + x \cdot \frac{dy}{dx} + 1 + \frac{dy}{dx} = 0.$$

Solving for dy/dx, we have

$$\frac{dy}{dx} = -\frac{1 + y}{1 + x}.$$

6.

$$\begin{aligned} 2xy + x^2\frac{dy}{dx} - 2\frac{dy}{dx} &= 0 \\ (x^2 - 2)\frac{dy}{dx} &= -2xy \\ \frac{dy}{dx} &= \frac{-2xy}{(x^2 - 2)} \end{aligned}$$

7. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} (x^2 \cdot 3y^2\frac{dy}{dx} + 2x \cdot y^3) - (x \cdot \frac{dy}{dx} + 1 \cdot y) &= 0 \\ 3x^2y^2\frac{dy}{dx} + 2xy^3 - x\frac{dy}{dx} - y &= 0 \\ 3x^2y^2\frac{dy}{dx} - x\frac{dy}{dx} &= y - 2xy^3 \\ (3x^2y^2 - x)\frac{dy}{dx} &= y - 2xy^3 \\ \frac{dy}{dx} &= \frac{y - 2xy^3}{3x^2y^2 - x}. \end{aligned}$$

8. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} x^{1/2} &= 5y^{1/2} \\ \frac{1}{2}x^{-1/2} &= \frac{5}{2}y^{-1/2}\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{\frac{1}{2}x^{-1/2}}{\frac{5}{2}y^{-1/2}} = \frac{1}{5}\sqrt{\frac{y}{x}} = \frac{1}{25}. \end{aligned}$$

We can also obtain this answer by realizing that the original equation represents part of the line $x = 25y$ which has slope $1/25$.

9. We differentiate implicitly both sides of the equation with respect to x.

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 25\,,$$
$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0\,,$$
$$\frac{dy}{dx} = -\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}y^{-\frac{1}{2}}} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}.$$

10. We differentiate implicitly with respect to x.

$$y + x\frac{dy}{dx} - 1 - \frac{3dy}{dx} = 0$$
$$(x-3)\frac{dy}{dx} = 1 - y$$
$$\frac{dy}{dx} = \frac{1-y}{x-3}$$

11.

$$12x + 8y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-12x}{8y} = \frac{-3x}{2y}$$

12.

$$2ax - 2by\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2ax}{-2by} = \frac{ax}{by}$$

13. We differentiate implicitly both sides of the equation with respect to x.

$$\ln x + \ln(y^2) = 3$$
$$\frac{1}{x} + \frac{1}{y^2}(2y)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-1/x}{2y/y^2} = -\frac{y}{2x}.$$

14. We differentiate implicitly both sides of the equation with respect to x.

$$\ln y + x\frac{1}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} = \frac{1}{x}$$
$$\frac{x}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} = \frac{1}{x} - \ln y$$
$$\frac{dy}{dx}\left(\frac{x}{y} + 3y^2\right) = \frac{1 - x\ln y}{x}$$
$$\frac{dy}{dx}\left(\frac{x + 3y^3}{y}\right) = \frac{1 - x\ln y}{x}$$
$$\frac{dy}{dx} = \frac{(1 - x\ln y)}{x} \cdot \frac{y}{(x + 3y^3)}$$

15. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned}\cos(xy)\left(y+x\frac{dy}{dx}\right)&=2\\ y\cos(xy)+x\cos(xy)\frac{dy}{dx}&=2\\ \frac{dy}{dx}&=\frac{2-y\cos(xy)}{x\cos(xy)}.\end{aligned}$$

16. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned}e^{\cos y}(-\sin y)\frac{dy}{dx}&=3x^2\arctan y+x^3\frac{1}{1+y^2}\frac{dy}{dx}\\ \frac{dy}{dx}\left(-e^{\cos y}\sin y-\frac{x^3}{1+y^2}\right)&=3x^2\arctan y\\ \frac{dy}{dx}&=\frac{3x^2\arctan y}{-e^{\cos y}\sin y-x^3(1+y^2)^{-1}}.\end{aligned}$$

17. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned}\arctan(x^2y)&=xy^2\\ \frac{1}{1+x^4y^2}(2xy+x^2\frac{dy}{dx})&=y^2+2xy\frac{dy}{dx}\\ 2xy+x^2\frac{dy}{dx}&=[1+x^4y^2][y^2+2xy\frac{dy}{dx}]\\ \frac{dy}{dx}[x^2-(1+x^4y^2)(2xy)]&=(1+x^4y^2)y^2-2xy\\ \frac{dy}{dx}&=\frac{y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}.\end{aligned}$$

18. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned}e^{x^2}+\ln y&=0\\ 2xe^{x^2}+\frac{1}{y}\frac{dy}{dx}&=0\\ \frac{dy}{dx}&=-2xye^{x^2}.\end{aligned}$$

19. We differentiate implicitly both sides of the equation with respect to x.

$$(x-a)^2+y^2=a^2$$

$$\begin{aligned}2(x-a)+2y\frac{dy}{dx}&=0\\ 2y\frac{dy}{dx}&=2a-2x\\ \frac{dy}{dx}&=\frac{2a-2x}{2y}=\frac{a-x}{y}.\end{aligned}$$

20. $\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx}=0,\ \frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}.$

21. Differentiating implicitly on both sides with respect to x,

$$\begin{aligned} a\cos(ay)\frac{dy}{dx} - b\sin(bx) &= y + x\frac{dy}{dx} \\ (a\cos(ay) - x)\frac{dy}{dx} &= y + b\sin(bx) \\ \frac{dy}{dx} &= \frac{y + b\sin(bx)}{a\cos(ay) - x}. \end{aligned}$$

22. Differentiating $x^2 + y^2 = 1$ with respect to x gives

$$2x + 2yy' = 0$$

so that

$$y' = -\frac{x}{y}$$

At the point $(0, 1)$ the slope is 0.

23. Differentiating $\sin(xy) = x$ with respect to x gives

$$(y + xy')\cos(xy) = 1$$

or

$$xy'\cos(xy) = 1 - y\cos(xy)$$

so that

$$y' = \frac{1 - y\cos(xy)}{x\cos(xy)}.$$

As we move along the curve to the point $(1, \frac{\pi}{2})$, the value of $dy/dx \to \infty$, which tells us the tangent to the curve at $(1, \frac{\pi}{2})$ has infinite slope; the tangent is the vertical line $x = 1$.

24. Differentiating with respect to x gives

$$3x^2 + 2xy' + 2y + 2yy' = 0$$

so that

$$y' = -\frac{3x^2 + 2y}{2x + 2y}$$

At the point $(1, 1)$ the slope is $-\frac{5}{4}$.

25. The slope is given by dy/dx, which we find using implicit differentiation. Notice that the product rule is needed for the second term. We differentiate to obtain:

$$\begin{aligned} 3x^2 + 5x^2\frac{dy}{dx} + 10xy + 4y\frac{dy}{dx} &= 4\frac{dy}{dx} \\ (5x^2 + 4y - 4)\frac{dy}{dx} &= -3x^2 - 10xy \\ \frac{dy}{dx} &= \frac{-3x^2 - 10xy}{5x^2 + 4y - 4}. \end{aligned}$$

At the point $(1, 2)$, we have $dy/dx = (-3 - 20)/(5 + 8 - 4) = -23/9$. The slope of this curve at the point $(1, 2)$ is $-23/9$.

26. First, we must find the slope of the tangent, i.e. $\left.\frac{dy}{dx}\right|_{(1,-1)}$. Differentiating implicitly, we have:

$$\begin{aligned} y^2 + x(2y)\frac{dy}{dx} &= 0, \\ \frac{dy}{dx} = -\frac{y^2}{2xy} &= -\frac{y}{2x}. \end{aligned}$$

Substitution yields $\left.\frac{dy}{dx}\right|_{(1,-1)} = -\frac{-1}{2} = \frac{1}{2}$. Using the point-slope formula for a line, we have that the equation for the tangent line is $y + 1 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{3}{2}$.

27. First we must find the slope of the tangent, $\frac{dy}{dx}$, at $(1, e^2)$. Differentiating implicitly, we have:

$$\frac{1}{xy}\left(x\frac{dy}{dx}+y\right) = 2$$
$$\frac{dy}{dx} = \frac{2xy-y}{x}.$$

Evaluating dy/dx at $(1, e^2)$ yields $(2(1)e^2 - e^2)/1 = e^2$. Using the point-slope formula for the equation of the line, we have:

$$y - e^2 = e^2(x-1),$$

or

$$y = e^2 x.$$

28. First, we must find the slope of the tangent, $\left.\frac{dy}{dx}\right|_{(4,2)}$. Implicit differentiation yields:

$$2y\frac{dy}{dx} = \frac{2x(xy-4) - x^2\left(x\frac{dy}{dx}+y\right)}{(xy-4)^2}.$$

Given the complexity of the above equation, we first want to substitute 4 for x and 2 for y (the coordinates of the point where we are constructing our tangent line), then solve for $\frac{dy}{dx}$. Substitution yields:

$$2\cdot 2\frac{dy}{dx} = \frac{(2\cdot 4)(4\cdot 2-4) - 4^2\left(4\frac{dy}{dx}+2\right)}{(4\cdot 2-4)^2} = \frac{8(4) - 16(4\frac{dy}{dx}+2)}{16} = -4\frac{dy}{dx}.$$

$$4\frac{dy}{dx} = -4\frac{dy}{dx},$$

Solving for $\frac{dy}{dx}$, we have:

$$\frac{dy}{dx} = 0.$$

The tangent is a horizontal line through $(4, 2)$, hence its equation is $y = 2$.

29. First, we must find the slope of the tangent at the origin, that is $\left.\frac{dy}{dx}\right|_{(0,0)}$. Rewriting $y = \frac{x}{y+a}$ as $y(y+a) = x$ so that we have

$$y^2 + ay = x$$

and differentiating implicitly gives

$$2y\frac{dy}{dx} + a\frac{dy}{dx} = 1$$
$$\frac{dy}{dx}(2y+a) = 1$$
$$\frac{dy}{dx} = \frac{1}{2y+a}.$$

Substituting $x = 0,\ y = 0$ yields $\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{1}{a}$. Using the point-slope formula for a line, we have that the equation for the tangent line is

$$y - 0 = \frac{1}{a}(x-0) \quad \text{or} \quad y = \frac{x}{a}.$$

30. First, we must find the slope of the tangent, $\left.\frac{dy}{dx}\right|_{(a,0)}$. We differentiate implicitly, obtaining:

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0\,,$$

$$\frac{dy}{dx} = -\frac{\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}.$$

Substitution yields, $\left.\frac{dy}{dx}\right|_{(a,0)} = \frac{\sqrt[3]{0}}{\sqrt[3]{a}} = 0$. The tangent is a horizontal line through $(a, 0)$, hence its equation is $y = 0$.

Problems

31. (a) By implicit differentiation, we have:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} - 4 + 7\frac{dy}{dx} &= 0 \\ (2y+7)\frac{dy}{dx} &= 4 - 2x \\ \frac{dy}{dx} &= \frac{4-2x}{2y+7}. \end{aligned}$$

(b) The curve has a horizontal tangent line when $dy/dx = 0$, which occurs when $4 - 2x = 0$ or $x = 2$. The curve has a horizontal tangent line at all points where $x = 2$.
The curve has a vertical tangent line when dy/dx is undefined, which occurs when $2y + 7 = 0$ or when $y = -7/2$. The curve has a vertical tangent line at all points where $y = -7/2$.

32. (a) Taking derivatives implicitly, we get

$$\begin{aligned} \frac{2}{25}x + \frac{2}{9}y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-9x}{25y}. \end{aligned}$$

(b) The slope is not defined anywhere along the line $y = 0$. This ellipse intersects that line in two places, $(-5, 0)$ and $(5, 0)$. (These are the "ends" of the ellipse where the tangent is vertical.)

33. Differentiating implicitly with respect to x, gives

$$\frac{d}{dx}\left(\frac{(x-2)^2}{16} + \frac{y^2}{4}\right) = \frac{d}{dx}(1).$$

Treating y as a function of x and using the chain rule, we get

$$\begin{aligned} \frac{x-2}{8} + \frac{y}{2}\cdot\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-\left(\frac{x-2}{8}\right)}{\frac{y}{2}} = \frac{2-x}{4y}. \end{aligned}$$

At $x = 2$, the value of dy/dx is 0 and the tangent lines are horizontal. The corresponding y-values satisfy

$$\begin{aligned} \frac{0^2}{16^2} + \frac{y^2}{4} &= 1 \\ y^2 &= 4 \\ y &= \pm 2. \end{aligned}$$

Thus the tangent line equations at $(2, 2)$ and $(2, -2)$ are the horizontal lines $y = 2$ and $y = -2$, respectively.

34. (a) If $x = 4$ then $16 + y^2 = 25$, so $y = \pm 3$. We find $\frac{dy}{dx}$ implicitly:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

So the slope at $(4,3)$ is $-\frac{4}{3}$ and at $(4,-3)$ is $\frac{4}{3}$. The tangent lines are:

$$(y-3) = -\frac{4}{3}(x-4) \quad \text{and} \quad (y+3) = \frac{4}{3}(x-4)$$

(b) The normal lines have slopes that are the negative of the reciprocal of the slopes of the tangent lines. Thus,

$$(y-3) = \frac{3}{4}(x-4) \quad \text{so} \quad y = \frac{3}{4}x$$

and

$$(y+3) = -\frac{3}{4}(x-4) \quad \text{so} \quad y = -\frac{3}{4}x$$

are the normal lines.

(c) These lines meet at the origin, which is the center of the circle.

35. (a) Solving for $\frac{dy}{dx}$ by implicit differentiation yields

$$3x^2 + 3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{y^2 - 3x^2}{3y^2 - 2xy}.$$

(b) We can approximate the curve near $x = 1$, $y = 2$ by its tangent line. The tangent line will have slope $\frac{(2)^2-3(1)^2}{3(2)^2-2(1)(2)} = \frac{1}{8} = 0.125$. Thus its equation is

$$y = 0.125x + 1.875$$

Using the y-values of the tangent line to approximate the y-values of the curve, we get:

x	0.96	0.98	1	1.02	1.04
approximate y	1.995	1.9975	2.000	2.0025	2.005

(c) When $x = 0.96$, we get the equation $0.96^3 + y^3 - 0.96y^2 = 5$, whose solution by numerical methods is 1.9945, which is close to the one above.

(d) The tangent line is horizontal when $\frac{dy}{dx}$ is zero and vertical when $\frac{dy}{dx}$ is undefined. These will occur when the numerator is zero and when the denominator is zero, respectively.

Thus, we know that the tangent is horizontal when $y^2 - 3x^2 = 0 \Rightarrow y = \pm\sqrt{3}x$. To find the points that satisfy this condition, we substitute back into the original equation for the curve:

$$x^3 + y^3 - xy^2 = 5$$
$$x^3 \pm 3\sqrt{3}x^3 - 3x^3 = 5$$
$$x^3 = \frac{5}{\pm 3\sqrt{3} - 2}$$

So $x \approx 1.1609$ or $x \approx -0.8857$.

Substituting,

$$y = \pm\sqrt{3}x \quad \text{so} \quad y \approx 2.0107 \quad \text{or} \quad y \approx 1.5341.$$

Thus, the tangent line is horizontal at $(1.1609, 2.0107)$ and $(-0.8857, 1.5341)$.

Also, we know that the tangent is vertical whenever $3y^2 - 2xy = 0$, that is, when $y = \frac{2}{3}x$ or $y = 0$. Substituting into the original equation for the curve gives us $x^3 + (\frac{2}{3}x)^3 - (\frac{2}{3})^2x^3 = 5$. This means $x^3 \approx 5.8696$, so $x \approx 1.8039$, $y \approx 1.2026$. The other vertical tangent is at $y = 0$, $x = \sqrt[3]{5}$.

36. The slope of the tangent to the curve $y = x^2$ at $x = 1$ is 2 so the equation of such a tangent will be of the form $y = 2x + c$. As the tangent must pass through $(1,1)$, $c = -1$ and so the required tangent is $y = 2x - 1$.

Any circle centered at $(8,0)$ will be of the form

$$(x-8)^2 + y^2 = R^2.$$

The slope of this curve at (x,y) is given by implicit differentiation:

$$2(x-8) + 2yy' = 0$$

or

$$y' = \frac{8-x}{y}$$

For the tangent to the parabola to be tangential to the circle we need

$$\frac{8-x}{y} = 2$$

so that at the point of contact of the circle and the line the coordinates are given by (x, y) when $y = 4 - x/2$. Substituting into the equation of the tangent line gives $x = 2$ and $y = 3$. From this we conclude that $R^2 = 45$ so that the equation of the circle is

$$(x-8)^2 + y^2 = 45.$$

37. (a) Differentiating both sides of the equation with respect to P gives

$$\frac{d}{dP}\left(\frac{4f^2P}{1-f^2}\right) = \frac{dK}{dP} = 0.$$

By the product rule

$$\begin{aligned}\frac{d}{dP}\left(\frac{4f^2P}{1-f^2}\right) &= \frac{d}{dP}\left(\frac{4f^2}{1-f^2}\right)P + \left(\frac{4f^2}{1-f^2}\right)\cdot 1\\ &= \left(\frac{(1-f^2)(8f) - 4f^2(-2f)}{(1-f^2)^2}\right)\frac{df}{dP}P + \left(\frac{4f^2}{1-f^2}\right)\\ &= \left(\frac{8f}{(1-f^2)^2}\right)\frac{df}{dP}P + \left(\frac{4f^2}{1-f^2}\right) = 0.\end{aligned}$$

So

$$\frac{df}{dP} = \frac{-4f^2/(1-f^2)}{8fP/(1-f^2)^2} = \frac{-1}{2P}f(1-f^2).$$

(b) Since f is a fraction of a gas, $0 \le f \le 1$. Also, in the equation relating f and P we can't have $f = 0$, since that would imply $K = 0$, and we can't have $f = 1$, since the left side is undefined there. So $0 < f < 1$. Thus $1 - f^2 > 0$. Also, pressure can't be negative, and from the equation relating f and P, we see that P can't be zero either, so $P > 0$. Therefore $df/dP = -(1/2P)f(1-f^2) < 0$ always. This means that at larger pressures less of the gas decomposes.

38. Let the point of intersection of the tangent line with the smaller circle be (x_1, y_1) and the point of intersection with the larger be (x_2, y_2). Let the tangent line be $y = mx + c$. Then at (x_1, y_1) and (x_2, y_2) the slopes of $x^2 + y^2 = 1$ and $y^2 + (x-3)^2 = 4$ are also m. The slope of $x^2 + y^2 = 1$ is found by implicit differentiation: $2x + 2yy' = 0$ so $y' = -x/y$. Similarly, the slope of $y^2 + (x-3)^2 = 4$ is $y' = -(x-3)/y$. Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1}{y_1} = -\frac{(x_2-3)}{y_2},$$

where $y_1 = \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4 - (x_2-3)^2}$. The positive values for y_1 and y_2 follow from Figure 3.9 and from our choice of $m > 0$. We obtain

$$\begin{aligned}\frac{x_1}{\sqrt{1-x_1^2}} &= \frac{x_2-3}{\sqrt{4-(x_2-3)^2}}\\ \frac{x_1^2}{1-x_1^2} &= \frac{(x_2-3)^2}{4-(x_2-3)^2}\\ x_1^2[4-(x_2-3)^2] &= (1-x_1^2)(x_2-3)^2\\ 4x_1^2 - (x_1^2)(x_2-3)^2 &= (x_2-3)^2 - x_1^2(x_2-3)^2\\ 4x_1^2 &= (x_2-3)^2\\ 2|x_1| &= |x_2-3|.\end{aligned}$$

From the picture $x_1 < 0$ and $x_2 < 3$. This gives $x_2 = 2x_1 + 3$ and $y_2 = 2y_1$. From

$$\frac{y_2-y_1}{x_2-x_1} = -\frac{x_1}{y_1},$$

substituting $y_1 = \sqrt{1 - x_1^2}$, $y_2 = 2y_1$ and $x_2 = 2x_1 + 3$ gives

$$x_1 = -\frac{1}{3}.$$

From $x_2 = 2x_1 + 3$ we get $x_2 = 7/3$. In addition, $y_1 = \sqrt{1 - x_1^2}$ gives $y_1 = 2\sqrt{2}/3$, and finally $y_2 = 2y_1$ gives $y_2 = 4\sqrt{2}/3$.

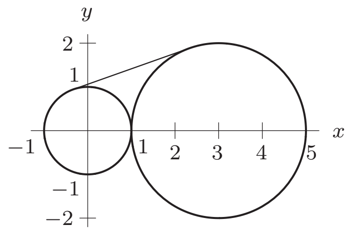


Figure 3.9

39. Using implicit differentiation we have

$$1 = (\cos y)\frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

Solving the trig identity $\sin^2 y + \cos^2 y = 1$ for $\cos y$ and substituting $x = \sin y$ gives $\cos y = \pm\sqrt{1 - x^2}$. However, because $y = \arcsin x$, we have $-\pi/2 \le y \le \pi/2$, so $\cos y \ge 0$ and thus we take the positive root:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

40. $y = x^{\frac{m}{n}}$. Taking n^{th} powers of both sides of this expression yields $(y)^n = (x^{\frac{m}{n}})^n$, or $y^n = x^m$.

$$\begin{aligned}
\frac{d}{dx}(y^n) &= \frac{d}{dx}(x^m) \\
ny^{n-1}\frac{dy}{dx} &= mx^{m-1} \\
\frac{dy}{dx} &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{m-\frac{m}{n}}} \\
&= \frac{m}{n}x^{(m-1)-(m-\frac{m}{n})} = \frac{m}{n}x^{\frac{m}{n}-1}.
\end{aligned}$$

41. Implicit differentiation gives

$$\begin{aligned}
\left(1 - \frac{2n^2a}{V^3}\frac{dV}{dP}\right)(V - nb) + \left(P + \frac{n^2a}{V^2}\right)\frac{dV}{dP} &= 0 \\
\frac{dV}{dP}\left(-\frac{2n^2a}{V^3}(V - nb) + \left(P + \frac{n^2a}{V^2}\right)\right) &= nb - V \\
\frac{dV}{dP}\left(P - \frac{n^2a}{V^2} + \frac{2n^3ab}{V^3}\right) &= nb - V \\
\frac{dV}{dP} &= \frac{nb - V}{P - n^2a/V^2 + 2n^3ab/V^3}.
\end{aligned}$$

Strengthen Your Understanding

42. Since we cannot solve for y in terms of x, we need to differentiate implicitly. This gives

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\sin(xy)\right),$$

so

$$\frac{dy}{dx} = \cos(xy)\left(1 \cdot y + x \cdot \frac{dy}{dx}\right).$$

Solving for dy/dx gives

$$\frac{dy}{dx} = \frac{y\cos(xy)}{1 - x\cos(xy)}.$$

43. The formula applies only when the point (x, y) is on the circle.

44. One possible format for the slope is

$$\frac{dy}{dx} = \frac{x^2 - 4}{y - 2}.$$

To confirm that the tangents exist, we would need to have the formula for the implicit function and check that there are points satisfying it with $y = 2$ and $x = \pm 2$.

45. A hyperbola opening up and down works here.

For example, $-x^2 + y^2 = 1$. Note that $\dfrac{dy}{dx} = \dfrac{x}{y}$. Although the denominator is zero when $y = 0$, the original equation is not satisfied by $y = 0$. Thus, there are no vertical tangents to the curve. Horizontal tangents occur when $x = 0$ and $y = \pm\sqrt{1}$.

46. True; differentiating the equation with respect to x, we get

$$2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0.$$

Solving for dy/dx, we get that

$$\frac{dy}{dx} = \frac{-y}{2y + x}.$$

Thus dy/dx exists where $2y + x \neq 0$. Now if $2y + x = 0$, then $x = -2y$. Substituting for x in the original equation, $y^2 + xy - 1 = 0$, we get

$$y^2 - 2y^2 - 1 = 0.$$

This simplifies to $y^2 + 1 = 0$, which has no solutions. Thus dy/dx exists everywhere.

Solutions for Section 3.8

Exercises

1. Using the chain rule, $\dfrac{d}{dz}\left(\sinh(3z + 5)\right) = \cosh(3z + 5) \cdot 3 = 3\cosh(3z + 5)$.

2. Using the chain rule, $\dfrac{d}{dx}\left(\cosh(2x)\right) = \left(\sinh(2x)\right) \cdot 2 = 2\sinh(2x)$.

3. Using the chain rule,

$$\frac{d}{dt}\left(\cosh^2 t\right) = 2\cosh t \cdot \sinh t.$$

.

4. Using the chain rule,

$$\frac{d}{dt}\left(\cosh(\sinh t)\right) = \sinh(\sinh t) \cdot \cosh t$$

.

5. Using the product rule,

$$\frac{d}{dt}\left(t^3 \sinh t\right) = 3t^2 \sinh t + t^3 \cosh t.$$

6. Using the product and chain rules, $\frac{d}{dt}(\cosh(3t)\sinh(4t)) = 3\sinh(3t)\sinh(4t) + 4\cosh(3t)\cosh(4t)$.

7. Using the chain rule, $\frac{d}{dx}(\tanh(3+\sinh x)) = \frac{1}{\cosh^2(3+\sinh x)} \cdot \cosh x$.

8. Using the chain rule twice, $\frac{d}{dt}\left(\cosh(e^{t^2})\right) = \sinh(e^{t^2}) \cdot e^{t^2} \cdot 2t = 2te^{t^2}\sinh(e^{t^2})$.

9. Using the chain rule,

$$\frac{d}{d\theta}(\ln(\cosh(1+\theta))) = \frac{1}{\cosh(1+\theta)} \cdot \sinh(1+\theta) = \frac{\sinh(1+\theta)}{\cosh(1+\theta)} = \tanh(1+\theta).$$

10. Using the chain rule twice,

$$\begin{aligned}\frac{d}{dy}(\sinh(\sinh(3y))) &= \cosh(\sinh(3y)) \cdot \cosh(3y) \cdot 3 \\ &= 3\cosh(3y) \cdot \cosh(\sinh(3y)).\end{aligned}$$

11. Using the chain rule, $f'(t) = 2\cosh t \sinh t - 2\sinh t \cosh t = 0$. This is to be expected since $\cosh^2 t - \sinh^2 t = 1$.

12. Using the formula for $\sinh x$ and the fact that $d(e^{-x})/dx = -e^{-x}$, we see that

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

13. Substitute $x = 0$ into the formula for $\sinh x$. This yields

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0.$$

14. Substituting $-x$ for x in the formula for $\sinh x$ gives

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x.$$

15. By definition $\cosh x = (e^x + e^{-x})/2$ so, since $e^{\ln t} = t$ and $e^{-\ln t} = 1/e^{\ln t} = 1/t$, we have

$$\cosh(\ln t) = \frac{e^{\ln t} + e^{-\ln t}}{2} = \frac{t + 1/t}{2} = \frac{t^2+1}{2t}.$$

16. By definition $\sinh x = (e^x - e^{-x})/2$ so, since $e^{\ln t} = t$ and $e^{-\ln t} = 1/e^{\ln t} = 1/t$, we have

$$\sinh(\ln t) = \frac{e^{\ln t} - e^{-\ln t}}{2} = \frac{t - 1/t}{2} = \frac{t^2-1}{2t}.$$

Problems

17. The graph of $\sinh x$ in the text suggests that

$$\text{As } x \to \infty, \quad \sinh x \to \frac{1}{2}e^x.$$
$$\text{As } x \to -\infty, \quad \sinh x \to -\frac{1}{2}e^{-x}.$$

Using the facts that

$$\text{As } x \to \infty, \quad e^{-x} \to 0,$$
$$\text{As } x \to -\infty, \quad e^{x} \to 0,$$

we can obtain the same results analytically:

$$\text{As } x \to \infty, \quad \sinh x = \frac{e^x - e^{-x}}{2} \to \frac{1}{2}e^x.$$
$$\text{As } x \to -\infty, \quad \sinh x = \frac{e^x - e^{-x}}{2} \to -\frac{1}{2}e^{-x}.$$

18. Using the identity

$$\cosh^2 t - \sinh^2 t = 1,$$

we see

$$x^2 - y^2 = 1.$$

19. First we observe that

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2}.$$

Now let's calculate

$$\begin{aligned}(\sinh x)(\cosh x) &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{(e^x)^2 - (e^{-x})^2}{4}\\ &= \frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{1}{2}\sinh(2x).\end{aligned}$$

Thus, we see that

$$\sinh(2x) = 2\sinh x \cosh x.$$

20. First, we observe that

$$\cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}.$$

Now let's use the fact that $e^x \cdot e^{-x} = 1$ to calculate

$$\begin{aligned}\cosh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2\\ &= \frac{(e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2}{4}\\ &= \frac{e^{2x} + 2 + e^{-2x}}{4}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sinh^2 x &= \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{(e^x)^2 - 2e^x \cdot e^{-x} + (e^{-x})^2}{4}\\ &= \frac{e^{2x} - 2 + e^{-2x}}{4}.\end{aligned}$$

Thus, to obtain $\cosh(2x)$, we need to add (rather than subtract) $\cosh^2 x$ and $\sinh^2 x$, giving

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{2e^{2x} + 2e^{-2x}}{4} \\ &= \frac{e^{2x} + e^{-2x}}{2} \\ &= \cosh(2x).\end{aligned}$$

Thus, we see that the identity relating $\cosh(2x)$ to $\cosh x$ and $\sinh x$ is

$$\cosh(2x) = \cosh^2 x + \sinh^2 x.$$

21. Recall that

$$\sinh A = \frac{1}{2}(e^A - e^{-A}) \quad \text{and} \quad \cosh A = \frac{1}{2}(e^A + e^{-A}).$$

Now substitute, expand and collect terms:

$$\begin{aligned}\cosh A \cosh B + \sinh A \sinh B &= \frac{1}{2}(e^A + e^{-A}) \cdot \frac{1}{2}(e^B + e^{-B}) + \frac{1}{2}(e^A - e^{-A}) \cdot \frac{1}{2}(e^B - e^{-B}) \\ &= \frac{1}{4}\left(e^{A+B} + e^{A-B} + e^{-A+B} + e^{-(A+B)}\right. \\ &\qquad \left. + e^{B+A} - e^{B-A} - e^{-B+A} + e^{-A-B}\right) \\ &= \frac{1}{2}\left(e^{A+B} + e^{-(A+B)}\right) \\ &= \cosh(A+B).\end{aligned}$$

22. Recall that

$$\sinh A = \frac{1}{2}(e^A - e^{-A}) \quad \text{and} \quad \cosh A = \frac{1}{2}(e^A + e^{-A}).$$

Now substitute, expand and collect terms:

$$\begin{aligned}\sinh A \cosh B + \cosh A \sinh B &= \frac{1}{2}(e^A - e^{-A}) \cdot \frac{1}{2}(e^B + e^{-B}) + \frac{1}{2}(e^A + e^{-A}) \cdot \frac{1}{2}(e^B - e^{-B}) \\ &= \frac{1}{4}\left(e^{A+B} + e^{A-B} - e^{-A+B} - e^{-(A+B)}\right. \\ &\qquad \left. + e^{B+A} + e^{B-A} - e^{-B+A} - e^{-A-B}\right) \\ &= \frac{1}{2}\left(e^{A+B} - e^{-(A+B)}\right) \\ &= \sinh(A+B).\end{aligned}$$

23. $\lim_{x\to\infty} \sinh(2x)/\cosh(3x) = \lim_{x\to\infty}(e^{2x} - e^{-2x})/(e^{3x} + e^{-3x}) = \lim_{x\to\infty}(1 - e^{-4x})/(e^x + e^{-5x}) = 0.$

24. Using the definition of $\sinh x$, we have $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{2x}}{\sinh(2x)} &= \lim_{x\to\infty} \frac{2e^{2x}}{e^{2x} - e^{-2x}} \\ &= \lim_{x\to\infty} \frac{2}{1 - e^{-4x}} \\ &= 2.\end{aligned}$$

25. Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh x^2 = \dfrac{e^{x^2}+e^{-x^2}}{2}$ and $\sinh x^2 = \dfrac{e^{x^2}-e^{-x^2}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to\infty}\frac{\sinh(x^2)}{\cosh(x^2)} &= \lim_{x\to\infty}\frac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}\\ &= \lim_{x\to\infty}\frac{e^{x^2}(1-e^{-2x^2})}{e^{x^2}(1+e^{-2x^2})}\\ &= \lim_{x\to\infty}\frac{1-e^{-2x^2}}{1+e^{-2x^2}}\\ &= 1.\end{aligned}$$

26. Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh 2x = \dfrac{e^{2x}+e^{-2x}}{2}$ and $\sinh 3x = \dfrac{e^{3x}-e^{-3x}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to\infty}\frac{\cosh(2x)}{\sinh(3x)} &= \lim_{x\to\infty}\frac{e^{2x}+e^{-2x}}{e^{3x}-e^{-3x}}\\ &= \lim_{x\to\infty}\frac{e^{2x}(1+e^{-4x})}{e^{2x}(e^{x}-e^{-5x})}\\ &= \lim_{x\to\infty}\frac{1+e^{-4x}}{e^{x}-e^{-5x}}\\ &= 0.\end{aligned}$$

27. Note that

$$\begin{aligned}e^{-3x}\cosh kx &= e^{-3x}\frac{e^{kx}+e^{-kx}}{2}\\ &= \frac{e^{(k-3)x}+e^{-(k+3)x}}{2}.\end{aligned}$$

If $|k| = 3$, then the limit as $x\to\infty$ is 1/2.
If $|k| > 3$, then the limit as $x\to\infty$ does not exist.
If $|k| < 3$, then the limit as $x\to\infty$ is 0.

28. Note that

$$\begin{aligned}\frac{\sinh kx}{\cosh 2x} &= \frac{e^{kx}-e^{-kx}}{e^{2x}+e^{-2x}}\\ &= \frac{e^{2x}(e^{(k-2)x}-e^{-(k+2)x})}{e^{2x}(1+e^{-4x})}\\ &= \frac{e^{(k-2)x}-e^{-(k+2)x}}{1+e^{-4x}}.\end{aligned}$$

If $k = 2$, then the limit as $x\to\infty$ is 1.
If $|k| > 2$, then the limit as $x\to\infty$ does not exist.
If $|k| < 2$, then the limit as $x\to\infty$ is 0.

29. (a) Since the cosh function is even, the height, y, is the same at $x = -T/w$ and $x = T/w$. The height at these endpoints is

$$y = \frac{T}{w}\cosh\left(\frac{w}{T}\cdot\frac{T}{w}\right) = \frac{T}{w}\cosh 1 = \frac{T}{w}\left(\frac{e^1+e^{-1}}{2}\right).$$

At the lowest point, $x = 0$, and the height is

$$y = \frac{T}{w}\cosh 0 = \frac{T}{w}.$$

Thus the "sag" in the cable is given by

$$\text{Sag} = \frac{T}{w}\left(\frac{e+e^{-1}}{2}\right) - \frac{T}{w} = \frac{T}{w}\left(\frac{e+e^{-1}}{2}-1\right) \approx 0.54\frac{T}{w}.$$

(b) To show that the differential equation is satisfied, take derivatives

$$\frac{dy}{dx} = \frac{T}{w} \cdot \frac{w}{T} \sinh\left(\frac{wx}{T}\right) = \sinh\left(\frac{wx}{T}\right)$$
$$\frac{d^2y}{dx^2} = \frac{w}{T} \cosh\left(\frac{wx}{T}\right).$$

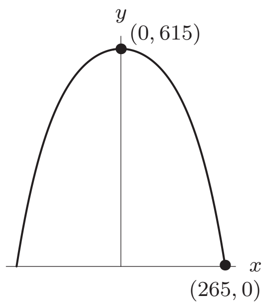
Therefore, using the fact that $1 + \sinh^2 a = \cosh^2 a$ and that cosh is always positive, we have:

$$\frac{w}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{w}{T}\sqrt{1 + \sinh^2\left(\frac{wx}{T}\right)} = \frac{w}{T}\sqrt{\cosh^2\left(\frac{wx}{T}\right)}$$
$$= \frac{w}{T}\cosh\left(\frac{wx}{T}\right).$$

So

$$\frac{w}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}.$$

30.



We know $x = 0$ and $y = 615$ at the top of the arch, so

$$615 = b - a\cosh(0/a) = b - a.$$

This means $b = a + 615$. We also know that $x = 265$ and $y = 0$ where the arch hits the ground, so

$$0 = b - a\cosh(265/a) = a + 615 - a\cosh(265/a).$$

We can solve this equation numerically on a calculator and get $a \approx 100$, which means $b \approx 715$. This results in the equation

$$y \approx 715 - 100\cosh\left(\frac{x}{100}\right).$$

31. (a) The graph in Figure 3.10 looks like the graph of $y = \cosh x$, with the minimum at about $(0.5, 6.3)$.

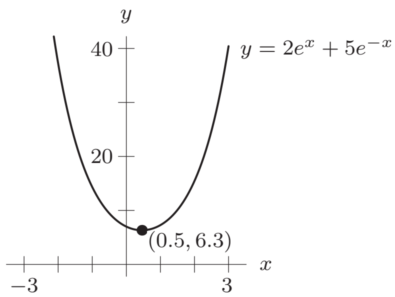


Figure 3.10

(b) We want to write

$$\begin{aligned} y = 2e^x + 5e^{-x} = A\cosh(x-c) &= \frac{A}{2}e^{x-c} + \frac{A}{2}e^{-(x-c)} \\ &= \frac{A}{2}e^x e^{-c} + \frac{A}{2}e^{-x}e^c \\ &= \left(\frac{Ae^{-c}}{2}\right)e^x + \left(\frac{Ae^c}{2}\right)e^{-x}. \end{aligned}$$

Thus, we need to choose A and c so that

$$\frac{Ae^{-c}}{2} = 2 \quad \text{and} \quad \frac{Ae^c}{2} = 5.$$

Dividing gives

$$\begin{aligned} \frac{Ae^c}{Ae^{-c}} &= \frac{5}{2} \\ e^{2c} &= 2.5 \\ c &= \frac{1}{2}\ln 2.5 \approx 0.458. \end{aligned}$$

Solving for A gives

$$A = \frac{4}{e^{-c}} = 4e^c \approx 6.325.$$

Thus,

$$y = 6.325\cosh(x - 0.458).$$

Rewriting the function in this way shows that the graph in part (a) is the graph of $\cosh x$ shifted to the right by 0.458 and stretched vertically by a factor of 6.325.

32. We want to show that for any $A,\ B$ with $A > 0,\ B > 0$, we can find K and c such that

$$\begin{aligned} y = Ae^x + Be^{-x} &= \frac{Ke^{(x-c)} + Ke^{-(x-c)}}{2} \\ &= \frac{K}{2}e^x e^{-c} + \frac{K}{2}e^{-x}e^c \\ &= \left(\frac{Ke^{-c}}{2}\right)e^x + \left(\frac{Ke^c}{2}\right)e^{-x}. \end{aligned}$$

Thus, we want to find K and c such that

$$\frac{Ke^{-c}}{2} = A \quad \text{and} \quad \frac{Ke^c}{2} = B.$$

Dividing, we have

$$\begin{aligned} \frac{Ke^c}{Ke^{-c}} &= \frac{B}{A} \\ e^{2c} &= \frac{B}{A} \\ c &= \frac{1}{2}\ln\left(\frac{B}{A}\right). \end{aligned}$$

If $A > 0,\ B > 0$, then there is a solution for c. Substituting to find K, we have

$$\begin{aligned} \frac{Ke^{-c}}{2} &= A \\ K &= 2Ae^c = 2Ae^{(\ln(B/A))/2} \\ &= 2Ae^{\ln\sqrt{B/A}} = 2A\sqrt{\frac{B}{A}} = 2\sqrt{AB}. \end{aligned}$$

Thus, if $A > 0,\ B > 0$, there is a solution for K also.

The fact that $y = Ae^x + Be^{-x}$ can be rewritten in this way shows that the graph of $y = Ae^x + Be^{-x}$ is the graph of $\cosh x$, shifted over by c and stretched (or shrunk) vertically by a factor of K.

33. (a) Substituting $x = 0$ gives
$$\tanh 0 = \frac{e^0 - e^{-0}}{e^0 + e^{-0}} = \frac{1-1}{2} = 0.$$

(b) Since $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ and $e^x + e^{-x}$ is always positive, $\tanh x$ has the same sign as $e^x - e^{-x}$. For $x > 0$, we have $e^x > 1$ and $e^{-x} < 1$, so $e^x - e^{-x} > 0$. For $x < 0$, we have $e^x < 1$ and $e^{-x} > 1$, so $e^x - e^{-x} < 0$. For $x = 0$, we have $e^x = 1$ and $e^{-x} = 1$, so $e^x - e^{-x} = 0$. Thus, $\tanh x$ is positive for $x > 0$, negative for $x < 0$, and zero for $x = 0$.

(c) Taking the derivative, we have
$$\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}.$$
Thus, for all x,
$$\frac{d}{dx}(\tanh x) > 0.$$
Thus, $\tanh x$ is increasing everywhere.

(d) As $x \to \infty$ we have $e^{-x} \to 0$; as $x \to -\infty$, we have $e^x \to 0$. Thus
$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1,$$
$$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -1.$$
Thus, $y = 1$ and $y = -1$ are horizontal asymptotes to the graph of $\tanh x$. See Figure 3.11.

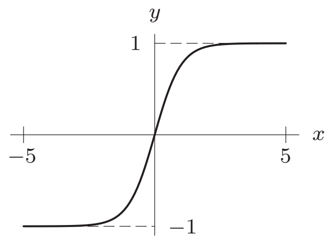


Figure 3.11: Graph of $y = \tanh x$

(e) The graph of $\tanh x$ suggests that $\tanh x$ is increasing everywhere; the fact that the derivative of $\tanh x$ is positive for all x confirms this. Since $\tanh x$ is increasing for all x, different values of x lead to different values of y, and therefore $\tanh x$ does have an inverse.

Strengthen Your Understanding

34. Since $f(x) = \cosh x = (e^x + e^{-x})/2$, the function is not periodic.

35. Since $f(x) = \cosh x = (e^x + e^{-x})/2$ we have
$$f'(x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

36. The required identity for hyperbolic functions is $\cosh^2 x - \sinh^2 x = 1$.

37. Since $\cosh x$ and $\sinh x$ behave like $e^x/2$ as $x \to \infty$ then $\tanh x = \sinh x/\cosh x \to 1$ as $x \to \infty$.

38. $f(x) = \cosh x$ is concave up. Other answers are possible.

39. We have
$$\lim_{x\to\infty} e^{kx}\cosh x = \lim_{x\to\infty} e^{kx}\left(\frac{e^x + e^{-x}}{2}\right) = \lim_{x\to\infty} \frac{e^{(k+1)x} + e^{(k-1)x}}{2}.$$
This limit does not exist for any value $k > -1$: for example, $k = 0$.

40. Since $\cosh 0 = 1$, we can shift $\cosh x$ to the right by 1 and up 2. Thus, $f(x) = \cosh(x-1) + 2$ is a possibility.

41. True. We have $\tanh x = (\sinh x)\,/\cosh x = (e^x - e^{-x})/(e^x + e^{-x})$. Replacing x by $-x$ in this expression gives $(e^{-x} - e^x)/(e^{-x} + e^x) = -\tanh x$.

42. False. The second, fourth and all even derivatives of $\sinh x$ are all $\sinh x$.

43. True. The definitions of $\sinh x$ and $\cosh x$ give

$$\sinh x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \frac{2e^x}{2} = e^x.$$

44. False. Since $(\sinh x)' = \cosh x > 0$, the function $\sinh x$ is increasing everywhere so can never repeat any of its values.

45. False. Since $(\sinh^2 x)' = 2\sinh x \cosh x$ and $(2\sinh x \cosh x)' = 2\sinh^2 x + 2\cosh^2 x > 0$, the function $\sinh^2 x$ is concave up everywhere.

Solutions for Section 3.9

Exercises

1. With $f(x) = \sqrt{1+x}$, the chain rule gives $f'(x) = 1/(2\sqrt{1+x})$, so $f(0) = 1$ and $f'(0) = 1/2$. Therefore the tangent line approximation of f near $x = 0$,

$$f(x) \approx f(0) + f'(0)(x - 0),$$

becomes

$$\sqrt{1+x} \approx 1 + \frac{x}{2}.$$

This means that, near $x = 0$, the function $\sqrt{1+x}$ can be approximated by its tangent line $y = 1 + x/2$. (See Figure 3.12.)

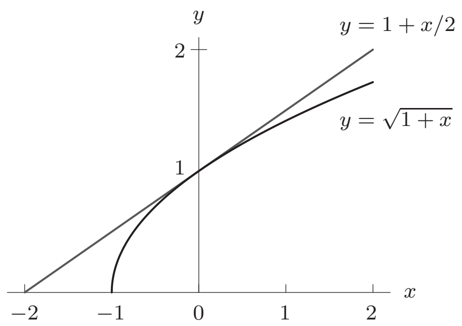


Figure 3.12

2. With $f(x) = e^x$, the tangent line approximation to f near $x = 0$ is $f(x) \approx f(0) + f'(0)(x - 0)$ which becomes $e^x \approx e^0 + e^0 x = 1 + 1x = 1 + x$. Thus, our local linearization of e^x near $x = 0$ is $e^x \approx 1 + x$.

3. With $f(x) = 1/x$, we see that the tangent line approximation to f near $x = 1$ is

$$f(x) \approx f(1) + f'(1)(x - 1),$$

which becomes

$$\frac{1}{x} \approx 1 + f'(1)(x - 1).$$

Since $f'(x) = -1/x^2$, $f'(1) = -1$. Thus our formula reduces to

$$\frac{1}{x} \approx 1 - (x - 1) = 2 - x.$$

This is the local linearization of $1/x$ near $x = 1$.

4. Since $f(1) = 1$ and we showed that $f'(1) = 2$, the local linearization is

$$f(x) \approx 1 + 2(x-1) = 2x - 1.$$

5. With $f(x) = e^{x^2}$, we get a tangent line approximation of $f(x) \approx f(1) + f'(1)(x-1)$ which becomes $e^{x^2} \approx e + \left(2xe^{x^2}\right)\Big|_{x=1}(x-1) = e + 2e(x-1) = 2ex - e$. Thus, our local linearization of e^{x^2} near $x = 1$ is $e^{x^2} \approx 2ex - e$.

6. With $f(x) = 1/(\sqrt{1+x})$, we see that the tangent line approximation to f near $x = 0$ is

$$f(x) \approx f(0) + f'(0)(x-0),$$

which becomes

$$\frac{1}{\sqrt{1+x}} \approx 1 + f'(0)x.$$

Since $f'(x) = (-1/2)(1+x)^{-3/2}$, $f'(0) = -1/2$. Thus our formula reduces to

$$\frac{1}{\sqrt{1+x}} \approx 1 - x/2.$$

This is the local linearization of $\dfrac{1}{\sqrt{1+x}}$ near $x = 0$.

7. Let $f(x) = e^{-x}$. Then $f'(x) = -e^{-x}$. So $f(0) = 1$, $f'(0) = -e^0 = -1$. Therefore, $e^{-x} \approx f(0) + f'(0)x = 1 - x$.

8. The graph of x^2 is concave up and lies above its tangent line; therefore, the linearization will always be too small. See Figure 3.13. The graph of $\sqrt{x}$ is concave down and lies below its tangent line, and therefore the linearization will be too large. See Figure 3.14.

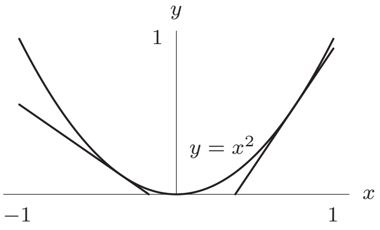


Figure 3.13

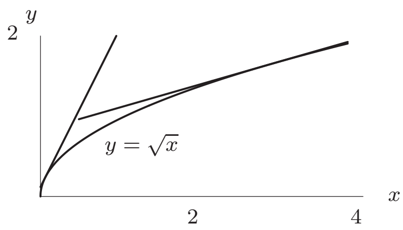


Figure 3.14

9. From Figure 3.15, we see that the error has its maximum magnitude at the end points of the interval, $x = \pm 1$. The magnitude of the error can be read off the graph as less than 0.2 or estimated as

$$|\text{Error}| \leq |1 - \sin 1| = 0.159 < 0.2.$$

The approximation is an overestimate for $x > 0$ and an underestimate for $x < 0$.

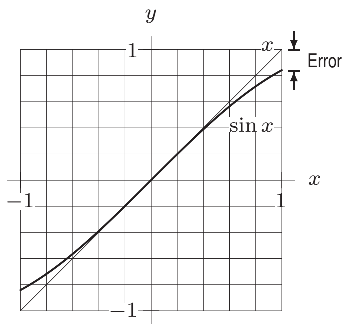


Figure 3.15

10. Figure 3.16 shows that $1 + x$ is an underestimate of e^x for $-1 \leq x \leq 1$. On this interval, the error has the largest magnitude at $x = 1$. Its magnitude can be estimated from the graph as less than 0.8, or estimated as

$$|\text{Error}| = e - 1 - 1 = 0.718 < 0.8.$$

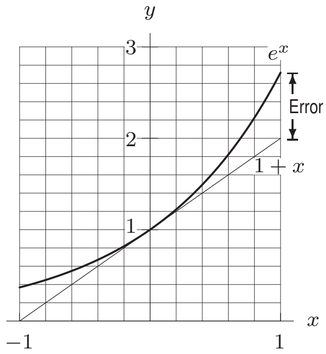


Figure 3.16

Problems

11. (a) If $f(x) = e^x$, then $f'(x) = e^x$, so $f'(0) = e^0 = 1$. Thus, the local linearization near $x = 0$ is

$$L(x) = f(0) + f'(0)(x - 0)$$
$$L(x) = 1 + x.$$

(b) Since the curve $f(x) = e^x$ is above the line $L(x) = 1 + x$ (see Figure 3.17), the error is positive everywhere except at $x = 0$, where it is 0.

(c) The true value of the function at $x = 1$ is $f(1) = e^1 = e = 2.718$. The approximation is $L(1) = 1 + 1 = 2$. The error is $E(1) = f(1) - L(1) = e - 2 = 0.718$. See Figure 3.17.

(d) We expect $E(1)$ to be larger than $E(0.1)$, because 0.1 is closer to $x = 0$, the point at which the linear approximation, $L(x)$, is exactly equal to the function, $f(x)$.

(e) We have

$$E(0.1) = f(0.1) - L(0.1) = e^{0.1} - (1 + 0.1) = 1.105 - 1.1 = 0.005.$$

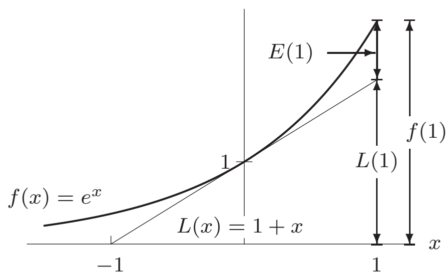


Figure 3.17

12. (a) Since

$$\frac{d}{dx}(\cos x) = -\sin x,$$

the slope of the tangent line is $-\sin(\pi/4) = -1/\sqrt{2}$. Since the tangent line passes through the point $(\pi/4, \cos(\pi/4)) = (\pi/4, 1/\sqrt{2})$, its equation is

$$y - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$
$$y = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + 1\right).$$

Thus, the tangent line approximation to $\cos x$ is

$$\cos x \approx -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}+1\right).$$

(b) From Figure 3.18, we see that the tangent line approximation is an overestimate.

(c) From Figure 3.18, we see that the maximum error for $0 \le x \le \pi/2$ is either at $x = 0$ or at $x = \pi/2$. The error can either be estimated from the graph, or as follows. At $x = 0$,

$$|\text{Error}| = \left|\cos 0 - \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}+1\right)\right| = 0.262 < 0.3.$$

At $x = \pi/2$,

$$|\text{Error}| = \left|\cos\frac{\pi}{2} + \frac{1}{\sqrt{2}}\frac{\pi}{2} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}+1\right)\right| = 0.152 < 0.2.$$

Thus, for $0 \le x \le \pi/2$, we have

$$|\text{Error}| < 0.3.$$

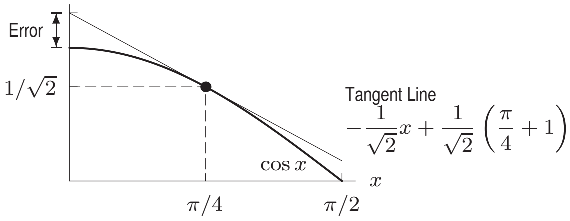


Figure 3.18

13. (a) See Figure 3.19.

(b) Since $f(2) = 2^3 - 3\cdot 2^2 + 3\cdot 2 + 1 = 3$, we have $f'(x) = 3x^2 - 6x + 3$, so $f'(2) = 3\cdot 2^2 - 6\cdot 2 + 3 = 3$, and the local linearization is $y = 3 + 3(x-2) = 3x - 3$.

(c) See Figure 3.19.

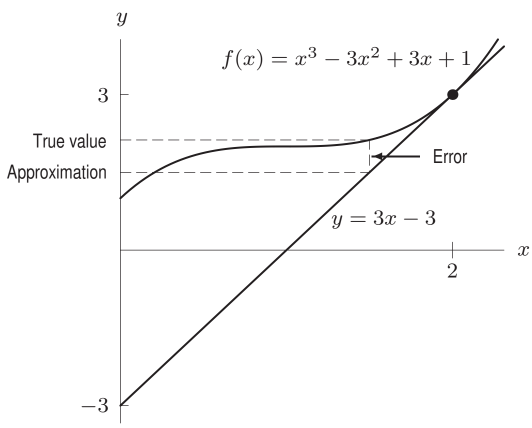


Figure 3.19

14. (a) Let $f(x) = (1+x)^k$. Then $f'(x) = k(1+x)^{k-1}$. Since

$$f(x) \approx f(0) + f'(0)(x-0)$$

is the tangent line approximation, and $f(0) = 1$, $f'(0) = k$, for small x we get

$$f(x) \approx 1 + kx.$$

(b) Since $\sqrt{1.1} = (1+0.1)^{1/2} \approx 1 + (1/2)0.1 = 1.05$ by the above method, this estimate is about right.

(c) The real answer is less than 1.05. Since $(1.05)^2 = (1+0.05)^2 = 1+2(1)(0.05)+(0.05)^2 = 1.1+(0.05)^2 > 1.1$, we have $(1.05)^2 > 1.1$ Therefore

$$\sqrt{1.1} < 1.05.$$

Graphically, this because the graph of $\sqrt{1+x}$ is concave down, so it bends below its tangent line. Therefore the true value ($\sqrt{1.1}$) which is on the curve is below the approximate value (1.05) which is on the tangent line.

15. Since the line meets the curve at $x = 1$, we have $a = 1$. Since the point with $x = 1$ lies on both the line and the curve, we have

$$f(a) = f(1) = 2 \cdot 1 - 1 = 1.$$

The approximation is an underestimate because the line lies under the curve. Since the linear function approximates $f(x)$, we have

$$f(1.2) \approx 2(1.2) - 1 = 1.4.$$

16. We have $f(x) = e^x + x$, so $f'(x) = e^x + 1$. Thus $f'(0) = 2$, so

$$\text{Local linearization near } x = 0 \qquad \text{is} \qquad f(x) \approx f(0) + f'(0)x = 1 + 2x.$$

We get an approximate solution using the local linearization instead of $f(x)$, so the equation becomes

$$1 + 2x = 2, \qquad \text{with solution} \qquad x = \frac{1}{2}.$$

A computer or calculator gives the actual solution as $x = 0.443$.

17. We have $f(x) = x + \ln(1+x)$, so $f'(x) = 1 + 1/(1+x)$. Thus $f'(0) = 2$ so

$$\text{Local linearization near } x = 0 \qquad \text{is} \qquad f(x) \approx f(0) + f'(0)x = 2x.$$

We get an approximate solution using the local linearization instead of $f(x)$, so the equation becomes

$$2x = 0.2, \qquad \text{with solution} \qquad x = 0.1.$$

A computer or calculator gives the actual value as $x = 0.102$.

18. (a) The line tangent to the graph of f at $x = 7$ is given by

$$\begin{aligned} y &= f(7) + f'(7)(x-7) \\ &= 13 - 0.38(x-7). \end{aligned}$$

We can use the tangent line to approximate the value of $f(7.1)$:

$$\begin{aligned} f(7.1) &\approx 13 - 0.38(7.1 - 7) \\ &= 13 - 0.38(0.1) \\ &= 12.962. \end{aligned}$$

(b) If $f''(x) < 0$, then the graph of f is everywhere concave down, so it lies below its tangent line. Thus, our tangent line approximation is an overestimate.

19. (a) Zooming in on the graphs of $y = e^t$ and $y = 0.02t + 1.098$ shows they cross just to the right of the origin. Numerically, we see that at $t = 0$

$$e^0 = 1 < 0.02 \cdot 0 + 1.098.$$

At $t = 0.2$, we have

$$e^{0.2} = 1.221 > 0.02 \cdot 2 + 1.098 = 1.102.$$

Therefore, somewhere between $t = 0$ and $t = 0.2$ the equation $e^t = 0.02t + 1.098$ has a solution.

(b) The linearization of e^t near 0 is $1 + t$, so the new equation is

$$\begin{aligned} 1 + t &= 0.02t + 1.098 \\ 0.98t &= 0.098 \\ t &= \frac{0.098}{0.98} = 0.1. \end{aligned}$$

20. We have

$$f(0) = 331.3$$
$$f'(T) = \frac{1}{2} \cdot 331.3 \left(1 + \frac{T}{273.15}\right)^{-1/2} \frac{1}{273.15}$$
$$f'(0) = 0.606.$$

Thus, for temperatures, T, near zero, we have

$$\text{Speed of sound} = f(T) \approx f(0) + f'(0)T = 331.3 + 0.606T \text{ meters/second.}$$

21. (a) See Figure 3.20.

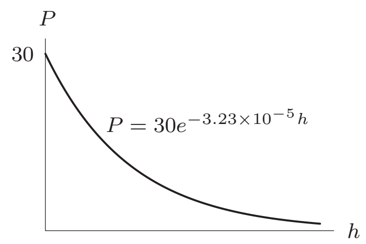


Figure 3.20

(b) Using the chain rule, we have

$$\frac{dP}{dh} = 30e^{-3.23\times 10^{-5}h}(-3.23 \times 10^{-5})$$

so

$$\left.\frac{dP}{dh}\right|_{h=0} = -30(3.23 \times 10^{-5}) = -9.69 \times 10^{-4}$$

Hence, at $h = 0$, the slope of the tangent line is -9.69×10^{-4}, so the equation of the tangent line is

$$y - 30 = (-9.69 \times 10^{-4})(h - 0)$$
$$y = (-9.69 \times 10^{-4})h + 30 = 30 - 0.000969h.$$

(c) The rule of thumb says

$$\begin{array}{c}\text{Drop in pressure from} \\ \text{sea level to height } h\end{array} = \frac{h}{1000}$$

But since the pressure at sea level is 30 inches of mercury, this drop in pressure is also $(30 - P)$, so

$$30 - P = \frac{h}{1000}$$

giving

$$P = 30 - 0.001h.$$

(d) The equations in (b) and (c) are almost the same: both have P intercepts of 30, and the slopes are almost the same $(9.69 \times 10^{-4} \approx 0.001)$. The rule of thumb calculates values of P which are very close to the tangent lines, and therefore yields values very close to the curve.

(e) The tangent line is slightly below the curve, and the rule of thumb line, having a slightly more negative slope, is slightly below the tangent line (for $h > 0$). Thus, the rule of thumb values are slightly smaller.

22. (a) The derivative gives the rate of change of the number of Android users. We estimate $A'(0)$ as the rate of change over the year:

$$A'(0) = \frac{10.9 - 0.866}{1} = 10.034 \text{ million users/year.}$$

(b) Similarly,

$$P'(0) = \frac{13.5 - 7.8}{1} = 5.7 \text{ million users/year.}$$

(c) Since $866{,}000 = 0.866$ million, the tangent line approximation to $A(t)$ near $t = 0$ is

$$A(t) \approx 0.866 + 10.034t.$$

The tangent line approximation for $P(t)$ is

$$P(t) \approx 7.8 + 5.7t.$$

The two groups of users are predicted to be the same size when

$$\begin{aligned} 0.866 + 10.034t &= 7.8 + 5.7t \\ 10.034t - 5.7t &= 7.8 - 0.866 \\ t &= \frac{7.8 - 0.866}{10.034 - 5.7} = 1.600 \text{ years.} \end{aligned}$$

Thus, Android and iPhones were predicted to have the same number of users 1.6 years after 2009; that is, in mid 2011.

(d) We are assuming that the rates of change of each group of users remain the same in the future.

23. (a) Suppose g is a constant and

$$T = f(l) = 2\pi\sqrt{\frac{l}{g}}.$$

Then

$$f'(l) = \frac{2\pi}{\sqrt{g}}\frac{1}{2}l^{-1/2} = \frac{\pi}{\sqrt{gl}}.$$

Thus, local linearity tells us that

$$f(l + \Delta l) \approx f(l) + \frac{\pi}{\sqrt{gl}}\Delta l.$$

Now $T = f(l)$ and $\Delta T = f(l + \Delta l) - f(l)$, so

$$\Delta T \approx \frac{\pi}{\sqrt{gl}}\Delta l = 2\pi\sqrt{\frac{l}{g}} \cdot \frac{1}{2}\frac{\Delta l}{l} = \frac{T}{2}\frac{\Delta l}{l}.$$

(b) Knowing that the length of the pendulum increases by 2% tells us that

$$\frac{\Delta l}{l} = 0.02.$$

Thus,

$$\Delta T \approx \frac{T}{2}(0.02) = 0.01T.$$

So

$$\frac{\Delta T}{T} \approx 0.01.$$

Thus, T increases by 1%.

24. (a) Considering l as a constant, we have

$$T = f(g) = 2\pi\sqrt{\frac{l}{g}}.$$

Then,

$$f'(g) = 2\pi\sqrt{l}\left(-\frac{1}{2}g^{-3/2}\right) = -\pi\sqrt{\frac{l}{g^3}}.$$

Thus, local linearity gives

$$f(g + \Delta g) \approx f(g) - \pi\sqrt{\frac{l}{g^3}}(\Delta g).$$

Since $T = f(g)$ and $\Delta T = f(g + \Delta g) - f(g)$, we have

$$\begin{aligned} \Delta T &\approx -\pi\sqrt{\frac{l}{g^3}}\Delta g = -2\pi\sqrt{\frac{l}{g}}\frac{\Delta g}{2g} = \frac{-T}{2}\frac{\Delta g}{g}. \\ \Delta T &\approx \frac{-T}{2}\frac{\Delta g}{g}. \end{aligned}$$

(b) If g increases by 1%, we know

$$\frac{\Delta g}{g} = 0.01.$$

Thus,

$$\frac{\Delta T}{T} \approx -\frac{1}{2}\frac{\Delta g}{g} = -\frac{1}{2}(0.01) = -0.005,$$

So, T decreases by 0.5%.

25. Since f has a positive second derivative, its graph is concave up, as in Figure 3.21 or 3.22. This means that the graph of $f(x)$ is above its tangent line. We see that in both cases

$$f(1+\Delta x) \geq f(1) + f'(1)\Delta x.$$

(The diagrams show Δx positive, but the result is also true if Δx is negative.)

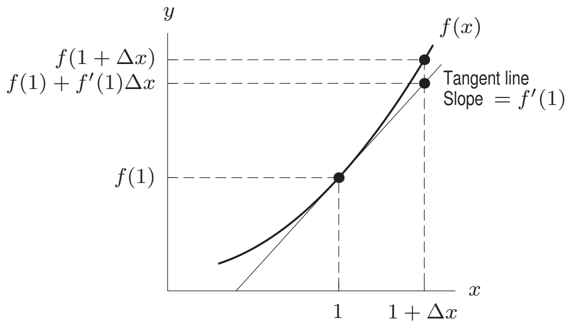


Figure 3.21

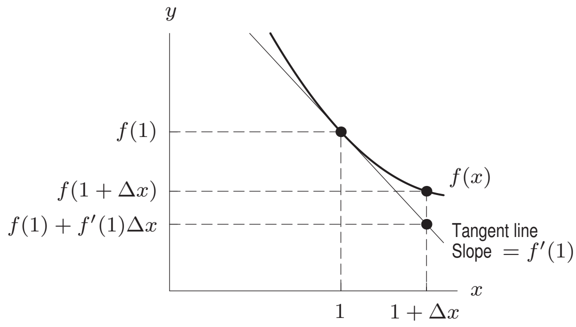


Figure 3.22

26. (a) Since f' is decreasing, $f'(5)$ is larger.
(b) Since f' is decreasing, its derivative, f'', is negative. Thus, $f''(5)$ is negative, so 0 is larger.
(c) Since $f''(x)$ is negative for all x, the graph of f is concave down. Thus the graph of $f(x)$ is below its tangent line. From Figure 3.23, we see that $f(5+\Delta x)$ is below $f(5) + f'(5)\Delta x$. Thus, $f(5) + f'(5)\Delta x$ is larger.

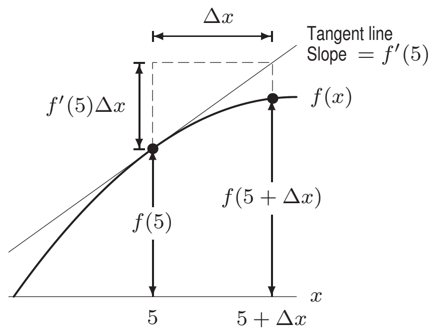


Figure 3.23

27. (a) The range is $f(20) = 16{,}398$ meters.
(b) We have

$$f'(\theta) = \frac{\pi}{90} 25510 \cos \frac{\pi\theta}{90}$$
$$f'(20) = 682.$$

Thus, for angles, θ, near 20°, we have

$$\text{Range} = f(\theta) \approx f(20) + f'(20)(\theta - 20) = 16398 + 682(\theta - 20) \text{ meters}.$$

(c) The true range for 21° is $f(21) = 17{,}070$ meters. The linear approximation gives

$$\text{Approximate range} = 16398 + 682(21 - 20) = 17080 \text{ meters}$$

which is a little too high.

28. (a) The time in the air is $t(20) = 34.9$ seconds
(b) We have

$$t'(\theta) = \frac{\pi}{180} 102 \cos\frac{\pi\theta}{180}$$
$$t'(20) = 1.67$$

Thus, for angles, θ, near 20°, we have

$$\text{Time in air} = t(\theta) \approx t(20) + t'(20)(\theta - 20) = 34.9 + 1.67(\theta - 20) \text{ seconds}.$$

(c) The true air time for 21° is $t(21) = 36.6$ seconds. The linear approximation gives

$$\text{Approximate peak altitude} = 34.9 + 1.67(21 - 20) = 36.6 \text{ seconds}$$

which is the correct value to three significant figures.

29. (a) The peak altitude is $h(20) = 1492$ meters
(b) We have

$$h'(\theta) = 2 \cdot \frac{\pi}{180} 12755 \sin\frac{\pi\theta}{180}\cos\frac{\pi\theta}{180}$$
$$h'(20) = 143.$$

Thus, for angles, θ, near 20°, we have

$$\text{Peak altitude} = h(\theta) \approx h(20) + h'(20)(\theta - 20) = 1490 + 143(\theta - 20) \text{ meters}.$$

(c) The true peak altitude for 21° is $h(21) = 1638$ meters. The linear approximation gives

$$\text{Approximate peak altitude} = 1492 + 143(21 - 20) = 1635 \text{ meters}$$

which is a little too low.

30. We have $f(x) = (1+x)^r$, so $f'(x) = r(1+x)^{r-1}$. Thus $f'(0) = r$ so the local linearization near $x = 0$ is

$$f(x) \approx f(0) + f'(0)x = 1 + rx.$$

Thus

$$(1+x)^r \approx 1 + rx \quad \text{for small values of } x.$$

Using the linearization with $r = 3/5$ and $x = 0.2$, we have

$$1.2^{3/5} = 1 + \frac{3}{5}0.2 = 1 + 0.12 = 1.12.$$

The actual value is $1.2^{3/5} = 1.117$.

31. We have $f(x) = e^{kx}$, so $f'(x) = ke^{kx}$. Thus $f'(0) = k$ so the local linearization near $x = 0$ is

$$f(x) \approx f(0) + f'(0)x = 1 + kx.$$

Thus

$$e^{kx} \approx 1 + kx \quad \text{for small values of } x.$$

Using the linearization with $k = 0.3$ and $x = 1$, we have

$$e^{0.3} \approx 1 + 0.3 = 1.3$$

The actual value is $e^{0.3} = 1.350$.

32. We have $f(x) = (b^2 + x)^{1/2}$, so $f'(x) = (1/2)(b^2 + x)^{-1/2}$. Thus $f'(0) = 1/(2b)$ so the local linearization near $x = 0$ is

$$f(x) \approx f(0) + f'(0)x = b + \frac{1}{2b}x.$$

Thus

$$\sqrt{b^2 + x} \approx b + \frac{1}{2b}x \quad \text{for small values of } x.$$

Using the linearization with $b = 5$ and $x = 1$, we have

$$\sqrt{26} \approx 5 + \frac{1}{10}1 = 5.1.$$

The actual value is $\sqrt{26} = 5.099$.

33. We have $f(1) = 1$ and $f'(1) = 4$. Thus

$$E(x) = x^4 - (1 + 4(x - 1)).$$

Values of $E(x)/(x - 1)$ near $x = 1$ are in Table 3.3.

Table 3.3

x	1.1	1.01	1.001
$E(x)/(x-1)$	0.641	0.060401	0.006004

From the table, we can see that

$$\frac{E(x)}{(x-1)} \approx 6(x - 1),$$

so $k = 6$ and

$$E(x) \approx 6(x - 1)^2.$$

In addition, $f''(1) = 12$, so

$$E(x) \approx 6(x - 1)^2 = \frac{f''(1)}{2}(x - 1)^2.$$

The same result can be obtained by rewriting the function x^4 using $x = 1 + (x - 1)$ and expanding:

$$x^4 = (1 + (x - 1))^4 = 1 + 4(x - 1) + 6(x - 1)^2 + 4(x - 1)^3 + (x - 1)^4.$$

Thus,

$$E(x) = x^4 - (1 + 4(x - 1)) = 6(x - 1)^2 + 4(x - 1)^3 + (x - 1)^4.$$

For x near 1, the value of $x - 1$ is small, so we ignore powers of $x - 1$ higher than the first, giving

$$E(x) \approx 6(x - 1)^2.$$

34. We have $f(0) = 1$ and $f'(0) = 0$. Thus

$$E(x) = \cos x - 1.$$

Values for $E(x)/(x - 0)$ near $x = 0$ are in Table 3.4.

Table 3.4

x	0.1	0.01	0.001
$E(x)/(x-0)$	-0.050	-0.0050	-0.00050

From the table, we can see that

$$\frac{E(x)}{(x-0)} \approx -0.5(x - 0),$$

so $k = -1/2$ and

$$E(x) \approx -\frac{1}{2}(x - 0)^2 = -\frac{1}{2}x^2.$$

In addition, $f''(0) = -1$, so

$$E(x) \approx -\frac{1}{2}x^2 = \frac{f''(0)}{2}x^2.$$

35. We have $f(0) = 1$ and $f'(0) = 1$. Thus

$$E(x) = e^x - (1 + x).$$

Values of $E(x)/(x - 0)$ near $x = 0$ are in Table 3.5.

Table 3.5

x	0.1	0.01	0.001
$E(x)/(x-0)$	0.052	0.0050	0.00050

From the table, we can see that

$$\frac{E(x)}{(x-0)} \approx 0.5(x - 0)$$

so $k = 1/2$ and

$$E(x) \approx \frac{1}{2}(x-0)^2 = \frac{1}{2}x^2.$$

In addition, $f''(0) = 1$, so

$$E(x) \approx \frac{1}{2}x^2 = \frac{f''(0)}{2}x^2$$

36. We have $f(1) = 1$ and $f'(1) = 1/2$. Thus

$$E(x) = \sqrt{x} - (1 + \frac{1}{2}(x - 1)).$$

Values of $E(x)/(x - 1)$ near $x = 1$ are in Table 3.6.

Table 3.6

x	1.1	1.01	1.001
$E(x)/(x-1)$	−0.0119	−0.00124	−0.000125

From the table, we can see that

$$\frac{E(x)}{(x-1)} \approx -0.125(x - 1)$$

so $k = -1/8$ and

$$E(x) \approx -\frac{1}{8}(x-1)^2.$$

In addition, $f''(1) = -1/4$, so

$$E(x) \approx -\frac{1}{8}(x-1)^2 = \frac{f''(1)}{2}(x-1)^2.$$

37. We have $f(1) = 0$ and $f'(1) = 1$. Thus

$$E(x) = \ln x - (x - 1).$$

Values of $E(x)/(x - 1)$ near $x = 1$ are in Table 3.7.

Table 3.7

x	1.1	1.01	1.001
$E(x)/(x-1)$	−0.047	−0.0050	−0.00050

From the table, we see that

$$\frac{E(x)}{(x-1)} \approx -0.5(x - 1),$$

so $k = -1/2$ and

$$E(x) \approx -\frac{1}{2}(x-1)^2.$$

In addition, $f''(1) = -1$, so

$$E(x) \approx -\frac{1}{2}(x-1)^2 = \frac{f''(1)}{2}(x-1)^2.$$

38. The local linearization of e^x near $x = 0$ is $1 + 1x$ so

$$e^x \approx 1 + x.$$

Squaring this yields, for small x,

$$e^{2x} = (e^x)^2 \approx (1 + x)^2 = 1 + 2x + x^2.$$

Local linearization of e^{2x} directly yields

$$e^{2x} \approx 1 + 2x$$

for small x. The two approximations are consistent because they agree: the tangent line approximation to $1 + 2x + x^2$ is just $1 + 2x$.

The first approximation is more accurate. One can see this numerically or by noting that the approximation for e^{2x} given by $1 + 2x$ is really the same as approximating e^y at $y = 2x$. Since the other approximation approximates e^y at $y = x$, which is twice as close to 0 and therefore a better general estimate, it's more likely to be correct.

39. (a) Let $f(x) = 1/(1+x)$. Then $f'(x) = -1/(1+x)^2$ by the chain rule. So $f(0) = 1$, and $f'(0) = -1$. Therefore, for x near 0, $1/(1 + x) \approx f(0) + f'(0)x = 1 - x$.

(b) We know that for small y, $1/(1 + y) \approx 1 - y$. Let $y = x^2$; when x is small, so is $y = x^2$. Hence, for small x, $1/(1 + x^2) \approx 1 - x^2$.

(c) Since the linearization of $1/(1 + x^2)$ is the line $y = 1$, and this line has a slope of 0, the derivative of $1/(1 + x^2)$ is zero at $x = 0$.

40. The local linearizations of $f(x) = e^x$ and $g(x) = \sin x$ near $x = 0$ are

$$f(x) = e^x \approx 1 + x$$

and

$$g(x) = \sin x \approx x.$$

Thus, the local linearization of $e^x \sin x$ is the local linearization of the product:

$$e^x \sin x \approx (1 + x)x = x + x^2 \approx x.$$

We therefore know that the derivative of $e^x \sin x$ at $x = 0$ must be 1. Similarly, using the local linearization of $1/(1 + x)$ near $x = 0$, $1/(1 + x) \approx 1 - x$, we have

$$\frac{e^x \sin x}{1+x} = (e^x)(\sin x)\left(\frac{1}{1+x}\right) \approx (1 + x)(x)(1 - x) = x - x^3$$

so the local linearization of the triple product $\dfrac{e^x \sin x}{1+x}$ at $x = 0$ is simply x. And therefore the derivative of $\dfrac{e^x \sin x}{1+x}$ at $x = 0$ is 1.

41. Note that

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}.$$

We use the hint: For small h, $f(x + h) \approx f(x) + f'(x)h$, and $g(x + h) \approx g(x) + g'(x)h$. Therefore

$$\begin{aligned} f(x + h)g(x + h) - f(x)g(x) &\approx [f(x) + hf'(x)][g(x) + hg'(x)] - f(x)g(x) \\ &= f(x)g(x) + hf'(x)g(x) + hf(x)g'(x) \\ &\quad + h^2 f'(x)g'(x) - f(x)g(x) \\ &= hf'(x)g(x) + hf(x)g'(x) + h^2 f'(x)g'(x). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{h\to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} &= \lim_{h\to 0} \frac{hf'(x)g(x) + hf(x)g'(x) + h^2 f'(x)g'(x)}{h} \\ &= \lim_{h\to 0} \frac{h\left(f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x)\right)}{h} \\ &= \lim_{h\to 0} \left(f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x)\right) \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

A more complete derivation can be given using the error term discussed in the section on Differentiability and Linear Approximation in Chapter 2. Adapting the notation of that section to this problem, we write

$$f(x+h) = f(x) + f'(x)h + E_f(h) \quad \text{and} \quad g(x+h) = g(x) + g'(x)h + E_g(h),$$

where $\lim_{h\to 0} \frac{E_f(h)}{h} = \lim_{h\to 0} \frac{E_g(h)}{h} = 0$. (This implies that $\lim_{h\to 0} E_f(h) = \lim_{h\to 0} E_g(h) = 0$.)

We have

$$\begin{aligned}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x)g(x)}{h} + f(x)g'(x) + f'(x)g(x) + f(x)\frac{E_g(h)}{h} + g(x)\frac{E_f(h)}{h} \\ &\quad + f'(x)g'(x)h + f'(x)E_g(h) + g'(x)E_f(h) + \frac{E_f(h)E_g(h)}{h} - \frac{f(x)g(x)}{h}\end{aligned}$$

The terms $f(x)g(x)/h$ and $-f(x)g(x)/h$ cancel out. All the remaining terms on the right, with the exception of the second and third terms, go to zero as $h \to 0$. Thus, we have

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x)g'(x) + f'(x)g(x).$$

42. Note that

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Using the local linearizations of f and g, we get that

$$\begin{aligned} f(g(x+h)) - f(g(x)) &\approx f\left(g(x) + g'(x)h\right) - f(g(x)) \\ &\approx f(g(x)) + f'(g(x))g'(x)h - f(g(x)) \\ &= f'(g(x))g'(x)h. \end{aligned}$$

Therefore,

$$\begin{aligned}[f(g(x))]' &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0} \frac{f'(g(x))g'(x)h}{h} \\ &= \lim_{h\to 0} f'(g(x))g'(x) = f'(g(x))g'(x).\end{aligned}$$

A more complete derivation can be given using the error term discussed in the section on Differentiability and Linear Approximation in Chapter 2. Adapting the notation of that section to this problem, we write

$$f(z+k) = f(z) + f'(z)k + E_f(k) \quad \text{and} \quad g(x+h) = g(x) + g'(x)h + E_g(h),$$

where $\lim_{h\to 0} \frac{E_g(h)}{h} = \lim_{k\to 0} \frac{E_f(k)}{k} = 0$.

Now we let $z = g(x)$ and $k = g(x+h) - g(x)$. Then we have $k = g'(x)h + E_g(h)$. Thus,

$$\begin{aligned}\frac{f(g(x+h)) - f(g(x))}{h} &= \frac{f(z+k) - f(z)}{h} \\ &= \frac{f(z) + f'(z)k + E_f(k) - f(z)}{h} = \frac{f'(z)k + E_f(k)}{h} \\ &= \frac{f'(z)g'(x)h + f'(z)E_g(h)}{h} + \frac{E_f(k)}{k}\cdot\left(\frac{k}{h}\right) \\ &= f'(z)g'(x) + \frac{f'(z)E_g(h)}{h} + \frac{E_f(k)}{k}\left[\frac{g'(x)h + E_g(h)}{h}\right] \\ &= f'(z)g'(x) + \frac{f'(z)E_g(h)}{h} + \frac{g'(x)E_f(k)}{k} + \frac{E_g(h)\cdot E_f(k)}{h\cdot k}\end{aligned}$$

Now, if $h \to 0$ then $k \to 0$ as well, and all the terms on the right except the first go to zero, leaving us with the term $f'(z)g'(x)$. Substituting $g(x)$ for z, we obtain

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x))g'(x).$$

43. We want to show that

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=L.$$

Substituting for $f(x)$ we have

$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a}&=\lim_{x\to a}\frac{f(a)+L(x-a)+E_L(x)-f(a)}{x-a}\\&=\lim_{x\to a}\left(L+\frac{E_L(x)}{x-a}\right)=L+\lim_{x\to 0}\frac{E_L(x)}{x-a}=L.\end{aligned}$$

Thus, we have shown that f is differentiable at $x=a$ and that its derivative is L, that is, $f'(a)=L$.

44. We know that the local linearization is $y=2x-1$. The fact that $f(x)$ is differentiable tells us that

$$\lim_{x\to 0}\frac{E(x)}{x-1}=\lim_{x\to 0}\frac{x^2-(2x-1)}{x-1}=0.$$

Suppose we take $\epsilon=1/10$, then there is a δ such that

$$\left|\frac{x^2-(2x-1)}{x-1}\right|<\frac{1}{10}\quad\text{for all}\quad |x-1|<\delta.$$

Thus

$$-\frac{1}{10}(x-1)<x^2-(2x-1)<\frac{1}{10}(x-1)\quad\text{for all}\quad 1-\delta<x<1+\delta.$$

To find this δ, we observe that

$$\left|\frac{x^2-(2x-1)}{x-1}\right|=\left|\frac{x^2-2x+1}{x-1}\right|=\left|\frac{(x-1)^2}{x-1}\right|=|x-1|.$$

Therefore we can take $\delta=1/10$. Then

$$\left|\frac{x^2-(2x-1)}{x-1}\right|<\frac{1}{10}\quad\text{for all}\quad |x-1|<\frac{1}{10}$$

so

$$-\frac{1}{10}(x-1)<|x^2-(2x-1)|<\frac{1}{10}(x-1).$$

Strengthen Your Understanding

45. The line $y=x+1$ is the linear approximation for $f(x)=e^x$ near $x=0$. If we move far from $x=0$, the approximation is useless.

For example, for $x=1$, the approximation gives $e^1\approx 2$ (instead of 2.718). For $x=2$, our estimate of 3 is not a good approximation for $e^2=7.389$.

This linear approximation is only useful near $x=0$.

46. The graph of F is concave down to the left of $x=0$ and concave up to the right. The tangent at $x=0$ lies beneath the graph for $x>0$ and above the graph for $x<0$. Thus, the linear approximation near $x=0$ is an overestimate for values less than zero and an underestimate for values greater than zero.

47. In order to have the same linear approximation, two functions f and g must go through the same point at $x=0$ and have the same slope at $x=0$. Thus, $f(0)=g(0)$ and $f'(0)=g'(0)$. While there are many answers, let $f(x)=x^3+1$ and $g(x)=x^4+1$. Then the linear approximation for both functions near $x=0$ is $y=1$.

48. One possible answer is $g(x)=\cos x$.

49. The linear approximation of a function f, for values of x near a, is given by $f(x)\approx f(a)+f'(a)(x-a)$. Since $f(x)=|x+1|$ does not have a derivative at $x=-1$ this function does not have a linear approximation for x near -1. Other answers are possible.

50. (a) False. Only if $k=f'(a)$ is L the local linearization of f.
(b) False. Since $f(a)=L(a)$ for any k, we have $\lim_{x\to a}(f(x)-L(x))=f(a)-L(a)=0$, but only if $k=f'(a)$ is L the local linearization of f.

Solutions for Section 3.10

Exercises

1. False. The derivative, $f'(x)$, is not equal to zero everywhere, because the function is not continuous at integral values of x, so $f'(x)$ does not exist there. Thus, the Constant Function Theorem does not apply.

2. True. If f' is positive on $[a, b]$, then f is continuous and the Increasing Function Theorem applies. Thus, f is increasing on $[a, b]$, so $f(a) < f(b)$.

3. False. Let $f(x) = x^3$ on $[-1, 1]$. Then $f(x)$ is increasing but $f'(x) = 0$ for $x = 0$.

4. False. The horse that wins the race may have been moving faster for some, but not all, of the race. The Racetrack Principle guarantees the converse—that if the horses start at the same time and one moves faster throughout the race, then that horse wins.

5. True. If $g(x)$ is the position of the slower horse at time x and $h(x)$ is the position of the faster, then $g'(x) \le h'(x)$ for $a < x < b$. Since the horses start at the same time, $g(a) = h(a)$, so, by the Racetrack Principle, $g(x) \le h(x)$ for $a \le x \le b$. Therefore, $g(b) \le h(b)$, so the slower horse loses the race.

6. Yes, it satisfies the hypotheses and the conclusion. This function has two points, c, at which the tangent to the curve is parallel to the secant joining $(a, f(a))$ to $(b, f(b))$, but this does not contradict the Mean Value Theorem. The function is continuous and differentiable on the interval $[a, b]$.

7. No, it does not satisfy the hypotheses. The function does not appear to be differentiable. There appears to be no tangent line, and hence no derivative, at the "corner."

No, it does not satisfy the conclusion as there is no horizontal tangent.

8. No. This function does not satisfy the hypotheses of the Mean Value Theorem, as it is not continuous.

However, the function has a point c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Thus, this satisfies the conclusion of the theorem.

9. No, it does not satisfy the hypotheses. This function does not appear to be continuous.

No, it does not satisfy the conclusion as there is no horizontal tangent.

Problems

10. The Mean Value Theorem tells us that

$$f'(4) = \frac{f(b) - f(a)}{b - a} = \frac{9 - 5}{7 - 2} = \frac{4}{5}.$$

Thus, the slope of the tangent line is $4/5$. Its equation is

$$y = b + \frac{4}{5}x.$$

Substituting $x = 4, y = 8$ gives

$$\begin{aligned} 8 &= b + \frac{4}{5}(4) \\ \frac{24}{5} &= b. \end{aligned}$$

Thus, the tangent line is

$$y = \frac{24}{5} + \frac{4}{5}x.$$

11. The Mean Value Theorem tells us that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{7 - 12}{13 - 3} = -0.5.$$

Since the slope at x_1 is negative but less steep than at c, we have

$$f'(x_1) > -0.5.$$

The slope at x_2 is negative and steeper than at c. Thus

$$f'(x_2) < -0.5.$$

12. We notice that $f(x) = p'(x)$. We have

$$p(1) = 1^5 + 8 \cdot 1^4 - 30 \cdot 1^3 + 30 \cdot 1^2 - 31 \cdot 1 + 22 = 0.$$
$$p(2) = 2^5 + 8 \cdot 2^4 - 30 \cdot 2^3 + 30 \cdot 2^2 - 31 \cdot 2 + 22 = 0.$$

Thus, by Rolle's Theorem, $f(x)$ has a zero between $x = 1$ and $x = 2$.

13. A polynomial $p(x)$ satisfies the conditions of Rolle's Theorem for all intervals $a \leq x \leq b$.

Suppose $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are the seven distinct zeros of $p(x)$ in increasing order. Thus $p(a_1) = p(a_2) = 0$, so by Rolle's Theorem, $p'(x)$ has a zero, c_1, between a_1 and a_2.

Similarly, $p'(x)$ has 6 distinct zeros, $c_1, c_2, c_3, c_4, c_5, c_6$, where

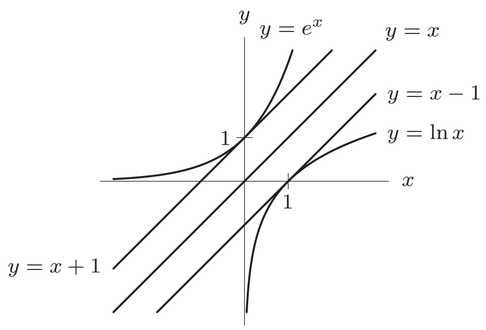
$$a_1 < c_1 < a_2$$
$$a_2 < c_2 < a_3$$
$$a_3 < c_3 < a_4$$
$$a_4 < c_4 < a_5$$
$$a_5 < c_5 < a_6$$
$$a_6 < c_6 < a_7.$$

The polynomial $p'(x)$ is of degree 6, so $p'(x)$ cannot have more than 6 zeros.

14. Let $f(x) = \sin x$ and $g(x) = x$. Then $f(0) = 0$ and $g(0) = 0$. Also $f'(x) = \cos x$ and $g'(x) = 1$, so for all $x \geq 0$ we have $f'(x) \leq g'(x)$. So the graphs of f and g both go through the origin and the graph of f climbs slower than the graph of g. Thus the graph of f is below the graph of g for $x \geq 0$ by the Racetrack Principle. In other words, $\sin x \leq x$ for $x \geq 0$.

15. Let $g(x) = \ln x$ and $h(x) = x - 1$. For $x \geq 1$, we have $g'(x) = 1/x \leq 1 = h'(x)$. Since $g(1) = h(1)$, the Racetrack Principle with $a = 1$ says that $g(x) \leq h(x)$ for $x \geq 1$, that is, $\ln x \leq x - 1$ for $x \geq 1$. For $0 < x \leq 1$, we have $h'(x) = 1 \leq 1/x = g'(x)$. Since $g(1) = h(1)$, the Racetrack Principle with $b = 1$ says that $g(x) \leq h(x)$ for $0 < x \leq 1$, that is, $\ln x \leq x - 1$ for $0 < x \leq 1$.

16.



Graphical solution: If f and g are inverse functions then the graph of g is just the graph of f reflected through the line $y = x$. But e^x and $\ln x$ are inverse functions, and so are the functions $x + 1$ and $x - 1$. Thus the equivalence is clear from the figure.

Algebraic solution: If $x > 0$ and

$$x + 1 \leq e^x,$$

then, replacing x by $x - 1$, we have

$$x \leq e^{x-1}.$$

Taking logarithms, and using the fact that $\ln$ is an increasing function, gives

$$\ln x \leq x - 1.$$

We can also go in the opposite direction, which establishes the equivalence.

17. The Decreasing Function Theorem is: Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) < 0$ on (a, b), then f is decreasing on $[a, b]$. If $f'(x) \leq 0$ on (a, b), then f is nonincreasing on $[a, b]$.

To prove the theorem, we note that if f is decreasing then $-f$ is increasing and vice-versa. Similarly, if f is nonincreasing, then $-f$ is nondecreasing. Thus if $f'(x) < 0$, then $-f'(x) > 0$, so $-f$ is increasing, which means f is decreasing. And if $f'(x) \leq 0$, then $-f'(x) \geq 0$, so $-f$ is nondecreasing, which means f is nonincreasing.

18. Dominic's trip lasted 88 minutes, so his average velocity for the trip was $116/88$ miles per minute, or $116/88 \cdot 60 \approx 79.1$ miles per hour. By the Mean Value Theorem, there must have been some time at which Dominic's instantaneous velocity was 79.1 miles per hour. Since the speed limit on I-10 between Phoenix and Tucson is never more than 75 miles per hour, Dominic must have been speeding at that time.

19. Use the Racetrack Principle, Theorem 3.10, with $g(x) = x$. Since $f'(x) \leq g'(x)$ for all x and $f(0) = g(0)$, then $f(x) \leq g(x) = x$ for all $x \geq 0$.

20. First apply the Racetrack Principle, Theorem 3.10, to $f'(t)$ and $g(t) = 3t$. Since $f''(t) \leq g'(t)$ for all t and $f'(0) = 0 = g(0)$, then $f'(t) \leq 3t$ for all $t \geq 0$. Next apply the Racetrack Principle again to $f(t)$ and $h(t) = \frac{3}{2}t^2$. Since $f'(t) \leq h'(t)$ for all $t \geq 0$ and $f(0) = 0 = h(0)$, then $f(t) \leq h(t) = \frac{3}{2}t^2$ for all $t \geq 0$.

21. Apply the Constant Function Theorem, Theorem 3.9, to $h(x) = f(x) - g(x)$. Then $h'(x) = 0$ for all x, so $h(x)$ is constant for all x. Since $h(5) = f(5) - g(5) = 0$, we have $h(x) = 0$ for all x. Therefore $f(x) - g(x) = 0$ for all x, so $f(x) = g(x)$ for all x.

22. By the Mean Value Theorem, Theorem 3.7, there is a number c, with $0 < c < 1$, such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}.$$

Since $f(1) - f(0) > 0$, we have $f'(c) > 0$.

Alternatively if $f'(c) \leq 0$ for all c in $(0, 1)$, then by the Increasing Function Theorem, $f(0) \geq f(1)$.

23. Since $f''(t) \leq 7$ for $0 \leq t \leq 2$, if we apply the Racetrack Principle with $a = 0$ to the functions $f'(t) - f'(0)$ and $7t$, both of which go through the origin, we get

$$f'(t) - f'(0) \leq 7t \quad \text{for } 0 \leq t \leq 2.$$

The left side of this inequality is the derivative of $f(t) - f'(0)t$, so if we apply the Racetrack Principle with $a = 0$ again, this time to the functions $f(t) - f'(0)t$ and $(7/2)t^2 + 3$, both of which have the value 3 at $t = 0$, we get

$$f(t) - f'(0)t \leq \frac{7}{2}t^2 + 3 \quad \text{for } 0 \leq t \leq 2.$$

That is,

$$f(t) \leq 3 + 4t + \frac{7}{2}t^2 \quad \text{for } 0 \leq t \leq 2.$$

In the same way, we can show that the lower bound on the acceleration, $5 \leq f''(t)$ leads to:

$$f(t) \geq 3 + 4t + \frac{5}{2}t^2 \quad \text{for } 0 \leq t \leq 2.$$

If we substitute $t = 2$ into these two inequalities, we get bounds on the position at time 2:

$$21 \leq f(2) \leq 25.$$

24. Consider the function $f(x) = h(x) - g(x)$. Since $f'(x) = h'(x) - g'(x) \geq 0$, we know that f is nondecreasing by the Increasing Function Theorem. This means $f(x) \leq f(b)$ for $a \leq x \leq b$. However, $f(b) = h(b) - g(b) = 0$, so $f(x) \leq 0$, which means $h(x) \leq g(x)$.

25. If $f'(x) = 0$, then both $f'(x) \geq 0$ and $f'(x) \leq 0$. By the Increasing and Decreasing Function Theorems, f is both nondecreasing and nonincreasing, so f is constant.

26. Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all x in (a, b). Hence, by the Constant Function Theorem, there is a constant C such that $h(x) = C$ on (a, b). Thus $f(x) = g(x) + C$.

27. We will show $f(x) = Ce^x$ by deducing that $f(x)/e^x$ is a constant. By the Constant Function Theorem, we need only show the derivative of $g(x) = f(x)/e^x$ is zero. By the quotient rule (since $e^x \neq 0$), we have

$$g'(x) = \frac{f'(x)e^x - e^x f(x)}{(e^x)^2}.$$

Since $f'(x) = f(x)$, we simplify and obtain

$$g'(x) = \frac{f(x)e^x - e^x f(x)}{(e^x)^2} = \frac{0}{e^{2x}} = 0,$$

which is what we needed to show.

28. Apply the Racetrack Principle to the functions $f(x) - f(a)$ and $M(x-a)$; we can do this since $f(a) - f(a) = M(a-a)$ and $f'(x) \leq M$. We conclude that $f(x) - f(a) \leq M(x-a)$. Similarly, apply the Racetrack Principle to the functions $m(x-a)$ and $f(x) - f(a)$ to obtain $m(x-a) \leq f(x) - f(a)$. If we substitute $x = b$ into these inequalities we get

$$m(b-a) \leq f(b) - f(a) \leq M(b-a).$$

Now, divide by $b - a$.

29. (a) Since $f''(x) \geq 0$, $f'(x)$ is nondecreasing on (a, b). Thus $f'(c) \leq f'(x)$ for $c \leq x < b$ and $f'(x) \leq f'(c)$ for $a < x \leq c$.

(b) Let $g(x) = f(c) + f'(c)(x-c)$ and $h(x) = f(x)$. Then $g(c) = f(c) = h(c)$, and $g'(x) = f'(c)$ and $h'(x) = f'(x)$. If $c \leq x < b$, then $g'(x) \leq h'(x)$, and if $a < x \leq c$, then $g'(x) \geq h'(x)$, by (a). By the Racetrack Principle, $g(x) \leq h'(x)$ for $c \leq x < b$ and for $a < x \leq c$, as we wanted.

Strengthen Your Understanding

30. The function f is not differentiable at $x = 0$, so the Mean value Theorem does not apply.

31. This function does not satisfy the conclusion of the Mean Value Theorem because it is not continuous at $x = 0$.

32. To apply the Constant Function Theorem, we need f to be continuous on $a \leq x \leq b$ and differentiable on $a < x < b$.

For example, the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

has a zero derivative on $0 < x < 1$ but is not constant on $0 \leq x \leq 1$.

33. The function f must be continuous on $a \leq x \leq b$ and differentiable on $a < x < b$. Any interval avoiding $x \leq 0$ will suffice, so, for example $1 \leq x \leq 2$.

34. The function f must be continuous on $a \leq x \leq b$ and differentiable on $a < x < b$. Any interval including $x = 0$ will suffice, so, for example $-1 \leq x \leq 1$.

35. The function $f(x) = |x|$ is continuous on $[-1, 1]$, but there is no number c, with $-1 < c < 1$, such that

$$f'(c) = \frac{|1| - |-1|}{1 - (-1)} = 0;$$

that is, the slope of $f(x) = |x|$ is never 0.

36. Let f be defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 2 \\ 19 & \text{if } x = 2 \end{cases}$$

Then f is differentiable on $(0, 2)$ and $f'(x) = 1$ for all x in $(0, 2)$. Thus there is no c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{19}{2}.$$

The reason that this function does not satisfy the conclusion of the Mean Value Theorem is that it is not continuous at $x = 2$.

37. Let f be defined by

$$f(x)=\begin{cases} x^2 & \text{if } 0\le x<1 \\ 1/2 & \text{if } x=1. \end{cases}$$

Then f is not continuous at $x=1$, but f is differentiable on $(0,1)$ and $f'(x)=2x$ for $0<x<1$. Thus, $c=1/4$ satisfies

$$f'(c)=\frac{f(1)-f(0)}{1-0}=\frac{1}{2},\quad \text{since}\quad f'\left(\frac{1}{4}\right)=2\cdot\frac{1}{4}=\frac{1}{2}.$$

38. True, by the Increasing Function Theorem, Theorem 3.8.

39. False. For example, let $f(x)=x+5$, and $g(x)=2x-3$. Then $f'(x)\le g'(x)$ for all x, but $f(0)>g(0)$.

40. False. For example, let $f(x)=3x+1$ and $g(x)=3x+7$.

41. False. For example, if $f(x)=-x$, then $f'(x)\le 1$ for all x, but $f(-2)=2$, so $f(-2)>-2$.

Solutions for Chapter 3 Review

Exercises

1. $w'=100(t^2+1)^{99}(2t)=200t(t^2+1)^{99}$.

2. $f'(t)=e^{3t}\cdot 3=3e^{3t}$.

3. Using the quotient rule gives $\dfrac{dz}{dt}=\dfrac{(2t+3)(t+1)-(t^2+3t+1)}{(t+1)^2}$ or $\dfrac{dz}{dt}=\dfrac{t^2+2t+2}{(t+1)^2}$.

4. $y'=\dfrac{\frac{1}{2\sqrt{t}}(t^2+1)-\sqrt{t}(2t)}{(t^2+1)^2}$.

5. Using the quotient rule,

$$h'(t)=\frac{(-1)(4+t)-(4-t)}{(4+t)^2}=-\frac{8}{(4+t)^2}.$$

6. $f'(x)=ex^{e-1}$.

7. $f'(x)=\dfrac{3x^2}{9}(3\ln x-1)+\dfrac{x^3}{9}\left(\dfrac{3}{x}\right)=x^2\ln x-\dfrac{x^2}{3}+\dfrac{x^2}{3}=x^2\ln x$

8.

$$\begin{aligned} f'(x)&=\frac{(2+3x+4x^2)(1)-(1+x)(3+8x)}{(2+3x+4x^2)^2}\\ &=\frac{2+3x+4x^2-3-11x-8x^2}{(2+3x+4x^2)^2}\\ &=\frac{-4x^2-8x-1}{(2+3x+4x^2)^2}. \end{aligned}$$

9. Using the chain rule, $g'(\theta)=(\cos\theta)e^{\sin\theta}$.

10. Since $y=\sqrt{\theta}\left(\sqrt{\theta}+\dfrac{1}{\sqrt{\theta}}\right)=\theta^{1/2}\theta^{1/2}+\dfrac{\sqrt{\theta}}{\sqrt{\theta}}=\theta+1$, we have $\frac{dy}{dx}=1$.

11. $f'(w)=\dfrac{1}{\cos(w-1)}[-\sin(w-1)]=-\tan(w-1)$.

[This could be done easily using the answer from Problem 20 and the chain rule.]

12. $\dfrac{d}{dy}\ln\ln(2y^3)=\dfrac{1}{\ln(2y^3)}\dfrac{1}{2y^3}6y^2=\dfrac{3}{y\ln(2y^3)}$.

13. $g'(x)=\dfrac{d}{dx}\left(x^k+k^x\right)=kx^{k-1}+k^x\ln k$.

14. $y'=0$

15. $\dfrac{dz}{d\theta}=3\sin^2\theta\cos\theta$

16.

$$\begin{aligned} f'(t) &= 2\cos(3t+5)\cdot(-\sin(3t+5))3 \\ &= -6\cos(3t+5)\cdot\sin(3t+5) \end{aligned}$$

17.

$$\begin{aligned} M'(\alpha) &= 2\tan(2+3\alpha)\cdot\frac{1}{\cos^2(2+3\alpha)}\cdot 3 \\ &= 6\cdot\frac{\tan(2+3\alpha)}{\cos^2(2+3\alpha)} \end{aligned}$$

18. $s'(\theta) = \frac{d}{d\theta}\sin^2(3\theta-\pi) = 6\cos(3\theta-\pi)\sin(3\theta-\pi)$.

19. $h'(t) = \frac{1}{e^{-t}-t}\left(-e^{-t}-1\right)$.

20.

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{\sin(5-\theta)}{\theta^2}\right) &= \frac{\cos(5-\theta)(-1)\theta^2-\sin(5-\theta)(2\theta)}{\theta^4} \\ &= -\frac{\theta\cos(5-\theta)+2\sin(5-\theta)}{\theta^3}. \end{aligned}$$

21. $w'(\theta) = \frac{1}{\sin^2\theta} - \frac{2\theta\cos\theta}{\sin^3\theta}$

22. $f'(\theta) = -1(1+e^{-\theta})^{-2}(e^{-\theta})(-1) = \frac{e^{-\theta}}{(1+e^{-\theta})^2}$.

23. $g'(w) = \frac{d}{dw}\left(\frac{1}{2^w+e^w}\right) = -\frac{2^w\ln 2+e^w}{(2^w+e^w)^2}$.

24. $f(t) = \frac{1}{t^2}+\frac{1}{t}-\frac{1}{t^4} = t^{-2}+t^{-1}-t^{-4}$
$f'(t) = -2t^{-3}-t^{-2}+4t^{-5}$.

25. Using the quotient rule and the chain rule,

$$\begin{aligned} h'(z) &= \frac{1}{2}\left(\frac{\sin(2z)}{\cos(2z)}\right)^{-1/2}\left[\frac{2\cos(2z)\cos(2z)-\sin(2z)(-2\sin(2z))}{\cos^2(2z)}\right] \\ &= \left(\frac{\cos(2z)}{\sin(2z)}\right)^{1/2}\left[\frac{\cos^2(2z)+\sin^2(2z)}{\cos^2(2z)}\right] \\ &= \frac{(\cos(2z))^{1/2}}{(\sin(2z))^{1/2}\cos^2(2z)} = \frac{1}{\sqrt{\sin(2z)}\sqrt{\cos^3(2z)}}. \end{aligned}$$

26. Using the chain rule and simplifying,

$$q'(\theta) = \frac{1}{2}(4\theta^2-\sin^2(2\theta))^{-1/2}(8\theta-2\sin(2\theta)(2\cos(2\theta))) = \frac{4\theta-2\sin(2\theta)\cos(2\theta)}{\sqrt{4\theta^2-\sin^2(2\theta)}}.$$

27. Using the product rule and factoring gives $\frac{dw}{dz} = 2^{-4z}\left[-4\ln(2)\sin(\pi z)+\pi\cos(\pi z)\right]$.

28. $g'(t) = \frac{3}{(3t-4)^2+1}$.

29. $r'(\theta) = \frac{d}{d\theta}\left(e^{\left(e^\theta+e^{-\theta}\right)}\right) = e^{\left(e^\theta+e^{-\theta}\right)}\left(e^\theta-e^{-\theta}\right)$.

30. Using the chain rule, we get:

$$m'(n) = \cos(e^n)\cdot(e^n)$$

31. Using the chain rule we get:

$$G'(\alpha) = e^{\tan(\sin\alpha)}(\tan(\sin\alpha))' = e^{\tan(\sin\alpha)} \cdot \frac{1}{\cos^2(\sin\alpha)} \cdot \cos\alpha.$$

32. Here we use the product rule, and then the chain rule, and then the product rule.

$$\begin{aligned} g'(t) &= \cos(\sqrt{t}e^t) + t(\cos\sqrt{t}e^t)' = \cos(\sqrt{t}e^t) + t(-\sin(\sqrt{t}e^t)\cdot(\sqrt{t}e^t)') \\ &= \cos(\sqrt{t}e^t) - t\sin(\sqrt{t}e^t)\cdot\left(\sqrt{t}e^t + \frac{1}{2\sqrt{t}}e^t\right) \end{aligned}$$

33. $f'(r) = e(\tan 2 + \tan r)^{e-1}(\tan 2 + \tan r)' = e(\tan 2 + \tan r)^{e-1}\left(\frac{1}{\cos^2 r}\right)$

34. $\frac{d}{dx}xe^{\tan x} = e^{\tan x} + xe^{\tan x}\frac{1}{\cos^2 x}.$

35. $\frac{dy}{dx} = 2e^{2x}\sin^2(3x) + e^{2x}(2\sin(3x)\cos(3x)3) = 2e^{2x}\sin(3x)(\sin(3x) + 3\cos(3x))$

36. $g'(x) = \frac{6x}{1+(3x^2+1)^2} = \frac{6x}{9x^4+6x^2+2}$

37. $\frac{dy}{dx} = (\ln 2)2^{\sin x}\cos x\cdot\cos x + 2^{\sin x}(-\sin x) = 2^{\sin x}\left((\ln 2)\cos^2 x - \sin x\right)$

38. Simplifying first gives $F(x) = ax\ln e + b = ax + b$. Thus $F'(x) = a$.
The same result is obtained by differentiating first and then simplifying.

39. $y = e^\theta e^{-1}$ $\quad y' = \frac{d}{d\theta}(e^\theta e^{-1}) = e^{-1}\frac{d}{d\theta}e^\theta = e^\theta e^{-1} = e^{\theta-1}.$

40. Using the product rule and factoring gives $f'(t) = e^{-4kt}(\cos t - 4k\sin t)$.

41. Using the product rule gives

$$\begin{aligned} H'(t) &= 2ate^{-ct} - c(at^2+b)e^{-ct} \\ &= (-cat^2 + 2at - bc)e^{-ct}. \end{aligned}$$

42. $\frac{d}{d\theta}\sqrt{a^2-\sin^2\theta} = \frac{1}{2\sqrt{a^2-\sin^2\theta}}(-2\sin\theta\cos\theta) = -\frac{\sin\theta\cos\theta}{\sqrt{a^2-\sin^2\theta}}.$

43. $y' = (\ln 5)5^x$.

44. Using the quotient rule gives

$$f'(x) = \frac{(-2x)(a^2+x^2) - (2x)(a^2-x^2)}{(a^2+x^2)^2} = \frac{-4a^2x}{(a^2+x^2)^2}.$$

45. Using the quotient rule gives

$$\begin{aligned} w'(r) &= \frac{2ar(b+r^3) - 3r^2(ar^2)}{(b+r^3)^2} \\ &= \frac{2abr - ar^4}{(b+r^3)^2}. \end{aligned}$$

46. Using the quotient rule gives

$$\begin{aligned} f'(s) &= \frac{-2s\sqrt{a^2+s^2} - \frac{s}{\sqrt{a^2+s^2}}(a^2-s^2)}{(a^2+s^2)} \\ &= \frac{-2s(a^2+s^2) - s(a^2-s^2)}{(a^2+s^2)^{3/2}} \\ &= \frac{-2a^2s - 2s^3 - a^2s + s^3}{(a^2+s^2)^{3/2}} \\ &= \frac{-3a^2s - s^3}{(a^2+s^2)^{3/2}}. \end{aligned}$$

47. $\dfrac{dy}{dx} = \dfrac{1}{1+\left(\frac{2}{x}\right)^2}\left(\dfrac{-2}{x^2}\right) = \dfrac{-2}{x^2+4}$

48. Using the chain rule gives $r'(t) = \dfrac{\cos(\frac{t}{k})}{\sin(\frac{t}{k})}\left(\dfrac{1}{k}\right)$.

49. Since $g(w) = 5(a^2 - w^2)^{-2}$, $g'(w) = -10(a^2-w^2)^{-3}(-2w) = \dfrac{20w}{(a^2-w^2)^3}$

50. $\dfrac{dy}{dx} = \dfrac{2e^{2x}(x^2+1) - e^{2x}(2x)}{(x^2+1)^2} = \dfrac{2e^{2x}(x^2+1-x)}{(x^2+1)^2}$

51. $g'(u) = \dfrac{ae^{au}}{a^2+b^2}$

52. Using the quotient and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(ae^{ax}+ae^{-ax})(e^{ax}+e^{-ax}) - (e^{ax}-e^{-ax})(ae^{ax}-ae^{-ax})}{(e^{ax}+e^{-ax})^2}\\ &= \frac{a(e^{ax}+e^{-ax})^2 - a(e^{ax}-e^{-ax})^2}{(e^{ax}+e^{-ax})^2}\\ &= \frac{a[(e^{2ax}+2+e^{-2ax}) - (e^{2ax}-2+e^{-2ax})]}{(e^{ax}+e^{-ax})^2}\\ &= \frac{4a}{(e^{ax}+e^{-ax})^2}\end{aligned}$$

53. Using the quotient rule gives

$$\begin{aligned}f'(x) &= \frac{1+\ln x - x(\frac{1}{x})}{(1+\ln x)^2}\\ &= \frac{\ln x}{(1+\ln x)^2}.\end{aligned}$$

54. Using the quotient and chain rules

$$\begin{aligned}\frac{dz}{dt} &= \frac{\frac{d}{dt}(e^{t^2}+t)\cdot\sin(2t) - (e^{t^2}+t)\frac{d}{dt}(\sin(2t))}{(\sin(2t))^2}\\ &= \frac{\left(e^{t^2}\cdot\frac{d}{dt}(t^2)+1\right)\sin(2t) - (e^{t^2}+t)\cos(2t)\frac{d}{dt}(2t)}{\sin^2(2t)}\\ &= \frac{(2te^{t^2}+1)\sin(2t) - (e^{t^2}+t)2\cos(2t)}{\sin^2(2t)}.\end{aligned}$$

55. Using the chain rule twice:

$$f'(t) = \cos\sqrt{e^t+1}\frac{d}{dt}\sqrt{e^t+1} = \cos\sqrt{e^t+1}\frac{1}{2\sqrt{e^t+1}}\cdot\frac{d}{dt}(e^t+1) = \cos\sqrt{e^t+1}\frac{1}{2\sqrt{e^t+1}}e^t = e^t\frac{\cos\sqrt{e^t+1}}{2\sqrt{e^t+1}}.$$

56. Using the chain rule twice:

$$g'(y) = e^{2e^{(y^3)}}\frac{d}{dy}\left(2e^{(y^3)}\right) = 2e^{2e^{(y^3)}}e^{(y^3)}\frac{d}{dy}(y^3) = 6y^2e^{(y^3)}e^{2e^{(y^3)}}.$$

57. $y' = 18x^2 + 8x - 2$.

58. $g(z) = z^5 + 5z^4 - z$
$g'(z) = 5z^4 + 20z^3 - 1$.

59. $f'(z) = (2\ln 3)z + (\ln 4)e^z$.

60. $\dfrac{dy}{dx} = 3 - 2(\ln 4)4^x$.

61. $f'(x) = 3x^2 + 3^x \ln 3$

62. $f'(\theta) = 2\theta\sin\theta + \theta^2\cos\theta + 2\cos\theta - 2\theta\sin\theta - 2\cos\theta = \theta^2\cos\theta$.

63.

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{1}{2}(\cos(5\theta))^{-\frac{1}{2}}(-\sin(5\theta)\cdot 5) + 2\sin(6\theta)\cos(6\theta)\cdot 6\\ &= -\frac{5}{2}\frac{\sin(5\theta)}{\sqrt{\cos(5\theta)}} + 12\sin(6\theta)\cos(6\theta)\end{aligned}$$

64. $r'(\theta) = \dfrac{d}{d\theta}\sin[(3\theta-\pi)^2] = \cos[(3\theta-\pi)^2]\cdot 2(3\theta-\pi)\cdot 3 = 6(3\theta-\pi)\cos[(3\theta-\pi)^2]$.

65. It is easier to do this by multiplying it out first, rather than using the product rule first: $z = s^4 - s, \quad z' = 4s^3 - 1$.

66. Since $\tan(\arctan(k\theta)) = k\theta$, because tangent and arctangent are inverse functions, we have $N'(\theta) = k$.

67. Using the product rule gives $h'(t) = ke^{kt}(\sin at + \cos bt) + e^{kt}(a\cos at - b\sin bt)$.

68. $f'(y) = (\ln 4)4^y(2-y^2) + 4^y(-2y) = 4^y((\ln 4)(2-y^2) - 2y)$.

69. $f'(t) = 4(\sin(2t) - \cos(3t))^3[2\cos(2t) + 3\sin(3t)]$

70. Since $\cos^2 y + \sin^2 y = 1$, we have $s(y) = \sqrt[3]{1+3} = \sqrt[3]{4}$. Thus $s'(y) = 0$.

71.

$$\begin{aligned}f'(x) &= (-2x+6x^2)(6-4x+x^7) + (4-x^2+2x^3)(-4+7x^6)\\ &= (-12x+44x^2-24x^3-2x^8+6x^9) + (-16+4x^2-8x^3+28x^6-7x^8+14x^9)\\ &= -16-12x+48x^2-32x^3+28x^6-9x^8+20x^9\end{aligned}$$

72.

$$\begin{aligned}h'(x) &= \left(-\frac{1}{x^2}+\frac{2}{x^3}\right)\left(2x^3+4\right) + \left(\frac{1}{x}-\frac{1}{x^2}\right)\left(6x^2\right)\\ &= -2x+4-\frac{4}{x^2}+\frac{8}{x^3}+6x-6\\ &= 4x-2-4x^{-2}+8x^{-3}\end{aligned}$$

73. Note: $f(z) = (5z)^{1/2} + 5z^{1/2} + 5z^{-1/2} - \sqrt{5}z^{-1/2} + \sqrt{5}$, so $f'(z) = \dfrac{5}{2}(5z)^{-1/2} + \dfrac{5}{2}z^{-1/2} - \dfrac{5}{2}z^{-3/2} + \dfrac{\sqrt{5}}{2}z^{-3/2}$.

74.

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} - 8xy - 4x^2\frac{dy}{dx} &= 0\\ (3y^2-4x^2)\frac{dy}{dx} &= 8xy - 3x^2\\ \frac{dy}{dx} &= \frac{8xy-3x^2}{3y^2-4x^2}\end{aligned}$$

75. Using the relation $\cos^2 y + \sin^2 y = 1$, the equation becomes:
$1 = y + 2$ or $y = -1$. Hence, $\dfrac{dy}{dx} = 0$.

76. We wish to find the slope $m = dy/dx$. To do this, we can implicitly differentiate the given formula in terms of x:

$$\begin{aligned}x^2 + 3y^2 &= 7\\ 2x + 6y\frac{dy}{dx} &= \frac{d}{dx}(7) = 0\\ \frac{dy}{dx} &= \frac{-2x}{6y} = \frac{-x}{3y}.\end{aligned}$$

Thus, at $(2,-1)$, $m = -(2)/3(-1) = 2/3$.

77. Taking derivatives implicitly, we find

$$\frac{dy}{dx} + \cos y \frac{dy}{dx} + 2x = 0$$
$$\frac{dy}{dx} = \frac{-2x}{1+\cos y}$$

So, at the point $x = 3, y = 0$,

$$\frac{dy}{dx} = \frac{(-2)(3)}{1+\cos 0} = \frac{-6}{2} = -3.$$

78. First, we differentiate with respect to x:

$$x \cdot \frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(x+2y) = -y$$
$$\frac{dy}{dx} = \frac{-y}{x+2y}.$$

At $x = 3$, we have

$$3y + y^2 = 4$$
$$y^2 + 3y - 4 = 0$$
$$(y-1)(y+4) = 0.$$

Our two points, then, are $(3, 1)$ and $(3, -4)$.

At $(3,1)$, $\frac{dy}{dx} = \frac{-1}{3+2(1)} = -\frac{1}{5}$; Tangent line: $(y-1) = -\frac{1}{5}(x-3)$.

At $(3,-4)$, $\frac{dy}{dx} = \frac{-(-4)}{3+2(-4)} = -\frac{4}{5}$; Tangent line: $(y+4) = -\frac{4}{5}(x-3)$.

Problems

79. $f'(t) = 6t^2 - 8t + 3$ and $f''(t) = 12t - 8$.

80.

$$f'(x) = -8 + 2\sqrt{2}x$$
$$f'(r) = -8 + 2\sqrt{2}r = 4$$
$$r = \frac{12}{2\sqrt{2}} = 3\sqrt{2}.$$

81. Since $f(x) = x^3 - 6x^2 - 15x + 20$, we have $f'(x) = 3x^2 - 12x - 15$. To find the points at which $f'(x) = 0$, we solve

$$3x^2 - 12x - 15 = 0$$
$$3(x^2 - 4x - 5) = 0$$
$$3(x+1)(x-5) = 0.$$

We see that $f'(x) = 0$ at $x = -1$ and at $x = 5$. The graph of $f(x)$ in Figure 3.24 appears to be horizontal at $x = -1$ and at $x = 5$, confirming what we found analytically.

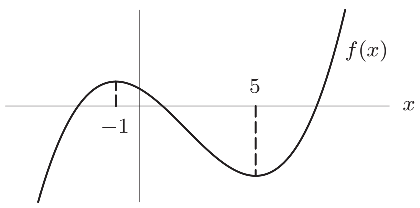


Figure 3.24

82. (a) Since the power of x will go down by one every time you take a derivative (until the exponent is zero after which the derivative will be zero), we can see immediately that $f^{(8)}(x) = 0$.
(b) $f^{(7)}(x) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^0 = 5040$.

83. (a) Applying the product rule to $h(x)$ we get $h'(1) = t'(1)s(1) + t(1)s'(1) \approx (-2) \cdot 3 + 0 \cdot 0 = -6$.
(b) Applying the product rule to $h(x)$ we get $h'(0) = t'(0)s(0) + t(0)s'(0) \approx (-2) \cdot 2 + 2 \cdot 2 = 0$.
(c) Applying the quotient rule to $p(x)$ we get $p'(0) = \dfrac{t'(0)s(0) - t(0)s'(0)}{(s(0))^2} \approx \dfrac{(-2) \cdot 2 - 2 \cdot 2}{2^2} = -2$.

Note that since $t(x)$ is a linear function whose slope looks like -2 from the graph, $t'(x) \approx -2$ everywhere. To find $s'(1)$, draw a line tangent to the curve at the point $(1, s(1))$, and estimate the slope.

84. Since $r(x) = s(t(x))$, the chain rule gives $r'(x) = s'(t(x)) \cdot t'(x)$. Thus,

$$r'(0) = s'(t(0)) \cdot t'(0) \approx s'(2) \cdot (-2) \approx (-2)(-2) = 4.$$

Note that since $t(x)$ is a linear function whose slope looks like -2 from the graph, $t'(x) \approx -2$ everywhere. To find $s'(2)$, draw a line tangent to the curve at the point $(2, s(2))$, and estimate the slope.

85. (a) Applying the chain rule we get $h'(1) = s'(s(1)) \cdot s'(1) \approx s'(3) \cdot 0 = 0$.
(b) Applying the chain rule we get $h'(2) = s'(s(2)) \cdot s'(2) \approx s'(2) \cdot s'(2) = (-2)^2 = 4$.
To find $s'(2)$, draw a line tangent to the curve at the point $(2, s(2))$, and estimate the slope.

86. We need to find all values for x such that

$$\frac{dy}{dx} = s'(s(x)) \cdot s'(x) = 0.$$

This is the case when either $s'(s(x)) = 0$ or $s'(x) = 0$. From the graph we see that $s'(x) = 0$ when $x \approx 1$. Also, $s'(s(x)) = 0$ when $s(x) \approx 1$, which happens when $x \approx -0.4$ or $x \approx 2.4$.

To find $s'(a)$, for any a, draw a line tangent to the curve at the point $(a, s(a))$, and estimate the slope.

87. (a) Applying the product rule we get $h'(-1) = 2 \cdot (-1) \cdot t(-1) + (-1)^2 \cdot t'(-1) \approx (-2) \cdot 4 + 1 \cdot (-2) = -10$.
(b) Applying the chain rule we get $p'(-1) = t'((-1)^2) \cdot 2 \cdot (-1) = -2 \cdot t'(1) \approx (-2) \cdot (-2) = 4$.
Note that since $t(x)$ is a linear function whose slope looks like -2 from the graph, $t'(x) \approx -2$ everywhere.

88. We have $r(1) = s(t(1)) \approx s(0) \approx 2$. By the chain rule, $r'(x) = s'(t(x)) \cdot t'(x)$, so

$$r'(1) = s'(t(1)) \cdot t'(1) \approx s'(0) \cdot (-2) \approx 2(-2) = -4.$$

Thus the equation of the tangent line is

$$y - 2 = -4(x - 1)$$
$$y = -4x + 6.$$

Note that since $t(x)$ is a linear function whose slope looks like -2 from the graph, $t'(x) \approx -2$ everywhere. To find $s'(0)$, draw a line tangent to the curve at the point $(0, s(0))$, and estimate the slope.

89. Estimates may vary. From the graphs, we estimate $g(1) \approx 2$, $g'(1) \approx 1$, and $f'(2) \approx 0.8$. Thus, by the chain rule,

$$h'(1) = f'(g(1)) \cdot g'(1) \approx f'(2) \cdot g'(1) \approx 0.8 \cdot 1 = 0.8.$$

90. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, and $g'(-0.4) \approx 2$. Thus, by the chain rule,

$$k'(1) = g'(f(1)) \cdot f'(1) \approx g'(-0.4) \cdot 0.5 \approx 2 \cdot 0.5 = 1.$$

91. Estimates may vary. From the graphs, we estimate $g(2) \approx 1.6$, $g'(2) \approx -0.5$, and $f'(1.6) \approx 0.8$. Thus, by the chain rule,

$$h'(2) = f'(g(2)) \cdot g'(2) \approx f'(1.6) \cdot g'(2) \approx 0.8(-0.5) = -0.4.$$

92. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, and $g'(0.3) \approx 1.7$. Thus, by the chain rule,

$$k'(2) = g'(f(2)) \cdot f'(2) \approx g'(0.3) \cdot f'(2) \approx 1.7 \cdot 1.1 \approx 1.9.$$

93. Taking the values of f, f', g, and g' from the table we get:

(a) $h(4) = f(g(4)) = f(3) = 1$.
(b) $h'(4) = f'(g(4))g'(4) = f'(3) \cdot 1 = 2$.
(c) $h(4) = g(f(4)) = g(4) = 3$.
(d) $h'(4) = g'(f(4))f'(4) = g'(4) \cdot 3 = 3$.
(e) $h'(4) = (f(4)g'(4) - g(4)f'(4))/f^2(4) = -5/16$.
(f) $h'(4) = f(4)g'(4) + g(4)f'(4) = 13$.

94. (a) $H'(2) = r'(2)s(2) + r(2)s'(2) = -1 \cdot 1 + 4 \cdot 3 = 11.$

(b) $H'(2) = \dfrac{r'(2)}{2\sqrt{r(2)}} = \dfrac{-1}{2\sqrt{4}} = -\dfrac{1}{4}.$

(c) $H'(2) = r'(s(2))s'(2) = r'(1) \cdot 3$, but we don't know $r'(1)$.

(d) $H'(2) = s'(r(2))r'(2) = s'(4)r'(2) = -3.$

95. (a) $f(x) = x^2 - 4g(x)$
$f'(x) = 2x - 4g'(x)$
$f'(2) = 2(2) - 4(-4) = 4 + 16 = 20$

(b) $f(x) = \frac{x}{g(x)}$
$f'(x) = \frac{g(x) - xg'(x)}{(g(x))^2}$
$f'(2) = \frac{g(2) - 2g'(2)}{(g(2))^2} = \frac{3 - 2(-4)}{(3)^2} = \frac{11}{9}$

(c) $f(x) = x^2 g(x)$
$f'(x) = 2xg(x) + x^2 g'(x)$
$f'(2) = 2(2)(3) + (2)^2(-4) = 12 - 16 = -4$

(d) $f(x) = (g(x))^2$
$f'(x) = 2g(x) \cdot g'(x)$
$f'(2) = 2(3)(-4) = -24$

(e) $f(x) = x \sin(g(x))$
$f'(x) = \sin(g(x)) + x\cos(g(x)) \cdot g'(x)$

$$\begin{aligned} f'(2) &= \sin(g(2)) + 2\cos(g(2)) \cdot g'(2) \\ &= \sin 3 + 2\cos(3) \cdot (-4) \\ &= \sin 3 - 8\cos 3 \end{aligned}$$

(f) $f(x) = x^2 \ln(g(x))$
$f'(x) = 2x \ln(g(x)) + x^2(\frac{g'(x)}{g(x)})$

$$\begin{aligned} f'(2) &= 2(2)\ln 3 + (2)^2(\tfrac{-4}{3}) \\ &= 4\ln 3 - \tfrac{16}{3} \end{aligned}$$

96. (a) $f(x) = x^2 - 4g(x)$
$f(2) = 4 - 4(3) = -8$
$f'(2) = 20$
Thus, we have a point $(2, -8)$ and slope $m = 20$. This gives

$$\begin{aligned} -8 &= 2(20) + b \\ b &= -48, \quad \text{so} \\ y &= 20x - 48. \end{aligned}$$

(b) $f(x) = \dfrac{x}{g(x)}$
$f(2) = \dfrac{2}{3}$
$f'(2) = \dfrac{11}{9}$
Thus, we have point $(2, \frac{2}{3})$ and slope $m = \frac{11}{9}$. This gives

$$\begin{aligned} \frac{2}{3} &= (\frac{11}{9})(2) + b \\ b &= \frac{2}{3} - \frac{22}{9} = \frac{-16}{9}, \quad \text{so} \\ y &= \frac{11}{9}x - \frac{16}{9}. \end{aligned}$$

(c) $f(x) = x^2 g(x)$
$f(2) = 4 \cdot g(2) = 4(3) = 12$
$f'(2) = -4$
Thus, we have point $(2, 12)$ and slope $m = -4$. This gives

$$12 = 2(-4) + b$$

$$b = 20, \quad \text{so}$$
$$y = -4x + 20.$$

(d) $f(x) = (g(x))^2$
$f(2) = (g(2))^2 = (3)^2 = 9$
$f'(2) = -24$
Thus, we have point $(2, 9)$ and slope $m = -24$. This gives

$$9 = 2(-24) + b$$
$$b = 57, \quad \text{so}$$
$$y = -24x + 57.$$

(e) $f(x) = x\sin(g(x))$
$f(2) = 2\sin(g(2)) = 2\sin 3$
$f'(2) = \sin 3 - 8\cos 3$
We will use a decimal approximation for $f(2)$ and $f'(2)$, so the point $(2, 2\sin 3) \approx (2, 0.28)$ and $m \approx 8.06$. Thus,

$$0.28 = 2(8.06) + b$$
$$b = -15.84, \quad \text{so}$$
$$y = 8.06x - 15.84.$$

(f) $f(x) = x^2 \ln g(x)$
$f(2) = 4\ln g(2) = 4\ln 3 \approx 4.39$
$f'(2) = 4\ln 3 - \dfrac{16}{3} \approx -0.94.$
Thus, we have point $(2, 4.39)$ and slope $m = -0.94$. This gives

$$4.39 = 2(-0.94) + b$$
$$b = 6.27, \quad \text{so}$$
$$y = -0.94x + 6.27.$$

97. When we zoom in on the origin, we find that two functions are not defined there. The other functions all look like straight lines through the origin. The only way we can tell them apart is their slope.

The following functions all have slope 0 and are therefore indistinguishable:
$\sin x - \tan x$, $\frac{x^2}{x^2+1}$, $x - \sin x$, and $\frac{1-\cos x}{\cos x}$.

These functions all have slope 1 at the origin, and are thus indistinguishable:
$\arcsin x$, $\frac{\sin x}{1+\sin x}$, $\arctan x$, $e^x - 1$, $\frac{x}{x+1}$, and $\frac{x}{x^2+1}$.

Now, $\frac{\sin x}{x} - 1$ and $-x\ln x$ both are undefined at the origin, so they are distinguishable from the other functions. In addition, while $\frac{\sin x}{x} - 1$ has a slope that approaches zero near the origin, $-x\ln x$ becomes vertical near the origin, so they are distinguishable from each other.

Finally, $x^{10} + \sqrt[10]{x}$ is the only function defined at the origin and with a vertical tangent there, so it is distinguishable from the others.

98. It makes sense to define the angle between two curves to be the angle between their tangent lines. (The tangent lines are the best linear approximations to the curves). See Figure 3.25. The functions $\sin x$ and $\cos x$ are equal at $x = \frac{\pi}{4}$.

$$\text{For } f_1(x) = \sin x, \quad f_1'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$
$$\text{For } f_2(x) = \cos x, \quad f_2'(\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}.$$

Using the point $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ for each tangent line we get $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4})$ and $y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 + \frac{\pi}{4})$, respectively.

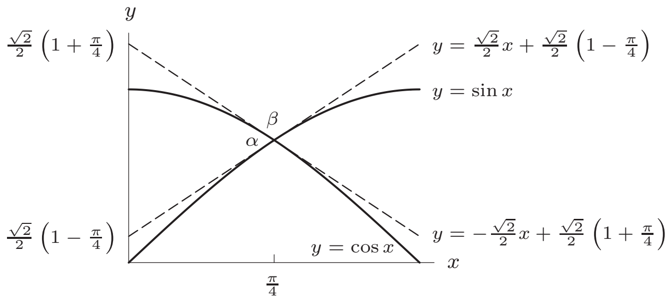


Figure 3.25

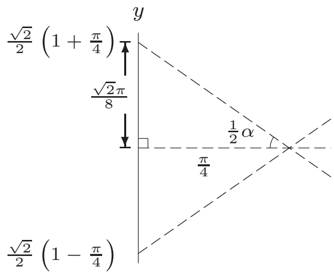


Figure 3.26

There are two possibilities of how to define the angle between the tangent lines, indicated by α and β above. The choice is arbitrary, so we will solve for both. To find the angle, α, we consider the triangle formed by these two lines and the y-axis. See Figure 3.26.

$$\tan\left(\frac{1}{2}\alpha\right)=\frac{\sqrt{2}\pi/8}{\pi/4}=\frac{\sqrt{2}}{2}$$
$$\frac{1}{2}\alpha=0.61548 \text{ radians}$$
$$\alpha=1.231 \text{ radians, or } 70.5^\circ.$$

Now let us solve for β, the other possible measure of the angle between the two tangent lines. Since α and β are supplementary, $\beta=\pi-1.231=1.909$ radians, or 109.4°.

99. The curves meet when $1+x-x^2=1-x+x^2$, that is when $2x(1-x)=0$ so that $x=1$ or $x=0$. Let

$$y_1(x)=1+x-x^2 \quad \text{and} \quad y_2(x)=1-x+x^2.$$

Then

$$y_1{}'=1-2x \quad \text{and} \quad y_2{}'=-1+2x.$$

At $x=0$, $y_1{}'=1$, $y_2{}'=-1$ so that $y_1{}'\cdot y_2{}'=-1$ and the curves are perpendicular. At $x=1$, $y_1{}'=-1$, $y_2{}'=1$ so that $y_1{}'\cdot y_2{}'=-1$ and the curves are perpendicular.

100. The curves meet when $1-x^3/3=x-1$, that is when $x^3+3x-6=0$. So the roots of this equation give us the x-coordinates of the intersection point. By numerical methods, we see there is one solution near $x=1.3$. See Figure 3.27. Let

$$y_1(x)=1-\frac{x^3}{3} \quad \text{and} \quad y_2(x)=x-1.$$

So we have

$$y_1{}'=-x^2 \quad \text{and} \quad y_2{}'=1.$$

However, $y_2{}'(x)=+1$, so if the curves are to be perpendicular when they cross, then $y_1{}'$ must be -1. Since $y_1{}'=-x^2$, $y_1{}'=-1$ only at $x=\pm1$ which is not the point of intersection. The curves are therefore not perpendicular when they cross.

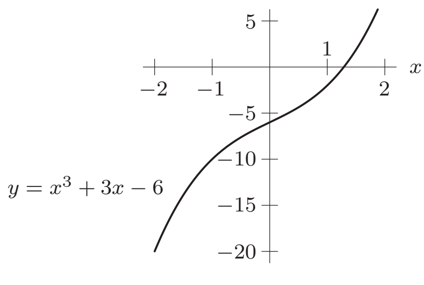


Figure 3.27

101. Differentiating gives $\dfrac{dy}{dx} = \ln x + 1 - b$.

To find the point at which the graph crosses the x-axis, set $y = 0$ and solve for x:

$$\begin{aligned} 0 &= x \ln x - bx \\ 0 &= x(\ln x - b). \end{aligned}$$

Since $x > 0$, we have

$$\begin{aligned} \ln x - b &= 0 \\ x &= e^b. \end{aligned}$$

At the point $(e^b, 0)$, the slope is

$$\frac{dy}{dx} = \ln\left(e^b\right) + 1 - b = b + 1 - b = 1.$$

Thus the equation of the tangent line is

$$\begin{aligned} y - 0 &= 1(x - e^b) \\ y &= x - e^b. \end{aligned}$$

102. Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2}$ and $\sinh 3x = \dfrac{e^{3x} - e^{-3x}}{2}$. Therefore

$$\begin{aligned} \lim_{x\to-\infty} \frac{\cosh(2x)}{\sinh(2x)} &= \lim_{x\to-\infty} \frac{e^{2x} + e^{-2x}}{e^{3x} - e^{-3x}} \\ &= \lim_{x\to-\infty} \frac{e^{-2x}(e^{4x} + 1)}{e^{-2x}(e^{5x} - e^{-x})} \\ &= \lim_{x\to-\infty} \frac{e^{4x} + 1}{e^{5x} - e^{-x}} \\ &= 0. \end{aligned}$$

103. Using the definition of $\sinh x$ we have $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$. Therefore

$$\begin{aligned} \lim_{x\to-\infty} \frac{e^{-2x}}{\sinh(2x)} &= \lim_{x\to-\infty} \frac{2e^{-2x}}{e^{2x} - e^{-2x}} \\ &= \lim_{x\to-\infty} \frac{2}{e^{4x} - 1} \\ &= -2. \end{aligned}$$

104. Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh x^2 = \dfrac{e^{x^2} + e^{-x^2}}{2}$ and $\sinh x^2 = \dfrac{e^{x^2} - e^{-x^2}}{2}$. Therefore

$$\begin{aligned} \lim_{x\to-\infty} \frac{\sinh(x^2)}{\cosh(x^2)} &= \lim_{x\to-\infty} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}} \\ &= \lim_{x\to-\infty} \frac{e^{x^2}(1 - e^{-2x^2})}{e^{x^2}(1 + e^{-2x^2})} \\ &= \lim_{x\to-\infty} \frac{1 - e^{-2x^2}}{1 + e^{-2x^2}} \\ &= 1. \end{aligned}$$

105. (a) Since $f(x) = \sqrt{x}$, we have $f'(x) = (1/2)x^{-1/2}$. So $f(4) = \sqrt{4} = 2$ and $f'(4) = (1/2)4^{-1/2} = 1/4$, and the tangent line approximation is

$$\begin{aligned} f(x) &\approx f(4) + f'(4)(x-4) \\ &\approx 2 + \frac{1}{4}(x-4). \end{aligned}$$

See Figure 3.28.

(b) For $x = 4.1$, the true value is

$$f(4.1) = \sqrt{4.1} = 2.02485...,$$

whereas the approximation is

$$f(4.1) \approx 2 + \frac{1}{4}(4.1 - 4) = 2.025.$$

Thus, the approximation differs from the true value by about 0.00015.

(c) For $x = 16$ the true value is:

$$f(16) = \sqrt{16} = 4$$

whereas the approximation is

$$f(16) \approx 2 + \frac{1}{4}(16 - 4) = 5.$$

Thus, the approximation differs from the true value by 1.

(d) The tangent line is a good approximation to the graph near $x = 4$, but not necessarily far away. Of course, there's no reason to expect that the curve will look like the tangent line if we go too far away, and usually it does not. (See Figure 3.28.) The problem is that we have traveled too far from the place where the curve looks like a line with slope $1/4$.

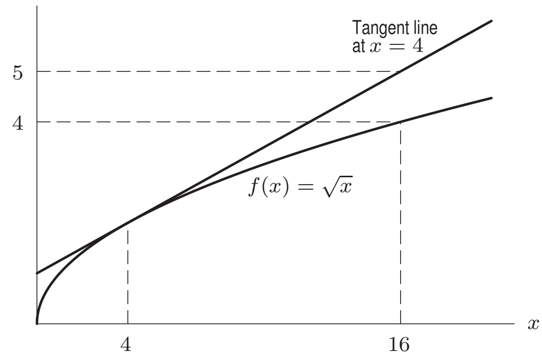


Figure 3.28: Local linearization: Approximating $f(x) = \sqrt{x}$ by its tangent line at $x = 4$

106. (a) From the figure, we see $a = 2$. The point with $x = 2$ lies on both the line and the curve. Since

$$y = -3 \cdot 2 + 7 = 1,$$

we have

$$f(a) = 1.$$

Since the slope of the line is -3, we have

$$f'(a) = -3.$$

(b) We use the line to approximate the function, so

$$f(2.1) \approx -3(2.1) + 7 = 0.7.$$

This is an underestimate, because the line is beneath the curve for $x > 2$. Similarly,

$$f(1.98) \approx -3(1.98) + 7 = 1.06.$$

This is an overestimate because the line is above the curve for $x < 2$.

The approximation $f(1.98) \approx 1.06$ is likely to be more accurate because 1.98 is closer to 2 than 2.1 is. Since the graph of $f(x)$ appears to bend away from the line at approximately the same rate on either side of $x = 2$, in this example, the error is larger for points farther from $x = 2$.

107. (a) The function $f(t)$ is linear; $g(t)$ is quadratic (polynomial of degree 2); and $h(t)$ is exponential.
(b) In 2010, we have $t = 130$. For $f(t)$, the rate of change is

$$f'(t) = 0.006$$
$$f'(130) = 0.006^\circ\text{C per year.}$$

For $g(t)$

$$g'(t) = 0.00012t - 0.0017$$
$$g'(130) = 0.00012 \cdot 130 - 0.0017 = 0.0139^\circ\text{C per year.}$$

For $h(t)$

$$h'(t) = 13.63(0.0004)e^{0.0004t} = 0.00545e^{0.0004t}$$
$$h'(130) = 0.00545e^{0.0004(130)} = 0.00574^\circ\text{C per year.}$$

(c) For $f(t)$: Change $= 0.006 \cdot 130 = 0.78^\circ$C.
For $g(t)$: Change $= 0.0139 \cdot 130 = 1.807^\circ$C.
For $h(t)$: Change $= 0.00574 \cdot 130 = 0.746^\circ$C.
(d) For $f(t)$: Predicted change $= f(130) - f(0) = (13.625 + 0.006 \cdot 130) - 13.625 = 0.78^\circ$C.
For $g(t)$: Predicted change $= g(130) - g(0) = (0.00006(130^2) - 0.0017(130) + 13.788) - 13.788 = 0.793^\circ$C
For $h(t)$: Predicted change $= h(130) - h(0) = 13.63e^{0.0004(130)} - 13.63 = 0.728^\circ$C.
(e) For the linear model, the answers in parts (c) and (d) are equal.
(f) For the quadratic model, the discrepancy is largest.

108. (a) We have $P = 9.906(0.997)^{11} = 9.584$ million.
(b) Differentiating, we have

$$\frac{dP}{dt} = 9.906(\ln 0.997)(0.997)^t$$
$$\text{so} \quad \left.\frac{dP}{dt}\right|_{t=11} = 9.906(\ln 0.997)(0.997)^{11} = -0.0288 \text{ million/year.}$$

Thus in 2020, Hungary's population will be decreasing by about 28,800 people per year.

109. $\dfrac{dF}{dr} = -\dfrac{2GMm}{r^3}$.

110. (a) If the distance $s(t) = 20e^{\frac{t}{2}}$, then the velocity, $v(t)$, is given by

$$v(t) = s'(t) = \left(20e^{\frac{t}{2}}\right)' = \left(\frac{1}{2}\right)\left(20e^{\frac{t}{2}}\right) = 10e^{\frac{t}{2}}.$$

(b) Observing the differentiation in (a), we note that

$$s'(t) = v(t) = \frac{1}{2}\left(20e^{\frac{t}{2}}\right) = \frac{1}{2}s(t).$$

Substituting $s(t)$ for $20e^{\frac{t}{2}}$, we obtain $s'(t) = \frac{1}{2}s(t)$.

111. (a) The rate of change of the population is $P'(t)$. If $P'(t)$ is proportional to $P(t)$, we have

$$P'(t) = kP(t).$$

(b) If $P(t) = Ae^{kt}$, then $P'(t) = kAe^{kt} = kP(t)$.

112. (a) Differentiating, we see

$$v = \frac{dy}{dt} = -2\pi\omega y_0 \sin(2\pi\omega t)$$
$$a = \frac{dv}{dt} = -4\pi^2\omega^2 y_0 \cos(2\pi\omega t).$$

(b) We have

$$y = y_0 \cos(2\pi\omega t)$$
$$v = -2\pi\omega y_0 \sin(2\pi\omega t)$$
$$a = -4\pi^2\omega^2 y_0 \cos(2\pi\omega t).$$

So

$$\text{Amplitude of } y \text{ is } |y_0|,$$
$$\text{Amplitude of } v \text{ is } |2\pi\omega y_0| = 2\pi\omega|y_0|,$$
$$\text{Amplitude of } a \text{ is } |4\pi^2\omega^2 y_0| = 4\pi^2\omega^2|y_0|.$$

The amplitudes are different (provided $2\pi\omega \neq 1$). The periods of the three functions are all the same, namely $1/\omega$.

(c) Looking at the answer to part (a), we see

$$\begin{aligned}\frac{d^2y}{dt^2} &= a = -4\pi^2\omega^2\left(y_0\cos(2\pi\omega t)\right)\\ &= -4\pi^2\omega^2 y.\end{aligned}$$

So we see that

$$\frac{d^2y}{dt^2} + 4\pi^2\omega^2 y = 0.$$

113. (a) Since $\lim_{t\to\infty} e^{-0.1t} = 0$, we see that $\lim_{t\to\infty}\frac{1000000}{1+5000e^{-0.1t}} = 1000000$. Thus, in the long run, close to 1,000,000 people will have had the disease. This can be seen in Figure 3.29.

(b) The rate at which people fall sick is given by the first derivative $N'(t)$.
$N'(t) \approx \frac{\Delta N}{\Delta t}$, where $\Delta t = 1$ day.

$$N'(t) = \frac{500{,}000{,}000}{e^{0.1t}(1+5000e^{-0.1t})^2} = \frac{500{,}000{,}000}{e^{0.1t} + 25{,}000{,}000e^{-0.1t} + 10^4}$$

In Figure 3.30, we see that the maximum value of $N'(t)$ is approximately 25,000. Therefore the maximum number of people to fall sick on any given day is 25,000. Thus there are no days on which a quarter million or more get sick.

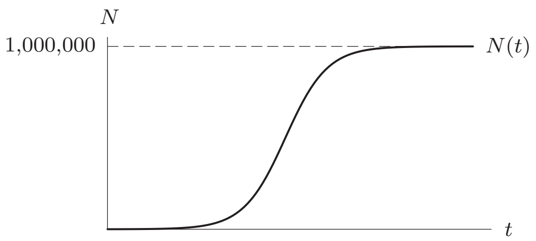


Figure 3.29

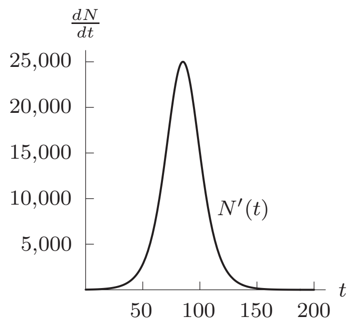


Figure 3.30

114. (a) We solve for t to find the time it takes for the population to reach 10 billion.

$$\begin{aligned}P(t) &= 10\\ 6.7e^{kt} &= 10\\ e^{kt} &= \frac{10}{6.7}\\ t &= \frac{\ln(10/6.7)}{k} = \frac{0.40048}{k} \text{ years}\end{aligned}$$

Thus the time is

$$f(k) = \frac{0.40048}{k} \text{ years.}$$

(b) The time to reach 10 billion with a growth rate of 1.2% is

$$f(0.012) = \frac{0.40048}{0.012} = 33.4 \text{ years.}$$

(c) We have

$$f'(k) = \frac{-0.40048}{k^2}$$
$$f'(0.012) = -2781.$$

Thus, for growth rates, k, near 1.2% we have

$$\text{Time for world population to reach 10 billion} = f(k) \approx f(0.012) + f'(0.012)(k - 0.012)$$
$$f(k) = 33.4 - 2781(k - 0.012) \text{ years}$$

(d) True time to reach 10 billion when the growth rate is 1.0% is $f(0.01) = 0.40048/0.01 = 40.0$ years. The linear approximation gives

$$\text{Approximate time} = 33.4 - 2781(0.01 - 0.012) = 39.0 \text{ years.}$$

115. (a) Suppose

$$g = f(r) = \frac{GM}{r^2}.$$

Then

$$f'(r) = \frac{-2GM}{r^3}.$$

So

$$f(r + \Delta r) \approx f(r) - \frac{2GM}{r^3}(\Delta r).$$

Since $f(r + \Delta r) - f(r) = \Delta g$, and $g = GM/r^2$, we have

$$\Delta g \approx -2\frac{GM}{r^3}(\Delta r) = -2g\frac{\Delta r}{r}.$$

(b) The negative sign tells us that the acceleration due to gravity decreases as the distance from the center of the earth increases.

(c) The fractional change in g is given by

$$\frac{\Delta g}{g} \approx -2\frac{\Delta r}{r}.$$

So, since $\Delta r = 4.315$ km and $r = 6400$ km, we have

$$\frac{\Delta g}{g} \approx -2\left(\frac{4.315}{6400}\right) = -0.00135 = -0.135\%.$$

116. Since g is the inverse of f, we know that $g(4) = f^{-1}(4) = 3$, so

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(3)} = \frac{1}{6}.$$

117. We must have

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(10)} = \frac{1}{8}.$$

118. We know that the velocity is given by

$$\frac{dx}{dt} = v(x).$$

By the chain rule,

$$\text{Acceleration } = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v'(x)v(x).$$

119. Since $f(x)$ is decreasing, its inverse function $f^{-1}(x)$ is also decreasing. Thus $(f^{-1})'(x) \leq 0$ for all x. Option (b) is incorrect.

120. (a) If $y = \ln x$, then

$$\begin{aligned} y' &= \frac{1}{x} \\ y'' &= -\frac{1}{x^2} \\ y''' &= \frac{2}{x^3} \\ y'''' &= -\frac{3\cdot 2}{x^4} \end{aligned}$$

and so

$$y^{(n)} = (-1)^{n+1}(n-1)!x^{-n}.$$

(b) If $y = xe^x$, then

$$\begin{aligned} y' &= xe^x + e^x \\ y'' &= xe^x + 2e^x \\ y''' &= xe^x + 3e^x \end{aligned}$$

so that

$$y^{(n)} = xe^x + ne^x.$$

(c) If $y = e^x \cos x$, then

$$\begin{aligned} y' &= e^x(\cos x - \sin x) \\ y'' &= -2e^x \sin x \\ y''' &= e^x(-2\cos x - 2\sin x) \\ y^{(4)} &= -4e^x \cos x \\ y^{(5)} &= e^x(-4\cos x + 4\sin x) \\ y^{(6)} &= 8e^x \sin x. \end{aligned}$$

Combining these results we get

$$\begin{aligned} y^{(n)} &= (-4)^{(n-1)/4}e^x(\cos x - \sin x), && n = 4m+1, \quad m = 0,1,2,3,\dots \\ y^{(n)} &= -2(-4)^{(n-2)/4}e^x \sin x, && n = 4m+2, \quad m = 0,1,2,3,\dots \\ y^{(n)} &= -2(-4)^{(n-3)/4}e^x(\cos x + \sin x), && n = 4m+3, \quad m = 0,1,2,3,\dots \\ y^{(n)} &= (-4)^{(n/4)}e^x \cos x, && n = 4m, \quad m = 1,2,3,\dots. \end{aligned}$$

121. (a) We multiply through by $h = f\cdot g$ and cancel as follows:

$$\begin{aligned} \frac{f'}{f} + \frac{g'}{g} &= \frac{h'}{h} \\ \left(\frac{f'}{f} + \frac{g'}{g}\right)\cdot fg &= \frac{h'}{h}\cdot fg \\ \frac{f'}{f}\cdot fg + \frac{g'}{g}\cdot fg &= \frac{h'}{h}\cdot h \\ f'\cdot g + g'\cdot f &= h', \end{aligned}$$

which is the product rule.

(b) We start with the product rule, multiply through by $1/(fg)$ and cancel as follows:

$$\begin{aligned} f'\cdot g + g'\cdot f &= h' \\ (f'\cdot g + g'\cdot f)\cdot\frac{1}{fg} &= h'\cdot\frac{1}{fg} \\ (f'\cdot g)\cdot\frac{1}{fg} + (g'\cdot f)\cdot\frac{1}{fg} &= h'\cdot\frac{1}{fg} \\ \frac{f'}{f} + \frac{g'}{g} &= \frac{h'}{h}, \end{aligned}$$

which is the additive rule shown in part (a).

122. This problem can be solved by using either the quotient rule or the fact that

$$\frac{f'}{f} = \frac{d}{dx}(\ln f) \quad \text{and} \quad \frac{g'}{g} = \frac{d}{dx}(\ln g).$$

We use the second method. The relative rate of change of f/g is $(f/g)'/(f/g)$, so

$$\frac{(f/g)'}{f/g} = \frac{d}{dx}\ln\left(\frac{f}{g}\right) = \frac{d}{dx}(\ln f - \ln g) = \frac{d}{dx}(\ln f) - \frac{d}{dx}(\ln g) = \frac{f'}{f} - \frac{g'}{g}.$$

Thus, the relative rate of change of f/g is the difference between the relative rates of change of f and of g.

CAS Challenge Problems

123. (a) Answers from different computer algebra systems may be in different forms. One form is:

$$\frac{d}{dx}(x+1)^x = x(x+1)^{x-1} + (x+1)^x\ln(x+1)$$
$$\frac{d}{dx}(\sin x)^x = x\cos x(\sin x)^{x-1} + (\sin x)^x\ln(\sin x)$$

(b) Both the answers in part (a) follow the general rule:

$$\frac{d}{dx}f(x)^x = xf'(x)\,(f(x))^{x-1} + (f(x))^x\ln(f(x)).$$

(c) Applying this rule to $g(x)$, we get

$$\frac{d}{dx}(\ln x)^x = x(1/x)(\ln x)^{x-1} + (\ln x)^x\ln(\ln x) = (\ln x)^{x-1} + (\ln x)^x\ln(\ln x).$$

This agrees with the answer given by the computer algebra system.

(d) We can write $f(x) = e^{\ln(f(x))}$. So

$$(f(x))^x = (e^{\ln(f(x))})^x = e^{x\ln(f(x))}.$$

Therefore, using the chain rule and the product rule,

$$\begin{aligned}\frac{d}{dx}(f(x))^x &= \frac{d}{dx}(x\ln(f(x)))\cdot e^{x\ln(f(x))} = \left(\ln(f(x)) + x\frac{d}{dx}\ln(f(x))\right)e^{x\ln(f(x))}\\ &= \left(\ln(f(x)) + x\frac{f'(x)}{f(x)}\right)(f(x))^x = \ln(f(x))\,(f(x))^x + xf'(x)\,(f(x))^{x-1}\\ &= xf'(x)\,(f(x))^{x-1} + (f(x))^x\ln(f(x)).\end{aligned}$$

124. (a) A CAS gives $f'(x) = 1$.

(b) By the chain rule,

$$f'(x) = \cos(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}.$$

Now $\cos t = \pm\sqrt{1-\sin^2 t}$. Furthermore, if $-\pi/2 \le t \le \pi/2$ then $\cos t \ge 0$, so we take the positive square root and get $\cos t = \sqrt{1-\sin^2 t}$. Since $-\pi/2 \le \arcsin x \le \pi/2$ for all x in the domain of arcsin, we have

$$\cos(\arcsin x) = \sqrt{1-(\sin(\arcsin x))^2} = \sqrt{1-x^2},$$

so

$$\frac{d}{dx}\sin(\arcsin(x)) = \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}} = 1.$$

(c) Since $\sin(\arcsin(x)) = x$, its derivative is 1.

125. (a) A CAS gives $g'(r) = 0$.

(b) Using the product rule,

$$\begin{aligned}g'(r) &= \frac{d}{dr}(2^{-2r})\cdot 4^r + 2^{-2r}\frac{d}{dr}(4^r) = -2\ln 2\cdot 2^{-2r}4^r + 2^{-2r}\ln 4\cdot 4^r\\ &= -\ln 4\cdot 2^{-2r}4^r + \ln 4\cdot 2^{-2r}4^r = (-\ln 4 + \ln 4)2^{-2r}4^r = 0\cdot 2^{-2r}4^r = 0.\end{aligned}$$

(c) By the laws of exponents, $4^r = (2^2)^r = 2^{2r}$, so $2^{-2r}4^r = 2^{-2r}2^{2r} = 2^0 = 1$. Therefore, its derivative is zero.

126. (a) A CAS gives $h'(t) = 0$

(b) By the chain rule

$$h'(t) = \frac{\frac{d}{dt}\left(1-\frac{1}{t}\right)}{1-\frac{1}{t}} + \frac{\frac{d}{dt}\left(\frac{t}{t-1}\right)}{\frac{t}{t-1}} = \frac{\frac{1}{t^2}}{\frac{t-1}{t}} + \frac{\frac{1}{t-1} - \frac{t}{(t-1)^2}}{\frac{t}{t-1}}$$
$$= \frac{1}{t^2-t} + \frac{(t-1)-t}{t^2-t} = \frac{1}{t^2-t} + \frac{-1}{t^2-t} = 0.$$

(c) The expression inside the first logarithm is $1-(1/t) = (t-1)/t$. Using the property $\log A + \log B = \log(AB)$, we get

$$\ln\left(1-\frac{1}{t}\right) + \ln\left(\frac{t}{t-1}\right) = \ln\left(\frac{t-1}{t}\right) + \ln\left(\frac{t}{t-1}\right)$$
$$= \ln\left(\frac{t-1}{t}\cdot\frac{t}{1-t}\right) = \ln 1 = 0.$$

Thus $h(t) = 0$, so $h'(t) = 0$ also.

PROJECTS FOR CHAPTER THREE

1. Let $r = i/100$. (For example if $i = 5\%$, $r = 0.05$.) Then the balance, $\$B$, after t years is given by

$$B = P(1+r)^t,$$

where $\$P$ is the original deposit. If we are doubling our money, then $B = 2P$, so we wish to solve for t in the equation $2P = P(1+r)^t$. This is equivalent to

$$2 = (1+r)^t.$$

Taking natural logarithms of both sides and solving for t yields

$$\ln 2 = t\ln(1+r),$$
$$t = \frac{\ln 2}{\ln(1+r)}.$$

We now approximate $\ln(1+r)$ near $r = 0$. Let $f(r) = \ln(1+r)$. Then $f'(r) = 1/(1+r)$. Thus, $f(0) = 0$ and $f'(0) = 1$, so

$$f(r) \approx f(0) + f'(0)r$$

becomes

$$\ln(1+r) \approx r.$$

Therefore,

$$t = \frac{\ln 2}{\ln(1+r)} \approx \frac{\ln 2}{r} = \frac{100\ln 2}{i} \approx \frac{70}{i},$$

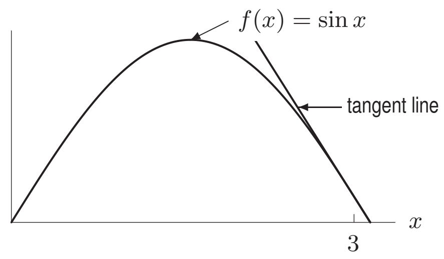
as claimed. We expect this approximation to hold for small values of i; it turns out that values of i up to 10 give good enough answers for most everyday purposes.

2. **(a)** (i) Set $f(x) = \sin x$, so $f'(x) = \cos x$. Guess $x_0 = 3$. Then

$$x_1 = 3 - \frac{\sin 3}{\cos 3} \approx 3.1425$$
$$x_2 = x_1 - \frac{\sin x_1}{\cos x_1} \approx 3.1415926533,$$

which is correct to one billionth!

(ii) Newton's method uses the tangent line at $x = 3$, i.e. $y - \sin 3 = \cos(3)(x - 3)$. Around $x = 3$, however, $\sin x$ is almost linear, since the second derivative $\sin''(\pi) = 0$. Thus using the tangent line to get an approximate value for the root gives us a very good approximation.



(iii) For $f(x) = \sin x$, we have

$$f(3) = 0.14112$$
$$f(4) = -0.7568,$$

so there is a root in $[3, 4]$. We now continue bisecting:

$$[3, 3.5] : f(3.5) = -0.35078 \text{ (bisection 1)}$$
$$[3, 3.25] : f(3.25) = -0.10819 \text{ (bisection 2)}$$
$$[3.125, 3.25] : f(3.125) = \quad 0.01659 \text{ (bisection 3)}$$
$$[3.125, 3.1875] : f(3.1875) = -0.04584 \text{ (bisection 4)}$$

We continue this process; after 11 bisections, we know the root lies between 3.1411 and 3.1416, which still is not as good an approximation as what we get from Newton's method in just two steps.

(b) (i) We have $f(x) = \sin x - \frac{2}{3}x$ and $f'(x) = \cos x - \frac{2}{3}$.
Using $x_0 = 0.904$,

$$x_1 = 0.904 - \frac{\sin(0.904) - \frac{2}{3}(0.904)}{\cos(0.904) - \frac{2}{3}} \approx 4.704,$$

$$x_2 = 4.704 - \frac{\sin(4.704) - \frac{2}{3}(4.704)}{\cos(4.704) - \frac{2}{3}} \approx -1.423,$$

$$x_3 = -1.433 - \frac{\sin(-1.423) - \frac{2}{3}(-1.423)}{\cos(-1.423) - \frac{2}{3}} \approx -1.501,$$

$$x_4 = -1.499 - \frac{\sin(-1.501) - \frac{2}{3}(-1.501)}{\cos(-1.501) - \frac{2}{3}} \approx -1.496,$$

$$x_5 = -1.496 - \frac{\sin(-1.496) - \frac{2}{3}(-1.496)}{\cos(-1.496) - \frac{2}{3}} \approx -1.496.$$

Using $x_0 = 0.905$,

$$x_1 = 0.905 - \frac{\sin(0.905) - \frac{2}{3}(0.905)}{\cos(0.905) - \frac{2}{3}} \approx 4.643,$$

$$x_2 = 4.643 - \frac{\sin(4.643) - \frac{2}{3}(4.643)}{\cos(4.643) - \frac{2}{3}} \approx -0.918,$$

$$x_3 = -0.918 - \frac{\sin(-0.918) - \frac{2}{3}(-0.918)}{\cos(-0.918) - \frac{2}{3}} \approx -3.996,$$

$$x_4 = -3.996 - \frac{\sin(-3.996) - \frac{2}{3}(-3.996)}{\cos(-3.996) - \frac{2}{3}} \approx -1.413,$$

$$x_5 = -1.413 - \frac{\sin(-1.413) - \frac{2}{3}(-1.413)}{\cos(-1.413) - \frac{2}{3}} \approx -1.502,$$

$$x_6 = -1.502 - \frac{\sin(-1.502) - \frac{2}{3}(-1.502)}{\cos(-1.502) - \frac{2}{3}} \approx -1.496.$$

Now using $x_0 = 0.906$,

$$x_1 = 0.906 - \frac{\sin(0.906) - \frac{2}{3}(0.906)}{\cos(0.906) - \frac{2}{3}} \approx 4.584,$$

$$x_2 = 4.584 - \frac{\sin(4.584) - \frac{2}{3}(4.584)}{\cos(4.584) - \frac{2}{3}} \approx -0.509,$$

$$x_3 = -0.510 - \frac{\sin(-0.509) - \frac{2}{3}(-0.509)}{\cos(-0.509) - \frac{2}{3}} \approx .207,$$

$$x_4 = -1.300 - \frac{\sin(.207) - \frac{2}{3}(.207)}{\cos(.207) - \frac{2}{3}} \approx -0.009,$$

$$x_5 = -1.543 - \frac{\sin(-0.009) - \frac{2}{3}(-0.009)}{\cos(-0.009) - \frac{2}{3}} \approx 0,$$

(ii) Starting with 0.904 and 0.905 yields the same value, but the two paths to get to the root are very different. Starting with 0.906 leads to a different root. Our starting points were near the maximum value of f. Consequently, a small change in x_0 makes a large change in x_1.

CHAPTER FOUR

Solutions for Section 4.1

Exercises

1. See Figure 4.1.

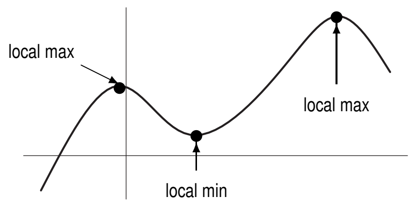


Figure 4.1

2. There are many possible answers. One possible graph is shown in Figure 4.2.

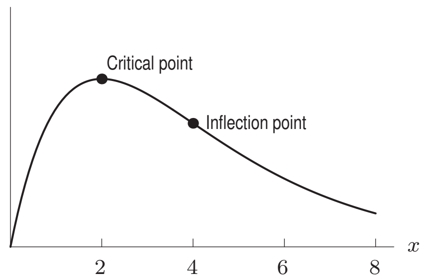


Figure 4.2

3. We sketch a graph which is horizontal at the two critical points. One possibility is shown in Figure 4.3.

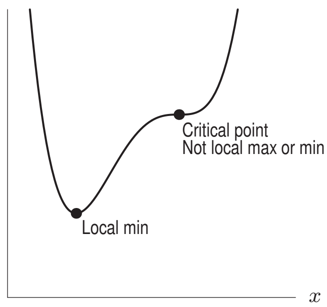


Figure 4.3

4. To find the critical points, we set $f'(x) = 0$. Since $f'(x) = 3x^2 - 18x + 24$, we have

$$\begin{aligned} 3x^2 - 18x + 24 &= 0 \\ 3(x^2 - 6x + 8) &= 0 \\ 3(x-2)(x-4) &= 0 \\ x &= 2, 4. \end{aligned}$$

There are two critical points: at $x = 2$, $x = 4$.
To find the inflection points, we look for points where f'' is zero or undefined. Since $f''(x) = 6x - 18$, it is defined everywhere, and setting $f''(x) = 0$ we get

$$\begin{aligned} 6x - 18 &= 0 \\ x &= 3. \end{aligned}$$

Furthermore, $6x - 18$ is positive when $x > 3$ and negative when $x < 3$, so f'' changes sign at $x = 3$. Thus, there is one inflection point: $x = 3$.

5. To find the critical points, we set $f'(x) = 0$. Since $f'(x) = 5x^4 - 30x^2$, we have

$$\begin{aligned} 5x^4 - 30x^2 &= 0 \\ 5x^2(x^2 - 6) &= 0 \\ x &= 0, -\sqrt{6}, \sqrt{6}. \end{aligned}$$

There are three critical points: $x = 0$, $x = -\sqrt{6}$, and $x = \sqrt{6}$.
To find the inflection points, we look for points where f'' is undefined or zero. Since $f''(x) = 20x^3 - 60x$, it is defined everywhere. Setting it equal to zero, we get

$$\begin{aligned} 20x^3 - 60x &= 0 \\ 20x(x^2 - 3) &= 0 \\ x &= 0, -\sqrt{3}, \sqrt{3}. \end{aligned}$$

Furthermore, $20x^3 - 60x$ is positive when $-\sqrt{3} < x < 0$ or $x > \sqrt{3}$ and negative when $x < -\sqrt{3}$ or $0 < x < \sqrt{3}$, so f'' changes sign at each of the solutions. Thus, there are three inflection points: $x = 0$, $x = -\sqrt{3}$, and $x = \sqrt{3}$.

6. To find the critical points, we set $f'(x) = 0$. Since $f'(x) = 5x^4 + 60x^3$, we have

$$\begin{aligned} 5x^4 + 60x^3 &= 0 \\ 5x^3(x + 12) &= 0 \\ x &= 0, -12. \end{aligned}$$

There are two critical points: $x = 0$, $x = -12$.
To find the inflection points, look for points where f'' is undefined or zero. Since $f''(x) = 20x^3 + 180x^2$, it is defined everywhere. Setting it equal to zero, we get

$$\begin{aligned} 20x^3 + 180x^2 &= 0 \\ 20x^2(x + 9) &= 0 \\ x &= 0, -9. \end{aligned}$$

Furthermore, $20x^2(x + 9)$ is negative when $x < -9$, and positive or zero when $x > -9$. So f'' does change sign at $x = -9$, but not at $x = 0$. Thus, there is one inflection point: $x = -9$.

7. To find the critical points, we set $f'(x) = 0$. Since $f'(x) = 5 - 3(1/x)$, we have

$$\begin{aligned} 5 - \frac{3}{x} &= 0 \\ 5 &= \frac{3}{x} \\ 5x &= 3 \\ x &= 3/5. \end{aligned}$$

There is one critical point, at $x = 3/5$.
To find the inflection points, we look for points where $f''(x)$ is undefined or zero. Since $f''(x) = 3x^{-2}$, it is undefined at $x = 0$. However, $f(x)$ is also undefined at $x = 0$, so this is not an inflection point. Also, $3/x^2$ is never equal to zero, so there are no inflection points.

8. To find the critical points, we set $f'(x) = 0$. Using the product rule, we have $f'(x) = 4x \cdot (e^{3x} \cdot 3) + 4 \cdot e^{3x}$.

$$\begin{aligned} 12xe^{3x} + 4e^{3x} &= 0 \\ 4e^{3x}(3x+1) &= 0 \\ x &= -1/3. \end{aligned}$$

There is one critical point, at $x = -1/3$.
To find the inflection points, we look for points where f'' is zero or undefined. Since $f''(x) = 12x \cdot (e^{3x} \cdot 3) + 12 \cdot e^{3x} + 4e^{3x} \cdot 3 = 12e^{3x}(3x+2)$, it is defined everywhere. Setting it equal to zero, we get

$$\begin{aligned} 12e^{3x}(3x+2) &= 0 \\ x &= -2/3. \end{aligned}$$

Since $12e^{3x}$ is always positive, $12e^{3x}(3x+2)$ is negative when $x < -2/3$ and positive when $x > -2/3$, so f'' changes sign at $x = -2/3$. Thus, there is one inflection point, at $x = -2/3$.

9. $f'(x) = 12x^3 - 12x^2$. To find critical points, we set $f'(x) = 0$. This implies $12x^2(x-1) = 0$. So the critical points of f are $x = 0$ and $x = 1$. To the left of $x = 0$, $f'(x) < 0$. Between $x = 0$ and $x = 1$, $f'(x) < 0$. To the right of $x = 1$, $f'(x) > 0$. Therefore, $f(1)$ is a local minimum, but $f(0)$ is not a local extremum. See Figure 4.4.

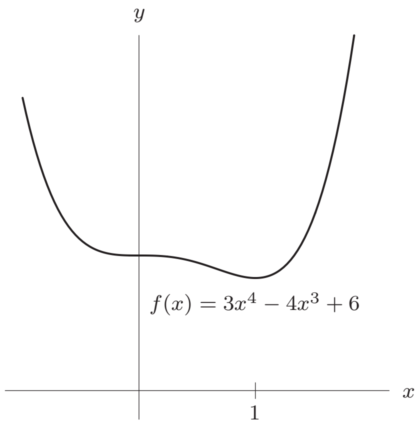


Figure 4.4

10. $f'(x) = 7(x^2-4)^6 2x = 14x(x-2)^6(x+2)^6$. The critical points of f are $x = 0$, $x = \pm 2$. To the left of $x = -2$, $f'(x) < 0$. Between $x = -2$ and $x = 0$, $f'(x) < 0$. Between $x = 0$ and $x = 2$, $f'(x) > 0$. To the right of $x = 2$, $f'(x) > 0$. Thus, $f(0)$ is a local minimum, whereas $f(-2)$ and $f(2)$ are not local extrema. See Figure 4.5.

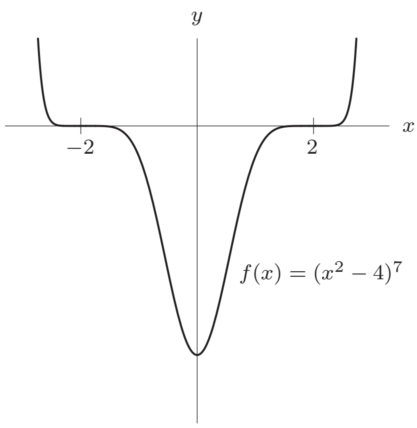


Figure 4.5

11. $f'(x) = 4(x^3-8)^3 3x^2$
$= 12x^2(x-2)^3(x^2+2x+4)^3$.
So the critical points are $x = 0$ and $x = 2$.
To the left of $x = 0$, $f'(x) < 0$.
Between $x = 0$ and $x = 2$, $f'(x) < 0$.
To the right of $x = 2$, $f'(x) > 0$.
Thus, $f(2)$ is a local minimum, whereas $f(0)$ is not a local extremum. See Figure 4.6.

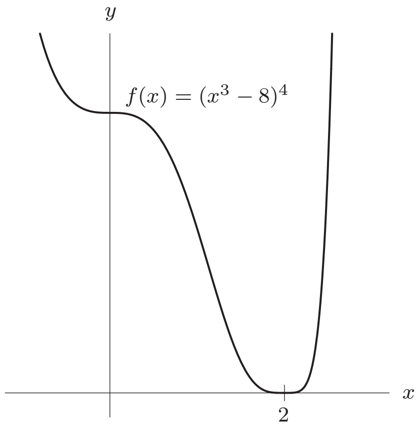


Figure 4.6

12.

$$f'(x) = \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

Critical points are $x = \pm 1$. To the left of $x = -1$, $f'(x) < 0$.
Between $x = -1$ and $x = 1$, $f'(x) > 0$.
To the right of $x = 1$, $f'(x) < 0$.
So, $f(-1)$ is a local minimum, $f(1)$ a local maximum. See Figure 4.7.

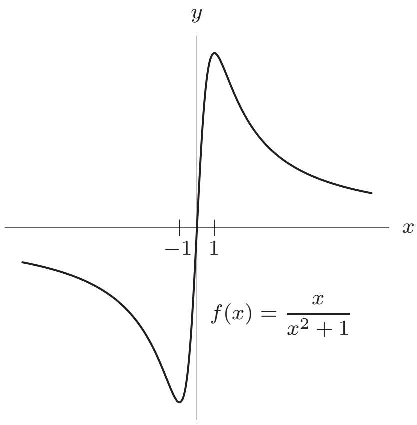


Figure 4.7

13. We have

$$g'(x) = e^{-3x} - 3xe^{-3x} = (1-3x)e^{-3x}.$$

To find critical points, we set $g'(x) = 0$. Then

$$(1-3x)e^{-3x} = 0.$$

Therefore, the critical point of g is $x = 1/3$. To the left of $x = 1/3$, we have $g'(x) > 0$. To the right of $x = 1/3$, we have $g'(x) < 0$. Thus $g(1/3)$ is a local maximum. See Figure 4.8.

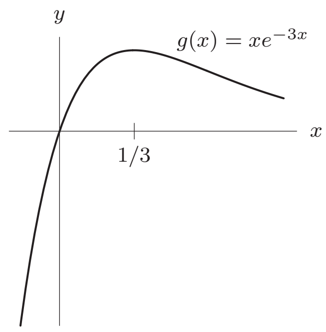


Figure 4.8

14. We have

$$h'(x) = 1 - 1/x^2.$$

To find critical points, we set $h'(x) = 0$. Then

$$\begin{aligned} 1 - \frac{1}{x^2} &= 0 \\ 1 &= \frac{1}{x^2} \\ x &= \pm 1. \end{aligned}$$

Therefore, the critical points of h are $x = -1$ and $x = 1$. For $0 < x < 1$, we have $h'(x) < 0$, and for $x > 1$, we have $h'(x) > 0$. Thus we have a local minimum at $x = 1$. For $x < -1$, we have $h'(x) > 0$ and for $-1 < x < 0$, we have $h'(x) < 0$. Thus $x = -1$ is a local maximum. See Figure 4.9.

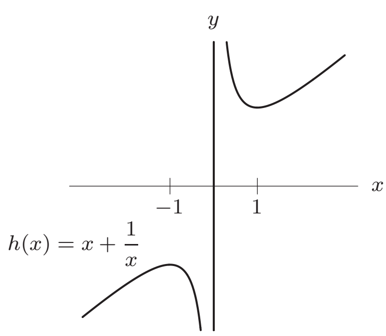


Figure 4.9

15. (a) A graph of $f(x) = e^{-x^2}$ is shown in Figure 4.10. It appears to have one critical point, at $x = 0$, and two inflection points, one between 0 and 1 and the other between 0 and -1.

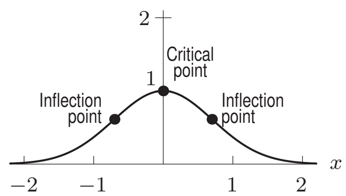


Figure 4.10

(b) To find the critical points, we set $f'(x) = 0$. Since $f'(x) = -2xe^{-x^2} = 0$, there is one solution, $x = 0$. The only critical point is at $x = 0$.
To find the inflection points, we first use the product rule to find $f''(x)$. We have

$$f''(x) = (-2x)(e^{-x^2}(-2x)) + (-2)(e^{-x^2}) = 4x^2e^{-x^2} - 2e^{-x^2}.$$

We set $f''(x) = 0$ and solve for x by factoring:

$$\begin{aligned} 4x^2e^{-x^2} - 2e^{-x^2} &= 0 \\ (4x^2 - 2)e^{-x^2} &= 0. \end{aligned}$$

Since e^{-x^2} is never zero, we have

$$\begin{aligned} 4x^2 - 2 &= 0 \\ x^2 &= \frac{1}{2} \\ x &= \pm 1/\sqrt{2}. \end{aligned}$$

There are exactly two inflection points, at $x = 1/\sqrt{2} \approx 0.707$ and $x = -1/\sqrt{2} \approx -0.707$.

16. (a) Increasing for $x > 0$, decreasing for $x < 0$.
(b) $f(0)$ is a local and global minimum, and f has no global maximum.

17. (a) Increasing for all x.
(b) No maxima or minima.

18. (a) Decreasing for $x < 0$, increasing for $0 < x < 4$, and decreasing for $x > 4$.
(b) $f(0)$ is a local minimum, and $f(4)$ is a local maximum.

19. (a) Decreasing for $x < -1$, increasing for $-1 < x < 0$, decreasing for $0 < x < 1$, and increasing for $x > 1$.
(b) $f(-1)$ and $f(1)$ are local minima, $f(0)$ is a local maximum.

Problems

20. (a) The function $f(x)$ is defined for $x \geq 0$. We set the derivative equal to zero to find critical points:

$$\begin{aligned} f'(x) = 1 + \frac{1}{2}ax^{-1/2} &= 0 \\ 1 + \frac{a}{2\sqrt{x}} &= 0. \end{aligned}$$

Since $a > 0$ and $\sqrt{x} > 0$ for $x > 0$, we have $f'(x) > 0$ for $x > 0$, so there is no critical point with $x > 0$. The only critical point is at $x = 0$ where $f'(x)$ is undefined.
(b) We see in part (a) that the derivative is positive for $x > 0$ so the function is increasing for all $x > 0$. The second derivative is

$$f''(x) = -\frac{1}{4}ax^{-3/2}.$$

Since a and $x^{-3/2}$ are positive for $x > 0$, the second derivative is negative for all $x > 0$. Thus the graph of f is concave down for all $x > 0$.

21. (a) This function is defined for $x > 0$. We set the derivative equal to zero and solve for x to find critical points:

$$\begin{aligned} f'(x) = 1 - b\frac{1}{x} &= 0 \\ 1 &= \frac{b}{x} \\ x &= b. \end{aligned}$$

The only critical point is at $x = b$.
(b) Since $f'(x) = 1 - bx^{-1}$, the second derivative is

$$f''(x) = bx^{-2} = \frac{b}{x^2}.$$

Since $b > 0$, the second derivative is always positive. Thus, the function is concave up everywhere and f has a local minimum at $x = b$.

22. (a) To find the critical points, we set the derivative of $f(x) = ax^{-2} + x$ equal to zero and solve for x.

$$\begin{aligned} f'(x) = -2ax^{-3} + 1 &= 0 \\ \frac{-2a}{x^3} + 1 &= 0 \\ x^3 &= 2a \\ x &= \sqrt[3]{2a}. \end{aligned}$$

There is one critical point, at $x = \sqrt[3]{2a}$. Although f' is undefined at $x = 0$, this is not a critical point since it is not in the domain of f.

(b) The second derivative is

$$f''(x) = 6ax^{-4} = \frac{6a}{x^4}.$$

If a and x^4 are positive, $f''(x)$ is positive and the graph is concave up everywhere, so the function has a local minimum at $x = \sqrt[3]{2a}$. Similarly, if a is negative, then $f''(x)$ is negative and the graph is concave down everywhere, so the function has a local maximum at $x = \sqrt[3]{2a}$.

23. To find the critical points, we set the derivative equal to zero and solve for t.

$$\begin{aligned} F'(t) = Ue^t + Ve^{-t}(-1) &= 0 \\ Ue^t - \frac{V}{e^t} &= 0 \\ Ue^t &= \frac{V}{e^t} \\ Ue^{2t} &= V \\ e^{2t} &= \frac{V}{U} \\ 2t &= \ln(V/U) \\ t &= \frac{\ln(V/U)}{2}. \end{aligned}$$

The derivative $F'(t)$ is never undefined, so the only critical point is $t = 0.5\ln(V/U)$.

24. The critical points of f are zeros of f'. Just to the left of the first critical point $f' > 0$, so f is increasing. Immediately to the right of the first critical point $f' < 0$, so f is decreasing. Thus, the first point must be a maximum. To the left of the second critical point, $f' < 0$, and to its right, $f' > 0$; hence it is a minimum. On either side of the last critical point, $f' > 0$, so it is neither a maximum nor a minimum. See the figure below. See Figure 4.11.

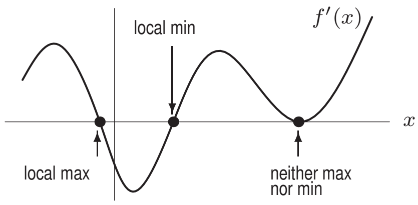


Figure 4.11

25. To find inflection points of the function f we must find points where f'' changes sign. However, because f'' is the derivative of f', any point where f'' changes sign will be a local maximum or minimum on the graph of f'. See Figure 4.12.

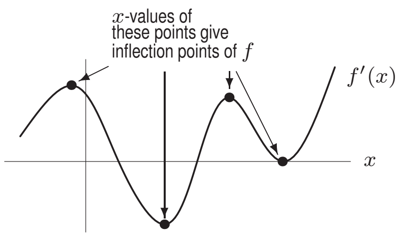


Figure 4.12

26. The inflection points of f are the points where f'' changes sign. See Figure 4.13.

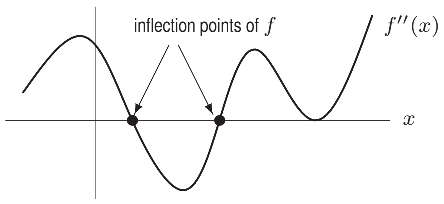


Figure 4.13

27. See Figure 4.14.

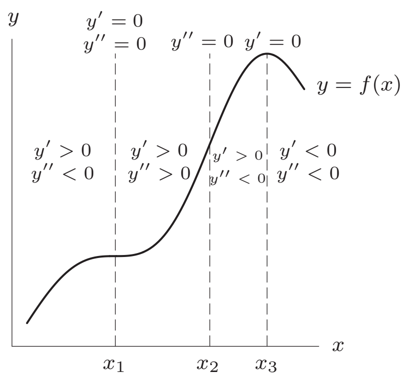


Figure 4.14

28. See Figure 4.15.

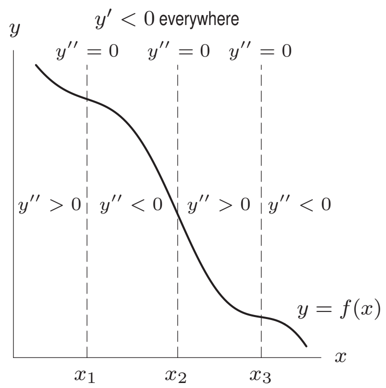


Figure 4.15

29. See Figure 4.16.

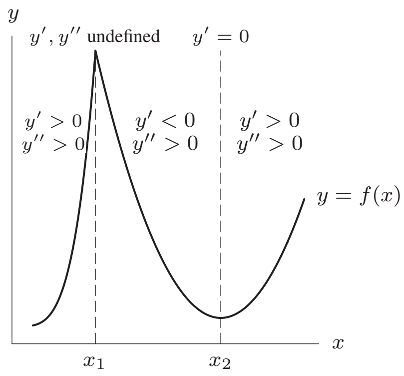


Figure 4.16

30. See Figure 4.17.

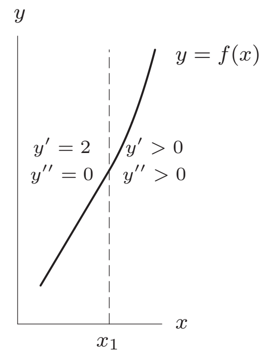


Figure 4.17

31. (a) A critical point occurs when $f'(x) = 0$. Since $f'(x)$ changes sign between $x = 2$ and $x = 3$, between $x = 6$ and $x = 7$, and between $x = 9$ and $x = 10$, we expect critical points at around $x = 2.5$, $x = 6.5$, and $x = 9.5$.

(b) Since $f'(x)$ goes from positive to negative at $x \approx 2.5$, a local maximum should occur there. Similarly, $x \approx 6.5$ is a local minimum and $x \approx 9.5$ a local maximum.

32. (a) It appears that this function has a local maximum at about $x = 1$, a local minimum at about $x = 4$, and a local maximum at about $x = 8$.

(b) The table now gives values of the derivative, so critical points occur where $f'(x) = 0$. Since f' is continuous, this occurs between 2 and 3, so there is a critical point somewhere around 2.5. Since f' is positive for values less than 2.5 and negative for values greater than 2.5, it appears that f has a local maximum at about $x = 2.5$. Similarly, it appears that f has a local minimum at about $x = 6.5$ and another local maximum at about $x = 9.5$.

33. See Figure 4.18.

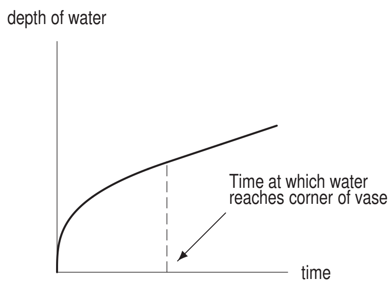


Figure 4.18

34. See Figure 4.19.

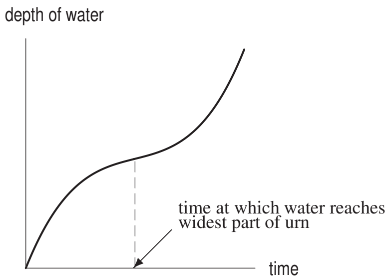


Figure 4.19

35. Differentiating using the product rule gives

$$f'(x) = 3x^2(1-x)^4 - 4x^3(1-x)^3 = x^2(1-x)^3(3(1-x)-4x) = x^2(1-x)^3(3-7x).$$

The critical points are the solutions to

$$f'(x) = x^2(1-x)^3(3-7x) = 0$$
$$x = 0, 1, \frac{3}{7}.$$

For $x < 0$, since $1-x > 0$ and $3-7x > 0$, we have $f'(x) > 0$.
For $0 < x < \frac{3}{7}$, since $1-x > 0$ and $3-7x > 0$, we have $f'(x) > 0$.
For $\frac{3}{7} < x < 1$, since $1-x > 0$ and $3-7x < 0$, we have $f'(x) < 0$.
For $1 < x$, since $1-x < 0$ and $3-7x < 0$, we have $f'(x) > 0$.

Thus, $x = 0$ is neither a local maximum nor a local minimum; $x = 3/7$ is a local maximum; $x = 1$ is a local minimum.

36. By the product rule

$$\begin{aligned} f'(x) = \frac{dx^m(1-x)^n}{dt} &= mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1} \\ &= x^{m-1}(1-x)^{n-1}(m(1-x)-nx) \\ &= x^{m-1}(1-x)^{n-1}(m-(m+n)x). \end{aligned}$$

We have $f'(x) = 0$ at $x = 0$, $x = 1$, and $x = m/(m+n)$, so these are the three critical points of f.

We can classify the critical points by determining the sign of $f'(x)$.
If $x < 0$, then $f'(x)$ has the same sign as $(-1)^{m-1}$: negative if m is even, positive if m is odd.
If $0 < x < m/(m+n)$, then $f'(x)$ is positive.
If $m/(m+n) < x < 1$, then $f'(x)$ is negative.
If $1 < x$, then $f'(x)$ has the same sign as $(-1)^n$: positive if n is even, negative if n is odd.
If m is even, then $f'(x)$ changes from negative to positive at $x = 0$, so f has a local minimum at $x = 0$.
If m is odd, then $f'(x)$ is positive to both the left and right of 0, so $x = 0$ is an inflection point of f.

At $x = m/(m+n)$, the derivative $f'(x)$ changes from positive to negative, so $x = m/(m+n)$ is a local maximum of f.

If n is even, then $f'(x)$ changes from negative to positive at $x = 1$, so f has a local minimum at $x = 1$. If n is odd, then $f'(x)$ is negative to both the left and right of 1, so $x = 1$ is an inflection point of f.

37. (a) From the graph of $P = \dfrac{2000}{1+e^{(5.3-0.4t)}}$ in Figure 4.20, we see that the population levels off at about 2000 rabbits.

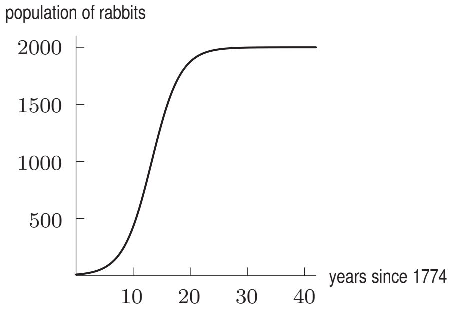


Figure 4.20

(b) The population appears to have been growing fastest when there were about 1000 rabbits, about 13 years after Captain Cook left them there.

(c) The rabbits reproduce quickly, so their population initially grew very rapidly. Limited food and space availability and perhaps predators on the island probably account for the population being unable to grow past 2000.

38. First, we wish to have $f'(6) = 0$, since $f(6)$ should be a local minimum:

$$\begin{aligned} f'(x) &= 2x + a = 0 \\ x &= -\frac{a}{2} = 6 \\ a &= -12. \end{aligned}$$

Next, we need to have $f(6) = -5$, since the point $(6, -5)$ is on the graph of $f(x)$. We can substitute $a = -12$ into our equation for $f(x)$ and solve for b:

$$\begin{aligned} f(x) &= x^2 - 12x + b \\ f(6) &= 36 - 72 + b = -5 \\ b &= 31. \end{aligned}$$

Thus, $f(x) = x^2 - 12x + 31$.

39. We wish to have $f'(3) = 0$. Differentiating to find $f'(x)$ and then solving $f'(3) = 0$ for a gives:

$$\begin{aligned} f'(x) &= x(ae^{ax}) + 1(e^{ax}) = e^{ax}(ax+1) \\ f'(3) &= e^{3a}(3a+1) = 0 \\ 3a + 1 &= 0 \\ a &= -\frac{1}{3}. \end{aligned}$$

Thus, $f(x) = xe^{-x/3}$.

40. Using the product rule on the function $f(x) = axe^{bx}$, we have $f'(x) = ae^{bx} + abxe^{bx} = ae^{bx}(1+bx)$. We want $f(\frac{1}{3}) = 1$, and since this is to be a maximum, we require $f'(\frac{1}{3}) = 0$. These conditions give

$$\begin{aligned} f(1/3) &= a(1/3)e^{b/3} = 1, \\ f'(1/3) &= ae^{b/3}(1 + b/3) = 0. \end{aligned}$$

Since $ae^{(1/3)b}$ is non-zero, we can divide both sides of the second equation by $ae^{(1/3)b}$ to obtain $0 = 1 + \frac{b}{3}$. This implies $b = -3$. Plugging $b = -3$ into the first equation gives us $a(\frac{1}{3})e^{-1} = 1$, or $a = 3e$. How do we know we have a maximum at $x = \frac{1}{3}$ and not a minimum? Since $f'(x) = ae^{bx}(1+bx) = (3e)e^{-3x}(1-3x)$, and $(3e)e^{-3x}$ is always positive, it follows that $f'(x) > 0$ when $x < \frac{1}{3}$ and $f'(x) < 0$ when $x > \frac{1}{3}$. Since f' is positive to the left of $x = \frac{1}{3}$ and negative to the right of $x = \frac{1}{3}$, $f(\frac{1}{3})$ is a local maximum.

41. The graph of $f(x) = x + \sin x$ is in Figure 4.21 and the graph of $f'(x) = 1 + \cos x$ is in Figure 4.22.

Where is f increasing most rapidly? At the points $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots,$ because these points are local maxima for f', and f' has the same value at each of them. Likewise f is increasing least rapidly at the points $x = \ldots, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots,$ since these points are local minima for f'. Notice that the points where f is increasing most rapidly and the points where it is increasing least rapidly are inflection points of f.

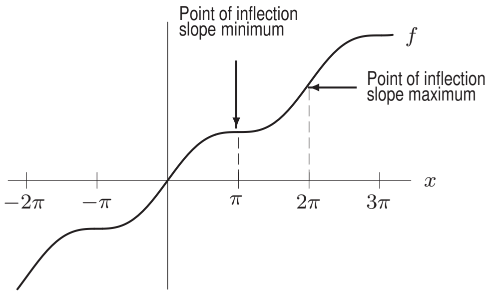


Figure 4.21: Graph of $f(x) = x + \sin x$

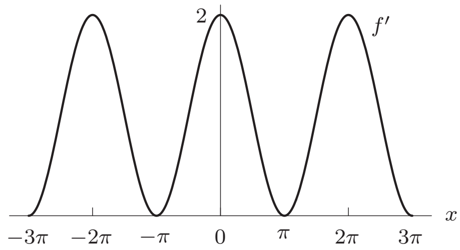


Figure 4.22: Graph of $f'(x) = 1 + \cos x$

42. Figure 4.23 contains the graph of $f(x) = x^2 + \cos x$.

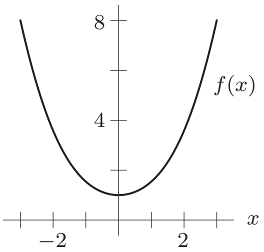


Figure 4.23

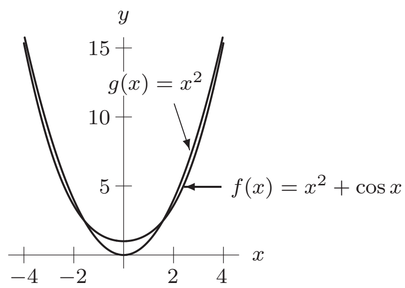


Figure 4.24

The graph looks like a parabola with no waves because $f''(x) = 2 - \cos x$, which is always positive. Thus, the graph of f is concave up everywhere; there are no waves. If you plot the graph of $f(x)$ together with the graph of $g(x) = x^2$, you see that the graph of f does wave back and forth across the graph of g, but never enough to change the concavity of f. See Figure 4.24.

43. Neither B nor C is 0 where A has its maxima and minimum. Therefore neither B nor C is the derivative of A, so $A = f''$. We can see B could be the derivative of C because where C has a maximum, B is 0. However, C is not the derivative of B because B is decreasing for some x-values and C is never negative. Thus, $C = f$, $B = f'$, and $A = f''$.

44. A has zeros where B has maxima and minima, so A could be a derivative of B. This is confirmed by comparing intervals on which B is increasing and A is positive. (They are the same.) So, C is either the derivative of A or the derivative of C is B. However, B does not have a zero at the point where C has a minimum, so B cannot be the derivative of C. Therefore, C is the derivative of A. So $B = f$, $A = f'$, and $C = f''$.

45. Since the derivative of an even function is odd and the derivative of an odd function is even, f and f'' are either both odd or both even, and f' is the opposite. Graphs I and III represent even functions; II represents an odd function, so II is f'. Since the maxima and minima of II occur where I crosses the x-axis, I must be the derivative of f', that is, f''. In addition, the maxima and minima of III occur where II crosses the x-axis, so III is f.

46. Since the derivative of an even function is odd and the derivative of an odd function is even, f and f'' are either both odd or both even, and f' is the opposite. Graphs I and II represent odd functions; III represents an even function, so III is f'. Since the maxima and minima of III occur where I crosses the x-axis, I must be the derivative of f', that is, f''. In addition, the maxima and minima of II occur where III crosses the x-axis, so II is f.

47. Using the quotient rule, since we have

$$f(x) = \frac{x + 50}{x^2 + 525},$$

$$f'(x) = \frac{x^2+525-2x(x+50)}{(x^2+525)^2} = \frac{-x^2-100x+525}{(x^2+525)^2}.$$

Since the denominator is positive for all x, setting $f'(x)=0$ gives:

$$x^2+100x-525=0$$
$$(x-5)(x+105)=0$$
$$x=5,-105.$$

These are the only two critical points. Checking signs using the formula for $f'(x)$, we find $f(x)$ is increasing for $-105 < x < 5$ and $f(x)$ is decreasing for $x < -105$ and $x > 5$.

It is difficult to solve this problem with a calculator for two reasons. First, the critical points are at -105 and 5, which are very far apart. Second, the range of values around the critical points is very small. For example,

$$f(-100) = -0.0047506$$
$$f(-105) = -0.0047619$$
$$f(-110) = -0.0047525$$

Thus, although $x=-105$ gives a local minimum, the graph of f is extremely flat in the neighborhood of -105. It is hard to see these small differences on a graph of f.

48. (a) When a number grows larger, its reciprocal grows smaller. Therefore, since f is increasing near x_0, we know that g (its reciprocal) must be decreasing. Another argument can be made using derivatives. We know that (since f is increasing) $f'(x) > 0$ near x_0. We also know (by the chain rule) that $g'(x) = (f(x)^{-1})' = -\frac{f'(x)}{f(x)^2}$. Since both $f'(x)$ and $f(x)^2$ are positive, this means $g'(x)$ is negative, which in turn means $g(x)$ is decreasing near $x=x_0$.

(b) Since f has a local maximum near x_1, $f(x)$ increases as x nears x_1, and then $f(x)$ decreases as x exceeds x_1. Thus the reciprocal of f, g, decreases as x nears x_1 and then increases as x exceeds x_1. Thus g has a local minimum at $x=x_1$. To put it another way, since f has a local maximum at $x=x_1$, we know $f'(x_1)=0$. Since $g'(x) = -\frac{f'(x)}{f(x)^2}$, $g'(x_1)=0$. To the left of x_1, $f'(x_1)$ is positive, so $g'(x)$ is negative. To the right of x_1, $f'(x_1)$ is negative, so $g'(x)$ is positive. Therefore, g has a local minimum at x_1.

(c) Since f is concave down at x_2, we know $f''(x_2) < 0$. We also know (from above) that

$$g''(x_2) = \frac{2f'(x_2)^2}{f(x_2)^3} - \frac{f''(x_2)}{f(x_2)^2} = \frac{1}{f(x_2)^2}\left(\frac{2f'(x_2)^2}{f(x_2)} - f''(x_2)\right).$$

Since $\frac{1}{f(x_2)^2} > 0$, $2f'(x_2)^2 > 0$, and $f(x_2) > 0$ (as f is assumed to be everywhere positive), we see that $g''(x_2)$ is positive. Thus g is concave up at x_2.

Note that for the first two parts of the problem, we did not need to require f to be positive (only non-zero). However, it was necessary here.

49. We know that $x=1$ is the only solution to $f(x)=0$, and that $x=2$ is the only solution to $f'(x)=0$.

(a) To find the zeros of this function, we solve the equation $y=0$:

$$\begin{aligned} f\left(x^2-3\right) &= 0 \\ x^2-3 &= 1 \qquad \text{because } f(1)=0 \\ x &= \pm 2. \end{aligned}$$

Thus, the zeros are $x=\pm 2$,

(b) To find the critical points of this function, we solve the equation $y'=0$:

$$\begin{aligned} \left(f\left(x^2-3\right)\right)' &= 0 \\ f'\left(x^2-3\right)(x^2-3)' &= 0 \qquad \text{Chain rule} \\ f'\left(x^2-3\right)\cdot 2x &= 0 \\ \text{so either} \quad 2x &= 0 \\ x &= 0 \\ \text{or} \quad f'\left(x^2-3\right) &= 0 \end{aligned}$$

$$\begin{aligned} x^2 - 3 &= 2 \qquad \text{because } f'(2) = 0 \\ x &= \pm\sqrt{5}. \end{aligned}$$

Thus, the critical points are $x = 0, \pm\sqrt{5}$.

50. We know that $x = 1$ is the only solution to $f(x) = 0$, and that $x = 2$ is the only solution to $f'(x) = 0$.

(a) To find the zeros of this function, we solve the equation $y = 0$:

$$\begin{aligned} (f(x))^2 + 3 &= 0 \\ (f(x))^2 &= -3. \quad \text{no solution} \end{aligned}$$

Thus, this function has no zeros.

(b) To find the critical points of this function, we solve the equation $y' = 0$:

$$\begin{aligned} \left((f(x))^2 + 3\right)' &= 0 \\ 2f(x)f'(x) &= 0 \quad \text{Chain rule} \\ \text{so either} \quad f(x) &= 0 \\ x &= 1 \quad \text{because } f(1) = 0 \\ \text{or} \quad f'(x) &= 0 \\ x &= 2. \quad \text{because } f'(2) = 0 \end{aligned}$$

Thus, this function has critical points at $x = 1, 2$.

51. We know $f'(x) < 0$ for all x. Since the graph of f lies entirely above the x-axis, we know $f(x) > 0$ for all x.

For $y = (f(x))^2$, we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(f(x))^2 \\ &= 2f(x)f'(x) \\ &= 2\underbrace{(\text{A positive number})}_{f(x)>0}\underbrace{(\text{A negative number})}_{f'(x)<0} \\ &= \text{A negative number.} \end{aligned}$$

We see that dy/dx is everywhere negative.

(a) Since the derivative is defined everywhere and has no zeros, there are no critical points.

(b) Since the derivative is everywhere negative, this is a decreasing function.

52. We know $f'(x) < 0$ for all x. Since the graph of f lies entirely above the x-axis, we know $f(x) > 0$ for all x.

For $y = f(x^2)$, we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}f\left(x^2\right) \\ &= 2xf'\left(x^2\right) \\ &= 2x \cdot \underbrace{(\text{A negative number})}_{f'\left(x^2\right)<0} \\ &= 2x \cdot (\text{A negative number}). \end{aligned}$$

We see that:

- $dy/dx = 0$ at $x = 0$
- $dy/dx < 0$ for $x > 0$
- $dy/dx > 0$ for $x < 0$

(a) Since $dy/dx = 0$ at $x = 0$, this means there is a critical point at $x = 0$.

(b) Since $dy/dx < 0$ for $x > 0$, the function decreases for $x > 0$. And since $dy/dx > 0$ for $x < 0$, the function increases for $x < 0$.

Strengthen Your Understanding

53. The function $f(x) = x^3$ is increasing and has an inflection point at $x = 0$.

54. Consider $f(x) = x^4$. We have $f''(x) = 12x^2$ so $f''(0) = 0$. However, f'' is positive for all $x \neq 0$. Thus, the graph of f does not change concavity at $x = 0$.

55. One possible example is $f(x) = x$, for which $f'(x) = 1$, so there are no critical points. There are many other examples.

56. One possible example is $f(x) = |x - 1|$. The function is defined at $x = 1$ since $f(1) = 0$, but f is not differentiable at $x = 1$. Thus, f has a critical point at $x = 1$.

57. One possible answer is $f(x) = \cos x$ which has local maxima at $0, \pm 2\pi, \pm 4\pi, \ldots$ (that is, at $x = 2n\pi$ for all integers n). The value at all the local maxima is 1. The local minima are at $\pm\pi, \pm 3\pi, \ldots$ (that is, at $x = (2n + 1)\pi$), all with value -1.

The function $f(x) = \sin x$ is another possibility, and there are many more.

58. True. Since the domain of f is all real numbers, all local minima occur at critical points.

59. True. Since the domain of f is all real numbers, all local maxima occur at critical points. Thus, if $x = p$ is a local maximum, $x = p$ must be a critical point.

60. False. A local maximum of f might occur at a point where f' does not exist. For example, $f(x) = -|x|$ has a local maximum at $x = 0$, but the derivative is not 0 (or defined) there.

61. False, because $x = p$ could be a local minimum of f. For example, if $f(x) = x^2$, then $f'(0) = 0$, so $x = 0$ is a critical point, but $x = 0$ is not a local maximum of f.

62. False. For example, if $f(x) = x^3$, then $f'(0) = 0$, but $f(x)$ does not have either a local maximum or a local minimum at $x = 0$.

63. True. Suppose f is increasing at some points and decreasing at others. Then $f'(x)$ takes both positive and negative values. Since $f'(x)$ is continuous, by the Intermediate Value Theorem, there would be some point where $f'(x)$ is zero, so that there would be a critical point. Since we are told there are no critical points, f must be increasing everywhere or decreasing everywhere.

64. True. Since f'' changes sign at the inflection point $x = p$, by the Intermediate Value Theorem, $f''(p) = 0$.

65. False. For example, if $f(x) = x^3$, then $f'(0) = 0$, so $x = 0$ is a critical point, but $x = 0$ is neither a local maximum nor a local minimum.

66. True. A cubic polynomial is a function of the form $f(x) = Ax^3 + Bx^2 + Cx + D$ with $A \neq 0$. We have $f''(x) = 6Ax + 2B$. There is an inflection point where $f''(x) = 0$, at $x = -B/(3A)$.

67. $f(x) = x^2 + 1$ is positive for all x and concave up.

68. This is impossible. If $f(a) > 0$, then the downward concavity forces the graph of f to cross the x-axis to the right or left of $x = a$, which means $f(x)$ cannot be positive for all values of x. More precisely, suppose that $f(x)$ is positive for all x and f is concave down. Thus there must be some value $x = a$ where $f(a) > 0$ and $f'(a)$ is not zero, since a constant function is not concave down. The tangent line at $x = a$ has nonzero slope and hence must cross the x-axis somewhere to the right or left of $x = a$. Since the graph of f must lie below this tangent line, it must also cross the x-axis, contradicting the assumption that $f(x)$ is positive for all x.

69. $f(x) = -x^2 - 1$ is negative for all x and concave down.

70. This is impossible. If $f(a) < 0$, then the upward concavity forces the graph of f to cross the x-axis to the right or left of $x = a$, which means $f(x)$ cannot be negative for all values of x. More precisely, suppose that $f(x)$ is negative for all x and f is concave up. Thus there must be some value $x = a$ where $f(a) < 0$ and $f'(a)$ is not zero, since a constant function is not concave up. The tangent line at $x = a$ has nonzero slope and hence must cross the x-axis somewhere to the right or left of $x = a$. Since the graph of f must lie above this tangent line, it must also cross the x-axis, contradicting the assumption that $f(x)$ is negative for all x.

71. This is impossible. Since f'' exists, so must f', which means that f is differentiable and hence continuous. If $f(x)$ were positive for some values of x and negative for other values, then by the Intermediate Value Theorem, $f(x)$ would have to be zero somewhere, but this is impossible since $f(x)f''(x) < 0$ for all x. Thus either $f(x) > 0$ for all values of x, in which case $f''(x) < 0$ for all values of x, that is f is concave down. But this is impossible by Problem 68. Or else $f(x) < 0$ for all x, in which case $f''(x) > 0$ for all x, that is f is concave up. But this is impossible by Problem 70.

72. This is impossible. Since f''' exists, f'' must be continuous. By the Intermediate Value Theorem, $f''(x)$ cannot change sign, since $f''(x)$ cannot be zero. In the same way, we can show that $f'(x)$ and $f(x)$ cannot change sign. Since the product of these three with $f'''(x)$ cannot change sign, $f'''(x)$ cannot change sign. Thus $f(x)f''(x)$ and $f'(x)f'''(x)$ cannot change sign. Since their product is negative for all x, one or the other must be negative for all x. By Problem 71, this is impossible.

73. (a) and (c). a is a critical point and $f(a)$ is a local minimum.

Solutions for Section 4.2

Exercises

1. See Figure 4.25.

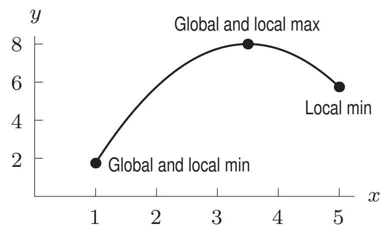


Figure 4.25

2. The global maximum is achieved at the two local maxima, which are at the same height. See Figure 4.26.

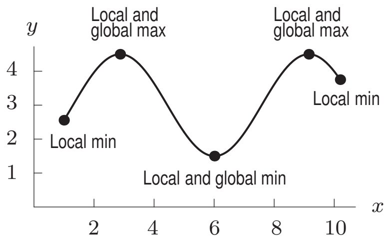


Figure 4.26

3. (a) We see in Figure 4.27 that the maximum occurs at about $x = 4$ and the maximum value of y is about $y = 60$.
(b) The maximum value occurs at a critical point, so we find all critical points of y:

$$\begin{aligned}\frac{dy}{dx} &= 12x - 3x^2 = 0\\ 3x(4-x) &= 0\\ x &= 0, x = 4.\end{aligned}$$

Since y is a cubic polynomial with negative leading coefficient, the critical point $x = 0$ gives a local minimum and the critical point $x = 4$ gives a local maximum. Since we require $x > 0$, the maximum value of the function occurs at $x = 4$. We find the maximum value of y by substituting $x = 4$:

$$\text{Maximum value of } y = 25 + 6 \cdot 4^2 - 4^3 = 57.$$

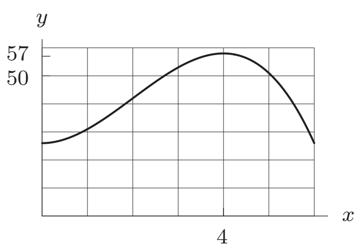


Figure 4.27

4. Since $f(x) = x^3 - 3x^2 + 20$ is continuous and the interval $-1 \leq x \leq 3$ is closed, there must be a global maximum and minimum. The candidates are critical points in the interval and endpoints. Since there are no points where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find all the critical points:

$$f'(x) = 3x^2 - 6x = 3x(x-2) = 0,$$

so $x = 0$ and $x = 2$ are the critical points; both are in the interval. We then compare the values of f at the critical points and the endpoints:

$$f(-1) = 16, \quad f(0) = 20, \quad f(2) = 16, \quad f(3) = 20.$$

Thus the global maximum is 20 at $x = 0$ and $x = 3$, and the global minimum is 16 at $x = -1$ and $x = 2$.

5. Since $f(x) = x^4 - 8x^2$ is continuous and the interval $-3 \leq x \leq 1$ is closed, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. Since there are no points where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find all the critical points:

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 0,$$

so $x = -2, 0, 2$ are the critical points; only $x = -2, 0$ are in the interval. We then compare the values of f at the critical points and the endpoints:

$$f(-3) = 9, \quad f(-2) = -16, \quad f(0) = 0, \quad f(1) = -7.$$

Thus the global maximum is 9 at $x = -3$ and the global minimum is -16 at $x = -2$.

6. Since $f(x) = xe^{-x^2/2}$ is continuous and the interval $-2 \leq x \leq 2$ is closed, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. Since there are no points where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find all the critical points:

$$f'(x) = e^{-x^2/2} + xe^{-x^2/2}(-x) = e^{-x^2/2}(1 - x^2).$$

Since $e^{-x^2/2} \neq 0$, the critical points are $x = -1, 1$; both are in the interval. We then compare the values of f at the critical points and the endpoints:

$$f(2) = 2e^{-2} = 0.271, \quad f(1) = e^{-1/2} = 0.607, \quad f(-1) = -f(1) = -0.607 \quad f(-2) = -f(2) = -0.271.$$

Thus, the global maximum is 0.607 at $x = 1$ and the global minimum is -0.607 at $x = -1$

7. Since $f(x) = 3x^{1/3} - x$ is continuous and the interval $-1 \leq x \leq 8$ is closed, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. One critical point is $x = 0$, since f' is undefined there. We solve $f'(x) = 0$ to find the other critical points:

$$f'(x) = x^{-2/3} - 1 = 0,$$

so $x = -1$ and $x = 1$. Thus the critical points are $x = -1, 0, 1$; all are in the interval. We then compare the values of f at the critical points and the endpoints:

$$f(-1) = -2, \quad f(0) = 0, \quad f(1) = 2, \quad f(8) = -2.$$

Thus, the global maximum is 2 at $x = 1$, and the global minimum is -2 at $x = -1$ and $x = 8$.

8. Since $f(x) = x - 2\ln(x+2)$ is continuous on the closed interval $0 \leq x \leq 2$, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. Since there are no points in the interval where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find the critical points:

$$f'(x) = 1 - \frac{2}{x+1} = 0,$$

so $x + 1 - 2 = 0$. Thus the only critical point is $x = 1$, which is in the interval. We then compare the values of f at the critical point and the endpoints:

$$f(0) = 0, \quad f(1) = 1 - 2\ln(2) = -0.386, \quad f(2) = 2 - 2\ln(3) = -0.197.$$

Thus, the global maximum is 0 at $x = 0$, and the global minimum is -0.386 at $x = 1$.

9. Since $f(x) = x^2 - 2|x|$ is continuous and the interval $-3 \le x \le 4$ is closed, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. The derivative f' is not defined at $x = 0$. To find the other critical points we solve $f'(x) = 0$. For $x > 0$ we have

$$f(x) = x^2 - 2x, \text{ so } f'(x) = 2x - 2 = 0.$$

Thus, $x = 1$ is the only critical point for $0 < x < 4$. For $x < 0$, we have:

$$f(x) = x^2 + 2x, \text{ so } f'(x) = 2x + 2 = 0.$$

Thus $x = -1$ is the only critical point for $-3 < x < 0$. We then compare the values of f at the critical points and the endpoints:

$$f(-3) = 3, \quad f(-1) = -1, \quad f(0) = 0, \quad f(1) = -1. \quad f(4) = 8.$$

Thus the global maximum is 8 at $x = 4$ and the global minimum is -1 at $x = -1$ and $x = 1$.

10. Since the denominator is never 0, we have that $f(x) = (x+1)/(x^2+3)$ is continuous. As the interval $-1 \le x \le 2$ is closed, there must be a global maximum and minimum. The candidates are critical points in the interval and endpoints. Since there are no points where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find all the critical points:

$$f'(x) = \frac{(x^2+3) - (x+1)(2x)}{(x^2+3)^2} = \frac{-x^2 - 2x + 3}{(x^2+3)^2} = 0.$$

Thus $-x^2 - 2x + 3 = -(x+3)(x-1) = 0$, so $x = -3$ and $x = 1$ are the critical points; only $x = 1$ is in the interval. We then compare the values of f at the critical points and the endpoints:

$$f(-1) = 0, \quad f(1) = \frac{1}{2}, \quad f(2) = \frac{3}{7}.$$

Thus the global maximum is $1/2$ at $x = 1$, and the global minimum is 0 at $x = -1$.

11. (a) We have $f'(x) = 10x^9 - 10 = 10(x^9 - 1)$. This is zero when $x = 1$, so $x = 1$ is a critical point of f. For values of x less than 1, x^9 is less than 1, and thus $f'(x)$ is negative when $x < 1$. Similarly, $f'(x)$ is positive for $x > 1$. Thus $f(1) = -9$ is a local minimum.

We also consider the endpoints $f(0) = 0$ and $f(2) = 1004$. Since $f'(0) < 0$ and $f'(2) > 0$, we see $x = 0$ and $x = 2$ are local maxima.

(b) Comparing values of f shows that the global minimum is at $x = 1$, and the global maximum is at $x = 2$.

12. (a) $f'(x) = 1 - 1/x$. This is zero only when $x = 1$. Now $f'(x)$ is positive when $1 < x \le 2$, and negative when $0.1 < x < 1$. Thus $f(1) = 1$ is a local minimum. The endpoints $f(0.1) \approx 2.4026$ and $f(2) \approx 1.3069$ are local maxima.

(b) Comparing values of f shows that $x = 0.1$ gives the global maximum and $x = 1$ gives the global minimum.

13. (a) Differentiating gives

$$f(x) = \sin^2 x - \cos x \quad \text{for } 0 \le x \le \pi$$
$$f'(x) = 2\sin x \cos x + \sin x = (\sin x)(2\cos x + 1),$$

so $f'(x) = 0$ when $\sin x = 0$ or when $2\cos x + 1 = 0$. Now, $\sin x = 0$ when $x = 0$ or when $x = \pi$. On the other hand, $2\cos x + 1 = 0$ when $\cos x = -1/2$, which happens when $x = 2\pi/3$. So the critical points are $x = 0$, $x = 2\pi/3$, and $x = \pi$.

Note that $\sin x > 0$ for $0 < x < \pi$. Also, $2\cos x + 1 < 0$ if $2\pi/3 < x \le \pi$ and $2\cos x + 1 > 0$ if $0 < x < 2\pi/3$. Therefore,

$$f'(x) < 0 \quad \text{for} \quad \frac{2\pi}{3} < x < \pi$$
$$f'(x) > 0 \quad \text{for} \quad 0 < x < \frac{2\pi}{3}.$$

Thus f has a local maximum at $x = 2\pi/3$ and local minima at $x = 0$ and $x = \pi$.

(b) We have

$$f(0) = [\sin(0)]^2 - \cos(0) = -1$$
$$f\left(\frac{2\pi}{3}\right) = \left[\sin\left(\frac{2\pi}{3}\right)\right]^2 - \cos\frac{2\pi}{3} = 1.25$$
$$f(\pi) = [\sin(\pi)]^2 - \cos(\pi) = 1.$$

Thus, the global maximum is at $x = 2\pi/3$, and the global minimum is at $x = 0$.

14. This is a parabola opening downward. We find the critical points by setting $g'(x) = 0$:

$$g'(x) = 4 - 2x = 0$$
$$x = 2.$$

Since $g'(x) > 0$ for $x < 2$ and $g'(x) < 0$ for $x > 2$, the critical point at $x = 2$ is a local maximum.

As $x \to \pm\infty$, the value of $g(x) \to -\infty$. Thus, the local maximum at $x = 2$ is a global maximum of $g(2) = 4 \cdot 2 - 2^2 - 5 = -1$. There is no global minimum. See Figure 4.28.

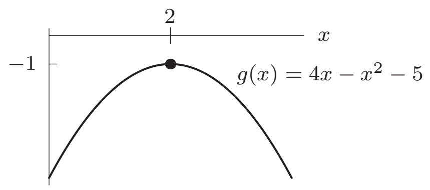


Figure 4.28

15. Differentiating gives

$$f'(x) = 1 - \frac{1}{x^2},$$

so the critical points satisfy

$$1 - \frac{1}{x^2} = 0$$
$$x^2 = 1$$
$$x = 1 \quad \text{(We want } x > 0\text{).}$$

Since f' is negative for $0 < x < 1$ and f' is positive for $x > 1$, there is a local minimum at $x = 1$.

Since $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$, the local minimum at $x = 1$ is a global minimum; there is no global maximum. See Figure 4.29. The the global minimum is $f(1) = 2$.

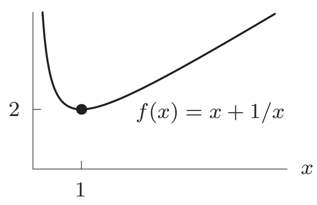


Figure 4.29

16. Differentiating using the product rule gives

$$g'(t) = 1 \cdot e^{-t} - te^{-t} = (1 - t)e^{-t},$$

so the critical point is $t = 1$.

Since $g'(t) > 0$ for $0 < t < 1$ and $g'(t) < 0$ for $t > 1$, the critical point is a local maximum.

As $t \to \infty$, the value of $g(t) \to 0$, and as $t \to 0^+$, the value of $g(t) \to 0$. Thus, the local maximum at $x = 1$ is a global maximum of $g(1) = 1e^{-1} = 1/e$. In addition, the value of $g(t)$ is positive for all $t > 0$; it tends to 0 but never reaches 0. Thus, there is no global minimum. See Figure 4.30.

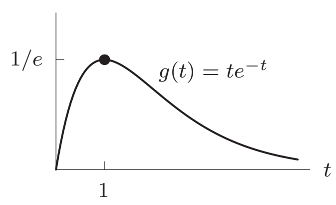


Figure 4.30

17. Differentiating gives

$$f'(x) = 1 - \frac{1}{x},$$

so the critical points satisfy

$$\begin{aligned} 1 - \frac{1}{x} &= 0 \\ \frac{1}{x} &= 1 \\ x &= 1. \end{aligned}$$

Since f' is negative for $0 < x < 1$ and f' is positive for $x > 1$, there is a local minimum at $x = 1$.

Since $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$, the local minimum at $x = 1$ is a global minimum; there is no global maximum. See Figure 4.31. Thus, the global minimum is $f(1) = 1$.

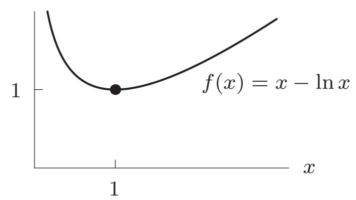


Figure 4.31

18. Differentiating using the quotient rule gives

$$f'(t) = \frac{1(1+t^2) - t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}.$$

The critical points are the solutions to

$$\begin{aligned} \frac{1-t^2}{(1+t^2)^2} &= 0 \\ t^2 &= 1 \\ t &= \pm 1. \end{aligned}$$

Since $f'(t) > 0$ for $-1 < t < 1$ and $f'(t) < 0$ otherwise, there is a local minimum at $t = -1$ and a local maximum at $t = 1$.

As $t \to \pm\infty$, we have $f(t) \to 0$. Thus, the local maximum at $t = 1$ is a global maximum of $f(1) = 1/2$, and the local minimum at $t = -1$ is a global minimum of $f(-1) = -1/2$. See Figure 4.32.

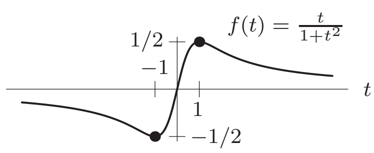


Figure 4.32

19. Differentiating using the product rule gives

$$\begin{aligned} f'(t) = 2\sin t\cos t \cdot \cos t - (\sin^2 t + 2)\sin t &= 0 \\ \sin t(2\cos^2 t - \sin^2 t - 2) &= 0 \\ \sin t(2(1-\sin^2 t) - \sin^2 t - 2) &= 0 \\ \sin t(-3\sin^2 t) = -3\sin^3 t &= 0. \end{aligned}$$

Thus, the critical points are where $\sin t = 0$, so

$$t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Since $f'(t) = -3\sin^3 t$ is negative for $-\pi < t < 0$, positive for $0 < t < \pi$, negative for $\pi < t < 2\pi$, and so on, we find that $t = 0, \pm 2\pi, \ldots$ give local minima, while $t = \pm\pi, \pm 3\pi, \ldots$ give local maxima. Evaluating gives

$$f(0) = f(\pm 2\pi) = (0+2)1 = 2$$
$$f(\pm\pi) = f(\pm 3\pi) = (0+2)(-1) = -2.$$

Thus, the global maximum of $f(t)$ is 2, occurring at $t = 0, \pm 2\pi, \ldots$, and the global minimum of $f(t)$ is -2, occurring at $t = \pm\pi, \pm 3\pi, \ldots$. See Figure 4.33.

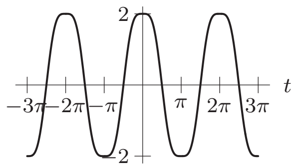


Figure 4.33

20. Let $y = x^3 - 4x^2 + 4x$. To locate the critical points, we solve $y' = 0$. Since $y' = 3x^2 - 8x + 4 = (3x-2)(x-2)$, the critical points are $x = 2/3$ and $x = 2$. To find the global minimum and maximum on $0 \le x \le 4$, we check the critical points and the endpoints: $y(0) = 0$; $y(2/3) = 32/27$; $y(2) = 0$; $y(4) = 16$. Thus, the global minimum is at $x = 0$ and $x = 2$, the global maximum is at $x = 4$, and $0 \le y \le 16$.

21. Let $y = e^{-x^2}$. Since $y' = -2xe^{-x^2}$, y is increasing for $x < 0$ and decreasing for $x > 0$. Hence $y = e^0 = 1$ is a global maximum.

When $x = \pm 0.3$, $y = e^{-0.09} \approx 0.9139$, which is a global minimum on the given interval. Thus $e^{-0.09} \le y \le 1$ for $|x| \le 0.3$.

22. Examination of the graph suggests that $0 \le x^3e^{-x} \le 2$. The lower bound of 0 is the best possible lower bound since

$$f(0) = (0)^3e^{-0} = 0.$$

To find the best possible upper bound, we find the critical points. Differentiating, using the product rule, yields

$$f'(x) = 3x^2e^{-x} - x^3e^{-x}$$

Setting $f'(x) = 0$ and factoring gives

$$3x^2e^{-x} - x^3e^{-x} = 0$$
$$x^2e^{-x}(3-x) = 0$$

So the critical points are $x = 0$ and $x = 3$. Note that $f'(x) < 0$ for $x > 3$ and $f'(x) > 0$ for $x < 3$, so $f(x)$ has a local maximum at $x = 3$. Examination of the graph tells us that this is the global maximum. So $0 \le x^3e^{-x} \le f(3)$.

$$f(3) = 3^3e^{-3} \approx 1.34425$$

So $0 \le x^3e^{-x} \le 3^3e^{-3} \approx 1.34425$ are the best possible bounds for the function.

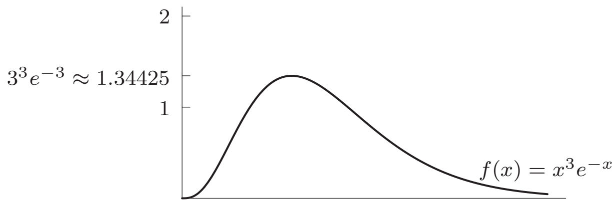


Figure 4.34

23. The graph of $y = x + \sin x$ in Figure 4.35 suggests that the function is nondecreasing over the entire interval. You can confirm this by looking at the derivative:

$$y' = 1 + \cos x$$

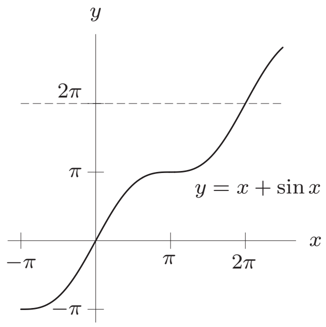


Figure 4.35: Graph of $y = x + \sin x$

Since $\cos x \geq -1$, we have $y' \geq 0$ everywhere, so y never decreases. This means that a lower bound for y is 0 (its value at the left endpoint of the interval) and an upper bound is 2π (its value at the right endpoint). That is, if $0 \leq x \leq 2\pi$:

$$0 \leq y \leq 2\pi.$$

These are the best bounds for y over the interval.

24. Let $y = \ln(1 + x)$. Since $y' = 1/(1 + x)$, y is increasing for all $x \geq 0$. The lower bound is at $x = 0$, so, $\ln(1) = 0 \leq y$. There is no upper bound.

25. Let $y = \ln(1 + x^2)$. Then $y' = 2x/(1 + x^2)$. Since the denominator is always positive, the sign of y' is determined by the numerator $2x$. Thus $y' > 0$ when $x > 0$, and $y' < 0$ when $x < 0$, and we have a local (and global) minimum for y at $x = 0$. Since $y(-1) = \ln 2$ and $y(2) = \ln 5$, the global maximum is at $x = 2$. Thus $0 \leq y \leq \ln 5$, or (in decimals) $0 \leq y < 1.61$. (Note that our upper bound has been rounded *up* from 1.6094.)

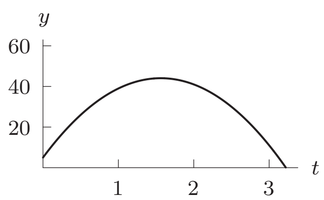
Problems

26. We want to maximize the height, y, of the grapefruit above the ground, as shown in the figure below. Using the derivative we can find exactly when the grapefruit is at the highest point. We can think of this in two ways. By common sense, at the peak of the grapefruit's flight, the velocity, dy/dt, must be zero. Alternately, we are looking for a global maximum of y, so we look for critical points where $dy/dt = 0$. We have

$$\frac{dy}{dt} = -32t + 50 = 0 \qquad \text{and so} \qquad t = \frac{-50}{-32} \approx 1.56 \text{ sec.}$$

Thus, we have the time at which the height is a maximum; the maximum value of y is then

$$y \approx -16(1.56)^2 + 50(1.56) + 5 = 44.1 \text{ feet.}$$



27. We find all critical points:

$$\frac{dy}{dx} = 2ax + b = 0$$
$$x = -\frac{b}{2a}.$$

Since y is a quadratic polynomial, its graph is a parabola which opens up if $a > 0$ and down if $a < 0$. The critical value is a maximum if $a < 0$ and a minimum if $a > 0$.

28. To find the value of w that minimizes S, we set dS/dw equal to zero and solve for w. To find dS/dw, we first solve for S:

$$S - 5pw = 3qw^2 - 6pq$$
$$S = 5pw + 3qw^2 - 6pq.$$

We now find the critical points:

$$\frac{dS}{dw} = 5p + 6qw = 0$$
$$w = -\frac{5p}{6q}.$$

There is one critical point. Since S is a quadratic function of w with a positive leading coefficient, the function has a minimum at this critical point.

29. To find the value of t that maximizes y, we set dy/dt equal to zero and solve for t. We use the product rule to find dy/dt.

$$\frac{dy}{dt} = at^2e^{-bt}(-b) + 2ate^{-bt} = 0$$
$$\frac{dy}{dt} = ate^{-bt}(-bt+2) = 0$$
$$t = 0, t = \frac{2}{b}.$$

Since a, t are nonnegative, ate^{-bt} is nonnegative. Thus dy/dt is positive for $-bt+2 > 0$ and negative for $-bt+2 < 0$. Also, $t \geq 0$, so

dy/dt $= 0$ for $t = 0, 2/b$
dy/dt is positive for $0 < t < 2/b$
dy/dt is negative for $t > 2/b$.

Thus, y has a local maximum, and also a global maximum, at $t = 2/b$. The graph of $y = at^2e^{-bt}$, using $a = 1$ and $b = 1$, is shown in Figure 4.36. We see that the function has a minimum at the critical point $t = 0$ and a maximum at $t = 2/b$.

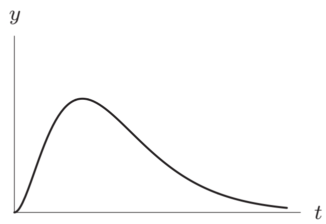


Figure 4.36

30. (a) We have

$$T(D) = \left(\frac{C}{2} - \frac{D}{3}\right)D^2 = \frac{CD^2}{2} - \frac{D^3}{3},$$

and

$$\frac{dT}{dD} = CD - D^2 = D(C - D).$$

Since, by this formula, dT/dD is zero when $D = 0$ or $D = C$, negative when $D > C$, and positive when $D < C$, we have (by the first derivative test) that the temperature change is maximized when $D = C$.

(b) The sensitivity is $dT/dD = CD - D^2$; its derivative is $d^2T/dD^2 = C - 2D$, which is zero if $D = C/2$, negative if $D > C/2$, and positive if $D < C/2$. Thus by the first derivative test the sensitivity is maximized at $D = C/2$.

31. (a) Since a/q decreases with q, this term represents the ordering cost. Since bq increases with q, this term represents the storage cost.

(b) At the minimum,

$$\frac{dC}{dq} = \frac{-a}{q^2} + b = 0$$

giving

$$q^2 = \frac{a}{b} \quad \text{so} \quad q = \sqrt{\frac{a}{b}}.$$

Since

$$\frac{d^2C}{dq^2} = \frac{2a}{q^3} > 0 \quad \text{for} \quad q > 0,$$

we know that $q = \sqrt{a/b}$ gives a local minimum. Since $q = \sqrt{a/b}$ is the only critical point, this must be the global minimum.

32. We look for critical points of M:

$$\frac{dM}{dx} = \frac{1}{2}wL - wx.$$

Now $dM/dx = 0$ when $x = L/2$. At this point $d^2M/dx^2 = -w$ so this point is a local maximum. The graph of $M(x)$ is a parabola opening downward, so the local maximum is also the global maximum.

33. (a) If we expect the rate to be nonnegative, then we must have $0 \le y \le a$. See Figure 4.37.

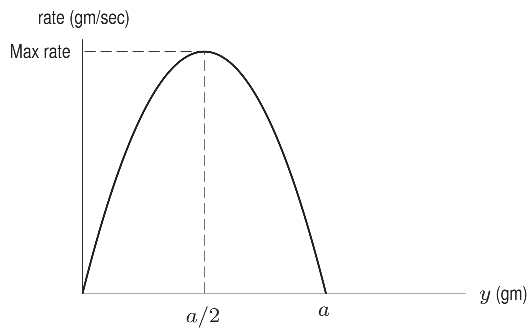


Figure 4.37

(b) The maximum value of the rate occurs at $y = a/2$, as can be seen from Figure 4.37, or by setting

$$\frac{d}{dy}(\text{rate}) = 0$$

$$\frac{d}{dy}(\text{rate}) = \frac{d}{dy}(kay - ky^2) = ka - 2ky = 0$$

$$y = \frac{a}{2}.$$

From the graph, we see that $y = a/2$ gives the global maximum.

34. We set $dU/dx = 0$ to find the critical points:

$$b\left(\frac{-2a^2}{x^3} + \frac{a}{x^2}\right) = 0$$

$$-2a^2 + ax = 0$$

$$x = 2a.$$

The only critical point is at $x = 2a$. When $x < 2a$ we have $dU/dx < 0$, and when $x > 2a$ we have $dU/dx > 0$. The potential energy, U, is minimized at $x = 2a$.

35. We set $f'(r) = 0$ to find the critical points:

$$\begin{aligned}\frac{2A}{r^3} - \frac{3B}{r^4} &= 0\\ \frac{2Ar - 3B}{r^4} &= 0\\ 2Ar - 3B &= 0\\ r &= \frac{3B}{2A}.\end{aligned}$$

The only critical point is at $r = 3B/(2A)$. If $r > 3B/(2A)$, we have $f' > 0$ and if $r < 3B/(2A)$, we have $f' < 0$. Thus, the force between the atoms is minimized at $r = 3B/(2A)$.

36. (a) To show that R is an increasing function of r_1, we show that $dR/dr_1 > 0$ for all values of r_1. We first solve for R:

$$\begin{aligned}\frac{1}{R} &= \frac{1}{r_1} + \frac{1}{r_2}\\ \frac{1}{R} &= \frac{r_2 + r_1}{r_1 r_2}\\ R &= \frac{r_1 r_2}{r_2 + r_1}.\end{aligned}$$

We use the quotient rule (and remember that r_2 is a constant) to find dR/dr_1:

$$\frac{dR}{dr_1} = \frac{(r_2 + r_1)(r_2) - (r_1 r_2)(1)}{(r_2 + r_1)^2} = \frac{(r_2)^2}{(r_2 + r_1)^2}.$$

Since dR/dr_1 is the square of a number, we have $dR/dr_1 > 0$ for all values of r_1, and thus R is increasing for all r_1.

(b) Since R is increasing on any interval $a \le r_1 \le b$, the maximum value of R occurs at the right endpoint $r_1 = b$.

37. (a) We want the maximum value of I. Using the properties of logarithms, we rewrite the expression for I as

$$I = k(\ln S - \ln S_0) - S + S_0 + I_0.$$

Since k and S_0 are constant, differentiating with respect to S gives

$$\frac{dI}{dS} = \frac{k}{S} - 1.$$

Thus, the critical point is at $S = k$. Since dI/dS is positive for $S < k$ and dI/dS is negative for $S > k$, we see that $S = k$ is a local maximum.

We only consider positive values of S. Since $S = k$ is the only critical point, it gives the global maximum value for I, which is

$$I = k(\ln k - \ln S_0) - k + S_0 + I_0.$$

(b) Since both k and S_0 are in the expression for the maximum value of I, both the particular disease and how it starts influence the maximum.

38. Suppose the points are given by x and $-x$, where $x \ge 0$. The function is odd, since

$$y = \frac{(-x)^3}{1 + (-x)^4} = -\frac{x^3}{1 + x^4},$$

so the corresponding y-coordinates are also opposite. See Figure 4.38. For $x > 0$, we have

$$m = \frac{\frac{x^3}{1+x^4} - \left(-\frac{x^3}{1+x^4}\right)}{x - (-x)} = \frac{1}{2x} \cdot \frac{2x^3}{1 + x^4} = \frac{x^2}{1 + x^4}.$$

For the maximum slope,

$$\begin{aligned}\frac{dm}{dx} = \frac{2x}{1 + x^4} - \frac{x^2(4x^3)}{(1 + x^4)^2} &= 0\\ \frac{2x(1 + x^4) - 4x^5}{(1 + x^4)^2} &= 0\end{aligned}$$

$$\begin{aligned}\frac{2x(1-x^4)}{(1+x^4)^2} &= 0\\ x\left(1-x^4\right) &= 0\\ x &= 0, \pm 1.\end{aligned}$$

For $x > 0$, there is one critical point, $x = 1$. Since m tends to 0 when $x \to 0$ and when $x \to \infty$, the critical point $x = 1$ gives the maximum slope. Thus, the maximum slope occurs when the line has endpoints

$$\left(-1, -\frac{1}{2}\right) \text{ and } \left(1, \frac{1}{2}\right).$$

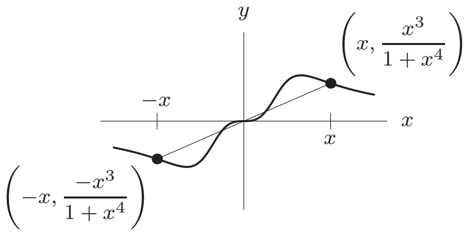


Figure 4.38

39. Since the function is positive, the graph lies above the x-axis. If there is a global maximum at $x = 3$, $t'(x)$ must be positive, then negative. Since $t'(x)$ and $t''(x)$ have the same sign for $x < 3$, they must both be positive, and thus the graph must be increasing and concave up. Since $t'(x)$ and $t''(x)$ have opposite signs for $x > 3$ and $t'(x)$ is negative, $t''(x)$ must again be positive and the graph must be decreasing and concave up. A possible sketch of $y = t(x)$ is shown in Figure 4.39.

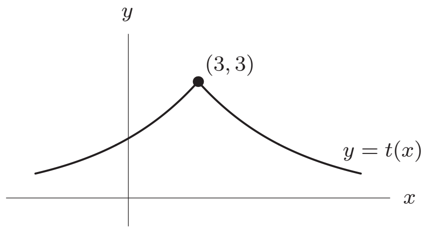


Figure 4.39

40. One possible graph of g is in Figure 4.40.

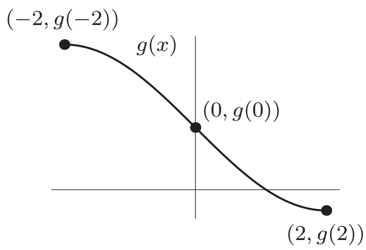


Figure 4.40

(a) From left to right, the graph of $g(x)$ starts "flat", decreases slowly at first then more rapidly, most rapidly at $x = 0$. The graph then continues to decrease but less and less rapidly until flat again at $x = 2$. The graph should exhibit symmetry about the point $(0, g(0))$.

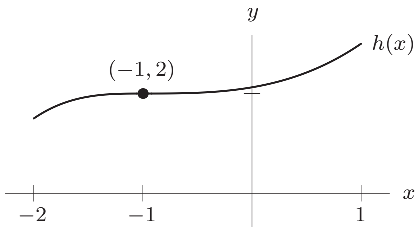
(b) The graph has an inflection point at $(0, g(0))$ where the slope changes from negative and decreasing to negative and increasing.
(c) The function has a global maximum at $x = -2$ and a global minimum at $x = 2$.
(d) Since the function is decreasing over the interval $-2 \le x \le 2$

$$g(-2) = 5 > g(0) > g(2).$$

Since the function appears symmetric about $(0, g(0))$, we have

$$g(-2) - g(0) = g(0) - g(2).$$

41. (a) We know that $h''(x) < 0$ for $-2 \le x < -1$, $h''(-1) = 0$, and $h''(x) > 0$ for $x > -1$. Thus, $h'(x)$ decreases to its minimum value at $x = -1$, which we know to be zero, and then increases; it is never negative.
(b) Since $h'(x)$ is non-negative for $-2 \le x \le 1$, we know that $h(x)$ is never decreasing on $[-2, 1]$. So a global maximum must occur at the right hand endpoint of the interval.
(c) The graph below shows a function that is increasing on the interval $-2 \le x \le 1$ with a horizontal tangent and an inflection point at $(-1, 2)$.



42. Suppose f has critical points $x = a$ and $x = b$. Suppose $a < b$. By the Extreme Value Theorem, we know that the derivative function, $f'(x)$, has global extrema on $[a, b]$. If both the maximum and minimum of $f'(x)$ occur at the endpoints of $[a, b]$, then $f'(a) = 0 = f'(b)$, so $f'(x) = 0$ for all x in $[a, b]$. In this case, f would have more than two critical points. Since f has only two critical points, there is a local maximum or minimum of f' inside the interval $[a, b]$.

43. To find the critical points, set $dD/dx = 0$:

$$\frac{dD}{dx} = 2(x - a_1) + 2(x - a_2) + 2(x - a_3) + \cdots + 2(x - a_n) = 0.$$

Dividing by 2 and solving for x gives

$$x + x + x + \cdots + x = a_1 + a_2 + a_3 + \cdots + a_n.$$

Since there are n terms on the left,

$$nx = a_1 + a_2 + a_3 + \cdots + a_n$$
$$x = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

The expression on the right is the average of $a_1, a_2, a_3, \cdots, a_n$.
Since D is a quadratic with positive leading coefficient, this critical point is a minimum.

44. (a) If both the global minimum and the global maximum are at the endpoints, then $f(x) = 0$ everywhere in $[a, b]$, since $f(a) = f(b) = 0$. In that case $f'(x) = 0$ everywhere as well, so any point in (a, b) will do for c.
(b) Suppose that either the global maximum or the global minimum occurs at an interior point of the interval. Let c be that point. Then c must be a local extremum of f, so, by the theorem concerning local extrema on page 188, we have $f'(c) = 0$, as required.

45. (a) The equation of the secant line between $x = a$ and $x = b$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

and

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a),$$

so $g(x)$ is the difference, or distance, between the graph of $f(x)$ and the secant line. See Figure 4.41.

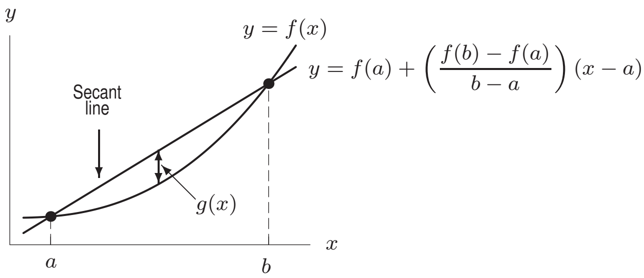


Figure 4.41: Value of $g(x)$ is the difference between the secant line and the graph of $f(x)$

(b) Figure 4.41 shows that $g(a) = g(b) = 0$. You can also easily check this from the formula for $g(x)$. By Rolle's Theorem, there must be a point c in (a, b) where $g'(c) = 0$.

(c) Differentiating the formula for $g(x)$, we have

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

So from $g'(c) = 0$, we get

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

as required.

Strengthen Your Understanding

46. The function f has a critical point at $x = 1$ where $f'' < 0$, so f has a local maximum at $(1, 0)$. However, checking the endpoints we find that f takes its largest value at the endpoint $x = 3$, where $f(3) = 4$.

47. Consider the interval $1 \le x \le 2$. Since f is increasing on this interval, its global minimum occurs at the left end-point of the interval, $x = 1$. If $x = 0$ is inside the interval $a \le x \le b$, then the global minimum of f occurs at $x = 0$. Otherwise, the global minimum of f will occur at an endpoint.

48. The function has a vertical asymptote at $x = 1$ where it is not defined. There is no global minimum nor maximum and therefore no upper or lower bound for $f(x)$.

49. One possible answer is $f(x) = 1 - x$ for which $f'(x) = -1$ so there are no critical points. Also $f(0) = 1$ is a global maximum and $f(1) = 0$ is a global minimum. There are many other possible answers.

50. The function must be a constant. One possible answer is $f(x) = 1$. Any constant function's global maximum is equal to its global minimum.

51. Solving $x^2 = 5$ and $x^2 = 2$, we see that one such interval is $\sqrt{2} \le x \le \sqrt{5}$. A second possible interval is $-\sqrt{5} \le x \le -\sqrt{2}$.

52. The function $f(x) = \sin x$ is differentiable and has 1 as its global maximum and -1 as its global minimum on any interval containing $-\pi/2 \le x \le \pi/2$ such as the interval $-4 \le x \le 4$. The function $f(x) = \cos x$ is another such function. There are many other possibilities.

53. True, $f(x) \le 4$ on the interval $(0, 2)$.

54. False. The values of $f(x)$ get arbitrarily close to 4, but $f(x) < 4$ for all x in the interval $(0, 2)$.

55. True. The values of $f(x)$ get arbitrarily close to 0, but $f(x) > 0$ for all x in the interval $(0, 2)$.

56. False. On the interval $(-1, 1)$, the global minimum is 0.

57. True, by the Extreme Value Theorem, Theorem 4.2.

58. (a) This is not implied; just because a function satisfies the conclusions of the statement, that does not mean it has to satisfy the conditions.

(b) This is not implied; if a function fails to satisfy the conditions of the statement, then the statement does not tell us anything about it.

(c) This is implied; if a function fails to satisfy the conclusions of the statement, then it could not satisfy the conditions of the statement, because if it did the statement would imply it also satisfied the conclusions.

59. False, since $f(x) = 1/x$ takes on arbitrarily large values as $x \to 0^+$. The Extreme Value Theorem requires the interval to be closed as well as bounded.

60. False. The Extreme Value Theorem says that continuous functions have global maxima and minima on every closed, bounded interval. It does not say that only continuous functions have such maxima and minima.

61. True. If the maximum is not at an endpoint, then it must be at critical point of f. But $x = 0$ is the only critical point of $f(x) = x^2$ and it gives a minimum, not a maximum.

62. True. For example, $A = 1$ and $A = 2$ are both upper bounds for $f(x) = \sin x$.

63. True. If $f'(0) > 0$, then f would be increasing at 0 and so $f(0) < f(x)$ for x just to the right of 0. Then $f(0)$ would not be a maximum for f on the interval $0 \le x \le 10$.

Solutions for Section 4.3

Exercises

1. Let the numbers be x and y. Then $x + y = 100$, so $y = 100 - x$.
Since both numbers are nonnegative, we restrict to $0 \le x \le 100$.

The product is

$$P = xy = x(100 - x) = 100x - x^2.$$

Differentiating to find the maximum,

$$\frac{dP}{dx} = 100 - 2x = 0$$
$$x = \frac{100}{2} = 50.$$

So there is a critical point at $x = 50$; the end points are at $x = 0, 100$.

Evaluating gives
At $x = 0$, we have $P = 0$.
At $x = 50$, we have $P = 50(100 - 50) = 2500$.
At $x = 100$, we have $P = 100(100 - 100) = 0$.
Thus the maximum value is 2500.

2. Let the numbers be x and y. Then $xy = 784$, so $y = 784/x$.
Since both numbers are positive, we restrict to $x > 0$.

The sum is

$$S = x + y = x + \frac{784}{x}.$$

Differentiating to find the minimum,

$$\frac{dS}{dx} = 1 - \frac{784}{x^2} = 0$$
$$x^2 = 784 \quad \text{so} \quad x = \pm\sqrt{784} = \pm 28.$$

For $x > 0$, there is only one critical point at $x = 28$. We find

$$\frac{d^2S}{dx^2} = 2\frac{784}{x^3}.$$

Since $d^2S/dx^2 > 0$ for $x > 0$, there is a local minimum at $x = 28$. The derivative dS/dx is negative for $0 < x < 28$ and dS/dx is positive for $x > 28$. Thus, $x = 28$ gives the global minimum for $x > 0$.

The minimum value of the sum is

$$S = 28 + \frac{784}{28} = 56.$$

3. Let the numbers be x, y, z and let $y = 2x$. Then

$$x + y + z = 3x + z = 36, \quad \text{so} \quad z = 36 - 3x.$$

Since all the numbers are nonnegative, we restrict to $0 \le x \le 12$.

The product is

$$P = xyz = x \cdot 2x \cdot (36 - 3x) = 72x^2 - 6x^3.$$

Differentiating to find the maximum,

$$\begin{aligned} \frac{dP}{dx} &= 144x - 18x^2 = 0 \\ -18x(x-8) &= 0 \\ x &= 0, 8. \end{aligned}$$

So there are critical points at $x = 0$ and $x = 8$; the end points are at $x = 0, 12$.

Evaluating gives:
At $x = 0$, we have $P = 0$.
At $x = 8$, we have $P = 8 \cdot 16 \cdot (36 - 3 \cdot 8) = 1536$.
At $x = 12$, we have $P = 12 \cdot 24 \cdot (36 - 3 \cdot 12) = 0$.
Thus, the maximum value of the product is 1536.

4. Let the sides be x and y cm. Then $2x + 2y = 64$, so $y = 32 - x$.
Since both sides are nonnegative, we restrict to $0 \le x \le 32$.

The area in cm^2 is

$$A = xy = x(32 - x) = 32x - x^2.$$

Differentiating to find the maximum,

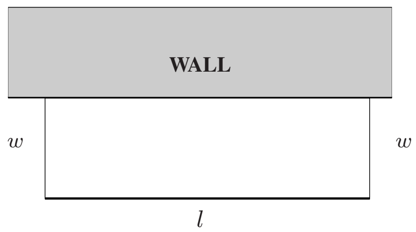
$$\begin{aligned} \frac{dA}{dx} &= 32 - 2x = 0 \\ x &= \frac{32}{2} = 16. \end{aligned}$$

So there is a critical point at $x = 16$ cm; the end points are at $x = 0$ and $x = 32$ cm.

Evaluating gives:
At $x = 0$, we have $A = 0\ \text{cm}^2$.
At $x = 16$, we have $A = 16(32 - 16) = 256\ \text{cm}^2$.
At $x = 32$, we have $A = 32(32 - 32) = 0\ \text{cm}^2$.
Thus, the maximum area occurs in a square of side 16 cm.

5. Let w and l be the width and length, respectively, of the rectangular area you wish to enclose. Then

$$\begin{aligned} w + w + l &= 100 \text{ feet} \\ l &= 100 - 2w \\ \text{Area} = w \cdot l &= w(100 - 2w) = 100w - 2w^2 \end{aligned}$$



To maximize area, we solve $A' = 0$ to find critical points. This gives $A' = 100 - 4w = 0$, so $w = 25$, $l = 50$. So the area is $25 \cdot 50 = 1250$ square feet. This is a local maximum by the second derivative test because $A'' = -4 < 0$. Since the graph of A is a parabola, the local maximum is in fact a global maximum.

6. We want to minimize the surface area S of the box, shown in Figure 4.42. The box has 6 faces: the top and bottom, each of which has area x^2 and the four sides, each of which has area xh. Thus we want to minimize

$$S = 2x^2 + 4xh.$$

The volume of the box is $8 = x^2h$, so $h = 8/x^2$. Substituting this expression in for h in the formula for S gives

$$S = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}.$$

Differentiating gives

$$\frac{dS}{dx} = 4x - \frac{32}{x^2}.$$

To minimize S we look for critical points, so we solve $0 = 4x - 32/x^2$. Multiplying by x^2 gives

$$0 = 4x^3 - 32,$$

so $x = 8^{1/3}$. Then we can find

$$h = \frac{8}{x^2} = \frac{8}{8^{2/3}} = 8^{1/3}.$$

Thus $x = h = 8^{1/3}$ cm (the box is a cube).

We can check that this critical point is a minimum of S by checking the sign of

$$\frac{d^2S}{dx^2} = 4 + \frac{64}{x^3}$$

which is positive when $x > 0$. So S is concave up at the critical point and therefore $x = 8^{1/3}$ gives a minimum value of S.

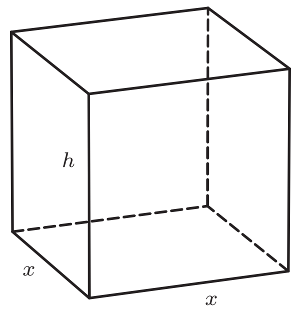


Figure 4.42

7. We want to minimize the surface area S of the box, shown in Figure 4.43. The box has 5 faces: the bottom which has area x^2 and the four sides, each of which has area xh. Thus we want to minimize

$$S = x^2 + 4xh.$$

The volume of the box is $8 = x^2h$, so $h = 8/x^2$. Substituting this expression in for h in the formula for S gives

$$S = x^2 + 4x \cdot \frac{8}{x^2} = x^2 + \frac{32}{x}.$$

Differentiating gives

$$\frac{dS}{dx} = 2x - \frac{32}{x^2}.$$

To minimize S we look for critical points, so we solve $0 = 2x - 32/x^2$. Multiplying by x^2 gives

$$0 = 2x^3 - 32,$$

so $x = 16^{1/3}$ cm. Then we can find

$$h = \frac{8}{x^2} = \frac{8}{16^{2/3}} = \frac{16}{2 \cdot 16^{2/3}} = \frac{16^{1/3}}{2}$$

cm.

We can check that this critical point is a minimum of S by checking the sign of

$$\frac{d^2S}{dx^2} = 2 + \frac{64}{x^3}$$

which is positive when $x > 0$. So S is concave up at the critical point and therefore $x = 16^{1/3}$ gives a minimum value of S.

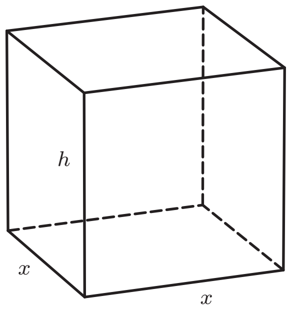


Figure 4.43

8. We want to minimize the surface area S of the cylinder, shown in Figure 4.44. The cylinder has 3 pieces: the top and bottom disks, each of which has area πr^2 and the tube, which has area $2\pi rh$. Thus we want to minimize

$$S = 2\pi r^2 + 2\pi rh.$$

The volume of the cylinder is $8 = \pi r^2 h$, so $h = 8/\pi r^2$. Substituting this expression for h in the formula for S gives

$$S = 2\pi r^2 + 2\pi r \cdot \frac{8}{\pi r^2} = 2\pi r^2 + \frac{16}{r}.$$

Differentiating gives

$$\frac{dS}{dr} = 4\pi r - \frac{16}{r^2}.$$

To minimize S we look for critical points, so we solve $0 = 4\pi r - 16/r^2$. Multiplying by r^2 gives

$$0 = 4\pi r^3 - 16,$$

so $r = (4/\pi)^{1/3}$ cm. Then we can find

$$h = \frac{8}{\pi r^2} = \frac{8}{\pi\left(\frac{4}{\pi}\right)^{2/3}} = \frac{2\left(\frac{4}{\pi}\right)}{\left(\frac{4}{\pi}\right)^{2/3}} = 2\left(\frac{4}{\pi}\right)^{1/3}.$$

We can check that this critical point is a minimum of S by checking the sign of

$$\frac{d^2S}{dr^2} = 4\pi + \frac{32}{r^3}$$

which is positive when $r > 0$. So S is concave up at the critical point and therefore $r = (4/\pi)^{1/3}$ cm is a minimum.

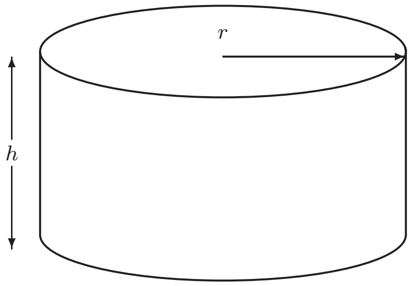


Figure 4.44

9. We want to minimize the surface area S of the cylinder, shown in Figure 4.45. The cylinder has two pieces: one disk (the base) which has area πr^2 and the tube (the sides), which has area $2\pi rh$. Thus we want to minimize

$$S = \pi r^2 + 2\pi rh.$$

The volume of the cylinder is $8 = \pi r^2 h$, so $h = 8/\pi r^2$. Substituting this expression in for h in the formula for S gives

$$S = \pi r^2 + 2\pi r \cdot \frac{8}{\pi r^2} = \pi r^2 + \frac{16}{r}.$$

Differentiating gives

$$\frac{dS}{dr} = 2\pi r - \frac{16}{r^2}.$$

To minimize S we look for critical points, so we solve $0 = 2\pi r - 16/r^2$. Multiplying by r^2 gives

$$0 = 2\pi r^3 - 16,$$

so $r = (8/\pi)^{1/3}$ cm. Then we can find

$$h = \frac{8}{\pi r^2} = \frac{8}{\pi\left(\frac{8}{\pi}\right)^{2/3}} = \frac{\frac{8}{\pi}}{\left(\frac{8}{\pi}\right)^{2/3}} = \left(\frac{8}{\pi}\right)^{1/3}.$$

We can check that this critical point is a minimum of S by checking the sign of

$$\frac{d^2S}{dr^2} = 2\pi + \frac{32}{r^3}$$

which is positive when $r > 0$. So S is concave up at the critical point and therefore $r = (8/\pi)^{1/3}$ cm is a minimum.

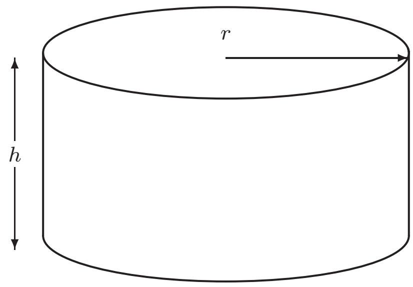


Figure 4.45

10. We want to maximize the area function

$$\begin{aligned} A(x) &= \text{Base} \times \text{Height} \\ &= xf(x) \\ &= \frac{1}{3}x^3 - 50x^2 + 1000x. \end{aligned}$$

Critical points of A are solutions of

$$A'(x) = x^2 - 100x + 1000 = 0.$$

There are two critical points, $x = 50 - 10\sqrt{15} = 11.27$ and $x = 50 + 10\sqrt{15} = 88.73$, but only the first is in the interval $0 \le x \le 20$.

The maximum occurs at the critical point in the interval or at one of the endpoints. Since

$$A(0) = 0 \qquad A(11.27) = 5396.5 \qquad A(20) = 2666.67$$

the maximum area occurs with $x = 50 - 10\sqrt{15} = 11.270$.

11. We want to maximize the area function

$$\begin{aligned} A(x) &= \frac{1}{2} \times \text{Base} \times \text{Height} \\ &= \frac{1}{2} \cdot 20 \cdot f(x) \\ &= \frac{10x^2}{3} - 500x + 10000. \end{aligned}$$

Critical points of A are solutions of

$$A'(x) = \frac{20x}{3} - 500 = 0.$$

There is one critical point, $x = 75$, but it is outside the interval $0 \le x \le 20$.

The maximum of A in the interval occurs at one of the endpoints. Since

$$A(0) = 10000 \qquad A(20) = 4000/3 = 1333.3$$

the maximum area occurs with $x = 0$.

We can also see that $x = 0$ gives te maximum area without calculus, since the base of the triangle is always 20 and the height is largest when $x = 0$.

12. The rectangle in Figure 4.46 has area, A, given by

$$A = 2xy = \frac{2x}{1+x^2} \qquad \text{for } x \ge 0.$$

At a critical point,

$$\begin{aligned} \frac{dA}{dx} = \frac{2}{1+x^2} + 2x\left(\frac{-2x}{(1+x^2)^2}\right) &= 0 \\ \frac{2(1+x^2-2x^2)}{(1+x^2)^2} &= 0 \\ 1 - x^2 &= 0 \\ x &= \pm 1. \end{aligned}$$

Since $A = 0$ for $x = 0$ and $A \to 0$ as $x \to \infty$, the critical point $x = 1$ is a local and global maximum for the area. Then $y = 1/2$, so the vertices are

$$(-1, 0)\,,\ (1, 0)\,,\ \left(1, \frac{1}{2}\right),\ \left(-1, \frac{1}{2}\right).$$

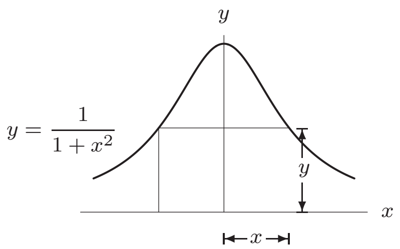


Figure 4.46

13. The triangle in Figure 4.47 has area, A, given by

$$A = \frac{1}{2}xy = \frac{x}{2}e^{-x/3}.$$

At a critical point,

$$\begin{aligned} \frac{dA}{dx} = \frac{1}{2}e^{-x/3} - \frac{x}{6}e^{-x/3} &= 0 \\ \frac{1}{6}e^{-x/3}(3 - x) &= 0 \\ x &= 3. \end{aligned}$$

Substituting the critical point and the endpoints into the formula for the area gives:

For $x = 1$, we have $A = \frac{1}{2}e^{-1/3} = 0.358$

For $x = 3$, we have $A = \frac{3}{2}e^{-1} = 0.552$

For $x = 5$, we have $A = \frac{5}{2}e^{-5/3} = 0.472$

Thus, the maximum area is 0.552 and the minimum area is 0.358.

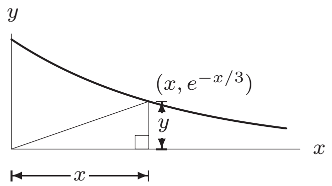


Figure 4.47

14. (a) The rectangle in Figure 4.48 has area, A, given by

$$A = xy = xe^{-2x}.$$

At a critical point, we have

$$\begin{aligned} \frac{dA}{dx} &= 1 \cdot e^{-2x} - 2xe^{-2x} = 0 \\ e^{-2x}(1-2x) &= 0 \\ x &= \frac{1}{2}. \end{aligned}$$

Since $A = 0$ when $x = 0$ and $A \to 0$ as $x \to \infty$, the critical point $x = 1/2$ is a local and global maximum. Thus the maximum area is

$$A = \frac{1}{2}e^{-2(1/2)} = \frac{1}{2e}.$$

(b) The rectangle in Figure 4.48 has perimeter, P, given by

$$P = 2x + 2y = 2x + 2e^{-2x}.$$

At a critical point, we have

$$\begin{aligned} \frac{dP}{dx} = 2 - 4e^{-2x} &= 0 \\ e^{-2x} &= \frac{1}{2} \\ -2x &= \ln\frac{1}{2} \\ x &= -\frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}\ln 2. \end{aligned}$$

To see if this critical point gives a maximum or minimum, we find

$$\frac{d^2P}{dx^2} = 8e^{-2x}.$$

Since $d^2P/dx^2 > 0$ for all x, including $x = \frac{1}{2}\ln 2$, the critical point is a local and global minimum. Thus, the minimum perimeter is

$$P = 2\left(\frac{1}{2}\ln 2\right) + 2e^{-2\left(\frac{1}{2}\ln 2\right)} = \ln 2 + 2e^{-\ln 2} = \ln 2 + 2 \cdot \frac{1}{2} = \ln 2 + 1.$$

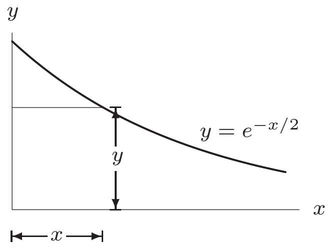


Figure 4.48

Problems

15. We take the derivative, set it equal to 0, and solve for x:

$$\begin{aligned}
\frac{dt}{dx} &= \frac{1}{6} - \frac{1}{4}\cdot\frac{1}{2}\left((2000-x)^2+600^2\right)^{-1/2}\cdot 2(2000-x) = 0\\
(2000-x) &= \frac{2}{3}\left((2000-x)^2+600^2\right)^{1/2}\\
(2000-x)^2 &= \frac{4}{9}\left((2000-x)^2+600^2\right)\\
\frac{5}{9}(2000-x)^2 &= \frac{4}{9}\cdot 600^2\\
2000-x &= \sqrt{\frac{4}{5}\cdot 600^2} = \frac{1200}{\sqrt{5}}\\
x &= 2000 - \frac{1200}{\sqrt{5}} \text{ feet.}
\end{aligned}$$

Note that $2000-(1200/\sqrt{5}) \approx 1463$ feet, as given in the example.

16. Let the sides of the rectangle have lengths a and b. We shall look for the minimum of the square s of the length of either diagonal, i.e. $s = a^2+b^2$. The area is $A = ab$, so $b = A/a$. This gives

$$s(a) = a^2 + \frac{A^2}{a^2}.$$

To find the minimum squared length we need to find the critical points of s. Differentiating s with respect to a gives

$$\frac{ds}{da} = 2a + (-2)A^2a^{-3} = 2a\left(1-\frac{A^2}{a^4}\right)$$

The derivative $ds/da = 0$ when $a = \sqrt{A}$, that is when $a = b$ and so the rectangle is a square. Because $\frac{d^2s}{da^2} = 2\left(1+\frac{3A^2}{a^4}\right) > 0$, this is a minimum.

17. From the triangle shown in Figure 4.49, we see that

$$\begin{aligned}
\left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2 &= 30^2\\
w^2+h^2 &= 4(30)^2 = 3600.
\end{aligned}$$

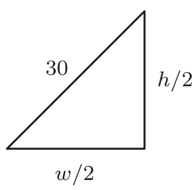


Figure 4.49

The strength, S, of the beam is given by

$$S = kwh^2,$$

for some constant k. To make S a function of only one variable, substitute for h^2, giving

$$S = kw(3600 - w^2) = k(3600w - w^3).$$

Differentiating and setting $dS/dw = 0$,

$$\frac{dS}{dw} = k(3600 - 3w^2) = 0.$$

Solving for w gives

$$w = \sqrt{1200} = 34.64 \text{ cm},$$

so

$$h^2 = 3600 - w^2 = 3600 - 1200 = 2400$$
$$h = \sqrt{2400} = 48.99 \text{ cm}.$$

Thus, $w = 34.64$ cm and $h = 48.99$ cm give a critical point. To check that this is a local maximum, we compute

$$\frac{d^2S}{dw^2} = -6w < 0 \quad \text{for} \quad w > 0.$$

Since $d^2S/dw^2 < 0$, we see that $w = 34.64$ cm is a local maximum. It is the only critical point, so it is a global maximum.

18. The area of the rectangle on the left is $2x$. The entire rectangle has area 2, so the area of the rectangle on the right is $2 - 2x$. We are to maximize the product

$$f(x) = 2x(2 - 2x) = 4x - 4x^2$$

of the two areas where $0 \leq x \leq 1$.

The critical points of f occur where

$$f'(x) = 4 - 8x = 0$$

at $x = 1/2$.

The maximum of f on the interval $0 \leq x \leq 1$ is at the critical point $x = 1/2$ or one of the endpoints $x = 0$ or $x = 1$. We have

$$f(0) = f(1) = 0 \text{ and } f\left(\frac{1}{2}\right) = 1.$$

The product of the areas is a maximum when $x = 1/2$.

19. First find the length h of the vertical segment at x. Since the top edge of the triangle has slope $h/x = 2/1$, we have $h = 2x$. Thus

$$\text{Area of smaller triangle} = \frac{1}{2} \times \text{Base} \times \text{Height} = \frac{1}{2} \cdot x \cdot 2x = x^2.$$

The entire triangle has area 1, so the area of the trapezoid on the right is $1 - x^2$. We are to maximize the product

$$f(x) = x^2(1 - x^2) = x^2 - x^4$$

of the two areas where $0 \leq x \leq 1$

The critical points of f occur where

$$f'(x) = 2x - 4x^3 = 2x(1 - 2x^2) = 0$$

at $x = 0$, $x = 1/\sqrt{2}$ or $x = -1/\sqrt{2}$. The only critical point in the interval $0 < x < 1$ is $x = 1/\sqrt{2}$.

The maximum of f on the interval $0 \leq x \leq 1$ is at one of the endpoints $x = 0$ or $x = 1$ or at the critical point $x = 1/\sqrt{2}$. We have

$$f(0) = f(1) = 0 \text{ and } f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}.$$

The product of the areas is a maximum when $x = 1/\sqrt{2}$.

20. (a) The rectangle on the left has area xy, and the semicircle on the right with radius $y/2$ has area $(1/2)\pi(y/2)^2 = \pi y^2/8$. We have

$$\text{Area of entire region} = xy + \frac{\pi}{8}y^2.$$

(b) The perimeter of the figure is made of two horizontal line segments of length x each, one vertical segment of length y, and a semicircle of radius $y/2$ of length $\pi y/2$. We have

$$\text{Perimeter of entire region} = 2x + y + \frac{\pi}{2}y = 2x + (1 + \frac{\pi}{2})y.$$

(c) We want to maximize the area $xy + \pi y^2/8$, with the perimeter condition $2x + (1 + \pi/2)y = 100$. Substituting

$$x = 50 - \left(\frac{1}{2} + \frac{\pi}{4}\right) y = 50 - \frac{2+\pi}{4}y$$

into the area formula, we must maximize

$$A(y) = \left(50 - \left(\frac{1}{2} + \frac{\pi}{4}\right) y\right) y + \frac{\pi}{8}y^2 = 50y - \left(\frac{1}{2} + \frac{\pi}{8}\right) y^2$$

on the interval

$$0 \le y \le 200/(2+\pi) = 38.8985$$

where $y \ge 0$ because y is a length and $y \le 200/(2+\pi)$ because $x \ge 0$ is a length.

The critical point of A occurs where

$$A'(y) = 50 - \left(1 + \frac{\pi}{4}\right) y = 0$$

at

$$y = \frac{200}{4+\pi} = 28.005.$$

A maximum for A must occur at the critical point $y = 200/(4+\pi)$ or at one of the endpoints $y = 0$ or $y = 200/(2+\pi)$. Since

$$A(0) = 0 \qquad A\left(\frac{200}{4+\pi}\right) = 700.124 \qquad A\left(\frac{200}{2+\pi}\right) = 594.2$$

the maximum is at

$$y = \frac{200}{4+\pi}.$$

Hence

$$x = 50 - \frac{2+\pi}{4}y = 50 - \frac{2+\pi}{4}\frac{200}{4+\pi} = \frac{100}{4+\pi}.$$

The dimensions giving maximum area with perimeter 100 are

$$x = \frac{100}{4+\pi} = 14.0 \qquad y = \frac{200}{4+\pi} = 28.0.$$

The length y is twice as great as x.

21. (a) The rectangle has area xy. The two semicircles together form a circle of radius $y/2$ and area $\pi(y/2)^2 = \pi y^2/4$. We have

$$\text{Area of entire region} = xy + \frac{\pi}{4}y^2.$$

(b) The perimeter of the figure is the sum $2x$ of the lengths of the two straight sections and the circumference πy of a circle of diameter y. Thus

$$\text{Perimeter of entire region} = 2x + \pi y.$$

(c) We want to maximize the area $xy + \pi y^2/4$, with the perimeter condition

$$2x + \pi y = 100.$$

Substituting

$$x = 50 - \frac{\pi}{2}y$$

into the area formula, we must maximize

$$A(y) = \left(50 - \frac{\pi}{2}y\right) y + \frac{\pi}{4}y^2 = 50y - \frac{\pi}{4}y^2$$

on the interval

$$0 \le y \le \frac{100}{\pi}$$

where $y \ge 0$ because y is a length and $y \le 100/\pi$ because $x \ge 0$ is a length.

The critical point of A occurs where

$$A'(y) = 50 - \frac{\pi}{2}y = 0$$

at $y = 100/\pi$, which is an endpoint. A maximum for A occurs at one of the endpoints $y = 0$ or $y = 100/\pi$. Since

$$A(0) = 0 \qquad A\left(\frac{100}{\pi}\right) = \frac{2500}{\pi}$$

the maximum is at $y = 100/\pi$. Hence $x = 0$.

The dimensions giving maximum area with perimeter 100 are $x = 0$, $y = 100/\pi$. The figure is a circle.

22. (a) The rectangle has area xy, the two semicircles with radius $x/2$ together have area $\pi(x/2)^2 = \pi x^2/4$, and the two semicircles with radius $y/2$ together have area $\pi y^2/4$. We have

$$\text{Area of entire figure} = xy + \frac{\pi}{4}x^2 + \frac{\pi}{4}y^2.$$

(b) The perimeter of the figure is the sum of the circumference of a circle of diameter x and a circle of diameter y. Thus

$$\text{Perimeter of entire figure} = \pi x + \pi y.$$

(c) We want to maximize the area $xy + \pi x^2/4 + \pi y^2/4$, with the perimeter condition

$$\pi x + \pi y = 100.$$

Substituting

$$x = \frac{100}{\pi} - y$$

into the area formula, we must maximize

$$A(y) = \left(\frac{100}{\pi} - y\right)y + \frac{\pi}{4}\left(\frac{100}{\pi} - y\right)^2 + \frac{\pi}{4}y^2$$

on the interval

$$0 \le y \le 100/\pi = 31.831$$

where $y \ge 0$ because y is a length and $y \le 100/\pi$ because $x \ge 0$ is a length.

The critical point of A occurs where

$$A'(y) = \frac{100}{\pi} - 50 + (\pi - 2)y = 0$$

at $y = 50/\pi = 15.916$. A maximum for A must occur at one of the endpoints $y = 0$ or $y = 100/\pi$ or at the critical point $y = 50/\pi$. Since

$$A(0) = A\left(\frac{100}{\pi}\right) = \frac{2500}{\pi} = 795.8 \qquad A\left(\frac{50}{\pi}\right) = 651.2,$$

the maximum is at $y = 0$, giving $x = 100/\pi$, or at $y = 100/\pi$, when $x = 0$.

The dimensions giving maximum area with perimeter 100 are $x = 0, y = 100/\pi$ and $x = 100/\pi, y = 0$. The figure is a circle.

23. (a) The length of the piece of wire made into a circle is x cm, so the length of the piece made into a square is $(L - x)$ cm. See Figure 4.50.

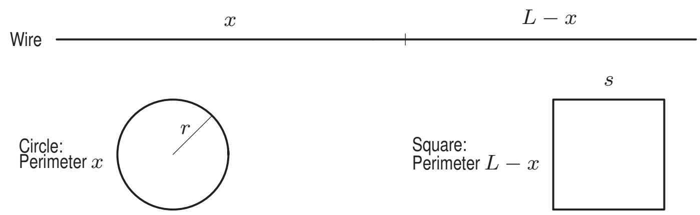


Figure 4.50

The circumference of the circle is x, so its radius, r cm, is given by

$$r = \frac{x}{2\pi} \text{ cm.}$$

The perimeter of the square is $(L - x)$, so the side length, s cm, is given by

$$s = \frac{L-x}{4} \text{ cm.}$$

Thus, the sum of areas is given by

$$A = \pi r^2 + s^2 = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{L-x}{4}\right)^2 = \frac{x^2}{4\pi} + \frac{(L-x)^2}{16}, \quad \text{for } 0 \le x \le L.$$

Setting $dA/dx = 0$ to find the critical points gives

$$\begin{aligned} \frac{dA}{dx} = \frac{x}{2\pi} - \frac{(L-x)}{8} &= 0 \\ 8x &= 2\pi L - 2\pi x \\ (8+2\pi)x &= 2\pi L \\ x &= \frac{2\pi L}{8+2\pi} = \frac{\pi L}{4+\pi} \approx 0.44L. \end{aligned}$$

To find the maxima and minima, we substitute the critical point and the endpoints, $x = 0$ and $x = L$, into the area function.

For $x = 0$, we have $A = \dfrac{L^2}{16}$.

For $x = \dfrac{\pi L}{4+\pi}$, we have $L - x = L - \dfrac{\pi L}{4+\pi} = \dfrac{4L}{4+\pi}$. Then

$$\begin{aligned} A &= \frac{\pi^2 L^2}{4\pi(4+\pi)^2} + \frac{1}{16}\left(\frac{4L}{4+\pi}\right)^2 = \frac{\pi L^2}{4(4+\pi)^2} + \frac{L^2}{(4+\pi)^2} \\ &= \frac{\pi L^2 + 4L^2}{4(4+\pi)^2} = \frac{L^2}{4(4+\pi)} = \frac{L^2}{16+4\pi}. \end{aligned}$$

For $x = L$, we have $A = \dfrac{L^2}{4\pi}$.

Thus, $x = \dfrac{\pi L}{4+\pi}$ gives the minimum value of $A = \dfrac{L^2}{16+4\pi}$.

Since $4\pi < 16$, we see that $x = L$ gives the maximum value of $A = \dfrac{L^2}{4\pi}$.

This corresponds to the situation in which we do not cut the wire at all and use the single piece to make a circle.

(b) At the maximum, $x = L$, so

$$\frac{\text{Length of wire in square}}{\text{Length of wire in circle}} = \frac{0}{L} = 0.$$
$$\frac{\text{Area of square}}{\text{Area of circle}} = \frac{0}{L^2/4\pi} = 0.$$

At the minimum, $x = \dfrac{\pi L}{4+\pi}$, so $L - x = L - \dfrac{\pi L}{4+\pi} = \dfrac{4L}{4+\pi}$.

$$\frac{\text{Length of wire in square}}{\text{Length of wire in circle}} = \frac{4L/(4+\pi)}{\pi L/(4+\pi)} = \frac{4}{\pi}.$$
$$\frac{\text{Area of square}}{\text{Area of circle}} = \frac{L^2/(4+\pi)^2}{\pi L^2/(4(4+\pi)^2)} = \frac{4}{\pi}.$$

(c) For a general value of x,

$$\frac{\text{Length of wire in square}}{\text{Length of wire in circle}} = \frac{L-x}{x}.$$
$$\frac{\text{Area of square}}{\text{Area of circle}} = \frac{(L-x)^2/16}{x^2/(4\pi)} = \frac{\pi}{4}\cdot\frac{(L-x)^2}{x^2}.$$

If the ratios are equal, we have

$$\frac{L-x}{x} = \frac{\pi}{4} \cdot \frac{(L-x)^2}{x^2}.$$

So either $L - x = 0$, giving $x = L$, or we can cancel $(L - x)$ and multiply through by $4x^2$, giving

$$\begin{aligned} 4x &= \pi(L-x) \\ x &= \frac{\pi L}{4+\pi}. \end{aligned}$$

Thus, the two values of x found in part (a) are the only values of x in $0 \le x \le L$ making the ratios in part (b) equal. (The ratios are not defined if $x = 0$.)

24. The minimum value of x in the interval $0 \le x \le 10$ is $x = 0$ and the maximum is $x = 10$.

25. The geometry shows that as x increases from 0 to 10 the length y decreases. The minimum value of y is 5 when $x = 10$, and the maximum value of y is $\sqrt{10^2 + 5^2} = \sqrt{125}$ when $x = 0$.

We can also use calculus. By the Pythagorean theorem we have

$$y = \sqrt{5^2 + (10-x)^2}.$$

The extreme values of

$$y = f(x) = \sqrt{5^2 + (10-x)^2}$$

for $0 \le x \le 10$ occur at the endpoints $x = 0$ or $x = 10$ or at a critical point of f. The only solution of the equation

$$f'(x) = \frac{x-10}{\sqrt{5^2 + (10-x)^2}} = 0$$

is $x = 10$, which is the only critical point of f. We have

$$f(0) = \sqrt{125} = 11.18 \qquad f(10) = 5.$$

Therefore the minimum value of y is 5 and the maximum value is $\sqrt{125} = 5\sqrt{5}$.

26. By the Pythagorean Theorem we have

$$y = \sqrt{5^2 + (10-x)^2}.$$

The extreme values of

$$f(x) = x + 2y = x + 2\sqrt{5^2 + (10-x)^2}$$

for $0 \le x \le 10$ occur at the endpoints $x = 0$ or $x = 10$ or at a critical point of f in the interval.

To find critical points, solve the equation

$$\begin{aligned} f'(x) &= 0 \\ 1 - \frac{2(10-x)}{\sqrt{5^2 + (10-x)^2}} &= 0 \\ 2(10-x) &= \sqrt{5^2 + (10-x)^2} \\ 3(10-x)^2 &= 5^2 \\ x &= 10 \pm \frac{5}{\sqrt{3}}. \end{aligned}$$

The only critical point of f in the interval $0 \le x \le 10$ is at $x = 10 - 5/\sqrt{3}$. Comparing values at the endpoints and critical points we have

$$f(0) = 10\sqrt{5} = 22.36 \qquad f(10) = 20 \qquad f(10 - \frac{5}{\sqrt{3}}) = 10 + 5\sqrt{3} = 18.66.$$

Therefore the minimum value of $x + 2y$ for $0 \le x \le 10$ is $10 + 5\sqrt{3}$ and the maximum value is $10\sqrt{5}$.

27. By the Pythagorean theorem we have

$$y = \sqrt{5^2 + (10-x)^2}.$$

The extreme values of

$$f(x) = 2x + y = 2x + \sqrt{5^2 + (10-x)^2}$$

for $0 \le x \le 10$ occur at the endpoints $x = 0$ or $x = 10$ or at a critical point of f in the interval.

To find critical points, solve the equation

$$\begin{aligned} f'(x) &= 0 \\ 2 - \frac{10-x}{\sqrt{5^2 + (10-x)^2}} &= 0 \\ 10 - x &= 2\sqrt{5^2 + (10-x)^2} \\ 3(10-x)^2 &= -100. \end{aligned}$$

The function f has no critical points. Comparing values at the endpoints we have

$$f(0) = 5\sqrt{5} = 11.18 \qquad f(10) = 25.$$

The minimum value of $2x + y$ for $0 \le x \le 10$ is $5\sqrt{5}$ and the maximum value is 25.

28. The distance $d(x)$ from the point $(x, \sqrt{1-x})$ on the curve to the origin is given by

$$d(x) = \sqrt{x^2 + (\sqrt{1-x})^2} = \sqrt{x^2 + (1-x)}.$$

Since x is in the domain of $y = \sqrt{1-x}$, we have $-\infty < x \le 1$. Differentiating gives

$$d'(x) = \frac{2x - 1}{2\sqrt{x^2 - x + 1}},$$

so $x = 1/2$ is the only critical point. Since $d'(x) < 0$ for $x < 1/2$ and $d'(x) > 0$ for $x > 1/2$, the point $x = 1/2$ is a minimum for x. The point $(1/2, 1/\sqrt{2})$ is the closest point on the curve to the origin.

29. (a) For points (x, y) on the ellipse, we have $y^2 = 1 - x^2/9$ and $-3 \le x \le 3$. We wish to minimize the distance

$$D = \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + 1 - \frac{x^2}{9}}.$$

To do so, we find the value of x minimizing $d = D^2$ for $-3 \le x \le 3$. This x also minimizes D. Since $d = (x-2)^2 + 1 - x^2/9$, we have

$$d'(x) = 2(x-2) - \frac{2x}{9} = \frac{16x}{9} - 4,$$

which is 0 when $x = 9/4$. Since $d''(9/4) = 16/9 > 0$, we see d has a local minimum at $x = 9/4$. Since the graph of d is a parabola, the local minimum is in fact a global minimum. Solving for y, we have

$$y^2 = 1 - \frac{x^2}{9} = 1 - \left(\frac{9}{4}\right)^2 \cdot \frac{1}{9} = \frac{7}{16},$$

so $y = \pm\sqrt{7}/4$. Therefore, the points on the ellipse closest to $(2, 0)$ are $\left(9/4, \pm\sqrt{7}/4\right)$.

(b) This time, we wish to minimize

$$D = \sqrt{(x - \sqrt{8})^2 + 1 - \frac{x^2}{9}}.$$

Again, let $d = D^2$ and minimize $d(x)$ for $-3 \le x \le 3$. Since $d = (x - \sqrt{8})^2 + 1 - x^2/9$,

$$d'(x) = 2(x - 2\sqrt{2}) - \frac{2x}{9} = \frac{16x}{9} - 4\sqrt{2}.$$

Therefore, $d'(x) = 0$ when $x = 9\sqrt{2}/4$. But $9\sqrt{2}/4 > 3$, so there are not any critical points on the interval $-3 \le x \le 3$. The minimum distance must be attained at an endpoint. Since $d'(x) < 0$ for all x between -3 and 3, the minimum is at $x = 3$. So $(3, 0)$ is the point on the ellipse closest to $(\sqrt{8}, 0)$.

30. Let the radius of the can be r and let its height be h. The surface area, S, and volume, V, of the can are given by

$$\begin{aligned} S &= \text{Area of the sides of can} + \text{Area of top and bottom} \\ S &= 2\pi rh + 2\pi r^2 \\ V &= \pi r^2 h. \end{aligned}$$

Since $S = 280$, we have $2\pi rh + 2\pi r^2 = 280$, we have

$$h = \frac{140 - \pi r^2}{\pi r}$$

and

$$V = \pi r^2 \frac{140 - \pi r^2}{\pi r} = 140r - \pi r^3.$$

Since $h \geq 0$, we have $\pi r^2 \leq 140$ and thus $0 < r \leq \sqrt{140/\pi} = 6.676$.

We have

$$V'(r) = 140 - 3\pi r^2,$$

so the only critical point of V with $r > 0$ is $r = \sqrt{140/3\pi} = 3.854$. The critical point is in the domain $0 < r \leq 6.676$.

Since $V''(r) = -6\pi r$ is negative for all $r > 0$, the critical point $r = 3.854$ cm gives the maximum value of the volume. For this value of r, we have $h = 7.708$ cm and $V = 359.721$ cm^3.

31. Figure 4.51 shows the vertical cross section through the cylinder and sphere. The circle has equation $y = \sqrt{1 - x^2}$, so if the cylinder has radius x and height y, its volume, V, is given by

$$V = \pi x^2 y = \pi x^2 \sqrt{1 - x^2} \qquad \text{for } 0 \leq x \leq 1.$$

At a critical point, $dV/dx = 0$, so

$$\begin{aligned} \frac{dV}{dx} = 2\pi x\sqrt{1-x^2} + \pi x^2\left(\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right) &= 0 \\ 2\pi x\sqrt{1-x^2} - \frac{\pi x^3}{\sqrt{1-x^2}} &= 0 \\ \frac{\pi x}{\sqrt{1-x^2}}\left(2\left(\sqrt{1-x^2}\right)^2 - x^2\right) &= 0 \\ x(2 - 3x^2) &= 0 \\ x &= 0, \ \pm\sqrt{\frac{2}{3}}. \end{aligned}$$

Since $V = 0$ at the endpoints $x = 0$ and $x = 1$, and V is positive at the only critical point, $x = \sqrt{2/3}$, in the interval, the critical point $x = \sqrt{2/3}$ is a local and global maximum. Thus, the cylinder with maximum volume has

$$\text{Radius } = x = \sqrt{\frac{2}{3}}$$

$$\text{Height } = y = \sqrt{1 - \left(\sqrt{\frac{2}{3}}\right)^2} = \sqrt{\frac{1}{3}}.$$

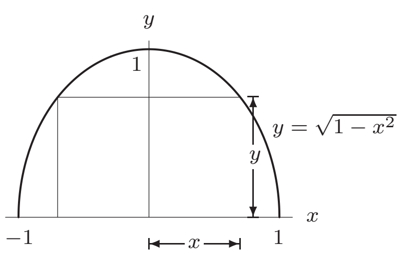


Figure 4.51

32. (a) If we expect the rate to be nonnegative, we must have $0 \le y \le a$ and $0 \le y \le b$. Since we assume $a < b$, we restrict y to $0 \le y \le a$.

In fact, the expression for the rate is nonnegative for y greater than b, but these values of y are not meaningful for the reaction. See Figure 4.52.

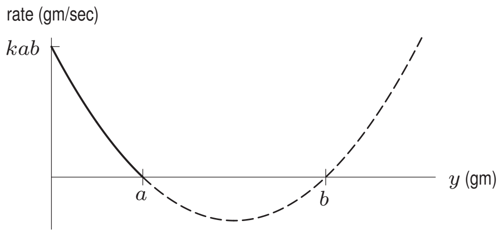
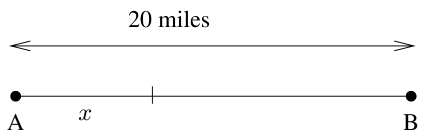


Figure 4.52

(b) From the graph, we see that the maximum rate occurs when $y = 0$; that is, at the start of the reaction.

33. Call the stacks A and B. (See below.) Assume that A corresponds to k_1, and B corresponds to k_2.



Suppose the point where the concentration of deposit is a minimum occurs at a distance of x miles from stack A. We want to find x such that

$$S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2} = k_2\left(\frac{7}{x^2} + \frac{1}{(20-x)^2}\right)$$

is a minimum, which is the same thing as minimizing $f(x) = 7x^{-2} + (20-x)^{-2}$ since k_2 is nonnegative.

We have

$$f'(x) = -14x^{-3} - 2(20-x)^{-3}(-1) = \frac{-14}{x^3} + \frac{2}{(20-x)^3} = \frac{-14(20-x)^3 + 2x^3}{x^3(20-x)^3}.$$

Thus we want to find x such that $-14(20-x)^3 + 2x^3 = 0$, which implies $2x^3 = 14(20-x)^3$. That's equivalent to $x^3 = 7(20-x)^3$, or $\frac{20-x}{x} = (1/7)^{1/3} \approx 0.523$. Solving for x, we have $20 - x = 0.523x$, whence $x = 20/1.523 \approx 13.13$.

To verify that this minimizes f, we take the second derivative:

$$f''(x) = 42x^{-4} + 6(20-x)^{-4} = \frac{42}{x^4} + \frac{6}{(20-x)^4} > 0$$

for any $0 < x < 20$, so by the second derivative test the concentration is minimized 13.13 miles from A.

34. We only consider $\lambda > 0$. For such λ, the value of $v \to \infty$ as $\lambda \to \infty$ and as $\lambda \to 0^+$. Thus, v does not have a maximum velocity. It will have a minimum velocity. To find it, we set $dv/d\lambda = 0$:

$$\frac{dv}{d\lambda} = k\frac{1}{2}\left(\frac{\lambda}{c} + \frac{c}{\lambda}\right)^{-1/2}\left(\frac{1}{c} - \frac{c}{\lambda^2}\right) = 0.$$

Solving, and remembering that $\lambda > 0$, we obtain

$$\begin{aligned} \frac{1}{c} - \frac{c}{\lambda^2} &= 0 \\ \frac{1}{c} &= \frac{c}{\lambda^2} \\ \lambda^2 &= c^2, \end{aligned}$$

so

$$\lambda = c.$$

Thus, we have one critical point. Since

$$\frac{dv}{d\lambda} < 0 \quad \text{for } \lambda < c$$

and

$$\frac{dv}{d\lambda} > 0 \quad \text{for } \lambda > c,$$

the first derivative test tells us that we have a local minimum of v at $x = c$. Since $\lambda = c$ is the only critical point, it gives the global minimum. Thus the minimum value of v is

$$v = k\sqrt{\frac{c}{c} + \frac{c}{c}} = \sqrt{2}k.$$

35. The domain for E is all real x. Note $E \to 0$ as $x \to \pm\infty$. The critical points occur where $dE/dx = 0$. The derivative is

$$\begin{aligned}\frac{dE}{dx} &= \frac{k}{(x^2 + r_0^2)^{3/2}} - \frac{3}{2} \cdot \frac{kx(2x)}{(x^2 + r_0^2)^{5/2}} \\ &= \frac{k\left(x^2 + r_0^2 - 3x^2\right)}{(x^2 + r_0^2)^{5/2}} \\ &= \frac{k\left(r_0^2 - 2x^2\right)}{(x^2 + r_0^2)^{5/2}}.\end{aligned}$$

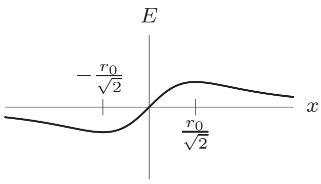
So $dE/dx = 0$ where

$$\begin{aligned} r_0^2 - 2x^2 &= 0 \\ x &= \pm\frac{r_0}{\sqrt{2}}.\end{aligned}$$

Looking at the formula for dE/dx shows

$$\begin{aligned}\frac{dE}{dx} &> 0 \text{ for } -\frac{r_0}{\sqrt{2}} < x < \frac{r_0}{\sqrt{2}} \\ \frac{dE}{dx} &< 0 \text{ for } x < -\frac{r_0}{\sqrt{2}} \\ \frac{dE}{dx} &< 0 \text{ for } x > \frac{r_0}{\sqrt{2}}.\end{aligned}$$

Therefore, $x = -r_0/\sqrt{2}$ gives the minimum value of E and $x = r_0/\sqrt{2}$ gives the maximum value of E.



36. A graph of F against θ is shown in Figure 4.53.

Taking the derivative:

$$\frac{dF}{d\theta} = -\frac{mg\mu(\cos\theta - \mu\sin\theta)}{(\sin\theta + \mu\cos\theta)^2}.$$

At a critical point, $dF/d\theta = 0$, so

$$\begin{aligned}\cos\theta - \mu\sin\theta &= 0 \\ \tan\theta &= \frac{1}{\mu} \\ \theta &= \arctan\left(\frac{1}{\mu}\right).\end{aligned}$$

If $\mu = 0.15$, then $\theta = \arctan(1/0.15) = 1.422 \approx 81.5^\circ$. To calculate the maximum and minimum values of F, we evaluate at this critical point and the endpoints:

$$\text{At } \theta = 0, \quad F = \frac{0.15mg}{\sin 0 + 0.15\cos 0} = 1.0mg \text{ newtons.}$$

$$\text{At } \theta = 1.422, \quad F = \frac{0.15mg}{\sin(1.422) + 0.15\cos(1.422)} = 0.148mg \text{ newtons.}$$

$$\text{At } \theta = \pi/2, \quad F = \frac{0.15mg}{\sin(\frac{\pi}{2}) + 0.15\cos(\frac{\pi}{2})} = 0.15mg \text{ newtons.}$$

Thus, the maximum value of F is $1.0mg$ newtons when $\theta = 0$ (her arm is vertical) and the minimum value of F is $0.148mg$ newtons is when $\theta = 1.422$ (her arm is close to horizontal). See Figure 4.54.

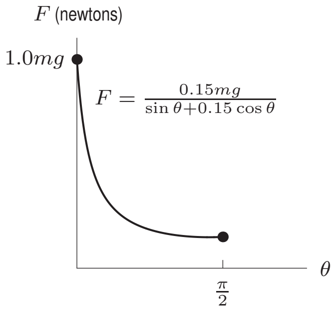


Figure 4.53

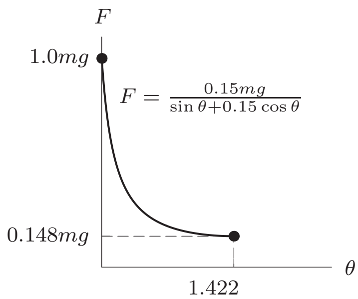


Figure 4.54

37. (a) We must locate the critical point(s) of $A(r)$ — i.e. the places where its derivative is zero. The derivative is

$$A'(r) = \frac{2r}{T} - \frac{T}{r^2},$$

so $A'(r) = 0$ when

$$\begin{aligned} A'(r) = \frac{2r}{T} - \frac{T}{r^2} &= 0 \\ \frac{2r}{T} &= \frac{T}{r^2} \\ 2r^3 &= T^2 \\ r^3 &= \frac{T^2}{2} \\ r &= \frac{T^{2/3}}{2^{1/3}}. \end{aligned}$$

The only critical point of $A(r)$ is thus $r = (T^{2/3})/(2^{1/3})$. To check that this is a local (and, since it is the only critical point, global) minimum, we employ the second derivative test. The second derivative $A''(r)$ is

$$A''(r) = \frac{2}{T} + \frac{2T}{r^3},$$

and at the critical point we have

$$A''\left(\frac{T^{2/3}}{2^{1/3}}\right) = \frac{2}{T} + \frac{4}{T} = \frac{6}{T} > 0,$$

which implies A is concave up at this critical point. It is therefore a local and global minimum.

(b) The new period is $2T$. To find the radius for this new period, substitute $2T$ for T in the expression for r:

$$\begin{aligned} r &= \frac{(2T)^{2/3}}{2^{1/3}} \\ &= 2^{2/3}\frac{T^{2/3}}{2^{1/3}}. \end{aligned}$$

Notice this is $2^{2/3}$ times the original radius. To find the percentage increase, compute $2^{2/3} \approx 1.5874$. The new radius therefore represents an increase of 58.74% over the original.

38. Let x equal the number of chairs ordered in excess of 300, so $0 \leq x \leq 100$.

$$\begin{aligned}\text{Revenue} = R &= (90 - 0.25x)(300 + x)\\ &= 27,000 - 75x + 90x - 0.25x^2 = 27,000 + 15x - 0.25x^2\end{aligned}$$

At a critical point $dR/dx = 0$. Since $dR/dx = 15 - 0.5x$, we have $x = 30$, and the maximum revenue is $\$27,225$ since the graph of R is a parabola which opens downward. The minimum is $\$0$ (when no chairs are sold).

39. If v is the speed of the boat in miles per hour, then

$$\text{Cost of fuel per hour (in \$/hour)} = kv^3,$$

where k is the constant of proportionality. To find k, use the information that the boat uses $\$100$ worth of fuel per hour when cruising at 10 miles per hour: $100 = k10^3$, so $k = 100/10^3 = 0.1$. Thus,

$$\text{Cost of fuel per hour (in \$/hour)} = 0.1v^3.$$

From the given information, we also have

$$\text{Cost of other operations (labor, maintenance, etc.) per hour (in \$/hour)} = 675.$$

So

$$\begin{aligned}\text{Total Cost per hour (in \$/hour)} &= \text{Cost of fuel (in \$/hour) + Cost of other (in \$/hour)}\\ &= 0.1v^3 + 675.\end{aligned}$$

However, we want to find the Cost per *mile*, which is the Total Cost per *hour* divided by the number of miles that the ferry travels in one hour. Since v is the speed in miles/hour at which the ferry travels, the number of miles that the ferry travels in one hour is simply v miles. Let C = Cost per *mile*. Then

$$\begin{aligned}\text{Cost per } \textit{mile} \text{ (in \$/mile)} &= \frac{\text{Total Cost per } \textit{hour} \text{ (in \$/hour)}}{\text{Distance traveled per hour (in miles/hour)}}\\ C &= \frac{0.1v^3 + 675}{v} = 0.1v^2 + \frac{675}{v}.\end{aligned}$$

We also know that $0 < v < \infty$. To find the speed at which Cost per *mile* is minimized, set

$$\frac{dC}{dv} = 2(0.1)v - \frac{675}{v^2} = 0$$

so

$$\begin{aligned}2(0.1)v &= \frac{675}{v^2}\\ v^3 &= \frac{675}{2(0.1)} = 3375\\ v &= 15 \text{ miles/hour.}\end{aligned}$$

Since

$$\frac{d^2C}{dv^2} = 0.2 + \frac{2(675)}{v^3} > 0$$

for $v > 0$, $v = 15$ gives a local minimum for C by the second-derivative test. Since this is the only critical point for $0 < v < \infty$, it must give a global minimum.

40. (a) The business must reorder often enough to keep pace with sales. If reordering is done every t months, then,

$$\begin{aligned}\text{Quantity sold in } t \text{ months} &= \text{Quantity reordered in each batch}\\ rt &= q\\ t &= \frac{q}{r} \text{ months.}\end{aligned}$$

(b) The amount spent on each order is $a + bq$, which is spent every q/r months. To find the monthly expenditures, divide by q/r. Thus, on average,

$$\text{Amount spent on ordering per month} = \frac{a + bq}{q/r} = \frac{ra}{q} + rb \text{ dollars.}$$

(c) The monthly cost of storage is $kq/2$ dollars, so

$$C = \text{Ordering costs} + \text{Storage costs}$$
$$C = \frac{ra}{q} + rb + \frac{kq}{2} \text{ dollars.}$$

(d) The optimal batch size minimizes C, so

$$\frac{dC}{dq} = \frac{-ra}{q^2} + \frac{k}{2} = 0$$
$$\frac{ra}{q^2} = \frac{k}{2}$$
$$q^2 = \frac{2ra}{k}$$

so

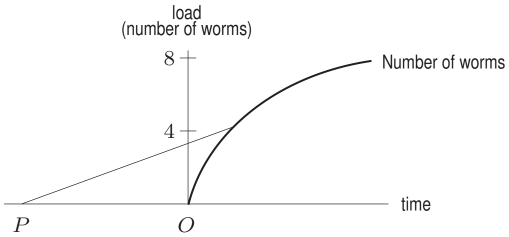
$$q = \sqrt{\frac{2ra}{k}} \text{ items per order.}$$

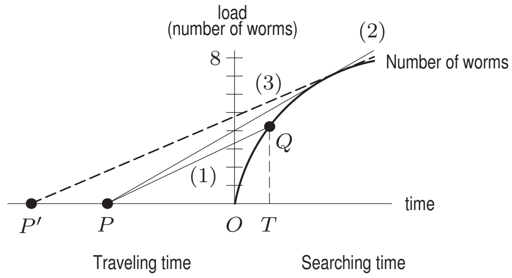
41. (a) The line in the left-hand figure has slope equal to the rate worms arrive. To understand why, see line (1) in the right-hand figure. (This is the same line.) For any point Q on the loading curve, the line PQ has slope

$$\frac{QT}{PT} = \frac{QT}{PO + OT} = \frac{\text{load}}{\text{traveling time} + \text{searching time}}.$$

(b) The slope of the line PQ is maximized when the line is tangent to the loading curve, which happens with line (2). The load is then approximately 7 worms.

(c) If the traveling time is increased, the point P moves to the left, to point P', say. If line (3) is tangent to the curve, it will be tangent to the curve further to the right than line (2), so the optimal load is larger. This makes sense: if the bird has to fly further, you'd expect it to bring back more worms each time.





42. Let x be as indicated in the figure in the text. Then the distance from S to Town 1 is $\sqrt{1+x^2}$ and the distance from S to Town 2 is $\sqrt{(4-x)^2+4^2} = \sqrt{x^2-8x+32}$.

$$\text{Total length of pipe } = f(x) = \sqrt{1+x^2} + \sqrt{x^2-8x+32}.$$

We want to look for critical points of f. The easiest way is to graph f and see that it has a local minimum at about $x = 0.8$ miles. Alternatively, we can use the formula:

$$\begin{aligned} f'(x) &= \frac{2x}{2\sqrt{1+x^2}} + \frac{2x-8}{2\sqrt{x^2-8x+32}} \\ &= \frac{x}{\sqrt{1+x^2}} + \frac{x-4}{\sqrt{x^2-8x+32}} \\ &= \frac{x\sqrt{x^2-8x+32} + (x-4)\sqrt{1+x^2}}{\sqrt{1+x^2}\sqrt{x^2-8x+32}} = 0. \end{aligned}$$

$f'(x)$ is equal to zero when the numerator is equal to zero.

$$x\sqrt{x^2-8x+32} + (x-4)\sqrt{1+x^2} = 0$$
$$x\sqrt{x^2-8x+32} = (4-x)\sqrt{1+x^2}.$$

Squaring both sides and simplifying, we get

$$\begin{aligned}x^2(x^2-8x+32) &= (x^2-8x+16)(14x^2)\\ x^4-8x^3+32x^2 &= x^4-8x^3+17x^2-8x+16\\ 15x^2+8x-16 &= 0,\\ (3x+4)(5x-4) &= 0.\end{aligned}$$

So $x = 4/5$. (Discard $x = -4/3$ since we are only interested in x between 0 and 4, between the two towns.) Using the second derivative test, we can verify that $x = 4/5$ is a local minimum.

43. (a) The distance the pigeon flies over water is

$$\overline{BP} = \frac{\overline{AB}}{\sin\theta} = \frac{500}{\sin\theta},$$

and over land is

$$\overline{PL} = \overline{AL} - \overline{AP} = 2000 - \frac{500}{\tan\theta} = 2000 - \frac{500\cos\theta}{\sin\theta}.$$

Therefore the energy required is

$$\begin{aligned}E &= 2e\left(\frac{500}{\sin\theta}\right) + e\left(2000 - \frac{500\cos\theta}{\sin\theta}\right)\\ &= 500e\left(\frac{2-\cos\theta}{\sin\theta}\right) + 2000e, \quad \text{for} \quad \arctan\left(\frac{500}{2000}\right) \le \theta \le \frac{\pi}{2}.\end{aligned}$$

(b) Notice that E and the function $f(\theta) = \dfrac{2-\cos\theta}{\sin\theta}$ must have the same critical points since the graph of E is just a stretch and a vertical shift of the graph of f. The graph of $\dfrac{2-\cos\theta}{\sin\theta}$ for $\arctan(\frac{500}{2000}) \le \theta \le \frac{\pi}{2}$ in Figure 4.55 shows that E has precisely one critical point, and that a minimum for E occurs at this point.

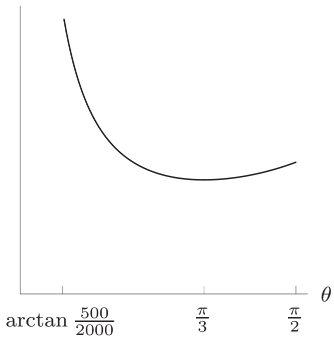


Figure 4.55: Graph of $f(\theta) = \frac{2-\cos\theta}{\sin\theta}$ for $\arctan(\frac{500}{2000}) \le \theta \le \frac{\pi}{2}$

To find the critical point θ, we solve $f'(\theta) = 0$ or

$$\begin{aligned}E' = 0 &= 500e\left(\frac{\sin\theta\cdot\sin\theta - (2-\cos\theta)\cdot\cos\theta}{\sin^2\theta}\right)\\ &= 500e\left(\frac{1-2\cos\theta}{\sin^2\theta}\right).\end{aligned}$$

Therefore $1 - 2\cos\theta = 0$ and so $\theta = \pi/3$.

(c) Letting $a = \overline{AB}$ and $b = \overline{AL}$, our formula for E becomes

$$\begin{aligned}E &= 2e\left(\frac{a}{\sin\theta}\right) + e\left(b - \frac{a\cos\theta}{\sin\theta}\right)\\ &= ea\left(\frac{2-\cos\theta}{\sin\theta}\right) + eb, \quad \text{for} \quad \arctan\left(\frac{a}{b}\right) \le \theta \le \frac{\pi}{2}.\end{aligned}$$

Again, the graph of E is just a stretch and a vertical shift of the graph of $\dfrac{2-\cos\theta}{\sin\theta}$. Thus, the critical point $\theta = \pi/3$ is independent of e, a, and b. But the maximum of E *on the domain* $\arctan(a/b) \le \theta \le \frac{\pi}{2}$ is dependent on the ratio $a/b = \dfrac{\overline{AB}}{\overline{AL}}$. In other words, the optimal angle is $\theta = \pi/3$ provided $\arctan(a/b) \le \frac{\pi}{3}$; otherwise, the optimal angle is $\arctan(a/b)$, which means the pigeon should fly over the lake for the entire trip—this occurs when $a/b > 1.733$.

44. We want to maximize the viewing angle, which is $\theta = \theta_1 - \theta_2$. See Figure 4.56. Now

$$\tan(\theta_1) = \frac{92}{x} \quad \text{so } \theta_1 = \arctan\left(\frac{92}{x}\right)$$
$$\tan(\theta_2) = \frac{46}{x} \quad \text{so } \theta_2 = \arctan\left(\frac{46}{x}\right).$$

Then

$$\theta = \arctan\left(\frac{92}{x}\right) - \arctan\left(\frac{46}{x}\right) \quad \text{for} \quad x > 0.$$

We look for critical points of the function by computing $d\theta/dx$:

$$\begin{aligned}\frac{d\theta}{dx} &= \frac{1}{1+(92/x)^2}\left(\frac{-92}{x^2}\right) - \frac{1}{1+(46/x)^2}\left(\frac{-46}{x^2}\right)\\ &= \frac{-92}{x^2+92^2} - \frac{-46}{x^2+46^2}\\ &= \frac{-92(x^2+46^2)+46(x^2+92^2)}{(x^2+92^2)\cdot(x^2+46^2)}\\ &= \frac{46(4232-x^2)}{(x^2+92^2)\cdot(x^2+46^2)}.\end{aligned}$$

Setting $d\theta/dx = 0$ gives

$$\begin{aligned}x^2 &= 4232\\ x &= \pm\sqrt{4232}.\end{aligned}$$

Since $x > 0$, the critical point is $x = \sqrt{4232} \approx 65.1$ meters. To verify that this is indeed where θ attains a maximum, we note that $d\theta/dx > 0$ for $0 < x < \sqrt{4232}$ and $d\theta/dx < 0$ for $x > \sqrt{4232}$. By the First Derivative Test, θ attains a maximum at $x = \sqrt{4232} \approx 65.1$.

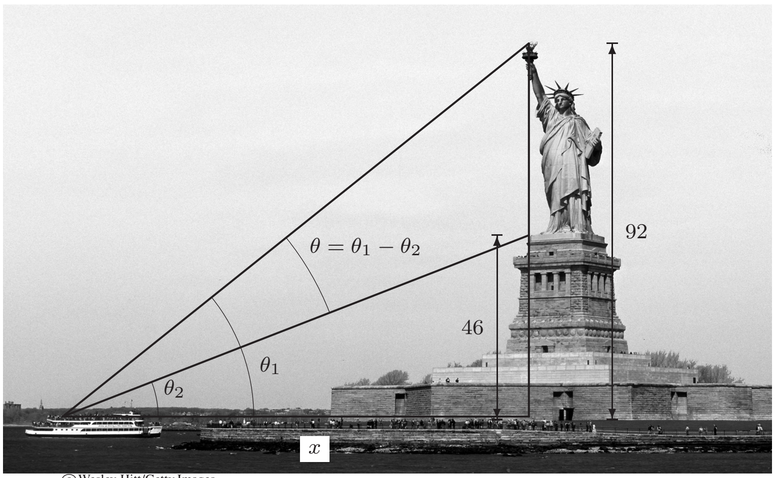


Figure 4.56

45. (a) Since the speed of light is a constant, the time of travel is minimized when the distance of travel is minimized. From Figure 4.57,

$$\text{Distance } \overrightarrow{OP} = \sqrt{x^2+1^2} = \sqrt{x^2+1}$$
$$\text{Distance } \overrightarrow{PQ} = \sqrt{(2-x)^2+1^2} = \sqrt{(2-x)^2+1}$$

Thus,

$$\text{Total distance traveled } = s = \sqrt{x^2+1} + \sqrt{(2-x)^2+1}.$$

The total distance is a minimum if

$$\frac{ds}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x + \frac{1}{2}((2-x)^2+1)^{-1/2} \cdot 2(2-x)(-1) = 0,$$

giving

$$\frac{x}{\sqrt{x^2+1}} - \frac{2-x}{\sqrt{(2-x)^2+1}} = 0$$
$$\frac{x}{\sqrt{x^2+1}} = \frac{2-x}{\sqrt{(2-x)^2+1}}$$

Squaring both sides gives

$$\frac{x^2}{x^2+1} = \frac{(2-x)^2}{(2-x)^2+1}.$$

Cross multiplying gives

$$x^2((2-x)^2+1) = (2-x)^2(x^2+1).$$

Multiplying out

$$x^2(4-4x+x^2+1) = (4-4x+x^2)(x^2+1)$$
$$4x^2-4x^3+x^4+x^2 = 4x^2-4x^3+x^4+4-4x+x^2.$$

Collecting terms and canceling gives

$$0 = 4-4x$$
$$x = 1.$$

We can see that this value of x gives a minimum by comparing the value of s at this point and at the endpoints, $x=0, x=2$.

At $x=1$,

$$s = \sqrt{1^2+1} + \sqrt{(2-1)^2+1} = 2.83.$$

At $x=0$,

$$s = \sqrt{0^2+1} + \sqrt{(2-0)^2+1} = 3.24.$$

At $x=2$,

$$s = \sqrt{2^2+1} + \sqrt{(2-2)^2+1} = 3.24.$$

Thus the shortest travel time occurs when $x=1$; that is, when P is at the point $(1,1)$.

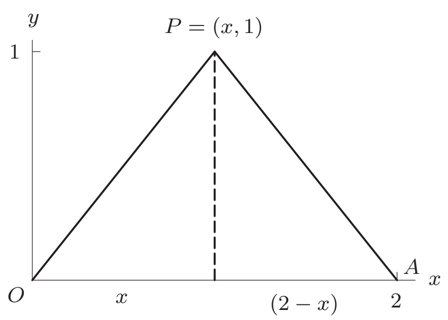


Figure 4.57

(b) Since $x=1$ is halfway between $x=0$ and $x=2$, the angles θ_1 and θ_2 are equal.

46. (a) We have

$$x^{1/x} = e^{\ln(x^{1/x})} = e^{(1/x)\ln x}.$$

Thus

$$\begin{aligned}\frac{d(x^{1/x})}{dx} &= \frac{d(e^{(1/x)\ln x})}{dx} = \frac{d(\frac{1}{x}\ln x)}{dx}e^{(1/x)\ln x} \\ &= \left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right)x^{1/x} \\ &= \frac{x^{1/x}}{x^2}(1-\ln x)\begin{cases} = 0 & \text{when } x = e \\ < 0 & \text{when } x > e \\ > 0 & \text{when } x < e.\end{cases}\end{aligned}$$

Hence $e^{1/e}$ is the global maximum for $x^{1/x}$, by the first derivative test.

(b) Since $x^{1/x}$ is increasing for $0 < x < e$ and decreasing for $x > e$, and 2 and 3 are the closest integers to e, either $2^{1/2}$ or $3^{1/3}$ is the maximum for $n^{1/n}$. We have $2^{1/2} \approx 1.414$ and $3^{1/3} \approx 1.442$, so $3^{1/3}$ is the maximum.

(c) Since $e < 3 < \pi$, and $x^{1/x}$ is decreasing for $x > e$, $3^{1/3} > \pi^{1/\pi}$.

47. (a) If, following the hint, we set $f(x) = (a+x)/2 - \sqrt{ax}$, then $f(x)$ represents the difference between the arithmetic and geometric means for some fixed a and any $x > 0$. We can find where this difference is minimized by solving $f'(x) = 0$. Since $f'(x) = \frac{1}{2} - \frac{1}{2}\sqrt{a}x^{-1/2}$, if $f'(x) = 0$ then $\frac{1}{2}\sqrt{a}x^{-1/2} = \frac{1}{2}$, or $x = a$. Since $f''(x) = \frac{1}{4}\sqrt{a}x^{-3/2}$ is positive for all positive x, by the second derivative test $f(x)$ has a minimum at $x = a$, and $f(a) = 0$. Thus $f(x) = (a+x)/2 - \sqrt{ax} \geq 0$ for all $x > 0$, which means $(a+x)/2 \geq \sqrt{ax}$. This means that the arithmetic mean is greater than the geometric mean unless $a = x$, in which case the two means are equal.

Alternatively, and without using calculus, we obtain

$$\begin{aligned}\frac{a+b}{2} - \sqrt{ab} &= \frac{a - 2\sqrt{ab} + b}{2} \\ &= \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0,\end{aligned}$$

and again we have $(a+b)/2 \geq \sqrt{ab}$.

(b) Following the hint, set $f(x) = \frac{a+b+x}{3} - \sqrt[3]{abx}$. Then $f(x)$ represents the difference between the arithmetic and geometric means for some fixed a, b and any $x > 0$. We can find where this difference is minimized by solving $f'(x) = 0$. Since $f'(x) = \frac{1}{3} - \frac{1}{3}\sqrt[3]{ab}x^{-2/3}$, $f'(x) = 0$ implies that $\frac{1}{3}\sqrt[3]{ab}x^{-2/3} = \frac{1}{3}$, or $x = \sqrt{ab}$. Since $f''(x) = \frac{2}{9}\sqrt[3]{ab}x^{-5/3}$ is positive for all positive x, by the second derivative test $f(x)$ has a minimum at $x = \sqrt{ab}$. But

$$f(\sqrt{ab}) = \frac{a+b+\sqrt{ab}}{3} - \sqrt[3]{ab\sqrt{ab}} = \frac{a+b+\sqrt{ab}}{3} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{3}.$$

By the first part of this problem, we know that $\frac{a+b}{2} - \sqrt{ab} \geq 0$, which implies that $a + b - 2\sqrt{ab} \geq 0$. Thus $f(\sqrt{ab}) = \frac{a+b-2\sqrt{ab}}{3} \geq 0$. Since f has a maximum at $x = \sqrt{ab}$, $f(x)$ is always nonnegative. Thus $f(x) = \frac{a+b+x}{3} - \sqrt[3]{abx} \geq 0$, so $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$. Note that equality holds only when $a = b = c$. (Part (b) may also be done without calculus, but it's harder than (a).)

48. For $x > 0$, the line in Figure 4.58 has

$$\text{Slope} = \frac{y}{x} = \frac{x^2e^{-3x}}{x} = xe^{-3x}.$$

If the slope has a maximum, it occurs where

$$\begin{aligned}\frac{d}{dx}(\text{Slope}) &= 1\cdot e^{-3x} - 3xe^{-3x} = 0 \\ e^{-3x}(1-3x) &= 0 \\ x &= \frac{1}{3}.\end{aligned}$$

For this x-value,

$$\text{Slope} = \frac{1}{3}e^{-3(1/3)} = \frac{1}{3}e^{-1} = \frac{1}{3e}.$$

Figure 4.58 shows that the slope tends toward 0 as $x \to \infty$; the formula for the slope shows that the slope tends toward 0 as $x \to 0$. Thus the only critical point, $x = 1/3$, must give a local and global maximum.

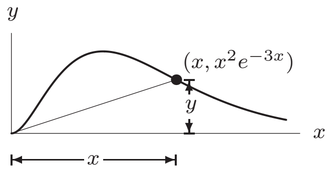


Figure 4.58

49. (a) For a point (t, s), the line from the origin has rise $= s$ and run $= t$; See Figure 4.59. Thus, the slope of the line OP is s/t.

(b) Sketching several lines from the origin to points on the curve (see Figure 4.60), we see that the maximum slope occurs at the point P, where the line to the origin is tangent to the graph. Reading from the graph, we see $t \approx 2$ hours at this point.

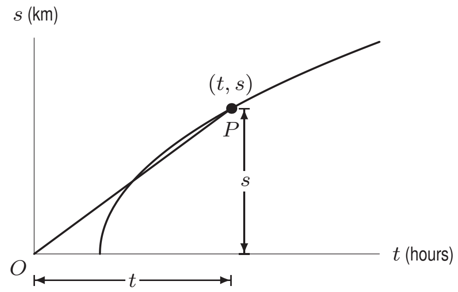


Figure 4.59

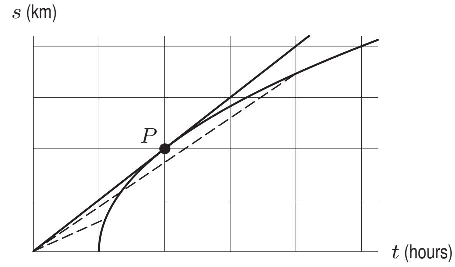


Figure 4.60

(c) The instantaneous speed of the cyclist at any time is given by the slope of the corresponding point on the curve. At the point P, the line from the origin is tangent to the curve, so the quantity s/t equals the cyclist's speed at the point P.

50. (a) To maximize benefit (surviving young), we pick 10, because that's the highest point of the benefit graph.

(b) To optimize the vertical distance between the curves, we can either do it by inspection or note that the slopes of the two curves will be the same where the difference is maximized. Either way, one gets approximately 9.

51. (a) At higher speeds, the bird uses more energy so the graph rises to the right. The initial drop is due to the fact that the energy it takes a bird to fly at very low speeds is greater than that needed to fly at a slightly higher speeds. (This is analogous to our swimming in a pool).

(b) The value of $f(v)$ measures energy per second; the value of $a(v)$ measures energy per meter. In one second, a bird traveling at rate v travels v meters and consumes $v \cdot a(v)$ joules. Thus $v \cdot a(v)$ represents the energy consumption per second, so $f(v) = v \cdot a(v)$.

(c) Since $v \cdot a(v) = f(v)$, we have
$$a(v) = \frac{f(v)}{v}.$$
This ratio is the slope of a line passing from the origin through the point $(v, f(v))$ on the curve. (See Figure 4.61.) Thus $a(v)$ is minimal when the slope of this line is minimal. This occurs where the line is tangent to the curve.

To find the value of v minimizing $a(v)$ symbolically, we solve $a'(v) = 0$. By the quotient rule,
$$a'(v) = \frac{vf'(v) - f(v)}{v^2}.$$
Thus $a'(v) = 0$ when $vf'(v) = f(v)$, or when $f'(v) = f(v)/v = a(v)$. Thus $a(v)$ is minimized when $a(v) = f'(v)$.

(d) Assuming the bird wants to go from one particular point to another, that is, when the distance is fixed, the bird should minimize $a(v)$. Then minimizing $a(v)$ minimizes the total energy used for the flight.

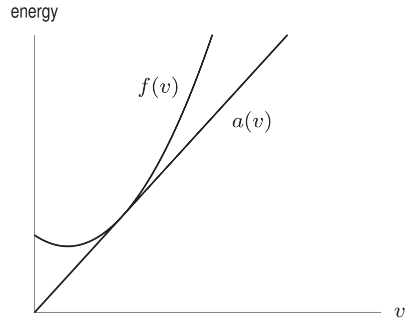


Figure 4.61

52. (a) Figure 4.62 contains the graph of total drag, plotted on the same coordinate system with induced and parasite drag. It was drawn by adding the vertical coordinates of Induced and Parasite drag.

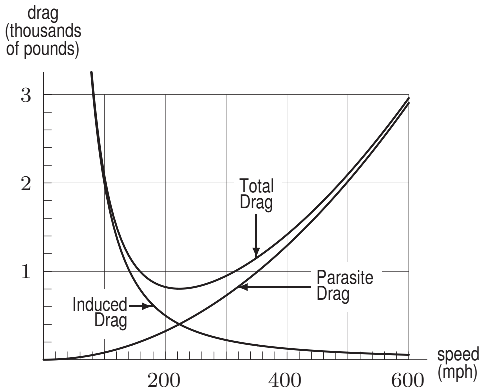


Figure 4.62

(b) Air speeds of approximately 160 mph and 320 mph each result in a total drag of 1000 pounds. Since two distinct air speeds are associated with a single total drag value, the total drag function does not have an inverse. The parasite and induced drag functions do have inverses, because they are strictly increasing and strictly decreasing functions, respectively.

(c) To conserve fuel, fly the at the air speed which minimizes total drag. This is the air speed corresponding to the lowest point on the total drag curve in part (a): that is, approximately 220 mph.

53. (a) To obtain $g(v)$, which is in gallons per mile, we need to divide $f(v)$ (in gallons per hour) by v (in miles per hour). Thus, $g(v) = f(v)/v$.

(b) By inspecting the graph, we see that $f(v)$ is minimized at approximately 220 mph.

(c) Note that a point on the graph of $f(v)$ has the coordinates $(v, f(v))$. The line passing through this point and the origin $(0, 0)$ has

$$\text{Slope } = \frac{f(v) - 0}{v - 0} = \frac{f(v)}{v} = g(v).$$

So minimizing $g(v)$ corresponds to finding the line of minimum slope from the family of lines which pass through the origin $(0, 0)$ and the point $(v, f(v))$ on the graph of $f(v)$. This line is the unique member of the family which is tangent to the graph of $f(v)$. The value of v corresponding to the point of tangency will minimize $g(v)$. This value of v will satisfy $f(v)/v = f'(v)$. From the graph in Figure 4.63, we see that $v \approx 300$ mph.

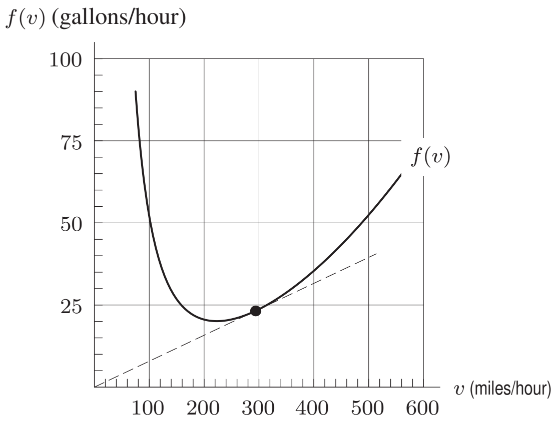


Figure 4.63

(d) The pilot's goal with regard to $f(v)$ and $g(v)$ would depend on the purpose of the flight, and might even vary within a given flight. For example, if the mission involved aerial surveillance or banner-towing over some limited area, or if the plane was flying a holding pattern, then the pilot would want to minimize $f(v)$ so as to remain aloft as long as possible. In a more normal situation where the purpose was economical travel between two fixed points, then the minimum net fuel expenditure for the trip would result from minimizing $g(v)$.

Strengthen Your Understanding

54. Since $A = 2x^2$, the area increases with x, and the maximum A occurs at the endpoint $x = 10$.

55. The function $V = h(20 - 2h)^2$ has domain $0 \le h \le 10$ for this model, or we cannot make the box from the cardboard. We maximize V on $0 \le h \le 10$.

56. The solution of an optimization problem depends on both the modeling function and the interval over which that function is optimized. The solution can occur either at a critical point of the function or at an endpoint of the interval. For example, if the modeling function is $f(x) = -(x-2)^2 + 6$ on the interval $5 \le x \le 10$, the absolute minimum occurs at $x = 10$ and the maximum occurs at $x = 5$; neither occurs at the vertex $x = 2$.

57. For example, if sides are 9 cm and 1 cm, the perimeter is 20 cm and the area is 9 cm^2.

58. If we interpret xy as the area of a rectangular field and $2x + 6y$ as a weighted perimeter of the field, one possible problem would be: A farmer would like to fence in a rectangular field of area 120 square feet. Fencing material along the north and south sides of the field costs \$1 per foot and along the east and west sides it costs \$3 per foot. What dimensions of the field minimize the cost for the fencing material?

59. The only additional information we have to provide is the cost of the material of the can. One possible choice is: The material for the bottom and the top of the can costs 2 cents per square centimeter and the material for the side of the can costs 1 cent per square centimeter.

Solutions for Section 4.4

Exercises

1. (a) See Figure 4.64.

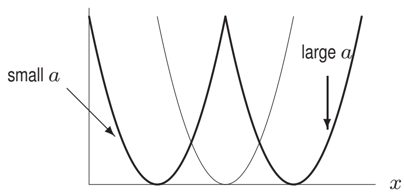


Figure 4.64

(b) We see in Figure 4.64 that in each case the graph of f is a parabola with one critical point, its vertex, on the positive x-axis. The critical point moves to the right along the x-axis as a increases.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$f'(x) = 2(x-a) = 0$$
$$x = a.$$

The only critical point is at $x = a$. As we saw in the graph, and as a increases, the critical point moves to the right.

2. (a) See Figure 4.65.

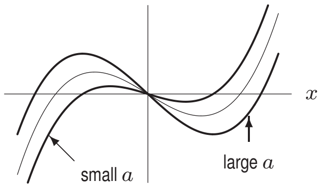


Figure 4.65

(b) We see in Figure 4.65 that in each case f has two critical points placed symmetrically about the origin, one in each of quadrants II and IV. As a increases, they appear to move farther apart, the one in quadrant II up and to the left, the one in quadrant IV down and to the right.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$f'(x) = 3x^2 - a = 0$$
$$x^2 = \frac{a}{3}$$
$$x = \pm\sqrt{\frac{a}{3}}.$$

There are two critical points, at $x = \sqrt{a/3}$ and $x = -\sqrt{a/3}$. (Since the parameter a is positive, the critical points exist.) As we saw in the graph, and as a increases, the critical points both move away from the vertical axis.

3. (a) See Figure 4.66.

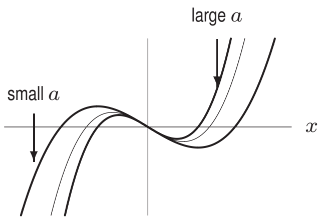


Figure 4.66

(b) We see in Figure 4.66 that in each case f has two critical points placed symmetrically about the origin, one in each of quadrants II and IV. As a increases, the critical points appear to move closer together, the one in quadrant II down and to the right, the one in quadrant IV up and to the left.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$f'(x) = 3ax^2 - 1 = 0$$
$$x^2 = \frac{1}{3a}$$
$$x = \pm\sqrt{\frac{1}{3a}}.$$

There are two critical points, at $x = \sqrt{1/(3a)}$ and $x = -\sqrt{1/(3a)}$. (Since the parameter a is positive, the critical points exist.) As we saw in the graph, and as a increases, the critical points both move toward the vertical axis.

4. (a) See Figure 4.67.

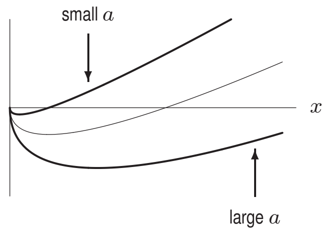


Figure 4.67

(b) The domain of f is $x \geq 0$ since the formula for $f(x)$ contains the square root of x. We see in Figure 4.67 that in each case f appears to have one critical point in quadrant IV where the function has a local minimum and possibly a second critical point at the origin where the graph appears to have a vertical tangent line. As the parameter a increases, the critical point in quadrant IV appears to move down and to the right.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$\begin{aligned} f'(x) = 1 - \frac{1}{2}ax^{-1/2} &= 0 \\ 1 - \frac{a}{2\sqrt{x}} &= 0 \\ 2\sqrt{x} &= a \\ \sqrt{x} &= \frac{a}{2} \\ x &= \frac{a^2}{4}. \end{aligned}$$

There is a critical point at $x = a^2/4$. Since f' is undefined at $x = 0$, there is also a critical point at $x = 0$. Increasing a has no effect on the critical point at $x = 0$. As we saw in the graph, as a increases the x-value of the other critical point increases and the critical point moves to the right.

5. (a) See Figure 4.68.

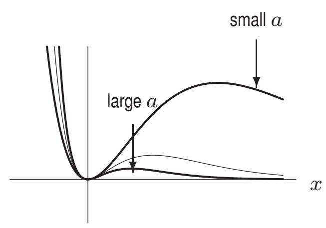


Figure 4.68

(b) We see in Figure 4.68 that in each case f appears to have two critical points. One critical point is a local minimum at the origin and the other is a local maximum in quadrant I. As the parameter a increases, the critical point in quadrant I appears to move down and to the left, closer to the origin.

(c) To find the critical points, we set the derivative equal to zero and solve for x. Using the product rule, we have:

$$\begin{aligned} f'(x) = x^2 \cdot e^{-ax}(-a) + 2x \cdot e^{-ax} &= 0 \\ xe^{-ax}(-ax + 2) &= 0 \\ x = 0 \quad \text{and} \quad x &= \frac{2}{a}. \end{aligned}$$

There are two critical points, at $x = 0$ and $x = 2/a$. As we saw in the graph, as a increases the nonzero critical point moves to the left.

6. (a) See Figure 4.69.

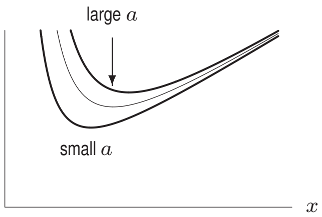


Figure 4.69

(b) We see in Figure 4.69 that in each case f appears to have one critical point, a local minimum in quadrant I. As the parameter a increases, the critical point appears to move up and to the right.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$\begin{aligned} f'(x) = -2ax^{-3} + 1 &= 0 \\ 1 &= \frac{2a}{x^3} \\ x^3 &= 2a \\ x &= \sqrt[3]{2a}. \end{aligned}$$

There is one critical point, at $x = \sqrt[3]{2a}$. As we saw in the graph, as a increases the critical point moves to the right.

7. (a) The larger the value of $|A|$, the steeper the graph (for the same x-value).

(b) The graph is shifted horizontally by B. The shift is to the left for positive B, to the right for negative B. There is a vertical asymptote at $x = -B$. See Figure 4.70.

(c)

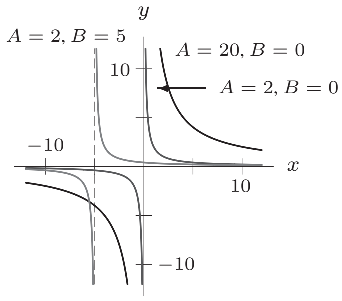


Figure 4.70

8. To find the critical points, we set the derivative of $f(w) = Aw^{-2} - Bw^{-1}$ equal to zero and solve for w.

$$\begin{aligned} f'(w) = -2Aw^{-3} - (-1)Bw^{-2} &= 0 \\ \frac{-2A}{w^3} + \frac{B}{w^2} &= 0 \\ \frac{-2A + Bw}{w^3} &= 0 \\ -2A + Bw &= 0 \\ w &= \frac{2A}{B}. \end{aligned}$$

Although f' is undefined at $w = 0$, this is not a critical point since it is not in the domain of f. The only critical point is $w = 2A/B$.

9. We have $f'(x) = -ae^{-ax}$, so that $f'(0) = -a$. We see that as a increases the slopes of the curves at the origin become more and more negative. Thus, A corresponds to $a = 1$, B to $a = 2$, and C to $a = 5$.

10. We have $f'(x) = (1 - ax)e^{-ax}$, so that the critical points occur when $1 - ax = 0$, that is, when $x = 1/a$. We see that as a increases the x-value of the extrema moves closer to the origin. Looking at the figure we see that A's local maximum is the farthest from the origin, while C's is the closest. Thus, A corresponds to $a = 1$, B to $a = 2$, and C to $a = 3$.

11. (a) Figure 4.71 shows the effect of varying a with $b = 1$.
(b) Figure 4.72 shows the effect of varying b with $a = 1$.

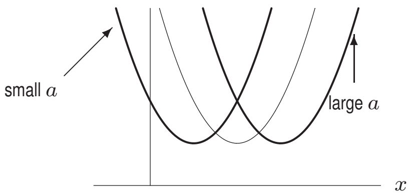


Figure 4.71

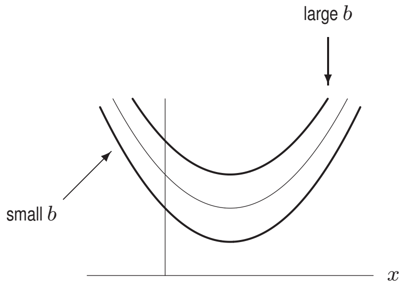


Figure 4.72

(c) From Figure 4.71 it appears that increasing a shifts the graph and its critical point to the right. From Figure 4.72 it appears that increasing b moves the graph and its critical point up.
(d) To find the critical points, we set the derivative equal to zero and solve for x.

$$f'(x) = 2(x - a) = 0$$
$$x = a.$$

There is one critical point, at $x = a$. As we saw in the graph, as a increases the critical point moves to the right. As b increases the critical point does not move horizontally.

12. (a) Figure 4.73 shows the effect of varying a with $b = 1$.
(b) Figure 4.74 shows the effect of varying b with $a = 1$.

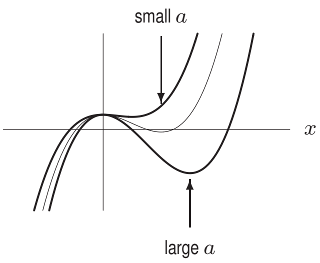


Figure 4.73

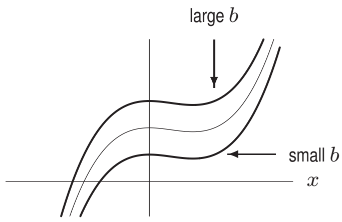


Figure 4.74

(c) From the graphs it appears that there are two critical points, a local maximum at $x = 0$ and a local minimum in the region $x > 0$. From Figure 4.73 it appears that increasing a moves the local minimum down and to the right, and does not move the local maximum. From Figure 4.74 it appears that increasing b moves both critical points up.
(d) To find the critical points, we set the derivative equal to zero and solve for x.

$$f'(x) = 3x^2 - 2ax = 0$$
$$x(3x - 2a) = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{2a}{3}.$$

There are two critical points, at $x = 0$ and at $x = 2a/3$. As a increases the critical point to the right moves farther right. As b increases, neither critical point moves horizontally. This matches what we saw in the graphs.

13. (a) Figure 4.75 shows the effect of varying a with $b = 1$.
(b) Figure 4.76 shows the effect of varying b with $a = 1$.

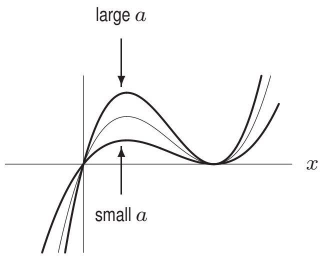


Figure 4.75

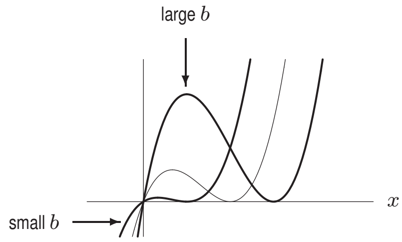


Figure 4.76

(c) In each case f appears to have two critical points, a local maximum in quadrant I and a local minimum on the positive x-axis. From Figure 4.75 it appears that increasing a moves the local maximum up and does not move the local minimum. From Figure 4.76 it appears that increasing b moves the local maximum up and to the right and moves the local minimum to the right along the x-axis.

(d) To find the critical points, we set the derivative equal to zero and solve for x. Using the product rule, we have:

$$\begin{aligned} f'(x) = ax \cdot 2(x-b) + a \cdot (x-b)^2 &= 0 \\ a(x-b)(2x + (x-b)) &= 0 \\ a(x-b)(3x-b) &= 0 \\ x = b \quad \text{or} \quad x &= \frac{b}{3}. \end{aligned}$$

There are two critical points, at $x = b$ and at $x = b/3$. Increasing a does not move either critical point horizontally. Increasing b moves both critical points to the right. This confirms what we saw in the graphs.

14. (a) Figure 4.77 shows the effect of varying a with $b = 1$.

(b) Figure 4.78 shows the effect of varying b with $a = 1$.

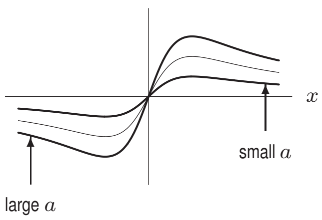


Figure 4.77

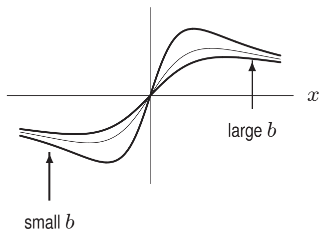


Figure 4.78

(c) From the graphs it appears that in all cases f has two critical points symmetrically placed about the origin, a local maximum in quadrant I and a local minimum in quadrant III. From Figure 4.77 it appears that increasing a moves the local maximum up and the local minimum down. From Figure 4.78 it appears that increasing b moves the local maximum down and to the right and the local minimum up and to the left. Thus increasing b appears to move both critical points closer to the horizontal axis and farther from the vertical axis.

(d) To find the critical points, we set the derivative equal to zero and solve for x. Using the quotient rule, we have:

$$\begin{aligned} f'(x) = \frac{(x^2+b)\cdot a - ax \cdot 2x}{(x^2+b)^2} &= 0 \\ \frac{ax^2 + ab - 2ax^2}{(x^2+b)^2} &= 0 \\ \frac{ab - ax^2}{(x^2+b)^2} &= 0 \\ ab - ax^2 &= 0 \\ a(b - x^2) &= 0 \\ x &= \pm\sqrt{b}. \end{aligned}$$

There are two critical points, at $x = \sqrt{b}$ and $x = -\sqrt{b}$. Increasing a does not move either critical point horizontally. Increasing b moves both critical points farther from the vertical axis. This confirms what we saw in the graphs.

15. (a) Figure 4.79 shows the effect of varying a with $b = 1$.
(b) Figure 4.80 shows the effect of varying b with $a = 1$.

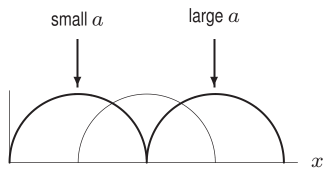


Figure 4.79

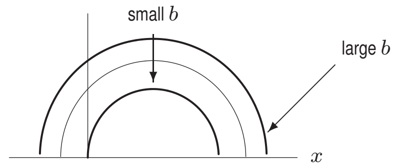


Figure 4.80

(c) The domain of $f(x) = \sqrt{b-(x-a)^2}$ is limited by the square root to $b-(x-a)^2 \geq 0$, so it is $a-\sqrt{b} < x < a+\sqrt{b}$.
From the graphs in parts (a) and (b) it appears that in all cases f has three critical points, one a local maximum and the other two on the x-axis at the endpoints of the domain of f where the graph of the function has a vertical tangent line. From Figure 4.79 it appears that increasing a moves all three critical points to the right. From Figure 4.80 it appears that increasing b moves the local maximum up and moves the other two critical points away from each other horizontally along the x-axis.
(d) To find the critical points, we set the derivative equal to zero and solve for x. Using the chain rule, we have:

$$\begin{aligned} f'(x) = \frac{1}{2}(b-(x-a)^2)^{-1/2}(-2(x-a)) &= 0 \\ \frac{-(x-a)}{\sqrt{b-(x-a)^2}} &= 0 \\ -(x-a) &= 0 \\ x &= a. \end{aligned}$$

There is a critical point of f at $x = a$. As a increases this critical point moves to the right. As b increases, it does not move horizontally.
The derivative of f is undefined when $\sqrt{b-(x-a)^2} = 0$. Solving for x, we have:

$$\begin{aligned} \sqrt{b-(x-a)^2} &= 0 \\ b-(x-a)^2 &= 0 \\ (x-a)^2 &= b \\ x-a &= \pm\sqrt{b} \\ x &= a \pm \sqrt{b}. \end{aligned}$$

The function f has two additional critical points, at $x = a+\sqrt{b}$ and $x = a-\sqrt{b}$, where the derivative f' is not defined and the graph of f has a vertical tangent. As a increases both these critical points move to the right, and as b increases they move in opposite directions away from the line $x = a$ containing the other critical point. We saw this in the graphs.

16. (a) Figure 4.81 shows the effect of varying a with $b = 1$.
(b) Figure 4.82 shows the effect of varying b with $a = 1$.

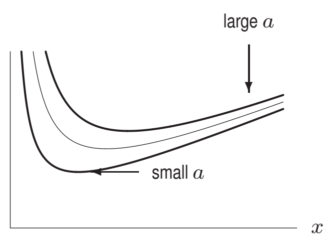


Figure 4.81

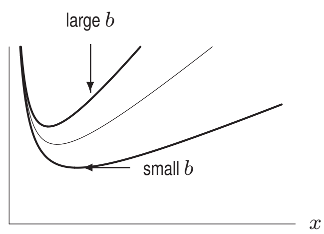


Figure 4.82

(c) It appears from the graphs that in all cases f has one critical point, a local minimum in quadrant I. From Figure 4.81 it appears that increasing a moves the critical point up and to the right. From Figure 4.82 it appears that increasing b moves the critical point up and to the left.

(d) To find the critical points, we set the derivative equal to zero and solve for x.

$$\begin{aligned} f'(x) = -ax^{-2} + b &= 0 \\ \frac{-a}{x^2} + b &= 0 \\ b &= \frac{a}{x^2} \\ x^2 &= \frac{a}{b} \\ x &= \pm\sqrt{\frac{a}{b}}. \end{aligned}$$

Since the domain of f is $x > 0$ we are only interested in positive values for x, so there is one critical point at $x = \sqrt{a/b}$. As a increases, this value gets larger and the critical point moves to the right. As b increases, this value gets smaller and the critical point moves to the left. This confirms what we see in the graphs.

Problems

17. We have $f'(x) = 1 - a^2/x^2$, so the critical points occur at $1 - a^2/x^2 = 0$, that is $x = \pm a$. We see that A has a minimum at $x = 3$, B at $x = 2$, and C at $x = 1$. Thus, C corresponds to $a = 1$, B to $a = 2$, and A to $a = 3$. The third value is 3.

18. For the curves to intersect in the same point for any positive value of a, say a and $b \neq a$, we need $x + a\sin x = x + b\sin x$, that is $(a-b)\sin x = 0$. Because $a - b \neq 0$, we find that $\sin x = 0$, which occurs at $x = 0, \pm\pi, \pm 2\pi \ldots$. For any integer n, the graphs go through the points $(n\pi, n\pi)$ for all a.

19. (a) Graphs of $y = xe^{-bx}$ for $b = 1, 2, 3, 4$ are in Figure 4.83. All the graphs rise at first, passing through the origin, reach a maximum and then decay toward 0. If b is small, the graph rises longer and to a higher maximum before the decay begins.

(b) Since

$$\frac{dy}{dx} = (1 - bx)e^{-bx},$$

we see

$$\frac{dy}{dx} = 0 \quad \text{at} \quad x = \frac{1}{b}.$$

The critical point has coordinates $(1/b, 1/(be))$. If b is small, the x and y-coordinates of the critical point are both large, indicating a higher maximum further to the right. See Figure 4.84.

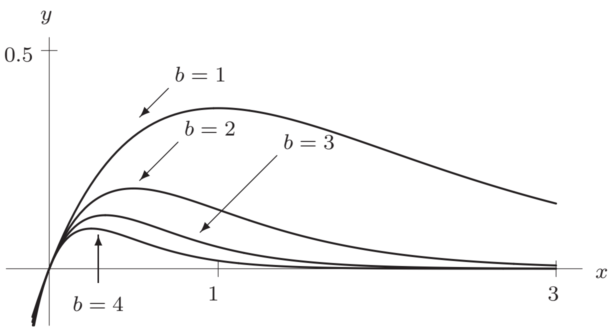


Figure 4.83

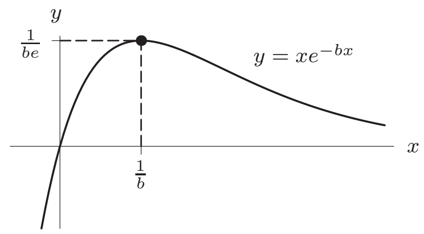


Figure 4.84

20. (a) Let $f(x) = axe^{-bx}$. To find the local maxima and local minima of f, we solve

$$f'(x) = ae^{-bx} - abxe^{-bx} = ae^{-bx}(1 - bx) \begin{cases} = 0 & \text{if } x = 1/b \\ < 0 & \text{if } x > 1/b \\ > 0 & \text{if } x < 1/b. \end{cases}$$

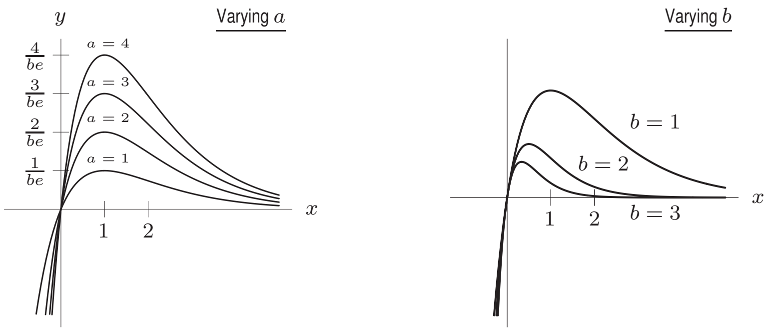
Therefore, f is increasing ($f' > 0$) for $x < 1/b$ and decreasing ($f' > 0$) for $x > 1/b$. A local maximum occurs at $x = 1/b$. There are no local minima. To find the points of inflection, we write

$$
\begin{aligned}
f''(x) &= -abe^{-bx} + ab^2xe^{-bx} - abe^{-bx} \\
&= -2abe^{-bx} + ab^2xe^{-bx} \\
&= ab(bx-2)e^{-bx},
\end{aligned}
$$

so $f'' = 0$ at $x = 2/b$. Therefore, f is concave up for $x < 2/b$ and concave down for $x > 2/b$, and the inflection point is $x = 2/b$.

(b) Varying a stretches or flattens the graph but does not affect the critical point $x = 1/b$ and the inflection point $x = 2/b$. Since the critical and inflection points are depend on b, varying b will change these points, as well as the maximum $f(1/b) = a/be$. For example, an increase in b will shift the critical and inflection points to the left, and also lower the maximum value of f.

(c)



21. Cubic polynomials are all of the form $f(x) = Ax^3 + Bx^2 + Cx + D$. There is an inflection point at the origin $(0, 0)$ if $f''(0) = 0$ and $f(0) = 0$. Since $f(0) = D$, we must have $D = 0$. Since $f''(x) = 6Ax + 2B$, giving $f''(0) = 2B$, we must have $B = 0$. The family of cubic polynomials with inflection point at the origin is the two parameter family $f(x) = Ax^3 + Cx$.

22. (a) Writing $y = L(1 + Ae^{-kt})^{-1}$, we find the first derivative by the chain rule

$$\frac{dy}{dt} = -L(1 + Ae^{-kt})^{-2}(-Ake^{-kt}) = \frac{LAke^{-kt}}{(1 + Ae^{-kt})^2}.$$

Using the quotient rule to calculate the second derivative gives

$$
\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{\left(-LAk^2e^{-kt}(1 + Ae^{-kt})^2 - 2LAke^{-kt}(1 + Ae^{-kt})(-Ake^{-kt})\right)}{(1 + Ae^{-kt})^4} \\
&= \frac{LAk^2e^{-kt}(-1 + Ae^{-kt})}{(1 + Ae^{-kt})^3}.
\end{aligned}
$$

(b) Since $L, A > 0$ and $e^{-kt} > 0$ for all t, the factor LAk^2e^{-kt} and the denominator are never zero. Thus, possible inflection points occur where

$$-1 + Ae^{-kt} = 0.$$

Solving for t gives

$$t = \frac{\ln A}{k}.$$

(c) The second derivative is positive to the left of $t = \ln(A)/k$ and negative to the right, so the function changes from concave up to concave down at $t = \ln(A)/k$.

23. (a) See Figure 4.85.

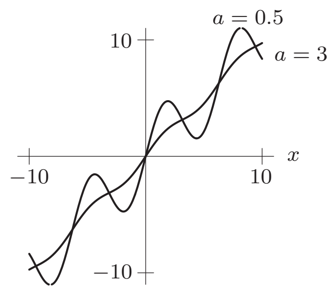


Figure 4.85

(b) The function $f(x) = x + a \sin x$ is increasing for all x if $f'(x) > 0$ for all x. We have $f'(x) = 1 + a \cos x$. Because $\cos x$ varies between -1 and 1, we have $1 + a \cos x > 0$ for all x if $-1 < a < 1$ but not otherwise. When $a = 1$, the function $f(x) = x + \sin x$ is increasing for all x, as is $f(x) = x - \sin x$, obtained when $a = -1$. Thus $f(x)$ is increasing for all x if $-1 \le a \le 1$.

24. (a) See Figure 4.86.

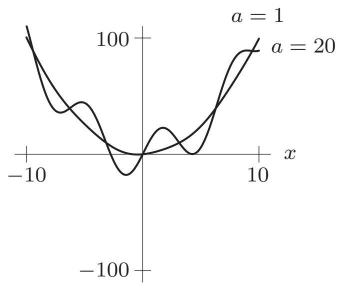
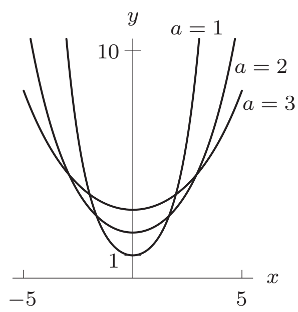


Figure 4.86

(b) The function $f(x) = x^2 + a \sin x$ is concave up for all x if $f''(x) > 0$ for all x. We have $f''(x) = 2 - a \sin x$. Because $\sin x$ varies between -1 and 1, we have $2 - a \sin x > 0$ for all x if $-2 < a < 2$ but not otherwise. Thus $f(x)$ is concave up for all x if $-2 < a < 2$.

25. For $-5 \le x \le 5$, we have the graphs of $y = a \cosh(x/a)$ shown below.



Increasing the value of a makes the graph flatten out and raises the minimum value. The minimum value of y occurs at $x = 0$ and is given by

$$y = a \cosh\left(\frac{0}{a}\right) = a\left(\frac{e^{0/a} + e^{-0/a}}{2}\right) = a.$$

26. Graphs of $y = e^{-ax}\sin(bx)$ for $b = 1$ and various values of a are shown in Figure 4.87. The parameter a controls the amplitude of the oscillations.

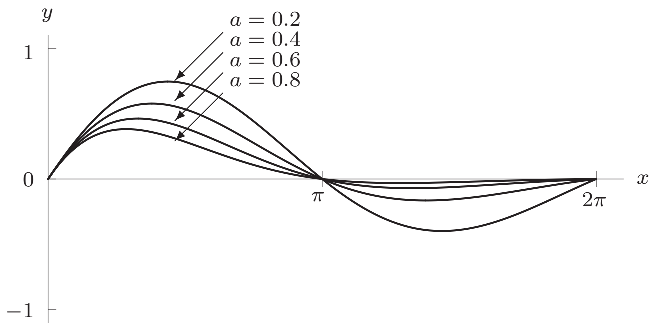
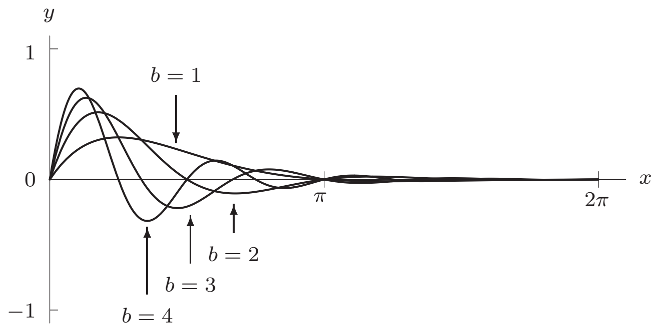


Figure 4.87

27.



The larger the value of b, the narrower the humps and more humps per given region there are in the graph.

28. Since $f'(x) = abe^{-bx}$, we have $f'(x) > 0$ for all x. Therefore, f is increasing for all x. Since $f''(x) = -ab^2e^{-bx}$, we have $f''(x) < 0$ for all x. Therefore, f is concave down for all x.

29. (a) The critical point will occur where $f'(x) = 0$. But by the product rule,

$$\begin{aligned} f'(x) = be^{1+bx} + b^2xe^{1+bx} &= 0 \\ be^{1+bx}(1+bx) &= 0 \\ 1 + bx &= 0 \\ x &= -\frac{1}{b}. \end{aligned}$$

Thus the critical point of f is located at $x = -\frac{1}{b}$.

(b) To determine whether this critical point is a local minimum or local maximum, we can use the first derivative test. Since both b and e^{1+bx} are positive for all x, the sign of $f'(x) = be^{1+bx}(1 + bx)$ depends on the sign of $1 + bx$.

$$\begin{aligned} &\text{for } x < -\tfrac{1}{b}, \quad f'(x) \text{ is negative} \\ &\text{for } x = \tfrac{1}{b}, \quad f'(x) = 0 \\ &\text{for } x > -\tfrac{1}{b}, \quad f'(x) \text{ is positive} \end{aligned}$$

Therefore, f goes from decreasing to increasing at $x = -\frac{1}{b}$, making this point a local minimum.

(c) We only need to substitute the critical point $x = -\frac{1}{b}$ into the original function f:

$$\begin{aligned} f\left(-\frac{1}{b}\right) &= b\left(-\frac{1}{b}\right)e^{1+b\left(-\frac{1}{b}\right)} \\ &= -1e^{1-1} \\ &= -1. \end{aligned}$$

This answer does not depend on the value of b. Though the x-coordinate of the critical point depends on b, the y-coordinate does not.

30. We study $h'(x) = k - e^{-x}$. Notice that the critical point of h, if it exists, is located where $h'(x) = 0$:

$$\begin{aligned} h'(x) = k - e^{-x} &= 0 \\ e^{-x} &= k \\ -x &= \ln k \\ x &= -\ln k. \end{aligned}$$

(a) Since the domain of the natural logarithm is all positive real numbers, $x = -\ln k$ does not exist (and hence there will be no critical point of h) if $k \leq 0$.
(b) In order for the critical point $x = -\ln k$ to exist, we must have $k > 0$.
(c) There is a horizontal asymptote for h only when $\lim_{x\to-\infty} h(x)$ or $\lim_{x\to+\infty} h(x)$ exists and is finite. But

$$\lim_{x\to-\infty} e^{-x} = +\infty \quad \text{and} \quad \lim_{x\to+\infty} e^{-x} = 0,$$

so

$$\lim_{x\to-\infty} h(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} kx.$$

The first limit is never finite and the only way to make the second one finite is to set $k = 0$.

31. The function g will have a critical point when $g'(x) = 0$. Solving this equation gives

$$\begin{aligned} g'(x) = 1 - ke^{x} &= 0 \\ ke^{x} &= 1 \\ e^{x} &= \frac{1}{k} \\ x &= \ln\frac{1}{k} = -\ln k. \end{aligned}$$

Since the natural logarithm has a domain of all positive real numbers, such a value for x may only exist for $k > 0$.

32. Since the horizontal asymptote is $y = 5$, we know $a = 5$. The value of b can be any number. Thus $y = 5(1 - e^{-bx})$ for any $b > 0$.

33. Since the maximum is on the y-axis, $a = 0$. At that point, $y = be^{-0^2/2} = b$, so $b = 3$.

34. The maximum of $y = e^{-(x-a)^2/b}$ occurs at $x = a$. (This is because the exponent $-(x-a)^2/b$ is zero when $x = a$ and negative for all other x-values. The same result can be obtained by taking derivatives.) Thus we know that $a = 2$.

Points of inflection occur where d^2y/dx^2 changes sign, that is, where $d^2y/dx^2 = 0$. Differentiating gives

$$\begin{aligned} \frac{dy}{dx} &= -\frac{2(x-2)}{b}e^{-(x-2)^2/b} \\ \frac{d^2y}{dx^2} &= -\frac{2}{b}e^{-(x-2)^2/b} + \frac{4(x-2)^2}{b^2}e^{-(x-2)^2/b} = \frac{2}{b}e^{-(x-2)^2/b}\left(-1 + \frac{2}{b}(x-2)^2\right). \end{aligned}$$

Since $e^{-(x-2)^2/b}$ is never zero, $d^2y/dx^2 = 0$ where

$$-1 + \frac{2}{b}(x-2)^2 = 0.$$

We know $d^2y/dx^2 = 0$ at $x = 1$, so substituting $x = 1$ gives

$$-1 + \frac{2}{b}(1-2)^2 = 0.$$

Solving for b gives

$$\begin{aligned} -1 + \frac{2}{b} &= 0 \\ b &= 2. \end{aligned}$$

Since $a = 2$, the function is

$$y = e^{-(x-2)^2/2}.$$

You can check that at $x = 2$, we have

$$\frac{d^2y}{dx^2} = \frac{2}{2}e^{-0}(-1+0) < 0$$

so the point $x = 2$ does indeed give a maximum. See Figure 4.88.

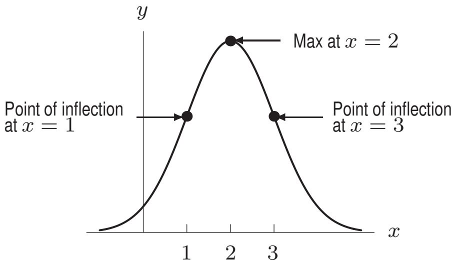


Figure 4.88: Graph of $y = e^{-(x-2)^2/2}$

35. We want a function of the form

$$y = \frac{L}{1 + Ae^{-kt}}.$$

Since the carrying capacity is 12, we have $L = 12$. The y-intercept is 4, so

$$\frac{L}{1+A} = \frac{12}{1+A} = 4$$
$$A = 2.$$

The point of inflection is at $(0.5, 6)$, so

$$0.5 = \frac{\ln(A)}{k} = \frac{\ln(2)}{k}$$
$$k = 1.386.$$

Thus, the function is

$$y = \frac{12}{1 + 2e^{-1.386x}}.$$

36. Since $y(0) = a/(1+b) = 2$, we have $a = 2 + 2b$. To find a point of inflection, we calculate

$$\frac{dy}{dt} = \frac{abe^{-t}}{(1+be^{-t})^2},$$

and using the quotient rule,

$$\frac{d^2y}{dx^2} = \frac{-abe^{-t}(1+be^{-t})^2 - abe^{-t}2(1+be^{-t})(-be^{-t})}{(1+be^{-t})^4}$$
$$= \frac{abe^{-t}(-1+be^{-t})}{(1+be^{-t})^3}.$$

The second derivative is equal to 0 when $be^{-t} = 1$, or $b = e^t$. When $t = 1$, we have $b = e$. The second derivative changes sign at this point, so we have an inflection point. Thus

$$y = \frac{2+2e}{1+e^{1-t}}.$$

37. Since the x^3 term has coefficient 1, the polynomial is of the form

$$y = x^3 + ax^2 + bx + c.$$

Differentiating gives

$$\frac{dy}{dx} = 3x^2 + 2ax + b.$$

There is a critical point at $x = 2$, so $dy/dx = 0$ at $x = 2$. Thus

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(2^2) + 2a(2) + b = 12 + 4a + b = 0, \text{ so } 4a + b = -12.$$

We take the second derivative to look for the inflection point. We find

$$\frac{d^2y}{dx^2} = 6x + 2a,$$

and for an inflection point at $x = 1$, we have $6 + 2a = 0$, so $a = -3$. We now use $a = -3$ and the relationship $4a + b = -12$, which gives $4(-3) + b = -12$, so $b = 0$.

We now have

$$y = x^3 - 3x^2 + c,$$

and using the point (1,4) gives

$$\begin{aligned} 1 - 3 + c &= 4, \\ c &= 6. \end{aligned}$$

Thus, $y = x^3 - 3x^2 + 6$.

38. Since the graph is symmetric about the y-axis, the polynomial must have only even powers. Also, since the y-intercept is 0, the constant term must be zero. Thus, the polynomial is of the form

$$y = ax^4 + bx^2.$$

Differentiating gives

$$\frac{dy}{dx} = 4ax^3 + 2bx = x(4ax^2 + 2b).$$

Thus $dy/dx = 0$ at $x = 0$ and when $4ax^2 + 2b = 0$. Since the maxima occur where $x = \pm 1$, we have $4a + 2b = 0$, so $b = -2a$.

We are given $y = 2$ when $x = \pm 1$, so using $y = ax^4 + bx^2$ gives

$$\begin{aligned} a + b &= 2, \\ a - 2a = -a &= 2, \end{aligned}$$

which gives $a = -2$ and $b = 4$. Thus

$$y = -2x^4 + 4x^2.$$

To see if the points $(1, 2)$ and $(-1, 2)$ are local maxima, we take the second derivative,

$$\frac{d^2y}{dx^2} = -24x^2 + 8,$$

which is negative if $x = \pm 1$. The critical points at $x = \pm 1$ are local maxima. Since the leading coefficient of this polynomial is negative, we know the graph decreases without bound as $|x|$ approaches $\pm\infty$, so these critical points are also global maxima. Note that at $x = 0$ the second derivative is positive, so the point $(0, 0)$ is a local minimum. There is no global minimum.

39. Differentiating gives

$$\frac{dy}{dt} = 2abt\cos(bt^2).$$

Since the first critical point for positive t occurs at $t = 1$, we have $b = \pi/2$. Then at $t = 2$, we have

$$\left.\frac{dy}{dt}\right|_{t=2} = \pi a(2)\cos(2\pi) = 2a\pi = 3,$$

so $a = \dfrac{3}{2\pi}$, and

$$y = \frac{3}{2\pi}\sin\left(\frac{\pi t^2}{2}\right).$$

40. Differentiating gives

$$\frac{dy}{dx} = -2abt\sin(bt^2).$$

Since the first critical point for positive t occurs at $t = 1$, we have $b = \pi$. Then at $t = 1/\sqrt{2}$, we have

$$\left.\frac{dy}{dt}\right|_{t=\frac{1}{\sqrt{2}}} = -\frac{2\pi a}{\sqrt{2}}\sin\left(\frac{\pi}{2}\right) = -\frac{2a\pi}{\sqrt{2}} = -2,$$

so $a = \dfrac{\sqrt{2}}{\pi}$, and

$$y = \frac{\sqrt{2}}{\pi}\cos(\pi t^2).$$

41. Differentiating $y = ae^{-x} + bx$ gives

$$\frac{dy}{dx} = -ae^{-x} + b.$$

Since the global minimum occurs for $x = 1$, we have $-a/e + b = 0$, so $b = a/e$.

The value of the function at $x = 1$ is 2, so we have $2 = a/e + b$, which gives

$$2 = \frac{a}{e} + \frac{a}{e} = \frac{2a}{e},$$

so $a = e$ and $b = 1$. Thus

$$y = e^{1-x} + x.$$

We compute $d^2y/dx^2 = e^{1-x}$, which is always positive, so this confirms that $x = 1$ is a local minimum. Because the value of $e^{1-x} + x$ as $x \to \pm\infty$ grows without bound, this local minimum is a global minimum.

42. Differentiating $y = bxe^{-ax}$ gives

$$\frac{dy}{dx} = be^{-ax} - abxe^{-ax} = be^{-ax}(1 - ax).$$

Since we have a critical point at $x = 3$, we know that $1 - 3a = 0$, so $a = 1/3$.

If $b > 0$, the first derivative goes from positive values to the left of $x = 3$ to negative values on the right of $x = 3$, so we know this critical point is a local maximum. Since the function value at this local maximum is 6, we have

$$6 = 3be^{-3/3} = \frac{3b}{e},$$

so $b = 2e$ and

$$y = 2xe^{1-x/3}.$$

43. Notice that since y is an odd function, choosing a and b so there is a minimum at $(3, 12)$ automatically results in a local maximum at $(-3, -12)$.

Differentiating $y = at + b/t$, we have

$$\frac{dy}{dt} = a - bt^{-2}, \text{ and } \frac{d^2y}{dt^2} = 2bt^{-3}.$$

The critical points require $dy/dt = 0$ at $t = \pm 3$. This gives $a - b/9 = 0$ so $b = 9a$.

Substituting the point $(3, 12)$ in the original equation, we have

$$12 = 3a + 9a/3 = 6a.$$

Thus $a = 2$, and $b = 18$, and

$$y = 2t + 18/t.$$

The second derivative test affirms that $(3, 12)$ is a local minimum, as the second derivative is positive at $t = 3$.

44. We begin by finding the intercepts, which occur where $f(x) = 0$, that is

$$x - k\sqrt{x} = 0$$
$$\sqrt{x}(\sqrt{x} - k) = 0$$

$$\text{so} \quad x = 0 \quad \text{or} \quad \sqrt{x} = k, \quad x = k^2.$$

So 0 and k^2 are the x-intercepts. Now we find the location of the critical points by setting $f'(x)$ equal to 0:

$$f'(x) = 1 - k\left(\frac{1}{2}x^{-(1/2)}\right) = 1 - \frac{k}{2\sqrt{x}} = 0.$$

This means

$$1 = \frac{k}{2\sqrt{x}}, \quad \text{so} \quad \sqrt{x} = \frac{1}{2}k, \quad \text{and} \quad x = \frac{1}{4}k^2.$$

We can use the second derivative to verify that $x = \frac{k^2}{4}$ is a local minimum. $f''(x) = 1 + \frac{k}{4x^{3/2}}$ is positive for all $x > 0$. So the critical point, $x = \frac{1}{4}k^2$, is $1/4$ of the way between the x-intercepts, $x = 0$ and $x = k^2$. Since $f''(x) = \frac{1}{4}kx^{-3/2}$, $f''(\frac{1}{4}k^2) = 2/k^2 > 0$, this critical point is a minimum.

45. (a) The x-intercept occurs where $f(x) = 0$, so

$$\begin{aligned} ax - x\ln x &= 0 \\ x(a - \ln x) &= 0. \end{aligned}$$

Since $x > 0$, we must have

$$\begin{aligned} a - \ln x &= 0 \\ \ln x &= a \\ x &= e^a. \end{aligned}$$

(b) See Figures 4.89 and 4.90.

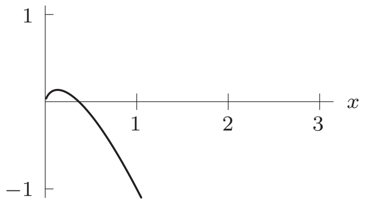


Figure 4.89: Graph of $f(x)$ with $a = -1$

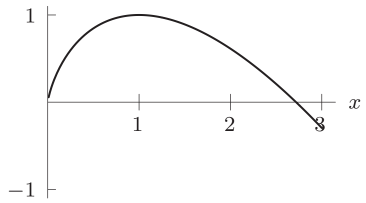


Figure 4.90: Graph of $f(x)$ with $a = 1$

(c) Differentiating gives $f'(x) = a - \ln x - 1$. Critical points are obtained by solving

$$\begin{aligned} a - \ln x - 1 &= 0 \\ \ln x &= a - 1 \\ x &= e^{a-1}. \end{aligned}$$

Since $e^{a-1} > 0$ for all a, there is no restriction on a. Now,

$$f(e^{a-1}) = ae^{a-1} - e^{a-1}\ln(e^{a-1}) = ae^{a-1} - (a-1)e^{a-1} = e^{a-1},$$

so the coordinates of the critical point are (e^{a-1}, e^{a-1}). From the graphs, we see that this critical point is a local maximum; this can be confirmed using the second derivative:

$$f''(x) = -\frac{1}{x} < 0 \qquad \text{for } x = e^{a-1}.$$

46. (a) Figures 4.91- 4.94 show graphs of $f(x) = x^2 + \cos(kx)$ for various values of k. For $k = 0.5$ and $k = 1$, the graphs look like parabolas. For $k = 3$, there is some waving in the parabola, which becomes more noticeable if $k = 5$. The waving begins to happen at about $k = 1.5$.

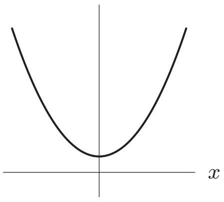


Figure 4.91: $k = 0.5$

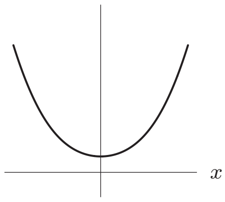


Figure 4.92: $k = 1$

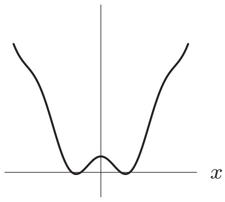


Figure 4.93: $k = 3$

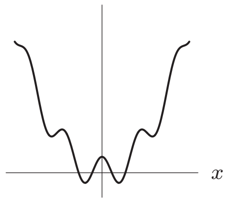


Figure 4.94: $k = 5$

(b) Differentiating, we have

$$f'(x) = 2x - k\sin(kx)$$
$$f''(x) = 2 - k^2\cos(kx).$$

If $k^2 \le 2$, then $f''(x) \ge 2 - 2\cos(kx) \ge 0$, since $\cos(kx) \le 1$. Thus, the graph is always concave up if $k \le \sqrt{2}$. If $k^2 > 2$, then $f''(x)$ changes sign whenever $\cos(kx) = 2/k^2$, which occurs for infinitely many values of x, since $0 < 2/k^2 < 1$.

(c) Since $f'(x) = 2x - k\sin(kx)$, we want to find all points where

$$2x - k\sin(kx) = 0.$$

Since

$$-1 \le \sin(kx) \le 1,$$

$f'(x) \ne 0$ if $x > k/2$ or $x < -k/2$. Thus, all the roots of $f'(x)$ must be in the interval $-k/2 \le x \le k/2$. The roots occur where the line $y = 2x$ intersects the curve $y = k\sin(kx)$, and there are only a finite number of such points for $-k/2 \le x \le k/2$.

47. (a) Figure 4.95 suggests that each graph decreases to a local minimum and then increases sharply. The local minimum appears to move to the right as k increases. It appears to move up until $k = 1$, and then to move back down.

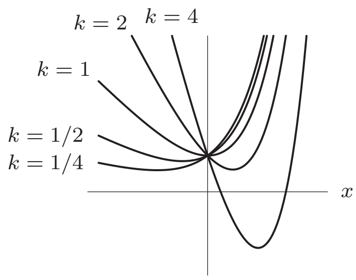


Figure 4.95

(b) $f'(x) = e^x - k = 0$ when $x = \ln k$. Since $f'(x) < 0$ for $x < \ln k$ and $f'(x) > 0$ for $x > \ln k$, f is decreasing to the left of $x = \ln k$ and increasing to the right, so f reaches a local minimum at $x = \ln k$.

(c) The minimum value of f is

$$f(\ln k) = e^{\ln k} - k(\ln k) = k - k\ln k.$$

Since we want to maximize the expression $k - k\ln k$, we can imagine a function $g(k) = k - k\ln k$. To maximize this function we simply take its derivative and find the critical points. Differentiating, we obtain

$$g'(k) = 1 - \ln k - k(1/k) = -\ln k.$$

Thus $g'(k) = 0$ when $k = 1$, $g'(k) > 0$ for $k < 1$, and $g'(k) < 0$ for $k > 1$. Thus $k = 1$ is a local maximum for $g(k)$. That is, the largest global minimum for f occurs when $k = 1$.

48. (a) The graph of r has a vertical asymptote if the denominator is zero. Since $(x-b)^2$ is nonnegative, the denominator can only be zero if $a \le 0$. Then

$$a + (x-b)^2 = 0$$
$$(x-b)^2 = -a$$
$$x - b = \pm\sqrt{-a}$$
$$x = b \pm \sqrt{-a}.$$

In order for there to be a vertical asymptote, a must be less than or equal to zero. There are no restrictions on b.

(b) Differentiating gives

$$r'(x) = \frac{-1}{(a+(x-b)^2)^2} \cdot 2(x-b),$$

so $r' = 0$ when $x = b$. If $a \le 0$, then r' is undefined at the same points at which r is undefined. Thus the only critical point is $x = b$. Since we want $r(x)$ to have a maximum at $x = 3$, we choose $b = 3$. Also, since $r(3) = 5$, we have

$$r(3) = \frac{1}{a + (3-3)^2} = \frac{1}{a} = 5 \qquad \text{so} \qquad a = \frac{1}{5}.$$

49. (a) $f'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$; so $x = 0$ and $x = \pm\sqrt{-a/2}$ (if $\pm\sqrt{-a/2}$ is real, i.e. if $-a/2 \geq 0$) are critical points.

(b) $x = 0$ is a critical point for any value of a. In order to guarantee that $x = 0$ is the only critical point, the factor $2x^2 + a$ should not have a root other than possibly $x = 0$. This means $a \geq 0$, since $2x^2 + a$ has only one root ($x = 0$) for $a = 0$, and no roots for $a > 0$. There is no restriction on the constant b.

Now $f''(x) = 12x^2 + 2a$ and $f''(0) = 2a$.

If $a > 0$, then by the second derivative test, $f(0)$ is a local minimum.

If $a = 0$, then $f(x) = x^4 + b$, which has a local minimum at $x = 0$.

So $x = 0$ is a local minimum when $a \geq 0$.

(c) Again, b will have no effect on the location of the critical points. In order for $f'(x) = 2x(2x^2 + a)$ to have three different roots, the constant a has to be negative. Let $a = -2c^2$, for some $c > 0$. Then

$f'(x) = 4x(x^2 - c^2) = 4x(x - c)(x + c)$.

The critical points of f are $x = 0$ and $x = \pm c = \pm\sqrt{-a/2}$.

To the left of $x = -c$, $f'(x) < 0$.

Between $x = -c$ and $x = 0$, $f'(x) > 0$.

Between $x = 0$ and $x = c$, $f'(x) < 0$.

To the right of $x = c$, $f'(x) > 0$.

So, $f(-c)$ and $f(c)$ are local minima and $f(0)$ is a local maximum.

(d) For $a \geq 0$, there is exactly one critical point, $x = 0$. For $a < 0$ there are exactly three different critical points. These exhaust all the possibilities. (Notice that the value of b is irrelevant here.)

50. (a) The graphs are shown in Figures 4.96–4.101.

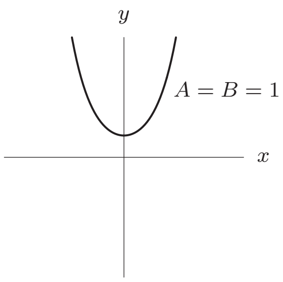


Figure 4.96: $A > 0, B > 0$

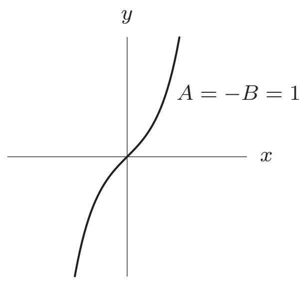


Figure 4.97: $A > 0, B < 0$

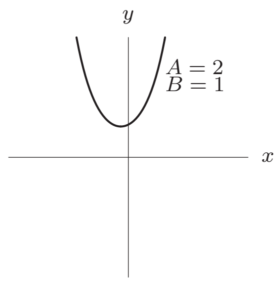


Figure 4.98: $A > 0, B > 0$

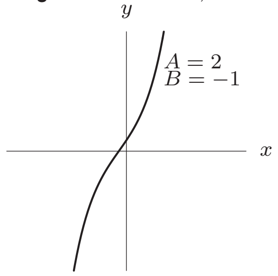


Figure 4.99: $A > 0, B < 0$

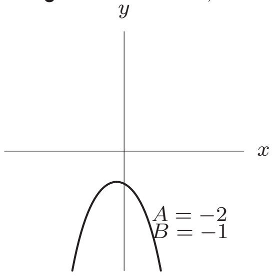


Figure 4.100: $A < 0, B < 0$

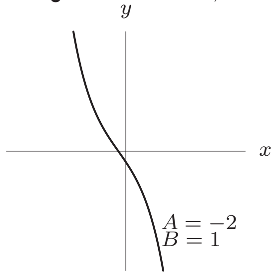


Figure 4.101: $A < 0, B > 0$

(b) If A and B have the same sign, the graph is U-shaped. If A and B are both positive, the graph opens upward. If A and B are both negative, the graph opens downward.

(c) If A and B have different signs, the graph appears to be everywhere increasing (if $A > 0, B < 0$) or decreasing (if $A < 0, B > 0$).

(d) The function appears to have a local maximum if $A < 0$ and $B < 0$, and a local minimum if $A > 0$ and $B > 0$.

To justify this, calculate the derivative

$$\frac{dy}{dx} = Ae^x - Be^{-x}.$$

Setting $dy/dx = 0$ gives

$$Ae^x - Be^{-x} = 0$$

$$\begin{aligned} Ae^x &= Be^{-x} \\ e^{2x} &= \frac{B}{A}. \end{aligned}$$

This equation has a solution only if B/A is positive, that is, if A and B have the same sign. In that case,

$$\begin{aligned} 2x &= \ln\left(\frac{B}{A}\right) \\ x &= \frac{1}{2}\ln\left(\frac{B}{A}\right). \end{aligned}$$

This value of x gives the only critical point.

To determine whether the critical point is a local maximum or minimum, we use the first derivative test. Since

$$\frac{dy}{dx} = Ae^x - Be^{-x},$$

we see that:

If $A > 0, B > 0$, we have $dy/dx > 0$ for large positive x and $dy/dx < 0$ for large negative x, so there is a local minimum.

If $A < 0, B < 0$, we have $dy/dx < 0$ for large positive x and $dy/dx > 0$ for large negative x, so there is a local maximum.

51. $T(t) =$ the temperature at time $t = a(1 - e^{-kt}) + b$.

(a) Since at time $t = 0$ the yam is at 20°C, we have

$$T(0) = 20^\circ = a\left(1 - e^0\right) + b = a(1-1) + b = b.$$

Thus $b = 20^\circ$C. Now, common sense tells us that after a period of time, the yam will heat up to about 200°, or oven temperature. Thus the temperature T should approach 200° as the time t grows large:

$$\lim_{t\to\infty} T(t) = 200^\circ\text{C} = a(1-0) + b = a + b.$$

Since $a + b = 200^\circ$, and $b = 20^\circ$C, this means $a = 180^\circ$C.

(b) Since we're talking about how quickly the yam is heating up, we need to look at the derivative, $T'(t) = ake^{-kt}$:

$$T'(t) = (180)ke^{-kt}.$$

We know $T'(0) = 2^\circ$C/min, so

$$2 = (180)ke^{-k(0)} = (180)(k).$$

So $k = (2^\circ\text{C/min})/180^\circ\text{C} = \frac{1}{90}\,\text{min}^{-1}$.

52. (a) Since

$$U = b\left(\frac{a^2 - ax}{x^2}\right) = 0 \quad \text{when} \quad x = a,$$

the x-intercept is $x = a$. There is a vertical asymptote at $x = 0$ and a horizontal asymptote at $U = 0$.

(b) Setting $dU/dx = 0$, we have

$$\frac{dU}{dx} = b\left(-\frac{2a^2}{x^3} + \frac{a}{x^2}\right) = b\left(\frac{-2a^2 + ax}{x^3}\right) = 0.$$

So the critical point is

$$x = 2a.$$

When $x = 2a$,

$$U = b\left(\frac{a^2}{4a^2} - \frac{a}{2a}\right) = -\frac{b}{4}.$$

The second derivative of U is

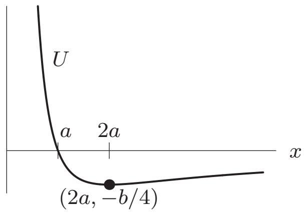
$$\frac{d^2U}{dx^2} = b\left(\frac{6a^2}{x^4} - \frac{2a}{x^3}\right).$$

When we evaluate this at $x = 2a$, we get

$$\frac{d^2U}{dx^2} = b\left(\frac{6a^2}{(2a)^4} - \frac{2a}{(2a)^3}\right) = \frac{b}{8a^2} > 0.$$

Since $d^2U/dx^2 > 0$ at $x = 2a$, we see that the point $(2a, -b/4)$ is a local minimum.

(c)



53. Both U and F have asymptotes at $x = 0$ and the x-axis. In Problem 52 we saw that U has intercept $(a, 0)$ and local minimum $(2a, -b/4)$. Differentiating U gives

$$F = b\left(\frac{2a^2}{x^3} - \frac{a}{x^2}\right).$$

Since

$$F = b\left(\frac{2a^2 - ax}{x^3}\right) = 0 \quad \text{for} \quad x = 2a,$$

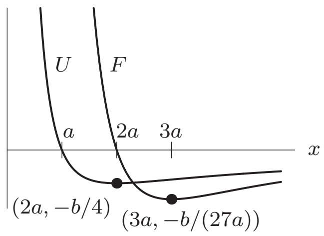
F has one intercept: $(2a, 0)$. Differentiating again to find the critical points:

$$\frac{dF}{dx} = b\left(-\frac{6a^2}{x^4} + \frac{2a}{x^3}\right) = b\left(\frac{-6a^2 + 2ax}{x^4}\right) = 0,$$

so $x = 3a$. When $x = 3a$,

$$F = b\left(\frac{2a^2}{27a^3} - \frac{a}{9a^2}\right) = -\frac{b}{27a}.$$

By the first or second derivative test, $x = 3a$ is a local minimum of F. See figure below.



54. (a) To find $\lim_{r\to 0^+} V(r)$, first rewrite $V(r)$ with a common denominator:

$$\begin{aligned}\lim_{r\to 0^+} V(r) &= \lim_{r\to 0^+} \frac{A}{r^{12}} - \frac{B}{r^6} \\ &= \lim_{r\to 0^+} \frac{A - Br^6}{r^{12}} \\ &\to \frac{A}{0^+} \to +\infty.\end{aligned}$$

As the distance between the two atoms becomes small, the potential energy diverges to $+\infty$.

(b) The critical point of $V(r)$ will occur where $V'(r) = 0$:

$$\begin{aligned}V'(r) = -\frac{12A}{r^{13}} + \frac{6B}{r^7} &= 0 \\ \frac{-12A + 6Br^6}{r^{13}} &= 0 \\ -12A + 6Br^6 &= 0 \\ r^6 &= \frac{2A}{B} \\ r &= \left(\frac{2A}{B}\right)^{1/6}\end{aligned}$$

To determine whether this is a local maximum or minimum, i we can use the first derivative test. Since r is positive, the sign of $V'(r)$ is determined by the sign of $-12A+6Br^6$. Notice that this is an increasing function of r for $r > 0$, so $V'(r)$ changes sign from $-$ to $+$ at $r = (2A/B)^{1/6}$. The first derivative test yields

r	$\leftarrow$	$\left(\frac{2A}{B}\right)^{1/6}$	$\rightarrow$
$V'(r)$	neg.	zero	pos.

Thus $V(r)$ goes from decreasing to increasing at the critical point $r = (2A/B)^{1/6}$, so this is a local minimum.

(c) Since $F(r) = -V'(r)$, the force is zero exactly where $V'(r) = 0$, i.e. at the critical points of V. The only critical point was the one found in part (b), so the only such point is $r = (2A/B)^{1/6}$.

(d) Since the numerator in $r = (2A/B)^{1/6}$ is proportional to $A^{1/6}$, the equilibrium size of the molecule increases when the parameter A is increased. Conversely, since B is in the denominator, when B is increased the equilibrium size of the molecules decrease.

55. (a) The force is zero where

$$\begin{aligned} f(r) = -\frac{A}{r^2} + \frac{B}{r^3} &= 0 \\ Ar^3 &= Br^2 \\ r &= \frac{B}{A}. \end{aligned}$$

The vertical asymptote is $r = 0$ and the horizontal asymptote is the r-axis.

(b) To find critical points, we differentiate and set $f'(r) = 0$:

$$\begin{aligned} f'(r) &= \frac{2A}{r^3} - \frac{3B}{r^4} = 0 \\ 2Ar^4 &= 3Br^3 \\ r &= \frac{3B}{2A}. \end{aligned}$$

Thus, $r = 3B/(2A)$ is the only critical point. Since $f'(r) < 0$ for $r < 3B/(2A)$ and $f'(r) > 0$ for $r > 3B/(2A)$, we see that $r = 3B/(2A)$ is a local minimum. At that point,

$$f\left(\frac{3B}{2A}\right) = -\frac{A}{9B^2/4A^2} + \frac{B}{27B^3/8A^3} = -\frac{4A^3}{27B^2}.$$

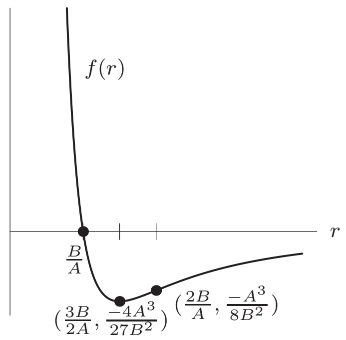
Differentiating again, we have

$$f''(r) = -\frac{6A}{r^4} + \frac{12B}{r^5} = -\frac{6}{r^5}(Ar - 2B).$$

So $f''(r) < 0$ where $r > 2B/A$ and $f''(r) > 0$ when $r < 2B/A$. Thus, $r = 2B/A$ is the only point of inflection. At that point

$$f\left(\frac{2B}{A}\right) = -\frac{A}{4B^2/A^2} + \frac{B}{8B^3/A^3} = -\frac{A^3}{8B^2}.$$

(c)



(d) (i) Increasing B means that the r-values of the zero, the minimum, and the inflection point increase, while the $f(r)$ values of the minimum and the point of inflection decrease in magnitude. See Figure 4.102.

(ii) Increasing A means that the r-values of the zero, the minimum, and the point of inflection decrease, while the $f(r)$ values of the minimum and the point of inflection increase in magnitude. See Figure 4.103.

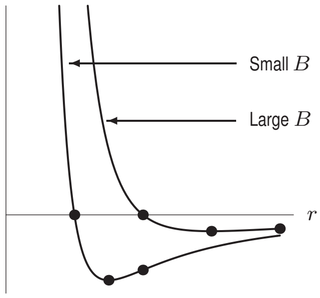


Figure 4.102: Increasing B

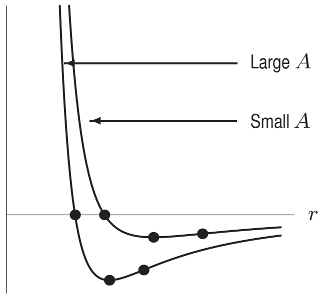


Figure 4.103: Increasing A

56. (a) The vertical intercept is $W = Ae^{-e^{b-c\cdot 0}} = Ae^{-e^b}$. There is no horizontal intercept since the exponential function is always positive. There is a horizontal asymptote. As $t \to \infty$, we see that $e^{b-ct} = e^b/e^{ct} \to 0$, since t is positive. Therefore $W \to Ae^0 = A$, so there is a horizontal asymptote at $W = A$.

(b) The derivative is

$$\frac{dW}{dt} = Ae^{-e^{b-ct}}(-e^{b-ct})(-c) = Ace^{-e^{b-ct}}e^{b-ct}.$$

Thus, dW/dt is always positive, so W is always increasing and has no critical points. The second derivative is

$$\begin{aligned}\frac{d^2W}{dt^2} &= \frac{d}{dt}(Ace^{-e^{b-ct}})e^{b-ct} + Ace^{-e^{b-ct}}\frac{d}{dt}(e^{b-ct}) \\ &= Ac^2e^{-e^{b-ct}}e^{b-ct}e^{b-ct} + Ace^{-e^{b-ct}}(-c)e^{b-ct} \\ &= Ac^2e^{-e^{b-ct}}e^{b-ct}(e^{b-ct} - 1).\end{aligned}$$

Now e^{b-ct} decreases from $e^b > 1$ when $t = 0$ toward 0 as $t \to \infty$. The second derivative changes sign from positive to negative when $e^{b-ct} = 1$, i.e., when $b - ct = 0$, or $t = b/c$. Thus the curve has an inflection point at $t = b/c$, where $W = Ae^{-e^{b-(b/c)c}} = Ae^{-1}$.

(c) See Figure 4.104.

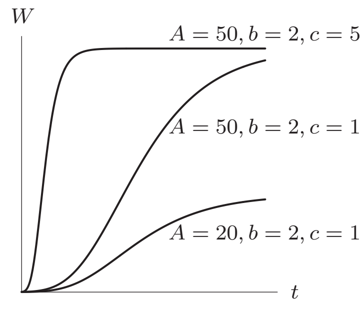


Figure 4.104

(d) The final size of the organism is given by the horizontal asymptote $W = A$. The curve is steepest at its inflection point, which occurs at $t = b/c$, $W = Ae^{-1}$. Since $e = 2.71828\ldots \approx 3$, the size the organism when it is growing fastest is about $A/3$, one third its final size. So yes, the Gompertz growth function is useful in modeling such growth.

Strengthen Your Understanding

57. As a counterexample, $f(x) = x^2 + 1$ has no zeros.

58. Differentiating f we see that $f'(x) = -a/x^2 + b$. Solving $f'(x) = 0$ we see that $x = \pm\sqrt{a/b}$ are critical points provided a and b have the same sign. So for example, $f(x) = 1/x + x$ has two critical points. But if a and b have opposite sign, f has no critical points. The function $f(x) = 1/x - x$ has no critical points, for example.

59. The family of functions $f(x) = kx(x - b)$ are all quadratics and satisfy $f(0) = f(b) = 0$, as required.

60. The condition for critical points is $f'(x) = 3ax^2 - b = 0$ or

$$x^2 = \frac{b}{3a}.$$

So there will be no critical points if

$$\frac{b}{3a} < 0.$$

As long as a and b have opposite signs, we have $b/(3a) < 0$. For example, we can take $a = 1, b = -1$ to get $f(x) = x^3 + x$.

61. Let $f(x) = ax^2$, with $a \neq 0$. Then $f'(x) = 2ax$, so f has a critical point only at $x = 0$.

62. Let $g(x) = ax^3 + bx^2$, where neither a nor b are allowed to be zero. Then

$$g'(x) = 3ax^2 + 2bx = x(3ax + 2b).$$

Then $g(x)$ has two distinct critical points, at $x = 0$ and at $x = -2b/3a$. Since

$$g''(x) = 6ax + 2b,$$

there is exactly one point of inflection, $x = -2b/6a = -b/3a$.

63. (a) and (c). Since the critical points of $f(x)$ are $x = \pm\sqrt{b/a}$, then as b increases $\left|\pm\sqrt{b/a}\right|$ increases. Therefore (a). The critical values of $f(x)$ are $f(\pm\sqrt{b/a}) = a(\pm\sqrt{b/a}) + \dfrac{b}{\pm\sqrt{b/a}} = \pm\sqrt{ab} \pm \sqrt{ab} = \pm 2\sqrt{ab}$. Therefore (c) is also true.

64. (b) and (c). Since the critical points of $f(x)$ are $x = \pm\sqrt{b/a}$, then as a increases $\left|\pm\sqrt{b/a}\right|$ decreases. Therefore (b). The critical values of $f(x)$ are $f(\pm\sqrt{b/a}) = a(\pm\sqrt{b/a}) + \dfrac{b}{\pm\sqrt{b/a}} = \pm\sqrt{ab} \pm \sqrt{ab} = \pm 2\sqrt{ab}$. Therefore (c) is also true.

Solutions for Section 4.5

Exercises

1. The profit function is positive when $R(q) > C(q)$, and negative when $C(q) > R(q)$. It's positive for $5.5 < q < 12.5$, and negative for $0 < q < 5.5$ and $12.5 < q$. Profit is maximized when $R(q) > C(q)$ and $R'(q) = C'(q)$ which occurs at about $q = 9.5$. See Figure 4.105.

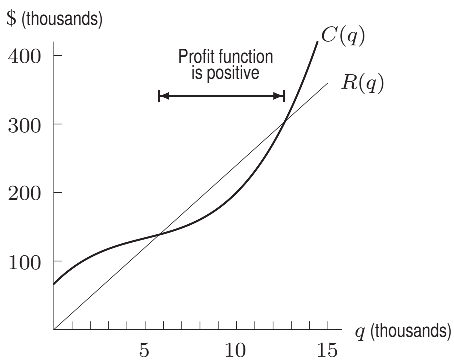


Figure 4.105

2. Since fixed costs are represented by the vertical intercept, they are $\$1.1$ million. The quantity that maximizes profit is about $q = 70$, and the profit achieved is $\$(3.7 - 2.5) = \$1.2$ million

3. The fixed costs are $\$5000$, the marginal cost per item is $\$2.40$, and the price per item is $\$4$.

4. The cost function $C(q) = b + mq$ satisfies $C(0) = 500$, so $b = 500$, and $MC = m = 6$. So

$$C(q) = 500 + 6q.$$

The revenue function is $R(q) = 12q$, so the profit function is

$$\pi(q) = R(q) - C(q) = 12q - 500 - 6q = 6q - 500.$$

5. The cost function $C(q) = b + mq$ satisfies $C(0) = 35{,}000$, so $b = 35{,}000$, and $MC = m = 10$. So

$$C(q) = 35{,}000 + 10q.$$

The revenue function is $R(q) = 15q$, so the profit function is

$$\pi(q) = R(q) - C(q) = 15q - 35{,}000 - 10q = 5q - 35{,}000.$$

6. The cost function $C(q) = b + mq$ satisfies $C(0) = 5000$, so $b = 5000$, and $MC = m = 15$. So

$$C(q) = 5000 + 15q.$$

The revenue function is $R(q) = 60q$, so the profit function is

$$\pi(q) = R(q) - C(q) = 60q - 5000 - 15q = 45q - 5000.$$

7. The cost function $C(q) = b + mq$ satisfies $C(0) = 0$, so $b = 0$. There are 32 ounces in a quart, so 160 in 5 quarts, or 20 cups at 8 ounces per cup. Thus each cup costs the operator 20 cents $= 0.20$ dollars, so $MC = m = 0.20$. So

$$C(q) = 0.20q.$$

The revenue function is $R(q) = 0.25q$. So the profit function is

$$\pi(q) = R(q) - C(q) = 0.25q - 0.20q = 0.05q.$$

8. The profit $\pi(q)$ is given by

$$\pi(q) = R(q) - C(q) = 500q - q^2 - (150 + 10q) = 490q - q^2 - 150.$$

The maximum profit occurs when

$$\pi'(q) = 490 - 2q = 0 \quad \text{so} \quad q = 245 \text{ items}.$$

Since $\pi''(q) = -2$, this critical point is a maximum. Alternatively, we obtain the same result from the fact that the graph of π is a parabola opening downward.

9. First find marginal revenue and marginal cost.

$$MR = R'(q) = 450$$

$$MC = C'(q) = 6q$$

Setting $MR = MC$ yields $6q = 450$, so marginal cost is equal to marginal revenue when

$$q = \frac{450}{6} = 75 \text{ units}.$$

Is profit maximized at $q = 75$? Profit $= R(q) - C(q)$;

$$\begin{aligned} R(75) - C(75) &= 450(75) - (10{,}000 + 3(75)^2) \\ &= 33{,}750 - 26{,}875 = \$6875. \end{aligned}$$

Testing $q = 74$ and $q = 76$:

$$\begin{aligned} R(74) - C(74) &= 450(74) - (10{,}000 + 3(74)^2) \\ &= 33{,}300 - 26{,}428 = \$6872. \end{aligned}$$

$$\begin{aligned} R(76) - C(76) &= 450(76) - (10{,}000 + 3(76)^2) \\ &= 34{,}200 - 27{,}328 = \$6872. \end{aligned}$$

Since profit at $q = 75$ is more than profit at $q = 74$ and $q = 76$, we conclude that profit is maximized locally at $q = 75$. The only endpoint we need to check is $q = 0$.

$$\begin{aligned} R(0) - C(0) &= 450(0) - (10{,}000 + 3(0)^2) \\ &= -\$10{,}000. \end{aligned}$$

This is clearly not a maximum, so we conclude that the profit is maximized globally at $q = 75$, and the total profit at this production level is \$6,875.

10. The marginal revenue, MR, is given by differentiating the total revenue function, R. We use the chain rule so

$$MR = \frac{dR}{dq} = \frac{1}{1 + 1000q^2} \cdot \frac{d}{dq}\left(1000q^2\right) = \frac{1}{1 + 1000q^2} \cdot 2000q.$$

When $q = 10$,

$$\text{Marginal revenue } = \frac{2000 \cdot 10}{1 + 1000 \cdot 10^2} = \$0.20\text{/item}.$$

When 10 items are produced, each additional item produced gives approximately \$0.20 in additional revenue.

11. (a) Profit is maximized when $R(q) - C(q)$ is as large as possible. This occurs at $q = 2500$, where profit $= 7500 - 5500 =$ \$2000.
(b) We see that $R(q) = 3q$ and so the price is $p = 3$, or \$3 per unit.
(c) Since $C(0) = 3000$, the fixed costs are \$3000.

12. (a) At $q = 5000$, $MR > MC$, so the marginal revenue to produce the next item is greater than the marginal cost. This means that the company will make money by producing additional units, and production should be increased.
(b) Profit is maximized where $MR = MC$, and where the profit function is going from increasing ($MR > MC$) to decreasing ($MR < MC$). This occurs at $q = 8000$.

Problems

13. (a) The value of $C(0)$ represents the fixed costs before production, that is, the cost of producing zero units, incurred for initial investments in equipment, and so on.
(b) The marginal cost decreases slowly, and then increases as quantity produced increases. See Problem 74, graph (b).
(c) Concave down implies decreasing marginal cost, while concave up implies increasing marginal cost.
(d) An inflection point of the cost function is (locally) the point of maximum or minimum marginal cost.
(e) One would think that the more of an item you produce, the less it would cost to produce extra items. In economic terms, one would expect the marginal cost of production to decrease, so we would expect the cost curve to be concave down. In practice, though, it eventually becomes more expensive to produce more items, because workers and resources may become scarce as you increase production. Hence after a certain point, the marginal cost may rise again. This happens in oil production, for example.

14. Since marginal revenue is larger than marginal cost around $q = 2000$, as you produce more of the product your revenue increases faster than your costs, so profit goes up, and maximal profit will occur at a production level above 2000.

15. Since for $q = 500$, we have $MC(500) = C'(500) = 75$ and $MR(500) = R'(500) = 100$, so $MR(500) > MC(500)$. Thus, increasing production from $q = 500$ increases profit.

16. Her marginal cost is at a minimum of \$25 when the quantity sold is 100, which is still below the marginal revenue of \$35. Thus she will continue to make a profit as the quantity increases, although her profit will start to decrease if the marginal cost goes above \$35. So the quantity that maximizes profit is greater than 100.

17. (a) We know that Profit = Revenue − Cost, so differentiating with respect to q gives:

$$\text{Marginal Profit} = \text{Marginal Revenue} - \text{Marginal Cost}.$$

We see from the figure in the problem that just to the left of $q = a$, marginal revenue is less than marginal cost, so marginal profit is negative there. To the right of $q = a$ marginal revenue is greater than marginal cost, so marginal profit is positive there. At $q = a$ marginal profit changes from negative to positive. This means that profit is decreasing to the left of a and increasing to the right. The point $q = a$ corresponds to a local minimum of profit, and does not maximize profit. It would be a terrible idea for the company to set its production level at $q = a$.

(b) We see from the figure in the problem that just to the left of $q = b$ marginal revenue is greater than marginal cost, so marginal profit is positive there. Just to the right of $q = b$ marginal revenue is less than marginal cost, so marginal profit is negative there. At $q = b$ marginal profit changes from positive to negative. This means that profit is increasing to the left of b and decreasing to the right. The point $q = b$ corresponds to a local maximum of profit. In fact, since the area between the MC and MR curves in the figure in the text between $q = a$ and $q = b$ is bigger than the area between $q = 0$ and $q = a$, $q = b$ is in fact a global maximum.

18. (a) The fixed cost is 0 because $C(0) = 0$.

(b) Profit, $\pi(q)$, is equal to money from sales, $7q$, minus total cost to produce those items, $C(q)$.

$$\pi = 7q - 0.01q^3 + 0.6q^2 - 13q$$
$$\pi' = -0.03q^2 + 1.2q - 6$$
$$\pi' = 0 \quad \text{if} \quad q = \frac{-1.2 \pm \sqrt{(1.2)^2 - 4(0.03)(6)}}{-0.06} \approx 5.9 \quad \text{or} \quad 34.1.$$

Now $\pi'' = -0.06q + 1.2$, so $\pi''(5.9) > 0$ and $\pi''(34.1) < 0$. This means $q = 5.9$ is a local min and $q = 34.1$ a local max. We now evaluate the endpoint, $\pi(0) = 0$, and the points nearest $q = 34.1$ with integer q-values:

$$\pi(35) = 7(35) - 0.01(35)^3 + 0.6(35)^2 - 13(35) = 245 - 148.75 = 96.25,$$
$$\pi(34) = 7(34) - 0.01(34)^3 + 0.6(34)^2 - 13(34) = 238 - 141.44 = 96.56.$$

So the (global) maximum profit is $\pi(34) = 96.56$. The money from sales is \$238, the cost to produce the items is \$141.44, resulting in a profit of \$96.56.

(c) The money from sales is equal to price×quantity sold. If the price is raised from \$7 by \$$x$ to \$$(7 + x)$, the result is a reduction in sales from 34 items to $(34 - 2x)$ items. So the result of raising the price by \$$x$ is to change the money from sales from $(7)(34)$ to $(7 + x)(34 - 2x)$ dollars. If the production level is fixed at 34, then the production costs are fixed at \$141.44, as found in part (b), and the profit is given by:

$$\pi(x) = (7 + x)(34 - 2x) - 141.44$$

This expression gives the profit as a function of change in price x, rather than as a function of quantity as in part (b). We set the derivative of π with respect to x equal to zero to find the change in price that maximizes the profit:

$$\frac{d\pi}{dx} = (1)(34 - 2x) + (7 + x)(-2) = 20 - 4x = 0$$

So $x = 5$, and this must give a maximum for $\pi(x)$ since the graph of π is a parabola which opens downward. The profit when the price is \$12 $(= 7 + x = 7 + 5)$ is thus $\pi(5) = (7 + 5)(34 - 2(5)) - 141.44 = \$146.56$. This is indeed higher than the profit when the price is \$7, so the smart thing to do is to raise the price by \$5.

19. For each month,

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$\pi = pq - wL = pcK^{\alpha}L^{\beta} - wL$$

The variable on the right is L, so at the maximum

$$\frac{d\pi}{dL} = \beta pcK^{\alpha}L^{\beta-1} - w = 0$$

Now $\beta - 1$ is negative, since $0 < \beta < 1$, so $1 - \beta$ is positive and we can write

$$\frac{\beta pcK^{\alpha}}{L^{1-\beta}} = w$$

giving

$$L = \left(\frac{\beta pcK^{\alpha}}{w}\right)^{\frac{1}{1-\beta}}$$

Since $\beta - 1$ is negative, when L is just above 0, the quantity $L^{\beta-1}$ is huge and positive, so $d\pi/dL > 0$. When L is large, $L^{\beta-1}$ is small, so $d\pi/dL < 0$. Thus the value of L we have found gives a global maximum, since it is the only critical point.

20. (a) We have $N(x) = 100 + 20x$, graphed in Figure 4.106.

(b) (i) Differentiating gives $N'(x) = 20$ and the graph of N' is a horizontal line. This means that rate of increase of the number of bees with acres of clover is constant—each acre of clover brings 20 more bees.

(ii) On the other hand, $N(x)/x = 100/x + 20$ means that the average number of bees per acre of clover approaches 20 as more acres are put under clover. See Figure 4.107. As x increases, $100/x$ decreases to 0, so $N(x)/x$ approaches 20 (i.e. $N(x)/x \to 20$). Since the total number of bees is 20 per acre plus the original 100, the average number of bees per acre is 20 plus the 100 shared out over x acres. As x increases, the 100 are shared out over more acres, and so its contribution to the average becomes less. Thus the average number of bees per acre approaches 20 for large x.

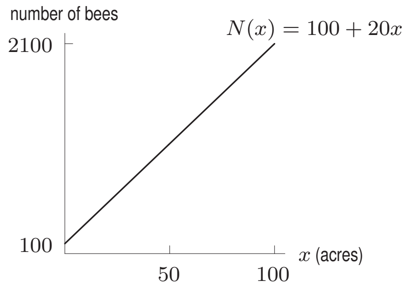


Figure 4.106

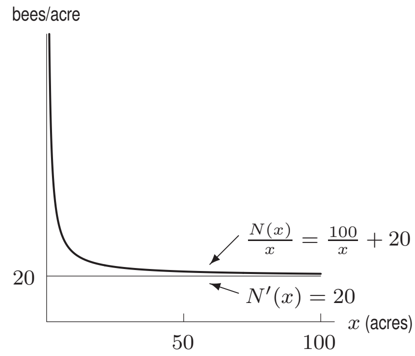


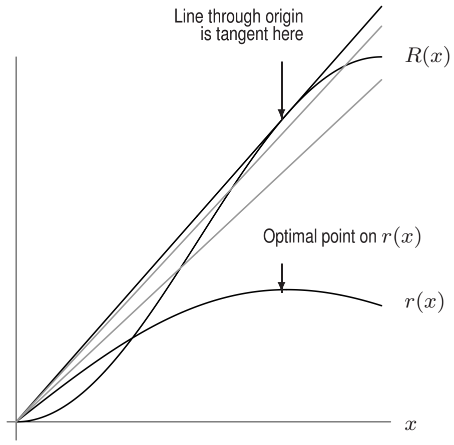
Figure 4.107

21. This question implies that the line from the origin to the point $(x, R(x))$ has some relationship to $r(x)$. The slope of this line is $R(x)/x$, which is $r(x)$. So the point x_0 at which $r(x)$ is maximal will also be the point at which the slope of this line is maximal. The question claims that the line from the origin to $(x_0, R(x_0))$ will be tangent to the graph of $R(x)$. We can understand this by trying to see what would happen if it were otherwise.

If the line from the origin to $(x_0, R(x_0))$ intersects the graph of $R(x)$, but is not tangent to the graph of $R(x)$ at x_0, then there are points of this graph on both sides of the line — and, in particular, there is some point x_1 such that the line from the origin to $(x_1, R(x_1))$ has larger slope than the line to $(x_0, R(x_0))$. (See the graph below.) But we picked x_0 so that no other line had larger slope, and therefore no such x_1 exists. So the original supposition is false, and the line from the origin to $(x_0, R(x_0))$ is tangent to the graph of $R(x)$.

(a) See (b).

(b)



(c)

$$r(x) = \frac{R(x)}{x}$$
$$r'(x) = \frac{xR'(x) - R(x)}{x^2}$$

So when $r(x)$ is maximized $0 = xR'(x) - R(x)$, the numerator of $r'(x)$, or $R'(x) = R(x)/x = r(x)$. i.e. when $r(x)$ is maximized, $r(x) = R'(x)$.

Let us call the x-value at which the maximum of r occurs x_m. Then the line passing through $R(x_m)$ and the origin is $y = x \cdot R(x_m)/x_m$. Its slope is $R(x_m)/x_m$, which also happens to be $r(x_m)$. In the previous paragraph, we showed that at x_m, this is also equal to the slope of the tangent to $R(x)$. So, the line through the origin is the tangent line.

22. (a) The value of MC is the slope of the tangent to the curve at q_0. See Figure 4.108.
(b) The line from the curve to the origin joins $(0, 0)$ and $(q_0, C(q_0))$, so its slope is $C(q_0)/q_0 = a(q_0)$.
(c) Figure 4.109 shows that the line whose slope is the minimum $a(q)$ is tangent to the curve $C(q)$. This line, therefore, also has slope MC, so $a(q) = MC$ at the q making $a(q)$ minimum.

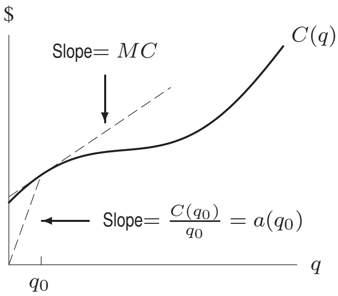


Figure 4.108

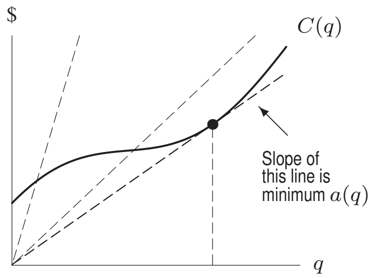


Figure 4.109

23. (a) The average cost is $a(q) = C(q)/q$, so the total cost is $C(q) = 0.01q^3 - 0.6q^2 + 13q$.
(b) Taking the derivative of $C(q)$ gives an expression for the marginal cost:

$$C'(q) = MC(q) = 0.03q^2 - 1.2q + 13.$$

To find the smallest MC we take its derivative and find the value of q that makes it zero. So: $MC'(q) = 0.06q - 1.2 = 0$ when $q = 1.2/0.06 = 20$. This value of q must give a minimum because the graph of $MC(q)$ is a parabola opening upward. Therefore the minimum marginal cost is $MC(20) = 1$. So the marginal cost is at a minimum when the additional cost per item is \$1.
(c) Differentiating gives $a'(q) = 0.02q - 0.6$.
Setting $a'(q) = 0$ and solving for q gives $q = 30$ as the quantity at which the average is minimized, since the graph of a is a parabola which opens upward. The minimum average cost is $a(30) = 4$ dollars per item.
(d) The marginal cost at $q = 30$ is $MC(30) = 0.03(30)^2 - 1.2(30) + 13 = 4$. This is the same as the average cost at this quantity. Note that since $a(q) = C(q)/q$, we have $a'(q) = (qC'(q) - C(q))/q^2$. At a critical point, q_0, of $a(q)$, we have

$$0 = a'(q_0) = \frac{q_0 C'(q_0) - C(q_0)}{q_0^2},$$

so $C'(q_0) = C(q_0)/q_0 = a(q_0)$. Therefore $C'(30) = a(30) = 4$ dollars per item.

Another way to see why the marginal cost at $q = 30$ must equal the minimum average cost $a(30) = 4$ is to view $C'(30)$ as the approximate cost of producing the 30^{th} or 31^{st} good. If $C'(30) < a(30)$, then producing the 31^{st} good would lower the average cost, i.e. $a(31) < a(30)$. If $C'(30) > a(30)$, then producing the 30^{th} good would raise the average cost, i.e. $a(30) > a(29)$. Since $a(30)$ is the global minimum, we must have $C'(30) = a(30)$.

24. (a) Since the company can produce more goods if it has more raw materials to use, the function $f(x)$ is increasing. Thus, we expect the derivative $f'(x)$ to be positive.
(b) The cost to the company of acquiring x units of raw material is wx, and the revenue from the sale of $f(x)$ units of the product is $pf(x)$. The company's profit is $\pi(x) = \text{Revenue} - \text{Cost} = pf(x) - wx$.
(c) Since profit $\pi(x)$ is maximized at $x = x^*$, we have $\pi'(x^*) = 0$. From $\pi'(x) = pf'(x) - w$, we have $pf'(x^*) - w = 0$. Thus $f'(x^*) = w/p$.
(d) Computing the second derivative of $\pi(x)$ gives $\pi''(x) = pf''(x)$. Since $\pi(x)$ has a maximum at $x = x^*$, the second derivative $\pi''(x^*) = pf''(x^*)$ is negative. Thus $f''(x^*)$ is negative.
(e) Differentiate both sides of $pf'(x^*) - w = 0$ with respect to w. The chain rule gives

$$p\frac{d}{dw}f'(x^*) - 1 = 0$$

$$pf''(x^*)\frac{dx^*}{dw} - 1 = 0$$
$$\frac{dx^*}{dw} = \frac{1}{pf''(x^*)}.$$

Since $f''(x^*) < 0$, we see dx^*/dw is negative.

(f) Since $dx^*/dw < 0$, the quantity x^* is a decreasing function of w. If the price w of the raw material goes up, the company should buy less.

25. (a) See Figure 4.110.

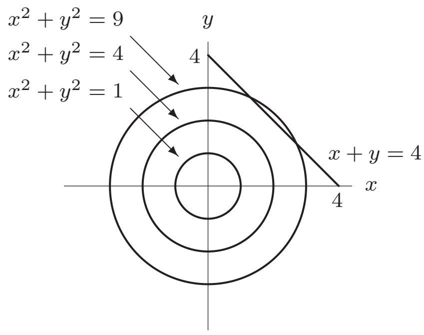


Figure 4.110

(b) As C increases in the equation $x^2 + y^2 = C$, the circle expands outward. For $C = 4$, the circle does not intersect the line. As C increases from 4, the circle expands until it touches the line. At $C = 9$, the circle cuts the line twice.

The minimum value of $C = x^2 + y^2$ occurs where a circle tangent to the line. For larger C-values, $x^2 + y^2 = C$ cuts the line twice, for smaller C-values, the circle does not touch the line.

(c) At the point at when the circle touches the line, the slope of the circle equals the slope of the line, namely -1. Implicit differentiation gives the slope of $x^2 + y^2 = C$:

$$2x + 2y \cdot y' = 0$$
$$y' = \frac{-x}{y}.$$

Thus, at the point where a circle touches the line, we have

$$-\frac{x}{y} = -1$$
$$x = y.$$

Substitution into $x + y = 4$ gives $2x = 4$, so $x = 2$ and $y = 2$. Thus, the minimum is $x^2 + y^2 = 2^2 + 2^2 = 8$.

26. (a) See Figure 4.111.

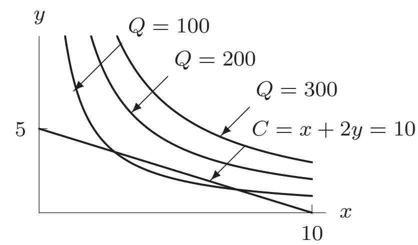


Figure 4.111

(b) Comparing the curves $Q = 100$, $Q = 200$, and $Q = 300$, we see that production increases as we move away from the origin. The curve $Q = 100$ cuts the line $C = x + 2y = 10$ twice while the curves $Q = 200$ and $Q = 300$ do not cut the line.

The maximum possible Q occurs where a curve touches the line. At this point, the slope of the production curve equals the slope of the budget line, namely $-1/2$.

(c) Using implicit differentiation, the slope of the curve $10xy = C$ is given by

$$10y + 10xy' = 0$$
$$y' = -\frac{y}{x}.$$

Thus, at the point where the curve touches the line, whose slope is $-1/2$, we have

$$-\frac{y}{x} = -\frac{1}{2}$$
$$y = \frac{x}{2}.$$

Substituting into $C = x + 2y = 10$ gives $2x = 10$, so $x = 5$ and $y = 5/2$. Thus, the maximum production is

$$Q = 10 \cdot 5 \cdot \frac{5}{2} = 125.$$

27. (a) See Figure 4.112.

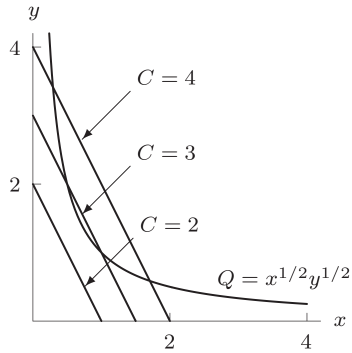


Figure 4.112

(b) Comparing the lines $C = 2,\ C = 3,\ C = 4$, we see that the cost increases as we move away from the origin. The line $C = 2$ does not cut the curve $Q = 1$; the lines $C = 3$ and $C = 4$ cut twice.

The minimum cost occurs where a cost line is tangent to the production curve.

(c) Using implicit differentiation, the slope of $x^{1/2}y^{1/2} = 1$ is given by

$$\frac{1}{2}x^{-1/2}y^{1/2} + \frac{1}{2}x^{1/2}y^{-1/2}y' = 0$$
$$y' = \frac{-x^{-1/2}y^{1/2}}{x^{1/2}y^{-1/2}} = -\frac{y}{x}.$$

The cost lines all have slope -2. Thus, if the curve is tangent to a line, we have

$$-\frac{y}{x} = -2$$
$$y = 2x.$$

Substituting into $Q = x^{1/2}y^{1/2} = 1$ gives

$$x^{1/2}(2x)^{1/2} = 1$$
$$\sqrt{2}x = 1$$
$$x = \frac{1}{\sqrt{2}}$$
$$y = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}.$$

Thus the minimum cost is

$$C = 2\frac{1}{\sqrt{2}} + \sqrt{2} = 2\sqrt{2}.$$

Strengthen Your Understanding

28. We are only given the cost and revenue when $q = 100$. Since we do not have any information about the marginal revenue and the marginal cost at $q = 100$, we do not know how an increase in production will affect either the revenue or the cost. Thus we cannot determine whether the profit will increase or decrease if production is increased from 100. We only know that when $q = 100$ the profit will be $R(100) - C(100) = \$60$.

29. The profit, π is given by

$$\pi(q) = R(q) - C(q).$$

Thus we have

$$\pi'(q) = R'(q) - C'(q),$$

and the critical points of $\pi(q)$ occur where $R'(q) = C'(q)$. In Figure 4.82, there are two such points, at approximately $q = 3.5$ and $q = 13$. To the left of $q = 10$, the graph shows that the costs exceed revenue, so the company is losing money. Profit is positive between $q = 10$ and $q = 15$. So the critical point at which profit is a maximum is $q = 13$. The maximum profit occurs when the quantity produced is about 13,000 units.

30. We want a quantity where the slope of the cost curve, C, is greater than the slope of the revenue curve, R. For example, small quantities, such as $q = 2$, or large quantities, such as $q = 15$.

31. The cost curve, C, lies above the revenue curve, R, for all quantities, q. See Figure 4.113.

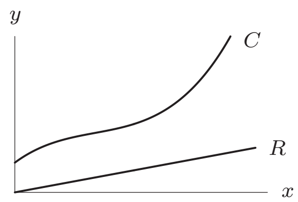


Figure 4.113

32. (e). A company should maximize profit, that is, it should maximize revenue minus cost.

33. (a) If $MR - MC > 0$, so $MR > MC$, then revenue is increasing faster than cost with production, so profit increases.

Solutions for Section 4.6

Exercises

1. The rate of change of temperature is

$$\frac{dH}{dt} = 16(-0.02)e^{-0.02t} = -0.32e^{-0.02t}.$$

When $t = 0$,

$$\frac{dH}{dt} = -0.32e^{0} = -0.32^\circ\text{C/min}.$$

When $t = 10$,

$$\frac{dH}{dt} = -0.32e^{-0.02(10)} = -0.262^\circ\text{C/min}.$$

2. The rate of growth, in billions of people per year, was

$$\frac{dP}{dt} = 6.7(0.011)e^{0.011t}.$$

On January 1, 2007, we have $t = 0$, so

$$\frac{dP}{dt} = 6.7(0.011)e^{0} = 0.0737 \text{ billion/year } = 73.7 \text{ million people/year}.$$

3. The rate of change of the power dissipated is given by

$$\frac{dP}{dR} = -\frac{81}{R^2}.$$

4. (a) The rate of change of the period is given by

$$\frac{dT}{dl} = \frac{2\pi}{\sqrt{9.8}}\frac{d}{dl}(\sqrt{l}) = \frac{2\pi}{\sqrt{9.8}} \cdot \frac{1}{2}l^{-1/2} = \frac{\pi}{\sqrt{9.8}} \cdot \frac{1}{\sqrt{l}} = \frac{\pi}{\sqrt{9.8l}}.$$

(b) The rate decreases since $\sqrt{l}$ is in the denominator.

5. Let $y(t)$ be the height of the aircraft, in feet, as a function of time, t, in minutes, and $H(y)$ be the outside temperature in $^\circ C$ as a function of height, y, in feet. We have

$$\frac{dy}{dt} = 500 \text{ feet/minute}, \qquad \frac{dH}{dy} = \frac{-2}{1000}{}^\circ C/\text{foot},$$

so we have

$$\begin{aligned} \frac{dH}{dt} &= \frac{dH}{dy} \cdot \frac{dy}{dt} \\ &= \frac{-2}{1000} \cdot 500 \\ &= -1^\circ C/\text{minute} \end{aligned}$$

6. (a) Since $P = 25$, we have

$$\begin{aligned} 25 &= 30e^{-3.23\times 10^{-5}h} \\ h &= \frac{\ln(25/30)}{-3.23 \times 10^{-5}} = 5644 \text{ ft.} \end{aligned}$$

(b) Both P and h are changing over time, and we differentiate with respect to time t, in minutes:

$$\frac{dP}{dt} = 30e^{-3.23\times 10^{-5}h}\left(-3.23 \times 10^{-5}\frac{dh}{dt}\right).$$

We see in part (a) that $h = 5644$ ft when $P = 25$ inches of mercury, and we know that $dP/dt = 0.1$. Substituting gives:

$$\begin{aligned} 0.1 &= 30e^{-3.23\times 10^{-5}\times 5644.630}\left(-3.23 \times 10^{-5}\frac{dh}{dt}\right) \\ \frac{dh}{dt} &= -124 \text{ ft/minute.} \end{aligned}$$

The glider's altitude is decreasing at a rate of about 124 feet per minute.

7. We differentiate $F = k/r^2$ with respect to t using the chain rule to give

$$\frac{dF}{dt} = -\frac{2k}{r^3} \cdot \frac{dr}{dt}.$$

We know that $k = 10^{13}$ newton $\cdot$ km^2 and that the rocket is moving at 0.2 km/sec when $r = 10^4$ km. In other words, $dr/dt = 0.2$ km/sec when $r = 10^4$. Substituting gives

$$\frac{dF}{dt} = -\frac{2 \cdot 10^{13}}{(10^4)^3} \cdot 0.2 = -4 \text{ newtons/sec.}$$

8. We know $dR/dt = 0.2$ when $R = 5$ and $V = 9$ and we want to know dI/dt. Differentiating $I = V/R$ with V constant gives

$$\frac{dI}{dt} = V\left(-\frac{1}{R^2}\frac{dR}{dt}\right),$$

so substituting gives

$$\frac{dI}{dt} = 9\left(-\frac{1}{5^2} \cdot 0.2\right) = -0.072 \text{ amps per second.}$$

9. We have

$$\frac{dA}{dt} = \frac{9}{16}[4 - 4\cos(4\theta)]\frac{d\theta}{dt}.$$

So

$$\left.\frac{dA}{dt}\right|_{\theta=\pi/4} = \frac{9}{16}\left(4 - 4\cos\left(4\cdot\frac{\pi}{4}\right)\right)0.2 = \frac{9}{16}(4+4)0.2 = 0.9\ \text{cm}^2/\text{min}.$$

10. Differentiating with respect to time t, in seconds, we have

$$\frac{d\phi}{dt} = 2\pi\left(\frac{1}{2}(x^2+4)^{-1/2}\left(2x\frac{dx}{dt}\right) - \frac{dx}{dt}\right).$$

Since the point is moving to the left, $dx/dt = -0.2$. Substituting $x = 3$ and $dx/dt = -0.2$, we have

$$\frac{d\phi}{dt} = 2\pi\left(\frac{1}{2}(3^2+4)^{-1/2}(2\cdot 3(-0.2)) - (-0.2)\right) = 0.211.$$

The potential is changing at a rate of 0.211 units per second.

11. (a) The rate of change of average cost as quantity increases is

$$\frac{dC}{dq} = -\frac{a}{q^2}\ \text{dollars/cell phone}.$$

(b) We are told that $dq/dt = 100$, and we want dC/dt. The chain rule gives

$$\frac{dC}{dt} = \frac{dC}{dq}\cdot\frac{dq}{dt} = -\frac{a}{q^2}\cdot 100 = -\frac{100a}{q^2}\ \text{dollars/week}.$$

Since a is positive, dC/dt is negative, so C is decreasing.

12. When we differentiate $x^2 + y^2 = 25$ with respect to time, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

(a) We use the fact that $x^2 + y^2 = 25$ to solve for y when $x = 0$. Since $0^2 + y^2 = 25$, we have $y = 5$ since y is positive. Substituting, we have

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ 2(0)(6) + 2(5)\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= 0. \end{aligned}$$

(b) We use the fact that $x^2 + y^2 = 25$ to solve for y when $x = 3$. Since $3^2 + y^2 = 25$, we have $y = 4$ since y is positive. Substituting, we have

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ 2(3)(6) + 2(4)\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -4.5. \end{aligned}$$

(c) We use the fact that $x^2 + y^2 = 25$ to solve for y when $x = 4$. Since $4^2 + y^2 = 25$, we have $y = 3$ since y is positive. Substituting, we have

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ 2(4)(6) + 2(3)\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -8. \end{aligned}$$

13. Since h is constant, we have
$$\frac{dV}{dt} = \frac{2}{3}xh\frac{dx}{dt}.$$
Thus for $h = 120$, $x = 150$, and $dx/dt = -0.002$, we have
$$\frac{dV}{dt} = \frac{2}{3}(150)(120)(-0.002) = -24\,\text{meters}^3/\text{yr}.$$
The volume is decreasing at 24 meters^3 per year.

14. (a) Treating a and l as constants, by the chain and quotient rules, we have
$$\frac{dF}{dt} = -\frac{K(2a+l)}{[a(a+l)]^2}\frac{da}{dt} = -\frac{K(35)}{[15(20)]^2}2 = -K(7.78\times 10^{-4})\,\text{newtons/min}.$$
(b) We have
$$\frac{dF}{dt} = -\frac{Ka}{[a(a+l)]^2}\frac{dl}{dt} = -\frac{K(15)}{[15(20)]^2}(-2) = K(3.33\times 10^{-4})\,\text{newtons/min}.$$

15. (a) We have
$$\text{Surface area} = 0.01(60^{0.25})(150^{0.75}) = 1.19\text{ meters}^2.$$
(b) Since $h = 150$ is constant, we have
$$s = 0.01w^{0.25}150^{0.75}.$$
Differentiating with respect to time t, in years,we have
$$\frac{ds}{dt} = 0.01\left(0.25w^{-0.75}\frac{dw}{dt}\right)150^{0.75}.$$
Substituting $dw/dt = 0.5$ and $w = 62$, we have
$$\frac{ds}{dt} = 0.01(0.25(62^{-0.75})(0.5))(150^{0.75}) = 0.0024\text{ meter}^2\text{/year}.$$
The surface area is increasing at a rate of 0.0024 meter^2 per year, or 24 cm^2 per year.

Problems

16. Let the other side of the rectangle be x cm. Then the area is $A = 10x$, so differentiating with respect to time gives
$$\frac{dA}{dt} = 10\frac{dx}{dt}.$$
We are interested in the instant when $x = 12$ and $dx/dt = 3$, giving
$$\frac{dA}{dt} = 10\frac{dx}{dt} = 10\cdot 3 = 30\text{ cm}^2\text{ per minute.}$$

17. Let the other side of the rectangle be x cm. Then the diagonal is
$$D = \sqrt{8^2+x^2} = \sqrt{64+x^2}.$$
Differentiating with respect to time gives
$$\frac{dD}{dt} = \frac{2x}{2\sqrt{64+x^2}}\frac{dx}{dt} = \frac{x}{\sqrt{64+x^2}}\frac{dx}{dt}.$$
We are interested in the instant when $x = 6$ and $dx/dt = 3$, giving
$$\frac{dD}{dt} = \frac{6}{\sqrt{64+6^2}}\cdot 3 = \frac{6}{10}\cdot 3 = 1.8\text{ cm per minute.}$$

18. Let the other leg be x cm. Then the area is

$$A = \frac{1}{2}7x = \frac{7}{2}x.$$

Differentiating with respect to time gives

$$\frac{dA}{dt} = \frac{7}{2}\frac{dx}{dt}.$$

We are interested in the instant when $x = 10$ and $dx/dt = -2$, where the negative sign reflects the fact that x is decreasing. Thus

$$\frac{dA}{dt} = \frac{7}{2}(-2) = -7 \text{ cm}^2 \text{ per second.}$$

The area is decreasing at 7 cm^2 per second.

19. The side length is $s = \sqrt{A}$ cm. Differentiating with respect to time gives

$$\frac{ds}{dt} = \frac{1}{2}\frac{1}{\sqrt{A}}\frac{dA}{dt}.$$

We are interested in the instant when $A = 576$ and $dA/dt = 3$, giving

$$\frac{ds}{dt} = \frac{1}{2}\frac{1}{\sqrt{576}}3 = \frac{1}{16}.$$

Thus, the side is increasing at $1/16$ cm per minute.

20. Since R_1 is constant, we know $dR_1/dt = 0$. We know that $dR_2/dt = 2$ ohms/min when $R_2 = 20$ ohms. At the moment we are interested in, $R_1 = 10$ and $R_2 = 20$, so

$$\frac{1}{R} = \frac{1}{10} + \frac{1}{20} = \frac{3}{20}, \quad \text{so} \quad R = \frac{20}{3} \text{ ohms.}$$

Differentiating with respect to time t,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2},$$

gives

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{{R_1}^2}\frac{dR_1}{dt} - \frac{1}{{R_2}^2}\frac{dR_2}{dt}.$$

Substituting gives

$$-\frac{1}{(20/3)^2}\frac{dR}{dt} = -\frac{1}{10^2}\cdot 0 - \frac{1}{20^2}\cdot 2$$

$$\frac{dR}{dt} = \left(\frac{20}{3}\right)^2 \cdot \frac{1}{20^2}\cdot 2 = \frac{2}{9} \text{ ohms/min.}$$

21. (a) The rate of change of temperature change is

$$\frac{dT}{dD} = \frac{d}{dD}\left(\frac{CD^2}{2} - \frac{D^3}{3}\right) = \frac{2CD}{2} - \frac{3D^2}{3} = CD - D^2.$$

(b) We want to know for what values of D the value of dT/dD is positive. This occurs when

$$\frac{dT}{dD} = (C - D)\,D > 0.$$

We only consider positive values of D, since a zero dosage obviously has no effect and a negative dosage does not make sense. If $D > 0$, then $(C - D)D > 0$ when $C - D > 0$, or $D < C$. So the rate of change of temperature change is positive for doses less than C.

22. (a) (i) Differentiating thinking of r as a constant gives

$$\frac{dP}{dt} = 500e^{rt/100}\cdot\frac{r}{100} = 5re^{rt/100}.$$

Substituting $t = 0$ gives

$$\frac{dP}{dt} = 5re^{r\cdot 0/100} = 5r \text{ dollars/yr.}$$

(ii) Substituting $t = 2$ gives
$$\frac{dP}{dt} = 5re^{r\cdot 2/100} = 5re^{0.02r} \text{ dollars/yr.}$$

(b) To differentiate thinking of r as variable, think of the function as
$$P = 500e^{r(t)\cdot t/100},$$
and use the chain rule (for $e^{rt/100}$) and the product rule (for $r(t)\cdot t$):
$$\frac{dP}{dt} = 500e^{r(t)\cdot t/100}\cdot\frac{1}{100}\cdot\frac{d}{dt}(r(t)\cdot t) = 5e^{r(t)\cdot t/100}\left(r(t)\cdot 1 + r'(t)\cdot t\right).$$
Substituting $t = 2$, $r = 4$, and $r'(2) = 0.3$ gives
$$\frac{dP}{dt} = 5e^{4\cdot 2/100}\left(4 + 0.3\cdot 2\right) = 24.916 \text{ dollars/year}$$
Thus, the price is increasing by about \$25 per year at that time.

23. (a) The rate of change of force with respect to distance is
$$\frac{dF}{dr} = \frac{2A}{r^3} - \frac{3B}{r^4}.$$
The units are units of force per units of distance.

(b) We are told that $dr/dt = k$ and we want dF/dt. By the chain rule
$$\frac{dF}{dt} = \frac{dF}{dr}\cdot\frac{dr}{dt} = \left(\frac{2A}{r^3} - \frac{3B}{r^4}\right)k.$$
The units are units of force per unit time.

24. The rate of change of velocity is given by
$$\frac{dv}{dt} = -\frac{mg}{k}\left(-\frac{k}{m}e^{-kt/m}\right) = ge^{-kt/m}.$$
When $t = 0$,
$$\left.\frac{dv}{dt}\right|_{t=0} = g.$$
When $t = 1$,
$$\left.\frac{dv}{dt}\right|_{t=1} = ge^{-k/m}.$$
These answers give the acceleration at $t = 0$ and $t = 1$. The acceleration at $t = 0$ is g, the acceleration due to gravity, and at $t = 1$, the acceleration is $ge^{-k/m}$, a smaller value.

25. We let x represent the distance from the base of the wall to the base of the ladder and y represent the distance from the top of the ladder to the base of the wall. Notice that x and y are changing over time, but that the length of the ladder is fixed at 10 m. Using the Pythagorean Theorem, we have
$$x^2 + y^2 = 10^2.$$
Differentiating with respect to time, we have
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$
We have $dx/dt = 0.5$ and we are finding dy/dt.

(a) When $x = 4$, we have $4^2 + y^2 = 10^2$ so $y = 9.165$. Substituting, we have
$$2(4)(0.5) + 2(9.165)\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = -0.218 \text{ m/sec.}$$
The top of the ladder is sliding down at 0.218 m/sec.

(b) When $x = 8$, we have $8^2 + y^2 = 10^2$ so $y = 6$. Substituting, we have
$$2(8)(0.5) + 2(6)\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = -0.667 \text{ m/sec.}$$
The top of the ladder is sliding down at 0.667 m/sec. Notice that the top slides down faster and faster as the bottom moves away at a constant speed.

26. Since the radius is 3 feet, the volume of the gas when the depth is h is given by

$$V = \pi 3^2 h = 9\pi h.$$

We want to find dV/dt when $h = 4$ and $dh/dt = 0.2$. Differentiating gives

$$\frac{dV}{dt} = 9\pi\frac{dh}{dt} = 9\pi(0.2) = 5.655 \text{ ft}^3/\text{sec}.$$

Notice that the value $H = 4$ is not used. This is because V is proportional to h so dV/dt does not depend on h.

27. The rate of 3 meters^3/min is a derivative and the units tell us it is dV/dt where V is volume in cubic meters and t is time in minutes. The rate at which the water level is rising is dh/dt where h is the height of the water in meters. We must relate V and h.

When the water level is h, the volume, V, of water in the cylinder is

$$V = \pi r^2 h$$

where $r = 5$ meters is the radius, so

$$V = 25\pi h.$$

Differentiating with respect to time gives

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt}.$$

We substitute $dV/dt = 3$ and solve for dh/dt:

$$3 = 25\pi\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{3}{25\pi} = 0.038 \text{ meter/min}.$$

The water level is rising 0.038 meters per minute. Notice that this rate does not depend on the fact that the cylinder is half full. We have $dh/dt = 0.038$ for all water levels h.

28. When the radius is r, the volume V of the snowball is

$$V = \frac{4}{3}\pi r^3.$$

We know that $dr/dt = -0.2$ when $r = 15$ and we want to know dV/dt at that time. Differentiating, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Substituting $dr/dt = -0.2$ gives

$$\left.\frac{dV}{dt}\right|_{r=15} = 4\pi(15)^2(-0.2) = -180\pi = -565.487 \text{ cm}^3/\text{hr}.$$

Thus, the volume is decreasing at 565.487 cm^3 per hour.

29. If V is the volume of the balloon and r is its radius, then

$$V = \frac{4}{3}\pi r^3.$$

We want to know the rate at which air is being blown into the balloon, which is the rate at which the volume is increasing, dV/dt. We are told that

$$\frac{dr}{dt} = 2\,\text{cm/sec} \quad \text{when} \quad r = 10\,\text{cm}.$$

Using the chain rule, we have

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Substituting gives

$$\frac{dV}{dt} = 4\pi(10)^2 2 = 800\pi = 2513.3\,\text{cm}^3/\text{sec}.$$

30. If C represents circulation time in seconds and B represent body mass in kilograms, we have

$$C = 17.40B^{1/4}.$$

As the child grows, both the body mass and the circulation time change. Differentiating with respect to time t, we have

$$\frac{dC}{dt} = 17.40\left(\frac{1}{4}B^{-3/4}\frac{dB}{dt}\right).$$

Substituting $B = 45$ and $dB/dt = 0.1$, we have

$$\frac{dC}{dt} = 17.40\left(\frac{1}{4}(45)^{-3/4}(0.1)\right) = 0.025.$$

The circulation time is increasing at a rate of 0.025 seconds per month.

31. (a) Since $P = 1$ when $V = 20$, we have

$$k = 1 \cdot (20^{1.4}) = 66.29.$$

Thus, we have

$$P = 66.29V^{-1.4}.$$

Differentiating gives

$$\frac{dP}{dV} = 66.29(-1.4V^{-2.4}) = -92.8V^{-2.4}\text{ atmospheres/cm}^3.$$

(b) We are given that $dV/dt = 2\,\text{cm}^3/\text{min}$ when $V = 30\,\text{cm}^3$. Using the chain rule, we have

$$\begin{aligned}\frac{dP}{dt} = \frac{dP}{dV}\cdot\frac{dV}{dt} &= \left(-92.8V^{-2.4}\,\frac{\text{atm}}{\text{cm}^3}\right)\left(2\frac{\text{cm}^3}{\text{min}}\right)\\ &= -92.8\left(30^{-2.4}\right)2\frac{\text{atm}}{\text{min}}\\ &= -0.0529\text{ atmospheres/min}\end{aligned}$$

Thus, the pressure is decreasing at 0.0529 atmospheres per minute.

32. If the length of a horizontal rod is x and the length of a vertical rod is h, the volume, V, is given by

$$V = x^2h.$$

Taking derivatives gives

$$\frac{dV}{dt} = 2xh\frac{dx}{dt} + x^2\frac{dh}{dt}.$$

When the area of the square base is 9 cm^2, we have $x = 3$ cm. Since the volume is then 180 cm^3, we know $h = 180/9 = 20$ cm. So

$$\frac{dV}{dt} = 2\cdot 3\cdot 20(0.001) + 3^2(0.002) = 0.138\,\text{cm}^3/\text{hr}.$$

33. (a) Since the slick is circular, if its radius is r meters, its area, A, is $A = \pi r^2$. Differentiating with respect to time using the chain rule gives

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$

We know $dr/dt = 0.1$ when $r = 150$, so

$$\frac{dA}{dt} = 2\pi 150(0.1) = 30\pi = 94.248\text{ m}^2/\text{min}.$$

(b) If the thickness of the slick is h, its volume, V, is given by

$$V = Ah.$$

Differentiating with respect to time using the product rule gives

$$\frac{dV}{dt} = \frac{dA}{dt}h + A\frac{dh}{dt}.$$

We know $h = 0.02$ and $A = \pi(150)^2$ and $dA/dt = 30\pi$. Since V is fixed, $dV/dt = 0$, so

$$0 = 0.02(30\pi) + \pi(150)^2 \frac{dh}{dt}.$$

Thus

$$\frac{dh}{dt} = -\frac{0.02(30\pi)}{\pi(150)^2} = -0.0000267 \text{ m/min},$$

so the thickness is decreasing at 0.0000267 meters per minute.

34. Let the volume of clay be V. The clay is in the shape of a cylinder, so $V = \pi r^2 L$. We know $dL/dt = 0.1$ cm/sec and we want to know dr/dt when $r = 1$ cm and $L = 5$ cm. Differentiating with respect to time t gives

$$\frac{dV}{dt} = \pi 2rL\frac{dr}{dt} + \pi r^2 \frac{dL}{dt}.$$

However, the amount of clay is unchanged, so $dV/dt = 0$ and

$$2rL\frac{dr}{dt} = -r^2\frac{dL}{dt},$$

therefore

$$\frac{dr}{dt} = -\frac{r}{2L}\frac{dL}{dt}.$$

When the radius is 1 cm and the length is 5 cm, and the length is increasing at 0.1 cm per second, the rate at which the radius is changing is

$$\frac{dr}{dt} = -\frac{1}{2\cdot 5}\cdot 0.1 = -0.01 \text{ cm/sec}.$$

Thus, the radius is decreasing at 0.01 cm/sec.

35. (a) When full

$$\text{Volume of water in filter } = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 6^2 \cdot 10 = 120\pi.$$

Water flows out at a rate of 1.5 cm^3 per second, so

$$\text{Time to empty } = \frac{120\pi}{1.5} = 80\pi = 251.327 \text{ secs}.$$

The time taken is 251.327 sec or just over 4 minutes.

(b) Let the radius of the surface of the water be r cm when the depth is h cm. See Figure 4.114. Then by similar triangles

$$\frac{6}{10} = \frac{r}{h}$$
$$r = \frac{3}{5}h.$$

Thus, when the depth of the water is h,

$$\text{Volume of water } = V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h = \frac{3}{25}\pi h^3 \text{ cm}^3.$$

(c) We know that water is flowing out at 1.5 cm^3 per second, so $dV/dt = -1.5$. We want to know dh/dt when $h = 8$. Differentiating the answer to part (b), we have

$$\frac{dV}{dt} = \frac{3}{25}\pi 3h^2\frac{dh}{dt} = \frac{9}{25}\pi h^2\frac{dh}{dt}.$$

Substituting $dV/dt = -1.5$ and $h = 8$ gives

$$-1.5 = \frac{9}{25}\pi 8^2 \cdot \frac{dh}{dt}$$
$$\frac{dh}{dt} = -\frac{1.5\cdot 25}{9\pi 8^2} = -0.0207 \text{ cm/sec}.$$

Thus, the water level is dropping by 0.0207 cm per second.

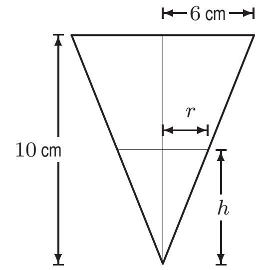


Figure 4.114

36. The surface area of an open cone is given by $A = \pi r s$ where s is the length of the side of the cone. Thus, if the height of the cone is h and the radius r, the side $s = \sqrt{r^2 + h^2}$. Since the cone has radius 4 in and depth 4 in, when the depth is h, the radius of the surface of the water is also h, so $r = h$. Thus

$$A = \pi r\sqrt{r^2 + h^2} = \pi r\sqrt{r^2 + r^2} = \sqrt{2}\pi r^2.$$

We want dA/dt when $r = 2.5$ and $dr/dt = 3$ in/min. Differentiating gives

$$\frac{dA}{dt} = 2\sqrt{2}\pi r\frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\sqrt{2}\pi \cdot 2.5 \cdot 3 = 66.643 \text{ in}^2/\text{min}.$$

37. Since the slant side of the cone makes an angle of 45° with the vertical, we have $r = h$, so the volume of the cone is

$$V = \frac{1}{3}\pi h^3.$$

Differentiating with respect to time gives

$$\frac{dV}{dt} = \pi h^2\frac{dh}{dt}.$$

We know $dV/dt = 2$ cubic meters/min, and we want to know dh/dt when $h = 0.5$ meters. Substituting gives

$$2 = \pi(0.5)^2\frac{dh}{dt},$$
$$\frac{dh}{dt} = \frac{2}{\pi(0.5)^2} = \frac{8}{\pi} \text{ meters/min.}$$

38. (a) Let $x,\ y$ be the distances, in miles, of the car and truck respectively, from the gas station. See Figure 4.115. If the car and truck are h miles apart, Pythagoras' Theorem gives

$$h^2 = x^2 + y^2.$$

We know that when $x = 3,\ dx/dt = -100$ (the negative sign represents the fact that the distance from the gas station is decreasing), and $y = 4,\ dy/dt = 80$. Thus

$$h^2 = 3^2 + 4^2 = 25 \text{ so } h = 5 \text{ miles.}$$

We want to find dh/dt. Differentiating $h^2 = x^2 + y^2$ gives

$$2h\frac{dh}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$5\frac{dh}{dt} = 3(-100) + 4(80)$$
$$\frac{dh}{dt} = \frac{-300 + 320}{5} = 4 \text{ mph.}$$

Thus, the distance is increasing at 4 mph.

(b) If $dy/dt = 70$, we have

$$5\frac{dh}{dt} = 3(-100) + 4(70)$$
$$\frac{dh}{dt} = \frac{-300+280}{5} = -4 \text{ mph.}$$

Thus, the distance is decreasing at 4 mph.

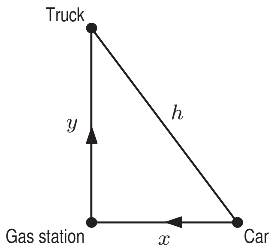


Figure 4.115

39. Let the origin be at the center of the wheel and (x, y) be the coordinates of a point on the wheel. Then $x^2 + y^2 = R^2$, where $R = 67.5$ meters is the radius of the wheel. One minute into the ride, we know the passenger is rising at 0.06 meters per second, so $dy/dt = 0.06$. We want to know dx/dt. Differentiating with respect to t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

so

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}.$$

Suppose we are looking at the wheel in such a way that it appears to be rotating counter clockwise. In one minute, the wheel travels through $360°/27 = 13.33°$. From Figure 4.116, we see that at this time the coordinates of the passenger are $x = R\sin 13.33°$ and $y = -R\cos 13.33°$. Since the vertical speed of the cabin is $dy/dt = 0.06$ meters per second, the horizontal speed of the wheel, dx/dt, is

$$\frac{dx}{dt} = -\frac{-R\cos 13.33°}{R\sin 13.33°}0.06 = 0.253 \text{ meters/second.}$$

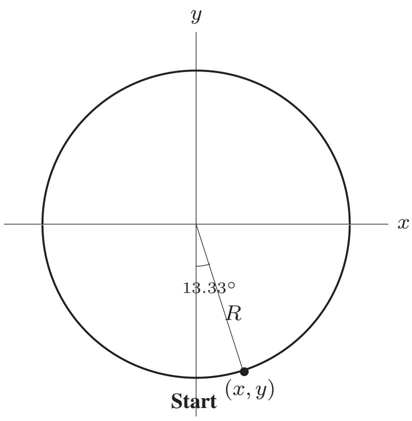


Figure 4.116

40. (a) From the second figure in the problem, we see that $\theta \approx 3.3$ when $t = 2$. The coordinates of P are given by $x = \cos\theta$, $y = \sin\theta$. When $t = 2$, the coordinates of P are

$$(x, y) \approx (\cos 3.3, \sin 3.3) = (-0.99, -0.16).$$

(b) Using the chain rule, the velocity in the x-direction is given by

$$v_x = \frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = -\sin\theta \cdot \frac{d\theta}{dt}.$$

From Figure 4.117, we estimate that when $t = 2$,

$$\left.\frac{d\theta}{dt}\right|_{t=2} \approx 2.$$

So

$$v_x = \frac{dx}{dt} \approx -(-0.16) \cdot (2) = 0.32.$$

Similarly, the velocity in the y-direction is given by

$$v_y = \frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt} = \cos\theta \cdot \frac{d\theta}{dt}.$$

When $t = 2$

$$v_y = \frac{dy}{dt} \approx (-0.99) \cdot (2) = -1.98.$$

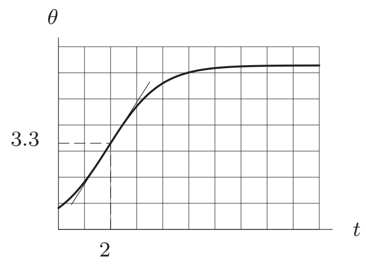


Figure 4.117

41. (a) On the interval $0 < M < 70$, we have

$$\text{Slope } = \frac{\Delta G}{\Delta M} = \frac{2.8}{70} = 0.04 \text{ gallons per mile.}$$

On the interval $70 < M < 100$, we have

$$\text{Slope } = \frac{\Delta G}{\Delta M} = \frac{4.6 - 2.8}{100 - 70} = \frac{1.8}{30} = 0.06 \text{ gallons per mile.}$$

(b) Gas consumption, in miles per gallon, is the reciprocal of the slope, in gallons per mile. On the interval $0 < M < 70$, gas consumption is $1/(0.04) = 25$ miles per gallon. On the interval $70 < M < 100$, gas consumption is $1/(0.06) = 16.667$ miles per gallon.

(c) In Figure 4.90 in the text, we see that the velocity for the first hour of this trip is 70 mph and the velocity for the second hour is 30 mph. The first hour may have been spent driving on an interstate highway and the second hour may have been spent driving in a city. The answers to part (b) would then tell us that this car gets 25 miles to the gallon on the highway and about 16 miles to the gallon in the city.

(d) Since $M = h(t)$, we have $G = f(M) = f(h(t)) = k(t)$. The function k gives the total number of gallons of gas used t hours into the trip. We have

$$G = k(0.5) = f(h(0.5)) = f(35) = 1.4 \text{ gallons.}$$

The car consumes 1.4 gallons of gas during the first half hour of the trip.

(e) Since $k(t) = f(h(t))$, by the chain rule, we have

$$\frac{dG}{dt} = k'(t) = f'(h(t)) \cdot h'(t).$$

Therefore:

$$\left.\frac{dG}{dt}\right|_{t=0.5} = k'(0.5) = f'(h(0.5)) \cdot h'(0.5) = f'(35) \cdot 70 = 0.04 \cdot 70 = 2.8 \text{ gallons per hour,}$$

and

$$\left.\frac{dG}{dt}\right|_{t=1.5} = k'(1.5) = f'(h(1.5)) \cdot h'(1.5) = f'(85) \cdot 30 = 0.06 \cdot 30 = 1.8 \text{ gallons per hour.}$$

Gas is being consumed at a rate of 2.8 gallons per hour at time $t = 0.5$ and is being consumed at a rate of 1.8 gallons per hour at time $t = 1.5$. Notice that gas is being consumed more quickly on the highway, even though the gas mileage is significantly better there.

42. The rate of change of temperature with distance, dH/dy, at altitude 4000 ft approximated by

$$\frac{dH}{dy} \approx \frac{\Delta H}{\Delta y} = \frac{38-52}{6-4} = -7^\circ\text{F/thousand ft.}$$

A speed of 3000 ft/min tells us $dy/dt = 3000$, so

$$\text{Rate of change of temperature with time} = \frac{dH}{dy} \cdot \frac{dy}{dt} \approx -7\frac{^\circ\text{F}}{\text{thousand ft}} \cdot \frac{3 \text{ thousand ft}}{\text{min}} = -21^\circ\text{F/min.}$$

Other estimates can be obtained by estimating the derivative as

$$\frac{dH}{dy} \approx \frac{\Delta H}{\Delta y} = \frac{52-60}{4-2} = -4^\circ\text{F/thousand ft}$$

or by averaging the two estimates

$$\frac{dH}{dy} \approx \frac{-7-4}{2} = -5.5^\circ\text{F/thousand ft.}$$

If the rate of change of temperature with distance is -4°/thousand ft, then

$$\text{Rate of change of temperature with time} = \frac{dH}{dy} \cdot \frac{dy}{dt} \approx -4\frac{^\circ\text{F}}{\text{thousand ft}} \cdot \frac{3 \text{ thousand ft}}{\text{min}} = -12^\circ\text{F/min.}$$

Thus, estimates for the rate at which temperature was decreasing range from 12°F/min to 21°F/min.

43. (a) Assuming that $T(1) = 98.6 - 2 = 96.6$, we get

$$\begin{aligned} 96.6 &= 68 + 30.6e^{-k\cdot 1} \\ 28.6 &= 30.6e^{-k} \\ 0.935 &= e^{-k}. \end{aligned}$$

So

$$k = -\ln(0.935) \approx 0.067.$$

(b) We're looking for a value of t which gives $T'(t) = -1$. First we find $T'(t)$:

$$\begin{aligned} T(t) &= 68 + 30.6e^{-0.067t} \\ T'(t) &= (30.6)(-0.067)e^{-0.067t} \approx -2e^{-0.067t}. \end{aligned}$$

Setting this equal to -1 per hour gives

$$\begin{aligned} -1 &= -2e^{-0.067t} \\ \ln(0.5) &= -0.067t \\ t &= -\frac{\ln(0.5)}{0.067} \approx 10.3. \end{aligned}$$

Thus, when $t \approx 10.3$ hours, we have $T'(t) \approx -1^\circ$F per hour.

(c) The coroner's rule of thumb predicts that in 24 hours the body temperature will decrease 25°F, to about 73.6°F. The formula predicts a temperature of

$$T(24) = 68 + 30.6e^{-0.067\cdot 24} \approx 74.1^\circ\text{F.}$$

44. (a) Using Pythagoras' theorem, we see

$$z^2 = 0.5^2 + x^2$$

so

$$z = \sqrt{0.25 + x^2}.$$

(b) We want to calculate dz/dt. Using the chain rule, we have

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} = \frac{2x}{2\sqrt{0.25 + x^2}} \frac{dx}{dt}.$$

Because the train is moving at 0.8 km/hr, we know that

$$\frac{dx}{dt} = 0.8 \text{ km/hr}.$$

At the moment we are interested in $z = 1$ km so

$$1^2 = 0.25 + x^2$$

giving

$$x = \sqrt{0.75} = 0.866 \text{ km}.$$

Therefore

$$\frac{dz}{dt} = \frac{2(0.866)}{2\sqrt{0.25 + 0.75}} \cdot 0.8 = 0.866 \cdot 0.8 = 0.693 \text{ km/min}.$$

(c) We want to know $d\theta/dt$, where θ is as shown in Figure 4.118. Since

$$\frac{x}{0.5} = \tan\theta$$

we know

$$\theta = \arctan\left(\frac{x}{0.5}\right),$$

so

$$\frac{d\theta}{dt} = \frac{1}{1 + (x/0.5)^2} \cdot \frac{1}{0.5} \frac{dx}{dt}.$$

We know that $dx/dt = 0.8$ km/min and, at the moment we are interested in, $x = \sqrt{0.75}$. Substituting gives

$$\frac{d\theta}{dt} = \frac{1}{1 + 0.75/0.25} \cdot \frac{1}{0.5} \cdot 0.8 = 0.4 \text{ radians/min}.$$

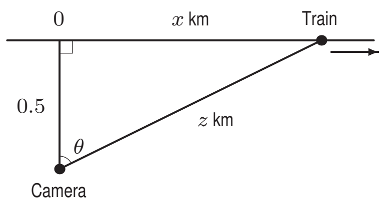


Figure 4.118

45. Using the triangle OSL in Figure 4.119, we label the distance x.

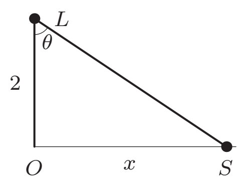


Figure 4.119

We want to calculate $dx/d\theta$. First we must find x as a function of θ. From the triangle, we see

$$\frac{x}{2} = \tan\theta \quad \text{so} \quad x = 2\tan\theta.$$

Thus,

$$\frac{dx}{d\theta} = \frac{2}{\cos^2\theta}.$$

46. The radius r is related to the volume by the formula $V = \frac{4}{3}\pi r^3$. By implicit differentiation, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

The surface area of a sphere is $4\pi r^2$, so we have

$$\frac{dV}{dt} = s \cdot \frac{dr}{dt},$$

but since $\frac{dV}{dt} = \frac{1}{3}s$ was given, we have

$$\frac{dr}{dt} = \frac{1}{3}.$$

47. The volume of a cube is $V = x^3$. So

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt},$$

and

$$\frac{1}{V}\frac{dV}{dt} = \frac{3}{x}\frac{dx}{dt}.$$

The surface area of a cube is $A = 6x^2$. So

$$\frac{dA}{dt} = 12x\frac{dx}{dt},$$

and

$$\frac{1}{A}\frac{dA}{dt} = \frac{2}{x}\frac{dx}{dt}.$$

Thus the percentage rate of change of the volume of the cube, $\frac{1}{V}\frac{dV}{dt}$, is larger.

48. (a) The end of the pipe sweeps out a circle of circumference $2\pi \cdot 20 = 40\pi$ meters in 5 minutes, so

$$\text{Speed } = \frac{40\pi}{5} = 8\pi = 25.133 \text{ meters/min.}$$

(b) The distance, h, between P and Q is given by the Law of Cosines:

$$h^2 = 50^2 + 20^2 - 2 \cdot 50 \cdot 20 \cos\theta.$$

When $\theta = \pi/2$, we have

$$h^2 = 50^2 + 20^2 - 2 \cdot 50 \cdot 20 \cdot 0.$$
$$h = \sqrt{2900} = 53.852 \text{ m.}$$

When $\theta = 0$, we have $h = 30$ m.

Since the pipe makes one rotation of 2π radians every 5 minutes, we know

$$\frac{d\theta}{dt} = \frac{2\pi}{5} \text{ radians/minute.}$$

Differentiating the relationship $h^2 = 50^2 + 20^2 - 2 \cdot 50 \cdot 20 \cos\theta$ gives

$$2h\frac{dh}{dt} = 2 \cdot 50 \cdot 20 \sin\theta\frac{d\theta}{dt}.$$

When $\theta = \pi/2$, we have

$$2\sqrt{2900}\frac{dh}{dt} = 2 \cdot 50 \cdot 20 \cdot 1 \cdot \frac{2\pi}{5}$$
$$\frac{dh}{dt} = \frac{50 \cdot 20}{\sqrt{2900}} \cdot \frac{2\pi}{5} = 23.335 \text{ meters/min.}$$

When $\theta = 0$, we have

$$2 \cdot 30\frac{dh}{dt} = 2 \cdot 50 \cdot 20 \cdot 0 \cdot \frac{2\pi}{5}$$
$$\frac{dh}{dt} = 0 \text{ meters/min.}$$

49. The volume, V, of a cone of radius r and height h is

$$V = \frac{1}{3}\pi r^2 h.$$

Figure 4.120 shows that $h/r = 10/8$, thus $r = 8h/10$, so

$$V = \frac{1}{3}\pi\left(\frac{8}{10}h\right)^2 h = \frac{64}{300}\pi h^3.$$

Differentiating V with respect to time, t, gives

$$\frac{dV}{dt} = \frac{64}{100}\pi h^2 \frac{dh}{dt}.$$

Since water is flowing into the tank at 0.1 cubic meters/min but leaking out at a rate of $0.004h^2$ cubic meters/min, we also have

$$\frac{dV}{dt} = 0.1 - 0.004h^2.$$

Equating the two expressions for dV/dt, we have

$$\frac{64}{100}\pi h^2 \frac{dh}{dt} = 0.1 - 0.004h^2.$$

Solving for dh/dt gives

$$\frac{dh}{dt} = \frac{0.1(100 - 4h^2)}{64\pi h^2}.$$

Notice dh/dt is positive when

$$100 - 4h^2 > 0 \text{ that is, when } 4h^2 < 100.$$

We conclude that if $h < 5$ then $dh/dt > 0$. Therefore, the depth increases until $h = 5$. For $h > 5$, we have $dh/dt < 0$, so the depth decreases whenever $h > 5$. Since the tank is more than 5 meters deep, it does not overflow.

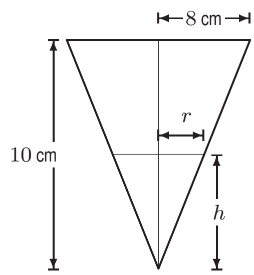


Figure 4.120

50. (a) Since the elevator is descending at 30 ft/sec, its height from the ground is given by $h(t) = 300 - 30t$, for $0 \leq t \leq 10$.

(b) From the triangle in the figure,

$$\tan\theta = \frac{h(t) - 100}{150} = \frac{300 - 30t - 100}{150} = \frac{200 - 30t}{150}.$$

Therefore

$$\theta = \arctan\left(\frac{200 - 30t}{150}\right)$$

and

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{200-30t}{150}\right)^2}\cdot\left(\frac{-30}{150}\right) = -\frac{1}{5}\left(\frac{150^2}{150^2 + (200 - 30t)^2}\right).$$

Notice that $\frac{d\theta}{dt}$ is always negative, which is reasonable since θ decreases as the elevator descends.

(c) If we want to know when θ changes (decreases) the fastest, we want to find out when $d\theta/dt$ has the largest magnitude. This will occur when the denominator, $150^2 + (200 - 30t)^2$, in the expression for $d\theta/dt$ is the smallest, or when $200 - 30t = 0$. This occurs when $t = \frac{200}{30}$ seconds, and so $h(\frac{200}{30}) = 100$ feet, i.e., when the elevator is at the level of the observer.

51. (a) We differentiate $a^2(t)+b^2(t)=c$ with respect to t to find

$$\frac{d}{dt}(a^2(t)+b^2(t))=\frac{d}{dt}c,$$

or

$$2a(t)\cdot a'(t)-2b(t)\cdot b'(t)=0,$$

giving

$$a(t)\cdot a'(t)=-b(t)\cdot b'(t).$$

(b) (i) If Angela likes Brian, then $a(t)>0$, so $b'(t)<0$. This means that $b(t)$ is decreasing, so Brian's affection decreases when Angela likes him.

(ii) If Angela dislikes Brian, then $a(t)<0$, so $b'(t)>0$. This means that $b(t)$ is increasing, so Brian's affection increases when Angela dislikes him.

(c) Substituting $b'(t)=-a(t)$ into $a(t)\cdot a'(t)=-b(t)\cdot b'(t)$ gives

$$a(t)\cdot a'(t)=-b(t)\cdot b'(t)=-b(t)(-a(t)),$$

so

$$a'(t)=b(t).$$

(i) If Brian likes Angela, then $b(t)>0$, so $a'(t)>0$. This means that $a(t)$ is increasing, so Angela's affection increases when Brian likes her.

(ii) If Brian dislikes Angela, then $b(t)<0$, so $a'(t)<0$. This means that $a(t)$ is decreasing, so Angela's affection decreases when Brian dislikes her.

(d) When $t=0$, they both like each other. This means that Angela's affection increases, while Brian's decreases. Eventually $b(t)<0$, when he dislikes her.

52. (a) We have either $x(0)=50$ and $y(0)=40$, or $y(0)=50$ and $x(0)=40$. In the first case $c=x^2(0)-y^2(0)=50^2-40^2=900$ whereas in the second $c=x^2(0)-y^2(0)=40^2-50^2=-900$. But $c>0$, so $c=900$ and we have $x^2(t)-y^2(t)=900$.

(b) Because $x^2(t)-y^2(t)=900$ we have $x^2(3)-y^2(3)=900$ so $x^2(3)=y^2(3)+900=16^2+900=1156$, giving $x(3)=\sqrt{1156}=34$. After 3 hours, y has 16 ships, and x has 34 ships.

(c) The condition $y(T)=0$ means that there are no more ships on that side, so the battle ends at time T hours.

(d) We have $x^2(T)-y^2(T)=900$ with $y(T)=0$ so $x(T)=30$ ships.

(e) The rate per hour at which y loses ships is $y'(t)$, so $y'(t)=kx$. Because y is decreasing, k is negative.

(f) We differentiate $x^2(t)-y^2(t)=900$ with respect to t to find

$$\frac{d}{dt}(x^2(t)-y^2(t))=\frac{d}{dt}900,$$

or

$$2x(t)\cdot x'(t)-2y(t)\cdot y'(t)=0,$$

giving

$$x'(t)=\frac{y(t)}{x(t)}y'(t).$$

But $y'(t)=kx(t)$ so

$$x'(t)=\frac{y(t)}{x(t)}kx(t)=ky(t).$$

(g) From part (b), we know that when $t=3$ we have $x(3)=34$, $y(3)=16$; we are now given that $x'(3)=32$. But $x'(t)=ky(t)$ so $32=ky(3)=16k$ giving $k=2$. In this case $y'(3)=kx(3)=2\cdot 34=68$ ships/hour.

Strengthen Your Understanding

53. The radius of a circle and the circle's diameter are related by the linear function $D=2R$. Thus, $\frac{dD}{dt}=\frac{dD}{dR}\frac{dR}{dt}=2\frac{dR}{dt}$. The rate of change of the diameter is twice the rate of change of the radius. For example, if the radius is increasing at a constant rate of 5 feet per second, then the circle's diameter is increasing at a constant rate of 10 feet per second.

54. Differentiating $y=1-x^2$ with respect to t gives

$$\frac{dy}{dt}=-2x\frac{dx}{dt}.$$

55. We want $dx/dt = g'(t)$ to be constant, so g can be any linear function of t, such as $g(t) = 5t$. Since $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, in order for dy/dt to be constant, we must also have dy/dx a constant. Choosing, for example, $f(x) = 2x + 1$ we have:

$$\frac{dx}{dt} = 5 \qquad \text{and} \qquad \frac{dy}{dt} = 2 \cdot \frac{dx}{dt} = 2(5) = 10.$$

There are many other possible solutions.

56. We want $dx/dt = g'(t)$ to be constant, so g can be any linear function of t, such as $g(t) = 5t$. If we choose f to be a linear function, then dy/dt will be constant. So we choose a function f which is not linear, such as $f(x) = x^2$. With $y = x^2$ and $x = 5t$, we have

$$\frac{dx}{dt} = 5 \qquad \text{and} \qquad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 2x(5) = 10t.$$

There are other possible examples.

57. True. The circumference C and radius r are related by $C = 2\pi r$, so $dC/dt = 2\pi dr/dt$. Thus if dr/dt is constant, so is dC/dt.

58. False. The circumference A and radius r are related by $A = \pi r^2$, so $dA/dt = 2\pi r dr/dt$. Thus dA/dt depends on r and since r is not constant, neither is dA/dt.

59. (c). Differentiating $x = 5\tan\theta$ gives

$$\frac{dx}{dt} = \frac{5}{\cos^2\theta}\frac{d\theta}{dt}.$$

Since the light rotates at 2 revolutions per minute $= 4\pi$ radians per minute, we know $d\theta/dt = 4\pi$. Thus, we can calculate dx/dt, the speed at which the spot is moving, for any angle θ.

Differentiating any of the other relationships introduces dr/dt, whose values we cannot find as easily as we can find $d\theta/dt$.

Solutions for Section 4.7

Exercises

1. Since $\lim_{x\to 2}(x-2) = \lim_{x\to 2}(x^2-4) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}.$$

2. Since $\lim_{x\to 1}(x^2+3x-4) = \lim_{x\to 1}(x-1) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to 1}\frac{x^2+3x-4}{x-1} = \lim_{x\to 1}\frac{2x+3}{1} = 5.$$

3. Since $\lim_{x\to 1}(x^6-1) = \lim_{x\to 1}(x^4-1) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to 1}\frac{x^6-1}{x^4-1} = \lim_{x\to 1}\frac{6x^5}{4x^3} = \frac{6}{4} = 1.5.$$

4. Since $\lim_{x\to 0}(e^x-1) = \lim_{x\to 0}(\sin x) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to 0}\frac{e^x-1}{\sin x} = \lim_{x\to 0}\frac{e^x}{\cos x} = \frac{1}{1} = 1.$$

5. Since the limit is not in the form $0/0$ or ∞/∞, l'Hopital's rule does not apply in this case. We have

$$\lim_{x\to 0}\frac{\sin x}{e^x} = \frac{0}{1} = 0.$$

6. Since $\lim_{x\to1}(\ln x) = \lim_{x\to1}(x-1) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to1} \frac{\ln x}{x-1} = \lim_{x\to1} \frac{(1/x)}{1} = \frac{1}{1} = 1.$$

7. Since $\lim_{x\to\infty}(\ln x) = \lim_{x\to\infty}(x) = \infty$, this is an ∞/∞ form, and l'Hopital's rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{(1/x)}{1} = 0.$$

8. Since $\lim_{x\to\infty}((\ln x)^3) = \lim_{x\to\infty}(x^2) = \infty$, this is an ∞/∞ form, and l'Hopital's rule applies. In fact, after applying l'Hopital's rule, we again obtain the ∞/∞ form. We apply l'Hopital's rule repeatedly, simplifying algebraically at each step:

$$\lim_{x\to\infty} \frac{(\ln x)^3}{x^2} = \lim_{x\to\infty} \frac{3(\ln x)^2(1/x)}{2x} = \lim_{x\to\infty} \frac{3(\ln x)^2}{2x^2} = \lim_{x\to\infty} \frac{6\ln x(1/x)}{4x} = \lim_{x\to\infty} \frac{6\ln x}{4x^2} = \lim_{x\to\infty} \frac{6/x}{8x} = \lim_{x\to\infty} \frac{6}{8x^2}.$$

The last limit has the form $6/\infty$ so the limit is 0.

9. Since $\lim_{x\to0}(e^{4x}-1) = 0$ and $\lim_{x\to0}(\cos x) = 1$, l'Hopital's rule does not apply in this case. We have

$$\lim_{x\to0} \frac{e^{4x}-1}{\cos x} = \frac{0}{1} = 0.$$

10. Since $\lim_{x\to1}(x^a-1) = \lim_{x\to1}(x^b-1) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to1} \frac{x^a-1}{x^b-1} = \lim_{x\to1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}.$$

11. Since $\lim_{x\to a}(\sqrt[3]{x} - \sqrt[3]{a}) = \lim_{x\to a}(x-a) = 0$, this is a $0/0$ form, and l'Hopital's rule applies:

$$\lim_{x\to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x-a} = \lim_{x\to a} \frac{(1/3)x^{-2/3}}{1} = \frac{1}{3}a^{-2/3}.$$

12. The larger power dominates. Using l'Hopital's rule

$$\begin{aligned} \lim_{x\to\infty} \frac{x^5}{0.1x^7} &= \lim_{x\to\infty} \frac{5x^4}{0.7x^6} = \lim_{x\to\infty} \frac{20x^3}{4.2x^5} \\ &= \lim_{x\to\infty} \frac{60x^2}{21x^4} = \lim_{x\to\infty} \frac{120x}{84x^3} = \lim_{x\to\infty} \frac{120}{252x^2} = 0 \end{aligned}$$

so $0.1x^7$ dominates.

13. We apply l'Hopital's rule twice to the ratio $50x^2/0.01x^3$:

$$\lim_{x\to\infty} \frac{50x^2}{0.01x^3} = \lim_{x\to\infty} \frac{100x}{0.03x^2} = \lim_{x\to\infty} \frac{100}{0.06x} = 0.$$

Since the limit is 0, we see that $0.01x^3$ is much larger than $50x^2$ as $x\to\infty$.

14. The power function dominates. Using l'Hopital's rule

$$\lim_{x\to\infty} \frac{\ln(x+3)}{x^{0.2}} = \lim_{x\to\infty} \frac{\frac{1}{(x+3)}}{0.2x^{-0.8}} = \lim_{x\to\infty} \frac{x^{0.8}}{0.2(x+3)}.$$

Using l'Hopital's rule again gives

$$\lim_{x\to\infty} \frac{x^{0.8}}{0.2(x+3)} = \lim_{x\to\infty} \frac{0.8x^{-0.2}}{0.2} = 0,$$

so $x^{0.2}$ dominates.

15. The exponential dominates. After 10 applications of l'Hopital's rule

$$\lim_{x\to\infty}\frac{x^{10}}{e^{0.1x}}=\lim_{x\to\infty}\frac{10x^9}{0.1e^{0.1x}}=\cdots=\lim_{x\to\infty}\frac{10!}{(0.1)^{10}e^{0.1x}}=0,$$

so $e^{0.1x}$ dominates.

Problems

16. Observe that both $f(4)$ and $g(4)$ are zero. Also, $f'(4)=1.4$ and $g'(4)=-0.7$, so by l'Hopital's rule,

$$\lim_{x\to 4}\frac{f(x)}{g(x)}=\frac{f'(4)}{g'(4)}=\frac{1.4}{-0.7}=-2.$$

17. Since $f'(a)>0$ and $g'(a)<0$, l'Hopital's rule tells us that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}<0.$$

18. Since $f'(a)<0$ and $g'(a)<0$, l'Hopital's rule tells us that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}>0.$$

19. Here $f(a)=g(a)=f'(a)=g'(a)=0$, and $f''(a)>0$ and $g''(a)<0$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f''(a)}{g''(a)}<0$$

20. Note that $f(0)=g(0)=0$ and $f'(0)=g'(0)$. Since $x=0$ looks like a point of inflection for each curve, $f''(0)=g''(0)=0$. Therefore, applying l'Hopital's rule successively gives us

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\lim_{x\to 0}\frac{f'''(x)}{g'''(x)}.$$

Now notice how the concavity of f changes: for $x<0$, it is concave up, so $f''(x)>0$, and for $x>0$ it is concave down, so $f''(x)<0$. Thus $f''(x)$ is a decreasing function at 0 and so $f'''(0)$ is negative. Similarly, for $x<0$, we see g is concave down and for $x>0$ it is concave up, so $g''(x)$ is increasing at 0 and so $g'''(0)$ is positive. Consequently,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'''(0)}{g'''(0)}<0.$$

21. The denominator approaches zero as x goes to zero and the numerator goes to zero even faster, so you should expect that the limit to be 0. You can check this by substituting several values of x close to zero. Alternatively, using l'Hopital's rule, we have

$$\lim_{x\to 0}\frac{x^2}{\sin x}=\lim_{x\to 0}\frac{2x}{\cos x}=0.$$

22. The numerator goes to zero faster than the denominator, so you should expect the limit to be zero. Using l'Hopital's rule, we have

$$\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{2\sin x\cos x}{1}=0.$$

23. The denominator goes to zero more slowly than x does, so the numerator goes to zero faster than the denominator, so you should expect the limit to be zero. With l'Hopital's rule,

$$\lim_{x\to 0}\frac{\sin x}{x^{1/3}}=\lim_{x\to 0}\frac{\cos x}{\frac{1}{3}x^{-2/3}}=\lim_{x\to 0}3x^{2/3}\cos x=0.$$

24. The denominator goes to zero more slowly than x. Therefore, you should expect that the limit to be 0. Using l'Hopital's rule,

$$\lim_{x\to 0}\frac{x}{(\sin x)^{1/3}} = \lim_{x\to 0}\frac{1}{\frac{1}{3}(\sin x)^{-2/3}\cos x} = \lim_{x\to 0}\frac{3(\sin x)^{2/3}}{\cos x} = 0,$$

since $\sin 0 = 0$ and $\cos 0 = 1$.

25. Since $\lim_{x\to\infty} x = \lim_{x\to\infty} e^x = \infty$, this is an ∞/∞ form, and l'Hopital's rule applies directly.

26. We have $\lim_{x\to 1} x = 1$ and $\lim_{x\to 1}(x-1) = 0$, so l'Hopital's rule does not apply.

27. We have $\lim_{t\to\infty}(1/t) - (2/t^2) = 0 - 0 = 0$. This is not an indeterminate form and l'Hopital's rule does not apply.

28. We have $\lim_{t\to 0^+} 1/t = \lim_{t\to 0^+} 1/(e^t - 1) = \infty$, so this is an $\infty - \infty$ form. Adding the fractions we get

$$\lim_{t\to 0^+}\frac{e^t - 1 - t}{t(e^t - 1)},$$

which is a $0/0$ form to which l'Hopital's rule can be applied.

29. We have $\lim_{x\to 0}(1+x)^x = (1+0)^0 = 1$. This is not an indeterminate form and l'Hopital's rule does not apply.

30. This is an ∞^0 form. With $y = \lim_{x\to\infty}(1+x)^{1/x}$, we take logarithms to get

$$\ln y = \lim_{x\to\infty}\frac{1}{x}\ln(1+x).$$

This limit is a $0\cdot\infty$ form,

$$\lim_{x\to\infty}\frac{1}{x}\ln(1+x),$$

which can be rewritten as the ∞/∞ form

$$\lim_{x\to\infty}\frac{\ln(1+x)}{x},$$

to which l'Hopital's rule applies.

31. Let $f(x) = \ln x$ and $g(x) = x^2 - 1$, so $f(1) = 0$ and $g(1) = 0$ and l'Hopital's rule can be used. To apply l'Hopital's rule, we first find $f'(x) = 1/x$ and $g'(x) = 2x$, then

$$\lim_{x\to 1}\frac{\ln x}{x^2 - 1} = \lim_{x\to 1}\frac{1/x}{2x} = \lim_{x\to 1}\frac{1}{2x^2} = \frac{1}{2}.$$

32. Let $f(t) = \sin^2 t$ and $g(t) = t - \pi$, then $f(\pi) = 0$ and $g(\pi) = 0$ but $f'(t) = 2\sin t\cos t$ and $g'(t) = 1$, so $f'(\pi) = 0$ and $g'(\pi) = 1$. L'Hopital's rule can be used, giving

$$\lim_{t\to\pi}\frac{\sin^2 t}{t - \pi} = \frac{0}{1} = 0.$$

33. Rewriting this as $\lim\limits_{n\to\infty} n^{1/n}$, we see that it is an ∞^0 form. The logarithm of the limit is

$$\lim_{n\to\infty}\frac{1}{n}\ln n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0.$$

Thus the original limit is $e^0 = 1$.

34. Let $f(x) = \ln x$ and $g(x) = 1/x$ so $f'(x) = 1/x$ and $g'(x) = -1/x^2$ and

$$\lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}\frac{x}{-1} = 0.$$

35. If $f(x) = \sinh(2x)$ and $g(x) = x$, then $f(0) = g(0) = 0$, so we use l'Hopital's Rule:

$$\lim_{x\to 0}\frac{\sinh 2x}{x} = \lim_{x\to 0}\frac{2\cosh 2x}{1} = 2.$$

36. If $f(x) = 1 - \cosh(3x)$ and $g(x) = x$, then $f(0) = g(0) = 0$, so we use l'Hopital's Rule:

$$\lim_{x\to 0} \frac{1-\cosh 3x}{x} = \lim_{x\to 0} \frac{-3\sinh 3x}{1} = 0.$$

37. Since $\cos 0 = 1$, we have $\cos^{-1}(1) = 0$ and $\lim_{x\to 1^{-1}} \cos^{-1} x = 0$. Therefore, both $\cos^{-1} x$ and $(x-1)$ tend to 0 as $x \to 1^-$, so l'Hopital's rule can be applied. Let $f(x) = \cos^{-1} x$ and $g(x) = x - 1$, and differentiate to get $f'(x) = -1/\sqrt{1-x^2}$ and $g'(x) = 1$. Applying l'Hopital's rule gives

$$\begin{aligned}\lim_{x\to 1^-} \frac{\cos^{-1} x}{x-1} &= \lim_{x\to 1^-} \frac{-1/\sqrt{1-x^2}}{1}\\ &= \lim_{x\to 1^-} \frac{-1}{\sqrt{1-x^2}}.\end{aligned}$$

However, $\sqrt{1-x^2} \to 0$ as $x \to 1^-$, so the limit does not exist.

38. To get this expression in a form in which l'Hopital's rule applies, we combine the fractions:

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}.$$

Letting $f(x) = \sin x - x$ and $g(x) = x\sin x$, we have $f(0) = 0$ and $g(0) = 0$ so l'Hopital's rule can be used. Differentiating gives $f'(x) = \cos x - 1$ and $g'(x) = x\cos x + \sin x$, so $f'(0) = 0$ and $g'(0) = 0$, so $f'(0)/g'(0)$ is undefined. Therefore, to apply l'Hopital's rule we differentiate again to obtain $f''(x) = -\sin x$ and $g''(x) = 2\cos x - x\sin x$, for which $f''(0) = 0$ and $g''(0) = 2 \neq 0$. Then

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right)\\ &= \lim_{x\to 0}\left(\frac{\cos x - 1}{x\cos x + \sin x}\right)\\ &= \lim_{x\to 0}\left(\frac{-\sin x}{2\cos x - x\sin x}\right)\\ &= \frac{0}{2} = 0.\end{aligned}$$

39. Adding the fractions, we get

$$\lim_{t\to 0^+}\left(\frac{2}{t} - \frac{1}{e^t - 1}\right) = \lim_{t\to 0^+} \frac{2e^t - 2 - t}{t(e^t-1)} = \lim_{t\to 0^+} \frac{2e^t - 1}{e^t + te^t - 1} = \frac{1}{0} = \infty.$$

Note that the limit is ∞, not $-\infty$ because $e^t > 1$ for $t > 0$, so the denominator is always positive.

40. Adding the fractions and applying l'Hopital's rule, we get

$$\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{e^t - 1}\right) = \lim_{t\to 0} \frac{e^t - 1 - t}{t(e^t - 1)} = \lim_{t\to 0} \frac{e^t - 1}{e^t + te^t - 1}.$$

This is again a $0/0$ form, so we apply l'Hopital's rule again to get

$$\lim_{t\to 0} \frac{e^t - 1}{e^t + te^t - 1} = \lim_{t\to 0} \frac{e^t}{2e^t + te^t} = \frac{1}{2}$$

41. Let $y = (1 + \sin 3/x)^x$. Taking logs gives

$$\ln y = x \ln\left(1 + \sin \frac{3}{x}\right).$$

To use l'Hopital's rule, we rewrite $\ln y$ as a fraction:

$$\begin{aligned}\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} x \ln\left(1 + \sin\left(\frac{3}{x}\right)\right) \\ &= \lim_{x\to\infty} \frac{\ln(1+\sin(3/x))}{1/x}.\end{aligned}$$

Let $f(x) = \ln(1 + \sin(3/x))$ and $g(x) = 1/x$ then

$$f'(x) = \frac{\cos(3/x)(-3/x^2)}{(1+\sin(3/x))} \quad \text{and} \quad g'(x) = -\frac{1}{x^2}.$$

Now apply l'Hopital's rule to get

$$\begin{aligned}\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{f(x)}{g(x)} \\ &= \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \\ &= \lim_{x\to\infty} \frac{\cos(3/x)(-3/x^2)/(1+\sin(3/x))}{-1/x^2} \\ &= \lim_{x\to\infty} \frac{3\cos(3/x)}{1+\sin(3/x)} = \frac{3\cos 0}{1+\sin 0} \\ &= 3.\end{aligned}$$

Since $\lim_{x\to\infty} \ln y = 3$, we have

$$\lim_{x\to\infty} y = e^3.$$

Thus,

$$\lim_{x\to\infty} \left(1 + \sin\frac{3}{x}\right)^x = e^3.$$

42. Since $\lim_{t\to 0} \sin^2 At = 0$ and $\lim_{t\to 0} \cos At - 1 = 1 - 1 = 0$, this is a $0/0$ form. Applying l'Hopital's rule we get

$$\lim_{t\to 0} \frac{\sin^2 At}{\cos At - 1} = \lim_{t\to 0} \frac{2A\sin At\cos At}{-A\sin At} = \lim_{t\to 0} -2\cos At = -2.$$

43. We rewrite this in the form

$$\lim_{t\to\infty} e^t - t^n = \lim_{t\to\infty} e^t\left(1 - \frac{t^n}{e^t}\right).$$

From Example 5 on page 244 of the text, with $k = A = 1$ and $p = n$, we know that e^t dominates t^n, so

$$\lim_{t\to\infty} e^t\left(1 - \frac{t^n}{e^t}\right) = \infty \cdot (1 - 0) = \infty.$$

44. To get this expression in a form in which l'Hopital's rule applies, we rewrite it as a fraction:

$$x^a \ln x = \frac{\ln x}{x^{-a}}.$$

Letting $f(x) = \ln x$ and $g(x) = x^{-a}$, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln x = -\infty, \quad \text{and} \quad \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{1}{x^a} = \infty.$$

So l'Hopital's rule can be used. To apply l'Hopital's rule we differentiate to get $f'(x) = 1/x$ and $g'(x) = -ax^{-a-1}$. Then

$$\begin{aligned}\lim_{x\to0^+} x^a \ln x &= \lim_{x\to0^+} \frac{\ln x}{x^{-a}}\\ &= \lim_{x\to0^+} \frac{1/x}{-ax^{-a-1}}\\ &= -\frac{1}{a}\lim_{x\to0^+} x^a\\ &= 0.\end{aligned}$$

45. Let $f(x) = \sin(2x)$ and $g(x) = x$. Observe that $f(1) = \sin 2 \neq 0$ and $g(1) = 1 \neq 0$. Therefore l'Hopital's rule does not apply. However,

$$\lim_{x\to1} \frac{\sin 2x}{x} = \frac{\sin 2}{1} = 0.909297.$$

46. Let $f(x) = \cos x$ and $g(x) = x$. Observe that since $f(0) = 1$, l'Hopital's rule does not apply. But since $g(0) = 0$,

$$\lim_{x\to0} \frac{\cos x}{x} \quad \text{does not exist.}$$

47. Let $f(x) = e^{-x}$ and $g(x) = \sin x$. Observe that as x increases, $f(x)$ approaches 0 but $g(x)$ oscillates between -1 and 1. Since $g(x)$ does not approach 0 in the limit, l'Hopital's rule does not apply. Because $g(x)$ is in the denominator and oscillates through 0 forever, the limit does not exist.

48. Let $n = 1/x$, so $n \to \infty$ as $x \to 0^+$. Thus

$$\lim_{x\to0^+} (1+x)^{1/x} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

49. Let $k = n/2$, so $k \to \infty$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{2k} = \lim_{k\to\infty} \left(\left(1+\frac{1}{k}\right)^k\right)^2 = e^2.$$

50. Let $n = 1/(kx)$, so $n \to \infty$ as $x \to 0^+$. Thus

$$\lim_{x\to0^+} (1+kx)^{t/x} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{nkt} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{kt} = e^{kt}.$$

51. Let

$$y = \left(1-\frac{1}{n}\right)^n.$$

Then

$$\ln y = \ln\left(1-\frac{1}{n}\right)^n = n\ln\left(1-\frac{1}{n}\right) = \frac{\ln(1-1/n)}{1/n}.$$

As $n \to \infty$, both the numerator and the denominator of the last fraction tend to 0. Thus, applying L'Hopital's rule, we have

$$\lim_{n\to\infty} \frac{\ln(1-1/n)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1-1/n}\cdot\frac{1}{n^2}}{-1/n^2} = \lim_{n\to\infty} -\frac{1}{1-1/n} = -1.$$

Thus,

$$\lim_{n\to\infty} \ln y = -1$$
$$\lim_{n\to\infty} y = e^{-1},$$

so

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right) = e^{-1}.$$

52. Let $k = n/\lambda$, so $k \to \infty$ as $n \to \infty$. Thus

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{k\to\infty}\left(1-\frac{1}{k}\right)^{k\lambda} = \lim_{k\to\infty}\left(\left(1-\frac{1}{k}\right)^k\right)^\lambda = (e^{-1})^\lambda = e^{-\lambda}.$$

53. This limit is of the form 0^0 so we apply l'Hopital's rule to

$$\ln f(t) = \frac{\ln\left((3^t+5^t)/2\right)}{t}.$$

We have

$$\begin{aligned}
\lim_{t\to-\infty}\ln f(t) &= \lim_{t\to-\infty}\frac{\left((\ln 3)3^t+(\ln 5)5^t\right)/\left(3^t+5^t\right)}{1}\\
&= \lim_{t\to-\infty}\frac{(\ln 3)3^t+(\ln 5)5^t}{3^t+5^t}\\
&= \lim_{t\to-\infty}\frac{\ln 3+(\ln 5)(5/3)^t}{1+(5/3)^t}\\
&= \frac{\ln 3+0}{1+0} = \ln 3.
\end{aligned}$$

Thus

$$\lim_{t\to-\infty} f(t) = \lim_{t\to-\infty} e^{\ln f(t)} = e^{\lim_{t\to-\infty}\ln f(t)} = e^{\ln 3} = 3.$$

54. This limit is of the form ∞^0 so we apply l'Hopital's rule to

$$\ln f(t) = \frac{\ln\left((3^t+5^t)/2\right)}{t}.$$

We have

$$\begin{aligned}
\lim_{t\to+\infty}\ln f(t) &= \lim_{t\to+\infty}\frac{\left((\ln 3)3^t+(\ln 5)5^t\right)/\left(3^t+5^t\right)}{1}\\
&= \lim_{t\to+\infty}\frac{(\ln 3)3^t+(\ln 5)5^t}{3^t+5^t}\\
&= \lim_{t\to+\infty}\frac{(\ln 3)(3/5)^t+\ln 5}{(3/5)^t+1}\\
&= \lim_{t\to+\infty}\frac{0+\ln 5}{0+1} = \ln 5.
\end{aligned}$$

Thus

$$\lim_{t\to-\infty} f(t) = \lim_{t\to-\infty} e^{\ln f(t)} = e^{\lim_{t\to-\infty}\ln f(t)} = e^{\ln 5} = 5.$$

55. This limit is of the form 1^∞ so we apply l'Hopital's rule to

$$\ln f(t) = \frac{\ln\left((3^t+5^t)/2\right)}{t}.$$

We have

$$\begin{aligned}
\lim_{t\to 0}\ln f(t) &= \lim_{t\to 0}\frac{\left((\ln 3)3^t+(\ln 5)5^t\right)/\left(3^t+5^t\right)}{1}\\
&= \lim_{t\to 0}\frac{(\ln 3)3^t+(\ln 5)5^t}{3^t+5^t}\\
&= \frac{\ln 3+\ln 5}{1+1} = \frac{\ln 15}{2}.
\end{aligned}$$

Thus

$$\lim_{t\to-\infty} f(t) = \lim_{t\to-\infty} e^{\ln f(t)} = e^{\lim_{t\to-\infty}\ln f(t)} = e^{(\ln 15)/2} = \sqrt{15}.$$

56. To evaluate, we use l'Hopital's Rule:

$$\lim_{x\to 0}\frac{\sinh 2x}{x}=\lim_{x\to 0}\frac{2\cosh 2x}{1}=2.$$

57. To evaluate, we use l'Hopital's Rule:

$$\lim_{x\to 0}\frac{1-\cosh 3x}{x}=\lim_{x\to 0}\frac{-3\sinh 3x}{1}=0.$$

58. To evaluate, we use l'Hopital's Rule:

$$\lim_{x\to 0}\frac{1-\cosh 5x}{x^2}=\lim_{x\to 0}\frac{-5\sinh 5x}{2x}=\lim_{x\to 0}\frac{-25\cosh 5x}{2}=-25/2.$$

59. To evaluate, we use l'Hopital's Rule:

$$\lim_{x\to 0}\frac{x-\sinh x}{x^3}=\lim_{x\to 0}\frac{1-\cosh x}{3x^2}=\lim_{x\to 0}\frac{-\sinh x}{6x}=\lim_{x\to 0}\frac{-\cosh x}{6}=-1/6.$$

60. Since the limit is of the form $0/0$, we can apply l'Hopital's rule. Doing it twice we have

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-\ln(1+x)}{x^2}&=\lim_{x\to 0}\frac{e^x-1/(1+x)}{2x}\\&=\lim_{x\to 0}\frac{e^x+1/(1+x)^2}{2}=\frac{1+1}{2}=1.\end{aligned}$$

61. Since the limit is of the form $0/0$, we can apply l'Hopital's rule. We have

$$\lim_{x\to \pi/2}\frac{1-\sin x+\cos x}{\sin x+\cos x-1}=\lim_{x\to \pi/2}\frac{-\cos x-\sin x}{\cos x-\sin x}=\frac{-1}{-1}=1.$$

62. Since the limit is of the form $0/0$, we can apply l'Hopital's rule. First note that

$$\frac{dx^x}{dx}=\frac{de^{x\ln x}}{dx}=e^{x\ln x}\left(x\cdot\frac{1}{x}+1\cdot\ln x\right)=x^x+x^x\ln x.$$

Applying l'Hopital's rule twice we have

$$\begin{aligned}\lim_{x\to 1}\frac{x^x-x}{1-x+\ln x}&=\lim_{x\to 1}\frac{x^x+x^x\ln x-1}{-1+1/x}\\&=\lim_{x\to 1}\frac{x^x+x^x\ln x+(x^x+x^x\ln x)\ln x+x^x(1/x)}{-1/x^2}=\frac{1+0+0+1}{-1}=-2.\end{aligned}$$

Strengthen Your Understanding

63. There is no such n because the function e^x dominates x^n for every n, no matter how large. This means that for any n, the values of e^x are much larger than the values of x^n for large x. Thus, x^n cannot dominate e^x.

To see why e^x dominates x^n for all positive integers n, let $f(x)=x^n$ and $g(x)=e^x$. Then $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to\infty} g(x)=\infty$ so we apply l'Hopital's rule:

$$\begin{aligned}\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{f(x)}{g(x)}&=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}\\&=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}\\&=\lim_{x\to\infty}\frac{n\cdot(n-1)x^{n-2}}{e^x}\\&\vdots\\&=\lim_{x\to\infty}\frac{n!}{e^x}=0\end{aligned}$$

so e^x dominates x^n for all positive integers n.

64. If $f(x) = 5x + \cos x$ and $g(x) = x$, we have $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, so we try to use l'Hopital's rule:

$$\lim_{x\to\infty} \frac{5x + \cos x}{x} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{5 - \sin x}{1}.$$

However, $\lim_{x\to\infty} 5 - \sin x$ does not exist, so l'Hopital's rule is not applicable.

65. When finding $\lim_{x\to a} f(x)/g(x)$, we can only apply l'Hopital's rule if $f(a) = g(a) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Thus, there are many examples of limits of rational functions to which l'Hopital's rule cannot be applied. One example is $\lim_{x\to 0} \frac{x+1}{x+2}$.

66. Since $\lim_{x\to\infty} \ln x = \infty$, if $\lim_{x\to\infty} f(x) = \pm\infty$ we could apply L'Hopital's rule to try to find $\lim_{x\to\infty} \frac{f(x)}{\ln x}$. One possible example is $f(x) = x$.

67. False. To use l'Hopital's rule, we need $f(a) = g(a) = 0$. For example, if $f(x) = 3$ and $g(x) = x$, then $g(1) = 1$ and $f'(1)/g'(1) = 0/1 = 0$, but $\lim_{x\to 1}(f(x)/g(x)) = 3/1 = 3$.

68. (b)

Solutions for Section 4.8

Exercises

1. Between times $t = 0$ and $t = 1$, x goes at a constant rate from 0 to 1 and y goes at a constant rate from 1 to 0. So the particle moves in a straight line from $(0, 1)$ to $(1, 0)$. Similarly, between times $t = 1$ and $t = 2$, it goes in a straight line to $(0, -1)$, then to $(-1, 0)$, then back to $(0, 1)$. So it traces out the diamond shown in Figure 4.121.

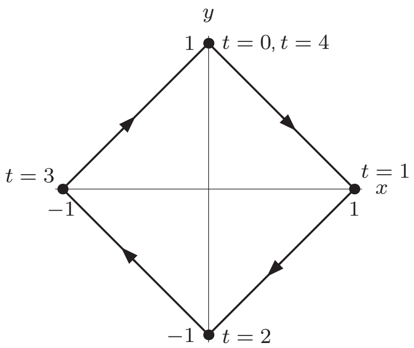


Figure 4.121

2. This is like Example 2, except that the x-coordinate goes all the way to 2 and back. So the particle traces out the rectangle shown in Figure 4.122.

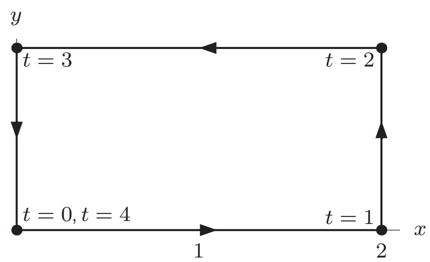


Figure 4.122

3. As the x-coordinate goes at a constant rate from 2 to 0, the y-coordinate goes from 0 to 1, then down to -1, then back to 0. So the particle zigs and zags from $(2, 0)$ to $(1.5, 1)$ to $(1, 0)$ to $(.5, -1)$ to $(0, 0)$. Then it zigs and zags back again, forming the shape in Figure 4.123.

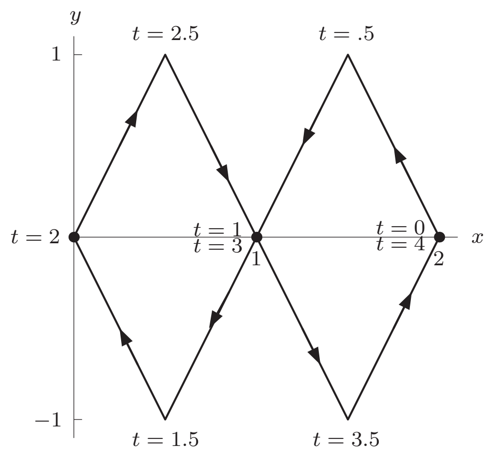


Figure 4.123

4. Between times $t = 0$ and $t = 1$, x goes from -1 to 1, while y stays fixed at 1. So the particle goes in a straight line from $(-1, 1)$ to $(1, 1)$. Then both the x- and y-coordinates decrease at a constant rate from 1 to -1. So the particle goes in a straight line from $(1, 1)$ to $(-1, -1)$. Then it moves across to $(1, -1)$, then back diagonally to $(-1, 1)$. See Figure 4.124.

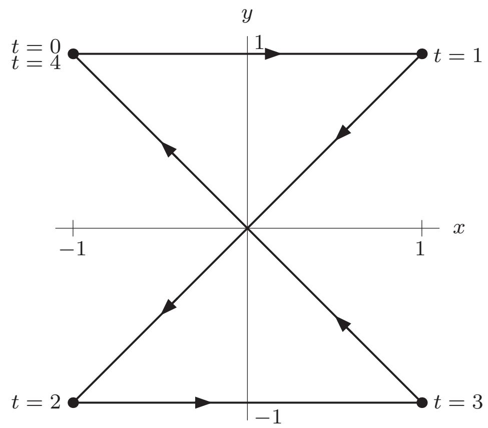


Figure 4.124

5. One possible answer is $x = 3\cos t, y = -3\sin t, 0 \le t \le 2\pi$.

6. One possible answer is $x = -2, y = t$.

7. One possible answer is $x = 2 + 5\cos t, y = 1 + 5\sin t, 0 \le t \le 2\pi$.

8. The parameterization $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, is a circle of radius 2 traced out counterclockwise starting at the point $(2, 0)$. To start at $(-2, 0)$, put a negative in front of the first coordinate

$$x = -2\cos t \quad y = 2\sin t, \qquad 0 \le t \le 2\pi.$$

Now we must check whether this parameterization traces out the circle clockwise or counterclockwise. Since when t increases from 0, $\sin t$ is positive, the point (x, y) moves from $(-2, 0)$ into the second quadrant. Thus, the circle is traced out clockwise and so this is one possible parameterization.

9. The slope of the line is

$$m = \frac{3-(-1)}{1-2} = -4.$$

The equation of the line with slope -4 through the point $(2,-1)$ is $y-(-1)=(-4)(x-2)$, so one possible parameterization is $x=t$ and $y=-4t+8-1=-4t+7$.

10. The ellipse $x^2/25+y^2/49=1$ can be parameterized by $x=5\cos t, y=7\sin t, 0\le t\le 2\pi$.

11. The parameterization $x=-3\cos t$, $y=7\sin t$, $0\le t\le 2\pi$, starts at the right point but sweeps out the ellipse in the wrong direction (the y-coordinate becomes positive as t increases). Thus, a possible parameterization is $x=-3\cos(-t)=-3\cos t, y=7\sin(-t)=-7\sin t, 0\le t\le 2\pi$.

12. For $0\le t\le \frac{\pi}{2}$, we have $x=\sin t$ increasing and $y=\cos t$ decreasing, so the motion is clockwise for $0\le t\le\frac{\pi}{2}$. Similarly, we see that the motion is clockwise for the time intervals $\frac{\pi}{2}\le t\le\pi, \pi\le t\le\frac{3\pi}{2}$, and $\frac{3\pi}{2}\le t\le 2\pi$.

13. The particle moves clockwise: For $0\le t\le\frac{\pi}{2}$, we have $x=\cos t$ decreasing and $y=-\sin t$ decreasing. Similarly, for the time intervals $\frac{\pi}{2}\le t\le\pi, \pi\le t\le\frac{3\pi}{2}$, and $\frac{3\pi}{2}\le t\le 2\pi$, we see that the particle moves clockwise.

14. Let $f(t)=t^2$. The particle is moving clockwise when $f(t)$ is decreasing, that is, when $f'(t)=2t<0$, so when $t<0$. The particle is moving counterclockwise when $f'(t)=2t>0$, so when $t>0$.

15. Let $f(t)=t^3-t$. The particle is moving clockwise when $f(t)$ is decreasing, that is, when $f'(t)=3t^2-1<0$, and counterclockwise when $f'(t)=3t^2-1>0$. That is, it moves clockwise when $-\sqrt{\frac{1}{3}}<t<\sqrt{\frac{1}{3}}$, between $(\cos((-\sqrt{\frac{1}{3}})^3+\sqrt{\frac{1}{3}}), \sin((-\sqrt{\frac{1}{3}})^3+\sqrt{\frac{1}{3}}))$ and $(\cos((\sqrt{\frac{1}{3}})^3-\sqrt{\frac{1}{3}}), \sin((\sqrt{\frac{1}{3}})^3-\sqrt{\frac{1}{3}})$, and counterclockwise when $t<-\sqrt{\frac{1}{3}}$ or $t>\sqrt{\frac{1}{3}}$.

16. Let $f(t)=\ln t$. Then $f'(t)=\frac{1}{t}$. The particle is moving counterclockwise when $f'(t)>0$, that is, when $t>0$. Any other time, when $t\le 0$, the position is not defined.

17. Let $f(t)=\cos t$. Then $f'(t)=-\sin t$. The particle is moving clockwise when $f'(t)<0$, or $-\sin t<0$, that is, when

$$2k\pi<t<(2k+1)\pi,$$

where k is an integer. The particle is otherwise moving counterclockwise, that is, when

$$(2k-1)\pi<t<2k\pi,$$

where k is an integer. Actually, the particle does not fully trace out a circle. The range of $f(t)$ is $[-1,1]$ so the particle oscillates between the points $(\cos(-1),\sin(-1))$ and $(\cos 1,\sin 1)$.

18. The circle $(x-2)^2+(y-2)^2=1$.

19. The line segment $y+x=4$, for $1\le x\le 3$.

20. The parabola $y=(x-2)^2$, for $1\le x\le 3$.

21. We see from the parametric equations that the particle moves along a line. It suffices to plot two points: at $t=0$, the particle is at point $(1,-4)$, and at $t=1$, the particle is at point $(4,-3)$. Since x increases as t increases, the motion is left to right on the line as shown in Figure 4.125.

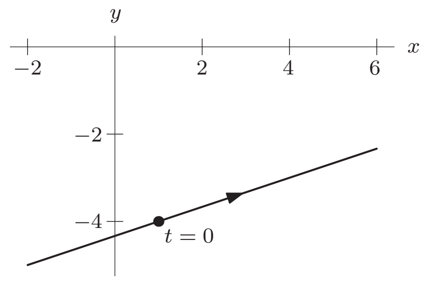


Figure 4.125

Alternately, we can solve the first equation for t, giving $t=(x-1)/3$, and substitute this into the second equation to get

$$y=\frac{x-1}{3}-4=\frac{1}{3}x-\frac{13}{3}.$$

The line is $y=\frac{1}{3}x-\frac{13}{3}$.

22. If we eliminate the parameter t, we see that

$$y = (x - 3) - 2 = x - 5.$$

Thus, the particle moves along a line. Notice, however, that $t^2 + 3 \geq 3$ and so $x \geq 3$. Similarly, $y \geq -2$. The graph is the ray, or half-line, shown in Figure 4.126.

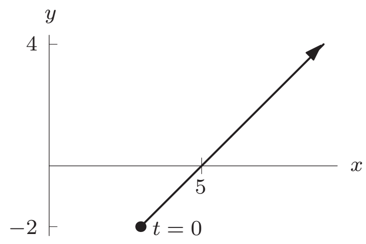


Figure 4.126

Notice that at $t = 0$, the particle is at point $(3, -2)$, at $t = 1$, the particle is at point $(4, -1)$, and at $t = -1$, the particle is at point $(4, -1)$. The particle is at the endpoint of the ray when $t = 0$. As t increases through negative t values, the particle moves down the ray toward the point $(3, -2)$. At $t = 0$, it changes direction. As t increases through positive t values, the particle moves up the ray again.

23. The Cartesian equation of the motion is obtained by eliminating the parameter t, giving

$$y = (x - 4)^2 - 3 = x^2 - 8x + 13.$$

The graph of this parabola is shown in Figure 4.127. At $t = 0$, the particle is at the vertex of the parabola $(4, -3)$, and the motion is left to right since x increases steadily as t increases.

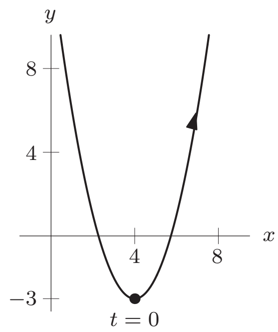


Figure 4.127

24. The graph is a circle centered at the origin with radius 1. The equation is

$$x^2 + y^2 = (\cos 3t)^2 + (\sin 3t)^2 = 1.$$

The particle is at the point $(1, 0)$ when $t = 0$, and motion is counterclockwise. See Figure 4.128.

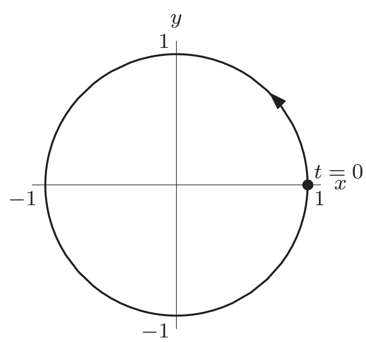


Figure 4.128

25. The graph is a circle centered at the origin with radius 3. The equation is

$$x^2 + y^2 = (3\cos t)^2 + (3\sin t)^2 = 9.$$

The particle is at the point $(3, 0)$ when $t = 0$, and motion is counterclockwise. See Figure 4.129.

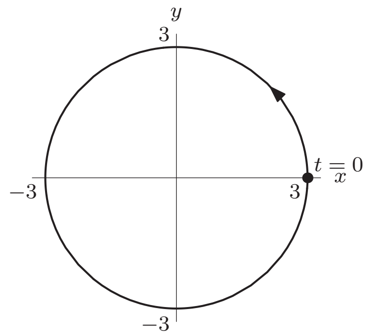


Figure 4.129

26. The graph is a circle centered at the point $(2, 7)$ with radius 5. The equation is

$$(x-2)^2 + (y-7)^2 = (5\cos t)^2 + (5\sin t)^2 = 25.$$

The particle is at the point $(7, 7)$ when $t = 0$, and motion is counterclockwise. See Figure 4.130.

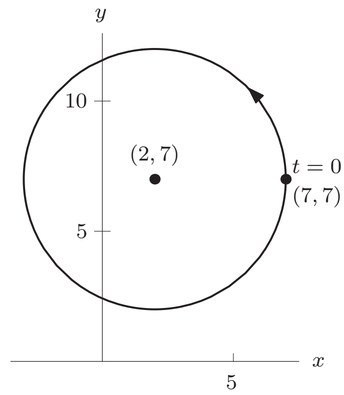


Figure 4.130

27. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 1}.$$

Thus when $t = 2$, the slope of the tangent line is $4/11$. Also when $t = 2$, we have

$$x = 2^3 - 2 = 6, \quad y = 2^2 = 4.$$

Therefore the equation of the tangent line is

$$(y - 4) = \frac{4}{11}(x - 6).$$

28. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+2}{2t-2}.$$

When $t = 1$, the denominator is zero and the numerator is nonzero, so the tangent line is vertical. Since $x = -1$ when $t = 1$, the equation of the tangent line is $x = -1$.

29. We have
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos(4t)}{3\cos(3t)}.$$
Thus when $t = \pi$, the slope of the tangent line is $-4/3$. Since $x = 0$ and $y = 0$ when $t = \pi$, the equation of the tangent line is $y = -(4/3)x$.

30. We have $dx/dt = 2t$ and $dy/dt = 3t^2$. Therefore, the speed of the particle is
$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{((2t)^2 + (3t^2)^2)} = |t| \cdot \sqrt{(4 + 9t^2)}.$$
The particle comes to a complete stop when its speed is 0, that is, if $t\sqrt{4 + 9t^2} = 0$, and so when $t = 0$.

31. We have $dx/dt = -2t\sin(t^2)$ and $dy/dt = 2t\cos(t^2)$. Therefore, the speed of the particle is given by
$$\begin{aligned} v &= \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} \\ &= \sqrt{4t^2(\sin(t^2))^2 + 4t^2(\cos(t^2))^2} \\ &= 2|t|\sqrt{\sin^2(t^2) + \cos^2(t^2)} \\ &= 2|t|. \end{aligned}$$
The particle comes to a complete stop when speed is 0, that is, if $2|t| = 0$, and so when $t = 0$.

32. We have
$$\frac{dx}{dt} = -2\sin 2t, \frac{dy}{dt} = \cos t.$$
The speed is
$$v = \sqrt{4\sin^2(2t) + \cos^2 t}.$$
Thus, $v = 0$ when $\sin(2t) = \cos t = 0$, and so the particle stops when $t = \pm\pi/2, \pm 3\pi/2, \ldots$ or $t = (2n+1)\frac{\pi}{2}$, for any integer n.

33. We have
$$\frac{dx}{dt} = 2t - 4, \quad \frac{dy}{dt} = 3t^2 - 12.$$
The speed is given by:
$$v = \sqrt{(2t-4)^2 + (3t^2 - 12)^2}.$$
The particle stops when $2t - 4 = 0$ and $3t^2 - 12 = 0$. Since these are both satisfied only by $t = 2$, this is the only time that the particle stops.

34. At $t = 2$, the position is $(2^2, 2^3) = (4, 8)$, the velocity in the x-direction is $2 \cdot 2 = 4$, and the velocity in the y-direction is $3 \cdot 2^2 = 12$. So we want the line going through the point $(4, 8)$, with the given x- and y-velocities:
$$x = 4 + 4t, \quad y = 8 + 12t.$$

Problems

35. The graphs are in Figure 4.131.

Figure 4.131

36. The graphs are in Figure 4.132.

Figure 4.132

37. (a) We get the part of the line with $x < 10$ and $y < 0$.
(b) We get the part of the line between the points $(10, 0)$ and $(11, 2)$.

38. (a) If $t \geq 0$, we have $x \geq 2, y \geq 4$, so we get the part of the line to the right of and above the point $(2, 4)$.
(b) When $t = 0$, $(x, y) = (2, 4)$. When $t = -1$, $(x, y) = (-1, -3)$. Restricting t to the interval $-1 \leq t \leq 0$ gives the part of the line between these two points.
(c) If $x < 0$, giving $2 + 3t < 0$ or $t < -2/3$. Thus $t < -2/3$ gives the points on the line to the left of the y-axis.

39. (a) Eliminating t between

$$x = 2 + t, \quad y = 4 + 3t$$

gives

$$y - 4 = 3(x - 2),$$
$$y = 3x - 2.$$

Eliminating t between

$$x = 1 - 2t, \quad y = 1 - 6t$$

gives

$$y - 1 = 3(x - 1),$$
$$y = 3x - 2.$$

Since both parametric equations give rise to the same equation in x and y, they both parameterize the same line.
(b) Slope $= 3$, $\quad y$-intercept $= -2$.

40. In all three cases, $y = x^2$, so that the motion takes place on the parabola $y = x^2$.

In case (a), the x-coordinate always increases at a constant rate of one unit distance per unit time, so the equations describe a particle moving to the right on the parabola at constant horizontal speed.

In case (b), the x-coordinate is never negative, so the particle is confined to the right half of the parabola. As t moves from $-\infty$ to $+\infty$, $x = t^2$ goes from ∞ to 0 to ∞. Thus the particle first comes down the right half of the parabola, reaching the origin $(0, 0)$ at time $t = 0$, where it reverses direction and goes back up the right half of the parabola.

In case (c), as in case (a), the particle traces out the entire parabola $y = x^2$ from left to right. The difference is that the horizontal speed is not constant. This is because a unit change in t causes larger and larger changes in $x = t^3$ as t approaches $-\infty$ or ∞. The horizontal motion of the particle is faster when it is farther from the origin.

41. (a) C_1 has center at the origin and radius 5, so $a = b = 0, k = 5$ or -5.
(b) C_2 has center at $(0, 5)$ and radius 5, so $a = 0, b = 5, k = 5$ or -5.
(c) C_3 has center at $(10, -10)$, so $a = 10, b = -10$. The radius of C_3 is $\sqrt{10^2 + (-10)^2} = \sqrt{200}$, so $k = \sqrt{200}$ or $k = -\sqrt{200}$.

42. (I) has a positive slope and so must be l_1 or l_2. Since its y-intercept is negative, these equations must describe l_2. (II) has a negative slope and positive x-intercept, so these equations must describe l_3.

43. It is a straight line through the point $(3, 5)$ with slope -1. A linear parameterization of the same line is $x = 3 + t$, $y = 5 - t$.

44. (a) The curve is a spiral as shown in Figure 4.133.

(b) At $t = 2$, the position is $(2\cos 2, 2\sin 2) = (-0.8323, 1.8186)$, and at $t = 2.01$ the position is $(2.01\cos 2.01, 2.01\sin 2.01) = (-0.8546, 1.8192)$. The distance between these points is

$$\sqrt{(-0.8546 - (-0.8323))^2 + (1.8192 - 1.8186)^2} \approx 0.022.$$

Thus the speed is approximately $0.022/0.01 \approx 2.2$. See Figure 4.134.

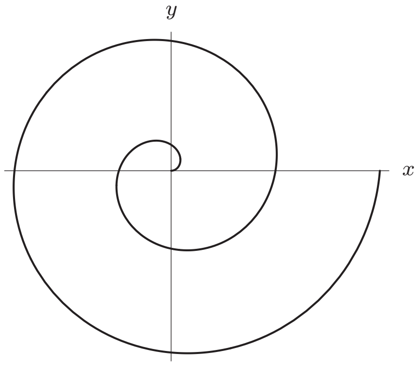


Figure 4.133: The spiral $x = t\cos t, y = t\sin t$ for $0 \le t \le 4\pi$

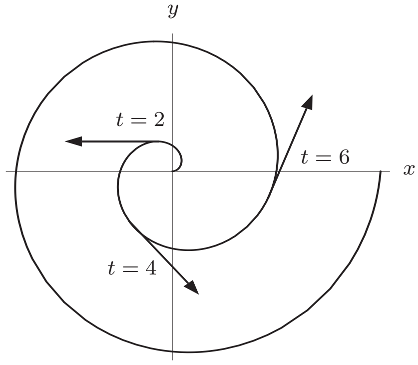


Figure 4.134: The spiral $x = t\cos t, y = t\sin t$ and three velocity vectors

(c) Evaluating the exact formula

$$v = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

gives :

$$v(2) = \sqrt{(-2.235)^2 + (0.077)^2} = 2.2363.$$

45. (a) The chain rule gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4e^{2t}}{e^t} = 4e^t.$$

(b) We are given $y = 2e^{2t}$ so $y = 2(e^t)^2$. Since $x = e^t$, we can substitute x for e^t. Thus $y = 2x^2$.

(c) Differentiating $y = 2x^2$ with respect to x, we get $dy/dx = 4x$. Notice that, since $x = e^t$, this is equivalent to the answer that we obtained in part (a).

46. (a) In order for the particle to stop, its velocity both dx/dt and dy/dt must be zero,

$$\frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1) = 0,$$
$$\frac{dy}{dt} = 2t - 2 = 2(t-1) = 0.$$

The value $t = 1$ is the only solution. Therefore, the particle stops when $t = 1$ at the point $(t^3 - 3t,\ t^2 - 2t)|_{t=1} = (-2, -1)$.

(b) In order for the particle to be traveling straight up or down, the velocity in the x-direction must be 0. Thus, we solve $dx/dt = 3t^2 - 3 = 0$ and obtain $t = \pm 1$. However, at $t = 1$ the particle has no vertical motion, as we saw in part (a). Thus, the particle is moving straight up or down only when $t = -1$. The position at that time is $(t^3 - 3t,\ t^2 - 2t)|_{t=-1} = (2, 3)$.

(c) For horizontal motion we need $dy/dt = 0$. That happens when $dy/dt = 2t - 2 = 0$, and so $t = 1$. But from part (a) we also have $dx/dt = 0$ also at $t = 1$, so the particle is not moving at all when $t = 1$. Thus, there is no time when the motion is horizontal.

47. (a) (i) A horizontal tangent occurs when $dy/dt = 0$ and $dx/dt \ne 0$. Thus,

$$\frac{dy}{dt} = 6e^{2t} - 2e^{-2t} = 0$$
$$6e^{2t} = 2e^{-2t}$$
$$e^{4t} = \frac{1}{3}$$
$$4t = \ln\frac{1}{3}$$
$$t = \frac{1}{4}\ln\frac{1}{3} = -0.25\ln 3 = -0.275.$$

We need to check that $dx/dt \neq 0$ when $t = -0.25 \ln 3$. Since $dx/dt = 2e^{2t} + 2e^{-2t}$ is always positive, dx/dt is never zero.

(ii) A vertical tangent occurs when $dx/dt = 0$ and $dy/dt \neq 0$. Since $dx/dt = 2e^{2t} + 2e^{-2t}$ is always positive, there is no vertical tangent.

(b) The chain rule gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6e^{2t} - 2e^{-2t}}{2e^{2t} + 2e^{-2t}} = \frac{3e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}}.$$

(c) As $t \to \infty$, we have $e^{-2t} \to 0$. Thus,

$$\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{3e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}} = \lim_{t\to\infty} \frac{3e^{2t}}{e^{2t}} = 3.$$

As $t \to \infty$, the fraction gets closer and closer to 3.

48. (a) Since $y = f'(t) > 0$ for all points on the curve, f is increasing.

(b) Since $x = f(t)$ is increasing by part (a), the motion is to the right from P to Q.

(c) As t increases and the curve is traced from P to Q, the values of the derivative $y = f'(t)$ decrease, so f is concave down.

49. (a) Figure 4.135 shows the path and the clockwise direction of motion. (The curve is an ellipse.)

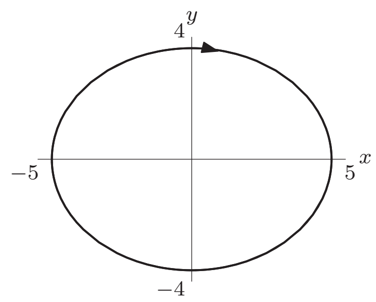


Figure 4.135

(b) At $t = \pi/4$, the position is given by

$$x(\pi/4) = 5\sin\frac{2\pi}{4} = 5\sin\frac{\pi}{2} = 5 \quad \text{and} \quad y(\pi/4) = 4\cos\frac{2\pi}{4} = 4\cos\frac{\pi}{2} = 0.$$

Differentiating, we get $x'(t) = 10\cos(2t)$ and $y'(t) = -8\sin(2t)$. At $t = \pi/4$, the velocity is given by

$$x'(\pi/4) = 10\cos\frac{2\pi}{4} = 10\cos\frac{\pi}{2} = 0 \quad \text{and} \quad y'(\pi/4) = -8\sin\frac{2\pi}{4} = -8\sin\frac{\pi}{2} = -8.$$

(c) As t increases from 0 to 2π, the ellipse is traced out twice. Thus, the particle passes through the point $(5, 0)$ twice.

(d) Since $x'(\pi/4) = 0$ and $y'(\pi/4) = -8$, when $t = \pi/4$, the particle is moving in the negative y-direction, parallel to the y-axis.

(e) At time t,

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(10\cos(2t))^2 + (-8\sin(2t))^2}.$$

When $t = \pi$,

$$\text{Speed} = \sqrt{(10\cos(2\pi))^2 + (-8\sin(2\pi))^2} = \sqrt{(10 \cdot 1)^2 + (-8 \cdot 0)^2} = 10.$$

50. (a) Substituting $\alpha = 36^\circ = \pi/5$ and $v_0 = 60$ into $x(t) = (v_0 \cos\alpha)t$ and $y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$, we get

$$x(t) = \left(60\cos\frac{\pi}{5}\right)t \quad \text{and} \quad y(t) = \left(60\sin\frac{\pi}{5}\right)t - \frac{1}{2}(32)t^2 = \left(60\sin\frac{\pi}{5}\right)t - 16t^2.$$

(b) Figure 4.136 shows the path and the direction of motion.

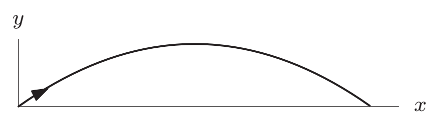


Figure 4.136

(c) When the football hits the ground, $y(t) = 0$, so

$$\begin{aligned}\left(60\sin\frac{\pi}{5}\right)t - 16t^2 &= 0\\ 4t\left(15\sin\frac{\pi}{5} - 4t\right) &= 0\\ t = 0 \quad \text{or} \quad t = \frac{15\sin(\pi/5)}{4} &= 2.204 \text{ seconds.}\end{aligned}$$

The ball hits the ground in approximately 2.204 seconds. The ball's distance from the spot where it was kicked is $x(2.204) = 106.994$ feet.

(d) At its highest point, the football is moving neither upward nor downward, so $y'(t)$ is zero. To find the time when the football reaches its maximum height, we set $y'(t) = 0$, giving

$$\begin{aligned}y'(t) = \left(60\sin\frac{\pi}{5}\right) - 32t &= 0\\ t = \frac{60\sin(\pi/5)}{32} &= 1.102 \text{ seconds.}\end{aligned}$$

This makes sense since this is half the time it took the football to reach the ground. The maximum height is $y(1.102) = 19.434$ feet. Thus the football reaches 19.434 feet.

(e) Since $x(t) = \left(60\cos\frac{\pi}{5}\right)t$ and $y(t) = \left(60\sin\frac{\pi}{5}\right)t - 16t^2$, we have

$$x'(t) = 60\cos\frac{\pi}{5} \quad \text{and} \quad y'(t) = 60\sin\frac{\pi}{5} - 32t.$$

Thus,

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{\left(60\cos\frac{\pi}{5}\right)^2 + \left(60\sin\frac{\pi}{5} - 32t\right)^2}.$$

At $t = 1$,

$$\text{Speed} = \sqrt{\left(60\cos\frac{\pi}{5}\right)^2 + \left(60\sin\frac{\pi}{5} - 32\right)^2} = 48.651 \text{ feet/sec.}$$

51. (a) To determine if the particles collide, we check whether they are ever at the same point at the same time. We first set the two x-coordinates equal to each other:

$$\begin{aligned}4t - 4 &= 3t\\ t &= 4.\end{aligned}$$

When $t = 4$, both x-coordinates are 12. Now we check whether the y-coordinates are also equal at $t = 4$:

$$\begin{aligned}y_A(4) &= 2\cdot 4 - 5 = 3\\ y_B(4) &= 4^2 - 2\cdot 4 - 1 = 7.\end{aligned}$$

Thus, the particles do not collide since they are not at the same point at the same time.

(b) For the particles to collide, we need both x- and y-coordinates to be equal. Since the x-coordinates are equal at $t = 4$, we find the k value making $y_A(4) = y_B(4)$.

Substituting $t = 4$ into $y_A(t) = 2t - k$ and $y_B(t) = t^2 - 2t - 1$, we have

$$\begin{aligned}8 - k &= 16 - 8 - 1\\ k &= 1.\end{aligned}$$

(c) To find the speed of the particles, we differentiate.

For particle A,
$x(t) = 4t - 4$, so $x'(t) = 4$, and $x'(4) = 4$
$y(t) = 2t - 1$, so $y'(t) = 2$, and $y'(4) = 2$

$$\text{Speed}_A = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{4^2 + 2^2} = \sqrt{20}.$$

For particle B,
$x(t) = 3t$, so $x'(t) = 3$, and $x'(4) = 3$
$y(t) = t^2 - 2t - 1$, so $y'(t) = 2t - 2$, and $y'(4) = 6$

$$\text{Speed}_B = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{3^2 + 6^2} = \sqrt{45}.$$

Thus, when $t = 4$, particle B is moving faster.

52. (a) Since $x = t^3 + t$ and $y = t^2$, we have

$$w = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2+1}.$$

Differentiating w with respect to t, we get

$$\frac{dw}{dt} = \frac{(3t^2+1)2 - (2t)(6t)}{(3t^2+1)^2} = \frac{-6t^2+2}{(3t^2+1)^2},$$

so

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw/dt}{dx/dt} = \frac{-6t^2+2}{(3t^2+1)^3}.$$

(b) When $t = 1$, we have $d^2y/dx^2 = -1/16 < 0$, so the curve is concave down.

53. (a) The x and y-coordinates of the point on the graph when $t = \pi/3$ are given by

$$x = 3 \cdot \frac{\pi}{3} = \pi \quad \text{and} \quad y = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

Thus when $t = \pi/3$, the particle is at the point $(\pi, -1/2)$.

To find the slope, we find dy/dx

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin(2t)}{3}.$$

When $t = \pi/3$,

$$\frac{dy}{dx} = \frac{-2\sin(2\pi/3)}{3} = -\frac{\sqrt{3}}{3}.$$

The equation of the tangent line when $t = \pi/3$ is:

$$y + \frac{1}{2} = -\frac{\sqrt{3}}{3}(x - \pi).$$

(b) To find the smallest positive value of t for which the y-coordinate is a local maximum, we set $dy/dt = 0$. We have

$$\frac{dy}{dt} = -2\sin(2t) = 0$$

$$2t = \pi \quad \text{or} \quad 2t = 2\pi$$

$$t = \frac{\pi}{2} \quad \text{or} \quad t = \pi.$$

There is a minimum of $y = \cos(2t)$ at $t = \pi/2$, and a maximum at $t = \pi$.

(c) To find d^2y/dx^2 when $t = 2$, we use the formula:

$$\frac{d^2y}{dx^2} = \frac{dw/dt}{dx/dt} \quad \text{where} \quad w = \frac{dy}{dx}.$$

Since $w = -2\sin(2t)/3$ from part (a), we have

$$\frac{d^2y}{dx^2} = \frac{-4\cos(2t)/3}{3}.$$

When $t = 2$, we have

$$\frac{d^2y}{dx^2} = \frac{-4\cos(4)/3}{3} = 0.291.$$

Since the second derivative is positive, the graph is concave up when $t = 2$.

54. (a) We differentiate for both x and y in terms of t, giving us:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t} + 6e^t}{e^t} = 2e^t + 6.$$

(b) To find d^2y/dx^2, we use the formula:

$$\frac{d^2y}{dx^2} = \frac{dw/dt}{dx/dt} \quad \text{where} \quad w = \frac{dy}{dx}$$
$$\frac{d^2y}{dx^2} = \frac{2e^t}{e^t} = 2.$$

Since the second derivative is always positive, the graph is concave up everywhere.

(c) We are given $y = e^{2t} + 6e^t + 9$. We can factor this to $y = (e^t + 3)^2$. Since $x = e^t + 3$, we can substitute x for $e^t + 3$. Thus, $y = x^2$ for $x > 3$.

(d) Since $y = x^2$, $dy/dx = 2x$ and $d^2y/dx^2 = 2$.

From part (a), we have $dy/dx = 2e^t + 6$. We can factor this to $dy/dx = 2(e^t + 3) = 2x$.

Part (b) tells us that $d^2y/dx^2 = 2$, which is what we have just determined.

Our graph is a parabola that is concave up everywhere.

55. (a) The particle touches the x-axis when $y = 0$. Since $y = \cos(2t) = 0$ for the first time when $2t = \pi/2$, we have $t = \pi/4$. To find the speed of the particle at that time, we use the formula

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\cos t)^2 + (-2\sin(2t))^2}.$$

When $t = \pi/4$,

$$\text{Speed} = \sqrt{(\cos(\pi/4))^2 + (-2\sin(\pi/2))^2} = \sqrt{(\sqrt{2}/2)^2 + (-2\cdot 1)^2} = \sqrt{9/2}.$$

(b) The particle is at rest when its speed is zero. Since $\sqrt{(\cos t)^2 + (-2\sin(2t))^2} \geq 0$, the speed is zero when

$$\cos t = 0 \quad \text{and} \quad -2\sin(2t) = 0.$$

Now $\cos t = 0$ when $t = \pi/2$ or $t = 3\pi/2$. Since $-2\sin(2t) = -4\sin t\cos t$, we see that this expression also equals zero when $t = \pi/2$ or $t = 3\pi/2$.

(c) We need to find d^2y/dx^2. First, we must determine dy/dx. We know

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\cos t} = \frac{-4\sin t\cos t}{\cos t} = -4\sin t.$$

Since $dy/dx = -4\sin t$, we can now use the formula:

$$\frac{d^2y}{dx^2} = \frac{dw/dt}{dx/dt} \quad \text{where} \quad w = \frac{dy}{dx}$$
$$\frac{d^2y}{dx^2} = \frac{-4\cos t}{\cos t} = -4.$$

Since d^2y/dx^2 is always negative, our graph is concave down everywhere.

Using the identity $y = \cos(2t) = 1 - 2\sin^2 t$, we can eliminate the parameter and write the original equation as $y = 1 - 2x^2$, which is a parabola that is concave down everywhere.

56. Let

$$w = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

We want to find

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw/dt}{dx}dt.$$

To find dw/dt, we use the quotient rule:

$$\frac{dw}{dt} = \frac{(dx/dt)(d^2y/dt^2) - (dy/dt)(d^2x/dt^2)}{(dx/dt)^2}.$$

We then divide this by dx/dt again to get the required formula, since

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw/dt}{dx/dt}.$$

57. For $0 \leq t \leq 2\pi$, we get Figure 4.137.

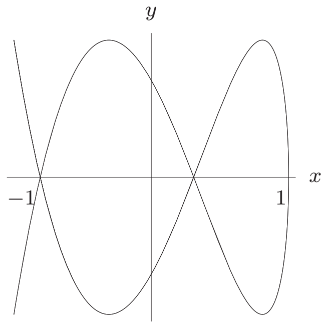


Figure 4.137

58. For $0 \leq t \leq 2\pi$, we get Figure 4.138.

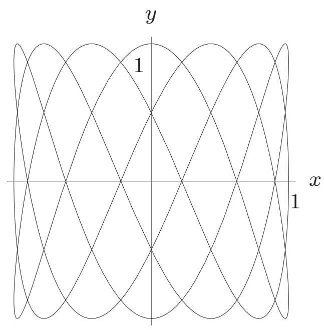


Figure 4.138

59. For $0 \leq t \leq 2\pi$, we get Figure 4.139.

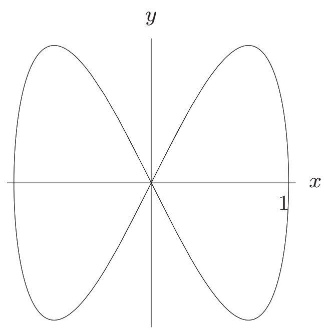


Figure 4.139

60. This curve never closes on itself. The plot for $0 \leq t \leq 8\pi$ is in Figure 4.140.

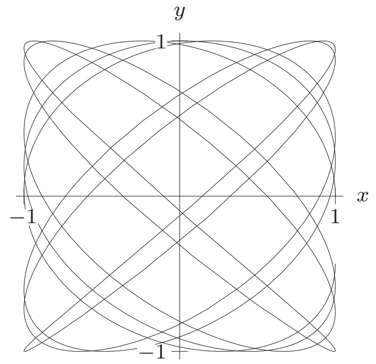


Figure 4.140

61. (a) To find the equations of the moon's motion relative to the star, you must first calculate the equation of the planet's motion relative to the star, and then the moon's motion relative to the planet, and then add the two together.

The distance from the planet to the star is R, and the time to make one revolution is one unit, so the parametric equations for the planet relative to the star are $x = R\cos t$, $y = R\sin t$.

The distance from the moon to the planet is 1, and the time to make one revolution is twelve units, therefore, the parametric equations for the moon relative to the planet are $x = \cos 12t$, $y = \sin 12t$.

Adding these together, we get:

$$x = R\cos t + \cos 12t,$$
$$y = R\sin t + \sin 12t.$$

(b) For the moon to stop completely at time t, the velocity of the moon must be equal to zero. Therefore,

$$\frac{dx}{dt} = -R\sin t - 12\sin 12t = 0,$$
$$\frac{dy}{dt} = R\cos t + 12\cos 12t = 0.$$

There are many possible values to choose for R and t that make both of these equations equal to zero. We choose $t = \pi$, and $R = 12$.

(c) The graph with $R = 12$ is shown in Figure 4.141.

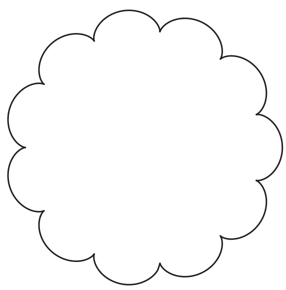


Figure 4.141

Strengthen Your Understanding

62. The parameterization does give the required line segment but traversed in the wrong direction.

63. The given parametric equations give a circle of radius 2 but centered at $(0, 0)$. We need $x = 2\cos\pi t$, $y = 1 + 2\sin\pi t$, $0 \le t \le 2$. There are many other possibilities.

64. One possible choice is $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \frac{\pi}{2}$. There are many other possibilities.

65. One possible choice is $x = t$, $y = 2t$, $0 \le t \le 1$. There are many other possibilities.

66. False. If the particle tracing out the curve comes to a complete stop, it can then head off in a completely new direction. For example, the curve given parametrically by $x = t^3$ and $y = t^2$ is the same as the graph of $y = x^{2/3}$ which has a cusp at $x = 0$.

67. False. The slope is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t\cos(t^2)}{-2t\sin(t^2)} = -\frac{\cos(t^2)}{\sin(t^2)}.$$

Solutions for Chapter 4 Review

Exercises

1. See Figure 4.142.

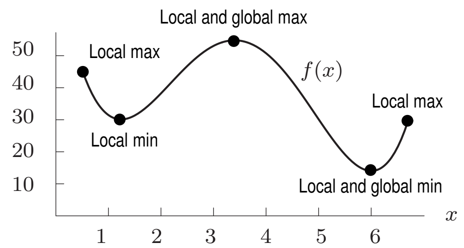


Figure 4.142

2. See Figure 4.143.

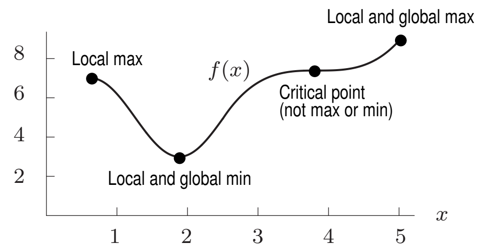


Figure 4.143

3. (a) We wish to investigate the behavior of $f(x) = x^3 - 3x^2$ on the interval $-1 \le x \le 3$. We find:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$
$$f''(x) = 6x - 6 = 6(x-1)$$

(b) The critical points of f are $x = 2$ and $x = 0$ since $f'(x) = 0$ at those points. Using the second derivative test, we find that $x = 0$ is a local maximum since $f'(0) = 0$ and $f''(0) = -6 < 0$, and that $x = 2$ is a local minimum since $f'(2) = 0$ and $f''(2) = 6 > 0$.

(c) There is an inflection point at $x = 1$ since f'' changes sign at $x = 1$.

(d) At the critical points, $f(0) = 0$ and $f(2) = -4$.
At the endpoints: $f(-1) = -4$, $f(3) = 0$.
So the global maxima are $f(0) = 0$ and $f(3) = 0$, while the global minima are $f(-1) = -4$ and $f(2) = -4$.
(e) See Figure 4.144.

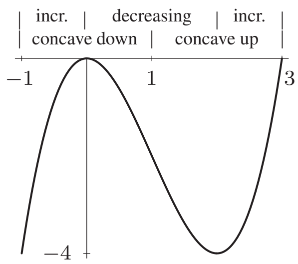


Figure 4.144

4. (a) First we find f' and f''; $f'(x) = 1 + \cos x$ and $f''(x) = -\sin x$.
(b) The critical point of f is $x = \pi$, since $f'(\pi) = 0$.
(c) Since f'' changes sign at $x = \pi$, it means that $x = \pi$ is an inflection point.
(d) Evaluating f at the critical point and endpoints, we find $f(0) = 0,\ f(\pi) = \pi,\ f(2\pi) = 2\pi$,. Therefore, the global maximum is $f(2\pi) = 2\pi$, and the global minimum is $f(0) = 0$. Note that $x = \pi$ is not a local maximum or minimum of f, and that the second derivative test is inconclusive here.
(e) See Figure 4.145.

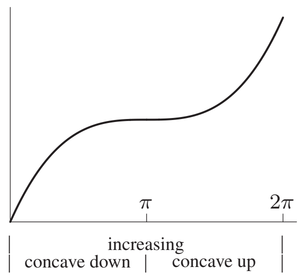


Figure 4.145

5. (a) First we find f' and f'':

$$\begin{aligned} f'(x) &= -e^{-x}\sin x + e^{-x}\cos x \\ f''(x) &= e^{-x}\sin x - e^{-x}\cos x \\ &\quad -e^{-x}\cos x - e^{-x}\sin x \\ &= -2e^{-x}\cos x \end{aligned}$$

(b) The critical points are $x = \pi/4, 5\pi/4$, since $f'(x) = 0$ here.
(c) The inflection points are $x = \pi/2, 3\pi/2$, since f'' changes sign at these points.
(d) At the endpoints, $f(0) = 0,\ f(2\pi) = 0$. So we have $f(\pi/4) = (e^{-\pi/4})(\sqrt{2}/2)$ as the global maximum; $f(5\pi/4) = -e^{-5\pi/4}(\sqrt{2}/2)$ as the global minimum.
(e) See Figure 4.146.

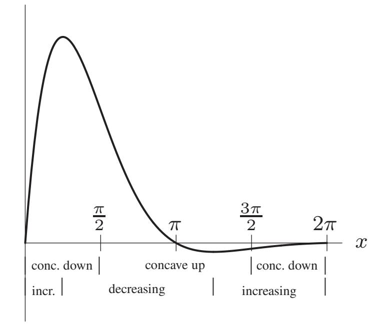


Figure 4.146

6. (a) We first find f' and f'':

$$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}} + \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3}x^{-\frac{5}{3}}(x-2)$$
$$f''(x) = \frac{10}{9}x^{-\frac{8}{3}} - \frac{2}{9}x^{-\frac{5}{3}} = -\frac{2}{9}x^{-\frac{8}{3}}(x-5)$$

(b) Critical point: $x = 2$.
(c) There are no inflection points, since f'' does not change sign on the interval $1.2 \leq x \leq 3.5$.
(d) At the endpoints, $f(1.2) \approx 1.94821$ and $f(3.5) \approx 1.95209$. So, the global minimum is $f(2) \approx 1.88988$ and the global maximum is $f(3.5) \approx 1.95209$.
(e) See Figure 4.147.

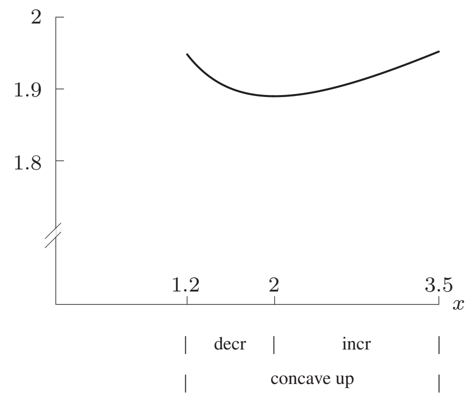


Figure 4.147

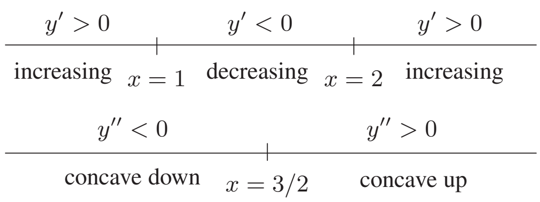
7. The polynomial $f(x)$ behaves like $2x^3$ as x goes to ∞. Therefore, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

We have $f'(x) = 6x^2 - 18x + 12 = 6(x-2)(x-1)$, which is zero when $x = 1$ or $x = 2$.

Also, $f''(x) = 12x - 18 = 6(2x-3)$, which is zero when $x = 3/2$. For $x < 3/2$, $f''(x) < 0$; for $x > 3/2$, $f''(x) > 0$. Thus $x = 3/2$ is an inflection point.

The critical points are $x = 1$ and $x = 2$, and $f(1) = 6$, $f(2) = 5$. By the second derivative test, $f''(1) = -6 < 0$, so $x = 1$ is a local maximum; $f''(2) = 6 > 0$, so $x = 2$ is a local minimum.

Now we can draw the diagrams below.



The graph of $f(x) = 2x^3 - 9x^2 + 12x + 1$ is shown in Figure 4.148. It has no global maximum or minimum.

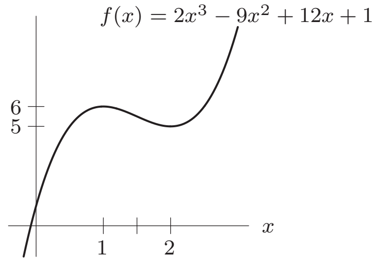


Figure 4.148

8. If we divide the denominator and numerator of $f(x)$ by x^2 we have

$$\lim_{x\to\pm\infty} \frac{4x^2}{x^2+1} = \lim_{x\to\pm\infty} \frac{4}{1+\frac{1}{x^2}} = 4$$

since

$$\lim_{x\to\pm\infty} \frac{1}{x^2} = 0.$$

Using the quotient rule we get

$$f'(x) = \frac{(x^2+1)8x - 4x^2(2x)}{(x^2+1)^2} = \frac{8x}{(x^2+1)^2},$$

which is zero when $x = 0$, positive when $x > 0$, and negative when $x < 0$. Thus $f(x)$ has a local minimum when $x = 0$, with $f(0) = 0$.

Because $f'(x) = 8x/(x^2+1)^2$, the quotient rule implies that

$$f''(x) = \frac{(x^2+1)^2 8 - 8x[2(x^2+1)2x]}{(x^2+1)^4}$$
$$= \frac{8x^2 + 8 - 32x^2}{(x^2+1)^3} = \frac{8(1-3x^2)}{(x^2+1)^3}.$$

The denominator is always positive, so $f''(x) = 0$ when $x = \pm\sqrt{1/3}$, positive when $-\sqrt{1/3} < x < \sqrt{1/3}$, and negative when $x > \sqrt{1/3}$ or $x < -\sqrt{1/3}$. This gives the diagram

$y' < 0$		$y' > 0$
decreasing	$x = 0$	increasing

$y'' < 0$		$y'' > 0$		$y'' < 0$
concave down	$x = -\sqrt{1/3}$	concave up	$x = \sqrt{1/3}$	concave down

and the graph of f looks Figure 4.149.

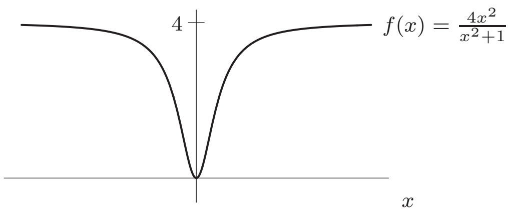


Figure 4.149

with inflection points $x = \pm\sqrt{1/3}$, a global minimum at $x = 0$, and no local or global maxima (since $f(x)$ never equals 4).

9. As $x \to -\infty$, $e^{-x} \to \infty$, so $xe^{-x} \to -\infty$. Thus $\lim_{x\to-\infty} xe^{-x} = -\infty$.

As $x \to \infty$, $\frac{x}{e^x} \to 0$, since e^x grows much more quickly than x. Thus $\lim_{x\to\infty} xe^{-x} = 0$.

Using the product rule,

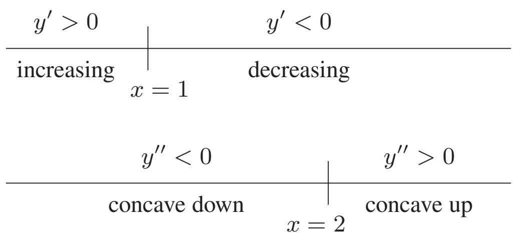
$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x},$$

which is zero when $x = 1$, negative when $x > 1$, and positive when $x < 1$. Thus $f(1) = 1/e^1 = 1/e$ is a local maximum.

Again, using the product rule,

$$\begin{aligned} f''(x) &= -e^{-x} - e^{-x} + xe^{-x} \\ &= xe^{-x} - 2e^{-x} \\ &= (x-2)e^{-x}, \end{aligned}$$

which is zero when $x = 2$, positive when $x > 2$, and negative when $x < 2$, giving an inflection point at $(2, \frac{2}{e^2})$. With the above, we have the following diagram:



The graph of f is shown in Figure 4.150.

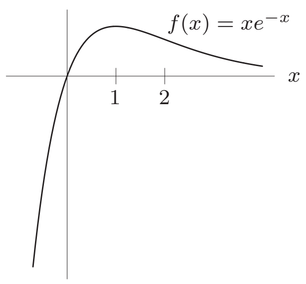


Figure 4.150

and $f(x)$ has one global maximum at $1/e$ and no local or global minima.

10. Since $f(x) = e^{-x}\sin x$ is continuous and the interval $0 \le x \le 2\pi$ is closed, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. Since there are no points where f' is undefined, we solve $f'(x) = 0$ to find all critical points:

$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(-\sin x + \cos x) = 0.$$

Since $e^{-x} \ne 0$, the critical points are when $\sin x = \cos x$; the only solutions in the given interval are $x = \pi/4$ and $x = 5\pi/4$. We then compare the values of f at the critical points and the endpoints:

$$f(0) = 0, \quad f(\pi/4) = e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right) = 0.322, \quad f(5\pi/4) = e^{-5\pi/4}\left(\frac{-\sqrt{2}}{2}\right) = -0.0139, \quad f(2\pi) = 0.$$

Thus the global maximum is 0.322 at $x = \pi/4$, and the global minimum is -0.0139 at $x = 5\pi/4$.

11. Since $f(x) = e^x + \cos x$ is continuous on the closed interval $0 \le x \le \pi$, there must be a global maximum and minimum. The possible candidates are critical points in the interval and endpoints. Since there are no points in the interval where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find the critical points:

$$f'(x) = e^x - \sin x = 0.$$

Since $e^x > 1$ for all $x > 0$ and $\sin x \le 1$ for all x, the only possibility is $x = 0$, but $e^0 - \sin 0 = 1$. Thus there are no critical points in the interval. We then compare the values of f at the endpoints:

$$f(0) = 2, \quad f(\pi) = e^\pi + \cos(\pi) = 22.141.$$

Thus, the global maximum is 22.141 at $x = \pi$, and the global minimum is 2 at $x = 0$.

12. Since $f(x) = x^2 + 2x + 1$ is continuous and the interval $0 \le x \le 3$ is closed, there must be a global maximum and minimum. The candidates are critical points in the interval and endpoints. Since there are no points where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find all the critical points:

$$f'(x) = 2x + 2 = 0,$$

so the only critical point, $x = -1$, is not in the interval $0 \le x \le 3$. At the endpoints, we have

$$f(0) = 1, \quad f(3) = 16.$$

Thus, the global maximum is 16 at $x = 3$, and the global minimum is 1 at $x = 0$.

13. Since $f(x) = e^{-x^2}$ is continuous and the interval $0 \le x \le 10$ is closed, there must be a global maximum and minimum. The candidates are critical points in the interval and endpoints. Since there are no points where $f'(x)$ is undefined, we solve $f'(x) = 0$ to find all the critical points:

$$f'(x) = -2xe^{-x^2} = 0,$$

so the only critical point, $x = 0$, is at an endpoint of the interval. At the endpoints, we have

$$f(0) = 1, \quad f(10) = e^{-100} \approx 0.$$

Thus, the global maximum is 1 at $x = 0$, and the global minimum is near 0 at $x = 10$.

14. We rewrite $h(z)$ as $h(z) = z^{-1} + 4z^2$.

Differentiating gives

$$h'(z) = -z^{-2} + 8z,$$

so the critical points satisfy

$$\begin{aligned} -z^{-2} + 8z &= 0 \\ z^{-2} &= 8z \\ 8z^3 &= 1 \\ z^3 &= \frac{1}{8} \\ z &= \frac{1}{2}. \end{aligned}$$

Since h' is negative for $0 < z < 1/2$ and h' is positive for $z > 1/2$, there is a local minimum at $z = 1/2$.

Since $h(z) \to \infty$ as $z \to 0^+$ and as $z \to \infty$, the local minimum at $z = 1/2$ is a global minimum; there is no global maximum. See Figure 4.151. Thus, the global minimum is $h(1/2) = 3$.

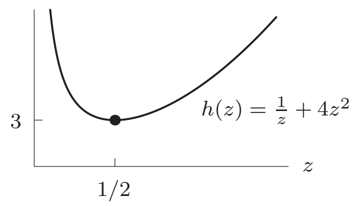


Figure 4.151

15. Since $g(t)$ is always decreasing for $t \geq 0$, we expect it to a global maximum at $t = 0$ but no global minimum. At $t = 0$, we have $g(0) = 1$, and as $t \to \infty$, we have $g(t) \to 0$.

Alternatively, rewriting as $g(t) = (t^3 + 1)^{-1}$ and differentiating using the chain rule gives

$$g'(t) = -(t^3 + 1)^{-2} \cdot 3t^2.$$

Since $3t^2 = 0$ when $t = 0$, there is a critical point at $t = 0$, and g decreases for all $t > 0$. See Figure 4.152.

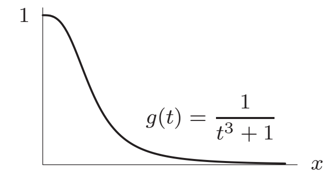


Figure 4.152

16. We begin by rewriting $f(x)$:

$$f(x) = \frac{1}{(x-1)^2 + 2} = ((x-1)^2 + 2)^{-1} = (x^2 - 2x + 3)^{-1}.$$

Differentiating using the chain rule gives

$$f'(x) = -(x^2 - 2x + 3)^{-2}(2x - 2) = \frac{2 - 2x}{(x^2 - 2x + 3)^2},$$

so the critical points satisfy

$$\begin{aligned} \frac{2 - 2x}{(x^2 - 2x + 3)^2} &= 0 \\ 2 - 2x &= 0 \\ 2x &= 2 \\ x &= 1. \end{aligned}$$

Since f' is positive for $x < 1$ and f' is negative for $x > 1$, there is a local maximum at $x = 1$.

Since $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$, the local maximum at $x = 1$ is a global maximum; there is no global minimum. See Figure 4.153. Thus, the global maximum is $f(1) = 1/2$.

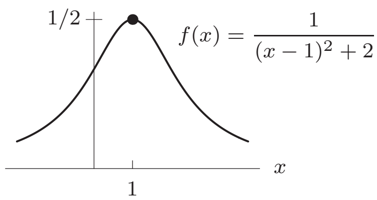


Figure 4.153

17. $\lim_{x \to \infty} f(x) = +\infty$, and $\lim_{x \to -\infty} f(x) = -\infty$.
There are no asymptotes.
$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$. Critical points are $x = -3$, $x = 1$.
$f''(x) = 6(x+1)$.

x		-3		-1		1	
f'	+	0	−	−	−	0	+
f''	−	−	−	0	+	+	+
f	↗⌢		↘⌢		↘⌣		↗⌣

Thus, $x = -1$ is an inflection point. $f(-3) = 12$ is a local maximum; $f(1) = -20$ is a local minimum. There are no global maxima or minima. See Figure 4.154.

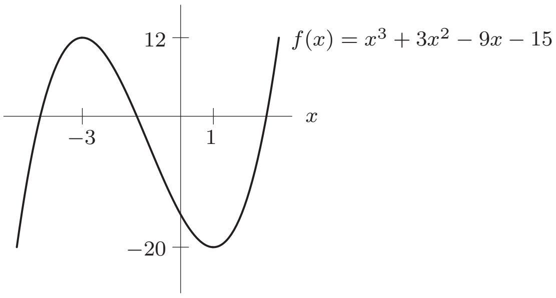
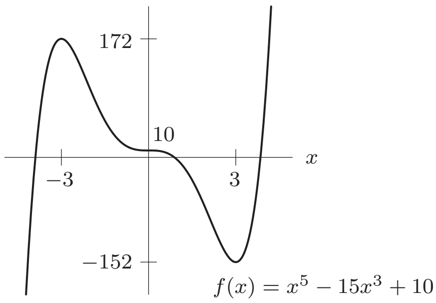


Figure 4.154

18. $\lim_{x\to+\infty} f(x) = +\infty$, and $\lim_{x\to-\infty} f(x) = -\infty$.
There are no asymptotes.
$f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9) = 5x^2(x+3)(x-3)$.
The critical points are $x = 0$, $x = \pm 3$. f' changes sign at 3 and -3 but not at 0.
$f''(x) = 20x^3 - 90x = 10x(2x^2 - 9)$. f'' changes sign at 0, $\pm 3/\sqrt{2}$.
So, inflection points are at $x = 0$, $x = \pm 3/\sqrt{2}$.

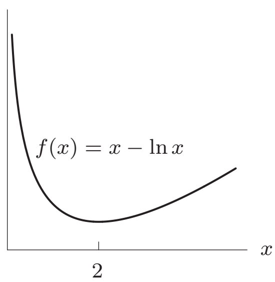
x		-3		$-3/\sqrt{2}$		0		$3/\sqrt{2}$		3	
f'	+	0	−		−	0	−		−	0	+
f''	−	−	−	0	+	0	−	0	+	+	+
f	↗⌢		↘⌢		↘⌣		↘⌢		↘⌣		↗⌣

Thus, $f(-3)$ is a local maximum; $f(3)$ is a local minimum. There are no global maxima or minima.



19. $\lim_{x\to+\infty} f(x) = +\infty$, and $\lim_{x\to 0^+} f(x) = +\infty$.
Hence, $x = 0$ is a vertical asymptote.
$f'(x) = 1 - \dfrac{2}{x} = \dfrac{x-2}{x}$, so $x = 2$ is the only critical point.
$f''(x) = \dfrac{2}{x^2}$, which can never be zero. So there are no inflection points.

x		2	
f'	−	0	+
f''	+	+	+
f	↘⌣		↗⌣



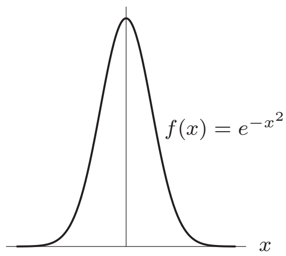
Thus, $f(2)$ is a local and global minimum.

20. Since $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote.
$f'(x) = -2xe^{-x^2}$. So, $x = 0$ is the only critical point.
$f''(x) = -2(e^{-x^2} + x(-2x)e^{-x^2}) = 2e^{-x^2}(2x^2 - 1) = 2e^{-x^2}(\sqrt{2}x - 1)(\sqrt{2}x + 1)$.
Thus, $x = \pm 1/\sqrt{2}$ are inflection points.

Table 4.1

x		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	
f'	+	+	+	0	–	–	–
f''	+	0	–	–	–	0	+
f	↗⌣		↗⌢		↘⌢		↘⌣

Thus, $f(0) = 1$ is a local and global maximum.

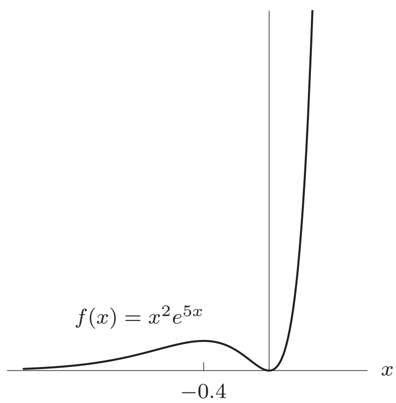


21. $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = 0$.
$y = 0$ is the horizontal asymptote.
$f'(x) = 2xe^{5x} + 5x^2e^{5x} = xe^{5x}(5x + 2)$.
Thus, $x = -\frac{2}{5}$ and $x = 0$ are the critical points.

$$\begin{aligned} f''(x) &= 2e^{5x} + 2xe^{5x}\cdot 5 + 10xe^{5x} + 25x^2e^{5x} \\ &= e^{5x}(25x^2 + 20x + 2). \end{aligned}$$

So, $x = \dfrac{-2 \pm \sqrt{2}}{5}$ are inflection points.

x		$\frac{-2-\sqrt{2}}{5}$		$-\frac{2}{5}$		$\frac{-2+\sqrt{2}}{5}$		0	
f'	+	+	+	0	–	–	–	0	+
f''	+	0	–	–	–	0	+	+	+
f	↗⌣		↗⌢		↘⌢		↘⌣		↗⌣



So, $f(-\frac{2}{5})$ is a local maximum; $f(0)$ is a local and global minimum.

22. $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 1.$
Thus, $y = 1$ is a horizontal asymptote. Since $x^2 + 1$ is never 0, there are no vertical asymptotes.

$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

So, $x = 0$ is the only critical point.

$$\begin{aligned} f''(x) &= \frac{2(x^2+1)^2 - 2x \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4} \\ &= \frac{2(x^2+1-4x^2)}{(x^2+1)^3} \\ &= \frac{2(1-3x^2)}{(x^2+1)^3}. \end{aligned}$$

So, $x = \pm\frac{1}{\sqrt{3}}$ are inflection points.

Table 4.2

x		$\frac{-1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$	
f'	−	−	−	0	+	+	+
f''	−	0	+	+	+	0	−
f	↘⌢		↘⌣		↗⌣		↗⌢

Thus, $f(0) = 0$ is a local and global minimum. A graph of $f(x)$ can be found in Figure 4.155.

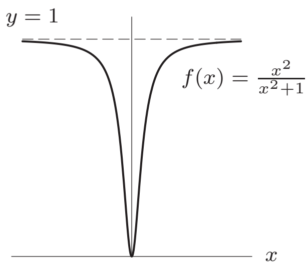


Figure 4.155

23. For $0 \le r \le a$, the speed at which air is expelled is given by

$$v(r) = k(a-r)r^2 = kar^2 - kr^3.$$

Thus, the derivative is defined for all r and given by

$$v'(r) = 2kar - 3kr^2 = 2kr\left(a - \frac{3}{2}r\right).$$

The derivative is zero if $r = \frac{2}{3}a$ or $r = 0$. These are the critical points of v. To decide if the critical points give global maxima or minima, we evaluate v at the critical point:

$$\begin{aligned} v\left(\frac{2}{3}a\right) &= k\left(a - \frac{2}{3}a\right)\left(\frac{2}{3}a\right)^2 \\ &= k\left(\frac{a}{3}\right)\frac{4a^2}{9} \\ &= \frac{4ka^3}{27}, \end{aligned}$$

and we evaluate v at the endpoints:

$$v(0) = v(a) = 0.$$

Thus, v has a global maximum at $r = \frac{2}{3}a$. The global minimum of $v = 0$ occurs at both endpoints $r = 0$ and $r = a$.

24. The slope of the curve, dy/dt, is given by

$$\frac{dy}{dt} = -50\left(1+6e^{-2t}\right)^{-2}\left(-12e^{-2t}\right) = 600e^{-2t}\left(1+6e^{-2t}\right)^{-2}.$$

If the slope has a maximum, it occurs at a critical point of dy/dt or at the endpoint $t=0$. We have

$$\begin{aligned}\frac{d^2y}{dt^2} &= 600\left(-2e^{-2t}\right)\left(1+6e^{-2t}\right)^{-2} - 1200e^{-2t}\left(1+6e^{-2t}\right)^{-3}\left(-12e^{-2t}\right)\\ &= \frac{1200e^{-2t}}{(1+6e^{-2t})^3}\left(6e^{-2t}-1\right).\end{aligned}$$

At a critical point of dy/dt, we have $d^2y/dt^2 = 0$, so

$$\begin{aligned}6e^{-2t}-1 &= 0\\ t &= \frac{1}{2}\ln 6 = 0.896.\end{aligned}$$

Since

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{600}{49} = 12.25 \quad\text{and}\quad \left.\frac{dy}{dt}\right|_{t=(1/2)\ln 6} = 25,$$

the maximum slope occurs at $t=\frac{1}{2}\ln 6$.

We see in Figure 4.156 that the slope increases as t increases from 0 and tends to 0 as $t\to\infty$, so the only critical point of slope, $t=\frac{1}{2}\ln 6$, is a local and global maximum for the slope. At $t=\frac{1}{2}\ln 6$, we have $y=25$. The point $\left(\frac{1}{2}\ln 6, 25\right)$ is the point where slope is maximum.

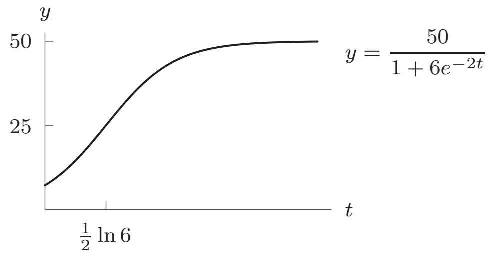


Figure 4.156

25. We have $f'(x) = (a^2-x^2)/(x^2+a^2)^2$, so that the critical points occur when $a^2-x^2=0$, that is, when $x=\pm a$. We see that as a increases the x-value of the extrema moves away the origin.

Looking at the figure we see that A's local maximum has the smallest x-value, while C's has the largest x-value. Thus, A corresponds to $a=1$, B to $a=2$, and C to $a=3$.

26. We have $f'(x) = ae^{-ax}$, so that $f'(0)=a$. We see that as a increases the slopes of the curves at the origin become more and more positive. Thus, C corresponds to $a=1$, B to $a=2$, and A to $a=3$.

27. (a) We set the derivative equal to zero and solve for x to find critical points:

$$\begin{aligned}f'(x) = 4x^3 - 4ax &= 0\\ 4x(x^2-a) &= 0.\end{aligned}$$

We see that there are three critical points:

$$\text{Critical points:}\quad x=0,\quad x=\sqrt{a},\quad x=-\sqrt{a}.$$

To find possible inflection points, we set the second derivative equal to zero and solve for x:

$$f''(x) = 12x^2 - 4a = 0.$$

There are two possible inflection points:

$$\text{Possible inflection points:}\quad x=\sqrt{\frac{a}{3}},\quad x=-\sqrt{\frac{a}{3}}.$$

To see if these are inflection points, we determine whether concavity changes by evaluating f'' at values on either side of each of the potential inflection points. We see that

$$f''(-2\sqrt{\frac{a}{3}}) = 12(4\frac{a}{3}) - 4a = 16a - 4a = 12a > 0,$$

so f is concave up to the left of $x = -\sqrt{a/3}$. Also,

$$f''(0) = -4a < 0,$$

so f is concave down between $x = -\sqrt{a/3}$ and $x = \sqrt{a/3}$. Finally, we see that

$$f''(2\sqrt{\frac{a}{3}}) = 12(4\frac{a}{3}) - 4a = 16a - 4a = 12a > 0,$$

so f is concave up to the right of $x = \sqrt{a/3}$. Since $f(x)$ changes concavity at $x = \sqrt{a/3}$ and $x = -\sqrt{a/3}$, both points are inflection points.

(b) The only positive critical point is at $x = \sqrt{a}$, so to have a critical point at $x = 2$, we substitute:

$$\begin{aligned} x &= \sqrt{a} \\ 2 &= \sqrt{a} \\ a &= 4. \end{aligned}$$

Since the critical point is at the point $(2, 5)$, we have

$$\begin{aligned} f(2) &= 5 \\ 2^4 - 2(4)2^2 + b &= 5 \\ 16 - 32 + b &= 5 \\ b &= 21. \end{aligned}$$

The function is $f(x) = x^4 - 8x^2 + 21$.

(c) We have seen that $a = 4$, so the inflection points are at $x = \sqrt{4/3}$ and $x = -\sqrt{4/3}$.

28. (a) The function $f(x)$ is defined for $x \geq 0$.

We set the derivative equal to zero and solve for x to find critical points:

$$\begin{aligned} f'(x) = 1 - \frac{1}{2}ax^{-1/2} &= 0 \\ 1 - \frac{a}{2\sqrt{x}} &= 0 \\ 2\sqrt{x} &= a \\ x &= \frac{a^2}{4}. \end{aligned}$$

Notice that f' is undefined at $x = 0$ so there are two critical points: $x = 0$ and $x = a^2/4$.

(b) We want the critical point $x = a^2/4$ to occur at $x = 5$, so we have:

$$\begin{aligned} 5 &= \frac{a^2}{4} \\ 20 &= a^2 \\ a &= \pm\sqrt{20}. \end{aligned}$$

Since a is positive, we use the positive square root. The second derivative,

$$f''(x) = \frac{1}{4}ax^{-3/2} = \frac{1}{4}\sqrt{20}x^{-3/2}$$

is positive for all $x > 0$, so the function is concave up and $x = 5$ gives a local minimum. See Figure 4.157.

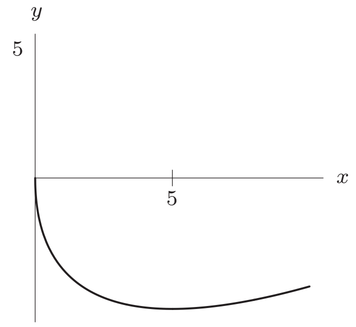


Figure 4.157

29. The domain is all real numbers except $x = b$. The function is undefined at $x = b$ and has a vertical asymptote there. To find the critical points, we set the derivative equal to zero and solve for x. Using the quotient rule, we have:

$$f'(x) = \frac{(x-b)2ax - (ax^2)1}{(x-b)^2} = 0$$
$$\frac{2ax^2 - 2abx - ax^2}{(x-b)^2} = 0$$
$$\frac{ax^2 - 2abx}{(x-b)^2} = 0.$$

The first derivative is equal to zero if

$$ax^2 - 2abx = 0$$
$$ax(x - 2b) = 0$$
$$x = 0 \quad \text{or} \quad x = 2b.$$

There are two critical points: at $x = 0$ and $x = 2b$.

30. Let the numbers be x and y. Then

$$\text{Average } = \frac{x+y}{2} = 180, \quad \text{so} \quad y = 360 - x.$$

Since both numbers are nonnegative, we restrict to $0 \le x \le 360$.

The product is

$$P = xy = x(360 - x) = 360x - x^2.$$

Differentiating to find the maximum,

$$\frac{dP}{dx} = 360 - 2x = 0$$
$$x = \frac{360}{2} = 180.$$

So there is a critical point at $x = 180$; the end points are at $x = 0, 360$.

Evaluating gives
At $x = 0$, we have $P = 0$.
At $x = 180$, we have $P = 180(360 - 180) = 32{,}400$.
At $x = 360$, we have $P = 360(360 - 360) = 0$.
Thus, the maximum value is 32,400.

31. Let the numbers be x, y, z and let $y = 2x$. Then

$$xyz = 2x^2 z = 192, \quad \text{so} \quad z = \frac{192}{2x^2} = \frac{96}{x^2}.$$

Since all the numbers are positive, we restrict to $x > 0$.

The sum is

$$S = x + y + z = x + 2x + \frac{96}{x^2} = 3x + \frac{96}{x^2}.$$

Differentiating to find the minimum,

$$\begin{aligned}\frac{dS}{dx} &= 3 - 2\frac{96}{x^3} = 0\\ 3 &= \frac{192}{x^3}\\ x^3 &= \frac{192}{3} = 64 \quad \text{so} \quad x = 4.\end{aligned}$$

There is only one critical point at $x = 4$. We find

$$\frac{d^2S}{dx^2} = 3\frac{192}{x^4}.$$

Since $d^2S/dx^2 > 0$ for all x, there is a local minimum at $x = 4$. The derivative dS/dx is negative for $0 < x < 4$ and dS/dx is positive for $x > 4$. Thus, $x = 4$ gives the global minimum for $x > 0$.

The minimum value of the sum obtained from the three numbers 4, 8, and 6 is

$$S = 3 \cdot 4 + \frac{96}{4^2} = 18.$$

32. Let x be the larger number and y be the smaller number. Then $x - y = 24$, so $y = x - 24$. Since both numbers are 100 or larger, we restrict to $x \geq 124$.

The product is

$$P = xy = x(x - 24) = x^2 - 24x.$$

Differentiating to find the minimum,

$$\begin{aligned}\frac{dP}{dx} &= 2x - 24 = 0\\ x &= \frac{24}{2} = 12.\end{aligned}$$

So there is a critical point at $x = 12$. There is an end point at $x = 124$; the domain $x \geq 124$ has no critical points.

Since the derivative, dP/dx, is positive for all x greater than 12, the minimum value of P occurs at the left end of the domain, $x = 124$.

The minimum value of the product is

$$P = 124(124 - 24) = 12{,}400.$$

33. (a) Total cost, in millions of dollars, $C(q) = 3 + 0.4q$.
(b) Revenue, in millions of dollars, $R(q) = 0.5q$.
(c) Profit, in millions of dollars, $\pi(q) = R(q) - C(q) = 0.5q - (3 + 0.4q) = 0.1q - 3$.

34. The square is $S = x^2$. Differentiating with respect to time gives

$$\frac{dS}{dt} = 2x\frac{dx}{dt}.$$

We are interested in the instant when $x = 10$ and $dS/dt = 5$, giving

$$5 = 2 \cdot 10\frac{dx}{dt},$$

so

$$\frac{dx}{dt} = \frac{5}{20} = \frac{1}{4} \text{ unit per second.}$$

35. We have

$$\frac{dM}{dt} = (3x^2 + 0.4x^3)\frac{dx}{dt}.$$

If $x = 5$, then

$$\frac{dM}{dt} = [3(5^2) + 0.4(5^3)](0.02) = 2.5 \text{ gm/hr}.$$

36. We have
$$\frac{dA}{dt} = (3 + 4\cos\theta + \cos(2\theta))\frac{d\theta}{dt}.$$
So
$$\left.\frac{dA}{dt}\right|_{\theta=\pi/2} = \left(3 + 4\cos\left(\frac{\pi}{2}\right) + \cos\pi\right) 0.3 = 0.6 \text{ cm}^2/min.$$

37. We see from the parametric equations that the particle moves along a line. It suffices to plot two points: at $t = 0$, the particle is at point $(4, 1)$, and at $t = 1$, the particle is at point $(2, 5)$. Since x decreases as t increases, the motion is right to left and the curve is shown in Figure 4.158.

Alternately, we can solve the first equation for t, giving $t = -(x-4)/2$, and substitute this into the second equation to get
$$y = 4\left(\frac{-(x-4)}{2}\right) + 1 = -2x + 9.$$
The line is $y = -2x + 9$.

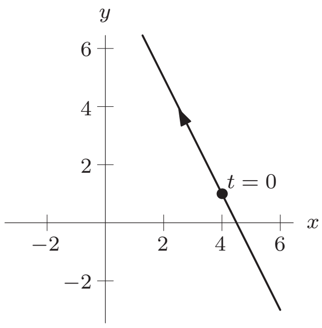


Figure 4.158

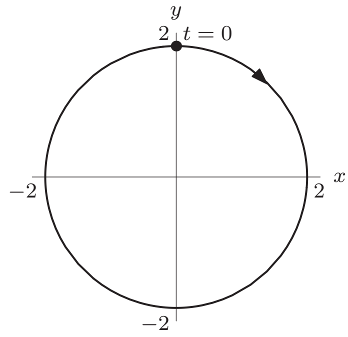


Figure 4.159

38. The graph is a circle centered at the origin with radius 2. The equation is
$$x^2 + y^2 = (2\sin t)^2 + (2\cos t)^2 = 4.$$
The particle is at the point $(0, 1)$ when $t = 0$, and motion is clockwise. See Figure 4.159.

Problems

39. (a) The function f is a local maximum where $f'(x) = 0$ and $f' > 0$ to the left, $f' < 0$ to the right. This occurs at the point x_3.

(b) The function f is a local minimum where $f'(x) = 0$ and $f' < 0$ to the left, $f' > 0$ to the right. This occurs at the points x_1 and x_5.

(c) The graph of f is climbing fastest where f' is a maximum, which is at the point x_2.

(d) The graph of f is falling most steeply where f' is the most negative, which is at the point 0.

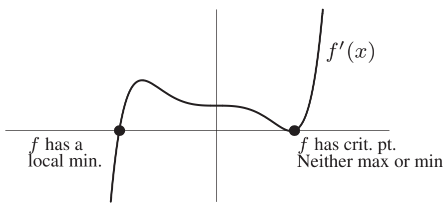
40. The function f has critical points at $x = 1, x = 3, x = 5$.

By the first derivative test, since f' is positive to the left of $x = 1$ and negative to the right, $x = 1$ is a local maximum.

Since f' is negative to the left of $x = 3$ and positive to the right, $x = 3$ is a local minimum.

Since f' does not change sign at $x = 5$, this point is neither a local maximum nor a local minimum.

41. The critical points of f occur where f' is zero. These two points are indicated in the figure below.



Note that the point labeled as a local minimum of f is not a critical point of f'.

42. From the first condition, we get that $x = 2$ is a local minimum for f. From the second condition, it follows that $x = 4$ is an inflection point. A possible graph is shown in Figure 4.160.

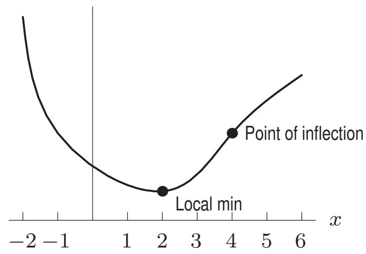


Figure 4.160

43. Since the x^3 term has coefficient of 1, the cubic polynomial is of the form $y = x^3 + ax^2 + bx + c$. We now find a, b, and c. Differentiating gives

$$\frac{dy}{dx} = 3x^2 + 2ax + b.$$

The derivative is 0 at local maxima and minima, so

$$\left.\frac{dy}{dx}\right|_{x=1} = 3(1)^2 + 2a(1) + b = 3 + 2a + b = 0$$

$$\left.\frac{dy}{dx}\right|_{x=3} = 3(3)^2 + 2a(3) + b = 27 + 6a + b = 0$$

Subtracting the first equation from the second and solving for a and b gives

$$24 + 4a = 0 \qquad \text{so} \qquad a = -6$$
$$b = -3 - 2(-6) = 9.$$

Since the y-intercept is 5, the cubic is

$$y = x^3 - 6x^2 + 9x + 5.$$

Since the coefficient of x^3 is positive, $x = 1$ is the maximum and $x = 3$ is the minimum. See Figure 4.161. To confirm that $x = 1$ gives a maximum and $x = 3$ gives a minimum, we calculate

$$\frac{d^2y}{dx^2} = 6x + 2a = 6x - 12.$$

At $x = 1$, $\dfrac{d^2y}{dx^2} = -6 < 0$, so we have a maximum.

At $x = 3$, $\dfrac{d^2y}{dx^2} = 6 > 0$, so we have a minimum.

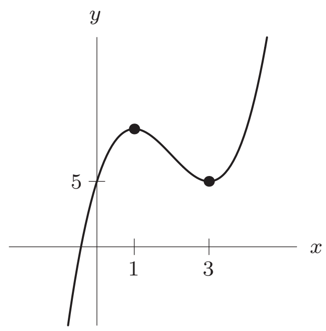


Figure 4.161: Graph of $y = x^3 - 6x^2 + 9x + 5$

44. Since the graph of the quartic polynomial is symmetric about the y-axis, the quartic must have only even powers and be of the form

$$y = ax^4 + bx^2 + c.$$

The y-intercept is 3, so $c = 3$. Differentiating gives

$$\frac{dy}{dx} = 4ax^3 + 2bx.$$

Since there is a maximum at $(1, 4)$, we have $dy/dx = 0$ if $x = 1$, so

$$4a(1)^3 + 2b(1) = 4a + 2b = 0 \qquad \text{so} \qquad b = -2a.$$

The fact that $dy/dx = 0$ if $x = -1$ gives us the same relationship

$$-4a - 2b = 0 \qquad \text{so} \qquad b = -2a.$$

We also know that $y = 4$ if $x = \pm 1$, so

$$a(1)^4 + b(1)^2 + 3 = a + b + 3 = 4 \qquad \text{so} \qquad a + b = 1.$$

Solving for a and b gives

$$a - 2a = 1 \qquad \text{so} \qquad a = -1 \text{ and } b = 2.$$

Finding d^2y/dx^2 so that we can check that $x = \pm 1$ are maxima, not minima, we see

$$\frac{d^2y}{dx^2} = 12ax^2 + 2b = -12x^2 + 4.$$

Thus $\dfrac{d^2y}{dx^2} = -8 < 0$ for $x = \pm 1$, so $x = \pm 1$ are maxima. See Figure 4.162.

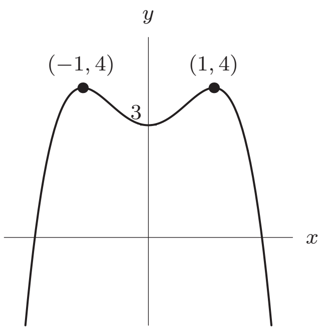


Figure 4.162: Graph of $y = -x^4 + 2x^2 + 3$

45. Differentiating $y = ax^b \ln x$, we have

$$\frac{dy}{dx} = abx^{b-1} \ln x + ax^b \cdot \frac{1}{x} = ax^{b-1}(b \ln x + 1).$$

Since the maximum occurs at $x = e^2$, we know that

$$a(e^2)^{b-1}(b \ln(e^2) + 1) = 0.$$

Since $a \neq 0$ and $(e^2)^{b-1} \neq 0$ for all b, we have

$$b \ln(e^2) + 1 = 0.$$

Since $\ln(e^2) = 2$, the equation becomes

$$\begin{aligned} 2b + 1 &= 0 \\ b &= -\frac{1}{2}. \end{aligned}$$

Thus $y = ax^{-1/2}\ln x$. When $x = e^2$, we know $y = 6e^{-1}$, so

$$y = a(e^2)^{-1/2}\ln e^2 = ae^{-1}(2) = 6e^{-1}$$
$$a = 3.$$

Thus $y = 3x^{-1/2}\ln x$. To check that $x = e^2$ gives a local maximum, we differentiate twice

$$\frac{dy}{dx} = -\frac{3}{2}x^{-3/2}\ln x + 3x^{-1/2}\cdot\frac{1}{x} = -\frac{3}{2}x^{-3/2}\ln x + 3x^{-3/2},$$
$$\frac{d^2y}{dx^2} = \frac{9}{4}x^{-5/2}\ln x - \frac{3}{2}x^{-3/2}\cdot\frac{1}{x} - \frac{3}{2}\cdot 3x^{-5/2}$$
$$= \frac{9}{4}x^{-5/2}\ln x - 6x^{-5/2} = \frac{3}{4}x^{-5/2}(3\ln x - 8).$$

At $x = e^2$, since $\ln(e^2) = 2$, we have a maximum because

$$\frac{d^2y}{dx^2} = \frac{3}{4}(e^2)^{-5/2}\left(3\ln(e^2) - 8\right) = \frac{3}{4}e^{-5}(3\cdot 2 - 8) < 0.$$

See Figure 4.163.

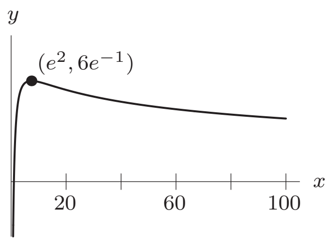


Figure 4.163: Graph of $y = 3x^{-1/2}\ln x$

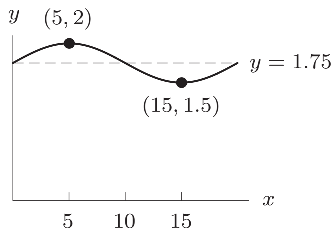


Figure 4.164: Graph of $y = 0.25\sin(\pi x/10) + 1.75$

46. Since the maximum is $y = 2$ and the minimum is $y = 1.5$, the amplitude is $A = (2 - 1.5)/2 = 0.25$. Between the maximum and the minimum, the x-value changes by 10. There is half a period between a maximum and the next minimum, so the period is 20. Thus

$$\frac{2\pi}{B} = 20 \qquad \text{so} \qquad B = \frac{\pi}{10}.$$

The mid-line is $y = C = (2 + 1.5)/2 = 1.75$. Figure 4.164 shows a graph of the function

$$y = 0.25\sin\left(\frac{\pi x}{10}\right) + 1.75.$$

47. First notice that since this function approaches 0 as x approaches either plus or minus infinity, any local extrema that we find are also global extrema.

Differentiating $y = axe^{-bx^2}$ gives

$$\frac{dy}{dx} = ae^{-bx^2} - 2abx^2e^{-bx^2} = ae^{-bx^2}(1 - 2bx^2).$$

Since we have a critical points at $x = 1$ and $x = -1$, we know $1 - 2b = 0$, so $b = 1/2$.

The global maximum is 2 at $x = 1$, so we have $2 = ae^{-1/2}$ which gives $a = 2e^{1/2}$. Notice that this value of a also gives the global minimum at $x = -1$.

Thus,

$$y = 2xe^{(\frac{1-x^2}{2})}.$$

48. We want to maximize the volume $V = x^2h$ of the box, shown in Figure 4.165. The box has 6 faces: the top and bottom, each of which has area x^2 and the four sides, each of which has area xh. Thus $8 = 2x^2 + 4xh$, so

$$h = \frac{8 - 2x^2}{4x}.$$

Substituting this expression in for h in the formula for V gives

$$V = x^2 \cdot \frac{8 - 2x^2}{4x} = \frac{1}{4}(8x - 2x^3).$$

Differentiating gives

$$\frac{dV}{dx} = \frac{1}{4}(8 - 6x^2).$$

To maximize V we look for critical points, so we solve $0 = (8 - 6x^2)/4$, getting $x = \pm\sqrt{4/3}$. We discard the negative solution, since x is a positive length. Then we can find

$$h = \frac{8 - 2x^2}{4x} = \frac{8 - 2\left(\frac{4}{3}\right)}{4\sqrt{\frac{4}{3}}} = \frac{\frac{16}{3}}{4\sqrt{\frac{4}{3}}} = \frac{\frac{4}{3}}{\sqrt{\frac{4}{3}}} = \sqrt{\frac{4}{3}}.$$

Thus $x = h = \sqrt{4/3}$ cm (the box is a cube).

We can check that this critical point is a maximum of V by checking the sign of

$$\frac{d^2V}{dx^2} = -3x^2$$

which is negative when $x \neq 0$. So V is concave down at the critical point and therefore $x = \sqrt{4/3}$ gives a maximum value of V.

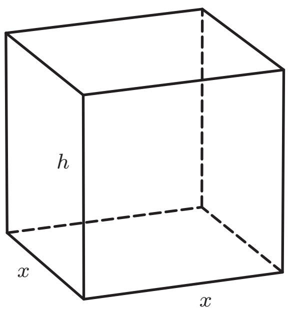


Figure 4.165

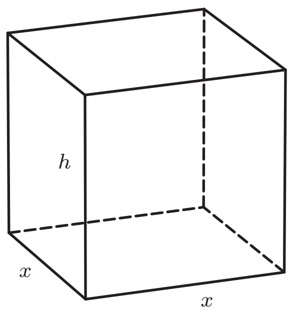


Figure 4.166

49. We want to maximize the volume $V = x^2h$ of the box, shown in Figure 4.166. The box has 5 faces: the bottom, which has area x^2 and the four sides, each of which has area xh. Thus $8 = x^2 + 4xh$, so

$$h = \frac{8 - x^2}{4x}.$$

Substituting this expression in for h in the formula for V gives

$$V = x^2 \cdot \frac{8 - x^2}{4x} = \frac{1}{4}(8x - x^3).$$

Differentiating gives

$$\frac{dV}{dx} = \frac{1}{4}(8 - 3x^2).$$

To maximize V we look for critical points, so we solve $0 = (8 - 3x^2)/4$, getting $x = \pm\sqrt{8/3}$. We discard the negative solution, since x is a positive length. Then we can find

$$h = \frac{8 - x^2}{4x} = \frac{8 - \frac{8}{3}}{4\sqrt{\frac{8}{3}}} = \frac{\frac{16}{3}}{4\sqrt{\frac{8}{3}}} = \frac{\frac{4}{3}}{2\sqrt{\frac{2}{3}}} = \sqrt{\frac{2}{3}}.$$

Thus $x = \sqrt{8/3}$ cm and $h = \sqrt{2/3}$ cm.

We can check that this critical point is a maximum of V by checking the sign of

$$\frac{d^2V}{dx^2} = -\frac{3}{2}x,$$

which is negative when $x > 0$. So V is concave down at the critical point and therefore $x = \sqrt{8/3}$ gives a maximum value of V.

50. We want to maximize the volume $V = \pi r^2 h$ of the cylinder, shown in Figure 4.167. The cylinder has 3 pieces: the top and bottom disks, each of which has area πr^2 and the tube, which has area $2\pi rh$. Thus $8 = 2\pi r^2 + 2\pi rh$. Solving for h gives

$$h = \frac{8 - 2\pi r^2}{2\pi r}.$$

Substituting this expression in for h in the formula for V gives

$$V = \pi r^2 \cdot \frac{8 - 2\pi r^2}{2\pi r} = 4r - \pi r^3.$$

Differentiating gives

$$\frac{dV}{dr} = 4 - 3\pi r^2.$$

To maximize V we look for critical points, so we solve $0 = 4 - 3\pi r^2$, thus $r = \pm 2/\sqrt{3\pi}$. We discard the negative solution, since r is a positive length. Substituting this value in for r in the formula for h gives

$$h = \frac{8 - 2\pi\left(\frac{4}{3\pi}\right)}{2\pi\frac{2}{\sqrt{3\pi}}} = \frac{\frac{16}{3}}{2\pi\frac{2}{\sqrt{3\pi}}} = \frac{\frac{8}{3\pi}}{\frac{2}{\sqrt{3\pi}}} = \frac{4}{\sqrt{3\pi}}.$$

We can check that this critical point is a maximum of V by checking the sign of

$$\frac{d^2V}{dr^2} = -6\pi r$$

which is negative when $r > 0$. So V is concave down at the critical point and therefore $r = 2/\sqrt{3\pi}$ is a maximum.

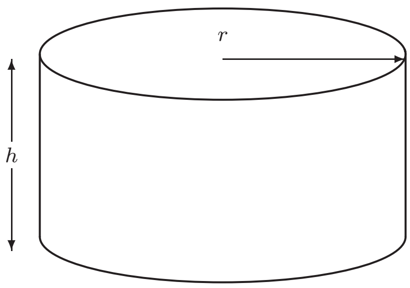


Figure 4.167

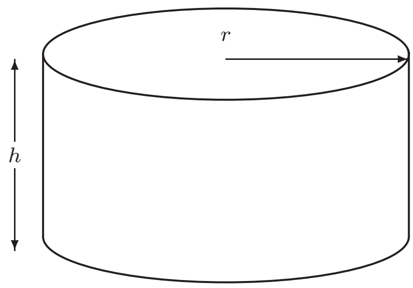


Figure 4.168

51. We want to maximize the volume $V = \pi r^2 h$ of the cylinder, shown in Figure 4.168. The cylinder has 2 pieces: the end disk, of area πr^2 and the tube, which has area $2\pi rh$. Thus $8 = \pi r^2 + 2\pi rh$. Solving for h gives

$$h = \frac{8 - \pi r^2}{2\pi r}.$$

Substituting this expression in for h in the formula for V gives

$$V = \pi r^2 \cdot \frac{8 - \pi r^2}{2\pi r} = \frac{1}{2}(8r - \pi r^3).$$

Differentiating gives

$$\frac{dV}{dr} = \frac{1}{2}(8 - 3\pi r^2).$$

To maximize V we look for critical points, so we solve $0 = (8-3\pi r^2)/2$, thus $r = \pm\sqrt{8/(3\pi)}$. We discard the negative solution, since r is a positive length. Substituting this value in for r in the formula for h gives

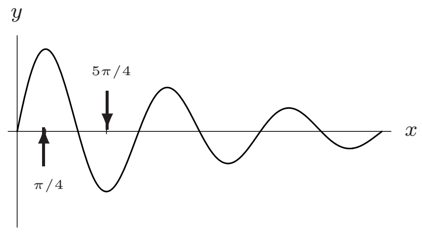
$$h = \frac{8 - \pi\left(\frac{8}{3\pi}\right)}{2\pi\sqrt{\frac{8}{3\pi}}} = \frac{\frac{16}{3}}{2\pi\sqrt{\frac{8}{3\pi}}} = \frac{\frac{8}{3\pi}}{\sqrt{\frac{8}{3\pi}}} = \sqrt{\frac{8}{3\pi}}.$$

We can check that this critical point is a maximum of V by checking the sign of

$$\frac{d^2V}{dr^2} = -3\pi r$$

which is negative when $r > 0$. So V is concave down at the critical point and therefore $r = \sqrt{8/(3\pi)}$ is a maximum.

52.



Letting $f(x) = e^{-x}\sin x$, we have

$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x.$$

Solving $f'(x) = 0$, we get $\sin x = \cos x$. This means $x = \arctan(1) = \pi/4$, and $\pi/4$ plus multiples of π, are the critical points of $f(x)$. By evaluating $f(x)$ at the points $k\pi + \pi/4$, where k is an integer, we can find:

$$e^{-5\pi/4}\sin(5\pi/4) \le e^{-x}\sin x \le e^{-\pi/4}\sin(\pi/4),$$

since $f(0) = 0$ at the endpoint. So

$$-0.014 \le e^{-x}\sin x \le 0.322.$$

53. Let $f(x) = x\sin x$. Then $f'(x) = x\cos x + \sin x$.
$f'(x) = 0$ when $x = 0, x \approx 2$, and $x \approx 5$. The latter two estimates we can get from the graph of $f'(x)$.

Zooming in (or using some other approximation method), we can find the zeros of $f'(x)$ with more precision. They are (approximately) 0, 2.029, and 4.913. We check the endpoints and critical points for the global maximum and minimum.

$$f(0) = 0, \qquad f(2\pi) = 0,$$
$$f(2.029) \approx 1.8197, \qquad f(4.914) \approx -4.814.$$

Thus for $0 \le x \le 2\pi$, $-4.81 \le f(x) \le 1.82$.

54. To find the best possible bounds for $f(x) = x^3 - 6x^2 + 9x + 5$ on $0 \le x \le 5$, we find the global maximum and minimum for the function on the interval. First, we find the critical points. Differentiating yields

$$f'(x) = 3x^2 - 12x + 9$$

Letting $f'(x) = 0$ and factoring yields

$$\begin{aligned} 3x^2 - 12x + 9 &= 0 \\ 3(x^2 - 4x + 3) &= 0 \\ 3(x-3)(x-1) &= 0 \end{aligned}$$

So $x = 1$ and $x = 3$ are critical points for the function on $0 \le x \le 5$. Evaluating the function at the critical points and endpoints gives us

$$\begin{aligned} f(0) &= (0)^3 - 6(0)^2 + 9(0) + 5 = 5 \\ f(1) &= (1)^3 - 6(1)^2 + 9(1) + 5 = 9 \\ f(3) &= (3)^3 - 6(3)^2 + 9(3) + 5 = 5 \\ f(5) &= (5)^3 - 6(5)^2 + 9(5) + 5 = 25 \end{aligned}$$

So the global minimum on this interval is $f(0) = f(3) = 5$ and the global maximum is $f(5) = 25$. From this we conclude

$$5 \le x^3 - 6x^2 + 9x + 5 \le 25$$

are the best possible bounds for the function on the interval $0 \le x \le 5$.

55. We first solve for P

$$P = -6jm^2 + 4jk - 5km,$$

and find the derivative

$$\frac{dP}{dm} = -12jm - 5k.$$

Since the derivative is defined for all m, we find the critical points by solving $dP/dm = 0$:

$$\frac{dP}{dm} = -12jm - 5k = 0,$$
$$m = -\frac{5k}{12j}.$$

There is one critical point at $m = -5k/(12j)$. Since P is a quadratic function of m with a negative leading coefficient $-6j$, the critical point gives the global maximum of P. There is no global minimum because $P \to -\infty$ as $m \to \pm\infty$.

56. Differentiating gives

$$\frac{dy}{dx} = a(e^{-bx} - bxe^{-bx}) = ae^{-bx}(1 - bx).$$

Thus, $dy/dx = 0$ when $x = 1/b$. Then

$$y = a\frac{1}{b}e^{-b \cdot 1/b} = \frac{a}{b}e^{-1}.$$

Differentiating again gives

$$\frac{d^2y}{dx^2} = -abe^{-bx}(1 - bx) - abe^{-bx}$$
$$= -abe^{-bx}(2 - bx)$$

When $x = 1/b$,

$$\frac{d^2y}{dx^2} = -abe^{-b \cdot 1/b}\left(2 - b \cdot \frac{1}{b}\right) = -abe^{-1}.$$

Therefore the point $(\frac{1}{b}, \frac{a}{b}e^{-1})$ is a maximum if a and b are positive. We can make $(2, 10)$ a maximum by setting

$$\frac{1}{b} = 2 \quad \text{so} \quad b = \frac{1}{2}$$

and

$$\frac{a}{b}e^{-1} = \frac{a}{1/2}e^{-1} = 2ae^{-1} = 10 \quad \text{so} \quad a = 5e.$$

Thus $a = 5e$, $b = 1/2$.

57. (a) We set the derivative equal to zero and solve for t to find critical points. Using the product rule, we have:

$$f'(t) = (at^2)(e^{-bt}(-b)) + (2at)e^{-bt} = 0$$
$$ate^{-bt}(-bt + 2) = 0$$
$$t = 0 \quad \text{or} \quad t = \frac{2}{b}.$$

There are two critical points: $t = 0$ and $t = 2/b$.

(b) Since we want a critical point at $t = 5$, we substitute and solve for b:

$$5 = 2/b$$
$$b = \frac{2}{5} = 0.4.$$

To find the value of a, we use the fact that $f(5) = 12$, so we have:

$$a(5^2)e^{-0.4(5)} = 12$$
$$a \cdot 25e^{-2} = 12$$
$$a = \frac{12e^2}{25} = 3.547.$$

(c) To show that $f(t)$ has a local minimum at $t = 0$ and a local maximum at $t = 5$, we can use the first derivative test or the second derivative test. Using the first derivative test, we evaluate f' at values on either side of $t = 0$ and $t = 5$. Since $f'(t) = 3.547te^{-0.4t}(-0.4t + 2)$, we have

$$f'(-1) = -3.547e^{0.4}(2.4) = -12.700 < 0$$

and

$$f'(1) = 3.547e^{-0.4}(1.6) = 3.804 > 0,$$

and

$$f'(6) = 3.547(6)e^{-2.4}(-0.4) = -0.772 < 0.$$

The function f is decreasing to the left of $t = 0$, increasing between $t = 0$ and $t = 5$, and decreasing to the right of $t = 5$. Therefore, $f(t)$ has a local minimum at $t = 0$ and a local maximum at $t = 5$. See Figure 4.169.

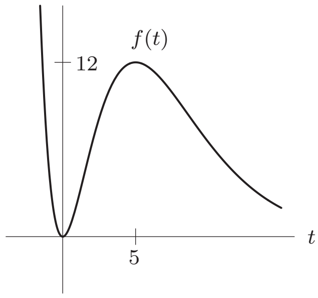


Figure 4.169

58. We have $f(x) = x^2 + 2ax = x(x + 2a) = 0$ when $x = 0$ or $x = -2a$.

$$f'(x) = 2x + 2a = 2(x + a) \begin{cases} = 0 & \text{when } x = -a \\ > 0 & \text{when } x > -a \\ < 0 & \text{when } x < -a. \end{cases}$$

See Figure 4.170. Furthermore, $f''(x) = 2$, so that $f(-a) = -a^2$ is a global minimum, and the graph is always concave up.

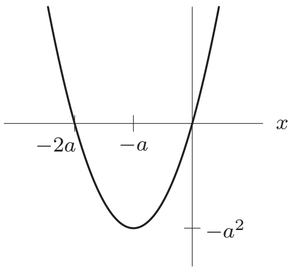
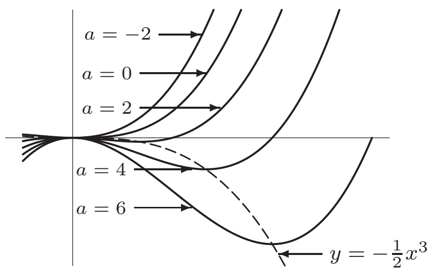


Figure 4.170

Increasing $|a|$ stretches the graph horizontally. Also, the critical value (the value of f at the critical point) drops further beneath the x-axis. Letting $a < 0$ would reflect the graph shown through the y-axis.

59.



To solve for the critical points, we set $\frac{dy}{dx} = 0$. Since $\frac{d}{dx}\left(x^3 - ax^2\right) = 3x^2 - 2ax$, we want $3x^2 - 2ax = 0$, so $x = 0$ or $x = \frac{2}{3}a$. At $x = 0$, we have $y = 0$. This first critical point is independent of a and lies on the curve $y = -\frac{1}{2}x^3$. At $x = \frac{2}{3}a$, we calculate $y = -\frac{4}{27}a^3 = -\frac{1}{2}\left(\frac{2}{3}a\right)^3$. Thus the second critical point also lies on the curve $y = -\frac{1}{2}x^3$.

60. We want the maximum value of $r(t) = ate^{-bt}$ to be 0.3 ml/sec and to occur at $t = 0.5$ sec. Differentiating gives

$$r'(t) = ae^{-bt} - abte^{-bt},$$

so $r'(t) = 0$ when

$$ae^{-bt}(1 - bt) = 0 \qquad \text{or} \qquad t = \frac{1}{b}.$$

Since the maximum occurs at $t = 0.5$, we have

$$\frac{1}{b} = 0.5 \qquad \text{so} \qquad b = 2.$$

Thus, $r(t) = ate^{-2t}$. The maximum value of r is given by

$$r(0.5) = a(0.5)e^{-2(0.5)} = 0.5ae^{-1}.$$

Since the maximum value of r is 0.3, we have

$$0.5ae^{-1} = 0.3 \qquad \text{so} \qquad a = \frac{0.3e}{0.5} = 1.63.$$

Thus, $r(t) = 1.63te^{-2t}$ ml/sec.

61. We know that $dp/dt = -3$ mm/sec when $p = 35$ mm and we want to know q and dq/dt at that time. We also know that $f = 15$ mm. Since

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f},$$

substituting $p = 35$ and $f = 15$ gives

$$\frac{1}{q} = \frac{1}{15} - \frac{1}{35} = \frac{4}{105}, \quad \text{so} \quad q = \frac{105}{4} = 26.25 \text{ mm}.$$

Differentiating with respect to time t gives,

$$-\frac{1}{p^2}\frac{dp}{dt} - \frac{1}{q^2}\frac{dq}{dt} = 0.$$

Substituting gives

$$-\frac{1}{35^2}(-3) - \frac{1}{(105/4)^2}\frac{dq}{dt} = 0,$$

so

$$\frac{dq}{dt} = \frac{(105/4)^2 3}{35^2} = \frac{27}{16} = 1.688 \text{ mm/sec}.$$

Thus, the image is moving away from the lens at 1.688 mm per second.

62.

$$\begin{aligned} r(\lambda) &= a(\lambda)^{-5}(e^{b/\lambda} - 1)^{-1} \\ r'(\lambda) &= a(-5\lambda^{-6})(e^{b/\lambda} - 1)^{-1} + a(\lambda^{-5})\left(\frac{b}{\lambda^2}e^{b/\lambda}\right)(e^{b/\lambda} - 1)^{-2} \end{aligned}$$

$(0.96, 3.13)$ is a maximum, so $r'(0.96) = 0$ implies that the following holds, with $\lambda = 0.96$:

$$\begin{aligned} 5\lambda^{-6}(e^{b/\lambda} - 1)^{-1} &= \lambda^{-5}\left(\frac{b}{\lambda^2}e^{b/\lambda}\right)(e^{b/\lambda} - 1)^{-2} \\ 5\lambda(e^{b/\lambda} - 1) &= be^{b/\lambda} \\ 5\lambda e^{b/\lambda} - 5\lambda &= be^{b/\lambda} \\ 5\lambda e^{b/\lambda} - be^{b/\lambda} &= 5\lambda \\ \left(\frac{5\lambda - b}{5\lambda}\right)e^{b/\lambda} &= 1 \\ \frac{4.8 - b}{4.8}e^{b/0.96} - 1 &= 0. \end{aligned}$$

Using Newton's method, or some other approximation method, we search for a root. The root should be near 4.8. Using our initial guess, we get $b \approx 4.7665$. At $\lambda = 0.96$, $r = 3.13$, so

$$\begin{aligned} 3.13 &= \frac{a}{0.96^5(e^{b/0.96}-1)} \qquad \text{or} \\ a &= 3.13(0.96)^5(e^{b/0.96}-1) \\ &\approx 363.23. \end{aligned}$$

As a check, we try $r(4) \approx 0.155$, which looks about right on the given graph.

63. Since $I(t)$ is a periodic function with period $2\pi/w$, it is enough to consider $I(t)$ for $0 \le wt \le 2\pi$. Differentiating, we find

$$\frac{dI}{dt} = -w\sin(wt) + \sqrt{3}w\cos(wt).$$

At a critical point

$$\begin{aligned} -w\sin(wt) + \sqrt{3}w\cos(wt) &= 0 \\ \sin(wt) &= \sqrt{3}\cos(wt) \\ \tan(wt) &= \sqrt{3}. \end{aligned}$$

So $wt = \pi/3$ or $4\pi/3$, or these values plus multiples of 2π. Substituting into I, we see

$$\text{At } wt = \frac{\pi}{3}: \quad I = \cos\left(\frac{\pi}{3}\right) + \sqrt{3}\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \sqrt{3}\cdot\left(\frac{\sqrt{3}}{2}\right) = 2.$$

$$\text{At } wt = \frac{4\pi}{3}: \quad I = \cos\left(\frac{4\pi}{3}\right) + \sqrt{3}\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \sqrt{3}\cdot\left(\frac{\sqrt{3}}{2}\right) = -2.$$

Thus, the maximum value is 2 amps and the minimum is -2 amps.

64.

$$\frac{dE}{d\theta} = \frac{(\mu+\theta)(1-2\mu\theta) - (\theta - \mu\theta^2)}{(\mu+\theta)^2} = \frac{\mu(1-2\mu\theta-\theta^2)}{(\mu+\theta)^2}.$$

Now $dE/d\theta = 0$ when $\theta = -\mu \pm \sqrt{1+\mu^2}$. Since $\theta > 0$, the only possible critical point is when $\theta = -\mu + \sqrt{\mu^2+1}$. Differentiating again gives $E'' < 0$ at this point and so it is a local maximum. Since $E(\theta)$ is continuous for $\theta > 0$ and $E(\theta)$ has only one critical point, the local maximum is the global maximum.

65. The top half of the circle has equation $y = \sqrt{1-x^2}$. The rectangle in Figure 4.171 has area, A, given by

$$A = 2xy = 2x\sqrt{1-x^2}, \qquad \text{for } 0 \le x \le 1.$$

At a critical point,

$$\begin{aligned} \frac{dA}{dt} = 2\sqrt{1-x^2} + 2x\left(\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\right) &= 0 \\ 2\sqrt{1-x^2} - \frac{2x^2}{\sqrt{1-x^2}} &= 0 \\ \frac{2\left(\sqrt{1-x^2}\right)^2 - 2x^2}{\sqrt{1-x^2}} &= 0 \\ \frac{2(1-x^2-x^2)}{\sqrt{1-x^2}} &= 0 \\ 2(1-2x^2) &= 0 \\ x &= \pm\frac{1}{\sqrt{2}}. \end{aligned}$$

Since $A = 0$ at the endpoints $x = 0$ and $x = 1$, and since A is positive at the only critical point, $x = 1/\sqrt{2}$, in the interval $0 \le x \le 1$, the critical point is a local and global maximum. The vertices on the circle have $y = \sqrt{1-(1/2)^2} = 1/\sqrt{2}$. Thus the coordinates of the rectangle with maximum area are

$$\left(\frac{1}{\sqrt{2}}, 0\right); \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right); \quad \left(-\frac{1}{\sqrt{2}}, 0\right); \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

and the maximum area is

$$A = 2\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1.$$

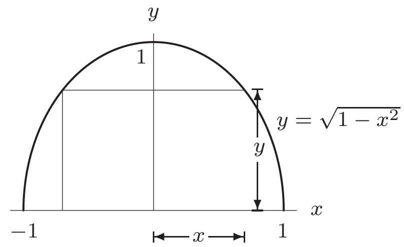


Figure 4.171

66. The triangle in Figure 4.172 has area, A, given by

$$A = \frac{1}{2}x \cdot y = \frac{1}{2}x^3e^{-3x}.$$

If the area has a maximum, it occurs where

$$\begin{aligned}\frac{dA}{dx} = \frac{3}{2}x^2e^{-3x} - \frac{3}{2}x^3e^{-3x} &= 0\\ \frac{3}{2}x^2\left(1-x\right)e^{-3x} &= 0\\ x &= 0, 1.\end{aligned}$$

The value $x = 0$ gives the minimum area, $A = 0$, for $x \geq 0$. Since

$$\frac{dA}{dx} = \frac{3}{2}x^2(1-x)e^{-3x},$$

we see that

$$\frac{dA}{dx} > 0 \text{ for } 0 < x < 1 \qquad \text{and} \qquad \frac{dA}{dx} < 0 \text{ for } x > 1.$$

Thus, $x = 1$ gives the local and global maximum of

$$A = \frac{1}{2}1^3e^{-3\cdot 1} = \frac{1}{2e^3}.$$

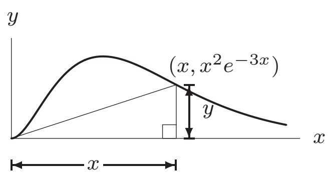


Figure 4.172

67. The distance from a given point on the parabola (x, x^2) to $(1, 0)$ is given by

$$D = \sqrt{(x-1)^2 + (x^2-0)^2}.$$

Minimizing this is equivalent to minimizing $d = (x-1)^2 + x^4$. (We can ignore the square root if we are only interested in minimizing because the square root is smallest when the thing it is the square root of is smallest.) To minimize d, we find its critical points by solving $d' = 0$. Since $d = (x-1)^2 + x^4 = x^2 - 2x + 1 + x^4$,

$$d' = 2x - 2 + 4x^3 = 2(2x^3 + x - 1).$$

By graphing $d' = 2(2x^3 + 2x - 1)$ on a calculator, we see that it has only 1 root, $x \approx 0.59$. This must give a minimum because $d \to \infty$ as $x \to -\infty$ and as $x \to +\infty$, and d has only one critical point. This is confirmed by the second derivative test: $d'' = 12x^2 + 2 = 2(6x^2 + 1)$, which is always positive. Thus the point $(0.59, 0.59^2) \approx (0.59, 0.35)$ is approximately the closest point of $y = x^2$ to $(1, 0)$.

68. Any point on the curve can be written (x, x^2). The distance between such a point and $(3, 0)$ is given by

$$s(x) = \sqrt{(3-x)^2 + (0-x^2)^2} = \sqrt{(3-x)^2 + x^4}.$$

Plotting this function in Figure 4.173, we see that there is a minimum near $x = 1$.

To find the value of x that minimizes the distance we can instead minimize the function $Q = s^2$ (the derivative is simpler). Then we have

$$Q(x) = (3-x)^2 + x^4.$$

Differentiating $Q(x)$ gives

$$\frac{dQ}{dx} = -6 + 2x + 4x^3.$$

Plotting the function $4x^3 + 2x - 6$ shows that there is one real solution at $x = 1$, which can be verified by substitution; the required coordinates are therefore $(1, 1)$. Because $Q''(x) = 2 + 12x^2$ is always positive, $x = 1$ is indeed the minimum. See Figure 4.174.

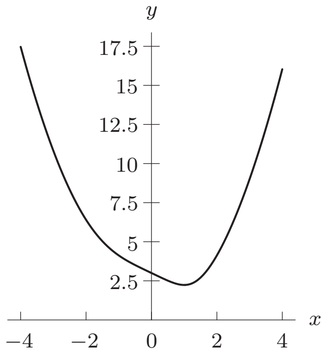


Figure 4.173

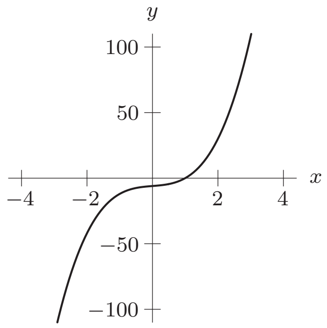


Figure 4.174

69. We see that the width of the tunnel is $2r$. The area of the rectangle is then $(2r)h$. The area of the semicircle is $(\pi r^2)/2$. The cross-sectional area, A, is then

$$A = 2rh + \frac{1}{2}\pi r^2$$

and the perimeter, P, is

$$P = 2h + 2r + \pi r.$$

From $A = 2rh + (\pi r^2)/2$ we get

$$h = \frac{A}{2r} - \frac{\pi r}{4}.$$

Thus,

$$P = 2\left(\frac{A}{2r} - \frac{\pi r}{4}\right) + 2r + \pi r = \frac{A}{r} + 2r + \frac{\pi r}{2}.$$

We now have the perimeter in terms of r and the constant A. Differentiating, we obtain

$$\frac{dP}{dr} = -\frac{A}{r^2} + 2 + \frac{\pi}{2}.$$

To find the critical points we set $P' = 0$:

$$\begin{aligned} -\frac{A}{r^2} + \frac{\pi}{2} + 2 &= 0 \\ \frac{r^2}{A} &= \frac{2}{4+\pi} \\ r &= \sqrt{\frac{2A}{4+\pi}}. \end{aligned}$$

Substituting this back into our expression for h, we have

$$h = \frac{A}{2} \cdot \frac{\sqrt{4+\pi}}{\sqrt{2A}} - \frac{\pi}{4} \cdot \frac{\sqrt{2A}}{\sqrt{4+\pi}}.$$

Since $P \to \infty$ as $r \to 0^+$ and as $r \to \infty$, this critical point must be a global minimum. Notice that the h-value simplifies to

$$h = \sqrt{\frac{2A}{4+\pi}} = r.$$

70. Consider the rectangle of sides x and y shown in Figure 4.175.

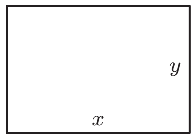


Figure 4.175

The total area is $xy = 3000$, so $y = 3000/x$. Suppose the left and right edges and the lower edge have the shrubs and the top edge has the fencing. The total cost is

$$\begin{aligned} C &= 45(x+2y) + 20(x) \\ &= 65x + 90y. \end{aligned}$$

Since $y = 3000/x$, this reduces to

$$C(x) = 65x + 90(3000/x) = 65x + 270{,}000/x.$$

Therefore, $C'(x) = 65 - 270{,}000/x^2$. We set this to 0 to find the critical points:

$$\begin{aligned} 65 - \frac{270{,}000}{x^2} &= 0 \\ \frac{270{,}000}{x^2} &= 65 \\ x^2 &= 4153.85 \\ x &= 64.450 \text{ ft} \end{aligned}$$

so that

$$y = 3000/x = 46.548 \text{ ft}.$$

Since $C(x) \to \infty$ as $x \to 0^+$ and $x \to \infty$, we see that $x = 64.450$ is a minimum. The minimum total cost is then

$$C(64.450) \approx \$8378.54.$$

71. Figure 4.176 shows the the pool has dimensions x by y and the deck extends 5 feet at either side and 10 feet at the ends of the pool.

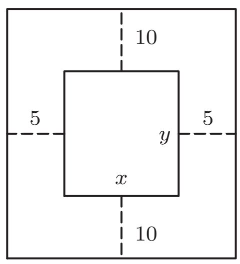


Figure 4.176

The dimensions of the plot of land containing the pool are then $(x+5+5)$ by $(y+10+10)$. The area of the land is then

$$A = (x+10)(y+20),$$

which is to be minimized. We also are told that the area of the pool is $xy = 1800$, so

$$y = 1800/x$$

and

$$\begin{aligned} A &= (x+10)\left(\frac{1800}{x}+20\right) \\ &= 1800 + 20x + \frac{18000}{x} + 200. \end{aligned}$$

We find dA/dx and set it to zero to get

$$\begin{aligned} \frac{dA}{dx} = 20 - \frac{18000}{x^2} &= 0 \\ 20x^2 &= 18000 \\ x^2 &= 900 \\ x &= 30 \text{ feet.} \end{aligned}$$

Since $A \to \infty$ as $x \to 0^+$ and as $x \to \infty$, this critical point must be a global minimum. Also, $y = 1800/30 = 60$ feet. The plot of land is therefore $(30 + 10) = 40$ by $(60 + 20) = 80$ feet.

72. (a) Suppose n passengers sign up for the cruise. If $n \le 100$, then the cruise's revenue is $R = 2000n$, so the maximum revenue is

$$R = 2000 \cdot 100 = 200{,}000.$$

If $n > 100$, then the price is

$$p = 2000 - 10(n - 100)$$

and hence the revenue is

$$R = n(2000 - 10(n - 100)) = 3000n - 10n^2.$$

To find the maximum revenue, we set $dR/dn = 0$, giving $20n = 3000$ or $n = 150$. Then the revenue is

$$R = (2000 - 10 \cdot 50) \cdot 150 = 225{,}000.$$

Since this is more than the maximum revenue when $n \le 100$, the boat maximizes its revenue with 150 passengers, each paying \$1500.

(b) We approach this problem in a similar way to part (a), except now we are dealing with the profit function π. If $n \le 100$, we have

$$\pi = 2000n - 80{,}000 - 400n,$$

so π is maximized with 100 passengers yielding a profit of

$$\pi = 1600 \cdot 100 - 80{,}000 = \$80{,}000.$$

If $n > 100$, we have

$$\pi = n(2000 - 10(n - 100)) - (80{,}000 + 400n).$$

We again set $d\pi/dn = 0$, giving $2600 = 20n$, so $n = 130$. The profit is then \$89,000. So the boat maximizes profit by boarding 130 passengers, each paying \$1700. This gives the boat \$89,000 in profit.

73. (a) $\pi(q)$ is maximized when $R(q) > C(q)$ and they are as far apart as possible. See Figure 4.177.

(b) $\pi'(q_0) = R'(q_0) - C'(q_0) = 0$ implies that $C'(q_0) = R'(q_0) = p$.

Graphically, the slopes of the two curves at q_0 are equal. This is plausible because if $C'(q_0)$ were greater than p or less than p, the maximum of $\pi(q)$ would be to the left or right of q_0, respectively. In economic terms, if the cost were rising more quickly than revenues, the profit would be maximized at a lower quantity (and if the cost were rising more slowly, at a higher quantity).

(c) See Figure 4.178.

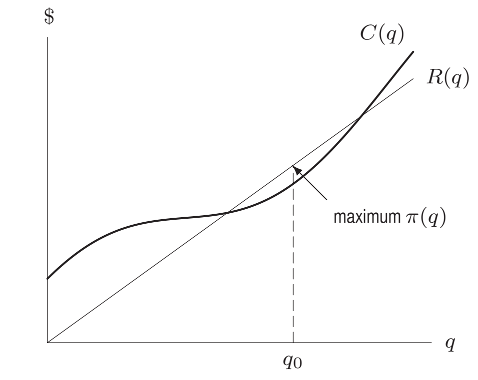


Figure 4.177

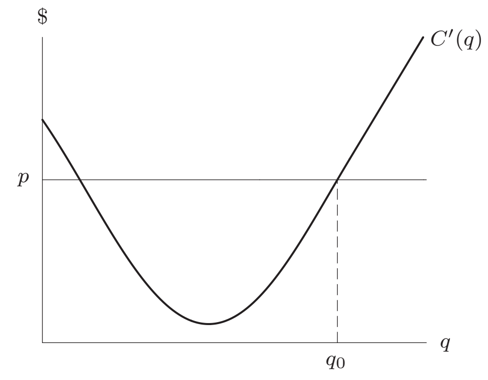
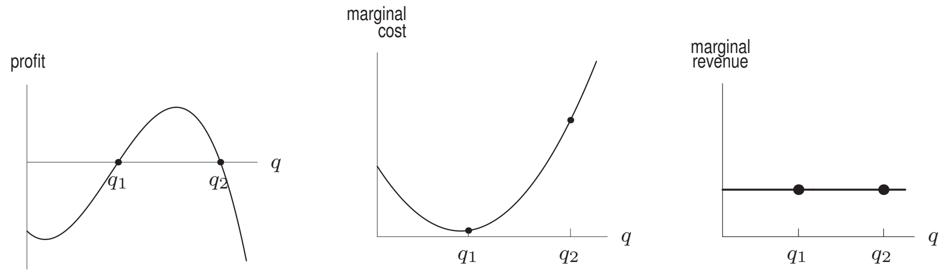


Figure 4.178

74.



75.

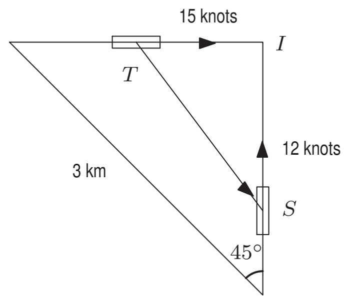


Figure 4.179: Position of the tanker and ship

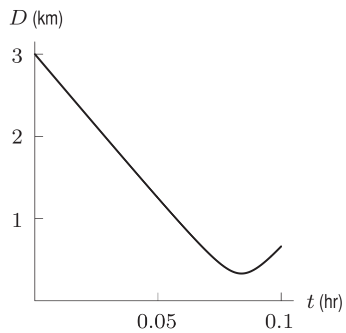


Figure 4.180: Distance between the ship at S and the tanker at T

Suppose t is the time, in hours, since the ships were 3 km apart. Then $\overline{TI} = \frac{3\sqrt{2}}{2} - (15)(1.85)t$ and $\overline{SI} = \frac{3\sqrt{2}}{2} - (12)(1.85)t$. So the distance, $D(t)$, in km, between the ships at time t is

$$D(t) = \sqrt{\left(\frac{3\sqrt{2}}{2} - 27.75t\right)^2 + \left(\frac{3\sqrt{2}}{2} - 22.2t\right)^2}.$$

Differentiating gives

$$\frac{dD}{dt} = \frac{-55.5\left(\frac{3}{\sqrt{2}} - 27.75\,t\right) - 44.4\left(\frac{3}{\sqrt{2}} - 22.2\,t\right)}{2\sqrt{\left(\frac{3}{\sqrt{2}} - 27.75\,t\right)^2 + \left(\frac{3}{\sqrt{2}} - 22.2\,t\right)^2}}.$$

Solving $dD/dt = 0$ gives a critical point at $t = 0.0839$ hours when the ships will be approximately 331 meters apart. So the ships do not need to change course. Alternatively, tracing along the curve in Figure 4.180 gives the same result. Note that this is after the eastbound ship crosses the path of the northbound ship.

76. Since the volume is fixed at 200 ml (i.e. 200 cm^3), we can solve the volume expression for h in terms of r to get (with h and r in centimeters)

$$h = \frac{200 \cdot 3}{7\pi r^2}.$$

Using this expression in the surface area formula we arrive at

$$S = 3\pi r\sqrt{r^2 + \left(\frac{600}{7\pi r^2}\right)^2}$$

By plotting $S(r)$ we see that there is a minimum value near $r = 2.7$ cm.

77. (a) We have $g'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$, which is zero if $t = e$, negative if $t > e$, and positive if $t < e$, since $\ln t$ is increasing. Thus $g(e) = \frac{1}{e}$ is a global maximum for g. Since $t = e$ was the only point at which $g'(t) = 0$, there is no minimum.

(b) Now $\ln t/t$ is increasing for $0 < t < e$, $\ln 1/1 = 0$, and $\ln 5/5 \approx 0.322 < \ln(e)/e$. Thus, for $1 < t < e$, $\ln t/t$ increases from 0 to above $\ln 5/5$, so there must be a t between 1 and e such that $\ln t/t = \ln 5/5$. For $t > e$, there is only one solution to $\ln t/t = \ln 5/5$, namely $t = 5$, since $\ln t/t$ is decreasing for $t > e$. For $0 < t < 1$, $\ln t/t$ is negative and so cannot equal $\ln 5/5$. Thus $\ln x/x = \ln t/t$ has exactly two solutions.

(c) The graph of $\ln t/t$ intersects the horizontal line $y = \ln 5/5$, at $x = 5$ and $x \approx 1.75$.

78. (a) x-intercept: $(a, 0)$, y-intercept: $(0, \frac{1}{a^2+1})$

(b) Area $= \frac{1}{2}(a)(\frac{1}{a^2+1}) = \frac{a}{2(a^2+1)}$

(c)

$$\begin{aligned} A &= \frac{a}{2(a^2+1)} \\ A' &= \frac{2(a^2+1) - a(4a)}{4(a^2+1)^2} \\ &= \frac{2(1-a^2)}{4(a^2+1)^2} \\ &= \frac{(1-a^2)}{2(a^2+1)^2}. \end{aligned}$$

If $A' = 0$, then $a = \pm 1$. We only consider positive values of a, and we note that A' changes sign from positive to negative at $a = 1$. Hence $a = 1$ is a local maximum of A which is a global maximum because $A' < 0$ for all $a > 1$ and $A' > 0$ for $0 < a < 1$.

(d) $A = \frac{1}{2}(1)(\frac{1}{2}) = \frac{1}{4}$

(e) Set $\frac{a}{2(a^2+a)} = \frac{1}{3}$ and solve for a:

$$\begin{aligned} 5a &= 2a^2 + 2 \\ 2a^2 - 5a + 2 &= 0 \\ (2a-1)(a-2) &= 0. \end{aligned}$$

79. (a) Since the volume of water in the container is proportional to its depth, and the volume is increasing at a constant rate,

$$d(t) = \text{ Depth at time } t = Kt,$$

where K is some positive constant. So the graph is linear, as shown in Figure 4.181. Since initially no water is in the container, we have $d(0) = 0$, and the graph starts from the origin.

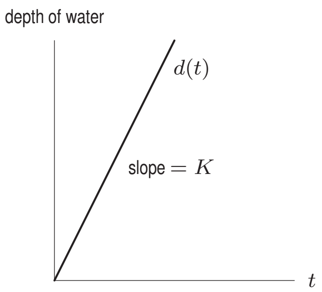


Figure 4.181

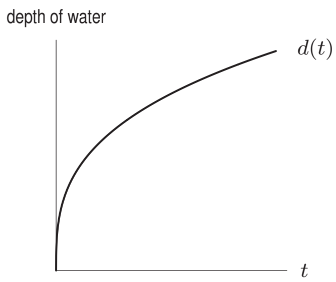


Figure 4.182

(b) As time increases, the additional volume needed to raise the water level by a fixed amount increases. Thus, although the depth, $d(t)$, of water in the cone at time t, continues to increase, it does so more and more slowly. This means $d'(t)$ is positive but decreasing, i.e., $d(t)$ is concave down. See Figure 4.182.

80. (a) The concavity changes at t_1 and t_3, as shown in Figure 4.183.

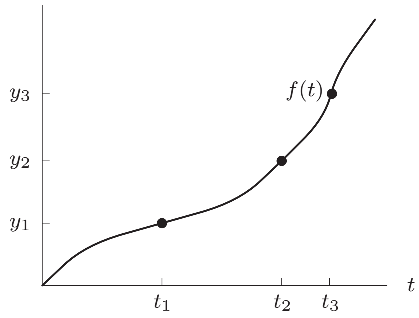


Figure 4.183

(b) $f(t)$ grows most quickly where the vase is skinniest (at y_3) and most slowly where the vase is widest (at y_1). The diameter of the widest part of the vase looks to be about 4 times as large as the diameter at the skinniest part. Since the area of a cross section is given by πr^2, where r is the radius, the ratio between areas of cross sections at these two places is about 4^2, so the growth rates are in a ratio of about 1 to 16 (the wide part being 16 times slower).

81. The volume, V, of a cone of radius r and height h is

$$V = \frac{1}{3}\pi r^2 h.$$

However, Figure 4.184 shows that $h/r = 12/5$, thus $r = 5h/12$, so

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h = \frac{25}{432}\pi h^3.$$

Differentiating with respect to time, t, gives

$$\frac{dV}{dt} = \frac{25}{144}\pi h^2 \frac{dh}{dt}.$$

When the depth of chemical in the tank is 1 meter, the level is falling at 0.1 meter/min so $h = 1$ and $dh/dt = -0.1$. Thus

$$\frac{dV}{dt} = -\frac{25}{144}\cdot\pi\cdot 1^2\cdot 0.1 = -0.0545 \text{ m}^3/\text{min}.$$

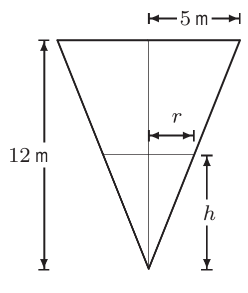


Figure 4.184

82. Evaluating the limits in the numerator and the denominator we get $0/e^0 = 0/1 = 0$, so this is not an indeterminate form. l'Hopital's rule does not apply.

83. We have $\lim_{x\to 1}\sin\pi x = \sin\pi = 0$, and $\lim_{x\to 1} x - 1 = 0$, so this is a $0/0$ form and l'Hopital's rule applies directly.

84. This is a $0/0$ form. Applying l'Hopital's rule twice, we get

$$\lim_{t\to 0}\frac{e^t-1-t}{t^2}=\lim_{t\to 0}\frac{e^t-0-1}{2t}=\lim_{t\to 0}\frac{e^t}{2}=\frac{1}{2}.$$

85. Let $f(t)=3\sin t-\sin 3t$ and $g(t)=3\tan t-\tan 3t$, then $f(0)=0$ and $g(0)=0$. Similarly,

$f'(t)=3\cos t-3\cos 3t$ $\qquad f'(0)=0$

$f''(t)=-3\sin t+9\sin 3t$ $\qquad f''(0)=0$

$f'''(t)=-3\cos t+27\cos 3t$ $\qquad f'''(0)=24$

$g'(t)=3\sec^2 t-3\sec^2 3t$ $\qquad g'(0)=0$

$g''(t)=6\sec^2 t\tan t-18\sec^2 3t\tan 3t$ $\qquad g''(0)=0$

$g'''(t)=-54\sec^4 3t-108\sec^2 37\tan^2 3t+6\sec^4 t+12\sec^2 t\tan^2 t$ $\qquad g'''(0)=-48$

Since the first and second derivatives of f and g are both 0 at $t=0$, we have to go as far as the third derivative to use l'Hopital's rule. Applying l'Hopital's rule gives

$$\lim_{t\to 0^+}\frac{3\sin t-\sin 3t}{3\tan t-\tan 3t}=\lim_{t\to 0^+}\frac{f'(t)}{g'(t)}=\lim_{t\to 0^+}\frac{f''(t)}{g''(t)}=\lim_{t\to 0^+}\frac{f'''(t)}{g'''(t)}=\frac{24}{-48}=-\frac{1}{2}.$$

86. If $f(x)=1-\cosh(5x)$ and $g(x)=x^2$, then $f(0)=g(0)=0$, so we use l'Hopital's Rule:

$$\lim_{x\to 0}\frac{1-\cosh 5x}{x^2}=\lim_{x\to 0}\frac{-5\sinh 5x}{2x}=\lim_{x\to 0}\frac{-25\cosh 5x}{2}=-\frac{25}{2}.$$

87. If $f(x)=x-\sinh x$ and $g(x)=x^3$, then $f(0)=g(0)=0$. However, $f'(0)=g'(0)=f''(0)=g''(0)=0$ also, so we use l'Hopital's Rule three times. Since $f'''(x)=-\cosh x$ and $g'''(x)=6$:

$$\lim_{x\to 0}\frac{x-\sinh x}{x^3}=\lim_{x\to 0}\frac{1-\cosh x}{3x^2}=\lim_{x\to 0}\frac{-\sinh x}{6x}=\lim_{x\to 0}\frac{-\cosh x}{6}=-\frac{1}{6}.$$

88. (a) The population is increasing if $dP/dt>0$, that is, if

$$kP(L-P)>0.$$

Since $P\geq 0$ and $k,L>0$, we must have $P>0$ and $L-P>0$ for this to be true. Thus, the population is increasing if $0<P<L$.

The population is decreasing if $dP/dt<0$, that is, if $P>L$.

The population remains constant if $dP/dt=0$, so $P=0$ or $P=L$.

(b) Differentiating with respect to t using the chain rule gives

$$\begin{aligned}\frac{d^2P}{dt^2}=\frac{d}{dt}\left(kP(L-P)\right)=\frac{d}{dP}(kLP-kP^2)\cdot\frac{dP}{dt}&=(kL-2kP)(kP(L-P))\\&=k^2P(L-2P)(L-P).\end{aligned}$$

89. We are given that the volume is increasing at a constant rate $\frac{dV}{dt}=400$. The radius r is related to the volume by the formula $V=\frac{4}{3}\pi r^3$. By implicit differentiation, we have

$$\frac{dV}{dt}=\frac{4}{3}\pi 3r^2\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$$

Plugging in $\frac{dV}{dt}=400$ and $r=10$, we have

$$400=400\pi\frac{dr}{dt}$$

so $\frac{dr}{dt}=\frac{1}{\pi}\approx 0.32\mu$m/day.

90. Let r be the radius of the raindrop. Then its volume $V=\frac{4}{3}\pi r^3$ cm^3 and its surface area is $S=4\pi r^2$ cm^2. It is given that

$$\frac{dV}{dt}=2S=8\pi r^2.$$

Furthermore,

$$\frac{dV}{dr} = 4\pi r^2,$$

so from the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad \text{and thus} \quad \frac{dr}{dt} = \frac{dV/dt}{dV/dr} = 2.$$

Since dr/dt is a constant, $dr/dt = 2$, the radius is increasing at a constant rate of 2 cm/sec.

91. (a) Since $d\theta/dt$ represents the rate of change of θ with time, $d\theta/dt$ represents the angular velocity of the disk.
(b) Suppose P is the point on the rim shown in Figure 4.185.

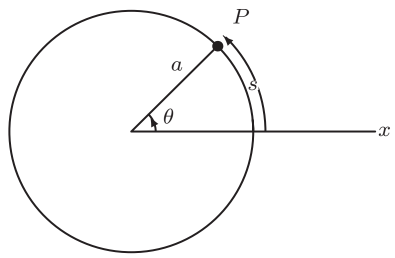


Figure 4.185

Any other point on the rim is moving at the same speed, though in a different direction. We know that since θ is in radians,

$$s = a\theta.$$

Since a is a constant, we know

$$\frac{ds}{dt} = a\frac{d\theta}{dt}.$$

But $ds/dt = v$, the speed of the point on the rim, so

$$v = a\frac{d\theta}{dt}.$$

92. We have

$$\frac{dD}{dt} = K[e^{-r} - (r+1)e^{-r}]\frac{dr}{dt} = -Kre^{-r}\frac{dr}{dt}.$$

93. We have

$$\frac{dM}{dt} = K\left(1 - \frac{1}{1+r}\right)\frac{dr}{dt}.$$

94. Let V be the volume of the ice, so that $V = 3\pi(r^2 - 1^2)$. Now,

$$\frac{dV}{dt} = 6\pi r\frac{dr}{dt}.$$

Thus for $r = 1.5$, we have

$$\frac{dV}{dt} = 6\pi(1.5)(0.03) = 0.848\,\text{cm}^3/\text{hr}.$$

95. The volume, V, of a cone of height h and radius r is

$$V = \frac{1}{3}\pi r^2 h.$$

Since the angle of the cone is $\pi/6$, so $r = h\tan(\pi/6) = h/\sqrt{3}$

$$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{1}{9}\pi h^3.$$

Differentiating gives

$$\frac{dV}{dh} = \frac{1}{3}\pi h^2.$$

To find dh/dt, use the chain rule to obtain

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}.$$

So,

$$\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{0.1\text{meters/hour}}{\pi h^2/3} = \frac{0.3}{\pi h^2} \text{ meters/hour}.$$

Since $r = h\tan(\pi/6) = h/\sqrt{3}$, we have

$$\frac{dr}{dt} = \frac{dh}{dt}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\frac{0.3}{\pi h^2} \text{ meters/hour}.$$

96. (a) The surface of the water is circular with radius r cm. Applying Pythagoras' Theorem to the triangle in Figure 4.186 shows that

$$(10-h)^2 + r^2 = 10^2$$

so

$$r = \sqrt{10^2 - (10-h)^2} = \sqrt{20h - h^2} \text{ cm}.$$

(b) We know $dh/dt = -0.1$ cm/hr and we want to know dr/dt when $h = 5$ cm. Differentiating

$$r = \sqrt{20h - h^2}$$

gives

$$\frac{dr}{dt} = \frac{1}{2}(20h - h^2)^{-1/2}\left(20\frac{dh}{dt} - 2h\frac{dh}{dt}\right) = \frac{10-h}{\sqrt{20h-h^2}} \cdot \frac{dh}{dt}.$$

Substituting $dh/dt = -0.1$ and $h = 5$ gives

$$\left.\frac{dr}{dt}\right|_{h=5} = \frac{5}{\sqrt{20\cdot 5 - 5^2}} \cdot (-0.1) = -\frac{1}{2\sqrt{75}} = -0.0577 \text{ cm/hr}.$$

Thus, the radius is decreasing at 0.0577 cm per hour.

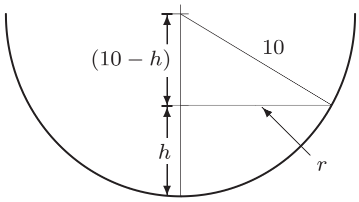


Figure 4.186

97. We have

$$\frac{dF}{dt} = \frac{K(a^2 - 2y^2)}{(a^2+y^2)^{5/2}}\frac{dy}{dt}.$$

(a) When $y = 0$, we have

$$\frac{dF}{dt} = Ka^{-3}\frac{dy}{dt}.$$

So, dF/dt is positive and F is increasing.

(b) When $y = a/\sqrt{2}$, we have

$$\frac{dF}{dt} = \frac{K(a^2 - a^2)}{(a^2 + (a^2/2))^{5/2}}\frac{dy}{dt} = 0.$$

So, $dF/dt = 0$ and F is not changing.

(c) When $y = 2a$, we have

$$\frac{dF}{dt} = \frac{K(a^2 - 8a^2)}{(a^2 + 4a^2)^{5/2}}\frac{dy}{dt} = -\frac{7Ka^2}{(5a^2)^{5/2}}\frac{dy}{dt}.$$

So, dF/dt is negative and F is decreasing.

98. The rate at which the voltage, V, is changing is obtained by differentiating $V = IR$ to get

$$\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}.$$

Since the voltage remains constant, $dV/dt = 0$. Thus

$$\frac{dR}{dt} = -\frac{R}{I}\frac{dI}{dt},$$

and the rate at which the resistance is changing is

$$\frac{dR}{dt} = -\frac{1000}{0.1}(0.001) = -10 \text{ ohms/min}.$$

We conclude that the resistance is falling by 10 ohms/min.

99. From Figure 4.187, Pythagoras' Theorem shows that the ground distance, d, between the train and the point, B, vertically below the plane is given by

$$d^2 = x^2 + y^2.$$

Figure 4.188 shows that

$$z^2 = d^2 + 4^2$$

so

$$z^2 = x^2 + y^2 + 4^2.$$

We know that when $x = 1,\ dx/dt = 80,\ y = 5,\ dy/dt = 500$, and we want to know dz/dt. First, we find z:

$$z^2 = 1^2 + 5^2 + 4^2 = 42, \text{ so } z = \sqrt{42}.$$

Differentiating $z^2 = x^2 + y^2 + 4^2$ gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Canceling 2s and substituting gives

$$\sqrt{42}\frac{dz}{dt} = 1(80) + 5(500)$$

$$\frac{dz}{dt} = \frac{2580}{\sqrt{42}} = 398.103 \text{ mph}.$$

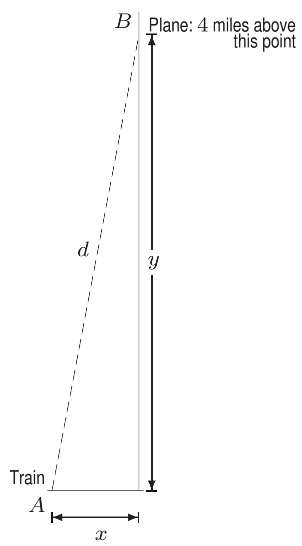


Figure 4.187: View from air

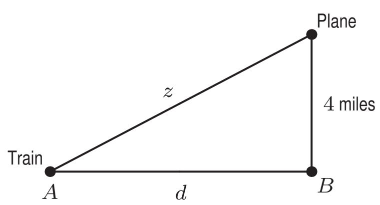


Figure 4.188: Vertical view

100. We want to find dP/dV. Solving $PV = k$ for P gives

$$P = k/V$$

so,

$$\frac{dP}{dV} = -\frac{k}{V^2}.$$

101. (a) Since $V = k/P$, the volume decreases.

(b) Since $PV = k$ and $P = 2$ when $V = 10$, we have $k = 20$, so

$$V = \frac{20}{P}.$$

We think of both P and V as functions of time, so by the chain rule

$$\frac{dV}{dt} = \frac{dV}{dP}\frac{dP}{dt},$$
$$\frac{dV}{dt} = -\frac{20}{P^2}\frac{dP}{dt}.$$

We know that $dP/dt = 0.05$ atm/min when $P = 2$ atm, so

$$\frac{dV}{dt} = -\frac{20}{2^2} \cdot (0.05) = -0.25 \text{ cm}^3\text{/min}.$$

CAS Challenge Problems

102. (a) Since $k > 0$, we have $\lim\limits_{t\to\infty} e^{-kt} = 0$. Thus

$$\lim_{t\to\infty} P = \lim_{t\to\infty} \frac{L}{1 + Ce^{-kt}} = \frac{L}{1 + C \cdot 0} = L.$$

The constant L is called the carrying capacity of the environment because it represents the long-run population in the environment.

(b) Using a CAS, we find

$$\frac{d^2P}{dt^2} = -\frac{LCk^2e^{-kt}(1 - Ce^{-kt})}{(1 + Ce^{-kt})^3}.$$

Thus, $d^2P/dt^2 = 0$ when

$$1 - Ce^{-kt} = 0$$
$$t = -\frac{\ln(1/C)}{k}.$$

Since e^{-kt} and $(1 + Ce^{-kt})$ are both always positive, the sign of d^2P/dt^2 is negative when $(1 - Ce^{-kt}) > 0$, that is, for $t > -\ln(1/C)/k$. Similarly, the sign of d^2P/dt^2 is positive when $(1 - Ce^{-kt}) < 0$, that is, for $t < -\ln(1/C)/k$. Thus, there is an inflection point at $t = -\ln(1/C)/k$.

For $t = -\ln(1/C)/k$,

$$P = \frac{L}{1 + Ce^{\ln(1/C)}} = \frac{L}{1 + C(1/C)} = \frac{L}{2}.$$

Thus, the inflection point occurs where $P = L/2$.

103. (a) The graph has a jump discontinuity whose position depends on a. The function is increasing, and the slope at a given x-value seems to be the same for all values of a. See Figure 4.189.

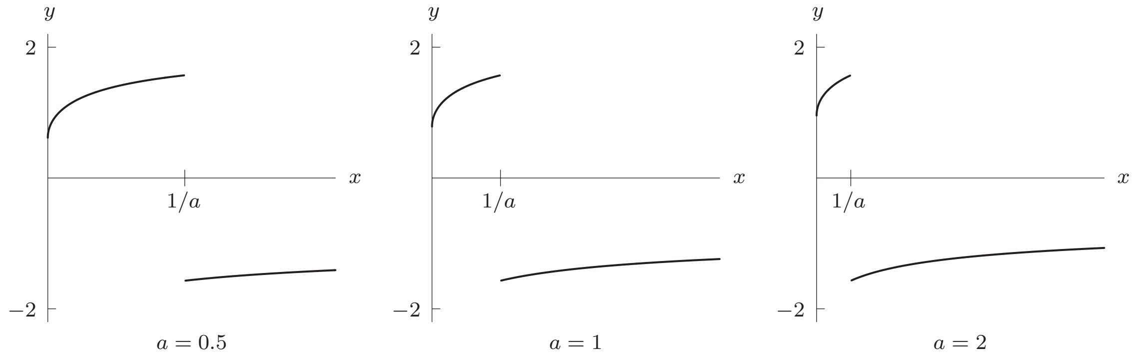


Figure 4.189

(b) Most computer algebra systems will give a fairly complicated answer for the derivative. Here is one example; others may be different.

$$\frac{dy}{dx} = \frac{\sqrt{x} + \sqrt{a}\,\sqrt{a\,x}}{2\,x\,\left(1 + a + 2\,\sqrt{a}\,\sqrt{x} + x + a\,x - 2\,\sqrt{a\,x}\right)}.$$

When we graph the derivative, it appears that we get the same graph for all values of a. See Figure 4.190.

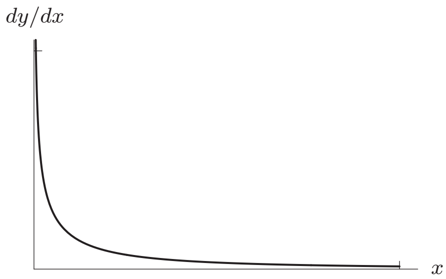


Figure 4.190

(c) Since a and x are positive, we have $\sqrt{ax} = \sqrt{a}\sqrt{x}$. We can use this to simplify the expression we found for the derivative:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sqrt{x} + \sqrt{a}\sqrt{ax}}{2x\left(1 + a + 2\sqrt{a}\sqrt{x} + x + ax - 2\sqrt{ax}\right)}\\ &= \frac{\sqrt{x} + \sqrt{a}\sqrt{a}\sqrt{x}}{2x\left(1 + a + 2\sqrt{a}\sqrt{x} + x + ax - 2\sqrt{a}\sqrt{x}\right)}\\ &= \frac{\sqrt{x} + a\sqrt{x}}{2x\,(1 + a + x + ax)} = \frac{(1+a)\sqrt{x}}{2x(1+a)(1+x)} = \frac{\sqrt{x}}{2x(1+x)}\end{aligned}$$

Since a has canceled out, the derivative is independent of a. This explains why all the graphs look the same in part (b). (In fact they are not exactly the same, because $f'(x)$ is undefined where $f(x)$ has its jump discontinuity. The point at which this happens changes with a.)

104. (a) A CAS gives

$$\frac{d}{dx}\text{arcsinh}\, x = \frac{1}{\sqrt{1+x^2}}$$

(b) Differentiating both sides of $\sinh(\text{arcsinh}\, x) = x$, we get

$$\begin{aligned}\cosh(\text{arcsinh}\, x)\frac{d}{dx}(\text{arcsinh}\, x) &= 1\\ \frac{d}{dx}(\text{arcsinh}\, x) &= \frac{1}{\cosh(\text{arcsinh}\, x)}.\end{aligned}$$

Since $\cosh^2 x - \sinh^2 x = 1$, $\cosh x = \pm\sqrt{1+\sinh^2 x}$. Furthermore, since $\cosh x > 0$ for all x, we take the positive square root, so $\cosh x = \sqrt{1+\sinh^2 x}$. Therefore, $\cosh(\text{arcsinh}\, x) = \sqrt{1+(\sinh(\text{arcsinh}\, x))^2} = \sqrt{1+x^2}$. Thus

$$\frac{d}{dx}\text{arcsinh}\, x = \frac{1}{\sqrt{1+x^2}}.$$

105. (a) A CAS gives

$$\frac{d}{dx}\text{arccosh}\, x = \frac{1}{\sqrt{x^2-1}}, \quad x \geq 1.$$

(b) Differentiating both sides of $\cosh(\text{arccosh}\, x) = x$, we get

$$\sinh(\text{arccosh}\, x)\frac{d}{dx}(\text{arccosh}\, x) = 1$$
$$\frac{d}{dx}(\text{arccosh}\, x) = \frac{1}{\sinh(\text{arccosh}\, x)}.$$

Since $\cosh^2 x - \sinh^2 x = 1$, $\sinh x = \pm\sqrt{\cosh^2 x - 1}$. If $x \geq 0$, then $\sinh x \geq 0$, so we take the positive square root. So $\sinh x = \sqrt{\cosh^2 x - 1}$, $x \geq 0$. Therefore, $\sinh(\text{arccosh}\, x) = \sqrt{(\cosh(\text{arccosh}\, x))^2 - 1} = \sqrt{x^2-1}$, for $x \geq 1$. Thus

$$\frac{d}{dx}\text{arccosh}\, x = \frac{1}{\sqrt{x^2-1}}.$$

106. (a) Using a computer algebra system or differentiating by hand, we get

$$f'(x) = \frac{1}{2\sqrt{a+x}(\sqrt{a}+\sqrt{x})} - \frac{\sqrt{a+x}}{2\sqrt{x}(\sqrt{a}+\sqrt{x})^2}.$$

Simplifying gives

$$f'(x) = \frac{-a+\sqrt{a}\,\sqrt{x}}{2\left(\sqrt{a}+\sqrt{x}\right)^2\sqrt{x}\,\sqrt{a+x}}.$$

The denominator of the derivative is always positive if $x > 0$, and the numerator is zero when $x = a$. Writing the numerator as $\sqrt{a}(\sqrt{x}-\sqrt{a})$, we see that the derivative changes from negative to positive at $x = a$. Thus, by the first derivative test, the function has a local minimum at $x = a$.

(b) As a increases, the local minimum moves to the right. See Figure 4.191. This is consistent with what we found in part (a), since the local minimum is at $x = a$.

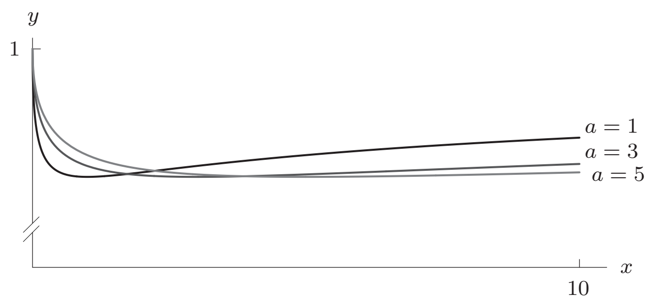


Figure 4.191

(c) Using a computer algebra system to find the second derivative when $a = 2$, we get

$$f''(x) = \frac{4\sqrt{2}+12\sqrt{x}+6\,x^{3/2}-3\sqrt{2}\,x^2}{4\left(\sqrt{2}+\sqrt{x}\right)^3 x^{3/2}\,(2+x)^{3/2}}.$$

Using the computer algebra system again to solve $f''(x) = 0$, we find that it has one zero at $x = 4.6477$. Graphing the second derivative, we see that it goes from positive to negative at $x = 4.6477$, so this is an inflection point.

107. (a) Different CASs give different answers. (In fact, their answers could be more complicated than what you get by hand.) One possible answer is

$$\frac{dy}{dx} = \frac{\tan\left(\frac{x}{2}\right)}{2\sqrt{\frac{1-\cos x}{1+\cos x}}}.$$

(b) The graph in Figure 4.192 is a step function:

$$f(x) = \begin{cases} 1/2 & 2n\pi < x < (2n+1)\pi \\ -1/2 & (2n+1)\pi < x < (2n+2)\pi. \end{cases}$$

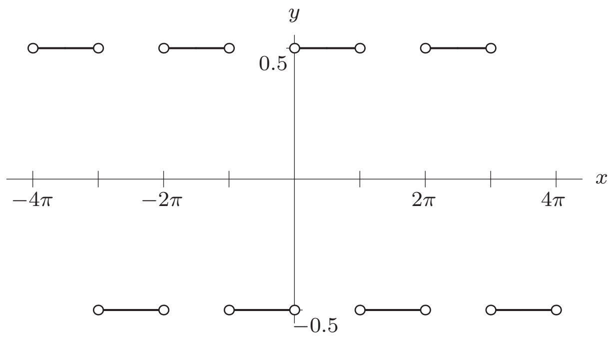


Figure 4.192

Figure 4.192, which shows the graph in disconnected line segments, is correct. However, unless you select certain graphing options in your CAS, it may join up the segments. Use the double angle formula $\cos(x) = \cos^2(x/2) - \sin^2(x/2)$ to simplify the answer in part (a). We find

$$\frac{dy}{dx} = \frac{\tan(x/2)}{2\sqrt{\frac{1-\cos x}{1+\cos x}}} = \frac{\tan(x/2)}{2\sqrt{\frac{1-\cos(2\cdot(x/2))}{1+\cos(2\cdot(x/2))}}} = \frac{\tan(x/2)}{2\sqrt{\frac{1-\cos^2(x/2)+\sin^2(x/2)}{1+\cos^2(x/2)-\sin^2(x/2)}}}$$

$$= \frac{\tan(x/2)}{2\sqrt{\frac{2\sin^2(x/2)}{2\cos^2(x/2)}}} = \frac{\tan(x/2)}{2\sqrt{\tan^2(x/2)}} = \frac{\tan(x/2)}{2\,|\tan(x/2)|}$$

Thus, $dy/dx = 1/2$ when $\tan(x/2) > 0$, i.e. when $0 < x < \pi$ (more generally, when $2n\pi < x < (2n+1)\pi$), and $dy/dx = -1/2$ when $\tan(x/2) < 0$, i.e., when $\pi < x < 2\pi$ (more generally, when $(2n+1)\pi < x < (2n+2)\pi$, where n is any integer).

108. (a)

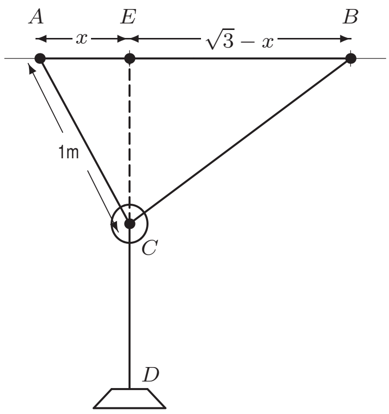


Figure 4.193

We want to maximize the sum of the lengths EC and CD in Figure 4.193. Let x be the distance AE. Then x can be between 0 and 1, the length of the left rope. By the Pythagorean theorem,

$$EC = \sqrt{1 - x^2}.$$

The length of the rope from B to C can also be found by the Pythagorean theorem:

$$BC = \sqrt{EC^2 + EB^2} = \sqrt{1 - x^2 + (\sqrt{3} - x)^2} = \sqrt{4 - 2\sqrt{3}x}.$$

Since the entire rope from B to D has length 3 m, the length from C to D is

$$CD = 3 - \sqrt{4 - 2\sqrt{3}x}.$$

The distance we want to maximize is

$$f(x) = EC + CD = \sqrt{1 - x^2} + 3 - \sqrt{4 - 2\sqrt{3}x}, \quad \text{for} \quad 0 \le x \le 1.$$

Differentiating gives

$$f'(x) = \frac{-2x}{2\sqrt{1 - x^2}} - \frac{-2\sqrt{3}}{2\sqrt{4 - 2\sqrt{3}x}}.$$

Setting $f'(x) = 0$ gives the cubic equation

$$2\sqrt{3}x^3 - 7x^2 + 3 = 0.$$

Using a computer algebra system to solve the equation gives three roots: $x = -1/\sqrt{3}, x = \sqrt{3}/2, x = \sqrt{3}$. We discard the negative root. Since x cannot be larger than 1 meter (the length of the left rope), the only critical point of interest is $x = \sqrt{3}/2$, that is, halfway between A and B.

To find the global maximum, we calculate the distance of the weight from the ceiling at the critical point and at the endpoints:

$$f(0) = \sqrt{1} + 3 - \sqrt{4} = 2$$
$$f\left(\frac{\sqrt{3}}{2}\right) = \sqrt{1 - \frac{3}{4}} + 3 - \sqrt{4 - 2\sqrt{3} \cdot \frac{\sqrt{3}}{2}} = 2.5$$
$$f(1) = \sqrt{0} + 3 - \sqrt{4 - 2\sqrt{3}} = 4 - \sqrt{3} = 2.27.$$

Thus, the weight is at the maximum distance from the ceiling when $x = \sqrt{3}/2$; that is, the weight comes to rest at a point halfway between points A and B.

(b) No, the equilibrium position depends on the length of the rope. For example, suppose that the left-hand rope was 1 cm long. Then there is no way for the pulley at its end to move to a point halfway between the anchor points.

PROJECTS FOR CHAPTER FOUR

1.

(a) (i) The slope, dV/dt, is the rate of change of the volume of air that has been exhaled over time. Thus, it is the rate of flow of the air out of the lungs, measured in liters per second. The greater the slope, the faster air is being expelled from the lungs. Notice that dV/dt is a volume flow; that is, the rate at which the volume of air exhaled is changing. It is not the speed at which air particles are leaving the lungs, which would be measured in meters per second.

(ii) As the horizontal asymptote on the graph, the vital capacity VC is the volume of air expelled at the end of the exhalation. It is the maximum volume of air the patient can exhale in one breath.

(b) (i) The slope of the flow-volume curve tells you how much the flow rate of air out of the lungs changes in response to exhalation of an additional volume of air.

(ii) For V near 0, when little air has been exhaled, the slope of the flow-volume curve is steep and positive. This means that the flow rate dV/dt is rapidly increasing as air is exhaled. The patient is expelling air faster and faster. The flow volume curve has zero slope when approximately 1 liter has been exhaled. This corresponds to the maximum flow rate dV/dt; air is expelled most rapidly at this stage. Thereafter, the slope of the flow-volume curve is negative, so the flow rate decreases and air is expelled slower and slower until approximately 5.5 liters of air have been exhaled. At that point, the patient has no more air to exhale and must take a breath.

(iii) Suppose the flow rate, dV/dt, is given as function of V by the function f, so $dV/dt = f(V)$; the flow-volume curve we were given is a graph of f. Now we are asked to plot $f'(V)$, whose graph is in Figure 4.194.

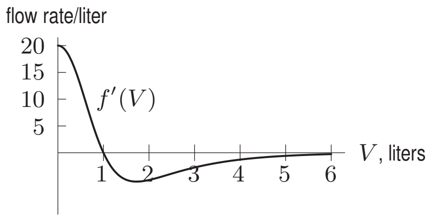


Figure 4.194

(iv) The maximal rate is identified as the highest point on the flow-volume curve, which occurs at about $V = 1$ liter. Thus, the patient's peak expiratory flow occurs when 1 liter of air has been exhaled. The peak expiratory flow is about 11.5 liters per second.

On the slope graph in Figure 4.194, a critical point occurs where the slope is 0; that is, where the slope graph intersects the horizontal axis. This occurs at $V = 1$. The first derivative test tells us that this critical point gives a maximum flow rate since the slope is positive to the left of $V = 1$ and negative to the right of $V = 1$.

(c) The increased resistance to airflow means that the patient exhales more slowly. This makes the volume-time curve rise more slowly. In fact, sometimes the patient exhales so slowly that they "run out of time" and cannot exhale all the volume they want to, leading to air trapping. This corresponds to a lowered vital capacity, VC. See Figure 4.195.

The peak expiratory flow will be lower. This means that the flow-volume curve does not rise as high. See Figure 4.196.

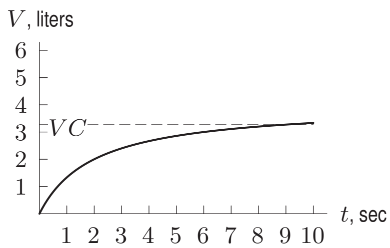


Figure 4.195: Volume-time curve for an asthmatic

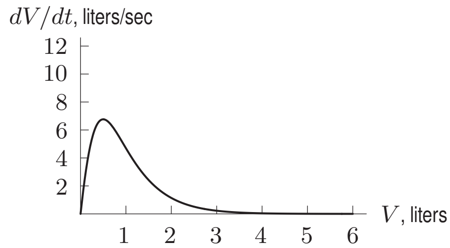


Figure 4.196: Flow-volume curve for an asthmatic

2.

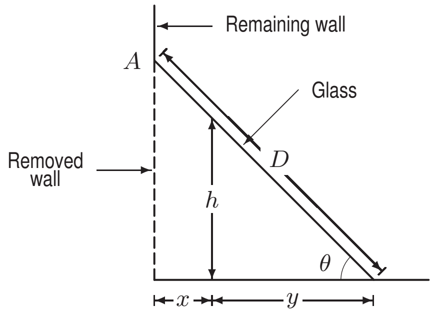


Figure 4.197: A Cross-section of the Projected Greenhouse

Suppose that the glass is at an angle θ (as shown in Figure 4.197), that the length of the wall is l, and that the glass has dimensions D ft by l ft. Since your parents will spend a fixed amount, the area of the glass, say k ft^2, is fixed:

$$Dl = k.$$

The width of the extension is $D\cos\theta$. If h is the height of your tallest parent, he or she can walk in a distance of x, and

$$\frac{h}{y} = \tan\theta, \quad \text{so} \quad y = \frac{h}{\tan\theta}.$$

Thus,

$$x = D\cos\theta - y = D\cos\theta - \frac{h}{\tan\theta} \quad \text{for } 0 < \theta < \frac{\pi}{2}.$$

We maximize x since doing so maximizes the usable area:

$$\begin{aligned}
\frac{dx}{d\theta} &= -D\sin\theta + \frac{h}{(\tan\theta)^2}\cdot\frac{1}{(\cos\theta)^2} = 0 \\
\sin^3\theta &= \frac{h}{D} \\
\theta &= \arcsin\left(\left(\frac{h}{D}\right)^{1/3}\right).
\end{aligned}$$

This is the only critical point, and $x \to 0$ when $\theta \to 0$ and when $\theta \to \pi/2$. Thus, the critical point is a global maximum. Since

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{h}{D}\right)^{2/3}},$$

the maximum value of x is

$$\begin{aligned}
x &= D\cos\theta - \frac{h}{\tan\theta} = D\cos\theta - \frac{h\cos\theta}{\sin\theta} \\
&= \left(D - \frac{h}{\sin\theta}\right)\cos\theta = \left(D - \frac{h}{(h/D)^{1/3}}\right)\cdot\left(1-\left(\frac{h}{D}\right)^{2/3}\right)^{1/2} \\
&= (D - h^{2/3}D^{1/3})\cdot\left(1-\frac{h^{2/3}}{D^{2/3}}\right)^{1/2} \\
&= D\left(1-\frac{h^{2/3}}{D^{2/3}}\right)\cdot\left(1-\frac{h^{2/3}}{D^{2/3}}\right)^{1/2} = D\left(1-\frac{h^{2/3}}{D^{2/3}}\right)^{3/2}.
\end{aligned}$$

This means

$$\begin{aligned}\text{Maximum Usable Area} &= lx \\ &= lD\left(1-\frac{h^{2/3}}{D^{2/3}}\right)^{3/2} \\ &= k\left(1-\left(\frac{hl}{k}\right)^{2/3}\right)^{3/2}\end{aligned}$$

3. **(a)** The point on the line $y = mx$ corresponding to the point $(2, 3.5)$ has y-coordinate given by $y = m(2) = 2m$. Thus, for the point $(2, 3.5)$

$$\text{Vertical distance to the line} = |2m - 3.5|.$$

We calculate the distance similarly for the other two points. We want to minimize the sum, S, of the squares of these vertical distances

$$S = (2m - 3.5)^2 + (3m - 6.8)^2 + (5m - 9.1)^2.$$

Differentiating with respect to m gives

$$\frac{dS}{dm} = 2(2m - 3.5)\cdot 2 + 2(3m - 6.8)\cdot 3 + 2(5m - 9.1)\cdot 5.$$

Setting $dS/dm = 0$ gives

$$2\cdot 2(2m - 3.5) + 2\cdot 3(3m - 6.8) + 2\cdot 5(5m - 9.1) = 0.$$

Canceling a 2 and multiplying out gives

$$\begin{aligned}4m - 7 + 9m - 20.4 + 25m - 45.5 &= 0 \\ 38m &= 72.9 \\ m &= 1.92.\end{aligned}$$

Thus, the best fitting line has equation $y = 1.92x$.

(b) To fit a line of the form $y = mx$ to the data, we take $y = V$ and $x = r^3$. Then k will be the slope m. So we make the following table of data:

r	2	5	7	8
$x = r^3$	8	125	343	512
$y = V$	8.7	140.3	355.8	539.2

To find the best fitting line of the form $y = mx$, we minimize the sums of the squares of the vertical distances from the line. For the point $(8, 8.7)$ the corresponding point on the line has $y = 8m$, so

$$\text{Vertical distance} = |8m - 8.7|.$$

We find distances from the other points similarly. Thus we want to minimize

$$S = (8m - 8.7)^2 + (125m - 140.3)^2 + (343m - 355.8)^2 + (512m - 539.2)^2.$$

Differentiating with respect to m, which is the variable, and setting the derivative to zero:

$$\frac{dS}{dm} = 2(8m - 8.7)\cdot 8 + 2(125m - 140.3)\cdot 125 + 2(343m - 355.8)\cdot 343 + 2(512m - 539.2)\cdot 512 = 0.$$

After canceling a 2, solving for m leads to the equation

$$8^2m + 125^2m + 343^2m + 512^2m = 8 \cdot 8.7 + 125 \cdot 140.3 + 343 \cdot 355.8 + 512 \cdot 539.2$$
$$m = 1.051.$$

Thus, $k = 1.051$ and the relationship between V and r is

$$V = 1.051r^3.$$

(In fact, the correct relationship is $V = \pi r^3/3$, so the exact value of k is $\pi/3 = 1.047$.)

(c) The best fitting line minimizes the sum of the squares of the vertical distances from points to the line. Since the point on the line $y = mx$ corresponding to (x_1, y_1) is the point with $y = mx_1$; for this point we have

$$\text{Vertical distance} = |mx_1 - y_1|.$$

We calculate the distance from the other points similarly. Thus we want to minimize

$$S = (mx_1 - y_1)^2 + (mx_2 - y_2)^2 \cdots + (mx_n - y_n)^2.$$

The variable is m (the x_is and y_is are all constants), so

$$\frac{dS}{dm} = 2(mx_1 - y_1)x_1 + 2(mx_2 - y_2)x_2 + \cdots + 2(mx_n - y_n)x_n = 0$$
$$2(m(x_1^2 + x_2^2 + \cdots + x_n^2) - (x_1y_1 + x_2y_2 + \cdots + x_ny_n)) = 0.$$

Solving for m gives

$$m = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{x_1^2 + x_2^2 + \cdots + x_n^2} = \frac{\sum_{i=1}^{n} x_iy_i}{\sum_{i=1}^{n} x_i^2}.$$

4. (a) (i) We want to minimize A, the total area lost to the forest, which is made up of n firebreaks and 1 stand of trees lying between firebreaks. The area of each firebreak is $(50\text{ km})(0.01\text{ km}) = 0.5\text{ km}^2$, so the total area lost to the firebreaks is $0.5n\text{ km}^2$. There are n total stands of trees between firebreaks. The area of a single stand of trees can be found by subtracting the firebreak area from the forest and dividing by n, so

$$\text{Area of one stand of trees} = \frac{2500 - 0.5n}{n}.$$

Thus, the total area lost is

$$A = \text{Area of one stand} + \text{Area lost to firebreaks}$$
$$= \frac{2500 - 0.5n}{n} + 0.5n = \frac{2500}{n} - 0.5 + 0.5n.$$

We assume that A is a differentiable function of a continuous variable, n. Differentiating this function yields

$$\frac{dA}{dn} = -\frac{2500}{n^2} + 0.5.$$

At critical points, $dA/dn = 0$, so $0.5 = 2500/n^2$ or $n = \sqrt{2500/0.5} \approx 70.7$. Since n must be an integer, we check that when $n = 71$, $A = 70.211$ and when $n = 70$, $A = 70.214$. Thus, $n = 71$ gives a smaller area lost.

We can check that this is a local minimum since the second derivative is positive everywhere

$$\frac{d^2A}{dn^2} = \frac{5000}{n^3} > 0.$$

Finally, we check the endpoints: $n = 1$ yields the entire forest lost after a fire, since there is only one stand of trees in this case and it all burns. The largest n is 5000, and in this case the firebreaks remove the entire forest. Both of these cases maximize the area of forest lost. Thus, $n = 71$ is a global minimum. So 71 firebreaks minimizes the area of forest lost.

(ii) Repeating the calculation using b for the width gives

$$A = \frac{2500}{n} - 50b + 50bn,$$

and

$$\frac{dA}{dn} = \frac{-2500}{n^2} + 50b,$$

with a critical point when $b = 50/n^2$ so $n = \sqrt{50/b}$. So, for example, if we make the width b four times as large we need half as many firebreaks.

(b) We want to minimize A, the total area lost to the forest, which is made up of n firebreaks in one direction, n firebreaks in the other, and one square of trees surrounded by firebreaks. The area of each firebreak is 0.5 km^2, and there are $2n$ of them, giving a total of $0.5 \cdot 2n$. But this is larger than the total area covered by the firebreaks, since it counts the small intersection squares, of size $(0.01)^2$, twice. Since there are n^2 intersections, we must subtract $(0.01)^2n^2$ from the total area of the $2n$ firebreaks. Thus,

$$\text{Area covered by the firebreaks } = 0.5 \cdot 2n - (0.01)^2n^2.$$

To this we must add the area of one square patch of trees lost in a fire. These are squares of side $(50 - 0.01n)/n = 50/n - 0.01$. Thus the total area lost is

$$A = n - 0.0001n^2 + (50/n - 0.01)^2$$

Treating n as a continuous variable and differentiating this function yields

$$\frac{dA}{dn} = 1 - 0.0002n + 2\left(\frac{50}{n} - 0.01\right)\left(\frac{-50}{n^2}\right).$$

Using a computer algebra system to find critical points we find that $dA/dn = 0$ when $n \approx 17$ and $n = 5000$. Thus $n = 17$ gives a minimum lost area, since the endpoints of $n = 1$ and $n = 5000$ both yield $A = 2500$ or the entire forest lost. So we use 17 firebreaks in each direction.

CHAPTER FIVE

Solutions for Section 5.1

Exercises

1. (a) Left sum
(b) Upper estimate
(c) 6
(d) $\Delta t = 2$
(e) Upper estimate is approximately $4 \cdot 2 + 2.9 \cdot 2 + 2 \cdot 2 + 1.5 \cdot 2 + 1 \cdot 2 + 0.8 \cdot 2 = 24.4$.

2. (a) (i) Since the velocity is increasing, for an upper estimate we use a right sum. Using $n = 4$, we have $\Delta t = 3$, so

$$\text{Upper estimate } = (37)(3) + (38)(3) + (40)(3) + (45)(3) = 480.$$

(ii) Using $n = 2$, we have $\Delta t = 6$, so

$$\text{Upper estimate } = (38)(6) + (45)(6) = 498.$$

(b) The answer using $n = 4$ is more accurate as it uses the values of $v(t)$ when $t = 3$ and $t = 9$.
(c) Since the velocity is increasing, for a lower estimate we use a left sum. Using $n = 4$, we have $\Delta t = 3$, so

$$\text{Lower estimate } = (34)(3) + (37)(3) + (38)(3) + (40)(3) = 447.$$

3. (a) Since the velocity is decreasing, for an upper estimate, we use a left sum. With $n = 5$,we have $\Delta t = 2$. Then

$$\text{Upper estimate } = (44)(2) + (42)(2) + (41)(2) + (40)(2) + (37)(2) = 408.$$

(b) For a lower estimate, we use a right sum, so

$$\text{Lower estimate } = (42)(2) + (41)(2) + (40)(2) + (37)(2) + (35)(2) = 390.$$

4. (a) Lower estimate $= 60 + 40 + 25 + 10 + 0 = 135$ feet. Upper estimate $= 88 + 60 + 40 + 25 + 10 = 223$ feet.
(b) See Figure 5.1.

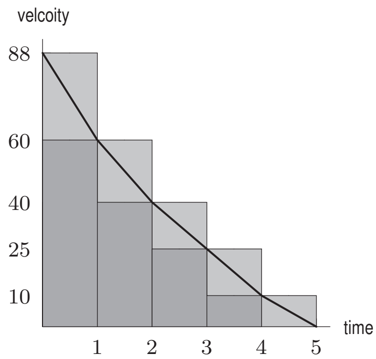


Figure 5.1

(c) The difference between the estimates $= 223 - 135 = 88$ feet. This is the sum of the lightly shaded areas in the graph, namely $(88 - 0) \cdot 1 = 88$ feet.

5. The distance traveled is represented by area under the velocity curve. We can approximate the area using left- and right-hand sums. Alternatively, counting the squares (each of which has area 10), and allowing for the broken squares, we can see that the area under the curve from 0 to 6 is between 140 and 150. Hence the distance traveled is between 140 and 150 meters.

6. Using $\Delta t = 2$,

$$\begin{aligned}\text{Lower estimate} &= v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2\\ &= 1(2) + 5(2) + 17(2)\\ &= 46\\ \text{Upper estimate} &= v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2\\ &= 5(2) + 17(2) + 37(2)\\ &= 118\\ \text{Average} &= \frac{46+118}{2} = 82\\ \text{Distance traveled} &\approx 82\,\text{meters.}\end{aligned}$$

7. (a) The velocity is always positive, so the particle is moving in the same direction throughout. However, the particle is speeding up until shortly before $t = 0$, and slowing down thereafter.

(b) The distance traveled is represented by the area under the curve. We can calculate over and understimates for the area using a combination of left- and right-hand sums. Alternatively, using whole grid squares, we can overestimate the area as $3+3+3+3+2+1 = 15$ cm, and we can underestimate the area as $1+2+2+1+0+0 = 6$ cm.

8. Using $\Delta t = 0.2$, our upper estimate is

$$\frac{1}{1+0}(0.2) + \frac{1}{1+0.2}(0.2) + \frac{1}{1+0.4}(0.2) + \frac{1}{1+0.6}(0.2) + \frac{1}{1+0.8}(0.2) \approx 0.75.$$

The lower estimate is

$$\frac{1}{1+0.2}(0.2) + \frac{1}{1+0.4}(0.2) + \frac{1}{1+0.6}(0.2) + \frac{1}{1+0.8}(0.2)\frac{1}{1+1}(0.2) \approx 0.65.$$

Since v is a decreasing function, the bug has crawled more than 0.65 meters, but less than 0.75 meters. We average the two to get a better estimate:

$$\frac{0.65+0.75}{2} = 0.70 \text{ meters.}$$

9. From $t = 0$ to $t = 3$ the velocity is constant and positive, so the change in position is $2\cdot 3$ cm, that is 6 cm to the right. From $t = 3$ to $t = 5$, the velocity is negative and constant, so the change in position is $-3\cdot 2$ cm, that is 6 cm to the left. Thus the total change in position is 0. The particle moves 6 cm to the right, followed by 6 cm to the left, and returns to where it started.

10. From $t = 0$ to $t = 5$ the velocity is positive so the change in position is to the right. The area under the velocity graph gives the distance traveled. The region is a triangle, and so has area $(1/2)bh = (1/2)5\cdot 10 = 25$. Thus the change in position is 25 cm to the right.

11. The velocity is constant and negative, so the change in position is $-3\cdot 5$ cm, that is 15 cm to the left.

12. From $t = 0$ to $t = 4$ the velocity is positive so the change in position is to the right. The area under the velocity graph gives the distance traveled. The region is a triangle, and so has area $(1/2)bh = (1/2)4\cdot 8 = 16$. Thus the change in position is 16 cm to the right for $t = 0$ to $t = 4$. From $t = 4$ to $t = 5$, the velocity is negative so the change in position is to the left. The distance traveled to the left is given by the area of the triangle, $(1/2)bh = (1/2)1\cdot 2 = 1$. Thus the total change in position is $16 - 1 = 15$ cm to the right.

13. (a) With $n = 4$, we have $\Delta t = 2$. Then

$$t_0 = 15, t_1 = 17, t_2 = 19, t_3 = 21, t_4 = 23 \quad\text{and}\quad f(t_0) = 10, f(t_1) = 13, f(t_2) = 18, f(t_3) = 20, f(t_4) = 30$$

(b)

$$\begin{aligned}\text{Left sum} &= (10)(2) + (13)(2) + (18)(2) + (20)(2) = 122\\ \text{Right sum} &= (13)(2) + (18)(2) + (20)(2) + (30)(2) = 162.\end{aligned}$$

(c) With $n = 2$,we have $\Delta t = 4$. Then

$$t_0 = 15, t_1 = 19, t_2 = 23 \quad \text{and} \quad f(t_0) = 10, f(t_1) = 18, f(t_2) = 30$$

(d)

$$\begin{aligned}\text{Left sum } &= (10)(4) + (18)(4) = 112\\ \text{Right sum } &= (18)(4) + (30)(4) = 192.\end{aligned}$$

14. (a) With $n = 4$, we have $\Delta t = 4$. Then

$$t_0 = 0, t_1 = 4, t_2 = 8, t_3 = 12, t_4 = 16 \quad \text{and} \quad f(t_0) = 25, f(t_1) = 23, f(t_2) = 22, f(t_3) = 20, f(t_4) = 17$$

(b)

$$\begin{aligned}\text{Left sum } &= (25)(4) + (23)(4) + (22)(4) + (20)(4) = 360\\ \text{Right sum } &= (23)(4) + (22)(4) + (20)(4) + (17)(4) = 328.\end{aligned}$$

(c) With $n = 2$,we have $\Delta t = 8$. Then

$$t_0 = 0, t_1 = 8, t_2 = 16 \quad \text{and} \quad f(t_0) = 25, f(t_1) = 22, f(t_2) = 17$$

(d)

$$\begin{aligned}\text{Left sum } &= (25)(8) + (22)(8) = 376\\ \text{Right sum } &= (22)(8) + (17)(8) = 312.\end{aligned}$$

Problems

15. (a) Note that 15 minutes equals 0.25 hours. Lower estimate $= 11(0.25) + 10(0.25) = 5.25$ miles. Upper estimate $= 12(0.25) + 11(0.25) = 5.75$ miles.

(b) Lower estimate $= 11(0.25) + 10(0.25) + 10(0.25) + 8(0.25) + 7(0.25) + 0(0.25) = 11.5$ miles. Upper estimate $= 12(0.25) + 11(0.25) + 10(0.25) + 10(0.25) + 8(0.25) + 7(0.25) = 14.5$ miles.

(c) The difference between Roger's pace at the beginning and the end of his run is 12 mph. If the time between the measurements is h, then the difference between the upper and lower estimates is $12h$. We want $12h < 0.1$, so

$$h < \frac{0.1}{12} \approx 0.0083 \text{ hours} = 30 \text{ seconds}$$

Thus Jeff would have to measure Roger's pace every 30 seconds.

16. The change in position is calculated from the area between the velocity graph and the t-axis, with the region below the axis corresponding to negatives velocities and counting negatively.

Figure 5.2 shows the graph of $f(t)$. From $t = 0$ to $t = 3$ the velocity is positive. The region under the graph of $f(t)$ is a triangle with height 6 cm/sec and base 3 seconds. Thus, from $t = 0$ to $t = 3$, the particle moves

$$\text{Distance moved to right } = \frac{1}{2} \cdot 3 \cdot 6 = 9 \text{ centimeters.}$$

From $t = 3$ to $t = 4$, the velocity is negative. The region between the graph of $f(t)$ and the t-axis is a triangle with height 2 cm/sec and base 1 second, so in this interval the particle moves

$$\text{Distance moved to left } = \frac{1}{2} \cdot 1 \cdot 2 = 1 \text{ centimeter.}$$

Thus, the total change in position is $9 - 1 = 8$ centimeters to the right.

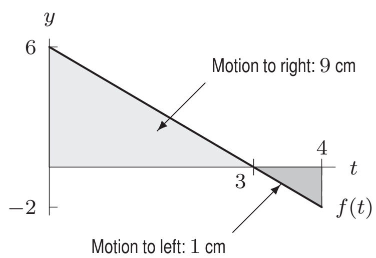


Figure 5.2

17. Since f is increasing, the right-hand sum is the upper estimate and the left-hand sum is the lower estimate. We have $f(a) = 13$, $f(b) = 23$ and $\Delta t = (b - a)/n = 2/100$. Thus,

$$\begin{aligned} |\text{Difference in estimates}| &= |f(b) - f(a)|\Delta t \\ &= |23 - 13|\frac{1}{50} = \frac{1}{5}. \end{aligned}$$

18. Since f is decreasing, the right-hand sum is the lower estimate and the left-hand sum is the upper estimate. We have $f(a) = 24$, $f(b) = 9$ and $\Delta t = (b - a)/n = 3/500$. Thus,

$$\begin{aligned} |\text{Difference in estimates}| &= |f(b) - f(a)|\Delta t \\ &= |9 - 24|\frac{3}{500} = 0.09. \end{aligned}$$

19. Since f is increasing, the right-hand sum is the upper estimate and the left-hand sum is the lower estimate. We have $f(0) = 0$, $f(\pi/2) = 1$ and $\Delta t = (b - a)/n = \pi/200$. Thus,

$$\begin{aligned} |\text{Difference in estimates}| &= |f(b) - f(a)|\Delta t \\ &= |1 - 0|\frac{\pi}{200} = 0.0157. \end{aligned}$$

20. Since f is decreasing, the right-hand sum is the lower estimate and the left-hand sum is the upper estimate. We have $f(0) = 1$, $f(2) = e^{-2}$ and $\Delta t = (b - a)/n = 2/20 = 1/10$. Thus,

$$\begin{aligned} |\text{Difference in estimates}| &= |f(b) - f(a)|\Delta t \\ &= |e^{-2} - 1|\frac{1}{10} = 0.086. \end{aligned}$$

21. (a) See Figure 5.3.
(b) The peak of the flight is when the velocity is 0, namely $t = 3$. The height at $t = 3$ is given by the area under the graph of the velocity from $t = 0$ to $t = 3$; see Figure 5.3. The region is a triangle of base 3 seconds and altitude 96 ft/sec, so the height is $(1/2)3 \cdot 96 = 144$ feet.
(c) The velocity is negative from $t = 3$ to $t = 5$, so the motion is downward then. The distance traveled downward can be calculated by the area of the triangular region which has base of 2 seconds and altitude of -64 ft/sec. Thus, the baseball travels $(1/2)2 \cdot 64 = 64$ feet downward from its peak height of 144 feet at $t = 3$. Thus, the height at time $t = 5$ is the total change in position, $144 - 64 = 80$ feet.

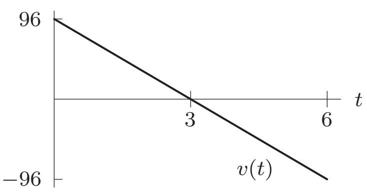


Figure 5.3

22. From $t = 0$ to $t = 3$, you are moving away from home ($v > 0$); thereafter you move back toward home. So you are the farthest from home at $t = 3$. To find how far you are then, we can measure the area under the v curve as about 9 squares, or $9 \cdot 10$ km/hr $\cdot$ 1 hr $= 90$ km. To find how far away from home you are at $t = 5$, we measure the area from $t = 3$ to $t = 5$ as about 25 km, except that this distance is directed toward home, giving a total distance from home during the trip of $90 - 25 = 65$ km.

23. (a) When the aircraft is climbing at v ft/min, it takes $1/v$ minutes to climb 1 foot. Therefore

$$\begin{aligned}\text{Lower estimate } &= \left(\frac{1 \text{ min}}{925 \text{ ft}}\right)(1000 \text{ ft}) + \left(\frac{1 \text{ min}}{875 \text{ ft}}\right)(1000 \text{ ft}) + \cdots + \left(\frac{1 \text{ min}}{490 \text{ ft}}\right)(1000 \text{ ft}) \\ &\approx 14.73 \text{ minutes.} \\ \text{Upper estimate } &= \left(\frac{1 \text{ min}}{875 \text{ ft}}\right)(1000 \text{ ft}) + \left(\frac{1 \text{ min}}{830 \text{ ft}}\right)(1000 \text{ ft}) + \cdots + \left(\frac{1 \text{ min}}{440 \text{ ft}}\right)(1000 \text{ ft}) \\ &\approx 15.93 \text{ minutes.}\end{aligned}$$

Note: The Pilot Operating Manual for this aircraft gives 16 minutes as the estimated time required to climb to $10{,}000$ ft.

(b) The difference between upper and lower sums with $\Delta x = 500$ ft would be

$$\text{Difference } = \left(\frac{1 \text{ min}}{440 \text{ ft}} - \frac{1 \text{ min}}{925 \text{ ft}}\right)(500 \text{ ft}) = 0.60 \text{ minutes.}$$

24. (a) At $t = 20$ minutes, she stops moving toward the lake (with $v > 0$) and starts to move away from the lake (with $v < 0$). So at $t = 20$ minutes the cyclist turns around.

(b) The cyclist is going the fastest when v has the greatest magnitude, either positive or negative. Looking at the graph, we can see that this occurs at $t = 40$ minutes, when $v = -25$ and the cyclist is pedaling at 25 km/hr away from the lake.

(c) From $t = 0$ to $t = 20$ minutes, the cyclist comes closer to the lake, since $v > 0$; thereafter, $v < 0$ so the cyclist moves away from the lake. So at $t = 20$ minutes, the cyclist comes the closest to the lake. To find out how close she is, note that between $t = 0$ and $t = 20$ minutes the distance she has come closer is equal to the area under the graph of v. Each box represents 5/6 of a kilometer, and there are about 2.5 boxes under the graph, giving a distance of about 2 km. Since she was originally 5 km away, she then is about $5 - 2 = 3$ km from the lake.

(d) At $t = 20$ minutes she turns around , since v changes sign then. Since the area below the t-axis is greater than the area above, the farthest she is from the lake is at $t = 60$ minutes. Between $t = 20$ and $t = 60$ minutes, the area under the graph is about 10.8 km. (Since 13 boxes $\cdot\, 5/6 = 10.8$.) So at $t = 60$ she will be about $3 + 10.8 = 13.8$ km from the lake.

25. (a) Since car B starts at $t = 2$, the tick marks on the horizontal axis (which we assume are equally spaced) are 2 hours apart. Thus car B stops at $t = 6$ and travels for 4 hours.
Car A starts at $t = 0$ and stops at $t = 8$, so it travels for 8 hours.

(b) Car A's maximum velocity is approximately twice that of car B, that is 100 km/hr.

(c) The distance traveled is given by the area of under the velocity graph. Using the formula for the area of a triangle, the distances are given approximately by

$$\begin{aligned}\text{Car } A \text{ travels} &= \frac{1}{2} \cdot \text{Base} \cdot \text{Height} = \frac{1}{2} \cdot 8 \cdot 100 = 400 \text{ km} \\ \text{Car } B \text{ travels} &= \frac{1}{2} \cdot \text{Base} \cdot \text{Height} = \frac{1}{2} \cdot 4 \cdot 50 = 100 \text{ km.}\end{aligned}$$

26. (a) Car A has the largest maximum velocity because the peak of car A's velocity curve is higher than the peak of B's.

(b) Car A stops first because the curve representing its velocity hits zero (on the t-axis) first.

(c) Car B travels farther because the area under car B's velocity curve is the larger.

27. (a) See Figure 5.4.

(b) The distance traveled is the area under the graph of the velocity in Figure 5.4. The region is a triangle of base 5 seconds and altitude 50 ft/sec, so the distance traveled is $(1/2)5 \cdot 50 = 125$ feet.

(c) The slope of the graph of the velocity function is the same, so the triangular region under it has twice the altitude and twice the base (it takes twice as long to stop). See Figure 5.5. Thus, the area is 4 times as large and the car travels 4 times as far.

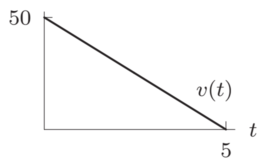


Figure 5.4

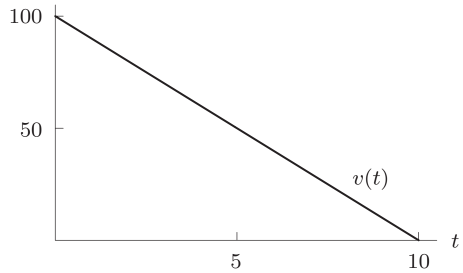


Figure 5.5

28. The graph of her velocity against time is a straight line from 0 mph to 60 mph; see Figure 5.6. Since the distance traveled is the area under the curve, we have

$$\text{Shaded area} = \frac{1}{2} \cdot t \cdot 60 = 10 \text{ miles}$$

Solving for t gives

$$t = \frac{1}{3}\text{hr} = 20 \text{ minutes} .$$

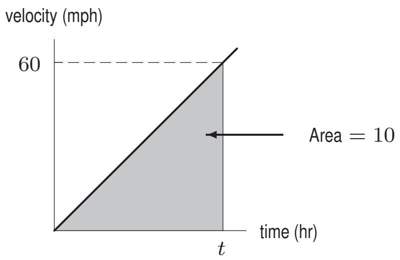


Figure 5.6

29. Since acceleration is the derivative of velocity $(a(t) = v'(t))$, and $a(t)$ is a constant, $v(t)$ must be a linear function. Its v-intercept is 0 since the object has zero initial velocity. Thus the graph of v is a line through the origin with slope 32 and equation $v(t) = 32t$ ft/sec. See Figure 5.7.

Using $\Delta t = 1$, the right sum, $32 + 64 + 96 + 128 = 320$ feet, is an upper bound on the distance traveled. The left sum, $0 + 32 + 64 + 96 = 192$ feet, is a lower bound.

To find the actual distance, we find the exact area of the region below the line—it is the triangle in Figure 5.7 with height $32 \cdot 4 = 128$ and base 4, so its area is 256 ft.

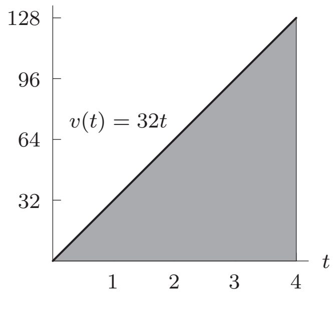


Figure 5.7

30. No, the 2010 Prius Plug-in Prototype cannot travel half a mile in EV-only mode in under 25 seconds, because the upper estimate for the distance traveled in 25 seconds is only about 0.3 miles. To see this, we convert seconds to hours (1 sec $= 1/3600$ hours) so that the upper distance estimate is in miles:

$$\begin{array}{c}\text{Total distance traveled}\\ \text{between } t = 0 \text{ and } t = 25\\ \text{(Upper estimate)}\end{array} = 20 \cdot \frac{5}{3600} + 33 \cdot \frac{5}{3600} + 45 \cdot \frac{5}{3600} + 50 \cdot \frac{5}{3600} + 59 \cdot \frac{5}{3600} < 0.3 \text{ miles}$$

31. To estimate the distance between the two cars at $t = 15$, we calculate upper and lower bound estimates for this distance and then average these estimates. Note that because the speed of the Prius is in miles per hour, we need to convert seconds to hours (1 sec $= 1/3600$ hours) to calculate the distance estimates in miles. We may then convert miles to feet (1 mile $= 5280$ feet).

$$\begin{aligned}\text{Distance between cars at } t = 15 \ (\text{Upper Estimate}) &\approx (\text{Difference in speeds at } t = 5)\cdot(\text{Travel time})\\ &+ (\text{Difference in speeds at } t = 10)\cdot(\text{Travel time})\\ &+ (\text{Difference in speeds at } t = 15)\cdot(\text{Travel time})\\ &= (33-20)\cdot\frac{5}{3600} + (53-33)\cdot\frac{5}{3600} + (70-45)\cdot\frac{5}{3600} = 0.081 \text{ miles}\end{aligned}$$

$$\begin{aligned}\text{Distance between cars at } t = 15 \ (\text{Lower Estimate}) &\approx (\text{Difference in speeds at } t = 0)\cdot(\text{Travel time})\\ &+ (\text{Difference in speeds at } t = 5)\cdot(\text{Travel time})\\ &+ (\text{Difference in speeds at } t = 10)\cdot(\text{Travel time})\\ &= (0-0)\cdot\frac{5}{3600} + (33-20)\cdot\frac{5}{3600} + (53-33)\cdot\frac{5}{3600} = 0.046 \text{ miles}\end{aligned}$$

Since:

$$\text{Average of Lower and Upper Estimates of distance between cars at } t = 15 = \frac{0.081 + 0.046}{2} = 0.0635 \text{ miles} = 335 \text{ feet},$$

the distance between the two cars is about 335 feet, 15 seconds after leaving the stoplight.

Strengthen Your Understanding

32. This is only true if the car accelerates at a constant rate, that is, if the graph of the velocity is a linear function increasing from 0 to 50 in 10 seconds. If the graph of velocity is above that line, the car travels further, and if the graph of velocity is below that line the car travels less far.

33. Recording the velocity every 0.1 seconds means there are 10 subdivisions, so if the velocity of the car in ft/sec is $f(t)$ after t seconds, and if f is increasing during the second, then the difference between the lower and upper estimates is

$$(f(b) - f(a))\frac{b-a}{10},$$

where $[a, b]$ is the interval during which the velocity is recorded. Since the interval is one second long, $b - a = 1$, so the difference is $(f(b) - f(a))/10$. This could be much bigger than 0.1 feet if the car is accelerating quickly. For example, if $f(a) = 0$ and $f(b) = 10$, then the difference between the estimates is 1 foot.

34. If f is decreasing on the interval then the right sum is less than the left sum, so $f(x) = x^2$, $[a, b] = [-2, -1]$ is an example.

35. If $f(t)$ is increasing on the interval $[a, b]$, then the difference between the upper and lower estimates is $|f(b)-f(a)|\cdot\Delta t = |f(b) - f(a)|(b - a)/n$, where n is the number of subdivisions. So if, for example, $f(b) - f(a) = 10$ and $b - a = 1$, then

$$|f(b) - f(a)|\frac{b-a}{n} = \frac{10}{n},$$

and n would have to be ≥ 100 to make this ≤ 0.1. A velocity function and interval that satisfy these conditions are $f(x) = 10x$ and $[a, b] = [0, 1]$.

36. True. Since the velocity is increasing, each term in the left-hand sum is less than the corresponding term in the right-hand sum.

37. True. If the velocity is $f(t)$ and the interval is $a \leq t \leq b$, then

$$\text{Difference between left and right sums } = (f(a) - f(b))\cdot\Delta t.$$

Since Δt is halved when the number of subdivisions is doubled, the difference is halved also.

38. False. For example, for a constant function, the difference does not get smaller, since it is always 0. Another example is the velocity function $f(x) = x^2$ on the interval $-1 \le x \le 1$. By the symmetry of the graph, for an even number of subdivisions the difference between the left and right sums is always 0.

39. (a) On $[3, 5]$ and on $[9, 10]$, since $v = 0$ there.
(b) 3600 feet to the east, since this is the area under the velocity curve between $t = 0$ and $t = 3$.
(c) At $t = 8$ minutes, since the areas above and below the curve between $t = 0$ and $t = 8$ are equal.
(d) It will take her 30 seconds longer. By calculating areas, we see that at $t = 11$,

$$\text{Distance from home } = 2 \cdot 30 \cdot 60 - 3 \cdot 30 \cdot 60 + 0.5 \cdot 30 \cdot 60 = -900 \text{ feet.}$$

Thus, at $t = 11$, she is 900 feet west of home. At a velocity of 30 ft/sec eastward, it takes $900/30 = 30$ seconds to get home.

Solutions for Section 5.2

Exercises

1. (a) Right sum
(b) Upper estimate
(c) 3
(d) $\Delta x = 2$

2. (a) Left sum
(b) Lower estimate
(c) 3
(d) $\Delta x = 2$

3. (a) Left-hand sum. Right-hand sum would be smaller.
(b) We have $a = 0$, $b = 2$, $n = 6$, $\Delta x = \frac{2}{6} = \frac{1}{3}$.

4.

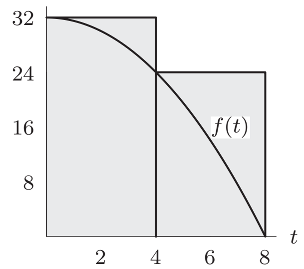


Figure 5.8: Left Sum, $\Delta t = 4$

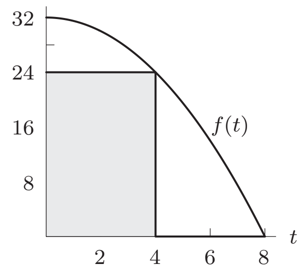


Figure 5.9: Right Sum, $\Delta t = 4$

(a) Left-hand sum $= 32 \cdot 4 + 24 \cdot 4 = 224$.
(b) Right-hand sum $= 24 \cdot 4 + 0 \cdot 4 = 96$.

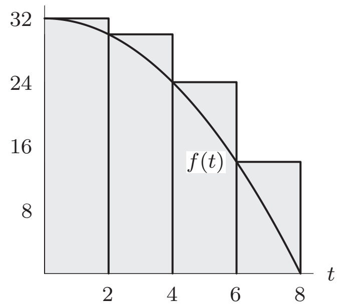


Figure 5.10: Left Sum, $\Delta t = 2$

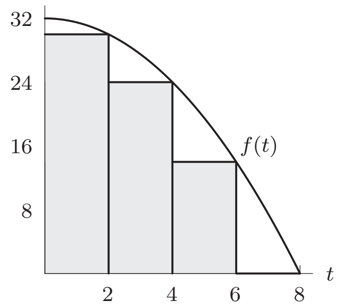


Figure 5.11: Right Sum, $\Delta t = 2$

(c) Left-hand sum $= 32 \cdot 2 + 30 \cdot 2 + 24 \cdot 2 + 14 \cdot 2 = 200$.
(d) Right-hand sum $= 30 \cdot 2 + 24 \cdot 2 + 14 \cdot 2 + 0 \cdot 2 = 136$.

5. We use a calculator or computer to see that $\int_1^4 (x^2+x)\,dx = 28.5$.

6. We use a calculator or computer to see that $\int_0^3 2^x\,dx = 10.0989$.

7. We use a calculator or computer to see that $\int_{-1}^1 e^{-x^2}\,dx = 1.4936$.

8. We use a calculator or computer to see that $\int_0^3 \ln(y^2+1)\,dy = 3.406$.

9. We use a calculator or computer to see that $\int_0^1 \sin(t^2)dt = 0.3103$.

10. We use a calculator or computer to see that $\int_3^4 \sqrt{e^z+z}\,dz = 6.111$.

11. We estimate $\int_0^{40} f(x)dx$ using left- and right-hand sums:

$$\text{Left sum} = 350\cdot 10 + 410\cdot 10 + 435\cdot 10 + 450\cdot 10 = 16{,}450.$$
$$\text{Right sum} = 410\cdot 10 + 435\cdot 10 + 450\cdot 10 + 460\cdot 10 = 17{,}550.$$

We estimate that

$$\int_0^{40} f(x)dx \approx \frac{16450+17550}{2} = 17{,}000.$$

In this estimate, we used $n=4$ and $\Delta x = 10$.

12. With $\Delta x = 3$, we have

$$\text{Left-hand sum } = 3(32+22+15+11) = 240,$$
$$\text{Right-hand sum } = 3(22+15+11+9) = 171.$$

The average of these two sums is our best guess for the value of the integral;

$$\int_0^{12} f(x)\,dx \approx \frac{240+171}{2} = 205.5.$$

13. We take $\Delta x = 3$. Then:

$$\begin{aligned}\text{Left-hand sum} &= 50(3)+48(3)+44(3)+36(3)+24(3)\\ &= 606\\ \text{Right-hand sum} &= 48(3)+44(3)+36(3)+24(3)+8(3)\\ &= 480\\ \text{Average} &= \frac{606+480}{2} = 543.\end{aligned}$$

So,

$$\int_0^{15} f(x)\,dx \approx 543.$$

14. Since we have 5 subdivisions,

$$\Delta x = \frac{b-a}{n} = \frac{7-3}{5} = 0.8.$$

The interval begins at $x=3$ and ends at $x=7$. Table 5.1 gives the value of $f(x)$ at the pertinent points.

Table 5.1

x	3.0	3.8	4.6	5.4	6.2	7.0
$f(x)$	$\frac{1}{1+3.0}$	$\frac{1}{1+3.8}$	$\frac{1}{1+4.6}$	$\frac{1}{1+5.4}$	$\frac{1}{1+6.2}$	$\frac{1}{1+7.0}$

So a right-hand sum is

$$\frac{1}{1+3.8}(0.8) + \frac{1}{1+4.6}(0.8) + \cdots + \frac{1}{1+7.0}(0.8).$$

15. $\int_0^{20} f(x)\,dx$ is equal to the area shaded in Figure 5.12. We estimate the area by counting boxes. There are about 15 boxes and each box represents 4 square units, so the area shaded is about 60. We have

$$\int_0^{20} f(x)\,dx \approx 60.$$

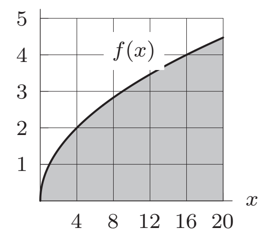


Figure 5.12

16. We know that

$$\int_{-10}^{15} f(x)dx = \text{Area under } f(x) \text{ between } x = -10 \text{ and } x = 15.$$

The area under the curve consists of approximately 14 boxes, and each box has area $(5)(5) = 25$. Thus, the area under the curve is about $(14)(25) = 350$, so

$$\int_{-10}^{15} f(x)dx \approx 350.$$

17. We know that

$$\int_{-3}^{5} f(x)dx = \text{Area above the axis} - \text{Area below the axis.}$$

The area above the axis is about 3 boxes. Since each box has area $(1)(5) = 5$, the area above the axis is about $(3)(5) = 15$. The area below the axis is about 11 boxes, giving an area of about $(11)(5) = 55$. We have

$$\int_{-3}^{5} f(x)dx \approx 15 - 55 = -40.$$

Problems

18. The graph given shows that f is positive for $0 \le t \le 1$. Since the graph is contained within a rectangle of height 100 and length 1, the answers -98.35 and 100.12 are both either too small or too large to represent $\int_0^1 f(t)dt$. Since the graph of f is above the horizontal line $y = 80$ for $0 \le t \le 0.95$, the best estimate is IV, 93.47 and not 71.84.

19. (a) The total area between $f(x)$ and the x-axis is the sum of the two given areas, so

$$\text{Area} = 7 + 6 = 13.$$

(b) To find the integral, we note that from $x = 3$ to $x = 5$, the function lies below the x-axis, and hence makes a negative contribution to the integral. So

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx = 7 - 6 = 1.$$

20. The graph of $y = 4 - x^2$ crosses the x-axis at $x = 2$ since solving $y = 4 - x^2 = 0$ gives $x = \pm 2$. See Figure 5.13. To find the total area, we find the area above the axis and the area below the axis separately. We have

$$\int_0^2 (4 - x^2)dx = 5.3333 \quad \text{and} \quad \int_2^3 (4 - x^2)dx = -2.3333.$$

As expected, the integral from 2 to 3 is negative. The area above the axis is 5.3333 and the area below the axis is 2.3333, so

$$\text{Total area } = 5.3333 + 2.3333 = 7.6666.$$

Thus the area is 7.667.

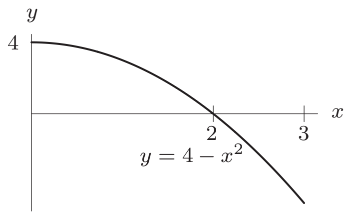


Figure 5.13

21. (a) The graph of $f(x) = x^3 - x$ crosses the x-axis at $x = 1$ since solving $f(x) = x^3 - x = 0$ gives $x = 0$ and $x = \pm 1$. See Figure 5.14. To find the total area, we find the area above the axis and the area below the axis separately. We have

$$\int_0^1 (x^3 - x)dx = -0.25 \quad \text{and} \quad \int_1^3 (x^3 - x)dx = 16.$$

As expected, the integral from 0 to 1 is negative. The area above the axis is 16 and the area below the axis is 0.25 so

$$\text{Total area } = 16.25.$$

(b) We have

$$\int_0^3 (x^3 - x)dx = 15.75.$$

Notice that the integral is equal to the area above the axis minus the area below the axis, as expected.

(c) No, they are not the same, since the integral counts area below the axis negatively while total area counts all area as positive. The two answers are not expected to be the same unless all the area lies above the horizontal axis.

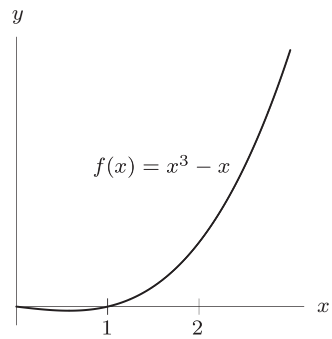


Figure 5.14

22. A graph of $y = 6x^3 - 2$ shows that this function is nonnegative on the interval $x = 5$ to $x = 10$. Thus,

$$\text{Area } = \int_5^{10} (6x^3 - 2)\, dx = 14{,}052.5.$$

The integral was evaluated on a calculator.

23. Since $\cos t \geq 0$ for $0 \leq t \leq \pi/2$, the area is given by

$$\text{Area} = \int_0^{\pi/2} \cos t\, dt = 1.$$

The integral was evaluated on a calculator.

24. A graph of $y = \ln x$ shows that this function is non-negative on the interval $x = 1$ to $x = 4$. Thus,

$$\text{Area} = \int_1^4 \ln x\, dx = 2.545.$$

The integral was evaluated on a calculator.

25. A graph of $y = 2\cos(t/10)$ shows that this function is nonnegative on the interval $t = 1$ to $t = 2$. Thus,

$$\text{Area} = \int_1^2 2\cos\frac{t}{10}\, dt = 1.977.$$

The integral was evaluated on a calculator.

26. Since $\cos\sqrt{x} > 0$ for $0 \leq x \leq 2$, the area is given by

$$\text{Area} = \int_0^2 \cos\sqrt{x}\, dx = 1.106.$$

The integral was evaluated on a calculator.

27. The graph of $y = 7 - x^2$ has intercepts $x = \pm\sqrt{7}$. See Figure 5.15. Therefore we have

$$\text{Area} = \int_{-\sqrt{7}}^{\sqrt{7}} (7 - x^2)\, dx = 24.694.$$

The integral was evaluated on a calculator.

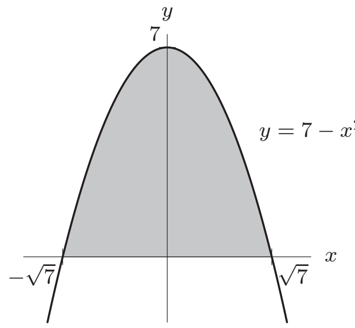


Figure 5.15

28. The graph of $y = x^4 - 8$ has intercepts $x = \pm\sqrt[4]{8}$. See Figure 5.16. Since the region is below the x-axis, the integral is negative, so

$$\text{Area} = -\int_{-\sqrt[4]{8}}^{\sqrt[4]{8}} (x^4 - 8)\, dx = 21.527.$$

The integral was evaluated on a calculator.

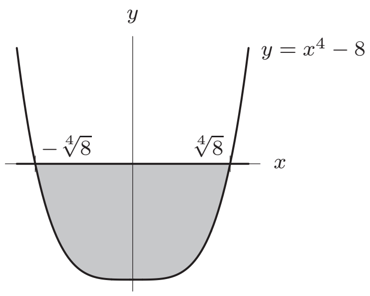


Figure 5.16

29. (a) The area between the graph of f and the x-axis between $x = a$ and $x = b$ is 13, so

$$\int_a^b f(x)\,dx = 13.$$

(b) Since the graph of $f(x)$ is below the x-axis for $b < x < c$,

$$\int_b^c f(x)\,dx = -2.$$

(c) Since the graph of $f(x)$ is above the x-axis for $a < x < b$ and below for $b < x < c$,

$$\int_a^c f(x)\,dx = 13 - 2 = 11.$$

(d) The graph of $|f(x)|$ is the same as the graph of $f(x)$ except that the part below the x-axis is reflected to be above it. See Figure 5.17. Thus

$$\int_a^c |f(x)|\,dx = 13 + 2 = 15.$$

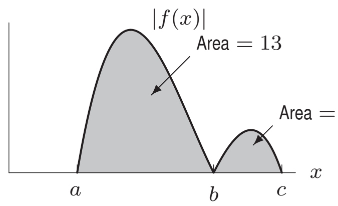


Figure 5.17

30. The region shaded between $x = 0$ and $x = 2$ appears to have approximately the same area as the region shaded between $x = -2$ and $x = 0$, but it lies below the axis. Since $\int_{-2}^{0} f(x)dx = 4$, we have the following results:

(a) $\int_0^2 f(x)dx \approx -\int_{-2}^{0} f(x)dx = -4.$
(b) $\int_{-2}^{2} f(x)dx \approx 4 - 4 = 0.$
(c) The total area shaded is approximately $4 + 4 = 8$.

31. (a) $\displaystyle\int_{-3}^{0} f(x)\,dx = -2.$

(b) $\displaystyle\int_{-3}^{4} f(x)\,dx = \int_{-3}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{4} f(x)\,dx = -2 + 2 - \frac{A}{2} = -\frac{A}{2}.$

32. We can compute each integral in this problem by finding the difference between the area that lies above the x-axis and the area that lies below the x-axis on the given interval.

(a) For $\int_0^2 f(x)\,dx$, on $0 \le x \le 1$ the area under the graph is 1; on $1 \le x \le 2$ the areas above and below the x-axis are equal and cancel each other out. Therefore, $\int_0^2 f(x)\,dx = 1$.

(b) On $3 \le x \le 7$ the graph of $f(x)$ is the upper half circle of radius 2 centered at $(5, 0)$. The integral is equal to the area between the graph of $f(x)$ and the x-axis, which is the area of a semicircle of radius 2. This area is 2π, and so

$$\int_3^7 f(x)\,dx = \frac{\pi 2^2}{2} = 2\pi.$$

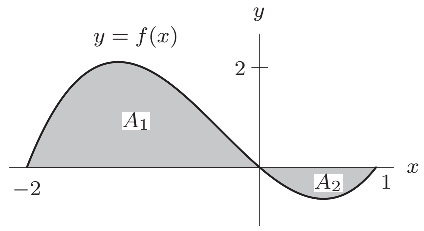
(c) On $2 \le x \le 7$ we are looking at two areas: We already know that the area of the semicircle on $3 \le x \le 7$ is 2π. On $2 \le x \le 3$, the graph lies below the x-axis and the area of the triangle is $\frac{1}{2}$. Therefore,

$$\int_2^7 f(x)\,dx = -\frac{1}{2} + 2\pi.$$

(d) For this portion of the problem, we can split the region between the graph and the x-axis into a quarter circle on $5 \le x \le 7$ and a trapezoid on $7 \le x \le 8$ below the x-axis. The semicircle has area π, the trapezoid has area $3/2$. Therefore,

$$\int_5^8 f(x)\,dx = \pi - \frac{3}{2}.$$

33. (a)



(b) $A_1 = \int_{-2}^{0} f(x)\,dx = 2.667.$

$A_2 = -\int_{0}^{1} f(x)\,dx = 0.417.$

So total area $= A_1 + A_2 \approx 3.084$. Note that while A_1 and A_2 are accurate to 3 decimal places, the quoted value for $A_1 + A_2$ is accurate only to 2 decimal places.

(c) $\int_{-2}^{1} f(x)\,dx = A_1 - A_2 = 2.250.$

34. $\int_0^4 \cos\sqrt{x}\,dx = 0.80 =$ Area A_1 – Area A_2. See Figure 5.18.

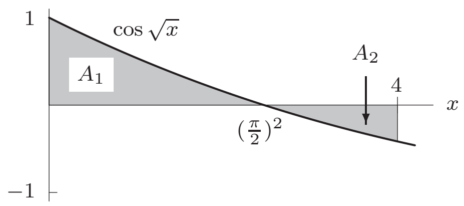


Figure 5.18

35. Looking at the graph of $e^{-x}\sin x$ for $0 \le x \le 2\pi$ in Figure 5.19, we see that the area, A_1, below the curve for $0 \le x \le \pi$ is much greater than the area, A_2, above the curve for $\pi \le x \le 2\pi$. Thus, the integral is

$$\int_0^{2\pi} e^{-x}\sin x\,dx = A_1 - A_2 > 0.$$

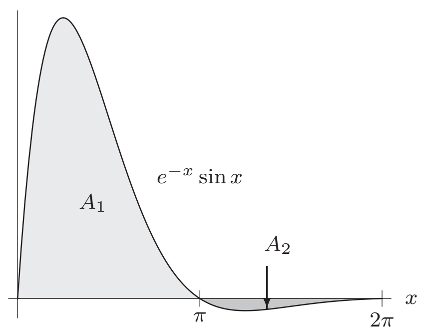


Figure 5.19

36. Since e^{-x^2} is decreasing between $x = 0$ and $x = 1$, the left sum is an overestimate and the right sum is an underestimate of the integral. Letting $f(x) = e^{-x^2}$, we divide the interval $0 \le x \le 1$ into $n = 5$ sub-intervals to create Table 5.2.

Table 5.2

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.000	0.961	0.852	0.698	0.527	0.368

(a) Letting $\Delta x = (1-0)/5 = 0.2$, we have:

$$\begin{aligned}\text{Left-hand sum} &= f(0)\Delta x + f(0.2)\Delta x + f(0.4)\Delta x + f(0.6)\Delta x + f(0.8)\Delta x \\ &= 1(0.2) + 0.961(0.2) + 0.852(0.2) + 0.698(0.2) + 0.527(0.2) \\ &= 0.808.\end{aligned}$$

(b) Again letting $\Delta x = (1-0)/5 = 0.2$, we have:

$$\begin{aligned}\text{Right-hand sum} &= f(0.2)\Delta x + f(0.4)\Delta x + f(0.6)\Delta x + f(0.8)\Delta x + f(1)\Delta x \\ &= 0.961(0.2) + 0.852(0.2) + 0.698(0.2) + 0.527(0.2) + 0.368(0.2) \\ &= 0.681.\end{aligned}$$

37. (a) See Figure 5.20.

$$\text{Left sum} = f(1)\Delta x + f(1.5)\Delta x = (\ln 1)0.5 + \ln(1.5)0.5 = (\ln 1.5)0.5.$$

(b) See Figure 5.21.

$$\text{Right sum} = f(1.5)\Delta x + f(2)\Delta x = (\ln 1.5)0.5 + (\ln 2)0.5.$$

(c) Right sum is an overestimate, left sum is an underestimate.

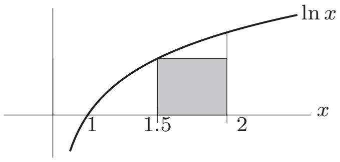


Figure 5.20: Left sum

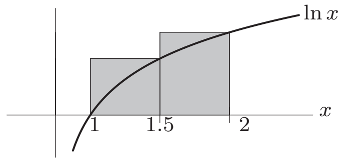


Figure 5.21: Right sum

38. (a) See Figure 5.22.
(b) See Figure 5.23.
(c) Since the rectangle above the curve in Figure 5.22 and the rectangle below the curve in Figure 5.23 look approximately equal in area (in fact, they are exactly equal), the left hand approximation to $\int_{-\pi}^{\pi} \sin x\, dx$ is 0.

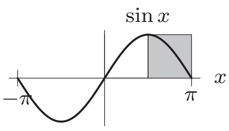


Figure 5.22

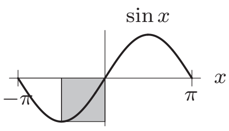


Figure 5.23

39. (a) The integral gives the shaded area in Figure 5.24. We find

$$\int_0^6 (x^2+1)\, dx = 78.$$

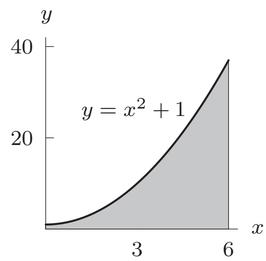


Figure 5.24

(b) Using $n = 3$, we have

$$\text{Left-hand sum} = f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 = 1\cdot 2 + 5\cdot 2 + 17\cdot 2 = 46.$$

This sum is an underestimate. See Figure 5.25.

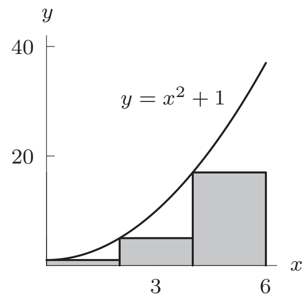


Figure 5.25

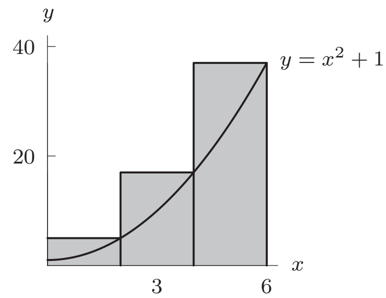


Figure 5.26

(c) Using $n = 3$ gives

$$\text{Right-hand sum} = f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 2 = 5\cdot 2 + 17\cdot 2 + 37\cdot 2 = 118.$$

This sum is an overestimate. See Figure 5.26.

40. (a) See Figure 5.27.

(b) Since each of the triangular regions in Figure 5.27 have area $1/2$, we have

$$\int_0^2 f(x)\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) Using $\Delta x = 1/2$ in the 4-term Riemann sum shown in Figure 5.28, we have

$$\begin{aligned}\text{Left hand sum } &= f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x\\ &= 1\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = 1.\end{aligned}$$

We notice that in this case the approximation is exactly equal to the exact value of the integral.

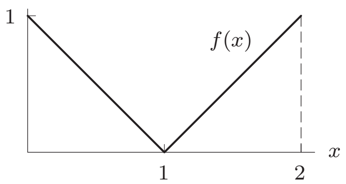


Figure 5.27

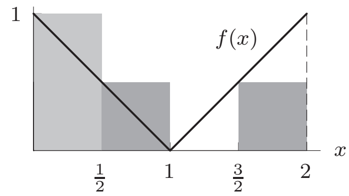


Figure 5.28

41. Left-hand sum gives: $1^2(1/4) + (1.25)^2(1/4) + (1.5)^2(1/4) + (1.75)^2(1/4) = 1.96875$.
Right-hand sum gives: $(1.25)^2(1/4) + (1.5)^2(1/4) + (1.75)^2(1/4) + (2)^2(1/4) = 2.71875$.

We estimate the value of the integral by taking the average of these two sums, which is 2.34375. Since x^2 is monotonic on $1 \le x \le 2$, the true value of the integral lies between 1.96875 and 2.71875. Thus the most our estimate could be off is 0.375. We expect it to be much closer. (And it is—the true value of the integral is $7/3 \approx 2.333$.)

42. We have $\Delta x = 2/500 = 1/250$. The formulas for the left- and right-hand Riemann sums give us that

$$\begin{aligned}\text{Left} &= \Delta x[f(-1) + f(-1+\Delta x) + \ldots + f(1-2\Delta x) + f(1-\Delta x)]\\ \text{Right} &= \Delta x[f(-1+\Delta x) + f(-1+2\Delta x) + \ldots + f(1-\Delta x) + f(1)].\end{aligned}$$

Subtracting these yields

$$\text{Right} - \text{Left} = \Delta x[f(1) - f(-1)] = \frac{1}{250}[6-2] = \frac{4}{250} = \frac{2}{125}.$$

43. See Figure 5.29.

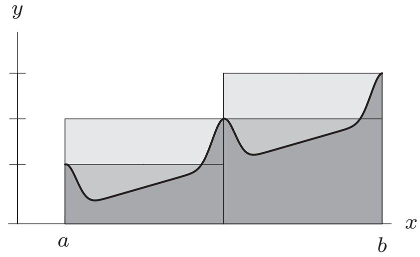


Figure 5.29: Integral vs. Left- and Right-Hand Sums

44. The statement is rarely true. The graph of almost any non-linear monotonic function, such as x^{10} for $0 < x < 1$, should provide convincing geometric evidence. Furthermore, if the statement were true, then (LHS+RHS)/2 would always give the exact value of the definite integral. This is not true.

45. (a) If the interval $1 \le t \le 2$ is divided into n equal subintervals of length $\Delta t = 1/n$, the subintervals are given by

$$1 \le t \le 1 + \frac{1}{n},\ 1 + \frac{1}{n} \le t \le 1 + \frac{2}{n},\ \ldots,\ 1 + \frac{n-1}{n} \le t \le 2.$$

The left-hand sum is given by

$$\text{Left sum } = \sum_{r=0}^{n-1} f\left(1 + \frac{r}{n}\right)\frac{1}{n} = \sum_{r=0}^{n-1} \frac{1}{1+r/n} \cdot \frac{1}{n} = \sum_{r=0}^{n-1} \frac{1}{n+r}$$

and the right-hand sum is given by

$$\text{Right sum } = \sum_{r=1}^{n} f\left(1 + \frac{r}{n}\right)\frac{1}{n} = \sum_{r=1}^{n} \frac{1}{n+r}.$$

Since $f(t) = 1/t$ is decreasing in the interval $1 \le t \le 2$, we know that the right-hand sum is less than $\int_1^2 1/t\,dt$ and the left-hand sum is larger than this integral. Thus we have

$$\sum_{r=1}^{n} \frac{1}{n+r} \quad < \quad \int_1^t \frac{1}{t}\,dt \quad < \quad \sum_{r=0}^{n-1} \frac{1}{n+r}.$$

(b) Subtracting the sums gives

$$\sum_{r=0}^{n-1} \frac{1}{n+r} - \sum_{r=1}^{n} \frac{1}{n+r} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

(c) Here we need to find n such that

$$\frac{1}{2n} \le 5 \times 10^{-6}, \quad \text{so} \quad n \ge \frac{1}{10} \times 10^{6} = 10^{5}.$$

Strengthen Your Understanding

46. This does not say that $f(x) \ge 0$. If $f(x) < 0$ on $[1, 3]$ then the integral is negative, but an area cannot be negative.

47. There are too many terms in the sum; last term should be $\sin(1.9)$. So the correct left-hand sum is

$$0.1\left(\sin(1) + \sin(1.1) + \cdots + \sin(1.9)\right).$$

48. Any function which is negative on the whole interval will do, for example $f(x) = -1$ and $[a, b] = [0, 1]$. There are also examples like $f(x) = -1 + x$ with $0 \le x \le 1.1$.

49. This is true if the function is negative on the interval $[2, 3]$; for example $f(x) = 2 - x$.

50. False. The integral is the change in position from $t = a$ to $t = b$. If the velocity changes sign in the interval, the total distance traveled and the change in position will not be the same.

51. False. A counterexample is given by $f(x) = 1$ on $[0, 2]$.Then

$$\int_0^2 f(x)\,dx = 2.$$

On the other hand, using $\Delta x = 1/2$ in the 4-term Riemann sum we have

$$\begin{aligned}\text{Left hand sum} &= f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x \\ &= 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 2.\end{aligned}$$

52. False. A counterexample is given by the functions f and g in Figure 5.30. The function f is decreasing, g is increasing, and we have

$$\int_1^2 f(x)\,dx = \int_1^2 g(x)\,dx,$$

because both integrals equal $1/2$, the area of of the same sized triangle.

Figure 5.30

53. False. Any function $f(x)$ that is negative between $x = 2$ and $x = 3$ has $\int_2^3 f(x)\,dx < 0$, so $\int_0^2 f(x)\,dx > \int_0^3 f(x)\,dx$.

54. True. Since $\int_0^2 f(x)\,dx$ is a number, if we use the variable t instead of the variable x in the function f, we get the same number for the definite integral.

55. False. Let $f(x) = x$ and $g(x) = 5$. Then $\int_2^6 f(x)\,dx = 16$ and $\int_2^6 g(x)\,dx = 20$, so $\int_2^6 f(x)\,dx \le \int_2^6 g(x)\,dx$, but $f(x) > g(x)$ for $5 < x < 6$.

56. An example is graphed in Figure 5.31.

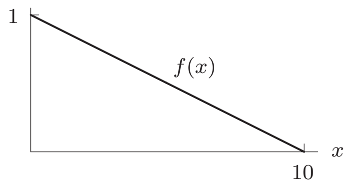


Figure 5.31

57. An example is graphed in Figure 5.32.

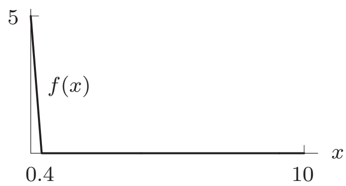


Figure 5.32

Solutions for Section 5.3

Exercises

1. The units of measurement are dollars.

2. The units of measurement are meters per second (which are units of velocity).

3. The units of measurement are foot-pounds (which are units of work).

4. The integral $\int_1^3 v(t)\,dt$ represents the change in position between time $t = 1$ and $t = 3$ seconds; it is measured in meters.

5. The integral $\int_0^6 a(t)\,dt$ represents the change in velocity between times $t = 0$ and $t = 6$ seconds; it is measured in km/hr.

6. The integral $\int_{2005}^{2011} f(t)\,dt$ represents the change in the world's population between the years 2005 and 2011. It is measured in billions of people.

7. The integral $\int_0^5 s(x)\,dx$ represents the change in salinity (salt concentration) in the first 5 cm of water; it is measured in gm/liter.

8. (a) One small box on the graph corresponds to moving at 750 ft/min for 15 seconds, which corresponds to a distance of 187.5 ft. Estimating the area beneath the velocity curves, we find:
Distance traveled by car 1 ≈ 5.5 boxes $= 1031.25$ ft.
Distance traveled by car 2 ≈ 3 boxes $= 562.5$ ft.

(b) The two cars will have gone the same distance when the areas beneath their velocity curves are equal. Since the two areas overlap, they are equal when the two shaded regions have equal areas, at $t \approx 1.6$ minutes. See Figure 5.33.

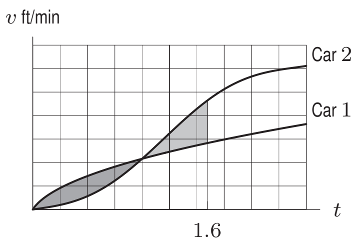


Figure 5.33

9. We have $f(t) = F'(t) = 2t$, so by the Fundamental Theorem of Calculus,

$$\int_1^3 2t\,dt = F(3) - F(1) = 9 - 1 = 8.$$

10. We have $f(t) = F'(t) = 6t + 4$, so by the Fundamental Theorem of Calculus,

$$\int_2^5 (6t+4)\,dt = F(5) - F(2) = 95 - 20 = 75.$$

11. We have $f(t) = F'(t) = 1/t$, so by the Fundamental Theorem of Calculus,

$$\int_1^5 \frac{1}{t}\,dt = F(5) - F(1) = \ln 5 - \ln 1 = \ln 5.$$

12. We have $f(t) = F'(t) = \cos t$, so by the Fundamental Theorem of Calculus,

$$\int_0^{\pi/2} \cos t\,dt = F(\pi/2) - F(0) = 1 - 0 = 1.$$

13. We have $f(t) = F'(t) = 7\ln(4)\cdot 4^t$, so by the Fundamental Theorem of Calculus,

$$\int_2^3 7\ln(4)\cdot 4^t\,dt = F(3) - F(2) = 448 - 112 = 336.$$

14. We have $f(t) = F'(t) = 1/(\cos^2 t)$, so by the Fundamental Theorem of Calculus,

$$\int_0^\pi 1/(\cos^2 t)\,dt = F(\pi) - F(0) = 0 - 0 = 0.$$

Problems

15. (a) We have

$$\frac{d}{dx}(x^3 + x) = 3x^2 + 1.$$

(b) By the Fundamental Theorem of Calculus with $F'(x) = 3x^2 + 1$ and $F(x) = x^3 + x$, we have

$$\int_0^2 (3x^2 + 1)\,dx = F(2) - F(0) = (2^3 + 2) - (0^3 + 0) = 10.$$

16. (a) We have

$$\frac{d}{dt}(\sin t) = \cos t.$$

(b) Since $v(t)$ is the derivative of distance traveled, by the Fundamental Theorem of Calculus with $F'(t) = v(t) = \cos t$ and $F(t) = \sin t$, we have

$$\text{Distance} = \int_0^{\pi/2} v(t)\,dt = \int_0^{\pi/2} \cos t\,dt = F\left(\frac{\pi}{2}\right) - F(0) = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1.$$

17. (a) By the chain rule,

$$\frac{d}{dx}\left(\frac{1}{2}\sin^2 t\right) = \frac{1}{2}\cdot 2\sin t\cos t = \sin t\cos t.$$

(i) Using a calculator, $\int_{0.2}^{0.4} \sin t\cos t\,dt = 0.056$

(ii) The Fundamental Theorem of Calculus tells us that the integral is

$$\int_{0.2}^{0.4} \sin t\cos t\,dt = F(0.4) - F(0.2) = \frac{1}{2}\left(\sin^2(0.4) - \sin^2(0.2)\right) = 0.05609.$$

18. (a) By the chain rule,

$$F'(x) = \frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}.$$

(i) Using a calculator, $\int_0^1 2xe^{x^2}\,dx = 1.718.$

(ii) The Fundamental Theorem of Calculus says we can get the exact value of the integral by looking at

$$\int_0^1 2xe^{x^2}\,dx = F(1) - F(0) = e^{1^2} - e^{0^2} = e - 1 = 1.71828.$$

19. (a) On day 12 pollution is removed from the lake at a rate of 500 kg/day.

(b) The limits of the integral are $t = 5$ and $t = 15$. Since t is time in days, the units of the 5 and 15 are days. The units of the integral are obtained by multiplying the units of $f(t)$, kg/day, by the units of dt, day. Thus the units of the integral are

$$\frac{\text{kg}}{\text{day}} \times \text{day} = \text{kg}.$$

The 4000 has units of kilograms.

(c) The integral of a rate gives the total change. Here $f(t)$ is the rate of change of the quantity of pollution that has been removed from the lake. The integral gives the change in the quantity of pollution that has been removed during the time interval; in other words, the total quantity removed during that time period. During the 10 days from day 5 to day 15, a total of 4000 kg were removed from the lake.

20. For any t, consider the interval $[t, t+\Delta t]$. During this interval, oil is leaking out at an approximately constant rate of $f(t)$ gallons/minute. Thus, the amount of oil which has leaked out during this interval can be expressed as

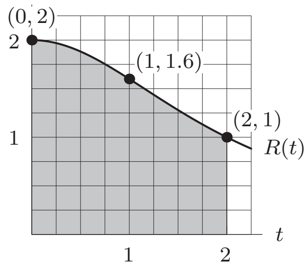
$$\text{Amount of oil leaked} = \text{Rate} \times \text{Time} = f(t)\,\Delta t$$

and the units of $f(t)\,\Delta t$ are gallons/minute $\times$ minutes $=$ gallons. The total amount of oil leaked is obtained by adding all these amounts between $t = 0$ and $t = 60$. (An hour is 60 minutes.) The sum of all these infinitesimal amounts is the integral

$$\begin{matrix}\text{Total amount of} \\ \text{oil leaked, in gallons}\end{matrix} = \int_0^{60} f(t)\,dt.$$

21. (a) The amount leaked between $t = 0$ and $t = 2$ is $\int_0^2 R(t)\,dt$.

(b)



(c) The rectangular boxes on the diagram each have area $\frac{1}{16}$. Of these 45 are wholly beneath the curve, hence the area under the curve is certainly more than $\frac{45}{16} > 2.81$. There are 9 more partially beneath the curve, and so the desired area is completely covered by 54 boxes. Therefore the area is less than $\frac{54}{16} < 3.38$.

These are very safe estimates but far apart. We can do much better by estimating what fractions of the broken boxes are beneath the curve. Using this method, we can estimate the area to be about 3.2, which corresponds to 3.2 gallons leaking over two hours.

22. (a) Using rectangles under the curve, we get

$$\text{Acres defaced } \approx (1)(0.2 + 0.4 + 1 + 2) = 3.6 \text{ acres.}$$

(b) Using rectangles above the curve, we get

$$\text{Acres defaced } \approx (1)(0.4 + 1 + 2 + 3.5) = 6.9 \text{ acres.}$$

(c) The number of acres defaced is between 3.6 and 6.9, so we estimate the average, 5.25 acres.

23. (a) Quantity used $= \int_0^5 f(t)\,dt$.

(b) Using a left sum, our approximation is

$$32e^{0.05(0)} + 32e^{0.05(1)} + 32e^{0.05(2)} + 32e^{0.05(3)} + 32e^{0.05(4)} = 177.270.$$

Since f is an increasing function, this represents an underestimate.

(c) Each term is a lower estimate of one year's consumption of oil.

24. (a) Solving $v(t) = -4t^2 + 16t = 0$ gives $t = 0$ and $t = 4$. At $t = 0$ the jump has just started; at $t = 4$ the jumper is momentarily stopped before starting back up. In Figure 5.34, we see the velocity is positive from $t = 0$ to $t = 4$ and negative from $t = 4$ to $t = 5$. The total number of meters traveled is given by the total area between the velocity curve and the t-axis. We find the total area by finding the areas above and below the axis separately. We have

$$\int_0^4 (-4t^2 + 16t)\,dt = 42.6667 \quad \text{and} \quad \int_4^5 (-4t^2 + 16t)\,dt = -9.3333.$$

The jumper traveled 42.6667 meters downward and then 9.3333 meters upward, so

$$\text{Total number of meters traveled } = 42.6667 + 9.3333 = 52 \text{ meters.}$$

(b) Since the jumper fell 42.6667 meters downward and came back up 9.3333 meters,

$$\text{Ending position } = 42.6667 - 9.3333 = 33.333 \text{ meters below the starting point.}$$

(c) We find that

$$\int_0^5 (-4t^2 + 16t)\,dt = 33.333 \text{ meters}.$$

Thus, the net change in position is a positive 33.333 meters, or 33.333 meters below the starting point. The net change is given by the integral, while the total number of meters traveled is given by the total area.

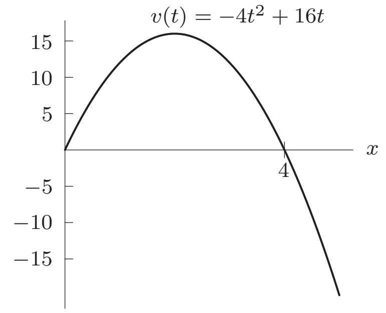


Figure 5.34

25. (a) Note that the rate $r(t)$ sometimes increases and sometimes decreases in the interval. We can calculate an upper estimate of the volume by choosing $\Delta t = 5$ and then choosing the highest value of $r(t)$ on each interval, and similarly a lower estimate by choosing the lowest value of $r(t)$ on each interval:

$$\text{Upper estimate} = 5[20 + 24 + 24] = 340 \text{ liters}.$$
$$\text{Lower estimate} = 5[12 + 20 + 16] = 240 \text{ liters}.$$

(b) A graph of $r(t)$ along with the areas represented by the choices of $r(t)$ in calculating the lower estimate is shown in Figure 5.35.

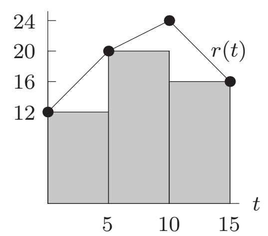


Figure 5.35

26. We use left- and right-hand sums to estimate the total amount of coal produced during this period:

$$\text{Left sum} = (1.090)2 + (1.094)2 + (1.121)2 + (1.072)2 + (1.132)2 + (1.147)2 = 13.312.$$

$$\text{Right sum} = (1.094)2 + (1.121)2 + (1.072)2 + (1.132)2 + (1.147)2 + (1.073)2 = 13.278.$$

We see that

$$\text{Total amount of coal produced} \approx \frac{13.312 + 13.278}{2} = 13.295 \text{ billion tons}.$$

The total amount of coal produced is the definite integral of the rate of coal production $r = f(t)$ given in the table. Since t is in years since 1997, the limits of integration are $t = 0$ and $t = 12$. We have

$$\text{Total amount of coal produced} = \int_0^{12} f(t)\,dt \text{ billion tons}.$$

27. Since W is in tons per week and t is in weeks since January 1, 2005, the integral $\int_0^{52} W\,dt$ gives the amount of waste, in tons, produced during the year 2005.

$$\text{Total waste during the year} = \int_0^{52} 3.75e^{-0.008t}dt = 159.5249 \text{ tons.}$$

Since waste removal costs $\$15/\text{ton}$, the cost of waste removal for the company is $159.5249 \cdot 15 = \$2392.87$.

28. The total number of "worker-hours" is equal to the area under the curve. The total area is about 14.5 boxes. Since each box represents (10 workers)(8 hours) $= 80$ worker-hours, the total area is 1160 worker-hours. At \$10 per hour, the total cost is \$11,600.

29. The time period 9am to 5pm is represented by the time $t = 0$ to $t = 8$ and $t = 24$ to $t = 32$. The area under the curve, or total number of worker-hours for these times, is about 9 boxes or $9(80) = 720$ worker-hours. The total cost for 9am to 5pm is $(720)(10) = \$7200$. The area under the rest of the curve is about 5.5 boxes, or $5.5(80) = 440$ worker-hours. The total cost for this time period is $(440)(15) = \$6600$. The total cost is about $7200 + 6600 = \$13{,}800$.

30. The area under the curve represents the number of cubic feet of storage times the number of days the storage was used. This area is given by

$$\begin{aligned}\text{Area under graph} &= \text{Area of rectangle} + \text{Area of triangle}\\ &= 30 \cdot 10{,}000 + \frac{1}{2} \cdot 30(30{,}000 - 10{,}000)\\ &= 600{,}000.\end{aligned}$$

Since the warehouse charges \$5 for every 10 cubic feet of storage used for a day, the company will have to pay $(5)(60{,}000) = \$300{,}000$.

31. If $H(t)$ is the temperature of the coffee at time t, by the Fundamental Theorem of Calculus

$$\text{Change in temperature} = H(10) - H(0) = \int_0^{10} H'(t)\,dt = \int_0^{10} -7e^{-0.1t}\,dt.$$

Therefore,

$$H(10) = H(0) + \int_0^{10} -7(0.9^t)\,dt \approx 90 - 44.2 = 45.8^\circ\text{C}.$$

32. The change in the amount of water is the integral of rate of change, so we have

$$\text{Number of liters pumped out} = \int_0^{60} (5 - 5e^{-0.12t})dt = 258.4 \text{ liters.}$$

Since the tank contained 1000 liters of water initially, we see that

$$\text{Amount in tank after one hour} = 1000 - 258.4 = 741.6 \text{ liters.}$$

33. By the Fundamental Theorem,

$$f(1) - f(0) = \int_0^1 f'(x)\,dx,$$

Since $f'(x)$ is negative for $0 \le x \le 1$, this integral must be negative and so $f(1) < f(0)$.

34. First rewrite each of the quantities in terms of f', since we have the graph of f'. If A_1 and A_2 are the positive areas shown in Figure 5.36:

$$\begin{aligned}f(3) - f(2) &= \int_2^3 f'(t)\,dt = -A_1\\ f(4) - f(3) &= \int_3^4 f'(t)\,dt = -A_2\\ \frac{f(4) - f(2)}{2} &= \frac{1}{2}\int_2^4 f'(t)\,dt = -\frac{A_1 + A_2}{2}\end{aligned}$$

Since Area $A_1 >$ Area A_2,

$$A_2 < \frac{A_1 + A_2}{2} < A_1$$

so

$$-A_1 < -\frac{A_1 + A_2}{2} < -A_2$$

and therefore

$$f(3) - f(2) < \frac{f(4) - f(2)}{2} < f(4) - f(3).$$

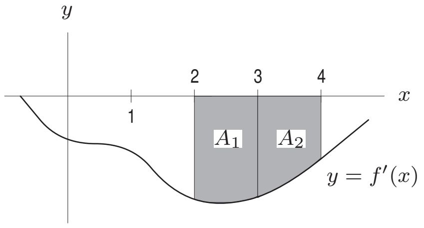


Figure 5.36

35. We know that the the integral of F, and therefore the work, can be obtained by computing the areas in Figure 5.37.

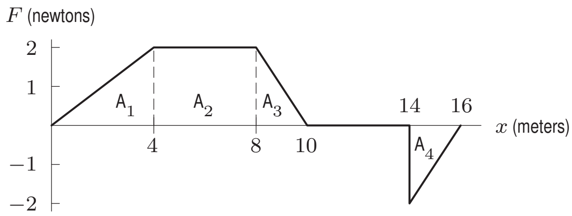


Figure 5.37

$$\begin{aligned} W = \int_0^{16} F(x)\,dx &= \text{Area above } x\text{-axis} - \text{Area below } x\text{-axis} \\ &= A_1 + A_2 + A_3 - A_4 \\ &= \frac{1}{2}\cdot 4\cdot 2 + 4\cdot 2 + \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 2\cdot 2 \\ &= 12 \text{ newton} \cdot \text{meters.} \end{aligned}$$

36. According to the Fundamental Theorem,

$$f(2) - \underbrace{f(1)}_{7} = \int_1^2 f'(t)\,dt$$

$$f(2) = 7 + \int_1^2 e^{-t^2}\,dt, \quad \text{since } f'(t) = e^{-t^2}.$$

We estimate the integral using left and right sums. Since $f'(t) = e^{-t^2}$ is decreasing between $t = 1$ and $t = 2$, the left sum overestimates and the right sum underestimates the integral.

To find left- and right-hand sums of 5 rectangles, we let $\Delta t = (2 - 1)/5 = 0.2$. The table gives values of $f'(t)$.

t	1.0	1.2	1.4	1.6	1.8	2.0
$f'(t)$	0.368	0.237	0.141	0.077	0.039	0.018

We have the following estimates:

$$LHS = \Delta t\left(f'(1.0) + f'(1.2) + f'(1.4) + f'(1.6) + f'(1.8)\right) = 0.2\,(0.862) = 0.1724$$
$$RHS = \Delta t\left(f'(1.2) + f'(1.4) + f'(1.6) + f'(1.8) + f'(2.0)\right) = 0.2(0.512) = 0.1024.$$

So

$$0.1024 < \int_1^2 e^{-t^2}\,dt < 0.1734.$$

Adding 7, we estimate that

$$7.1024 < f(2) < 7.1724.$$

37. All the integrals have positive values, since $f \geq 0$. The integral in (ii) is about one-half the integral in (i), due to the apparent symmetry of f. The integral in (iv) will be much larger than the integral in (i), since the two peaks of f^2 rise to 10,000. The integral in (iii) will be smaller than half of the integral in (i), since the peaks in $f^{1/2}$ will only rise to 10. So

$$\int_0^2 (f(x))^{1/2}\,dx < \int_0^1 f(x)\,dx < \int_0^2 f(x)\,dx < \int_0^2 (f(x))^2\,dx.$$

38. (a) V, since the slope is constant.
(b) IV, since the net area under this curve is the most negative.
(c) III, since the area under the curve is largest.
(d) II, since the steepest ascent at $t = 0$ occurs on this curve.
(e) III, since average velocity is (total distance)/5, and III moves the largest total distance.
(f) I, since average acceleration is $\frac{1}{5}\int_0^5 v'(t)\,dt = \frac{1}{5}(v(5) - v(0))$, and in I, the velocity increases the most from start ($t = 0$) to finish ($t = 5$).

39. To make this estimate, we first observe that, for constant speed,

$$\text{Fuel used } = \frac{\text{Miles traveled}}{\text{Miles/gallon}} = \frac{\text{Speed} \cdot \text{Time}}{\text{Miles/gallon}}.$$

We make estimates at the start and end of each of the six intervals given. The 60 mph increase in speed (from 10 mph to 70 mph) takes half an hour, or 30 minutes. Since the speed increases at a constant rate, each 10 mph increase takes 5 minutes.

At the start of the first 5 minutes, the speed is 10 mph and at the end, the speed is 20 mph. Since 5min $= 5/60$ hour, during the first 5 minutes the distance traveled is between

$$10 \cdot \frac{5}{60} \text{ mile} \qquad \text{and} \qquad 20 \cdot \frac{5}{60} \text{ mile.}$$

During this first five minute period, the speed is between 10 and 20 mph, so the fuel efficiency is between 15 mpg and 18 mpg. So the fuel used during this period is between

$$\frac{1}{18} \cdot 10 \cdot \frac{5}{60} \text{ gallons} \qquad \text{and} \qquad \frac{1}{15} \cdot 20 \cdot \frac{5}{60} \text{ gallons.}$$

Thus, an underestimate of the fuel used is

$$\text{Fuel } = \left(\frac{1}{18} \cdot 10 + \frac{1}{21} \cdot 20 + \frac{1}{23} \cdot 30 + \frac{1}{24} \cdot 40 + \frac{1}{25} \cdot 50 + \frac{1}{26} \cdot 60\right)\frac{5}{60} = 0.732 \text{ gallons.}$$

An overestimate of the fuel used is

$$\text{Fuel } = \left(\frac{1}{15} \cdot 20 + \frac{1}{18} \cdot 30 + \frac{1}{21} \cdot 40 + \frac{1}{23} \cdot 50 + \frac{1}{24} \cdot 60 + \frac{1}{25} \cdot 70\right)\frac{5}{60} = 1.032 \text{ gallons.}$$

We can get better bounds by using the actual distance traveled during each five minute period. For example, in the first five minutes the average speed is 15 mph (halfway between 10 and 20 mph because the speed is increasing at a constant rate). Since 5 minutes is 5/60 of an hour, in the first five minutes the car travels 15(5/60)=5/4 miles. Thus the fuel used during this period was between

$$\frac{5}{4} \cdot \frac{1}{18} \qquad \text{and} \qquad \frac{5}{4} \cdot \frac{1}{15}.$$

Using this method for each five minute period gives a lower estimate of 0.843 gallons and an upper estimate of 0.909 gallons.

40. The value of this integral tells us how much oil is pumped from the well between day $t = 0$ and day $t = t_0$.

41. The time period from t_0 to $2t_0$ is shorter (and contained within) the time period from 0 to $2t_0$. Thus, the amount of oil pumped out during the shorter time period, $\int_{t_0}^{2t_0} r(t)\,dt$, is less than the amount of oil pumped out in the longer timer period, $\int_0^{2t_0} r(t)\,dt$. This means

$$\int_{t_0}^{2t_0} r(t)\,dt < \int_0^{2t_0} r(t)\,dt.$$

The length of the time period from $2t_0$ to $3t_0$ is the same as the length from t_0 to $2t_0$: both are t_0 days. But the rate at which oil is pumped is going down, since $r'(t) < 0$. Thus, less oil is pumped out during the later time period, so

$$\int_{2t_0}^{3t_0} r(t)\,dt < \int_{t_0}^{2t_0} r(t)\,dt.$$

We conclude that $\int_{2t_0}^{3t_0} r(t)\,dt < \int_{t_0}^{2t_0} r(t)\,dt < \int_0^{2t_0} r(t)\,dt.$

42. (a) The black curve is for boys, the colored one for girls. The area under each curve represents the change in growth in centimeters. Since men are generally taller than women, the curve with the larger area under it is the height velocity of the boys.

(b) Each square below the height velocity curve has area 1 cm/yr · 1 yr = 1 cm. Counting squares lying below the black curve gives about 43 cm. Thus, on average, boys grow about 43 cm between ages 3 and 10.

(c) Counting squares lying below the black curve gives about 23 cm growth for boys during their growth spurt. Counting squares lying below the colored curve gives about 18 cm for girls during their growth spurt.

(d) We can measure the difference in growth by counting squares that lie between the two curves. Between ages 2 and 12.5, the average girl grows faster than the average boy. Counting squares yields about 5 cm between the colored and black curves for $2 \leq x \leq 12.5$. Counting squares between the curves for $12.5 \leq x \leq 18$ gives about 18 squares. Thus, there is a net increase of boys over girls by about $18 - 5 = 13$ cm.

43. Using a left-hand Riemann sum with $\Delta t = 0.1$, we have:

$$\begin{aligned}\int_0^{0.5} f(t)\,dt &\approx f(0)\Delta t + f(0.1)\Delta t + \cdots + f(0.4)\Delta t\\ &= 0.3(0.1) + 0.2(0.1) + 0.2(0.1) + 0.3(0.1) + 0.4(0.1) \quad \text{using values from the table}\\ &= 0.14.\end{aligned}$$

44. From the Fundamental Theorem of Calculus, we have:

$$\begin{aligned}\int_{0.2}^{0.5} g'(t)\,dt &= g(0.5) - g(0.2) \quad \text{since } g \text{ is an antiderivative of } g'\\ &= 0.8 - 5.1 \quad \text{using values from the table}\\ &= -4.3.\end{aligned}$$

45. Referring to the table, we see that:

$$\begin{aligned}g\left(f(0.0)\right) &= g(0.3) = 5.1\\ g\left(f(0.1)\right) &= g(0.2) = 5.1\\ g\left(f(0.2)\right) &= g(0.2) = 5.1\end{aligned}$$

Using a left-hand Riemann sum with $\Delta t = 0.1$, we have:

$$\begin{aligned}\int_0^{0.3} g\left(f(t)\right)\,dt &\approx g\left(f(0)\right)\Delta t + g\left(f(0.1)\right)\Delta t + g\left(f(0.2)\right)\Delta t.\\ &= 5.1(0.1) + 5.1(0.1) + 5.1(0.1)\\ &= 1.53.\end{aligned}$$

46. Since C is an antiderivative of c, we know by the Fundamental Theorem that

$$\int_{15}^{24} c(n)\,dn = C(24) - C(15) = 13 - 8 = 5.$$

Since $C(24)$ is the cost to plow a 24 inch snowfall, and $C(15)$ is the cost to plow a 15 inch snowfall, this tells us that it costs \$5 million more to plow a 24 inch snowfall than a 15 inch snowfall.

47. Since C is an antiderivative of c, we know from the Fundamental Theorem that

$$\underbrace{\int_{0}^{15} c(n)\,dn}_{7.5} = \underbrace{C(15)}_{8} - C(0)$$
$$\text{so}\quad C(0) = 8 - 7.5 = 0.5.$$

This tells us the cost of preparing for a storm, even if no snow falls, is \$0.5 million, or \$500,000.

48. By the Fundamental Theorem, we have

$$\begin{aligned} c(15) + \int_{15}^{24} c'(n)\,dn &= c(15) + c(24) - c(15) \\ &= c(24) = 0.4. \end{aligned}$$

Since $c(n) = C'(n)$, we have $C'(24) = 0.4$, which tells us the cost of plowing increases at the instantaneous rate of \$0.4 million/inch, or by \$400,000 per each additional inch, after 24 inches have already fallen.

49. The expression $\int_0^2 r(t)\,dt$ represents the amount of water that leaked from the ruptured pipe during the first two hours. Likewise, the expression $\int_2^4 r(t)\,dt$ represents the amount of water that leaked from the ruptured pipe during the next two hours. Since the leak "worsened throughout the afternoon," we know that r is an increasing function, which means that more water leaked out during the second two hours than during the first two hours. Therefore,

$$\int_0^2 r(t)\,dt < \int_2^4 r(t)\,dt.$$

50. The expression $\int_0^4 r(t)\,dt$ represents the amount of water that leaked from the ruptured pipe during the first four hours.

The expression $r(4)$ represents the rate water was leaking from the pipe at time $t = 4$, or four hours after the leak began. Had water leaked at this rate during the whole four-hour time period,

$$\begin{array}{c}\text{Total amount of water leaked} \\ \text{at a constant rate of } r(4)\end{array} = \underbrace{\text{Rate}}_{r(4)} \times \underbrace{\text{Time elapsed}}_{4} = 4r(4) \text{ gallons.}$$

According to the article, the leak "worsened throughout the afternoon," so the rate that water leaked out initially was less than $r(4)$. Thus, $4r(4)$ overestimates the total amount of water leaked during the first four hours.

We conclude that

$$\int_0^4 r(t)\,dt < 4r(4).$$

51. According to the article, the leak "worsened throughout the afternoon," eventually reaching (before being shut off) 8 million gallons/hour. Thus, $0 < r(t) < 8$ million gal/hr. The leak began around 10 am and ended after 6 pm. The expression $\int_0^8 r(t)\,dt$ represents the total amount of water leaked during the first 8 hours (or almost the total amount of water leaked altogether). Since $r(t) < 8$ million gal/hr,

$$\begin{array}{c}\text{Total amount of water leaked} \\ \text{during first 8 hours}\end{array} < \underbrace{\text{Rate (million gal/hr)}}_{8} \times \underbrace{\text{Time elapsed}}_{8} = 64 \text{ million gallons.}$$

Thus,

$$\int_0^8 r(t)\,dt < 64 \text{ million gallons.}$$

Strengthen Your Understanding

52. Since $f(t)$ represents the rate of change of the weight of the dog, $\int_0^4 f(t)dt$ represents the total change in the weight of the dog from the time the dog is born to its fourth birthday. This is not the same as the weight of the dog when it is four years old, because the dog has a nonzero weight when it is born.

53. In the Fundamental Theorem of Calculus, the integrand is the derivative of the function that is evaluated at the limits of the integral. Since $\sqrt{x}$ is not the derivative of $\sqrt{x}$, the statement is not correct. One way to write a correct statement is $\int_4^9 1/(2\sqrt{x})\,dx = \sqrt{9} - \sqrt{4}$.

54. Since $\frac{d}{dx}e^x = e^x$, $f(x) = e^x$ with $a = 2$ and $b = 4$ is one example.

55. If the car travels at a constant velocity of 50 miles per hour, it travels 200 miles in 4 hours as shown in Figure 5.38.

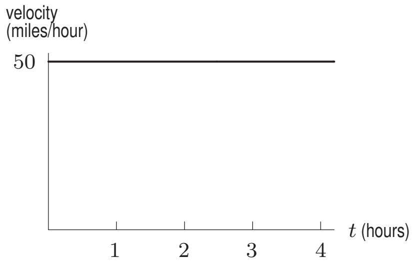


Figure 5.38

56. False. The units of the integral are the product of the units for $f(x)$ times the units for x.

Solutions for Section 5.4

Exercises

1. Note that $\int_a^b g(x)\,dx = \int_a^b g(t)\,dt$. Thus, we have

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = 8 + 2 = 10.$$

2. Note that $\int_a^b f(z)\,dz = \int_a^b f(x)\,dx$. Thus, we have

$$\int_a^b cf(z)\,dz = c\int_a^b f(z)\,dz = 8c.$$

3. Note that $\int_a^b (g(x))^2\,dx = \int_a^b (g(t))^2\,dt$. Thus, we have

$$\int_a^b \left((f(x))^2 - (g(x))^2\right)\,dx = \int_a^b (f(x))^2\,dx - \int_a^b (g(x))^2\,dx = 12 - 3 = 9.$$

4. We have

$$\int_a^b (f(x))^2\,dx - \left(\int_a^b f(x)\,dx\right)^2 = 12 - 8^2 = -52.$$

5. We write

$$\begin{aligned}\int_a^b \left(c_1g(x)+(c_2f(x))^2\right)\,dx &= \int_a^b \left(c_1g(x)+c_2^2(f(x))^2\right)\,dx\\ &= \int_a^b c_1g(x)\,dx+\int_a^b c_2^2(f(x))^2\,dx\\ &= c_1\int_a^b g(x)\,dx + c_2^2\int_a^b (f(x))^2\,dx\\ &= c_1(2)+c_2^2(12) = 2c_1+12c_2^2.\end{aligned}$$

6. The graph of $y = f(x-5)$ is the graph of $y = f(x)$ shifted to the right by 5. Since the limits of integration have also shifted by 5 (to $a+5$ and $b+5$), the areas corresponding to $\int_{a+5}^{b+5} f(x-5)\,dx$ and $\int_a^b f(x)\,dx$ are the same. Thus,

$$\int_{a+5}^{b+5} f(x-5)\,dx = \int_a^b f(x)\,dx = 8.$$

7. Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 (1+t)\,dt = \dfrac{1}{2}(4) = 2.$

8. Average value $= \dfrac{1}{10-0}\displaystyle\int_0^{10} e^t\,dt = \dfrac{1}{10}(22025) = 2202.5$

9. We have

$$\begin{aligned}\text{Average value} &= \frac{1}{b-a}\int_a^b f(x)\,dx\\ &= \frac{1}{b-a}\int_a^b 2\,dx = \frac{1}{b-a}\cdot\begin{pmatrix}\text{Area of rectangle}\\ \text{of height 2 and base } b-a\end{pmatrix}\\ &= \frac{1}{b-a}\left[2(b-a)\right] = 2.\end{aligned}$$

10. Sketch the graph of f on $1 \le x \le 3$. The integral is the area under the curve, which is a trapezoidal area. So the average value is

$$\frac{1}{3-1}\int_1^3 (4x+7)\,dx = \frac{1}{2}\cdot\frac{11+19}{2}\cdot 2 = \frac{30}{2} = 15.$$

11. The integral represents the area below the graph of $f(x)$ but above the x-axis.

(a) Since each square has area 1, by counting squares and half-squares we find

$$\int_1^6 f(x)\,dx = 8.5.$$

(b) The average value is $\dfrac{1}{6-1}\displaystyle\int_1^6 f(x)\,dx = \dfrac{8.5}{5} = \dfrac{17}{10} = 1.7.$

12. Since the average value is given by

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

the units for dx inside the integral are canceled by the units for $1/(b-a)$ outside the integral, leaving only the units for $f(x)$. This is as it should be, since the average value of f should be measured in the same units as $f(x)$.

13. The graph of $y = e^x$ is above the line $y = 1$ for $0 \le x \le 2$. See Figure 5.39. Therefore

$$\text{Area} = \int_0^2 (e^x - 1)\,dx = 4.389.$$

The integral was evaluated on a calculator.

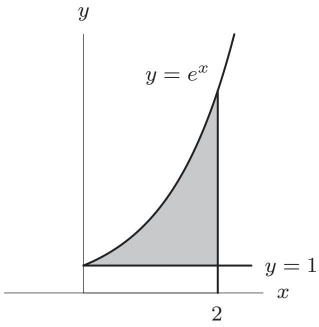


Figure 5.39

14. The graph of $y = 5\ln(2x)$ is above the line $y = 3$ for $3 \leq x \leq 5$. See Figure 5.40. Therefore

$$\text{Area } = \int_3^5 (5\ln(2x) - 3)\,dx = 14.688.$$

The integral was evaluated on a calculator.

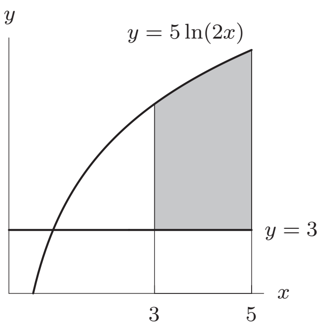


Figure 5.40

15. Since $x^3 \leq x^2$ for $0 \leq x \leq 1$, we have

$$\text{Area } = \int_0^1 (x^2 - x^3)\,dx = 0.083.$$

The integral was evaluated on a calculator.

16. Since $x^{1/2} \leq x^{1/3}$ for $0 \leq x \leq 1$, we have

$$\text{Area } = \int_0^1 (x^{1/3} - x^{1/2})\,dx = 0.0833.$$

The integral was evaluated on a calculator.

17. The graph of $y = \sin x + 2$ is above the line $y = 0.5$ for $6 \leq x \leq 10$. See Figure 5.41. Therefore

$$\text{Area } = \int_6^{10} \sin x + 2 - 0.5\,dx = 7.799.$$

The integral was evaluated on a calculator.

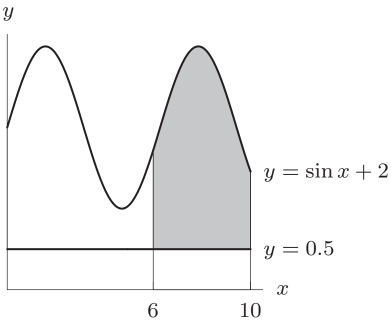


Figure 5.41

18. The graph of $y = \cos t$ is above the graph of $y = \sin t$ for $0 \le t \le \pi/4$ and $y = \cos t$ is below $y = \sin t$ for $\pi/4 < t < \pi$. See Figure 5.42. Therefore, we find the area in two pieces:

$$\text{Area } = \int_0^{\pi/4} (\cos t - \sin t)\,dt + \int_{\pi/4}^{\pi} (\sin t - \cos t)\,dt = 2.828.$$

The integral was evaluated on a calculator.

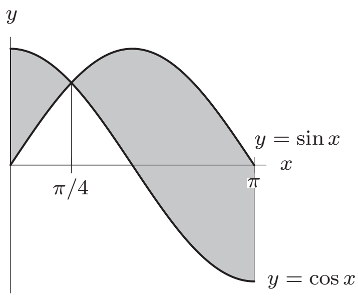


Figure 5.42

19. To find the points of intersection of the two curves, we must solve the equation

$$e^{-x} = 4(x - x^2).$$

This equation cannot be solved exactly, but numerical methods (for example, zooming in on the graph on a calculator) give $x = 0.261$ and $x = 0.883$. See Figure 5.43. Since $y = 4(x - x^2)$ is above $y = e^{-x}$ on this interval,

$$\text{Area } = \int_{0.261}^{0.883} (4(x - x^2) - e^{-x})\,dx = 0.172.$$

The integral was evaluated numerically.

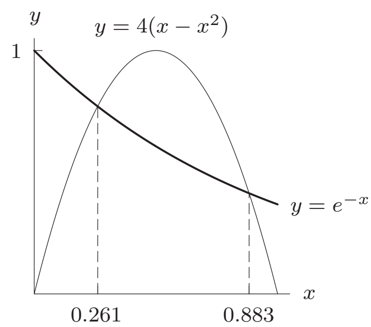


Figure 5.43

20. The curves $y = e^{-x}$ and $y = \ln x$ cross where

$$e^{-x} = \ln x.$$

Numerical methods (for example, zooming in on the graph on a calculator) give $x = 1.310$. See Figure 5.44. Since $y = e^{-x}$ is above $y = \ln x$ for $1 \le x \le 1.310$ and $y = e^{-x}$ is below $y = \ln x$ for $1.310 < x \le 2$, we break the area into two parts:

$$\text{Area} = \int_1^{1.310} (e^{-x} - \ln x)\, dx + \int_{1.310}^{2} (\ln x - e^{-x})\, dx.$$

Evaluating the integrals numerically gives

$$\text{Area} = 0.054 + 0.208 = 0.262.$$

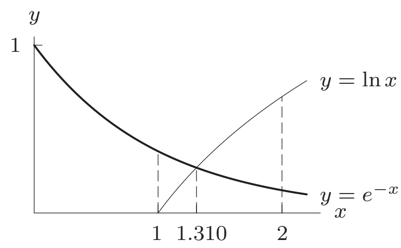


Figure 5.44

Problems

21. (a) For $0 \le x \le 3$, we have

$$\text{Average value} = \frac{1}{3-0}\int_0^3 f(x)dx = \frac{1}{3}(6) = 2.$$

(b) If $f(x)$ is even, the graph is symmetric about the x-axis. For example, see Figure 5.45. By symmetry, the area between $x = -3$ and $x = 3$ is twice the area between $x = 0$ and $x = 3$, so

$$\int_{-3}^{3} f(x)dx = 2(6) = 12.$$

Thus for $-3 \le x \le 3$, we have

$$\text{Average value} = \frac{1}{3-(-3)}\int_{-3}^{3} f(x)dx = \frac{1}{6}(12) = 2.$$

The graph confirms that the average value between $x = -3$ and $x = 3$ is the same as the average value between $x = 0$ and $x = 3$, which is 2.

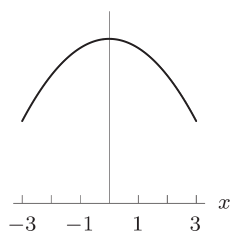


Figure 5.45

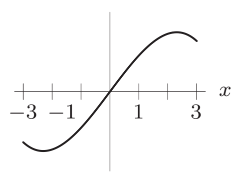


Figure 5.46

(c) If $f(x)$ is odd, then the graph is symmetric about the origin. For example, see Figure 5.46. By symmetry, the area above the x-axis cancels out the area below the x-axis, so

$$\int_{-3}^{3} f(x)dx = 0.$$

Thus for $-3 \le x \le 3$, we have

$$\text{Average value } = \frac{1}{3-(-3)}\int_{-3}^{3} f(x)dx = \frac{1}{6}(0) = 0.$$

The graph confirms that the average value between $x = -3$ and $x = 3$ is zero.

22. We know that we can divide the integral up as follows:

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx.$$

The graph suggests that f is an even function for $-1 \le x \le 1$, so $\int_{-1}^{1} f(x)\,dx = 2\int_0^1 f(x)\,dx$. Substituting this in to the preceding equation, we have

$$\int_0^3 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)\,dx + \int_1^3 f(x)\,dx.$$

23. (a) The integral represents the area of a rectangle with height 1 and base $b - a$. See Figure 5.47. Thus $\int_a^b 1\,dx = b - a$.

(b) (i) $\int_2^5 1\,dx = 3$.

(ii) $\int_{-3}^{8} 1\,dx = 11$.

(iii) $\int_1^3 23\,dx = 23\int_1^3 1\,dx = 46$.

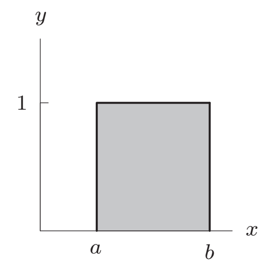


Figure 5.47

24. Using properties of the definite integral, we have:

$$\begin{aligned}
\int_2^5 (2f(x)+3)\,dx &= 17\\
2\int_2^5 f(x)\,dx + 3\int_2^5 1\,dx &= 17\\
2\int_2^5 f(x)\,dx + 3\cdot 3 &= 17\\
2\int_2^5 f(x)\,dx &= 8\\
\int_2^5 f(x)\,dx &= 4.
\end{aligned}$$

25. Since $t = 0$ in 1975 and $t = 35$ in 2010, we want:

$$\begin{aligned}
\text{Average Value } &= \frac{1}{35-0}\int_0^{35} 225(1.15)^t\,dt\\
&= \frac{1}{35}(212{,}787) = \$6080.
\end{aligned}$$

26. (a) The integral represents the area above the x-axis and below the line $y = x$ between $x = a$ and $x = b$. See Figure 5.48. This area is

$$A_1 + A_2 = a(b-a) + \frac{1}{2}(b-a)^2 = \left(a + \frac{b-a}{2}\right)(b-a) = \frac{b+a}{2}(b-a) = \frac{b^2-a^2}{2}.$$

The formula holds similarly for negative values.

(b) (i) $\int_2^5 x\,dx = 21/2$.
(ii) $\int_{-3}^8 x\,dx = 55/2$.
(iii) $\int_1^3 5x\,dx = 5\int_1^3 x\,dx = 20$.

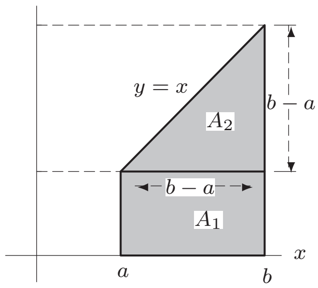


Figure 5.48

27. We have

$$30 = \int_{-2}^3 f\,dx = \int_{-2}^2 f(x)\,dx + \int_2^3 f(x)\,dx.$$

Since f is odd, $\int_{-2}^2 f(x)\,dx = 0$, so $\int_2^3 f(x)\,dx = 30$.

28. We have

$$8 = \int_{-2}^2 (f(x)-3)\,dx = \int_{-2}^2 f(x)\,dx - 3\int_{-2}^2 1\,dx.$$

Thus $\int_{-2}^2 f(x)\,dx = 8 + 3(2-(-2)) = 20$. Since f is even, $\int_0^2 f(x)\,dx = (1/2)20 = 10$.

29. This integral is 0 because the function $x^3\cos(x^2)$ is odd (meaning $f(-x) = -f(x)$), and so the negative contribution to the integral from $-\frac{\pi}{4} < x < 0$ exactly cancels the positive contribution from $0 < x < \frac{\pi}{4}$. See Figure 5.49.

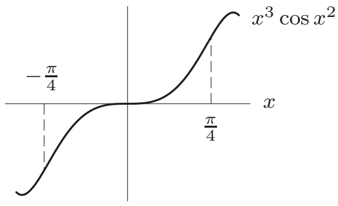


Figure 5.49

30. On the interval $2 \le x \le 5$,

$$\text{Average value of } f = \frac{1}{5-2}\int_2^5 f(x)\,dx = 4,$$

so

$$\int_2^5 f(x)\,dx = 12.$$

Thus

$$\int_2^5 (3f(x)+2)\,dx = 3\int_2^5 f(x)\,dx + 2\int_2^5 1\,dx = 3(12) + 2(5-2) = 42.$$

31. We know

$$\int_1^3 (x^2 - x)\,dx = \int_1^3 x^2\,dx - \int_1^3 x\,dx.$$

Since $\int_1^3 3x^2 dx = 3\int_1^3 x^2 dx$ and we are given $\int_1^3 3x^2\,dx = 26$, we have

$$\int_1^3 x^2\,dx = \frac{26}{3}.$$

Similarly, since $\int_1^3 2x dx = 2\int_1^3 x dx$ and $\int_1^3 2x\,dx = 8$, we have

$$\int_1^3 x\,dx = 4.$$

Thus,

$$\int_1^3 (x^2 - x)\,dx = \frac{26}{3} - 4 = \frac{14}{3}.$$

32. (a) Over the interval $[-1, 3]$, we estimate that the total change of the population is about 1.5, by counting boxes between the curve and the x-axis; we count about 1.5 boxes below the x-axis from $x = -1$ to $x = 1$ and about 3 above from $x = 1$ to $x = 3$. So the average rate of change is just the total change divided by the length of the interval, that is $1.5/4 = 0.375$ thousand/hour.

(b) We can estimate the total change of the algae population by counting boxes between the curve and the x-axis. Here, there is about 1 box above the x-axis from $x = -3$ to $x = -2$, about 0.75 of a box below the x-axis from $x = -2$ to $x = -1$, and a total change of about 1.5 boxes thereafter (as discussed in part (a)). So the total change is about $1 - 0.75 + 1.5 = 1.75$ thousands of algae.

33. (a) The integral is the area above the x-axis minus the area below the x-axis. Thus, we can see that $\int_{-3}^3 f(x)\,dx$ is about $-6 + 2 = -4$ (the negative of the area from $t = -3$ to $t = 1$ plus the area from $t = 1$ to $t = 3$.)

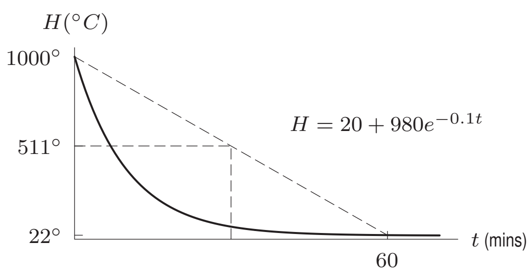
(b) Since the integral in part (a) is negative, the average value of $f(x)$ between $x = -3$ and $x = 3$ is negative. From the graph, however, it appears that the average value of $f(x)$ from $x = 0$ to $x = 3$ is positive. Hence (ii) is the larger quantity.

34. (a) At the end of one hour $t = 60$, and $H = 22°$C.

(b)

$$\begin{aligned}\text{Average temperature} &= \frac{1}{60}\int_0^{60} (20 + 980e^{-0.1t})dt \\ &= \frac{1}{60}(10976) = 183°\text{C}.\end{aligned}$$

(c) Average temperature at beginning and end of hour $= (1000 + 22)/2 = 511°$C. The average found in part (b) is smaller than the average of these two temperatures because the bar cools quickly at first and so spends less time at high temperatures. Alternatively, the graph of H against t is concave up.



35. (a) We have

$$\text{Average population} = \frac{1}{40}\int_0^{40} 112(1.011)^t\,dt.$$

Evaluating the integral numerically gives

$$\text{Average population} = 140.508 \text{ million}.$$

(b) In 2010, $t = 0$, and $P = 112(1.011)^0 = 112$
In 2050, $t = 40$, and $P = 112(1.011)^{40} = 173.486$.

$$\text{Average} = \frac{1}{2}(112 + 173.486) = 142.743 \text{ million.}$$

(c) If P had been linear, the average value found in part (a) would have been the same as we found in part (b). Since the population graph is concave up, it is below the secant line. Thus, the actual values of P are less than the corresponding values on the secant line, so the average found in part (a) is smaller than that found in part (b).

36. (a) See Figure 5.50. Since the shaded region lies within a rectangle of area 1, the area is less than 1.
(b) Since the area is given by the integral

$$\text{Area} = \int_0^1 e^{-x^2/2}\,dx = 0.856.$$

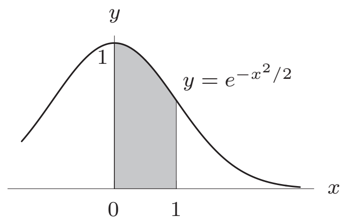


Figure 5.50

37. Notice that $f(x) = \sqrt{1+x^3}$ is increasing for $0 \le x \le 2$, since x^3 gets bigger as x increases. This means that $f(0) \le f(x) \le f(2)$. For this function, $f(0) = 1$ and $f(2) = 3$. Thus, the area under $f(x)$ lies between the area under the line $y = 1$ and the area under the line $y = 3$ on the interval $0 \le x \le 2$. See Figure 5.51. That is,

$$1(2-0) \le \int_0^2 \sqrt{1+x^3}\,dx \le 3(2-0).$$

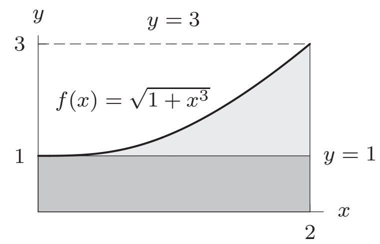


Figure 5.51

38. (a) The integrand is positive, so the integral cannot be negative.
(b) The integrand ≥ 0. If the integral $= 0$, then the integrand must be identically 0, which is not true.

39. See Figure 5.52.

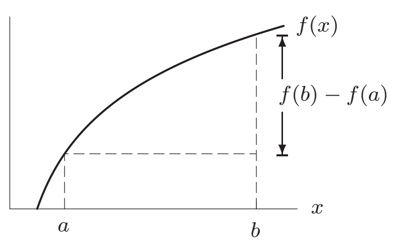


Figure 5.52

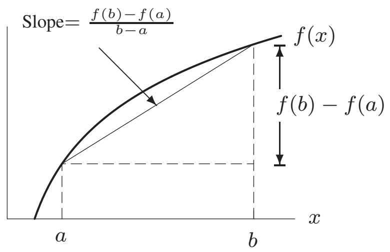


Figure 5.53

40. See Figure 5.53.

41. See Figure 5.54.

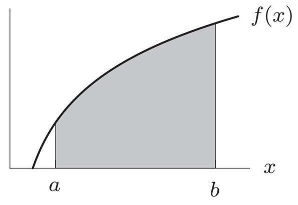


Figure 5.54

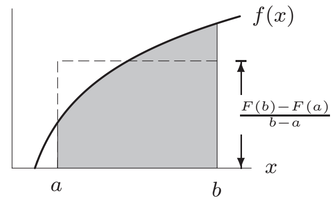


Figure 5.55

42. See Figure 5.55. Note that we are using the interpretation of the definite integral as the length of the interval times the average value of the function on that interval, which we developed in Section 5.3.

43. See Figure 5.56. Since $\int_0^1 f(x)\,dx = A_1$ and $\int_1^2 f(x)\,dx = -A_2$ and $\int_2^3 f(x)\,dx = A_3$, we know that

$$0 < \int_0^1 f(x)\,dx < -\int_1^2 f(x)\,dx < \int_2^3 f(x)\,dx.$$

In addition, $\int_0^2 f(x)\,dx = A_1 - A_2$, which is negative, but smaller in magnitude than $\int_1^2 f(x)\,dx$. Thus

$$\int_1^2 f(x)\,dx < \int_0^2 f(x)\,dx < 0.$$

The area A_3 lies inside a rectangle of height 20 and base 1, so $A_3 < 20$. The area A_2 lies inside a rectangle below the x-axis of height 10 and width 1, so $-10 < A_2$. Thus:

$$\text{(viii)} < \text{(ii)} < \text{(iii)} < \text{(vi)} < \text{(i)} < \text{(v)} < \text{(iv)} < \text{(vii)}.$$

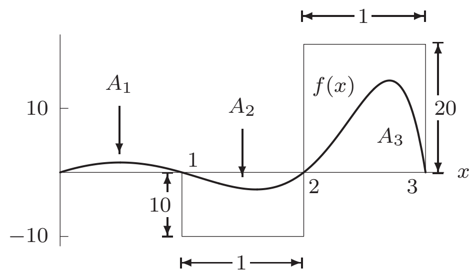


Figure 5.56

44. (a) (i) Since the triangular region under the graphs of $f(x)$ has area $1/2$, we have

$$\text{Average}(f) = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

(ii) Similarly,

$$\text{Average}(g) = \frac{1}{2-0}\int_0^2 g(x)\,dx = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

(iii) Since $f(x)$ is nonzero only for $0 \le x < 1$ and $g(x)$ is nonzero only for $1 < x \le 2$, the product $f(x)g(x) = 0$ for all x. Thus

$$\text{Average}(f\cdot g) = \frac{1}{2-0}\int_0^2 f(x)g(x)\,dx = \frac{1}{2}\int_0^2 0\,dx = 0.$$

(b) Since the average values of $f(x)$ and $g(x)$ are nonzero, their product is nonzero. Thus the left side of the statement is nonzero. However, the average of the product $f(x)g(x)$ is zero. Thus, the right side of the statement is zero, so the statement is not true.

45. (a) Since $f(x) = \sin x$ over $[0, \pi]$ is between 0 and 1, the average of $f(x)$ must itself be between 0 and 1. Furthermore, since the graph of $f(x)$ is concave down on this interval, the average value must be greater than the average height of the triangle shown in Figure 5.57, namely, 0.5.

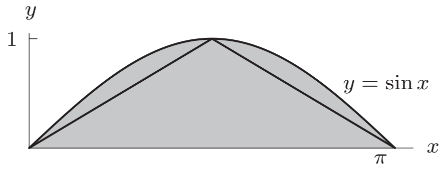


Figure 5.57

(b) Average $= \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi \sin x\, dx = 0.64.$

46. (a) Splitting the integral in order to make use of the values in the table gives:

$$\frac{1}{\sqrt{2\pi}}\int_1^3 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^3 e^{-x^2/2}\,dx - \frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx = 0.4987 - 0.3413 = 0.1574.$$

(b) Using the symmetry of $e^{x^2/2}$, we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-2}^3 e^{-x^2/2}\,dx &= \frac{1}{\sqrt{2\pi}}\int_{-2}^0 e^{-x^2/2}\,dx + \frac{1}{\sqrt{2\pi}}\int_0^3 e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^2 e^{-x^2/2}\,dx + \frac{1}{\sqrt{2\pi}}\int_0^3 e^{-x^2/2}\,dx \\ &= 0.4772 + 0.4987 = 0.9759.\end{aligned}$$

47. There is not enough information to decide. If we had more information—say, for example, we knew that g was periodic or had even or odd symmetry—we might be able to evaluate this expression.

48. The graph of $y = g(-x)$ is the graph of $y = g(x)$ reflected horizontally across the y-axis. This does not change the total area. Thus, the two integrals are equal, so $\displaystyle\int_{-4}^4 g(-x)\,dx = 12.$

49. Since $\displaystyle\int_0^7 f(x)\,dx = 25$, we have $\sqrt{\displaystyle\int_0^7 f(x)\,dx} = \sqrt{25} = 5.$

50. There is not enough information to decide. In terms of area, knowing the area under the graph of f on $0 \le x \le 7$ does not help us find the area on $0 \le x \le 3.5$. It depends on the shape of the graph and where it lies above and below the x-axis.

51. The graph of $g(x) = f(x+2)$ is the graph of $y = f(x)$ shifted left by 2 units. Thus, the area under the graph of g on $-2 \le x \le 5$ is the same as the area under the graph of f on $0 \le x \le 7$. Therefore,

$$\int_{-2}^5 f(x+2)\,dx = 25.$$

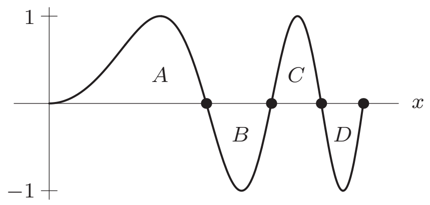
52. We have

$$\int_0^7 (f(x)+2)\,dx = \underbrace{\int_0^7 f(x)\,dx}_{25} + \underbrace{\int_0^7 2\,dx}_{7\cdot 2 = 14} = 39.$$

Note that, thinking graphically, $\int_0^7 2\,dx$ is the area of a rectangle of height 2 and width 7.

53. (a) Clearly, the points where $x = \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \sqrt{4\pi}$ are where the graph intersects the x-axis because $f(x) = \sin(x^2) = 0$ where x is the square root of some multiple of π.

(b) Let $f(x) = \sin(x^2)$, and let A, B, C, and D be the areas of the regions indicated in the figure below. Then we see that $A > B > C > D$.



Note that

$$\int_0^{\sqrt{\pi}} f(x)\,dx = A, \quad \int_0^{\sqrt{2\pi}} f(x)\,dx = A - B,$$

$$\int_0^{\sqrt{3\pi}} f(x)\,dx = A - B + C, \quad \text{and} \quad \int_0^{\sqrt{4\pi}} f(x)\,dx = A - B + C - D.$$

It follows that

$$\int_0^{\sqrt{\pi}} f(x)\,dx = A > \int_0^{\sqrt{3\pi}} f(x)\,dx = A - (B - C) = A - B + C >$$

$$\int_0^{\sqrt{4\pi}} f(x)\,dx = A - B + C - D > \int_0^{\sqrt{2\pi}} f(x)\,dx = (A - B) > 0.$$

And thus the ordering is $n = 1$, $n = 3$, $n = 4$, and $n = 2$ from largest to smallest. All the numbers are positive.

54. See Figure 5.58.

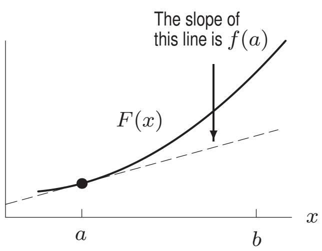


Figure 5.58

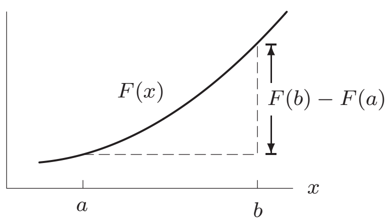


Figure 5.59

55. See Figure 5.59.

56. See 5.60.

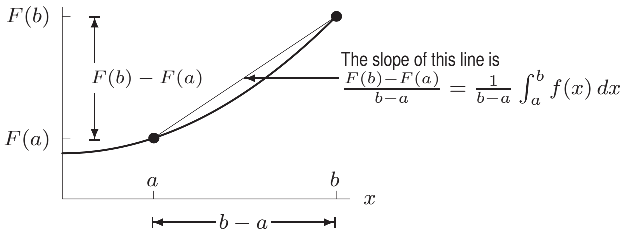


Figure 5.60

57. On the interval $a \leq t \leq b$, we have

$$\text{Average value of } v(t) = \frac{1}{b-a}\int_a^b v(t)\,dt.$$

Since $v(t) = s'(t)$, by the Fundamental Theorem of Calculus, we get:

$$\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{b-a}(s(b) - s(a)) = \text{ Average velocity}.$$

Strengthen Your Understanding

58. It is possible that $f(x)$ has both positive and negative values on the interval $[a, b]$, even if the integral is positive.

For example, suppose that $a = 0$ and $b = 3$, and let $f(x) = 2 - x$ for all x. Then by finding areas, we can see that $\int_0^3 f(x)dx = 3/2$, which is nonnegative. However, $f(x) < 0$ for x in the interval $(2, 3]$.

59. Using the properties of definite integrals, we have

$$\begin{aligned}\int_a^b (5 + 3f(x))\,dx &= \int_a^b 5\,dx + \int_a^b 3f(x)\,dx\\ &= 5(b-a) + 3\int_a^b f(x)\,dx.\end{aligned}$$

The value of the integral $\int_a^b 5\,dx$ is $5(b-a)$ because the integral represents the area of a rectangle with height 5 and width $b - a$.

60. The time needs to be expressed in days in the definite integral, since the population is a function of time t in days. If the six-month period contains 181 days, the correct integral is

$$\frac{1}{181}\int_0^{181} f(t)\,dt.$$

For a different six-month period, the number 181 may be different.

61. If we let $A = \int_0^1 f(x)\,dx$, then we have $\int_0^1 2f(x)\,dx = 2\int_0^1 f(x)\,dx = 2A$. We want to define $f(x)$ so that $2A < A$. In order for this to happen, A must be negative. So we'll define $f(x)$ to be negative on $[0, 1]$; for example, $f(x) = -1$. Then $\int_0^1 f(x)\,dx = -1$, and $\int_0^1 2f(x)\,dx = \int_0^1 -2\,dx = -2$.

62. Let's try the function $f(x) = 2 - x$. The area between the graph of the function $f(x) = 2 - x$ and the x-axis on the interval $[0, 4]$ consists of two isosceles right triangles with legs of length 2: one above the x-axis and one below. When we compute the integral $\int_0^4 (2 - x)\,dx$, the areas of these two congruent triangles cancel out, and the integral is zero.

63. If the car is going $v(t)$ miles per hour at time t hours, then

$$\text{Average speed } = \frac{1}{5-0}\int_0^5 v(t)\,dt.$$

64. True, since $\int_0^2 (f(x) + g(x))\,dx = \int_0^2 f(x)\,dx + \int_0^2 g(x)\,dx$.

65. False. It is possible that $\int_0^2 (f(x) + g(x))\,dx = 10$ and $\int_0^2 f(x)\,dx = 4$ and $\int_0^2 g(x)\,dx = 6$, for instance. For example, if $f(x) = 5x - 3$ and $g(x) = 3$, then $\int_0^2 (f(x) + g(x))\,dx = \int_0^2 5x\,dx = 10$, but $\int_0^2 f(x)\,dx = 4$ and $\int_0^2 g(x)\,dx = 6$.

66. False. We know that $\int_0^4 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx$, but it is not true that $\int_2^4 f(x)\,dx$ must be the same value as $\int_0^2 f(x)\,dx$. For example, if $f(x) = 3x$, then $\int_0^2 f(x)\,dx = 6$, but $\int_0^4 f(x)\,dx = 24$.

67. True, since $\int_0^2 2f(x)\,dx = 2\int_0^2 f(x)\,dx$.

68. False. This would be true if $h(x) = 5f(x)$. However, we cannot assume that $f(5x) = 5f(x)$, so for many functions this statement is false. For example, if f is the constant function $f(x) = 3$, then $h(x) = 3$ as well, so $\int_0^2 f(x)\,dx = \int_0^2 h(x)\,dx = 6$.

69. True. If $a = b$, then $\Delta x = 0$ for any Riemann sum for f on the interval $[a, b]$, so every Riemann sum has value 0. Thus, the limit of the Riemann sums is 0.

70. False. For example, let $a = -1$ and $b = 1$ and $f(x) = x$.Then the areas bounded by the graph of f and the x-axis on the two halves of the interval $[-1, 1]$ cancel with each other and make $\int_{-1}^1 f(x)\,dx = 0$. See Figure 5.61.

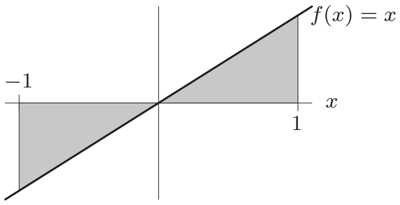


Figure 5.61

71. False. Let $f(x) = 7$ and $g(x) = 9$ for all x.
Then $\int_1^2 f(x)\,dx + \int_2^3 g(x)\,dx = 7 + 9 = 16$, but $\int_1^3 (f(x) + g(x))\,dx = \int_1^3 16\,dx = 32$.

72. False. If the graph of f is symmetric about the y-axis, this is true, but otherwise it is usually not true. For example, if $f(x) = x + 1$ the area under the graph of f for $-1 \le x \le 0$ is less than the area under the graph of f for $0 \le x \le 1$, so $\int_{-1}^1 f(x)\,dx < 2\int_0^1 f(x)\,dx$. See Figure 5.62.

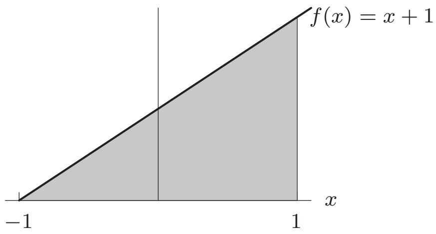


Figure 5.62

73. True, by Theorem 5.4 on Comparison of Definite Integrals:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{b-a}\int_a^b g(x)\,dx.$$

74. True. We have

$$\begin{aligned}
\text{Average value of } f \text{ on } [0,10] &= \frac{1}{10-0}\int_0^{10} f(x)\,dx \\
&= \frac{1}{10}\left(\int_0^5 f(x)\,dx + \int_5^{10} f(x)\,dx\right) \\
&= \frac{1}{2}\left(\frac{1}{5}\int_0^5 f(x)\,dx + \frac{1}{5}\int_5^{10} f(x)\,dx\right) \\
&= \begin{array}{l}\text{The average of the average value of } f \text{ on } [0,5] \text{ and} \\ \text{the average value of } f \text{ on } [5,10].\end{array}
\end{aligned}$$

75. False. If the values of $f(x)$ on the interval $[c, d]$ are larger than the values of $f(x)$ in the rest of the interval $[a, b]$, then the average value of f on the interval $[c, d]$ is larger than the average value of f on the interval $[a, b]$. For example, suppose

$$f(x) = \begin{cases} 0 & x < 1 \text{ or } x > 2 \\ 1 & 1 \le x \le 2. \end{cases}$$

Then the average value of f on the interval $[1, 2]$ is 1, whereas the average value of f on the interval $[0, 3]$ is $(1/(3 - 0))\int_0^3 f(x)\,dx = 1/3$.

76. True. We have by the properties of integrals in Theorem 5.3,

$$\int_1^9 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^9 f(x)\,dx.$$

Since $(1/(4-1))\int_1^4 f(x)\,dx = A$ and $(1/(9-4))\int_4^9 f(x)\,dx = B$, we have

$$\int_1^9 f(x)\,dx = 3A + 5B.$$

Dividing this equation through by 8, we get that the average value of f on the interval $[1,9]$ is $(3/8)A + (5/8)B$.

77. True. By the properties of integrals in Theorem 5.3, we have:

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Dividing both sides of this equation through by $b-a$, we get that the average value of $f(x)+g(x)$ is average value of $f(x)$ plus the average value of $g(x)$:

$$\frac{1}{b-a}\int_a^b (f(x)+g(x))\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx.$$

78. False. A counterexample is given by

$$f(x) = \begin{cases} 3 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & 0 \le x \le 1 \\ 3 & 1 < x \le 2. \end{cases}$$

Since $f(x)$ is nonzero for $0 \le x < 1$ and $g(x)$ is nonzero for $1 < x \le 2$, the product $f(x)g(x) = 0$ for all x. Thus the average values of $f(x)$ and $g(x)$ are nonzero, but the average of the product is zero.

Specifically, on $[0,2]$ we have

$$\begin{aligned} \text{Average}(f) &= \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{3}{2} \\ \text{Average}(g) &= \frac{1}{2-0}\int_0^2 g(x)\,dx = \frac{3}{2} \\ \text{Average}(f\cdot g) &= \frac{1}{2-0}\int_0^2 f(x)g(x)\,dx = \frac{1}{2}\int_0^2 0\,dx = 0. \end{aligned}$$

but

$$\text{Average}(f)\cdot\text{Average}(g) = \frac{3}{2}\cdot\frac{3}{2} = \frac{9}{4}.$$

79. (a) Does not follow; the statement implies that

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = 5 + 7 = 12,$$

but the fact that the two integrals add to 12 does not tell us what the integrals are individually. For example, we could have $\int_a^b f(x)\,dx = 10$ and $\int_a^b g(x)\,dx = 2$.

(b) This follows:

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = 7 + 7 = 14.$$

(c) This follows: rearranging the original statement by subtracting $\int_a^b g(x)\,dx$ from both sides gives

$$\int_a^b (f(x)+g(x))\,dx - \int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

Since $f(x)+g(x) = h(x)$, we have $f(x) = h(x) - g(x)$. Substituting for $f(x)$, we get

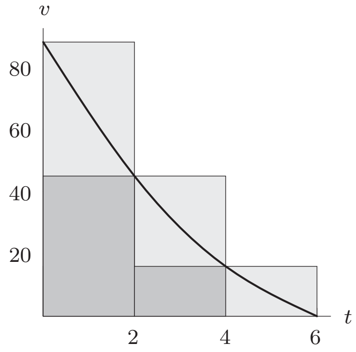
$$\int_a^b h(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (h(x)-g(x))\,dx.$$

Solutions for Chapter 5 Review

Exercises

1. (a) Lower estimate $= (45)(2) + (16)(2) + (0)(2) = 122$ feet.
Upper estimate $= (88)(2) + (45)(2) + (16)(2) = 298$ feet.

(b)



2. To find the distance the car moved before stopping, we estimate the distance traveled for each two-second interval. Since speed decreases throughout, we know that the left-handed sum will be an overestimate to the distance traveled and the right-hand sum an underestimate. Applying the formulas for these sums with $\Delta t = 2$ gives:

$$\begin{aligned} \text{LEFT} &= 2(100 + 80 + 50 + 25 + 10) = 530 \text{ ft.} \\ \text{RIGHT} &= \;\; 2(80 + 50 + 25 + 10 + 0) \;\; = 330 \text{ ft.} \end{aligned}$$

(a) The best estimate of the distance traveled will be the average of these two estimates, or

$$\text{Best estimate} = \frac{530 + 330}{2} = 430 \text{ ft.}$$

(b) All we can be sure of is that the distance traveled lies between the upper and lower estimates calculated above. In other words, all the black-box data tells us for sure is that the car traveled between 330 and 530 feet before stopping. So we don't know whether it hit the skunk or not; the answer is (ii).

3. $\int_0^3 f(x)\,dx$ is equal to the area shaded. We estimate the area by counting shaded rectangles. There are 3 fully shaded and about 4 partially shaded rectangles, for a total of approximately 5 shaded rectangles. Since each rectangle represents 4 square units, our estimated area is $5(4) = 20$. We have

$$\int_0^3 f(x)\,dx \approx 20.$$

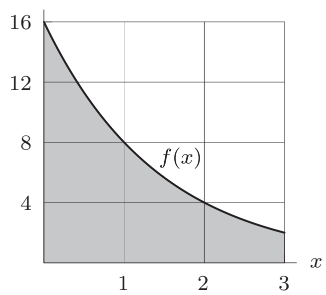


Figure 5.63

4. With $\Delta x = 5$, we have

$$\text{Left-hand sum } = 5(0+100+200+100+200+250+275) = 5625,$$

$$\text{Right-hand sum } = 5(100+200+100+200+250+275+300) = 7125.$$

The average of these two sums is our best guess for the value of the integral;

$$\int_{-15}^{20} f(x)\,dx \approx \frac{5625+7125}{2} = 6375.$$

5. We take $\Delta t = 20$. Then:

$$\begin{aligned}\text{Left-hand sum} &= 1.2(20)+2.8(20)+4.0(20)+4.7(20)+5.1(20)\\ &= 356.\\ \text{Right-hand sum} &= 2.8(20)+4.0(20)+4.7(20)+5.1(20)+5.2(20)\\ &= 436.\\ \int_0^{100} f(t)\,dt \approx \text{Average} &= \frac{356+436}{2} = 396.\end{aligned}$$

6. (a) Suppose $f(t)$ is the flow rate in m^3/hr at time t. We are only given two values of the flow rate, so in making our estimates of the flow, we use one subinterval, with $\Delta t = 3/1 = 3$:

$$\text{Left estimate} = 3[f(6\text{ am})] = 3\cdot 100 = 300\text{ m}^3 \quad \text{(an underestimate)}$$
$$\text{Right estimate} = 3[f(9\text{ am})] = 3\cdot 280 = 840\text{ m}^3 \quad \text{(an overestimate).}$$

The best estimate is the average of these two estimates,

$$\text{Best estimate} = \frac{\text{Left}+\text{Right}}{2} = \frac{300+840}{2} = 570\text{ m}^3.$$

(b) Since the flow rate is increasing throughout, the error, i.e., the difference between over- and under-estimates, is given by

$$\text{Error} \le \Delta t\,[f(9\text{ am}) - f(6\text{ am})] = \Delta t[280-100] = 180\Delta t.$$

We wish to choose Δt so that the the error $180\Delta t \le 6$, or $\Delta t \le 6/180 = 1/30$. So the flow rate gauge should be read every $1/30$ of an hour, or every 2 minutes.

7. (a) An upper estimate is $9.81+8.03+6.53+5.38+4.41 = 34.16$ m/sec. A lower estimate is $8.03+6.53+5.38+4.41+3.61 = 27.96$ m/sec.

(b) The average is $\frac{1}{2}(34.16+27.96) = 31.06$ m/sec. Because the graph of acceleration is concave up, this estimate is too high, as can be seen in Figure 5.64. The area of the shaded region is the average of the areas of the rectangles $ABFE$ and $CDFE$.

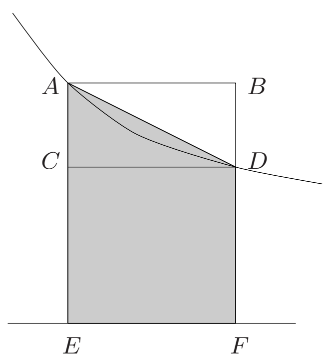


Figure 5.64

8. We have $f(t) = F'(t) = 4t^3$, so by the Fundamental Theorem of Calculus,

$$\int_{-1}^{1} 4t^3\,dt = F(1) - F(-1) = 1 - 1 = 0.$$

Notice in this case the integral is 0 because the function being integrated, $f(t) = 4t^3$, is odd: the negative contribution to the integral from $a = -1$ to $b = 1$ exactly cancels the positive.

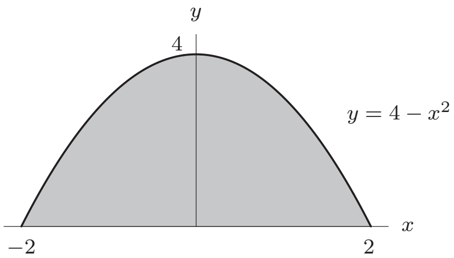
9. We have $f(t) = F'(t) = 12t^3 - 15t^2 + 5$, so by the Fundamental Theorem of Calculus,

$$\int_{-2}^{1} (12t^3 - 15t^2 + 5)\,dt = F(1) - F(-2) = 3 - 78 = -75.$$

10. The x intercepts of $y = 4 - x^2$ are $x = -2$ and $x = 2$, and the graph is above the x-axis on the interval $[-2, 2]$.

$$\text{Area} = \int_{-2}^{2} (4 - x^2)\,dx = 10.667.$$

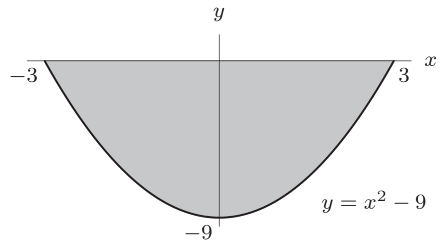
The integral was evaluated on a calculator.



11. The x intercepts of $y = x^2 - 9$ are $x = -3$ and $x = 3$, and since the graph is below the x axis on the interval $[-3, 3]$.

$$\text{Area} = -\int_{-3}^{3} (x^2 - 9)\,dx = 36.$$

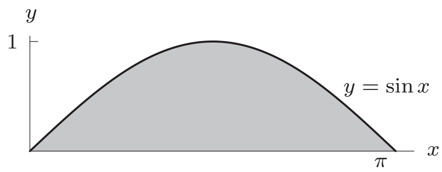
The integral was evaluated on a calculator.



12. Since x intercepts are $x = 0, \pi, 2\pi, \ldots$,

$$\text{Area} = \int_{0}^{\pi} \sin x\,dx = 2.$$

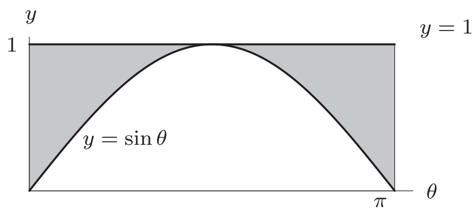
The integral was evaluated on a calculator.



13. Since the θ intercepts of $y = \sin\theta$ are $\theta = 0, \pi, 2\pi, \ldots$,

$$\text{Area} = \int_{0}^{\pi} (1 - \sin\theta)\,d\theta = 1.142.$$

The integral was evaluated on a calculator.



14. The graph of $y = -x^2 + 5x - 4$ is shown in Figure 5.65. We wish to find the area shaded. Since the graph crosses the x-axis at $x = 1$, we must split the integral at $x = 1$. For $x < 1$, the graph is below the x-axis, so the area is the negative of the integral. Thus

$$\text{Area shaded} = -\int_{0}^{1} (-x^2 + 5x - 4)dx + \int_{1}^{3} (-x^2 + 5x - 4)dx.$$

Using a calculator or computer, we find

$$\int_{0}^{1} (-x^2 + 5x - 4)dx = -1.8333 \quad\text{and}\quad \int_{1}^{3} (-x^2 + 5x - 4)dx = 3.3333.$$

Thus,

$$\text{Area shaded} = 1.8333 + 3.3333 = 5.167.$$

(Notice that $\int_0^3 f(x)dx = -1.8333 + 3.333 = 1.5$, but the value of this integral is not the area shaded.)

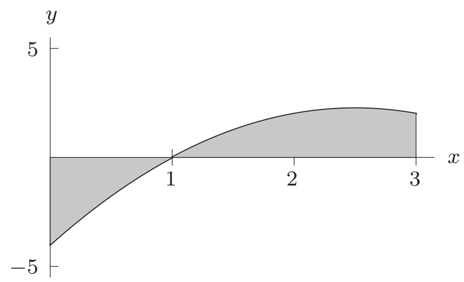


Figure 5.65

15. The graph of $y = \cos x + 7$ is above $y = \ln(x - 3)$ for $5 \le x \le 7$. See Figure 5.66. Therefore

$$\text{Area } = \int_5^7 \cos x + 7 - \ln(x - 3)\, dx = 13.457.$$

The integral was evaluated on a calculator.

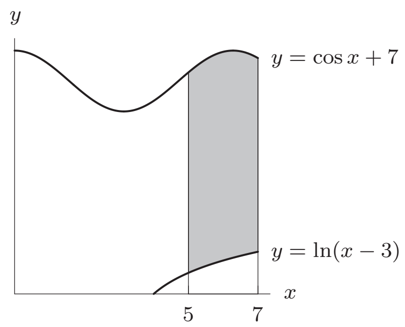


Figure 5.66

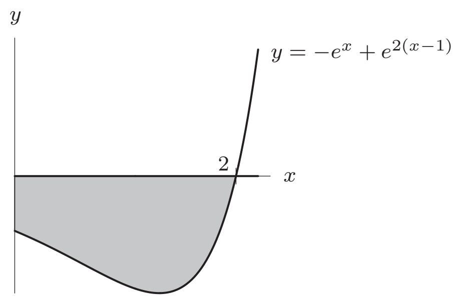


Figure 5.67

16. The graph of $y = -e^x + e^{2(x-1)}$ has intercepts where $e^x = e^{2(x-1)}$, or where $x = 2(x - 1)$, so $x = 2$. See Figure 5.67. Since the region is below the x-axis, the integral is negative, so

$$\text{Area } = -\int_0^2 -e^x + e^{2(x-1)}\, dx = 2.762.$$

The integral was evaluated on a calculator.

Problems

17. This integral represents the area of two triangles, each of base 1 and height 1. See Figure 5.68. Therefore:

$$\int_{-1}^1 |x|\, dx = \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 1 = 1.$$

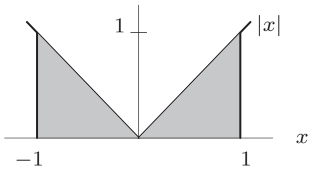


Figure 5.68

18. The distance traveled is represented by area under the velocity curve. We can approximate the area using left- and right-hand sums. Alternatively, by counting squares and fractions of squares, we find that the area under the graph appears to be around 310 (miles/hour) sec, within about 10. So the distance traveled was about 310 $\left(\frac{5280}{3600}\right) \approx 455$ feet, within about 10 $\left(\frac{5280}{3600}\right) \approx 15$ feet. (Note that 455 feet is about 0.086 miles)

19. (a) We calculate the right- and left-hand sums as follows:

$$\text{Left} = 2[80 + 52 + 28 + 10] = 340 \text{ ft.}$$
$$\text{Right} = 2[52 + 28 + 10 + 0] = 180 \text{ ft.}$$

Our best estimate will be the average of these two sums,

$$\text{Best} = \frac{\text{Left} + \text{Right}}{2} = \frac{340 + 180}{2} = 260 \text{ ft.}$$

(b) Since v is decreasing throughout,

$$\begin{aligned}\text{Left} - \text{Right} &= \Delta t \cdot [f(0) - f(8)] \\ &= 80\Delta t.\end{aligned}$$

Since our best estimate is the average of Left and Right, the maximum error is $(80)\Delta t/2$. For $(80)\Delta t/2 \leq 20$, we must have $\Delta t \leq 1/2$. In other words, we must measure the velocity every 0.5 second.

20. As illustrated in Figure 5.69, the left- and right-hand sums are both equal to $(4\pi) \cdot 3 = 12\pi$, while the integral is smaller. Thus we have:

$$\int_0^{4\pi} (2 + \cos x)\, dx < \text{Left-hand sum} = \text{Right-hand sum}.$$

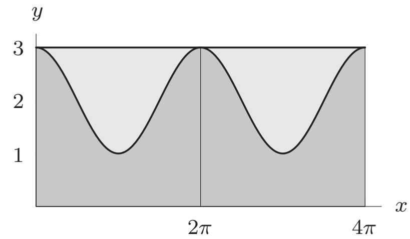


Figure 5.69: Integral vs. Left- and Right-Hand Sums

21. Distance traveled = $\displaystyle\int_0^{1.1} \sin(t^2)\, dt = 0.399$ miles.

22. Since $v(t) \geq 0$ for $0 \leq t \leq 3$, we can find the total distance traveled by integrating the velocity from $t = 0$ to $t = 3$. Using a calculator to compute the integral gives

$$\text{Distance} = \int_0^3 \ln(t^2 + 1)\, dt = 3.406 \text{ ft.}$$

23. (a) The integral $\int_0^{30} f(t)dt$ represents the total emissions of nitrogen oxides, in millions of metric tons, during the period 1970 to 2000.

(b) We estimate the integral using left- and right-hand sums:

$$\text{Left sum} = (26.9)5 + (26.4)5 + (27.1)5 + (25.8)5 + (25.5)5 + (25.0)5 = 783.5.$$

$$\text{Right sum} = (26.4)5 + (27.1)5 + (25.8)5 + (25.5)5 + (25.0)5 + (22.6)5 = 762.0.$$

We average the left- and right-hand sums to find the best estimate of the integral:

$$\int_0^{30} f(t)dt \approx \frac{783.5 + 762.0}{2} = 772.8 \text{ million metric tons.}$$

Between 1970 and 2000, about 772.8 million metric tons of nitrogen oxides were emitted.

24. (a) If the level first becomes acceptable at time t_1, then $R_0 = 4R(t_1)$, and

$$\frac{1}{4}R_0 = R_0 e^{-0.004t_1}$$
$$\frac{1}{4} = e^{-0.004t_1}.$$

Taking natural logs on both sides yields

$$\ln\frac{1}{4} = -0.004t_1$$
$$t_1 = \frac{\ln\frac{1}{4}}{-0.004} \approx 346.574 \text{ hours.}$$

(b) Since the initial radiation was four times the acceptable limit of 0.6 millirems/hour, we have $R_0 = 4(0.6) = 2.4$. The rate at which radiation is emitted is $R(t) = R_0 e^{-0.004t}$, so

$$\text{Total radiation emitted} = \int_0^{346.574} 2.4e^{-0.004t}\,dt.$$

Evaluating the integral numerically, we find that 450 millirems were emitted during this time.

25. (a) An overestimate is 7 tons. An underestimate is 5 tons.
(b) An overestimate is $7+8+10+13+16+20 = 74$ tons. An underestimate is $5+7+8+10+13+16 = 59$ tons.
(c) If measurements are made every Δt months, then the error is $|f(6) - f(0)| \cdot \Delta t$. So for this to be less than 1 ton, we need $(20-5)\cdot\Delta t < 1$, or $\Delta t < 1/15$. So measurements every 2 days or so will guarantee an error in over- and underestimates of less than 1 ton.

26. Suppose $F(t)$ represents the total quantity of water in the water tower at time t, where t is in days since April 1. Then the graph shown in the problem is a graph of $F'(t)$. By the Fundamental Theorem,

$$F(30) - F(0) = \int_0^{30} F'(t)dt.$$

We can calculate the change in the quantity of water by calculating the area under the curve. If each box represents about 300 liters, there is about one box, or -300 liters, from $t = 0$ to $t = 12$, and 6 boxes, or about $+1800$ liters, from $t = 12$ to $t = 30$. Thus

$$\int_0^{30} F'(t)dt = 1800 - 300 = 1500,$$

so the final amount of water is given by

$$F(30) = F(0) + \int_0^{30} F'(t)dt = 12{,}000 + 1500 = 13{,}500 \text{ liters.}$$

27. (a) Average value of $f = \frac{1}{5}\int_0^5 f(x)\,dx$.
(b) Average value of $|f| = \frac{1}{5}\int_0^5 |f(x)|\,dx = \frac{1}{5}\left(\int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx\right)$.

28. We'll show that in terms of the average value of f,

$$\text{I} > \text{II} = \text{IV} > \text{III}$$

Using the definition of average value and the fact that f is even, we have

$$\begin{aligned}\text{Average value of } f \text{ on II} &= \frac{\int_0^2 f(x)dx}{2} = \frac{\frac{1}{2}\int_{-2}^2 f(x)dx}{2}\\ &= \frac{\int_{-2}^2 f(x)dx}{4}\\ &= \text{Average value of } f \text{ on IV.}\end{aligned}$$

Since f is decreasing on [0,5], the average value of f on the interval $[0, c]$, where $0 \le c \le 5$, is decreasing as a function of c. The larger the interval the more low values of f are included. Hence

$$\begin{array}{c}\text{Average value of } f \\ \text{on } [0,1]\end{array} > \begin{array}{c}\text{Average value of } f \\ \text{on } [0,2]\end{array} > \begin{array}{c}\text{Average value of } f \\ \text{on } [0,5]\end{array}$$

29. (a) A graph of $f'(x) = \sin(x^2)$ is shown in Figure 5.70. Since the derivative $f'(x)$ is positive between $x = 0$ and $x = 1$, the change in $f(x)$ is positive, so $f(1)$ is larger than $f(0)$. Between $x = 2$ and $x = 2.5$, we see that $f'(x)$ is negative, so the change in $f(x)$ is negative; thus, $f(2)$ is greater than $f(2.5)$.

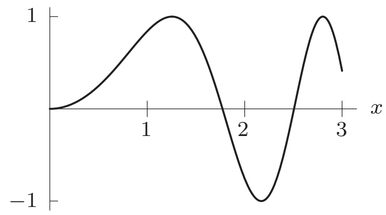


Figure 5.70: Graph of $f'(x) = \sin(x^2)$

(b) The change in $f(x)$ between $x = 0$ and $x = 1$ is given by the Fundamental Theorem of Calculus:

$$f(1) - f(0) = \int_0^1 \sin(x^2)dx = 0.310.$$

Since $f(0) = 2$, we have

$$f(1) = 2 + 0.310 = 2.310.$$

Similarly, since

$$f(2) - f(0) = \int_0^2 \sin(x^2)dx = 0.805,$$

we have

$$f(2) = 2 + 0.805 = 2.805.$$

Since

$$f(3) - f(0) = \int_0^3 \sin(x^2)dx = 0.774,$$

we have

$$f(3) = 2 + 0.774 = 2.774.$$

The results are shown in the table.

x	0	1	2	3
$f(x)$	2	2.310	2.805	2.774

30. (a) By the product rule

$$F'(t) = 1 \cdot \ln t + t \cdot \frac{1}{t} - 1 = \ln t.$$

(b) (i) Using a calculator, we find $\int_{10}^{12} \ln t\, dt = 4.793$.

(ii) The Fundamental Theorem of Calculus tells us that we can get the exact value of the integral by looking at

$$\int_{10}^{12} \ln t\, dt = F(12) - F(10) = (12\ln 12 - 12) - (10\ln 10 - 10) = 12\ln 12 - 10\ln 10 - 2 = 4.79303.$$

31. (a) $F(0) = 0$.
(b) F increases because $F'(x) = e^{-x^2}$ is positive.
(c) Using a calculator, we get $F(1) \approx 0.7468$, $F(2) \approx 0.8821$, $F(3) \approx 0.8862$.

32. We have

$$8 = \int_{-2}^{5} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx.$$

Since f is odd, $\int_{-2}^{2} f(x)\,dx = 0$, so $\int_{-2}^{5} f(x)\,dx = 8$.

33. Since f is even, $\int_0^2 f(x)\,dx = (1/2)6 = 3$ and $\int_0^5 f(x)\,dx = (1/2)14 = 7$. Therefore

$$\int_2^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^2 f(x)\,dx = 7 - 3 = 4.$$

34. We have

$$18 = \int_2^5 (3f(x) + 4)\,dx = 3\int_2^5 f(x)\,dx + \int_2^5 4\,dx.$$

Thus, since $\int_2^5 4\,dx = 4(5-2) = 12$, we have

$$3\int_2^5 f(x)\,dx = 18 - 12 = 6,$$

so

$$\int_2^5 f(x)\,dx = 2.$$

35. We have $\int_2^4 f(x)\,dx = 8/2 = 4$ and $\int_4^5 f(x)\,dx = -\int_5^4 f(x)\,dx = -1$. Thus

$$\int_2^5 f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^5 f(x)\,dx = 4 - 1 = 3.$$

36. (a) 0, since the integrand is an odd function and the limits are symmetric around 0.
(b) 0, since the integrand is an odd function and the limits are symmetric around 0.

37. Let $A = \int_0^p f(x)\,dx$. We are told $A > 0$. We can interpret A as the area under f on this interval. Since the graph of f repeats periodically every p units, the area under the graph also repeats. This means

$$\underbrace{\int_0^p f(x)\,dx}_{\text{area under one full period}} = \underbrace{\int_p^{2p} f(x)\,dx}_{\text{area under one full period}} = \underbrace{\int_{2p}^{3p} f(x)\,dx}_{\text{area under one full period}} = \cdots = A.$$

We have that

$$\int_0^{2p} f(x)\,dx = \underbrace{\int_0^p f(x)\,dx}_{A} + \underbrace{\int_p^{2p} f(x)\,dx}_{A} = 2A$$

$$\int_p^{5p} f(x)\,dx = \underbrace{\int_p^{2p} f(x)\,dx}_{A} + \cdots + \underbrace{\int_{4p}^{5p} f(x)\,dx}_{A} = 4A$$

$$\int_{5p}^{7p} f(x)\,dx = \underbrace{\int_{5p}^{6p} f(x)\,dx}_{A} + \underbrace{\int_{6p}^{7p} f(x)\,dx}_{A} = 2A,$$

so

$$\frac{\int_0^{2p} f(x)\,dx + \int_p^{5p} f(x)\,dx}{\int_{5p}^{7p} f(x)\,dx} = \frac{2A + 4A}{2A} = 3.$$

38. Since the graph is symmetrical, we know that

$$\int_{-r}^{0} f(x)\,dx = \int_{0}^{r} f(x)\,dx = \frac{1}{2}\int_{-r}^{r} f(x)\,dx.$$

This means that if we let $A = \int_{-r}^{0} f(x)\,dx$ then

$$\frac{\overbrace{\int_{-r}^{r} f(x)\,dx}^{2A} + \overbrace{\int_{0}^{r} f(x)\,dx}^{A}}{\underbrace{\int_{-r}^{0} f(x)\,dx}_{A}} = \frac{2A + A}{A} = 3.$$

39. (a) Since the function is odd, the areas above and below the x-axis cancel. Thus,

$$\int_{-3}^{0} xe^{-x^2}\,dx = -\int_{0}^{3} xe^{-x^2}\,dx,$$

so

$$\int_{-3}^{3} xe^{-x^2}\,dx = \int_{-3}^{0} xe^{-x^2}\,dx + \int_{0}^{3} xe^{-x^2}\,dx = 0.$$

(b) For $0 \le x \le 3$ with $n = 3$, we have $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $\Delta x = 1$. See Figure 5.71. Thus,

$$\text{Left sum } = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x = 0e^{-0^2} \cdot 1 + 1e^{-1^2} \cdot 1 + 2e^{-2^2} \cdot 1 = 0.4045.$$

(c) For $-3 \le x \le 0$, with $n = 3$, we have $x_0 = -3$, $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, and $\Delta x = 1$. See Figure 5.71. Thus,

$$\text{Left sum } = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x = -3e^{-(-3)^2} \cdot 1 - 2e^{-(-2)^2} \cdot 1 - 1e^{-(-1)^2} \cdot 1 = -0.4049.$$

(d) No. The rectangles between -3 and 0 are not the same size as those between 0 and 3. See Figure 5.71. There are three rectangles with nonzero height on $[-3, 0]$ and only two on $[0, 3]$.

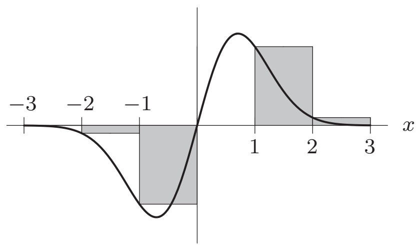


Figure 5.71

40. (a) Train A starts earlier than Train B, and stops later. At every moment Train A is going faster than Train B. Both trains increase their speed at a constant rate through the first half of their trip and slow down during the second half. Both trains reach their maximum speed at the same time. The area under the velocity graph for Train A is larger than the area under the velocity graph for Train B, meaning that Train A travels farther—as would be expected, given that its speed is always higher than B's.

(b) (i) The maximum velocity is read off the vertical axis. The graph for Train A appears to go about twice as high as the graph for Train B; see Figure 5.72. So

$$\frac{\text{Maximum velocity of Train } A}{\text{Maximum velocity of Train } B} = \frac{v_A}{v_B} \approx 2.$$

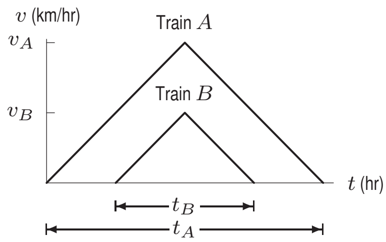


Figure 5.72

(ii) The time of travel is the horizontal distance between the start and stop times (the two t-intercepts). The horizontal distance for Train A appears to be about twice the corresponding distance for Train B; see Figure 5.72. So

$$\frac{\text{Time traveled by Train } A}{\text{Time traveled by Train } B} = \frac{t_A}{t_B} \approx 2.$$

(iii) The distance traveled by each train is given by the area under its graph. Since the area of triangle is $\frac{1}{2}$ · Base · Height, and since the base and height for Train A is approximately twice that for Train B, we have

$$\frac{\text{Distance traveled by Train } A}{\text{Distance traveled by Train } B} = \frac{\frac{1}{2} \cdot v_A \cdot t_A}{\frac{1}{2} \cdot v_B \cdot t_B} \approx 2 \cdot 2 = 4.$$

41. (a) The model indicates that wind capacity was 21,000 in 2000 and $21{,}000e^{0.22 \cdot 10} = 189{,}525.284$ in 2010.
(b) Since the continuous growth rate was 0.22, the wind energy generating capacity was increasing by 22% per year.
(c) We use the formula for average value between $t = 0$ and $t = 10$:

$$\text{Average value } = \frac{1}{10-0}\int_0^{10} 21{,}000e^{0.22t}\,dt = \frac{1}{10}(766{,}024.02) = 76{,}602.402 \text{ megawatts.}$$

42. We know that T is the time at which all the fuel has burned, so $\int_0^T r(t)\,dt$ is the total amount of fuel burned during the entire time period. We also know that Q is the initial amount of fuel.

Since the fuel is completely burned at time $t = T$, we conclude that $\int_0^T r(t)\,dt = Q$.

43. The expression $\int_0^{0.5T} r(t)\,dt$ is the amount of fuel burned during the first half of the total time.

The expression $\int_{0.5T}^{T} r(t)\,dt$ is the amount of fuel burned during the second half of the total time.

Since $r'(t) < 0$, we know the rate fuel burns is going down. Thus, fuel is burned faster at first, then slower later on, so more fuel is burned during the first half of the total time than during the second half.

We conclude that $\int_0^{0.5T} r(t)\,dt > \int_{0.5T}^{T} r(t)\,dt$.

44. The expression $\int_0^{T/3} r(t)\,dt$ is the amount of fuel burned during the first one-third of the total time. The expression $Q/3$ is one-third of the total fuel burned.

Since $r'(t) < 0$, we know the rate fuel burns is going down. Thus, fuel is burned faster at first, then slower later on, so more than one-third of the total fuel burned is burned during the first one-third of the total time.

We conclude that $\int_0^{T/3} r(t)\,dt > Q/3$.

45. We know that T_h is the time at which half the fuel has burned, so $\int_0^{T_h} r(t)\,dt$ equals half the fuel burned.

We know that T is the time at which all the fuel has burned, so $0.5T$ is halfway through the entire time period. This means $\int_0^{0.5T} r(t)\,dt$ is the amount of fuel burned halfway through the total time period.

Since $r'(t) < 0$, we know the rate fuel burns is going down. This means more than half the fuel is burned during the first half of the total time period than the second half.

We conclude that more than half the fuel is burned during the first half of the time period, so $\int_0^{T_h} r(t)\,dt. < \int_0^{0.5T} r(t)\,dt$

46. By the given property, $\int_a^a f(x)\,dx = -\int_a^a f(x)\,dx$, so $2\int_a^a f(x)\,dx = 0$. Thus $\int_a^a f(x)\,dx = 0$.

47. We know that the average value of $v(x) = 4$, so

$$\frac{1}{6-1}\int_1^6 v(x)\,dx = 4, \text{ and thus } \int_1^6 v(x)\,dx = 20.$$

Similarly, we are told that

$$\frac{1}{8-6}\int_6^8 v(x)\,dx = 5, \text{ so } \int_6^8 v(x)\,dx = 10.$$

The average value for $1 \le x \le 8$ is given by

$$\text{Average value} = \frac{1}{8-1}\int_1^8 v(x)\,dx = \frac{1}{7}\left(\int_1^6 v(x)\,dx + \int_6^8 v(x)\,dx\right) = \frac{20+10}{7} = \frac{30}{7}.$$

48. (a) Yes.
(b) No, because the sum of the left sums has 20 subdivisions. The result is the left-sum approximation with 20 subdivisions to $\int_1^3 f(x)\,dx$.

49. In Figure 5.73 the area A_1 is largest, A_2 is next, and A_3 is smallest. We have

$$\text{I} = \int_a^b f(x)\,dx = A_1, \quad \text{II} = \int_a^c f(x)\,dx = A_1 - A_2, \quad \text{III} = \int_a^e f(x)\,dx = A_1 - A_2 + A_3,$$
$$\text{IV} = \int_b^e f(x)\,dx = -A_2 + A_3, \quad \text{V} = \int_b^c f(x)\,dx = -A_2.$$

The relative sizes of A_1, A_2, and A_3 mean that I is positive and largest, III is next largest (since $-A_2 + A_3$ is negative, but less negative than $-A_2$), II is next largest, but still positive (since A_1 is larger than A_2). The integrals IV and V are both negative, but V is more negative. Thus

$$\text{V} < \text{IV} < 0 < \text{II} < \text{III} < \text{I}.$$

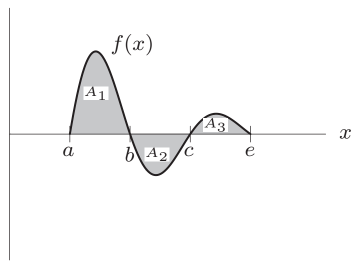


Figure 5.73

50. Since $f(t) = F'(t)$, we know from the Fundamental Theorem of Calculus that

$$\int_7^9 f(t)\,dt = \underbrace{F(9)}_{5} - \underbrace{F(7)}_{4} = 1.$$

This tells us that between the hours of 7 am and 9 am, the ice thickness increased by 1 inch.

51. Since $f(t) = F'(t)$, we know from the Fundamental Theorem of Calculus that

$$\int_4^7 f(t)\,dt = F(7) - F(4)$$
$$\text{so} \quad F(4) = \underbrace{F(7)}_{4} - \underbrace{\int_4^7 f(t)\,dt}_{1.5} = 2.5.$$

This tells us that at 4 am the ice is 2.5 inches thick.

52. We know that $f(t) = F'(t)$, so $F'(3.5) = f(3.5)$. We have:

$$\int_2^{3.5} f'(t)\,dt = 0.75$$
$$f(3.5) - f(2) = 0.75 \qquad \text{Fundamental Theorem of Calculus}$$
$$f(3.5) = \underbrace{f(2)}_{0.5} + 0.75 = 1.25,$$

so $F'(3.5) = 1.25$. This tells us that the ice thickness is growing by 1.25 inches/hour at 3:30 am.

53. (a) For $-2 \le x \le 2$, f is symmetrical about the y-axis, so $\int_{-2}^0 f(x)\,dx = \int_0^2 f(x)\,dx$ and $\int_{-2}^2 f(x)\,dx = 2\int_0^2 f(x)\,dx$.

(b) For any function f, $\int_0^2 f(x)\,dx = \int_0^5 f(x)\,dx - \int_2^5 f(x)\,dx$.

(c) Note that $\int_{-2}^0 f(x)\,dx = \frac{1}{2}\int_{-2}^2 f(x)\,dx$, so $\int_0^5 f(x)\,dx = \int_{-2}^5 f(x)\,dx - \int_{-2}^0 f(x)\,dx = \int_{-2}^5 f(x)\,dx - \frac{1}{2}\int_{-2}^2 f(x)\,dx$.

54. (a) We know that $\int_2^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^2 f(x)\,dx$. By symmetry, $\int_0^2 f(x)\,dx = \frac{1}{2}\int_{-2}^2 f(x)\,dx$, so $\int_2^5 f(x)\,dx = \int_0^5 f(x)\,dx - \frac{1}{2}\int_{-2}^2 f(x)\,dx$.

(b) $\int_2^5 f(x)\,dx = \int_{-2}^5 f(x)\,dx - \int_{-2}^2 f(x)\,dx = \int_{-2}^5 f(x)\,dx - 2\int_{-2}^0 f(x)\,dx$.

(c) Using symmetry again, $\int_0^2 f(x)\,dx = \frac{1}{2}\left(\int_{-2}^5 f(x)\,dx - \int_2^5 f(x)\,dx\right)$.

55. (a) The mouse changes direction (when its velocity is zero) at about times 17, 23, and 27.

(b) The mouse is moving most rapidly to the right at time 10 and most rapidly to the left at time 40.

(c) The mouse is farthest to the right when the integral of the velocity, $\int_0^t v(t)\,dt$, is most positive. Since the integral is the sum of the areas above the axis minus the areas below the axis, the integral is largest when the velocity is zero at about 17 seconds. The mouse is farthest to the left of center when the integral is most negative at 40 seconds.

(d) The mouse's speed decreases during seconds 10 to 17, from 20 to 23 seconds, and from 24 seconds to 27 seconds.

(e) The mouse is at the center of the tunnel at any time t for which the integral from 0 to t of the velocity is zero. This is true at time 0 and again somewhere around 35 seconds.

56. To estimate the distance traveled during the first 15 seconds, we calculate upper and lower bound estimates for this distance, and then average these estimates. Note that because the speed of the Prius is in miles per hour, we need to convert seconds to hours (1 sec $= 1/3600$ hours) to calculate the distance estimates in miles. We then convert miles to feet (1 mile $= 5280$ feet).

$$\begin{array}{c}\text{Total distance traveled}\\ \text{between } t=0 \text{ and } t=15\\ \text{(Lower Estimate)}\end{array} = 0 \cdot \frac{5}{3600} + 20 \cdot \frac{5}{3600} + 33 \cdot \frac{5}{3600} = 0.0736 \text{ miles or 389 feet}$$

$$\begin{array}{c}\text{Total distance traveled}\\ \text{between } t=0 \text{ and } t=15\\ \text{(Upper Estimate)}\end{array} = 20 \cdot \frac{5}{3600} + 33 \cdot \frac{5}{3600} + 45 \cdot \frac{5}{3600} = 0.1361 \text{ miles or 719 feet}$$

Since:

$$\begin{array}{c}\text{Average of Lower and Upper}\\ \text{Estimates}\end{array} = \frac{389 + 719}{2} = 554 \text{ feet},$$

the 2010 Prius Prototype travels about 554 feet, roughly a tenth of a mile, during the first 15 seconds of movement in EV-only mode.

57. To estimate the distance between the two cars at $t = 5$, we calculate upper and lower bound estimates for this distance and then average these estimates. Note that because the speed of the Prius is in miles per hour, we need to convert seconds to hours (1 sec $= 1/3600$ hours) to calculate the distance estimates in miles. We may then convert miles to feet (1 mile $= 5280$ feet).

$$\begin{aligned}\text{Distance between cars at } t=5 \text{ (Upper Estimate)} &\approx (\text{Difference in speeds at } t=5)\cdot(\text{Travel time})\\ &= (33-20)\cdot\frac{5}{3600} = 0.018 \text{ miles}\end{aligned}$$

$$\begin{aligned}\text{Distance between cars at } t=5 \text{ (Lower Estimate)} &\approx (\text{Difference in speeds at } t=0)\cdot(\text{Travel time})\\ &= (0-0)\cdot\frac{5}{3600} = 0 \text{ miles}\end{aligned}$$

Since:

$$\text{Average of Lower and Upper Estimates of distance between cars at } t=5 = \frac{0.018+0}{2} = 0.009 \text{ miles} = 48 \text{ feet},$$

the distance between the two cars is about 48 feet, five seconds after leaving the stoplight.

58. (a) The acceleration is positive for $0 \le t < 40$ and for a tiny period before $t = 60$, since the slope is positive over these intervals. Just to the left of $t = 40$, it looks like the acceleration is approaching 0. Between $t = 40$ and a moment just before $t = 60$, the acceleration is negative.

(b) The maximum altitude was about 500 feet, when t was a little greater than 40 (here we are estimating the area under the graph for $0 \le t \le 42$).

(c) The acceleration is greatest when the slope of the velocity is most positive. This happens just before $t = 60$, where the magnitude of the velocity is plunging and the direction of the acceleration is positive, or up.

(d) The deceleration is greatest when the slope of the velocity is most negative. This happens just after $t = 40$.

(e) After the Montgolfier Brothers hit their top climbing speed (at $t = 40$), they suddenly stopped climbing and started to fall. This suggests some kind of catastrophe—the flame going out, the balloon ripping, etc. (In actual fact, in their first flight in 1783, the material covering their balloon, held together by buttons, ripped and the balloon landed in flames.)

(f) The total change in altitude for the Montgolfiers and their balloon is the definite integral of their velocity, or the total area under the given graph (counting the part after $t = 42$ as negative, of course). As mentioned before, the total area of the graph for $0 \le t \le 42$ is about 500. The area for $t > 42$ is about 220. So subtracting, we see that the balloon finished 280 feet or so higher than where it began.

59. (a) About 300 meter^3/sec.

(b) About 250 meter^3/sec.

(c) Looking at the graph, we can see that the 1996 flood reached its maximum just between March and April, for a high of about 1250 meter^3/sec. Similarly, the 1957 flood reached its maximum in mid-June, for a maximum flow rate of 3500 meter^3/sec.

(d) The 1996 flood lasted about $1/3$ of a month, or about 10 days. The 1957 flood lasted about 4 months.

(e) The area under the controlled flood graph is about $2/3$ box. Each box represents 500 meter^3/sec for one month. Since

$$\begin{aligned}1 \text{ month} &= 30\frac{\text{days}}{\text{month}}\cdot 24\frac{\text{hours}}{\text{day}}\cdot 60\frac{\text{minutes}}{\text{hour}}\cdot 60\frac{\text{seconds}}{\text{minute}}\\ &= 2.592\cdot 10^6 \approx 2.6\cdot 10^6 \text{seconds},\end{aligned}$$

each box represents

$$\text{Flow} \approx (500 \text{ meter}^3/\text{sec})\cdot(2.6\cdot 10^6 \text{ sec}) = 13\cdot 10^8 \text{ meter}^3 \text{of water.}$$

So, for the artificial flood,

$$\text{Additional flow} \approx \frac{2}{3}\cdot 13\cdot 10^8 = 8.7\cdot 10^8 \text{ meter}^3 \approx 10^9 \text{ meter}^3.$$

(f) The 1957 flood released a volume of water represented by about 12 boxes above the 250 meter/sec baseline. Thus, for the natural flood,

$$\text{Additional flow} \approx 12\cdot 13\cdot 10^8 = 1.95\cdot 10^{10} \approx 2\cdot 10^{10} \text{ meter}^3.$$

So, the natural flood was nearly 20 times larger than the controlled flood and lasted much longer.

60. In (a), $f'(1)$ is the slope of a tangent line at $x = 1$, which is negative. As for (c), the rate of change in $f(x)$ is given by $f'(x)$, and the average value of this over $0 \le x \le a$ is

$$\frac{1}{a-0}\int_0^a f'(x)\,dx = \frac{f(a)-f(0)}{a-0}.$$

This is the slope of the line through the points $(0,1)$ and $(a,0)$, which is less negative that the tangent line at $x = 1$. Therefore, $(a) < (c) < 0$. The quantity (b) is $\left(\int_0^a f(x)\,dx\right)/a$ and (d) is $\int_0^a f(x)\,dx$, which is the net area under the graph of f (counting the area as negative for f below the x-axis). Since $a > 1$ and $\int_0^a f(x)\,dx > 0$, we have 0 <(b)<(d). Therefore

$$\text{(a)} < \text{(c)} < \text{(b)} < \text{(d)}.$$

61. This expression gives the tangent-line approximation to $f(3.1)$. Since $f''(x) < 0$, this tells us that the graph of f is concave-down, which means that the graph of f lies below its tangent line. Thus, this value overestimates $f(3.1)$.

62. From the Fundamental Theorem of Calculus, we see that

$$f(3) + \int_3^{3.1} f'(t)\,dt = f(3) + f(3.1) - f(3) = f(3.1),$$

so this value is exact.

63. The expression $\dfrac{F(3.11) - F(3.1)}{0.01}$ gives the slope of the secant line to the graph of F from $x = 3.1$ to $x = 3.11$. This approximates the value of $F'(3.1)$. Since $f(x) = F'(x)$, this means the expressions approximates the value of $f(3.1)$.

Notice that $f'(x) = F''(x)$; since $f'(x) > 0$, this means $F''(x) > 0$, which tells us that the graph of F is concave-up. This means the slope of the secant line from $x = 3.1$ to $x = 3.11$ is greater (either "less negative" or "more positive") than the slope of the tangent line at $x = 3.1$. Thus, this expression overestimates $F'(3.1)$, so it is an overestimate of $f(3.1)$.

64. We see that $\displaystyle\sum_{i=1}^{n} f'(x_i)\,\Delta x$ is a right-hand Riemann sum. Since $f''(x) < 0$ for all x, we know that $f'(x)$ is a decreasing function, which means the right-hand sum is an underestimate of the true value. Thus,

$$\begin{aligned} f(3) + \sum_{i=1}^{n} f'(x_i)\,\Delta x &< f(3) + \int_3^{3.1} f'(x)\,dx \\ &= f(3) + f(3.1) - f(3) \quad \text{Fundamental Theorem of Calculus} \\ &= f(3.1). \end{aligned}$$

This means the expression is an underestimate of $f(3.1)$.

65. (a) Divide the interval $0 \le t \le T$ into small subintervals of length Δt on which the temperature is approximately constant. Then if t_i is the i^{th} interval, on that interval

$$\Delta S \approx (H - H_{\text{min}})\Delta t = (f(t_i) - H_{\text{min}})\Delta t.$$

Thus, the total number of degree-days is approximated by

$$S \approx \sum_{i=1}^{n} (f(t_i) - H_{\text{min}})\Delta t.$$

As $n \to \infty$, we have

$$S = \int_0^T (f(t) - H_{\text{min}})\,dt.$$

(b) Since $H_{\text{min}} = 15^\circ\text{C}$ and $S = 125$ degree-days, the formula in part (a) gives

$$125 = \int_0^T (f(t) - 15)dt.$$

We approximate the definite integral using a Riemann sum:

$$\int_0^T (f(t)-15)dt = \sum_{i=1}^{n}(f(t_i)-15)\Delta t.$$

In the table, we have $\Delta t = 1$. We want to find how many terms in the sum are required to (approximately) equal the required number of degree-days, $S = 125$. From the table, we have

$$\sum_{i=1}^{2}(f(t_i)-15)\Delta t = (20-15)1 + (22-15)1 = 12,$$

so the sum of degree-days over a two-day period is much smaller than the required sum of 125. Likewise, the sum over a three-day period is only 24:

$$\sum_{i=1}^{3}(f(t_i)-15)\Delta t = (20-15)1 + (22-15)1 + (27-15)1 = 24.$$

Summing over a ten-day period and an eleven-day period, we find, respectively, that

$$\begin{aligned}\sum_{i=1}^{10}(f(t_i)-15)\Delta t &= (20-15)1 + (22-15)1 + (27-15)1 + (28-15)1 + (27-15)1 + (31-15)1\\ &\quad +(29-15)1 + (30-15)1 + (28-15)1 + (25-15)1 = 117\\ \sum_{i=1}^{11}(f(t_i)-15)\Delta t &= (20-15)1 + (22-15)1 + (27-15)1 + (28-15)1 + (27-15)1 + (31-15)1\\ &\quad +(29-15)1 + (30-15)1 + (28-15)1 + (25-15)1 + (24-15)1 = 126.\end{aligned}$$

Thus, the sum reaches 125 degree-days somewhere just short of $T = 11$ days.

66. (a) The slope of a line is the tangent of the angle the line makes with the horizontal. Thus the slope, $f'(x)$, of the tangent line at x is the tangent of the angle θ.

(b) Using the identity

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

and part (a) we have

$$1 + (f'(x))^2 = \frac{1}{\cos^2\theta}.$$

(c) Using the chain rule, we have

$$\frac{d}{dx}(\tan\theta) = \frac{1}{\cos^2\theta}\frac{d\theta}{dx}.$$

Therefore, using parts (a) and (b), we have

$$\begin{aligned}\frac{d\theta}{dx} &= \cos^2\theta \frac{d}{dx}f'(x)\\ &= \frac{1}{1+(f'(x))^2}f''(x).\end{aligned}$$

(d) By the Fundamental Theorem of Calculus we have

$$\theta(b) - \theta(a) = \int_a^b \theta'(x)\,dx = \int_a^b \frac{f''(x)}{1+(f'(x))^2}\,dx.$$

CAS Challenge Problems

67. (a) We have $\Delta x = (1-0)/n = 1/n$ and $x_i = 0 + i\cdot\Delta x = i/n$. So we get

$$\text{Right-hand sum } = \sum_{i=1}^{n}(x_i)^4\Delta x = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^4\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\frac{i^4}{n^5}.$$

(b) The CAS gives

$$\text{Right-hand sum} = \sum_{i=1}^{n} \frac{i^4}{n^5} = \frac{6n^4 + 15n^3 + 10n^2 - 1}{30n^4}.$$

(The results may look slightly different depending on the CAS you use.)

(c) Using a CAS or by hand, we get

$$\lim_{n\to\infty} \frac{6n^4 + 15n^3 + 10n^2 - 1}{30n^4} = \lim_{n\to\infty} \frac{6n^4}{30n^4} = \frac{1}{5}.$$

The numerator is dominated by the highest power term, which is $6n^4$, so when n is large, the ratio behaves like $6n^4/30n^4 = 1/5$ as $n \to \infty$. Thus we see that

$$\int_0^1 x^4 dx = \frac{1}{5}.$$

68. (a) A Riemann sum with n subdivisions of $[0,1]$ has $\Delta x = 1/n$ and $x_i = i/n$. Thus,

$$\text{Right-hand sum } = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^5 \left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{i^5}{n^6}.$$

(b) A CAS gives

$$\text{Right-hand sum } = \sum_{i=1}^{n} \frac{i^5}{n^6} = \frac{2n^4 + 6n^3 + 5n^2 - 1}{12n^4}.$$

(c) Taking the limit by hand or using a CAS gives

$$\lim_{n\to\infty} \frac{2n^4 + 6n^3 + 5n^2 - 1}{12n^4} = \lim_{n\to\infty} \frac{2n^4}{12n^4} = \frac{1}{6}.$$

The numerator is dominated by the highest power term, which is $2n^4$, so the ratio behaves like $2n^4/12n^4 = 1/6$, as $n \to \infty$. Thus we see that

$$\int_0^1 x^5 dx = \frac{1}{6}.$$

69. (a) Since the length of the interval of integration is $2 - 1 = 1$, the width of each subdivision is $\Delta t = 1/n$. Thus the endpoints of the subdivision are

$$t_0 = 1, \quad t_1 = 1 + \Delta t = 1 + \frac{1}{n}, \quad t_2 = 1 + 2\Delta t = 1 + \frac{2}{n}, \ldots,$$
$$t_i = 1 + i\Delta t = 1 + \frac{i}{n}, \ldots, \quad t_{n-1} = 1 + (n-1)\Delta t = 1 + \frac{n-1}{n}.$$

Thus, since the integrand is $f(t) = t$,

$$\text{Left-hand sum} = \sum_{i=0}^{n-1} f(t_i)\Delta t = \sum_{i=0}^{n-1} t_i \Delta t = \sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right) \frac{1}{n} = \sum_{i=0}^{n-1} \frac{n+i}{n^2}.$$

(b) The CAS finds the formula for the Riemann sum

$$\sum_{i=0}^{n-1} \frac{n+i}{n^2} = \frac{\frac{(-1+n)\,n}{2} + n^2}{n^2} = \frac{3}{2} - \frac{1}{2n}.$$

(c) Taking the limit as $n \to \infty$

$$\lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{2n}\right) = \lim_{n\to\infty} \frac{3}{2} - \lim_{n\to\infty} \frac{1}{2n} = \frac{3}{2} + 0 = \frac{3}{2}.$$

(d) The shape under the graph of $y = t$ between $t = 1$ and $t = 2$ is a trapezoid of width 1, height 1 on the left and 2 on the right. So its area is $1 \cdot (1 + 2)/2 = 3/2$. This is the same answer we got by computing the definite integral.

70. (a) Since the length of the interval of integration is $2 - 1 = 1$, the width of each subdivision is $\Delta t = 1/n$. Thus the endpoints of the subdivision are

$$t_0 = 1, \quad t_1 = 1 + \Delta t = 1 + \frac{1}{n}, \quad t_2 = 1 + 2\Delta t = 1 + \frac{2}{n}, \ldots,$$
$$t_i = 1 + i\Delta t = 1 + \frac{i}{n}, \ldots, \quad t_{n-1} = 1 + (n-1)\Delta t = 1 + \frac{n-1}{n}.$$

Thus, since the integrand is $f(t) = t^2$,

$$\text{Left-hand sum} = \sum_{i=0}^{n-1} f(t_i)\Delta t = \sum_{i=0}^{n-1} t_i^2 \Delta t = \sum_{i=0}^{n-1} \left(1 + \frac{i}{n}\right)^2 \frac{1}{n} = \sum_{i=0}^{n-1} \frac{(n+i)^2}{n^3}.$$

(b) Using a CAS to find the sum, we get

$$\sum_{i=0}^{n-1} \frac{(n+i)^2}{n^3} = \frac{(-1+2n)(-1+7n)}{6n^2} = \frac{7}{3} + \frac{1}{6n^2} - \frac{3}{2n}.$$

(c) Taking the limit as $n \to \infty$

$$\lim_{n\to\infty}\left(\frac{7}{3} + \frac{1}{6n^2} - \frac{3}{2n}\right) = \lim_{n\to\infty}\frac{7}{3} + \lim_{n\to\infty}\frac{1}{6n^2} - \lim_{n\to\infty}\frac{3}{2n} = \frac{7}{3} + 0 + 0 = \frac{7}{3}.$$

(d) We have calculated $\int_1^2 t^2\,dt$ using Riemann sums. Since t^2 is above the t-axis between $t = 1$ and $t = 2$, this integral is the area; so the area is 7/3.

71. (a) Since the length of the interval of integration is π, the width of each subdivision is $\Delta x = \pi/n$. Thus the endpoints of the subdivision are

$$x_0 = 0, \quad x_1 = 0 + \Delta x = \frac{\pi}{n}, \quad x_2 = 0 + 2\Delta x = \frac{2\pi}{n}, \ldots,$$
$$x_i = 0 + i\Delta x = \frac{i\pi}{n}, \quad \ldots, \quad x_n = 0 + n\Delta x = \frac{n\pi}{n} = \pi.$$

Thus, since the integrand is $f(x) = \sin x$,

$$\text{Right-hand sum} = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \sin(x_i)\Delta x = \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)\frac{\pi}{n}.$$

(b) If the CAS can evaluate this sum, we get

$$\sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)\frac{\pi}{n} = \frac{\pi\cot(\pi/2n)}{n} = \frac{\pi\cos(\pi/2n)}{n\sin(\pi/2n)}.$$

(c) Using the computer algebra system, we find that

$$\lim_{n\to\infty}\frac{\pi\cos(\pi/2n)}{n\sin(\pi/2n)} = 2.$$

(d) The computer algebra system gives

$$\int_0^\pi \sin x\,dx = 2.$$

72. (a) A CAS gives

$$\int_a^b \sin(cx)\,dx = \frac{\cos(ac)}{c} - \frac{\cos(bc)}{c}.$$

(b) If $F(x)$ is an antiderivative of $\sin(cx)$, then the Fundamental Theorem of Calculus says that

$$\int_a^b \sin(cx)\,dx = F(b) - F(a).$$

Comparing this with the answer to part (a), we see that

$$F(b) - F(a) = \frac{\cos(ac)}{c} - \frac{\cos(bc)}{c} = \left(-\frac{\cos(cb)}{c}\right) - \left(-\frac{\cos(ca)}{c}\right).$$

This suggests that

$$F(x) = -\frac{\cos(cx)}{c}.$$

Taking the derivative confirms this:

$$\frac{d}{dx}\left(-\frac{\cos(cx)}{c}\right) = \sin(cx).$$

73. (a) Different systems may give different answers. A typical answer is

$$\int_a^c \frac{x}{1+bx^2}\,dx = \frac{\ln\left(\frac{|c^2b+1|}{|a^2b+1|}\right)}{2b}.$$

Some CASs may not have the absolute values in the answer; since $b > 0$, the answer is correct without the absolute values.

(b) Using the properties of logarithms, we can rewrite the answer to part (a) as

$$\int_a^c \frac{x}{1+bx^2}\,dx = \frac{\ln|c^2b+1| - \ln|a^2b+1|}{2b} = \frac{\ln|c^2b+1|}{2b} - \frac{\ln|a^2b+1|}{2b}.$$

If $F(x)$ is an antiderivative of $x/(1+bx^2)$, then the Fundamental Theorem of Calculus says that

$$\int_a^c \frac{x}{1+bx^2}\,dx = F(c) - F(a).$$

Thus

$$F(c) - F(a) = \frac{\ln|c^2b+1|}{2b} - \frac{\ln|a^2b+1|}{2b}.$$

This suggests that

$$F(x) = \frac{\ln|1+bx^2|}{2b}.$$

(Since $b > 0$, we know $|1+bx^2| = 1+bx^2$.) Taking the derivative confirms this:

$$\frac{d}{dx}\left(\frac{\ln(1+b\,x^2)}{2\,b}\right) = \frac{x}{1+b\,x^2}.$$

PROJECTS FOR CHAPTER FIVE

1. (a) The work done by the ventricle wall on the blood is the force exerted on the blood times the distance the wall pushes the blood; that is, the distance the wall moves. In Phases 1 and 3 there is no work done because the ventricle wall is motionless.

The wall moves in Phases 2 and 4. The work done by the wall is positive if the force from the wall to the blood is in the direction of the movement, and it is negative if the force and movement are in opposite directions. In Phase 4, the ventricle is contracting, pushing the blood, so the work is positive. In Phase 2, the ventricle is expanding, being pushed out by the blood, so the work is negative.

(b) Since pressure is expressed in units of force per unit area (newtons/cm^2) and volume is expressed in units of length cubed (cm^3), area in the pressure-volume plane has units of

$$\frac{\text{newtons}}{(\text{cm})^2} \times (\text{cm})^3 = \text{newtons} \times \text{cm}.$$

The units of area in the pressure-volume plane are force times distance, which are the units of work and energy.

(c) (i) Let the area of the LV wall be A. The force exerted by the wall in Phase 4 when the LV has volume V is the pressure, $f(V)$, times the area A. When the wall moves inward during Phase 4 a small distance Δx, the volume, which decreases, changes by approximately $\Delta V = -A\Delta x$. The work done during this movement is approximately

$$\text{Force} \cdot \text{Distance} = f(V)A \cdot \Delta x = -f(V)\,\Delta V.$$

Thus, the total work done in Phase 4 as the volume decreases from b to a is $\int_b^a -f(V)\,dV = \int_a^b f(V)\,dV$.

(ii) The force exerted by the wall in Phase 2 when the LV has volume V is the pressure, $g(V)$, times the area of the LV wall, A. When the wall moves outward during Phase 2 a small distance Δx, the volume, which increases, changes by approximately $\Delta V = A\Delta x$. Since the force from the wall to the blood is in the direction opposite the direction of motion, the work done during this movement is approximately

$$-\text{Force} \cdot \text{Distance} = -g(V)A \cdot \Delta x = -g(V)\,\Delta V.$$

Thus, the total work done in Phase 2 as the volume increases from a to b is $\int_a^b -g(V)\,dV = -\int_a^b g(V)\,dV$.

(d) Since Phases 1 and 3 are vertical lines, the area of the pressure-volume loop is the area between $f(V)$ and $g(V)$ from $V = a$ to $V = b$. Thus,

$$\text{Area enclosed} = \int_a^b (f(V) - g(V))\,dV.$$

Since the LV does no work in Phases 1 and 3, using part (c), we have

$$\begin{aligned}\text{Area enclosed} &= \int_a^b f(V)\,dV - \int_a^b g(V)\,dV \\ &= \text{Work done in Phase 4} + \text{Work done in Phase 2} \\ &= \text{Work done by LV in one cardiac cycle.}\end{aligned}$$

2. To find distances from the velocity graph, we use the fact that if t is the time measured from noon, and v is the velocity, then

$$\begin{matrix}\text{Distance traveled} \\ \text{by car up to time } T\end{matrix} = \int_0^T v\,dt = \begin{matrix}\text{Area under velocity} \\ \text{graph between 0 and } T.\end{matrix}$$

The truck's motion can be represented on the same graph by the horizontal line $v = 50$, starting at $t = 1$. The distance traveled by the truck is then the rectangular area under this line, and the distance between the two vehicles is the difference between these areas. Note each small rectangle on the graph corresponds to moving at 10 mph for a half hour (that is, to a distance of 5 miles).

(a) The distance traveled by the car when the truck starts is represented by the shaded area in Figure 5.74, which totals about seven rectangles or about 35 miles.

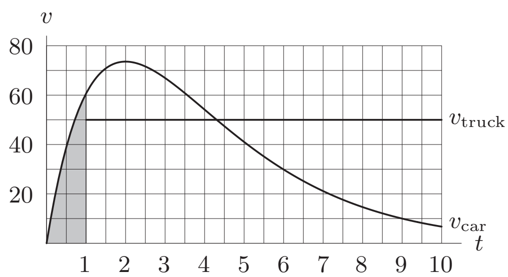


Figure 5.74: Shaded area represents distance traveled by car from noon to 1pm

(b) At 3 pm, the car is traveling with a velocity of about 67 mph, while the truck has a velocity of 50 mph. Because the car is ahead of the truck at 3 pm and is traveling at a greater velocity, the distance between the car and the truck is increasing at this time. If d_{car} and d_{truck} represent the distance traveled by the car and the truck respectively, then

$$\text{Distance apart} = d_{\text{car}} - d_{\text{truck}}.$$

The rate of change of the distance apart is given by its derivative:

$$\begin{aligned}(\text{Distance apart})' &= (d_{\text{car}})' - (d_{\text{truck}})' \\ &= v_{\text{car}} - v_{\text{truck}}\end{aligned}$$

At 3 pm, we have $(\text{Distance apart})' = 67$ mph $- 50$ mph $= 17$ mph. Thus, at 3 pm the car is traveling with a velocity 17 mph greater than the truck's velocity, and the distance between them is increasing at 17 miles per hour.

At 2 pm, the car's velocity is greatest. Because the truck's velocity is constant, $v_{\text{car}} - v_{\text{truck}}$ is largest when the car's velocity is largest. Thus, at 2 pm the distance between the car and the truck is increasing fastest—that is, the car is pulling away at the greatest rate.

(Note: This only takes into account the time when the truck is moving. When the truck is not moving (from 12:00 to 1:00), the car pulls away from the truck at an even greater rate.)

(c) The car starts ahead of the truck, and the distance between them increases as long as the velocity of the car is greater than the velocity of the truck. Later, when the truck's velocity exceeds the car's, the truck starts to gain on the car. In other words, the distance between the car and the truck increases as long as $v_{\text{car}} > v_{\text{truck}}$, and it decreases when $v_{\text{car}} < v_{\text{truck}}$. Therefore, the maximum distance occurs when $v_{\text{car}} = v_{\text{truck}}$, that is, when $t \approx 4.3$ hours (at about 4:20 pm). (See Figure 5.75.)

The distance traveled by the car is the area under the v_{car} graph between $t = 0$ and $t = 4.3$; the distance traveled by the truck is the area under the v_{truck} line between $t = 1$ (when it started) and $t = 4.3$. So the distance between the car and truck is represented by the shaded area in Figure 5.75. Thus, approximately

$$\text{Distance between car and truck } = 35 \text{ miles } + 50 \text{ miles} = 85 \text{ miles}.$$

(d) The truck overtakes the car when both have traveled the same distance. This occurs when the area under the curve (v_{car}) up to that time equals the area under the line v_{truck} up to that time. Since the areas under v_{car} and v_{truck} overlap (see Figure 5.76), they are equal when the lightly shaded area equals the heavily shaded area (which we know is about 85 miles). This happens when $t \approx 8.3$ hours, or about 8:20 pm. At this time, each has traveled about 365 miles.

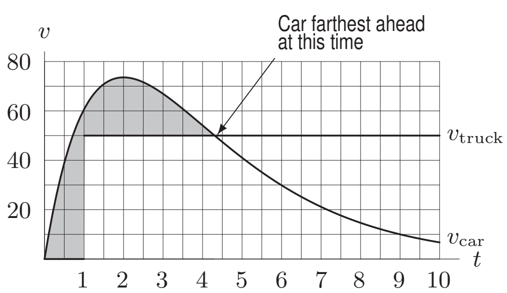


Figure 5.75: Shaded area = distance by which car is ahead at about 4:20 pm

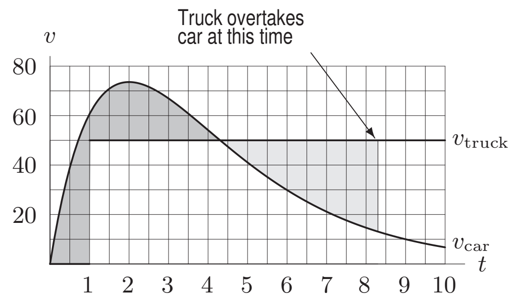


Figure 5.76: Truck overtakes car when dark and light shaded areas are equal

(e) See Figure 5.77.

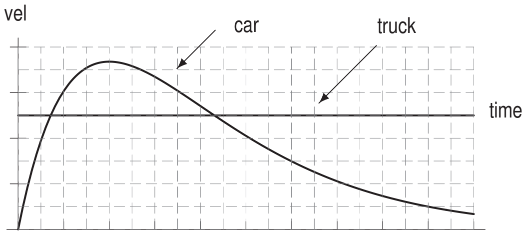


Figure 5.77

(f) The graphs intersect twice, at about 0.7 hours and 4.3 hours. At each intersection point, the velocity of the car is equal to the velocity of the truck, so $v_{\text{car}} = v_{\text{truck}}$. From the time they start until 0.7 hours later, the truck is traveling at a greater velocity than the car, so the truck is ahead of the car and is pulling farther away. At 0.7 hours they are traveling at the same velocity, and after 0.7 hours the car is traveling faster than the truck, so that the car begins to gain on the truck. Thus, at 0.7 hours the truck is farther from the car than it is immediately before or after 0.7 hours.

Similarly, because the car's velocity is greater than the truck's after 0.7 hours, it will catch up with the truck and eventually pass and pull away from the truck until 4.3 hours, at which point the two are again traveling at the same velocity. After 4.3 hours the truck travels faster than the car, so that it now gains on the car. Thus, 4.3 hours represents the point where the car is farthest ahead of the truck.

3. (a) For Operation 1, we have the following:

(i) A plot of current use versus time is given in Figure 5.78.

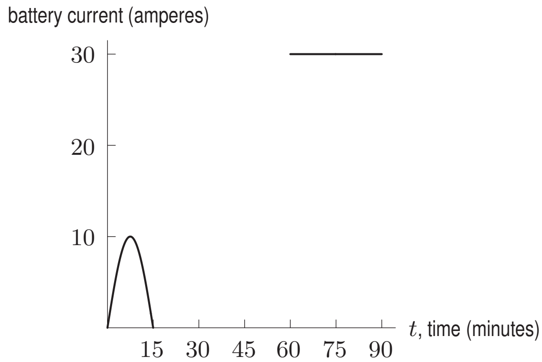


Figure 5.78: Operation 1

The battery current function is given by the formulas

$$D(t) = \begin{cases} 10\sin\dfrac{2\pi t}{30} & 0 \le t \le 15 \\ 0 & 15 \le t \le 60 \\ 30 & 60 \le t \le 90. \end{cases}$$

(ii) The battery current function gives the rate at which the current is flowing. Thus, the total discharge is given by the integral of the battery current function:

$$\begin{aligned} \text{Total discharge } &= \int_0^{90} D(t)\,dt = \int_0^{15} 10\sin\frac{2\pi t}{30}\,dt + \int_{15}^{60} 0\,dt + \int_{60}^{90} 30\,dt \\ &\approx 95.5 + 900 \\ &= 995.5 \text{ ampere-minutes} = 16.6 \text{ ampere-hours.} \end{aligned}$$

(iii) The battery can discharge up to 40% of 50 ampere-hours, which is 20 ampere-hours, without damage. Since Operation 1 can be performed with just 16.6 ampere-hours, it is safe.

(b) For Operation 2, we have the following:

(i) The total battery discharge is given by the area under the battery current curve in Figure 5.79. The area under the right-most portion of the curve, (when the satellite is shadowed by the earth), is easily calculated as 30 amps$\cdot$ 30 minutes $=$ 900 ampere-minutes $=$ 15 ampere-hours. For the other part we estimate by trapezoids, which are the average of left and right rectangles on each subinterval. Estimated values of the function are in Table 5.3.

Table 5.3 *Estimated values of the battery current*

Time	0	5	10	15	20	25	30
Current	5	16	18	12	5	12	0

Using $\Delta t = 5$, we see

$$\begin{aligned}\text{Total discharge } &= \frac{1}{2}(5+16)\cdot 5 + \frac{1}{2}(16+18)\cdot 5 + \frac{1}{2}(18+12)\cdot 5 \\ &\quad + \frac{1}{2}(12+5)\cdot 5 + \frac{1}{2}(5+12)\cdot 5 + \frac{1}{2}(12+0)\cdot 5 \\ &\approx 330 \text{ ampere-minutes} = 5.5 \text{ ampere-hours.}\end{aligned}$$

The total estimated discharge is 20.5 ampere-hours.

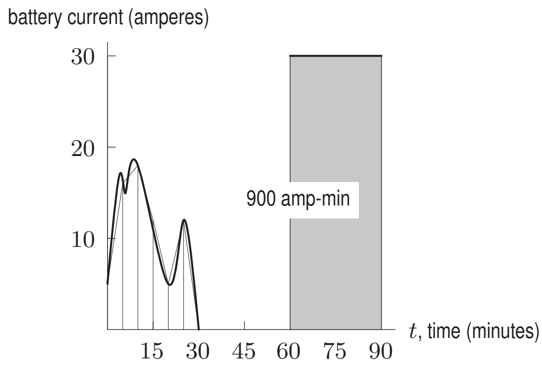


Figure 5.79: Operation 2

(ii) Since the estimated discharge appears to be an underestimate, Operation 2 probably should not be performed.

CHAPTER SIX

Solutions for Section 6.1

Exercises

1. (a) If $f(x)$ is positive over an interval, then $F(x)$ is increasing over the interval.
(b) If $f(x)$ is increasing over an interval, then $F(x)$ is concave up over the interval.

2. Since dP/dt is positive for $t < 3$ and negative for $t > 3$, we know that P is increasing for $t < 3$ and decreasing for $t > 3$. Between each two integer values, the magnitude of the change is equal to the area between the graph dP/dt and the t-axis. For example, between $t = 0$ and $t = 1$, we see that the change in P is 1. Since $P = 0$ at $t = 0$, we must have $P = 1$ at $t = 1$. The other values are found similarly, and are shown in Table 6.1.

Table 6.1

t	0	1	2	3	4	5
P	0	1	2	2.5	2	1

3. Since dP/dt is negative for $t < 3$ and positive for $t > 3$, we know that P is decreasing for $t < 3$ and increasing for $t > 3$. Between each two integer values, the magnitude of the change is equal to the area between the graph dP/dt and the t-axis. For example, between $t = 0$ and $t = 1$, we see that the change in P is -1. Since $P = 2$ at $t = 0$, we must have $P = 1$ at $t = 1$. The other values are found similarly, and are shown in Table 6.2.

Table 6.2

t	1	2	3	4	5
P	1	0	$-1/2$	0	1

4. See Figure 6.1.

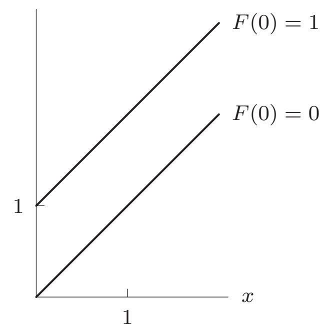


Figure 6.1

5. See Figure 6.2.

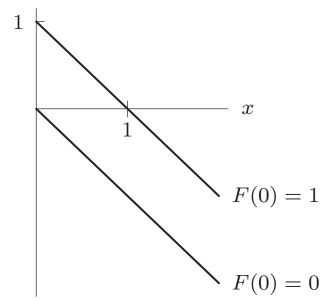


Figure 6.2

6. See Figure 6.3.

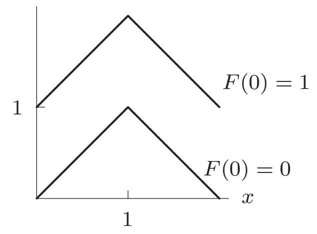


Figure 6.3

7. See Figure 6.4.

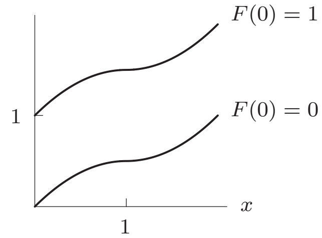


Figure 6.4

8. See Figure 6.5.

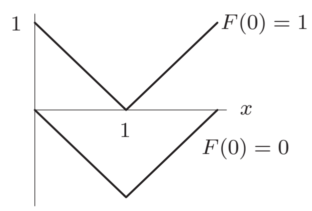


Figure 6.5

9. See Figure 6.6.

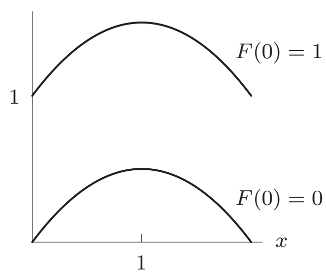


Figure 6.6

10. See Figure 6.7.

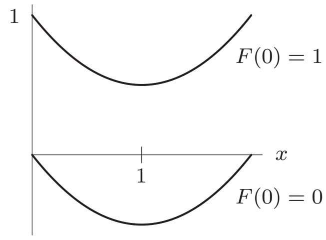


Figure 6.7

11. See Figure 6.8

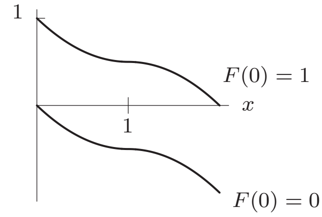


Figure 6.8

Problems

12. (a) If $\int_2^5 f(x)\,dx = 4$, and $F(5) = 10$ then $F(2) = 6$, since $\int_2^5 f(x)\,dx$ gives the total change in $F(x)$ between $x = 2$ and $x = 5$.

(b) If $\int_0^{100} f(x)\,dx = 0$, then $F(100) = F(0)$, since the total change in $F(x)$ from $x = 0$ to $x = 100$ is 0.

13. By the Fundamental Theorem of Calculus, we know that

$$f(2) - f(0) = \int_0^2 f'(x)dx.$$

Using a left-hand sum, we estimate $\int_0^2 f'(x)dx \approx (10)(2) = 20$. Using a right-hand sum, we estimate $\int_0^2 f'(x)dx \approx (18)(2) = 36$. Averaging, we have

$$\int_0^2 f'(x)dx \approx \frac{20+36}{2} = 28.$$

We know $f(0) = 100$, so

$$f(2) = f(0) + \int_0^2 f'(x)dx \approx 100 + 28 = 128.$$

Similarly, we estimate

$$\int_2^4 f'(x)dx \approx \frac{(18)(2) + (23)(2)}{2} = 41,$$

so

$$f(4) = f(2) + \int_2^4 f'(x)dx \approx 128 + 41 = 169.$$

Similarly,

$$\int_4^6 f'(x)dx \approx \frac{(23)(2) + (25)(2)}{2} = 48,$$

so

$$f(6) = f(4) + \int_4^6 f'(x)dx \approx 169 + 48 = 217.$$

The values are shown in the table.

x	0	2	4	6
$f(x)$	100	128	169	217

14. The change in $f(x)$ between 0 and 2 is equal to $\int_0^2 f'(x)\,dx$. A left-hand estimate for this integral is $(17)(2) = 34$ and a right hand estimate is $(15)(2) = 30$. Our best estimate is the average, 32. The change in $f(x)$ between 0 and 2 is $+32$. Since $f(0) = 50$, we have $f(2) = 82$. We find the other values similarly. The results are shown in Table 6.3.

Table 6.3

x	0	2	4	6
$f(x)$	50	82	107	119

15. Between $t = 0$ and $t = 1$, the particle moves at 10 km/hr for 1 hour. Since it starts at $x = 5$, the particle is at $x = 15$ when $t = 1$. See Figure 6.9. The graph of distance is a straight line between $t = 0$ and $t = 1$ because the velocity is constant then.

Between $t = 1$ and $t = 2$, the particle moves 10 km to the left, ending at $x = 5$. Between $t = 2$ and $t = 3$, it moves 10 km to the right again. See Figure 6.9.

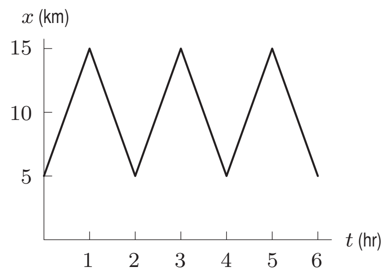


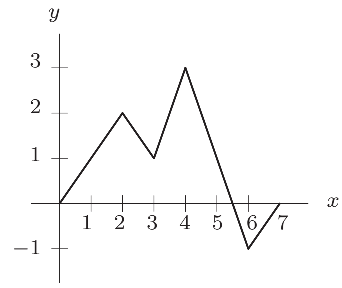
Figure 6.9

As an aside, note that the original velocity graph is not entirely realistic as it suggests the particle reverses direction instantaneously at the end of each hour. In practice this means the reversal of direction occurs over a time interval that is short in comparison to an hour.

16. (a) We know that $\int_0^3 f'(x)dx = f(3) - f(0)$ from the Fundamental Theorem of Calculus. From the graph of f' we can see that $\int_0^3 f'(x)dx = 2 - 1 = 1$ by subtracting areas between f' and the x-axis. Since $f(0) = 0$, we find that $f(3) = 1$. Similar reasoning gives $f(7) = \int_0^7 f'(x)dx = 2 - 1 + 2 - 4 + 1 = 0$.

(b) We have $f(0) = 0$, $f(2) = 2$, $f(3) = 1$, $f(4) = 3$, $f(6) = -1$, and $f(7) = 0$. So the graph, beginning at $x = 0$, starts at zero, increases to 2 at $x = 2$, decreases to 1 at $x = 3$, increases to 3 at $x = 4$, then passes through a zero as it decreases to -1 at $x = 6$, and finally increases to 0 at 7. Thus, there are three zeroes: $x = 0$, $x = 5.5$, and $x = 7$.

(c)



17. Let $y'(t) = dy/dt$. Then y is the antiderivative of y' such that $y(0) = 0$. We know that

$$y(x) = \int_0^x y'(t)\,dt.$$

Thus, $y(x)$ is the area under the graph of dy/dt from $t = 0$ to $t = x$, where regions below the t-axis contribute negatively to the integral. We see that $y(t_1) = 2$, $y(t_3) = 2 - 2 = 0$, and $y(t_5) = 2$. See Figure 6.10.

Since y' is positive on the intervals $(0, t_1)$ and (t_3, ∞), we know that y is increasing on those intervals. Since y' is negative on the interval (t_1, t_3), we know that y is decreasing on that interval.

Since y' is increasing on the interval (t_2, t_4), we know that y is concave up on that interval; since y' is decreasing on $(0, t_2)$, we know that y is concave down there. The point where the concavity changes, t_2, is an inflection point. Finally, since y' is constant and positive on the interval (t_4, ∞), the graph of y is linear with positive slope on this interval. The value $y(t_1) = 2$ is a local maximum and $y(t_3) = 0$ is a local minimum.

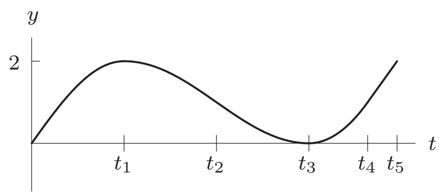


Figure 6.10

18. Let $y'(t) = dy/dt$. Then y is the antiderivative of y' such that $y(0) = 0$. We know that

$$y(x) = \int_0^x y'(t)\,dt.$$

Thus $y(x)$ is the area under the graph of dy/dt from $t = 0$ to $t = x$, with regions below the t-axis contributing negatively to the integral. We see that $y(t_1) = -2$, $y(t_3) = -2 + 2 = 0$, and $y(t_5) = -2$. See Figure 6.11.

Since y' is positive on the interval (t_1, t_3), we know that y is increasing on that interval. Since y' is negative on the intervals $(0, t_1)$ and (t_3, ∞), we know y is decreasing on those intervals.

Since y' is increasing on $(0, t_2)$, we know that y is concave up on that interval. Since y' is decreasing on (t_2, t_4), we know that y is concave down there. The point where concavity changes, t_2, is an inflection point. In addition, since y' is a negative constant on the interval (t_4, ∞), the graph of y is a line with negative slope on this interval. The value $y(t_1) = -2$ is a local minimum and $y(t_3) = 0$ is a local maximum.

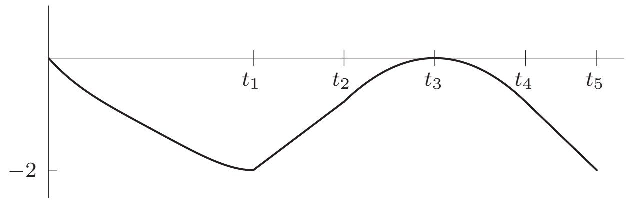


Figure 6.11

19. The critical points are at $(0, 5)$, $(2, 21)$, $(4, 13)$, and $(5, 15)$. A graph is given in Figure 6.12.

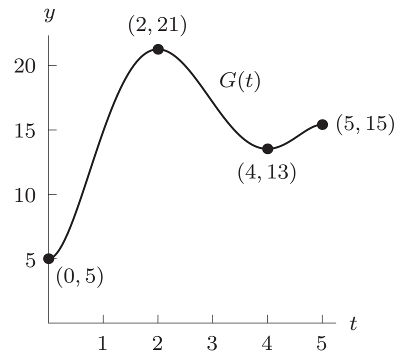


Figure 6.12

20. Looking at the graph of g' in Figure 6.13, we see that the critical points of g occur when $x = 15$ and $x = 40$, since $g'(x) = 0$ at these values. Inflection points of g occur when $x = 10$ and $x = 20$, because $g'(x)$ has a local maximum or minimum at these values. Knowing these four key points, we sketch the graph of $g(x)$ in Figure 6.14.

We start at $x = 0$, where $g(0) = 50$. Since g' is negative on the interval $[0, 10]$, the value of $g(x)$ is decreasing there. At $x = 10$ we have

$$\begin{aligned} g(10) &= g(0) + \int_0^{10} g'(x)\,dx \\ &= 50 - (\text{area of shaded trapezoid } T_1) \\ &= 50 - \left(\frac{10+20}{2} \cdot 10\right) = -100. \end{aligned}$$

Similarly,

$$\begin{aligned} g(15) &= g(10) + \int_{10}^{15} g'(x)\,dx \\ &= -100 - (\text{area of triangle } T_2) \\ &= -100 - \frac{1}{2}(5)(20) = -150. \end{aligned}$$

Continuing,

$$g(20) = g(15) + \int_{15}^{20} g'(x)\,dx = -150 + \frac{1}{2}(5)(10) = -125,$$

and

$$g(40) = g(20) + \int_{20}^{40} g'(x)\,dx = -125 + \frac{1}{2}(20)(10) = -25.$$

We now find concavity of $g(x)$ in the intervals $[0, 10]$, $[10, 15]$, $[15, 20]$, $[20, 40]$ by checking whether $g'(x)$ increases or decreases in these same intervals. If $g'(x)$ increases, then $g(x)$ is concave up; if $g'(x)$ decreases, then $g(x)$ is concave down. Thus we finally have the graph of $g(x)$ in Figure 6.14.

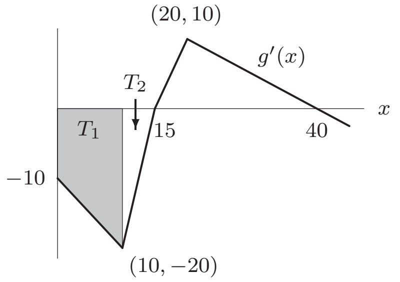


Figure 6.13

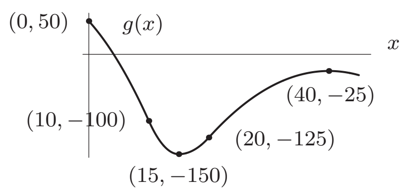


Figure 6.14

21. The rate of change is negative for $t < 5$ and positive for $t > 5$, so the concentration of adrenaline decreases until $t = 5$ and then increases. Since the area under the t-axis is greater than the area over the t-axis, the concentration of adrenaline goes down more than it goes up. Thus, the concentration at $t = 8$ is less than the concentration at $t = 0$. See Figure 6.15.

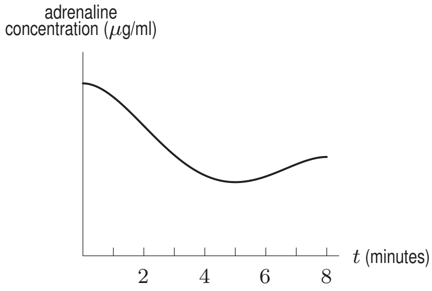


Figure 6.15

22. (a) The total volume emptied must increase with time and cannot decrease. The smooth graph (I) that is always increasing is therefore the volume emptied from the bladder. The jagged graph (II) that increases then decreases to zero is the flow rate.

(b) The total change in volume is the integral of the flow rate. Thus, the graph giving total change (I) shows an antiderivative of the rate of change in graph (II).

23. See Figure 6.16. Note that since $f(x_1) = 0$ and $f'(x_1) < 0$, $F(x_1)$ is a local maximum; since $f(x_3) = 0$ and $f'(x_3) > 0$, $F(x_3)$ is a local minimum. Also, since $f'(x_2) = 0$ and f changes from decreasing to increasing about $x = x_2$, F has an inflection point at $x = x_2$.

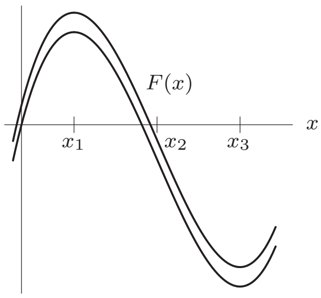


Figure 6.16

24. See Figure 6.17. Note that since $f(x_1) = 0$ and $f'(x_1) > 0$, $F(x_1)$ is a local minimum; since $f(x_3) = 0$ and $f'(x_3) < 0$, $F(x_3)$ is a local maximum. Also, since $f'(x_2) = 0$ and f changes from decreasing to increasing about $x = x_2$, F has an inflection point at $x = x_2$.

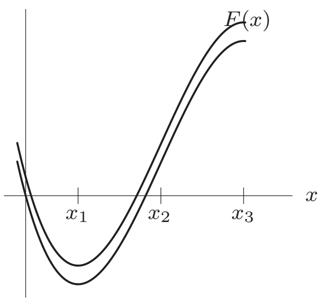


Figure 6.17

25. See Figure 6.18. Note that since $f(x_1) = 0$, $F(x_1)$ is either a local minimum or a point of inflection; it is impossible to tell which from the graph. Since $f'(x_3) = 0$, and f' changes sign around $x = x_3$, $F(x_3)$ is an inflection point. Also, since $f'(x_2) = 0$ and f changes from increasing to decreasing about $x = x_2$, F has another inflection point at $x = x_2$.

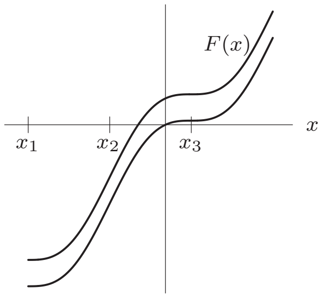


Figure 6.18

26. See Figure 6.19. Since $f(x_1) = 0$ and $f'(x_1) < 0$, we see that $F(x_1)$ is a local maximum. Since $f(x_3) = 0$ and $f'(x_3) > 0$, we see that $F(x_3)$ is a local minimum. Since $f'(x_2) = 0$ and f changes from decreasing to increasing at $x = x_2$, we see that F has an inflection point at $x = x_2$.

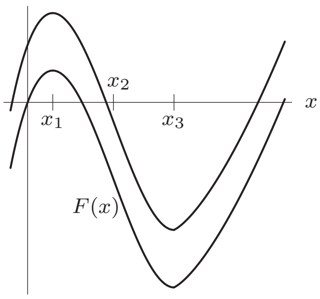


Figure 6.19

27. The graph of $f(x) = 2\sin(x^2)$ is shown in Figure 6.20. We see that there are roots at $x = 1.77$ and $x = 2.51$. These are the critical points of $F(x)$. Looking at the graph, it appears that of the three areas marked, A_1 is the largest, A_2 is next, and A_3 is smallest. Thus, as x increases from 0 to 3, the function $F(x)$ increases (by A_1), decreases (by A_2), and then increases again (by A_3). Therefore, the maximum is attained at the critical point $x = 1.77$.

What is the value of the function at this maximum? We know that $F(1) = 5$, so we need to find the change in F between $x = 1$ and $x = 1.77$. We have

$$\text{Change in } F = \int_1^{1.77} 2\sin(x^2)\,dx = 1.17.$$

We see that $F(1.77) = 5 + 1.17 = 6.17$, so the maximum value of F on this interval is 6.17.

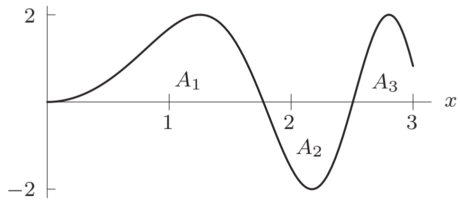
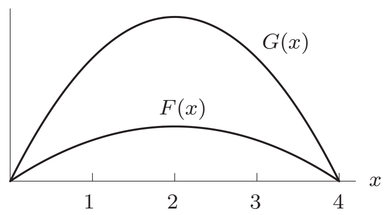


Figure 6.20

28. Both $F(x)$ and $G(x)$ have roots at $x = 0$ and $x = 4$. Both have a critical point (which is a local maximum) at $x = 2$. However, since the area under $g(x)$ between $x = 0$ and $x = 2$ is larger than the area under $f(x)$ between $x = 0$ and $x = 2$, the y-coordinate of $G(x)$ at 2 will be larger than the y-coordinate of $F(x)$ at 2. See below.



29. (a) Let $r(t)$ be the leakage rate in liters per second at time t minutes, shown in the graph. Since time is in minutes, it is helpful to express leakage in units of liters per minute. The leakage rate in liters per minute is $60r(t)$. The quantity leaked during the first b minutes, in liters, is given by $\int_0^b 60r(t)\,dt$.

We can evaluate the quantity leaked by computing area under the rate graph, by counting grid squares. Each grid square contributes area

$$\text{1 grid square} = \left(10\,\frac{\text{liter}}{\text{sec}}\right) \times (10\text{ minutes}) = 60 \cdot 10\frac{\text{liters}}{\text{minute}} \times (10\text{ minutes}) = 6000\text{ liters}.$$

Thus each grid square represents 6000 liters of leakage. So

$$\begin{matrix}\text{Total spill over} \\ \text{first 10 minutes}\end{matrix} = \frac{1}{2}\text{ grid square } \cdot 6000 = 3000\text{ liters}.$$

$$\begin{matrix}\text{Total spill over} \\ \text{20 minutes}\end{matrix} = 2\text{ grid squares } \cdot 6000 = 12{,}000\text{ liters}.$$

Continuing, we have

Time t (minutes)	0	10	20	30	40	50
Total spill, over t minutes (liters)	0	3000	12,000	21,000	27,000	30,000

(b) Plotting the values from part (a) and connecting with a smooth curve gives Figure 6.21.

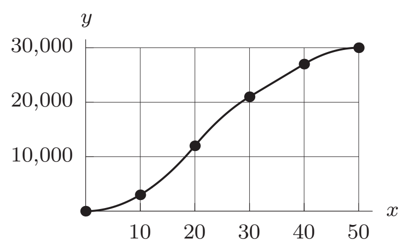


Figure 6.21

30. (a) Suppose $Q(t)$ is the amount of water in the reservoir at time t. Then

$$Q'(t) = \begin{matrix}\text{Rate at which water} \\ \text{in reservoir is changing}\end{matrix} = \begin{matrix}\text{Inflow} \\ \text{rate}\end{matrix} - \begin{matrix}\text{Outflow} \\ \text{rate}\end{matrix}$$

Thus the amount of water in the reservoir is increasing when the inflow curve is above the outflow, and decreasing when it is below. This means that $Q(t)$ is a maximum where the curves cross in July 2007 (as shown in Figure 6.22), and $Q(t)$ is decreasing fastest when the outflow is farthest above the inflow curve, which occurs about October 2007 (see Figure 6.22).

To estimate values of $Q(t)$, we use the Fundamental Theorem which says that the change in the total quantity of water in the reservoir is given by

$$Q(t) - Q(\text{Jan 2007}) = \int_{\text{Jan 07}}^{t} (\text{inflow rate} - \text{outflow rate})\, dt$$

or

$$Q(t) = Q(\text{Jan 2007}) + \int_{\text{Jan 07}}^{t} (\text{Inflow rate} - \text{Outflow rate})\, dt.$$

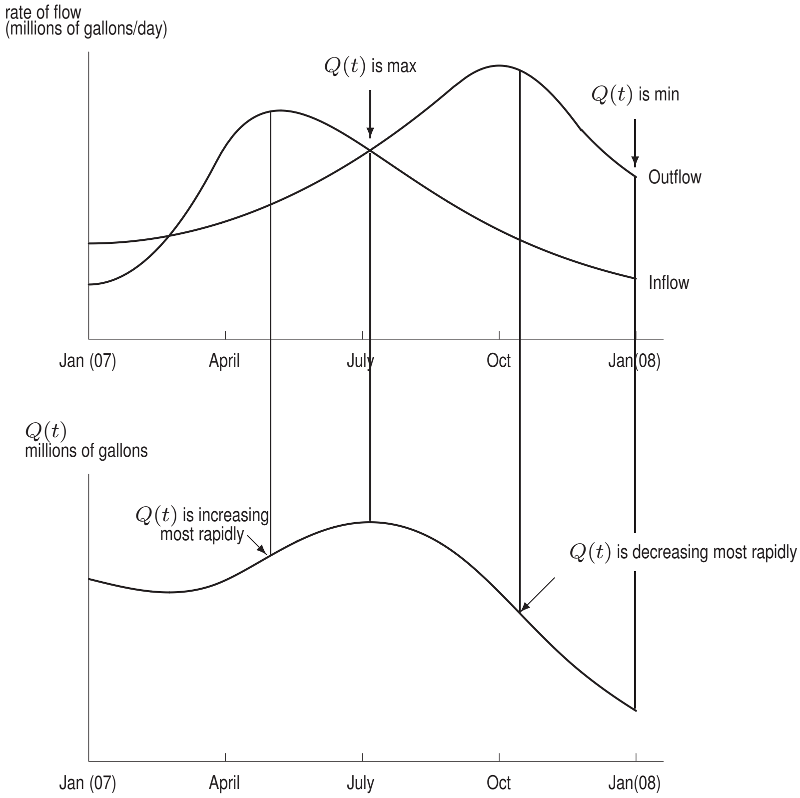


Figure 6.22

(b) See Figure 6.22. Maximum in July 2007. Minimum in Jan 2008.
(c) See Figure 6.22. Increasing fastest in May 2007. Decreasing fastest in Oct 2007.
(d) In order for the water to be the same as Jan 2007 the total amount of water which has flowed into the reservoir minus the total amount of water which has flowed out of the reservoir must be 0. Referring to Figure 6.23, we have

$$\int_{\text{Jan 07}}^{\text{July 08}} (\text{Inflow} - \text{Outflow})\, dt = -A_1 + A_2 - A_3 + A_4 = 0$$

giving $A_1 + A_3 = A_2 + A_4$

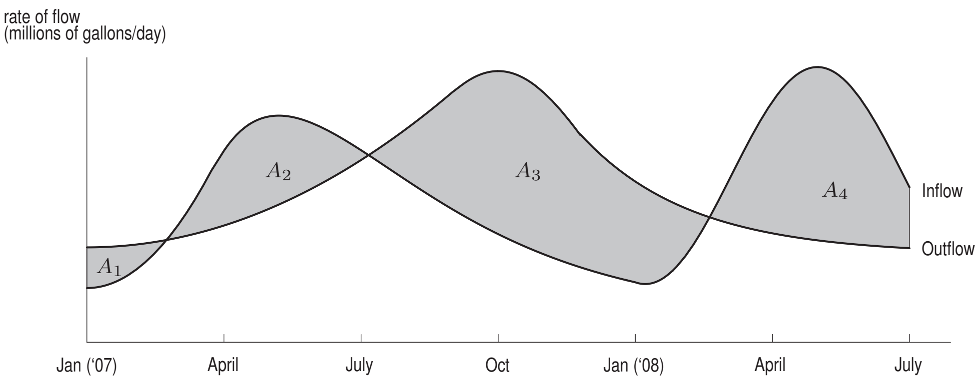


Figure 6.23

Strengthen Your Understanding

31. The statement has $f(x)$ and $F(x)$ reversed. Namely if an antiderivative $F(x)$ is increasing, then $F'(x) = f(x) \geq 0$.

We can see the statement given is not always true by looking for a counterexample. The function $f(x) = 2x$ is always increasing, but it has antiderivatives that are less than 0. For example, the antiderivative $F(x)$ with $F(0) = -1$ is negative at 0. See Figure 6.24.

A correct statement is: If $f(x) > 0$ everywhere, then $F(x)$ is increasing everywhere.

Figure 6.24: $f(x) = 2x$ increasing and its antiderivative $F(x)$ is not always positive

32. Consider $F(x) = x^2$ and $G(x) = x^2 + 1$ which are both antiderivatives of $f(x) = 2x$. Then $H(x) = F(x) + G(x) = 2x^2 + 1$, and $H'(x) = 4x \neq f(x)$. Thus, $H(x) = F(x) + G(x)$ is not an antiderivative of $f(x)$.

33. Any function for which the area between the function's graph above the x-axis is equal to the area between the function's graph below the x-axis will work. One such function, $f(x) = 1 - x$, is shown in Figure 6.25.

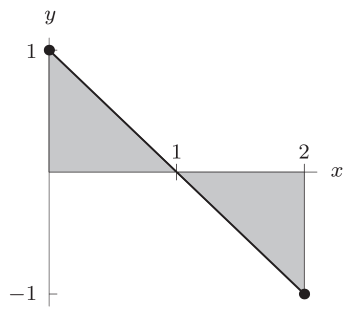


Figure 6.25: $f(x) = 1 - x$

34. Any positive function will work. One such function is $f(x) = 1$; see Figure 6.26.

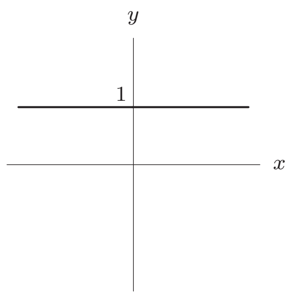


Figure 6.26

35. True. A function can have only one derivative.

36. False. If $f(t)$ is an increasing function, then F' is increasing, so F changes at an increasing rate. Thus F is not linear.

Solutions for Section 6.2

Exercises

1. $5x$

2. $\frac{5}{2}t^2$

3. $\frac{1}{3}x^3$

4. $\frac{1}{3}t^3 + \frac{1}{2}t^2$

5. $\frac{2}{3}z^{\frac{3}{2}}$

6. $\ln|z|$

7. $-\dfrac{1}{t}$

8. $\sin t$

9. $\dfrac{d}{dz}\left(\dfrac{1}{z^3}\right) = \dfrac{d}{dz}\left(z^{-3}\right) = -\dfrac{1}{2z^2}$

10. $\dfrac{y^5}{5} + \ln|y|$

11. e^z

12. $-\cos t$

13. $\frac{2}{3}t^3 + \frac{3}{4}t^4 + \frac{4}{5}t^5$

14. $\dfrac{t^4}{4} - \dfrac{t^3}{6} - \dfrac{t^2}{2}$

15. $\dfrac{t^2+1}{t} = t + \dfrac{1}{t}$, which has antiderivative $\dfrac{t^2}{2} + \ln|t|$

16. $\dfrac{5}{2}x^2 - \dfrac{2}{3}x^{\frac{3}{2}}$

17. $F(t) = \displaystyle\int 6t\,dt = 3t^2 + C$

18. $H(x) = \displaystyle\int (x^3 - x)\,dx = \dfrac{x^4}{4} - \dfrac{x^2}{2} + C$

19. $F(x) = \displaystyle\int (x^2 - 4x + 7)\,dx = \dfrac{x^3}{3} - 2x^2 + 7x + C$

20. $G(t) = \displaystyle\int \sqrt{t}\,dt = \dfrac{2}{3}t^{3/2} + C$

21. $R(t) = \displaystyle\int (t^3 + 5t - 1)\,dt = \dfrac{t^4}{4} + \dfrac{5}{2}t^2 - t + C$

22. $F(z) = \displaystyle\int (z + e^z)\,dz = \dfrac{z^2}{2} + e^z + C$

23. $G(x) = \displaystyle\int (\sin x + \cos x)\,dx = -\cos x + \sin x + C$

24. $H(x) = \displaystyle\int (4x^3 - 7)\,dx = x^4 - 7x + C$

25. $P(t) = \displaystyle\int \dfrac{1}{\sqrt{t}}\,dt = 2t^{1/2} + C$

26. $P(t) = \displaystyle\int (2 + \sin t)\,dt = 2t - \cos t + C$

27. $G(x) = \displaystyle\int \dfrac{5}{x^3}\,dx = -\dfrac{5}{2x^2} + C$

28. $H(t) = \int \frac{7}{\cos^2 t}\,dt = 7\tan t + C$

29. $f(x) = 3$, so $F(x) = 3x + C$. $F(0) = 0$ implies that $3 \cdot 0 + C = 0$, so $C = 0$. Thus $F(x) = 3x$ is the only possibility.

30. $f(x) = 2x$, so $F(x) = x^2 + C$. $F(0) = 0$ implies that $0^2 + C = 0$, so $C = 0$. Thus $F(x) = x^2$ is the only possibility.

31. $f(x) = -7x$, so $F(x) = \frac{-7x^2}{2} + C$. $F(0) = 0$ implies that $-\frac{7}{2} \cdot 0^2 + C = 0$, so $C = 0$. Thus $F(x) = -7x^2/2$ is the only possibility.

32. $f(x) = 2 + 4x + 5x^2$, so $F(x) = 2x + 2x^2 + \frac{5}{3}x^3 + C$. $F(0) = 0$ implies that $C = 0$. Thus $F(x) = 2x + 2x^2 + \frac{5}{3}x^3$ is the only possibility.

33. $f(x) = \frac{1}{4}x$, so $F(x) = \frac{x^2}{8} + C$. $F(0) = 0$ implies that $\frac{1}{8} \cdot 0^2 + C = 0$, so $C = 0$. Thus $F(x) = x^2/8$ is the only possibility.

34. $f(x) = x^2$, so $F(x) = \dfrac{x^3}{3} + C$. $F(0) = 0$ implies that $\dfrac{0^3}{3} + C = 0$, so $C = 0$. Thus $F(x) = \dfrac{x^3}{3}$ is the only possibility.

35. $f(x) = x^{1/2}$, so $F(x) = \frac{2}{3}x^{3/2} + C$. $F(0) = 0$ implies that $\frac{2}{3} \cdot 0^{3/2} + C = 0$, so $C = 0$. Thus $F(x) = \frac{2}{3}x^{3/2}$ is the only possibility.

36. $f(x) = \sin x$, so $F(x) = -\cos x + C$. $F(0) = 0$ implies that $-\cos 0 + C = 0$, so $C = 1$. Thus $F(x) = -\cos x + 1$ is the only possibility.

37. $\frac{5}{2}x^2 + 7x + C$

38. $\int (4t + \frac{1}{t})\,dt = 2t^2 + \ln|t| + C$

39. $\int (2 + \cos t)\,dt = 2t + \sin t + C$

40. $\int 7e^x\,dx = 7e^x + C$

41. $\int (3e^x + 2\sin x)\,dx = 3e^x - 2\cos x + C$

42. We have:

$$\begin{aligned}\int (4e^x - 3\sin x)\ dx &= \int 4e^x\,dx - \int 3\sin x\,dx \\ &= 4\int e^x\,dx - 3\int \sin x\,dx \\ &= 4e^x - 3(-\cos x) + C \\ &= 4e^x + 3\cos x + C.\end{aligned}$$

43. We have

$$\begin{aligned}\int \left(5x^2 + 2\sqrt{x}\right)\ dx &= \int 5x^2\,dx + \int 2\sqrt{x}\,dx \\ &= 5\int x^2\,dx + 2\int x^{1/2}\,dx \\ &= 5\left(\frac{1}{3}x^3\right) + 2\left(\frac{2}{3}\cdot x^{3/2}\right) + C \\ &= \frac{5}{3}\cdot x^3 + \frac{4}{3}\cdot x^{3/2} + C.\end{aligned}$$

44. $\int (x+3)^2\,dx = \int (x^2 + 6x + 9)\,dx = \dfrac{x^3}{3} + 3x^2 + 9x + C$

45. $\int \frac{8}{\sqrt{x}}\,dx = 16x^{1/2} + C$

46. $3\ln|t| + \dfrac{2}{t} + C$

47. $e^x + 5x + C$

48. Expand the integrand and then integrate

$$\int t^3(t^2+1)\,dt = \int (t^5+t^3)\,dt = \frac{1}{6}t^6 + \frac{1}{4}t^4 + C.$$

49. $\frac{2}{5}x^{5/2} - 2\ln|x| + C$

50. Since $f(x) = \dfrac{x+1}{x} = 1 + \dfrac{1}{x}$, the indefinite integral is $x + \ln|x| + C$

51. $\displaystyle\int_0^3 (x^2+4x+3)\,dx = \left(\frac{x^3}{3} + 2x^2 + 3x\right)\Big|_0^3 = (9+18+9) - 0 = 36$

52. $\displaystyle\int_1^3 \frac{1}{t}\,dt = \ln|t|\Big|_1^3 = \ln|3| - \ln|1| = \ln 3 \approx 1.0986.$

53. $\displaystyle\int_0^{\pi/4} \sin x\,dx = -\cos x\Big|_0^{\pi/4} = -\cos\frac{\pi}{4} - (-\cos 0) = -\frac{\sqrt{2}}{2} + 1 = 0.293.$

54. $\displaystyle\int_0^1 2e^x\,dx = 2e^x\Big|_0^1 = 2e - 2 \approx 3.437.$

55. $\displaystyle\int_0^2 3e^x\,dx = 3e^x\Big|_0^2 = 3e^2 - 3e^0 = 3e^2 - 3 = 19.167.$

56. $\displaystyle\int_2^5 (x^3 - \pi x^2)\,dx = \left(\frac{x^4}{4} - \frac{\pi x^3}{3}\right)\Big|_2^5 = \frac{609}{4} - 39\pi \approx 29.728.$

57. $\displaystyle\int_0^1 \sin\theta\,d\theta = -\cos\theta\Big|_0^1 = 1 - \cos 1 \approx 0.460.$

58. Since $\dfrac{1+y^2}{y} = \dfrac{1}{y} + y,$

$$\int_1^2 \frac{1+y^2}{y}\,dy = \left(\ln|y| + \frac{y^2}{2}\right)\Big|_1^2 = \ln 2 + \frac{3}{2} \approx 2.193.$$

59. $\displaystyle\int_0^2 \left(\frac{x^3}{3} + 2x\right)dx = \left(\frac{x^4}{12} + x^2\right)\Big|_0^2 = \frac{4}{3} + 4 = 16/3 \approx 5.333.$

60. $\displaystyle\int_0^{\pi/4} (\sin t + \cos t)\,dt = (-\cos t + \sin t)\Big|_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1+0) = 1.$

Problems

61. The rate at which water is entering the tank (in volume per unit time) is

$$\frac{dV}{dt} = 120 - 6t \text{ ft}^3/\text{min}.$$

Thus, the total quantity of water in the tank at time $t = 4$, in ft^3, is

$$V = \int_0^4 (120 - 6t)\,dt.$$

Since an antiderivative to $120 - 6t$ is

$$120t - 3t^2,$$

we have

$$\begin{aligned} V &= \int_0^4 (120-6t)\,dt = (120t - 3t^2)\Big|_0^4 \\ &= (120\cdot 4 - 3\cdot 4^2) - (120\cdot 0 - 3\cdot 0^2) \\ &= 432 \text{ ft}^3. \end{aligned}$$

The radius is 5 feet, so if the height is h ft, the volume is $V = \pi 5^2 h = 25\pi h$. Thus, at time $t = 4$, we have $V = 432$, so

$$\begin{aligned} 432 &= 25\pi h \\ h &= \frac{432}{25\pi} = 5.500 \text{ ft.} \end{aligned}$$

62. (a) The formula $v = 6 - 2t$ implies that $v > 0$ (the car is moving forward) if $0 \le t < 3$ and that $v < 0$ (the car is moving backward) if $t > 3$. When $t = 3$, $v = 0$, so the car is not moving at the instant $t = 3$. The car is decelerating when $|v|$ is decreasing; since v decreases (from 6 to 0) on the interval $0 \le t < 3$, the car decelerates on that interval. The car accelerates when $|v|$ is increasing, which occurs on the domain $t > 3$.

(b) The car moves forward on the interval $0 \le t < 3$, so it is furthest to the right at $t = 3$. For all $t > 3$, the car is decelerating. There is no upper bound on the car's distance behind its starting point since it is decelerating for all $t > 3$.

(c) Let $s(t)$ be the position of the car at time t. Then

$$v(t) = s'(t),$$

so $s(t)$ is an antiderivative of $v(t)$. Thus,

$$s(t) = \int v(t)\,dt = \int (6 - 2t)\,dt = 6t - t^2 + C.$$

Since the car's position is measured from its starting point, we have $s(0) = 0$, so $C = 0$. Thus, $s(t) = 6t - t^2$.

63. (a) Since the rotor is slowing down at a constant rate,

$$\text{Angular acceleration } = \frac{260 - 350}{1.5} = -60 \text{ revs/min}^2.$$

Units are revolutions per minute per minute, or revs/min^2.

(b) To decrease from 350 to 0 revs/min at a deceleration of 60 revs/min^2,

$$\text{Time needed } = \frac{350}{60} \approx 5.83 \text{ min.}$$

(c) We know angular acceleration is the derivative of angular velocity. Since

$$\text{Angular acceleration } = -60 \text{ revs/min}^2,$$

we have

$$\text{Angular velocity } = -60t + C.$$

Measuring time from the moment when angular velocity is 350 revs/min, we get $C = 350$. Thus

$$\text{Angular velocity } = -60t + 350.$$

So, the total number of revolutions made between the time the angular speed is 350 revs/min and stopping is given by:

$$\begin{aligned} \text{Number of revolutions } &= \int_0^{5.83} (\text{Angular velocity})\,dt \\ &= \int_0^{5.83} (-60t + 350)dt = -30t^2 + 350t \Big|_0^{5.83} \\ &= 1020.83 \text{ revolutions.} \end{aligned}$$

64. Since $C'(x) = 4000 + 10x$ we want to evaluate the indefinite integral

$$\int (4000 + 10x)\,dx = 4000x + 5x^2 + K$$

where K is a constant. Thus $C(x) = 5x^2 + 4000x + K$, and the fixed cost of 1,000,000 riyal means that $C(0) = 1{,}000{,}000 = K$. Therefore, the total cost is

$$C(x) = 5x^2 + 4000x + 1{,}000{,}000.$$

Since $C(x)$ depends on x^2, the square of the depth drilled, costs will increase dramatically when x grows large.

65. $\displaystyle\int_0^3 x^2\,dx = \frac{x^3}{3}\Bigg|_0^3 = 9 - 0 = 9.$

66. Since $y = x^3 - x = x(x-1)(x+1)$, the graph crosses the axis at the three points shown in Figure 6.27. The two regions have the same area (by symmetry). Since the graph is below the axis for $0 < x < 1$, we have

$$\begin{aligned}\text{Area } &= 2\left(-\int_0^1 \left(x^3 - x\right)\,dx\right)\\ &= -2\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = -2\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}.\end{aligned}$$

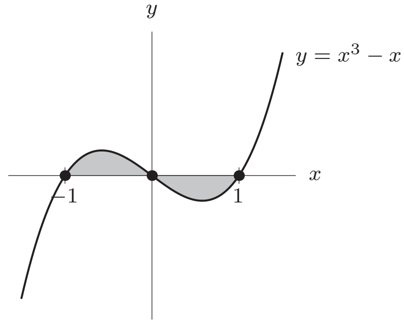


Figure 6.27

67. The area we want (the shaded area in Figure 6.28) is symmetric about the y-axis and so is given by

$$\begin{aligned}\text{Area } &= 2\int_0^{\pi/3}\left(\cos x - \frac{1}{2}\left(\frac{3}{\pi}x\right)^2\right)\,dx\\ &= 2\int_0^{\pi/3}\cos x\,dx - \int_0^{\pi/3}\frac{9}{\pi^2}x^2\,dx\\ &= 2\sin x\Bigg|_0^{\pi/3} - \frac{9}{\pi^2}\cdot\frac{x^3}{3}\Bigg|_0^{\pi/3}\\ &= 2\cdot\frac{\sqrt{3}}{2} - \frac{3}{\pi^2}\cdot\frac{\pi^3}{3^3} = \sqrt{3} - \frac{\pi}{9}.\end{aligned}$$

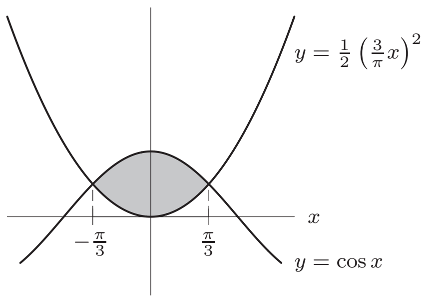


Figure 6.28

68. Since $y < 0$ from $x = 0$ to $x = 1$ and $y > 0$ from $x = 1$ to $x = 3$, we have

$$\begin{aligned}\text{Area} &= -\int_0^1 \left(3x^2 - 3\right)\,dx + \int_1^3 \left(3x^2 - 3\right)\,dx\\ &= -\left(x^3 - 3x\right)\Bigg|_0^1 + \left(x^3 - 3x\right)\Bigg|_1^3\\ &= -(-2 - 0) + (18 - (-2)) = 2 + 20 = 22.\end{aligned}$$

69. (a) See Figure 6.29. Since $f(x) > 0$ for $0 < x < 2$ and $f(x) < 0$ for $2 < x < 5$, we have

$$\begin{aligned}
\text{Area} &= \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx \\
&= \int_0^2 (x^3 - 7x^2 + 10x)\,dx - \int_2^5 (x^3 - 7x^2 + 10x)\,dx \\
&= \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Bigg|_0^2 - \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Bigg|_2^5 \\
&= \left[\left(4 - \frac{56}{3} + 20\right) - (0 - 0 + 0)\right] - \left[\left(\frac{625}{4} - \frac{875}{3} + 125\right) - \left(4 - \frac{56}{3} + 20\right)\right] \\
&= \frac{253}{12}.
\end{aligned}$$

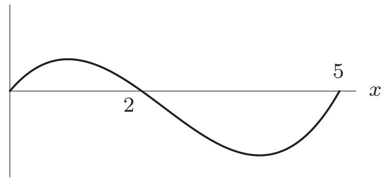


Figure 6.29: Graph of $f(x) = x^3 - 7x^2 + 10x$

(b) Calculating $\int_0^5 f(x)\,dx$ gives

$$\begin{aligned}
\int_0^5 f(x)\,dx &= \int_0^5 (x^3 - 7x^2 + 10x)\,dx \\
&= \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Bigg|_0^5 \\
&= \left(\frac{625}{4} - \frac{875}{3} + 125\right) - (0 - 0 + 0) \\
&= -\frac{125}{12}.
\end{aligned}$$

This integral measures the difference between the area above the x-axis and the area below the x-axis. Since the definite integral is negative, the graph of $f(x)$ lies more below the x-axis than above it. Since the function crosses the axis at $x = 2$,

$$\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = \frac{16}{3} - \frac{63}{4} = \frac{-125}{12},$$

whereas

$$\text{Area} = \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx = \frac{16}{3} + \frac{64}{4} = \frac{253}{12}.$$

70. The graph of $y = e^x - 2$ is below the x-axis at $x = 0$ and above the x-axis at $x = 2$. The graph crosses the axis where

$$\begin{aligned}
e^x - 2 &= 0 \\
x &= \ln 2.
\end{aligned}$$

See Figure 6.30. Thus we find the area by dividing the region at $x = \ln 2$:

$$\begin{aligned}
\text{Area} &= -\int_0^{\ln 2} (e^x - 2)\,dx + \int_{\ln 2}^2 (e^x - 2)\,dx \\
&= (-e^x + 2x)\Bigg|_0^{\ln 2} + (e^x - 2x)\Bigg|_{\ln 2}^2 \\
&= -e^{\ln 2} + 2\ln 2 + e^0 + e^2 - 4 - (e^{\ln 2} - 2\ln 2) \\
&= -2e^{\ln 2} + 4\ln 2 - 3 + e^2 \\
&= -2 \cdot 2 + 4\ln 2 - 3 + e^2 = e^2 + 4\ln 2 - 7.
\end{aligned}$$

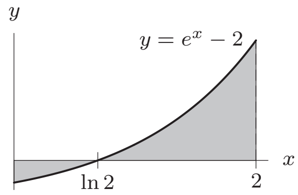


Figure 6.30

71. Solving $x^2 = 2 - x^2$ shows that the curves intersect at $x = \pm 1$. To find the area between the curves for $-1 \le x \le 1$ we integrate the top curve $y = 2 - x^2$ minus the bottom curve $y = x^2$. Thus

$$\text{Area between curves} = \int_{-1}^{1} (2 - x^2 - x^2)\,dx = 2x - \frac{2}{3}x^3\Big|_{-1}^{1} = 2 - \frac{2}{3} - \left(-2 + \frac{2}{3}\right) = \frac{8}{3}.$$

72. The function is a cubic polynomial which crosses the x-axis at $x = 1, 2, 3$, with $f(x) \ge 0$ for $1 \le x \le 2$ and $f(x) \le 0$ for $2 \le x \le 3$. Thus the total area is given by

$$\text{Total area} = \int_1^2 f(x)\,dx + \left|\int_2^3 f(x)\,dx\right|.$$

Since $f(x) = x^3 - 6x^2 + 11x - 6$, we can evaluate each integral to get

$$\text{Total area} = \left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)\Big|_1^2 + \left|\left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)\Big|_2^3\right| = \frac{1}{4} + \left|-\frac{1}{4}\right| = \frac{1}{2}.$$

73. The graph of $y = e^x - 2$ starts below the x-axis at $x = 0$ and climbs. See Figure 6.31. The graph crosses the axis where

$$e^x = 2$$
$$x = \ln 2.$$

Since the integral

$$\int_0^c (e^x - 2)\,dx = -A_1 + A_2,$$

we choose c to make the integral 0:

$$\int_0^c (e^x - 2)\,dx = (e^x - 2x)\Big|_0^c = e^c - 2c - 1 = 0$$
$$e^c - 2c = 1.$$

Solving numerically (for example, by zooming in on the graph on a calculator), we get

$$c = 0 \quad \text{and} \quad c = 1.257.$$

The solution we want is $c = 1.257$.

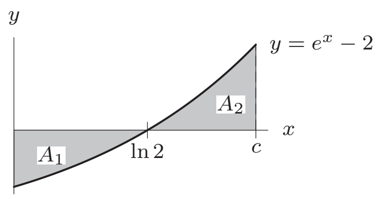


Figure 6.31

74. Since the area under the curve is 6, we have

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = 2x^{1/2}\Big|_1^b = 2b^{1/2} - 2(1) = 6.$$

Thus $b^{1/2} = 4$ and $b = 16$.

75. The graph of $y = c(1-x^2)$ has x-intercepts of $x = \pm 1$. See Figure 6.32. Since it is symmetric about the y-axis, we have

$$\begin{aligned}\text{Area } &= \int_{-1}^1 c(1-x^2)\,dx = 2c\int_0^1 (1-x^2)\,dx \\ &= 2c\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{4c}{3}.\end{aligned}$$

We want the area to be 1, so

$$\frac{4c}{3} = 1, \quad \text{giving} \quad c = \frac{3}{4}.$$

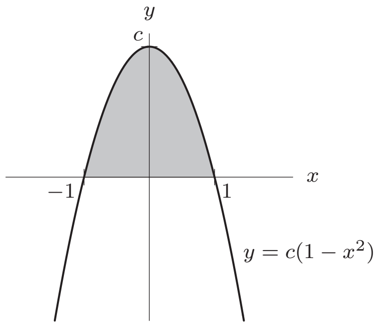


Figure 6.32

76. The curves intersect at $(0,0)$ and $(\pi, 0)$. At any x-coordinate the "height" between the two curves is $\sin x - x(x-\pi)$. See Figure 6.33.

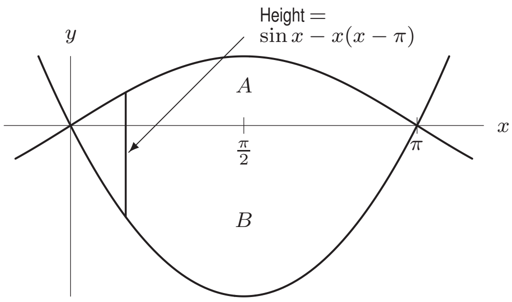


Figure 6.33

Thus the total area is

$$\begin{aligned}\int_0^\pi [\sin x - x(x-\pi)]\,dx = &= \int_0^\pi (\sin x - x^2 + \pi x)\,dx \\ &= \left(-\cos x - \frac{x^3}{3} + \frac{\pi x^2}{2}\right)\Big|_0^\pi \\ &= \left(1 - \frac{\pi^3}{3} + \frac{\pi^3}{2}\right) - (-1) \\ &= 2 + \frac{\pi^3}{6}.\end{aligned}$$

Another approach is to notice that the area between the two curves is (area A) + (area B).

$$\begin{aligned}\text{Area B} &= -\int_0^\pi x(x-\pi)\,dx \text{ since the function is negative on } 0 \le x \le \pi \\ &= -\left(\frac{x^3}{3} - \frac{\pi x^2}{2}\right)\Bigg|_0^\pi = \frac{\pi^3}{2} - \frac{\pi^3}{3} = \frac{\pi^3}{6}; \\ \text{Area A} &= \int_0^\pi \sin x\,dx = -\cos x\Bigg|_0^\pi = 2.\end{aligned}$$

Thus the area is $2 + \dfrac{\pi^3}{6}$.

77. See Figure 6.34. The average value of $f(x)$ is given by

$$\text{Average } = \frac{1}{9-0}\int_0^9 \sqrt{x}\,dx = \frac{1}{9}\left(\frac{2}{3}x^{3/2}\Bigg|_0^9\right) = \frac{1}{9}\left(\frac{2}{3}9^{3/2} - 0\right) = \frac{1}{9}18 = 2.$$

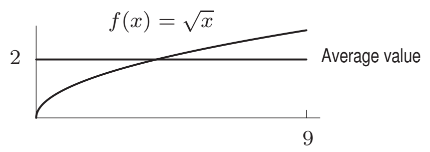


Figure 6.34

78. We have

$$\begin{aligned} f\left(x^{-1}\right) &= 4\left(x^{-1}\right)^{-3} \qquad \text{because } f(x) = 4x^{-3} \\ &= 4x^3 \\ \text{so} \quad \int_1^3 f\left(x^{-1}\right)\,dx &= \int_1^3 4x^3\,dx \\ &= 4\left(\frac{1}{4}\right)x^4\Bigg|_1^3 \\ &= 3^4 - 1^4 = 80.\end{aligned}$$

79. By the Fundamental theorem,

$$\int_1^3 f'(x)\,dx = f(3) - f(1) = 4x^{-3}\Bigg|_1^3 = 4\cdot 3^{-3} - 4\cdot 1^{-3} = -3.8519.$$

80. The curves $y = x$ and $y = x^n$ cross at $x = 0$ and $x = 1$. For $0 < x < 1$, the curve $y = x$ is above $y = x^n$. Thus the area is given by

$$A_n = \int_0^1 (x - x^n)d\,x = \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{2} - \frac{1}{n+1} \to \frac{1}{2}.$$

Since $x^n \to 0$ for $0 \le x < 1$, as $n \to \infty$, the area between the curves approaches the area under the line $y = x$ between $x = 0$ and $x = 1$.

81. (a) Recall that $x = e^{\ln x}$. Thus $x^x = (e^{\ln x})^x = e^{x\ln x}$.

(b) Using part (a) and the chain rule

$$\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x\ln x}) = e^{x\ln x}\frac{d}{dx}(x\ln x) = e^{x\ln x}(\ln x + 1) = x^x(\ln x + 1).$$

(c) By the Fundamental Theorem of Calculus and part (b),

$$\int x^x(1 + \ln x)\,dx = x^x + C.$$

(d) By the Fundamental Theorem of Calculus,

$$\int_1^2 x^x(1+\ln x)dx = x^x\Big|_1^2 = 2^2 - 1^1 = 3.$$

Using a calculator, we get $\int_1^2 x^x(1+\ln x)\,dx = 3$.

82. (a) The average value of $f(t) = \sin t$ over $0 \le t \le 2\pi$ is given by the formula

$$\begin{aligned}\text{Average} &= \frac{1}{2\pi - 0}\int_0^{2\pi} \sin t\,dt \\ &= \frac{1}{2\pi}(-\cos t)\Big|_0^{2\pi} \\ &= \frac{1}{2\pi}(-\cos 2\pi - (-\cos 0)) = 0.\end{aligned}$$

We can check this answer by looking at the graph of $\sin t$ in Figure 6.35. The area below the curve and above the t-axis over the interval $0 \le t \le \pi$, A_1, is the same as the area above the curve but below the t-axis over the interval $\pi \le t \le 2\pi$, A_2. When we take the integral of $\sin t$ over the entire interval $0 \le t \le 2\pi$, we get $A_1 - A_2 = 0$.

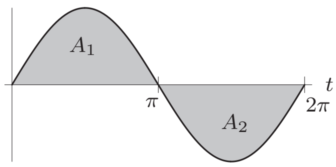


Figure 6.35

(b) Since

$$\int_0^{\pi} \sin t\,dt = -\cos t\Big|_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2,$$

the average value of $\sin t$ on $0 \le t \le \pi$ is given by

$$\text{Average value } = \frac{1}{\pi}\int_0^{\pi} \sin t\,dt = \frac{2}{\pi}.$$

83. The area beneath the curve in Figure 6.36 is given by

$$\int_0^a y\,dx = \int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \left[a\,x - \frac{4\sqrt{a}\,x^{3/2}}{3} + \frac{x^2}{2}\right]_0^a = \frac{a^2}{6}.$$

The area of the square is a^2 so the area above the curve is $5a^2/6$. Thus, the ratio of the areas is 5 to 1.

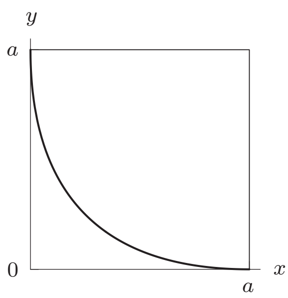


Figure 6.36: The curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Strengthen Your Understanding

84. We cannot integrate the numerator and denominator separately. Using the quotient rule to differentiate we see that

$$\frac{d}{dx}\left(\frac{x^3+x}{x^2}+C\right)=\frac{(3x^2+1)(x^2)-(x^3+x)(2x)}{x^4},$$

which is not equal to $\frac{3x^2+1}{2x}$.

To compute the correct integral, we can split the integrand into two terms:

$$\int\frac{3x^2+1}{2x}\,dx=\int\left(\frac{3}{2}x+\frac{1}{2x}\right)\,dx=\frac{3}{4}x^2+\frac{1}{2}\ln|x|+C.$$

85. The statement is true for all $n\neq-1$, since $\int x^{-1}\,dx=\ln|x|+C$.

86. If two functions $F(x)$ and $G(x)$ differ by a constant, they have the same derivative. Therefore, $F(x)=x^4$ and $G(x)=x^4-8$ both have the same derivative.

87. We are given $F(x)=mx+C$ with m a negative constant. Since $F'(x)=f(x)=m$, any negative constant function will work, for example $f(x)=-1$.

88. True. Check by differentiating $\frac{d}{dx}(2(x+1)^{3/2})=2\cdot\frac{3}{2}(x+1)^{1/2}=3\sqrt{x+1}$.

89. True. Any antiderivative of $3x^2$ is obtained by adding a constant to x^3.

90. True. Any antiderivative of $1/x$ is obtained by adding a constant to $\ln|x|$.

91. False. Differentiating using the product and chain rules gives

$$\frac{d}{dx}\left(\frac{-1}{2x}e^{-x^2}\right)=\frac{1}{2x^2}e^{-x^2}+e^{-x^2}.$$

92. False. It is not true in general that $\int xf(x)\,dx=x\int f(x)\,dx$, so this statement is false for many functions $f(x)$. For example, if $f(x)=1$, then $\int xf(x)\,dx=x^2/2+C$, but $x\int f(x)\,dx=x(x+C)$.

93. True. Adding a constant to an antiderivative gives another antiderivative.

94. True. Since $F(x)$ and $G(x)$ are antiderivatives of the same function on an interval, $F(x)-G(x)$ is a constant function. Thus $F(10)-G(10)=F(5)-G(5)>0$.

95. False. For a counterexample, take $f(x)=g(x)=1$. Then $F(x)=x$ and $G(x)=x$ are antiderivatives of $f(x)$ and $g(x)$, but $F(x)\cdot G(x)=x^2$ is not an antiderivative of $f(x)\cdot g(x)=1$.

96. True. The derivative of $F(x)-G(x)$ is $(F(x)-G(x))'=f(x)-f(x)=0$, so $F(x)-G(x)$ is a constant function.

Solutions for Section 6.3

Exercises

1. We differentiate $y=xe^{-x}+2$ using the product rule to obtain

$$\begin{aligned}\frac{dy}{dx}&=x\left(e^{-x}(-1)\right)+(1)e^{-x}+0\\&=-xe^{-x}+e^{-x}\\&=(1-x)e^{-x},\end{aligned}$$

and so $y=xe^{-x}+2$ satisfies the differential equation. We now check that $y(0)=2$:

$$\begin{aligned}y&=xe^{-x}+2\\y(0)&=0e^0+2=2.\end{aligned}$$

2. First, we check that $y = \sin(2t)$ satisfies the initial condition $y(0) = 0$:

$$\sin(2 \cdot 0) = 0.$$

Next, we substitute $y = \sin(2t)$ into each side of the differential equation and check that we get the same result. The left-hand side gives

$$\frac{dy}{dt} = 2\cos(2t).$$

For $0 \le t \le \pi/4$, the right-hand side gives

$$2\sqrt{1-y^2} = 2\sqrt{1-\sin^2(2t)} = 2\cos(2t).$$

Thus for $0 \le t \le \pi/4$ the initial value problem is satisfied.

3. We need to find a function whose derivative is $2x$, so one antiderivative is x^2. The general solution is

$$y = x^2 + C.$$

To check, we differentiate to get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + C\right) = 2x,$$

as required.

4. We need to find a function whose derivative is t^2, so one antiderivative is $t^3/3$. The general solution is

$$y = \frac{t^3}{3} + C.$$

To check, we differentiate to get

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{t^3}{3} + C\right) = t^2,$$

as required.

5. We need to find a function whose derivative is $x^3 + 5x^4$, so one antiderivative is $x^4/4 + x^5$. The general solution is

$$y = \frac{x^4}{4} + x^5 + C.$$

To check, we differentiate to get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^4}{4} + x^5 + C\right) = x^3 + 5x^4,$$

as required.

6. We need to find a function whose derivative is e^t, so one antiderivative is e^t. The general solution is

$$y = e^t + C.$$

To check, we differentiate to get

$$\frac{dy}{dt} = \frac{d}{dt}\left(e^t + C\right) = e^t,$$

as required.

7. We need to find a function whose derivative is $\cos x$, so one antiderivative is $\sin x$. The general solution is

$$y = \sin x + C.$$

To check, we differentiate to get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + C) = \cos x,$$

as required.

8. We need to find a function whose derivative is $1/x$, so one antiderivative is $\ln x$. The general solution is

$$y = \ln x + C.$$

To check, we differentiate to get

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x + C) = \frac{1}{x},$$

as required.

9. Integrating gives
$$\int \frac{dy}{dx}\,dx = \int (3x^2)\,dx = x^3 + C.$$
If $y = 5$ when $x = 0$, then $0^3 + C = 5$ so $C = 5$. Thus $y = x^3 + 5$.

10. Integrating gives
$$\int \frac{dy}{dx}\,dx = \int (x^5 + x^6)\,dx = \frac{x^6}{6} + \frac{x^7}{7} + C.$$
If $y = 2$ when $x = 1$, then $1^6/6 + 1^7/7 + C = 2$ so
$$C = 2 - \frac{1}{6} - \frac{1}{7} = \frac{71}{42}.$$
Thus
$$y = \frac{x^6}{6} + \frac{x^7}{7} + \frac{71}{42}.$$

11. Integrating gives
$$\int \frac{dy}{dx}\,dx = \int (e^x)\,dx = e^x + C.$$
If $y = 7$ when $x = 0$, then $e^0 + C = 7$ so $C = 6$. Thus $y = e^x + 6$.

12. Integrating gives
$$\int \frac{dy}{dx}\,dx = \int (\sin x)\,dx = -\cos x + C.$$
If $y = 3$ when $x = 0$, then $-\cos 0 + C = 3$ so $C = 4$. Thus $y = -\cos x + 4$.

Problems

13. The acceleration is $a(t) = -32$, so the velocity is $v(t) = -32t + C$. We find C using $v(0) = -10$, (negative velocity is downward) so
$$v(t) = -32t - 10.$$
Then, the height of the rock above the water is
$$s(t) = -16t^2 - 10t + D.$$
We find D using $s(0) = 100$, so
$$s(t) = -16t^2 - 10t + 100.$$
Now we can find when the rock hits the water by solving the quadratic equation
$$0 = -16t^2 - 10t + 100.$$
There are two solutions: $t = 2.207$ seconds and $t = -2.832$ seconds. We discard the negative solution. So, at $t = 2.207$ the rock is traveling with a velocity of
$$v(2.207) = -32(2.207) - 10 = -80.624 \text{ ft/sec}.$$
Thus the rock is traveling with a speed of 80.624 ft/sec downward when it hits the water.

14. (a) To find the height of the balloon, we integrate its velocity with respect to time:
$$\begin{aligned} h(t) &= \int v(t)\,dt \\ &= \int (-32t + 40)\,dt \\ &= -32\frac{t^2}{2} + 40t + C. \end{aligned}$$
Since at $t = 0$, we have $h = 30$, we can solve for C to get $C = 30$, giving us a height of
$$h(t) = -16t^2 + 40t + 30.$$

(b) To find the average velocity between $t = 1.5$ and $t = 3$, we find the total displacement and divide by time.

$$\text{Average velocity} = \frac{h(3) - h(1.5)}{3 - 1.5} = \frac{6 - 54}{1.5} = -32 \text{ ft/sec}.$$

The balloon's average velocity is 32 ft/sec downward.

(c) First, we must find the time when $h(t) = 6$. Solving the equation $-16t^2 + 40t + 30 = 6$, we get

$$\begin{aligned}6 &= -16t^2 + 40t + 30\\ 0 &= -16t^2 + 40t + 24\\ 0 &= 2t^2 - 5t - 3\\ 0 &= (2t + 1)(t - 3).\end{aligned}$$

Thus, $t = -1/2$ or $t = 3$. Since $t = -1/2$ makes no physical sense, we use $t = 3$ to calculate the balloon's velocity. At $t = 3$, we have a velocity of $v(3) = -32(3) + 40 = -56$ ft/sec. So the balloon's velocity is 56 ft/sec downward at the time of impact.

15. Since the acceleration $a = dv/dt$, where v is the velocity of the car, we have

$$\frac{dv}{dt} = -0.6t + 4.$$

Integrating gives

$$v = -0.6\frac{t^2}{2} + 4t + C.$$

The car starts from rest, so $v = 0$ when $t = 0$, and therefore $C = 0$. If x is the distance from the starting point, $v = dx/dt$ and

$$\frac{dx}{dt} = -0.3t^2 + 4t,$$

so

$$x = -\frac{0.3}{3}t^3 + \frac{4}{2}t^2 + C = -0.1t^3 + 2t^2 + C.$$

Since $x = 0$ when $t = 0$, we have $C = 0$, so

$$x = -0.1t^3 + 2t^2.$$

We want to solve for t when $x = 100$:

$$100 = -0.1t^3 + 2t^2.$$

This equation can be rewritten as

$$\begin{aligned}0.1t^3 - 2t^2 + 100 &= 0\\ t^3 - 20t^2 + 1000 &= 0.\end{aligned}$$

The equation can be solved numerically, or by tracing along a graph, or by factoring

$$(t - 10)(t^2 - 10t - 100) = 0.$$

The solutions are $t = 10$ and $t = \frac{10 \pm \sqrt{500}}{2} = -6.18,\ 16.18$. Since we are told $0 \le t \le 12$, the solution we want is $t = 10$ sec.

16.

$$\begin{aligned}\frac{dy}{dt} &= k\sqrt{t} = kt^{1/2}\\ y &= \frac{2}{3}kt^{3/2} + C.\end{aligned}$$

Since $y = 0$ when $t = 0$, we have $C = 0$, so

$$y = \frac{2}{3}kt^{3/2}.$$

17. (a) Since

$$R'(p) = 25 - 2p$$
$$R(p) = \int (25 - 2p)\,dp = 25p - p^2 + C.$$

We assume that the revenue is 0 when the price is 0. Substituting gives

$$0 = 25 \cdot 0 - 0^2 + C$$
$$C = 0.$$

Thus

$$R(p) = 25p - p^2.$$

(b) The revenue increases with price if $R'(p) > 0$, that is $25 - 2p > 0$, so $p < 12.5$ dollars. The revenue decreases with price if $R'(p) < 0$, that is $25 - 2p < 0$, so $p > 12.5$.

18. (a) The marginal cost, MC, is found by differentiating the total cost function, C, with respect to q so $MC = C'(q)$. Thus the differential equation is

$$C'(q) = 3q^2 + 6q + 9.$$

(b) Solving the differential equation gives

$$C(q) = \int \left(3q^2 + 6q + 9\right)\,dq$$
$$= q^3 + 3q^2 + 9q + D,$$

where D is a constant. We can check this by noting

$$C'(q) = \frac{d}{dq}\left(q^3 + 3q^2 + 9q + D\right) = 3q^2 + 6q + 9 = MC.$$

The fixed costs are 400, so $C = 400$ when $q = 0$. Thus,

$$400 = 0^3 + 3 \cdot 0^2 + 9 \cdot 0 + D,$$

so $D = 400$. The total cost function is

$$C(q) = q^3 + 3q^2 + 9q + 400.$$

19. (a) Acceleration = $a(t) = -9.8$ m/sec^2
Velocity = $v(t) = -9.8t + 40$ m/sec
Height = $h(t) = -4.9t^2 + 40t + 25$ m
.

(b) At the highest point,

$$v(t) = -9.8t + 40 = 0,$$

so

$$t = \frac{40}{9.8} = 4.082 \text{ seconds.}$$

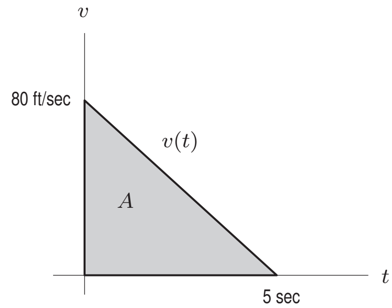
At that time, $h(4.082) = 106.633$ m. We see that the tomato reaches a height of 106.633 m, at 4.082 seconds after it is thrown.

(c) The tomato lands when $h(t) = 0$, so

$$-4.9t^2 + 40t + 25 = 0.$$

The solutions are $t = -0.583$ and $t = 8.747$ seconds. We see that it lands 8.747 seconds after it is thrown.

20. (a)



(b) The total distance is represented by the shaded region A, the area under the graph of $v(t)$.
(c) The area A, a triangle, is given by

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(5\,\text{sec})(80\,\text{ft/sec}) = 200\,\text{ft}.$$

(d) Using integration and the Fundamental Theorem of Calculus, we have $A = \int_0^5 v(t)\,dt$ or $A = s(5) - s(0)$, where $s(t)$ is an antiderivative of $v(t)$.

We have that $a(t)$, the acceleration, is constant: $a(t) = k$ for some constant k. Therefore $v(t) = kt + C$ for some constant C. We have $80 = v(0) = k(0) + C = C$, so that $v(t) = kt + 80$. Putting in $t = 5$, $0 = v(5) = (k)(5) + 80$, or $k = -80/5 = -16$.

Thus $v(t) = -16t + 80$, and an antiderivative for $v(t)$ is $s(t) = -8t^2 + 80t + C$. Since the total distance traveled at $t = 0$ is 0, we have $s(0) = 0$ which means $C = 0$. Finally, $A = \int_0^5 v(t)\,dt = s(5) - s(0) = (-8(5)^2 + (80)(5)) - (-8(0)^2 + (80)(0)) = 200\,\text{ft}$, which agrees with the previous part.

21. (a) The velocity is decreasing at 32 ft/sec^2, the acceleration due to gravity.
(b) The graph is a line because the velocity is decreasing at a constant rate.
(c) The highest point is reached when the velocity is 0, which occurs when

$$\text{Time} = \frac{160}{32} = 5 \text{ sec}.$$

(d) The object hits the ground at $t = 10$ seconds, since by symmetry if the object takes 5 seconds to go up, it takes 5 seconds to come back down.
(e) See Figure 6.37.
(f) The maximum height is the distance traveled when going up, which is represented by the area A of the triangle above the time axis.

$$\text{Area} = \frac{1}{2}(160 \text{ ft/sec})(5 \text{ sec}) = 400 \text{ feet}.$$

(g) The slope of the line is -32 so

$$v(t) = -32t + 160.$$

Antidifferentiating, we get

$$s(t) = -16t^2 + 160t + s_0.$$

Since the object starts on the ground, $s_0 = 0$, so

$$s(t) = -16t^2 + 160t.$$

At $t = 5$, we have

$$s(t) = -400 + 800 = 400 \text{ ft}.$$

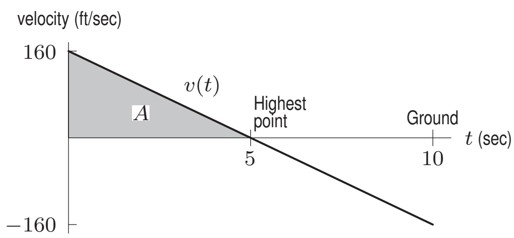


Figure 6.37

22. The equation of motion is $y = -\frac{gt^2}{2} + v_0 t + y_0 = -16t^2 + 128t + 320$. Taking the first derivative, we get $v = -32t + 128$. The second derivative gives us $a = -32$.

(a) At its highest point, the stone's velocity is zero:
$v = 0 = -32t + 128$, so $t = 4$.

(b) At $t = 4$, the height is $y = -16(4)^2 + 128(4) + 320 = 576$ ft

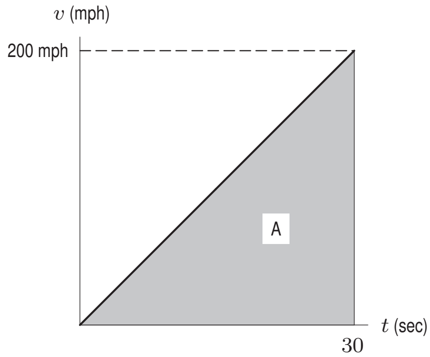
(c) When the stone hits the beach,

$$\begin{aligned} y = 0 &= -16t^2 + 128t + 320 \\ 0 &= -t^2 + 8t + 20 = (10 - t)(2 + t). \end{aligned}$$

So $t = 10$ seconds.

(d) Impact is at $t = 10$. The velocity, v, at this time is $v(10) = -32(10) + 128 = -192$ ft/sec. Upon impact, the stone's velocity is 192 ft/sec downward.

23. Since the acceleration is constant, a graph of the velocity versus time looks like this:



The distance traveled in 30 seconds, which is how long the runway must be, is equal to the area represented by A. We have $A = \frac{1}{2}(\text{base})(\text{height})$. First we convert the required velocity into miles per second.

$$\begin{aligned} 200 \text{ mph} &= \frac{200 \text{ miles}}{\text{hour}} \left(\frac{1 \text{ hour}}{60 \text{ minutes}}\right)\left(\frac{1 \text{ minute}}{60 \text{ seconds}}\right) \\ &= \frac{200}{3600} \frac{\text{miles}}{\text{second}} \\ &= \frac{1}{18} \text{ miles/second.} \end{aligned}$$

Therefore $A = \frac{1}{2}(30 \text{ sec})(200 \text{ mph}) = \frac{1}{2}(30 \text{ sec})\left(\frac{1}{18} \text{ miles/sec}\right) = \frac{5}{6}$ miles.

24. The height of an object above the ground which begins at rest and falls for t seconds is

$$s(t) = -16t^2 + K,$$

where K is the initial height. Here the flower pot falls from 200 ft, so $K = 200$. To see when the pot hits the ground, solve $-16t^2 + 200 = 0$. The solution is

$$t = \sqrt{\frac{200}{16}} \approx 3.54 \text{ seconds.}$$

Now, velocity is given by $s'(t) = v(t) = -32t$. So, the velocity when the pot hits the ground is

$$v(3.54) \approx -113.1 \text{ ft/sec},$$

which is approximately 77 mph downward.

25. The first thing we should do is convert our units. We'll bring everything into feet and seconds. Thus, the initial speed of the car is

$$\frac{70 \text{ miles}}{\text{hour}}\left(\frac{1 \text{ hour}}{3600 \text{ sec}}\right)\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right) \approx 102.7 \text{ ft/sec.}$$

We assume that the acceleration is constant as the car comes to a stop. A graph of its velocity versus time is given in Figure 6.38. We know that the area under the curve represents the distance that the car travels before it comes to a stop, 157 feet. But this area is a triangle, so it is easy to find t_0, the time the car comes to rest. We solve

$$\frac{1}{2}(102.7)t_0 = 157,$$

which gives

$$t_0 \approx 3.06 \text{ sec}.$$

Since acceleration is the rate of change of velocity, the car's acceleration is given by the slope of the line in Figure 6.38. Thus, the acceleration, k, is given by

$$k = \frac{102.7 - 0}{0 - 3.06} \approx -33.56 \text{ ft/sec}^2.$$

Notice that k is negative because the car is slowing down.

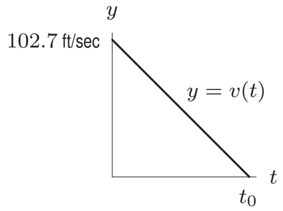
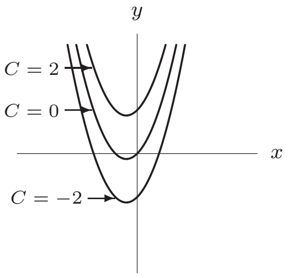
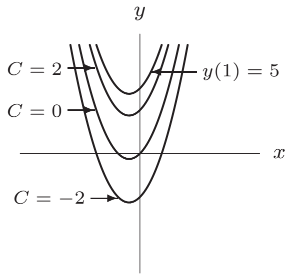


Figure 6.38: Graph of velocity versus time

26. (a) $y = \int (2x + 1)\, dx$, so the solution is $y = x^2 + x + C$.

(b)





(c) At $y(1) = 5$, we have $1^2 + 1 + C = 5$ and so $C = 3$. Thus we have the solution $y = x^2 + x + 3$.

27. (a) We are asking for a function whose derivative is $\sin x + 2$. One antiderivative of $\sin x + 2$ is

$$y = -\cos x + 2x.$$

The general solution is therefore

$$y = -\cos x + 2x + C,$$

where C is any constant. Figure 6.39 shows several curves in this family.

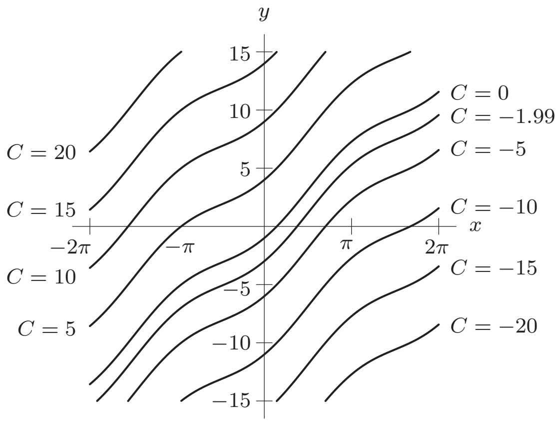


Figure 6.39

(b) We have already seen that the general solution to the differential equation is $y = -\cos x + 2x + C$. The initial condition allows us to determine the constant C. Substituting $y(3) = 5$ gives

$$5 = y(3) = -\cos 3 + 2 \cdot 3 + C,$$

so C is given by

$$C = 5 + \cos 3 - 6 \approx -1.99.$$

Thus, the (unique) solution is

$$y = -\cos x + 2x - 1.99.$$

Figure 6.39 shows this particular solution, marked $C = -1.99$.

28. (a) $a(t) = 1.6$, so $v(t) = 1.6t + v_0 = 1.6t$, since the initial velocity is 0.
(b) $s(t) = 0.8t^2 + s_0$, where s_0 is the rock's initial height.

29. (a) $s = v_0 t - 16t^2$, where $v_0 =$ initial velocity, and $v = s' = v_0 - 32t$. At the maximum height, $v = 0$, so $v_0 = 32t_{\text{max}}$. Plugging into the distance equation yields $100 = 32t_{\text{max}}^2 - 16t_{\text{max}}^2 = 16t_{\text{max}}^2$, so $t_{\text{max}} = \frac{5}{2}$ seconds, from which we get $v_0 = 32\left(\frac{5}{2}\right) = 80$ ft/sec.
(b) This time $g = 5$ ft/sec^2, so $s = v_0 t - 2.5t^2 = 80t - 2.5t^2$, and $v = s' = 80 - 5t$. At the highest point, $v = 0$, so $t_{\text{max}} = \frac{80}{5} = 16$ seconds. Plugging into the distance equation yields $s = 80(16) - 2.5(16)^2 = 640$ ft.

30. The velocity as a function of time is given by: $v = v_0 + at$. Since the object starts from rest, $v_0 = 0$, and the velocity is just the acceleration times time: $v = -32t$. Integrating this, we get position as a function of time: $y = -16t^2 + y_0$, where the last term, y_0, is the initial position at the top of the tower, so $y_0 = 400$ feet. Thus we have a function giving position as a function of time: $y = -16t^2 + 400$.

To find at what time the object hits the ground, we find t when $y = 0$. We solve $0 = -16t^2 + 400$ for t, getting $t^2 = 400/16 = 25$, so $t = 5$. Therefore the object hits the ground after 5 seconds. At this time it is moving with a velocity $v = -32(5) = -160$ feet/second.

31. In Problem 30 we used the equation $0 = -16t^2 + 400$ to learn that the object hits the ground after 5 seconds. In a more general form this is the equation $y = -\frac{g}{2}t^2 + v_0 t + y_0$, and we know that $v_0 = 0$, $y_0 = 400$ ft. So the moment the object hits the ground is given by $0 = -\frac{g}{2}t^2 + 400$. In Problem 30 we used $g = 32$ ft/sec^2, but in this case we want to find a g that results in the object hitting the ground after only 5/2 seconds. We put in 5/2 for t and solve for g:

$$0 = -\frac{g}{2}\left(\frac{5}{2}\right)^2 + 400, \text{ so } g = \frac{2(400)}{(5/2)^2} = 128 \text{ ft/sec}^2.$$

32. $a(t) = -32$. Since $v(t)$ is the antiderivative of $a(t)$, $v(t) = -32t + v_0$. But $v_0 = 0$, so $v(t) = -32t$. Since $s(t)$ is the antiderivative of $v(t)$, $s(t) = -16t^2 + s_0$, where s_0 is the height of the building. Since the ball hits the ground in 5 seconds, $s(5) = 0 = -400 + s_0$. Hence $s_0 = 400$ feet, so the window is 400 feet high.

33. Let time $t = 0$ be the moment when the astronaut jumps up. If acceleration due to gravity is 5 ft/sec^2 and initial velocity is 10 ft/sec, then the velocity of the astronaut is described by

$$v(t) = 10 - 5t.$$

Suppose $y(t)$ describes his distance from the surface of the moon. By the Fundamental Theorem,

$$y(t) - y(0) = \int_0^t (10 - 5x)\, dx$$
$$y(t) = 10t - \frac{1}{2}5t^2.$$

since $y(0) = 0$ (assuming the astronaut jumps off the surface of the moon).

The astronaut reaches the maximum height when his velocity is 0, i.e. when

$$\frac{dy}{dt} = v(t) = 10 - 5t = 0.$$

Solving for t, we get $t = 2$ sec as the time at which he reaches the maximum height from the surface of the moon. At this time his height is

$$y(2) = 10(2) - \frac{1}{2}5(2)^2 = 10 \text{ ft}.$$

When the astronaut is at height $y = 0$, he either just landed or is about to jump. To find how long it is before he comes back down, we find when he is at height $y = 0$. Set $y(t) = 0$ to get

$$\begin{aligned} 0 &= 10t - \frac{1}{2}5t^2 \\ 0 &= 20t - 5t^2 \\ 0 &= 4t - t^2 \\ 0 &= t(t-4). \end{aligned}$$

So we have $t = 0$ sec (when he jumps off) and $t = 4$ sec (when he lands, which gives the time he spent in the air).

34.
- For $[0, t_1]$, the acceleration is constant and positive and the velocity is positive so the displacement is positive. Thus, the work done is positive.
- For $[t_1, t_2]$, the acceleration, and therefore the force, is zero. Therefore, the work done is zero.
- For $[t_2, t_3]$, the acceleration is negative and thus the force is negative. The velocity, and thus the displacement, is positive; therefore the work done is negative.
- For $[t_3, t_4]$, the acceleration (and thus the force) and the velocity (and thus the displacement) are negative. Thus, the work done is positive.
- For $[t_2, t_4]$, the acceleration and thus the force is constant and negative. Velocity both positive and negative; total displacement is 0. Since force is constant, work is 0.

35. Since

$$\text{Acceleration } = \frac{dv}{dt} = -g,$$

velocity is the antiderivative of $-g$, so

$$v = -gt + C.$$

Since the initial velocity is v_0, then $C = v_0$, so

$$v = -gt + v_0.$$

We know that

$$\frac{ds}{dt} = v = -gt + v_0.$$

Therefore, we can find s by antidifferentiating again, giving:

$$s = -\frac{gt^2}{2} + v_0 t + C.$$

If the initial position is s_0, then we must have

$$s = -\frac{gt^2}{2} + v_0 t + s_0.$$

36. Let the acceleration due to gravity equal $-k$ meters/sec^2, for some positive constant k, and suppose the object falls from an initial height of $s(0)$ meters. We have $a(t) = dv/dt = -k$, so that

$$v(t) = -kt + v_0.$$

Since the initial velocity is zero, we have

$$v(0) = -k(0) + v_0 = 0,$$

which means $v_0 = 0$. Our formula becomes

$$v(t) = \frac{ds}{dt} = -kt.$$

This means

$$s(t) = \frac{-kt^2}{2} + s_0.$$

Since

$$s(0) = \frac{-k(0)^2}{2} + s_0,$$

we have $s_0 = s(0)$, and our formula becomes

$$s(t) = \frac{-kt^2}{2} + s(0).$$

Suppose that the object falls for t seconds. Assuming it has not hit the ground, its height is

$$s(t) = \frac{-kt^2}{2} + s(0),$$

so that the distance traveled is

$$s(0) - s(t) = \frac{kt^2}{2} \text{ meters},$$

which is proportional to t^2.

37. (a) $t = \dfrac{s}{\frac{1}{2}v_{\max}}$, where t is the time it takes for an object to travel the distance s, starting from rest with uniform acceleration a. $v_{\max}$ is the highest velocity the object reaches. Since its initial velocity is 0, the mean of its highest velocity and initial velocity is $\frac{1}{2}v_{\max}$.

(b) By Problem 36, $s = \frac{1}{2}gt^2$, where g is the acceleration due to gravity, so it takes $\sqrt{200/32} = 5/2$ seconds for the body to hit the ground. Since $v = gt, v_{\max} = 32(\frac{5}{2}) = 80$ ft/sec. Galileo's statement predicts $(100 \text{ ft})/(40 \text{ ft/sec}) = 5/2$ seconds, and so Galileo's result is verified.

(c) If the acceleration is a constant a, then $s = \frac{1}{2}at^2$, and $v_{\max} = at$. Thus

$$\frac{s}{\frac{1}{2}v_{\max}} = \frac{\frac{1}{2}at^2}{\frac{1}{2}at} = t.$$

38. (a) Since the gravitational force is acting downward

$$-\frac{GMm}{r^2} = m\frac{d^2s}{dt^2}.$$

Hence,

$$\frac{d^2s}{dt^2} = -\frac{GM}{r^2} = \text{Constant}.$$

If we define $g = GM/r^2$, then

$$\frac{d^2s}{dt^2} = -g.$$

(b) The fact that the mass cancels out of Newton's equations of motion reflects Galileo's experimental observation that the acceleration due to gravity is independent of the mass of the body.

39. (a) Since $s(t) = -\frac{1}{2}gt^2$, the distance a body falls in the first second is

$$s(1) = -\frac{1}{2}\cdot g\cdot 1^2 = -\frac{g}{2}.$$

In the second second, the body travels

$$s(2) - s(1) = -\frac{1}{2}\left(g\cdot 2^2 - g\cdot 1^2\right) = -\frac{1}{2}(4g - g) = -\frac{3g}{2}.$$

In the third second, the body travels

$$s(3) - s(2) = -\frac{1}{2}\left(g\cdot 3^2 - g\cdot 2^2\right) = -\frac{1}{2}(9g - 4g) = -\frac{5g}{2},$$

and in the fourth second, the body travels

$$s(4) - s(3) = -\frac{1}{2}\left(g\cdot 4^2 - g\cdot 3^2\right) = -\frac{1}{2}(16g - 9g) = -\frac{7g}{2}.$$

(b) Galileo seems to have been correct. His observation follows from the fact that the differences between consecutive squares are consecutive odd numbers. For, if n is any number, then $n^2 - (n-1)^2 = 2n - 1$, which is the n^{th} odd number (where 1 is the first).

Strengthen Your Understanding

40. The dropped rock has constant acceleration, $dv/dt = -32$ ft/sec^2. Assuming that the rock is not thrown, we let initial velocity be zero, so $v = ds/dt = -32t$. Thus $s = -16t^2 + K$, where K is the initial height, in feet, from which the rock was dropped. We compare the two formulas for s:

$$s_1(t) = -16t^2 + 400$$

and

$$s_2(t) = -16t^2 + 200.$$

The rock dropped from the 400-foot cliff hits the ground when $s_1(t) = 0$ or at $t = 5$ seconds, and the second rock hits the ground when $s_2(t) = 0$, approximately $t = 3.5$ seconds. Thus, the rock dropped from a 400-foot cliff takes less than twice as long to hit the ground as the rock dropped from a 200-foot cliff.

41. If $y = \cos(t^2)$, then $dy/dt = -2t\sin(t^2)$. Thus, $y = \cos(t^2)$ does not satisfy the differential equation.

42. The differential equation has solutions of the form $y = F(x) + C$ with $F'(x) = f(x)$. Two solutions correspond to different values of C and do not cross at any point.

43. The differential equation $dy/dx = 0$ has general solution $y = C$, which is a family of constant solutions.

44. Solutions are of the form

$$y = \frac{t^2}{2} + 3t + C.$$

Different solutions correspond to different values of C. For example

$$y = \frac{t^2}{2} + 3t + 1$$
$$y = \frac{t^2}{2} + 3t + 2.$$

45. If $y = \cos(5x)$, we have $dy/dx = -5\sin(5x)$. The equation $dy/dx = -5\sin(5x)$ is therefore a differential equation that has $y = \cos(5x)$ as a solution.

46. True. If $F(x)$ is an antiderivative of $f(x)$, then $F'(x) = f(x)$, so $dy/dx = f(x)$. Therefore, $y = F(x)$ is a solution to this differential equation.

47. True. If $y = F(x)$ is a solution to the differential equation $dy/dx = f(x)$, then $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

48. True. If acceleration is $a(t) = k$ for some constant k, $k \neq 0$, then we have

$$\text{Velocity} = v(t) = \int a(t)\,dt = \int k\,dt = kt + C_1,$$

for some constant C_1. We integrate again to find position as a function of time:

$$\text{Position} = s(t) = \int v(t)\,dt = \int (kt + C_1)\,dt = \frac{kt^2}{2} + C_1 t + C_2,$$

for some constant C_2. Since $k \neq 0$, this is a quadratic polynomial.

49. False. In an initial value problem the value of y is specified at one value of x, but it does not have to be $x = 0$.

50. False. The solution of the initial value problem $dy/dx = 1$ with $y(0) = -5$ is a solution of the differential equation that is not positive at $x = 0$.

51. True. If $dy/dx = f(x) > 0$, then all solutions $y(x)$ have positive derivative and thus are increasing functions.

52. True. Two solutions $y = F(x)$ and $y = G(x)$ of the same differential equation $dy/dx = f(x)$ are both antiderivatives of $f(x)$ and hence they differ by a constant: $F(x) - G(x) = C$ for all x. Since $F(3) \neq G(3)$ we have $C \neq 0$.

53. True. If $y = f(x)$ satisfies the differential equation $dy/dx = \sin x/x$, then $f'(x) = \sin x/x$. Since $(f(x) + 5)' = f'(x) = \sin x/x$, the function $y = f(x) + 5$ is also a solution of the same differential equation.

54. True. All solutions of the differential equation $dy/dt = 3t^2$ are in the family $y(t) = t^3 + C$ of antiderivatives of $3t^2$. The initial condition $y(1) = \pi$ tells us that $y(1) = \pi = 1^3 + C$, so $C = \pi - 1$. Thus $y(t) = t^3 + \pi - 1$ is the only solution of the initial value problem.

Solutions for Section 6.4

Exercises

1.

Table 6.4

x	0	0.5	1	1.5	2
$I(x)$	0	0.50	1.09	2.03	3.65

2. Using the Fundamental Theorem, we know that the change in F between $x = 0$ and $x = 0.5$ is given by

$$F(0.5) - F(0) = \int_0^{0.5} \sin t \cos t \, dt \approx 0.115.$$

Since $F(0) = 1.0$, we have $F(0.5) \approx 1.115$. The other values are found similarly, and are given in Table 6.5.

Table 6.5

b	0	0.5	1	1.5	2	2.5	3
$F(b)$	1	1.11492	1.35404	1.4975	1.41341	1.17908	1.00996

3. (a) Again using 0.00001 as the lower limit, because the integral is improper, gives $\text{Si}(4) = 1.76$, $\text{Si}(5) = 1.55$.
(b) $\text{Si}(x)$ decreases when the integrand is negative, which occurs when $\pi < x < 2\pi$.

4. If $f'(x) = \sin(x^2)$, then $f(x)$ is of the form

$$f(x) = C + \int_a^x \sin(t^2)\, dt.$$

Since $f(0) = 7$, we take $a = 0$ and $C = 7$, giving

$$f(x) = 7 + \int_0^x \sin(t^2)\, dt.$$

5. If $f'(x) = \dfrac{\sin x}{x}$, then $f(x)$ is of the form

$$f(x) = C + \int_a^x \frac{\sin t}{t}\, dt.$$

Since $f(1) = 5$, we take $a = 1$ and $C = 5$, giving

$$f(x) = 5 + \int_1^x \frac{\sin t}{t}\, dt.$$

6. If $f'(x) = \text{Si}(x)$, then $f(x)$ is of the form

$$f(x) = C + \int_a^x \text{Si}(t)\, dt.$$

Since $f(0) = 2$, we take $a = 0$ and $C = 2$, giving

$$f(x) = 2 + \int_0^x \text{Si}(t)\, dt.$$

7. By the Fundamental Theorem, $f(x) = F'(x)$. Since f is positive and increasing, F is increasing and concave up. Since $F(0) = \int_0^0 f(t)dt = 0$, the graph of F must start from the origin. See Figure 6.40.

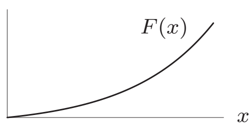


Figure 6.40

8. By the Fundamental Theorem, $f(x) = F'(x)$. Since f is positive and decreasing, F is increasing and concave down. Since $F(0) = \int_0^0 f(t)dt = 0$, the graph of F must start from the origin. See Figure 6.41.

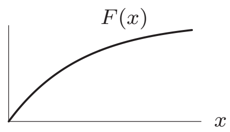


Figure 6.41

9. Since f is always positive, F is always increasing. F has an inflection point where $f' = 0$. Since $F(0) = \int_0^0 f(t)dt = 0$, F goes through the origin. See Figure 6.42.

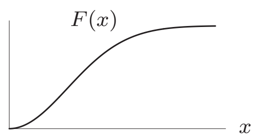


Figure 6.42

10. Since f is always non-negative, F is increasing. F is concave up where f is increasing and concave down where f is decreasing; F has inflection points at the critical points of f. Since $F(0) = \int_0^0 f(t)dt = 0$, the graph of F goes through the origin. See Figure 6.43.

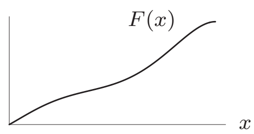


Figure 6.43

11. $\cos(x^2)$.

12. From

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$$

we have

$$\frac{d}{dt}\int_4^t \sin(\sqrt{x})\,dx = \sin(\sqrt{t}).$$

13. $(1+x)^{200}$.

14. From

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

we have

$$\frac{d}{dx}\int_2^x \ln(t^2+1)\,dt = \ln(x^2+1).$$

15. $\arctan(x^2)$.

16. Considering $\mathrm{Si}(x^2)$ as the composition of $\mathrm{Si}(u)$ and $u(x) = x^2$, we may apply the chain rule to obtain

$$\begin{aligned}\frac{d}{dx} &= \frac{d(\mathrm{Si}(u))}{du} \cdot \frac{du}{dx} \\ &= \frac{\sin u}{u} \cdot 2x \\ &= \frac{2\sin(x^2)}{x}.\end{aligned}$$

Problems

17. We need to find where $F''(x)$ is positive or negative. First, we compute

$$F'(x) = e^{-x^2},$$

then

$$F''(x) = -2xe^{-x^2}.$$

Since $e^{-x^2} > 0$ for all x, we see that

$$F''(x) > 0 \text{ for } x < 0$$

and

$$F''(x) < 0 \text{ for } x > 0.$$

Thus the graph of $F(x)$ is concave up for $x < 0$ and concave down for $x > 0$.

18. The graph of $f(x) = xe^{-x}$ is shown in Figure 6.44. We see that $f(x) < 0$ when $x < 0$ and $f(x) > 0$ when $x > 0$. There is some value a with $-1 < a < 0$, for which the area of $f(x)$ below the x-axis from $x = a$ to $x = 0$ equals the area above the x-axis from $x = 0$ to $x = 1$. Then for this value of a,

$$F(a) = \int_1^a f(t)\,dt = \int_1^a te^{-t}\,dt = 0.$$

Also

$$F(1) = \int_1^1 te^{-t}\,dt = 0.$$

There are no other values of x for which $F(x) = 0$, since for $x > 1$, the area under the graph of $f(x)$ increases, because $f(x) > 0$. Similarly, for $x < a$, we have $f(x) < 0$ and the area under the graph of $f(x)$ increases as x gets more negative.

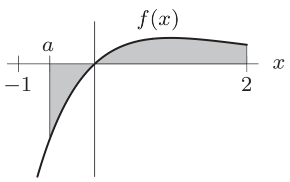


Figure 6.44

19. By the Fundamental Theorem of Calculus,

$$F(x) - F(0) = \int_0^x f(t)\,dt = \int_0^x \sin(t^2)\,dt.$$

Since $F(0) = 0$, we have

$$F(x) = \int_0^x \sin(t^2)\,dt.$$

Using a calculator, we get

$$\begin{aligned} F(0) &= 0 \\ F(0.5) &= 0.041 \\ F(1) &= 0.310 \\ F(1.5) &= 0.778 \\ F(2) &= 0.805 \\ F(2.5) &= 0.431. \end{aligned}$$

20. The graph of $f(x) = F'(x) = \sin(x^2)$ is in Figure 6.45. The function $F(x)$ is increasing where $F'(x) = f(x) > 0$, that is, for $0 < x < \sqrt{\pi} \approx 1.772$. The function $F(x)$ is decreasing where $F'(x) = f(x) < 0$, that is, for $1.772 < x \le 2.5$.

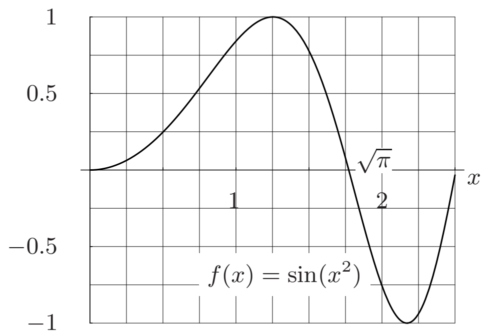


Figure 6.45

21. Using the solution to Problem 20, we see that $F(x)$ is increasing for $x < \sqrt{\pi}$ and $F(x)$ is decreasing for $x > \sqrt{\pi}$. Thus $F(x)$ has its maximum value when $x = \sqrt{\pi}$.

By the Fundamental Theorem of Calculus,

$$F(\sqrt{\pi}) - F(0) = \int_0^{\sqrt{\pi}} f(t)\,dt = \int_0^{\sqrt{\pi}} \sin(t^2)\,dt.$$

Since $F(0) = 0$, calculating Riemann sums gives

$$F(\sqrt{\pi}) = \int_0^{\sqrt{\pi}} \sin(t^2)\,dt = 0.895.$$

22. See Figure 6.46.

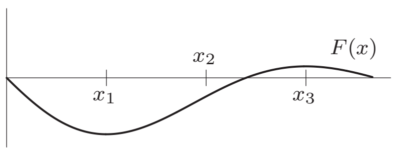


Figure 6.46

23. We know that $F(x)$ increases for $x < 50$ because the derivative of F is positive there. See Figure 6.47. Similarly, $F(x)$ decreases for $x > 50$. Therefore, the graph of F rises until $x = 50$, and then it begins to fall. Thus, the maximum value attained by F is $F(50)$. To evaluate $F(50)$, we use the Fundamental Theorem:

$$F(50) - F(20) = \int_{20}^{50} F'(x)\,dx,$$

which gives

$$F(50) = F(20) + \int_{20}^{50} F'(x)\,dx = 150 + \int_{20}^{50} F'(x)\,dx.$$

The definite integral equals the area of the shaded region under the graph of F', which is roughly 350. Therefore, the greatest value attained by F is $F(50) \approx 150 + 350 = 500$.

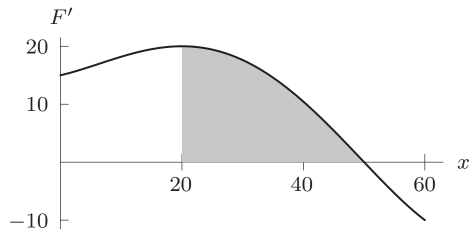
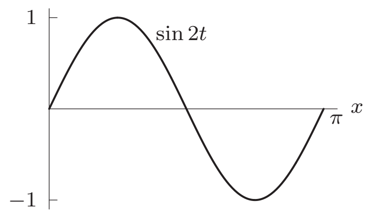


Figure 6.47

24. (a) The definition of g gives $g(0) = \int_0^0 f(t)\,dt = 0$.
(b) The Fundamental Theorem gives $g'(1) = f(1) = -2$.
(c) The function g is concave upward where g'' is positive. Since $g'' = f'$, we see that g is concave up where f is increasing. This occurs on the interval $1 \le x \le 6$.
(d) The function g decreases from $x = 0$ to $x = 3$ and increases for $3 < x \le 8$, and the magnitude of the increase is more than the magnitude of the decrease. Thus g takes its maximum value at $x = 8$.

25. (a) Since $\frac{d}{dt}(\cos(2t)) = -2\sin(2t)$, we have $F(\pi) = \int_0^\pi \sin(2t)\,dt = -\frac{1}{2}\cos(2t)\Big|_0^\pi = -\frac{1}{2}(1-1) = 0$.
(b) $F(\pi) =$ (Area above t-axis) $-$ (Area below t-axis) $= 0$. (The two areas are equal.)

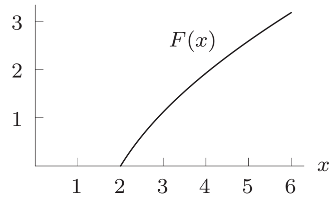


(c) $F(x) \ge 0$ everywhere. $F(x) = 0$ only at integer multiples of π. This can be seen for $x \ge 0$ by noting $F(x) =$ (Area above t-axis) $-$ (Area below t-axis), which is always non-negative and only equals zero when x is an integer multiple of π. For $x > 0$

$$\begin{aligned} F(-x) &= \int_0^{-x} \sin 2t\,dt \\ &= -\int_{-x}^{0} \sin 2t\,dt \\ &= \int_0^{x} \sin 2t\,dt = F(x), \end{aligned}$$

since the area from $-x$ to 0 is the negative of the area from 0 to x. So we have $F(x) \ge 0$ for all x.

26. (a) $F'(x) = \dfrac{1}{\ln x}$ by the Construction Theorem.
(b) For $x \ge 2$, $F'(x) > 0$, so $F(x)$ is increasing. Since $F''(x) = -\dfrac{1}{x(\ln x)^2} < 0$ for $x \ge 2$, the graph of $F(x)$ is concave down.
(c)



27. (a) The definition of R gives

$$R(0) = \int_0^0 \sqrt{1+t^2}\,dt = 0$$

and

$$R(-x) = \int_0^{-x} \sqrt{1+t^2}\,dt.$$

Changing the variable of integration by letting $t = -z$ gives

$$\int_0^{-x} \sqrt{1+t^2}\,dt = \int_0^{x} \sqrt{1+(-z)^2}(-dz) = -\int_0^{x} \sqrt{1+z^2}\,dz.$$

Thus R is an odd function.

(b) Using the Second Fundamental Theorem gives $R'(x) = \sqrt{1+x^2}$, which is always positive, so R is increasing everywhere.

(c) Since

$$R''(x) = \frac{x}{\sqrt{1+x^2}},$$

then R is concave up if $x > 0$ and concave down if $x < 0$.

(d) See Figure 6.48.

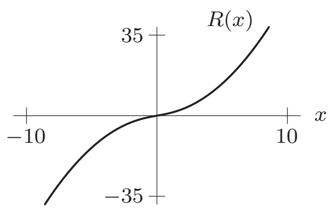


Figure 6.48

(e) We have

$$\lim_{x\to\infty} \frac{R(x)}{x^2} = \lim_{x\to\infty} \frac{\int_0^x \sqrt{1+t^2}\,dt}{x^2}.$$

Using l'Hopital's rule gives

$$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{2x} = \lim_{x\to\infty} \frac{\sqrt{1/x^2+1}}{2} = \frac{1}{2}.$$

Thus the limit exists; its value is 1/2.

28. (a) We have $F(x) = \int_a^x f(t)\,dx = 0$ for all x.

(b) We have $F'(x) = f(x)$ by the Second Fundamental Theorem. We have $F'(x) = 0$ by part (a). Hence $f(x) = 0$.

29. Since $F'(x) = e^{-x^2}$ and $F(0) = 2$, we have

$$F(x) = F(0) + \int_0^x e^{-t^2}\,dt = 2 + \int_0^x e^{-t^2}\,dt.$$

Substituting $x = 1$ and evaluating the integral numerically gives

$$F(1) = 2 + \int_0^1 e^{-t^2}\,dt = 2.747.$$

30. Since $G'(x) = \cos(x^2)$ and $G(0) = -3$, we have

$$G(x) = G(0) + \int_0^x \cos(t^2)\,dt = -3 + \int_0^x \cos(t^2)\,dt.$$

Substituting $x = -1$ and evaluating the integral numerically gives

$$G(-1) = -3 + \int_0^{-1} \cos(t^2)\,dt = -3.905.$$

31. We have

$$\begin{aligned} w(0.4) &= \int_0^{0.4} q(x)\,dx \\ &\approx 0.1q(0.0) + 0.1q(0.1) + 0.1q(0.2) + 0.1q(0.3) \qquad \text{left-hand sum with } \Delta t = 0.1 \\ &= 0.1(5.3) + 0.1(5.2) + 0.1(4.9) + 0.1(4.5) = 1.99. \end{aligned}$$

Since $q'(x) < 0$, we know a left-hand sum provides an overestimate.

Alternatively, we could use a right-hand sum to provide an underestimate:

$$\begin{aligned} w(0.4) &= \int_0^{0.4} q(x)\,dx \\ &\approx 0.1q(0.1) + 0.1q(0.2) + 0.1q(0.3) + 0.1q(0.4) \qquad \text{right-hand sum with } \Delta t = 0.1 \\ &= 0.1(5.2) + 0.1(4.9) + 0.1(4.5) + 0.1(3.9) = 1.85. \end{aligned}$$

32. We have:

$$\begin{aligned} v(0.4) &= \int_0^{0.4} q'(x)\,dx, \\ &= q(0.4) - q(0) \qquad \text{Fundamental Theorem of Calculus} \\ &= 3.9 - 5.3 \qquad \text{from the table} \\ &= -1.4. \end{aligned}$$

This answer is exact.

33. Since $w(t) = \int_0^t q(x)\,dx$, we know from the Construction Theorem that w is an antiderivative of q. This means:

$$\begin{aligned} w'(t) &= q(t), \\ \text{so} \quad w'(0.4) &= q(0.4) = 3.9 \quad \text{from the table.} \end{aligned}$$

This answer is exact.

34. Since $v(t) = \int_0^t q'(x)\,dx$, we know from the Construction Theorem that v is an antiderivative of q'. This means:

$$\begin{aligned} v'(t) &= q'(t) \\ \text{so} \quad v'(0.4) &= q'(0.4) \\ &\approx \frac{q(0.5) - q(0.4)}{0.1} \qquad \text{average rate of change of } q \\ &= \frac{3.1 - 3.9}{0.1} = -8. \end{aligned}$$

Since q' and q'' are both negative, we know q is decreasing and its graph is concave-down. This tells us that the secant line from $x = 0.4$ to $x = 0.5$ is steeper (its slope is "more negative") than the tangent line, making -8 an underestimate.

Alternatively we can write

$$\begin{aligned} v'(0.4) = q'(0.4) &\approx \frac{q(0.4) - q(0.3)}{0.1} \qquad \text{average rate of change of } q \\ &= \frac{3.9 - 4.5}{0.1} = -6. \end{aligned}$$

Here, the secant line from $x = 0.3$ to $x = 0.4$ is less steep (its slope is "less negative") than the tangent line, making -6 an overestimate.

35. If we let $F(x) = \int_0^x \ln(1 + t^2)\,dt$, using the chain rule gives

$$\frac{d}{dx}F(x^2) = 2xF'(x^2) = 2x\ln(1 + (x^2)^2) = 2x\ln(1 + x^4).$$

36. If we let $f(t) = \int_1^t \cos(x^2)\,dx$ and $g(t) = \sin t$, using the chain rule gives

$$\frac{d}{dt}\int_1^{\sin t} \cos(x^2)\,dx = f'(g(t)) \cdot g'(t) = \cos((\sin t)^2) \cdot \cos t = \cos(\sin^2 t)(\cos t).$$

37. We first write

$$\int_{2t}^{4} \sin(\sqrt{x})\,dx$$

as

$$-\int_{4}^{2t} \sin(\sqrt{x})\,dx.$$

Letting

$$F(t) = -\int_4^t \sin(\sqrt{x})\,dx,$$

and using the chain rule gives

$$\frac{d}{dt}F(2t) = 2F'(2t) = 2[-\sin(\sqrt{2t})] = -2\sin(\sqrt{2t}).$$

38. If we split the integral at $x = 0$, we have

$$\int_{-x^2}^{x^2} e^{t^2}\,dt = \int_{-x^2}^{0} e^{t^2}\,dt + \int_0^{x^2} e^{t^2}\,dt = -\int_0^{-x^2} e^{t^2}\,dt + \int_0^{x^2} e^{t^2}\,dt.$$

If we let

$$F(x) = \int_0^x e^{t^2}\,dt,$$

using the chain rule on each part separately gives

$$\frac{d}{dx}[-F(-x^2) + F(x^2)] = -(-2x)F'(-x^2) + (2x)F'(x^2) = (2x)e^{(-x^2)^2} + (2x)e^{(x^2)^2} = 4xe^{x^4}.$$

39.

$$\begin{aligned}\frac{d}{dx}[x\,\text{erf}(x)] &= \text{erf}(x)\frac{d}{dx}(x) + x\frac{d}{dx}[\text{erf}(x)] \\ &= \text{erf}(x) + x\frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right) \\ &= \text{erf}(x) + \frac{2}{\sqrt{\pi}}xe^{-x^2}.\end{aligned}$$

40. If we let $f(x) = \text{erf}(x)$ and $g(x) = \sqrt{x}$, then we are looking for $\frac{d}{dx}[f(g(x))]$. By the chain rule, this is the same as $g'(x)f'(g(x))$. Since

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right) \\ &= \frac{2}{\sqrt{\pi}}e^{-x^2}\end{aligned}$$

and $g'(x) = \dfrac{1}{2\sqrt{x}}$, we have

$$f'(g(x)) = \frac{2}{\sqrt{\pi}}e^{-x},$$

and so

$$\frac{d}{dx}[\text{erf}(\sqrt{x})] = \frac{1}{2\sqrt{x}}\frac{2}{\sqrt{\pi}}e^{-x} = \frac{1}{\sqrt{\pi x}}e^{-x}.$$

41. If we let $f(x) = \int_0^x e^{-t^2}\,dt$ and $g(x) = x^3$, then we use the chain rule because we are looking for $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$. Since $f'(x) = e^{-x^2}$, we have

$$\frac{d}{dx}\left(\int_0^{x^3} e^{-t^2}\,dt\right) = f'(x^3) \cdot 3x^2 = e^{-(x^3)^2} \cdot 3x^2 = 3x^2 e^{-x^6}.$$

42. We split the integral $\int_x^{x^3} e^{-t^2}\,dt$ into two pieces, say at $t = 1$ (though it could be at any other point):

$$\int_x^{x^3} e^{-t^2}\,dt = \int_1^{x^3} e^{-t^2}\,dt + \int_x^1 e^{-t^2}\,dt = \int_1^{x^3} e^{-t^2}\,dt - \int_1^x e^{-t^2}\,dt.$$

We have used the fact that $\int_x^1 e^{-t^2}\,dt = -\int_1^x e^{-t^2}\,dt$. Differentiating gives

$$\frac{d}{dx}\left(\int_x^{x^3} e^{-t^2}\,dt\right) = \frac{d}{dx}\left(\int_1^{x^3} e^{-t^2}\,dt\right) - \frac{d}{dx}\left(\int_1^{x} e^{-t^2}\,dt\right)$$

For the first integral, we use the chain rule with $g(x) = x^3$ as the inside function, so the final answer is

$$\frac{d}{dx}\left(\int_x^{x^3} e^{-t^2}\,dt\right) = e^{-(x^3)^2} \cdot 3x^2 - e^{-x^2} = 3x^2 e^{-x^6} - e^{-x^2}.$$

Strengthen Your Understanding

43. Note that $\displaystyle\int_0^5 t^2\,dt$ is a constant, therefore its derivative is zero.

44. $f(x) = x^2$ has a minimum at $x = 0$. However, $F(x)$ is non-decreasing for all x since $F'(x) = x^2 \geq 0$ everywhere.

45. The derivative of $F(x)$, which is $f(x)$, has a local minimum at $x = 2$. The function $F(x)$ has local minimums at $x = -1$ and $x = 3$, where $f(x)$ goes from negative to positive.

46. From the Second Fundamental Theorem of Calculus, we have $F(x) = \int_0^x f(t)dt$ (so that $F(0) = 0$). In order to assure that F is a nondecreasing function, we need only pick a function $f(t)$ that is nonnegative for all t. The choices are many, but one possible example is $f(t) = t^2$, giving $F(x) = \int_0^x t^2\,dt$.

47. From the Second Fundamental Theorem of Calculus, we have $G(x) = \int_a^x g(t)dt$. Since $G(7) = 0$, we let $a = 7$, and in order to ensure that G is concave up, we need a function $g(t)$ that has a positive derivative. One possible example is $g(t) = e^t$, giving $G(x) = \int_7^x e^t dt$.

48. True. The Construction Theorem for Antiderivatives gives a method for building an antiderivative with a definite integral.

49. True, by the Second Fundamental Theorem of Calculus.

50. True. We see that

$$F(5) - F(3) = \int_0^5 f(t)\,dt - \int_0^3 f(t)\,dt = \int_3^5 f(t)\,dt.$$

51. False. If f is positive then F is increasing, but if f is negative then F is decreasing.

52. True. Since F and G are both antiderivatives of f, they must differ by a constant. In fact, we can see that the constant C is equal to $\int_0^2 f(t)\,dt$ since

$$F(x) = \int_0^x f(t)\,dt = \int_2^x f(t)\,dt + \int_0^2 f(t)\,dt = G(x) + C.$$

53. True, since $\int_0^x (f(t) + g(t))\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$.

Solutions for Chapter 6 Review

Exercises

1. We find the changes in $f(x)$ between any two values of x by counting the area between the curve of $f'(x)$ and the x-axis. Since $f'(x)$ is linear throughout, this is quite easy to do. From $x = 0$ to $x = 1$, we see that $f'(x)$ outlines a triangle of area $1/2$ below the x-axis (the base is 1 and the height is 1). By the Fundamental Theorem,

$$\int_0^1 f'(x)\,dx = f(1) - f(0),$$

so

$$f(0) + \int_0^1 f'(x)\,dx = f(1)$$
$$f(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

Similarly, between $x = 1$ and $x = 3$ we can see that $f'(x)$ outlines a rectangle below the x-axis with area -1, so $f(2) = 3/2 - 1 = 1/2$. Continuing with this procedure (note that at $x = 4$, $f'(x)$ becomes positive), we get the table below.

x	0	1	2	3	4	5	6
$f(x)$	2	$3/2$	$1/2$	$-1/2$	-1	$-1/2$	$1/2$

2. Since $F(0) = 0$, $F(b) = \int_0^b f(t)\,dt$. For each b we determine $F(b)$ graphically as follows:
$F(0) = 0$
$F(1) = F(0) +$ Area of 1×1 rectangle $= 0 + 1 = 1$
$F(2) = F(1) +$ Area of triangle $(\frac{1}{2} \cdot 1 \cdot 1) = 1 + 0.5 = 1.5$
$F(3) = F(2) +$ Negative of area of triangle $= 1.5 - 0.5 = 1$
$F(4) = F(3) +$ Negative of area of rectangle $= 1 - 1 = 0$
$F(5) = F(4) +$ Negative of area of rectangle $= 0 - 1 = -1$
$F(6) = F(5) +$ Negative of area of triangle $= -1 - 0.5 = -1.5$
The graph of $F(t)$, for $0 \le t \le 6$, is shown in Figure 6.49.

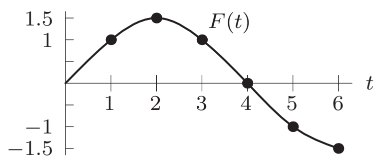


Figure 6.49

3. F is increasing because f is positive; F is concave up because f is increasing. See Figure 6.50.

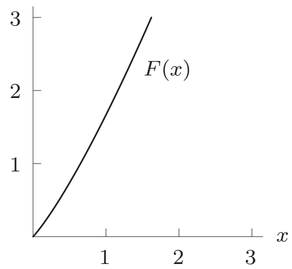


Figure 6.50

4. F is increasing because f is positive; F is concave up because f is increasing. See Figure 6.51.

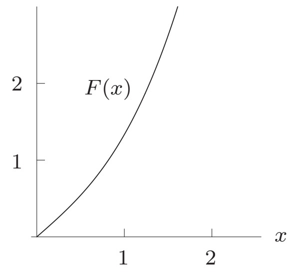


Figure 6.51

5. (a) The value of the integral is negative since the area below the x-axis is greater than the area above the x-axis. We count boxes: The area below the x-axis includes approximately 11.5 boxes and each box has area $(2)(1) = 2$, so

$$\int_0^5 f(x)dx \approx -23.$$

The area above the x-axis includes approximately 2 boxes, each of area 2, so

$$\int_5^7 f(x)dx \approx 4.$$

So we have

$$\int_0^7 f(x)dx = \int_0^5 f(x)dx + \int_5^7 f(x)dx \approx -23 + 4 = -19.$$

(b) By the Fundamental Theorem of Calculus, we have

$$F(7) - F(0) = \int_0^7 f(x)dx$$

so,

$$F(7) = F(0) + \int_0^7 f(x)dx = 25 + (-19) = 6.$$

6. $\int 5x\,dx = \frac{5}{2}x^2 + C.$

7. $\int x^3\,dx = \frac{x^4}{4} + C$

8. $\int \sin\theta\,d\theta = -\cos\theta + C$

9. $\int (x^3 - 2)\,dx = \frac{x^4}{4} - 2x + C$

10. $\int (t^2 + \frac{1}{t^2})\,dt = \frac{t^3}{3} - \frac{1}{t} + C$

11. $\int \frac{4}{t^2}\,dt = -\frac{4}{t} + C$

12. $\int (x^2 + 5x + 8)\,dx = \frac{x^3}{3} + \frac{5x^2}{2} + 8x + C$

13. $\int 4\sqrt{w}\,dw = \frac{8}{3}w^{3/2} + C$

14. $2t^2 + 7t + C$

15. $\sin\theta + C$

16. $\int \left(t^{3/2} + t^{-3/2}\right)\, dt = \frac{2t^{5/2}}{5} - 2t^{-1/2} + C$

17. $\frac{x^2}{2} + 2x^{1/2} + C$

18. $\pi x + \frac{x^{12}}{12} + C$

19. $3\sin t + 2t^{3/2} + C$

20. $\int \left(y - \frac{1}{y}\right)^2 dy = \int \left(y^2 - 2 + \frac{1}{y^2}\right) dy = \frac{y^3}{3} - 2y - \frac{1}{y} + C$

21. $\tan x + C$

22. $2\ln|x| - \pi\cos x + C$

23. Since $f(x) = x + 1 + \frac{1}{x}$, the indefinite integral is $\frac{1}{2}x^2 + x + \ln|x| + C$

24. $5e^z + C$

25. $\frac{1}{\ln 2}2^x + C$, since $\frac{d}{dx}(2^x) = (\ln 2)\cdot 2^x$

26. $3\sin x + 7\cos x + C$

27. $2e^x - 8\sin x + C$

28. $\int_{-3}^{-1} \frac{2}{r^3}\, dr = -r^{-2}\Big|_{-3}^{-1} = -1 + \frac{1}{9} = -8/9 \approx -0.889.$

29. We have

$$\int_{-\pi/2}^{\pi/2} 2\cos\phi d\phi = 2\left(\sin\phi\Big|_{-\pi/2}^{\pi/2}\right) = 2\left(\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right) = 2\left(1 - (-1)\right). = 4.$$

30. $F(x) = \int f(x)\, dx = \int x^2\, dx = \frac{x^3}{3} + C$. If $F(0) = 4$, then $F(0) = 0 + C = 4$ and thus $C = 4$. So $F(x) = \frac{x^3}{3} + 4$.

31. We have $F(x) = \frac{x^4}{4} + 2x^3 - 4x + C$. Since $F(0) = 4$, we have $4 = 0 + C$, so $C = 4$. So $F(x) = \frac{x^4}{4} + 2x^3 - 4x + 4$.

32. $F(x) = \int \sqrt{x}\, dx = \frac{2}{3}x^{3/2} + C$. If $F(0) = 4$, then $F(0) = 0 + C = 4$ and thus $C = 4$. So $F(x) = \frac{2}{3}x^{3/2} + 4$.

33. $F(x) = \int e^x\, dx = e^x + C$. If $F(0) = 4$, then $F(0) = 1 + C = 4$ and thus $C = 3$. So $F(x) = e^x + 3$.

34. $F(x) = \int \sin x\, dx = -\cos x + C$. If $F(0) = 4$, then $F(0) = -1 + C = 4$ and thus $C = 5$. So $F(x) = -\cos x + 5$.

35. $F(x) = \int \cos x\, dx = \sin x + C$. If $F(0) = 4$, then $F(0) = 0 + C = 4$ and thus $C = 4$. So $F(x) = \sin x + 4$.

36. Since $(x^x)' = x^x(1 + \ln x)$, we have

$$\int_1^3 x^x(1 + \ln x)\, dx = x^x\Big|_1^3 = 3^3 - 1^1 = 26.$$

37. Since $y = x + \sin x - \pi$, we differentiate to see that $dy/dx = 1 + \cos x$, so y satisfies the differential equation. To show that it also satisfies the initial condition, we check that $y(\pi) = 0$:

$$y = x + \sin x - \pi$$
$$y(\pi) = \pi + \sin\pi - \pi = 0.$$

38. Differentiating y with respect to x gives

$$y' = nx^{n-1}$$

for all values of A.

39. $y = \int (x^3 + 5)\,dx = \frac{x^4}{4} + 5x + C$

40. $y = \int \left(8x + \frac{1}{x}\right)\,dx = 4x^2 + \ln|x| + C$

41. $W = \int 4\sqrt{t}\,dt = \frac{8}{3}t^{3/2} + C$

42. $r = \int 3\sin p\,dp = -3\cos p + C$

43. $y = \int (6x^2 + 4x)\,dx = 2x^3 + 2x^2 + C$. If $y(2) = 10$, then $2(2)^3 + 2(2)^2 + C = 10$ and $C = 10 - 16 - 8 = -14$. Thus, $y = 2x^3 + 2x^2 - 14$.

44. $P = \int 10e^t\,dt = 10e^t + C$. If $P(0) = 25$, then $10e^0 + C = 25$ so $C = 15$. Thus, $P = 10e^t + 15$.

45. $s = \int (-32t + 100)\,dt = -16t^2 + 100t + C$. If $s = 50$ when $t = 0$, then $-16(0)^2 + 100(0) + C = 50$, so $C = 50$. Thus $s = -16t^2 + 100t + 50$.

46. Integrating gives

$$\int \frac{dq}{dz}\,dz = \int (2 + \sin z)\,dz = 2z - \cos z + C.$$

If $q = 5$ when $z = 0$, then $2(0) - \cos(0) + C = 5$ so $C = 6$. Thus $q = 2z - \cos z + 6$.

47. $\frac{d}{dt}\int_t^\pi \cos(z^3)\,dz = \frac{d}{dt}\left(-\int_\pi^t \cos(z^3)\,dz\right) = -\cos(t^3)$.

48. $\frac{d}{dx}\int_x^1 \ln t\,dt = \frac{d}{dx}\left(-\int_1^x \ln t\,dt\right) = -\ln x$.

Problems

49. We can start by finding four points on the graph of $F(x)$. The first one is given: $F(2) = 3$. By the Fundamental Theorem of Calculus, $F(6) = F(2) + \int_2^6 F'(x)dx$. The value of this integral is -7 (the area is 7, but the graph lies below the x-axis), so $F(6) = 3 - 7 = -4$. Similarly, $F(0) = F(2) - 2 = 1$, and $F(8) = F(6) + 4 = 0$. We sketch a graph of $F(x)$ by connecting these points, as shown in Figure 6.52.

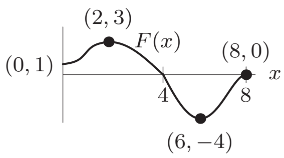


Figure 6.52

50. Between time $t = 0$ and time $t = B$, the velocity of the cork is always positive, which means the cork is moving upward. At time $t = B$, the velocity is zero, and so the cork has stopped moving altogether. Since shortly thereafter the velocity of the cork becomes negative, the cork will next begin to move downward. Thus when $t = B$ the cork has risen as far as it ever will, and is riding on top of the crest of the wave.

From time $t = B$ to time $t = D$, the velocity of the cork is negative, which means it is falling. When $t = D$, the velocity is again zero, and the cork has ceased to fall. Thus when $t = D$ the cork is riding on the bottom of the trough of the wave.

Since the cork is on the crest at time B and in the trough at time D, it is probably midway between crest and trough when the time is midway between B and D. Thus at time $t = C$ the cork is moving through the equilibrium position on its way down. (The equilibrium position is where the cork would be if the water were absolutely calm.) By symmetry, $t = A$ is the time when the cork is moving through the equilibrium position on the way up.

Since acceleration is the derivative of velocity, points where the acceleration is zero would be critical points of the velocity function. Since point A (a maximum) and point C (a minimum) are critical points, the acceleration is zero there.

A possible graph of the height of the cork is shown in Figure 6.53. The horizontal axis represents a height equal to the average depth of the ocean at that point (the equilibrium position of the cork).

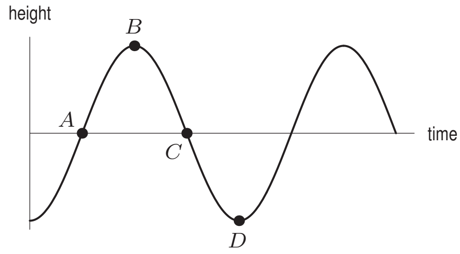


Figure 6.53

51. (a) Critical points of $F(x)$ are the zeros of f: $x = 1$ and $x = 3$.
(b) $F(x)$ has a local minimum at $x = 1$ and a local maximum at $x = 3$.
(c) See Figure 6.54.

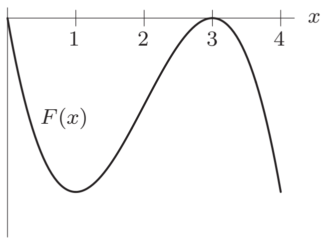


Figure 6.54

Notice that the graph could also be above or below the x-axis at $x = 3$.

52. (a) Critical points of $F(x)$ are $x = -1$, $x = 1$ and $x = 3$.
(b) $F(x)$ has a local minimum at $x = -1$, a local maximum at $x = 1$, and a local minimum at $x = 3$.
(c) See Figure 6.55.

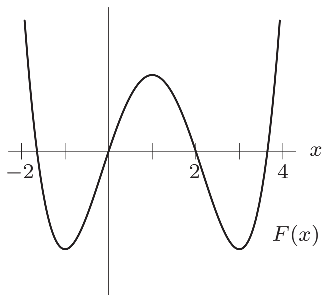
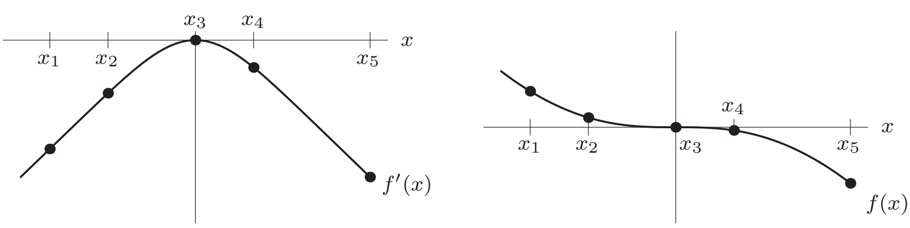


Figure 6.55

53.



(a) $f(x)$ is greatest at x_1.

(b) $f(x)$ is least at x_5.
(c) $f'(x)$ is greatest at x_3..
(d) $f'(x)$ is least at x_5.
(e) $f''(x)$ is greatest at x_1.
(f) $f''(x)$ is least at x_5.

54. (a) Starting at $x = 3$, we are given that $f(3) = 0$. Moving to the left on the interval $2 < x < 3$, we have $f'(x) = -1$, so $f(2) = f(3) - (1)(-1) = 1$. On the interval $0 < x < 2$, we have $f'(x) = 1$, so

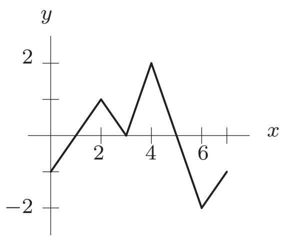
$$f(0) = f(2) + 1(-2) = -1.$$

Moving to the right from $x = 3$, we know that $f'(x) = 2$ on $3 < x < 4$. So $f(4) = f(3) + 2 = 2$. On the interval $4 < x < 6$, $f'(x) = -2$ so

$$f(6) = f(4) + 2(-2) = -2.$$

On the interval $6 < x < 7$, we have $f'(x) = 1$, so

$$f(7) = f(6) + 1 = -2 + 1 = -1.$$



(b) In part (a) we found that $f(0) = -1$ and $f(7) = -1$.
(c) The integral $\int_0^7 f'(x)\,dx$ is given by the sum

$$\int_0^7 f'(x)\,dx = (1)(2) + (-1)(1) + (2)(1) + (-2)(2) + (1)(1) = 0.$$

Alternatively, knowing $f(7)$ and $f(0)$ and using the Fundamental Theorem of Calculus, we have

$$\int_0^7 f'(x)\,dx = f(7) - f(0) = -1 - (-1) = 0.$$

55. We have

$$\text{Area} = \int_1^4 x^2\,dx = \left.\frac{x^3}{3}\right|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = 21.$$

56. The graph crosses the x-axis where

$$7 - 8x + x^2 = 0$$
$$(x-7)(x-1) = 0;$$

so $x = 1$ and $x = 7$. See Figure 6.56. The parabola opens upward and the region is below the x-axis, so

$$\begin{aligned}\text{Area} &= -\int_1^7 (7 - 8x + x^2)\,dx \\ &= -\left(7x - 4x^2 + \frac{x^3}{3}\right)\Bigg|_1^7 = 36.\end{aligned}$$

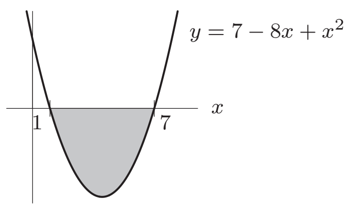


Figure 6.56

57. Since $y = x^3(1-x)$ is positive for $0 \le x \le 1$ and $y = 0$, when $x = 0, 1$, the area is given by

$$\text{Area } = \int_0^1 x^3(1-x)\,dx = \int_0^1 (x^3 - x^4)\,dx = \frac{x^4}{4} - \frac{x^5}{5}\bigg|_0^1 = \frac{1}{20}.$$

58. Since $y = 0$ only when $x = 0$ and $x = 1$, the area lies between these limits and is given by

$$\begin{aligned}\text{Area } &= \int_0^1 x^2(1-x)^2 dx = \int_0^1 x^2(1-2x+x^2)\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx \\ &= \frac{x^3}{3} - \frac{2}{4}x^4 + \frac{x^5}{5}\bigg|_0^1 = \frac{1}{30}.\end{aligned}$$

59. The curves cross at the origin and at $(1, 1)$. See Figure 6.57. Since the upper half of $x = y^2$ is given by $y = \sqrt{x}$ and this curve is above $y = x^2$, we have

$$\text{Area } = \int_0^1 (\sqrt{x} - x^2)\,dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)\bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

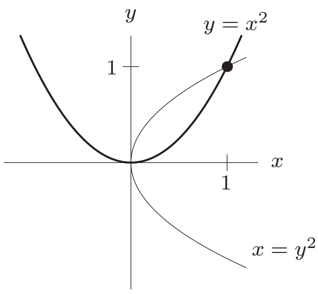


Figure 6.57

60. The graph is shown in Figure 6.58. Since $\cos\theta \ge \sin\theta$ for $0 \le \theta \le \pi/4$, we have

$$\begin{aligned}\text{Area } &= \int_0^{\pi/4} (\cos\theta - \sin\theta)\,d\theta \\ &= (\sin\theta + \cos\theta)\bigg|_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1.\end{aligned}$$

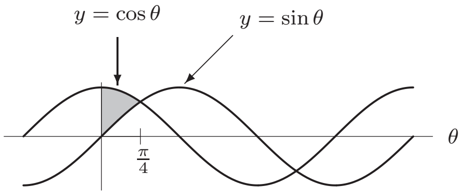


Figure 6.58

61. The graphs of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ cross at $\theta = \pi/4$ and $\theta = 5\pi/4$. See Figure 6.59. Since $\sin\theta$ is above $\cos\theta$ between these two crossing points and $\cos\theta$ is above $\sin\theta$ outside, we have

$$\text{Area } = \int_0^{\pi/4} (\cos\theta - \sin\theta)\,d\theta + \int_{\pi/4}^{5\pi/4} (\sin\theta - \cos\theta)\,d\theta + \int_{5\pi/4}^{2\pi} (\cos\theta - \sin\theta)\,d\theta$$

$$= (\sin\theta + \cos\theta)\Big|_0^{\pi/4} + (-\cos\theta - \sin\theta)\Big|_{\pi/4}^{5\pi/4} + (\sin\theta + \cos\theta)\Big|_{5\pi/4}^{2\pi}$$
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) + \left(-\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right) + \left(1 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right)$$
$$= \frac{8}{\sqrt{2}} = 4\sqrt{2}.$$

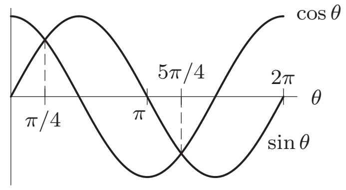


Figure 6.59

62. Since the graph of $y = e^x$ is above the graph of $y = \cos x$ (see Figure 6.60) we have

$$\begin{aligned}\text{Area } &= \int_0^1 (e^x - \cos x)\,dx \\ &= \int_0^1 e^x\,dx - \int_0^1 \cos x\,dx \\ &= e^x\Big|_0^1 - \sin x\Big|_0^1 \\ &= e^1 - e^0 - \sin 1 + \sin 0 \\ &= e - 1 - \sin 1.\end{aligned}$$

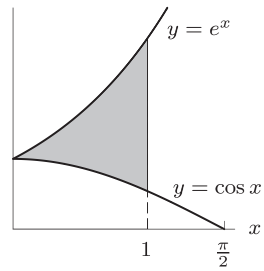


Figure 6.60

63. The area is given by

$$\begin{aligned}A = \int_{-1}^1 (\cosh x - \sinh x)\,dx &= (\sinh x - \cosh x)\Big|_{-1}^1 \\ &= \sinh 1 - \cosh 1 - (\sinh(-1) - \cosh(-1)) \\ &= 2\sinh 1.\end{aligned}$$

64. The area under $f(x) = 8x$ between $x = 1$ and $x = b$ is given by $\int_1^b (8x)dx$. Using the Fundamental Theorem to evaluate the integral:

$$\text{Area } = 4x^2\Big|_1^b = 4b^2 - 4.$$

Since the area is 192, we have

$$\begin{aligned} 4b^2 - 4 &= 192 \\ 4b^2 &= 196 \\ b^2 &= 49 \\ b &= \pm 7. \end{aligned}$$

Since b is larger than 1, we have $b = 7$.

65. The graph of $y = x^2 - c^2$ has x-intercepts of $x = \pm c$. See Figure 6.61. The shaded area is given by

$$\begin{aligned} \text{Area } &= -\int_{-c}^{c} (x^2 - c^2)\, dx \\ &= -2\int_0^c (x^2 - c^2)\, dx \\ &= -2\left(\frac{x^3}{3} - c^2 x\right)\Big|_0^c = -2\left(\frac{c^3}{3} - c^3\right) = \frac{4}{3}c^3. \end{aligned}$$

We want c to satisfy $(4c^3)/3 = 36$, so $c = 3$.

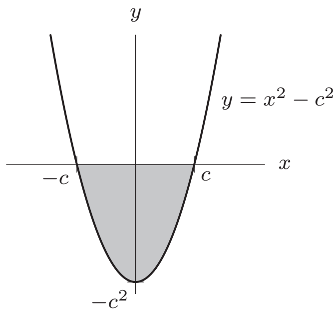


Figure 6.61

66. We have

$$\text{Average value } = \frac{1}{10-0}\int_0^{10} (x^2+1)dx = \frac{1}{10}\left(\frac{x^3}{3} + x\right)\Big|_0^{10} = \frac{1}{10}\left(\frac{10^3}{3} + 10 - 0\right) = \frac{103}{3}.$$

We see in Figure 6.62 that the average value of $103/3 \approx 34.33$ for $f(x)$ looks right.

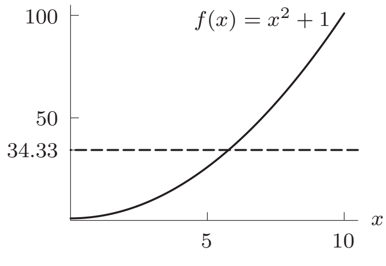


Figure 6.62

67. The average value of $v(x)$ on the interval $1 \le x \le c$ is

$$\frac{1}{c-1}\int_1^c \frac{6}{x^2}\,dx = \frac{1}{c-1}\left(-\frac{6}{x}\right)\Big|_1^c = \frac{1}{c-1}\left(\frac{-6}{c}+6\right) = \frac{6}{c}.$$

Since $\frac{1}{c-1}\int_1^c \frac{6}{x^2}dx = 1$, we have $\frac{6}{c} = 1$, so $c = 6$.

68. Letting $y = f(x)$, we have $x = f^{-1}(y)$:

$$\begin{aligned} y &= 5\sqrt{x} \\ \sqrt{x} &= 0.2y \\ x &= 0.04y^2 \\ \text{so}\quad f^{-1}(x) &= 0.04x^2. \end{aligned}$$

Letting $k = 0.04$ and $n = 2$, we have

$$\begin{aligned} \int_1^4 f^{-1}(x)\,dx &= \frac{0.04}{2+1}\cdot x^{2+1}\Big|_1^4 \\ &= \frac{0.04}{3}\cdot x^3\Big|_1^4 \\ &= \frac{0.04}{3}\left(4^3 - 1^3\right) \\ &= 0.84. \end{aligned}$$

69. We have

$$\begin{aligned} (f(x))^{-1} &= \left(5\sqrt{x}\right)^{-1} \\ &= \frac{1}{5x^{0.5}} \\ &= 0.2x^{-0.5}. \end{aligned}$$

Letting $k = 0.2$ and $n = -0.5$, we have

$$\begin{aligned} \int_1^4 (f(x))^{-1}\,dx &= \frac{0.2}{-0.5+1}\cdot x^{-0.5+1}\Big|_1^4 \\ &= \frac{0.2}{0.5}\cdot x^{0.5}\Big|_1^4 \\ &= 0.4\left(4^{0.5} - 1^{0.5}\right) \\ &= 0.4. \end{aligned}$$

70. Letting $k = 5$ and $n = 0.5$, we have

$$\begin{aligned} \int_1^4 f(x)\,dx &= \frac{5}{0.5+1}\cdot x^{0.5+1}\Big|_1^4 \\ &= \frac{5}{3/2}\cdot x^{3/2}\Big|_1^4 \\ &= \frac{10}{3}\cdot\left(4^{3/2} - 1^{3/2}\right) \\ &= \frac{10}{3}\cdot 7 = \frac{70}{3}. \end{aligned}$$

Thus,

$$\left(\int_1^4 f(x)\,dx\right)^{-1} = \left(\frac{70}{3}\right)^{-1} = \frac{3}{70}.$$

71. We have:

$$\begin{aligned} f(2) &= \int_0^2 2x^2\,dx \\ &= 2\int_0^2 x^2\,dx \\ &= 2\cdot\frac{1}{3}x^3\Big|_0^2 \\ &= \frac{16}{3}. \end{aligned}$$

72. We have:

$$\begin{aligned} f(n) &= \int_0^n nx^2\,dx \\ &= n\int_0^n x^2\,dx \quad n \text{ is a constant} \\ &= n\cdot\frac{1}{3}x^3\Big|_0^n \\ &= \frac{1}{3}n^4. \end{aligned}$$

73. If we let $f(x) = \int_2^x \sin(t^2)\,dt$ and $g(x) = x^3$, using the chain rule gives

$$\frac{d}{dx}\int_2^{x^3} \sin(t^2)\,dt = f'(g(x))\cdot g'(x) = \sin((x^3)^2)\cdot 3x^2 = 3x^2\sin(x^6).$$

74. Since $\int_{\cos x}^3 e^{t^2}\,dt = -\int_3^{\cos x} e^{t^2}\,dt$, if we let $f(x) = \int_3^x e^{t^2}\,dt$ and $g(x) = \cos x$, using the chain rule gives

$$\frac{d}{dx}\int_{\cos x}^3 e^{t^2}\,dt = -\frac{d}{dx}\int_3^{\cos x} e^{t^2}\,dt = -f'(g(x))\cdot g'(x) = -e^{(\cos x)^2}(-\sin x) = \sin x e^{\cos^2 x}.$$

75. If we split the integral at $x = 0$, we have

$$\int_{-x}^x e^{-t^4}\,dt = \int_{-x}^0 e^{-t^4}\,dt + \int_0^x e^{-t^4}\,dt = -\int_0^{-x} e^{-t^4}\,dt + \int_0^x e^{-t^4}\,dt.$$

If we let

$$F(x) = \int_0^x e^{-t^4}\,dt,$$

using the chain rule on each part separately gives

$$\frac{d}{dx}[-F(-x) + F(x)] = -(-1)F'(-x) + (1)F'(x) = e^{-(-x)^4} + e^{-x^4} = 2e^{-x^4}.$$

76. We split the integral at $x = 1$ (or any other point we choose):

$$\int_{e^t}^{t^3} \sqrt{1+x^2}\,dx = \int_1^{t^3} \sqrt{1+x^2}\,dx + \int_{e^t}^1 \sqrt{1+x^2}\,dx = \int_1^{t^3} \sqrt{1+x^2}\,dx - \int_1^{e^t} \sqrt{1+x^2}\,dx.$$

Differentiating each part separately and using the chain rule gives

$$\begin{aligned} \frac{d}{dt}\int_{e^t}^{t^3} \sqrt{1+x^2}\,dx &= \frac{d}{dt}\int_1^{t^3} \sqrt{1+x^2}\,dx - \frac{d}{dt}\int_1^{e^t} \sqrt{1+x^2}\,dx \\ &= \sqrt{1+(t^3)^2}\cdot 3t^2 - \sqrt{1+(e^t)^2}\cdot e^t \\ &= 3t^2\sqrt{1+t^6} - e^t\sqrt{1+e^{2t}}. \end{aligned}$$

77. (a) Inventory at time t is $f(t) = Q - \frac{Q}{A}t$. See Figure 6.63.

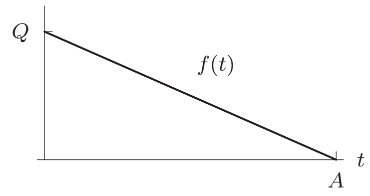


Figure 6.63

(b) Over the period $0 \le t \le A$,

$$\begin{aligned}\text{Average inventory} &= \frac{1}{A}\int_0^A f(t)\,dt \\ &= \frac{1}{A}\int_0^A \left(Q - \frac{Q}{A}t\right)\,dt \\ &= \frac{1}{A}\left(Qt - \frac{Q}{2A}t^2\right)\bigg|_0^A = \frac{1}{A}\left(QA - \frac{QA}{2}\right) = \frac{1}{2}Q.\end{aligned}$$

Graphically, the average is the y coordinate of the midpoint of the line in the graph above. Thus, the answer should be $\frac{Q+0}{2} = \frac{Q}{2}$.

Common sense tells us that since the rate is constant, the average amount should equal the amount at the midpoint of the interval, which is indeed $Q/2$.

78. (a) The distance traveled is equal to the area of the region under the graph of $v(t)$ between $t = 0$ and $t = 10$, or, the area of the trapezoid T in Figure 6.64.

$$\text{Area of } T = \frac{v(0) + v(10)}{2}\Delta t = \frac{(2 + 10\cdot 0) + (2 + 10\cdot 10)}{2}10 = 520 \text{ ft.}$$

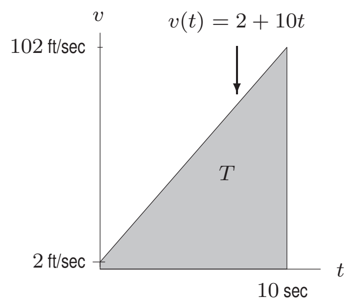


Figure 6.64

(b) Since $v(t) = 2 + 10t > 0$ for all $t \ge 0$, the car is always moving in the same direction. If the car's initial position is $s(0)$, then its position at time t is simply $s(0)+$ distance traveled in t seconds. To find this distance we calculate the area of the trapezoid where the right-hand limit is t rather than 10.

$$\text{Area of trapezoid} = \frac{v(0) + v(t)}{2}\Delta t = \frac{(2 + 10\cdot 0) + (2 + 10t)}{2}t = \frac{4 + 10t}{2}t = 5t^2 + 2t.$$

Thus $s(t) = s(0) + 5t^2 + 2t$ feet.

(c) The distance traveled by the car between $t = 0$ and $t = 10$ seconds is found by substituting into $s(t)$:

$$s(10) - s(0) = \left(s(0) + 5(10)^2 + 2\cdot 10\right) - s(0) = 5(10)^2 + 2\cdot 10 = 520 \text{ ft.}$$

This is the same answer we found in part (a).

(d) The distance traveled by the car between $t = 0$ and $t = 10$ is the area of the region under the graph of $v(t)$ and between $t = 0$ and $t = 10$, that is

$$\text{Total distance} = \int_0^{10} v(t)\,dt = \int_0^{10} (2 + 10t)\,dt.$$

The Fundamental Theorem of Calculus asserts that since $s(t)$ is an antiderivative for $v(t)$,

$$\int_0^{10} (2 + 10t)\,dt = s(10) - s(0).$$

The integral on the left is given by the area we found in part (a) and the difference on the right is what we found in part (c). Thus the Fundamental Theorem guarantees that the distances found in parts (a) and (c) are the same.

79. (a) Since $f'(t)$ is positive on the interval $0 < t < 2$ and negative on the interval $2 < t < 5$, the function $f(t)$ is increasing on $0 < t < 2$ and decreasing on $2 < t < 5$. Thus $f(t)$ attains its maximum at $t = 2$. Since the area under the t-axis is greater than the area above the t-axis, the function $f(t)$ decreases more than it increases. Thus, the minimum is at $t = 5$.

(b) To estimate the value of f at $t = 2$, we see that the area under $f'(t)$ between $t = 0$ and $t = 2$ is about 1 box, which has area 5. Thus,

$$f(2) = f(0) + \int_0^2 f'(t)dt \approx 50 + 5 = 55.$$

The maximum value attained by the function is $f(2) \approx 55$.

The area between $f'(t)$ and the t-axis between $t = 2$ and $t = 5$ is about 3 boxes, each of which has an area of 5. Thus

$$f(5) = f(2) + \int_2^5 f'(t)dt \approx 55 + (-15) = 40.$$

The minimum value attained by the function is $f(5) = 40$.

(c) Using part (b), we have $f(5) - f(0) = 40 - 50 = -10$. Alternately, we can use the Fundamental Theorem:

$$f(5) - f(0) = \int_0^5 f'(t)dt \approx 5 - 15 = -10.$$

80. Let v be the velocity and s be the position of the particle at time t. We know that $a = dv/dt$, so acceleration is the slope of the velocity graph. Similarly, velocity is the slope of the position graph. Graphs of v and s are shown in Figures 6.65 and 6.66, respectively.

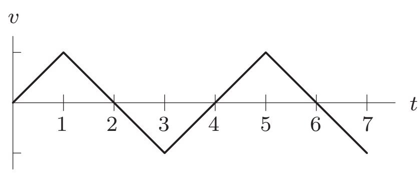


Figure 6.65: Velocity against time

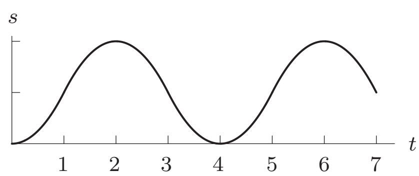


Figure 6.66: Position against time

81. (a) Since 6 sec = $1/10$ min,

$$\text{Angular acceleration} = \frac{2500 - 1100}{1/10} = 14{,}000 \text{ revs/min}^2.$$

(b) We know angular acceleration is the derivative of angular velocity. Since

$$\text{Angular acceleration} = 14{,}000,$$

we have

$$\text{Angular velocity} = 14{,}000t + C.$$

Measuring time from the moment at which the angular velocity is 1100 revs/min, we have $C = 1100$. Thus,

$$\text{Angular velocity} = 14{,}000t + 1100.$$

Thus the total number of revolutions performed during the period from $t = 0$ to $t = 1/10$ min is given by

$$\begin{array}{c}\text{Number of}\\ \text{revolutions}\end{array} = \int_0^{1/10} (14000t + 1100)dt = 7000t^2 + 1100t\Big|_0^{1/10} = 180 \text{ revolutions}.$$

82. (a) We find F for each piece, $0 \leq x \leq 1$ and $1 \leq x \leq 2$.
For $0 \leq x \leq 1$, we have $f(x) = -x + 1$, so F is of the form

$$\int (-x+1)\,dx = -\frac{x^2}{2} + x + C.$$

Since we want $F(1) = 1$, we need $C = 1/2$. See Figure 6.67.
For $1 \leq x \leq 2$, we have $f(x) = x - 1$, so F is of the form

$$\int (x-1)\,dx = \frac{x^2}{2} - x + C.$$

Again, since we want $F(1) = 1$, we have $C = 3/2$. See Figure 6.67.

(b) Evaluating

$$F(2) - F(0) = \left(\frac{2^2}{2} - 2 + \frac{3}{2}\right) - \left(-\frac{0^2}{2} + 0 + \frac{1}{2}\right) = 1.$$

The region under the graph of f consists of two triangles, whose area is

$$\text{Area } = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) The Fundamental Theorem of Calculus says

$$\int_0^2 f(x)\,dx = F(2) - F(0).$$

Since the value of the integral is just the area under the curve, we have shown this in part (b).

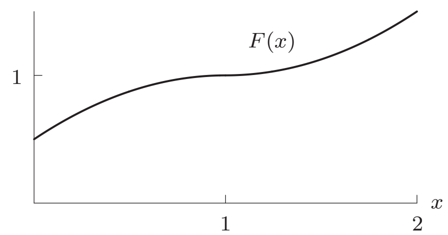
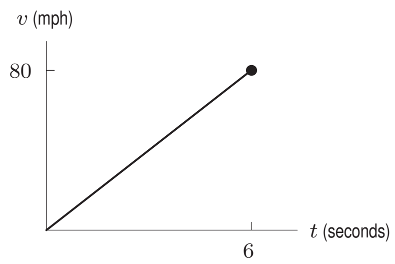


Figure 6.67

83. Since the car's acceleration is constant, a graph of its velocity against time t is linear, as shown below.



The acceleration is just the slope of this line:

$$\frac{dv}{dt} = \frac{80 - 0 \text{ mph}}{6 \text{ sec}} = \frac{40}{3} = 13.33\frac{\text{mph}}{\text{sec}}.$$

To convert our units into ft/sec^2,

$$\frac{40}{3} \cdot \frac{\text{mph}}{\text{sec}} \cdot \frac{5280 \text{ ft}}{1 \text{ mile}} \frac{1 \text{ hour}}{3600 \text{ sec}} = 19.55\frac{\text{ft}}{\text{sec}^2}$$

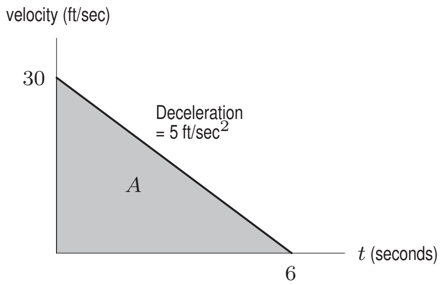
84. (a) Since the velocity is constantly decreasing, and $v(6) = 0$, the car stops after 6 seconds.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$v(t)$ (ft/sec)	30	27.5	25	22.5	20	17.5	15	12.5	10	7.5	5	2.5	0

(b) Over the interval $a \le t \le a + \frac{1}{2}$, the left-hand velocity is $v(a)$, and the right-hand velocity is $v(a + \frac{1}{2})$. Since we are considering half-second intervals, $\Delta t = \frac{1}{2}$, and $n = 12$. The left sum is 97.5 ft., and the right sum is 82.5 ft.

(c) Area A in the figure below represents distance traveled.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \cdot 6 \cdot 30 = 90 \text{ ft.}$$



(d) The velocity is constantly decreasing at a rate of 5 ft/sec per second, i.e. after each second the velocity has dropped by 5 units. Therefore $v(t) = 30 - 5t$.

An antiderivative for $v(t)$ is $s(t)$, where $s(t) = 30t - \frac{5}{2}t^2$. Thus by the Fundamental Theorem of Calculus, the distance traveled $= s(6) - s(0) = (30(6) - \frac{5}{2}(6)^2) - (30(0) - \frac{5}{2}(0)^2) = 90$ ft. Since $v(t)$ is decreasing, the left-hand sum in part (b) overestimates the distance traveled, while the right-hand sum underestimates it.

The area A is equal to the average of the left-hand and right-hand sums: 90 ft $= \frac{1}{2}(97.5 \text{ ft} + 82.5 \text{ ft})$. The left-hand sum is an overestimate of A; the right-hand sum is an underestimate.

85. (a) Using $g = -32$ ft/sec^2, we have

t (sec)	0	1	2	3	4	5
$v(t)$ (ft/sec)	80	48	16	−16	−48	−80

(b) The object reaches its highest point when $v = 0$, which appears to be at $t = 2.5$ seconds. By symmetry, the object should hit the ground again at $t = 5$ seconds.

(c) Left sum $= 80(1) + 48(1) + 16(\frac{1}{2}) = 136$ ft , which is an overestimate.
Right sum $= 48(1) + 16(1) + (-16)(\frac{1}{2}) = 56$ ft , which is an underestimate.
Note that we used a smaller third rectangle of width $1/2$ to end our sum at $t = 2.5$.

(d) We have $v(t) = 80 - 32t$, so antidifferentiation yields $s(t) = 80t - 16t^2 + s_0$.
But $s_0 = 0$, so $s(t) = 80t - 16t^2$.
At $t = 2.5$, $s(t) = 100$ ft., so 100 ft. is the highest point.

86. Since $A'(r) = C(r)$ and $C(r) = 2\pi r$, we have

$$A'(r) = 2\pi r.$$

Thus, we have, for some arbitrary constant K:

$$A(r) = \int 2\pi r \, dr = 2\pi \int r \, dr = 2\pi \frac{r^2}{2} + K = \pi r^2 + K.$$

Since a circle of radius $r = 0$ has area $= 0$, we substitute to find K:

$$0 = \pi 0^2 + K$$
$$K = 0.$$

Thus

$$A(r) = \pi r^2.$$

87. Since $V'(r) = S(r)$ and $S(r) = 4\pi r^2$, we have

$$V'(r) = 4\pi r^2.$$

Thus, we have, for some arbitrary constant K:

$$V(r) = \int 4\pi r^2\,dr = 4\pi \int r^2\,dr = 4\pi \frac{r^3}{3} + K = \frac{4}{3}\pi r^3 + K.$$

Since a sphere of radius $r = 0$ has volume $= 0$, we substitute to find K:

$$0 = \frac{4}{3}\pi 0^3 + K$$
$$K = 0.$$

Thus

$$V(r) = \frac{4}{3}\pi r^3.$$

88. The velocity of the car decreases at a constant rate, so we can write: $dv/dt = -a$. Integrating this gives $v = -at + C$. The constant of integration C is the velocity when $t = 0$, so $C = 60$ mph $= 88$ ft/sec, and $v = -at + 88$. From this equation we can see the car comes to rest at time $t = 88/a$.

Integrating the expression for velocity we get $s = -\frac{a}{2}t^2 + 88t + C$, where C is the initial position, so $C = 0$. We can use fact that the car comes to rest at time $t = 88/a$ after traveling 200 feet. Start with

$$s = -\frac{a}{2}t^2 + 88t,$$

and substitute $t = 88/a$ and $s = 200$:

$$200 = -\frac{a}{2}\left(\frac{88}{a}\right)^2 + 88\left(\frac{88}{a}\right) = \frac{88^2}{2a}$$
$$a = \frac{88^2}{2(200)} = 19.36 \text{ ft/sec}^2$$

89. (a) In the beginning, both birth and death rates are small; this is consistent with a very small population. Both rates begin climbing, the birth rate faster than the death rate, which is consistent with a growing population. The birth rate is then high, but it begins to decrease as the population increases.

(b)

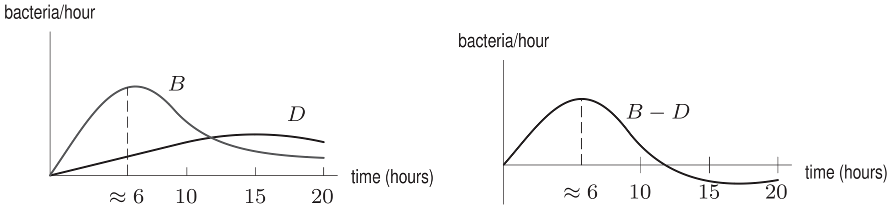


Figure 6.68: Difference between B and D is greatest at $t \approx 6$

The bacteria population is growing most quickly when $B - D$, the rate of change of population, is maximal; that happens when B is farthest above D, which is at a point where the slopes of both graphs are equal. That point is $t \approx 6$ hours.

(c) Total number born by time t is the area under the B graph from $t = 0$ up to time t. See Figure 6.69.

Total number alive at time t is the number born minus the number that have died, which is the area under the B graph minus the area under the D graph, up to time t. See Figure 6.70.

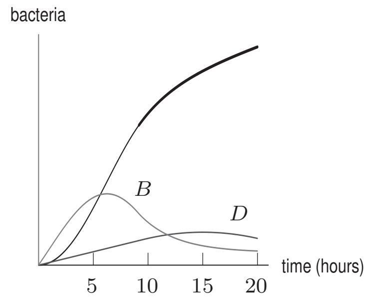


Figure 6.69: Number born by time t is $\int_0^t B(x)\,dx$

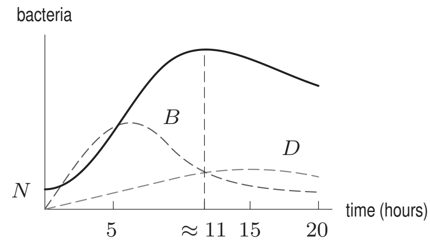


Figure 6.70: Number alive at time t is $\int_0^t (B(x) - D(x))\,dx$

From Figure 6.70, we see that the population is at a maximum when $B = D$, that is, after about 11 hours. This stands to reason, because $B - D$ is the rate of change of population, so population is maximized when $B - D = 0$, that is, when $B = D$.

90. See Figure 6.71.

Suppose t_1 is the time to fill the left side to the top of the middle ridge. Since the container gets wider as you go up, the rate dH/dt decreases with time. Therefore, for $0 \le t \le t_1$, graph is concave down.

At $t = t_1$, water starts to spill over to right side and so depth of left side does not change. It takes as long for the right side to fill to the ridge as the left side, namely t_1. Thus the graph is horizontal for $t_1 \le t \le 2t_1$.

For $t \ge 2t_1$, water level is above the central ridge. The graph is climbing because the depth is increasing, but at a slower rate than for $t \le t_1$ because the container is wider. The graph is concave down because width is increasing with depth. Time t_3 represents the time when container is full.

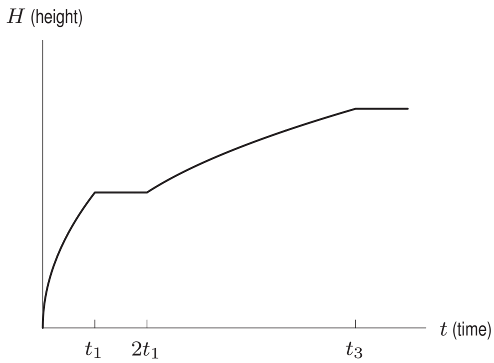


Figure 6.71

91. By the Second Fundamental Theorem, we know that $N'(t) = r(t)$. Since $r(t) > 0$, this means $N'(t) > 0$, so N is an increasing function. Since $r'(t) < 0$, this means $N''(t) < 0$, so the graph of N is concave down.

92. Since $r(x) > 0$ and $r'(x) < 0$, we know that the graph of r lies above the axis and is dropping from left to right. From the definition of N, we see that:

$$\begin{aligned}
N(20) &= \int_0^{20} r(x)\,dx = \text{Area from } t = 0 \text{ to } t = 20\\
N(10) &= \int_0^{10} r(x)\,dx = \text{Area from } t = 0 \text{ to } t = 10\\
N(20) - N(10) &= \int_{10}^{20} r(x)\,dx = \text{Area from } t = 10 \text{ to } t = 20\\
N(15) - N(5) &= \int_5^{15} r(x)\,dx = \text{ Area from } t = 5 \text{ to } t = 15.
\end{aligned}$$

- The largest area is described by $N(20) = \int_0^{20} r(x)\,dx$. This corresponds to the amount of pollutant leeched out during the first 20 days.

- The remaining three areas all describe 10-day intervals. Since the function is decreasing, areas to the left are larger than areas to the right. Thus,

$$\underbrace{\int_{10}^{20} r(x)\,dx}_{N(20)-N(10)} < \underbrace{\int_{5}^{15} r(x)\,dx}_{N(15)-N(5)} < \underbrace{\int_{0}^{10} r(x)\,dx}_{N(10)}.$$

These correspond (from least to greatest) the amount of pollutant leeched out from day 10 to day 20, from day 5 to day 15, and from day 0 to day 10.
- In conclusion: $N(20) - N(10) < N(15) - N(5) < N(10) < N(20)$.

93. (a) We know that $F(0) = \int_0^0 f(t)\,dt = 0$. so F has a zero at $x = 0$.

We know that

$$F(5) = \int_0^5 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^5 f(t)\,dt = 0 \text{ since } \int_0^3 f(t)\,dt = -\int_3^5 f(t)\,dt.$$

So F has a zero at $x = 5$.

We know that $f(x) = F'(x)$, so $F'(3) = f(3) = 0$.

This means F has at least two zeros, at $x = 0, 5$, and one critical point, at $x = 3$.

(b) We know that $G(1) = \int_1^1 F(t)\,dt = 0$, so G has a zero at $x = 1$.

We know that $F(x) = G'(x)$, so $G'(5) = F(5) = 0$ and $G'(0) = F(0) = 0$.

This means G has at least one zero, at $x = 1$, and two critical points, at $x = 0, 5$.

94. (a) The definition of P gives

$$P(0) = \int_0^0 \arctan(t^2)\,dt = 0$$

and

$$P(-x) = \int_0^{-x} \arctan(t^2)\,dt.$$

Changing the variable of integration by letting $t = -z$ gives

$$\int_0^{-x} \arctan(t^2)\,dt = \int_0^x \arctan((-z)^2)(-dz) = -\int_0^x \arctan(z^2)\,dz.$$

Thus P is an odd function.

(b) Using the Second Fundamental Theorem gives $P'(x) = \arctan(x^2)$, which is greater than 0 for $x \neq 0$. Thus P is increasing everywhere.

(c) Since

$$P''(x) = \frac{2x}{1 + x^4},$$

we have P concave up if $x > 0$ and concave down if $x < 0$.

(d) See Figure 6.72.

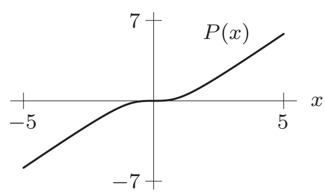


Figure 6.72

CAS Challenge Problems

95. (a) We have $\Delta x = \dfrac{(b-a)}{n}$ and $x_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$, so, since $f(x_i) = {x_i}^3$,

$$\text{Riemann sum } = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[a + i\left(\frac{b-a}{n}\right)\right]^3\left(\frac{b-a}{n}\right).$$

(b) A CAS gives

$$\sum_{i=1}^{n}\left[a + \frac{i(b-a)}{n}\right]^3\frac{(b-a)}{n} = -\frac{(a-b)(a^3(n-1)^2 + (a^2b+ab^2)(n^2-1) + b^3(n+1)^3)}{4n^2}.$$

Taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[a + i\left(\frac{b-a}{n}\right)\right]^3\left(\frac{b-a}{n}\right) = -\frac{(a+b)(a-b)(a^2+b^2)}{4}.$$

(c) The answer to part (b) simplifies to $\dfrac{b^4}{4} - \dfrac{a^4}{4}$. Since $\dfrac{d}{dx}\left(\dfrac{x^4}{4}\right) = x^3$, the Fundamental Theorem of Calculus says that

$$\int_a^b x^3 dx = \left.\frac{x^4}{4}\right|_a^b = \frac{b^4}{4} - \frac{a^4}{4}.$$

96. (a) A CAS gives

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} \qquad \int e^{3x}\,dx = \frac{1}{3}e^{3x} \qquad \int e^{3x+5}\,dx = \frac{1}{3}e^{3x+5}.$$

(b) The three integrals in part (a) obey the rule

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}.$$

(c) Checking the formula by calculating the derivative

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) &= \frac{1}{a}\frac{d}{dx}e^{ax+b} && \text{by the constant multiple rule}\\ &= \frac{1}{a}e^{ax+b}\frac{d}{dx}(ax+b) && \text{by the chain rule}\\ &= \frac{1}{a}e^{ax+b}\cdot a = e^{ax+b}.\end{aligned}$$

97. (a) A CAS gives

$$\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) \qquad \int \sin(4x)\,dx = -\frac{1}{4}\cos(4x) \qquad \int \sin(3x-2)\,dx = -\frac{1}{3}\cos(3x-2).$$

(b) The three integrals in part (a) obey the rule

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b).$$

(c) Checking the formula by calculating the derivative

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{a}\cos(ax+b)\right) &= -\frac{1}{a}\frac{d}{dx}\cos(ax+b) && \text{by the constant multiple rule}\\ &= -\frac{1}{a}(-\sin(ax+b))\frac{d}{dx}(ax+b) && \text{by the chain rule}\\ &= -\frac{1}{a}(-\sin(ax+b))\cdot a = \sin(ax+b).\end{aligned}$$

98. (a) A CAS gives

$$\int \frac{x-2}{x-1}\,dx = x - \ln|x-1|$$
$$\int \frac{x-3}{x-1}\,dx = x - 2\ln|x-1|$$
$$\int \frac{x-1}{x-2}\,dx = x + \ln|x-2|$$

Although the absolute values are needed in the answer, some CASs may not include them.

(b) The three integrals in part (a) obey the rule

$$\int \frac{x-a}{x-b}\,dx = x + (b-a)\ln|x-b|.$$

(c) Checking the formula by calculating the derivative

$$\frac{d}{dx}\left(x + (b-a)\ln|x-b|\right) = 1 + (b-a)\frac{1}{x-b} \quad \text{by the sum and constant multiple rules}$$
$$= \frac{(x-b)+(b-a)}{x-b} = \frac{x-a}{x-b}$$

99. (a) A CAS gives

$$\int \frac{1}{(x-1)(x-3)}\,dx = \frac{1}{2}(\ln|x-3| - \ln|x-1|)$$
$$\int \frac{1}{(x-1)(x-4)}\,dx = \frac{1}{3}(\ln|x-4| - \ln|x-1|)$$
$$\int \frac{1}{(x-1)(x+3)}\,dx = \frac{1}{4}(\ln|x+3| - \ln|x-1|).$$

Although the absolute values are needed in the answer, some CASs may not include them.

(b) The three integrals in part (a) obey the rule

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{b-a}(\ln|x-b| - \ln|x-a|).$$

(c) Checking the formula by calculating the derivative

$$\frac{d}{dx}\left(\frac{1}{b-a}(\ln|x-b| - \ln|x-a|)\right) = \frac{1}{b-a}\left(\frac{1}{x-b} - \frac{1}{x-a}\right)$$
$$= \frac{1}{b-a}\left(\frac{(x-a)-(x-b)}{(x-a)(x-b)}\right)$$
$$= \frac{1}{b-a}\left(\frac{b-a}{(x-a)(x-b)}\right) = \frac{1}{(x-a)(x-b)}.$$

PROJECTS FOR CHAPTER SIX

1. (a) If the poorest $p\%$ of the population has exactly $p\%$ of the goods, then $F(x) = x$.

(b) Any such F is increasing. For example, the poorest 50% of the population includes the poorest 40%, and so the poorest 50% must own more than the poorest 40%. Thus $F(0.4) \le F(0.5)$, and so, in general, F is increasing. In addition, it is clear that $F(0) = 0$ and $F(1) = 1$.

The graph of F is concave up by the following argument. Consider $F(0.05) - F(0.04)$. This is the fraction of resources the fifth poorest percent of the population has. Similarly, $F(0.20) - F(0.19)$ is the fraction of resources that the twentieth poorest percent of the population has. Since the twentieth poorest percent owns more than the fifth poorest percent, we have

$$F(0.05) - F(0.04) \le F(0.20) - F(0.19).$$

More generally, we can see that

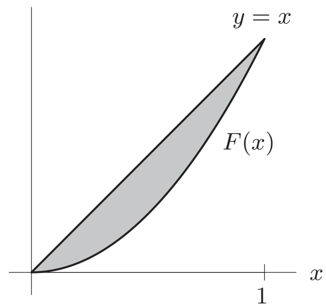
$$F(x_1 + \Delta x) - F(x_1) \le F(x_2 + \Delta x) - F(x_2)$$

for any x_1 smaller than x_2 and for any increment Δx. Dividing this inequality by Δx and taking the limit as $\Delta x \to 0$, we get

$$F'(x_1) \le F'(x_2).$$

So, the derivative of F is an increasing function, i.e. F is concave up.

(c) G is twice the shaded area below in the following figure. If the resource is distributed evenly, then G is zero. The larger G is, the more unevenly the resource is distributed. The maximum possible value of G is 1.



2. (a) In Figure 6.73, the area of the shaded region is $F(M)$. Thus, $F(M) = \int_0^M y(t)\,dt$ and, by the Fundamental Theorem, $F'(M) = y(M)$.

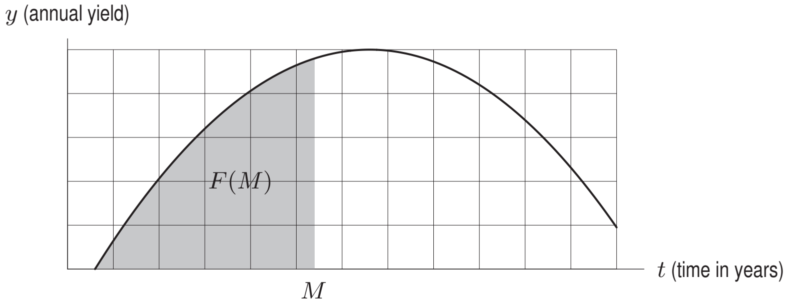


Figure 6.73

(b) Figure 6.74 is a graph of $F(M)$. Note that the graph of y looks like the graph of a quadratic function. Thus, the graph of F looks like a cubic.

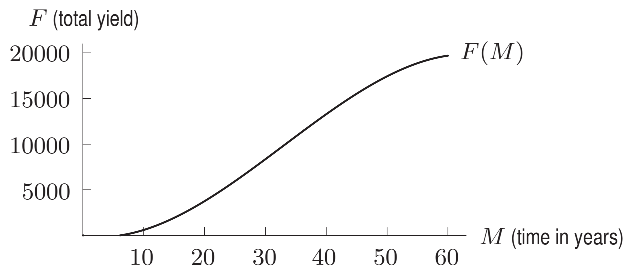


Figure 6.74

(c) We have

$$a(M) = \frac{1}{M}F(M) = \frac{1}{M}\int_0^M y(t)\,dt.$$

(d) If the function $a(M)$ takes on its maximum at some point M, then $a'(M) = 0$. Since

$$a(M) = \frac{1}{M}F(M),$$

differentiating using the quotient rule gives

$$a'(M) = \frac{MF'(M) - F(M)}{M^2} = 0,$$

so $MF'(M) = F(M)$. Since $F'(M) = y(M)$, the condition for a maximum may be written as

$$My(M) = F(M)$$

or as

$$y(M) = a(M).$$

To estimate the value of M which satisfies $My(M) = F(M)$, use the graph of $y(t)$. Notice that $F(M)$ is the area under the curve from 0 to M, and that $My(M)$ is the area of a rectangle of base M and height $y(M)$. Thus, we want the area under the curve to be equal to the area of the rectangle, or $A = B$ in Figure 6.75. This happens when $M \approx 50$ years. In other words, the orchard should be cut down after about 50 years.

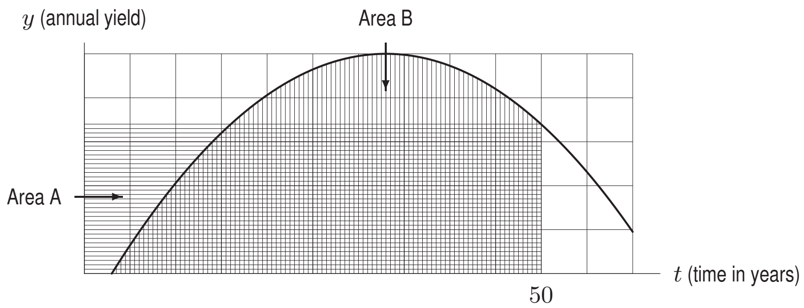
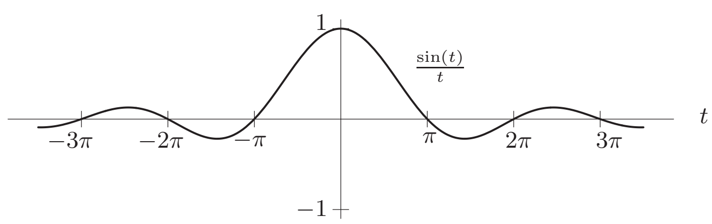


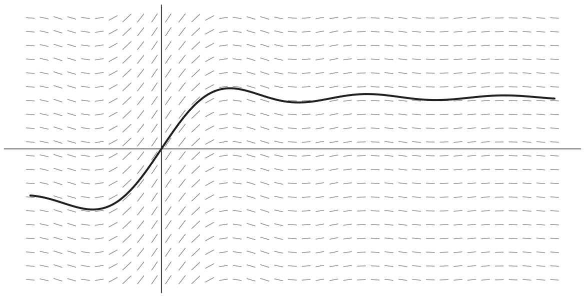
Figure 6.75

3. (a) (i)



(ii) $\text{Si}(x)$ neither always decreases nor always increases, since its derivative, $x^{-1}\sin x$, has both positive and negative values for $x > 0$. For positive x, $\text{Si}(x)$ is the area under the curve $\frac{\sin t}{t}$ and above the t-axis from $t = 0$ to $t = x$, minus the area above the curve and below the t-axis. Looking at the graph above, one can see that this difference of areas is going to always be positive.

(iii)

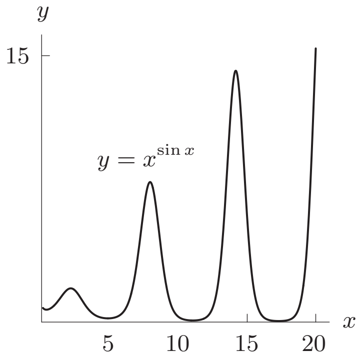


It seems that the limit exists: the curve drawn in the slope field,

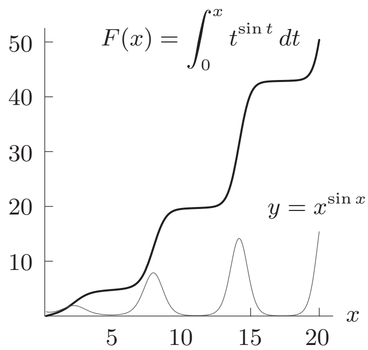
$$y = Si(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

seems to approach some limiting height as $x \to \infty$. (In fact, the limiting height is $\pi/2$.)

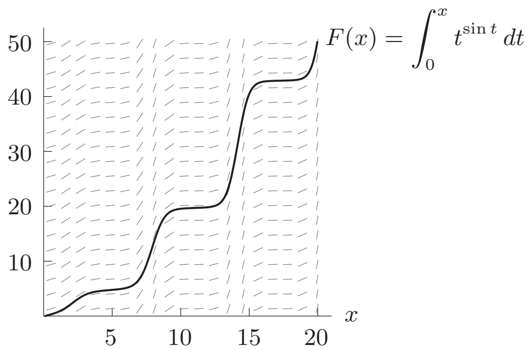
(b) (i)



(ii)



(iii)



(c) (i) The most obvious feature of the graph of $y = \sin(x^2)$ is its symmetry about the y-axis. This means the function $g(x) = \sin(x^2)$ is an even function, i.e. for all x, we have $g(x) = g(-x)$. Since $\sin(x^2)$ is even, its antiderivative F must be odd, that is $F(-x) = -F(-x)$. To see this, set $F(t) = \int_0^t \sin(x^2)\,dx$, then

$$F(-t) = \int_0^{-t} \sin(x^2)\,dx = -\int_{-t}^{0} \sin(x^2)\,dx = -\int_0^t \sin(x^2)\,dx = -F(t),$$

since the area from $-t$ to 0 is the same as the area from 0 to t. Thus $F(t) = -F(-t)$ and F is odd.

The second obvious feature of the graph of $y = \sin(x^2)$ is that it oscillates between -1 and 1 with a "period" which goes to zero as $|x|$ increases. This implies that $F'(x)$ alternates between intervals where it is positive or negative, and increasing or decreasing, with frequency growing arbitrarily large as $|x|$ increases. Thus $F(x)$ itself similarly alternates between intervals where it is increasing or decreasing, and concave up or concave down.

Finally, since $y = \sin(x^2) = F'(x)$ passes through $(0, 0)$, and $F(0) = 0$, F is tangent to the x-axis at the origin.

(ii)

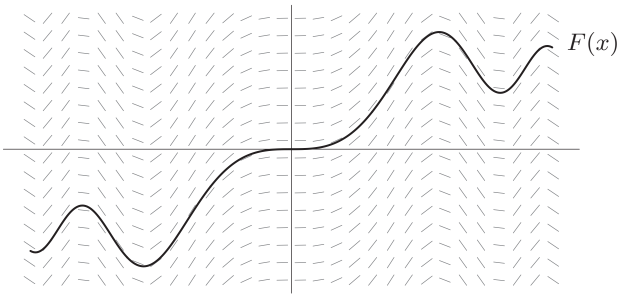


Figure 6.76

F never crosses the x-axis in the region $x > 0$, and $\lim_{x\to\infty} F(x)$ exists. One way to see these facts is to note that by the Construction Theorem,

$$F(x) = F(x) - F(0) = \int_0^x F'(t)\, dt.$$

So $F(x)$ is just the area between the curve $y = \sin(t^2)$ and the t-axis for $0 \le t \le x$ (with area above the t-axis counting positively, and area below the t-axis counting negatively). Now looking at the graph of curve, we see that this area will include alternating pieces above and below the t-axis. We can also see that the area of these pieces is approaching 0 as we go further out. So we add a piece, take a piece away, add another piece, take another piece away, and so on.

It turns out that this means that the sums of the pieces converge. To see this, think of walking from point A to point B. If you walk almost to B, then go a smaller distance toward A, then a yet smaller distance back toward B, and so on, you will eventually approach some point between A and B. So we can see that $\lim_{x\to\infty} F(x)$ exists. Also, since we always subtract a smaller piece than we just added, and the first piece is added instead of subtracted, we see that we never get a negative sum; thus $F(x)$ is never negative in the region $x > 0$, so $F(x)$ never crosses the x-axis there.

CHAPTER SEVEN

Solutions for Section 7.1

Exercises

1. (a) We substitute $w = 1 + x^2$, $dw = 2x\,dx$.

$$\int_{x=0}^{x=1} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{w=1}^{w=2} \frac{1}{w}\,dw = \frac{1}{2}\ln|w|\bigg|_1^2 = \frac{1}{2}\ln 2.$$

(b) We substitute $w = \cos x$, $dw = -\sin x\,dx$.

$$\int_{x=0}^{x=\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = -\int_{w=1}^{w=\sqrt{2}/2} \frac{1}{w}\,dw$$
$$= -\ln|w|\bigg|_1^{\sqrt{2}/2} = -\ln\frac{\sqrt{2}}{2} = \frac{1}{2}\ln 2.$$

2. (a) $\frac{d}{dx}\sin(x^2+1) = 2x\cos(x^2+1)$; $\quad \frac{d}{dx}\sin(x^3+1) = 3x^2\cos(x^3+1)$
(b) (i) $\frac{1}{2}\sin(x^2+1) + C$ (ii) $\frac{1}{3}\sin(x^3+1) + C$
(c) (i) $-\frac{1}{2}\cos(x^2+1) + C$ (ii) $-\frac{1}{3}\cos(x^3+1) + C$

3. We use the substitution $w = 3x$, $dw = 3\,dx$.

$$\int e^{3x}dx = \frac{1}{3}\int e^w dw = \frac{1}{3}e^w + C = \frac{1}{3}e^{3x} + C.$$

Check: $\frac{d}{dx}(\frac{1}{3}e^{3x} + C) = \frac{1}{3}e^{3x}(3) = e^{3x}$.

4. Make the substitution $w = t^2$, $dw = 2t\,dt$. The general antiderivative is $\int te^{t^2}\,dt = (1/2)e^{t^2} + C$.

5. We use the substitution $w = -x$, $dw = -\,dx$.

$$\int e^{-x}dx = -\int e^w dw = -e^w + C = -e^{-x} + C.$$

Check: $\frac{d}{dx}(-e^{-x} + C) = -(-e^{-x}) = e^{-x}$.

6. We use the substitution $w = -0.2t$, $dw = -0.2\,dt$.

$$\int 25e^{-0.2t}dt = \frac{25}{-0.2}\int e^w dw = -125e^w + C = -125e^{-0.2t} + C.$$

Check: $\frac{d}{dt}(-125e^{-0.2t} + C) = -125e^{-0.2t}(-0.2) = 25e^{-0.2t}$.

7. We use the substitution $w = 2x$, $dw = 2\,dx$.

$$\int \sin(2x)dx = \frac{1}{2}\int \sin(w)dw = -\frac{1}{2}\cos(w) + C = -\frac{1}{2}\cos(2x) + C.$$

Check: $\frac{d}{dx}(-\frac{1}{2}\cos(2x) + C) = \frac{1}{2}\sin(2x)(2) = \sin(2x)$.

8. We use the substitution $w = t^2$, $dw = 2t\,dt$.

$$\int t\cos(t^2)dt = \frac{1}{2}\int \cos(w)dw = \frac{1}{2}\sin(w) + C = \frac{1}{2}\sin(t^2) + C.$$

Check: $\dfrac{d}{dt}(\dfrac{1}{2}\sin(t^2) + C) = \dfrac{1}{2}\cos(t^2)(2t) = t\cos(t^2)$.

9. We use the substitution $w = 3 - t$, $dw = -\,dt$.

$$\int \sin(3-t)dt = -\int \sin(w)dw = -(-\cos(w)) + C = \cos(3-t) + C.$$

Check: $\frac{d}{dt}(\cos(3-t) + C) = -\sin(3-t)(-1) = \sin(3-t)$.

10. We use the substitution $w = -x^2$, $dw = -2x\,dx$.

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^{-x^2}(-2x\,dx) = -\frac{1}{2}\int e^w\,dw$$
$$= -\frac{1}{2}e^w + C = -\frac{1}{2}e^{-x^2} + C.$$

Check: $\frac{d}{dx}(-\frac{1}{2}e^{-x^2} + C) = (-2x)(-\frac{1}{2}e^{-x^2}) = xe^{-x^2}$.

11. Either expand $(r+1)^3$ or use the substitution $w = r + 1$. If $w = r + 1$, then $dw = dr$ and

$$\int (r+1)^3\,dr = \int w^3\,dw = \frac{1}{4}w^4 + C = \frac{1}{4}(r+1)^4 + C.$$

12. We use the substitution $w = y^2 + 5$, $dw = 2y\,dy$.

$$\int y(y^2+5)^8\,dy = \frac{1}{2}\int (y^2+5)^8(2y\,dy)$$
$$= \frac{1}{2}\int w^8\,dw = \frac{1}{2}\frac{w^9}{9} + C$$
$$= \frac{1}{18}(y^2+5)^9 + C.$$

Check: $\frac{d}{dy}(\frac{1}{18}(y^2+5)^9 + C) = \frac{1}{18}[9(y^2+5)^8(2y)] = y(y^2+5)^8$.

13. We use the substitution $w = 1 + 2x^3$, $dw = 6x^2\,dx$.

$$\int x^2(1+2x^3)^2\,dx = \int w^2(\frac{1}{6}\,dw) = \frac{1}{6}(\frac{w^3}{3}) + C = \frac{1}{18}(1+2x^3)^3 + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{18}(1+2x^2)^3 + C\right] = \frac{1}{18}[3(1+2x^3)^2(6x^2)] = x^2(1+2x^3)^2$.

14. We use the substitution $w = t^3 - 3$, $dw = 3t^2\,dt$.

$$\int t^2(t^3-3)^{10}\,dt = \frac{1}{3}\int (t^3-3)^{10}(3t^2dt) = \int w^{10}\left(\frac{1}{3}\,dw\right)$$
$$= \frac{1}{3}\frac{w^{11}}{11} + C = \frac{1}{33}(t^3-3)^{11} + C.$$

Check: $\frac{d}{dt}[\frac{1}{33}(t^3-3)^{11} + C] = \frac{1}{3}(t^3-3)^{10}(3t^2) = t^2(t^3-3)^{10}$.

15. We use the substitution $w = x^2 + 3$, $dw = 2x\,dx$.

$$\int x(x^2+3)^2\,dx = \int w^2(\frac{1}{2}\,dw) = \frac{1}{2}\frac{w^3}{3} + C = \frac{1}{6}(x^2+3)^3 + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{6}(x^2+3)^3 + C\right] = \frac{1}{6}\left[3(x^2+3)^2(2x)\right] = x(x^2+3)^2$.

16. We use the substitution $w = x^2 - 4$, $dw = 2x\,dx$.

$$\int x(x^2-4)^{7/2}\,dx = \frac{1}{2}\int (x^2-4)^{7/2}(2xdx) = \frac{1}{2}\int w^{7/2}\,dw$$
$$= \frac{1}{2}\left(\frac{2}{9}w^{9/2}\right) + C = \frac{1}{9}(x^2-4)^{9/2} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{9}(x^2-4)^{9/2} + C\right) = \frac{1}{9}\left(\frac{9}{2}(x^2-4)^{7/2}\right)2x = x(x^2-4)^{7/2}$.

17. In this case, it seems easier not to substitute.

$$\int y^2(1+y)^2\,dy = \int y^2(y^2+2y+1)\,dy = \int (y^4+2y^3+y^2)\,dy$$
$$= \frac{y^5}{5}+\frac{y^4}{2}+\frac{y^3}{3}+C.$$

Check: $\dfrac{d}{dy}\left(\dfrac{y^5}{5}+\dfrac{y^4}{2}+\dfrac{y^3}{3}+C\right) = y^4+2y^3+y^2 = y^2(y+1)^2.$

18. We use the substitution $w = 2t-7,\ dw = 2\,dt.$

$$\int (2t-7)^{73}\,dt = \frac{1}{2}\int w^{73}\,dw = \frac{1}{(2)(74)}w^{74}+C = \frac{1}{148}(2t-7)^{74}+C.$$

Check: $\dfrac{d}{dt}\left[\dfrac{1}{148}(2t-7)^{74}+C\right] = \dfrac{74}{148}(2t-7)^{73}(2) = (2t-7)^{73}.$

19. We use the substitution $w = x^3+1$, $dw = 3x^2\,dx$, to get

$$\int x^2e^{x^3+1}\,dx = \frac{1}{3}\int e^w\,dw = \frac{1}{3}e^w+C = \frac{1}{3}e^{x^3+1}+C.$$

Check: $\dfrac{d}{dx}\left(\dfrac{1}{3}e^{x^3+1}+C\right) = \dfrac{1}{3}e^{x^3+1}\cdot 3x^2 = x^2e^{x^3+1}.$

20. We use the substitution $w = y+5$, $dw = dy$, to get

$$\int \frac{dy}{y+5} = \int \frac{dw}{w} = \ln|w|+C = \ln|y+5|+C.$$

Check: $\dfrac{d}{dy}(\ln|y+5|+C) = \dfrac{1}{y+5}.$

21. We use the substitution $w = 4-x,\ dw = -dx.$

$$\int \frac{1}{\sqrt{4-x}}\,dx = -\int \frac{1}{\sqrt{w}}\,dw = -2\sqrt{w}+C = -2\sqrt{4-x}+C.$$

Check: $\dfrac{d}{dx}(-2\sqrt{4-x}+C) = -2\cdot\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{4-x}}\cdot -1 = \dfrac{1}{\sqrt{4-x}}.$

22. In this case, it seems easier not to substitute.

$$\int (x^2+3)^2\,dx = \int (x^4+6x^2+9)\,dx = \frac{x^5}{5}+2x^3+9x+C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{x^5}{5}+2x^3+9x+C\right] = x^4+6x^2+9 = (x^2+3)^2.$

23. We use the substitution $w = \cos\theta+5,\ dw = -\sin\theta\,d\theta.$

$$\int \sin\theta(\cos\theta+5)^7\,d\theta = -\int w^7\,dw = -\frac{1}{8}w^8+C$$
$$= -\frac{1}{8}(\cos\theta+5)^8+C.$$

Check:

$$\frac{d}{d\theta}\left[-\frac{1}{8}(\cos\theta+5)^8+C\right] = -\frac{1}{8}\cdot 8(\cos\theta+5)^7\cdot(-\sin\theta)$$
$$= \sin\theta(\cos\theta+5)^7$$

24. We use the substitution $w = \cos 3t,\ dw = -3\sin 3t\,dt$.

$$\int \sqrt{\cos 3t}\sin 3t\,dt = -\frac{1}{3}\int \sqrt{w}\,dw$$
$$= -\frac{1}{3}\cdot\frac{2}{3}w^{\frac{3}{2}} + C = -\frac{2}{9}(\cos 3t)^{\frac{3}{2}} + C.$$

Check:

$$\frac{d}{dt}\left[-\frac{2}{9}(\cos 3t)^{\frac{3}{2}} + C\right] = -\frac{2}{9}\cdot\frac{3}{2}(\cos 3t)^{\frac{1}{2}}\cdot(-\sin 3t)\cdot 3$$
$$= \sqrt{\cos 3t}\sin 3t.$$

25. We use the substitution $w = \sin\theta,\ dw = \cos\theta\,d\theta$.

$$\int \sin^6\theta\cos\theta\,d\theta = \int w^6\,dw = \frac{w^7}{7} + C = \frac{\sin^7\theta}{7} + C.$$

Check: $\dfrac{d}{d\theta}\left[\dfrac{\sin^7\theta}{7} + C\right] = \sin^6\theta\cos\theta.$

26. We use the substitution $w = \sin\alpha,\ dw = \cos\alpha\,d\alpha$.

$$\int \sin^3\alpha\cos\alpha\,d\alpha = \int w^3\,dw = \frac{w^4}{4} + C = \frac{\sin^4\alpha}{4} + C.$$

Check: $\dfrac{d}{d\alpha}\left(\dfrac{\sin^4\alpha}{4} + C\right) = \dfrac{1}{4}\cdot 4\sin^3\alpha\cdot\cos\alpha = \sin^3\alpha\cos\alpha.$

27. We use the substitution $w = \sin 5\theta,\ dw = 5\cos 5\theta\,d\theta$.

$$\int \sin^6 5\theta\cos 5\theta\,d\theta = \frac{1}{5}\int w^6\,dw = \frac{1}{5}(\frac{w^7}{7}) + C = \frac{1}{35}\sin^7 5\theta + C.$$

Check: $\dfrac{d}{d\theta}(\dfrac{1}{35}\sin^7 5\theta + C) = \dfrac{1}{35}[7\sin^6 5\theta](5\cos 5\theta) = \sin^6 5\theta\cos 5\theta.$

Note that we could also use Problem 25 to solve this problem, substituting $w = 5\theta$ and $dw = 5\,d\theta$ to get:

$$\int \sin^6 5\theta\cos 5\theta\,d\theta = \frac{1}{5}\int \sin^6 w\cos w\,dw$$
$$= \frac{1}{5}(\frac{\sin^7 w}{7}) + C = \frac{1}{35}\sin^7 5\theta + C.$$

28. We use the substitution $w = \cos 2x,\ dw = -2\sin 2x\,dx$.

$$\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\int \frac{dw}{w}$$
$$= -\frac{1}{2}\ln|w| + C = -\frac{1}{2}\ln|\cos 2x| + C.$$

Check:

$$\frac{d}{dx}\left[-\frac{1}{2}\ln|\cos 2x| + C\right] = -\frac{1}{2}\cdot\frac{1}{\cos 2x}\cdot -2\sin 2x$$
$$= \frac{\sin 2x}{\cos 2x} = \tan 2x.$$

29. We use the substitution $w = \ln z, dw = \frac{1}{z}\,dz$.

$$\int \frac{(\ln z)^2}{z}\,dz = \int w^2\,dw = \frac{w^3}{3} + C = \frac{(\ln z)^3}{3} + C.$$

Check: $\dfrac{d}{dz}\left[\dfrac{(\ln z)^3}{3} + C\right] = 3\cdot\dfrac{1}{3}(\ln z)^2\cdot\dfrac{1}{z} = \dfrac{(\ln z)^2}{z}.$

30. We use the substitution $w = e^t + t,\ dw = (e^t + 1)\,dt$.

$$\int \frac{e^t + 1}{e^t + t}\,dt = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|e^t + t| + C.$$

Check: $\frac{d}{dt}(\ln|e^t + t| + C) = \frac{e^t + 1}{e^t + t}$.

31. It seems easier not to substitute.

$$\int \frac{(t+1)^2}{t^2}\,dt = \int \frac{(t^2 + 2t + 1)}{t^2}\,dt$$
$$= \int \left(1 + \frac{2}{t} + \frac{1}{t^2}\right)\,dt = t + 2\ln|t| - \frac{1}{t} + C.$$

Check: $\frac{d}{dt}(t + 2\ln|t| - \frac{1}{t} + C) = 1 + \frac{2}{t} + \frac{1}{t^2} = \frac{(t+1)^2}{t^2}$.

32. We use the substitution $w = y^2 + 4,\ dw = 2y\,dy$.

$$\int \frac{y}{y^2 + 4}\,dy = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(y^2 + 4) + C.$$

(We can drop the absolute value signs since $y^2 + 4 \geq 0$ for all y.)

Check: $\frac{d}{dy}\left[\frac{1}{2}\ln(y^2 + 4) + C\right] = \frac{1}{2} \cdot \frac{1}{y^2 + 4} \cdot 2y = \frac{y}{y^2 + 4}$.

33. Let $w = \sqrt{2}x$, so $dw = \sqrt{2}dx$. Then, since $\frac{d}{dw}\arctan w = \frac{1}{1 + w^2}$, we have

$$\int \frac{dx}{1 + 2x^2} = \frac{1}{\sqrt{2}}\int \frac{dw}{1 + w^2} = \frac{1}{\sqrt{2}}\arctan w + C = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + C.$$

34. Let $w = 2x$, then $dw = 2\,dx$ so that

$$\int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2}\int \frac{dw}{\sqrt{1 - w^2}} = \frac{1}{2}\arcsin w + C = \frac{1}{2}\arcsin(2x) + C.$$

35. We use the substitution $w = \sqrt{x},\ dw = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \cos w(2\,dw) = 2\sin w + C = 2\sin\sqrt{x} + C.$$

Check: $\frac{d}{dx}(2\sin\sqrt{x} + C) = 2\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos\sqrt{x}}{\sqrt{x}}$.

36. We use the substitution $w = \sqrt{y},\ dw = \frac{1}{2\sqrt{y}}\,dy$.

$$\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = 2\int e^w\,dw = 2e^w + C = 2e^{\sqrt{y}} + C.$$

Check: $\frac{d}{dy}(2e^{\sqrt{y}} + C) = 2e^{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} = \frac{e^{\sqrt{y}}}{\sqrt{y}}$.

37. We use the substitution $w = x + e^x,\ dw = (1 + e^x)\,dx$.

$$\int \frac{1 + e^x}{\sqrt{x + e^x}}\,dx = \int \frac{dw}{\sqrt{w}} = 2\sqrt{w} + C = 2\sqrt{x + e^x} + C.$$

Check: $\frac{d}{dx}(2\sqrt{x + e^x} + C) = 2 \cdot \frac{1}{2}(x + e^x)^{-\frac{1}{2}} \cdot (1 + e^x) = \frac{1 + e^x}{\sqrt{x + e^x}}$.

38. We use the substitution $w = 2 + e^x$, $dw = e^x\,dx$.

$$\int \frac{e^x}{2+e^x}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(2+e^x) + C.$$

(We can drop the absolute value signs since $2 + e^x \geq 0$ for all x.)

Check: $\frac{d}{dx}[\ln(2+e^x) + C] = \frac{1}{2+e^x} \cdot e^x = \frac{e^x}{2+e^x}$.

39. We use the substitution $w = x^2 + 2x + 19$, $dw = 2(x+1)dx$.

$$\int \frac{(x+1)dx}{x^2+2x+19} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(x^2+2x+19) + C.$$

(We can drop the absolute value signs, since $x^2 + 2x + 19 = (x+1)^2 + 18 > 0$ for all x.)

Check: $\frac{1}{dx}[\frac{1}{2}\ln(x^2+2x+19)] = \frac{1}{2}\frac{1}{x^2+2x+19}(2x+2) = \frac{x+1}{x^2+2x+19}$.

40. We use the substitution $w = 1 + 3t^2$, $dw = 6t\,dt$.

$$\int \frac{t}{1+3t^2}\,dt = \int \frac{1}{w}(\frac{1}{6}\,dw) = \frac{1}{6}\ln|w| + C = \frac{1}{6}\ln(1+3t^2) + C.$$

(We can drop the absolute value signs since $1 + 3t^2 > 0$ for all t).

Check: $\frac{d}{dt}\left[\frac{1}{6}\ln(1+3t^2) + C\right] = \frac{1}{6}\frac{1}{1+3t^2}(6t) = \frac{t}{1+3t^2}$.

41. We use the substitution $w = e^x + e^{-x}$, $dw = (e^x - e^{-x})\,dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(e^x + e^{-x}) + C.$$

(We can drop the absolute value signs since $e^x + e^{-x} > 0$ for all x).

Check: $\frac{d}{dx}[\ln(e^x + e^{-x}) + C] = \frac{1}{e^x + e^{-x}}(e^x - e^{-x})$.

42. We use the substitution $w = \sin(x^2)$, $dw = 2x\cos(x^2)\,dx$.

$$\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\,dx = \frac{1}{2}\int w^{-\frac{1}{2}}\,dw = \frac{1}{2}(2w^{\frac{1}{2}}) + C = \sqrt{\sin(x^2)} + C.$$

Check: $\frac{d}{dx}(\sqrt{\sin(x^2)} + C) = \frac{1}{2\sqrt{\sin(x^2)}}[\cos(x^2)]2x = \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}$.

43. Since $d(\cosh 3t)/dt = 3\sinh 3t$, we have

$$\int \sinh 3t\,dt = \frac{1}{3}\cosh 3t + C.$$

44. Since $d(\sinh x)/dx = \cosh x$, we have

$$\int \cosh x\,dx = \sinh x + C.$$

45. Since $d(\sinh(2w+1))/dw = 2\cosh(2w+1)$, we have

$$\int \cosh(2w+1)\,dw = \frac{1}{2}\sinh(2w+1) + C.$$

46. Since $d(\cosh z)/dz = \sinh z$, the chain rule shows that

$$\frac{d}{dz}(e^{\cosh z}) = (\sinh z)e^{\cosh z}.$$

Thus,

$$\int (\sinh z)e^{\cosh z}\,dz = e^{\cosh z} + C.$$

47. Use the substitution $w = \cosh x$ and $dw = \sinh x\, dx$ so

$$\int \cosh^2 x \sinh x\, dx = \int w^2\, dw = \frac{1}{3}w^3 + C = \frac{1}{3}\cosh^3 x + C.$$

Check this answer by taking the derivative: $\frac{d}{dx}\left[\frac{1}{3}\cosh^3 x + C\right] = \cosh^2 x \sinh x.$

48. We use the substitution $w = x^2$ and $dw = 2x dx$ so

$$\int x \cosh x^2\, dx = \frac{1}{2}\int \cosh w\, dw = \frac{1}{2}\sinh w + C = \frac{1}{2}\sinh x^2 + C.$$

Check this answer by taking the derivative: $\frac{d}{dx}\left[\frac{1}{2}\sinh x^2 + C\right] = x\cosh x^2.$

49. The general antiderivative is $\int(\pi t^3 + 4t)\, dt = (\pi/4)t^4 + 2t^2 + C.$

50. Make the substitution $w = 3x$, $dw = 3\, dx$. We have
$$\int \sin 3x\, dx = \frac{1}{3}\int \sin w\, dw = \frac{1}{3}(-\cos w) + C = -\frac{1}{3}\cos 3x + C.$$

51. Make the substitution $w = x^2$, $dw = 2x\, dx$. We have
$$\int 2x\cos(x^2)\, dx = \int \cos w\, dw = \sin w + C = \sin x^2 + C.$$

52. Make the substitution $w = t^3$, $dw = 3t^2\, dt$. The general antiderivative is $\int 12t^2\cos(t^3)\, dt = 4\sin(t^3) + C.$

53. Make the substitution $w = 2 - 5x$, then $dw = -5dx$. We have
$$\int \sin(2-5x)dx = \int \sin w\left(-\frac{1}{5}\right) dw = -\frac{1}{5}(-\cos w) + C = \frac{1}{5}\cos(2-5x) + C.$$

54. Make the substitution $w = \sin x$, $dw = \cos x\, dx$. We have
$$\int e^{\sin x}\cos x\, dx = \int e^w\, dw = e^w + C = e^{\sin x} + C.$$

55. Make the substitution $w = x^2 + 1$, $dw = 2x\, dx$. We have
$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(x^2+1) + C.$$
(Notice that since $x^2 + 1 \geq 0$, $|x^2+1| = x^2 + 1$.)

56. Make the substitution $w = 2x$, then $dw = 2dx$. We have

$$\begin{aligned}\int \frac{1}{3\cos^2 2x}\, dx &= \frac{1}{3}\int \frac{1}{\cos^2 w}\left(\frac{1}{2}\right) dw\\ &= \frac{1}{6}\int \frac{1}{\cos^2 w}\, dw = \frac{1}{6}\tan w + C = \frac{1}{6}\tan 2x + C.\end{aligned}$$

57. $\int_0^\pi \cos(x+\pi)\, dx = \sin(x+\pi)\Big|_0^\pi = \sin(2\pi) - \sin(\pi) = 0 - 0 = 0$

58. We substitute $w = \pi x$. Then $dw = \pi\, dx$.

$$\int_{x=0}^{x=\frac{1}{2}} \cos\pi x\, dx = \int_{w=0}^{w=\pi/2} \cos w\left(\frac{1}{\pi}\, dw\right) = \frac{1}{\pi}(\sin w)\Big|_0^{\pi/2} = \frac{1}{\pi}$$

59. $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\, d\theta = e^{-\cos\theta}\Big|_0^{\pi/2} = e^{-\cos(\pi/2)} - e^{-\cos(0)} = 1 - \frac{1}{e}$

60. $\int_1^2 2xe^{x^2}\, dx = e^{x^2}\Big|_1^2 = e^{2^2} - e^{1^2} = e^4 - e = e(e^3 - 1)$

61. Let $\sqrt{x} = w$, $\frac{1}{2}x^{-\frac{1}{2}}dx = dw$, $\frac{dx}{\sqrt{x}} = 2\, dw$. If $x = 1$ then $w = 1$, and if $x = 4$ so $w = 2$. So we have

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \int_1^2 e^w \cdot 2\, dw = 2e^w\Big|_1^2 = 2(e^2 - e) \approx 9.34.$$

62. We substitute $w = t + 2$, so $dw = dt$.

$$\int_{t=-1}^{t=e-2} \frac{1}{t+2} dt = \int_{w=1}^{w=e} \frac{dw}{w} = \ln|w| \Big|_1^e = \ln e - \ln 1 = 1.$$

63. We substitute $w = \sqrt{x}$. Then $dw = \frac{1}{2}x^{-1/2}dx$.

$$\int_{x=1}^{x=4} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = \int_{w=1}^{w=2} \cos w(2\, dw)$$
$$= 2(\sin w)\Big|_1^2 = 2(\sin 2 - \sin 1).$$

64. We substitute $w = 1 + x^2$. Then $dw = 2x\, dx$.

$$\int_{x=0}^{x=2} \frac{x}{(1+x^2)^2}\, dx = \int_{w=1}^{w=5} \frac{1}{w^2}\left(\frac{1}{2}\, dw\right) = -\frac{1}{2}\left(\frac{1}{w}\right)\Big|_1^5 = \frac{2}{5}.$$

65. $\displaystyle\int_{-1}^{3} (x^3 + 5x)\, dx = \frac{x^4}{4}\Big|_{-1}^{3} + \frac{5x^2}{2}\Big|_{-1}^{3} = 40.$

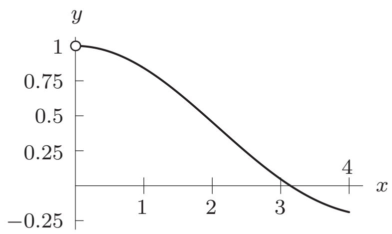
66. $\displaystyle\int_{-1}^{1} \frac{1}{1+y^2}\, dy = \tan^{-1} y\Big|_{-1}^{1} = \frac{\pi}{2}.$

67. $\displaystyle\int_{1}^{3} \frac{1}{x}\, dx = \ln x\Big|_{1}^{3} = \ln 3.$

68. $\displaystyle\int_{1}^{3} \frac{dt}{(t+7)^2} = \frac{-1}{t+7}\Big|_{1}^{3} = \left(-\frac{1}{10}\right) - \left(-\frac{1}{8}\right) = \frac{1}{40}$

69. $\displaystyle\int_{-1}^{2} \sqrt{x+2}\, dx = \frac{2}{3}(x+2)^{3/2}\Big|_{-1}^{2} = \frac{2}{3}\left[(4)^{3/2} - (1)^{3/2}\right] = \frac{2}{3}(7) = \frac{14}{3}$

70. It turns out that $\frac{\sin x}{x}$ cannot be integrated using elementary methods. However, the function is decreasing on [1,2]. One way to see this is to graph the function on a calculator or computer, as has been done below:



So since our function is monotonic, the error for our left- and right-hand sums is less than or equal to $\left|\frac{\sin 2}{2} - \frac{\sin 1}{1}\right| \Delta t \approx 0.61\Delta t$. So with 13 intervals, our error will be less than 0.05. With $n = 13$, the left sum is about 0.674, and the right sum is about 0.644. For more accurate sums, with $n = 100$ the left sum is about 0.6613 and the right sum is about 0.6574. The actual integral is about 0.6593.

71. Let $w = \sqrt{y+1}$, so $y = w^2 - 1$ and $dy = 2w\, dw$. Thus

$$\int y\sqrt{y+1}\, dy = \int (w^2 - 1)w2w\, dw = 2\int w^4 - w^2\, dw$$
$$= \frac{2}{5}w^5 - \frac{2}{3}w^3 + C = \frac{2}{5}(y+1)^{5/2} - \frac{2}{3}(y+1)^{3/2} + C.$$

72. Let $w = (z+1)^{1/3}$, so $z = w^3 - 1$ and $dz = 3w^2\,dw$. Thus

$$\int z(z+1)^{1/3}\,dz = \int (w^3-1)w3w^2\,dw = 3\int w^6 - w^3\,dw$$
$$= \frac{3}{7}w^7 - \frac{3}{4}w^4 + C = \frac{3}{7}(z+1)^{7/3} - \frac{3}{4}(z+1)^{4/3} + C.$$

73. Let $w = \sqrt{t+1}$, so $t = w^2 - 1$ and $dt = 2w\,dw$. Thus

$$\int \frac{t^2+t}{\sqrt{t+1}}\,dt = \int \frac{(w^2-1)^2 + (w^2-1)}{w}2w\,dw = 2\int w^4 - w^2\,dw$$
$$= \frac{2}{5}w^5 - \frac{2}{3}w^3 + C = \frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2} + C.$$

74. Let $w = 2 + 2\sqrt{x}$, so $x = ((w-2)/2)^2 = (w/2-1)^2$, and $dx = 2(w/2-1)(1/2)\,dw = (w/2-1)\,dw$. Thus

$$\int \frac{dx}{2+2\sqrt{x}} = \int \frac{(w/2-1)\,dw}{w} = \int \left(\frac{1}{2} - \frac{1}{w}\right)\,dw$$
$$= \frac{w}{2} - \ln|w| + C = \frac{1}{2}(2+2\sqrt{x}) - \ln|2+2\sqrt{x}| + C$$
$$= 1 + \sqrt{x} - \ln|2+2\sqrt{x}| + C = \sqrt{x} - \ln|2+2\sqrt{x}| + C.$$

In the last line, the 1 has been combined with the C.

75. Let $w = \sqrt{x-2}$, so $x = w^2 + 2$ and $dx = 2w\,dw$. Thus

$$\int x^2\sqrt{x-2}\,dx = \int (w^2+2)^2 w2w\,dw = 2\int w^6 + 4w^4 + 4w^2\,dw$$
$$= \frac{2}{7}w^7 + \frac{8}{5}w^5 + \frac{8}{3}w^3 + C$$
$$= \frac{2}{7}(x-2)^{7/2} + \frac{8}{5}(x-2)^{5/2} + \frac{8}{3}(x-2)^{3/2} + C.$$

76. Let $w = \sqrt{1-z}$, so $z = 1 - w^2$ and $dz = -2w\,dw$. Thus

$$\int (z+2)\sqrt{1-z}\,dz = \int (1-w^2+2)w(-2w)\,dw = 2\int w^4 - 3w^2\,dw$$
$$= \frac{2}{5}w^5 - 2w^3 + C = \frac{2}{5}(1-z)^{5/2} - 2(1-z)^{3/2} + C.$$

77. Let $w = \sqrt{t+1}$, so $t = w^2 - 1$ and $dt = 2w\,dw$. Thus

$$\int \frac{t}{\sqrt{t+1}}\,dt = \int \frac{w^2-1}{w}2w\,dw = 2\int w^2 - 1\,dw$$
$$= \frac{2}{3}w^3 - 2w + C = \frac{2}{3}(t+1)^{3/2} - 2(t+1)^{1/2} + C.$$

78. Let $w = \sqrt{2x+1}$, so $x = \frac{1}{2}(w^2-1)$ and $dx = w\,dw$. Thus

$$\int \frac{3x-2}{\sqrt{2x+1}}\,dx = \int \frac{3\cdot\frac{1}{2}(w^2-1) - 2}{w}w\,dw = \int \frac{3}{2}w^2 - \frac{7}{2}\,dw$$
$$= \frac{1}{2}w^3 - \frac{7}{2}w + C = \frac{1}{2}(2x+1)^{3/2} - \frac{7}{2}(2x+1)^{1/2} + C.$$

Problems

79. If we let $y = 3x$ in the first integral, we get $dy = 3dx$. Also, the limits $x = 0$ and $x = \pi/3$ become $y = 0$ and $y = 3(\pi/3) = \pi$. Thus

$$\int_0^{\pi/3} 3\sin^2(3x)\,dx = \int_0^{\pi/3} \sin^2(3x)\,3dx = \int_0^{\pi} \sin^2(y)\,dy.$$

80. If we let $t = s^2$ in the first integral, we get $dt = 2s\,ds$, so

$$2\,ds = \frac{1}{s}dt = \frac{1}{\sqrt{t}}dt.$$

Also, the limits $s = 1$ and $s = 2$ become $t = 1$ and $t = 4$. Thus

$$\int_1^2 2\ln(s^2+1)\,ds = \int_1^2 \ln(s^2+1)2\,ds = \int_1^4 \ln(t+1)\frac{1}{\sqrt{t}}\,dt.$$

81. If we let $w = e^z$ in the first integral, we get $dw = e^z dz$ and $z = \ln w$. Also, the limits $w = 1$ and $w = e$ become $z = 0$ and $z = 1$. Thus

$$\int_1^e (\ln w)^3\,dw = \int_0^1 z^3 e^z\,dz.$$

82. If we let $t = \pi - x$ in the first integral, we get $dt = -dx$ and $x = \pi - t$. Also, the limits $x = 0$ and $x = \pi$ become $t = \pi$ and $t = 0$. Thus

$$\int_0^\pi x\cos(\pi - x)\,dx = -\int_\pi^0 (\pi - t)\cos t\,dt = \int_0^\pi (\pi - t)\cos t\,dt.$$

83. As x goes from $\sqrt{a}$ to $\sqrt{b}$ the values of $w = x^2$ increase from a to b. Since $x = \sqrt{w}$ we have $dx = dw/(2\sqrt{w})$. Hence

$$\int_{\sqrt{a}}^{\sqrt{b}} dx = \int_a^b \frac{1}{2\sqrt{w}}dw = \int_a^b g(w)dw$$

with

$$g(w) = \frac{1}{2\sqrt{w}}.$$

84. As x goes from a to b the values of $w = e^x$ increase from e^a to e^b. We have $e^{-x} = 1/w$. Since $x = \ln w$ we have $dx = dw/w$. Hence

$$\int_a^b e^{-x}dx = \int_{e^a}^{e^b} \frac{1}{w}\frac{dw}{w} = \int_{e^a}^{e^b} g(w)dw$$

with

$$g(w) = \frac{1}{w^2}.$$

85. After the substitution $w = e^x$ and $dw = e^x\,dx$, the first integral becomes

$$\frac{1}{2}\int \frac{1}{1+w^2}\,dw,$$

while after the substitution $w = \sin x$ and $dw = \cos x\,dx$, the second integral becomes

$$\int \frac{1}{1+w^2}\,dw.$$

86. The substitution $w = \ln x$, $dw = \frac{1}{x}\,dx$ transforms the first integral into $\int w\,dw$, which is just a respelling of the integral $\int x\,dx$.

87. For the first integral, let $w = \sin x$, $dw = \cos x\,dx$. Then

$$\int e^{\sin x}\cos x\,dx = \int e^w\,dw.$$

For the second integral, let $w = \arcsin x$, $dw = \frac{1}{\sqrt{1-x^2}}dx$. Then

$$\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx = \int e^w\,dw.$$

88. The substitutions $w = \sin x$, $dw = \cos w$ and $w = x^3 + 1$, $dw = 3x^2\,dx$ transform the integrals into

$$\int w^3\,dw \quad \text{and} \quad \frac{1}{3}\int w^3\,dw.$$

89. For the first integral, let $w = x + 1$, $dw = dx$. Then

$$\int \sqrt{x+1}\,dx = \int \sqrt{w}\,dw.$$

For the second integral, let $w = 1 + \sqrt{x}$, $dw = \frac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{1}{2\sqrt{x}}\,dx$. Then, $\frac{dx}{\sqrt{x}} = 2\,dw$, and

$$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int \sqrt{1+\sqrt{x}}\left(\frac{dx}{\sqrt{x}}\right) = 2\int \sqrt{w}\,dw.$$

90. Using substitution, we can show

$$\int_0^1 f'(x)\sin f(x)\,dx = -\cos f(x)\bigg|_0^1 = -\cos f(1) + \cos f(0) = \cos 5 - \cos 7.$$

91. Using substitution, we have

$$\int_1^3 f'(x)e^{f(x)}\,dx = e^{f(x)}\bigg|_1^3 = e^{f(3)} - e^{f(1)} = e^{11} - e^7.$$

92. Using substitution, we have

$$\int_1^3 \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|\bigg|_1^3 = \ln|f(3)| - \ln|f(1)| = \ln 11 - \ln 7 = \ln(11/7).$$

93. Using substitution, we have

$$\int_0^1 e^x f'(e^x)\,dx = f(e^x)\bigg|_0^1 = f(e^1) - f(e^0) = f(e) - f(1) = 10 - 7 = 3.$$

94. Using substitution, we have

$$\int_1^e \frac{f'(\ln x)}{x}\,dx = f(\ln x)\bigg|_1^e = f(\ln e) - f(\ln 1) = f(1) - f(0) = 7 - 5 = 2.$$

95. Using substitution, we can show

$$\int_0^1 f'(x)(f(x))^2\,dx = \frac{(f(x))^3}{3}\bigg|_0^1 = \frac{(f(1))^3}{3} - \frac{(f(0))^3}{3} = \frac{7^3}{3} - \frac{5^3}{3} = \frac{218}{3}.$$

96. Using substitution, we have

$$\int_0^{\pi/2} \sin x \cdot f'(\cos x)\,dx = -f(\cos x)\bigg|_0^{\pi/2} = -f\left(\cos\left(\frac{\pi}{2}\right)\right) + f(\cos 0) = -f(0) + f(1) = -5 + 7 = 2.$$

97. By substitution with $w = g(x)$, we have

$$\int g'(x)(g(x))^4\,dx = \int w^4 dw = \frac{w^5}{5} + C = \frac{(g(x))^5}{5} + C.$$

98. By substitution with $w = g(x)$, we have

$$\int g'(x)e^{g(x)}\,dx = \int e^w dw = e^w + C = e^{g(x)} + C.$$

99. By substitution with $w = g(x)$, we have

$$\int g'(x)\sin g(x)\,dx = \int \sin w\,dw = -\cos w + C = -\cos g(x) + C.$$

100. By substitution with $w = 1 + g(x)$, we have

$$\int g'(x)\sqrt{1+g(x)}\,dx = \int \sqrt{w}\,dw = \int w^{1/2}dw = \frac{w^{3/2}}{3/2} + C = \frac{2}{3}(1+g(x))^{3/2} + C.$$

101. Letting $w = 1 - 4x^3, dw = -12x^2\,dx$, we have

$$\int x^2\sqrt{1-4x^3}\,dx = \int (\underbrace{1-4x^3}_{w})^{1/2}\left(-\frac{1}{12}\right)\underbrace{(-12x^2\,dx)}_{dw} = \int \left(-\frac{1}{12}\right)w^{1/2}\,dw,$$

so $w = 1 - 4x^3, k = -1/12, n = 1/2$.

102. Letting $w = \sin t, dw = \cos t\,dt$, we have

$$\int \frac{\cos t}{\sin t}\,dt = \int (\underbrace{\sin t}_{w})^{-1}\underbrace{\cos t\,dt}_{dw} = \int 1\cdot w^{-1}\,dw,$$

so $w = \sin t, k = 1, n = -1$.

103. Letting $w = x^2 - 3, dw = 2x\,dx$, we have

$$\int \frac{\overbrace{2x\,dx}^{dw}}{(\underbrace{x^2-3}_{w})^2} = \int 1\cdot w^{-2}\,dw,$$

so $w = x^2 - 3, k = 1, n = -2$.

104. Since $w = 5x^2 + 7$ we have $dw = 10x\,dx$ so $x\,dx = 0.1\,dw$. Finding the limits of integration in terms of w gives:

$$w_0 = 5(1)^2 + 7 = 12$$
$$w_1 = 5(5)^2 + 7 = 132.$$

This gives

$$\int_1^5 \frac{3x\,dx}{\sqrt{5x^2+7}} = \int_{12}^{132}\frac{3(0.1\,dw)}{w^{1/2}} = \int_{12}^{132} 0.3w^{-1/2}\,dw,$$

so $k = 0.3, n = -1/2, w_0 = 12, w_1 = 132$.

105. Since $w = 2^x + 3$ we have $dw = 2^x\ln 2\,dx$, so $2^x\,dx = dw/\ln 2$. Finding the limits of integration in terms of w gives:

$$w_0 = 2^0 + 3 = 4$$
$$w_1 = 2^5 + 3 = 35.$$

This gives

$$\int_0^5 \frac{2^x\,dx}{2^x+3} = \int_4^{35}\frac{dw/\ln 2}{2^x+3} = \int_4^{35}\frac{1}{\ln 2}w^{-1}\,dw,$$

so $k = 1/\ln 2, n = -1, w_0 = 4, w_1 = 35$.

106. Since $w = \sin 2x$ we have $dw = 2\cos 2x\,dx$, so $\cos 2x\,dx = 0.5\,dw$. Finding the limits of integration in terms of w gives:

$$w_0 = \sin\left(2\cdot\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right) = 0.5$$
$$w_1 = \sin\left(2\cdot\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

This gives

$$\int_{\pi/12}^{\pi/4} \sin^7(2x)\cos(2x)\,dx = \int_{0.5}^{1} 0.5w^7\,dw,$$

so $k = 0.5, n = 7, w_0 = 0.5, w_1 = 1$.

107. Letting $w = -x^2, dw = -2x\,dx$, we have

$$\int xe^{-x^2}\,dx = \int\left(-\frac{1}{2}\right)e^w\,dw,$$

so $w = -x^2, k = -1/2$.

108. Letting $w = \sin\phi, dw = \cos\phi$, we have

$$\int e^{\sin\phi}\cos\phi d\phi = \int 1\cdot e^w\,dw,$$

so $w = \sin\phi, k = 1$.

109. We have

$$\int\sqrt{e^r}\,dr = \int (e^r)^{1/2}\,dr$$
$$= \int e^{0.5r}\,dr.$$

Letting $w = 0.5r, dw = 0.5\,dr$, and $dr = 2\,dw$, we have

$$\int\sqrt{e^r}\,dr = \int 2e^w\,dw,$$

so $w = 0.5r, k = 2$.

110. We have

$$\int\frac{z^2\,dz}{e^{-z^3}} = \int e^{z^3}z^2\,dz.$$

Letting $w = z^3$, so $dw = 3z^2\,dz$, we have

$$\int\frac{z^2\,dz}{e^{-z^3}} = \int\frac{1}{3}e^w\,dw,$$

so $w = z^3, k = 1/3$.

111. We have

$$\int e^{2t}e^{3t-4}\,dt = \int e^{-4}e^{5t}\,dt$$
$$= \int e^{-4}\cdot\frac{1}{5}e^w\,dw \quad \text{where } w = 5t, dw = 5\,dt,$$
$$= \int\frac{1}{5e^4}\cdot e^w\,dw,$$

so $w = 5t$ and $k = 1/(5e^4)$.

112. Let $w = 2t - 3, dw = 2\,dt$. Then

$$\int_3^7 e^{2t-3}\,dt = \int_{t=3}^{t=7} e^w 0.5\,dw = \int_{w=3}^{w=11} 0.5e^w\,dw,$$

so $a = 3, b = 11, A = 0.5, w = 2t - 3$.

113. Let $w = \cos(\pi t)$, $dw = -\pi \sin(\pi t)\, dt$. Then $\sin(\pi t)\, dt = -(1/\pi)\, dw$, giving

$$\int_0^1 e^{\cos(\pi t)} \sin(\pi t)\, dt = \int_{t=0}^{t=1} e^w \left(\frac{-1}{\pi}\right) dw = \int_{w=1}^{w=-1} \left(\frac{-1}{\pi}\right) e^w\, dw,$$

so $a = 1, b = -1, A = -1/\pi, w = \cos(\pi t)$. Note that since

$$\int_1^{-1} \frac{-1}{\pi} e^w\, dw = -\int_{-1}^1 \frac{-1}{\pi} e^w\, dw = \int_{-1}^1 \frac{1}{\pi} e^w\, dw,$$

another possible answer is $a = -1, b = 1, A = 1/\pi, w = \cos(\pi t)$.

114. (a) $2\sqrt{x} + C$
(b) $2\sqrt{x+1} + C$
(c) To get this last result, we make the substitution $w = \sqrt{x}$. Normally we would like to substitute $dw = \frac{1}{2\sqrt{x}}\, dx$, but in this case we cannot since there are no spare $\frac{1}{\sqrt{x}}$ terms around. Instead, we note $w^2 = x$, so $2w\, dw = dx$. Then

$$\begin{aligned}\int \frac{1}{\sqrt{x}+1}\, dx &= \int \frac{2w}{w+1}\, dw \\ &= 2\int \frac{(w+1)-1}{w+1}\, dw \\ &= 2\int \left(1 - \frac{1}{w+1}\right) dw \\ &= 2(w - \ln|w+1|) + C \\ &= 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C.\end{aligned}$$

We also note that we can drop the absolute value signs, since $\sqrt{x} + 1 \geq 0$ for all x.

115. (a) This integral can be evaluated using integration by substitution. We use $w = x^2$, $dw = 2xdx$.

$$\int x \sin x^2 dx = \frac{1}{2}\int \sin(w) dw = -\frac{1}{2}\cos(w) + C = -\frac{1}{2}\cos(x^2) + C.$$

(b) This integral cannot be evaluated using a simple integration by substitution.
(c) This integral cannot be evaluated using a simple integration by substitution.
(d) This integral can be evaluated using integration by substitution. We use $w = 1 + x^2$, $dw = 2xdx$.

$$\int \frac{x}{(1+x^2)^2} dx = \frac{1}{2}\int \frac{1}{w^2} dw = \frac{1}{2}\left(\frac{-1}{w}\right) + C = \frac{-1}{2(1+x^2)} + C.$$

(e) This integral cannot be evaluated using a simple integration by substitution.
(f) This integral can be evaluated using integration by substitution. We use $w = 2 + \cos x$, $dw = -\sin x dx$.

$$\int \frac{\sin x}{2+\cos x} dx = -\int \frac{1}{w} dw = -\ln|w| + C = -\ln|2+\cos x| + C.$$

116. To find the area under the graph of $f(x) = xe^{x^2}$, we need to evaluate the definite integral

$$\int_0^2 xe^{x^2}\, dx.$$

This is done in Example 9, Section 7.1, using the substitution $w = x^2$, the result being

$$\int_0^2 xe^{x^2}\, dx = \frac{1}{2}(e^4 - 1).$$

117. Since $f(x) = 1/(x+1)$ is positive on the interval $x = 0$ to $x = 2$, we have

$$\text{Area} = \int_0^2 \frac{1}{x+1} dx = \ln(x+1)\Big|_0^2 = \ln 3 - \ln 1 = \ln 3.$$

The area is $\ln 3 \approx 1.0986$.

118. The area under $f(x) = \sinh(x/2)$ between $x = 0$ and $x = 2$ is given by

$$A = \int_0^2 \sinh\left(\frac{x}{2}\right)\, dx = 2\cosh\left(\frac{x}{2}\right)\Big|_0^2 = 2\cosh 1 - 2.$$

119. Since $(e^{\theta+1})^3 = e^{3\theta+3} = e^{3\theta} \cdot e^3$, we have

$$\begin{aligned}\text{Area } &= \int_0^2 (e^{\theta+1})^3\, d\theta = \int_0^2 e^{3\theta} \cdot e^3\, d\theta \\ &= e^3 \int_0^2 e^{3\theta}\, d\theta = e^3 \frac{1}{3} e^{3\theta}\Big|_0^2 = \frac{e^3}{3}(e^6 - 1).\end{aligned}$$

120. Since e^{t+1} is larger than e^t, we have

$$\text{Area } = \int_0^2 (e^{t+1} - e^t)\, dt = (e^{t+1} - e^t)\Big|_0^2 = e^3 - e^2 - e + 1.$$

(The integral $\int e^{t+1}\, dt = e^{t+1} + C$ can be done by substitution or guess and check.)

121. The curves $y = e^x$ and $y = 3$ cross where

$$e^x = 3 \quad \text{so} \quad x = \ln 3,$$

and the graph of $y = e^x$ is below the line $y = 3$ for $0 \le x \le \ln 3$. (See Figure 7.1.) Thus

$$\begin{aligned}\text{Area } &= \int_0^{\ln 3} (3 - e^x)\, dx = (3x - e^x)\Big|_0^{\ln 3} \\ &= 3\ln 3 - e^{\ln 3} - (0 - 1) = 3\ln 3 - 2.\end{aligned}$$

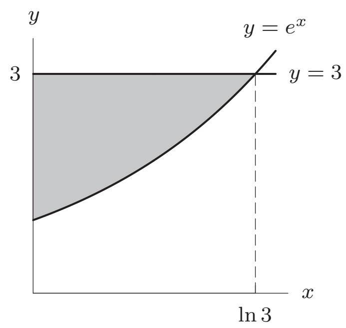


Figure 7.1

122. See Figure 7.2. The period of $V = V_0 \sin(\omega t)$ is $2\pi/\omega$, so the area under the first arch is given by

$$\begin{aligned}\text{Area} &= \int_0^{\pi/\omega} V_0 \sin(\omega t)\, dt \\ &= -\frac{V_0}{\omega} \cos(\omega t)\Big|_0^{\pi/\omega} \\ &= -\frac{V_0}{\omega}\cos(\pi) + \frac{V_0}{\omega}\cos(0) \\ &= -\frac{V_0}{\omega}(-1) + \frac{V_0}{\omega}(1) = \frac{2V_0}{\omega}.\end{aligned}$$

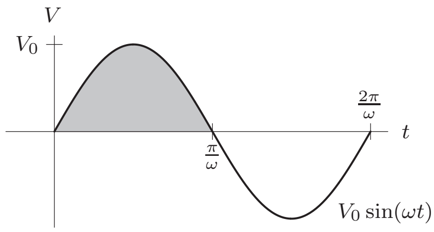


Figure 7.2

123. If $f(x) = \dfrac{1}{x+1}$, the average value of f on the interval $0 \le x \le 2$ is defined to be

$$\frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 \frac{dx}{x+1}.$$

We'll integrate by substitution. We let $w = x+1$ and $dw = dx$, and we have

$$\int_{x=0}^{x=2} \frac{dx}{x+1} = \int_{w=1}^{w=3} \frac{dw}{w} = \ln w\bigg|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

Thus, the average value of $f(x)$ on $0 \le x \le 2$ is $\frac{1}{2}\ln 3 \approx 0.5493$. See Figure 7.3.

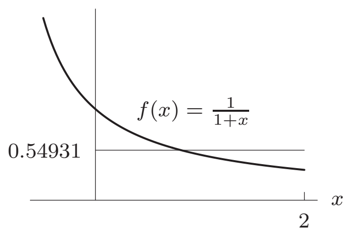


Figure 7.3

124. On the interval $[0, 2b]$

$$\text{Average value of } f = \frac{1}{2b}\int_0^{2b} f(x)\,dx.$$

If we let $x = 2u$ in this integral, we get

$$\frac{1}{2b}\int_0^{b} f(2u)\,2du = \frac{1}{b}\int_0^{b} g(u)\,du = \text{ Average value of } g \text{ on } [0, b].$$

125. (a) If $w = t/2$, then $dw = (1/2)dt$. When $t = 0$, $w = 0$; when $t = 4$, $w = 2$. Thus,

$$\int_0^4 g(t/2)\,dt = \int_0^2 g(w)\,2dw = 2\int_0^2 g(w)\,dw = 2 \cdot 5 = 10.$$

(b) If $w = 2 - t$, then $dw = -dt$. When $t = 0$, $w = 2$; when $t = 2$, $w = 0$. Thus,

$$\int_0^2 g(2-t)\,dt = \int_2^0 g(w)\,(-dw) = +\int_0^2 g(w)\,dw = 5.$$

126. (a) If $w = 2t$, then $dw = 2dt$. When $t = 0$, $w = 0$; when $t = 0.5$, $w = 1$. Thus,

$$\int_0^{0.5} f(2t)\,dt = \int_0^1 f(w)\,\frac{1}{2}dw = \frac{1}{2}\int_0^1 f(w)\,dw = \frac{3}{2}.$$

(b) If $w = 1 - t$, then $dw = -dt$. When $t = 0$, $w = 1$; when $t = 1$, $w = 0$. Thus,

$$\int_0^1 f(1-t)\,dt = \int_1^0 f(w)\,(-dw) = +\int_0^1 f(w)\,dw = 3.$$

(c) If $w = 3 - 2t$, then $dw = -2dt$. When $t = 1$, $w = 1$; when $t = 1.5$, $w = 0$. Thus,

$$\int_1^{1.5} f(3-2t)\,dt = \int_1^0 f(w)\left(-\frac{1}{2}dw\right) = +\frac{1}{2}\int_0^1 f(w)\,dw = \frac{3}{2}.$$

127. (a) The Fundamental Theorem gives

$$\int_{-\pi}^{\pi} \cos^2\theta\sin\theta\,d\theta = -\frac{\cos^3\theta}{3}\Big|_{-\pi}^{\pi} = \frac{-(-1)^3}{3} - \frac{-(-1)^3}{3} = 0.$$

This agrees with the fact that the function $f(\theta) = \cos^2\theta\sin\theta$ is odd and the interval of integration is centered at $x = 0$, thus we must get 0 for the definite integral.

(b) The area is given by

$$\text{Area } = \int_0^{\pi} \cos^2\theta\sin\theta\,d\theta = -\frac{\cos^3\theta}{3}\Big|_0^{\pi} = \frac{-(-1)^3}{3} - \frac{-(1)^3}{3} = \frac{2}{3}.$$

128. (a) $\displaystyle\int 4x(x^2+1)\,dx = \int (4x^3+4x)\,dx = x^4 + 2x^2 + C.$

(b) If $w = x^2 + 1$, then $dw = 2x\,dx$.

$$\int 4x(x^2+1)\,dx = \int 2w\,dw = w^2 + C = (x^2+1)^2 + C.$$

(c) The expressions from parts (a) and (b) look different, but they are both correct. Note that $(x^2+1)^2 + C = x^4 + 2x^2 + 1 + C$. In other words, the expressions from parts (a) and (b) differ only by a constant, so they are both correct antiderivatives.

129. (a) We first try the substitution $w = \sin\theta,\ dw = \cos\theta\,d\theta$. Then
$\displaystyle\int \sin\theta\cos\theta\,d\theta = \int w\,dw = \frac{w^2}{2} + C = \frac{\sin^2\theta}{2} + C.$

(b) If we instead try the substitution $w = \cos\theta,\ dw = -\sin\theta\,d\theta$, we get
$\displaystyle\int \sin\theta\cos\theta\,d\theta = -\int w\,dw = -\frac{w^2}{2} + C = -\frac{\cos^2\theta}{2} + C.$

(c) Once we note that $\sin 2\theta = 2\sin\theta\cos\theta$, we can also say
$\displaystyle\int \sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \sin 2\theta\,d\theta.$
Substituting $w = 2\theta,\ dw = 2\,d\theta$, the above equals
$\displaystyle\frac{1}{4}\int \sin w\,dw = -\frac{\cos w}{4} + C = -\frac{\cos 2\theta}{4} + C.$

(d) All these answers are correct. Although they have different forms, they differ from each other only in terms of a constant, and thus they are all acceptable antiderivatives. For example, $1 - \cos^2\theta = \sin^2\theta$, so $\frac{\sin^2\theta}{2} = -\frac{\cos^2\theta}{2} + \frac{1}{2}$. Thus the first two expressions differ only by a constant C.
Similarly, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$, so $-\frac{\cos 2\theta}{4} = -\frac{\cos^2\theta}{2} + \frac{1}{4}$, and thus the second and third expressions differ only by a constant. Of course, if the first two expressions and the last two expressions differ only in the constant C, then the first and last only differ in the constant as well.

130. Letting $w = 6x\sqrt{x} = 6x^{3/2}$, we have $dw = (3/2)6x^{1/2}dx = 9\sqrt{x}dx$, so $\sqrt{x}dx = (1/9)dw$. This means we can write

$$\int_1^9 \overbrace{f\left(6x\sqrt{x}\right)}^{f(w)}\ \overbrace{\sqrt{x}dx}^{(1/9)\,dw} = \int_{x=1}^{x=9} f(w)\cdot\frac{1}{9}\,dw$$

$$= \int_{w=6}^{w=162} \frac{1}{9} \cdot f(w)\,dw,$$

so $k = 1/9, a = 6, b = 162$.

131. Letting $w = \ln\left(x^2+1\right)$, we have $dw = \left(x^2+1\right)^{-1} \cdot 2x dx$, so

$$\frac{x dx}{x^2+1} = \frac{1}{2} dw.$$

This means we can write

$$\begin{aligned}\int_2^5 \frac{f\left(\ln\left(x^2+1\right)\right) x\,dx}{x^2+1} &= \int_2^5 \overbrace{f\left(\ln\left(x^2+1\right)\right)}^{f(w)} \overbrace{\frac{x dx}{x^2+1}}^{(1/2)\,dw} \\ &= \int_{x=2}^{x=5} f(w) \cdot \frac{1}{2}\,dw \\ &= \int_{w=\ln 5}^{w=\ln 26} \frac{1}{2} \cdot f(w)\,dw,\end{aligned}$$

so $k = 1/2, a = \ln 5, b = \ln 26$.

132. Since $\tan x = \sin x / \cos x$, we use the substitution $w = \cos x$. Then we have

$$y = \int \tan x + 1\,dx = \int \frac{\sin x}{\cos x}\,dx + \int 1\,dx = -\ln|\cos x| + x + C.$$

Therefore, the general solution of the differential equation is

$$y = -\ln|\cos x| + x + C.$$

The initial condition allows us to determine the constant C. Substituting $y(0) = 1$ gives

$$1 = -\ln|\cos 0| + 0 + C.$$

Since $\ln|\cos 0| = \ln 1 = 0$, we have $C = 1$. The solution is

$$y = -\ln|\cos x| + x + 1.$$

133. We substitute $w = 1 - x$ into $I_{m,n}$. Then $dw = -dx$, and $x = 1 - w$.
When $x = 0$, $w = 1$, and when $x = 1$, $w = 0$, so

$$\begin{aligned}I_{m,n} &= \int_0^1 x^m (1-x)^n dx = \int_1^0 (1-w)^m w^n (-dw) \\ &= -\int_1^0 w^n (1-w)^m dw = \int_0^1 w^n (1-w)^m dw = I_{n,m}.\end{aligned}$$

134. (a) Since change of position is the integral of velocity, for t in seconds we have

$$\text{Change in position } = \int_0^{60} f(t)\,dt \text{ meters.}$$

(b) Since 1 minute is 60 seconds, $t = 60T$. The constant 60 has units sec/min, so $60T$ has units sec/min $\times$ min $=$ sec. Applying the substitution $t = 60T$ to the integral in part (a), we get

$$\text{Change in position} = \int_0^1 f(60T)\,60\,dT \text{ meters.}$$

135. (a) Integrating I, we have

$$C(t) = 1.3t + C_0.$$

Substituting $t = 0$ gives $C_0 = 311$, so

$$C(t) = 1.3t + 311.$$

Integrating II, we have

$$C(t) = 0.5t + 0.03\frac{t^2}{2} + C_0.$$

Substituting $t = 0$ gives $C_0 = 311$, so

$$C(t) = 0.5t + 0.015t^2 + 311.$$

Integrating III, we have

$$C(t) = \frac{0.5}{0.02}e^{0.02t} + C_0$$

Substituting $t = 0$ and $C_0 = 311$, we have

$$311 = 25e^{0.02(0)} + C_0$$
$$311 = 25 + C_0$$
$$C_0 = 286.$$

Thus

$$C(t) = 25e^{0.02t} + 286.$$

(b) In 2020, we have $t = 70$, so
I $C(70) = 1.3 \cdot 70 + 311 = 402$ ppm.
II $C(70) = 0.5 \cdot 70 + 0.015 \cdot 70^2 + 311 = 419.5$ ppm.
III $C(70) = 25e^{0.02(70)} + 286 = 387.380$ ppm.

136. (a) A time period of Δt hours with flow rate of $f(t)$ cubic meters per hour has a flow of $f(t)\Delta t$ cubic meters. Summing the flows, we get total flow $\approx \Sigma f(t)\Delta t$, so

$$\text{Total flow} = \int_0^{72} f(t)\,dt \text{ cubic meters.}$$

(b) Since 1 day is 24 hours, $t = 24T$. The constant 24 has units hours per day, so $24T$ has units hours/day $\times$ day $=$ hours. Applying the substitution $t = 24T$ to the integral in part (a), we get

$$\text{Total flow} = \int_0^{3} f(24T)\,24\,dT \text{ cubic meters.}$$

137. (a) In 2010, we have $P = 6.1e^{0.012\cdot 10} = 6.9$ billion people.
In 2020, we have $P = 6.1e^{0.012\cdot 20} = 7.8$ billion people.
(b) We have

$$\text{Average population} = \frac{1}{10-0}\int_0^{10} 6.1e^{0.012t}\,dt = \frac{1}{10}\cdot\frac{6.1}{0.012}e^{0.012t}\Big|_0^{10}$$
$$= \frac{1}{10}\left(\frac{6.1}{0.012}(e^{0.12} - e^0)\right) = 6.5.$$

The average population of the world between 2000 and 2010 is predicted to be 6.5 billion people.

138. (a) At time $t = 0$, the rate of oil leakage $= r(0) = 50$ thousand liters/minute.
At $t = 60$, rate $= r(60) = 15.06$ thousand liters/minute.
(b) To find the amount of oil leaked during the first hour, we integrate the rate from $t = 0$ to $t = 60$:

$$\text{Oil leaked} = \int_0^{60} 50e^{-0.02t}\,dt = \left(-\frac{50}{0.02}e^{-0.02t}\right)\Big|_0^{60}$$
$$= -2500e^{-1.2} + 2500e^0 = 1747 \text{ thousand liters.}$$

139. (a) $E(t) = 1.4e^{0.07t}$

(b)

$$\begin{aligned}
\text{Average Yearly Consumption} &= \frac{\text{Total Consumption for the Century}}{100 \text{ years}} \\
&= \frac{1}{100}\int_0^{100} 1.4e^{0.07t}\,dt \\
&= (0.014)\left[\frac{1}{0.07}e^{0.07t}\Big|_0^{100}\right] \\
&= (0.014)\left[\frac{1}{0.07}(e^7 - e^0)\right] \\
&= 0.2(e^7 - 1) \approx 219 \text{ million megawatt-hours.}
\end{aligned}$$

(c) We are looking for t such that $E(t) \approx 219$:

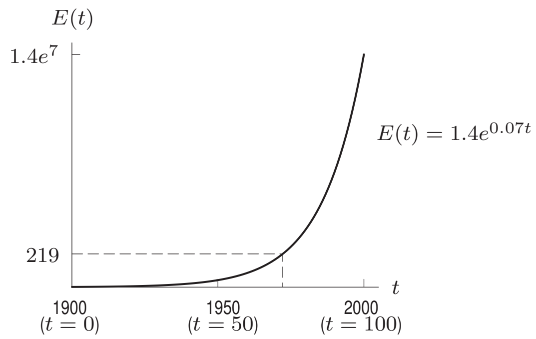
$$\begin{aligned}
1.4e^{0.07t} &\approx 219 \\
e^{0.07t} &= 156.4.
\end{aligned}$$

Taking natural logs,

$$\begin{aligned}
0.07t &= \ln 156.4 \\
t &\approx \frac{5.05}{0.07} \approx 72.18.
\end{aligned}$$

Thus, consumption was closest to the average during 1972.

(d) Between the years 1900 and 2000 the graph of $E(t)$ looks like



From the graph, we can see the t value such that $E(t) = 219$. It lies to the right of $t = 50$, and is thus in the second half of the century.

140. We have $Q'(t) = -I(t)$ and $I(t) = I_0e^{-t}$, so

$$Q'(t) = -I_0e^{-t}.$$

We have

$$Q(t) = I_0\int -e^{-t}\,dt = I_0e^{-t} + C.$$

Since $Q(0) = Q_0$, we have

$$Q_0 = I_0e^{-0} + C = I_0 + C,$$

so

$$C = Q_0 - I_0$$

Thus,

$$Q(t) = I_0e^{-t} + Q_0 - I_0.$$

141. Since $v = \dfrac{dh}{dt}$, it follows that $h(t) = \int v(t)\,dt$ and $h(0) = h_0$. Since

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) = \frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t},$$

we have

$$h(t) = \int v(t)\,dt = \frac{mg}{k}\int dt - \frac{mg}{k}\int e^{-\frac{k}{m}t}\,dt.$$

The first integral is simply $\dfrac{mg}{k}t + C$. To evaluate the second integral, make the substitution $w = -\frac{k}{m}t$. Then

$$dw = -\frac{k}{m}\,dt,$$

so

$$\int e^{-\frac{k}{m}t}\,dt = \int e^{w}\left(-\frac{m}{k}\right)dw = -\frac{m}{k}e^{w} + C = -\frac{m}{k}e^{-\frac{k}{m}t} + C.$$

Thus

$$\begin{aligned} h(t) &= \int v\,dt = \frac{mg}{k}t - \frac{mg}{k}\left(-\frac{m}{k}e^{-\frac{k}{m}t}\right) + C \\ &= \frac{mg}{k}t + \frac{m^2g}{k^2}e^{-\frac{k}{m}t} + C. \end{aligned}$$

Since $h(0) = h_0$,

$$\begin{aligned} h_0 &= \frac{mg}{k}\cdot 0 + \frac{m^2g}{k^2}e^{0} + C; \\ C &= h_0 - \frac{m^2g}{k^2}. \end{aligned}$$

Thus

$$\begin{aligned} h(t) &= \frac{mg}{k}t + \frac{m^2g}{k^2}e^{-\frac{k}{m}t} - \frac{m^2g}{k^2} + h_0 \\ h(t) &= \frac{mg}{k}t - \frac{m^2g}{k^2}\left(1 - e^{-\frac{k}{m}t}\right) + h_0. \end{aligned}$$

142. Since v is given as the velocity of a falling body, the height h is decreasing, so $v = -\frac{dh}{dt}$, and it follows that $h(t) = -\int v(t)\,dt$ and $h(0) = h_0$. Let $w = e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}$. Then

$$dw = \sqrt{gk}\left(e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}\right)dt,$$

so $\dfrac{dw}{\sqrt{gk}} = (e^{t\sqrt{gk}} - e^{-t\sqrt{gk}})\,dt$. Therefore,

$$\begin{aligned} -\int v(t)dt &= -\int \sqrt{\frac{g}{k}}\left(\frac{e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}\right)dt \\ &= -\sqrt{\frac{g}{k}}\int \frac{1}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}\left(e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}\right)dt \\ &= -\sqrt{\frac{g}{k}}\int \left(\frac{1}{w}\right)\frac{dw}{\sqrt{gk}} \\ &= -\sqrt{\frac{g}{gk^2}}\ln|w| + C \\ &= -\frac{1}{k}\ln\left(e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}\right) + C. \end{aligned}$$

Since

$$h(0) = -\frac{1}{k}\ln(e^{0} + e^{0}) + C = -\frac{\ln 2}{k} + C = h_0,$$

we have $C = h_0 + \dfrac{\ln 2}{k}$. Thus,

$$h(t) = -\frac{1}{k}\ln\left(e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}\right) + \frac{\ln 2}{k} + h_0 = -\frac{1}{k}\ln\left(\frac{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}{2}\right) + h_0.$$

143. (a) In the first case, we are given that $R_0 = 1000$ widgets/year. So we have $R = 1000e^{0.125t}$. To determine the total number sold, we need to integrate this rate over the time period from 0 to 10. Therefore

$$\text{Total number of widgets sold} = \int_0^{10} 1000e^{0.125t}\,dt = \frac{1000}{0.125}e^{0.125t}\bigg|_0^{10} = 8000(e^{1.25} - 1) = 19{,}923 \text{ widgets.}$$

In the second case,

$$\text{Total number of widgets sold} = \int_0^{10} 1{,}000{,}000e^{0.125t}\,dt = 1{,}000{,}000e^{0.125t}\bigg|_0^{10} = 19.9 \text{ million widgets.}$$

(b) We want to determine T such that

$$\int_0^T 1000e^{0.125t}\,dt = \frac{19{,}923}{2}.$$

Evaluating both sides, we get

$$\begin{aligned}
\frac{1000}{0.125}e^{0.0125t}\bigg|_0^T &= 8000(e^{0.125T} - 1) = 9{,}961\\
8000e^{0.125T} &= 17{,}961\\
e^{0.125T} &= 2.245\\
0.125T &= 0.8088,\\
T &= 6.47 \text{ years.}
\end{aligned}$$

Similarly, in the second case, we want T so that

$$\int_0^T 1{,}000{,}000e^{0.125t}\,dt = \frac{19{,}900{,}000}{2}$$

Evaluating both sides, we get

$$\begin{aligned}
\frac{1{,}000{,}000}{0.125}e^{0.125t}\bigg|_0^T &= 9{,}950{,}000\\
8{,}000{,}000(e^{0.125T} - 1) &= 9{,}950{,}000\\
8{,}000{,}000e^{0.125T} &= 17{,}950{,}000\\
e^{0.125T} &= 2.2438\\
T &= 6.47 \text{ years.}
\end{aligned}$$

So the half way mark is reached at the same time regardless of the initial rate.

(c) Since half the widgets are sold in the last $3\frac{1}{2}$ years of the decade, if each widget is expected to last at least 3.5 years, their claim could easily be true.

144. (a) Amount of water entering tank in a short period of time = Rate×Time = $r(t)\Delta t$.

(b) Summing the contribution from each of the small intervals Δt:

$$\begin{array}{c}\text{Amount of water entering the tank}\\ \text{between } t = 0 \text{ and } t = 5\end{array} \approx \sum_{i=0}^{n-1} r(t_i)\Delta t, \qquad \text{where } \Delta t = 5/n.$$

Taking a limit as $\Delta t \to 0$:

$$\begin{array}{c}\text{Amount of water entering the tank}\\ \text{between } t = 0 \text{ and } t = 5\end{array} = \int_0^5 r(t)\,dt.$$

(c) If $Q(t)$ is the amount of water in the tank at time t, then $Q'(t) = r(t)$. We want to calculate $Q(5) - Q(0)$. By the Fundamental Theorem,

$$\begin{aligned}\text{Amount which has entered tank} &= Q(5) - Q(0) = \int_0^5 r(t)\,dt = \int_0^5 20e^{0.02t}\,dt = \frac{20}{0.02}e^{0.02t}\Big|_0^5 \\ &= 1000(e^{0.02(5)} - 1) \approx 105.17 \text{ gallons.}\end{aligned}$$

(d) By the Fundamental Theorem again,

$$\begin{aligned}\text{Amount which has entered tank} &= Q(t) - Q(0) = \int_0^t r(t)\,dt \\ Q(t) - 3000 &= \int_0^t 20e^{0.02t}\,dt\end{aligned}$$

so

$$\begin{aligned}Q(t) &= 3000 + \int_0^t 20e^{0.02t}\,dt = 3000 + \frac{20}{0.02}e^{0.02t}\Big|_0^t \\ &= 3000 + 1000(e^{0.02t} - 1) \\ &= 1000e^{0.02t} + 2000.\end{aligned}$$

Strengthen Your Understanding

145. To do guess-and-check or substitution, we need an extra factor of $f'(x)$ in the integrand. For example, if $f(x) = 2x$, the left side is

$$\int (2x)^2\,dx = \int 4x^2\,dx = \frac{4x^3}{3} + C,$$

while the right side is $8x^3/3 + C$.

146. If we do guess-and-check and our guess is off by a constant factor, we can fix our guess by dividing our guess by that constant. This does not work if we are off by a factor which is not constant. In this case, if we check our answer by differentiating, we get, using the quotient rule:

$$\frac{d}{dx}\left(\frac{\sin(x^2)}{2x}\right) = \frac{\cos(x^2)(2x)(2x) - 2\sin x^2}{4x^2} \neq \cos x^2.$$

147. When we make a substitution in a definite integral, we must either change the limits of integration or return to the original variable before evaluating. Since $w = 3x$, the right side should be $(1/3)\int_0^{3\pi/2} \cos w\,dw$.

148. The inside function is $\cos\theta$, so $f(\theta)$ should be the derivative of $\cos\theta$ (or a constant multiple of the derivative). Thus, $f(\theta) = -\sin\theta$ works.

149. The integrand should have a factor that is a constant multiple of $3x^2 - 3$, for example $\int \sin(x^3 - 3x)(x^2 - 1)\,dx$.

150. True. Let $w = f(x)$, so $dw = f'(x)\,dx$, then

$$\int f'(x)\cos(f(x))\,dx = \int \cos w\,dw = \sin w + C = \sin(f(x)) + C.$$

151. False. Differentiating gives

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)} \cdot f'(x),$$

so, in general

$$\int \frac{1}{f(x)}\,dx \neq \ln|f(x)| + C.$$

152. True. Let $w = 5 - t^2$, then $dw = -2t\,dt$.

Solutions for Section 7.2

Exercises

1. (a) Since we can change x^2 into a multiple of x^3 by integrating, let $v' = x^2$ and $u = e^x$. Using $v = x^3/3$ and $u' = e^x$ we get

$$\int x^2 e^x dx = \int uv' dx = uv - \int u'v dx$$
$$= \frac{x^3 e^x}{3} - \frac{1}{3}\int x^3 e^x dx.$$

(b) Since we can change x^2 into a multiple of x by differentiating, let $u = x^2$ and $v' = e^x$. Using $u' = 2x$ and $v = e^x$ we have

$$\int x^2 e^x dx = \int uv' dx = uv - \int u'v dx$$
$$= x^2 e^x - 2\int xe^x dx.$$

2. Let $u = \arctan x$, $v' = 1$. Then $v = x$ and $u' = \dfrac{1}{1+x^2}$. Integrating by parts, we get:

$$\int 1 \cdot \arctan x\, dx = x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2}\, dx.$$

To compute the second integral use the substitution, $z = 1 + x^2$.

$$\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\ln|z| + C = \frac{1}{2}\ln(1+x^2) + C.$$

Thus,

$$\int \arctan x\, dx = x \cdot \arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

3. Let $u = t$, $v' = \sin t$. Thus, $v = -\cos t$ and $u' = 1$. With this choice of u and v, integration by parts gives:

$$\int t\sin t\, dt = -t\cos t - \int(-\cos t)\, dt$$
$$= -t\cos t + \sin t + C.$$

4. Let $u = t^2$, $v' = \sin t$ implying $v = -\cos t$ and $u' = 2t$. Integrating by parts, we get:

$$\int t^2 \sin t\, dt = -t^2\cos t - \int 2t(-\cos t)\, dt.$$

Again, applying integration by parts with $u = t$, $v' = \cos t$, we have:

$$\int t\cos t\, dt = t\sin t + \cos t + C.$$

Thus

$$\int t^2 \sin t\, dt = -t^2\cos t + 2t\sin t + 2\cos t + C.$$

5. Let $u = t$ and $v' = e^{5t}$, so $u' = 1$ and $v = \frac{1}{5}e^{5t}$.
Then $\int te^{5t}\, dt = \frac{1}{5}te^{5t} - \int \frac{1}{5}e^{5t}\, dt = \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} + C.$

6. Let $u = t^2$ and $v' = e^{5t}$, so $u' = 2t$ and $v = \frac{1}{5}e^{5t}$.
Then $\int t^2 e^{5t}\,dt = \frac{1}{5}t^2e^{5t} - \frac{2}{5}\int te^{5t}\,dt$.
Using Problem 5, we have $\int t^2e^{5t}\,dt = \frac{1}{5}t^2e^{5t} - \frac{2}{5}(\frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}) + C$
$= \frac{1}{5}t^2e^{5t} - \frac{2}{25}te^{5t} + \frac{2}{125}e^{5t} + C$.

7. Let $u = p$ and $v' = e^{(-0.1)p}$, $u' = 1$. Thus, $v = \int e^{(-0.1)p}\,dp = -10e^{(-0.1)p}$. With this choice of u and v, integration by parts gives:

$$\begin{aligned}\int pe^{(-0.1)p}\,dp &= p(-10e^{(-0.1)p}) - \int(-10e^{(-0.1)p})\,dp\\ &= -10pe^{(-0.1)p} + 10\int e^{(-0.1)p}\,dp\\ &= -10pe^{(-0.1)p} - 100e^{(-0.1)p} + C.\end{aligned}$$

8. Let $u = z+1$, $v' = e^{2z}$. Thus, $v = \frac{1}{2}e^{2z}$ and $u' = 1$. Integrating by parts, we get:

$$\begin{aligned}\int(z+1)e^{2z}\,dz &= (z+1)\cdot\frac{1}{2}e^{2z} - \int\frac{1}{2}e^{2z}\,dz\\ &= \frac{1}{2}(z+1)e^{2z} - \frac{1}{4}e^{2z} + C\\ &= \frac{1}{4}(2z+1)e^{2z} + C.\end{aligned}$$

9. Integration by parts with $u = \ln x$, $v' = x$ gives

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int\frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

Or use the integral table, III-13, with $n = 1$.

10. Let $u = \ln x$ and $v' = x^3$, so $u' = \frac{1}{x}$ and $v = \frac{x^4}{4}$. Then

$$\int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \int\frac{x^3}{4}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C.$$

11. Let $u = \ln 5q$, $v' = q^5$. Then $v = \frac{1}{6}q^6$ and $u' = \dfrac{1}{q}$. Integrating by parts, we get:

$$\begin{aligned}\int q^5\ln 5q\,dq &= \frac{1}{6}q^6\ln 5q - \int(5\cdot\frac{1}{5q})\cdot\frac{1}{6}q^6\,dq\\ &= \frac{1}{6}q^6\ln 5q - \frac{1}{36}q^6 + C.\end{aligned}$$

12. Let $u = \theta^2$ and $v' = \cos 3\theta$, so $u' = 2\theta$ and $v = \frac{1}{3}\sin 3\theta$.
Then $\int\theta^2\cos 3\theta\,d\theta = \frac{1}{3}\theta^2\sin 3\theta - \frac{2}{3}\int\theta\sin 3\theta\,d\theta$. The integral on the right hand side is simpler than our original integral, but to evaluate it we need to again use integration by parts.
To find $\int\theta\sin 3\theta\,d\theta$, let $u = \theta$ and $v' = \sin 3\theta$, so $u' = 1$ and $v = -\frac{1}{3}\cos 3\theta$.
This gives

$$\int\theta\sin 3\theta\,d\theta = -\frac{1}{3}\theta\cos 3\theta + \frac{1}{3}\int\cos 3\theta\,d\theta = -\frac{1}{3}\theta\cos 3\theta + \frac{1}{9}\sin 3\theta + C.$$

Thus,

$$\int\theta^2\cos 3\theta d\theta = \frac{1}{3}\theta^2\sin 3\theta + \frac{2}{9}\theta\cos 3\theta - \frac{2}{27}\sin 3\theta + C.$$

13. Let $u = \sin\theta$ and $v' = \sin\theta$, so $u' = \cos\theta$ and $v = -\cos\theta$. Then

$$\begin{aligned}\int \sin^2\theta\,d\theta &= -\sin\theta\cos\theta + \int \cos^2\theta\,d\theta \\ &= -\sin\theta\cos\theta + \int (1-\sin^2\theta)\,d\theta \\ &= -\sin\theta\cos\theta + \int 1\,d\theta - \int \sin^2\theta\,d\theta.\end{aligned}$$

By adding $\int \sin^2\theta\,d\theta$ to both sides of the above equation, we find that $2\int \sin^2\theta\,d\theta = -\sin\theta\cos\theta + \theta + C$, so $\int \sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C'$.

14. Let $u = \cos(3\alpha+1)$ and $v' = \cos(3\alpha+1)$, so $u' = -3\sin(3\alpha+1)$, and $v = \frac{1}{3}\sin(3\alpha+1)$. Then

$$\begin{aligned}\int \cos^2(3\alpha+1)\,d\alpha &= \int (\cos(3\alpha+1))\cos(3\alpha+1)\,d\alpha \\ &= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \int \sin^2(3\alpha+1)\,d\alpha \\ &= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \int \left(1-\cos^2(3\alpha+1)\right)\,d\alpha \\ &= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \alpha - \int \cos^2(3\alpha+1)\,d\alpha.\end{aligned}$$

By adding $\int \cos^2(3\alpha+1)\,d\alpha$ to both sides of the above equation, we find that

$$2\int \cos^2(3\alpha+1)\,d\alpha = \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \alpha + C,$$

which gives

$$\int \cos^2(3\alpha+1)\,d\alpha = \frac{1}{6}\cos(3\alpha+1)\sin(3\alpha+1) + \frac{\alpha}{2} + C.$$

15. Let $u = (\ln t)^2$ and $v' = 1$, so $u' = \dfrac{2\ln t}{t}$ and $v = t$. Then

$$\int (\ln t)^2\,dt = t(\ln t)^2 - 2\int \ln t\,dt = t(\ln t)^2 - 2t\ln t + 2t + C.$$

(We use the fact that $\displaystyle\int \ln x\,dx = x\ln x - x + C$, a result which can be derived using integration by parts.)

16. Remember that $\ln(x^2) = 2\ln x$. Therefore,

$$\int \ln(x^2)\,dx = 2\int \ln x\,dx = 2x\ln x - 2x + C.$$

Check:

$$\frac{d}{dx}(2x\ln x - 2x + C) = 2\ln x + \frac{2x}{x} - 2 = 2\ln x = \ln(x^2).$$

17. Let $u = y$ and $v' = (y+3)^{1/2}$, so $u' = 1$ and $v = \frac{2}{3}(y+3)^{3/2}$:

$$\int y\sqrt{y+3}\,dy = \frac{2}{3}y(y+3)^{3/2} - \int \frac{2}{3}(y+3)^{3/2}\,dy = \frac{2}{3}y(y+3)^{3/2} - \frac{4}{15}(y+3)^{5/2} + C.$$

18. Let $u = t+2$ and $v' = \sqrt{2+3t}$, so $u' = 1$ and $v = \frac{2}{9}(2+3t)^{3/2}$. Then

$$\begin{aligned}\int (t+2)\sqrt{2+3t}\,dt &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt \\ &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{4}{135}(2+3t)^{5/2} + C.\end{aligned}$$

19. Let $u = \theta + 1$ and $v' = \sin(\theta + 1)$, so $u' = 1$ and $v = -\cos(\theta + 1)$.

$$\int (\theta + 1)\sin(\theta + 1)\,d\theta = -(\theta + 1)\cos(\theta + 1) + \int \cos(\theta + 1)\,d\theta$$
$$= -(\theta + 1)\cos(\theta + 1) + \sin(\theta + 1) + C.$$

20. Let $u = z$, $v' = e^{-z}$. Thus $v = -e^{-z}$ and $u' = 1$. Integration by parts gives:

$$\int ze^{-z}\,dz = -ze^{-z} - \int (-e^{-z})\,dz$$
$$= -ze^{-z} - e^{-z} + C$$
$$= -(z + 1)e^{-z} + C.$$

21. Let $u = \ln x$, $v' = x^{-2}$. Then $v = -x^{-1}$ and $u' = x^{-1}$. Integrating by parts, we get:

$$\int x^{-2}\ln x\,dx = -x^{-1}\ln x - \int (-x^{-1})\cdot x^{-1}\,dx$$
$$= -x^{-1}\ln x - x^{-1} + C.$$

22. Let $u = y$ and $v' = \frac{1}{\sqrt{5-y}}$, so $u' = 1$ and $v = -2(5 - y)^{1/2}$.

$$\int \frac{y}{\sqrt{5 - y}}\,dy = -2y(5 - y)^{1/2} + 2\int (5 - y)^{1/2}\,dy = -2y(5 - y)^{1/2} - \frac{4}{3}(5 - y)^{3/2} + C.$$

23. $\displaystyle\int \frac{t + 7}{\sqrt{5 - t}}\,dt = \int \frac{t}{\sqrt{5 - t}}\,dt + 7\int (5 - t)^{-1/2}\,dt.$

To calculate the first integral, we use integration by parts. Let $u = t$ and $v' = \frac{1}{\sqrt{5-t}}$, so $u' = 1$ and $v = -2(5-t)^{1/2}$. Then

$$\int \frac{t}{\sqrt{5 - t}}\,dt = -2t(5 - t)^{1/2} + 2\int (5 - t)^{1/2}\,dt = -2t(5 - t)^{1/2} - \frac{4}{3}(5 - t)^{3/2} + C.$$

We can calculate the second integral directly: $7\displaystyle\int (5 - t)^{-1/2} = -14(5 - t)^{1/2} + C_1$. Thus

$$\int \frac{t + 7}{\sqrt{5 - t}}\,dt = -2t(5 - t)^{1/2} - \frac{4}{3}(5 - t)^{3/2} - 14(5 - t)^{1/2} + C_2.$$

24. Let $u = (\ln x)^4$ and $v' = x$, so $u' = \frac{4(\ln x)^3}{x}$ and $v = \frac{x^2}{2}$. Then

$$\int x(\ln x)^4\,dx = \frac{x^2(\ln x)^4}{2} - 2\int x(\ln x)^3\,dx.$$

$\int x(\ln x)^3\,dx$ is somewhat less complicated than $\int x(\ln x)^4\,dx$. To calculate it, we again try integration by parts, this time letting $u = (\ln x)^3$ (instead of $(\ln x)^4$) and $v' = x$. We find

$$\int x(\ln x)^3\,dx = \frac{x^2}{2}(\ln x)^3 - \frac{3}{2}\int x(\ln x)^2\,dx.$$

Once again, express the given integral in terms of a less-complicated one. Using integration by parts two more times, we find that

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int x(\ln x)\,dx$$

and that

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

Putting this all together, we have

$$\int x(\ln x)^4\,dx = \frac{x^2}{2}(\ln x)^4 - x^2(\ln x)^3 + \frac{3}{2}x^2(\ln x)^2 - \frac{3}{2}x^2\ln x + \frac{3}{4}x^2 + C.$$

25. Integrate by parts letting $u=(\ln r)^2$ and $dv=rdr$, then $du=(2/r)\ln r dr$ and $v=r^2/2$. We get

$$\int r(\ln r)^2\,dr=\frac{1}{2}r^2(\ln r)^2-\int r\ln r\,dr.$$

Then using integration by parts again with $u=\ln r$ and $dv=rdr$, so $du=dr/r$ and $v=r^2/2$, we get

$$\int r\ln^2 r\,dr=\frac{1}{2}r^2(\ln r)^2-\left[\frac{1}{2}r^2\ln r-\frac{1}{2}\int r\,dr\right]=\frac{1}{2}r^2(\ln r)^2-\frac{1}{2}r^2\ln r+\frac{1}{4}r^2+C.$$

26. Let $u=\arcsin w$ and $v'=1$, so $u'=\frac{1}{\sqrt{1-w^2}}$ and $v=w$. Then

$$\int \arcsin w\,dw=w\arcsin w-\int\frac{w}{\sqrt{1-w^2}}\,dw=w\arcsin w+\sqrt{1-w^2}+C.$$

27. Let $u=\arctan 7z$ and $v'=1$, so $u'=\frac{7}{1+49z^2}$ and $v=z$. Now $\int\frac{7z\,dz}{1+49z^2}$ can be evaluated by the substitution $w=1+49z^2$, $dw=98z\,dz$, so

$$\int\frac{7z\,dz}{1+49z^2}=7\int\frac{\frac{1}{98}\,dw}{w}=\frac{1}{14}\int\frac{dw}{w}=\frac{1}{14}\ln|w|+C=\frac{1}{14}\ln(1+49z^2)+C$$

So

$$\int\arctan 7z\,dz=z\arctan 7z-\frac{1}{14}\ln(1+49z^2)+C.$$

28. This integral can first be simplified by making the substitution $w=x^2$, $dw=2x\,dx$. Then

$$\int x\arctan x^2\,dx=\frac{1}{2}\int\arctan w\,dw.$$

To evaluate $\int\arctan w\,dw$, we'll use integration by parts. Let $u=\arctan w$ and $v'=1$, so $u'=\frac{1}{1+w^2}$ and $v=w$. Then

$$\int\arctan w\,dw=w\arctan w-\int\frac{w}{1+w^2}\,dw=w\arctan w-\frac{1}{2}\ln|1+w^2|+C.$$

Since $1+w^2$ is never negative, we can drop the absolute value signs. Thus, we have

$$\begin{aligned}\int x\arctan x^2\,dx&=\frac{1}{2}\left(x^2\arctan x^2-\frac{1}{2}\ln(1+(x^2)^2)+C\right)\\&=\frac{1}{2}x^2\arctan x^2-\frac{1}{4}\ln(1+x^4)+C.\end{aligned}$$

29. Let $u=x^2$ and $v'=xe^{x^2}$, so $u'=2x$ and $v=\frac{1}{2}e^{x^2}$. Then

$$\int x^3e^{x^2}\,dx=\frac{1}{2}x^2e^{x^2}-\int xe^{x^2}\,dx=\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C.$$

Note that we can also do this problem by substitution and integration by parts. If we let $w=x^2$, so $dw=2x\,dx$, then $\int x^3e^{x^2}\,dx=\frac{1}{2}\int we^w\,dw$. We could then perform integration by parts on this integral to get the same result.

30. To simplify matters, let us try the substitution $w=x^3$, $dw=3x^2\,dx$. Then

$$\int x^5\cos x^3\,dx=\frac{1}{3}\int w\cos w\,dw.$$

Now we integrate by parts. Let $u=w$ and $v'=\cos w$, so $u'=1$ and $v=\sin w$. Then

$$\begin{aligned}\frac{1}{3}\int w\cos w\,dw&=\frac{1}{3}[w\sin w-\int\sin w\,dw]\\&=\frac{1}{3}[w\sin w+\cos w]+C\\&=\frac{1}{3}x^3\sin x^3+\frac{1}{3}\cos x^3+C\end{aligned}$$

31. Let $u = x, u' = 1$ and $v' = \sinh x, v = \cosh x$. Integrating by parts, we get

$$\begin{aligned}\int x \sinh x \, dx &= x \cosh x - \int \cosh x \, dx \\ &= x \cosh x - \sinh x + C.\end{aligned}$$

32. Let $u = x - 1, u' = 1$ and $v' = \cosh x, v = \sinh x$. Integrating by parts, we get

$$\begin{aligned}\int (x-1) \cosh x \, dx &= (x-1) \sinh x - \int \sinh x \, dx \\ &= (x-1) \sinh x - \cosh x + C.\end{aligned}$$

33. $\displaystyle\int_1^5 \ln t \, dt = (t \ln t - t)\Big|_1^5 = 5 \ln 5 - 4 \approx 4.047$

34. $\displaystyle\int_3^5 x \cos x \, dx = (\cos x + x \sin x)\Big|_3^5 = \cos 5 + 5 \sin 5 - \cos 3 - 3 \sin 3 \approx -3.944.$

35. We use integration by parts. Let $u = z$ and $v' = e^{-z}$, so $u' = 1$ and $v = -e^{-z}$. Then

$$\begin{aligned}\int_0^{10} ze^{-z} \, dz &= -ze^{-z}\Big|_0^{10} + \int_0^{10} e^{-z} \, dz \\ &= -10e^{-10} + (-e^{-z})\Big|_0^{10} \\ &= -11e^{-10} + 1 \\ &\approx 0.9995.\end{aligned}$$

36. $\displaystyle\int_1^3 t \ln t \, dt = \left(\frac{1}{2}t^2 \ln t - \frac{1}{4}t^2\right)\Big|_1^3 = \frac{9}{2} \ln 3 - 2 \approx 2.944.$

37. We use integration by parts. Let $u = \arctan y$ and $v' = 1$, so $u' = \frac{1}{1+y^2}$ and $v = y$. Thus

$$\begin{aligned}\int_0^1 \arctan y \, dy &= (\arctan y)y\Big|_0^1 - \int_0^1 \frac{y}{1+y^2} \, dy \\ &= \frac{\pi}{4} - \frac{1}{2} \ln |1 + y^2|\Big|_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \approx 0.439.\end{aligned}$$

38. $\displaystyle\int_0^5 \ln(1+t) \, dt = ((1+t)\ln(1+t) - (1+t))\Big|_0^5 = 6 \ln 6 - 5 \approx 5.751.$

39. We use integration by parts. Let $u = \arcsin z$ and $v' = 1$, so $u' = \dfrac{1}{\sqrt{1-z^2}}$ and $v = z$. Then

$$\int_0^1 \arcsin z \, dz = z \arcsin z\Big|_0^1 - \int_0^1 \frac{z}{\sqrt{1-z^2}} \, dz = \frac{\pi}{2} - \int_0^1 \frac{z}{\sqrt{1-z^2}} \, dz.$$

To find $\int_0^1 \frac{z}{\sqrt{1-z^2}}\,dz$, we substitute $w = 1-z^2$, so $dw = -2z\,dz$.
Then

$$\int_{z=0}^{z=1} \frac{z}{\sqrt{1-z^2}}\,dz = -\frac{1}{2}\int_{w=1}^{w=0} w^{-\frac{1}{2}}\,dw = \frac{1}{2}\int_{w=0}^{w=1} w^{-\frac{1}{2}}\,dw = w^{\frac{1}{2}}\Big|_0^1 = 1.$$

Thus our final answer is $\frac{\pi}{2} - 1 \approx 0.571$.

40. To simplify the integral, we first make the substitution $z = u^2$, so $dz = 2u\,du$. Then

$$\int_{u=0}^{u=1} u \arcsin u^2\,du = \frac{1}{2}\int_{z=0}^{z=1} \arcsin z\,dz.$$

From Problem 39, we know that $\int_0^1 \arcsin z\,dz = \frac{\pi}{2} - 1$. Thus,

$$\int_0^1 u \arcsin u^2\,du = \frac{1}{2}(\frac{\pi}{2} - 1) \approx 0.285.$$

41. (a) This integral can be evaluated using integration by parts with $u = x$, $v' = \sin x$.
(b) We evaluate this integral using the substitution $w = 1 + x^3$.
(c) We evaluate this integral using the substitution $w = x^2$.
(d) We evaluate this integral using the substitution $w = x^3$.
(e) We evaluate this integral using the substitution $w = 3x + 1$.
(f) This integral can be evaluated using integration by parts with $u = x^2$, $v' = \sin x$.
(g) This integral can be evaluated using integration by parts with $u = \ln x$, $v' = 1$.

42. A calculator gives $\int_1^2 \ln x\,dx = 0.386$. An antiderivative of $\ln x$ is $x\ln x - 1$, so the Fundamental Theorem of Calculus gives

$$\int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 1.$$

Since $2\ln 2 - 1 = 0.386$, the value from the Fundamental Theorem agrees with the numerical answer.

Problems

43. Using a log property, we have

$$\int \ln\left((5-3x)^2\right)\,dx = \int 2\ln(5-3x)\,dx.$$

So we let

$$\begin{aligned} w &= 5 - 3x \\ dw &= -3dx \\ dx &= -\frac{1}{3}\,dw, \end{aligned}$$

which gives,

$$\begin{aligned} \int \ln(5-3x)^2\,dx &= \int 2\ln\overbrace{(5-3x)}^{w}\ \overbrace{dx}^{-\frac{1}{3}dw} \\ &= \int -\frac{2}{3}\cdot \ln w\,dw. \end{aligned}$$

Thus, $w = 5 - 3x, k = -2/3$.

44. We have

$$\begin{aligned}\int \ln \frac{1}{\sqrt{4-5x}}\,dx &= \int \ln \frac{1}{(4-5x)^{1/2}}\,dx \\ &= \int \ln\left((4-5x)^{-1/2}\right)\,dx \\ &= \int -\frac{1}{2}\cdot \ln(4-5x)\,dx \qquad \text{log property.}\end{aligned}$$

So we let

$$\begin{aligned} w &= 4-5x \\ dw &= -5dx \\ dx &= -\frac{1}{5}\,dw,\end{aligned}$$

which gives

$$\begin{aligned}\int \ln \frac{1}{\sqrt{4-5x}}\,dx &= \int -\frac{1}{2}\cdot \ln \overbrace{(4-5x)}^{w}\ \overbrace{dx}^{-\frac{1}{5}\,dw} \\ &= \int \frac{1}{10}\ln w\,dw.\end{aligned}$$

Thus $w = 4-5x, k = 1/10$.

45. We have

$$\begin{aligned}\int \frac{\ln\left((\ln x)^3\right)}{x}\,dx &= \int \frac{3\ln(\ln x)}{x}\,dx \qquad \text{log property} \\ &= \int 3\ln(\ln x)\,x^{-1}\,dx.\end{aligned}$$

So we let

$$\begin{aligned} w &= \ln x \\ dw &= x^{-1}\,dx\end{aligned}$$

which gives

$$\begin{aligned}\int \frac{\ln\left((\ln x)^3\right)}{x}\,dx &= \int 3\ln \overbrace{(\ln x)}^{w}\ \overbrace{x^{-1}\,dx}^{dw} \\ &= \int 3\ln w\,dw.\end{aligned}$$

Thus $w = \ln x, k = 3$.

46. Using integration by parts with $u' = e^{-t}$, $v = t$, so $u = -e^{-t}$ and $v' = 1$, we have

$$\begin{aligned}\text{Area } &= \int_0^2 te^{-t}\,dt = -te^{-t}\Big|_0^2 - \int_0^2 -1\cdot e^{-t}\,dt \\ &= (-te^{-t} - e^{-t})\Big|_0^2 = -2e^{-2} - e^{-2} + 1 = 1 - 3e^{-2}.\end{aligned}$$

47. Since $\int \arctan z\,dz = \int 1\cdot\arctan z\,dz$, we take $u = \arctan z$, $v' = 1$, so $u' = 1/(1+z^2)$ and $v = z$. Then

$$\int \arctan z\,dz = z\arctan z - \int \frac{z}{1+z^2}\,dz = z\arctan z - \frac{1}{2}\ln(1+z^2) + C.$$

Thus, we have

$$\int_0^2 \arctan z\,dz = \left(z\arctan z - \frac{1}{2}\ln(z^2+1)\right)\Bigg|_0^2 = 2\arctan 2 - \frac{1}{2}\ln 5.$$

48. Since $\int \arcsin y\,dy = \int 1\cdot\arcsin y\,dy$, we take $u = \arcsin y$, $v' = 1$, so $u = 1/\sqrt{1-y^2}$ and $v = y$. Thus

$$\int \arcsin y\,dy = y\arcsin y - \int \frac{y}{\sqrt{1-y^2}}\,dy.$$

Substituting $w = 1-y^2, dw = -2y\,dy$, we have

$$\int \frac{y}{\sqrt{1-y^2}}\,dy = \int \frac{-1/2}{\sqrt{w}}\,dw = -\frac{1}{2}\int w^{-1/2}\,dw = w^{1/2} + C = \sqrt{1-y^2} + C.$$

Thus

$$\text{Area } = \int_0^1 \arcsin y\,dy = (y\arcsin y + \sqrt{1-y^2})\Bigg|_0^1 = 1\cdot\arcsin 1 - 1 = \frac{\pi}{2} - 1.$$

49. Since $\ln(x^2) = 2\ln x$ and $\int \ln x\,dx = x\ln x - x + C$, we have

$$\begin{aligned}\text{Area } &= \int_1^2 (\ln(x^2) - \ln x)\,dx = \int_1^2 (2\ln x - \ln x)\,dx \\ &= \int_1^2 \ln x\,dx = (x\ln x - x)\Bigg|_1^2 = 2\ln 2 - 2 - (1\ln 1 - 1) = 2\ln 2 - 1.\end{aligned}$$

50. Since the graph of $f(t) = \ln(t^2-1)$ is above the graph of $g(t) = \ln(t-1)$ for $t > 1$, we have

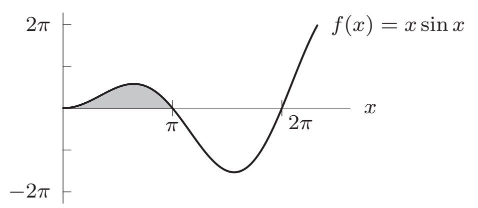
$$\text{Area } = \int_2^3 (\ln(t^2-1) - \ln(t-1))\,dt = \int_2^3 \ln\left(\frac{t^2-1}{t-1}\right)\,dt = \int_2^3 \ln(t+1)\,dt.$$

We can cancel the factor of $(t-1)$ in the last step above because the integral is over $2 \le t \le 3$, where $(t-1)$ is not zero.

We use $\int \ln x\,dx = x\ln x - 1$ with the substitution $x = t+1$. The limits $t = 2, t = 3$ become $x = 3, x = 4$, respectively. Thus

$$\begin{aligned}\text{Area } &= \int_2^3 \ln(t+1)\,dt = \int_3^4 \ln x\,dx = (x\ln x - x)\Bigg|_3^4 \\ &= 4\ln 4 - 4 - (3\ln 3 - 3) = 4\ln 4 - 3\ln 3 - 1.\end{aligned}$$

51.



The graph of $f(x) = x\sin x$ is shown above. The first positive zero is at $x = \pi$, so, using integration by parts,

$$\text{Area } = \int_0^\pi x\sin x\,dx$$

$$
\begin{aligned}
&= -x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx \\
&= -x\cos x\Big|_0^{\pi} + \sin x\Big|_0^{\pi} \\
&= -\pi\cos\pi - (-0\cos 0) + \sin\pi - \sin 0 = \pi.
\end{aligned}
$$

52. From integration by parts in Problem 13, we obtain

$$\int \sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C.$$

Using the identity given in the book, we have

$$\int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C.$$

Although the answers differ in form, they are really the same, since (by one of the standard double angle formulas) $-\frac{1}{4}\sin 2\theta = -\frac{1}{4}(2\sin\theta\cos\theta) = -\frac{1}{2}\sin\theta\cos\theta$.

53. Integration by parts: let $u = \cos\theta$ and $v' = \cos\theta$, so $u' = -\sin\theta$ and $v = \sin\theta$.

$$
\begin{aligned}
\int \cos^2\theta\,d\theta &= \sin\theta\cos\theta - \int(-\sin\theta)(\sin\theta)\,d\theta \\
&= \sin\theta\cos\theta + \int \sin^2\theta\,d\theta.
\end{aligned}
$$

Now use $\sin^2\theta = 1 - \cos^2\theta$.

$$
\begin{aligned}
\int \cos^2\theta\,d\theta &= \sin\theta\cos\theta + \int(1-\cos^2\theta)\,d\theta \\
&= \sin\theta\cos\theta + \int d\theta - \int\cos^2\theta\,d\theta.
\end{aligned}
$$

Adding $\int\cos^2\theta\,d\theta$ to both sides, we have

$$
\begin{aligned}
2\int \cos^2\theta\,d\theta &= \sin\theta\cos\theta + \theta + C \\
\int \cos^2\theta\,d\theta &= \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C'.
\end{aligned}
$$

Use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.

$$\int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C.$$

The only difference is in the two terms $\frac{1}{2}\sin\theta\cos\theta$ and $\frac{1}{4}\sin 2\theta$, but since $\sin 2\theta = 2\sin\theta\cos\theta$, we have $\frac{1}{4}\sin 2\theta = \frac{1}{4}(2\sin\theta\cos\theta) = \frac{1}{2}\sin\theta\cos\theta$, so there is no real difference between the formulas.

54. First, let $u = e^x$ and $v' = \sin x$, so $u' = e^x$ and $v = -\cos x$.
Thus $\int e^x\sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx$. To calculate $\int e^x\cos x\,dx$, we again need to use integration by parts.
Let $u = e^x$ and $v' = \cos x$, so $u' = e^x$ and $v = \sin x$.
Thus

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx.$$

This gives

$$\int e^x\sin x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx.$$

By adding $\int e^x \sin x\,dx$ to both sides, we obtain

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x) + C.$$

$$\text{Thus } \int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

This problem could also be done in other ways; for example, we could have started with $u = \sin x$ and $v' = e^x$ as well.

55. Let $u = e^\theta$ and $v' = \cos\theta$, so $u' = e^\theta$ and $v = \sin\theta$. Then $\int e^\theta \cos\theta\,d\theta = e^\theta \sin\theta - \int e^\theta \sin\theta\,d\theta$.

In Problem 54 we found that $\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.

$$\begin{aligned}\int e^\theta \cos\theta\,d\theta &= e^\theta \sin\theta - \left[\frac{1}{2}e^\theta(\sin\theta - \cos\theta)\right] + C\\ &= \frac{1}{2}e^\theta(\sin\theta + \cos\theta) + C.\end{aligned}$$

56. We integrate by parts. Since in Problem 54 we found that $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x)$, we let $u = x$ and $v' = e^x \sin x$, so $u' = 1$ and $v = \frac{1}{2}e^x(\sin x - \cos x)$.

$$\begin{aligned}\text{Then } \int xe^x \sin x\,dx &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\int e^x(\sin x - \cos x)\,dx\\ &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\int e^x \sin x\,dx + \frac{1}{2}\int e^x \cos x\,dx.\end{aligned}$$

Using Problems 54 and 55, we see that this equals

$$\begin{aligned}&\frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{4}e^x(\sin x - \cos x) + \frac{1}{4}e^x(\sin x + \cos x) + C\\ &\quad = \frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}e^x \cos x + C.\end{aligned}$$

57. Again we use Problems 54 and 55. Integrate by parts, letting $u = \theta$ and $v' = e^\theta \cos\theta$, so $u' = 1$ and $v = \frac{1}{2}e^\theta(\sin\theta + \cos\theta)$. Then

$$\begin{aligned}\int \theta e^\theta \cos\theta\,d\theta &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{2}\int e^\theta(\sin\theta + \cos\theta)\,d\theta\\ &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{2}\int e^\theta \sin\theta\,d\theta - \frac{1}{2}\int e^\theta \cos\theta\,d\theta\\ &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{4}e^\theta(\sin\theta - \cos\theta) - \frac{1}{4}(\sin\theta + \cos\theta) + C\\ &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{2}e^\theta \sin\theta + C.\end{aligned}$$

58. Using integration by parts on the first integral with $u(x) = \ln x$ and $v'(x) = f''(x)$, we have $u'(x) = 1/x$ and $v(x) = f'(x)$, so

$$\int f''(x)\ln x\,dx = f'(x)\ln x - \int \frac{f'(x)}{x}\,dx$$

Using integration by parts on the second integral with $u(x) = f(x)$ and $v'(x) = 1/x^2$, we have $u'(x) = f'(x)$ and $v(x) = -1/x$, so

$$\int \frac{f(x)}{x^2} = -\frac{f(x)}{x} + \int \frac{f'(x)}{x}\,dx.$$

Adding the results gives

$$\int f''(x)\ln x\,dx + \int \frac{f(x)}{x^2}\,dx = f'(x)\ln x - \int \frac{f'(x)}{x}\,dx - \frac{f(x)}{x} + \int \frac{f'(x)}{x}\,dx = f'(x)\ln x - \frac{f(x)}{x} + C$$

59. Using integration by parts with $u(x) = x$ and $v'(x) = f''(x)$ gives $u'(x) = 1$ and $v(x) = f'(x)$, so

$$\int x f''(x)\,dx = x f'(x) - \int f'(x)\,dx = x f'(x) - f(x) + C.$$

60. Using integration by parts, we let $u = x, v' = f'(x)$. This gives $u' = 1, v = f(x)$, and we have

$$\begin{aligned}
\int uv'\,dx &= uv - \int u'v\,dx \\
&= \overbrace{xf(x)}^{uv} - \int \overbrace{1 \cdot f(x)}^{u'v}\,dx \\
&= xf(x) - F(x) + C \qquad \text{since } f(x) = F'(x).
\end{aligned}$$

$$\begin{aligned}
\text{Therefore,} \quad \int_0^5 uv'\,dx &= (xf(x) - F(x))\Big|_0^5 \\
&= 5f(5) - F(5) - (0 \cdot f(0) - F(0)) \\
&= 5 \cdot 27 - 20 - (0 \cdot 2 - 10) = 125.
\end{aligned}$$

61. We integrate by parts. Since we know what the answer is supposed to be, it's easier to choose u and v'. Let $u = x^n$ and $v' = e^x$, so $u' = nx^{n-1}$ and $v = e^x$. Then

$$\int x^n e^x\,dx = x^n e^x - n \int x^{n-1} e^x\,dx.$$

62. We integrate by parts. Let $u = x^n$ and $v' = \cos ax$, so $u' = nx^{n-1}$ and $v = \frac{1}{a}\sin ax$. Then

$$\begin{aligned}
\int x^n \cos ax\,dx &= \frac{1}{a}x^n \sin ax - \int (nx^{n-1})(\frac{1}{a}\sin ax)\,dx \\
&= \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1} \sin ax\,dx.
\end{aligned}$$

63. We integrate by parts. Let $u = x^n$ and $v' = \sin ax$, so $u' = nx^{n-1}$ and $v = -\frac{1}{a}\cos ax$. Then

$$\begin{aligned}
\int x^n \sin ax\,dx &= -\frac{1}{a}x^n \cos ax - \int (nx^{n-1})(-\frac{1}{a}\cos ax)\,dx \\
&= -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1} \cos ax\,dx.
\end{aligned}$$

64. We integrate by parts. Since we know what the answer is supposed to be, it's easier to choose u and v'. Let $u = \cos^{n-1} x$ and $v' = \cos x$, so $u' = (n-1)\cos^{n-2} x(-\sin x)$ and $v = \sin x$. Then

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)\,dx \\
&= \cos^{n-1} x \sin x - (n-1)\int \cos^n x\,dx + (n-1)\int \cos^{n-2} x\,dx.
\end{aligned}$$

Thus, by adding $(n-1)\int \cos^n x\,dx$ to both sides of the equation, we find

$$\begin{aligned}
n\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx, \\
\text{so} \int \cos^n\,dx &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.
\end{aligned}$$

65. (a) One way to avoid integrating by parts is to take the derivative of the right hand side instead. Since $\int e^{ax}\sin bx\,dx$ is the antiderivative of $e^{ax}\sin bx$,

$$\begin{aligned} e^{ax}\sin bx &= \frac{d}{dx}[e^{ax}(A\sin bx + B\cos bx) + C] \\ &= ae^{ax}(A\sin bx + B\cos bx) + e^{ax}(Ab\cos bx - Bb\sin bx) \\ &= e^{ax}[(aA - bB)\sin bx + (aB + bA)\cos bx]. \end{aligned}$$

Thus $aA - bB = 1$ and $aB + bA = 0$. Solving for A and B in terms of a and b, we get

$$A = \frac{a}{a^2+b^2},\ B = -\frac{b}{a^2+b^2}.$$

Thus

$$\int e^{ax}\sin bx = e^{ax}\left(\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\right) + C.$$

(b) If we go through the same process, we find

$$ae^{ax}[(aA - bB)\sin bx + (aB + bA)\cos bx] = e^{ax}\cos bx.$$

Thus $aA - bB = 0$, and $aB + bA = 1$. In this case, solving for A and B yields

$$A = \frac{b}{a^2+b^2},\ B = \frac{a}{a^2+b^2}.$$

Thus $\int e^{ax}\cos bx = e^{ax}(\frac{b}{a^2+b^2}\sin bx + \frac{a}{a^2+b^2}\cos bx) + C$.

66. Since $f'(x) = 2x$, integration by parts tells us that

$$\begin{aligned} \int_0^{10} f(x)g'(x)\,dx &= f(x)g(x)\Big|_0^{10} - \int_0^{10} f'(x)g(x)\,dx \\ &= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx. \end{aligned}$$

We can use left and right Riemann Sums with $\Delta x = 2$ to approximate $\int_0^{10} xg(x)\,dx$:

$$\begin{aligned} \text{Left sum} &\approx 0\cdot g(0)\Delta x + 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x \\ &= (0(2.3) + 2(3.1) + 4(4.1) + 6(5.5) + 8(5.9))\,2 = 205.6. \\ \text{Right sum} &\approx 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x + 10\cdot g(10)\Delta x \\ &= (2(3.1) + 4(4.1) + 6(5.5) + 8(5.9) + 10(6.1))\,2 = 327.6. \end{aligned}$$

A good estimate for the integral is the average of the left and right sums, so

$$\int_0^{10} xg(x)\,dx \approx \frac{205.6 + 327.6}{2} = 266.6.$$

Substituting values for f and g, we have

$$\begin{aligned} \int_0^{10} f(x)g'(x)\,dx &= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx \\ &\approx 10^2(6.1) - 0^2(2.3) - 2(266.6) = 76.8 \approx 77. \end{aligned}$$

67. Using integration by parts we have

$$\begin{aligned} \int_0^1 xf''(x)\,dx &= xf'(x)\Big|_0^1 - \int_0^1 f'(x)\,dx \\ &= 1\cdot f'(1) - 0\cdot f'(0) - [f(1) - f(0)] \\ &= 2 - 0 - 5 + 6 = 3. \end{aligned}$$

68. Letting $u=\sqrt{x}, u'=\dfrac{1}{2\sqrt{x}}, v=f(x), v'=f'(x)$, we have:

$$\begin{aligned}\int \underbrace{f'(x)\sqrt{x}}_{uv'}\,dx &= \underbrace{f(x)\sqrt{x}}_{uv} - \int \underbrace{f(x)\cdot\frac{1}{2\sqrt{x}}}_{vu'}\,dx\\ &= \underbrace{f(x)\sqrt{x}}_{h(x)} - \frac{1}{2}\int \underbrace{f(x)\cdot\frac{1}{\sqrt{x}}}_{g'(x)}\,dx\\ &= h(x) - \frac{1}{2}\int g'(x)\,dx\\ &= h(x) - \frac{1}{2}g(x) + C.\end{aligned}$$

69. Letting $u=x, v'=f'(x)$, we have $u'=1$ and $v=f(x)$. Integration by parts gives:

$$\int \overbrace{xf'(x)}^{uv'}\,dx = \overbrace{xf(x)}^{uv} - \int \overbrace{f(x)}^{vu'}\,dx.$$

This means

$$\begin{aligned}\int_0^7 xf'(x)\,dx &= xf(x)\Big|_0^7 - \underbrace{\int_0^7 f(x)\,dx}_{5}\\ &= 7\underbrace{f(7)}_{0} - 0\cdot f(0) - 5 \qquad x=7 \text{ is a zero of } f\\ &= -5.\end{aligned}$$

70. (a) We have

$$\begin{aligned}F(a) &= \int_0^a x^2e^{-x}\,dx\\ &= -x^2e^{-x}\Big|_0^a + \int_0^a 2xe^{-x}\,dx\\ &= (-x^2e^{-x} - 2xe^{-x})\Big|_0^a + 2\int_0^a e^{-x}\,dx\\ &= (-x^2e^{-x} - 2xe^{-x} - 2e^{-x})\Big|_0^a\\ &= -a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2.\end{aligned}$$

(b) $F(a)$ is increasing because x^2e^{-x} is positive, so as a increases, the area under the curve from 0 to a also increases and thus the integral increases.

(c) We have $F'(a)=a^2e^{-a}$, so

$$F''(a) = 2ae^{-a} - a^2e^{-a} = a(2-a)e^{-a}.$$

We see that $F''(a) > 0$ for $0 < a < 2$, so F is concave up on this interval.

71. We have

$$\text{Bioavailability} = \int_0^3 15te^{-0.2t}dt.$$

We first use integration by parts to evaluate the indefinite integral of this function. Let $u=15t$ and $v'=e^{-0.2t}dt$, so $u'=15dt$ and $v=-5e^{-0.2t}$. Then,

$$\begin{aligned}\int 15te^{-0.2t}dt &= (15t)(-5e^{-0.2t}) - \int(-5e^{-0.2t})(15dt)\\ &= -75te^{-0.2t} + 75\int e^{-0.2t}dt = -75te^{-0.2} - 375e^{-0.2t} + C.\end{aligned}$$

Thus,

$$\int_0^3 15te^{-0.2t}dt = (-75te^{-0.2t} - 375e^{-0.2t})\Big|_0^3 = -329.29 + 375 = 45.71.$$

The bioavailability of the drug over this time interval is 45.71 (ng/ml)-hours.

72. (a) Increasing V_0 increases the maximum value of V, since this maximum is V_0. Increasing ω or ϕ does not affect the maximum of V.

(b) Since

$$\frac{dV}{dt} = -\omega V_0 \sin(\omega t + \phi),$$

the maximum of dV/dt is ωV_0. Thus, the maximum of dV/dt is increased if V_0 or ω is increased, and is unaffected if ϕ is increased.

(c) The period of $V = V_0 \cos(\omega t + \phi)$ is $2\pi/\omega$, so

$$\text{Average value} = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} (V_0\cos(\omega t + \phi))^2\, dt.$$

Substituting $x = \omega t + \phi$, we have $dx = \omega dt$. When $t = 0$, $x = \phi$, and when $t = 2\pi/\omega$, $x = 2\pi + \phi$. Thus,

$$\begin{aligned}\text{Average value} &= \frac{\omega}{2\pi}\int_\phi^{2\pi+\phi} V_0^2(\cos x)^2 \frac{1}{\omega}\, dx\\ &= \frac{V_0^2}{2\pi}\int_\phi^{2\pi+\phi} (\cos x)^2\, dx.\end{aligned}$$

Using integration by parts and the fact that $\sin^2 x = 1 - \cos^2 x$, we see that

$$\begin{aligned}\text{Average value} &= \frac{V_0^2}{2\pi}\left[\frac{1}{2}(\cos x \sin x + x)\right]_\phi^{2\pi+\phi}\\ &= \frac{V_0^2}{4\pi}\left[\cos(2\pi+\phi)\sin(2\pi+\phi) + (2\pi+\phi) - \cos\phi\sin\phi - \phi\right]\\ &= \frac{V_0^2}{4\pi}\cdot 2\pi = \frac{V_0^2}{2}.\end{aligned}$$

Thus, increasing V_0 increases the average value; increasing ω or ϕ has no effect.
However, it is not in fact necessary to compute the integral to see that ω does not affect the average value, since all ω's dropped out of the average value expression when we made the substitution $x = \omega t + \phi$.

73. (a) We know that $\dfrac{dE}{dt} = r$, so the total energy E used in the first T hours is given by $E = \displaystyle\int_0^T te^{-at}\, dt$. We use integration by parts. Let $u = t$, $v' = e^{-at}$. Then $u' = 1$, $v = -\frac{1}{a}e^{-at}$.

$$\begin{aligned}E &= \int_0^T te^{-at}\, dt\\ &= -\frac{t}{a}e^{-at}\Big|_0^T - \int_0^T \left(-\frac{1}{a}e^{-at}\right) dt\\ &= -\frac{1}{a}Te^{-aT} + \frac{1}{a}\int_0^T e^{-at}\, dt\\ &= -\frac{1}{a}Te^{-aT} + \frac{1}{a^2}(1 - e^{-aT}).\end{aligned}$$

(b)

$$\lim_{T\to\infty} E = -\frac{1}{a}\lim_{T\to\infty}\left(\frac{T}{e^{aT}}\right) + \frac{1}{a^2}\left(1 - \lim_{T\to\infty}\frac{1}{e^{aT}}\right).$$

Since $a > 0$, the second limit on the right hand side in the above expression is 0. In the first limit, although both the numerator and the denominator go to infinity, the denominator e^{aT} goes to infinity more quickly than T does. So in the end the denominator e^{aT} is much greater than the numerator T. Hence $\displaystyle\lim_{T\to\infty}\frac{T}{e^{aT}} = 0$. (You can check this by graphing $y = \dfrac{T}{e^{aT}}$ on a calculator or computer for some values of a.) Thus $\displaystyle\lim_{T\to\infty} E = \frac{1}{a^2}$.

74. Letting $u = \ln|x|, u' = \dfrac{1}{x}, v = f(x), v' = f'(x)$, we have:

$$\begin{aligned}\int \underbrace{f'(x)\ln|x|}_{uv'}\,dx &= \underbrace{f(x)\ln|x|}_{uv} - \int \underbrace{\frac{1}{x}\cdot f(x)}_{u'v}\,dx\\ &= \underbrace{f(x)\ln|x|}_{h(x)} - \int \underbrace{\frac{f(x)}{x}}_{g'(x)}\,dx\\ &= h(x) - g(x) + C.\end{aligned}$$

75. (a) We have $u = \operatorname{erf}(x), v' = 1$, which means $v = x$. From the Construction Theorem, we see that $\operatorname{erf}(x)$ is an antiderivative of $\dfrac{2}{\sqrt{\pi}}e^{-x^2}$, so $u' = \dfrac{2}{\sqrt{\pi}}e^{-x^2}$. Using integration by parts, we have

$$\begin{aligned}\int \overbrace{\operatorname{erf}(x)}^{uv'}\,dx &= \overbrace{x\operatorname{erf}(x)}^{uv} - \int \overbrace{x\frac{2}{\sqrt{\pi}}e^{-x^2}}^{u'v}\,dx\\ &= x\operatorname{erf}(x) - \int \frac{2}{\sqrt{\pi}}e^{-x^2}x\,dx.\end{aligned}$$

(b) One possibility is to let $w = -x^2, dw = -2xdx$, which gives:

$$\int \frac{2}{\sqrt{\pi}}e^{-x^2}x\,dx = \frac{2}{\sqrt{\pi}}\int e^w\left(-\frac{1}{2}\right)dw = -\frac{1}{\sqrt{\pi}}e^w + C = -\frac{1}{\sqrt{\pi}}e^{-x^2} + C.$$

Another possibility is to let $w = e^{-x^2}, dw = -2xe^{-x^2}dx$, which leads to the same result.

(c) Based on our answers to parts (a) and (b), we have:

$$\begin{aligned}\int \operatorname{erf}(x)\,dx &= x\operatorname{erf}(x) - \int \frac{2}{\sqrt{\pi}}e^{-x^2}x\,dx\\ &= x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}e^{-x^2} + C.\end{aligned}$$

Checking our answer, we see that:

$$\begin{aligned}\frac{d}{dx}\left(x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}e^{-x^2} + C\right) &= \frac{d}{dx}\left(x\operatorname{erf}(x)\right) + \frac{d}{dx}\left(\frac{1}{\sqrt{\pi}}e^{-x^2}\right)\\ &= 1\cdot\operatorname{erf}(x) + x\cdot\underbrace{(\operatorname{erf}(x))'}_{\frac{2}{\sqrt{\pi}}e^{-x^2}} + \frac{1}{\sqrt{\pi}}e^{-x^2}(-2x)\\ &= \operatorname{erf}(x) + \frac{2x}{\sqrt{\pi}}e^{-x^2} - \frac{2x}{\sqrt{\pi}}e^{-x^2}\\ &= \operatorname{erf}(x). \qquad \text{as required}\end{aligned}$$

76. Since $\operatorname{Li}(x) = \displaystyle\int_2^x \frac{1}{\ln t}\,dt$, we see from the Construction Theorem that $\operatorname{Li}(x)$ is an antiderivative of $1/\ln(x)$, which means that $(\operatorname{Li}(x))' = 1/\ln(x)$. We need to perform integration by parts where $u = \operatorname{Li}(x), v = \ln x$. We have:

$$u' = \frac{1}{\ln x} \quad \text{and} \quad v' = \frac{1}{x},$$

which means

$$\begin{aligned}\int \underbrace{\operatorname{Li}(x)x^{-1}}_{uv'}\,dx &= uv - \int u'v\,dx \qquad \text{integration by parts}\\ &= \underbrace{\operatorname{Li}(x)\ln x}_{uv} - \int \underbrace{\frac{1}{\ln(x)}\cdot\ln x}_{u'v}\,dx\\ &= \operatorname{Li}(x)\ln x - \int dx\\ &= \operatorname{Li}(x)\ln x - x + C.\end{aligned}$$

Strengthen Your Understanding

77. To use integration by parts on $\int t\ln t\,dt$, use $u=\ln t$, $v'=t$.

78. We can write $\arctan x = (1)(\arctan x)$. Then using $u=\arctan x$ and $v'=1$, we can evaluate the integral using integration by parts.

79. Using integration by parts with $u=f(x)$ and $v'=1$ gives $u'=f'(x)$, $v=x$, so we get

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx.$$

80. The integral $\int \theta^2 \sin\theta\,d\theta$ requires integration by parts twice. Many other answers are possible.

81. $\int x^3e^x\,dx$

82. $\int e^x \sin x\,dx$, can be evaluated by applying integration by parts twice, both times with $u=e^x$, or by applying integration by parts once with $u=\sin x$ and then a second time with $u=\cos x$.

83. True. Let $u=t, v'=\sin(5-t)$, so $u'=1, v=\cos(5-t)$. Then the integral $\int 1\cdot\cos(5-t)\,dt$ can be done by guess-and-check or by substituting $w=5-t$.

84. True. If we let $u=t^2$ and $v'=e^{3-t}$ and integrate by parts twice, we obtain

$$\int t^2e^{3-t}\,dt = -t^2e^{3-t} + 2(-te^{3-t} - e^{3-t}) + C.$$

85. False. Suppose we have $\int x^2\sin x\,dx$. If we choose $u=\sin x$ and $v'=x^2$, the resulting integral is

$$\frac{x^3}{3}\sin x - \int \frac{x^3}{3}\cos x\,dx,$$

which is no simpler than the original integral. Choosing $u=x^2$ and $v'=\sin x$ make the integral simpler. After integrating twice, we get $-x^2\cos x + 2(x\sin x + 2\cos x) + C$.

Solutions for Section 7.3

Exercises

1. $\frac{1}{6}x^6\ln x - \frac{1}{36}x^6 + C$. (Let $n=5$ in III-13.)

2. $\frac{1}{10}e^{(-3\theta)}(-3\cos\theta + \sin\theta) + C$.
(Let $a=-3, b=1$ in II-9.)

3. The integrand, a polynomial, x^3, multiplied by $\sin 5x$, is in the form of III-15. There are only three successive derivatives of x^3 before 0 is reached (namely, $3x^2$, $6x$, and 6), so there will be four terms. The signs in the terms will be $-++-$, as given in III-15, so we get

$$\int x^3\sin 5x\,dx = -\frac{1}{5}x^3\cos 5x + \frac{1}{25}\cdot 3x^2\sin 5x + \frac{1}{125}\cdot 6x\cos 5x - \frac{1}{625}\cdot 6\sin 5x + C.$$

4. Formula III-13 applies only to functions of the form $x^n\ln x$, so we'll have to multiply out and separate into two integrals.

$$\int (x^2+3)\ln x\,dx = \int x^2\ln x\,dx + 3\int \ln x\,dx.$$

Now we can use formula III-13 on each integral separately, to get

$$\int (x^2+3)\ln x\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + 3(x\ln x - x) + C.$$

5. Note that you can't use substitution here: letting $w = x^3 + 5$ does not work, since there is no $dw = 3x^2\,dx$ in the integrand. What will work is simply multiplying out the square: $(x^3+5)^2 = x^6 + 10x^3 + 25$. Then use I-1:

$$\int (x^3+5)^2\,dx = \int x^6\,dx + 10\int x^3\,dx + 25\int 1\,dx = \frac{1}{7}x^7 + 10\cdot\frac{1}{4}x^4 + 25x + C.$$

6. $-\frac{1}{5}\cos^5 w + C$
(Let $x = \cos w$, as suggested in IV-23. Then $-\sin w\,dw = dx$, and $\int \sin w \cos^4 w\,dw = -\int x^4\,dx$.)

7. $-\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C.$
(Use IV-17.)

8. $\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right)e^{2x} + C.$
(Let $a = 2, p(x) = x^3$ in III-14.)

9. $\left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x} + C.$
(Let $a = 3, p(x) = x^2$ in III-14.)

10. $\frac{1}{3}e^{x^3} + C.$
(Substitute $w = x^3$, $dw = 3x^2\,dx$. It is not necessary to use the table.)

11. $\left(\frac{1}{3}x^4 - \frac{4}{9}x^3 + \frac{4}{9}x^2 - \frac{8}{27}x + \frac{8}{81}\right)e^{3x} + C.$
(Let $a = 3, p(x) = x^4$ in III-14.)

12. Substitute $w = 5u,\ dw = 5\,du$. Then

$$\begin{aligned}\int u^5\ln(5u)\,du &= \frac{1}{5^6}\int w^5 \ln w\,dw \\ &= \frac{1}{5^6}\left(\frac{1}{6}w^6\ln w - \frac{1}{36}w^6 + C\right) \\ &= \frac{1}{6}u^6\ln 5u - \frac{1}{36}u^6 + C.\end{aligned}$$

Or use $\ln 5u = \ln 5 + \ln u$.

$$\begin{aligned}\int u^5 \ln 5u\,du &= \ln 5\int u^5\,du + \int u^5\ln u\,du \\ &= \frac{u^6}{6}\ln 5 + \frac{1}{6}u^6\ln u - \frac{1}{36}u^6 + C \quad \text{(using III-13)} \\ &= \frac{u^6}{6}\ln 5u - \frac{1}{36}u^6 + C.\end{aligned}$$

13. $\frac{1}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}} + C.$
(Let $a = \sqrt{3}$ in V-24).

14. We first factor out the 9 and then use formula V-24:

$$\begin{aligned}\int \frac{dx}{9x^2+16} = \int \frac{dx}{9(x^2+16/9)} &= \frac{1}{9}\cdot\frac{1}{4/3}\arctan\left(\frac{x}{4/3}\right) + C \\ &= \frac{1}{12}\arctan\left(\frac{3x}{4}\right) + C.\end{aligned}$$

15. We first factor out the 16 and then use formula V-28 to get

$$\int \frac{dx}{\sqrt{25-16x^2}} = \int \frac{dx}{\sqrt{16(25/16 - x^2)}} = \frac{1}{4}\int \frac{dx}{\sqrt{(5/4)^2 - x^2}}$$

$$= \frac{1}{4}\arcsin\left(\frac{x}{5/4}\right) + C$$
$$= \frac{1}{4}\arcsin\left(\frac{4x}{5}\right) + C.$$

16. The integral suggests formula VI-29, but is not a perfect match because of the coefficient of 9. One way to deal with the 9 is to factor it out, so that, using formula IV-29,

$$\int \frac{dx}{\sqrt{9x^2+25}} = \int \frac{dx}{\sqrt{9(x^2+25/9)}} = \frac{1}{3}\int \frac{dx}{\sqrt{x^2+(5/3)^2}} = \frac{1}{3}\ln\left|x+\sqrt{x^2+\left(\frac{5}{3}\right)^2}\right| + C.$$

Alternatively, we can write

$$\int \frac{dx}{\sqrt{9x^2+25}} = \int \frac{dx}{\sqrt{(3x)^2+25}}.$$

We now use the substitution $w = 3x$, so that $dw = 3dx$, and the integral becomes

$$\int \frac{\frac{1}{3}dw}{\sqrt{w^2+25}} = \frac{1}{3}\ln\left|w+\sqrt{w^2+25}\right| + C = \frac{1}{3}\ln|3x+\sqrt{9x^2+25}| + C.$$

17. $\frac{5}{16}\sin 3\theta \sin 5\theta + \frac{3}{16}\cos 3\theta \cos 5\theta + C.$
(Let $a = 3, b = 5$ in II-12.)

18. $\frac{3}{16}\cos 3\theta \sin 5\theta - \frac{5}{16}\sin 3\theta \cos 5\theta + C.$
(Let $a = 3, b = 5$ in II-10.)

19. Let $m = 3$ in IV-21.

$$\int \frac{1}{\cos^3 x}\,dx = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\int \frac{1}{\cos x}\,dx$$
$$= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{4}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C \text{ by IV-22.}$$

20. Use long division to reorganize the integral:

$$\int \frac{t^2+1}{t^2-1}\,dt = \int \left(1 + \frac{2}{t^2-1}\right)\,dt = \int dt + \int \frac{2}{(t-1)(t+1)}\,dt.$$

To get this second integral, let $a = 1, b = -1$ in V-26, so

$$\int \frac{t^2+1}{t^2-1}\,dt = t + \ln|t-1| - \ln|t+1| + C.$$

21. $\frac{1}{34}e^{5x}(5\sin 3x - 3\cos 3x) + C.$
(Let $a = 5, b = 3$ in II-8.)

22. $\frac{1}{45}(7\cos 2y \sin 7y - 2\sin 2y \cos 7y) + C.$
(Let $a = 2, b = 7$ in II-11.)

23.

$$\int y^2 \sin 2y\,dy = -\frac{1}{2}y^2\cos 2y + \frac{1}{4}(2y)\sin 2y + \frac{1}{8}(2)\cos 2y + C$$
$$= -\frac{1}{2}y^2\cos 2y + \frac{1}{2}y\sin 2y + \frac{1}{4}\cos 2y + C.$$

(Use $a = 2, p(y) = y^2$ in III-15.)

24. Substitute $w = x^2, dw = 2x\,dx$. Then $\int x^3 \sin x^2\,dx = \frac{1}{2}\int w \sin w\,dw$. By III-15, we have

$$\int w \sin w\,dw = -\frac{1}{2}w\cos w + \frac{1}{2}\sin w + C = -\frac{1}{2}x^2\cos x^2 + \frac{1}{2}\sin x^2 + C.$$

25. Substitute $w = 7x,\ dw = 7\,dx$. Then use IV-21.

$$\begin{aligned}\int \frac{1}{\cos^4 7x}\,dx = \frac{1}{7}\int \frac{1}{\cos^4 w}\,dw &= \frac{1}{7}\left[\frac{1}{3}\frac{\sin w}{\cos^3 w} + \frac{2}{3}\int \frac{1}{\cos^2 w}\,dw\right]\\ &= \frac{1}{21}\frac{\sin w}{\cos^3 w} + \frac{2}{21}\left[\frac{\sin w}{\cos w} + C\right]\\ &= \frac{1}{21}\frac{\tan w}{\cos^2 w} + \frac{2}{21}\tan w + C\\ &= \frac{1}{21}\frac{\tan 7x}{\cos^2 7x} + \frac{2}{21}\tan 7x + C.\end{aligned}$$

26. Substitute $w = 3\theta,\ dw = 3\,d\theta$. Then use IV-19, letting $m = 3$.

$$\begin{aligned}\int \frac{1}{\sin^3 3\theta}\,d\theta = \frac{1}{3}\int \frac{1}{\sin^3 w}\,dw &= \frac{1}{3}\left[-\frac{1}{2}\frac{\cos w}{\sin^2 w} + \frac{1}{2}\int \frac{1}{\sin w}\,dw\right]\\ &= -\frac{1}{6}\frac{\cos w}{\sin^2 w} + \frac{1}{6}\left[\frac{1}{2}\ln\left|\frac{\cos(w)-1}{\cos(w)+1}\right| + C\right] \quad \text{by IV-20}\\ &= -\frac{1}{6}\frac{\cos 3\theta}{\sin^2 3\theta} + \frac{1}{12}\ln\left|\frac{\cos(3\theta)-1}{\cos(3\theta)+1}\right| + C.\end{aligned}$$

27. Substitute $w = 2\theta,\ dw = 2\,d\theta$. Then use IV-19, letting $m = 2$.

$$\int \frac{1}{\sin^2 2\theta}\,d\theta = \frac{1}{2}\int \frac{1}{\sin^2 w}\,dw = \frac{1}{2}(-\frac{\cos w}{\sin w}) + C = -\frac{1}{2\tan w} + C = -\frac{1}{2\tan 2\theta} + C.$$

28. Use IV-21 twice to get the exponent down to 1:

$$\begin{aligned}\int \frac{1}{\cos^5 x}\,dx &= \frac{1}{4}\frac{\sin x}{\cos^4 x} + \frac{3}{4}\int \frac{1}{\cos^3 x}\,dx\\ \int \frac{1}{\cos^3 x}\,dx &= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\int \frac{1}{\cos x}\,dx.\end{aligned}$$

Now use IV-22 to get

$$\int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + C.$$

Putting this all together gives

$$\int \frac{1}{\cos^5 x}\,dx = \frac{1}{4}\frac{\sin x}{\cos^4 x} + \frac{3}{8}\frac{\sin x}{\cos^2 x} + \frac{3}{16}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + C.$$

29.

$$\int \frac{1}{x^2+4x+3}\,dx = \int \frac{1}{(x+1)(x+3)}\,dx = \frac{1}{2}(\ln|x+1| - \ln|x+3|) + C.$$

(Let $a = -1$ and $b = -3$ in V-26).

30.

$$\int \frac{1}{x^2+4x+4}\,dx = \int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2} + C.$$

You need not use the table.

31.

$$\int \frac{dz}{z(z-3)} = -\frac{1}{3}(\ln|z| - \ln|z-3|) + C.$$

(Let $a = 0, b = 3$ in V-26.)

32.

$$\int \frac{dy}{4-y^2} = -\int \frac{dy}{(y+2)(y-2)} = -\frac{1}{4}(\ln|y-2| - \ln|y+2|) + C.$$

(Let $a = 2, b = -2$ in V-26.)

33. $\arctan(z+2) + C$.
(Substitute $w = z + 2$ and use V-24, letting $a = 1$.)

34.

$$\int \frac{1}{y^2+4y+5}\,dy = \int \frac{1}{1+(y+2)^2}\,dy = \arctan(y+2) + C.$$

(Substitute $w = y + 2$, and let $a = 1$ in V-24).

35. We use the method of IV-23 in the table. Using the Pythagorean Identity, we rewrite the integrand:

$$\sin^3 x = \left(\sin^2 x\right)\sin x = \left(1 - \cos^2 x\right)\sin x = \sin x - \cos^2 x \sin x.$$

Thus, we have

$$\begin{aligned}\int \sin^3 x\,dx &= \int \left(\sin x - \cos^2 x \sin x\right)dx \\ &= \int \sin x\,dx - \int \cos^2 x \sin x\,dx.\end{aligned}$$

The first of these new integrals can be easily found. The second can be found using the substitution $w = \cos x$ so $dw = -\sin x\,dx$. The second integral becomes

$$\begin{aligned}\int \cos^2 x \sin x\,dx &= -\int w^2 dw \\ &= -\frac{1}{3}w^3 + C \\ &= -\frac{1}{3}\cos^3 x + C,\end{aligned}$$

so the final answer is

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin x\,dx - \int \cos^2 x \sin x\,dx \\ &= -\cos x + (1/3)\cos^3 x + C.\end{aligned}$$

36. Using the advice in IV-23, since both m and n are even and since n is negative, we convert everything to cosines, since $\cos x$ is in the denominator.

$$\begin{aligned}\int \tan^4 x\,dx &= \int \frac{\sin^4 x}{\cos^4 x}\,dx \\ &= \int \frac{(1-\cos^2 x)^2}{\cos^4 x}\,dx \\ &= \int \frac{1}{\cos^4 x}\,dx - 2\int \frac{1}{\cos^2 x}\,dx + \int 1\,dx.\end{aligned}$$

By IV-21

$$\begin{aligned}\int \frac{1}{\cos^4 x}\,dx &= \frac{1}{3}\frac{\sin x}{\cos^3 x} + \frac{2}{3}\int \frac{1}{\cos^2 x}\,dx, \\ \int \frac{1}{\cos^2 x}\,dx &= \frac{\sin x}{\cos x} + C.\end{aligned}$$

Substituting back in, we get

$$\int \tan^4 x\,dx = \frac{1}{3}\frac{\sin x}{\cos^3 x} - \frac{4}{3}\frac{\sin x}{\cos x} + x + C.$$

37. Since $\cosh^2 x - \sinh^2 x = 1$ we rewrite $\sinh^3 x$ as $\sinh x \sinh^2 x = \sinh x(\cosh^2 x - 1)$. Then

$$\begin{aligned}\int \sinh^3 x \cosh^2 x\,dx &= \int \sinh x(\cosh^2 x - 1)\cosh^2 x\,dx \\ &= \int (\cosh^4 x - \cosh^2 x)\sinh x\,dx.\end{aligned}$$

Now use the substitution $w = \cosh x$, $dw = \sinh x\,dx$ to find

$$\int (w^4 - w^2)\,dw = \frac{1}{5}w^5 - \frac{1}{3}w^3 + C = \frac{1}{5}\cosh^5 x - \frac{1}{3}\cosh^3 x + C.$$

38. Since $\cosh^2 x - \sinh^2 x = 1$ we rewrite $\cosh^3 x$ as $\cosh x \cosh^2 x = \cosh x(1 + \sinh^2 x)$. This gives

$$\begin{aligned}\int \sinh^2 x \cosh^3 x\,dx &= \int \sinh^2 x(\sinh^2 x + 1)\cosh x\,dx \\ &= \int (\sinh^4 x + \sinh^2 x)\cosh x\,dx.\end{aligned}$$

Now use the substitution $w = \sinh x$, $dw = \cosh x\,dx$ to find

$$\int (w^4 - w^2)\,dw = \frac{1}{5}w^5 - \frac{1}{3}w^3 + C = \frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x + C.$$

39.

$$\begin{aligned}\int \sin^3 3\theta \cos^2 3\theta\,d\theta &= \int (\sin 3\theta)(\cos^2 3\theta)(1 - \cos^2 3\theta)\,d\theta \\ &= \int \sin 3\theta(\cos^2 3\theta - \cos^4 3\theta)\,d\theta.\end{aligned}$$

Using an extension of the tip given in rule IV-23, we let $w = \cos 3\theta,\ dw = -3\sin 3\theta\,d\theta$.

$$\begin{aligned}\int \sin 3\theta(\cos^2 3\theta - \cos^4 3\theta)\,d\theta &= -\frac{1}{3}\int (w^2 - w^4)\,dw \\ &= -\frac{1}{3}(\frac{w^3}{3} - \frac{w^5}{5}) + C \\ &= -\frac{1}{9}(\cos^3 3\theta) + \frac{1}{15}(\cos^5 3\theta) + C.\end{aligned}$$

40. If we make the substitution $w = 2z^2$ then $dw = 4z\,dz$, and the integral becomes:

$$\int ze^{2z^2}\cos(2z^2)\,dz = \frac{1}{4}\int e^w \cos w\,dw$$

Now we can use Formula 9 from the table of integrals to get:

$$\begin{aligned}\frac{1}{4}\int e^w \cos w\,dw &= \frac{1}{4}\left[\frac{1}{2}e^w(\cos w + \sin w) + C\right] \\ &= \frac{1}{8}e^w(\cos w + \sin w) + C \\ &= \frac{1}{8}e^{2z^2}(\cos 2z^2 + \sin 2z^2) + C\end{aligned}$$

41. Substitute $w = 3\alpha$, $dw = 3\,d\alpha$. Then $d\alpha = \frac{1}{3}\,dw$. We have

$$\begin{aligned}\int_{\alpha=0}^{\alpha=\frac{\pi}{12}} \sin 3\alpha\, d\alpha &= \frac{1}{3}\int_{w=0}^{w=\frac{\pi}{4}} \sin w\, dw \\ &= -\frac{1}{3}\cos w\Big|_0^{\frac{\pi}{4}} \\ &= -\frac{1}{3}\left(\frac{\sqrt{2}}{2}-1\right) = \frac{1}{3}\left(1-\frac{\sqrt{2}}{2}\right).\end{aligned}$$

42. Let $a = 5$ and $b = 6$ in II-12. Then

$$\int_{-\pi}^{\pi} \sin 5x \cos 6x\, dx = \frac{1}{11}(6\sin 5x \sin 6x + 5\cos 5x \cos 6x)\Big|_{-\pi}^{\pi} = 0.$$

This answer makes sense since $\sin 5x \cos 6x$ is odd, so its integral over any interval $-a \le x \le a$ should be 0.

43. Using III-13:

$$\begin{aligned}\int_1^2 (x-2x^3)\ln x\, dx &= \int_1^2 x\ln x\, dx - 2\int_1^2 x^3 \ln x\, dx \\ &= \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^2 - \left(\frac{1}{2}x^4\ln x - \frac{1}{8}x^4\right)\Big|_1^2 \\ &= 2\ln 2 - \frac{3}{4} - \left(8\ln 2 - \frac{15}{8}\right) \\ &= \frac{9}{8} - 6\ln 2 = -3.034.\end{aligned}$$

44. Let $a = \sqrt{3}$ in VI-30 and VI-28:

$$\int_0^1 \sqrt{3-x^2}\, dx = \left(\frac{1}{2}x\sqrt{3-x^2} + \frac{3}{2}\arcsin\frac{x}{\sqrt{3}}\right)\Big|_0^1 \approx 1.630.$$

45. $\displaystyle\int_0^1 \frac{1}{x^2+2x+1}\, dx = \int_0^1 \frac{1}{(x+1)^2}\, dx.$

We substitute $w = x+1$, so $dw = dx$. Note that when $x = 1$, we have $w = 2$, and when $x = 0$, we have $w = 1$.

$$\int_{x=0}^{x=1} \frac{1}{(x+1)^2}\, dx = \int_{w=1}^{w=2} \frac{1}{w^2}\, dw = -\frac{1}{w}\Big|_{w=1}^{w=2} = -\frac{1}{2}+1 = \frac{1}{2}.$$

46. Substitute $w = x+1$ and use V-24:

$$\begin{aligned}\int_0^1 \frac{dx}{x^2+2x+5} &= = \int_0^1 \frac{dx}{(x+1)^2+4} \\ &= \frac{1}{2}\arctan\frac{x+1}{2}\Big|_0^1 = \frac{1}{2}\arctan 1 - \frac{1}{2}\arctan\frac{1}{2} \approx 0.1609.\end{aligned}$$

47. Let $w = x^2$, $dw = 2x\,dx$. When $x = 0$, $w = 0$, and when $x = \frac{1}{\sqrt{2}}$, $w = \frac{1}{2}$. Then

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x\,dx}{\sqrt{1-x^4}} = \int_0^{\frac{1}{2}} \frac{\frac{1}{2}\,dw}{\sqrt{1-w^2}} = \frac{1}{2}\arcsin w\Big|_0^{\frac{1}{2}} = \frac{1}{2}\left(\arcsin\frac{1}{2} - \arcsin 0\right) = \frac{\pi}{12}.$$

48. Let $w = x + 2$, giving $dw = dx$. When $x = 0$, $w = 2$, and when $x = 1$, $w = 3$. Thus,

$$\int_0^1 \frac{(x+2)}{(x+2)^2+1} dx = \int_2^3 \frac{w}{w^2+1}\, dw.$$

For the last integral, we make the substitution $u = w^2 + 1$, $du = 2w\, dw$. Then, we have

$$\begin{aligned} \int_2^3 \frac{w}{w^2+1}\, dw &= \frac{1}{2} \ln |w^2+1| \Big|_2^3 \\ &= \frac{1}{2}(\ln|10| - \ln|5|) \\ &= \frac{1}{2} \ln\left(\frac{10}{5}\right) = \frac{1}{2}\ln(2) \end{aligned}$$

49. Using IV-19 with $m = 3$, followed by IV-20, we find that

$$\int_{\pi/4}^{\pi/3} \frac{dx}{\sin^3 x} = -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{4} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| \Bigg|_{\pi/4}^{\pi/3} = 0.5398.$$

50. Substitute $w = x + 3$ and then use VI-29:

$$\begin{aligned} \int_{-3}^{-1} \frac{dx}{\sqrt{x^2+6x+10}} &= \int_{-3}^{-1} \frac{dx}{\sqrt{(x+3)^2+1}} = \ln|x+3+\sqrt{x^2+6x+10}| \Big|_{-3}^{-1}. \\ &= \ln|2+\sqrt{5}| \approx 1.4436. \end{aligned}$$

Problems

51. Letting $w = 2x + 1, dw = 2\, dx, dx = 0.5\, dw$, we find that $k = 0.5, w = 2x + 1, n = 3$:

$$\int (2x+1)^3 \ln(2x+1)\, dx = \int w^3 \ln(w) 0.5\, dw = \int 0.5 w^3 \ln w\, dw.$$

52. Letting $w = 2x + 1, dw = 2\, dx, dx = 0.5\, dw$, we see that $k = -1/4, w = 2x + 1, n = 3$:

$$\begin{aligned} \int (2x+1)^3 \ln \frac{1}{\sqrt{2x+1}}\, dx &= \int w^3 \ln\left(\frac{1}{w^{1/2}}\right) 0.5\, dw \\ &= \int 0.5 w^3 \ln\left(w^{-1/2}\right)\, dw \\ &= \int 0.5 w^3 \left(-\frac{1}{2}\right) \ln w\, dw \quad \text{log property} \\ &= \int -\frac{1}{4} w^3 \ln w\, dw. \end{aligned}$$

53. We can use form (i) by writing this as

$$\int \frac{dx}{5 - \dfrac{x}{4} - \dfrac{x^2}{6}} = \int \frac{dx}{-\dfrac{1}{6}x^2 + \left(-\dfrac{1}{4}\right)x + 5},$$

where $a = -1/6, b = -1/4, c = 5$.

54. We can use form (ii) by writing this as

$$\int \frac{dx}{2x+\frac{5}{7+3x}} = \int \frac{dx}{2x+\frac{5}{7+3x}} \cdot \frac{7+3x}{7+3x} \quad \text{simplify}$$

$$= \int \frac{3x+7}{2x(3x+7)+\frac{5}{7+3x}\cdot(3x+7)}\,dx \quad \text{distribute}$$

$$= \int \frac{3x+7}{6x^2+14x+5}\,dx,$$

with $m = 3, n = 7, a = 6, b = 14, c = 5$.

55. We can use form (iii) by writing this as:

$$\int \frac{dx}{(x^2-5x+6)^3(x^2-4x+4)^2(x^2-6x+9)^2} = \int \frac{dx}{(x^2-5x+6)^3\left((x-2)(x-2)\right)^2\left((x-3)(x-3)\right)^2} \quad \text{factor}$$

$$= \int \frac{dx}{(x^2-5x+6)^3(x-2)^4(x-3)^4} \quad \text{regroup}$$

$$= \int \frac{dx}{(x^2-5x+6)^3\left((x-2)(x-3)\right)^4} \quad \text{regroup}$$

$$= \int \frac{dx}{(x^2-5x+6)^3\left(x^2-5x+6\right)^4} \quad \text{multiply out}$$

$$= \int \frac{dx}{(x^2-5x+6)^7} \quad \text{simplify,}$$

where $a = 1, b = -5, c = 6, n = 7$.

56. We see that $a = e, b = 4, \lambda = 3$: $\int \frac{e^{6x}}{4+e^{3x+1}}\,dx = \int \frac{e^{6x}}{4+e^{3x}e^1}\,dx = \int \frac{e^{2\cdot 3x}}{e\cdot e^{3x}+4}\,dx.$

57. Multiplying the numerator and denominator by e^{-4x}, we see that $a = 5, b = 4, \lambda = 2$:

$$\int \frac{e^{8x}}{4e^{4x}+5e^{6x}}\,dx = \int \frac{e^{8x}}{4e^{4x}+5e^{6x}}\cdot\frac{e^{-4x}}{e^{-4x}}\,dx$$

$$= \int \frac{e^{8x-4x}}{4e^{4x}e^{-4x}+5e^{6x}e^{-4x}}\,dx$$

$$= \int \frac{e^{4x}}{4e^{4x-4x}+5e^{6x-4x}}\,dx$$

$$= \int \frac{e^{2\cdot 2x}}{5e^{2x}+4}\,dx,$$

58. (a) Let $w = x^2, dw = 2xdx$. Then

$$\int x^5 e^{bx^2}\,dx = \int x^4 e^{bx^2} x\,dx$$

$$= \int w^2 e^{bw}\frac{1}{2}\,dw \quad \text{since } x^4 = w^2 \text{ and } xdx = \tfrac{1}{2}dw$$

$$= \int kw^2 e^{bw}\,dw. \quad \text{where } k = \tfrac{1}{2}$$

(b) Using the identity from the table, we have

$$\int kw^2 e^{bw}\,dw = ke^{bw}\left(\frac{w^2}{b}-\frac{2w}{b^2}+\frac{2}{b^3}\right)+C,$$

so, since $k = \frac{1}{2}, w = x^2$,

$$\int kw^2 e^{bw}\,dw = \frac{1}{2}e^{bx^2}\left(\frac{x^4}{b}-\frac{2x^2}{b^2}+\frac{2}{b^3}\right)+C.$$

59. Using II-10 in the integral table, if $m \neq \pm n$, then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta &= \frac{1}{n^2 - m^2}[m \cos m\theta \sin n\theta - n \sin m\theta \cos n\theta]\Big|_{-\pi}^{\pi} \\ &= \frac{1}{n^2 - m^2}[(m \cos m\pi \sin n\pi - n \sin m\pi \cos n\pi) - \\ &\quad (m \cos(-m\pi) \sin(-n\pi) - n \sin(-m\pi) \cos(-n\pi))]\end{aligned}$$

But $\sin k\pi = 0$ for all integers k, so each term reduces to 0, making the whole integral reduce to 0.

60. Using formula II-11, if $m \neq \pm n$, then

$$\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = \frac{1}{n^2 - m^2}(n \cos m\theta \sin n\theta - m \sin m\theta \cos n\theta)\Big|_{-\pi}^{\pi}.$$

We see that in the evaluation, each term will have a $\sin k\pi$ term, so the expression reduces to 0.

61. (a)

$$\begin{aligned}\frac{1}{1-0}\int_0^1 V_0 \cos(120\pi t) dt &= \frac{V_0}{120\pi}\sin(120\pi t)\Big|_0^1 \\ &= \frac{V_0}{120\pi}[\sin(120\pi) - \sin(0)] \\ &= \frac{V_0}{120\pi}[0 - 0] = 0.\end{aligned}$$

(b) Let's find the average of V^2 first.

$$\begin{aligned}\overline{V}^2 = \text{Average of } V^2 &= \frac{1}{1-0}\int_0^1 V^2 dt \\ &= \frac{1}{1-0}\int_0^1 (V_0 \cos(120\pi t))^2 dt \\ &= V_0^2 \int_0^1 \cos^2(120\pi t) dt\end{aligned}$$

Now, let $120\pi t = x$, and $dt = \dfrac{dx}{120\pi}$. So

$$\begin{aligned}\overline{V}^2 &= \frac{V_0^2}{120\pi}\int_0^{120\pi} \cos^2 x dx. \\ &= \frac{V_0^2}{120\pi}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right)\Big|_0^{120\pi} \qquad \text{II-18} \\ &= \frac{V_0^2}{120\pi}60\pi = \frac{V_0^2}{2}.\end{aligned}$$

So, the average of V^2 is $\dfrac{V_0^2}{2}$ and $\overline{V} = \sqrt{\text{average of } V^2} = \dfrac{V_0}{\sqrt{2}}$.

(c) $V_0 = \sqrt{2} \cdot \overline{V} = 110\sqrt{2} \approx 156$ volts.

62. (a) Since $R(T)$ is the rate or production, we find the total production by integrating:

$$\begin{aligned}\int_0^N R(t)\, dt &= \int_0^N (A + Be^{-t}\sin(2\pi t))\, dt \\ &= NA + B\int_0^N e^{-t}\sin(2\pi t)\, dt.\end{aligned}$$

Let $a = -1$ and $b = 2\pi$ in II-8.

$$= NA + \frac{B}{1+4\pi^2} e^{-t}(-\sin(2\pi t) - 2\pi\cos(2\pi t))\Big|_0^N.$$

Since N is an integer (so $\sin 2\pi N = 0$ and $\cos 2\pi N = 1$),

$$\int_0^N R(t)\,dt = NA + B\frac{2\pi}{1+4\pi^2}(1-e^{-N}).$$

Thus the total production is $NA + \frac{2\pi B}{1+4\pi^2}(1-e^{-N})$ over the first N years.

(b) The average production over the first N years is

$$\int_0^N \frac{R(t)\,dt}{N} = A + \frac{2\pi B}{1+4\pi^2}\left(\frac{1-e^{-N}}{N}\right).$$

(c) As $N \to \infty$, $A + \frac{2\pi B}{1+4\pi^2}\frac{1-e^{-N}}{N} \to A$, since the second term in the sum goes to 0. This is why A is called the average!

(d) When t gets large, the term $Be^{-t}\sin(2\pi t)$ gets very small. Thus, $R(t) \approx A$ for most t, so it makes sense that the average of $\int_0^N R(t)\,dt$ is A as $N \to \infty$.

(e) This model is not reasonable for long periods of time, since an oil well has finite capacity and will eventually "run dry." Thus, we cannot expect average production to be close to constant over a long period of time.

Strengthen Your Understanding

63. To find the integral, we factor the denominator

$$7 - t^2 = -(t-\sqrt{7})(t+\sqrt{7})$$

and use V-26 with $a = \sqrt{7}$, $b = -\sqrt{7}$.

64. If $a = 3$, then $x^2 + 4x + 3 = (x+1)(x+3)$, so by Formula V-26, the answer involves $\ln$ not $\arctan$. In general, the antiderivative involves an $\arctan$ only if the quadratic has no real roots.

65. To use the table, we first make the substitution $w = 2x+1$. Then $dw = 2\,dx$. So the integral becomes

$$\int e^{2x+1}\sin(2x+1)\,dx = \frac{1}{2}\int e^w \sin w\,dw + C.$$

Now using Formula II-8 we get

$$\int e^{2x+1}\sin(2x+1)\,dx = \frac{1}{4}e^{2x+1}(\sin(2x+1) - \cos(2x+1)) + C.$$

66. Formula II-12 only holds for $a \neq b$; the integral is defined.

The integral can be solved using substitution. Let $\sin x = w$. Then $\cos x\,dx = dw$. We have

$$\int \sin x\cos x\,dx = \int w\,dw = \frac{w^2}{2} + C.$$

Substituting $w = \sin x$ back we get

$$\int \sin x\cos x\,dx = \frac{\sin^2 x}{2} + C.$$

67. The table only gives formulas for $\int p(x)\sin x\,dx$, where $p(x)$ is a polynomial. Since $1/x$ is not a polynomial, the table is no help. In fact, it is known that $\sin x/x$ does not have an elementary function as an antiderivative.

68. To evaluate $\int 1/\sqrt{2x-x^2}\,dx$, we must first complete the square for $2x - x^2$ to get $1-(x-1)^2$. Then we can substitute $w = x-1$ and use Formula V-28 in the table.

69. $\int 1/\sin^4 x\,dx$ can be evaluated by using Formula IV-19 twice.

70. True. Rewrite $\sin^7\theta = \sin\theta\sin^6\theta = \sin\theta(1-\cos^2\theta)^3$. Expanding, substituting $w = \cos\theta, dw = -\sin\theta\, d\theta$, and integrating gives a polynomial in w, which is a polynomial in $\cos\theta$.

71. False. Completing the square gives

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \arctan(x+2) + C.$$

72. False. Factoring gives

$$\int \frac{dx}{x^2+4x-5} = \int \frac{dx}{(x+5)(x-1)} = \frac{1}{6}\int \left(\frac{1}{x-1} - \frac{1}{x+5}\right) dx = \frac{1}{6}(\ln|x-1| - \ln|x+5|) + C.$$

73. True. Let $w = \ln x, dw = x^{-1}\,dx$. Then

$$\int x^{-1}((\ln x)^2 + (\ln x)^3)\,dx = \int (w^2+w^3)\,dw = \frac{w^3}{3} + \frac{w^4}{4} + C = \frac{(\ln x)^3}{3} + \frac{(\ln x)^4}{4} + C.$$

Solutions for Section 7.4

Exercises

1. Since $6x + x^2 = x(6+x)$, we take

$$\frac{x+1}{6x+x^2} = \frac{A}{x} + \frac{B}{6+x}.$$

So,

$$x+1 = A(6+x) + Bx$$
$$x+1 = (A+B)x + 6A,$$

giving

$$A + B = 1$$
$$6A = 1.$$

Thus $A = 1/6$, and $B = 5/6$ so

$$\frac{x+1}{6x+x^2} = \frac{1/6}{x} + \frac{5/6}{6+x}.$$

2. Since $25 - x^2 = (5-x)(5+x)$, we take

$$\frac{20}{25-x^2} = \frac{A}{5-x} + \frac{B}{5+x}.$$

So,

$$20 = A(5+x) + B(5-x)$$
$$20 = (A-B)x + 5A + 5B,$$

giving

$$A - B = 0$$
$$5A + 5B = 20.$$

Thus $A = B = 2$ and

$$\frac{20}{25-x^2} = \frac{2}{5-x} + \frac{2}{5+x}.$$

3. Since $w^4 - w^3 = w^3(w-1)$, we have

$$\frac{1}{w^4 - w^3} = \frac{A}{w-1} + \frac{B}{w} + \frac{C}{w^2} + \frac{D}{w^3}.$$

Thus,

$$\begin{aligned} 1 &= Aw^3 + B(w-1)w^2 + C(w-1)w + D(w-1) \\ 1 &= (A+B)w^3 + (-B+C)w^2 + (-C+D)w + (-D), \end{aligned}$$

giving

$$\begin{aligned} A + B &= 0 \\ -B + C &= 0 \\ -C + D &= 0 \\ -D &= 1. \end{aligned}$$

Thus $A = 1$, $B = -1$, $C = -1$ and $D = -1$ so

$$\frac{1}{w^4 - w^3} = \frac{1}{w-1} - \frac{1}{w} - \frac{1}{w^2} - \frac{1}{w^3}.$$

4. Since $y^3 - y^2 + y - 1 = (y-1)(y^2+1)$, we take

$$\frac{2y}{y^3 - y^2 + y - 1} = \frac{A}{y-1} + \frac{By + C}{y^2 + 1}$$

So,

$$\begin{aligned} 2y &= A(y^2+1) + (By+C)(y-1) \\ 2y &= (A+B)y^2 + (C-B)y + A - C, \end{aligned}$$

giving

$$\begin{aligned} A + B &= 0 \\ -B + C &= 2 \\ A - C &= 0. \end{aligned}$$

Thus $A = C = 1$, $B = -1$ so

$$\frac{2y}{y^3 - y^2 + y - 1} = \frac{1}{y-1} + \frac{1-y}{y^2+1}.$$

5. Since $y^3 - 4y = y(y-2)(y+2)$, we take

$$\frac{8}{y^3 - 4y} = \frac{A}{y} + \frac{B}{y-2} + \frac{C}{y+2}.$$

So,

$$\begin{aligned} 8 &= A(y-2)(y+2) + By(y+2) + Cy(y-2) \\ 8 &= (A+B+C)y^2 + (2B-2C)y - 4A, \end{aligned}$$

giving

$$\begin{aligned} A + B + C &= 0 \\ 2B - 2C &= 0 \\ -4A &= 8. \end{aligned}$$

Thus $A = -2$, $B = C = 1$ so

$$\frac{8}{y^3 - 4y} = \frac{-2}{y} + \frac{1}{y-2} + \frac{1}{y+2}.$$

6. Since $s^2+3s+2=(s+2)(s+1)$, we have

$$\frac{2(1+s)}{s(s^2+3s+2)} = \frac{2(1+s)}{s(s+2)(s+1)} = \frac{2}{s(s+2)},$$

so we take

$$\frac{2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}.$$

Thus,

$$\begin{aligned} 2 &= A(s+2) + Bs \\ 2 &= (A+B)s + 2A, \end{aligned}$$

giving

$$\begin{aligned} A+B &= 0 \\ 2A &= 2. \end{aligned}$$

Thus $A=1$ and $B=-1$ and

$$\frac{2(1+s)}{s(s^2+3s+2)} = \frac{1}{s} - \frac{1}{s+2}.$$

7. Since $s^4-1=(s^2-1)(s^2+1)=(s-1)(s+1)(s^2+1)$, we have

$$\frac{2}{s^4-1} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}.$$

Thus,

$$\begin{aligned} 2 &= A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1) \\ 2 &= (A+B+C)s^3 + (A-B+D)s^2 + (A+B-C)s + (A-B-D), \end{aligned}$$

giving

$$\begin{aligned} A+B+C &= 0 \\ A-B+D &= 0 \\ A+B-C &= 0 \\ A-B-D &= 2. \end{aligned}$$

From the first and third equations we find $A+B=0$ and $C=0$. From the second and fourth we find $A-B=1$ and $D=-1$. Thus $A=1/2$ and $B=-1/2$ and

$$\frac{2}{s^4-1} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2+1}.$$

8. Using the result of Problem 1, we have

$$\int \frac{x+1}{6x+x^2}\,dx = \int \frac{1/6}{x}\,dx + \int \frac{5/6}{6+x}\,dx = \frac{1}{6}\left(\ln|x| + 5\ln|6+x|\right) + C.$$

9. Using the result of Problem 2, we have

$$\int \frac{20}{25-x^2}\,dx = \int \frac{2}{5-x}\,dx + \int \frac{2}{5+x}\,dx = -2\ln|5-x| + 2\ln|5+x| + C.$$

10. Using the result of Exercise 3, we have

$$\int \frac{1}{w^4-w^3}\,dw = \int \left(\frac{1}{w-1} - \frac{1}{w} - \frac{1}{w^2} - \frac{1}{w^3}\right) dw = \ln|w-1| - \ln|w| + \frac{1}{w} + \frac{1}{2w^2} + C.$$

11. Using the result of Problem 4, we have

$$\int \frac{2y}{y^3-y^2+y-1}\,dy = \int \frac{1}{y-1}\,dy + \int \frac{1-y}{y^2+1}\,dy = \ln|y-1| + \arctan y - \frac{1}{2}\ln\left|y^2+1\right| + C.$$

12. Using the result of Problem 5, we have

$$\int \frac{8}{y^3-4y}\,dy = \int \frac{-2}{y}\,dy + \int \frac{1}{y-2}\,dy + \int \frac{1}{y+2}\,dy = -2\ln|y| + \ln|y-2| + \ln|y+2| + C.$$

13. Using the result of Exercise 6, we have

$$\int \frac{2(1+s)}{s(s^2+3s+2)}\,ds = \int \left(\frac{1}{s} - \frac{1}{s+2}\right)\,ds = \ln|s| - \ln|s+2| + C.$$

14. Using the result of Exercise 7, we have

$$\int \frac{2}{s^4-1}\,ds = \int \left(\frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2+1}\right)\,ds = \frac{1}{2}\ln|s-1| - \frac{1}{2}\ln|s+1| - \arctan s + C.$$

15. We let

$$\frac{3x^2-8x+1}{x^3-4x^2+x+6} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

giving

$$\begin{aligned} 3x^2-8x+1 &= A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1) \\ 3x^2-8x+1 &= (A+B+C)x^2 - (2A+5B+C)x - 3A + 6B - 2C \end{aligned}$$

so

$$\begin{aligned} A+B+C &= 3 \\ -2A-5B-C &= -8 \\ -3A+6B-2C &= 1. \end{aligned}$$

Thus, $A=B=C=1$, so

$$\int \frac{3x^2-8x+1}{x^3-4x^2+x+6}\,dx = \int \frac{dx}{x-2} + \int \frac{dx}{x+1} + \int \frac{dx}{x-3} = \ln|x-2| + \ln|x+1| + \ln|x-3| + K.$$

We use K as the constant of integration, since we already used C in the problem.

16. We let

$$\frac{1}{x^3-x^2} = \frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

giving

$$\begin{aligned} 1 &= Ax(x-1) + B(x-1) + Cx^2 \\ 1 &= (A+C)x^2 + (B-A)x - B \end{aligned}$$

so

$$\begin{aligned} A+C &= 0 \\ B-A &= 0 \\ -B &= 1. \end{aligned}$$

Thus, $A=B=-1$, $C=1$, so

$$\int \frac{dx}{x^3-x^2} = -\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x-1} = -\ln|x| + x^{-1} + \ln|x-1| + K.$$

We use K as the constant of integration, since we already used C in the problem.

17. We let

$$\frac{10x+2}{x^3-5x^2+x-5} = \frac{10x+2}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$

giving

$$10x+2 = A(x^2+1) + (Bx+C)(x-5)$$
$$10x+2 = (A+B)x^2 + (C-5B)x + A - 5C$$

so

$$A+B = 0$$
$$C-5B = 10$$
$$A-5C = 2.$$

Thus, $A = 2$, $B = -2$, $C = 0$, so

$$\int \frac{10x+2}{x^3-5x^2+x-5}\,dx = \int \frac{2}{x-5}\,dx - \int \frac{2x}{x^2+1}\,dx = 2\ln|x-5| - \ln\left|x^2+1\right| + K.$$

18. Division gives

$$\frac{x^4+12x^3+15x^2+25x+11}{x^3+12x^2+11x} = x + \frac{4x^2+25x+11}{x^3+12x^2+11x}.$$

Since $x^3+12x^2+11x = x(x+1)(x+11)$, we write

$$\frac{4x^2+25x+11}{x^3+12x^2+11x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+11}$$

giving

$$4x^2+25x+11 = A(x+1)(x+11) + Bx(x+11) + Cx(x+1)$$
$$4x^2+25x+11 = (A+B+C)x^2 + (12A+11B+C)x + 11A$$

so

$$A+B+C = 4$$
$$12A+11B+C = 25$$
$$11A = 11.$$

Thus, $A = B = 1$, $C = 2$ so

$$\int \frac{x^4+12x^3+15x^2+25x+11}{x^3+12x^2+11x}\,dx = \int x\,dx + \int \frac{dx}{x} + \int \frac{dx}{x+1} + \int \frac{2dx}{x+11}$$
$$= \frac{x^2}{2} + \ln|x| + \ln|x+1| + 2\ln|x+11| + K.$$

We use K as the constant of integration, since we already used C in the problem.

19. Division gives

$$\frac{x^4+3x^3+2x^2+1}{x^2+3x+2} = x^2 + \frac{1}{x^2+3x+2}.$$

Since $x^2+3x+2 = (x+1)(x+2)$, we write

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

giving

$$1 = A(x+2) + B(x+1)$$
$$1 = (A+B)x + 2A + B$$

so

$$A + B = 0$$
$$2A + B = 1.$$

Thus, $A = 1$, $B = -1$ so

$$\begin{aligned}\int \frac{x^4+3x^3+2x^2+1}{x^2+3x+2}\,dx &= \int x^2\,dx + \int \frac{dx}{x+1} - \int \frac{dx}{x+2} \\ &= \frac{x^3}{3} + \ln|x+1| - \ln|x+2| + C.\end{aligned}$$

20. Since $x = (3/2)\sin t$, we have $dx = (3/2)\cos t\,dt$. Substituting into the integral gives

$$\int \frac{1}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{9-9\sin^2 t}}\left(\frac{3}{2}\cos t\right)dt = \int \frac{1}{2}dt = \frac{1}{2}t + C = \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + C.$$

21. Since $x = \sin t + 2$, we have

$$4x - 3 - x^2 = 4(\sin t + 2) - 3 - (\sin t + 2)^2 = 1 - \sin^2 t = \cos^2 t$$

and $dx = \cos t\,dt$, so substitution gives

$$\int \frac{1}{\sqrt{4x-3-x^2}} = \int \frac{1}{\sqrt{\cos^2 t}}\cos t\,dt = \int dt = t + C = \arcsin(x-2) + C.$$

22. Completing the square gives $x^2 + 4x + 5 = 1 + (x+2)^2$. Since $x + 2 = \tan t$ and $dx = (1/\cos^2 t)dt$, we have

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{1+\tan^2 t}\cdot\frac{1}{\cos^2 t}dt = \int dt = t + C = \arctan(x+2) + C.$$

23. (a) Yes, use $x = 3\sin\theta$.
(b) No; better to substitute $w = 9 - x^2$, so $dw = -2x\,dx$.

24. (a) Substitute $w = x^2 + 10$, so $dw = 2x\,dx$.
(b) Substitute $x = \sqrt{10}\tan\theta$.

Problems

25. We have

$$\begin{aligned} w &= (3x+2)(x-1) \\ &= 3x^2 - x - 2 \\ \text{so}\quad dw &= (6x-1)\,dx \\ \text{which means}\quad (12x-2)\,dx &= 2\,dw, \\ \text{giving}\quad \int \frac{12x-2}{(3x+2)(x-1)}\,dx &= 2\int \frac{dw}{w}, \end{aligned}$$

so $w = (3x+2)(x-1) = 3x^2 - x - 2$ and $k = 2$.

26. Using partial fractions, we have

$$\begin{aligned} \frac{12x-2}{(3x+2)(x-1)} &= \frac{A}{3x+2} + \frac{B}{x-1} \\ 12x - 2 &= \left(\frac{A}{3x+2} + \frac{B}{x-1}\right)(3x+2)(x-1) && \text{multiply by denominator} \\ 12x - 2 &= A(x-1) + B(3x+2) && \text{simplify} \\ &= (A+3B)x + 2B - A && \text{regroup} \\ \text{so}\quad A + 3B &= 12 && \text{matching coefficients of } x \end{aligned}$$

$$\begin{aligned}
\text{and} \quad 2B - A &= -2 && \text{matching constants}\\
\text{giving} \quad A + 3B + (2B - A) &= 12 + (-2) && \text{adding above two equations}\\
5B &= 10\\
B &= 2\\
\text{which means} \quad A + 3 \cdot 2 &= 12 && \text{letting } B = 2 \text{ in } A + 3B = 12\\
A &= 6
\end{aligned}$$

$$\text{yielding} \int \frac{12x-2}{(3x+2)(x-1)}\,dx = \int \frac{6}{3x+2} + \int \frac{2}{x-1}\,dx.$$

Checking our answer, we see that:

$$\begin{aligned}
\frac{6}{3x+2} + \frac{2}{x-1} &= \frac{6}{3x+2} \cdot \frac{x-1}{x-1} + \frac{2}{x-1} \cdot \frac{3x+2}{3x+2}\\
&= \frac{6(x-1) + 2(3x+2)}{(3x+2)(x-1)}\\
&= \frac{6x - 6 + 6x + 4}{(3x+2)(x-1)}\\
&= \frac{12x-2}{(3x+2)(x-1)},
\end{aligned}$$

as required.

27. Focusing on the integrand, we have:

$$\begin{aligned}
\frac{2x+9}{(3x+5)(4-5x)} &= \frac{2x+9}{3\left(x+\frac{5}{3}\right)(-5)\left(x-\frac{4}{5}\right)}\\
&= \frac{2x+9}{-15\left(x+\frac{5}{3}\right)\left(x-\frac{4}{5}\right)}\\
&= \frac{-\frac{2}{15} \cdot x + \left(-\frac{3}{5}\right)}{\left(x+\frac{5}{3}\right)\left(x-\frac{4}{5}\right)}.
\end{aligned}$$

So

$$\int \frac{2x+9}{(3x-5)(4-5x)}\,dx = \int \frac{\overbrace{-\frac{2}{15}}^{c} \cdot x + \overbrace{\left(-\frac{3}{5}\right)}^{d}}{\underbrace{\left(x - \left(-\frac{5}{3}\right)\right)}_{x-a}\underbrace{\left(x - \frac{4}{5}\right)}_{x-b}}\,dx.$$

We see that $a = -5/3, b = 4/5, c = -2/15, d = -3/5$.

28. We have:

$$\begin{aligned}
\sqrt{12 - 4x^2} &= \sqrt{4\,(3 - x^2)}\\
&= 2\sqrt{\left(\sqrt{3}\right)^2 - x^2},\\
\text{so} \quad \int \frac{dx}{\sqrt{12 - 4x^2}} &= \int \frac{dx}{2\sqrt{\left(\sqrt{3}\right)^2 - x^2}}\\
&= \int \frac{0.5\,dx}{\sqrt{\left(\sqrt{3}\right)^2 - x^2}}.
\end{aligned}$$

Thus, $a = \sqrt{3}, k = 0.5$.

29. Taking the hint, we see that

$$\begin{aligned} e^{2x} - 4e^x + 3 &= (e^x)^2 - 4e^x + 3 \\ &= w^2 - 4w + 3 \qquad \text{letting } w = e^x \\ &= (w-3)(w-1). \quad \text{factoring} \end{aligned}$$

Notice that $dw = e^x dx$ and that the numerator of the integrand can be written $2e^x + 1 = 2w + 1$. Using the method of partial fractions, we have:

$$\begin{aligned} \frac{2w+1}{(w-3)(w-1)} &= \frac{A}{w-3} + \frac{B}{w-1} \\ 2w+1 &= A(w-1) + B(w-3) \qquad \text{multiply through by } (w-3)(w-1) \\ &= (A+B)w + (-A-3B). \quad \text{regroup} \end{aligned}$$

Identifying the coefficients of w and the constant terms on both sides, we have:

$$\begin{aligned} A + B &= 2 \qquad \text{so } A = 2 - B \\ \text{and} \quad -A - 3B &= 1 \\ \text{so} \quad -(2-B) - 3B &= 1 \qquad \text{since } A = 2 - B \\ -2 - 2B &= 1 \\ \text{giving} \quad B &= -\frac{3}{2} \\ \text{and} \quad A &= 2 - B = \frac{7}{2}. \end{aligned}$$

This means $\dfrac{2w+1}{(w-3)(w-1)} = \dfrac{7/2}{w-3} + \dfrac{-3/2}{w-1}$, so:

$$\begin{aligned} \int_0^{\ln 7} \frac{2e^x+1}{e^{2x} - 4e^x + 3} e^x\, dx &= \int_{x=0}^{x=\ln 7} \frac{2w+1}{(w-3)(w-1)}\, dw \\ &= \int_{w=1}^{w=7} \frac{2w+1}{(w-3)(w-1)}\, dw \qquad \text{since } w = e^x \\ &= \int_1^7 \left(\frac{7/2}{w-3} + \frac{-3/2}{w-1} \right) dw. \end{aligned}$$

Thus,

$$A = \frac{7}{2}, B = -\frac{3}{2}, w = e^x, dw = e^x dx, r = 1, s = 7.$$

30. (a) We have

$$\frac{3x+6}{x^2+3x} = \frac{2(x+3)}{x(x+3)} + \frac{x}{x(x+3)} = \frac{2}{x} + \frac{1}{x+3}.$$

Thus

$$\int \frac{3x+6}{x^2+3x}\, dx = \int \left(\frac{2}{x} + \frac{1}{x+3} \right) dx = 2\ln|x| + \ln|x+3| + C.$$

(b) Let $a = 0, b = -3, c = 3$, and $d = 6$ in V-27.

$$\begin{aligned} \int \frac{3x+6}{x^2+3x}\, dx &= \int \frac{3x+6}{x(x+3)}\, dx \\ &= \frac{1}{3}(6\ln|x| + 3\ln|x+3|) + C = 2\ln|x| + \ln|x+3| + C. \end{aligned}$$

31. Since $x^2 + 2x + 2 = (x+1)^2 + 1$, we have

$$\int \frac{1}{x^2+2x+2}\, dx = \int \frac{1}{(x+1)^2+1} dx.$$

Substitute $x + 1 = \tan\theta$, so $x = (\tan\theta) - 1$.

32. Since $x^2 + 6x + 9$ is a perfect square, we write

$$\int \frac{1}{x^2+6x+25}dx = \int \frac{1}{(x^2+6x+9)+16}dx = \int \frac{1}{(x+3)^2+16}dx.$$

We use the trigonometric substitution $x + 3 = 4\tan\theta$, so $x = 4\tan\theta - 3$.

33. Since $y^2 + 3y + 3 = (y + 3/2)^2 + (3 - 9/4) = (y + 3/2)^2 + 3/4$, we have

$$\int \frac{dy}{y^2+3y+3} = \int \frac{dy}{(y+3/2)^2+3/4}.$$

Substitute $y + 3/2 = \tan\theta$, so $y = (\tan\theta) - 3/2$.

34. Since $x^2 + 2x + 2 = (x + 1)^2 + 1$, we have

$$\int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{x+1}{(x+1)^2+1}dx.$$

Substitute $w = (x + 1)^2$, so $dw = 2(x + 1)\,dx$.

This integral can also be calculated without completing the square, by substituting $w = x^2 + 2x + 2$, so $dw = 2(x + 1)\,dx$.

35. Since $2z - z^2 = 1 - (z - 1)^2$, we have

$$\int \frac{4}{\sqrt{2z-z^2}}\,dz = 4\int \frac{1}{\sqrt{1-(z-1)^2}}\,dz.$$

Substitute $z - 1 = \sin\theta$, so $z = (\sin\theta) + 1$.

36. Since $2z - z^2 = 1 - (z - 1)^2$, we have

$$\int \frac{z-1}{\sqrt{2z-z^2}}\,dz = \int \frac{z-1}{\sqrt{1-(z-1)^2}}\,dz.$$

Substitute $w = 1 - (z - 1)^2$, so $dw = -2(z - 1)\,dz$.

37. Since $t^2 + 4t + 7 = (t + 2)^2 + 3$, we have

$$\int (t+2)\sin(t^2+4t+7)\,dt = \int (t+2)\sin((t+2)^2+3)\,dt.$$

Substitute $w = (t + 2)^2 + 3$, so $dw = 2(t + 2)\,dt$.

This integral can also be computed without completing the square, by substituting $w = t^2 + 4t + 7$, so $dw = (2t + 4)\,dt$.

38. Since $\theta^2 - 4\theta = (\theta - 2)^2 - 4$, we have

$$\int (2-\theta)\cos(\theta^2-4\theta)\,d\theta = \int -(\theta-2)\cos((\theta-2)^2-4)\,d\theta.$$

Substitute $w = (\theta - 2)^2 - 4$, so $dw = 2(\theta - 2)\,d\theta$.

This integral can also be computed without completing the square, by substituting $w = \theta^2 - 4\theta$, so $dw = (2\theta - 4)\,d\theta$.

39. We write

$$\frac{1}{(x-5)(x-3)} = \frac{A}{x-5} + \frac{B}{x-3},$$

giving

$$1 = A(x-3) + B(x-5)$$
$$1 = (A+B)x - (3A+5B)$$

so

$$A + B = 0$$
$$-3A - 5B = 1.$$

Thus, $A = 1/2$, $B = -1/2$, so

$$\int \frac{1}{(x-5)(x-3)}\,dx = \int \frac{1/2}{x-5}\,dx - \int \frac{1/2}{x-3}\,dx = \frac{1}{2}\ln|x-5| - \frac{1}{2}\ln|x-3| + C.$$

40. We write

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3},$$

giving

$$\begin{aligned} 1 &= A(x+3) + B(x+2) \\ 1 &= (A+B)x + (3A+2B) \end{aligned}$$

so

$$\begin{aligned} A + B &= 0 \\ 3A + 2B &= 1. \end{aligned}$$

Thus, $A = 1$, $B = -1$, so

$$\int \frac{1}{(x+2)(x+3)}\,dx = \int \frac{1}{x+2}\,dx - \int \frac{1}{x+3}\,dx = \ln|x+2| - \ln|x+3| + C.$$

41. We write

$$\frac{1}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2},$$

giving

$$\begin{aligned} 1 &= A(x-2) + B(x+7) \\ 1 &= (A+B)x + (-2A+7B) \end{aligned}$$

so

$$\begin{aligned} A + B &= 0 \\ -2A + 7B &= 1. \end{aligned}$$

Thus, $A = -1/9$, $B = 1/9$, so

$$\int \frac{1}{(x+7)(x-2)}\,dx = -\int \frac{1/9}{x+7}\,dx + \int \frac{1/9}{x-2}\,dx = -\frac{1}{9}\ln|x+7| + \frac{1}{9}\ln|x-2| + C.$$

42. The denominator $x^2 - 3x + 2$ can be factored as $(x-1)(x-2)$. Splitting the integrand into partial fractions with denominators $(x-1)$ and $(x-2)$, we have

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Multiplying by $(x-1)(x-2)$ gives the identity

$$x = A(x-2) + B(x-1)$$

so

$$x = (A+B)x - 2A - B.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} -2A - B &= 0 \\ A + B &= 1. \end{aligned}$$

Solving these equations gives $A = -1$, $B = 2$ and the integral becomes

$$\int \frac{x}{x^2-3x+2}dx = -\int \frac{1}{x-1}dx + 2\int \frac{1}{x-2}dx = -\ln|x-1| + 2\ln|x-2| + C.$$

43. This can be done by formula V–26 in the integral table or by partial fractions

$$\int \frac{dz}{z^2+z} = \int \frac{dz}{z(z+1)} = \int \left(\frac{1}{z} - \frac{1}{z+1}\right) dz = \ln|z| - \ln|z+1| + C.$$

Check:

$$\frac{d}{dz}(\ln|z| - \ln|z+1| + C) = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z^2+z}.$$

44. We know $x^2 + 5x + 4 = (x+1)(x+4)$, so we can use V-26 of the integral table with $a = -1$ and $b = -4$ to write

$$\int \frac{dx}{x^2+5x+4} = \frac{1}{3}(\ln|x+1| - \ln|x+4|) + C.$$

45. We use partial fractions and write

$$\frac{1}{3P-3P^2} = \frac{A}{3P} + \frac{B}{1-P},$$

multiply through by $3P(1-P)$, and then solve for A and B, getting $A = 1$ and $B = 1/3$. So

$$\begin{aligned}\int \frac{dP}{3P-3P^2} &= \int \left(\frac{1}{3P} + \frac{1}{3(1-P)}\right) dP = \frac{1}{3}\int \frac{dP}{P} + \frac{1}{3}\int \frac{dP}{1-P} \\ &= \frac{1}{3}\ln|P| - \frac{1}{3}\ln|1-P| + C = \frac{1}{3}\ln\left|\frac{P}{1-P}\right| + C.\end{aligned}$$

46. Using partial fractions, we have:

$$\frac{3x+1}{x^2-3x+2} = \frac{3x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Multiplying by $(x-1)$ and $(x-2)$, this becomes

$$\begin{aligned}3x+1 &= A(x-2) + B(x-1) \\ &= (A+B)x - 2A - B\end{aligned}$$

which produces the system of equations

$$\begin{cases} A + B = 3 \\ -2A - B = 1. \end{cases}$$

Solving this system yields $A = -4$ and $B = 7$. So,

$$\begin{aligned}\int \frac{3x+1}{x^2-3x+2}dx &= \int \left(-\frac{4}{x-1} + \frac{7}{x-2}\right) dx \\ &= -4\int \frac{dx}{x-1} + 7\int \frac{dx}{x-2} \\ &= -4\ln|x-1| + 7\ln|x-2| + C.\end{aligned}$$

47. Since $2y^2 + 3y + 1 = (2y+1)(y+1)$, we write

$$\frac{y+2}{2y^2+3y+1} = \frac{A}{2y+1} + \frac{B}{y+1},$$

giving

$$\begin{aligned}y+2 &= A(y+1) + B(2y+1) \\ y+2 &= (A+2B)y + A + B\end{aligned}$$

so

$$\begin{aligned}A + 2B &= 1 \\ A + B &= 2.\end{aligned}$$

Thus, $A = 3$, $B = -1$, so

$$\int \frac{y+2}{2y^2+3y+1}\,dy = \int \frac{3}{2y+1}\,dy - \int \frac{1}{y+1}\,dy = \frac{3}{2}\ln|2y+1| - \ln|y+1| + C.$$

48. Since $x^3 + x = x(x^2+1)$ cannot be factored further, we write

$$\frac{x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiplying by $x(x^2+1)$ gives

$$x+1 = A(x^2+1) + (Bx+C)x$$
$$x+1 = (A+B)x^2 + Cx + A,$$

so

$$A+B=0$$
$$C=1$$
$$A=1.$$

Thus, $A = C = 1$, $B = -1$, and we have

$$\int \frac{x+1}{x^3+x}\,dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+1}\right) = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$$
$$= \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + \arctan x + K.$$

49. Since $x^2 + x^4 = x^2(1+x^2)$ cannot be factored further, we write

$$\frac{x-2}{x^2+x^4} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}.$$

Multiplying by $x^2(1+x^2)$ gives

$$x-2 = Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2$$
$$x-2 = (A+C)x^3 + (B+D)x^2 + Ax + B,$$

so

$$A+C=0$$
$$B+D=0$$
$$A=1$$
$$B=-2.$$

Thus, $A = 1$, $B = -2$, $C = -1$, $D = 2$, and we have

$$\int \frac{x-2}{x^2+x^4}\,dx = \int \left(\frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}\right)dx = \int \frac{dx}{x} - 2\int \frac{dx}{x^2} - \int \frac{x\,dx}{1+x^2} + 2\int \frac{dx}{1+x^2}$$
$$= \ln|x| + \frac{2}{x} - \frac{1}{2}\ln\left|1+x^2\right| + 2\arctan x + K.$$

We use K as the constant of integration, since we already used C in the problem.

50. Let $y = 5\tan\theta$ so $dy = (5/\cos^2\theta)\,d\theta$. Since $1 + \tan^2\theta = 1/\cos^2\theta$, we have

$$\int \frac{y^2}{25+y^2}\,dy = \int \frac{25\tan^2\theta}{25(1+\tan^2\theta)} \cdot \frac{5}{\cos^2\theta}\,d\theta = 5\int \tan^2\theta\,d\theta.$$

Using $1 + \tan^2\theta = 1/\cos^2\theta$ again gives

$$\int \frac{y^2}{25+y^2}\,dy = 5\int \tan^2\theta\,d\theta = 5\int \left(\frac{1}{\cos^2\theta} - 1\right)d\theta = 5\tan\theta - 5\theta + C.$$

In addition, since $\theta = \arctan(y/5)$, we get

$$\int \frac{y^2}{25+y^2}\,dy = y - 5\arctan\left(\frac{y}{5}\right) + C.$$

51. Since $(4-z^2)^{3/2} = (\sqrt{4-z^2})^3$, we substitute $z = 2\sin\theta$, so $dz = 2\cos\theta\, d\theta$. We get

$$\int \frac{dz}{(4-z^2)^{3/2}} = \int \frac{2\cos\theta\, d\theta}{(4-4\sin^2\theta)^{3/2}} = \int \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\tan\theta + C$$

Since $\sin\theta = z/2$, we have $\cos\theta = \sqrt{1-(z/2)^2} = (\sqrt{4-z^2})/2$, so

$$\int \frac{dz}{(4-z^2)^{3/2}} = \frac{1}{4}\tan\theta + C = \frac{1}{4}\frac{\sin\theta}{\cos\theta} + C = \frac{1}{4}\frac{z/2}{(\sqrt{4-z^2})/2} + C = \frac{z}{4\sqrt{4-z^2}} + C$$

52. We have

$$\frac{10}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}.$$

Thus,

$$10 = A(s^2+1) + (Bs+C)(s+2)$$
$$10 = (A+B)s^2 + (2B+C)s + (A+2C),$$

giving

$$A + B = 0$$
$$2B + C = 0$$
$$A + 2C = 10.$$

Thus, from the first two equations we have $C = -2B = 2A$, which, when used in the third, gives $5A = 10$, so that $A = 2$, $B = -2$, and $C = 4$. We now have

$$\frac{10}{(s+2)(s^2+1)} = \frac{2}{s+2} + \frac{-2s+4}{s^2+1} = \frac{2}{s+2} - \frac{2s}{s^2+1} + \frac{4}{s^2+1},$$

so

$$\int \frac{10}{(s+2)(s^2+1)}\, ds = \int \left(\frac{2}{s+2} - \frac{2s}{s^2+1} + \frac{4}{s^2+1}\right) ds = 2\ln|s+2| - \ln\left|s^2+1\right| + 4\arctan s + K.$$

We use K as the constant of integration, since we already used C in the problem.

53. Completing the square, we get

$$x^2 + 4x + 13 = (x+2)^2 + 9.$$

We use the substitution $x + 2 = 3\tan t$, then $dx = (3/\cos^2 t)\, dt$. Since $\tan^2 t + 1 = 1/\cos^2 t$, the integral becomes

$$\int \frac{1}{(x+2)^2+9}\, dx = \int \frac{1}{9\tan^2 t + 9} \cdot \frac{3}{\cos^2 t}\, dt = \int \frac{1}{3}\, dt = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C.$$

54. Using the substitution $w = e^x$, we get $dw = e^x dx$, so we have

$$\int \frac{e^x}{(e^x-1)(e^x+2)}\, dx = \int \frac{dw}{(w-1)(w+2)}.$$

But

$$\frac{1}{(w-1)(w+2)} = \frac{1}{3}\left(\frac{1}{w-1} - \frac{1}{w+2}\right),$$

so

$$\int \frac{e^x}{(e^x-1)(e^x+2)}\, dx = \int \frac{1}{3}\left(\frac{1}{w-1} - \frac{1}{w+2}\right) dw$$
$$= \frac{1}{3}(\ln|w-1| - \ln|w+2|) + C$$
$$= \frac{1}{3}(\ln|e^x-1| - \ln|e^x+2|) + C.$$

55. Let $x = \tan\theta$ so $dx = (1/\cos^2\theta)d\theta$. Since $\sqrt{1+\tan^2\theta} = 1/\cos\theta$, we have

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int \frac{1/\cos^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}\,d\theta = \int \frac{\cos\theta}{\tan^2\theta\cos^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

The last integral can be evaluated by guess-and-check or by substituting $w = \sin\theta$. The result is

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\frac{1}{\sin\theta} + C.$$

We need to know $\sin\theta$ in terms of x. From the triangle in Figure 7.4, which shows $x = \tan\theta$, we see that $\sin\theta = x/\sqrt{1+x^2}$. Thus

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{1+x^2}}{x} + C.$$

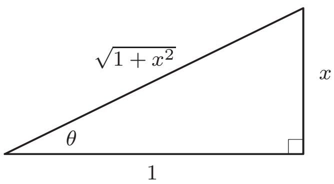


Figure 7.4: In this triangle, $\tan\theta = x$

56. Let $x = 3\sin\theta$ so $dx = 3\cos\theta\,d\theta$, giving

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}3\cos\theta\,d\theta = \int \frac{(9\sin^2\theta)(3\cos\theta)}{3\cos\theta}\,d\theta = 9\int \sin^2\theta\,d\theta.$$

From formula IV-18, we get

$$\int \sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C.$$

We have $\sin\theta = x/3$ and $\theta = \arcsin(x/3)$. From the triangle in Figure 7.5, which shows $\sin\theta = x/3$, we see that $\cos\theta = \sqrt{9-x^2}/3$. Therefore

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= 9\int \sin^2\theta\,d\theta = -\frac{9}{2}\sin\theta\cos\theta + \frac{9}{2}\theta + C \\ &= -\frac{9}{2}\cdot\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C = -\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C\end{aligned}$$

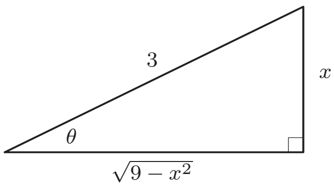


Figure 7.5: In this triangle, $\sin\theta = x/3$

57. Let $2x = \sin\theta$. Then $dx = (1/2)\cos\theta\,d\theta$ and $\sqrt{1-4x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$, so

$$\begin{aligned}\int \frac{\sqrt{1-4x^2}}{x^2}\,dx &= \int \frac{\cos\theta}{(\sin^2\theta)/4}\left(\frac{1}{2}\cos\theta\,d\theta\right) = 2\int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \\ &= 2\int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = 2\int \left(\frac{1}{\sin^2\theta} - 1\right)\,d\theta.\end{aligned}$$

From formula IV-18, we get

$$\int \left(\frac{1}{\sin^2\theta} - 1\right) d\theta = -\frac{\cos\theta}{\sin\theta} - \theta + C.$$

We have $\sin\theta = 2x$ and $\theta = \arcsin(2x)$. From the triangle in Figure 7.6, which shows $\sin\theta = 2x$, we see that $\cos\theta = \sqrt{1-4x^2}$. Therefore

$$\int \frac{\sqrt{1-4x^2}}{x^2}\,dx = 2\left(-\frac{\cos\theta}{\sin\theta} - \theta\right) + C = 2\left(-\frac{\sqrt{1-4x^2}}{2x} - \arcsin(2x)\right) + C = -\frac{\sqrt{1-4x^2}}{x} - 2\arcsin(2x) + C$$

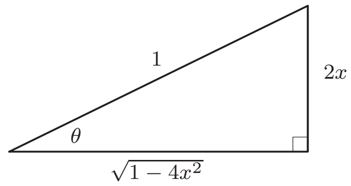


Figure 7.6: In this triangle, $\sin\theta = 2x$

58. Let $3x = 5\sin\theta$. Then $dx = (5/3)\cos\theta\,d\theta$ and $\sqrt{25-9x^2} = \sqrt{25 - 25\sin^2\theta} = 5\cos\theta$, so

$$\begin{aligned}\int \frac{\sqrt{25-9x^2}}{x}\,dx &= \int \frac{5\cos\theta}{(5/3)\sin\theta}\left(\frac{5}{3}\cos\theta\,d\theta\right) = 5\int \frac{\cos^2\theta}{\sin\theta}\,d\theta\\ &= 5\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 5\int\left(\frac{1}{\sin\theta} - \sin\theta\right)d\theta.\end{aligned}$$

From formula IV-21, we get

$$\int \left(\frac{1}{\sin\theta} - \sin\theta\right) d\theta = \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + \cos\theta + C.$$

We have $\sin\theta = 3x/5$ and $\theta = \arcsin(3x/5)$. From the triangle in Figure 7.7, which shows $\sin\theta = 3x/5$, we see that $\cos\theta = \sqrt{25-9x^2}/5$. Therefore

$$\int \frac{\sqrt{25-9x^2}}{x}\,dx = 5\left(\frac{1}{2}\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right| + \cos\theta\right) + C = \frac{5}{2}\ln\left|\frac{5-\sqrt{25-9x^2}}{5+\sqrt{25-9x^2}}\right| + \sqrt{25-9x^2} + C$$

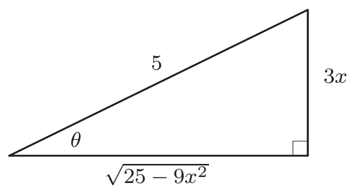


Figure 7.7: In this triangle, $\sin\theta = 3x/5$

59. Let $2x = 3\sin\theta$. Then $dx = (3/2)\cos\theta\,d\theta$ and $\sqrt{9-4x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$, so

$$\int \frac{1}{x\sqrt{9-4x^2}}\,dx = \int \frac{1}{(3\sin\theta/2)(3\cos\theta)}\left(\frac{3}{2}\cos\theta\,d\theta\right) = \frac{1}{3}\int\frac{1}{\sin\theta}\,d\theta.$$

From formula IV-21, we get

$$\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{2}\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right| + C$$

From the triangle in Figure 7.8, which shows $\sin\theta = 2x/3$, we see that $\cos\theta = \sqrt{9-4x^2}/3$. Therefore

$$\int \frac{1}{x\sqrt{9-4x^2}}\,dx = \frac{1}{6}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + C = \frac{1}{6}\ln\left|\frac{\sqrt{9-4x^2}/3 - 1}{\sqrt{9-4x^2}/3 + 1}\right| + C = \frac{1}{6}\ln\left|\frac{\sqrt{9-4x^2} - 3}{\sqrt{9-4x^2} + 3}\right| + C.$$

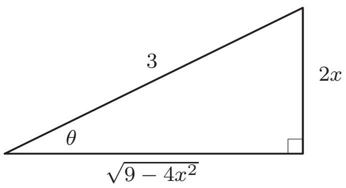


Figure 7.8: In this triangle, $\sin\theta = 2x/3$

60. Let $4x = \tan\theta$. Then $dx = (1/4)/\cos^2\theta\,d\theta$ and $\sqrt{1+16x^2} = \sqrt{1+\tan^2\theta} = 1/\cos\theta$, so

$$\int \frac{1}{x\sqrt{1+16x^2}}\,dx = \int \left(\frac{\cos\theta}{(1/4)\tan\theta}\right)\frac{1/4}{\cos^2\theta}\,d\theta = \int \frac{1}{\sin\theta}\,d\theta.$$

From formula IV-21, we get

$$\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + C$$

From the triangle in Figure 7.9, which shows $\tan\theta = 4x$, we see that $\cos\theta = 1/\sqrt{1+16x^2}$. Therefore

$$\int \frac{1}{x\sqrt{1+4x^2}}\,dx = \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + C = \frac{1}{2}\ln\left|\frac{1/\sqrt{1+16x^2} - 1}{1/\sqrt{1+16x^2} + 1}\right| + C = \frac{1}{2}\ln\left|\frac{1 - \sqrt{1+16x^2}}{1 + \sqrt{1+16x^2}}\right| + C.$$

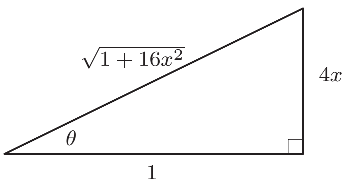


Figure 7.9: In this triangle, $\tan\theta = 4x$

61. Let $x = 2\sin\theta$ so $dx = 2\cos\theta d\theta$. Since $\sqrt{4 - 4\sin^2\theta} = 2\cos\theta$, we have

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{(4\sin^2\theta)(2\cos\theta)} 2\cos\theta\,d\theta = \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta.$$

From formula IV-20 with $m = 2$, we get

$$\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{\cos\theta}{\sin\theta} + C.$$

We need to know $\cos\theta$ in terms of x. From the triangle in Figure 7.10, which shows $x/2 = \sin\theta$, we see that $\cos\theta = \sqrt{4-x^2}/2$. Thus

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C = -\frac{1}{4}\frac{\sqrt{4-x^2}/2}{x/2} + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C.$$

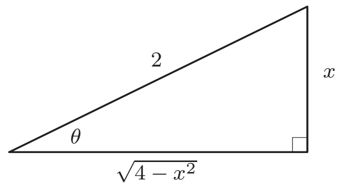


Figure 7.10: In this triangle, $\sin\theta = x/2$

62. Let $2x = 5\tan\theta$ so $dx = (5/2)(1/\cos^2\theta)d\theta$. Since

$$(25+4x^2)^{3/2} = (25+25\tan^2\theta)^{3/2} = (25/\cos^2\theta)^{3/2} = 125/\cos^3\theta,$$

we have

$$\int \frac{1}{(25+4x^2)^{3/2}}\,dx = \int \frac{\cos^3\theta}{125}\left(\frac{5}{2}\frac{1}{\cos^2\theta}\,d\theta\right) = \frac{1}{50}\int \cos\theta\,d\theta = \frac{1}{50}\sin\theta + C.$$

We need to know $\sin\theta$ in terms of x. From the triangle in Figure 7.11, which shows $2x/5 = \tan\theta$, we see that $\sin\theta = 2x/\sqrt{25+4x^2}$. Thus

$$\int \frac{1}{(25+4x^2)^{3/2}}\,dx = \frac{1}{50}\sin\theta + C = \frac{1}{50}\frac{2x}{\sqrt{25+4x^2}} + C = \frac{1}{25}\frac{x}{\sqrt{25+4x^2}} + C$$

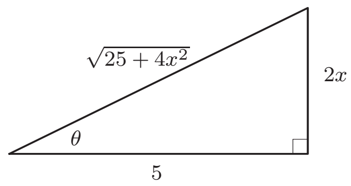


Figure 7.11: In this triangle, $\tan\theta = 2x/5$

63. Let $x = 4\sin\theta$ so $dx = 4\cos\theta\,d\theta$. Since

$$(16-x^2)^{3/2} = (16-16\sin^2\theta)^{3/2} = (16\cos^2\theta)^{3/2} = 64\cos^3\theta,$$

we have

$$\int \frac{1}{(16-x^2)^{3/2}}\,dx = \int \frac{1}{64\cos^3\theta}(4\cos\theta\,d\theta) = \frac{1}{16}\int \frac{1}{\cos^2\theta}\,d\theta.$$

From formula IV-22 with $m = 2$, we get

$$\int \frac{1}{\cos^2\theta}\,d\theta = \frac{\sin\theta}{\cos\theta} + C = \tan\theta + C.$$

We need to know $\tan\theta$ in terms of x. From the triangle in Figure 7.12, which shows $x/4 = \sin\theta$, we see that $\tan\theta = x/\sqrt{16-x^2}$. Thus

$$\int \frac{1}{(16-x^2)^{3/2}}\,dx = \frac{1}{16}\tan\theta + C = \frac{1}{16}\frac{x}{\sqrt{16-x^2}} + C$$

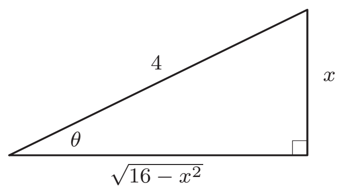


Figure 7.12: In this triangle, $\sin\theta = x/4$

64. Let $3x = \tan\theta$ so $dx = (1/3)(1/\cos^2\theta)\,d\theta$. Since $(1+9x^2)^{3/2} = (1+\tan^2\theta)^{3/2} = 1/\cos^3\theta$, we have

$$\int \frac{x^2}{(1+9x^2)^{3/2}}\,dx = \int \frac{1}{9}\frac{\sin^2\theta}{\cos^2\theta}(\cos^3\theta)\frac{1}{3\cos^2\theta}\,d\theta = \frac{1}{27}\int \frac{\sin^2\theta}{\cos\theta}\,d\theta.$$

From formula IV-22 with $m = 2$, we get

$$\int \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta = \int \left(\frac{1}{\cos\theta} - \cos\theta\right)\,d\theta = \frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| - \sin\theta + C.$$

We need to know $\sin\theta$ in terms of x. From the triangle in Figure 7.13, which shows $3x = \tan\theta$, we see that $\sin\theta = 3x/\sqrt{1+9x^2}$. Thus

$$\int \frac{x^2}{(1+9x^2)^{3/2}}\,dx = \frac{1}{27}\left(\frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| - \sin\theta\right) + C = \frac{1}{27}\left(\frac{1}{2}\ln\left|\frac{3x+\sqrt{1+9x^2}}{3x-\sqrt{1+9x^2}}\right| - \frac{3x}{\sqrt{1+9x^2}}\right) + C.$$

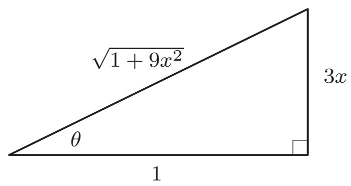


Figure 7.13: In this triangle, $\tan\theta = 3x$

65. Notice that because $\frac{3x}{(x-1)(x-4)}$ is negative for $2 \le x \le 3$,

$$\text{Area} = -\int_2^3 \frac{3x}{(x-1)(x-4)}dx.$$

Using partial fractions gives

$$\frac{3x}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4} = \frac{(A+B)x - B - 4A}{(x-1)(x-4)}.$$

Multiplying through by $(x-1)(x-4)$ gives

$$3x = (A+B)x - B - 4A$$

so $A = -1$ and $B = 4$. Thus

$$-\int_2^3 \frac{3x}{(x-1)(x-4)}dx = -\int_2^3 \left(\frac{-1}{x-1} + \frac{4}{x-4}\right)\,dx = (\ln|x-1| - 4\ln|x-4|)\Big|_2^3 = 5\ln 2.$$

66. We have

$$\text{Area} = \int_0^1 \frac{3x^2+x}{(x^2+1)(x+1)}dx.$$

Using partial fractions gives

$$\begin{aligned}\frac{3x^2+x}{(x^2+1)(x+1)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1}\\ &= \frac{(Ax+B)(x+1) + C(x^2+1)}{(x^2+1)(x+1)}\\ &= \frac{(A+C)x^2 + (A+B)x + B + C}{(x^2+1)(x+1)}.\end{aligned}$$

Thus

$$3x^2 + x = (A+C)x^2 + (A+B)x + B + C,$$

giving

$$3 = A + C, \quad 1 = A + B, \quad \text{and} \quad 0 = B + C,$$

with solution

$$A = 2, B = -1, C = 1.$$

Thus

$$\begin{aligned}
\text{Area} &= \int_0^1 \frac{3x^2 + x}{(x^2+1)(x+1)} dx \\
&= \int_0^1 \left(\frac{2x}{x^2+1} - \frac{1}{x^2+1} + \frac{1}{x+1} \right) dx \\
&= \ln(x^2+1) - \arctan x + \ln|x+1| \Big|_0^1 \\
&= 2\ln 2 - \pi/4.
\end{aligned}$$

67. We have

$$\text{Area} = \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx.$$

Let $x = \sin\theta$ so $dx = \cos\theta\, d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. When $x = 0, \theta = 0$. When $x = 1/2, \theta = \pi/6$.

$$\begin{aligned}
\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx &= \int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta = \int_0^{\pi/6} \sin^2\theta\, d\theta \\
&= \left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2} \right) \Bigg|_0^{\pi/6} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.
\end{aligned}$$

The integral $\int \sin^2\theta\, d\theta$ is done using parts and the identity $\cos^2\theta + \sin^2\theta = 1$.

68. We have

$$\text{Area} = \int_0^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}}\, dx.$$

Let $x = 2\sin\theta$ so $dx = 2\cos\theta\, d\theta$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$. When $x = 0, \theta = 0$ and when $x = \sqrt{2}, \theta = \pi/4$.

$$\begin{aligned}
\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}} dx &= \int_0^{\pi/4} \frac{(2\sin\theta)^3}{\sqrt{4-(2\sin\theta)^2}} 2\cos\theta\, d\theta \\
&= 8 \int_0^{\pi/4} \sin^3\theta\, d\theta = 8 \int_0^{\pi/4} (\sin\theta - \sin\theta\cos^2\theta)\, d\theta \\
&= 8 \left(-\cos\theta + \frac{\cos^3\theta}{3} \right) \Bigg|_0^{\pi/4} = 8 \left(\frac{2}{3} - \frac{5}{6\sqrt{2}} \right).
\end{aligned}$$

69. We have

$$\text{Area} = \int_0^3 \frac{1}{\sqrt{x^2+9}} dx.$$

Let $x = 3\tan\theta$ so $dx = (3/\cos^2\theta) d\theta$ and

$$\sqrt{x^2+9} = \sqrt{\frac{9\sin^2\theta}{\cos^2\theta} + 9} = \frac{3}{\cos\theta}.$$

When $x = 0, \theta = 0$ and when $x = 3, \theta = \pi/4$. Thus

$$\int_0^3 \frac{1}{\sqrt{x^2+9}} dx = \int_0^{\pi/4} \frac{1}{\sqrt{9\tan^2\theta + 9}} \frac{3}{\cos^2\theta} d\theta = \int_0^{\pi/4} \frac{1}{3/\cos\theta} \cdot \frac{3}{\cos^2\theta} d\theta = \int_0^{\pi/4} \frac{1}{\cos\theta} d\theta$$
$$= \frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right|\Bigg|_0^{\pi/4} = \frac{1}{2}\ln\left|\frac{1/\sqrt{2}+1}{1/\sqrt{2}-1}\right| = \frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right).$$

This answer can be simplified to $\ln(1+\sqrt{2})$ by multiplying the numerator and denominator of the fraction by $(\sqrt{2}+1)$ and using the properties of logarithms. The integral $\int(1/\cos\theta)d\theta$ is done using the Table of Integrals.

70. We have

$$\text{Area} = \int_{\sqrt{3}}^3 \frac{1}{x\sqrt{x^2+9}} dx.$$

Let $x = 3\tan\theta$ so $dx = (3/\cos^2\theta)d\theta$ and

$$x\sqrt{x^2+9} = 3\frac{\sin\theta}{\cos\theta}\sqrt{\frac{9\sin^2\theta}{\cos^2\theta}+9} = \frac{9\sin\theta}{\cos^2\theta}.$$

When $x = \sqrt{3}, \theta = \pi/6$ and when $x = 3, \theta = \pi/4$. Thus

$$\int_{\sqrt{3}}^3 \frac{1}{x\sqrt{x^2+9}} dx = \int_{\pi/6}^{\pi/4} \frac{1}{9\sin\theta/\cos^2\theta} \cdot \frac{3}{\cos^2\theta} d\theta = \frac{1}{3}\int_{\pi/6}^{\pi/4} \frac{1}{\sin\theta} d\theta$$
$$= \frac{1}{3}\cdot\frac{1}{2}\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right|\Bigg|_{\pi/6}^{\pi/4} = \frac{1}{6}\left(\ln\left|\frac{1/\sqrt{2}-1}{1/\sqrt{2}+1}\right| - \ln\left|\frac{\sqrt{3}/2-1}{\sqrt{3}/2+1}\right|\right)$$
$$= \frac{1}{6}\left(\ln\left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right| + \ln\left|\frac{\sqrt{3}+2}{\sqrt{3}-2}\right|\right).$$

This answer can be simplified by multiplying the first fraction by $(1-\sqrt{2})$ in numerator and denominator and the second one by $(\sqrt{3}+2)$. This gives

$$\text{Area} = \frac{1}{6}(\ln(3-2\sqrt{2}) + \ln(7+4\sqrt{3})) = \frac{1}{6}\ln((3-2\sqrt{2})(7+4\sqrt{3})).$$

The integral $\int(1/\sin\theta)d\theta$ is done using the Table of Integrals.

71. Using partial fractions, we write

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$
$$1 = A(1-x) + B(1+x) = (B-A)x + A + B.$$

So, $B - A = 0$ and $A + B = 1$, giving $A = B = 1/2$. Thus

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int\left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx = \frac{1}{2}(\ln|1+x| - \ln|1-x|) + C.$$

Using the substitution $x = \sin\theta$, we get $dx = \cos\theta\, d\theta$, we have

$$\int \frac{dx}{1-x^2} = \int \frac{\cos\theta}{1-\sin^2\theta} d\theta = \int \frac{\cos\theta}{\cos^2\theta} d\theta = \int \frac{1}{\cos\theta} d\theta.$$

The Table of Integrals Formula IV-22 gives

$$\int \frac{dx}{1-x^2} = \int \frac{1}{\cos\theta} d\theta = \frac{1}{2}\ln\left|\frac{(\sin\theta)+1}{(\sin\theta)-1}\right| + C = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C.$$

The properties of logarithms and the fact that $|x-1| = |1-x|$ show that the two results are the same:

$$\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| = \frac{1}{2}(\ln|1+x| - \ln|1-x|).$$

72. Using partial fractions, we write

$$\frac{2x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$2x = A(x-1) + B(x+1) = (A+B)x - A + B.$$

So, $A + B = 2$ and $-A + B = 0$, giving $A = B = 1$. Thus

$$\int \frac{2x}{x^2-1}\,dx = \int \left(\frac{1}{x+1} + \frac{1}{x-1}\right) dx = \ln|x+1| + \ln|x-1| + C.$$

Using the substitution $w = x^2 - 1$, we get $dw = 2x\,dx$, so we have

$$\int \frac{2x}{x^2-1}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln\left|x^2-1\right| + C.$$

The properties of logarithms show that the two results are the same:

$$\ln|x+1| + \ln|x-1| = \ln|(x+1)(x-1)| = \ln\left|x^2-1\right|.$$

73. Using partial fractions, we write

$$\frac{3x^2+1}{x^3+x} = \frac{3x^2+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$3x^2 + 1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A.$$

So, $A + B = 3$, $C = 0$ and $A = 1$, giving $B = 2$. Thus

$$\int \frac{3x^2+1}{x^3+x}\,dx = \int \left(\frac{1}{x} + \frac{2x}{x^2+1}\right) dx = \ln|x| + \ln\left|x^2+1\right| + K.$$

Using the substitution $w = x^3 + x$, we get $dw = (3x^2+1)dx$, so we have

$$\int \frac{3x^2+1}{x^3+x}\,dx = \int \frac{dw}{w} = \ln|w| + K = \ln\left|x^3+x\right| + K.$$

The properties of logarithms show that the two results are the same:

$$\ln|x| + \ln\left|x^2+1\right| + K = \ln\left|x(x^2+1)\right| + K = \ln\left|x^3+x\right| + K.$$

We use K as the constant of integration, since we already used C in the problem.

74. (a) We differentiate:

$$\frac{d}{d\theta}\left(-\frac{1}{\tan\theta}\right) = \frac{1}{\tan^2\theta}\cdot\frac{1}{\cos^2\theta} = \frac{1}{\frac{\sin^2\theta}{\cos^2\theta}}\cdot\frac{1}{\cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Thus,

$$\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{1}{\tan\theta} + C.$$

(b) Let $y = \sqrt{5}\sin\theta$ so $dy = \sqrt{5}\cos\theta\,d\theta$ giving

$$\int \frac{dy}{y^2\sqrt{5-y^2}} = \int \frac{\sqrt{5}\cos\theta}{5\sin^2\theta\sqrt{5-5\sin^2\theta}}\,d\theta = \frac{1}{5}\int \frac{\sqrt{5}\cos\theta}{\sin^2\theta\sqrt{5}\cos\theta}\,d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{1}{5\tan\theta} + C.$$

Since $\sin\theta = y/\sqrt{5}$, we have $\cos\theta = \sqrt{1-(y/\sqrt{5})^2} = \sqrt{5-y^2}/\sqrt{5}$. Thus,

$$\int \frac{dy}{y^2\sqrt{5-y^2}} = -\frac{1}{5\tan\theta} + C = -\frac{\sqrt{5-y^2}/\sqrt{5}}{5(y/\sqrt{5})} + C = -\frac{\sqrt{5-y^2}}{5y} + C.$$

75. (a) If $a \neq b$, we have

$$\int \frac{1}{(x-a)(x-b)}\,dx = \int \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C.$$

(b) If $a = b$, we have

$$\int \frac{1}{(x-a)(x-a)}\,dx = \int \frac{1}{(x-a)^2}\,dx = -\frac{1}{x-a} + C.$$

76. (a) If $a \neq b$, we have

$$\int \frac{x}{(x-a)(x-b)}\,dx = \int \frac{1}{a-b}\left(\frac{a}{x-a} - \frac{b}{x-b}\right)\,dx = \frac{1}{a-b}(a\ln|x-a| - b\ln|x-b|) + C.$$

(b) If $a = b$, we have

$$\int \frac{x}{(x-a)^2}\,dx = \int \left(\frac{1}{x-a} + \frac{a}{(x-a)^2}\right)\,dx = \ln|x-a| - \frac{a}{x-a} + C.$$

77. (a) If $a > 0$, then

$$x^2 - a = (x-\sqrt{a})(x+\sqrt{a}).$$

This means that we can use partial fractions:

$$\frac{1}{x^2-a} = \frac{A}{x-\sqrt{a}} + \frac{B}{x+\sqrt{a}},$$

giving

$$1 = A(x+\sqrt{a}) + B(x-\sqrt{a}),$$

so $A + B = 0$ and $(A-B)\sqrt{a} = 1$. Thus, $A = -B = 1/(2\sqrt{a})$.

So

$$\int \frac{1}{x^2-a}\,dx = \int \frac{1}{2\sqrt{a}}\left(\frac{1}{x-\sqrt{a}} - \frac{1}{x+\sqrt{a}}\right)\,dx = \frac{1}{2\sqrt{a}}\left(\ln\left|x-\sqrt{a}\right| - \ln\left|x+\sqrt{a}\right|\right) + C.$$

(b) If $a = 0$, we have

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C.$$

(c) If $a < 0$, then $-a > 0$ so $x^2 - a = x^2 + (-a)$ cannot be factored. Thus

$$\int \frac{1}{x^2-a}\,dx = \int \frac{1}{x^2+(-a)}\,dx = \frac{1}{\sqrt{-a}}\arctan\left(\frac{x}{\sqrt{-a}}\right) + C.$$

78. (a) We integrate to find

$$\int \frac{b}{x(1-x)}\,dx = b\int\left(\frac{1}{x} + \frac{1}{1-x}\right)\,dx = b(\ln|x| - \ln|1-x|) + C = b\ln\left|\frac{x}{1-x}\right| + C,$$

so

$$t(p) = \int_a^p \frac{b}{x(1-x)}\,dx = b\ln\left(\frac{p}{1-p}\right) - b\ln\left(\frac{a}{1-a}\right) = b\ln\left(\frac{p(1-a)}{a(1-p)}\right).$$

(b) We know that $t(0.01) = 0$ so

$$0 = b\ln\left(\frac{0.01(1-a)}{0.99a}\right).$$

But $b > 0$ and $\ln x = 0$ means $x = 1$, so

$$\begin{aligned}\frac{0.01(1-a)}{0.99a} &= 1\\ 0.01(1-a) &= 0.99a\\ 0.01 - 0.01a &= 0.99a\\ a &= 0.01.\end{aligned}$$

(c) We know that $t(0.5) = 1$ so

$$1 = b\ln\left(\frac{0.5\cdot 0.99}{0.5\cdot 0.01}\right) = b\ln 99, b = \frac{1}{\ln 99} = 0.218.$$

(d) We have

$$t(0.9) = \int_{0.01}^{0.9} \frac{0.218}{x(1-x)}\,dx = \frac{1}{\ln 99}\ln\left(\frac{0.9(1-0.01)}{0.01(1-0.9)}\right) = 1.478.$$

79. (a) We want to evaluate the integral

$$T = \int_0^{a/2} \frac{k\,dx}{(a-x)(b-x)}.$$

Using partial fractions, we have

$$\begin{aligned}\frac{k}{(a-x)(b-x)} &= \frac{C}{a-x} + \frac{D}{b-x}\\ k &= C(b-x) + D(a-x)\\ k &= -(C+D)x + Cb + Da\end{aligned}$$

so

$$\begin{aligned}0 &= -(C+D)\\ k &= Cb + Da,\end{aligned}$$

giving

$$C = -D = \frac{k}{b-a}.$$

Thus, the time is given by

$$\begin{aligned}T = \int_0^{a/2} \frac{k\,dx}{(a-x)(b-x)} &= \frac{k}{b-a}\int_0^{a/2}\left(\frac{1}{a-x} - \frac{1}{b-x}\right)dx\\ &= \frac{k}{b-a}\left(-\ln|a-x| + \ln|b-x|\right)\Big|_0^{a/2}\\ &= \frac{k}{b-a}\ln\left|\frac{b-x}{a-x}\right|\Big|_0^{a/2}\\ &= \frac{k}{b-a}\left(\ln\left(\frac{2b-a}{a}\right) - \ln\left(\frac{b}{a}\right)\right)\\ &= \frac{k}{b-a}\ln\left(\frac{2b-a}{b}\right).\end{aligned}$$

(b) A similar calculation with x_0 instead of $a/2$ leads to the following expression for the time

$$\begin{aligned}T = \int_0^{x_0} \frac{k\,dx}{(a-x)(b-x)} &= \frac{k}{b-a}\ln\left|\frac{b-x}{a-x}\right|\Big|_0^{x_0}\\ &= \frac{k}{b-a}\left(\ln\left|\frac{b-x_0}{a-x_0}\right| - \ln\left(\frac{b}{a}\right)\right).\end{aligned}$$

As $x_0 \to a$, the value of $|a-x_0| \to 0$, so $|b-x_0|/|a-x_0| \to \infty$. Thus, $T \to \infty$ as $x_0 \to a$. In other words, the time taken tends to infinity.

80. We complete the square in the exponent so that we can make a substitution:

$$\begin{aligned}m(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-x^2/2}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2-2tx)/2}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-((x^2-2tx+t^2)-t^2)/2}\,dx\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} \cdot e^{t^2/2}\, dx$$
$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\, dx.$$

Substitute $w = x - t$, then $dw = dx$ and $w = \infty$ when $x = \infty$, and $w = -\infty$ when $x = -\infty$. Thus

$$m(t) = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2/2}\, dw = \frac{e^{t^2/2}}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$
$$m(t) = e^{t^2/2}.$$

Strengthen Your Understanding

81. The partial fractions required are of the form

$$\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

82. We use $x = 2\tan\theta$.

83. Since $\int 1/(1+x^2)\, dx = \arctan x + C$, so $f(x) = 1/(1+x^2)$ works.

84. To evaluate $\int 1/(x^3 - x)\, dx$, we first factor $x^3 - x = x(x-1)(x+1)$ and then use partial fractions.

85. If $Q(x)$ does not decompose into linear factors, then we cannot decompose $P(x)/Q(x)$ into partial fractions. So for example, if $P(x) = x$ and $Q(x) = x^2 + 1$, then the rational function $P(x)/Q(x)$ does not have a partial fraction decomposition. Alternatively, if $Q(x)$ contains a repeated linear factor, then the partial fraction decomposition may not be in the given form. For example, if $P(x) = x + 1$ and $Q(x) = x^2 + 6x + 9$, then the partial fraction decomposition of $P(x)/Q(x)$ is

$$\frac{x+1}{x^2+6x+9} = \frac{1}{x+3} - \frac{2}{(x+3)^2}.$$

86. An example is the integral

$$\int \frac{dx}{\sqrt{9-4x^2}}.$$

If we make the substitution $x = \frac{3}{2}\sin\theta$, $dx = \frac{3}{2}\cos\theta\, d\theta$, then we obtain

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{\frac{3}{2}\cos\theta}{\sqrt{9-9\sin^2\theta}}\, d\theta = \int \frac{\frac{3}{2}\cos\theta}{\sqrt{9\cos^2\theta}}\, d\theta = \int \frac{\frac{3}{2}\cos\theta}{3\cos\theta}\, d\theta$$
$$= \int \frac{1}{2}\, d\theta = \frac{1}{2}\theta + C = \frac{1}{2} \cdot \arcsin\left(\frac{2}{3}x\right) + C$$

87. False. If we use the given substitution the radicand becomes $9 - 9\tan^2\theta = 9(1 - \tan^2\theta)$. However, $1 - \tan^2\theta$ is not equal to a monomial.

88. True. Since the denominator can be factored into $x^2(x+1)$, involving the repeated factor x^2, we use partial fractions of the form

$$\int \frac{1}{x^3 + x^2}\, dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}\, dx.$$

89. (e). $x = \sin\theta$ is appropriate.

90. (c) and (b). $\dfrac{x^2}{1-x^2} = -1 + \dfrac{1}{1-x^2} = 1 + \dfrac{1}{(1-x)(1+x)}$.

Solutions for Section 7.5

Exercises

1. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.14. We see that this approximation is an underestimate.

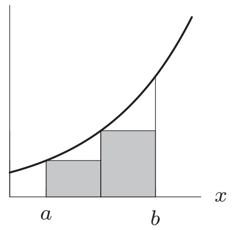


Figure 7.14

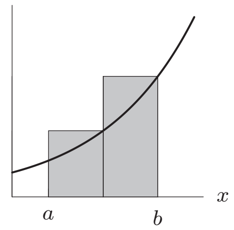


Figure 7.15

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.15. We see that this approximation is an overestimate.

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.16. We see that this approximation is an overestimate.

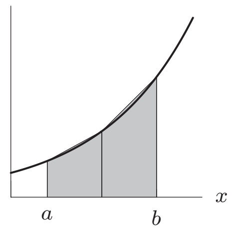


Figure 7.16

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.17. We see from the tangent line interpretation that this approximation is an underestimate

Figure 7.17

2. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.18. We see that this approximation is an overestimate.

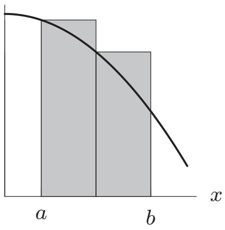


Figure 7.18

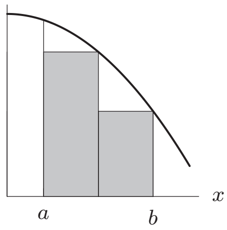


Figure 7.19

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.19. We see that this approximation is an underestimate.

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.20. We see that this approximation is an underestimate.

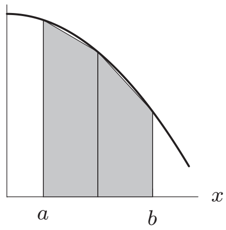


Figure 7.20

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.21. We see from the tangent line interpretation that this approximation is an overestimate.

Figure 7.21

3. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.22. We see that this approximation is an underestimate.

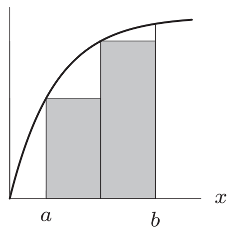


Figure 7.22

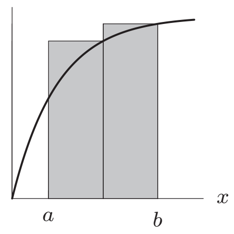


Figure 7.23

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.23. We see that this approximation is an overestimate.

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.24. We see that this approximation is an underestimate.

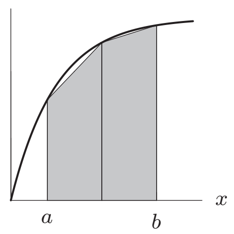
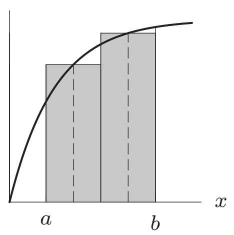


Figure 7.24

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.25. We see from the tangent line interpretation that this approximation is an overestimate.



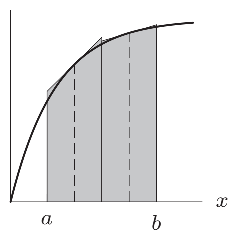


Figure 7.25

4. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.26. We see that this approximation is an overestimate.

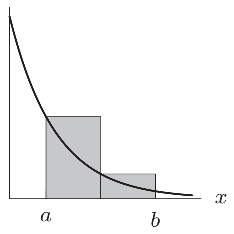


Figure 7.26

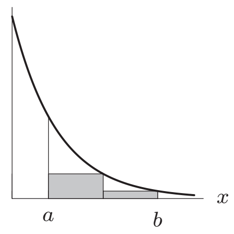


Figure 7.27

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.27. We see that this approximation is an underestimate.

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.28. We see that this approximation is an overestimate.

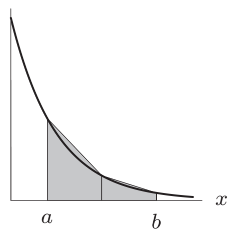


Figure 7.28

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.29. We see from the tangent line interpretation that this approximation is an underestimate.

Figure 7.29

5. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.30. We see that this approximation is an underestimate (that is, it is more negative).

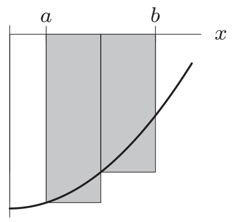


Figure 7.30

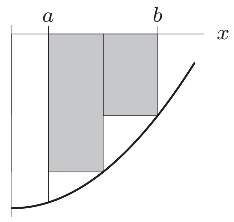


Figure 7.31

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.31. We see that this approximation is an overestimate (that is, it is less negative).

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.32. We see that this approximation is an overestimate (that is, it is less negative).

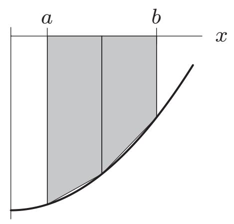


Figure 7.32

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.33. We see from the tangent line interpretation that this approximation is an underestimate (that is, it is more negative).

Figure 7.33

6. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.34. We see that this approximation is an overestimate (that is, it is less negative).

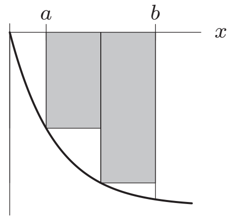


Figure 7.34

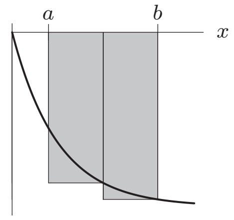


Figure 7.35

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.35. We see that this approximation is an underestimate (that is, it is more negative).

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.36. We see that this approximation is an overestimate (that is, it is less negative).

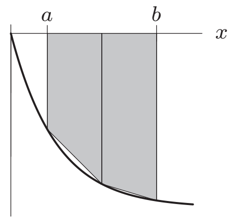


Figure 7.36

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.37. We see from the tangent line interpretation that this approximation is an underestimate (that is, the approximation is more negative).

Figure 7.37

7. (a) Since two rectangles are being used, the width of each rectangle is 3. The height is given by the left-hand endpoint so we have

$$\text{LEFT(2)} = f(0)\cdot 3 + f(3)\cdot 3 = 0^2\cdot 3 + 3^2\cdot 3 = 27.$$

(b) Since two rectangles are being used, the width of each rectangle is 3. The height is given by the right-hand endpoint so we have

$$\text{RIGHT(2)} = f(3)\cdot 3 + f(6)\cdot 3 = 3^2\cdot 3 + 6^2\cdot 3 = 135.$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$\text{TRAP(2)} = \frac{27+135}{2} = 81.$$

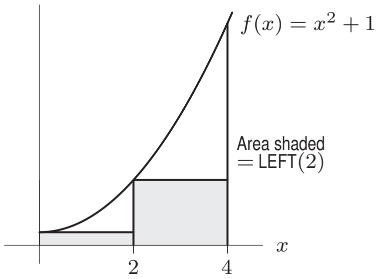
(d) Since two rectangles are being used, the width of each rectangle is 3. The height is given by the height at the midpoint so we have

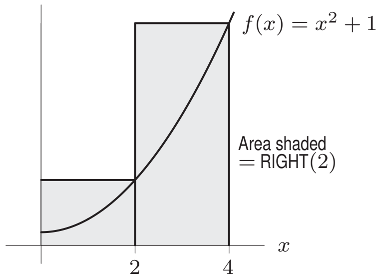
$$\text{MID(2)} = f(1.5)\cdot 3 + f(4.5)\cdot 3 = (1.5)^2\cdot 3 + (4.5)^2\cdot 3 = 67.5.$$

8. (a)

$$\begin{aligned}\text{LEFT}(2) &= 2 \cdot f(0) + 2 \cdot f(2) \\ &= 2 \cdot 1 + 2 \cdot 5 \\ &= 12 \\ \text{RIGHT}(2) &= 2 \cdot f(2) + 2 \cdot f(4) \\ &= 2 \cdot 5 + 2 \cdot 17 \\ &= 44\end{aligned}$$

(b)



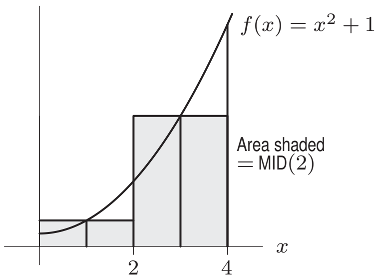


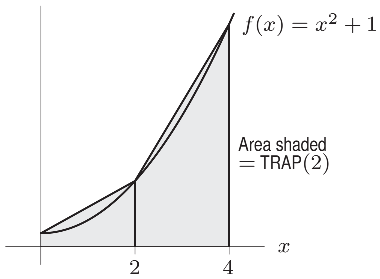
LEFT(2) is an underestimate, while RIGHT(2) is an overestimate.

9. (a)

$$\begin{aligned}\text{MID}(2) &= 2 \cdot f(1) + 2 \cdot f(3) \\ &= 2 \cdot 2 + 2 \cdot 10 \\ &= 24 \\ \text{TRAP}(2) &= \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} \\ &= \frac{12 + 44}{2} \quad \text{(see Problem 8)} \\ &= 28\end{aligned}$$

(b)





MID(2) is an underestimate, since $f(x) = x^2 + 1$ is concave up and a tangent line will be below the curve. TRAP(2) is an overestimate, since a secant line lies above the curve.

10. (a) Since two rectangles are being used, the width of each rectangle is $\pi/2$. The height is given by the left-hand endpoint so we have

$$\text{LEFT}(2) = f(0) \cdot \frac{\pi}{2} + f(\pi/2) \cdot \frac{\pi}{2} = \sin 0 \cdot \frac{\pi}{2} + \sin(\pi/2) \cdot \frac{\pi}{2} = \frac{\pi}{2}.$$

(b) Since two rectangles are being used, the width of each rectangle is $\pi/2$. The height is given by the right-hand endpoint so we have

$$\text{RIGHT}(2) = f(\pi/2) \cdot \frac{\pi}{2} + f(\pi) \cdot \frac{\pi}{2} = \sin(\pi/2) \cdot \frac{\pi}{2} + \sin(\pi) \cdot \frac{\pi}{2} = \frac{\pi}{2}.$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$\text{TRAP}(2) = \frac{\frac{\pi}{2} + \frac{\pi}{2}}{2} = \frac{\pi}{2}.$$

(d) Since two rectangles are being used, the width of each rectangle is $\pi/2$. The height is given by the height at the midpoint so we have

$$\text{MID}(2) = f(\pi/4)\cdot\frac{\pi}{2} + f(3\pi/4)\cdot\frac{\pi}{2} = \sin(\pi/4)\cdot\frac{\pi}{2} + \sin(3\pi/4)\cdot\frac{\pi}{2} = \frac{\sqrt{2}\pi}{2}.$$

Problems

11. For $n = 5$, we have $\Delta t = (2-1)/5 = 0.2$, so

$$\begin{aligned}\text{MID}(5) &= 0.2f(1.1) + 0.2f(1.3) + 0.2f(1.5) + 0.2f(1.7) + 0.2f(1.9)\\ &= 0.2\,(-2.9 - 3.7 - 3.2 - 1.7 + 0.5)\\ &= -2.2.\end{aligned}$$

12. For $n = 4$, we have $\Delta x = (2-0)/4 = 0.5$, so

$$\begin{aligned}\text{MID}(4) &= f(0.25)0.5 + f(0.75)(0.5) + f(1.25)(0.5) + f(1.75)(0.5)\\ &= 0.5(5.8 + 9.3 + 10.8 + 10.3)\\ &= 18.1.\end{aligned}$$

13. We have $n = 2$ and $\Delta x = 0.5$, so

$$\begin{aligned}\text{LEFT}(2) &= \Delta x\,(f(2) + f(2.5))\\ &= 0.5\left(\frac{1}{2^2} + \frac{1}{2.5^2}\right)\\ &= 0.5\left(\frac{1}{4} + \frac{4}{25}\right)\\ &= \frac{41}{200} = 0.205\\ \text{RIGHT}(2) &= \Delta x\,(f(2.5) + f(3))\\ &= 0.5\left(\frac{1}{2.5^2} + \frac{1}{3^2}\right)\\ &= 0.5\left(\frac{4}{25} + \frac{1}{9}\right)\\ &= \frac{61}{450} = 0.1356\\ \text{TRAP}(2) &= \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2}\\ &= \frac{\frac{41}{200} + \frac{61}{450}}{2} = 0.1703.\end{aligned}$$

14. We have $n = 2$ and $\Delta x = 0.5$, so

$$\begin{aligned}\text{MID}(2) &= \Delta x\,(f(2.25) + f(2.75))\\ &= 0.5\left(\frac{1}{2.25^2} + \frac{1}{2.75^2}\right)\\ &= 0.1649.\end{aligned}$$

15. (a) (i) Let $f(x) = \frac{1}{1+x^2}$. The left-hand Riemann sum is

$$\begin{aligned}&\frac{1}{8}\left(f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right)\\ &= \frac{1}{8}\left(\frac{64}{64} + \frac{64}{65} + \frac{64}{68} + \frac{64}{73} + \frac{64}{80} + \frac{64}{89} + \frac{64}{100} + \frac{64}{113}\right)\\ &\approx 8(0.1020) = 0.8160.\end{aligned}$$

(ii) Let $f(x) = \frac{1}{1+x^2}$. The right-hand Riemann sum is

$$\begin{aligned}&\frac{1}{8}\left(f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f(1)\right)\\ &= \frac{1}{8}\left(\frac{64}{65} + \frac{64}{68} + \frac{64}{73} + \frac{64}{80} + \frac{64}{89} + \frac{64}{100} + \frac{64}{113} + \frac{64}{128}\right)\\ &\approx 0.8160 - \frac{1}{16} = 0.7535.\end{aligned}$$

(iii) The trapezoid rule gives us that

$$\text{TRAP}(8) = \frac{\text{LEFT}(8) + \text{RIGHT}(8)}{2} \approx 0.7847.$$

(b) Since $1 + x^2$ is increasing for $x > 0$, so $\dfrac{1}{1+x^2}$ is decreasing over the interval. Thus

$$\begin{aligned} \text{RIGHT}(8) < \int_0^1 \frac{1}{1+x^2}\,dx &< \text{LEFT}(8) \\ 0.7535 < \frac{\pi}{4} &< 0.8160 \\ 3.014 < \pi &< 3.264. \end{aligned}$$

16. Let $s(t)$ be the distance traveled at time t and $v(t)$ be the velocity at time t. Then the distance traveled during the interval $0 \le t \le 6$ is

$$\begin{aligned} s(6) - s(0) &= s(t)\Big|_0^6 \\ &= \int_0^6 s'(t)\,dt \quad \text{(by the Fundamental Theorem)} \\ &= \int_0^6 v(t)\,dt. \end{aligned}$$

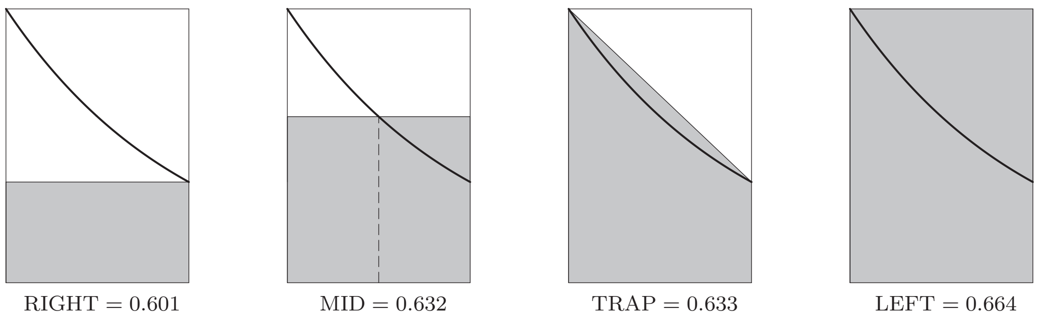
We estimate the distance by estimating this integral.

From the table, we find: $\text{LEFT}(6) = 31$, $\text{RIGHT}(6) = 39$, $\text{TRAP}(6) = 35$.

17. Since the function is decreasing, LEFT is an overestimate and RIGHT is an underestimate. Since the graph is concave down, secant lines lie below the graph so TRAP is an underestimate and tangent lines lie above the graph so MID is an overestimate. We can see that MID and TRAP are closer to the exact value than LEFT and RIGHT. In order smallest to largest, we have:
RIGHT(n) < TRAP(n) < Exact value < MID(n) < LEFT(n).

18. For a decreasing function whose graph is concave up, the diagrams below show that RIGHT < MID < TRAP < LEFT. Thus,

(a) $0.664 =$ LEFT, $0.633 =$ TRAP, $0.632 =$ MID, and $0.601 =$ RIGHT.
(b) $0.632 <$ true value < 0.633.



19. $f(x)$ is increasing, so RIGHT gives an overestimate and LEFT gives an underestimate.

20. $f(x)$ is concave down, so MID gives an overestimate and TRAP gives an underestimate.

21. $f(x)$ is decreasing and concave up, so LEFT and TRAP give overestimates and RIGHT and MID give underestimates.

22. $f(x)$ is concave up, so TRAP gives an overestimate and MID gives an underestimate.

23. (a) Since we are using $n = 2$ on the interval $x = 0$ to $x = 4$, we have $\Delta x = 2$. The midpoint of the first subinterval is 1 and the midpoint of the second subinterval is 3, so we have

$$\text{MID(2)} = 3\sqrt{1} \cdot 2 + 3\sqrt{3} \cdot 2 = 16.392.$$

(b) We have

$$\int_0^4 3x^{1/2} dx = 2x^{3/2}\Big|_0^4 = 2(4^{3/2}) = 16.$$

(c) Error $=$ Actual $-$ Approximation $= 16 - 16.392 = -0.392$.

(d) Since n is multiplied by 10 in going from $n = 2$ to $n = 20$, we expect the error to be multiplied by approximately $1/10^2$. We estimate that

$$\text{Error for MID(20)} \approx \frac{1}{100} \cdot (-0.392) = -0.00392.$$

(e) We expect the approximation for MID(20) to be an overestimate (since $f(x) = 3\sqrt{x}$ is concave down) and to be 0.00392 away from the exact value of 16. We have

$$\text{MID(20)} \approx 16.00392.$$

24. (a) Since $f(x)$ is closer to horizontal (that is, $|f'| < |g'|$), LEFT and RIGHT will be more accurate with $f(x)$.

(b) Since $g(x)$ has more curvature, MID and TRAP will be more accurate with $f(x)$.

25. (a) TRAP(4) gives probably the best estimate of the integral. We cannot calculate MID(4).

$$\text{LEFT(4)} = 3 \cdot 100 + 3 \cdot 97 + 3 \cdot 90 + 3 \cdot 78 = 1095$$
$$\text{RIGHT(4)} = 3 \cdot 97 + 3 \cdot 90 + 3 \cdot 78 + 3 \cdot 55 = 960$$
$$\text{TRAP(4)} = \frac{1095 + 960}{2} = 1027.5.$$

(b) Because there are no points of inflection, the graph is either concave down or concave up. By plotting points, we see that it is concave down. So TRAP(4) is an underestimate.

26. (a) $\displaystyle\int_0^{2\pi} \sin\theta \, d\theta = -\cos\theta\Big|_0^{2\pi} = 0.$

(b) See Figure 7.38. $\text{MID}(1)$ is 0 since the midpoint of 0 and 2π is π, and $\sin\pi = 0$. Thus $\text{MID}(1) = 2\pi(\sin\pi) = 0$. The midpoints we use for $\text{MID}(2)$ are $\pi/2$ and $3\pi/2$, and $\sin(\pi/2) = -\sin(3\pi/2)$. Thus $\text{MID}(2) = \pi\sin(\pi/2) + \pi\sin(3\pi/2) = 0$.

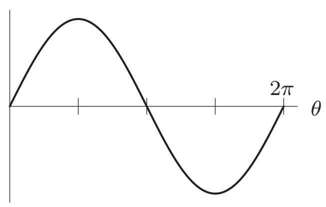


Figure 7.38

(c) $\text{MID}(3) = 0$.

In general, $\text{MID}(n) = 0$ for all n, even though your calculator (because of round-off error) might not return it as such. The reason is that $\sin(x) = -\sin(2\pi - x)$. If we use $\text{MID}(n)$, we will always take sums where we are adding pairs of the form $\sin(x)$ and $\sin(2\pi - x)$, so the sum will cancel to 0. (If n is odd, we will get a $\sin\pi$ in the sum which does not pair up with anything — but $\sin\pi$ is already 0.)

27. See Figure 7.39. We observe that the error in the left or right rule depends on how steeply the graph of f rises or falls. A steep curve makes the triangular regions missed by the left or right rectangles tall and hence large in area. This observation suggests that the error in the left or right rules depends on the magnitude of the derivative of f.

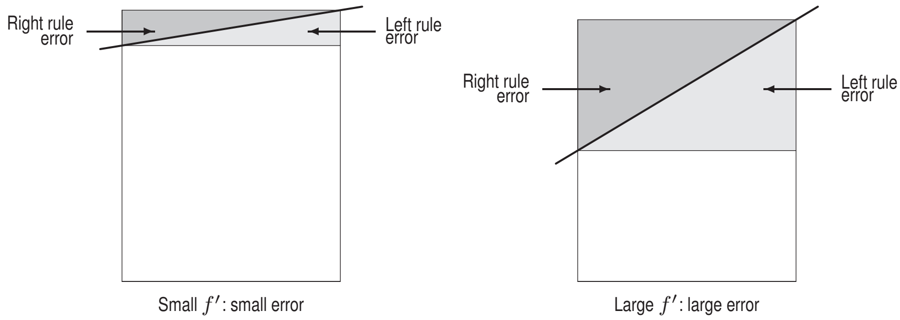


Figure 7.39: The error in the left and right rules depends on the steepness of the curve

28. From Figure 7.40 it appears that the errors in the trapezoid and midpoint rules depend on how much the curve is bent up or down. In other words, the concavity, and hence the magnitude of the second derivative, f'', has an effect on the errors of these two rules.

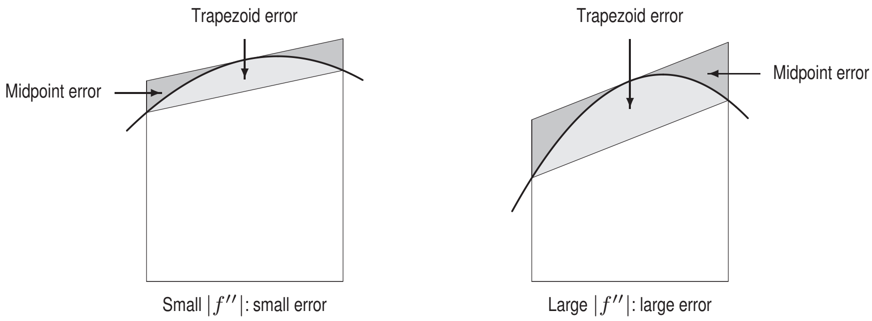


Figure 7.40: The error in the trapezoid and midpoint rules depends on how bent the curve is

29. (a) The graph of $y = \sqrt{2 - x^2}$ is the upper half of a circle of radius $\sqrt{2}$ centered at the origin. The integral represents the area under this curve between the lines $x = 0$ and $x = 1$. From Figure 7.41, we see that this area can be split into 2 parts, A_1 and A_2. Notice since $OQ = QP = 1$, $\triangle OQP$ is isosceles. Thus $\angle POQ = \angle ROP = \frac{\pi}{4}$, and A_1 is exactly $\frac{1}{8}$ of the entire circle. Thus the total area is

$$\text{Area} = A_1 + A_2 = \frac{1}{8}\pi(\sqrt{2})^2 + \frac{1 \cdot 1}{2} = \frac{\pi}{4} + \frac{1}{2}.$$

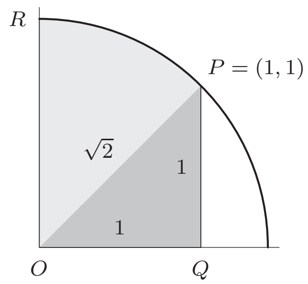


Figure 7.41

(b) $\text{LEFT}(5) \approx 1.32350$, $\text{RIGHT}(5) \approx 1.24066, T$
$\text{TRAP}(5) \approx 1.28208$, $\text{MID}(5) \approx 1.28705$

Exact value ≈ 1.285398163

Left-hand error ≈ -0.03810, Right-hand error ≈ 0.04474,
Trapezoidal error ≈ 0.00332, Midpoint error ≈ -0.001656

Thus right-hand error $>$ trapezoidal error $> 0 >$ midpoint error $>$ left-hand error, and |midpt error| $<$ |trap error| $<$ |left-error| $<$ |right-error|.

30. We approximate the area of the playing field by using Riemann sums. From the data provided,

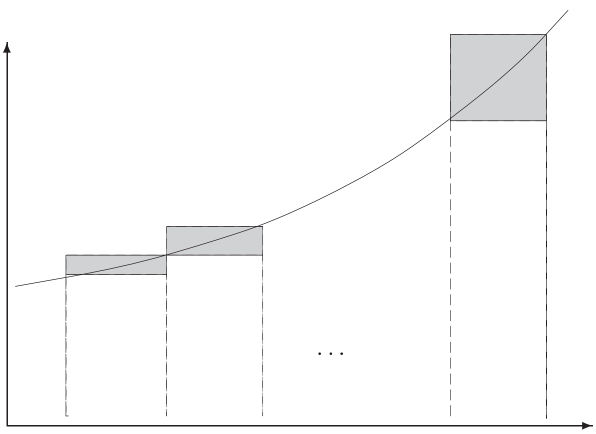
$$\text{LEFT}(10) = \text{RIGHT}(10) = \text{TRAP}(10) = 89{,}000 \text{ square feet.}$$

Thus approximately

$$\frac{89{,}000 \text{ sq. ft.}}{200 \text{ sq. ft./lb.}} = 445 \text{ lbs. of fertilizer}$$

should be necessary.

31.



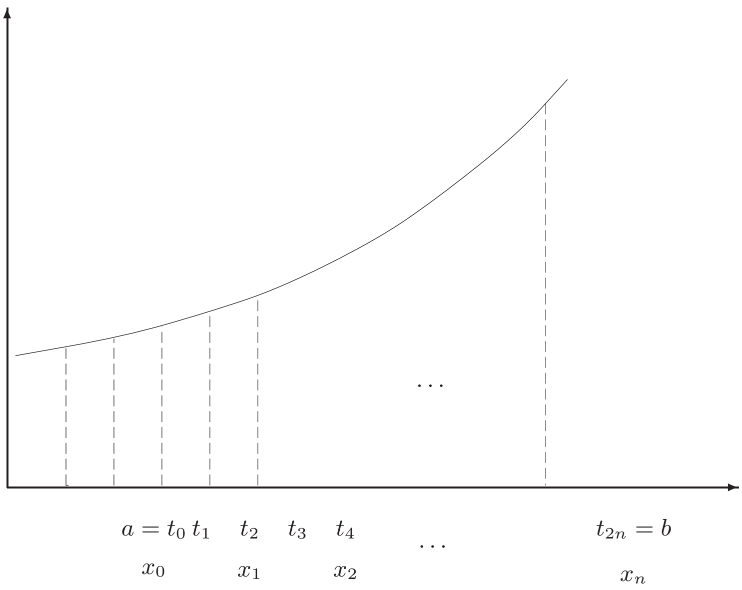
From the diagram, the difference between RIGHT(n) and LEFT(n) is the area of the shaded rectangles.
RIGHT$(n) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$
LEFT$(n) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$
Notice that the terms in these two sums are the same, except that RIGHT(n) contains $f(x_n)\Delta x$ $(= f(b)\Delta x)$, and LEFT(n) contains $f(x_0)\Delta x$ $(= f(a)\Delta x)$. Thus

$$\begin{aligned}\text{RIGHT}(n) &= \text{LEFT}(n) + f(x_n)\Delta x - f(x_0)\Delta x \\ &= \text{LEFT}(n) + f(b)\Delta x - f(a)\Delta x\end{aligned}$$

32.

$$\begin{aligned}\text{TRAP}(n) &= \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2} \\ &= \frac{\text{LEFT}(n) + \text{LEFT}(n) + f(b)\Delta x - f(a)\Delta x}{2} \\ &= \text{LEFT}(n) + \frac{1}{2}(f(b) - f(a))\Delta x\end{aligned}$$

33.



Divide the interval $[a, b]$ into n pieces, by $x_0, x_1, x_2, \ldots, x_n$, and also into $2n$ pieces, by $t_0, t_1, t_2, \ldots, t_{2n}$. Then the x's coincide with the even t's, so $x_0 = t_0,\ x_1 = t_2,\ x_2 = t_4,\ \ldots,\ x_n = t_{2n}$ and $\Delta t = \frac{1}{2}\Delta x$.

$$\text{LEFT}(n) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

Since $\text{MID}(n)$ is obtained by evaluating f at the midpoints $t_1, t_3, t_5, \ldots$ of the x intervals, we get

$$\text{MID}(n) = f(t_1)\Delta x + f(t_3)\Delta x + \cdots + f(t_{2n-1})\Delta x$$

Now

$$\text{LEFT}(2n) = f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_{2n-1})\Delta t.$$

Regroup terms, putting all the even t's first, the odd t's last:

$$\begin{aligned}\text{LEFT}(2n) &= f(t_0)\Delta t + f(t_2)\Delta t + \cdots + f(t_{2n-2})\Delta t + f(t_1)\Delta t + f(t_3)\Delta t + \cdots + f(t_{2n-1})\Delta t \\ &= \underbrace{f(x_0)\frac{\Delta x}{2} + f(x_1)\frac{\Delta x}{2} + \cdots + f(x_{n-1})\frac{\Delta x}{2}}_{\text{LEFT}(n)/2} + \underbrace{f(t_1)\frac{\Delta x}{2} + f(t_3)\frac{\Delta x}{2} + \cdots + f(t_{2n-1})\frac{\Delta x}{2}}_{\text{MID}(n)/2}\end{aligned}$$

So

$$\text{LEFT}(2n) = \frac{1}{2}(\text{LEFT}(n) + \text{MID}(n))$$

34. When $n = 10$, we have $a = 1; b = 2; \Delta x = \frac{1}{10}; f(a) = 1; f(b) = \frac{1}{2}$.
$\text{LEFT}(10) \approx 0.71877, \text{RIGHT}(10) \approx 0.66877, \text{TRAP}(10) \approx 0.69377$
We have
$\text{RIGHT}(10) = \text{LEFT}(10) + f(b)\Delta x - f(a)\Delta x = 0.71877 + \frac{1}{10}(\frac{1}{2}) - \frac{1}{10}(1) = 0.66877$, and $\text{TRAP}(10) = \text{LEFT}(10) + \frac{\Delta x}{2}(f(b) - f(a)) = 0.71877 + \frac{1}{10}\frac{1}{2}(\frac{1}{2} - 1) = 0.69377$,
so the equations are verified.

35. First, we compute:

$$\begin{aligned}(f(b) - f(a))\Delta x &= (f(b) - f(a))\left(\frac{b-a}{n}\right) \\ &= (f(5) - f(2))\left(\frac{3}{n}\right) \\ &= (21 - 13)\left(\frac{3}{n}\right) \\ &= \frac{24}{n}\end{aligned}$$

RIGHT(10) = LEFT(10) + 24 = 3.156 + 2.4 = 5.556.
TRAP(10) = LEFT(10) + $\frac{1}{2}$(2.4) = 3.156 + 1.2 = 4.356.
LEFT(20) = $\frac{1}{2}$(LEFT(10) + MID(10)) = $\frac{1}{2}$(3.156 + 3.242) = 3.199.
RIGHT(20) = LEFT(20) + 2.4 = 3.199 + 1.2 = 4.399.
TRAP(20) = LEFT(20) + $\frac{1}{2}$(1.2) = 3.199 + 0.6 = 3.799.

36. (a) If $f(x) = 1$, then

$$\int_a^b f(x)\,dx = (b-a).$$

Also,

$$\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left(\frac{1}{2} + 2 + \frac{1}{2}\right) = (b-a).$$

So the equation holds for $f(x) = 1$.

If $f(x) = x$, then

$$\int_a^b f(x)\,dx = \left.\frac{x^2}{2}\right|_a^b = \frac{b^2-a^2}{2}.$$

Also,

$$\begin{aligned}\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) &= \frac{b-a}{3}\left(\frac{a}{2} + 2\frac{a+b}{2} + \frac{b}{2}\right)\\ &= \frac{b-a}{3}\left(\frac{a}{2} + a + b + \frac{b}{2}\right)\\ &= \frac{b-a}{3}\left(\frac{3}{2}b + \frac{3}{2}a\right)\\ &= \frac{(b-a)(b+a)}{2}\\ &= \frac{b^2-a^2}{2}.\end{aligned}$$

So the equation holds for $f(x) = x$.

If $f(x) = x^2$, then $\int_a^b f(x)\,dx = \left.\frac{x^3}{3}\right|_a^b = \frac{b^3-a^3}{3}$. Also,

$$\begin{aligned}\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) &= \frac{b-a}{3}\left(\frac{a^2}{2} + 2\left(\frac{a+b}{2}\right)^2 + \frac{b^2}{2}\right)\\ &= \frac{b-a}{3}\left(\frac{a^2}{2} + \frac{a^2+2ab+b^2}{2} + \frac{b^2}{2}\right)\\ &= \frac{b-a}{3}\left(\frac{2a^2+2ab+2b^2}{2}\right)\\ &= \frac{b-a}{3}\left(a^2+ab+b^2\right)\\ &= \frac{b^3-a^3}{3}.\end{aligned}$$

So the equation holds for $f(x) = x^2$.

(b) For any quadratic function, $f(x) = Ax^2 + Bx + C$, the "Facts about Sums and Constant Multiples of Integrands" give us:

$$\int_a^b f(x)\,dx = \int_a^b (Ax^2 + Bx + C)\,dx = A\int_a^b x^2\,dx + B\int_a^b x\,dx + C\int_a^b 1\,dx.$$

Now we use the results of part (a) to get:

$$\begin{aligned}\int_a^b f(x)\,dx &= A\frac{h}{3}\left(\frac{a^2}{2} + 2m^2 + \frac{b^2}{2}\right) + B\frac{h}{3}\left(\frac{a}{2} + 2m + \frac{b}{2}\right) + C\frac{h}{3}\left(\frac{1}{2} + 2\cdot 1 + \frac{1}{2}\right)\\ &= \frac{h}{3}\left(\frac{Aa^2+Ba+C}{2} + 2(Am^2+Bm+C) + \frac{Ab^2+Bb+C}{2}\right)\end{aligned}$$

$$= \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right)$$

37. (a) Suppose $q_i(x)$ is the quadratic function approximating $f(x)$ on the subinterval $[x_i, x_{i+1}]$, and m_i is the midpoint of the interval, $m_i = (x_i + x_{i+1})/2$. Then, using the equation in Problem 36, with $a = x_i$ and $b = x_{i+1}$ and $h = \Delta x = x_{i+1} - x_i$:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \int_{x_i}^{x_{i+1}} q_i(x)dx = \frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2}\right).$$

(b) Summing over all subintervals gives

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} q_i(x)dx = \sum_{i=0}^{n-1}\frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2}\right).$$

Splitting the sum into two parts:

$$\begin{aligned} &= \frac{2}{3}\sum_{i=0}^{n-1} q_i(m_i)\Delta x + \frac{1}{3}\sum_{i=0}^{n-1}\frac{q_i(x_i)+q_i(x_{i+1})}{2}\Delta x \\ &= \frac{2}{3}\text{MID}(n) + \frac{1}{3}\text{TRAP}(n) \\ &= \text{SIMP}(n). \end{aligned}$$

Strengthen Your Understanding

38. The midpoint rule is exact if the integrand is a linear function.

39. As $n \to \infty$, the error approaches 0, so $\text{TRAP}(n) \to 0$ only if the value of the definite integral is 0.

40. It depends on the concavity. If f is concave up, then $\text{TRAP}(n) \geq \text{MID}(n)$, but if f is concave down, then $\text{TRAP}(n) \leq \text{MID}(n)$.

41. Since the total time for each extra two digits of TRAP goes up by a factor of 10, the total time grows exponentially with the number of digits. Thus there is much more additional time to go from 8 digits to 10 digits than from 1 digit to 3 digits.

42. Since we want RIGHT(10) to be an underestimate of $\int_0^1 f(x)dx$, we use a function $f(x)$ that is decreasing on $[0, 1]$. Since we want MID(10) to be an overestimate of the integral, we make $f(x)$ concave down on $[0, 1]$. The function $f(x) = 1 - x^2$ has both of these features. Since $f(x) = 1 - x^2$ is both decreasing and concave down on $[0, 1]$, we have

$$\text{RIGHT}(10) < \int_0^1 f(x)dx < \text{MID}(10).$$

43. In most cases, increasing the number of subintervals in a Trapezoid Rule approximation makes the approximation more accurate. Therefore, we try a function $f(x)$ for which the Trapezoid Rule overestimates the integral, since it is likely in this case that TRAP(40) is a greater overestimate of $\int_0^{10} f(x)dx$ than TRAP(80). Suppose that $f(x) = e^x$. Then $f(x)$ is concave up, and TRAP(40) and TRAP(80) are both overestimates. However, TRAP(80) is smaller, since all of the trapezoids that make up the estimate TRAP(80) fit inside the trapezoids that make up TRAP(40).

44. True. $y^2 - 1$ is concave up, and the midpoint rule always underestimates for a function that is concave up.

45. False. If the function $f(x)$ is a line, then the trapezoid rule gives the exact answer to $\int_a^b f(x)\,dx$.

46. False. The subdivision size $\Delta x = (1/10)(6 - 2) = 4/10$.

47. True, since $\Delta x = (6 - 2)/n = 4/n$.

48. False. If f is decreasing, then on each subinterval the value of $f(x)$ at the left endpoint is larger than the value at the right endpoint, which means that LEFT(n) >RIGHT(n) for any n.

49. False. As n approaches infinity, LEFT(n) approaches the value of the integral $\int_2^6 f(x)\,dx$, which is generally not 0.

50. True. We have

$$\text{LEFT}(n) - \text{RIGHT}(n) = (f(x_0) + f(x_1) + \cdots + f(x_{n-1}))\Delta x - (f(x_1) + f(x_2) + \cdots + f(x_n))\Delta x.$$

On the right side of the equation, all terms cancel except the first and last, so:

$$\text{LEFT}(n) - \text{RIGHT}(n) = (f(x_0) - f(x_n))\Delta x = (f(2) - f(6))\Delta x.$$

This is also discussed in Section 5.1.

51. True. This follows from the fact that $\Delta x = (6-2)/n = 4/n$.

52. False. Since $\text{LEFT}(n) - \text{RIGHT}(n) = (f(2) - f(6))\Delta x$, we have $\text{LEFT}(n) = \text{RIGHT}(n)$ for any function such that $f(2) = f(6)$. Such a function, for example $f(x) = (x-4)^2$, need not be a constant function.

53. False. Although TRAP(n) is usually a better estimate, it is not always better. If $f(2) = f(6)$, then $\text{LEFT}(n) = \text{RIGHT}(n)$ and hence $\text{TRAP}(n) = \text{LEFT}(n) = \text{RIGHT}(n)$, so in this case TRAP$(n)$ is no better.

54. False. This is true if f is an increasing function or if f is a decreasing function, but it is not true in general. For example, suppose that $f(2) = f(6)$. Then $\text{LEFT}(n) = \text{RIGHT}(n)$ for all n, which means that if $\int_2^6 f(x)\,dx$ lies between LEFT(n) and RIGHT(n), then it must equal LEFT(n), which is not always the case.

For example, if $f(x) = (x-4)^2$ and $n = 1$, then $f(2) = f(6) = 4$, so

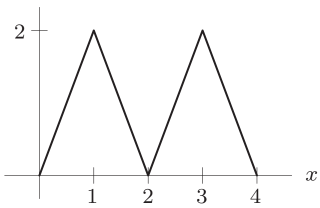
$$\text{LEFT}(1) = \text{RIGHT}(1) = 4 \cdot (6-2) = 16.$$

However

$$\int_2^6 (x-4)^2\,dx = \left.\frac{(x-4)^3}{3}\right|_2^6 = \frac{2^3}{3} - \left(-\frac{2^3}{3}\right) = \frac{16}{3}.$$

In this example, since $\text{LEFT}(n) = \text{RIGHT}(n)$, we have $\text{TRAP}(n) = \text{LEFT}(n)$. However trapezoids overestimate the area, since the graph of f is concave up. This is also discussed in Section 7.5.

55. False. Suppose f is the following:



Then $\text{LEFT}(2) = 0$, $\text{LEFT}(4) = 4$, and

$$\int_a^b f(x)\,dx = 4.$$

56. True. Since f' and g' are greater than 0, all left rectangles give underestimates. The bigger the derivative, the bigger the underestimate, so the bigger the error. (Note: if we did not have $0 < f' < g'$, but instead just had $f' < g'$, the statement would not necessarily be true. This is because some left rectangles could be overestimates and some could be underestimates–so, for example, it could be that the error in approximating g is 0! If $0 < f' < g'$, however, this can't happen.

Solutions for Section 7.6

Exercises

1. (a) See Figure 7.42. The area extends out infinitely far along the positive x-axis.

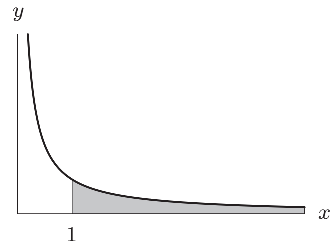


Figure 7.42

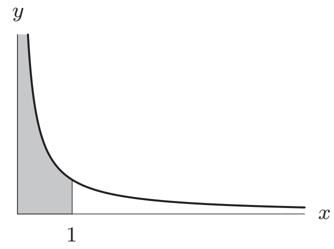


Figure 7.43

(b) See Figure 7.43. The area extends up infinitely far along the positive y-axis.

2. We have

$$\int_0^\infty e^{-0.4x}dx = \lim_{b\to\infty}\int_0^b e^{-0.4x}dx = \lim_{b\to\infty}(-2.5e^{-0.4x})\big|_0^b = \lim_{b\to\infty}(-2.5e^{-0.4b} + 2.5).$$

As $b \to \infty$, we know $e^{-0.4b} \to 0$ and so we see that the integral converges to 2.5. See Figure 7.44. The area continues indefinitely out to the right.

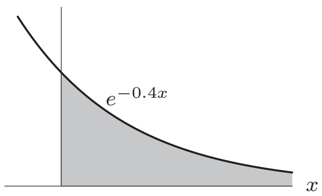


Figure 7.44

3. (a) We use a calculator or computer to evaluate the integrals.
When $b = 5$, we have $\int_0^5 xe^{-x}dx = 0.9596$.
When $b = 10$, we have $\int_0^{10} xe^{-x}dx = 0.9995$.
When $b = 20$, we have $\int_0^{20} xe^{-x}dx = 0.99999996$.

(b) It appears from the answers to part (a) that $\int_0^\infty xe^{-x}dx = 1.0$.

4. (a) See Figure 7.45. The total area under the curve is shaded.

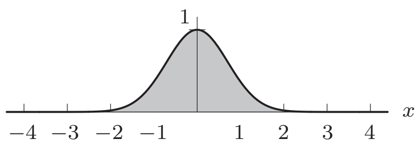


Figure 7.45

(b) When $a = 1$, we use a calculator or computer to see that $\int_{-1}^1 e^{-x^2}dx = 1.49365$.
Similarly, we have:
When $a = 2$, the value of the integral is 1.76416.
When $a = 3$, the value of the integral is 1.77241.
When $a = 4$, the value of the integral is 1.77245.
When $a = 5$, the value of the integral is 1.77245.

(c) It appears that the integral $\int_{-\infty}^\infty e^{-x^2}dx$ converges to approximately 1.77245.

5. We have

$$\int_1^\infty \frac{1}{5x+2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{5x+2}dx = \lim_{b\to\infty}\left(\frac{1}{5}\ln(5x+2)\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{1}{5}\ln(5b+2)-\frac{1}{5}\ln(7)\right).$$

As $b \leftarrow \infty$, we know that $\ln(5b+2)\to\infty$, and so this integral diverges.

6. We have

$$\int_1^\infty \frac{1}{(x+2)^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{(x+2)^2}dx = \lim_{b\to\infty}\left(\frac{-1}{x+2}\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{-1}{b+2}-\frac{-1}{3}\right) = 0+\frac{1}{3}=\frac{1}{3}.$$

This integral converges to $1/3$.

7. This integral is improper at the lower end, so

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{a\to0^+}\int_a^1 \ln x\,dx\\ &= \lim_{a\to0^+}(x\ln x - x)\Big|_a^1\\ &= \lim_{a\to0^+}(1\ln 1-1)-(a\ln a-a))\\ &= -1+\lim_{a\to0^+} a(1-\ln a)\\ &= -1+\lim_{a\to0^+}\frac{1-\ln a}{a^{-1}}\\ &= -1+\lim_{a\to0^+}\frac{-1/a}{-1/a^2}\quad\text{by l'Hopital}\\ &= -1+\lim_{a\to0^+} a\\ &= -1.\end{aligned}$$

If the integral converges, we'd expect it to have a negative value because the logarithm graph is below the x-axis for $0<x<1$.

8. Use the substitution $w=\sqrt{x}$. Since $dw=\frac{1}{2}x^{-1/2}\,dx$, we have $dx=2w\,dw$, so

$$\int e^{-\sqrt{x}}\,dx = 2\int we^{-w}\,dw.$$

Note that this substitution leaves the limits unchanged. Using integration by parts with $u=w$ and $v'=e^{-w}$, we find that

$$\begin{aligned}2\int we^{-w}\,dw &= 2\left[-we^{-w}+\int e^{-w}\,dw\right]\\ &= 2[-we^{-w}-e^{-w}].\end{aligned}$$

So,

$$\begin{aligned}\int_0^\infty we^{-w}\,dx &= \lim_{b\to\infty}2\left[-we^{-w}-e^{-w}\right]_0^b\\ &= 2\lim_{b\to\infty}(-be^{-b}-e^{-b}+1)\\ &= 2\left[\lim_{b\to\infty}-\frac{b+1}{e^b}+1\right]\\ &= 2\left[\lim_{b\to\infty}-\frac{1}{e^b}+1\right]\quad\text{by l'Hopital}\\ &= 2.\end{aligned}$$

9. We have

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty}\left(\frac{-1}{2}e^{-x^2}\right)\Big|_0^b = \lim_{b\to\infty}\left(\frac{-1}{2}e^{-b^2}-\frac{-1}{2}\right) = 0+\frac{1}{2}=\frac{1}{2}.$$

This integral converges to $1/2$.

10.

$$\int_1^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_1^b e^{-2x}\,dx = \lim_{b\to\infty} -\frac{e^{-2x}}{2}\bigg|_1^b$$
$$= \lim_{b\to\infty}(-e^{-2b}/2 + e^{-2}/2) = 0 + e^{-2}/2 = e^{-2}/2,$$

where the first limit is 0 because $\lim_{x\to\infty} e^{-x} = 0$.

11. Using integration by parts with $u = x$ and $v' = e^{-x}$, we find that

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}dx = -(1+x)e^{-x}$$

so

$$\begin{aligned}\int_0^\infty \frac{x}{e^x}\,dx &= \lim_{b\to\infty}\int_0^b \frac{x}{e^x}\,dx \\ &= \lim_{b\to\infty} -1(1+x)e^{-x}\bigg|_0^b \\ &= \lim_{b\to\infty}\left[1-(1+b)e^{-b}\right] \\ &= 1.\end{aligned}$$

12.

$$\int_1^\infty \frac{x}{4+x^2} = \lim_{b\to\infty}\int_1^b \frac{x}{4+x^2}\,dx = \lim_{b\to\infty}\frac{1}{2}\ln|4+x^2|\bigg|_1^b = \lim_{b\to\infty}\frac{1}{2}\ln|4+b^2| - \frac{1}{2}\ln 5.$$

As $b\to\infty$, $\ln|4+b^2|\to\infty$, so the limit diverges.

13.

$$\begin{aligned}\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx &= \lim_{b\to-\infty}\int_b^0 \frac{e^x}{1+e^x}\,dx \\ &= \lim_{b\to-\infty}\ln|1+e^x|\bigg|_b^0 \\ &= \lim_{b\to-\infty}[\ln|1+e^0| - \ln|1+e^b|] \\ &= \ln(1+1)-\ln(1+0) = \ln 2.\end{aligned}$$

14. First, we note that $1/(z^2+25)$ is an even function. Therefore,

$$\int_{-\infty}^\infty \frac{dz}{z^2+25} = \int_{-\infty}^0 \frac{dz}{z^2+25} + \int_0^\infty \frac{dz}{z^2+25} = 2\int_0^\infty \frac{dz}{z^2+25}.$$

We'll now evaluate this improper integral by using a limit:

$$\int_0^\infty \frac{dz}{z^2+25} = \lim_{b\to\infty}\left(\frac{1}{5}\arctan(b/5) - \frac{1}{5}\arctan(0)\right) = \frac{1}{5}\cdot\frac{\pi}{2} = \frac{\pi}{10}.$$

So the original integral is twice that, namely $\pi/5$.

15. This integral is improper because $1/\sqrt{x}$ is undefined at $x=0$. Then

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to0^+}\int_b^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to0^+}\left(2\sqrt{x}\bigg|_b^4\right) = \lim_{b\to0^+}\left(4-2\sqrt{b}\right) = 4.$$

The integral converges.

16.

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx &= \lim_{b\to\pi/2^-} \int_{\pi/4}^{b} \frac{\sin x}{\sqrt{\cos x}}\,dx \\ &= \lim_{b\to\pi/2^-} -\int_{\pi/4}^{b} (\cos x)^{-1/2}(-\sin x)\,dx \\ &= \lim_{b\to\pi/2^-} -2(\cos x)^{1/2}\Big|_{\pi/4}^{b} \\ &= \lim_{b\to\pi/2^-} [-2(\cos b)^{1/2} + 2(\cos\pi/4)^{1/2}] \\ &= 2\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{2}} = 2^{\frac{3}{4}}.\end{aligned}$$

17. This integral is improper because $1/v$ is undefined at $v = 0$. Then

$$\int_0^1 \frac{1}{v}\,dv = \lim_{b\to 0^+} \int_b^1 \frac{1}{v}\,dv = \lim_{b\to 0^+} \left(\ln v\Big|_b^1\right) = -\ln b.$$

As $b \to 0^+$, this goes to infinity and the integral diverges.

18.

$$\lim_{a\to 0^+} \int_a^1 \frac{x^4+1}{x}\,dx = \lim_{a\to 0^+} \left(\frac{x^4}{4} + \ln x\right)\Big|_a^1 = \lim_{a\to 0^+} [1/4 - (a^4/4 + \ln a)],$$

which diverges as $a \to 0$, since $\ln a \to -\infty$.

19.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+1}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to\infty} \arctan(x)\Big|_1^b \\ &= \lim_{b\to\infty} [\arctan(b) - \arctan(1)] \\ &= \pi/2 - \pi/4 = \pi/4.\end{aligned}$$

20.

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x^2+1}}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x^2+1}}\,dx \\ &= \lim_{b\to\infty} \ln|x + \sqrt{x^2+1}|\Big|_1^b \\ &= \lim_{b\to\infty} \ln(b + \sqrt{b^2+1}) - \ln(1+\sqrt{2}).\end{aligned}$$

As $b \to \infty$, this limit does not exist, so the integral diverges.

21. We use V-26 with $a = 4$ and $b = -4$:

$$\begin{aligned}\int_0^4 \frac{-1}{u^2-16}\,du &= \lim_{b\to 4^-} \int_0^b \frac{-1}{u^2-16}\,du \\ &= \lim_{b\to 4^-} \int_0^b \frac{-1}{(u-4)(u+4)}\,du \\ &= \lim_{b\to 4^-} \frac{-(\ln|u-4| - \ln|u+4|)}{8}\Big|_0^b \\ &= \lim_{b\to 4^-} -\frac{1}{8}\left(\ln|b-4| + \ln 4 - \ln|b+4| - \ln 4\right).\end{aligned}$$

As $b \to 4^-$, $\ln|b-4| \to -\infty$, so the limit does not exist and the integral diverges.

22.

$$\int_1^\infty \frac{y}{y^4+1}\,dy = \lim_{b\to\infty} \frac{1}{2}\int_1^b \frac{2y}{(y^2)^2+1}\,dy$$
$$= \lim_{b\to\infty} \frac{1}{2}\arctan(y^2)\Big|_1^b$$
$$= \lim_{b\to\infty} \frac{1}{2}[\arctan(b^2) - \arctan 1]$$
$$= (1/2)[\pi/2 - \pi/4] = \pi/8.$$

23. With the substitution $w = \ln x$, $dw = \frac{1}{x}dx$,

$$\int \frac{dx}{x\ln x} = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|\ln x| + C$$

so

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln x}$$
$$= \lim_{b\to\infty} \ln|\ln x|\Big|_2^b$$
$$= \lim_{b\to\infty} [\ln|\ln b| - \ln|\ln 2|].$$

As $b \to \infty$, the limit goes to ∞ and hence the integral diverges.

24. With the substitution $w = \ln x$, $dw = \frac{1}{x}dx$,

$$\int \frac{\ln x}{x}\,dx = \int w\,dw = \frac{1}{2}w^2 + C = \frac{1}{2}(\ln x)^2 + C$$

so

$$\int_0^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+} \frac{1}{2}[\ln(x)]^2\Big|_a^1 = \lim_{a\to 0^+} -\frac{1}{2}[\ln(a)]^2.$$

As $a \to 0^+$, $\ln a \to -\infty$, so the integral diverges.

25. This is a proper integral; use V-26 in the integral table with $a = 4$ and $b = -4$.

$$\int_{16}^{20} \frac{1}{y^2-16}\,dy = \int_{16}^{20} \frac{1}{(y-4)(y+4)}\,dy$$
$$= \frac{\ln|y-4| - \ln|y+4|}{8}\Big|_{16}^{20}$$
$$= \frac{\ln 16 - \ln 24 - (\ln 12 - \ln 20)}{8}$$
$$= \frac{\ln 320 - \ln 288}{8} = \frac{1}{8}\ln(10/9) = 0.01317.$$

26. Using the substitution $w = -x^{\frac{1}{2}}$, $-2dw = x^{-\frac{1}{2}}\,dx$,

$$\int e^{-x^{\frac{1}{2}}} x^{-\frac{1}{2}}\,dx = -2\int e^w\,dw = -2e^{-x^{\frac{1}{2}}} + C.$$

So

$$\int_0^\pi \frac{1}{\sqrt{x}}e^{-\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^\pi \frac{1}{\sqrt{x}}e^{-\sqrt{x}}\,dx$$
$$= \lim_{b\to 0^+} -2e^{-\sqrt{x}}\Big|_b^\pi$$
$$= 2 - 2e^{-\sqrt{\pi}}.$$

27. Letting $w = \ln x$, $dw = \frac{1}{x}dx$,

$$\int \frac{dx}{x(\ln x)^2} = \int w^{-2}dw = -w^{-1} + C = -\frac{1}{\ln x} + C,$$

so

$$\begin{aligned}\int_3^\infty \frac{dx}{x(\ln x)^2} &= \lim_{b\to\infty}\int_3^b \frac{dx}{x(\ln x)^2}\\ &= \lim_{b\to\infty}\left(-\frac{1}{\ln b} + \frac{1}{\ln 3}\right)\\ &= \frac{1}{\ln 3}.\end{aligned}$$

28.

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx &= \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}\,dx\\ &= \lim_{b\to 2^-} \arcsin\frac{x}{2}\bigg|_0^b\\ &= \lim_{b\to 2^-} \arcsin\frac{b}{2} = \arcsin 1 = \frac{\pi}{2}.\end{aligned}$$

29. $\displaystyle\int_4^\infty \frac{dx}{(x-1)^2} = \lim_{b\to\infty}\int_4^b \frac{dx}{(x-1)^2} = \lim_{b\to\infty} -\frac{1}{(x-1)}\bigg|_4^b = \lim_{b\to\infty}\left[-\frac{1}{b-1} + \frac{1}{3}\right] = \frac{1}{3}.$

30. $\displaystyle\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C = \frac{1}{2}\left(\ln\frac{|x-1|}{|x+1|}\right) + C$, so

$$\begin{aligned}\int_4^\infty \frac{dx}{x^2-1} &= \lim_{b\to\infty}\int_4^b \frac{dx}{x^2-1}\\ &= \lim_{b\to\infty}\frac{1}{2}\left(\ln\frac{|x-1|}{|x+1|}\right)\bigg|_4^b\\ &= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{b-1}{b+1}\right) - \frac{1}{2}\ln\frac{3}{5}\right]\\ &= -\frac{1}{2}\ln\frac{3}{5} = \frac{1}{2}\ln\frac{5}{3}.\end{aligned}$$

31.

$$\begin{aligned}\int_7^\infty \frac{dy}{\sqrt{y-5}} &= \lim_{b\to\infty}\int_7^b \frac{dy}{\sqrt{y-5}}\\ &= \lim_{b\to\infty} 2\sqrt{y-5}\bigg|_7^b\\ &= \lim_{b\to\infty}(2\sqrt{b-5} - 2\sqrt{2}).\end{aligned}$$

As $b \to \infty$, this limit goes to ∞, so the integral diverges.

32. The integrand is undefined at $y = 3$.

$$\begin{aligned}\int_0^3 \frac{y\,dy}{\sqrt{9-y^2}} &= \lim_{b\to 3^-}\int_0^b \frac{y}{\sqrt{9-y^2}}\,dy = \lim_{b\to 3^-}\left(-(9-y^2)^{1/2}\right)\bigg|_0^b\\ &= \lim_{b\to 3^-}\left(3 - (9-b^2)^{1/2}\right) = 3.\end{aligned}$$

33. The integrand is undefined at $\theta = 4$, so we must split the integral there.

$$\int_4^6 \frac{d\theta}{(4-\theta)^2} = \lim_{a\to 4^+} \int_a^6 \frac{d\theta}{(4-\theta)^2} = \lim_{a\to 4^+} (4-\theta)^{-1}\Big|_a^6 = \lim_{a\to 4^+} \left(\frac{1}{-2} - \frac{1}{4-a}\right).$$

Since $1/(4-a) \to -\infty$ as $a \to 4$ from the right, the integral does not converge. It is not necessary to check the convergence of $\int_3^4 \frac{d\theta}{(4-\theta)^2}$. However, we could have started with $\int_3^4 \frac{d\theta}{(4-\theta)^2}$, instead of $\int_4^6 \frac{d\theta}{(4-\theta)^2}$, and arrived at the same conclusion.

Problems

34. We have

$$\begin{aligned} f(x) &= \int_{-\infty}^{x} e^t\,dt \\ &= \lim_{a\to-\infty} \int_a^x e^t\,dt \\ &= \lim_{a\to-\infty} (e^x - e^a) \\ &= e^x - \underbrace{\lim_{a\to-\infty} e^a}_{0} \\ &= e^x. \end{aligned}$$

35. (a) There is no simple antiderivative for this integrand, so we use numerical methods. We find

$$P(1) = \frac{1}{\sqrt{\pi}} \int_0^1 e^{-t^2}\,dt = 0.421.$$

(b) To calculate this improper integral, use numerical methods. If you cannot input infinity into your calculator, increase the upper limit until the value of the integral settles down. We find

$$P(\infty) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\,dt = 0.500.$$

36. Since the graph is above the x-axis for $x \ge 0$, we have

$$\begin{aligned} \text{Area } &= \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x}\,dx \\ &= \lim_{b\to\infty} \left(-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx \right) \\ &= \lim_{b\to\infty} \left(-be^{-b} - e^{-x}\Big|_0^b \right) \\ &= \lim_{b\to\infty} (-be^{-b} - e^{-b} + e^0) = 1. \end{aligned}$$

37. The curve has an asymptote at $t = \frac{\pi}{2}$, and so the area integral is improper there.

$$\text{Area } = \int_0^{\frac{\pi}{2}} \frac{dt}{\cos^2 t} = \lim_{b\to\frac{\pi}{2}} \int_0^b \frac{dt}{\cos^2 t} = \lim_{b\to\frac{\pi}{2}} \tan t \Big|_0^b,$$

which diverges. Therefore the area is infinite.

38. We have:

$$f(3) = \int_0^\infty 3^{-t}dt = \lim_{b\to\infty} \int_0^b 3^{-t}dt = \lim_{b\to\infty} -\frac{1}{\ln 3} 3^{-t}\Big|_0^b = \lim_{b\to\infty} -\frac{1}{\ln 3}\left(3^{-b} - 3^0\right) = \frac{1}{\ln 3} = 0.910.$$

(Since $\lim_{b\to\infty} 3^{-b} = 0$.)

39. We have:

$$f(3) = \int_1^\infty t^{-3}dt = \lim_{b\to\infty}\int_1^b t^{-3}dt = \lim_{b\to\infty} -\frac{1}{2}t^{-2}\bigg|_1^b = \lim_{b\to\infty} -\frac{1}{2}\left(b^{-2}-1\right) = \frac{1}{2}.$$

(Since $\lim\limits_{b\to\infty} b^{-2} = 0$.)

40. We have:

$$f(3) = \int_0^\infty 3e^{-3t}\,dt = 3\lim_{b\to\infty}\int_0^b e^{-3t}\,dt = 3\left(-\frac{1}{3}\right)\lim_{b\to\infty} e^{-3t}\bigg|_0^b = -\lim_{b\to\infty}\left(e^{-3b}-1\right) = -(-1) = 1.$$

41. We have:

$$f(3) = \int_0^\infty 2t\cdot 3e^{-t(3)^2}\,dt = \int_0^\infty 6te^{-9t}\,dt = 6\lim_{b\to\infty}\int_0^b te^{-9t}\,dt.$$

Integration by parts with $u = t, du = dt, dv = e^{-9t}dt, v = (-1/9)e^{-9t}$ gives:

$$\int te^{-9t}\,dt = t\left(-\frac{1}{9}\right)e^{-9t} - \int\left(-\frac{1}{9}\right)e^{-9t}dt = -\frac{1}{9}te^{-9t} - \frac{1}{81}e^{-9t} + C.$$

Thus:

$$f(3) = 6\lim_{b\to\infty}\left(-\frac{1}{9}te^{-9t} - \frac{1}{81}e^{-9t}\right)\bigg|_0^b = 6\left(0+\frac{1}{81}\right) = \frac{2}{27}.$$

42. (a) We have

$$\int_0^\infty \frac{e^{-y/\alpha}}{\alpha}\,dy = \lim_{b\to\infty} -e^{-y/\alpha}\bigg|_0^b = \lim_{b\to\infty}(1-e^{-b/\alpha}) = 1.$$

(b) Using integration by parts with $u = y$ and $v' = (1/\alpha)e^{-y/\alpha}$, so $u' = 1$, $v = -e^{-y/\alpha}$, we have

$$\begin{aligned}\int_0^\infty \frac{ye^{-y/\alpha}}{\alpha}\,dy &= \lim_{b\to\infty}\int_0^b \frac{ye^{-y/\alpha}}{\alpha}\,dy\\ &= \lim_{b\to\infty}\left(-ye^{-y/\alpha}\bigg|_0^b + \int_0^b e^{-y/\alpha}\,dy\right)\\ &= \lim_{b\to\infty}\left(-be^{-b/\alpha} - \alpha e^{-y/\alpha}\bigg|_0^b\right)\\ &= \lim_{b\to\infty}(-be^{-b/\alpha} - \alpha e^{-b/\alpha} + \alpha)\end{aligned}$$

Since $\lim_{b\to\infty} -be^{-b/\alpha} = \lim_{b\to\infty} e^{-b/\alpha} = 0$, we have

$$\int \frac{ye^{-y/\alpha}}{\alpha}\,dy = \alpha.$$

(c) Using integration by parts, this time with $u = y^2$, $v' = (1/\alpha)e^{-y/\alpha}$, so $u' = 2y$, $v = -e^{-y/\alpha}$, we have

$$\begin{aligned}\int_0^\infty \frac{y^2e^{-y/\alpha}}{\alpha}\,dy &= \lim_{b\to\infty}\int_0^b \frac{y^2e^{-y/\alpha}}{\alpha}\,dy\\ &= \lim_{b\to\infty}\left(-y^2e^{-y/\alpha}\bigg|_0^b + 2\int_0^b ye^{-y/\alpha}\,dy\right)\\ &= \lim_{b\to\infty} -b^2e^{-b/\alpha} + 2\int_0^\infty ye^{-y/\alpha}\,dy\end{aligned}$$

Now $\lim_{b\to\infty} -b^2e^{-b/\alpha} = 0$ and in part (b) we found

$$\int_0^\infty \frac{ye^{-y/\alpha}}{\alpha}\,dy = \alpha,$$

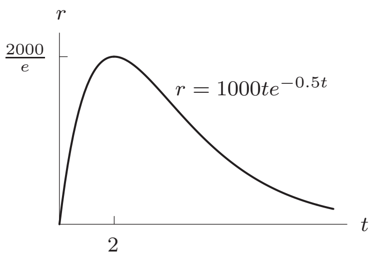
so

$$\int_0^\infty ye^{-y/\alpha}\,dy = \alpha^2.$$

Thus,

$$\int_0^\infty \frac{y^2e^{-y/\alpha}}{\alpha}\,dy = 2\int_0^\infty ye^{-y/\alpha}\,dy = 2\alpha^2.$$

43. (a) Using a calculator or a computer, the graph is:



(b) People are getting sick fastest when the rate of infection is highest, i.e. when r is at its maximum. Since

$$\begin{aligned} r' &= 1000e^{-0.5t} - 1000(0.5)te^{-0.5t} \\ &= 500e^{-0.5t}(2-t) \end{aligned}$$

this must occur at $t = 2$.

(c) The total number of sick people $= \displaystyle\int_0^\infty 1000te^{-0.5t}\,dt$.

Using integration by parts, with $u = t$, $v' = e^{-0.5t}$:

$$\begin{aligned} \text{Total} &= \lim_{b\to\infty} 1000\left(\frac{-t}{0.5}e^{-0.5t}\Big|_0^b - \int_0^b \frac{-1}{0.5}e^{-0.5t}\,dt\right) \\ &= \lim_{b\to\infty} 1000\left(-2be^{-0.5b} - \frac{2}{0.5}e^{-0.5b}\right)\Big|_0^b \\ &= \lim_{b\to\infty} 1000\left(-2be^{-0.5b} - 4e^{-0.5b} + 4\right) \\ &= 4000 \text{ people.} \end{aligned}$$

44. The energy required is

$$\begin{aligned} E &= \int_1^\infty \frac{kq_1q_2}{r^2}\,dr = kq_1q_2 \lim_{b\to\infty} -\frac{1}{r}\Big|_1^b \\ &= (9\times 10^9)(1)(1)(1) = 9\times 10^9 \text{ joules} \end{aligned}$$

45. We let $t = (x-a)/\sqrt{b}$. This means that $dt = dx/\sqrt{b}$, and that $t = \pm\infty$ when $x = \pm\infty$. We have

$$\int_{-\infty}^\infty e^{-(x-a)^2/b}\,dx = \int_{-\infty}^\infty e^{-t^2}\,(\sqrt{b}\,dt) = \sqrt{b}\int_{-\infty}^\infty e^{-t^2}\,dt = \sqrt{b}\sqrt{\pi} = \sqrt{b\pi}.$$

46. Applying integration by parts with $u = e^{-x^2/2}$ and $v' = g'(x)$, so $u' = -xe^{-x^2/2}$ and $v = g(x)$, we have

$$\int_{-\infty}^\infty g'(x)e^{-x^2/2}\,dx = g(x)e^{-x^2/2}\Big|_{-\infty}^\infty - \int_{-\infty}^\infty -xe^{-x^2/2}g(x)\,dx.$$

Since $g(x)$ is bounded, $\lim_{x\to\infty} g(x)e^{-x^2/2} = \lim_{x\to-\infty} g(x)e^{-x^2/2} = 0$. Thus

$$\int_{-\infty}^\infty g'(x)e^{-x^2/2}\,dx = \int_{-\infty}^\infty xg(x)e^{-x^2/2}\,dx.$$

47. Make the substitution $w = 2x$, so $dw = 2\,dx$ and $x = w/2$. When $x = 0$, $w = 0$ and when $x \to \infty$, we have $w \to \infty$. We have

$$\begin{aligned}\int_0^\infty \frac{x^4 e^{2x}}{(e^{2x}-1)^2}\,dx &= \int_0^\infty \frac{(w/2)^4 e^w}{(e^w-1)^2}\,dw/2\\ &= \frac{1}{32}\int_0^\infty \frac{w^4 e^w}{(e^w-1)^2}\,dw\\ &= \frac{1}{32}\frac{4\pi^4}{15} = \frac{\pi^4}{120}.\end{aligned}$$

48. (a)

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}\,dt\\ &= \lim_{b\to\infty}\int_0^b e^{-t}\,dt\\ &= \lim_{b\to\infty} -e^{-t}\Big|_0^b\\ &= \lim_{b\to\infty}[1-e^{-b}] = 1.\end{aligned}$$

Using Problem 11,

$$\Gamma(2) = \int_0^\infty te^{-t}\,dt = 1.$$

(b) We integrate by parts. Let $u = t^n$, $v' = e^{-t}$. Then $u' = nt^{n-1}$ and $v = -e^{-t}$, so

$$\int t^n e^{-t}\,dt = -t^n e^{-t} + n\int t^{n-1}e^{-t}\,dt.$$

So

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty t^n e^{-t}\,dt\\ &= \lim_{b\to\infty}\int_0^b t^n e^{-t}\,dt\\ &= \lim_{b\to\infty}\Big[-t^n e^{-t}\Big|_0^b + n\int_0^b t^{n-1}e^{-t}\,dt\Big]\\ &= \lim_{b\to\infty} -b^n e^{-b} + \lim_{b\to\infty} n\int_0^b t^{n-1}e^{-t}\,dt\\ &= 0 + n\int_0^\infty t^{n-1}e^{-t}\,dt\\ &= n\Gamma(n).\end{aligned}$$

(c) We already have $\Gamma(1) = 1$ and $\Gamma(2) = 1$. Using $\Gamma(n+1) = n\Gamma(n)$ we can get

$$\begin{aligned}\Gamma(3) &= 2\Gamma(2) = 2\\ \Gamma(4) &= 3\Gamma(3) = 3\cdot 2\\ \Gamma(5) &= 4\Gamma(4) = 4\cdot 3\cdot 2.\end{aligned}$$

So it appears that $\Gamma(n)$ is just the first $n-1$ numbers multiplied together, so $\Gamma(n) = (n-1)!$.

Strengthen Your Understanding

49. Let $f(x) = g(x) = 1/x$. Then both $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ diverge, but $\int_1^\infty f(x)g(x)\,dx = \int_1^\infty 1/x^2\,dx$, which converges.

50. If $f(x) = 1/x$, then $\int_1^\infty f(x)\,dx$ diverges, but $\lim_{x\to\infty} f(x) = 0$.

51. The function $f(x) = 1/x$ is an example. We know that $\lim_{x\to\infty} 1/x = 0$, but

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

Thus the improper integral diverges.

52. Since the integral is over a bounded interval, and $f(x)$ is continuous at $x = 2$ and at $x = 5$, we need to find a function $f(x)$ that has a discontinuity in the interval $(2, 5)$. Let's try $f(x) = 1/(x-3)^2$. Because this function is discontinuous at $x = 3$, we must split the improper integral $\int_2^5 1/(x-3)^2\,dx$ into two integrals:

$$\begin{aligned}\int_2^5 \frac{1}{(x-3)^2}\,dx &= \int_2^3 \frac{1}{(x-3)^2}\,dx + \int_3^5 \frac{1}{(x-3)^2}\,dx \\ &= \lim_{b\to 3^-} \int_2^b \frac{1}{(x-3)^2}\,dx + \lim_{a\to 3^+} \int_a^5 \frac{1}{(x-3)^2}\,dx \\ &= \lim_{b\to 3^-} \left(-\frac{1}{(b-3)} + \frac{1}{2-3}\right) + \lim_{a\to 3^+} \left(-\frac{1}{(5-3)} + \frac{1}{(a-3)}\right) \\ &= \lim_{b\to 3^-} \left(-\frac{1}{(b-3)} - 1\right) + \lim_{a\to 3^+} \left(\frac{1}{(a-3)} - \frac{1}{2}\right)\end{aligned}$$

Both of these limits go to ∞. Thus the improper integral $\int_2^5 f(x)dx$ diverges.

53. True. Since

$$\lim_{b\to\infty} \int_0^b f(x)\,dx = \int_0^a f(x)\,dx + \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

the limit on the left side of the equation is finite exactly when the limit on the right side is finite. Thus, if $\int_0^\infty f(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$.

54. diverges.

True. Suppose that f has period p. Then $\int_0^p f(x)\,dx$, $\int_p^{2p} f(x)\,dx$, $\int_{2p}^{3p} f(x)\,dx$,... are all equal. If we let $k = \int_0^p f(x)\,dx$, then $\int_0^{np} f(x)\,dx = nk$, for any positive integer n. Since $f(x)$ is positive, so is k. Thus as n approaches ∞, the value of $\int_0^{np} f(x)\,dx = nk$ approaches ∞. That means that $\lim_{b\to\infty} \int_0^b f(x)\,dx$ is not finite; that is, the integral diverges.

55. False. Let $f(x) = 1/(x+1)$. Then

$$\int_0^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty} \ln|x+1|\Big|_0^b = \lim_{b\to\infty} \ln(b+1),$$

but $\lim_{b\to\infty} \ln(b+1)$ does not exist.

56. False. Let $f(x) = x + 1$. Then

$$\int_0^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty} \ln|x+1|\Big|_0^b = \lim_{b\to\infty} \ln(b+1),$$

but $\lim_{b\to\infty} \ln(b+1)$ does not exist.

57. True. By properties of integrals and limits,

$$\lim_{b\to\infty} \int_0^b (f(x) + g(x))\,dx = \lim_{b\to\infty} \int_0^b f(x)\,dx + \lim_{b\to\infty} \int_0^b g(x)\,dx.$$

Since the two limits on the right side of the equation are finite, the limit on the left side is also finite, that is, $\int_0^\infty (f(x) + g(x))\,dx$ converges.

58. False. For example, let $f(x) = x$ and $g(x) = -x$. Then $f(x) + g(x) = 0$, so $\int_0^\infty (f(x) + g(x))\,dx$ converges, even though $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ diverge.

59. True. By properties of integrals and limits,

$$\lim_{b\to\infty}\int_0^b af(x)\,dx = a\lim_{b\to\infty}\int_0^b f(x)\,dx.$$

Thus, the limit on the left of the equation is finite exactly when the limit on the right side of the equation is finite. Thus $\int_0^\infty af(x)\,dx$ converges if $\int_0^\infty f(x)\,dx$ converges.

60. True. Make the substitution $w = ax$. Then $dw = a\,dx$, so

$$\int_0^b f(ax)\,dx = \frac{1}{a}\int_0^c f(w)\,dw,$$

where $c = ab$. As b approaches infinity, so does c, since a is constant. Thus the limit of the left side of the equation as b approaches infinity is finite exactly when the limit of the right side of the equation as c approaches infinity is finite. That is, $\int_0^\infty f(ax)\,dx$ converges exactly when $\int_0^\infty f(x)\,dx$ converges.

61. True. Make the substitution $w = a + x$, so $dw = dx$. Then $w = a$ when $x = 0$, and $w = a + b$ when $x = b$, so

$$\int_0^b f(a+x)\,dx = \int_a^{b+a} f(w)\,dw = \int_a^c f(w)\,dw$$

where $c = b + a$. As b approaches infinity, so does c, since a is constant. Thus the limit of the left side of the equation as b approaches infinity is finite exactly when the limit of the right side of the equation as c approaches infinity is finite. Since $\int_0^\infty f(x)\,dx$ converges, we know that $\lim_{c\to\infty}\int_0^c f(w)\,dw$ is finite, so $\lim_{c\to\infty}\int_a^c f(w)\,dw$ is finite for any positive a. Thus, $\int_0^\infty f(a+x)\,dx$ converges.

62. False. We have

$$\int_0^b (a + f(x))\,dx = \int_0^b a\,dx + \int_0^b f(x)\,dx.$$

Since $\int_0^\infty f(x)\,dx$ converges, the second integral on the right side of the equation has a finite limit as b approaches infinity. But the first integral on the right side has an infinite limit as b approaches infinity, since $a \neq 0$. Thus the right side all together has an infinite limit, which means that $\int_0^\infty (a + f(x))\,dx$ diverges.

Solutions for Section 7.7

Exercises

1. For large x, the integrand behaves like $1/x^2$ because

$$\frac{x^2}{x^4+1} \approx \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Since $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, we expect our integral to converge. More precisely, since $x^4 + 1 > x^4$, we have

$$\frac{x^2}{x^4+1} < \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Since $\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent, the comparison test tells us that $\displaystyle\int_1^\infty \frac{x^2}{x^4+1}\,dx$ converges also.

2. For large x, the integrand behaves like $1/x$ because

$$\frac{x^3}{x^4-1} \approx \frac{x^3}{x^4} = \frac{1}{x}.$$

Since $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ does not converge, we expect our integral not to converge. More precisely, since $x^4 - 1 < x^4$, we have

$$\frac{x^3}{x^4-1} > \frac{x^3}{x^4} = \frac{1}{x}.$$

Since $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ does not converge, the comparison test tells us that $\displaystyle\int_2^\infty \frac{x^3}{x^4-1}\,dx$ does not converge either.

3. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x^2+1}{x^3+3x+2}$ behaves like $\frac{x^2}{x^3}$ or $\frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, we predict that the given integral will diverge.

4. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{1}{x^2+5x+1}$ behaves like $\frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, we predict that the given integral will converge.

5. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x}{x^2+2x+4}$ behaves like $\frac{x}{x^2}$ or $\frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, we predict that the given integral will diverge.

6. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x^2-6x+1}{x^2+4}$ behaves like $\frac{x^2}{x^2}$ or 1. Since $\int_1^\infty 1\,dx$ diverges, we predict that the given integral will diverge.

7. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{5x+2}{x^4+8x^2+4}$ behaves like $\frac{5x}{x^4}$ or $\frac{5}{x^3}$. Since $\int_1^\infty \frac{5}{x^3}\,dx$ converges, we predict that the given integral will converge.

8. For large t, the 2 is negligible in comparison to e^{5t}, so the integrand behaves like e^{-5t}. Thus

$$\frac{1}{e^{5t}+2} \approx \frac{1}{e^{5t}} = e^{-5t}.$$

More precisely, since $e^{5t}+2 > e^{5t}$, we have

$$\frac{1}{e^{5t}+2} < \frac{1}{e^{5t}} = e^{-5t}.$$

Since $\int_1^\infty e^{-5t}\,dt$ converges, by the Comparison Theorem $\int_1^\infty \frac{1}{e^{5t}+2}\,dt$ converges also.

9. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x^2+4}{x^4+3x^2+11}$ behaves like $\frac{x^2}{x^4}$ or $\frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, we predict that the given integral will converge.

10. It converges:

$$\int_{50}^\infty \frac{dz}{z^3} = \lim_{b\to\infty}\int_{50}^b \frac{dz}{z^3} = \lim_{b\to\infty}\left(-\frac{1}{2}z^{-2}\Big|_{50}^b\right) = \frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{50^2}-\frac{1}{b^2}\right) = \frac{1}{5000}$$

11. Since $\frac{1}{1+x} \geq \frac{1}{2x}$ and $\frac{1}{2}\int_0^\infty \frac{1}{x}\,dx$ diverges, we have that $\int_1^\infty \frac{dx}{1+x}$ diverges.

12. If $x \geq 1$, we know that $\frac{1}{x^3+1} \leq \frac{1}{x^3}$, and since $\int_1^\infty \frac{dx}{x^3}$ converges, the improper integral $\int_1^\infty \frac{dx}{x^3+1}$ converges.

13. The integrand is unbounded as $t \to 5$. We substitute $w = t-5$, so $dw = dt$. When $t = 5$, $w = 0$ and when $t = 8$, $w = 3$.

$$\int_5^8 \frac{6}{\sqrt{t-5}}\,dt = \int_0^3 \frac{6}{\sqrt{w}}\,dw.$$

Since

$$\int_0^3 \frac{6}{\sqrt{w}}\,dw = \lim_{a\to0^+} 6\int_a^3 \frac{1}{\sqrt{w}}\,dw = 6\lim_{a\to0^+} 2w^{1/2}\Big|_a^3 = 12\lim_{a\to0^+}(\sqrt{3}-\sqrt{a}) = 12\sqrt{3},$$

our integral converges.

14. The integral converges.

$$\int_0^1 \frac{1}{x^{19/20}}\,dx = \lim_{a\to0}\int_a^1 \frac{1}{x^{19/20}}\,dx = \lim_{a\to0} 20x^{1/20}\Big|_a^1 = \lim_{a\to0} 20\left(1-a^{1/20}\right) = 20.$$

15. This integral diverges. To see this, substitute $t+1=w$, $dt=dw$. So,

$$\int_{t=-1}^{t=5} \frac{dt}{(t+1)^2} = \int_{w=0}^{w=6} \frac{dw}{w^2},$$

which diverges.

16. Since we know the antiderivative of $\dfrac{1}{1+u^2}$, we can use the Fundamental Theorem of Calculus to evaluate the integral. Since the integrand is even, we write

$$\int_{-\infty}^{\infty} \frac{du}{1+u^2} = 2\int_0^{\infty} \frac{du}{1+u^2} = 2\lim_{b\to\infty}\int_0^b \frac{du}{1+u^2}$$
$$= 2\lim_{b\to\infty} \arctan b = 2\left(\frac{\pi}{2}\right) = \pi.$$

Thus, the integral converges to π.

17. Since $\dfrac{1}{u+u^2} < \dfrac{1}{u^2}$ for $u \geq 1$, and since $\displaystyle\int_1^{\infty} \frac{du}{u^2}$ converges, $\displaystyle\int_1^{\infty} \frac{du}{u+u^2}$ converges.

18. This improper integral diverges. We expect this because, for large θ, $\dfrac{1}{\sqrt{\theta^2+1}} \approx \dfrac{1}{\sqrt{\theta^2}} = \dfrac{1}{\theta}$ and $\displaystyle\int_1^{\infty} \frac{d\theta}{\theta}$ diverges. More precisely, for $\theta \geq 1$

$$\frac{1}{\sqrt{\theta^2+1}} \geq \frac{1}{\sqrt{\theta^2+\theta^2}} = \frac{1}{\sqrt{2}\sqrt{\theta^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\theta}$$

and $\displaystyle\int_1^{\infty} \frac{d\theta}{\theta}$ diverges. (The factor $\frac{1}{\sqrt{2}}$ does not affect the divergence.)

19. For $\theta \geq 2$, we have $\dfrac{1}{\sqrt{\theta^3+1}} \leq \dfrac{1}{\sqrt{\theta^3}} = \dfrac{1}{\theta^{\frac{3}{2}}}$, and $\displaystyle\int_2^{\infty} \frac{d\theta}{\theta^{3/2}}$ converges (check by integration), so $\displaystyle\int_2^{\infty} \frac{d\theta}{\sqrt{\theta^3+1}}$ converges.

20. This integral is improper at $\theta = 0$. For $0 \leq \theta \leq 1$, we have $\dfrac{1}{\sqrt{\theta^3+\theta}} \leq \dfrac{1}{\sqrt{\theta}}$, and since $\displaystyle\int_0^1 \frac{1}{\sqrt{\theta}}\,d\theta$ converges, $\displaystyle\int_0^1 \frac{d\theta}{\sqrt{\theta^3+\theta}}$ converges.

21. Since $\dfrac{1}{1+e^y} \leq \dfrac{1}{e^y} = e^{-y}$ and $\displaystyle\int_0^{\infty} e^{-y}\,dy$ converges, the integral $\displaystyle\int_0^{\infty} \frac{dy}{1+e^y}$ converges.

22. This integral is convergent because, for $\phi \geq 1$,

$$\frac{2+\cos\phi}{\phi^2} \leq \frac{3}{\phi^2},$$

and $\displaystyle\int_1^{\infty} \frac{3}{\phi^2}d\phi = 3\int_1^{\infty} \frac{1}{\phi^2}d\phi$ converges.

23. Since $\dfrac{1}{e^z+2^z} < \dfrac{1}{e^z} = e^{-z}$ for $z \geq 0$, and $\displaystyle\int_0^{\infty} e^{-z}dz$ converges, $\displaystyle\int_0^{\infty} \frac{dz}{e^z+2^z}$ converges.

24. Since $\dfrac{1}{\phi^2} \leq \dfrac{2-\sin\phi}{\phi^2}$ for $0 < \phi \leq \pi$, and since $\displaystyle\int_0^{\pi} \frac{1}{\phi^2}d\phi$ diverges, $\displaystyle\int_0^{\pi} \frac{2-\sin\phi}{\phi^2}d\phi$ must diverge.

25. Since $\dfrac{3+\sin\alpha}{\alpha} \geq \dfrac{2}{\alpha}$ for $\alpha \geq 4$, and since $\displaystyle\int_4^{\infty} \frac{2}{\alpha}d\alpha$ diverges, then $\displaystyle\int_4^{\infty} \frac{3+\sin\alpha}{\alpha}d\alpha$ diverges.

Problems

26. (a) The area is infinite. The area under $1/x$ is infinite and the area under $1/x^2$ is 1. So the area between the two has to be infinite also.

(b) Since $f(x)$ is bounded between 0 and $1/x^2$, and the area under $1/x^2$ is finite, $f(x)$ will have finite area by the comparison test. Similarly, $h(x)$ lies above $1/x$, whose area is infinite, so $h(x)$ must have infinite area as well. We can tell nothing about the area of $g(x)$, because the comparison test tells us nothing about a function larger than a function with finite area but smaller than one with infinite area. Finally, $k(x)$ will certainly have infinite area, because it has a lower bound m, for some $m > 0$. Thus, $\int_0^a k(x)\,dx \geq ma$, and since the latter does not converge as $a \to \infty$, neither can the former.

27. The convergence or divergence of an improper integral depends on the long-term behavior of the integrand, not on its short-term behavior. Figure 7.46 suggests that $g(x) \leq f(x)$ for all values of x beyond $x = k$. Since $\int_k^\infty f(x)\,dx$ converges, we expect $\int_k^\infty g(x)\,dx$ converges also.

However we are interested in $\int_a^\infty g(x)\,dx$. Breaking the integral into two parts enables us to use the fact that $\int_k^\infty g(x)\,dx$ is finite:

$$\int_a^\infty g(x)\,dx = \int_a^k g(x)\,dx + \int_k^\infty g(x)\,dx.$$

The first integral is also finite because the interval from a to k is finite. Therefore, we expect $\int_a^\infty g(x)\,dx$ converges.

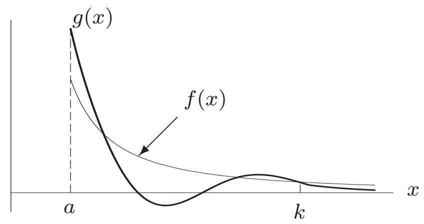


Figure 7.46

28. First let's calculate the indefinite integral $\displaystyle\int \frac{dx}{x(\ln x)^p}$. Let $\ln x = w$, then $\dfrac{dx}{x} = dw$. So

$$\begin{aligned}
\int \frac{dx}{x(\ln x)^p} &= \int \frac{dw}{w^p} \\
&= \begin{cases} \ln|w| + C, & \text{if } p = 1 \\ \frac{1}{1-p} w^{1-p} + C, & \text{if } p \neq 1 \end{cases} \\
&= \begin{cases} \ln|\ln x| + C, & \text{if } p = 1 \\ \frac{1}{1-p} (\ln x)^{1-p} + C, & \text{if } p \neq 1. \end{cases}
\end{aligned}$$

Notice that $\lim_{x\to\infty} \ln x = +\infty$.

(a) $p = 1$:

$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{b\to\infty} \Big(\ln|\ln b| - \ln|\ln 2| \Big) = +\infty.$$

(b) $p < 1$:

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \frac{1}{1-p} \left(\lim_{b\to\infty} (\ln b)^{1-p} - (\ln 2)^{1-p} \right) = +\infty.$$

(c) $p > 1$:

$$\begin{aligned}
\int_2^\infty \frac{dx}{x(\ln x)^p} &= \frac{1}{1-p} \left(\lim_{b\to\infty} (\ln b)^{1-p} - (\ln 2)^{1-p} \right) \\
&= \frac{1}{1-p} \left(\lim_{b\to\infty} \frac{1}{(\ln b)^{p-1}} - (\ln 2)^{1-p} \right) \\
&= -\frac{1}{1-p} (\ln 2)^{1-p}.
\end{aligned}$$

Thus, $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$ is convergent for $p > 1$, divergent for $p \leq 1$.

29. The indefinite integral $\int \frac{dx}{x(\ln x)^p}$ is computed in Problem 28. Let $\ln x = w$, then $\frac{dx}{x} = dw$. Notice that $\lim_{x\to 1} \ln x = 0$, and $\lim_{x\to 0^+} \ln x = -\infty$.

For this integral notice that $\ln 1 = 0$, so the integrand blows up at $x = 1$.

(a) $p = 1$:

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{a\to 1^+}(\ln|\ln 2| - \ln|\ln a|)$$

Since $\ln a \to 0$ as $a \to 1$, $\ln|\ln a| \to -\infty$ as $b \to 1$. So the integral is divergent.

(b) $p < 1$:

$$\begin{aligned}\int_1^2 \frac{dx}{x(\ln x)^p} &= \frac{1}{1-p}\lim_{a\to 1^+}\left((\ln 2)^{1-p} - (\ln a)^{1-p}\right)\\ &= \frac{1}{1-p}(\ln 2)^{1-p}.\end{aligned}$$

(c) $p > 1$:

$$\int_1^2 \frac{dx}{x(\ln x)^p} = \frac{1}{1-p}\lim_{a\to 1^+}\left((\ln 2)^{1-p} - (\ln a)^{1-p}\right)$$

As $\lim_{a\to 1^+}(\ln a)^{1-p} = \lim_{a\to 1^+}\frac{1}{(\ln a)^{p-1}} = +\infty$, the integral diverges.

Thus, $\int_1^2 \frac{dx}{x(\ln x)^p}$ is convergent for $p < 1$, divergent for $p \geq 1$.

30. (a) Since $e^{-x^2} \leq e^{-3x}$ for $x \geq 3$,

$$\int_3^\infty e^{-x^2}\,dx \leq \int_3^\infty e^{-3x}\,dx$$

Now

$$\begin{aligned}\int_3^\infty e^{-3x}\,dx &= \lim_{b\to\infty}\int_3^b e^{-3x}\,dx = \lim_{b\to\infty} -\frac{1}{3}e^{-3x}\Big|_3^b\\ &= \lim_{b\to\infty}\frac{e^{-9}}{3} - \frac{e^{-3b}}{3} = \frac{e^{-9}}{3}.\end{aligned}$$

Thus

$$\int_3^\infty e^{-x^2}\,dx \leq \frac{e^{-9}}{3}.$$

(b) By reasoning similar to part (a),

$$\int_n^\infty e^{-x^2}\,dx \leq \int_n^\infty e^{-nx}\,dx,$$

and

$$\int_n^\infty e^{-nx}\,dx = \frac{1}{n}e^{-n^2},$$

so

$$\int_n^\infty e^{-x^2}\,dx \leq \frac{1}{n}e^{-n^2}.$$

31. (a) The tangent line to e^t has slope $(e^t)' = e^t$. Thus at $t = 0$, the slope is $e^0 = 1$. The line passes through $(0, e^0) = (0, 1)$. Thus the equation of the tangent line is $y = 1 + t$. Since e^t is everywhere concave up, its graph is always above the graph of any of its tangent lines; in particular, e^t is always above the line $y = 1 + t$. This is tantamount to saying

$$1 + t \leq e^t,$$

with equality holding only at the point of tangency, $t = 0$.

(b) If $t = \dfrac{1}{x}$, then the above inequality becomes

$$1 + \frac{1}{x} \leq e^{1/x}, \text{ or } e^{1/x} - 1 \geq \frac{1}{x}.$$

Since $t = \dfrac{1}{x}$, t is never zero. Therefore, the inequality is strict, and we write

$$e^{1/x} - 1 > \frac{1}{x}.$$

(c) Since $e^{1/x} - 1 > \dfrac{1}{x}$,

$$\frac{1}{x^5\left(e^{1/x} - 1\right)} < \frac{1}{x^5\left(\frac{1}{x}\right)} = \frac{1}{x^4}.$$

Since $\displaystyle\int_1^\infty \frac{dx}{x^4}$ converges, $\displaystyle\int_1^\infty \frac{dx}{x^5\left(e^{1/x} - 1\right)}$ converges.

Strengthen Your Understanding

32. We cannot compare the integrals, because the first integrand is sometimes less than the second and sometimes greater than the second, depending on the sign of $\sin x$.

33. We can use the comparison test with $\dfrac{1}{x^{\sqrt{2}}}$. Since $x > 0$, we have

$$0 < \frac{1}{x^{\sqrt{2}} + 1} < \frac{1}{x^{\sqrt{2}}},$$

and $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\, dx$ is convergent ($p = \sqrt{2} > 1$), the original integral, $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}} + 1}\, dx$, converges.

34. The integral $\displaystyle\int_0^\infty f(x)\, dx$ might converge or diverge; the comparison test can not be used here.

35. Let $f(x) = \dfrac{1}{x^2}$. Then $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx$ converges. However $\displaystyle\int_1^\infty \frac{1}{f(x)}\, dx = \int_1^\infty x^2\, dx$ diverges.

36. We know that the integral $\int_1^\infty 3/(2x^2)\, dx$ converges because

$$\int_1^\infty \frac{3}{2x^2} dx = \lim_{b\to\infty} \int_1^b \frac{3}{2x^2} dx = \lim_{b\to\infty} \left(-\frac{3}{2b} + \frac{3}{2}\right) = \frac{3}{2}.$$

So we know that if $f(x)$ is positive and $f(x) \leq 3/(2x^2)$ for all $x \geq 1$, then $\int_1^\infty f(x)dx$ converges. So the function $f(x) = 3/(2x^2 + 1)$ is a good example.

37. Since $-1 \leq \sin x \leq 1$ for all x, we know that, for all x,

$$7x - 2 \leq 7x - 2\sin x \leq 7x + 2.$$

So since $7x - 2\sin x \geq 0$ for all $x \geq 1$, we know that for all $x \geq 1$,

$$\frac{3}{7x - 2\sin x} \geq \frac{3}{7x + 2}.$$

So let

$$f(x) = \frac{3}{7x + 2}.$$

We then have

$$\int_1^\infty f(x)dx = \lim_{b\to\infty} \int_1^b \frac{3}{7x + 2} dx = \lim_{b\to\infty} \left(\frac{3}{7}\ln|7b + 2| - \frac{3}{7}\ln 9\right) = \infty$$

So the integral $\int_1^\infty f(x)dx$ diverges, as desired.

38. True. Since $\dfrac{1}{e^x + x} < \dfrac{1}{e^x}$ and $\displaystyle\int_0^\infty \frac{1}{e^x}\, dx$ converges, $\displaystyle\int_0^\infty \frac{dx}{e^x + x}$ converges.

39. True. Since $\dfrac{1}{x^2 - 3} > \dfrac{1}{x^2}$ and $\displaystyle\int_0^1 \frac{dx}{x^2}$ diverges, $\displaystyle\int_0^1 \frac{dx}{x^2 - 3}$ diverges.

Solutions for Chapter 7 Review

Exercises

1. $\frac{1}{3}(t+1)^3$
2. $-\cos 2\theta$
3. $\dfrac{5^x}{\ln 5}$
4. $e^t + 5\dfrac{1}{5}e^{5t} = e^t + e^{5t}$
5. Using the power rule gives $\dfrac{3}{2}w^2 + 7w + C$.
6. $\frac{1}{2}e^{2r} + C$
7. Since $\dfrac{d}{dt}\cos t = -\sin t$, we have

$$\int \sin t\, dt = -\cos t + C, \text{ where } C \text{ is a constant.}$$

8. Let $2t = w$, then $2dt = dw$, so $dt = \frac{1}{2}dw$, so

$$\int \cos 2t\, dt = \int \frac{1}{2}\cos w\, dw = \frac{1}{2}\sin w + C = \frac{1}{2}\sin 2t + C,$$

where C is a constant.

9. Let $5z = w$, then $5dz = dw$, which means $dz = \frac{1}{5}dw$, so

$$\int e^{5z}\, dz = \int e^w \cdot \frac{1}{5}dw = \frac{1}{5}\int e^w\, dw = \frac{1}{5}e^w + C = \frac{1}{5}e^{5z} + C,$$

where C is a constant.

10. $\sin(x+1) + C$
11. Since $\displaystyle\int \sin w\, d\theta = -\cos w + C$, the substitution $w = 2\theta$, $dw = 2\, d\theta$ gives $\displaystyle\int \sin 2\theta\, d\theta = -\frac{1}{2}\cos 2\theta + C$.
12. Let $w = x^3 - 1$, then $dw = 3x^2 dx$ so that

$$\int (x^3-1)^4 x^2 dx = \frac{1}{3}\int w^4 dw = \frac{1}{15}w^5 + C = \frac{1}{15}(x^3-1)^5 + C.$$

13. The power rule gives $\dfrac{2}{5}x^{5/2} + \dfrac{3}{5}x^{5/3} + C$
14. From the rule for antidifferentiation of exponentials, we get

$$\int \left(e^x + 3^x\right) dx = e^x + \frac{1}{\ln 3}\cdot 3^x + C.$$

15. $\displaystyle\int \frac{1}{e^z}\, dz = \int e^{-z}\, dz = -e^{-z} + C$
16. Rewrite the integrand as

$$\int \left(\frac{4}{x^2} - \frac{3}{x^3}\right) dx = 4\int x^{-2}\, dx - 3\int x^{-3}\, dx = -4x^{-1} + \frac{3}{2}x^{-2} + C.$$

17. Dividing by x^2 gives

$$\int \left(\frac{x^3+x+1}{x^2}\right) dx = \int \left(x + \frac{1}{x} + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 + \ln|x| - \frac{1}{x} + C.$$

18. Let $w = 1 + \ln x$, then $dw = dx/x$ so that

$$\int \frac{(1+\ln x)^2}{x}\,dx = \int w^2 dw = \frac{1}{3}w^3 + C = \frac{1}{3}(1+\ln x)^3 + C.$$

19. Substitute $w = t^2$, so $dw = 2t\,dt$.

$$\int te^{t^2}\,dt = \frac{1}{2}\int e^{t^2} 2t\,dt = \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{t^2} + C.$$

Check:

$$\frac{d}{dt}\left(\frac{1}{2}e^{t^2} + C\right) = 2t\left(\frac{1}{2}e^{t^2}\right) = te^{t^2}.$$

20. Integration by parts with $u = x$, $v' = \cos x$ gives

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx + C = x\sin x + \cos x + C.$$

Or use III-16 with $p(x) = x$ and $a = 1$ in the integral table.

21. Integration by parts twice gives

$$\int x^2 e^{2x}\,dx = \frac{x^2 e^{2x}}{2} - \int 2xe^{2x}\,dx = \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C$$
$$= (\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4})e^{2x} + C.$$

Or use the integral table, III-14 with $p(x) = x^2$ and $a = 1$.

22. Using substitution with $w = 1 - x$ and $dw = -dx$, we get

$$\int x\sqrt{1-x}\,dx = -\int (1-w)\sqrt{w}dw = \frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2} + C = \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C.$$

23. Let $u = \ln y$, $v' = y$. Then, $v = \frac{1}{2}y^2$ and $u' = \dfrac{1}{y}$. Integrating by parts, we get:

$$\int y\ln y\,dy = \frac{1}{2}y^2\ln y - \int \frac{1}{2}y^2 \cdot \frac{1}{y}\,dy$$
$$= \frac{1}{2}y^2\ln y - \frac{1}{2}\int y\,dy$$
$$= \frac{1}{2}y^2\ln y - \frac{1}{4}y^2 + C.$$

24. We integrate by parts, with $u = y$, $v' = \sin y$. We have $u' = 1$, $v = -\cos y$, and

$$\int y\sin y\,dy = -y\cos y - \int (-\cos y)\,dy = -y\cos y + \sin y + C.$$

Check:

$$\frac{d}{dy}(-y\cos y + \sin y + C) = -\cos y + y\sin y + \cos y = y\sin y.$$

25. We integrate by parts, using $u = (\ln x)^2$ and $v' = 1$. Then $u' = 2\frac{\ln x}{x}$ and $v = x$, so

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$

But, integrating by parts or using the integral table, $\int \ln x\,dx = x\ln x - x + C$. Therefore,

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C.$$

Check:

$$\frac{d}{dx}\left[x(\ln x)^2 - 2x\ln x + 2x + C\right] = (\ln x)^2 + x\frac{2\ln x}{x} - 2\ln x - 2x\frac{1}{x} + 2 = (\ln x)^2.$$

26. Using the exponent rules and the chain rule, we have

$$\int e^{0.5-0.3t}\,dt = e^{0.5}\int e^{-0.3t}\,dt = -\frac{e^{0.5}}{0.3}e^{-0.3t} + C = -\frac{e^{0.5-0.3t}}{0.3} + C.$$

27. Let $\sin\theta = w$, then $\cos\theta\,d\theta = dw$, so

$$\int \sin^2\theta\cos\theta\,d\theta = \int w^2\,dw = \frac{1}{3}w^3 + C = \frac{1}{3}\sin^3\theta + C,$$

where C is a constant.

28. Substitute $w = 4 - x^2$, $dw = -2x\,dx$:

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int \sqrt{w}\,dw = -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(4-x^2)^{3/2} + C.$$

Check

$$\frac{d}{dx}\left[-\frac{1}{3}(4-x^2)^{3/2} + C\right] = -\frac{1}{3}\left[\frac{3}{2}(4-x^2)^{1/2}(-2x)\right] = x\sqrt{4-x^2}.$$

29. Expanding the numerator and dividing, we have

$$\int \frac{(u+1)^3}{u^2}\,du = \int \frac{(u^3+3u^2+3u+1)}{u^2}\,du = \int \left(u + 3 + \frac{3}{u} + \frac{1}{u^2}\right)du$$
$$= \frac{u^2}{2} + 3u + 3\ln|u| - \frac{1}{u} + C.$$

Check:

$$\frac{d}{du}\left(\frac{u^2}{2} + 3u + 3\ln|u| - \frac{1}{u} + C\right) = u + 3 + 3/u + 1/u^2 = \frac{(u+1)^3}{u^2}.$$

30. Substitute $w = \sqrt{y}$, $dw = 1/(2\sqrt{y})\,dy$. Then

$$\int \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy = 2\int \cos w\,dw = 2\sin w + C = 2\sin\sqrt{y} + C.$$

Check:

$$\frac{d}{dy}2\sin\sqrt{y} + C = \frac{2\cos\sqrt{y}}{2\sqrt{y}} = \frac{\cos\sqrt{y}}{\sqrt{y}}.$$

31. Since $\frac{d}{dz}(\tan z) = \frac{1}{\cos^2 z}$, we have

$$\int \frac{1}{\cos^2 z}\,dz = \tan z + C.$$

Check:

$$\frac{d}{dz}(\tan z + C) = \frac{d}{dz}\frac{\sin z}{\cos z} = \frac{(\cos z)(\cos z) - (\sin z)(-\sin z)}{\cos^2 z} = \frac{1}{\cos^2 z}.$$

32. Denote $\int \cos^2\theta\,d\theta$ by A. Let $u = \cos\theta$, $v' = \cos\theta$. Then, $v = \sin\theta$ and $u' = -\sin\theta$. Integrating by parts, we get:

$$A = \cos\theta\sin\theta - \int(-\sin\theta)\sin\theta\,d\theta.$$

Employing the identity $\sin^2\theta = 1 - \cos^2\theta$, the equation above becomes:

$$A = \cos\theta\sin\theta + \int d\theta - \int \cos^2\theta\,d\theta$$
$$= \cos\theta\sin\theta + \theta - A + C.$$

Solving this equation for A, and using the identity $\sin 2\theta = 2\cos\theta\sin\theta$ we get:

$$A = \int \cos^2\theta\, d\theta = \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C.$$

[Note: An alternate solution would have been to use the identity $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$.]

33. Multiplying out and integrating term by term:

$$\int t^{10}(t-10)\, dt = \int (t^{11} - 10t^{10})\, dt = \int t^{11} dt - 10\int t^{10}\, dt = \frac{1}{12}t^{12} - \frac{10}{11}t^{11} + C.$$

34. Substitute $w = 2x - 6$. Then $dw = 2\, dx$ and

$$\begin{aligned}\int \tan(2x-6)\, dx &= \frac{1}{2}\int \tan w\, dw = \frac{1}{2}\int \frac{\sin w}{\cos w}\, dw \\ &= -\frac{1}{2}\ln|\cos w| + C \text{ by substitution or by I-7 of the integral table.} \\ &= -\frac{1}{2}\ln|\cos(2x-6)| + C.\end{aligned}$$

35. Let $\ln x = w$, then $\frac{1}{x}\, dx = dw$, so

$$\int \frac{(\ln x)^2}{x}\, dx = \int w^2\, dw = \frac{1}{3}w^3 + C = \frac{1}{3}(\ln x)^3 + C, \text{ where } C \text{ is a constant.}$$

36. Multiplying out, dividing, and then integrating yields

$$\int \frac{(t+2)^2}{t^3}\, dt = \int \frac{t^2+4t+4}{t^3}\, dt = \int \frac{1}{t}\, dt + \int \frac{4}{t^2}\, dt + \int \frac{4}{t^3}\, dt = \ln|t| - \frac{4}{t} - \frac{2}{t^2} + C,$$

where C is a constant.

37. Integrating term by term:

$$\int \left(x^2 + 2x + \frac{1}{x}\right)\, dx = \frac{1}{3}x^3 + x^2 + \ln|x| + C,$$

where C is a constant.

38. Dividing and then integrating, we obtain

$$\int \frac{t+1}{t^2}\, dt = \int \frac{1}{t}\, dt + \int \frac{1}{t^2}\, dt = \ln|t| - \frac{1}{t} + C, \text{ where } C \text{ is a constant.}$$

39. Let $t^2 + 1 = w$, then $2t\, dt = dw$, $t dt = \frac{1}{2}\, dw$, so

$$\int te^{t^2+1}\, dt = \int e^w \cdot \frac{1}{2}\, dw = \frac{1}{2}\int e^w\, dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{t^2+1} + C,$$

where C is a constant.

40. Let $\cos\theta = w$, then $-\sin\theta\, d\theta = dw$, so

$$\begin{aligned}\int \tan\theta\, d\theta &= \int \frac{\sin\theta}{\cos\theta}\, d\theta = \int \frac{-1}{w}\, dw \\ &= -\ln|w| + C = -\ln|\cos\theta| + C,\end{aligned}$$

where C is a constant.

41. If $u = \sin(5\theta)$, $du = \cos(5\theta) \cdot 5\,d\theta$, so

$$\begin{aligned}\int \sin(5\theta)\cos(5\theta)d\theta &= \frac{1}{5}\int \sin(5\theta)\cdot 5\cos(5\theta)d\theta = \frac{1}{5}\int u\,du \\ &= \frac{1}{5}\left(\frac{u^2}{2}\right) + C = \frac{1}{10}\sin^2(5\theta) + C\end{aligned}$$

or

$$\begin{aligned}\int \sin(5\theta)\cos(5\theta)d\theta &= \frac{1}{2}\int 2\sin(5\theta)\cos(5\theta)d\theta = \frac{1}{2}\int \sin(10\theta)d\theta \quad \text{(using } \sin(2x) = 2\sin x\cos x) \\ &= \frac{-1}{20}\cos(10\theta) + C.\end{aligned}$$

42. Using substitution,

$$\begin{aligned}\int \frac{x}{x^2+1}\,dx &= \int \frac{1/2}{w}\,dw \qquad (x^2+1 = w, 2x\,dx = dw, xdx = \frac{1}{2}\,dw) \\ &= \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|x^2+1| + C,\end{aligned}$$

where C is a constant.

43. Since $\dfrac{d}{dz}(\arctan z) = \dfrac{1}{1+z^2}$, we have

$$\int \frac{dz}{1+z^2} = \arctan z + C, \text{ where } C \text{ is a constant.}$$

44. Let $w = 2z$, so $dw = 2dz$. Then, since $\dfrac{d}{dw}\arctan w = \dfrac{1}{1+w^2}$, we have

$$\int \frac{dz}{1+4z^2} = \int \frac{\frac{1}{2}dw}{1+w^2} = \frac{1}{2}\arctan w + C = \frac{1}{2}\arctan 2z + C.$$

45. Let $w = \cos 2\theta$. Then $dw = -2\sin 2\theta\,d\theta$, hence

$$\int \cos^3 2\theta \sin 2\theta\,d\theta = -\frac{1}{2}\int w^3\,dw = -\frac{w^4}{8} + C = -\frac{\cos^4 2\theta}{8} + C.$$

Check:

$$\frac{d}{d\theta}\left(-\frac{\cos^4 2\theta}{8}\right) = -\frac{(4\cos^3 2\theta)(-\sin 2\theta)(2)}{8} = \cos^3 2\theta \sin 2\theta.$$

46. Let $\cos 5\theta = w$, then $-5\sin 5\theta\,d\theta = dw$, $\sin 5\theta\,d\theta = -\frac{1}{5}dw$. So

$$\begin{aligned}\int \sin 5\theta \cos^3 5\theta\,d\theta &= \int w^3\cdot(-\frac{1}{5})\,dw = -\frac{1}{5}\int w^3\,dw = -\frac{1}{20}w^4 + C \\ &= -\frac{1}{20}\cos^4 5\theta + C,\end{aligned}$$

where C is a constant.

47.

$$\begin{aligned}\int \sin^3 z\cos^3 z\,dz &= \int \sin z(1-\cos^2 z)\cos^3 z\,dz \\ &= \int \sin z\cos^3 z\,dz - \int \sin z\cos^5 z\,dz\end{aligned}$$

$$
\begin{aligned}
&= \int w^3\,(-dw) - \int w^5\,(-dw) \quad (\text{let } \cos z = w, \text{ so } -\sin z\,dz = dw)\\
&= -\int w^3\,dw + \int w^5\,dw\\
&= -\frac{1}{4}w^4 + \frac{1}{6}w^6 + C\\
&= -\frac{1}{4}\cos^4 z + \frac{1}{6}\cos^6 z + C,
\end{aligned}
$$

where C is a constant.

48. If $u = t - 10$, $t = u + 10$ and $dt = 1\,du$, so substituting we get

$$
\begin{aligned}
\int (u+10)u^{10} du = \int (u^{11} + 10u^{10})\,du &= \frac{1}{12}u^{12} + \frac{10}{11}u^{11} + C\\
&= \frac{1}{12}(t-10)^{12} + \frac{10}{11}(t-10)^{11} + C.
\end{aligned}
$$

49. Let $\sin\theta = w$, then $\cos\theta\,d\theta = dw$, so

$$
\begin{aligned}
\int \cos\theta\sqrt{1+\sin\theta}\,d\theta &= \int \sqrt{1+w}\,dw\\
&= \frac{(1+w)^{3/2}}{3/2} + C = \frac{2}{3}(1+\sin\theta)^{3/2} + C,
\end{aligned}
$$

where C is a constant.

50.

$$
\begin{aligned}
\int xe^x\,dx &= xe^x - \int e^x\,dx \qquad (\text{let } x = u, e^x = v', e^x = v)\\
&= xe^x - e^x + C,
\end{aligned}
$$

where C is a constant.

51.

$$
\begin{aligned}
\int t^3e^t\,dt &= t^3e^t - \int 3t^2e^t\,dt \qquad (\text{let } t^3 = u, e^t = v', 3t^2 = u', e^t = v)\\
&= t^3e^t - 3\int t^2e^t\,dt \qquad (\text{let } t^2 = u, e^t = v')\\
&= t^3e^t - 3(t^2e^t - \int 2te^t\,dt)\\
&= t^3e^t - 3t^2e^t + 6\int te^t\,dt \qquad (\text{let } t = u, e^t = v')\\
&= t^3e^t - 3t^2e^t + 6(te^t - \int e^t\,dt)\\
&= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C,
\end{aligned}
$$

where C is a constant.

52. Let $x^2 = w$, then $2xdx = dw$, $x = 1 \Rightarrow w = 1$, $x = 3 \Rightarrow w = 9$. Thus,

$$
\begin{aligned}
\int_1^3 x(x^2+1)^{70}\,dx &= \int_1^9 (w+1)^{70}\frac{1}{2}\,dw\\
&= \frac{1}{2}\cdot\frac{1}{71}(w+1)^{71}\Big|_1^9\\
&= \frac{1}{142}(10^{71} - 2^{71}).
\end{aligned}
$$

53. Let $w = 3z + 5$ and $dw = 3\,dz$. Then

$$\int (3z+5)^3\,dz = \frac{1}{3}\int w^3 dw = \frac{1}{12}w^4 + C = \frac{1}{12}(3z+5)^4 + C.$$

54. Rewrite $9 + u^2$ as $9[1 + (u/3)^2]$ and let $w = u/3$, then $dw = du/3$ so that

$$\int \frac{du}{9+u^2} = \frac{1}{3}\int \frac{dw}{1+w^2} = \frac{1}{3}\arctan w + C = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C.$$

55. Let $u = \sin w$, then $du = \cos w\,dw$ so that

$$\int \frac{\cos w}{1+\sin^2 w}\,dw = \int \frac{du}{1+u^2} = \arctan u + C = \arctan(\sin w) + C.$$

56. Let $w = \ln x$, then $dw = (1/x)dx$ which gives

$$\int \frac{1}{x}\tan(\ln x)\,dx = \int \tan w\,dw = \int \frac{\sin w}{\cos w}\,dw = -\ln(|\cos w|) + C = -\ln(|\cos(\ln x)|) + C.$$

57. Let $w = \ln x$, then $dw = (1/x)dx$ so that

$$\int \frac{1}{x}\sin(\ln x)\,dx = \int \sin w\,dw = -\cos w + C = -\cos(\ln x) + C.$$

58. Let $u = 16 - w^2$, then $du = -2w\,dw$ so that

$$\int \frac{wdw}{\sqrt{16-w^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{16-w^2} + C.$$

59. Dividing and then integrating term by term, we get

$$\int \frac{e^{2y}+1}{e^{2y}}\,dy = \int \left(\frac{e^{2y}}{e^{2y}} + \frac{1}{e^{2y}}\right)dy = \int (1+e^{-2y})\,dy = \int dy + \left(-\frac{1}{2}\right)\int e^{-2y}(-2)\,dy$$
$$= y - \frac{1}{2}e^{-2y} + C.$$

60. Let $u = 1 - \cos w$, then $du = \sin w\,dw$ which gives

$$\int \frac{\sin w\,dw}{\sqrt{1-\cos w}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1-\cos w} + C.$$

61. Let $w = \ln x$. Then $dw = (1/x)dx$ which gives

$$\int \frac{dx}{x\ln x} = \int \frac{dw}{w} = \ln|w| + C = \ln|\ln x| + C.$$

62. Let $w = 3u + 8$, then $dw = 3du$ and

$$\int \frac{du}{3u+8} = \int \frac{dw}{3w} = \frac{1}{3}\ln|3u+8| + C.$$

63. Let $w = \sqrt{x^2+1}$, then $dw = \dfrac{xdx}{\sqrt{x^2+1}}$ so that

$$\int \frac{x}{\sqrt{x^2+1}}\cos\sqrt{x^2+1}\,dx = \int \cos w\,dw = \sin w + C = \sin\sqrt{x^2+1} + C.$$

64. Integrating by parts using $u = t^2$ and $dv = \frac{t\,dt}{\sqrt{1+t^2}}$ gives $du = 2t\,dt$ and $v = \sqrt{1+t^2}$. Now

$$\begin{aligned}\int \frac{t^3}{\sqrt{1+t^2}}\,dt &= t^2\sqrt{1+t^2} - \int 2t\sqrt{1+t^2}\,dt\\ &= t^2\sqrt{1+t^2} - \frac{2}{3}(1+t^2)^{3/2} + C\\ &= \sqrt{1+t^2}(t^2 - \frac{2}{3}(1+t^2)) + C\\ &= \sqrt{1+t^2}\frac{(t^2-2)}{3} + C.\end{aligned}$$

65. Using integration by parts, let $r = u$ and $dt = e^{ku}du$, so $dr = du$ and $t = (1/k)e^{ku}$. Thus

$$\int ue^{ku}\,du = \frac{u}{k}e^{ku} - \frac{1}{k}\int e^{ku}\,du = \frac{u}{k}e^{ku} - \frac{1}{k^2}e^{ku} + C.$$

66. Let $u = w + 5$, then $du = dw$ and noting that $w = u - 5$ we obtain

$$\begin{aligned}\int (w+5)^4 w\,dw &= \int u^4(u-5)\,du\\ &= \int \left(u^5 - 5u^4\right)\,du\\ &= \frac{1}{6}u^6 - u^5 + C\\ &= \frac{1}{6}(w+5)^6 - (w+5)^5 + C.\end{aligned}$$

67. $\int e^{\sqrt{2}x+3}dx = \frac{1}{\sqrt{2}}\int e^{\sqrt{2}x+3}\sqrt{2}dx$. If $u = \sqrt{2}x + 3$, $du = \sqrt{2}dx$, so

$$\frac{1}{\sqrt{2}}\int e^u du = \frac{1}{\sqrt{2}}e^u + C = \frac{1}{\sqrt{2}}e^{\sqrt{2}x+3} + C.$$

68. $\int (e^x + x)^2 dx = \int (e^{2x} + 2xe^x + x^2)dy$. Separating into three integrals, we have

$$\int e^{2x}dx = \frac{1}{2}\int e^{2x}2\,dx = \frac{1}{2}e^{2x} + C_1,$$

$$\int 2xe^x dx = 2\int xe^x dx = 2xe^x - 2e^x + C_2$$

from Formula II-13 of the integral table or integration by parts, and

$$\int x^2\,dx = \frac{x^3}{3} + C_3.$$

Combining the results and writing $C = C_1 + C_2 + C_3$, we get

$$\frac{1}{2}e^{2x} + 2xe^x - 2e^x + \frac{x^3}{3} + C.$$

69. Integrate by parts, $r = \ln u$ and $dt = u^2\,du$, so $dr = (1/u)\,du$ and $t = (1/3)u^3$. We have

$$\int u^2 \ln u\,du = \frac{1}{3}u^3\ln u - \frac{1}{3}\int u^2\,du = \frac{1}{3}u^3\ln u - \frac{1}{9}u^3 + C.$$

70. The integral table yields

$$\int \frac{5x+6}{x^2+4}\,dx = \frac{5}{2}\ln|x^2+4| + \frac{6}{2}\arctan\frac{x}{2} + C$$
$$= \frac{5}{2}\ln|x^2+4| + 3\arctan\frac{x}{2} + C.$$

Check:

$$\frac{d}{dx}\left(\frac{5}{2}\ln|x^2+4| + \frac{6}{2}\arctan\frac{x}{2} + C\right) = \frac{5}{2}\left(\frac{1}{x^2+4}(2x) + 3\frac{1}{1+(x/2)^2}\frac{1}{2}\right)$$
$$= \frac{5x}{x^2+4} + \frac{6}{x^2+4} = \frac{5x+6}{x^2+4}.$$

71. Using Table IV-19, let $m = 3$, $w = 2x$, and $dw = 2dx$. Then

$$\int \frac{1}{\sin^3(2x)}\,dx = \frac{1}{2}\int \frac{1}{\sin^3 w}\,dw$$
$$= \frac{1}{2}\left[\frac{-1}{(3-1)}\frac{\cos w}{\sin^2 w}\right] + \frac{1}{4}\int \frac{1}{\sin w}\,dw,$$

and using Table IV-20, we have

$$\int \frac{1}{\sin w}\,dw = \frac{1}{2}\ln\left|\frac{\cos w - 1}{\cos w + 1}\right| + C.$$

Thus,

$$\int \frac{1}{\sin^3(2x)}\,dx = -\frac{\cos 2x}{4\sin^2 2x} + \frac{1}{8}\ln\left|\frac{\cos 2x - 1}{\cos 2x + 1}\right| + C.$$

72. We can factor $r^2 - 100 = (r-10)(r+10)$ so we can use Table V-26 (with $a = 10$ and $b = -10$) to get

$$\int \frac{dr}{r^2 - 100} = \frac{1}{20}\left[\ln|r-10| + \ln|r+10|\right] + C.$$

73. Integration by parts will be used twice here. First let $u = y^2$ and $dv = \sin(cy)\,dy$, then $du = 2y\,dy$ and $v = -(1/c)\cos(cy)$. Thus

$$\int y^2\sin(cy)\,dy = -\frac{y^2}{c}\cos(cy) + \frac{2}{c}\int y\cos(cy)\,dy.$$

Now use integration by parts to evaluate the integral in the right hand expression. Here let $u = y$ and $dv = \cos(cy)dy$ which gives $du = dy$ and $v = (1/c)\sin(cy)$. Then we have

$$\int y^2\sin(cy)\,dy = -\frac{y^2}{c}\cos(cy) + \frac{2}{c}\left(\frac{y}{c}\sin(cy) - \frac{1}{c}\int \sin(cy)\,dy\right)$$
$$= -\frac{y^2}{c}\cos(cy) + \frac{2y}{c^2}\sin(cy) + \frac{2}{c^3}\cos(cy) + C.$$

74. Integration by parts will be used twice. First let $u = e^{-ct}$ and $dv = \sin(kt)dt$, then $du = -ce^{-ct}dt$ and $v = (-1/k)\cos kt$. Then

$$\int e^{-ct}\sin kt\,dt = -\frac{1}{k}e^{-ct}\cos kt - \frac{c}{k}\int e^{-ct}\cos kt\,dt$$
$$= -\frac{1}{k}e^{-ct}\cos kt - \frac{c}{k}\left(\frac{1}{k}e^{-ct}\sin kt + \frac{c}{k}\int e^{-ct}\sin kt\,dt\right)$$
$$= -\frac{1}{k}e^{-ct}\cos kt - \frac{c}{k^2}e^{-ct}\sin kt - \frac{c^2}{k^2}\int e^{-ct}\sin kt\,dt$$

Solving for $\int e^{-ct}\sin kt\,dt$ gives

$$\frac{k^2+c^2}{k^2}\int e^{-ct}\sin kt\,dt = -\frac{e^{-ct}}{k^2}\left(k\cos kt + c\sin kt\right),$$

so

$$\int e^{-ct}\sin kt\,dt = -\frac{e^{-ct}}{k^2+c^2}\left(k\cos kt + c\sin kt\right) + C.$$

75. Using II-9 from the integral table, with $a = 5$ and $b = 3$, we have

$$\begin{aligned}\int e^{5x}\cos(3x)\,dx &= \frac{1}{25+9}e^{5x}\left[5\cos(3x)+3\sin(3x)\right]+C\\ &= \frac{1}{34}e^{5x}\left[5\cos(3x)+3\sin(3x)\right]+C.\end{aligned}$$

76. Since $\int (x^{\sqrt{k}} + (\sqrt{k})^x)dx = \int x^{\sqrt{k}}dx + \int (\sqrt{k})^x\,dx$, for the first integral, use Formula I-1 with $n = \sqrt{k}$. For the second integral, use Formula I-3 with $a = \sqrt{k}$. The result is

$$\int (x^{\sqrt{x}} + (\sqrt{k})^x)\,dx = \frac{x^{(\sqrt{k})+1}}{(\sqrt{k})+1} + \frac{(\sqrt{k})^x}{\ln\sqrt{k}} + C.$$

77. Factor $\sqrt{3}$ out of the integrand and use VI-30 of the integral table with $u = 2x$ and $du = 2dx$ to get

$$\begin{aligned}\int \sqrt{3+12x^2}\,dx &= \int \sqrt{3}\sqrt{1+4x^2}\,dx\\ &= \frac{\sqrt{3}}{2}\int \sqrt{1+u^2}\,du\\ &= \frac{\sqrt{3}}{4}\left(u\sqrt{1+u^2} + \int \frac{1}{\sqrt{1+u^2}}\,du\right).\end{aligned}$$

Then from VI-29, simplify the integral on the right to get

$$\begin{aligned}\int \sqrt{3+12x^2}\,dx &= \frac{\sqrt{3}}{4}\left(u\sqrt{1+u^2} + \ln|u+\sqrt{1+u^2}|\right) + C\\ &= \frac{\sqrt{3}}{4}\left(2x\sqrt{1+(2x)^2} + \ln|2x+\sqrt{1+(2x)^2}|\right) + C.\end{aligned}$$

78. By completing the square, we get

$$x^2-3x+2 = (x^2-3x+(-\frac{3}{2})^2)+2-\frac{9}{4} = (x-\frac{3}{2})^2-\frac{1}{4}.$$

Then

$$\int \frac{1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{1}{\sqrt{(x-\frac{3}{2})^2-\frac{1}{4}}}\,dx.$$

Let $w = (x-(3/2))$, then $dw = dx$ and $a^2 = 1/4$. Then we have

$$\int \frac{1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{1}{\sqrt{w^2-a^2}}\,dw$$

and from VI-29 of the integral table we have

$$\begin{aligned}\int \frac{1}{\sqrt{w^2-a^2}}\,dw &= \ln\left|w+\sqrt{w^2-a^2}\right| + C\\ &= \ln\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}\right| + C\\ &= \ln\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right| + C.\end{aligned}$$

79. First divide x^2+3x+2 into x^3 to obtain

$$\frac{x^3}{x^2+3x+2} = x-3+\frac{7x+6}{x^2+3x+2}.$$

Since $x^2+3x+2=(x+1)(x+2)$, we can use V-27 of the integral table (with $c=7$, $d=6$, $a=-1$, and $b=-2$) to get

$$\int \frac{7x+6}{x^2+3x+2}\,dx = -\ln|x+1| + 8\ln|x+2| + C.$$

Including the terms $x-3$ from the long division and integrating them gives

$$\int \frac{x^3}{x^2+3x+2}\,dx = \int\left(x-3+\frac{7x+6}{x^2+3x+6}\right)dx = \frac{1}{2}x^2 - 3x - \ln|x+1| + 8\ln|x+2| + C.$$

80. First divide x^2+1 by x^2-3x+2 to obtain

$$\frac{x^2+1}{x^2-3x+2} = 1+\frac{3x-1}{x^2-3x+2}.$$

Factoring $x^2-3x+2=(x-2)(x-1)$ we can use V-27 (with $c=3$, $d=-1$, $a=2$ and $b=1$) to write

$$\int \frac{3x-1}{x^2-3x+2}\,dx = 5\ln|x-2| - 2\ln|x-1| + C.$$

Remembering to include the extra term of $+1$ we got when dividing, we get

$$\int \frac{x^2+1}{x^2-3x+2}\,dx = \int\left(1+\frac{3x-1}{x^2-3x+2}\right)dx = x + 5\ln|x-2| - 2\ln|x-1| + C.$$

81. We can factor the denominator into $ax(x+\frac{b}{a})$, so

$$\int \frac{dx}{ax^2+bx} = \frac{1}{a}\int \frac{1}{x(x+\frac{b}{a})}$$

Now we can use V-26 (with $A=0$ and $B=-\frac{b}{a}$ to give

$$\frac{1}{a}\int \frac{1}{x(x+\frac{b}{a})} = \frac{1}{a}\cdot\frac{a}{b}\left(\ln|x| - \ln\left|x+\frac{b}{a}\right|\right) + C = \frac{1}{b}\left(\ln|x| - \ln\left|x+\frac{b}{a}\right|\right) + C.$$

82. Let $w = ax^2+2bx+c$, then $dw=(2ax+2b)dx$ so that

$$\int \frac{ax+b}{ax^2+2bx+c}\,dx = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|ax^2+2bx+c| + C.$$

83. Multiplying out and integrating term by term,

$$\int\left(\frac{x}{3}+\frac{3}{x}\right)^2 dx = \int\left(\frac{x^2}{9}+2+\frac{9}{x^2}\right)dx = \frac{1}{9}\left(\frac{x^3}{3}\right) + 2x + 9\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{27} + 2x - \frac{9}{x} + C.$$

84. If $u = 2^t+1$, $du = 2^t(\ln 2)\,dt$, so

$$\int \frac{2^t}{2^t+1}\,dt = \frac{1}{\ln 2}\int \frac{2^t\ln 2}{2^t+1}\,dt = \frac{1}{\ln 2}\int \frac{1}{u} = \frac{1}{\ln 2}\ln|u| + C = \frac{1}{\ln 2}\ln|2^t+1| + C.$$

85. If $u = 1-x$, $du = -1\,dx$, so

$$\int 10^{1-x}dx = -1\int 10^{1-x}(-1\,dx) = -1\int 10^u\,du = -1\frac{10^u}{\ln 10} + C = -\frac{1}{\ln 10}10^{1-x} + C.$$

86. Multiplying out and integrating term by term gives

$$\int (x^2+5)^3 dx = \int (x^6+15x^4+75x^2+125)dx = \frac{1}{7}x^7 + 15\frac{x^5}{5} + 75\frac{x^3}{3} + 125x + C$$
$$= \frac{1}{7}x^7 + 3x^5 + 25x^3 + 125x + C.$$

87. Integrate by parts letting $r = v$ and $dt = \arcsin v\, dv$ then $dr = dv$ and to find t we integrate $\arcsin v\, dv$ by parts letting $x = \arcsin v$ and $dy = dv$. This gives

$$t = v \arcsin v - \int (1/\sqrt{1-v^2}) v\, dv = v \arcsin v + \sqrt{1-v^2}.$$

Now, back to the original integration by parts, and we have

$$\int v \arcsin v\, dv = v^2 \arcsin v + v\sqrt{1-v^2} - \int \left[v \arcsin v + \sqrt{1-v^2} \right] dv.$$

Adding $\int v \arcsin v\, dv$ to both sides of the above line we obtain

$$2\int v \arcsin v\, dv = v^2 \arcsin v + v\sqrt{1-v^2} - \int \sqrt{1-v^2}\, dv$$
$$= v^2 \arcsin v + v\sqrt{1-v^2} - \frac{1}{2} v\sqrt{1-v^2} - \frac{1}{2}\arcsin v + C.$$

Dividing by 2 gives

$$\int v \arcsin v\, dv = \left(\frac{v^2}{2} - \frac{1}{4} \right) \arcsin v + \frac{1}{4} v\sqrt{1-v^2} + K,$$

where $K = C/2$.

88. By VI-30 in the table of integrals, we have

$$\int \sqrt{4-x^2}\, dx = \frac{x\sqrt{4-x^2}}{2} + 2\int \frac{1}{\sqrt{4-x^2}}\, dx.$$

The same table informs us in formula VI-28 that

$$\int \frac{1}{\sqrt{4-x^2}}\, dx = \arcsin \frac{x}{2} + C.$$

Thus

$$\int \sqrt{4-x^2}\, dx = \frac{x\sqrt{4-x^2}}{2} + 2 \arcsin \frac{x}{2} + C.$$

89. By long division, $\dfrac{z^3}{z-5} = z^2 + 5z + 25 + \dfrac{125}{z-5}$, so

$$\int \frac{z^3}{z-5} dz = \int \left(z^2 + 5z + 25 + \frac{125}{z-5} \right) dz = \frac{z^3}{3} + \frac{5z^2}{2} + 25z + 125\int \frac{1}{z-5} dz$$
$$= \frac{z^3}{3} + \frac{5}{2}z^2 + 25z + 125 \ln|z-5| + C.$$

90. If $u = 1 + \cos^2 w$, $du = 2(\cos w)^1(-\sin w)\, dw$, so

$$\int \frac{\sin w \cos w}{1+\cos^2 w}\, dw = -\frac{1}{2}\int \frac{-2\sin w \cos w}{1+\cos^2 w}\, dw = -\frac{1}{2}\int \frac{1}{u}\, du = -\frac{1}{2}\ln|u| + C$$
$$= -\frac{1}{2}\ln|1+\cos^2 w| + C.$$

91. $\int \frac{1}{\tan(3\theta)} d\theta = \int \frac{1}{\left(\frac{\sin(3\theta)}{\cos(3\theta)}\right)} d\theta = \int \frac{\cos(3\theta)}{\sin(3\theta)} d\theta$. If $u = \sin(3\theta)$, $du = \cos(3\theta) \cdot 3d\theta$, so

$$\int \frac{\cos(3\theta)}{\sin(3\theta)} d\theta = \frac{1}{3}\int \frac{3\cos(3\theta)}{\sin(3\theta)} d\theta = \frac{1}{3}\int \frac{1}{u} du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|\sin(3\theta)| + C.$$

92. $\int \frac{x}{\cos^2 x} dx = \int x \frac{1}{\cos^2 x} dx$. Using integration by parts with $u = x$, $du = dx$ and $dv = \frac{1}{\cos^2 x} dx$, $v = \tan x$, we have

$$\int x\left(\frac{1}{\cos^2 x} dx\right) = x\tan x - \int \tan x dx.$$

Formula I-7 gives the final result of $x\tan x - (-\ln|\cos x|) + C = x\tan x + \ln|\cos x| + C$.

93. Dividing and integrating term by term gives

$$\int \frac{x+1}{\sqrt{x}} dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx = \int (x^{1/2} + x^{-1/2}) dx = \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}x^{3/2} + 2\sqrt{x} + C.$$

94. If $u = \sqrt{x+1}$, $u^2 = x + 1$ with $x = u^2 - 1$ and $dx = 2u\,du$. Substituting, we get

$$\int \frac{x}{\sqrt{x+1}} dx = \int \frac{(u^2-1)2u\,du}{u} = \int (u^2-1)2\,du = 2\int (u^2-1)\,du$$
$$= \frac{2u^3}{3} - 2u + C = \frac{2(\sqrt{x+1})^3}{3} - 2\sqrt{x+1} + C.$$

95. $\int \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}} = \int (\sqrt{x}+1)^{1/2} \frac{1}{\sqrt{x}} dx$; if $u = \sqrt{x} + 1$, $du = \frac{1}{2\sqrt{x}} dx$, so we have

$$2\int (\sqrt{x}+1)^{1/2} \frac{1}{2\sqrt{x}} dx = 2\int u^{1/2} du = 2\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = \frac{4}{3}u^{3/2} + C = \frac{4}{3}(\sqrt{x}+1)^{3/2} + C.$$

96. If $u = e^{2y} + 1$, then $du = e^{2y} 2\,dy$, so

$$\int \frac{e^{2y}}{e^{2y}+1} dy = \frac{1}{2}\int \frac{2e^{2y}}{e^{2y}+1} dy = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|e^{2y}+1| + C.$$

97. If $u = z^2 - 5$, $du = 2z\,dz$, then

$$\int \frac{z}{(z^2-5)^3} dz = \int (z^2-5)^{-3} z\,dz = \frac{1}{2}\int (z^2-5)^{-3} 2z\,dz = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C$$
$$= \frac{1}{-4(z^2-5)^2} + C.$$

98. Letting $u = z - 5$, $z = u + 5$, $dz = du$, and substituting, we have

$$\int \frac{z}{(z-5)^3} dz = \int \frac{u+5}{u^3} du = \int (u^{-2} + 5u^{-3}) du = \frac{u^{-1}}{-1} + 5\left(\frac{u^{-2}}{-2}\right) + C$$
$$= \frac{-1}{(z-5)} + \frac{-5}{2(z-5)^2} + C.$$

99. If $u = 1 + \tan x$ then $du = \frac{1}{\cos^2 x} dx$, and so

$$\int \frac{(1+\tan x)^3}{\cos^2 x} dx = \int (1+\tan x)^3 \frac{1}{\cos^2 x} dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(1+\tan x)^4}{4} + C.$$

100. $\int \frac{(2x-1)e^{x^2}}{e^x}dx = \int e^{x^2-x}(2x-1)dx$. If $u = x^2 - x$, $du = (2x-1)dx$, so

$$\begin{aligned}\int e^{x^2-x}(2x-1)dx &= \int e^u du \\ &= e^u + c \\ &= e^{x^2-x} + C.\end{aligned}$$

101. We use the substitution $w = x^2 + x,\ dw = (2x+1)\,dx$.

$$\begin{aligned}\int (2x+1)e^{x^2}e^x\,dx &= \int (2x+1)e^{x^2+x}\,dx = \int e^w\,dw \\ &= e^w + C = e^{x^2+x} + C.\end{aligned}$$

Check: $\frac{d}{dx}(e^{x^2+x} + C) = e^{x^2+x}\cdot(2x+1) = (2x+1)e^{x^2}e^x$.

102. Let $w = y^2 - 2y + 1$, so $dw = 2(y-1)\,dy$. Then

$$\int \sqrt{y^2-2y+1}(y-1)\,dy = \int w^{1/2}\frac{1}{2}\,dw = \frac{1}{2}\cdot\frac{2}{3}w^{3/2} + C = \frac{1}{3}(y^2-2y+1)^{3/2} + C.$$

Alternatively, notice the integrand can be written $\sqrt{(y-1)^2}(y-1) = (y-1)^2$. This leads to a different-looking but equivalent answer.

103. Let $w = 2 + 3\cos x$, so $dw = -3\sin x\,dx$, giving $-\frac{1}{3}\,dw = \sin x\,dx$. Then

$$\begin{aligned}\int \sin x\left(\sqrt{2+3\cos x}\right)\,dx &= \int \sqrt{w}\left(-\frac{1}{3}\right)\,dw = -\frac{1}{3}\int \sqrt{w}\,dw \\ &= \left(-\frac{1}{3}\right)\frac{w^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{2}{9}(2+3\cos x)^{\frac{3}{2}} + C.\end{aligned}$$

104. Using Table III-14, with $a = -4$ we have

$$\begin{aligned}\int (x^2-3x+2)e^{-4x}\,dx &= -\frac{1}{4}(x^2-3x+2)e^{-4x} \\ &\quad -\frac{1}{16}(2x-3)e^{-4x} - \frac{1}{64}(2)e^{-4x} + C. \\ &= \frac{1}{32}e^{-4x}(-11+20x-8x^2) + C.\end{aligned}$$

105. Let $x = 2\theta$, then $dx = 2d\theta$. Thus

$$\int \sin^2(2\theta)\cos^3(2\theta)\,d\theta = \frac{1}{2}\int \sin^2 x\cos^3 x\,dx.$$

We let $w = \sin x$ and $dw = \cos x\,dx$. Then

$$\begin{aligned}\frac{1}{2}\int \sin^2 x\cos^3 x\,dx &= \frac{1}{2}\int \sin^2 x\cos^2 x\cos x\,dx \\ &= \frac{1}{2}\int \sin^2 x(1-\sin^2 x)\cos x\,dx \\ &= \frac{1}{2}\int w^2(1-w^2)\,dw = \frac{1}{2}\int (w^2-w^4)\,dw \\ &= \frac{1}{2}\left(\frac{w^3}{3} - \frac{w^5}{5}\right) + C = \frac{1}{6}\sin^3 x - \frac{1}{10}\sin^5 x + C \\ &= \frac{1}{6}\sin^3(2\theta) - \frac{1}{10}\sin^5(2\theta) + C.\end{aligned}$$

106. If $u = 2\sin x$, then $du = 2\cos x\,dx$, so

$$\begin{aligned}\int \cos(2\sin x)\cos x\,dx &= \frac{1}{2}\int \cos(2\sin x)2\cos x\,dx = \frac{1}{2}\int \cos u\,du \\ &= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2\sin x) + C.\end{aligned}$$

107. Let $w = x + \sin x$, then $dw = (1+\cos x)\,dx$ which gives

$$\int (x+\sin x)^3(1+\cos x)\,dx = \int w^3\,dw = \frac{1}{4}w^4 + C = \frac{1}{4}(x+\sin x)^4 + C.$$

108. Using Table III-16,

$$\begin{aligned}\int \left(2x^3+3x+4\right)\cos(2x)\,dx &= \frac{1}{2}(2x^3+3x+4)\sin(2x) \\ &\quad +\frac{1}{4}(6x^2+3)\cos(2x) \\ &\quad -\frac{1}{8}(12x)\sin(2x) - \frac{3}{4}\cos(2x) + C. \\ &= 2\sin(2x) + x^3\sin(2x) + \frac{3x^2}{2}\cos(2x) + C.\end{aligned}$$

109. Use the substitution $w = \sinh x$ and $dw = \cosh x dx$ so

$$\int \sinh^2 x\cosh x\,dx = \int w^2\,dw = \frac{w^3}{3} + C = \frac{1}{3}\sinh^3 x + C.$$

Check this answer by taking the derivative: $\dfrac{d}{dx}\left[\dfrac{1}{3}\sinh^3 x + C\right] = \sinh^2 x\cosh x.$

110. We use the substitution $w = x^2 + 2x$ and $dw = (2x+2)\,dx$ so

$$\int (x+1)\sinh(x^2+2x)\,dx = \frac{1}{2}\int \sinh w\,dw = \frac{1}{2}\cosh w + C = \frac{1}{2}\cosh(x^2+2x) + C.$$

Check this answer by taking the derivative: $\dfrac{d}{dx}\left[\dfrac{1}{2}\cosh(x^2+2x) + C\right] = \dfrac{1}{2}(2x+2)\sinh(x^2+2x) = (x+1)\sinh(x^2+2x)$.

111. Substitute $w = 1 + x^2$, $dw = 2x\,dx$. Then $x\,dx = \dfrac{1}{2}\,dw$, and

$$\int_{x=0}^{x=1} x(1+x^2)^{20}\,dx = \frac{1}{2}\int_{w=1}^{w=2} w^{20}\,dw = \left.\frac{w^{21}}{42}\right|_1^2 = \frac{299593}{6} = 49932\frac{1}{6}.$$

112. Substitute $w = x^2 + 4,\ dw = 2x\,dx$. Then,

$$\begin{aligned}\int_{x=4}^{x=1} x\sqrt{x^2+4}\,dx &= \frac{1}{2}\int_{w=20}^{w=5} w^{\frac{1}{2}}\,dw = \left.\frac{1}{3}w^{\frac{3}{2}}\right|_{20}^{5} \\ &= \frac{1}{3}\left(5^{\frac{3}{2}} - 8\cdot 5^{\frac{3}{2}}\right) = -\frac{7}{3}\cdot 5^{3/2} = -\frac{35}{3}\sqrt{5}\end{aligned}$$

113. We substitute $w = \cos\theta + 5$, $dw = -\sin\theta\,d\theta$. Then

$$\int_{\theta=0}^{\theta=\pi} \sin\theta\,d\theta(\cos\theta+5)^7 = -\int_{w=6}^{w=4} w^7 dw = \int_{w=4}^{w=6} w^7 dw = \left.\frac{w^8}{8}\right|_4^6 = 201{,}760.$$

114. Let $w = 1 + 5x^2$. We have $dw = 10x\,dx$, so $\dfrac{dw}{10} = x\,dx$. When $x = 0$, $w = 1$. When $x = 1$, $w = 6$.

$$\begin{aligned}\frac{x\,dx}{1+5x^2} &= \int_1^6 \frac{\frac{1}{10}\,dw}{w} = \frac{1}{10}\int_1^6 \frac{dw}{w} = \frac{1}{10}\ln|w|\Big|_1^6 \\ &= \frac{1}{10}(\ln 6 - \ln 1) = \frac{\ln 6}{10}\end{aligned}$$

115.

$$\int_1^2 \frac{x^2+1}{x}dx = \int_1^2 (x + \frac{1}{x})dx = \left(\frac{x^2}{2} + \ln|x|\right)\Bigg|_1^2 = \frac{3}{2} + \ln 2.$$

116. Using integration by parts, we have

$$\int_1^3 \ln(x^3)\,dx = 3\int_1^3 \ln x\,dx = 3(x\ln x - x)\Big|_1^3 = 9\ln 3 - 6 \approx 3.8875.$$

This matches the approximation given by Simpson's rule with 10 intervals.

117. In Problem 25, we found that

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C.$$

Thus

$$\int_1^e (\ln x)^2\,dx = [x(\ln x)^2 - 2x\ln x + 2x]\Big|_1^e = e - 2 \approx 0.71828.$$

This matches the approximation given by Simpson's rule with 10 intervals.

118. Integrating by parts, we take $u = e^{2x}$, $u' = 2e^{2x}$, $v' = \sin 2x$, and $v = -\frac{1}{2}\cos 2x$, so

$$\int e^{2x}\sin 2x\,dx = -\frac{e^{2x}}{2}\cos 2x + \int e^{2x}\cos 2x\,dx.$$

Integrating by parts again, with $u = e^{2x}$, $u' = 2e^{2x}$, $v' = \cos 2x$, and $v = \frac{1}{2}\sin 2x$, we get

$$\int e^{2x}\cos 2x\,dx = \frac{e^{2x}}{2}\sin 2x - \int e^{2x}\sin 2x\,dx.$$

Substituting into the previous equation, we obtain

$$\int e^{2x}\sin 2x\,dx = -\frac{e^{2x}}{2}\cos 2x + \frac{e^{2x}}{2}\sin 2x - \int e^{2x}\sin 2x\,dx.$$

Solving for $\int e^{2x}\sin 2x\,dx$ gives

$$\int e^{2x}\sin 2x\,dx = \frac{1}{4}e^{2x}(\sin 2x - \cos 2x) + C.$$

This result can also be obtained using II-8 in the integral table. Thus

$$\int_{-\pi}^{\pi} e^{2x}\sin 2x = [\frac{1}{4}e^{2x}(\sin 2x - \cos 2x)]\Big|_{-\pi}^{\pi} = \frac{1}{4}(e^{-2\pi} - e^{2\pi}) \approx -133.8724.$$

We get -133.37 using Simpson's rule with 10 intervals. With 100 intervals, we get -133.8724. Thus our answer matches the approximation of Simpson's rule.

119.

$$\begin{aligned}\int_0^{10} ze^{-z}\,dz &= [-ze^{-z}]\Big|_0^{10} - \int_0^{10} -e^{-z}\,dz \qquad (\text{let } z=u, e^{-z}=v', -e^{-z}=v)\\ &= -10e^{-10} - [e^{-z}]\Big|_0^{10}\\ &= -10e^{-10} - e^{-10} + 1\\ &= -11e^{-10} + 1.\end{aligned}$$

120. Let $\sin\theta = w$, $\cos\theta\,d\theta = dw$. So, if $\theta = -\frac{\pi}{3}$, then $w = -\frac{\sqrt{3}}{2}$, and if $\theta = \frac{\pi}{4}$, then $w = \frac{\sqrt{2}}{2}$. So we have

$$\int_{-\pi/3}^{\pi/4} \sin^3\theta\cos\theta\,d\theta = \int_{-\sqrt{3}/2}^{\sqrt{2}/2} w^3\,dw = \frac{1}{4}w^4\Big|_{-\sqrt{3}/2}^{\sqrt{2}/2} = \frac{1}{4}\left[\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{-\sqrt{3}}{2}\right)^4\right] = -\frac{5}{64}.$$

121. We substitute $w = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then $dw = \frac{1}{3}x^{-\frac{2}{3}}\,dx = \frac{1}{3\sqrt[3]{x^2}}\,dx$.

$$\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx = \int_{x=1}^{x=8} e^w(3\,dw) = 3e^w\Big|_{x=1}^{x=8} = 3e^{\sqrt[3]{x}}\Big|_1^8 = 3(e^2 - e).$$

122.

$$\int_0^1 \frac{dx}{x^2+1} = \tan^{-1}x\Big|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

123. We put the integral in a convenient form for a substitution by using the fact that $\sin^2\theta = 1 - \cos^2\theta$. Thus: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2\theta\sin^5\theta\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2\theta(1-\cos^2\theta)^2\sin\theta\,d\theta.$

Now, we can make a substitution which helps. We let $w = \cos\theta$, so $dw = -\sin\theta\,d\theta$.

Note that $w = \frac{\sqrt{2}}{2}$ when $\theta = -\frac{\pi}{4}$ and when $\theta = \frac{\pi}{4}$. Thus after our substitution, we get

$$-\int_{w=\frac{\pi}{4}}^{w=\frac{\pi}{4}} w^2(1-w^2)^2\,dw.$$

Since the upper and lower limits of integration are the same, this definite integral must equal 0. Notice that we could have deduced this fact immediately, since $\cos^2\theta$ is even and $\sin^5\theta$ is odd, so $\cos^2\theta\sin^5\theta$ is odd.

Thus $\int_{-\frac{\pi}{4}}^{0} \cos^2\theta\sin^5\theta\,d\theta = -\int_0^{\frac{\pi}{4}} \cos^2\theta\sin^5\theta\,d\theta$, and the given integral must evaluate to 0.

124. We substitute $w = x^2 + 4x + 5$, so $dw = (2x+4)\,dx$. Notice that when $x = -2, w = 1$, and when $x = 0, w = 5$.

$$\int_{x=-2}^{x=0} \frac{2x+4}{x^2+4x+5}\,dx = \int_{w=1}^{w=5} \frac{1}{w}\,dw = \ln|w|\Big|_{w=1}^{w=5} = \ln 5.$$

125. Since $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$, let's imagine that our fraction is the result of adding together two terms, one with a denominator of $x-1$, the other with a denominator of $x+1$:

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To find A and B, we multiply by the least common multiple of both sides to clear the fractions. This yields

$$\begin{aligned}1 &= A(x+1) + B(x-1)\\ &= (A+B)x + (A-B).\end{aligned}$$

Since the two sides are equal for all values of x in the domain, and there is no x term on the left-hand side, $A + B = 0$. Similarly, since A and $-B$ are constant terms on the right-hand side, and 1 is the constant term on the left-hand side, $A - B = 1$. Therefore, we have the system of equations

$$\begin{aligned} A + B &= 0 \\ A - B &= 1. \end{aligned}$$

Solving this gives us $A = 1/2$ and $B = -1/2$, so

$$\frac{1}{x^2 - 1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

Now, we find the integral

$$\begin{aligned} \int \frac{1}{x^2-1} dx &= \int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} \right) dx \\ &= \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C. \end{aligned}$$

126. (a) We split $\frac{1}{x^2-x} = \frac{1}{x(x-1)}$ into partial fractions:

$$\frac{1}{x^2-x} = \frac{A}{x} + \frac{B}{x-1}.$$

Multiplying by $x(x-1)$ gives

$$1 = A(x-1) + Bx = (A+B)x - A,$$

so $-A = 1$ and $A + B = 0$, giving $A = -1$, $B = 1$. Therefore,

$$\int \frac{1}{x^2-x}\, dx = \int \left(\frac{1}{x-1} - \frac{1}{x} \right) dx = \ln|x-1| - \ln|x| + C.$$

(b) $\displaystyle\int \frac{1}{x^2-x}\, dx = \int \frac{1}{(x-1)(x)}\, dx$. Using $a = 1$ and $b = 0$ in V-26, we get $\ln|x-1| - \ln|x| + C$.

127. Split the integrand into partial fractions, giving

$$\begin{aligned} \frac{1}{x(L-x)} &= \frac{A}{x} + \frac{B}{L-x} \\ 1 &= A(L-x) + Bx = (B-A)x + AL. \end{aligned}$$

We have $B - A = 0$ and $AL = 1$, so $A = B = 1/L$. Thus,

$$\int \frac{1}{x(L-x)}\, dx = \int \frac{1}{L}\left(\frac{1}{x} + \frac{1}{L-x} \right) dx = \frac{1}{L}\left(\ln|x| - \ln|L-x| \right) + C.$$

128. Splitting the integrand into partial fractions with denominators $(x-2)$ and $(x+2)$, we have

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Multiplying by $(x-2)(x+2)$ gives the identity

$$1 = A(x+2) + B(x-2)$$

so

$$1 = (A+B)x + 2A - 2B.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} 2A - 2B &= 1 \\ A + B &= 0. \end{aligned}$$

Solving these equations gives $A = 1/4$, $B = -1/4$ and the integral becomes

$$\int \frac{1}{(x-2)(x+2)} dx = \frac{1}{4} \int \frac{1}{x-2} dx - \frac{1}{4} \int \frac{1}{x+2} dx = \frac{1}{4}\left(\ln|x-2| - \ln|x+2| \right) + C.$$

129. Let $x = 5\sin t$. Then $dx = 5\cos t\,dt$, so substitution gives

$$\int \frac{1}{\sqrt{25-x^2}} = \int \frac{5\cos t}{\sqrt{25-25\sin^2 t}}dt = \int dt = t + C = \arcsin\left(\frac{x}{5}\right) + C.$$

130. Splitting the integrand into partial fractions with denominators x and $(x+5)$, we have

$$\frac{1}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}.$$

Multiplying by $x(x+5)$ gives the identity

$$1 = A(x+5) + Bx$$

so

$$1 = (A+B)x + 5A.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} 5A &= 1 \\ A + B &= 0. \end{aligned}$$

Solving these equations gives $A = 1/5$, $B = -1/5$ and the integral becomes

$$\int \frac{1}{x(x+5)}dx = \frac{1}{5}\int \frac{1}{x}dx - \frac{1}{5}\int \frac{1}{x+5}dx = \frac{1}{5}\left(\ln|x| - \ln|x+5|\right) + C.$$

131. We use the trigonometric substitution $3x = \sin\theta$. Then $dx = \frac{1}{3}\cos\theta\,d\theta$ and substitution gives

$$\begin{aligned} \int \frac{1}{\sqrt{1-9x^2}}\,dx &= \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{3}\cos\theta\,d\theta = \frac{1}{3}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta \\ &= \frac{1}{3}\int 1\,d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\arcsin(3x) + C. \end{aligned}$$

132. Splitting the integrand into partial fractions with denominators x, $(x+2)$ and $(x-1)$, we have

$$\frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}.$$

Multiplying by $x(x+2)(x-1)$ gives the identity

$$2x + 3 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

so

$$2x + 3 = (A+B+C)x^2 + (A-B+2C)x - 2A.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} -2A &= 3 \\ A - B + 2C &= 2 \\ A + B + C &= 0. \end{aligned}$$

Solving these equations gives $A = -3/2$, $B = -1/6$ and $C = 5/3$. The integral becomes

$$\begin{aligned} \int \frac{2x+3}{x(x+2)(x-1)}dx &= -\frac{3}{2}\int \frac{1}{x}dx - \frac{1}{6}\int \frac{1}{x+2} + \frac{5}{3}\int \frac{1}{x-1}dx \\ &= -\frac{3}{2}\ln|x| - \frac{1}{6}\ln|x+2| + \frac{5}{3}\ln|x-1| + K. \end{aligned}$$

We use K as the constant of integration, since we already used C in the problem.

133. The denominator can be factored to give $x(x-1)(x+1)$. Splitting the integrand into partial fractions with denominators x, $x-1$, and $x+1$, we have

$$\frac{3x+1}{x(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x}.$$

Multiplying by $x(x-1)(x+1)$ gives the identity

$$3x+1 = Ax(x+1) + Bx(x-1) + C(x-1)(x+1)$$

so

$$3x+1 = (A+B+C)x^2 + (A-B)x - C.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x and x^2 on both sides must be equal. So

$$\begin{aligned} -C &= 1 \\ A-B &= 3 \\ A+B+C &= 0. \end{aligned}$$

Solving these equations gives $A = 2$, $B = -1$ and $C = -1$. The integral becomes

$$\begin{aligned} \int \frac{3x+1}{x(x+1)(x-1)}dx &= \int \frac{2}{x-1}dx - \int \frac{1}{x+1}dx - \int \frac{1}{x}dx \\ &= 2\ln|x-1| - \ln|x+1| - \ln|x| + K. \end{aligned}$$

We use K as the constant of integration, since we already used C in the problem.

134. Splitting the integrand into partial fractions with denominators $(1+x)$, $(1+x)^2$ and x, we have

$$\frac{1+x^2}{x(1+x)^2} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{x}.$$

Multiplying by $x(1+x)^2$ gives the identity

$$1+x^2 = Ax(1+x) + Bx + C(1+x)^2$$

so

$$1+x^2 = (A+C)x^2 + (A+B+2C)x + C.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x and x^2 on both sides must be equal. So

$$\begin{aligned} C &= 1 \\ A+B+2C &= 0 \\ A+C &= 1. \end{aligned}$$

Solving these equations gives $A = 0$, $B = -2$ and $C = 1$. The integral becomes

$$\int \frac{1+x^2}{(1+x)^2 x}dx = -2\int \frac{1}{(1+x)^2}dx + \int \frac{1}{x}dx = \frac{2}{1+x} + \ln|x| + K.$$

We use K as the constant of integration, since we already used C in the problem.

135. Completing the square, we get

$$x^2 + 2x + 2 = (x+1)^2 + 1.$$

We use the substitution $x + 1 = \tan t$, so $dx = (1/\cos^2 t)dt$. Since $\tan^2 t + 1 = 1/\cos^2 t$, the integral becomes

$$\int \frac{1}{(x+1)^2+1}dx = \int \frac{1}{\tan^2 t + 1} \cdot \frac{1}{\cos^2 t}dt = \int dt = t + C = \arctan(x+1) + C.$$

136. Completing the square in the denominator gives

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}.$$

We make the substitution $\tan\theta = x+2$. Then $dx = \frac{1}{\cos^2\theta}\,d\theta$.

$$\begin{aligned}\int \frac{dx}{(x+2)^2+1} &= \int \frac{d\theta}{\cos^2\theta(\tan^2\theta+1)}\\ &= \int \frac{d\theta}{\cos^2\theta(\frac{\sin^2\theta}{\cos^2\theta}+1)}\\ &= \int \frac{d\theta}{\sin^2\theta+\cos^2\theta}\\ &= \int d\theta = \theta + C\end{aligned}$$

But since $\tan\theta = x+2$, $\theta = \arctan(x+2)$, and so $\theta + C = \arctan(x+2)+C$.

137. We use the trigonometric substitution $bx = a\sin\theta$. Then $dx = \frac{a}{b}\cos\theta\,d\theta$, and we have

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx &= \int \frac{1}{\sqrt{a^2-(a\sin\theta)^2}}\cdot\frac{a}{b}\cos\theta\,d\theta = \int \frac{1}{a\sqrt{1-\sin^2\theta}}\cdot\frac{a}{b}\cos\theta\,d\theta\\ &= \frac{1}{b}\int\frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \frac{1}{b}\int 1\,d\theta = \frac{1}{b}\theta + C = \frac{1}{b}\arcsin\left(\frac{bx}{a}\right)+C.\end{aligned}$$

138. Using the substitution $w = \sin x$, we get $dw = \cos x dx$, so we have

$$\int \frac{\cos x}{\sin^3 x+\sin x}\,dx = \int \frac{dw}{w^3+w}.$$

But

$$\frac{1}{w^3+w} = \frac{1}{w(w^2+1)} = \frac{1}{w} - \frac{w}{w^2+1},$$

so

$$\begin{aligned}\int \frac{\cos x}{\sin^3 x+\sin x}\,dx &= \int\left(\frac{1}{w}-\frac{w}{w^2+1}\right)dw\\ &= \ln|w| - \frac{1}{2}\ln\left|w^2+1\right| + C\\ &= \ln|\sin x| - \frac{1}{2}\ln\left|\sin^2 x+1\right| + C.\end{aligned}$$

139. Using the substitution $w = e^x$, we get $dw = e^x dx$, so we have

$$\int \frac{e^x}{e^{2x}-1}\,dx = \int \frac{dw}{w^2-1}.$$

But

$$\frac{1}{w^2-1} = \frac{1}{(w-1)(w+1)} = \frac{1}{2}\left(\frac{1}{w-1}-\frac{1}{w+1}\right),$$

so

$$\begin{aligned}\int \frac{e^x}{e^{2x}-1}\,dx &= \int \frac{1}{2}\left(\frac{1}{w-1}-\frac{1}{w+1}\right)dw\\ &= \frac{1}{2}(\ln|w-1| - \ln|w+1|) + C\\ &= \frac{1}{2}(\ln|e^x-1| - \ln|e^x+1|) + C.\end{aligned}$$

140. This is an improper integral because $\sqrt{16-x^2}=0$ at $x=4$. So

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{16-x^2}} &= \lim_{b\to 4^-}\int_0^b \frac{dx}{\sqrt{16-x^2}}\\ &= \lim_{b\to 4^-}(\arcsin x/4)\Big|_0^b\\ &= \lim_{b\to 4^-}[\arcsin(b/4)-\arcsin(0)] = \pi/2-0=\pi/2.\end{aligned}$$

141. This integral is improper because $5/x^2$ is undefined at $x=0$. Then

$$\int_0^3 \frac{5}{x^2}\,dx = \lim_{b\to 0^+}\int_b^3 \frac{5}{x^2}\,dx = \lim_{b\to 0^+}\left(-5x^{-1}\Big|_b^3\right) = \lim_{b\to 0^+}\left(\frac{-5}{3}+\frac{5}{b}\right).$$

As $b\to 0^+$, this goes to infinity and the integral diverges.

142. This integral is improper because $1/(x-2)$ is undefined at $x=2$. Then

$$\int_0^2 \frac{1}{x-2}\,dx = \lim_{b\to 2^-}\int_0^b \frac{1}{x-2}\,dx = \lim_{b\to 2^-}\left(\ln|x-2|\Big|_0^b\right) = \lim_{b\to 2^-}\left(\ln|b-2|-\ln 2\right).$$

As $b\to 2^-$, this goes to negative infinity and the integral diverges.

143. This integral is improper because $1/\sqrt[3]{8-x}$ is undefined at $x=8$. Then

$$\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx = \lim_{b\to 8^-}\int_0^b (8-x)^{-1/3}\,dx = \lim_{b\to 8^-}\left(-1.5(8-x)^{2/3}\Big|_0^b\right) = \lim_{b\to 8^-}\left(-1.5(8-b)^{2/3}+1.5(8^{2/3})\right)=6.$$

The integral converges.

144. $\displaystyle\int_4^\infty \frac{dt}{t^{3/2}}$ should converge, since $\displaystyle\int_1^\infty \frac{dt}{t^n}$ converges for $n>1$.
We calculate its value.
$$\int_4^\infty \frac{dt}{t^{3/2}} = \lim_{b\to\infty}\int_4^b t^{-3/2}\,dt = \lim_{b\to\infty} -2t^{-1/2}\Big|_4^b = \lim_{b\to\infty}\left(1-\frac{2}{\sqrt{b}}\right)=1.$$

145. $\displaystyle\int \frac{dx}{x\ln x} = \ln|\ln x| + C$. (Substitute $w=\ln x$, $dw=\frac{1}{x}\,dx$).
Thus
$$\int_{10}^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_{10}^b \frac{dx}{x\ln x} = \lim_{b\to\infty}\ln|\ln x|\Big|_{10}^b = \lim_{b\to\infty}\ln(\ln b)-\ln(\ln 10).$$

As $b\to\infty$, $\ln(\ln b)\to\infty$, so this diverges.

146. To find $\displaystyle\int we^{-w}\,dw$, integrate by parts, with $u=w$ and $v'=e^{-w}$. Then $u'=1$ and $v=-e^{-w}$.
Then
$$\int we^{-w}\,dw = -we^{-w} + \int e^{-w}\,dw = -we^{-w}-e^{-w}+C.$$

Thus
$$\int_0^\infty we^{-w}\,dw = \lim_{b\to\infty}\int_0^b we^{-w}\,dw = \lim_{b\to\infty}(-we^{-w}-e^{-w})\Big|_0^b = 1.$$

147. The trouble spot is at $x=0$, so we write

$$\int_{-1}^1 \frac{1}{x^4}\,dx = \int_{-1}^0 \frac{1}{x^4}\,dx + \int_0^1 \frac{1}{x^4}\,dx.$$

However, both these integrals diverge. For example,

$$\int_0^1 \frac{1}{x^4}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^4}\,dx = \lim_{a\to 0^+} -\frac{x^{-3}}{3}\Big|_a^1 = \lim_{a\to 0^+}\left(\frac{1}{3a^3}-\frac{1}{3}\right).$$

Since this limit does not exist, $\int_0^1 \frac{1}{x^4}\,dx$ diverges and so the original integral diverges.

148. Since the value of $\tan\theta$ is between -1 and 1 on the interval $-\pi/4 \le \theta \le \pi/4$, our integral is not improper and so converges. Moreover, since $\tan\theta$ is an odd function, we have

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan\theta\,d\theta &= \int_{-\frac{\pi}{4}}^{0} \tan\theta\,d\theta + \int_0^{\frac{\pi}{4}} \tan\theta\,d\theta \\ &= -\int_{-\frac{\pi}{4}}^{0} \tan(-\theta)\,d\theta + \int_0^{\frac{\pi}{4}} \tan\theta\,d\theta \\ &= -\int_0^{\frac{\pi}{4}} \tan\theta\,d\theta + \int_0^{\frac{\pi}{4}} \tan\theta\,d\theta = 0.\end{aligned}$$

149. It is easy to see that this integral converges:

$$\frac{1}{4+z^2} < \frac{1}{z^2}, \quad \text{and so} \quad \int_2^\infty \frac{1}{4+z^2}\,dz < \int_2^\infty \frac{1}{z^2}\,dz = \frac{1}{2}.$$

We can also find its exact value.

$$\begin{aligned}\int_2^\infty \frac{1}{4+z^2}\,dz &= \lim_{b\to\infty}\int_2^b \frac{1}{4+z^2}\,dz \\ &= \lim_{b\to\infty}\left(\frac{1}{2}\arctan\frac{z}{2}\Big|_2^b\right) \\ &= \lim_{b\to\infty}\left(\frac{1}{2}\arctan\frac{b}{2} - \frac{1}{2}\arctan 1\right) \\ &= \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}.\end{aligned}$$

Note that $\frac{\pi}{8} < \frac{1}{2}$.

150. We find the exact value:

$$\begin{aligned}\int_{10}^\infty \frac{1}{z^2-4}\,dz &= \int_{10}^\infty \frac{1}{(z+2)(z-2)}\,dz \\ &= \lim_{b\to\infty}\int_{10}^b \frac{1}{(z+2)(z-2)}\,dz \\ &= \lim_{b\to\infty}\frac{1}{4}\left(\ln|z-2| - \ln|z+2|\right)\Big|_{10}^b \\ &= \frac{1}{4}\lim_{b\to\infty}\left[(\ln|b-2| - \ln|b+2|) - (\ln 8 - \ln 12)\right] \\ &= \frac{1}{4}\lim_{b\to\infty}\left[\left(\ln\frac{b-2}{b+2}\right) + \ln\frac{3}{2}\right] \\ &= \frac{1}{4}(\ln 1 + \ln 3/2) = \frac{\ln 3/2}{4}.\end{aligned}$$

151. Substituting $w = t+5$, we see that our integral is just $\int_0^{15} \frac{dw}{\sqrt{w}}$. This will converge, since $\int_0^b \frac{dw}{w^p}$ converges for $0 < p < 1$. We find its exact value:

$$\int_0^{15} \frac{dw}{\sqrt{w}} = \lim_{a\to 0^+}\int_a^{15} \frac{dw}{\sqrt{w}} = \lim_{a\to 0^+} 2w^{\frac{1}{2}}\Big|_a^{15} = 2\sqrt{15}.$$

152. Since $\sin\phi < \phi$ for $\phi > 0$,

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin\phi}\, d\phi > \int_0^{\frac{\pi}{2}} \frac{1}{\phi}\, d\phi,$$

The integral on the right diverges, so the integral on the left must also. Alternatively, we use IV-20 in the integral table to get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1}{\sin\phi}\, d\phi &= \lim_{b\to 0^+} \int_b^{\frac{\pi}{2}} \frac{1}{\sin\phi}\, d\phi \\ &= \lim_{b\to 0^+} \frac{1}{2}\ln\left|\frac{\cos\phi - 1}{\cos\phi + 1}\right|\Bigg|_b^{\frac{\pi}{2}} \\ &= -\frac{1}{2}\lim_{b\to 0^+} \ln\left|\frac{\cos b - 1}{\cos b + 1}\right|.\end{aligned}$$

As $b \to 0^+$, $\cos b - 1 \to 0$ and $\cos b + 1 \to 2$, so $\ln\left|\frac{\cos b - 1}{\cos b + 1}\right| \to -\infty$. Thus the integral diverges.

153. Let $\phi = 2\theta$. Then $d\phi = 2\, d\theta$, and

$$\begin{aligned}\int_0^{\pi/4} \tan 2\theta\, d\theta &= \int_0^{\pi/2} \frac{1}{2}\tan\phi\, d\phi = \int_0^{\pi/2} \frac{1}{2}\frac{\sin\phi}{\cos\phi}\, d\phi \\ &= \lim_{b\to(\pi/2)^-} \int_0^b \frac{1}{2}\frac{\sin\phi}{\cos\phi}\, d\phi = \lim_{b\to(\pi/2)^-} -\frac{1}{2}\ln|\cos\phi|\Bigg|_0^b.\end{aligned}$$

As $b \to \pi/2$, $\cos\phi \to 0$, so $\ln|\cos\phi| \to -\infty$. Thus the integral diverges.

One could also see this by noting that $\cos x \approx \pi/2 - x$ and $\sin x \approx 1$ for x close to $\pi/2$: therefore, $\tan x \approx 1/(\frac{\pi}{2} - x)$, the integral of which diverges.

154. The integrand $\dfrac{x}{x+1} \to 1$ as $x \to \infty$, so there's no way $\displaystyle\int_1^{\infty} \frac{x}{x+1}\, dx$ can converge.

155. This function is difficult to integrate, so instead we try to compare it with some other function. Since $\frac{\sin^2\theta}{\theta^2+1} \geq 0$, we see that $\int_0^{\infty} \frac{\sin^2\theta}{\theta^2+1}\, d\theta \geq 0$. Also, since $\sin^2\theta \leq 1$,

$$\int_0^{\infty} \frac{\sin^2\theta}{\theta^2+1}\, d\theta \leq \int_0^{\infty} \frac{1}{\theta^2+1}\, d\theta = \lim_{b\to\infty} \arctan\theta\Bigg|_0^b = \frac{\pi}{2}.$$

Thus $\int_0^{\infty} \frac{\sin^2\theta}{\theta^2+1}\, d\theta$ converges, and its value is between 0 and $\frac{\pi}{2}$.

156. $\displaystyle\int_0^{\pi} \tan^2\theta d\theta = \tan\theta - \theta + C$, by formula IV-23. The integrand blows up at $\theta = \frac{\pi}{2}$, so

$$\int_0^{\pi} \tan^2\theta d\theta = \int_0^{\frac{\pi}{2}} \tan^2\theta d\theta + \int_{\frac{\pi}{2}}^{\pi} \tan^2\theta d\theta = \lim_{b\to\frac{\pi}{2}} [\tan\theta - \theta]_0^b + \lim_{a\to\frac{\pi}{2}} [\tan\theta - \theta]_a^{\pi}$$

which is undefined.

157. Since $0 \leq \sin x < 1$ for $0 \leq x \leq 1$, we have

$$\begin{aligned} &(\sin x)^{\frac{3}{2}} < (\sin x) \\ \text{so} \quad &\frac{1}{(\sin x)^{\frac{3}{2}}} > \frac{1}{(\sin x)} \\ \text{or} \quad &(\sin x)^{-\frac{3}{2}} > (\sin x)^{-1}\end{aligned}$$

Thus $\displaystyle\int_0^1 (\sin x)^{-1} dx = \lim_{a\to 0} \ln\left|\frac{1}{\sin x} - \frac{1}{\tan x}\right|_a^1$, which is infinite.

Hence, $\int_0^1 (\sin x)^{-\frac{3}{2}} dx$ is infinite.

Problems

158. Since $(e^x)^2 = e^{2x}$, we have

$$\text{Area } = \int_0^1 (e^x)^2\,dx = \int_0^1 e^{2x}\,dx = \frac{1}{2}e^{2x}\bigg|_0^1 = \frac{1}{2}(e^2-1).$$

159. Since $(e^x)^3 = e^{3x}$ and $(e^x)^2 = e^{2x}$, and the graph of $(e^x)^3$ is above the graph of $(e^x)^2$ for $x > 0$, we have

$$\begin{aligned}\text{Area } &= \int_0^e (e^x)^3 - (e^x)^2\,dx = \int_0^3 (e^{3x} - e^{2x})\,dx = \left(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\right)\bigg|_0^3\\ &= \frac{1}{3}e^9 - \frac{1}{2}e^6 - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{3}e^9 - \frac{1}{2}e^6 + \frac{1}{6}.\end{aligned}$$

160. The curves $y = e^x$ and $y = 5e^{-x}$ cross where

$$\begin{aligned}e^x &= 5e^{-x}\\ e^{2x} &= 5\\ x &= \frac{1}{2}\ln 5.\end{aligned}$$

Since the graph of $y = 5e^{-x}$ is above the graph of $y = e^x$ for $0 \le x \le \frac{1}{2}\ln 5$ (see Figure 7.47), we have

$$\begin{aligned}\text{Area } &= \int_0^{(\ln 5)/2} (5e^{-x} - e^x)\,dx\\ &= (-5e^{-x} - e^x)\bigg|_0^{(\ln 5)/2}\\ &= -5e^{-(\ln 5)/2} - e^{(\ln 5)/2} - (-5\cdot 1 - 1)\\ &= -5(e^{\ln 5})^{-1/2} - (e^{\ln 5})^{1/2} + 6\\ &= -5(5^{-1/2}) - (5)^{1/2} + 6\\ &= -\frac{5}{\sqrt{5}} - \sqrt{5} + 6 = -2\sqrt{5} + 6.\end{aligned}$$

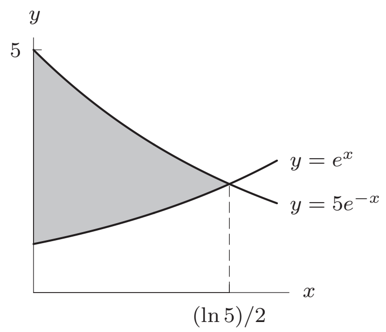


Figure 7.47

161. As is evident from Figure 7.48 showing the graphs of $y = \sin x$ and $y = \cos x$, the crossings occur at $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$, and the regions bounded by any two consecutive crossings have the same area. So picking two consecutive crossings, we get an area of

$$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx\\ &= 2\sqrt{2}.\end{aligned}$$

(Note that we integrated $\sin x - \cos x$ here because for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$, $\sin x \ge \cos x$.)

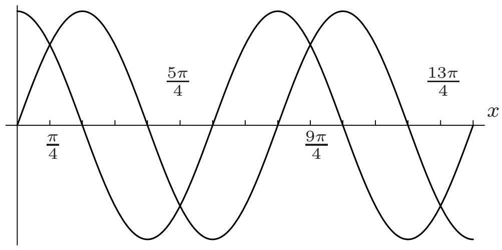


Figure 7.48

162. In the interval of $0 \le x \le 2$, the equation $y = \sqrt{4-x^2}$ represents one quadrant of the radius 2 circle centered at the origin. Interpreting the integral as an area gives a value of $\frac{1}{4}\pi 2^2 = \pi$.

163. Letting $w = 3x^5 + 2, dw = 15x^4\,dx$, we see that $x^4\,dx = dw/15$, so:

$$\begin{aligned}\int 3x^4\sqrt{3x^5+2}\,dx &= \int 3\left(3x^5+2\right)^{1/2}\cdot x^4\,dx\\ &= \int 3w^{1/2}\cdot\frac{1}{15}\cdot dw\\ &= \int \frac{1}{5}w^{1/2}\,dw,\end{aligned}$$

so $w = 3x^5+2, p = 1/2, k = 1/5$.

164. Letting $w = \cos(3\theta), dw = -3\sin(3\theta)\,d\theta$, we see that $\sin(3\theta)\,d\theta = (-1/3)\,dw$, so:

$$\begin{aligned}\int \frac{5\sin(3\theta)\,d\theta}{\cos^3(3\theta)} &= \int 5\left(\cos(3\theta)\right)^{-3}\sin(3\theta)\,d\theta\\ &= \int 5w^{-3}\cdot\frac{-dw}{3}\\ &= \int -\frac{5}{3}w^{-3}\,dw,\end{aligned}$$

so $w = \cos(3\theta), p = -3, k = -5/3$.

Alternatively, if we write

$$\frac{5\sin(3\theta)}{\cos^3(3\theta)} = \frac{5\sin(3\theta)}{\cos(3\theta)}\cdot\frac{1}{\cos^2(3\theta)}$$

and let $w = \tan(3\theta)$, so $dw = 3\,d\theta/\cos^2(3\theta)$, and we have

$$\int \frac{5\sin(3\theta)\,d\theta}{\cos^3(3\theta)} = \int 5\tan(3\theta)\frac{1}{\cos^2(3\theta)}\,d\theta = \int \frac{5}{3}w\,dw,$$

so here $w = \tan(3\theta), p = 1, k = 5/3$.

165. Using partial fractions we have

$$\begin{aligned}\frac{1}{(2x-3)(3x-2)} &= \frac{A}{2x-3} + \frac{B}{3x-2}\\ A(3x-2) + B(2x-3) &= 1\\ 3Ax - 2A + 2Bx - 3B &= 1\\ (3A+2B)x - 2A - 3B &= 1.\end{aligned}$$

We see that:

$$3A + 2B = 0$$

$$\begin{aligned} \text{so} \quad A &= -\frac{2}{3}\cdot B \\ \text{and} \quad -2A - 3B &= 1 \\ -2\left(-\frac{2}{3}\cdot B\right) - 3B &= 1 \qquad \text{since } A = -\tfrac{2}{3}B \\ \frac{4}{3}\cdot B - 3B &= 1 \\ -\frac{5}{3}B &= 1 \\ B &= -\frac{3}{5} = -0.6 \\ A &= -\frac{2}{3}\left(-\frac{3}{5}\right) \qquad \text{since } A = -\tfrac{2}{3}B \\ &= \frac{2}{5} = 0.4. \end{aligned}$$

Thus, $\displaystyle\int \frac{dx}{(2x-3)(3x-2)} = \int \left(\frac{0.4}{2x-3} + \frac{-0.6}{3x-2}\right)dx.$

166. Let $u = 0.5x - 1, du = 0.5\,dx$. Then $dx = 2\,du$ and, solving for x,

$$\begin{aligned} 0.5x - 1 &= u \\ 0.5x &= u + 1 \\ x &= 2u + 2. \end{aligned}$$

This means

$$\begin{aligned} x^2 + x &= (2u+2)^2 + 2u + 2 \\ &= 4u^2 + 10u + 6 \end{aligned}$$

so

$$\begin{aligned} \int (x^2+x)\cos(0.5x-1)\,dx &= \int (4u^2+10u+6)\cos u(2\,du) \\ &= \int (8u^2+20u+12)\cos u\,du, \end{aligned}$$

where $p(u) = 8u^2 + 20u + 12$.

167. Let $u = -x^2, du = -2x\,dx$, so that $x\,dx = -0.5du$. We have:

$$\begin{aligned} \int x^3 e^{-x^2}\,dx &= \int \underbrace{x^2}_{-u}\, e^{\overbrace{-x^2}^{u}} \cdot \underbrace{x\,dx}_{-0.5du} \\ &= \int 0.5ue^u\,du, \end{aligned}$$

so $k = 0.5, u = -x^2$.

168. Letting $u = \cos\sqrt{x}$, we have

$$\begin{aligned} du &= -\frac{1}{2}x^{-1/2}\sin\sqrt{x}\,dx \\ &= -\frac{\sin\sqrt{x}\,dx}{2\sqrt{x}}. \end{aligned}$$

So

$$\frac{\sin\sqrt{x}\,dx}{\sqrt{x}} = -2du.$$

Thus,

$$\int \frac{\cos^4\left(\sqrt{x}\right)\sin\sqrt{x}\,dx}{\sqrt{x}} = \int \frac{\left(\cos\left(\sqrt{x}\right)\right)^4\sin\sqrt{x}\,dx}{\sqrt{x}}$$

$$= \int \left(\cos\left(\sqrt{x}\right)\right)^4 \cdot \frac{\sin\sqrt{x}\,dx}{\sqrt{x}}$$
$$= \int -2u^4\,du,$$

where $k = -2, n = 4$, and $u = \cos\sqrt{x}$.

169. After the substitution $w = x^2$, the second integral becomes

$$\frac{1}{2}\int \frac{dw}{\sqrt{1-w^2}}.$$

170. After the substitution $w = x + 2$, the first integral becomes

$$\int w^{-2}\,dw.$$

After the substitution $w = x^2 + 1$, the second integral becomes

$$\frac{1}{2}\int w^{-2}\,dw.$$

171. After the substitution $w = 1 - x^2$, the first integral becomes

$$-\frac{1}{2}\int w^{-1}\,dw.$$

After the substitution $w = \ln x$, the second integral becomes

$$\int w^{-1}\,dw.$$

172. *First solution*: After the substitution $w = x + 1$, the first integral becomes

$$\int \frac{w-1}{w}\,dw = w - \int w^{-1}\,dw.$$

With this same substitution, the second integral becomes

$$\int w^{-1}\,dw.$$

Second solution: We note that the sum of the integrands is 1, so the sum of the integrals is x. Thus

$$\int \frac{x}{x+1}\,dx = x - \int \frac{1}{x+1}\,dx.$$

173. If we let $w = 2x$ in the first integral, we get $dw = 2dx$. Also, the limits $w = 0$ and $w = 2$ become $x = 0$ and $x = 1$. Thus

$$\int_0^2 e^{-w^2}\,dw = \int_0^1 e^{-4x^2}\,2\,dx.$$

174. If we let $t = 3u$ in the first integral, we get $dt = 3du$, so $dt/t = du/u$. Also, the limits $t = 0$ and $t = 3$ become $u = 0$ and $u = 1$. Thus

$$\int_0^3 \frac{\sin t}{t}\,dt = \int_0^1 \frac{\sin 3u}{u}\,du.$$

If we rename the variable u as t in the last integral, we have the equality we want.

175. Since the definition of f is different on $0 \le t \le 1$ than it is on $1 \le t \le 2$, break the definite integral at $t = 1$.

$$\begin{aligned}\int_0^2 f(t)\,dt &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \\ &= \int_0^1 t^2\,dt + \int_1^2 (2-t)\,dt \\ &= \frac{t^3}{3}\bigg|_0^1 + \left(2t - \frac{t^2}{2}\right)\bigg|_1^2 \\ &= 1/3 + 1/2 = 5/6 \approx 0.833\end{aligned}$$

176. (a) (i) Multiplying out gives

$$\int (x^2 + 10x + 25)\,dx = \frac{x^3}{3} + 5x^2 + 25x + C.$$

(ii) Substituting $w = x + 5$, so $dw = dx$, gives

$$\int (x+5)^2\,dx = \int w^2\,dw = \frac{w^3}{3} + C = \frac{(x+5)^3}{3} + C.$$

(b) The results of the two calculations are not the same since

$$\frac{(x+5)^3}{3} + C = \frac{x^3}{3} + \frac{15x^2}{3} + \frac{75x}{3} + \frac{125}{3} + C.$$

However they differ only by a constant, $125/3$, as guaranteed by the Fundamental Theorem of Calculus.

177. (a) Since $h(z)$ is even, we know that $\int_0^1 h(z)\,dz = \int_{-1}^0 h(z)\,dz$. Since $\int_{-1}^1 h(z)\,dz = \int_{-1}^0 h(z)\,dz + \int_0^1 h(z)\,dz$, we see that $\int_{-1}^1 h(z)\,dz = 2\int_0^1 h(z)\,dz = 7$. Thus $\int_0^1 h(z)\,dz = 3.5$

(b) If $w = z + 3$, then $dw = dz$. When $z = -4$, $w = -1$; when $z = -2$, $w = 1$. Thus,

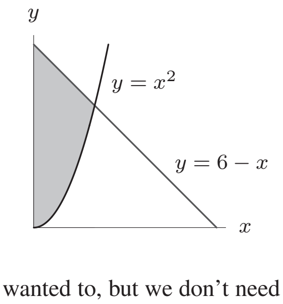
$$\int_{-4}^{-2} 5h(z+3)\,dt = 5\int_{-1}^{1} h(w)\,(dw) = 5 \cdot 7 = 35.$$

178. The point of intersection of the two curves $y = x^2$ and $y = 6 - x$ is at (2,4). The average height of the shaded area is the average value of the difference between the functions:

$$\frac{1}{(2-0)}\int_0^2 ((6-x) - x^2)\,dx = (3x - \frac{x^2}{4} - \frac{x^3}{6})\bigg|_0^2 = \frac{11}{3}.$$

179. The average width of the shaded area in the figure below is the average value of the horizontal distance between the two functions. If we call this horizontal distance $h(y)$, then the average width is

$$\frac{1}{(6-0)}\int_0^6 h(y)\,dy.$$

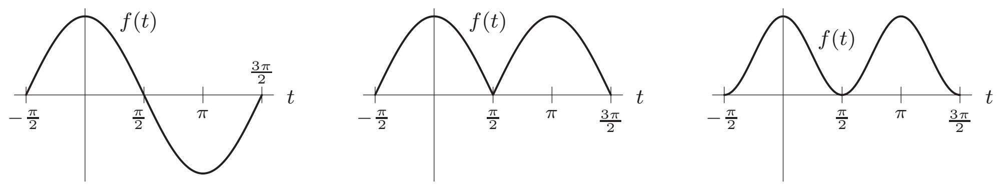


We could compute this integral if we wanted to, but we don't need to. We can simply note that the integral (without the $\frac{1}{6}$ term) is just the area of the shaded region; similarly, the integral in Problem 178 is *also* just the area of the shaded region. So they are the same. Now we know that our average width is just $\frac{1}{3}$ as much as the average height, since we divide by 6 instead of 2. So the answer is $\frac{11}{9}$.

180. (a) i. 0 ii. $\frac{2}{\pi}$ iii. $\frac{1}{2}$

(b) Average value of $f(t) <$ Average value of $k(t) <$ Average value of $g(t)$

We can look at the three functions in the range $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, since they all have periods of 2π ($|\cos t|$ and $(\cos t)^2$ also have a period of π, but that does not hurt our calculation). It is clear from the graphs of the three functions below that the average value for $\cos t$ is 0 (since the area above the x-axis is equal to the area below it), while the average values for the other two are positive (since they are everywhere positive, except where they are 0).



It is also fairly clear from the graphs that the average value of $g(t)$ is greater than the average value of $k(t)$; it is also possible to see this algebraically, since

$$(\cos t)^2 = |\cos t|^2 \le |\cos t|$$

because $|\cos t| \le 1$ (and both of these $\le$'s are $<$'s at all the points where the functions are not 0 or 1).

181. This calculation cannot be correct because the integrand is positive everywhere, yet the value given for the integral is negative.

The calculation is incorrect because the integral is improper but has not been treated as such. The integral is improper because the integrand $1/x^2$ is undefined at $x = 0$. To determine whether the integral converges we split the integral into two improper integrals:

$$\int_{-2}^{2} \frac{1}{x^2} dx = \int_{-2}^{0} \frac{1}{x^2} dx + \int_{0}^{2} \frac{1}{x^2} dx.$$

To decide whether the second integral converges, we compute

$$\int_{0}^{2} \frac{1}{x^2} dx = \lim_{a \to 0^+} \int_{a}^{2} \frac{1}{x^2} dx = \lim_{a \to 0^+} \left(-\frac{1}{2} + \frac{1}{a}\right).$$

The limit does not exist, and $\int_0^2 (1/x^2)\,dx$ diverges, so the original the integral $\int_{-2}^{2} 1/x^2\,dx$ diverges.

182. (a) We have

$$E(x) + F(x) = \int \frac{e^x}{e^x + e^{-x}}\,dx + \int \frac{e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{e^x + e^{-x}}{e^x + e^{-x}}\,dx = \int 1\,dx = x + C_1.$$

(b) We have

$$E(x) - F(x) = \int \frac{e^x}{e^x + e^{-x}}\,dx - \int \frac{e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln\left|e^x + e^{-x}\right| + C_2,$$

by using the substitution $w = e^x + e^{-x}$ in the final integral.

(c) We have

$$E(x) + F(x) = x + C_1$$
$$E(x) - F(x) = \ln\left|e^x + e^{-x}\right| + C_2.$$

Adding and subtracting we find

$$E(x) = \frac{x}{2} + \frac{1}{2}\ln\left|e^x + e^{-x}\right| + C,$$

where the arbitrary constant $C = (C_1 + C_2)/2$, and

$$F(x) = \frac{x}{2} - \frac{1}{2}\ln\left|e^x + e^{-x}\right| + C,$$

where $C = (C_1 - C_2)/2$.

183. Since $f(x)$ is decreasing on $[a, b]$, the left-hand Riemann sums are all overestimates and the right-hand sums are all underestimates. Because increasing the number of subintervals generally brings an approximation closer to the actual value, LEFT(10) is closer to the actual value (i.e., smaller, since the left sums are overestimates) than LEFT(5), and analogously for RIGHT(10) and RIGHT(5). Since the graph of $f(x)$ is concave down, a secant line lies below the curve and a tangent line lies above the curve. Therefore, TRAP is an underestimate and MID is an overestimate. Putting these observations together, we have

$$\text{RIGHT}(5) < \text{RIGHT}(10) < \text{TRAP}(10) < \text{Exact value} < \text{MID}(10) < \text{LEFT}(10) < \text{LEFT}(5).$$

184. (a) $f(x) = 1 + e^{-x}$ is concave up for $0 \le x \le 0.5$, so trapezoids will overestimate $\int_0^{0.5} f(x)dx$, and the midpoint rule will underestimate.

(b) $f(x) = e^{-x^2}$ is concave down for $0 \le x \le 0.5$, so trapezoids will underestimate $\int_0^{0.5} f(x)dx$ and midpoint will overestimate the integral.

(c) Both the trapezoid rule and the midpoint rule will give the exact value of the integral. Note that upper and lower sums will not, unless the line is horizontal.

185. (a) For the left-hand rule, error is approximately proportional to $\frac{1}{n}$. If we let n_p be the number of subdivisions needed for accuracy to p places, then there is a constant k such that

$$\begin{aligned} 5\times 10^{-5} &= \frac{1}{2}\times 10^{-4} \approx \frac{k}{n_4} \\ 5\times 10^{-9} &= \frac{1}{2}\times 10^{-8} \approx \frac{k}{n_8} \\ 5\times 10^{-13} &= \frac{1}{2}\times 10^{-12} \approx \frac{k}{n_{12}} \\ 5\times 10^{-21} &= \frac{1}{2}\times 10^{-20} \approx \frac{k}{n_{20}} \end{aligned}$$

Thus the ratios $n_4 : n_8 : n_{12} : n_{20} \approx 1 : 10^4 : 10^8 : 10^{16}$, and assuming the computer time necessary is proportional to n_p, the computer times are approximately

4 places:	2 seconds	
8 places:	2×10^4 seconds	$\approx$ 6 hours
12 places:	2×10^8 seconds	$\approx$ 6 years
20 places:	2×10^{16} seconds	$\approx$ 600 million years

(b) For the trapezoidal rule, error is approximately proportional to $\frac{1}{n^2}$. If we let N_p be the number of subdivisions needed for accuracy to p places, then there is a constant C such that

$$\begin{aligned} 5\times 10^{-5} &= \frac{1}{2}\times 10^{-4} \approx \frac{C}{{N_4}^2} \\ 5\times 10^{-9} &= \frac{1}{2}\times 10^{-8} \approx \frac{C}{{N_8}^2} \\ 5\times 10^{-13} &= \frac{1}{2}\times 10^{-12} \approx \frac{C}{{N_{12}}^2} \\ 5\times 10^{-21} &= \frac{1}{2}\times 10^{-20} \approx \frac{C}{{N_{20}}^2} \end{aligned}$$

Thus the ratios ${N_4}^2 : {N_8}^2 : {N_{12}}^2 : {N_{20}}^2 \approx 1 : 10^4 : 10^8 : 10^{16}$, and the ratios $N_4 : N_8 : N_{12} : N_{20} \approx 1 : 10^2 : 10^4 : 10^8$. So the computer times are approximately

4 places:	2 seconds	
8 places:	2×10^2 seconds	$\approx$ 3 minutes
12 places:	2×10^4 seconds	$\approx$ 6 hours
20 places:	2×10^8 seconds	$\approx$ 6 years

186. Use integration by parts, with $u = x$ and $dv = xe^{-x^2}$. Then $v = -(1/2)e^{-x^2}$, and

$$\int_0^b x^2 e^{-x^2}\,dx = -\frac{1}{2}xe^{-x^2}\Big|_0^b + \int_0^b \frac{1}{2}e^{-x^2}\,dx$$
$$= -\frac{1}{2}be^{-b^2} + \frac{1}{2}\int_0^b e^{-x^2}\,dx.$$

Since the exponential grows faster than any power,

$$\lim_{b\to\infty} be^{-b^2} = \lim_{b\to\infty}\frac{b}{e^{b^2}} = 0.$$

So

$$\int_0^b x^2 e^{-x^2}\,dx = 0 + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}.$$

187. (a) We calculate the integral using partial fractions with denominators P and $L - P$:

$$\frac{k}{P(L-P)} = \frac{A}{P} + \frac{B}{L-P}$$
$$k = A(L-P) + BP$$
$$k = (B-A)P + AL.$$

Thus,

$$B - A = 0$$
$$AL = k,$$

so $A = B = k/L$, and the time is given by

$$T = \int_{L/4}^{L/2} \frac{k\,dP}{P(L-P)} = \frac{k}{L}\int_{L/4}^{L/2}\left(\frac{1}{P} + \frac{1}{L-P}\right)dP = \frac{k}{L}(\ln|P| - \ln|L-P|)\Big|_{L/4}^{L/2}$$
$$= \frac{k}{L}\left(\ln\left(\frac{L}{2}\right) - \ln\left(\frac{L}{2}\right) - \ln\left(\frac{L}{4}\right) + \ln\left(\frac{3L}{4}\right)\right)$$
$$= \frac{k}{L}\ln\left(\frac{3L/4}{L/4}\right) = \frac{k}{L}\ln(3).$$

(b) A similar calculation gives the following expression for the time:

$$T = \frac{k}{L}(\ln|P| - \ln|L-P|)\Big|_{P_1}^{P_2} = \frac{k}{L}\left(\ln|P_2| - \ln|L-P_2| - \ln|P_1| + \ln|L-P_1|\right).$$

If $P_2 \to L$, then $L - P_2 \to 0$, so $\ln P_2 \to \ln L$, and $\ln(L - P_2) \to -\infty$. Thus the time tends to infinity.

188. If $I(t)$ is average per-capita income t years after 2005, then $I'(t) = r(t)$.

(a) Since $t = 10$ in 2015, by the Fundamental Theorem,

$$I(10) - I(0) = \int_0^{10} r(t)\,dt = \int_0^{10} 1556.37e^{0.045t}\,dt = \frac{1556.37}{0.045}e^{0.045t}\Big|_0^{10} = 19{,}655.65 \text{ dollars.}$$

so $I(10) = 34{,}586 + 19{,}655.65 = 54{,}241.65$.

(b) We have

$$I(t) - I(0) = \int_0^t r(t)\,dt = \int_0^t 1556.37e^{0.045t}\,dt = \frac{1556.37}{0.045}e^{0.045t}\Big|_0^t = 34{,}586\left(e^{0.045t} - 1\right).$$

Thus, since $I(0) = 34{,}586$,

$$I(t) = 34{,}586 + 34{,}586(e^{0.045t} - 1) = 34{,}586e^{0.045t} \text{ dollars.}$$

189. (a) Since the rate is given by $r(t) = 2te^{-2t}$ ml/sec, by the Fundamental Theorem of Calculus, the total quantity is given by the definite integral:

$$\text{Total quantity} \approx \int_0^\infty 2te^{-2t}\,dt = 2\lim_{b\to\infty}\int_0^b te^{-2t}\,dt.$$

Integration by parts with $u = t$, $v' = e^{-2t}$ gives

$$\begin{aligned}\text{Total quantity} &\approx 2\lim_{b\to\infty}\left(-\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t}\right)\Big|_0^b \\ &= 2\lim_{b\to\infty}\left(\frac{1}{4} - \left(\frac{b}{2}+\frac{1}{4}\right)e^{-2b}\right) = 2\cdot\frac{1}{4} = 0.5 \text{ ml.}\end{aligned}$$

(b) At the end of 5 seconds,

$$\text{Quantity received} = \int_0^5 2te^{-2t}\,dt \approx 0.49975 \text{ ml.}$$

Since $0.49975/0.5 = 0.9995 = 99.95\%$, the patient has received 99.95% of the dose in the first 5 seconds.

190. The rate at which petroleum is being used t years after 1990 is given by

$$r(t) = 1.4\cdot 10^{20}(1.02)^t \text{ joules/year.}$$

Between 1990 and M years later

$$\begin{aligned}\text{Total quantity of petroleum used } &= \int_0^M 1.4\cdot 10^{20}(1.02)^t\,dt = 1.4\cdot 10^{20}\left.\frac{(1.02)^t}{\ln(1.02)}\right|_0^M \\ &= \frac{1.4\cdot 10^{20}}{\ln(1.02)}((1.02)^M - 1) \text{ joules.}\end{aligned}$$

Setting the total quantity used equal to 10^{22} gives

$$\begin{aligned}\frac{1.4\cdot 10^{20}}{\ln(1.02)}\left((1.02)^M - 1\right) &= 10^{22} \\ (1.02)^M &= \frac{100\ln(1.02)}{1.4} + 1 = 2.41 \\ M &= \frac{\ln(2.41)}{\ln(1.02)} \approx 45 \text{ years.}\end{aligned}$$

So we will run out of petroleum in 2035.

191. (a) We have

$$S = \int_0^{18}(30-10)dt = 20\int_0^{18} dt = 20(18-0) = 360.$$

The units of S are degree-days, because the integrand $f(t) - 10$ has units of °C, and dt has units of days.

(b) In Figure 7.49, $f(t)$ and H_{min} are represented by the horizontal lines at $H = 30$ and $H = 10$, and T is represented by a vertical line at $t = 18$. The value of S, given by the definite integral, is represented by the area of the rectangle bounded by the vertical lines $t = 0$ (the H-axis) and $t = 18$, and the horizontal lines $H = 10$ and $H = 30$.

(c) The temperature cycles from a high of 30°C to a low of 10°C once every 6 days. During the 18-day period the temperature completes 3 complete cycles. The area between this curve and the horizontal line $H = H_{\text{min}} = 10$ gives the value of the definite integral. See Figure 7.50. In order to get the same area as before, (namely $S = 360$), we see that T_2 must be larger than $T = 18$. Thus we want T_2 to satisfy:

$$S = \int_0^{T_2}(g(t)-10)dt = \int_0^{T_2}\left(10\cos\left(\frac{2\pi t}{6}\right)+10\right)\,dt = 360.$$

Notice that, by symmetry, the area on the interval $0 \le t \le 18$ is half the area shown in Figure 7.49. Thus, we expect that $T_2 = 2T = 36$. We can check this by calculation, using a substitution to evaluate the integral:

$$S = \int_0^{36}\left(10\cos\left(\frac{2\pi t}{6}\right)+10\right)\,dt = \left(10\cdot\frac{6}{2\pi}\sin\left(\frac{2\pi t}{6}\right)+10t\right)\Big|_0^{T_2} = 360.$$

Thus, if $T_2 = 36$, the integral evaluates to $S = 360$, as required. Thus, 36 days are required for development for with these temperatures.

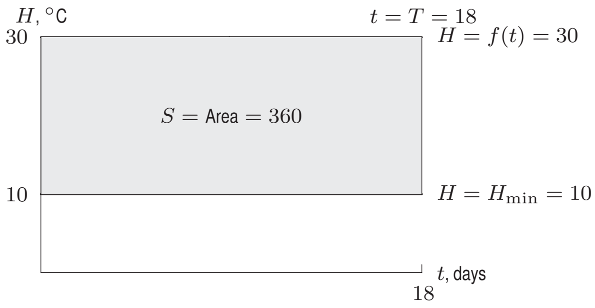


Figure 7.49

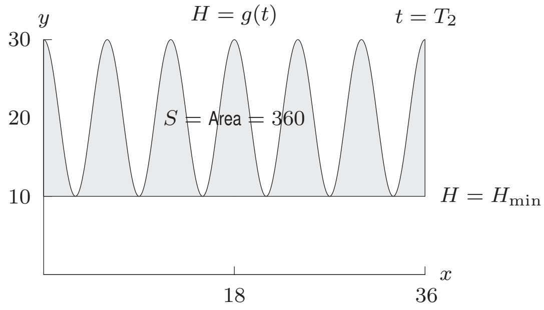


Figure 7.50

192. We want to calculate

$$\int_0^1 C_n \sin(n\pi x) \cdot C_m \sin(m\pi x)\, dx.$$

We use II-11 from the table of integrals with $a = n\pi$, $b = m\pi$. Since $n \neq m$, we see that

$$\begin{aligned}\int_0^1 \Psi_n(x) \cdot \Psi_m(x)\, dx &= C_n C_m \int_0^1 \sin(n\pi x)\sin(m\pi x)\, dx \\ &= \frac{C_n C_m}{m^2\pi^2 - n^2\pi^2} \left(n\pi \cos(n\pi x)\sin(m\pi x) - \ m\pi \sin(n\pi x)\cos(m\pi x)\right)\Big|_0^1 \\ &= \frac{C_n C_m}{(m^2 - n^2)\pi^2} \Big(n\pi \cos(n\pi)\sin(m\pi) - m\pi \sin(n\pi)\cos(m\pi) \\ &\qquad - n\pi \cos(0)\sin(0) + m\pi \sin(0)\cos(0)\Big) \\ &= 0\end{aligned}$$

since $\sin(0) = \sin(n\pi) = \sin(m\pi) = 0$.

CAS Challenge Problems

193. (a) A CAS gives

$$\int \frac{\ln x}{x}\, dx = \frac{(\ln x)^2}{2}$$
$$\int \frac{(\ln x)^2}{x}\, dx = \frac{(\ln x)^3}{3}$$
$$\int \frac{(\ln x)^3}{x}\, dx = \frac{(\ln x)^4}{4}$$

(b) Looking at the answers to part (a),

$$\int \frac{(\ln x)^n}{x}\, dx = \frac{(\ln x)^{n+1}}{n+1} + C.$$

(c) Let $w = \ln x$. Then $dw = (1/x)dx$, and

$$\int \frac{(\ln x)^n}{x}\, dx = \int w^n\, dw = \frac{w^{n+1}}{n+1} + C = \frac{(\ln x)^{n+1}}{n} + C.$$

194. (a) A CAS gives

$$\int \ln x\, dx = -x + x\ln x$$
$$\int (\ln x)^2\, dx = 2x - 2x\ln x + x(\ln x)^2$$
$$\int (\ln x)^3\, dx = -6x + 6x\ln x - 3x(\ln x)^2 + x(\ln x)^3$$
$$\int (\ln x)^4\, dx = 24x - 24x\ln x + 12x(\ln x)^2 - 4x(\ln x)^3 + x(\ln x)^4$$

(b) In each of the cases in part (a), the expression for the integral $\int (\ln x)^n\, dx$ has two parts. The first part is simply a multiple of the expression for $\int (\ln x)^{n-1}\, dx$. For example, $\int (\ln x)^2\, dx$ starts out with $2x - 2x\ln x = -2\int \ln x\, dx$. Similarly, $\int (\ln x)^3\, dx$ starts out with $-6x + 6x\ln x - 3(\ln x)^2 = -3\int (\ln x)^2\, dx$, and $\int (\ln x)^4\, dx$ starts out with $-4\int (\ln x)^3\, dx$. The remaining part of each antiderivative is a single term: it's $x(\ln x)^2$ in the case $n = 2$, it's $x(\ln x)^3$ for $n = 3$, and it's $x(\ln x)^4$ for $n = 4$. The general pattern is

$$\int (\ln x)^n\, dx = -n\int (\ln x)^{n-1}\, dx + x(\ln x)^n.$$

To check this formula, we use integration by parts. Let $u = (\ln x)^n$ so $u' = n(\ln x)^{n-1}/x$ and $v' = 1$ so $v = x$. Then

$$\int (\ln x)^n\, dx = x(\ln x)^n - \int n\frac{(\ln x)^{n-1}}{x}\cdot x\, dx$$
$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx.$$

This is the result we obtained before.

Alternatively, we can check our result by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(-n\int (\ln x)^{n-1}\, dx + x(\ln x)^n\right) &= -n(\ln x)^{n-1} + \frac{d}{dx}(x(\ln x)^n)\\ &= -n(\ln x)^{n-1} + (\ln x)^n + x\cdot n(\ln x)^{n-1}\frac{1}{x}\\ &= -n(\ln x)^{n-1} + (\ln x)^n + n(\ln x)^{n-1} = (\ln x)^n.\end{aligned}$$

Therefore,

$$\int (\ln x)^n\, dx = -n\int (\ln x)^{n-1}\, dx + x(\ln x)^n.$$

195. (a) A possible answer from the CAS is

$$\int \sin^3 x\, dx = \frac{-9\,\cos(x) + \cos(3\,x)}{12}.$$

(b) Differentiating

$$\frac{d}{dx}\left(\frac{-9\,\cos(x) + \cos(3\,x)}{12}\right) = \frac{9\,\sin(x) - 3\,\sin(3\,x)}{12} = \frac{3\sin x - \sin(3x)}{4}.$$

(c) Using the identities, we get

$$\begin{aligned}\sin(3x) = \sin(x + 2x) &= \sin x\cos 2x + \cos x\sin 2x\\ &= \sin x(1 - 2\sin^2 x) + \cos x(2\sin x\cos x)\\ &= \sin x - 2\sin^3 x + 2\sin x(1 - \sin^2 x)\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

Thus,

$$3\sin x - \sin(3x) = 3\sin x - (3\sin x - 4\sin^3 x) = 4\sin^3 x,$$

so

$$\frac{3\sin x - \sin(3x)}{4} = \sin^3 x.$$

196. (a) A possible answer is

$$\int \sin x \cos x \cos(2x)\, dx = -\frac{\cos(4x)}{16}.$$

Different systems may give the answer in a different form.

(b)

$$\frac{d}{dx}\left(-\frac{\cos(4x)}{16}\right) = \frac{\sin(4x)}{4}.$$

(c) Using the double angle formula $\sin 2A = 2\sin A \cos A$ twice, we get

$$\frac{\sin(4x)}{4} = \frac{2\sin(2x)\cos(2x)}{4} = \frac{2 \cdot 2\sin x \cos x \cos(2x)}{4} = \sin x \cos x \cos(2x).$$

197. (a) A possible answer from the CAS is

$$\int \frac{x^4}{(1+x^2)^2}\, dx = x + \frac{x}{2\,(1+x^2)} - \frac{3}{2}\arctan(x).$$

Different systems may give the answer in different form.

(b) Differentiating gives

$$\frac{d}{dx}\left(x + \frac{x}{2\,(1+x^2)} - \frac{3}{2}\arctan(x)\right) = 1 - \frac{x^2}{\left(1+x^2\right)^2} - \frac{1}{1+x^2}.$$

(c) Putting the result of part (b) over a common denominator, we get

$$\begin{aligned} 1 - \frac{x^2}{\left(1+x^2\right)^2} - \frac{1}{1+x^2} &= \frac{\left(1+x^2\right)^2 - x^2 - (1+x^2)}{(1+x^2)^2} \\ &= \frac{1 + 2x^2 + x^4 - x^2 - 1 - x^2}{(1+x^2)^2} = \frac{x^4}{(1+x^2)^2}. \end{aligned}$$

PROJECTS FOR CHAPTER SEVEN

1. (a) If $e^t \geq 1 + t$, then

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\, dt \\ &\geq 1 + \int_0^x (1+t)\, dt = 1 + x + \frac{1}{2}x^2. \end{aligned}$$

We can keep going with this idea. Since $e^t \geq 1 + t + \frac{1}{2}t^2$,

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\, dt \\ &\geq 1 + \int_0^x (1 + t + \frac{1}{2}t^2)\, dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3. \end{aligned}$$

We notice that each term in our summation is of the form $\frac{x^n}{n!}$. Furthermore, we see that if we have a sum $1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$ such that

$$e^x \geq 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!},$$

then

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\, dt \\ &\geq 1 + \int_0^x \left(1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!}\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^{n+1}}{(n+1)!}. \end{aligned}$$

Thus we can continue this process as far as we want, so

$$e^x \geq 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n = \sum_{j=0}^{n} \frac{x^j}{j!} \text{ for any } n.$$

(In fact, it turns out that if you let n get larger and larger and keep adding up terms, your values approach exactly e^x.)

(b) We note that $\sin x = \int_0^x \cos t\, dt$ and $\cos x = 1 - \int_0^x \sin t\, dt$. Thus, since $\cos t \leq 1$, we have

$$\sin x \leq \int_0^x 1\, dt = x.$$

Now using $\sin t \leq t$, we have

$$\cos x \leq 1 - \int_0^x t\, dt = 1 - \frac{1}{2}x^2.$$

Then we just keep going:

$$\sin x \leq \int_0^x \left(1 - \frac{1}{2}t^2\right)\, dt = x - \frac{1}{6}x^3.$$

Therefore

$$\cos x \leq 1 - \int_0^x \left(t - \frac{1}{6}t^3\right)\, dt = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

CHAPTER EIGHT

Solutions for Section 8.1

Exercises

1. (a) The strip stretches between $y = 0$ and $y = 2x$, so

$$\text{Area of region } \approx \sum (2x)\Delta x.$$

(b) We have

$$\text{Area } = \int_0^3 (2x)\,dx = x^2\Big|_0^3 = 9.$$

2. (a) The strip stretches between $y = 0$ and $y = -x^2 + 6x$, so

$$\text{Area of region } \approx \sum (-x^2 + 6x)\Delta x.$$

(b) We have

$$\text{Area } = \int_0^6 (-x^2 + 6x)\,dx = -\frac{x^3}{3} + 3x^2\Big|_0^6 = 36.$$

3. (a) The strip stretches from $x = y/2$ to $x = 3$, so

$$\text{Area of region } \approx \sum \left(3 - \frac{y}{2}\right)\,\Delta y.$$

(b) We have

$$\text{Area } = \int_0^6 \left(3 - \frac{y}{2}\right)\,dy = \left(3y - \frac{y^2}{4}\right)\Big|_0^6 = 9.$$

4. (a) The equation $y = -x^2 + 6x$ can be solved for x as

$$x^2 - 6x + y = 0$$
$$x = \frac{6 \pm \sqrt{36 - 4y}}{2} = 3 \pm \sqrt{9 - y}.$$

The left end of the strip is given by $x = 3 - \sqrt{9 - y}$, and the right end is given by $x = 3 + \sqrt{9 - y}$. Thus,

$$\text{Area of region } \approx \sum ((3 + \sqrt{9 - y}) - (3 - \sqrt{9 - y}))\Delta y = \sum 2\sqrt{9 - y}\,\Delta y.$$

(b) We have

$$\text{Area } = \int_0^9 2\sqrt{9 - y}\,dy = \frac{2(9 - y)^{3/2}}{-3/2}\Big|_0^9 = 36.$$

5. Each strip is a rectangle of length 3 and width Δx, so

$$\text{Area of strip } = 3\Delta x, \quad \text{so}$$
$$\text{Area of region } = \int_0^5 3\,dx = 3x\Big|_0^5 = 15.$$

Check: This area can also be computed using Length $\times$ Width $= 5 \cdot 3 = 15$.

6. Using similar triangles, the height, y, of the strip is given by

$$\frac{y}{3} = \frac{x}{6} \quad \text{so} \quad y = \frac{x}{2}.$$

Thus,

$$\text{Area of strip } \approx y\Delta x = \frac{x}{2}\Delta x,$$

so

$$\text{Area of region } = \int_0^6 \frac{x}{2}\,dx = \left.\frac{x^2}{4}\right|_0^6 = 9.$$

Check: This area can also be computed using the formula $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2} \cdot 6 \cdot 3 = 9$.

7. By similar triangles, if w is the length of the strip at height h, we have

$$\frac{w}{3} = \frac{5-h}{5} \quad \text{so} \quad w = 3\left(1 - \frac{h}{5}\right).$$

Thus,

$$\text{Area of strip } \approx w\Delta h = 3\left(1 - \frac{h}{5}\right)\Delta h.$$

$$\text{Area of region } = \int_0^5 3\left(1 - \frac{h}{5}\right)\,dh = \left.\left(3h - \frac{3h^2}{10}\right)\right|_0^5 = \frac{15}{2}.$$

Check: This area can also be computed using the formula $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2} \cdot 3 \cdot 5 = \frac{15}{2}$.

8. Suppose the length of the strip shown is w. Then the Pythagorean theorem gives

$$h^2 + \left(\frac{w}{2}\right)^2 = 3^2 \quad \text{so} \quad w = 2\sqrt{3^2 - h^2}.$$

Thus

$$\text{Area of strip } \approx w\Delta h = 2\sqrt{3^2 - h^2}\Delta h,$$

$$\text{Area of region } = \int_{-3}^{3} 2\sqrt{3^2 - h^2}\,dh.$$

Using VI-30 in the Table of Integrals, we have

$$\text{Area } = \left.\left(h\sqrt{3^2 - h^2} + 3^2 \arcsin\left(\frac{h}{3}\right)\right)\right|_{-3}^{3} = 9(\arcsin 1 - \arcsin(-1)) = 9\pi.$$

Check: This area can also be computed using the formula $\pi r^2 = 9\pi$.

9. The strip has width Δy, so the variable of integration is y. The length of the strip is x. Since $x^2 + y^2 = 10$ and the region is in the first quadrant, solving for x gives $x = \sqrt{10 - y^2}$. Thus

$$\text{Area of strip } \approx x\Delta y = \sqrt{10 - y^2}\,dy.$$

The region stretches from $y = 0$ to $y = \sqrt{10}$, so

$$\text{Area of region } = \int_0^{\sqrt{10}} \sqrt{10 - y^2}\,dy.$$

Evaluating using VI-30 from the Table of Integrals, we have

$$\text{Area } = \frac{1}{2}\left.\left(y\sqrt{10 - y^2} + 10\arcsin\left(\frac{y}{\sqrt{10}}\right)\right)\right|_0^{\sqrt{10}} = 5(\arcsin 1 - \arcsin 0) = \frac{5}{2}\pi.$$

Check: This area can also be computed using the formula $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(\sqrt{10})^2 = \frac{5}{2}\pi$.

10. The strip has width Δy, so the variable of integration is y. The length of the strip is $2x$ for $x \geq 0$. For positive x, we have $x = y$. Thus,

$$\text{Area of strip } \approx 2x\Delta y = 2y\Delta y.$$

Since the region extends from $y = 0$ to $y = 4$,

$$\text{Area of region } = \int_0^4 2y\,dy = y^2\Big|_0^4 = 16.$$

Check: The area of the region can be computed by $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2} \cdot 8 \cdot 4 = 16$.

11. The width of the strip is Δy, so the variable of integration is y. Since the graphs are $x = y$ and $x = y^2$, the length of the strip is $y - y^2$, and

$$\text{Area of strip } \approx (y - y^2)\Delta y.$$

The curves cross at the points $(0, 0)$ and $(1, 1)$, so

$$\text{Area of region } = \int_0^1 (y - y^2)\,dy = \frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1 = \frac{1}{6}.$$

12. The width of the strip is Δx, so the variable of integration is x. The line has equation $y = 6 - 3x$. The length of the strip is $6 - 3x - (x^2 - 4) = 10 - 3x - x^2$. (Since $x^2 - 4$ is negative where the graph is below the x-axis, subtracting $x^2 - 4$ there adds the length below the x-axis.) Thus

$$\text{Area of strip } \approx (10 - 3x - x^2)\Delta x.$$

Both graphs cross the x-axis where $x = 2$, so

$$\text{Area of region } = \int_0^2 (10 - 3x - x^2)\,dx = 10x - \frac{3}{2}x^2 - \frac{x^3}{3}\Big|_0^2 = \frac{34}{3}.$$

13. Each slice is a circular disk with radius $r = 2$ cm.

$$\text{Volume of disk } = \pi r^2 \Delta x = 4\pi\Delta x \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum 4\pi\Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum 4\pi\Delta x = \int_0^9 4\pi\,dx \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume } = 4\pi x\Big|_0^9 = 36\pi \text{ cm}^3.$$

Check: The volume of the cylinder can also be calculated using the formula $V = \pi r^2 h = \pi 2^2 \cdot 9 = 36\pi$ cm^3.

14. Each slice is a circular disk. Since the radius of the cone is 2 cm and the length is 6 cm, the radius is one-third of the distance from the vertex. Thus, the radius at x is $r = x/3$ cm. See Figure 8.1.

$$\text{Volume of slice } \approx \pi r^2 \Delta x = \frac{\pi x^2}{9}\Delta x \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \pi\frac{x^2}{9}\Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum \pi\frac{x^2}{9}\Delta x = \int_0^6 \pi\frac{x^2}{9}\,dx \text{ cm}^3.$$

Evaluating, we get

$$\text{Total volume } = \frac{\pi}{9}\frac{x^3}{3}\bigg|_0^6 = \frac{\pi}{9}\cdot\frac{6^3}{3} = 8\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 2^2 \cdot 6 = 8\pi \text{ cm}^3$.

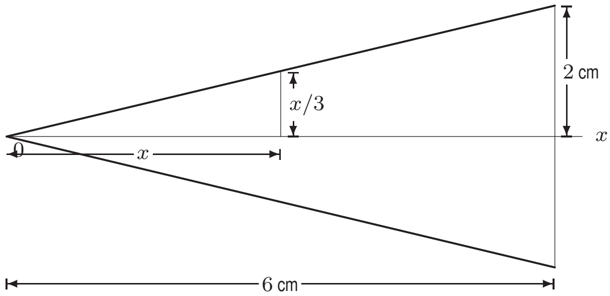


Figure 8.1

15. Each slice is a circular disk. From Figure 8.2, we see that the radius at height y is $r = \frac{2}{5}y$ cm. Thus

$$\text{Volume of disk } \approx \pi r^2 \Delta y = \pi\left(\frac{2}{5}y\right)^2 \Delta y = \frac{4}{25}\pi y^2 \Delta y \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \frac{4\pi}{25} y^2 \Delta y \text{ cm}^3.$$

Taking the limit as $\Delta y \to 0$, we get

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum \frac{4\pi}{25} y^2 \Delta y = \int_0^5 \frac{4\pi}{25} y^2 dy \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume } = \frac{4\pi}{25}\frac{y^3}{3}\bigg|_0^5 = \frac{20}{3}\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} 2^2 \cdot 5 = \frac{20}{3}\pi \text{ cm}^3$.

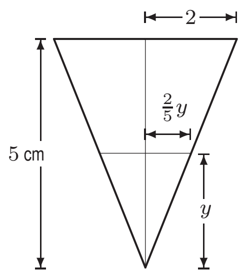


Figure 8.2

16. Each slice is a rectangular slab of length 10 m and width that decreases with height. See Figure 8.3. At height y, the length x is given by the Pythagorean Theorem

$$y^2 + x^2 = 7^2.$$

Solving gives $x = \sqrt{7^2 - y^2}$ m. Thus the width of the slab is $2x = 2\sqrt{7^2 - y^2}$ and

$$\text{Volume of slab } = \text{ Length } \cdot \text{ Width } \cdot \text{ Height } = 10 \cdot 2\sqrt{7^2 - y^2} \cdot \Delta y = 20\sqrt{7^2 - y^2}\Delta y \text{ m}^3.$$

Summing over all slabs, we have

$$\text{Total volume } \approx \sum 20\sqrt{7^2 - y^2}\Delta y \text{ m}^3.$$

Taking a limit as $\Delta y \to 0$, we get

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum 20\sqrt{7^2 - y^2}\Delta y = \int_0^7 20\sqrt{7^2 - y^2}\, dy \text{ m}^3.$$

To evaluate, we use the table of integrals or the fact that $\int_0^7 \sqrt{7^2 - y^2}\, dy$ represents the area of a quarter circle of radius 7, so

$$\text{Total volume } = \int_0^7 20\sqrt{7^2 - y^2}\, dy = 20 \cdot \frac{1}{4}\pi 7^2 = 245\pi \text{ m}^3.$$

Check: the volume of a half cylinder can also be calculated using the formula $V = \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi 7^2 \cdot 10 = 245\pi$ m^3.

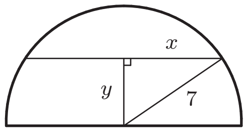


Figure 8.3

17. Each slice is a circular disk. See Figure 8.4. The radius of the sphere is 5 mm, and the radius r at height y is given by the Pythagorean Theorem

$$y^2 + r^2 = 5^2.$$

Solving gives $r = \sqrt{5^2 - y^2}$ mm. Thus,

$$\text{Volume of disk } \approx \pi r^2 \Delta y = \pi(5^2 - y^2)\Delta y \text{ mm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \pi(5^2 - y^2)\Delta y \text{ mm}^3.$$

Taking the limit as $\Delta y \to 0$, we get

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum \pi(5^2 - y^2)\Delta y = \int_0^5 \pi(5^2 - y^2)\, dy \text{ mm}^3.$$

Evaluating gives

$$\text{Total volume } = \pi \left(25y - \frac{y^3}{3}\right)\Bigg|_0^5 = \frac{250}{3}\pi \text{ mm}^3.$$

Check: The volume of a hemisphere can be calculated using the formula $V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi 5^3 = \frac{250}{3}\pi$ mm^3.

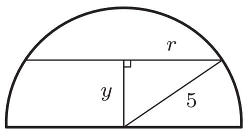


Figure 8.4

18. Each slice is a square; the side length decreases as we go up the pyramid. See Figure 8.5. Since the base of the pyramid is equal to its vertical height, the slice at distance y from the base, or $(2-y)$ from the top, has side $(2-y)$. Thus

$$\text{Volume of slice } \approx (2-y)^2 \Delta y \text{ m}^3.$$

Summing over all slices, we get

$$\text{Total volume } \approx \sum (2-y)^2 \Delta y \text{ m}^3.$$

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum (2-y)^2 \Delta y = \int_0^2 (2-y)^2 dy \text{ m}^3.$$

Evaluating, we find

$$\text{Total volume } = \int_0^2 (4 - 4y + y^2)\, dy = \left(4y - 2y^2 + \frac{y^3}{3}\right)\bigg|_0^2 = \frac{8}{3} \text{ m}^3.$$

Check: The volume of the pyramid can also be calculated using the formula $V = \frac{1}{3}b^2h = \frac{1}{3}2^2 \cdot 2 = \frac{8}{3}$ m^3.

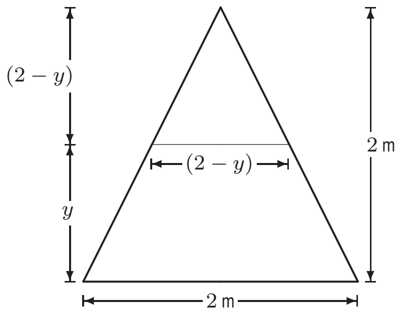


Figure 8.5

Problems

19. Triangle of base and height 1 and 3. See Figure 8.6. (Either 1 or 3 can be the base. A non-right triangle is also possible.)

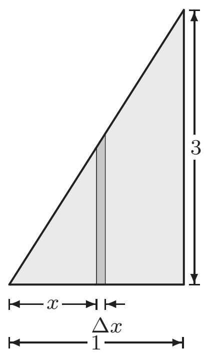


Figure 8.6

20. Semicircle of radius $r = 9$. See Figure 8.7.

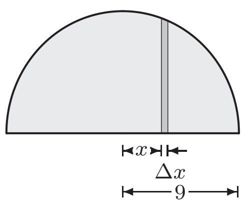


Figure 8.7

21. Quarter circle of radius $r = \sqrt{15}$. See Figure 8.8.

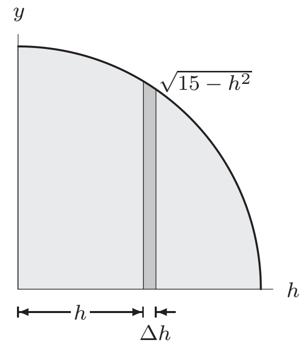


Figure 8.8

22. Triangle of base and height 7 and 5. See Figure 8.9. (Either 7 or 5 can be the base. A non-right triangle is also possible.)

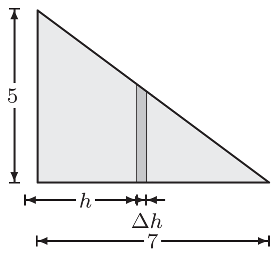


Figure 8.9

23. There are at least two possible answers. Since $y = x - x^2 \geq 0$ when $0 \leq x \leq 1$, one possibility is that the integral gives the area between the parabola $y = x - x^2$ and the line $y = 0$ as shown in Figure 8.10. Alternatively, since $x \geq x^2$ when $0 \leq x \leq 1$, the integral gives the area between the curves $y = x^2$ and $y = x$, as shown in Figure 8.11.

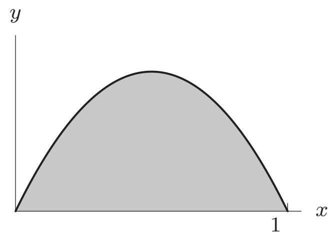


Figure 8.10

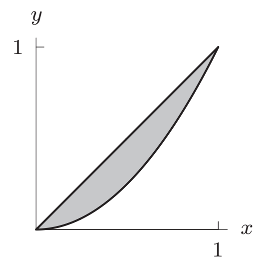


Figure 8.11

24. (a) The area is shown in Figure 8.12(a) with one vertical slice shown. Each slice is bounded on the top by $y = 3x$ and on the bottom by $y = x^2$, so the height of each slice is $3x - x^2$. The width of each slice is Δx, so the area is represented by

$$A = \int_0^3 (3x - x^2)\,dx = \frac{3}{2}x^2 - \frac{1}{3}x^3\bigg|_0^3$$
$$= \frac{3}{2} \cdot 3^2 - \frac{1}{3}\dot{3}^3 - (0 - 0)$$
$$= 4.5.$$

Figure 8.12

(b) The area is shown in Figure 8.12(b) with one horizontal slice shown. Each slice is bounded on the right by $y = x^2$, which is $x = \sqrt{y}$ distance out from the y-axis, and is bounded on the left by $y = 3x$, which is $x = y/3$ distance out from the y-axis. The length of each slice is $\sqrt{y} - y/3$. The width of each slice is Δy, and the area extends from $y = 0$ to $y = 9$ so the area is represented by

$$\begin{aligned} A = \int_0^9 \left(\sqrt{y} - \frac{y}{3}\right) dy &= \frac{2}{3}y^{3/2} - \frac{1}{6}y^2\Big|_0^9 \\ &= \frac{2}{3}\cdot 9^{3/2} - \frac{1}{6}\cdot 9^2 - (0-0) \\ &= 4.5. \end{aligned}$$

Notice that the two ways of computing the area give the same answer, as we expect.

25. (a) The area is shown in Figure 8.13(a) with one vertical slice shown. Each slice is bounded on the top by $y = 12 - x$ and on the bottom by $y = 2x$, so the height of each slice is $(12 - x) - 2x$. The width of each slice is Δx and the area ranges from $x = 0$ to $x = 4$, so the area is represented by

$$\begin{aligned} A = \int_0^4 ((12-x) - 2x)\, dx &= \int_0^4 12 - 3x\, dx \\ &= 12x - \frac{3}{2}x^2\Big|_0^4 \\ &= 12\cdot 4 - \frac{3}{2}\cdot 4^2 - (0-0) \\ &= 24. \end{aligned}$$

Figure 8.13

(b) The area is shown in Figure 8.13(b) with two horizontal slices shown. Because some of the slices are bounded on the right by $y = 12 - x$ while others are bounded on the right by $y = 2x$, we use two separate integrals to calculate this area using horizontal slices. The slices between $y = 0$ and $y = 8$ are bounded on the right by $y = 2x$, which is $x = y/2$ distance from the y-axis, and are bounded on the left by the y-axis, which is $x = 0$. This part of the area is given by

$$\text{Bottom part of the area } = \int_0^8 \frac{y}{2}\, dy = \frac{1}{4}y^2\Big|_0^8 = \frac{1}{4}\cdot 8^2 - 0 = 16.$$

The slices between $y = 8$ and $y = 12$ are bounded on the right by $y = 12 - x$, which is $x = 12 - y$ distance from the y-axis, and are bounded on the left by the y-axis, which is $x = 0$. This part of the area is given by

$$\text{Top part of the area } = \int_8^{12} (12-y)\, dy = 12y - \frac{1}{2}y^2\Big|_8^{12} = 12\cdot 12 - \frac{1}{2}\cdot 12^2 - \left(12\cdot 8 - \frac{1}{2}\cdot 8^2\right) = 8.$$

The total area is given by

$$\text{Total area } = \int_0^8 \frac{y}{2}\,dy + \int_8^{12} (12-y)\,dy = 16 + 8 = 24.$$

Notice that the two ways of computing the area give the same answer, as we expect.

26. (a) The area is shown in Figure 8.14(a) with two vertical slices shown. Because some of the slices are bounded on the top by $y = x^2$ while others are bounded on the top by $y = 6 - x$, we use two separate integrals to calculate this area using vertical slices. The slices between $x = 0$ and $x = 2$ are bounded on the top by $y = x^2$ and are bounded on the bottom by the x-axis, which is $y = 0$. This part of the area is given by

$$\text{Left part of the area } = \int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}\cdot 2^3 - 0 = 2.667.$$

The slices between $x = 2$ and $x = 6$ are bounded on the top by $y = 6 - x$ and are bounded on the bottom by the x-axis, which is $y = 0$. This part of the area is given by

$$\begin{aligned}\text{Right part of the area } &= \int_2^6 (6-x)\,dx = 6x - \frac{1}{2}x^2\Big|_2^6 \\ &= 6\cdot 6 - \frac{1}{2}\cdot 6^2 - \left(6\cdot 2 - \frac{1}{2}\cdot 2^2\right) = 8.\end{aligned}$$

The total area is given by

$$\text{Total area } = \int_0^2 x^2\,dx + \int_2^6 (6-x)\,dx = 2.667 + 8 = 10.667.$$

Figure 8.14

(b) The area is shown in Figure 8.14(b) with one horizontal slice shown. The slices are all bounded on the right by $y = 6 - x$, which is $x = 6 - y$ distance from the y-axis, and are bounded on the left by $y = x^2$, which is $x = \sqrt{y}$ distance from the y-axis. The area ranges from $y = 0$ to $y = 4$ so the area is given by

$$\begin{aligned}A = \int_0^4 ((6-y) - \sqrt{y})\,dy &= 6y - \frac{1}{2}y^2 - \frac{2}{3}\cdot y^{3/2}\Big|_0^4 \\ &= 6\cdot 4 - \frac{1}{2}\cdot 4^2 - \frac{2}{3}\cdot 4^{3/2} - 0 \\ &= 10.667.\end{aligned}$$

Notice that the two ways of computing the area give the same answer, as we expect.

27. (a) The area is shown in Figure 8.15(a) with two vertical slices shown. Because some of the slices are bounded on the top by $y = 2x$ while others are bounded on the top by $y = 6$, we need to use two separate integrals to calculate this area using vertical slices. The slices between $x = 0$ and $x = 3$ are bounded on the top by $y = 2x$ and are bounded on the bottom by the x-axis, which is $y = 0$. This part of the area is given by

$$\text{Left part of the area } = \int_0^3 2x\,dx = x^2\Big|_0^3 = 3^2 - 0 = 9.$$

The slices between $x = 3$ and $x = 5$ are bounded on the top by $y = 6$ and are bounded on the bottom by the x-axis, which is $y = 0$. This part of the area is given by

$$\text{Right part of the area } = \int_3^5 6\,dx = 6x\Big|_3^5 = 6 \cdot 5 - 6 \cdot 3 = 12.$$

The total area is given by

$$\text{Total area } = \int_0^3 2x\,dx + \int_3^5 6\,dx = 9 + 12 = 21.$$

Figure 8.15

(b) The area is shown in Figure 8.15(b) with one horizontal slice shown. The slices are all bounded on the right by $x = 5$ and are bounded on the left by $y = 2x$, which is $x = y/2$ distance from the y-axis. The area ranges from $y = 0$ to $y = 6$ so the area is given by

$$A = \int_0^6 (5 - (y/2))\,dy = 5x - \frac{1}{4}y^2\Big|_0^6 = 5 \cdot 6 - \frac{1}{4} \cdot 6^2 - 0 = 21.$$

Notice that the two ways of computing the area give the same answer, as we expect.

28. Hemisphere with radius 12. See Figure 8.16.

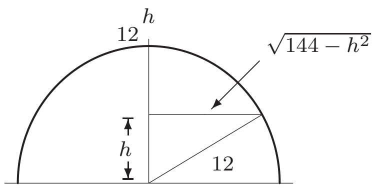


Figure 8.16

29. Cone with height 12 and radius $12/3 = 4$. See Figure 8.17.

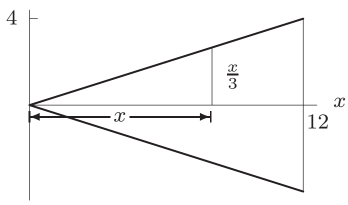


Figure 8.17

30. Cone with height 6 and radius 3. See Figure 8.18.

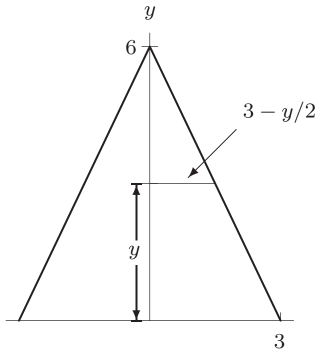


Figure 8.18

31. Hemisphere with radius 2. See Figure 8.19.

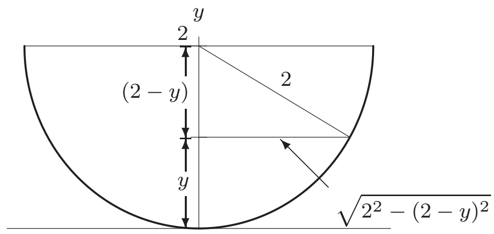
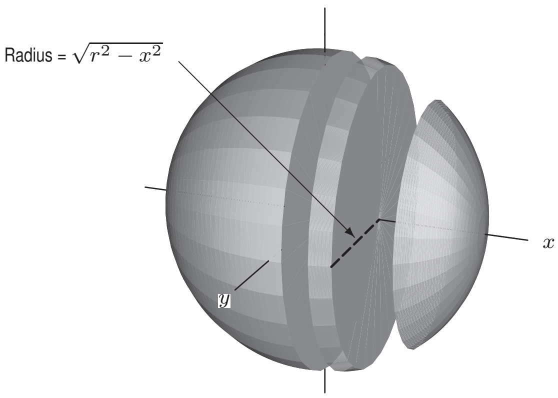


Figure 8.19

32.



We slice up the sphere in planes perpendicular to the x-axis. Each slice is a circle, with radius $y = \sqrt{r^2 - x^2}$; that's the radius because $x^2 + y^2 = r^2$ when $z = 0$. Then the volume is

$$V \approx \sum \pi(y^2)\,\Delta x = \sum \pi(r^2 - x^2)\,\Delta x.$$

Therefore, as Δx tends to zero, we get

$$\begin{aligned} V &= \int_{x=-r}^{x=r} \pi(r^2 - x^2)\,dx \\ &= 2\int_{x=0}^{x=r} \pi(r^2 - x^2)\,dx \\ &= 2\left(\pi r^2 x - \frac{\pi x^3}{3}\right)\Bigg|_0^r \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

33. We slice the cone horizontally into cylindrical disks with radius r and thickness Δh. See Figure 8.20. The volume of each disk is $\pi r^2 \Delta h$. We use the similar triangles in Figure 8.21 to write r as a function of h:

$$\frac{r}{h} = \frac{3}{12} \quad \text{so} \quad r = \frac{1}{4}h.$$

The volume of the disk at height h is $\pi(\frac{1}{4}h)^2\Delta h$. To find the total volume, we integrate this quantity from $h = 0$ to $h = 12$.

$$V = \int_0^{12} \pi\left(\frac{1}{4}h\right)^2 dh = \frac{\pi}{16}\frac{h^3}{3}\Bigg|_0^{12} = 36\pi = 113.097 \text{ m}^3.$$

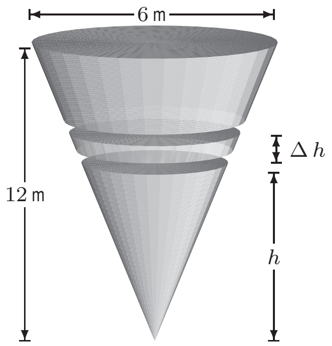


Figure 8.20

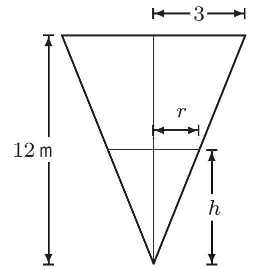
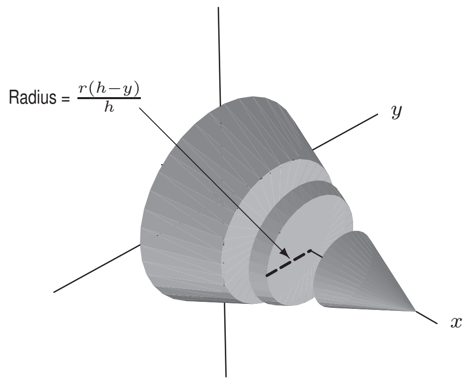


Figure 8.21

34.



This cone is what you get when you rotate the line $x = r(h-y)/h$ about the y–axis. So slicing perpendicular to the y–axis yields

$$V = \int_{y=0}^{y=h} \pi x^2\,dy = \pi \int_0^h \left(\frac{(h-y)r}{h}\right)^2 dy$$
$$= \pi \frac{r^2}{h^2} \int_0^h (h^2 - 2hy + y^2)\,dy$$
$$= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{y^3}{3}\right]\Bigg|_0^h = \frac{\pi r^2 h}{3}.$$

35. (a) A vertical slice has a triangular shape and thickness Δx. See Figure 8.22.

$$\text{Volume of slice} = \text{Area of triangle} \cdot \Delta x = \frac{1}{2}\text{ Base} \cdot \text{Height} \cdot \Delta x = \frac{1}{2}\cdot 2 \cdot 3\Delta x = 3\Delta x \text{ cm}^3.$$

Thus,

$$\text{Total volume} = \lim_{\Delta x \to 0} \sum 3\Delta x = \int_0^4 3\,dx = 3x\Big|_0^4 = 12 \text{ cm}^3.$$

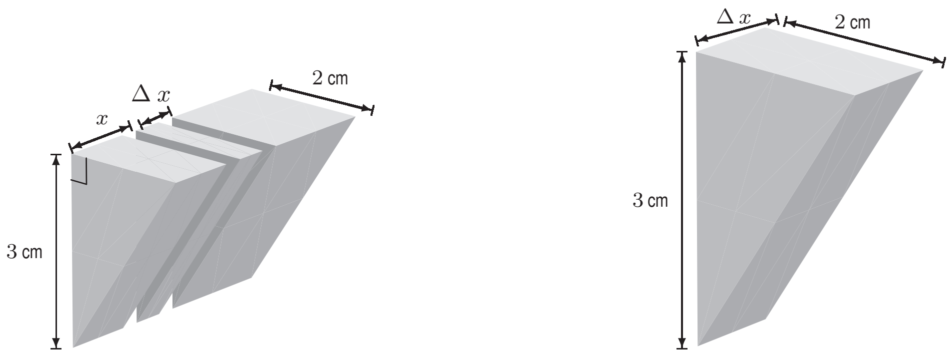


Figure 8.22

(b) A horizontal slice has a rectangular shape and thickness Δh. See Figure 8.23. Using similar triangles, we see that

$$\frac{w}{2} = \frac{3-h}{3},$$

so

$$w = \frac{2}{3}(3-h) = 2 - \frac{2}{3}h.$$

Thus

$$\text{Volume of slice } \approx 4w\Delta h = 4\left(2 - \frac{2}{3}h\right)\Delta h = \left(8 - \frac{8}{3}h\right)\Delta h.$$

So,

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum \left(8 - \frac{8}{3}h\right)\Delta h = \int_0^3 \left(8 - \frac{8}{3}h\right) dh = \left(8h - \frac{4h^2}{3}\right)\Bigg|_0^3 = 12 \text{ cm}^3.$$

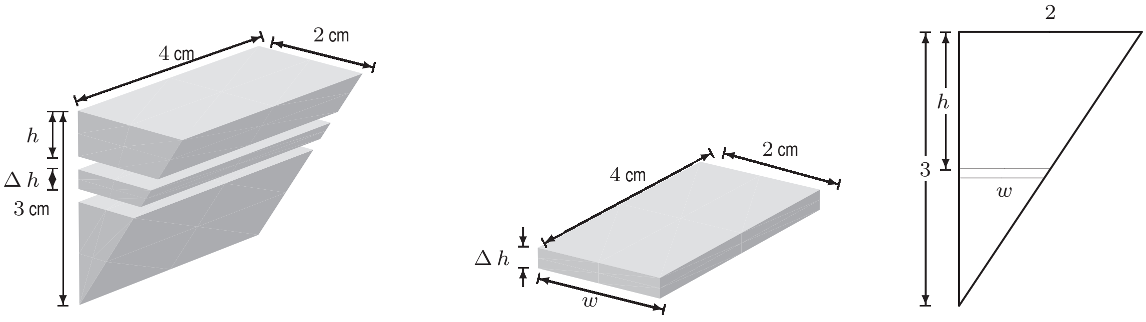


Figure 8.23

36. We slice the water into horizontal slices, each of which is a rectangle. See Figure 8.24.

$$\text{Volume of slice } \approx 150w\Delta h \text{ km}^3.$$

To find w in terms of h, we use the similar triangles in Figure 8.25:

$$\frac{w}{3} = \frac{h}{0.2} \quad \text{so} \quad w = 15h.$$

So

$$\text{Volume of slice } \approx 150 \cdot 15h\Delta h = 2250h\Delta h \text{ km}^3.$$

Summing over all slices and letting $\Delta h \to 0$ gives

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum 2250h\Delta h = \int_0^{0.2} 2250h \, dh \text{ km}^3.$$

Evaluating the integral gives

$$\text{Total volume } = 2250 \frac{h^2}{2}\Bigg|_0^{0.2} = 45 \text{ km}^3.$$

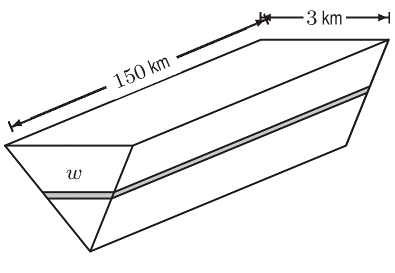


Figure 8.24

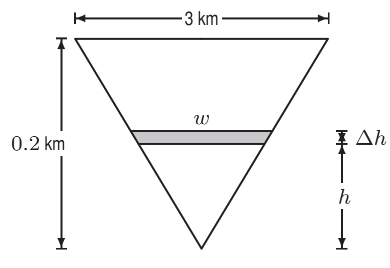


Figure 8.25

37. To calculate the volume of material, we slice the dam horizontally. See Figure 8.26. The slices are rectangular, so

$$\text{Volume of slice } \approx 1400w\Delta h \text{ m}^3.$$

Since w is a linear function of h, and $w = 160$ when $h = 0$, and $w = 10$ when $h = 150$, this function has slope $= (10 - 160)/150 = -1$. Thus

$$w = 160 - h \text{ meters,}$$

so

$$\text{Volume of slice } \approx 1400(160 - h)\Delta h \text{ m}^3.$$

Summing over all slices and taking the limit as $\Delta h \to 0$ gives

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum 1400(160 - h)\Delta h = \int_0^{150} 1400(160 - h)\, dh \text{ m}^3.$$

Evaluating the integral gives

$$\text{Total volume } = 1400 \left(160h - \frac{h^2}{2}\right)\Bigg|_0^{150} = 1.785 \cdot 10^7 \text{ m}^3.$$

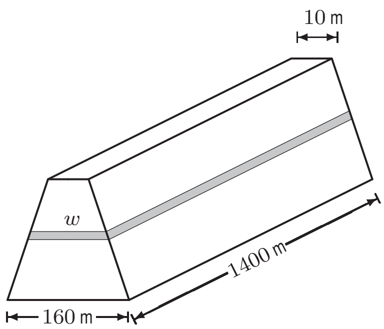


Figure 8.26

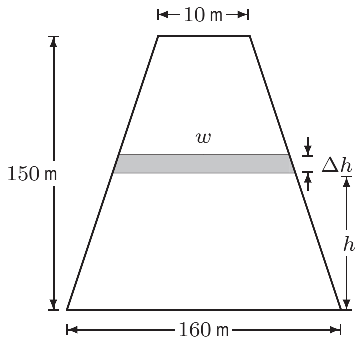


Figure 8.27

Strengthen Your Understanding

38. The horizontal slice at height y goes from $x = 0$ to a point on the line $y = 2x$. At this point, $x = y/2$. The correct integral for the area is $\int_0^8 y/2\, dy$.

39. Slice the sphere $x^2+y^2 = 10^2$ into slices of thickness Δx perpendicular to the x-axis. The typical slice is approximately a cylinder of radius $y = \sqrt{10^2 - x^2}$ and height $h = \Delta x$. The volume of the slice is approximately $\pi y^2 h = \pi(10^2 - x^2)\Delta x$. The volume of the sphere is thus $\int_{-10}^{10} \pi \left(10^2 - x^2\right)\, dx$.

40. It would be hard to use vertical slices to find the area of the shaded region in Figure 8.28 because the vertical slices on the right portion of the figure are in two pieces.

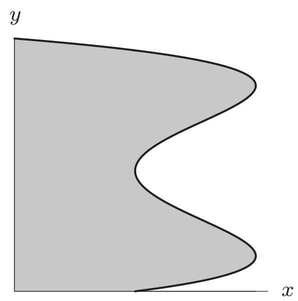


Figure 8.28

41. One possible answer is the region between the positive x-axis, the positive y-axis and the line $y = 1 - x$. For horizontal slices, the width of a slice at height y is $x = 1 - y$, and for vertical slices the height of a slice at position x is $y = 1 - x$.

42. True. Since $y = \pm\sqrt{9 - x^2}$ represent the top and bottom halves of the sphere, slicing disks perpendicular to the x-axis gives

$$\text{Volume of slice } \approx \pi y^2 \Delta x = \pi(9 - x^2)\Delta x$$
$$\text{Volume} = \int_{-3}^{3} \pi(9 - x^2)\, dx.$$

43. False. Evaluating does not give the volume of a cone $\pi r^2 h/3$:

$$\int_0^h \pi(r - y)\, dy = \pi\left(ry - \frac{y^2}{2}\right)\Bigg|_0^h = \pi\left(rh - \frac{h^2}{2}\right).$$

Alternatively, you can show by slicing that the integral representing this volume is $\int_0^h \pi r^2 (1 - y/h)^2\, dy$.

44. False. Using the table of integrals (VI-28 and VI-30) or a trigonometric substitution gives

$$\int_0^r \pi\sqrt{r^2 - y^2}\, dy = \frac{\pi}{2}\left(y\sqrt{r^2 - y^2} + r^2 \arcsin\left(\frac{y}{r}\right)\right)\Bigg|_0^r = \frac{\pi r^2}{2}(\arcsin 1 - \arcsin 0) = \frac{\pi^2 r^2}{4}.$$

The volume of a hemisphere is $2\pi r^3/3$.

Alternatively, you can show by slicing that the integral representing this volume is $\int_0^r \pi(r^2 - y^2)\, dy$.

45. True. Horizontal slicing gives rectangular slabs of length l, thickness Δy, and width $w = 2\sqrt{r^2 - y^2}$. So the volume of one slab is $2l\sqrt{r^2 - y^2}\Delta y$, and the integral is $\int_{-r}^{r} 2l\sqrt{r^2 - y^2}\, dy$.

Solutions for Section 8.2

Exercises

1. (a) The volume of a disk is given by

$$\Delta V \approx \pi(2x)^2 \Delta x = 4\pi x^2\, \Delta x,$$

so

$$\text{Volume } = \int_0^3 4\pi x^2\, dx.$$

(b) We have

$$\text{Volume } = 4\pi \frac{x^3}{3}\Bigg|_0^3 = 36\pi.$$

2. (a) The volume of a disk is given by

$$\Delta V \approx \pi(-x^2 + 6x)^2 \Delta x,$$

so

$$\text{Volume } = \int_0^6 \pi(-x^2 + 6x)^2\, dx.$$

(b) We have

$$\text{Volume } = \int_0^6 \pi(x^4 - 12x^3 + 36x^2)\, dx = \pi\left(\frac{x^5}{5} - 3x^4 + 12x^3\right)\Bigg|_0^6 = \frac{1296\pi}{5}.$$

3. (a) The strip stretches from $x = y/2$ to $x = 3$. The volume of a disk with a hole in it is

$$\Delta V \approx \pi\left(3^2 - \left(\frac{y}{2}\right)^2\right)\Delta y = \frac{\pi}{4}(36 - y^2)\,\Delta y,$$

so

$$\text{Volume } = \int_0^6 \frac{\pi}{4}(36 - y^2)\,dy.$$

(b) We have

$$\text{Volume } = \frac{\pi}{4}\left(36y - \frac{y^3}{3}\right)\Bigg|_0^6 = 36\pi.$$

4. (a) The equation $y = -x^2 + 6x$ can be solved for x as

$$\begin{aligned} x^2 - 6x + y &= 0 \\ x &= \frac{6 \pm \sqrt{36 - 4y}}{2} = 3 \pm \sqrt{9 - y}. \end{aligned}$$

The left end of the strip is given by $x = 3 - \sqrt{9-y}$, and the right end is given by $x = 3 + \sqrt{9-y}$. Thus, the volume of a disk with a hole is

$$\begin{aligned} \Delta V &\approx \pi((3 + \sqrt{9-y})^2 - (3 - \sqrt{9-y})^2)\,\Delta y \\ &= \pi((9 + 6\sqrt{9-y} + 9 - y) - (9 - 6\sqrt{9-y} + 9 - y))\,\Delta y \\ &= 12\pi\sqrt{9-y}\,\Delta y, \end{aligned}$$

so

$$\text{Volume } = 12\pi\int_0^9 \sqrt{9-y}\,dy.$$

(b) We have

$$\text{Volume } = \frac{12\pi(9-y)^{3/2}}{-3/2}\Bigg|_0^9 = 216\pi.$$

5. The volume is given by

$$V = \int_0^1 \pi y^2 dx = \int_0^1 \pi x^4 dx = \pi\frac{x^5}{5}\Bigg|_0^1 = \frac{\pi}{5}.$$

6. The volume is given by

$$V = \int_1^2 \pi y^2\,dx = \int_1^2 \pi(x+1)^4\,dx = \frac{\pi(x+1)^5}{5}\Bigg|_1^2 = \frac{211\pi}{5}.$$

7. The volume is given by

$$V = \int_{-2}^0 \pi(4 - x^2)^2\,dx = \pi\int_{-2}^0 (16 - 8x^2 + x^4)\,dx = \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Bigg|_{-2}^0 = \frac{256\pi}{15}.$$

8. The volume is given by

$$V = \int_{-1}^1 \pi(\sqrt{x+1})^2\,dx = \pi\int_{-1}^1 (x+1)\,dx = \pi\left(\frac{x^2}{2} + x\right)\Bigg|_{-1}^1 = 2\pi.$$

9. The volume is given by

$$V = \int_{-1}^1 \pi y^2\,dx = \int_{-1}^1 \pi(e^x)^2\,dx = \int_{-1}^1 \pi e^{2x}\,dx = \frac{\pi}{2}e^{2x}\Bigg|_{-1}^1 = \frac{\pi}{2}(e^2 - e^{-2}).$$

10. The volume is given by

$$V = \int_0^{\pi/2} \pi y^2\, dx = \int_0^{\pi/2} \pi \cos^2 x\, dx.$$

Integration by parts gives

$$V = \frac{\pi}{2}(\cos x \sin x + x)\bigg|_0^{\pi/2} = \frac{\pi^2}{4}.$$

11. The volume is given by

$$V = \int_0^1 \pi \left(\frac{1}{x+1}\right)^2 dx = \pi \int_0^1 \frac{dx}{(x+1)^2} = -\pi(x+1)^{-1}\bigg|_0^1 = \pi\left(1 - \frac{1}{2}\right) = \frac{\pi}{2}.$$

12. The volume is given by

$$V = \pi \int_0^1 (\sqrt{\cosh 2x})^2\, dx = \pi \int_0^1 \cosh 2x\, dx = \frac{\pi}{2}\sinh 2x\bigg|_0^1 = \frac{\pi}{2}\sinh 2.$$

13. Since the graph of $y = x^2$ is below the graph of $y = x$ for $0 \le x \le 1$, the volume is given by

$$V = \int_0^1 \pi x^2\, dx - \int_0^1 \pi (x^2)^2\, dx = \pi \int_0^1 (x^2 - x^4)\, dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\bigg|_0^1 = \frac{2\pi}{15}.$$

14. Since the graph of $y = e^{3x}$ is above the graph of $y = e^x$ for $0 \le x \le 1$, the volume is given by

$$V = \int_0^1 \pi (e^{3x})^2\, dx - \int_0^1 \pi (e^x)^2\, dx = \int_0^1 \pi(e^{6x} - e^{2x})\, dx = \pi\left(\frac{e^{6x}}{6} - \frac{e^{2x}}{2}\right)\bigg|_0^1 = \pi\left(\frac{e^6}{6} - \frac{e^2}{2} + \frac{1}{3}\right).$$

15. Since $f'(x) = x$, we evaluate the integral numerically or using the table to get

$$\text{Arc length } = \int_0^2 \sqrt{1 + x^2}\, dx = \frac{\ln(\sqrt{5} + 2)}{2} + \sqrt{5} = 2.958.$$

16. Since $f'(x) = -\sin x$, we evaluate the integral numerically to get

$$\text{Arc length } = \int_0^2 \sqrt{1 + \sin^2 x}\, dx = 2.508.$$

17. Since $f'(x) = 1/(x + 1)$, we evaluate the integral numerically to get

$$\text{Arc length } = \int_0^2 \sqrt{1 + \left(\frac{1}{x+1}\right)^2}\, dx = 2.302.$$

18. Note that this function is actually $x^{3/2}$ in disguise. So

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left[\frac{3}{2}x^{\frac{1}{2}}\right]^2}\, dx = \int_{x=0}^{x=2} \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \frac{4}{9}\int_{w=1}^{w=\frac{11}{2}} w^{\frac{1}{2}}\, dw \\ &= \frac{8}{27}w^{\frac{3}{2}}\bigg|_1^{\frac{11}{2}} = \frac{8}{27}\left(\left(\frac{11}{2}\right)^{\frac{3}{2}} - 1\right) \approx 3.526, \end{aligned}$$

where we set $w = 1 + \frac{9}{4}x$, so $dx = \frac{4}{9}dw$.

19. This is a one-quarter of the circumference of a circle of radius 2. That circumference is $2 \cdot 2\pi = 4\pi$, so the length is $\frac{4\pi}{4} = \pi$.

20. Since $f'(x) = \sinh x$, the arc length is given by

$$L = \int_0^2 \sqrt{1+\sinh^2 x}\,dx = \int_0^2 \sqrt{\cosh^2 x}\,dx = \int_0^2 \cosh x\,dx = \sinh x\Big|_0^2 = \sinh 2.$$

21. The length is

$$\int_1^2 \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt = \int_1^2 \sqrt{5^2+4^2+(-1)^2}\,dt = \sqrt{42}.$$

This is the length of a straight line from the point $(8,5,2)$ to $(13,9,1)$.

22. We have

$$\begin{aligned} D &= \int_0^1 \sqrt{(-e^t\sin(e^t))^2+(e^t\cos(e^t))^2}\,dt \\ &= \int_0^1 \sqrt{e^{2t}}\,dt = \int_0^1 e^t\,dt \\ &= e-1. \end{aligned}$$

This is the length of the arc of a unit circle from the point $(\cos 1, \sin 1)$ to $(\cos e, \sin e)$—in other words between the angles $\theta = 1$ and $\theta = e$. The length of this arc is $(e-1)$.

23. We have

$$D = \int_0^{2\pi} \sqrt{(-3\sin 3t)^2+(5\cos 5t)^2}\,dt.$$

We cannot find this integral symbolically, but numerical methods show $D \approx 24.6$.

24. Since $dx/dt = -3\cos^2 t\sin t$, $dy/dt = 3\sin^2 t\cos t$, we have

$$\begin{aligned} \text{Arc length } &= \int_0^{2\pi} \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}\,dt = 3\int_0^{2\pi} |\cos t\sin t|\sqrt{cos^2 t+\sin^2 t}\,dt \\ &= 3\int_0^{2\pi} |\cos t\sin t|\,dt. \end{aligned}$$

We calculate the integral over the interval $0 \le t \le \pi/2$, where both $\cos t$ and $\sin t$ are positive, and multiply by 4:

$$\text{Arc length } = 12\int_0^{\pi/2} \cos t\sin t\,dt = 6\sin^2 t\Big|_0^{\pi/2} = 6.$$

Problems

25. The two functions intersect at $(0,0)$ and $(8,2)$. We slice the volume with planes perpendicular to the line $x = 9$. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi r_{out}^2\Delta y - \pi r_{in}^2\Delta y.$$

The outer radius is the horizontal distance from the line $x = 9$ to the curve $x = y^3$, so $r_{out} = 9 - y^3$. Similarly, the inner radius is the horizontal distance from the line $x = 9$ to the curve $x = 4y$, so $r_{in} = 9 - 4y$. Integrating from $y = 0$ to $y = 2$ we have

$$V = \int_0^2 [\pi(9-y^3)^2 - \pi(9-4y)^2]\,dy.$$

26. The two functions intersect at $(0,0)$ and $(8,2)$. We slice the volume with planes perpendicular to the line $y = 3$. This divides the solid into thin washers with

$$\text{Volume of slice } = \pi r_{out}^2\Delta x - \pi r_{in}^2\Delta x.$$

The outer radius is the vertical distance from the line $y = 3$ to the curve $y = \frac{1}{4}x$, so $r_{out} = 3 - \frac{1}{4}x$. Similarly, the inner radius is the vertical distance from the line $y = 3$ to the curve $y = \sqrt[3]{x}$, so $r_{in} = 3 - \sqrt[3]{x}$. Integrating from $x = 0$ to $x = 8$ we have

$$V = \int_0^8 \left[\pi(3-\frac{1}{4}x)^2 - \pi(3-\sqrt[3]{x})^2\right]\,dx.$$

27. Note that the lines $x = 9$ and $y = \frac{1}{3}x$ intersect at $(9, 3)$. We slice the volume with planes that are perpendicular to the line $y = -2$. This divides the solid into thin washers with

$$\text{Volume of slice } = \pi r_{out}^2 \Delta x - \pi r_{in}^2 \Delta x.$$

Note that the inner radius is the vertical distance from the line $y = -2$ to the x-axis, so $r_{in} = 2$. Similarly, the outer radius is the vertical distance from the line $y = -2$ to the line $y = \frac{1}{3}x$, so $r_{out} = 2 + \frac{1}{3}x$. Integrating from $x = 0$ to $x = 9$ we have

$$V = \int_0^9 \left[\pi(2 + \frac{1}{3}x)^2 - \pi 2^2\right] dx.$$

28. Note that the lines $x = 9$ and $y = \frac{1}{3}x$ intersect at $(9, 3)$. We slice the volume with planes that are perpendicular to the line $x = -1$. This divides the solid into thin washers with

$$\text{Volume of slice } = \pi r_{out}^2 \Delta y - \pi r_{in}^2 \Delta y.$$

Note that the inner radius is the horizontal distance from the line $x = -1$ to the line $x = 3y$, so $r_{in} = 1 + 3y$. Similarly, the outer radius is the horizontal distance from the line $x = -1$ to the line $x = 9$, so $r_{out} = 1 + 9$. Integrating from $y = 0$ to $y = 3$ we have

$$V = \int_0^3 [\pi(1 + 9)^2 - \pi(1 + 3y)^2]\, dy.$$

29. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the x-axis. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi((r_{out})^2 - (r_{in})^2)\Delta x.$$

The outer radius is the vertical distance from the x-axis to the curve $y = 5x$, so $r_{out} = 5x$. Similarly, the inner radius is the vertical distance from the x-axis to the curve $y = x^2$, so $r_{in} = x^2$. Integrating from $x = 0$ to $x = 5$ we have

$$V = \int_0^5 \pi((5x)^2 - (x^2)^2)\, dx.$$

30. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the y-axis. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi((r_{out})^2 - (r_{in})^2)\Delta y.$$

The outer radius is the horizontal distance from the y-axis to the curve $x = \sqrt{y}$, so $r_{out} = \sqrt{y}$. Similarly, the inner radius is the horizontal distance from the y-axis to the curve $x = y/5$, so $r_{in} = y/5$. Integrating from $y = 0$ to $y = 25$ we have

$$V = \int_0^{25} \pi((\sqrt{y})^2 - (y/5)^2)\, dy.$$

31. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the horizontal line $y = -4$. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi((r_{out})^2 - (r_{in})^2)\Delta x.$$

The outer radius is the vertical distance from the line $y = -4$ to the curve $y = 5x$, so $r_{out} = 4 + 5x$. Similarly, the inner radius is the vertical distance from the line $y = -4$ to the curve $y = x^2$, so $r_{in} = 4 + x^2$. Integrating from $x = 0$ to $x = 5$ we have

$$V = \int_0^5 \pi((4 + 5x)^2 - (4 + x^2)^2)\, dx.$$

32. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the vertical line $x = -3$. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi((r_{out})^2 - (r_{in})^2)\Delta y.$$

The outer radius is the horizontal distance from the line $x = -3$ to the curve $x = \sqrt{y}$, so $r_{out} = 3 + \sqrt{y}$. Similarly, the inner radius is the horizontal distance from the line $x = -3$ to the curve $x = y/5$, so $r_{in} = 3 + y/5$. Integrating from $y = 0$ to $y = 25$ we have

$$V = \int_0^{25} \pi((3 + \sqrt{y})^2 - (3 + y/5)^2)\, dy.$$

33. One arch of the sine curve lies between $x = 0$ and $x = \pi$. Since $d(\sin x)/dx = \cos x$, evaluating the integral numerically gives

$$\text{Arc length } = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx = 3.820.$$

34. The curves cross at $(1, 1)$; see Figure 8.29. The straight side has length $\sqrt{2} = 1.414$; as this is the hypotenuse of a right triangle with sides $1, 1, \sqrt{2}$. The curved side has

$$\text{Arc length } = \int_0^1 \sqrt{1 + (2x)^2}\, dx = \frac{\ln(\sqrt{5} + 2)}{4} + \frac{\sqrt{5}}{2} = 1.479.$$

Thus the perimeter is $1.4142 + 1.479 = 2.893$.

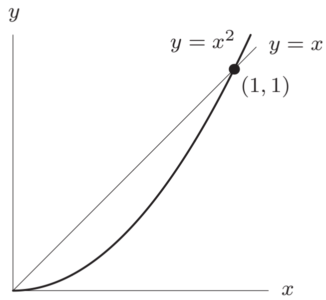


Figure 8.29

35. (a) Slicing the region perpendicular to the x-axis gives disks of radius y. See Figure 8.30.

$$\text{Volume of slice} \approx \pi y^2 \Delta x = \pi(x^2 - 1)\Delta x.$$

Thus,

$$\text{Total volume} = \lim_{\Delta x \to 0} \sum \pi(x^2 - 1)\Delta x = \int_2^3 \pi(x^2 - 1)\, dx = \pi \left(\frac{x^3}{3} - x \right) \Bigg|_2^3$$
$$= \pi \left(9 - 3 - \left(\frac{8}{3} - 2 \right) \right) = \frac{16\pi}{3}.$$

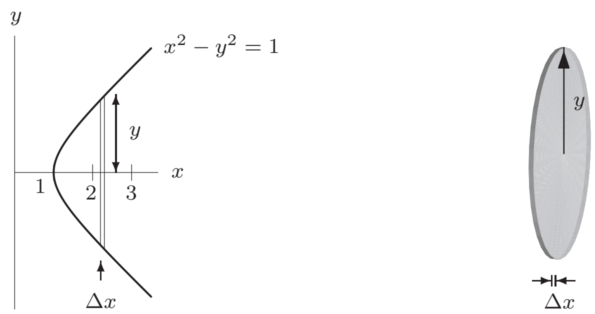


Figure 8.30

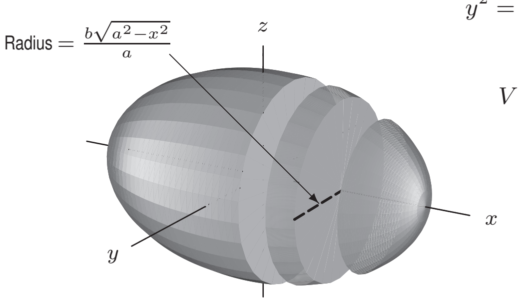
(b) The arc length, L, of the curve $y = f(x)$ is given by $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$. In this problem y is an implicit function of x. Solving for y gives $y = \sqrt{x^2 - 1}$ as the equation of the top half of the hyperbola. Differentiating gives

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}.$$

Thus

$$\text{Arc length} = \int_2^3 \sqrt{1 + \left(\frac{x}{\sqrt{x^2 - 1}} \right)^2}\, dx = \int_2^3 \sqrt{1 + \frac{x^2}{x^2 - 1}}\, dx = \int_2^3 \sqrt{\frac{2x^2 - 1}{x^2 - 1}}\, dx = 1.48.$$

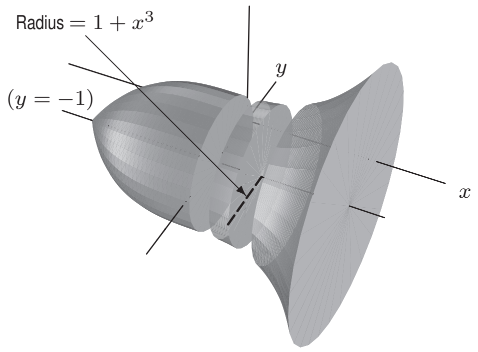
36.



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right).$$

$$\begin{aligned} V &= \int_{-a}^{a} \pi y^2 dx = \pi \int_{-a}^{a} b^2\left(1 - \frac{x^2}{a^2}\right) dx \\ &= 2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a \\ &= 2\pi b^2\left(a - \frac{a^3}{3a^2}\right) = 2\pi b^2 \left(a - \frac{1}{3}a\right) \\ &= \frac{4}{3}\pi a b^2. \end{aligned}$$

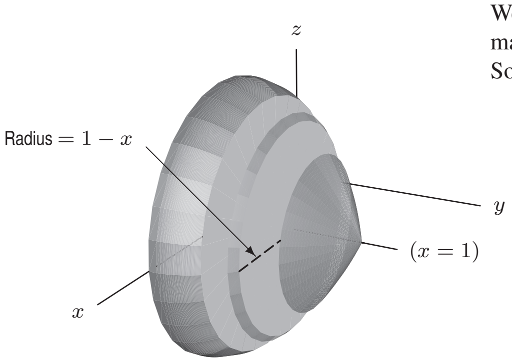
37.



We slice the region perpendicular to the x–axis. The Riemann sum we get is $\sum \pi(x^3+1)^2\Delta x$. So the volume V is the integral

$$\begin{aligned} V &= \int_{-1}^{1} \pi(x^3+1)^2\, dx \\ &= \pi \int_{-1}^{1} (x^6 + 2x^3 + 1)\, dx \\ &= \pi\left(\frac{x^7}{7} + \frac{x^4}{2} + x\right)\Bigg|_{-1}^{1} \\ &= (16/7)\pi \approx 7.18. \end{aligned}$$

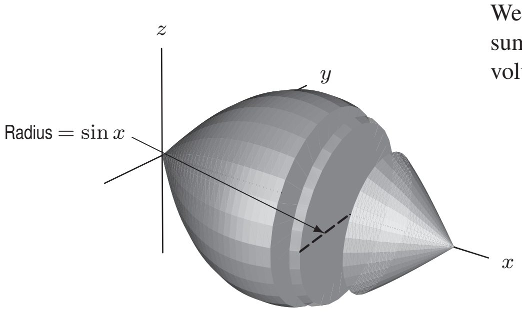
38.



We slice the region perpendicular to the y–axis. The Riemann sum we get is $\sum \pi(1-x)^2\Delta y = \sum \pi(1-y^2)^2 \Delta y$. So the volume V is the integral

$$\begin{aligned} V &= \int_0^1 \pi(1-y^2)^2\, dy \\ &= \pi\int_0^1 (1 - 2y^2 + y^4)\, dy \\ &= \pi\left(y - \frac{2y^3}{3} + \frac{y^5}{5}\right)\Bigg|_0^1 \\ &= (8/15)\pi \approx 1.68. \end{aligned}$$

39.



We take slices perpendicular to the x–axis. The Riemann sum for approximating the volume is $\sum \pi \sin^2 x \Delta x$. The volume is the integral corresponding to that sum, namely

$$\begin{aligned} V &= \int_0^\pi \pi \sin^2 x\, dx \\ &= \pi\left[-\frac{1}{2}\sin x \cos x + \frac{1}{2}x\right]\Bigg|_0^\pi = \frac{\pi^2}{2} \approx 4.935. \end{aligned}$$

40. Slice the object into disks horizontally, as in Figure 8.31. A typical disk has thickness Δy and radius $x = \sqrt{y}$. Thus

$$\text{Volume of slice } \approx \pi x^2 \Delta y = \pi y\, \Delta y.$$

$$\text{Volume of solid } = \lim_{\Delta y \to 0} \sum \pi y\, \Delta y = \int_0^1 \pi y\, dy = \pi \frac{y^2}{2}\Bigg|_0^1 = \frac{\pi}{2}.$$

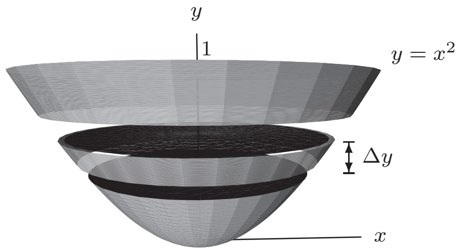


Figure 8.31

41. Slice the object into rings vertically, as is Figure 8.32. A typical ring has thickness Δx and outer radius $y = 1$ and inner radius $y = x^2$.

$$\text{Volume of slice } \approx \pi 1^2 \Delta x - \pi y^2 \Delta x = \pi(1 - x^4)\,\Delta x.$$

$$\text{Volume of solid } = \lim_{\Delta x \to 0} \sum \pi(1 - x^4)\,\Delta x = \int_0^1 \pi(1 - x^4)\,dx = \pi\left(x - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{4}{5}\pi.$$

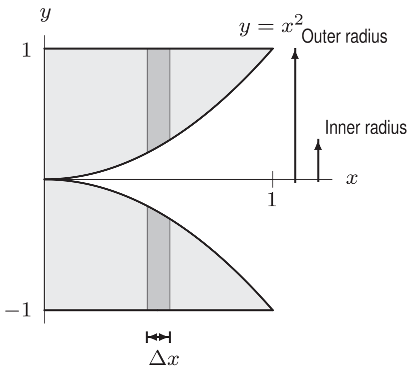


Figure 8.32: Cross-section of solid

42. The region is cylindrical with a hole around the axis of rotation, $y = -2$. Slice it into rings vertically, as in Figure 8.33. A typical ring has thickness Δx and outer radius $1 + 2 = 3$ and inner radius $y + 2 = x^2 + 2$. Thus

$$\text{Volume of slice } \approx \pi 3^2 \Delta x - \pi(x^2 + 2)^2 \Delta x = \pi(5 - x^4 - 4x^2)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 \pi(5 - x^4 - 4x^2)\,\Delta x = \pi\left(5x - \frac{x^5}{5} - \frac{4}{3}x^3\right)\Bigg|_0^1 = \frac{52\pi}{15}.$$

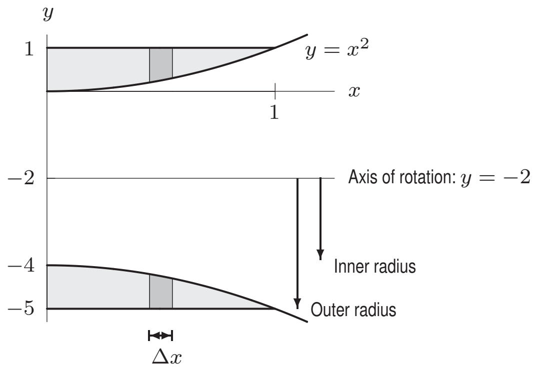


Figure 8.33: Cross-section of solid

43. Slicing perpendicularly to the x-axis gives squares whose thickness is Δx and whose side is $1 - y = 1 - x^2$. See Figure 8.34. Thus

$$\text{Volume of square slice } \approx (1-x^2)^2 \Delta x = (1 - 2x^2 + x^4)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 (1 - 2x^2 + x^4)\,dx = x - \frac{2}{3}x^3 + \frac{x^5}{5}\Bigg|_0^1 = \frac{8}{15}.$$

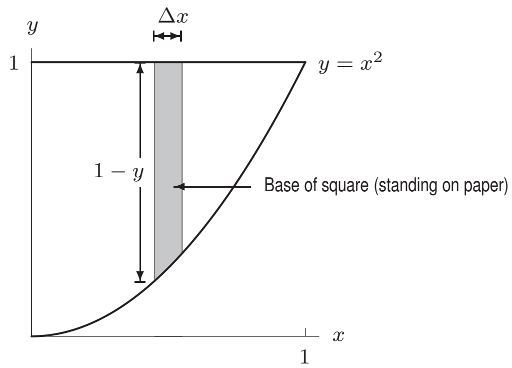


Figure 8.34: Base of solid

44. Slicing perpendicularly to the x-axis gives semicircles whose thickness is Δx and whose diameter is $1 - y = 1 - x^2$. See Figure 8.35. Thus

$$\text{Volume of semicircular slice } \approx \pi \left(\frac{1-x^2}{2}\right)^2 \Delta x = \frac{\pi}{4}(1 - 2x^2 + x^4)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 \frac{\pi}{4}(1 - 2x^2 + x^4)\,dx = \frac{\pi}{4}\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{\pi}{4} \cdot \frac{8}{15} = \frac{2\pi}{15}.$$

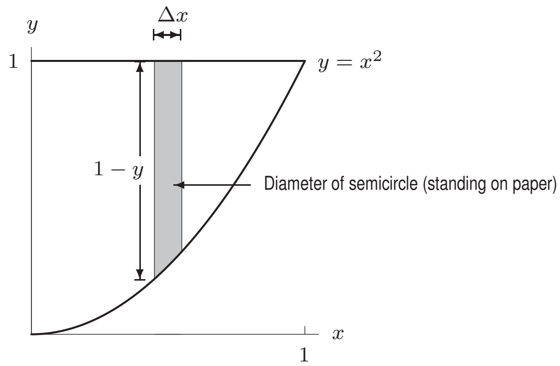


Figure 8.35: Base of solid

45. An equilateral triangle of side s has height $\sqrt{3}s/2$ and

$$\text{Area } = \frac{1}{2} \cdot s \cdot \frac{\sqrt{3}s}{2} = \frac{\sqrt{3}}{4}s^2.$$

Slicing perpendicularly to the y-axis gives equilateral triangles whose thickness is Δy and whose side is $x = \sqrt{y}$. See Figure 8.36. Thus

$$\text{Volume of triangular slice } \approx \frac{\sqrt{3}}{4}(\sqrt{y})^2 \Delta y = \frac{\sqrt{3}}{4} y\,\Delta y.$$

$$\text{Volume of solid } = \int_0^1 \frac{\sqrt{3}}{4} y\,dy = \frac{\sqrt{3}}{4}\frac{y^2}{2}\bigg|_0^1 = \frac{\sqrt{3}}{8}.$$

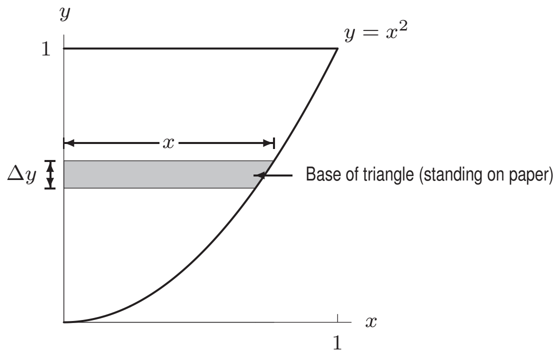
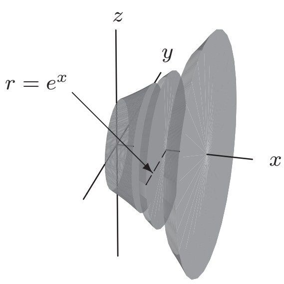


Figure 8.36: Base of solid

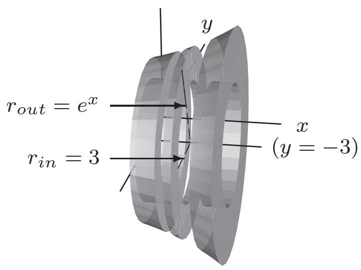
46.



This is the volume of revolution gotten from the rotating the curve $y = e^x$. Take slices perpendicular to the x-axis. They will be circles with radius e^x, so

$$V = \int_{x=0}^{x=1} \pi y^2\,dx = \pi \int_0^1 e^{2x}\,dx$$
$$= \frac{\pi e^{2x}}{2}\bigg|_0^1 = \frac{\pi(e^2-1)}{2} \approx 10.036.$$

47.



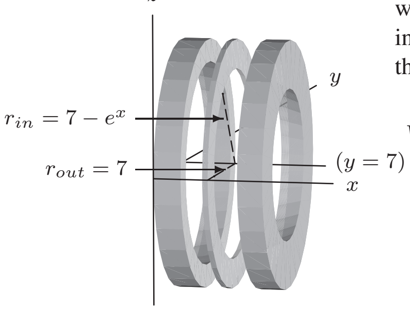
We slice the volume with planes perpendicular to the line $y = -3$. This divides the curve into thin washers, as in Example 3 on page 424 of the text, whose volumes are

$$\pi r_{\text{out}}^2 dx - \pi r_{\text{in}}^2 dx = \pi(3+y)^2 dx - \pi 3^2 dx.$$

So the integral we get from adding all these washers up is

$$V = \int_{x=0}^{x=1} [\pi(3+y)^2 - \pi 3^2]\,dx$$
$$= \pi \int_0^1 [(3+e^x)^2 - 9]\,dx$$
$$= \pi \int_0^1 [e^{2x} + 6e^x]\,dx = \pi[\frac{e^{2x}}{2} + 6e^x]\bigg|_0^1$$
$$= \pi[(e^2/2 + 6e) - (1/2 + 6)] \approx 42.42.$$

48.



This problem can be done by slicing the volume into washers with planes perpendicular to the axis of rotation, $y = 7$, just like in Example 3. This time the outside radius of a washer is 7, and the inside radius is $7 - e^x$. Therefore, the volume V is

$$V = \int_{x=0}^{x=1} [\pi 7^2 - \pi(7-e^x)^2]\,dx = \pi \int_0^1 (14e^x - e^{2x})\,dx$$
$$= \pi\left[14e^x - \frac{1}{2}e^{2x}\right]\bigg|_0^1 = \pi\left[14e - \frac{1}{2}e^2 - \left(14 - \frac{1}{2}\right)\right]$$
$$\approx 65.54.$$

49. We now slice perpendicular to the x-axis. As stated in the problem, the cross-sections obtained thereby will be squares, with base length e^x. The volume of one square slice is $(e^x)^2\,dx$. (See Figure 8.37.) Adding the volumes of the slices yields

$$\text{Volume } = \int_{x=0}^{x=1} y^2\,dx = \int_0^1 e^{2x}\,dx = \left.\frac{e^{2x}}{2}\right|_0^1 = \frac{e^2-1}{2} = 3.195.$$

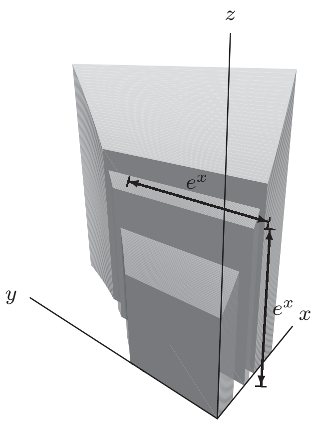
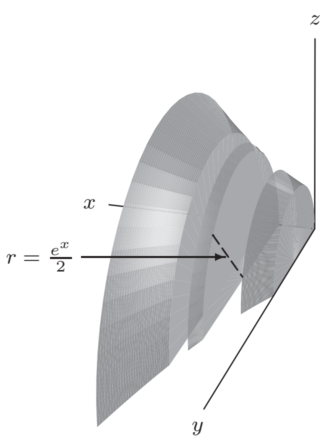


Figure 8.37

50.



We slice perpendicular to the x-axis. As stated in the problem, the cross-sections obtained thereby will be semicircles, with radius $\frac{e^x}{2}$. The volume of one semicircular slice is $\frac{1}{2}\pi\left(\frac{e^x}{2}\right)^2\,dx$. (Look at the picture.) Adding up the volumes of the slices yields

$$\begin{aligned}\text{Volume } &= \int_{x=0}^{x=1} \pi\frac{y^2}{2}\,dx = \frac{\pi}{8}\int_0^1 e^{2x}\,dx \\ &= \left.\frac{\pi e^{2x}}{16}\right|_0^1 = \frac{\pi(e^2-1)}{16} \approx 1.25.\end{aligned}$$

51. If we revolve the region $0 \le y \le g(x)$ around the x-axis, the volume of the resulting solid is given by $\displaystyle\int_0^\pi \pi\,(g(x))^2\,dx$.

Taking the hint, we can write our integral in this form:

$$\int_0^\pi \pi(4-4\cos^2 x)\,dx = \int_0^\pi 4\pi\left(1-\cos^2 x\right)\,dx = \int_0^\pi 4\pi\sin^2 x\,dx = \int_0^\pi \pi\underbrace{(2\sin x)^2}_{(g(x))^2}\,dx.$$

Therefore, $g(x) = 2\sin x$ or $g(x) = -2\sin x$.

52. At time $t = 6$, the particle has traveled $3 \cdot 6 = 18$ cm. Suppose it is then at the point $x = b$. Then the arc length from the origin to this point is 18 cm. Since

$$\frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = x^{1/2},$$

we have

$$\text{Arc length } = \int_0^b \sqrt{1+(x^{1/2})^2}\,dx = \int_0^b \sqrt{1+x}\,dx = 18$$

$$\begin{aligned}
\frac{2}{3}(1+x)^{3/2}\Big|_0^b &= 18 \\
\frac{2}{3}(1+b)^{3/2} - \frac{2}{3} &= 18 \\
(1+b)^{3/2} &= 28 \\
b &= 28^{2/3} - 1 = 8.221.
\end{aligned}$$

When $x = 8.221$, $y = 15.714$.

53. We want to approximate $\int_0^{120} A(h)\,dh$, where h is height, and $A(h)$ represents the cross-sectional area of the trunk at height h. Since $A = \pi r^2$ (circular cross-sections), and $c = 2\pi r$, where c is the circumference, we have $A = \pi r^2 = \pi[c/(2\pi)]^2 = c^2/(4\pi)$. We make a table of $A(h)$ based on this:

Table 8.1

height (feet)	0	20	40	60	80	100	120
Area (square feet)	53.79	38.52	28.73	15.60	2.865	0.716	0.080

We now form left & right sums using the chart:

$$\begin{aligned}
\text{LEFT}(6) &= 53.79\cdot 20 + 38.52\cdot 20 + 28.73\cdot 20 + 15.60\cdot 20 + 2.865\cdot 20 + 0.716\cdot 20 \\
&= 2804.42. \\
\text{RIGHT}(6) &= 38.52\cdot 20 + 28.73\cdot 20 + 15.60\cdot 20 + 2.865\cdot 20 + 0.716\cdot 20 + 0.080\cdot 20 \\
&= 1730.22
\end{aligned}$$

So

$$\text{TRAP}(6) = \frac{\text{RIGHT}(6) + \text{LEFT}(6)}{2} = \frac{2804.42 + 1730.22}{2} = 2267.32 \text{ cubic feet.}$$

54. If $y = e^{-x^2/2}$, then $x = \sqrt{-2\ln y}$. (Note that since $0 < y \le 1$, $\ln y \le 0$.) A typical slice has thickness Δy and radius x. See Figure 8.38. So

$$\text{Volume of slice } = \pi x^2\,\Delta y = -2\pi \ln y\,\Delta y.$$

Thus,

$$\text{Total volume } = -2\pi \int_0^1 \ln y\,dy.$$

Since $\ln y$ is not defined at $y = 0$, this is an improper integral:

$$\begin{aligned}
\text{Total Volume } &= -2\pi \int_0^1 \ln y\,dy = -2\pi \lim_{a\to 0} \int_a^1 \ln y\,dy \\
&= -2\pi \lim_{a\to 0} (y\ln y - y)\Big|_a^1 = -2\pi \lim_{a\to 0}(-1 - a\ln a + a).
\end{aligned}$$

By looking at the graph of $x \ln x$ on a calculator, we see that $\lim_{a\to 0} a\ln a = 0$. Thus,

$$\text{Total volume } = -2\pi(-1) = 2\pi.$$

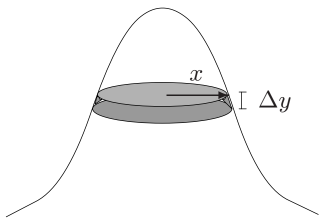


Figure 8.38

55. (a) We can begin by slicing the pie into horizontal slabs of thickness Δh located at height h. To find the radius of each slice, we note that radius increases linearly with height. Since $r = 4.5$ when $h = 3$ and $r = 3.5$ when $h = 0$, we should have $r = 3.5 + h/3$. Then the volume of each slab will be $\pi r^2\ \Delta h = \pi(3.5 + h/3)^2\ \Delta h$. To find the total volume of the pie, we integrate this from $h = 0$ to $h = 3$:

$$\begin{aligned} V &= \pi \int_0^3 \left(3.5 + \frac{h}{3}\right)^2\ dh \\ &= \pi \left[\frac{h^3}{27} + \frac{7h^2}{6} + \frac{49h}{4}\right]\Bigg|_0^3 \\ &= \pi \left[\frac{3^3}{27} + \frac{7(3^2)}{6} + \frac{49(3)}{4}\right] \approx 152 \text{ in}^3. \end{aligned}$$

(b) We use 1.5 in as a rough estimate of the radius of an apple. This gives us a volume of $(4/3)\pi(1.5)^3 \approx 10$ in^3. Since $152/10 \approx 15$, we would need about 15 apples to make a pie.

56. (a) The volume can be computed by several methods, not all of them requiring integration. We will slice horizontally, forming rectangular slabs of length 100 cm, height Δy, width w and integrate. See Figure 8.39.

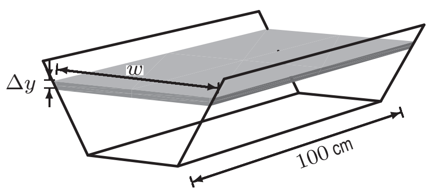


Figure 8.39

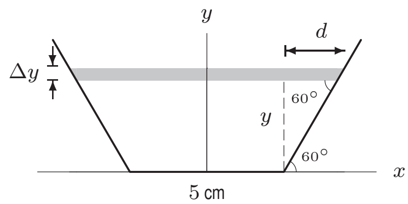


Figure 8.40

From the right triangle, we see

$$\frac{y}{d} = \tan 60^\circ = \sqrt{3}$$

so

$$d = \frac{y}{\sqrt{3}}.$$

Thus

$$w = 5 + 2d = 5 + \frac{2y}{\sqrt{3}}.$$

The volume of the slab is

$$\Delta V \approx 100 w \Delta y = 100\left(5 + \frac{2y}{\sqrt{3}}\right)\Delta y,$$

so the total volume is given by

$$\begin{aligned} \text{Volume} &= \lim_{\Delta y \to 0} \sum \Delta V = \lim_{\Delta y \to 0} \sum 100\left(5 + \frac{2y}{\sqrt{3}}\right)\Delta y \\ &= \int_0^h 100\left(5 + \frac{2y}{\sqrt{3}}\right) dy = 100\left(5y + \frac{y^2}{\sqrt{3}}\right)\Bigg|_0^h = 100\left(5h + \frac{h^2}{\sqrt{3}}\right) \text{ cm}^3. \end{aligned}$$

(b) The maximum value of h is $h = 5\sin 60^\circ = 5\sqrt{3}/2$ cm ≈ 4.33 cm.

(c) The maximum volume of water that the gutter can hold is given by substituting $h = 5\sqrt{3}/2$ into the volume:

$$\text{Maximum volume} = 100\left(5 \cdot \frac{5\sqrt{3}}{2} + \left(\frac{5\sqrt{3}}{2}\right)^2 \Big/ \sqrt{3}\right) = \frac{2500}{4}(2\sqrt{3} + \sqrt{3}) = 1875\sqrt{3} \approx 3247.6 \text{ cm}^3.$$

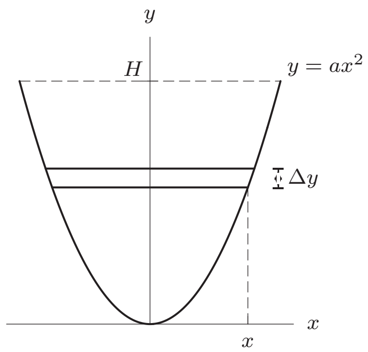
(d) Because the gutter is narrower at the bottom than the top, if it is filled with half the maximum possible volume of water, the gutter will be filled to a depth of more than half of 4.33 cm.

(e) We want to solve for the value of h such that

$$\text{Volume} = 100\left(5h + \frac{h^2}{\sqrt{3}}\right) = \frac{1}{2}\cdot 1875\sqrt{3} = \frac{1}{2}V_{\max}$$
$$5h + \frac{h^2}{\sqrt{3}} = 16.238.$$

Solving gives $h = 2.52$ and $h = -11.18$. Since only positive values of h are meaningful, $h = 2.52$ cm.

57.



We divide the interior of the boat into flat slabs of thickness Δy and width $2x = 2\sqrt{y/a}$. (See above.) We have

$$\text{Volume of slab } \approx 2xL\Delta y = 2L\sqrt{\frac{y}{a}}\Delta y.$$

We are interested in the total volume of the region $0 \le y \le H$, so

$$\begin{aligned}\text{Total volume } &= \lim_{\Delta y\to 0}\sum 2L\left(\frac{y}{a}\right)^{(1/2)}\Delta y = \int_0^H 2L\left(\frac{y}{a}\right)^{(1/2)} dy\\ &= \frac{2L}{\sqrt{a}}\int_0^H y^{(1/2)}dy = \frac{4LH^{(3/2)}}{3\sqrt{a}}.\end{aligned}$$

If L and H are in meters,

$$\text{Buoyancy force } = \frac{40,000LH^{(3/2)}}{3\sqrt{a}} \text{ newtons.}$$

58. We can find the volume of the tree by slicing it into a series of thin horizontal cylinders of height dh and circumference C. The volume of each cylindrical disk will then be

$$V = \pi r^2\, dh = \pi\left(\frac{C}{2\pi}\right)^2 dh = \frac{C^2\, dh}{4\pi}.$$

Summing all such cylinders, we have the total volume of the tree as

$$\text{Total volume} = \frac{1}{4\pi}\int_0^{120} C^2\, dh.$$

We can estimate this volume using a trapezoidal approximation to the integral with $\Delta h = 20$:

$$\begin{aligned}\text{LEFT estimate } &= \frac{1}{4\pi}[20(31^2+28^2+21^2+17^2+12^2+8^2)] = \frac{1}{4\pi}(53660).\\ \text{RIGHT estimate } &= \frac{1}{4\pi}[20(28^2+21^2+17^2+12^2+8^2+2^2)] = \frac{1}{4\pi}(34520).\\ \text{TRAP } &= \frac{1}{4\pi}(44090) \approx 3509 \text{ cubic inches.}\end{aligned}$$

59. (a) The volume, V, contained in the bowl when the surface has height h is

$$V = \int_0^h \pi x^2\, dy.$$

However, since $y = x^4$, we have $x^2 = \sqrt{y}$ so that

$$V = \int_0^h \pi\sqrt{y}\, dy = \frac{2}{3}\pi h^{3/2}.$$

Differentiating gives $dV/dh = \pi h^{1/2} = \pi\sqrt{h}$. We are given that $dV/dt = -6\sqrt{h}$, where the negative sign reflects the fact that V is decreasing. Using the chain rule we have

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{1}{dV/dh}\cdot\frac{dV}{dt} = \frac{1}{\pi\sqrt{h}}\cdot(-6\sqrt{h}) = -\frac{6}{\pi}.$$

Thus, $dh/dt = -6/\pi$, a constant.

(b) Since $dh/dt = -6/\pi$ we know that $h = -6t/\pi + C$. However, when $t = 0$, $h = 1$, therefore $h = 1 - 6t/\pi$. The bowl is empty when $h = 0$, that is when $t = \pi/6$ units.

60. The problem appears complicated, because we are now working in three dimensions. However, if we take one dimension at a time, we will see that the solution is not too difficult. For example, let's just work at a constant depth, say 0. We apply the trapezoid rule to find the approximate area along the length of the boat. For example, by the trapezoid rule the approximate area at depth 0 from the front of the boat to 10 feet toward the back is $\frac{(2+8)\cdot 10}{2} = 50$. Overall, at depth 0 we have that the area for each length span is as follows:

Table 8.2

length span:	0–10	10–20	20–30	30–40	40–50	50–60
depth 0	50	105	145	165	165	130

We can fill in the whole chart the same way:

Table 8.3

length span:		0–10	10–20	20–30	30–40	40–50	50–60
	0	50	105	145	165	165	130
	2	25	60	90	105	105	90
depth	4	15	35	50	65	65	50
	6	5	15	25	35	35	25
	8	0	5	10	10	10	10

Now, to find the volume, we just apply the trapezoid rule to the depths and areas. For example, according to the trapezoid rule the approximate volume as the depth goes from 0 to 2 and the length goes from 0 to 10 is $\frac{(50+25)\cdot 2}{2} = 75$. Again, we fill in a chart:

Table 8.4

length span:		0–10	10–20	20–30	30–40	40–50	50–60
	0–2	75	165	235	270	270	220
depth	2–4	40	95	140	170	170	140
span	4–6	20	50	75	100	100	75
	6–8	5	20	35	45	45	35

Adding all this up, we find the volume is approximately 2595 cubic feet.

You might wonder what would have happened if we had done our trapezoids along the depth axis first instead of along the length axis. If you try this, you'd find that you come up with the same answers in the volume chart! For the trapezoid rule, it does not matter which axis you choose first.

61. (a) The equation of a circle of radius r around the origin is $x^2+y^2=r^2$. This means that $y^2=r^2-x^2$, so $2y(dy/dx) = -2x$, and $dy/dx = -x/y$. Since the circle is symmetric about both axes, its arc length is 4 times the arc length in the first quadrant, namely

$$4\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^r \sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx.$$

(b) Evaluating this integral yields

$$4\int_0^r \sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx = 4\int_0^r \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 4\int_0^r \sqrt{\frac{r^2}{r^2-x^2}}\,dx$$
$$= 4r\int_0^r \sqrt{\frac{1}{r^2-x^2}}\,dx = 4r(\arcsin(x/r))\Big|_0^r = 2\pi r.$$

This is the expected answer.

62. As can be seen in Figure 8.41, the region has three straight sides and one curved one. The lengths of the straight sides are 1, 1, and e. The curved side is given by the equation $y = f(x) = e^x$. We can find its length by the formula

$$\int_0^1 \sqrt{1+f'(x)^2}\,dx = \int_0^1 \sqrt{1+(e^x)^2}\,dx = \int_0^1 \sqrt{1+e^{2x}}\,dx.$$

Evaluating the integral numerically gives 2.0035. The total length, therefore, is about $1+1+e+2.0035 \approx 6.722$.

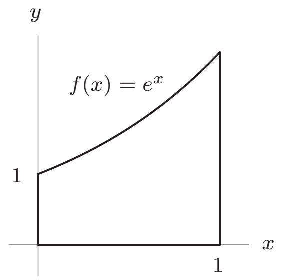


Figure 8.41

63. The graph of f is a downward-opening parabola with x-intercepts at $x = 0, 4$, and $f'(x) = (-x(x-4))' = 4-2x$. The arc length of the graph of f from $x=0$ to $x=4$ is given by

$$\text{Arc length} = \int_0^4 \sqrt{1+(f'(x))^2}\,dx$$
$$= \int_0^4 \sqrt{1+(4-2x)^2}\,dx. \quad \text{because } f'(x) = 4-2x$$

This integral can be simplified as $\int_0^4 \sqrt{4x^2-16x+17}\,dx$.

64. We would like to write the integrand in the form

$$\sqrt{1+\sqrt{x}} = \sqrt{1+(f'(x))^2}.$$

This suggests:

$$\begin{aligned}
\left(f'(x)\right)^2 &= x^{1/2} \\
f'(x) &= x^{1/4} && \text{alternatively } f'(x) = -x^{1/4} \\
f(x) &= \frac{4}{5}\cdot x^{5/4} + C \\
&= 0.8x^{5/4} && \text{letting } C = 0 \text{ for convenience.}
\end{aligned}$$

Thus,

$$\int_1^4 \sqrt{1+\sqrt{x}}\,dx = \int_1^4 \sqrt{1+\left(\left(0.8x^{5/4}\right)'\right)^2}\,dx,$$

which is the arc length of the curve $y = 0.8x^{5/4}$ from $x = 1$ to $x = 4$. Other possible solutions are any of the curves $y = \pm 0.8x^{5/4} + C$.

65. The graph of f is concave down where $f'' < 0$:

$$\begin{aligned} f(x) &= e^{-x^2}. \\ f'(x) &= -2xe^{-x^2} \\ f''(x) &= -2e^{-x^2} + 4x^2e^{-x^2} \\ &= 2e^{-x^2}\left(2x^2-1\right). \end{aligned}$$

This means that $f''(x) < 0$ where

$$\begin{aligned} 2x^2 - 1 &< 0 \\ x^2 &< \frac{1}{2}. \end{aligned}$$

Thus, the graph is concave down for $-\sqrt{1/2} < x < \sqrt{1/2}$. The arc length of this portion of the graph of f is given by

$$\begin{aligned} \text{Arc length} &= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \sqrt{1+(f'(x))^2}\,dx \\ &= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \sqrt{1+\left(-2xe^{-x^2}\right)^2}\,dx \\ &= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \sqrt{1+4x^2e^{-2x^2}}\,dx. \end{aligned}$$

66. The graph of f is concave down where $f''(x) < 0$:

$$\begin{aligned} f(x) &= x^4 - 8x^3 + 18x^2 + 3x + 7 \\ f'(x) &= 4x^3 - 24x^2 + 36x + 3 \\ f''(x) &= 12x^2 - 48x + 36 \\ &= 12(x-1)(x-3). \end{aligned}$$

Hence $f''(x) < 0$ for $1 < x < 3$. The arc length of this portion of the graph of f is given by

$$\begin{aligned} \text{Arc length} &= \int_1^3 \sqrt{1+(f'(x))^2}\,dx \\ &= \int_1^3 \sqrt{1+(4x^3-24x^2+36x+3)^2}\,dx. \end{aligned}$$

67. The arc length of the catenary between $x = -b$ and $x = b$ is 10 meters. Since

$$\frac{d}{dx}(\cosh x) = \sinh x = \frac{1}{2}(e^x - e^{-x}),$$

we have

$$\begin{aligned} \text{Arc length } &= \int_{-b}^{b} \sqrt{1+\frac{1}{4}(e^x-e^{-x})^2}\,dx = \int_{-b}^{b} \frac{1}{2}\sqrt{4+e^{2x}-2+e^{-2x}}\,dx \\ &= \int_{-b}^{b} \frac{1}{2}\sqrt{(e^x+e^{-x})^2}\,dx = \int_{-b}^{b} \frac{1}{2}(e^x+e^{-x})\,dx = \frac{1}{2}(e^x-e^{-x})\Big|_{-b}^{b} = e^b - e^{-b}. \end{aligned}$$

Thus

$$e^b - e^{-b} = 10.$$

Solving numerically gives $b = 2.312$ meters. Thus, the ends of the chain are $2(2.312) = 4.624$ meters apart.

68. Here are many functions which "work."

- Any linear function $y = mx + b$ "works." This follows because $\frac{dy}{dx} = m$ is constant for such functions. So

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + m^2}\, dx = (b - a)\sqrt{1 + m^2}.$$

- The function $y = \frac{x^4}{8} + \frac{1}{4x^2}$ "works": $\frac{dy}{dx} = \frac{1}{2}(x^3 - 1/x^3)$, and

$$\begin{aligned}\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int \sqrt{1 + \frac{\left(x^3 - \frac{1}{x^3}\right)^2}{4}}\, dx = \int \sqrt{1 + \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}}\, dx \\ &= \int \sqrt{\frac{1}{4}\left(x^3 + \frac{1}{x^3}\right)^2}\, dx = \int \frac{1}{2}\left(x^3 + \frac{1}{x^3}\right) dx \\ &= \left[\frac{x^4}{8} - \frac{1}{4x^2}\right] + C.\end{aligned}$$

- One more function that "works" is $y = \ln(\cos x)$; we have $\frac{dy}{dx} = -\sin x / \cos x$. Hence

$$\begin{aligned}\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. &= \int \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\, dx = \int \sqrt{1 + \frac{\sin^2 x}{\cos^2 x}}\, dx \\ &= \int \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\, dx = \int \sqrt{\frac{1}{\cos^2 x}}\, dx \\ &= \int \frac{1}{\cos x}\, dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C,\end{aligned}$$

where the last integral comes from IV-22 of the integral tables.

69. (a) If $f(x) = \int_0^x \sqrt{g'(t)^2 - 1}\, dt$, then, by the Fundamental Theorem of Calculus, $f'(x) = \sqrt{g'(x)^2 - 1}$. So the arc length of f from 0 to x is

$$\begin{aligned}\int_0^x \sqrt{1 + (f'(t))^2}\, dt &= \int_0^x \sqrt{1 + (\sqrt{g'(t)^2 - 1})^2}\, dt \\ &= \int_0^x \sqrt{1 + g'(t)^2 - 1}\, dt \\ &= \int_0^x g'(t)\, dt = g(x) - g(0) = g(x).\end{aligned}$$

(b) If g is the arc length of any function f, then by the Fundamental Theorem of Calculus, $g'(x) = \sqrt{1 + f'(x)^2} \geq 1$. So if $g'(x) < 1$, g cannot be the arc length of a function.

(c) We find a function f whose arc length from 0 to x is $g(x) = 2x$. Using part (a), we see that

$$f(x) = \int_0^x \sqrt{(g'(t))^2 - 1}\, dt = \int_0^x \sqrt{2^2 - 1}\, dt = \sqrt{3}x.$$

This is the equation of a line. Does it make sense to you that the arc length of a line segment depends linearly on its right endpoint?

70. (a) For $n = 1$, we have

$$|x| + |y| = 1.$$

In the first quadrant, the equation is the line

$$x + y = 1.$$

By symmetry, the graph in the other quadrants gives the square in Figure 8.42.

For $n = 2$, the equation is of a circle of radius 1, centered at the origin:

$$x^2 + y^2 = 1.$$

For $n = 4$, the equation is

$$x^4 + y^4 = 1.$$

The graph is similar to a circle, but bulging out more. See Figure 8.42.

(b) For $n = 1$, the arc length is the perimeter of the square. Each side is the hypotenuse of a right triangle of sides $1, 1, \sqrt{2}$. Thus

$$\text{Arc length } = 4\sqrt{2} = 5.657.$$

For $n = 2$, the arc length is the perimeter of the circle of radius 1. Thus

$$\text{Arc length } = 2\pi \cdot 1 = 2\pi = 6.283.$$

For $n = 4$, we find the arc length using the formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

We find the arc length of the top half of the curve, given by $y = (1 - x^4)^{1/4}$, and double it. Since

$$\frac{dy}{dx} = \frac{1}{4}(1 - x^4)^{-3/4}(-4x^3) = \frac{-x^3}{(1 - x^4)^{3/4}},$$

$$\text{Arc length } = 2\int_{-1}^{1} \sqrt{1 + \left(\frac{-x^3}{(1 - x^4)^{3/4}}\right)^2}\, dx = 2\int_{-1}^{1} \sqrt{1 + \frac{x^6}{(1 - x^4)^{3/2}}}\, dx.$$

The integral is improper because the integral is not defined at $x = \pm 1$. Using numerical methods, we find

$$\text{Arc length } = 2\int_{-1}^{1} \sqrt{1 + \frac{x^6}{(1 - x^4)^{3/2}}}\, dx = 7.018.$$

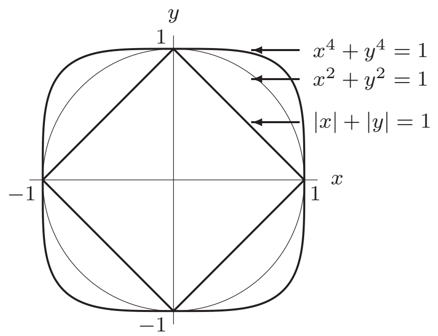


Figure 8.42

Strengthen Your Understanding

71. The volume is $\int_0^5 \left(\pi(3x)^2 - \pi(2x)^2\right)\, dx$, since the slices are disks-with-holes.

72. The arc length is given by $\int_0^{\pi/4} \sqrt{1 + (f'(x))^2}\, dx$ with $f(x) = \sin x$. Thus, the correct formula for the arc length is $\int_0^{\pi/4} \sqrt{1 + \cos^2 x}\, dx$.

73. The curve begins at the point $(0, 0)$ and ends at the point $(2, 32)$. The distance between these points is $\sqrt{2^2 + 32^2} > 32$. Since the arc length of any curve is greater than or equal to the length of the straight line connecting the points, the arc length of $y = x^5$ between $x = 0$ and $x = 2$ must be greater than 32.

74. The triangular region with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ gives the exact same solid shape whether rotated about the x- or y-axis.

75. One example is the region bounded by $y = 2x$ and the x-axis for $0 \le x \le 1$. When the region is rotated about the x-axis, we get a cone of radius 2 and height 1. When it is rotated about the y-axis, we get a cone of radius 1 and height 2.

Since

$$\text{Volume around } x\text{-axis } = \frac{1}{3}\pi \cdot 2^2 \cdot 1 = \frac{4\pi}{3}$$
$$\text{Volume around } y\text{-axis } = \frac{1}{3}\pi \cdot 1^2 \cdot 2 = \frac{2\pi}{3},$$

the volume is greater around the x-axis than around the y-axis.

76. The circle of radius 5 centered at $(5, 0)$ goes through both the points $(0, 0)$ and $(10, 0)$. The upper semicircle and the lower semicircle are different curves between these two points that have the same arc length, namely half the circumference of the circle.

77. The distance from $(0, 0)$ to $(1, 1)$ is $\sqrt{2}$, so any curve other than a straight line between the points has an arc length greater than $\sqrt{2}$. One possible example is $f(x) = x^2$.

78. False. The volume also depends on how far away the region is from the axis of revolution. For example, let R be the rectangle $0 \le x \le 8, 0 \le y \le 1$ and let S be the rectangle $0 \le x \le 3, 0 \le y \le 2$. Then rectangle R has area greater than rectangle S. However, when you revolve R about the x-axis you get a cylinder, lying on its side, of radius 1 and length 8, which has volume 8π. When you revolve S about the x-axis, you get a cylinder of radius 2 and length 3, which has volume 12π. Thus the second volume is larger, even though the region revolved has smaller area.

79. False. Suppose that the graph of f starts at the point $(0, 100)$ and then goes down to $(1, 0)$ and from there on goes along the x-axis. For example, if $f(x) = 100(x-1)^2$ on the interval $[0, 1]$ and $f(x) = 0$ on the interval $[1, 10]$, then f is differentiable on the interval $[0, 10]$. The arc length of the graph of f on the interval $[0, 1]$ is at least 100, while the arc length on the interval $[1, 10]$ is 9.

80. True. Since f is concave up, f' is an increasing function, so $f'(x) \ge f'(0) = 3/4$ on the interval $[0, 4]$. Thus $\sqrt{1 + (f'(x))^2} \ge \sqrt{1 + 9/16} = 5/4$. Then we have:

$$\text{Arc length} = \int_0^4 \sqrt{1 + (f'(x))^2}\, dx \ge \int_0^4 \frac{5}{4}\, dx = 5.$$

81. False. Since f is concave down, this means that $f'(x)$ is decreasing, so $f'(x) \le f'(0) = 3/4$ on the interval $[0, 4]$. However, it could be that $f'(x)$ becomes negative so that $(f'(x))^2$ becomes large, making the integral for the arc length large also. For example, $f(x) = (3/4)x - x^2$ is concave down and $f'(0) = 3/4$, but $f(0) = 0$ and $f(4) = -13$, so the graph of f on the interval $[0, 4]$ has arc length at least 13.

Solutions for Section 8.3

Exercises

1. With $r = 1$ and $\theta = 2\pi/3$, we find $x = r\cos\theta = 1 \cdot \cos(2\pi/3) = -1/2$ and $y = r\sin\theta = 1 \cdot \sin(2\pi/3) = \sqrt{3}/2$.
The rectangular coordinates are $(-1/2, \sqrt{3}/2)$.

2. With $r = \sqrt{3}$ and $\theta = -3\pi/4$, we find $x = r\cos\theta = \sqrt{3}\cos(-3\pi/4) = \sqrt{3}(-\sqrt{2}/2) = -\sqrt{6}/2$ and $y = r\sin\theta = \sqrt{3}\sin(-3\pi/4) = \sqrt{3}(-\sqrt{2}/2) = -\sqrt{6}/2$.
The rectangular coordinates are $(-\sqrt{6}/2, -\sqrt{6}/2)$.

3. With $r = 2\sqrt{3}$ and $\theta = -\pi/6$, we find $x = r\cos\theta = 2\sqrt{3}\cos(-\pi/6) = 2\sqrt{3} \cdot \sqrt{3}/2 = 3$ and $y = r\sin\theta = 2\sqrt{3}\sin(-\pi/6) = 2\sqrt{3}(-1/2) = -\sqrt{3}$.
The rectangular coordinates are $(3, -\sqrt{3})$.

4. With $r = 2$ and $\theta = 5\pi/6$, we find $x = r\cos\theta = 2\cos(5\pi/6) = 2(-\sqrt{3}/2) = -\sqrt{3}$ and $y = r\sin\theta = 2\sin(5\pi/6) = 2(1/2) = 1$.
The rectangular coordinates are $(-\sqrt{3}, 1)$.

5. With $x = 1$ and $y = 1$, find r from $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$. Find θ from $\tan\theta = y/x = 1/1 = 1$. Thus, $\theta = \tan^{-1}(1) = \pi/4$. Since $(1, 1)$ is in the first quadrant this is a correct θ. The polar coordinates are $(\sqrt{2}, \pi/4)$.

6. With $x = -1$ and $y = 0$, find $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 0^2} = 1$. Find θ from $\tan\theta = y/x = 0/(-1) = 0$. Thus, $\theta = \tan^{-1}(0) = 0$. Since $(-1, 0)$ is on the x-axis between the second and third quadrant, $\theta = \pi$. The polar coordinates are $(1, \pi)$.

7. With $x = \sqrt{6}$ and $y = -\sqrt{2}$, find $r = \sqrt{(\sqrt{6})^2 + (-\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$. Find θ from $\tan\theta = y/x = -\sqrt{2}/\sqrt{6} = -1/\sqrt{3}$. Thus, $\theta = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Since $(\sqrt{6}, -\sqrt{2})$ is in the fourth quadrant, this is the correct θ. The polar coordinates are $(2\sqrt{2}, -\pi/6)$.

8. With $x = -\sqrt{3}$ and $y = 1$, find $r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Find θ from $\tan\theta = y/x = 1/(-\sqrt{3})$. Thus, $\theta = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Since $(-\sqrt{3}, 1)$ is in the second quadrant, $\theta = -\pi/6 + \pi = 5\pi/6$. The polar coordinates are $(2, 5\pi/6)$.

9. (a) Table 8.5 contains values of $r = 1 - \sin\theta$, both exact and rounded to one decimal.

Table 8.5

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$3\pi/2$	$5\pi/3$	2π	$7\pi/3$	$5\pi/2$	$8\pi/3$
r	1	$1-\sqrt{3}/2$	0	$1-\sqrt{3}/2$	1	$1+\sqrt{3}/2$	2	$1+\sqrt{3}/2$	1	$1-\sqrt{3}/2$	0	$1-\sqrt{3}/2$
r	1	0.134	0	0.134	1	1.866	2	1.866	1	0.134	0	0.134

(b) See Figure 8.43.

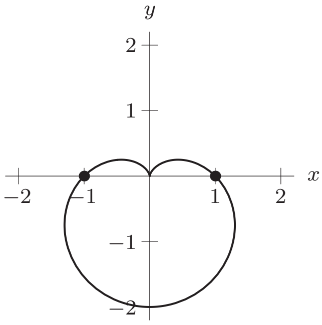


Figure 8.43

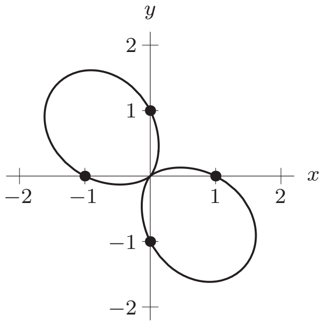


Figure 8.44

(c) The circle has equation $r = 1/2$. The cardioid is $r = 1 - \sin\theta$. Solving these two simultaneously gives

$$1/2 = 1 - \sin\theta,$$

or

$$\sin\theta = 1/2.$$

Thus, $\theta = \pi/6$ or $5\pi/6$. This gives the points $(x, y) = ((1/2)\cos\pi/6, (1/2)\sin\pi/6) = (\sqrt{3}/4, 1/4)$ and $(x, y) = ((1/2)\cos 5\pi/6, (1/2)\sin 5\pi/6) = (-\sqrt{3}/4, 1/4)$ as the location of intersection.

(d) The curve $r = 1 - \sin 2\theta$, pictured in Figure 8.44, has two regions instead of the one region that $r = 1 - \sin\theta$ has. This is because $1 - \sin 2\theta$ will be 0 twice for every 2π cycle in θ, as opposed to once for every 2π cycle in θ for $1 - \sin\theta$.

10. There will be n loops. See Figures 8.45-8.48.

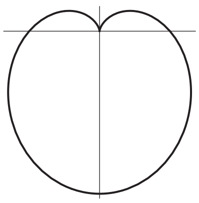


Figure 8.45: $n = 1$

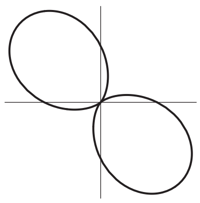


Figure 8.46: $n = 2$

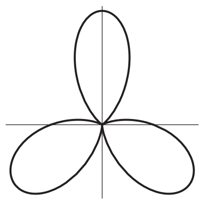


Figure 8.47: $n = 3$

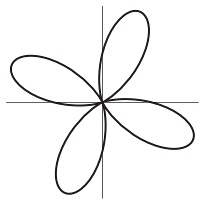


Figure 8.48: $n = 4$

11. The graph will begin to draw over itself for any $\theta \geq 2\pi$ so the graph will look the same in all three cases. See Figure 8.49.

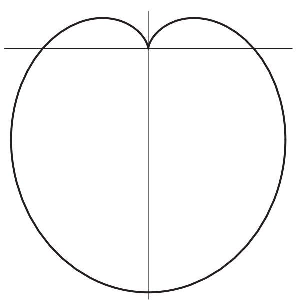


Figure 8.49

12. The curve will be a smaller loop inside a larger loop with an intersection point at the origin. Larger n values increase the size of the loops. See Figures 8.50-8.52.

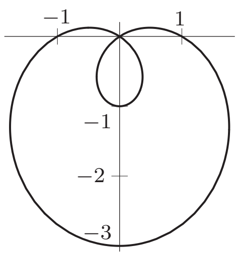


Figure 8.50: $n = 2$

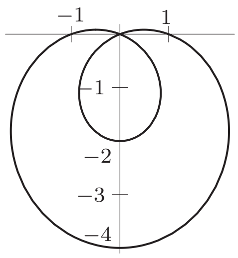


Figure 8.51: $n = 3$

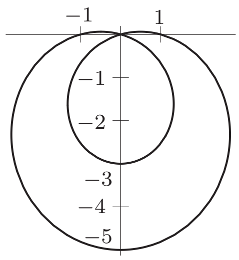


Figure 8.52: $n = 4$

13. See Figures 8.53 and 8.54. The first curve will be similar to the second curve, except the cardioid (heart) will be rotated clockwise by 90° ($\pi/2$ radians). This makes sense because of the identity $\sin\theta = \cos(\theta - \pi/2)$.

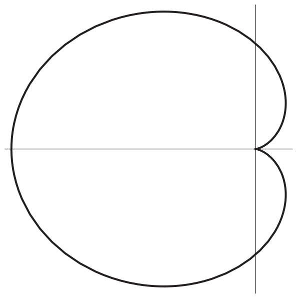


Figure 8.53: $r = 1 - \cos\theta$

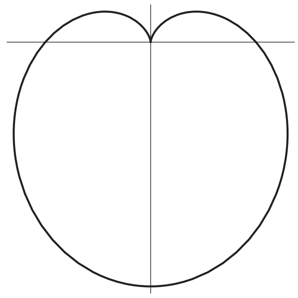


Figure 8.54: $r = 1 - \sin\theta$

14. Let $0 \le \theta \le 2\pi$ and $3/16 \le r \le 1/2$.

15. A loop starts and ends at the origin, that is, when $r = 0$. This happens first when $\theta = \pi/4$ and next when $\theta = 5\pi/4$. This can also be seen by using a trace mode on a calculator. Thus restricting θ so that $\pi/4 \le \theta \le 5\pi/4$ will graph the upper loop only. See Figure 8.55. To show only the other loop use $0 \le \theta \le \pi/4$ and $5\pi/4 \le \theta \le 2\pi$. See Figure 8.56.

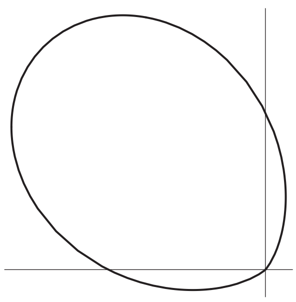


Figure 8.55: $\pi/4 \le \theta \le 5\pi/4$

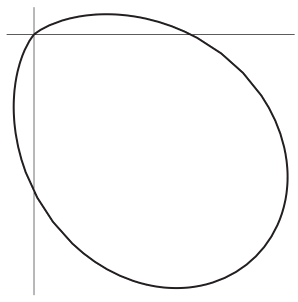


Figure 8.56: $0 \le \theta \le \pi/4$ and $5\pi/4 \le \theta \le 2\pi$

16. (a) Let $0 \le \theta \le \pi/4$ and $0 \le r \le 1$.
(b) Break the region into two pieces: one with $0 \le x \le \sqrt{2}/2$ and $0 \le y \le x$, the other with $\sqrt{2}/2 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$.

17. The region is given by $\sqrt{8} \le r \le \sqrt{18}$ and $\pi/4 \le \theta \le \pi/2$.

18. The region is given by $0 \le r \le 2$ and $-\pi/6 \le \theta \le \pi/6$.

19. The circular arc has equation $r = 1$, for $0 \le \theta \le \pi/2$. the vertical line $x = 2$ has polar equation $r\cos\theta = 2$, or $r = 2/\cos\theta$. So the region is described by $0 \le \theta \le \pi/2$ and $1 \le r \le 2/\cos\theta$.

20. Expressing x and y in terms of θ, we have

$$x = 2\cos\theta \quad \text{and} \quad y = 2\sin\theta.$$

The slope is given by

$$\frac{dy}{dx} = \frac{-2\cos\theta}{2\sin\theta} = -\frac{\cos\theta}{\sin\theta}.$$

At $\theta = \pi/4$, we have

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = -1.$$

21. Expressing x and y in terms of θ, we have

$$x = e^{\theta}\cos\theta \quad \text{and} \quad y = e^{\theta}\sin\theta.$$

The slope is given by

$$\frac{dy}{dx} = \frac{e^{\theta}\sin\theta + e^{\theta}\cos\theta}{e^{\theta}\cos\theta - e^{\theta}\sin\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}.$$

At $\theta = \pi/2$, we have $\sin\theta = 1$ and $\cos\theta = 0$, so

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{1+0}{0-1} = -1.$$

22. Expressing x and y in terms of θ, we have

$$x = (1-\cos\theta)\cos\theta \quad \text{and} \quad y = (1-\cos\theta)\sin\theta.$$

The slope is given by

$$\frac{dy}{dx} = \frac{-(1-\cos\theta)\cos\theta - \sin\theta\sin\theta}{(1-\cos\theta)\sin\theta - \sin\theta\cos\theta}.$$

At $\theta = \pi/2$, we have $\cos\theta = 0$ and $\sin\theta = 1$, so that

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = -1.$$

23. The curve is given parametrically by

$$x = e^{\theta}\cos\theta \quad \text{and} \quad y = e^{\theta}\sin\theta.$$

Thus, calculating $dx/d\theta$ and $dy/d\theta$, gives

$$\begin{aligned}
\text{Arc length } &= \int_{\pi/2}^{\pi} \sqrt{(e^{\theta}\cos\theta - e^{\theta}\sin\theta)^2 + (e^{\theta}\sin\theta + e^{\theta}\cos\theta)^2}\, d\theta \\
&= \int_{\pi/2}^{\pi} e^{\theta}\sqrt{(\cos\theta - \sin\theta)^2 + (\sin\theta + \cos\theta)^2}\, d\theta \\
&= \int_{\pi/2}^{\pi} e^{\theta}\sqrt{2}\, d\theta \\
&= \sqrt{2}(e^{\pi} - e^{\pi/2}).
\end{aligned}$$

24. The curve is given parametrically by

$$x = \theta^2 \cos\theta \quad \text{and} \quad y = \theta^2 \sin\theta.$$

Thus, calculating $dx/d\theta$ and $dy/d\theta$, gives

$$\begin{aligned}
\text{Arc length } &= \int_0^{2\pi} \sqrt{(2\theta\cos\theta - \theta^2\sin\theta)^2 + (2\theta\sin\theta + \theta^2\cos\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4\theta^2 + \theta^4}\, d\theta \\
&= \int_0^{2\pi} \theta\sqrt{4+\theta^2}\, d\theta \\
&= \frac{1}{3}(4+4\pi^2)^{3/2} - \frac{1}{3}4^{3/2}.
\end{aligned}$$

Problems

25. The formula for area is

$$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

Therefore, since

$$A = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/3} (\sin 3\theta)^2\, d\theta,$$

we have $r = \sin 3\theta$. The integral represents the shaded area inside one petal of the three-petaled rose curve, $r = \sin 3\theta$, in Figure 8.57.

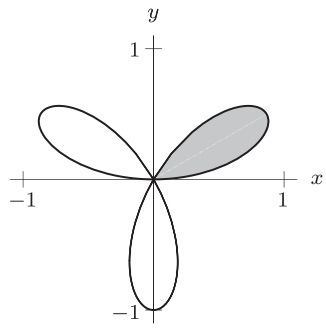


Figure 8.57: Graph of $r = \sin 3\theta$

26. The spiral is shown in Figure 8.58.

$$\text{Area } = \frac{1}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{1}{6}\theta^3\Big|_0^{2\pi} = \frac{8\pi^3}{6}.$$

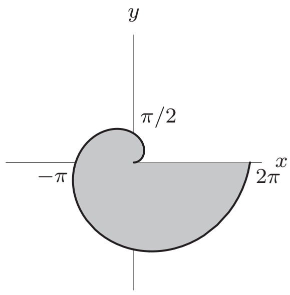


Figure 8.58: Spiral $r = \theta$

27. The region between the spirals is shaded in Figure 8.59.

$$\text{Area } = \frac{1}{2}\int_0^{2\pi} ((2\theta)^2 - \theta^2)\, d\theta = \frac{1}{2}\int_0^{2\pi} 3\theta^2\, d\theta = \frac{1}{2}\theta^3\Big|_0^{2\pi} = 4\pi^3.$$

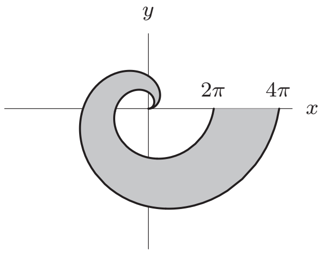


Figure 8.59: Region between the inner spiral, $r = \theta$, and the outer spiral, $r = 2\theta$

28. The cardioid is shown in Figure 8.60. The following integral can be evaluated using a calculator or by parts or using the table of integrals.

$$\begin{aligned}\text{Area } &= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right)\, d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\Big|_0^{2\pi} = \frac{1}{2}(2\pi + 0 + 0 + \pi) = \frac{3\pi}{2}.\end{aligned}$$

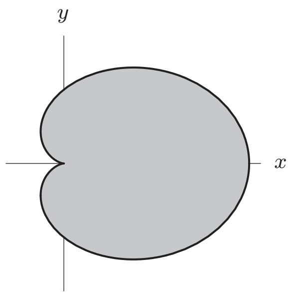


Figure 8.60: Cardioid $r = 1 + \cos\theta$

29. (a) See Figure 8.61. In polar coordinates, the line $x = 1$ is $r\cos\theta = 1$, so its equation is

$$r = \frac{1}{\cos\theta}.$$

The circle of radius 2 centered at the origin has equation

$$r = 2.$$

(b) The line and circle intersect where

$$\begin{aligned}\frac{1}{\cos\theta} &= 2 \\ \cos\theta &= \frac{1}{2} \\ \theta &= -\frac{\pi}{3}, \frac{\pi}{3}.\end{aligned}$$

Thus,

$$\text{Area } = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(2^2 - \left(\frac{1}{\cos\theta}\right)^2\right)\, d\theta.$$

(c) Evaluating gives

$$\text{Area } = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4-\frac{1}{\cos^2\theta}\right)d\theta = \frac{1}{2}(4\theta-\tan\theta)\Big|_{-\pi/3}^{\pi/3} = \frac{4\pi}{3}-\sqrt{3}.$$

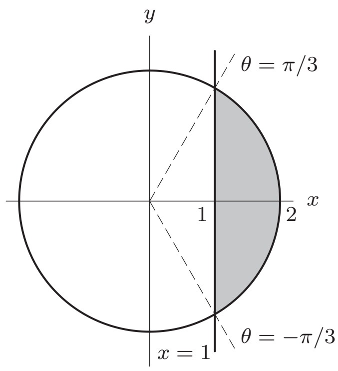


Figure 8.61

30. Using the formula Area$= (1/2)\int_\alpha^\beta (f(\theta))^2\,d\theta$ gives

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \frac{a^2}{2}\theta\Big|_0^{2\pi} = \pi a^2,$$

which is the formula for the area of a disk of radius a.

31. Using the formula Arc length $= \int_\alpha^\beta \sqrt{(dx/d\theta)^2+(dy/d\theta)^2}d\theta$, where $x = a\cos\theta$, $y = a\sin\theta$ and $a > 0$ gives

$$\text{Arc length} = \int_0^{2\pi}\sqrt{(-a\sin\theta)^2+(a\cos\theta)^2}\,d\theta = \int_0^{2\pi} a\,d\theta = a\theta|_0^{2\pi} = 2\pi a,$$

which is the formula for the circumference of a circle of radius a.

32. See Figure 8.62. Notice that the curves intersect at $(1,0)$, where $\theta = 0, 2\pi$, and at $(-1,0)$, where $\theta = \pi$, so

$$\text{Area } = \frac{1}{2}\int_\pi^{2\pi}(1^2-(1+\sin\theta)^2)\,d\theta = \frac{1}{2}\int_\pi^{2\pi}(-2\sin\theta-\sin^2\theta)\,d\theta.$$

Using a calculator, integration by parts, or formula IV-17 in the integral table, we have

$$\text{Area } = \frac{1}{2}\left(2\cos\theta+\frac{1}{2}\sin\theta\cos\theta-\frac{1}{2}\theta\right)\Big|_\pi^{2\pi} = \frac{1}{2}\left(2\cdot 2+0-\frac{1}{2}\pi\right) = 2-\frac{\pi}{4}.$$

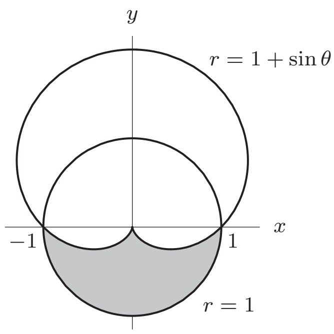


Figure 8.62

33. The two curves intersect where

$$\begin{aligned}1-\sin\theta &= \frac{1}{2}\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6},\frac{5\pi}{6}.\end{aligned}$$

See Figure 8.63. We find the area of the right half and multiply that answer by 2 to get the entire area. The integrals can be computed numerically with a calculator or, as we show, using integration by parts or formula IV-17 in the integral tables.

$$\begin{aligned}\text{Area of right half } &= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left((1-\sin\theta)^2-\left(\frac{1}{2}\right)^2\right)d\theta\\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left(1-2\sin\theta+\sin^2\theta-\frac{1}{4}\right)d\theta\\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left(\frac{3}{4}-2\sin\theta+\sin^2\theta\right)d\theta\\ &= \frac{1}{2}\left(\frac{3}{4}\theta+2\cos\theta-\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta\right)\Bigg|_{-\pi/2}^{\pi/6}\\ &= \frac{1}{2}\left(\frac{5\pi}{6}+\frac{7\sqrt{3}}{8}\right).\end{aligned}$$

Thus,

$$\text{Total area } = \frac{5\pi}{6}+\frac{7\sqrt{3}}{8}.$$

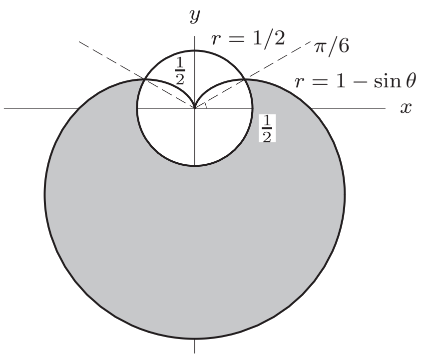


Figure 8.63

34. Figure 8.64 shows the curves which touch at $(2,0)$ and the origin. However, the circle lies entirely inside the cardioid, so we find the area by subtracting the area of the circle from that of the cardioid. To find the areas, we take the integrals.

The cardioid, $r=1+\cos\theta$, starts at $(2,0)$ when $\theta=0$ and traces the top half, reaching the origin when $\theta=\pi$. Thus

$$\text{Area of cardioid } = 2\cdot\frac{1}{2}\int_0^{\pi}(1+\cos\theta)^2\,d\theta.$$

The circle starts at $(2,0)$ when $\theta=0$ and traces the top half, reaching the origin when $\theta=\pi/2$. Thus

$$\text{Area of circle } = 2\cdot\frac{1}{2}\int_0^{\pi/2}(2\cos\theta)^2\,d\theta.$$

The area, A, we want is therefore

$$\begin{aligned}
\text{Area } &= 2\cdot\frac{1}{2}\int_0^{\pi}(1+\cos\theta)^2\,d\theta - 2\cdot\frac{1}{2}\int_0^{\pi/2}(2\cos\theta)^2\,d\theta\\
&= \int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta - \int_0^{\pi/2}4\cos^2\theta\,d\theta\\
&= \left(\theta+2\sin\theta+\frac{1}{2}(\sin\theta\cos\theta+\theta)\right)\Big|_0^{\pi} - \frac{4}{2}(\sin\theta\cos\theta+\theta)\Big|_0^{\pi/2}\\
&= \frac{3}{2}\pi - 2\cdot\frac{\pi}{2} = \frac{\pi}{2}.
\end{aligned}$$

Alternatively, we could compute the area of the cardioid and subtract the area of the circle of radius 1 from it.

The integrals can be computed numerically using a calculator, or, as we show, using integration by parts or formula IV-18 from the integral tables.

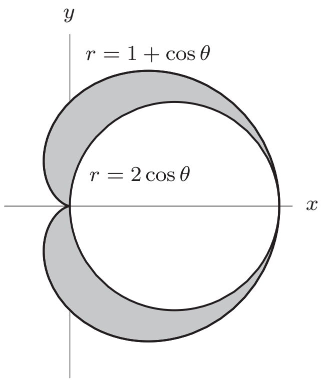


Figure 8.64

35. (a) The graph of $r = 2\cos\theta$ is a circle of radius 1 centered at $(1, 0)$; the graph of $r = 2\sin\theta$ is a circle of radius 1 centered at $(0, 1)$. See Figure 8.65.

(b) The Cartesian coordinates of the points of intersection are at $(0, 0)$ and $(1, 1)$.

The origin corresponds to $\theta = \pi/2$ on $r = 2\cos\theta$ and to $\theta = 0$ on $r = 2\sin\theta$. The point $(1, 1)$ has polar coordinates $r = \sqrt{2}, \theta = \pi/4$.

We find the area below the line $\theta = \pi/4$ and above $r = 2\sin\theta$ and double it:

$$\text{Area } = 2\cdot\frac{1}{2}\int_0^{\pi/4}(2\sin\theta)^2\,d\theta = 4\int_0^{\pi/4}\sin^2\theta\,d\theta.$$

Using a calculator, integration by parts or formula IV-17 from the integral tables,

$$\text{Area } = 4\left(-\frac{1}{2}\sin\theta\cos\theta+\frac{\theta}{2}\right)\Big|_0^{\pi/4} = -2\cdot\frac{1}{2}+2\frac{\pi}{4} = \frac{\pi}{2}-1.$$

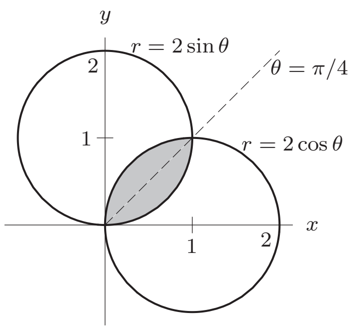


Figure 8.65

36. The area is

$$
\begin{aligned}
A = \frac{1}{2}\int_0^a r^2\,d\theta = \frac{1}{2}\int_0^a \theta^2\,d\theta &= 1 \\
\frac{1}{2}\left(\frac{\theta^3}{3}\right)\bigg|_0^a &= 1 \\
\frac{a^3}{6} &= 1 \\
a^3 &= 6 \\
a &= \sqrt[3]{6}.
\end{aligned}
$$

37. (a) See Figure 8.66.
(b) The curves intersect when $r^2 = 2$

$$
\begin{aligned}
4\cos 2\theta &= 2 \\
\cos 2\theta &= \frac{1}{2}.
\end{aligned}
$$

In the first quadrant:

$$2\theta = \frac{\pi}{3} \quad \text{so} \quad \theta = \frac{\pi}{6}.$$

Using symmetry, the area in the first quadrant can be multiplied by 4 to find the area of the total bounded region.

$$
\begin{aligned}
\text{Area } &= 4\left(\frac{1}{2}\right)\int_0^{\pi/6}(4\cos 2\theta - 2)\,d\theta \\
&= 2\left(\frac{4\sin 2\theta}{2} - 2\theta\right)\bigg|_0^{\pi/6} \\
&= 4\sin\frac{\pi}{3} - \frac{2}{3}\pi \\
&= 4\frac{\sqrt{3}}{2} - \frac{2}{3}\pi \\
&= 2\sqrt{3} - \frac{2}{3}\pi = 1.370.
\end{aligned}
$$

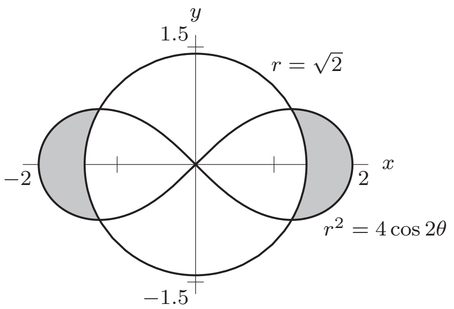


Figure 8.66

38. The slope of the tangent line at $\theta = \pi/3$ is $dy/dx = \sqrt{3}/5$. Since $x = 3\sin(2\theta)\cos\theta$ and $y = 3\sin(2\theta)\sin\theta$, when $\theta = \pi/3$, we have $x = 3\sqrt{3}/4$ and $y = 9/4$. Thus, the equation of the tangent line is

$$
\begin{aligned}
y - \frac{9}{4} &= \frac{\sqrt{3}}{5}\left(x - \frac{3\sqrt{3}}{4}\right) \\
y &= \frac{\sqrt{3}}{5}x - \frac{9}{20} + \frac{9}{4} \\
y &= \frac{\sqrt{3}}{5}x + \frac{9}{5}.
\end{aligned}
$$

39. We first find the points with horizontal and vertical tangents in the first quadrant and then use symmetry to obtain the points in other quadrants.

The slope of the tangent line is

$$\frac{dy}{dx} = \frac{6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta}{6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta}.$$

The curve has a horizontal tangent where

$$6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta = 0.$$

Solving this equation numerically for $0 < \theta < \pi/2$, we have $\theta = 0.9553$; in addition $\theta = 0$ is a solution. Thus, there are horizontal tangents where $x = 1.633$ and $y = 2.309$ and where $x = 0$, $y = 0$. Thus, the five points with horizontal tangents are

$$(1.633, 2.309); \quad (-1.633, 2.309); \quad (-1.633, -2.309); \quad (1.633, -2.309); \quad (0, 0).$$

The curve has vertical tangents where

$$6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta = 0.$$

Solving this equation numerically for $0 < \theta < \pi/2$, we have $\theta = 0.6155$; in addition $\theta = \pi/2$ is a solution. Thus, there are vertical tangents where $x = 2.309$, $y = 1.633$, and where $x = 0, y = 0$. Thus, there are five points with vertical tangents:

$$(2.309, 1.633); \quad (-2.309, 1.633); \quad (-2.309, -1.633); \quad (2.309, -1.633); \quad (0, 0).$$

40. We can express x and y in terms of θ as a parameter. Since $r = \theta$, we have

$$x = r\cos\theta = \theta\cos\theta \quad \text{and} \quad y = r\sin\theta = \theta\sin\theta.$$

Calculating the slope using the parametric formula,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta},$$

we have

$$\frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$$

Horizontal tangents occur where $dy/dx = 0$, so

$$\sin\theta + \theta\cos\theta = 0$$
$$\theta = -\tan\theta.$$

Solving this equation numerically gives

$$\theta = 0, 2.029, 4.913.$$

Vertical tangents occur where dy/dx is undefined, so

$$\cos\theta - \theta\sin\theta = 0$$
$$\theta = \frac{1}{\tan\theta} = \cot\theta.$$

Solving this equation numerically gives

$$\theta = 0.860, 3.426.$$

41. (a) Expressing x and y parametrically in terms of θ, we have

$$x = r\cos\theta = \frac{\cos\theta}{\theta} \quad \text{and} \quad y = r\sin\theta = \frac{\sin\theta}{\theta}.$$

The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \left(\frac{\theta\cos\theta - \sin\theta}{\theta^2}\right) \Big/ \left(\frac{-\theta\sin\theta - \cos\theta}{\theta^2}\right) = \frac{\sin\theta - \theta\cos\theta}{\cos\theta + \theta\sin\theta}.$$

At $\theta = \pi/2$, we have

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{1 - (\pi/2)0}{0 + (\pi/2)1} = \frac{2}{\pi}.$$

At $\theta = \pi/2$, we have $x = 0$, $y = 2/\pi$, so the equation of the tangent line is

$$y = \frac{2}{\pi}x + \frac{2}{\pi}.$$

(b) As $\theta \to 0$,

$$x = \frac{\cos\theta}{\theta} \to \infty \quad \text{and} \quad y = \frac{\sin\theta}{\theta} \to 1.$$

Thus, $y = 1$ is a horizontal asymptote. See Figure 8.67.

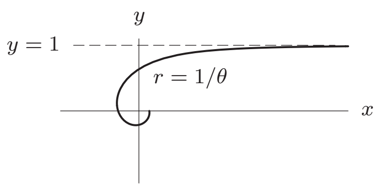


Figure 8.67

42. The limaçon is given by $r = 1 + 2\cos\theta$; see Figure 8.68. At $\theta = 0$, the graph is at $(3, 0)$; as θ increases, the graph sweeps out the top arc (on which the maximum value of y occurs), reaching the origin when

$$\begin{aligned} 1 + 2\cos\theta &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}. \end{aligned}$$

Thus, we want to find the maximum value of y on the interval $0 \le \theta \le 2\pi/3$. Since $y = r\sin\theta$, we want to find the maximum value of

$$y = (1 + 2\cos\theta)\sin\theta = \sin\theta + 2\cos\theta\sin\theta.$$

At a critical point

$$\begin{aligned} \frac{dy}{d\theta} &= \cos\theta - 2\sin^2\theta + 2\cos^2\theta = 0 \\ \cos\theta - 2(1 - \cos^2\theta) + 2\cos^2\theta &= 0 \\ 4\cos^2\theta + \cos\theta - 2 &= 0 \\ \cos\theta &= \frac{-1 \pm \sqrt{33}}{8} = 0.593, -0.843. \end{aligned}$$

Thus, $\theta = \cos^{-1}(0.593) = 0.936$ and $\theta = \cos^{-1}(-0.843) = 2.574$ are the critical values. Since 2.574 is outside the interval $0 \le \theta \le 2\pi/3$, there is one critical point $\theta = 0.963$.

At the endpoints of the interval, $y = 0$. At $\theta = 0.936$, we have $y = 1.760$, which is the maximum value.

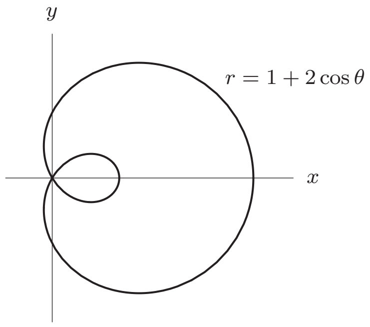


Figure 8.68: The inner loop has $r < 0$

43. Since $x = \theta\cos\theta$ and $y = \theta\sin\theta$, we have

$$\text{Arc length } = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta = 21.256.$$

44. Since $x = \cos\theta/\theta$ and $y = \sin\theta/\theta$, we have

$$\begin{aligned}\text{Arc length } &= \int_\pi^{2\pi} \sqrt{\left(\frac{-\theta\sin\theta - \cos\theta}{\theta^2}\right)^2 + \left(\frac{\theta\cos\theta - \sin\theta}{\theta^2}\right)^2}\, d\theta \\ &= \int_\pi^{2\pi} \sqrt{\frac{\theta^2+1}{\theta^4}}\, d\theta = 0.712.\end{aligned}$$

45. Parameterized by θ, the curve $r = f(\theta)$ is given by $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Then

$$\begin{aligned}\text{Arc length } &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + (f(\theta))^2\sin^2\theta} \\ &\qquad \overline{+(f'(\theta))^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + (f(\theta))^2\cos^2\theta}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\sin^2\theta + \cos^2\theta)}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta.\end{aligned}$$

46. Let $f(\theta) = \theta$. Then $f'(\theta) = 1$. Using the formula derived in Problem 45, we have

$$\text{Arclength } = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta = \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln|\theta + \sqrt{1+\theta^2}|\right]_0^\pi = \frac{1}{2}\pi\sqrt{1+\pi^2} + \frac{1}{2}\ln|\pi + \sqrt{1+\pi^2}|$$

47. Let $f(\theta) = 1 + \cos\theta$. Then $f'(\theta) = -\sin\theta$. Using numerical methods, we have

$$\text{Arclength } = \int_0^{\pi/2} \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}\, d\theta = \int_0^{\pi/2} \sqrt{2+2\cos\theta}\, d\theta = 2.828.$$

Strengthen Your Understanding

48. The formula $\theta = \tan^{-1}(y/x)$ is incorrect if $x \le 0$. If $x = 0$, then $\tan^{-1}(y/x)$ is not defined and so cannot be used. If $x < 0$, then the point (x, y) is in quadrant II or III, but $\tan^{-1}(y/x)$ is an angle in quadrant I or IV, because $-\pi/2 < \tan^{-1}(y/x) < \pi/2$.

49. The points on the polar curve with $\pi/2 < \theta < \pi$ are in quadrant IV, because $r = \sin(2\theta) < 0$.

50. Since $dr/d\theta$ is the rate of change of r with respect to θ, it cannot be interpreted as the slope of the polar curve. The circle $r = 1$ has positive slope at $\theta = 3\pi/4$, yet $dr/d\theta = 0$ everywhere, since the radius does not change as θ increases.

51. The rose with four petals, $r = 3\sin 2\theta$, shown in Figure 8.69 is symmetric about both axes.

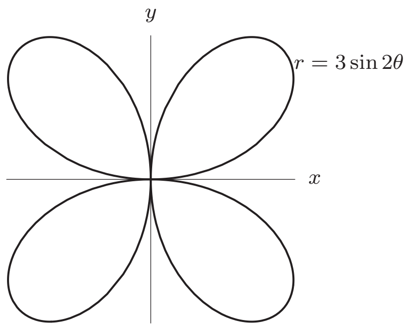


Figure 8.69

52. Changing the polar coordinate θ by adding 2π does not change the point. For example, $(r, \theta) = (10, \pi)$ and $(r, \theta) = (10, 3\pi)$ correspond to the same point $(x, y) = (-10, 0)$.

53. The circle of radius k centered at the origin has equation $r = k$. For example, $r = 100$.

54. One example is the Archimedean spiral $r = \theta$.

55. A polar curve is symmetric about the x-axis if replacing θ by $-\theta$ in its formula makes no difference, so any formula involving only $\cos\theta$ will work. One such example is the limaçon $r = 1 + \cos\theta$.

Solutions for Section 8.4

Exercises

1. Since density is e^{-x} gm/cm,

$$\text{Mass } = \int_0^{10} e^{-x}\,dx = -e^{-x}\Big|_0^{10} = 1 - e^{-10} \text{ gm.}$$

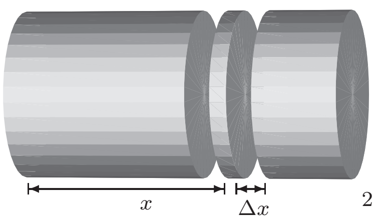
2. Strips perpendicular to the x-axis have length 3, area $3\Delta x$, and mass $5 \cdot 3\Delta x$ gm. Thus

$$\text{Mass } = \int_0^2 5 \cdot 3\,dx = \int_0^2 15\,dx.$$

Strips perpendicular to the y-axis have length 2, area $2\Delta y$, and mass $5 \cdot 2\Delta y$ gm. Thus

$$\text{Mass } = \int_0^3 5 \cdot 2\,dy = \int_0^3 10\,dy.$$

3. (a) Suppose we choose an x, $0 \le x \le 2$. If Δx is a small fraction of a meter, then the density of the rod is approximately $\delta(x)$ anywhere from x to $x + \Delta x$ meters from the left end of the rod (see below). The mass of the rod from x to $x + \Delta x$ meters is therefore approximately $\delta(x)\Delta x = (2 + 6x)\Delta x$. If we slice the rod into N pieces, then a Riemann sum is $\sum_{i=1}^{N}(2 + 6xi)\Delta x$.



(b) The definite integral is

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (2+6x)\,dx = (2x+3x^2)\Big|_0^2 = 16 \text{ grams.}$$

4. We have

$$\begin{aligned}\text{Moment} &= \int_0^2 x\delta(x)\,dx = \int_0^2 x(2+6x)\,dx \\ &= \int_0^2 (6x^2+2x)\,dx = (2x^3+x^2)\Big|_0^2 = 20 \text{ gram-meters.}\end{aligned}$$

Now, using this and Problem 3 (b), we have

$$\text{Center of mass} = \frac{\text{Moment}}{\text{Mass}} = \frac{20 \text{ gram-meters}}{16 \text{ grams}} = \frac{5}{4} \text{ meters (from its left end).}$$

5. (a) Figure 8.70 shows a graph of the density function.

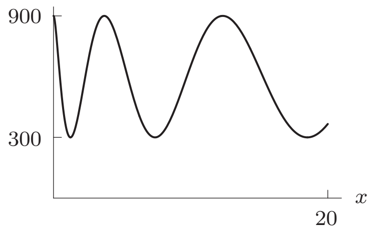


Figure 8.70

(b) Suppose we choose an x, $0 \le x \le 20$. We approximate the density of the number of the cars between x and $x+\Delta x$ miles as $\delta(x)$ cars per mile. Therefore, the number of cars between x and $x+\Delta x$ is approximately $\delta(x)\Delta x$. If we slice the 20 mile strip into N slices, we get that the total number of cars is

$$C \approx \sum_{i=1}^{N} \delta(x_i)\Delta x = \sum_{i=1}^{N} \left[600 + 300\sin(4\sqrt{x_i+0.15})\right]\Delta x,$$

where $\Delta x = 20/N$. (This is a right-hand approximation; the corresponding left-hand approximation is $\sum_{i=0}^{N-1} \delta(x_i)\Delta x$.)

(c) As $N \to \infty$, the Riemann sum above approaches the integral

$$C = \int_0^{20} (600 + 300\sin 4\sqrt{x+0.15})\,dx.$$

If we calculate the integral numerically, we find $C \approx 11513$. We can also find the integral exactly as follows:

$$\begin{aligned}C &= \int_0^{20} (600 + 300\sin 4\sqrt{x+0.15})\,dx \\ &= \int_0^{20} 600\,dx + \int_0^{20} 300\sin 4\sqrt{x+0.15}\,dx \\ &= 12000 + 300\int_0^{20} \sin 4\sqrt{x+0.15}\,dx.\end{aligned}$$

Let $w = \sqrt{x+0.15}$, so $x = w^2 - 0.15$ and $dx = 2w\,dw$. Then

$$\begin{aligned}\int_{x=0}^{x=20} \sin 4\sqrt{x+0.15}\,dx &= 2\int_{w=\sqrt{0.15}}^{w=\sqrt{20.15}} w\sin 4w\,dw, \text{ (using integral table III-15)} \\ &= 2\left[-\frac{1}{4}w\cos 4w + \frac{1}{16}\sin 4w\right]\Bigg|_{\sqrt{0.15}}^{\sqrt{20.15}} \\ &\approx -1.624.\end{aligned}$$

Using this, we have $C \approx 12000 + 300(-1.624) \approx 11513$, which matches our numerical approximation.

6. (a) Orient the rectangle in the coordinate plane in such a way that the side referred to in the problem—call it S—lies on the y-axis from $y = 0$ to $y = 5$, as shown in Figure 8.71. We may subdivide the rectangle into strips of width Δx and length 5. If the left side of a given strip is a distance x away from S (i.e., the y-axis), its density 2 is $1/(1+x^4)$. If Δx is small enough, the density of the strip is approximately constant—i.e., the density of the whole strip is about $1/(1+x^4)$. The mass of the strip is just its density times its area, or $5\Delta x/(1+x^4)$. Thus the mass of the whole rectangle is approximated by the Riemann sum

$$\sum \frac{5\Delta x}{1+x^4},$$

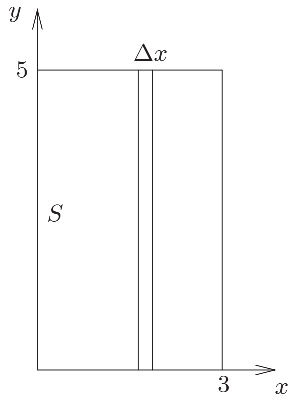


Figure 8.71

(b) The exact mass of the rectangle is obtained by letting $\Delta x \to 0$ in the Riemann sums above, giving us the integral

$$\int_0^3 \frac{5\,dx}{1+x^4}.$$

Since it is not easy to find an antiderivative of $5/(1+x^4)$, we evaluate this integral numerically, getting 5.5.

7. The total mass is 7 grams. The center of mass is given by

$$\overline{x} = \frac{2(-3)+5(4)}{7} = 2 \text{ cm to right of origin.}$$

8. The total mass is 9 gm, and so the center of mass is located at $\overline{x} = \frac{1}{9}(-10\cdot 5 + 1\cdot 3 + 2\cdot 1) = -5$.

9. We slice the block horizontally. A slice has area $10\cdot 3 = 30$ and thickness Δz. On such a slice, the density is approximately constant. Thus

$$\text{Mass of slice} \approx \text{Density} \cdot \text{Volume} \approx (2-z)\cdot 30\Delta z,$$

so we have

$$\text{Mass of block} \approx \sum (2-z)30\Delta z.$$

In the limit as $\Delta z \to 0$, the sum becomes an integral and the approximation becomes exact. Thus

$$\text{Mass of block} = \int_0^1 (2-z)30\,dz = 30\left(2z - \frac{z^2}{2}\right)\Bigg|_0^1 = 30\left(2-\frac{1}{2}\right) = 45.$$

Problems

10. According to the table, the number of bats/ha drops from 117 at 3 km to 85 at 4 km. To make an overestimate, we can assume there are 117 bats/ha everywhere between 3 and 4 km from the cave. We have

$$\text{Area of ring (km}^2) = \underbrace{\text{Area of outer circle}}_{\pi\cdot 4^2} - \underbrace{\text{Area of inner circle}}_{\pi\cdot 3^2} = 7\pi.$$

There are 100 ha per km^2, so the area is $7\pi(100) = 700\pi$ ha. Assuming 117 bats/ha, this gives

$$\text{Overestimate for number of bats} = \underbrace{\text{Area of ring (ha)}}_{700\pi} \times \underbrace{\text{Number bats per ha}}_{117} = 257{,}296.$$

11. According to the table, the number of bats/ha drops from 117 at 3 km to 85 at 4 km. To make an underestimate, we can assume there are 85 bats/ha everywhere between 3 and 4 km from the cave. We have

$$\text{Area of ring (km}^2) = \underbrace{\text{Area of outer circle}}_{\pi\cdot 4^2} - \underbrace{\text{Area of inner circle}}_{\pi\cdot 3^2} = 7\pi.$$

There are 100 ha per km^2, so the area is $7\pi(100) = 700\pi$ ha. Assuming 85 bats/ha, this gives

$$\text{Underestimate for number of bats} = \underbrace{\text{Area of ring (ha)}}_{700\pi} \times \underbrace{\text{Number bats per ha}}_{85} = 186{,}925.$$

12. The number of bats in a ring of inner radius r and width Δr is given by

$$\begin{aligned}\text{Number of bats} &= \text{Area of ring (ha)} \times \text{Bats per ha} \\ &\approx \underbrace{100 \cdot \overbrace{2\pi r\Delta r}^{\text{area (km}^2)}}_{\text{area (ha)}} \times \underbrace{f(r)}_{\text{bats per ha}} \\ &= 200\pi f(r) r\Delta r.\end{aligned}$$

Thus, adding up the contributions from each ring gives:

$$\text{Total number of bats} \approx \sum 200\pi f(r) r\Delta r.$$

Taking the limit of the sum as $\Delta r \to 0$, we get

$$\text{Total number of bats} = \int_0^5 200\pi f(r) r dr.$$

13. Since the density varies with x, the region must be sliced perpendicular to the x-axis. This has the effect of making the density approximately constant on each strip. See Figure 8.72. Since a strip is of height y, its area is approximately $y\Delta x$. The density on the strip is $\delta(x) = 1 + x$ gm/cm^2. Thus

$$\text{Mass of strip} \approx \text{Density} \cdot \text{Area} \approx (1 + x)y\Delta x \text{ gm}.$$

Because the tops of the strips end on two different lines, one for $x \geq 0$ and the other for $x < 0$, the mass is calculated as the sum of two integrals. See Figure 8.72. For the left part of the region, $y = x + 1$, so

$$\begin{aligned}\text{Mass of left part} &= \lim_{\Delta x\to 0} \sum (1 + x)y\Delta x = \int_{-1}^{0} (1 + x)(x + 1)\, dx \\ &= \int_{-1}^{0} (1 + x)^2\, dx = \left.\frac{(x + 1)^3}{3}\right|_{-1}^{0} = \frac{1}{3} \text{ gm}.\end{aligned}$$

From Figure 8.72, we see that for the right part of the region, $y = -x + 1$, so

$$\begin{aligned}\text{Mass of right part} &= \lim_{\Delta x\to 0} \sum (1 + x)y\Delta x = \int_{0}^{1} (1 + x)(-x + 1)\, dx \\ &= \int_{0}^{1} (1 - x^2)\, dx = x - \left.\frac{x^3}{3}\right|_{0}^{1} = \frac{2}{3} \text{ gm}.\end{aligned}$$

$$\text{Total mass} = \frac{1}{3} + \frac{2}{3} = 1 \text{ gm}.$$

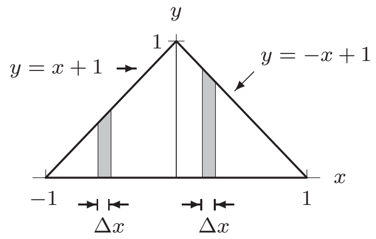


Figure 8.72

14. Since the density $\delta(y) = 2 + y^2$ is constant for fixed y, we take slices parallel to the x-axis of height Δy and length 2. Then a slice has area $= 2\Delta y$ and approximate mass of $(2 + y^2) \cdot 2\Delta y$. Adding the slices and letting $\Delta y \to 0$ gives

$$\text{Mass} = \int_0^3 2(2 + y^2)\, dy = 4y + \frac{2}{3}y^3 \Big|_0^3 = 30 \text{ gm}.$$

15. (a) We must find where the population density is zero. The density is given by the function

$$10{,}000(3 - r);$$

if $10{,}000(3 - r) = 0$, then we must have $r = 3$. We thus conclude that the radius of Circle City is 3 miles. (Note that for $r > 3$, $10{,}000(3 - r)$ becomes negative, so at that point, our function no longer gives a meaningful representation of population density.)

(b) We refer to Example 4 in this section, with $f(r) = 10{,}000(3 - r)$. The population is approximated by a sum

$$\text{Population} \approx \sum 2\pi r \cdot 10{,}000(3 - r)\Delta r.$$

Since the city radius is 3 miles, r ranges from 0 to 3. Hence as $\Delta r \to 0$, the sum is given by the integral

$$\text{Population} = \int_0^3 2\pi r \cdot 10{,}000(3 - r)\, dr.$$

This integral evaluates to $9\pi \cdot 10{,}000 \approx 282{,}743$. So we can say that the population of Circle City is approximately 282,743.

16. (a) Partition $[0, 10{,}000]$ into N subintervals of width Δr. The area in the i^{th} subinterval is $\approx 2\pi r_i \Delta r$. So the total mass in the slick $= M \approx \sum_{i=1}^{N} 2\pi r_i \left(\frac{50}{1+r_i}\right) \Delta r$.

(b) $M = \displaystyle\int_0^{10{,}000} 100\pi \frac{r}{1+r} dr$. We may rewrite $\dfrac{r}{1+r}$ as $\frac{1+r}{1+r} - \frac{1}{1+r} = 1 - \frac{1}{1+r}$, so that

$$M = \int_0^{10{,}000} 100\pi\left(1 - \frac{1}{1+r}\right) dr = 100\pi \left(r - \ln|1+r| \Big|_0^{10{,}000} \right)$$
$$= 100\pi(10{,}000 - \ln(10{,}001)) \approx 3.14 \times 10^6 \text{ kg}.$$

(c) We wish to find an R such that

$$\int_0^R 100\pi \frac{r}{1+r} dr = \frac{1}{2} \int_0^{10{,}000} 100\pi \frac{r}{1+r} dr \approx 1.57 \times 10^6.$$

So $100\pi(R - \ln|R + 1|) \approx 1.57 \times 10^6$; $R - \ln|R + 1| \approx 5000$. By trial and error, we find $R \approx 5009$ meters.

17. (a) We form a Riemann sum by slicing the region into concentric rings of radius r and width Δr. Then the volume deposited on one ring will be the height $H(r)$ multiplied by the area of the ring. A ring of width Δr will have an area given by

$$\begin{aligned}\text{Area} &= \pi(r + \Delta r)^2 - \pi(r^2) \\ &= \pi(r^2 + 2r\Delta r + (\Delta r)^2 - r^2) \\ &= \pi(2r\Delta r + (\Delta r)^2).\end{aligned}$$

Since Δr is approaching zero, we can approximate

$$\text{Area of ring} \approx \pi(2r\Delta r + 0) = 2\pi r\Delta r.$$

From this, we have

$$\Delta V \approx H(r) \cdot 2\pi r\Delta r.$$

Thus, summing the contributions from all rings we have

$$V \approx \sum H(r) \cdot 2\pi r\Delta r.$$

Taking the limit as $\Delta r \to 0$, we get

$$V = \int_0^5 2\pi r \left(0.115e^{-2r}\right) dr.$$

(b) We use integration by parts:

$$\begin{aligned} V &= 0.23\pi \int_0^5 \left(re^{-2r}\right) dr \\ &= 0.23\pi \left(\frac{re^{-2r}}{-2} - \frac{e^{-2r}}{4}\right)\Bigg|_0^5 \\ &\approx 0.181(\text{millimeters}) \cdot (\text{kilometers})^2 = 0.181 \cdot 10^{-3} \cdot 10^6 \text{ meters}^3 = 181 \text{ cubic meters.} \end{aligned}$$

18. Partition $a \le x \le b$ into N subintervals of width $\Delta x = \dfrac{(b-a)}{N}$; $a = x_0 < x_1 < \cdots < x_N = b$. The mass of the strip on the ith subinterval is approximately $m_i = \delta(x_i)[f(x_i) - g(x_i)]\Delta x$. If we use a right-hand Riemann sum, the approximation for the total mass is
$\displaystyle\sum_{i=1}^{N} \delta(x_i)[f(x_i) - g(x_i)]\Delta x$, and the exact mass is $M = \displaystyle\int_a^b \delta(x)[f(x) - g(x)]dx$.

19. (a) Use the formula for the volume of a cylinder:

$$\text{Volume} = \pi r^2 l.$$

Since it is only a half cylinder

$$\text{Volume of shed} = \frac{1}{2}\pi r^2 l.$$

(b) Set up the axes as shown in Figure 8.73. The density can be defined as

$$\text{Density} = ky.$$

Now slice the sawdust horizontally into slabs of thickness Δy as shown in Figure 8.74, and calculate

$$\text{Volume of slab} \approx 2xl\Delta y = 2l(\sqrt{r^2 - y^2})\Delta y.$$

$$\text{Mass of slab} = \text{Density} \cdot \text{Volume} \approx 2kly\sqrt{r^2 - y^2}\Delta y.$$

Finally, we compute the total mass of sawdust:

$$\text{Total mass of sawdust} = \int_0^r 2kly\sqrt{r^2 - y^2}\, dy = -\frac{2}{3}kl(r^2 - y^2)^{3/2}\Bigg|_0^r = \frac{2klr^3}{3}.$$

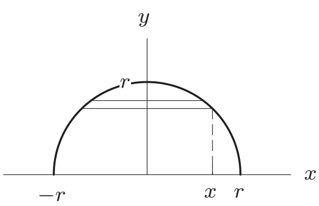


Figure 8.73

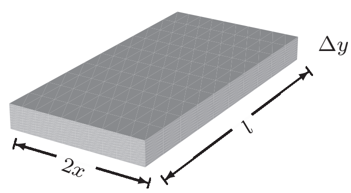


Figure 8.74

20. First we rewrite the chart, listing the density with the corresponding distance from the center of the earth (x km below the surface is equivalent to $6370 - x$ km from the center):

This gives us spherical shells whose volumes are $\frac{4}{3}\pi(r_i^3 - r_{i+1}^3)$ for any two consecutive distances from the origin. We will assume that the density of the earth is increasing with depth. Therefore, the average density of the i^{th} shell is between D_i and D_{i+1}, the densities at top and bottom of shell i. So $\frac{4}{3}\pi D_{i+1}(r_i^3 - r_{i+1}^3)$ and $\frac{4}{3}\pi D_i(r_i^3 - r_{i+1}^3)$ are upper and lower bounds for the mass of the shell.

Table 8.6

i	x_i	$r_i = 6370 - x_i$	D_i
0	0	6370	3.3
1	1000	5370	4.5
2	2000	4370	5.1
3	2900	3470	5.6
4	3000	3370	10.1
5	4000	2370	11.4
6	5000	1370	12.6
7	6000	370	13.0
8	6370	0	13.0

To get a rough approximation of the mass of the earth, we don't need to use all the data. Let's just use the densities at $x = 0$, 2900, 5000 and 6370 km. Calculating an upper bound on the mass,

$$M_U = \frac{4}{3}\pi[13.0(1370^3 - 0^3) + 12.6(3470^3 - 1370^3) + 5.6(6370^3 - 3470^3)] \cdot 10^{15} \approx 7.29 \times 10^{27} \text{ g}.$$

The factor of 10^{15} may appear unusual. Remember the radius is given in kilometers and the density is given in g/cm^3, so we must convert kilometers to centimeters: 1 km $= 10^5$ cm , so 1 km$^3 = 10^{15}$ cm^3.

The lower bound is

$$M_L = \frac{4}{3}\pi[12.6(1370^3 - 0^3) + 5.6(3470^3 - 1370^3) + 3.3(6370^3 - 3470^3)] \cdot 10^{15} \approx 4.05 \times 10^{27} \text{ g}.$$

Here, our upper bound is just under 2 times our lower bound.

Using all our data, we can find a more accurate estimate. The upper and lower bounds are

$$M_U = \frac{4}{3}\pi \sum_{i=0}^{7} D_{i+1}(r_i^3 - r_{i+1}^3) \cdot 10^{15} \text{ g}$$

and

$$M_L = \frac{4}{3}\pi \sum_{i=0}^{7} D_i(r_i^3 - r_{i+1}^3) \cdot 10^{15} \text{ g}.$$

We have

$$\begin{aligned} M_U &= \frac{4}{3}\pi[4.5(6370^3 - 5370^3) + 5.1(5370^3 - 4370^3) + 5.6(4370^3 - 3470^3) \\ &+ 10.1(3470^3 - 3370^3) + 11.4(3370^3 - 2370^3) + 12.6(2370^3 - 1370^3) \\ &+ 13.0(1370^3 - 370^3) + 13.0(370^3 - 0^3)] \cdot 10^{15} \text{ g} \\ &\approx 6.50 \times 10^{27} \text{ g} \end{aligned}$$

and

$$\begin{aligned} M_L &= \frac{4}{3}\pi[3.3(6370^3 - 5370^3) + 4.5(5370^3 - 4370^3) + 5.1(4370^3 - 3470^3) \\ &+ 5.6(3470^3 - 3370^3) + 10.1(3370^3 - 2370^3) + 11.4(2370^3 - 1370^3) \\ &+ 12.6(1370^3 - 370^3) + 13.0(370^3 - 0^3)] \cdot 10^{15} \text{ g} \\ &\approx 5.46 \times 10^{27} \text{ g}. \end{aligned}$$

21. We slice time into small intervals. Since t is given in seconds, we convert the minute to 60 seconds. We consider water loss over the time interval $0 \leq t \leq 60$. We also need to convert inches into feet since the velocity is given in ft/sec. Since 1 inch = $1/12$ foot, the square hole has area $1/144$ square feet. For water flowing through a hole with constant velocity v, the amount of water which has passed through in some time, Δt, can be pictured as the rectangular solid in Figure 8.75, which has volume

$$\text{Area} \cdot \text{Height} = \text{Area} \cdot \text{Velocity} \cdot \text{Time}.$$

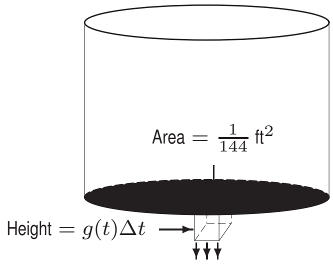


Figure 8.75: Volume of water passing through hole

Over a small time interval of length Δt, starting at time t, water flows with a nearly constant velocity $v = g(t)$ through a hole $1/144$ square feet in area. In Δt seconds, we know that

$$\text{Water lost } \approx \left(\frac{1}{144}\text{ ft}^2\right)(g(t)\text{ ft/sec})(\Delta t\text{ sec}) = \left(\frac{1}{144}\right) g(t)\,\Delta t\text{ ft}^3.$$

Adding the water from all subintervals gives

$$\text{Total water lost } \approx \sum \frac{1}{144} g(t)\,\Delta t\text{ ft}^3.$$

As $\Delta t \to 0$, the sum tends to the definite integral:

$$\text{Total water lost} = \int_0^{60} \frac{1}{144} g(t)\,dt\text{ ft}^3.$$

22. (a) Divide the atmosphere into spherical shells of thickness Δh. See Figure 8.76. The density on a typical shell, $\rho(h)$, is approximately constant. The volume of the shell is approximately the surface area of a sphere of radius $r_e + h$ meters times Δh, where $r_e = 6.4 \cdot 10^6$ meters is the radius of the earth,

$$\text{Volume of Shell } \approx 4\pi(r_e + h_i)^2 \Delta h.$$

A Riemann sum for the total mass is

$$\text{Mass } \approx \sum 4\pi(r_e + h)^2 \times 1.28e^{-0.000124h_i}\,\Delta h\text{ kg}.$$

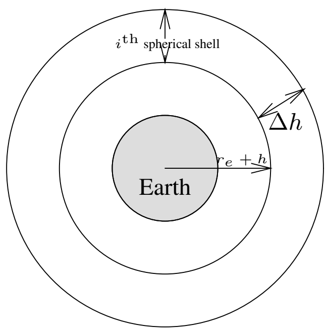


Figure 8.76

(b) This Riemann sum becomes the integral

$$\begin{aligned} \text{Mass } &= 4\pi \int_0^{100} (r_e + h)^2 \cdot 1.28e^{-0.000124h}\, dh \\ &= 4\pi \int_0^{100} (6.4 \cdot 10^6 + h)^2 \cdot 1.28e^{-0.000124h}\, dh. \end{aligned}$$

Evaluating the integral using numerical methods gives $M = 6.5 \cdot 10^{16}$ kg.

23. We need the numerator of $\overline{x}$, to be zero, i.e. $\sum x_i m_i = 0$. Since all of the masses are the same, we can factor them out and write $4\sum x_i = 0$. Thus the fourth mass needs to be placed so that all of the positions sum to zero. The first three positions sum to $(-6+1+3) = -2$, so the fourth mass needs to be placed at $x = 2$.

24. We have

$$\text{Total mass of the rod } = \int_0^3 (1+x^2)\, dx = \left[x + \frac{x^3}{3}\right]\Bigg|_0^3 = 12 \text{ grams.}$$

In addition,

$$\text{Moment } = \int_0^3 x(1+x^2)\, dx = \left[\frac{x^2}{2} + \frac{x^4}{4}\right]\Bigg|_0^3 = \frac{99}{4} \text{ gram-meters.}$$

Thus, the center of mass is at the position $\bar{x} = \frac{99/4}{12} = 2.06$ meters.

25. The center of mass is

$$\bar{x} = \frac{\int_0^\pi x(2+\sin x)\, dx}{\int_0^\pi (2+\sin x)\, dx}.$$

The numerator is $\int_0^\pi (2x + x\sin x)\, dx = (x^2 - x\cos x + \sin x)\Big|_0^\pi = \pi^2 + \pi$.

The denominator is $\int_0^\pi (2+\sin x)\, dx = (2x - \cos x)\Big|_0^\pi = 2\pi + 2$. So the center of mass is at

$$\overline{x} = \frac{\pi^2+\pi}{2\pi+2} = \frac{\pi(\pi+1)}{2(\pi+1)} = \frac{\pi}{2}.$$

26. (a) We find that

$$\text{Moment } = \int_0^1 x(1+kx^2)\, dx = \left(\frac{x^2}{2} + \frac{kx^4}{4}\right)\Bigg|_0^1 = \frac{1}{2} + \frac{k}{4} \text{ gram-meters,}$$

and that

$$\text{Total mass } = \int_0^1 (1+kx^2)\, dx = \left(x + \frac{kx^3}{3}\right)\Bigg|_0^1 = 1 + \frac{k}{3} \text{ grams.}$$

Thus, the center of mass is

$$\bar{x} = \frac{\frac{1}{2} + \frac{k}{4}}{1 + \frac{k}{3}} = \frac{3}{4}\left(\frac{2+k}{3+k}\right) \text{ meters.}$$

(b) Let $f(k) = \frac{3}{4}\left(\frac{2+k}{3+k}\right)$. Then $f'(k) = \frac{3}{4}\left(\frac{1}{(3+k)^2}\right)$, which is always positive, so f is an increasing function of k. Since $f(0) = 0.5$, this is the smallest value of f. As $k \to \infty$, $f(k) \to 3/4 = 0.75$. So $f(k)$ is always between 0.5 and 0.75.

27. (a) The density is minimum at $x = -1$ and increases as x increases, so more of the mass of the rod is in the right half of the rod. We thus expect the balancing point to be to the right of the origin.

(b) We need to compute

$$\begin{aligned} \int_{-1}^1 x(3-e^{-x})\, dx &= \left(\frac{3}{2}x^2 + xe^{-x} + e^{-x}\right)\Bigg|_{-1}^1 \quad \text{(using integration by parts)} \\ &= \frac{3}{2} + e^{-1} + e^{-1} - \left(\frac{3}{2} - e^1 + e^1\right) = \frac{2}{e}. \end{aligned}$$

We must divide this result by the total mass, which is given by

$$\int_{-1}^{1}(3-e^{-x})\,dx = (3x+e^{-x})\Big|_{-1}^{1} = 6-e+\frac{1}{e}.$$

We therefore have

$$\bar{x} = \frac{2/e}{6-e+(1/e)} = \frac{2}{1+6e-e^2} \approx 0.2.$$

28. Since the region is symmetric about the x-axis, $\bar{y}=0$.

To find $\bar{x}$, we first find the density. The area of the disk is $\pi/2$ m^2, so it has density $3/(\pi/2)=6/\pi$ kg/m^2. We find the mass of the small strip of width Δx in Figure 8.77. The height of the strip is $\sqrt{1-x^2}$, so

$$\text{Area of the small strip } \approx A_x(x)\Delta x = 2\cdot\sqrt{1-x^2}\Delta x \text{ m}^2.$$

When multiplied by the density $6/\pi$, we get

$$\text{Mass of the strip } \approx \frac{12}{\pi}\cdot\sqrt{1-x^2}\Delta x \text{ kg}.$$

We then sum the product of these masses with x, and take the limit as $\Delta x \to 0$ to get

$$\text{Moment } = \int_0^1 \frac{12}{\pi}x\sqrt{1-x^2}\,dx = -\frac{4}{\pi}(1-x^2)^{3/2}\Big|_0^1 = \frac{4}{\pi} \text{ meter}.$$

Finally, we divide by the total mass 3 kg to get the result $\bar{x}=4/(3\pi)$ meters.

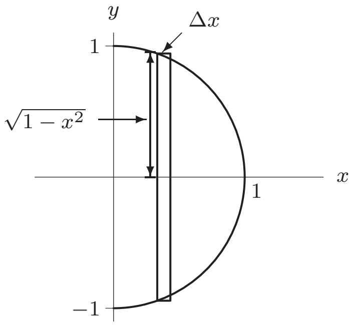


Figure 8.77: Area of a small strip

29. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

$$\text{Area of the plate } = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\text{ cm}^2.$$

Thus the mass of the plate is $2\cdot 1/3 = 2/3$ gm.

(b) See Figure 8.78. Since the region is "fatter" closer to $x=1$, $\bar{x}$ is greater than $1/2$.

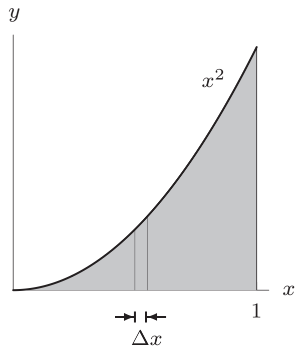


Figure 8.78

(c) To find the center of mass along the x-axis, we slice the region into vertical strips of width Δx. See Figure 8.78. Then

$$\text{Area of strip } = A_x(x)\Delta x \approx x^2\Delta x$$

Then, since the density is 2 gm/cm^2, we have

$$\overline{x} = \frac{\int_0^1 2x^3\,dx}{2/3} = \frac{3}{2}\cdot\left.\frac{2x^4}{4}\right|_0^1 = 3\left(\frac{1}{4}\right) = \frac{3}{4}\text{cm}.$$

This is greater than $1/2$, as predicted in part (b).

30. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

$$\text{Area of the plate } = \int_0^1 \sqrt{x}\,dx = \left.\frac{2}{3}x^{3/2}\right|_0^1 = \frac{2}{3}\text{ cm}^2.$$

Thus the mass of the plate is $5\cdot 2/3 = 10/3$ gm.

(b) To find $\bar{x}$, we slice the region into vertical strips of width Δx. See Figure 8.79. Then

$$\text{Area of strip } = A_x(x)\Delta x \approx \sqrt{x}\Delta x\text{ cm}^2.$$

Then, since the density is 5 gm/cm^2, we have

$$\overline{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} = \frac{\int_0^1 5x^{3/2}\,dx}{10/3} = \frac{3}{10}\left. 2x^{5/2}\right|_0^1 = \frac{3}{5}\text{ cm}.$$

To find $\bar{y}$, we slice the region into horizontal strips of width Δy

$$\text{Area of horizontal strip } = A_y(y)\Delta y \approx (1-x)\Delta y = (1-y^2)\Delta y\text{ cm}^2.$$

Then, since the density is 5 gm/cm^2, we have

$$\overline{y} = \frac{\int y\delta A_y(y)\,dy}{\text{Mass}} = \frac{\int_0^1 5(y-y^3)\,dy}{10/3} = \frac{3}{10}\,5\left.\left(\frac{y^2}{2}-\frac{y^4}{4}\right)\right|_0^1 = \frac{3}{10}\cdot\frac{5}{4} = \frac{3}{8}\text{ cm}.$$

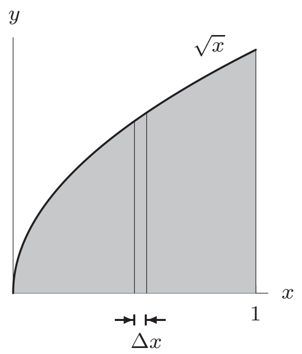


Figure 8.79

31. The triangle is symmetric about the x axis, so $\bar{y} = 0$.

To find $\bar{x}$, we first calculate the density. The area of the triangle is $ab/2$, so it has density $2m/(ab)$ where m is the total mass of the triangle. We need to find the mass of a small strip of width Δx located at x_i (see Figure 8.80).

$$\text{Area of the small strip } \approx A_x(x)\Delta x = 2\cdot\frac{b(a-x)}{2a}\Delta x.$$

Multiplying by the density $2m/(ab)$ gives

$$\text{Mass of the strip } \approx 2m\frac{(a-x)}{a^2}\Delta x.$$

We then sum the product of these masses with x_i, and take the limit as $\Delta x \to 0$ to get

$$\text{Moment } = \int_0^a \frac{2mx(a-x)}{a^2}\,dx = \frac{2m}{a^2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{2m}{a^2}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{ma}{3}.$$

Finally, we divide by the total mass m to get the desired result $\bar{x} = a/3$, which is independent of the length of the base b.

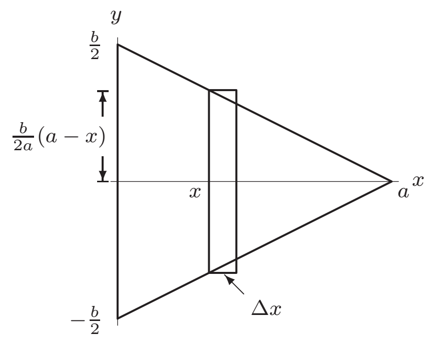


Figure 8.80

32. Stand the cone with the base horizontal, with center at the origin. Symmetry gives us that $\bar{x} = \bar{y} = 0$. Since the cone is fatter near its base we expect the center of mass to be nearer to the base.

Slice the cone into disks parallel to the xy-plane.

As we saw in Example 3 on page 416, a disk of thickness Δz at height z above the base has

$$\text{Volume of disk } = A_z(z)\Delta z \approx \pi(5-z)^2\Delta z \text{ cm}^3.$$

Thus, since the density is δ,

$$\overline{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}} = \frac{\int_0^5 z\cdot\delta\pi(5-z)^2\,dz}{\text{Mass}} \text{ cm}.$$

To evaluate the integral in the numerator, we factor out the constant density δ and π to get

$$\int_0^5 z\cdot\delta\pi(5-z)^2\,dz = \delta\pi\int_0^5 z(25-10z+z^2)\,dz = \delta\pi\left(\frac{25z^2}{2} - \frac{10z^3}{3} + \frac{z^4}{4}\right)\Bigg|_0^5 = \frac{625}{12}\delta\pi.$$

We divide this result by the total mass of the cone, which is $\left(\frac{1}{3}\pi 5^2\cdot 5\right)\delta$:

$$\overline{z} = \frac{\frac{625}{12}\delta\pi}{\frac{1}{3}\pi 5^3\delta} = \frac{5}{4} = 1.25\,\text{cm}.$$

As predicted, the center of mass is closer to the base of the cone than its top.

33. Since the density is constant, the total mass of the solid is the product of the volume of the solid and its density: $\delta\pi(1-e^{-2})/2$. By symmetry, $\bar{y} = 0$. To find $\bar{x}$, we slice the solid into disks of width Δx, perpendicular to the x-axis. See Figure 8.81. A disk at x has radius $y = e^{-x}$, so

$$\text{Volume of disk } = A_x(x)\Delta x = \pi y^2\Delta x = \pi e^{-2x}\Delta x.$$

Since the density is δ, we have

$$\overline{x} = \frac{\int_0^1 x\cdot\delta\pi e^{-2x}dx}{\text{Total mass}} = \frac{\delta\pi\int_0^1 xe^{-2x}dx}{\delta\pi(1-e^{-2})/2} = \frac{2}{1-e^{-2}}\int_0^1 xe^{-2x}\,dx.$$

The integral $\int xe^{-2x}\,dx$ can be done by parts: let $u = x$ and $v' = e^{-2x}$. Then $u' = 1$ and $v = e^{-2x}/(-2)$. So

$$\int xe^{-2x}\,dx = \frac{xe^{-2x}}{-2} - \int\frac{e^{-2x}}{-2}\,dx = \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}.$$

and then

$$\int_0^1 xe^{-2x}\,dx = \left(\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right)\Bigg|_0^1 = \left(\frac{e^{-2}}{-2} - \frac{e^{-2}}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{1-3e^{-2}}{4}.$$

The final result is:

$$\overline{x} = \frac{2}{1-e^{-2}} \cdot \frac{1-3e^{-2}}{4} = \frac{1-3e^{-2}}{2-2e^{-2}} \approx 0.343.$$

Notice that $\bar{x}$ is less that $1/2$, as we would expect from the fact that the solid is wider near the origin.

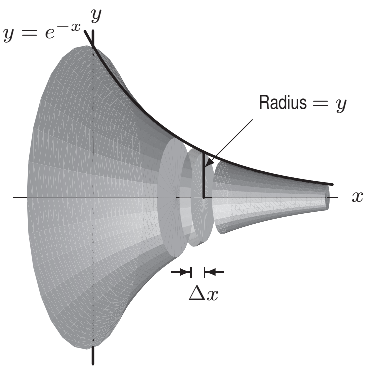


Figure 8.81

34. (a) Position the pyramid so that the center of its base lies at the origin on the xy-plane. Slice the pyramid into square slabs parallel to its base. We compute the mass of the pyramid by adding the masses of the slabs.

The mass of a slab is its volume multiplied by the density δ. To compute the volume of a slab, we need to get an expression for the side s of the slab in terms of its height z. Using the similar triangles in Figure 8.82, we see that

$$\frac{s}{40} = \frac{(10-z)}{10}.$$

Thus $s = 4(10-z)$. Since the area of the square slab's face is s^2,

$$\text{Volume of the slab } \approx A_z(z)\Delta z = s^2\Delta z = 16(10-z)^2\Delta z.$$

$$\text{Mass of slab } = 16\delta(10-z)^2\Delta z.$$

The mass of the pyramid can be found by summing all of the masses of the slabs, and letting the thickness Δz approach zero:

$$\text{Total mass } = \lim_{\Delta z\to 0}\sum 16\delta(10-z)^2\Delta z = \int_0^{10} 16\delta(10-z)^2\,dz = \frac{-16\delta(10-z)^3}{3}\Bigg|_0^{10} = \frac{16000\delta}{3}\text{gm.}$$

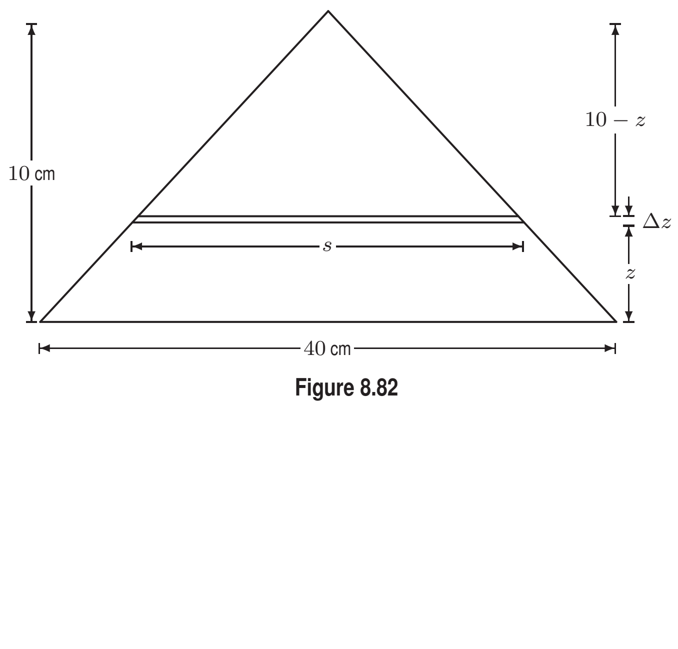


Figure 8.82

(b) From symmetry, we have $\bar{x} = \bar{y} = 0$. Since the pyramid is fatter near its base we expect the center of mass to be nearer to the base. Since

$$\text{Volume of slab } = A_z(z)\Delta z = 16(10-z)^2\Delta z,$$

$$\overline{z} = \frac{\int_0^{10} z \cdot 16\delta(10-z)^2\,dz}{\text{Total mass}}.$$

To evaluate the integral in the numerator, we factor out the constant 16δ and expand the integrand to get

$$16\delta \int_0^{10} (100z - 20z^2 + z^3)\,dz = 16\delta \left(50z^2 + \frac{-20z^3}{3} + \frac{z^4}{4}\right)\Bigg|_0^{10} = \frac{40000\delta}{3}.$$

We divide this result by the total mass $16000\delta/3$ of the pyramid

$$\overline{z} = \frac{40000\delta/3}{16000\delta/3} = \frac{40000}{16000} = 2.5\,\text{cm}.$$

As predicted, the center of mass is closer to the base of the pyramid than its top.

Strengthen Your Understanding

35. Mass density can never be negative, but $f(x) < 0$ for $0 < x < 5$.

36. The center of mass with density $\delta(x)$ for $0 \le x \le 10$ is given by

$$\text{Center of mass} = \frac{\int_0^{10} x\delta(x)\,dx}{\int_0^{10} \delta(x)\,dx}.$$

The correct formula for $\delta(x) = x^2$ is

$$\text{Center of mass} = \frac{\int_0^{10} x^3\,dx}{\int_0^{10} x^2\,dx}.$$

37. Any density function that is symmetric about the center of the rod will also give a center of mass in the center of the rod. For example, if a rod has ends at $x = -1$ and $x = 1$ and density $\delta(x) = x^2$ then its center of mass is at the origin.

38. When the density is constant, for example, if $\delta(r) = 3$ everywhere on the disk, the mass of the disk is the density times the area of the disk, or $3 \cdot 9\pi = 27\pi$ gm. Since the density is not constant and is less than 3 everywhere except at the center of the disk, the mass of the disk must be less than 27π gm.

39. Let the rod be the interval $0 \le x \le 10$. Suppose the density is given by

$$\delta(x) = \begin{cases} 5, & 0 \le x \le 1 \\ 1, & 1 < x < 5 \\ 4, & 5 \le x \le 10 \end{cases}.$$

The rod is denser at the end $x = 0$ than at the end $x = 10$.
We have

$$\int_0^{10} x\delta(x)\,dx = \int_0^1 5x\,dx + \int_1^5 x\,dx + \int_5^{10} 4x\,dx = 2.5 + 12 + 150 = 164.5$$
$$\int_0^{10} \delta(x)\,dx = \int_0^1 5\,dx + \int_1^5 1\,dx + \int_5^{10} 4\,dx = 5 + 4 + 20 = 29.$$

Therefore

$$\text{Center of Mass} = \frac{\int_0^{10} x\delta(x)\,dx}{\int_0^{10} \delta(x)\,dx} = \frac{164.5}{29} = 5.672.$$

The center of mass at $x = 5.672$ is nearer to the end $x = 10$ than to the end $x = 0$.

40. Any rod whose mass increases towards one end will skew the center of mass towards that end. As an example, if $\delta(x) = x$, then the total mass of the rod is

$$\text{Mass} = \int_0^2 x\,dx = 2$$

and the center of mass is

$$\text{Center of mass} = \frac{1}{2}\int_0^2 x \cdot \delta(x)\,dx = \frac{1}{2} \cdot \frac{x^3}{3}\bigg|_0^2 = \frac{4}{3}\text{ cm}$$

which is not at the center of the rod.

41. Taking a constant density $\delta(x) = 1$ gives the desired result, since then the center of mass is

$$\text{Center of mass} = \frac{\int_0^2 x \cdot \delta(x)\,dx}{\int_0^2 \delta(x)\,dx} = \frac{2}{2} = 1\text{ cm.}$$

42. False. The population density needs to be approximately constant on each ring. This is only true if the population density is a function of r, the distance from the center of the city.

43. False. Since the density varies with y, the region must be sliced perpendicular to the y-axis, along the lines of constant y.

44. False. Although the density is greater near the center, the area of the suburbs is much larger than the area of the inner city, and population is determined by both area and density. In fact, the population of the inner city:

$$\int_0^1 (10 - 3r)2\pi r\,dr = 2\pi(5r^2 - r^3)\bigg|_0^1 = 8\pi$$

is less than the population of the suburbs:

$$\int_1^2 (10 - 3r)2\pi r\,dr = 2\pi(5r^2 - r^3)\bigg|_1^2 = 16\pi.$$

45. True. One way to look at it is that the center of mass should not change if you change the units by which you measure the masses. If you double the masses, that is no different than using as a new unit of mass half the old unit. Alternatively, let the masses be m_1, m_2, and m_3 located at x_1, x_2, and x_3. Then the center of mass is given by:

$$\bar{x} = \frac{x_1m_1 + x_2m_2 + x_3m_3}{m_1 + m_2 + m_3}.$$

Doubling the masses does not change the center of mass, since it doubles both the numerator and the denominator.

46. False. The center of mass of a circular ring (for example, a coin with a hole in it) is at the center.

47. True. The density of particles hitting the target is approximately constant on concentric rings.

48. False. If the density were constant this would be true, but suppose that all the mass on the left half is concentrated at $x = 0$ and all the mass on the right side is concentrated at $x = 3$. In order for the rod to balance at $x = 2$, the weight on the left side must be half the weight on the right side.

Solutions for Section 8.5

Exercises

1. This is in British units, so the distance raised, d, is first converted to feet: $d = 0.75.$ feet.

$$\text{Work done } = F \cdot d = 40\text{ lb } \cdot 0.75\text{ ft } = 30\text{ ft-lb.}$$

2. This is in International units, so the distance raised, d, is first converted to meters: $d = 0.3$ m. The force due to gravity is $mg = (20\text{ kg})(g\text{ m/sec}^2)$, so

$$W = F \cdot d = [(20\text{ kg})(9.8\text{ m/sec}^2)] \cdot (0.3\text{ m}) = 58.8\text{ joules.}$$

3. Because the force is a function $F(x)$ of position x, then in moving from $x = a$ to $x = b$,

$$\text{Work done } = \int_a^b F(x)\,dx.$$

In this case

$$\text{Work done } = \int_0^1 x^2 + 2x\,dx = \frac{1}{3}x^3 + x^2\Big|_0^1 = 1.333 \text{ ft-lb.}$$

4. The work done is given by

$$W = \int_1^2 3x\,dx = \frac{3}{2}x^2\Big|_1^2 = \frac{9}{2} \text{ joules.}$$

5. The work done is given by

$$W = \int_0^3 3x\,dx = \frac{3}{2}x^2\Big|_0^3 = \frac{27}{2} \text{ joules.}$$

6. (a) For compression from $x = 0$ to $x = 1$,

$$\text{Work } = \int_0^1 3x\,dx = \frac{3}{2}x^2\Big|_0^1 = \frac{3}{2} = 1.5 \text{ joules.}$$

For compression from $x = 4$ to $x = 5$,

$$\text{Work } = \int_4^5 3x\,dx = \frac{3}{2}x^2\Big|_4^5 = \frac{3}{2}(25 - 16) = \frac{27}{2} = 13.5 \text{ joules.}$$

(b) The second answer is larger. Since the force increases with x, for a given displacement, the work done is larger for larger x values. Thus, we expect more work to be done in moving from $x = 4$ to $x = 5$ than from $x = 0$ to $x = 1$.

7. Since all parts of the plate are at the same depth, the force is constant on all parts of the plate. The force is the pressure, $62.4 \cdot 150$ lb/ft^2 multiplied by the area of the plate, 400π ft^2, so

$$\text{Force} = 1.176 \cdot 10^7 \text{pounds.}$$

8. Because the bottom of the fish tank is horizontal, the pressure is the same at every point on the bottom, so

$$\text{Pressure } = 62.4 \text{ lb/ft}^3 \cdot 1 \text{ ft } = 62.4 \text{ lb/ft}^2.$$

and

$$\text{Force } = \text{Pressure} \cdot \text{Area} = 62.4 \text{ lb/ft}^2 \cdot 2 \text{ ft} \cdot 1 \text{ ft} = 124.8 \text{ pounds.}$$

9. The only work is done by lifting the bucket initially, since the motion is parallel to the force of gravity, so the work is $2 \cdot 10 = 20$ ft-lb. When the child is walking and holding the bucket at a constant height, the force of gravity and the motion are at right angles, and the work done is zero. Thus, the total work done is 20 ft-lb.

10. Since the gravitational force is

$$F = \frac{4 \cdot 10^{14}}{r^2} \text{ newtons}$$

and r varies between $6.4 \cdot 10^6$ and $7.4 \cdot 10^6$ meters,

$$\begin{aligned}\text{Work done } &= \int_{6.4\cdot 10^6}^{7.4\cdot 10^6} \frac{4 \cdot 10^{14}}{r^2}\,dr = -4.10^{14}\frac{1}{r}\Big|_{6.4\cdot 10^6}^{7.4\cdot 10^6} \\ &= 4 \cdot 10^{14}\left(\frac{1}{6.4 \cdot 10^6} - \frac{1}{7.4 \cdot 10^6}\right) = 8.4 \cdot 10^6 \text{ joules.}\end{aligned}$$

Problems

11. The force exerted on the satellite by the earth (and vice versa!) is GMm/r^2, where r is the distance from the center of the earth to the center of the satellite, m is the mass of the satellite, M is the mass of the earth, and G is the gravitational constant. So the total work done is

$$\int_{6.4\cdot 10^6}^{8.4\cdot 10^6} F\,dr = \int_{6.4\cdot 10^6}^{8.4\cdot 10^6} \frac{GMm}{r^2}\,dr = \left(\frac{-GMm}{r}\right)\Bigg|_{6.4\cdot 10^6}^{8.4\cdot 10^6} \approx 1.489\cdot 10^{10} \text{ joules.}$$

12. Let x be the distance from ground to the bucket of cement. At height x, if the bucket is lifted by Δx, the work done is $500\Delta x + 0.5(75-x)\Delta x$. See Figure 8.83. The $500\Delta x$ term is due to the bucket of cement; the $0.5(75-x)\Delta x$ term is due to the remaining cable. So the total work, W, required to lift the bucket is

$$\begin{aligned} W &= \int_0^{30} 500dx + \int_0^{30} 0.5(75-x)dx \\ &= 500\cdot 30 + 0.5(75\cdot 30 - \frac{1}{2}30^2) \\ &= 15{,}900 \text{ ft-lb.} \end{aligned}$$

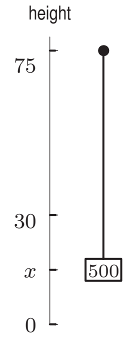


Figure 8.83

13. When the anchor has been lifted through h feet, the length of chain in the water is $25-h$ feet, so the total weight of the anchor and chain in the water is $50+3(25-h)$ lb. Then

$$\begin{aligned} \text{Work to lift the anchor and chain } \Delta h \text{ higher} &= \text{Weight}\cdot\text{Distance lifted} \\ &= (100+3(25-h))\Delta h. \end{aligned}$$

To find the total work, we integrate from $h=0$ to $h=25$:

$$W = \int_0^{25} (100+3(25-h))dh = \int_0^{25} (175-3h)dh = \left(175h - \frac{3h^2}{2}\right)\Bigg|_0^{25} = 3437.5\text{ft-lbs.}$$

14. To lift the weight an additional height Δh off the ground from a height of h, we must do work on the weight and the amount of rope not yet pulled onto the roof. Since the roof is 30 ft off the ground, there will be $30-h$ feet remaining of rope, for a weight of $4(30-h)$. So the work required to raise the weight and the rope a height Δh will be $\Delta h(1000+4(30-h))$. To find the total work, we integrate this quantity from $h=0$ to $h=10$:

$$\begin{aligned} \text{Work} &= \int_0^{10} (1000+4(30-h))\,dh \\ &= \int_0^{10} (1120-4h)\,dh \\ &= (1120h-2h^2)\Big|_0^{10} \\ &= 11{,}200-200 \\ &= 11{,}000 \text{ ft-lbs.} \end{aligned}$$

15. The bucket moves upward at $40/10 = 4$ meters/minute. If time is in minutes, at time t the bucket is at a height of $x = 4t$ meters above the ground. See Figure 8.84.

The water drips out at a rate of $5/10 = 0.5$ kg/minute. Initially there is 20 kg of water in the bucket, so at time t minutes, the mass of water remaining is

$$m = 20 - 0.5t \text{ kg.}$$

Consider the time interval between t and $t+\Delta t$. During this time the bucket moves a distance $\Delta x = 4\Delta t$ meters. So, during this interval,

$$\text{Work done} \approx mg\Delta x = (20 - 0.5t)g4\Delta t \text{ joules.}$$

$$\text{Total work done} = \lim_{\Delta t \to 0} \sum (20 - 0.5t)g4\Delta t = 4g\int_0^{10} (20 - 0.5t)\,dt$$
$$= 4g(20t - 0.25t^2)\Big|_0^{10} = 700g = 700(9.8) = 6860 \text{ joules.}$$

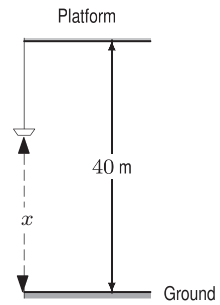


Figure 8.84

16. Let x be the height (in feet) from ground to the cube of ice. It follows that $0 \le x \le 100$. At height x, the ice cube w weighs $2000 - 4x$ since it's being lifted at a rate 1 ft./min. and it's melting at a rate of 4 lb/min. To lift it Δx more the work required is $(2000 - 4x)\Delta x$. So the total work done is

$$\begin{aligned} W &= \int_0^{100} (2000 - 4x)dx \\ &= \left(2000x - 2x^2\right)\Big|_0^{100} \\ &= 2000(100) - 2\cdot(100)^2 \\ &= 180{,}000 \quad \text{ft-lb.} \end{aligned}$$

17. Let y represent depth below the surface of the can. Slice the water in the can horizontally into cylinders of height Δy and radius 1. See Figure 8.85. To find the weight of such a slice, we multiply its volume by its weight per cubic foot: we obtain

$$\text{Weight} = \pi 1^2(\Delta y)(62.4) \approx 196.04\,\Delta y.$$

So the work required to lift a slice at depth y to the surface is $196.04y\,\Delta y$.

To find the total work required to empty the can, we integrate:

$$\text{Total Work} \approx \int_1^3 196.04y\,dy = 196.04\frac{y^2}{2}\Big|_1^3 \approx 784.14 \text{ ft-lb.}$$

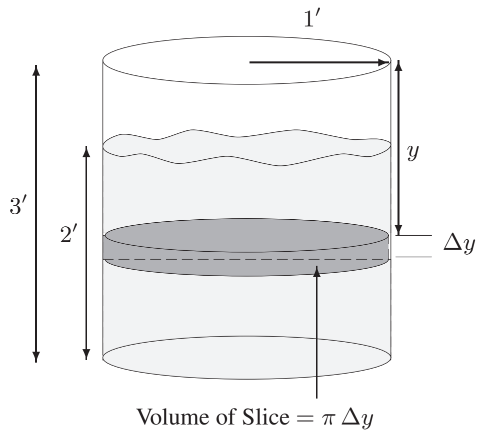


Figure 8.85

18. We slice the water horizontally and find the work required to pump each horizontal slice of water over the top. See Figure 8.86. At a distance h ft above the bottom, a slice of thickness Δh has

$$\text{Volume } \approx 50 \cdot 20\Delta h \text{ ft}^3.$$

Since the density of water is ρ lb/ft^3,

$$\text{Weight of the slice } \approx \rho(50 \cdot 20 \cdot \Delta h) \text{ lbs.}$$

The distance to lift the slice of water at height h ft is $10 - h$ ft, so

$$\begin{aligned}\text{Work to move one slice} &= \rho \cdot \text{Volume} \cdot \text{Distance lifted}\\ &\approx \rho(50 \cdot 20 \cdot \Delta h)(10-h)\\ &= 100\rho(10-h)\Delta h \text{ ft-lb.}\end{aligned}$$

The work done, W, to pump all the water is the sum of the work done on the pieces:

$$W \approx \sum 100\rho(10-h)\Delta h.$$

As $\Delta h \to 0$, we obtain a definite integral. Since h varies from $h = 0$ to $h = 9$ and $\rho = 62.4$ lb/ft^3, the total work is:

$$W = \int_0^9 100\rho(10-h)dh = 62400\left(10h - \frac{h^2}{2}\right)\Bigg|_0^9 = 62400(49.5) = 3{,}088{,}800.$$

The work to pump all the water out is $3{,}088{,}800$ ft-lbs.

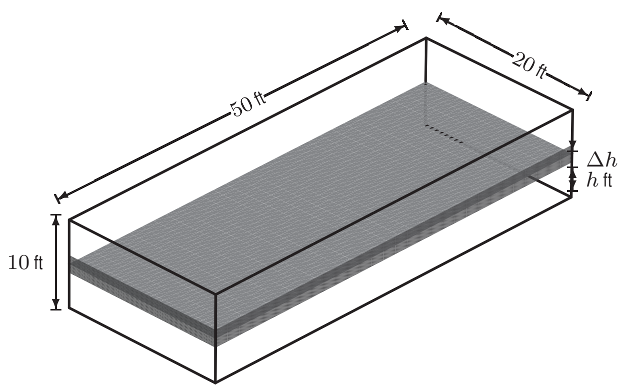


Figure 8.86

19. Let x be the distance measured from the bottom the tank. See Figure 8.87. To pump a layer of water of thickness Δx at x feet from the bottom, the work needed is

$$(62.4)\pi 6^2(20-x)\Delta x.$$

Therefore, the total work is

$$\begin{aligned} W &= \int_0^{10} 36\cdot(62.4)\pi(20-x)dx \\ &= 36\cdot(62.4)\pi(20x-\frac{1}{2}x^2)\Big|_0^{10} \\ &= 36\cdot(62.4)\pi(200-50) \\ &\approx 1{,}058{,}591.1 \text{ ft-lb.} \end{aligned}$$

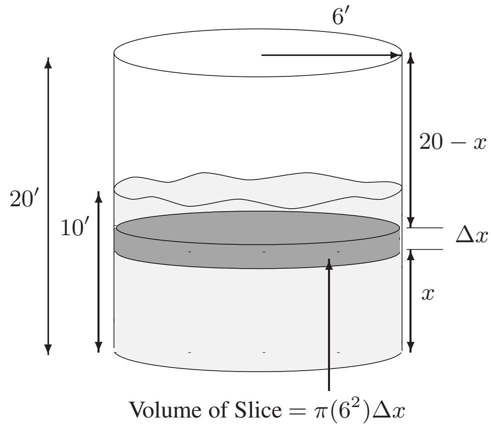


Figure 8.87

20. Let x be the distance from the bottom of the tank. See Figure 8.88. To pump a layer of water of thickness Δx at x feet from the bottom to 10 feet above the tank, the work done is $(62.4)\pi 6^2(30-x)\Delta x$. Thus the total work is

$$\begin{aligned} &\int_0^{20} 36\cdot(62.4)\pi(30-x)dx \\ &= 36\cdot(62.4)\pi\left(30x-\frac{1}{2}x^2\right)\Big|_0^{20} \\ &= 36\cdot(62.4)\pi(30(20)-\frac{1}{2}20^2) \\ &\approx 2{,}822{,}909.50 \text{ ft-lb.} \end{aligned}$$

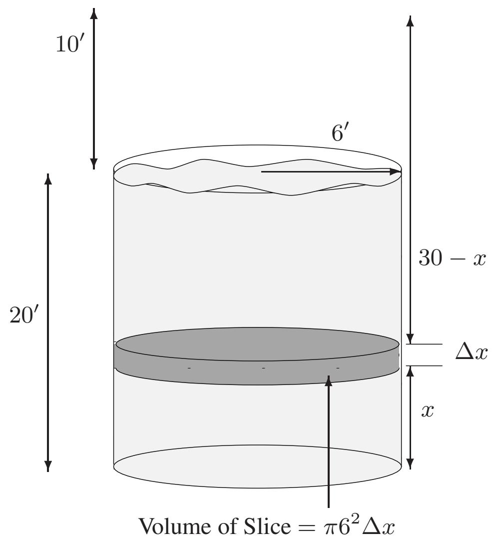


Figure 8.88

21. (a) We slice the water horizontally. Each slice is a cylindrical slab of radius 4 and thickness Δh, so

$$\text{Volume of each slab } \approx \pi 4^2\Delta h \text{ ft}^3.$$

See Figure 8.89. The density of water is δ lb/ft^3, so

$$\text{Weight of slab } \approx \delta\pi 4^2\Delta h \text{ lb.}$$

Water at a height of h ft must be lifted a distance of $10-h$ ft.

$$\begin{aligned} \text{Work to move one slice} &= \delta\cdot\text{Volume}\cdot\text{Distance lifted} \\ &\approx \delta(\pi(4)^2\Delta h)(10-h) \text{ ft-lb.} \end{aligned}$$

Since the density of water is $\delta = 62.4$ lb/ft^3 and since h varies from $h=0$ to $h=10$, the total work, W, is:

$$W = \int_0^{10}\delta(\pi 4^2)(10-h)dh = 16\delta\pi\int_0^{10}(10-h)dh = 998.4\pi\left(10h-\frac{h^2}{2}\right)\Big|_0^{10} = 156{,}828 \text{ ft-lb.}$$

The total work required is 156,828 ft-lbs.

(b) This is the same as part (a) except the water must be lifted a distance of $15 - h$ ft. The total work is:

$$W = \int_0^{10} \delta(\pi 4^2)(15-h)dh = 16\delta\pi \int_0^{10} (15-h)dh = 998.4\pi \left(15h - \frac{h^2}{2}\right)\Bigg|_0^{10} = 313{,}656 \text{ ft-lb.}$$

The total work required is 313,656 ft-lbs.

(c) This is the same as part (a) except that h varies from $h = 0$ to $h = 8$. The total work is:

$$W = \int_0^{8} \delta(\pi 4^2)(10-h)dh = 16\delta\pi \int_0^{8} (10-h)dh = 998.4\pi \left(10h - \frac{h^2}{2}\right)\Bigg|_0^{8} = 150{,}555.$$

The total work required is 150,555 ft-lbs.

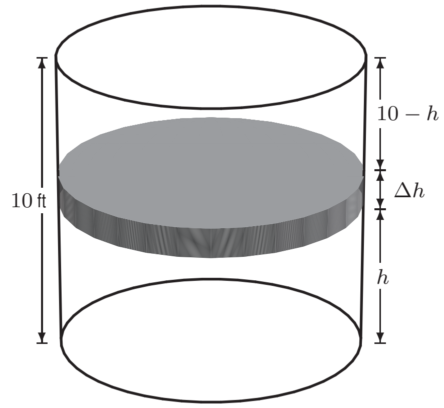


Figure 8.89

22.

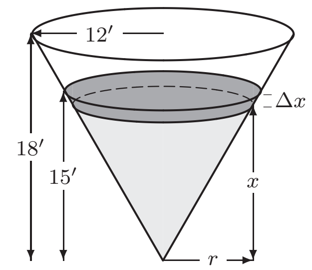


Figure 8.90

Let x be the depth of the water measured from the bottom of the tank. See Figure 8.90. It follows that $0 \le x \le 15$. Let r be the radius of the section of the cone with height x. By similar triangles, $\frac{r}{x} = \frac{12}{18}$, so $r = \frac{2}{3}x$. Then the work required to pump a layer of water with thickness of Δx at depth x over the top of the tank is $62.4\pi \left(\frac{2}{3}x\right)^2 \Delta x(18 - x)$. So the total work done by pumping the water over the top of the tank is

$$\begin{aligned} W &= \int_0^{15} 62.4\pi \left(\frac{2}{3}x\right)^2 (18-x)dx \\ &= \frac{4}{9}\, 62.4\pi \int_0^{15} x^2(18-x)dx \\ &= \frac{4}{9}\, 62.4\pi \left(6x^3 - \frac{1}{4}x^4\right)\Bigg|_0^{15} \\ &= \frac{4}{9}\, 62.4\pi(7593.75) \approx 661{,}619.41 \text{ ft-lb.} \end{aligned}$$

23. We slice the water horizontally as in Figure 8.91. We use similar triangles to find the radius r of the slice at height h in terms of h:

$$\frac{r}{h} = \frac{4}{12} \quad \text{so} \quad r = \frac{1}{3}h.$$

At height h,

$$\text{Volume of slice} \approx \pi r^2 \Delta h = \pi \left(\frac{1}{3}h\right)^2 \Delta h \text{ ft}^3.$$

The density of water is δ lb/ft^3, so

$$\text{Weight of slice } \approx \delta\pi \left(\frac{1}{3}h\right)^3 \Delta h \text{ lb}.$$

The water at height h must be lifted a distance of $12 - h$ ft, so

$$\begin{aligned}\text{Work to move slice} &= \delta \cdot \text{Volume} \cdot \text{Distance lifted}\\ &\approx \delta \left(\pi \left(\frac{1}{3}h\right)^2 \Delta h\right)(12-h) \text{ ft-lb}.\end{aligned}$$

The work done, W, to lift all the water is the sum of the work done on the pieces:

$$W \approx \sum \delta\pi(\frac{1}{3}h)^2 \Delta h(12-h) \text{ ft-lb}.$$

As $\Delta h \to 0$, we obtain a definite integral. Since h varies from $h = 0$ to $h = 9$, and $\delta = 62.4$, we have:

$$W = \int_0^9 \delta\pi(\frac{1}{3}h)^2(12-h)dh = \frac{62.4\pi}{9}\int_0^9 (12h^2 - h^3)dh = \frac{62.4\pi}{9}\left(4h^3 - \frac{h^4}{4}\right)\Bigg|_0^9 = 27{,}788 \text{ ft-lb}.$$

The work to pump all the water out is 27,788 ft-lbs.

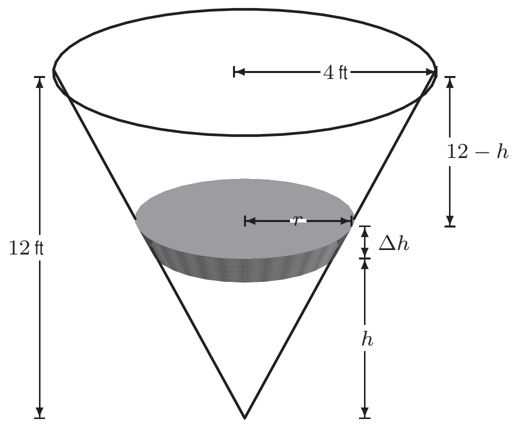


Figure 8.91

24. Let h represent distance below the surface in feet. We slice the tank up into horizontal slabs of thickness Δh. From looking at Figure 8.92, we can see that the slabs will be rectangular. The length of any slab is 12 feet. The width w of a slab h units below the ground will equal $2x$, where $(14-h)^2 + x^2 = 16$, so $w = 2\sqrt{4^2 - (14-h)^2}$. The volume of such a slab is therefore $12w\,\Delta h = 24\sqrt{16-(14-h)^2}\,\Delta h$ cubic feet; the slab weighs $42 \cdot 24\sqrt{16-(14-h)^2}\,\Delta h = 1008\sqrt{16-(14-h)^2}\,\Delta h$ pounds. So the total work done in pumping out all the gasoline is

$$\int_{10}^{18} 1008h\sqrt{16-(14-h)^2}\,dh = 1008\int_{10}^{18} h\sqrt{16-(14-h)^2}\,dh.$$

Substitute $s = 14 - h$, $ds = -dh$. We get

$$1008\int_{10}^{18} h\sqrt{16-(14-h)^2}\,dh = -1008\int_4^{-4} (14-s)\sqrt{16-s^2}\,ds$$

$$= 1008 \cdot 14 \int_{-4}^{4} \sqrt{16 - s^2}\, ds - 1008 \int_{-4}^{4} s\sqrt{16 - s^2}\, ds.$$

The first integral represents the area of a semicircle of radius 4, which is 8π. The second is the integral of an odd function, over the interval $-4 \le s \le 4$, and is therefore 0. Hence, the total work is $1008 \cdot 14 \cdot 8\pi \approx 354{,}673$ foot-pounds.

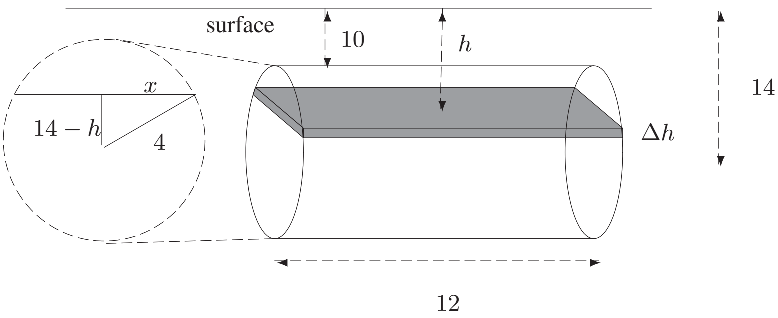


Figure 8.92

25. Divide the muddy water into horizontal slabs of thickness Δh. See Figure 8.93. Then for a typical slab

$$\text{Volume of slab} = \pi(0.5)^2 \Delta h \text{ m}^3$$
$$\text{Mass of slab} \approx \delta(h)\pi(0.5)^2 \Delta h = 0.25\pi(1 + kh)\Delta h \text{ kg}$$

The water in this slab is moved a distance of $h + 0.3$ meters to the rim of the barrel. Now

$$\text{Work done} = \text{Mass} \cdot g \cdot \text{Distance moved},$$

and work is measured in newtons if mass is in kilograms and distance is in meters, so

$$\text{Work done in moving slab} \approx 0.25\pi(1 + kh)g(h + 0.3)\Delta h \text{ joules}.$$

Since the slices run from $h = 0$ to $h = 1.5$, we have

$$\text{Total work done} = \int_0^{1.5} 0.25\pi(1 + kh)g(h + 0.3)\, dh$$
$$= 0.366(k + 1.077)g\pi \text{ joules}$$

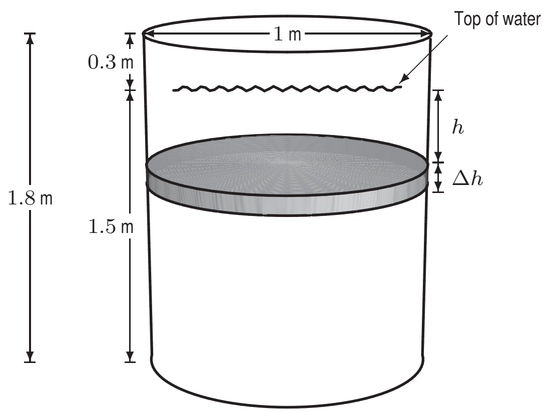


Figure 8.93

26. (a) We divide the triangular end into horizontal strips of thickness Δh. The length of a strip, s, depends on its height h from the bottom. See Figure 8.94. We use similar triangles to see that

$$\frac{s}{h} = \frac{2}{3} \quad \text{so} \quad s = \frac{2}{3}h.$$

Since each strip is approximately a rectangle, at height h,

$$\text{Area of strip } \approx \frac{2}{3}h\Delta h \text{ ft}^2.$$

Since the depth at height h is $3 - h$, writing δ for the density of water, we have:

$$\begin{aligned}\text{Force on one strip} &= \delta \cdot \text{Depth} \cdot \text{Area}\\ &\approx \delta(3-h)\left(\frac{2}{3}h\Delta h\right) \text{ lb.}\end{aligned}$$

To find the total force, F, we integrate the force on a strip from $h = 0$ to $h = 3$, using $\delta = 62.4$ lb/ft^3:

$$F = \int_0^3 \delta(3-h)\frac{2}{3}h dh = \frac{2}{3}\delta\int_0^3 (3h - h^2)dh = \frac{2}{3}62.4\left(\frac{3h^2}{2} - \frac{h^3}{3}\right)\Bigg|_0^3 = 187.2 \text{ lbs.}$$

(b) To find the work, we slice the water horizontally. Each slice is a rectangular slab with thickness Δh, length 15 ft, and width s as in Figure 8.95. As we saw in part (a), at a height of h we have $s = \frac{2}{3}h$. At height h,

$$\text{Volume of slab } \approx 15\left(\frac{2}{3}h\right)\Delta h \text{ ft}^3.$$

The distance to lift the slice at height h is $3 - h$, so if δ is the density of water, we have:

$$\begin{aligned}\text{Work to lift one slice} &= \delta \cdot \text{Volume} \cdot \text{Distance lifted}\\ &\approx \delta(15(\frac{2}{3}h)\Delta h)(3-h) \text{ ft-lb.}\end{aligned}$$

To find the total work, W, we integrate the work to lift a slice from $h = 0$ to $h = 3$, using $\delta = 62.4$ lb/ft^3.

$$W = \int_0^3 \delta 15\left(\frac{2}{3}h\right)(3-h)\,dh = 10\delta\int_0^3 (3h - h^2)dh = 10 \cdot 62.4\left(\frac{3h^2}{2} - \frac{h^3}{3}\right)\Bigg|_0^3 = 10\delta(4.5) = 2808 \text{ ft-lbs.}$$

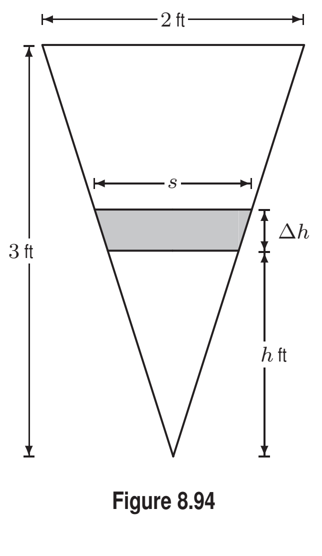


Figure 8.94

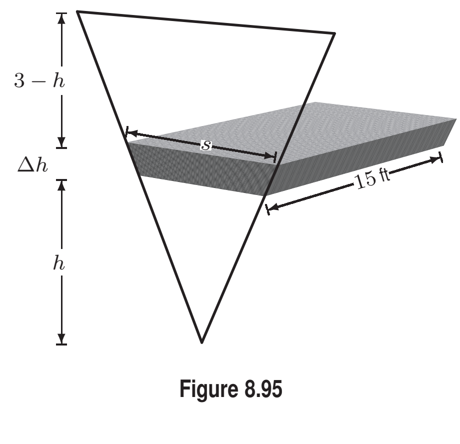


Figure 8.95

27. (a) Divide the wall into horizontal strips, each of height Δh. See Figure 8.96. The area of each strip is $1000\Delta h$, and the pressure at depth h is $62.4h$, so

$$\text{Force on strip } \approx 1000(62.4h)\Delta h$$
$$\text{Force on dam } \approx \sum 1000(62.4h)\Delta h.$$

(b) The force on the dam is given by the integral

$$\text{Force on dam } = \int_0^{50} 1000(62.4h)\,dh = 62400\frac{h^2}{2}\bigg|_0^{50} = 78{,}000{,}000 \text{ pounds.}$$

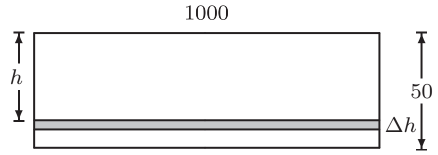


Figure 8.96

28. See Figure 8.97.

For the bottom: The bottom of the tank is at constant depth 15 feet, and therefore is under constant pressure, $15 \cdot 62.4 = 936$ lb/ft^2. The area of the base is 200 ft^2, so

$$\text{Total force on bottom } = 200 \text{ ft}^2 \cdot 936 \text{ lb/ft}^2 = 187200 \text{ lb.}$$

For the 15×10 side: The area of a horizontal strip of width dh is $10\,dh$ square feet, and the pressure at height h is $62.4h$ pounds per square foot. Therefore, the force on such a strip is $62.4h(10\,dh)$ pounds. Hence,

$$\text{Total force on the } 15 \times 10 \text{ side } = \int_0^{15} (62.4h)(10)\,dh = 624\frac{h^2}{2}\bigg|_0^{15} = 70200 \text{ lbs.}$$

For the 15×20 side: Similarly,

$$\text{Total force on the } 15 \times 20 \text{ side } = \int_0^{15} (62.4h)(20)\,dh = 1248\frac{h^2}{2}\bigg|_0^{15} = 140400 \text{ lbs.}$$

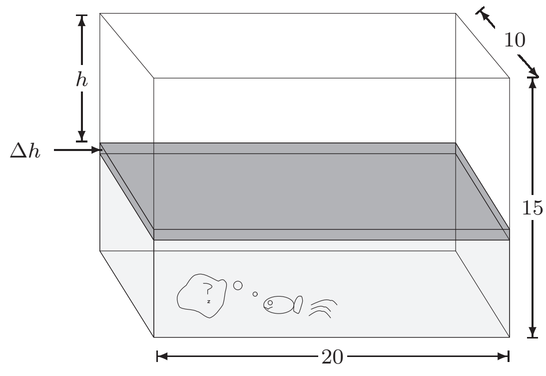


Figure 8.97

29. We divide the water against the dam into horizontal strips, each of thickness Δh and length 100.

$$\text{Area of each strip } \approx 100\Delta h \text{ ft}^2.$$

See Figure 8.98. The strip at height h ft from the bottom is at a water depth of $40-h$, so, if δ lb/ft^3 is the density of water, we have:

$$\begin{aligned}\text{Force of one strip} &= \delta \cdot \text{Depth} \cdot \text{Area}\\ &\approx \delta(40-h)(100\Delta h) \text{ lb.}\end{aligned}$$

To find the total force, F, we integrate the force on a strip from $h=0$ to $h=40$, using $\delta = 62.4$ lb/ft^3:

$$F = \int_0^{40} \delta(40-h)100dh = 100\cdot 62.4\left(40h - \frac{h^2}{2}\right)\Bigg|_0^{40} = 6240(800) = 4{,}992{,}000\text{lbs.}$$

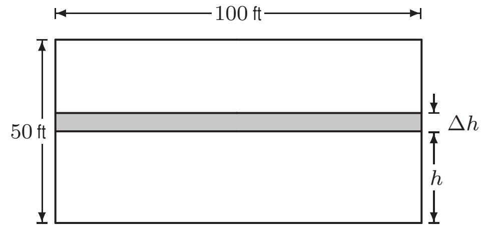


Figure 8.98

30. Bottom:

$$\text{Water force } = 62.4(2)(12) = 1497.6 \text{ lbs.}$$

Front and back:

$$\begin{aligned}\text{Water force } &= (62.4)(4)\int_0^2 (2-x)\,dx = (62.4)(4)(2x - \frac{1}{2}x^2)\Bigg|_0^2\\ &= (62.4)(4)(2) = 499.2 \text{ lbs.}\end{aligned}$$

Both sides:

$$\text{Water force} = (62.4)(3)\int_0^2 (2-x)\,dx = (62.4)(3)(2) = 374.4 \text{ lbs.}$$

31. (a) Since the density of water is $\delta = 1000$ kg/m^3, at the base of the dam, water pressure $\delta g h = 1000\cdot 9.8\cdot 180 = 1.76\cdot 10^6$ nt/m^2.

(b) To set up a definite integral giving the force, we divide the dam into horizontal strips. We use horizontal strips because the pressure along each strip is approximately constant, since each part is at approximately the same depth. See Figure 8.99.

$$\text{Area of strip } = 2000\Delta h \text{ m}^2.$$

Pressure at depth of h meters $= \delta g h = 9800h$ nt/m^2. Thus,

$$\text{Force on strip } \approx \text{ Pressure } \times \text{ Area } = 9800h\cdot 2000\Delta h = 1.96\cdot 10^7 h\Delta h \text{ nt.}$$

Summing over all strips and letting $\Delta h \to 0$ gives:

$$\text{Total force } = \lim_{\Delta h\to 0}\sum 1.96\cdot 10^7 h\Delta h = 1.96\cdot 10^7\int_0^{180} h\,dh \text{ newtons.}$$

Evaluating gives

$$\text{Total force } = 1.96\cdot 10^7 \frac{h^2}{2}\Bigg|_0^{180} = 3.2\cdot 10^{11} \text{ newtons.}$$

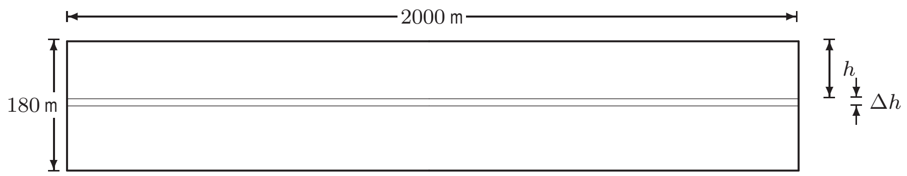


Figure 8.99

32. (a) At a depth of 350 feet,

$$\text{Pressure } = 62.4 \cdot 350 = 21{,}840 \text{ lb/ft}^2.$$

To imagine this pressure, we convert to pounds per square inch, giving a pressure of $21{,}840/144 = 151.7$ lb/in^2.

(b) (i) When the square is held horizontally, the pressure is constant at $21{,}840$ lbs/ft^2, so

$$\text{Force } = \text{ Pressure } \cdot \text{ Area } = 21{,}840 \cdot 5^2 = 546{,}000 \text{ pounds.}$$

(ii) When the square is held vertically, only the bottom is at 350 feet. Dividing into horizontal strips, as in Figure 8.100, we have

$$\text{Area of strip } = 5\Delta h \text{ ft}^2.$$

Since the pressure on a strip at a depth of h feet is $62.4h$ lb/ft^2,

$$\text{Force on strip } \approx 62.4h \cdot 5\Delta h = 312h\Delta h \text{ pounds.}$$

Summing over all strips and taking the limit as $\Delta h \to 0$ gives a definite integral. The strips vary between a depth of 350 feet and 345 feet, so

$$\text{Total force } = \lim_{\Delta h \to 0} \sum 312h\Delta h = \int_{345}^{350} 312h\, dh \text{ pounds.}$$

Evaluating gives

$$\text{Total force } = 312\frac{h^2}{2}\bigg|_{345}^{350} = 156(350^2 - 345^2) = 542{,}100 \text{ pounds.}$$

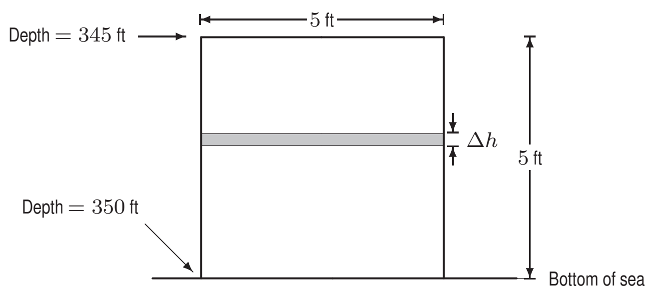


Figure 8.100

33. (a) Since water has density 62.4 lb/ft^3, at a depth of $12{,}500$ feet,

$$\text{Pressure } = \text{ Density } \times \text{ Depth } = 62.4 \cdot 12{,}500 = 780{,}000 \text{ lb/square foot.}$$

To imagine this pressure, observe that it is equivalent to $780{,}000/144 \approx 5400$ pounds per square inch.

(b) To calculate the pressure on the porthole (window), we slice it into horizontal strips, as the pressure remains approximately constant along each one. See Figure 8.101. Since each strip is approximately rectangular

$$\text{Area of strip } \approx 2r\Delta h \text{ ft}^2.$$

To calculate r in terms of h, we use the Pythagorean Theorem:

$$\begin{aligned} r^2 + h^2 &= 9 \\ r &= \sqrt{9 - h^2}, \end{aligned}$$

so

$$\text{Area of strip } \approx 2\sqrt{9 - h^2}\Delta h \text{ ft}^2.$$

The center of the porthole is at a depth of $12{,}500$ feet below the surface, so the strip shown in Figure 8.101 is at a depth of $(12{,}500 - h)$ feet. Thus, pressure on the strip is $62.4(12{,}500 - h)$ lb/ft^2, so

$$\begin{aligned} \text{Force on strip } = \text{ Pressure } \times \text{ Area } &\approx 62.4(12{,}500 - h)2\sqrt{9 - h^2}\Delta h \text{ lb} \\ &= 124.8(12{,}500 - h)\sqrt{9 - h^2}\Delta h \text{ lb.} \end{aligned}$$

To get the total force, we sum over all strips and take the limit as $\Delta h \to 0$. Since h ranges from -3 to 3, we get the integral

$$\begin{aligned}\text{Total force } &= \lim_{\Delta h\to 0}\sum 124.8(12{,}500-h)\sqrt{9-h^2}\Delta h\\ &= 124.8\int_{-3}^{3}(12{,}500-h)\sqrt{9-h^2}\,dh \text{ lb.}\end{aligned}$$

Evaluating the integral numerically, we obtain a total force of $2.2\cdot 10^7$ pounds.

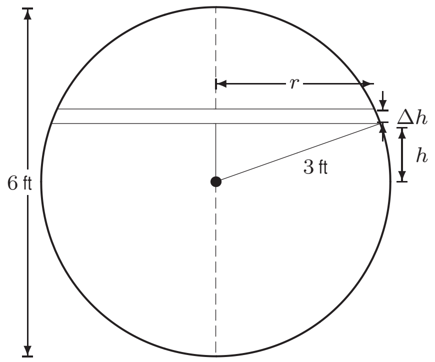


Figure 8.101: Center of circle is 12,500 ft below the surface of ocean

34. We divide the dam into horizontal strips since the pressure is then approximately constant on each one. See Figure 8.102.

$$\text{Area of strip } \approx w\Delta h \text{ m}^2.$$

Since w is a linear function of h, and $w = 3600$ when $h = 0$, and $w = 3000$ when $h = 100$, the function has slope $(3000 - 3600)/100 = -6$. Thus,

$$w = 3600 - 6h,$$

so

$$\text{Area of strip } \approx (3600-6h)\Delta h \text{ m}^2.$$

The density of water is $\delta = 1000$ kg/m^3, so the pressure at depth h meters $= \delta gh = 1000\cdot 9.8h = 9800h$ nt/m^2. Thus,

$$\text{Total force } = \lim_{\Delta h\to 0}\sum 9800h(3600-6h)\Delta h = 9800\int_0^{100} h(3600-6h)\,dh \text{ newtons.}$$

Evaluating the integral gives

$$\text{Total force } = 9800(1800h^2 - 2h^3)\Big|_0^{100} = 1.6\cdot 10^{11} \text{ newtons.}$$

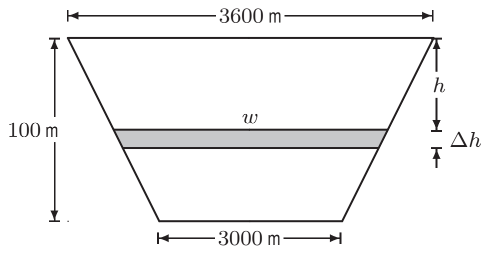


Figure 8.102

35. We need to divide the disk up into circular rings of charge and integrate their contributions to the potential (at P) from 0 to a. These rings, however, are not uniformly distant from the point P. A ring of radius z is $\sqrt{R^2+z^2}$ away from point P (see Figure 8.103).

The ring has area $2\pi z\,\Delta z$, and charge $2\pi z\sigma\,\Delta z$. The potential of the ring is then $\dfrac{2\pi z\sigma\,\Delta z}{\sqrt{R^2+z^2}}$ and the total potential at point P is

$$\int_0^a \frac{2\pi z\sigma\,dz}{\sqrt{R^2+z^2}} = \pi\sigma\int_0^a \frac{2z\,dz}{\sqrt{R^2+z^2}}.$$

We make the substitution $u=z^2$. Then $du=2z\,dz$. We obtain

$$\pi\sigma\int_0^a \frac{2z\,dz}{\sqrt{R^2+z^2}} = \pi\sigma\int_0^{a^2}\frac{du}{\sqrt{R^2+u}} = \pi\sigma(2\sqrt{R^2+u})\Big|_0^{a^2}$$
$$= \pi\sigma(2\sqrt{R^2+z^2})\Big|_0^a = 2\pi\sigma(\sqrt{R^2+a^2}-R).$$

(The substitution $u=R^2+z^2$ or $\sqrt{R^2+z^2}$ works also.)

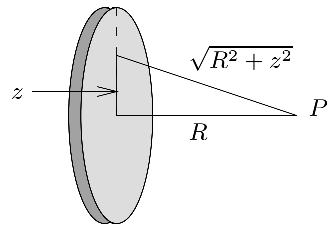


Figure 8.103

36. The density of the rod is $10\text{ kg}/6\text{ m} = \frac{5}{3}\frac{\text{kg}}{\text{m}}$. A little piece, dx m, of the rod thus has mass $5/3\,dx$ kg. If this piece has an angular velocity of 2 rad/sec, then its actual velocity is $2|x|$ m/sec. This is because a radian angle sweeps out an arc length equal to the radius of the circle, and in this case the little piece moves in circles about the origin of radius $|x|$. See Figure 8.104. The kinetic energy of the little piece is $mv^2/2 = (5/3\,dx)(2|x|)^2/2 = \frac{10}{3}x^2\,dx$.

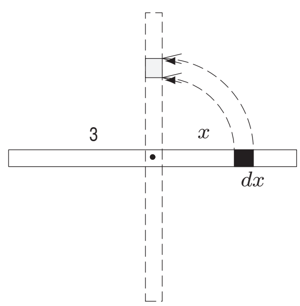


Figure 8.104

Therefore,

$$\text{Total Kinetic Energy} = \int_{-3}^{3}\frac{10x^2}{3}\,dx = \frac{20}{3}\left[\frac{x^3}{3}\right]\Big|_0^3 = 60\text{ kg}\cdot\text{m}^2/\text{sec}^2 = 60\text{ joules}.$$

37. We slice the record into rings in such a way that every point has approximately the same speed: use concentric circles around the hole. See Figure 8.105. We assume the record is a flat disk of uniform density: since its mass is 50 grams and its area is $\pi(10\text{cm})^2 = 100\pi\ \text{cm}^2$, the record has density $\frac{50}{100\pi} = \frac{1}{2\pi}\frac{\text{gram}}{\text{cm}^2}$. So a ring of width dr, having area about $2\pi r\, dr\ \text{cm}^2$, has mass approximately $(2\pi r\, dr)(1/2\pi) = r\, dr$ gm. At radius r, the velocity of the ring is

$$33\tfrac{1}{3}\,\frac{\text{rev}}{\text{min}}\cdot\frac{1\ \text{min}}{60\ \text{sec}}\cdot\frac{2\pi r\ \text{cm}}{1\ \text{rev}} = \frac{10\pi r}{9}\,\frac{\text{cm}}{\text{sec}}.$$

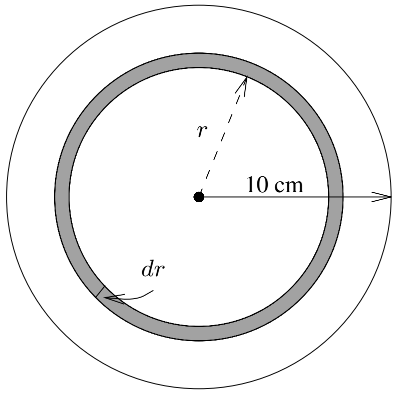


Figure 8.105

The kinetic energy of the ring is

$$\frac{1}{2}mv^2 = \frac{1}{2}(r\, dr\ \text{grams})\left(\frac{10\pi r}{9}\,\frac{\text{cm}}{\text{sec}}\right)^2 = \frac{50\pi^2 r^3\, dr}{81}\,\frac{\text{gram}\cdot\text{cm}^2}{\text{sec}^2}.$$

So the kinetic energy of the record, summing the energies of all these rings, is

$$\int_0^{10}\frac{50\pi^2 r^3\, dr}{81} = \left.\frac{25\pi^2 r^4}{162}\right|_0^{10} \approx 15231\,\frac{\text{gram}\cdot\text{cm}^2}{\text{sec}^2} = 15231\ \text{ergs}.$$

38. The density of the rod, in mass per unit length, is M/l (see Figure 8.106). So a slice of size dr has mass $\frac{M\, dr}{l}$. It pulls the small mass m with force $Gm\frac{M\, dr}{l}/r^2 = \frac{GmM\, dr}{lr^2}$. So the total gravitational attraction between the rod and point is

$$\begin{aligned}\int_a^{a+l}\frac{GmM\, dr}{lr^2} &= \frac{GmM}{l}\left(-\frac{1}{r}\right)\Bigg|_a^{a+l}\\ &= \frac{GmM}{l}\left(\frac{1}{a}-\frac{1}{a+l}\right)\\ &= \frac{GmM}{l}\,\frac{l}{a(a+l)} = \frac{GmM}{a(a+l)}.\end{aligned}$$

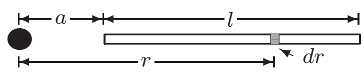


Figure 8.106

39. This time, let's split the second rod into small slices of length dr. See Figure 8.107. Each slice is of mass $\frac{M_2}{l_2}\, dr$, since the density of the second rod is $\frac{M_2}{l_2}$. Since the slice is small, we can treat it as a particle at distance r away from the end of the first rod, as in Problem 38. By that problem, the force of attraction between the first rod and particle is

$$\frac{GM_1\frac{M_2}{l_2}\, dr}{(r)(r+l_1)}.$$

So the total force of attraction between the rods is

$$\begin{aligned}\int_a^{a+l_2}\frac{GM_1\frac{M_2}{l_2}\, dr}{(r)(r+l_1)} &= \frac{GM_1M_2}{l_2}\int_a^{a+l_2}\frac{dr}{(r)(r+l_1)}\\ &= \frac{GM_1M_2}{l_2}\int_a^{a+l_2}\frac{1}{l_1}\left(\frac{1}{r}-\frac{1}{r+l_1}\right)dr.\end{aligned}$$

$$= \frac{GM_1M_2}{l_1l_2}(\ln|r| - \ln|r+l_1|)\bigg|_a^{a+l_2}$$
$$= \frac{GM_1M_2}{l_1l_2}[\ln|a+l_2| - \ln|a+l_1+l_2| - \ln|a| + \ln|a+l_1|]$$
$$= \frac{GM_1M_2}{l_1l_2}\ln\left[\frac{(a+l_1)(a+l_2)}{a(a+l_1+l_2)}\right].$$

This result is symmetric: if you switch l_1 and l_2 or M_1 and M_2, you get the same answer. That means it's not important which rod is "first," and which is "second."

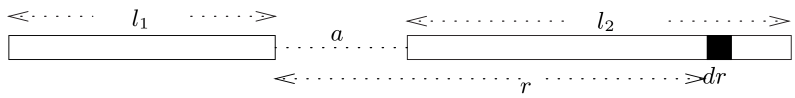


Figure 8.107

40. In Figure 8.108, consider a small piece of the ring of length Δl and mass

$$\Delta M = \frac{\Delta l M}{2\pi a}.$$

The gravitational force exerted by the small piece of the ring is along the line QP. As we sum over all pieces of the ring, the components perpendicular to the line OP cancel. The components of the force toward the point O are all in the same direction, so the net force is in this direction. The small piece of length Δl and mass $\Delta l M/2\pi a$ is at a distance of $\sqrt{a^2+y^2}$ from P, so

$$\text{Gravitational force from small piece} = \Delta F = \frac{G\frac{\Delta l M}{2\pi a}m}{(\sqrt{a^2+y^2})^2} = \frac{GMm\Delta l}{2\pi a(a^2+y^2)}.$$

Thus the force toward O exerted by the small piece is given by

$$\Delta F\cos\theta = \Delta F\frac{y}{\sqrt{a^2+y^2}} = \frac{GMm\Delta l}{2\pi a(a^2+y^2)}\frac{y}{\sqrt{a^2+y^2}} = \frac{GMmy\Delta l}{2\pi a(a^2+y^2)^{3/2}}.$$

The total force toward O is given by $F \approx \sum \Delta F\cos\theta$, so

$$F = \frac{GMmy\cdot\text{Total length}}{2\pi a(a^2+y^2)^{3/2}} = \frac{GMmy2\pi a}{2\pi a(a^2+y^2)^{3/2}} = \frac{GMmy}{(a^2+y^2)^{3/2}}.$$

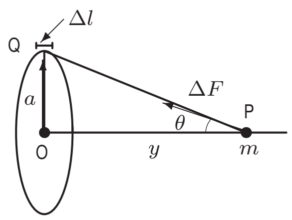


Figure 8.108

41. Divide the disk into rings of radius r, width Δr, as shown in Figure 8.109.
Then

$$\text{Area of ring} \approx 2\pi r\Delta r.$$

Since total area of disk is πa^2,

$$\text{Mass of ring} \approx \frac{2\pi r\Delta r}{\pi a^2}M = \frac{2rM}{a^2}\Delta r.$$

Thus, calculating the gravitational force due to the ring, we have

$$\begin{array}{c}\text{Gravitational force}\\ \text{on } m \text{ due to ring}\end{array} = G\left(\frac{2rM}{a^2}\Delta r\right)\frac{my}{(r^2+y^2)^{3/2}} = \frac{2GMmyr}{a^2(r^2+y^2)^{3/2}}\Delta r.$$

Summing over all rings, we get

$$\begin{array}{c}\text{Total gravitational force}\\ \text{on } m \text{ due to disk}\end{array} \approx \sum \frac{2GMmyr}{a^2(r^2+y^2)^{3/2}}\Delta r.$$

As $\Delta r \to 0$, we get

$$\begin{aligned}\begin{array}{c}\text{Gravitational force}\\ \text{on } m \text{ due to disk}\end{array} &= \int_0^a \frac{2GMmyr}{a^2(r^2+y^2)^{3/2}}dr = \frac{2GMmy}{a^2}\cdot\left.\frac{-1}{(r^2+y^2)^{1/2}}\right|_0^a\\ &= \frac{2GMmy}{a^2}\left(\frac{1}{y}-\frac{1}{(a^2+y^2)^{1/2}}\right).\end{aligned}$$

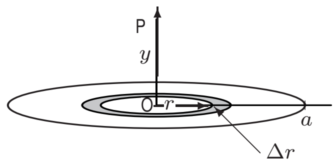


Figure 8.109

Strengthen Your Understanding

42. The calculation given is correct when the entire rope is lifted 20 m. But the portions of the rope nearer the top are raised a shorter distance, so less work is required.

43. It takes less work to pump the oil out of the top half of the tank than to pump it out of the bottom half of the tank because the oil in the top half has to be lifted a shorter distance. Emptying the full tank takes less than twice as much work as emptying the half-filled tank.

44. The force exerted lifting the rock is equal to 10 kg $\times$ 9.8 m/sec^2 $= 98$ nt. The work is

$$\text{Work } = \text{Force } \times \text{Distance} = 98\cdot 2 \text{ joules}.$$

45. Raising a 2 kg book 1 meter off the ground requires work Force $\times$ Distance where Force $= (2 \text{ kg})\left(9.8 \text{ m/sec}^2\right)$ and Distance $= 1$ meter .

46. Let tank A be a cylinder of radius 1 meter and height 4 meters and tank B be a cylinder of radius 2 meters and height 1 meter as shown in Figure 8.110. The volume of each tank is $4\pi = 12.56$ meters^3. All the water in tank B has to be pumped at least 9 meters, whereas the water in the top part of tank A has to be pumped a shorter distance. Therefore, it takes less work to pump the water from tank A to a height of 10 meters than the water from tank B.

Figure 8.110

47. False. Work is the product of force and distance moved, so the work done in either case is 200 ft-lb.

48. True. Displacement in the same direction as the force gives positive work; displacement in the opposite direction as the force gives negative work.

49. False. Since the water pressure increases with depth, the force on the lower half of the new dam is greater than the force on the upper half of the new dam, which is the same as the force on the old dam. Thus the force on the new dam is more than double the force on the old dam.

50. True. Since pressure increases with depth and we want the pressure to be approximately constant on each strip, we use horizontal strips.

51. False. The pressure is positive and when integrated gives a positive force.

52. True. Although work is expressed in an integral, the average value is also expressed in an integral. We have:

$$\text{Average value of the force} = \frac{1}{4-1}\int_1^4 F(x)\,dx.$$

Thus if we multiply the average force by 3, we get $\int_1^4 F(x)\,dx$, which is the work done.

Solutions for Section 8.6

Exercises

1. The future value, in dollars, is

$$C(1+0.03)^{25}.$$

2. The present value, in dollars, is

$$\frac{C}{(1+0.03)^{25}}.$$

3. The present value, in dollars, is

$$\frac{C}{e^{0.03(5)}}.$$

4. The present value, in dollars, is

$$\int_0^{15} C\,e^{-0.02t}\,dt.$$

5. The future value, in dollars, is

$$\int_0^{15} C\,e^{0.02(15-t)}\,dt.$$

6. The future value, in dollars, is

$$500e^{0.02(C)} + 500e^{0.02(C-1)} + 500e^{0.02(C-2)}.$$

7. We want

$$Ce^{r\cdot 30} = 25{,}000.$$

Thus,

$$e^{30r} = \frac{25{,}000}{C},$$

so, taking logs, we find the continuous interest rate is

$$r = \frac{\ln(25{,}000/C)}{30} \text{ per year.}$$

8. At any time t, in a time interval Δt, an amount of $1000\Delta t$ is deposited into the account. This amount earns interest for $(10-t)$ years giving a future value of $1000e^{(0.08)(10-t)}$. Summing all such deposits, we have

$$\text{Future value} = \int_0^{10} 1000e^{0.08(10-t)}\, dt = \$15{,}319.30.$$

9. (a)

$$\begin{aligned}\text{Future Value } &= \int_0^{20} 100e^{0.10(20-t)}\,dt \\ &= 100\int_0^{20} e^2 e^{-0.10t}\,dt \\ &= \frac{100e^2}{-0.10}e^{-0.10t}\Big|_0^{20} \\ &= \frac{100e^2}{0.10}(1-e^{-0.10(20)}) \approx \$6389.06.\end{aligned}$$

The present value of the income stream is

$$\begin{aligned}\int_0^{20} 100e^{-0.10t}\,dt &= 100\left(\frac{1}{-0.10}\right)e^{-0.10t}\Big|_0^{20} \\ &= 1000\left(1-e^{-2}\right) = \$864.66.\end{aligned}$$

Note that this is also the present value of the sum \$6389.06.

(b) Let T be the number of years for the balance to reach \$5000. Then

$$\begin{aligned}5000 &= \int_0^{T} 100e^{0.10(T-t)}\,dt \\ 50 &= e^{0.10T}\int_0^{T} e^{-0.10t}\,dt \\ &= \frac{e^{0.10T}}{-0.10}e^{-0.10t}\Big|_0^{T} \\ &= 10e^{0.10T}(1-e^{-0.10T}) = 10e^{0.10}T - 10.\end{aligned}$$

So, $60 = 10e^{0.10T}$, and $T = 10\ln 6 \approx 17.92$ years.

10. We compute the future value first: we have

$$\text{Future value} = \int_0^{5} 2000e^{0.08(5-t)}\,dt = \$12,295.62.$$

We can compute the present value using an integral and the income stream or using the future value. We compute the present value, P, from the future value:

$$12295.62 = Pe^{0.08(5)} \quad \text{so} \quad P = 8242.00.$$

The future value of this income stream is $\$12,295.62$ and the present value of this income stream is $\$8,242.00$.

11. (a) We compute the future value of this income stream:

$$\text{Future value} = \int_0^{20} 1000e^{0.07(20-t)}\,dt = \$43,645.71.$$

After 20 years, the account will contain $\$43,645.71$.

(b) The person has deposited \$1000 every year for 20 years, for a total of $\$20,000$.

(c) The total interest earned is $\$43,645.71 - \$20,000 = \$23,645.71$.

12. Since we want to have \$20,000 in 60 years starting with a single initial deposit of \$10,000, we have

$$20{,}000 = 10{,}000\, e^{r(60)}.$$

Solving for r, we get:

$$\begin{aligned} 2 &= e^{r(60)} \\ r &= \frac{\ln(2)}{60} \\ r &= 0.01155 \qquad \text{or} \qquad r = 1.155\% \text{ per year.} \end{aligned}$$

13. Since we want to have \$20,000 in 15 years starting with a single initial investment of \$10,000, we have

$$20{,}000 = 10{,}000e^{r(15)}.$$

Solving for r, we get:

$$\begin{aligned} 2 &= e^{r(15)} \\ r &= \frac{\ln(2)}{15} \\ r &= 0.04621 \qquad \text{or} \qquad r = 4.621\% \text{ per year.} \end{aligned}$$

14. Since we want to have \$20,000 in 5 years starting with a single initial deposit of \$10,000, we have

$$20{,}000 = 10{,}000e^{r(5)}.$$

Solving for r, we get:

$$\begin{aligned} 2 &= e^{r(5)} \\ r &= \frac{\ln(2)}{5} \\ r &= 0.13863 \qquad \text{or} \qquad r = 13.863\% \text{ per year.} \end{aligned}$$

Notice that the goal of doubling the initial \$10,000 deposit in a short period of time (5 years) leads to a high interest rate (13.863%).

15. A constant income stream that pays 20,000 in 5 years has a yearly rate amount of $20{,}000/5 = 4000$ dollars/year. Thus, the future value, B, of the stream at the end of 5 years is

$$B = \int_0^5 4000\, e^{0.02(5-t)}\, dt = 4000\, \frac{-e^{0.02(5-t)}}{0.02}\bigg|_0^5 = 200{,}000(e^{0.1} - 1) = 21{,}034.18 \text{ dollars.}$$

16. A constant income stream that pays 20,000 in 10 years has a yearly rate amount of $20{,}000/10 = 2000$ dollars/year. Thus, the future value, B, of the stream at the end of 10 years is

$$B = \int_0^{10} 2000\, e^{0.02(10-t)}\, dt = 2000\, \frac{-e^{0.02(10-t)}}{0.02}\bigg|_0^{10} = 100{,}000(e^{0.2} - 1) = 22{,}140.28 \text{ dollars.}$$

17. A constant income stream that pays 20,000 in 20 years has a yearly rate amount of $20{,}000/20 = 1000$ dollars/year. Thus, the future value, B, of the stream at the end of 20 years is

$$B = \int_0^{20} 1000\, e^{0.02(20-t)}\, dt = 1000\, \frac{-e^{0.02(20-t)}}{0.02}\bigg|_0^{20} = 50{,}000(e^{0.4} - 1) = 24{,}591.23 \text{ dollars.}$$

Problems

18. If $S(t) = S$ is the constant income stream, then we want

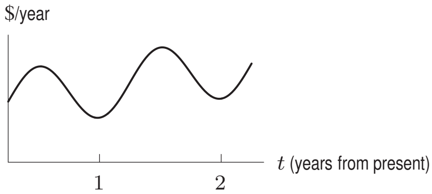
$$\text{Future value } = 20000 = \int_0^{10} Se^{0.03(10-t)}\, dt = S\int_0^{10} e^{0.03(10-t)}\, dt.$$

Evaluating the integral gives

$$\int_0^{10} e^{0.03(10-t)}\, dt = \frac{-1}{0.03}e^{0.03(10-t)}\bigg|_0^{10} = 11.662.$$

Then $S = 20000/11.662 = 1714.97$ dollars per year.

19.



The graph reaches a peak each summer, and a trough each winter. The graph shows sunscreen sales increasing from cycle to cycle. This gradual increase may be due in part to inflation and to population growth.

20. (a) The lump sum payment has a present value of 120 million dollars . We compute the present value of the other option at 6% and 3%. An award of \$195 million paid out continuously over 20 years works out to an income stream of 9.75 million dollars per year.

If the interest rate is 6%, compounded continuously, we have

$$\text{Present value at } 6\% = \int_0^{20} 9.75e^{-0.06t}\,dt = \frac{9.75}{-0.06}(e^{-0.06\cdot 20} - 1) = 113.556 \text{ million dollars.}$$

Since this amount is less than the lump sum payment of 120 million dollars, the lump sum payment is preferable if the interest rate is 6%.

If the interest rate is 3%, we have

$$\text{Present value at } 3\% = \int_0^{20} 9.75e^{-0.03t}\,dt = \frac{9.75}{-0.03}(e^{-0.03\cdot 20} - 1) = 146.636 \text{ million dollars.}$$

Since this amount is greater than the lump sum payment of 120 million dollars, taking payments continuously over 20 years is the better option if the interest rate is 3%.

(b) The assumption is that the interest rates would stay relatively high (closer to 6%). The interest rate at which the decision changes is the solution to the equation

$$\frac{9.75}{-x}(e^{-20x} - 1) = 120.$$

Using a graph or numerical methods gives $x = 5.3\%$. Thus if Mr. Nabors chooses the lump sum, he is assuming interests rates stay above 5.3%.

21. (a) Solve for $P(t) = P$.

$$\begin{aligned} 100000 &= \int_0^{10} Pe^{0.10(10-t)}\,dt = Pe\int_0^{10} e^{-0.10t}\,dt \\ &= \frac{Pe}{-0.10}e^{-0.10t}\Big|_0^{10} = Pe(-3.678 + 10) \\ &= P \cdot 17.183. \end{aligned}$$

So, $P \approx \$5820$ per year.

(b) To answer this, we'll calculate the present value of \$100,000:

$$\begin{aligned} 100000 &= Pe^{0.10(10)} \\ P &\approx \$36{,}787.94. \end{aligned}$$

22. (a) Let L be the number of years for the balance to reach \$10,000. Since our income stream is \$1000 per year, the future value of this income stream should equal (in L years) \$10,000. Thus

$$\begin{aligned} 10000 &= \int_0^{L} 1000e^{0.05(L-t)}\,dt = 1000e^{0.05L}\int_0^{L} e^{-0.05t}\,dt \\ &= 1000e^{0.05L}\left(-\frac{1}{0.05}\right)e^{-0.05t}\Big|_0^{L} = 20000e^{0.05L}\left(1 - e^{-0.05L}\right) \\ &= 20000e^{0.05L} - 20000 \\ \text{so}\quad e^{0.05L} &= \frac{3}{2} \\ L &= 20\ln\left(\frac{3}{2}\right) \approx 8.11 \text{ years.} \end{aligned}$$

(b) We want

$$10000 = 2000e^{0.05L} + \int_0^L 1000e^{0.05(L-t)}\,dt.$$

The first term on the right hand side is the future value of our initial balance. The second term is the future value of our income stream. We want this sum to equal \$10,000 in L years. We solve for L:

$$\begin{aligned}
10000 &= 2000e^{0.05L} + 1000e^{0.05L}\int_0^L e^{-0.05t}\,dt \\
&= 2000e^{0.05L} + 1000e^{0.05L}\left(\frac{1}{-0.05}\right)e^{-0.05t}\Big|_0^L \\
&= 2000e^{0.05L} + 20000e^{0.05L}\left(1 - e^{-0.05L}\right) \\
&= 2000e^{0.05L} + 20000e^{0.05L} - 20000.
\end{aligned}$$

So,

$$\begin{aligned}
22000e^{0.05L} &= 30000 \\
e^{0.05L} &= \frac{30000}{22000} \\
L &= 20\ln\frac{15}{11} \approx 6.203 \text{ years.}
\end{aligned}$$

23. You should choose the payment which gives you the highest present value. The immediate lump-sum payment of \$2800 obviously has a present value of exactly \$2800, since you are getting it now. We can calculate the present value of the installment plan as:

$$\begin{aligned}
\text{PV} &= 1000e^{-0.06(0)} + 1000e^{-0.06(1)} + 1000e^{-0.06(2)} \\
&\approx \$2828.68.
\end{aligned}$$

Since the installment payments offer a (slightly) higher present value, you should accept this option.

24. The initial \$9000 deposit earns interest for 20 years. Since we want to have \$18,000 in 20 years, we have:

$$18{,}000 = 9000e^{r(20)}.$$

Solving for r, we get:

$$\begin{aligned}
\frac{18{,}000}{9000} &= e^{r(20)} \\
r &= \frac{\ln\ 2}{20} \\
r &= 0.03466 \qquad \text{or} \qquad r = 3.466\% \text{ per year.}
\end{aligned}$$

25. The initial \$6000 deposit earns interest for 20 years. The second deposit only earns interest for 17 years. Since we want to have \$18,000 in 20 years, we have:

$$18{,}000 = 6000e^{r(20)} + 3000e^{r(17)},$$

or

$$\frac{18{,}000}{3000} = 2e^{r(20)} + e^{r(17)}.$$

We can solve for r numerically or by graphing the functions on the left- and right-hand sides of the equation, and finding the intersection point. This yields $r = 0.03641$, or $r = 3.641\%$ per year.

26. The initial \$3000 deposit earns interest for 20 years. The second deposit earns interest for 17 years. Since we want to have \$18,000 in 20 years, we have:

$$18{,}000 = 3000e^{r(20)} + 6000e^{r(17)},$$

or

$$\frac{18{,}000}{3000} = e^{r(20)} + 2e^{r(17)}.$$

We can solve for r numerically or by graphing the functions on the left- and right-hand sides of the equation, and finding the intersection point. This yields $r = 0.03843$, or $r = 3.843\%$ per year.

27. We want to reach a target of $18,000 in 20 years, so we have:

$$18{,}000 = \int_0^{20} 300\, e^{r(20-t)}\, dt.$$

In order to solve for r, we integrate, then solve the resulting equation graphically or numerically. We have:

$$\begin{aligned}\frac{18{,}000}{300} &= \left.\frac{-e^{r(20-t)}}{r}\right|_0^{20}\\ 60 &= \frac{e^{20r}-1}{r}.\end{aligned}$$

We can solve for r by graphing the functions on the left- and right-hand sides of the last equation, and finding the intersection point. This yields $r = 0.09519$, or $r = 9.519\%$ per year.

28. Since we want the future value of the income stream of 4800 per year to be 100,000 in 15 years, we have

$$100{,}000 = \int_0^{15} 4800 e^{r(15-t)}\, dt.$$

Evaluating the integral gives

$$100{,}000 = \left.\frac{4800}{-r} e^{r(15-t)}\right|_0^{15} = \frac{4800}{-r}(e^0 - e^{15r}) = \frac{4800}{r}(e^{15r}-1).$$

This equation is not solvable by algebraic means. However, using a computer or calculator to solve for r numerically or graphically we obtain $r = 0.0416$. So a continuous interest rate of 4.16% per year is required.

29. (a) We calculate the future values of the two options:

$$\begin{aligned}\text{FV}_1 &= 6e^{0.1(3)} + 2e^{0.1(2)} + 2e^{0.1(1)} + 2e^{0.1(0)}\\ &\approx 8.099 + 2.443 + 2.210 + 2\\ &= \$14.752 \text{ million.}\end{aligned}$$

$$\begin{aligned}\text{FV}_2 &= e^{0.1(3)} + 2e^{0.1(2)} + 4e^{0.1(1)} + 6e^{0.1(0)}\\ &\approx 1.350 + 2.443 + 4.421 + 6\\ &= \$14.214 \text{ million.}\end{aligned}$$

As we can see, the first option gives a higher future value, so he should choose Option 1.

(b) From the future value we can easily derive the present value using the formula $\text{PV} = \text{FV}e^{-rt}$. So the present value is

$$\text{Option 1: PV} = 14.752e^{0.1(-3)} \approx \$10.929 \text{ million.}$$

$$\text{Option 2: PV} = 14.214e^{0.1(-3)} \approx \$10.530 \text{ million.}$$

30. At any time t, the company receives income of $s(t)$ per year. It will then invest this money for a length of $2 - t$ years at 6% interest, giving it future value of $s(t)e^{(0.06)(2-t)}$ from this income. If we sum all such incomes over the two-year period, we can find the total value of the sales:

$$\begin{aligned}\text{Value} &= \int_0^2 s(t)e^{(0.06)(2-t)}\, dt = \int_0^2 \left[50e^{-t}e^{(0.06)(2-t)}\right] dt\\ &= \int_0^2 \left[50e^{0.12-1.06t}\right] dt = \left.\left(\frac{-53.1838}{e^{1.06t}}\right)\right|_0^2 = \$46{,}800.\end{aligned}$$

31. Price in future $= P(1+20\sqrt{t})$.
The present value V of price satisfies $V = P(1+20\sqrt{t})e^{-0.05t}$.
We want to maximize V. To do so, we find the critical points of $V(t)$ for $t \geq 0$. (Recall that $\sqrt{t}$ is nondifferentiable at $t = 0$.)

$$\begin{aligned}\frac{dV}{dt} &= P\left[\frac{20}{2\sqrt{t}}e^{-0.05t} + (1+20\sqrt{t})(-0.05e^{-0.05t})\right] \\ &= Pe^{-0.05t}\left[\frac{10}{\sqrt{t}} - 0.05 - \sqrt{t}\right].\end{aligned}$$

Setting $\frac{dV}{dt} = 0$ gives $\frac{10}{\sqrt{t}} - 0.05 - \sqrt{t} = 0$. Using a calculator, we find $t \approx 10$ years. Since $V'(t) > 0$ for $0 < t < 10$ and $V'(t) < 0$ for $t > 10$, we confirm that this is a maximum. Thus, the best time to sell the wine is in 10 years.

32. (a) Suppose the oil extracted over the time period $[0, M]$ is S. (See Figure 8.111.) Since $q(t)$ is the rate of oil extraction, we have:

$$S = \int_0^M q(t)dt = \int_0^M (a - bt)dt = \int_0^M (10 - 0.1t)\,dt.$$

To calculate the time at which the oil is exhausted, set $S = 100$ and try different values of M. We find $M = 10.6$ gives

$$\int_0^{10.6} (10 - 0.1t)\,dt = 100,$$

so the oil is exhausted in 10.6 years.

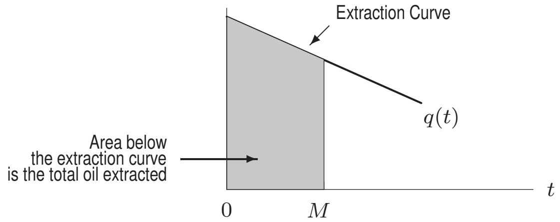


Figure 8.111

(b) Suppose p is the oil price, C is the extraction cost per barrel, and r is the interest rate. We have the present value of the profit as

$$\begin{aligned}\text{Present value of profit} &= \int_0^M (p - C)q(t)e^{-rt}dt \\ &= \int_0^{10.6} (20 - 10)(10 - 0.1t)e^{-0.1t}\,dt \\ &= 624.9 \text{ million dollars.}\end{aligned}$$

33. (a) Let's split the time interval into n parts, each of length Δt. During the interval from t_i to t_{i+1}, profit is earned at a rate of approximately $(2 - 0.1t_i)$ thousand dollars per year, or $(2000 - 100t_i)$ dollars per year. Thus during this period, a total profit of $(2000 - 100t_i)\Delta t$ dollars is earned. Since this profit is earned t_i years in the future, its present value is $(2000 - 100t_i)\Delta t e^{-0.1t_i}$ dollars. Thus

$$\text{Total present value} \approx \sum_{i=0}^{n-1}(2000 - 100t_i)e^{-0.1t_i}\Delta t.$$

(b) The Riemann sum corresponds to the integral:

$$\text{Present value} = \int_0^M e^{-0.10t}(2000 - 100t)\,dt.$$

(c) To find where the present value is maximized, we take the derivative of

$$P(M) = \int_0^M e^{-0.10t}(2000 - 100t)\,dt,$$

with respect to M, and obtain

$$P'(M) = e^{-0.10M}(2000 - 100M).$$

This is 0 when $2000 - 100M = 0$, that is, when $M = 20$ years. The value $M = 20$ maximizes $P(M)$, since $P'(M) > 0$ for $M < 20$, and $P'(M) < 0$ for $M > 20$. To determine what the maximum is, we evaluate the integral representation for $P(20)$ by III-14 in the integral table:

$$\begin{aligned} P(20) &= \int_0^{20} e^{-0.10t}(2000 - 100t)\,dt \\ &= \left[\frac{(2000 - 100t)}{-0.10}e^{-0.10t} + 10000e^{-0.10t}\right]\Bigg|_0^{20} \approx \$11353.35. \end{aligned}$$

34. One good way to approach the problem is in terms of present values. In 1980, the present value of Germany's loan was 20 billion DM. Now let's figure out the rate that the Soviet Union would have to give money to Germany to pay off 10% interest on the loan by using the formula for the present value of a continuous stream. Since the Soviet Union sends gas at a constant rate, the rate of deposit, $P(t)$, is a constant c. Since they don't start sending the gas until after 5 years have passed, the present value of the loan is given by:

$$\text{Present Value} = \int_5^\infty P(t)e^{-rt}\,dt.$$

We want to find c so that

$$\begin{aligned} 20{,}000{,}000{,}000 &= \int_5^\infty ce^{-rt}\,dt = c\int_5^\infty e^{-rt}\,dt \\ &= c\lim_{b\to\infty}(-10e^{-0.10t})\Big|_5^b = ce^{-0.10(5)} \\ &\approx 6.065c. \end{aligned}$$

Dividing, we see that c should be about 3.3 billion DM per year. At 0.10 DM per m^3 of natural gas, the Soviet Union must deliver gas at the constant, continuous rate of about 33 billion m^3 per year.

35. Measuring money in thousands of dollars, the equation of the line representing the demand curve passes through (50, 980) and (350, 560). Its slope is $(560 - 980)/(350 - 50) = -420/300$. See Figure 8.112. So the equation is $y - 560 = -\frac{420}{300}(x - 350)$, i.e. $y - 560 = -\frac{7}{5}x + 490$. Thus

$$\begin{aligned} \text{Consumer surplus } &= \int_0^{350}\left(-\frac{7}{5}x + 1050\right)dx - 350\cdot 560 = -\frac{7}{10}x^2 + 1050x\Big|_0^{350} - 196000 \\ &= 85{,}750. \end{aligned}$$

(Note that $85{,}750 = \frac{1}{2}\cdot 490\cdot 350$, the area of the triangle in Figure 8.112. We could have used this instead of the integral to find the consumer surplus.)

Recalling that our unit measure for the price axis is \$1000/car, the consumer surplus is \$85,750,000.

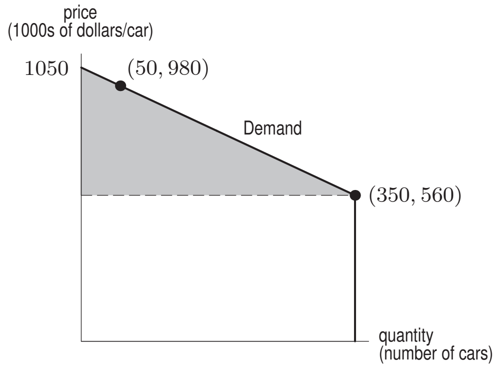


Figure 8.112

36. The supply curve, $S(q)$, represents the minimum price p per unit that the suppliers will be willing to supply some quantity q of the good for. See Figure 8.113. If the suppliers have q^* of the good and q^* is divided into subintervals of size Δq, then if the consumers could offer the suppliers for each Δq a price increase just sufficient to induce the suppliers to sell an additional Δq of the good, the consumers' total expenditure on q^* goods would be

$$p_1\Delta q + p_2\Delta q + \cdots = \sum p_i \Delta q.$$

As $\Delta q \to 0$ the Riemann sum becomes the integral $\int_0^{q^*} S(q)\,dq$. Thus $\int_0^{q^*} S(q)\,dq$ is the amount the consumers would pay if suppliers could be forced to sell at the lowest price they would be willing to accept.

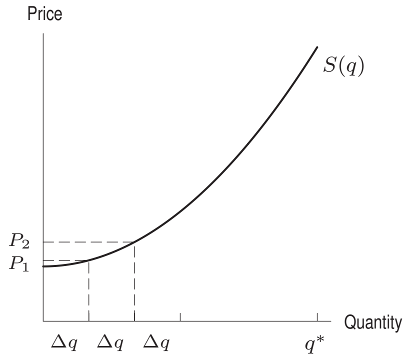


Figure 8.113

37.

$$\begin{aligned}\int_0^{q^*} (p^* - S(q))\,dq &= \int_0^{q^*} p^*\,dq - \int_0^{q^*} S(q)\,dq \\ &= p^*q^* - \int_0^{q^*} S(q)\,dq.\end{aligned}$$

Using Problem 36, this integral is the extra amount consumers pay (i.e., suppliers earn over and above the minimum they would be willing to accept for supplying the good). It results from charging the equilibrium price.

38. (a) p^*q^* = the total amount paid for q^* of the good at equilibrium. See Figure 8.114.

(b) $\int_0^{q^*} D(q)\,dq$ = the maximum consumers would be willing to pay if they had to pay the highest price acceptable to them for each additional unit of the good. See Figure 8.115.

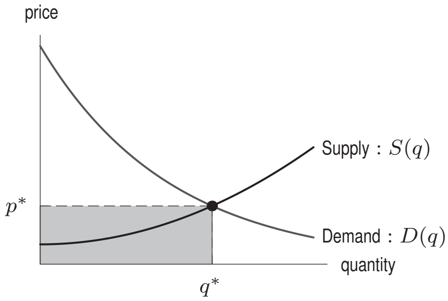


Figure 8.114

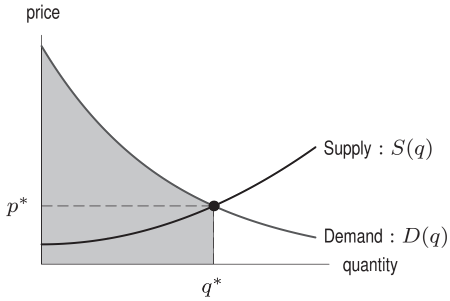


Figure 8.115

(c) $\int_0^{q^*} S(q)\,dq =$ the minimum suppliers would be willing to accept if they were paid the minimum price acceptable to them for each additional unit of the good. See Figure 8.116.

(d) $\int_0^{q^*} D(q)\,dq - p^*q^* =$ consumer surplus. See Figure 8.117.

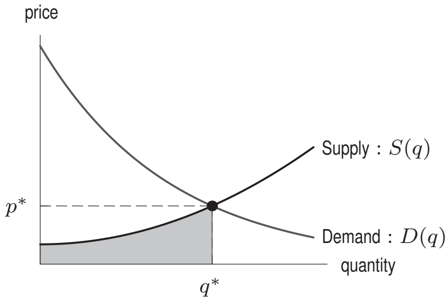


Figure 8.116

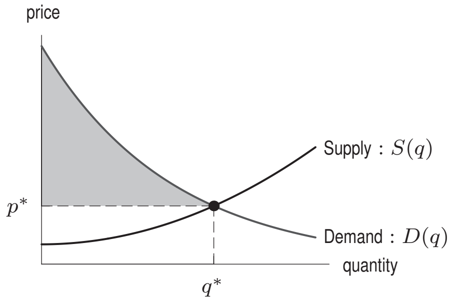


Figure 8.117

(e) $p^*q^* - \int_0^{q^*} S(q)\,dq =$ producer surplus. See Figure 8.118.

(f) $\int_0^{q^*} (D(q) - S(q))\,dq =$ producer surplus and consumer surplus. See Figure 8.119.

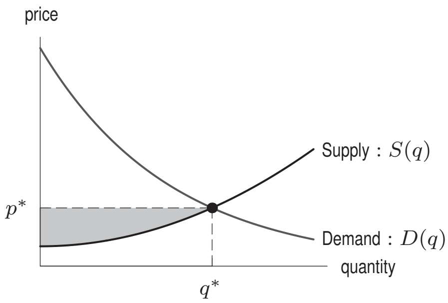


Figure 8.118

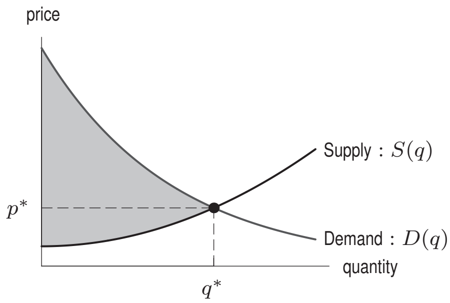


Figure 8.119

39.

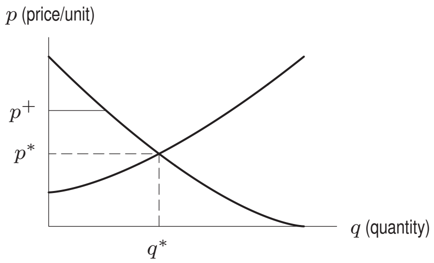


Figure 8.120: What effect does the artificially high price, p^+, have?

(a) A graph of possible demand and supply curves for the milk industry is given in Figure 8.120, with the equilibrium price and quantity labeled p^* and q^* respectively. Suppose that the price is fixed at the artificially high price labeled p^+ in Figure 8.120. Recall that the consumer surplus is the difference between the amount the consumers did pay (p^+) and the amount they would have been willing to pay (given on the demand curve). This is the area shaded in Figure 8.121(i). Notice that this consumer surplus is clearly less than the consumer surplus at the equilibrium price, shown in Figure 8.121(ii).

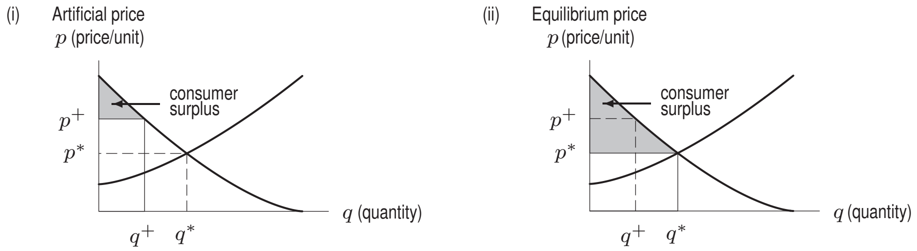


Figure 8.121: Consumer surplus for the milk industry

(b) At a price of p^+, the quantity sold, q^+, is less than it would have been at the equilibrium price. The producer surplus is the area between p^+ and the supply curve *at this reduced demand*. This area is shaded in Figure 8.122(i). Compare this producer surplus (at the artificially high price) to the producer surplus in Figure 8.122(ii) (at the equilibrium price). It appears that in this case, producer surplus is greater at the artificial price than at the equilibrium price. (Different supply and demand curves might have led to a different answer.)

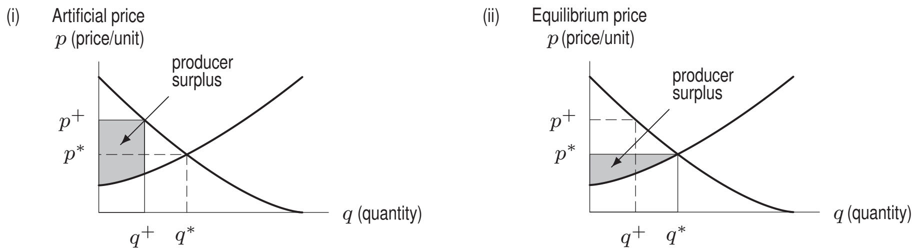


Figure 8.122: Producer surplus for the milk industry

(c) The total gains from trade (Consumer surplus + Producer surplus) at the artificially high price of p^+ is the area shaded in Figure 8.123(i). The total gains from trade at the equilibrium price of p^* is the area shaded in Figure 8.123(ii). It is clear that, under artificial price conditions, total gains from trade go down. The total financial effect of the artificially high price on all producers and consumers combined is a negative one.

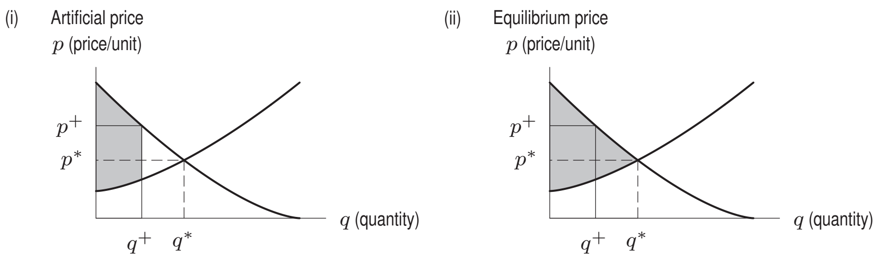
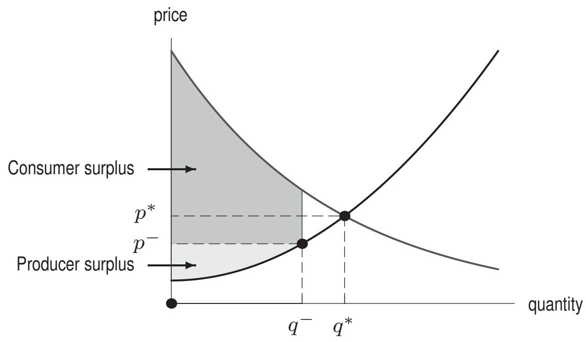


Figure 8.123: Total gains from trade

40.



(a) The producer surplus is the area on the graph between p^- and the supply function. Lowering the price also lowers the producer surplus.

(b) Note that the consumer surplus—the area between the line p^- and the supply curve—increases or decreases depending on the functions describing the supply and demand and on the lowered price. (For example, the consumer surplus seems to be increased in the graph above, but if the price were brought down to \$0 then the consumer surplus would be zero, and hence clearly less than the consumer surplus at equilibrium.)

(c) The graph above shows that the total gains from the trade are decreased.

Strengthen Your Understanding

41. With no interest, the future value of the stream is equal to the total amount deposited over 10 years, which is \$20,000. So the future value is more than \$20,000.

42. The present value of S at a 4% interest rate, compounded annually, is

$$PV = \frac{S}{1.04}.$$

The present value of S at a 3% interest rate, compounded annually, is

$$PV = \frac{S}{1.03}.$$

Thus, the present value is smaller with a 4% interest rate.

43. The present value of a continuous stream of payments is found using an integral:

$$\text{PV} = \int_0^2 Pe^{-rt}dt = -\frac{P}{r}e^{-rt}\Big|_0^2 = \frac{P}{r}(1 - e^{-2r}).$$

44. Producer surplus is measured in dollars.

45. See Figure 8.124.

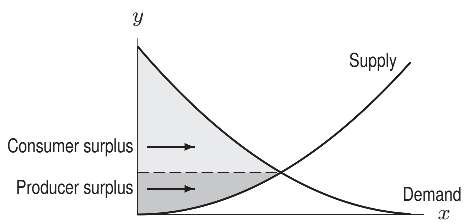
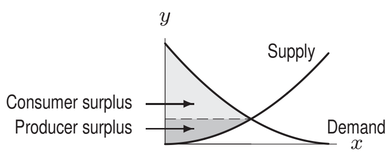


Figure 8.124



46. We want $10{,}000e^{-r\cdot 10} < 5000$. Solving

$$e^{-10r} = 0.5$$
$$r = -\frac{1}{10}\ln(0.5) = 0.0693.$$

We want an interest rate larger than 0.0693, such as 7% or larger.
Larger than 6.93%/ yr

47. A present value of A with an interest rate r, compounded annually, has future value $A(1+r)$ after one year. Since we want the future value to be \$5000, we have

$$A(1+r) = 5000$$

or

$$A = \frac{5000}{1+r}.$$

Every choice of r gives a choice of A. For example, a 5% interest rate corresponds to $r = 0.05$ and gives $A = 5000/1.05 = 4761.90$. If the annual interest rate is 5%, then \$4761.90 now has future value \$5000 one year from now.

48. Suppose there is an annual interest rate of 10%. Then the present value, A, of the \$10,000 is:

$$A = \frac{10,000}{(1+0.10)^{10}} = 3855.$$

Thus, a deposit of roughly \$3855 is needed now to have 10,000 dollars in 10 years, assuming a 10% interest rate.

There are two ways to think about the remaining values in the table. We can ask how much the amount needed now, \$3855, will grow to in t years. This means the remaining values in the table arise by calculating:

$$3855(1+0.10)^t \quad \text{for} \quad t = 1, 2, 3, 4.$$

Alternatively, we realize that a single deposit made t years from now would have $10 - t$ years to grow to reach our target investment of \$10,000 in 10 years. This means the remaining values in the table arise by calculating:

$$\frac{10,000}{(1+0.10)^{10-t}}, \quad \text{for} \quad t = 1, 2, 3, 4.$$

If we ignore rounding errors, we see that two approaches yield the same table values:

t (years from now)	0	1	2	3	4
\$ (dollars)	3855	4241	4665	5131	5644

Other possible answers to this problem can arise from alternate interest rate choices.

Solutions for Section 8.7

Exercises

1. The two humps of probability in density (a) correspond to two intervals on which its cumulative distribution function is increasing. Thus (a) and (II) correspond.

A density function increases where its cumulative distribution function is concave up, and it decreases where its cumulative distribution function is concave down. Density (b) matches the distribution with both concave up and concave down sections, which is (I). Density (c) matches (III) which has a concave down section but no interval over which it is concave up.

2.

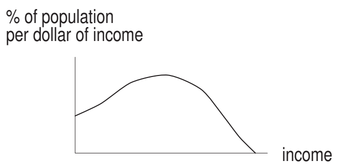


Figure 8.125: Density function

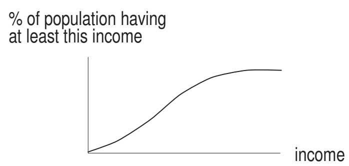


Figure 8.126: Cumulative distribution function

3.

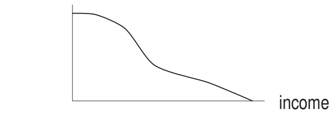


Figure 8.127: Density function

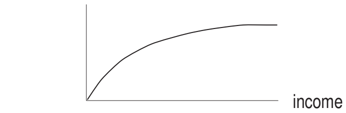


Figure 8.128: Cumulative distribution function

4.

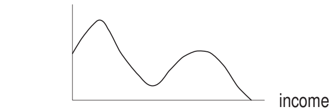


Figure 8.129: Density function

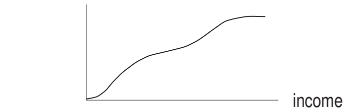


Figure 8.130: Cumulative distribution function

5. Since the function takes on the value of 4, it cannot be a cdf (whose maximum value is 1). In addition, the function decreases for $x > c$, which means that it is not a cdf. Thus, this function is a pdf. The area under a pdf is 1, so $4c = 1$ giving $c = \frac{1}{4}$. The pdf is $p(x) = 4$ for $0 \le x \le \frac{1}{4}$, so the cdf is given in Figure 8.131 by

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x & \text{for} \quad 0 \le x \le \dfrac{1}{4} \\ 1 & \text{for} \quad x > \dfrac{1}{4} \end{cases}$$

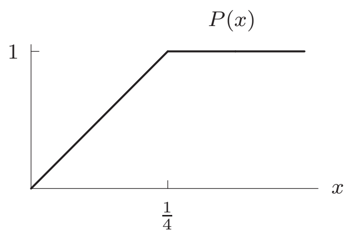


Figure 8.131

6. Since the function is decreasing, it cannot be a cdf (whose values never decrease). Thus, the function is a pdf.

The area under a pdf is 1, so, using the formula for the area of a triangle, we have

$$\frac{1}{2}4c = 1, \quad \text{giving} \quad c = \frac{1}{2}.$$

The pdf is

$$p(x) = \frac{1}{2} - \frac{1}{8}x \quad \text{for} \quad 0 \le x \le 4,$$

so the cdf is given in Figure 8.132 by

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \dfrac{x}{2} - \dfrac{x^2}{16} & \text{for} \quad 0 \le x \le 4 \\ 1 & \text{for} \quad x > 4. \end{cases}$$

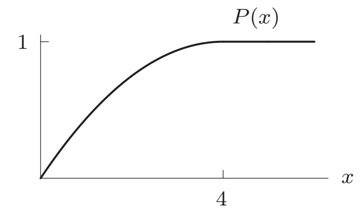


Figure 8.132

7. Since the function levels off at the value of c, the area under the graph is not finite, so it is not 1. Thus, this function cannot be a pdf.

It is a cdf and $c = 1$. The cdf is given by

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \dfrac{x}{5} & \text{for} \quad 0 \le x \le 5 \\ 1 & \text{for} \quad x > 5. \end{cases}$$

The pdf in Figure 8.133 is given by

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1/5 & \text{for} \quad 0 \le x \le 5 \\ 0 & \text{for} \quad x > 5. \end{cases}$$

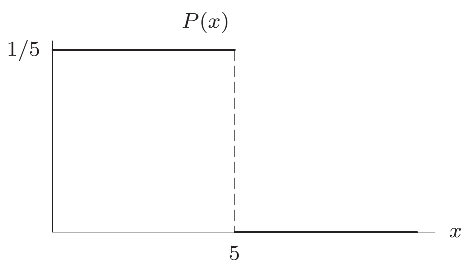


Figure 8.133

8. This function decreases, so it cannot be a cdf. Since the graph must represent a pdf, the area under it is 1. The region consists of two rectangles, each of base 0.5, and one of height $2c$ and one of height c, so

$$\text{Area} = 2c(0.5) + c(0.5) = 1$$
$$c = \frac{1}{1.5} = \frac{2}{3}$$

The pdf is therefore

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4/3 & \text{for} \quad 0 \le x \le 0.5 \\ 2/3 & \text{for} \quad 0.5 < x \le 1 \\ 0 & \text{for} \quad x > 1. \end{cases}$$

The cdf $P(x)$ is the antiderivative of this function with $P(0) = 0$. See Figure 8.134. The formula for $P(x)$ is

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x/3 & \text{for} \quad 0 \le x \le 0.5 \\ 2/3 + (2/3)(x - 0.5) & \text{for} \quad 0.5 < x \le 1 \\ 1 & \text{for} \quad x > 1. \end{cases}$$

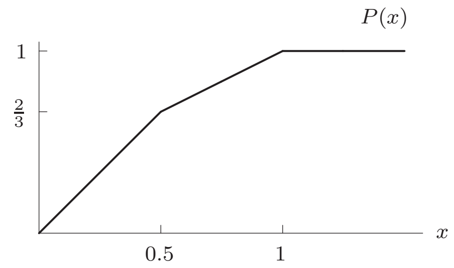


Figure 8.134

9. This function increases and levels off to c. The area under the curve is not finite, so it is not 1. Thus, the function must be a cdf, not a pdf, and $3c = 1$, so $c = 1/3$.

The pdf, $p(x)$ is the derivative, or slope, of the function shown, so, using $c = 1/3$,

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ (1/3 - 0)/(2 - 0) = 1/6 & \text{for} \quad 0 \le x \le 2 \\ (1 - 1/3)/(4 - 2) = 1/3 & \text{for} \quad 2 < x \le 4 \\ 0 & \text{for} \quad x > 4. \end{cases}$$

See Figure 8.135.

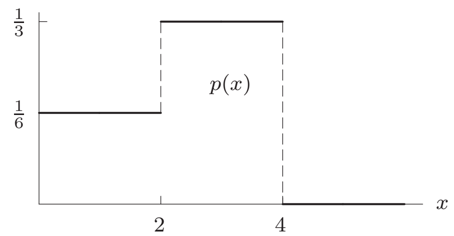


Figure 8.135

10. This function does not level off to 1, and it is not always increasing. Thus, the function is a pdf. Since the area under the curve must be 1, using the formula for the area of a triangle,

$$\frac{1}{2} \cdot c \cdot 1 = 1 \quad \text{so} \quad c = 2.$$

Thus, the pdf is given by

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x & \text{for} \quad 0 \le x \le 0.5 \\ 2 - 4(x - 0.5) = 4 - 4x & \text{for} \quad 0.5 < x \le 1 \\ 0 & \text{for} \quad x > 0. \end{cases}$$

To find the cdf, we integrate each part of the function separately, making sure that the constants of integration are arranged so that the cdf is continuous.

Since $\int 4x dx = 2x^2 + C$ and $P(0) = 0$, we have $2(0)^2 + C = 0$ so $C = 0$. Thus $P(x) = 2x^2$ on $0 \le x \le 0.5$. At $x = 0.5$, the cdf has value $P(0.5) = 2(0.5)^2 = 0.5$. Thus, we arrange that the integral of $4 - 4x$ goes through the point $(0.5, 0.5)$. Since $\int (4 - 4x)\, dx = 4x - 2x^2 + C$, we have

$$4(0.5) - 2(0.5)^2 + C = 0.5 \quad \text{giving} \quad C = -1.$$

Thus

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 2x^2 & \text{for} \quad 0 \le x \le 0.5 \\ 4x - 2x^2 - 1 & \text{for} \quad 0.5 < x \le 1 \\ 1 & \text{for} \quad x > 1. \end{cases}$$

See Figure 8.136.

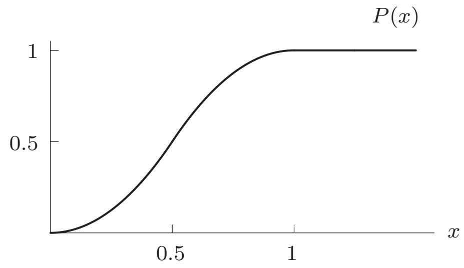


Figure 8.136

11. The statement $p(70) = 0.05$ means that for some small interval Δx around 70, the fraction of families with incomes in that interval around \$70,000 is about $0.05\Delta x$.

12. We need to find a nonnegative function $p(x)$ for which the area between $y = p(x)$ and the x axis is 1. One example is a decreasing linear function $p(x) = b + mx$ with $m < 0$. We choose b and m so that $p(5) = 0$ and the triangular area under the graph of $p(x)$ for $0 \le x \le 5$ is 1. Since b is the height of the triangle, we have

$$\text{Area} = \frac{1}{2} b \cdot 5 = 1$$

or $b = 2/5$. Then $p(5) = 0$ gives

$$\frac{2}{5} + 5m = 0$$

or $m = -2/25$. Thus

$$p(x) = \frac{2}{5} - \frac{2}{25}x \quad \text{when} \quad 0 \le x \le 5 \quad \text{and} \quad p(x) = 0 \text{ otherwise.}$$

Problems

13. For a given energy E, Figure 8.137 shows that the area under the graph to the right of E is larger for graph B than it is for graph A. Therefore graph B has more molecules at higher kinetic energies, so it is the hotter gas. So graph A corresponds to 300 kelvins and graph B corresponds to 500 kelvins.

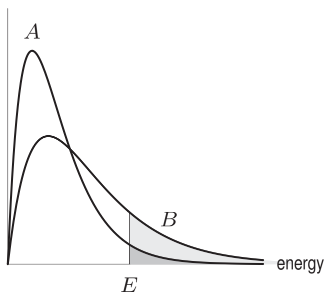


Figure 8.137

14. No. Though the density function has its maximum value at 50, this does not mean that a large fraction of the population receives scores near 50. The value $p(50)$ can not be interpreted as a probability. Probability corresponds to *area* under the graph of a density function. Most of the area in this case is in the broad hump covering the range $0 \le x \le 40$, very little in the peak around $x = 50$. Most people score in the range $0 \le x \le 40$.

15. (a) Let $P(x)$ be the cumulative distribution function of the heights of the unfertilized plants. As do all cumulative distribution functions, $P(x)$ rises from 0 to 1 as x increases. The greatest number of plants will have heights in the range where $P(x)$ rises the most. The steepest rise appears to occur at about $x = 1$ m. Reading from the graph we see that $P(0.9) \approx 0.2$ and $P(1.1) \approx 0.8$, so that approximately $P(1.1) - P(0.9) = 0.8 - 0.2 = 0.6 = 60\%$ of the unfertilized plants grow to heights between 0.9 m and 1.1 m. Most of the plants grow to heights in the range 0.9 m to 1.1 m.

(b) Let $P_A(x)$ be the cumulative distribution function of the plants that were fertilized with A. Since $P_A(x)$ rises the most in the range 0.7 m $\leq x \leq$ 0.9 m, many of the plants fertilized with A will have heights in the range 0.7 m to 0.9 m. Reading from the graph of P_A, we find that $P_A(0.7) \approx 0.2$ and $P_A(0.9) \approx 0.8$, so $P_A(0.9) - P_A(0.7) \approx 0.8 - 0.2 = 0.6 = 60\%$ of the plants fertilized with A have heights between 0.7 m and 0.9 m. Fertilizer A had the effect of stunting the growth of the plants.

On the other hand, the cumulative distribution function $P_B(x)$ of the heights of the plants fertilized with B rises the most in the range 1.1 m $\leq x \leq$ 1.3 m, so most of these plants have heights in the range 1.1 m to 1.3 m. Fertilizer B caused the plants to grow about 0.2 m taller than they would have with no fertilizer.

16. (a) $F(7) = 0.6$ tells us that 60% of the trees in the forest have height 7 meters or less.

(b) $F(7) > F(6)$. There are more trees of height less than 7 meters than trees of height less than 6 meters because every tree of height ≤ 6 meters also has height ≤ 7 meters.

17. For a small interval Δx around 68, the fraction of the population of American men with heights in this interval is about $(0.2)\Delta x$. For example, taking $\Delta x = 0.1$, we can say that approximately $(0.2)(0.1) = 0.02 = 2\%$ of American men have heights between 68 and 68.1 inches.

18. We want to find the cumulative distribution function for the age density function. We see that $P(10)$ is equal to 0.15 since the table shows that 15% of the population is between 0 and 10 years of age. Also,

$$P(20) = \frac{\text{Fraction of the population}}{\text{between 0 and 20 years old}} = 0.15 + 0.14 = 0.29$$

and

$$P(30) = 0.15 + 0.14 + 0.14 = 0.43.$$

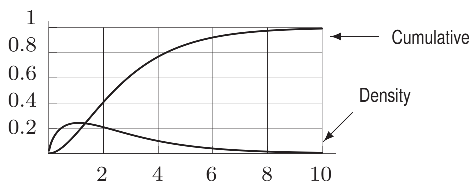
Continuing in this way, we obtain the values for $P(t)$ shown in Table 8.7.

Table 8.7 *Cumulative distribution function of ages in the US*

t	0	10	20	30	40	50	60	70	80	90	100
$P(t)$	0	0.15	0.29	0.43	0.60	0.74	0.84	0.92	0.97	0.99	1.00

19. (a) The two functions are shown below. The choice is based on the fact that the cumulative distribution does not decrease.

(b) The cumulative distribution levels off to 1, so the top mark on the vertical scale must be 1.



The total area under the density function must be 1. Since the area under the density function is about 2.5 boxes, each box must have area $1/2.5 = 0.4$. Since each box has a height of 0.2, the base must be 2.

20. (a) The area under the graph of the height density function $p(x)$ is concentrated in two humps centered at 0.5 m and 1.1 m. The plants can therefore be separated into two groups, those with heights in the range 0.3 m to 0.7 m, corresponding to the first hump, and those with heights in the range 0.9 m to 1.3 m, corresponding to the second hump. This grouping of the grasses according to height is probably close to the species grouping. Since the second hump contains more area than the first, there are more plants of the tall grass species in the meadow.

(b) As do all cumulative distribution functions, the cumulative distribution function $P(x)$ of grass heights rises from 0 to 1 as x increases. Most of this rise is achieved in two spurts, the first as x goes from 0.3 m to 0.7 m, and the second as x goes from 0.9 m to 1.3 m. The plants can therefore be separated into two groups, those with heights in the range 0.3 m to 0.7 m, corresponding to the first spurt, and those with heights in the range 0.9 m to 1.3 m, corresponding to the second spurt. This grouping of the grasses according to height is the same as the grouping we made in part (a), and is probably close to the species grouping.

(c) The fraction of grasses with height less than 0.7 m equals $P(0.7) = 0.25 = 25\%$. The remaining 75% are the tall grasses.

21. (a) The percentage of calls lasting from 1 to 2 minutes is given by the integral

$$\int_1^2 p(x)\,dx \int_1^2 0.4e^{-0.4x}\,dx = e^{-0.4} - e^{-0.8} \approx 22.1\%.$$

(b) A similar calculation (changing the limits of integration) gives the percentage of calls lasting 1 minute or less as

$$\int_0^1 p(x)\,dx = \int_0^1 0.4e^{-0.4x}\,dx = 1 - e^{-0.4} \approx 33.0\%.$$

(c) The percentage of calls lasting 3 minutes or more is given by the improper integral

$$\int_3^\infty p(x)\,dx = \lim_{b\to\infty} \int_3^b 0.4e^{-0.4x}\,dx = \lim_{b\to\infty} (e^{-1.2} - e^{-0.4b}) = e^{-1.2} \approx 30.1\%.$$

(d) The cumulative distribution function is the integral of the probability density; thus,

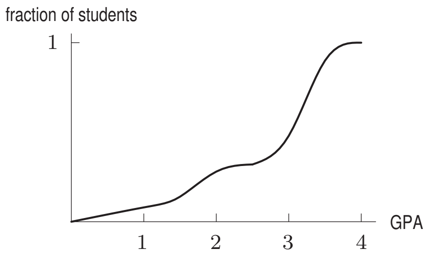
$$C(h) = \int_0^h p(x)\,dx = \int_0^h 0.4e^{-0.4x}\,dx = 1 - e^{-0.4h}.$$

22. (a) The fraction of students passing is given by the area under the curve from 2 to 4 divided by the total area under the curve. This appears to be about $\frac{2}{3}$.

(b) The fraction with honor grades corresponds to the area under the curve from 3 to 4 divided by the total area. This is about $\frac{1}{3}$.

(c) The peak around 2 probably exists because many students work to get just a passing grade.

(d)



23. (a) Most of the earth's surface is below sea level. Much of the earth's surface is either around 3 miles below sea level or exactly at sea level. It appears that essentially all of the surface is between 4 miles below sea level and 2 miles above sea level. Very little of the surface is around 1 mile below sea level.

(b) The fraction below sea level corresponds to the area under the curve from -4 to 0 divided by the total area under the curve. This appears to be about $\frac{3}{4}$.

24. (a) We must have $\displaystyle\int_0^\infty f(t)dt = 1$, for even though it is possible that any given person survives the disease, everyone eventually dies. Therefore,

$$\int_0^\infty cte^{-kt}dt = 1.$$

Integrating by parts gives

$$\begin{aligned}
\int_0^b cte^{-kt}dt &= -\frac{c}{k}te^{-kt}\Big|_0^b + \int_0^b \frac{c}{k}e^{-kt}dt \\
&= \left(-\frac{c}{k}te^{-kt} - \frac{c}{k^2}e^{-kt}\right)\Big|_0^b \\
&= \frac{c}{k^2} - \frac{c}{k}be^{-kb} - \frac{c}{k^2}e^{-kb}.
\end{aligned}$$

As $b \to \infty$, we see

$$\int_0^{\infty} cte^{-kt} dt = \frac{c}{k^2} = 1 \quad \text{so} \quad c = k^2.$$

(b) We are told that $\displaystyle\int_0^5 f(t)dt = 0.4$, so using the fact that $c = k^2$ and the antiderivatives from part (a), we have

$$\begin{aligned}\int_0^5 k^2 te^{-kt} dt &= \left(-\frac{k^2}{k}te^{-kt} - \frac{k^2}{k^2}e^{-kt}\right)\Big|_0^5 \\ &= 1 - 5ke^{-5k} - e^{-5k} = 0.4\end{aligned}$$

so

$$5ke^{-5k} + e^{-5k} = 0.6.$$

Since this equation cannot be solved exactly, we use a calculator or computer to find $k = 0.275$. Since $c = k^2$, we have $c = (0.275)^2 = 0.076$.

(c) The cumulative death distribution function, $C(t)$, represents the fraction of the population that have died up to time t. Thus,

$$\begin{aligned}C(t) &= \int_0^t k^2 xe^{-kx} dx = \left(-kxe^{-kx} - e^{-kx}\right)\Big|_0^t \\ &= 1 - kte^{-kt} - e^{-kt}.\end{aligned}$$

Strengthen Your Understanding

25. The fact that $p(1) = 0.02$ tells us that the probability that x falls in a small interval of length Δx around 1 is $0.02\Delta x$.

26. The cumulative distribution gives the probability that x is less than some value. Thus, $P(5) = 0.4$ means that the probability that x is less that 5 is 0.4.

27. Every density function $p(t)$ satisfies the equation $\int_{-\infty}^{\infty} p(t)\,dt = 1$. Since $\int_{-\infty}^{\infty} t^2\,dt = \infty$, the function $p(t) = t^2$ is not a density function.

28. Since this function is increasing for $x > 0$, the area under its graph increases without bound. So the area under the graph of this function is not equal to 1.

29. As $x \to \infty$ the function $P(x) = x^2 e^x$ grows without bound, whereas a cumulative distribution function must approach 1.

30. Every cumulative distribution is a nondecreasing function, but $P(t) = e^{-t^2}$ is decreasing for $t > 0$.

31. A cumulative distribution function is increasing for all values of x; a probability density function is not.

32. We can take

$$p(x) = \begin{cases} 1/20, & 0 \le x \le 20 \\ 0, & \text{otherwise.} \end{cases}$$

It is a density function because $p(x) \ge 0$ for $0 \le x \le 20$ and $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

33. We can use

$$P(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t \le 1 \\ 1, & t > 1. \end{cases}$$

It is a cumulative distribution function because $P(t)$ is an increasing function that increases from 0 to 1.

34. One possible density function is given by

$$p(x) = \begin{cases} 0 & x < 2 \\ 1/5 & 2 \le x \le 7 \\ 0 & 7 < x. \end{cases}$$

This function is nonzero only between $x = 2$ and $x = 7$ and has $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

35. The cumulative distribution function increases between $x = 3$ and $x = 7$. One possible cumulative distribution function is

$$P(x) = \begin{cases} 0 & x < 3 \\ 1/4(x-3) & 3 \le x \le 7 \\ 1 & x > 7. \end{cases}$$

36. False. Since $p(x) < 0$ for $x < 0$, it cannot be a probability density function.

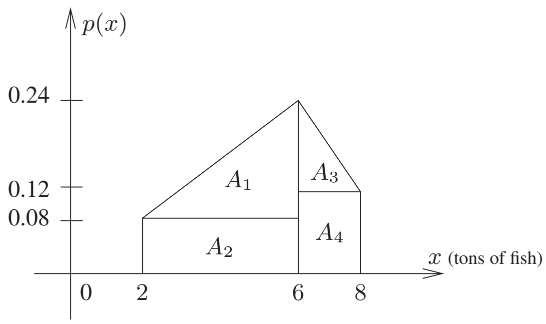
37. False. It is true that $p(x) \ge 0$ for all x, but we also need $\int_{-\infty}^{\infty} p(x)dx = 1$. Since $p(x) = 0$ for $x \le 0$, we need only check the integral from 0 to ∞. We have

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_0^b = \frac{1}{2}.$$

Solutions for Section 8.8

Exercises

1.



Splitting the figure into four pieces, we see that

$$\begin{aligned}\text{Area under the curve} &= A_1 + A_2 + A_3 + A_4 \\ &= \frac{1}{2}(0.16)4 + 4(0.08) + \frac{1}{2}(0.12)2 + 2(0.12) \\ &= 1.\end{aligned}$$

We expect the area to be 1, since $\displaystyle\int_{-\infty}^{\infty} p(x)\,dx = 1$ for any probability density function, and $p(x)$ is 0 except when $2 \le x \le 8$.

2. Recall that the mean is $\displaystyle\int_{-\infty}^{\infty} xp(x)\,dx$. In the fishing example, $p(x) = 0$ except when $2 \le x \le 8$, so the mean is

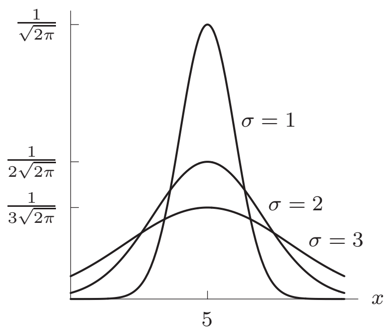
$$\int_2^8 xp(x)\,dx.$$

Using the equation for $p(x)$ from the graph,

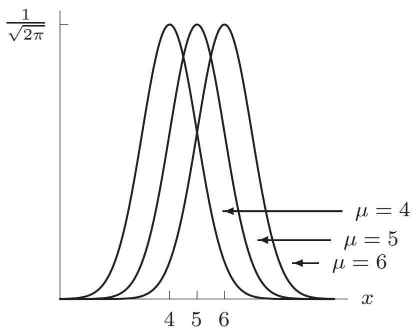
$$\begin{aligned}\int_2^8 xp(x)\,dx &= \int_2^6 xp(x)\,dx + \int_6^8 xp(x)\,dx \\ &= \int_2^6 x(0.04x)\,dx + \int_6^8 x(-0.06x + 0.6)\,dx \\ &= \frac{0.04x^3}{3}\Bigg|_2^6 + \left(-0.02x^3 + 0.3x^2\right)\Bigg|_6^8 \\ &\approx 5.253 \text{ tons.}\end{aligned}$$

3. (a)

(i)



(ii)



(b) Recall that the mean is the "balancing point." In other words, if the area under the curve was made of cardboard, we'd expect it to balance at the mean. All of the graphs are symmetric across the line $x = \mu$, so μ is the "balancing point" and hence the mean.

As the graphs also show, increasing σ flattens out the graph, in effect lessening the concentration of the data near the mean. Thus, the smaller the σ value, the more data is clustered around the mean.

Problems

4. The mean is the value of the integral

$$\int_0^2 x \cdot 0.5(2-x)\,dx = \frac{2}{3}.$$

The median is the value of T such that

$$\int_0^T 0.5(2-x)\,dx = 0.5.$$

Integrating gives the equation

$$x - 0.5\frac{x^2}{2}\bigg|_0^T = T - \frac{T^2}{4} = 0.5$$

which is a quadratic with solutions $T = 2 \pm \sqrt{2}$. Since the median must be between 0 and 2, the solution we want is $T = 2 - \sqrt{2} = 0.586$.

5. The median is the value of T such that $P(T) = 0.5$, so we solve

$$T - \frac{T^2}{4} = \frac{1}{2}$$

to get $T = 2 \pm \sqrt{2}$. Since the median is between 0 and 2 we discard the larger solution, so the median is $T = 2 - \sqrt{2}$.
To find the mean, we first calculate the probability density

$$\text{Density } = p(x) = P'(x) = 1 - \frac{x}{2}$$

and then evaluate the integral

$$\text{Mean } = \int_0^2 xp(x)\,dx = \int_0^2 x - \frac{x^2}{2}\,dx = \frac{2}{3}.$$

So the mean is $2/3$.

6. (a) Since $d(e^{-ct})/dt = ce^{-ct}$, we have

$$c\int_0^6 e^{-ct}dt = -e^{-ct}\Big|_0^6 = 1 - e^{-6c} = 0.1,$$

so

$$c = -\frac{1}{6}\ln 0.9 \approx 0.0176.$$

(b) Similarly, with $c = 0.0176$, we have

$$c\int_6^{12} e^{-ct}dt = -e^{-ct}\Big|_6^{12}$$
$$= e^{-6c} - e^{-12c} = 0.9 - 0.81 = 0.09,$$

so the probability is 9%.

7. (a) We can find the proportion of students by integrating the density $p(x)$ between $x = 1.5$ and $x = 2$:

$$P(2) - P(1.5) = \int_{1.5}^{2} \frac{x^3}{4}\,dx$$
$$= \frac{x^4}{16}\Big|_{1.5}^{2}$$
$$= \frac{(2)^4}{16} - \frac{(1.5)^4}{16} = 0.684,$$

so that the proportion is $0.684 : 1$ or 68.4%.

(b) We find the mean by integrating x times the density over the relevant range:

$$\text{Mean} = \int_0^2 x\left(\frac{x^3}{4}\right)\,dx$$
$$= \int_0^2 \frac{x^4}{4}\,dx$$
$$= \frac{x^5}{20}\Big|_0^2$$
$$= \frac{2^5}{20} = 1.6 \text{ hours.}$$

(c) The median will be the time T such that exactly half of the students are finished by time T, or in other words

$$\frac{1}{2} = \int_0^T \frac{x^3}{4}\,dx$$
$$\frac{1}{2} = \frac{x^4}{16}\Big|_0^T$$
$$\frac{1}{2} = \frac{T^4}{16}$$
$$T = \sqrt[4]{8} = 1.682 \text{ hours.}$$

8. (a) Since $\int_0^\infty p(x)\,dx = 1$, we have

$$1 = \int_0^\infty ae^{-0.122x}\,dx$$
$$= \frac{a}{-0.122}e^{-0.122x}\Big|_0^\infty = \frac{a}{0.122}.$$

So $a = 0.122$.

(b)

$$P(x) = \int_0^x p(t)\,dt$$
$$= \int_0^x 0.122e^{-0.122t}\,dt$$
$$= -e^{0.122t}\Big|_0^x = 1 - e^{-0.122x}.$$

(c) Median is the x such that

$$P(x) = 1 - e^{-0.122x} = 0.5.$$

So $e^{-0.122x} = 0.5$. Thus,

$$x = -\frac{\ln 0.5}{0.122} \approx 5.68 \text{ seconds}$$

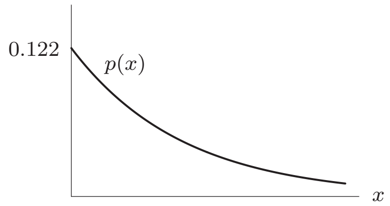
and

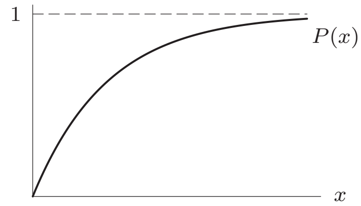
$$\text{Mean} = \int_0^\infty x(0.122)e^{-0.122x}\,dx = -\int_0^\infty x\left(-0.122e^{-0.122x}\right)\,dx.$$

We now use integration by parts. Let $u = -x$ and $v' = -0.122e^{-0.122x}$. Then $u' = -1$, and $v = e^{-0.122x}$. Therefore,

$$\text{Mean} = -xe^{-0.122x}\Big|_0^\infty + \int_0^\infty e^{-0.122x}\,dx = \frac{1}{0.122} \approx 8.20 \text{ seconds.}$$

(d)





9. (a) The cumulative distribution function

$$\begin{aligned} P(t) = \int_0^t p(x)dx &= \text{Area under graph of density function } p(x) \text{ for } 0 \le x \le t \\ &= \text{Fraction of population who survive } t \text{ years or less after treatment} \\ &= \text{Fraction of population who survive up to } t \text{ years after treatment.} \end{aligned}$$

(b) The probability that a randomly selected person survives for at least t years is the probability that he lives t years or longer, so

$$\begin{aligned} S(t) &= \int_t^\infty p(x)\,dx = \lim_{b\to\infty}\int_t^b Ce^{-Ct}\,dx \\ &= \lim_{b\to\infty} -e^{-Ct}\Big|_t^b = \lim_{b\to\infty} -e^{-Cb} - (-e^{-Ct}) = e^{-Ct}, \end{aligned}$$

or equivalently,

$$S(t) = 1 - \int_0^t p(x)\,dx = 1 - \int_0^t Ce^{-Ct}\,dx = 1 + e^{-Ct}\Big|_0^t = 1 + (e^{-Ct} - 1) = e^{-Ct}.$$

(c) The probability of surviving at least two years is

$$S(2) = e^{-C(2)} = 0.70$$

so

$$\begin{aligned} \ln e^{-C(2)} &= \ln 0.70 \\ -2C &= \ln 0.7 \\ C &= -\frac{1}{2}\ln 0.7 \approx 0.178. \end{aligned}$$

10. (a) The probability you dropped the glove within a kilometer of home is given by

$$\int_0^1 2e^{-2x}dx = -e^{-2x}\Big|_0^1 = -e^{-2}+1 \approx 0.865.$$

(b) Since the probability that the glove was dropped within y km $= \int_0^y p(x)dx = 1-e^{-2y}$, we solve

$$\begin{aligned} 1-e^{-2y} &= 0.95 \\ e^{-2y} &= 0.05 \\ y &= \frac{\ln 0.05}{-2} \approx 1.5 \text{ km.} \end{aligned}$$

11. (a) Since $\mu = 100$ and $\sigma = 15$:

$$p(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-100}{15}\right)^2}.$$

(b) The fraction of the population with IQ scores between 115 and 120 is (integrating numerically)

$$\begin{aligned} \int_{115}^{120} p(x)\,dx &= \int_{115}^{120} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{450}}\,dx \\ &= \frac{1}{15\sqrt{2\pi}} \int_{115}^{120} e^{-\frac{(x-100)^2}{450}}\,dx \\ &\approx 0.067 = 6.7\% \text{ of the population.} \end{aligned}$$

12. (a) The normal distribution of car speeds with $\mu = 58$ and $\sigma = 4$ is shown in Figure 8.138.

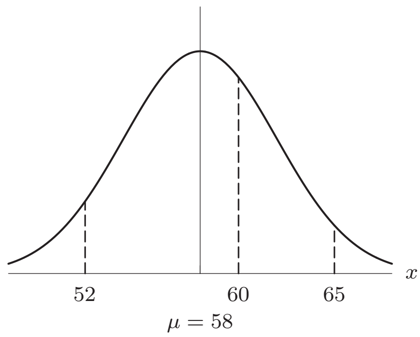


Figure 8.138

The probability that a randomly selected car is going between 60 and 65 is equal to the area under the curve from $x = 60$ to $x = 65$,

$$\text{Probability} = \frac{1}{4\sqrt{2\pi}} \int_{60}^{65} e^{-(x-58)^2/(2\cdot 4^2)}\,dx \approx 0.2685.$$

We obtain the value 0.2685 using a calculator or computer.

(b) To find the fraction of cars going under 52 km/hr, we evaluate the integral

$$\text{Fraction} = \frac{1}{4\sqrt{2\pi}} \int_0^{52} e^{-(x-58)^2/32} dx \approx 0.067.$$

Thus, approximately 6.7% of the cars are going less than 52 km/hr.

13. (a) First, we find the critical points of $p(x)$:

$$\begin{aligned}\frac{d}{dx}p(x) &= \frac{1}{\sigma\sqrt{2\pi}}\left[\frac{-2(x-\mu)}{2\sigma^2}\right]e^{-\frac{(x-\mu)^2}{2\sigma^2}}\\ &= -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.\end{aligned}$$

This implies $x=\mu$ is the only critical point of $p(x)$.

To confirm that $p(x)$ is maximized at $x=\mu$, we rely on the first derivative test. As $-\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is always negative, the sign of $p'(x)$ is the opposite of the sign of $(x-\mu)$; thus $p'(x)>0$ when $x<\mu$, and $p'(x)<0$ when $x>\mu$.

(b) To find the inflection points, we need to find where $p''(x)$ changes sign; that will happen only when $p''(x)=0$. As

$$\frac{d^2}{dx^2}p(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left[-\frac{(x-\mu)^2}{\sigma^2}+1\right],$$

$p''(x)$ changes sign when $\left[-\frac{(x-\mu)^2}{\sigma^2}+1\right]$ does, since the sign of the other factor is always negative. This occurs when

$$\begin{aligned}-\frac{(x-\mu)^2}{\sigma^2}+1 &= 0,\\ -(x-\mu)^2 &= -\sigma^2,\\ x-\mu &= \pm\sigma.\end{aligned}$$

Thus, $x=\mu+\sigma$ or $x=\mu-\sigma$. Since $p''(x)>0$ for $x<\mu-\sigma$ and $x>\mu+\sigma$ and $p''(x)<0$ for $\mu-\sigma\le x\le\mu+\sigma$, these are in fact points of inflection.

(c) μ represents the mean of the distribution, while σ is the standard deviation. In other words, σ gives a measure of the "spread" of the distribution, i.e., how tightly the observations are clustered about the mean. A small σ tells us that most of the data are close to the mean; a large σ tells us that the data is spread out.

14. The fraction of the population within one standard deviation of the mean is given by

$$\text{Fraction within } \sigma \text{ of mean } = \int_{-\sigma}^{\sigma}\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/(2\sigma^2)}dx.$$

Let us substitute $w=\frac{x}{\sigma}$ so that $dw=\frac{1}{\sigma}dx$, and when $x=\pm\sigma$, $w=\pm1$. Then we have

$$\text{Fraction } = \int_{-\sigma}^{\sigma}\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/(2\sigma^2)}dx = \int_{-1}^{1}\frac{1}{\sqrt{2\pi}\sigma}e^{-w^2/2}\cdot\sigma dw = \int_{-1}^{1}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}dw.$$

This integral is independent of σ. Evaluating the integral numerically gives 0.68, showing that about 68% of the population lies within one standard deviation of the mean.

15. It is not (a) since a probability density must be a non-negative function; not (c) since the total integral of a probability density must be 1; (b) and (d) are probability density functions, but (d) is not a good model. According to (d), the probability that the next customer comes after 4 minutes is 0. In real life there should be a positive probability of not having a customer in the next 4 minutes. So (b) is the best answer.

16. (a) Since P is the cumulative distribution function, the percentage of households that made between \$40,000 and \$60,000 is

$$P(60)-P(40) = 66.8\% - 50.1\% = 16.7\%.$$

Therefore 16.7% of the households made between \$40,000 and \$60,000.

The percentage of households making over \$100,000 is $100\% - 87.1\% = 12.9\%$.

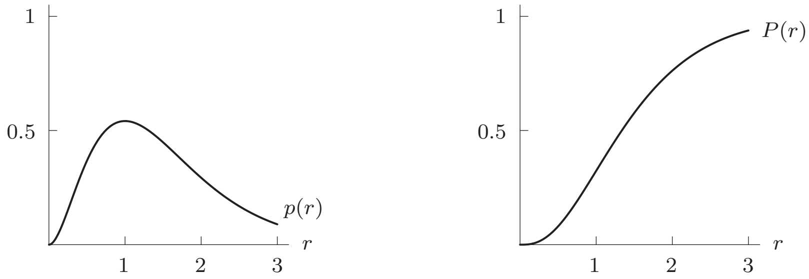
(b) The median income is the income such that half the households make less than this amount. Looking at the table, we see that the 50% mark occurs between \$20,000 and \$40,000. Since $P(20)=29.5\%$, we know 29.5% of the households made less than \$20,000. Assuming local linearity we calculate the slope of the line connecting $(20, 29.5)$ with $(40, 50.1)$ as $(50.1-29.5)/20 = 1.03$. If x is the additional income it takes to achieve the median, then $1.03x = 50.0-29.5$, so $x=19.90$. Since $20+19.90=39.90$, the median income is approximately \$39,900. This looks sensible since just over 50% of the population earn \$40,000.

(c) The percentage of households that made between \$40,000 and \$75,000 is $76.2-50.1=26.1$. Since this percentage is less than $1/3$, the statement is false.

17. (a) Let the $p(r)$ be the density function. Then $P(r) = \int_0^r p(x)\,dx$, and from the Fundamental Theorem of Calculus, $p(r) = \frac{d}{dr}P(r) = \frac{d}{dr}(1-(2r^2+2r+1)e^{-2r}) = -(4r+2)e^{-2r} + 2(2r^2+2r+1)e^{-2r}$, or $p(r) = 4r^2e^{-2r}$.

We have that $p'(r) = 8r(e^{-2r}) - 8r^2e^{-2r} = e^{-2r}\cdot 8r(1-r)$, which is zero when $r=0$ or $r=1$, negative when $r>1$, and positive when $r<1$. Thus $p(1) = 4e^{-2} \approx 0.54$ is a relative maximum.

Here are sketches of $p(r)$ and the cumulative position $P(r)$:



(b) The median distance is the distance r such that $P(r) = 1-(2r^2+2r+1)e^{-2r} = 0.5$, or equivalently, $(2r^2+2r+1)e^{-2r} = 0.5$.

By experimentation with a calculator, we find that $r \approx 1.33$ Bohr radii is the median distance.

The mean distance is equal to the value of the integral $\int_0^\infty rp(r)\,dr = \lim_{x\to\infty}\int_0^x rp(r)\,dr$. We have that $\int_0^x rp(r)\,dr = \int_0^x 4r^3e^{-2r}\,dr$. Using the integral table, we get

$$\begin{aligned}\int_0^x 4r^3e^{-2r}\,dr &= \left[\left(-\frac{1}{2}\right)4r^3 - \frac{1}{4}(12r^2) - \frac{1}{8}(24r) - \frac{1}{16}(24)\right]e^{-2x}\Bigg|_0^x \\ &= \frac{3}{2} - \left[2x^3+3x^2+3x+\frac{3}{2}\right]e^{-2x}.\end{aligned}$$

Taking the limit of this expression as $x\to\infty$, we see that all terms involving (powers of x or constants) $\cdot\, e^{-2x}$ have limit 0, and thus the mean distance is 1.5 Bohr radii.

The most likely distance is obtained by maximizing $p(r) = 4r^2e^{-2r}$; as we have already seen this corresponds to $r=1$ Bohr unit.

(c) Because it is the most likely distance of the electron from the nucleus.

Strengthen Your Understanding

18. A median satisfies $P(T) = 0.5$ where P is the cumulative distribution function.

19. The median is the value which divides the area under the density function graph into halves. The median of this function cannot be 1 since all of the area is to the left of $x=1$. The median is between 0 and 1.

20. We can use the normal distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

with $\mu = \sigma = 1/2$ to get

$$p(x) = \frac{2}{\sqrt{2\pi}}e^{-2(x-1/2)^2}.$$

21. We can take a constant, or uniform distribution

$$p(x) = \begin{cases} 0 & \text{for} \quad x<0 \\ 1 & \text{for} \quad 0\le x\le 1 \\ 0 & \text{for} \quad x>1. \end{cases}$$

Since

$$\int_0^1 x\,dx = \frac{1}{2}, \quad \text{the mean is } \frac{1}{2}.$$

Since

$$\int_0^{1/2} p(x)\,dx = \int_0^{1/2} 1\,dx = \frac{1}{2}, \quad \text{the median is also } \frac{1}{2}.$$

22. False. Note that p is the density function for the population, not the cumulative density function. Thus $p(10) = 1/2$ means that the probability of x lying in a small interval of length Δx around $x = 10$ is about $(1/2)\Delta x$.

23. True. This follows directly from the definition of the cumulative density function.

24. True. The interval from $x = 9.98$ to $x = 10.04$ has length 0.06. Assuming that the value of $p(x)$ is near $1/2$ for $9.98 < x < 10.04$, the fraction of the population in that interval is $\int_{9.98}^{10.04} p(x)\,dx \approx (1/2)(0.06) = 0.03$.

25. False. Note that p is the density function for the population, not the cumulative density function. Thus $p(10) = p(20)$ means that x values near 10 are as likely as x values near 20.

26. True. By the definition of the cumulative distribution function, $P(20) - P(10) = 0$ is the fraction of the population having x values between 10 and 20.

Solutions for Chapter 8 Review

Exercises

1. Vertical slices are circular. Horizontal slices would be similar to ellipses in cross-section, or at least ovals (a word derived from *ovum*, the Latin word for egg).

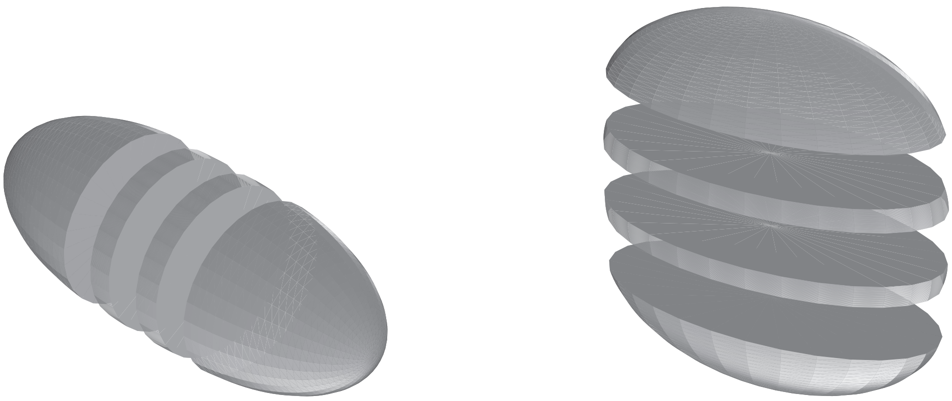
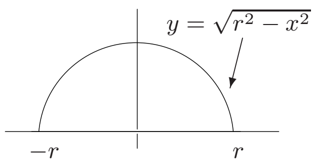


Figure 8.139

2. The limits of integration are 0 and b, and the rectangle represents the region under the curve $f(x) = h$ between these limits. Thus,

$$\text{Area of rectangle} = \int_0^b h\,dx = hx\Big|_0^b = hb.$$

3. The circle $x^2 + y^2 = r^2$ cannot be expressed as a function $y = f(x)$, since for every x with $-r < x < r$, there are two corresponding y values on the circle. However, if we consider the top half of the circle only, as shown below, we have $x^2 + y^2 = r^2$, or $y^2 = r^2 - x^2$, and taking the positive square root, we have that $y = \sqrt{r^2 - x^2}$ is the equation of the top semicircle.



Then

$$\text{Area of Circle} = 2(\text{Area of semicircle}) = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

We evaluate this using integral table formula 30.

$$\begin{aligned}2\int_{x=-r}^{x=r}\sqrt{r^2-x^2}\,dx &= 2\left[\frac{1}{2}\left(x\sqrt{r^2-x^2}+r^2\arcsin\frac{x}{r}\right)\right]\Bigg|_{-r}^{r}\\ &= r^2(\arcsin 1-\arcsin(-1))\\ &= r^2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\pi r^2.\end{aligned}$$

4. Name the slanted line $y = f(x)$. Then the triangle is the region under the line $y = f(x)$ and between the lines $y = 0$ and $x = b$. Thus,

$$\text{Area of triangle} = \int_0^b f(x)\,dx.$$

Since $f(x)$ is a line of slope h/b which passes through the origin, its equation is $f(x) = hx/b$. Thus,

$$\text{Area of triangle} = \int_0^b \frac{hx}{b}\,dx = \frac{hx^2}{2b}\Bigg|_0^b = \frac{hb^2}{2b} = \frac{hb}{2}.$$

5. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = x^2 + 1$ and thickness Δx. See Figure 8.140. The volume of a typical slice is $\pi(x^2+1)^2\Delta x$. The volume, V, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(x^2+1)^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_0^4 \pi(x^2+1)^2dx = \pi\int_0^4 (x^4+2x^2+1)dx = \pi\left(\frac{x^5}{5}+\frac{2x^3}{3}+x\right)\Bigg|_0^4 = \frac{3772\pi}{15} = 790.006.$$

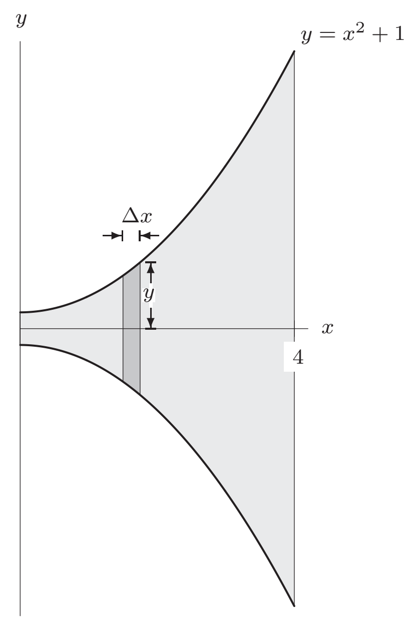


Figure 8.140

6. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = \sqrt{x}$ and thickness Δx. See Figure 8.141. The volume of a typical slice is $\pi(\sqrt{x})^2\Delta x$. The volume, V, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(\sqrt{x})^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_1^2 \pi(\sqrt{x})^2 dx = \pi \int_1^2 x dx = \pi \left(\frac{x^2}{2}\right)\bigg|_1^2 = \frac{3\pi}{2} = 4.712.$$

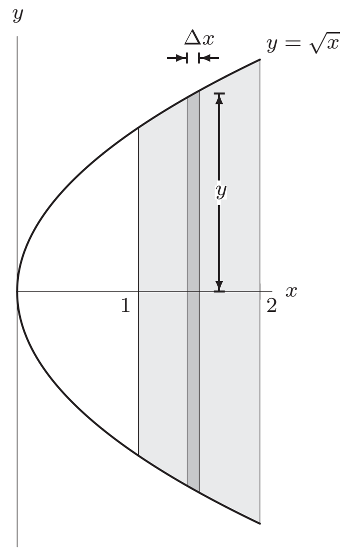


Figure 8.141

7. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = e^{-2x}$ and thickness Δx. See Figure 8.142. The volume of a typical slice is $\pi(e^{-2x})^2\Delta x$. The volume, V, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(e^{-2x})^2 \Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_0^1 \pi(e^{-2x})^2 dx = \pi \int_0^1 e^{-4x} dx = \pi\left(-\frac{1}{4}\right)(e^{-4x})\bigg|_0^1 = -\frac{\pi}{4}(e^{-4} - 1) = 0.771.$$

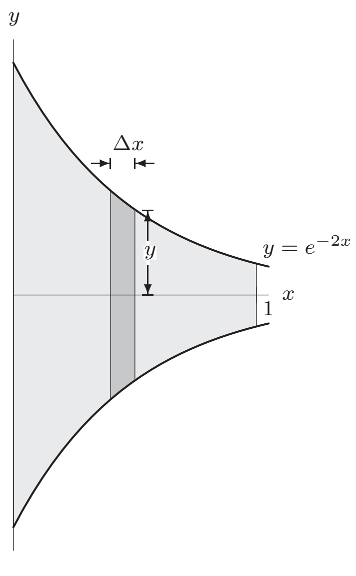


Figure 8.142

8. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = 4 - x^2$ and thickness Δx. See Figure 8.143. The volume, V, of a typical slice is $\pi(4-x^2)^2\Delta x$. The volume of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(4 - x^2)^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral. Since the region lies between $x = -2$ and $x = 2$, we have:

$$V = \int_{-2}^{2} \pi(4-x^2)^2 dx = \pi \int_{-2}^{2} (16 - 8x^2 + x^4)dx = \pi \left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Bigg|_{-2}^{2} = \frac{512\pi}{15} = 107.233.$$

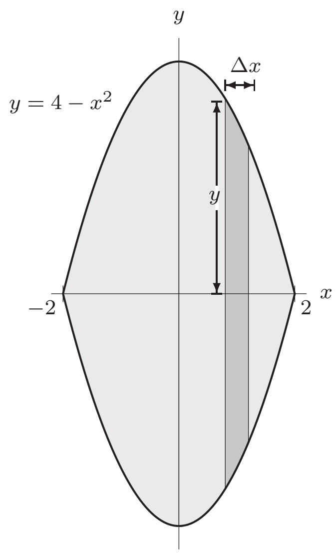


Figure 8.143

9. We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a smaller circular disk of inside radius r_{in}. We see in Figure 8.144 that $r_{\text{out}} = 2x$ and $r_{\text{in}} = x$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(2x)^2\Delta x - \pi(x)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi(2x)^2 - \pi(x)^2)dx = \pi \int_0^3 (4x^2 - x^2)\, dx = \pi \int_0^3 3x^2 dx = \pi x^3\Big|_0^3 = 27\pi = 84.823.$$

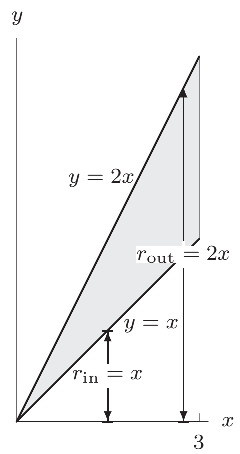


Figure 8.144

10. The two functions intersect at $(0,0)$ and $(8,2)$. We slice the volume with planes perpendicular to the x-axis. This divides the solid into thin washers with

$$\text{Volume of slice } = \pi r_{out}^2 \Delta x - \pi r_{in}^2 \Delta x.$$

The inner radius is the vertical distance from the x-axis to the curve $y = \frac{1}{4}x$. Similarly, the outer radius is the vertical distances from the x-axis to the curve $y = \sqrt[3]{x}$. Integrating from $x = 0$ to $x = 8$ we have

$$V = \int_0^8 \left[\pi(\sqrt[3]{x})^2 - \pi(\frac{1}{4}x)^2\right] dx.$$

11. The region is bounded by $y = 2$, the y-axis and $y = x^{1/3}$. The two functions $y = 2$ and $y = x^{1/3}$ intersect at $(8,2)$. We slice the volume with planes that are perpendicular to the y-axis. This divides the solid into thin cylinders with

$$\text{Volume } \approx \pi r^2 \Delta y.$$

The radius is the distance from the y-axis to the curve $x = y^3$. Integrating from $y = 0$ to $y = 2$ we have

$$V = \int_0^2 \pi(y^3)^2 \, dy.$$

12. The region is bounded by $y = 2$, the y-axis and $y = x^{1/3}$. The two functions $y = 2$ and $y = x^{1/3}$ intersect at $(8,2)$. We slice the volume with planes that are perpendicular to the line $y = -2$. This divides the solid into thin washers with

$$\text{Volume } \approx \pi r_{out}^2 \Delta x - \pi r_{in}^2 \Delta x.$$

The inner radius is the distance from the line $y = -2$ to the curve $y = x^{1/3}$ and the outer radius is the distance from the line $y = -2$ to the line $y = 2$. Integrating from $x = 0$ to $x = 8$ we have

$$V = \int_0^8 \left[\pi(2-(-2))^2 - \pi(x^{1/3} - (-2))^2\right] dx.$$

13. The region is bounded by $x = 4y$, the x-axis and $x = 8$. The two lines $x = 4y$ and $x = 8$ intersect at $(8,2)$. We slice the volume with planes that are perpendicular to the line $x = 10$. This divides the solid into thin washers with

$$\text{Volume } \approx \pi r_{out}^2 dy - \pi r_{in}^2 dy.$$

The inner radius is the distance from the line $x = 10$ to the line $x = 8$ and the outer radius is the distance from the line $x = 10$ to the line $x = 4y$. Integrating from $y = 0$ to $y = 2$ we have

$$V = \int_0^2 \left[\pi(10 - 4y)^2 - \pi(2)^2\right] dy.$$

14. The region is bounded by $y = \frac{1}{4}x$, the x-axis and $x = 8$. The two lines $y = \frac{1}{4}x$ and $x = 8$ intersect at $(8,2)$. We slice the volume with planes that are perpendicular to the line $y = 3$. This divides the solid into thin washers with

$$\text{Volume } \approx \pi r_{out}^2 \Delta x - \pi r_{in}^2 \Delta x.$$

The inner radius is the distance from the line $y = 3$ to the line $y = \frac{1}{4}x$ and the outer radius is the distance from the line $y = 3$ to the x-axis. Integrating from $x = 0$ to $x = 8$ we have

$$V = \int_0^8 \left[\pi(1 - \frac{1}{4}x)^2 - \pi(3)^2\right] dx.$$

15. The region is bounded by $x = 4x$ and $x = y^3$. The two functions intersect at $(0,0)$ and $(8,2)$. We slice the volume with planes that are perpendicular to the line $x = -3$. This divides the solid into thin washers with

$$\text{Volume } = \pi r_{out}^2 \Delta y - \pi r_{in}^2 \Delta y.$$

The inner radius is the distance from the line $x = -3$ to $x = y^3$ and the outer radius is the distance from the line $x = -3$ to the line $x = 4y$. Integrating from $y = 0$ to $y = 2$ we have

$$V = \int_0^2 \left[\pi(4y + 3)^2 - \pi(y^3 + 3)^2\right] dx.$$

16. Each slice is a circular disk. The radius, r, of the disk increases with h and is given in the problem by $r = \sqrt{h}$. Thus

$$\text{Volume of slice } \approx \pi r^2 \Delta h = \pi h \Delta h.$$

Summing over all slices, we have

$$\text{Total volume } \approx \sum \pi h \Delta h.$$

Taking a limit as $\Delta h \to 0$, we get

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum \pi h \Delta h = \int_0^{12} \pi h \, dh.$$

Evaluating gives

$$\text{Total volume } = \pi \frac{h^2}{2} \bigg|_0^{12} = 72\pi.$$

17.

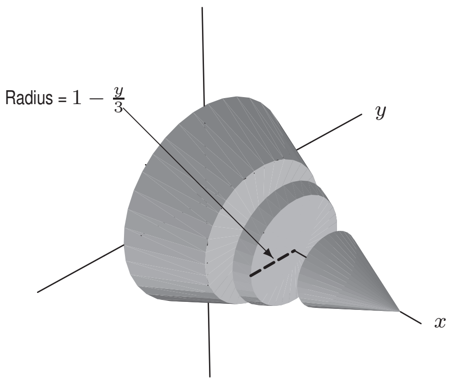


Figure 8.145

Slice parallel to the base of the cone, or, equivalently, rotate the line $x = (3 - y)/3$ about the y–axis. (One can also slice the other way.) See Figure 8.145. The volume V is given by

$$\begin{aligned} V &= \int_{y=0}^{y=3} \pi x^2 \, dy = \int_0^3 \pi \left(\frac{3-y}{3}\right)^2 dy \\ &= \pi \int_0^3 \left(1 - \frac{2y}{3} + \frac{y^2}{9}\right) dy \\ &= \pi \left(y - \frac{y^2}{3} + \frac{y^3}{27}\right) \bigg|_0^3 = \pi. \end{aligned}$$

18. (a) We slice the pyramid horizontally. See Figure 8.146. Each slice is a square slab of thickness Δh, so the volume of a slice at height h is $s^2 \Delta h$, where s is the length of a side. We use the similar triangles in Figure 8.147 to write s as a function of h:

$$\frac{s}{10-h} = \frac{8}{10} \quad \text{so} \quad s = 0.8(10-h).$$

The volume of the slice at height h is $(0.8(10-h))^2 \Delta h$. To find the total volume, we integrate this quantity from $h = 0$ to $h = 10$.

$$V = \int_0^{10} (0.8(10-h))^2 dh = 0.64 \int_0^{10} (h-10)^2 dh = \frac{16}{75}(h-10)^3 \bigg|_0^{10} = \frac{640}{3} = 213.333 \text{ m}^3.$$

(b) As in part (a),

$$\text{Volume of a slice at height } h \approx s^2 \Delta h = (0.8(10-h))^2 \Delta h.$$

The height h ranges from $h = 0$ to $h = 6$. We have

$$V = \int_0^6 (0.8(10-h))^2 dh = 0.64 \int_0^6 (h-10)^2 \, dh = \frac{16}{75}(h-10)^3 \bigg|_0^6 = \frac{4992}{25} = 199.680 \text{ m}^3.$$

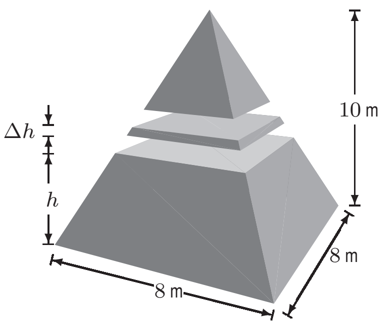


Figure 8.146

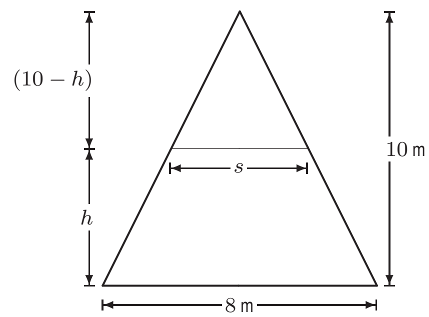


Figure 8.147

19. We slice the tank horizontally. There is an outside radius r_{out} and an inside radius r_{in} and, at height h,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta h - \pi(r_{\text{in}})^2\Delta h.$$

See Figure 8.148. We see that $r_{\text{out}} = 3$ for every slice. We use similar triangles to find r_{in} in terms of the height h:

$$\frac{r_{\text{in}}}{h} = \frac{3}{6} \quad \text{so} \quad r_{\text{in}} = \frac{1}{2}h.$$

At height h,

$$\text{Volume of slice } \approx \pi(3)^2\Delta h - \pi\left(\frac{1}{2}h\right)^2\Delta h.$$

To find the total volume, we integrate this quantity from $h = 0$ to $h = 6$.

$$V = \int_0^6 \left(\pi(3)^2 - \pi\left(\frac{1}{2}h\right)^2\right) dh = \pi\int_0^6 \left(9 - \frac{1}{4}h^2\right) dh = \pi\left(9h - \frac{h^3}{12}\right)\Bigg|_0^6 = 36\pi = 113.097 \text{ m}^3.$$

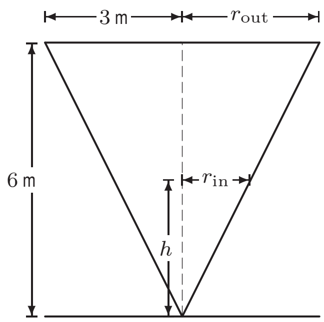


Figure 8.148

20. Since $f(x) = \sin x$, $f'(x) = \cos(x)$, so

$$\text{Arc Length} = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx.$$

21. We'll find the arc length of the top half of the ellipse, and multiply that by 2. In the top half of the ellipse, the equation $(x^2/a^2) + (y^2/b^2) = 1$ implies

$$y = +b\sqrt{1 - \frac{x^2}{a^2}}.$$

Differentiating $(x^2/a^2) + (y^2/b^2) = 1$ implicitly with respect to x gives us

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0,$$

so

$$\frac{dy}{dx} = \frac{\frac{-2x}{a^2}}{\frac{2y}{b^2}} = -\frac{b^2x}{a^2y}.$$

Substituting this into the arc length formula, we get

$$\begin{aligned}\text{Arc Length} &= \int_{-a}^{a} \sqrt{1 + \left(-\frac{b^2x}{a^2y}\right)^2}\, dx \\ &= \int_{-a}^{a} \sqrt{1 + \left(\frac{b^4x^2}{a^4(b^2)(1 - \frac{x^2}{a^2})}\right)}\, dx \\ &= \int_{-a}^{a} \sqrt{1 + \left(\frac{b^2x^2}{a^2(a^2 - x^2)}\right)}\, dx.\end{aligned}$$

Hence the arc length of the entire ellipse is

$$2\int_{-a}^{a}\sqrt{1+\left(\frac{b^2x^2}{a^2(a^2-x^2)}\right)}\,dx.$$

22. Since $f'(x)=\cos x$, we have

$$L=\int_0^3\sqrt{1+(f'(x))^2}\,dx=\int_0^3\sqrt{1+\cos^2 x}\,dx=3.621.$$

We see in Figure 8.149 that the length of the curve is slightly longer than the length of the x-axis from $x=0$ to $x=3$, so the answer of 3.621 makes sense.

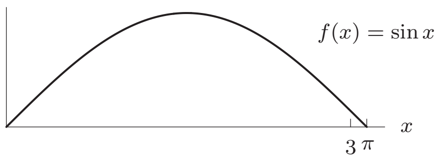


Figure 8.149

23. Since $f'(x)=10x$, we have

$$L=\int_0^3\sqrt{1+(f'(x))^2}\,dx=\int_0^3\sqrt{1+(10x)^2}\,dx=\int_0^3\sqrt{1+100x^2}\,dx=45.230.$$

We see in Figure 8.150 that the length of the curve is definitely longer than 45 and slightly longer than $\sqrt{45^2+3^2}=45.10$, so the answer of 45.230 is reasonable.

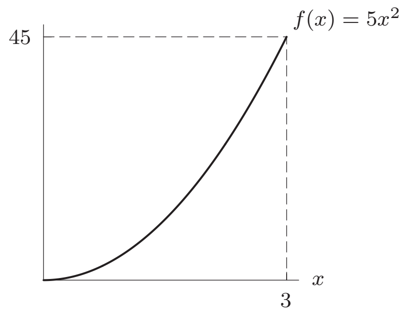


Figure 8.150

24. The arc length of $\sqrt{1-x^2}$ from $x=0$ to $x=1$ is one quarter of the perimeter of the unit circle. Hence the length is $\frac{2\pi}{4}=\frac{\pi}{2}$.

25. The arc length is given by

$$L=\int_1^2\sqrt{1+e^{2x}}\,dx\approx 4.785.$$

Note that $\sqrt{1+e^{2x}}$ does not have an obvious elementary antiderivative, so we use an approximation method to find an approximate value for L.

26. The arc length is given by

$$L=\int_1^2\sqrt{1+\left(x^4+\frac{1}{(16x^4)}-\frac{1}{2}\right)}\,dx=\int_1^2\sqrt{\left(x^2+\frac{1}{(4x^2)}\right)^2}\,dx$$

$$=\int_1^2\left(x^2+\frac{1}{4}x^{-2}\right)dx=\left[\frac{x^3}{3}-\frac{1}{4x}\right]\Bigg|_1^2=\frac{59}{24}.$$

27. We have $dx/dt = -3\sin t$, $dy/dt = 2\cos t$, so, evaluating the integral numerically, we have

$$\text{Arc length } = \int_0^{2\pi} \sqrt{9\cos^2 t + 4\sin^2 t}\, dt = 15.865.$$

The curve is a ellipse.

28. We have $dx/dt = -2\sin(2t)$, $dy/dt = 2\cos(2t)$, so, simplifying the integrand, we have

$$\text{Arc length } = \int_0^{\pi} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}\, dt = 2\int_0^{\pi} dt = 2\pi.$$

The curve is a circle of radius 1.

29.
- $\int_0^1 f(x)\, dx$ gives the area under the graph of f from 0 to 1.
- The graph of f is concave up and passes through the points $(0,0)$ and $(1,1)$, so it lies below the line $y = x$.
- The area under $y = x$ from 0 to 1 is half the area of a square of side 1, or $1/2$. Thus, $\int_0^1 f(x)\, dx < \frac{1}{2}$.

30.
- Since $f(0) = 0$, the Fundamental Theorem gives $\int_0^{0.5} f'(x)\, dx = f(0.5) - f(0) = f(0.5)$.
- The graph of f is concave up and passes through the points $(0,0)$ and $(1,1)$, so it lies below the line $y = x$. This means $f(x) < x$ for $0 < x < 1$.
- Since $f(x) < x$, we have $f(0.5) < 0.5$. Hence $\int_0^{0.5} f'(x)dx < \frac{1}{2}$.

31.
- Since $f(x) = x^p$, we know $f^{-1}(x) = x^{1/p}$, because $(x^p)^{1/p} = \left(x^{1/p}\right)^p = x$.
- Since $f(0) = 0$ and $f(1) = 1$, we know $f^{-1}(0) = 0$ and $f^{-1}(1) = 1$. Likewise, since f is increasing, so is f^{-1}.
- Since $p > 1$, we know $0 < 1/p < 1$, so the graph of f^{-1} is concave down, and therefore lies above the line $y = x$ on $0 < x < 1$.
- $\int_0^1 f^{-1}(x)\, dx$ gives the area under f^{-1} from 0 to 1, so, since the graph lies above $y = x$ on this interval, we have $\int_0^1 f^{-1}(x)\, dx > \frac{1}{2}$.

32.
- $\int_0^1 \pi\left(f(x)\right)^2\, dx$ gives the volume of the region formed by rotating the graph of f on $0 \le x \le 1$ about the x-axis.
- The graph of f is concave up and contains $(0,0)$ and $(1,1)$, so it lies below the line $y = x$ on $0 < x < 1$.
- This means the region formed by rotating the graph of f lies within the region formed by rotating the line segment $y = x$, which is a cone of base $r = 1$ and height $h = 1$. The volume of this cone is $(1/3)\pi r^2 h = \pi/3$.
- Since this cone contains the region formed by rotating the graph of f, we have $\int_0^1 \pi\left(f(x)\right)^2\, dx < \frac{\pi}{3}$.

33.
- $\int_0^1 \sqrt{1 + (f'(x))^2}\, dx$ gives the arc length of the graph of f from $x = 0$ to $x = 1$.
- The graph of f is concave up and contains $(0,0)$ and $(1,1)$, so it lies below the line $y = x$ on $0 < x < 1$. The arc length of the line between $(0,0)$, and $(1,1)$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$.
- The line segment between $(0,0)$ and $(1,1)$ is shorter than the arc length of f, so $\int_0^1 \sqrt{1 + (f'(x))^2}\, dx > \sqrt{2}$.

Problems

34. (a) The points of intersection are $x = 0$ to $x = 2$, so we have

$$\text{Area} = \int_0^2 (2x - x^2)dx = x^2 - \frac{x^3}{3}\bigg|_0^2 = \frac{4}{3} = 1.333.$$

(b) The outside radius is $2x$ and the inside radius is x^2, so we have

$$\text{Volume} = \int_0^2 (\pi(2x)^2 - \pi(x^2)^2)dx = \pi\int_0^2 (4x^2 - x^4)dx = \frac{\pi}{15}(20x^3 - 3x^5)\bigg|_0^2 = \frac{64\pi}{15} = 13.404.$$

(c) The length of the perimeter is equal to the length of the top plus the length of the bottom. Using the arclength formula, and the fact that the derivative of $2x$ is 2 and the derivative of x^2 is $2x$, we have

$$L = \int_0^2 \sqrt{1+2^2}dx + \int_0^2 \sqrt{1+(2x)^2}dx = 4.4721 + 4.6468 = 9.119.$$

35. There are at least two possible answers. Since $\sqrt{4-x^2} - (-\sqrt{4-x^2}) \geq 0$ when $0 \leq x \leq 2$, one possibility is that the integral gives the area between the curve $y = 2\sqrt{4-x^2}$ and the line $y = 0$ as shown in Figure 8.151.

Alternatively, since $\sqrt{4-x^2} \geq -\sqrt{4-x^2}$ when $0 \leq x \leq 2$, the integral gives the area between the quarter circles $y = \sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$, as shown in Figure 8.152.

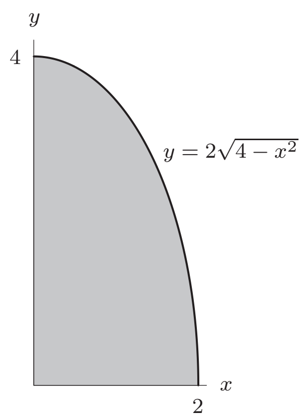


Figure 8.151

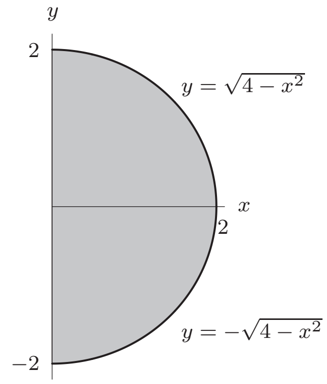


Figure 8.152

36. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the horizontal line $y = 30$. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi((r_{out})^2 - (r_{in})^2)\Delta x.$$

The outer radius is the vertical distance from the line $y = 30$ to the curve $y = x^2$, so $r_{out} = 30 - x^2$. Similarly, the inner radius is the vertical distance from the line $y = 30$ to the curve $y = 5x$, so $r_{in} = 30 - 5x$. Integrating from $x = 0$ to $x = 5$ we have

$$V = \int_0^5 \pi((30-x^2)^2 - (30-5x)^2)\, dx.$$

37. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the vertical line $x = 8$. This divides the solid into thin washers with volume

$$\text{Volume of slice } = \pi((r_{out})^2 - (r_{in})^2)\Delta y.$$

The outer radius is the horizontal distance from the line $x = 8$ to the curve $x = y/5$, so $r_{out} = 8 - y/5$. Similarly, the inner radius is the horizontal distance from the line $x = 8$ to the curve $x = \sqrt{y}$, so $r_{in} = 8 - \sqrt{y}$. Integrating from $y = 0$ to $y = 25$ we have

$$V = \int_0^{25} \pi((8-y/5)^2 - (8-\sqrt{y})^2)\, dy.$$

38. (a) See Figure 8.153

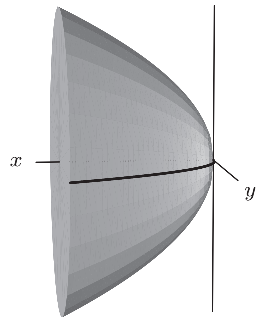


Figure 8.153: Rotated Region

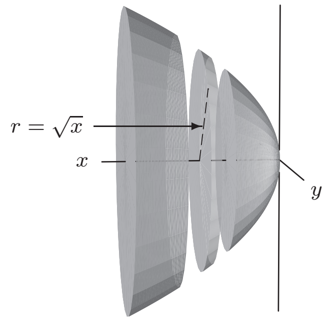


Figure 8.154: Cutaway View

(b) Divide [0,1] into N subintervals of width $\Delta x = \frac{1}{N}$. The volume of the i^{th} disc is $\pi(\sqrt{x_i})^2 \Delta x = \pi x_i \Delta x$. So, $V \approx \sum_{i=1}^{N} \pi x_i \Delta x$. See Figure 8.154

(c)

$$\text{Volume} = \int_0^1 \pi x \, dx = \frac{\pi}{2} x^2 \Big|_0^1 = \frac{\pi}{2} \approx 1.57.$$

39. (a) See Figure 8.155.

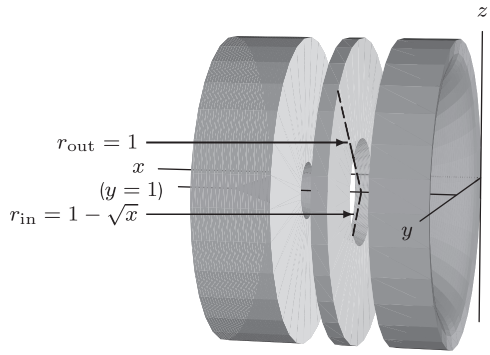


Figure 8.155

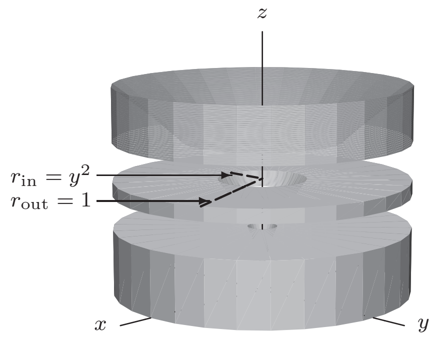


Figure 8.156

Slice the figure perpendicular to the x–axis. One gets washers of inner radius $1 - \sqrt{x}$ and outer radius 1. Therefore,

$$\begin{aligned} V &= \int_0^1 \left(\pi 1^2 - \pi(1-\sqrt{x})^2\right) dx \\ &= \pi \int_0^1 (1 - [1 - 2\sqrt{x} + x]) \, dx \\ &= \pi \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2\right]_0^1 = \frac{5\pi}{6} \approx 2.62. \end{aligned}$$

(b) See Figure 8.156. Note that $x = y^2$. We now integrate over y instead of x, slicing perpendicular to the y–axis. This gives us washers of inner radius x and outer radius 1. So

$$\begin{aligned} V &= \int_{y=0}^{y=1} (\pi 1^2 - \pi x^2) \, dy \\ &= \int_0^1 \pi(1 - y^4) dy \\ &= \left(\pi y - \frac{\pi}{5} y^5\right) \Big|_0^1 = \pi - \frac{\pi}{5} = \frac{4\pi}{5} \approx 2.51. \end{aligned}$$

40. (a) Since $y = ax^2$ is non-negative, we integrate to find the area:

$$\text{Area} = \int_0^2 (ax^2)dx = a\frac{x^3}{3}\bigg|_0^2 = \frac{8a}{3}.$$

(b) Each slice of the object is approximately a cylinder with radius ax^2 and thickness Δx. We have

$$\text{Volume} = \int_0^2 \pi(ax^2)^2 dx = \pi a^2 \frac{x^5}{5}\bigg|_0^2 = \frac{32}{5}a^2\pi.$$

41. (a) Since $y = e^{-bx}$ is non-negative, we integrate to find the area:

$$\text{Area} = \int_0^1 (e^{-bx})dx = \frac{-1}{b}e^{-bx}\bigg|_0^1 = \frac{1}{b}(1 - e^{-b}).$$

(b) Each slice of the object is approximately a cylinder with radius e^{-bx} and thickness Δx. We have

$$\text{Volume} = \int_0^1 \pi(e^{-bx})^2 dx = \pi\int_0^1 e^{-2bx}dx = \frac{-\pi}{2b}e^{-2bx}\bigg|_0^1 = \frac{\pi}{2b}(1 - e^{-2b}).$$

42. (a) We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a smaller circular disk of radius r_{in}. We see in Figure 8.157 that $r_{\text{out}} = \sin x$ and $r_{\text{in}} = 0.5x$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(\sin x)^2\Delta x - \pi(0.5x)^2\Delta x.$$

To find the total volume, we integrate this quantity between the points of intersection $x = 0$ and $x = 1.9$:

$$V = \int_0^{1.9} (\pi(\sin x)^2 - \pi(0.5x)^2)dx = \pi\left(-\frac{\sin x\cos x}{2} - \frac{x^3}{12} + x\right)\bigg|_0^{1.9} = 1.669.$$

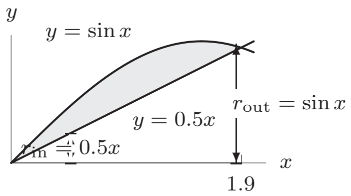


Figure 8.157

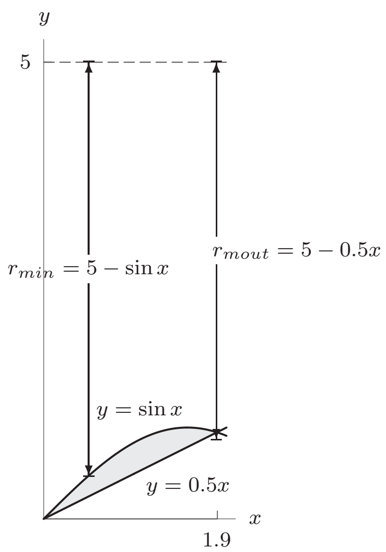


Figure 8.158

(b) We see in Figure 8.158 that $r_{\text{out}} = 5 - 0.5x$ and $r_{\text{in}} = 5 - \sin x$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(5 - 0.5x)^2\Delta x - \pi(5 - \sin x)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between the points of intersection $x = 0$ and $x = 1.9$:

$$V = \int_0^{1.9} \left(\pi(5 - 0.5x)^2 - \pi(5 - \sin x)^2\right)dx = \frac{\pi}{12}(6(\sin x - 20)\cos x + x(x^2 - 30x - 6))\bigg|_0^{1.9} = 11.550.$$

43. We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a disk of radius r_{in}. We see in Figure 8.159 that $r_{\text{out}} = 5 + 2x$ and $r_{\text{in}} = 5$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(5+2x)^2\Delta x - \pi(5)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 4$:

$$V = \int_0^4 (\pi(5+2x)^2 - \pi(5)^2)dx = \pi\int_0^4 ((5+2x)^2 - 25)\,dx = \pi\left(\frac{4}{3}x^3 + 10x^2\right)\Bigg|_0^4 = \frac{736\pi}{3} = 770.737.$$

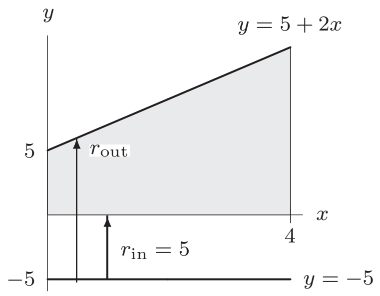


Figure 8.159

44. (a) We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a disk of radius r_{in}. We see in Figure 8.160 that $r_{\text{out}} = 2 + x^2$ and $r_{\text{in}} = 2$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(2+x^2)^2\Delta x - \pi(2)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi(2+x^2)^2 - \pi(2)^2)dx = \pi\int_0^3 ((2+x^2)^2 - 4)\,dx = \frac{\pi}{15}(3x^5 + 20x^3)\Bigg|_0^3 = \frac{423\pi}{5} = 265.778.$$

(b) We see in Figure 8.161 that $r_{\text{out}} = 10$ and $r_{\text{in}} = 10 - x^2$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(10)^2\Delta x - \pi(10-x^2)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi(10)^2 - \pi(10-x^2)^2)dx = \pi\int_0^3 (100 - (10-x^2)^2)\,dx = \frac{\pi}{15}(100x^3 - 3x^5)\Bigg|_0^3 = \frac{657\pi}{5} = 412.805.$$

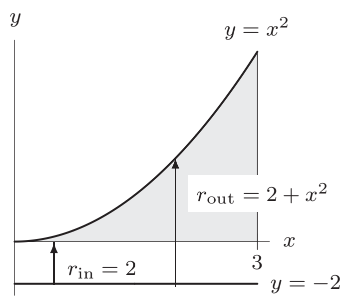


Figure 8.160

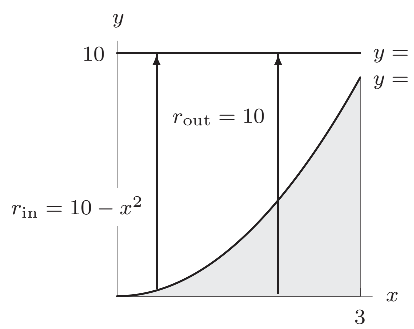


Figure 8.161

45. Slice the object into disks vertically, as in Figure 8.162. A typical disk has thickness Δx and radius $y = \sqrt{1-x^2}$. Thus

$$\text{Volume of disk } \approx \pi y^2 \Delta x = \pi(1-x^2)\,\Delta x.$$

$$\text{Volume of solid } = \lim_{\Delta x \to 0} \sum \pi(1-x^2)\,\Delta x = \int_0^1 \pi(1-x^2)\,dx = \pi\left(x - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2\pi}{3}.$$

Note: As we expect, this is the volume of a half sphere.

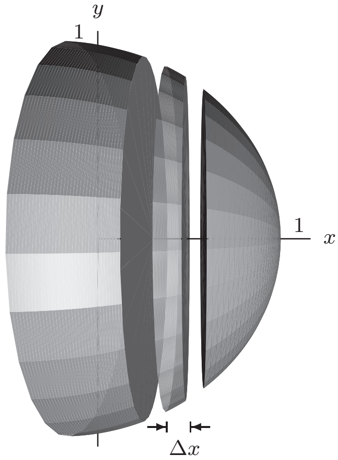


Figure 8.162

46. Slice the object into rings horizontally, as in Figure 8.163. A typical ring has thickness Δy, inner radius 2 and outer radius $2 + x = 2 + \sqrt{1-y^2}$. Thus,

$$\text{Volume of ring } \approx \pi(2+\sqrt{1-y^2})^2\Delta y - \pi 2^2 \Delta y = \pi(4\sqrt{1-y^2} + 1 - y^2)\,\Delta y.$$

$$\begin{aligned}
\text{Volume of solid } &= \int_0^1 \pi(4\sqrt{1-y^2} + 1 - y^2)\,dy \\
&= 4\pi \int_0^1 \sqrt{1-y^2}\,dy + \pi \int_0^1 1\,dy - \pi \int_0^1 y^2\,dy \\
&= 4\pi\left(\frac{1}{2}\left(y\sqrt{1-y^2}\Big|_0^1 + \int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy\right)\right) + \pi y\Big|_0^1 - \frac{\pi y^3}{3}\Big|_0^1 \\
&= 2\pi y\sqrt{1-y^2} + 2\pi \arcsin y + \pi y - \frac{\pi y^3}{3}\Big|_0^1 \\
&= 0 + 2\pi \arcsin 1 + \pi - \frac{\pi}{3} - 0 - 2\pi \arcsin 0 - 0 + 0 \\
&= \pi^2 + \frac{2\pi}{3} = 11.964.
\end{aligned}$$

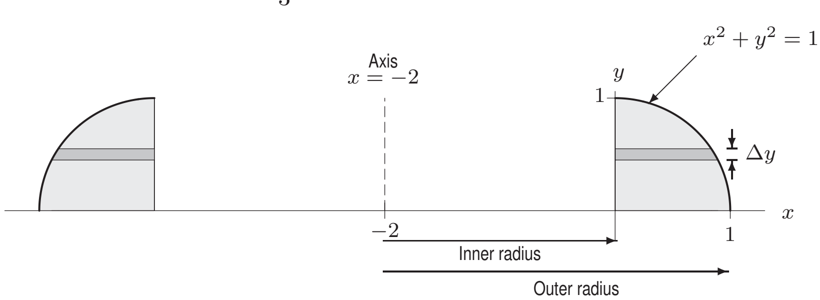


Figure 8.163: Cross-section of solid

47. Slice the object into rings horizontally, as in Figure 8.164. A typical ring has thickness Δy, outer radius 1, and inner radius $1 - x = 1 - \sqrt{1-y^2}$. Thus,

$$\text{Volume of ring } \approx \pi 1^2 \Delta y - \pi(1 - \sqrt{1-y^2})^2 \Delta y = \pi(2\sqrt{1-y^2} - (1-y^2))\,\Delta y.$$

$$\begin{aligned}
\text{Volume of solid } &= \int_0^1 \pi(2\sqrt{1-y^2} - 1 + y^2)\,dy \\
&= 2\pi \int_0^1 \sqrt{1-y^2}\,dy - \pi \int_0^1 1\,dy + \pi \int_0^1 y^2\,dy \\
&= 2\pi \cdot \frac{1}{2}\left(y\sqrt{1-y^2}\Big|_0^1 + 1^2 \int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy \right) - \pi y\Big|_0^1 + \frac{\pi y^3}{3}\Big|_0^1 \\
&= \pi y\sqrt{1-y^2} + \pi \arcsin y - \pi y + \frac{\pi y^3}{3}\Big|_0^1 \\
&= 0 + \frac{\pi^2}{2} - \pi + \frac{\pi}{3} - 0 - 0 + 0 - 0 \\
&= \frac{\pi^2}{2} - \frac{2\pi}{3} = 2.840.
\end{aligned}$$

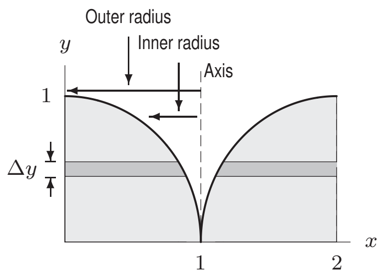


Figure 8.164: Cross-section of solid

48. Slicing perpendicularly to the x-axis gives squares whose thickness is Δx and whose side is $y = \sqrt{1-x^2}$. See Figure 8.165. Thus,

$$\text{Volume of square slice } \approx (\sqrt{1-x^2})^2 \Delta x = (1-x^2)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 (1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}.$$

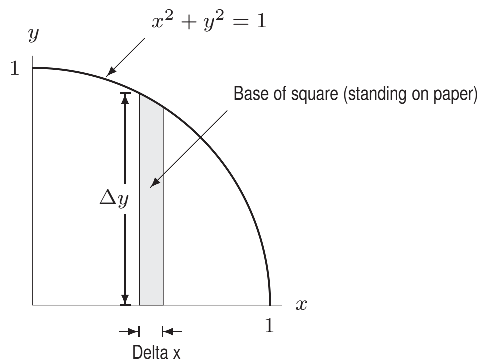


Figure 8.165: Base of solid

49. Slicing perpendicularly to the y-axis gives semicircles whose thickness is Δy and whose diameter is $x = \sqrt{1-y^2}$. See Figure 8.166. Thus

$$\text{Volume of semicircular slice } \approx \frac{\pi}{2}\left(\frac{\sqrt{1-y^2}}{2}\right)^2 \Delta y = \frac{\pi}{8}(1-y^2)\,\Delta y.$$

$$\text{Volume of solid } = \int_0^1 \frac{\pi}{8}(1-y^2)\,dy = \frac{\pi}{8}\left(y - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{\pi}{8}\cdot\frac{2}{3} = \frac{\pi}{24}.$$

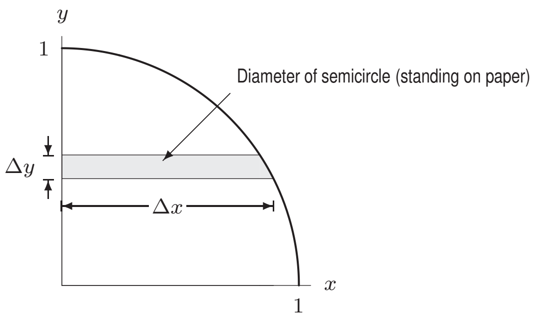


Figure 8.166: Base of Solid

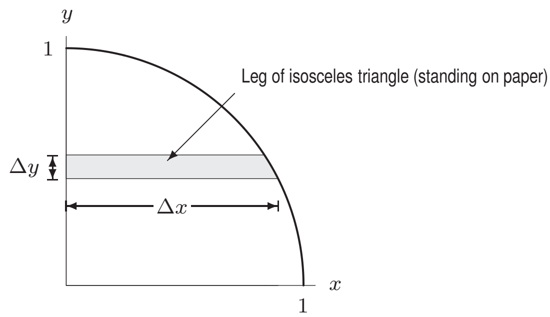


Figure 8.167: Base of solid

50. An isosceles triangle with legs of length s has

$$\text{Area } = \frac{1}{2}s^2.$$

Slicing perpendicularly to the y-axis gives isosceles triangles whose thickness is Δy and whose leg is $x = \sqrt{1-y^2}$. See Figure 8.167. Thus

$$\text{Volume of triangular slice } \approx \frac{1}{2}(\sqrt{1-y^2})\,\Delta y = \frac{1}{2}(1-y^2)\,\Delta y.$$

$$\text{Volume of solid } = \int_0^1 \frac{1}{2}(1-y^2)\,dy = \frac{1}{2}\left(y - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}.$$

51. The curve $y = x(x-3)^2$ has x-intercepts at $x = 0, 3$ and lies above the x-axis on this interval.

Thus, $\int_0^3 x(x-3)^2\,dx$ gives the area under the graph of f from $x = 0$ to $x = 3$.

52. The curve $y = x(x-3)^2$ has x-intercepts at $x = 0, 3$ and lies above the x-axis on this interval. Rotating the curve about the x-axis forms a solid of revolution with

$$\text{Volume} = \int_0^3 \pi\,(f(x))^2\;dx = \int_0^3 \pi\left(x(x-3)^2\right)^2\;dx = \int_0^3 \pi x^2(x-3)^4\,dx.$$

Thus, this expression represents a volume of revolution about the x-axis between $x = 0$ and $x = 3$.

53. Since $y = (e^x + e^{-x})/2$, $y' = (e^x - e^{-x})/2$. The length of the catenary is

$$\begin{aligned}\int_{-1}^1 \sqrt{1+(y')^2}\,dx &= \int_{-1}^1 \sqrt{1+\left[\frac{e^x - e^{-x}}{2}\right]^2}\,dx = \int_{-1}^1 \sqrt{1+\frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}}\,dx \\ &= \int_{-1}^1 \sqrt{\left[\frac{e^x+e^{-x}}{2}\right]^2}\,dx = \int_{-1}^1 \frac{e^x+e^{-x}}{2}\,dx \\ &= \left[\frac{e^x - e^{-x}}{2}\right]\Bigg|_{-1}^1 = e - e^{-1}.\end{aligned}$$

54. (a) Slice the headlight into N disks of height Δx by cutting perpendicular to the x–axis. The radius of each disk is y; the height is Δx. The volume of each disk is $\pi y^2 \Delta x$. Therefore, the Riemann sum approximating the volume of the headlight is

$$\sum_{i=1}^{N} \pi y_i^2 \Delta x = \sum_{i=1}^{N} \pi \frac{9x_i}{4} \Delta x.$$

(b)

$$\pi \int_0^4 \frac{9x}{4}\, dx = \pi \frac{9}{8} x^2 \bigg|_0^4 = 18\pi.$$

55. (a) The line $y = ax$ must pass through (l, b). Hence $b = al$, so $a = b/l$.

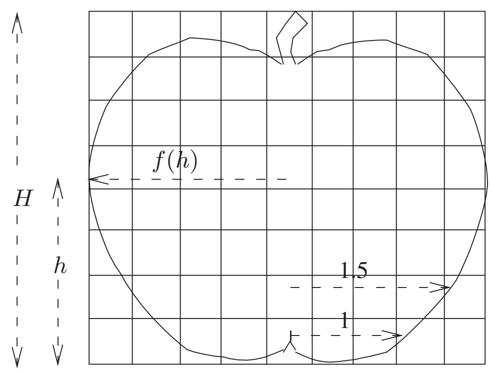
(b) Cut the cone into N slices, slicing perpendicular to the x–axis. Each piece is almost a cylinder. The radius of the ith cylinder is $r(x_i) = \frac{bx_i}{l}$, so the volume

$$V \approx \sum_{i=1}^{N} \pi \left(\frac{bx_i}{l}\right)^2 \Delta x.$$

Therefore, as $N \to \infty$, we get

$$\begin{aligned} V &= \int_0^l \pi b^2 l^{-2} x^2 dx \\ &= \pi \frac{b^2}{l^2} \left[\frac{x^3}{3}\right]_0^l = \left(\pi \frac{b^2}{l^2}\right)\left(\frac{l^3}{3}\right) = \frac{1}{3}\pi b^2 l. \end{aligned}$$

56. (a) If you slice the apple perpendicular to the core, you expect that the cross section will be approximately a circle.



If $f(h)$ is the radius of the apple at height h above the bottom, and H is the height of the apple, then

$$\text{Volume} = \int_0^H \pi f(h)^2\, dh.$$

Ignoring the stem, $H \approx 3.5$. Although we do not have a formula for $f(h)$, we can estimate it at various points. (Remember, we measure here from the bottom of the *apple*, which is not quite the bottom of the graph.)

h	0	0.5	1	1.5	2	2.5	3	3.5
$f(h)$	1	1.5	2	2.1	2.3	2.2	1.8	1.2

Now let $g(h) = \pi f(h)^2$, the area of the cross-section at height h. From our approximations above, we get the following table.

h	0	0.5	1	1.5	2	2.5	3	3.5
$g(h)$	3.14	7.07	12.57	13.85	16.62	13.85	10.18	4.52

We can now take left- and right-hand sum approximations. Note that $\Delta h = 0.5$ inches. Thus

$$\text{LEFT}(9) = (3.14 + 7.07 + 12.57 + 13.85 + 16.62 + 13.85 + 10.18)(0.5) = 38.64.$$
$$\text{RIGHT}(9) = (7.07 + 12.57 + 13.85 + 16.62 + 13.85 + 10.18 + 4.52)(0.5) = 39.33.$$

Thus the volume of the apple is ≈ 39 cu.in.

(b) The apple weighs $0.03 \times 39 \approx 1.17$ pounds, so it costs about 94¢.

57.

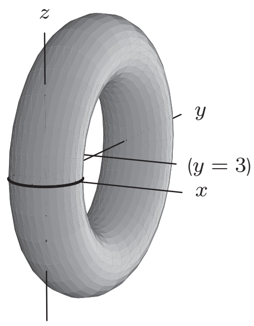


Figure 8.168: The Torus

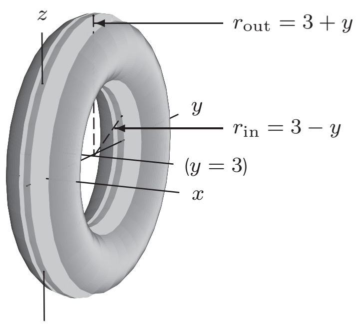


Figure 8.169: Slice of Torus

As shown in Figure 8.169, we slice the torus perpendicular to the line $y = 3$. We obtain washers with width dx, inner radius $r_{\text{in}} = 3 - y$, and outer radius $r_{\text{out}} = 3 + y$. Therefore, the area of the washer is $\pi r_{\text{out}}^2 - \pi r_{\text{in}}^2 = \pi[(3+y)^2 - (3-y)^2] = 12\pi y$. Since $y = \sqrt{1-x^2}$, the volume is gotten by summing up the volumes of the washers: we get

$$\int_{-1}^{1} 12\pi\sqrt{1-x^2}\,dx = 12\pi \int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

But $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ is the area of a semicircle of radius 1, which is $\frac{\pi}{2}$. So we get $12\pi \cdot \frac{\pi}{2} = 6\pi^2 \approx 59.22$. (Or, you could use

$$\int \sqrt{1-x^2}\,dx = \left[x\sqrt{1-x^2} + \arcsin(x)\right],$$

by VI-30 and VI-28.)

58. The arc length of the curve $y = f(t)$ from $t = 3$ to $t = 8$ is $\int_3^8 \sqrt{1 + (f'(t))^2}\,dt$. Thus, we want a function f such that

$$\int_3^8 \sqrt{1 + (f'(t))^2}\,dt = \int_3^8 \sqrt{1 + e^{6t}}\,dt.$$

Thus, we have

$$\left(f'(t)\right)^2 = e^{6t}.$$

One possibility is

$$f'(t) = e^{3t}$$
$$f(t) = \frac{1}{3}e^{3t} + C.$$

For any constant C, the original integral is the arc length of the curve $y = \frac{1}{3}e^{3t} + C$ from $t = 3$ to $t = 8$.

Another solution to $\left(f'(t)\right)^2 = e^{6t}$ is $f'(t) = -e^{3t}$, which gives $f(t) = -\frac{1}{3}e^{3t} + C$.

59. We take a cross-section of the pipe and cut it up in such a way that the speed of the water is nearly uniform on each slice. See Figure 8.170.

We use thin rings around the pipe's center; if a given ring is narrow enough, all points on it will be roughly equidistant from the center. Since the water speed is a function of the distance from the center, the speed is nearly constant on the entire ring.

Let r be the distance from the center to the inner boundary of the ring, and let Δr be the width of the ring, as in Figure 8.170. By straightening the ring into a thin rectangle, we find that its area is approximately given by the quantity $2\pi r\Delta r$. The speed across any part of the ring is roughly equal to the speed across the inner boundary, $10(1-r^2)$ inches per second. The flow is defined as the speed times the area; thus on any given ring we have

$$\text{Flow} \approx 10(1-r^2)\cdot 2\pi r\Delta r.$$

The total flow across the pipe cross-section is approximated by a Riemann sum incorporating all of the rings:

$$\text{Total Flow} \approx 20\pi \sum (1-r^2)r\Delta r,$$

where r is in between 0 and 1. Letting $\Delta r \to 0$, we obtain the exact solution:

$$\text{Total Flow} = 20\pi \int_0^1 (1-r^2)r\,dr = 20\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_0^1 = 5\pi \text{ cubic inches/second.}$$

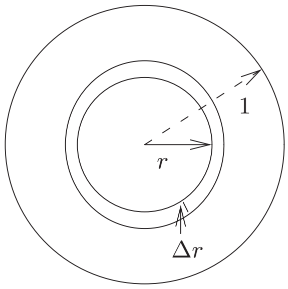


Figure 8.170

60. Multiplying $r = 2a\cos\theta$ by r, converting to Cartesian coordinates, and completing the square gives

$$\begin{aligned} r^2 &= 2ar\cos\theta \\ x^2+y^2 &= 2ax \\ x^2-2ax+a^2+y^2 &= a^2 \\ (x-a)^2+y^2 &= a^2. \end{aligned}$$

This is the standard form of the equation of a circle with radius a and center $(x,y)=(a,0)$.

To check the limits on θ note that the circle is in the right half plane, where $-\pi/2 \le \theta \le \pi/2$. Rays from the origin at all these angles meet the circle because the circle is tangent to the y-axis at the origin.

61. The area is given by

$$\int_{-\pi/2}^{\pi/2} \frac{1}{2}r^2\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(2a\cos\theta\right)^2\,d\theta = 2a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = 2a^2\left(\frac{1}{2}\cos\theta\sin\theta+\frac{\theta}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = \pi a^2.$$

(We have used formula IV-18 from the integral table. The integral can also be done using a calculator or integration by parts.)

62. See Figure 8.171. The circles meet where

$$\begin{aligned} 2a\cos\theta &= a \\ \cos\theta &= \frac{1}{2} \\ \theta &= \pm\frac{\pi}{3}. \end{aligned}$$

The area is obtained by subtraction:

$$\begin{aligned}
\text{Area} &= \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2}(2a\cos\theta)^2 - \frac{1}{2}a^2\right) d\theta \\
&= \int_{-\pi/3}^{\pi/3} \left(2a^2\cos^2\theta - \frac{1}{2}a^2\right) d\theta \\
&= \left(2a^2\left(\frac{1}{2}\cos\theta\sin\theta + \frac{\theta}{2}\right) - \frac{a^2}{2}\theta\right)\Bigg|_{-\pi/3}^{\pi/3} \\
&= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)a^2.
\end{aligned}$$

Since

$$\frac{\left(\pi/3 + \sqrt{3}/2\right)a^2}{\pi a^2} = 61\%$$

the shaded region covers 61% of circle C.

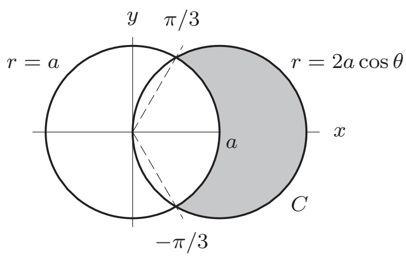


Figure 8.171

63. (a) Writing C in parametric form gives

$$x = 2a\cos^2\theta \quad \text{and} \quad y = 2a\cos\theta\sin\theta,$$

so the slope is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2a\sin^2\theta + 2a\cos^2\theta}{-4a\cos\theta\sin\theta} = \frac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta}.$$

(b) The maximum y-value occurs where $dy/dx = 0$, so

$$\begin{aligned}
\sin^2\theta - \cos^2\theta &= 0 \\
\theta &= \pm\frac{\pi}{4}.
\end{aligned}$$

The value $\theta = \pi/4$ gives the maximum y-value; $\theta = -\pi/4$ gives the minimum y-value.

64. Writing C in parametric form gives

$$x = 2a\cos^2\theta \quad \text{and} \quad y = 2a\cos\theta\sin\theta,$$

so

$$\begin{aligned}
\text{Arc length} &= \int_{-\pi/2}^{\pi/2} \sqrt{(-4a\cos\theta\sin\theta)^2 + (-2a\sin^2\theta + 2a\cos^2\theta)^2}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{4\cos^2\theta\sin^2\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{(\sin^2\theta + \cos^2\theta)^2}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} d\theta = 2\pi a.
\end{aligned}$$

65. This function has zeros at $x = -2$ and $x = 1$. The bounded region lies between these two zeros. Thus,

$$\text{Volume} = \int_{-2}^{1} \pi \left((x-1)^2(x+2)\right)^2 \, dx.$$

66. The total mass is 12 gm, so the center of mass is located at $\overline{x} = \frac{1}{12}(-5\cdot 3 - 3\cdot 3 + 2\cdot 3 + 7\cdot 3) = \frac{1}{4}$.

67. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

$$\text{Area of the plate } = \int_0^1 (\sqrt{x} - x^2)\, dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{3}\text{ cm}^2.$$

Thus the mass of the plate is $2\cdot 1/3 = 2/3$ gm.

(b) See Figure 8.172. Since the region is "fatter" closer to the origin, $\bar{x}$ is less than $1/2$.

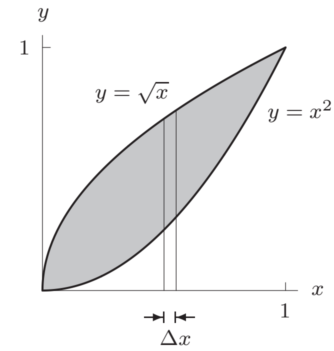


Figure 8.172

(c) To find $\bar{x}$, we slice the region into vertical strips of width Δx. See Figure 8.172.

$$\text{Area of strip } = A_x(x)\Delta x \approx (\sqrt{x} - x^2)\Delta x \text{ cm}^2.$$

Then we have

$$\overline{x} = \frac{\int x\delta A_x(x)\, dx}{\text{Mass}} = \frac{\int_0^1 2x(\sqrt{x} - x^2)\, dx}{2/3} = \frac{3}{2}\int_0^1 2(x^{3/2} - x^3)\, dx = \frac{3}{2}\cdot 2\left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{9}{20}\text{ cm}.$$

This is less than $1/2$, as predicted in part (b). So $\bar{x} = \bar{y} = 9/20$ cm.

68. Let x be the height from ground to the weight. It follows that $0 \le x \le 20$. At height x, to lift the weight Δx more, the work needed is $200\Delta x + 2(20 - x)\Delta x = (240 - 2x)\Delta x$. So the total work is

$$W = \int_0^{20} (240 - 2x)dx = (240x - x^2)\Bigg|_0^{20} = 240(20) - 20^2 = 4400 \text{ ft-lb}.$$

69. Imagine the pole is divided into n segments of length Δx. The heights of the segments are given by $x_1, x_2, \ldots, x_i, \ldots x_n$. A segment of length Δx weighs $\dfrac{20\text{ lb}}{10\text{ ft}} \cdot \Delta x = 2\Delta x$. The work required to raise a segment a vertical distance of x_i ft is

$$\text{Work to raise segment } x_i \text{ ft} = \underbrace{\text{Weight}}_{2\Delta x} \cdot \underbrace{\text{Distance}}_{x_i} = 2x_i\Delta x.$$

The total work is therefore

$$\begin{aligned}\text{Total work} &= \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i\Delta x \\ &= \int_0^{10} 2x\, dx = x^2\Bigg|_0^{10} = 100 \text{ ft-lbs}.\end{aligned}$$

To check our answer, notice that the work required to raise the entire 20 lb pole so that it is suspended horizontally 10 ft above the ground is:

$$\text{Work} = \underbrace{\text{Weight}}_{20\text{ lbs}} \cdot \underbrace{\text{Distance}}_{10\text{ ft}} = 200 \text{ ft-lbs}.$$

This is more than 100 lbs, because it should take more work to raise the entire pole 10 ft than to stand it upright.

70. Let x be the distance from the bucket to the surface of the water. It follows that $0 \le x \le 40$. At x feet, the bucket weighs $(30 - \frac{1}{4}x)$, where the $\frac{1}{4}x$ term is due to the leak. When the bucket is x feet from the surface of the water, the work done by raising it Δx feet is $(30 - \frac{1}{4}x)\,\Delta x$. So the total work required to raise the bucket to the top is

$$\begin{aligned} W &= \int_0^{40} (30 - \frac{1}{4}x)dx \\ &= \left(30x - \frac{1}{8}x^2\right)\Big|_0^{40} \\ &= 30(40) - \frac{1}{8}40^2 = 1000 \text{ ft-lb.} \end{aligned}$$

71. Consider lifting a rectangular slab of water h feet from the top up to the top. See Figure 8.173. The area of such a slab is $(10)(20) = 200$ square feet; if the thickness is Δh, then the volume of such a slab is $200\,\Delta h$ cubic feet. This much water weighs 62.4 pounds per ft^3, so the weight of such a slab is $(200\,\Delta h)(62.4) = 12480\,\Delta h$ pounds. To lift that much water h feet requires $12480h\,\Delta h$ foot-pounds of work. To lift the whole tank, we lift one plate at a time; integrating over the slabs yields

$$\int_0^{15} 12480h\,dh = \frac{12480h^2}{2}\Big|_0^{15} = \frac{12480 \cdot 15^2}{2} = 1{,}404{,}000 \text{ foot-pounds.}$$

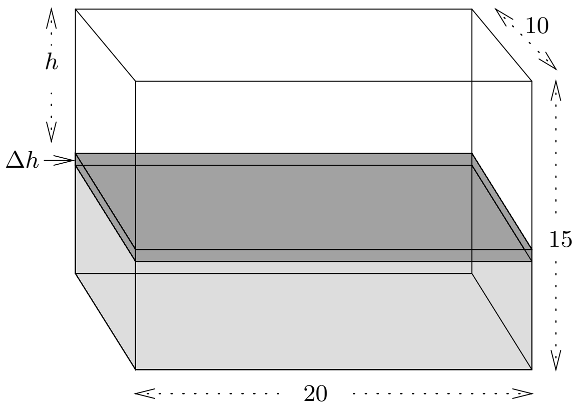


Figure 8.173

72. We begin by slicing the oil into slabs at a distance h below the surface with thickness Δh. We can then calculate the volume of the slab and the work needed to raise this slab to the surface, a distance of h.

$$\begin{aligned} \text{Volume of } \Delta h \text{ disk} &= \pi r^2 \Delta h = 25\pi\Delta h \\ \text{Weight of } \Delta h \text{ disk} &= (25\pi)(50)\Delta h \\ \text{Distance to raise} &= h \\ \text{Work to raise} &= (25\pi)(50)(h)\Delta h. \end{aligned}$$

Integrating the work over all such slabs, we have

$$\begin{aligned} \text{Work} &= \int_{19}^{25} (50)(25\pi)(h)\,dh \\ &= 625\pi h^2 \Big|_{19}^{25} \\ &= 390{,}625\pi - 225{,}625\pi \\ &\approx 518{,}363 \text{ ft-lbs.} \end{aligned}$$

A diagram of this tank is shown in Figure 8.174.

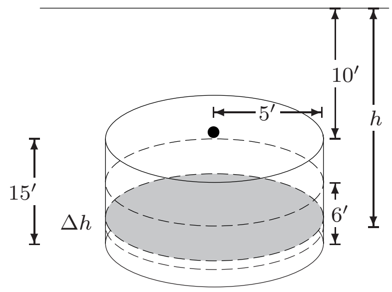


Figure 8.174

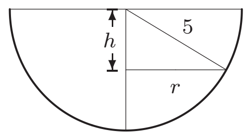


Figure 8.175

73. We slice the gasoline horizontally. At a distance h feet below the surface, the horizontal slab is a cylinder with radius r and thickness Δh, so

$$\text{Volume of one slab } \approx \pi r^2 \Delta h.$$

To find the radius r at a depth h from the top as in Figure 8.175, we note that $h^2 + r^2 = 5^2$, so $r = \sqrt{25 - h^2}$. At depth h

$$\text{Volume of one slice} \approx \pi(\sqrt{25 - h^2})^2 \Delta h = \pi(25 - h^2)\Delta h \text{ ft}^3.$$

The gasoline at depth h must be lifted a distance of h ft, so

$$\begin{aligned}\text{Work to move one slice} &= \rho \cdot \text{Volume} \cdot \text{Distance lifted} \\ &\approx \rho(\pi(25 - h^2)\Delta h)(h) \text{ ft-lb.}\end{aligned}$$

The work done, W, to lift all the gasoline is the sum of the work done on the pieces:

$$W \approx \sum \rho(\pi(25 - h^2)\Delta h)h \text{ ft-lb..}$$

As $\Delta h \to 0$, we obtain a definite integral. Since h varies from $h = 0$ to $h = 5$ and $\rho = 42$, we have:

$$W = \int_0^5 \rho\pi(25h - h^3)dh = 42\pi \left(25\frac{h^2}{2} - \frac{h^4}{4}\right)\Bigg|_0^5 = \frac{13125\pi}{2} = 20{,}617 \text{ ft-lb.}$$

The work to pump all the gasoline out is 20,617 ft-lbs.

74. Let h be height above the bottom of the dam. Then

$$\begin{aligned}\text{Water force } &= \int_0^{25} (62.4)(25 - h)(60)\, dh \\ &= (62.4)(60)\left(25h - \frac{h^2}{2}\right)\Bigg|_0^{25} \\ &= (62.4)(60)(625 - 312.5) \\ &= (62.4)(60)(312.5) \\ &= 1{,}170{,}000 \text{ lbs.}\end{aligned}$$

75. If the weight of the chain were negligible, the work required would be $1000 \cdot 20 = 20{,}000$ ft-lbs. Because of the chain, the total work is slightly more than 20,000 ft-lbs. When the object is h ft off the ground, the length of chain is $50 - h$ so the total weight being lifted is $1000 + 2(50 - h)$ lb. See Figure 8.176. Thus

$$\begin{aligned}\text{Work to lift the weight an addition } \Delta h \text{ higher} &= \text{Weight} \cdot \text{Distance lifted} \\ &\approx (1000 + 2(50 - h))\Delta h \text{ ft-lb.}\end{aligned}$$

To find the total work, we integrate this quantity from $h = 0$ to $h = 20$:

$$W = \int_0^{20} (1000 + 2(50 - h))dh = \int_0^{20} (1100 - 2h)dh = (1100h - h^2)\Big|_0^{20} = 21{,}600\text{ft-lbs}.$$

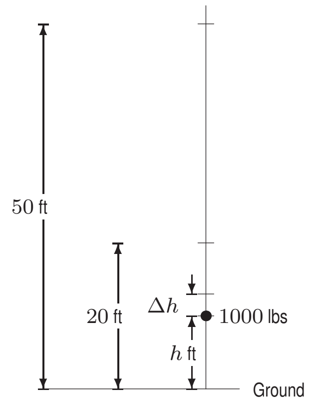


Figure 8.176

76.

$$\begin{aligned}\text{Future Value } &= \int_0^{15} 3000e^{0.06(15-t)}\,dt = 3000e^{0.9}\int_0^{15} e^{-0.06t}dt \\ &= 3000e^{0.9}\left(\frac{1}{-0.06}e^{-0.06t}\right)\Big|_0^{15} = 3000e^{0.9}\left(\frac{1}{-0.06}e^{-0.9} + \frac{1}{0.06}e^{0}\right) \\ &\approx \$72{,}980.16\end{aligned}$$

$$\begin{aligned}\text{Present Value } &= \int_0^{15} 3000e^{-0.06t}\,dt = 3000\left(-\frac{1}{0.06}\right)e^{-0.06t}\Big|_0^{15} \\ &\approx \$29{,}671.52.\end{aligned}$$

There's a quicker way to calculate the present value of the income stream, since the future value of the income stream is (as we've shown) \$72,980.16, the present value of the income stream must be the present value of \$72,980.16. Thus,

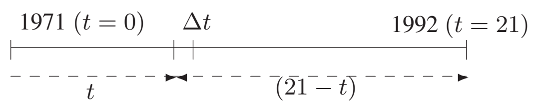
$$\begin{aligned}\text{Present Value} &= \$72{,}980.16(e^{-.06\cdot 15}) \\ &\approx \$29{,}671.52,\end{aligned}$$

which is what we got before.

77. We divide up time between 1971 and 1992 into intervals of length Δt, and calculate how much of the strontium-90 produced during that time interval is still around.

Strontium-90 decays exponentially, so if a quantity S_0 was produced t years ago, and S is the quantity around today, $S = S_0e^{-kt}$. Since the half-life is 28 years, $\frac{1}{2} = e^{-k(28)}$, giving $k = -\ln(1/2)/28 \approx 0.025$.

We measure t in years from 1971, so that 1992 is $t = 21$.



Since strontium-90 is produced at a rate of 3 kg/year, during the interval Δt, a quantity $3\Delta t$ kg was produced. Since this was $(21 - t)$ years ago, the quantity remaining now is $(3\Delta t)e^{-0.025(21-t)}$. Summing over all such intervals gives

$$\begin{matrix}\text{Strontium remaining} \\ \text{in 1992}\end{matrix} \approx \int_0^{21} 3e^{-0.025(21-t)}\,dt = \frac{3e^{-0.025(21-t)}}{0.025}\Big|_0^{21} = 49\text{ kg}.$$

[Note: This is like a future value problem from economics, but with a negative interest rate.]

78. (a) Slice the mountain horizontally into N cylinders of height Δh. The sum of the volumes of the cylinders will be

$$\sum_{i=1}^{N} \pi r^2 \Delta h = \sum_{i=1}^{N} \pi \left(\frac{3.5 \cdot 10^5}{\sqrt{h+600}}\right)^2 \Delta h.$$

(b)

$$\begin{aligned}
\text{Volume} &= \int_{400}^{14400} \pi \left(\frac{3.5 \cdot 10^5}{\sqrt{h+600}}\right)^2 dh \\
&= 1.23 \cdot 10^{11} \pi \int_{400}^{14400} \frac{1}{(h+600)} dh \\
&= 1.23 \cdot 10^{11} \pi \ln(h+600)\Big|_{400}^{14400} dh \\
&= 1.23 \cdot 10^{11} \pi \left[\ln 15000 - \ln 1000\right] \\
&= 1.23 \cdot 10^{11} \pi \ln(15000/1000) \\
&= 1.23 \cdot 10^{11} \pi \ln 15 \approx 1.05 \cdot 10^{12} \text{ cubic feet.}
\end{aligned}$$

79. Look at the disc-shaped slab of water at height y and of thickness Δy in Figure 8.177. The rate at which water is flowing out when it is at depth y is $k\sqrt{y}$ (Torricelli's Law, with k constant). Then, if $x = g(y)$, we have

$$\Delta t = \begin{pmatrix}\text{Time for water to} \\ \text{drop by this amount}\end{pmatrix} = \frac{\text{Volume}}{\text{Rate}} = \frac{\pi (g(y))^2 \Delta y}{k\sqrt{y}}.$$

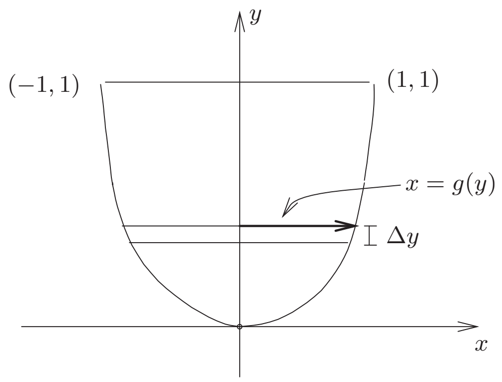


Figure 8.177

If the rate at which the depth of the water is dropping is constant, then dy/dt is constant, so we want

$$\frac{\pi (g(y))^2}{k\sqrt{y}} = \text{constant},$$

so $g(y) = c\sqrt[4]{y}$, for some constant c. Since $x = 1$ when $y = 1$, we have $c = 1$ and so $x = \sqrt[4]{y}$, or $y = x^4$.

80. The statement $P(70) = 0.76$ means that 76% of the population has ages less than 70.

81. Graph B is more spread out to the right, and so it represents a gas in which more of the molecules are moving at faster velocities. Thus the average velocity in gas B is larger.

82. Every photon which falls a given distance from the center of the detector has the same probability of being detected. This suggests that we divide the plate up into concentric rings of thickness Δr. Consider one such ring having inner radius r and outer radius $r + \Delta r$. For this ring,

$$\text{Number of photons hitting ring per unit time } \approx N \cdot \text{ Area of ring } \approx N \cdot 2\pi r \Delta r.$$

Then,

$$\text{Number of photons detected on ring per unit time} \approx \text{Number hitting} \cdot S(r) \approx N \cdot 2\pi r\Delta r \cdot S(r).$$

Summing over all rings gives us

$$\text{Total number of photons detected per unit time} \approx \sum 2\pi NrS(r)\Delta r.$$

Taking the limit as $\Delta r \to 0$ gives

$$\text{Total number of photons detected per unit time} = \int_0^R 2\pi NrS(r)dr.$$

83. The number of houses in a ring of width Δr a distance r_i from the city center is given by:

$$\begin{aligned}\text{Number houses} &= 1000 \text{ houses/mi}^2 \times \text{Area of ring}\\ &= 1000 \cdot 2\pi r_i \Delta r = 2000\pi r_i \Delta r.\end{aligned}$$

The value of the houses in this ring is given by:

$$\begin{aligned}\text{Value} &= \text{Price per house} \times \text{Number of houses}\\ &= p(r_i) \cdot 1000 \cdot 2\pi r_i \Delta r = 2000\pi r_i p(r_i)\Delta r.\end{aligned}$$

The total value of the houses within 7 miles of the city center is therefore

$$\begin{aligned}\text{Total value} &= \lim_{n\to\infty} \sum_{i=1}^{n} 2000\pi r_i p(r_i)\Delta r\\ &= \int_0^7 2000\pi r p(r)\,dr\\ &= 2000\pi \int_0^7 400e^{-0.2r^2} r dr\\ &= 800{,}000\pi \int_{r=0}^{r=7} e^{w^2}(-2.5)\ dw \qquad \text{let } w = -0.2r^2, dw = -0.4rdr\\ &= -2{,}000{,}000\pi \int_{r=0}^{r=7} e^{w}\,dw\\ &= -2{,}000{,}000\pi e^{-0.2r^2}\Big|_0^7\\ &= -2{,}000{,}000\pi\left(e^{-0.2(7)^2} - e^{-0.2(0)^2}\right) = 6{,}282{,}837.\end{aligned}$$

This figure is in \$1000s, so the total value of homes is approximately \$6.2828 billion.

84. (a) Divide the cross-section of the blood into rings of radius r, width Δr. See Figure 8.178.

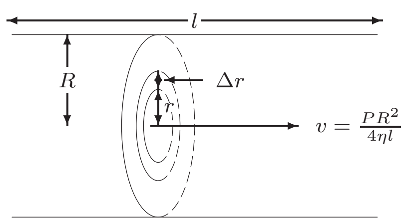


Figure 8.178

Then

$$\text{Area of ring} \approx 2\pi r\Delta r.$$

The velocity of the blood is approximately constant throughout the ring, so

$$\begin{aligned}\text{Rate blood flows through ring} &\approx \text{Velocity}\cdot\text{Area}\\ &= \frac{P}{4\eta l}(R^2-r^2)\cdot 2\pi r\Delta r.\end{aligned}$$

Thus, summing over all rings, we find the total blood flow:

$$\text{Rate blood flowing through blood vessel} \approx \sum \frac{P}{4\eta l}(R^2-r^2)2\pi r\Delta r.$$

Taking the limit as $\Delta r \to 0$, we get

$$\text{Rate blood flowing through blood vessel} = \int_0^R \frac{\pi P}{2\eta l}(R^2 r - r^3)dr$$

$$= \frac{\pi P}{2\eta l}\left(\frac{R^2r^2}{2}-\frac{r^4}{4}\right)\Big|_0^R = \frac{\pi P R^4}{8\eta l}.$$

(b) Since

$$\text{Rate of blood flow} = \frac{\pi P R^4}{8\eta l},$$

if we take $k = \pi P/(8\eta l)$, then we have

$$\text{Rate of blood flow} = kR^4,$$

that is, rate of blood flow is proportional to R^4, in accordance with Poiseuille's Law.

85. Pick a small interval of time Δt which takes place at time t. Fuel is consumed at a rate of $(25+0.1v)^{-1}$ gallons per mile. In the time Δt, the car moves $v\,\Delta t$ miles, so it consumes $v\,\Delta t/(25+0.1v)$ gallons during the instant Δt. Since $v = 50\frac{t}{t+1}$, the car consumes

$$\frac{v\,\Delta t}{25+0.1v} = \frac{50\frac{t}{t+1}\,\Delta t}{25+0.1\left(50\frac{t}{t+1}\right)} = \frac{50t\,\Delta t}{25(t+1)+5t} = \frac{10t\,\Delta t}{6t+5}$$

gallons of gas, in terms of the time t at which the instant occurs. To find the total gas consumed, sum up the instants in an integral:

$$\text{Gas consumed } = \int_2^3 \frac{10t}{6t+5}dt \approx 1.25 \text{ gallons.}$$

86. (a) Slicing horizontally, as shown in Figure 8.179, we see that the volume of one disk-shaped slab is

$$\Delta V \approx \pi x^2\Delta y = \frac{\pi y}{a}\Delta y.$$

Thus, the volume of the water is given by

$$V = \int_0^h \frac{\pi}{a} y dy = \frac{\pi}{a}\frac{y^2}{2}\Big|_0^h = \frac{\pi h^2}{2a}.$$

(b) The surface of the water is a circle of radius x. Since at the surface, $y = h$, we have $h = ax^2$. Thus, at the surface, $x = \sqrt{(h/a)}$. Therefore the area of the surface of water is given by

$$A = \pi x^2 = \frac{\pi h}{a}.$$

(c) If the rate at which water is evaporating is proportional to the surface area, we have

$$\frac{dV}{dt} = -kA.$$

(The negative sign is included because the volume is decreasing.) By the chain rule, $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$. We know $\frac{dV}{dh} = \frac{\pi h}{a}$ and $A = \frac{\pi h}{a}$ so

$$\frac{\pi h}{a}\frac{dh}{dt} = -k\frac{\pi h}{a} \quad \text{giving} \quad \frac{dh}{dt} = -k.$$

(d) Integrating gives

$$h = -kt + h_0.$$

Solving for t when $h = 0$ gives

$$t = \frac{h_0}{k}.$$

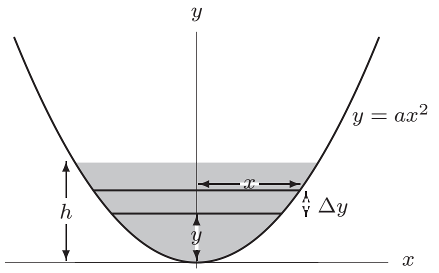


Figure 8.179

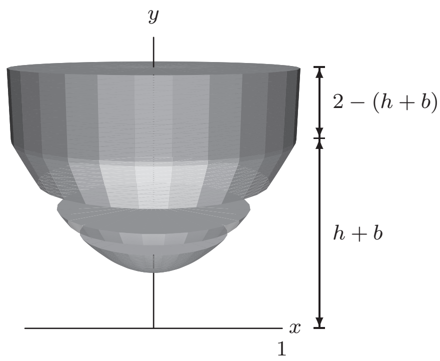


Figure 8.180

87. (a) The volume of water in the centrifuge is $\pi(1^2)\cdot 1 = \pi$ cubic meters. The centrifuge has total volume 2π cubic meters, so the volume of the air in the centrifuge is π cubic meters. Now suppose the equation of the parabola is $y = h + bx^2$. We know that the volume of air in the centrifuge is the volume of the top part (a cylinder) plus the volume of the middle part (shaped like a bowl). See Figure 8.180.

To find the volume of the cylinder of air, we find the maximum water depth. If $x = 1$, then $y = h + b$. Therefore the height of the water at the edge of the bowl, 1 meter away from the center, is $h + b$. The volume of the cylinder of air is therefore $[2 - (h + b)] \cdot \pi \cdot (1)^2 = [2 - h - b]\pi$.

To find the volume of the bowl of air, we note that the bowl is a volume of rotation with radius x at height y, where $y = h + bx^2$. Solving for x^2 gives $x^2 = (y - h)/b$. Hence, slicing horizontally as shown in the picture:

$$\text{Bowl Volume} = \int_h^{h+b} \pi x^2\, dy = \int_h^{h+b} \pi \frac{y-h}{b}\, dy = \left.\frac{\pi(y-h)^2}{2b}\right|_h^{h+b} = \frac{b\pi}{2}.$$

So the volume of both pieces together is $[2 - h - b]\pi + b\pi/2 = (2 - h - b/2)\pi$. But we know the volume of air should be π, so $(2 - h - b/2)\pi = \pi$, hence $h + b/2 = 1$ and $b = 2 - 2h$. Therefore, the equation of the parabolic cross-section is $y = h + (2 - 2h)x^2$.

(b) The water spills out the top when $h + b = h + (2 - 2h) = 2$, or when $h = 0$. The bottom is exposed when $h = 0$. Therefore, the two events happen simultaneously.

88. Any small piece of mass ΔM on either of the two spheres has kinetic energy $\frac{1}{2}v^2\Delta M$. Since the angular velocity of the two spheres is the same, the actual velocity of the piece ΔM will depend on how far away it is from the axis of revolution. The further away a piece is from the axis, the faster it must be moving and the larger its velocity v. This is because if ΔM is at a distance r from the axis, in one revolution it must trace out a circular path of length $2\pi r$ about the axis. Since every piece in either sphere takes 1 minute to make 1 revolution, pieces farther from the axis must move faster, as they travel a greater distance.

Thus, since the thin spherical shell has more of its mass concentrated farther from the axis of rotation than does the solid sphere, the bulk of it is traveling faster than the bulk of the solid sphere. So, it has the higher kinetic energy.

89. Any small piece of mass ΔM on either of the two hoops has kinetic energy $\frac{1}{2}v^2\Delta M$. Since the angular velocity of the two hoops is the same, the actual velocity of the piece ΔM will depend on how far away it is from the axis of revolution. The further away a piece is from the axis, the faster it must be moving and the larger its velocity v. This is because if ΔM is at a distance r from the axis, in one revolution it must trace out a circular path of length $2\pi r$ about the axis. Since every piece in either hoop takes 1 minute to make 1 revolution, pieces farther from the axis must move faster, as they travel a greater distance.

The hoop rotating about the cylindrical axis has all of its mass at a distance R from the axis, whereas the other hoop has a good bit of its mass close (or on) the axis of rotation. So, since the bulk of the hoop rotating about the cylindrical axis is traveling faster than the bulk of the other hoop, it must have the higher kinetic energy.

CAS Challenge Problems

90. (a) We need to check that the point with the given coordinates is on the curve, i.e., that

$$x = a\sin^2 t, \quad y = \frac{a\sin^3 t}{\cos t}$$

satisfies the equation

$$y = \sqrt{\frac{x^3}{a-x}}.$$

This can be done by substituting into the computer algebra system and asking it to simplify the difference between the two sides, or by hand calculation:

$$\begin{aligned}\text{Right-hand side} &= \sqrt{\frac{(a\sin^2 t)^3}{a - a\sin^2 t}} = \sqrt{\frac{a^3\sin^6 t}{a(1-\sin^2 t)}} \\ &= \sqrt{\frac{a^3\sin^6 t}{a\cos^2 t}} = \sqrt{\frac{a^2\sin^6 t}{\cos^2 t}} \\ &= \frac{a\sin^3 t}{\cos t} = y = \text{Left-hand side.}\end{aligned}$$

We chose the positive square root because both $\sin t$ and $\cos t$ are nonnegative for $0 \le t \le \pi/2$. Thus the point always lies on the curve. In addition, when $t = 0$, $x = 0$ and $y = 0$, so the point starts at $x = 0$. As t approaches $\pi/2$, the value of $x = a\sin^2 t$ approaches a and the value of $y = a\sin^3 t/\cos t$ increases without bound (or approaches ∞), so the point on the curve approaches the vertical asymptote at $x = a$.

(b) We calculate the volume using horizontal slices. See the graph of $y = \sqrt{x^3/(a-x)}$ in Figure 8.181.

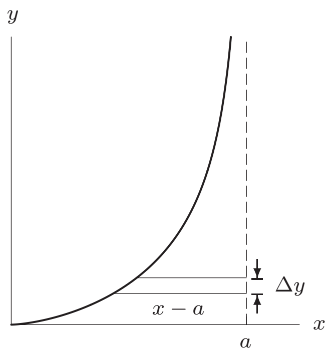


Figure 8.181

The slice at y is a disk of thickness Δy and radius $x - a$, hence it has volume $\pi(x-a)^2\Delta y$. So the volume is given by the improper integral

$$\text{Volume} = \int_0^\infty \pi(x-a)^2\,dy.$$

(c) We substitute

$$x = a\sin^2 t, \quad y = \frac{a\sin^3 t}{\cos t}$$

and

$$dy = \frac{d}{dt}\left(\frac{a\sin^3 t}{\cos t}\right)dt = a\left(3\sin^2 t + \frac{\sin^4 t}{\cos^2 t}\right)dt.$$

Since $t = 0$ where $y = 0$ and $t = \pi/2$ at the asymptote where $y \to \infty$, we get

$$\begin{aligned}\text{Volume} &= \int_0^{\pi/2} \pi(a\sin^2 t - a)^2 a\left(3\sin^2 t + \frac{\sin^4 t}{\cos^2 t}\right)dt \\ &= \pi a^3 \int_0^{\pi/2}(3\sin^2 t\cos^4 t + \sin^4 t\cos^2 t)\,dt = \frac{\pi^2 a^3}{8}.\end{aligned}$$

You can use a CAS to calculate this integral; it can also be done using trigonometric identities.

91. (a) The expression for arc length in terms of a definite integral gives

$$A(t) = \int_0^t \sqrt{1+4x^2}\,dx = \frac{2\,t\,\sqrt{1+4\,t^2} + \operatorname{arcsinh}(2\,t)}{4}.$$

The integral was evaluated using a computer algebra system; different systems may give the answer in different forms. Here arcsinh is the inverse function of the hyperbolic sine function.

(b) Figure 8.182 shows that the graphs of $A(t)$ and t^2 look very similar. This suggests that $A(t) \approx t^2$.

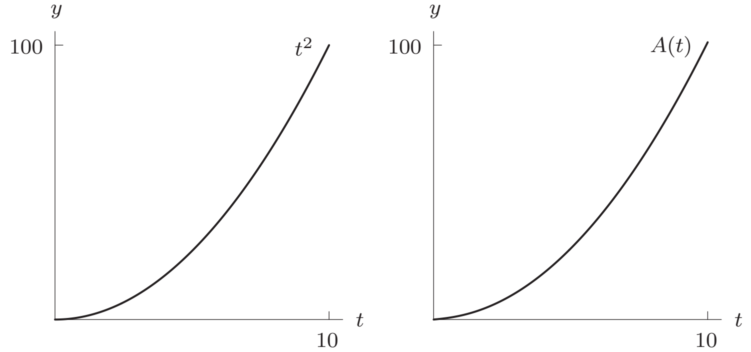


Figure 8.182

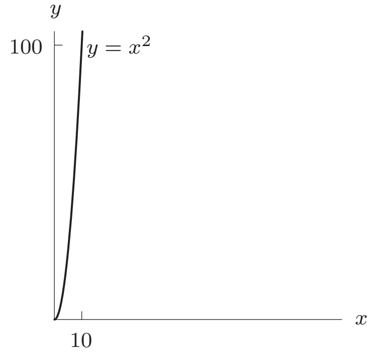


Figure 8.183

(c) The graph in Figure 8.183 is approximately vertical and close to the y axis. Thus, if we measure the arc length up to a certain y-value, the answer is approximately the same as if we had measured the length straight up the y-axis. Hence

$$A(t) \approx y = f(t) = t^2.$$

So

$$A(t) \approx t^2.$$

92. (a) The expression for arc length in terms of a definite integral gives

$$A(t) = \int_0^t \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = \frac{2\sqrt{t}\sqrt{1+4t} + \operatorname{arcsinh}(2\sqrt{t})}{4}.$$

The integral was evaluated using a computer algebra system; different systems may give the answer in different forms. Some may involve ln instead of arcsinh, which is the inverse function of the hyperbolic sine function.

(b) Figure 8.185 shows that the graphs of $A(t)$ and the graph of $y = t$ look very similar. This suggests that $A(t) \approx t$.

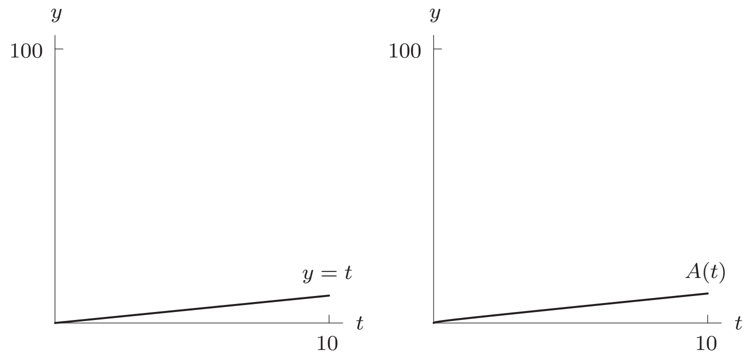


Figure 8.184

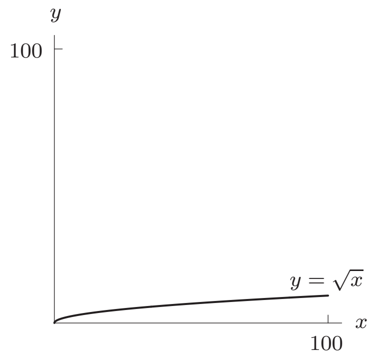


Figure 8.185

(c) The graph in Figure 8.185 is approximately horizontal and close to the x-axis. Thus, if we measure the arc length up to a certain x-value, the answer is approximately the same as if we had measured the length straight along the x-axis. Hence

$$A(t) \approx x = t.$$

So

$$A(t) \approx t.$$

93. (a) Slice the sphere at right angles to the axis of the cylinder. Consider a slice of thickness Δx at distance x from the center of the sphere. The cross-section is an annulus (ring) with internal radius $r_i = a$ and outer radius $r_o = \sqrt{r^2 - x^2}$. Thus

$$\text{Area of annulus } = \pi {r_o}^2 - \pi {r_i}^2 = \pi\left(\sqrt{r^2 - x^2}\right)^2 - \pi a^2 = \pi(r^2 - x^2 - a^2).$$

$$\text{Volume of slice} \approx \pi(r^2 - x^2 - a^2)\Delta x.$$

The lower and upper limits of the integral are where the cylinder meets the sphere, i.e., where $x^2 + a^2 = r^2$, or $x = \pm\sqrt{r^2 - a^2}$. Thus

$$\text{Volume of bead} = \int_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} \pi(r^2 - x^2 - a^2)\,dx.$$

(b) Using a computer algebra system to evaluate the integral, we have

$$\text{Volume of bead} = \frac{4\pi}{3}\left(r^2 - a^2\right)^{3/2}.$$

PROJECTS FOR CHAPTER EIGHT

1. (a) The hydrostatic pressure p_h is a linear function of distance x, so $p_h = b + mx$. At the artery end, $x = 0$, and $p_h = 35$. At the vein end, $x = L = 0.1$ and $p_h = 15$. This means the points $(0, 35)$ and $(0.1, 15)$ are on the graph of p_h, so the slope is given by

$$m = \frac{15 - 35}{0.1 - 0} = -200\,\frac{\text{mm Hg}}{\text{cm}}.$$

We know that $b = 35$, so $p_h = 35 - 200x$ mm Hg.

(b) The net pressure, p, is the difference between the hydrostatic pressure and the oncotic pressure. Using part (a), we have

$$p = p_h - p_o = (35 - 200x) - 23 = 12 - 200x \text{ mm Hg}.$$

(c) Since $j = k \cdot p$, we have

$$j \text{ units} = (k \text{ units}) \times (p \text{ units}) = \frac{\text{cm}}{\text{sec} \cdot \text{mm Hg}} \times \text{mm Hg} = \frac{\text{cm}}{\text{sec}}.$$

In addition

$$\text{Volume/time/area units} = \frac{\text{Volume units}}{\text{Time units}} \times \frac{1}{\text{Area units}} = \frac{\text{cm}^3}{\text{sec}} \times \frac{1}{\text{cm}^2} = \frac{\text{cm}}{\text{sec}}.$$

Thus, j has units of volume per time per area.

(d) To find the flow rate through a small section of length Δx of the capillary, we multiply j, the flow rate per capillary wall area, by the area, $A = 2\pi r\,\Delta x$ of the section. To find the flow through the wall of the entire capillary, we integrate. Thus,

$$\text{Flow rate } = \int_0^L j \cdot 2\pi r\,dx = \int_0^L kp \cdot 2\pi r\,dx$$

Using the formula for p from part (b), and the values $L = 0.1$ cm, $r = 0.0004$ cm, and $k = 10^{-7}$ cm/(sec$\cdot$ mm Hg), evaluating the integral symbolically or numerically, we have

$$\text{Flow rate } = \int_0^{0.1} 2\pi(10^{-7})(0.0004)(12 - 200x)\,dx = 8 \cdot 10^{-11}\pi(12x - 100x^2)\Big|_0^{0.1} = 5.03 \cdot 10^{-11}\frac{\text{cm}^3}{\text{sec}}.$$

2. (a) The volume of a cylinder of radius r and height h is $V = \pi r^2 h$. Since the total volume of the urine sample is 2000 ml and the diameter is 10 cm (so the radius is 5 cm), we have:

$$\pi r^2 h = \pi 5^2 h = 2000$$
$$h = \frac{2000}{25\pi} = 25.46 \text{ cm.}$$

Thus, the depth of urine in the cylinder is 25.46 cm.

(i) The protein concentration c is a linear function of height y, so $c = b + my$. At the bottom of the sample, $y = 0$, and $c = 0.96$. At the top, $y = 25.46$ and $c = 0.14$. This means the points $(0, 0.96)$ and $(25.46, 0.14)$ are on the graph of c, so the slope is given by

$$m = \frac{0.14 - 0.96}{25.46 - 0} = -0.0322 \, \frac{\text{mg}}{\text{ml} \cdot \text{cm}}.$$

We know that $b = 0.96$, so $c = 0.96 - 0.0322y$ mg/ml.

(ii) A cross sectional slice of the cylinder has volume $5^2\pi\Delta y = 25\pi\Delta y$. This means the quantity of protein in a given slice is

$$\Delta Q = \underbrace{\text{Volume}}_{25\pi\Delta y} \times \underbrace{\text{Concentration}}_{0.96-0.0322y} = 25\pi\Delta y\,(0.96 - 0.0322y) \text{ mg.}$$

(iii) To find the total quantity of protein, we integrate:

$$\begin{aligned}\text{Total mass} &= \int_{y=0}^{25.46} 25\pi\,(0.96 - 0.0322y)\,dy \\ &= 25\pi\left(0.96y - 0.0161y^2\right)\Big|_0^{25.46} \\ &= 1099.98 \text{ mg.}\end{aligned}$$

Thus, the patient excreted approximately 1100 mg, or 1.1 gm, of protein during the 24 hour period in which urine was collected. This is above the threshold of 1 gm per 24 hour period at which active treatment is recommended.

(b) Let the radius of the collection container be r, the volume of the urine be V, and the concentrations of protein at the top and bottom of the collected fluid be c_t and c_b. We compute the quantity of protein by following the procedure of part (a).

If h is the depth of the sample, we have

$$V = \pi r^2 h.$$

Assuming that protein concentration, c, is a linear function of the distance, y, from the bottom of the container, we have

$$c = c_b + my$$

where the slope m is given by

$$m = \frac{c_t - c_b}{h}.$$

The quantity of protein in a horizontal slice is

$$\Delta Q = \text{Volume} \times \text{Concentration} = \pi r^2 \Delta y \cdot c.$$

To find the total quantity of protein in the container, we integrate:

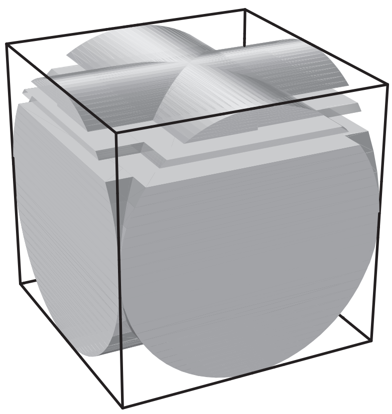
$$\text{Total mass} = \int_0^h \pi r^2 c(y)\,dy = \int_0^h \pi r^2 (c_b + my)\,dy$$

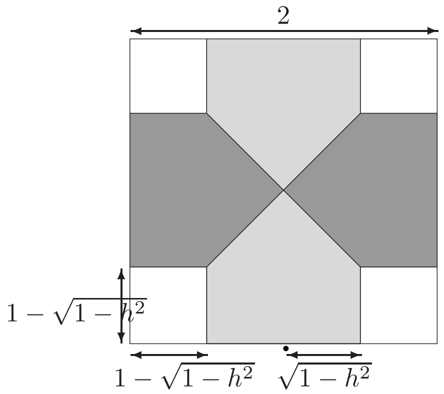
$$= \pi r^2 \left(c_b y + m\frac{y^2}{2} \right) \bigg|_0^h = \pi r^2 \left(c_b h + m\frac{h^2}{2} \right)$$
$$= \pi r^2 \left(c_b h + \frac{c_t - c_b}{h} \cdot \frac{h^2}{2} \right) = \pi r^2 h \left(c_b + \frac{c_t - c_b}{2} \right)$$
$$= \frac{c_t + c_b}{2} V.$$

Thus, the quantity of protein in the sample is the product of the average of the top and bottom protein concentrations by the volume of urine collected.

3. Let us make coordinate axes with the origin at the center of the box. The x and y axes will lie along the central axes of the cylinders, and the (height) axis will extend vertically to the top of the box. If one slices the cylinders horizontally, one gets a cross. The cross is what you get if you cut out four corner squares from a square of side length 2. If h is the height of the cross above (or below) the xy plane, the equation of a cylinder is $h^2 + y^2 = 1$ (or $h^2 + x^2 = 1$). Thus the "armpits" of the cross occur where $y^2 - 1 = -h^2 = x^2 - 1$ for some fixed height h—that is, out $\sqrt{1-h^2}$ units from the center, or $1 - \sqrt{1-h^2}$ units away from the edge. Each corner square has area $(1 - \sqrt{1-h^2})^2 = 2 - h^2 - 2\sqrt{1-h^2}$. The whole big square has area 4. Therefore, the area of the cross is

$$4 - 4(2 - h^2 - 2\sqrt{1-h^2}) = -4 + 4h^2 + 8\sqrt{1-h^2}.$$





We integrate this from $h = -1$ to $h = 1$, and obtain the volume, V:

$$V = \int_{-1}^{1} -4 + 4h^2 + 8\sqrt{1-h^2}\, dh$$
$$= \left[-4h + \frac{4h^3}{3} + 8 \cdot \frac{1}{2} \left(h\sqrt{1-h^2} + \arcsin h \right) \right] \bigg|_{-1}^{1}$$
$$= -8 + \frac{8}{3} + 4\pi = 4\pi - \frac{16}{3} \approx 7.23.$$

This is a reasonable answer, as the volume of the cube is 8, and the volume of one cylinder alone is $2\pi \approx 6.28$.

4. (a) Let y represent height, and let x represent horizontal distance from the lowest point of the cable. Then the stretched cable is a parabola of the form $y = kx^2$ passing through the point $(1280/2, 143) = (640, 143)$. Therefore, $143 = k(640)^2$ so $k \approx 3.491 \times 10^{-4}$. To find the arc length of the parabola, we take twice the arc length of the part to the right of the lowest point. Since $dy/dx = 2kx$,

$$\text{Arc Length} = 2\int_0^{640} \sqrt{1 + (2kx)^2}\, dx = 2\int_0^{640} \sqrt{1 + 4k^2x^2}\, dx.$$

The easiest way to find this integral is to substitute the value of k and find the integral's value numerically, giving

$$\text{Arc Length} \approx 1321.4 \text{ meters.}$$

Alternatively, we can make the substitution $w = 2kx$:

$$\begin{aligned}\text{Arc Length} &= \frac{2}{2k}\int_0^{1280k}\sqrt{1+w^2}\,dw\\ &= \frac{1}{k}\int_0^{1280k}\sqrt{1+w^2}\,dw\\ &= \frac{1}{2k}\left(w\sqrt{1+w^2}\Big|_0^{1280k}\right)+\frac{1}{2k}\left(\int_0^{1280k}\frac{1}{\sqrt{1+w^2}}\,dw\right)\\ &\quad\text{[Using the integral table, Formula VI-29, or substitute } w=\tan\theta]\\ &= \frac{1}{2k}\left(1280k\sqrt{1+(1280k)^2}\right)+\frac{1}{2k}\left(\ln\left|x+\sqrt{1+x^2}\right|\Big|_0^{1280k}\right)\\ &= \frac{1}{2k}\left(1280k\sqrt{1+(1280k)^2}\right)+\frac{1}{2k}\left(\ln\left|1280k+\sqrt{1+(1280k)^2}\right|\right)\\ &\approx 1321.4 \text{ meters.}\end{aligned}$$

(b) Adding 0.05% to the length of the cable gives a cable length of $(1321.4)(1.0005) = 1322.1$. We now want to calculate the new shape of the parabola; that is, we want to find a new k so that the arc length is 1322.1. Since

$$\text{Arc Length} = 2\int_0^{640}\sqrt{1+4k^2x^2}\,dx$$

we can find k numerically by trial and error. Trying values close to our original value of k, we find $k \approx 3.52 \times 10^{-4}$. To find the sag for this new k, we find the height $y = kx^2$ for which the cable hangs from the towers. This is

$$y = k(640)^2 \approx 144.2.$$

Thus the cable sag is 144.2 meters, over a meter more than on a cold winter day. Notice, though, that although the length increases by 0.05%, the sag increases by more:$144.2/143 \approx 1.0084$, an increase of 0.84%.

5. (a) Revolving the semi-circle $y = \sqrt{r^2 - x^2}$ around the x-axis yields the sphere of radius r. See Figure 8.186. Differentiating yields:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{r^2-x^2}}\cdot x = -\frac{x}{y}.$$

Thus, substituting $-x/y$ for $f'(x)$, we get

$$\begin{aligned}\text{Surface area} &= 2\pi\int_{-r}^{r} y\sqrt{1+\frac{x^2}{y^2}}\,dx = 2\pi\int_{-r}^{r}\sqrt{x^2+y^2}\,dx\\ &= 2\pi r\int_{-r}^{r}dx = 4\pi r^2.\end{aligned}$$

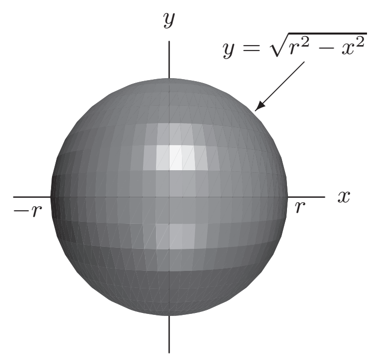


Figure 8.186

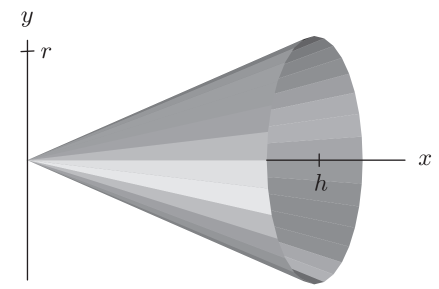


Figure 8.187

(b) Revolving the line $y = rx/h$ around the x-axis yields the cone. The base of the cone is a circle with area πr^2. See Figure 8.187. The area of the rest of the cone is

$$\begin{aligned}\text{Surface area} &= 2\pi \int_0^h y\sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\sqrt{1+\frac{r^2}{h^2}}\left(\frac{r}{h}\int_0^h x\,dx\right)\\ &= 2\pi\frac{r}{h}\frac{h^2}{2}\sqrt{1+\frac{r^2}{h^2}} = \pi r\sqrt{r^2+h^2}\end{aligned}$$

Adding the area of the base, we get

$$\text{Total surface area of cone } = \pi r^2 + \pi r\sqrt{r^2+h^2}.$$

(c) We find the volume of $y = 1/x$ revolved about the x-axis as x runs from 1 to ∞. See Figure 8.188.

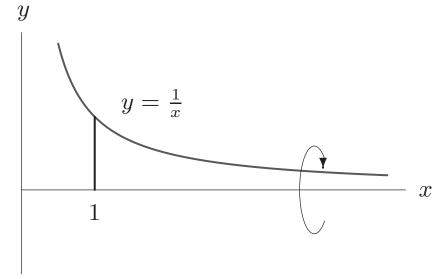


Figure 8.188

$$\text{Volume } = \int_1^\infty \pi y^2\,dx = \pi\int_1^\infty \frac{1}{x^2}\,dx = \pi\lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \pi\lim_{b\to\infty}\left.\frac{-1}{x}\right|_1^b = \pi$$

Thus, the volume of this solid is finite and equal to π.

(d) Now we show the surface area of this solid is unbounded. We have

$$\text{Surface area } = 2\pi\int_1^\infty y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

We cannot easily compute the antiderivative of $\frac{1}{x}\sqrt{1+\frac{1}{x^4}}$, so we bound the integral from below by noticing that

$$\sqrt{1+\frac{1}{x^4}} \geq 1.$$

Thus we see that

$$\text{Surface area } \geq 2\pi\int_1^\infty \frac{1}{x}\,dx = 2\pi\lim_{b\to\infty}\int_1^b\frac{dx}{x} = 2\pi\lim_{b\to\infty}\left.\ln x\right|_1^b.$$

Since $\ln x$ goes to infinity as x goes to infinity, the surface area is unbounded.

Alternatively, we can try calculating

$$2\pi\int_1^b \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

for larger and larger values of b. We would see that the integral seems to diverge.

(e) For a solid generated by the revolution of a curve $y = f(x)$ for $a \le x \le b$,

$$\text{Volume} = \int_a^b \pi y^2\, dx$$

and

$$\text{Surface area} = \int_a^b 2\pi y \sqrt{1 + (f'(x))^2}\, dx.$$

The volume and the surface area will be equal if

$$f(x) = 2\sqrt{1 + (f'(x))^2}.$$

We find a function $y = f(x)$ which satisfies this relation:

$$\begin{aligned}
y &= 2\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\
\frac{y^2}{4} &= 1 + \left(\frac{dy}{dx}\right)^2 \\
\frac{dy}{dx} &= \sqrt{\frac{y^2}{4} - 1} \\
\frac{dy}{\sqrt{y^2 - 4}} &= \frac{1}{2}\, dx \\
\int \frac{dy}{\sqrt{y^2 - 4}} &= \int \frac{1}{2}\, dx \\
\ln|y + \sqrt{y^2 - 4}| &= \frac{x}{2} + C \\
y + \sqrt{y^2 - 4} &= Ae^{x/2}
\end{aligned}$$

Notice in the third line we have used the fact that $dy/dx \ge 0$. Any function, $y = f(x)$, which satisfies this relationship has the required property.

6. (a) We want to find a such that $\int_0^\infty p(v)\, dv = \lim_{r\to\infty} a \int_0^r v^2 e^{-mv^2/2kT}\, dv = 1$. Therefore,

$$\frac{1}{a} = \lim_{r\to\infty} \int_0^r v^2 e^{-mv^2/2kT}\, dv.$$

To evaluate the integral, use integration by parts with the substitutions $u = v$ and $w' = ve^{-mv^2/2kT}$:

$$\begin{aligned}
\int_0^r \underbrace{v}_{u} \underbrace{ve^{-mv^2/2kT}}_{w'}\, dv &= \underbrace{v}_{u} \underbrace{\frac{e^{-mv^2/2kT}}{-m/kT}}_{w}\Bigg|_0^r - \int_0^r \underbrace{1}_{u'} \underbrace{\frac{e^{-mv^2/2kT}}{-m/kT}}_{w}\, dv \\
&= -\frac{kTr}{m} e^{-mr^2/2kT} + \frac{kT}{m}\int_0^r e^{-mv^2/2kT}\, dv.
\end{aligned}$$

From the normal distribution we know that $\displaystyle\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{2}$, so

$$\int_0^\infty e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{2}.$$

Therefore in the above integral, make the substitution $x = \sqrt{\frac{m}{kT}}v$, so that $dx = \sqrt{\frac{m}{kT}}\, dv$, or $dv = \sqrt{\frac{kT}{m}}\, dx$. Then

$$\frac{kT}{m}\int_0^r e^{-mv^2/2kT}\, dv = \left(\frac{kT}{m}\right)^{3/2} \int_0^{\sqrt{\frac{m}{kT}}r} e^{-x^2/2}\, dx.$$

Substituting this into Equation 6a we get

$$\frac{1}{a} = \lim_{r\to\infty}\left(-\frac{kTr}{m}e^{-mr^2/2kT} + \left(\frac{kT}{m}\right)^{3/2}\int_0^{\sqrt{\frac{m}{kT}}r} e^{-x^2/2}\,dx\right) = 0 + \left(\frac{kT}{m}\right)^{3/2}\cdot\frac{\sqrt{2\pi}}{2}.$$

Therefore, $a = \frac{2}{\sqrt{2\pi}}(\frac{m}{kT})^{3/2}$. Substituting the values for k, T, and m gives $a \approx 3.4\times 10^{-8}$.

(b) To find the median, we wish to find the speed x such that

$$\int_0^x p(v)\,dv = \int_0^x av^2e^{-\frac{mv^2}{2kT}}\,dv = \frac{1}{2},$$

where $a = \frac{2}{\sqrt{2\pi}}(\frac{m}{kT})^{3/2}$. Using a calculator, by trial and error we get $x \approx 441$ m/sec.

To find the mean, we find

$$\int_0^\infty vp(v)\,dv = \int_0^\infty av^3e^{-\frac{mv^2}{2kT}}\,dv.$$

This integral can be done by substitution. Let $u = v^2$, so $du = 2vdv$. Then

$$\begin{aligned}\int_0^\infty av^3e^{-\frac{mv^2}{2kT}}\,dv &= \frac{a}{2}\int_{v=0}^{v=\infty} v^2e^{-\frac{mv^2}{2kT}}2v\,dv\\ &= \frac{a}{2}\int_{u=0}^{u=\infty} ue^{-\frac{mu}{2kT}}\,du\\ &= \lim_{r\to\infty}\frac{a}{2}\int_0^r ue^{-\frac{mu}{2kT}}\,du.\end{aligned}$$

Now, using the integral table, we have

$$\begin{aligned}\int_0^\infty av^3e^{-\frac{mv^2}{2kT}}\,dv &= \lim_{r\to\infty}\frac{a}{2}\left[-\frac{2kT}{m}ue^{-\frac{mu}{2kT}} - \left(-\frac{2kT}{m}\right)^2e^{-\frac{mu}{2kT}}\right]\Bigg|_0^r\\ &= \frac{a}{2}\left(-\frac{2kT}{m}\right)^2\\ &\approx 457.7 \text{ m/sec.}\end{aligned}$$

The maximum for $p(v)$ will be at a point where $p'(v) = 0$.

$$\begin{aligned}p'(v) &= a(2v)e^{-\frac{mv^2}{2kT}} + av^2\left(-\frac{2mv}{2kT}\right)e^{-\frac{mv^2}{2kT}}\\ &= ae^{-\frac{mv^2}{2kT}}\left(2v - v^3\frac{m}{kT}\right).\end{aligned}$$

Thus $p'(v) = 0$ at $v = 0$ and at $v = \sqrt{\frac{2kT}{m}} \approx 405$. It's obvious that $p(0) = 0$, and that $p \to 0$ as $v \to \infty$. So $v = 405$ gives us a maximum: $p(405) \approx 0.002$.

(c) The mean, as we found in part (b), is $\frac{a}{2}\frac{4k^2T^2}{m^2} = \frac{4}{\sqrt{2\pi}}\frac{k^{1/2}T^{1/2}}{m^{1/2}}$. It is clear, then, that as T increases so does the mean. We found in part (b) that $p(v)$ reached its maximum at $v = \sqrt{\frac{2kT}{m}}$. Thus

$$\begin{aligned}\text{The maximum value of } p(v) &= \frac{2}{\sqrt{2\pi}}\left(\frac{m}{kT}\right)^{3/2}\frac{2kT}{m}e^{-1}\\ &= \frac{4}{e\sqrt{2\pi}}\frac{m^{1/2}}{kT^{1/2}}.\end{aligned}$$

Thus as T increases, the maximum value decreases.

CHAPTER NINE

Solutions for Section 9.1

Exercises

1. The first term is $2^1 + 1 = 3$. The second term is $2^2 + 1 = 5$. The third term is $2^3 + 1 = 9$, the fourth is $2^4 + 1 = 17$, and the fifth is $2^5 + 1 = 33$. The first five terms are $3, 5, 9, 17, 33$.

2. The first term is $1 + (-1)^1 = 1 - 1 = 0$. The second term is $2 + (-1)^2 = 2 + 1 = 3$. The third term is $3 - 1 = 2$ and the fourth is $4 + 1 = 5$. The first five terms are $0, 3, 2, 5, 4$.

3. The first term is $2 \cdot 1/(2 \cdot 1 + 1) = 2/3$. The second term is $2 \cdot 2/(2 \cdot 2 + 1) = 4/5$. The first five terms are

$$2/3, 4/5, 6/7, 8/9, 10/11.$$

4. The first term is $(-1)^1(1/2)^1 = -1/2$. The second term is $(-1)^2(1/2)^2 = 1/4$. The first five terms are

$$-1/2, 1/4, -1/8, 1/16, -1/32.$$

5. The first term is $(-1)^2(1/2)^0 = 1$. The second term is $(-1)^3(1/2)^1 = -1/2$. The first five terms are

$$1, -1/2, 1/4, -1/8, 1/16.$$

6. The first term is $(1 - 1/(1 + 1))^{(1+1)} = (1/2)^2$. The second term is $(1 - 1/3)^3 = (2/3)^3$. The first five terms are

$$(1/2)^2, (2/3)^3, (3/4)^4, (4/5)^5, (5/6)^6.$$

7. The terms look like powers of 2 so we guess $s_n = 2^n$. This makes the first term $2^1 = 2$ rather than 4. We try instead $s_n = 2^{n+1}$. If we now check, we get the terms $4, 8, 16, 32, 64, \ldots$, which is right.

8. We compare with positive powers of 2, which are $2, 4, 8, 16, 32, \ldots$. Each term is one less, so we take $s_n = 2^n - 1$.

9. We observe that if we subtract 1 from each term of the sequence, we get $1, 4, 9, 16, 25, \ldots$, namely the squares $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$. Thus $s_n = n^2 + 1$.

10. First notice that $s_n = 2n - 1$ is a formula for the general term of the sequence

$$1,\ 3,\ 5,\ 7,\ 9, \ldots.$$

To obtain the alternating signs in the original sequence, we try multiplying by $(-1)^n$. However, checking $(-1)^n(2n-1)$ for $n = 1, 2, 3, \ldots$ gives

$$-1,\ 3,\ -5,\ 7,\ -9, \ldots.$$

To get the correct signs, we multiply by $(-1)^{n+1}$ and take

$$s_n = (-1)^{n+1}(2n - 1).$$

11. The numerator is n. The denominator is then $2n + 1$, so $s_n = n/(2n + 1)$.

12. The denominators are the even numbers, so we try $s_n = 1/(2n)$. To get the signs to alternate, we try multiplying by $(-1)^n$. That gives

$$-1/2, 1/4, -1/6, 1/8, -1/10, \ldots,$$

so we multiply by $(-1)^{n+1}$ instead. Thus $s_n = (-1)^{n+1}/(2n)$.

Problems

13. Since 2^n increases without bound as n increases, the sequence diverges.

14. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|0.2| < 1$, we have $\lim_{n\to\infty} (0.2)^n = 0$, so the sequence converges to 0

15. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|e^{-2}| < 1$, we have $\lim_{n\to\infty} (e^{-2n}) = \lim_{n\to\infty} (e^{-2})^n = 0$, so $\lim_{n\to\infty} (3+e^{-2n}) = 3+0 = 3$, so the sequence converges to 3.

16. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|-0.3| < 1$, we have $\lim_{n\to\infty} (-0.3)^n = 0$, so the sequence converges to 0.

17. We have:
$$\lim_{n\to\infty}\left(\frac{n}{10}+\frac{10}{n}\right) = \lim_{n\to\infty}\frac{n}{10} + \lim_{n\to\infty} 10n.$$
Since $n/10$ gets arbitrarily large and $10/n$ approaches 0 as $n \to \infty$, the sequence diverges.

18. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $\left|\frac{2}{3}\right| < 1$, we have $\lim_{n\to\infty}\left(\frac{2^n}{3^n}\right) = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$, so the sequence converges to 0.

19. We have
$$\lim_{n\to\infty}\frac{2n+1}{n} = \lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 2,$$
so the sequence converges to 2.

20. We have:
$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$
The terms of the sequence alternate in sign, but they approach 0, so the sequence converges to 0.

21. Since $1/n$ approaches zero and $\ln n$ becomes arbitrarily large as $n \to \infty$, the sequence diverges.

22. As n increases, the term $2n$ is much larger in magnitude than $(-1)^n 5$ and the term $4n$ is much larger in magnitude than $(-1)^n 3$. Thus dividing the numerator and denominator by n and using the fact that $\lim_{n\to\infty} 1/n = 0$, we have
$$\lim_{n\to\infty}\frac{2n+(-1)^n 5}{4n-(-1)^n 3} = \lim_{n\to\infty}\frac{2+(-1)^n 5/n}{4-(-1)^n 3/n} = \frac{1}{2}.$$
Thus, the sequence converges to $1/2$.

23. Since $\lim_{n\to\infty} 1/n = 0$ and $-1 \le \sin n \le 1$, the terms approach zero and the sequence converges to 0.

24. Since $s_n = \cos(\pi n) = 1$ if n is even and $s_n = \cos(\pi n) = -1$ if n is odd, the values of s_n alternate between 1 and -1, so the limit does not exist. Thus, the sequence diverges.

25. (a) matches (IV), since the sequence increases toward 1.
(b) matches (III), since the odd terms increase toward 1 and the even terms decrease toward 1.
(c) matches (II), since the sequence decreases toward 0.
(d) matches (I), since the sequence decreases toward 1.

26. (a) matches (II), since $\lim_{n\to\infty}(n(n+1)-1) = \infty$.
(b) matches (III), since $\lim_{n\to\infty}(1/(n+1)) = 0$ and $1/(n+1)$ is always positive.
(c) matches (I), since $\lim_{n\to\infty}(1-n^2) = -\infty$.
(d) matches (IV), since $\lim_{n\to\infty}\cos(1/n) = \cos 0 = 1$.
(e) matches (V), since $\sin n$ is bounded above and below by ± 1, so $\lim_{n\to\infty}((\sin n)/n) = 0$ and the sign of $\sin n$ varies as $n \to \infty$.

27. (a) matches (II), since the sequence increases toward 2.
(b) matches (III), since the even terms decrease toward 2 and odd terms decrease toward -2.
(c) matches (IV), since the even terms decrease toward 2 and odd terms increase toward 2.
(d) matches (I), since the sequence decreases toward 2.
(e) matches (V), since the even terms decrease toward 2 and odd terms increase toward -2.

28. We have $s_2 = 2s_1 + 3 = 2\cdot 1 + 3 = 5$ and $s_3 = 2s_2 + 3 = 2\cdot 5 + 3 = 13$. Continuing, we get
$$1, 5, 13, 29, 61, 125.$$

29. We have $s_2 = s_1 + 2 = 3$ and $s_3 = s_2 + 3 = 6$. Continuing, we get
$$1,\ 3,\ 6,\ 10,\ 15,\ 21.$$

30. We have $s_2 = s_1 + 1/2 = 0 + (1/2)^1 = 1/2$ and $s_3 = s_2 + (1/2)^2 = 1/2 + 1/4 = 3/4$. Continuing, we get

$$0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \frac{31}{32}.$$

31. We have $s_3 = s_2 + 2s_1 = 5 + 2 \cdot 1 = 7$ and $s_4 = s_3 + 2s_2 = 7 + 2 \cdot 5 = 17$. Continuing, we get

$$1,\ 5,\ 7,\ 17,\ 31,\ 65.$$

32. We have

$$\begin{aligned} a_2 &= a_{2-1} + 3 \cdot 2 = a_1 + 6 &&= 8 + 6 &&= 14 \\ a_3 &= a_{3-1} + 3 \cdot 3 = a_2 + 9 &&= 14 + 9 &&= 23 \\ a_4 &= a_{4-1} + 3 \cdot 4 = a_3 + 12 &&= 23 + 12 &&= 35. \end{aligned}$$

33. We have

$$\begin{aligned} a_2 &= a_{2-1} + 3 \cdot 2 = a_1 + 6 = 8 + 6 = 14 \\ a_3 &= a_{3-1} + 3 \cdot 3 = a_2 + 9 = 14 + 9 = 23 \\ a_4 &= a_{4-1} + 3 \cdot 4 = a_3 + 12 = 23 + 12 = 35 \end{aligned}$$

$$\begin{aligned} b_2 &= b_{2-1} + a_{2-1} = b_1 + a_1 = 5 + 8 = 13 \\ b_3 &= b_{3-1} + a_{3-1} = b_2 + a_2 = 13 + 14 = 27 \\ b_4 &= b_{4-1} + a_{4-1} = b_3 + a_3 = 27 + 23 = 50 \\ b_5 &= b_{5-1} + a_{5-1} = b_4 + a_4 = 50 + 35 = 85. \end{aligned}$$

34. We have:

$$\begin{aligned} s_1 & &&= 0 \\ s_2 & &&= 0 \\ s_3 & &&= 1 \\ s_4 &= s_3 + s_2 + s_1 = 1 + 0 + 0 &&= 1 \\ s_5 &= s_4 + s_3 + s_2 = 1 + 1 + 0 &&= 2 \\ s_6 &= s_5 + s_4 + s_3 = 2 + 1 + 1 &&= 4 \\ s_7 &= s_6 + s_5 + s_4 = 4 + 2 + 1 &&= 7 \\ s_8 &= s_7 + s_6 + s_5 = 7 + 4 + 2 &&= 13 \\ s_9 &= s_8 + s_7 + s_6 = 13 + 7 + 4 &&= 24 \\ s_{10} &= s_9 + s_8 + s_7 = 24 + 13 + 7 &&= 44. \end{aligned}$$

35. The first 6 terms of the sequence for the sampling is

$$(-0.5)^2,\ (0.0)^2,\ (0.5)^2,\ (1.0)^2,\ (1.5)^2,\ (2.0)^2,$$
$$= 0.25,\ 0.00,\ 0.25,\ 1.00,\ 2.25,\ 4.00.$$

36. The first 6 terms of the sequence for the sampling is

$$\cos 0.5,\ \cos 1.0,\ \cos 1.5,\ \cos 2.0,\ \cos 2.5,\ \cos 3.0$$
$$= 0.878,\ 0.540,\ 0.071,\ -0.416,\ -0.801,\ -0.990.$$

37. The first 6 terms of the sequence for the sampling are

$$\frac{\sin 1}{1},\ \frac{\sin 2}{2},\ \frac{\sin 3}{3},\ \frac{\sin 4}{4},\ \frac{\sin 5}{5},\ \frac{\sin 6}{6}$$
$$= 0.841,\ 0.455,\ 0.047,\ -0.189,\ -0.192,\ -0.047.$$

38. The first smoothing gives

$$0,\ 6,\ -6,\ 6,\ -6,\ 6, \ldots$$

The second smoothing gives

$$3,\ 0,\ 2,\ -2,\ 2, \ldots$$

The smoothing process diminishes the peaks and valleys of this alternating sequence.

39. The first smoothing gives

$$0,\ 0,\ 6,\ 6,\ 6,\ 0,\ 0 \ldots$$

The second smoothing gives

$$0,\ 2,\ 4,\ 6,\ 4,\ 2 \ldots$$

The smoothing process spreads out the spike at the fourth term to the neighboring terms.

40. The first smoothing gives

$$1.5,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7 \ldots$$

The second smoothing gives

$$1.75,\ 2.17,\ 3,\ 4,\ 5,\ 6 \ldots$$

Terms which are already the same as their average with their neighbors are not changed.

41. Each term is 2 more than the previous term, so a recursive definition is $s_n = s_{n-1} + 2$ for $n > 1$ and $s_1 = 1$.

42. Each term is 2 more than the previous term, so a recursive definition is $s_n = s_{n-1} + 2$ for $n > 1$ and $s_1 = 2$. Notice that the even positive integers and odd positive integers have the same recursive definition except for the starting term.

43. Each term is twice the previous term minus one, so a recursive definition is $s_n = 2s_{n-1} - 1$ for $n > 1$ and $s_1 = 3$. We also notice that the differences of consecutive terms are powers of 2, so $s_2 = s_1 + 2$, $s_3 = s_2 + 2^2$, and so on. Thus another recursive definition is $s_n = s_{n-1} + 2^{n-1}$ for $n > 1$ and $s_1 = 3$.

44. The differences between consecutive terms are $4, 9, 16, 25$, so, for example, $s_2 = s_1 + 4$ and $s_3 = s_2 + 9$. Thus, a possible recursive definition is $s_n = s_{n-1} + n^2$ for $n > 1$ and $s_1 = 1$.

45. The differences are $2, 3, 4, 5$, so, for example, $s_2 = s_1 + 2$, $s_3 = s_2 + 3$, and $s_4 = s_3 + 4$. Thus, a recursive definition is $s_n = s_{n-1} + n$ for $n > 1$ and $s_1 = 1$.

46. The numerator and denominator of each term are related to the numerator and denominator of the previous term. The denominator is the previous numerator and the numerator is the sum of the previous numerator and previous denominator. For example,

$$\frac{5}{3} = \frac{2+3}{3} \text{ and } \frac{8}{5} = \frac{3+5}{5}.$$

If we simplify, we get

$$\frac{5}{3} = \frac{2}{3} + 1, \text{ and } \frac{8}{5} = \frac{3}{5} + 1.$$

In words, we turn the previous term upside down and add 1. Thus, a recursive definition is $s_n = \dfrac{1}{s_{n-1}} + 1$ for $n > 1$ and $s_1 = 1$.

47. For $n > 1$, if $s_n = 3n - 2$, then $s_{n-1} = 3(n-1) - 2 = 3n - 5$, so

$$s_n - s_{n-1} = (3n-2) - (3n-5) = 3,$$

giving

$$s_n = s_{n-1} + 3.$$

In addition, $s_1 = 3 \cdot 1 - 2 = 1$.

48. For $n > 1$, if $s_n = n(n+1)/2$, then $s_{n-1} = (n-1)(n-1+1)/2 = n(n-1)/2$. Since

$$s_n = \frac{1}{2}(n^2 + n) = \frac{n^2}{2} + \frac{n}{2} \quad \text{and} \quad s_{n-1} = \frac{1}{2}(n^2 - n) = \frac{n^2}{2} - \frac{n}{2},$$

we have

$$s_n - s_{n-1} = \frac{n}{2} + \frac{n}{2} = n,$$

so

$$s_n = s_{n-1} + n.$$

In addition, $s_1 = 1(2)/2 = 1$.

49. For $n > 1$, if $s_n = 2n^2 - n$, then $s_{n-1} = 2(n-1)^2 - (n-1) = 2n^2 - 5n + 3$, so

$$s_n - s_{n-1} = (2n^2 - n) - (2n^2 - 5n + 3) = 4n - 3,$$

giving

$$s_n = s_{n-1} + 4n - 3.$$

In addition, $s_1 = 2 \cdot 1^2 - 1 = 1$.

50. The sequence seems to converge. By the 25^{th} term it stabilizes to four decimal places at $L = 0.7391$.

51. The sequence oscillates up and down, but by the 20^{th} term it stabilizes to 4 decimal places at $L = 0.5671$.

52. The sequence appears to be decreasing toward 0, but at a slower and slower rate. Even after 100 terms, it is hard to guess what the series will eventually do. It can be shown that it converges to 0.

53. The sequence converges to 1. By the tenth term, it stabilizes to three decimal places at 1.000.

54. (a) Since month 10 is October, V_{10} is the number of SUVs sold in the US in October 2004.
(b) The difference $V_n - V_{n-1}$ represents the increase in sales between month $(n-1)$ and month n.
(c) The sum $\sum_{i=1}^{12} V_i$ represents the total sales of SUVs in the year 2004 (twelve months). The sum $\sum_{i=1}^{n} V_i$ represents the total sales in the n months starting from January 1, 2004.

55. (a) Since you have two parents and four grandparents, $s_1 = 2$ and $s_2 = 4$. In general, $s_n = 2^n$.
(b) Solving $s_n = 6 \cdot 10^9$ gives

$$\begin{aligned} 2^n &= 6 \cdot 10^9 \\ n &= \frac{\ln(6 \cdot 10^9)}{\ln 2} = 32.482. \end{aligned}$$

Thus, 33 or more generations ago, the number of ancestors is greater than the current population of the world. Since the population of the world 33 generations ago was much smaller than it is now, there must have been overlap among our ancestors.

56. In year 1, the payment is

$$p_1 = 10{,}000 + 0.05(100{,}000) = 15{,}000.$$

The balance in year 2 is $100{,}000 - 10{,}000 = 90{,}000$, so

$$p_2 = 10{,}000 + 0.05(90{,}000) = 14{,}500.$$

The balance in year 3 is 80,000, so

$$p_3 = 10{,}000 + 0.05(80{,}000) = 14{,}000.$$

Thus,

$$\begin{aligned} p_n &= 10{,}000 + 0.05(100{,}000 - (n-1) \cdot 10{,}000) \\ &= 15{,}500 - 500n. \end{aligned}$$

57. (a) The bottom row contains k cans, the next one contains $(k-1)$ cans, then $(k-2)$ and so on. Thus, there are k rows. Since the top row contains 1 can, the second contains 2 cans, etc, we have $a_n = n$.
(b) Since the n^{th} row contains n cans, $a_n = n$,

$$T_n = T_{n-1} + a_n$$

gives

$$T_n = T_{n-1} + n, \quad \text{for } n > 1.$$

In addition, $T_1 = 1$.
(c) If $T_n = \frac{1}{2}n(n+1)$, then $T_{n-1} = \frac{1}{2}(n-1)n$, so

$$T_n - T_{n-1} = \frac{1}{2}n(n+1) - \frac{1}{2}n(n-1) = \frac{n}{2}(n+1-(n-1)) = n.$$

In addition, $T_1 = \frac{1}{2} \cdot 1(2) = 1$.

58. (a) In the first year, $d_1 = 20{,}000(0.12)$, so the car's value at the end of the first year is \$$20{,}000(0.88)$. In the second year, $d_2 = 20{,}000(0.88)(0.12)$, so the car's value at the end of the second year is \$$20{,}000(0.88)^2$. Similarly, $d_3 = 20{,}000(0.88)^2(0.12)$. In general

$$d_n = 20{,}000(0.88)^{n-1}(0.12).$$

(b) The first year $r_1 = 400$; the second year $r_2 = 400(1.18)$, the third year $r_3 = 400(1.18)^2$. In general, $r_n = 400(1.18)^{n-1}$.

(c) We have

$$\begin{aligned}\text{Total cost} &= d_1 + d_2 + d_3 + r_1 + r_2 + r_3 \\ &= 20{,}000(0.12)(1 + 0.88 + (0.88)^2) + 400(1 + 1.18 + (1.18)^2) \\ &= 7799.52 \text{ dollars.}\end{aligned}$$

(d) A two-year old car has the same pattern of expenses except that the initial price is \$$20{,}000(0.88)^2$ instead of \$20,000 and that the repair costs start at \$$400(1.18)^2$ instead of \$400. Then

$$\begin{aligned}\text{Total cost} &= 20{,}000(0.88)^2(0.12)(1 + 0.88 + (0.88)^2) + 400(1.18)^2(1 + 1.18 + (1.18)^2) \\ &= 6923.05 \text{ dollars.}\end{aligned}$$

Thus, the two-year-old car costs you less and you should buy it.

59. (a) The first 12 terms are

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144.$$

(b) The sequence of ratios is

$$1,\ 2,\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{21}{13},\ \frac{34}{21},\ \frac{55}{34},\ \frac{89}{55},\ldots.$$

To three decimal places, the first ten ratios are

$$1,\ 2,\ 1.500,\ 1.667,\ 1.600,\ 1.625,\ 1.615,\ 1.619,\ 1.618,\ 1.618.$$

It appears that the sequence of ratios is converging to $r = 1.618$. We find $(1.618)^2 = 2.618 = 1.618 + 1$ so r seems to satisfy $r^2 = r + 1$. Alternatively, by the quadratic formula, the positive root of $x^2 - x - 1 = 0$ is $(1 + \sqrt{5})/2 = 1.618$.

(c) If we multiply both sides of the equation $r^2 = r + 1$ by Ar^{n-2}, we obtain

$$Ar^n = Ar^{n-1} + Ar^{n-2}.$$

Thus, if $s_n = Ar^n$, then $s_{n-1} = Ar^{n-1}$ and $s_{n-2} = Ar^{n-2}$, so the sequence satisfies $s_n = s_{n-1} + s_{n-2}$.

60. (a) Since $25/8 = 3 + 1/8$, we have $f(25/8) = 3 + (1 - 1/8) = 3 + 7/8 = 31/8$.
Since $13/9 = 1 + 4/9$, we have $f(13/9) = 1 + (1 - 4/9) = 1 + 5/9 = 14/9$.
Since $\pi = 3 + (\pi - 3)$, we have $f(\pi) = 3 + (1 - (\pi - 3)) = 7 - \pi$.

(b) The terms are given by

$$\begin{aligned}s_1 &= 1 \\ s_2 &= \frac{1}{f(1)} = \frac{1}{1 + (1 - 0)} = \frac{1}{2} \\ s_3 &= \frac{1}{f(1/2)} = \frac{1}{0 + (1 - 1/2)} = 2 \\ s_4 &= \frac{1}{f(2)} = \frac{1}{2 + (1 - 0)} = \frac{1}{3} \\ s_5 &= \frac{1}{f(1/3)} = \frac{1}{0 + (1 - 1/3)} = \frac{3}{2} \\ s_6 &= \frac{1}{f(3/2)} = \frac{1}{1 + (1 - 1/2)} = \frac{2}{3}.\end{aligned}$$

The first six terms of the sequence are

$$1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}.$$

61. We want to define $\lim_{n\to\infty} s_n = L$ so that s_n is as close to L as we please for all sufficiently large n. Thus, the definition says that for any positive ϵ, there is a value N such that

$$|s_n - L| < \epsilon \quad \text{whenever} \quad n \geq N.$$

62. We use Theorem 9.1, so we must show that s_n is bounded. Since t_n converges, it is bounded so there is a number M, such that $t_n \leq M$ for all n. Therefore $s_n \leq t_n \leq M$ for all n. Since s_n is increasing, $s_1 \leq s_n$ for all n. Thus if we let $K = s_1$, we have $K \leq s_n \leq M$ for all n, so s_n is bounded. Therefore, s_n converges.

Strengthen Your Understanding

63. A decreasing sequence does not have to converge to 0; in fact, it does not have to converge at all (consider the sequence $s_n = -n$, for example). In this case, the limit of the sequence is

$$\lim_{n\to\infty} \frac{3n+10}{7n+3} = \lim_{n\to\infty} \frac{3+10/n}{7+3/n} = \frac{3+0}{7+0} = \frac{3}{7}.$$

64. Even though each term of the sequence is greater than 2, the terms could be getting progressively closer to 2 and the limit could equal 2. For example, consider the sequence

$$s_n = 2 + \frac{1}{n}.$$

The limit of this sequence is

$$\lim_{n\to\infty} \left(2 + \frac{1}{n}\right) = 2 + 0 = 2.$$

Thus, the limit of the sequence could be 2.

65. Since we want our sequence to be increasing and converging to 0, all of the terms of the sequence must be negative. So we need to find a sequence whose terms are negative and approaching zero. An example is $s_n = -1/n$.

66. One example is the sequence $s_n = n$. This sequence is increasing and therefore monotone, but it does not converge because the terms do not get closer and closer to any specific finite value.

67. False. The first 1000 terms could be the same for two different sequences and yet one sequence converges and the other diverges. For example, $s_n = 0$ for all n is a convergent sequence, but

$$t_n = \begin{cases} 0 & \text{if } n \leq 1000 \\ n & \text{if } n > 1000 \end{cases}$$

is a divergent sequence.

68. False. The limit could be zero. For example, $s_n = 1/n$ is a convergent sequence of positive terms and $\lim_{n\to\infty} s_n = 0$.

69. True. If there is no term greater than a million, then the sequence is bounded by $0 < s_n < 10^6$ for all n.

70. True. If there is only a finite number of terms greater than a million, then we can choose the largest of them to be an upper bound M for the sequence. Thus the sequence is bounded by $0 < s_n \leq M$ for all n.

71. False. The terms s_n tend to the limit of the sequence which may not be zero. For example, $s_n = 1 + 1/n$ is a convergent sequence and s_n tends to 1 as n increases.

72. False. For example the sequence $-2, -1, 0, 1, 2, 3, \ldots$ with $s_n = n - 3$ is monotone increasing and has both positive and negative terms.

73. True. If a monotone sequence does not converge, then it is unbounded. If moreover the sequence contains only positive terms then it is bounded below by zero. Thus it is not bounded above, and in particular it is not bounded above by a million.

74. False. The decreasing sequence $-1, -2, -3, \ldots$ has all terms less than a million, but it has no lower bound. Thus it is unbounded.

75. (b). Sequence (I) is monotone increasing, bounded between 9 and 10.

Sequence (II) is monotone decreasing and bounded between 10 and 11.

Sequence (III) is bounded between -1 and 1 but it is not monotone because it begins $0.54, -0.42, -0.99, -0.65, \ldots$, which contains both a jump down, from 0.54 to -0.42 and a jump up, from -0.99 to -0.65. Terms in a monotone sequence always jump in the same direction.

Sequence (IV) is monotone increasing but is not bounded.

Solutions for Section 9.2

Exercises

1. Sequence, because the terms are not added together.
2. Series, because the terms are added together.
3. Sequence, because the terms are not added together.
4. Sequence, because the terms are not added together.
5. Series, because the terms are added together.
6. Series, because the terms are added together.
7. Series, because the terms are added together.
8. Yes, $a = 5$, ratio $= -2$.
9. No. Ratio between successive terms is not constant: $\dfrac{1/3}{1/2} = 0.66\ldots$, while $\dfrac{1/4}{1/3} = 0.75$.
10. Yes, $a = 2$, ratio $= 1/2$.
11. Yes, $a = 1$, ratio $= -1/2$.
12. No. Ratio between successive terms is not constant: $\dfrac{2x^2}{x} = 2x$, while $\dfrac{3x^3}{2x^2} = \dfrac{3}{2}x$.
13. Yes, $a = 1$, ratio $= 2z$.
14. No. Ratio between successive terms is not constant: $\dfrac{6z^2}{3z} = 2z$, while $\dfrac{9z^3}{6z^2} = \dfrac{3}{2}z$.
15. Yes, $a = 1$, ratio $= -x$.
16. Yes, $a = 1$, ratio $= -y^2$.
17. Yes, $a = y^2$, ratio $= y$.
18. No. Ratio between successive terms is not constant: $\dfrac{-z^4}{z^2} = -z^2$, while $\dfrac{z^8}{-z^4} = -z^4$.
19. The series has 26 terms. The first term is $a = 2$ and the constant ratio is $x = 0.1$, so

$$\text{Sum } = \frac{a(1-x^{26})}{(1-x)} = \frac{2(1-(0.1)^{26})}{0.9} = 2.222.$$

20. The series has 10 terms. The first term is $a = 0.2$ and the constant ratio is $x = 0.1$, so

$$\text{Sum } = \frac{0.2(1-x^{10})}{(1-x)} = \frac{0.2(1-(0.1)^{10})}{0.9} = 0.222.$$

21. The series has 9 terms. The first term is $a = 0.00002$ and the constant ratio is $x = 0.1$, so

$$\text{Sum } = \frac{0.00002(1-x^{9})}{(1-x)} = \frac{0.00002(1-(0.1)^{9})}{0.9} = 0.0000222.$$

22. The series has 14 terms. The first term is $a = 8$ and the constant ratio is $x = 1/2 = 0.5$, so

$$\text{Sum } = \frac{8(1-x^{14})}{(1-x)} = \frac{8(1-(0.5)^{14})}{0.5} = 15.999$$

23. We have

$$\begin{aligned}\text{Sum} &= 36 + 12 + 4 + \frac{4}{3} + \frac{4}{9} + \cdots \\ &= 36 + 36\cdot\frac{1}{3} + 36\left(\frac{1}{3}\right)^2 + 36\left(\frac{1}{3}\right)^3 + \cdots \\ &= \frac{36}{1-1/3} = 54.\end{aligned}$$

24. We have

$$\begin{aligned}\text{Sum} &= -810 + 540 - 360 + 240 - 160 + \cdots \\ &= -810 + (-810)\cdot\left(-\frac{2}{3}\right) + (-810)\cdot\left(-\frac{2}{3}\right)^2 + (-810)\cdot\left(-\frac{2}{3}\right)^3 + \cdots \\ &= \frac{-810}{1-(-2/3)} = -486.\end{aligned}$$

25. We have

$$\begin{aligned}\text{Sum} &= 80 + \frac{80}{\sqrt{2}} + 40 + \frac{40}{\sqrt{2}} + 20 + \frac{20}{\sqrt{2}} + \cdots \\ &= 80 + 80\left(\frac{1}{\sqrt{2}}\right) + 80\left(\frac{1}{\sqrt{2}}\right)^2 + +80\left(\frac{1}{\sqrt{2}}\right)^3 + \cdots \\ &= \frac{80}{1-1/\sqrt{2}} = 273.137.\end{aligned}$$

26. This is a geometric series with first term 1 and ratio $z/2$:

$$1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots = \frac{1}{1-(z/2)}.$$

This series converges for $|z/2| < 1$, that is for $-2 < z < 2$.

27. This is a geometric series with first term 1 and ratio $3x$:

$$1 + 3x + 9x^2 + 27x^3 + \cdots = 1 + (3x) + (3x)^2 + (3x)^3 + \cdots = \frac{1}{1-(3x)}.$$

This series converges for $|3x| < 1$, that is for $-1/3 < x < 1/3$.

28. This is a geometric series with first term y and ratio $-y$:

$$y - y^2 + y^3 - y^4 + \cdots = \frac{y}{1-(-y)} = \frac{y}{1+y}.$$

This series converges for $|-y| < 1$, that is for $-1 < y < 1$.

29. This is a geometric series with first term 2 and ratio $-2z$,

$$2 - 4z + 8z^2 - 16z^3 + \cdots = \frac{2}{1-(-2z)} = \frac{2}{1+2z}.$$

This series converges for $|-2z| < 1$, that is for $-1/2 < z < 1/2$.

30. We can rewrite the series as $3 + (x + x^2 + x^3 + \cdots)$. The terms after the first term define a geometric series with first term x and ratio x. Therefore, we have

$$3 + x + x^2 + x^3 + \cdots = 3 + (x + x^2 + x^3 + \cdots) = 3 + \frac{x}{1-x}.$$

This series converges for $|x| < 1$, that is for $-1 < x < 1$.

31. We can rewrite the series as $4 + (y + y^2/3 + y^3/9 + \cdots)$. The terms after the first term define a geometric series with first term y and ratio $y/3$. Therefore, we have

$$4 + y + y^2/3 + y^3/9 + \cdots = 4 + (y + y^2/3 + y^3/9 + \cdots) = 4 + \frac{y}{1-(y/3)}.$$

This series converges for $|y/3| < 1$, that is for $-3 < y < 3$.

Problems

32. Since the amount of ampicillin excreted during the time interval between tablets is 250 mg, we have

$$\begin{aligned}\text{Amount of ampicillin excreted} &= \text{Original quantity} - \text{Final quantity} \\ 250 &= Q - (0.04)Q.\end{aligned}$$

Solving for Q gives, as before,

$$Q = \frac{250}{1-0.04} \approx 260.42.$$

33. (a)

$$\begin{aligned}P_1 &= 0\\ P_2 &= 250(0.04)\\ P_3 &= 250(0.04) + 250(0.04)^2\\ P_4 &= 250(0.04) + 250(0.04)^2 + 250(0.04)^3\\ &\vdots\\ P_n &= 250(0.04) + 250(0.04)^2 + 250(0.04)^3 + \cdots + 250(0.04)^{n-1}\end{aligned}$$

(b) $P_n = 250(0.04)\left(1 + (0.04) + (0.04)^2 + (0.04)^3 + \cdots + (0.04)^{n-2}\right) = 250\dfrac{0.04(1-(0.04)^{n-1})}{1-0.04}$

(c)

$$\begin{aligned}P &= \lim_{n\to\infty} P_n\\ &= \lim_{n\to\infty} 250\frac{0.04(1-(0.04)^{n-1})}{1-0.04}\\ &= \frac{(250)(0.04)}{0.96} = 0.04Q \approx 10.42\end{aligned}$$

Thus, $\lim_{n\to\infty} P_n = 10.42$ and $\lim_{n\to\infty} Q_n = 260.42$. We would expect these limits to differ because one is right before taking a tablet, one is right after. We would expect the difference between them to be 250 mg, the amount of ampicillin in one tablet.

34. (a) The quantity of atenolol in the blood is given by $Q(t) = Q_0e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 6.3 hours,

$$\frac{1}{2} = e^{-6.3k}, \quad k = -\frac{1}{6.3}\ln\frac{1}{2} \approx 0.11.$$

After 24 hours

$$Q = Q_0e^{-k(24)} \approx Q_0e^{-0.11(24)} \approx Q_0(0.07).$$

Thus, the percentage of the atenolol that remains after 24 hours $\approx 7\%$.

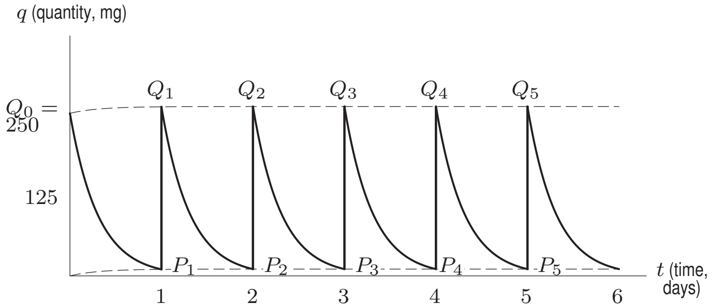
(b)

$$\begin{aligned}Q_0 &= 50\\ Q_1 &= 50 + 50(0.07)\\ Q_2 &= 50 + 50(0.07) + 50(0.07)^2\\ Q_3 &= 50 + 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\ &\vdots\\ Q_n &= 50 + 50(0.07) + 50(0.07)^2 + \cdots + 50(0.07)^n = \frac{50(1-(0.07)^{n+1})}{1-0.07}\end{aligned}$$

(c)

$$\begin{aligned}P_1 &= 50(0.07)\\ P_2 &= 50(0.07) + 50(0.07)^2\\ P_3 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\ P_4 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + 50(0.07)^4\\ &\vdots\\ P_n &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + \cdots + 50(0.07)^n\\ &= 50(0.07)\left(1 + (0.07) + (0.07)^2 + \cdots + (0.07)^{n-1}\right) = \frac{0.07(50)(1-(0.07)^n)}{1-0.07}\end{aligned}$$

35.



36. Let n be the number of days elapsed. As $n \to \infty$, the amount of pollutants right after a dump is given by the sum of an infinite geometric series:

$$B = 8 + 8(0.75) + 8(0.75)^2 + 8(0.75)^3 + \cdots = \frac{8}{1 - 0.75} = 32.$$

Thus, the level of pollutants in the bay approaches 32 tons over time.

37. (a) In the first year, consumption is 25 million tons. In the second year this grows to $25(1.01)$ million tons, and in subsequent years the consumption grows by a factor of 1.01. During the next n years

$$\text{Total consumption} = 25 + 25(1.01) + 25(1.01)^2 + \cdots 25(1.01)^{n-1}\text{million tons.}$$

This is a geometric series with $a = 25$ and $x = 1.01$, thus we have

$$\text{Total consumption} = 25\left(\frac{1.01^n - 1}{1.01 - 1}\right).$$

The reserves are exhausted when the total consumption reaches 400 so we find n by solving the equation

$$25\left(\frac{1.01^n - 1}{1.01 - 1}\right) = 400.$$

Simplifying, gives

$$\frac{1.01^n - 1}{0.01} = 16$$

so

$$1.01^n - 1 = 0.16$$

and

$$1.01^n = 1.16.$$

Taking natural logs of both sides gives

$$\ln 1.01^n = \ln 1.16$$

so

$$n \ln 1.01 = \ln 1.16$$

and

$$n = \frac{\ln 1.16}{\ln 1.01} = 14.916.$$

The reserves are exhausted after 15 years.

(b) If an $r\%$ annual reduction is imposed then

$$\text{Total consumption} = 25 + 25\left(1 - \frac{r}{100}\right) + 25\left(1 - \frac{r}{100}\right)^2 + \cdots \text{million tons.}$$

This is an infinite geometric series with $a = 25$ and $x = 1 - r/100$, thus we have

$$\begin{aligned}\text{Total consumption} &= 25\frac{1}{1 - (1 - r/100)} \\ &= \frac{2500}{r}.\end{aligned}$$

For existing reserves never to be exhausted, we need $2500/r < 400$, which gives $r > 6.25\%$. Consumption must be reduced by at least 6.25% annually.

38.

$$\begin{aligned}\text{Total present value, in dollars} &= 1000 + 1000e^{-0.01} + 1000e^{-0.01(2)} + 1000e^{-0.01(3)} + \cdots \\ &= 1000 + 1000(e^{-0.01}) + 1000(e^{-0.01})^2 + 1000(e^{-0.01})^3 + \cdots\end{aligned}$$

This is an infinite geometric series with $a = 1000$ and $x = e^{(-0.01)}$, and sum

$$\text{Total present value, in dollars} = \frac{1000}{1 - e^{-0.01}} = 100{,}500.833.$$

39. (a) (i) On the night of December 31, 1999:

First deposit will have grown to $2(1.04)^7$ million dollars.
Second deposit will have grown to $2(1.04)^6$ million dollars.
...
Most recent deposit (Jan.1, 1999) will have grown to $2(1.04)$ million dollars.

Thus

$$\begin{aligned}\text{Total amount} &= 2(1.04)^7 + 2(1.04)^6 + \cdots + 2(1.04) \\ &= 2(1.04)\underbrace{(1 + 1.04 + \cdots + (1.04)^6)}_{\text{finite geometric series}} \\ &= 2(1.04)\left(\frac{1-(1.04)^7}{1-1.04}\right) \\ &= 16.43 \text{ million dollars.}\end{aligned}$$

(ii) Notice that if 10 payments were made, there are 9 years between the first and the last. On the day of the last payment:

First deposit will have grown to $2(1.04)^9$ million dollars.
Second deposit will have grown to $2(1.04)^8$ million dollars.
...
Last deposit will be 2 million dollars.

Therefore

$$\begin{aligned}\text{Total amount} &= 2(1.04)^9 + 2(1.04)^8 + \cdots + 2 \\ &= 2\underbrace{(1 + 1.04 + (1.04)^2 + \cdots + (1.04)^9)}_{\text{finite geometric series}} \\ &= 2\left(\frac{1-(1.04)^{10}}{1-1.04}\right) \\ &= 24.01 \text{ million dollars.}\end{aligned}$$

(b) In part (a) (ii) we found the future value of the contract 9 years in the future. Thus

$$\text{Present Value} = \frac{24.01}{(1.04)^9} = 16.87 \text{ million dollars.}$$

Alternatively, we can calculate the present value of each of the payments separately:

$$\begin{aligned}\text{Present Value} &= 2 + \frac{2}{1.04} + \frac{2}{(1.04)^2} + \cdots + \frac{2}{(1.04)^9} \\ &= 2\left(\frac{1-(1/1.04)^{10}}{1-1/1.04}\right) = 16.87 \text{ million dollars.}\end{aligned}$$

Notice that the present value of the contract (\$16.87 million) is considerably less than the face value of the contract, \$20 million.

40. The first \$200 is invested for 24 months so it has a future value of $\$200(1+0.5/100)^{24} = \$200 \cdot 1.005^{24}$. The second installment has a future value of $\$\ 200 \cdot 1.005^{23}$, since it is invested for only 23 months. The final installment, paid after 23 months, has a future value of $\$\ 200 \cdot 1.005$. So,

$$\begin{aligned}\text{Total investment} &= 200 \cdot 1.005^{24} + 200 \cdot 1.005^{23} + \ldots + 200 \cdot 1.005 \\ &= 200 \cdot 1.005\left(1.005^{23} + 1.005^{22} + \ldots + 1\right) \\ &= 200 \cdot 1.005\left(\frac{1-1.005^{24}}{1-1.005}\right) \\ &= 200 \cdot 1.005 \cdot 25.432 \\ &= \$5111.82.\end{aligned}$$

41. To get \$20,000 the day he retires he needs to invest a present value P such that $P(1+5/100)^{20} = \$20{,}000$. Solving for P gives the present value $P = \$20{,}000 \cdot 1.05^{-20}$. To fund the second payment he needs to invest $\$\ 20{,}000 \cdot 1.05^{-21}$, and so on. To fund the payment 10 years after his retirement he needs to invest $\$\ 20{,}000 \cdot 1.05^{-30}$. There are 11 payments in all, so

$$\begin{aligned}\text{Total investment} &= 20{,}000 \cdot 1.05^{-20} + 20{,}000 \cdot 1.05^{-21} + \ldots + 20{,}000 \cdot 1.05^{-30} \\ &= 20{,}000 \cdot 1.05^{-20}\left(1 + 1.05^{-1} + \ldots + 1.05^{-10}\right) \\ &= 20{,}000 \cdot 1.05^{-20}\left(\frac{1-1.05^{-11}}{1-1.05^{-1}}\right) \\ &= 20{,}000 \cdot 1.05^{-20} \cdot 8.7217 \\ &= \$65{,}742.60.\end{aligned}$$

42. If the half-life is T hours, then the exponential decay formula $Q = Q_0e^{-kt}$ gives $k = \ln 2/T$. If we start with $Q_0 = 1$ tablet, then the amount of drug present in the body after $5T$ hours is

$$Q = e^{-5kT} = e^{-5\ln 2} = 0.03125,$$

so 3.125% of a tablet remains. Thus, immediately after taking the first tablet, there is one tablet in the body. Five half-lives later, this has reduced to $1 \cdot 0.03125 = 0.03125$ tablets, and immediately after the second tablet there are $1 + 0.03125$ tablets in the body. Continuing this forever leads to

$$\text{Number of tablets in body } = 1 + 0.03125 + (0.03125)^2 + \cdots + (0.03125)^n + \cdots.$$

This is an infinite geometric series, with common ratio $x = 0.03125$, and sum $1/(1-x)$. Thus

$$\text{Number of tablets in body } = \frac{1}{1-0.03125} = 1.0323.$$

43. The amount of additional income generated directly by people spending their extra money is $\$100(0.8) = \$80$ million. This additional money in turn is spent, generating another $(\$100(0.8))\,(0.8) = \$100(0.8)^2$ million. This continues indefinitely, resulting in

$$\text{Total additional income} = 100(0.8) + 100(0.8)^2 + 100(0.8)^3 + \cdots = \frac{100(0.8)}{1-0.8} = \$400 \text{ million}$$

44. (a)

$$\begin{aligned}\text{Present value of first coupon } &= \frac{50}{1.05} \\ \text{Present value of second coupon } &= \frac{50}{(1.05)^2}, \text{ etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value} &= \underbrace{\frac{50}{1.05} + \frac{50}{(1.05)^2} + \cdots + \frac{50}{(1.05)^{10}}}_{\text{coupons}} + \underbrace{\frac{1000}{(1.05)^{10}}}_{\text{principal}} \\ &= \frac{50}{1.05}\left(1 + \frac{1}{1.05} + \cdots + \frac{1}{(1.05)^9}\right) + \frac{1000}{(1.05)^{10}} \\ &= \frac{50}{1.05}\left(\frac{1-\left(\frac{1}{1.05}\right)^{10}}{1-\frac{1}{1.05}}\right) + \frac{1000}{(1.05)^{10}} \\ &= 386.087 + 613.913 \\ &= \$1000\end{aligned}$$

(b) When the interest rate is 5%, the present value equals the principal.
(c) When the interest rate is more than 5%, the present value is smaller than it is when interest is 5% and must therefore be less than the principal. Since the bond will sell for around its present value, it will sell for less than the principal; hence the description *trading at discount*.
(d) When the interest rate is less than 5%, the present value is more than the principal. Hence the bond will be selling for more than the principal, and is described as *trading at a premium*.

45. The total of the spending and respending of the additional income is given by the series: Total additional income $= 100(0.9) + 100(0.9)^2 + 100(0.9)^3 + \cdots = \frac{100(0.9)}{1-0.9} = \$900$ million.
Notice the large effect of changing the assumption about the fraction of money spent has: the additional spending more than doubles.

46. (a) Let h_n be the height of the n^{th} bounce after the ball hits the floor for the n^{th} time. Then from Figure 9.1,

$$\begin{aligned} h_0 &= \text{height before first bounce } = 10 \text{ feet}, \\ h_1 &= \text{height after first bounce } = 10\left(\frac{3}{4}\right) \text{ feet}, \\ h_2 &= \text{height after second bounce } = 10\left(\frac{3}{4}\right)^2 \text{ feet}. \end{aligned}$$

Generalizing gives

$$h_n = 10\left(\frac{3}{4}\right)^n.$$

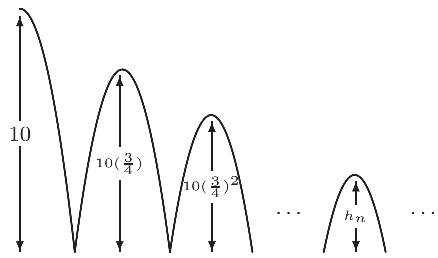


Figure 9.1

(b) When the ball hits the floor for the first time, the total distance it has traveled is just $D_1 = 10$ feet. (Notice that this is the same as $h_0 = 10$.) Then the ball bounces back to a height of $h_1 = 10\left(\frac{3}{4}\right)$, comes down and hits the floor for the second time. See Figure 9.1. The total distance it has traveled is

$$D_2 = h_0 + 2h_1 = 10 + 2\cdot 10\left(\frac{3}{4}\right) = 25 \text{ feet}.$$

Then the ball bounces back to a height of $h_2 = 10\left(\frac{3}{4}\right)^2$, comes down and hits the floor for the third time. It has traveled

$$D_3 = h_0 + 2h_1 + 2h_2 = 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 = 25 + 2\cdot 10\left(\frac{3}{4}\right)^2 = 36.25 \text{ feet}.$$

Similarly,

$$\begin{aligned} D_4 &= h_0 + 2h_1 + 2h_2 + 2h_3 \\ &= 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 \\ &= 36.25 + 2\cdot 10\left(\frac{3}{4}\right)^3 \\ &\approx 44.69 \text{ feet.} \end{aligned}$$

(c) When the ball hits the floor for the n^{th} time, its last bounce was of height h_{n-1}. Thus, by the method used in part (b), we get

$$\begin{aligned} D_n &= h_0 + 2h_1 + 2h_2 + 2h_3 + \cdots + 2h_{n-1} \\ &= 10 + \underbrace{2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 + \cdots + 2\cdot 10\left(\frac{3}{4}\right)^{n-1}}_{\text{finite geometric series}} \\ &= 10 + 2\cdot 10\cdot\left(\frac{3}{4}\right)\left(1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-2}\right) \\ &= 10 + 15\left(\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\left(\frac{3}{4}\right)}\right) \\ &= 10 + 60\left(1-\left(\frac{3}{4}\right)^{n-1}\right). \end{aligned}$$

47. (a) The acceleration of gravity is 32 ft/sec^2 so acceleration $= 32$ and velocity $v = 32t + C$. Since the ball is dropped, its initial velocity is 0 so $v = 32t$. Thus the position is $s = 16t^2 + C$. Calling the initial position $s = 0$, we have $s = 6t$. The distance traveled is h so $h = 16t$. Solving for t we get $t = \frac{1}{4}\sqrt{h}$.

(b) The first drop from 10 feet takes $\frac{1}{4}\sqrt{10}$ seconds. The first full bounce (to $10\cdot(\frac{3}{4})$ feet) takes $\frac{1}{4}\sqrt{10\cdot(\frac{3}{4})}$ seconds to rise, therefore the same time to come down. Thus, the full bounce, up and down, takes $2(\frac{1}{4})\sqrt{10\cdot(\frac{3}{4})}$ seconds. The next full bounce takes $2(\frac{1}{4})10\cdot(\frac{3}{4})^2 = 2(\frac{1}{4})\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^2$ seconds. The n^{th} bounce takes $2(\frac{1}{4})\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^n$ seconds. Therefore the

$$\begin{aligned} &\text{Total amount of time} \\ &= \frac{1}{4}\sqrt{10} + \underbrace{\frac{2}{4}\sqrt{10}\sqrt{\frac{3}{4}} + \frac{2}{4}\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^2 + \frac{2}{4}\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^3}_{\text{Geometric series with } a = \frac{2}{4}\sqrt{10}\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}} \text{ and } x = \sqrt{\frac{3}{4}}} + \cdots \\ &= \frac{1}{4}\sqrt{10} + \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}}\left(\frac{1}{1-\sqrt{3/4}}\right) \text{ seconds.} \end{aligned}$$

Strengthen Your Understanding

48. The formula $4/(1-1/4)$ used is for computing the sum of a geometric series, not the limit of a sequence. The sequence is given by the formula

$$s_n = 4\left(\frac{1}{4}\right)^{n-1}, \quad n \geq 1,$$

so since $(1/4)$ is between 0 and 1, the sequence converges to 0.

49. The formula for the sum of an infinite geometric series does not apply because the common ratio is not between -1 and 1.

50. If the common ratio, x, of a geometric series satisfies $|x| \geq 1$ then the series diverges. If $x = 3$ then the series $3 + 6 + 12 + 24 + \cdots$ diverges.

51. Two possible examples are the series $1+1+1+1+\ldots$, which has common ratio 1, and the series $1+(-1)+1+(-1)+\ldots$, which has common ratio -1.

52. One way to find an example is to start with a geometric series with four distinct terms, and then rescale the terms so that their sum is 10.

For example, we might start with the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8},$$

whose sum is $15/8$. Since we want a sum of 10, we scale these terms up by a factor of

$$\frac{10}{15/8}=\frac{16}{3}.$$

This gives the series

$$\frac{16}{3}+\frac{8}{3}+\frac{4}{3}+\frac{2}{3}=10.$$

This series is geometric with common ratio $1/2$.

53. We know that $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$ for $|x|<1$. Letting $x=1/2$ we have $\frac{a}{1-1/2}=2a$. Thus $a=5$ gives a limit of 10 for the geometric series.

54. (c). The common ratio in series (I) and (IV) is between -1 and 1. The common ratio in (II) and (III) is greater than one.

Solutions for Section 9.3

Exercises

1. The series is $1+2+3+4+5+\cdots$. The sequence of partial sums is

$$S_1=1,\quad S_2=1+2,\quad S_3=1+2+3,\quad S_4=1+2+3+4,\quad S_5=1+2+3+4+5,\ldots$$

which is

$$1,\quad 3,\quad 6,\quad 10,\quad 15\ldots.$$

2. The series is $-1+1/2-1/3+1/4-1/5+\cdots$. The sequence of partial sums is

$$S_1=-1,\quad S_2=-1+\frac{1}{2},\quad S_3=-1+\frac{1}{2}-\frac{1}{3},\quad S_4=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4},\quad S_5=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5},\ldots$$

which is

$$-1,\quad -\frac{1}{2},\quad -\frac{5}{6},\quad -\frac{7}{12},\quad -\frac{47}{60},\ldots.$$

3. The series is $1/2+1/6+1/12+1/20+1/30+\cdots$. The sequence of partial sums is

$$S_1=\frac{1}{2},\quad S_2=\frac{1}{2}+\frac{1}{6},\quad S_3=\frac{1}{2}+\frac{1}{6}+\frac{1}{12},\quad S_4=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20},\quad S_5=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30},\ldots$$

which is

$$\frac{1}{2},\quad \frac{2}{3},\quad \frac{3}{4},\quad \frac{4}{5},\quad \frac{5}{6}.$$

4. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{1}{(x+2)^2}\,dx$ converges or diverges:

$$\begin{aligned}\int_1^\infty \frac{1}{(x+2)^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{(x+2)^2}\,dx \\ &= \lim_{b\to\infty}\int_3^b \frac{1}{w^2}\,dw \qquad \text{(Substitute } w = x+2\text{)} \\ &= \lim_{b\to\infty} -\frac{1}{w}\bigg|_3^b \\ &= \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{3}\right) = \frac{1}{3}.\end{aligned}$$

Since the integral $\int_1^\infty \frac{1}{(x+2)^2}\,dx$ converges, we conclude from the integral test that the series $\sum_{n=1}^\infty \frac{1}{(n+2)^2}$ converges.

5. We use the integral test with $f(x) = x/(x^2+1)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_1^\infty \frac{x}{x^2+1}dx$ converges or diverges:

$$\int_1^\infty \frac{x}{x^2+1}dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}dx = \lim_{b\to\infty}\frac{1}{2}\ln(x^2+1)\bigg|_1^b = \lim_{b\to\infty}\left(\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2\right) = \infty.$$

Since the integral $\int_1^\infty \frac{x}{x^2+1}dx$ diverges, we conclude from the integral test that the series $\sum_{n=1}^\infty \frac{n}{n^2+1}$ diverges.

6. We use the integral test with $f(x) = 1/e^x$ to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{1}{e^x}dx$ converges or diverges:

$$\int_1^\infty \frac{1}{e^x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty} -e^{-x}\bigg|_1^b = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = e^{-1}.$$

Since the integral $\int_1^\infty \frac{1}{e^x}dx$ converges, we conclude from the integral test that the series $\sum_{n=1}^\infty \frac{1}{e^n}$ converges. We can also observe that this is a geometric series with ratio $x = 1/e < 1$, and hence it converges.

7. We use the integral test with $f(x) = 1/(x(\ln x)^2)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges or diverges:

$$\int_2^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\frac{-1}{\ln x}\bigg|_2^b = \lim_{b\to\infty}\left(\frac{-1}{\ln b}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges, we conclude from the integral test that the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges.

8. The improper integral $\int_1^\infty x^{-3}\,dx$ converges to $\frac{1}{2}$, since

$$\int_1^b x^{-3}\,dx = \frac{x^{-2}}{-2}\bigg|_1^b = \frac{b^{-2}}{-2} - \frac{1^{-2}}{-2} = \frac{1}{-2b^2}+\frac{1}{2}$$

and

$$\lim_{b\to\infty}\left(\frac{1}{-2b^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

The terms of the series $\sum_{n=2}^{\infty} n^{-3}$ form a right hand sum for the improper integral; each term represents the area of a rectangle of width 1 fitting completely under the graph of the function x^{-3}. (See Figure 9.2.) Thus the sequence of partial sums is bounded above by $1/2$. Since the partial sums are increasing (every new term added is positive) the series is guaranteed to converge to some number less than or equal to $1/2$ by Theorem 9.1.

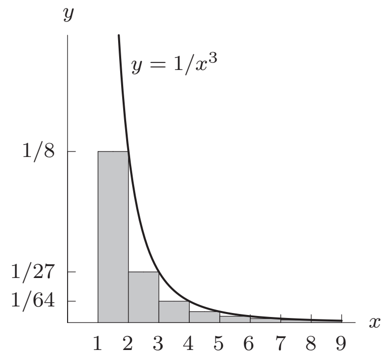


Figure 9.2

9. The improper integral $\int_0^\infty \frac{1}{x^2+1}\,dx$ converges to $\frac{\pi}{2}$, since

$$\int_0^b \frac{1}{x^2+1}\,dx = \arctan x\big|_0^b = \arctan b - \arctan 0 = \arctan b,$$

and $\lim_{b\to\infty} \arctan b = \frac{\pi}{2}$. The terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ form a right hand sum for the improper integral; each term represents the area of a rectangle of width 1 fitting completely under the graph of the function $\frac{1}{x^2+1}$. (See Figure 9.3.) Thus the sequence of partial sums is bounded above by $\frac{\pi}{2}$. Since the partial sums are increasing (every new term added is positive), the series is guaranteed to converge to some number less than or equal to $\pi/2$ by Theorem 9.1.

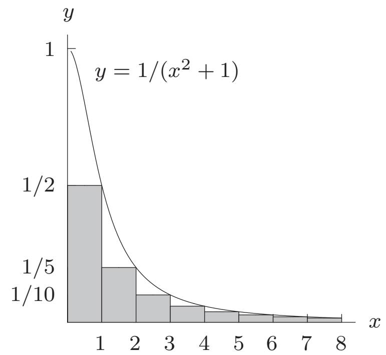


Figure 9.3

10. The integral test requires that $f(x) = x^2$, which is not decreasing.

11. The integral test requires that $f(x) = (-1)^x/x$. However $(-1)^x$ is not defined for all x.

12. The integral test requires that $f(x) = e^{-x}\sin x$, which is not positive, nor is it decreasing.

Problems

13. Using the integral test, we compare the series with

$$\int_0^\infty \frac{3}{x+2}\,dx = \lim_{b\to\infty}\int_0^b \frac{3}{x+2}\,dx = 3\ln|x+2|\Big|_0^b.$$

Since $\ln(b+2)$ is unbounded as $b\to\infty$, the integral diverges and therefore so does the series.

14. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral:

$$\int_0^\infty \frac{4}{2x+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{4}{2x+1}\,dx = \lim_{b\to\infty} 2\ln(2x+1)\Big|_0^b = \lim_{b\to\infty} 2\ln(2b+1).$$

Since the limit does not exist, the integral diverges, so the series $\sum_{n=0}^{\infty}\frac{4}{2n+1}$ diverges.

15. We use the integral test and calculate the corresponding improper integral, $\int_0^\infty 2/\sqrt{2+x}\,dx$:

$$\int_0^\infty \frac{2}{\sqrt{2+x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{2\,dx}{\sqrt{2+x}} = \lim_{b\to\infty} 4(2+x)^{1/2}\Big|_0^b = \lim_{b\to\infty} 4\left((2+b)^{1/2}-2^{1/2}\right).$$

Since the limit does not exist, the integral diverges, so the series $\sum_{n=1}^{\infty}\frac{2}{\sqrt{2+n}}$ diverges.

16. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral using the substitution $w=x^2$

$$\int_0^\infty \frac{2x}{1+x^4}\,dx = \lim_{b\to\infty}\int_0^b \frac{2x}{1+x^4}\,dx = \lim_{b\to\infty}\arctan x^2\Big|_0^b = \lim_{b\to\infty}\arctan b^2 = \frac{\pi}{2}.$$

Since the limit exists, the integral converges, so the series $\sum_{n=0}^{\infty}\frac{2n}{1+n^4}$ converges.

17. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral using the substitution $w=1+x^2$:

$$\int_0^\infty \frac{2x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{2x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\frac{-1}{(1+x^2)}\Big|_0^b = \lim_{b\to\infty}\left(\frac{-1}{1+b^2}+1\right) = 1.$$

Since the limit exists, the integral converges, so the series $\sum_{n=0}^{\infty}\frac{2n}{(1+n^2)^2}$ converges.

18. The terms in this series do not tend to 0. We have

$$\lim_{n\to\infty}\frac{2n}{\sqrt{4+n^2}} = \lim_{n\to\infty}\frac{2}{\sqrt{4/n^2+1}} = 2.$$

Thus the series diverges. It is also possible (although much slower) to use the integral test. We calculate the corresponding improper integral using the substitution $w=x^2$

$$\int_0^\infty \frac{2x}{\sqrt{4+x^2}}\,dx = \lim_{b\to\infty}\int_0^b \frac{2x}{\sqrt{4+x^2}}\,dx = \lim_{b\to\infty} 2\sqrt{4+x^2}\Big|_0^b = \lim_{b\to\infty}(2\sqrt{4+b^2}-4).$$

Since the limit does not exist, the integral diverges, so the series $\sum_{n=0}^{\infty}\frac{2n}{\sqrt{4+n^2}}$ diverges.

19. We use the integral test and calculate the corresponding improper integral, $\int_1^\infty 3/(2x-1)^2\,dx$:

$$\int_1^\infty \frac{3\,dx}{(2x-1)^2} = \lim_{b\to\infty}\int_1^b \frac{3\,dx}{(2x-1)^2} = \lim_{b\to\infty} \frac{-3/2}{(2x-1)}\bigg|_1^b = \lim_{b\to\infty}\left(\frac{-3/2}{(2b-1)} + \frac{3}{2}\right) = \frac{3}{2}.$$

Since the integral converges, the series $\sum_{n=1}^{\infty} \frac{3}{(2n-1)^2}$ converges.

20. We use the integral test and calculate the corresponding improper integral, $\int_1^\infty 4/(2x+1)^3\,dx$. Using the substitution $w = 2x+1$, we have

$$\int_1^\infty \frac{4\,dx}{(2x+1)^3} = \lim_{b\to\infty}\int_1^b \frac{4\,dx}{(2x+1)^3} = \lim_{b\to\infty} -\frac{1}{(2x+1)^2}\bigg|_1^b = \lim_{b\to\infty}\left(-\frac{1}{(2b+1)^2} + \frac{1}{9}\right) = \frac{1}{9}.$$

Since the integral converges, the series $\sum_{n=1}^{\infty} \frac{4}{(2n+1)^3}$ converges.

21. Using the integral test, we compare the series with

$$\int_0^\infty \frac{3}{x^2+4}\,dx = \lim_{b\to\infty}\int_0^b \frac{3}{x^2+4}\,dx = \frac{3}{2}\lim_{b\to\infty} \arctan\left(\frac{x}{2}\right)\bigg|_0^b = \frac{3}{2}\lim_{b\to\infty}\arctan\left(\frac{b}{2}\right) = \frac{3\pi}{4},$$

by integral table V-24. Since the integral converges so does the series.

22. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral using the substitution $w = 2x$:

$$\int_0^\infty \frac{2}{1+4x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{2}{1+4x^2}\,dx = \lim_{b\to\infty}\arctan(2x)\bigg|_0^b = \lim_{b\to\infty}\arctan(2b) = \pi/2.$$

Since the limit exists, the integral converges, so the series $\sum_{n=0}^{\infty} \frac{2}{1+4n^2}$ converges.

23. Writing $a_n = n/(n+1)$, we have $\lim_{n\to\infty} a_n = 1$ so the series diverges by Property 3 of Theorem 9.2.

24. Writing $a_n = (n+1)/(2n+3)$, we have $\lim_{n\to\infty} a_n = 1/2$, so the limit of individual terms is not 0. The series diverges by Property 3 of Theorem 9.2.

25. Both $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ and $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ are convergent geometric series. Therefore, by Property 1 of Theorem 9.2, the series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n$ converges.

26. The series $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is a convergent geometric series, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series.
If $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n + \frac{1}{n}\right)$ converged, then $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n + \frac{1}{n}\right) - \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ would converge by Theorem 9.2. Therefore $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n + \frac{1}{n}\right)$ diverges.

27. The series can be written as

$$\sum_{n=1}^{\infty}\frac{n+2^n}{n2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{1}{n}\right).$$

If this series converges, then $\sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{1}{n}\right) - \sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ would converge by Theorem 9.2. Since this is the harmonic series, which diverges, then the series $\sum_{n=1}^{\infty}\frac{n+2^n}{n}$ diverges.

28. Let $a_n = (\ln n)/n$ and $f(x) = (\ln x)/x$. We use the integral test and consider the improper integral

$$\int_c^{\infty} \frac{\ln x}{x} dx.$$

Since

$$\int_c^R \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2\Big|_c^R = \frac{1}{2}\left((\ln R)^2 - (\ln c)^2\right),$$

and $\ln R$ grows without bound as $R \to \infty$, the integral diverges. Therefore, the integral test tells us that the series, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$, also diverges.

29. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral using the substitution $w = 1 + \ln x$:

$$\int_1^{\infty} \frac{1}{x(1+\ln x)} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x(1+\ln x)} dx = \lim_{b\to\infty} \ln(1+\ln x)\Big|_1^b = \lim_{b\to\infty} \ln(1+\ln b).$$

Since the limit does not exist, the integral diverges, so the series $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$ diverges.

30. We use the integral test and calculate the corresponding improper integral, $\int_3^{\infty} (x+1)/(x^2+2x+2)\, dx$:

$$\int_3^{\infty} \frac{x+1}{x^2+2x+2} dx = \lim_{b\to\infty} \int_3^b \frac{x+1}{x^2+2x+2} dx = \lim_{b\to\infty} \frac{1}{2} \ln|x^2+2x+2|\Big|_3^b = \lim_{b\to\infty} \frac{1}{2}(\ln(b^2+2b+2) - \ln 17).$$

Since the limit does not exist (it is ∞), the integral diverges, so the series $\sum_{n=3}^{\infty} \frac{n+1}{n^2+2n+2}$ diverges.

31. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral using the substitution $w = 1 + x$:

$$\int_0^{\infty} \frac{1}{x^2+2x+2} dx = \lim_{b\to\infty} \int_0^b \frac{1}{x^2+2x+2} dx = \lim_{b\to\infty} \frac{1}{1+(x+1)^2} dx$$

$$= \lim_{b\to\infty} \arctan(x+1)\Big|_0^b = \lim_{b\to\infty} (\arctan(b+1) - \arctan 1) = \pi/2 - \arctan 1.$$

Since the limit exists, the integral converges, so the series $\sum_{n=0}^{\infty} \frac{1}{n^2+2n+2}$ converges.

32. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding improper integral:

$$\int_2^{\infty} \frac{x\ln x + 4}{x^2} dx = \lim_{b\to\infty} \int_2^b \frac{x\ln x+4}{x^2} dx = \lim_{b\to\infty} \left[\int_2^b \frac{\ln x}{x} dx + \int_2^b \frac{4}{x^2} dx\right].$$

Using the substitution $w = \ln x$ on the first integral, we have

$$\lim_{b\to\infty} \int_2^b \frac{x\ln x+4}{x^2} dx = \lim_{b\to\infty} \left(\frac{(\ln x)^2}{2} - \frac{4}{x}\right)\Big|_2^b = \lim_{b\to\infty} \frac{(\ln b)^2}{2} - \frac{4}{b} - \frac{(\ln 2)^2}{2} + 2.$$

Since the limit does not exist, the integral diverges, so the series $\sum_{n=2}^{\infty} \frac{n\ln n + 4}{n^2}$ diverges.

33. Using $\ln(2^n) = n\ln 2$, we see that

$$\sum \frac{1}{\ln(2^n)} = \sum \frac{1}{(\ln 2)n}.$$

The series on the right is the harmonic series multiplied by $1/\ln 2$. Since the harmonic series diverges, $\sum_{n=1}^{\infty} 1/\ln(2^n)$ diverges.

34. Using $\ln(2^n) = n\ln 2$, we see that
$$\sum_{n=1}^{\infty} \frac{1}{(\ln(2^n))^2} = \sum_{n=1}^{\infty} \frac{1}{(\ln 2)^2 n^2}.$$
Since $\sum 1/n^2$ converges, $\sum 1/((\ln(2))^2 n^2)$ converges by property 1 of Theorem 9.2.

35. (a) With $a_n = \ln((n+1)/n)$ we have
$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \\ &= \ln(2/1) + \ln(3/2) + \ln(4/3) + \cdots + \ln(n/(n-1)) + \ln((n+1)/n) \\ &= \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\cdot\frac{n+1}{n}\right) = \ln(n+1). \end{aligned}$$
(b) Since the limit of the partial sums, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln(n+1)$, does not exist, the series diverges.

36. (a) Using $r = e^{\ln r}$ and $n = e^{\ln n}$ we have $r^{\ln n} = e^{(\ln r)(\ln n)} = n^{\ln r}$.
(b) By part (a) we have $r^{\ln n} = n^{\ln r} = 1/n^{-\ln r}$. Since the p-series $\sum 1/n^p$ converges if and only if $p > 1$, the series $\sum_{n=1}^{\infty} 1/n^{-\ln r}$ converges if and only if $-\ln r > 1$, which is equivalent to $\ln r < -1$ or $r < 1/e$. Thus $\sum_{n=1}^{\infty} r^{\ln n}$ converges if $0 < r < 1/e$ and diverges if $r \geq 1/e$.

37. (a) A common denominator is $k(k+1)$ so
$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}.$$
(b) Using the result of part (a), the partial sum can be written as
$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}.$$
All of the intermediate terms cancel out, leaving only the first and last terms. Thus $S_{10} = 1 - \frac{1}{11}$ and $S_n = 1 - \frac{1}{n+1}$.
(c) The limit of S_n as $n \to \infty$ is $\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$. Thus the series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converges to 1.

38. (a) The partial sum
$$S_3 = \ln\left(\frac{1\cdot 3}{2\cdot 2}\right) + \ln\left(\frac{2\cdot 4}{3\cdot 3}\right) + \ln\left(\frac{3\cdot 5}{4\cdot 4}\right).$$
Using the property $\ln(A) + \ln(B) = \ln(AB)$, we get
$$S_3 = \ln\left(\frac{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5}{2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4}\right).$$
The intermediate factors cancel out, leaving only $\ln\left(\frac{1\cdot 5}{2\cdot 4}\right)$, so $S_3 = \ln\left(\frac{5}{8}\right)$.
(b) For the partial sum S_n, similar steps yield
$$S_n = \ln\left(\frac{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdots n(n+2)}{2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdots(n+1)\cdot(n+1)}\right).$$
As before, most of the factors cancel, leaving $S_n = \ln\left(\frac{n+2}{2(n+1)}\right)$.
(c) The limit of $S_n = \ln\left(\frac{n+2}{2(n+1)}\right)$ as $n\to\infty$ is $\lim_{n\to\infty}\ln\left(\frac{n+2}{2(n+1)}\right) = \ln\left(\frac{1}{2}\right)$. Thus the series $\sum_{k=1}^{\infty}\ln\left(\frac{k(k+2)}{(k+1)^2}\right)$ converges to $\ln\left(\frac{1}{2}\right)$.

39. Let S_n be the n^{th} partial sum for $\sum a_n$ and let T_n be the n^{th} partial sum for $\sum b_n$. Then the n^{th} partial sums for $\sum(a_n + b_n)$, $\sum(a_n - b_n)$, and $\sum ka_n$ are $S_n + T_n$, $S_n - T_n$, and kS_n, respectively. To show that these series converge, we have to show that the limits of their partial sums exist. By the properties of limits,
$$\begin{aligned} \lim_{n\to\infty}(S_n + T_n) &= \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n \\ \lim_{n\to\infty}(S_n - T_n) &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} T_n \\ \lim_{n\to\infty} kS_n &= k\lim_{n\to\infty} S_n. \end{aligned}$$
This proves that the limits of the partial sums exist, so the series converge.

40. Let S_n be the n-th partial sum for $\sum a_n$ and let T_n be the n-th partial sum for $\sum b_n$. Suppose that $S_N = T_N + k$. Since $a_n = b_n$ for $n \geq N$, we have $S_n = T_n + k$ for $n \geq N$. Hence if S_n converges to a limit, so does T_n, and vice versa. Thus, $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

41. We have $a_n = S_n - S_{n-1}$. If $\sum a_n$ converges, then $S = \lim_{n\to\infty} S_n$ exists. Hence $\lim_{n\to\infty} S_{n-1}$ exists and is equal to S also. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

42. From Property 1 in Theorem 9.2, we know that if $\sum a_n$ converges, then so does $\sum ka_n$.

Now suppose that $\sum a_n$ diverges and $\sum ka_n$ converges for $k \neq 0$. Thus using Property 1 and replacing $\sum a_n$ by $\sum ka_n$, we know that the following series converges:

$$\sum \frac{1}{k}(ka_n) = \sum a_n.$$

Thus, we have arrived at a contradiction, which means our original assumption, that $\sum_{n=1}^{\infty} ka_n$ converged, must be wrong.

43. A typical partial sum of the series $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$, say S_5, shows what happens in the general case:

$$S_5 = (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + (a_5 - a_4) + (a_6 - a_5) = a_6 - a_1$$

as all of the intermediate terms cancel out. The same thing will happen in the general partial sum: $S_n = a_{n+1} - a_1$.

Now the series $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ converges if the sequence of partial sums S_n has a limit as $n \to \infty$. Since we're assuming that the original series $\sum_{n=1}^{\infty} a_n$ converges, we know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = 0$ by property 3 of Theorem 9.2. Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (a_{n+1} - a_1) = 0 - a_1 = -a_1.$$

Since the sequence of partial sums converges (to $-a_1$), the series $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ converges (also to $-a_1$).

44. If $a_n = 1$ for all n, then $\sum a_n$ diverges but $\sum(a_{n+1} - a_n) = \sum 0$ converges. If $a_n = n$ for all n, then $\sum a_n$ diverges, and $\sum a_{n+1} - a_n = \sum 1$ diverges

45. (a) Suppose $\displaystyle\int_c^{\infty} f(x)\,dx$ diverges. In Figure 9.4 we see that for each positive integer k

$$\int_N^{N+k+1} f(x)\,dx \leq f(N) + f(N+1) + \cdots + f(N+k).$$

Since $f(n) = a_n$ for all n, we have

$$\int_N^{N+k+1} f(x)\,dx \leq a_N + a_{N+1} + \cdots + a_{N+k}.$$

Since $f(x)$ is defined for all $x \geq c$, if $\int_c^{\infty} f(x)\,dx$ is divergent, then $\int_N^{\infty} f(x)\,dx$ is divergent. So as $k \to \infty$, the integral $\int_N^{N+k+1} f(x)\,dx$ diverges, so the partial sums of the series $\sum_{i=N}^{\infty} a_i$ diverge. Thus, the series $\sum_{i=1}^{\infty} a_i$ diverges.

More precisely, suppose the series converged. Then the partial sums would be bounded. (The partial sums would be less than the sum of the series, since all the terms in the series are positive.) But that would imply that the integral converged, by Theorem 9.1 on Convergence of Monotone Bounded Sequences. This contradicts the assumption that $\int_N^{\infty} f(x)\,dx$ is divergent.

(b) Suppose $\int_c^\infty f(x)\,dx$ converges. Let N an integer with $N \geq c$. Consider the series $\sum_{i=N+1}^\infty a_i$. The partial sums of this series are increasing because all the terms in the series are positive. We show the partial sums are bounded using the right-hand sum in Figure 9.5. We see that for each positive integer k

$$f(N+1)+f(N+2)+\cdots+f(N+k) \leq \int_N^{N+k} f(x)\,dx.$$

Since $f(n) = a_n$ for all n, and $c \leq N$, we have

$$a_{N+1}+a_{N+2}+\cdots+a_{N+k} \leq \int_c^{N+k} f(x)\,dx.$$

Since $f(x)$ is a positive function, $\int_c^{N+k} f(x)\,dx \leq \int_c^b f(x)\,dx$ for all $b \geq N+k$. Since f is positive and $\int_c^\infty f(x)\,dx$ is convergent, $\int_c^{N+k} f(x)\,dx < \int_c^\infty f(x)\,dx$, so we have

$$a_{N+1}+a_{N+2}+\cdots+a_{N+k} \leq \int_c^\infty f(x)\,dx \quad \text{for all } k.$$

Thus, the partial sums of the series $\sum_{i=N+1}^\infty a_i$ are bounded and increasing, so this series converges by Theorem 9.1.

Now use Theorem 9.2, property 2, to conclude that $\sum_{i=1}^\infty a_i$ converges.

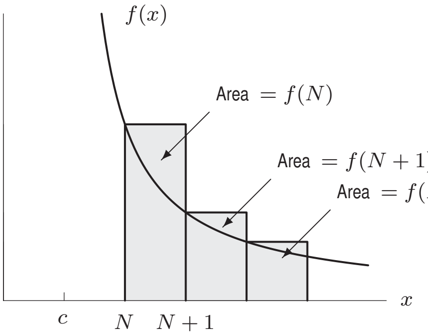


Figure 9.4

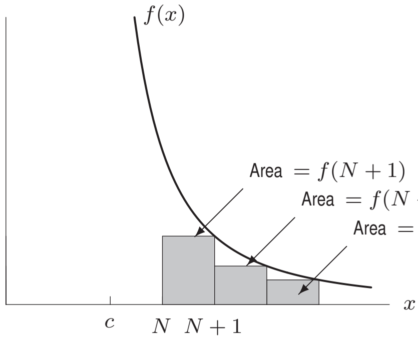


Figure 9.5

46. (a) Show that the sum of each group of fractions is more than $1/2$.
(b) Explain why this shows that the harmonic series does not converge.

(a) Notice that

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$
$$\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16} > \frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16} = \frac{8}{16} = \frac{1}{2}.$$

In the same way, we can see that the sum of the fractions in each grouping is greater than $1/2$.

(b) Since the sum of the first n groups is greater than $n/2$, it follows that the partial sums of the harmonic series are not bounded. Thus, the harmonic series diverges.

47. (a) Since for $x > 0$,

$$\int \frac{1}{x \ln x} dx = \ln(\ln x) + C$$

we have

$$\int_2^\infty \frac{1}{x \ln x} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x} dx = \lim_{b\to\infty} (\ln(\ln b) - \ln(\ln 2)) = \infty.$$

The series diverges by the integral test.

(b) The terms in each group are decreasing so we can bound each group as follows:

$$\frac{1}{3\ln 3} + \frac{1}{4 \ln 4} > \frac{1}{4\ln 4} + \frac{1}{4\ln 4} = \frac{1}{2\ln 4}$$

and

$$\frac{1}{5\ln 5} + \frac{1}{6\ln 6} + \frac{1}{7\ln 7} + \frac{1}{8\ln 8} > 4\frac{1}{8\ln 8} = \frac{1}{2\ln 8}.$$

Similarly, the group whose final term is $1/(2^n \ln(2^n))$ is greater than $1/(2\ln(2^n)) = 1/(2(\ln 2)n)$. Thus

$$\sum_{n=2}^{2^N} \frac{1}{n \ln n} > \sum_{n=1}^{N} \frac{1}{2(\ln 2)n}.$$

The series on the right is the harmonic series multiplied by the constant $1/(2\ln 2)$. Since the harmonic series diverges, $\sum 1/(n \ln n)$ diverges.

48. (a) See Figure 9.6. We draw the left-hand sum approximation to $\int_1^{n+1}(1/x)\,dx$ using $\Delta x = 1$. Since the rectangles lie everywhere above the curve, for every positive integer k, we conclude that

$$\int_k^{k+1} \frac{dx}{x} < \frac{1}{k}.$$

We see that

$$a_{n+1} - a_n = \frac{1}{n+1} - \ln(n+2) + \ln(n+1) = \frac{1}{n+1} - \int_{n+1}^{n+2} \frac{dx}{x},$$

so we see that $a_n < a_{n+1}$ for every positive integer n.

(b) For every positive integer n, the value of a_n represents the difference between the left-hand approximation of $\int_1^{n+1}(1/x)\,dx$ and the integral itself.

This difference can be viewed as a sum of areas of n roughly triangular pieces, with the k^{th} piece having a vertical side running from $y = 1/k$ to $y = 1/(k+1)$. Each piece has width 1. All the pieces can be moved to the left and stacked up to fit inside the first rectangle of the left-hand approximation, which runs from $x = 1$ to $x = 2$ and from $y = 0$ to $y = 1$. Thus the sum of all these stacked areas is less than 1, so $a_n < 1$ for all n. See Figure 9.7.

(c) Since a_n is an increasing sequence bounded above by 1, Theorem 9.1 ensures that $\lim_{n\to\infty} a_n$ exists.

(d) The sequence converges slowly, but a calculator or computer gives $a_{200} = 0.5747$. For comparison, $a_{100} = 0.5723$, $a_{500} = 0.5762$. Thus, $\gamma \approx 0.58$. More extensive calculations show that $\gamma \approx 0.577216$.

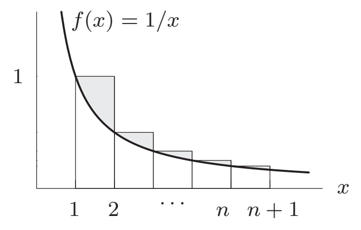


Figure 9.6

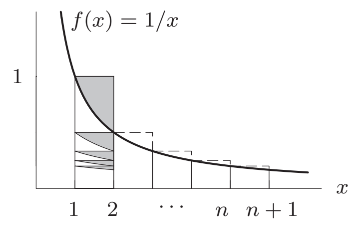


Figure 9.7

49. (a) A calculator or computer gives

$$\sum_{1}^{20} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{20^2} = 1.596.$$

(b) Since $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, the answer to part (a) gives

$$\frac{\pi^2}{6} \approx 1.596$$
$$\pi \approx \sqrt{6 \cdot 1.596} = 3.09$$

(c) A calculator or computer gives

$$\sum_{1}^{100} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{100^2} = 1.635,$$

so

$$\frac{\pi^2}{6} \approx 1.635$$
$$\pi \approx \sqrt{6 \cdot 1.635} = 3.13.$$

(d) The error in approximating $\pi^2/6$ by $\sum_1^{20} 1/n^2$ is the tail of the series $\sum_{21}^{\infty} 1/n^2$. From Figure 9.8, we see that

$$\sum_{21}^{\infty} \frac{1}{n^2} < \int_{20}^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{20}^{\infty} = \frac{1}{20} = 0.05.$$

A similar argument leads to a bound for the error in approximating $\pi^2/6$ by $\sum_1^{100} 1/n^2$ as

$$\sum_{101}^{\infty} \frac{1}{n^2} < \int_{100}^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{100}^{\infty} = \frac{1}{100} = 0.01.$$

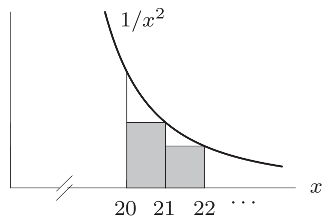


Figure 9.8

50. (a) We have $e > 1 + 1 + 1/2 + 1/6 + 1/24 = 65/24 = 2.708$.

(b) We have

$$\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} \leq \frac{1}{1 \cdot 2 \cdot 2 \cdot 2 \cdots 2} = \frac{1}{2^{n-1}}.$$

(c) The inequality in part (b) can be used to replace the given series with a geometric series that we can sum.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} < 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{1 - 1/2} = 3.$$

51. (a) The right-hand sum for $\int_0^N x^N dx$ with $\Delta x = 1$ is the sum $1^5 \cdot 1 + 2^5 \cdot 1 + 3^5 \cdot 1 + \cdots + N^5 \cdot 1 = S_N$. This sum is greater than the integral because the integrand x^5 is increasing on the interval $0 < x < N$. Since $\int_0^N x^5 dx = N^6/6$, we have $S_N > N^6/6$.

(b) The left-hand sum for $\int_1^{N+1} x^N dx$ with $\Delta x = 1$ is the sum $1^5 \cdot 1 + 2^5 \cdot 1 + 3^5 \cdot 1 + \cdots + N^5 \cdot 1 = S_N$. This sum is less than the integral because the integrand x^5 is increasing on the interval $1 < x < N+1$. Since $\int_1^{N+1} x^5 dx = ((N+1)^6 - 1)/6$, we have $S_N < ((N+1)^6 - 1)/6$.

(c) By parts (a) and (b) we have

$$\frac{N^6/6}{N^6/6} = 1 < \frac{S_N}{N^6/6} < \frac{((N+1)^6 - 1)/6}{N^6/6} = (1 + \frac{1}{N})^6 - \frac{1}{N^6}.$$

Since both $\lim_{N\to\infty} 1 = 1$ and $\lim_{N\to\infty}((1+\frac{1}{N})^6 - \frac{1}{N^6}) = 1$, we conclude that the limit in the middle also equals 1, $\lim_{N\to\infty} S_N/(N^6/6) = 1$.

52. (a) Neglecting signs, the table reveals a regular pattern reminiscent of a linear function with slope 4. We see that term n is given by $4n - 3$:

Term	1	2	3	4	n
Value	1	5	9	13	$4n-3$

The odd-numbered terms are positive and the even-numbered terms are negative, so

$$c_n = \begin{cases} 4n - 3 & \text{for } n \text{ odd} \\ -1 \cdot (4n-3) & \text{for } n \text{ even.} \end{cases}$$

Notice that $-(-1)^n$ is negative if n is even, and positive if n is odd. Thus we can write $c_n = -(-1)^n(4n-3)$. Since $-(-1)^n = (-1)^1(-1)^n = (-1)^{n+1}$, we can also write this $c_n = (-1)^{n+1}(4n-3)$.

(b) Using double factorial (!!) notation, we can write: $b_2 = \left(\frac{1!!}{2!!}\right)^3, b_3 = \left(\frac{3!!}{4!!}\right)^3, b_4 = \left(\frac{5!!}{6!!}\right)^3, b_5 = \left(\frac{7!!}{8!!}\right)^3, \ldots$. Focusing on the patterns in the table of $1, 3, 5, 7$ in the numerator and $2, 4, 6, 8$ in the denominator, we see that both are reminiscent of linear functions with slope 2:

Term	2	3	4	5	n
Numerator	1	3	5	7	$2n-3$
Denominator	2	4	6	8	$2n-2$

This means we can write $b_n = \left(\frac{(2n-3)!!}{(2n-2)!!}\right)^3$.

(c) Putting together our answers for parts (a) and (b), we have $a_n = -(-1)^n(4n-3)\left(\frac{(2n-3)!!}{(2n-2)!!}\right)^3$, or (equivalently) $a_n = (-1)^{n+1}(4n-3)\left(\frac{(2n-3)!!}{(2n-2)!!}\right)^3$. Checking our answer, we have:

$$\begin{aligned}
a_1 &= -(-1)^1(4\cdot 1 - 3)\left(\frac{(2\cdot 1-3)!!}{(2\cdot 1-2)!!}\right)^3 = 1\left(\frac{(-1)!!}{0!!}\right)^3 = 1 && \text{since } (-1)!! = 0!! = 1 \\
a_2 &= -(-1)^2(4\cdot 2 - 3)\left(\frac{(2\cdot 2-3)!!}{(2\cdot 2-2)!!}\right)^3 = -5\left(\frac{1!!}{2!!}\right)^2 = -5\left(\frac{1}{2}\right)^3 && \text{since } 1!! = 1 \text{ and } 2!! = 2 \\
a_3 &= -(-1)^3(4\cdot 3 - 3)\left(\frac{(2\cdot 3-3)!!}{(2\cdot 3-2)!!}\right)^3 = 9\left(\frac{3!!}{4!!}\right)^2 = 9\left(\frac{1\times 3}{2\times 4}\right)^3 \\
s_4 &= -(-1)^4(4\cdot 4 - 3)\left(\frac{(2\cdot 4-3)!!}{(2\cdot 4-2)!!}\right)^3 = -13\left(\frac{5!!}{6!!}\right)^2 = -13\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^3,
\end{aligned}$$

and so on, as required.

Strengthen Your Understanding

53. If the terms of a series do not approach zero, the series does not converge. But just because the terms approach zero does not mean the series converges. For example, $\sum(1/n)$ diverges even though the terms approach zero.

54. The integral $\int_1^\infty 1/x^3\,dx$ converges to $\frac{1}{2}$, because

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^3} = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

However, the series $\sum_{n=1}^\infty 1/n^3 = 1 + 1/2^3 + 1/3^3 + \cdots$ has a sum which is larger than 1 as its first term is 1 and all the subsequent terms are positive. Thus, the sum of the series is not $\frac{1}{2}$.

In general, the sum of a series and the value of an improper integral used to test it for convergence are not the same. Both converge or both diverge, but if they converge, usually they converge to different values.

55. The series $\sum_{n=1}^\infty 1/n$ is an example. The terms of the series converge to zero, but the series is a p-series with $p \le 1$, and therefore the series diverges.

56. One example is the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$. This series is convergent because it is a p-series with $p = 2 > 1$.

If $a_n = 1/n^2$, then $\sqrt{a_n} = 1/n$, and the series $\sum_{n=1}^\infty 1/n$ diverges because it is a p-series with $p = 1$.

57. True. Writing out the terms of this series, we have

$$\begin{aligned}&(1+(-1)^1)+(1+(-1)^2)+(1+(-1)^3)+(1+(-1)^4)+\cdots\\ &=(1-1)+(1+1)+(1-1)+(1+1)+\cdots\\ &=0+2+0+2+\cdots.\end{aligned}$$

58. True. The definition of convergence of a series is that its partial sums are a convergent sequence.

59. False. For example, if $a_n = 1/n$ and $b_n = -1/n$, then $|a_n + b_n| = 0$, so $\sum |a_n + b_n|$ converges. However $\sum |a_n|$ and $\sum |b_n|$ are the harmonic series, which diverge.

60. False. The terms in the series do not go to zero:

$$\begin{aligned}2^{(-1)^1}+2^{(-1)^2}+2^{(-1)^3}+2^{(-1)^4}+2^{(-1)^5}+\cdots &= 2^{-1}+2^1+2^{-1}+2^1+2^{-1}+\cdots\\ &= 1/2+2+1/2+2+1/2+\cdots.\end{aligned}$$

61. True. If the terms do not tend to zero, the partial sums do not tend to a limit. For example, if the terms are all greater than 0.1, the partial sums will grow without bound.

62. False. Consider the series $\displaystyle\sum_{n=1}^\infty 1/n$. This series does not converge, but $1/n \to 0$ as $n \to \infty$.

63. False. If $a_n = b_n = 1/n$, then $\sum a_n$ and $\sum b_n$ do not converge. However, $a_n b_n = 1/n^2$, so $\sum a_n b_n$ does converge.

64. False. If $a_n b_n = 1/n^2$ and $a_n = b_n = 1/n$, then $\sum a_n b_n$ converges, but $\sum a_n$ and $\sum b_n$ do not converge.

65. (d)

Solutions for Section 9.4

Exercises

1. Let $a_n = 1/(n-3)$, for $n \ge 4$. Since $n - 3 < n$, we have $1/(n-3) > 1/n$, so

$$a_n > \frac{1}{n}.$$

The harmonic series $\displaystyle\sum_{n=4}^\infty \frac{1}{n}$ diverges, so the comparison test tells us that the series $\displaystyle\sum_{n=4}^\infty \frac{1}{n-3}$ also diverges.

2. Let $a_n = 1/(n^2+2)$. Since $n^2+2 > n^2$, we have $1/(n^2+2) < 1/n^2$, so

$$0 < a_n < \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ also converges.

3. Let $a_n = e^{-n}/n^2$. Since $e^{-n} < 1$, for $n \geq 1$,we have $\frac{e^{-n}}{n^2} < \frac{1}{n^2}$, so

$$0 < a_n < \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ also converges.

4. As n gets large, polynomials behave like the leading term, so for large n,

$$\frac{n^3+1}{n^4+2n^3+2n} \quad \text{behaves like} \quad \frac{n^3}{n^4} = \frac{1}{n}.$$

Since the series $\sum_{n=1}^{\infty} 1/n$ diverges, we predict that the given series will diverge.

5. As n gets large, polynomials behave like the leading term, so for large n,

$$\frac{n+4}{n^3+5n-3} \quad \text{behaves like} \quad \frac{n}{n^3} = \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, we predict that the given series will converge.

6. As n gets large, polynomials behave like the leading term, so for large n,

$$\frac{1}{n^4+3n^3+7} \quad \text{behaves like} \quad \frac{1}{n^4}.$$

Since the series $\sum_{n=1}^{\infty} 1/n^4$ converges, we predict that the given series will converge.

7. As n gets large, polynomials behave like the leading term, so for large n,

$$\frac{n-4}{\sqrt{n^3+n^2+8}} \quad \text{behaves like} \quad \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}.$$

Since the series $\sum_{n=1}^{\infty} 1/n^{1/2}$ diverges, we predict that the given series will diverge.

8. Let $a_n = 1/(3^n+1)$. Since $3^n+1 > 3^n$, we have $1/(3^n+1) < 1/3^n = \left(\frac{1}{3}\right)^n$, so

$$0 < a_n < \left(\frac{1}{3}\right)^n.$$

Thus we can compare the series $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$. This geometric series converges since $|1/3| < 1$, so the comparison test tells us that $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ also converges.

9. Let $a_n = 1/(n^4+e^n)$. Since $n^4+e^n > n^4$, we have

$$\frac{1}{n^4+e^n} < \frac{1}{n^4},$$

so

$$0 < a_n < \frac{1}{n^4}.$$

Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$ also converges.

10. Since $\ln n \le n$ for $n \ge 2$, we have $1/\ln n \ge 1/n$, so the series diverges by comparison with the harmonic series, $\sum 1/n$.

11. Let $a_n = n^2/(n^4+1)$. Since $n^4+1 > n^4$, we have $\dfrac{1}{n^4+1} < \dfrac{1}{n^4}$, so

$$a_n = \frac{n^2}{n^4+1} < \frac{n^2}{n^4} = \frac{1}{n^2},$$

therefore

$$0 < a_n < \frac{1}{n^2}.$$

Since the p-series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, the comparison test tells us that the series $\sum_{n=1}^{\infty} \dfrac{n^2}{n^4+1}$ converges also.

12. We know that $|\sin n| < 1$, so

$$\frac{n\sin^2 n}{n^3+1} \le \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since the p-series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, comparison gives that $\sum_{n=1}^{\infty} \dfrac{n\sin^2 n}{n^3+1}$ converges.

13. Let $a_n = (2^n+1)/(n2^n-1)$. Since $n2^n - 1 < n2^n + n = n(2^n+1)$, we have

$$\frac{2^n+1}{n2^n-1} > \frac{2^n+1}{n(2^n+1)} = \frac{1}{n}.$$

Therefore, we can compare the series $\sum_{n=1}^{\infty} \dfrac{2^n+1}{n2^n-1}$ with the divergent harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$. The comparison test tells us that $\sum_{n=1}^{\infty} \dfrac{2^n+1}{n2^n-1}$ also diverges.

14. Since $a_n = \dfrac{n}{2^n}$, replacing n by $n+1$ gives $a_{n+1} = \dfrac{n+1}{2^{n+1}}$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$ converges.

15. Since $a_n = 1/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 1/(2n+2)!$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \dfrac{1}{(2n)!}$ converges.

16. Since $a_n = (n!)^2/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = ((n+1)!)^2/(2n+2)!$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}.$$

However, since $(n+1)! = (n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2(n!)^2(2n)!}{(2n+2)(2n+1)(2n)!(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{4n+2},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

17. Since $a_n = n!(n+1)!/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = (n+1)!(n+2)!/(2n+2)!$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+1)!(n+2)!}{(2n+2)!}}{\frac{n!(n+1)!}{(2n)!}} = \frac{(n+1)!(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{n!(n+1)!}.$$

However, since $(n+2)! = (n+2)(n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)(n+1)}{(2n+2)(2n+1)} = \frac{n+2}{2(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ converges.

18. Since $a_n = 1/(r^n n!)$, replacing n by $n+1$ gives $a_{n+1} = 1/(r^{n+1}(n+1)!)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{r^{n+1}(n+1)!}}{\frac{1}{r^n n!}} = \frac{r^n n!}{r^{n+1}(n+1)!} = \frac{1}{r(n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{r} \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{r^n n!}$ converges for all $r > 0$.

19. Since $a_n = 1/(ne^n)$, replacing n by $n+1$ gives $a_{n+1} = 1/(n+1)e^{n+1}$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)e^{n+1}}}{\frac{1}{ne^n}} = \frac{ne^n}{(n+1)e^{n+1}} = \left(\frac{n}{n+1}\right)\frac{1}{e}.$$

Therefore

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{e} < 1.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{ne^n}$ converges.

20. Since $a_n = 2^n/(n^3+1)$, replacing n by $n+1$ gives $a_{n+1} = 2^{n+1}/((n+1)^3+1)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{(n+1)^3+1}}{\frac{2^n}{n^3+1}} = \frac{2^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{2^n} = 2\frac{n^3+1}{(n+1)^3+1},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2.$$

Since $L > 1$ the ratio test tells us that the series $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$ diverges.

21. Even though the first term is negative, the terms alternate in sign, so it is an alternating series.

22. Since $\cos(n\pi) = (-1)^n$, this is an alternating series.

23. Since $(-1)^n \cos(n\pi) = (-1)^{2n} = 1$, this is not an alternating series.

24. Since $a_n = \cos n$ is not always positive, this is not an alternating series.

25. Let $a_n = 1/\sqrt{n}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{n+1}$. Since $\sqrt{n+1} > \sqrt{n}$, we have $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

26. Let $a_n = 1/(2n+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(2n+3)$. Since $2n+3 > 2n+1$, we have

$$0 < a_{n+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ converges.

27. Let $a_n = 1/(n^2+2n+1) = 1/(n+1)^2$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(n+2)^2$. Since $n+2 > n+1$, we have

$$\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2}$$

so

$$0 < a_{n+1} < a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$ converges.

28. Let $a_n = 1/e^n$. Then replacing n by $n+1$ we have $a_{n+1} = 1/e^{n+1}$. Since $e^{n+1} > e^n$, we have $\frac{1}{e^{n+1}} < \frac{1}{e^n}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ converges by the alternating series test. We can also observe that the series is geometric with ratio $x = -1/e$ can hence converges since $|x| < 1$.

29. Both $\sum \frac{(-1)^n}{2^n} = \sum \left(\frac{-1}{2}\right)^n$ and $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ are convergent geometric series. Thus $\sum \frac{(-1)^n}{2^n}$ is absolutely convergent.

30. The series $\sum \frac{(-1)^n}{2n}$ converges by the alternating series test. However $\sum \frac{1}{2n}$ diverges because it is a multiple of the harmonic series. Thus $\sum \frac{(-1)^n}{2n}$ is conditionally convergent.

31. Since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

the series $\sum (-1)^n \frac{n}{n+1}$ does not converge. It is a divergent series.

32. The series $\sum \frac{(-1)^n}{n^4+7}$ converges by the alternating series test. Moreover, the series $\sum \frac{1}{n^4+7}$ converges by comparison with the convergent p-series $\sum \frac{1}{n^4}$. Thus $\sum \frac{(-1)^n}{n^4+7}$ is absolutely convergent.

33. We first check absolute convergence by deciding whether $\sum 1/(n \ln n)$ converges by using the integral test. Since

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b\to\infty} \ln(\ln(x))\Big|_2^b = \lim_{b\to\infty} (\ln(\ln(b)) - \ln(\ln(2))),$$

and since this limit does not exist, $\sum \frac{1}{n \ln n}$ diverges.

We now check conditional convergence. The original series is alternating so we check whether $a_{n+1} < a_n$. Consider $a_n = f(n)$, where $f(x) = 1/(x \ln x)$. Since

$$\frac{d}{dx}\left(\frac{1}{x \ln x}\right) = \frac{-1}{x^2 \ln x}\left(1 + \frac{1}{\ln x}\right)$$

is negative for $x > 1$, we know that a_n is decreasing for $n \geq 2$. Thus, for $n \geq 2$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n} = a_n.$$

Since $1/(n \ln n) \to 0$ as $n \to \infty$, we see that $\sum \frac{(-1)^{n-1}}{n \ln n}$ is conditionally convergent.

34. Although $\cos n$ is sometimes positive and sometimes negative, the series is not alternating because is does not change sign every term. For example $\cos 1 > 0$, whereas $\cos 2$ and $\cos 3$ are negative. If we use the comparison test on

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|,$$

we find

$$\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} 1/n^2$ is a p-series with $p = 2$, and therefore converges. Thus the original series is absolutely convergent.

35. This is an alternating series that converges by the alternating series test. However

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a p-series with $p = 1/2$, so this series diverges. Thus the original series is conditionally convergent.

36. We first check absolute convergence by deciding whether $\sum \arcsin(1/n)$ converges. Since $\sin\theta \approx \theta$ for small angles θ, writing $x = \sin\theta$ we see that

$$\arcsin x \approx x.$$

Since $\arcsin x$ "behaves like" x for small x, we expect $\arcsin(1/n)$ to "behave like" $1/n$ for large n. To confirm this we calculate

$$\lim_{n\to\infty} \frac{\arcsin(1/n)}{1/n} = \lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1.$$

Thus $\sum \arcsin(1/n)$ diverges by limit comparison with the harmonic series $\sum 1/n$.

We now check conditional convergence. The original series is alternating, so we check whether $a_{n+1} < a_n$. Consider $a_n = f(n)$, where $f(x) = \arcsin(1/x)$. Since

$$\frac{d}{dx} \arcsin \frac{1}{x} = \frac{1}{\sqrt{1 - (1/x)^2}}\left(-\frac{1}{x^2}\right)$$

is negative for $x > 1$, we know that a_n is decreasing for $n > 1$. Therefore, for $n > 1$

$$a_{n+1} = \arcsin(1/(n+1)) < \arcsin(1/n) = a_n.$$

Since $\arcsin(1/n) \to 0$ as $n \to \infty$, we see that $\sum(-1)^{n-1}\arcsin(1/n)$ is conditionally convergent.

37. We first check absolute convergence by deciding whether $\sum \frac{\arctan(1/n)}{n^2}$ converges. Since $\arctan x$ is the angle between $-\pi/2$ and $\pi/2$, we have $\arctan(1/n) < \pi/2$ for all n. We compare

$$\frac{\arctan(1/n)}{n^2} < \frac{\pi/2}{n^2},$$

and conclude that since $(\pi/2)\sum 1/n^2$ converges, $\sum \frac{\arctan(1/n)}{n^2}$ converges. Thus $\sum \frac{(-1)^{n-1}\arctan(1/n)}{n^2}$ is absolutely convergent.

38. We have

$$\frac{a_n}{b_n} = \frac{(5n+1)/(3n^2)}{1/n} = \frac{5n+1}{3n},$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n+1}{3n} = \frac{5}{3} = c \neq 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series, the original series diverges.

39. We have

$$\frac{a_n}{b_n} = \frac{((1+n)/(3n))^n}{(1/3)^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n,$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = c \neq 0.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series, the original series converges.

40. The n^{th} term is $a_n = 1 - \cos(1/n)$ and we are taking $b_n = 1/n^2$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2}.$$

This limit is of the indeterminate form $0/0$ so we evaluate it using l'Hopital's rule. We have

$$\lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2} = \lim_{n\to\infty} \frac{\sin(1/n)(-1/n^2)}{-2/n^3} = \lim_{n\to\infty} \frac{1}{2}\frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{1}{2}\frac{\sin x}{x} = \frac{1}{2}.$$

The limit comparison test applies with $c = 1/2$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore $\sum(1-\cos(1/n))$ also converges.

41. The n^{th} term $a_n = 1/(n^4 - 7)$ behaves like $1/n^4$ for large n, so we take $b_n = 1/n^4$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^4-7)}{1/n^4} = \lim_{n\to\infty} \frac{n^4}{n^4-7} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^4$ converges because $p = 4 > 1$. Therefore $\sum 1/(n^4-7)$ also converges.

42. The n^{th} term $a_n = (n+1)/(n^2+2)$ behaves like $n/n^2 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+1)/(n^2+2)}{1/n} = \lim_{n\to\infty} \frac{n^2+n}{n^2+2} = 1.$$

The limit comparison test applies with $c = 1$. Since the harmonic series $\sum 1/n$ diverges, the series $\sum(n+1)/(n^2+2)$ also diverges.

43. The n^{th} term $a_n = (n^3 - 2n^2 + n + 1)/(n^4 - 2)$ behaves like $n^3/n^4 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3-2n^2+n+1)/(n^4-2)}{1/n} = \lim_{n\to\infty} \frac{n^4-2n^3+n^2+n}{n^4-2} = 1.$$

The limit comparison test applies with $c = 1$. The harmonic series $\sum 1/n$ diverges. Thus $\sum \left(n^3 - 2n^2 + n + 1\right) / \left(n^4 - 2\right)$ also diverges.

44. The n^{th} term $a_n = 2^n/(3^n - 1)$ behaves like $2^n/3^n$ for large n, so we take $b_n = 2^n/3^n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n/(3^n-1)}{2^n/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-1} = \lim_{n\to\infty} \frac{1}{1-3^{-n}} = 1.$$

The limit comparison test applies with $c = 1$. The geometric series $\sum 2^n/3^n = \sum(2/3)^n$ converges. Therefore $\sum 2^n/(3^n-1)$ also converges.

45. The n^{th} term $a_n = 1/(2\sqrt{n}+\sqrt{n+2})$ behaves like $1/(3\sqrt{n})$ for large n, so we take $b_n = 1/(3\sqrt{n})$. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(2\sqrt{n}+\sqrt{n+2})}{1/(3\sqrt{n})} &= \lim_{n\to\infty}\frac{3\sqrt{n}}{2\sqrt{n}+\sqrt{n+2}}\\ &= \lim_{n\to\infty}\frac{3\sqrt{n}}{\sqrt{n}\left(2+\sqrt{1+2/n}\right)}\\ &= \lim_{n\to\infty}\frac{3}{2+\sqrt{1+2/n}} = \frac{3}{2+\sqrt{1+0}}\\ &= 1.\end{aligned}$$

The limit comparison test applies with $c = 1$. The series $\sum 1/(3\sqrt{n})$ diverges because it is a multiple of a p-series with $p = 1/2 < 1$. Therefore $\sum 1/(2\sqrt{n}+\sqrt{n+2})$ also diverges.

46. The n^{th} term,

$$a_n = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{4n^2-2n},$$

behaves like $1/(4n^2)$ for large n, so we take $b_n = 1/(4n^2)$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(4n^2-2n)}{1/(4n^2)} = \lim_{n\to\infty}\frac{4n^2}{4n^2-2n} = \lim_{n\to\infty}\frac{1}{1-1/(2n)} = 1.$$

The limit comparison test applies with $c = 1$. The series $\sum 1/(4n^2)$ converges because it is a multiple of a p-series with $p = 2 > 1$. Therefore $\sum\left(\frac{1}{2n-1} - \frac{1}{2n}\right)$ also converges.

47. The n^{th} term,

$$a_n = \frac{n}{\cos n + e^n},$$

behaves like n/e^n for large n, so we take $b_n = n/e^n$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n/\left(\cos n + e^n\right)}{n/e^n} = \lim_{n\to\infty}\frac{e^n}{\cos n + e^n} = \lim_{n\to\infty}\frac{1}{\cos n/e^n + 1} = \frac{1}{0+1} = 1.$$

The limit comparison test applies with $c = 1$. The series $\sum n/e^n$ converges by the ratio test because

$$\lim_{n\to\infty}\frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty}\frac{(n+1)/e^{n+1}}{n/e^n} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{e} = \frac{1}{e},$$

which is less than 1. Therefore $\sum\left(\dfrac{n}{\cos n + e^n}\right)$ also converges.

48. The n^{th} term,

$$a_n = \frac{4\sin n + n}{n^2},$$

behaves like $1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(4\sin n + n\right)/n^2}{1/n} = \lim_{n\to\infty}\frac{4\sin n + n}{n} = \lim_{n\to\infty}\frac{4\sin n/n + 1}{1} = \frac{0+1}{1} = 1.$$

The limit comparison test applies with $c = 1$. The series $\sum 1/n$ diverges because it is a p-series with $p \le 1$. Therefore $\sum\left(\dfrac{4\sin n + n}{n^2}\right)$ also diverges.

Problems

49. The comparison test requires that $a_n = (-1)^n/n^2$ be positive. It is not.

50. The comparison test requires that $a_n = \sin n$ be positive for all n. It is not.

51. With $a_n = (-1)^n$, we have $|a_{n+1}/a_n| = 1$, and $\lim_{n\to\infty}|a_{n+1}/a_n| = 1$, so the test gives no information.

52. With $a_n = \sin n$, we have $|a_{n+1}/a_n| = |\sin(n+1)/\sin n|$, which does not have a limit as $n \to \infty$, so the test does not apply.

53. The sequence $a_n = n$ does not satisfy either $a_{n+1} < a_n$ or $\lim_{n\to\infty} a_n = 0$.

54. The alternating series test requires $a_n = \sin n$ be positive, which it is not. This is not an alternating series.

55. The alternating series test requires $a_n = 2 - 1/n$ which is positive and satisfies $a_{n+1} < a_n$ but $\lim_{n\to\infty} a_n = 2$. Since $\lim_{n\to\infty} a_n \neq 0$, we cannot use the alternating series test.

56. We cannot use the alternating series test in this case because the absolute values of the terms are not decreasing; that is, we do not have a series of the form $\sum(-1)^{n-1}a_n$ where $a_{n+1} < a_n$ for every n. For example, if we look at the fourth and fifth terms of the series, we notice that $\frac{1}{2} < \frac{2}{3}$.

In fact if the terms of this series are combined in pairs, we have the harmonic series, which is divergent:

$$(2-1)/1 + (2-1)/2 + (2-1)/3 + \cdots = 1 + 1/2 + 1/3 + \cdots.$$

57. The partial sums are $S_1 = 1$, $S_2 = -1$, $S_3 = 2$, $S_{10} = -5$, $S_{11} = 6$, $S_{100} = -50$, $S_{101} = 51$, $S_{1000} = -500$, $S_{1001} = 501$, which appear to be oscillating further and further from 0. This series does not converge.

58. The partial sums look like: $S_1 = 1$, $S_2 = 0.9$, $S_3 = 0.91$, $S_4 = 0.909$, $S_5 = 0.9091$, $S_6 = 0.90909$. The series appears to be converging to $0.909090\ldots$ or $10/11$.

Since $a_n = 10^{-k}$ is positive and decreasing and $\lim\limits_{n\to\infty} 10^{-n} = 0$, the alternating series test confirms the convergence of the series.

59. The partial sums look like: $S_1 = 1$, $S_2 = 0$, $S_3 = 0.5$, $S_4 = 0.3333$, $S_5 = 0.375$, $S_{10} = 0.3679$, $S_{20} = 0.3679$, and higher partial sums agree with these first 4 decimal places. The series appears to be converging to about 0.3679.

Since $a_n = 1/n!$ is positive and decreasing and $\lim_{n\to\infty} 1/n! = 0$, the alternating series test confirms the convergence of this series.

60. We use the ratio test with $a_n = \dfrac{8^n}{n!}$. Replacing n by $n+1$ gives $a_{n+1} = \dfrac{8^{n+1}}{(n+1)!}$ and

$$\frac{|a_{n+1}|}{|a_n|} = \frac{8^{n+1}/(n+1)!}{8^n/n!} = \frac{8n!}{(n+1)!} = \frac{8}{n+1}.$$

Thus

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{8}{n+1} = 0.$$

Since $L < 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{8^n}{n!}$ converges.

61. We use the ratio test with $a_n = \dfrac{n2^n}{3^n}$. Replacing n by $n+1$ gives $a_{n+1} = \dfrac{(n+1)2^{n+1}}{3^{n+1}}$ and

$$\frac{|a_{n+1}|}{|a_n|} = \frac{((n+1)2^{n+1})/3^{n+1}}{n2^n/3^n} = \frac{2(n+1)}{3n}.$$

Thus

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2(n+1)}{3n} = \lim_{n\to\infty} \frac{2(1+1/n)}{3} = \frac{2}{3}.$$

Since $L < 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{3^n}$ converges.

62. We use the ratio test and calculate

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(0.1)^{n+1}/(n+1)!}{(0.1)^n/n!} = \lim_{n\to\infty} \frac{0.1}{n+1} = 0.$$

Since the limit is less than 1, the series converges.

63. We use the ratio test and calculate

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!/(n+1)^2}{(n-1)!/n^2} = \lim_{n\to\infty} \left(\frac{n!}{(n-1)!} \cdot \frac{n^2}{(n+1)^2}\right) = \lim_{n\to\infty} \left(n \cdot \frac{n^2}{(n+1)^2}\right).$$

Since the limit does not exist (it is ∞), the series diverges.

64. The first few terms of the series may be written

$$1 + e^{-1} + e^{-2} + e^{-3} + \cdots;$$

this is a geometric series with $a = 1$ and $x = e^{-1} = 1/e$. Since $|x| < 1$, the geometric series converges to $S = \dfrac{1}{1-x} = \dfrac{1}{1-e^{-1}} = \dfrac{e}{e-1}$.

65. The first few terms of the series may be written

$$e + e^2 + e^3 + \cdots = e + e\cdot e + e\cdot e^2 + \cdots;$$

this is a geometric series with $a = e$ and $x = e$. Since $|x| > 1$, this geometric series diverges.

66. We use the ratio test with $a_n = \dfrac{(2n)!}{(n!)^2}$. Replacing n by $n+1$ gives $a_{n+1} = \dfrac{(2n+2)!}{((n+1)!)^2}$ and

$$\frac{|a_{n+1}|}{|a_n|} = \frac{((2n+2)!)/((n+1)!)^2}{(2n)!/(n!)^2} = \frac{(2n+2)!(n!)^2}{(2n)!((n+1)!)^2} = \frac{(2n+2)(2n+1)(2n)!(n!)^2}{(2n)!(n+1)^2(n!)^2} = \frac{2(2n+1)}{n+1}.$$

Thus

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2(2n+1)}{n+1} = \lim_{n\to\infty} \frac{2(2+1/n)}{1+1/n} = 4.$$

Since $L > 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges.

67. We compare the series with the convergent series $\sum 1/n^2$. From the graph of $\tan x$, we see that $\tan x < 2$ for $0 \le x \le 1$, so $\tan(1/n) < 2$ for all n. Thus

$$\frac{1}{n^2}\tan\left(\frac{1}{n}\right) < \frac{1}{n^2}2,$$

so the series converges, since $2\sum 1/n^2$ converges. Alternatively, we try the integral test. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the corresponding integral using the substitution $w = 1/x$:

$$\int_1^{\infty} \frac{1}{x^2}\tan\left(\frac{1}{x}\right)dx = \lim_{b\to\infty}\int_1^{b} \frac{1}{x^2}\tan\left(\frac{1}{x}\right)dx = \lim_{b\to\infty} \ln\left(\cos\frac{1}{x}\right)\Bigg|_1^b = \lim_{b\to\infty}\left(\ln\left(\cos\left(\frac{1}{b}\right)\right) - \ln(\cos 1)\right) = -\ln(\cos 1).$$

Since the limit exists, the integral converges, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\tan(1/n)$ converges.

68. We use the limit comparison test with $a_n = \dfrac{n+1}{n^3+6}$. Because a_n behaves like $\dfrac{n}{n^3} = \dfrac{1}{n^2}$ as $n \to \infty$, we take $b_n = 1/n^2$.

We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2(n+1)}{n^3+6} = 1.$$

By the limit comparison test (with $c = 1$) since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^3+6}$ also converges.

69. We use the limit comparison test with $a_n = \dfrac{5n+2}{2n^2+3n+7}$. Because a_n behaves like $\dfrac{5n}{2n^2} = \dfrac{5}{2n}$ as $n \to \infty$, we take $b_n = 1/n$.

We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n(5n+2)}{2n^2+3n+7} = \frac{5}{2}.$$

By the limit comparison test (with $c = 5/2$) since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty}\frac{5n+2}{2n^2+3n+7}$ also diverges.

70. Let $a_n = 1/\sqrt{3n-1}$. Then replacing n by $n+1$ gives $a_{n+1} = 1/\sqrt{3(n+1)-1}$. Since

$$\sqrt{3(n+1)-1} > \sqrt{3n-1},$$

we have

$$a_{n+1} < a_n.$$

In addition, $\lim_{n\to\infty} a_n = 0$ so the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{3n-1}}$ converges.

71. Since the exponential, 2^n, grows faster than the power, n^2, the terms are growing in size. Thus, $\lim_{n\to\infty} a_n \neq 0$. We conclude that this series diverges.

72. Since $0 \leq |\sin n| \leq 1$ for all n, we may be able to compare with $1/n^2$. We have $0 \leq |\sin n/n^2| \leq 1/n^2$ for all n. So $\sum |\sin n/n^2|$ converges by comparison with the convergent series $\sum(1/n^2)$. Therefore $\sum(\sin n/n^2)$ also converges, since absolute convergence implies convergence by Theorem 9.6.

73. We take $a_n = \left|\dfrac{\sin n^2}{n^2}\right|$. Since $|\sin n^2| \leq 1$ for all n, $\left|\dfrac{\sin n^2}{n^2}\right| \leq \dfrac{1}{n^2}$.

We take $b_n = \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By the comparison test, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n^2}{n^2}\right|$ also converges. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n^2}{n^2}$ converges by Theorem 9.6.

74. Note that $\cos(n\pi)/n = (-1)^n/n$, so this is an alternating series. Therefore, since $1/(n+1) < 1/n$ and $\lim_{n\to\infty} 1/n = 0$, we see that $\sum(\cos(n\pi)/n)$ converges by the alternating series test.

75. As $n \to \infty$, we see that

$$\frac{n+2}{n^2-1} \to \frac{n}{n^2} = \frac{1}{n}.$$

Since $\sum(1/n)$ diverges, we expect our series to have the same behavior.

More precisely, for all $n \geq 2$, we have

$$0 \leq \frac{1}{n} = \frac{n}{n^2} \leq \frac{n+2}{n^2-1},$$

so $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$ diverges by comparison with the divergent series $\displaystyle\sum \frac{1}{n}$.

76. Since

$$\frac{3}{\ln n^2} = \frac{3}{2\ln n},$$

our series behaves like the series $\sum 1/\ln n$. More precisely, for all $n \geq 2$, we have

$$0 \leq \frac{1}{n} \leq \frac{1}{\ln n} \leq \frac{3}{2\ln n} = \frac{3}{\ln n^2},$$

so $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\ln n^2}$ diverges by comparison with the divergent series $\displaystyle\sum \frac{1}{n}$.

77. Let $a_n = 1/\sqrt{n^2(n+2)}$. Since $n^2(n+2) = n^3 + 2n^2 > n^3$, we have

$$0 < a_n < \frac{1}{n^{3/2}}.$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, the comparison test tells us that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2(n+2)}}$$

also converges.

78. Let $a_n = n(n+1)/\sqrt{n^3+2n^2}$. Since $n^3 + 2n^2 = n^2(n+2)$, we have

$$a_n = \frac{n(n+1)}{n\sqrt{n+2}} = \frac{n+1}{\sqrt{n+2}}$$

so a_n grows without bound as $n \to \infty$, therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$ diverges.

79. Factoring gives $\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^a = 2^a\sum_{n=1}^{\infty}\frac{1}{n^a}$. This is a constant times a p-series that converges if $a > 1$ and diverges if $a \le 1$.

80. This is a geometric series for all $a > 0$, with ratio $2/a$. Therefore, the series converges when $2/a < 1$ and diverges when $2/a \ge 1$. Thus $\sum_{n=1}^{\infty}\left(\frac{2}{a}\right)^n$ converges for $a > 2$ and diverges for $a \le 2$.

81. This is a geometric series with ratio $\ln a$, so it converges when $|\ln a| < 1$ and diverges when $|\ln a| \ge 1$. We have $|\ln a| < 1$ if $1/e < a < e$ and $|\ln a| \ge 1$ if $a \ge$ or $a \le 1/e$. Thus, the series $\sum_{n=1}^{\infty}(\ln a)^n$ converges for $1/e < a < e$ and diverges for $a \ge e$ and $0 < a \le 1/e$.

82. For $a > 0$, the terms of the series are positive and eventually decreasing. We use the integral test and calculate the corresponding improper integral:

$$\int_1^\infty \frac{\ln x}{x^a}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^a}\,dx.$$

For $a \ne 1$, use integration by parts with $u = \ln x$ and $v' = x^{-a}$:

$$\int \frac{\ln x}{x^a}\,dx = \frac{x^{-a+1}}{-a+1}\ln x - \int \frac{x^{-a}}{-a+1}\,dx = \frac{x^{-a+1}}{-a+1}\ln x - \frac{x^{-a+1}}{(-a+1)^2} = \frac{x^{-a+1}}{-a+1}\left(\ln x - \frac{1}{-a+1}\right).$$

Thus,

$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^a}\,dx = \lim_{b\to\infty}\left[\frac{b^{-a+1}}{-a+1}\left(\ln b - \frac{1}{-a+1}\right)\right] + \frac{1}{(-a+1)^2} = \frac{1}{-a+1}\lim_{b\to\infty}\frac{\ln b - 1/(-a+1)}{b^{a-1}} + \frac{1}{(-a+1)^2}.$$

Use l'Hopital's Rule to obtain

$$\lim_{b\to\infty}\left(\frac{\ln b - 1/(-a+1)}{b^{a-1}}\right) = \lim_{b\to\infty}\left(\frac{1/b}{(a-1)b^{a-2}}\right) = \lim_{b\to\infty}\frac{1}{(a-1)b^{a-1}}.$$

This limit exists for $a - 1 > 0$ and does not exist for $a - 1 < 0$. Thus the series converges for $a > 1$ and diverges for $0 < a < 1$.

For $a = 1$, the series $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverges because $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left.\frac{(\ln x)^2}{2}\right|_1^b$, and this limit diverges. For $a \le 0$, $\lim_{n\to\infty}\frac{\ln n}{n^a}$ does not exist, so the series diverges by Property 3 of Theorem 9.2. Thus $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ converges for $a > 1$ and diverges for $a \le 1$.

83. To use the alternating series test, consider $a_n = f(n)$, where $f(x) = \arctan(a/x)$. We need to show that $f(x)$ is decreasing. Since

$$f'(x) = \frac{1}{1+(a/x)^2}\left(-\frac{a}{x^2}\right),$$

we have $f'(x) < 0$ for $a > 0$, so $f(x)$ is decreasing for all x. Thus $a_{n+1} < a_n$ for all n, and as $\lim_{n\to\infty}\arctan(a/n) = 0$ for all a, by the alternating series test,

$$\sum_{n=1}^{\infty}(-1)^n\arctan(a/n)$$

converges.

84. The n^{th} partial sum of the series is given by

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n},$$

so the absolute value of the first term omitted is $1/(n+1)$. By Theorem 9.9, we know that the value, S, of the sum differs from S_n by less than $1/(n+1)$. Thus, we want to choose n large enough so that $1/(n+1) \le 0.01$. Solving this inequality for n yields $n \ge 99$, so we take 99 or more terms in our partial sum.

85. The n^{th} partial sum of the series is given by

$$S_n = 1 - \frac{2}{3} + \frac{4}{9} - \cdots + (-1)^{(n-1)}\left(\frac{2}{3}\right)^{(n-1)},$$

so the absolute value of the first term omitted is $(2/3)^n$. By Theorem 9.9, we know that the value, S, of the sum differs from S_n by less than $(2/3)^n$. Thus, we want to choose n large enough so that $(2/3)^n \le 0.01$. Solving this inequality for n yields $n \ge 11.358$, so taking 12 or more terms in our partial sum is guaranteed to be within 0.01 of the sum of the series.

Note: Since this is a geometric series, we know the exact sum to be $1/(1+2/3) = 0.6$. The partial sum S_{12} is 0.595, which is indeed within 0.01 of the sum of the series. Note, however, that $S_{11} = 0.6069$, which is also within 0.01 of the exact sum of the series. Theorem 9.9 gives us a value of n for which S_n is guaranteed to be within a small tolerance of the sum of an alternating series, but not necessarily the *smallest* such value.

86. The n^{th} partial sum of the series is given by

$$S_n = \frac{1}{2} - \frac{1}{24} + \frac{1}{720} - \cdots + \frac{(-1)^{n-1}}{(2n)!},$$

so the absolute value of the first term omitted is $1/(2n+2)!$. By Theorem 9.9, we know that the value, S, of the sum differs from S_n by less than $1/(2n+2)!$. Thus, we want to choose n large enough so that $1/(2n+2)! \le 0.01$. Substituting $n = 2$ into the expression $1/(2n+2)!$ yields $1/720$ which is less than 0.01. We therefore take 2 or more terms in our partial sum.

87. Since $0 \le c_n \le 2^{-n}$ for all n, and since $\sum 2^{-n}$ is a convergent geometric series, $\sum c_n$ converges by the Comparison Test. Similarly, since $2^n \le a_n$, and since $\sum 2^n$ is a divergent geometric series, $\sum a_n$ diverges by the Comparison Test. We do not have enough information to determine whether or not $\sum b_n$ and $\sum d_n$ converge.

88. (a) The sum $\sum a_n \cdot b_n = \sum 1/n^5$, which converges, as a p-series with $p = 5$, or by the integral test:

$$\int_1^\infty \frac{1}{x^5}\,dx = \lim_{b\to\infty} \left.\frac{x^{-4}}{(-4)}\right|_1^b = \lim_{b\to\infty} \frac{b^{-4}}{(-4)} + \frac{1}{4} = \frac{1}{4}.$$

Since this improper integral converges, $\sum a_n \cdot b_n$ also converges.

(b) This is an alternating series that satisfies the conditions of the alternating series test: the terms are decreasing and have limit 0, so $\sum (-1)^n/\sqrt{n}$ converges.

(c) We have $a_n b_n = 1/n$, so $\sum a_n b_n$ is the harmonic series, which diverges.

89. Since $\lim_{n\to\infty} a_n/b_n = 0$, for large enough n we have $|a_n/b_n| < 1/2$ and thus $0 \le |a_n| < b_n/2 < b_n$. By the comparison test applied to $\sum |a_n|$ and $\sum b_n$, the series $\sum |a_n|$ converges. The series $\sum a_n$ converges absolutely and thus it converges.

90. Since $\lim_{n\to\infty} a_n/b_n = \infty$, for large enough n we have $a_n/b_n > 1$ and thus $a_n > b_n$. By the comparison test applied to $\sum a_n$ and $\sum b_n$, the series $\sum a_n$ diverges.

91. Each term in $\sum b_n$ is greater than or equal to a_1 times a term in the harmonic series:

$$\begin{aligned} b_1 &= a_1 \cdot 1 \\ b_2 &= \frac{a_1 + a_2}{2} > a_1 \cdot \frac{1}{2} \\ b_3 &= \frac{a_1 + a_2 + a_3}{3} > a_1 \cdot \frac{1}{3} \\ &\vdots \\ b_n &= \frac{a_1 + a_2 + \cdots + a_n}{n} > a_1 \cdot \frac{1}{n} \end{aligned}$$

Adding these inequalities gives

$$\sum b_n > a_1 \sum \frac{1}{n}.$$

Since the harmonic series $\sum 1/n$ diverges, a_1 times the harmonic series also diverges. Then, by the comparison test, the series $\sum b_n$ diverges.

92. Suppose we let $c_n = (-1)^n a_n$. (We have just given the terms of the series $\sum(-1)^n a_n$ a new name.) Then

$$|c_n| = |(-1)^n a_n| = |a_n|.$$

Thus $\sum |c_n|$ converges, and by Theorem 9.6,

$$\sum c_n = \sum(-1)^n a_n \quad \text{converges.}$$

93. (a) Since

$$|a_n| = a_n \quad \text{if } a_n \geq 0$$
$$|a_n| = -a_n \text{ if } a_n < 0,$$

we have

$$a_n + |a_n| = 2|a_n| \text{ if } a_n \geq 0$$
$$a_n + |a_n| = 0 \qquad \text{if } a_n < 0.$$

Thus, for all n,

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

(b) If $\sum |a_n|$ converges, then $\sum 2|a_n|$ is convergent, so, by comparison, $\sum (a_n + |a_n|)$ is convergent. Then

$$\sum ((a_n + |a_n|) - |a_n|) = \sum a_n$$

is convergent, as it is the difference of two convergent series.

94. The limit

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1,$$

so the series converges.

95. The limit

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{5n+1}{3n^2} = 0 < 1,$$

so the series converges.

Strengthen Your Understanding

96. The series is not alternating since $(-1)^{2n} = 1$ for all $n > 0$, so we cannot use the alternating series test. The series is the same as $\sum 1/n^2$ which converges by the p-test with $p = 2$.

97. We could show the series converges by comparison with the convergent series $\sum 1/n^2$. The ratio test gives

$$\lim_{n\to\infty} \frac{(1/((n+1)^2+1)}{1/(n^2+1)} = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = \lim_{n\to\infty} \frac{1+1/n^2}{((n+1)/n)^2+1/n^2} = \frac{1+0}{1+0} = 1.$$

The ratio test is inconclusive when the limit of the ratios is 1.

98. It is true that the comparison series $\sum 1/n^2$ converges, but in order to make the comparison, we need $1/n^{3/2} \leq 1/n^2$ for all n; we have instead $1/n^{3/2} > 1/n^2$ for all $n > 1$.

99. If we want the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ to behave differently, then we need a_n to be negative for some values of n. Thus we try an alternating series.

The series $\sum_{n=1}^{\infty}(-1)^n/n$ works. Because this series is of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ where $b_n > 0$ for all n and $b_{n+1} < b_n$ for all n and also $b_n \to 0$, the series converges by the alternating series test.

However, the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} 1/n$ diverges by the p-series test.

100. The following series is alternating:

$$\sum a_n = \sum(-1)^n n,$$

but since the terms do not tend to 0 as $n \to \infty$, the series does not converge.

101. The geometric series with $a_n = 3^n$ satisfies this condition.

102. False. The sequence $-1, 1, -1, 1, \ldots$ given by $s_n = (-1)^n$ alternates in sign but does not converge.

103. False. It does not tell us anything to know that b_n is larger than a convergent series. For example, if $a_n = 1/n^2$ and $b_n = 1$, then $0 \le a_n \le b_n$ and $\sum a_n$ converges, but $\sum b_n$ diverges. Since this statement is not true for all a_n and b_n, the statement is false.

104. True. This is one of the statements of the comparison test.

105. True. Consider the series $\sum(-b_n)$ and $\sum(-a_n)$. The series $\sum(-b_n)$ converges, since $\sum b_n$ converges, and

$$0 \le -a_n \le -b_n.$$

By the comparison test, $\sum(-a_n)$ converges, so $\sum a_n$ converges.

106. False. It is true that if $\sum |a_n|$ converges, then we know that $\sum a_n$ converges. However, knowing that $\sum a_n$ converges does *not* tell us that $\sum |a_n|$ converges.

For example, if $a_n = (-1)^{n-1}/n$, then $\sum a_n$ converges by the alternating series test. However, $\sum |a_n|$ is the harmonic series which diverges.

107. False. For example, if $a_n = 1/n^2$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

However, $\sum 1/n^2$ converges.

108. True, since if we write out the terms of the series, using the fact that $\cos(2\pi n) = 1$ for all n, we have

$$\begin{aligned} &(-1)^0\cos 0 + (-1)^1\cos(2\pi) + (-1)^2\cos(4\pi) + (-1)^3\cos(6\pi) + \cdots \\ &= 1\cdot 1 - 1\cdot 1 + 1\cdot 1 - 1\cdot 1 + \cdots \\ &= 1 - 1 + 1 - 1 + \cdots. \end{aligned}$$

This is an alternating series.

109. False. This is an alternating series, but since the terms do not go to zero, it does not converge.

110. False. For example, if $a_n = (-1)^{n-1}/n$, then $\sum a_n$ converges by the alternating series test. But $(-1)^n a_n = (-1)^n(-1)^{n-1}/n = (-1)^{2n-1}/n = -1/n$. Thus, $\sum(-1)^n a_n$ is the negative of the harmonic series and does not converge.

111. This is true. It is a restatement of Theorem 9.9.

112. This statement is false. The statement is true if the series converges by the alternating series test, but not in general. Consider, for example, the alternating series

$$S = 10 - 0.01 + 0.8 - 0.7 - 0 + 0 - 0 + \cdots.$$

Since the later terms are all 0, we can find the sum exactly:

$$S = 10.69.$$

If we approximated the sum by the first term, $S_1 = 10$, the magnitude of the first term omitted would be 0.01. Thus, if the statement in this problem were true, we would say that the true value of the sum lay between $10 + 0.01 = 10.01$ and $10 - 0.01 = 9.99$ which it does not.

113. True. Let $c_n = (-1)^n|a_n|$. Then $|c_n| = |a_n|$ so $\sum |c_n|$ converges, and therefore $\sum c_n = \sum(-1)^n|a_n|$ converges.

114. True. Since the series is alternating, Theorem 9.9 gives the error bound. Summing the first 100 terms gives S_{100}, and if the true sum is S,

$$|S - S_{100}| < a_{101} = \frac{1}{101} < 0.01.$$

115. True. If $\sum |a_n|$ is convergent, then so is $\sum a_n$.

116. False. The alternating harmonic series $\sum \frac{(-1)^n}{n}$ is conditionally convergent because it converges by the Alternating Series test, but the harmonic series $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is divergent. The alternating harmonic series is not absolutely convergent.

117. True. By the comparison test, if $\sum a_n$ is larger term-by-term than a divergent series, then $\sum a_n$ diverges. If $\sum b_n$ diverges, then so does $\sum 0.5b_n$.

118. (b). This series should be compared with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

Solutions for Section 9.5

Exercises

1. Yes.

2. No, because it contains negative powers of x.

3. No, each term is a power of a different quantity.

4. Yes. It's a polynomial, or a series with all coefficients beyond the 7th being zero.

5. The general term can be written as $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^n$ for $n \geq 1$. Other answers are possible.

6. The general term can be written as $\dfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!} x^n$ for $n \geq 1$. Other answers are possible.

7. The general term can be written as $\dfrac{(-1)^k (x-1)^{2k}}{(2k)!}$ for $k \geq 0$. Other answers are possible.

8. The general term can be written as $\dfrac{(-1)^{k+1}(x-1)^{2k+1}}{(2(k-1))!}$ for $k \geq 1$ or as $\dfrac{(-1)^k (x-1)^{2k+3}}{(2k)!}$ for $k \geq 0$. Other answers are possible.

9. The general term can be written as $\dfrac{(x-a)^n}{2^{n-1} \cdot n!}$ for $n \geq 1$. Other answers are possible.

10. The general term can be written as $\dfrac{(k+1)(x+5)^{2k+1}}{(k-1)!}$ for $k \geq 1$ or as $\dfrac{(k+2)(x+5)^{2k+3}}{k!}$ for $k \geq 0$. Other answers are possible.

11. Since $C_n = n$, replacing n by $n+1$ gives $C_{n+1} = n+1$. Using the ratio test with $a_n = nx^n$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{n+1}{n} = |x|.$$

Thus the radius of convergence is $R = 1$.

12. This series may be written as

$$1 + 5x + 25x^2 + \cdots$$

so $C_n = 5^n$. Using the ratio test, with $a_n = 5^n x^n$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{5^{n+1}}{5^n} = 5|x|.$$

Thus the radius of convergence is $R = 1/5$.

13. Since $C_n = n^3$, replacing n by $n+1$ gives $C_{n+1} = (n+1)^3$. Using the ratio test, with $a_n = n^3 x^n$, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x| \frac{|C_{n+1}|}{|C_n|} = |x| \frac{(n+1)^3}{n^3} = |x| \left(\frac{n+1}{n} \right)^3.$$

We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|.$$

Thus the radius of convergence is $R = 1$.

14. Let $C_n = 2^n + n^2$. Then replacing n by $n+1$ gives $C_{n+1} = 2^{n+1} + (n+1)^2$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x| \frac{|C_{n+1}|}{|C_n|} = |x| \frac{2^{n+1} + (n+1)^2}{2^n + n^2} = 2|x| \left(\frac{2^n + \frac{1}{2}(n+1)^2}{2^n + n^2} \right).$$

Since 2^n dominates n^2 as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2|x|.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

15. Since $C_n = (n+1)/(2^n+n)$, replacing n by $n+1$ gives $C_{n+1} = (n+2)/(2^{n+1}+n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(n+2)/(2^{n+1}+n+1)}{(n+1)/(2^n+n)} = |x|\frac{n+2}{2^{n+1}+n+1}\cdot\frac{2^n+n}{n+1} = |x|\frac{n+2}{n+1}\cdot\frac{2^n+n}{2^{n+1}+n+1}.$$

Since

$$\lim_{n\to\infty}\frac{n+2}{n+1} = 1$$

and

$$\lim_{n\to\infty}\left(\frac{2^n+n}{2^{n+1}+n+1}\right) = \frac{1}{2}\lim_{n\to\infty}\left(\frac{2^n+n}{2^n+(n+1)/2}\right) = \frac{1}{2},$$

because 2^n dominates n as $n\to\infty$, we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \frac{1}{2}|x|.$$

Thus the radius of convergence is $R = 2$.

16. Since $C_n = 2^n/n$, replacing n by $n+1$ gives $C_{n+1} = 2^{n+1}/(n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x-1|\frac{|C_{n+1}|}{|C_n|} = |x-1|\frac{2^{n+1}/(n+1)}{2^n/n} = |x-1|\frac{2^{n+1}}{(n+1)}\cdot\frac{n}{2^n} = 2|x-1|\left(\frac{n}{n+1}\right),$$

so

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x-1|.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

17. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-3)^n}\right| = |x-3|\frac{n}{2(n+1)}.$$

Therefore, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-3|\lim_{n\to\infty}\frac{n}{2(n+1} = \frac{|x-3|}{2}.$$

Thus by the ratio test, the series converges if $|x-3|/2 < 1$, that is $|x-3| < 2$. The radius of convergence is $R = 2$.

18. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right| = |x^2|\frac{1}{(2n+2)(2n+1)}.$$

Therefore we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0.$$

Thus, by the ratio test, the series converges for all x, so the radius of convergence is $R = \infty$.

19. The coefficient of the n^{th} term is $C_n = (-1)^{n+1}/n^2$. Now consider the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n^2x^{n+1}}{(n+1)^2x^n}\right| \to |x| \quad\text{as}\quad n\to\infty.$$

Thus, the radius of convergence is $R = 1$.

20. Here the coefficient of the n^{th} term is $C_n = (2^n/n!)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2^{n+1}/(n+1)!)x^{n+1}}{(2^n/n!)x^n}\right| = \frac{2|x|}{n+1} \to 0 \text{ as } n\to\infty.$$

Thus, the radius of convergence is $R = \infty$, and the series converges for all x.

21. Here $C_n = (2n)!/(n!)^2$. We have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1))!/((n+1)!)^2x^{n+1}}{(2n)!/(n!)^2x^n}\right| = \frac{(2(n+1))!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}|x|$$

$$= \frac{(2n+2)(2n+1)|x|}{(n+1)^2} \to 4|x| \text{ as } n\to\infty.$$

Thus, the radius of convergence is $R = 1/4$.

22. Here the coefficient of the n^{th} term is $C_n = (2n+1)/n$. Applying the ratio test, we consider:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((2n+3)/(n+1))x^{n+1}}{((2n+1)/n)x^n}\right| = |x|\frac{2n+3}{2n+1}\cdot\frac{n}{n+1} \to |x| \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = 1$.

23. We write the series as

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,$$

so

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}.$$

Replacing n by $n+1$, we have

$$a_{n+1} = (-1)^{n+1-1}\frac{x^{2(n+1)-1}}{2(n+1)-1} = (-1)^n\frac{x^{2n+1}}{2n+1}.$$

Thus

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(-1)^n x^{2n+1}}{2n+1}\right| \cdot \left|\frac{2n-1}{(-1)^{n-1}x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2,$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{2n-1}{2n+1}x^2 = x^2.$$

By the ratio test, this series converges if $L < 1$, that is, if $x^2 < 1$, so $R = 1$.

24. As seen in Example 7, we can write the general term of the series as

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!},$$

so that, replacing n by $n+1$, we have

$$a_{n+1} = (-1)^{(n+1)-1}\frac{x^{2(n+1)-1}}{(2(n+1)-1)!} = (-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right|}{\left|(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}\right|} = \left|\frac{(-1)^n x^{2n+1}(2n-1)!}{(-1)^{n-1}x^{2n-1}(2n+1)!}\right| = \left|\frac{(-1)\,x^2}{(2n+1)\,2n}\right| = \frac{x^2}{(2n+1)\,2n}.$$

Because

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{x^2}{(2n+1)\,2n} = 0,$$

we have $\lim_{n\to\infty}|a_{n+1}|/|a_n| = 0 < 1$ for all x. Thus, the ratio test guarantees that the power series converges for every real number x. The radius of convergence is infinite and the interval of convergence is all x.

Problems

25. (a) The general term of the series is x^n/n if n is odd and $-x^n/n$ if n is even, so $C_n = (-1)^{n-1}/n$, and we can use the ratio test. We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|\lim_{n\to\infty}\frac{|(-1)^n/(n+1)|}{|(-1)^{n-1}/n|} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|.$$

Therefore the radius of convergence is $R = 1$. This tells us that the power series converges for $|x| < 1$ and does not converge for $|x| > 1$. Notice that the radius of convergence does not tell us what happens at the endpoints, $x = \pm 1$.

(b) The endpoints of the interval of convergence are $x = \pm 1$. At $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

This is an alternating series with $a_n = 1/n$, so by the alternating series test, it converges. At $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$$

This is the negative of the harmonic series, so it does not converge. Therefore the right endpoint is included, and the left endpoint is not included in the interval of convergence, which is $-1 < x \le 1$.

26. Let $C_n = 2^n/n$. Then replacing n by $n+1$ gives $C_{n+1} = 2^{n+1}/(n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{2^{n+1}/(n+1)}{2^n/n} = |x|\frac{2^{n+1}}{n+1}\cdot\frac{n}{2^n} = 2|x|\left(\frac{n}{n+1}\right).$$

Thus

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x|.$$

The radius of convergence is $R = 1/2$.

For $x = 1/2$ the series becomes the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges.

For $x = -1/2$ the series becomes the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges. See Example 8 on page 517.

27. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3^{n+1}}\cdot\frac{3^n}{x^n}\right| = \frac{|x|}{3}.$$

Since $|x|/3 < 1$ when $|x| < 3$, the radius of convergence is 3 and the series converges for $-3 < x < 3$.

We check the endpoints:

$$x = 3: \quad \sum_{n=0}^{\infty}\frac{x^n}{3^n} = \sum_{n=0}^{\infty}\frac{3^n}{3^n} = \sum_{n=0}^{\infty}1^n \quad \text{which diverges.}$$

$$x = -3: \quad \sum_{n=0}^{\infty}\frac{x^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty}(-1)^n \quad \text{which diverges.}$$

The series diverges at both the endpoints, so the interval of convergence is $-3 < x < 3$.

28. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right| = \frac{n}{n+1}\cdot|x-3|.$$

Since $n/(n+1)\to 1$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-3|.$$

The series converges for $|x-3| < 1$. The radius of convergence is 1 and the series converges for $2 < x < 4$.

We check the endpoints. For $x = 2$, we have

$$\sum_{n=2}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=2}^{\infty}\frac{(2-3)^n}{n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}.$$

This is the alternating harmonic series and converges. For $x = 4$, we have

$$\sum_{n=2}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=2}^{\infty}\frac{(4-3)^n}{n} = \sum_{n=2}^{\infty}\frac{1}{n}.$$

This is the harmonic series and diverges. The series converges at $x = 2$ and diverges at $x = 4$. Therefore, the interval of convergence is $2 \le x < 4$.

29. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2x^{2(n+1)}}{2^{2(n+1)}}\cdot\frac{2^{2n}}{n^2x^{2n}}\right| = \left(\frac{n+1}{n}\right)^2\cdot\frac{x^2}{4}.$$

Since $(n+1)/n\to 1$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{4}.$$

We have $x^2/4 < 1$ when $|x| < 2$. The radius of convergence is 2 and the series converges for $-2 < x < 2$.

We check the endpoints. For $x = -2$, we have

$$\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{n^2(-2)^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} n^2,$$

which diverges. Similarly, for $x = 2$, we have

$$\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{n^2 2^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} n^2,$$

which diverges. The series diverges at both endpoints, so the interval of convergence is $-2 < x < 2$.

30. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{(-1)^n (x-5)^n}\right| = \left(\frac{n}{n+1}\right)^2 \cdot \frac{|x-5|}{2}.$$

Since $n/(n+1) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-5|}{2}.$$

We have $|x-5|/2 < 1$ when $|x-5| < 2$. The radius of convergence is 2 and the series converges for $3 < x < 7$.

We check the endpoints. For $x = 3$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n(3-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a p-series with $p = 2$ and it converges. For $x = 7$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n(7-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Since $\sum \frac{1}{n^2}$ converges, the alternating series $\sum \frac{(-1)^n}{n^2}$ also converges. The series converges at both its endpoints, so the interval of convergence is $3 \le x \le 7$.

31. We use the ratio test to find the radius of convergence;

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)+1}}{(n+1)!} \cdot \frac{n!}{x^{2n+1}}\right| = \left|\frac{x^2}{n+1}\right|$$

Since $\lim_{n\to\infty} |x^2|/(n+1) = 0$ for all x, the radius of convergence is $R = \infty$. There are no endpoints to check. The interval of convergence is all real numbers $-\infty < x < \infty$.

32. We use the ratio test to find the radius of convergence

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = |x|(n+1).$$

Since $\lim_{n\to\infty} |x|(n+1) = \infty$ for all $x \neq 0$, the radius of convergence is $R = 0$. There are no endpoints to check. The series converges only for $x = 0$, so the interval of convergence is the single point $x = 0$.

33. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(5x)^{n+1}/\sqrt{n+1}}{(5x)^n/\sqrt{n}}\right| = 5|x|\sqrt{\frac{n}{n+1}}.$$

Since $\sqrt{\frac{n}{n+1}} \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 5|x|.$$

We have $5|x| < 1$ when $|x| < 1/5$. The radius of convergence is $1/5$ and the series converges for $-1/5 < x < 1/5$.

We check the endpoints. For $x = -1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This is an alternating series. Since we have $0 < \dfrac{1}{\sqrt{n+1}} < \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$, by the alternating series test, the series converges at the endpoint $x = -1/5$. For $x = 1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a p-series with $p = 1/2$ and it diverges. Therefore, the interval of convergence is $-1/5 \leq x < 1/5$.

34. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(5x)^{2n+2}/\sqrt{n+1}}{(5x)^{2n}/\sqrt{n}}\right| = 5^2x^2\sqrt{\frac{n}{n+1}}.$$

Since $\sqrt{\dfrac{n}{n+1}} \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 25x^2.$$

We have $25x^2 < 1$ when $|x| < 1/5$. The radius of convergence is $1/5$ and the series converges for $-1/5 < x < 1/5$.

We check the endpoints. For $x = \pm 1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(5x)^{2n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a p-series with $p = 1/2$ and it diverges. The series diverges at both endpoints $x = \pm 1/5$, so the interval of convergence is $-1/5 \leq x < 1/5$.

35. The expression $1/(1+2z)$ has the form $a/(1-x)$, which is the sum of a geometric series with $a = 1$, $x = -2z$. The power series is

$$1 + (-2z) + (-2z)^2 + (-2z)^3 + \cdots = \sum_{n=0}^{\infty}(-2z)^n.$$

This series converges for $|-2z| < 1$, that is, for $-1/2 < z < 1/2$.

36. The expression $2/(1+y^2)$ has the form $a/(1-x)$, which is the sum of a geometric series with $a = 2$, $x = -y^2$. The power series is

$$2 + (-y^2) + (-y^2)^2 + (-y^2)^3 + \cdots = \sum_{n=0}^{\infty} 2(-y^2)^n.$$

This series converges for $|-y^2| < 1$, that is, for $-1 < y < 1$.

37. The expression $3/(1-z/2)$ has the form $a/(1-x)$, which is the sum of a geometric series with $a = 3$, $x = z/2$. The power series is

$$3 + 3(z/2) + 3(z/2)^2 + 3(z/2)^3 + \cdots = \sum_{n=0}^{\infty} 3(z/2)^n.$$

This series converges for $|z/2| < 1$, that is, for $-2 < z < 2$.

38. To compare $8/(4+y)$ with $a/(1-x)$, divide the numerator and denominator by 4. This gives $8/(4+y) = 2/(1+y/4)$, which is the sum of a geometric series with $a = 2$, $x = -y/4$. The power series is

$$2 + 2(-y/4) + 2(-y/4)^2 + 2(-y/4)^3 + \cdots = \sum_{n=0}^{\infty} 2(-y/4)^n.$$

This series converges for $|y/4| < 1$, that is, for $-4 < y < 4$.

39. The coefficient of the n^{th} term of the binomial power series is given by

$$C_n = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

To apply the ratio test, consider

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= |x|\left|\frac{p(p-1)(p-2)\cdots(p-(n-1))(p-n)/(n+1)!}{p(p-1)(p-2)\cdots(p-(n-1))/n!}\right| \\ &= |x|\left|\frac{p-n}{n+1}\right| = |x|\left|\frac{p}{n+1} - \frac{n}{n+1}\right| \to |x| \text{ as } n \to \infty.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

40. The k^{th} coefficient in the series $\sum kC_k x^k$ is $D_k = k \cdot C_k$. We are given that the series $\sum C_k x^k$ has radius of convergence R by the ratio test, so

$$|x| \lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \frac{|x|}{R}.$$

Thus, applying the ratio test to the new series, we have

$$\lim_{k\to\infty}\left|\frac{D_{k+1}x^{k+1}}{D_k x^k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)C_{k+1}}{kC_k}\right||x| = \frac{|x|}{R}.$$

Hence the new series has radius of convergence R.

41. The radius of convergence, R, is between 5 and 7.

42. The series is centered at $x = -7$. Since the series converges at $x = 0$, which is a distance of 7 from $x = -7$, the radius of convergence, R, is at least 7. Since the series diverges at $x = -17$, which is a distance of 10 from $x = -7$, the radius of convergence is no more than 10. That is, $7 \le R \le 10$.

43. The radius of convergence of the series, R, is at least 4 but no larger than 7.

(a) False. Since $10 > R$ the series diverges.
(b) True. Since $3 < R$ the series converges.
(c) False. Since $1 < R$ the series converges.
(d) Not possible to determine since the radius of convergence may be more or less than 6.

44. The series is centered at $x = 3$. Since the series converges at $x = 7$, which is a distance of 4 from $x = 3$, we know $R \ge 4$. Since the series diverges at $x = 10$, which is a distance of 7 from $x = 3$, we know $R \le 7$. That is, $4 \le R \le 7$.

Since $x = 11$ is a distance of 8 from $x = 3$, the series diverges at $x = 11$.
Since $x = 5$ is a distance of 2 from $x = 3$, the series converges there.
Since $x = 0$ is a distance of 3 from $x = 3$, the series converges at $x = 3$.

45. (a) We use the ratio test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\ &= \frac{x^2}{4(n+1)^2}.\end{aligned}$$

For a fixed value of x, we have

$$\frac{x^2}{4(n+1)^2} \to 0 \quad \text{as} \quad n \to \infty.$$

The series converges for all x, so the domain of $J(x)$ is all real numbers.

(b) Since

$$J(x) = 1 - \frac{x^2}{4} + \cdots,$$

we have $J(0) = 1$.

(c) We have

$$\begin{aligned}
S_0(x) &= 1\\
S_1(x) &= 1-\frac{x^2}{4}\\
S_2(x) &= 1-\frac{x^2}{4}+\frac{x^4}{64}\\
S_3(x) &= 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}\\
S_4(x) &= 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147{,}456}.
\end{aligned}$$

(d) The value of $J(1)$ can be approximated using partial sums. Substituting $x=1$ into the partial sum polynomials, we have

$$\begin{aligned}
S_0(1) &= 1\\
S_1(1) &= 0.75\\
S_2(1) &= 0.765625\\
S_3(1) &= 0.765191\\
S_4(1) &= 0.765198.
\end{aligned}$$

We estimate that $J(1)\approx 0.765$. Theorem 9.9 can be used to bound the error.

(e) We see from the series that $J(x)$ is an even function, so $J(-1)=J(1)$. Thus, $J(-1)\approx 0.765$.

46. (a) We have

$$f(x)=1+x+\frac{x^2}{2}+\cdots,$$

so

$$f(0)=1+0+0+\cdots=1.$$

(b) To find the domain of f, we find the interval of convergence.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|}=\lim_{n\to\infty}\left(\frac{|x|^{n+1}n!}{|x|^n(n+1)!}\right)=|x|\lim_{n\to\infty}\frac{1}{n+1}=0.$$

Thus the series converges for all x, so the domain of f is all real numbers.

(c) Differentiating term-by-term gives

$$\begin{aligned}
f'(x)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) &= \frac{d}{dx}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\\
&= 0+1+2\frac{x}{2!}+3\frac{x^2}{3!}+4\frac{x^3}{4!}+\cdots\\
&= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.
\end{aligned}$$

Thus, the series for f and f' are the same, so

$$f(x)=f'(x).$$

(d) We guess $f(x)=e^x$.

47. (a) Since only odd powers are involved in the series for $g(x)$,

$$g(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots,$$

we see that $g(x)$ is odd. Substituting $x=0$ gives $g(0)=0$.

(b) Differentiating term by term gives

$$\begin{aligned}
g'(x) &= 1-3\frac{x^2}{3!}+5\frac{x^4}{5!}-7\frac{x^6}{7!}+\cdots\\
&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.
\end{aligned}$$

$$g''(x) = 0 - 2\frac{x}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots$$
$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots.$$

So we see $g''(x) = -g(x)$.

(c) We guess $g(x) = \sin x$ since then $g'(x) = \cos x$ and $g''(x) = -\sin x = g(x)$. We check $g(0) = 0 = \sin 0$ and $g'(0) = 1 = \cos 0$.

48. (a) We have

$$(p(x))^2 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2$$
$$= 1 - 2\cdot\frac{x^2}{2} + \left(-\frac{x^2}{2!}\right)^2 + 2\frac{x^4}{4!} - 2\frac{x^6}{6!} - 2\frac{x^2}{2!}\cdot\frac{x^4}{4!}\cdots$$
$$= 1 - x^2 + \left(\frac{1}{4} + \frac{1}{12}\right)x^4 - x^6\left(\frac{1}{3\cdot 5\cdot 4!} + \frac{1}{4!}\right)\cdots$$
$$= 1 - x^2 + \frac{x^4}{3} - \frac{2}{45}x^6\cdots.$$

$$(q(x))^2 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 = x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^2$$
$$= x^2\left(1 - 2\frac{x^2}{3!} + \left(-\frac{x^2}{3!}\right)^2 + 2\frac{x^4}{5!}\cdots\right)$$
$$= x^2\left(1 - \frac{x^2}{3} + x^4\left(\frac{1}{(3!)^2} + \frac{1}{5\cdot 4\cdot 3}\right)\cdots\right)$$
$$= x^2\left(1 - \frac{x^2}{3} + \frac{2}{45}x^4\cdots\right)$$
$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6\cdots.$$

Thus, up to terms in x^6, we have

$$(p(x))^2 + (q(x))^2 = 1.$$

(b) The result of part (a) suggests that $p(x)$ and $q(x)$ could be the sine and cosine. Since $p(x)$ is even and $q(x)$ is odd, we guess that $p(x) = \cos x$ and $q(x) = \sin x$.

Strengthen Your Understanding

49. To find the radius of convergence, we calculate

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{C_{n+1}|x|^{n+1}}{C_n|x|^n} = |x|\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = 0.$$

Thus, the radius of convergence is ∞, not 0.

50. The series has an interval of convergence centered at $x = 0$. Since the series diverges at $x = 2$, the radius of convergence is 2 or less. This means that the series diverges for all points at a distance of more than 2 from the origin. Thus, the series cannot converge at $x = 3$.

51. In order to get a power series that does not converge at $x = 0$, we need to construct a power series about a point other than $x = 0$. We'll use $a = 2$, and the series

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{n}.$$

If we let $x = 0$ and use the ratio test to determine the convergence or divergence of the series, we get

$$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(n+1)}{(-2)^n/n}\right| = \lim_{n\to\infty}\left|\frac{-2n}{n+1}\right| = 2.$$

Since the limit is greater than 1, the series diverges at $x = 0$ by the ratio test.

52. The series $\sum_{n=1}^{\infty} n!(x-5)^n$ converges at $x=5$. The ratio test shows that the series diverges for any other value of x, since

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x-5)^{n+1}}{n!(x-5)^n}\right| = |x-5|\lim_{n\to\infty}(n+1) > 1$$

for $x \neq 5$.

53. The series with $\sum x^n/n^2$ satisfies this condition. The limit

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x|^{n+1}/(n+1)^2}{|x|^n/n^2} = |x|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = |x|.$$

Thus, the radius of convergence is 1.

At the endpoints, $\sum |x^n/n^2| = \sum 1/n^2$, which is a convergent p-series. Thus, the series is absolutely convergent at the endpoints $x = \pm 1$.

54. False. Writing out terms, we have

$$(x-1) + (x-2)^2 + (x-3)^3 + \cdots.$$

A power series is a sum of powers of $(x-a)$ for constant a. In this case, the value of a changes from term to term, so it is not a power series.

55. True. This power series has an interval of convergence about $x=0$. If the power series converges for $x=2$, the radius of convergence is 2 or more. Thus, $x=1$ is well within the interval of convergence, so the series converges at $x=1$.

56. False. This power series has an interval of convergence about $x=0$. Knowing the power series converges for $x=1$ does not tell us whether the series converges for $x=2$. Since the series converges at $x=1$, we know the radius of convergence is at least 1. However, we do not know whether the interval of convergence extends as far as $x=2$, so we cannot say whether the series converges at $x=2$.

For example, $\sum \frac{x^n}{2^n}$ converges for $x=1$ (it is a geometric series with ratio of $1/2$), but does not converge for $x=2$ (the terms do not go to 0).

Since this statement is not true for all C_n, the statement is false.

57. True. This power series has an interval of convergence centered on $x=0$. If the power series does not converge for $x=1$, then the radius of convergence is less than or equal to 1. Thus, $x=2$ lies outside the interval of convergence, so the series does not converge there.

58. True. The radius of convergence, R, is given by $\lim_{n\to\infty}|C_{n+1}|/|C_n| = 1/R$, if this limit exists, and since these series have the same coefficients, C_n, the radii of convergence are the same.

59. False. Two series can have the same radius of convergence without having the same coefficients. For example, $\sum x^n$ and $\sum nx^n$ both have radius of convergence of 1:

$$\lim_{n\to\infty}\frac{C_{n+1}}{C_n} = \lim_{n\to\infty}\frac{1}{1} = 1 \quad\text{and}\quad \lim_{n\to\infty}\frac{B_{n+1}}{B_n} = \lim_{n\to\infty}\frac{n+1}{n} = 1.$$

60. False. There are power series, such as $\sum x^n/n$, which converge at one endpoint, -1, but not at the other, 1.

61. True. The power series $\sum C_n(x-a)^n$ converges at $x=a$.

62. True. Since the power series converges at $x=10$, the radius of convergence is at least 10. Thus, $x=-9$ must be within the interval of convergence.

63. False. If $\sum C_n x^n$ converges at $x=10$, the radius of convergence is at least 10. However, if the radius of convergence were exactly 10, then $x=10$ is the endpoint of the interval of convergence and convergence there does not guarantee convergence at the other endpoint.

64. True. Intervals of convergence can be of any length and centered at any point and can include one endpoint and not the other.

65. False. The interval of convergence of $\sum C_n x^n$ is centered at the origin.

66. True. The interval of convergence is centered on $x=a$, so $a = (-11+1)/2 = -5$.

67. (d). Since the series is centered at $x=0$ and diverges at $x=7$, the radius of convergence is $R \le 7$. The series converges at $x=-3$, so $R \ge 3$. Since $|-9| = 9 > R$, the series diverges at $x=-9$.

Solutions for Chapter 9 Review

Exercises

1. $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}\cdots+\frac{3}{2^{10}}=3\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{10}}\right)=\frac{3\left(1-\frac{1}{2^{11}}\right)}{1-\frac{1}{2}}=\frac{3\left(2^{11}-1\right)}{2^{10}}$

2. We have

$$\begin{aligned}\text{Sum} &= -2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots\\ &= -2+(-2)\left(-\frac{1}{2}\right)+(-2)\left(-\frac{1}{2}\right)^2+(-2)\left(-\frac{1}{2}\right)^3+\cdots\\ &= \frac{-2}{1+1/2}=-\frac{4}{3}.\end{aligned}$$

3. This is the sum of a finite geometric series with $a=125, n=21, x=0.8$:

$$S_{21}=\underbrace{125}_{a}+\underbrace{125(0.8)}_{ax}+\underbrace{125(0.8)^2}_{ax^2}+\cdots+\underbrace{125(0.8)^{20}}_{ax^{n-1}}=\underbrace{\frac{125\left(1-0.8^{21}\right)}{1-0.8}}_{\frac{a(1-x^n)}{1-x}}=619.235.$$

4. This is a finite geometric series with $k-2$ terms in it, so $n=k-2$. The initial term is $a=(0.5)^3=0.125$ and the constant ratio is $x=0.5$. Using the formula for the sum of a finite geometric series, we get

$$\text{Sum}=\frac{a(1-x^n)}{1-x}=\frac{0.125(1-(0.5)^{k-2})}{1-0.5}=0.25(1-(0.5)^{k-2}).$$

5. If $b=1$, then the sum is 6. If $b\neq 1$, we use the formula for the sum of a finite geometric series. We can write the series as

$$b^5+b^5\cdot b+b^5\cdot b^2+b^5\cdot b^3+b^5\cdot b^4+b^5\cdot b^5.$$

This is a six-term geometric series ($n=6$) with initial term $a=b^5$ and constant ratio $x=b$:

$$\text{Sum}=\frac{a(1-x^n)}{1-x}=\frac{b^5(1-b^6)}{1-b}.$$

6. Using the formula for the sum of an infinite geometric series,

$$\sum_{n=4}^{\infty}\left(\frac{1}{3}\right)^n=\left(\frac{1}{3}\right)^4+\left(\frac{1}{3}\right)^5+\cdots=\left(\frac{1}{3}\right)^4\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\right)=\frac{(\frac{1}{3})^4}{1-\frac{1}{3}}=\frac{1}{54}$$

7. Using the formula for the sum of a finite geometric series,

$$\sum_{n=4}^{20}\left(\frac{1}{3}\right)^n=\left(\frac{1}{3}\right)^4+\left(\frac{1}{3}\right)^5+\cdots+\left(\frac{1}{3}\right)^{20}=\left(\frac{1}{3}\right)^4\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\left(\frac{1}{3}\right)^{16}\right)=\frac{(1/3)^4(1-(1/3)^{17})}{1-(1/3)}=\frac{3^{17}-1}{2\cdot 3^{20}}.$$

8. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n+5}{4^n}=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n+\sum_{n=0}^{\infty}\frac{5}{4^n}$, a sum of two geometric series.

We have $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n=\frac{1}{1-3/4}=4$ and $\displaystyle\sum_{n=0}^{\infty}\frac{5}{4^n}=\frac{5}{1-1/4}=\frac{20}{3}$, so

$$\sum_{n=0}^{\infty}\frac{3^n+5}{4^n}=4+\frac{20}{3}=\frac{32}{3}$$

.

9. We have

$$\begin{aligned}
S_1 &= 36.\\
S_2 &= 36 + 36\left(\frac{1}{3}\right) = 48.\\
S_3 &= 36 + 36\left(\frac{1}{3}\right) + 36\left(\frac{1}{3}\right)^2 = 52.\\
S_4 &= 36 + 36\left(\frac{1}{3}\right) + 36\left(\frac{1}{3}\right)^2 + 36\left(\frac{1}{3}\right)^3 = 53.333.
\end{aligned}$$

Here we have $a = 36$ and $x = 1/3$, so

$$S_n = \frac{a(1-x^n)}{1-x} = \frac{36(1-(1/3)^n)}{1-1/3}.$$

As $n \to \infty$, we see that $S_n \to 36/(2/3) = 54$.

10. We have

$$\begin{aligned}
S_1 &= 1280.\\
S_2 &= 1280 + 1280\left(-\frac{3}{4}\right) = 320.\\
S_3 &= 1280 + 1280\left(-\frac{3}{4}\right) + 1280\left(-\frac{3}{4}\right)^2 = 1040.\\
S_4 &= 1280 + 1280\left(-\frac{3}{4}\right) + 1280\left(-\frac{3}{4}\right)^2 + 1280\left(-\frac{3}{4}\right)^3 = 500.
\end{aligned}$$

Here we have $a = 1280$ and $x = -3/4$, so

$$S_n = \frac{a(1-x^n)}{1-x} = \frac{1280(1-(-3/4)^n)}{1+3/4}.$$

As $n \to \infty$, we see that $S_n \to 1280/(7/4) = 731.429$.

11. We have

$$\begin{aligned}
S_1 &= -810.\\
S_2 &= -810 - 810\left(-\frac{2}{3}\right) = -270.\\
S_3 &= -810 - 810\left(-\frac{2}{3}\right) - 810\left(-\frac{2}{3}\right)^2 = -630.\\
S_4 &= -810 - 810\left(-\frac{2}{3}\right) - 810\left(-\frac{2}{3}\right)^2 - 810\left(-\frac{2}{3}\right)^3 = -390.
\end{aligned}$$

Here we have $a = -810$ and $x = -2/3$, so

$$S_n = \frac{a(1-x^n)}{1-x} = \frac{-810(1-(-2/3)^n)}{1+2/3}.$$

As $n \to \infty$, we see that $S_n \to -810/(5/3) = -486$.

12. We have

$$\begin{aligned}
S_1 &= 2.\\
S_2 &= 2 + 2(3z).\\
S_3 &= 2 + 2(3z) + 2(3z)^2.\\
S_4 &= 2 + 2(3z) + 2(3z)^2 + 2(3z)^3.
\end{aligned}$$

Here we have $a = 2$ and $x = 3z$, so

$$S_n = \frac{a(1-x^n)}{1-x} = \frac{2(1-(3z)^n)}{1-3z}.$$

As $n \to \infty$, we see that $(3z)^n \to 0$ if $|3z| < 1$ but diverges if $|3z| \geq 1$. Thus,

$$S_n \to \frac{2}{1-3z} \text{ if } |z| < 1/3, \text{ but diverges otherwise.}$$

13. As n increases, the term $4n$ is much larger than 3 and $7n$ is much larger than 5. Thus dividing the numerator and denominator by n and using the fact that $\lim_{n\to\infty} 1/n = 0$, we have

$$\lim_{n\to\infty} \frac{3+4n}{5+7n} = \lim_{n\to\infty} \frac{(3/n)+4}{(5/n)+7} = \frac{4}{7}.$$

Thus, the sequence converges to $4/7$.

14. We have:

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1.$$

The terms of the sequence do not approach 0, and oscillate between values that are getting closer to $+1$ and -1. Thus the sequence diverges.

15. The first eight terms of the sequence are:

$$\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0.$$

The sequence then repeats this pattern, so it diverges.

16. Since the exponential function 2^n dominates the power function n^3 as $n \to \infty$, the series diverges.

17. We use the integral test with $f(x) = 1/x^3$ to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{1}{x^3}dx$ converges or diverges:

$$\int_1^\infty \frac{1}{x^3}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3}dx = \lim_{b\to\infty} \left.\frac{-1}{2x^2}\right|_1^b = \lim_{b\to\infty} \left(\frac{-1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\int_1^\infty \frac{1}{x^3}dx$ converges, we conclude from the integral test that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

18. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{3x^2+2x}{x^3+x^2+1}dx$ converges or diverges. The integral can be calculated using the substitution $w = x^3 + x^2 + 1$, $dw = (3x^2 + 2x)\,dx$.

$$\begin{aligned}\int_1^\infty \frac{3x^2+2x}{x^3+x^2+1}dx &= \lim_{b\to\infty} \int_1^b \frac{3x^2+2x}{x^3+x^2+1}dx \\ &= \lim_{b\to\infty} \ln|x^3+x^2+1|\Big|_1^b \\ &= \lim_{b\to\infty} \left(\ln|b^3+b^2+1| - \ln 3\right) = \infty.\end{aligned}$$

Since the integral $\int_1^\infty \frac{3x^2+2x}{x^3+x^2+1}dx$ diverges, we conclude from the integral test that the series $\sum_{n=1}^{\infty} \frac{3n^2+2n}{n^3+n^2+1}$ diverges.

19. We use the integral test to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_0^\infty xe^{-x^2}dx$ converges or diverges:

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty} \int_0^b xe^{-x^2}dx = \lim_{b\to\infty} \left.-\frac{1}{2}e^{-x^2}\right|_0^b = \lim_{b\to\infty} \left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\int_0^\infty xe^{-x^2}dx$ converges, we conclude from the integral test that the series $\sum_{n=0}^{\infty} ne^{-n^2}$ converges.

20. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_2^\infty \frac{2}{x^2-1}\,dx$ converges or diverges:

$$\begin{aligned}
\int_2^\infty \frac{2}{x^2-1}\,dx &= \lim_{b\to\infty}\int_2^b \frac{2}{x^2-1}\,dx \\
&= \lim_{b\to\infty}\left(\int_2^b \left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx\right) \qquad \text{(Using partial fractions)} \\
&= \lim_{b\to\infty}\left(\ln|x-1| - \ln|x+1| \Big|_2^b\right) \\
&= \lim_{b\to\infty}\left(\ln\left|\frac{x-1}{x+1}\right| \Big|_2^b\right) \\
&= \lim_{b\to\infty}\left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left(\frac{1}{3}\right)\right) = \ln 1 - \ln\frac{1}{3} = \ln 3.
\end{aligned}$$

Since the integral $\int_2^\infty \frac{2}{x^2-1}\,dx$ converges, we conclude that the series $\sum_{n=2}^\infty \frac{2}{n^2-1}$ converges.

21. Since $a_n = 1/(2^n n!)$, replacing n by $n+1$ gives $a_{n+1} = 1/(2^{n+1}(n+1)!)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} = \frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2(n+1)},$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{1}{2n+2} = 0.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^\infty \frac{1}{2^n n!}$ converges.

22. Since $a_n = (n-1)!/5^n$, replacing n by $n+1$ gives $a_{n+1} = n!/5^{n+1}$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n!/5^{n+1}}{(n-1)!/5^n} = \frac{n!}{5(n-1)!} = \frac{n(n-1)!}{5(n-1)!} = \frac{n}{5},$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{n}{5} = \infty.$$

Since $L > 1$, the ratio test tells us that $\sum_{n=1}^\infty \frac{(n-1)!}{5^n}$ diverges.

23. Since $a_n = (2n)!/(n!(n+1)!)$, replacing n by $n+1$ gives $a_{n+1} = (2n+2)!/((n+1)!(n+2)!)$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(2n+2)!}{(n+1)!(n+2)!}}{\frac{(2n)!}{n!(n+1)!}} = \frac{(2n+2)!}{(n+1)!(n+2)!}\cdot\frac{n!(n+1)!}{(2n)!}.$$

However, since $(n+2)! = (n+2)(n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \frac{2(2n+1)}{n+2},$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 4.$$

Since $L > 1$, the ratio test tells us that $\sum_{n=1}^\infty \frac{(2n)!}{n!(n+1)!}$ diverges.

24. Let $a_n = 1/(n^2+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/((n+1)^2+1)$. Since $(n+1)^2+1 > n^2+1$, we have

$$0 < \frac{1}{(n+1)^2+1} < \frac{1}{n^2+1},$$

so

$$0 < a_{n+1} < a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$, therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges.

25. Let $a_n = 1/\sqrt{n^2+1}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{(n+1)^2+1}$. Since $\sqrt{(n+1)^2+1} > \sqrt{n^2+1}$, we have

$$\frac{1}{\sqrt{(n+1)^2+1}} < \frac{1}{\sqrt{n^2+1}},$$

so

$$0 < a_{n+1} < a_n.$$

In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges by the alternating series test.

26. The series $\sum \frac{(-1)^n}{n^{1/2}}$ converges by the alternating series test. However $\sum \frac{1}{n^{1/2}}$ diverges because it is a p-series with $p = 1/2 \le 1$. Thus $\sum \frac{(-1)^n}{n^{1/2}}$ is conditionally convergent.

27. Since

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right) = 1,$$

the n^{th} term $a_n = (-1)^n\left(1+\frac{1}{n^2}\right)$ does not tend to zero as $n \to \infty$. Thus, the series $\sum(-1)^n\left(1+\frac{1}{n^2}\right)$ is divergent.

28. We first check absolute convergence by deciding whether $\sum \ln n/n$ converges. Since $\ln(n) > 1$ for all $n > 2$, we compare

$$\frac{1}{n} < \frac{\ln n}{n}.$$

The harmonic series $\sum 1/n$ diverges, so $\sum \ln n/n$ diverges. Alternatively, we can use the integral test. Since

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \left.\frac{(\ln x)^2}{2}\right|_1^b = \lim_{b\to\infty}\frac{(\ln b)^2}{2},$$

and since this limit does not exist, $\sum \frac{\ln n}{n}$ diverges.

We now check conditional convergence. The original series is alternating, so we check whether $a_{n+1} < a_n$. Consider $a_n = f(n)$, where $f(x) = \ln n/n$. Since

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1}{x^2}(1-\ln x)$$

is negative for $x > e$, we know that a_n is decreasing for $n > e$. Thus for $n \ge 3$,

$$a_{n+1} = \frac{\ln(n+1)}{n+1} < \frac{\ln(n)}{n} = a_n.$$

Since $\ln n/n \to 0$ as $n \to \infty$, we see that $\sum \frac{(-1)^{n-1}\ln n}{n}$ is conditionally convergent.

29. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\arctan n} = \frac{2}{\pi} \ne 0$$

we know that $\sum \frac{(-1)^{n-1}}{\arctan n}$ diverges by Property 3 of Theorem 9.2.

30. Let $a_n = n^2/(3n^2+4)$. Since $3n^2+4 > 3n^2$, we have $\dfrac{n^2}{3n^2+4} < \dfrac{1}{3}$, so

$$0 < a_n < \left(\frac{1}{3}\right)^n.$$

The geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2}{3n^2+4}\right)^n$ also converges.

31. Let $a_n = 1/(n\sin^2 n)$. Since $0 < \sin^2 n < 1$, for any positive integer n, we have $n\sin^2 n < n$, so $\dfrac{1}{n\sin^2 n} > \dfrac{1}{n}$, thus

$$a_n > \frac{1}{n}.$$

The harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sin^2 n}$ also diverges.

32. The n^{th} term $a_n = \sqrt{n-1}/(n^2+3)$ behaves like $\sqrt{n}/n^2 = 1/n^{3/2}$ for large n, so we take $b_n = 1/n^{3/2}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n-1}/(n^2+3)}{1/n^{3/2}} = \lim_{n\to\infty}\frac{n^{3/2}\sqrt{n-1}}{n^2+3} = \lim_{n\to\infty}\frac{n^2\sqrt{1-1/n}}{n^2(1+3/n^2)} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^{3/2}$ converges because $p = 3/2 > 1$. Therefore $\sum\sqrt{n-1}/(n^2+3)$ also converges.

33. The n^{th} term $a_n = (n^3-2n^2+n+1)/(n^5-2)$ behaves like $n^3/n^5 = 1/n^2$ for large n, so we take $b_n = 1/n^2$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^3-2n^2+n+1)/(n^5-2)}{1/n^2} = \lim_{n\to\infty}\frac{n^5-2n^4+n^3+n^2}{n^5-2} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore the series $\sum\left(n^3-2n^2+n+1\right)/\left(n^5-2\right)$ also converges.

34. The n^{th} term is $a_n = \sin(1/n^2)$. When n is large, $1/n^2$ is near zero, so $\sin(1/n^2)$ is near $1/n^2$. We see that $\sin(1/n^2)$ behaves like $1/n^2$ for large n, so we take $b_n = 1/n^2$. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\sin(1/n^2)}{1/n^2}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\\ &= 1.\end{aligned}$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore $\sum\sin(1/n^2)$ also converges.

35. The n^{th} term $a_n = 1/(\sqrt{n^3-1})$ behaves like $1/\sqrt{n^3} = 1/n^{3/2}$ for large n, so we take $b_n = 1/n^{3/2}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/\sqrt{n^3-1}}{1/n^{3/2}} = \lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3-1}} = \lim_{n\to\infty}\frac{n^{3/2}}{n^{3/2}\sqrt{1-1/n^3}} = \lim_{n\to\infty}\frac{1}{\sqrt{1-1/n^3}} = \frac{1}{\sqrt{1-0}} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^{3/2}$ converges because $p = 3/2 > 1$. Therefore $\sum 1/\sqrt{n^3-1}$ also converges.

36. Since $f(x) = 1/(x+1)$ is continuous, positive and decreasing, we apply the integral test, and we obtain

$$\int_1^{\infty}\frac{1}{x+1}dx = \lim_{b\to\infty}\int_1^b\frac{1}{1+x}\,dx = \lim_{b\to\infty}(\ln(b+1)-\ln 2) = \infty.$$

Since this improper integral diverges, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+1}$ also diverges. We can also observe the series is the harmonic series, with the first term missing, and hence diverges by Property 2 of Theorem 9.2.

37. This is a p-series with $p > 1$, so it converges.

38. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_3^\infty \frac{2}{\sqrt{x-2}}\,dx$ converges or diverges:

$$\begin{aligned}\int_3^\infty \frac{2}{\sqrt{x-2}}\,dx &= \lim_{b\to\infty}\int_3^b \frac{2}{\sqrt{x-2}}\,dx \\ &= \lim_{b\to\infty}\int_1^b \frac{2}{\sqrt{w}}\,dw \qquad \text{(Substitute } w = x-2.) \\ &= \lim_{b\to\infty} 4\sqrt{w}\bigg|_1^b = \infty.\end{aligned}$$

Since the limit does not exist, the integral $\int_3^\infty \frac{2}{\sqrt{x-2}}\,dx$ diverges, and we conclude from the integral test that the series $\sum_{n=3}^{\infty}\frac{2}{\sqrt{n-2}}$ diverges. The limit comparison test with $b_n = 1/\sqrt{n}$ can also be used.

39. This is an alternating series. Let $a_n = 1/(\sqrt{n}+1)$. Then $\lim_{n\to\infty} a_n = 0$. Now replace n by $n+1$ to give $a_{n+1} = 1/(\sqrt{n+1}+1)$. Since $\sqrt{n+1}+1 > \sqrt{n}+1$, we have $\frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1}$, so

$$0 < a_{n+1} = \frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1} = a_n.$$

Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}+1}$ converges.

40. Writing $a_n = n^2/(n^2+1)$, we have $\lim_{n\to\infty} a_n = 1$ so the series diverges by Property 3 of Theorem 9.2.

41. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{x^2}{x^3+1}dx$ converges or diverges:

$$\int_1^\infty \frac{x^2}{x^3+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x^2}{x^3+1}\,dx = \lim_{b\to\infty}\frac{1}{3}\ln|x^3+1|\bigg|_1^b = \lim_{b\to\infty}\left(\frac{1}{3}\ln(b^3+1) - \frac{1}{3}\ln 2\right).$$

Since the limit does not exist, the integral $\int_1^\infty \frac{x^2}{x^3+1}dx$ diverges an so we conclude from the integral test that the series $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$ diverges. The limit comparison test with $b_n = 1/n$ can also be used.

42. We use the ratio test. Since $a_n = 3^n/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 3^{n+1}/(2n+2)!$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}/(2n+2)!}{3^n/(2n)!} = \frac{3^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{3^n}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{3}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0.$$

Since $L < 1$, the ratio test tells us that the series $\sum_{n=1}^{\infty}\frac{3^n}{(2n)!}$ converges.

43. We use the ratio test. Since $a_n = (2n)!/(n!)^2$, replacing n by $n+1$ gives $a_{n+1} = (2n+2)!/((n+1)!)^2$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)!}{(n+1)!(n+1)!}\cdot\frac{n!n!}{(2n)!}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$ and $(n+1)! = (n+1)n!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)},$$

therefore

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4.$$

As $L > 1$ the ratio test tells us that the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges.

44. The series can be written as

$$\sum_{n=1}^{\infty} \frac{n^2+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p > 1$, we see $\sum_{n=1}^{\infty} \frac{n^2+2^n}{n^2 2^n}$ converges by Theorem 9.2.

45. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{3^{2n}}\right| = 3^2 \frac{1}{(2n+2)(2n+1)}.$$

Therefore we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Thus by the ratio test, the series converges.

46. Let $a_n = 2^{-n}\frac{(n+1)}{(n+2)} = \left(\frac{n+1}{n+2}\right)\left(\frac{1}{2^n}\right)$. Since $\frac{(n+1)}{(n+2)} < 1$ and $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$, we have

$$0 < a_n < \left(\frac{1}{2}\right)^n,$$

so that we can compare the series $\sum_{n=1}^{\infty} 2^{-n}\frac{(n+1)}{(n+2)}$ with the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. The comparison test tells us that

$$\sum_{n=1}^{\infty} 2^{-n}\frac{(n+1)}{(n+2)}$$

also converges.

47. We have

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^n} = \lim_{n\to\infty} \frac{2}{(2n+3)(2n+2)} = 0,$$

so the series converges by the ratio test, since $L < 1$.

48. Since there is an n in the numerator and a $\sqrt{n}$ in the denominator, the terms in this series are increasing in magnitude. We have

$$\lim_{n\to\infty} \left|\frac{n+1}{\sqrt{n}}(-1)^n\right| = \lim_{n\to\infty} \frac{n+1}{\sqrt{n}} = \infty,$$

so $\lim_{n\to\infty}(-1)^n(n+1)/\sqrt{n}$ does not approach zero. Therefore, the series diverges by Property 3 of Theorem 9.2.

49. The series can be written as

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2}{5^n} + \frac{3^n}{5^n}\right) = \sum_{n=0}^{\infty} \left(2\left(\frac{1}{5}\right)^n + \left(\frac{3}{5}\right)^n\right).$$

The series $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$ is a geometric series which converges because $|\frac{1}{5}| < 1$. Likewise, the geometric series $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ converges because $|\frac{3}{5}| < 1$. Since both series converge, Property 1 of Theorem 9.2 tells us that the series $\sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$ also converges.

50. The n^{th} term $a_n = ((1+5n)/(4n))^n$ behaves like $(5/4)^n$ for large n, so we take $b_n = (5/4)^n$ and use the limit comparison test. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{((1+5n)/4n)^n}{(5/4)^n} = \lim_{n\to\infty} \left(\frac{1+5n}{5n}\right)^n,$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\left(1+\frac{1}{5n}\right)^{5n}\right)^{1/5} = e^{1/5}.$$

The limit comparison test applies with $c = e^{1/5}$. The geometric series $\sum(5/4)^n$ diverges because $x = 5/4 \geq 1$. Therefore $\sum((1+5n)/(4n))^n$ also diverges.

51. Writing $a_n = 1/(2+\sin n)$, we have $\lim_{n\to\infty} a_n$ does not exist, so the series diverges by Property 3 of Theorem 9.2.

52. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_3^\infty \frac{1}{(2x-5)^3}\,dx$ converges or diverges:

$$\begin{aligned}\int_3^\infty \frac{1}{(2x-5)^3}\,dx &= \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{1}{w^3}\,dw \qquad \text{(Substitute } w = 2x-5\text{)}\\ &= -\frac{1}{2}\lim_{b\to\infty} \frac{1}{2w^2}\bigg|_1^b\\ &= -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{2b^2}-\frac{1}{2}\right) = \frac{1}{4}.\end{aligned}$$

Since the integral $\int_3^\infty \frac{1}{(2x-5)^3}\,dx$ converges, we conclude from the integral test that the series $\sum_{n=3}^{\infty} \frac{1}{(2n-5)^3}$ converges. The limit comparison test, with $b_n = 1/n^3$ can also be used.

53. The n^{th} term $a_n = 1/(n^3-3)$ behaves like $1/n^3$ for large n, so we take $b_n = 1/n^3$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^3-3)}{1/n^3} = \lim_{n\to\infty} \frac{n^3}{n^3-3} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^3$ converges because $p = 3 > 1$. Therefore $\sum 1/(n^3 - 3$ also converges.

54. Note that

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

is an alternating series with the absolute values of the terms decreasing to 0. Thus, the series converges by the alternating series test.

55. Since $\ln(1+1/k) = \ln((k+1)/k) = \ln(k+1) - \ln k$, the n^{th} partial sum of this series is

$$\begin{aligned}S_n &= \sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right)\\ &= \sum_{k=1}^{n}\ln(k+1) - \sum_{k=1}^{n}\ln k\\ &= (\ln 2 + \ln 3 + \cdots + \ln(n+1)) - (\ln 1 + \ln 2 + \cdots + \ln n)\\ &= \ln(n+1) - \ln 1\\ &= \ln(n+1).\end{aligned}$$

Thus, the partial sums, S_n, grow without bound as $n \to \infty$, so the series diverges by the definition.

56. The ratio test gives

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)/2^{n+1}}{n/2^n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2},$$

so the series converges since $L < 1$.

57. Since $\ln n$ grows much more slowly than n, we suspect that $(\ln n)^2 < n$ for large n. This can be confirmed with L'Hopital's rule.

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)/n}{1} = \lim_{n\to\infty} \frac{2(\ln n)}{n} = 0.$$

Therefore, for large n, we have $(\ln n)^2/n < 1$, and hence for large n,

$$\frac{1}{n} < \frac{1}{(\ln n)^2}.$$

Thus $\sum_{n=2}^{\infty} 1/(\ln n)^2$ diverges by comparison with the divergent harmonic series $\sum 1/n$.

58. Let $C_n = \dfrac{(2n)!}{(n!)^2}$. Then replacing n by $n+1$, we have $C_{n+1} = \dfrac{(2n+2)!}{((n+1)!)^2}$. Thus, with $a_n = (2n)!x^n/(n!)^2$, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(2n+2)!/((n+1)!)^2}{(2n)!/(n!)^2} = |x|\frac{(2n+2)!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$ and $(n+1)! = (n+1)n!$ we have

$$\frac{|C_{n+1}|}{|C_n|} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)},$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = |x| \lim_{n\to\infty} \frac{4n+2}{n+1} = 4|x|,$$

so the radius of convergence of this series is $R = 1/4$.

59. Let $C_n = 1/(n!+1)$. Then replacing n by $n+1$ gives $C_{n+1} = 1/((n+1)!+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{1/((n+1)!+1)}{1/(n!+1)} = |x|\frac{n!+1}{(n+1)!+1}.$$

Since $n!$ and $(n+1)!$ dominate the constant term 1 as $n \to \infty$ and $(n+1)! = (n+1)\cdot n!$ we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0.$$

Thus the radius of convergence is $R = \infty$.

60. To find R, we consider the following limit, where the coefficient of the n^{th} term is given by $C_n = n^2$.

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{(n+1)^2x^{n+1}}{n^2x^n}\right| &= \lim_{n\to\infty} |x|\frac{n^2+2n+1}{n^2} \\ &= |x| \lim_{n\to\infty} \left(\frac{1+(2/n)+(1/n^2)}{1}\right) = |x|.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

61. Here the coefficient of the n^{th} term is $C_n = n/(2n+1)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)/(2n+3))x^{n+1}}{(n/(2n+1))x^n}\right| = \frac{(n+1)(2n+1)}{n(2n+3)}|x| \to |x| \text{ as } n \to \infty.$$

Thus, by the ratio test, the radius of convergence is $R = 1$.

62. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3^{n+1}(n+1)^2} \cdot \frac{3^n n^2}{x^n}\right| = \left(\frac{n}{n+1}\right)^2 \cdot \frac{|x|}{3}.$$

Since $n/(n+1) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{3}.$$

We have $|x|/3 < 1$ when $|x| < 3$. The radius of convergence is 3 and the series converges for $-3 < x < 3$.

We check the endpoints. For $x = -3$, we have

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

We know $\sum \frac{1}{n^2}$ is a p-series with $p = 2$ so it converges. Therefore the alternating series $\sum \frac{(-1)^n}{n^2}$ also converges. For $x = 3$, we have

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a p-series with $p = 2$ and it converges. The series converges at both its endpoints and the interval of convergence is $-3 \le x \le 3$.

63. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(-1)^n(x-2)^n}\right| = \frac{|x-2|}{5}.$$

Since $|x-2|/5 < 1$ when $|x-2| < 5$, the radius of convergence is 5 and the series converges for $-3 < x < 7$.

We check the endpoints:

$$x = -3: \quad \sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-3-2)^n}{5^n} = \sum_{n=0}^{\infty} 1 \quad \text{which diverges.}$$

$$x = 7: \quad \sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(7-2)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \quad \text{which diverges.}$$

The series diverges at both the endpoints, so the interval of convergence is $-3 < x < 7$.

64. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n}\right| = \frac{n}{n+1} \cdot |x|.$$

Since $n/(n+1) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|.$$

The series converges for $|x| < 1$. The radius of convergence is 1 and the series converges for $-1 < x < 1$.

We check the endpoints. For $x = -1$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series and diverges. For $x = 1$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is the alternating harmonic series and converges. The series diverges at $x = -1$ and converges at $x = 1$. Therefore, interval of convergence is $-1 < x \le 1$.

65. We use the ratio test to find the radius of convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right|$$

Since $\lim_{n\to\infty} |x|/(n+1) = 0$ for all x, the radius of convergence is $R = \infty$. There are no endpoints to check. The interval of convergence is all real numbers $-\infty < x < \infty$.

Problems

66. We have

$$s_1 = \frac{(-1)^1(2\cdot 1+1)^2}{2^{2\cdot 1-1} + (-1)^{1+1}} = \frac{(-1)3^2}{2^1+1} = \frac{-9}{3} = -3$$

$$s_2 = \frac{(-1)^2(2\cdot 2+1)^2}{2^{2\cdot 2-1}+(-1)^{2+1}} = \frac{5^2}{2^3+(-1)} = \frac{25}{7}$$
$$s_3 = \frac{(-1)^3(2\cdot 3+1)^2}{2^{2\cdot 3-1}+(-1)^{3+1}} = \frac{(-1)7^2}{2^5+1} = \frac{-49}{33}$$
$$s_4 = \frac{(-1)^4(2\cdot 4+1)^2}{2^{2\cdot 4-1}+(-1)^{4+1}} = \frac{9^2}{2^7+(-1)} = \frac{81}{127},$$

so the first four terms are $-3, 25/7, -49/33, 81/127$.

67. We see that $s_1, s_2, s_3, \ldots$ forms an arithmetic sequence—that is, the values go up by the same amount, 2, each time. We conclude that

$$s_n = 2n + 3.$$

68. We see that $t_1, t_2, t_3, \ldots$ is a sequence of consecutive odd square numbers. Since $t_1 = 3$, we conclude that

$$t_n = (2n+1)^2.$$

69. We have

$$\begin{aligned} a_2 &= a_1 + 2\cdot 2 = 5+4 &= 9 \quad &\text{because } a_1 = 5 \\ a_3 &= a_2 + 2\cdot 3 = 9+6 &= 15 & \\ a_4 &= a_3 + 2\cdot 4 = 15+8 &= 23. & \end{aligned}$$

70. First, we have

$$\begin{aligned} a_2 &= a_1 + 2\cdot 2 = 5+4 &= 9 \quad &\text{because } a_1 = 5 \\ a_3 &= a_2 + 2\cdot 3 = 9+6 &= 15 & \\ a_4 &= a_3 + 2\cdot 4 = 15+8 &= 23. & \end{aligned}$$

Next, we find that

$$\begin{aligned} b_2 &= b_1 + a_1 = 10+5 &= 15 \quad &\text{because } b_1 = 10 \\ b_3 &= b_2 + a_2 = 15+9 &= 24 & \\ b_4 &= b_3 + a_3 = 24+15 &= 39 & \\ b_5 &= b_4 + a_4 = 39+23 &= 62. & \end{aligned}$$

71. The series can be written as

$$\sum_{n=1}^{\infty} \frac{n^r + r^n}{n^r r^n} = \sum_{n=1}^{\infty} \frac{1}{r^n} + \sum_{n=1}^{\infty} \frac{1}{n^r}.$$

If $0 < r < 1$, both series diverge, but if $r > 1$ both series converge.

If $r = 1$ the given series becomes $\sum_{n=1}^{\infty} \frac{n+1}{n}$ which diverges.

By Theorem 9.2 the given series converges if $r > 1$.

72. The series converges for $|x-2| = 2$ and diverges for $|x-2| = 4$, thus the radius of convergence of the series, R, is at least 2 but no larger than 4.

(a) False. If $x = 7$ then $|x-2| = 5$, so the series diverges.
(b) False. If $x = 1$ then $|x-2| = 1$, so the series converges.
(c) True. If $x = 0.5$ then $|x-2| = 1.5$, so the series converges.
(d) If $x = 5$ then $|x-2| = 3$ and it is not possible to determine whether or not the series converges at this point.
(e) False. If $x = -3$ then $|x-2| = 5$, so the series diverges.

73. (a) Using an argument similar to Example 7 in Section 9.5, we take

$$a_n = (-1)^n \frac{t^{2n}}{(2n)!},$$

so, replacing n by $n+1$,

$$a_{n+1} = (-1)^{n+1} \frac{t^{2(n+1)}}{(2(n+1))!} = (-1)^{n+1} \frac{t^{2n+2}}{(2n+2)!}.$$

Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(-1)^{n+1} t^{2n+2}/(2n+2)!|}{|(-1)^n t^{2n}/(2n)!|} = \frac{t^2}{(2n+2)(2n+1)},$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{t^2}{(2n+2)(2n+1)} = 0.$$

The radius of convergence is therefore ∞, so the series converges for all t. Therefore, the domain of h is all real numbers.

(b) Since h involves only even powers,

$$h(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots,$$

h is an even function.

(c) Differentiating term by term, we have

$$\begin{aligned} h'(t) &= 0 - 2\frac{t}{2!} + 4\frac{t^3}{4!} - 6\frac{t^6}{6!} + \cdots \\ &= -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots . \end{aligned}$$

$$\begin{aligned} h''(t) &= -1 + 3\frac{t^2}{3!} - 5\frac{t^4}{5!} + \cdots \\ &= -1 + \frac{t^2}{2!} - \frac{t^4}{4!} + \cdots . \end{aligned}$$

So we see $h''(t) = -h(t)$.

74. If a payment M in the future has present value P, then

$$M = P(1+r)^t,$$

where t is the number of periods in the future and r is the interest rate. Thus

$$P = \frac{M}{(1+r)^t}.$$

The monthly interest rate here is $0.09/12 = 0.0075$, so the present value of first payment, made at the end of the first month, is $M/(1.0075)$. The present value of the second payment is $M/(1.0075)^2$. Continuing in this way, the sum of the present value of all of the payments is

$$\frac{M}{(1.0075)} + \frac{M}{(1.0075)^2} + \cdots + \frac{M}{(1.0075)^{240}}.$$

This is a finite geometric series with 240 terms, with sum

$$\frac{M}{1.0075}\left(\frac{1 - 1.0075^{-240}}{1 - 1.0075^{-1}}\right) = 111.145M.$$

Setting this equal to the loan amount of 200,000 gives a monthly payment of $M = \$200{,}000/111.145 = \$1799.45$.

75. In the first year, extraction is 12 million tons. In the second year this falls to $12(1-0.05) = 12(0.95)$ million tons, and in subsequent years the extraction falls by a factor of 0.95. During the next n years

$$\text{Total extraction} = 12 + 12(0.95) + 12(0.95)^2 + \cdots 12(0.95)^{n-1}, \text{million tons.}$$

This is a geometric series with $a = 12$ and $x = 0.95$, so

$$\text{Total extraction} = 12\left(\frac{1-0.95^n}{1-0.95}\right).$$

Since $|x| < 1$, when $n \to \infty$

$$\text{Total extraction} \to \frac{12}{1-0.95} = 240 \text{ million tons.}$$

World reserves of the mineral must exceed 240 million tons if extraction is to continue indefinitely.

76. (a) It is easier to work with the value of the car first and then find the yearly losses. The value of the car goes down by 10% a year. Thus, the value at the end of the first years is $v_1 = 30{,}000(0.9)$. The value at the end of the second year is $v_2 = 30{,}000(0.9)^2$. The value at the end of n years is $v_n = 30{,}000(0.9)^n$. Thus, the losses in the first four years are

$$\begin{aligned}
l_1 &= 30{,}000(0.1)\\
l_2 &= v_1 - v_2 = 30{,}000(0.9) - 30{,}000(0.9)^2 = 30{,}000(0.9)(0.1)\\
l_3 &= v_2 - v_3 = 30{,}000(0.9)^2 - 30{,}000(0.9)^3 = 30{,}000(0.9)^2(0.1)\\
l_4 &= v_3 - v_4 = 30{,}000(0.9)^3 - 30{,}000(0.9)^4 = 30{,}000(0.9)^3(0.1).
\end{aligned}$$

Thus,

$$l_n = v_{n-1} - v_n = 30{,}000(0.9)^{n-1}(0.1) = 3000(0.9)^{n-1}.$$

(b) In the first year, $m_1 = 500$; in the second year, $m_2 = 500(1.2)$; in the third year, $m_3 = 500(1.2)^2$. Thus

$$m_n = 500(1.2)^{n-1}.$$

(c) We want to find n such that $m_n \geq l_n$, so

$$500(1.2)^{n-1} \geq 3000(0.9)^{n-1}.$$

We solve

$$\begin{aligned}
500(1.2)^{n-1} &= 3000(0.9)^{n-1}\\
\frac{(1.2)^{n-1}}{(0.9)^{n-1}} &= \frac{3000}{500}\\
\left(\frac{1.2}{0.9}\right)^{n-1} &= 6\\
(n-1)\ln\left(\frac{1.2}{0.9}\right) &= \ln 6\\
n-1 &= \frac{\ln 6}{\ln(1.2/0.9)}\\
n &= 6.228 + 1 = 7.228.
\end{aligned}$$

So, maintenance first exceeds losses in year 8. In year 7,

$$l_7 = 3000(0.9)^6 = \$1594, \quad m_7 = 500(1.2)^6 = \$1493.$$

In year 8,

$$l_8 = 3000(0.9)^7 = \$1435, \quad m_8 = 500(1.2)^7 = \$1792.$$

77.

$$\text{Present value of first coupon} = \frac{50}{1.06}$$
$$\text{Present value of second coupon} = \frac{50}{(1.06)^2}, \text{ etc.}$$

$$\begin{aligned}
\text{Total present value} &= \underbrace{\frac{50}{1.06} + \frac{50}{(1.06)^2} + \cdots + \frac{50}{(1.06)^{10}}}_{\text{coupons}} + \underbrace{\frac{1000}{(1.06)^{10}}}_{\text{principal}} \\
&= \frac{50}{1.06}\left(1 + \frac{1}{1.06} + \cdots + \frac{1}{(1.06)^9}\right) + \frac{1000}{(1.06)^{10}} \\
&= \frac{50}{1.06}\left(\frac{1 - \left(\frac{1}{1.06}\right)^{10}}{1 - \frac{1}{1.06}}\right) + \frac{1000}{(1.06)^{10}} \\
&= 368.004 + 558.395 \\
&= \$926.40
\end{aligned}$$

78.

$$\text{Present value of first coupon} = \frac{50}{1.04}$$
$$\text{Present value of second coupon} = \frac{50}{(1.04)^2}, \text{etc.}$$

$$\begin{aligned}
\text{Total present value} &= \underbrace{\frac{50}{1.04} + \frac{50}{(1.04)^2} + \cdots + \frac{50}{(1.04)^{10}}}_{\text{coupons}} + \underbrace{\frac{1000}{(1.04)^{10}}}_{\text{principal}} \\
&= \frac{50}{1.04}\left(1 + \frac{1}{1.04} + \cdots + \frac{1}{(1.04)^9}\right) + \frac{1000}{(1.04)^{10}} \\
&= \frac{50}{1.04}\left(\frac{1 - \left(\frac{1}{1.04}\right)^{10}}{1 - \frac{1}{1.04}}\right) + \frac{1000}{(1.04)^{10}} \\
&= 405.545 + 675.564 \\
&= \$1081.11
\end{aligned}$$

79. (a) The present value is given by the finite series:

$$\begin{aligned}
\text{Total present value} &= \underbrace{\frac{5}{1.04} + \frac{5}{(1.04)^2} + \cdots + \frac{5}{(1.04)^{100}}}_{\text{coupons}} + \underbrace{\frac{100}{(1.04)^{100}}}_{\text{principal}} \\
&= \frac{5}{1.04}\left(1 + \frac{1}{1.04} + \cdots + \frac{1}{(1.04)^{99}}\right) + \frac{100}{(1.04)^{100}} \\
&= \frac{5}{1.04}\left(\frac{1 - \left(\frac{1}{1.04}\right)^{100}}{1 - \frac{1}{1.04}}\right) + \frac{100}{(1.04)^{100}} \\
&= \$124.50.
\end{aligned}$$

(b) The present value is now given by the infinite series

$$\begin{aligned}
\text{Total present value} &= \frac{5}{1.04} + \frac{5}{(1.04)^2} + \frac{5}{(1.04)^3} + \cdots \\
&= \frac{5}{1.04}\left(1 + \frac{1}{1.04} + \frac{1}{(1.04)^2} + \cdots\right) \\
&= \frac{5}{1.04}\left(\frac{1}{1 - \frac{1}{1.04}}\right) = \$125.
\end{aligned}$$

Notice how little difference there is between the worth of the bond which pays for 100 years and the one which pays forever.

80. The quantity of cephalexin in the body is given by $Q(t) = Q_0 e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 0.9 hours,

$$\frac{1}{2} = e^{-0.9k}, \quad k = -\frac{1}{0.9}\ln\frac{1}{2} \approx 0.8.$$

(a) After 6 hours

$$Q = Q_0 e^{-k(6)} \approx Q_0 e^{-0.8(6)} = Q_0(0.01).$$

Thus, the percentage of the cephalexin that remains after 6 hours $\approx 1\%$.

(b)

$$\begin{aligned} Q_1 &= 250 \\ Q_2 &= 250 + 250(0.01) \\ Q_3 &= 250 + 250(0.01) + 250(0.01)^2 \\ Q_4 &= 250 + 250(0.01) + 250(0.01)^2 + 250(0.01)^3 \end{aligned}$$

(c)

$$\begin{aligned} Q_3 &= \frac{250(1-(0.01)^3)}{1-0.01} \\ &\approx 252.5 \\ Q_4 &= \frac{250(1-(0.01)^4)}{1-0.01} \\ &\approx 252.5 \end{aligned}$$

Thus, by the time a patient has taken three cephalexin tablets, the quantity of drug in the body has leveled off to 252.5 mg.

(d) Looking at the answers to part (b) shows that

$$\begin{aligned} Q_n &= 250 + 250(0.01) + 250(0.01)^2 + \cdots + 250(0.01)^{n-1} \\ &= \frac{250(1-(0.01)^n)}{1-0.01}. \end{aligned}$$

(e) In the long run, $n \to \infty$. So,

$$Q = \lim_{n\to\infty} Q_n = \frac{250}{1-0.01} = 252.5.$$

81. A person should expect to pay the present value of the bond on the day it is bought.

$$\begin{aligned} \text{Present value of first payment} &= \frac{10}{1.04} \\ \text{Present value of second payment} &= \frac{10}{(1.04)^2}, \text{ etc.} \end{aligned}$$

Therefore,

$$\text{Total present value} = \frac{10}{1.04} + \frac{10}{(1.04)^2} + \frac{10}{(1.04)^3} + \cdots.$$

This is a geometric series with $a = \frac{10}{1.04}$ and $x = \frac{1}{1.04}$, so

$$\text{Total present value} = \frac{\frac{10}{1.04}}{1-\frac{1}{1.04}} = £250.$$

82. (a)

$$\begin{aligned} \text{Total amount of money deposited} &= 100 + 92 + 84.64 + \cdots \\ &= 100 + 100(0.92) + 100(0.92)^2 + \cdots \\ &= \frac{100}{1-0.92} = 1250 \quad \text{dollars} \end{aligned}$$

(b) Credit multiplier $= 1250/100 = 12.50$
The 12.50 is the factor by which the bank has increased its deposits, from \$100 to \$1250.

83. (a) (i) Since the number of bacteria doubles every half hour, the number quadruples every hour. Thus

$$\begin{aligned} R_1 &= B_0 \cdot 4 \\ R_2 &= B_0 \cdot 4^2 \\ &\vdots \\ R_n &= B_0 \cdot 4^n. \end{aligned}$$

(ii) Each hour, the number of bacteria is multiplied by a factor a, so

$$F_n = B_0 a^n.$$

The bacteria doubles in number in 10 hours, so

$$F_{10} = 2B_0.$$

Thus,

$$\begin{aligned} B_0 a^{10} &= 2B_0 \\ a &= 2^{1/10}, \end{aligned}$$

so

$$F_n = B_0(2^{1/10})^n = B_0 2^{n/10}.$$

(iii) The ratio is

$$Y_n = \frac{R_n}{F_n} = \frac{B_0 4^n}{B_0 2^{n/10}} = \left(\frac{4}{2^{1/10}}\right)^n = \left(2^{1.9}\right)^n.$$

(b) We want to solve for n making $Y_n = 1{,}000{,}000$:

$$\begin{aligned} \left(2^{1.9}\right)^n &= 1{,}000{,}000 \\ n &= \frac{\ln(1{,}000{,}000)}{\ln(2^{1.9})} = 10.490. \end{aligned}$$

Thus, in about ten and a half hours, there are a million times as many bacteria in the baby formula kept at room temperature.

84. (a) We have:

$$\begin{aligned} s_5 &= \frac{1}{5}\left(\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{5}{5}\right)^2\right) \\ &= \frac{1}{5}\left(\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{25}{25}\right) \\ &= \frac{1}{5} \cdot \frac{55}{25} \\ &= \frac{11}{25} = 0.44. \end{aligned}$$

(b) From the pattern, we have:

$$\begin{aligned} s_n &= \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2. \end{aligned}$$

This can also be written as $s_n = \displaystyle\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$.

(c) We think of s_n as a right-hand Riemann sum by writing it as

$$s_n = \sum_{i=1}^{n} \underbrace{\left(\frac{i}{n}\right)^2}_{f(x_i)} \cdot \underbrace{\frac{1}{n}}_{\Delta x} = \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

where $f(x) = x^2, \Delta x = (1-0)/n = 1/n$, and where $x_1, x_2, \ldots, x_n$ is given by $1/n, 2/n, \ldots, 1$. We see that $x_1, x_2, \ldots$ are evenly-spaced points on the interval from $a = 0$ to $b = 1$, separated by gaps of Δx, as in any standard right-hand sum. Taking limits, we have:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}x^3\bigg|_0^1 = \frac{1}{3}.$$

85. This series converges by the alternating series test, so we can use Theorem 9.9. The n^{th} partial sum of the series is given by

$$S_n = 1 - \frac{1}{6} + \frac{1}{120} - \cdots + \frac{(-1)^{n-1}}{(2n-1)!},$$

so the absolute value of the first term omitted is $1/(2n+1)!$. By Theorem 9.9, we know that the true value of the sum differs from S_n by less than $1/(2n+1)!$. Thus, we want to choose n large enough so that $1/(2n+1)! \leq 0.01$. Substituting $n = 2$ into the expression $1/(2n+1)!$ yields $1/720$ which is less than 0.01, so $S_2 = 1 - (1/6) = 5/6$ approximates the sum to within 0.01 of the actual sum.

86. No. If the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges then, using Theorem 9.2, part 3, we have $\lim_{n\to\infty}(-1)^{n-1}a_n = 0$, which cannot happen if $\lim_{n\to\infty} a_n \neq 0$.

87. Let $b_n = (3/2)^n$. Then $0 \leq b_n \leq 2^n$, and $\sum b_n$ diverges since it is a geometric series with a common ratio greater than one. Now, let $b_n = (1/2)^n$. Then $0 \leq b_n \leq 2^n$, but this time $\sum b_n$ converges because it is a geometric series with a common ratio less than one. Other answers are possible.

88. If $\sum(a_n + b_n)$ converged, then $\sum(a_n + b_n) - \sum a_n = \sum b_n$ would converge by Theorem 9.2. Since we know that $\sum b_n$ does not converge, we conclude that $\sum(a_n + b_n)$ diverges.

89. We have $0 \leq a_n/n \leq a_n$ for all $n \geq 1$. Therefore, since $\sum a_n$ converges, $\sum a_n/n$ converges by the Comparison Test.

90. Since $\sum a_n$ converge, we know that $\lim_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty}(1/a_n)$ does not exist, and it follows that $\sum(1/a_n)$ diverges by Property 3 of Theorem 9.2.

91. There is not enough information to determine whether or not na_n converges. To see that this is the case, note that if $a_n = 1/n^2$, then $\sum na_n = \sum(1/n)$, which diverges. However, if $a_n = 1/n^3$ then $\sum na_n = \sum(1/n^2)$, which converges.

92. We have $a_n + (a_n/2) = (3/2)a_n$, so the series $\sum(a_n + a_n/2)$ converges since it is a constant multiple of the convergent series $\sum a_n$.

93. Since $\sum a_n$ converges, we know that $\lim_{n\to\infty} a_n = 0$. Therefore, we can choose a positive integer N large enough so that $|a_n| \leq 1$ for all $n \geq N$, so we have $0 \leq a_n^2 \leq a_n$ for all $n \geq N$. Thus, by Property 2 of Theorem 9.2, $\sum a_n^2$ converges by comparison with the convergent series $\sum a_n$.

94. The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{2}{n}$$

diverges by Theorem 9.2 and the fact that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} 0 = 0$$

converges. But $\sum_{n=1}^{\infty} -\frac{1}{n}$ diverges by Theorem 9.2 and the fact that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Thus, if $a_n = 1/n$ and $b_n = 1/n$, so that $\sum a_n$ and $\sum b_n$ both diverge, we see that $\sum(a_n + b_n)$ may diverge.

If, on the other hand, $a_n = 1/n$ and $b_n = -1/n$, so that $\sum a_n$ and $\sum b_n$ both diverge, we see that $\sum(a_n + b_n)$ may converge.

Therefore, if $\sum a_n$ and $\sum b_n$ both diverge, we cannot tell whether $\sum(a_n + b_n)$ converges or diverges. Thus the statement is true.

95. (a) See Figure 9.9.

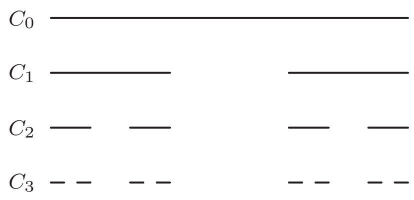


Figure 9.9

(b) At the first stage, we remove one segment of length $1/3$

$$\text{At stage 1, length removed } = \frac{1}{3}.$$

At the second stage, we remove $2 = 2^1$ segments of length $\frac{1}{3}\left(\frac{1}{3}\right) = \frac{1}{3^2}$ each.

$$\text{At stage 2, length removed } = \frac{2^1}{3^2}.$$

At the third stage, we remove $4 = 2^2$ segments of length $\frac{1}{3}\left(\frac{1}{9}\right) = \frac{1}{3^3}$ each.

$$\text{At stage 3, length removed } = \frac{2^2}{3^3}.$$

The same reasoning shows that

$$\text{At stage } n\text{, length removed } = \frac{2^{n-1}}{3^n} = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}.$$

To find the length of all segments removed by the n^{th} stage, we add

$$\text{Total length } = \frac{1}{3} + \frac{1}{3}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right)^2 + \cdots + \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}.$$

(c) Using part (b), we have an infinite geometric series with $a = 1/3$ and $x = 2/3$. Therefore

$$\text{Total length removed } = \sum_{i=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^i = \frac{1/3}{1-2/3} = 1.$$

Notice that even though segments of total length 1 are removed from the initial segment of length 1, there is still an infinite number of points remaining. The remaining points form a famous fractal, called the Cantor Set.

96. Using a right-hand sum, we have

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n.$$

If a computer could add a million terms in one second, then it could add

$$60\frac{\text{sec}}{\text{min}} \cdot 60\frac{\text{min}}{\text{hour}} \cdot 24\frac{\text{hour}}{\text{day}} \cdot 365\frac{\text{days}}{\text{year}} \cdot 1 \text{ million } \frac{\text{terms}}{\text{sec}}$$

terms per year. Thus,

$$1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} < 1 + \ln n = 1 + \ln(60 \cdot 60 \cdot 24 \cdot 365 \cdot 10^6) \approx 32.082 < 33.$$

So the sum after one year is about 32.

97. We want to estimate $\displaystyle\sum_{k=1}^{100{,}000} \frac{1}{k}$ using left and right Riemann sum approximations to $f(x) = 1/x$ on the interval $1 \le x \le 100{,}000$. Figure 9.10 shows a left Riemann sum approximation with 99,999 terms. Since $f(x)$ is decreasing, the left Riemann sum overestimates the area under the curve. Figure 9.10 shows that the first term in the sum is $f(1)\cdot 1$ and the last is $f(99{,}999)\cdot 1$, so we have

$$\int_1^{100{,}000} \frac{1}{x}\,dx < \text{ LHS } = f(1)\cdot 1 + f(2)\cdot 1 + \cdots + f(99{,}999)\cdot 1.$$

Since $f(x) = 1/x$, the left Riemann sum is

$$\text{LHS } = \frac{1}{1}\cdot 1 + \frac{1}{2}\cdot 1 + \cdots + \frac{1}{99{,}999}\cdot 1 = \sum_{k=1}^{99{,}999}\frac{1}{k},$$

so

$$\int_1^{100{,}000} \frac{1}{x}\,dx < \sum_{k=1}^{99{,}999}\frac{1}{k}.$$

Since we want the sum to go $k = 100{,}000$ rather than $k = 99{,}999$, we add $1/100{,}000$ to both sides:

$$\int_1^{100{,}000} \frac{1}{x}\,dx + \frac{1}{100{,}000} < \sum_{k=1}^{99{,}999}\frac{1}{k} + \frac{1}{100{,}000} = \sum_{k=1}^{100{,}000}\frac{1}{k}.$$

The left Riemann sum has therefore given us an underestimate for our sum. We now use the right Riemann sum in Figure 9.11 to get an overestimate for our sum.

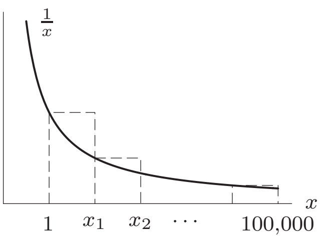


Figure 9.10

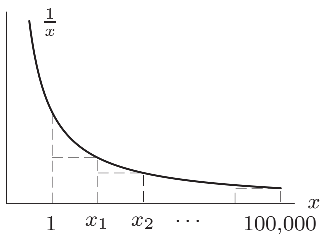


Figure 9.11

The right Riemann sum again has 99,999 terms, but this time the sum underestimates the area under the curve. Figure 9.11 shows that the first rectangle has area $f(2)\cdot 1$ and the last $f(100{,}000)\cdot 1$, so we have

$$\text{RHS } = f(2)\cdot 1 + f(3)\cdot 1 + \cdots + f(100{,}000)\cdot 1 < \int_1^{100{,}000}\frac{1}{x}\,dx.$$

Since $f(x) = 1/x$, the right Riemann sum is

$$\text{RHS } = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \cdots + \frac{1}{100{,}000}\cdot 1 = \sum_{k=2}^{100{,}000}\frac{1}{k}.$$

So

$$\sum_{k=2}^{100{,}000}\frac{1}{k} < \int_1^{100{,}000}\frac{1}{x}\,dx.$$

Since we want the sum to start at $k = 1$, we add 1 to both sides:

$$\sum_{k=1}^{100{,}000}\frac{1}{k} = \frac{1}{1} + \sum_{k=2}^{100{,}000}\frac{1}{k} < 1 + \int_1^{100{,}000}\frac{1}{x}\,dx.$$

Putting these under- and overestimates together, we have

$$\int_1^{100{,}000} \frac{1}{x}\,dx + \frac{1}{100{,}000} < \sum_{k=1}^{100{,}000} \frac{1}{k} < 1 + \int_1^{100{,}000} \frac{1}{x}\,dx.$$

Since $\displaystyle\int_1^{100{,}000} \frac{1}{x}\,dx = \ln 100{,}000 - \ln 1 = 11.513$, we have

$$11.513 < \sum_{k=1}^{100{,}000} \frac{1}{k} < 12.513.$$

Therefore we have $\displaystyle\sum_{k=1}^{100{,}000} \frac{1}{k} \approx 12$.

98. The argument is false. Property 1 of Theorem 9.2 only applies to convergent series. In addition, by the limits comparison test with $b_n = 1/n^2$, the series converges.

PROJECTS FOR CHAPTER NINE

1. (a) The undiluted strength is 2 mg/ml, so the concentration of a 10-fold dilution is 0.2 mg/ml and the concentration of a 100-fold dilution is 0.02 mg/ml. Since each of the first 11 steps lasts 15 minutes $= 0.25$ hour, the volume infused at each step is given, in ml, by

$$\text{Volume} = \text{Rate} \times \text{Time} = (\text{Rate, in ml/hr}) \times (0.25 \text{ hr}).$$

The dose administered is given, in mg, by

$$\text{Dose} = \text{Concentration} \times \text{Volume} = (\text{Concentration, in mg/ml}) \times (\text{Rate, in ml/hr}) \times (0.25 \text{ hr}).$$

Thus in the first step, the concentration is 0.02 mg/ml and

$$\text{Dose administered} = 0.02 \times 2.5 \times 0.25 = 0.0125 \text{ mg}.$$

At the second step,

$$\text{Dose administered} = 0.02 \times 5.0 \times 0.25 = 0.0250 \text{ mg},$$

and so on. See the first 11 rows in Table 9.1.

(b) Using the spreadsheet to find the cumulative dose given in the first 11 steps, we find that 25.5875 mg have been administered. To reach the target dose of 500 mg,

$$\text{Dose at } 12^{\text{th}} \text{ step} = 500 - 25.5875 = 474.4125 \text{ mg}.$$

The time required to deliver the 12^{th} dose is

$$\text{Time} = \frac{\text{Dose}}{\text{Concentration} \times \text{Rate}} = \frac{474.4125 \text{ mg}}{2 \times 51.2 \text{ mg/hr}} = 4.633 \text{ hours} = 278 \text{ minutes}.$$

The volume infused at the last step is $4.633 \cdot 51.2 = 237.21$ ml. See the last row in Table 9.1.

Table 9.1

Step	Solution	Concentration (mg/ml)	Rate (ml/hr)	Time (min)	Volume infused per step (ml)	Dose infused per step (mg)	Cumulative dose (mg)	Ratio of dose administered in this step to dose administered in previous step
1	100-fold dilution	0.02	2.5	15	0.63	0.0125	0.0125	
2	100-fold dilution	0.02	5.0	15	1.25	0.0250	0.0375	2
3	100-fold dilution	0.02	10.0	15	2.50	0.0500	0.0875	2
4	100-fold dilution	0.02	20.0	15	5.00	0.1000	0.1875	2
5	10-fold dilution	0.2	4.0	15	1.00	0.2000	0.3875	2
6	10-fold dilution	0.2	8.0	15	2.00	0.4000	0.7875	2
7	10-fold dilution	0.2	16.0	15	4.00	0.8000	1.5875	2
8	10-fold dilution	0.2	32.0	15	8.00	1.6000	3.1875	2
9	undiluted	2	6.4	15	1.60	3.2000	6.3875	2
10	undiluted	2	12.8	15	3.20	6.4000	12.7875	2
11	undiluted	2	25.6	15	6.40	12.8000	25.5875	2
12	undiluted	2	51.2	278	237.21	474.4125	500	

(c) The dose starts at 0.0125 mg and increases by a factor of 2 at each step. Hence if D_n is the dose at step n, we have a geometric series with terms $D_1 = 0.0125$, $D_2 = 0.0125 \cdot 2, \ldots$, and $D_n = 0.0125 \cdot 2^{n-1}$. Thus,

$$\begin{aligned}\text{Total dose in 11 steps } &= 0.0125 + 0.0125 \cdot 2 + \cdots + 0.0125 \cdot 2^{10} \\ &= 0.0125 \frac{(2^{11} - 1)}{2 - 1} = 25.5875 \text{ mg.}\end{aligned}$$

which agrees with the value computed in the spreadsheet in part (a).

(d) The first 11 fifteen minute doses take 2 hours 45 minutes. Adding the 4.63 hours, or 4 hours and 38 minutes, for the 12^{th} dose, we see that to administer the full target dose of 500 mg requires 7 hours 23 minutes, about 7 and a half hours.

2. (a) To show f is decreasing for $x > 1$, we look at $f'(x)$:

$$f'(x) = n(n+1)x^{n-1} - n(n+1)x^n = n(n+1)x^{n-1}(1 - x).$$

Thus, for $x > 1$, we have $f'(x) < 0$, so f is decreasing. Since $f(1) = 1$, this means $f(x) < 1$ for $x > 1$. Factoring x^n out of $f(x)$, we get

$$f(x) = (n+1)x^n - nx^{n+1} = x^n(n + 1 - nx) < 1.$$

(b) We simplify the value of x

$$x = \frac{1 + 1/n}{1 + 1/(n+1)} = \frac{(n+1)/n}{(n+2)/(n+1)} = \frac{(n+1)^2}{n(n+2)}.$$

Before substituting into $x^n(n + 1 - nx) < 1$, we calculate

$$\begin{aligned}n + 1 - nx &= n + 1 - n\frac{(n+1)^2}{n(n+2)} \\ &= \frac{(n+1)(n+2) - (n+1)^2}{n+2} \\ &= \frac{(n+1)(n+2-(n+1))}{n+2} = \frac{n+1}{n+2}.\end{aligned}$$

Thus, substituting into the inequality from part (a), $x^n(n + 1 - nx) < 1$, gives

$$x^n\left(\frac{n+1}{n+2}\right) < 1.$$

(c) We want to show $s_n < s_{n+1}$. Since $s_n = (1+1/n)^n$ and $s_{n+1} = (1+1/(n+1))^{n+1}$, using the definition of x, we have

$$\begin{aligned}\frac{s_n}{s_{n+1}} &= \frac{(1+1/n)^n}{(1+1/(n+1))^n} \cdot \frac{1}{(1+1/(n+1))} \\ &= x^n \left(\frac{n+1}{n+2}\right).\end{aligned}$$

Thus, by part (b), we have

$$\frac{s_n}{s_{n+1}} = x^n \left(\frac{n+1}{n+2}\right) < 1,$$

so

$$s_n < s_{n+1}.$$

Thus, the sequence is increasing.

(d) Substituting $x = 1 + 1/2n$ into the inequality from part (a) gives

$$\left(1+\frac{1}{2n}\right)^n \left(n+1-n\left(1+\frac{1}{2n}\right)\right) = \left(1+\frac{1}{2n}\right)^n \left(1-\frac{1}{2}\right) = \frac{1}{2}\left(1+\frac{1}{2n}\right)^n < 1.$$

Thus

$$\left(1+\frac{1}{2n}\right)^n < 2.$$

(e) When we square this inequality, we get

$$\left(1+\frac{1}{2n}\right)^{2n} < 4,$$

that is, for all n

$$s_{2n} < 4.$$

Thus, the even terms are bounded above by 4. Because we have shown the sequence is increasing, for each odd term, we have

$$s_{2n-1} < 2_{2n} < 4,$$

so the odd terms are also bounded above by 4. Since all terms are bounded below by 0, the sequence is bounded.

(f) From parts (c) and (e), we know that the sequence is increasing and bounded, and therefore, by Theorem 9.1, it has a limit.

3. (a) (i) p^2

(ii) There are two ways to do this. One way is to compute your opponent's probability of winning two in a row, which is $(1-p)^2$. Then the probability that neither of you win the next points is:

$$\begin{aligned}1 - (&\text{Probability you win next two} + \text{Probability opponent wins next two}) \\ &= 1-(p^2+(1-p)^2) \\ &= 1-(p^2+1-2p+p^2) \\ &= 2p^2-2p \\ &= 2p(1-p).\end{aligned}$$

The other way to compute this is to observe either you win the first point and lose the second or vice versa. Both have probability $p(1-p)$, so the probability you split the points is $2p(1-p)$.

(iii)

$$\begin{aligned}\text{Probability} &= (\text{Probability of splitting next two}) \cdot (\text{Probability of winning two after that}) \\ &= 2p(1-p)p^2\end{aligned}$$

(iv)

$$\begin{aligned}\text{Probability} &= (\text{Probability of winning next two}) + (\text{Probability of splitting next two,} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{winning two after that}) \\ &= p^2 + 2p(1-p)p^2\end{aligned}$$

(v) The probability is:

$$\begin{aligned}w &= (\text{Probability of winning first two}) \\ &\quad + (\text{Probability of splitting first two})\cdot(\text{Probability of winning next two}) \\ &\quad + (\text{Prob. of split. first two})\cdot(\text{Prob. of split. next two})\cdot(\text{Prob. of winning next two}) \\ &\quad + \cdots \\ &= p^2 + 2p(1-p)p^2 + (2p(1-p))^2\, p^2 + \cdots.\end{aligned}$$

This is an infinite geometric series with a first term of p^2 and a ratio of $2p(1-p)$. Therefore the probability of winning is

$$w = \frac{p^2}{1-2p(1-p)}.$$

(vi) For $p = 0.5$, $w = \frac{(0.5)^2}{1-2(0.5)(1-(0.5))} = 0.5$. This is what we would expect. If you and your opponent are equally likely to score the next point, you and your opponent are equally likely to win the next game.

For $p = 0.6$, $w = \frac{(0.6)^2}{1-2(0.6)(0.4)} = 0.69$. Here your probability of winning the next point has been magnified to a probability 0.69 of winning the game. Thus it gives the better player an advantage to have to win by two points, rather than the "sudden death" of winning by just one point. This makes sense: when you have to win by two, the stronger player always gets a second chance to overcome the weaker player's winning the first point on a "fluke."

For $p = 0.7$, $w = \frac{(0.7)^2}{1-2(0.7)(0.3)} = 0.84$. Again, the stronger player's probability of winning is magnified.

For $p = 0.4$, $w = \frac{(0.4)^2}{1-2(0.4)(0.6)} = 0.31$. We already computed that for $p = 0.6$, $w = 0.69$. Thus the value for w when $p = 0.4$, should be the same as the probability of your opponent winning for $p = 0.6$, namely $1 - 0.69 = 0.31$.

(b) (i)

$$\begin{aligned}S &= (\text{Prob. you score first point}) \\ &\qquad + (\text{Prob. you lose first point, your opponent loses the next,} \\ &\qquad\quad \text{you win the next}) \\ &\qquad + (\text{Prob. you lose a point, opponent loses, you lose,} \\ &\qquad\quad \text{opponent loses, you win}) \\ &\qquad + \cdots \\ &= (\text{Prob. you score first point}) \\ &\qquad + (\text{Prob. you lose})\cdot(\text{Prob. opponent loses})\cdot(\text{Prob. you win}) \\ &\qquad + (\text{Prob. you lose})\cdot(\text{Prob. opponent loses})\cdot(\text{Prob. you lose}) \\ &\qquad\quad \cdot(\text{Prob. opponent loses})\cdot(\text{Prob. you win}) + \cdots \\ &= p + (1-p)(1-q)p + ((1-p)(1-q))^2\, p + \cdots \\ &= \frac{p}{1-(1-p)(1-q)}\end{aligned}$$

(ii) Since S is your probability of winning the next point, we can use the formula computed in part (v) of (a) for winning two points in a row, thereby winning the game:

$$w = \frac{S^2}{1 - 2S(1-S)}.$$

- When $p = 0.5$ and $q = 0.5$,

$$S = \frac{0.5}{1-(0.5)(0.5)} = 0.67.$$

Therefore

$$w = \frac{S^2}{1-2S(1-S)} = \frac{(0.67)^2}{1-2(0.67)(1-0.67)} = 0.80.$$

- When $p = 0.6$ and $q = 0.5$,

$$S = \frac{0.6}{1-(0.4)(0.5)} = 0.75 \quad \text{and} \quad w = \frac{(0.75)^2}{1-2(0.75)(1-0.75)} = 0.9.$$

4. (a) (i) Amount remaining $=$ Amount taken $\cdot x = 5 \cdot 8x$ mg.

(ii) Amount remaining $=$ Amount remaining from first dose $+$ Amount of second dose $= 5 \cdot 8x + 5 \cdot 7 = 5(8x + 7)$ mg.

(iii) Amount remaining $=$ Sum of amounts remaining from previous doses $= 5 \cdot 8x^2 + 5 \cdot 7x + 5 \cdot 6 = 5(8x^2 + 7x + 6)$ mg.

(iv) Amount remaining = $5(8x^7 + 7x^6 + 6x^5 + \cdots + 2x + 1)$ mg.

(v) Amount remaining = $5(8x^7 + 7x^6 + 6x^5 + \cdots + 2x + 1)x$ mg.

(vi) Amount remaining = $5(8x^7 + 7x^6 + 6x^5 + \cdots + 2x + 1)x^n$ mg.

(b) Notice that the sum looks like a geometric series, except that each term has been differentiated. If we differentiate the finite geometric series

$$S_9 = 1 + x + x^2 + \cdots + x^8 = \frac{1-x^9}{1-x}$$

we have

$$\frac{dS_9}{dx} = 0 + 1 + 2x + \cdots + 8x^7 = \frac{d}{dx}\left(\frac{1-x^9}{1-x}\right) = \frac{-9x^8(1-x) - (1-x^9)(-1)}{(1-x)^2} = \frac{8x^9 - 9x^8 + 1}{(1-x)^2}.$$

Thus, we see that

$$T = 8x^7 + 7x^6 + 6x^5 + \cdots + 2x + 1 = \frac{8x^9 - 9x^8 + 1}{(1-x)^2}.$$

(c) Let n be the number of days since the eighth dose was taken. Then

$$\text{Amount of prednisone in the body } = 5(8x^7 + 7x^6 + 6x^5 + \cdots + 2x + 1)x^n = 5\frac{8x^9 - 9x^8 + 1}{(1-x)^2}x^n \text{ mg.}$$

If the half-life of prednisone is 24 hours, then $x = 1/2 = 0.5$, so

$$\text{Amount of prednisone in the body after } n \text{ days } = 5\frac{8(0.5)^9 - 9(0.5)^8 + 1}{(1-(0.5))^2}(0.5)^n = 19.6094(0.5)^n \text{ mg.}$$

We want this to be 3% of one tablet, that is $0.03 \times 5 = 0.15$ mg, so we need to solve for n:

$$19.6094(0.5)^n = 0.15.$$

This leads to

$$(0.5)^n = \frac{0.15}{19.6094},$$

so

$$n = \frac{\ln(0.15/19.6094)}{\ln(0.5)} = 7.037 \approx 7 \text{ days.}$$

(d) The amount of prednisone in the body is the sum of the amounts remaining from the previous doses. The first dose of n tablets was $(n-1)$ days before, so

$$\text{Number of tablets remaining from first dose } = nx^{n-1}.$$

The dose of $(n-1)$ tablets was $(n-2)$ days before, so

$$\text{Number of tablets remaining from second dose } = (n-1)x^{n-2}.$$

Adding similar terms, ending with the last dose of 1 table which just occurred, gives the total number of tablets remaining:

$$T_n = nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x + 1.$$

To find a closed form for the sum, we differentiate the finite geometric series

$$S_{n+1} = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x},$$

giving

$$\frac{dS_{n+1}}{dx} = 0 + 1 + 2x + \cdots + nx^{n-1} = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right) = \frac{-(n+1)x^n(1-x) - (1-x^{n+1})(-1)}{(1-x)^2}.$$

Thus, we see that

$$T_n = nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x + 1 = \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}.$$

CHAPTER TEN

Solutions for Section 10.1

Exercises

1. Let $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$. Then $f(0) = 1$.

$$\begin{aligned} f'(x) &= 1!(1-x)^{-2} & f'(0) &= 1!, \\ f''(x) &= 2!(1-x)^{-3} & f''(0) &= 2!, \\ f'''(x) &= 3!(1-x)^{-4} & f'''(0) &= 3!, \\ f^{(4)}(x) &= 4!(1-x)^{-5} & f^{(4)}(0) &= 4!, \\ f^{(5)}(x) &= 5!(1-x)^{-6} & f^{(5)}(0) &= 5!, \\ f^{(6)}(x) &= 6!(1-x)^{-7} & f^{(6)}(0) &= 6!, \\ f^{(7)}(x) &= 7!(1-x)^{-8} & f^{(7)}(0) &= 7!. \end{aligned}$$

$$\begin{aligned} P_3(x) &= 1 + x + x^2 + x^3, \\ P_5(x) &= 1 + x + x^2 + x^3 + x^4 + x^5, \\ P_7(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7. \end{aligned}$$

2. Let $\dfrac{1}{1+x} = (1+x)^{-1}$. Then $f(0) = 1$.

$$\begin{aligned} f'(x) &= -1!(1+x)^{-2} & f'(0) &= -1, \\ f''(x) &= 2!(1+x)^{-3} & f''(0) &= 2!, \\ f'''(x) &= -3!(1+x)^{-4} & f'''(0) &= -3!, \\ f^{(4)}(x) &= 4!(1+x)^{-5} & f^{(4)}(0) &= 4!, \\ f^{(5)}(x) &= -5!(1+x)^{-6} & f^{(5)}(0) &= -5!, \\ f^{(6)}(x) &= 6!(1+x)^{-7} & f^{(6)}(0) &= 6!, \\ f^{(7)}(x) &= -7!(1+x)^{-8} & f^{(7)}(0) &= -7!, \\ f^{(8)}(x) &= 8!(1+x)^{-9} & f^{(8)}(0) &= 8!. \end{aligned}$$

$$\begin{aligned} P_4(x) &= 1 - x + x^2 - x^3 + x^4, \\ P_6(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6, \\ P_8(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8. \end{aligned}$$

3. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Then $f(0) = 1$, and

$$\begin{aligned} f'(x) &= \tfrac{1}{2}(1+x)^{-1/2} & f'(0) &= \tfrac{1}{2}, \\ f''(x) &= -\tfrac{1}{4}(1+x)^{-3/2} & f''(0) &= -\tfrac{1}{4}, \\ f'''(x) &= \tfrac{3}{8}(1+x)^{-5/2} & f'''(0) &= \tfrac{3}{8}, \\ f^{(4)}(x) &= -\tfrac{15}{16}(1+x)^{-7/2} & f^{(4)}(0) &= -\tfrac{15}{16}. \end{aligned}$$

Thus,

$$\begin{aligned}
P_2(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2, \\
P_3(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3, \\
P_4(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4.
\end{aligned}$$

4. Let $f(x) = \sqrt[3]{1-x} = (1-x)^{1/3}$. Then $f(0) = 1$, and

$$\begin{array}{ll}
f'(x) = -\frac{1}{3}(1-x)^{-2/3} & f'(0) = -\frac{1}{3}, \\
f''(x) = -\frac{2}{3^2}(1-x)^{-5/3} & f''(0) = -\frac{2}{3^2}, \\
f'''(x) = -\frac{10}{3^3}(1-x)^{-8/3} & f'''(0) = -\frac{10}{3^3}, \\
f^{(4)}(x) = -\frac{80}{3^4}(1-x)^{-11/3} & f^{(4)}(0) = -\frac{80}{3^4}.
\end{array}$$

Then,

$$\begin{aligned}
P_2(x) &= 1 - \frac{1}{3}x - \frac{1}{2!}\frac{2}{3^2}x^2 = 1 - \frac{1}{3}x - \frac{1}{9}x^2, \\
P_3(x) &= P_2(x) - \frac{1}{3!}\left(\frac{10}{3^3}\right)x^3 = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3, \\
P_4(x) &= P_3(x) - \frac{1}{4!}\frac{80}{3^4}x^4 = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 - \frac{10}{243}x^4.
\end{aligned}$$

5. Let $f(x) = \cos x$. Then $f(0) = \cos(0) = 1$, and

$$\begin{array}{ll}
f'(x) = -\sin x & f'(0) = 0, \\
f''(x) = -\cos x & f''(0) = -1, \\
f'''(x) = \sin x & f'''(0) = 0, \\
f^{(4)}(x) = \cos x & f^{(4)}(0) = 1, \\
f^{(5)}(x) = -\sin x & f^{(5)}(0) = 0, \\
f^{(6)}(x) = -\cos x & f^{(6)}(0) = -1.
\end{array}$$

Thus,

$$\begin{aligned}
P_2(x) &= 1 - \frac{x^2}{2!}, \\
P_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \\
P_6(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.
\end{aligned}$$

6. Let $f(x) = \ln(1+x)$. Then $f(0) = \ln 1 = 0$, and

$$\begin{array}{ll}
f'(x) = (1+x)^{-1} & f'(0) = 1, \\
f''(x) = (-1)(1+x)^{-2} & f''(0) = -1, \\
f'''(x) = 2(1+x)^{-3} & f'''(0) = 2, \\
f^{(4)}(x) = -3!(1+x)^{-4} & f^{(4)}(0) = -3!, \\
f^{(5)}(x) = 4!(1+x)^{-5} & f^{(5)}(0) = 4!, \\
f^{(6)}(x) = -5!(1+x)^{-6} & f^{(6)}(0) = -5!, \\
f^{(7)}(x) = 6!(1+x)^{-7} & f^{(7)}(0) = 6!, \\
f^{(8)}(x) = -7!(1+x)^{-8} & f^{(8)}(0) = -7!, \\
f^{(9)}(x) = 8!(1+x)^{-9} & f^{(9)}(0) = 8!.
\end{array}$$

So,

$$\begin{aligned}
P_5(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5},\\
P_7(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7},\\
P_9(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9}.
\end{aligned}$$

7. Let $f(x) = \arctan x$. Then $f(0) = \arctan 0 = 0$, and

$$\begin{aligned}
f'(x) &= 1/(1+x^2) = (1+x^2)^{-1} && f'(0) = 1,\\
f''(x) &= (-1)(1+x^2)^{-2}2x && f''(0) = 0,\\
f'''(x) &= 2!(1+x^2)^{-3}2^2x^2 + (-1)(1+x^2)^{-2}2 && f'''(0) = -2,\\
f^{(4)}(x) &= -3!(1+x^2)^{-4}2^3x^3 + 2!(1+x^2)^{-3}2^3x &&\\
&\quad + 2!(1+x^2)^{-3}2^2x && f^{(4)}(0) = 0.
\end{aligned}$$

Therefore,

$$P_3(x) = P_4(x) = x - \frac{1}{3}x^3.$$

8. Let $f(x) = \tan x$. So $f(0) = \tan 0 = 0$, and

$$\begin{aligned}
f'(x) &= 1/\cos^2 x && f'(0) = 1,\\
f''(x) &= 2\sin x/\cos^3 x && f''(0) = 0,\\
f'''(x) &= (2/\cos^2 x) + (6\sin^2 x/\cos^4 x) && f'''(0) = 2,\\
f^{(4)}(x) &= (16\sin x/\cos^3 x) + (24\sin^3 x/\cos^5 x) && f^{(4)}(0) = 0.
\end{aligned}$$

Thus,

$$P_3(x) = P_4(x) = x + \frac{x^3}{3}.$$

9. Let $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$. Then $f(0) = 1$.

$$\begin{aligned}
f'(x) &= -\tfrac{1}{2}(1+x)^{-3/2} && f'(0) = -\tfrac{1}{2},\\
f''(x) &= \tfrac{3}{2^2}(1+x)^{-5/2} && f''(0) = \tfrac{3}{2^2},\\
f'''(x) &= -\tfrac{3\cdot 5}{2^3}(1+x)^{-7/2} && f'''(0) = -\tfrac{3\cdot 5}{2^3},\\
f^{(4)}(x) &= \tfrac{3\cdot 5\cdot 7}{2^4}(1+x)^{-9/2} && f^{(4)}(0) = \tfrac{3\cdot 5\cdot 7}{2^4}
\end{aligned}$$

Then,

$$\begin{aligned}
P_2(x) &= 1 - \frac{1}{2}x + \frac{1}{2!}\frac{3}{2^2}x^2 = 1 - \frac{1}{2}x + \frac{3}{8}x^2,\\
P_3(x) &= P_2(x) - \frac{1}{3!}\frac{3\cdot 5}{2^3}x^3 = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3,\\
P_4(x) &= P_3(x) + \frac{1}{4!}\frac{3\cdot 5\cdot 7}{2^4}x^4 = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.
\end{aligned}$$

10. Let $f(x) = (1+x)^p$.

(a) Suppose that $p = 0$. Then $f(x) = 1$ and $f^{(k)}(x) = 0$ for any $k \geq 1$. Thus $P_2(x) = P_3(x) = P_4(x) = 1$.
(b) If $p = 1$ then $f(x) = 1 + x$, so

$$\begin{aligned}
f(0) &= 1,\\
f'(x) &= 1,\\
f^{(k)}(x) &= 0 \qquad k \geq 2.
\end{aligned}$$

Thus $P_2(x) = P_3(x) = P_4(x) = 1 + x$.

(c) In general:

$$\begin{aligned}
f(x) &= (1+x)^p, \\
f'(x) &= p(1+x)^{p-1}, \\
f''(x) &= p(p-1)(1+x)^{p-2}, \\
f'''(x) &= p(p-1)(p-2)(1+x)^{p-3}, \\
f^{(4)}(x) &= p(p-1)(p-2)(p-3)(1+x)^{p-4}.
\end{aligned}$$

$$\begin{aligned}
f(0) &= 1, \\
f'(0) &= p, \\
f''(0) &= p(p-1), \\
f'''(0) &= p(p-1)(p-2), \\
f^{(4)}(0) &= p(p-1)(p-2)(p-3).
\end{aligned}$$

$$\begin{aligned}
P_2(x) &= 1 + px + \frac{p(p-1)}{2}x^2, \\
P_3(x) &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3, \\
P_4(x) &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 \\
&\quad + \frac{p(p-1)(p-2)(p-3)}{24}x^4.
\end{aligned}$$

11. Let $f(x) = \sqrt{1-x} = (1-x)^{1/2}$. Then $f'(x) = -\frac{1}{2}(1-x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1-x)^{-3/2}$, $f'''(x) = -\frac{3}{8}(1-x)^{-5/2}$. So $f(0) = 1$, $f'(0) = -\frac{1}{2}$, $f''(0) = -\frac{1}{4}$, $f'''(0) = -\frac{3}{8}$, and

$$\begin{aligned}
P_3(x) &= 1 - \frac{1}{2}x - \frac{1}{4}\frac{1}{2!}x^2 - \frac{3}{8}\frac{1}{3!}x^3 \\
&= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16}.
\end{aligned}$$

12. Let $f(x) = e^x$. Since $f^{(k)}(x) = e^x = f(x)$ for all $k \geq 1$, the Taylor polynomial of degree 4 for $f(x) = e^x$ about $x = 1$ is

$$\begin{aligned}
P_4(x) &= e^1 + e^1(x-1) + \frac{e^1}{2!}(x-1)^2 + \frac{e^1}{3!}(x-1)^3 + \frac{e^1}{4!}(x-1)^4 \\
&= e\left[1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right].
\end{aligned}$$

13. Let $f(x) = \dfrac{1}{1+x} = (1+x)^{-1}$. Then $f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, $f'''(x) = -6(1+x)^{-4}$, $f^{(4)}(x) = 24(1+x)^{-5}$. So $f(2) = \frac{1}{3}$, $f'(2) = -\frac{1}{3^2}$, $f''(2) = \frac{2}{3^3}$, $f'''(2) = -\frac{6}{3^4}$, and $f^{(4)}(2) = \frac{24}{3^5}$. Therefore,

$$\begin{aligned}
P_4(x) &= \frac{1}{3} - \frac{1}{3^2}(x-2) + \frac{2}{3^3}\frac{1}{2!}(x-2)^2 - \frac{6}{3^4}\frac{1}{3!}(x-2)^3 + \frac{24}{3^5}\frac{1}{4!}(x-2)^4 \\
&= \frac{1}{3}\left(1 - \frac{x-2}{3} + \frac{(x-2)^2}{3^2} - \frac{(x-2)^3}{3^3} + \frac{(x-2)^4}{3^4}\right).
\end{aligned}$$

14. Let $f(x) = \cos x$. $f(\frac{\pi}{2}) = 0$.

$$\begin{aligned}
f'(x) &= -\sin x & f'(\tfrac{\pi}{2}) &= -1, \\
f''(x) &= -\cos x & f''(\tfrac{\pi}{2}) &= 0, \\
f'''(x) &= \sin x & f'''(\tfrac{\pi}{2}) &= 1, \\
f^{(4)}(x) &= \cos x & f^{(4)}(\tfrac{\pi}{2}) &= 0.
\end{aligned}$$

So,

$$\begin{aligned}P_4(x) &= 0 - \left(x - \frac{\pi}{2}\right) + 0 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3\\ &= -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3.\end{aligned}$$

15. Let $f(x) = \sin x$.
Then $f'(x) = \cos x$, $f''(x) = -\sin x$, and $f'''(x) = -\cos x$, so the Taylor polynomial for $\sin x$ of degree three about $x = -\pi/4$ is

$$\begin{aligned}P_3(x) &= \sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)\left(x + \frac{\pi}{4}\right)\\ &+ \frac{-\sin\left(-\frac{\pi}{4}\right)}{2!}\left(x + \frac{\pi}{4}\right)^2 + \frac{-\cos\left(-\frac{\pi}{4}\right)}{3!}\left(x + \frac{\pi}{4}\right)^3\\ &= \frac{\sqrt{2}}{2}\left(-1 + \left(x + \frac{\pi}{4}\right) + \frac{1}{2}\left(x + \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x + \frac{\pi}{4}\right)^3\right).\end{aligned}$$

16. Let $f(x) = \ln(x^2)$. Then $\ln(1^2) = \ln 1 = 0$.
Then $f'(x) = 2x^{-1}$, $f''(x) = -2x^{-2}$, $f'''(x) = 4x^{-3}$, and $f^{(4)}(x) = -12x^{-4}$.
The Taylor polynomial of degree 4 for $f(x) = \ln(x^2)$ about $x = 1$ is

$$\begin{aligned}P_4(x) &= \ln(1^2) + 2\cdot 1^{-1}(x-1) + \frac{-2\cdot 1^{-2}}{2!}(x-1)^2 + \frac{4\cdot 1^{-3}}{3!}(x-1)^3 + \frac{-12\cdot 1^{-4}}{4!}(x-1)^4\\ &= 0 + 2(x-1) - (x-1)^2 + \frac{4}{6}(x-1)^3 - \frac{12}{24}(x-1)^4\\ &= 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{2}(x-1)^4.\end{aligned}$$

Problems

17. The third degree Taylor polynomial of $f(x)$ will have the same terms as the seventh degree polynomial but only up to the x^3 term. So the third degree Taylor polynomial of $f(x)$ is given by

$$P_3(x) = 1 - \frac{x}{3} + \frac{5x^2}{7} + 8x^3.$$

18. Using the fact that

$$f(x) \approx P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

and identifying coefficients with those given for $P_3(x)$, we obtain the following:

(a) $f(0) =$ constant term which equals 2, so $f(0) = 2$.
(b) $f'(0) =$ coefficient of x which equals -1, so $f'(0) = -1$.
(c) $\frac{f''(0)}{2!} =$ coefficient of x^2 which equals $-1/3$, so $f''(0) = -2/3$.
(d) $\frac{f'''(0)}{3!} =$ coefficient of x^3 which equals 2, so $f'''(0) = 12$.

19.

$$\begin{aligned}f(x) &= 4x^2 - 7x + 2 & f(0) &= 2\\ f'(x) &= 8x - 7 & f'(0) &= -7\\ f''(x) &= 8 & f''(0) &= 8,\end{aligned}$$

so $P_2(x) = 2 + (-7)x + \frac{8}{2}x^2 = 4x^2 - 7x + 2$. We notice that $f(x) = P_2(x)$ in this case.

20. $f'(x) = 3x^2 + 14x - 5$, $f''(x) = 6x + 14$, $f'''(x) = 6$. Thus, about $a = 0$,

$$\begin{aligned}P_3(x) &= 1 + \frac{-5}{1!}x + \frac{14}{2!}x^2 + \frac{6}{3!}x^3\\ &= 1 - 5x + 7x^2 + x^3\\ &= f(x).\end{aligned}$$

21. (a) We'll make the following conjecture:
"If $f(x)$ is a polynomial of degree n, i.e.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n,$$

then $P_n(x)$, the n^{th} degree Taylor polynomial for $f(x)$ about $x = 0$, is $f(x)$ itself."

(b) All we need to do is to calculate $P_n(x)$, the n^{th} degree Taylor polynomial for f about $x = 0$ and see if it is the same as $f(x)$.

$$\begin{aligned} f(0) &= a_0; \\ f'(0) &= \left(a_1 + 2a_2 x + \cdots + na_n x^{n-1}\right)\Big|_{x=0} \\ &= a_1; \\ f''(0) &= \left(2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2}\right)\Big|_{x=0} \\ &= 2!a_2. \end{aligned}$$

If we continue doing this, we'll see in general

$$f^{(k)}(0) = k!a_k, \qquad k = 1, 2, 3, \cdots, n.$$

Therefore,

$$\begin{aligned} P_n(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ &= f(x). \end{aligned}$$

22. Since the coefficient of $(x-1)^5$ term of $p(x)$ is given by

$$C_5 = \frac{f^{(5)}(1)}{5!},$$

we know that $f^{(5)}(1) = 5!C_5$. Note that

$$p(x) = \sum_{n=0}^{10} \frac{(x-1)^n}{n!} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \frac{(x-1)^5}{5!} + \cdots + \frac{(x-1)^{10}}{10!},$$

so $C_5 = 1/5!$. Therefore

$$f^{(5)}(1) = \frac{1}{5!}5! = 1.$$

23. Referring to the table, we have:

$$\begin{aligned} P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\ &= -3 + 5x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-1}{4!}x^4 + \frac{4}{5!}x^5 \\ &= -3 + 5x - x^2 - \frac{1}{24}x^4 + \frac{1}{30}x^5. \end{aligned}$$

24. Referring to the formula for $f^{(n)}(0)$, we have:

$$\begin{aligned} f'(0) = f^{(1)}(0) &= -(-2)^1 = 2 \\ f''(0) = f^{(2)}(0) &= -(-2)^2 = -4 \\ f'''(0) = f^{(3)}(0) &= -(-2)^3 = 8 \\ f^{(4)}(0) &= -(-2)^4 = -16 \\ f^{(5)}(0) &= -(-2)^5 = 32. \end{aligned}$$

Since $f(0) = -1$, we have

$$\begin{aligned} P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\ &= -1 + 2x + \frac{-4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{-16}{4!}x^4 + \frac{32}{5!}x^5 \\ &= -1 + 2x - 2x^2 + \frac{4}{3}x^3 - \frac{2}{3}x^4 + \frac{4}{15}x^5. \end{aligned}$$

25. Since $P_2(x)$ is the second degree Taylor polynomial for $f(x)$ about $x = 0$, $P_2(0) = f(0)$, which says $a = f(0)$. Since

$$\frac{d}{dx}P_2(x)\bigg|_{x=0} = f'(0),$$

$b = f'(0)$; and since

$$\frac{d^2}{dx^2}P_2(x)\bigg|_{x=0} = f''(0),$$

$2c = f''(0)$. In other words, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$. So $c < 0$ since f is concave down; $b > 0$ since f is increasing; $a > 0$ since $f(0) > 0$.

26. As we can see from Problem 25, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.
So $a > 0$, $b < 0$ and $c < 0$.

27. As we can see from Problem 25, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.
So $a < 0$, $b > 0$ and $c > 0$.

28. As we can see from Problem 25, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.
So $a < 0$, $b < 0$ and $c > 0$.

29.

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x - \frac{x^3}{3!}}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!}\right) = 1.$$

30.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-(1-\frac{x^2}{2!}+\frac{x^4}{4!})}{x^2} = \lim_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{4!}\right) = \frac{1}{2}.$$

31. For $f(h) = e^h$, $P_4(h) = 1 + h + \dfrac{h^2}{2} + \dfrac{h^3}{3!} + \dfrac{h^4}{4!}$. So

(a)

$$\begin{aligned} \lim_{h\to 0}\frac{e^h - 1 - h}{h^2} &= \lim_{h\to 0}\frac{P_4(h) - 1 - h}{h^2} \\ &= \lim_{h\to 0}\frac{\frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!}}{h^2} \\ &= \lim_{h\to 0}\left(\frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!}\right) \\ &= \frac{1}{2}. \end{aligned}$$

(b)

$$\lim_{h\to 0}\frac{e^h - 1 - h - \frac{h^2}{2}}{h^3} = \lim_{h\to 0}\frac{P_4(h) - 1 - h - \frac{h^2}{2}}{h^3}$$

$$= \lim_{h\to 0} \frac{\frac{h^3}{3!} + \frac{h^4}{4!}}{h^3} = \lim_{h\to 0} \left(\frac{1}{3!} + \frac{h}{4!}\right)$$
$$= \frac{1}{3!} = \frac{1}{6}.$$

Using Taylor polynomials of higher degree would not have changed the results since the terms with higher powers of h all go to zero as $h \to 0$.

32. (a) We use the Taylor polynomial of degree two for f and h about $x = 2$.

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \frac{3}{2}(x-2)^2$$
$$h(x) \approx h(2) + h'(2)(x-2) + \frac{h''(2)}{2!}(x-2)^2 = \frac{7}{2}(x-2)^2$$

Thus, using the fact that near $x = 2$ we can approximate a function by Taylor polynomials

$$\lim_{x\to 2} \frac{f(x)}{h(x)} = \lim_{x\to 2} \frac{\frac{3}{2}(x-2)^2}{\frac{7}{2}(x-2)^2} = \frac{3}{7}.$$

(b) We use the Taylor polynomial of degree two for f and g about $x = 2$.

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \frac{3}{2}(x-2)^2$$
$$g(x) \approx g(2) + g'(2)(x-2) + \frac{g''(2)}{2!}(x-2)^2 = 22(x-2) + \frac{5}{2}(x-2)^2.$$

Thus,

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \left(\frac{\frac{3}{2}(x-2)^2}{22(x-2) + 5(x-2)^2}\right) = \lim_{x\to 2} \left(\frac{\frac{3}{2}(x-2)}{22 + 5(x-2)}\right) = \frac{0}{22} = 0.$$

33. (a) Since the coefficient of the x-term of each f is 1, we know $f_1'(0) = f_2'(0) = f_3'(0) = 1$. Thus, each of the fs slopes upward near 0, and are in the second figure.

The coefficient of the x-term in g_1 and in g_2 is 1, so $g_1'(0) = g_2'(0) = 1$. For g_3 however, $g_3'(0) = -1$. Thus, g_1 and g_2 slope up near 0, but g_3 slopes down. The gs are in the first figure.

(b) Since $g_1(0) = g_2(0) = g_3(0) = 1$, the point A is $(0, 1)$.
Since $f_1(0) = f_2(0) = f_3(0) = 2$, the point B is $(0, 2)$.

(c) Since g_3 slopes down, g_3 is I. Since the coefficient of x^2 for g_1 is 2, we know

$$\frac{g_1''(0)}{2!} = 2 \qquad \text{so} \qquad g_1''(0) = 4.$$

By similar reasoning $g_2''(0) = 2$. Since g_1 and g_2 are concave up, and g_1 has a larger second derivative, g_1 is III and g_2 is II.

Calculating the second derivatives of the fs from the coefficients x^2, we find

$$f_1''(0) = 4 \qquad f_2''(0) = -2 \qquad f_3''(0) = 2.$$

Thus, f_1 and f_3 are concave up, with f_1 having the larger second derivative, so f_1 is III and f_3 is II. Then f_2 is concave down and is I.

34. Let $f(x)$ be a function that has derivatives up to order n at $x = a$. Let

$$P_n(x) = C_0 + C_1(x-a) + \cdots + C_n(x-a)^n$$

be the polynomial of degree n that approximates $f(x)$ about $x = a$. We require that $P_n(x)$ and all of its first n derivatives agree with those of the function $f(x)$ at $x = a$, i.e., we want

$$f(a) = P_n(a),$$
$$f'(a) = P_n'(a),$$
$$f''(a) = P_n''(a),$$
$$\vdots$$
$$f^{(n)}(a) = P_n^{(n)}(a).$$

When we substitute $x = a$ in $P_n(x)$, all the terms except the first drop out, so

$$f(a) = C_0.$$

Now differentiate $P_n(x)$:

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots + nC_n(x-a)^{n-1}.$$

Substitute $x = a$ again, which yields

$$f'(a) = P_n'(a) = C_1.$$

Differentiate $P_n'(x)$:

$$P_n''(x) = 2C_2 + 3 \cdot 2C_3(x-a) + \cdots + n(n-1)C_n(x-a)^{n-2}$$

and substitute $x = a$ again:

$$f''(a) = P_n''(a) = 2C_2.$$

Differentiating and substituting again gives

$$f'''(a) = P_n'''(a) = 3 \cdot 2C_3.$$

Similarly,

$$f^{(k)}(a) = P_n{}^{(k)}(a) = k!C_k.$$

So, $C_0 = f(a)$, $C_1 = f'(a)$, $C_2 = \frac{f''(a)}{2!}$, $C_3 = \frac{f'''(a)}{3!}$, and so on.

If we adopt the convention that $f^{(0)}(a) = f(a)$ and $0! = 1$, then

$$C_k = \frac{f^{(k)}(a)}{k!}, \; k = 0, 1, 2, \cdots, n.$$

Therefore,

$$\begin{aligned} f(x) \approx P_n(x) &= C_0 + C_1(x-a) + C_2(x-a)^2 \cdots + C_n(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

35. (a) The first degree Taylor polynomials $P(x)$ and $Q(x)$ for $f(x)$ and $g(x)$ near $x = 0$ and their product are given by

$$\begin{aligned} P(x) &= 1 + x \\ Q(x) &= 1 + 2x \\ P(x)Q(x) &= 1 + 3x + 2x^2. \end{aligned}$$

(b) The Taylor polynomial $R(x)$ of degree 2 of $h(x) = 1/\big((1-x)(1-2x)\big)$ near $x = 0$ is

$$R(x) = 1 + 3x + 7x^2.$$

(c) The two polynomials $R(x)$ and $P(x)Q(x)$ are not the same. The product of Taylor polynomials of two functions is usually not a Taylor polynomial of the product.

36. (a) The first degree Taylor polynomials $P(x)$ and $Q(x)$ for $f(x)$ and $g(x)$ near $x = 0$ and their product are given by

$$\begin{aligned} P(x) &= f(0) + xf'(0) \\ Q(x) &= g(0) + xg'(0) \\ P(x)Q(x) &= f(0)g(0) + (f'(0)g(0) + f(0)g'(0))x + f'(0)g'(0)x^2. \end{aligned}$$

(b) The Taylor polynomial $R(x)$ of degree 2 of $h(x) = f(x)g(x)$ near $x = 0$ is

$$R(x) = f(0)g(0) + (f'(0)g(0) + f(0)g'(0))x + (f''(0)g(0) + 2f'(0)g'(0) + f(0)g''(0))x^2/2.$$

(c) The two polynomials $R(x)$ and $P(x)Q(x)$ are the same if the coefficients of x^2 are identical, that is

$$f'(0)g'(0) = (f''(0)g(0) + 2f'(0)g'(0) + f(0)g''(0))/2,$$

or

$$f''(0)g(0) + f(0)g''(0) = 0.$$

37. (a) $f(x) = e^{x^2}$.
$f'(x) = 2xe^{x^2}$, $f''(x) = 2(1+2x^2)e^{x^2}$, $f'''(x) = 4(3x+2x^3)e^{x^2}$,
$f^{(4)}(x) = 4(3+6x^2)e^{x^2} + 4(3x+2x^3)2xe^{x^2}$.
The Taylor polynomial about $x = 0$ is

$$\begin{aligned} P_4(x) &= 1 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 \\ &= 1 + x^2 + \frac{1}{2}x^4. \end{aligned}$$

(b) $f(x) = e^x$. The Taylor polynomial of degree 2 is

$$Q_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2.$$

If we substitute x^2 for x in the Taylor polynomial for e^x of degree 2, we will get $P_4(x)$, the Taylor polynomial for e^{x^2} of degree 4:

$$\begin{aligned} Q_2(x^2) &= 1 + x^2 + \frac{1}{2}(x^2)^2 \\ &= 1 + x^2 + \frac{1}{2}x^4 \\ &= P_4(x). \end{aligned}$$

(c) Let $Q_{10}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!}$ be the Taylor polynomial of degree 10 for e^x about $x = 0$. Then

$$\begin{aligned} P_{20}(x) &= Q_{10}(x^2) \\ &= 1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \cdots + \frac{(x^2)^{10}}{10!} \\ &= 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots + \frac{x^{20}}{10!}. \end{aligned}$$

(d) Let $e^x \approx Q_5(x) = 1 + \frac{x}{1!} + \cdots + \frac{x^5}{5!}$. Then

$$\begin{aligned} e^{-2x} &\approx Q_5(-2x) \\ &= 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} \\ &= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5. \end{aligned}$$

38. (a) $\dfrac{\sin t}{t} \approx \dfrac{t - \frac{t^3}{3!}}{t} = 1 - \dfrac{t^2}{6}$

$$\int_0^1 \frac{\sin t}{t}\,dt \approx \int_0^1 \left(1 - \frac{t^2}{6}\right) dt = t - \frac{t^3}{18}\bigg|_0^1 = 0.94444\cdots$$

(b) $\dfrac{\sin t}{t} \approx \dfrac{t - \frac{t^3}{3!} + \frac{t^5}{5!}}{t} = 1 - \dfrac{t^2}{6} + \dfrac{t^4}{120}$

$$\int_0^1 \frac{\sin t}{t}\,dt \approx \int_0^1 \left(1 - \frac{t^2}{6} + \frac{t^4}{120}\right) dt = t - \frac{t^3}{18} + \frac{t^5}{600}\bigg|_0^1 = 0.94611\cdots$$

39. (a) The equation $\sin x = 0.2$ has one solution near $x = 0$ and infinitely many others, one near each multiple of π. See Figure 10.1. The equation $x - \dfrac{x^3}{3!} = 0.2$ has three solutions, one near $x = 0$ and two others. See Figure 10.2.

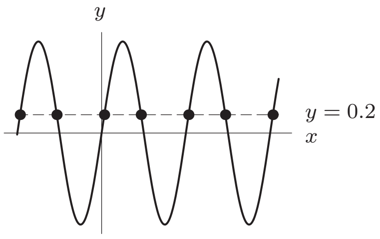


Figure 10.1: Graph of $y = \sin x$ and $y = 0.2$

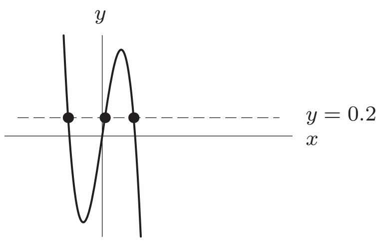


Figure 10.2: Graph of $y = x - \frac{x^3}{3!}$ and $y = 0.2$

(b) Near $x = 0$, the cubic Taylor polynomial $x - x^3/3! \approx \sin x$. Thus, the solutions to the two equations near $x = 0$ are approximately equal. The other solutions are not close. The reason is that $x - x^3/3!$ only approximates $\sin x$ near $x = 0$ but not further away. See Figure 10.3.

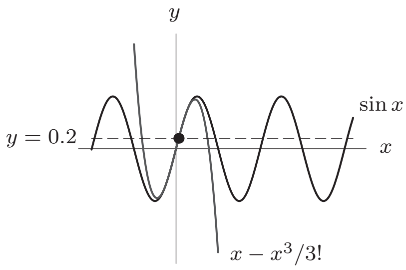


Figure 10.3

40. Changing $\sin\theta$ into θ makes sense if the two values are almost equal. If we measure θ in radians, this is true for values of θ close to zero. (Recall the first degree Taylor polynomial: $\sin\theta \approx \theta$.) In other words, the switch is justified when the pendulum does not swing very far from the vertical.

41. (a) The graphs of $y = \cos x$ and $y = 1 - 0.1x$ cross at $x = 0$ and for another x-value just to the right of $x = 0$. (There are other crossings much further to the right.)

(b) Since

$$\cos x \approx 1 - \frac{x^2}{2}$$

the equation becomes

$$\begin{aligned} 1 - \frac{x^2}{2} &= 1 - 0.1x \\ \frac{x^2}{2} &= 0.1x \\ x &= 0, 0.2. \end{aligned}$$

The solution $x = 0$ is an exact solution to the original equation; $x = 0.2$ is an approximate solution to the original equation.

Strengthen Your Understanding

42. The constant term of the Taylor polynomial is $f(0)$, hence substituting 0 into $f(x)$ gives $\ln(2)$ which is positive.

43. The coefficient of the x term is given by $f'(0)$ and $f'(0) = 1$.

44. An example is $f(x) = \sin x$. Another example is $f(x) = \dfrac{x}{1+x^2}$. Many other examples are possible.

45. An example is $p(x) = 1 + 3(x-1) + (x-1)^3$. Another example is $p(x) = 5 + 3(x-1) + 2(x-1)^2 + (x-1)^3$. Many other examples are possible.

46. False. For example, both $f(x) = x^2$ and $g(x) = x^2 + x^3$ have $P_2(x) = x^2$.

47. False. The approximation $\sin\theta \approx \theta - \theta^3/3!$ holds for θ in radians, not degrees.

48. False. $P_2(x) = f(5) + f'(5)(x-5) + (f''(5)/2)(x-5)^2 = e^5 + e^5(x-5) + (e^5/2)(x-5)^2$.

49. False. Since -1 is the coefficient of x^2 in $P_2(x)$, we know that $f''(0)/2! = -1$, so $f''(0) < 0$, which implies that f is concave down near $x = 0$.

50. False. For example the quadratic approximation to $\cos x$ for x near 0 is $1 - x^2/2$, whereas the linear approximation is the constant function 1. Although the quadratic approximation is better near 0, for large values of x it takes large negative values, whereas the linear approximation stays equal to 1. Since $\cos x$ oscillates between 1 and -1, the linear approximation is better than the quadratic for large x (although it is not very good).

51. False. For example, if $a = 0$ and $f(x) = \cos x$, then $P_1(x) = 1$, and $P_1(x)$ touches $\cos x$ at $x = 0, 2\pi, 4\pi, \ldots$.

52. False. Since $f(-1) = g(-1)$ the graphs of f and g intersect at $x = -1$. Since $f'(-1) < g'(-1)$, the slope of f is less than the slope of g at $x = -1$. Thus $f(x) > g(x)$ for all x sufficiently close to -1 on the left, and $f(x) < g(x)$ for all x sufficiently close to -1 on the right.

53. True. If

$$P_2(x) = \text{Quadratic approximation to } f = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2}(x+1)^2$$
$$Q_2(x) = \text{Quadratic approximation to } g = g(-1) + g'(-1)(x+1) + \frac{g''(-1)}{2}(x+1)^2$$

then $P_2(x) - Q_2(x) = (f''(-1) - g''(-1))(x+1)^2/2 < 0$ for all $x \neq -1$. Thus $P_2(x) < Q_2(x)$ for all $x \neq -1$. This implies that for x sufficiently close to -1 (but not equal to -1), we have $f(x) < g(x)$.

Solutions for Section 10.2

Exercises

1. Differentiating $(1+x)^{3/2}$:

$$\begin{aligned} f(x) &= (1+x)^{3/2} & f(0) &= 1, \\ f'(x) &= (3/2)(1+x)^{1/2} & f'(0) &= \tfrac{3}{2}, \\ f''(x) &= (1/2)(3/2)(1+x)^{-1/2} = (3/4)(1+x)^{-1/2} & f''(0) &= \tfrac{3}{4}, \\ f'''(x) &= (-1/2)(3/4)(1+x)^{-3/2} = (-3/8)(1+x)^{-3/2} & f'''(0) &= -\tfrac{3}{8}. \end{aligned}$$

$$\begin{aligned} f(x) = (1+x)^{3/2} &= 1 + \frac{3}{2}\cdot x + \frac{(3/4)x^2}{2!} + \frac{(-3/8)x^3}{3!} + \cdots \\ &= 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \cdots \end{aligned}$$

2. Differentiating $\sqrt[4]{x+1}$:

$$\begin{aligned} f(x) &= \sqrt[4]{x+1} = (x+1)^{1/4} & f(0) &= 1, \\ f'(x) &= (1/4)(x+1)^{-3/4} & f'(0) &= \tfrac{1}{4}, \\ f''(x) &= (-3/4)(1/4)(x+1)^{-7/4} = (-3/16)(x+1)^{-7/4} & f''(0) &= -\tfrac{3}{16}, \\ f'''(x) &= (-7/4)(-3/16)(x+1)^{-11/4} = (21/64)(x+1)^{-11/4} & f'''(0) &= \tfrac{21}{64}. \end{aligned}$$

$$\begin{aligned} f(x) = \sqrt[4]{x+1} &= 1 + \frac{1}{4}\cdot x + \frac{(-3/16)x^2}{2!} + \frac{(21/64)x^3}{3!} + \cdots \\ &= 1 + \frac{x}{4} - \frac{3x^2}{32} + \frac{7x^3}{128} - \cdots. \end{aligned}$$

3. Differentiating $\sin(-x)$:

$$\begin{aligned}
f(x) &= \sin(-x) & f(0) &= 0,\\
f'(x) &= \cos(-x)(-1) = -\cos(-x) & f'(0) &= -1,\\
f''(x) &= -(-\sin(-x))(-1) = -\sin(-x) & f''(0) &= 0,\\
f'''(x) &= -\cos(-x)(-1) = \cos(-x) & f'''(0) &= 1\\
f^{(4)}(x) &= -\sin(-x)(-1) = \sin(-x) & f^{(4)}(0) &= 0,\\
f^{(5)}(x) &= \cos(-x)(-1) = -\cos(-x) & f^{(5)}(0) &= -1,\\
f^{(6)}(x) &= -(-\sin(-x))(-1) = -\sin(-x) & f^{(6)}(0) &= 0,\\
f^{(7)}(x) &= -\cos(-x)(-1) = \cos(-x) & f^{(7)}(0) &= 1.
\end{aligned}$$

$$\begin{aligned}
f(x) = \sin(-x) &= 0 - 1\cdot x + \frac{0x^2}{2!} + \frac{1x^3}{3!} + \frac{0x^4}{4!} + \frac{-1x^5}{5!} + \frac{0x^6}{6!} + \frac{1x^7}{7!} + \cdots\\
&= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.
\end{aligned}$$

Notice that the series for $\sin(-x)$ is obtained from the series for $\sin x$ by changing the signs. This is expected since $\sin(-x) = -\sin x$.

4. Differentiating $\ln(1-x)$

$$\begin{aligned}
f(x) &= \ln(1-x) & f(0) &= 0,\\
f'(x) &= \tfrac{1}{1-x}(-1) = -(1-x)^{-1} & f'(0) &= -1,\\
f''(x) &= -(-(1-x)^{-2})(-1) = -(1-x)^{-2} & f''(0) &= -1,\\
f'''(x) &= -2(-(1-x)^{-3})(-1) = -2(1-x)^{-3} & f'''(0) &= -2\\
f^{(4)}(x) &= -3(-2(1-x)^{-4})(-1) = -6(1-x)^{-4} & f^{(4)}(0) &= -6.
\end{aligned}$$

$$\begin{aligned}
f(x) = \ln(1-x) &= 0 - 1\cdot x + \frac{(-1)x^2}{2!} + \frac{(-2)x^3}{3!} + \frac{(-6)x^4}{4!} + \cdots\\
&= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots.
\end{aligned}$$

5.

$$\begin{aligned}
f(x) &= \tfrac{1}{1-x} = (1-x)^{-1} & f(0) &= 1,\\
f'(x) &= -(1-x)^{-2}(-1) = (1-x)^{-2} & f'(0) &= 1,\\
f''(x) &= -2(1-x)^{-3}(-1) = 2(1-x)^{-3} & f''(0) &= 2,\\
f'''(x) &= -6(1-x)^{-4}(-1) = 6(1-x)^{-4} & f'''(0) &= 6.
\end{aligned}$$

$$\begin{aligned}
f(x) = \frac{1}{1-x} &= 1 + 1\cdot x + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \cdots\\
&= 1 + x + x^2 + x^3 + \cdots
\end{aligned}$$

6.

$$\begin{aligned}
f(x) &= \tfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} & f(0) &= 1\\
f'(x) &= -\tfrac{1}{2}(1+x)^{-\frac{3}{2}} & f'(0) &= -\tfrac{1}{2}\\
f''(x) &= \tfrac{3}{4}(1+x)^{-\frac{5}{2}} & f''(0) &= \tfrac{3}{4}\\
f'''(x) &= -\tfrac{15}{8}(1+x)^{-\frac{7}{2}} & f'''(0) &= -\tfrac{15}{8}
\end{aligned}$$

$$\begin{aligned}
f(x) = \frac{1}{\sqrt{1+x}} &= 1 + \left(-\frac{1}{2}\right)x + \frac{(\frac{3}{4})x^2}{2!} + \frac{(-\frac{15}{8})x^3}{3!} + \cdots\\
&= 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots
\end{aligned}$$

7.

$$\begin{aligned} f(y) &= \sqrt[3]{1-y} = (1-y)^{\frac{1}{3}} & f(0) &= 1 \\ f'(y) &= \tfrac{1}{3}(1-y)^{-\frac{2}{3}}(-1) = -\tfrac{1}{3}(1-y)^{-\frac{2}{3}} & f'(0) &= -\tfrac{1}{3} \\ f''(y) &= \tfrac{2}{9}(1-y)^{-\frac{5}{3}}(-1) = -\tfrac{2}{9}(1-y)^{-\frac{5}{3}} & f''(0) &= \tfrac{2}{9} \\ f'''(y) &= \tfrac{10}{27}(1-y)^{-\frac{8}{3}}(-1) = -\tfrac{10}{27}(1-y)^{-\frac{8}{3}} & f'''(0) &= -\tfrac{10}{27} \end{aligned}$$

$$\begin{aligned} f(y) = \sqrt[3]{1-y} &= 1 + \left(-\frac{1}{3}\right)y + \frac{(-\frac{2}{9})y^2}{2!} + \frac{(-\frac{10}{27})y^3}{3!} + \cdots \\ &= 1 - \frac{y}{3} - \frac{y^2}{9} - \frac{5y^3}{81} - \cdots \end{aligned}$$

8.

$$\begin{aligned} f(x) &= \sin x & f(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'(x) &= \cos x & f'(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f''(x) &= -\sin x & f''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'''(x) &= -\cos x & f'''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned} \sin x &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^2}{2!} - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{3!} - \cdots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 - \cdots \end{aligned}$$

9.

$$\begin{aligned} f(\theta) &= \cos\theta & f(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'(\theta) &= -\sin\theta & f'(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f''(\theta) &= -\cos\theta & f''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'''(\theta) &= \sin\theta & f'''(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\theta - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{(\theta-\frac{\pi}{4})^2}{2!} + \frac{\sqrt{2}}{2}\frac{(\theta-\frac{\pi}{4})^3}{3!} - \cdots \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\theta - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(\theta - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(\theta - \frac{\pi}{4}\right)^3 - \cdots \end{aligned}$$

10. Differentiating gives

$$\begin{aligned} f(t) &= \cos t & f(\tfrac{\pi}{6}) &= \tfrac{\sqrt{3}}{2}, \\ f'(t) &= -\sin t & f'(\tfrac{\pi}{6}) &= -\tfrac{1}{2}, \\ f''(t) &= -\cos t & f''(\tfrac{\pi}{6}) &= -\tfrac{\sqrt{3}}{2}, \\ f'''(t) &= \sin t & f'''(\tfrac{\pi}{6}) &= \tfrac{1}{2}. \end{aligned}$$

$$\begin{aligned} \cos t &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(t - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}\frac{(t-\frac{\pi}{6})^2}{2!} + \frac{1}{2}\frac{(t-\frac{\pi}{6})^3}{3!} - \cdots \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(t - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(t - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(t - \frac{\pi}{6}\right)^3 - \cdots \end{aligned}$$

11.

$$\begin{aligned} f(\theta) &= \sin\theta & f(-\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'(\theta) &= \cos\theta & f'(-\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f''(\theta) &= -\sin\theta & f''(-\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'''(\theta) &= -\cos\theta & f'''(-\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned}\sin\theta &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\theta + \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\frac{(\theta + \frac{\pi}{4})^2}{2!} - \frac{\sqrt{2}}{2}\frac{(\theta + \frac{\pi}{4})^3}{3!} + \cdots \\ &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\theta + \frac{\pi}{4}\right) + \frac{\sqrt{2}}{4}\left(\theta + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(\theta + \frac{\pi}{4}\right)^3 + \cdots .\end{aligned}$$

12.

$$\begin{aligned} f(x) &= \tan x & f(\tfrac{\pi}{4}) &= 1, \\ f'(x) &= \tfrac{1}{\cos^2 x} & f'(\tfrac{\pi}{4}) &= 2, \\ f''(x) &= \tfrac{-2(-\sin x)}{\cos^3 x} = \tfrac{2\sin x}{\cos^3 x} & f''(\tfrac{\pi}{4}) &= 4, \\ f'''(x) &= \tfrac{-6\sin x(-\sin x)}{\cos^4 x} + \tfrac{2}{\cos^2 x} & f'''(\tfrac{\pi}{4}) &= 16. \end{aligned}$$

$$\begin{aligned}\tan x &= 1 + 2\left(x - \frac{\pi}{4}\right) + 4\frac{(x - \frac{\pi}{4})^2}{2!} + 16\frac{\left(x - \frac{\pi}{4}\right)^3}{3!} + \cdots \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \cdots\end{aligned}$$

13.

$$\begin{aligned} f(x) &= \tfrac{1}{x} & f(1) &= 1 \\ f'(x) &= -\tfrac{1}{x^2} & f'(1) &= -1 \\ f''(x) &= \tfrac{2}{x^3} & f''(1) &= 2 \\ f'''(x) &= -\tfrac{6}{x^4} & f'''(1) &= -6 \end{aligned}$$

$$\begin{aligned}\frac{1}{x} &= 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots .\end{aligned}$$

14. Again using the derivatives found in Problem 13, we have

$$f(2) = \frac{1}{2}, \qquad f'(2) = -\frac{1}{4}, \qquad f''(2) = \frac{1}{4}, \qquad f'''(2) = -\frac{3}{8}.$$

$$\begin{aligned}\frac{1}{x} &= \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{4\cdot 2!} - \frac{3(x-2)^3}{8\cdot 3!} + \cdots \\ &= \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \cdots\end{aligned}$$

15. Using the derivatives from Problem 13, we have

$$f(-1) = -1, \quad f'(-1) = -1, \quad f''(-1) = -2, \quad f'''(-1) = -6.$$

Hence,

$$\begin{aligned}\frac{1}{x} &= -1 - (x+1) - \frac{2(x+1)^2}{2!} - \frac{6(x+1)^3}{3!} - \cdots \\ &= -1 - (x+1) - (x+1)^2 - (x+1)^3 - \cdots\end{aligned}$$

16. The general term can be written as x^n for $n \geq 0$.

17. The general term can be written as $(-1)^n x^n$ for $n \geq 0$.

18. The general term can be written as $-x^n/n$ for $n \geq 1$.

19. The general term can be written as $(-1)^{n-1}x^n/n$ for $n \geq 1$.

20. The general term can be written as $(-1)^k x^{2k+1}/(2k+1)!$ for $k \geq 0$.

21. The general term can be written as $(-1)^k x^{2k+1}/(2k+1)$ for $k \geq 0$.

22. The general term can be written as $x^{2k}/k!$ for $k \geq 0$.

23. The general term can be written as $(-1)^k x^{4k+2}/(2k)!$ for $k \geq 0$.

24. The series is

$$(1+x)^3 = 1 + 3x + \frac{3\cdot 2}{2!}x^2 + \frac{3\cdot 2\cdot 1}{3!}x^3 + \frac{3\cdot 2\cdot 1\cdot 0}{4!}x^4 + \cdots.$$

The x^4 term and all terms beyond it turn out to be zero, because each coefficient contains a factor of 0. Simplifying gives

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$

which is the usual expansion obtained by multiplying out $(1+x)^3$.

Problems

25. By looking at Figure 10.4 we can that the Taylor polynomials are reasonable approximations for the function $f(x) = \frac{1}{\sqrt{1+x}}$ between $x = -1$ and $x = 1$. Thus a good guess is that the interval of convergence is $-1 < x < 1$.

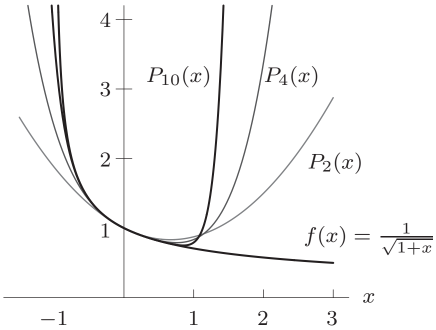


Figure 10.4

26. By looking at Figure 10.5, we see that the Taylor polynomials are reasonable approximations for the function $f(x) = \sqrt{1+x}$ between $x = -1$ and $x = 1$. Thus a good guess is that the Taylor series converges to $f(x)$ for $-1 < x < 1$.

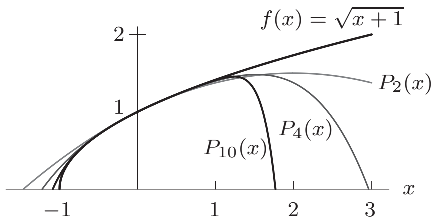


Figure 10.5

27. (a) Figure 10.6 suggests that the Taylor polynomials converge to $f(x) = \dfrac{1}{1-x}$ on the interval $-1 < x < 1$.

(b) Since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots.$$

the ratio test gives

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = |x|.$$

Thus, the radius of convergence is $R = 1$. The series converges if $|x| < 1$; that is, $-1 < x < 1$.

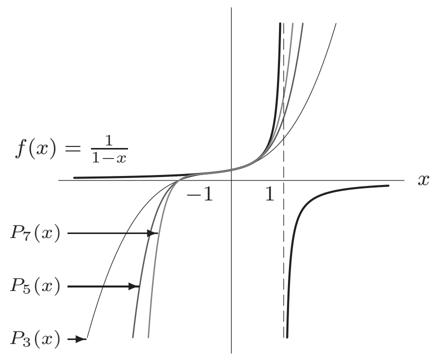


Figure 10.6

28. The Taylor series of e^x around $x = 0$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

To find the radius of convergence, we apply the ratio test with $a_k = x^k/k!$.

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|x|^{k+1}/(k+1)!}{|x|^k/k!} = \lim_{k\to\infty} \frac{|x|}{k+1} = 0.$$

Hence the radius of convergence is $R = \infty$.

29. The Taylor series for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots,$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = |x| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |x|.$$

Thus the series converges for $|x| < 1$, and the radius of convergence is 1. Note: This series can be obtained from the series for $\ln(1+x)$ by replacing x by $-x$ and has the same radius of convergence as the series for $\ln(1+x)$.

30. (a) We have shown that the series is

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

so the general term is

$$\frac{p(p-1)\ldots(p-(n-1))}{n!}x^n.$$

(b) We use the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \left| \frac{p(p-1)\ldots(p-(n-1))(p-n)\cdot n!}{(n+1)!p(p-1)\ldots(p-(n-1))} \right| = |x| \lim_{n\to\infty} \left| \frac{p-n}{n+1} \right|.$$

Since p is fixed, we have

$$\lim_{n\to\infty} \left| \frac{p-n}{n+1} \right| = 1, \quad \text{so} \quad R = 1.$$

31. We know that the Taylor series for e^x around 0 is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Using the right hand side of the above equation for e^x in the expression $\dfrac{e^x - 1}{x}$, we have

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1}{x}.$$

Simplifying we get

$$\lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = \lim_{x\to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots = 1.$$

Hence this limit is equal to 1.

32. The second coefficient of the Taylor expansion is

$$\frac{g''(0)}{2} = 1, \quad \text{so} \quad g''(0) = 2.$$

Similarly, the third coefficient is

$$\frac{g'''(0)}{3!} = 0 \quad \text{so} \quad g'''(0) = 0.$$

Finally, the tenth coefficient is

$$\frac{g^{(10)}(0)}{10!} = \frac{1}{5!} \quad \text{so} \quad g^{(10)}(0) = \frac{10!}{5!}.$$

33. Let C_n be the coefficient of the n^{th} term in the series. Note that

$$0 = C_1 = \frac{d}{dx}(x^2 e^{x^2})\bigg|_{x=0},$$

and since

$$\frac{1}{2} = C_6 = \frac{\frac{d^6}{dx^6}(x^2 e^{x^2})\Big|_{x=0}}{6!},$$

we have

$$\frac{d^6}{dx^6}(x^2 e^{x^2})\bigg|_{x=0} = \frac{6!}{2} = 360.$$

34. (a) From the coefficients of the $(x-1)$ terms of the fs, we see that

$$f_1'(1) = 1, \qquad f_2'(1) = -1 \qquad f_3'(1) = -2.$$

From the $(x-1)^2$ terms of the fs, we see that

$$\frac{f_1''(1)}{2!} = -1, \qquad \frac{f_2''(1)}{2!} = 1, \qquad \frac{f_3''(1)}{2!} = 1,$$

so $f_1''(1) = -2$, $f_2''(1) = 2$, $f_3''(1) = 2$.

Thus, f_1 slopes up at $x = 1$ and f_2 and f_3 slope down; f_3 slopes down more steeply than f_2. This means that the fs are in the first figure, since graphs II and III in the second figure have the same negative slope at point B.

By a similar argument, we find

$$g_1'(4) = -1, \quad g_2'(4) = -1, \quad g_3'(4) = 1, \text{ and } g_1''(4) = -2, \quad g_2''(4) = 2, \quad g_3''(4) = 2.$$

Thus, two of the gs slope down, one of which is concave up and one is concave down; the third g slopes up and is concave up. This confirms that the gs are in the second figure.

(b) Since $f_1(1) = f_2(1) = f_3(1) = 3$, the point A is $(1, 3)$.
Since $g_1(4) = g_2(4) = g_3(4) = 5$, the point B is $(4, 5)$.

(c) In the first figure, graph I is f_1 since it slopes up. Graph II is f_2 since it slopes down, but less steeply than graph III, which is f_3.

In the second figure, graph I is g_3, since it slopes up. Graph II is g_2 since it slopes down and is concave up. Graph III is g_1 since it slopes down and is concave down.

35. This is the series for e^x with x replaced by 2, so the series converges to e^2.

36. This is the series for $\sin x$ with x replaced by 1, so the series converges to $\sin 1$.

37. This is the series for $1/(1-x)$ with x replaced by $1/4$, so the series converges to $1/(1-(1/4)) = 4/3$.

38. This is the series for $\cos x$ with x replaced by 10, so the series converges to $\cos 10$.

39. This is the series for $\ln(1+x)$ with x replaced by $1/2$, so the series converges to $\ln(3/2)$.

40. The Taylor series for $f(x) = 1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

Substituting $x = 0.1$ gives

$$1 - 0.1 + (0.1)^2 - (0.1)^3 + \cdots = \frac{1}{1+0.1} = \frac{1}{1.1}.$$

Alternatively, this is a geometric series with $a = 1$, $x = -0.1$.

41. This is the series for e^x with $x = 3$ substituted. Thus

$$1 + 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots = e^3.$$

42. This is the series for $\cos x$ with $x = 1$ substituted. Thus

$$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots = \cos 1.$$

43. This is the series for e^x with -0.1 substituted for x, so

$$1 - 0.1 + \frac{0.01}{2!} - \frac{0.001}{3!} + \cdots = e^{-0.1}.$$

44. Since $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$, a geometric series, we solve $\dfrac{1}{1-x} = 5$ giving $\dfrac{1}{5} = 1 - x$, so $x = \dfrac{4}{5}$.

45. Since $x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \cdots = \ln(1+x)$, we solve $\ln(1+x) = 0.2$, giving $1 + x = e^{0.2}$, so $x = e^{0.2} - 1$.

46. We define $e^{i\theta}$ to be

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

Suppose we consider the expression $\cos\theta + i\sin\theta$, with $\cos\theta$ and $\sin\theta$ replaced by their Taylor series:

$$\cos\theta + i\sin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

Reordering terms, we have

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots$$

Using the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$, we can rewrite the series as

$$\cos\theta + i\sin\theta = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

Amazingly enough, this series is the Taylor series for e^x with $i\theta$ substituted for x. Therefore, we have shown that

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

Strengthen Your Understanding

47. The left hand side of the equation is finite, namely -1, whereas the right hand side of the equation is infinite. The statement is wrong, since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

only for $-1 < x < 1$.

48. First note that this Taylor series is convergent at 3. Substituting $x = 4$ into the series we get

$$1 + 1 + \cdots,$$

which is divergent, similarly substituting $x = 2$, we get the alternating geometric series

$$1 + (-1) + (-1)^2 + (-1)^3 + \cdots,$$

which is divergent. Since $x = 3$ is the center of the interval of convergence of the Taylor series, we see that the radius of convergence cannot be greater or equal to 1.

49. The function $f(x) = \cos x$ is an example. The Taylor series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The third-degree term of this series is zero.

50. A possible example is $1 + (x+1) + (x+1)^2 + \ldots$. Many other examples are possible.

51. False. The Taylor series for $\sin x$ about $x = \pi$ is calculated by taking derivatives and using the formula

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots.$$

The series for $\sin x$ about $x = \pi$ turns out to be

$$-(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \cdots.$$

52. True. Since f is even, $f(-x) = f(x)$ for all x. Taking the derivative of both sides of this equation, we get $f'(-x)(-1) = f'(x)$, which at $x = 0$ gives $-f'(0) = f'(0)$, so $f'(0) = 0$. Taking the derivative again gives $f''(-x) = f''(x)$, i.e., f'' is even. Using the same reasoning again, we get that $f'''(0) = 0$, and, continuing in this way, we get $f^{(n)}(0) = 0$ for all odd n. Thus, for all odd n, the coefficient of x^n in the Taylor series is $f^{(n)}(0)/n! = 0$, so all the terms with odd exponent are zero.

53. True. The coefficient of x^7 is $-8/7!$, so

$$\frac{f^{(7)}(0)}{7!} = \frac{-8}{7!}$$

giving $f^{(7)}(0) = -8$.

54. False. For example, the Taylor series

$$1 + x + x^2 + x^3 + \cdots$$

for $f(x) = 1/(1-x)$ diverges for $|x| > 1$, but $1/(1-x)$ is defined for $|x| > 1$.

55. True. For large x, the graph of $P_{10}(x)$ looks like the graph of its highest powered term, $x^{10}/10!$. But e^x grows faster than any power, so e^x gets further and further away from $x^{10}/10! \approx P_{10}(x)$.

Solutions for Section 10.3

Exercises

1. Substitute $y = -x$ into $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$. We get

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots. \end{aligned}$$

2. We'll use

$$\sqrt{1+y} = (1+y)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right) y + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{y^2}{2!} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{y^3}{3!} + \cdots = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} - \cdots.$$

Substitute $y = -2x$.

$$\sqrt{1-2x} = 1 + \frac{(-2x)}{2} - \frac{(-2x)^2}{8} + \frac{(-2x)^3}{16} - \cdots = 1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \cdots$$

3. Substitute $x = \theta^2$ into series for $\cos x$:

$$\cos(\theta^2) = 1 - \frac{(\theta^2)^2}{2!} + \frac{(\theta^2)^4}{4!} - \frac{(\theta^2)^6}{6!} + \cdots = 1 - \frac{\theta^4}{2!} + \frac{\theta^8}{4!} - \frac{\theta^{12}}{6!} + \cdots.$$

4. Substituting $x = -2y$ into $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ gives

$$\ln(1-2y) = (-2y) - \frac{(-2y)^2}{2} + \frac{(-2y)^3}{3} - \frac{(-2y)^4}{4} + \cdots = -2y - 2y^2 - \frac{8}{3}y^3 - 4y^4 - \cdots.$$

5. Since $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$, integrating gives

$$\arcsin x = c + x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots.$$

Since $\arcsin 0 = 0$, $c = 0$.

6. We substitute $3t$ into the series for $\sin x$ and multiply by t. Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

substituting $3t$ gives

$$\sin(3t) = (3t) - \frac{(3t)^3}{3!} + \frac{(3t)^5}{5!} - \frac{(3t)^7}{7!} + \cdots = 3t + \frac{-9}{2}t^3 + \frac{81}{40}t^5 + \frac{-243}{560}t^7 + \cdots,$$

so

$$t\sin(3t) = 3t^2 - \frac{9}{2}t^4 + \frac{81}{40}t^6 - \frac{243}{560}t^8 + \cdots.$$

7. Substituting $x = -z^2$ into $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$ gives

$$\frac{1}{\sqrt{1-z^2}} = 1 - \frac{(-z^2)}{2} + \frac{3(-z^2)^2}{8} - \frac{5(-z^2)^3}{16} + \cdots = 1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \cdots.$$

8.

$$\frac{z}{e^{z^2}} = ze^{-z^2} = z\left(1 + (-z^2) + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!} + \cdots\right)$$
$$= z - z^3 + \frac{z^5}{2!} - \frac{z^7}{3!} + \cdots$$

9.

$$\phi^3 \cos(\phi^2) = \phi^3\left(1 - \frac{(\phi^2)^2}{2!} + \frac{(\phi^2)^4}{4!} - \frac{(\phi^2)^6}{6!} + \cdots\right)$$
$$= \phi^3 - \frac{\phi^7}{2!} + \frac{\phi^{11}}{4!} - \frac{\phi^{15}}{6!} + \cdots$$

10. From Example 3, we know the Taylor series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots.$$

Substituting $x = r^2$, we get:

$$\arctan(r^2) = r^2 - \frac{r^6}{3} + \frac{r^{10}}{5} - \frac{r^{14}}{7} + \cdots.$$

11. Multiplying out gives $(1+x)^3 = 1 + 3x + 3x^2 + x^3$. Since this polynomial equals the original function for all x, it must be the Taylor series. The general term is $0 \cdot x^n$ for $n \geq 4$.

12. Substituting t^2 into the series for $\sin x$ gives

$$\sin(t^2) = t^2 - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} + \cdots + \frac{(-1)^k (t^2)^{2k+1}}{(2k+1)!} + \cdots$$
$$= t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} + \cdots + \frac{(-1)^k t^{4k+2}}{(2k+1)!} + \cdots$$

Therefore

$$t\sin(t^2) - t^3 = \left(t^3 - \frac{t^7}{3!} + \frac{t^{11}}{5!} + \cdots + \frac{(-1)^k t^{4k+3}}{(2k+1)!} + \cdots\right) - t^3$$
$$= -\frac{t^7}{3!} + \frac{t^{11}}{5!} + \cdots + \frac{(-1)^k t^{4k+3}}{(2k+1)!} + \cdots \quad \text{for } k \geq 1.$$

13. Using the Binomial theorem:

$$\frac{1}{\sqrt{1-x}}$$
$$= (1-x)^{-1/2}$$
$$= 1 + \left(-\frac{1}{2}\right)(-x) + \frac{(-1/2)(-3/2)(-x)^2}{2!} + \cdots + \frac{(-1/2)(-3/2)\cdots(-\frac{1}{2} - n + 1)(-x)^n}{n!} + \cdots \text{ for } n \geq 1.$$

Substituting y^2 for x:

$$\frac{1}{\sqrt{1-y^2}} = (1-y^2)^{-1/2}$$
$$= 1 + \frac{1}{2}y^2 + \frac{3}{8}y^4 + \cdots + \frac{(1/2)(3/2)\cdots(\frac{1}{2} + n - 1)y^{2n}}{n!} + \cdots \text{ for } n \geq 1.$$

14.

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$
$$= \frac{1}{2}\left(1-\frac{x}{2}+\left(\frac{x}{2}\right)^2-\left(\frac{x}{2}\right)^3+\cdots\right)$$

15. Using the binomial expansion for $(1+x)^{1/2}$ with $x = h/T$:

$$\sqrt{T+h} = \left(T+\frac{T}{T}h\right)^{1/2} = \left(T\left(1+\frac{h}{T}\right)\right)^{1/2} = \sqrt{T}\left(1+\frac{h}{T}\right)^{1/2}$$
$$= \sqrt{T}\left(1+(1/2)\left(\frac{h}{T}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{h}{T}\right)^2+\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{h}{T}\right)^3\cdots\right)$$
$$= \sqrt{T}\left(1+\frac{1}{2}\left(\frac{h}{T}\right)-\frac{1}{8}\left(\frac{h}{T}\right)^2+\frac{1}{16}\left(\frac{h}{T}\right)^3\cdots\right).$$

16. Using the binomial expansion for $(1+x)^{-1}$ with $x = -r/a$:

$$\frac{1}{a-r} = \frac{1}{a-a\left(\frac{r}{a}\right)} = \frac{1}{a\left(1-\frac{r}{a}\right)} = \frac{1}{a}\left(1+\left(-\frac{r}{a}\right)\right)^{-1}$$
$$= \frac{1}{a}\left(1+(-1)\left(-\frac{r}{a}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{r}{a}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(-\frac{r}{a}\right)^3+\cdots\right)$$
$$= \frac{1}{a}\left(1-\left(-\frac{r}{a}\right)+\left(-\frac{r}{a}\right)^2-\left(-\frac{r}{a}\right)^3+\cdots\right)$$
$$= \frac{1}{a}\left(1+\left(\frac{r}{a}\right)+\left(\frac{r}{a}\right)^2+\left(\frac{r}{a}\right)^3+\cdots\right).$$

17. Using the binomial expansion for $(1+x)^{-2}$ with $x = r/a$:

$$\frac{1}{(a+r)^2} = \frac{1}{\left(a+a\left(\frac{r}{a}\right)\right)^2} = \frac{1}{\left(a\left(1+\frac{r}{a}\right)\right)^2} = \frac{1}{a^2}\left(1+\left(\frac{r}{a}\right)\right)^{-2}$$
$$= \frac{1}{a^2}\left(1+(-2)\left(\frac{r}{a}\right)+\frac{(-2)(-3)}{2!}\left(\frac{r}{a}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{r}{a}\right)^3+\cdots\right)$$
$$= \frac{1}{a^2}\left(1-2\left(\frac{r}{a}\right)+3\left(\frac{r}{a}\right)^2-4\left(\frac{r}{a}\right)^3+\cdots\right).$$

18.

$$\sqrt[3]{P+t} = \left(P+P\left(\frac{t}{P}\right)\right)^{1/3} = \left(P\left(1+\frac{t}{P}\right)\right)^{1/3} = \sqrt[3]{P}\left(1+\frac{t}{P}\right)^{1/3}$$
$$= \sqrt[3]{P}\left(1+(1/3)\left(\frac{t}{P}\right)+\frac{(1/3)(-2/3)}{2!}\left(\frac{t}{P}\right)^2+\frac{(1/3)(-2/3)(-5/3)}{3!}\left(\frac{t}{P}\right)^3\cdots\right)$$
$$= \sqrt[3]{P}\left(1+\frac{1}{3}\left(\frac{t}{P}\right)-\frac{1}{9}\left(\frac{t}{P}\right)^2+\frac{5}{81}\left(\frac{t}{P}\right)^3\cdots\right).$$

19.

$$\frac{a}{\sqrt{a^2+x^2}} = \frac{a}{a(1+\frac{x^2}{a^2})^{\frac{1}{2}}} = \left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$$
$$= 1 + \left(-\frac{1}{2}\right)\frac{x^2}{a^2} + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^2}{a^2}\right)^2$$
$$+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x^2}{a^2}\right)^3 + \cdots$$
$$= 1 - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{3}{8}\left(\frac{x}{a}\right)^4 - \frac{5}{16}\left(\frac{x}{a}\right)^6 + \cdots$$

Problems

20.

$$\sqrt{(1+t)}\sin t = \left(1+\frac{t}{2}-\frac{t^2}{8}+\frac{t^3}{16}-\cdots\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)$$

Multiplying and collecting terms yields

$$\sqrt{(1+t)}\sin t = t + \frac{t^2}{2} - \left(\frac{t^3}{3!}+\frac{t^3}{8}\right) + \left(\frac{t^4}{16}-\frac{t^4}{12}\right) + \cdots$$
$$= t + \frac{1}{2}t^2 - \frac{7}{24}t^3 - \frac{1}{48}t^4 + \cdots.$$

21.

$$e^t\cos t = \left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right)$$

Multiplying out and collecting terms gives

$$e^t\cos t = 1+t+\left(\frac{t^2}{2!}-\frac{t^2}{2!}\right)+\left(\frac{t^3}{3!}-\frac{t^3}{2!}\right)+\left(\frac{t^4}{4!}+\frac{t^4}{4!}-\frac{t^4}{(2!)^2}\right)+\cdots$$
$$= 1+t-\frac{t^3}{3}-\frac{t^4}{6}+\cdots.$$

22. Substituting the series for $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ into

$$\sqrt{1+y} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 - \cdots$$

gives

$$\sqrt{1+\sin\theta} = 1 + \frac{1}{2}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right) - \frac{1}{8}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)^2$$
$$+\frac{1}{16}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)^3 - \cdots$$
$$= 1 + \frac{1}{2}\theta - \frac{\theta^2}{8} + \left(\frac{\theta^3}{16}-\frac{\theta^3}{2\cdot 3!}\right) + \cdots$$
$$= 1 + \frac{1}{2}\theta - \frac{1}{8}\theta^2 - \frac{1}{48}\theta^3 + \cdots$$

23. We know that

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

So, treating the given function as the sum of a geometric series with initial term 1 and common ratio $\ln(1+t)$, we have

$$\begin{aligned}\frac{1}{1-\ln(1+t)} &= 1 + \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right) + \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right)^2 \\ &\quad + \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right)^3 + \cdots \\ &= 1 + \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right) + \left(t^2 - t^3 + \cdots\right) + \left(t^3 - \cdots\right) + \cdots \\ &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots.\end{aligned}$$

24. (a) Since the Taylor series for e^x and e^{-x} are given by

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,\end{aligned}$$

we have

$$e^x + e^{-x} = 2 + 0x + 2\frac{x^2}{2!} + 0\frac{x^3}{3!} + 2\frac{x^4}{4!} + \cdots = 2 + x^2 + \frac{x^4}{12} + \cdots.$$

(b) For x near 0, we can approximate $e^x + e^{-x}$ by its second degree Taylor polynomial, $P_2(x)$, whose graph is a parabola:

$$e^x + e^{-x} \approx P_2(x) = 2 + x^2.$$

25. (a) Since the Taylor series for e^x and e^{-x} are given by

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots,\end{aligned}$$

we have

$$e^x - e^{-x} = 0 + 2x + 0\frac{x^2}{2!} + 2\frac{x^3}{3!} + 0\frac{x^4}{4!} + 2\frac{x^5}{5!} \cdots = 2x + \frac{x^3}{3} + \frac{x^5}{60} + \cdots.$$

(b) For x near 0, we can approximate $e^x - e^{-x}$ by its third degree Taylor polynomial, $P_3(x)$:

$$e^x - e^{-x} \approx P_3(x) = 2x + \frac{x^3}{3}.$$

The function $P_3(x)$ is a cubic polynomial whose graph is symmetric about the origin.

26. Recall that the Taylor series for e^x around 0 is given by

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Hence the Taylor series for e^{x^2} around 0 is then

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots.$$

The first three terms are then

$$P_4(x) = 1 + x^2 + \frac{x^4}{2!}.$$

We approximate

$$\int_0^1 e^{x^2}\,dx \approx \int_0^1 \left(1 + x^2 + \frac{x^4}{2!}\right) dx.$$

Computing the integral

$$\int_0^1 \left(1 + x^2 + \frac{x^4}{2!}\right) dx = x + \frac{x^3}{3} + \frac{x^5}{10}\bigg|_0^1 = 1 + \frac{1}{3} + \frac{1}{10} \approx 1.433.$$

27. Notice that $\sum px^{p-1}$, is the derivative, term-by-term, of a geometric series:

$$\sum_{p=1}^{\infty} px^{p-1} = 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + \cdots = \frac{d}{dx}(\underbrace{x + x^2 + x^3 + \cdots}_{\text{Geometric series}}).$$

For $|x| < 1$, the sum of the geometric series with first term x and common ratio x is

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}.$$

Differentiating gives

$$\sum_{p=1}^{\infty} px^{p-1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

28. From the series for $\ln(1+y)$,

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots,$$

we get

$$\ln(1+y^2) = y^2 - \frac{y^4}{2} + \frac{y^6}{3} - \frac{y^8}{4} + \cdots$$

The Taylor series for $\sin y$ is

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$$

So

$$\sin y^2 = y^2 - \frac{y^6}{3!} + \frac{y^{10}}{5!} - \frac{y^{14}}{7!} + \cdots$$

The Taylor series for $\cos y$ is

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

So

$$1 - \cos y = \frac{y^2}{2!} - \frac{y^4}{4!} + \frac{y^6}{6!} + \cdots$$

Near $y = 0$, we can drop terms beyond the fourth degree in each expression:

$$\ln(1+y^2) \approx y^2 - \frac{y^4}{2}$$
$$\sin y^2 \approx y^2$$
$$1 - \cos y \approx \frac{y^2}{2!} - \frac{y^4}{4!}.$$

(Note: These functions are all even, so what holds for negative y will hold for positive y.)
Clearly $1 - \cos y$ is smallest, because the y^2 term has a factor of $\frac{1}{2}$. Thus, for small y,

$$\frac{y^2}{2!} - \frac{y^4}{4!} < y^2 - \frac{y^4}{2} < y^2$$

so

$$1 - \cos y < \ln(1+y^2) < \sin(y^2).$$

29. The Taylor series for $1+\sin\theta$ near $\theta=0$ is

$$1+\sin\theta = 1+\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots$$

The Taylor series for e^θ is

$$e^\theta = 1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\cdots$$

The Taylor series for $\frac{1}{\sqrt{1+\theta}}$ is

$$\frac{1}{\sqrt{1+\theta}} = 1-\frac{\theta}{2}+\frac{3\theta^2}{8}-\frac{5\theta^3}{16}+\cdots$$

Substituting -2θ into this formula yields

$$\frac{1}{\sqrt{1-2\theta}} = 1+\theta+\frac{3}{2}\theta^2+\frac{5}{2}\theta^3+\cdots$$

These three series are identical in the constant and linear terms, but they differ in their quadratic terms. For values of θ near zero, the quadratic terms dominate all of the subsequent terms, so we can use the approximations

$$\begin{aligned}1+\sin\theta &\approx 1+\theta\\ e^\theta &\approx 1+\theta+\frac{1}{2}\theta^2\\ \frac{1}{\sqrt{1-2\theta}} &\approx 1+\theta+\frac{3}{2}\theta^2.\end{aligned}$$

Clearly $1+\sin\theta$ is smallest, because the θ^2 term is zero, and the θ^2 terms of the other two series are positive. The function $1/\sqrt{1-2\theta}$ is largest, because the coefficient of its θ^2 term is the greatest. Therefore, for θ near zero,

$$1+\sin\theta \le e^\theta \le \frac{1}{\sqrt{1-2\theta}}.$$

30. We have

$$\begin{aligned}P_9(x) &= \sum_{n=0}^{4}\frac{x^{2n+1}}{2n+1}\\ &= \frac{x^1}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}.\end{aligned}$$

So,

$$\begin{aligned}P_9(2x) &= 2x+\frac{1}{3}(2x)^3+\frac{1}{5}(2x)^5+\frac{1}{7}(2x)^7+\frac{1}{9}(2x)^9\\ &= 2x+\frac{8}{3}x^3+\frac{32}{5}x^5+\frac{128}{7}x^7+\frac{512}{9}x^9.\end{aligned}$$

31. The Taylor series about 0 for $y=\dfrac{1}{1-x^2}$ is

$$y = 1+x^2+x^4+x^6+\cdots.$$

The series for $y=(1+x)^{1/4}$ is, using the binomial expansion,

$$y = 1+\frac{1}{4}x+\frac{1}{4}\left(-\frac{3}{4}\right)\frac{x^2}{2!}+\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\frac{x^3}{3!}+\cdots.$$

The series for $y=\sqrt{1+\dfrac{x}{2}}=(1+\dfrac{x}{2})^{1/2}$ is, again using the binomial expansion,

$$y = 1+\frac{1}{2}\cdot\frac{x}{2}+\frac{1}{2}\left(-\frac{1}{2}\right)\cdot\frac{x^2}{8}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{x^3}{48}+\cdots.$$

Similarly for $y = \dfrac{1}{\sqrt{1-x}} = (1-x)^{-(1/2)}$,

$$y = 1 + \left(-\frac{1}{2}\right)(-x) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{x^2}{2!} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdot\frac{-x^3}{3!} + \cdots.$$

Near 0, let's truncate these series after their x^2 terms:

$$\begin{aligned}\frac{1}{1-x^2} &\approx 1 + x^2,\\ (1+x)^{1/4} &\approx 1 + \frac{1}{4}x - \frac{3}{32}x^2,\\ \sqrt{1+\frac{x}{2}} &\approx 1 + \frac{1}{4}x - \frac{1}{32}x^2,\\ \frac{1}{\sqrt{1-x}} &\approx 1 + \frac{1}{2}x + \frac{3}{8}x^2.\end{aligned}$$

Thus $\frac{1}{1-x^2}$ looks like a parabola opening upward near the origin, with y-axis as the axis of symmetry, so (a) = I.

Now $\frac{1}{\sqrt{1-x}}$ has the largest positive slope ($\frac{1}{2}$), and is concave up (because the coefficient of x^2 is positive). So (d) = II.

The last two both have positive slope ($\frac{1}{4}$) and are concave down. Since $(1+x)^{\frac{1}{4}}$ has the smallest second derivative (i.e., the most negative coefficient of x^2), (b) = IV and therefore (c) = III.

32. From the Taylor series for the sine function we know that

$$\sin t \approx t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}.$$

Thus, from the definition of $\mathrm{Si}(x)$, we have:

$$\begin{aligned}\mathrm{Si}(2) &= \int_0^2 \frac{\sin t}{t}\,dt\\ &\approx \int_0^2 \frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right)dt\\ &= \int_0^2 \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!}\right)dt\\ &= t - \frac{1}{3}\cdot\frac{t^3}{3!} + \frac{1}{5}\cdot\frac{t^5}{5!} - \frac{1}{7}\cdot\frac{t^7}{7!}\Big|_0^2\\ &= 2 - \frac{2^3}{3\cdot 3!} + \frac{2^5}{5\cdot 5!} - \frac{2^7}{7\cdot 7!}\\ &= 1.60526.\end{aligned}$$

This is very close to the actual value of $\mathrm{Si}(2) = 1.60541$, found using a computer algebra system.

33. The Taylor series for $\sin t$ about $t = 0$ is $\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots$. This gives the Taylor series for $\sin\left(t^2\right)$ about $t = 0$:

$$\sin\left(t^2\right) = t^2 - \frac{1}{3!}\left(t^2\right)^3 + \frac{1}{5!}\left(t^2\right)^5 - \frac{1}{7!}\left(t^2\right)^7 + \cdots = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \frac{1}{7!}t^{14} + \cdots,$$

so

$$\begin{aligned}f(x) &= \int_0^x \sin\left(t^2\right)dt\\ &= \int_0^x \left(t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \frac{1}{7!}t^{14} + \cdots\right)dt\\ &= \frac{1}{3}x^3 - \frac{1}{7}\frac{1}{3!}x^7 + \frac{1}{11}\frac{1}{5!}x^{11} - \frac{1}{15}\frac{1}{7!}x^{15} + \cdots\\ &= \frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11} - \frac{1}{75{,}600}x^{15} + \cdots\end{aligned}$$

34. (a) $f(t) = te^t$.
Use the Taylor expansion for e^t :

$$\begin{aligned} f(t) &= t\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \\ &= t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \cdots \end{aligned}$$

(b)

$$\begin{aligned} \int_0^x f(t)\,dt = \int_0^x te^t\,dt &= \int_0^x \left(t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \cdots\right) dt \\ &= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4\cdot 2!} + \frac{t^5}{5\cdot 3!} + \cdots \Big|_0^x \\ &= \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4\cdot 2!} + \frac{x^5}{5\cdot 3!} + \cdots \end{aligned}$$

(c) Substitute $x = 1$:

$$\int_0^1 te^t\,dt = \frac{1}{2} + \frac{1}{3} + \frac{1}{4\cdot 2!} + \frac{1}{5\cdot 3!} + \cdots$$

In the integral above, to integrate by parts, let $u = t,\ dv = e^t\,dt$, so $du = dt$, $v = e^t$.

$$\int_0^1 te^t\,dt = te^t\Big|_0^1 - \int_0^1 e^t\,dt = e - (e-1) = 1$$

Hence

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4\cdot 2!} + \frac{1}{5\cdot 3!} + \cdots = 1.$$

35. Since it does not depend on n, we can factor out e^{-k}, giving

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!}e^{-k} &= e^{-k}\sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!} \\ &= e^{-k}\underbrace{\left(\frac{1}{0!} + \frac{k}{1!} + \frac{k^2}{2!} + \frac{k^3}{3!} + \cdots\right)}_{\text{This is the series for } e^k} \\ &= e^{-k}\cdot e^k \\ &= 1. \end{aligned}$$

36. (a) The Taylor approximation to $f(x) = \cosh x$ about $x = 0$ is of the form

$$\cosh x \approx \cosh(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!}.$$

We have the following results:

$$\begin{aligned} f(x) &= \cosh x \quad \text{so } f(0) = 1, \\ f'(x) &= \sinh x \quad \text{so } f'(0) = 0, \\ f''(x) = \frac{d}{dx}(\sinh x) &= \cosh x \quad \text{so } f''(0) = 1, \\ f'''(x) &= \sinh x \quad \text{so } f'''(0) = 0. \end{aligned}$$

The derivatives continue to alternate between $\cosh x$ and $\sinh x$, so their values at 0 continue to alternate between 0 and 1. Therefore

$$\cosh x \approx 1 + 0\cdot x + 1\cdot\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} + 1\cdot\frac{x^4}{4!} + \cdots,$$

so the degree 8 Taylor approximation is given by

$$\cosh x \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}.$$

(b) We use the polynomial obtained from part (a) to estimate $\cosh 1$,

$$\cosh 1 \approx 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} = 1.543080357.$$

Compared to the actual value of $\cosh 1 = 1.543080635\ldots$, the error is less than 10^{-6}.

(c) Since $\frac{d}{dx}(\cosh x) = \sinh x$, we have

$$\begin{aligned}\sinh x &\approx \frac{d}{dx}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}\right)\\ &= \frac{2x}{2!} + \frac{4x^3}{4!} + \frac{6x^5}{6!} + \frac{8x^7}{8!}\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}.\end{aligned}$$

37. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\sinh 2x = (e^{2x} - e^{-2x})/2$, the Taylor expansion for $\sinh 2x$ is

$$\begin{aligned}\sinh 2x = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}\right) &= \frac{1}{2}\left(\sum_{n=0}^{\infty}(1-(-1)^n)\frac{(2x)^n}{n!}\right)\\ &= \sum_{m=0}^{\infty}\frac{(2x)^{2m+1}}{(2m+1)!}.\end{aligned}$$

Since $\cosh 2x = (e^{2x} + e^{-2x})/2$, we have

$$\begin{aligned}\cosh 2x = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}\right) &= \frac{1}{2}\left(\sum_{n=0}^{\infty}(1+(-1)^n)\frac{(2x)^n}{n!}\right)\\ &= \sum_{m=0}^{\infty}\frac{(2x)^{2m}}{(2m)!}.\end{aligned}$$

38. (a) The Taylor series for $1/(1-x) = 1 + x + x^2 + x^3 + \ldots$, so

$$\begin{aligned}\frac{1}{0.98} = \frac{1}{1-0.02} &= 1 + (0.02) + (0.02)^2 + (0.02)^3 + \ldots\\ &= 1.020408\ldots\end{aligned}$$

(b) Since $d/dx(1/(1-x)) = (1/(1-x))^2$, the Taylor series for $1/(1-x)^2$ is

$$\frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Thus

$$\begin{aligned}\frac{1}{(0.99)^2} = \frac{1}{(1-0.01)^2} &= 1 + 2(0.01) + 3(0.0001) + 4(0.000001) + \cdots\\ &= 1.0203040506\ldots\end{aligned}$$

39. (a)
$$f(x) = (1+ax)(1+bx)^{-1} = (1+ax)\left(1 - bx + (bx)^2 - (bx)^3 + \cdots\right)$$
$$= 1 + (a-b)x + (b^2 - ab)x^2 + \cdots$$

(b) $e^x = 1 + x + \frac{x^2}{2} + \cdots$
Equating coefficients:
$$a - b = 1,$$
$$b^2 - ab = \frac{1}{2}.$$

Solving gives $a = \frac{1}{2}$, $b = -\frac{1}{2}$.

40. (a) If $\phi = 0$,
$$\text{left side} = b(1+1+1) = 3b \approx 0$$
so the equation is almost satisfied and there could be a solution near $\phi = 0$.

(b) We have
$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots$$
$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots$$
So
$$\cos^2\phi = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots\right)\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots\right).$$
Neglecting terms of order ϕ^2 and higher, we get
$$\sin\phi \approx \phi$$
$$\cos\phi \approx 1$$
$$\cos^2\phi \approx 1.$$
So $\phi + b(1+1+1) \approx 0$, whence $\phi \approx -3b$.

41. (a) $\mu = \dfrac{mM}{m+M}$.
If $M >> m$, then the denominator $m + M \approx M$, so $\mu \approx \dfrac{mM}{M} = m$.

(b)
$$\mu = m\left(\frac{M}{m+M}\right) = m\left(\frac{\frac{1}{M}M}{\frac{m}{M} + \frac{M}{M}}\right) = m\left(\frac{1}{1 + \frac{m}{M}}\right)$$
We can use the binomial expansion since $\frac{m}{M} < 1$.
$$\mu = m\left[1 - \frac{m}{M} + \left(\frac{m}{M}\right)^2 - \left(\frac{m}{M}\right)^3 + \cdots\right]$$

(c) If $m \approx \dfrac{1}{1836}M$, then $\frac{m}{M} \approx \frac{1}{1836} \approx 0.000545$.
So a first order approximation to μ would give $\mu = m(1 - 0.000545)$. The percentage difference from $\mu = m$ is -0.0545%.

42. (a) Solving for ω_0 gives
$$\left(\omega_0 L - \frac{1}{\omega_0 C}\right)^2 = 0$$
$$\omega_0 L - \frac{1}{\omega_0 C} = 0$$
$$\omega_0 L = \frac{1}{\omega_0 C}$$
$$\omega_0^2 = \frac{1}{LC}$$
$$\omega_0 = \sqrt{\frac{1}{LC}}.$$
(Note: We discarded the negative root, because we need positive frequencies.)

(b) Set $\omega = \omega_0 + \Delta\omega$ and get

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = \left((\omega_0 + \Delta\omega)L - \frac{1}{(\omega_0 + \Delta\omega)C}\right)^2.$$

We need to find a Taylor expansion for the term $\dfrac{1}{C(\omega_0 + \Delta\omega)}$. Using the binomial expansion we have

$$\begin{aligned}\frac{1}{C(\omega_0 + \Delta\omega)} &= \frac{1}{C\omega_0}\left(1 + \frac{\Delta\omega}{\omega_0}\right)^{-1} \\ &= \frac{1}{C\omega_0}\left(1 - \frac{\Delta\omega}{\omega_0} + \cdots\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\omega L - \frac{1}{\omega C} &= \omega_0 L + L\Delta\omega - \frac{1}{C\omega_0}\left(1 - \frac{\Delta\omega}{\omega_0} + \cdots\right) \\ &= \sqrt{\frac{1}{LC}} \cdot L + L\Delta\omega - \frac{1}{C}\sqrt{\frac{LC}{1}} + \frac{\Delta\omega}{C} \cdot \frac{LC}{1} \\ &= \sqrt{\frac{L}{C}} + L\Delta\omega - \sqrt{\frac{L}{C}} + L\Delta\omega - \cdots \\ &= 2L\Delta\omega - \cdots.\end{aligned}$$

So

$$\left(\omega L - \frac{1}{\omega C}\right)^2 \approx 4L^2(\Delta\omega)^2.$$

Notice that if $\Delta\omega = 0$ in the above expression, we get 0, which is what we expected, since in this case $\omega = \omega_0$.

43. (a) Factoring the expression for $t_1 - t_2$, we get

$$\begin{aligned}\Delta t = t_1 - t_2 &= \frac{2l_2}{c(1 - v^2/c^2)} - \frac{2l_1}{c\sqrt{1 - v^2/c^2}} - \frac{2l_2}{c\sqrt{1 - v^2/c^2}} + \frac{2l_1}{c(1 - v^2/c^2)} \\ &= \frac{2(l_1 + l_2)}{c(1 - v^2/c^2)} - \frac{2(l_1 + l_2)}{c\sqrt{1 - v^2/c^2}} \\ &= \frac{2(l_1 + l_2)}{c}\left(\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}}\right).\end{aligned}$$

Expanding the two terms within the parentheses in terms of v^2/c^2 gives

$$\begin{aligned}\left(1 - \frac{v^2}{c^2}\right)^{-1} &= 1 + \frac{v^2}{c^2} + \frac{(-1)(-2)}{2!}\left(\frac{-v^2}{c^2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{-v^2}{c^2}\right)^3 + \cdots \\ &= 1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots \\ \left(1 - \frac{v^2}{c^2}\right)^{-1/2} &= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(\frac{-v^2}{c^2}\right)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left(\frac{-v^2}{c^2}\right)^3 + \cdots \\ &= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\end{aligned}$$

Thus, we have

$$\begin{aligned}\Delta t &= \frac{2(l_1 + l_2)}{c}\left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots - 1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} - \frac{5}{16}\frac{v^6}{c^6} - \cdots\right) \\ &= \frac{2(l_1 + l_2)}{c}\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{5}{8}\frac{v^4}{c^4} + \frac{11}{16}\frac{v^6}{c^6} + \cdots\right) \\ \Delta t &\approx \frac{(l_1 + l_2)}{c}\left(\frac{v^2}{c^2} + \frac{5}{4}\frac{v^4}{c^4}\right).\end{aligned}$$

(b) For small v. we can neglect all but the first nonzero term, so

$$\Delta t \approx \frac{(l_1+l_2)}{c}\cdot\frac{v^2}{c^2} = \frac{(l_1+l_2)}{c^3}v^2.$$

Thus, Δt is proportional to v^2 with constant of proportionality $(l_1+l_2)/c^3$.

44. (a) Since the expression under the square root sign, $1-\frac{v^2}{c^2}$ must be positive in order to give a real value of m, we have

$$\begin{aligned} 1-\frac{v^2}{c^2} &> 0 \\ \frac{v^2}{c^2} &< 1 \\ v^2 &< c^2, \\ \text{so} \quad -c < v &< c. \end{aligned}$$

In other words, the object can never travel faster that the speed of light.

(b) See Figure 10.7.

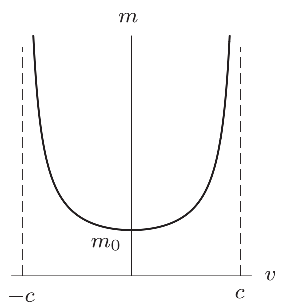


Figure 10.7

(c) Notice that $m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}$. If we substitute $u = -\frac{v^2}{c^2}$, we get $m = m_0(1+u)^{-1/2}$ and we can use the binomial expansion to get:

$$\begin{aligned} m &= m_0\left(1-\frac{1}{2}u+\frac{(-1/2)(-3/2)}{2!}u^2+\cdots\right) \\ &= m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right). \end{aligned}$$

(d) We would expect this series to converge only for values of the original function that exist, namely when $|v| < c$.

45. (a) To find when V takes on its minimum values, set $\frac{dV}{dr} = 0$. So

$$\begin{aligned} -V_0\frac{d}{dr}\left(2\left(\frac{r_0}{r}\right)^6-\left(\frac{r_0}{r}\right)^{12}\right) &= 0 \\ -V_0\left(-12r_0^6r^{-7}+12r_0^{12}r^{-13}\right) &= 0 \\ 12r_0^6r^{-7} &= 12r_0^{12}r^{-13} \\ r_0^6 &= r^6 \\ r &= r_0. \end{aligned}$$

Rewriting $V'(r)$ as $\frac{12r_0^6V_0}{r_7}\left(1-\left(\frac{r_0}{r}\right)^6\right)$, we see that $V'(r) > 0$ for $r > r_0$ and $V'(r) < 0$ for $r < r_0$. Thus, $V = -V_0(2(1)^6-(1)^{12}) = -V_0$ is a minimum.

(Note: We discard the negative root $-r_0$ since the distance r must be positive.)

(b)

$$V(r) = -V_0\left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right) \qquad V(r_0) = -V_0$$
$$V'(r_0) = 0$$
$$V'(r) = -V_0(-12r_0^6 r^{-7} + 12r_0^{12}r^{-13}) \qquad V''(r_0) = 72V_0 r_0^{-2}$$
$$V''(r) = -V_0(84r_0^6 r^{-8} - 156r_0^{12}r^{-14})$$

The Taylor series is thus:

$$V(r) = -V_0 + 72V_0 r_0^{-2} \cdot (r - r_0)^2 \cdot \frac{1}{2} + \cdots$$

(c) The difference between V and its minimum value $-V_0$ is

$$V - (-V_0) = 36V_0 \frac{(r - r_0)^2}{r_0^2} + \cdots$$

which is approximately proportional to $(r - r_0)^2$ since terms containing higher powers of $(r - r_0)$ have relatively small values for r near r_0.

(d) From part (a) we know that $dV/dr = 0$ when $r = r_0$, hence $F = 0$ when $r = r_0$. Since, if we discard powers of $(r - r_0)$ higher than the second,

$$V(r) \approx -V_0\left(1 - 36\frac{(r - r_0)^2}{r_0^2}\right)$$

giving

$$F = -\frac{dV}{dr} \approx 72 \cdot \frac{r - r_0}{r_0^2}(-V_0) = -72V_0\frac{r - r_0}{r_o^2}.$$

So F is approximately proportional to $(r - r_0)$.

46. Solving for P gives

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}.$$

Expanding P in terms of $1/V$ gives

$$P = \frac{nRT}{V}\left(1 - \frac{nb}{V}\right)^{-1} - \frac{n^2 a}{V^2}$$
$$P = \frac{nRT}{V}\left(1 + \frac{nb}{V} + \cdots\right) - \frac{n^2 a}{V^2}$$
$$P = \frac{nRT}{V} + \frac{n^2 bRT}{V^2} - \frac{n^2 a}{V^2}$$
$$P = nRT\left(\frac{1}{V}\right) + n^2(bRT - a)\left(\frac{1}{V}\right)^2 + \cdots.$$

Strengthen Your Understanding

47. By substituting $x = 0$, we get $1/2 = 1$. So this equation is not correct. The Taylor series for $\frac{1}{2+x}$ can be obtained using the Taylor series of $\frac{1}{1+x}$ and writing

$$\frac{1}{2+x} = \frac{1}{2(1 + x/2)}$$
$$= \frac{1}{2}\left(1 + \frac{x}{2}\right)^{-1}$$
$$= \frac{1}{2}\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \cdots\right)$$

Hence the equation given is missing a factor of $1/2$ on the right.

48. The order of operations has not been respected. In e^{-x} the function e^x is composed with the function $-x$. Therefore we have: $e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$.

49. If a function or some of its derivatives are not defined at $x = 0$ then we cannot write a Taylor series for that function. An example is $\ln x$. Many other examples are possible.

50. An example is $f(x) = |x|$. The function $|x|$ is not differentiable at $x = 0$, hence we cannot write its Taylor series around 0. Many other examples are possible.

51. True. Since the derivative of a sum is the sum of the derivatives, Taylor series add.

52. True. Since the Taylor series for $\cos x$ has only even powers, multiplying by x^3 gives only odd powers.

53. False. The derivative of $f(x)g(x)$ is not $f'(x)g'(x)$. If this statement were true, the Taylor series for $(\cos x)(\sin x)$ would have all zero terms.

54. True. We have

$$L_1(x) + L_2(x) = (f_1(0) + f_1'(0)x) + (f_2(0) + f_2'(0)x) = (f_1(0) + f_2(0)) + (f_1'(0) + f_2'(0))x.$$

The right hand side is the linear approximation to $f_1 + f_2$ near $x = 0$.

55. False. The quadratic approximation to $f_1(x)f_2(x)$ near $x = 0$ is

$$f_1(0)f_2(0) + (f_1'(0)f_2(0) + f_1(0)f_2'(0))x + \frac{f_1''(0)f_2(0) + 2f_1'(0)f_2'(0) + f_1(0)f_2''(0)}{2}x^2.$$

On the other hand, we have

$$L_1(x) = f_1(0) + f_1'(0)x, \quad L_2(x) = f_2(0) + f_2'(0)x,$$

so

$$L_1(x)L_2(x) = (f_1(0) + f_1'(0)x)(f_2(0) + f_2'(0)x) = f_1(0)f_2(0) + (f_1'(0)f_2(0) + f_2'(0)f_1(0))x + f_1'(0)f_2'(0)x^2.$$

The first two terms of the right side agree with the quadratic approximation to $f_1(x)f_2(x)$ near $x = 0$, but the term of degree 2 does not.

For example, the linear approximation to e^x is $1+x$, but the quadratic approximation to $(e^x)^2 = e^{2x}$ is $1+2x+2x^2$, not $(1+x)^2 = 1 + 2x + x^2$.

56. (a) $\ln(4 - x) = \ln(4(1 - x/4)) = \ln(4) + \ln(1 - x/4)$, so the Taylor series converges for $-1 < x/4 < 1$, or $-4 < x < 4$. The radius of convergence is 4.

(b) $\ln(4 + x) = \ln(4(1 - (-x/4))) = \ln(4) + \ln(1 - (-x/4))$, so the Taylor series converges for $-1 < -x/4 < 1$, or $4 > x > -4$. The radius convergence is 4.

(c) $\ln(1 + 4x^2) = \ln(1 - (-4x^2))$, so the Taylor series converges for $-1 < -4x^2 < 1$. This gives $x^2 < 1/4$, or $-1/2 < x < 1/2$. The radius of convergence is $1/2$.

57. (c). The Taylor series for $3\tan(x/3) = 3(x/3 + (x/3)^3/3 + 21(x/3)^5/120 + \cdots) = x + x^2/27 + 7x^5/3240 + \cdots$

Solutions for Section 10.4

Exercises

1. The error bound in approximating e^{01} using the Taylor polynomial of degree 3 for $f(x) = e^x$ about $x = 0$ is:

$$|E_3| = |f(0.1) - P_3(0.1)| \leq \frac{M \cdot |0.1 - 0|^4}{4!} = \frac{M(0.1)^4}{24},$$

where $|f^{(4)}(x)| \leq M$ for $0 \leq x \leq 0.1$. Now, $f^{(4)}(x) = e^x$. Since e^x is increasing for all x, we see that $|f^{(4)}(x)|$ is maximized for x between 0 and 0.1 when $x = 0.1$. Thus,

$$|f^{(4)}| \leq e^{0.1},$$

so

$$|E_3| \leq \frac{e^{0.1} \cdot (0.1)^4}{24} = 0.00000460.$$

The Taylor polynomial of degree 3 is

$$P_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.$$

The approximation is $P_3(0.1)$, so the actual error is

$$E_3 = e^{0.1} - P_3(0.1) = 1.10517092 - 1.10516667 = 0.00000425,$$

which is slightly less than the bound.

2. The error bound in approximating $\sin(0.2)$ using the Taylor polynomial of degree 3 for $f(x) = \sin x$ about $x = 0$ is:

$$|E_3| = |f(0.2) - P_3(0.2)| \le \frac{M \cdot |0.2 - 0|^4}{4!} = \frac{M(0.2)^4}{24},$$

where $|f^{(4)}(x)| \le M$ for $0 \le x \le 0.2$. Now, $f^{(4)}(x) = \sin x$. By looking at the graph of $\sin x$, we see that $|f^{(4)}(x)|$ is maximized for x between 0 and 0.2 when $x = 0.2$. Thus,

$$|f^{(4)}| \le \sin(0.2),$$

so

$$|E_3| \le \frac{\sin(0.2) \cdot (0.2)^4}{24} = 0.0000132.$$

The Taylor polynomial of degree 3 is

$$P_3(x) = x - \frac{1}{3!}x^3.$$

The approximation is $P_3(0.1)$, so the actual error is

$$E_3 = \sin(0.2) - P_3(0.2) = 0.19866933 - 0.19866667 = 0.00000266,$$

which is much less than the bound.

3. The error bound in approximating $\cos(-0.3)$ using the Taylor polynomial of degree 3 for $f(x) = \cos x$ about $x = 0$ is:

$$|E_3| = |f(-0.3) - P_3(-0.3)| \le \frac{M \cdot |-0.3 - 0|^4}{4!} = \frac{M(-0.3)^4}{24},$$

where $|f^{(4)}(x)| \le M$ for $0 \ge x \ge -0.3$. Now, $f^{(4)}(x) = \cos x$, which has its largest value of 1 at $x = 0$. Thus,

$$|f^{(4)}| \le \cos 0 = 1,$$

so

$$|E_3| \le \frac{1 \cdot (-0.3)^4}{24} = 0.000338.$$

The Taylor polynomial of degree 3 is

$$P_3(x) = 1 - \frac{1}{2!}x^2.$$

The approximation is $P_3(-0.3)$, so the actual error is

$$E_3 = \cos(-0.3) - P(-0.3) = 0.955336 - 0.955000 = 0.000336,$$

which is slightly less than the bound.

4. The error bound in approximating $\sqrt{0.9}$ using the Taylor polynomial of degree three for $f(x) = \sqrt{1+x}$ about $x = 0$ is:

$$|E_3| = |f(-0.1) - P_3(-0.1)| \le \frac{M \cdot |-0.1 - 0|^4}{4!} = \frac{M \cdot (-0.1)^4}{24},$$

where $|f^{(4)}| \le M$ for $0 \ge x \ge -0.1$. Since

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2},$$

we see that if x is between 0 and -0.1, the maximum is at $x = -0.1$. Thus $|f^{(4)}x)| \le (15/16)(1 - 0.1)^{-7/2}$, so

$$|E_3| \le \frac{15}{16}(1 - 0.1)^{-7/2} \cdot \frac{(-0.1)^4}{24} = 0.00000565.$$

The Taylor polynomial of degree 3 is

$$\begin{aligned}P_3(x) &= 1 + \frac{1}{2}x + \frac{1}{2}(-\frac{1}{2})\frac{x^2}{2!} + \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\frac{x^3}{3!}\\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.\end{aligned}$$

The approximation is $P_3(-0.1)$, so the actual error is

$$E_3 = \sqrt{0.9} - P_3(-0.1) = 0.94868330 - 0.94868750 = -0.00000420,$$

which is slightly less, in absolute value, than the bound.

5. The error bound in approximating $\ln(1.5)$ using the Taylor polynomial of degree 3 for $f(x) = \ln(1+x)$ about $x = 0$ is:

$$|E_4| = |f(0.5) - P_3(0.5)| \le \frac{M \cdot |0.5 - 0|^4}{4!} = \frac{M(0.5)^4}{24},$$

where $|f^{(4)}(x)| \le M$ for $0 \le x \le 0.5$. Since

$$f^{(4)}(x) = \frac{-3!}{(1+x)^4}$$

and the denominator attains its minimum when $x = 0$, we have $|f^{(4)}(x)| \le 3!$, so

$$|E_4| \le \frac{3!\,(0.5)^4}{24} = 0.0156.$$

The Taylor polynomial of degree 3 is

$$\begin{aligned}P_3(x) &= 0 + x + (-1)\frac{x^2}{2!} + (-1)(-2)\frac{x^3}{3!}\\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3.\end{aligned}$$

The approximation is $P_3(0.5)$, so the actual error is

$$E_3 = \ln(1.5) - P_3(0.5) = 0.4055 - 0.4167 = -0.0112$$

which is slightly less, in absolute value, than the bound.

6. The error bound in approximating $1/\sqrt{3}$ using the Taylor polynomial of degree three for $f(x) = (1+x)^{-1/2}$ about $x = 0$ is:

$$|E_3| = |f(2) - P_3(2)| \le \frac{M \cdot |2 - 0|^4}{4!} = \frac{M \cdot 2^4}{24},$$

where $|f^{(4)}| \le M$ for $0 \le x \le 2$. Since

$$f^{(4)}(x) = \frac{105}{16}(1+x)^{-9/2},$$

we see that if x is between 0 and 2, then $|f^{(4)}x)| \le \frac{105}{16}$. Thus,

$$|E_3| \le \frac{105}{16} \cdot \frac{2^4}{24} = \frac{105}{24} = 4.375.$$

This is not a very helpful bound on the error, but that is to be expected as the Taylor series does not converge at $x = 2$. (At $x = 2$, we are outside the interval of convergence.)

The Taylor polynomial of degree 3 is

$$\begin{aligned}P_3(x) &= 1 + (\frac{1}{2})x + (-\frac{1}{2})(-\frac{3}{2})\frac{x^2}{2!} + (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\frac{x^3}{3!}\\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3.\end{aligned}$$

The approximation is $P_3(2)$, so the actual error is

$$E_3 = \frac{1}{\sqrt{3}} - P_3(2) = 0.577 - (-1.0) = 1.577,$$

which is much less than the bound, but still very large.

7. The error bound in approximating $\tan(1)$ using the Taylor polynomial of degree three for $f(x) = \tan x$ about $x = 0$ is:

$$|E_3| = |f(1) - P_3(x)| \leq \frac{M \cdot |1-0|^4}{4!} = \frac{M}{24}$$

where $|f^{(4)}(x)| \leq M$ for $0 \leq x \leq 1$. Now,

$$f^{(4)}(x) = \frac{16 \sin x}{\cos^3 x} + \frac{24 \sin^3 x}{\cos^5 x}.$$

From a graph of $f^{(4)}(x)$, we see that $f^{(4)}(x)$ is increasing for x between 0 and 1. Thus,

$$|f^{(4)}(x)| \leq |f^{(4)}(1)| = 396,$$

so

$$|E_3| \leq \frac{396}{24} = 16.5.$$

This is not a very helpful error bound! The reason the error bound is so huge is that $x = 1$ is getting near the vertical asymptote of the tangent graph, and the fourth derivative is enormous there.

Observing that any term in $f''(x)$ or $f'''(x)$ involving $\tan x$ is zero at $x = 0$, we calculate the Taylor polynomial of degree 3:

$$P_3(x) = 0 + x + (0)\frac{x^2}{2!} + (2)\frac{x^3}{3!}$$
$$= x + \frac{1}{3}x^3.$$

The approximation is $P_3(1)$, so the actual error is

$$E_3 = \tan 1 - P_3(1) = 1.557 - 1.333 = 0.224,$$

which is much less than the bound.

8. The error bound in approximating $0.5^{1/3}$ using the Taylor polynomial of degree 3 for $f(x) = (1-x)^{1/3}$ about $x = 0$ is:

$$|E_3| = |f(0.5) - P_3(0.5)| \leq \frac{M \cdot |0.5-0|^4}{4!} = \frac{M(0.5)^4}{24},$$

where $|f^{(4)}(x)| \leq M$ for $0 \leq x \leq 0.5$. Now,

$$f^{(4)}(x) = -\frac{80}{81}(1-x)^{-11/3}.$$

By looking at the graph of $(1-x)^{-11/3}$, we see that $|f^{(4)}(x)|$ is maximized for x between 0 and 0.5 when $x = 0.5$. Thus,

$$|f^{(4)}| \leq \frac{80}{81}\left(\frac{1}{2}\right)^{-11/3} = \frac{80}{81} \cdot 2^{11/3},$$

so

$$|E_3| \leq \frac{80 \cdot 2^{11/3} \cdot (0.5)^4}{81 \cdot 24} = 0.033.$$

The Taylor polynomial of degree 3 is

$$P_3(x) = 1 + \frac{1}{3}(-x) + \frac{1}{3}\frac{-2}{3}\frac{(-x)^2}{2!} + \frac{1}{3}\frac{-2}{3}\frac{-5}{3}\frac{(-x)^3}{3!}$$
$$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3.$$

The approximation is $P_3(0.5)$, so the actual error is

$$E_3 = (0.5)^{1/3} - P_3(0.5) = 0.79370 - 0.79784 = -0.00414,$$

which is much less than the bound.

Problems

9. (a) See the middle row of Table 10.1.

(b) See the bottom row of Table 10.1.

Table 10.1

x	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.5
$\sin x$	-0.4794	-0.3894	-0.2955	-0.1987	-0.0998	0	0.0998	0.1987	0.2955	0.3894	0.4794
E_1	0.0206	0.0106	0.0045	0.0013	0.0002	0	-0.0002	-0.0013	-0.0045	-0.0106	-0.0206

(c) See Figure 10.8.

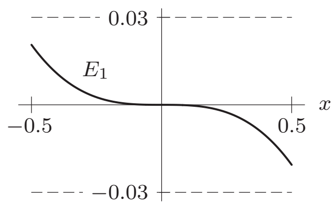


Figure 10.8

The fact that the graph of E_1 lies between the horizontal lines at ± 0.03 shows that $|E_1| < 0.03$ for $-0.5 \leq x \leq 0.5$.

10. Let $f(x) = \sqrt{1+x}$. We use a Taylor polynomial with $x = 1$ to approximate $\sqrt{2}$. The error bound for the Taylor approximation of degree three for $f(x) = \sqrt{2}$ about $x = 0$ is:

$$|E_3| = |f(1) - P_3(1)| \leq \frac{M \cdot |1-0|^4}{4!} = \frac{M}{24},$$

where $|f^{(4)}(x)| \leq M$ for $0 \leq x \leq 1$.

Now,

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2} = \frac{-15}{16(1+x)^{7/2}}.$$

Since $1 \leq (1+x)^{7/2}$ for x between 0 and 1, we see that

$$|f^{(4)}(x)| = \frac{15}{16(1+x)^{\frac{7}{2}}} \leq \frac{15}{16}$$

for x between 0 and 1. Thus,

$$|E_3| \leq \frac{15}{16 \cdot 24} < 0.039$$

11. (a) The third degree Taylor approximation of degree 3 of e^x around 0 is

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Using the third-degree error bound, if $|f^{(4)}(x)| \leq M$ for $-2 \leq x \leq 2$, then

$$|E_3(x)| = |f(x) - P_3(x)| \leq \frac{M}{4!} \cdot |x|^4 \leq \frac{M2^4}{4!}.$$

Since $|f^{(4)}(x)| = e^x$, and e^x is increasing on $[-2, 2]$,

$$f^{(4)}(x) \leq e^2 \text{ for all } x \in [-2, 2].$$

So we can let $M = e^2$ and we get

$$|E_3(x)| < \frac{e^2 \cdot 2^4}{4!} \approx 5$$

(b) The actual maximum error is $|e^2 - P_3(2)| = 1.06$.

12. For $f(x) = \cos x$, note that $f^{n+1}(x)$, is $\pm\cos x$ or $\pm\sin x$, no matter what n is. So

$$\left|f^{n+1}(x)\right| < 1 \text{ for any } x.$$

By the Lagrange error bound with $M = 1$, we have

$$E_n(x) = |f(x) - P_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

The right hand side of the equation above goes to zero as $n \to \infty$.

Hence we can always find n large enough that $\dfrac{|x|^{n+1}}{(n+1)!} < 10^{-9}$ for all x.

Note here that 10^{-9} in the question does not play any role. We could have replaced it by any small number.

13. (a) θ is the first degree approximation of $f(\theta) = \sin\theta$; it is also the second degree approximation, since the next term in the Taylor expansion is 0.
$P_1(\theta) = \theta$ is an overestimate for $0 < \theta \leq 1$, and is an underestimate for $-1 \leq \theta < 0$. (This can be seen easily from a graph.)

(b) Using the second degree error bound, if $|f^{(3)}(\theta)| \leq M$ for $-1 \leq \theta \leq 1$, then

$$|E_2| \leq \frac{M \cdot |\theta|^3}{3!} \leq \frac{M}{6}.$$

For what value of M is $|f^{(3)}(\theta)| \leq M$ for $-1 \leq \theta \leq 1$? Well, $|f^{(3)}(\theta)| = |-\cos\theta| \leq 1$. So $|E_2| \leq \frac{1}{6} = 0.17$.

14. (a) $\theta - \dfrac{\theta^3}{3!}$ is the third degree Taylor approximation of $f(\theta) = \sin\theta$; it is also the fourth degree approximation, since the next term in the Taylor expansion is 0.
$P_3(\theta)$ is an underestimate for $0 < \theta \leq 1$, and is an overestimate for $-1 \leq \theta < 0$. (This can be checked with a calculator.)

(b) Using the fourth degree error bound, if $|f^{(5)}(\theta)| \leq M$ for $-1 \leq \theta \leq 1$, then

$$|E_4| \leq \frac{M \cdot |\theta|^5}{5!} \leq \frac{M}{120}.$$

For what value of M is $|f^{(5)}(\theta)| \leq M$ for $-1 \leq \theta \leq 1$? Since $f^{(5)}(\theta) = \cos\theta$ and $|\cos\theta| \leq 1$, we have

$$|E_4| \leq \frac{1}{120} \leq 0.0084.$$

15. (a) The Taylor polynomial of degree 0 about $t = 0$ for $f(t) = e^t$ is simply $P_0(x) = 1$. Since $e^t \geq 1$ on $[0, 0.5]$, the approximation is an underestimate.

(b) Using the zero degree error bound, if $|f'(t)| \leq M$ for $0 \leq t \leq 0.5$, then

$$|E_0| \leq M \cdot |t| \leq M(0.5).$$

Since $|f'(t)| = |e^t| = e^t$ is increasing on $[0, 0.5]$,

$$|f'(t)| \leq e^{0.5} < \sqrt{4} = 2.$$

Therefore

$$|E_0| \leq (2)(0.5) = 1.$$

(Note: By looking at a graph of $f(t)$ and its 0^{th} degree approximation, it is easy to see that the greatest error occurs when $t = 0.5$, and the error is $e^{0.5} - 1 \approx 0.65 < 1$. So our error bound works.)

16. (a) The second-degree Taylor polynomial for $f(t) = e^t$ is $P_2(t) = 1 + t + t^2/2$. Since the full expansion of $e^t = 1 + t + t^2/2 + t^3/6 + t^4/24 + \cdots$ is clearly larger than $P_2(t)$ for $t > 0$, $P_2(t)$ is an underestimate on $[0, 0.5]$.

(b) Using the second-degree error bound, if $|f^{(3)}(t)| \leq M$ for $0 \leq t \leq 0.5$, then

$$|E_2| \leq \frac{M}{3!} \cdot |t|^3 \leq \frac{M(0.5)^3}{6}.$$

Since $|f^{(3)}(t)| = e^t$, and e^t is increasing on $[0, 0.5]$,

$$f^{(3)}(t) \leq e^{0.5} < \sqrt{4} = 2.$$

So

$$|E_2| \leq \frac{(2)(0.5)^3}{6} < 0.047.$$

17. (a) (i) The vertical distance between the graph of $y = \cos x$ and $y = P_{10}(x)$ at $x = 6$ is no more than 4, so

$$|\text{Error in } P_{10}(6)| \le 4.$$

Since at $x = 6$ the $\cos x$ and $P_{20}(x)$ graphs are indistinguishable in this figure, the error must be less than the smallest division we can see, which is about 0.2 so,

$$|\text{Error in } P_{20}(6)| \le 0.2.$$

(ii) The maximum error occurs at the ends of the interval, that is, at $x = -9, x = 9$. At $x = 9$, the graphs of $y = \cos x$ and $y = P_{20}(x)$ are no more than 1 apart, so

$$\left|\begin{array}{l}\text{Maximum error in } P_{20}(x)\\ \text{for } -9 \le x \le 9\end{array}\right| \le 1.$$

(b) We are looking for the largest x-interval on which the graphs of $y = \cos x$ and $y = P_{10}(x)$ are indistinguishable. This is hard to estimate accurately from the figure, though $-4 \le x \le 4$ certainly satisfies this condition.

18. The maximum possible error for the n^{th}-degree Taylor polynomial about $x = 0$ approximating $\cos x$ is $|E_n| \le \frac{M \cdot |x-0|^{n+1}}{(n+1)!}$, where $|\cos^{(n+1)} x| \le M$ for $0 \le x \le 1$. Now the derivatives of $\cos x$ are simply $\cos x, \sin x, -\cos x$, and $-\sin x$. The largest magnitude these ever take is 1, so $|\cos^{(n+1)}(x)| \le 1$, and thus $|E_n| \le \frac{|x|^{n+1}}{(n+1)!} \le \frac{1}{(n+1)!}$. The same argument works for $\sin x$.

19. By the results of Problem 18, if we approximate $\cos 1$ using the n^{th} degree polynomial, the error is at most $\frac{1}{(n+1)!}$. For the answer to be correct to four decimal places, the error must be less than 0.00005. Thus, the first n such that $\frac{1}{(n+1)!} < 0.00005$ will work. In particular, when $n = 7$, $\frac{1}{8!} = \frac{1}{40370} < 0.00005$, so the 7^{th} degree Taylor polynomial will give the desired result. For six decimal places, we need $\frac{1}{(n+1)!} < 0.0000005$. Since $n = 9$ works, the 9^{th} degree Taylor polynomial is sufficient.

20. The graph of E_0 looks like a parabola, and the graph shows

$$|E_0| < 0.01 \quad \text{for} \quad |x| \le 0.1.$$

(In fact $|E_0| < 0.005$ on this interval.) Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$
$$E_0 = \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

So, for small x,

$$E_0 \approx -\frac{x^2}{2},$$

and therefore the graph of E_0 is parabolic. See Figure 10.9.

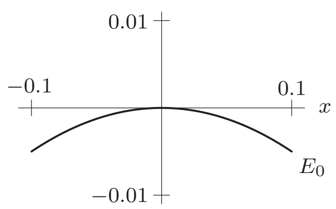


Figure 10.9

21. Since $f(x) = e^x$, the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ is also e^x, no matter what n is. Now fix a number x and let $M = e^x$, then $|f^{(n+1)}(t)| \le e^t \le e^x$ on the interval $0 \le t \le x$. (This works for $x \ge 0$; if $x < 0$ then we can take $M = 1$.) The important observation is that for any x the *same* number M bounds all the higher derivatives $f^{(n+1)}(x)$.

By the error bound formula, we now have

$$|E_n(x)| = |e^x - P_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!} \quad \text{for every } n.$$

To show that the errors go to zero, we must show that for a fixed x and a fixed number M,

$$\frac{M}{(n+1)!}|x|^{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Since M is fixed, we need only show that

$$\frac{1}{(n+1)!}|x|^{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

This was shown in the text on page 562. Therefore, the Taylor series $1 + x + x^2/2! + \cdots$ does converge to e^x.

22.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Write the error in approximating $\sin x$ by the Taylor polynomial of degree $n = 2k+1$ as E_n so that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots (-1)^k \frac{x^{2k+1}}{(2k+1)!} + E_n.$$

(Notice that $(-1)^k = 1$ if k is even and $(-1)^k = -1$ if k is odd.) We want to show that if x is fixed, $E_n \to 0$ as $k \to \infty$. Since $f(x) = \sin x$, all the derivatives of $f(x)$ are $\pm \sin x$ or $\pm \cos x$, so we have for all n and all x

$$|f^{(n+1)}(x)| \leq 1.$$

Using the bound on the error given in the text on page 562, we see that

$$|E_n| \leq \frac{1}{(2k+2)!}|x|^{2k+2}.$$

By the argument in the text on page 562, we know that for all x,

$$\frac{|x|^{2k+2}}{(2k+2)!} = \frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n = 2k+1 \to \infty.$$

Thus the Taylor series for $\sin x$ does converge to $\sin x$ for every x.

23. (a) Since $4 \arctan 1 = \pi$, we approximate π by approximating $4 \arctan x$ by Taylor polynomials with $x = 1$. Let $f(x) = 4 \arctan x$. We find the Taylor polynomial of f about $x = 0$.

$$\begin{aligned}
f(0) &= 0 \\
f'(x) &= \frac{4}{1+x^2} \quad \text{so} \quad f'(0) = 4 \\
f''(x) &= -\frac{8x}{(1+x^2)^2} \quad \text{so} \quad f''(0) = 0 \\
f'''(x) &= -\frac{8}{(1+x^2)^2} + \frac{32x^2}{(1+x^2)^3} \quad \text{so} \quad f'''(0) = -8.
\end{aligned}$$

Thus, the third degree Taylor polynomial for f is

$$F_3(x) = \frac{4x}{1!} - \frac{8}{3!}x^3 = 4x - \frac{4}{3}x^3.$$

In particular,

$$F_3(1) = 4 - \frac{4}{3} = \frac{8}{3} \approx 2.67.$$

Note: If you already have the Taylor series for $1/(1+x^2)$, the Taylor polynomial for $\arctan x$ can also be found by integration.

(b) We now approximate π by looking at $g(x) = 2\arcsin x$ about $x = 0$ and substituting $x = 1$.

$$\begin{aligned} g(0) &= 0 \\ g'(x) &= \frac{2}{\sqrt{1-x^2}} \quad \text{so} \quad g'(0) = 2 \\ g''(x) &= \frac{2x}{(1-x^2)^{\frac{3}{2}}} \quad \text{so} \quad g''(0) = 0 \\ g'''(x) &= \frac{2}{(1-x^2)^{\frac{3}{2}}} + \frac{6x^2}{(1-x^2)^{\frac{5}{2}}} \quad \text{so} \quad g'''(0) = 2. \end{aligned}$$

Thus, the third degree Taylor polynomial for g is

$$G_3(x) = \frac{2x}{1!} + \frac{2x^3}{3!} = 2x + \frac{1}{3}x^3.$$

In particular,

$$G_3(1) = \frac{7}{3} \approx 2.33.$$

Note: If you already have the Taylor series for $1/\sqrt{1-x^2}$, the Taylor polynomial for $\arcsin x$ can also be found by integration.

(c) To estimate the maximum possible error, $|E_3|$, in the approximation using the arctangent, we need a bound on the fourth derivative of $f(x) = \arctan x$ on $0 \le x \le 1$. Since

$$f^{(4)}(x) = -\frac{192x^3}{(1+x^2)^4} + \frac{96x}{(1+x^2)^3},$$

now use a graphing calculator to see that the maximum value of $|f^{(4)}(x)|$, on $0 \le x \le 1$ is about 18.6. Thus,

$$|E_3| \le \frac{18.6}{4!} \approx 0.78.$$

(Notice that $\pi \approx 3.14$ is within 0.78 of 2.67.)

(d) To estimate the maximum possible error, $|E_n|$, in an approximation using the arcsine, we need a bound on the derivatives of $g(x) = \arcsin x$ on $0 \le x \le 1$. The derivatives of $\arcsin x$ contain terms of the form $(1-x^2)^{-a}$, for some positive a. As x gets close to 1, the value of $(1-x^2)^{-a}$ approaches ∞. Thus, we cannot get a bound on the derivatives of $\arcsin x$, so the error formula does not give us a bound on $|E_n|$.

Strengthen Your Understanding

24. This statement is not correct, since $f(x)$ and its Taylor approximation $P_n(x)$ around a have the same value at $x = a$. So $f(a) = P_n(a)$ for all n.

25. We can make the error $|f(x) - P_n(x)|$ as small as we want for any given x by picking a large n. However, we may not be able to find a Taylor polynomial that produces a small error for all values of x simultaneously.

For example, if $f(x) = \cos x$, then every Taylor polynomial $P_n(x)$ goes to $\pm\infty$ as $x \to \infty$ and as $x \to -\infty$. This means that no matter what n we choose, for large values of x, the value of $P_n(x)$ is more than 1 away from $f(x)$.

26. A possible answer is $\sin x$, or e^x. Other examples are possible.

27. We'll try using Taylor polynomials for the function $f(x) = 1/x$ about $x = 1$. We have

$$f(x) = \frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n \cdot (x-1)^n$$

We know that, if $P_n(x)$ is the n^{th}-degree Taylor polynomial for $f(x)$ at $x = 1$, then

$$|f(x) - P_n(x)| \le \frac{M}{(n+1)!} \cdot |x-1|^{n+1},$$

where M is the maximum value of the absolute value of the $(n+1)^{\text{st}}$ derivative of $f(x)$ on $[1, 1.5]$. The $(n+1)^{\text{st}}$ derivative of $f(x)$ is

$$f^{(n+1)}(x) = \frac{(-1)^{n+1} \cdot (n+1)!}{x^{n+2}},$$

and the absolute value of this derivative is never more than $(n+1)!$ on $[1, 1.5]$. So $M = (n+1)!$, and therefore

$$|f(x) - P_n(x)| \leq |x-1|^{n+1}.$$

We want to make sure this is less than 0.1 on the interval $[1, 1.5]$, so we want $|1.5-1|^{n+1} < 0.1$. This is true if $n = 3$, so we use the Taylor polynomial

$$P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

28. According to the error formula we need $\max |f^{(n+1)}| \leq M$ on the interval between 0 and x. Since we are working with the second-degree Taylor polynomial we have $n = 2$. Therefore, we need a function and an interval such that the third derivative of the function on that interval is less or equal to 4. We can choose, for example, $f(x) = e^x$ and c small enough such that $f^{(3)}(x) = e^x \leq 4$ on $[-c, c]$. There are many choices of c, for example $c = 1$ would work.

29. False. If f is itself a polynomial of degree n then it is equal to its n^{th} Taylor polynomial.

30. True. By Theorem 10.1, $|E_n(x)| < 10|x|^{n+1}/(n+1)!$. Since $\lim_{n\to\infty} |x|^{n+1}/(n+1)! = 0$, $E_n(x) \to 0$ as $n \to \infty$, so the Taylor series converges to $f(x)$ for all x.
True

31. False. The Taylor series for f near $x = 0$ always converges at $x = 0$, since $\sum_{n=0}^{\infty} C_n x^n$ at $x = 0$ is just the constant C_0.

32. True. When $x = 1$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}.$$

Since $f^{(n)}(0) \geq n!$, the terms of this series are all greater than 1. So the series cannot converge

33. False. For example, if $f^{(n)}(0) = n!$, then the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n,$$

which converges at $x = 1/2$.

Solutions for Section 10.5

Exercises

1. No, a Fourier series has terms of the form $\cos nx$, not $\cos^n x$.

2. Not a Fourier series because terms are not of the form $\sin nx$.

3. Yes. Terms are of the form $\sin nx$ and $\cos nx$.

4. Yes. This is a Fourier series where the $\cos nx$ terms all have coefficients of zero.

5.

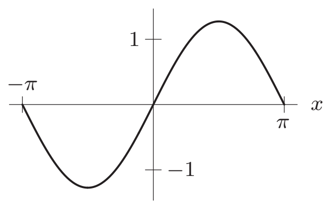
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\,dx + \int_{0}^{\pi} 1\,dx\right] = 0$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x\,dx + \int_{0}^{\pi} \cos x\,dx\right]$$
$$= \frac{1}{\pi}\left[-\sin x\Big|_{-\pi}^{0} + \sin x\Big|_{0}^{\pi}\right] = 0.$$

Similarly, a_2 and a_3 are both 0. See Figure 10.10.
(In fact, notice $f(x)\cos nx$ is an odd function, so $\int_{-\pi}^{\pi} f(x)\cos nx = 0$.)

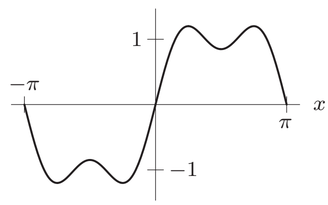
$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin x\,dx + \int_{0}^{\pi}\sin x\,dx\right]$$
$$= \frac{1}{\pi}\left[\cos x\Big|_{-\pi}^{0} + (-\cos x)\Big|_{0}^{\pi}\right] = \frac{4}{\pi}$$
$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin 2x\,dx + \int_{0}^{\pi}\sin 2x\,dx\right]$$
$$= \frac{1}{\pi}\left[\frac{1}{2}\cos 2x\Big|_{-\pi}^{0} + (-\frac{1}{2}\cos 2x)\Big|_{0}^{\pi}\right] = 0.$$
$$b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 3x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin 3x\,dx + \int_{0}^{\pi}\sin 3x\,dx\right]$$
$$= \frac{1}{\pi}\left[\frac{1}{3}\cos 3x\Big|_{-\pi}^{0} + (-\frac{1}{3}\cos 3x)\Big|_{0}^{\pi}\right] = \frac{4}{3\pi}.$$

Thus, $F_1(x) = F_2(x) = \frac{4}{\pi}\sin x$ and $F_3(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$. See Figure 10.11.



$F_1(x) = F_2(x) = \frac{4}{\pi}\sin x$

Figure 10.10



$F_3(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$

Figure 10.11

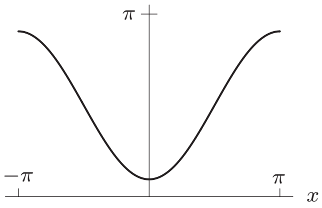
6. First,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right] = \frac{1}{2\pi}\left[-\frac{x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi}\right] = \frac{\pi}{2}.$$

To find the a_i's, we use the integral table. For $n \geq 1$,

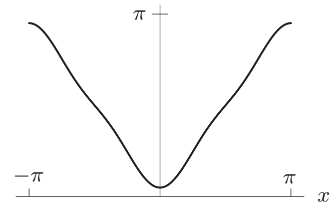
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right]$$
$$= \frac{1}{\pi}\left[\left(-\frac{x}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right)\Big|_{-\pi}^{0} + \left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\Big|_{0}^{\pi}\right]$$
$$= \frac{1}{\pi}\left(-\frac{1}{n^2} + \frac{1}{n^2}\cos(-n\pi) + \frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right)$$
$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

Thus, $a_1 = -\frac{4}{\pi}, a_2 = 0$, and $a_3 = -\frac{4}{9\pi}$. See Figure 10.12. To find the b_i's, note that $f(x)$ is even, so for $n \geq 1$, $f(x)\sin(nx)$ is odd. Thus, $\int_{-\pi}^{\pi} f(x)\sin(nx) = 0$, so all the b_i's are 0. $F_1 = F_2 = \frac{\pi}{2} - \frac{4}{\pi}\cos x$, $F_3 = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$. See Figure 10.13.



$F_1(x) = F_2(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x$

Figure 10.12



$F_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$

Figure 10.13

7. The energy of the function $f(x)$ is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{1}{3\pi}x^3\Big|_{-\pi}^{\pi}$$
$$= \frac{1}{3\pi}(\pi^3 - (-\pi^3)) = \frac{2\pi^3}{3\pi} = \frac{2}{3}\pi^2 = 6.57974.$$

From Problem 6, we know all the b_i's are 0 and $a_0 = \frac{\pi}{2}$, $a_1 = -\frac{4}{\pi}$, $a_2 = 0$, $a_3 = -\frac{4}{9\pi}$. Therefore the energy in the constant term and first three harmonics is

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 = 2a_0^2 + a_1^2 + a_2^2 + a_3^2$$
$$= 2\left(\frac{\pi^2}{4}\right) + \frac{16}{\pi^2} + 0 + \frac{16}{81\pi^2} = 6.57596$$

which means that they contain $\dfrac{6.57596}{6.57974} = 0.99942 \approx 99.942\%$ of the total energy.

8. First, we find a_0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{1}{2\pi}\left(\frac{x^3}{3}\Big|_{-\pi}^{\pi}\right) = \frac{\pi^2}{3}$$

To find $a_n, n \geq 1$, we use the integral table (III-15 and III-16).

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx = \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]\Big|_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos(n\pi) + \frac{2\pi}{n^2}\cos(-n\pi)\right]$$
$$= \frac{4}{n^2}\cos(n\pi)$$

Again, $\cos(n\pi) = (-1)^n$ for all integers n, so $a_n = (-1)^n\frac{4}{n^2}$. Note that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\,dx.$$

x^2 is an even function, and $\sin nx$ is odd, so $x^2\sin nx$ is odd. Thus $\int_{-\pi}^{\pi}x^2\sin nx\,dx = 0$, and $b_n = 0$ for all n. We deduce that the n^{th} Fourier polynomial for f (where $n \geq 1$) is

$$F_n(x) = \frac{\pi^2}{3} + \sum_{i=1}^{n}(-1)^i\frac{4}{i^2}\cos(ix).$$

In particular, we have the graphs in Figure 10.14.

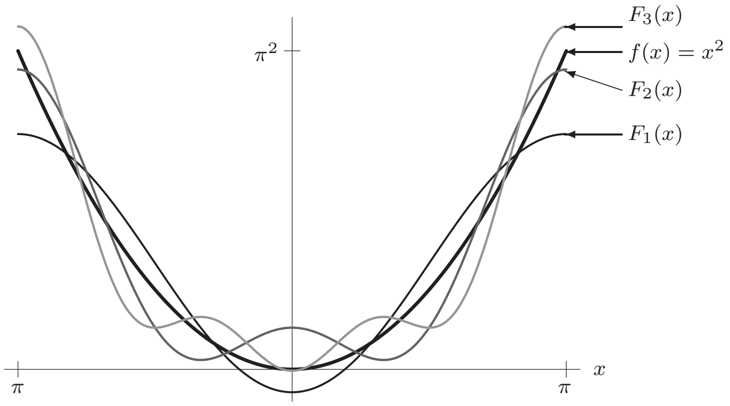


Figure 10.14

9.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}$$

As in Problem 10, we use the integral table (III-15 and III-16) to find formulas for a_n and b_n.

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx &= \frac{1}{\pi}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\Bigg|_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right) \\ &= \frac{1}{n^2\pi}\Bigg(\cos(n\pi) - 1\Bigg). \end{aligned}$$

Note that since $\cos(n\pi) = (-1)^n$, $a_n = 0$ if n is even and $a_n = -\frac{2}{n^2\pi}$ if n is odd.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx \\ &= \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Bigg|_0^{\pi} \\ &= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi)\right) \\ &= -\frac{1}{n}\cos(n\pi) \\ &= \frac{1}{n}(-1)^{n+1} \quad \text{if } n \geq 1 \end{aligned}$$

We have that the n^{th} Fourier polynomial for h (for $n \geq 1$) is

$$H_n(x) = \frac{\pi}{4} + \sum_{i=1}^{n}\left(\frac{1}{i^2\pi}\Bigg(\cos(i\pi) - 1\Bigg)\cdot\cos(ix) + \frac{(-1)^{i+1}\sin(ix)}{i}\right).$$

This can also be written as

$$H_n(x) = \frac{\pi}{4} + \sum_{i=1}^{n}\frac{(-1)^{i+1}\sin(ix)}{i} + \sum_{i=1}^{\left[\frac{n}{2}\right]}\frac{-2}{(2i-1)^2\pi}\cos((2i-1)x)$$

where $\left[\frac{n}{2}\right]$ denotes the biggest integer smaller than or equal to $\frac{n}{2}$. In particular, we have the graphs in Figure 10.15.

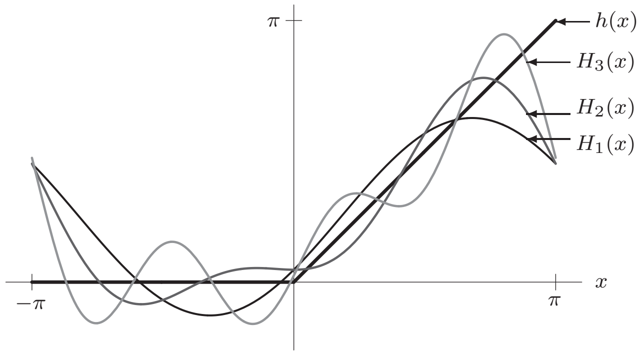


Figure 10.15

10. To find the n^{th} Fourier polynomial, we must come up with a general formula for a_n and b_n. First, we find a_0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\bigg|_{-\pi}^{\pi}\right] = 0$$

Now we use the integral table (III-15 and III-16) to find a_n and b_n for $n \geq 1$.

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx &= \frac{1}{\pi}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\cos(-n\pi)\right) = 0 \end{aligned}$$

(Note that since $x\cos nx$ is odd, we could have deduced that $\int_{-\pi}^{\pi} x\cos nx = 0$.)

$$\begin{aligned} b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx &= \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) - \frac{\pi}{n}\cos(-n\pi)\right) \\ &= -\frac{2}{n}\cos(n\pi) \end{aligned}$$

Notice that $\cos(n\pi) = (-1)^n$ for all integers n, so $b_n = (-1)^{n+1}(\frac{2}{n})$.
Thus the n^{th} Fourier polynomial for g is

$$G_n(x) = \sum_{i=1}^{n}(-1)^{i+1}\frac{2}{i}\sin(ix).$$

In particular, we have the graphs in Figure 10.16.

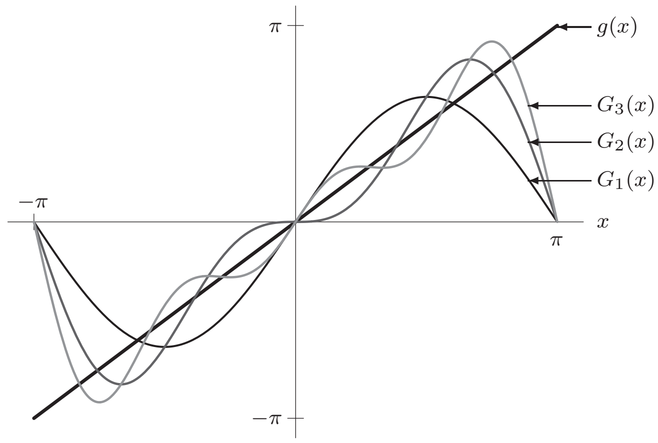


Figure 10.16

Problems

11. The period of $f(x)$ is equal to 2. Since a_0, the constant term of the Fourier series of $f(x)$ is the average value of f over the interval $[-1, 1]$, it is given by the following integral:

$$a_0 = \frac{1}{2}\int_{-1}^{1} |x| dx.$$

By looking at the graph of $f(x)$ between -1 and 1, we see that the area under f is equal to 1. Multiplying this by $1/2$ we have

$$a_0 = \frac{1}{2}.$$

We can check this analytically:

$$a_0 = \frac{1}{2}\int_{-1}^{1} |x| dx = \frac{1}{2}\left(\int_{-1}^{0}(-x)dx + \int_{0}^{1} x dx\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}.$$

12. In Problem 10 we found that the Fourier polynomial of $g(x) = x$ of degree n is:

$$G_n(x) = \sum_{i=1}^{n}(-1)^{i+1}\frac{2}{i}\sin(ix).$$

Hence the Fourier series of $g(x)$ is:

$$g(x) = x = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{2}{i}\sin(ix).$$

By substituting $x = \pi/2$ and expanding we get

$$\frac{\pi}{2} = 2 - \frac{2}{3} + \frac{2}{5} - \cdots$$

Factoring 2 out from the right hand side and dividing both sides by 2, we have

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{2k-1} = \frac{\pi}{4}.$$

13. (a) (i) The graph of $y = \sin x + \frac{1}{3}\sin 3x$ is in Figure 10.17.

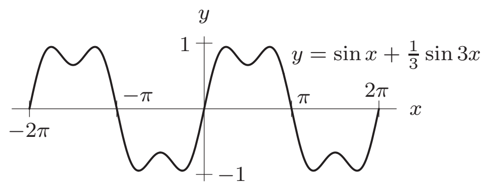


Figure 10.17

(ii) The graph of $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$ is in Figure 10.17.

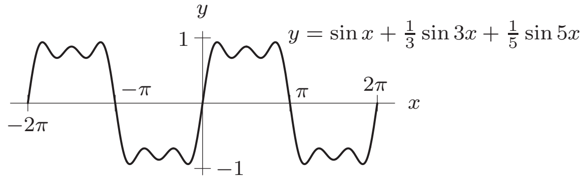


Figure 10.18

(b) Following the pattern, we add the term $\frac{1}{7}\sin 7x$ to get Figure 10.19.

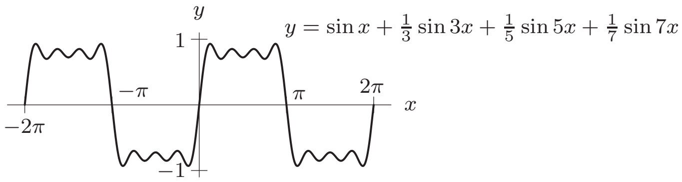


Figure 10.19

(c) The equation is

$$f(x) = \begin{cases} \vdots & \vdots \\ 1 & -2\pi \le x < -\pi \\ -1 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \\ -1 & \pi \le x < 2\pi \\ \vdots & \vdots \end{cases}$$

The square wave function is not continuous at $x = 0, \ \pm\pi, \ \pm 2\pi, \ldots$. See Figure 10.20.

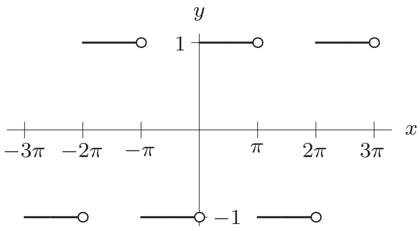


Figure 10.20

14. (a) The graph of $g(x)$ is in Figure 10.21.

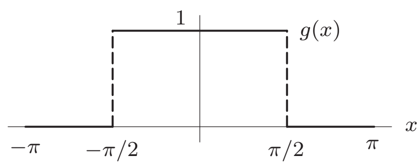


Figure 10.21

First find the Fourier coefficients: a_0 is the average value of g on $[-\pi, \pi]$ so from the graph, it is clear that

$$a_0 = \frac{1}{2\pi}(\pi \times 1) = \frac{1}{2},$$

or analytically,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{2\pi}x\Big|_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$
$$= \frac{1}{2\pi}(\pi) = \frac{1}{2},$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{k\pi}\left(\sin\frac{k\pi}{2} - \sin\left(-\frac{k\pi}{2}\right)\right) = \frac{1}{k\pi}\left(2\sin\frac{k\pi}{2}\right),$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-\pi/2}^{\pi/2}$$
$$= -\frac{1}{k\pi}\left(\cos\frac{k\pi}{2} - \cos\left(-\frac{k\pi}{2}\right)\right). = -\frac{1}{k\pi}(0) = 0$$

So,

$$a_1 = \frac{1}{\pi}\left(2\sin\frac{\pi}{2}\right) = \frac{2}{\pi},$$
$$a_2 = \frac{1}{2\pi}\left(2\sin\frac{2\pi}{2}\right) = 0,$$
$$a_3 = \frac{1}{3\pi}\left(2\sin\frac{3\pi}{2}\right) = -\frac{2}{3\pi},$$

which gives

$$F_3(x) = \frac{1}{2} + \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x.$$

See Figure 10.22.

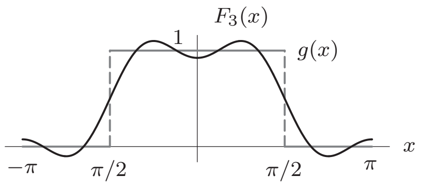


Figure 10.22

(b) There are cosines instead of sines (but the energy spectrum remains the same).

15. We have $f(x) = x$, $0 \le x < 1$. Let $t = 2\pi x - \pi$. Notice that as x varies from 0 to 1, t varies from $-\pi$ to π. Thus if we rewrite the function in terms of t, we can find the Fourier series in terms of t in the usual way. To do this, let $g(t) = f(x) = x = \frac{t+\pi}{2\pi}$ on $-\pi \le t < \pi$. We now find the fourth degree Fourier polynomial for g.

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t+\pi}{2\pi}\,dt = \frac{1}{(2\pi)^2}\left(\frac{t^2}{2} + \pi t\right)\Big|_{-\pi}^{\pi} = \frac{1}{2}$$

Notice, a_0 is the average value of both f and g. For $n \ge 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t+\pi}{2\pi}\cos(nt)dt = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}(t\cos(nt) + \pi\cos(nt))dt$$
$$= \frac{1}{2\pi^2}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt) + \frac{\pi}{n}\sin(nt)\right]\Big|_{-\pi}^{\pi}$$
$$= 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t+\pi}{2\pi}\sin(nt)\,dt = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}(t\sin(nt) + \pi\sin(nt))\,dt$$
$$= \frac{1}{2\pi^2}\left[-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt) - \frac{\pi}{n}\cos(nt)\right]\Big|_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi^2}(-\frac{4\pi}{n}\cos(\pi n)) = -\frac{2}{\pi n}\cos(\pi n) = \frac{2}{\pi n}(-1)^{n+1}.$$

We get the integrals for a_n and b_n using the integral table (formulas III-15 and III-16).

Thus, the Fourier polynomial of degree 4 for g is:

$$G_4(t) = \frac{1}{2} + \frac{2}{\pi}\sin t - \frac{1}{\pi}\sin 2t + \frac{2}{3\pi}\sin 3t - \frac{1}{2\pi}\sin 4t.$$

Now, since $g(t) = f(x)$, the Fourier polynomial of degree 4 for f can be found by replacing t in terms of x again. Thus,

$$F_4(x) = \frac{1}{2} + \frac{2}{\pi}\sin(2\pi x - \pi) - \frac{1}{\pi}\sin(4\pi x - 2\pi) + \frac{2}{3\pi}\sin(6\pi x - 3\pi) - \frac{1}{2\pi}\sin(8\pi x - 4\pi).$$

Now, using the fact that $\sin(x - \pi) = -\sin x$ and $\sin(x - 2\pi) = \sin x$, etc., we have:

$$F_4(x) = \frac{1}{2} - \frac{2}{\pi}\sin(2\pi x) - \frac{1}{\pi}\sin(4\pi x) - \frac{2}{3\pi}\sin(6\pi x) - \frac{1}{2\pi}\sin(8\pi x).$$

See Figure 10.23.

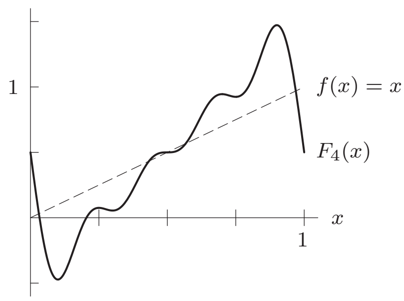


Figure 10.23

16. Since the period is 2, we make the substitution $t = \pi x - \pi$. Thus, $x = \frac{t+\pi}{\pi}$. We find the Fourier coefficients. Notice that all of the integrals are the same as in Problem 15 except for an extra factor of 2. Thus, $a_0 = 1$, $a_n = 0$, and $b_n = \frac{4}{\pi n}(-1)^{n+1}$, so:

$$G_4(t) = 1 + \frac{4}{\pi}\sin t - \frac{2}{\pi}\sin 2t + \frac{4}{3\pi}\sin 3t - \frac{1}{\pi}\sin 4t.$$

Again, we substitute back in to get a Fourier polynomial in terms of x:

$$\begin{aligned} F_4(x) &= 1 + \frac{4}{\pi}\sin(\pi x - \pi) - \frac{2}{\pi}\sin(2\pi x - 2\pi) \\ &\quad + \frac{4}{3\pi}\sin(3\pi x - 3\pi) - \frac{1}{\pi}\sin(4\pi x - 4\pi) \\ &= 1 - \frac{4}{\pi}\sin(\pi x) - \frac{2}{\pi}\sin(2\pi x) - \frac{4}{3\pi}\sin(3\pi x) - \frac{1}{\pi}\sin(4\pi x). \end{aligned}$$

See Figure 10.24.

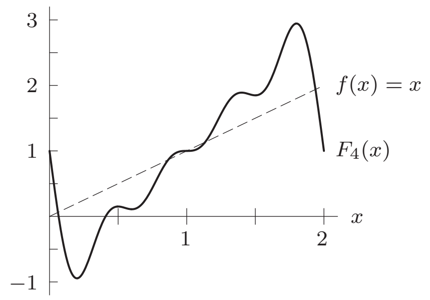


Figure 10.24

Notice in this case, the terms in our series are $\sin(n\pi x)$, not $\sin(2\pi n x)$, as in Problem 15. In general, the terms will be $\sin(n\frac{2\pi}{b}x)$, where b is the period.

17. The signal received on earth is in the form of a periodic function $h(t)$, which can be expanded in a Fourier series

$$\begin{aligned} h(t) &= a_0 + a_1\cos t + a_2\cos 2t + a_3\cos 3t + \cdots \\ &\quad + b_1\sin t + b_2\sin 2t + b_3\sin 3t + \cdots \end{aligned}$$

If the periodic noise consists of *only* the second and higher harmonics of the Fourier series, then the original signal contributed the fundamental harmonic plus the constant term, i.e.,

$$\underbrace{a_0}_{\text{constant term}} + \underbrace{a_1\cos t + b_1\sin t}_{\text{fundamental harmonic}} = \underbrace{A\cos t}_{\text{original signal}}.$$

In order to find A, we need to find a_0, a_1, and b_1. Looking at the graph of $h(t)$, we see

$$\begin{aligned} a_0 = \text{average value of } h(t) &= \frac{1}{2\pi}(\text{Area above the } x\text{-axis – Area below the } x\text{-axis}) \\ &= \frac{1}{2\pi}\left[80\left(\frac{\pi}{2}\right) - \left(50\left(\frac{\pi}{4}\right) + 30\left(\frac{\pi}{4}\right) + 30\left(\frac{\pi}{4}\right) + 50\left(\frac{\pi}{4}\right)\right)\right] \\ &= \frac{1}{2\pi}\left[80\left(\frac{\pi}{2}\right) - 80\left(\frac{\pi}{2}\right)\right] = \frac{1}{2\pi}\cdot 0, = 0 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\cos t\,dt \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{-3\pi/4} -50\cos t\,dt + \int_{-3\pi/4}^{-\pi/2} 0\cos t\,dt + \int_{-\pi/2}^{-\pi/4} -30\cos t\,dt\right. \\ &\quad \left. + \int_{-\pi/4}^{\pi/4} 80\cos t\,dt + \int_{\pi/4}^{\pi/2} -30\cos t\,dt + \int_{\pi/2}^{3\pi/4} 0\cos t\,dt + \int_{3\pi/4}^{\pi} -50\cos t\,dt\right] \\ &= \frac{1}{\pi}\left[-50\sin t\Big|_{-\pi}^{-3\pi/4} - 30\sin t\Big|_{-\pi/2}^{-\pi/4}\right. \\ &\quad \left. +80\sin t\Big|_{-\pi/4}^{\pi/4} - 30\sin t\Big|_{\pi/4}^{\pi/2} - 50\sin t\Big|_{3\pi/4}^{\pi}\right] \\ &= \frac{1}{\pi}\left[-50\left(-\frac{\sqrt{2}}{2} - 0\right) - 30\left(-\frac{\sqrt{2}}{2} - (-1)\right) + 80\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right)\right. \\ &\quad \left. -30\left(1 - \frac{\sqrt{2}}{2}\right) - 50\left(0 - \frac{\sqrt{2}}{2}\right)\right] \\ &= \frac{1}{\pi}[25\sqrt{2} + 15\sqrt{2} - 30 + 40\sqrt{2} + 40\sqrt{2} - 30 + 15\sqrt{2} + 25\sqrt{2}] \\ &= \frac{1}{\pi}[160\sqrt{2} - 60] = 52.93, \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\sin t\,dt \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{-3\pi/4} -50\sin t\,dt + \int_{-3\pi/4}^{-\pi/2} 0\sin t\,dt + \int_{-\pi/2}^{-\pi/4} -30\sin t\,dt\right. \\ &\quad \left. + \int_{-\pi/4}^{\pi/4} 80\sin t\,dt + \int_{\pi/4}^{\pi/2} -30\sin t\,dt + \int_{\pi/2}^{3\pi/4} 0\sin t\,dt + \int_{3\pi/4}^{\pi} -50\sin t\,dt\right] \\ &= \frac{1}{\pi}\left[50\cos t\Big|_{-\pi}^{-3\pi/4} + 30\cos t\Big|_{-\pi/2}^{-\pi/4} - 80\cos t\Big|_{-\pi/4}^{\pi/4} + 30\cos t\Big|_{\pi/4}^{\pi/2} + 50\cos t\Big|_{3\pi/4}^{\pi}\right] \\ &= \frac{1}{\pi}\left[50\left(-\frac{\sqrt{2}}{2} - (-1)\right) + 30\left(\frac{\sqrt{2}}{2} - 0\right) - 80\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right. \end{aligned}$$

$$+30\left(0-\frac{\sqrt{2}}{2}\right)+50\left(-1-(-\frac{\sqrt{2}}{2})\right)\Bigg]$$
$$=\frac{1}{\pi}\left[-25\sqrt{2}+50+15\sqrt{2}-0-15\sqrt{2}-50+25\sqrt{2}\right]=\frac{1}{\pi}(0)=0.$$

Also, we could have just noted that $b_1=\frac{1}{\pi}\int_{-\pi}^{\pi}h(t)\sin t\,dt=0$ because $h(t)\sin t$ is an odd function.
Substituting in, we get

$$a_0+a_1\cos t+b_1\sin t=0+52.93\cos t+0=A\cos t.$$

So $A=52.93$.

18. The energy spectrum of the flute shows that the first two harmonics have equal energies and contribute the most energy by far. The higher harmonics contribute relatively little energy. In contrast, the energy spectrum of the bassoon shows the comparative weakness of the first two harmonics to the third harmonic which is the strongest component.

19. Let $f(x)=a_k\cos kx+b_k\sin kx$. Then the energy of f is given by

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx&=\frac{1}{\pi}\int_{-\pi}^{\pi}(a_k\cos kx+b_k\sin kx)^2\,dx\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}(a_k^2\cos^2 kx-2a_kb_k\cos kx\sin kx+b_k^2\sin^2 kx)\,dx\\&=\frac{1}{\pi}\left[a_k^2\int_{-\pi}^{\pi}\cos^2 kx\,dx-2a_kb_k\int_{-\pi}^{\pi}\cos kx\sin kx\,dx+b_k^2\int_{-\pi}^{\pi}\sin^2 kx\,dx\right]\\&=\frac{1}{\pi}\left[a_k^2\pi-2a_kb_k\cdot 0+b_k^2\pi\right]=a_k^2+b_k^2.\end{aligned}$$

20. Since each square in the graph has area $(\frac{\pi}{4})\cdot(0.2)$,

$$\begin{aligned}a_0&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx\\&=\frac{1}{2\pi}\cdot(\frac{\pi}{4})\cdot(0.2)\,[\text{Number of squares under graph above }x\text{-axis}\\&\qquad\qquad-\text{Number of squares above graph below }x\text{ axis}]\\&\approx\frac{1}{2\pi}\cdot(\frac{\pi}{4})\cdot(0.2)\cdot[13+11-14]=0.25.\end{aligned}$$

Approximate the Fourier coefficients using Riemann sums.

$$\begin{aligned}a_1&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos x\,dx\\&\approx\frac{1}{\pi}\left[f(-\pi)\cos(-\pi)+f\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right)+f(0)\cos(0)+f\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)\right]\cdot\frac{\pi}{2}\\&=\frac{1}{\pi}\left[(0.92)(-1)+(1)(0)+(-1.7)(1)+(0.7)(0)\right]\cdot\frac{\pi}{2}\\&=-1.31\end{aligned}$$

Similarly for b_1:

$$\begin{aligned}b_1&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin x\,dx\\&\approx\frac{1}{\pi}\left[f(-\pi)\sin(-\pi)+f\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right)+f(0)\sin(0)+f\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\right]\cdot\frac{\pi}{2}\\&=\frac{1}{\pi}\left[(0.92)(0)+(1)(-1)+(-1.7)(0)+(0.7)(1)\right]\cdot\frac{\pi}{2}\\&=-0.15.\end{aligned}$$

So our first Fourier approximation is

$$F_1(x)=0.25-1.31\cos x-0.15\sin x.$$

See Figure 10.25

Similarly for a_2:

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx$$
$$\approx \frac{1}{\pi}\left[f(-\pi)\cos(-2\pi) + f\left(-\frac{\pi}{2}\right)\cos(-\pi) + f(0)\cos(0) + f\left(\frac{\pi}{2}\right)\cos(-\pi)\right]\cdot\frac{\pi}{2}$$
$$= \frac{1}{\pi}\left[(0.92)(1) + (1)(-1) + (-1.7)(1) + (0.7)(-1)\right]\cdot\frac{\pi}{2}$$
$$= -1.24$$

Similarly for b_2:

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx$$
$$\approx \frac{1}{\pi}\left[f(-\pi)\sin(-2\pi) + f\left(-\frac{\pi}{2}\right)\sin(-\pi) + f(0)\sin(0) + f\left(\frac{\pi}{2}\right)\sin(-\pi)\right]\cdot\frac{\pi}{2}$$
$$= \frac{1}{\pi}\left[(0.92)(0) + (1)(0) + (-1.7)(0) + (0.7)(0)\right]\cdot\frac{\pi}{2}$$
$$= 0.$$

So our second Fourier approximation is

$$F_2(x) = 0.25 - 1.31\cos x - 0.15\sin x - 1.24\cos 2x.$$

See Figure 10.26.

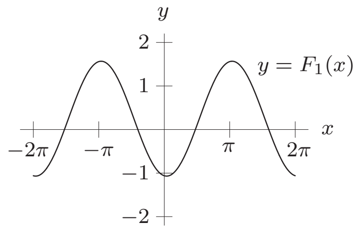


Figure 10.25

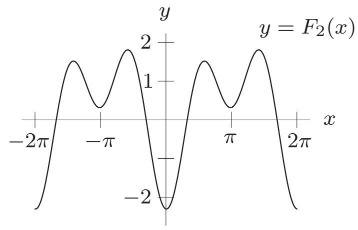


Figure 10.26

As you can see from comparing our graphs of F_1 and F_2 to the original, our estimates of the Fourier coefficients are not very accurate.

There are other methods of estimating the Fourier coefficients such as taking other Riemann sums, using Simpson's rule, and using the trapezoid rule. With each method, the greater the number of subdivisions, the more accurate the estimates of the Fourier coefficients.

The actual function graphed in the problem was

$$y = \frac{1}{4} - 1.3\cos x - \frac{\sin(\frac{3}{5})}{\pi}\sin x - \frac{2}{\pi}\cos 2x - \frac{\cos 1}{3\pi}\sin 2x$$

$$= 0.25 - 1.3\cos x - 0.18\sin x - 0.63\cos 2x - 0.057\sin 2x.$$

21. The Fourier series for f is

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos kx + \sum_{k=1}^{\infty} b_k\sin kx.$$

Pick any positive integer m. Then multiply through by $\sin mx$, to get

$$f(x)\sin mx = a_0\sin mx + \sum_{k=1}^{\infty} a_k\cos kx\sin mx + \sum_{k=1}^{\infty} b_k\sin kx\sin mx.$$

Now, integrate term-by-term on the interval $[-\pi, \pi]$ to get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin mx\,dx &= \int_{-\pi}^{\pi}\left(a_0\sin mx + \sum_{k=1}^{\infty} a_k\cos kx\sin mx + \sum_{k=1}^{\infty} b_k\sin kx\sin mx\right)dx\\ &= a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\sin mx\,dx\right)\\ &\quad + \sum_{k=1}^{\infty}\left(b_k\int_{-\pi}^{\pi}\sin kx\sin mx\,dx\right).\end{aligned}$$

Since m is a positive integer, we know that the first term of the above expression is zero (because $\int_{-\pi}^{\pi}\sin mx\,dx = 0$). Since $\int_{-\pi}^{\pi}\cos kx\sin mx\,dx = 0$, we know that everything in the first infinite sum is zero. Since $\int_{-\pi}^{\pi}\sin kx\sin mx\,dx = 0$ where $k \neq m$, the second infinite sum reduces down to the case where $k = m$ so

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_m\int_{-\pi}^{\pi}\sin mx\sin mx\,dx = b_m\pi.$$

Divide by π to get

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

22. (a) See Figure 10.27.

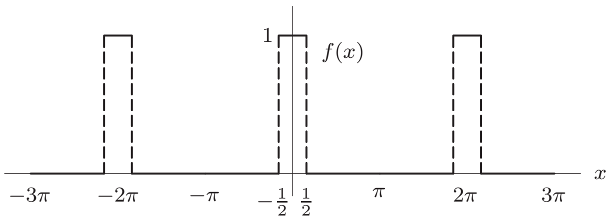


Figure 10.27

The energy of the pulse train f is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-1/2}^{1/2} 1^2\,dx = \frac{1}{\pi}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{\pi}.$$

Next, find the Fourier coefficients:

$$a_0 = \text{average value of } f \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{ Area}) = \frac{1}{2\pi}(1) = \frac{1}{2\pi},$$

$$\begin{aligned}a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1/2}^{1/2}\cos kx\,dx = \frac{1}{k\pi}\sin kx\bigg|_{-1/2}^{1/2}\\ &= \frac{1}{k\pi}\left(\sin\left(\frac{k}{2}\right) - \sin\left(-\frac{k}{2}\right)\right) = \frac{1}{k\pi}\left(2\sin\left(\frac{k}{2}\right)\right),\end{aligned}$$

$$\begin{aligned}b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1/2}^{1/2}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\bigg|_{-1/2}^{1/2}\\ &= -\frac{1}{k\pi}\left(\cos\left(\frac{k}{2}\right) - \cos\left(-\frac{k}{2}\right)\right) = \frac{1}{k\pi}(0) = 0.\end{aligned}$$

The energy of f contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{2\pi}\right)^2 = \frac{1}{2\pi^2}$$

which is

$$\frac{A_0^2}{E} = \frac{1/2\pi^2}{1/\pi} = \frac{1}{2\pi} \approx 0.159155 = 15.9155\% \quad \text{of the total.}$$

The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} = \frac{\left(\frac{2\sin\frac{1}{2}}{\pi}\right)^2}{\frac{1}{\pi}} \approx 0.292653.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 0.159155 + 0.292653 = 0.451808\%.$$

(b) The formula for the energy of the k^{th} harmonic is

$$A_k^2 = a_k^2 + b_k^2 = \left(\frac{2\sin\frac{k}{2}}{k\pi}\right)^2 + 0^2 = \frac{4\sin^2\frac{k}{2}}{k^2\pi^2}.$$

By graphing it as a continuous function for $k \geq 1$, we see its overall behavior as k gets larger. See Figure 10.28. The energy spectrum for the first five terms is graphed below as well in Figure 10.29.

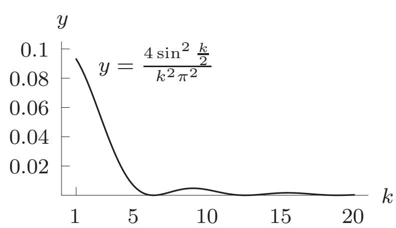


Figure 10.28

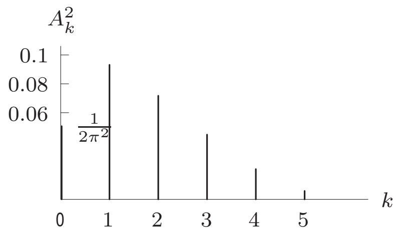


Figure 10.29

(c) The constant term and the first five harmonics are needed to capture 90% of the energy of$f$. This was determined by adding the fractions of energy of f contained in each harmonic until the sum reached at least 90% of the total energy of f:

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} + \frac{A_2^2}{E} + \frac{A_3^2}{E} + \frac{A_4^2}{E} + \frac{A_5^2}{E} \approx 90.1995\%.$$

(d) $F_5(x) = \frac{1}{2\pi} + \frac{2\sin(\frac{1}{2})}{\pi}\cos x + \frac{\sin 1}{\pi}\cos 2x + \frac{2\sin(\frac{3}{2})}{3\pi}\cos 3x + \frac{\sin 2}{2\pi}\cos 4x + \frac{2\sin(\frac{5}{2})}{5\pi}\cos 5x$. See Figure 10.30.

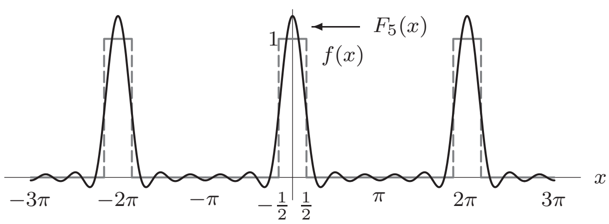


Figure 10.30

23. (a) See Figure 10.31.

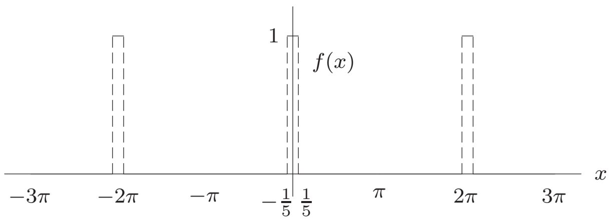


Figure 10.31

The energy of the pulse train f is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-1/5}^{1/5}1^2\,dx = \frac{1}{\pi}\left(\frac{1}{5}-\left(-\frac{1}{5}\right)\right) = \frac{2}{5\pi}.$$

Next, find the Fourier coefficients:

$$a_0 = \text{average value of } f \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{ Area}) = \frac{1}{2\pi}\left(\frac{2}{5}\right) = \frac{1}{5\pi},$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1/5}^{1/5}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-1/5}^{1/5}$$
$$= \frac{1}{k\pi}\left(\sin\left(\frac{k}{5}\right) - \sin\left(-\frac{k}{5}\right)\right) = \frac{1}{k\pi}\left(2\sin\left(\frac{k}{5}\right)\right),$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1/5}^{1/5}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-1/5}^{1/5}$$
$$= -\frac{1}{k\pi}\left(\cos\left(\frac{k}{5}\right) - \cos\left(-\frac{k}{5}\right)\right) = \frac{1}{k\pi}(0) = 0.$$

The energy of f contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{5\pi}\right)^2 = \frac{2}{25\pi^2}$$

which is

$$\frac{A_0^2}{E} = \frac{2/25\pi^2}{2/5\pi} = \frac{1}{5\pi} \approx 0.063662 = 6.3662\% \quad \text{of the total.}$$

The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} = \frac{\left(\frac{2\sin\frac{1}{5}}{\pi}\right)^2}{\frac{2}{5\pi}} \approx 0.12563.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 0.06366 + 0.12563 = 0.18929 = 18.929\%.$$

(b) The formula for the energy of the k^{th} harmonic is

$$A_k^2 = a_k^2 + b_k^2 = \left(\frac{2\sin\frac{k}{5}}{k\pi}\right)^2 + 0^2 = \frac{4\sin^2\frac{k}{5}}{k^2\pi^2}.$$

By graphing this formula as a continuous function for $k \geq 1$, we see its overall behavior as k gets larger in Figure 10.32. The energy spectrum for the first five terms is shown in Figure 10.33.

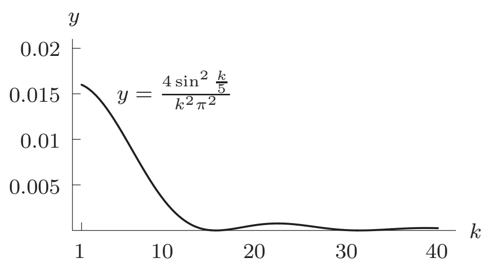


Figure 10.32

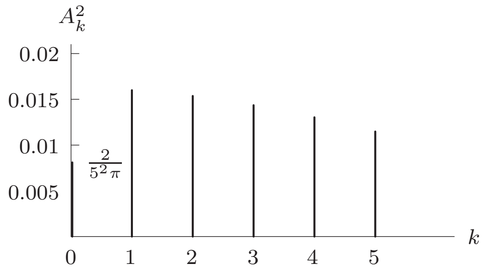


Figure 10.33

(c) The constant term and the first five harmonics contain

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} + \frac{A_2^2}{E} + \frac{A_3^2}{E} + \frac{A_4^2}{E} + \frac{A_5^2}{E} \approx 61.5255\%$$

of the total energy of f.

(d) The fifth Fourier approximation to f is

$F_5(x) = \frac{1}{5\pi} + \frac{2\sin(\frac{1}{5})}{\pi}\cos x + \frac{\sin(\frac{2}{5})}{\pi}\cos 2x + \frac{2\sin(\frac{3}{5})}{3\pi}\cos 3x + \frac{\sin(\frac{4}{5})}{2\pi}\cos 4x + \frac{2\sin 1}{5\pi}\cos 5x$. See Figure 10.34.

For comparison, Figure 10.35 shows the thirteenth Fourier approximation to f.

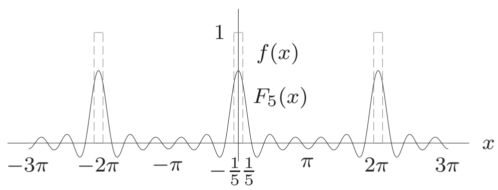


Figure 10.34

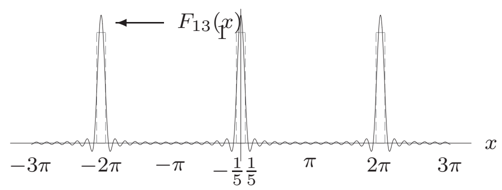


Figure 10.35

24. (a) See Figure 10.36.

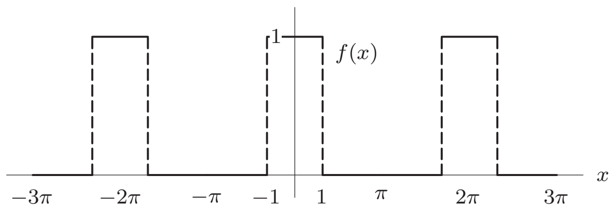


Figure 10.36

The energy of the pulse train f is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-1}^{1} 1^2 = \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}.$$

Next, find the Fourier coefficients:

$$a_0 = \text{average value of } f \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{ Area}) = \frac{1}{2\pi}(2) = \frac{1}{\pi},$$

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1}^{1}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-1}^{1} \\
&= \frac{1}{k\pi}(\sin k - \sin(-k)) = \frac{1}{k\pi}(2\sin k),
\end{aligned}
$$

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1}^{1}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-1}^{1} \\
&= -\frac{1}{k\pi}(\cos k - \cos(-k)) = \frac{1}{k\pi}(0) = 0.
\end{aligned}
$$

The energy of f contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{\pi}\right)^2 = \frac{2}{\pi^2}$$

which is

$$\frac{A_0^2}{E} = \frac{2/\pi^2}{2/\pi} = \frac{1}{\pi} \approx 0.3183 = 31.83\% \quad \text{of the total.}$$

The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} = \frac{\left(\frac{2\sin 1}{\pi}\right)^2}{\frac{2}{\pi}} \approx 0.4508 = 45.08\%.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 0.7691 = 76.91\%.$$

(b) The fraction of energy contained in the second harmonic is

$$\frac{A_2^2}{E} = \frac{a_2^2}{E} = \frac{\left(\frac{\sin 2}{\pi}\right)^2}{\frac{2}{\pi}} \approx 0.1316 = 13.16\%$$

so the fraction of energy contained in the constant term and first two harmonics is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} + \frac{A_2^2}{E} \approx 0.7691 + 0.1316 = 0.9007 = 90.07\%.$$

Therefore, the constant term and the first two harmonics are needed to capture 90% of the energy of f.

(c)

$$F_3(x) = \frac{1}{\pi} + \frac{2\sin 1}{\pi}\cos x + \frac{\sin 2}{\pi}\cos 2x + \frac{2\sin 3}{3\pi}\cos 3x.$$

See Figure 10.37.

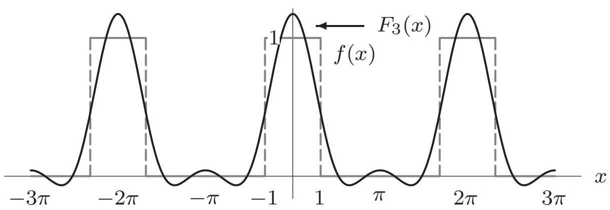


Figure 10.37

25. As c gets closer and closer to 0, the energy of the pulse train will also approach 0, since

$$E=\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx=\frac{1}{\pi}\int_{-c/2}^{c/2}1^2\,dx=\frac{1}{\pi}\left(\frac{c}{2}-\left(-\frac{c}{2}\right)\right)=\frac{c}{\pi}.$$

The energy spectrum shows the *relative* distribution of the energy of f among its harmonics. The fraction of energy carried by each harmonic gets smaller as c gets closer to 0, as shown by comparing the k^{th} terms of the Fourier series for pulse trains with $c=2,1,0.4$. For instance, notice that the *fraction* or *percentage* of energy carried by the constant term gets smaller as c gets smaller; the same is true for the energy carried by the first harmonic.

If each harmonic contributes less energy, then more harmonics are needed to capture a fixed percentage of energy. For example, if $c=2$, only the constant term and the first two harmonics are needed to capture 90% of the total energy of that pulse train. If $c=1$, the constant term and the first five harmonics are needed to get 90% of the energy of that pulse train. If $c=0.4$, the constant term and the first thirteen harmonics are needed to get 90% of the energy of that pulse train. This means that more harmonics, or more terms in the series, are needed to get an accurate approximation. Compare the graphs of the fifth and thirteenth Fourier approximations of f in Problem 23.

26. By formula II-11 of the integral table,

$$\int_{-\pi}^{\pi}\cos kx\cos mx\,dx=\frac{1}{m^2-k^2}\left(m\cos(kx)\sin(mx)-k\sin(kx)\cos(mx)\right)\Bigg|_{-\pi}^{\pi}.$$

Again, since $\sin(n\pi)=0$ for any integer n, it is easy to see that this expression is simply 0.

27. We make the substitution $u=mx$, $dx=\frac{1}{m}du$. Then

$$\int_{-\pi}^{\pi}\cos^2 mx\,dx=\frac{1}{m}\int_{u=-m\pi}^{u=m\pi}\cos^2 u\,du.$$

By Formula IV-18 of the integral table, this equals

$$\begin{aligned}\frac{1}{m}\left[\frac{1}{2}\cos u\sin u\right]\Bigg|_{-m\pi}^{m\pi}+\frac{1}{m}\frac{1}{2}\int_{-m\pi}^{m\pi}1\,du&=0+\frac{1}{2m}u\Bigg|_{-m\pi}^{m\pi}=\frac{1}{2m}u\Bigg|_{-m\pi}^{m\pi}\\&=\frac{1}{2m}(2m\pi)=\pi.\end{aligned}$$

28. The easiest way to do this is to use Problem 27.

$$\begin{aligned}\int_{-\pi}^{\pi}\sin^2 mx\,dx=\int_{-\pi}^{\pi}(1-\cos^2 mx)\,dx&=\int_{-\pi}^{\pi}dx-\int_{-\pi}^{\pi}\cos^2 mx\,dx\\&=2\pi-\pi\quad\text{using Problem 27}\\&=\pi.\end{aligned}$$

29. By formula II-12 of the integral table,

$$\begin{aligned}&\int_{-\pi}^{\pi}\sin kx\cos mx\,dx\\&=\frac{1}{m^2-k^2}\left(m\sin(kx)\sin(mx)+k\cos(kx)\cos(mx)\right)\Bigg|_{-\pi}^{\pi}\\&=\frac{1}{m^2-k^2}\Big[m\sin(k\pi)\sin(m\pi)+k\cos(k\pi)\cos(m\pi)\\&\qquad -m\sin(-k\pi)\sin(-m\pi)-k\cos(-k\pi)\cos(-m\pi)\Big].\end{aligned}$$

Since k and m are positive integers, $\sin(k\pi)=\sin(m\pi)=\sin(-k\pi)=\sin(-m\pi)=0$. Also, $\cos(k\pi)=\cos(-k\pi)$ since $\cos x$ is even. Thus this expression reduces to 0. [Note: since $\sin kx\cos mx$ is odd, so $\int_{-\pi}^{\pi}\sin kx\cos mx\,dx$ must be 0.]

30. Using formula II-10 in the integral table,

$$\int_{-\pi}^{\pi} \sin kx \sin mx\, dx = \frac{1}{m^2 - k^2}\left[k\cos(kx)\sin(mx) - m\sin(kx)\cos(mx)\right]\Bigg|_{-\pi}^{\pi}.$$

Again, since $\sin(n\pi) = 0$ for all integers n, this expression reduces to 0.

31. (a) To show that $g(t)$ is periodic with period 2π, we calculate

$$g(t+2\pi) = f\left(\frac{b(t+2\pi)}{2\pi}\right) = f\left(\frac{bt}{2\pi} + b\right) = f\left(\frac{bt}{2\pi}\right) = g(t).$$

Since $g(t+2\pi) = g(t)$ for all t, we know that $g(t)$ is periodic with period 2π. In addition

$$g\left(\frac{2\pi x}{b}\right) = f\left(\frac{b(2\pi x/b)}{2\pi}\right) = f(x).$$

(b) We make the change of variable $t = 2\pi x/b$, $dt = (2\pi/b)dx$ in the usual formulas for the Fourier coefficients of $g(t)$, as follows:

$$a_0 = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t)dt = \frac{1}{2\pi}\int_{x=-b/2}^{b/2} g\left(\frac{2\pi x}{b}\right)\frac{2\pi}{b}dx = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\,dx$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{t=-\pi}^{\pi} g(t)\cos(kt)\,dt = \frac{1}{\pi}\int_{x=-b/2}^{b/2} g\left(\frac{2\pi x}{b}\right)\cos\left(\frac{2\pi kx}{b}\right)\frac{2\pi}{b}\,dx \\ &= \frac{2}{b}\int_{-b/2}^{b/2} f(x)\cos\left(\frac{2\pi kx}{b}\right)\,dx \\ b_k &= \frac{1}{\pi}\int_{t=-\pi}^{\pi} g(t)\sin(kt)\,dt = \frac{1}{\pi}\int_{x=-b/2}^{b/2} g\left(\frac{2\pi x}{b}\right)\sin\left(\frac{2\pi kx}{b}\right)\frac{2\pi}{b}\,dx \\ &= \frac{2}{b}\int_{-b/2}^{b/2} f(x)\sin\left(\frac{2\pi kx}{b}\right)\,dx \end{aligned}$$

(c) By part (a), the Fourier series for $f(x)$ can be obtained by substituting $t = 2\pi x/b$ into the Fourier series for $g(t)$ which was found in part (b).

Strengthen Your Understanding

32. Since $\sin(kx)$ is an odd function and $\cos(mx)$ is an even function, we know $\sin(kx)\cos(mx)$ is an odd function. Hence $\int_{-\pi}^{\pi}\sin(kx)\cos(mx)\,dx$ is equal to 0.

33. Unlike Taylor series, Fourier series are good global approximations rather than local ones. Thus, a_0 is the average of $f(x)$ on the interval of approximation.

34. Since $\cos kx$ is even for each $k > 0$, we expect f to be even if there are no sine terms. So let f be any even function with period 2π. Then the coefficients of the sine terms are

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = 0, \quad \text{for } k > 0,$$

since $f(x)\sin kx$ is odd. For example, we could let $f(x) = \cos x$.

35. Since $\sin kx$ is an odd function for all $k > 0$, we expect f to be odd if there are no cosine terms. So let f be any odd function.with period 2π. Then the coefficients of the cosine terms are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = 0, \text{ for } k \geq 1,$$

since $f(x)\cos kx$ is odd. For example, we could let $f(x) = \sin x$.

36. True. Since f is even, $f(x)\sin(mx)$ is odd for any m, so

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x(mx)\,dx = 0.$$

37. (b). The graph describes an even function, which eliminates (a) and (c). The Fourier series for (d) would have values near π for x close to 0.

Solutions for Chapter 10 Review

Exercises

1. $e^x \approx 1 + e(x-1) + \frac{e}{2}(x-1)^2$

2. $\ln x \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$

3. $\sin x \approx -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x+\frac{\pi}{4}\right) + \frac{1}{2\sqrt{2}}\left(x+\frac{\pi}{4}\right)^2$

4. Differentiating $f(x) = \tan x$, we get $f'(x) = 1/\cos^2 x$, $f''(x) = 2\sin x/\cos^3 x$.

Since $\tan(\pi/4) = 1$, $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, we have $f(\pi/4) = 1$, $f'(\pi/4) = 1/(1/\sqrt{2})^2 = 2$, $f''(\pi/4) = \frac{2(1/\sqrt{2})}{(1/\sqrt{2})^3} = 4$, so

$$\begin{aligned}\tan x &\approx f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''(\frac{\pi}{4})}{2!}\left(x-\frac{\pi}{4}\right)^2\\ &= 1 + 2\left(x-\frac{\pi}{4}\right) + \frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2 = 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2.\end{aligned}$$

5. $f'(x) = 3x^2 + 14x - 5$, $f''(x) = 6x + 14$, $f'''(x) = 6$. The Taylor polynomial about $x = 1$ is

$$\begin{aligned}P_3(x) &= 4 + \frac{12}{1!}(x-1) + \frac{20}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3\\ &= 4 + 12(x-1) + 10(x-1)^2 + (x-1)^3.\end{aligned}$$

Notice that if you multiply out and collect terms in $P_3(x)$, you will get $f(x)$ back.

6. Let $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$. Then $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, $f'''(x) = 6(1-x)^{-4}$, and $f^{(4)}(x) = 24(1-x)^{-5}$. The Taylor polynomial of degree 4 about $a = 2$ is thus

$$\begin{aligned}P_4(x) &= (1-2)^{-1} + (1-2)^{-2}(x-2) + \frac{2(1-2)^{-3}}{2!}(x-2)^2\\ &\quad + \frac{6(1-2)^{-4}}{3!}(x-2)^3 + \frac{24(1-2)^{-5}}{4!}(x-4)^4\\ &= -1 + (x-2) - (x-2)^2 + (x-2)^3 - (x-2)^4.\end{aligned}$$

7. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$.

Then $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, and $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$. The Taylor polynomial of degree three about $x = 1$ is thus

$$\begin{aligned}P_3(x) &= (1+1)^{1/2} + \frac{1}{2}(1+1)^{-1/2}(x-1) + \frac{-\frac{1}{4}(1+1)^{-3/2}}{2!}(x-1)^2\\ &\quad + \frac{\frac{3}{8}(1+1)^{-5/2}}{3!}(x-1)^3\\ &= \sqrt{2}\left(1 + \frac{x-1}{4} - \frac{(x-1)^2}{32} + \frac{(x-1)^3}{128}\right).\end{aligned}$$

8. Let $f(x) = \ln x$. Then $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, and $f^{(4)}(x) = -3 \cdot 2x^{-4}$. So,

$$\begin{aligned} P_4(x) &= \ln 2 + 2^{-1}(x-2) + \frac{-2^{-2}}{2!}(x-2)^2 \\ &\quad + \frac{2 \cdot 2^{-3}}{3!}(x-2)^3 + \frac{-3 \cdot 2 \cdot 2^{-4}}{4!}(x-2)^4 \\ &= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64}. \end{aligned}$$

9. The first four nonzero terms of P_7 are given by:

$$\begin{aligned} i=1: &\quad \frac{(-1)^{1+1}3^1}{(1-1)!}x^{2\cdot 1-1} = 3x \\ i=2: &\quad \frac{(-1)^{2+1}3^2}{(2-1)!}x^{2\cdot 2-1} = -9x^3 \\ i=3: &\quad \frac{(-1)^{3+1}3^3}{(3-1)!}x^{2\cdot 3-1} = \frac{27}{2}\cdot x^5 \\ i=4: &\quad \frac{(-1)^{4+1}3^4}{(4-1)!}x^{2\cdot 4-1} = -\frac{27}{2}\cdot x^7. \end{aligned}$$

Thus, $P_7 = 3x - 9x^3 + \frac{27}{2}\cdot x^5 - \frac{27}{2}\cdot x^7$.

10. We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Therefore, using the hint,

$$\begin{aligned} f(x) &= 0.5(1 + \cos 2x) \\ &= 0.5 + 0.5(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \\ &= 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \frac{2^5}{6!}x^6 + \cdots \\ &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots \end{aligned}$$

11. We multiply the series for e^t by t^2. Since

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,$$

multiplying by t^2 gives

$$\begin{aligned} t^2 e^t &= t^2 + t^3 + \frac{t^4}{2!} + \frac{t^5}{3!} + \cdots \\ &= t^2 + t^3 + \frac{1}{2}t^4 + \frac{1}{6}t^5 + \cdots. \end{aligned}$$

12. We substitute $3y$ into the series for $\cos x$. Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

substituting $x = 3y$ gives

$$\begin{aligned} \cos(3y) &= 1 - \frac{(3y)^2}{2!} + \frac{(3y)^4}{4!} - \frac{(3y)^6}{6!} + \cdots \\ &= 1 - \frac{9}{2}y^2 + \frac{27}{8}y^4 - \frac{81}{80}y^6 + \cdots. \end{aligned}$$

13.

$$\begin{aligned}\theta^2\cos\theta^2 &= \theta^2\left(1-\frac{(\theta^2)^2}{2!}+\frac{(\theta^2)^4}{4!}-\frac{(\theta^2)^6}{6!}+\cdots\right)\\ &= \theta^2-\frac{\theta^6}{2!}+\frac{\theta^{10}}{4!}-\frac{\theta^{14}}{6!}+\cdots\end{aligned}$$

14. Substituting $y=t^2$ in $\sin y = y-\frac{y^3}{3!}+\frac{y^5}{5!}-\frac{y^7}{7!}+\cdots$ gives

$$\sin t^2 = t^2-\frac{t^6}{3!}+\frac{t^{10}}{5!}-\frac{t^{14}}{7!}+\cdots$$

15.

$$\begin{aligned}\frac{t}{1+t} = t(1+t)^{-1} &= t\left(1+(-1)t+\frac{(-1)(-2)}{2!}t^2+\frac{(-1)(-2)(-3)}{3!}t^3+\cdots\right)\\ &= t-t^2+t^3-t^4+\cdots\end{aligned}$$

16. Substituting $y=-4z^2$ into $\frac{1}{1+y}=1-y+y^2-y^3+\cdots$ gives

$$\frac{1}{1-4z^2}=1+4z^2+16z^4+64z^6+\cdots$$

17.

$$\begin{aligned}\frac{1}{\sqrt{4-x}} = \frac{1}{2\sqrt{1-\frac{x}{2}}} &= \frac{1}{2}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}\\ &= \frac{1}{2}\left(1-\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^2\right.\\ &\quad\left.-\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{2}\right)^3+\cdots\right)\\ &= \frac{1}{2}+\frac{1}{8}x+\frac{3}{64}x^2+\frac{5}{256}x^3+\cdots\end{aligned}$$

18. We use the binomial series to expand $1/\sqrt{1-z^2}$ and multiply by z^2. Since

$$\begin{aligned}\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} &= 1-\frac{1}{2}x+\frac{(-1/2)(-3/2)}{2!}x^2+\frac{(-1/2)(-3/2)(-5/2)}{3!}x^3+\cdots\\ &= 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\cdots.\end{aligned}$$

Substituting $x=-z^2$ gives

$$\begin{aligned}\frac{z^2}{\sqrt{1-z^2}} &= 1-\frac{1}{2}(-z^2)+\frac{3}{8}(-z^2)^2-\frac{5}{16}(-z^2)^3+\cdots\\ &= 1+\frac{1}{2}z^2+\frac{3}{8}z^4+\frac{5}{16}z^6+\cdots.\end{aligned}$$

Multiplying by z^2, we have

$$\frac{z^2}{\sqrt{1-z^2}} = z^2+\frac{1}{2}z^4+\frac{3}{8}z^6+\frac{5}{16}z^8+\cdots.$$

19.

$$\frac{a}{a+b} = \frac{a}{a(1+\frac{b}{a})} = \left(1+\frac{b}{a}\right)^{-1} = 1 - \frac{b}{a} + \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^3 + \cdots$$

20. Using the binomial expansion for $(1+x)^{-3/2}$ with $x = r/a$:

$$\begin{aligned}\frac{1}{(a+r)^{3/2}} &= \frac{1}{\left(a+a\left(\frac{r}{a}\right)\right)^{3/2}} = \frac{1}{\left(a\left(1+\frac{r}{a}\right)\right)^{3/2}} = \frac{1}{a^{3/2}}\left(1+\left(\frac{r}{a}\right)\right)^{-3/2} \\ &= \frac{1}{a^{3/2}}\left(1+(-3/2)\left(\frac{r}{a}\right)+\frac{(-3/2)(-5/2)}{2!}\left(\frac{r}{a}\right)^2+\frac{(-3/2)(-5/2)(-7/2)}{3!}\left(\frac{r}{a}\right)^3+\cdots\right) \\ &= \frac{1}{a^{3/2}}\left(1-\frac{3}{2}\left(\frac{r}{a}\right)+\frac{15}{8}\left(\frac{r}{a}\right)^2-\frac{35}{16}\left(\frac{r}{a}\right)^3+\cdots\right).\end{aligned}$$

21. Using the binomial expansion for $(1+x)^{3/2}$ with $x = y/B$.

$$\begin{aligned}(B^2+y^2)^{3/2} &= \left(B^2+B^2\left(\frac{y^2}{B^2}\right)\right)^{3/2} = \left(B^2\left(1+\left(\frac{y}{B}\right)^2\right)\right)^{3/2} = B^3\left(1+\left(\frac{y}{B}\right)^2\right)^{3/2} \\ &= B^3\left(1+(3/2)\left(\left(\frac{y}{B}\right)^2\right)^1+\frac{(3/2)(1/2)}{2!}\left(\left(\frac{y}{B}\right)^2\right)^2+\frac{(3/2)(1/2)(-1/2)}{3!}\left(\left(\frac{y}{B}\right)^2\right)^3\cdots\right) \\ &= B^3\left(1+\frac{3}{2}\left(\frac{y}{B}\right)^2+\frac{3}{8}\left(\frac{y}{B}\right)^4-\frac{1}{16}\left(\frac{y}{B}\right)^6\cdots\right).\end{aligned}$$

22.

$$\begin{aligned}\sqrt{R-r} &= \sqrt{R}\left(1-\frac{r}{R}\right)^{\frac{1}{2}} \\ &= \sqrt{R}\left(1+\frac{1}{2}\left(-\frac{r}{R}\right)+\frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{r}{R}\right)^2\right. \\ &\quad \left.+\frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{r}{R}\right)^3+\cdots\right) \\ &= \sqrt{R}\left(1-\frac{1}{2}\frac{r}{R}-\frac{1}{8}\frac{r^2}{R^2}-\frac{1}{16}\frac{r^3}{R^3}-\cdots\right)\end{aligned}$$

Problems

23. The second degree Taylor polynomial for $f(x)$ around $x = 3$ is

$$\begin{aligned}f(x) &\approx f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2 \\ &= 1+5(x-3)-\frac{10}{2!}(x-3)^2 = 1+5(x-3)-5(x-3)^2.\end{aligned}$$

Substituting $x = 3.1$, we get

$$f(3.1) \approx 1+5(3.1-3)-5(3.1-3)^2 = 1+5(0.1)-5(0.01) = 1.45.$$

24. Factoring out a 3, we see

$$3\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots\right) = 3e^1 = 3e.$$

25. Infinite geometric series with $a = 1$, $x = -1/3$, so

$$\text{Sum} = \frac{1}{1-(-1/3)} = \frac{3}{4}.$$

26. This is the series for e^x with $x = -2$ substituted. Thus

$$1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} + \cdots = 1 + (-2) + \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!} + \cdots = e^{-2}.$$

27. This is the series for $\sin x$ with $x = 2$ substituted. Thus

$$2 - \frac{8}{3!} + \frac{32}{5!} - \frac{128}{7!} + \cdots = 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots = \sin 2.$$

28. Factoring out a 0.1, we see

$$0.1\left(0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \frac{(0.1)^7}{7!} + \cdots\right) = 0.1\sin(0.1).$$

29. (a) Factoring out $7(1.02)^3$ and using the formula for the sum of a finite geometric series with $a = 7(1.02)^3$ and $r = 1/1.02$, we see

$$\begin{aligned}
\text{Sum} &= 7(1.02)^3 + 7(1.02)^2 + 7(1.02) + 7 + \frac{7}{(1.02)} + \frac{7}{(1.02)^3} + \cdots + \frac{7}{(1.02)^{100}} \\
&= 7(1.02)^3\left(1 + \frac{1}{(1.02)} + \frac{1}{(1.02)^2} + \cdots + \frac{1}{(1.02)^{103}}\right) \\
&= 7(1.02)^3\frac{\left(1 - \frac{1}{(1.02)^{104}}\right)}{1 - \frac{1}{1.02}} \\
&= 7(1.02)^3\left(\frac{(1.02)^{104} - 1}{(1.02)^{104}}\frac{1.02}{0.02}\right) \\
&= \frac{7(1.02^{104} - 1)}{0.02(1.02)^{100}}.
\end{aligned}$$

(b) Using the Taylor expansion for e^x with $x = (0.1)^2$, we see

$$\begin{aligned}
\text{Sum} &= 7 + 7(0.1)^2 + \frac{7(0.1)^4}{2!} + \frac{7(0.1)^6}{3!} + \cdots \\
&= 7\left(1 + (0.1)^2 + \frac{(0.1)^4}{2!} + \frac{(0.1)^6}{3!} + \cdots\right) \\
&= 7e^{(0.1)^2} \\
&= 7e^{0.01}.
\end{aligned}$$

30. Let C_n be the coefficient of the n^{th} term in the series. $C_1 = f'(0)/1!$, so $f'(0) = 1!C_1 = 1\cdot 1 = 1$.
Similarly, $f''(0) = 2!C_2 = 2!\cdot\frac{1}{2} = 1$;
$f'''(0) = 3!C_3 = 3!\cdot\frac{1}{3} = 2! = 2$;
$f^{(10)}(0) = 10!C_{10} = 10!\cdot\frac{1}{10} = \frac{10!}{10} = 9! = 362880$.

31. Write out series expansions about $x = 0$, and compare the first few terms:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\
1 - \cos x &= 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\
e^x - 1 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\arctan x &= \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - \cdots)\,dx
\end{aligned}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \qquad \text{(note that the arbitrary constant is 0)}$$

$$x\sqrt{1-x} = x(1-x)^{1/2} = x\left(1 - \frac{1}{2}x + \frac{(1/2)(-1/2)}{2}x^2 + \cdots\right)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{8} + \cdots$$

So, considering just the first term or two (since we are interested in small x)

$$1 - \cos x < x\sqrt{1-x} < \ln(1+x) < \arctan x < \sin x < x < e^x - 1.$$

32. The graph in Figure 10.38 suggests that the Taylor polynomials converge to $f(x) = \dfrac{1}{1+x}$ on the interval $(-1, 1)$. The Taylor expansion is

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots,$$

so the ratio test gives

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(-1)^{n+1}x^{n+1}|}{|(-1)^n x^n|} = |x|.$$

Thus, the series converges if $|x| < 1$; that is $-1 < x < 1$.

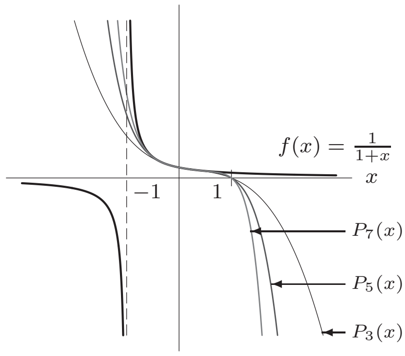


Figure 10.38

33. The Taylor series of $\dfrac{1}{1-2x}$ around $x = 0$ is

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \cdots = \sum_{k=0}^{\infty} (2x)^k.$$

To find the radius of convergence, we apply the ratio test with $a_k = (2x)^k$.

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{2^{k+1}|x|^{k+1}}{2^k|x|^k} = 2|x|.$$

Hence the radius of convergence is $R = 1/2$.

34. First we use the Taylor series expansion for $\ln(1+t)$,

$$\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots$$

to find the Taylor series expansion of $\ln(1+x+x^2)$ by putting $t = x + x^2$. We get

$$\ln(1+x+x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \cdots.$$

Next we use the Taylor series for $\sin x$ to get

$$\sin^2 x = (\sin x)^2 = (x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots)^2 = x^2 - \frac{1}{3}x^4 + \cdots.$$

Finally,

$$\frac{\ln(1+x+x^2) - x}{\sin^2 x} = \frac{\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \cdots}{x^2 - \frac{1}{3}x^4 + \cdots} \to \frac{1}{2}, \quad \text{as} \quad x \to 0.$$

35. The fourth-degree Taylor polynomial for f at $x = 0$ is

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 \\ &= 0 + 1x + \frac{-3}{2}x^2 + \frac{7}{6}x^3 + \frac{-15}{24}x^4 \\ &= x - \frac{3}{2}x^2 + \frac{7}{6}x^3 - \frac{5}{8}x^4. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{0.6} f(x)\,dx &\approx \int_0^{0.6} P_4(x)\,dx \\ &= \left(\frac{1}{2}x^2 - \frac{3}{2}\cdot\frac{1}{3}x^3 + \frac{7}{6}\cdot\frac{1}{4}x^4 - \frac{5}{8}\cdot\frac{1}{5}x^5\right)\Bigg|_0^{0.6} \\ &= \left(\frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{7}{24}x^4 - \frac{1}{8}x^5\right)\Bigg|_0^{0.6} = 0.10008. \end{aligned}$$

36. (a) Using the Taylor series for e^x, we have:

$$\begin{aligned} e^{-x^3} &= 1 + \left(-x^3\right) + \frac{\left(-x^3\right)^2}{2} + \frac{\left(-x^3\right)^3}{6} + \frac{\left(-x^3\right)^4}{24} + \cdots \\ &= 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \frac{x^{12}}{24} + \cdots. \end{aligned}$$

(b) Using the Taylor polynomial of degree 12 from part (a), we have:

$$\begin{aligned} f'(x) &= \left(1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \frac{x^{12}}{24} + \cdots\right)' \\ &= -3x^2 + \frac{6x^5}{2} - \frac{9x^8}{6} + \frac{12x^{11}}{24} + \cdots \\ &= -3x^2 + 3x^5 - \frac{3x^8}{2} + \frac{x^{11}}{2} + \cdots \\ f''(x) &= \left(f'(x)\right)' \\ &= \left(-3x^2 + 3x^5 - \frac{3x^8}{2} + \frac{x^{11}}{2} + \cdots\right)' \\ &= -6x + 15x^4 - 12x^7 + \frac{11x^{10}}{2} + \cdots. \end{aligned}$$

37. We find the Taylor polynomial for $\cos(x^2)$ by substituting into the series for $\cos x$:

$$\cos\left(x^2\right) \approx 1 - \frac{1}{2}\left(x^2\right)^2 + \frac{1}{4!}\left(x^2\right)^4 = 1 - \frac{x^4}{2} + \frac{x^8}{24}.$$

This means that

$$\int_0^1 \cos\left(x^2\right)\,dx \approx \int_0^1 \left(1 - \frac{x^4}{2} + \frac{x^8}{24}\right)dx = \left(x - \frac{x^5}{10} + \frac{x^9}{216}\right)\Bigg|_0^1 = 1 - \frac{1}{10} + \frac{1}{216} = 0.90463.$$

This is a very good estimate; the actual value (found using a computer) is $0.90452\ldots$.

38. (a) The series for $\frac{\sin 2\theta}{\theta}$ is

$$\frac{\sin 2\theta}{\theta} = \frac{1}{\theta}\left(2\theta - \frac{(2\theta)^3}{3!} + \frac{(2\theta)^5}{5!} - \cdots\right) = 2 - \frac{4\theta^2}{3} + \frac{4\theta^4}{15} - \cdots$$

so $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = 2$.

(b) Near $\theta = 0$, we make the approximation

$$\frac{\sin 2\theta}{\theta} \approx 2 - \frac{4}{3}\theta^2$$

so the parabola is $y = 2 - \frac{4}{3}\theta^2$.

39. (a) Since $\int(1-x^2)^{-1/2}dx = \arcsin x$, we use the Taylor series for $(1-x^2)^{-1/2}$ to find the Taylor series for $\arcsin x$:

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots$$

so

$$\arcsin x = \int (1-x^2)^{-1/2}dx = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots$$

(b) From Example 3 in Section 10.3, we know

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

so that

$$\frac{\arctan x}{\arcsin x} = \frac{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots}{x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots} \to 1, \quad \text{as} \quad x \to 0.$$

40. (a) The Taylor series is given by

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{1}{2!}\cdot f''(0)x^2 + \frac{1}{3!}\cdot f'''(0)x^3 + \frac{1}{4!}\cdot f^{(4)}(0)x^4 + \cdots \\
&= 1 + \underbrace{\frac{(1+1)!}{2^1}}_{f'(0)}x + \frac{1}{2!}\cdot\underbrace{\frac{(2+1)!}{2^2}}_{f''(0)}x^2 + \frac{1}{3!}\cdot\underbrace{\frac{(3+1)!}{2^3}}_{f'''(0)}x^3 + \frac{1}{4!}\cdot\underbrace{\frac{(4+1)!}{2^4}}_{f^{(4)}(0)}x^4 + \cdots \\
&= 1 + 2!\cdot\frac{1}{2^1}\cdot x + \frac{3!}{2!}\cdot\frac{1}{2^2}\cdot x^2 + \frac{4!}{3!}\cdot\frac{1}{2^3}\cdot x^3 + \frac{5!}{4!}\cdot\frac{1}{2^4}\cdot x^4 + \cdots \\
&= 1 + \frac{2}{2^1}\cdot x^1 + \frac{3}{2^2}\cdot x^2 + \frac{4}{2^3}\cdot x^3 + \frac{5}{2^4}\cdot x^4 + \cdots \\
&= 1 + \frac{1+1}{2^1}\cdot x^1 + \frac{2+1}{2^2}\cdot x^2 + \frac{3+1}{2^3}\cdot x^3 + \frac{4+1}{2^4}\cdot x^4 + \cdots + \frac{k+1}{2^k}\cdot x^k + \cdots
\end{aligned}$$

We see that the coefficient of x^k is $(k+1)!/2^k$, so

$$f(x) = \sum_{k=0}^{\infty}\frac{k+1}{2^k}x^k.$$

Note that the general term works for $k=0$, since $(0+1)!/2^0 = 1/1 = 1$.

(b) Each successive term in this series involves a higher power of $3/2$. Since $3/2 > 1$, this means each successive term is larger than the one before. Therefore the series does not converge.

(c) Given that $\int_0^1\left(\sum_{n=1}^{\infty}a_nx^n\right)dx = \sum_{n=1}^{\infty}\left(\int_0^1 a_nx^n\,dx\right)$, we have:

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1\left(1 + \frac{2}{2^1}\cdot x^1 + \frac{3}{2^2}\cdot x^2 + \frac{4}{2^3}\cdot x^3 + \frac{5}{2^4}\cdot x^4 + \cdots\right)dx \\
&= \int_0^1 1\,dx + \int_0^1\frac{2}{2^1}\cdot x\,dx + \int_0^1\frac{3}{2^2}\cdot x^2\,dx + \int_0^1\frac{4}{2^3}\cdot x^3\,dx + \int_0^1\frac{5}{2^4}\cdot x^4\,dx + \cdots \\
&= x\Big|_0^1 + \frac{2}{2^1}\cdot\frac{1}{2}x^2\Big|_0^1 + \frac{3}{2^2}\cdot\frac{1}{3}x^3\Big|_0^1 + \frac{4}{2^3}\cdot\frac{1}{4}x^4\Big|_0^1 + \frac{5}{2^4}\cdot\frac{1}{5}x^5\Big|_0^1 + \cdots \\
&= 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots.
\end{aligned}$$

This is the sum of a geometric series, which we know to equal $\dfrac{1}{1-1/2} = 2$.

41. (a) See Figure 10.39. The graph of E_1 looks like a parabola. Since the graph of E_1 is sandwiched between the graph of $y = x^2$ and the x axis, we have

$$|E_1| \le x^2 \quad \text{for} \quad |x| \le 0.1.$$

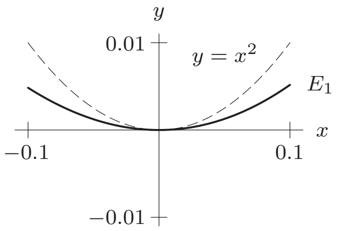


Figure 10.39

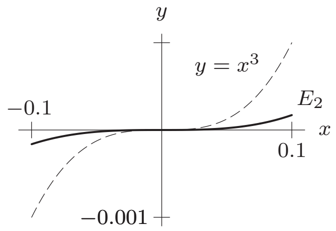


Figure 10.40

(b) See Figure 10.40. The graph of E_2 looks like a cubic, sandwiched between the graph of $y = x^3$ and the x axis, so

$$|E_2| \le x^3 \quad \text{for} \quad |x| \le 0.1.$$

(c) Using the Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we see that

$$E_1 = e^x - (1+x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Thus for small x, the $x^2/2!$ term dominates, so

$$E_1 \approx \frac{x^2}{2!},$$

and so E_1 is approximately a quadratic.

Similarly

$$E_2 = e^x - (1 + x + \frac{x^2}{2}) = \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Thus for small x, the $x^3/3!$ term dominates, so

$$E_2 \approx \frac{x^3}{3!}$$

and so E_2 is approximately a cubic.

42. (a) We have

$$\begin{aligned} P_7(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \cdots \\ &= 0 + 1\cdot x + \frac{1}{2}\cdot 0\cdot x^2 + \frac{1}{3!}(2!)x^3 + \frac{1}{4!}\cdot 0\cdot x^4 + \frac{1}{5!}(4!)x^5 + \frac{1}{6!}\cdot 0\cdot x^6 + \frac{1}{7!}(6!)x^7 \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}. \end{aligned}$$

(b) We infer from the pattern in part (a) that:

$$\begin{aligned} f(x) &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{(\text{odd number})}}{\text{same odd number}} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}. \end{aligned}$$

43. (a) The Taylor polynomial of degree 2 is

$$V(x) \approx V(0) + V'(0)x + \frac{V''(0)}{2}x^2.$$

Since $x = 0$ is a minimum, $V'(0) = 0$ and $V''(0) > 0$. We can not say anything about the sign or value of $V(0)$. Thus

$$V(x) \approx V(0) + \frac{V''(0)}{2}x^2.$$

(b) Differentiating gives an approximation to $V'(x)$ at points near the origin

$$V'(x) \approx V''(0)x.$$

Thus, the force on the particle is approximated by $-V''(0)x$.

$$\text{Force} = -V'(x) \approx -V''(0)x.$$

Since $V''(0) > 0$, the force is approximately proportional to x with negative proportionality constant, $-V''(0)$. This means that when x is positive, the force is negative, which means pointing toward the origin. When x is negative, the force is positive, which means pointing toward the origin. Thus, the force always points toward the origin.

Physical principles tell us that the particle is at equilibrium at the minimum potential. The direction of the force toward the origin supports this, as the force is tending to restore the particle to the origin.

44. (a) For reference, Figure 10.41 shows the graphs of the two functions.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

Notice that the first two terms are the same in both series.

(b) $\dfrac{1}{1+x^2}$ is greater.

(c) Even, because the only terms involved are of even degree.

(d) The coefficients for e^{-x^2} become extremely small for higher powers of x, and we can "counteract" the effect of these powers for large values of x. The series for $\frac{1}{1+x^2}$ has no such coefficients.

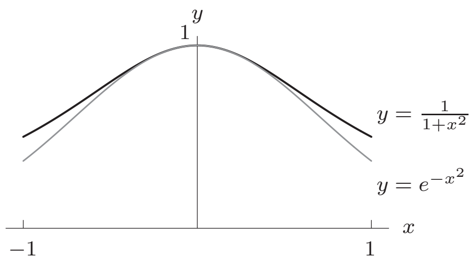


Figure 10.41

45. We have:

$$\begin{aligned}
P_4(x) &= \sum_{n=1}^{4} \frac{(-n)^{n-1}}{n!} x^n \\
&= \frac{(-1)^{1-1}}{1!}x^1 + \frac{(-2)^{2-1}}{2!}x^2 + \frac{(-3)^{3-1}}{3!}x^3 + \frac{(-4)^{4-1}}{4!}x^4 \\
&= \frac{(-1)^0}{1}x + \frac{(-2)^1}{2}x^2 + \frac{(-3)^2}{6}x^3 + \frac{(-4)^3}{24}x^4 \\
&= x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4.
\end{aligned}$$

46. We can approximate $f(x)$ using the Taylor polynomial of degree 5:

$$\begin{aligned}
P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\
&= 2 + 0\cdot x + \frac{-1}{2}x^2 + \frac{0}{6}x^3 - \frac{3}{24}x^4 + \frac{6}{120}x^5 \\
&= 2 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{20}x^5.
\end{aligned}$$

Thus,

$$\int_0^2 f(x)\,dx \approx \int_0^2 \left(2 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{20}x^5\right) dx = 2x - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{120}x^6 \bigg|_0^2 = 2\cdot 2 - \frac{1}{6}\cdot 2^3 - \frac{1}{40}\cdot 2^5 + \frac{1}{120}\cdot 2^6 = 2.4.$$

47. We have

$$\begin{aligned} f'(t) &= t^{-1}e^t \\ &\approx \frac{1}{t}\left(1+t+\frac{1}{2}\cdot t^2+\frac{1}{6}\cdot t^3\right) \\ &= \frac{1}{t}+1+\frac{1}{2}\cdot t+\frac{1}{6}\cdot t^2. \end{aligned}$$

Since f is an antiderivative of this function:

$$\begin{aligned} f(t) = \int f'(t)\,dt &\approx \int \left(\frac{1}{t}+1+\frac{1}{2}\cdot t+\frac{1}{6}\cdot t^2\right)\,dt \\ &= \ln t + t + \frac{1}{2}\cdot\frac{1}{2}t^2+\frac{1}{6}\cdot\frac{1}{3}t^3+C \\ &= \ln t + \underbrace{t+\frac{1}{4}t^2+\frac{1}{18}t^3}_{P_3(t)} + C \end{aligned}$$

$$\text{so}\quad P_3(t) = t+\frac{1}{4}t^2+\frac{1}{18}t^3.$$

Note that the problem does not give enough information for us to find C. The actual definition of f is given in terms of an improper integral; using this definition, it can be shown that C equals the so-called *Euler-Mascheroni constant* $\lambda = 0.57721\ldots$.

48. This time we are interested in how a function behaves at large values in its domain. Therefore, we don't want to expand $V = 2\pi\sigma(\sqrt{R^2+a^2}-R)$ about $R=0$. We want to find a variable which becomes small as R gets large. Since $R > a$, it is helpful to write

$$V = R2\pi\sigma\left(\sqrt{1+\frac{a^2}{R^2}}-1\right).$$

We can now expand a series in terms of $(\frac{a}{R})^2$. This may seem strange, but suspend your disbelief. The Taylor series for $\sqrt{1+\frac{a^2}{R^2}}$ is

$$1+\frac{1}{2}\frac{a^2}{R^2}+\frac{(1/2)(-1/2)}{2}\left(\frac{a^2}{R^2}\right)^2+\cdots$$

So $V = R2\pi\sigma\left(1+\frac{1}{2}\frac{a^2}{R^2}-\frac{1}{8}\left(\frac{a^2}{R^2}\right)^2+\cdots-1\right)$. For large R, we can drop the $-\frac{1}{8}\frac{a^4}{R^4}$ term and terms of higher order, so

$$V \approx \frac{\pi\sigma a^2}{R}.$$

Notice that what we really did by expanding around $(\frac{a}{R})^2 = 0$ was expanding around $R=\infty$. We then get a series that converges for large R.

49. (a) $F = \frac{GM}{R^2}+\frac{Gm}{(R+r)^2}$

(b) $F = \frac{GM}{R^2}+\frac{Gm}{R^2}\frac{1}{(1+\frac{r}{R})^2}$

Since $\frac{r}{R} < 1$, use the binomial expansion:

$$\frac{1}{(1+\frac{r}{R})^2} = \left(1+\frac{r}{R}\right)^{-2} = 1-2\left(\frac{r}{R}\right)+(-2)(-3)\frac{(\frac{r}{R})^2}{2!}+\cdots$$

$$F = \frac{GM}{R^2}+\frac{Gm}{R^2}\left[1-2\left(\frac{r}{R}\right)+3\left(\frac{r}{R}\right)^2-\cdots\right].$$

(c) Discarding higher power terms, we get

$$\begin{aligned} F &\approx \frac{GM}{R^2}+\frac{Gm}{R^2}-\frac{2Gmr}{R^3} \\ &= \frac{G(M+m)}{R^2}-\frac{2Gmr}{R^3}. \end{aligned}$$

Looking at the expression, we see that the term $\frac{G(M+m)}{R^2}$ is the field strength at a distance R from a single particle of mass $M+m$. The correction term, $-\frac{2Gmr}{R^3}$, is negative because the field strength exerted by a particle of mass $(M+m)$ at a distance R would clearly be larger than the field strength at P in the question.

50. (a) For $a/h < 1$, we have

$$\frac{1}{(a^2+h^2)^{1/2}} = \frac{1}{h(1+a^2/h^2)^{1/2}} = \frac{1}{h}\left(1 - \frac{1}{2}\frac{a^2}{h^2} + \frac{3}{8}\frac{a^4}{h^4} - \dots\right).$$

Thus

$$\begin{aligned} F &= \frac{2GMmh}{a^2}\left(\frac{1}{h} - \frac{1}{h}\left(1 - \frac{1}{2}\frac{a^2}{h^2} + \frac{3}{8}\frac{a^4}{h^4} - \dots\right)\right) \\ &= \frac{2GMmh}{a^2h}\left(1 - 1 + \frac{1}{2}\frac{a^2}{h^2} - \frac{3}{8}\frac{a^4}{h^4} - \dots\right) \\ &= \frac{2GMm}{a^2}\frac{1}{2}\frac{a^2}{h^2}\left(1 - \frac{3}{4}\frac{a^2}{h^2}\dots\right) = \frac{GMm}{h^2}\left(1 - \frac{3}{4}\frac{a^2}{h^2} - \dots\right). \end{aligned}$$

(b) Taking only the first nonzero term gives

$$F \approx \frac{GMm}{h^2}.$$

Notice that this approximation to F is independent of a.

(c) If $a/h = 0.02$, then $a^2/h^2 = 0.0004$, so

$$F \approx \frac{GMm}{h^2}(1 - \frac{3}{4}(0.0004)) = \frac{GMm}{h^2}(1 - 0.0003).$$

Thus, the approximations differ by 0.0003 = 0.03%.

51. (a) If h is much smaller than R, we can say that $(R+h) \approx R$, giving the approximation

$$F = \frac{mgR^2}{(R+h)^2} \approx \frac{mgR^2}{R^2} = mg.$$

(b)

$$\begin{aligned} F &= \frac{mgR^2}{(R+h)^2} = \frac{mg}{(1+h/R)^2} = mg(1+h/R)^{-2} \\ &= mg\left(1 + \frac{(-2)}{1!}\left(\frac{h}{R}\right) + \frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{h}{R}\right)^3 + \cdots\right) \\ &= mg\left(1 - \frac{2h}{R} + \frac{3h^2}{R^2} - \frac{4h^3}{R^3} + \cdots\right) \end{aligned}$$

(c) The first order correction comes from term $-2h/R$. The approximation for F is then given by

$$F \approx mg\left(1 - \frac{2h}{R}\right).$$

If the first order correction alters the estimate for F by 10%, we have

$$\frac{2h}{R} = 0.10 \qquad \text{so} \quad h = 0.05R \approx 0.05(6400) = 320 \text{ km}.$$

The approximation $F \approx mg$ is good to within 10% — that is, up to about 300 km.

52. Since expanding $f(x+h)$ and $g(x+h)$ in Taylor series gives

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots, \\ g(x+h) &= g(x) + g'(x)h + \frac{f''(x)}{2!}h^2 + \dots, \end{aligned}$$

we substitute to get

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{(f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + ...)(g(x) + g'(x)h + \frac{1}{2}g''(x)h^2 + ...) - f(x)g(x)}{h}$$
$$= \frac{f(x)g(x) + (f'(x)g(x) + f(x)g'(x))h + \text{ Terms in } h^2 \text{ and higher powers } - f(x)g(x)}{h}$$
$$= \frac{h(f'(x)g(x) + f(x)g'(x) + \text{ Terms in } h \text{ and higher powers})}{h}$$
$$= f'(x)g(x) + f(x)g'(x) + \text{ Terms in } h \text{ and higher powers.}$$

Thus, taking the limit as $h \to 0$, we get

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= f'(x)g(x) + f(x)g'(x).$$

53. Expanding $f(y+k)$ and $g(x+h)$ in Taylor series gives

$$f(y+k) = f(y) + f'(y)k + \frac{f''(y)}{2!}k^2 + \cdots,$$
$$g(x+h) = g(x) + g'(x)h + \frac{g''(x)}{2!}h^2 + \cdots.$$

Now let $y = g(x)$ and $y + k = g(x+h)$. Then $k = g(x+h) - g(x)$ so

$$k = g'(x)h + \frac{g''(x)}{2!}h^2 + \cdots.$$

Substituting $g(x+h) = y + k$ and $y = g(x)$ in the series for $f(y+k)$ gives

$$f(g(x+h)) = f(g(x)) + f'(g(x))k + \frac{f''(g(x))}{2!}k^2 + \cdots.$$

Now, substituting for k, we get

$$f(g(x+h)) = f(g(x)) + f'(g(x))\cdot(g'(x)h + \frac{g''(x)}{2!}h^2 + \cdots) + \frac{f''(g(x))}{2!}(g'(x)h + \ldots)^2 + \cdots$$
$$= f(g(x)) + (f'(g(x)))\cdot g'(x)h + \text{Terms in } h^2 \text{ and higher powers.}$$

So, substituting for $f(g(x+h))$ and dividing by h, we get

$$\frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x))\cdot g'(x) + \text{ Terms in } h \text{ and higher powers,}$$

and thus, taking the limit as $h \to 0$,

$$\frac{d}{dx}f(g(x)) = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}$$
$$= f'(g(x))\cdot g'(x).$$

54. (a) Notice $g'(0) = 0$ because g has a critical point at $x = 0$. So, for $n \geq 2$,

$$g(x) \approx P_n(x) = g(0) + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots + \frac{g^{(n)}(0)}{n!}x^n.$$

(b) The Second Derivative test says that if $g''(0) > 0$, then 0 is a local minimum and if $g''(0) < 0$, 0 is a local maximum.

(c) Let $n = 2$. Then $P_2(x) = g(0) + \frac{g''(0)}{2!}x^2$. So, for x near 0,

$$g(x) - g(0) \approx \frac{g''(0)}{2!}x^2.$$

If $g''(0) > 0$, then $g(x) - g(0) \geq 0$, as long as x stays near 0. In other words, there exists a small interval around $x = 0$ such that for any x in this interval $g(x) \geq g(0)$. So $g(0)$ is a local minimum.
The case when $g''(0) < 0$ is treated similarly; then $g(0)$ is a local maximum.

55. The situation is more complicated. Let's first consider the case when $g'''(0) \neq 0$. To be specific let $g'''(0) > 0$. Then

$$g(x) \approx P_3(x) = g(0) + \frac{g'''(0)}{3!}x^3.$$

So, $g(x) - g(0) \approx \dfrac{g'''(0)}{3!}x^3$. (Notice that $\dfrac{g'''(0)}{3!} > 0$ is a constant.) Now, no matter how small an open interval I around $x = 0$ is, there are always some x_1 and x_2 in I such that $x_1 < 0$ and $x_2 > 0$, which means that $\dfrac{g'''(0)}{3!}x_1^3 < 0$ and $\dfrac{g'''(0)}{3!}x_2^3 > 0$, i.e. $g(x_1) - g(0) < 0$ and $g(x_2) - g(0) > 0$. Thus, $g(0)$ is neither a local minimum nor a local maximum. (If $g'''(0) < 0$, the same conclusion still holds. Try it! The reasoning is similar.)

Now let's consider the case when $g'''(0) = 0$. If $g^{(4)}(0) > 0$, then by the fourth degree Taylor polynomial approximation to g at $x = 0$, we have

$$g(x) - g(0) \approx \frac{g^{(4)}(0)}{4!}x^4 > 0$$

for x in a small open interval around $x = 0$. So $g(0)$ is a local minimum. (If $g^{(4)}(0) < 0$, then $g(0)$ is a local maximum.)

In general, suppose that $g^{(k)}(0) \neq 0$, $k \geq 2$, and all the derivatives of g with order less than k are 0. In this case g looks like cx^k near $x = 0$, which determines its behavior there. Then $g(0)$ is neither a local minimum nor a local maximum if k is odd. For k even, $g(0)$ is a local minimum if $g^{(k)}(0) > 0$, and $g(0)$ is a local maximum if $g^{(k)}(0) < 0$.

56. Let us begin by finding the Fourier coefficients for $f(x)$. Since f is odd, $\int_{-\pi}^{\pi} f(x)\,dx = 0$ and $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$. Thus $a_i = 0$ for all $i \geq 0$. On the other hand,

$$\begin{aligned} b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin(nx)\,dx + \int_{0}^{\pi} \sin(nx)\,dx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{n}\cos(nx)\Big|_{-\pi}^{0} - \frac{1}{n}\cos(nx)\Big|_{0}^{\pi}\right] \\ &= \frac{1}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0\right] \\ &= \frac{2}{n\pi}\Big(1 - \cos(n\pi)\Big). \end{aligned}$$

Since $\cos(n\pi) = (-1)^n$, this is 0 if n is even, and $\frac{4}{n\pi}$ if n is odd. Thus the n^{th} Fourier polynomial (where n is odd) is

$$F_n(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \cdots + \frac{4}{n\pi}\sin(nx).$$

Evaluating at $x = \pi/2$, we get

$$\begin{aligned} F_n(\pi/2) &= \frac{4}{\pi}\sin\frac{\pi}{2} + \frac{4}{3\pi}\sin\frac{3\pi}{2} + \frac{4}{5\pi}\sin\frac{5\pi}{2} + \frac{4}{7\pi}\sin\frac{7\pi}{2} + \cdots + \frac{4}{n\pi}\sin\frac{n\pi}{2} \\ &= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{2n+1}\frac{1}{2n+1}\right). \end{aligned}$$

But we are assuming $F_n(\pi/2)$ approaches $f(\pi/2) = 1$ as $n \to \infty$, so

$$\frac{\pi}{4}F_n(\pi/2) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{2n+1}\frac{1}{2n+1} \to \frac{\pi}{4}\cdot 1 = \frac{\pi}{4}.$$

57. (a) Expand $f(x)$ into its Fourier series:

$$\begin{aligned} f(x) = {} & a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + a_k\cos kx + \cdots \\ & + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots + b_k\sin kx + \cdots \end{aligned}$$

Then differentiate term-by-term:

$$\begin{aligned} f'(x) = {} & -a_1\sin x - 2a_2\sin 2x - 3a_3\sin 3x - \cdots - ka_k\sin kx - \cdots \\ & + b_1\cos x + 2b_2\cos 2x + 3b_3\cos 3x + \cdots + kb_k\cos kx + \cdots \end{aligned}$$

Regroup terms:

$$\begin{aligned} f'(x) = &+b_1 \cos x + 2b_2 \cos 2x + 3b_3 \cos 3x + \cdots + kb_k \cos kx + \cdots \\ &-a_1 \sin x - 2a_2 \sin 2x - 3a_3 \sin 3x - \cdots - ka_k \sin kx - \cdots \end{aligned}$$

which forms a Fourier series for the derivative $f'(x)$. The Fourier coefficient of $\cos kx$ is kb_k and the Fourier coefficient of $\sin kx$ is $-ka_k$. Note that there is no constant term as you would expect from the formula ka_k with $k = 0$. Note also that if the k^{th} harmonic f is absent, so is that of f'.

(b) If the amplitude of the k^{th} harmonic of f is

$$A_k = \sqrt{a_k^2 + b_k^2}, \quad k \geq 1,$$

then the amplitude of the k^{th} harmonic of f' is

$$\sqrt{(kb_k)^2 + (-ka_k)^2} = \sqrt{k^2(b_k^2 + a_k^2)} = k\sqrt{a_k^2 + b_k^2} = kA_k.$$

(c) The energy of the k^{th} harmonic of f' is k^2 times the energy of the k^{th} harmonic of f.

58. Let r_k and s_k be the Fourier coefficients of $Af + Bg$. Then

$$\begin{aligned} r_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big[Af(x) + Bg(x)\Big]\, dx \\ &= A\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx\right] + B\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, dx\right] \\ &= Aa_0 + Bc_0. \end{aligned}$$

Similarly,

$$\begin{aligned} r_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \Big[Af(x) + Bg(x)\Big] \cos(kx)\, dx \\ &= A\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\, dx\right] + B\left[\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)\, dx\right] \\ &= Aa_k + Bc_k. \end{aligned}$$

And finally,

$$\begin{aligned} s_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \Big[Af(x) + Bg(x)\Big] \sin(kx)\, dx \\ &= A\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\, dx\right] + B\left[\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(kx)\, dx\right] \\ &= Ac_k + Bd_k. \end{aligned}$$

59. Since $g(x) = f(x + c)$, we have that $[g(x)]^2 = [f(x + c)]^2$, so g^2 is f^2 shifted horizontally by c. Since f has period 2π, so does f^2 and g^2. If you think of the definite integral as an area, then because of the periodicity, integrals of f^2 over any interval of length 2π have the same value. So

$$\text{Energy of } f = \int_{-\pi}^{\pi} (f(x))^2\, dx = \int_{-\pi+c}^{\pi+c} (f(x))^2\, dx.$$

Now we know that

$$\begin{aligned} \text{Energy of } g &= \frac{1}{\pi}\int_{-\pi}^{\pi} (g(x))^2\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x + c))^2\, dx. \end{aligned}$$

Using the substitution $t = x + c$, we see that the two energies are equal.

CAS Challenge Problems

60. (a) The Taylor polynomials of degree 10 are

$$\text{For } \sin^2 x, \qquad P_{10}(x) = x^2 - \frac{x^4}{3} + \frac{2\,x^6}{45} - \frac{x^8}{315} + \frac{2\,x^{10}}{14175}$$
$$\text{For } \cos^2 x, \qquad Q_{10}(x) = 1 - x^2 + \frac{x^4}{3} - \frac{2\,x^6}{45} + \frac{x^8}{315} - \frac{2\,x^{10}}{14175}$$

(b) The coefficients in $P_{10}(x)$ are the negatives of the corresponding coefficients of $Q_{10}(x)$. The constant term of $P_{10}(x)$ is 0 and the constant term of $Q_{10}(x)$ is 1. Thus, $P_{10}(x)$ and $Q_{10}(x)$ satisfy

$$Q_{10}(x) = 1 - P_{10}(x).$$

This makes sense because $\cos^2 x$ and $\sin^2 x$ satisfy the identity

$$\cos^2 x = 1 - \sin^2 x.$$

61. (a) The Taylor polynomials of degree 7 are

$$\text{For } \sin x, \qquad P_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$
$$\text{For } \sin x\cos x, \qquad Q_7(x) = x - \frac{2\,x^3}{3} + \frac{2\,x^5}{15} - \frac{4\,x^7}{315}$$

(b) The coefficient of x^3 in $Q_7(x)$ is $-2/3$, and the coefficient of x^3 in $P_7(x)$ is $-1/6$, so the ratio is

$$\frac{-2/3}{-1/6} = 4.$$

The corresponding ratios for x^5 and x^7 are

$$\frac{2/15}{1/120} = 16 \quad \text{and} \quad \frac{-4/315}{-1/5040} = 64.$$

(c) It appears that the ratio is always a power of 2. For x^3, it is $4 = 2^2$; for x^5, it is $16 = 2^4$; for x^7, it is $64 = 2^6$. This suggests that in general, for the coefficient of x^n, it is 2^{n-1}.

(d) From the identity $\sin(2x) = 2\sin x\cos x$, we expect that $P_7(2x) = 2Q_7(x)$. So, if a_n is the coefficient of x^n in $P_7(x)$, and if b_n is the coefficient of x^n in $Q_7(x)$, then, since the x^n terms $P_7(2x)$ and $2Q_7(x)$ must be equal, we have

$$a_n(2x)^n = 2b_n x^n.$$

Dividing both sides by x^n and combining the powers of 2, this gives the pattern we observed. For $a_n \neq 0$,

$$\frac{b_n}{a_n} = 2^{n-1}.$$

62. (a) For $f(x) = x^2$ we have $f'(x) = 2x$ so the tangent line is

$$y = f(2) + f'(2)(x-2) = 4 + 4(x-2)$$
$$y = 4x - 4.$$

For $g(x) = x^3 - 4x^2 + 8x - 7$, we have $g'(x) = 3x^2 - 8x + 8$, so the tangent line is

$$y = g(1) + g'(1)(x-1) = -2 + 3(x-1)$$
$$y = 3x - 5.$$

For $h(x) = 2x^3 + 4x^2 - 3x + 7$, we have $h'(x) = 6x^2 + 8x - 3$. So the tangent line is

$$y = h(-1) + h'(-1)(x+1) = 12 - 5(x+1)$$
$$y = -5x + 7.$$

(b) Division by a CAS or by hand gives

$$\frac{f(x)}{(x-2)^2} = \frac{x^2}{(x-2)^2} = 1 + \frac{4x-4}{(x-2)^2} \quad \text{so} \quad r(x) = 4x-4,$$
$$\frac{g(x)}{(x-1)^2} = \frac{x^3-4x^2+8x-7}{(x-1)^2} = x-2+\frac{3x-5}{(x-1)^2} \quad \text{so} \quad r(x) = 3x-5,$$
$$\frac{h(x)}{(x+1)^2} = \frac{2x^3+4x^2-3x+7}{(x+1)^2} = 2x + \frac{-5x+7}{(x+1)^2} = \quad \text{so} \quad r(x) = -5x+7.$$

(c) In each of these three cases, $y = r(x)$ is the equation of the tangent line. We conjecture that this is true in general.

(d) The Taylor expansion of a function $p(x)$ is

$$p(x) = p(a) + p'(a)(x-a) + \frac{p''(a)}{2!}(x-a)^2 + \frac{p'''(a)}{3!}(x-a)^3 + \cdots$$

Now divide $p(x)$ by $(x-a)^2$. On the right-hand side, all terms from $p''(a)(x-a)^2/2!$ onward contain a power of $(x-a)^2$ and divide exactly by $(x-a)^2$ to give a polynomial $q(x)$, say. So the remainder is $r(x) = p(a)+p'(a)(x-a)$, the tangent line.

63. (a) The Taylor polynomial is

$$P_{10}(x) = 1 + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160}$$

(b) All the terms have even degree. A polynomial with only terms of even degree is an even function. This suggests that f might be an even function.

(c) To show that f is even, we must show that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= \frac{-x}{e^{-x}-1} + \frac{-x}{2} = \frac{x}{1-\frac{1}{e^x}} - \frac{x}{2} = \frac{xe^x}{e^x-1} - \frac{x}{2} \\ &= \frac{xe^x - \frac{1}{2}x(e^x-1)}{e^x-1} \\ &= \frac{xe^x - \frac{1}{2}xe^x + \frac{1}{2}x}{e^x-1} = \frac{\frac{1}{2}xe^x + \frac{1}{2}x}{e^x-1} = \frac{\frac{1}{2}x(e^x-1)+x}{e^x-1} \\ &= \frac{1}{2}x + \frac{x}{e^x-1} = \frac{x}{e^x-1} + \frac{x}{2} = f(x) \end{aligned}$$

64. (a) The Taylor polynomial is

$$P_{11}(x) = \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}.$$

(b) Evaluating, we get

$$P_{11}(1) = \frac{1^3}{3} - \frac{1^7}{42} + \frac{1^{11}}{1320} = 0.310281$$
$$S(1) = \int_0^1 \sin(t^2)\,dt = 0.310268.$$

We need to take about 6 decimal places in the answer as this allows us to see the error. (The values of $P_{11}(1)$ and $S(1)$ start to differ in the fifth decimal place.) Thus, the percentage error is $(0.310281 - 0.310268)/0.310268 = 0.000013/0.310268 = 0.000042 = 0.0042\%$. On the other hand,

$$P_{11}(2) = \frac{2^3}{3} - \frac{2^7}{42} + \frac{2^{11}}{1320} = 1.17056$$
$$S(2) = \int_0^2 \sin(t^2)\,dt = 0.804776.$$

The percentage error in this case is $(1.17056 - 0.804776)/0.804776 = 0.365784/0.804776 = 0.454517$, or about 45%.

PROJECTS FOR CHAPTER TEN

1. (a) We rewrite the denominator as $(1+(A^2-2A)\cos^2\theta)^{-1/2}$ and expand using a binomial series. Since we do not want terms beyond the quadratic, we omit terms with powers higher than A^2.

$$\begin{aligned}(1+(A^2-2A)\cos^2\theta)^{-1/2} &= 1-\frac{1}{2}(A^2-2A)\cos^2\theta-\frac{\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}(A^2-2A)^2\cos^4\theta+\cdots\\ &= 1-\frac{1}{2}(A^2-2A)\cos^2\theta+\frac{3}{8}(A^4-4A^3+4A^2)\cos^4\theta+\cdots\\ &= 1+A\cos^2\theta+A^2\left(\frac{3}{2}\cos^4\theta-\frac{1}{2}\cos^2\theta\right)+\cdots\end{aligned}$$

Now multiply by $(1-A\cos^2\theta)$, again omitting powers higher than A^2:

$$\begin{aligned}\cos\alpha &= (1-A\cos^2\theta)(1+(A^2-2A)\cos^2\theta)^{-1/2}\\ &= (1-A\cos^2\theta)\left(1+A\cos^2\theta+A^2\left(\frac{3}{2}\cos^4\theta-\frac{1}{2}\cos^2\theta\right)+\cdots\right)\\ &= 1+A^2\left(\frac{3}{2}\cos^4\theta-\frac{1}{2}\cos^2\theta-\cos^4\theta\right)+\cdots\\ &= 1+A^2\left(\frac{1}{2}\cos^4\theta-\frac{1}{2}\cos^2\theta\right)+\cdots\\ &= 1+\frac{1}{2}A^2\cos^2\theta(\cos^2\theta-1)+\cdots\\ &\approx 1-\frac{1}{2}A^2\cos^2\theta\sin^2\theta.\end{aligned}$$

The approximation of part (a) is therefore the approximation of $\cos\alpha$ by its quadratic Taylor polynomial.

(b) Since the bulge in the earth is so small, we have $\alpha\approx 0$. Therefore to a good accuracy we can approximate $\cos\alpha$ by its Taylor series about 0,

$$\cos\alpha\approx 1-\frac{\alpha^2}{2}.$$

By part (a) we have

$$\begin{aligned}1-\frac{\alpha^2}{2} &\approx 1-\frac{1}{2}A^2\cos^2\theta\sin^2\theta\\ \alpha^2 &\approx A^2\cos^2\theta\sin^2\theta.\end{aligned}$$

Using the identity $\sin(2\theta)=2\sin\theta\cos\theta$ we get

$$\alpha^2\approx A^2\cos^2\theta\sin^2\theta=\frac{1}{4}A^2\sin^2(2\theta),$$

and therefore, since $A>0$, and taking α and θ to be positive, we have

$$\alpha\approx\frac{1}{2}A\sin(2\theta).$$

(c) The value of α is the exact value and $(1/2)A\sin(2\theta)$ is the approximation. Thus, we have

$$\text{Error }=\frac{(1/2)A\sin(2\theta)-\alpha}{\alpha}.$$

Letting $A=0.0034$ and expressing all angles in degrees, we have Table 10.2:

Table 10.2

θ	$0°$	$20°$	$40°$	$60°$	$80°$
α	$0°$	$0.06280°$	$0.09611°$	$0.084425°$	$0.033317°$
$(1/2)A\sin(2\theta)$	$0°$	$0.06261°$	$0.09592°$	$0.084353°$	$0.033314°$
Error	0	0.30%	0.20%	0.085%	0.010%

2. (a) A calculator gives $4\tan^{-1}(1/5) - \tan^{-1}(1/239) = 0.7853981634$, which agrees with $\pi/4$ to ten decimal places. Notice that you cannot verify that Machin's formula is *exactly* true numerically (because any calculator has only a finite number of digits.) Showing that the formula is exactly true requires a theoretical argument.

(b) The Taylor polynomial of degree 5 approximating $\arctan x$ is

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5}.$$

Thus,

$$\begin{aligned}\pi &= 4\left(4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)\right)\\ &\approx 4\left(4\left(\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5\right) - \left(\frac{1}{239} - \frac{1}{3}\left(\frac{1}{239}\right)^3 + \frac{1}{5}\left(\frac{1}{239}\right)^5\right)\right)\\ &\approx 3.141621029.\end{aligned}$$

The true value is $\pi = 3.141592653\ldots$.

(c) Because the values of x, namely $x = 1/5$ and $x = 1/239$, are much smaller than 1, the terms in the series get smaller much faster.

(d) (i) If $A = \arctan(120/119)$ and $B = -\arctan(1/239)$, then

$$\tan A = \frac{120}{119} \quad \text{and} \quad \tan B = -\frac{1}{239}.$$

Substituting

$$\tan(A + B) = \frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)} = 1.$$

Thus

$$A + B = \arctan 1,$$

so

$$\arctan\left(\frac{120}{119}\right) - \arctan\left(\frac{1}{239}\right) = \arctan 1.$$

(ii) If $A = B = \arctan(1/5)$, then

$$\tan(A + B) = \frac{(1/5) + (1/5)}{1 - (1/5)(1/5)} = \frac{5}{12}.$$

Thus

$$A + B = \arctan\left(\frac{5}{12}\right),$$

so

$$2\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{5}{12}\right).$$

If $A = B = 2\arctan(1/5)$, then $\tan A = \tan B = 5/12$, so

$$\tan(A+B) = \frac{(5/12)+(5/12)}{1-(5/12)(5/12)} = \frac{120}{119}.$$

Thus

$$A + B = \arctan\left(\frac{120}{119}\right),$$

so

$$4\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{120}{119}\right).$$

(iii) Using the result of part (a) and substituting the results of part (b), we obtain

$$4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \arctan 1 = \frac{\pi}{4}.$$

3. (a) (i) Using a Taylor series expansion, we have

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{3!}h^3 + \cdots.$$

So we have

$$\frac{f(x_0) - f(x_0-h)}{h} - f'(x_0) \approx \frac{f''(x_0)}{2}h + \cdots.$$

This suggests the following bound for small h:

$$\left|\frac{f(x_0) - f(x_0-h)}{h} - f'(x_0)\right| \le \frac{Mh}{2},$$

where $|f''(x)| \le M$ for $|x - x_0| < |h|$.

(ii) We use Taylor series expansions:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots$$
$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{3!}h^3 + \cdots.$$

Subtracting gives

$$\begin{aligned} f(x_0+h) - f(x_0-h) &= 2f'(x_0)h + \frac{2f'''(x_0)}{3!}h^3 + \cdots \\ &= 2f'(x_0)h + \frac{1}{3}f'''(x_0)h^3 + \cdots. \end{aligned}$$

So

$$\frac{f(x_0+h) - f(x_0-h)}{2h} = f'(x_0) + \frac{f'''(x_0)}{6}h^2 + \cdots.$$

This suggests the following bound for small h:

$$\left|\frac{f(x_0+h) - f(x_0-h)}{2h} - f'(x_0)\right| \le \frac{Mh^2}{6},$$

where $|f'''(x)| \le M$ for $|x - x_0| < |h|$.

(iii) Expanding each term in the numerator is a Taylor series, we have

$$\begin{aligned}
f(x_0+2h) &= f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + \frac{4}{3}f'''(x_0)h^3 \\
&\quad + \frac{2}{3}f^{(4)}(x_0)h^4 + \frac{4}{15}f^{(5)}(x_0)h^5 + \cdots \\
f(x_0+h) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 \\
&\quad + \frac{f^{(4)}(x_0)}{4!}h^4 + \frac{f^{(5)}(x_0)}{5!}h^5 + \cdots, \\
f(x_0-h) &= f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{3!}h^3 \\
&\quad + \frac{f^{(4)}(x_0)}{4!}h^4 - \frac{f^{(5)}(x_0)}{5!}h^5 + \cdots, \\
f(x_0-2h) &= f(x_0) - 2f'(x_0)h + 2f''(x_0)h^2 - \frac{4}{3}f'''(x_0)h^3 \\
&\quad + \frac{2}{3}f^{(4)}(x_0)h^4 - \frac{4}{15}f^{(5)}(x_0)h^5 + \cdots.
\end{aligned}$$

Combining the expansions in pairs, we have

$$\begin{aligned}
8f(x_0+h) - 8f(x_0-h) &= 16f'(x_0)h + \frac{8}{3}f'''(x_0)h^3 + \frac{2}{15}f^{(5)}(x_0)h^5 + \cdots \\
f(x_0+2h) - f(x_0-2h) &= 4f'(x_0)h + \frac{8}{3}f'''(x_0)h^3 + \frac{8}{15}f^{(5)}(x_0)h^5 + \cdots.
\end{aligned}$$

Thus,

$$-f(x_0+2h) + 8f(x_0+h) - 8f(x_0-h) + f(x_0-2h) = 12f'(x_0)h - \frac{6}{15}f^{(5)}(x_0)h^5 + \cdots$$

so

$$\frac{-f(x_0+2h) + 8f(x_0+h) - 8f(x_0-h) + f(x_0-2h)}{12h} = f'(x_0) - \frac{f^{(5)}(x_0)}{30}h^4 + \cdots.$$

This suggests the following bound for small h,

$$\left| \frac{-f(x_0+2h) + 8f(x_0+h) - 8f(x_0-h) + f(x_0-2h)}{12h} - f'(x_0) \right| \le \frac{Mh^4}{30},$$

where $|f^{(5)}(x)| \le M$ for $|x - x_0| \le |h|$.

(b) (i)

h	$(f(x_0) - f(x_0-h))/h$	Error
10^{-1}	0.951626	4.837×10^{-2}
10^{-2}	0.995017	4.983×10^{-3}
10^{-3}	0.9995	4.998×10^{-4}
10^{-4}	0.99995	5×10^{-5}

The errors are roughly proportional to h, agreeing with part (a).

(ii)

h	$(f(x_0+h) - f(x_0-h))/(2h)$	Error
10^{-1}	1.00167	1.668×10^{-3}
10^{-2}	1.00001667	1.667×10^{-5}
10^{-3}	1.0000001667	1.667×10^{-7}
10^{-4}	1.000000001667	1.667×10^{-9}

The errors are roughly proportional to h^2, agreeing with part (a).

(iii)

h	$(-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h))/(12h)$	Error
10^{-1}	0.99999667	3.337×10^{-6}
10^{-2}	0.9999999999667	3.333×10^{-10}
10^{-3}	0.99999999999999667	3.333×10^{-14}
10^{-4}	0.999999999999999999667	3.333×10^{-18}

The errors are roughly proportional to h^4, agreeing with part (a). This is the most accurate formula.

(c) (i)

h	$(f(x_0)-f(x_0-h))/h$	Error
10^{-1}	1.0001×10^6	1.00×10^{10}
10^{-2}	1.0001×10^7	1.00×10^{10}
10^{-3}	1.0101×10^8	1.01×10^{10}
10^{-4}	1.11111×10^9	1.11×10^{10}
10^{-5}	Undefined	Undefined
10^{-6}	-1.11111×10^{10}	-1.11×10^9
10^{-7}	-1.0101×10^{10}	-1.01×10^8
10^{-8}	-1.001×10^{10}	-1.00×10^7
10^{-9}	-1.0001×10^{10}	-1.00×10^6

(ii)

h	$(f(x_0+h)-f(x_0-h))/(2h)$	Error
10^{-1}	1×10^2	1×10^{10}
10^{-2}	1×10^4	1×10^{10}
10^{-3}	1.0001×10^6	1.0001×10^{10}
10^{-4}	1.0101×10^8	1.0101×10^{10}
10^{-5}	Undefined	Undefined
10^{-6}	-1.0101×10^{10}	-1.01×10^8
10^{-7}	-1.0001×10^{10}	-1.00×10^6
10^{-8}	-1.000001×10^{10}	-1.00×10^4
10^{-9}	$-1.00000001 \times 10^{10}$	-1.00×10^2

(iii)

h	$(-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h))/(12h)$	Error
10^{-1}	1.25×10^2	1.00×10^{10}
10^{-2}	1.25×10^4	1.00×10^{10}
10^{-3}	1.25013×10^6	1.00×10^{10}
10^{-4}	1.26326×10^8	1.01×10^{10}
10^{-5}	Undefined	Undefined
10^{-6}	-9.99579×10^9	4.21×10^6
10^{-7}	$-9.9999995998 \times 10^{10}$	4.00×10^2
10^{-8}	$-9.99999999996 \times 10^{10}$	4.00×10^{-2}
10^{-9}	$-9.999999999999996 \times 10^{10}$	4.00×10^{-6}

For relatively large values of h, these approximation formulas fail miserably. The main reason is that $f(x) = 1/x$ changes very quickly at $x_0 = 10^{-5}$. In fact, $f(x) \to \pm\infty$ as $x \to 0$. So we must use very small values for h when estimating a limit (involving f and $x_0 = 10^{-5}$) as $h \to 0$. Here, $h > 10^{-5}$ is too big, since the values of $x_0 - h$ cross over the discontinuity at $x = 0$. For smaller values of h, that make sure we stay on the good side of the abyss, these formulas work quite well. Already by $h = 10^{-6}$, formula (c) is the best approximation.

CHAPTER ELEVEN

Solutions for Section 11.1

Exercises

1. Since $y = x^3$, we know that $y' = 3x^2$. Substituting $y = x^3$ and $y' = 3x^2$ into the differential equation we get

$$\begin{aligned} 0 &= xy' - 3y \\ &= x(3x^2) - 3(x^3) \\ &= 3x^3 - 3x^3 \\ &= 0. \end{aligned}$$

Since this equation is true for all x, we see that $y = x^3$ is in fact a solution.

2. (a) We substitute $y = x^4$ and its derivative $dy/dx = 4x^3$ into the differential equation:

$$\begin{aligned} x\frac{dy}{dx} &= 4y \\ x(4x^3) &= 4(x^4)? \\ \text{Yes:}\quad 4x^4 &= 4x^4. \end{aligned}$$

The function $y = x^4$ satisfies the differential equation so it is a solution.

(b) We substitute $y = x^4 + 3$ and its derivative $dy/dx = 4x^3$ into the differential equation:

$$\begin{aligned} x\frac{dy}{dx} &= 4y \\ x(4x^3) &= 4(x^4 + 3)? \\ 4x^4 &\neq 4x^4 + 12. \end{aligned}$$

The function $y = x^4 + 3$ does not satisfy the differential equation so it is not a solution.

(c) We substitute $y = x^3$ and its derivative $dy/dx = 3x^2$ into the differential equation:

$$\begin{aligned} x\frac{dy}{dx} &= 4y \\ x(3x^2) &= 4(x^3)? \\ 3x^3 &\neq 4x^3. \end{aligned}$$

The function $y = x^3$ does not satisfy the differential equation so it is not a solution.

(d) We substitute $y = 7x^4$ and its derivative $dy/dx = 28x^3$ into the differential equation:

$$\begin{aligned} x\frac{dy}{dx} &= 4y \\ x(28x^3) &= 4(7x^4)? \\ \text{Yes:}\quad 28x^4 &= 28x^4. \end{aligned}$$

The function $y = 7x^4$ satisfies the differential equation so it is a solution.

3. (a) We substitute $y = 4x^3$ and its derivative $dy/dx = 12x^2$ into the differential equation:

$$\begin{aligned} y\frac{dy}{dx} &= 6x^2 \\ (4x^3)\cdot(12x^2) &= 6x^2? \\ 48x^5 &\neq 6x^2. \end{aligned}$$

The function $y = 4x^3$ does not satisfy the differential equation so it is not a solution.

(b) We substitute $y = 2x^{3/2}$ and its derivative $dy/dx = 3x^{1/2}$ into the differential equation:

$$\begin{aligned} y\frac{dy}{dx} &= 6x^2 \\ (2x^{3/2}) \cdot (3x^{1/2}) &= 6x^2? \\ \text{Yes:} \quad 6x^2 &= 6x^2. \end{aligned}$$

The function $y = 2x^{3/2}$ satisfies the differential equation so it is a solution.

(c) We substitute $y = 6x^{3/2}$ and its derivative $dy/dx = 9x^{1/2}$ into the differential equation:

$$\begin{aligned} y\frac{dy}{dx} &= 6x^2 \\ (6x^{3/2}) \cdot (9x^{1/2}) &= 6x^2? \\ 54x^2 &\neq 6x^2. \end{aligned}$$

The function $y = 6x^{3/2}$ does not satisfy the differential equation so it is not a solution.

4. Differentiating $y(x) = Ae^{\lambda x}$ gives

$$y'(x) = A\lambda e^{\lambda x} = \lambda(Ae^{\lambda x}) = \lambda y.$$

Therefore, $y(x)$ is a solution of $y' = \lambda y$ for any value of A.

5. If $y = \sin 2t$, then $\frac{dy}{dt} = 2\cos 2t$, and $\frac{d^2y}{dt^2} = -4\sin 2t$.
Thus $\frac{d^2y}{dt^2} + 4y = -4\sin 2t + 4\sin 2t = 0$.

6. If $P = P_0e^t$, then

$$\frac{dP}{dt} = \frac{d}{dt}(P_0e^t) = P_0e^t = P.$$

7. Differentiating $x^2 + y^2 = r^2$ implicitly, with r a constant, gives

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

8. (a) To determine whether Q is increasing or decreasing, we check to see whether dQ/dt is positive or negative. Substituting $Q = 8$ and $t = 2$ into the differential equation, we have:

$$\begin{aligned} \frac{dQ}{dt} &= \frac{t}{Q} - 0.5 \\ \frac{dQ}{dt} &= \frac{2}{8} - 0.5 \\ \frac{dQ}{dt} &= -0.25 < 0. \end{aligned}$$

Since dQ/dt is negative, we see that Q is decreasing at $t = 2$.

(b) Since $dQ/dt = -0.25$, the rate of change of Q at $t = 2$ is -0.25 per unit of t. If the rate of change stays approximately constant over the interval, then Q changes by approximately -0.25 in going from $t = 2$ to $t = 3$. We have:

$$\begin{aligned} \text{Value of } Q \text{ at } 3 &\approx \text{Value of } Q \text{ at } 2 + \text{Change in } Q \\ &= 8 + (-0.25) \\ &= 7.75. \end{aligned}$$

9. We know that at time $t = 0$ the value of y is 8. Since we are told that $dy/dt = 0.5y$, we know that at time $t = 0$ the derivative of y is $.5(8) = 4$. Thus, as t goes from 0 to 1, y will increase by 4, so at $t = 1$, $y = 8 + 4 = 12$.

Likewise, at $t = 1$, we get $dy/dt = 0.5(12) = 6$ so that at $t = 2$, we obtain $y = 12 + 6 = 18$.
At $t = 2$, we have $dy/dt = 0.5(18) = 9$ so that at $t = 3$, we obtain $y = 18 + 9 = 27$.
At $t = 3$, we have $dy/dt = 0.5(27) = 13.5$ so that at $t = 4$, we obtain $y = 27 + 13.5 = 40.5$.
Thus, we get the values in the following table

t	0	1	2	3	4
y	8	12	18	27	40.5

10. At the end of 5 days, $\frac{dy}{dt} = 100 - 67.2 = 32.8\%$ per week. Thus during the next working day, which is the first day of the second week, the amount learned is about $32.8(\frac{1}{5}) = 6.6\%$.
At the end of 6 working days,

$$y \approx 67.2\% + 6.6\% = 73.8\%.$$

Continuing in this manner gives the data in Table 11.1.

Table 11.1

Time (days)	6	7	8	9	10	11	12
learned (approximate)	73.8	79.0	83.2	86.6	89.3	91.4	93.1
Time (days)	13	14	15	16	17	18	19
learned (approximate)	94.5	95.6	96.5	97.2	97.7	98.2	98.6

11. Since $x(0) = 5$, we have $Ce^{3\cdot 0} = 5$; that is, $C = 5$. So the particular solution is $x(t) = 5e^{3t}$.

12. Since $P = 5$ when $t = 3$, we have $5 = C/3$; therefore, $C = 15$. So the particular solution is $P = 15/t$.

13. Because $y = 3$ when $t = 1$, we know that $3 = \sqrt{2\cdot 1 + C}$. Therefore, $2 + C = 9$, and thus $C = 7$. So the particular solution is $y = \sqrt{2t+7}$.

14. Because $Q = 4$ when $t = 2$, we have $4 = 1/(2C + C)$; therefore, $3C = 1/4$. So $C = 1/12$. Thus, the particular solution is $Q = 12/(t+1)$.

Problems

15. In order to prove that $y = A + Ce^{kt}$ is a solution to the differential equation

$$\frac{dy}{dt} = k(y - A),$$

we must show that the derivative of y with respect to t is in fact equal to $k(y - A)$:

$$\begin{aligned} y &= A + Ce^{kt} \\ \frac{dy}{dt} &= 0 + (Ce^{kt})(k) \\ &= kCe^{kt} \\ &= k(Ce^{kt} + A - A) \\ &= k\left((Ce^{kt} + A) - A\right) \\ &= k(y - A). \end{aligned}$$

16. If $y = \cos\omega t$, then

$$\frac{dy}{dt} = -\omega\sin\omega t, \qquad \frac{d^2y}{dt^2} = -\omega^2\cos\omega t.$$

Thus, if $\frac{d^2y}{dt^2} + 9y = 0$, then

$$\begin{aligned} -\omega^2\cos\omega t + 9\cos\omega t &= 0 \\ (9 - \omega^2)\cos\omega t &= 0. \end{aligned}$$

Thus $9 - \omega^2 = 0$, or $\omega^2 = 9$, so $\omega = \pm 3$.

17. Differentiating and using the fact that

$$\frac{d}{dt}(\cosh t) = \sinh t \quad \text{and} \quad \frac{d}{dt}(\sinh t) = \cosh t,$$

we see that

$$\begin{aligned}\frac{dx}{dt} &= \omega C_1 \sinh \omega t + \omega C_2 \cosh \omega t \\ \frac{d^2x}{dt^2} &= \omega^2 C_1 \cosh \omega t + \omega^2 C_2 \sinh \omega t \\ &= \omega^2 \left(C_1 \cosh \omega t + C_2 \sinh \omega t\right).\end{aligned}$$

Therefore, we see that

$$\frac{d^2x}{dt^2} = \omega^2 x.$$

18. If $Q = Ce^{kt}$, then

$$\frac{dQ}{dt} = Cke^{kt} = k(Ce^{kt}) = kQ.$$

We are given that $\frac{dQ}{dt} = -0.03Q$, so we know that $kQ = -0.03Q$. Thus we either have $Q = 0$ (in which case $C = 0$ and k is anything) or $k = -0.03$. Notice that if $k = -0.03$, then C can be any number.

19. Since $y = x^2 + k$, we know that $y' = 2x$. Substituting $y = x^2 + k$ and $y' = 2x$ into the differential equation, we get

$$\begin{aligned}10 &= 2y - xy' \\ &= 2(x^2 + k) - x(2x) \\ &= 2x^2 + 2k - 2x^2 \\ &= 2k.\end{aligned}$$

Thus, $k = 5$ is the only solution.

20. If y satisfies the differential equation, then we must have

$$\begin{aligned}\frac{d\left(5 + 3e^{kx}\right)}{dx} &= 10 - 2(5 + 3e^{kx}) \\ 3ke^{kx} &= 10 - 10 - 6e^{kx} \\ 3ke^{kx} &= -6e^{kx} \\ k &= -2.\end{aligned}$$

So, if $k = -2$ the formula for y solves the differential equation.

21. (a) If $y = Cx^n$ is a solution to the given differential equation, then we must have

$$\begin{aligned}x\frac{d\left(Cx^n\right)}{dx} - 3(Cx^n) &= 0 \\ x(Cnx^{n-1}) - 3(Cx^n) &= 0 \\ Cnx^n - 3Cx^n &= 0 \\ C(n-3)x^n &= 0.\end{aligned}$$

Thus, if $C = 0$, we get $y = 0$ is a solution, for every n. If $C \neq 0$, then $n = 3$, and so $y = Cx^3$ is a solution.

(b) Because $y = 40$ for $x = 2$, we cannot have $C = 0$. Thus, by part (a), we get $n = 3$. The solution to the differential equation is

$$y = Cx^3.$$

To determine C if $y = 40$ when $x = 2$, we substitute these values into the equation.

$$\begin{aligned}40 &= C \cdot 2^3 \\ 40 &= C \cdot 8 \\ C &= 5.\end{aligned}$$

So, now both C and n are fixed at specific values.

22. (a) Differentiating $y(x) = A + Be^{x^2/2}$ gives $y'(x) = Bxe^{x^2/2}$ so that

$$y' - xy - x = Bxe^{x^2/2} - x\left(A + Be^{x^2/2}\right) - x = -x(A+1).$$

For $y(x) = A + Be^{x^2/2}$ to be a solution we need $-x(A+1) = 0$ so $A = -1$.

(b) Given the solution $y(x) = -1 + Be^{x^2/2}$, we have $y(0) = -1 + B$. Since $y(0) = 1$, we have $B = 2$.

23. (a) If

$$y = \frac{e^x + e^{-x}}{2},$$

then

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2},$$

and

$$\frac{d^2y}{dx^2} = \frac{e^x + e^{-x}}{2}.$$

If $k = 1$, then

$$\begin{aligned} k\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} = \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}} \\ &= \sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}} = \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2} \\ &= \left|\frac{e^x + e^{-x}}{2}\right| = \frac{e^x + e^{-x}}{2} \quad \text{(since } e^x + e^{-x} > 0) \\ &= \frac{d^2y}{dx^2}. \end{aligned}$$

(b) $y = \frac{e^{Ax} + e^{-Ax}}{2A}$, so

$$\frac{dy}{dx} = \frac{e^{Ax} - e^{-Ax}}{2} \quad \text{and} \quad \frac{d^2y}{dx^2} = A\left(\frac{e^{Ax} + e^{-Ax}}{2}\right).$$

Therefore we have

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{e^{Ax} - e^{-Ax}}{2}\right)^2 = 1 + \frac{1}{4}\left(e^{2Ax} + e^{-2Ax} - 2\right) \\ &= \frac{1}{4}\left(e^{2Ax} + e^{-2Ax} + 2\right) = \frac{1}{4}\left(e^{Ax} + e^{-Ax}\right)^2. \end{aligned}$$

This means

$$\begin{aligned} k\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= k\sqrt{\frac{1}{4}\left(e^{Ax} + e^{-Ax}\right)^2} = \frac{k}{2} \cdot \left|e^{Ax} + e^{-Ax}\right| \\ &= k\frac{\left(e^{Ax} + e^{-Ax}\right)}{2} \quad \text{(since } e^{Ax} + e^{-Ax} > 0). \end{aligned}$$

Since we want $\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, we must have $A = k$.

24.

$$\begin{array}{lll} \text{(I) } y = e^x, & y' = e^x, & y'' = e^x \\ \text{(II) } y = x^3, & y' = 3x^2, & y'' = 6x \\ \text{(III) } y = e^{-x}, & y' = -e^{-x}, & y'' = e^{-x} \\ \text{(IV) } y = x^{-2}, & y' = -2x^{-3}, & y'' = 6x^{-4} \end{array}$$

and so:

(a) (I),(III) because $y'' = y$ in each case.
(b) (IV) because $x^2y'' + 2xy' - 2y = x^2(6x^{-4}) + 2x(-2x^{-3}) - 2x^{-2} = 6x^{-2} - 4x^{-2} - 2x^{-2} = 0$.
(c) (II),(IV) because $x^2y'' = 6y$ in each case.

25.

$$\begin{array}{llll}
\text{(I)} & y = 2\sin x, & dy/dx = 2\cos x, & d^2y/dx^2 = -2\sin x \\
\text{(II)} & y = \sin 2x, & dy/dx = 2\cos 2x, & d^2y/dx^2 = -4\sin 2x \\
\text{(III)} & y = e^{2x}, & dy/dx = 2e^{2x}, & d^2y/dx^2 = 4e^{2x} \\
\text{(IV)} & y = e^{-2x}, & dy/dx = -2e^{-2x}, & d^2y/dx^2 = 4e^{-2x}
\end{array}$$

and so:

(a) (IV)
(b) (III)
(c) (III), (IV)
(d) (II)

26. (a) (IV) because $dy/dx = k = kx/x = y/x$.
(b) (III) because $dy/dx = ke^{kx} = ky$.
(c) (I) because $dy/dx = kxe^{kx} + e^{kx} = ky + xe^{kx}/x = ky + y/x$.
(d) (II) because $dy/dx = kx^{k-1} = kx^k/x = ky/x$.

27. No, it is not the general solution since it does not contain an arbitrary constant. It is a particular solution since $y' = 3e^{3x} = 3y$, but it is not the general solution.

28. (a) We substitute $y = A + Be^{-2t}$ and its derivative $dy/dt = -2Be^{-2t}$ into the differential equation:

$$\begin{aligned}
\frac{dy}{dt} &= 100 - 2y \\
-2Be^{-2t} &= 100 - 2(A + Be^{-2t}) \\
-2Be^{-2t} &= 100 - 2A - 2Be^{-2t} \\
0 &= 100 - 2A \\
A &= 50.
\end{aligned}$$

If $A = 50$, the function satisfies the differential equation. There are no conditions on B. The function $y = 50+Be^{-2t}$ is the general solution to the differential equation.

(b) We substitute $y = 85$ and $t = 0$ into the general solution:

$$\begin{aligned}
y &= 50 + Be^{-2t} \\
85 &= 50 + Be^{-2(0)} \\
85 &= 50 + B \\
B &= 35.
\end{aligned}$$

We see that $B = 35$. The particular solution satisfying the differential equation and the initial condition is $y = 50 + 35e^{-2t}$.

Strengthen Your Understanding

29. Although $Q = 6e^{4t}$ satisfies the differential equation $dQ/dt = 4Q$, it is not the general solution. The general solution is the family of all possible solutions to the differential equation and contains an arbitrary constant. $Q = 6e^{4t}$ is only one particular solution, not a general solution.

30. The differential equation $dx/dt = 1/x$ represents a function $x(t)$ whose rate of change with respect to t is $1/x$. Since $x > 0$ when $t = 0$, $1/x$ is also positive, and thus, x is increasing when t is near 0.

31. An example is $dy/dx = x/y$ with the condition that $y = 100$ when $x = 0$. Many other examples are possible.

32. A second-order differential equation is one involving the second derivative, so an example is $d^2y/dx^2 = 3y$ or $y'' = xy^2$ or $y'' + 2y' + 4 = 0$. Many other examples are possible.

33. An example is $dy/dx = 2x$ with solutions $y = x^2$ and $y = x^2 + 5$. Other examples are possible.

34. An example is $dy/dx = \sin x$. We find the solution by integrating and see that the solution is the function $y = -\cos x + C$. One solution is the trigonometric function $y = -\cos x$. Other examples are possible.

35. An example is $dy/dx = 1/x$. We find the solution by integrating and see that the solution is the function $y = \ln|x| + C$. One solution is the logarithmic function $y = \ln|x|$. Other examples are possible.

36. An example is $dy/dx = e^x$. In fact, if $f(x)$ is any increasing positive function, then the solutions of $dy/dx = f(x)$ are increasing since $f(x) > 0$ and concave up since $d^2y/dx^2 = f'(x) > 0$.

37. We want to have $dy/dx = 0$ when $y - x^2 = 0$, so let $dy/dx = y - x^2$.

38. False. The general solution contains an arbitrary constant, but the constant may not be added to the particular solution.
For example, the differential equation $dy/dt = y$ has $y = e^t$ as a particular solution, but the general solution is $y = Ce^t$, which cannot be rewritten in the form $y = e^t + C$.

39. False. The function $y = t^2$ is a solution to $y'' = 2$.

40. True. Since $f'(x) = g(x)$, we have $f''(x) = g'(x)$. Since $g(x)$ is increasing, $g'(x) > 0$ for all x, so $f''(x) > 0$ for all x. Thus the graph of f is concave up for all x.

41. False. We just need an example of a function $f(x)$ which is decreasing for $x > 0$, but whose derivative $f'(x) = g(x)$ is increasing for $x > 0$. An example is $f(x) = 1/x$. Clearly $f(x)$ is decreasing for $x > 0$ but its derivative $f'(x) = -1/x^2$ is clearly increasing for $x > 0$.

42. True. Since $g(x)$ is increasing, $g(x) \geq g(0) = 1$ for all $x \geq 0$. Since $f'(x) = g(x)$, this means that $f'(x) > 0$ for all $x \geq 0$. Therefore $f(x)$ is increasing for all $x \geq 0$.

43. False. If $g(x) > 0$ for all x, then $f(x)$ would have to be increasing for all x so $f(x + p) = f(x)$ would be impossible. For example, let $g(x) = 2 + \cos x$. Then a possibility for f is $f(x) = 2x + \sin x$. Then $g(x)$ is periodic, but $f(x)$ is not.

44. False. Let $g(x) = 0$ for all x and let $f(x) = 17$. Then $f'(x) = g(x)$ and $\lim_{x\to\infty} g(x) = 0$, but $\lim_{x\to\infty} f(x) = 17$.

45. True. Since $\lim_{x\to\infty} g(x) = \infty$, there must be some value $x = a$ such that $g(x) > 1$ for all $x > a$. Then $f'(x) > 1$ for all $x > a$. Thus, for some constant C, we have $f(x) > x + C$ for all $x > a$, which implies that $\lim_{x\to\infty} f(x) = \infty$. More precisely, let $C = f(a) - a$ and let $h(x) = f(x) - x - C$. Then $h(a) = 0$ and $h'(x) = f'(x) - 1 > 0$ for all $x > a$. Thus h is increasing so $h(x) > 0$ for all $x > a$, which means that $f(x) > x + C$ for all $x > a$.

46. False. Let $f(x) = x^3$ and $g(x) = 3x^2$. Then $y = f(x)$ satisfies $dy/dx = g(x)$ and $g(x)$ is even while $f(x)$ is odd.

47. False. The example $f(x) = x^3$ and $g(x) = 3x^2$ shows that you might expect $f(x)$ to be odd. However, the additive constant C can mess things up. For example, still let $g(x) = 3x^2$, but let $f(x) = x^3 + 1$ instead. Then $g(x)$ is still even, but $f(x)$ is not odd (for example, $f(-1) = 0$ but $-f(1) = -2$).

Solutions for Section 11.2

Exercises

1. (a) The slope at any point is given by the derivative, so we find the slope by substituting the x- and y-coordinates into the differential equation $dy/dx = x^2 - y^2$ to find dy/dx.

$$\text{The slope at } (1,0) \text{ is } \frac{dy}{dx} = 1^2 - 0^2 = 1.$$
$$\text{The slope at } (0,1) \text{ is } \frac{dy}{dx} = 0^2 - 1^2 = -1.$$
$$\text{The slope at } (1,1) \text{ is } \frac{dy}{dx} = 1^2 - 1^2 = 0.$$
$$\text{The slope at } (2,1) \text{ is } \frac{dy}{dx} = 2^2 - 1^2 = 3.$$
$$\text{The slope at } (1,2) \text{ is } \frac{dy}{dx} = 1^2 - 2^2 = -3.$$
$$\text{The slope at } (2,2) \text{ is } \frac{dy}{dx} = 2^2 - 2^2 = 0.$$

(b) See Figure 11.1.

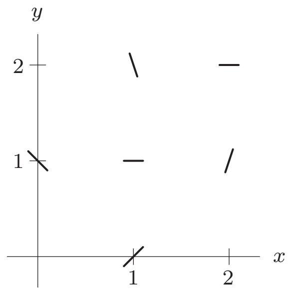


Figure 11.1

2. We substitute x- and y-coordinates into the differential equation to determine the slopes. At the point $(1, 1)$, we have $dy/dx = 1/1 = 1$ so the slope is 1. At the point $(1, 0)$, we have $dy/dx = 1/0$, which is undefined. An undefined slope corresponds to a vertical line segment. Continuing in this way, we create Figure 11.2.

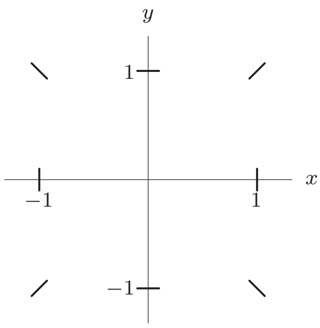


Figure 11.2

3. We substitute x- and y-coordinates into the differential equation to determine the slopes. At the point $(-1, 2)$, we have $dy/dx = 2^2 = 4$ so the slope is 4. At the point $(0, 2)$, we have $dy/dx = 2^2 = 4$, so the slope is again 4. Continuing in this way, we create Figure 11.3.

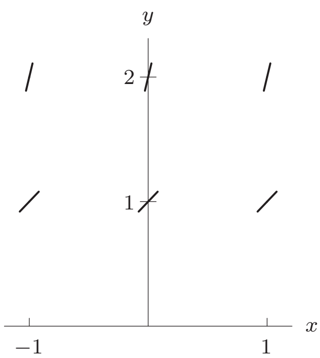


Figure 11.3

4. Figure (I) shows a line segment at $(4, 0)$ with positive slope. The only possible differential equation is (b), since at $(4, 0)$ we have $y' = \cos 0 = 1$. Note that (a) is not possible as $y'(4, 0) = e^{-16} = 0.0000001$, a much smaller positive slope than that shown.

Figure (II) shows a line segment at $(0, 4)$ with zero slope. The possible differential equations are (d), since at $(0, 4)$ we have $y' = 4(4 - 4) = 0$, and (f), since at $(0, 4)$ we have $y' = 0(3 - 0) = 0$.

Figure (III) shows a line segment at $(4, 0)$ with negative slope of large magnitude. The only possible differential equation is (f), since at $(4, 0)$ we have $y' = 4(3-4) = -4$. Note that (c) is not possible as $y'(4, 0) = \cos(4-0) = -0.65$, a negative slope of smaller magnitude than that shown.

Figure (IV) shows a line segment at $(4, 0)$ with a negative slope of small magnitude. The only possible differential equation is (c), since at $(4, 0)$ we have $y' = \cos(4-0) = -0.65$. Note that (f) is not possible as $y'(4, 0) = 4(3-4) = -4$, a negative slope of larger magnitude than that shown.

Figure (V) shows a line segment at $(0, 4)$ with positive slope. Possible differential equations are (a), since at $(0, 4)$ we have $y' = e^{0^2} = 1$, and (c), since at $(0, 4)$ we have $y' = \cos(4 - 4) = 1$.

Figure (VI) shows a line segment at $(0, 4)$ with a negative slope of large magnitude. The only possible differential equation is (e), since at $(0, 4)$ we have $y' = 4(3 - 4) = -4$. Note that (b) is not possible as $y'(0, 4) = \cos 4 = -0.65$, a negative slope of smaller magnitude than that shown.

5. There are many possible answers. One possibility is shown in Figures 11.4 and 11.5.

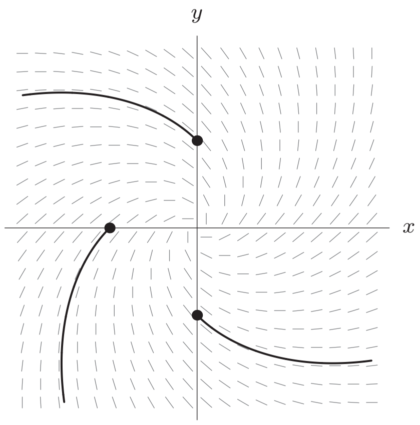


Figure 11.4

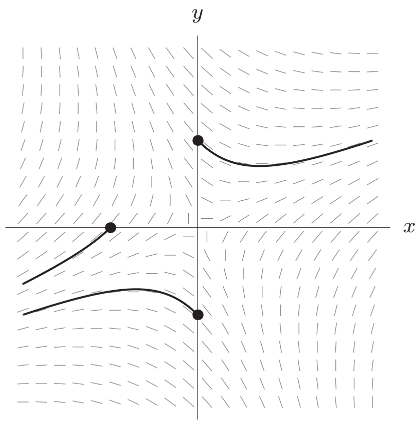


Figure 11.5

6. See Figure 11.6. Other choices of solution curves are, of course, possible.

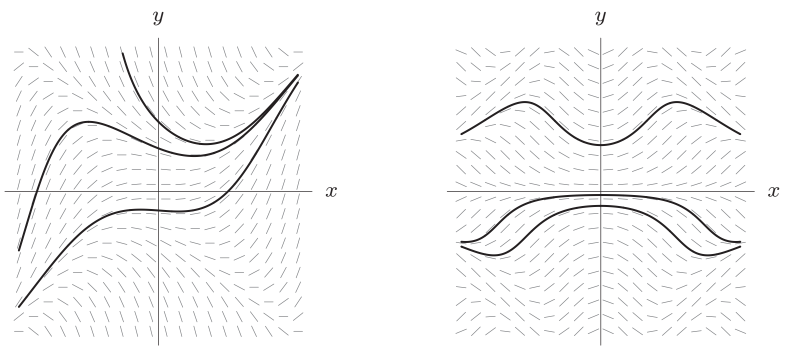


Figure 11.6

7. The first graph has the equation $y' = x^2 - y^2$. We can see this by looking along the line $y = x$. On the first slope field, it seems that $y' = 0$ along this line, as it should if $y' = x^2 - y^2$. This is not the case for the second graph. Another way to see this is to look along the y-axis on both graphs. The slope lines change with y on the first graph, but are constant on the second. But at $x = 0$, the slope should be $-y^2$, which varies with y, so the first graph is the one that fits.

At $(0, 1)$, $y' = -1$, and at $(1, 0)$, $y' = 1$, so we are looking for points on the axes where the line segment is sloped at $45°$. See Figure 11.7.

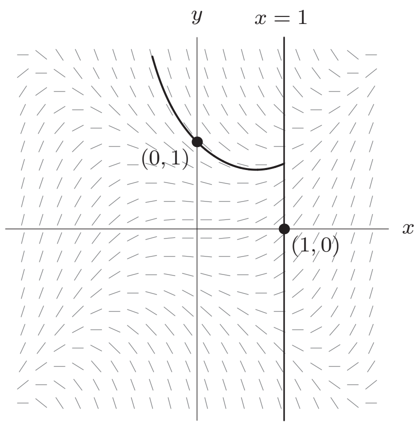


Figure 11.7

8. (a) See Figure 11.8.

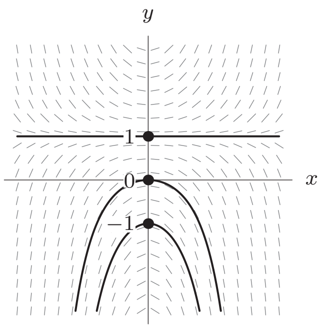


Figure 11.8

(b) The solution is $y(x) = 1$.
(c) Since $y' = 0$ and $x(y - 1) = 0$, this is a solution.

9. (a) See Figure 11.9.

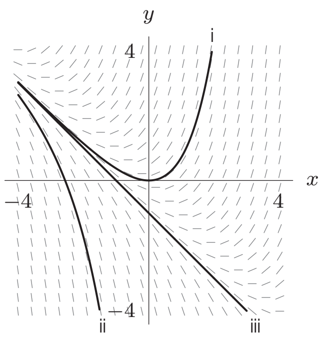


Figure 11.9

(b) The solution through $(-1, 0)$ appears to be linear, so its equation is $y = -x - 1$.
(c) If $y = -x - 1$, then $y' = -1$ and $x + y = x + (-x - 1) = -1$, so this checks as a solution.

Problems

10. We draw line segments with positive slope to the left of the y-axis, line segments with negative slope to the right of the y-axis, and horizontal line segments on the y-axis. A slope field satisfying these conditions is shown in Figure 11.10. Other answers are possible.

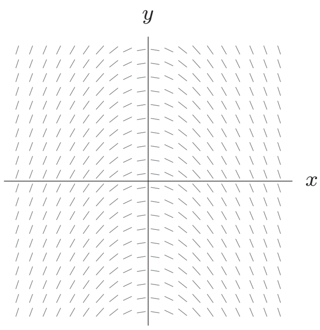


Figure 11.10

11. We draw line segments with positive slope when P is between 2 and 5, line segments with negative slope in the sections below $P = 2$ and above $P = 5$, and horizontal line segments on the horizontal lines $P = 2$ and $P = 5$. A slope field satisfying these conditions is shown in Figure 11.11. Other answers are possible.

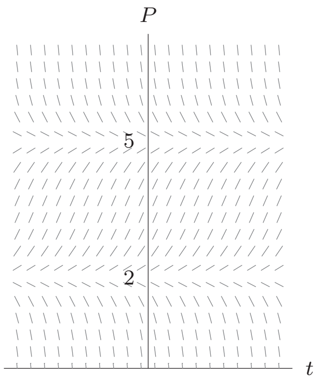


Figure 11.11

12. (a) See Figure 11.12.

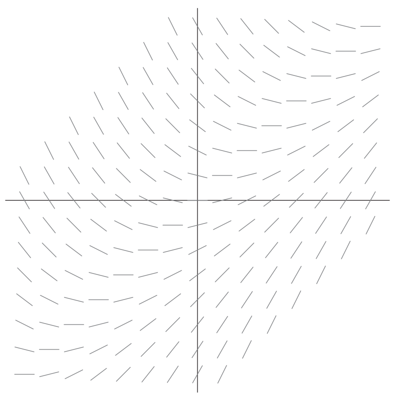


Figure 11.12

(b) From the graph, the solution through $(1, 0)$ appears linear with the equation approximately $y = x - 1$.
In fact, if $y = x - 1$, then $x - y = x - (x - 1) = 1 = y'$, so $y = x - 1$ is the solution through $(1, 0)$.

13. (a) See Figure 11.13.

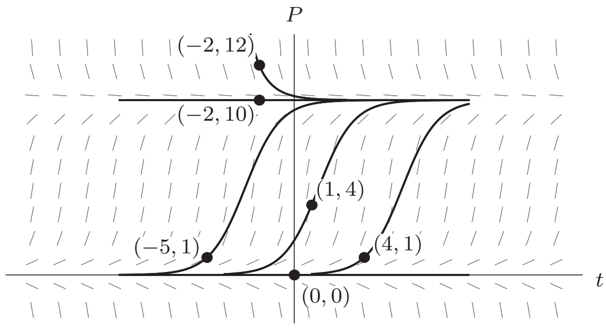


Figure 11.13

(b) If $0 < P < 10$, the solution is increasing; if $P > 10$, it is decreasing. If $P(0) = 5$, then P tends to 10.

14. (a) and (b) See Figure 11.14

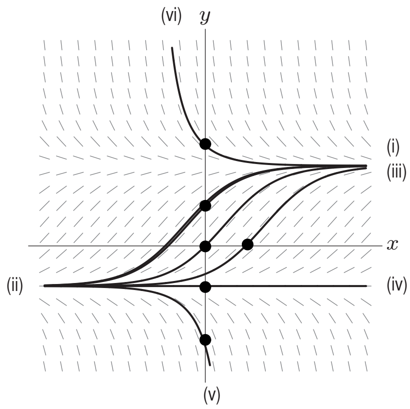


Figure 11.14

(c) Figure 11.14 shows that a solution will be increasing if its y-values fall in the range $-1 < y < 2$. This makes sense since if we examine the equation $y' = 0.5(1 + y)(2 - y)$, we will find that $y' > 0$ if $-1 < y < 2$. Notice that if the y-value ever gets to 2, then $y' = 0$ and the function becomes constant, following the line $y = 2$. (The same is true if ever $y = -1$.)

From the graph, the solution is decreasing if $y > 2$ or $y < -1$. Again, this also follows from the equation, since in either case $y' < 0$.

The curve has a horizontal tangent if $y' = 0$, which only happens if $y = 2$ or $y = -1$. This also can be seen on the graph in Figure 11.14.

15. Notice that $y' = \dfrac{x + y}{x - y}$ is zero when $x = -y$ and is undefined when $x = y$. A solution curve will be horizontal (slope$= 0$) when passing through a point with $x = -y$, and will be vertical (slope undefined) when passing through a point with $x = y$. The only slope field for which this is true is slope field (b).

16. (a) See Figure 11.15.

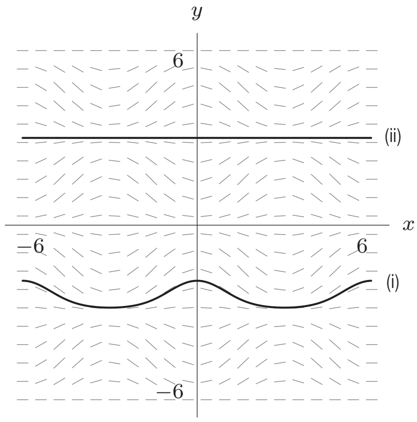


Figure 11.15

(b) We can see that the slope lines are horizontal when y is an integer multiple of π. We conclude from Figure 11.15 that the solution is $y = n\pi$ in this case.

To check this, we note that if $y = n\pi$, then $(\sin x)(\sin y) = (\sin x)(\sin n\pi) = 0 = y'$. Thus $y = n\pi$ is a solution to $y' = (\sin x)(\sin y)$, and it passes through $(0, n\pi)$.

17. (a) Since $y' = -y$, the slope is negative above the x-axis (when y is positive) and positive below the x-axis (when y is negative). The only slope field for which this is true is II.

(b) Since $y' = y$, the slope is positive for positive y and negative for negative y. This is true of both I and III. As y gets larger, the slope should get larger, so the correct slope field is I.

(c) Since $y' = x$, the slope is positive for positive x and negative for negative x. This corresponds to slope field V.

(d) Since $y' = \dfrac{1}{y}$, the slope is positive for positive y and negative for negative y. As y approaches 0, the slope becomes larger in magnitude, which corresponds to solution curves close to vertical. The correct slope field is III.

(e) Since $y' = y^2$, the slope is always positive, so this must correspond to slope field IV.

18. The slope fields in (I) and (II) appear periodic. (I) has zero slope at $x = 0$, so (I) matches $y' = \sin x$, whereas (II) matches $y' = \cos x$. The slope in (V) tends to zero as $x \to \pm\infty$, so this must match $y' = e^{-x^2}$. Of the remaining slope fields, only (III) shows negative slopes, matching $y' = xe^{-x}$. The slope in (IV) is zero at $x = 0$, so it matches $y' = x^2e^{-x}$. This leaves field (VI) to match $y' = e^{-x}$.

19. Judging from the figure, we see that:

- The slope depends on y, not on x.
- The slope is positive for $y < 0$ and $y > 10$.
- The slope is zero at $y = 0$ and $y = 10$.

This corresponds to equation (e): $y' = 0.05y(y - 10)$.

20. Judging from the figure, we see that:

- The slope depends on y, not on x.
- The slope is positive for $0 < y < 10$.
- The slope is zero at $y = 0$ and $y = 10$.

This corresponds to equation (a): $y' = 0.05y(10 - y)$.

21. Judging from the figure, we see that:

- The slope depends on x, not on y.
- The slope is positive for $0 < x < 10$.
- The slope is zero at $x = 0$ and $x = 10$.

This corresponds to equation (b): $y' = 0.05x(10 - x)$.

22. Judging from the figure, we see that:

- The slope depends on x, not on y.
- The slope is positive for $0 < x < 5$.
- The slope is zero at $x = 0$ and $x = 5$.

This corresponds to equation (d): $y' = 0.05x(5 - x)$.

23. (a) We know that $\frac{dy}{dx} = e^{x^2} > 0$ for all x. As $x \to \infty$, the slopes tend toward infinity. Similarly, as $x \to -\infty$, the slopes tend to infinity. Thus the appropriate slope field is (IV).

(b) We know that $\frac{dy}{dx} > 0$ for all x. As $x \to \pm\infty$, the slopes tend toward 0. Only slope fields I and III meet these conditions. We know that when $x = 0$, $\frac{dy}{dx} = e^{-2x^2} = e^0 = 1$.

We know that when $x = 1$, $\frac{dy}{dx} = e^{-2x^2} = e^{-2} \approx 0.14$. Thus the appropriate slope field is (I).

(c) We know that $\frac{dy}{dx} > 0$ for all x. As $x \to \pm\infty$, the slopes tend toward 0. Only slope fields I and III meet these conditions. We know that when $x = 0$, $\frac{dy}{dx} = e^{-x^2/2} = e^0 = 1$. We know that when $x = 1$, $\frac{dy}{dx} = e^{-x^2/2} = e^{-1/2} \approx 0.61$. Thus the appropriate slope field is (III).

(d) The slope field is both positive and negative. In fact, when $x = 0$, $\frac{dy}{dx} = e^{-0.5x}\cos x = e^0 \cos 0 = 1$.

Also, when $x = 1$, $\frac{dy}{dx} = e^{-0.5x}\cos x = e^{-1/2}\cos 1 \approx 0.33$, and when $x = 2$, $\frac{dy}{dx} = e^{-0.5x}\cos x = e^{-1}\cos 2 \approx -0.153$. Thus the appropriate slope field is (V).

(e) The slope field is positive for all values of x. When $x = 0$, $\frac{dy}{dx} = \frac{1}{(1 + 0.5\cos x)^2} \approx 0.44$. Thus the appropriate slope field is (II).

(f) The slope field is negative for all values of x. Thus the appropriate slope field is (VI).

24. When $a = 1$ and $b = 2$, the Gompertz equation is $y' = -y\ln(y/2) = y\ln(2/y) = y(\ln 2 - \ln y)$. This differential equation is similar to the differential equation $y' = y(2-y)$ in certain ways. For example, in both equations y' is positive for $0 < y < 2$ and negative for $y > 2$. Also, for y-values close to 2, the quantities $(\ln 2 - \ln y)$ and $(2 - y)$ are both close to 0, so $y(\ln 2 - \ln y)$ and $y(2-y)$ are approximately equal to zero. Thus around $y = 2$ the slope fields look almost the same. This happens again around $y = 0$, since around $y = 0$ both $y(2-y)$ and $y(\ln 2 - \ln y)$ go to 0. (Note that $\lim_{y\to 0^+}(y\ln y) = 0$.) For y values close to 1 the slope fields look similar since the local linearization of $\ln y$ near $y = 1$ is $y - 1$; hence, near $y = 1$, $y(\ln 2 - \ln y) \approx y(\ln 2 - (y-1)) \approx y(1.69 - y) \approx y(2-y)$. Finally, for $y > 2$, $\ln y$ grows much slower than y, so the slope field for $y' = y(\ln 2 - \ln y)$ is less steep, negatively, than for $y' = y(2-y)$.

Strengthen Your Understanding

25. Since the slope is zero at the point $(1, 1)$, the slope field has a horizontal line segment there. On the other hand, the solution $y = x$ goes through the point $(1, 1)$ with a slope of 1. See Figure 11.16. This is impossible since the slope field is tangent to the solution curves.

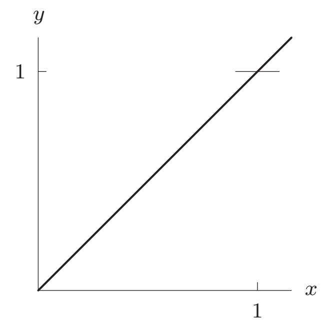


Figure 11.16

26. Note that $y' = y$ has a slope field with all positive slopes when $y > 0$. The given slope field, however, has negative slopes in the second quadrant.

27. If the slopes are all positive, then dy/dx is always positive. Some examples are $dy/dx = x^2 + 1$ or $dy/dx = y^2 + 1$ or $dy/dx = x^2 + y^2 + 1$ or $dy/dx = e^x$.

28. The sign of the derivative depends on the y-coordinate, since the y-coordinate determines whether a point is above or below the x-axis. Since we want a positive derivative when y is positive and a negative derivative when y is negative, one example is $dy/dx = y$.

29. If a derivative dy/dx depends only on x, then the derivative is constant for any fixed value of x. In other words, the slope is the same for all points on any vertical line. One possibility is the slope field for $dy/dx = x$ in Figure 11.17.

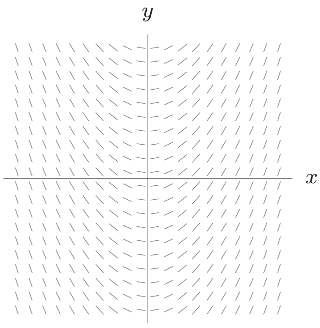


Figure 11.17

30. If a derivative dy/dx depends only on y, then the derivative is constant for any fixed value of y. In other words, the slope is the same for all points on any horizontal line. One possibility is the slope field for $dy/dx = y$ in Figure 11.18.

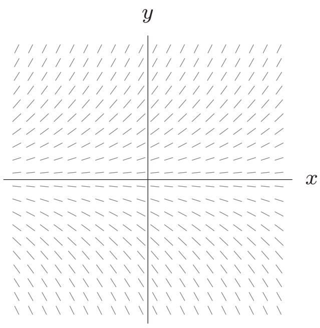


Figure 11.18

31. False. If $y(0) \leq 0$, then $\lim_{x\to\infty} y = -\infty$.

32. True. No matter what initial value you pick, the solution curve has the x-axis as an asymptote.

33. False. There appear to be two equilibrium values dividing the plane into regions with different limiting behavior.

34. True. We have $dy/dx > 0$ at every point because $x^2+y^2+1 > 0$, and a positive derivative indicates increasing function.

35. False. We have

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d(x^2+y^2+1)}{dx}\\ &= 2x + 2y\frac{dy}{dx}\\ &= 2x + 2y(x^2+y^2+1)\\ &= 2x + 2y + 2x^2y + 2y^3.\end{aligned}$$

At the point $(x, y) = (-1, 0)$ we have $d^2y/dx^2 = -2 < 0$. A negative second derivative indicates function concave down. The solution curve of the differential equation that passes through the point $(-1, 0)$ is concave down at $(-1, 0)$.

36. True. The slope of the graph of f is $dy/dx = 2x - y$. Thus when $x = a$ and $y = b$, the slope is $2a - b$.

37. True. Saying $y = f(x)$ is a solution for the differential equation $dy/dx = 2x - y$ means that if we substitute $f(x)$ for y, the equation is satisfied. That is, $f'(x) = 2x - f(x)$.

38. False. Since $f'(x) = 2x - f(x)$, we would have $1 = 2x - 5$ so $x = 3$ is the only possibility.

39. True. Differentiate $dy/dx = 2x - y$, to get:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x - y) = 2 - \frac{dy}{dx} = 2 - (2x - y).$$

40. False. Since $f'(1) = 2(1) - 5 = -3$, the point $(1, 5)$ could not be a critical point of f.

41. True. Since $dy/dx = 2x - y$, the slope of the graph of f is negative at any point satisfying $2x < y$, that is any point lying above the line $y = 2x$. The slope of the graph of f is positive at any point satisfying $2x > y$, that is any point lying below the line $y = 2x$.

42. True. When we differentiate $dy/dx = 2x - y$, we get:

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y).$$

Thus at any inflection point of $y = f(x)$, we have $d^2y/dx^2 = 2 - (2x - y) = 0$. That is, any inflection point of f must satisfy $y = 2x - 2$.

43. False. Suppose that $g(x) = f(x) + C$, where $C \neq 0$. In order to be a solution of $dy/dx = 2x - y$ we would need $g'(x) = 2x - g(x)$. Instead we have:

$$g'(x) = f'(x) = 2x - f(x) = 2x - (g(x) - C) = 2x - g(x) + C.$$

Since $C \neq 0$, this means $g(x)$ is not a solution of $dy/dx = 2x - y$.

44. True. We will use the hint. Let $w = g(x) - f(x)$. Then:

$$\frac{dw}{dx} = g'(x) - f'(x) = (2x - g(x)) - (2x - f(x)) = f(x) - g(x) = -w.$$

Thus $dw/dx = -w$. This equation is the equation for exponential decay and has the general solution $w = Ce^{-x}$. Thus,

$$\lim_{x\to\infty} (g(x) - f(x)) = \lim_{x\to\infty} Ce^{-x} = 0.$$

Solutions for Section 11.3

Exercises

1. Using Euler's method, we have:

$$\begin{aligned}
\text{At } (x,y) = (0,4):&\quad y' = (0-2)(4-3) &&= -2 \\
\text{so at } x = 0.1:&\quad y = 4.0 - 2(0.1) &&= 3.8 &&\text{because } \Delta x = 0.1 \\
\text{At } (x,y) = (0.1,3.8):&\quad y' = (0.1-2)(3.8-3) &&= -1.52 \\
\text{so at } x = 0.2:&\quad y = 3.8 - 1.52(0.1) &&= 3.648 &&\text{because } \Delta x = 0.1 \\
\text{At } (x,y) = (0.2,3.648):&\quad y' = (0.2-2)(3.648-3) &&= -1.1664.
\end{aligned}$$

Thus, the completed table is

x	y	y'
0.0	4.0	-2
0.1	3.8	-1.52
0.2	3.648	-1.1664

2. Using Euler's method, we have:

$$\begin{aligned}
\text{At } (x,y) = (1,-3)\text{:}\quad & y' = 4(1)(-3) && = -12 \\
\text{so at } x = 1.01\text{:}\quad & y = -3 - 12(0.01) && = -3.12 && \text{because } \Delta x = 0.01 \\
\text{At } (x,y) = (1.01,-3.12)\text{:}\quad & y' = 4(1.01)(-3.12) && = -12.6048 \\
\text{so at } x = 1.02\text{:}\quad & y = -3.12 - 12.6048(0.01) && = -3.246 && \text{because } \Delta x = 0.01 \\
\text{At } (x,y) = (1.02,-3.246)\text{:}\quad & y' = 4(1.02)(-3.246) && = -13.244.
\end{aligned}$$

Thus, the completed table is

x	y	y'
1.00	-3	-12
1.01	-3.12	-12.6048
1.02	-3.246	-13.244

3. We know $P = 1500$ at time $t = 0$. This means

$$\frac{dP}{dt} = 0.00008(1500)(1900 - 1500) = 48.$$

From time $t = 0$ to $t = 1$ we have $\Delta t = 1$, so the new value of P is given by

$$\begin{array}{c}\text{Value of } P \\ \text{at } t = 1\end{array} = 1500 + 48(1) = 1548.$$

We repeat this process to find the values of P at $t = 2$ and $t = 3$. See the table.

t	P	dP/dt
0	1500	$0.00008(1500)(1900 - 1500) = 48$
1	$1500 + 48(1) = 1548$	$0.00008(1548)(1900 - 1548) = 43.59168$
2	$1548 + 43.59168(1) = 1591.59168$	$0.00008(1591.59168)(1900 - 1591.59168) = 39.26881$
3	$1591.59168 + 39.26881 = 1630.86049$	(no calculation necessary)

Rounding gives values of $P = 1548$, $P = 1591.5917$, $P = 1630.860$ at $t = 1, 2, 3$.

4. (a) Since dy/dx is always 3 and $\Delta x = 0.2$, at every step we have

$$\Delta y = \frac{dy}{dx} \cdot \Delta x = 3(0.2) = 0.6.$$

The results are in Table 11.2. We see that Euler's method gives an approximate value of $y = 5$ at $x = 1$.

Table 11.2

x	0	0.2	0.4	0.6	0.8	1.0
y	2	2.6	3.2	3.8	4.4	5

(b) The general solution to $dy/dx = 3$ is $y = 3x + C$. We use the initial condition to see that $C = 2$ so the particular solution is $y = 3x + 2$.

(c) The exact value of the solution at $x = 1$ is $y = 3(1) + 2 = 5$. Since the approximate value from Euler's method is also 5, the error is 0.

(d) Euler's method approximates a solution using line segments. Since in this case the exact solution is itself a line, Euler's method gives exact values.

5. (a) At point $P_0 = (0, 10)$, we have

$$\Delta y = \text{slope at } P_0 \cdot \Delta x = (10 - 0)(0.2) = 2.$$

Therefore, point P_1 is $(0.2, 12)$. At point P_1, we have

$$\Delta y = \text{slope at } P_1 \cdot \Delta x = (12 - 0.2)(0.2) = 2.36.$$

So point P_2 is $(0.4, 14.36)$. Continuing in this way, we obtain the approximate solution shown in Table 11.3.

Table 11.3

x	0	0.2	0.4	0.6	0.8	1.0
y	10	12	14.36	17.15	20.46	24.39

(b) Since the slopes are getting larger, we expect the solution to be concave up.
(c) Since the solution is concave up, we expect our approximations to be underestimates. It can be shown that the true value of y at $x = 1$ is 26.46.

6. (a) The results from Euler's method with $\Delta x = 0.1$ are in Table 11.4.
(b) We have

$$y(x) = \frac{x^4}{4} + C,$$

so that $y(0) = 0$ gives $C = 0$, and the required solution is therefore

$$y(x) = \frac{x^4}{4}.$$

This is shown in the 3rd column of Table 11.4.
(c) The computed solution underestimates the real solution since the solution is concave up and is approximated in every interval by the tangent which is beneath the curve. See Figure 11.19.

Table 11.4

	Computed Solution	
x_n	Approx. $y(x_n)$	$y(x_n)$
0	0	0
0.1	0	0.000025
0.2	0.0001	0.0004
0.3	0.0009	0.002025
0.4	0.0036	0.0064
0.5	0.01	0.015625
0.6	0.0225	0.0324
0.7	0.0441	0.060025
0.8	0.0784	0.1024
0.9	0.1296	0.164025
1.0	0.2025	0.25

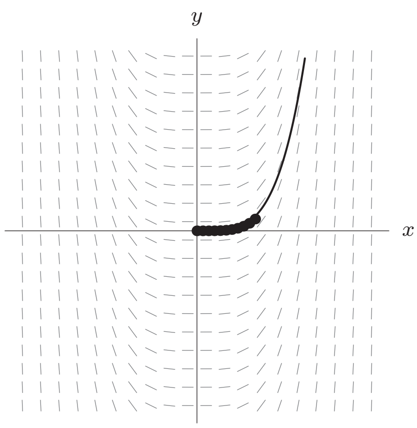


Figure 11.19

7. (a) In Table 11.5, we see that $y(0.4) \approx 1.5282$.
(b) In Table 11.6, we see that $y(0.4) = -1.4$. (This answer is exact.)

Table 11.5 *Euler's method for* $y' = x + y$ *with* $y(0) = 1$

x	y	$\Delta y =$(slope)Δx
0	1	$0.1 = (1)(0.1)$
0.1	1.1	$0.12 = (1.2)(0.1)$
0.2	1.22	$0.142 = (1.42)(0.1)$
0.3	1.362	$0.1662 = (1.662)(0.1)$
0.4	1.5282	

Table 11.6 *Euler's method for* $y' = x + y$ *with* $y(-1) = 0$

x	y	$\Delta y =$(slope)Δx
-1	0	$-0.1 = (-1)(0.1)$
-0.9	-0.1	$-0.1 = (-1)(0.1)$
-0.8	-0.2	$-0.1 = (-1)(0.1)$
-0.7	-0.3	
⋮	⋮	Notice that y
0	-1	decreases by 0.1
⋮	⋮	for every step
0.4	-1.4	

8. (a) See Figure 11.20.
(b) See Table 11.7. At $x = 1$, Euler's method gives $y \approx 0.16$.
(c) Our answer to (a) appears to be an underestimate. This is as we would expect, since the solution curve is concave up.

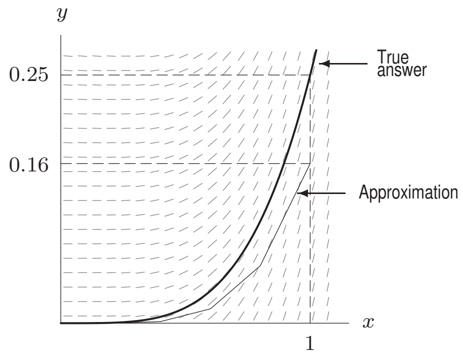


Figure 11.20

Table 11.7

x	y	$\Delta y =$ (slope)Δx
0	0	0
0.2	0	0.0016
0.4	0.0016	0.0128
0.6	0.0144	0.0432
0.8	0.0576	0.1024
1	0.1600	

9. (a) See Figure 11.21.
(b) $y(0) = 1$,

$$y(0.1) \approx y(0) + 0.1y(0) = 1 + 0.1(1) = 1.1$$
$$y(0.2) \approx y(0.1) + 0.1y(0.1) = 1.1 + 0.1(1.1) = 1.21$$
$$y(0.3) \approx y(0.2) + 0.1y(0.2) = 1.21 + 0.1(1.21) = 1.331$$
$$y(0.4) \approx 1.4641$$
$$y(0.5) \approx 1.61051$$
$$y(0.6) \approx 1.77156$$
$$y(0.7) \approx 1.94872$$
$$y(0.8) \approx 2.14359$$
$$y(0.9) \approx 2.35795$$
$$y(1.0) \approx 2.59374$$

(c) See Figure 11.21. A smooth curve drawn through the solution points seems to match the slope field.
(d) For $y = e^x$, we have $y' = e^x = y$ and $y(0) = e^0 = 1$. See Table 11.8.

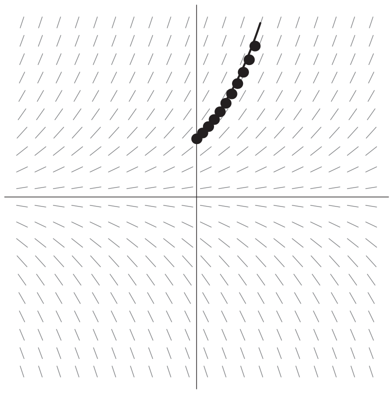


Figure 11.21

Table 11.8

Computed Solution		
x_n	Approx. $y(x_n)$	$y(x_n)$
0	1	1
0.1	1.1	1.10517
0.2	1.21	1.22140
0.3	1.331	1.34986
0.4	1.4641	1.49182
0.5	1.61051	1.64872
0.6	1.77156	1.82212
0.7	1.94872	2.01375
0.8	2.14359	2.22554
0.9	2.35795	2.45960
1.0	2.59374	2.71828

Problems

10. (a) (i)

Table 11.9 *Euler's method for* $y' = (\sin x)(\sin y)$, *starting at* $(0, 2)$

x	y	$\Delta y =$(slope)Δx
0	2	$0 = (\sin 0)(\sin 2)(0.1)$
0.1	2	$0.009 = (\sin 0.1)(\sin 2)(0.1)$
0.2	2.009	$0.018 = (\sin 0.2)(\sin 2.009)(0.1)$
0.3	2.027	

(ii)

Table 11.10 *Euler's method for* $y' = (\sin x)(\sin y)$, *starting at* $(0, \pi)$

x	y	$\Delta y =$(slope)Δx
0	π	$0 = (\sin 0)(\sin \pi)(0.1)$
0.1	π	$0 = (\sin 0.1)(\sin \pi)(0.1)$
0.2	π	$0 = (\sin 0.2)(\sin \pi)(0.1)$
0.3	π	

(b) The slope field shows that the slope of the solution curve through $(0, \pi)$ is always 0. Thus the solution curve is the horizontal line with equation $y = \pi$.

11. (a)

Table 11.11

x	y	$\Delta y =$ (slope)Δx
0	1.000	-0.200
0.2	0.800	-0.120
0.4	0.680	-0.060
0.6	0.620	-0.005
0.8	0.615	0.052
1	0.667	

(b)

Table 11.12

x	y	$\Delta y = (\text{slope})\Delta x$
0	1.000	-0.100
0.1	0.900	-0.080
0.2	0.820	-0.063
0.3	0.757	-0.048
0.4	0.709	-0.034
0.5	0.674	-0.020
0.6	0.654	-0.007
0.7	0.647	0.007
0.8	0.654	0.021
0.9	0.675	0.035
1	0.710	

12. By looking at the slope fields, or by computing the second derivative

$$\frac{d^2y}{dx^2} = 2x - 2y\frac{dy}{dx} = 2x - 2x^2y + 2y^3,$$

we see that the solution curve is concave up, so Euler's method gives an underestimate.

13. Since the error is proportional to one over the number of subintervals, the error using 10 intervals should be roughly half the error obtained using 5 intervals. Since both the estimates are underestimates, if we let A be the actual value we have:

$$\begin{aligned}\frac{1}{2}(A - 0.667) &= A - 0.710\\ A - 0.667 &= 2A - 1.420\\ A &= 0.753\end{aligned}$$

Therefore, 0.753 should be a better approximation.

14. (a)

Table 11.13

t	y	slope $= \frac{1}{t}$	$\Delta y = (\text{slope})\Delta t = \frac{1}{t}(0.1)$
1	0	1	0.1
1.1	0.1	0.909	0.091
1.2	0.191	0.833	0.083
1.3	0.274	0.769	0.077
1.4	0.351	0.714	0.071
1.5	0.422	0.667	0.067
1.6	0.489	0.625	0.063
1.7	0.552	0.588	0.059
1.8	0.610	0.556	0.056
1.9	0.666	0.526	0.053
2	0.719		

(b) If $\frac{dy}{dt} = \frac{1}{t}$, then $y = \ln|t| + C$.
Starting at $(1, 0)$ means $y = 0$ when $t = 1$, so $C = 0$ and $y = \ln|t|$.
After ten steps, $t = 2$, so $y = \ln 2 \approx 0.693$.

(c) Approximate $y = 0.719$, Exact $y = 0.693$.
Thus the approximate answer is too big. This is because the solution curve is concave down, and so the tangent lines are above the curve. Figure 11.22 shows the slope field of $y' = 1/t$ with the solution curve $y = \ln t$ plotted on top of it.

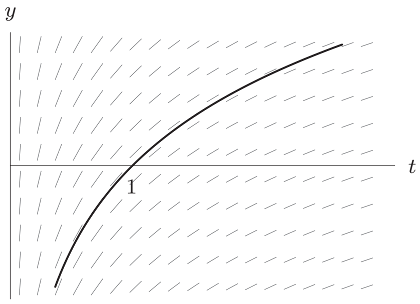


Figure 11.22

15. (a) $\Delta x = 0.5$

Table 11.14 *Euler's method for $y' = 2x$, with $y(0) = 1$*

x	y	$\Delta y =$(slope)Δx
0	1	$0 = (2 \cdot 0)(0.5)$
0.5	1	$0.5 = (2 \cdot 0.5)(0.5)$
1	1.5	

$\Delta x = 0.25$

Table 11.15 *Euler's method for $y' = 2x$, with $y(0) = 1$*

x	y	$\Delta y =$(slope)Δx
0	1	$0 = (2 \cdot 0)(0.25)$
0.25	1	$0.125 = (2 \cdot 0.25)(0.25)$
0.50	1.125	$0.25 = (2 \cdot 0.5)(0.25)$
0.75	1.375	$0.375 = (2 \cdot 0.75)(0.25)$
1	1.75	

(b) General solution is $y = x^2 + C$, and $y(0) = 1$ gives $C = 1$. Thus, the solution is $y = x^2 + 1$. So the true value of y when $x = 1$ is $y = 1^2 + 1 = 2$.

(c) When $\Delta x = 0.5$, error $= 0.5$.
When $\Delta x = 0.25$, error $= 0.25$.
Thus, decreasing Δx by a factor of 2 has decreased the error by a factor of 2, as expected.

16. For $\Delta x = 0.2$, we get the following results.

$$\begin{aligned}
y(1.2) &\approx y(1) + 0.2\sin(1 \cdot y(1)) = 1.168294 \\
y(1.4) &\approx y(1.2) + 0.2\sin(1.2 \cdot y(1.2)) = 1.365450 \\
y(1.6) &\approx y(1.4) + 0.2\sin(1.4 \cdot y(1.4)) = 1.553945 \\
y(1.8) &\approx y(1.6) + 0.2\sin(1.6 \cdot y(1.6)) = 1.675822 \\
y(2.0) &\approx y(1.8) + 0.2\sin(1.8 \cdot y(1.8)) = 1.700779
\end{aligned}$$

Repeating this with $\Delta x = 0.1$ and 0.05 gives the results in Table 11.16 below

Table 11.16

Computed Solution			
x-value	$\Delta x = 0.2$	$\Delta x = 0.1$	$\Delta x = 0.05$
1.0	1	1	1
1.1		1.084147	1.086501
1.2	1.168294	1.177079	1.181232
1.3		1.275829	1.280619
1.4	1.365450	1.375444	1.379135
1.5		1.469214	1.469885
1.6	1.553945	1.549838	1.546065
1.7		1.611296	1.602716
1.8	1.675822	1.650458	1.637809
1.9		1.667451	1.652112
2.0	1.700779	1.664795	1.648231

The computed approximations for $y(2)$ using step sizes $\Delta x = 0.2, 0.1, 0.05$ are 1.700779, 1.664795, and 1.648231, respectively. Plotting these points we see that they lie approximately on a straight line.

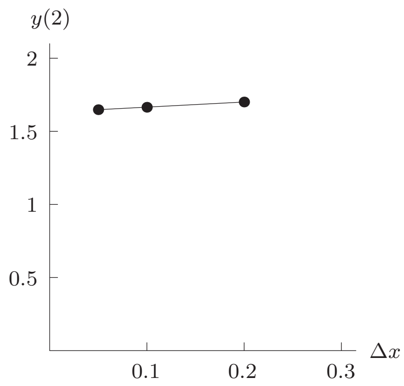


Figure 11.23

In the limit, as Δx tends to zero, the results produced by Euler's method should converge to the exact value of $y(2)$. This limiting value is the vertical intercept of the line drawn in Figure 11.23. This gives $y(2) \approx 1.632$.

17. (a) Using one step, $\frac{\Delta B}{\Delta t} = 0.05$, so $\Delta B = \left(\frac{\Delta B}{\Delta t}\right)\Delta t = 50$. Therefore we get an approximation of $B \approx 1050$ after one year.

(b) With two steps, $\Delta t = 0.5$ and we have

Table 11.17

t	B	$\Delta B = (0.05B)\Delta t$
0	1000	25
0.5	1025	25.63
1.0	1050.63	

(c) Keeping track to the nearest hundredth with $\Delta t = 0.25$, we have

Table 11.18

t	B	$\Delta B = (0.05B)\Delta t$
0	1000	12.5
0.25	1012.5	12.66
0.5	1025.16	12.81
0.75	1037.97	12.97
1	1050.94	

(d) In part (a), we get our approximation by making a single increment, ΔB, where ΔB is just $0.05B$. If we think in terms of interest, ΔB is just like getting one end of the year interest payment. Since ΔB is 0.05 times the balance B, it is like getting 5% interest at the end of the year.

(e) Part (b) is equivalent to computing the final amount in an account that begins with \$1000 and earns 5% interest compounded twice annually. Each step is like computing the interest after 6 months. When $t = 0.5$, for example, the interest is $\Delta B = (0.05B) \cdot \frac{1}{2}$, and we add this to \$1000 to get the new balance.

Similarly, part (c) is equivalent to the final amount in an account that has an initial balance of \$1000 and earns 5% interest compounded quarterly.

18. Assume that $x > 0$ and that we use n steps in Euler's method. Label the x-coordinates we use in the process $x_0, x_1, \ldots, x_n$, where $x_0 = 0$ and $x_n = x$. Then using Euler's method to find $y(x)$, we get

Table 11.19

	x	y	$\Delta y =$ (slope)Δx
P_0	$0 = x_0$	0	$f(x_0)\Delta x$
P_1	x_1	$f(x_0)\Delta x$	$f(x_1)\Delta x$
P_2	x_2	$f(x_0)\Delta x + f(x_1)\Delta x$	$f(x_2)\Delta x$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
P_n	$x = x_n$	$\sum_{i=0}^{n-1} f(x_i)\Delta x$	

Thus the result from Euler's method is $\sum_{i=0}^{n-1} f(x_i)\Delta x$. We recognize this as the left-hand Riemann sum that approximates $\int_0^x f(t)\,dt$.

Strengthen Your Understanding

19. For differential equations of the form $\dfrac{dy}{dx} = k$, where k is constant, Euler's method traces the exact solution for any initial condition.

20. If $x(0) > 0$, then the statement is true. However, if $x(0) < 0$, then the solution curve is decreasing. Since we are using the value of x at the beginning of a subinterval to estimate the rate of change of x on an interval, we obtain an overestimate for $x(1)$.

21. The approximate values lie on a line if the slope $\Delta y/\Delta x$ is constant, which occurs if dy/dx is constant. One example is $dy/dx = 5$. The approximate values found using Euler's method lie on a line for any initial condition and value of Δx. Other examples are possible.

22. One step of Euler's method gives an underestimate when the solution is concave up. Thus, we look for a differential equation whose solution curves are concave up everywhere. An example is given by the equation $dy/dx = y$ with $y(0) = 1$ which has solution $y = e^x$.

23. False. Euler's method approximates y-values of points on the solution curve.

24. True. Both lead to

$$y(1) \approx f(0) \cdot 0.2 + f(0.2) \cdot 0.2 + f(0.4) \cdot 0.2 + f(0.6) \cdot 0.2 + f(0.8) \cdot 0.2.$$

Solutions for Section 11.4

Exercises

1. **(a)** Yes **(b)** No **(c)** Yes
(d) No **(e)** Yes **(f)** Yes
(g) No **(h)** Yes **(i)** No
(j) Yes **(k)** Yes **(l)** No

2. Separating variables gives

$$\int \frac{1}{P}dP = -\int 2dt,$$

so

$$\ln|P| = -2t + C.$$

Therefore

$$P = \pm e^{-2t+C} = Ae^{-2t}.$$

The initial value $P(0) = 1$ gives $1 = A$, so

$$P = e^{-2t}.$$

3. Separating variables gives

$$\int \frac{dP}{P} = \int 0.02\,dt,$$

so

$$\ln|P| = 0.02t + C.$$

Thus

$$|P| = e^{0.02t+C}$$

and

$$P = Ae^{0.02t}, \text{ where } A = \pm e^C.$$

We are given $P(0) = 20$. Therefore, $P(0) = Ae^{(0.02)\cdot 0} = A = 20$. So the solution is

$$P = 20e^{0.02t}.$$

4. Separating variables and integrating both sides gives

$$\int \frac{1}{L}dL = \frac{1}{2}\int dp$$

or

$$\ln|L| = \frac{1}{2}p + C.$$

This can be written

$$L = \pm e^{(1/2)p+C} = Ae^{p/2}.$$

The initial condition $L(0) = 100$ gives $100 = A$, so

$$L = 100e^{p/2}.$$

5. Separating variables gives

$$\int \frac{dQ}{Q} = \int \frac{dt}{5},$$

so

$$\ln|Q| = \frac{1}{5}t + C.$$

So

$$|Q| = e^{\frac{1}{5}t+C} = e^{\frac{1}{5}t}e^C$$

and

$$Q = Ae^{\frac{1}{5}t}, \text{ where } A = \pm e^C.$$

From the initial conditions we know that $Q(0) = 50$, so $Q(0) = Ae^{(\frac{1}{5})\cdot 0} = A = 50$. Thus

$$Q = 50e^{\frac{1}{5}t}.$$

6. Separating variables gives

$$\int P dP = \int dt$$

so that

$$\frac{P^2}{2} = t + C$$

or

$$P = \pm\sqrt{2t + D}$$

(where $D = 2C$).

The initial condition $P(0) = 1$ implies we must take the positive root and that $1 = D$, so

$$P = \sqrt{2t + 1}.$$

7. Separating variables gives

$$\int \frac{dm}{m} = \int 3\,dt$$
$$\ln|m| = 3t + C$$
$$m = \pm e^C e^{3t} = Ae^{3t}.$$

Since $m = 5$ when $t = 1$, we have $5 = Ae^3$, so $A = 5/e^3$. Thus

$$m = \frac{5}{e^3}e^{3t} = 5e^{3t-3}.$$

8. Separating variables gives

$$\int \frac{dI}{I} = \int 0.2\,dx,$$

so

$$\ln|I| = 0.2x + C.$$

Thus,

$$I = Ae^{0.2x}, \text{ where } A = \pm e^C.$$

According to the given boundary condition, $I(-1) = 6$. Therefore, $I(-1) = Ae^{0.2(-1)} = Ae^{-0.2} = 6$, so $A = 6e^{0.2}$. Thus

$$I = 6e^{0.2}e^{0.2x} = 6e^{0.2(x+1)}.$$

9. Separating variables gives

$$\int \frac{dz}{z} = \int 5\,dt$$
$$\ln|z| = 5t + C.$$

Solving for z, we have

$$z = Ae^{5t}, \text{ where } A = \pm e^C.$$

Using the fact that $z(1) = 5$, we have $z(1) = Ae^5 = 5$, so $A = 5/e^5$. Therefore,

$$z = \frac{5}{e^5}e^{5t} = 5e^{5t-5}.$$

10. Separating variables gives

$$\int \frac{1}{m} dm = \int ds.$$

Hence

$$\ln|m| = s + C$$

which gives

$$m = \pm e^{s+C} = Ae^s.$$

The initial condition $m(1) = 2$ gives $2 = Ae^1$ or $A = 2/e$, so

$$m = \frac{2}{e}e^s = 2e^{s-1}.$$

11. Separating variables gives

$$\int \frac{1}{u^2}du = \int \frac{1}{2}dt$$

or

$$-\frac{1}{u} = \frac{1}{2}t + C.$$

The initial condition gives $C = -1$ and so

$$u = \frac{1}{1-(1/2)t}.$$

12. Separating variables and integrating gives

$$\int \frac{1}{z}dz = \int y dy$$

which gives

$$\ln|z| = \frac{1}{2}y^2 + C$$

or

$$z = \pm e^{(1/2)y^2+C} = Ae^{y^2/2}.$$

The initial condition $y = 0$, $z = 1$ gives $A = 1$. Therefore

$$z = e^{y^2/2}.$$

13. Separating variables gives

$$\int \frac{dy}{y} = -\int \frac{1}{3}\,dx$$

$$\ln|y| = -\frac{1}{3}x + C.$$

Solving for y, we have

$$y = Ae^{-\frac{1}{3}x}, \text{ where } A = \pm e^C.$$

Since $y(0) = A = 10$, we have

$$y = 10e^{-\frac{1}{3}x}.$$

14. Separating variables gives

$$\int \frac{dy}{y-200} = \int 0.5dt$$

$$\ln|y-200| = 0.5t + C$$

$$y = 200 + Ae^{0.5t}, \quad \text{where } A = \pm e^C.$$

The initial condition, $y(0) = 50$, gives

$$50 = 200 + A, \quad \text{so} \quad A = -150.$$

Thus,

$$y = 200 - 150e^{0.5t}.$$

15. Separating variables gives

$$\int \frac{dP}{P+4} = \int dt,$$

so

$$\begin{aligned}\ln|P+4| &= t + C\\ P+4 &= Ae^t\\ P &= Ae^t - 4.\end{aligned}$$

Since $P = 100$ when $t = 0$, we have $P(0) = Ae^0 - 4 = 100$, and $A = 104$. Therefore

$$P = 104e^t - 4.$$

16. Factoring out a 2 on the right makes the integration easier:

$$\begin{aligned}\frac{dy}{dx} &= 2y - 4 = 2(y-2)\\ \int \frac{dy}{y-2} &= \int 2\,dx,\end{aligned}$$

giving

$$\ln|y-2| = 2x + C.$$

Thus,

$$|y-2| = e^{2x+C},$$

so

$$y - 2 = Ae^{2x}, \text{ where } A = \pm e^C.$$

The curve passes through $(2,5)$, which means $3 = Ae^4$, so $A = 3/e^4$. Thus,

$$y = 2 + \frac{3}{e^4}e^{2x} = 2 + 3e^{2x-4}.$$

17. Factoring and separating variables gives

$$\begin{aligned}\frac{dQ}{dt} &= 0.3(Q-400)\\ \int \frac{dQ}{Q-400} &= \int 0.3\,dt\\ \ln|Q-400| &= 0.3t + C\\ Q &= 400 + Ae^{0.3t}, \quad \text{where } A = \pm e^C.\end{aligned}$$

The initial condition, $Q(0) = 50$, gives

$$50 = 400 + A \quad \text{so} \quad A = -350.$$

Thus

$$Q = 400 - 350e^{0.3t}.$$

18. Factoring out the 0.1 gives

$$\begin{aligned}\frac{dm}{dt} &= 0.1m + 200 = 0.1(m+2000)\\ \int \frac{dm}{m+2000} &= \int 0.1\,dt,\end{aligned}$$

so

$$\ln|m+2000| = 0.1t + C,$$

and

$$m = Ae^{0.1t} - 2000, \text{ where } A = \pm e^C.$$

Using the initial condition, $m(0) = Ae^{(0.1)\cdot 0} - 2000 = 1000$, gives $A = 3000$. Thus

$$m = 3000e^{0.1t} - 2000.$$

19. Rearrange and write

$$\int \frac{1}{1-R} dR = \int dy$$

or

$$-\ln|1-R| = y + C$$

which can be written as

$$1 - R = \pm e^{-C-y} = Ae^{-y}$$

or

$$R = 1 - Ae^{-y}.$$

The initial condition $R(1) = 0.1$ gives $0.1 = 1 - Ae^{-1}$ and so

$$A = 0.9e.$$

Therefore

$$R = 1 - 0.9e^{1-y}.$$

20. Rewriting gives

$$\frac{dB}{dt} + 2B = 50$$

and

$$\frac{dB}{dt} = -2B + 50 = -2(B - 25),$$

so

$$\int \frac{dB}{B - 25} = -\int 2\,dt$$
$$\ln|B - 25| = -2t + C.$$

Thus, we have

$$B - 25 = Ae^{-2t}, \text{ where } A = \pm e^C.$$

Using the initial condition, $B(1) = 100$, we have $75 = Ae^{-2}$, so $A = 75e^2$. Thus

$$B = 25 + 75e^2e^{-2t} = 25 + 75e^{2-2t}.$$

21. Write

$$\int \frac{1}{y} dy = \int \frac{1}{3+t} dt$$

and so

$$\ln|y| = \ln|3+t| + C$$

or

$$\ln|y| = \ln D|3+t|$$

where $\ln D = C$. Therefore

$$y = D(3+t).$$

The initial condition $y(0) = 1$ gives $D = \frac{1}{3}$, so

$$y = \frac{1}{3}(3+t).$$

22. Separating variables gives

$$\begin{aligned}\frac{dz}{dt} &= te^z \\ e^{-z}dz &= tdt \\ \int e^{-z}\,dz &= \int t\,dt,\end{aligned}$$

so

$$-e^{-z} = \frac{t^2}{2} + C.$$

Since the solution passes through the origin, $z = 0$ when $t = 0$, we must have

$$-e^{-0} = \frac{0}{2} + C, \text{ so } C = -1.$$

Thus

$$-e^{-z} = \frac{t^2}{2} - 1,$$

or

$$z = -\ln\left(1 - \frac{t^2}{2}\right).$$

23. Separating variables gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{5y}{x} \\ \int \frac{dy}{y} &= \int \frac{5}{x}dx \\ \ln|y| &= 5\ln|x| + C.\end{aligned}$$

Thus

$$|y| = e^{5\ln|x|}e^C = e^C e^{\ln|x|^5} = e^C|x|^5,$$

giving

$$y = Ax^5, \quad \text{where} \quad A = \pm e^C.$$

Since $y = 3$ when $x = 1$, so $A = 3$. Thus

$$y = 3x^5.$$

24. Separating variables gives

$$\begin{aligned}\frac{dy}{dt} &= y^2(1+t) \\ \int \frac{dy}{y^2} &= \int (1+t)\,dt,\end{aligned}$$

so

$$-\frac{1}{y} = t + \frac{t^2}{2} + C,$$

giving

$$y = -\frac{1}{t + t^2/2 + C}.$$

Since $y = 2$ when $t = 1$, we have

$$2 = -\frac{1}{1 + 1/2 + C}, \quad \text{so} \quad 2C + 3 = -1, \quad \text{and} \quad C = -2.$$

Thus

$$y = -\frac{1}{t^2/2 + t - 2} = -\frac{2}{t^2 + 2t - 4}.$$

25. Separating variables gives

$$\frac{dz}{dt} = z + zt^2 = z(1+t^2)$$
$$\int \frac{dz}{z} = \int (1+t^2)dt,$$

so

$$\ln|z| = t + \frac{t^3}{3} + C,$$

giving

$$z = Ae^{t+t^3/3}.$$

We have $z = 5$ when $t = 0$, so $A = 5$ and

$$z = 5e^{t+t^3/3}.$$

26. Separating variables gives

$$\frac{dw}{d\theta} = \theta w^2 \sin\theta^2$$
$$\int \frac{dw}{w^2} = \int \theta \sin\theta^2 \, d\theta,$$

so

$$-\frac{1}{w} = -\frac{1}{2}\cos\theta^2 + C.$$

According to the initial conditions, $w(0) = 1$, so $-1 = -\frac{1}{2} + C$ and $C = -\frac{1}{2}$. Thus,

$$-\frac{1}{w} = -\frac{1}{2}\cos\theta^2 - \frac{1}{2}$$
$$\frac{1}{w} = \frac{\cos\theta^2 + 1}{2}$$
$$w = \frac{2}{\cos\theta^2 + 1}.$$

27. Separating variables and integrating gives

$$\int \frac{1}{w^2} dw = -\int \tan\psi d\psi.$$

To integrate the right side, write $\tan\psi = \sin\psi/\cos\psi$ and use the substitution $w = \cos\psi$ giving

$$-\frac{1}{w} = \ln|\cos\psi| + C$$

so

$$w = \frac{-1}{\ln|\cos\psi| + C}.$$

Using the initial condition $w(0) = 2$ we have

$$2 = \frac{-1}{\ln|\cos 0| + C} = \frac{-1}{\ln 1 + C} = \frac{-1}{0 + C}$$

so

$$C = -\frac{1}{2}.$$

Thus the solution is

$$w = \frac{-1}{\ln|\cos\psi| - 1/2}.$$

28. Separating variables gives

$$x(x+1)\frac{du}{dx} = u^2$$
$$\int \frac{du}{u^2} = \int \frac{dx}{x(x+1)} = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx,$$

so

$$-\frac{1}{u} = \ln|x| - \ln|x+1| + C.$$

We have $u(1) = 1$, so $-\frac{1}{1} = \ln|1| - \ln|1+1| + C$. So $C = \ln 2 - 1$. Solving for u yields

$$-\frac{1}{u} = \ln|x| - \ln|x+1| + \ln 2 - 1 = \ln\frac{2|x|}{|x+1|} - 1,$$

so

$$u = \frac{-1}{\ln|\frac{2x}{x+1}| - 1}.$$

Problems

29. (a) We separate variables and integrate:

$$\frac{dy}{dx} = \frac{4x}{y^2}$$
$$y^2\,dy = 4x\,dx$$
$$\int y^2\,dy = \int 4x\,dx$$
$$\frac{y^3}{3} = 2x^2 + C$$
$$y = \sqrt[3]{6x^2 + B}.$$

Here, we use B as the arbitrary constant, replacing $3C$ when we multiply through by 3.

(b) When we substitute $y = 1$ when $x = 0$, we have:

$$1 = \sqrt[3]{6(0^2) + B}$$
$$1 = \sqrt[3]{B}$$
$$B = 1^3 = 1.$$

The particular solution satisfying $y(0) = 1$ is $y = \sqrt[3]{6x^2 + 1}$.

When we substitute $y = 2$ when $x = 0$, we have:

$$2 = \sqrt[3]{6(0^2) + B}$$
$$2 = \sqrt[3]{B}$$
$$B = 2^3 = 8.$$

The particular solution satisfying $y(0) = 2$ is $y = \sqrt[3]{6x^2 + 8}$.

When we substitute $y = 3$ when $x = 0$, we have:

$$3 = \sqrt[3]{6(0^2) + B}$$
$$3 = \sqrt[3]{B}$$
$$B = 3^3 = 27.$$

The particular solution satisfying $y(0) = 3$ is $y = \sqrt[3]{6x^2 + 27}$.

The three solutions are shown in Figure 11.24.

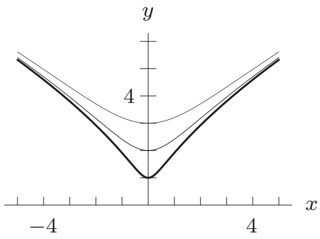


Figure 11.24

30. (a) We separate variables and integrate:

$$\begin{aligned}
\frac{dP}{dt} &= 0.2(P-50)\\
\frac{1}{P-50}\,dP &= 0.2\,dt\\
\int \frac{1}{P-50}\,dP &= \int 0.2\,dt\\
\ln|P-50| &= 0.2t + C\\
|P-50| &= e^{0.2t+C}\\
P-50 &= Be^{0.2t}\\
P &= 50 + Be^{0.2t}.
\end{aligned}$$

Here, we use B as the arbitrary constant,replacing $\pm e^C$.

(b) When we substitute $P = 40$ when $t = 0$, we have:

$$\begin{aligned}
40 &= 50 + Be^{0.2(0)}\\
40 &= 50 + B\\
B &= -10.
\end{aligned}$$

The particular solution satisfying $P(0) = 40$ is $P = 50 - 10e^{0.2t}$.

When we substitute $P = 50$ when $t = 0$, we have:

$$\begin{aligned}
50 &= 50 + Be^{0.2(0)}\\
50 &= 50 + B\\
B &= 0.
\end{aligned}$$

The particular solution satisfying $P(0) = 50$ is the constant solution $P = 50$.

When we substitute $P = 60$ when $t = 0$, we have:

$$\begin{aligned}
60 &= 50 + Be^{0.2(0)}\\
60 &= 50 + B\\
B &= 10.
\end{aligned}$$

The particular solution satisfying $P(0) = 60$ is $P = 50 + 10e^{0.2t}$.

The three solutions are shown in Figure 11.25.

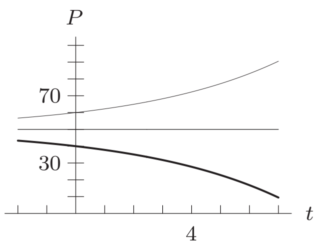


Figure 11.25

31. (a) Separating variables and integrating gives

$$\int \frac{1}{100-y}dy = \int dt$$

so that

$$-\ln|100-y| = t + C$$

or

$$y(t) = 100 - Ae^{-t}.$$

(b) See Figure 11.26.

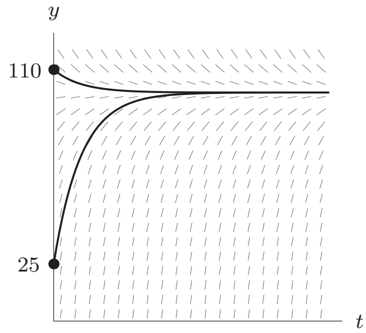


Figure 11.26

(c) The initial condition $y(0) = 25$ gives $A = 75$, so the solution is

$$y(t) = 100 - 75e^{-t}.$$

The initial condition $y(0) = 110$ gives $A = -10$ so the solution is

$$y(t) = 100 + 10e^{-t}.$$

(d) The increasing function, $y(t) = 100 - 75e^{-t}$.

32. By separating variables, we obtain $r\,dr = k\,dt$. Integrating yields

$$\frac{r^2}{2} = kt + C,$$

where C is a constant. If $t = 0$ is the time when the spill begins, then $r = 0$ when $t = 0$; therefore, we must have $C = 0$. Therefore, we have

$$r = \sqrt{2kt}.$$

Using the fact that $r = 400$ when $t = 16$, we obtain $400 = \sqrt{2k \cdot 16}$; this yields $k = 5000$. Since the units of dr/dt are feet/hour and r is measured in feet, the units of k must be feet^2/hour.

33. (a) See Figure 11.27.

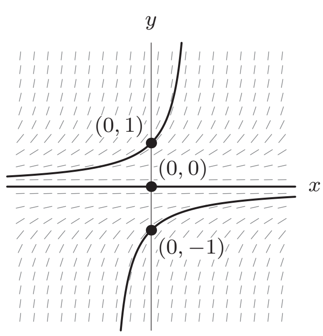


Figure 11.27

(b) It appears that the solution curves in the upper half plane are asymptotic to $y = 0$ as x tends to $-\infty$, and that they are unbounded above as x increases.

It appears that the solution curves in the lower half plane are asymptotic to $y = 0$ as x tends to $+\infty$, and that they are unbounded below as x decreases.

The solution curve through the origin is the x-axis, so it is asymptotic to $y = 0$ on both ends.

(c) We separate variables. From $dy/dx = y^2$ we get, for $y \neq 0$,

$$\begin{aligned} \int \frac{1}{y^2} dy &= \int dx \\ -\frac{1}{y} &= x + C \\ y &= \frac{-1}{x + C}. \end{aligned}$$

(d) For a given number C, the y-value is not defined for $x = -C$, so the formula gives two solution curves. The first is

$$y = \frac{-1}{x + C} \quad \text{for } x < -C.$$

This curve is in the upper half plane, has a vertical asymptote at $x = -C$ and satisfies $\lim_{x \to -\infty} y = 0$. The second curve is

$$y = \frac{-1}{x + C} \quad \text{for } x > -C,$$

which is in the lower half plane, has a vertical asymptote at $x = -C$, and satisfies $\lim_{x \to +\infty} y = 0$.

34. Separating variables gives

$$\int \frac{dR}{R} = \int k\,dt.$$

Integrating gives

$$\ln |R| = kt + C,$$

so

$$\begin{aligned} |R| &= e^{kt+C} = e^{kt} e^C \\ R &= Ae^{kt}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0. \end{aligned}$$

35. Separating variables gives

$$\begin{aligned} \frac{dQ}{dt} &= \frac{Q}{k} \\ \int \frac{dQ}{Q} &= \int \frac{1}{k}\,dt \end{aligned}$$

Integrating gives

$$\begin{aligned} \ln |Q| &= \frac{t}{k} + C \\ Q &= Ae^{t/k}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0. \end{aligned}$$

36. Separating variables gives

$$\int \frac{dP}{P - a} = \int dt.$$

Integrating yields

$$\ln |P - a| = t + C,$$

so

$$\begin{aligned} |P - a| &= e^{t+C} = e^t e^C \\ P &= a + Ae^t, \quad \text{where } A = \pm e^C \quad \text{or } A = 0. \end{aligned}$$

37. Separating variables gives

$$\int \frac{dQ}{b-Q} = \int dt.$$

Integrating yields

$$-\ln|b-Q| = t + C,$$

so

$$\begin{aligned} |b-Q| &= e^{-(t+C)} = e^{-t}e^{-C} \\ Q &= b - Ae^{-t}, \quad \text{where } A = \pm e^{-C} \quad \text{or } A = 0. \end{aligned}$$

38. Separating variables gives

$$\int \frac{dP}{P-a} = \int k\,dt.$$

Integrating yields

$$\ln|P-a| = kt + C,$$

so

$$P = a + Ae^{kt} \quad \text{where } A = \pm e^{C} \quad \text{or } A = 0.$$

39. Factoring and separating variables gives

$$\begin{aligned} \frac{dR}{dt} &= a\left(R + \frac{b}{a}\right) \\ \int \frac{dR}{R + b/a} &= \int a\,dt \\ \ln\left|R + \frac{b}{a}\right| &= at + C \\ R &= -\frac{b}{a} + Ae^{at}, \quad \text{where } A \text{ can be any constant.} \end{aligned}$$

40. Separating variables and integrating gives

$$\int \frac{1}{aP+b}dP = \int dt.$$

This gives

$$\begin{aligned} \frac{1}{a}\ln|aP+b| &= t + C \\ \ln|aP+b| &= at + aC \\ aP + b &= \pm e^{at+aC} = Ae^{at}, \quad \text{where } A = \pm e^{aC} \quad \text{or} \quad A = 0, \end{aligned}$$

or

$$P = \frac{1}{a}(Ae^{at} - b).$$

41. Separating variables and integrating gives

$$\int \frac{1}{y^2}dy = \int k(1+t^2)dt$$

or

$$-\frac{1}{y} = k\left(t + \frac{1}{3}t^3\right) + C.$$

Hence,

$$y = \frac{-1}{k(t + \frac{1}{3}t^3) + C}.$$

42. Separating variables and integrating gives

$$\int \frac{1}{R^2+1}dR = \int a dx$$

or

$$\arctan R = ax + C$$

so that

$$R = \tan(ax + C).$$

43. Separating variables and integrating gives

$$\int \frac{1}{L-b}dL = \int k(x+a)dx$$

or

$$\ln|L-b| = k\left(\frac{1}{2}x^2 + ax\right) + C.$$

Solving for L gives

$$L = b + Ae^{k(\frac{1}{2}x^2+ax)}, \quad \text{where } A \text{ can be any constant.}$$

44. Separating variables gives

$$\frac{dy}{dt} = y(2-y),$$

so

$$\int \frac{dy}{y(y-2)} = -\int dt,$$

so

$$-\frac{1}{2}\int\left(\frac{1}{y} - \frac{1}{y-2}\right)dy = -\int dt.$$

Integrating yields

$$\frac{1}{2}(\ln|y-2| - \ln|y|) = -t + C,$$

so

$$\ln\frac{|y-2|}{|y|} = -2t + 2C.$$

Exponentiating both sides yields

$$\left|1 - \frac{2}{y}\right| = e^{-2t+2C}$$

$$\frac{2}{y} = 1 - Ae^{-2t}, \quad \text{where } A = \pm e^{2C}$$

$$y = \frac{2}{1-Ae^{-2t}}.$$

But

$$y(0) = \frac{2}{1-A} = 1,$$

so $A = -1$, and

$$y = \frac{2}{1+e^{-2t}}.$$

45. Separating variables gives

$$\frac{dx}{dt} = \frac{x\ln x}{t},$$

so

$$\int \frac{dx}{x\ln x} = \int \frac{dt}{t},$$

and thus

$$\ln|\ln x| = \ln t + C,$$

so

$$|\ln x| = e^C e^{\ln t} = e^C t.$$

Therefore

$$\ln x = At, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0, \quad \text{so} \quad x = e^{At}.$$

46. Separating variables gives

$$\begin{aligned} t\frac{dx}{dt} &= (1 + 2\ln t)\tan x \\ \frac{dx}{\tan x} &= \left(\frac{1 + 2\ln t}{t}\right) dt \\ \int \frac{\cos x}{\sin x}\, dx &= \int \left(\frac{1}{t} + \frac{2\ln t}{t}\right) dt. \end{aligned}$$

Integrating gives

$$\begin{aligned} \ln|\sin x| &= \ln t + (\ln t)^2 + C \\ |\sin x| &= e^{\ln t + (\ln t)^2 + C} = t(e^{\ln t})^{\ln t} e^C = t(t^{\ln t})e^C. \end{aligned}$$

So

$$\sin x = At^{(\ln t)+1}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0.$$

Therefore

$$x = \arcsin(At^{(\ln t)+1}).$$

47. Since

$$\frac{dy}{dt} = -y\ln\left(\frac{y}{2}\right),$$

we have

$$\frac{dy}{y\ln(y/2)} = -\,dt,$$

so that

$$\int \frac{dy}{y\ln(y/2)} = \int (-\,dt).$$

Substituting $w = \ln(y/2),\ dw = \dfrac{1}{y}\,dy$ gives:

$$\int \frac{dw}{w} = -\int dt$$

so

$$\begin{aligned} \ln|w| &= -t + C \\ \ln\left|\ln\left(\frac{y}{2}\right)\right| &= -t + C. \end{aligned}$$

Since $y(0) = 1$, we have $C = \ln|\ln\frac{1}{2}| = \ln(|-\ln 2|) = \ln(\ln 2)$. Thus

$$\ln\left|\ln\left(\frac{y}{2}\right)\right| = -t + \ln(\ln 2),$$

or

$$\left|\ln\left(\frac{y}{2}\right)\right| = e^{-t+\ln(\ln 2)}$$

Since $e^{\ln(\ln 2)} = \ln 2$, this simplifies to

$$\left|\ln\left(\frac{y}{2}\right)\right| = (\ln 2)e^{-t},$$

so

$$\ln\left(\frac{y}{2}\right) = \pm(\ln 2)e^{-t}.$$

Since $y(0) = 1$, and $\ln(1/2) = -\ln 2$, we take the $-$ sign, giving

$$\ln\left(\frac{y}{2}\right) = -(\ln 2)e^{-t}.$$

Thus,

$$y = 2e^{-(\ln 2)e^{-t}}$$
$$y = 2(e^{-\ln 2})^{e^{-t}} = 2(2^{-1})^{e^{-t}}$$
$$y = 2(2^{-e^{-t}}).$$

(Note that $\ln(y/2) = (\ln 2)e^{-t}$ does not satisfy $y(0) = 1$.)

48. (a) See Figure 11.28.

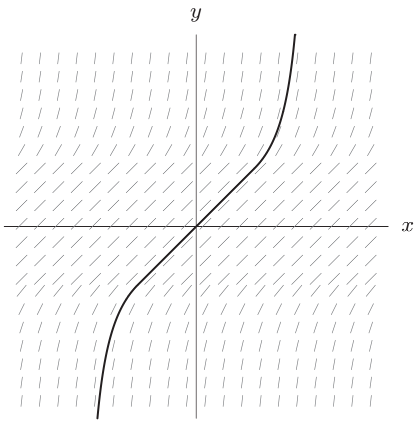


Figure 11.28

(b) It appears that the solution curves are unbounded below as x decreases and unbounded above as x increases.

(c) In the region $-1 \le y \le 1$, we have $dy/dx = 1$, a constant slope, so the solution curves in this region are straight lines of slope 1: the equations are $y = x + C$. This formula holds for $-C - 1 \le x \le -C + 1$.

In the regions where $|y| \ge 1$, we solve the differential equation by separation of variables,

$$\int \frac{1}{y^2} dy = \int dx$$
$$-\frac{1}{y} = x + C$$
$$y = \frac{-1}{x+C}.$$

Thus, the formula for the general solution is $y = -1/(x + C)$ for both $y \le -1$ and $y \ge 1$. The corresponding x-values are determined as follows. We have $y \le -1$ if

$$\frac{-1}{x+C} \le -1$$
$$\frac{1}{x+C} \ge 1$$
$$0 < x + C \le 1$$
$$-C < x \le -C + 1.$$

We have $y \ge 1$ if

$$\frac{-1}{x+C} \ge 1$$
$$\frac{1}{x+C} \le -1$$
$$-1 \le x + C < 0$$
$$-C - 1 \le x < -C.$$

Summarizing, with different constants for the three regions to avoid confusion, we have the following explicit solutions:

If $y \le -1$, then $y = -1/(x + C_1)$ for $-C_1 < x \le -C_1 + 1$.
If $-1 \le y \le 1$, then $y = x + C_2$ for $-C_2 - 1 \le x \le -C_2 + 1$.
If $1 \le y$, then $y = -1/(x + C_3)$ for $-C_3 - 1 \le x < -C_3$.

(d) If a solution curve touches the region $y \le -1$, then the formula $y = -1/(x + C_1)$ found in part (c) shows that $x = -C_1$ is a vertical asymptote. Moreover, the curve touches the line $y = -1$ at $x = 1 - C_1$. The curve continues, crossing the region $-1 \le y \le 1$ as a straight line of slope 1, until it reaches the horizontal line $y = 1$. After that the solution curve enters the region $y \ge 1$ where there is a vertical asymptote at $x = -C_3$.

A similar argument applies if a solution curves touches the region $y \ge 1$.

Finally, if a solution curve touches the region $-1 \le y \le 1$ then it has a straight line section that runs to the two horizontal lines $y = -1$ and $y = 1$ where the solution curve enters the two outer regions where $|y| \ge 1$ and approaches vertical asymptotes.

(e) A complete solution curve has the formula

$$y = \begin{cases} -1/(x + C_1) & \text{if } -C_1 < x \le -C_1 + 1 \\ x + C_2 & \text{if } -C_2 - 1 \le x \le -C_2 + 1 \\ -1/(x + C_3) & \text{if } -C_3 - 1 \le x < -C_3 \end{cases}$$

where the conditions that the first two formulas match up where $y = -1$ and the last two match up where $y = 1$ are given by

$$-C_1 + 1 = -C_2 - 1$$
$$-C_2 + 1 = -C_3 - 1.$$

Thus $C_2 = C_1 - 2$ and $C_3 = C_2 - 2 = (C_1 - 2) - 2 = C_1 - 4$.

The two vertical asymptotes are thus

$$x = -C_1$$
$$x = -C_3 = -C_1 + 4.$$

The second asymptote is 4 units to the right of the first.

49. (a), (b) See Figure 11.29.

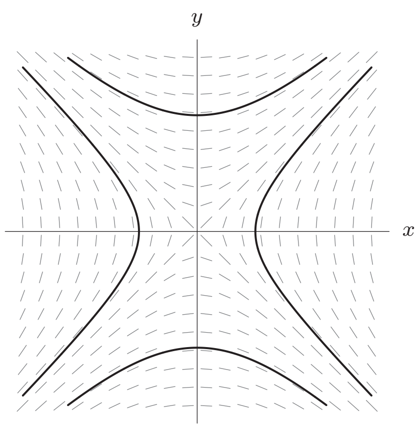


Figure 11.29

(c) Since $dy/dx = x/y$, we have

$$\int y\,dy = \int x\,dx,$$

and thus

$$\frac{y^2}{2} = \frac{x^2}{2} + C,$$

or

$$y^2 - x^2 = 2C.$$

This is the equation of the hyperbolas in part (b).

50. **(a), (b)** See Figure 11.30.

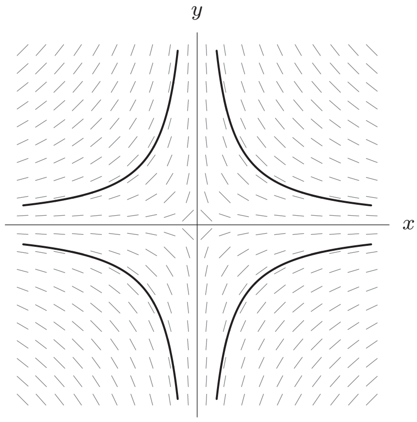


Figure 11.30

(c) Since

$$\frac{dy}{dx} = -\frac{y}{x},$$

we have

$$\int \frac{dy}{y} = -\int \frac{dx}{x},$$

so

$$\ln|y| = -\ln|x| + C,$$

giving

$$|y| = e^{-\ln|x|+C} = (|x|)^{-1}e^C.$$

Thus,

$$y = \frac{A}{x}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0.$$

51. By looking at the slope fields, we see that any solution curve of $y' = x/y$ intersects any solution curve to $y' = -y/x$. Now if the two curves intersect at (x, y), then the two slopes at (x, y) are negative reciprocals of each other, because

$$-\frac{1}{x/y} = -\frac{y}{x}.$$

Hence, the two curves intersect at right angles.

Strengthen Your Understanding

52. It is impossible to separate variables in the differential equation $dy/dx = x + y$. If we subtract y from both sides, we obtain $dy/dx - y = x$. If we then try to separate the dx, we have $dy - y\,dx = x\,dx$. The variables cannot be separated in this differential equation.

53. The solution to $dP/dt = 0.2t$ is $P = 0.1t^2 + C$.

Exponential growth occurs when the derivative is a constant multiple of the dependent variable, not the independent variable.

54. We have $dy/dx = e^{x+y} = e^x e^y$. Dividing both sides by e^y, we have $e^{-y}dy/dx = e^x$. Then multiplying both sides by dx, we have $e^{-y}\,dy = e^x\,dx$. Since e^{-y} is certainly not equal to $-e^y$, the statement is false.

55. There are many differential equations which are not separable. One possible example is $dy/dx = x + y$. Other examples are possible.

56. An expression such as $f(x) = \cos x$ satisfies the requirement since $dy/dx = \cos x + xy - \cos x = xy$ is a separable differential equation. Other examples are possible.

57. This family has $f'(x) = 2x$, so let $dy/dx = 2x$.

58. If we differentiate implicitly the equation for the family, we get $2x - 2y dy/dx = 0$. Solving for dy/dx, we get the differential equation we want: $dy/dx = x/y$.

59. True. The general solution to $y' = -ky$ is $y = Ce^{-kt}$.

60. False. In order to be solved using separation of variables, a differential equation must have the form $dy/dx = f(x)g(y)$, so we would need $x + y = f(x)g(y)$. This certainly does not appear to be true. If it were, setting $x = 0$ and $y = 0$, we would have $f(0)g(0) = 0$ so either $f(0) = 0$ or $g(0) = 0$. If $f(0) = 0$, then substituting in $x = 0$ and $y = 1$, we have $0 + 1 = f(0)g(1) = 0$, which is absurd. We get the same contradiction if we assume $g(0) = 0$.

61. True. Rewrite the equation as $dy/dx = xy + x = x(y + 1)$. Since the equation now has the form $dy/dx = f(x)g(y)$, it can be solved by separation of variables.

62. False. It is true that $y = x^3$ is a solution of the differential equation, since $dy/dx = 3x^2 = 3y^{2/3}$, but it is not the only solution passing through $(0, 0)$. Another solution is the constant function $y = 0$. Usually there is only one solution curve to a differential equation passing through a given point, but not always.

Solutions for Section 11.5

Exercises

1. (a) = (III), (b) = (IV), (c) = (I), (d) = (II).

2. (a) (I)
(b) (IV)
(c) (II) and (IV)
(d) (II) and (III)

3. (a) The equilibrium solutions occur where the slope $y' = 0$, which occurs on the slope field where the lines are horizontal, or (looking at the equation) at $y = 2$ and $y = -1$. Looking at the slope field, we can see that $y = 2$ is stable, since the slopes at nearby values of y point toward it, whereas $y = -1$ is unstable.
(b) Draw solution curves passing through the given points by starting at these points and following the flow of the slopes, as shown in Figure 11.31.

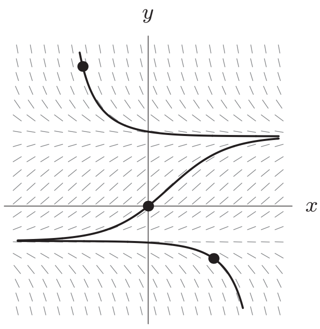


Figure 11.31

4. The equilibrium solutions of a differential equation are those functions satisfying the differential equation whose derivative is everywhere 0. Graphically, this means that a function is an equilibrium solution if it is a horizontal line that lies on the slope field. Looking at the figure in the problem, it appears that the equilibrium solutions for this problem are at $y = 1$ and $y = 3$. An equilibrium solution is stable if a small change in the initial value conditions gives a solution which tends toward equilibrium as $t \to \infty$. We see that $y = 3$ is a stable solution, while $y = 1$ is an unstable solution. See Figure 11.32.

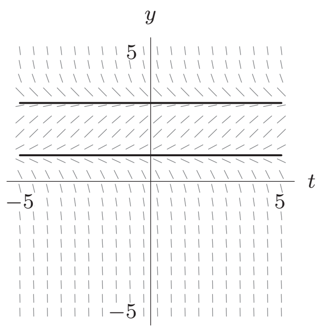


Figure 11.32

5. For $y = \alpha$, we have $dy/dt = 0$, so the solution curve is horizontal. This is an equilibrium solution.
For $y > \alpha$, we have $dy/dt < 0$, so the function is decreasing.
For $y < \alpha$, we have $dy/dt > 0$, so the function is increasing. See Figure 11.33.

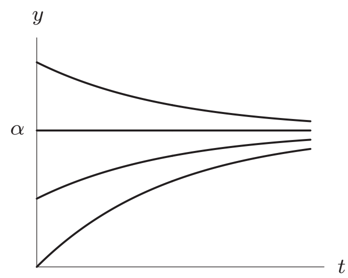


Figure 11.33

6. When the derivative $dw/dt = 0$, the function is constant. This occurs when $w = 3$ or $w = 7$. So, $w = 3$ and $w = 7$ are solutions. Their graphs are horizontal lines. When $3 < w < 7$, the derivative is negative. Therefore, solutions, w, are decreasing in this region. For $w < 3$ or $w > 7$, the derivative is positive. Thus, solutions, w, are increasing in these regions. See Figure 11.34.

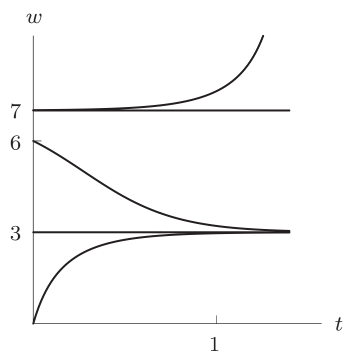


Figure 11.34

7. (a) Separating variables, we have $\frac{dH}{H-200} = -k\,dt$, so $\int \frac{dH}{H-200} = \int -k\,dt$, whence $\ln|H-200| = -kt + C$, and $H - 200 = Ae^{-kt}$, where $A = \pm e^C$. The initial condition is that the yam is $20°C$ at the time $t = 0$. Thus $20 - 200 = A$, so $A = -180$. Thus $H = 200 - 180e^{-kt}$.

(b) Using part (a), we have $120 = 200 - 180e^{-k(30)}$. Solving for k, we have $e^{-30k} = \frac{-80}{-180}$, giving

$$k = \frac{\ln\frac{4}{9}}{-30} \approx 0.027.$$

Note that this k is correct if t is given in *minutes*. (If t is given in hours, $k = \frac{\ln\frac{4}{9}}{-\frac{1}{2}} \approx 1.62$.)

8. (a) We know that the equilibrium solution is the solution satisfying the differential equation whose derivative is everywhere 0. Thus we have

$$\begin{aligned}\frac{dy}{dt} &= 0\\ 0.5y - 250 &= 0\\ y &= 500.\end{aligned}$$

(b) We use separation of variables. Since

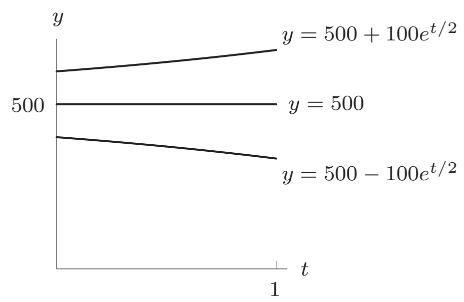
$$\frac{dy}{dt} = 0.5y - 250,$$

we have

$$\begin{aligned}\int \frac{1}{0.5y-250}dy &= \int dt\\ 2\ln|0.5y-250| &= t + C\\ 0.5y - 250 &= e^{(t+C)/2}\\ y &= Ae^{t/2} + 500,\end{aligned}$$

where $A = 2e^{C/2}$.

(c) Using initial value $y(0) = 500$, we have $y = 500$, the equilibrium solution. Using initial value $y(0) = 400$, we have $A = -100$ and so $y = 500 - 100e^{t/2}$. Using initial value $y(0) = 600$, we have $A = 100$ and so $y = 500 + 100e^{t/2}$. These three solutions are shown below.



(d) We see above that the equilibrium solution $y = 500$ is unstable.

9. (a) To find the equilibrium solutions, we must set

$$dy/dx = 0.5y(y-4)(2+y) = 0$$

which gives three solutions: $y = 0$, $y = 4$, and $y = -2$.

(b) From Figure 11.35, we see that $y = 0$ is stable and $y = 4$ and $y = -2$ are both unstable.

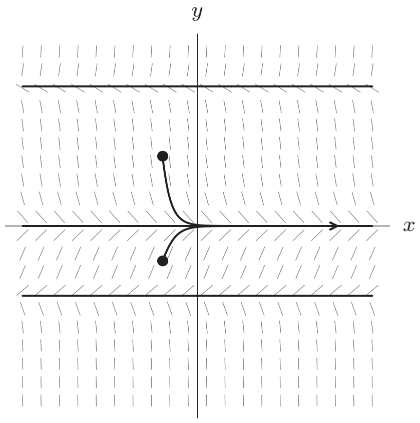


Figure 11.35

10. (a) A very hot cup of coffee cools faster than one near room temperature. The differential equation given says that the rate at which the coffee cools is proportional to the difference between the temperature of the surrounding air and the temperature of the coffee. Since $dH/dt < 0$ (the coffee is cooling) and $H - 20 > 0$ (the coffee is warmer than room temperature), k must be positive.

(b) Separating variables gives

$$\int \frac{1}{H-20}dH = \int -kdt$$

and so

$$\ln|H-20| = -kt + C$$

and

$$H(t) = 20 + Ae^{-kt}.$$

If the coffee is initially boiling (100°C), then $A = 80$ and so

$$H(t) = 20 + 80e^{-kt}.$$

When $t = 2$, the coffee is at $90°C$ and so $90 = 20 + 80e^{-2k}$ so that $k = \frac{1}{2}\ln\frac{8}{7}$.
Let the time when the coffee reaches $60°C$ be H_d, so that

$$60 = 20 + 80e^{-kH_d}$$

$$e^{-kH_d} = \frac{1}{2}.$$

Therefore,

$$H_d = \frac{1}{k}\ln 2 = \frac{2\ln 2}{\ln\frac{8}{7}} \approx 10\text{minutes}.$$

11. Since it takes 6 years to reduce the pollution to 10%, another 6 years would reduce the pollution to 10% of 10%, which is equivalent to 1% of the original. Therefore it takes 12 years for 99% of the pollution to be removed. (Note that the value of Q_0 does not affect this.) Thus the second time is double the first because the fraction remaining, 0.01, in the second instance is the square of the fraction remaining, 0.1, in the first instance.

12. Michigan:

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{158}{4.9\times 10^3}Q \approx -0.032Q$$

so

$$Q = Q_0e^{-0.032t}.$$

We want to find t such that

$$0.1Q_0 = Q_0e^{-0.032t}$$

so

$$t = \frac{-\ln(0.1)}{0.032} \approx 72 \text{ years}.$$

Ontario:

$$\frac{dQ}{dt} = -\frac{r}{V}Q = \frac{-209}{1.6 \times 10^3}Q = -0.131Q$$

so

$$Q = Q_0 e^{-0.131t}.$$

We want to find t such that

$$0.1Q_0 = Q_0 e^{-0.131t}$$

so

$$t = \frac{-\ln(0.1)}{0.131} \approx 18 \text{ years.}$$

Lake Michigan will take longer because it is larger (4900 km^3 compared to 1600 km^3) and water is flowing through it at a slower rate (158 km^3/year compared to 209 km^3/year).

13. Lake Superior will take the longest, because the lake is largest (V is largest) and water is moving through it most slowly (r is smallest). Lake Erie looks as though it will take the least time because V is smallest and r is close to the largest. For Erie, $k = r/V = 175/460 \approx 0.38$. The lake with the largest value of r is Ontario, where $k = r/V = 209/1600 \approx 0.13$. Since e^{-kt} decreases faster for larger k, Lake Erie will take the shortest time for any fixed fraction of the pollution to be removed.

For Lake Superior,

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{65.2}{12{,}200}Q \approx -0.0053Q$$

so

$$Q = Q_0 e^{-0.0053t}.$$

When 80% of the pollution has been removed, 20% remains so $Q = 0.2Q_0$. Substituting gives us

$$0.2Q_0 = Q_0 e^{-0.0053t}$$

so

$$t = -\frac{\ln(0.2)}{0.0053} \approx 301 \text{ years.}$$

(Note: The 301 is obtained by using the exact value of $\frac{r}{V} = \frac{65.2}{12{,}200}$, rather than 0.0053. Using 0.0053 gives 304 years.)
For Lake Erie, as in the text

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{175}{460}Q \approx -0.38Q$$

so

$$Q = Q_0 e^{-0.38t}.$$

When 80% of the pollution has been removed

$$0.2Q_0 = Q_0 e^{-0.38t}$$
$$t = -\frac{\ln(0.2)}{0.38} \approx 4 \text{ years.}$$

So the ratio is

$$\frac{\text{Time for Lake Superior}}{\text{Time for Lake Erie}} \approx \frac{301}{4} \approx 75.$$

In other words it will take about 75 times as long to clean Lake Superior as Lake Erie.

Problems

14. (a) We define P to be the population of India, in billions of people, in year t, where t represents the number of years since 2010.
(b) We have $dP/dt = 0.0135P$ with initial condition $P(0) = 1.15$.
(c) The general solution is $P = Ce^{0.0135t}$ and the particular solution satisfying the initial condition is $P = 1.15e^{0.0135t}$.

15. (a) We define N to be the amount of nicotine in the body, in mg, at time t, where t represents the number of hours since smoking the cigarette.
(b) We have $dN/dt = -0.347N$ with initial condition $N(0) = 0.4$. Notice that the constant -0.347 is negative since the quantity of nicotine is decreasing.
(c) The general solution is $N = Ce^{-0.347t}$ and the particular solution satisfying the initial condition is $N = 0.4e^{-0.347t}$.

16. (a) We define S to be world solar PV market installations, in megawatts, in year t, where t represents the number of years since 2007.
(b) We have $dS/dt = 0.48S$ with initial condition $S(0) = 2826$.
(c) The general solution is $S = Ce^{0.48t}$ and the particular solution satisfying the initial condition is $S = 2826e^{0.48t}$.

17. (a) We define G to be the size, in acres, of Grinnell Glacier in year t, where t represents the number of years since 2007.
(b) We have $dG/dt = -0.043G$ with initial condition $G(0) = 142$. Notice that the constant -0.043 is negative because the size is decreasing.
(c) The general solution is $G = Ce^{-0.043t}$ and the particular solution satisfying the initial condition is $G = 142e^{-0.043t}$.

18. (a) The differential equation is

$$\frac{1}{B}\frac{dB}{dt} = 0.067 \quad \text{or} \quad \frac{dB}{dt} = 0.067B.$$

(b) The differential equation is

$$\frac{1}{P}\frac{dP}{dt} = 0.033 \quad \text{or} \quad \frac{dP}{dt} = 0.033P.$$

(c) For initial prices B_0 and P_0, solving the differential equations in part (a) and part (b) gives

$$B = B_0e^{0.067t},$$

and

$$P = P_0e^{0.033t}.$$

(d) Doubling time for textbook price is given by

$$\begin{aligned} 2B_0 &= B_0e^{0.067t} \\ t &= \frac{\ln 2}{0.067} = 10.345 \text{ years.} \end{aligned}$$

(e) Doubling time for inflation is given by

$$\begin{aligned} 2P_0 &= P_0e^{0.033t} \\ t &= \frac{\ln 2}{0.033} = 21.004 \text{ years.} \end{aligned}$$

(f) The doubling times are in the ratio

$$\frac{\text{Doubling time:Textbook}}{\text{Doubling time:Inflation}} = \frac{10.345}{21.004} = \frac{(\ln 2)/0.067}{(\ln 2)/0.033} = \frac{0.033}{0.067} = \frac{\text{Inflation growth rate}}{\text{Textbook growth rate}}.$$

The ratio of the doubling times is the reciprocal of the ratio of the growth rates.

19. We find the temperature of the orange juice as a function of time. Newton's Law of Heating says that the rate of change of the temperature is proportional to the temperature difference. If S is the temperature of the juice, this gives us the equation

$$\frac{dS}{dt} = -k(S - 65) \qquad \text{for some constant } k.$$

Notice that the temperature of the juice is increasing, so the quantity dS/dt is positive. In addition, $S = 40$ initially, making the quantity $(S - 65)$ negative.
Separating variables gives:

$$\begin{aligned} \int \frac{dS}{S-65} &= -\int k\,dt \\ \ln|S - 65| &= -kt + C \\ S - 65 &= Ae^{-kt}, \text{ where } A = \pm e^C. \end{aligned}$$

So

$$S = 65 + Ae^{-kt}.$$

Since at $t = 0$, $S = 40$, we have $40 = 65 + C$, so $C = -25$. Thus, $S = 65 - 25e^{-kt}$ for some positive constant k. See Figure 11.36 for the graph.

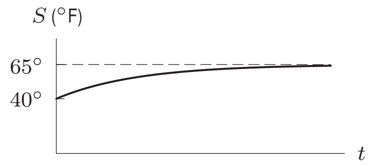


Figure 11.36: Graph of
$S = 65 - 25e^{-kt}$ for $k > 0$

20. According to Newton's Law of Cooling, the temperature, T, of the roast as a function of time, t, satisfies

$$T'(t) = k(350 - T)$$
$$T(0) = 40.$$

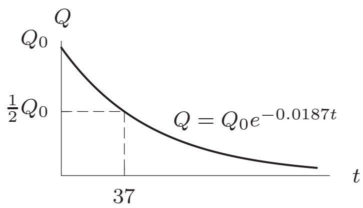
Solving this differential equation, we get that $T = 350 - 310e^{-kt}$ for some $k > 0$. To find k, we note that at $t = 1$ we have $T = 90$, so

$$\begin{aligned} 90 &= 350 - 310e^{-k(1)} \\ \frac{260}{310} &= e^{-k} \\ k &= -\ln\left(\frac{260}{310}\right) \\ &\approx 0.17589. \end{aligned}$$

Thus, $T = 350 - 310e^{-0.17589t}$. Solving for t when $T = 140$, we have

$$\begin{aligned} 140 &= 350 - 310e^{-0.17589t} \\ \frac{210}{310} &= e^{-0.17589t} \\ t &= \frac{\ln(210/310)}{-0.17589} \\ t &\approx 2.21 \text{ hours.} \end{aligned}$$

21. (a)



(b) $\dfrac{dQ}{dt} = -kQ$

(c) Since $25\% = 1/4$, it takes two half-lives $= 74$ hours for the drug level to be reduced to 25%. Alternatively, $Q = Q_0e^{-kt}$ and $\frac{1}{2} = e^{-k(37)}$, we have

$$k = -\frac{\ln(1/2)}{37} \approx 0.0187.$$

Therefore $Q = Q_0e^{-0.0187t}$. We know that when the drug level is 25% of the original level that $Q = 0.25Q_0$. Setting these equal, we get

$$0.25 = e^{-0.0187t}.$$

giving

$$t = -\frac{\ln(0.25)}{0.0187} \approx 74 \text{ hours} \approx 3 \text{ days.}$$

22. (a) Since the amount leaving the blood is proportional to the quantity in the blood,

$$\frac{dQ}{dt} = -kQ \quad \text{for some positive constant } k.$$

Thus $Q = Q_0 e^{-kt}$, where Q_0 is the initial quantity in the bloodstream. Only 20% is left in the blood after 3 hours. Thus $0.20 = e^{-3k}$, so $k = \frac{\ln 0.20}{-3} \approx 0.5365$. Therefore $Q = Q_0 e^{-0.5365t}$.

(b) Since 20% is left after 3 hours, after 6 hours only 20% of that 20% will be left. Thus after 6 hours only 4% will be left, so if the patient is given 100 mg, only 4 mg will be left 6 hours later.

23. (a) Suppose $Y(t)$ is the quantity of oil in the well at time t. We know that the oil in the well decreases at a rate proportional to $Y(t)$, so

$$\frac{dY}{dt} = -kY.$$

Integrating, and using the fact that initially $Y = Y_0 = 10^6$, we have

$$Y = Y_0 e^{-kt} = 10^6 e^{-kt}.$$

In six years, $Y = 500,000 = 5 \cdot 10^5$, so

$$5 \cdot 10^5 = 10^6 e^{-k \cdot 6}$$

so

$$0.5 = e^{-6k}$$
$$k = -\frac{\ln 0.5}{6} = 0.1155.$$

When $Y = 600,000 = 6 \cdot 10^5$,

$$\text{Rate at which oil decreasing } = \left|\frac{dY}{dt}\right| = kY = 0.1155(6 \cdot 10^5) = 69{,}300 \text{ barrels/year.}$$

(b) We solve the equation

$$5 \cdot 10^4 = 10^6 e^{-0.1155t}$$
$$0.05 = e^{-0.1155t}$$
$$t = \frac{\ln 0.05}{-0.1155} = 25.9 \text{ years.}$$

24. (a) If $C' = -kC$, and then $C = C_0 e^{-kt}$. Since the half-life is 5730 years, $\frac{1}{2}C_0 = C_0 e^{-5730k}$. Solving for k, we have $-5730k = \ln(1/2)$ so $k = \frac{-\ln(1/2)}{5730} \approx 0.000121$.

(b) From the given information, we have $0.91 = e^{-kt}$, where t is the age of the shroud. Solving for t, we have $t = \frac{-\ln 0.91}{k} \approx 779.4$ years.

25. The rate of disintegration is proportional to the quantity of carbon-14 present. Let Q be the quantity of carbon-14 present at time t, with $t = 0$ in 1977. Then

$$Q = Q_0 e^{-kt},$$

where Q_0 is the quantity of carbon-14 present in 1977 when $t = 0$. Then we know that

$$\frac{Q_0}{2} = Q_0 e^{-k(5730)}$$

so that

$$k = -\frac{\ln(1/2)}{5730} = 0.000121.$$

Thus

$$Q = Q_0 e^{-0.000121t}.$$

The quantity present at any time is proportional to the rate of disintegration at that time so

$$Q_0 = c8.2 \qquad \text{and} \qquad Q = c13.5$$

where c is a constant of proportionality. Thus substituting for Q and Q_0 in

$$Q = Q_0 e^{-0.000121t}$$

gives

$$c13.5 = c8.2e^{-0.000121t}$$

so

$$t = -\frac{\ln(13.5/8.2)}{0.000121} \approx -4120.$$

Thus Stonehenge was built about 4120 years before 1977, in about 2150 B.C.

26. (a) $\frac{dT}{dt} = -k(T - A)$, where $A = 68°$F is the temperature of the room, and t is time since 9 am.

(b)

$$\int \frac{dT}{T-A} = -\int k dt$$
$$\ln|T - A| = -kt + C$$
$$T = A + Be^{-kt}.$$

Using $A = 68$, and $T(0) = 90.3$, we get $B = 22.3$. Thus

$$T = 68 + 22.3e^{-kt}.$$

At $t = 1$, we have

$$89.0 = 68 + 22.3e^{-k}$$
$$21 = 22.3e^{-k}$$
$$k = -\ln\frac{21}{22.3} \approx 0.06.$$

Thus $T = 68 + 22.3e^{-0.06t}$.

We want to know when T was equal to $98.6°$F, the temperature of a live body, so

$$98.6 = 68 + 22.3e^{-0.06t}$$
$$\ln\frac{30.6}{22.3} = -0.06t$$
$$t = \left(-\frac{1}{0.06}\right)\ln\frac{30.6}{22.3}$$
$$t \approx -5.27.$$

The victim was killed approximately $5\frac{1}{4}$ hours prior to 9 am, at 3:45 am.

27. (a) The differential equation is

$$\frac{dT}{dt} = -k(T - A),$$

where $A = 10°$F is the outside temperature.

(b) Integrating both sides yields

$$\int \frac{dT}{T-A} = -\int k\,dt.$$

Then $\ln|T - A| = -kt + C$, so $T = A + Be^{-kt}$. Thus

$$T = 10 + 58e^{-kt}.$$

Since 10:00 pm corresponds to $t = 9$,

$$57 = 10 + 58e^{-9k}$$
$$\frac{47}{58} = e^{-9k}$$
$$\ln\frac{47}{58} = -9k$$
$$k = -\frac{1}{9}\ln\frac{47}{58} \approx 0.0234.$$

At 7:00 the next morning ($t = 18$) we have

$$\begin{aligned} T &\approx 10 + 58e^{18(-0.0234)} \\ &= 10 + 58(0.66) \\ &\approx 48^\circ\text{F}, \end{aligned}$$

so the pipes won't freeze.

(c) We assumed that the temperature outside the house stayed constant at 10°F. This is probably incorrect because the temperature was most likely warmer during the day (between 1 pm and 10 pm) and colder after (between 10 pm and 7 am). Thus, when the temperature in the house dropped from 68°F to 57°F between 1 pm and 10 pm, the outside temperature was probably higher than 10°F, which changes our calculation of the value of the constant k. The house temperature will most certainly be lower than 48°F at 7 am, but not by much—not enough to freeze.

28. (a) Since speed is the derivative of distance, Galileo's mistaken conjecture was $\frac{dD}{dt} = kD$.

(b) We know that if Galileo's conjecture were true, then $D(t) = D_0 e^{kt}$, where D_0 would be the initial distance fallen. But if we drop an object, it starts out not having traveled any distance, so $D_0 = 0$. This would lead to $D(t) = 0$ for all t.

29. (a) Letting k be the constant of proportionality, by Newton's Law of Cooling, we have

$$\frac{dH}{dt} = k(68 - H).$$

(b) To find the equilibrium solution, we solve $dH/dt = k(68 - H) = 0$. We find that $H = 68$ is an equilibrium solution. This makes sense because the temperature of an object at 68°F in a 68°F room will not change. To determine whether the equilibrium is stable, we consider a solution with an initial condition near $H = 68$. If $H = 69$, then $dH/dt = k(68 - 69) = -k$, which is negative; therefore, H will decrease and move towards the equilibrium. Similarly, a solution with an initial condition less than $H = 68$ will increase towards the equilibrium. Therefore, the equilibrium is stable.

(c) We solve this equation by separating variables:

$$\begin{aligned} \int \frac{dH}{68 - H} &= \int k\, dt \\ -\ln|68 - H| &= kt + C \\ 68 - H &= \pm e^{C - kt} \\ H &= 68 - Ae^{-kt}. \end{aligned}$$

(d) We are told that $H = 40$ when $t = 0$; this tells us that

$$\begin{aligned} 40 &= 68 - Ae^{-k(0)} \\ 40 &= 68 - A \\ A &= 28. \end{aligned}$$

Knowing A, we can solve for k using the fact that $H = 48$ when $t = 1$:

$$\begin{aligned} 48 &= 68 - 28e^{-k(1)} \\ \frac{20}{28} &= e^{-k} \\ k &= -\ln\left(\frac{20}{28}\right) = 0.33647. \end{aligned}$$

So the formula is $H(t) = 68 - 28e^{-0.33647t}$. We calculate H when $t = 3$, by

$$H(3) = 68 - 28e^{-0.33647(3)} = 57.8^\circ\text{F}.$$

30. (a) Since we are told that the rate at which the quantity of the drug decreases is proportional to the amount of the drug left in the body, we know the differential equation modeling this situation is

$$\frac{dQ}{dt} = kQ.$$

Since we are told that the quantity of the drug is decreasing, we know that $k < 0$.

(b) To find the equilibrium, we solve $dQ/dt = 0$ and get $Q = 0$. Thus, $Q = 0$ is an equilibrium. This makes sense because patients with no hydrocodone bitartrate in their body will remain that way unless they take the drug. We expect the equilibrium to be stable because patients who have taken the drug will gradually lose it, moving towards the equilibrium. We can confirm this by checking solutions with initial conditions near $Q = 0$.

(c) We know that the general solution to the differential equation

$$\frac{dQ}{dt} = kQ$$

is

$$Q = Ce^{kt}.$$

(d) We are told that the half life of the drug is 3.8 hours. This means that at $t = 3.8$, the amount of the drug in the body is half the amount that was in the body at $t = 0$, or, in other words,

$$0.5Q(0) = Q(3.8).$$

Solving this equation gives

$$\begin{aligned} 0.5Q(0) &= Q(3.8) \\ 0.5Ce^{k(0)} &= Ce^{k(3.8)} \\ 0.5C &= Ce^{k(3.8)} \\ 0.5 &= e^{k(3.8)} \\ \ln(0.5) &= k(3.8) \\ \frac{\ln(0.5)}{3.8} &= k \\ k &\approx -0.182. \end{aligned}$$

(e) From part (c) we know that the formula for Q is

$$Q = Ce^{-0.182t}.$$

We are told that initially there are 10 mg of the drug in the body. Thus, at $t = 0$, we get

$$10 = Ce^{-0.182(0)}$$

so

$$C = 10.$$

Thus, our equation becomes

$$Q(t) = 10e^{-0.182t}.$$

Substituting $t = 12$, we get

$$\begin{aligned} Q(t) &= 10e^{-0.182t} \\ Q(12) &= 10e^{-0.182(12)} \\ &= 10e^{-2.184} \\ Q(12) &\approx 1.126 \text{ mg}. \end{aligned}$$

31. (a) The differential equation is

$$\frac{dB}{dt} = \frac{r}{100} \cdot B.$$

The constant of proportionality is

$$\frac{r}{100}.$$

(b) To find the equilibrium, we solve

$$\frac{dB}{dt} = \frac{r}{100} \cdot B = 0$$

to get $B = 0$.

If we have a solution with initial condition $B > 0$, then $dB/dt > 0$, and the solution increases and moves away from the equilibrium. Thus, the equilibrium is unstable. In terms of a bank account, this means that an empty bank account remains empty unless somebody deposits money into it, but a bank account with money in it accumulates more money.

(c) Solving, we have

$$\begin{aligned}\frac{dB}{B} &= \frac{r\,dt}{100}\\ \int \frac{dB}{B} &= \int \frac{r}{100}\,dt\\ \ln|B| &= \frac{r}{100}t + C\\ B &= \pm e^{(r/100)t+C} = Ae^{(r/100)t}, \qquad A = \pm e^C.\end{aligned}$$

A is the initial amount in the account, since A is the amount at time $t = 0$.

(d) See Figure 11.37.

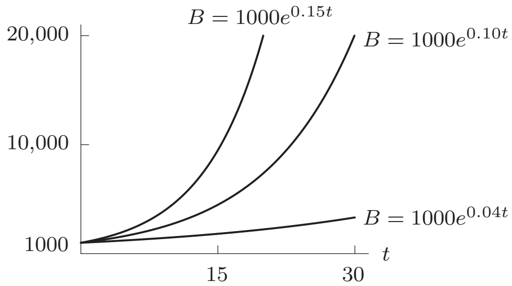


Figure 11.37

Strengthen Your Understanding

32. When $y = 2$, we have $dy/dx = 8 - 8x \neq 0$. An equilibrium solution must have derivative equal to zero for all values of x.

33. An equilibrium is a constant solution, so $y = x^2$ cannot be an equilibrium solution.

34. Newton's Law of Heating and Cooling says that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding air. Once the roast is in the oven, the temperature of the surrounding air is 350°F, so the correct differential equation is $dH/dt = k(H - 350)$.

35. The differential equation is of the form $dy/dt = ky$, where k is negative. One possible answer is $dy/dt = -2y$.

36. Some possible answers are $dQ/dt = Q - 500$ or $dQ/dx = 2(Q - 500)$. Other answers are possible. We must have Q as the dependent variable and the derivative equal to zero when $Q = 500$.

37. Since the graph has an equilibrium solution at $P = 25$, the solution with initial condition $P_0 = 25$ must be a horizontal line. Since the equilibrium solution is unstable, it is likely that the solutions with $P_0 = 20$ and $P_0 = 30$ bend away from the horizontal line. See Figure 11.38. Other answers are possible.

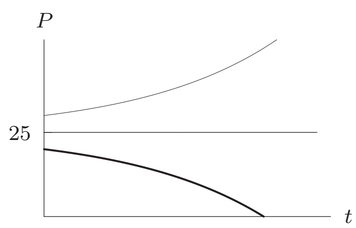


Figure 11.38

Solutions for Section 11.6

Exercises

1. (a) (III) An island can only sustain the population up to a certain size. The population will grow until it reaches this limiting value.
(b) (V) The ingot will get hot and then cool off, so the temperature will increase and then decrease.
(c) (I) The speed of the car is constant, and then decreases linearly when the breaks are applied uniformly.
(d) (II) Carbon-14 decays exponentially.
(e) (IV) Tree pollen is seasonal, and therefore cyclical.

2. The balance is increasing at a rate of 0.05 times the current balance and is decreasing at a rate of $12{,}000$ dollars per year. The differential equation is

$$\frac{dB}{dt} = 0.05B - 12{,}000.$$

3. The balance is increasing at a rate of 0.037 times the current balance and is also increasing at a rate of 5000 dollars per year. The differential equation is

$$\frac{dB}{dt} = 0.037B + 5000.$$

4. The balance is decreasing at a rate of 0.08 times the current balance and is increasing at a rate of 2000 dollars per year. The differential equation is

$$\frac{dB}{dt} = -0.08B + 2000.$$

5. The balance is decreasing at a rate of 0.065 times the current balance and is also decreasing at a rate of $50{,}000$ dollars per year. The differential equation is

$$\frac{dB}{dt} = -0.065B - 50{,}000.$$

6. (a) If $B = f(t)$, where t is in years,

$$\frac{dB}{dt} = \text{Rate of money earned from interest} + \text{Rate of money deposited}$$
$$\frac{dB}{dt} = 0.10B + 1000.$$

(b) We use separation of variables to solve the differential equation

$$\frac{dB}{dt} = 0.1B + 1000.$$

$$\begin{aligned}\int \frac{1}{0.1B + 1000} dB &= \int dt \\ \frac{1}{0.1} \ln|0.1B + 1000| &= t + C_1 \\ 0.1B + 1000 &= C_2 e^{0.1t} \\ B &= Ce^{0.1t} - 10{,}000\end{aligned}$$

For $t = 0$, $B = 0$, hence $C = 10{,}000$. Therefore, $B = 10{,}000e^{0.1t} - 10{,}000$.

7. (a) By Newton's Law of Cooling, we have

$$\frac{dH}{dt} = k(H - 50)$$

for some k. Furthermore, we know the juice's original temperature $H(0) = 90$.

(b) Separating variables, we get

$$\int \frac{dH}{(H-50)} = \int k\,dt.$$

We then integrate:

$$\begin{aligned}\ln|H-50| &= kt + C\\ H-50 &= e^{kt}\cdot A\\ H &= 50 + Ae^{kt}.\end{aligned}$$

Thus, $H(0) = 90$ gives $A = 40$, and $H(5) = 80$ gives

$$\begin{aligned}50 + 40e^{5k} &= 80\\ e^{5k} &= \frac{30}{40}\\ 5k &= \ln(0.75)\\ k &= \frac{1}{5}\ln(0.75) \approx -0.05754.\end{aligned}$$

Therefore

$$H(t) = 50 + 40e^{-0.05754t}.$$

(c) We now solve for t at which $H(t) = 60$:

$$\begin{aligned}60 &= 50 + 40e^{-0.05754t}\\ \frac{1}{4} &= e^{-0.05754t}\\ \ln(0.25) &= -0.05754t\\ t &= 24 \text{ minutes.}\end{aligned}$$

8. Since mg is constant and $a = dv/dt$, differentiating $ma = mg - kv$ gives

$$m\frac{da}{dt} = -k\frac{dv}{dt} = -ma.$$

Thus, the differential equation is

$$\frac{da}{dt} = -\frac{k}{m}a.$$

Solving for a gives

$$a = a_0 e^{-kt/m}.$$

At $t = 0$, we have $a = g$, the acceleration due to gravity. Thus, $a_0 = g$, so

$$a = ge^{-kt/m}.$$

Problems

9. (a) There are two factors that are affecting B: the money leaving the account, which is at a constant rate of -2000 per year, and the interest accumulating in it, which accrues at a rate of $(0.08)B$. Since

Rate of change of balance = Rate in − Rate out,

the differential equation for B is

$$\frac{dB}{dt} = 0.08B - 2000.$$

(b) To find the equilibrium solution, we solve $dB/dt = 0$ to get $0.08B - 2000 = 0$, or $B = 25000$. To determine whether the equilibrium is stable, we consider what happens to a solution with initial condition near the equilibrium. If the initial condition is $B = 26000$, then initially $dB/dt = 0.08(26000) - 2000 > 0$. Thus, the interest accumulated by the account exceeds the amount being spent, and the balance increases. If a solution has initial condition $B = 24000$, then $dB/dt = 0.08(24000) - 2000 < 0$. Thus, more money is being spent than the interest can replenish, so the balance decreases. The equilibrium is unstable.

(c) We solve the differential equation by separating variables and then integrating:

$$\begin{aligned}\int \frac{dB}{0.08B-2000} &= \int dt\\ 12.5\ln|0.08B-2000| &= t+C\\ \ln|0.08B-2000| &= \frac{t}{12.5}+C\\ 0.08B-2000 &= \pm e^{0.08t+C}\\ B &= 25{,}000+Ae^{0.08t}.\end{aligned}$$

(d) (i) If the initial deposit is 20,000, then we have $B = 20{,}000$ when $t = 0$, which leads to $A = -5000$. Knowing A, we can find $B(5)$ as:

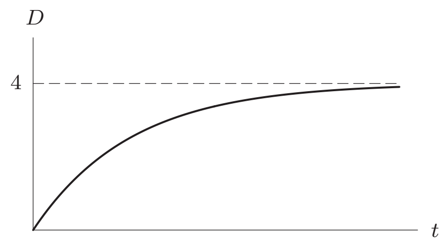
$$B(5) = 25{,}000 - 5000e^{0.08(5)} = \$17{,}540.88.$$

(ii) Now $B = 30{,}000$ when $t = 0$ leads to $A = 5000$, giving $B(5) = \$32{,}459.12$.

10. Let $D(t)$ be the quantity of dead leaves, in grams per square centimeter. Then $\frac{dD}{dt} = 3 - 0.75D$, where t is in years. We factor out -0.75 and then separate variables.

$$\begin{aligned}\frac{dD}{dt} &= -0.75(D-4)\\ \int \frac{dD}{D-4} &= \int -0.75\,dt\\ \ln|D-4| &= -0.75t+C\\ |D-4| &= e^{-0.75t+C} = e^{-0.75t}e^{C}\\ D &= 4+Ae^{-0.75t},\ \text{where } A=\pm e^{C}.\end{aligned}$$

If initially the ground is clear, the solution looks like the following graph:



The equilibrium level is 4 grams per square centimeter, regardless of the initial condition.

11. (a) Letting P represent the quantity of pollutant in the lake, in metric tons, in year t, we have

$$\frac{dP}{dt} = -0.16P + 8.$$

Substituting $P_0 = 45$ and $P_0 = 55$, we get

$$\left.\frac{dP}{dt}\right|_{P=45} = -0.16(45)+8 = 0.8 \qquad \text{and} \qquad \left.\frac{dP}{dt}\right|_{P=55} = -0.16(55)+8 = -0.8.$$

Since the derivative is positive for $P_0 = 45$, the quantity of pollutant is increasing then. For $P_0 = 55$, the derivative is negative, so the quantity of pollutant is decreasing then.

(b) Writing the differential equation as

$$\frac{dP}{dt} = -0.16(P-50),$$

we use separation of variables to see that the general solution is

$$P = 50 + Ce^{-0.16t}.$$

In all cases, the quantity of pollutant levels off at 50 metric tons after a long time.

12. Caffeine is leaving the body at a rate of 17% per hour and is entering the body at a rate of 130 mg per hour, so the differential equation is

$$\frac{dA}{dt} = -0.17A + 130.$$

Writing the differential equation as

$$\frac{dA}{dt} = -0.17(A - 764.706),$$

we use separation of variables to see that the general solution is $A = 764.706 + Ce^{-0.17t}$. Since the initial condition is $A_0 = 0$, we have $C = -764.706$ so the particular solution is

$$A = 764.706 - 764.706e^{-0.17t}.$$

We substitute $t = 10$ to find the amount of caffeine at 5 pm:

$$A = 764.706 - 764.706e^{-0.17(10)} = 625.007.$$

At 5 pm, the person has about 625 mg of caffeine in the body.

13. (a) If I is intensity and l is the distance traveled through the water, then for some $k > 0$,

$$\frac{dI}{dl} = -kI.$$

(The proportionality constant is negative because intensity decreases with distance). Thus $I = Ae^{-kl}$. Since $I = A$ when $l = 0$, A represents the initial intensity of the light.

(b) If 50% of the light is absorbed in 10 feet, then $0.50A = Ae^{-10k}$, so $e^{-10k} = \frac{1}{2}$, giving

$$k = \frac{-\ln \frac{1}{2}}{10} = \frac{\ln 2}{10}.$$

In 20 feet, the percentage of light left is

$$e^{-\frac{\ln 2}{10}\cdot 20} = e^{-2\ln 2} = (e^{\ln 2})^{-2} = 2^{-2} = \frac{1}{4},$$

so $\frac{3}{4}$ or 75% of the light has been absorbed. Similarly, after 25 feet,

$$e^{-\frac{\ln 2}{10}\cdot 25} = e^{-2.5\ln 2} = (e^{\ln 2})^{-\frac{5}{2}} = 2^{-\frac{5}{2}} \approx 0.177.$$

Approximately 17.7% of the light is left, so 82.3% of the light has been absorbed.

14. (a) The differential equation for the population has the form

$$\text{Rate of change} = \text{Birth rate} - \text{Death rate}$$

The birth rate is 140 m/yr $= 0.14$ bn/yr. Since we assume the death rate is increasing linearly from 57 m/year to 80 m/year over 30 years, we have

$$\text{Rate of change of death rate} = \frac{80 - 57}{30} = 0.767\text{(million/year)/year}.$$

Then at time t, the death rate is given by

$$\text{Death rate} = 57 + 0.767t \text{ million/year} = 0.057 + 0.000767t \text{ billion/year}.$$

The differential equation is

$$\frac{dP}{dt} = 0.14 - (0.057 + 0.000767t)$$
$$\frac{dP}{dt} = 0.083 - 0.000767t.$$

(b) The differential equation can be solved by direct integration:

$$P = 0.083t - 0.000767\frac{t^2}{2} + P_0.$$

Since $P_0 = 6.9$ billion is the population at $t = 0$, we have

$$P = 0.083t - 0.000767\frac{t^2}{2} + 6.9.$$

(c) In 2050, we have $t = 40$, so

$$P = 0.083(40) - 0.000767\left(\frac{40^2}{2}\right) + 6.9 = 9.61 \text{ bn.}$$

15. (a) If $P =$ pressure and $h =$ height, $\frac{dP}{dh} = -3.7 \times 10^{-5}P$, so $P = P_0 e^{-3.7\times10^{-5}h}$. Now $P_0 = 29.92$, since pressure at sea level (when $h = 0$) is 29.92, so $P = 29.92e^{-3.7\times10^{-5}h}$. At the top of Mt. Whitney, the pressure is

$$P = 29.92e^{-3.7\times10^{-5}(14500)} \approx 17.50 \text{ inches of mercury.}$$

At the top of Mt. Everest, the pressure is

$$P = 29.92e^{-3.7\times10^{-5}(29000)} \approx 10.23 \text{ inches of mercury.}$$

(b) The pressure is 15 inches of mercury when

$$15 = 29.92e^{-3.7\times10^{-5}h}$$

Solving for h gives $h = \frac{-1}{3.7\times10^{-5}}\ln(\frac{15}{29.92}) \approx 18{,}661.5$ feet.

16. (a) Since the rate of change of the weight is given by

$$\frac{dW}{dt} = \frac{1}{3500}(\text{Intake} - \text{Amount to maintain weight})$$

we have

$$\frac{dW}{dt} = \frac{1}{3500}(I - 20W).$$

(b) To find the equilibrium, we solve $dW/dt = 0$, or

$$\frac{1}{3500}(I - 20W) = 0.$$

Solving for W, we get $W = I/20$.

This means that if an athletic adult male weighing $I/20$ pounds has a constant caloric intake of I calories per day, his weight remains constant. We expect the equilibrium to be stable because an athletic adult male slightly over the equilibrium weight loses weight because his caloric intake is too low to maintain the higher weight. Similarly, an adult male slightly under the equilibrium weight gains weight because his caloric intake is higher than required to maintain his weight.

(c) Factoring the right side of the differential equation

$$\frac{dW}{dt} = \frac{1}{3500}(I - 20W) = -\frac{1}{175}\left(W - \frac{I}{20}\right),$$

we separate variables and integrate:

$$\int \frac{dW}{W - I/20} = -\int \frac{1}{175}\,dt.$$

Thus we have

$$\ln\left|W - \frac{I}{20}\right| = -\frac{1}{175}t + C$$

so that

$$W - \frac{I}{20} = Ae^{-(1/175)t}$$

or in other words

$$W = \frac{I}{20} + Ae^{-(1/175)t}.$$

Let us call the person's initial weight W_0 at $t = 0$. Then $W_0 = I/20 + Ce^0$, so $C = W_0 - I/20$. Thus

$$W = \frac{I}{20} + \left(W_0 - \frac{I}{20}\right)e^{-(1/175)t}.$$

(d) Using part (c), we have $W = 150 + 10e^{-(1/175)t}$. This means that $W \to 150$ as $t \to \infty$. See Figure 11.39.

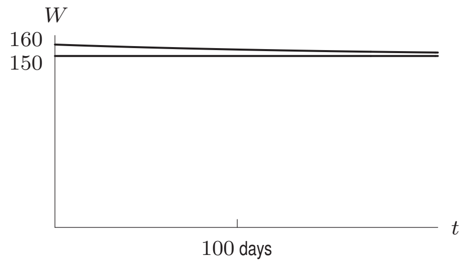


Figure 11.39

17. (a) We know that the rate at which morphine leaves the body is proportional to the amount of morphine in the body at that particular instant. If we let Q be the amount of morphine in the body, we get that

$$\text{Rate of morphine leaving the body} = kQ,$$

where k is the rate of proportionality. The solution is $Q = Q_0e^{kt}$ (neglecting the continuously incoming morphine). Since the half-life is 2 hours, we have

$$\frac{1}{2}Q_0 = Q_0e^{k\cdot 2},$$

so

$$k = \frac{\ln(1/2)}{2} = -0.347.$$

(b) Since

$$\text{Rate of change of quantity } = \text{Rate in } - \text{Rate out},$$

we have

$$\frac{dQ}{dt} = -0.347Q + 2.5.$$

(c) Equilibrium occurs when $dQ/dt = 0$, that is, when $0.347Q = 2.5$ or $Q = 7.2$ mg.

18. Let the depth of the water at time t be y. Then $\dfrac{dy}{dt} = -k\sqrt{y}$, where k is a positive constant. Separating variables,

$$\int \frac{dy}{\sqrt{y}} = -\int k\,dt,$$

so

$$2\sqrt{y} = -kt + C\,.$$

When $t = 0$, $y = 36$; $2\sqrt{36} = -k \cdot 0 + C$, so $C = 12$.
When $t = 1$, $y = 35$; $2\sqrt{35} = -k + 12$, so $k \approx 0.17$.
Thus, $2\sqrt{y} \approx -0.17t + 12$. We are looking for t such that $y = 0$; this happens when $t \approx \frac{12}{0.17} \approx 71$ hours, or about 3 days.

19. We are given that the rate of change of pressure with respect to volume, dP/dV is proportional to P/V, so that

$$\frac{dP}{dV} = k\frac{P}{V}.$$

Using separation of variables and integrating gives

$$\int \frac{dP}{P} = k\int \frac{dV}{V}.$$

Evaluating these integral gives

$$\ln P = k\ln V + c$$

or equivalently,

$$P = AV^k.$$

20. (a) If A is surface area, we know that for some constant K

$$\frac{dV}{dt} = -KA.$$

If r is the radius of the sphere, $V = 4\pi r^3/3$ and $A = 4\pi r^2$. Solving for r in terms of V gives $r = (3V/4\pi)^{1/3}$, so

$$\frac{dV}{dt} = -K(4\pi r^2) = -K4\pi\left(\frac{3V}{4\pi}\right)^{2/3} \quad \text{so} \quad \frac{dV}{dt} = -kV^{2/3}$$

where k is another constant, $k = K(4\pi)^{1/3}3^{2/3}$.

(b) Separating variables gives

$$\int \frac{dV}{V^{2/3}} = -\int k\,dt$$
$$3V^{1/3} = -kt + C.$$

Since $V = V_0$ when $t = 0$, we have $3V_0^{1/3} = C$, so

$$3V^{1/3} = -kt + 3V_0^{1/3}.$$

Solving for V gives

$$V = \left(-\frac{k}{3}t + V_0^{1/3}\right)^3.$$

This function is graphed in Figure 11.40.

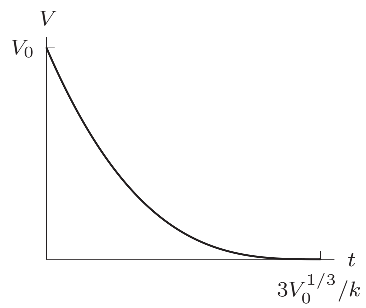


Figure 11.40

(c) The snowball disappears when $V = 0$, that is when

$$-\frac{k}{3}t + V_0^{1/3} = 0$$

giving

$$t = \frac{3V_0^{1/3}}{k}.$$

21. Let $V(t)$ be the volume of water in the tank at time t, then

$$\frac{dV}{dt} = k\sqrt{V}$$

This is a separable equation which has the solution

$$V(t) = (\frac{kt}{2} + C)^2$$

Since $V(0) = 200$ this gives $200 = C^2$ so

$$V(t) = (\frac{kt}{2} + \sqrt{200})^2.$$

However, $V(1) = 180$ therefore

$$180 = (\frac{k}{2} + \sqrt{200})^2,$$

so that $k = 2\left(\sqrt{180} - \sqrt{200}\right) = -1.45146$. Therefore,

$$V(t) = (-0.726t + \sqrt{200})^2.$$

The tank will be half-empty when $V(t) = 100$, so we solve

$$100 = (-0.726t + \sqrt{200})^2$$

to obtain $t = 5.7$ days. The tank will be half empty in 5.7 days.

The volume after 4 days is $V(4)$ which is approximately 126.32 liters.

22. (a) We have $\frac{dy}{dt} = -k(y - a)$, where $k > 0$ and a are constants.

(b) Separating variables and integrating we have

$$\int \frac{dy}{y - a} = \int -k\,dt,$$

so

$$\ln|y - a| = \ln(y - a) = -kt + C.$$

Thus,

$$y - a = Ae^{-kt} \quad \text{where } A = e^C.$$

Initially nothing has been forgotten, so $y(0) = 1$. Therefore, $1 - a = Ae^0 = A$, so $y - a = (1 - a)e^{-kt}$ or $y = (1 - a)e^{-kt} + a$.

(c) As $t \to \infty$, $e^{-kt} \to 0$, so $y \to a$.

Thus, a represents the fraction of material which is remembered in the long run. The constant k tells us about the rate at which material is forgotten.

23. (a) We have

$$\frac{dp}{dt} = -k(p - p^*),$$

where k is constant. Notice that $k > 0$, since if $p > p^*$ then dp/dt should be negative, and if $p < p^*$ then dp/dt should be positive.

(b) Separating variables, we have

$$\int \frac{dp}{p - p^*} = \int -k\,dt.$$

Solving, we find $p = p^* + (p_0 - p^*)e^{-kt}$, where p_0 is the initial price.

(c) See Figure 11.41.

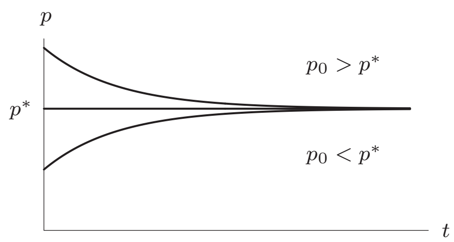


Figure 11.41

(d) As $t \to \infty$, $p \to p^*$. We see this in the solution in part (b), since as $t \to \infty$, $e^{-kt} \to 0$. In other words, as $t \to \infty$, p approaches the equilibrium price p^*.

24. We have

$$\underbrace{\text{The rate that people first go see the movie}}_{dN/dt} = k \times \underbrace{\left(\text{The number of people who would like to see it but haven't yet}\right)}_{L-N}$$

$$\begin{aligned}
\frac{dN}{dt} &= k(L-N) \\
\int \frac{dN}{N-L} &= -\int k\,dt && \text{separation of variables} \\
\ln|N-L| &= -kt + C \\
|N-L| &= Be^{-kt} && B = e^C \\
N - L &= Ae^{-kt} && A = -B \\
N &= L + Ae^{-kt}.
\end{aligned}$$

Presumably no one sees the movie before it is released, so $N = 0$ on day $t = 0$, and we have

$$\begin{aligned}
0 &= L + Ae^{-k\cdot 0} \\
A &= -L \\
\text{so} \quad N &= L - Le^{-kt}.
\end{aligned}$$

25. (a)

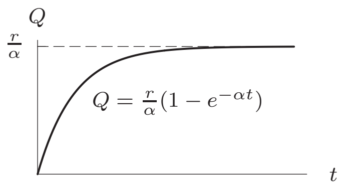
$$\begin{aligned}
\frac{dQ}{dt} &= r - \alpha Q = -\alpha(Q - \frac{r}{\alpha}) \\
\int \frac{dQ}{Q - r/\alpha} &= = -\alpha \int dt \\
\ln\left|Q - \frac{r}{\alpha}\right| &= -\alpha t + C \\
Q - \frac{r}{\alpha} &= Ae^{-\alpha t}
\end{aligned}$$

When $t = 0$, $Q = 0$, so $A = -\frac{r}{\alpha}$ and

$$Q = \frac{r}{\alpha}(1 - e^{-\alpha t})$$

So,

$$Q_\infty = \lim_{t\to\infty} Q = \frac{r}{\alpha}.$$



(b) Doubling r doubles Q_∞. Since $Q_\infty = r/\alpha$, the time to reach $\frac{1}{2}Q_\infty$ is obtained by solving

$$\begin{aligned}
\frac{r}{2\alpha} &= \frac{r}{\alpha}(1 - e^{-\alpha t}) \\
\frac{1}{2} &= 1 - e^{-\alpha t} \\
e^{-\alpha t} &= \frac{1}{2} \\
t &= -\frac{\ln(1/2)}{\alpha} = \frac{\ln 2}{\alpha}.
\end{aligned}$$

So altering r does not alter the time it takes to reach $\frac{1}{2}Q_\infty$. See Figure 11.42.

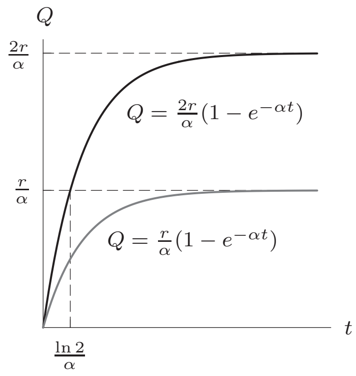


Figure 11.42

(c) Q_∞ is halved by doubling α, and so is the time, $t = \frac{\ln 2}{\alpha}$, to reach $\frac{1}{2}Q_\infty$.

26. (a) Concentration of carbon monoxide $= \dfrac{\text{Quantity in room}}{\text{Volume}}$.
If $Q(t)$ represents the quantity of carbon monoxide in the room at time t, $c(t) = Q(t)/60$.

$$\begin{array}{c}\text{Rate quantity of}\\ \text{carbon monoxide in room}\\ \text{changes}\end{array} = \text{rate in} - \text{rate out}$$

Now

$$\text{Rate in} = 5\%(0.002\text{m}^3/\text{min}) = 0.05(0.002) = 0.0001\text{m}^3/\text{min}.$$

Since smoky air is leaving at $0.002\text{m}^3/\text{min}$, containing a concentration $c(t) = Q(t)/60$ of carbon monoxide

$$\text{Rate out} = 0.002\frac{Q(t)}{60}$$

Thus

$$\frac{dQ}{dt} = 0.0001 - \frac{0.002}{60}Q$$

Since $c = Q/60$, we can substitute $Q = 60c$, giving

$$\begin{aligned}\frac{d(60c)}{dt} &= 0.0001 - \frac{0.002}{60}(60c)\\ \frac{dc}{dt} &= \frac{0.0001}{60} - \frac{0.002}{60}c\end{aligned}$$

(b) Factoring the right side of the differential equation and separating gives

$$\begin{aligned}\frac{dc}{dt} &= -\frac{0.0001}{3}(c - 0.05) \approx 3 \times 10^{-5}(c - 0.05)\\ \int \frac{dc}{c - 0.05} &= -\int 3 \times 10^{-5} dt\\ \ln|c - 0.05| &= -3 \times 10^{-5}t + K\\ c - 0.05 &= Ae^{-3\times 10^{-5}t} \quad \text{where} A = \pm e^K.\end{aligned}$$

Since $c = 0$ when $t = 0$, we have $A = -0.05$, so

$$c = 0.05 - 0.05e^{-3\times 10^{-5}t}$$

(c) As $t \to \infty$, $e^{-3\times 10^{-5}t} \to 0$ so $c \to 0.05$.
Thus in the long run, the concentration of carbon monoxide tends to 5%, the concentration of the incoming air.

27. We have $c = 0.05 - 0.05e^{-3\times10^{-5}t}$. We want to solve for t when $c = 0.0002$:

$$\begin{aligned} 0.0002 &= 0.05 - 0.05e^{-3\times10^{-5}t} \\ -0.0498 &= -0.05e^{-3\times10^{-5}t} \\ e^{-3\times10^{-5}t} &= 0.996 \\ t &= \frac{-\ln(0.996)}{3\times10^{-5}} = 133.601 \text{ min.} \end{aligned}$$

28. (a) Now

$$\frac{dS}{dt} = (\text{Rate at which salt enters the pool}) - (\text{Rate at which salt leaves the pool}),$$

and, for example,

$$\begin{pmatrix}\text{Rate at which salt}\\ \text{enters the pool}\end{pmatrix} = \begin{pmatrix}\text{Concentration of}\\ \text{salt solution}\end{pmatrix} \times \begin{pmatrix}\text{Flow rate of}\\ \text{salt solution}\end{pmatrix}$$
$$(\text{grams/minute}) = (\text{grams/liter}) \times (\text{liters/minute})$$

so

$$\text{Rate at which salt enters the pool} = (10 \text{ grams/liter}) \times (60 \text{ liters/minute}) = (600 \text{ grams/minute})$$

The rate at which salt leaves the pool depends on the concentration of salt in the pool. At time t, the concentration is $\frac{S(t)}{2\times10^6 \text{ liters}}$, where $S(t)$ is measured in grams.
Thus

$$\text{Rate at which salt leaves the pool} = \frac{S(t)\text{ grams}}{2\times10^6 \text{ liters}} \times \frac{60 \text{ liters}}{\text{minute}} = \frac{3S(t)\text{ grams}}{10^5 \text{ minutes}}.$$

Thus

$$\frac{dS}{dt} = 600 - \frac{3S}{100{,}000}.$$

(b) $\frac{dS}{dt} = -\frac{3}{100{,}000}(S - 20{,}000{,}000)$
$\int \frac{dS}{S - 20{,}000{,}000} = \int -\frac{3}{100{,}000}\,dt$
$\ln|S - 20{,}000{,}000| = -\frac{3}{100{,}000}t + C$
$S = 20{,}000{,}000 - Ae^{-\frac{3}{100{,}000}t}$
Since $S = 0$ at $t = 0$, $A = 20{,}000{,}000$. Thus $S(t) = 20{,}000{,}000 - 20{,}000{,}000e^{-\frac{3}{100{,}000}t}$.

(c) As $t \to \infty$, $e^{-\frac{3}{100{,}000}t} \to 0$, so $S(t) \to 20{,}000{,}000$ grams. The concentration approaches 10 grams/liter. Note that this makes sense; we'd expect the concentration of salt in the pool to become closer and closer to the concentration of salt being poured into the pool as $t \to \infty$.

29. We are given that

$$BC = 2OC.$$

If the point A has coordinates (x, y) then $OC = x$ and $AC = y$. The slope of the tangent line, y', is given by

$$y' = \frac{AC}{BC} = \frac{y}{BC},$$

so

$$BC = \frac{y}{y'}.$$

Substitution into $BC = 2OC$ gives

$$\frac{y}{y'} = 2x,$$

so

$$\frac{y'}{y} = \frac{1}{2x}.$$

Separating variables to integrate this differential equation gives

$$\begin{aligned}\int \frac{dy}{y} &= \int \frac{dx}{2x}\\ \ln|y| &= \frac{1}{2}\ln|x| + C = \ln\sqrt{|x|} + \ln A\\ |y| &= A\sqrt{|x|}\\ y &= \pm(A\sqrt{x}).\end{aligned}$$

Thus, in the first quadrant, the curve has equation $y = A\sqrt{x}$.

30. (a) Newton's Law of Motion says that

$$\text{Force} = (\text{mass}) \times (\text{acceleration}).$$

Since acceleration, dv/dt, is measured upward and the force due to gravity acts downward,

$$-\frac{mgR^2}{(R+h)^2} = m\frac{dv}{dt}$$

so

$$\frac{dv}{dt} = -\frac{gR^2}{(R+h)^2}.$$

(b) Since $v = \frac{dh}{dt}$, the chain rule gives

$$\frac{dv}{dt} = \frac{dv}{dh}\cdot\frac{dh}{dt} = \frac{dv}{dh}\cdot v.$$

Substituting into the differential equation in part (a) gives

$$v\frac{dv}{dh} = -\frac{gR^2}{(R+h)^2}.$$

(c) Separating variables gives

$$\begin{aligned}\int v\,dv &= -\int \frac{gR^2}{(R+h)^2}\,dh\\ \frac{v^2}{2} &= \frac{gR^2}{(R+h)} + C\end{aligned}$$

Since $v = v_0$ when $h = 0$,

$$\frac{{v_0}^2}{2} = \frac{gR^2}{(R+0)} + C \quad \text{gives} \quad C = \frac{{v_0}^2}{2} - gR,$$

so the solution is

$$\begin{aligned}\frac{v^2}{2} &= \frac{gR^2}{(R+h)} + \frac{{v_0}^2}{2} - gR\\ v^2 &= {v_0}^2 + \frac{2gR^2}{(R+h)} - 2gR\end{aligned}$$

(d) The escape velocity v_0 ensures that $v^2 \geq 0$ for all $h \geq 0$. Since the positive quantity $\frac{2gR^2}{(R+h)} \to 0$ as $h \to \infty$, to ensure that $v^2 \geq 0$ for all h, we must have

$${v_0}^2 \geq 2gR.$$

When ${v_0}^2 = 2gR$ so $v_0 = \sqrt{2gR}$, we say that v_0 is the escape velocity.

31. (a) For this situation,

$$\begin{pmatrix}\text{Rate money added}\\ \text{to account}\end{pmatrix} = \begin{pmatrix}\text{Rate money added}\\ \text{via interest}\end{pmatrix} + \begin{pmatrix}\text{Rate money}\\ \text{deposited}\end{pmatrix}.$$

Translating this into an equation yields

$$\frac{dB}{dt} = 0.05B + 1200.$$

(b) Solving this equation via separation of variables gives

$$\begin{aligned}\frac{dB}{dt} &= 0.05B + 1200 \\ &= 0.05(B + 24000).\end{aligned}$$

So

$$\int \frac{dB}{B + 24000} = \int 0.05\, dt,$$

and

$$\ln |B + 24000| = 0.05t + C.$$

Solving for B gives

$$|B + 24000| = e^{0.05t+C} = e^{C} e^{0.05t},$$

so

$$B = Ae^{0.05t} - 24000, \text{ (where } A = e^{c}).$$

We may find A using the initial condition $B_0 = f(0) = 0$

$$A - 24000 = 0 \quad \text{or} \quad A = 24000.$$

The solution is

$$B = 24{,}000e^{0.05t} - 24{,}000.$$

(c) After 5 years, the balance is

$$B = f(5) = 24{,}000(e^{0.05(5)} - 1) = 6816.61 \text{ dollars.}$$

32. (a) The balance in the account at the beginning of the month is given by the following sum

$$\left(\begin{array}{c}\text{balance in}\\ \text{account}\end{array}\right) = \left(\begin{array}{c}\text{previous month's}\\ \text{balance}\end{array}\right) + \left(\begin{array}{c}\text{interest on}\\ \text{previous month's balance}\end{array}\right) + \left(\begin{array}{c}\text{monthly deposit}\\ \text{of \$100}\end{array}\right)$$

Denote month i's balance by B_i. Assuming the interest is compounded continuously, we have

$$\left(\begin{array}{c}\text{previous month's}\\ \text{balance}\end{array}\right) + \left(\begin{array}{c}\text{interest on previous}\\ \text{month's balance}\end{array}\right) = B_{i-1}e^{0.1/12}.$$

Since the interest rate is $10\% = 0.1$ per year, interest is $\frac{0.1}{12}$ per month. So at month i, the balance is

$$B_i = B_{i-1}e^{\frac{0.1}{12}} + 100$$

Explicitly, we have for the five years (60 months) the equations:

$$\begin{aligned}B_0 &= 0 \\ B_1 &= B_0 e^{\frac{0.1}{12}} + 100 \\ B_2 &= B_1 e^{\frac{0.1}{12}} + 100 \\ B_3 &= B_2 e^{\frac{0.1}{12}} + 100 \\ \vdots\ & \quad \vdots \\ B_{60} &= B_{59} e^{\frac{0.1}{12}} + 100\end{aligned}$$

In other words,

$$\begin{aligned}B_1 &= 100 \\ B_2 &= 100e^{\frac{0.1}{12}} + 100 \\ B_3 &= (100e^{\frac{0.1}{12}} + 100)e^{\frac{0.1}{12}} + 100 \\ &= 100e^{\frac{(0.1)2}{12}} + 100e^{\frac{0.1}{12}} + 100\end{aligned}$$

$$B_4 = 100e^{\frac{(0.1)3}{12}} + 100e^{\frac{(0.1)2}{12}} + 100e^{\frac{(0.1)}{12}} + 100$$
$$\vdots \quad \vdots$$
$$B_{60} = 100e^{\frac{(0.1)59}{12}} + 100e^{\frac{(0.1)58}{12}} + \cdots + 100e^{\frac{(0.1)1}{12}} + 100$$
$$B_{60} = \sum_{k=0}^{59} 100e^{\frac{(0.1)k}{12}}$$

(b) The sum $B_{60} = \sum_{k=0}^{59} 100e^{\frac{(0.1)k}{12}}$ can be written as $B_{60} = \sum_{k=0}^{59} 1200e^{\frac{(0.1)k}{12}}(\frac{1}{12})$ which is the left Riemann sum for $\int_0^5 1200e^{0.1t}\,dt$, with $\Delta t = \frac{1}{12}$ and $N = 60$. Evaluating the sum on a calculator gives $B_{60} = 7752.26$.

(c) The situation described by this problem is almost the same as that in Problem 31, except that here the money is being deposited once a month rather than continuously; however the nominal yearly rates are the same. Thus we would expect the balance after 5 years to be approximately the same in each case. This means that the answer to part (b) of this problem should be approximately the same as the answer to part (c) to Problem 31. Since the deposits in this problem start at the end of the first month, as opposed to right away, we would expect the balance after 5 years to be slightly smaller than in Problem 31, as is the case.

Alternatively, we can use the Fundamental Theorem of Calculus to show that the integral can be computed exactly

$$\int_0^5 1200e^{0.1t}\,dt = 12000(e^{(0.1)5} - 1) = 7784.66$$

Thus $\int_0^5 1200e^{0.1t}\,dt$ represents the exact solution to Problem 31. Since $1200e^{0.1t}$ is an increasing function, the left hand sum we calculated in part (b) of this problem underestimates the integral. Thus the answer to part (b) of this problem should be less than the answer to part (c) of Problem 31.

Strengthen Your Understanding

33. When we substitute $B = 5000$ into the differential equation, we see that $dB/dt = 0.08(5000) - 250 = 150 > 0$. Since the rate of change is positive, the balance is increasing.

34. The units do not match up. If time t is measured in days, we need to convert 25 mg per hour to the equivalent rate of 600 mg per day. If time t is measured in hours, we need to convert the rate 15% per day into the equivalent rate of 0.625% per hour.

35. We use Q for the quantity of drug in the body at time t. The rate in is constant (such as 50) and the rate out is proportional (such as $0.08Q$). One possible answer is $dQ/dt = 50 - 0.08Q$. Other answers are possible.

36. We use Q for the size of the quantity at time t. The rate due to its growth on its own is proportional to the cube root (such as $0.2\sqrt[3]{Q}$) and the rate due to the external contribution is constant (such as 100). One possible answer is $dQ/dt = 0.2\sqrt[3]{Q} + 100$.

37. We use Q for the size of the quantity at time t. Since the rate of growth dQ/dt goes down as Q goes up, one possible model is that the rate of growth is inversely proportional to Q. We have $dQ/dt = k/Q$ with k positive. One example is

$$\frac{dQ}{dt} = \frac{0.5}{Q}.$$

Other answers are possible.

Solutions for Section 11.7

Exercises

1. (a) $P = \frac{1}{1+e^{-t}} = (1+e^{-t})^{-1}$
$\frac{dP}{dt} = -(1+e^{-t})^{-2}(-e^{-t}) = \frac{e^{-t}}{(1+e^{-t})^2}$.
Then $P(1-P) = \frac{1}{1+e^{-t}}\left(1 - \frac{1}{1+e^{-t}}\right) = \left(\frac{1}{1+e^{-t}}\right)\left(\frac{e^{-t}}{1+e^{-t}}\right) = \frac{e^{-t}}{(1+e^{-t})^2} = \frac{dP}{dt}$.

(b) As t tends to ∞, e^{-t} goes to 0. Thus $\lim_{t\to\infty} \frac{1}{1+e^{-t}} = 1$.

2. We know that there is an equilibrium solution at the carrying capacity $P = 100$ and that P increases toward 100 if $0 < P < 100$ and P decreases toward 100 if $P > 100$. Furthermore, we know that there is an inflection point at a height of $P = 100/2 = 50$ and that the graph is concave up for $0 < P < 50$ and is concave down for $50 < P < 100$. See Figure 11.43.

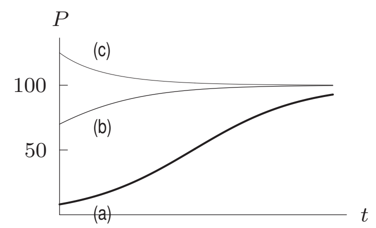


Figure 11.43

3. We know that there is an equilibrium solution at the carrying capacity $Q = 1/0.0004 = 2500$ and that Q increases toward 2500 if $0 < Q < 2500$ and Q decreases toward 2500 if $Q > 2500$. Furthermore, we know that there is an inflection point at a height of $Q = 2500/2 = 1250$ and that the graph is concave up for $0 < Q < 1250$ and is concave down for $1250 < Q < 2500$. See Figure 11.44.

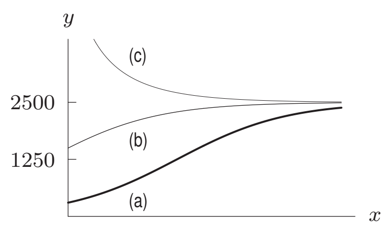


Figure 11.44

4. Since dP/dt is a quadratic function of P with negative leading coefficient, the graph of dP/dt against P is a parabola opening down. Since $dP/dt = 0$ when $P = 0$ and when $P = 250$, the horizontal intercepts of the parabola are at 0 and 250. See Figure 11.45. Since we don't know the value of k, we can't put a scale on the vertical axis.

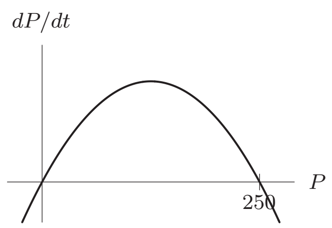


Figure 11.45

5. Since dA/dt is a quadratic function of A with negative leading coefficient, the graph of dA/dt against A is a parabola opening down. Since $dA/dt = 0$ when $A = 0$ and when $A = 1/0.0002 = 5000$, the horizontal intercepts of the parabola are at 0 and 5000. See Figure 11.46. Since we don't know the value of k, we can't put a scale on the vertical axis.

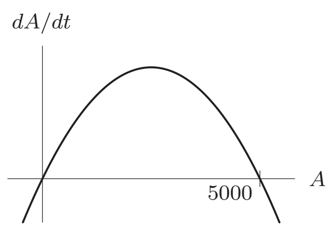


Figure 11.46

6. We see in Figure 11.69 that $dP/dt = 0$ when $P = 0$ and when $P = 45$. Thus, there are equilibrium solutions at $P = 0$ and at $P = 45$. If $0 < P < 45$, the derivative dP/dt is positive so P is an increasing function of t. If $P < 0$ or $P > 45$, the derivative dP/dt is negative so P is a decreasing function of t. See Figure 11.47.

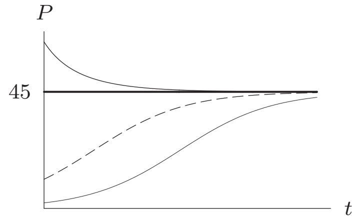


Figure 11.47

7. We see in Figure 11.70 that there appear to be equilibrium solutions at $Q = 0$ and $Q = 800$. Thus, the graph of dQ/dt against Q will have horizontal intercepts at $Q = 0$ and $Q = 800$. Since Q is growing logistically, the graph of dQ/dt against Q is a parabola opening down. See Figure 11.48.

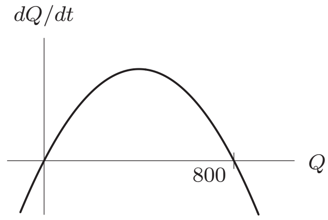


Figure 11.48

8. (a) See Figure 11.49.
(b) The value $P = 1$ is a stable equilibrium. (See part (d) below for a more detailed discussion.)
(c) Looking at the solution curves, we see that P is increasing for $0 < P < 1$ and decreasing for $P > 1$. The values of $P = 0$, $P = 1$ are equilibria. In the long run, P tends to 1, unless you start with $P = 0$. The solution curves with initial populations of less than $P = 1/2$ have inflection points at $P = 1/2$. (This will be demonstrated algebraically in part (d) below.) At the inflection point, the population is growing fastest.
(d) See Figure 11.50.

Since $dP/dt = 3P - 3P^2 = 3P(1 - P)$, the graph of dP/dt against P is a parabola, opening downward with P intercepts at 0 and 1. The quantity dP/dt is positive for $0 < P < 1$, negative for $P > 1$ (and $P < 0$). The quantity dP/dt is 0 at $P = 0$ and $P = 1$, and maximum at $P = 1/2$. The fact that $dP/dt = 0$ at $P = 0$ and $P = 1$ tells us that these are equilibria. Further, since $dP/dt > 0$ for $0 < P < 1$, solution curves starting here increase toward $P = 1$.

If the population starts at a value $P < 1/2$, it increases at an increasing rate up to $P = 1/2$. After this, P continues to increase, but at a decreasing rate. The fact that dP/dt has a maximum at $P = 1/2$ tells us that there is a point of inflection when $P = 1/2$. Similarly, since $dP/dt < 0$ for $P > 1$, solution curves starting with $P > 1$ will decrease to $P = 1$. Thus, $P = 1$ is a stable equilibrium.

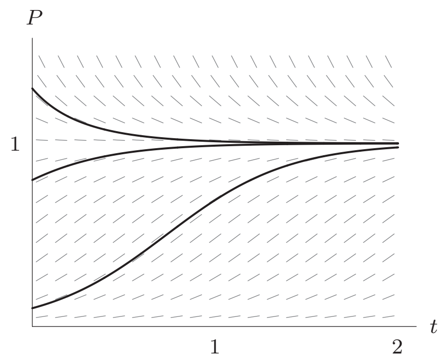


Figure 11.49

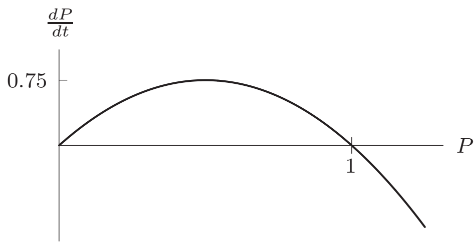


Figure 11.50

9. (a) Equilibrium values are values where $dP/dt = 0$, so the equilibrium values are at $P = 0$ and $P = 2000$.
(b) At $P = 500$, we see that dP/dt is positive so P is increasing.

10. (a) Equilibrium values are values where $dP/dt = 0$, so the equilibrium values are at $P = 0$ and $P = 400$.
(b) At $P = 500$, we see that dP/dt is negative so P is decreasing.

11. (a) We see that $k = 0.035$ which tells us that the quantity P grows by about 3.5% per unit time when P is very small relative to L. We also see that $L = 6000$ which tells us the upper limit on the value of P if P is initially below 6000.
(b) The largest rate of change occurs when $P = L/2 = 3000$.

12. (a) We factor out $0.1P$ on the right hand side to obtain $dP/dt = 0.1P(1 - 0.0008P)$. We see that $k = 0.1$ which tells us that the quantity P grows by about 10% per unit time when P is very small relative to L. We also see that $L = 1/0.0008 = 1250$ which tells us the upper limit on the value of P if P is initially below 1250.
(b) The largest rate of change occurs when $P = L/2 = 625$.

13. We see from the differential equation that $k = 0.05$ and $L = 2800$, so the general solution is

$$P = \frac{2800}{1 + Ae^{-0.05t}}.$$

14. We see from the differential equation that $k = 0.012$ and $L = 5700$, so the general solution is

$$P = \frac{5700}{1 + Ae^{-0.012t}}.$$

15. We see from the differential equation that $k = 0.68$ and $L = 1/0.00025 = 4000$, so the general solution is

$$P = \frac{4000}{1 + Ae^{-0.68t}}.$$

16. We factor out $0.2P$ to obtain $dP/dt = 0.2P(1 - 0.004P)$. We see that $k = 0.2$ and $L = 1/0.004 = 250$, so the general solution is

$$P = \frac{250}{1 + Ae^{-0.2t}}.$$

17. We rewrite

$$10P - 5P^2 = 10P\left(1 - \frac{P}{2}\right),$$

so $k = 10$ and $L = 2$. Since $P_0 = L/4$, we have $A = (L - P_0)/P_0 = 3$. Thus

$$P = \frac{2}{1 + 3e^{-10t}}.$$

The time to peak dP/dt is

$$t = \frac{1}{k}\ln A = \ln(3)/10.$$

18. We rewrite

$$0.02P - 0.0025P^2 = 0.02P\left(1 - \frac{P}{8}\right),$$

so $k = 0.02$ and $L = 8$. Since $P_0 = 1$, we have $A = (8 - 1)/1 = 7$. Thus

$$P = \frac{8}{1 + 7e^{-0.02t}}.$$

The time of peak dP/dt is

$$t = \frac{1}{k}\ln A = \frac{1}{0.02}\ln 7 = 50\ln 7.$$

19. We can immediately read off $k = 0.3$, $L = 100$. Since $P_0 = 75$, we have $A = (L - P_0)/P_0 = (100 - 75)/75 = 1/3$. Thus

$$P = \frac{100}{1 + (1/3)e^{-0.3t}}.$$

The time to peak dP/dt is

$$t = \frac{1}{k}\ln A = -\ln(3)/0.3.$$

Note that the time to peak dP/dt is negative since $P_0 > L/2$.

20. Rewriting the differential equation:

$$\frac{1}{P}\frac{dP}{dt} = 0.12 - 0.02P,$$

so $k = 0.12$ and $-k/L = -0.02$. Thus $L = 0.12/0.02 = 6$. Since $A = (6 - 2)/2 = 2$, we have

$$P = \frac{6}{1 + 2e^{-0.12t}}.$$

The time to peak dP/dt is

$$t = \frac{1}{k}\ln A = \frac{\ln 2}{0.12}.$$

21. We see from the differential equation that $k = 0.8$ and $L = 8500$, so the general solution is

$$P = \frac{8500}{1 + Ae^{-0.8t}}.$$

We substitute $t = 0$ and $P = 500$ to solve for the constant A:

$$\begin{aligned} 500 &= \frac{8500}{1 + Ae^{0}} \\ \frac{1}{500} &= \frac{1 + A}{8500} \\ 17 &= 1 + A \\ A &= 16. \end{aligned}$$

The solution to this initial value problem is

$$P = \frac{8500}{1 + 16e^{-0.8t}}.$$

22. We see from the differential equation that $k = 0.04$ and $L = 1/0.0001 = 10{,}000$, so the general solution is

$$P = \frac{10000}{1 + Ae^{-0.04t}}.$$

We substitute $t = 0$ and $P = 200$ to solve for the constant A:

$$\begin{aligned} 200 &= \frac{10000}{1 + Ae^{0}} \\ \frac{1}{200} &= \frac{1 + A}{10000} \\ 50 &= 1 + A \\ A &= 49. \end{aligned}$$

The solution to this initial value problem is

$$P = \frac{10{,}000}{1 + 49e^{-0.04t}}.$$

Problems

23. (a) Substituting the value $t = 0$ we get

$$N(0) = \frac{400}{1 + 399e^{-0.4(0)}} = \frac{400}{1 + 399(1)} = 1.$$

The fact that $N(0) = 1$ tells us that at the moment the rumor begins spreading, there is only one person who knows the content of the rumor.

(b) Substituting $t = 2$ we get

$$N(2) = \frac{400}{1 + 399e^{-0.4(2)}} = \frac{400}{1 + 399(0.449)} = 2.$$

Substituting in $t = 10$ we get

$$N(10) = \frac{400}{1 + 399e^{-0.4(10)}} = \frac{400}{1 + 7.308} = 48.$$

(c) The graph of $N(t)$ is shown in Figure 11.51.

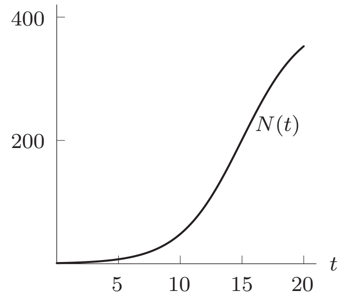


Figure 11.51

(d) We are asked to find the time t at which 200 people will have heard the rumor. We can use the formula in the text

$$t = \frac{1}{k}\ln A = \frac{1}{0.4}\ln 399 = 14.972.$$

Alternatively we can solve the equation

$$200 = \frac{400}{1 + 399e^{-0.4t}}.$$

We get

$$\begin{aligned} 1 + 399e^{-0.4t} &= \frac{400}{200} = 2 \\ 399e^{-0.4t} &= 1 \\ e^{-0.4t} &= \frac{1}{399} \\ \ln e^{-0.4t} &= \ln\frac{1}{399} \\ -0.4t &= \ln(1/399) \\ t &= \frac{\ln(1/399)}{-0.4} = 14.972. \end{aligned}$$

Thus, after about 15 hours half the people have heard the rumor.

We are asked to solve for the time at which 399 people (that is, virtually everyone) will have heard the rumor. We solve

$$\begin{aligned} 399 = N(t) &= \frac{400}{1 + 399e^{-0.4t}} \\ 1 + 399e^{-0.4t} &= \frac{400}{399} \\ 399e^{-0.4t} &= \frac{400}{399} - 1 \\ e^{-0.4t} &= \frac{1}{399}\left(\frac{400}{399} - 1\right) = \frac{400/399 - 1}{399} \\ \ln e^{-0.4t} &= \ln\frac{400/399 - 1}{399} \\ -0.4t &= \ln\frac{400/399 - 1}{399} \\ t &= \frac{\ln((400/399 - 1)/399)}{-0.4} = 29.945. \end{aligned}$$

Thus, after approximately 30 hours 399 people (virtually everyone) will have heard the rumor.

(e) The rumor is spreading fastest at $L/2 = 400/2 = 200$ or when 200 people have already heard the rumor, so after about 15 hours.

24. (a) At $t = 0$, which corresponds to 1935, we have

$$P = \frac{1}{1 + 2.968e^{-0.0275(0)}} = 0.252$$

showing that about 25% of the land was in use in 1935.

(b) This model predicts that as t gets very large, P will approach 1. That is, the model predicts that in the long run, all the land will be used for farming.

(c) The time when half the land is in use is the time at which dP/dt is greatest. This occurs when

$$t = \frac{1}{k} \ln A = \frac{1}{0.0275} \ln(2.968) = 39.6 \text{ years.}$$

Since $t = 0$ corresponds to 1935, $t = 39.6$ corresponds to $1935 + 39.6 = 1974.6$. According to this model, the Tojolobal were using half their land in 1974.

(d) The inflection point occurs when $P = L/2$ or at one-half the carrying capacity. In this case, $P = \frac{1}{2}$ in 1974, as shown in part (c).

25. (a) The equilibrium population occurs when dP/dt is zero. Solving

$$\frac{dP}{dt} = 1 - 0.0004P = 0$$

gives $P = 2500$ fish as the equilibrium population.

(b) The solution of the differential equation is

$$P(t) = \frac{2500}{1 + Ae^{-0.25t}}$$

subject to $P(-10) = 1000$ if $t = 0$ represents the present time. So we have

$$1000 = \frac{2500}{(1 + Ae^{2.5})}$$

from which $A = 0.123127$ and

$$P(0) = \frac{2500}{(1 + 0.123127)} \approx 2230.$$

Therefore, the current population is approximately 2230 fish.

(c) The effect of losing 10% of the fish each year gives the revised differential equation

$$\frac{dP}{dt} = (0.25 - 0.0001P)P - 0.1P$$

or

$$\frac{dP}{dt} = (0.15 - 0.0001P)P.$$

The revised equilibrium population occurs where $dP/dt = 0$, or about 1500 fish.

26. (a) The maximum rate of change occurs at approximately $t = 50$ and the rate of change at that point is approximately $(270 - 195)/(50 - 40) = 7.5$.

(b) The maximum rate of change occurs at the point where $P = 270$ so we estimate that the carrying capacity is approximately $270 \cdot 2 = 540$.

27. (a) The limiting value of $f(t)$ is about 36, so the total number of infected computers is about 36,000.

(b) The curve has an inflection point at about $t \approx 16$ hours, and then $n = f(16) \approx 18$.

(c) The virus was spreading fastest at about when $t = 16$, that is at 4 pm on July 19, 2001. At that time, about 18,000 computers were infected.

(d) At the inflection point, the number of computers infected is about half the number infected in the long run.

28. Let r be the relative growth rate, that is $r = (1/N)dN/\,dt$. Since r is a linear function of N, its graph contains the points $(N, r) = (5, 15\%)$ and $(N, r) = (10, 14.5\%)$, so its slope is

$$m = \frac{\Delta r}{\Delta N} = \frac{0.145 - 0.15}{10 - 5} = \frac{-0.005}{5} = -0.001.$$

Using the point $(5, 0.15)$, we have

$$r = 0.15 - 0.001(N - 5)$$
$$r = 0.155 - 0.001N.$$

This gives us the differential equation

$$\frac{1}{N}\frac{dN}{dt} = 0.155 - 0.001N$$
$$\frac{dN}{dt} = N\left(0.155 - 0.001N\right).$$

We see that $dN/\,dt = 0$ where $N = 0$ or where

$$0.155 - 0.001N = 0$$
$$0.001N = 0.155$$
$$N = 155.$$

Our model predicts the spread of pigweed will halt when 155 million acres are afflicted.

29. (a) Using the approximation

$$\frac{dP}{dt} \approx \frac{P(t+1) - P(t)}{1},$$

we see that dP/dt and annual production are approximately equal.

For 1993 we have $dP/dt = 22.0$ billion barrels per year.

For 2008 we have $dP/dt = 26.9$ billion barrels per year.

(b) For 1993 we have

$$\frac{1}{P}\frac{dP}{dt} = \frac{22.0}{724} = 0.0304/\text{year} = 3.04\% \text{ per year}.$$

For 2008 we have

$$\frac{1}{P}\frac{dP}{dt} = \frac{26.9}{1100} = 0.0245/\text{year} = 2.45\% \text{ per year}.$$

(c) We are finding the equation of the line that contains the two points $(724, 0.0304)$ and $(1100, 0.0245)$. The slope between these points is

$$\text{Slope } = \frac{0.0245 - 0.0304}{1100 - 724} = -0.0000157.$$

Thus, the fitted line has an equation of the form

$$\frac{1}{P}\frac{dP}{dt} = k - 0.0000157P.$$

Using the point $P = 724,\ (1/P)dP/dt = 0.0304$, we solve for the vertical intercept k:

$$0.0304 = k - 0.0000157(724)$$
$$k = 0.0304 + 0.0000157(724) = 0.0418.$$

Thus, the equation of the line is

$$\frac{1}{P}\frac{dP}{dt} = -0.0000157P + 0.0418.$$

(d) The total quantity, L. of world oil reserves in 1859 is the value of P making $dP/dt = 0$, so it is the horizontal intercept of the line we computed in (c). We solve

$$0 = -0.0000157P + 0.0418$$

for P to obtain $P = 2662$. This model estimates the total world oil reserves in 1859 to be $L = 2662$ billion barrels.

(e) Using $t = 0$ for 1993, we have the differential equation

$$\frac{dP}{dt} = 0.0418P\left(1 - \frac{P}{2662}\right).$$

The solution is

$$P = \frac{2662}{1 + Ae^{-0.0418t}},$$

where $A = (2662 - 724)/724 = 2.677$. So, in billions of barrels, we have

$$P = \frac{2662}{1 + 2.677e^{-0.0418t}}.$$

30. (a) Using the formula on page 632 of the text for the time which gives the maximum value of dP/dt we have

$$t = \frac{1}{k}\ln A = \frac{1}{0.0418}\ln 2.677 = 23.6 \text{ years}.$$

So peak worldwide oil production is projected to occur about the year $1993 + 24 = 2017$.

(b) Using the formula on page 632 of the text for the maximum value of $(1/P)dP/dt$ with $k = 0.0418$ and $L = 3500$ we have

$$t = \frac{1}{k}\ln\frac{L - P_0}{P_0} = \frac{1}{0.0418}\ln\frac{3500 - 724}{724} = 32.2 \text{ years}.$$

So peak worldwide oil production is projected to occur about the year $1993 + 32 = 2025$.

31. (a) The differential equation is

$$\frac{dP}{dt} = 0.0418P\left(1 - \frac{P}{2662}\right).$$

The predicted values of dP/dt for 1998 and 2003 are

$$\left.\frac{dP}{dt}\right|_{1998} = 0.0418 \cdot 841\left(1 - \frac{841}{2662}\right) = 24.0 \text{ billion barrels per year}$$

$$\left.\frac{dP}{dt}\right|_{2003} = 0.0418 \cdot 964\left(1 - \frac{964}{2662}\right) = 25.7 \text{ billion barrels per year.}$$

Comparison with the actual annual production values, 24.4 bn barrel in 1998 and 25.3 bn barrel in 2003, we see the model fits the data well. In each case, the prediction is about $0.4/25 \approx 0.016 \approx 2\%$ from the the actual values. See Figure 11.52.

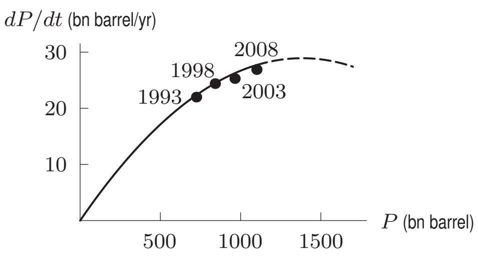


Figure 11.52

(b) The point $(0, 724)$ lies exactly on the curve, since the data for 1993 was used to find the solution to the differential equation. The predicted values are

$$\text{In 1998, } P = \frac{2662}{1 + 2.677e^{-0.0418 \cdot 5}} = 839.2 \text{ billion barrels}$$

$$\text{In 2003, } P = \frac{2662}{1 + 2.677e^{-0.0418 \cdot 10}} = 963.6 \text{ billion barrels}$$

$$\text{In 2008, } P = \frac{2662}{1 + 2.677e^{-0.0418 \cdot 15}} = 1095.5 \text{ billion barrels.}$$

These values very close to the actual values of 841, 940, 1100 billion barrels—less than $1/2\%$ difference. See Figure 11.53.

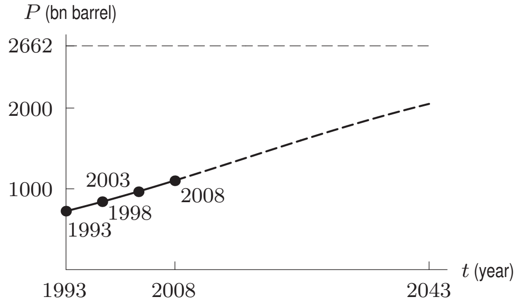


Figure 11.53

32. (a) Since $t = 0$ is in 1993, for 2010, we substitute $t = 17$ into the logistic function

$$P = \frac{2662}{1 + 2.677e^{-0.0418t}} \quad \text{so} \quad P = \frac{2662}{1 + 2.677e^{-0.0418 \cdot 17}} = 1149.722 \text{ bn barrels.}$$

(b) To estimate the quantity of oil produced during 2010, we can use a difference quotient or a derivative. We can find $P = 1177.098$ billion barrels when $t = 18$ (in 2011) and use the approximation

$$\left.\frac{dP}{dt}\right|_{2010} \approx \left.\frac{P(t+1) - P(t)}{1}\right|_{t=17} = \frac{P(18) - P(17)}{1} = \frac{1177.098 - 1149.722}{1} = 27.376 \text{ bn barrels.}$$

The derivative gives a similar value:

$$\left.\frac{dP}{dt}\right|_{t=17} = 27.302 \text{ bn barrels.}$$

(c) Since the original reserves were estimated to be $L = 2662$ bn barrels, the oil projected to remain is $2662 - 1149.722 = 1512.278$ billion barrels.

(d) The difference between the projection and the actual value is $27.376 - 26.9 = 0.476$ billion barrels, or about half a billion in 2010.

More precisely, the estimate is $0.476/26.9 = 0.018 \approx 2\%$ too high—quite close.

33. (a) Since $t = 0$ is in 1993, for 2020, we substitute $t = 27$ into the logistic function

$$P = \frac{2662}{1 + 2.677e^{-0.0418t}} \quad \text{so} \quad P = \frac{2662}{1 + 2.677e^{-0.0418 \cdot 27}} = 1426.603 \text{ bn barrels.}$$

(b) When 300 billion barrels of oil remain, then $2662 - 300 = 2362$ billion barrels of oil have been produced. We are solving $P(t) = 2362$ for t:

$$\begin{aligned}
\frac{2662}{1 + 2.677e^{-0.0418t}} &= 2362 \\
2362 + 2362(2.677e^{-0.0418t}) &= 2662 \\
2362(2.677e^{-0.0418t}) &= 300 \\
e^{-0.0418t} &= 0.0474 \\
-0.0418t &= \ln 0.0474 \\
t &= 72.9 \text{ years.}
\end{aligned}$$

So, in about $1993 + 73 = 2066$, only 300 billion barrels of oil are projected to remain in the ground.

34. (a) We know that a logistic curve can be modeled by the function

$$P = \frac{L}{1 + Ae^{-kt}}$$

where $A = (L - P_0)/(P_0)$ and P is the number of people infected by the virus at a particular time t. We know that L is the limiting value, or the maximal number of people infected with the virus, so in our case

$$L = 5000.$$

We are also told that initially there are only ten people infected with the virus so that we get

$$P_0 = 10.$$

Thus we have

$$\begin{aligned}
A &= \frac{L - P_0}{P_0} \\
&= \frac{5000 - 10}{10} \\
&= 499.
\end{aligned}$$

We are also told that in the early stages of the virus, infection grows exponentially with $k = 1.78$. Thus we get that the logistic function for people infected is

$$P = \frac{5000}{1 + 499e^{-1.78t}}.$$

(b) See Figure 11.54.

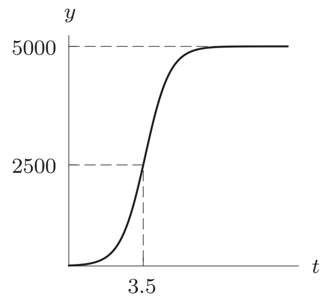


Figure 11.54

(c) Looking at the graph we see that the the point at which the rate changes from increasing to decreasing, the inflection point, occurs at roughly $t = 3.5$ giving a value of $P = 2500$. Thus after roughly 2500 people have been infected, the rate of infection starts dropping. See above.

35. (a) Let I be the number of informed people at time t, and I_0 the number who know initially. Then this model predicts that $dI/dt = k(M - I)$ for some positive constant k. Solving this, we find the solution is

$$I = M - (M - I_0)e^{-kt}.$$

We sketch the solution with $I_0 = 0$. Notice that dI/dt is largest when I is smallest, so the information spreads fastest in the beginning, at $t = 0$. In addition, Figure 11.55 shows that $I \to M$ as $t \to \infty$, meaning that everyone gets the information eventually.

(b) In this case, the model suggests that $dI/dt = kI(M - I)$ for some positive constant k. This is a logistic model with carrying capacity M. We sketch the solutions for three different values of I_0 in Figure 11.56.

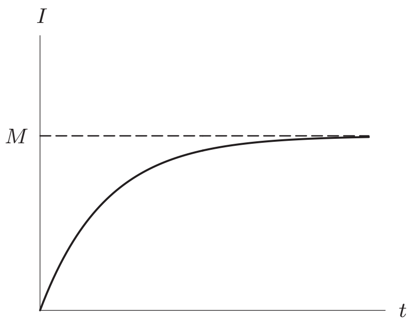


Figure 11.55

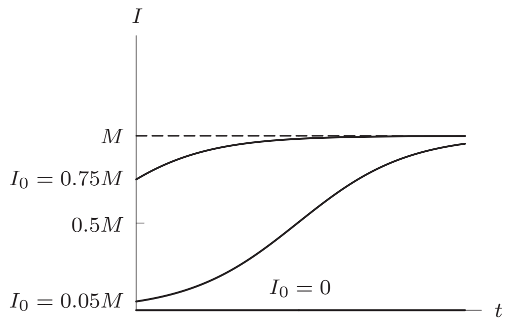


Figure 11.56

(i) If $I_0 = 0$ then $I = 0$ for all t. In other words, if nobody knows something, it does not spread by word of mouth!
(ii) If $I_0 = 0.05M$, then dI/dt is increasing up to $I = M/2$. Thus, the information is spreading fastest at $I = M/2$.
(iii) If $I_0 = 0.75M$, then dI/dt is always decreasing for $I > M/2$, so dI/dt is largest when $t = 0$.

36. (a) Figure 11.57 shows that the yeast population seems to stabilize at about 13, so we take this to be the limiting value, L.

(b) Solving $\frac{dP}{dt} = kP(1 - \frac{P}{L})$ for k, we get:

$$k = \frac{dP/dt}{P \cdot (1 - P/L)}.$$

We now find dP/dt from the first two data points.

$$\frac{dP}{dt} \approx \frac{\Delta P}{\Delta t} = \frac{P(10) - P(0)}{10 - 0} = \frac{8.87 - 0.37}{10} = 0.85.$$

Putting in our values for dP/dt, L, and $P(10)$, we get:

$$k \approx \frac{dP/dt}{P(10) \cdot (1 - P(10)/L)} = \frac{0.85}{(8.87)(1 - 8.87/13)} = 0.30.$$

(c) For $k = 0.3$ and $L = 13$,

$$A = \frac{L - P_0}{P_0} = \frac{13 - 0.37}{0.37} = 34.1.$$

Putting this into the equation for P we get:

$$P = \frac{13}{1 + Ae^{-kt}} = \frac{13}{1 + 34.1e^{-0.3t}},$$

which is plotted in Figure 11.58.

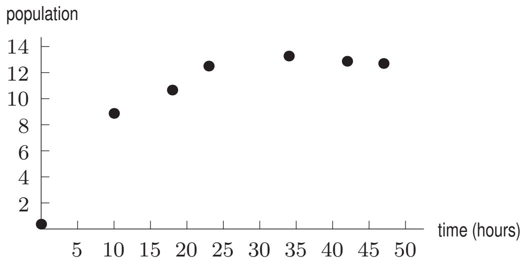


Figure 11.57

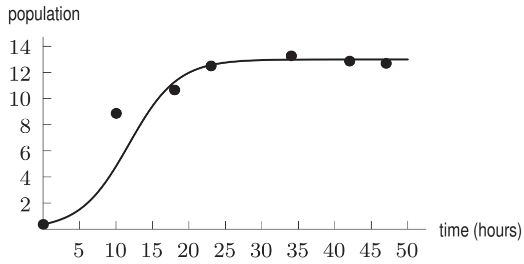


Figure 11.58: $P = 13/\left(1 + 34.1e^{-0.3t}\right)$

37. (a) The population seems to level off around 5.8, which leads us to believe that the population is growing logistically. If it were growing exponentially, we would expect the rate of increase to continue increasing with time.

(b) To find k, we solve the logistic equation, $dP/dt = kP(1 - P/L)$, for k:

$$k = \frac{dP/dt}{P \cdot (1 - P/L)}.$$

We now need to estimate dP/dt and L from the data. The population seems to level off at 5.8, so we take this as as the carrying capacity, L. We use the first two data points to find dP/dt:

$$\frac{dP}{dt} \approx \frac{\Delta P}{\Delta t} = \frac{P(13) - P(0)}{13 - 0} = \frac{1.7 - 1.00}{13} = 0.054$$

Putting in our values for dP/dt, L, and $P(13)$, we get:

$$k \approx \frac{dP/dt}{P(13) \cdot (1 - P(13)/L)} = \frac{0.054}{(1.7)(1 - 1.7/5.8)} = 0.045$$

(c) The solution curve for $dP/dt = kP(1 - P/L)$ with $k = 0.045$, and $L = 5.8$ has equation

$$P = \frac{5.8}{(1 + Ae^{-kt})} \quad \text{where} \quad A = \frac{L - P_0}{P_0}.$$

Since $P_0 = 1$ then $A = \frac{5.8-1}{1} = 4.8$, we have

$$P = \frac{5.8}{1 + 4.8e^{-0.045t}}$$

The data and the curve are sketched in Figure 11.59.

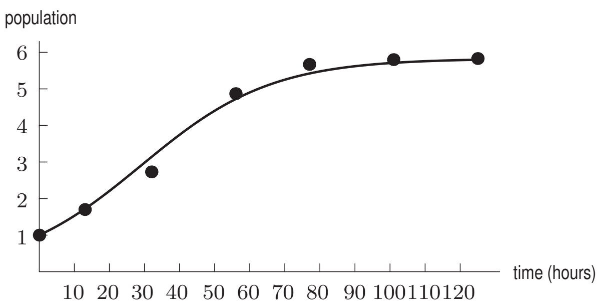


Figure 11.59: $P = 5.8/\left(1 + 4.8e^{-0.045t}\right)$

38. Let I be the number of infected people. Then, the number of healthy people in the population is $M - I$. The rate of infection is

$$\text{Infection rate} = \frac{0.01}{M}(M-I)I.$$

and the rate of recovery is

$$\text{Recovery rate} = 0.009I.$$

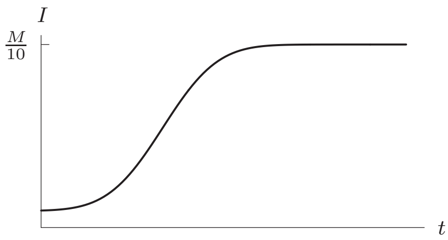
Therefore,

$$\frac{dI}{dt} = \frac{0.01}{M}(M-I)I - 0.009I$$

or

$$\frac{dI}{dt} = 0.001I(1 - 10\frac{I}{M}).$$

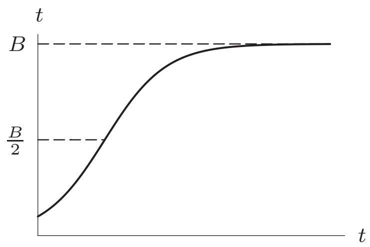
This is a logistic differential equation, and so the solution will look like the following graph:



The limiting value for I is $\frac{1}{10}M$, so $1/10$ of the population is infected in the long run.

39. (a) $\frac{dp}{dt} = kp(B-p)$, where $k > 0$.

(b) To find when $\frac{dp}{dt}$ is largest, we notice that $\frac{dp}{dt} = kp(B-p)$, as a function of p, is a parabola opening downward with the maximum at $p = \frac{B}{2}$, i.e. when $\frac{1}{2}$ the tin has turned to powder. This is the time when the tin is crumbling fastest.



(c) If $p = 0$ initially, then $\frac{dp}{dt} = 0$, so we would expect p to remain 0 forever. However, since many organ pipes get tin pest, we must reconcile the model with reality. There are two possible ideas which solve this problem. First, we could assume that p is never 0. In other words, we assume that all tin pipes, no matter how new, must contain some small amount of tin pest. Assuming this means that all organ pipes must deteriorate due to tin pest eventually. Another explanation is that the powder forms at a slow rate even if there was none present to begin with. Since not all organ pipes suffer, it is possible that the conversion is catalyzed by some other impurities not present in all pipes.

40. (a) At equilibrium $dP/dt = 0$, so

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) - cP = 0,$$

Thus if $P \neq 0$,

$$kL - kP - cL = 0$$
$$P = \frac{k-c}{k}L.$$

At equilibrium, the annual harvest is

$$H = cP = \frac{c(k-c)}{k}L.$$

(b) Since k and L are constant and the annual harvest is $H = c(k-c)L/k$, at the maximum where $dH/dc = 0$, we have

$$\frac{dH}{dc} = \frac{(k-2c)}{k}L = 0$$
$$c = \frac{k}{2}.$$

The maximum value of H is then

$$H = \left(\frac{k}{2}\right)\frac{(k-k/2)}{k}L = \frac{k}{4}L.$$

(c) The equilibrium population is represented by the P-value (horizontal coordinate) at the point of intersection of the line and parabola. The annual harvest at the equilibrium, cP, is represented by the vertical coordinate at the point of intersection.

As c increases toward k, the slope of the line $dP/dt = cP$ gets steeper and the intersection point between the line and parabola moves closer to the origin. For $c > k/2$, the equilibrium population, $P = (k-c)L/k$, and the annual harvest, $H = c(k-c)L/k$, get smaller and smaller. At $c = k$, the population becomes extinct.

41. The population dies out if H is large enough that $dP/dt < 0$ for all P. The largest value for $dP/dt = kP(1 - P/L)$ occurs when $P = L/2$; then

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) = k\frac{L}{2}\left(1 - \frac{L/2}{L}\right) = \frac{kL}{4}.$$

Thus if $H > kL/4$, we have $dP/dt < 0$ for all P and the population dies out if the quota is met.

Strengthen Your Understanding

42. Since

$$\frac{dP}{dt} = 0.08P - 0.0032P^2 = 0.08P\left(1 - \frac{P}{25}\right),$$

we see that when we set dP/dt equal to zero, there are two solutions: one at $P = 25$ and one at $P = 0$. There are two equilibrium solutions, with the second one at $P = 0$.

43. The maximum rate of change occurs at $Q = 25$ not at $t = 25$.

44. The curve has the shape we expect of a logistic curve. However, since the graph is leveling off at a carrying capacity of 100, we expect the inflection point to be at a height of 50, but in the graph the inflection point is above 60.

45. Any quantity that might increase exponentially at first and then eventually level off at a maximum value is a reasonable answer: for example, the growth of a population in a confined space or sales of a new product with a saturation level.

46. If the maximum rate of change of a logistic function occurs at $P = 75$, then the carrying capacity is $2 \cdot 75 = 150$. An answer is in the form

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{150}\right),$$

where k is any positive constant.

47. Since Q is growing logistically, the graph of dQ/dt against Q is a parabola opening down which passes through the origin. Since Q has an equilibrium value at $Q = 500$, the other horizontal intercept is at $Q = 500$. See Figure 11.60.

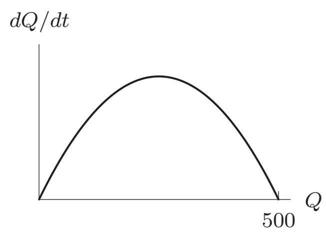


Figure 11.60

48. Since P is growing logistically, the graph of dP/dt against P is a parabola opening down which passes through the origin. Since P increases when $0 < P < 20$, we know that dP/dt is positive when $0 < P < 20$. Since P decreases when $P < 0$ or $P > 20$, we know that dP/dt is negative for those values of P. See Figure 11.61.

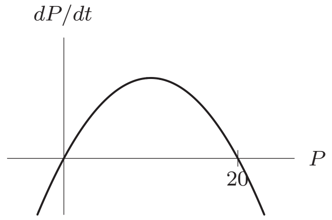


Figure 11.61

49. False. This is a logistic equation with equilibrium values $P = 0$ and $P = 2$. Solution curves do not cross the line $P = 2$ and do not go from $(0, 1)$ to $(1, 3)$.

50. True. This is a logistic differential equation. Any solution with $P(0) > 0$ tends toward the carrying capacity, L, as $t \to \infty$.

Solutions for Section 11.8

Exercises

1. Here x and y both increase at about the same rate.

2. Initially $x = 0$, so we start with only y. Then y decreases while x increases. Then x continues to increase while y starts to increase as well. Finally y continues to increase while x decreases.

3. x decreases quickly while y increases more slowly.

4. The closed trajectory represents populations which oscillate repeatedly.

5. We set each derivative equal to zero and solve:

$$\begin{aligned}\frac{dx}{dt} &= 0\\ -3x + xy &= 0\\ x(-3 + y) &= 0\\ x &= 0 \text{ or } y = 3.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dt} &= 0\\ 5y - xy &= 0\\ y(5 - x) &= 0\\ y &= 0 \text{ or } x = 5.\end{aligned}$$

Since both derivatives must be zero at an equilibrium point, the equilibrium points are ordered pairs for which $x = 0$ or $y = 3$ *and* $y = 0$ or $x = 5$. The equilibrium points are $(0, 0)$ and $(5, 3)$.

6. We set each derivative equal to zero and solve:

$$\begin{aligned}\frac{dx}{dt} &= 0\\ -2x + 4xy &= 0\\ x(-2 + 4y) &= 0\\ x &= 0 \text{ or } y = 0.5.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dt} &= 0\\ -8y + 2xy &= 0\\ y(-8+2x) &= 0\\ y = 0 &\text{ or } x = 4.\end{aligned}$$

Since both derivatives must be zero at an equilibrium point, the equilibrium points are ordered pairs for which $x = 0$ or $y = 0.5$ *and* $y = 0$ or $x = 4$. The equilibrium points are $(0, 0)$ and $(4, 0.5)$.

7. We set each derivative equal to zero and solve:

$$\begin{aligned}\frac{dx}{dt} &= 0\\ 15x - 5xy &= 0\\ x(15-5y) &= 0\\ x = 0 &\text{ or } y = 3.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dt} &= 0\\ 10y + 2xy &= 0\\ y(10+2x) &= 0\\ y = 0 &\text{ or } x = -5.\end{aligned}$$

Since both derivatives must be zero at an equilibrium point, the equilibrium points are ordered pairs for which $x = 0$ or $y = 3$ *and* $y = 0$ or $x = -5$. The equilibrium points are $(0, 0)$ and $(-5, 3)$. (While a negative value might not make sense if x represents the size of a population, it is reasonable in other scenarios in which systems of differential equations are relevant such as if x represents the net worth of a company.)

8. We set each derivative equal to zero and solve:

$$\begin{aligned}\frac{dx}{dt} &= 0\\ x^2 - xy &= 0\\ x(x-y) &= 0\\ x = 0 &\text{ or } y = x.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dt} &= 0\\ 15y - 3y^2 &= 0\\ y(15-3y) &= 0\\ y = 0 &\text{ or } y = 5.\end{aligned}$$

Since both derivatives must be zero at an equilibrium point, the equilibrium points are ordered pairs for which $x = 0$ or $y = x$ *and* $y = 0$ or $y = 5$. The equilibrium points are $(0, 0)$ and $(0, 5)$ and $(5, 5)$.

9. (a) At the point $x = 3$ and $y = 2$, we have

$$\begin{aligned}\frac{dx}{dt} &= 5(3) - 3(3)(2) = -3 < 0\\ \frac{dy}{dt} &= -8(2) + (3)(2) = -10 < 0,\end{aligned}$$

so both x and y are decreasing.

(b) At the point $x = 5$ and $y = 1$, we have

$$\begin{aligned}\frac{dx}{dt} &= 5(5) - 3(5)(1) = 10 > 0\\ \frac{dy}{dt} &= -8(1) + (5)(1) = -3 < 0,\end{aligned}$$

so x is increasing and y is decreasing.

10. (a) At the point $P = 2$ and $Q = 3$, we have

$$\frac{dP}{dt} = 2(2) - 10 = -6 < 0$$
$$\frac{dQ}{dt} = 3 - 0.2(2)(3) = 1.8 > 0,$$

so P is decreasing and Q is increasing.

(b) At the point $P = 6$ and $Q = 5$, we have

$$\frac{dP}{dt} = 2(6) - 10 = 2 > 0$$
$$\frac{dQ}{dt} = 5 - 0.2(6)(5) = -1 < 0,$$

so P is increasing and Q is decreasing.

Problems

11. (a) The values are given by

Time t (days)	8	12	16	20	22	24	28	32	36	40	44
Susceptibles	1950	1850	1550	1000	750	550	350	250	200	200	200
Infecteds	20	80	240	460	500	460	320	180	100	40	20

(b) The peak of the epidemic is on day 22 when about 500 people are infected.

(c) About 2000 people are susceptible at the onset of the epidemic and only 200 are still susceptible at its end. About 1800 catch the disease and about 200 are spared.

12. (a) On day 20 about 750 are infected.

(b) On day 20 about 2600 are susceptible. Since there are 4000 people in the population, there are $4000 - 2600 = 1400$ who have already had the disease.

(c) At day 60 the epidemic is over. There are about 400 people who are still susceptible, because they never got the disease. The rest, $4000 - 400 = 3600$ caught the disease sometime during the epidemic.

13. (a) The human population shrinks as humans are turned into zombies, so parameter a is negative. The interaction term for zombies is positive, since an interaction between a human and a zombie increases the zombie population, so c is positive. The zombie population shrinks by a certain percentage each time interval because some zombies will starve to death, so b is negative.

(b) The terms aHZ and cHZ both indicate the rate of human to zombie conversions. Since each loss of one human is directly a gain of one zombie, $aHZ = -cHZ$. Therefore, the parameter a is exactly the negative of the parameter c. Thus, $a = -c$.

14. (a) This is an example of a predator-prey relationship, so this system goes with IV, with x representing the fox population and y representing the hare population.

(b) This system models the relationship in II, with x representing the tree population and y representing the owl population.

(c) This system models the relationship in I with x and y being either bees or flowers.

(d) We write a system of differential equations for III with elk and buffalo. In each case, the species would do fine on its own but is hurt by the other. The terms have signs as follows, with proportionality constants included as desired.

$$\frac{dx}{dt} = x - xy$$
$$\frac{dy}{dt} = y - xy$$

Notice that x and y can each represent either the elk or the buffalo population.

15. Since

$$\frac{dS}{dt} = -aSI,$$
$$\frac{dI}{dt} = aSI - bI,$$
$$\frac{dR}{dt} = bI$$

we have

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -aSI + aSI - bI + bI = 0.$$

Thus $\frac{d}{dt}(S + I + R) = 0$, so $S + I + R =$ constant.

16. This is an example of a predator-prey relationship. Normally, we would expect the worm population, in the absence of predators, to increase without bound. As the number of worms w increases, so would the rate of increase dw/dt; in other words, the relation $dw/dt = w$ might be a reasonable model for the worm population in the absence of predators.

However, since there are predators (robins), dw/dt won't be that big. We must lessen dw/dt. It makes sense that the more interaction there is between robins and worms, the more slowly the worms are able to increase their numbers. Hence we lessen dw/dt by the amount wr to get $dw/dt = w - wr$. The term $-wr$ reflects the fact that more interactions between the species means slower reproduction for the worms.

Similarly, we would expect the robin population to decrease in the absence of worms. We'd expect the population decrease at a rate related to the current population, making $dr/dt = -r$ a reasonable model for the robin population in absence of worms. The negative term reflects the fact that the greater the population of robins, the more quickly they are dying off. The wr term in $dr/dt = -r + wr$ reflects the fact that the more interactions between robins and worms, the greater the tendency for the robins to increase in population.

17. If there are no worms, then $w = 0$, and $\frac{dr}{dt} = -r$ giving $r = r_0e^{-t}$, where r_0 is the initial robin population. If there are no robins, then $r = 0$, and $\frac{dw}{dt} = w$ giving $w = w_0e^{t}$, where w_0 is the initial worm population.

18. There is symmetry across the line $r = w$. Indeed, since $\frac{dr}{dw} = \frac{r(w-1)}{w(1-r)}$, if we switch w and r we get $\frac{dw}{dr} = \frac{w(r-1)}{r(1-w)}$, so $\frac{dr}{dw} = \frac{r(1-w)}{w(r-1)}$. Since switching w and r changes nothing, the slope field must be symmetric across the line $r = w$. The slope field shows that the solution curves are either spirals or closed curves. Since there is symmetry about the line $r = w$, the solutions must in fact be closed curves.

19. If $w = 2$ and $r = 2$, then $\frac{dw}{dt} = -2$ and $\frac{dr}{dt} = 2$, so initially the number of worms decreases and the number of robins increases. In the long run, however, the populations will oscillate; they will even go back to $w = 2$ and $r = 2$. See Figure 11.62.

Figure 11.62

20. Sketching the trajectory through the point $(2, 2)$ on the slope field given shows that the maximum robin population is about 2500, and the minimum robin population is about 500. When the robin population is at its maximum, the worm population is about 1,000,000.

21.

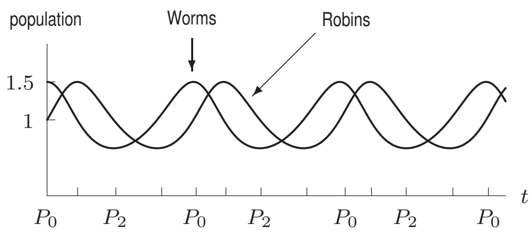


Figure 11.63

22. It will work somewhat; the maximum number the robins reach will increase. However, the minimum number the robins reach will decrease as well. (See graph of slope field.) In the long term, the robin-worm populations will again fall into a cycle. Notice, however, if the extra robins are added during the part of the cycle where there are the fewest robins, the new cycle will have smaller variation. See Figure 11.64.

Note that if too many robins are added, the minimum number may get so small the model may fail, since a small number of robins are more susceptible to disaster.

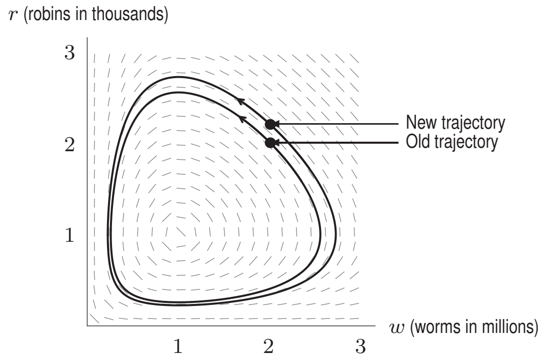


Figure 11.64

23. The numbers of robins begins to increase while the number of worms remains approximately constant. See Figure 11.65.

The numbers of robins and worms oscillate periodically between 0.2 and 3, with the robin population lagging behind the worm population.

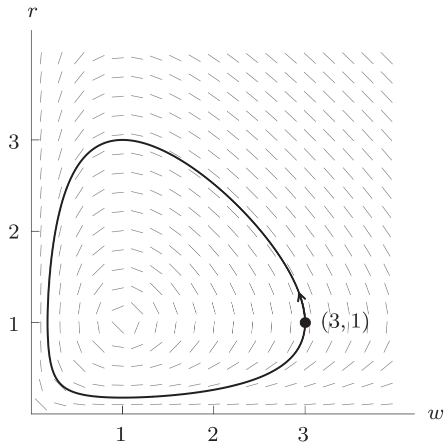


Figure 11.65

24. Estimating from the phase plane, we have

$$0.18 < r < 3$$

so the robin population lies between 180 and 3000. Similarly

$$0.2 < w < 3,$$

so the worm population lies between 200,000 and 3,000,000.

When the robin population is at its minimum $r \approx 0.2$, then $w \approx 0.87$, so that there are approximately 870,000 worms.

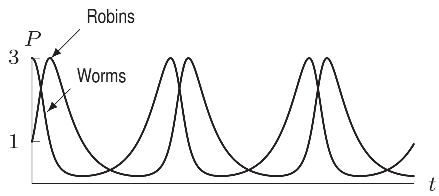
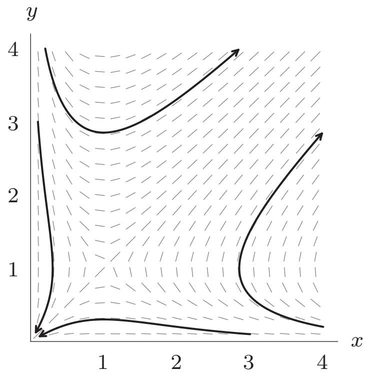


Figure 11.66

25. (a) Symbiosis, because both populations decrease while alone but are helped by the presence of the other.
(b)



Both populations tend to infinity or both tend to zero.

26. (a) Competition, because both populations grow logistically when alone, but are harmed by the presence of the other.
(b) See Figure 11.67. In the long run, $x \to 2$, $y \to 0$. In other words, y becomes extinct.

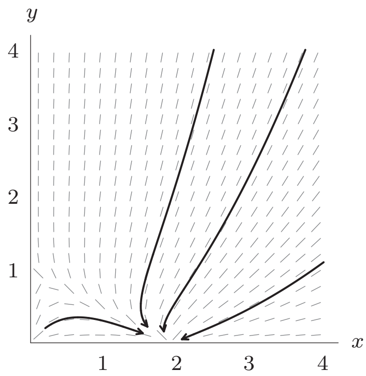


Figure 11.67

27. (a) Predator-prey, because x decreases while alone, but is helped by y, whereas y increases logistically when alone, and is harmed by x. Thus x is predator, y is prey.
(b) See Figure 11.68. Provided neither initial population is zero, both populations tend to about 1. If x is initially zero, but y is not, then $y \to \infty$. If y is initially zero, but x is not, then $x \to 0$.

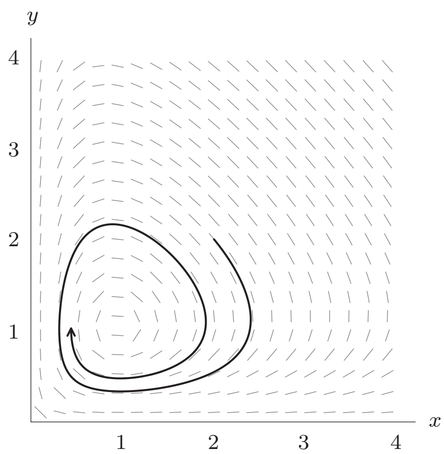
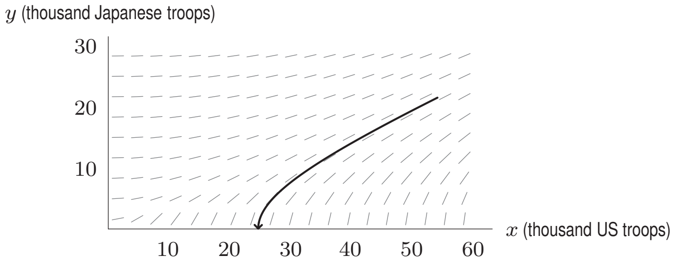


Figure 11.68

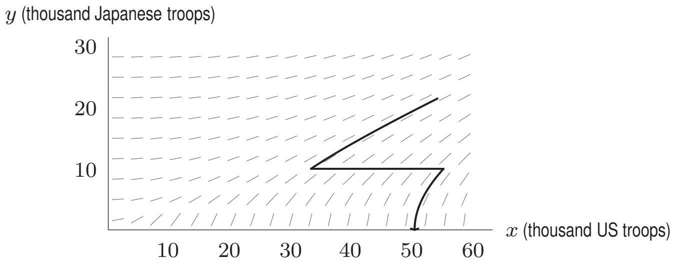
28. (a) Thinking of y as a function of x and x as a function of t, then by the chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-0.01x}{-0.05y} = \frac{x}{5y}$$



(b) The figure above shows the slope field for this differential equation and the trajectory starting at $x_0 = 54$, $y_0 = 21.5$. The trajectory goes to the x-axis, where $y = 0$, meaning that the Japanese troops were all killed or wounded before the US troops were, and thus predicts the US victory (which did occur). Since the trajectory meets the x-axis at $x \approx 25$, the differential equation predicts that about 25,000 US troops would survive the battle.

(c) The fact that the US got reinforcements, while the Japanese did not, does not alter the predicted outcome (a US victory). The US reinforcements have the effect of changing the trajectory, altering the number of troops surviving the battle. See the graph below.



29. (a) Thinking of y as a function of x and x as a function of t, then by the chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-bx}{-ay} = \frac{bx}{ay}$$

(b) Separating variables,

$$\int ay\,dy = \int bx\,dx$$
$$a\frac{y^2}{2} = b\frac{x^2}{2} + k$$
$$ay^2 - bx^2 = C \quad \text{where } C = 2k$$

30. (a) Lanchester's square law for the battle of Iwo Jima is

$$0.05y^2 - 0.01x^2 = C.$$

If we measure x and y in thousands, $x_0 = 54$ and $y_0 = 21.5$, so $0.05(21.5)^2 - 0.01(54)^2 = C$ giving $C = -6.0475$. Thus the equation of the trajectory is

$$0.05y^2 - 0.01x^2 = -6.0475$$

giving

$$x^2 - 5y^2 = 604.75.$$

(b) Assuming that the battle did not end until all the Japanese were dead or wounded, that is, $y = 0$, then the number of US soldiers remaining is given by $x^2 - 5(0)^2 = 604.75$. This gives $x = 24.59$, or about 25,000 troops. This is approximately what happened.

31. (a) Since the guerrillas are hard to find, the rate at which they are put out of action is proportional to the number of chance encounters between a guerrilla and a conventional soldier, which is in turn proportional to the number of guerrillas and to the number of conventional soldiers. Thus the rate at which guerrillas are put out of action is proportional to the product of the strengths of the two armies.

(b)
$$\frac{dx}{dt} = -xy$$
$$\frac{dy}{dt} = -x.$$

(c) Thinking of y as a function of x and x a function of of t, then by the chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{-xy} = \frac{1}{y}.$$

Separating variables:

$$\int y\,dy = \int dx$$
$$\frac{y^2}{2} = x + C.$$

The value of C is determined by the initial strengths of the two armies. Note that C could be written on the opposite side of the equation, giving it the opposite sign.

(d) The sign of C determines which side wins the battle. Looking at the general solution $\frac{y^2}{2} = x + C$, we see that if $C > 0$ the y-intercept is at $\sqrt{2C}$, so y wins the battle by virtue of the fact that it still has troops when $x = 0$. If $C < 0$ then the curve intersects the axes at $x = -C$, so x wins the battle because it has troops when $y = 0$. If $C = 0$, then the solution goes to the point $(0, 0)$, which represents the case of mutual annihilation.

(e) We assume that an army wins if the opposing force goes to 0 first. Figure 11.69 shows that in our formulation, the conventional force wins if $C > 0$ and the guerrillas win if $C < 0$. Neither side wins if $C = 0$ (all soldiers on both sides are killed in this case).

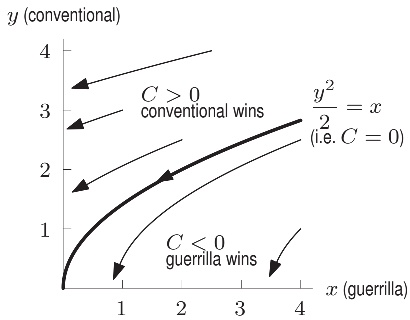


Figure 11.69

32. (a) Taking the constants of proportionality to be a and b, with $a > 0$ and $b > 0$, the equations are

$$\frac{dx}{dt} = -axy$$
$$\frac{dy}{dt} = -bxy.$$

(b) Thinking of x and y as functions of t, then by the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-bxy}{-axy} = \frac{b}{a}.$$

Solving the differential equation gives

$$y = \frac{b}{a}x + C,$$

where C depends on the initial sizes of the two armies. Note that C could be written on the opposite side of the equation, giving it the opposite sign.

(c) The sign of C determines which side wins the battle. Looking at the general solution $y = \frac{b}{a}x + C$, we see that if $C > 0$ the y-intercept is at C, so y wins the battle by virtue of the fact that it still has troops when $x = 0$. If $C < 0$ then the curve intersects the axes at $x = -\frac{a}{b}C$, so x wins the battle because it has troops when $y = 0$. If $C = 0$, then the solution goes to the point $(0, 0)$, which represents the case of mutual annihilation.

(d) We assume that an army wins if the opposing force goes to 0 first. Figure 11.70 shows that in our formulation, y wins if $C > 0$ and x wins if $C < 0$.

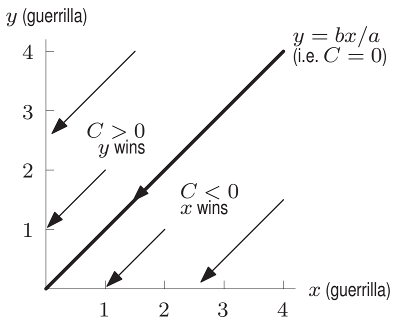


Figure 11.70

33. (a) At an equilibrium point both w and r are constant, so

$$\frac{dw}{dt} = 0 \text{ and } \frac{dr}{dt} = 0.$$

Therefore, we need to solve

$$aw - cwr = 0 \text{ and } -br + kwr = 0.$$

Rearranging gives

$$w(a - cr) = 0 \text{ and } -r(b - kw) = 0$$

so the only equilibrium points are $w = 0, r = 0$ and

$$w = \frac{b}{k} \text{ and } r = \frac{a}{c}.$$

(b) If the insecticide causes a decline in the number of worms then the model becomes

$$\frac{dw}{dt} = aw - cwr - pw \text{ and } \frac{dr}{dt} = -br + kwr$$

where p is a positive constant. Solving as before,

$$w(a - p - cr) = 0 \text{ and } -r(b - kw) = 0,$$

so the equilibrium points are $w = 0, r = 0$ and

$$w = \frac{b}{k} \text{ and } r = \frac{a - p}{c}.$$

So, the equilibrium worm population is unchanged but the equilibrium robin population falls.

Strengthen Your Understanding

34. It is possible for dx/dt and dy/dt to be positive simultaneously. For example, if $x = y = 1$, then $dx/dt = 2.6$ and $dy/dt = 3.5$, meaning that both x and y are increasing.

35. Using the chain rule, we know that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Thus, if $dx/dt < 0$ and $dy/dt > 0$, we must have $dy/dx < 0$.

36. Since X is indifferent to Y and thrives on its own, we have $dx/dt = kx$ with k positive. Since Y needs X to survive, we have $dy/dt = -k_1 y + k_2 xy$, with k_1 and k_2 positive. One possible example is

$$\frac{dx}{dt} = 0.5x$$
$$\frac{dy}{dt} = -0.1y + 0.3xy.$$

37. We have $dx/dt = k_1 x - k_2 xy$ with k_1 and k_2 positive. Similarly for dy/dt. One possible example is

$$\frac{dx}{dt} = 0.5x - 0.2xy$$
$$\frac{dy}{dt} = 0.1y - 0.3xy.$$

38. The parameter a is larger if a disease is more contagious, so we pick a very contagious disease for D_1 and a disease that is not very contagious for D_2. The flu is spread easily through such things as sneezing, whereas a disease such as HIV requires something like an exchange of bodily fluids, so we might pick a flu for D_1 and HIV for D_2.

39. True. Specifying $x(0)$ and $y(0)$ corresponds to picking a starting point in the plane and thereby picking the unique solution curve through that point.

40. False. Competitive exclusion, in which one population drives out another, is modeled by a system of differential equations.

Solutions for Section 11.9

Exercises

1. Equilibrium points occur where both derivatives dx/dt and dy/dt are zero. Thus, equilibrium are located at the intersection of a nullcline with vertical line segments and a nullcline with horizontal line segments. We see in Figure 11.71 there is only one equilibrium point located at the point $(4, 10)$.

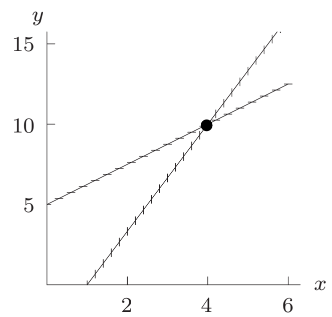


Figure 11.71

2. (a) Since the point $(4, 7)$ is in Region I, $dx/dt < 0$ and $dy/dt > 0$.
(b) Since the point $(4, 10)$ is an equilibrium point, $dx/dt = 0$ and $dy/dt = 0$.
(c) Since the point $(6, 15)$ is in Region II, $dx/dt < 0$ and $dy/dt < 0$.

3. Since $(2, 5)$ is in Region IV, $dx/dt > 0$ and $dy/dt > 0$. Thus, the trajectory goes up and to the right toward the equilibrium $(4, 10)$. See Figure 11.72.

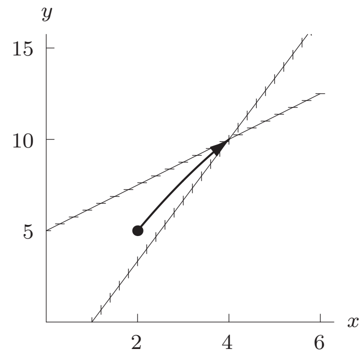


Figure 11.72

4. Since $dx/dt > 0$ and $dy/dt < 0$ in Region III, a trajectory that starts at $(2, 10)$ goes down and to the right toward the equilibrium $(4, 10)$ located at the intersection of the nullclines. See Figure 11.73.

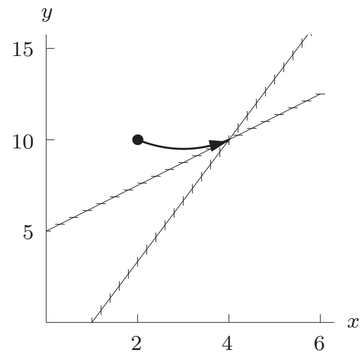


Figure 11.73

5. We can see in Figure 11.89 that any trajectory eventually tends toward the equilibrium point $(4, 10)$.

6. Equilibrium points occur where both derivatives dx/dt and dy/dt are zero. Thus, equilibrium are located at the intersection of a nullcline with vertical line segments and a nullcline with horizontal line segments. We see in Figure 11.71 there are three equilibrium points located at $(0, 0)$, $(0, 6)$, and approximately $(13, 0)$.

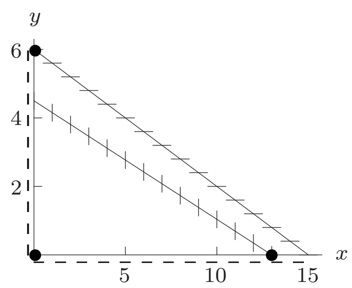


Figure 11.74

7. (a) The point $(5, 2)$ has $dx/dt > 0$ and $dy/dt > 0$.
(b) The point $(10, 2)$ has $dx/dt < 0$. Since the point $(10, 2)$ is on a nullcline horizontal line segments, $dy/dt = 0$.
(c) The point $(10, 1)$ has $dy/dt > 0$. Since the point $(10, 1)$ is on a nullcline with vertical line segments, $dx/dt = 0$.

8. At the point $(2, 5)$, $dx/dt < 0$ and $dy/dt > 0$. The trajectory goes up and to the left, tending toward the equilibrium point $(0, 6)$. See Figure 11.75.

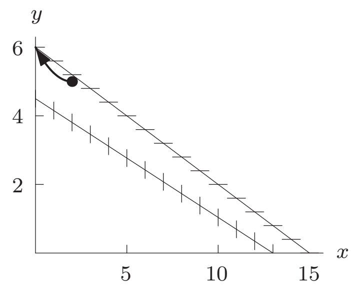


Figure 11.75

9. At the point $(10, 4)$, we have $dx/dt < 0$ and $dy/dt < 0$. The trajectory goes down and to the left, crossing the top nullcline with $dy/dt = 0$. The trajectory then goes up and to the left since dy/dt has changed from negative to positive. The trajectory eventually tends toward the equilibrium point $(0, 6)$. See Figure 11.76.

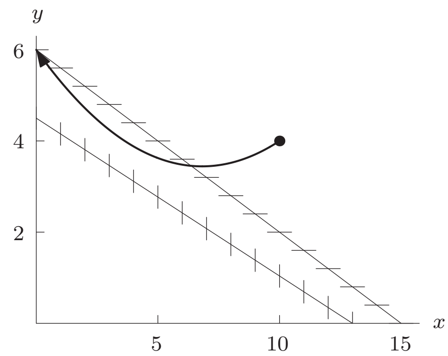


Figure 11.76

10. Any trajectory eventually tends toward the equilibrium point $(0, 6)$.

11. On the phase plane, it appears that slopes are vertical along the lines $y = 3$ and $x = 0$ (the y-axis) and that the slopes are horizontal along the lines $x = 5$ and $y = 0$ (the x-axis). See Figure 11.77.

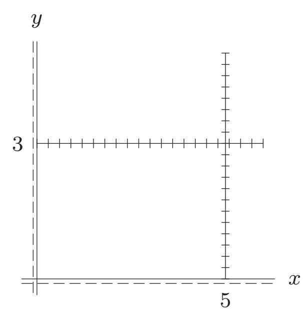


Figure 11.77

12. (a) To find the equilibrium points we set

$$20x - 10xy = 0$$
$$25y - 5xy = 0.$$

So, $x = 0$, $y = 0$ is an equilibrium point. Another one is given by

$$10y = 20$$
$$5x = 25.$$

Therefore, $x = 5$, $y = 2$ is the other equilibrium point.

(b) At $x = 2$, $y = 4$,

$$\frac{dx}{dt} = 20x - 10xy = 40 - 80 = -40$$
$$\frac{dy}{dt} = 25y - 5xy = 100 - 40 = 60.$$

Since these are not both zero, this point is not an equilibrium point.

Problems

13. We first find the nullclines. Again, we assume $x,\ y \geq 0$.
Vertical nullclines occur where $dx/dt = 0$, which happens when $\frac{dx}{dt} = x(2 - x - y) = 0$, i.e. when $x = 0$ or $x + y = 2$.
Horizontal nullclines occur where $dy/dt = 0$, which happens when $\frac{dy}{dt} = y(1-x-y) = 0$, i.e. when $y = 0$ or $x+y = 1$.
These nullclines are shown in Figure 11.78.

Equilibrium points (also shown in Figure 11.78) occur where both dy/dt and dx/dt are 0, i.e. at the intersections of vertical and horizontal nullclines. There are three such points for these equations: $(0, 0)$, $(0, 1)$, and $(2, 0)$.

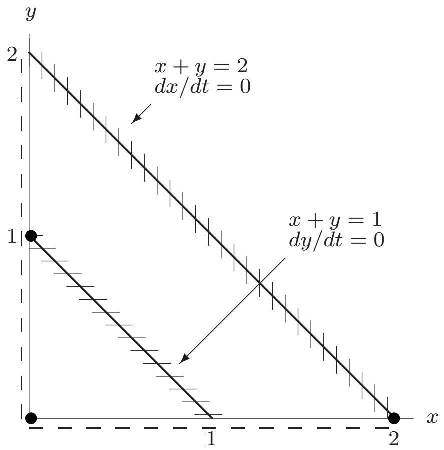


Figure 11.78: Nullclines and equilibrium points (dots)

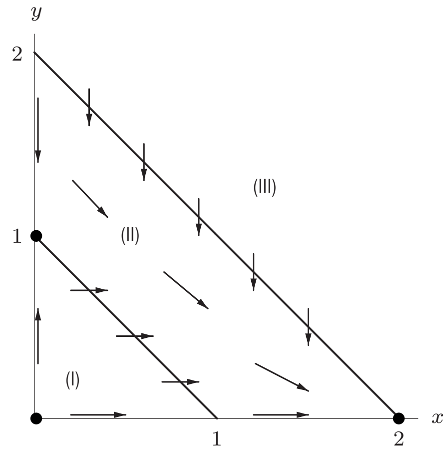


Figure 11.79: General directions of trajectories and equilibrium points (dots)

Looking at sectors in Figure 11.79, we see that no matter in what sector the initial point lies, the trajectory will head toward the equilibrium point $(2, 0)$.

14. We first find the nullclines. Vertical nullclines occur where $\frac{dx}{dt} = 0$, which happens when $x = 0$ or $y = \frac{1}{3}(2 - x)$. Horizontal nullclines occur where $\frac{dy}{dt} = y(1 - 2x) = 0$, which happens when $y = 0$ or $x = \frac{1}{2}$. These nullclines are shown in Figure 11.80.

Equilibrium points (also shown in Figure 11.80) occur at the intersections of vertical and horizontal nullclines. There are three such points for this system of equations; $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(2, 0)$.

The nullclines divide the positive quadrant into four regions as shown in Figure 11.80. Trajectory directions for these regions are shown in Figure 11.81.

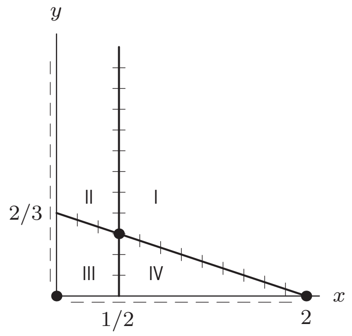


Figure 11.80: Nullclines and equilibrium points (dots)

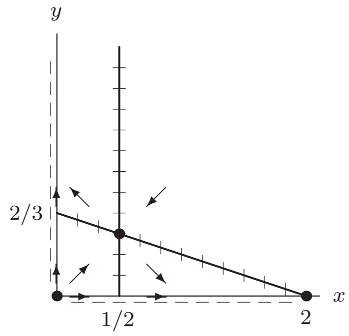


Figure 11.81: General directions of trajectories and equilibrium points (dots)

15. We first find nullclines. Vertical nullclines occur where $\frac{dx}{dt} = x(2 - x - 2y) = 0$, which happens when $x = 0$ or $y = \frac{1}{2}(2 - x)$. Horizontal nullclines occur where $\frac{dy}{dt} = y(1 - 2x - y) = 0$, which happens when $y = 0$ or $y = 1 - 2x$. These nullclines are shown in Figure 11.82.

Equilibrium points (also shown in the figure below) occur at the intersections of vertical and horizontal nullclines. There are three such points for this system; $(0, 0)$, $(0, 1)$, and $(2, 0)$.

The nullclines divide the positive quadrant into three regions as shown in the figure below. Trajectory directions for these regions are shown in Figure 11.83.

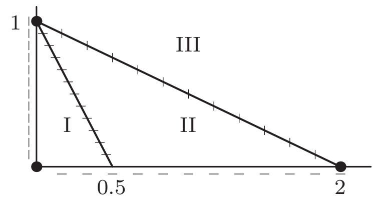


Figure 11.82: Nullclines and equilibrium points (dots)

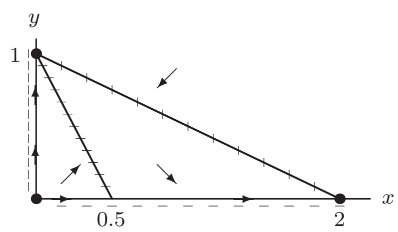


Figure 11.83: General directions of trajectories and equilibrium points (dots)

16. We first find the nullclines. Vertical nullclines occur where $\frac{dx}{dt} = x(1 - y - \frac{x}{3}) = 0$, which happens when $x = 0$ or $y = 1 - \frac{x}{3}$. Horizontal nullclines occur where $\frac{dy}{dt} = y(1 - \frac{y}{2} - x) = 0$, which happens when $y = 0$ or $y = 2(1 - x)$. These nullclines are shown in Figure 11.84.

Equilibrium points (also shown in Figure 11.84) occur at the intersections of vertical and horizontal nullclines. There are four such points for this system: $(0, 0)$, $(0, 2)$, $(3, 0)$, and $(\frac{3}{5}, \frac{4}{5})$.

The nullclines divide the positive quadrant into four regions as shown in Figure 11.84. Trajectory directions for these regions are shown in Figure 11.85.

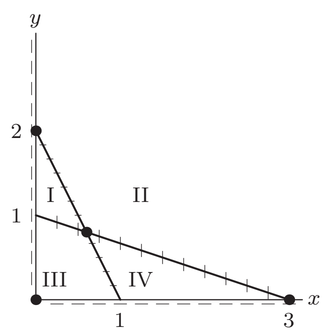


Figure 11.84: Nullclines and equilibrium points (dots)

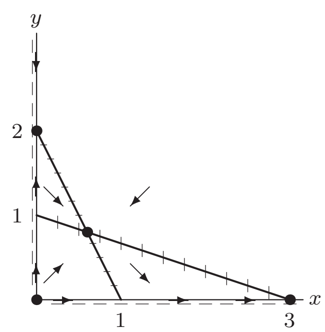


Figure 11.85: General directions of trajectories and equilibrium points (dots)

17. We first find the nullclines. Again, we assume $x,\ y \geq 0$.
$\frac{dx}{dt} = x(1 - x - \frac{y}{3}) = 0$ when $x = 0$ or $x + y/3 = 1$.
$\frac{dy}{dt} = y(1 - y - \frac{x}{2}) = 0$ when $y = 0$ or $y + x/2 = 1$.
These nullclines are shown in Figure 11.86. There are four equilibrium points for these equations. Three of them are the points, $(0, 0)$, $(0, 1)$, and $(1, 0)$. The fourth is the intersection of the two lines $x + y/3 = 1$ and $y + x/2 = 1$. This point is $(\frac{4}{5}, \frac{3}{5})$.

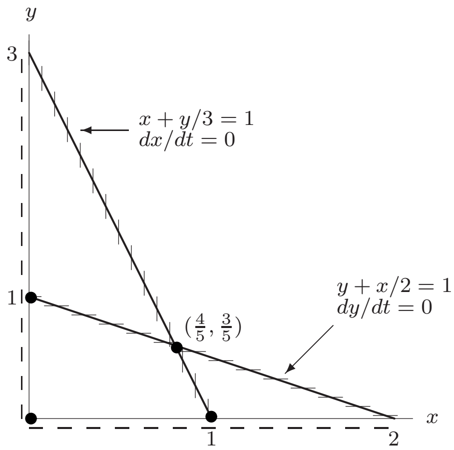


Figure 11.86: Nullclines and equilibrium points (dots)

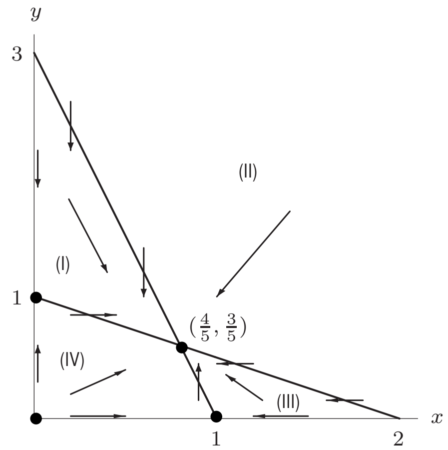


Figure 11.87: General directions of trajectories and equilibrium points (dots)

Looking at sectors in Figure 11.87, we see that no matter in what sector the initial point lies, the trajectory will head toward the equilibrium point $(\frac{4}{5}, \frac{3}{5})$. Only if the initial point lies on the x- or y-axis, will the trajectory head toward the equilibrium points at $(1, 0)$, $(0, 1)$, or $(0, 0)$. In fact, the trajectory will go to $(0, 0)$ only if it starts there, in which case $x(t) = y(t) = 0$ for all t. From direction of the trajectories in Figure 11.87, it appears that if the initial point is in sectors (I) or (III), then it will remain in that sector as it heads toward the equilibrium.

18. We assume that $x,\ y \geq 0$ and then find the nullclines. $\frac{dx}{dt} = x(1 - \frac{x}{2} - y) = 0$ when $x = 0$ or $y + \frac{x}{2} = 1$.
$\frac{dy}{dt} = y(1 - \frac{y}{3} - x) = 0$ when $y = 0$ or $x + \frac{y}{3} = 1$.
We find the equilibrium points. They are $(2, 0)$, $(0, 3)$, $(0, 0)$, and $(\frac{4}{5}, \frac{3}{5})$. The nullclines and equilibrium points are shown in Figure 11.88.

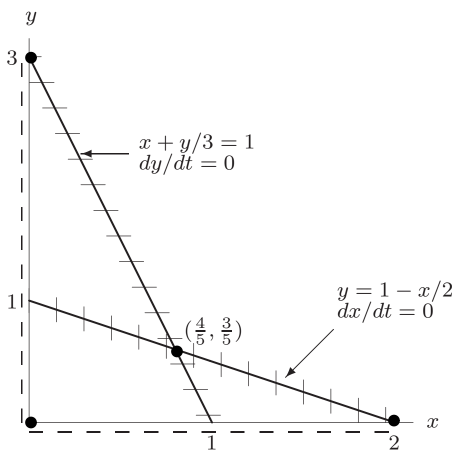


Figure 11.88: Nullclines and equilibrium points (dots)

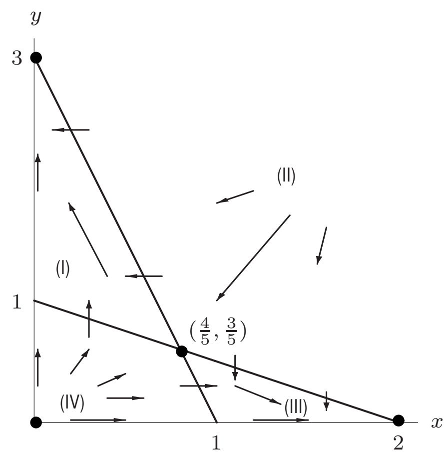


Figure 11.89: General directions of trajectories and equilibrium points (dots)

Figure 11.89 shows that if the initial point is in sector (I), the trajectory heads toward the equilibrium point $(0, 3)$. Similarly, if the trajectory begins in sector (III), then it heads toward the equilibrium $(2, 0)$ over time. If the trajectory begins in sector (II) or (IV), it can go to any of the three equilibrium points $(2, 0)$, $(0, 3)$, or $(\frac{4}{5}, \frac{3}{5})$.

19. (a) $dS/dt = 0$ where $S = 0$ or $I = 0$ (both axes).
$dI/dt = 0.0026I(S - 192)$, so $dI/dt = 0$ where $I = 0$ or $S = 192$.
Thus every point on the S axis is an equilibrium point (corresponding to no one being sick).

(b) In region I, where $S > 192$, $\frac{dS}{dt} < 0$ and $\frac{dI}{dt} > 0$.
In region II, where $S < 192$, $\frac{dS}{dt} < 0$ and $\frac{dI}{dt} < 0$. See Figure 11.90.

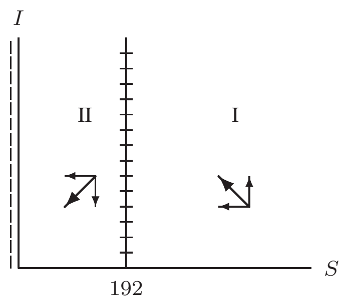


Figure 11.90

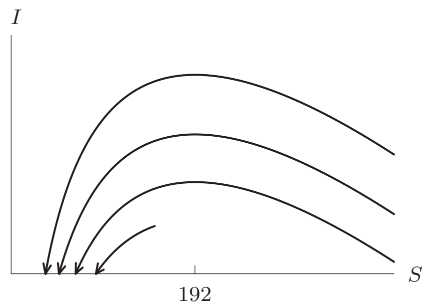


Figure 11.91

(c) If the trajectory starts with $S_0 > 192$, then I increases to a maximum when $S = 192$. If $S_0 < 192$, then I always decreases. See Figure 11.90. Regardless of the initial conditions, the trajectory always goes to a point on the S-axis (where $I = 0$). The S-intercept represents the number of students who never get the disease. See Figure 11.91.

20. The nullclines are where $\frac{dw}{dt} = 0$ or $\frac{dr}{dt} = 0$.
$\frac{dw}{dt} = 0$ when $w - wr = 0$, so $w(1 - r) = 0$ giving $w = 0$ or $r = 1$.
$\frac{dr}{dt} = 0$ when $-r + rw = 0$, so $r(w - 1) = 0$ giving $r = 0$ or $w = 1$.

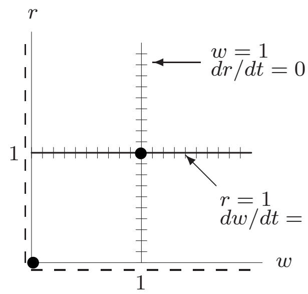


Figure 11.92: Nullclines and equilibrium points (dots)

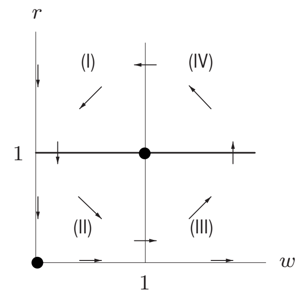


Figure 11.93

The equilibrium points are where the nullclines intersect: $(0, 0)$ and $(1, 1)$. The nullclines split the first quadrant into four sectors. See Figure 11.92. We can get a feel for how the populations interact by seeing the direction of the trajectories in each sector. See Figure 11.93. If the populations reach an equilibrium point, they will stay there. If the worm population dies out, the robin population will also die out, too. However, if the robin population dies out, the worm population will continue to grow.

Otherwise, it seems that the populations cycle around the equilibrium $(1, 1)$. The trajectory moves from sector to sector: trajectories in sector (I) move to sector (II); trajectories in sector (II) move to sector (III); trajectories in sector (III) move to sector (IV); trajectories in sector (IV) move back to sector (I). The robins keep the worm population down by feeding on them, but the robins need the worms (as food) to sustain the population. These conflicting needs keep the populations moving in a cycle around the equilibrium.

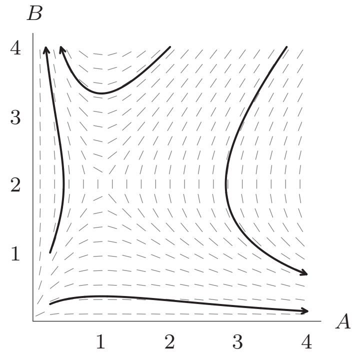
21. (a) If B were not present, then we'd have $A' = 2A$, so company A's net worth would grow exponentially. Similarly, if A were not present, B would grow exponentially. The two companies restrain each other's growth, probably by competing for the market.

(b) To find equilibrium points, find the solutions of the pair of equations

$$A' = 2A - AB = 0$$
$$B' = B - AB = 0$$

The first equation has solutions $A = 0$ or $B = 2$. The second has solutions $B = 0$ or $A = 1$. Thus the equilibrium points are (0,0) and (1,2).

(c) In the long run, one of the companies will go out of business. Two of the trajectories in the figure below go toward the A axis; they represent A surviving and B going out of business. The trajectories going toward the B axis represent A going out of business. Notice both the equilibrium points are unstable.



22. (a) See Figure 11.94.

$$\frac{dx}{dt} = 0 \text{ when } x = \frac{10.5}{0.45} = 23.3$$
$$\frac{dy}{dt} = 0 \text{ when } 8.2x - 0.8y - 142 = 0$$

There is an equilibrium point where the trajectories cross at $x = 23.3$, $y = 61.7$

In region I, $\frac{dx}{dt} > 0$, $\frac{dy}{dt} < 0$.

In region II, $\frac{dx}{dt} < 0$, $\frac{dy}{dt} < 0$.

In region III, $\frac{dx}{dt} < 0$, $\frac{dy}{dt} > 0$.

In region IV, $\frac{dx}{dt} > 0$, $\frac{dy}{dt} > 0$.

(b) See Figure 11.95.

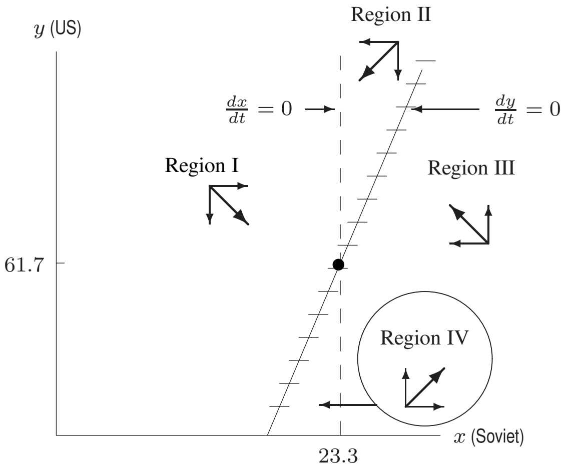


Figure 11.94: Nullclines and equilibrium point (dot) for US-Soviet arms race

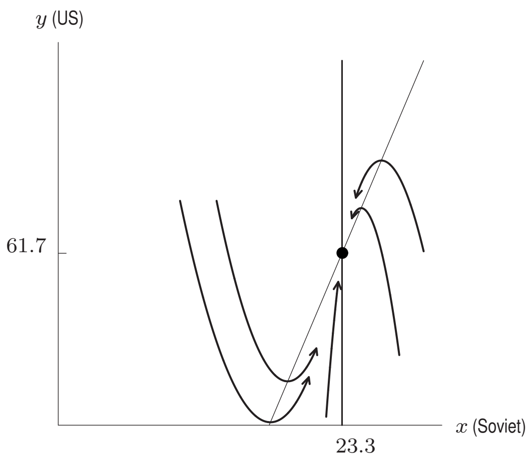


Figure 11.95: Trajectories for US-Soviet arms race.

(c) All the trajectories tend toward the equilibrium point $x = 23.3$, $y = 61.7$. Thus the model predicts that in the long-run the arms race will level off with the Soviet Union spending 23.3 billion dollars a year on arms and the US 61.7 billion dollars.

(d) As the model predicts, yearly arms expenditure did tend toward 23 billion for the Soviet Union and 62 billion for the US.

23. (a) The nullclines are $P = 0$ or $P_1 + 3P_2 = 13$ (where $dP_1/dt = 0$) and $P = 0$ or $P_2 + 0.4P_1 = 6$ (where $dP_2/dt = 0$).
(b) The phase plane in Figure 11.96 shows that P_2 will eventually exclude P_1 regardless of where the experiment starts so long as there were some P_2 originally. Consequently, the data points would have followed a trajectory that starts at the origin, crosses the first nullcline and goes left and upward between the two nullclines to the point $P_1 = 0$, $P_2 = 6$.

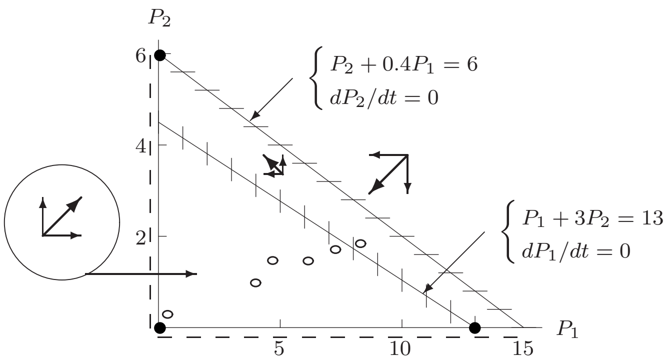


Figure 11.96: Nullclines and equilibrium points (dots) for Gause's yeast data (hollow dots)

Strengthen Your Understanding

24. The trajectory has no arrow to indicate direction.

25. Equilibrium occurs when both $dx/dt = 0$ and $dy/dt = 0$. Both nullclines have horizontal line segments indicating points that $dy/dt = 0$. There are no nullclines that indicate where $dx/dt = 0$. Thus, the point $(6, 6)$ has $dy/dt = 0$ but $dx/dt \neq 0$ and is not an equilibrium point.

26. An equilibrium point is a point where a nullcline with vertical trajectories intersects a nullcline with horizontal trajectories. There are many possible ways to draw this graph. One example is shown in Figure 11.97.

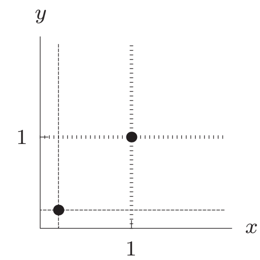


Figure 11.97

27. Nullclines show where trajectories are vertical or horizontal. We see that the trajectory shown is vertical at approximately the points $(0.4, 1)$ and $(2.1, 1)$ so it is reasonable to draw a nullcline showing vertical trajectories on the line $y = 1$. The trajectory shown is horizontal at approximately the points $(1, 2.1)$ and $(1, 0.4)$ so it is reasonable to draw a nullcline showing horizontal trajectories on the line $x = 1$. See Figure 11.98. Other answers are possible.

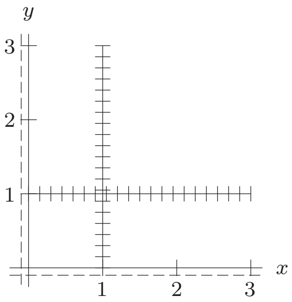


Figure 11.98

28. We follow the directions, and use the fact that if the trajectory crosses the nullcline $x = 2$, it is horizontal there and then bends down. See Figure 11.99. Other answers are possible.

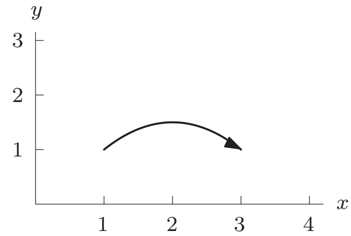


Figure 11.99

Solutions for Chapter 11 Review

Exercises

1. For (I): $y = xe^x$, $y' = e^x + xe^x$, and $y'' = 2e^x + xe^x$.
For (II): $y = xe^{-x}$, $y' = e^{-x} - xe^{-x}$, and $y'' = -2e^{-x} + xe^{-x}$.
Thus (I) satisfies equation (d).
(II) satisfies equation (c).

2. (a) Since $y' = 1 + y^2$, the slope is everywhere positive. This is true for slope field (III).
(b) Since $y = x$, the slopes are positive to the right of the y-axis and negative to the left of the y-axis. Solution curves are parabolas, as in slope field (VI).
(c) Since $y' = \sin x$, the slopes oscillate between -1 and 1. This corresponds to slope field (V).
(d) Since $y' = y$, the slopes are positive above the x-axis and negative below the x-axis. This is true for slope field (I).
(e) Since $y' = x - y$, the slopes are zero when $x = y$. This corresponds to slope field (IV).
(f) Since $y' = 4 - y$, the slopes are positive when $y < 4$ and negative when $y > 4$. This corresponds to slope field (II).

3. (a) = (I), (b) = (IV), (c) = (III). Graph (II) represents an egg originally at 0° C which is moved to the kitchen table (20° C) two minutes after the egg in part (a) is moved.

4. This equation is separable, so we integrate, giving

$$\int dP = \int t\,dt$$

so

$$P(t) = \frac{t^2}{2} + C.$$

5. This equation is separable, so we integrate, giving

$$\int \frac{1}{0.2y-8}\,dy = \int dx$$

so

$$\frac{1}{0.2}\ln|0.2y-8| = x + C.$$

Thus

$$y(x) = 40 + Ae^{0.2x}.$$

6. This equation is separable, so we integrate, giving

$$\int \frac{1}{10-2P}\,dP = \int dt$$

so

$$\frac{1}{-2}\ln|10-2P| = t + C.$$

Thus

$$P = 5 + Ae^{-2t}.$$

7. This equation is separable, so we integrate, giving

$$\int \frac{1}{10+0.5H}\,dH = \int dt$$

so

$$\frac{1}{0.5}\ln|10+0.5H| = t + C.$$

Thus

$$H = Ae^{0.5t} - 20.$$

8. This equation is separable, so we integrate, using the table of integrals or partial fractions, to get

$$\int \frac{1}{R-3R^2}\,dR = 2\int dt$$

$$\int \frac{1}{R}\,dR + \int \frac{3}{1-3R}\,dR = 2\int dt$$

so

$$\begin{aligned}
\ln|R| - \ln|1-3R| &= 2t + C\\
\ln\left|\frac{R}{1-3R}\right| &= 2t + C\\
\frac{R}{1-3R} &= Ae^{2t}\\
R &= \frac{Ae^{2t}}{1+3Ae^{2t}}.
\end{aligned}$$

9. This equation is separable, so we integrate, using the table of integrals or partial fractions, to get:

$$\int \frac{250}{100P-P^2}\,dP = \int dt$$

$$\frac{250}{100}\left(\int \frac{1}{P}\,dP + \int \frac{1}{100-P}\,dP\right) = \int dt$$

so

$$\begin{aligned}
2.5(\ln|P| - \ln|100-P|) &= t + C\\
2.5\ln\left|\frac{P}{100-P}\right| &= t + C
\end{aligned}$$

$$\frac{P}{100-P} = Ae^{0.4t}$$
$$P = \frac{100Ae^{0.4t}}{1+Ae^{0.4t}}$$

10. $\frac{dy}{dx} + xy^2 = 0$ means $\frac{dy}{dx} = -xy^2$, so $\int \frac{dy}{y^2} = \int -x\,dx$ giving $-\frac{1}{y} = -\frac{x^2}{2} + C$. Since $y(1) = 1$ we have $-1 = -\frac{1}{2} + C$ so $C = -\frac{1}{2}$. Thus, $-\frac{1}{y} = -\frac{x^2}{2} - \frac{1}{2}$ giving $y = \frac{2}{x^2+1}$.

11. $\frac{dP}{dt} = 0.03P + 400$ so $\int \frac{dP}{P+\frac{40000}{3}} = \int 0.03dt$.
$\ln|P + \frac{40000}{3}| = 0.03t + C$ giving $P = Ae^{0.03t} - \frac{40000}{3}$. Since $P(0) = 0$, $A = \frac{40000}{3}$, therefore $P = \frac{40000}{3}(e^{0.03t} - 1)$.

12. $1 + y^2 - \frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = y^2 + 1$, so $\int \frac{dy}{1+y^2} = \int dx$ and $\arctan y = x + C$. Since $y(0) = 0$ we have $C = 0$, giving $y = \tan x$.

13. $2\sin x - y^2\frac{dy}{dx} = 0$ giving $2\sin x = y^2\frac{dy}{dx}$. $\int 2\sin x\,dx = \int y^2\,dy$ so $-2\cos x = \frac{y^3}{3} + C$. Since $y(0) = 3$ we have $-2 = 9 + C$, so $C = -11$. Thus, $-2\cos x = \frac{y^3}{3} - 11$ giving $y = \sqrt[3]{33 - 6\cos x}$.

14. $\frac{dk}{dt} = (1 + \ln t)k$ gives $\int \frac{dk}{k} = \int (1 + \ln t)dt$ so $\ln|k| = t\ln t + C$. $k(1) = 1$, so $0 = 0 + C$, or $C = 0$. Thus, $\ln|k| = t\ln t$ and $|k| = e^{t\ln t} = t^t$, giving $k = \pm t^t$.
But recall $k(1) = 1$, so $k = t^t$ is the solution.

15. $\frac{dy}{dx} = \frac{y(3-x)}{x(\frac{1}{2}y-4)}$ gives $\int \frac{(\frac{1}{2}y-4)}{y}dy = \int \frac{(3-x)}{x}dx$ so $\int(\frac{1}{2} - \frac{4}{y})dy = \int(\frac{3}{x} - 1)dx$. Thus $\frac{1}{2}y - 4\ln|y| = 3\ln|x| - x + C$. Since $y(1) = 5$, we have $\frac{5}{2} - 4\ln 5 = \ln|1| - 1 + C$ so $C = \frac{7}{2} - 4\ln 5$. Thus,

$$\frac{1}{2}y - 4\ln|y| = 3\ln|x| - x + \frac{7}{2} - 4\ln 5.$$

We cannot solve for y in terms of x, so we leave the equation in this form.

16. $\frac{dy}{dx} = \frac{0.2y(18+0.1x)}{x(100+0.5y)}$ giving $\int \frac{(100+0.5y)}{0.2y}\,dy = \int \frac{18+0.1x}{x}\,dx$, so

$$\int \left(\frac{500}{y} + \frac{5}{2}\right) dy = \int \left(\frac{18}{x} + \frac{1}{10}\right) dx.$$

Therefore, $500\ln|y| + \frac{5}{2}y = 18\ln|x| + \frac{1}{10}x + C$. Since the curve passes through (10,10), $500\ln 10 + 25 = 18\ln 10 + 1 + C$, so $C = 482\ln 10 + 24$. Thus, the solution is

$$500\ln|y| + \frac{5}{2}y = 18\ln|x| + \frac{1}{10}x + 482\ln 10 + 24.$$

We cannot solve for y in terms of x, so we leave the answer in this form.

17. This equation is separable and so we write it as

$$\frac{1}{z(z-1)}\frac{dz}{dt} = 1.$$

We integrate with respect to t, giving

$$\int \frac{1}{z(z-1)}dz = \int dt$$
$$\int \frac{1}{z-1}dz - \int \frac{1}{z}dz = \int dt$$
$$\ln|z-1| - \ln|z| = t + C$$
$$\ln|\frac{z-1}{z}| = t + C,$$

so that

$$\frac{z-1}{z} = e^{t+C} = ke^t.$$

Solving for z gives

$$z(t) = \frac{1}{1-ke^t}.$$

The initial condition $z(0) = 10$ gives
$$\frac{1}{1-k} = 10$$
or $k = 0.9$. The solution is therefore
$$z(t) = \frac{1}{1-0.9e^t}.$$

18. Using the solution of the logistic equation given on page 631 in Section 11.7, and using $y(0) = 1$, we get $y = \frac{10}{1+9e^{-10t}}$.

19. $\frac{dy}{dx} = \frac{y(100-x)}{x(20-y)}$ gives $\int (\frac{20-y}{y})\,dy = \int \left(\frac{100-x}{x}\right)\,dx$. Thus, $20\ln|y| - y = 100\ln|x| - x + C$. The curve passes through $(1,20)$, so $20\ln 20 - 20 = -1 + C$ giving $C = 20\ln 20 - 19$. Therefore, $20\ln|y| - y = 100\ln|x| - x + 20\ln 20 - 19$. We cannot solve for y in terms of x, so we leave the equation in this form.

20. $\frac{df}{dx} = \sqrt{xf(x)}$ gives $\int \frac{df}{\sqrt{f(x)}} = \int \sqrt{x}\,dx$, so $2\sqrt{f(x)} = \frac{2}{3}x^{\frac{3}{2}} + C$. Since $f(1) = 1$, we have $2 = \frac{2}{3} + C$ so $C = \frac{4}{3}$. Thus, $2\sqrt{f(x)} = \frac{2}{3}x^{\frac{3}{2}} + \frac{4}{3}$, so $f(x) = (\frac{1}{3}x^{\frac{3}{2}} + \frac{2}{3})^2$.
(Note: this is only defined for $x \geq 0$.)

21. $\frac{dy}{dx} = e^{x-y}$ giving $\int e^y\,dy = \int e^x\,dx$ so $e^y = e^x + C$. Since $y(0) = 1$, we have $e^1 = e^0 + C$ so $C = e - 1$. Thus, $e^y = e^x + e - 1$, so $y = \ln(e^x + e - 1)$.
[Note: $e^x + e - 1 > 0$ always.]

22. $\frac{dy}{dx} = e^{x+y} = e^x e^y$ implies $\int e^{-y}\,dy = \int e^x\,dx$ implies $-e^{-y} = e^x + C$. Since $y = 0$ when $x = 1$, we have $-1 = e + C$, giving $C = -1 - e$. Therefore $-e^{-y} = e^x - 1 - e$ and $y = -\ln(1 + e - e^x)$.

23. $e^{-\cos\theta}\frac{dz}{d\theta} = \sqrt{1-z^2}\sin\theta$ implies $\int \frac{dz}{\sqrt{1-z^2}} = \int e^{\cos\theta}\sin\theta\,d\theta$ implies $\arcsin z = -e^{\cos\theta} + C$. According to the initial conditions: $z(0) = \frac{1}{2}$, so $\arcsin\frac{1}{2} = -e^{\cos 0} + C$, therefore $\frac{\pi}{6} = -e + C$, and $C = \frac{\pi}{6} + e$. Thus $z = \sin(-e^{\cos\theta} + \frac{\pi}{6} + e)$.

24. $(1+t^2)y\frac{dy}{dt} = 1-y$ implies that $\int \frac{y\,dy}{1-y} = \int \frac{dt}{1+t^2}$ implies that $\int \left(-1 + \frac{1}{1-y}\right)\,dy = \int \frac{dt}{1+t^2}$. Therefore $-y - \ln|1-y| = \arctan t + C$. $y(1) = 0$, so $0 = \arctan 1 + C$, and $C = -\frac{\pi}{4}$, so $-y - \ln|1-y| = \arctan t - \frac{\pi}{4}$. We cannot solve for y in terms of t.

25. We have
$$\frac{dy}{dt} = 2^y \sin^3 t,$$
so
$$\int 2^{-y}\,dy = \int \sin^3 t\,dt.$$
Using Integral Table Formula 17, gives
$$-\frac{1}{\ln 2}2^{-y} = -\frac{1}{3}\sin^2 t\cos t - \frac{2}{3}\cos t + C.$$
According to the initial conditions: $y(0) = 0$ so
$$-\frac{1}{\ln 2} = -\frac{2}{3} + C, \quad \text{and} \quad C = \frac{2}{3} - \frac{1}{\ln 2}.$$
Thus,
$$-\frac{1}{\ln 2}2^{-y} = -\frac{1}{3}\sin^2 t\cos t - \frac{2}{3}\cos t + \frac{2}{3} - \frac{1}{\ln 2}.$$
Solving for y gives:
$$2^{-y} = \frac{\ln 2}{3}\sin^2 t\cos t + \frac{2\ln 2}{3}\cos t - \frac{2\ln 2}{3} + 1.$$
It can be shown that the right side is always > 0, so we can take natural logs.
$$y\ln 2 = -\ln\left(\frac{\ln 2}{3}\sin^2 t\cos t + \frac{2\ln 2}{3}\cos t - \frac{2\ln 2}{3} + 1\right),$$
so
$$y = \frac{-\ln\left(\frac{\ln 2}{3}\sin^2 t\cos t + \frac{2\ln 2}{3}\cos t - \frac{2\ln 2}{3} + 1\right)}{\ln 2}.$$

Problems

26. (a) To find equilibrium values, we solve

$$\begin{aligned} \frac{dP}{dt} &= 0 \\ 0.025P - 0.00005P^2 &= 0 \\ 0.025P(1 - 0.002P) &= 0 \\ P = 0 \quad \text{and} \quad P &= 500. \end{aligned}$$

The equilibrium values are $P = 0$ and $P = 500$.

(b) Between 0 and 500, P is increasing. Above 500, P is decreasing. We have:

(i) For $P_0 = 100$, the population will increase to a limiting value of 500.
(ii) For $P_0 = 400$, the population will increase to a limiting value of 500.
(iii) For $P_0 = 500$, the population is at equilibrium and will stay at 500.
(iv) For $P_0 = 800$, the population will decrease to a limiting value of 500.

27. (a) We know that the equilibrium solutions are the functions satisfying the differential equation whose derivative everywhere is 0. Thus we have

$$\begin{aligned} \frac{dy}{dt} &= 0 \\ 0.2(y-3)(y+2) &= 0 \\ (y-3)(y+2) &= 0. \end{aligned}$$

The solutions are $y = 3$ and $y = -2$.

(b)

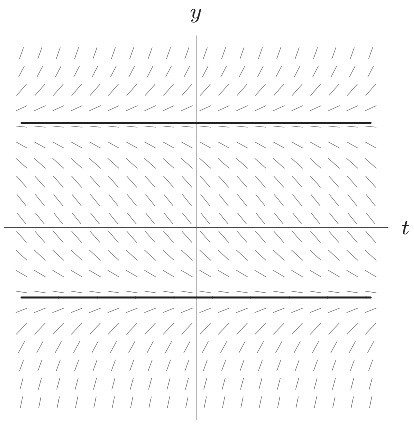


Figure 11.100

Looking at Figure 11.100, we see that the line $y = 3$ is an unstable solution, while the line $y = -2$ is a stable solution.

28. Since $f(x)$ is a solution to $y' = xy - y$, we know that

$$f'(x) = xf(x) - f(x).$$

Letting $y = 2f(x)$, we want to find out if the left-hand side of this differential equation, y', equals the right-hand side, $xy - y$. First we consider the left-hand side:

$$\begin{aligned} y' &= (2f(x))' \\ &= 2f'(x) \\ &= 2\left(xf(x) - f(x)\right) \quad \text{since } f'(x) = xf(x) - f(x) \\ &= 2xf(x) - 2f(x). \end{aligned}$$

Turning to the right-hand side, we see that:

$$\begin{aligned} xy - y &= x\,(2f(x)) - 2f(x) \quad \text{because } y = 2f(x) \\ &= 2xf(x) - 2f(x), \end{aligned}$$

We see that the left-hand side and the right-hand side both equal $2xf(x) - 2f(x)$, so $y = 2f(x)$ is also a solution to the equation.

29. Since $y = f(x)$ is a solution to $y' = xy - y$, we know that

$$f'(x) = xf(x) - f(x).$$

Letting $y = 2 + f(x)$, we want to find out if the left-hand side of this differential equation, y', equals the right-hand side, $xy - y$. First we consider the left-hand side:

$$\begin{aligned} y' &= (f(x) + 2)' \\ &= f'(x) \\ &= xf(x) - f(x) \quad \text{since } f'(x) = xf(x) - f(x). \end{aligned}$$

Turning to the right-hand side, we see that:

$$\begin{aligned} xy - y &= x\,(2 + f(x)) - (2 + f(x)) \quad \text{because } y = 2 + f(x) \\ &= xf(x) - f(x) + 2x - 2. \end{aligned}$$

We see that the left-hand side, $xf(x)-f(x)$, is not the same as the right-hand side, $xf(x)-f(x)+2x-2$, so $y = 2+f(x)$ is not a solution to the equation.

30. (a) (i) 1 step: $\Delta y = \dfrac{1}{(\cos x)(\cos y)}\Delta x = \dfrac{1}{(\cos 0)(\cos 0)}\dfrac{1}{2} = \dfrac{1}{2}.$

Thus, using 1 step, we get $(\frac{1}{2}, \frac{1}{2})$ as our approximation.

(ii) 2 steps: $\Delta x = \frac{1}{4}$.

x	y	$\Delta y = \frac{1}{(\cos x)(\cos y)}\Delta x$
0	0	0.25
0.25	0.25	0.266
0.5	0.516	

Thus, using 2 steps, we get (0.5,0.516) as our approximation.

(iii) 4 steps: $\Delta x = \frac{1}{8}$

x	y	$\Delta y = \frac{1}{(\cos x)(\cos y)}\Delta x$
0	0	0.125
0.125	0.125	0.127
0.25	0.252	0.133
0.375	0.385	0.145
0.5	0.530	

Thus, using 4 steps, we get (0.5,0.530) as our approximation.

(b) We have

$$\frac{dy}{dx} = \frac{1}{(\cos x)(\cos y)},$$

so

$$\int \cos y\, dy = \int \frac{dx}{\cos x}.$$

The integral table gives

$$\sin y = \frac{1}{2}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + C.$$

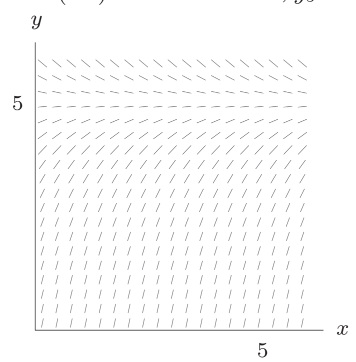
Our curve passes through (0,0), so, $0 = 0 + C$, and $C = 0$. Therefore

$$y = \arcsin\left(\frac{1}{2}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right|\right).$$

When $x = \frac{1}{2}$, $y \approx 0.549$. Our answers in parts (a)-(c) are all underestimates. In each case, the error is about $\frac{1}{n+1}$, where n is the number of steps. We expect the error to be approximately proportional to $\frac{1}{n}$, so this seems reasonable.

31. (a) $\Delta x = \frac{1}{5} = 0.2$.
At $x = 0$:
$y_0 = 1, y' = 4$; so $\Delta y = 4(0.2) = 0.8$. Thus, $y_1 = 1 + 0.8 = 1.8$.
At $x = 0.2$:
$y_1 = 1.8, y' = 3.2$; so $\Delta y = 3.2(0.2) = 0.64$. Thus, $y_2 = 1.8 + 0.64 = 2.44$.
At $x = 0.4$:
$y_2 = 2.44, y' = 2.56$; so $\Delta y = 2.56(0.2) = 0.512$. Thus, $y_3 = 2.44 + 0.512 = 2.952$.
At $x = 0.6$:
$y_3 = 2.952, y' = 2.048$; so $\Delta y = 2.048(0.2) = 0.4096$. Thus, $y_4 = 3.3616$.
At $x = 0.8$:
$y_4 = 3.3616, y' = 1.6384$; so $\Delta y = 1.6384(0.2) = 0.32768$. Thus, $y_5 = 3.68928$. So $y(1) \approx 3.689$.

(b)



Since solution curves are concave down for $0 \le y \le 5$, and $y(0) = 1 < 5$, the estimate from Euler's method will be an overestimate.

(c) Solving by separation:
$\displaystyle\int \frac{dy}{5-y} = \int dx$, so $-\ln|5-y| = x + C$.
Then $5 - y = Ae^{-x}$ where $A = \pm e^{-C}$. Since $y(0) = 1$, we have $5 - 1 = Ae^0$, so $A = 4$.
Therefore, $y = 5 - 4e^{-x}$, and $y(1) = 5 - 4e^{-1} \approx 3.528$.
(Note: as predicted, the estimate in (a) is too large.)

(d) Doubling the value of n will probably halve the error and, therefore, give a value half way between 3.528 and 3.689, which is approximately 3.61.

32. A continuous growth rate of 0.38% means that

$$\frac{1}{P}\frac{dP}{dt} = 0.38\% = 0.0038.$$

Separating variables and integrating gives

$$\int \frac{dP}{P} = \int 0.0038\, dt$$
$$P = P_0 e^{0.0038t} = (7.5 \cdot 10^6)e^{0.0038t}.$$

33. (a) Assuming that the world's population grows exponentially, satisfying $dP/dt = cP$, and that the land in use for crops is proportional to the population, we expect A to satisfy $dA/dt = kA$.

(b) We have $A(t) = A_0e^{kt} = 4.55 \cdot 10^9 e^{kt}$, where t is the number of years after 1966. Since 30 years later the amount of land in use is 4.93 billion hectares, we have

$$4.93 \cdot 10^9 = (4.55 \cdot 10^9)e^{k(30)},$$

so

$$e^{30k} = \frac{4.93}{4.55}.$$

Solving for k gives

$$k = \frac{\ln(4.93/4.55)}{30} = 0.00267.$$

Thus,

$$A = (4.55 \cdot 10^9)e^{0.00267t}.$$

We want to find t such that

$$6 \cdot 10^9 = A(t) = (4.55 \cdot 10^9)e^{0.00267t}.$$

Taking logarithms gives

$$t = \frac{\ln(6/4.55)}{0.00267} = 103.608 \text{ years.}$$

This model predicts land will have run out 104 years after 1966, that is by the year 2070.

34. (a) Since the growth rate of the tumor is proportional to its size, we should have

$$\frac{dS}{dt} = kS.$$

(b) We can solve this differential equation by separating variables and then integrating:

$$\int \frac{dS}{S} = \int k\,dt$$
$$\ln|S| = kt + B$$
$$S = Ce^{kt}.$$

(c) This information is enough to allow us to solve for C:

$$5 = Ce^{0t}$$
$$C = 5.$$

(d) Knowing that $C = 5$, this second piece of information allows us to solve for k:

$$8 = 5e^{3k}$$
$$k = \frac{1}{3}\ln\left(\frac{8}{5}\right) \approx 0.1567.$$

So the tumor's size is given by

$$S = 5e^{0.1567t}.$$

35. (a) The rate of growth of the money in the account is proportional to the amount of money in the account. Thus

$$\frac{dM}{dt} = rM.$$

(b) Solving, we have $dM/M = r\,dt$.

$$\int \frac{dM}{M} = \int r\,dt$$
$$\ln|M| = rt + C$$
$$M = e^{rt+C} = Ae^{rt}, \qquad A = e^C.$$

When $t = 0$ (in 2010), $M = 2000$, so $A = 2000$ and $M = 2000e^{rt}$.

(c) See Figure 11.101.

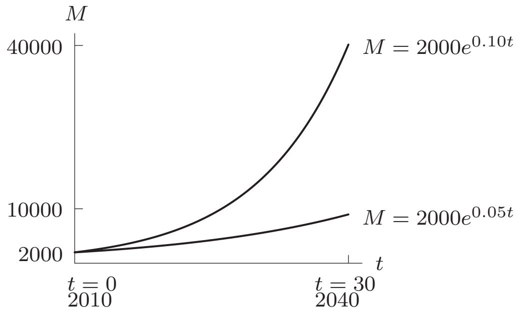


Figure 11.101

36. We know that

$$\text{Rate at which quantity of carbon-14 is changing} = -k(\text{current quantity}).$$

If Q is the quantity of carbon-14 at time t (in years),

$$\text{Rate at which quantity is changing} = \frac{dQ}{dt} = -kQ.$$

This differential equation has solution

$$Q = Q_0 e^{-kt}$$

where Q_0 is the initial quantity. Since at the end of one year 9999 parts are left out of 10,000, we know that

$$9999 = 10{,}000e^{-k(1)}.$$

Solving for k gives

$$k = \ln 0.9999 \approx 0.0001.$$

Thus $Q = Q_0 e^{-0.0001t}$. See Figure 11.102.

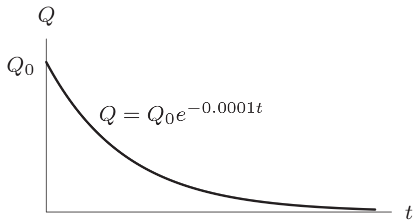


Figure 11.102: Exponential decay

37. Using (Rate balance changes) = (Rate interest is added)− (Rate payments are made), when the interest rate is i, we have

$$\frac{dB}{dt} = iB - 100.$$

Solving this equation, we find:

$$\begin{aligned}
\frac{dB}{dt} &= i\left(B - \frac{100}{i}\right) \\
\int \frac{dB}{B - \frac{100}{i}} &= \int i\,dt \\
\ln\left|B - \frac{100}{i}\right| &= it + C \\
B - \frac{100}{i} &= Ae^{it}, \text{ where } A = \pm e^C.
\end{aligned}$$

At time $t = 0$ we start with a balance of \$1000. Thus
$1000 - \frac{100}{i} = Ae^0$, so $A = 1000 - \frac{100}{i}$.
Thus $B = \frac{100}{i} + (1000 - \frac{100}{i})e^{it}$.
When $i = 0.05$, $B = 2000 - 1000e^{0.05t}$.
When $i = 0.1$, $B = 1000$.
When $i = 0.15$, $B = 666.67 + 333.33e^{0.15t}$.
We now look at the graph in Figure 11.103 when $i = 0.05$, $i = 0.1$, and $i = 0.15$.

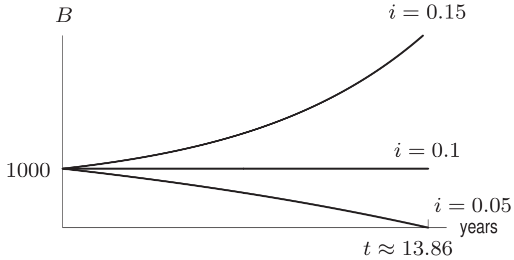


Figure 11.103

38. (a) $c'(t) = a(k - c(t))$ where $a > 0$ is a constant.
(b)

$$\int \frac{dc}{k-c} = \int a\,dt$$

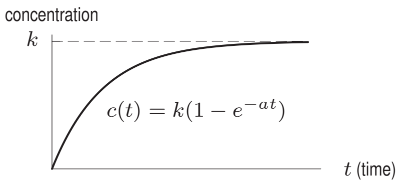
$$-\ln|k-c| = at + C, \quad C \text{ is a constant of integration}$$

$$k - c = Ae^{-at}$$

If $c = c_0$ when $t = 0$, then $k - c_0 = A$, so

$$k - c = (k - c_0)e^{-at}$$
$$c = k + (c_0 - k)e^{-at}$$

(c) If $c_0 = 0$, then $c = k - ke^{-at} = k(1 - e^{-at})$.



39. (a) Use the fact that

$$\text{Rate balance changing} = \text{Rate interest accrued} - \text{Rate payments made}.$$

Thus

$$\frac{dB}{dt} = 0.05B - 12{,}000.$$

(b) We solve the equation by separation of variables. First, however, we factor out a 0.05 on the right hand side of the equation to make the work easier.

$$\frac{dB}{dt} = 0.05(B - 240000)$$

$$\frac{dB}{B - 240000} = 0.05\,dt$$

$$\int \frac{dB}{B - 240000} = \int 0.05\,dt,$$

so

$$\ln|B - 240000| = 0.05t + C$$
$$|B - 240000| = e^{0.05t+C} = e^{0.05t}e^{C},$$

so $B - 240000 = Ae^{0.05t}$, where $A = \pm e^{C}$.

If the initial balance is B_0, then $B_0 - 240000 = Ae^{0} = A$, thus $B - 240000 = (B_0 - 240000)e^{0.05t}$, so $B = (B_0 - 240000)e^{0.05t} + 240000$.

(c) To find the initial balance such that the account has a 0 balance after 20 years, we solve

$$0 = (B_0 - 240{,}000)e^{(0.05)20} + 240{,}000 = (B_0 - 240{,}000)e^{1} + 240{,}000,$$

$$B_0 = 240{,}000 - \frac{240{,}000}{e} \approx \$151{,}708.93.$$

40. (a) Since the rate of change is proportional to the amount present, $dy/dt = ky$ for some constant k.

(b) Solving the differential equation, we have $y = Ae^{kt}$, where A is the initial amount. Since 100 grams become 54.9 grams in one hour, $54.9 = 100e^{k}$, so $k = \ln(54.9/100) \approx -0.5997$.

Thus, after 10 hours, there remains $100e^{(-0.5997)10} \approx 0.2486$ grams.

41. Let $C(t)$ be the current flowing in the circuit at time t, then

$$\frac{dC}{dt} = -\alpha C$$

where $\alpha > 0$ is the constant of proportionality between the rate at which the current decays and the current itself.

The general solution of this differential equation is $C(t) = Ae^{-\alpha t}$ but since $C(0) = 30$, we have that $A = 30$, and so we get the particular solution $C(t) = 30e^{-\alpha t}$.

When $t = 0.01$, the current has decayed to 11 amps so that $11 = 30e^{-\alpha 0.01}$ which gives $\alpha = -100\ln(11/30) = 100.33$ so that,

$$C(t) = 30e^{-100.33t}.$$

42. Since the rate at which the volume, V, is decreasing is proportional to the surface area, A, we have

$$\frac{dV}{dt} = -kA,$$

where the negative sign reflects the fact that V is decreasing. Suppose the radius of the sphere is r. Then $V = \frac{4}{3}\pi r^3$ and, using the chain rule, $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. The surface area of a sphere is given by $A = 4\pi r^2$. Thus

$$4\pi r^2\frac{dr}{dt} = -k4\pi r^2$$

so

$$\frac{dr}{dt} = -k.$$

Since the radius decreases from 1 cm to 0.5 cm in 1 month, we have $k = 0.5$ cm/month. Thus

$$\frac{dr}{dt} = -0.5$$

so

$$r = -0.5t + r_0.$$

Since $r = 1$ when $t = 0$, we have $r_0 = 1$, so

$$r = -0.5t + 1.$$

We want to find t when $r = 0.2$, so

$$0.2 = -0.5t + 1$$

and

$$t = \frac{0.8}{0.5} = 1.6 \text{ months}.$$

43. By rewriting the equation, we see that it is logistic:

$$\frac{1}{P}\frac{dP}{dt} = \frac{(100 - P)}{1000}.$$

Before looking at its solution, we explain why there must always be at least 100 individuals. Since the population begins at 200, the quantity dP/dt is initially negative, so the population initially decreases. It continues to do so while $P > 100$. If the population ever reached 100, then dP/dt would be 0. This would mean the population stopped changing—so if the population ever decreased to 100, that's where it would stay. The fact that dP/dt is always negative for $P > 100$ also shows that the population is always under 200, as shown in Figure 11.104.

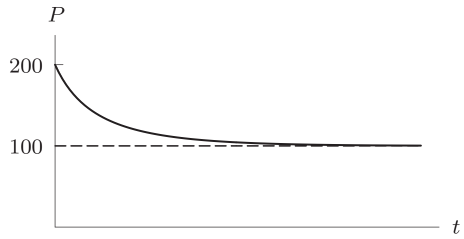


Figure 11.104

The solution, as given by the formula derived in the chapter, is

$$P = \frac{100}{1 - 0.5e^{-0.1t}}$$

44. (a) The logistic model is a reasonable one because in the 1990s, very few households had DVD players. Since DVD disc media is more convenient than VHS tape media, as DVD players dropped in price, more people bought them. However, the rate of DVD-player ownership had to start slowing down at some point as it is impossible for more than 100% of households to have DVD players.

(b) To find the point of inflection, we find the year at which the increase in DVD-player ownership alters from increasing to decreasing. The following table from 1998 to 2006.

Year	1998-99	1999-2000	2000-01	2001-02	2002-03	2003-04	2004-05	2005-06
Change in %	4	8	8	14	15	20	5	6

Looking at the table, we see that the change in percent alters from increasing to decreasing in about 2003. At this time 50% of households own DVD players. The inflection point is approximately $(2003, 50)$. Since at the inflection point on a logistic curve, the vertical coordinate is $L/2$, we have

$$L/2 = 50$$
$$L = 100.$$

Thus the limiting value is about 100%. Ownership may be expected to rise until the next new media format replaces it.

(c) Since the general form of a logistic function is

$$P = \frac{L}{1 + Ae^{-kt}}$$

where L is the limiting value, in this case we have $L = 86.395$, predicting a limiting value of 86.395%.

45. (a) Quantity of A present at time t equals $(a - x)$.
Quantity of B present at time t equals $(b - x)$.
So

$$\text{Rate of formation of } C = k(\text{Quantity of } A)(\text{Quantity of } B)$$

gives

$$\frac{dx}{dt} = k(a - x)(b - x)$$

(b) Separating gives

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt.$$

Rewriting the denominator as $(a-x)(b-x) = (x-a)(x-b)$ enables us to use Formula 26 in the Table of Integrals provided $a \neq b$. For some constant K, this gives

$$\frac{1}{a-b}\left(\ln|x-a| - \ln|x-b|\right) = kt + K.$$

Thus

$$\begin{aligned}
\ln\left|\frac{x-a}{x-b}\right| &= (a-b)kt + K(a-b)\\
\left|\frac{x-a}{x-b}\right| &= e^{K(a-b)}e^{(a-b)kt}\\
\frac{x-a}{x-b} &= Me^{(a-b)kt} \quad \text{where } M = \pm e^{K(a-b)}.
\end{aligned}$$

Since $x = 0$ when $t = 0$, we have $M = \frac{a}{b}$. Thus

$$\frac{x-a}{x-b} = \frac{a}{b}e^{(a-b)kt}.$$

Solving for x, we have

$$\begin{aligned}
bx - ba &= ae^{(a-b)kt}(x-b)\\
x(b - ae^{(a-b)kt}) &= ab - abe^{(a-b)kt}\\
x &= \frac{ab(1-e^{(a-b)kt})}{b - ae^{(a-b)kt}} = \frac{ab(e^{bkt} - e^{akt})}{be^{bkt} - ae^{akt}}.
\end{aligned}$$

46. Quantity of A left at time t = Quantity of B left at time t equals $(a-x)$.
Thus

$$\text{Rate of formation of } C = k(\text{Quantity of } A)(\text{Quantity of } B)$$

gives

$$\frac{dx}{dt} = k(a-x)(a-x) = k(a-x)^2.$$

Separating gives

$$\int \frac{dx}{(x-a)^2} = \int k\,dt$$

Integrating gives, for some constant K,

$$-(x-a)^{-1} = kt + K.$$

When $t = 0$, $x = 0$ so $K = a^{-1}$. Solving for x:

$$\begin{aligned}
-(x-a)^{-1} &= kt + a^{-1}\\
x - a &= -\frac{1}{kt + a^{-1}}\\
x &= a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}
\end{aligned}$$

47. (a) If alone, the x population grows exponentially, since if $y = 0$ we have $dx/dt = 0.01x$. If alone, the y population decreases to 0 exponentially, since if $x = 0$ we have $dy/dt = -0.2y$.

(b) This is a predator-prey relationship: interaction between populations x and y decreases the x population and increases the y population. The interaction of lions and gazelles might be modeled by these equations.

48. (a) If alone, the x and y populations each grow exponentially, because the equations become $dx/dt = 0.01x$ and $dy/dt = 0.2y$.

(b) For each population, the presence of the other decreases their growth rate. The two populations are therefore competitors—they may be eating each other's food, for instance. The interaction of gazelles and zebras might be modeled by these equations.

49. (a) The x population is unaffected by the y population—it grows exponentially no matter what the y population is, even if $y = 0$. If alone, the y population decreases to zero exponentially, because its equation becomes $dy/dt = -0.1y$.
(b) Here, interaction between the two populations helps the y population but does not effect the x population. This is not a predator-prey relationship; instead, this is a one-way relationship, where the y population is helped by the existence of x's. These equations could, for instance, model the interaction of rhinoceroses (x) and dung beetles (y).

50. (a) See Figure 11.105.
(b) The two equilibrium values are $P = 0$ and $P = 100$. Given any positive initial condition, the shrimp population will level off at 100 tons of shrimp in the bay.
(c) The new differential equation is

$$\frac{dP}{dt} = 0.8P(1 - 0.01P) - 10.$$

(d) Notice that subtracting 10 just moves the graph of dP/dt against P down 10 units. See Figure 11.106.
(e) We see in Figure 11.106 that the equilibrium values are at approximately $P = 14.6$ and $P = 85.4$.
(f) We see in Figure 11.106 that if $P = 12$, then dP/dt is negative. The shrimp population will decrease from $P = 12$. If $P = 25$ or $P = 75$, we see in Figure 11.106 that dP/dt is positive, so the shrimp populations will increase from either of these populations.

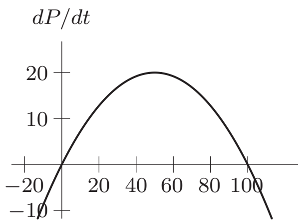


Figure 11.105

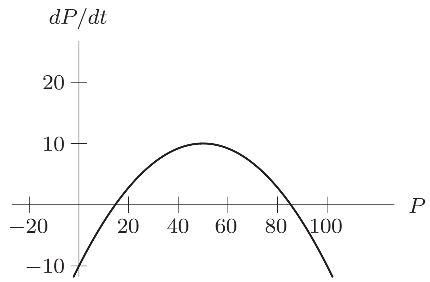


Figure 11.106

51. (a) We have

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{-3y - xy}{-2x - xy} = \frac{y(x+3)}{x(y+2)}.$$

Thus,

$$\left(\frac{y+2}{y}\right) dy = \left(\frac{x+3}{x}\right) dx$$

so

$$\int (1 + \frac{2}{y})\, dy = \int (1 + \frac{3}{x})\, dx.$$

So,

$$y + 2\ln|y| = x + 3\ln|x| + C.$$

Since x and y are non-negative,

$$y + 2\ln y = x + 3\ln x + C.$$

This is as far as we can go with this equation – we cannot solve for y in terms of x, for example. We can, however, put it in the form

$$e^{y+2\ln y} = e^{x+3\ln x+C}, \quad \text{or} \quad y^2 e^y = Ax^3 e^x.$$

(b) An equilibrium state satisfies

$$\frac{dx}{dt} = -2x - xy = 0 \quad \text{and} \quad \frac{dy}{dt} = -3y - xy = 0.$$

Solving the first equation, we have

$$-x(y+2) = 0, \quad \text{so} \quad x = 0 \quad \text{or} \quad y = -2.$$

The second equation has solutions

$$y = 0 \quad \text{or} \quad x = -3.$$

Since $x, y \geq 0$, the only equilibrium point is $(0, 0)$.

(c) We can use either of our forms for the solution. Looking at

$$y^2 e^y = Ax^3 e^x,$$

we see that if x and y are very small positive numbers, then

$$e^x \approx e^y \approx 1.$$

Thus,

$$y^2 \approx Ax^3, \quad \text{or} \quad \frac{y^2}{x^3} \approx A, \text{ a constant.}$$

Looking at

$$y + 2\ln y = x + 3\ln x + C,$$

we note that if x and y are small, then they are negligible compared to $\ln y$ and $\ln x$. Thus,

$$2\ln y \approx 3\ln x + C,$$

giving

$$\ln y^2 - \ln x^3 \approx C,$$

so

$$\ln \frac{y^2}{x^3} \approx C$$

and therefore

$$\frac{y^2}{x^3} \approx e^C, \text{ a constant.}$$

(d) If

$$x(0) = 4 \quad \text{and} \quad y(0) = 8,$$

then

$$8 + 2\ln 8 = 4 + 3\ln 4 + C.$$

Note that

$$2\ln 8 = 3\ln 4 = \ln 64,$$

giving

$$4 = C.$$

So the phase trajectory is

$$y + 2\ln y = x + 3\ln x + 4.$$

(Or equivalently, $y^2 e^y = e^4 x^3 e^x = x^3 e^{x+4}$.)

(e) If the concentrations are equal, then

$$y + 2\ln y = y + 3\ln y + 4,$$

giving

$$-\ln y = 4 \quad \text{or} \quad y = e^{-4}.$$

Thus, they are equal when $y = x = e^{-4} \approx 0.0183$.

(f) Using part (c), we have that if x is small,

$$\frac{y^2}{x^3} \approx e^4.$$

Since $x = e^{-10}$ is certainly small,

$$\frac{y^2}{e^{-30}} \approx e^4, \quad \text{and} \quad y \approx e^{-13}.$$

52. (a) Using the chain-rule we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

giving

$$\frac{dy}{dx} = \frac{-ax}{-ay} = \frac{x}{y}.$$

so

$$\int y\,dy = \int x\,dx,$$

giving

$$y^2 - x^2 = C.$$

Figure 11.107 shows the graph of the slope field of this equation with the solution satisfying $y = 46$ when $x = 40$ sketched in.

When $t = 0$, $y = 46$, $x = 40$, so $C = 46^2 - 40^2 = 516$. Therefore, the solution is $y^2 - x^2 = 516$.

(b) The battle is over when $x = 0$. The number of French/Spanish ships remaining is the y-intercept, thus, $y^2 - 0^2 = 516$ giving $y \approx 22.7$ French/Spanish ships remaining.

(c) For both sub-battles, the solution trajectory will have the form $y^2 - x^2 = C$. Each sub-battle will have a different value of C.

For the 32 versus 23 sub-battle, $C = 23^2 - 32^2 = -495$, so the trajectory is:

$$y^2 - x^2 = -495$$

or

$$x^2 - y^2 = 495.$$

This has no y-intercept, but an x-intercept of $x = \sqrt{495} \approx 22.2$, meaning that the model predicts the British won the sub-battle with about 22.2 ships remaining.

For the 8 versus 23 sub-battle, $C = 23^2 - 8^2 = 465$ so the trajectory is $y^2 - x^2 = 465$. This has no x-intercept, but a y-intercept of $y = \sqrt{465} \approx 21.6$, meaning that the model predicts a French/Spanish victory with about 21.6 ships remaining.

(d) If the remaining ships from these two sub-battles then fight each other, the British have a slight advantage (22.2 versus 21.6). Thus the British could be expected to win the overall battle, although they started with a weaker fleet. This is in fact what happened.

The trajectory for this last battle has $C = 495 - 460 = 30$, so the equation is

$$x^2 - y^2 = 30.$$

This has an x-intercept of $x = \sqrt{30} \approx 5.5$, so the model predicts a British victory with about $5\frac{1}{2}$ ships remaining.

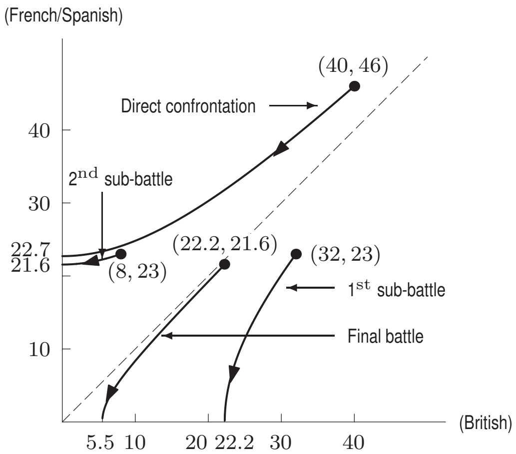


Figure 11.107

53. (a) The insects grow exponentially with no birds around (the equation becomes $dx/dt = 3x$); the birds die out exponentially with no insects to feed on $dy/dt = -10y$). The interaction increases the birds' growth rate (the $+0.001xy$ term is positive), but decreases the insects' growth rate (the $-0.02xy$ term is negative). This is as we would expect: having the insects around helps the birds; having birds around hurts the insects.

(b) Equilibrium solutions occur where both derivatives are zero:

$$(3 - 0.02y)x = 0$$

$$-(10 - 0.001x)y = 0.$$

We see that the solutions are $(0, 0)$ and $(10,000, 150)$

(c) The chain rule gives an equation for dy/dx:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-10+0.001x)}{x(3-0.02y)}.$$

Separation of variables gives

$$\int \frac{-10+0.001x}{x}\,dx = \int \frac{3-0.02y}{y}\,dy,$$

which yields $3\ln y - 0.02y = -10\ln x + 0.001x + C$.
Using the initial point $A = (10,000, 160)$, we have

$$3\ln 160 - 0.02(160) = -10\ln 10{,}000 + 0.001(10{,}000) + C.$$

Thus, $C \approx 94.13$ and the solution is $3\ln y - 0.02y = -10\ln x + 0.001x + 94.1$

(d) We can check that the equation is satisfied by points B, C, D by substituting the coordinates into the equation $3\ln y - 0.02y = -10\ln x + 0.001x + 94.1$.

(e) The phase plane and the trajectory is in Figure 11.108.

(f) Consider point A. We have

$$\frac{dy}{dt} = 0, \quad \text{and} \quad \frac{dx}{dt} = 3(10,000) - 0.02(10,000)(160) = -2000 < 0.$$

Thus x is decreasing at point A. Hence the rotation is counterclockwise in the phase plane, and the order of traversal is $A \longrightarrow B \longrightarrow C \longrightarrow D$.

(g) The graphs of x and y versus t are in Figure 11.109.

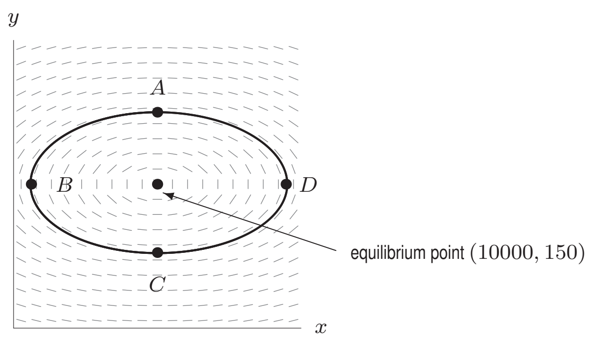


Figure 11.108

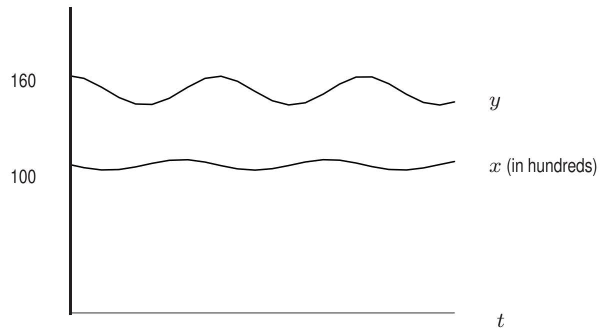


Figure 11.109

(h) At points A and C, we have $dy/dx = 0$, and at B and D, we have $dx/dy = 0$, so these points are extrema: y is maximized at A, minimized at C; x is maximized at D, minimized at B.

54. (a) In the equation for dx/dt, the term involving x, namely $-0.2x$, is negative meaning that as x increases, dx/dt decreases. This corresponds to the statement that the more a country spends on armaments, the less it wants to increase spending.

On the other hand, since $+0.15y$ is positive, as y increases, dx/dt increases, corresponding to the fact that the more a country's opponent arms, the more the country will arm itself.

The constant term, 20, is positive means that if both countries are unarmed initially, (so $x = y = 0$), then dx/dt is positive and so the country will start to arm. In other words, disarmament is not an equilibrium situation in this model.

(b) The nullclines are shown in Figure 11.110. When $dx/dt = 0$, the trajectories are vertical (on the line $-0.2x + 0.15y + 20 = 0$); when $dy/dt = 0$ the trajectories are horizontal (on $0.1x - 0.2y + 40 = 0$). There is only one equilibrium point, $x = y = 400$.

(c) In region I, try $x = 400$, $y = 0$, giving

$$\frac{dx}{dt} = -0.2(400) + 0.15(0) + 20 < 0$$
$$\frac{dy}{dt} = 0.1(400) - 0.2(0) + 4 - 0 > 0$$

In region II, try $x = 500$, $y = 500$, giving

$$\frac{dx}{dt} = -0.2(500) + 0.15(500) + 20 < 0$$
$$\frac{dy}{dt} = 0.1(500) - 0.2(500) + 40 < 0$$

In region III, try $x = 0$, $y = 400$, giving

$$\frac{dx}{dt} = -0.2(0) + 0.15(400) + 20 > 0$$
$$\frac{dy}{dt} = 0.1(0) - 0.2(400) + 40 < 0$$

In region IV, try $x = 0$, $y = 0$, giving

$$\frac{dx}{dt} = -0.2(0) + 0.15(0) + 20 > 0$$
$$\frac{dy}{dt} = 0.1(0) - 0.2(0) + 40 > 0$$

See Figure 11.110.

(d) The one equilibrium point is stable.

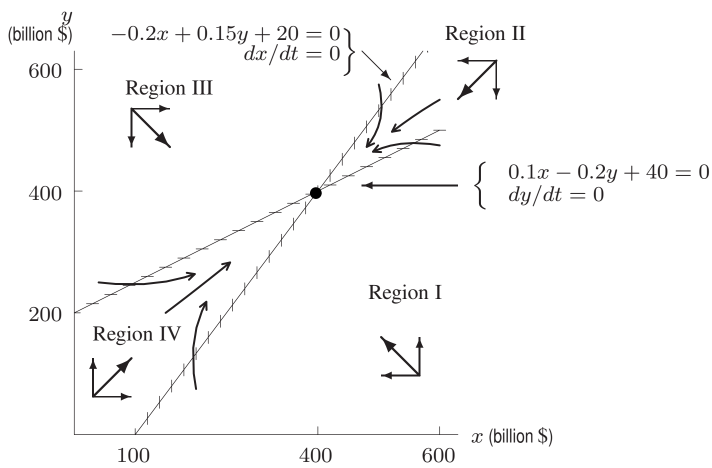


Figure 11.110: Nullclines and equilibrium point(dot) for arms race

(e) If both sides disarm, then both sides spend \$0. Thus initially $x = y = 0$, and $dx/dt = 20$ and $dy/dt = 40$. Since both dx/dt and dy/dt are positive, both sides start arming. Figure 11.110 shows that they will both arm until each is spending about \$400 billion.

(f) If the country spending $\$y$ billion is unarmed, then $y = 0$ and the corresponding point on the phase plane is on the x-axis. Any trajectory starting on the x-axis tends toward the equilibrium point $x = y = 400$. Similarly, a trajectory starting on the y-axis represents the other country being unarmed; such a trajectory also tends to the same equilibrium point.

Thus, if either side disarms unilaterally, that is, if we start out with one of the countries spending nothing, then over time, they will still both end up spending roughly \$400 billion.

(g) This model predicts that, in the long run, both countries will spend near to \$400 billion, no matter where they start.

55. If $0 < P < L$, then $P/L < 1$ and $P/(2L) < 1$:

$$\frac{dP}{dt} = -\underbrace{k}_{+}\underbrace{\left(1-\frac{P}{L}\right)}_{+}\underbrace{\left(1-\frac{P}{2L}\right)}_{+} < 0.$$

Thus, if initially there are fewer than L animals, $dP/dt < 0$ and the population will decrease.

56. If $L < P < 2L$, then $P/L > 1$ and $P/(2L) < 1$:

$$\frac{dP}{dt} = -\underbrace{k}_{+}\underbrace{\left(1-\frac{P}{L}\right)}_{-}\underbrace{\left(1-\frac{P}{2L}\right)}_{+} > 0.$$

Thus, if initially there are between L and $2L$ animals, $dP/dt > 0$ and the population will increase. Since $P = 2L$ is an equilibrium solution, the population will increase towards $P = 2L$.

57. If $P > 2L$, then $P/L > 1$ and $P/(2L) > 1$:

$$\frac{dP}{dt} = -\underbrace{k}_{+}\underbrace{\left(1-\frac{P}{L}\right)}_{-}\underbrace{\left(1-\frac{P}{2L}\right)}_{-} < 0.$$

Thus, if initially there are more than $2L$ animals, $dP/dt < 0$ and the population will decrease. Since $P = 2L$ is an equilibrium solution, the population will decrease towards $P = 2L$.

CAS Challenge Problems

58. (a) We find the equilibrium solutions by setting $dP/dt = 0$, that is, $P(P-1)(2-P) = 0$, which gives three solutions, $P = 0$, $P = 1$, and $P = 2$.

(b) To get your computer algebra system to check that P_1 and P_2 are solutions, substitute one of them into the equation and form an expression consisting of the difference between the right and left hand sides, then ask the CAS to simplify that expression. Do the same for the other function. In order to avoid too much typing, define P_1 and P_2 as functions in your system.

(c) Substituting $t = 0$ gives

$$P_1(0) = 1 - \frac{1}{\sqrt{4}} = 1/2$$
$$P_2(0) = 1 + \frac{1}{\sqrt{4}} = 3/2.$$

We can find the limits using a computer algebra system. Alternatively, setting $u = e^t$, we can use the limit laws to calculate

$$\begin{aligned}\lim_{t\to\infty} \frac{e^t}{\sqrt{3+e^{2t}}} &= \lim_{u\to\infty} \frac{u}{\sqrt{3+u^2}} = \lim_{u\to\infty} \sqrt{\frac{u^2}{3+u^2}} \\ &= \sqrt{\lim_{u\to\infty} \frac{u^2}{3+u^2}} = \sqrt{\lim_{u\to\infty} \frac{1}{\frac{3}{u^2}+1}} \\ &= \sqrt{\frac{1}{\lim\limits_{u\to\infty} \dfrac{3}{u^2}+1}} = \sqrt{\frac{1}{0+1}} = 1.\end{aligned}$$

Therefore, we have

$$\lim_{t\to\infty} P_1(t) = 1 - 1 = 0$$
$$\lim_{t\to\infty} P_2(t) = 1 + 1 = 2.$$

To predict these limits without having a formula for P, looking at the original differential equation. We see if $0 < P < 1$, then $P(P-1)(2-P) < 0$, so $P' < 0$. Thus, if $0 < P(0) < 1$, then $P'(0) < 0$, so P is initially decreasing, and tends toward the equilibrium solution $P = 0$. On the other hand, if $1 < P < 2$, then $P(P-1)(2-P) > 0$, so $P' > 0$. So, if $1 < P(0) < 2$, then $P'(0) > 0$, so P is initially increasing and tends toward the equilibrium solution $P = 2$.

59. (a) Using the integral equation with $n+1$ replaced by n, we have

$$y_n(a) = b + \int_a^a (y_{n-1}(t)^2 + t^2)\,dt = b + 0 = b.$$

(b) We have $a = 1$ and $b = 0$, so the integral equation tells us that

$$y_{n+1}(s) = \int_1^s (y_n(t)^2 + t^2)\,dt.$$

With $n = 0$, since $y_0(s) = 0$, the CAS gives

$$y_1(s) = \int_1^s 0 + t^2\,dt = -\frac{1}{3} + \frac{s^3}{3}.$$

Then

$$y_2(s) = \int_1^s (y_1(t)^2 + t^2)\,dt = -\frac{17}{42} + \frac{s}{9} + \frac{s^3}{3} - \frac{s^4}{18} + \frac{s^7}{63},$$

and

$$\begin{aligned} y_3(s) &= \int_1^s (y_2(t)^2 + t^2)\,dt \\ &= -\frac{157847}{374220} + \frac{289\,s}{1764} - \frac{17\,s^2}{378} + \frac{82\,s^3}{243} - \frac{17\,s^4}{252} + \frac{s^5}{42} - \frac{s^6}{486} + \frac{s^7}{63} - \frac{11\,s^8}{1764} + \\ &\quad \frac{5\,s^9}{6804} + \frac{2\,s^{11}}{2079} - \frac{s^{12}}{6804} + \frac{s^{15}}{59535}. \end{aligned}$$

(c) The solution y, and the approximations y_1, y_2, y_3 are graphed in Figure 11.111. The approximations appear to be accurate on the range $0.5 \le s \le 1.5$.

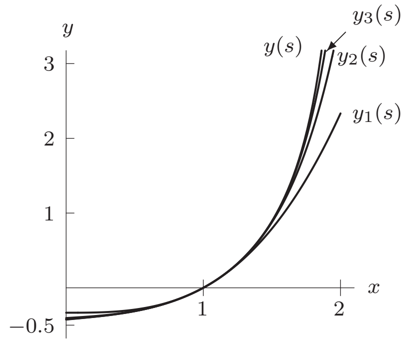


Figure 11.111

60. (a) See Figure 11.112.

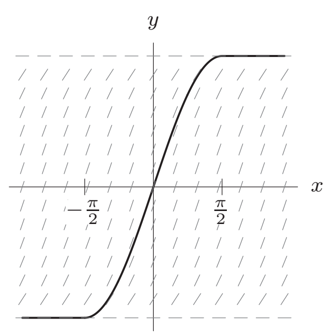


Figure 11.112

(b) Different CASs give different answers, for example they might say $y = \sin x$, or they might say

$$y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

(c) Both the sample CAS answers in part (b) are wrong. The first one, $y = \sin x$, is wrong because $\sin x$ starts decreasing at $x = \pi/2$, where the slope field clearly shows that y should be increasing at all times. The second answer is better, but it does not give the solution outside the range $-\pi/2 \le x \le \pi/2$. The correct answer is the one sketched in Figure 11.112, which has formula

$$y = \begin{cases} -1 & x \le -\frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & x \le \frac{\pi}{2} \le x. \end{cases}$$

PROJECTS FOR CHAPTER ELEVEN

1. Assuming dT_r/dt is proportional to T_r, we have:

$$\frac{dT_r}{dt} = -kT_r \qquad \text{where } k \text{ is a positive constant of proportionality.}$$

The solution to this differential equation is the exponential decay function,

$$T_r = Ae^{-kt}.$$

The solution was derived by separating variables:

$$\begin{aligned} \int \frac{1}{T_r}\, dT_r &= \int -k\, dt \\ \ln|T_r| &= -kt + C \\ |T_r| &= e^C e^{-kt} \\ T_r &= Ae^{-kt}. \quad \text{where } |A| = e^C. \end{aligned}$$

We find the values of A and k from the data in Table 11.12.We know that:

$$\begin{aligned} Ae^{-k\cdot 4} &= 37 && \text{since } T_r = 37 \text{ at } t = 4 \\ Ae^{-k\cdot 19.5} &= 13 && \text{since } T_r = 13 \text{ at } t = 19.5 \\ \text{so} \quad \frac{Ae^{-k\cdot 4}}{Ae^{-k\cdot 19.5}} &= \frac{37}{13} && \text{dividing} \\ e^{15.5k} &= \frac{37}{13} \\ k = \frac{1}{15.5}\ln\left(\frac{37}{13}\right) &= 0.06748. \end{aligned}$$

Having found k, we can now find A:

$$\begin{aligned} Ae^{-0.06748(4)} &= 37 \qquad \text{since } T_r = 37 \text{ at } t = 4 \\ A &= 48.5 \text{ ng/ml}. \end{aligned}$$

Thus, the concentration of tryptase t hours after surgery is modeled by the exponential decay function

$$T_r = 48.5e^{-0.06748t} \text{ ng/ml}.$$

The greatest concentration is 48.5 ng/ml, which occurs at $t = 0$ when the patient leaves surgery. We can diagnose anaphylaxis since this peak level is above 45 ng/ml.

2. (a) (i) Integrating we have

$$\frac{dP}{dt} = 30.2$$
$$P = 30.2t + C.$$

Since $P(0) = 95$, we have $C = 95$, so

$$P = 30.2t + 95.$$

Substituting $t = 87$ and rounding to the nearest person gives

$$P = 30.2 \cdot 87 + 95 = 2722.$$

The linear model predicts that 2722 people would have contracted SARS by June 12, 2003.

(ii) Separating variables, we have

$$\frac{1}{P}\frac{dP}{dt} = 0.12$$
$$\int \frac{dP}{P} = \int 0.12\,dt$$
$$\ln|P| = 0.12t + C$$
$$P = Ae^{0.12t}.$$

Since $P(0) = 95$, we have $A = 95$, so

$$P = 95e^{0.12t}.$$

Substituting $t = 87$ and rounding to the nearest person gives

$$P = 95e^{0.12 \cdot 87} = 3,249,062.$$

The exponential model predicts that 3.249 million people would have contracted SARS by June 12, 2003.

(iii) Writing the differential equation in the form

$$\frac{dP}{dt} = 0.19P - 0.0002P^2$$
$$\frac{dP}{dt} = 0.19P\left(1 - \frac{0.0002}{0.19}P\right)$$
$$\frac{dP}{dt} = 0.19P\left(1 - \frac{P}{950}\right),$$

we use the analytic solution derived on page 630 of the text to obtain

$$P = \frac{950}{1 + Ae^{-0.19t}}, \quad \text{with } A = \frac{950 - 95}{95} = 9$$

so

$$P = \frac{950}{1 + 9e^{-0.19t}}.$$

Substituting $t = 87$ and rounding to the nearest person gives

$$P = \frac{950}{1 + 9e^{-0.19 \cdot 87}} = 950.$$

The logistic model predicts that 950 people will have contracted SARS by June 12, 2003.

(b) (i) The three methods give very different predictions. The linear and logistic are about 3000 and 1000, respectively, while the exponential model is 3 million, nearly half the population of Hong Kong.

(ii) The number of new cases per day is approximated by the derivative, dP/dt. The linear model predicts a constant number of new cases each day; the exponential model predicts an increasing number of new cases each day; the logistic model predicts that the number of new cases per day will first increase and then decrease.

(iii) The general trend in the figure shows that the number of new cases per day first climbed and then fell, suggesting that the logistic model fits best. The high values are largely Mondays, and represent two days of data recorded as one, since no new cases were reported on Sundays.

(c) (i) The formula

$$P = \frac{950}{1 + 9e^{-0.19t}}$$

has limiting value $P = 950$ as $t \to \infty$. Thus, this formula predicts that the maximum number of cases expected is 950.

(ii) The graph allows us to estimate (very roughly) when the daily increase was largest, namely about April 10. Since the maximum rate of change of P (and the maximum daily increase in P) occurs at $L/2$, where L is the maximum value of P, we expect the maximum value of P to be about $2 \cdot 998 \approx 2000$.

(d) See Figures 11.113–11.115. The dots represent the actual data.

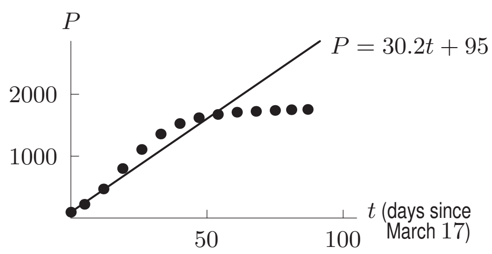


Figure 11.113: Linear predictions and actual data

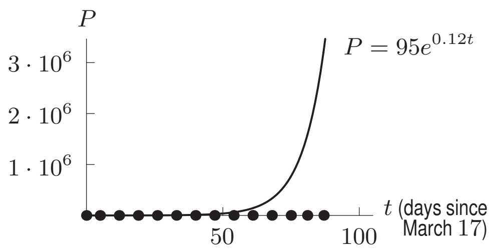


Figure 11.114: Exponential predictions and actual data

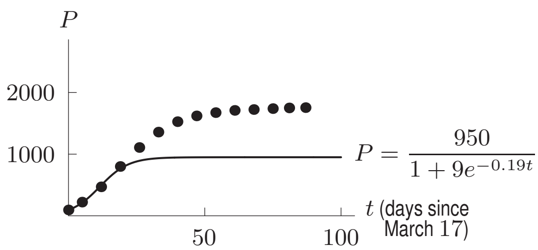


Figure 11.115: Logistic predictions and actual data

3. **(a)** Since I_0 is the number of infecteds on day $t = 0$, March 17, we have $I_0 = 95$. Since S_0 is the initial number of susceptibles, which is the whole population of Hong Kong, $S_0 \approx 6.8$ million.

(b) For $a = 1.25 \cdot 10^{-8}$ and $b = 0.06$, the system of equations is

$$\frac{dS}{dt} = -1.25 \cdot 10^{-8} SI$$
$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.06I.$$

So, by the chain rule,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{1.25 \cdot 10^{-8} SI - 0.06I}{-1.25 \cdot 10^{-8} SI} = -1 + \frac{4.8 \cdot 10^6}{S}.$$

The slope field and trajectory are in Figure 11.116.

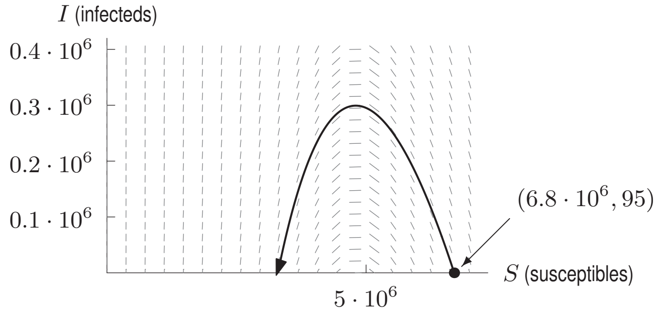


Figure 11.116

(c) The maximum value of I is about 300,000; this gives us the maximum number of infecteds at any one time—the total number of people infected during the course of the disease is much greater than this. The trajectory meets the S-axis at about 3.3 million; this tells us that when the disease dies out, there are still 3.3 million susceptibles who have never had the disease. Therefore $6.8 - 3.3 = 3.5$ million people are predicted to have had the disease.

The threshold value of S occurs where $dI/dt = 0$ and $I \neq 0$, so, for $a = 1.25 \cdot 10^{-8}$ and $b = 0.06$,

$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.06I = 0,$$

giving

$$\text{Threshold value} = S = \frac{0.06}{1.25 \cdot 10^{-8}} = 4.8 \cdot 10^6 \text{ people.}$$

The threshold value tells us that if the initial susceptible population, S_0 is more than 4.8 million, there will be an epidemic. If S_0 is less than 4.8 million, there will not be an epidemic. Since the population of Hong Kong is over 4.8 million, an epidemic is predicted.

(d) The value of b represents the rate at which infecteds are removed from circulation. Quarantine increases the rate people are removed and thus increases b.

(e) For $a = 1.25 \cdot 10^{-8}$ and $b = 0.24$, the system of differential equations is

$$\frac{dS}{dt} = -1.25 \cdot 10^{-8} SI$$
$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.24I.$$

So, by the chain rule,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{1.25 \cdot 10^{-8} SI - 0.24I}{-1.25 \cdot 10^{-8} SI} = -1 + \frac{19.2 \cdot 10^6}{S}.$$

The slope field is in Figure 11.117. The solution trajectory does not show as the disease dies out right away.

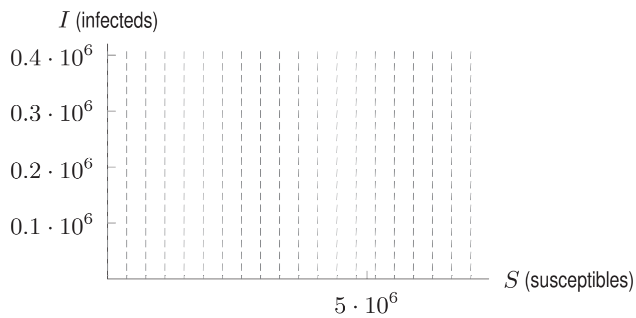


Figure 11.117

(f) The threshold value of S occurs where $dI/dt = 0$ and $I \neq 0$, so, for $b = 0.24$ and the same value of a,

$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.24I = 0,$$

giving

$$\text{Threshold value } = S = \frac{0.24}{1.25 \cdot 10^{-8}} = 19.2 \cdot 10^6 \text{ people.}$$

The threshold value tells us that if S_0 is less than 19.2 million, there will be no epidemic. The population of Hong Kong is 6.8 million, so S_0 is below this value. Thus no epidemic is predicted.

Policies, such as quarantine, which raise the value of b can be effective in preventing an epidemic. In this case, the value of b increased sufficiently that the population of Hong Kong fell below the threshold value, and a potential epidemic was averted. However, we do not have evidence that the quarantine policy was responsible for the increase in b.

(g) Policy I: Closing off the city changes the initial values of S_0 and I_0 but not the values of a and b. If not one infected person enters the city, then $I_0 = 0$ and the solution trajectory is an equilibrium point on the S-axis. However, in practice it is almost impossible to cut off a city completely, so usually $I_0 > 0$. Also, by the time a policy to close off a city is put into effect, there may already be infected people inside the city, so again $I_0 > 0$. Thus, whether or not there is an epidemic depends on whether S_0 is greater than the threshold value, not on the value of I_0 (provided $I_0 > 0$).

For example, in the case of Hong Kong with the March values of a and b, changing the value of I_0 to 1 leaves the solution trajectory much as before; see Figure 11.118. The main difference is that the epidemic occurs slightly later. So a policy of isolating a city only works if it keeps the disease out of the city of the city entirely. Thus, Policy I does not help the city except in the exceptional case that *every* infected person is kept out.

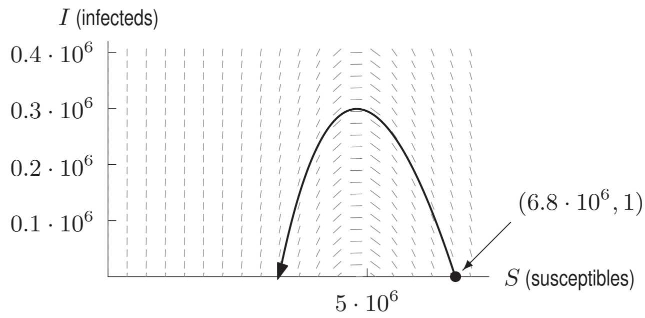


Figure 11.118

Policy II: From the analysis of the Hong Kong data, we see that a quarantine policy can help prevent an epidemic if the value of b is increased enough to bring S_0 below the threshold value. Thus, Policy II can be very effective.

4. (a)

$$p(x) = \text{the number of people with incomes } \geq x.$$

$$p(x + \Delta x) = \text{the number of people with incomes } \geq x + \Delta x.$$

So the number of people with incomes between x and $x + \Delta x$ is

$$p(x) - p(x + \Delta x) = -\Delta p.$$

Since all the people with incomes between x and $x + \Delta x$ have incomes of about x (if Δx is small), the total amount of money earned by people in this income bracket is approximately $x(-\Delta p) = -x\Delta p$.

(b) Pareto's law claims that the average income of all the people with incomes $\geq x$ is kx. Since there are $p(x)$ people with income $\geq x$, the total amount of money earned by people in this group is $kxp(x)$.

The total amount of money earned by people with incomes $\geq (x+\Delta x)$ is therefore $k(x+\Delta x)p(x+\Delta x)$. Then the total amount of money earned by people with incomes between x and $x+\Delta x$ is

$$kxp(x) - k(x+\Delta x)p(x+\Delta x).$$

Since $\Delta p = p(x+\Delta x) - p(x)$, we can substitute $p(x+\Delta x) = p(x) + \Delta p$. Thus the total amount of money earned by people with incomes between x and $x+\Delta x$ is

$$kxp(x) - k(x+\Delta x)(p(x)+\Delta p).$$

Multiplying out, we have

$$kxp(x) - kxp(x) - k(\Delta x)p(x) - kx\Delta p - k\Delta x\Delta p$$

Simplifying and dropping the second order term $\Delta x\Delta p$ gives the total amount of money earned by people with incomes between x and $x+\Delta x$ as

$$-kp\Delta x - kx\Delta p.$$

(c) Setting the answers to parts (a) and (b) equal gives

$$-x\Delta p = -kp\Delta x - kx\Delta p.$$

Dividing by Δx, and letting $\Delta x \to 0$ so that $\frac{\Delta p}{\Delta x} \to p'$, we have

$$\begin{aligned} x\frac{\Delta p}{\Delta x} &= kp + kx\frac{\Delta p}{\Delta x} \\ xp' &= kp + kxp' \end{aligned}$$

so

$$(1-k)xp' = kp.$$

(d) We solve this equation by separating variables

$$\begin{aligned} \int \frac{dp}{p} &= \int \frac{k}{(1-k)}\frac{dx}{x} \\ \ln p &= \frac{k}{(1-k)}\ln x + C \quad \text{(no absolute values needed since } p, x > 0) \\ \ln p &= \ln x^{k/(1-k)} + \ln A \quad \text{(writing } C = \ln A) \\ \ln p &= \ln[Ax^{k/(1-k)}] \quad \text{(using } \ln(AB) = \ln A + \ln B) \\ p &= Ax^{k/(1-k)} \end{aligned}$$

(e) We take $A = 1$. For $k = 10$, $p = x^{-10/9} \approx x^{-1}$. For $k = 1.1$, $p = x^{-11}$. The functions are graphed in Figure 11.119. Notice that the larger the value of k, the less negative the value of $k/(1-k)$ (remember $k > 1$), and the slower $p(x) \to 0$ as $x \to \infty$.

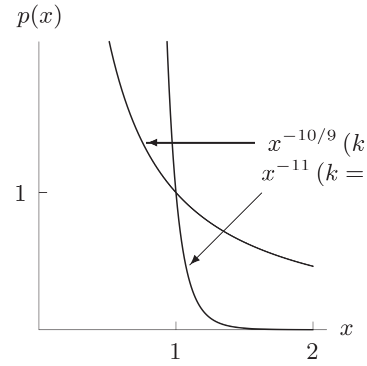


Figure 11.119

5. (a) Writing $F = b\left(\frac{a^2-ar}{r^3}\right) = 0$ shows $F = 0$ when $r = a$, so $r = a$ gives the equilibrium position.

(b) Expanding $1/r^3$ about $r = a$ gives

$$\begin{aligned}\frac{1}{r^3} = \frac{1}{(a+r-a)^3} &= \frac{1}{a^3}\left(1+\frac{r-a}{a}\right)^{-3} \\ &= \frac{1}{a^3}\left(1-3\left(\frac{r-a}{a}\right)+\frac{(-3)(-4)}{2!}\left(\frac{r-a}{a}\right)^2-\cdots\right) \\ &= \frac{1}{a^3}\left(1-\frac{3(r-a)}{a}+\frac{6(r-a)^2}{a^2}-\cdots\right).\end{aligned}$$

Similarly, expanding $1/r^2$ about $r = a$ gives

$$\begin{aligned}\frac{1}{r^2} = \frac{1}{(a+r-a)^2} &= \frac{1}{a^2}\left(1+\frac{r-a}{a}\right)^{-2} \\ &= \frac{1}{a^2}\left(1-2\left(\frac{r-a}{a}\right)+\frac{(-2)(-3)}{2!}\left(\frac{r-a}{a}\right)^2-\cdots\right) \\ &= \frac{1}{a^2}\left(1-2\left(\frac{r-a}{a}\right)+3\left(\frac{r-a}{a}\right)^2-\cdots\right).\end{aligned}$$

Thus, combining gives

$$\begin{aligned}F &= b\left(\frac{1}{a}\left(1-\frac{3(r-a)}{a}+\frac{6(r-a)^2}{a^2}-\cdots\right)-\frac{1}{a}\left(1-\frac{2(r-a)}{a}+\frac{3(r-a)^2}{a^2}-\cdots\right)\right) \\ &= \frac{b}{a}\left(-\frac{(r-a)}{a}+\frac{3(r-a)^2}{a^2}-\cdots\right) \\ &= \frac{b}{a^2}\left(-(r-a)+\frac{3(r-a)^2}{a}-\cdots\right).\end{aligned}$$

(c) Setting $x = r - a$ gives

$$F \approx \frac{b}{a^2}\left(-x+\frac{3x^2}{a}\right).$$

(d) For small x, we discard the quadratic term in part (c), giving

$$F \approx \frac{-b}{a^2}x.$$

The acceleration is d^2x/dt^2. Thus, using Newton's Second Law:

$$\text{Force} = \text{Mass}\cdot\text{Acceleration}$$

we get

$$\frac{-bx}{a^2} = m\frac{d^2x}{dt^2}.$$

So

$$\frac{d^2x}{dt^2}+\frac{b}{a^2m}x = 0.$$

This differential equation represents an oscillation of the form $x = C_1\cos\omega t + C_2\sin\omega t$, where $\omega^2 = b/(a^2m)$ so $\omega = \sqrt{b/(a^2m)}$. Thus, we have

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi a\sqrt{\frac{m}{b}}.$$

CHAPTER TWELVE

Solutions for Section 12.1

Exercises

1. The point P is $\sqrt{1^2+2^2+1^2} = \sqrt{6} = 2.45$ units from the origin, and Q is $\sqrt{2^2+0^2+0^2} = 2$ units from the origin. Since $2 < \sqrt{6}$, the point Q is closer.

2. The distance formula: $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ gives us the distance between any pair of points (x_1,y_1,z_1) and (x_2,y_2,z_2). Thus, we find

$$\begin{aligned}\text{Distance from } P_1 \text{ to } P_2 &= 2\sqrt{2}\\ \text{Distance from } P_2 \text{ to } P_3 &= \sqrt{6}\\ \text{Distance from } P_1 \text{ to } P_3 &= \sqrt{10}\end{aligned}$$

So P_2 and P_3 are closest to each other.

3. The distance of a point $P = (x, y, z)$ from the yz-plane is $|x|$, from the xz-plane is $|y|$, and from the xy-plane is $|z|$. So, B is closest to the yz-plane, since it has the smallest x-coordinate in absolute value. B lies on the xz-plane, since its y-coordinate is 0. B is farthest from the xy-plane, since it has the largest z-coordinate in absolute value.

4. Your final position is $(1, -1, 1)$. This places you in front of the yz-plane, to the left of the xz-plane, and above the xy-plane.

5. An example is the line $y = z$ in the yz-plane. See Figure 12.1.

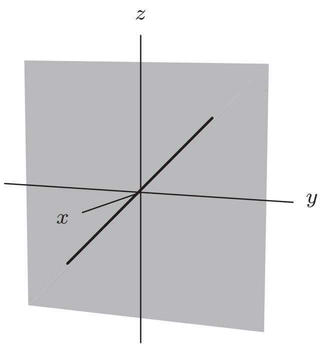


Figure 12.1

6. The midpoint is found by averaging coordinates:

$$\text{Midpoint } = \left(\frac{-1+5}{2}, \frac{3+6}{2}, \frac{9-3}{2}\right) = (2, 4.5, 3).$$

7. The graph is a horizontal plane at height 4 above the xy-plane. See Figure 12.2.

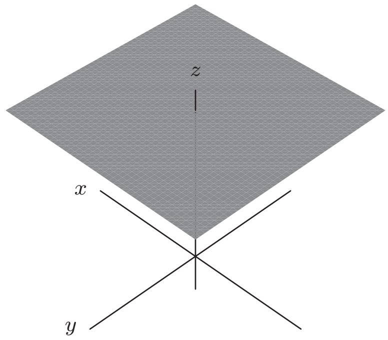


Figure 12.2

8. The graph is a plane parallel to the yz-plane, and passing through the point $(-3, 0, 0)$. See Figure 12.3.

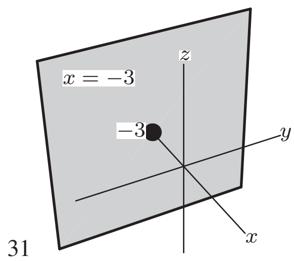


Figure 12.3

9. The graph is a plane parallel to the xz-plane, and passing through the point $(0, 1, 0)$. See Figure 12.4.

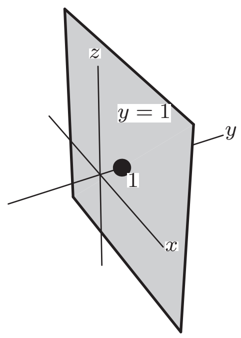


Figure 12.4

10. The graph is all points with $y = 4$ and $z = 2$, i.e., a line parallel to the x-axis and passing through the points $(0, 4, 2)$; $(2, 4, 2)$; $(4, 4, 2)$ etc. See Figure 12.5.

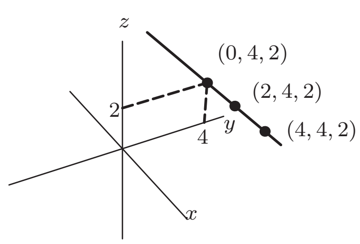


Figure 12.5

11. The radius is $7-(-1)=8$, so the highest point is at $(2,3,15)$.

12. The equation is $x^2+y^2+z^2=25$

13. The sphere has equation $(x-1)^2+y^2+z^2=4$.

14. The plane has equation $y=3$.

15. (a) 80-90°F
(b) 60-72°F
(c) 60-100°F

16.

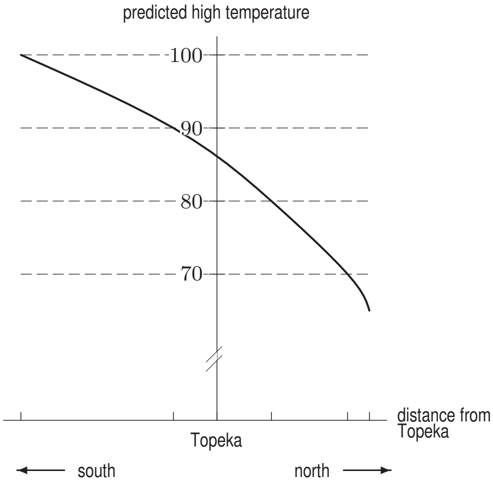


Figure 12.6

17.

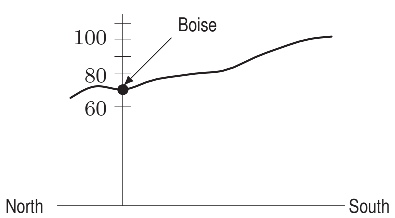


Figure 12.7

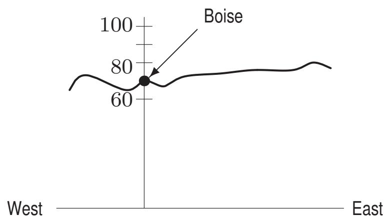


Figure 12.8

18. Beef consumption by households making \$20,000/year is given by Row 1 of Table 12.1 on page 667 of the text.

Table 12.1

p	3.00	3.50	4.00	4.50
$f(20,p)$	2.65	2.59	2.51	2.43

For households making \$20,000/year, beef consumption decreases as price goes up.
Beef consumption by households making $\$100,000$/year is given by Row 5 of Table 12.1.

Table 12.2

p	3.00	3.50	4.00	4.50
$f(100,p)$	5.79	5.77	5.60	5.53

For households making \$100,000/year, beef consumption also decreases as price goes up.
Beef consumption by households when the price of beef is \$3.00/lb is given by Column 1 of Table 12.1.

Table 12.3

I	20	40	60	80	100
$f(I, 3.00)$	2.65	4.14	5.11	5.35	5.79

When the price of beef is \$3.00/lb, beef consumption increases as income increases.
Beef consumption by households when the price of beef is \$4.00/lb is given by Column 3 of Table 12.1.

Table 12.4

I	20	40	60	80	100
$f(I, 4.00)$	2.51	3.94	4.97	5.19	5.60

When the price of beef is \$4.00/lb, beef consumption increases as income increases.

19. Table 12.5 gives the amount M spent on beef per household per week. Thus, the amount the household spent on beef in a year is $52M$. Since the household's annual income is I thousand dollars, the proportion of income spent on beef is

$$P = \frac{52M}{1000I} = 0.052\frac{M}{I}.$$

Thus, we need to take each entry in Table 12.5, divide it by the income at the left, and multiply by 0.052. Table 12.6 shows the results.

Table 12.5 *Money spent on beef (\$/household/week)*

Income (\$1,000)	Price of Beef (\$) 3.00	3.50	4.00	4.50
20	7.95	9.07	10.04	10.94
40	12.42	14.18	15.76	17.46
60	15.33	17.50	19.88	21.78
80	16.05	18.52	20.76	22.82
100	17.37	20.20	22.40	24.89

Table 12.6 *Proportion of annual income spent on beef*

Income (\$1,000)	Price of Beef (\$) 3.00	3.50	4.00	4.50
20	0.021	0.024	0.026	0.028
40	0.016	0.018	0.020	0.023
60	0.013	0.015	0.017	0.019
80	0.010	0.012	0.013	0.015
100	0.009	0.011	0.012	0.013

20. If the price of beef is held constant, beef consumption for households with various incomes can be read from a fixed column in Table 12.1 on page 667 of the text. For example, the column corresponding to $p = 3.00$ gives the function $h(I) = f(I, 3.00)$; it tells you how much beef a household with income I will buy at \$3.00/lb. Looking at the column from the top down, you can see that it is an increasing function of I. This is true in every column. This says that at any fixed price for beef, consumption goes up as household income goes up—which makes sense. Thus, f is an increasing function of I for each value of p.

Problems

21. (a) According to Table 12.2 of the problem, it feels like $-19°$F.

(b) A wind of 20 mph, according to Table 12.2.

(c) About 17.5 mph. Since at a temperature of 25°F, when the wind increases from 15 mph to 20 mph, the temperature adjusted for wind chill decreases from 13°F to 11°F, we can say that a 5 mph increase in wind speed causes a 2°F decrease in the temperature adjusted for wind chill. Thus, each 2.5 mph increase in wind speed brings *about* a 1°F drop in the temperature adjusted for wind chill. If the wind speed at 25°F increases from 15 mph to 17.5 mph, then the temperature you feel will be $13 - 1 = 12°$F.

(d) Table 12.2 shows that with wind speed 20 mph the temperature will feel like 0°F when the air temperature is somewhere between 15°F and 20°F. When the air temperature drops 5°F from 20°F to 15°F, the temperature adjusted for wind-chill drops 6°F from 4°F to $-2°$F. We can say that for every 1°F decrease in air temperature there is *about* a $6/5 = 1.2°$F drop in the temperature you feel. To drop the temperature you feel from 4°F to 0°F will take an air temperature drop of about $4/1.2 = 3.3°$F from 20°F. With a wind of 20 mph, approximately $20 - 3.3 = 16.7°$F would feel like 0°F.

22.

Table 12.7 *Temperature adjusted for wind chill at $20^\circ F$*

Wind speed (mph)	5	10	15	20	25
Adjusted temperature (°F)	13	9	6	4	3

Table 12.8 *Temperature adjusted for wind chill at $0^\circ F$*

Wind speed (mph)	5	10	15	20	25
Adjusted temperature (°F)	−11	−16	−19	−22	−24

23.

Table 12.9 *Temperature adjusted for wind chill at 5 mph*

Temperature (°F)	35	30	25	20	15	10	5	0
Adjusted temperature (°F)	31	25	19	13	7	1	−5	−11

Table 12.10 *Temperature adjusted for wind chill at 20 mph*

Temperature (°F)	35	30	25	20	15	10	5	0
Adjusted temperature (°F)	24	17	11	4	−2	−9	−15	−22

24. (a) The total cost in dollars of renting a car is 40 times the number of days plus 0.15 times the number of miles driven, so

$$C = f(d, m) = 40d + 0.15m.$$

(b) We have

$$f(5, 300) = 40(5) + 0.15(300) = \$245.$$

Renting a car for 5 days and driving it 300 miles costs $245.

25. The gravitational force on a 100 kg object which is $7,000,000$ meters from the center of the earth (or about 600 km above the earth's surface) is about 820 newtons.

26. (a) The acceleration due to gravity decreases as h increases, because the gravitational force gets weaker the farther away you are from the planet. (In fact, g is inversely proportional to the square of the distance from the center of the planet.)

(b) The acceleration due to gravity increases as m increases. The more massive the planet, the larger the gravitational force. (In fact, g is proportional to m.)

27. By drawing the top four corners, we find that the length of the edge of the cube is 5. See Figure 12.9. We also notice that the edges of the cube are parallel to the coordinate axis. So the x-coordinate of the the center equals

$$-1 + \frac{5}{2} = 1.5.$$

The y-coordinate of the center equals

$$-2 + \frac{5}{2} = 0.5.$$

The z-coordinate of the center equals

$$2 - \frac{5}{2} = -0.5.$$

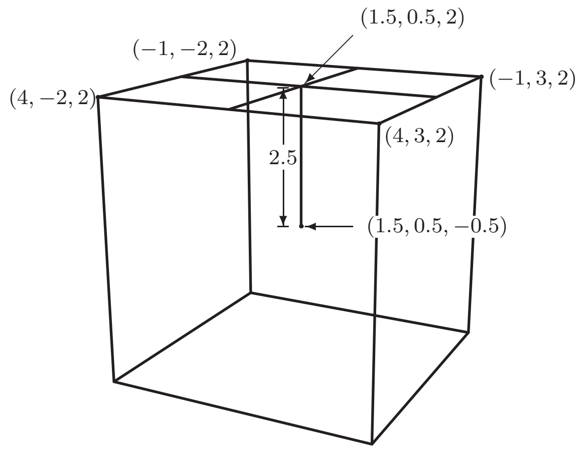


Figure 12.9

28. The equation for the points whose distance from the x-axis is 2 is given by $\sqrt{y^2+z^2}=2$, i.e. $y^2+z^2=4$. It specifies a cylinder of radius 2 along the x-axis. See Figure 12.10.

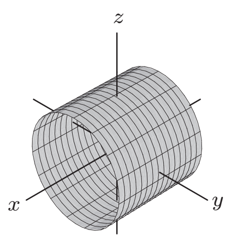


Figure 12.10

29. The distance of any point with coordinates (x, y, z) from the x-axis is $\sqrt{y^2+z^2}$. The distance of the point from the xy-plane is $|x|$. Since the condition states that these distances are equal, the equation for the condition is

$$\sqrt{y^2+z^2}=|x| \qquad \text{i.e.} \qquad y^2+z^2=x^2.$$

This is the equation of a cone whose tip is at the origin and which opens along the x-axis with a slope of 1 as shown in Figure 12.11.

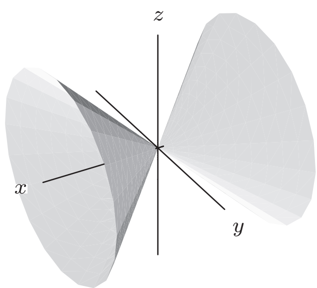


Figure 12.11

30. The coordinates of points on the y-axis are $(0, y, 0)$. The distance from any such point $(0, y, 0)$ to the point (a, b, c) is $d=\sqrt{a^2+(b-y)^2+c^2}$. Therefore, the closest point will have $y=b$ in order to minimize d. The resulting distance is then: $d=\sqrt{a^2+c^2}$.

31. (a) The sphere has center at $(2, 3, 3)$ and radius 4. The planes parallel to the xy-plane just touching the sphere are 4 above and 4 below the center. Thus, the planes $z = 7$ and $z = -1$ are both parallel to the xy-plane and touch the sphere at the points $(2, 3, 7)$ and $(2, 3, -1)$.

(b) The planes $x = 6$ and $x = -2$ just touch the sphere at $(6, 3, 3)$ and at $(-2, 3, 3)$ respectively and are parallel to the yz-plane.

(c) The planes $y = 7$ and $y = -1$ just touch the sphere at $(2, 7, 3)$ and at $(2, -1, 3)$ respectively and are parallel to the xz-plane.

32. The edges of the cube have length 4. Thus, the center of the sphere is the center of the cube which is the point $(4, 7, 1)$ and the radius is $r = 2$. Thus an equation of this sphere is

$$(x - 4)^2 + (y - 7)^2 + (z - 1)^2 = 4.$$

33. (a) The vertex at the opposite end of a diagonal across the base is $(12, 7, 2)$. The other two points are $(5, 7, 2)$ and $(12, 1, 2)$.

(b) The vertex at the opposite end of a diagonal across the top is $(5, 1, 4)$. The other two points are $(5, 7, 4)$ and $(12, 1, 4)$.

34. Using the distance formula, we find that

$$\begin{aligned}
\text{Distance from } P_1 \text{ to } P &= \sqrt{206}\\
\text{Distance from } P_2 \text{ to } P &= \sqrt{152}\\
\text{Distance from } P_3 \text{ to } P &= \sqrt{170}\\
\text{Distance from } P_4 \text{ to } P &= \sqrt{113}
\end{aligned}$$

So $P_4 = (-4, 2, 7)$ is closest to $P = (6, 0, 4)$.

35. (a) To find the intersection of the sphere with the yz-plane, substitute $x = 0$ into the equation of the sphere:

$$(-1)^2 + (y + 3)^2 + (z - 2)^2 = 4,$$

therefore

$$(y + 3)^2 + (z - 2)^2 = 3$$

This equation represents a circle of radius $\sqrt{3}$.
On the xz-plane $y = 0$:

$$(x - 1)^2 + 3^2 + (z - 2)^2 = 4,$$

therefore

$$(x - 1)^2 + (z - 2)^2 = -5$$

The negative sign on the right side of this equation shows that the sphere does not intersect the xz-plane, since the left side of the equation is always non-negative.
On the xy-plane, $z = 0$:

$$(x - 1)^2 + (y + 3)^2 + (-2)^2 = 4,$$

therefore

$$(x - 1)^2 + (y + 3)^2 = 0.$$

This equation has the unique solution $x = 1$, $y = -3$, so the xy-plane intersects the sphere in the single point $(1, -3, 0)$.

(b) Since the sphere does not intersect the xz-plane, it cannot intersect the x or z axes. On the y-axis, we have $x = z = 0$. Substituting this into the equation for the sphere we get

$$(-1)^2 + (y + 3)^2 + (-2)^2 = 4,$$

therefore

$$(y + 3)^2 = -1.$$

This equation has no solutions because the right hand side is negative, and the left-hand side is always non-negative. Thus the sphere does not intersect any of the coordinate axes.

36. The length corresponds to the y-axis, therefore the y-coordinates of the corners must be $1 \pm \frac{13}{2} = -5.5, 7.5$. See Figure 12.12. The height corresponds to the z-axis, therefore the z-coordinates of the corners must be $-2 \pm \frac{5}{2} = 0.5, -4.5$. The width corresponds to the x-axis, therefore the x-coordinates of the corners must be $1 \pm 3 = 4, -2$. The coordinates of those eight corners are therefore

$$(4, 7.5, 0.5),\ (-2, 7.5, 0.5), (-2, -5.5, 0.5), (4, -5.5, 0.5),$$
$$(4, 7.5, -4.5),\ (-2, 7.5, -4.5), (-2, -5.5, -4.5), (4, -5.5, -4.5).$$

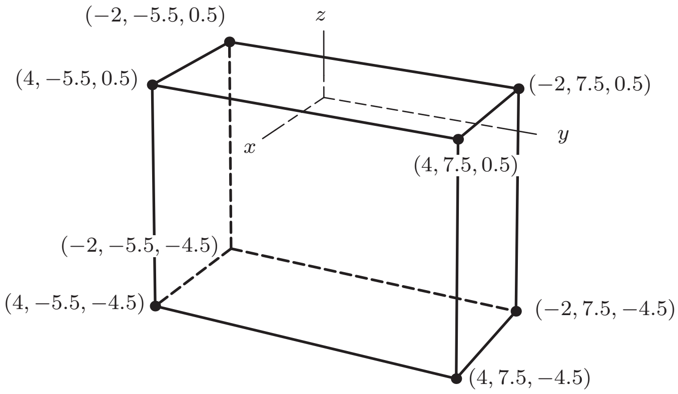


Figure 12.12

37. The length of the side of the triangle is 2, so its height is $\sqrt{3}$. The coordinates of the highest point are $(8, 0, \sqrt{3})$.

38. (a) We find the midpoint by averaging

$$\text{Midpoint} = \left(\frac{1+5}{2}, \frac{5+13}{2}, \frac{7+19}{2}\right) = (3, 9, 13).$$

(b) We use a weighted average, with the coordinates of point A weighted three times more heavily than point B:

$$\text{Point} = \left(\frac{3\cdot 1+5}{4}, \frac{3\cdot 5+13}{4}, \frac{3\cdot 7+19}{4}\right) = (2, 7, 10).$$

(c) We find the point in a similar way to part (b), but weighting B more heavily

$$\text{Point} = \left(\frac{1+3\cdot 5}{4}, \frac{5+3\cdot 13}{4}, \frac{7+3\cdot 19}{4}\right) = (4, 11, 16).$$

Strengthen Your Understanding

39. The graph of the equation $y = 1$ is a plane perpendicular to the y-axis, not a line. The x-axis is parallel to the plane.

40. The xy-plane has equation $z = 0$,

The equation $xy = 0$ means either $x = 0$ (the equation of the yz-plane) or $y = 0$ (the equation of the xz-plane). Points on the xy-plane all have $z = 0$; this is its equation.

41. The closest point on the x-axis to $(2, 3, 4)$ is $(2, 0, 0)$. The distance from $(2, 3, 4)$ to this point is

$$d = \sqrt{(2-2)^2 + (3-0)^2 + (4-0)^2} = \sqrt{25} = 5.$$

42. One possible function that is increasing in x and decreasing in y is given by the formula $f(x, y) = x - y$. For a fixed value of x, the value of $x - y$ decreases as y increases, and for a fixed value of y, the value of $x - y$ increases as x increases. There are many other possible answers.

43. If we pick a point with $z = -5$, its distance from the plane $z = -5$ is zero. The distance of a point from the xz-plane is the magnitude of the y-coordinate. So the point $(-2, -1, -5)$ is a distance of 1 from the xz-plane and a distance of zero from the plane $z = -5$. There are many other possible points.

44. True. Since each choice of x and y determines a unique value for $f(x, y)$, choosing $x = 10$ yields a unique value of $f(10, y)$ for any choice of y.

45. True. Since each choice of $h > 0$ and $s > 0$ determines a unique value for the volume V, we can say V is a function of h and s. In fact, this function has a formula: $V(h, s) = h \cdot s^2$.

46. False. If, for example, $d = 2$ meters and $H = 57°$C, there could be many times t at which the water temperature is 57 °C at 2 meters depth.

47. False. A function may have different inputs that yield equal outputs.

48. True. Since each of $f(x)$ and $g(y)$ has at most one output for each input, so does their product.

49. True. All points in the $z = 2$ plane have z-coordinate 2, hence are below any point of the form $(a, b, 3)$.

50. False. The plane $z = 2$ is parallel to the xy-plane.

51. True. Both are distance $\sqrt{2}$ from the origin.

52. False. The point $(2, -1, 3)$ does not satisfy the equation. It is at the center of the sphere, and does not lie on the graph.

53. True. The origin is the closest point in the yz-plane to the point $(3, 0, 0)$, and its distance to $(3, 0, 0)$ is 3.

54. False. There is an entire circle (of radius 4) of points in the yz-plane that are distance 5 from $(3, 0, 0)$.

55. False. The value of b can be ± 4.

56. True. Otherwise f would have more than one value for a given pair (x, y), which cannot happen if f is a function.

57. False. For example, the y-axis intersects the graph of $f(x, y) = 1 - x^2 - y^2$ twice, at $y = \pm 1$.

Solutions for Section 12.2

Exercises

1. (a) The value of z decreases as x increases. See Figure 12.13.
(b) The value of z increases as y increases. See Figure 12.14.

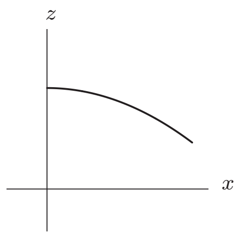


Figure 12.13

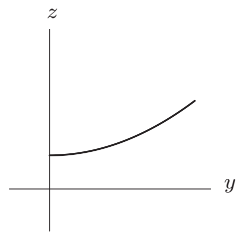


Figure 12.14

2. (a) is (IV), since $z = 2 + x^2 + y^2$ is a paraboloid opening upward with a positive z-intercept.
(b) is (II), since $z = 2 - x^2 - y^2$ is a paraboloid opening downward.
(c) is (I), since $z = 2(x^2 + y^2)$ is a paraboloid opening upward and going through the origin.
(d) is (V), since $z = 2 + 2x - y$ is a slanted plane.
(e) is (III), since $z = 2$ is a horizontal plane.

3. (a) The value of z only depends on the distance from the point (x, y) to the origin. Therefore the graph has a circular symmetry around the z-axis. There are two such graphs among those depicted in the figure in the text: I and V. The one corresponding to $z = \frac{1}{x^2+y^2}$ is I since the function blows up as (x, y) gets close to $(0, 0)$.
(b) For similar reasons as in part (a), the graph is circularly symmetric about the z-axis, hence the corresponding one must be V.
(c) The graph has to be a plane, hence IV.
(d) The function is independent of x, hence the corresponding graph can only be II. Notice that the cross-sections of this graph parallel to the yz-plane are parabolas, which is a confirmation of the result.
(e) The graph of this function is depicted in III. The picture shows the cross-sections parallel to the zx-plane, which have the shape of the cubic curves $z = x^3 -$ constant.

4. The graph is a horizontal plane 3 units above the xy-plane. See Figure 12.15.

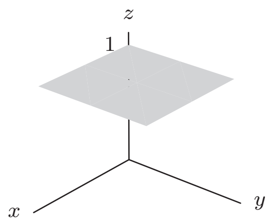


Figure 12.15

5. The graph is a sphere of radius 3, centered at the origin. See Figure 12.16.

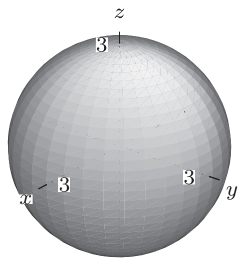


Figure 12.16

6. The graph is a bowl opening up, with vertex at the point $(0, 0, 4)$. See Figure 12.17.

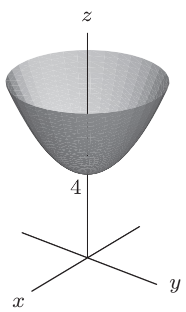


Figure 12.17

7. Since $z = 5-(x^2+y^2)$, the graph is an upside-down bowl moved up 5 units and with vertex at $(0, 0, 5)$. See Figure 12.18.

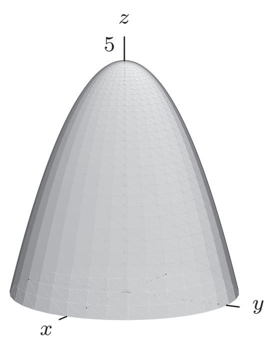


Figure 12.18

8. In the yz-plane, the graph is a parabola opening up. Since there are no restrictions on x, we extend this parabola along the x-axis. The graph is a parabolic cylinder opening up, extended along the x-axis. See Figure 12.19.

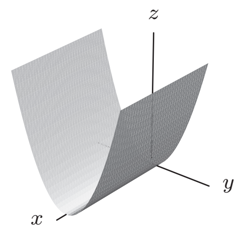


Figure 12.19

9. The graph is a plane with x-intercept 6, and y-intercept 3, and z-intercept 4. See Figure 12.20.

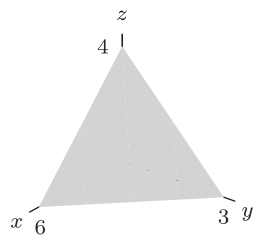


Figure 12.20

10. In the xy-plane, the graph is a circle of radius 2. Since there are no restrictions on z, we extend this circle along the z-axis. The graph is a circular cylinder extended in the z-direction. See Figure 12.21.

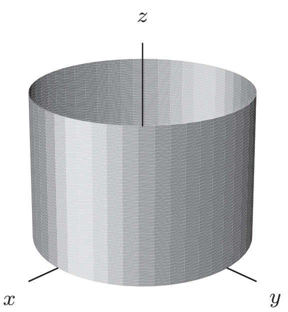


Figure 12.21

11. In the xz-plane, the graph is a circle of radius 2. Since there are no restrictions on y, we extend this circle along the y-axis. The graph is a circular cylinder extended in the y-direction. See Figure 12.22.

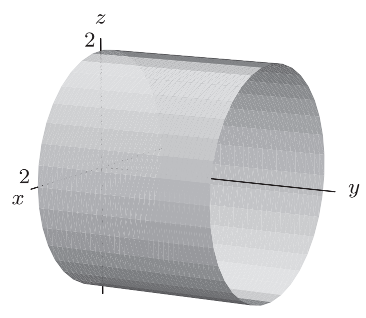


Figure 12.22

12. All the points on the cylinder are at a distance $\sqrt{7}$ from the y-axis. Since this distance is given by $\sqrt{x^2+z^2}$, we have

$$\sqrt{x^2+z^2} = \sqrt{7}$$
$$x^2+z^2 = 7.$$

13. A sphere of radius 3 centered at the origin has equation $x^2+y^2+z^2 = 3^2 = 9$, so shifting the center to $(0.\sqrt{7}, 0)$ gives

$$x^2 + (y-\sqrt{7})^2 + z^2 = 9.$$

14. A paraboloid with vertex at the origin but opening along the positive x-axis is

$$x = y^2 + z^2.$$

A parabola opening toward the negative x-axis is

$$x = -y^2 - z^2$$

so moving the vertex to $(1, 3, 5)$ gives

$$x = 1 - (y-3)^2 - (z-5)^2.$$

Problems

15. (a) Cross-sections with x fixed at $x = b$ are in Figure 12.23.

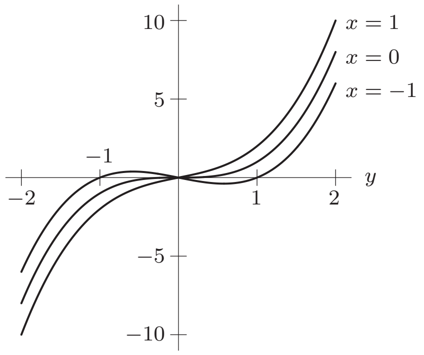


Figure 12.23: Cross-section $f(a, y) = y^3 + ay$, with $a = -1,\ 0,\ 1$

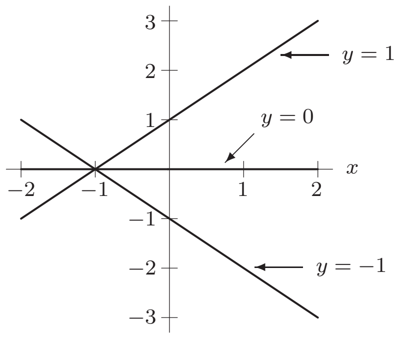


Figure 12.24: Cross-section $f(x, b) = b^3 + bx$, with $b = -1,\ 0,\ 1$

(b) Cross-section with y fixed at $y = 6$ are in Figure 12.24.

16. We have $f(3, 2) = 2e^{-2(5-3)} = 0.037$. We see that 2 hours after the injection of 3 mg of this drug, the concentration of the drug in the blood is 0.037 mg per liter.

17. (a) Holding x fixed at 4 means that we are considering an injection of 4 mg of the drug; letting t vary means we are watching the effect of this dose as time passes. Thus the function $f(4, t)$ describes the concentration of the drug in the blood resulting from a 4 mg injection as a function of time. Figure 12.25 shows the graph of $f(4, t) = te^{-t}$. Notice that the concentration in the blood from this dose is at a maximum at 1 hour after injection, and that the concentration in the blood eventually approaches zero.

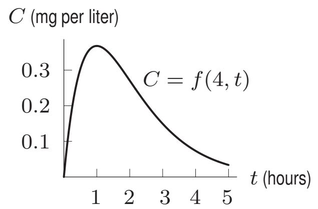


Figure 12.25: The function $f(4, t)$ shows the concentration in the blood resulting from a 4 mg injection

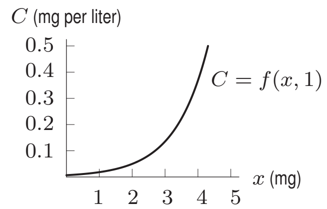


Figure 12.26: The function $f(x, 1)$ shows the concentration in the blood 1 hour after the injection

(b) Holding t fixed at 1 means that we are focusing on the blood 1 hour after the injection; letting x vary means we are considering the effect of different doses at that instant. Thus, the function $f(x,1)$ gives the concentration of the drug in the blood 1 hour after injection as a function of the amount injected. Figure 12.26 shows the graph of $f(x,1) = e^{-(5-x)} = e^{x-5}$. Notice that $f(x,1)$ is an increasing function of x. This makes sense: If we administer more of the drug, the concentration in the bloodstream is higher.

18. The one-variable function $f(a,t)$ represents the effect of an injection of a mg at time t. Figure 12.27 shows the graphs of the four functions $f(1,t) = te^{-4t}$, $f(2,t) = te^{-3t}$, $f(3,t) = te^{-2t}$, and $f(4,t) = te^{-t}$ corresponding to injections of 1, 2, 3, and 4 mg of the drug. The general shape of the graph is the same in every case: The concentration in the blood is zero at the time of injection $t = 0$, then increases to a maximum value, and then decreases toward zero again. We see that if a larger dose of the drug is administered, the peak of the graph is later and higher. This makes sense, since a larger dose will take longer to diffuse fully into the bloodstream and will produce a higher concentration when it does.

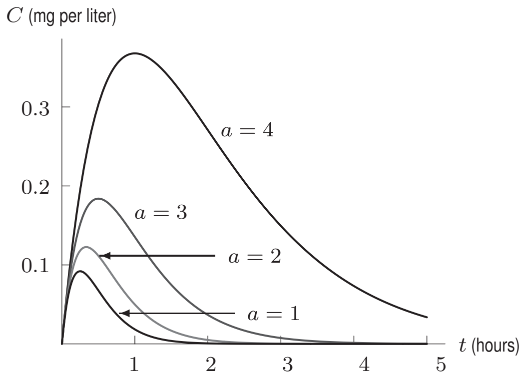


Figure 12.27: Concentration $C = f(a,t)$ of the drug resulting from an a mg injection

19. (a) is (IV), (b) is (IX), (c) is (VII), (d) is (I), (e) is (VIII), (f) is (II), (g) is (VI), (h) is (III), (i) is (V).

20. (a) This is a bowl; z increases as the distance from the origin increases, from a minimum of 0 at $x = y = 0$.
(b) Neither. This is an upside-down bowl. This function decreases from 1, at $x = y = 0$, to arbitrarily large negative values as x and y increase due to the negative squared terms of x and y. It looks like the bowl in part (a) except flipped over and raised up slightly.
(c) This is a plate. Solving the equation for z gives $z = 1 - x - y$ which describes a plane whose x and y slopes are -1. It is perfectly flat, but not horizontal.
(d) Within its domain, this function is a bowl. It is undefined at points at which $x^2 + y^2 > 5$, but within those limits it describes the bottom half of a sphere of radius $\sqrt{5}$ centered at the origin.
(e) This function is a plate. It is perfectly flat and horizontal.

21. (a)

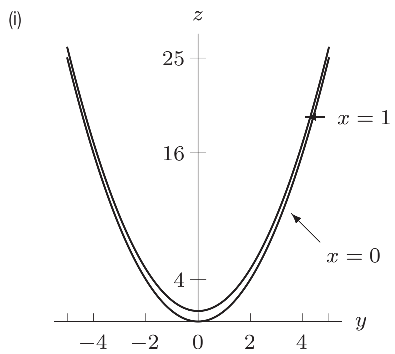


Figure 12.30: Cross-sections of $z = x^2 + y^2$

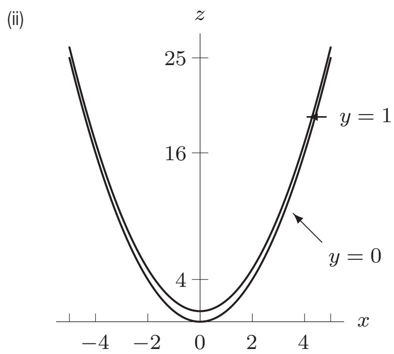


Figure 12.31: Cross-sections of $z = x^2 + y^2$

(b)

(i)

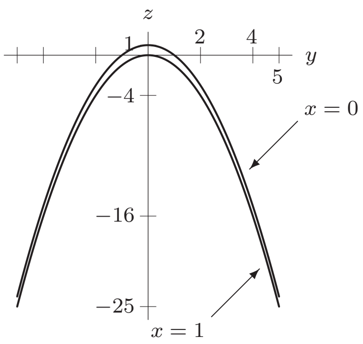


Figure 12.34: Cross-sections of $z = 1 - x^2 - y^2$

(ii)

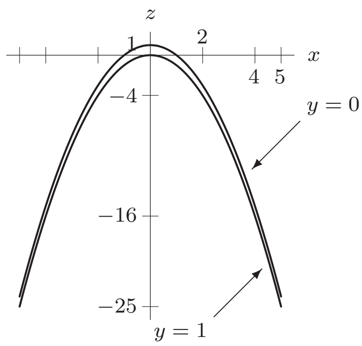


Figure 12.35: Cross-sections of $z = 1 - x^2 - y^2$

(c)

(i)

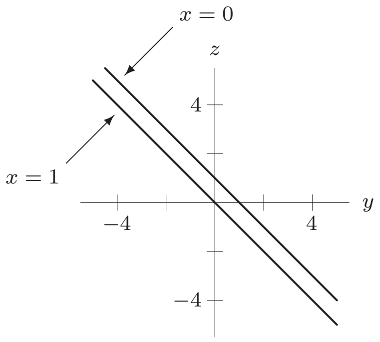


Figure 12.38: Cross-sections of $x + y + z = 1$

(ii)

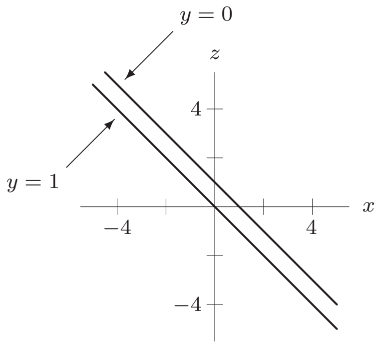


Figure 12.39: Cross-sections of $x + y + z = 1$

(d)

(i)

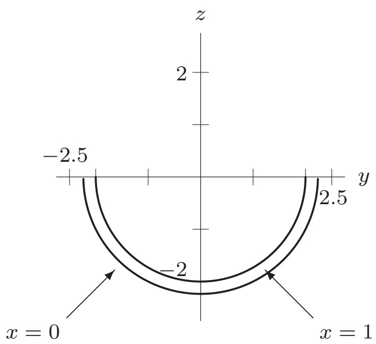


Figure 12.42: Cross-sections of $z = -\sqrt{5 - x^2 - y^2}$

(ii)

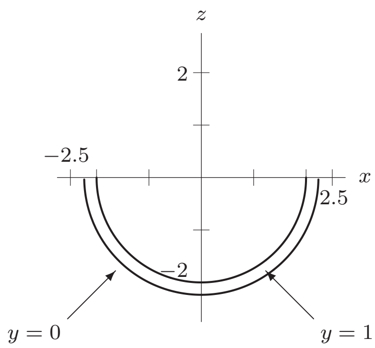


Figure 12.43: Cross-sections of $z = -\sqrt{5 - x^2 - y^2}$

(e)

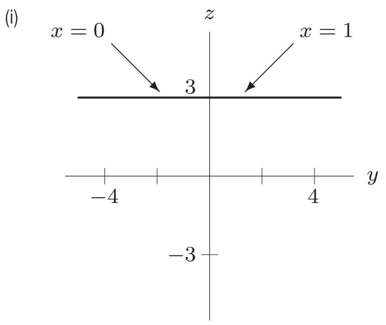


Figure 12.46: Cross-section of $z = 3$

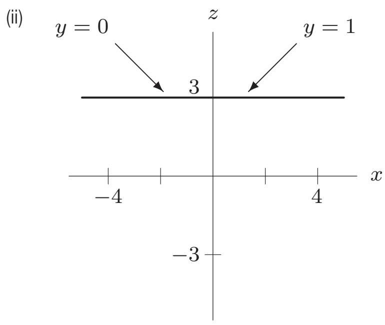


Figure 12.47: Cross-section of $z = 3$

22. (a) If we have iron stomachs and can consume cola and pizza endlessly without ill effects, then we expect our happiness to increase without bound as we get more cola and pizza. Graph (IV) shows this since it increases along both the pizza and cola axes throughout.

(b) If we get sick upon eating too many pizzas or drinking too much cola, then we expect our happiness to decrease once either or both of those quantities grows past some optimum value. This is depicted in graph (I) which increases along both axes until a peak is reached, and then decreases along both axes.

(c) If we do get sick after too much cola, but are always able to eat more pizza, then we expect our happiness to decrease after we drink some optimum amount of cola, but continue to increase as we get more pizza. This is shown by graph (III) which increases continuously along the pizza axis but, after reaching a maximum, begins to decrease along the cola axis.

23. One possible equation: $z = x^2 + y^2 + 5$. See Figure 12.48.

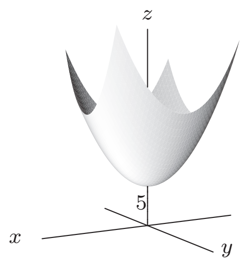


Figure 12.48

24. One possible equation: $x + y + z = 1$. See Figure 12.49.

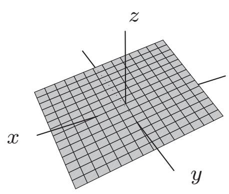


Figure 12.49

25. One possible equation: $z = (x - y)^2$. See Figure 12.50.

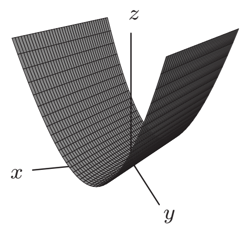


Figure 12.50

26. One possible equation: $z = -\sqrt{x^2 + y^2}$. See Figure 12.51.

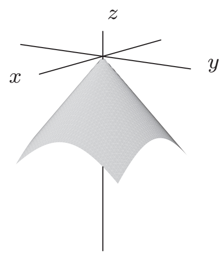


Figure 12.51

27. When h is fixed, say $h = 1$, then

$$V = f(r, 1) = \pi r^2 1 = \pi r^2$$

Similarly,

$$f(r, \frac{2}{3}) = \frac{4}{9}\pi r^2 \quad \text{and} \quad f(r, \frac{1}{3}) = \frac{\pi}{9} r^2$$

When r is fixed, say $r = 1$, then

$$f(1, h) = \pi(1)^2 h = \pi h$$

Similarly,

$$f(2, h) = 4\pi \quad \text{and} \quad f(3, h) = 9\pi h.$$

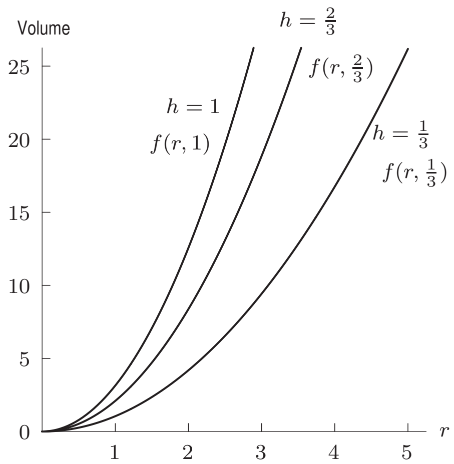


Figure 12.52

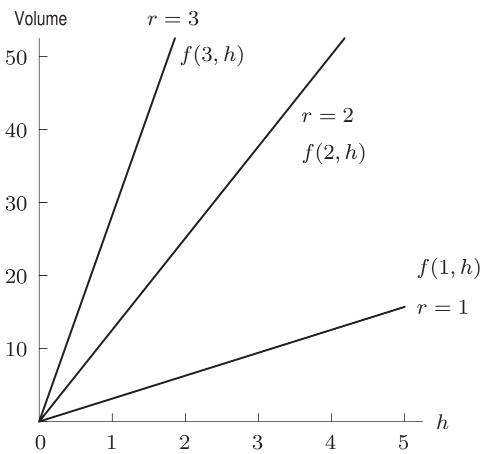


Figure 12.53

28. (a) The plane $y = 0$ intersects the graph in the curve $z = 4x^2 + 1$, which is a parabola opening upward.
(b) The plane $x = 0$ intersects the graph in $z = -y^2 + 1$, which is a parabola opening downward because of the negative coefficient of y^2.
(c) The plane $z = 1$ intersects the graph in $4x^2 - y^2 = 0$. Since this factors as $(2x - y)(2x + y) = 0$, it is the equation for the two lines $y = 2x$ and $y = -2x$.

29.

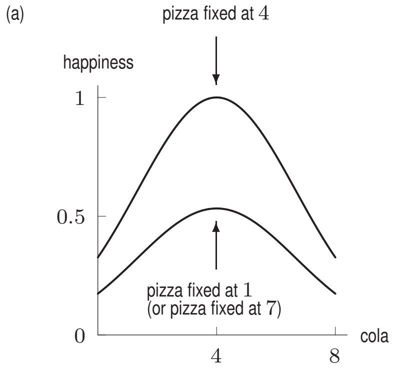


Figure 12.54: Cross-sections of graph I

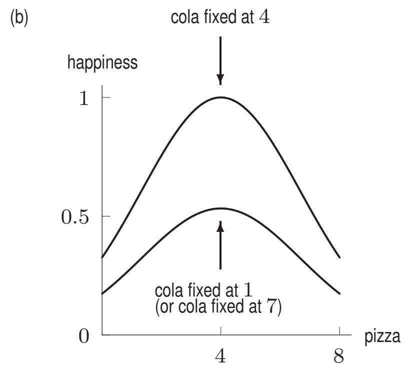


Figure 12.55: Cross-sections of graph I

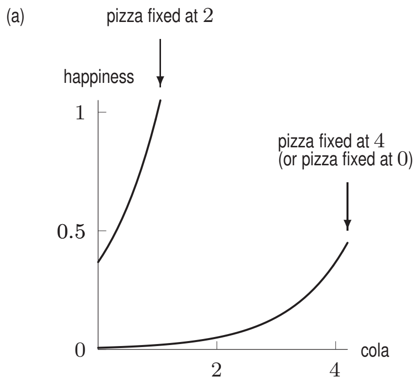


Figure 12.56: Cross-sections of graph II

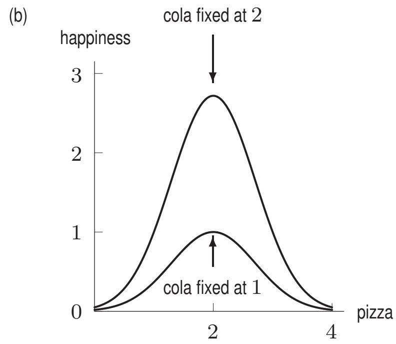


Figure 12.57: Cross-sections of graph II

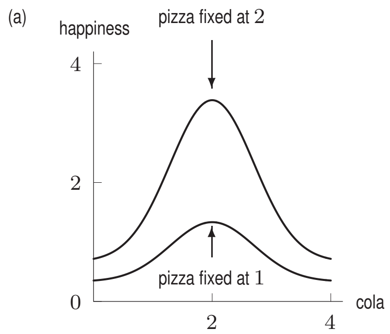


Figure 12.58: Cross-sections of graph III

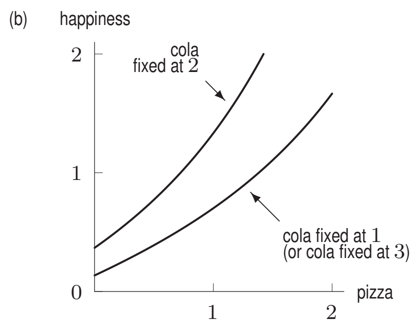


Figure 12.59: Cross-sections of graph III

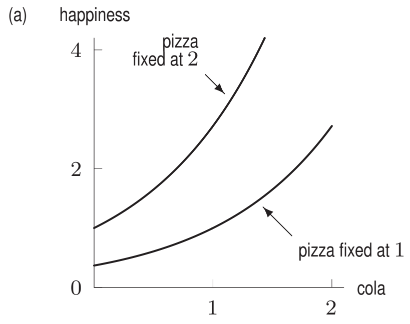


Figure 12.60: Cross-sections of graph IV

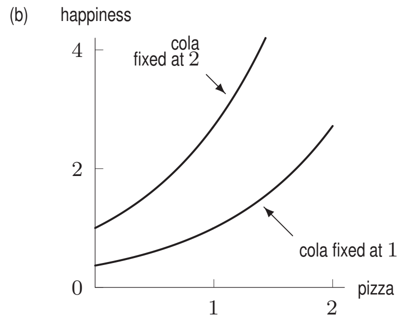


Figure 12.61: Cross-sections of graph IV

30. (a) Figures 12.62-12.65 show the wave profile at time $t = -1, 0, 1, 2$.

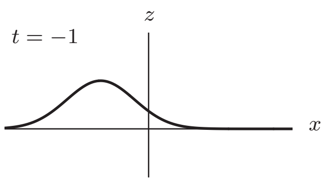


Figure 12.62

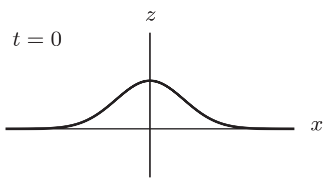


Figure 12.63

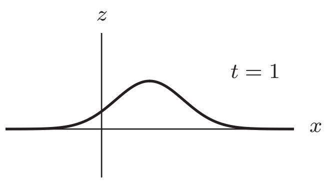


Figure 12.64

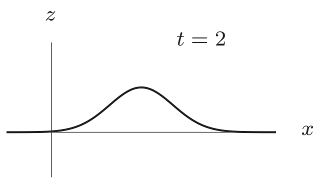


Figure 12.65

(b) Increasing x

(c) The graph in Figure 12.66 represents a wave traveling in the opposite direction.

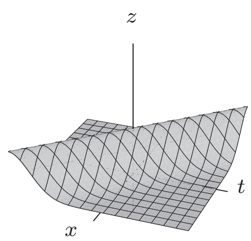


Figure 12.66

31. (a) Cross-sections with t fixed are in Figure 12.67. The equations are

$$f(x,0) = \cos 0 \sin x = \sin x,$$
$$f(x,\pi/4) = \cos(\pi/4)\sin x = \frac{1}{\sqrt{2}}\sin x.$$

Cross-sections with t fixed are in Figure 12.68. The equations are

$$f(\pi/4, t) = \cos t \sin(\pi/4) = \frac{1}{\sqrt{2}} \cos t,$$
$$f(\pi/2, t) = \cos t \sin(\pi/2) = \cos t.$$

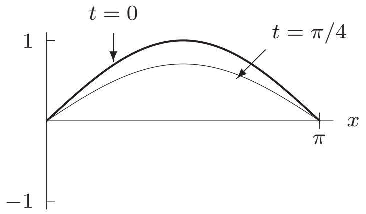


Figure 12.67: Cross-sections with t fixed

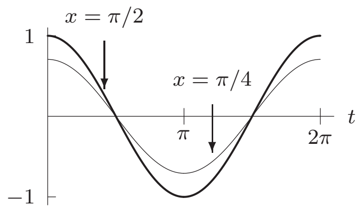


Figure 12.68: Cross-sections with x fixed

(b) If $x = 0, \pi$, then $f(0, t) = \cos t \cdot \sin 0 = 0 = f(\pi, t)$. The ends of the string are at $x = 0, \pi$, which do not move so the displacement is 0 there for all t.

(c) The cross-sections with t fixed are snapshots of the string at different instants in time. Graphs of these cross-sections are the curves obtained when the the graph of f in Figure 12.69 is sliced perpendicular to the t axis. Every plane perpendicular to the t-axis intersects the surface in one arch of a sine curve. The amplitude of the arch changes with t as a cosine curve.

The cross-sections with x fixed show how a single point on the string moves as time goes by. Graphs of these cross-sections are obtained by slicing the graph perpendicular to the x axis. Notice in Figure 12.69 that the cross-sections with $x = 0$ and $x = \pi$ are flat lines since the endpoints of the string don't move. The cross-section with $x = \pi/2$ is a cosine curve with amplitude 1, because the midpoint of the string oscillates back and forth. Cross-sections with x fixed between 0 and $\pi/2$ and between $\pi/2$ and π are cosine curves with amplitude between 0 and 1, representing the fact that these points on the string oscillate back and forth with the same period as $x = \pi/2$, but a smaller amplitude.

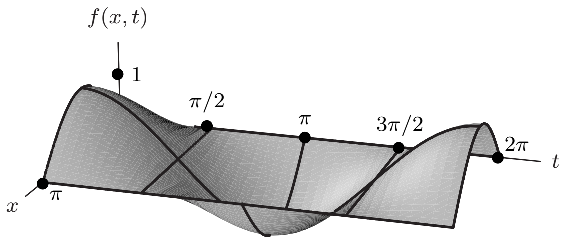


Figure 12.69: Graph of vibrating string function $f(x, t) = \cos t \sin x$

Strengthen Your Understanding

32. The graph of a function $f(x, y)$ is a parabolic surface in 3-space, not a circle.

33. If we hold x fixed, then $z = f(x, y) = x^2$ is also fixed, so the cross-section is a line parallel to the y-axis.

34. We know that $z = x^2 + y^2 + 2$ is positive everywhere and that the surface intersects the plane $z = 2$ only at $(0, 0, 2)$. So let $f(x, y) = x^2 + y^2 + 2$.

35. The function $f(x, y) = x^2 - 1$ intersects the xz-plane (and any plane parallel to the xz-plane) in the parabola $z = x^2 - 1$. It also intersect the yz-plane in the line $z = -1$. So f is a possible example. The function $g(x, y) = x^2 + y$ intersects the xz-plane in the parabola $z = x^2$ and the yz-plane in the line $y = z$. So g is another possible example. There are many others.

36. The function $f(x, y) = 1 - x^2 - y^2$ intersects the xy-plane in the circle $x^2 + y^2 = 1$. So f is a possible example. There are many others.

37. False. Fixing $w = k$ gives the one-variable function $g(v) = e^v/k$, which is an increasing exponential function if $k > 0$, but is decreasing if $k < 0$.

38. True. For example, consider the weekly beef consumption C of a household as a function of total income I and the cost of beef per pound p. It is possible that consumption increases as income increases (for fixed p) and consumption decreases as the price of beef increases (for fixed I).

39. True. For example, consider $f(x, y) = e^x \cdot (6 - y)$. Then $g(x) = f(x, 5) = e^x$, which is an increasing function of x. On the other hand, $h(x) = f(x, 10) = -4e^x$, which is a decreasing function of x.

40. False. The point $(0, 0, 0)$ does not satisfy the equation.

41. True. The x-axis is where $y = z = 0$.

42. False. If $x = 10$, substituting gives $10^2 + y^2 + z^2 = 10$, so $y^2 + z^2 = -90$. Since $y^2 + z^2$ cannot be negative, a point with $x = 10$ cannot satisfy the equation.

43. True. The cross-section with $y = 1$ is the line $z = x + 1$.

44. True. The cross-sections with $x = c$ are all of the form $z = 1 - y^2$.

45. True. The cross-sections with $y = c$ are of the form $z = 1 - c^2$, which are horizontal lines.

46. True. For any a and b, we have $f(a, b) \neq g(a, b)$. The graph of g is same as the graph of f, except it is shifted 2 units vertically.

47. True. The intersection, where $f(x, y) = g(x, y)$, is given by $x^2 + y^2 = 1 - x^2 - y^2$, or $x^2 + y^2 = 1/2$. This is a circle of radius $1/\sqrt{2}$ parallel to the xy-plane at height $z = 1/2$.

48. False. For example, $f(x, y) = x^2$ (or any cylinder along the y-axis) is not a plane but has lines for $x = c$ cross-sections.

49. False. Wherever $f(x, y) = 0$ the graphs of $f(x, y)$ and $-f(x, y)$ will intersect.

50. True. The graph is the bowl-shaped $g(x, y) = x^2 + y^2$ turned upside-down and shifted upward by 10 units.

51. (c), a plane. While x is fixed at 2, y and z can vary freely.

Solutions for Section 12.3

Exercises

1. We'll set $z = 4$ at the peak. See Figure 12.70.

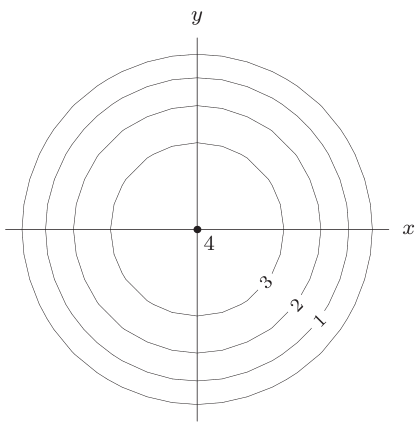


Figure 12.70

2. See Figure 12.71.

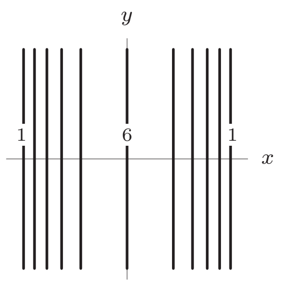


Figure 12.71

3. We will take $z = 4$ to be the flat area. See Figure 12.72.

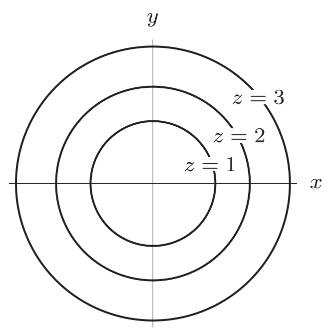


Figure 12.72

4. See Figure 12.73.

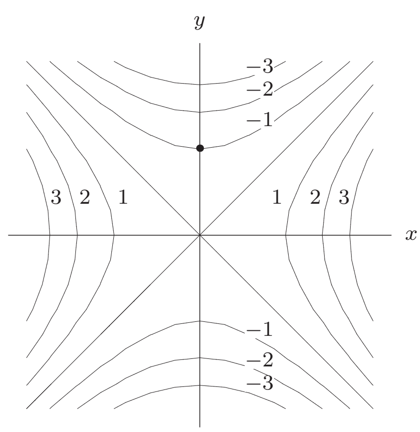


Figure 12.73

5. The contour where $f(x, y) = x + y = c$, or $y = -x + c$, is the graph of the straight line with slope -1 as shown in Figure 12.74. Note that we have plotted the contours for $c = -3, -2, -1, 0, 1, 2, 3$. The contours are evenly spaced.

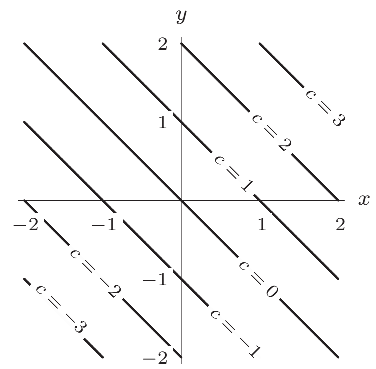


Figure 12.74

6. The contour where $f(x, y) = 3x + 3y = c$ or $y = -x + c/3$ is the graph of the straight line of slope -1 as shown in Figure 12.75. Note that we have plotted the contours for $c = -9, -6, -3, 0, 3, 6, 9$. The contours are evenly spaced.

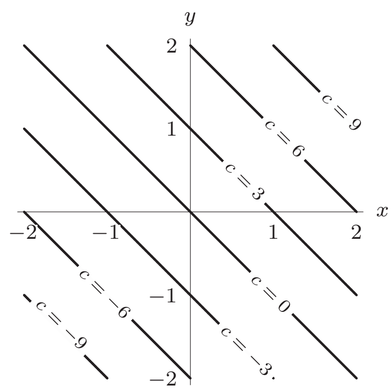


Figure 12.75

7. The contour where $f(x, y) = x^2 + y^2 = c$, where $c \geq 0$, is the graph of the circle centered at $(0, 0)$, with radius $\sqrt{c}$ as shown in Figure 12.76. Note that we have plotted the contours for $c = 0, 1, 2, 3, 4$. The contours become more closely packed as we move further from the origin.

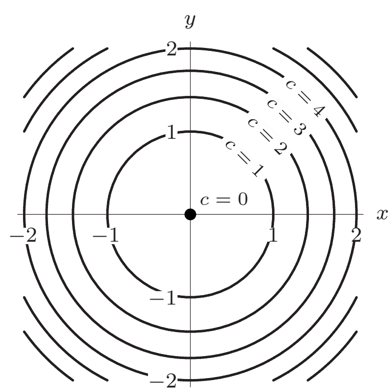


Figure 12.76

8. The contour where $f(x, y) = -x^2 - y^2 + 1 = c$, where $c \leq 1$, is the graph of the circle centered at $(0, 0)$, with radius $\sqrt{1-c}$ as shown in Figure 12.77. Note that we have plotted the contours for $c = -3, -2, -1, 0, 1$. The contours become more closely packed as we move further from the origin.

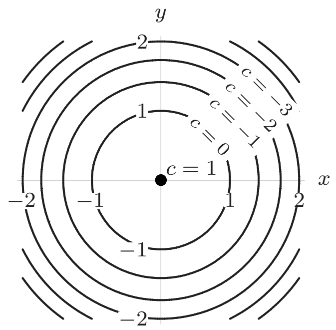


Figure 12.77

9. The contour where $f(x, y) = xy = c$, is the graph of the hyperbola $y = c/x$ if $c \neq 0$ and the coordinate axes if $c = 0$, as shown in Figure 12.78. Note that we have plotted contours for $c = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$. The contours become more closely packed as we move further from the origin.

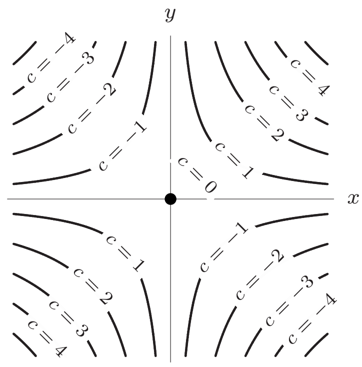


Figure 12.78

10. The contour where $f(x, y) = y - x^2 = c$ is the graph of the parabola $y = x^2 + c$, with vertex $(0, c)$ and symmetric about the y-axis, shown in Figure 12.79. Note that we have plotted the contours for $c = -2, -1, 0, 1$. The contours become more closely packed as we move farther from the y-axis.

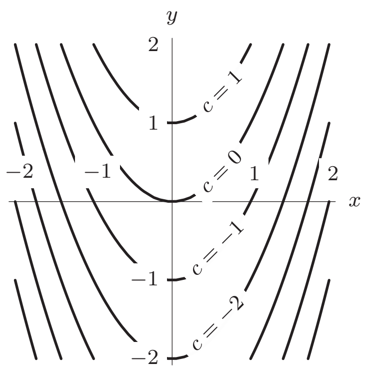


Figure 12.79

11. The contour where $f(x, y) = x^2 + 2y^2 = c$, where $c \geq 0$, is the graph of the ellipse with focuses $(-\sqrt{\frac{c}{2}}, 0)$, $(\sqrt{\frac{c}{2}}, 0)$ and axes lying on x- and y-axes as shown in Figure 12.80. Note that we have plotted the contours for $c = 0, 1, 2, 3, 4$. The contours become more closely packed as we move further from the origin.

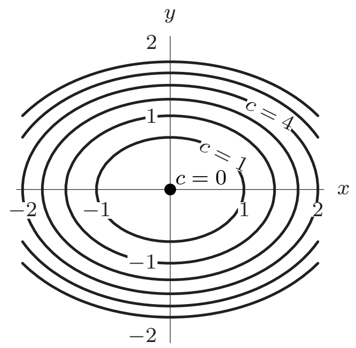


Figure 12.80

12. The contour where $f(x, y) = \sqrt{x^2 + 2y^2} = c$, where $c \geq 0$, is the graph of the ellipse with focuses $(-\frac{c\sqrt{2}}{2}, 0)$, $(\frac{c\sqrt{2}}{2}, 0)$ and axes lying on x- and y-axes as shown in Figure 12.81. Note that we have plotted the contours for $c = 0, 1, 2, 3, 4$. See Figure 12.81.

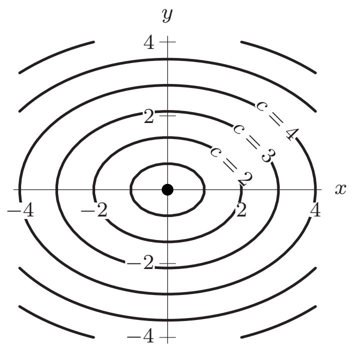


Figure 12.81

13. The contour where $f(x, y) = \cos(\sqrt{x^2 + y^2}) = c$, where $-1 \leq c \leq 1$, is a set of circles centered at $(0, 0)$, with radius $\cos^{-1} c + 2k\pi$ with $k = 0, 1, 2, ..$ and $-\cos^{-1} c + 2k\pi$, with $k = 1, 2, 3, ...$ as shown in Figure 12.82. Note that we have plotted contours for $c = 0, 0.2, 0.4, 0.6, 0.8, 1$.

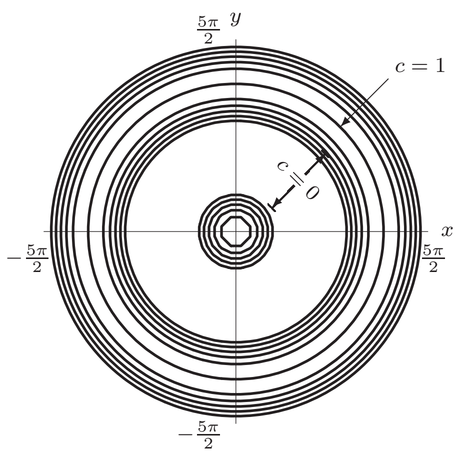


Figure 12.82

14. Since $f(5,10) = 3 \cdot 5^2 \cdot 10 + 7 \cdot 5 + 20 = 805$, an equation for the contour is

$$3x^2y + 7x + 20 = 805.$$

15. (a) Level curves are in Figure 12.83.

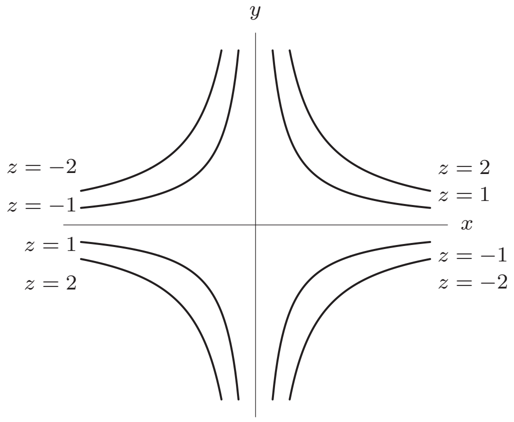


Figure 12.83

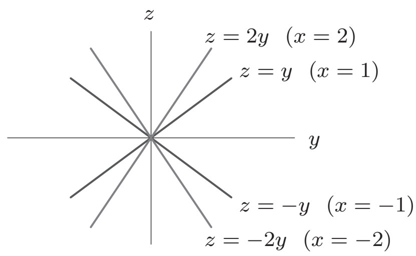


Figure 12.84

(b) Cross-sections with x constant are in Figure 12.84
(c) Setting $y = x$ gives the curve $z = x^2$ in Figure 12.85

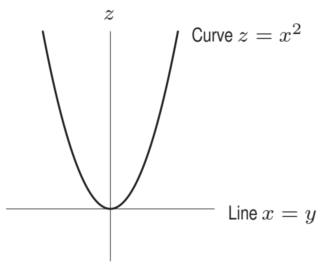


Figure 12.85

16. (a) (III)
(b) (I)
(c) (V)
(d) (II)
(e) (IV)

17. The values in Table 12.5 are not constant along rows or columns and therefore cannot be the lines shown in (I) or (IV). Also observe that as you move away from the origin, whose contour value is 0, the z-values on the contours increase. Thus, this table corresponds to diagram (II).

The values in Table 12.6 are also not constant along rows or columns. Since the contour values are decreasing as you move away from the origin, this table corresponds to diagram (III).

Table 12.7 shows that for each fixed value of x, we have constant contour value, suggesting a straight vertical line at each x-value, as in diagram (IV).

Table 12.8 also shows lines, however these are horizontal since for each fixed value of y we have constant contour values. Thus, this table matches diagram (I).

18. Superimposing the surface $z = 1/2$ on the graph of $f(x, y)$ gives Figure 12.86. The contour $f(x, y) = 1/2$ is the intersection of the two surfaces; that is, the collection of closed curves as shown in Figure 12.87

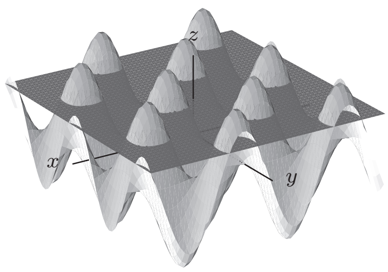


Figure 12.86

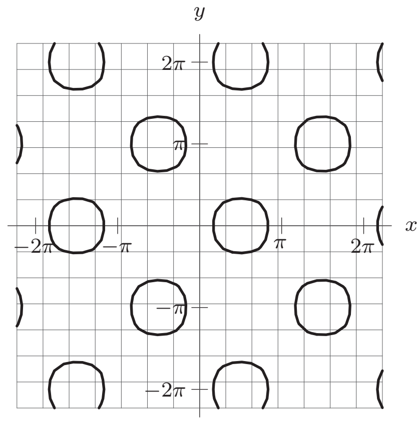


Figure 12.87

Problems

19. We expect total sales to decrease as the price increases and to increase as advertising expenditures increase. Moving parallel to the x-axis, the Q-values on the contours decrease, whereas moving parallel to the y-axis, the Q-values increase. Thus, x is the price and y is advertising expenditures

20. To find a value, evaluate $f(x, y) = 100e^x - 50y^2$ at any point (x, y) on the contour. Check by evaluating the function at a couple of points on each contour. Starting from the left and estimating points on the contour, we have

First contour: $f(0, 0) = 100$, $f(0.2, 0.65) = 101$
Second contour: $f(0.4, 0) = 149$, $f(0.6, 0.8) = 150$
Third contour: $f(0.7, 0) = 201$, $f(0.8, 0.65) = 201$
Fourth contour: $f(0.92, 0) = 251$, $f(1, 0.65) = 251$.

Since the true values of f are equally spaced multiples of 10, it seems that they must be 100, 150, 200, and 250. See Figure 12.88.

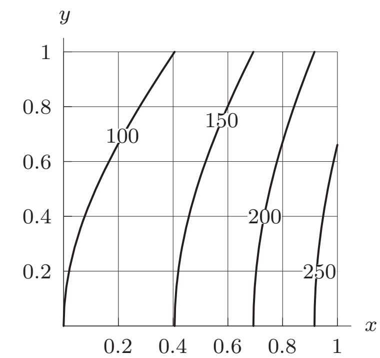


Figure 12.88

21. (a) We have

$$f(x, y) = x^2 - y^2 - 2x + 4y - 3 = (x - 1)^2 - (y - 2)^2.$$

Thus, the graph of f has the same saddle shape as that of $z = x^2 - y^2$ but centered at $x = 1$, $y = 2$. The function increases in the x-direction and decreases in the y-direction, so f corresponds to III.

(b) We have

$$g(x, y) = x^2 + y^2 - 2x - 4y + 15 = (x - 1)^2 + (y - 2)^2 + 10.$$

Thus, the graph of g is a paraboloid opening upward, with vertex at $(1, 2, 10)$. So h corresponds to VI.

(c) We have

$$h(x, y) = -x^2 - y^2 + 2x + 4y - 8 = -(x - 1)^2 - (y - 2)^2 - 3.$$

Thus, the graph of h is a paraboloid opening downward, with vertex at $(1, 2, -3)$. So h corresponds to I.

(d) We have
$$j(x,y) = -x^2 + y^2 + 2x - 4y + 3 = -(x-1)^2 + (y-2)^2.$$
Thus, the graph of j has the same saddle shape as that of $z = -x^2 + y^2$ but centered at $x = 1$, $y = 2$. The function decreases in the x-direction and increases in the y-direction, so j corresponds to IV.

(e) Since $k(x,y) = \sqrt{(x-1)^2 + (y-2)^2}$, the graph of k is a cone opening upward with vertex at $(1,2,0)$. Thus, the graph of k corresponds to II.

(f) Since $l(x,y) = -\sqrt{(x-1)^2 + (y-2)^2}$, the graph of l is a cone opening downward with vertex at $(1,2,0)$. Thus, the graph of l corresponds to V.

22. (a) Find the point where the horizontal line for 15 mph meets the contour for $-20°$F wind chill. The actual temperature is about $0°$F.

(b) The horizontal line for 10 mph meets the vertical line for $0°$F about $1/5$ of the way from the contour for $-20°$F to the contour for $0°$F wind chill. We estimate the wind chill to be about $-16°$F.

(c) We look for the point on the vertical line for $-20°$F where the wind chill is $-50°$F, the danger point for humans. This is a point on the line that is about half way between the contours for $-60°$F and $-40°$F. The point can not be determined exactly, but we estimate that it occurs where the wind speed is about 23 mph.

(d) A temperature drop of $20°$F corresponds to moving left from one vertical grid line to the next on the horizontal line for 15 mph. This horizontal movement appears to correspond to about $1\ 1/4$ the horizontal distance between contours crossing the line. Since contours are spaced at $20°$F wind chill, we estimate that the wind chill drops about $25°$F when the air temperature goes down $20°$F during a 15 mph wind.

23. To sketch the curve, first put dots on the point where an f contour crosses a g contour of the same value. Then connect the dots with a smooth curve. See Figure 12.89.

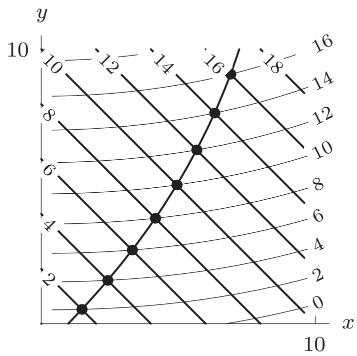


Figure 12.89: Black: $f(x,y)$. Blue; $g(x,y)$

24. Many different answers are possible. Answers are in degrees Celsius.

(a) Minnesota in winter. See Figure 12.90.

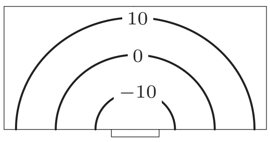


Figure 12.90

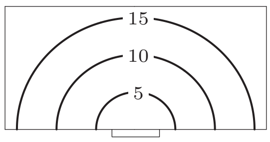


Figure 12.91

(b) San Francisco in winter. See Figure 12.91.

(c) Houston in summer. See Figure 12.92.

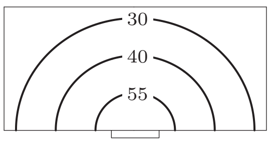


Figure 12.92

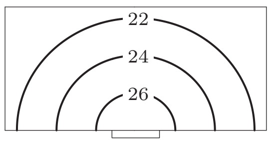


Figure 12.93

(d) Oregon in summer. See Figure 12.93.

25. The point $x = 10$, $t = 5$ is between the contours $H = 70$ and $H = 75$, a little closer to the former. Therefore, we estimate $H(10, 5) \approx 72$, i.e., it is about 72°F. Five minutes later we are at the point $x = 10$, $t = 10$, which is just above the contour $H = 75$, so we estimate that it has warmed up to 76°F by then.

26. The line $t = 5$ crosses the contour $H = 80$ at about $x = 4$; this means that $H(4, 5) \approx 80$, and so the point $(4, 80)$ is on the graph of the one-variable function $y = H(x, 5)$. Each time the line crosses a contour, we can plot another point on the graph of $H(x, 5)$, and thus get a sketch of the graph. See Figure 12.94. Each data point obtained from the contour map has been indicated by a dot on the graph. The graph of $H(x, 20)$ was obtained in a similar way.

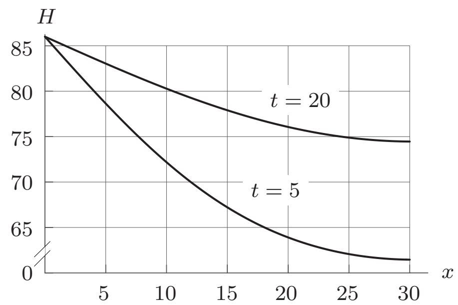


Figure 12.94: Graph of $H(x, 5)$ and $H(x, 20)$: heat as a function of distance from the heater at $t = 5$ and $t = 20$ minutes

These two graphs describe the temperature at different positions as a function of x for $t = 5$ and $t = 20$.

Notice that the graph of $H(x, 5)$ descends more steeply than the graph of $H(x, 20)$; this is because the contours are quite close together along the line $t = 5$, whereas they are more spread out along the line $t = 20$. In practical terms the shape of the graph of $H(x, 5)$ tells us that the temperature drops quickly as you move away from the heater, which makes sense, since the heater was turned on just five minutes ago. On the other hand, the graph of $H(x, 20)$ descends more slowly, which makes sense, because the heater has been on for 20 minutes and the heat has had time to diffuse throughout the room.

27. (a) The contour lines are much closer together on path A, so path A is steeper.
(b) If you are on path A and turn around to look at the countryside, you find hills to your left and right, obscuring the view. But the ground falls away on either side of path B, so you are likely to get a much better view of the countryside from path B.
(c) There is more likely to be a stream alongside path A, because water follows the direction of steepest descent.

28. (a) The point representing 13% and \$6000 on the graph lies between the 120 and 140 contours. We estimate the monthly payment to be about \$137.
(b) Since the interest rate has dropped, we will be able to borrow more money and still make a monthly payment of \$137. To find out how much we can afford to borrow, we find where the interest rate of 11% intersects the \$137 contour and read off the loan amount to which these values correspond. Since the \$137 contour is not shown, we estimate its position from the \$120 and \$140 contours. We find that we can borrow an amount of money that is more than \$6000 but less than \$6500. So we can borrow about \$250 more without increasing the monthly payment.
(c) The entries in the table will be the amount of loan at which each interest rate intersects the 137 contour. Using the \$137 contour from (b) we make table 12.11.

Table 12.11 *Amount borrowed at a monthly payment of \$137.*

Interest Rate (%)	0	1	2	3	4	5	6	7
Loan Amount (\$)	8200	8000	7800	7600	7400	7200	7000	6800
Interest rate (%)	8	9	10	11	12	13	14	15
Loan Amount (\$)	6650	6500	6350	6250	6100	6000	5900	5800

(a) The point representing 8% and \$6000 on the graph lies between the 120 and 140 contours. We estimate the monthly payment to be about \$122.

(b) Since the interest rate has dropped, we will be able to borrow more money and still make a monthly payment of \$122. To find out how much we can afford to borrow, we find where the interest rate of 6% intersects the \$122 contour and read off the loan amount to which these values correspond. Since the \$122 contour is not shown, we estimate its position from the \$120 and \$140 contours. We find that we can borrow an amount of money that is more than \$6000 but less than \$6500. So we can borrow about \$350 more without increasing the monthly payment.

(c) The entries in the table will be the amount of loan at which each interest rate intersects the 122 contour. Using the \$122 contour from (b) we make table 12.12.

Table 12.12 *Amount borrowed at a monthly payment of \$122.*

Interest Rate (%)	0	1	2	3	4	5	6	7
Loan Amount (\$)	7400	7200	7000	6800	6650	6500	6350	6200
Interest rate (%)	8	9	10	11	12	13	14	15
Loan Amount (\$)	6000	5850	5700	5600	5500	5400	5300	5200

29. The vertical spacing between the contours just north and just south of the trail increases as you move eastward along the trail. A possible contour diagram is in Figure 12.95.

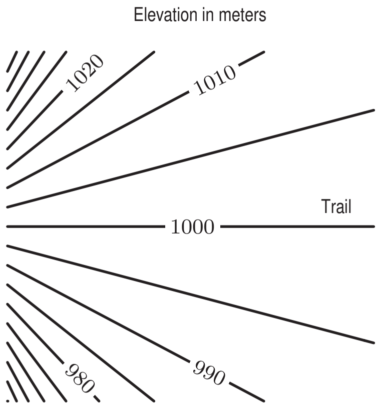


Figure 12.95

30. (a) I
(b) IV
(c) II
(d) III
See Figure 12.96.

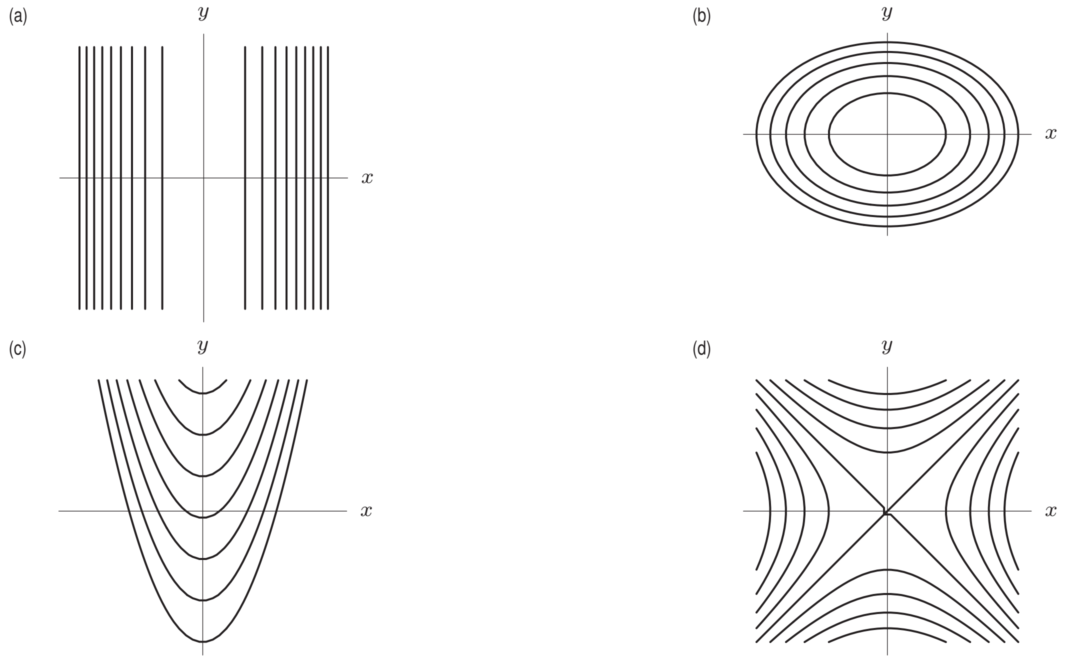


Figure 12.96

31. Figure 12.97 shows an east-west cross-section along the line $N = 50$ kilometers.
Figure 12.98 shows an east-west cross-section along the line $N = 100$ kilometers.

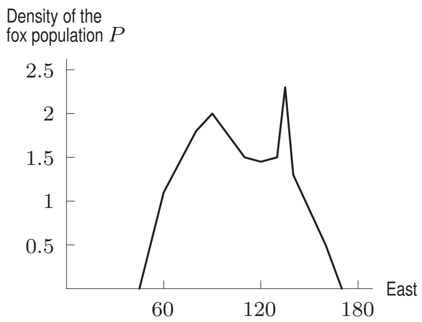


Figure 12.97

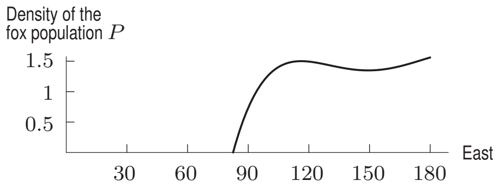


Figure 12.98

Figure 12.99 shows a north-south cross-section along the line $E = 60$ kilometers.
Figure 12.100 shows a north-south cross-section along the line $E = 120$ kilometers.

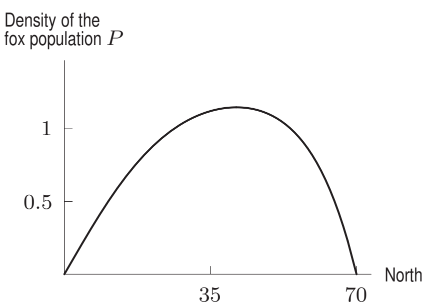


Figure 12.99

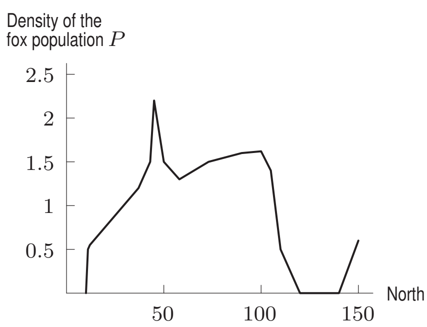


Figure 12.100

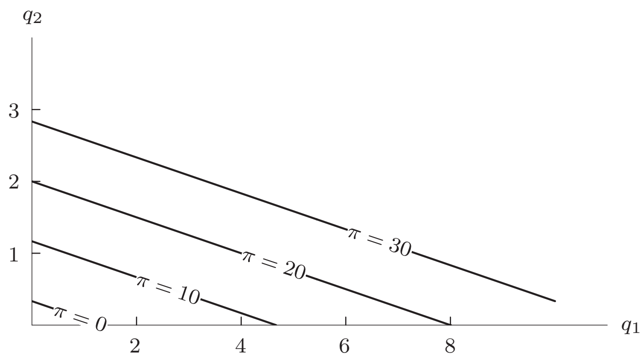
32. (a) The profit is given by the following:

$$\pi = \text{Revenue from } q_1 + \text{ Revenue from } q_2 - \text{Cost}.$$

Measuring π in thousands, we obtain:

$$\pi = 3q_1 + 12q_2 - 4.$$

(b) A contour diagram of π follows. Note that the units of π are in thousands.



33. For any Cobb-Douglass function $F(K, L) = bL^\alpha K^\beta$, if we increase the inputs by a factor of m, from (K, L) to (mK, mL) we get:

$$\begin{aligned} F(mK, mL) &= b(mL)^\alpha (mK)^\beta \\ &= m^{\alpha+\beta} bL^\alpha K^\beta \\ &= m^{\alpha+\beta} F(K, L) \end{aligned}$$

Thus we see that increasing inputs by a factor of m increases outputs by a factor of $m^{\alpha+\beta}$.

If $\alpha + \beta < 1$, then increasing each input by a factor of m will result in an increase in output of less than a factor of m. This applies to statements (a) and (E). In statement (a), $\alpha + \beta = 0.25 + 0.25 = 0.5$, so increasing inputs by a factor of $m = 4$, as in statement (E), increases output by a factor of $4^{0.5} = 2$. We can match statements (a) and (E) to graph (II) by noting that when $(K, L) = (1, 1)$, we have $F = 1$ and when we double the inputs $(m = 2)$ to $(K, L) = (2, 2)$, F increases by *less than* a factor of 2. This is called decreasing returns to scale.

If $\alpha + \beta = 1$, then increasing K and L by a factor of m will result in an increase in F by the same factor m. This applies to statements (b) and (D). In statement (b), $\alpha + \beta = 0.5 + 0.5 = 1$, and in statement (D), an increase in inputs by a factor of 3 results in an increase in F by the same factor. We match these statements to graph (I) where we see that increasing (K, L) from $(1, 1)$ to $(3, 3)$ results in an increase in F from $F = 1$ to $F = 3$. This is called constant returns to scale.

If $\alpha + \beta > 1$, then we have increasing returns to scale, i.e. an increase in K and L by a factor of m results in an increase in F by more than a factor of m. This is the case for equation (c), where $\alpha + \beta = 0.75 + 0.75 = 1.5$. Statement (G) also applies an increase in inputs by a factor of $m = 2$ results in an increase in output by *more than* 2, in this case by a factor of almost 3. We can match statements (c) and (G) to graph (III), where we see that increasing (K, L) from $(1, 1)$ to $(2, 2)$ results in a change in F by more than a factor of 2 (but less than a factor of 3). This is called increasing returns to scale.

This information is summarized in Table 12.13.

Table 12.13

Function	Graph	Statement
$F(L, K) = L^{0.25}K^{0.25}$	(II)	(E)
$F(L, K) = L^{0.5}K^{0.5}$	(I)	(D)
$F(L, K) = L^{0.75}K^{0.75}$	(III)	(G)

34. Suppose P_0 is the production given by L_0 and K_0, so that

$$P_0 = f(L_0, K_0) = cL_0^\alpha K_0^\beta.$$

We want to know what happens to production if L_0 is increased to $2L_0$ and K_0 is increased to $2K_0$:

$$\begin{aligned} P &= f(2L_0, 2K_0) \\ &= c(2L_0)^\alpha (2K_0)^\beta \\ &= c2^\alpha L_0^\alpha 2^\beta K_0^\beta \\ &= 2^{\alpha+\beta} cL_0^\alpha K_0^\beta \\ &= 2^{\alpha+\beta} P_0. \end{aligned}$$

Thus, doubling L and K has the effect of multiplying P by $2^{\alpha+\beta}$. Notice that if $\alpha+\beta > 1$, then $2^{\alpha+\beta} > 2$, if $\alpha+\beta = 1$, then $2^{\alpha+\beta} = 2$, and if $\alpha+\beta < 1$, then $2^{\alpha+\beta} < 2$. Thus, $\alpha+\beta > 1$ gives increasing returns to scale, $\alpha+\beta = 1$ gives constant returns to scale, and $\alpha+\beta < 1$ gives decreasing returns to scale.

35. (a) If $f(x, y) = x^{0.2}y^{0.8} = c$, then solving for y gives

$$y^{0.8} = \frac{c}{x^{0.2}}$$

$$y = \left(\frac{c}{x^{0.2}}\right)^{1/0.8} = \frac{A}{x^{1/4}}.$$

Here A is another constant, $A = c^{1/0.8}$.

Similarly, if $g(x, y) = x^{0.8}y^{0.2} = k$, then solving for y gives

$$y^{0.2} = \frac{k}{x^{0.8}}$$

$$y = \left(\frac{k}{x^{0.8}}\right)^{1/0.2} = \frac{B}{x^4}.$$

Since the y-values in Figure (I) decay more quickly than those in Figure (II), we see that Figure (I) is $g(x, y)$ and Figure (II) is $f(x, y)$.

(b) Since the y-values in Figure (III) decrease slower than in Figure (I) and faster than in Figure (II), we have $0.2 < \alpha < 0.8$.

36. Using the rules of logarithms on f and g gives

$$f(x, y) = \ln(x^{0.7}y^{0.3})$$

$$g(x, y) = \ln(x^{0.3}y^{0.7}).$$

Thus the level curves of f are of the form

$$\ln(x^{0.7}y^{0.3}) = c \qquad \text{so} \qquad x^{0.7}y^{0.3} = e^c = A \qquad \text{or} \qquad y = \frac{A^{1/0.3}}{x^{7/3}}.$$

The level curves of g are of the form

$$\ln(x^{0.3}y^{0.7}) = c \qquad \text{so} \qquad x^{0.3}y^{0.7} = e^c = A \qquad \text{or} \qquad y = \frac{A^{1/0.7}}{x^{3/7}}.$$

The level curves of h and j are ellipses. For any constant c, the level curve

$$h(x, y) = 0.3x^2 + 0.7y^2 = c$$

cuts the x-axis at $x = \sqrt{c/0.3}$ and the y-axis at $y = \sqrt{c/0.7}$. Thus the x-intercept is larger than the y-intercept. A similar argument tells us that the x-intercept of $j(x, y) = 0.7x^2 + 0.3y^2 = c$ is smaller than its y-intercept.

Thus Graph (I) is $h(x, y)$; Graph (II) is $j(x, y)$; Graph (III) is $f(x, y)$; Graph (IV) is $g(x, y)$.

37. (a) Multiply the values on each contour of the original contour diagram by 3. See Figure 12.101.

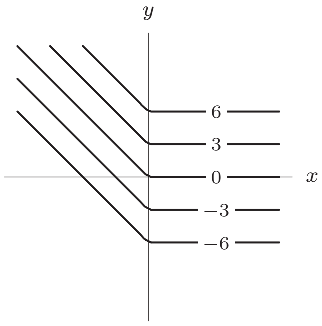


Figure 12.101: $3f(x, y)$

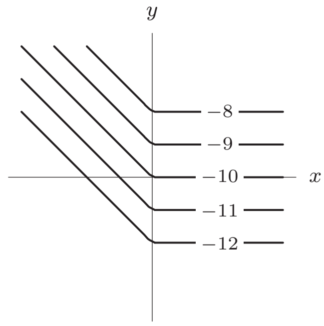


Figure 12.102: $f(x, y) - 10$

(b) Subtract 10 from the values on each contour. See Figure 12.102.
(c) Shift the diagram 2 units to the right and 2 units up. See Figure 12.103.

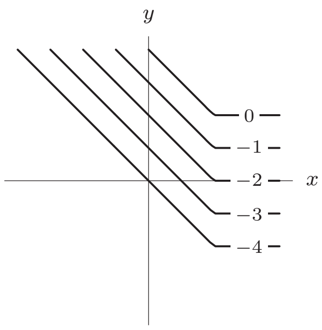


Figure 12.103: $f(x - 2, y - 2)$

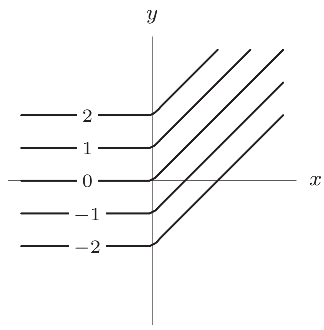


Figure 12.104: $f(-x, y)$

(d) Reflect the diagram about the y-axis. See Figure 12.104.

38. (a) See Figure 12.105.

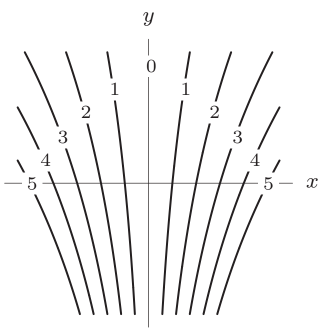


Figure 12.105

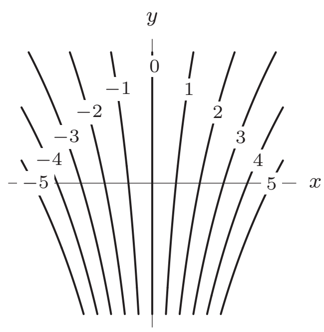


Figure 12.106

(b) See Figure 12.106.

39. Since $f(x, y) = x^2 - y^2 = (x - y)(x + y) = 0$ gives $x - y = 0$ or $x + y = 0$, the contours $f(x, y) = 0$ are the lines $y = x$ or $y = -x$. In the regions between them, $f(x, y) > 0$ or $f(x, y) < 0$ as shown in Figure 12.107. The surface $z = f(x, y)$ is above the xy-plane where $f > 0$ (that is on the shaded regions containing the x-axis) and is below the

xy-plane where $f < 0$. This means that a person could sit on the surface facing along the positive or negative x-axis, and with his/her legs hanging down the sides below the y-axis. Thus, the graph of the function is saddle-shaped at the origin.

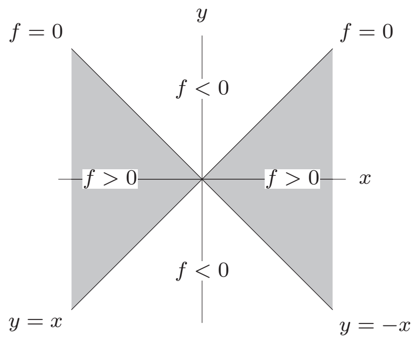


Figure 12.107

40. We need three lines with $g(x, y) = 0$, so that the xy-plane is divided into six regions. For example

$$g(x, y) = y(x - y)(x + y)$$

has the contour map in Figure 12.108. (Many other answers to this question are possible.)

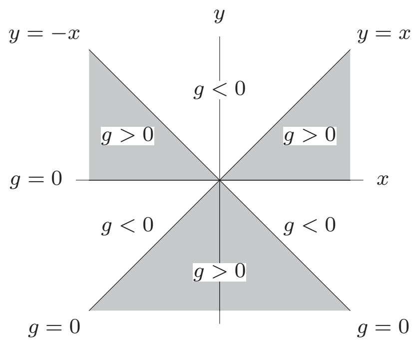


Figure 12.108

41. To read off the cross-sections of f with t fixed, we choose a t value and move horizontally across the diagram looking at the values on the contours. For $t = 0$, as we move from the left at $x = 0$ to the right at $x = \pi$, we cross contours of 0.25, 0.50, 0.75 and reach a maximum at $x = \pi/2$, and then decrease back to 0. That is because if time is fixed at $t = 0$, then $f(x, 0)$ is the displacement of the string at that time: no displacement at $x = 0$ and $x = \pi$ and greatest displacement at $x = \pi/2$. For cross-sections with t fixed at larger values, as we move along a horizontal line, we cross fewer contours and reach a smaller maximum value: the string is becoming less curved. At time $t = \pi/2$, the string is straight so we see a value of 0 all the way across the diagram, namely a contour with value 0. For $t = \pi$, the string has vibrated to the other side and the displacements are negative as we read across the diagram reaching a minimum at $x = \pi/2$.

The cross-sections of f with x fixed are read vertically. At $x = 0$ and $x = \pi$, we see vertical contours of value 0 because the end points of the string have 0 displacement no matter what time it is. The cross-section for $x = \pi/2$ is found by moving vertically up the diagram at $x = \pi/2$. As we expect, the contour values are largest at $t = 0$, zero at $t = \pi/2$, and a minimum at $t = \pi$.

Notice that the spacing of the contours is also important. For example, for the $t = 0$ cross-section, contours are most closely spaced at the end points at $x = 0$ and $x = \pi$ and most spread out at $x = \pi/2$. That is because the shape of the string at time $t = 0$ is a sine curve, which is steepest at the end points and relatively flat in the middle. Thus, the contour diagram shows the steepest terrain at the end points and flattest terrain in the middle.

42. (a) Since P is proportional to d^2 and to v^3, a formula for P is $P(d, v) = kd^2v^3$, where k is the constant of proportionality.

(b) Let d be the diameter of the original windmill, and let v_1 be the wind speed at which the windmill produces 100 kW. Then

$$kd^2v_1^3 = 100, \quad \text{and thus} \quad k = \frac{100}{d^2v_1^3}.$$

The second windmill has diameter $2d$ and we want to find a speed v_2 such that $k(2d)^2v_2^3 = 100$. We solve for v_2:

$$v_2^3 = \frac{100}{k(2d)^2} = \frac{100}{100/(d^2v_1^3)\cdot 4d^2} = \frac{d^2v_1^3}{4d^2} = \frac{v_1^3}{4}$$

$$v_2 = \frac{v_1}{\sqrt[3]{4}}$$

So v_2 needs to be $1/\sqrt[3]{4}$ of v_1, or about 63% of v_1.

(c) Contours of P are curves of the form $kd^2v^3 = c$, or $v = \sqrt[3]{c/(kd^2)}$. Thus, a contour diagram for P looks like the diagram in Figure 12.109.

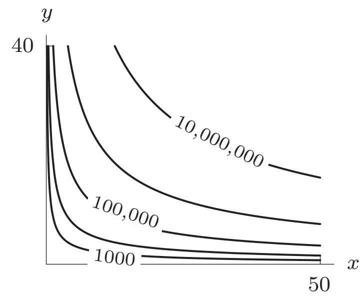


Figure 12.109

Strengthen Your Understanding

43. A contour diagram for $z = f(x, y)$ is a collection of curves in the xy-plane. The contour diagram is like a 2-dimensional map of the graph of $f(x, y)$, which is a surface in 3-space.

44. The contours of both functions are concentric circles centered at $(x, y) = (0, 0)$. However, for the equally spaced z-values, such as $z = 1, 2, 3, 4\ldots$, the contour diagram of f consists of equally spaced concentric circles, whereas the contour diagram of g consists of circles that get closer and closer together as z increases in value.

45. The function $f(x, y) = x^2$ has contours that are two parallel lines for positive values of z. In particular for $z = 10$, the contour of f consists of two parallel lines: $x = \pm\sqrt{10}$. The functions $g(x, y) = |y|$ and $h(x, y) = x^3 - 9x + 5$ also work. The $z = 10$ contour of h consists of three parallel lines. There are many others possibilities.

46. If $z = f(x, y) = y - x^2$, then all the contours have the form $y - x^2 = c$, so $y = x^2 + c$, which are parabolas for every value of c.

47. Could not be true. If the origin is on the level curve $z = 1$, then $z = f(0, 0) = 1 \neq -1$. So $(0, 0)$ cannot be on both $z = 1$ and $z = -1$.

48. Might be true. One may consider the function

$$z = f(x, y) = (x^2 + y^2 - 2)(x^2 + y^2 - 3) + 1$$

49. Might be true. The function $z = x^2 - y^2 + 1$ has this property. The level curve $z = 1$ is the lines $y = x$ and $y = -x$.

50. Not true. There are no level curves for $z > 1$ or $z \le 0$.

51. True. For every point (x, y), compute the value $z = e^{-(x^2+y^2)}$ at that point. The level curve obtained by getting z equal to that value goes through the point (x, y).

52. True. If there were such an intersection point, that point would have two different temperatures simultaneously.

53. True. Different regions that are isolated from each other can have the same temperature.

54. True. If $f = c$ then the contours are of the form $c = y^2 + (x-2)^2$, which are circles centered at $(2, 0)$ if $c > 0$. But if $c = 0$ the contour is the single point $(2, 0)$.

55. False. The graph could be a hemisphere, a bowl-shape, or any surface formed by rotating a curve about a vertical line.

56. False. Contours get closer together in a direction if the function is increasing or decreasing *at an increasing rate* in that direction.

57. False. As a counterexample, consider any function with one variable missing, e.g. $f(x, y) = x^2$. The graph of this is not a plane (it is a *parabolic cylinder*) but has contours which are lines of the form $x = c$.

58. False. The fact that the $f = 10$ and $g = 10$ contours are identical only says that one horizontal slice through each graph is the same, but does not imply that the entire graphs are the same. A counterexample is given by $f(x, y) = x^2 + y^2$ and $g(x, y) = 20 - x^2 - y^2$.

59. True. The graph of g is the same as the graph of f translated down by 5 units, so the horizontal slice of f at height 5 is the same as the horizontal slice of g at height 0.

Solutions for Section 12.4

Exercises

1.

Table 12.14

$x\backslash y$	0.0	1.0
0.0	−1.0	1.0
2.0	3.0	5.0

2.

Table 12.15

$x\backslash y$	−1.0	0.0	1.0
2.0	4.0	6.0	8.0
3.0	1.0	3.0	5.0

3. A table of values is linear if the rows are all linear and have the same slope and the columns are all linear and have the same slope. The table does not represent a linear function since none of the rows or columns is linear.

4. A table of values is linear if the rows are all linear and have the same slope and the columns are all linear and have the same slope. We see that the table might represent a linear function since the slope in each row is 3 and the slope in each column is -4.

5. A table of values is linear if the rows are all linear and have the same slope and the columns are all linear and have the same slope. The table might represent a linear function since the slope in each row is 5 and the slope in each column is 2.

6. A table of values is linear if the rows are all linear and have the same slope and the columns are all linear and have the same slope. The table does not represent a linear function since different rows have different slopes.

7. A contour diagram is linear if the contours are parallel straight lines, equally spaced for equally spaced values of z. This contour diagram could represent a linear function.

8. A contour diagram is linear if the contours are parallel straight lines, equally spaced for equally spaced values of z. We see that the contour diagram in the problem does not represent a linear function.

9. Since

$$\begin{aligned} 0 &= c + m\cdot 0 + n\cdot 0 \quad && c = 0\\ -1 &= c + m\cdot 0 + n\cdot 2 && c + 2n = -1\\ -4 &= c + m\cdot(-3) + n\cdot 0 && c - 3m = -4 \end{aligned}$$

we get:

$$c = 0, m = \frac{4}{3}, n = -\frac{1}{2}.$$

Thus, $z = \frac{4}{3}x - \frac{1}{2}y$.

10. Let the equation of the plane be

$$z = c + mx + ny$$

Since we know the points: $(4, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$ are all on the plane, we know that they satisfy the same equation. We can use these values of (x, y, z) to find c, m, and n. Putting these points into the equation we get:

$$0 = c + m \cdot 4 + n \cdot 0 \quad \text{so } c = -4m$$

$$0 = c + m \cdot 0 + n \cdot 3 \quad \text{so } c = -3n$$

$$2 = c + m \cdot 0 + n \cdot 0 \quad \text{so } c = 2$$

Because we have a value for c, we can solve for m and n to get

$$c = 2, m = -\frac{1}{2}, n = -\frac{2}{3}.$$

So the linear function is

$$z = 2 - \frac{1}{2}x - \frac{2}{3}y.$$

11. Figure 12.110 shows the two lines the plane must contain.

Both lines are parallel to the x-axis; thus our plane must have x-slope zero. On the other hand, the line in the xy-plane is 2 units down and one unit to the right of the line in the xz-plane; hence the y-slope of our plane must be -2. Thus the equation is

$$z = 0x - 2y + c = -2y + c,$$

for some constant c. Since the plane contains the point $(0, 0, 2)$, the value of c must be 2. So the equation is

$$z = -2y + 2.$$

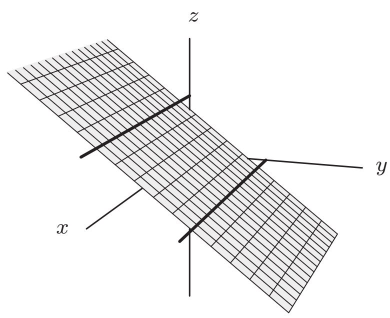


Figure 12.110

12. When $y = 0$, $c + mx = 3x + 4$, so $c = 4$, $m = 3$. Thus, when $x = 0$, we have $4 + ny = y + 4$, so $n = 1$. Thus, $z = 4 + 3x + y$.

13. (a) Since z is a linear function of x and y with slope 2 in the x-direction, and slope 3 in the y-direction, we have:

$$z = 2x + 3y + c$$

We can write an equation for changes in z in terms of changes in x and y:

$$\begin{aligned}\Delta z &= (2(x + \Delta x) + 3(y + \Delta y) + c) - (2x + 3y + c) \\ &= 2\Delta x + 3\Delta y\end{aligned}$$

Since $\Delta x = 0.5$ and $\Delta y = -0.2$, we have

$$\Delta z = 2(0.5) + 3(-0.2) = 0.4$$

So a 0.5 change in x and a -0.2 change in y produces a 0.4 change in z.

(b) As we know that $z = 2$ when $x = 5$ and $y = 7$, the value of z when $x = 4.9$ and $y = 7.2$ will be

$$z = 2 + \Delta z = 2 + 2\Delta x + 3\Delta y$$

where Δz is the change in z when x changes from 4.9 to 5 and y changes from 7.2 to 7. We have $\Delta x = 4.9 - 5 = -0.1$ and $\Delta y = 7.2 - 7 = 0.2$. Therefore, when $x = 4.9$ and $y = 7.2$, we have

$$z = 2 + 2 \cdot (-0.1) + 5 \cdot 0.2 = 2.4$$

14. (a) Substituting in the values for the slopes, we see that the formula for the plane is $z = c + 5x - 3y$ for some value of c. Substituting the point $(4, 3, -2)$ gives $c = -13$. The formula for the plane is

$$z = -13 + 5x - 3y.$$

(b) When $z = 0$, we have

$$\begin{aligned} 0 &= -13 + 5x - 3y \\ 3y &= 5x - 13 \\ y &= \frac{5}{3}x - \frac{13}{3}. \end{aligned}$$

The contour for $z = 0$ is a line with slope 5/3 and y-intercept 13/3. Similarly we find other contours. See Figure 12.111.

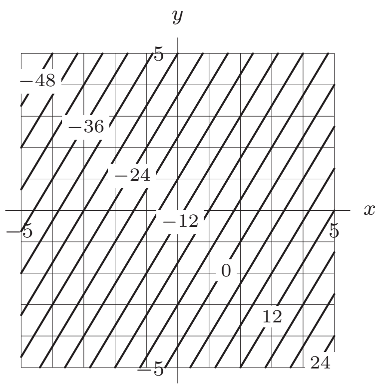


Figure 12.111

Problems

15. The revenue function, R, is linear and so we may write it as:

$$R = (p_1)c + (p_2)d$$

where p_1 is the price of CDs and p_2 is the price of DVDs, in dollars. From the diagram, we can pick two points, such as $c = 100, d = 100$ on the contour $R = 2000$, and $c = 50, d = 300$ on the contour $R = 4000$. These points give the following system of linear equations:

$$\begin{aligned} 2000 &= 100p_1 + 100p_2 \\ 4000 &= 50p_1 + 300p_2. \end{aligned}$$

Solving gives $p_1 = 8$ dollars and $p_2 = 12$ dollars.

16. (a) Yes.
(b) The coefficient of m is 15 dollars per month. It represents the monthly charge to use this service. The coefficient of t is 0.05 dollars per minute. Each minute the customer is on-line costs 5 cents.
(c) The intercept represents the base charge. It costs \$35 just to get hooked up to this service.
(d) We have $f(3, 800) = 120$. A customer who uses this service for three months and is on-line for a total of 800 minutes is charged \$120.

17. (a) Expenditure, E, is given by the equation:

$$E = (\text{price of raw material } 1)m_1 + (\text{price of raw material } 2)m_2 + C$$

where C denotes all the other expenses (assumed to be constant). Since the prices of the raw materials are constant, but m_1 and m_2 are variables, we have a linear function.

(b) Revenue, R, is given by the equation:

$$R = (p_1)q_1 + (p_2)q_2.$$

Since p_1 and p_2 are constant, while q_1 and q_2 are variables, we again have a linear function.

(c) Revenue is again given by the equation,

$$R = (p_1)q_1 + (p_2)q_2.$$

Since p_2 and q_2 are now constant, the term $(p_2)q_2$ is also constant. However, since p_1 and q_1 are variables, the $(p_1)q_1$ term means that the function is not linear.

18. The data in Table 12.10 is apparently linear with a slope in the w direction of about 0.9 calories burned for every extra 20 lbs of weight, and a slope in the s direction of about 1.6 calories burned for every extra mile per hour of speed. Since $B = 4.2$ when $w = 120$ and $s = 8$, a formula for B is

$$B = 4.2 + 0.9(w - 120) + 1.6(s - 8).$$

The formula does not make sense for low weights or speeds. For example, it says that a person weighing 120 pounds going 5 mph burns a negative number of calories per minute, as would a person (child) weighing 60 lbs and going 7 mph.

19. The time in minutes to go 10 miles at a speed of s mph is $(10/s)(60) = 600/s$. Thus the 120 lb person going 10 mph uses $(7.4)(600/10) = 444$ calories, and the 180 lb person going 8 mph uses $(7.0)(600/8) = 525$ calories. The 120 lb person burns $444/120 = 3.7$ calories per pound for the trip, while the 180 lb person burns $525/180 = 2.9$ calories per pound for the trip.

20. A trip of 10 miles at s mph takes $10/s$ hours $= 600/s$ minutes. Since the number of calories burned per minute is B, the total number of calories burned on the trip is $B \cdot 600/s$. Thus

$$P = \frac{B(600/s)}{w} = \frac{600(4.2 + 0.9(w - 120) + 1.6(s - 8))}{sw}$$

21. The function, g, has a slope of 3 in the x direction and a slope of 1 in the y direction, so $g(x, y) = c + 3x + y$. Since $g(0, 0) = 0$, the formula is $g(x, y) = 3x + y$.

22. The function h decreases as y increases: each increase of y by 2 takes you down one contour and hence changes the function by 2, so the slope in the y direction is -1. The slope in the x direction is 2, so the formula is $h(x, y) = c+2x-y$. From the diagram we see that $h(0, 0) = 4$, so $c = 4$. Therefore, the formula for this linear function is $h(x, y) = 4+2x-y$.

23. For each column in the table, we find that as x increases by 1, $f(x, y)$ increases by 2, so the x slope is 2. For each row in the table, we find that as y increases by 1, $f(x, y)$ decreases by 0.5, so the y slope is -0.5. So the function has the form $f(x, y) = 2x - 0.5y + c$. Also note that $f(0, 0) = 1$, so $c = 1$. Therefore, the function is $f(x, y) = 2x - 0.5y + 1$.

24. For each column in the table, we find that as x increases by 100, $f(x, y)$ decreases by 1, so the x slope is -0.01. For each row in the table, we find that as y increases by 10, $f(x, y)$ increases by 3, so the y slope is 0.3. So the function has the form $f(x, y) = -0.01x + 0.3y + c$. Also note that $f(100, 10) = 3$, so $c = 1$. Therefore, the function is $f(x, y) = -0.01x + 0.3y + 1$

25. See Figure 12.112.

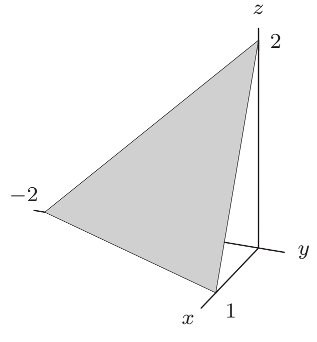


Figure 12.112

26. See Figure 12.113.

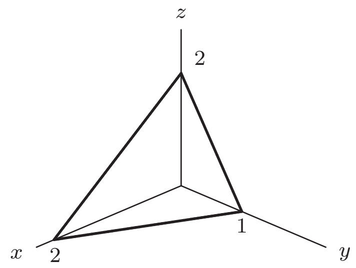


Figure 12.113

27. See Figure 12.114.

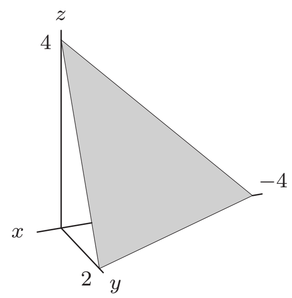


Figure 12.114

28. See Figure 12.115.

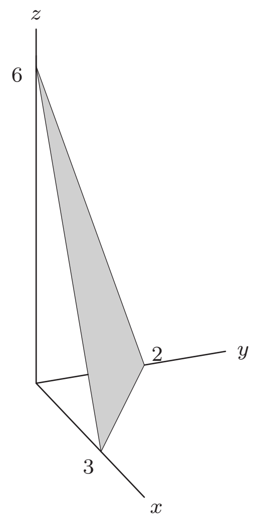


Figure 12.115

29. (a) The contours of f have equation

$$k = c + mx + ny, \quad \text{where } k \text{ is a constant.}$$

Solving for y gives:

$$y = -\frac{m}{n}x + \frac{k-c}{n}$$

Since c, m, n and k are constants, this is the equation of a line. The coefficient of x is the slope and is equal to $-m/n$.

(b) Substituting $x+n$ for x and $y-m$ for y into $f(x,y)$ gives

$$f(x+n, y-m) = c + m(x+n) + n(y-m)$$

Multiplying out and simplifying gives

$$f(x+n, y-m) = c + mx + mn + ny - nm$$

$$f(x+n, y-m) = c + mx + ny = f(x,y)$$

(c) Part (b) tells us that if we move n units in the x direction and $-m$ units in the y direction, the value of the function $f(x,y)$ remains constant. Since contours are lines where the function has a constant value, this implies that we remain on the same contour. This agrees with part (a) which tells us that the slope of any contour line will be $-m/n$. Since the slope is $\Delta y/\Delta x$, it follows that changing y by $-m$ and x by n will keep us on the same contour.

30. (a) We see always the same change in z, namely $\Delta z = 7$, for each step through the table in this diagonal direction. For example, in the third step of the diagonal starting at 3 we get $24 - 17 = 7$, and in the second step of the diagonal starting at 6 we get $20 - 13 = 7$.

(b) We see always the same change in z, namely $\Delta z = -5$, for each step in this direction. For example, in the second step starting from 19 we get $9 - 14 = -5$, and in the first step starting at 22 we get $17 - 22 = -5$.

(c) For a linear function, $z = mx + ny + c$, we have:

$$z_1 - z_2 = (mx_1 + ny_1 + c) - (mx_2 + ny_2 + c) = m(x_1 - x_2) + n(y_1 - y_2).$$

Writing $\Delta z = z_1 - z_2$, and $\Delta x = x_1 - x_2$, and $\Delta y = y_1 - y_2$, we have

$$\Delta z = m\Delta x + n\Delta y.$$

For the particular linear function in this problem, we have

$$\Delta z = \frac{4}{5}\Delta x + \frac{3}{2}\Delta y.$$

In part (a), as we move down the diagonal, we are taking steps with the same $\Delta x = 5$ and same $\Delta y = 2$. Therefore we will get the same change in z for each step,

$$\Delta z = \frac{4}{5}(5) + \frac{3}{2}(2) = 7.$$

In part (b), for each step we have $\Delta x = -10$ and $\Delta y = 2$, so for each step

$$\Delta z = \frac{4}{5}(-10) + \frac{3}{2}(2) = -5.$$

31. (a) We have $\Delta z = 7$. Thus

$$\text{Slope} = \frac{7}{\sqrt{5^2 + 2^2}} = \frac{7}{\sqrt{29}}.$$

(b) We have $\Delta z = -5$. Thus

$$\text{Slope} = \frac{-5}{\sqrt{(-10)^2 + 2^2}} = \frac{-5}{\sqrt{104}}.$$

32. Graph (I) has contour lines that slope upward from left to right, so it corresponds to h, j, k, or m. Since the values on the contour lines are increasing with x and decreasing with y, Graph (I) corresponds to h or j. Since $(0,0,12)$ is a point on the contours of h but not of j for $-2 \le x, y \le 2$, the values on the contour lines show that Graph (I) corresponds to h.

Graph (II) has contour lines that slope downward from left to right, so it corresponds to f, g, n, or p. Since the values on the contour lines are decreasing with x and with y, Graph (II) corresponds to n or p. Since $(0,0,14)$ is a point on the contours of n but not of p for $-2 \le x, y \le 2$, the values on the contour lines show that Graph (II) corresponds to n.

Graph (III) has contour lines that slope downward from left to right, so it corresponds to f, g, n, or p. Since the values on the contour lines are increasing with x and with y, Graph (III) corresponds to f or g. Since $(0,0,10)$ is a point on the contours of f but not of g for $-2 \le x, y \le 2$, the values on the contour lines show that Graph (III) corresponds to f.

Graph (IV) has contour lines that slope upward from left to right, so it corresponds to h, j, k, or m. Since the values on the contour lines are increasing with y and decreasing with x, Graph (IV) corresponds to k or m. Since $(0,0,60)$ is a point on the contours of m but not of k for $-2 \le x, y \le 2$, the values on the contour lines show that Graph (IV) corresponds to m.

Strengthen Your Understanding

33. The function $f(x, y) = e^{x+y}$ has contours that are parallel lines $x + y = c$, but it is not linear. This example generalizes to $g(x + y)$ for any function $g(t)$. The family of functions $h(x, y) = r(x)$ also works, for any function r. There are other examples.

34. The function $f(x, y) = xy$ has linear cross-sections for both x and y fixed, but it is not linear. Any function of the form $g(x, y) = (mx + b)(ny + c)$ also satisfies this condition.

35. A possible example is in Table 12.16, where the rows have slopes 1, 2, 3, respectively, and the columns have slopes 1, 2, 3. Notice that the function is not linear since the slopes in each row (and in each column) are different.

Table 12.16

$x\backslash y$	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

36. If the linear function is $z = mx + ny + c$, then the contour for $z = 0$ is:

$$mx + ny + c = 0.$$

We want this line to have slope 2, so we rewrite it in slope-intercept form:

$$y = -\frac{m}{n}x - \frac{c}{n}.$$

Thus, we want $-m/n = 2$, for example $m = -2, n = 1$. So $z = -2x + y$ is one example. There are others.

37. False. At every point (x, y) the z coordinate on the first plane is 2 units lower than the second so these planes are parallel and do not intersect.

38. False. The first row is linear with slope $1/0.1 = 10$. The second row is linear with slope $1.07/0.1 = 10.7$. Since the slope of the first row is not the same as the slope of the second row, the function is not linear.

39. False. The contours are of the form $c = 3x + 2y$ which are lines with slope $-3/2$.

40. True. The contours of a linear function $f(x, y) = c + mx + ny$ look like $k = c + mx + ny$ which are parallel lines with slope $-m/n$.

41. True. $f(0, 0) = 0, f(0, 1) = 4$ give a y slope of 4, but $f(0, 0) = 0, f(0, 3) = 5$ give a y slope of $5/3$. Since linearity means the y slope must be the same between any two points, this function cannot be linear.

42. True. A linear function has constant slopes in the x and y directions, so its graph is a plane.

43. True. Since the graph of a linear function is a plane, any vertical slice parallel to the yz-plane will yield a line.

44. False. Any function of the form $f(x, y) = c$ is linear (with zero slope in both the x and y directions) and has a graph which is parallel to the xy-plane.

45. True. Functions can have only one value for a given input, so their graphs can intersect a vertical line at most once. A vertical plane would not satisfy this property, so cannot be the graph of a function.

46. False. There is at least one point where $f(a, b) = 0$, for example $(a, b) = (1, 1)$. There are an infinite number of other points lying on the straight-line contour $f(a, b) = 0$.

47. False. All of the columns have to have the same slope, as do the rows, but the row slopes can differ from the column slopes.

48. False. Simply knowing where the plane intersects the xy-plane does not determine the plane uniquely. There are an infinite number of linear functions whose graph intersects the xy-plane in this line. Two examples: $f(x, y) = -1 - 2x + y$ and $g(x, y) = -2 - 4x + 2y$.

Solutions for Section 12.5

Exercises

1. (a) Observe that setting $f(x, y, z) = c$ gives a cylinder about the x-axis, with radius $\sqrt{c}$. These surfaces are in graph (I).
(b) By the same reasoning the level curves for $h(x, y, z)$ are cylinders about the y-axis, so they are represented in graph (II).

2. Points on one of the nested spheres in II have constant distance from the origin, so these spheres are level surfaces $f(x, y, z) = x^2 + y^2 + z^2 = c$. Points on one of the nested cylinders in I have constant distance from the y-axis, so these cylinders are level surfaces $g(x, y, z) = x^2 + z^2 = k$.

3. If we solve for z, we get $z = \frac{1}{3}(5 - x - 2y)$, so the level surface is the graph of $f(x, y) = \frac{1}{3}(5 - x - 2y)$.

4. We are looking for all points (x, y, z) whose distance from the origin is 2, that is, $(x - 0)^2 + (y - 0)^2 + (z - 0)^2 = 4$, or $x^2 + y^2 + z^2 = 4$, which is a level surface of $f(x, y, z) = x^2 + y^2 + z^2$.

5. If we solve for z, we get $z = (1 - x^2 - y)^2$, so the level surface is the graph of $f(x, y) = (1 - x^2 - y)^2$.

6. We are looking for all points (x, y, z) whose distance from (a, b, c) is a constant k, that is, $(x-a)^2+(y-b)^2+(z-c)^2 = k^2$, which is a level surface of $f(x, y, z) = (x - a)^2 + (y - b)^2 + (z - c)^2$.

7. Only the elliptical paraboloid, the hyperbolic paraboloid and the plane. These are the only surfaces in the catalog that satisfy the "vertical line test," that is, they have at most one z-value for each x and y.

8. An elliptic paraboloid.

9. A hyperboloid of two sheets.

10. A plane.

11. An ellipsoid.

12. Yes,
$$z = f(x, y) = x^2 + 3y^2.$$

13. Yes,
$$z = f(x, y) = \frac{2}{5}x + \frac{3}{5}y - 2.$$

14. No, because some z values correspond to two points on the surface.

15. No, because $z = \sqrt{x^2 + 3y^2}$ and $z = -\sqrt{x^2 + 3y^2}$, so some z-values correspond to two points on the surface.

Problems

16. The plane is represented by
$$z = f(x, y) = 2x - \frac{y}{2} - 3$$
and
$$g(x, y, z) = 4x - y - 2z = 6.$$
Other answers are possible

17. The top half of the sphere is represented by
$$z = f(x, y) = \sqrt{10 - x^2 - y^2}$$
and
$$g(x, y, z) = x^2 + y^2 + z^2 = 10, \quad z \geq 0.$$
Other answers are possible.

18. The bottom half of the ellipsoid is represented by
$$z = f(x, y) = -\sqrt{2(1 - x^2 - y^2)}$$
$$g(x, y, z) = x^2 + y^2 + \frac{z^2}{2} = 1, \quad z \leq 0.$$
Other answers are possible

19. (a) The isothermal surfaces of f are parallel planes. Each plane is described by the equation

$$2x - 3y + z = c + 20,$$

for each value of the constant c.

(b) We have:

$$f_z(0,0,0) = 1.$$

This means that if we start at the point $(0,0,0)$ and move slightly upwards in the direction of the positive z-axis, our temperature is increasing by one degree Fahrenheit for each additional unit we move.

(c) To increase our temperature the fastest we should move away from the isothermal plane passing through $(0,0,0)$ in a direction that allows us to reach the warmer isothermal planes as fast as possible. This means that we should follow a normal vector to the isothermal plane passing through $(0,0,0)$ that has a positive $\vec{k}$ component (temperature increases with c and $c+20$ gives the z-intercept of each isothermal plane). We have:

$$\text{Isothermal plane through } (0,0,0): \; 2x - 3y + z = 20,$$
$$\text{Normal vector to isothermal plane through } (0,0,0): \; \vec{n} = 2\vec{i} - 3\vec{j} + \vec{k}.$$

So, we must move away from the origin in the direction of the vector $2\vec{i} - 3\vec{j} + \vec{k}$.

(d) Isothermal surfaces of f are of the form

$$z = c + 20 - 2x + 3y,$$

so, setting $c = -3$, we see that $f(x,y) = -2x + 3y + 17$ is an isothermal surface of f. On this surface the temperature is -3 degrees Fahrenheit.

20. (a) We expect B to be an increasing function of all three variables.

(b) A deposit of \$1250 at a 1% annual interest rate leads to a balance of \$1276 after 25 months.

21. We expect P to be an increasing function of A and r. (If you borrow more, your payments go up; if the interest rates go up, your payments go up.) However, P is a decreasing function of t. (If you spread out your payments over more years, you pay less each month.)

22. The graph of $g(x,y) = x + 2y$ is the set of all points (x,y,z) satisfying $z = x + 2y$, or $x + 2y - z = 0$. This is a level surface, but we want the surface equal to the constant value 1, not 0, so we can add 1 to both sides to get $x+2y-z+1 = 1$. Thus, $f(x,y,z) = x + 2y - z + 1$ has level surface $f = 1$ identical to the graph of $g(x,y) = x + 2y$.

23. If we solve $x^2+y^2/4+z^2/9 = 1$ for z we get $z = \pm 3\sqrt{1 - x^2 - y^2/4}$. Thus we can take $f(x,y) = 3\sqrt{1 - x^2 - y^2/4}$ and $g(x,y) = -3\sqrt{1 - x^2 - y^2/4}$.

24. The equation of any plane parallel to the plane $z = 2x+3y-5$ has x-slope 2 and y-slope 3, so has equation $z = 2x+3y-c$ for any constant c, or $2x + 3y - z = c$. Thus we could take $g(x,y,z) = 2x + 3y - z$. Other answers are possible.

25. (a) The graph of $f(x,y)$ is obtained by plotting points (x,y,z), where $z = f(x,y)$. Since the square root function is never negative, we have $z \ge 0$. Setting $z = \sqrt{1 - x^2 - y^2}$ and squaring both sides leads to $x^2 + y^2 + z^2 = 1$, which is the equation for a sphere of radius 1. The graph of the function includes only those points where $z \ge 0$, that is, the upper hemisphere of radius 1, centered at the origin.

(b) If we take $g(x,y,z) = f(x,y) - z = \sqrt{1 - x^2 - y^2} - z$, then the level surface $g(x,y,z) = 0$ is the surface S.

26. (a) The graph of $f(x,y)$ is obtained by plotting points (x,y,z), where $z = f(x,y)$. Since the square root function is never negative, we have $z \ge 0$. Setting $z = \sqrt{1 - y^2}$ and squaring both sides leads to $y^2 + z^2 = 1$, which is the equation for a circular cylinder of radius 1 lying along the x-axis (since x is missing from the equation). The graph of the function includes only those points where $z \ge 0$, that is, the upper half of the cylinder.

(b) If we take $g(x,y,z) = f(x,y) - z = \sqrt{1 - y^2} - z$, then the level surface $g(x,y,z) = 0$ is the surface S.

27. Starting with the equation $z = \sqrt{x^2 + y^2}$, we flip the cone and shift it up one, yielding $z = 1 - \sqrt{x^2 + y^2}$. This is a cone with vertex at $(0,0,1)$ that intersects the xy-plane in a circle of radius 1. Interchanging the variables, we see that $y = 1 - \sqrt{x^2 + z^2}$ is an equation whose graph includes the desired cone C. Finally, we express this equation as a level surface $g(x,y,z) = 1 - \sqrt{x^2 + z^2} - y = 0$.

28. In the xz-plane, the equation $x^2/4 + z^2 = 1$ is an ellipse, with widest points at $x = \pm 2$ on the x-axis and crossing the z-axis at $z = \pm 1$. Since the equation has no y term, the level surface is a cylinder of elliptical cross-section, centered along the y-axis.

29. Setting y to a constant c yields the equation $x^2 + z^2 = 1 - c^2/4$, which, for $-2 \le c \le 2$ gives circular cross-sections. Fixing $x = c$ yields the equation $y^2/4 + z^2 = 1 - c^2$, which for $-1 \le c \le 1$ yields elliptical cross-sections. A similar result is true for cross-sections with constant z. Thus the level surface appears to be a unit sphere, centered at the origin, that has been stretched by a factor of two in the y-direction (this shape is called an ellipsoid).

30. The level surfaces are graphs of the equations $x + y + z = c$ for different values of the constant c. These are all parallel planes.

31. The level surfaces are the graphs of $\sin(x + y + z) = k$ for constant k (with $-1 \le k \le 1$). This means $x + y + z = \sin^{-1}(k) + 2\pi n$, or $\pi - \sin^{-1}(k) + 2n\pi$ for all integers n. Therefore for each value of k, with $-1 \le k \le 1$, we get an infinite family of parallel planes. So the level surfaces are families of parallel planes.

32. Let's consider the function $y = 2 + \sin z$ drawn in the yz-plane in Figure 12.116.

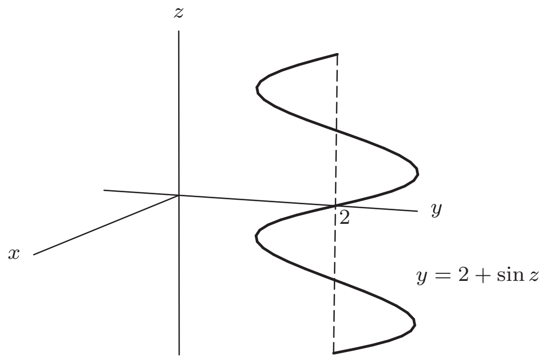


Figure 12.116

Now rotate this graph around the z-axis. Then, a point (x, y, z) is on the surface if and only if $x^2 + y^2 = (2+\sin z)^2$. Thus, the surface generated is a surface of rotation with the profile shown in Figure 12.116.

Similarly, the surface with equation $x^2 + y^2 = (f(z))^2$ is the surface obtained rotating the graph of $y = f(z)$ around the z-axis.

33. $f(x, y, z) = x^2 - y^2 + z^2$ has 3 types of level surfaces depending on the values of c in the equation $x^2 - y^2 + z^2 = c$. We write this as $x^2 + z^2 = y^2 + c$ and think of what happens as we take a cross-section of the surface, perpendicular to the y-axis by holding y fixed.
(i) For $c > 0$, the level surface is a hyperboloid of 1 sheet.
(ii) For $c < 0$, the level surface is a hyperboloid of 2 sheets.
(iii) For $c = 0$, the level surface is a cone.

34. The level surfaces are the graphs of $g(x, y, z) = e^{-(x^2+y^2+z^2)} = k$ for constant values of k such that $0 < k \le 1$. So $x^2 + y^2 + z^2 = -\ln k$, which is the graph of a sphere since $-\ln k \ge 0$.

35. The level surfaces are all planes described as follows:

When $h(x, y, z) = 1$, the plane is given by

$$e^{z-y} = 1, \quad \text{so} \quad z - y = \ln 1 = 0.$$

When $h(x, y, z) = e$, the plane is given by

$$e^{z-y} = e, \quad \text{so} \quad z - y = \ln e = 1.$$

When $h(x, y, z) = e^2$, the plane is given by

$$e^{z-y} = e^2, \quad \text{so} \quad z - y = \ln e^2 = 2.$$

See Figure 12.117.

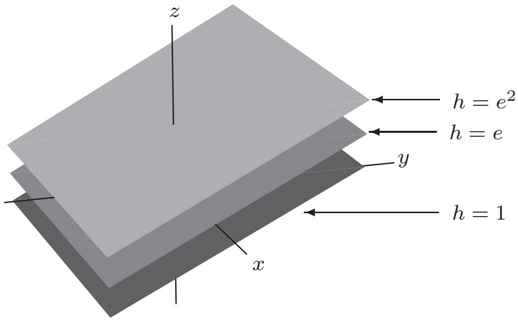


Figure 12.117

36. For values of $f < 4$, the level surfaces are spheres, with larger f giving smaller radii. See Figure 12.118.

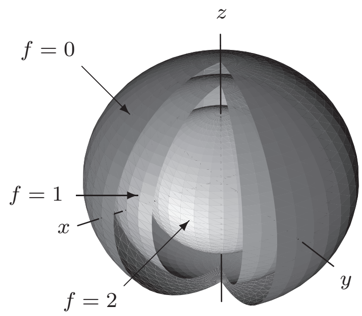


Figure 12.118

37. For values of $g < 1$, the level surfaces are cylinders centered on the z-axis, with larger g values giving smaller radii. See Figure 12.119.

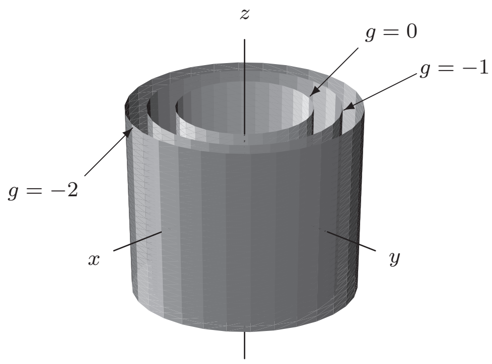


Figure 12.119

Strengthen Your Understanding

38. The graph of $f(x, y, z)$ is all points (x, y, z, w) in 4-space such that $w = f(x, y, z)$. This graph cannot be drawn in 3-space; it would need 4 dimensions.

39. Since z is missing in the formula for $f(x, y, z)$, the level surface $f(x, y, z) = x^2 - y^2 = c$ is a hyperbolic cylinder. All its cross-sections perpendicular to the z-axis are the same hyperbola $x^2 - y^2 = c$.

40. Since z is missing in the formula for $f(x,y,z)$, the level surface $f(x,y,z)=x^2+y^2=c$ is a cylinder running along the z-axis.

41. A linear function of three variables has level surfaces that are equally spaced planes. Choosing a linear function that does not depend on x gives level surfaces perpendicular to the yz-plane. The function $f(x,y,z)=y+z$, for example, works. Its level surfaces are the planes: $c=y+z$, or $z=-y+c$.

42. A cylinder centered on the y-axis has equation $x^2+z^2=c$, so we take $f(x,y,z)=x^2+z^2$. There are other possible answers.

43. Let $f(x,y,z)=(x+y+z)^2$. Then $f(x,y,z)=c\geq 0$ means $x+y+z=\sqrt{c}$, which for different c are parallel planes.

44. One family of paraboloids is given by equations of the form $z=x^2+y^2-c$, where c is a constant. Rearranging this equation, we obtain $x^2+y^2-z=c$. Therefore, the level sets of the function $f(x,y,z)=x^2+y^2-z$ are paraboloids.

45. True. Both are the set of all points (x,y,z) in 3-space satisfying $z=x^2+y^2$.

46. False. The graph of $f(x,y)=\sqrt{1-x^2-y^2}$ is the upper unit hemisphere, while the graph of $g=1$ is $x^2+y^2+z^2=1$, which is the entire unit sphere (both spheres with center at the origin).

47. True. The graph of $f(x,y)$ is the set of all points (x,y,z) satisfying $z=f(x,y)$. If we define the three-variable function g by $g(x,y,z)=f(x,y)-z$, then the level surface $g=0$ is exactly the same as the graph of $f(x,y)$.

48. False. For example, the function $g(x,y,z)=x^2+y^2+z^2$ has level surface $g=1$ which is a sphere of radius 1, centered at the origin. This surface cannot be the graph of any function $f(x,y)$, since a vertical line intersects it in more than one place.

49. True. The level surfaces are of the form $x+2y+z=k$, or $z=k-x-2y$. These are the graphs of the linear functions $f(x,y)=k-x-2y$, each of which has x-slope of -1 and y-slope equal to -2. Thus they form parallel planes.

50. False. The level surfaces are of the form $x^2+y+z^2=k$, or $y=k-x^2-z^2$. These are paraboloids centered on the y-axis, not cylinders.

51. False. The level surface $g=0$ of the function $g(x,y,z)=x^2+y^2+z^2$ consists of only the origin.

52. True. The level surfaces $g=k$ are of the form $ax+by+cz+d=k$, or

$$z=\frac{1}{c}(-ax-by+(k-d)).$$

Thus z is a linear function of x and y, whose graph is a plane.

53. False. For example, the function $g(x,y,z)=\sin(x+y+z)$ has level surfaces of the form $x+y+z=k$, where $k=\arcsin(c)+n\pi$, for $n=0,\pm1,\pm2,\ldots$. These surfaces are planes (for $-1\leq c\leq 1$).

54. True. If there is a point (a,b,c) lying on both $g(x,y,z)=k_1$ and $g(x,y,z)=k_2$, then we must have $g(a,b,c)=k_1$ and $g(a,b,c)=k_2$. Since g is a function, it can only have a single value at a point, so $k_1=k_2$.

Solutions for Section 12.6

Exercises

1. No, $1/(x^2+y^2)$ is not defined at the origin, so is not continuous at all points in the square $-1\leq x\leq 1, -1\leq y\leq 1$.

2. The function $1/(x^2+y^2)$ is continuous on the square $1\leq x\leq 2, 1\leq y\leq 2$. The functions x^2 and y^2 are continuous everywhere, and so is their sum. The constant function 1 is continuous, and thus so is the ratio $1/(x^2+y^2)$, as long as $x^2+y^2\neq 0$. Since the only place $x^2+y^2=0$ is at the origin, and the origin is not included in the square, the function is continuous in the square.

3. The function $y/(x^2+2)$ is continuous on the disk $x^2+y^2\leq 1$. The functions x^2+2 and y are continuous everywhere, and so is their ratio, as long as the denominator is not 0. But x^2+2 is always at least 2, so the function is continuous on the disk (actually at all points in the plane).

4. The function $e^{\sin x}/\cos y$ is continuous on the rectangle $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}, 0\leq y\leq\frac{\pi}{4}$. The functions $\sin x$ and e^x are continuous everywhere, and so is their composition $e^{\sin x}$. Then the ratio $e^{\sin x}/\cos y$ is continuous as long as the denominator is not 0. But $\cos y$ is not 0 in the interval $0\leq y\leq\frac{\pi}{4}$, so the function is continuous on the given rectangle.

5. The function $\tan(\theta)$ is undefined when $\theta=\pi/2\approx 1.57$. Since there are points in the square $-2\leq x\leq 2, -2\leq y\leq 2$ with $x\cdot y=\pi/2$ (e.g. $x=1, y=\pi/2$) the function $\tan(xy)$ is not defined inside the square, hence not continuous.

6. The function $\sqrt{2x-y}$ is undefined when $2x-y<0$. Since there are points in the disk $x^2+y^2\le 4$ with $2x-y<0$ (e.g. $x=0, y=1$) the function $\sqrt{2x-y}$ is not defined at all points inside the disk and hence is not continuous.

7. Since the composition of continuous functions is continuous, the function f is continuous at $(0,0)$ and we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} e^{-x-y} = e^{-0-0} = 1$$

8. Since the composition of continuous functions is continuous, the function f is continuous at $(0,0)$. We have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0+0 = 0.$$

9. Since f does not depend on y we have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x}{x^2+1} = \frac{0}{0+1} = 0.$$

10. Since the composition of continuous functions is continuous, the function f is continuous at $(0,0)$. We have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x+y}{\sin y+2} = \frac{0+0}{0+2} = 0.$$

11. We want to compute

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

As $r=\sqrt{x^2+y^2}$ is the distance from (x,y) to $(0,0)$ we have that $(x,y)\to(0,0)$ is equivalent to $r\to 0$. Hence the limit becomes:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{\sin r^2}{r^2} = 1.$$

Problems

12. We want to show that f does not have a limit as (x,y) approaches $(0,0)$. So let us suppose that (x,y) tends to $(0,0)$ along the line $y=mx$, where the slope $m\neq 1$. Then

$$f(x,y) = f(x,mx) = \frac{x+mx}{x-mx} = \frac{(1+m)x}{(1-m)x} = \frac{1+m}{1-m}.$$

Therefore

$$\lim_{x\to 0} f(x,mx) = \frac{1+m}{1-m}$$

and so for $m=2$ we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=2x}} f(x,y) = \frac{1+2}{1-2} = \frac{3}{-1} = -3$$

and for $m=3$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=3x}} f(x,y) = \frac{1+3}{1-3} = \frac{4}{-2} = -2.$$

Thus no matter how close they are to the origin, there will be points (x,y) where the value $f(x,y)$ is close to -3 and points (x,y) where $f(x,y)$ is close to -2. So the limit:

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

13. We want to show that f does not have a limit as (x, y) approaches $(0, 0)$. Let us suppose that (x, y) tends to $(0, 0)$ along the line $y = mx$. Then

$$f(x, y) = f(x, mx) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}.$$

Therefore

$$\lim_{x\to 0} f(x, mx) = \frac{1 - m^2}{1 + m^2}$$

and so for $m = 1$ we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x, y) = \frac{1-1}{1+1} = \frac{0}{2} = 0$$

and for $m = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x, y) = \frac{1-0}{1+0} = 1.$$

Thus no matter how close they are to the origin, there will be points (x, y) such that $f(x, y)$ is close to 0 and points (x, y) where $f(x, y)$ is close to 1. So the limit:

$$\lim_{(x,y)\to(0,0)} f(x, y) \text{ does not exist.}$$

14. Points along the positive x-axis are of the form $(x, 0)$; at these points the function looks like $2x/2x = 1$ everywhere (except at the origin, where it is undefined). On the other hand, along the y-axis, the function looks like $-y^2/y^2 = -1$. Since approaching the origin along two different paths yields numbers that are not the same, the limit does not exist.

15. We want to show that f does not have a limit as (x, y) approaches $(0, 0)$. For this let us consider $x > 0$, $y > 0$, which gives

$$\lim_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} f(x, y) = \lim_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} \frac{xy}{|xy|} = 1.$$

On the other hand, if $x > 0$, $y < 0$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ x>0,y<0}} f(x, y) = \lim_{\substack{(x,y)\to(0,0)\\ x>0,y<0}} \frac{xy}{|xy|} = \lim_{\substack{(x,y)\to(0,0)\\ x>0,y<0}} \frac{xy}{-xy} = -1.$$

Thus no matter how close to the origin they are, there will be points (x, y) such that $f(x, y)$ is close to 1 and points (x, y) such that $f(x, y)$ is close to -1. So the limit

$$\lim_{(x,y)\to(0,0)} f(x, y) \text{ does not exist.}$$

16. Let us suppose that (x, y) tends to $(0, 0)$ along the curve $y = kx^2$, where $k \neq -1$. We get

$$f(x, y) = f(x, kx^2) = \frac{x^2}{x^2 + kx^2} = \frac{1}{1 + k}.$$

Therefore:

$$\lim_{x\to 0} f(x, kx^2) = \frac{1}{1 + k}$$

and so for $k = 0$ we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x, y) = 1$$

and for $k = 1$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} f(x, y) = \frac{1}{2}.$$

Thus no matter how close they are to the origin, there will be points (x, y) where the value $f(x, y)$ is close to 1 and points (x, y) where $f(x, y)$ is close to $\frac{1}{2}$. So the limit:

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist.

17. For $x > 0$, we have
$$f(x, y) = y.$$
Thus, the surface representing f for $x > 0$ is the plane $z = y$.

For $x < 0$, we have
$$f(x, y) = -y.$$
Thus, the surface representing f for $x < 0$ is the plane $z = -y$.

For $x = 0$, we have
$$f(x, y) = 0.$$
Thus, the surface representing f is two half-planes and the y-axis.

(a) The function is continuous at every point on the x-axis.
(b) The function is not continuous at any point on the y-axis, except at the origin, because $f(x, y) = 0$ on the y-axis and not nearby unless $y = 0$.
(c) The function is continuous at the origin.

(a) Yes
(b) No
(c) Yes

18. The function, f is continuous at all points (x, y) with $x \neq 3$. We analyze the continuity of f at the point $(3, a)$. We have:
$$\lim_{(x,y)\to(3,a),x<3} f(x, y) = \lim_{y\to a}(c + y) = c + a$$
$$\lim_{(x,y)\to(3,a),x>3} f(x, y) = \lim_{x>3,x\to 3}(5 - x) = 2.$$
We want to see if we can find one value of c such that $c + a = 2$ for all a. This would mean that $c = 2 - a$, but then c would be dependent on a. Therefore, we cannot make the function continuous everywhere.

19. The function f is continuous at all points (x, y) with $x \neq 3$. So let's analyze the continuity of f at the point $(3, a)$. We have
$$\lim_{\substack{(x,y)\to(3,a)\\x<3}} f(x, y) = \lim_{y\to a}(c + y) = c + a$$
$$\lim_{\substack{(x,y)\to(3,a)\\x>3}} f(x, y) = \lim_{y\to a}(5 - y) = 5 - a.$$
So we need to see if we can find one value for c such that $c + a = 5 - a$ for all a. This would require that $c = 5 - 2a$, but then c would depend on a, which is exactly what we don't want. Therefore, we cannot make the function continuous everywhere.

20. It is not continuous at $(0, 0)$. The function $f(x, y) = x^2 + y^2$ gets closer and closer to 0 as (x, y) gets closer to the origin; but the value of $f(0, 0)$ is not 0, it is 2. Since the value of the function is not equal to the limit, the function is not continuous at the origin.

21. The function $f(x, y) = x^2 + y^2 + 1$ gets closer and closer to 1 as (x, y) gets closer to the origin. To make f continuous at the origin, we need to have $f(0, 0) = 1$. Thus $c = 1$ will make the function continuous at the origin.

22. (a) The graphs are shown in Figure 12.120.

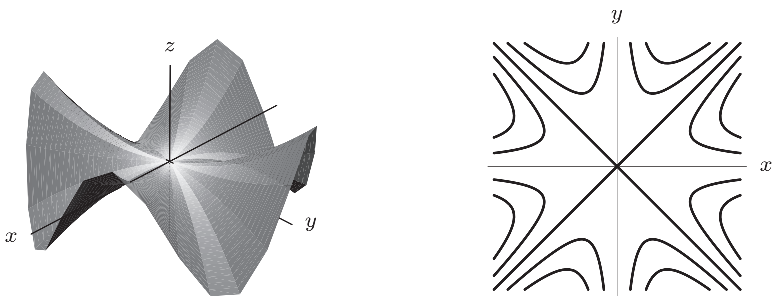


Figure 12.120

(b) Yes, it seems that if x and y are both close to 0, the values of the function are both close to $0 = f(0, 0)$.

23. (a) We have $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y, so these are both continuous (constant) functions of one variable.

(b) The contour diagram suggests that the contours of f are lines through the origin. Providing it is not vertical, the equation of such a line is

$$y = mx.$$

To confirm that such lines are contours of f, we must show that f is constant along these lines. Substituting into the function, we get

$$f(x,y) = f(x,mx) = \frac{x(mx)}{x^2 + (mx)^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1+m^2} = \text{constant}.$$

Since $f(x,y)$ is constant along the line $y = mx$, such lines are contained in contours of f.

(c) We consider the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the line $y = mx$. We can see that

$$\lim_{x\to 0} f(x,mx) = \frac{m}{1+m^2}.$$

Therefore, if $m = 1$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \frac{1}{2}$$

whereas if $m = 0$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = 0.$$

Thus, no matter how close we are to the origin, we can find points (x,y) where the value $f(x,y)$ is $1/2$ and points (x,y) where the value $f(x,y)$ is 0. So the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus, f is not continuous at $(0,0)$, even though the one-variable functions $f(x,0)$ and $f(0,y)$ are continuous at $(0,0)$. See Figures 23 and 23

Strengthen Your Understanding

24. For continuity, one also needs the value of the limit to be the same as $f(a,b)$.

25. For the quotient f/g, one also needs $g(a,b) \neq 0$.

26. Let

$$f(x,y) = \frac{1}{x^2+y^2} + \frac{1}{(x-1)^2 + (y-2)^2}.$$

27. Let

$$f(x,y) = \frac{x^2+2y^2}{x^2+y^2}.$$

Approaching along the x-axis means setting $y = 0$, so then

$$f(x,y) = \frac{x^2}{x^2} = 1 \qquad \text{for all } x \neq 0.$$

Thus, the limit approaching $(0,0)$ along the x-axis is 1.

Approaching along the y-axis means setting $x = 0$, so then

$$f(x,y) = \frac{2y^2}{y^2} = 2, \qquad \text{for all } y \neq 0.$$

Thus, the limit approaching $(0,0)$ along the y-axis is 2.

28. One possible answer is $f(x,y) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } x \geq 2. \end{cases}$

29. One possible answer if $f(x,y) = 1/((x-2)^2 + y^2)$.

30. One possible answer is $f(x,y) = 1/(x^2 + y^2 - 1)$.

Solutions for Chapter 12 Review

Exercises

1. The distance of a point $P = (x, y, z)$ from the yz-plane is $|x|$, from the xz-plane is $|y|$, and from the xy-plane is $|z|$. So A is closest to the yz-plane, since it has the smallest x-coordinate in absolute value. B lies on the xz-plane, since its y-coordinate is 0. C is farthest from the xy-plane, since it has the largest z-coordinate in absolute value.

2. Your final position is $(1, -1, -3)$. Therefore, you are in front of the yz-plane, to the left of the xz-plane, and below the xy-plane.

3. An example is the line $z = -x$ in the xz-plane. See Figure 12.121.

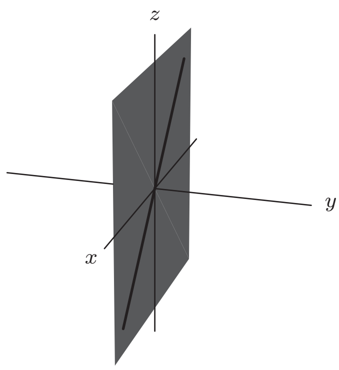


Figure 12.121

4. Given (x, y) we can solve uniquely for z, namely $z = 5 - 3x + 2y$. Thus, z is a function of x and y:

$$z = f(x, y) = 5 - 3x + 2y.$$

5. The equation $x^2 + y^2 + z^2 = 100$ does not determine z uniquely from x and y. For example, the points $(0, 0, 10)$ and $(0, 0, -10)$ both satisfy the equation. Therefore z is not a function of (x, y).

6. Given (x, y) we can solve uniquely for z, namely $z = 2 + \frac{x}{5} + \frac{y}{5} - \frac{3x^2}{5} + y^2$. Thus, z is a function of x and y:

$$z = f(x, y) = 2 + \frac{x}{5} + \frac{y}{5} - \frac{3x^2}{5} + y^2.$$

7. Planes perpendicular to the positive y-axis should yield the graphs of upright parabolas $f(x, y)$, which widen as y decreases (giving $f(x, 2)$ and $f(x, 1)$). When $y = 0$, the parabola flattens out, creating a horizontal line for $f(x, 0)$. The graphs then turn downward, creating the parabolas $f(x, -1)$ and $f(x, -2)$ which become narrower as y decreases. So the graph (IV) bests fits this information.

8. (a) is (IV). The level curves of f and g are lines, with slope of $f = -1$ and slope of $g = 1$. See Figure 12.122.
(b) is (II). The level curves of f and g are lines, with slope of $f = -2/3$ and slope of $g = 2/3$. See Figure 12.123.
(c) is (I). The level curves of f are parabolas opening upward; the level curves of g are the shape of $\ln x$, but upside down and for both positive and negative x-values. See Figure 12.124.
(d) is (III). The level curves of f are hyperbolas centered on the x- or y-axes; the level curves of g are rectangular hyperbolas in quadrants (I) and (III) or quadrants (II) and (IV). See Figure 12.125.

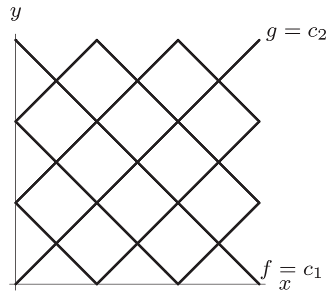


Figure 12.122

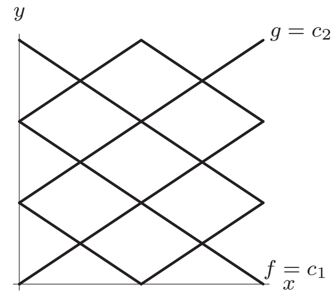


Figure 12.123

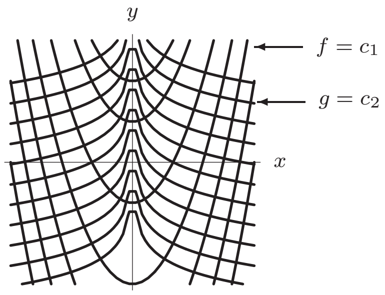


Figure 12.124

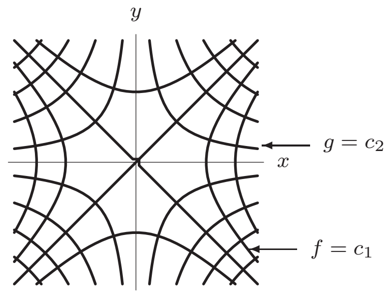


Figure 12.125

9. (a) is (I), because there is a minimum at the origin and the surface slopes steadily upward.

(b) is (IV), because there is a maximum at the origin and the surface slopes increasingly steeply downward as we move away from the origin.

(c) is (II), because there is a maximum at the origin and the surface slopes steadily downward.

(d) is (III), because there is a minimum at the origin and the surface slopes increasingly fast upward as we move away from the origin.

10. Contours are lines of the form $3x - 5y + 1 = c$ as shown in Figure 12.126. Note that for the regions of x and y given, the c values range from $-12 < c < 12$ and are evenly spaced.

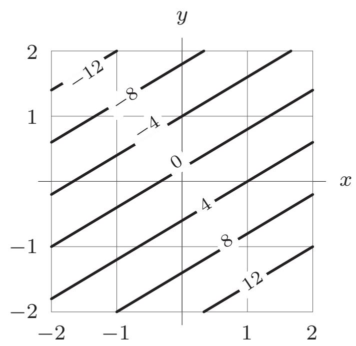


Figure 12.126

11. Since setting $z = c$, with $-1 \leq c \leq 1$ gives $y = \sin^{-1} c + 2n\pi$ or $y = \pi - \sin^{-1} c + 2n\pi$ =constant, where n is any integer, contours are horizontal lines as shown in Figure 12.127.

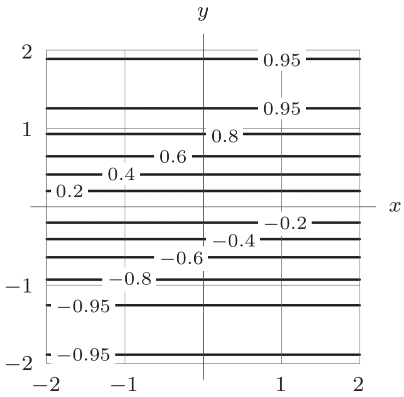


Figure 12.127

12. Contours are ellipses of the form $2x^2 + y^2 = c$ as shown in Figure 12.128. Note that for the ranges of x and y given, the range of c value is $1 \le c < 9$ and are closer together farther from the origin.

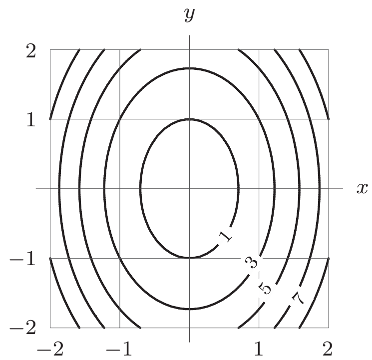


Figure 12.128

13. The contours are ellipses of the form $2x^2 + y^2 = -\ln c$ as shown in Figure 12.129. For the ranges of x and y given, the c values range from just above 0 to 1.

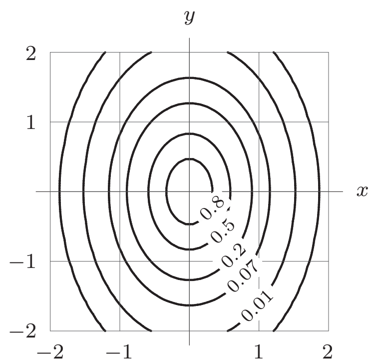


Figure 12.129

14. These conditions describe a line parallel to the z-axis which passes through the xy-plane at $(2, 1, 0)$.

15. The equation is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 25$

16. The equation will be of the form $mx + ny + ez = d$, but you can divide through by d to get an equation of the form $ax + by + cz = 1$ (d can not be zero, as the origin is not in the plane). Now plug in the points: From $(0, 0, 2)$, we get $a(0) + b(0) + c(2) = 1$. From this we get $c = \frac{1}{2}$. Similarly we get $a = \frac{1}{5}$, and $b = \frac{1}{3}$. So the equation that fits these points is

$$\frac{x}{5} + \frac{y}{3} + \frac{z}{2} = 1.$$

The equation of this plane can also be obtained by calculating the normal as the cross product of two vectors lying in the plane.

17. We complete the square

$$\begin{aligned} x^2 + 4x + y^2 - 6y + z^2 + 12z &= 0 \\ x^2 + 4x + 4 + y^2 - 6y + 9 + z^2 + 12z + 36 &= 4 + 9 + 36 \\ (x + 2)^2 + (y - 3)^2 + (z + 6)^2 &= 49 \end{aligned}$$

The center is $(-2, 3, -6)$ and the radius is 7.

18. A contour diagram is linear if the contours are parallel straight lines, equally spaced for equally spaced values of z. This contour diagram does not represent a linear function.

19. A contour diagram is linear if the contours are parallel straight lines, equally spaced for equally spaced values of z. This contour diagram could represent a linear function.

20. (a) Since the function is linear, the increment between successive entries in the same column is constant. From the third column we see that the increment is $2 - 8 = -6$. Subtract 6 to go from any entry in the table to the entry below it, and add 6 to get the entry above it. See Table 12.17.

Table 12.17

		y		
		2.5	3.0	3.50
x	-1	6	7	8
	1	0	1	2
	3	-6	-5	-4

(b) From the third column of the table we calculate

$$\text{Slope in } x\text{-direction } = m = \frac{2-8}{1-(-1)} = -3.$$

From the first row of the table we calculate

$$\text{Slope in } y\text{-direction } = n = \frac{8-6}{3.5-2.5} = 2.$$

The equation of the linear function is

$$\begin{aligned} f(x, y) &= z_0 + m(x - x_0) + n(y - y_0) \\ &= f(-1, 2.5) - 3(x - (-1)) + 2(y - 2.5) = -2 - 3x + 2y. \end{aligned}$$

21. The level surfaces appear to be circular cylinders centered on the z-axis. Since they don't change with z, there is no z in the formula, and we can use the formula for a circle in the xy-plane, $x^2 + y^2 = r^2$. Thus the level surfaces are of the form $f(x, y, z) = x^2 + y^2 = c$ for $c > 0$.

22. The paraboloid is $z = x^2 + y^2 + 5$, so it is represented by

$$z = f(x, y) = x^2 + y^2 + 5$$

and

$$g(x, y, z) = x^2 + y^2 + 5 - z = 0.$$

Other answers are possible.

23. Plane is $(x/2) + (y/3) + (z/4) = 1$, so it is represented by

$$z = f(x, y) = 4 - 2x - \frac{4}{3}y$$

and

$$g(x, y, z) = \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1.$$

Other answers are possible.

24. The upper half of the sphere is represented by

$$z = f(x, y) = \sqrt{1 - x^2 - y^2}$$

and

$$g(x, y, z) = x^2 + y^2 + z^2 = 1.$$

Other answers are possible.

25. The sphere is $(x - 3)^2 + y^2 + z^2 = 4$, so the lower half is represented by

$$z = f(x, y) = -\sqrt{4 - (x - 3)^2 - y^2}$$

and

$$g(x, y, z) = (x - 3)^2 + y^2 + z^2 = 4.$$

Other answers are possible.

26. The level surfaces have equation $\cos(x+y+z) = c$. For each value of c between -1 and 1, the level surface is an infinite family of planes parallel to $x + y + z = \arccos(c)$. For example, the level surface $\cos(x + y + z) = 0$ is the family of planes

$$x + y + z = \frac{\pi}{2} \pm 2n\pi, \quad n = 0, 1, 2, \ldots.$$

27. A cylindrical surface.

28. A cone.

29. (a) The contours of g are parallel straight lines, and equally spaced function values correspond to equally spaced contours. These are the characteristics of the contour diagram of a linear function.

(b) The zero contour goes through the origin, so $g(0, 0) = 0$ is one value of the function.

The slope m in the x-direction, obtained from the function values at $(0, 0)$ and $(50, 0)$, is

$$m = \frac{g(50, 0) - g(0, 0)}{50 - 0} = \frac{10000 - 0}{50 - 0} = 200.$$

The slope n in the y-direction, obtained from the function values at $(0, 0)$ and $(0, 50)$, is

$$n = \frac{g(0, 50) - g(0, 0)}{50 - 0} = \frac{5000 - 0}{50 - 0} = 100.$$

We have the formula

$$\begin{aligned} g(x, y) &= z_0 + m(x - x_0) + n(y - y_0) \\ &= g(0, 0) + m(x - 0) + n(y - 0) = 200x + 100y. \end{aligned}$$

Problems

30. The cross-sections perpendicular to the t-axis are sine curves of the form $g(x, b) = (\cos b) \sin 2x$; these have period π. The cross-sections perpendicular to the x-axis are cosine curves of the form $g(a, t) = (\sin 2a) \cos t$; these have period 2π.

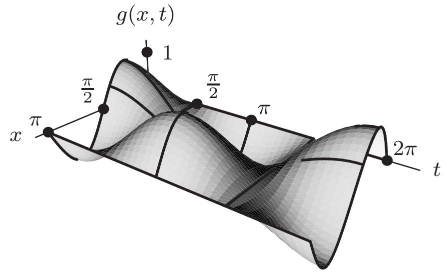


Figure 12.130: Graph $g(x,t) = \cos t \sin 2x$

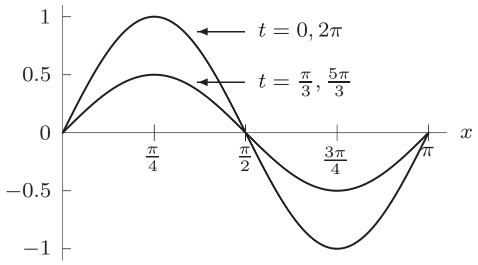


Figure 12.131: Cross-section $g(x,b) = (\cos b) \sin 2x$, with $b = 0, \pi/3, 5\pi/3, 2\pi$

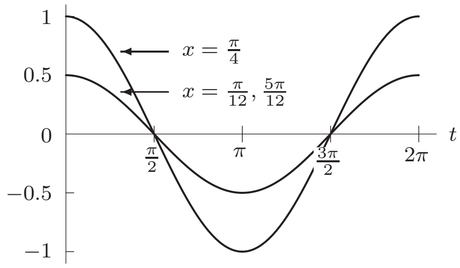


Figure 12.132: Cross-section $g(a,t) = (\sin 2a) \cos t$ with $a = \pi/12, \pi/4, 5\pi/12$

31. If

$$P_0 = f(L_0, K_0) = 1.01 L_0^{0.75} K_0^{0.25}$$

then replacing L_0 and K_0 by $2L_0$ and $2K_0$ gives

$$\begin{aligned} f(2L_0, 2K_0) &= 1.01(2L_0)^{0.75}(2K_0)^{0.25} \\ &= 2^{0.75} 2^{0.25} \cdot 1.01 L_0^{0.75} K_0^{0.25} \\ &= 2f(L_0, K_0) \\ &= 2P_0. \end{aligned}$$

So, doubling labor and capital doubles production.

32. (a) The level curve $f = 1$ is given by

$$\begin{aligned} \sqrt{x^2+y^2} + x &= 1 \\ \sqrt{x^2+y^2} &= 1 - x. \end{aligned}$$

Since $\sqrt{x^2+y^2} \geq 0$, we must have $x \leq 1$. Squaring gives

$$x^2 + y^2 = (1-x)^2 = 1 - 2x + x^2$$

So the level curve is given by

$$x = -\frac{1}{2}y^2 + \frac{1}{2}$$

with $x \leq 1$. Looking at the equation for the level curve, x always satisfies $x \leq 1$ since $x \leq \frac{1}{2}$. This means the level curve $f = 1$ is the parabola $x = -\frac{1}{2}y^2 + \frac{1}{2}$. See Figure 12.133.

Similarly, the level curve $f = 2$ has equation, valid for $x \leq 2$,

$$\begin{aligned} \sqrt{x^2+y^2} &= 2 - x \\ x^2 + y^2 &= 4 - 4x + x^2 \\ x &= -\frac{1}{4}y^2 + 1 \end{aligned}$$

The level curve $f = 3$ has equation, valid for $x \leq 3$,

$$\begin{aligned}\sqrt{x^2+y^2} &= 3 - x\\ x^2 + y^2 &= 9 - 6x + x^2\\ x &= -\frac{1}{6}y^2 + \frac{3}{2}.\end{aligned}$$

Both $f = 2$ and $f = 3$ are valid for all x and y satisfying the respective equations, so the level curves are parabolas. See Figure 12.133.

(b) The level curve $f = c$ has equation, valid for $x \leq c$,

$$\begin{aligned}\sqrt{x^2+y^2} &= c - x\\ x^2 + y^2 &= c^2 - 2cx + x^2\\ x &= -\frac{1}{2c}y^2 + \frac{c}{2}.\end{aligned}$$

If $c > 0$, then any x satisfying this equation satisfies $x \leq \frac{c}{2}$, so we have $x < c$. Thus, the level curve exists for $c > 0$. If $c < 0$, then any x satisfying the level curve equation also satisfies $x \geq \frac{c}{2}$, so $x > c$ (since c is negative). Thus, the level curves do not exist for $c < 0$. If $c = 0$, we get the level curve $y = 0$ with $x \leq 0$. Summarizing, we have that level curves exist only for $c \geq 0$.

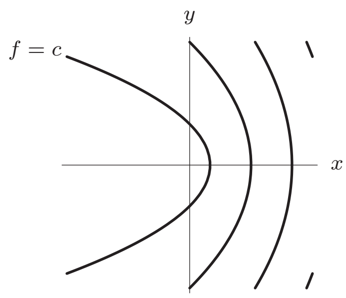


Figure 12.133

33. (a) You can see the sequence of values 1, 2, 3, 4, 5, 6, ... as you follow diagonal paths in the table upward to the right, changing to the next lower diagonal after reaching the top $x = 1$ row. The pattern continues in the same way, giving Table 12.18

Table 12.18

x \ y	1	2	3	4	5	6
1	1	3	6	10	15	21
2	2	5	9	14	20	27
3	4	8	13	19	26	34
4	7	12	18	25	33	42
5	11	17	24	32	41	51
6	16	23	31	40	50	61

(b) It appears that the value of f increases by 1 whenever x is decreased by 1 and y is increased by 1. To check this, compute

$$\begin{aligned}f(x-1, y+1) &= (1/2)((x-1)+(y+1)-2)((x-1)+(y+1)-1)+(y+1)\\ &= (1/2)(x+y-2)(x+y-1)+y+1\\ &= f(x,y)+1\end{aligned}$$

It appears that the value of f increases by 1 when moving from a point $(1, y)$ to the point $(y+1, 1)$. To check this, compute

$$\begin{aligned} f(y+1,1) &= (1/2)((y+1)+1-2)((y+1)+1-1)+1 \\ &= \frac{1}{2}y^2 + \frac{1}{2}y + 1 \\ &= (1/2)(1+y-2)(1+y-1)+y+1 \\ &= f(1,y)+1 \end{aligned}$$

34. Let us suppose that (x, y) approaches $(0, 0)$ along the line $y = x$. Then

$$f(x,y) = f(x,x) = \frac{x^3}{x^4+x^2} = \frac{x}{x^2+1}.$$

Therefore

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

On the other hand, if (x, y) approaches $(0, 0)$ along the parabola $y = x^2$ we have

$$f(x,y) = f(x,x^2) = \frac{x^4}{2x^4} = \frac{1}{2}$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} f(x,y) = \lim_{x\to 0} f(x,x^2) = \frac{1}{2}.$$

Thus no matter how close they are to the origin, there will be points (x, y) such that $f(x, y)$ is close to 0 and points (x, y) such that $f(x, y)$ is close to $\frac{1}{2}$. So the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

35. Points along the positive x-axis are of the form $(x, 0)$; at these points the function looks like $x/2x = 1/2$ everywhere (except at the origin, where it is undefined). On the other hand, along the y-axis, the function looks like $y^2/y = y$, which approaches 0 as we get closer to the origin. Since approaching the origin along two different paths yields numbers that are not the same, the limit does not exist.

36. We will study the continuity of f at $(a, 0)$. Now $f(a, 0) = 1 - a$. In addition:

$$\begin{aligned} \lim_{\substack{(x,y)\to(a,0)\\ y>0}} f(x,y) &= \lim_{x\to a}(1-x) = 1-a \\ \lim_{\substack{(x,y)\to(a,0)\\ y<0}} f(x,y) &= \lim_{x\to a} -2 = -2. \end{aligned}$$

If $a = 3$, then

$$\lim_{\substack{(x,y)\to(3,0)\\ y>0}} f(x,y) = 1-3 = -2 = \lim_{\substack{(x,y)\to(3,0)\\ y<0}} f(x,y)$$

and so $\lim_{(x,y)\to(3,0)} f(x,y) = -2 = f(3,0)$. Therefore f is continuous at $(3, 0)$.

On the other hand, if $a \neq 3$, then

$$\lim_{\substack{(x,y)\to(a,0)\\ y>0}} f(x,y) = 1-a \neq -2 = \lim_{\substack{(x,y)\to(a,0)\\ y<0}} f(x,y)$$

so $\lim_{(x,y)\to(a,0)} f(x,y)$ does not exist. Thus f is not continuous at $(a, 0)$ if $a \neq 3$.

Thus, f is not continuous along the line $y = 0$. (In fact the only point on this line where f is continuous is the point $(3, 0)$.)

37. (a) A student with SATs of 1050 and a GPA of 3.0 has a z-value given by

$$z = 0.003 \cdot 1050 + 0.8 \cdot 3.0 - 4 = 1.55$$

Since $1.55 < 2.3$, this student will not be admitted.

(b) A student with SATs of 1600 and GPA of y has a z-value given by

$$z = 0.003 \cdot 1600 + 0.8y - 4 = 0.8 + 0.8y = 0.8(y+1)$$

Since $0.8(y+1)$ may be greater than or less than 2.3, not all of the students with SAT scores of 1600 will be admitted.

(c) A student with GPA of 4.3 and SATs of x has a z-value given by

$$z = 0.003x + 0.8 \cdot 4.3 - 4 = 0.003x - 0.56$$

Since $0.003x - 0.56$ may be greater than or less than 2.3, not all of the students with a high school GPA of 4.3 will be admitted.

(d) See Figure 12.134.

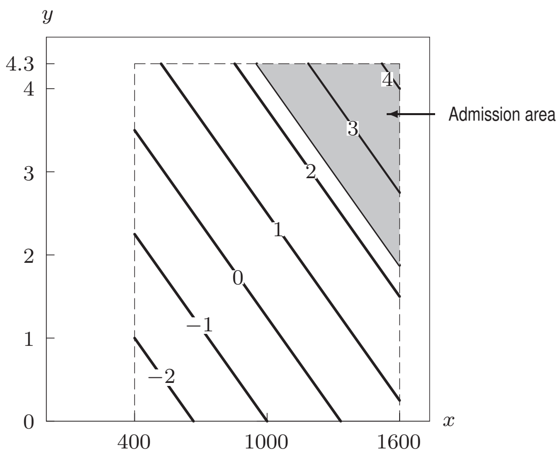


Figure 12.134

(e) If $\Delta x = 100$, then $\Delta z = 0.003\Delta x = 0.003 \cdot 100 = 0.3$.
If $\Delta y = 0.5$, then $\Delta z = 0.8 \cdot 0.5 = 0.4$.
An extra 0.5 of high school GPA increases a student's z-value by more than an extra 100 points on the SAT. Thus, the increase in GPA is more important.

38. (a) The plane $y = 1$ intersects the graph in the parabola $z = (x^2+1)\sin(1)+x = x^2\sin(1)+x+\sin(1)$. Since $\sin(1)$ is a constant, $z = x^2\sin(1) + x + \sin(1)$ is a quadratic function whose graph is a parabola.

Any plane of the form $y = a$ will do as long as a is not a multiple of π.

(b) The plane $y = \pi$ intersects the graph in the straight line $z = \pi^2 x$. (Since $\sin\pi = 0$, the equation becomes linear, $z = \pi^2 x$ if $y = \pi$.)

(c) The plane $x = 0$ intersects the graph in the curve $z = \sin y$.

39. (a) To find the level curves, we let T be a constant.

$$T = 100 - x^2 - y^2$$
$$x^2 + y^2 = 100 - T,$$

which is an equation for a circle of radius $\sqrt{100-T}$ centered at the origin. At $T = 100^\circ$, we have a circle of radius 0 (a point). At $T = 75^\circ$, we have a circle of radius 5. At $T = 50^\circ$, we have a circle of radius $5\sqrt{2}$. At $T = 25^\circ$, we have a circle of radius $5\sqrt{3}$. At $T = 0^\circ$, we have a circle of radius 10.

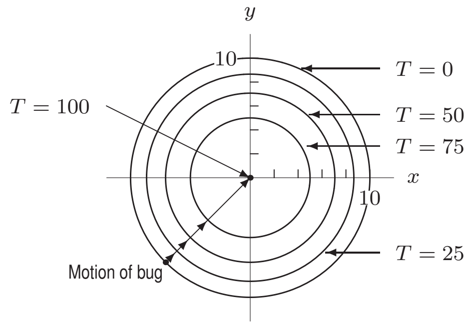


Figure 12.135

(b) No matter where we put the bug, it should go straight toward the origin—the hottest point on the xy-plane. Its direction of motion is perpendicular to the tangent lines of the level curves, as can be seen in Figure 12.135.

40. Let the equation of the plane be $z = ax + by + c$. When $z = 0$, the line on the xy-plane is $ax + by + c = 0$. Since we know that the plane intersects the xy-plane along the line $y = 2x + 2$ we have $b \neq 0$ and

$$-\frac{a}{b} = 2 \quad -\frac{c}{b} = 2$$

Since $(1, 2, 2)$ lies on the plane, we can use the equation $z = ax + by + c$ to get

$$2 = a + 2b + c$$

Solving the equations gives

$$a = 2,$$
$$b = -1,$$
$$c = 2.$$

Hence $z = 2x - y + 2$ and the linear function is $f(x, y) = 2x - y + 2$.

41. (a) Since $z = c$, where $-1 \leq c \leq 1$ is a constant, gives $\sqrt{x^2 + y^2} = \pm\cos^{-1}(c) + 2k\pi$, where k is any integer such that $\pm\cos^{-1}(c) + 2k\pi$ is non-negative, or $x^2 + y^2 = r^2$, where $r = \pm\cos^{-1}(c) + 2k\pi$, which represents a family of circles of radius r centered at $(0, 0)$, the level curves of the function are families of circles, as shown in Figure 12.136.

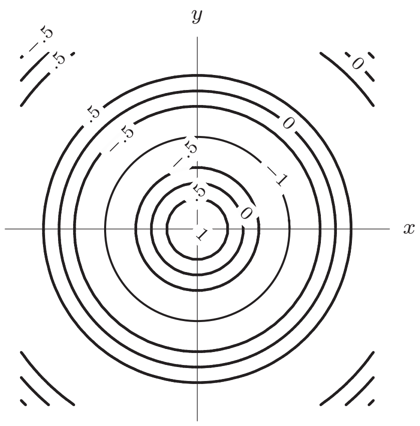


Figure 12.136

(b) The plane containing the x- and z-axes is the plane $y = 0$. Thus the cross-section is $z = \cos\sqrt{x^2 + 0^2} = \cos(|x|) = \cos x$, as shown in Figure 12.137.

(c) Denote the line $y = x$ in the xy-plane as r-axis and put units on it such that the units on the r-axis coincide with the units on the x-axis and y-axis, namely, $r^2 = x^2 + y^2$. Thus, the cross-section is $z = \cos\sqrt{r^2} = \cos(|r|) = \cos r$, as shown in Figure 12.138.

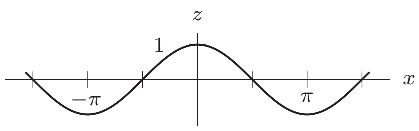


Figure 12.137

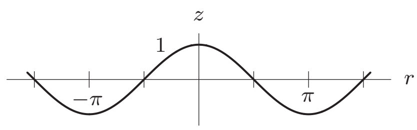


Figure 12.138

42. The function $y = f(x, 0) = \cos 0 \sin x = \sin x$ gives the displacement of each point of the string when time is held fixed at $t = 0$. The function $f(x, 1) = \cos 1 \sin x = 0.54 \sin x$ gives the displacement of each point of the string at time $t = 1$. Graphing $f(x, 0)$ and $f(x, 1)$ gives in each case an arch of the sine curve, the first with amplitude 1 and the second with amplitude 0.54. For each different fixed value of t, we get a different snapshot of the string, each one a sine curve with amplitude given by the value of $\cos t$. The result looks like the sequence of snapshots shown in Figure 12.139.

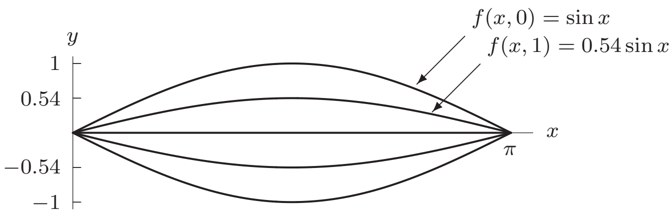


Figure 12.139

43. The function $f(0, t) = \cos t \sin 0 = 0$ gives the displacement of the left end of the string as time varies. Since that point remains stationary, the displacement is zero. The function $f(1, t) = \cos t \sin 1 = 0.84 \cos t$ gives the displacement of the point at $x = 1$ as time varies. Since $\cos t$ oscillates back and forth between 1 and -1, this point moves back and forth with maximum displacement of 0.84 in either direction. Notice the maximum displacements are greatest at $x = \pi/2$ where $\sin x = 1$.

44. (a) For $t = 0$, we have $y = f(x, 0) = \sin x, \quad 0 \le x \le \pi$, as in Figure 12.140.

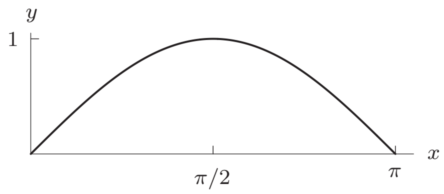


Figure 12.140

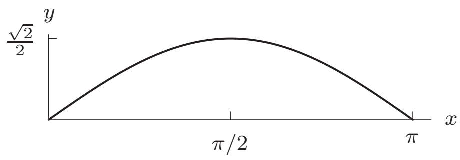


Figure 12.141

For $t = \pi/4$, we have $y = f(x, \pi/4) = \frac{\sqrt{2}}{2} \sin x, \quad 0 \le x \le \pi$, as in Figure 12.141.
For $t = \pi/2$, we have $y = f(x, \pi/2) = 0$, as in Figure 12.142.

Figure 12.142

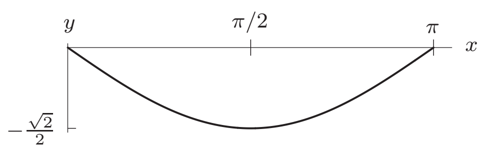


Figure 12.143

For $t = 3\pi/4$, we have $y = f(x, 3\pi/4) = \frac{-\sqrt{2}}{2}\sin x, \quad 0 \le x \le \pi$, as in Figure 12.143.
For $t = \pi$, we have $y = f(x, \pi) = -\sin x, \quad 0 \le x \le \pi$, as in Figure 12.144.

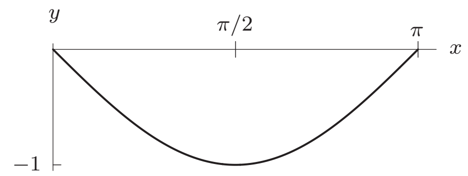


Figure 12.144

(b) The graphs show an arch of a sine wave which is above the x-axis, concave down at $t = 0$, is straight along the x-axis at $t = \pi/2$, and below the x-axis, concave up at $t = \pi$, like a guitar string vibrating up and down.

45. (a) For $g(x,t) = \cos 2t \sin x$, our snapshots for fixed values of t are still one arch of the sine curve. The amplitudes, which are governed by the $\cos 2t$ factor, now change twice as fast as before. That is, the string is vibrating twice as fast.

(b) For $y = h(x,t) = \cos t \sin 2x$, the vibration of the string is more complicated. If we hold t fixed at any value, the snapshot now shows one full period, i.e. one crest and one trough, of the sine curve. The magnitude of the sine curve is time dependent, given by $\cos t$. Now the center of the string, $x = \pi/2$, remains stationary just like the end points. This is a vibrating string with the center held fixed, as shown in Figure 12.145.

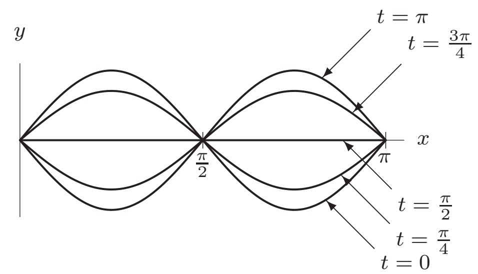


Figure 12.145: Another vibrating string: $y = h(x,t) = \cos t \sin 2x$

CAS Challenge Problems

46. (a) Let $C = (x, y, 0)$. Since distance $AC = 2$ we have $x^2 + y^2 = 2^2$, and since distance $BC = 2$ we have $(x-2)^2 + y^2 = 2^2$. Solving these two equations, we have $C = (1, \sqrt{3}, 0)$ or $C = (1, -\sqrt{3}, 0)$. We will pick the first choice (the second choice gives different answers in the next part).

(b) Let $D = (x, y, z)$. Distance $DA = 2$ implies that $x^2 + y^2 + z^2 = 4$. Distance $DB = 2$ implies that $(x-2)^2 + y^2 + z^2 = 4$. Distance $DC = 2$ implies that $(x-1)^2 + (y - \sqrt{3})^2 + z^2 = 4$. Solving these three equations, we have: $x = 1$, $y = 1/\sqrt{3}$, $z = 2\sqrt{2}/\sqrt{3}$ or $x = 1$, $y = 1/\sqrt{3}$, $z = -2\sqrt{2}/\sqrt{3}$. Picking the first choice we have $D = (1, 1/\sqrt{3}, 2\sqrt{2}/\sqrt{3})$.

(c) The figure is a tetrahedron, that is, a polyhedron with four faces, each of which is an equilateral triangle: ABC, ABD, ACD, BCD.

47. (a)

$$\begin{aligned}
f(x, f(x,y)) &= 3 + x + 2(3 + x + 2y) = (3 + 2\cdot 3) + (1+2)x + 2^2 y = 9 + 3x + 4y \\
f(x, f(x, f(x,y))) &= 3 + x + 2(3 + x + 2(3 + x + 2y)) \\
&= (3 + 2\cdot 3 + 2^2\cdot 3) + (1 + 2 + 2^2)x + 2^3 y = 21 + 7x + 8y
\end{aligned}$$

(b) From part (a) we guess that the general pattern for k nested fs is

$$(3 + 2\cdot 3 + 2^2\cdot 3 + \cdots + 2^{k-1}\cdot 3) + (1 + 2 + 2^2 + \cdots + 2^{k-1})x + 2^k y$$

Thus

$$\begin{aligned}
&f(x, f(x, f(x, f(x, f(x, f(x,y)))))) = \\
&(3 + 2\cdot 3 + 2^2\cdot 3 + \cdots + 2^5\cdot 3) + (1 + 2 + 2^2 + \cdots + 2^5)x + 2^6 y = 189 + 63x + 64y.
\end{aligned}$$

48. (a) Since $f(1,1,1) = 16$, $f(1,1,2) = 21$, an increase of 1 in z increases the value of f by 5. Thus we estimate $f(1,1,3) \approx 21+5 = 26$. Similarly, since $f(1,0,1) = 20$, $f(1,1,1) = 16$, an increase of 1 in y decreases the value of f by 4. So we estimate $f(1,2,1) \approx 16-4 = 12$.

(b) When x and y are fixed at 1, f is a linear function of z, thus the linear approximation will give a precise answer for $f(1,1,3)$. However, when x and z are fixed at 1, f is the sum of an exponential function of y and a linear function, thus the linear approximation will not be accurate for $f(1,2,1)$.

(c) Since $f(x,y,z) = ax^2+byz+czx^3+d2^{x-y}$ and $f(1,0,1) = 20$, $f(1,1,1) = 16$, $f(1,1,2) = 21$, $f(0,0,1) = 6$, we have

$$\begin{aligned} a+c+2d &= 20 \\ a+b+c+d &= 16 \\ a+2b+2c+d &= 21 \\ d &= 6 \end{aligned}$$

Solving for a, b, c, d, we get $f(x,y,z) = 5x^2+2yz+3zx^3+6\cdot 2^{x-y}$.

(d) $f(1,1,3) = 26$, which matches the estimate in part (a). $f(1,2,1) = 15$, which does not agree with the estimate in part (a).

PROJECTS FOR CHAPTER TWELVE

1. (a) The Leq is greatest near the runways. The largest contour marked is 72 dB; Heathrow's two main runways are located within this contour and run east-west. The noise level on the runways exceeds 72 dB. For comparison, the noise level 50 feet from the edge of a freeway in mid-morning is about 76 dB.

(b) Since the prevailing wind is from the west, the planes take off towards the west and land coming in from the east, which explains why the contours are aligned east-west. Many of the planes taking off want eventually to go east, so they turn off to the southwest to start a U-turn back to the east. A limited number, particularly those heading for transatlantic flights, turn right; fewer still head due west on a straight-out departure. When planes are approaching or departing, they have to use considerable power at low altitude, and hence are significantly noisier; this noise is concentrated at the end of the runways.

(c) The noise level falls off rapidly to the north and south of the runways. This is reflected in the fact that the contours are very close together along the length of both runways.

(d) Suppose the decibel measure of the sound, B_1, on one contour is given by

$$B_1 = 10\log_{10}\left(\frac{L_1}{L_0}\right)$$

and the decibel measure of the sound on the next higher contour, B_2, is

$$B_2 = 10\log_{10}\left(\frac{L_2}{L_0}\right).$$

Since contours are labeled at 3 dB intervals,

$$3 = B_2 - B_1 = 10\left(\log_{10}\left(\frac{L_2}{L_0}\right) - \log_{10}\left(\frac{L_1}{L_0}\right)\right) = 10\log_{10}\left(\frac{L_2/L_0}{L_1/L_0}\right) = 10\log_{10}\left(\frac{L_2}{L_1}\right).$$

Solving for L_2/L_1 gives

$$\frac{L_2}{L_1} = 10^{3/10} \approx 2.$$

Thus, moving from one contour to the next at 3 dB higher corresponds to approximately doubling the sound intensity.

(e) We have shown that an increase of 3 dB corresponds approximately to doubling the sound intensity, so a decrease of 3 dB corresponds approximately to halving the sound intensity. We are told that the new jets will make 50% less noise, so sound intensity will be halved. Thus, we will subtract 3 dB from each contour value; for example, the present 57 dB contour will be labeled as 54 dB.

2. **(a)** About 15 feet along the wall, because that's where there are regions of cold air (55°F and 65°F).

(b) Roughly between 10 am and 12 noon, and between 4 pm and 6 pm.

(c) Roughly between midnight and 2 am, between 10 am and 1 pm, and between 4 pm and 9 pm, since that is when the temperature near the heater is greater than 80°F.

(d)

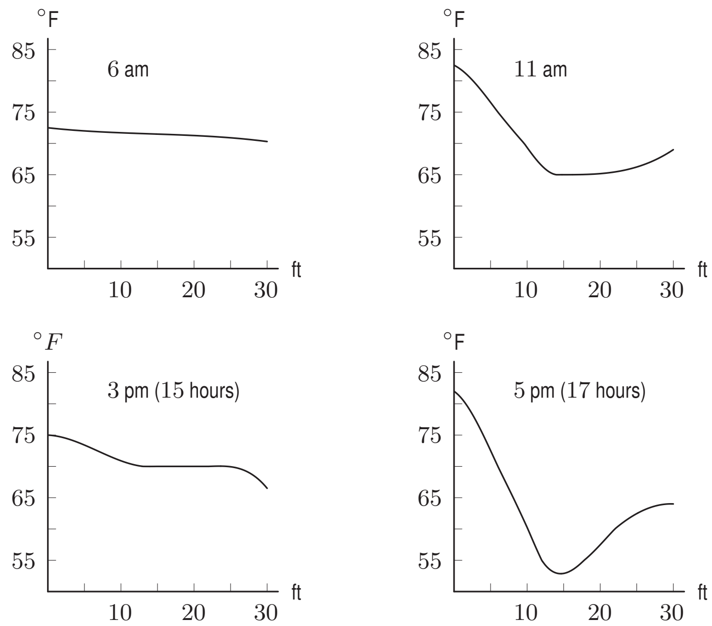


Figure 12.146

(e)

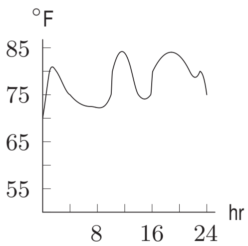


Figure 12.147: Temp. vs. Time at heater

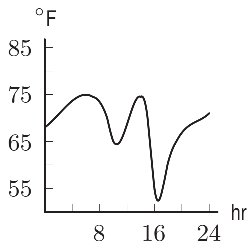


Figure 12.148: Temp. vs. Time at window

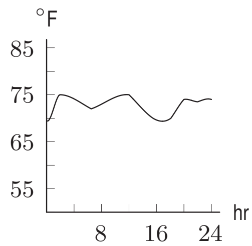


Figure 12.149: Temp. vs. Time midway between heater and window

(f) The temperature at the window is colder at 5 pm than at 11 am because the outside temperature is colder at 5 pm than at 11 am.

(g) The thermostat is set to roughly 70°F. We know this because the temperature in the room stays close to 70°F until we get close (a couple of feet) to the window.

(h) We are told that the thermostat is about 2 feet from the window. Thus, the thermostat is either about 13 feet or about 17 feet from the wall. If the thermostat is set to 70°F, every time the temperature at the thermostat goes over or under 70°F, the heater turns off or on. Look at the point at which the vertical lines at 13 feet or about 17 feet cross the 70°F contours. We need to decide which of these crossings correspond best with the times that the heater turns on and off. (These times can be seen along the wall.) Notice that the 17 foot line does not cross the 70°F contour after 16 hours (4 pm). Thus, if the thermostat were 17 feet from the wall, the heater would not turn off after 4 pm. However, the heater does turn off at about 21 hours (9 pm). Since this is the time that the 13 foot line crosses the 70°F contour, we estimate that the thermostat is about 13 feet away from the wall.

3. (a) Let x = distance (microns) from center of waveguide, t = time (nanoseconds) as shown in the problem, and I = intensity of light as marked on the given level curves.

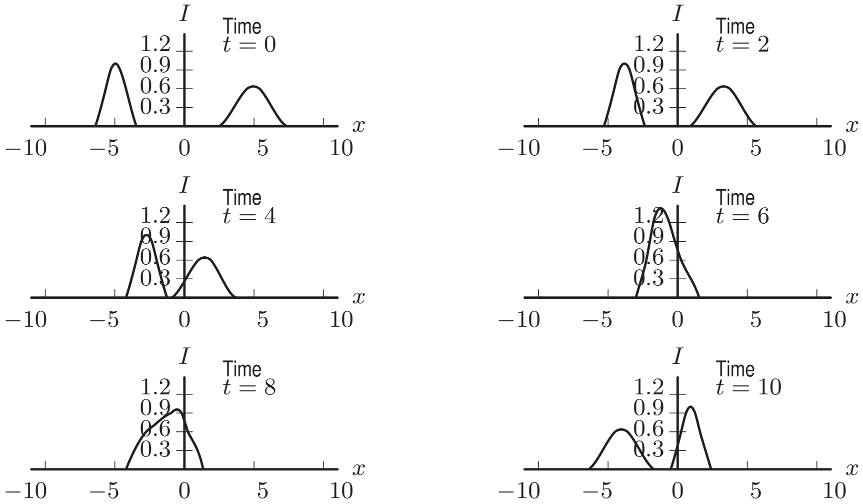


Figure 12.150

(b) Two waves would start out at opposite ends of the screen. The wave on the left would be slightly taller and narrower than the wave on the right. The waves would move toward one another, the wave on the right moving a little faster. They would meet to the left of the center and appear to merge, becoming taller. They would then proceed in the directions they were initially going, ultimately leaving the screen on the side opposite to where they began.

(c) Let x = distance (microns), t = time (nanoseconds), and I = intensity.

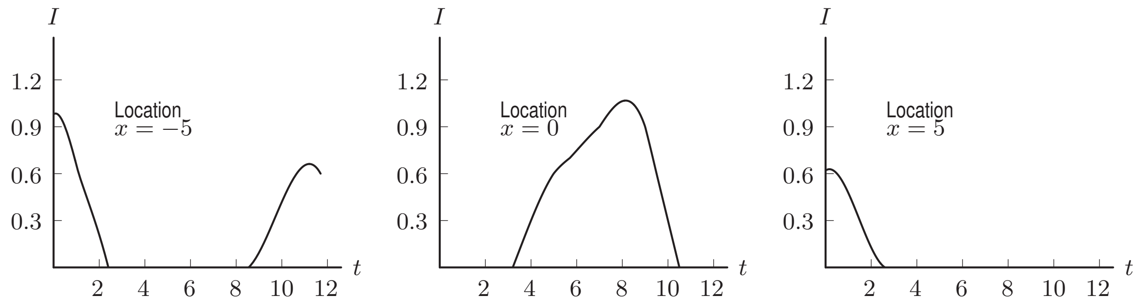


Figure 12.151

(d) Two pulses of light are traveling down a wave-guide toward one another. They meet in the center and, as they pass through one another, appear brighter. They then continue along in the wave-guide in the directions they were going.

CHAPTER THIRTEEN

Solutions for Section 13.1

Exercises

1. The vectors are $\vec{a} = \vec{i} + 3\vec{j}$, $\vec{b} = 3\vec{i} + 2\vec{j}$, $\vec{v} = -2\vec{i} - 2\vec{j}$, and $\vec{w} = -\vec{i} + 2\vec{j}$.

2. $\vec{a} = 2\vec{i} + \vec{j}$,
$\vec{b} = 2\vec{i}$,
$\vec{c} = -2\vec{i}$,
$\vec{d} = -2\vec{i} + 2\vec{j}$,
$\vec{e} = -2\vec{i} - \vec{j}$

3. The vector we want is the displacement from Q to P, which is given by

$$\vec{QP} = (1-4)\vec{i} + (2-6)\vec{j} = -3\vec{i} - 4\vec{j}$$

4. The vector we want is the displacement from P to Q, which is given by

$$\vec{PQ} = (4-1)\vec{i} + (6-2)\vec{j} = 3\vec{i} + 4\vec{j}$$

5. $\vec{a} = \vec{b} = \vec{c} = 3\vec{k}, \quad \vec{d} = 2\vec{i} + 3\vec{k}, \quad \vec{e} = \vec{j}, \quad \vec{f} = -2\vec{i}$

6. $\vec{u} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{v} = -\vec{i} + 2\vec{k}$.

7. $4\vec{i} + 2\vec{j} - 3\vec{i} + \vec{j} = \vec{i} + 3\vec{j}$

8. $\vec{i} + 2\vec{j} - 6\vec{i} - 3\vec{j} = -5\vec{i} - \vec{j}$

9. $-4\vec{i} + 8\vec{j} - 0.5\vec{i} + 0.5\vec{k} = -4.5\vec{i} + 8\vec{j} + 0.5\vec{k}$

10. $(0.9\vec{i} - 1.8\vec{j} - 0.02\vec{k}) - (0.6\vec{i} - 0.05\vec{k}) = 0.3\vec{i} - 1.8\vec{j} + 0.03\vec{k}$

11. $3\vec{i} - 4\vec{j} + 2\vec{k} - 6\vec{i} - 8\vec{j} + \vec{k} = -3\vec{i} - 12\vec{j} + 3\vec{k}$

12. $4\vec{i} - 3\vec{j} + 7\vec{k} - 10\vec{i} - 2\vec{j} + 4\vec{k} = -6\vec{i} - 5\vec{j} + 11\vec{k}$

13. $0.6\vec{i} + 0.2\vec{j} - \vec{k} + 0.3\vec{i} + 0.3\vec{k} = 0.9\vec{i} + 0.2\vec{j} - 0.7\vec{k}$

14. $\vec{i} - \frac{1}{2}\vec{j} + \frac{3}{2}\vec{k} + 3\vec{i} - \frac{1}{2}\vec{j} + \frac{3}{2}\vec{k} = 4\vec{i} - \vec{j} + 3\vec{k}$

15. $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$.

16. The length is given by

$$\|\vec{z}\| = \sqrt{(1)^2 + (-3)^2 + (-1)^2} = \sqrt{1+9+1} = \sqrt{11}.$$

17. $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$.

18. $\|\vec{v}\| = \sqrt{7.2^2 + (-1.5)^2 + 2.1^2} = \sqrt{58.5} \approx 7.6$.

19. $\|\vec{v}\| = \sqrt{1.2^2 + (-3.6)^2 + 4.1^2} = \sqrt{31.21} \approx 5.6$.

20. $4\vec{z} = 4(\vec{i} - 3\vec{j} - \vec{k}) = 4\vec{i} - 12\vec{j} - 4\vec{k}$.

21.

$$\begin{aligned}
5\vec{a} + 2\vec{b} &= 5(2\vec{j} + \vec{k}) + 2(-3\vec{i} + 5\vec{j} + 4\vec{k}) \\
&= (5(2)\vec{j} + 5(1)\vec{k}) + (2(-3)\vec{i} + 2(5)\vec{j} + 2(4)\vec{k}) \\
&= (10\vec{j} + 5\vec{k}) + (-6\vec{i} + 10\vec{j} + 8\vec{k}) = (0-6)\vec{i} + (10+10)\vec{j} + (5+8)\vec{k} \\
&= -6\vec{i} + 20\vec{j} + 13\vec{k}.
\end{aligned}$$

22. $\vec{a}+\vec{z}=(2\vec{j}+\vec{k})+(\vec{i}-3\vec{j}-\vec{k})=(0+1)\vec{i}+(2-3)\vec{j}+(1-1)\vec{k}=\vec{i}-\vec{j}$

23. $2\vec{c}+\vec{x}=2(\vec{i}+6\vec{j})+(-2\vec{i}+9\vec{j})=(2\vec{i}+12\vec{j})+(-2\vec{i}+9\vec{j})=(2-2)\vec{i}+(12+9)\vec{j}=21\vec{j}$.

24.

$$\begin{aligned}2\vec{a}+7\vec{b}-5\vec{z}&=2(2\vec{j}+\vec{k})+7(-3\vec{i}+5\vec{j}+4\vec{k})-5(\vec{i}-3\vec{j}-\vec{k})\\&=(4\vec{j}+2\vec{k})+(-21\vec{i}+35\vec{j}+28\vec{k})-(5\vec{i}-15\vec{j}-5\vec{k})\\&=(-21-5)\vec{i}+(4+35+15)\vec{j}+(2+28+5)\vec{k}=-26\vec{i}+54\vec{j}+35\vec{k}.\end{aligned}$$

25.

$$\begin{aligned}\|\vec{y}-\vec{x}\|&=\|(4\vec{i}-7\vec{j})-(-2\vec{i}+9\vec{j})\|=\|(4-(-2))\vec{i}+(-7-9)\vec{j}\|=\|6\vec{i}-16\vec{j}\|\\&=\sqrt{6^2+(-16)^2}=\sqrt{36+256}=\sqrt{292}=2\sqrt{73}.\end{aligned}$$

26. (a) See Figure 13.1.
(b) $\|\vec{v}\|=\sqrt{5^2+7^2}=\sqrt{74}=8.602$.
(c) We see in Figure 13.2 that $\tan\theta=\frac{7}{5}$ and so $\theta=54.46°$.

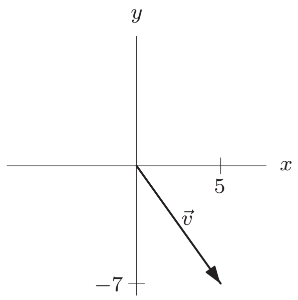


Figure 13.1

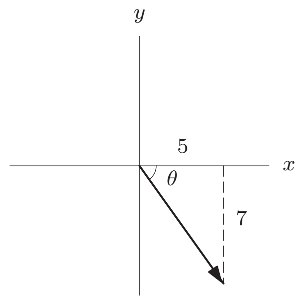


Figure 13.2

27. Since $\sqrt{(0.06)^2+(0.08)^2}=0.1$, the vector is

$$\frac{1}{0.1}(0.06\vec{i}-0.08\vec{k})=0.6\vec{i}-0.8\vec{k}.$$

28. To find a vector in the opposite direction to $\vec{v}=\vec{i}-\vec{j}+\vec{k}$, we can take the scalar multiple $(-1)\vec{v}=-\vec{i}+\vec{j}-\vec{k}$. The magnitude of the vector $(-1)\vec{v}$ is $\sqrt{(-1)^2+1^2+(-1)^2}=\sqrt{3}$. So

$$(-\vec{i}+\vec{j}-\vec{k})/\sqrt{3}$$

is the unit vector in the opposite direction to $\vec{i}-\vec{j}+\vec{k}$.

29. Since $||\vec{v}||^2=\sqrt{2^2+(-1)^2+(-\sqrt{11})^2}=16$, we have $||\vec{v}||=4$. Thus, the vector we want is

$$-\frac{\vec{v}}{||\vec{v}||}=-\frac{1}{4}(2\vec{i}-\vec{j}-\sqrt{11}\vec{k})=-\frac{1}{2}\vec{i}+\frac{1}{4}\vec{j}+\frac{\sqrt{11}}{4}\vec{k}.$$

30. The length of the vector $\vec{i}-\vec{j}+2\vec{k}$ is $\sqrt{1^2+(-1)^2+2^2}=\sqrt{6}$. We can scale the vector down to length 2 by multiplying it by $\frac{2}{\sqrt{6}}$. So the answer is $\frac{2}{\sqrt{6}}\vec{i}-\frac{2}{\sqrt{6}}\vec{j}+\frac{4}{\sqrt{6}}\vec{k}$.

Problems

31. If two vectors are parallel, they are scalar multiples of one another. Thus

$$\frac{a^2}{5a}=\frac{6}{-3}.$$

Solving for a gives

$$a^2=-2\cdot 5a \quad \text{so} \quad a=0,-10.$$

32. (a) The displacement from P to Q is given by

$$\overrightarrow{PQ} = (4\vec{i} + 6\vec{j}) - (\vec{i} + 2\vec{j}) = 3\vec{i} + 4\vec{j}.$$

Since

$$\|\overrightarrow{PQ}\| = \sqrt{3^2 + 4^2} = 5,$$

a unit vector $\vec{u}$ in the direction of $\overrightarrow{PQ}$ is given by

$$\vec{u} = \frac{1}{5}\overrightarrow{PQ} = \frac{1}{5}(3\vec{i} + 4\vec{j}) = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

(b) A vector of length 10 pointing in the same direction is given by

$$10\vec{u} = 10(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}) = 6\vec{i} + 8\vec{j}.$$

33. The vector $\vec{v} = -\vec{i} + \vec{j}$ points northwest. Since $\|\vec{v}\| = \sqrt{2}$, the unit vector pointing northwest is $\vec{u} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.

34. (a) The components are $v_1 = 2\cos\pi/4 = \sqrt{2}, v_2 = 2\sin\pi/4 = \sqrt{2}$. See Figure 13.3. Thus $\vec{v} = \sqrt{2}\vec{i} + \sqrt{2}\vec{j}$.

(b) Since the vector lies in the xz-plane, its y-component is 0. Its x-component is $1\cos(\frac{\pi}{6})\vec{i} = \frac{\sqrt{3}}{2}\vec{i}$ and its z-component is $1\sin(\frac{\pi}{6})\vec{j} = \frac{1}{2}\vec{j}$. See Figure 13.4. So the vector is $\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k}$.

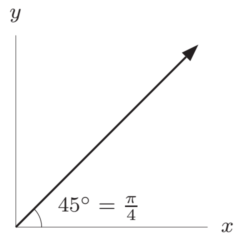


Figure 13.3

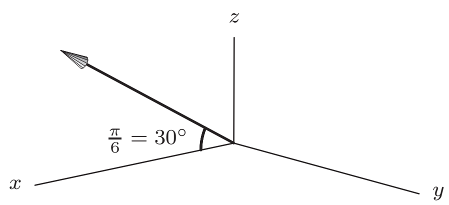


Figure 13.4

35. The coordinates of the points are:

$$A = (0, 0), \quad B = (2, 2), \quad C = (7, 0), \quad D = (3, 4), \quad E = (4, 2).$$

(a) We find $\overrightarrow{AB} = 2\vec{i} + 2\vec{j}$, $\overrightarrow{CD} = -4\vec{i} + 4\vec{j}$. Therefore,

$$\vec{u} = (2.5)(2\vec{i} + 2\vec{j}) + (-0.8)(-4\vec{i} + 4\vec{j}) = 5\vec{i} + 5\vec{j} + 3.2\vec{i} - 3.2\vec{j} = 8.2\vec{i} + 1.8\vec{j},$$
$$\vec{v} = (2.5)(-2\vec{i} - 2\vec{j}) - (-0.8)(-4\vec{i} + 4\vec{j}) = -5\vec{i} - 5\vec{j} - 3.2\vec{i} + 3.2\vec{j} = -8.2\vec{i} - 1.8\vec{j}.$$

(b) We see that $\vec{v} = -\vec{u}$. We know that $-\overrightarrow{AB}$ is equivalent to $\overrightarrow{BA}$. In other words,

$$\vec{v} = -(2.5)\overrightarrow{AB} + (0.8)\overrightarrow{CD}.$$

By factoring out a -1, we get

$$\vec{v} = -((2.5)\overrightarrow{AB} + (-0.8)\overrightarrow{CD}) = -\vec{u}.$$

36. We find $\overrightarrow{EA}$ to be $-4\vec{i} - 2\vec{j}$. A unit vector on the $\overrightarrow{EA}$ direction is

$$\vec{n} = \frac{-4i - 2j}{\sqrt{4^2 + 2^2}} = \frac{-2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}.$$

So a vector of length 2 in this direction is

$$\vec{p} = 2\vec{n} = \frac{-4}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}.$$

Thus, $\vec{p} = -\frac{4\sqrt{5}}{5}\vec{i} - \frac{2\sqrt{5}}{5}\vec{j}$.

37. (a) True, by the property of commutativity.
(b) It does not make sense, we cannot add vectors and scalars.
(c) True, by the property of commutativity.
(d) This is not always true. For example, let $\vec{a} = \vec{i} + 2\vec{j}$, $\vec{b} = -2\vec{i} + \vec{j}$. Then

$$\begin{aligned} \|\vec{a} + \vec{b}\| &= \|\vec{i} + 2\vec{j} - 2\vec{i} + \vec{j}\| = \|-\vec{i} + 3\vec{j}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10} \\ \|\vec{a}\| &= \sqrt{1^2 + 2^2} = \sqrt{5} \\ \|\vec{b}\| &= \sqrt{(-2)^2 + 1^2} = \sqrt{5}. \end{aligned}$$

So, $\|\vec{a}\| + \|\vec{b}\| = \sqrt{5} + \sqrt{5} = 2\sqrt{5}$. But $\sqrt{10} \neq 2\sqrt{5}$ and $\|\vec{a} + \vec{b}\| \neq \|\vec{a}\| + \|\vec{b}\|$.

38.

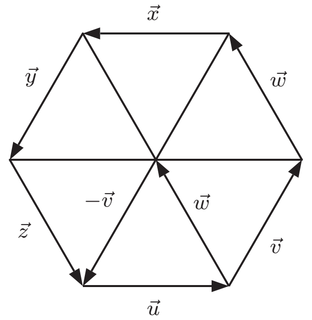


Figure 13.5

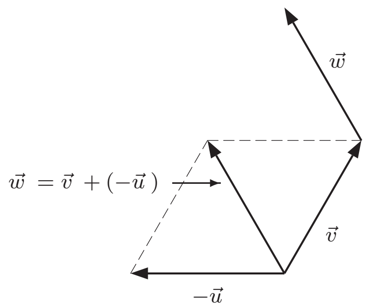


Figure 13.6

Break the hexagon up into 6 equilateral triangles, as shown in Figure 13.5.
Then $\vec{u} - \vec{v} + \vec{w} = \vec{0}$, so $\vec{w} = \vec{v} - \vec{u}$
Similarly, $\vec{x} = -\vec{u}$, $\vec{y} = -\vec{v}$, $\vec{z} = -\vec{w} = \vec{u} - \vec{v}$.

39. (a) We need $6\vec{i} + 8\vec{j} + 3\vec{k} = \lambda(2\vec{i} + (t^2 + \frac{2}{3}t + 1)\vec{j} + t\vec{k})$ for some λ. This gives

$$\begin{aligned} 6 &= 2\lambda \\ 8 &= (t^2 + \frac{2}{3}t + 1)\lambda \\ 3 &= t\lambda \end{aligned}$$

From the first equation, we have $\lambda = 3$. Substituting $\lambda = 3$ into the third equation gives $t = 1$. Check the second equation, it says $8 = 8$, if $t = 1$ and $\lambda = 3$. So for $t = 1$, the two vectors are parallel to each other.

(b) Similar to part (a), we need to solve

$$\begin{aligned} 2 &= t\lambda \\ -4 &= \lambda \\ 1 &= \lambda(t-1) \end{aligned}$$

From the first two equations we have $\lambda = -4$ and $t = -\frac{1}{2}$. Substituting this into the third equation gives $1 = 6$. Thus this system of equations has no solution, so the pair of vectors is not parallel to each other for any value of t.

(c) $2t\vec{i} + t\vec{j} + t\vec{k} = \frac{t}{3}(6\vec{i} + 3\vec{j} + 3\vec{k})$. For any t, the two vectors are parallel to each other.

40. Since the component of $\vec{v}$ in the $\vec{i}$-direction is 3, we have $\vec{v} = 3\vec{i} + b\vec{j}$ for some b. Since $\|\vec{v}\| = 5$, we have $\sqrt{3^2 + b^2} = 5$, so $b = 4$ or $b = -4$. There are two vectors satisfying the properties given: $\vec{v} = 3\vec{i} + 4\vec{j}$ and $\vec{v} = 3\vec{i} - 4\vec{j}$.

41. Let $\vec{v} = x\vec{i} + y\vec{j}$.
We want $\|\vec{v}\| = 1$ and $\|\vec{v} + \vec{i}\| = 1$, i.e.,

$$\sqrt{x^2 + y^2} = 1 \quad \text{and} \quad \sqrt{(x+1)^2 + y^2} = 1.$$

Setting these equations equal and solving for x gives:

$$\begin{aligned}\sqrt{x^2+y^2} &= \sqrt{(x+1)^2+y^2}\\ x^2+y^2 &= (x+1)^2+y^2 \quad \text{(after squaring both sides)}\\ x^2+y^2 &= x^2+2x+1+y^2\\ 0 &= 2x+1\\ x &= -\frac{1}{2}.\end{aligned}$$

Since we know $x=-\frac{1}{2}$, we can use the fact that $\|\vec v\|=1$ to solve for y:

$$\begin{aligned}\|\vec v\| &= 1\\ \sqrt{x^2+y^2} &= 1\\ x^2+y^2 &= 1\\ y^2 &= 1-x^2\\ y &= \pm\sqrt{1-x^2}\\ &= \pm\sqrt{1-\left(\frac{1}{2}\right)^2}\\ &= \pm\frac{\sqrt{3}}{2}.\end{aligned}$$

Thus the vectors we are looking for are:

$$\vec v = -\frac{1}{2}\vec i + \frac{\sqrt{3}}{2} \quad \text{and} \quad \vec v = -\frac{1}{2}\vec i - \frac{\sqrt{3}}{2}\vec j\,.$$

42. We must check that all the points are the same distance apart, i.e., the magnitude of the displacement vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{BA}$, $\overrightarrow{CB}$ and $\overrightarrow{CA}$ is the same. Here goes:

$$\begin{aligned}\|\overrightarrow{OA}\| &= \|(2\vec i+0\vec j+0\vec k)-(0\vec i+0\vec j+0\vec k)\| = \sqrt{2^2+0^2+0^2}=2\\ \|\overrightarrow{OB}\| &= \|(1\vec i+\sqrt{3}\vec j+0\vec k)-(0\vec i+0\vec j+0\vec k)\| = \sqrt{1^2+(\sqrt{3})^2+0^2}=2\\ \|\overrightarrow{OC}\| &= \|(1\vec i+1/\sqrt{3}\vec j+2\sqrt{2/3}\vec k)-(0\vec i+0\vec j+0\vec k)\| = \sqrt{1+1/3+4(2/3)}=2\\ \|\overrightarrow{BA}\| &= \|(2\vec i+0\vec j+0\vec k)-(1\vec i+\sqrt{3}\vec j+0\vec k)\| = \sqrt{1+3+0}=2\\ \|\overrightarrow{CB}\| &= \|(1\vec i+\sqrt{3}\vec j+0\vec k)-(1\vec i+1/\sqrt{3}\vec j+2\sqrt{2/3}\vec k)\|\\ &= \sqrt{0^2+(\sqrt{3}-1/\sqrt{3})^2+4(2/3)} = \sqrt{3-2+1/3+8/3}=2\\ \|\overrightarrow{CA}\| &= \|(2\vec i+0\vec j+0\vec k)-(1\vec i+1/\sqrt{3}\vec j+2\sqrt{2/3}\vec k)\| = \sqrt{1+1/3+4(2/3)}=2.\end{aligned}$$

43. In Figure 13.7 let O be the origin, points A, B, and C be the vertices of the triangle, point D be the midpoint of $\overline{BC}$, and Q be the point in the line segment $\overline{DA}$ that is $\frac{1}{3}|DA|$ away from D.

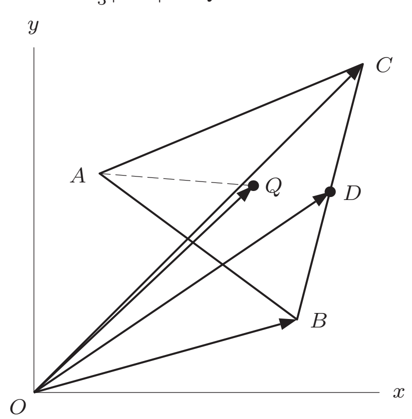


Figure 13.7

From Figure 13.7 we see that

$$\begin{aligned}\overrightarrow{OQ} = \overrightarrow{OD} + \overrightarrow{DQ} = \overrightarrow{OD} + \frac{1}{3}\overrightarrow{DA} \\ = \overrightarrow{OD} + \frac{1}{3}(\overrightarrow{OA} - \overrightarrow{OD}) \\ = \overrightarrow{OD} + \frac{1}{3}\overrightarrow{OA} - \frac{1}{3}\overrightarrow{OD} \\ = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD}.\end{aligned}$$

Because the diagonals of a parallelogram meet at their midpoint, and $2\overrightarrow{OD}$ is a diagonal of the parallelogram formed by $\overrightarrow{OB}$ and $\overrightarrow{OC}$, we have:

$$\overrightarrow{OD} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}),$$

so we can write:

$$\overrightarrow{OQ} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\left(\frac{1}{2}\right)(\overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

Thus a vector from the origin to a point $\frac{1}{3}$ of the way along median AD from D, the midpoint, is given by $\frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.

In a similar manner we can show that the vector from the origin to the point $\frac{1}{3}$ of the way along any median from the midpoint of the side it bisects is also $\frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. See Figure 13.8 and 13.9.

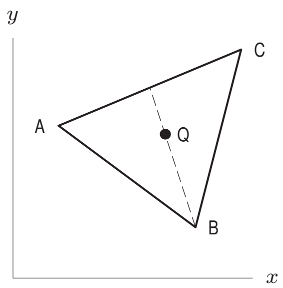


Figure 13.8

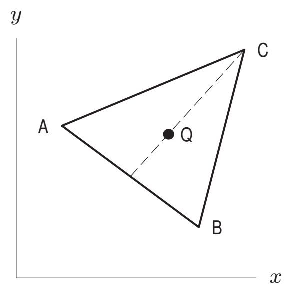


Figure 13.9

Thus the medians of a triangle intersect at a point $\frac{1}{3}$ of the way along each median from the side that each bisects.

Strengthen Your Understanding

44. The number $\|\vec{u} + \vec{v}\|$ could be less than 1. For example, for $\vec{u} = \vec{i}$ and $\vec{v} = -0.5\vec{i}$, $\vec{u} + \vec{v} = 0.5\vec{i}$ and $\|\vec{u} + \vec{v}\| = 0.5 < 1$.

45. If $c < 0$, then $c\vec{u}$ has the opposite direction of $\vec{u}$. Also, if $c = 0$, then $c\vec{v} = \vec{0}$, which has no direction at all.

46. If the angle between the vectors $\vec{u}$ and $\vec{v}$ is more than 90°, then $\|\vec{v} - \vec{u}\|$ is the length of the longer diagonal of the parallelogram.

47. Since $\vec{u} + \vec{w} = \vec{u}$, we see that $\vec{w} = \vec{u} - \vec{u} = \vec{0}$. Since $\vec{w}$ is the zero vector, this means $\vec{v} + \vec{w} = \vec{v} + \vec{0} = \vec{v}$.

48. Since $\vec{v}$ lies on a plane parallel to the yz-plane it must have a zero $\vec{i}$-component. So $\vec{v} = a\vec{j} + b\vec{k}$. If we choose $a = 1$, then $\vec{v} = \vec{j} + b\vec{k}$. Now we set:

$$\|\vec{v}\| = \sqrt{1 + b^2} = 2, \quad \text{which gives} \quad b = \pm\sqrt{3}.$$

So, $\vec{v} = \vec{j} + \sqrt{3}\,\vec{k}$ is a possible answer.

49. Take any equilateral triangle PQR whose sides have length one. Let $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{PR}$, which are unit vectors. Then $\vec{v} - \vec{u} = \overrightarrow{QR}$, which is also a unit vector. The vectors $\vec{u} = \vec{i}$ and $\vec{v} = (1/2)\vec{i} + (\sqrt{3}/2)\vec{j}$ work, for example.

50. Since $\vec{w}$ is the difference between $\vec{u}$ and $\vec{v}$ we have $\vec{w} = \vec{v} - \vec{u}$, or $2\vec{i} + 3\vec{j} = \vec{v} - \vec{u}$. Solving for $\vec{v}$ we get $\vec{v} = 2\vec{i} + 3\vec{j} + \vec{u}$. We can choose any vector we wish for $\vec{u}$ and compute the corresponding vector $\vec{v}$. For example, if we let $u = \vec{i}$ then we get $\vec{v} = 3\vec{i} + 3\vec{j}$.

51. False. There are exactly two unit vectors: one in the same direction as $\vec{v}$ and the other in the opposite direction. Explicitly, the unit vectors parallel to $\vec{v}$ are $\pm\dfrac{1}{\|\vec{v}\|}\vec{v}$.

52. False. The length of this vector is $\sqrt{1/3+1/3+4/3}=\sqrt{2}$, not 1.

53. True. Multiplying by a scalar greater than one stretches the length of the vector by the scalar.

54. False. If $\vec{v}$ and $\vec{w}$ are not parallel, the three vectors $\vec{v}$, $\vec{w}$ and $\vec{v}+\vec{w}$ can be thought of as three sides of a triangle. (If the tail of $\vec{w}$ is placed at the head of $\vec{v}$, then $\vec{v}+\vec{w}$ is a vector from the tail of $\vec{v}$ to the head of $\vec{w}$.) The length of one side of a triangle is less than the sum of the lengths of the other two sides. Alternatively, a counterexample is $\vec{v}=\vec{i}$ and $\vec{w}=\vec{j}$. Then $\|\vec{i}+\vec{j}\|=\sqrt{2}$ but $\|\vec{i}\|+\|\vec{j}\|=2$.

55. False. If $\vec{v}$ and $\vec{w}$ are not parallel, the three vectors $\vec{v}$, $\vec{w}$ and $\vec{v}-\vec{w}$ can be thought of as three sides of a triangle. (If the tails of $\vec{v}$ and $\vec{w}$ are placed together, then $\vec{v}-\vec{w}$ is a vector from the head of $\vec{w}$ to the head of $\vec{v}$.) The length of one side of a triangle is less than the sum of the lengths of the other two sides. Alternatively, a counterexample is $\vec{v}=\vec{i}$ and $\vec{w}=\vec{j}$. Then $\|\vec{i}-\vec{j}\|=\sqrt{2}$ but $\|\vec{i}\|-\|\vec{j}\|=0$.

56. False. Two vectors are parallel if and only if one is a nonzero scalar multiple of the other. If $c(\vec{i}-2\vec{j}+\vec{k})=2\vec{i}-\vec{j}+\vec{k}$, then $c=2$ so that the $\vec{i}$ components are equal, but multiplication by 2 does not make the $\vec{j}$ or $\vec{k}$ components equal. Thus, there is no scalar multiple of $\vec{i}-2\vec{j}+\vec{k}$ that is equal to $2\vec{i}-\vec{j}+\vec{k}$.

57. False. As a counterexample, take $3\vec{i}$ and $-\vec{i}$. Then the sum is $2\vec{i}$ which has magnitude 2 (smaller than $\|3\vec{i}\|=3$).

58. False. Since magnitudes are nonnegative this cannot be true when $c<0$. The correct statement is $\|c\vec{v}\|=|c|\|\vec{v}\|$.

59. False. To find the displacement vector *from* $(1,1,1)$ *to* $(1,2,3)$ we subtract $\vec{i}+\vec{j}+\vec{k}$ from $\vec{i}+2\vec{j}+3\vec{k}$ to get $(1-1)\vec{i}+(2-1)\vec{j}+(3-1)\vec{k}=\vec{j}+2\vec{k}$.

60. False. The displacement vector from (a,b) to (c,d) has the same magnitude but opposite direction as the displacement vector from (c,d) to (a,b).

Solutions for Section 13.2

Exercises

1. Scalar

2. Scalar

3. Temperature is measured by a single number, and so is a scalar.

4. The magnetic field is a vector because it has both a magnitude (the strength of the field) and a direction (the direction of the compass).

5. Writing $\vec{P}=(P_1,P_2,\cdots,P_{50})$ where P_i is the population of the i^{th} state, shows that $\vec{P}$ can be thought of as a vector with 50 components.

6. In components, we have $\vec{v}=10\cos(45^\circ)\vec{i}-10\sin(45^\circ)\vec{j}=(5\sqrt{2})\vec{i}-(5\sqrt{2})\vec{j}=7.07\vec{i}-7.07\vec{j}$. Notice that the coefficient in the $\vec{j}$-direction must be negative. The components are $5\sqrt{2}\,\vec{i}$ and $-5\sqrt{2}\,\vec{j}$.

7. In components, we have $\vec{v}=-40\cos(20^\circ)\vec{i}-40\sin(20^\circ)\vec{j}=-37.59\vec{i}-13.68\vec{j}$. Notice that both coefficients are negative. The components are $-37.59\vec{i}$ and $-13.68\vec{j}$.

8. (a) If the car is going east, it is going solely in the positive x direction, so its velocity vector is $50\vec{i}$.
(b) If the car is going south, it is going solely in the negative y direction, so its velocity vector is $-50\vec{j}$.
(c) If the car is going southeast, the angle between the x-axis and the velocity vector is -45°. Therefore

$$\begin{aligned}\text{velocity vector} &= 50\cos(-45^\circ)\vec{i}+50\sin(-45^\circ)\vec{j}\\ &= 25\sqrt{2}\vec{i}-25\sqrt{2}\vec{j}.\end{aligned}$$

(d) If the car is going northwest, the velocity vector is at a 45° angle to the y-axis, which is 135° from the x-axis. Therefore:

$$\text{velocity vector}=50(\cos 135^\circ)\vec{i}+50(\sin 135^\circ)\vec{j}=-25\sqrt{2}\vec{i}+25\sqrt{2}\vec{j}.$$

9. We need to calculate the length of each vector.

$$\|21\vec{i} + 35\vec{j}\| = \sqrt{21^2 + 35^2} = \sqrt{1666} \approx 40.8,$$
$$\|40\vec{i}\| = \sqrt{40^2} = 40.$$

So the first car is faster.

10. See Figure 13.10. Since

$$\tan\theta = \frac{18}{15},$$

we have

$$\theta = \arctan\left(\frac{18}{15}\right) = 50.194^\circ.$$

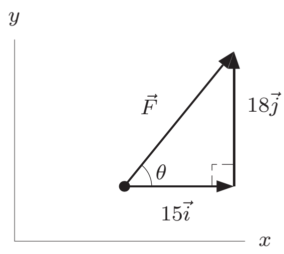


Figure 13.10

Problems

11. (a) We start by finding the velocity, $\vec{w}$, of the boat relative to the riverbed. It is given by

$$\vec{w} = \vec{v} + \vec{c} = 8\vec{i} + (0.6\vec{i} + 0.8\vec{j}) = 8.6\vec{i} + 0.8\vec{j}.$$

The speed of the boat relative to the riverbed is the magnitude of $\vec{w}$:

$$\text{Speed} = \|\vec{w}\| = \sqrt{8.6^2 + 0.8^2} \approx 8.64 \text{ km/hr}.$$

(b) Since the velocity, $\vec{v} = 8\vec{i}$, is parallel to the x-axis, we want to find the angle between $\vec{w}$ and the x-axis. From Figure 13.11, this angle is

$$\theta = \arctan\left(\frac{0.8}{8.6}\right) \approx 0.093 \text{ radians}.$$

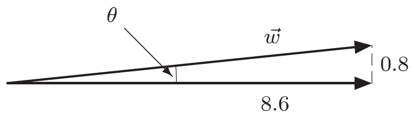


Figure 13.11: Angle between $\vec{v}$ and $\vec{w}$

In practical terms, this angle tells us that if you set your boat parallel to the x-axis at 8 km/hr, the current will take you about 0.093 radians $\approx 5^\circ$ off course.

12. Now $\vec{c} = -2(0.6\vec{i} + 0.8\vec{j}) = -1.2\vec{i} - 1.6\vec{j}$. The velocity vector for the boat relative to the riverbed is

$$\vec{w} = \vec{v} + \vec{c} = (8 - 1.2)\vec{i} - 1.6\vec{j} = 6.8\vec{i} - 1.6\vec{j}$$

so the speed is $\sqrt{6.8^2 + (-1.6)^2} = 6.986$ km/hr.

13. (a) The velocity vector for the boat is $\vec{b} = 25\vec{i}$ and the velocity vector for the current is

$$\vec{c} = -10\cos(45^\circ)\vec{i} - 10\sin(45^\circ)\vec{j} = -7.07\vec{i} - 7.07\vec{j}\,.$$

The actual velocity of the boat is

$$\vec{b} + \vec{c} = 17.93\vec{i} - 7.07\vec{j}\,.$$

(b) $\|\vec{b} + \vec{c}\| = 19.27$ km/hr.

(c) We see in Figure 13.12 that $\tan\theta = \dfrac{7.07}{17.93}$, so $\theta = 21.52^\circ$ south of east.

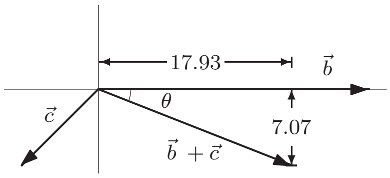


Figure 13.12

14.

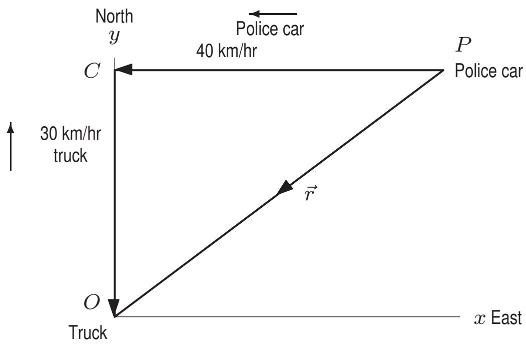


Figure 13.13

Since both vehicles reach the crossroad in exactly one hour, at the present the truck is at O in Figure 13.13; the police car is at P and the crossroads is at C. If $\vec{r}$ is the vector representing the line of sight of the truck with respect to the police car.

$$\vec{r} = -40\vec{i} - 30\vec{j}$$

15. Suppose $\vec{u}$ represents the velocity of the plane relative to the air and $\vec{w}$ represents the velocity of the wind. We can add these two vectors by adding their components. Suppose north is in the y-direction and east is the x-direction. The vector representing the airplane's velocity makes an angle of 45° with north; the components of $\vec{u}$ are

$$\vec{u} = 700\sin 45^\circ\vec{i} + 700\cos 45^\circ\vec{j} \approx 495\vec{i} + 495\vec{j}\,.$$

Since the wind is blowing from the west, $\vec{w} = 60\vec{i}$. By adding these we get a resultant vector $\vec{v} = 555\vec{i} + 495\vec{j}$. The direction relative to the north is the angle θ shown in Figure 13.14 given by

$$\begin{aligned}\theta &= \tan^{-1}\frac{x}{y} = \tan^{-1}\frac{555}{495}\\ &\approx 48.3^\circ\end{aligned}$$

The magnitude of the velocity is

$$\begin{aligned}\|\vec{v}\| &= \sqrt{495^2 + 555^2} = \sqrt{553{,}050}\\ &= 744 \text{ km/hr.}\end{aligned}$$

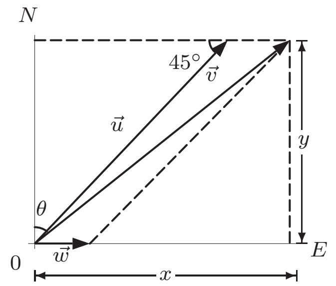


Figure 13.14: Note that θ is the angle between north and the vector $\vec v$

16. We want the total force on the object to be zero. We must choose the third force $\vec F_3$ so that $\vec F_1 + \vec F_2 + \vec F_3 = 0$. Since $\vec F_1 + \vec F_2 = 11\vec i - 4\vec j$, we need $\vec F_3 = -11\vec i + 4\vec j$.

17. The velocity vector for the wind is $\vec w = 60\cos(45°)\vec i - 60\sin(45°)\vec j = 42.43\vec i - 42.43\vec j$. If the airplane is to head due east, then the component in the $\vec j$-direction of $\vec p + \vec w$ must be zero (where $\vec p$ represents the velocity vector for the airspeed of the plane.) Thus, we have $\vec p = A\vec i + 42.43\vec j$, for some value of A. Since the airplane is flying at an airspeed of 500 km/hr, we have

$$\begin{aligned} \|\vec p\| &= 500 \\ \sqrt{A^2 + 42.43^2} &= 500 \\ A &= 498.20. \end{aligned}$$

We have

$$\vec p = 498.20\vec i + 42.43\vec j.$$

This is the direction the plane should head in order to go due east. We use $\tan\theta = \dfrac{42.43}{498.20}$, so $\theta = 4.87°$. The plane should head $4.87°$ north of east. Since

$$\vec p + \vec w = 540.63\vec i,$$

the airplane's speed relative to the ground is $\|\vec p + \vec w\| = 540.63$ km/hr.

18. Let the x-axis point east and the y-axis point north. Since the wind is blowing from the northeast at a speed of 50 km/hr, the velocity of the wind is

$$\vec w = -50\cos 45°\vec i - 50\sin 45°\vec j \approx -35.4\vec i - 35.4\vec j.$$

Let $\vec a$ be the velocity of the airplane, relative to the air, and let ϕ be the angle from the x-axis to $\vec a$; since $\|\vec a\| = 600$ km/hr, we have $\vec a = 600\cos\phi\vec i + 600\sin\phi\vec j$. (See Figure 13.15.)

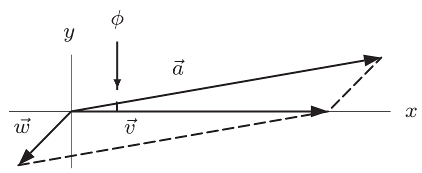


Figure 13.15

Now the resultant velocity, $\vec v$, is given by

$$\begin{aligned} \vec v = \vec a + \vec w &= (600\cos\phi\vec i + 600\sin\phi\vec j) + (-35.4\vec i - 35.4\vec j) \\ &= (600\cos\phi - 35.4)\vec i + (600\sin\phi - 35.4)\vec j. \end{aligned}$$

Since the airplane is to fly due east, i.e., in the x direction, then the y-component of the velocity must be 0, so we must have

$$\begin{aligned} 600\sin\phi - 35.4 &= 0 \\ \sin\phi &= \frac{35.4}{600}. \end{aligned}$$

Thus $\phi = \arcsin(35.4/600) \approx 3.4°$.

19. Let the x-axis point east and the y-axis point north. We use $\vec{C}$, $\vec{W}$, and $\vec{E}$ to represent the current, wind, and engine vectors, respectively. We resolve the current and wind velocity vectors into components. Since the current points 25° north of east with a speed of 12, we have

$$\vec{C} = 12\cos(25^\circ)\vec{i} + 12\sin(25^\circ)\vec{j} = 10.876\vec{i} + 5.071\vec{j}.$$

Since $\vec{C}$ lies in the first quadrant, both coefficients are positive.

The wind points 80° south of east with a speed of 7 km/hr, so we have

$$\vec{W} = 7\cos(80^\circ)\vec{i} - 7\sin(80^\circ)\vec{j} = 1.216\vec{i} - 6.894\vec{j}.$$

Since $\vec{W}$ lies in the fourth quadrant, the coefficient of $\vec{i}$ is positive and the coefficient of $\vec{j}$ is negative.

The combined velocity on the boat is due east at a speed of 40 km/hr, so we want

$$\vec{C} + \vec{W} + \vec{E} = 40\vec{i}.$$

We solve for $\vec{E}$:

$$\begin{aligned}\vec{E} &= 40\vec{i} - (\vec{C} + \vec{W})\\ &= 40\vec{i} - ((10.876\vec{i} + 5.071\vec{j}) + (1.216\vec{i} - 6.894\vec{j}))\\ &= 40\vec{i} - (12.092\vec{i} - 1.823\vec{j})\\ &= 27.908\vec{i} + 1.823\vec{j}.\end{aligned}$$

The engine should push the boat with a speed of $\|\vec{E}\| = \sqrt{27.908^2 + 1.823^2} = 27.97$ km/hr, and in direction $\arctan(1.823/27.908) = 3.74^\circ$ north of east.

20. (a) Let x-axis be the East direction and y-axis be the North direction. From Figure 13.16,

$$\theta = \sin^{-1}(4/5) = 53.1^\circ.$$

That is, he should steer at 53.1° east of south.

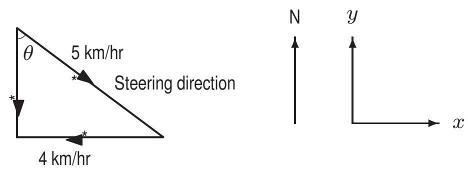


Figure 13.16

(b)

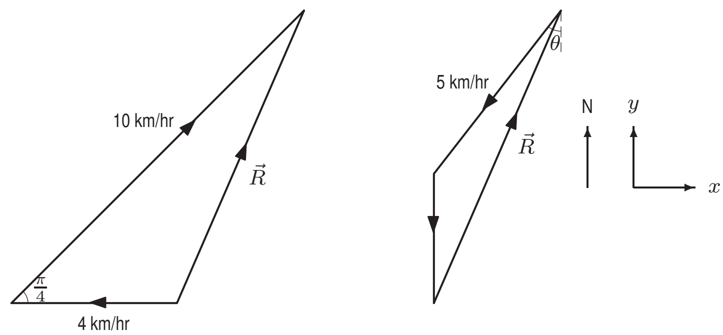


Figure 13.17

Let $\vec{R}$ be the resultant of the wind and river velocities, that is

$$\begin{aligned}\vec{R} &= -4\vec{i} + (10\cos(\frac{\pi}{4})\vec{i} + 10\cos(\frac{\pi}{4})\vec{j}) \\ &= (-4+5\sqrt{2})\vec{i} + 5\sqrt{2}\vec{j}.\end{aligned}$$

From Figure 13.17, we see that to get the the x-component of his rowing velocity and the x-component of $\vec{R}$ to cancel each other, we must have

$$\begin{aligned}5\sin\theta &= -4+5\sqrt{2} \\ \theta &= \sin^{-1}\left(\frac{-4+5\sqrt{2}}{5}\right) = 37.9^\circ.\end{aligned}$$

However for this value of θ, the y-component of the velocity is

$$5\sqrt{2} - 5\cos(37.9^\circ) = 3.1.$$

Since the y-component is positive, the man will not move across the river in a southward direction.

21. Let $\vec{R}$ be the resultant force, and let $\vec{F}_1$ and $\vec{F}_2$ be the forces exerted by the larger and smaller tugs. See Figure 13.18. Then $\|\vec{F_1}\| = \frac{5}{4}\|\vec{F_2}\|$. The y components of the vectors $\vec{F}_1$ and $\vec{F}_2$ must cancel each other in order to ensure that the ship travels due east, hence

$$\|\vec{F_1}\|\sin 30^\circ = \|\vec{F_2}\|\sin\theta,$$

so

$$\frac{5}{4}\|\vec{F_2}\|\sin 30^\circ = \|\vec{F_2}\|\sin\theta,$$

giving $\sin\theta = \frac{5}{8}$, and hence $\theta = \sin^{-1}\frac{5}{8} = 38.7^\circ$.

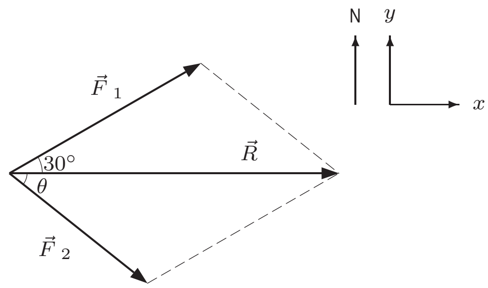


Figure 13.18

22. The known force is

$$\vec{F_1} = 15\cos 20^\circ\vec{i} + 15\sin 20^\circ\vec{j} = 14.1\vec{i} + 5.13\vec{j} \text{ pounds}.$$

Let the unknown force be $\vec{F_2} = a\vec{i} + b\vec{j}$.

Because $\vec{F_1}$ and $\vec{F_2}$ together pull due east, we have $\vec{F_1} + \vec{F_2} = k\vec{i}$ for a positive constant k. Thus

$$(15\cos 20^\circ + a)\vec{i} + (15\sin 20^\circ + b)\vec{j} = k\vec{i} + 0\vec{j}.$$

Hence

$$b = -15\sin 20^\circ = -5.1 \text{ pounds}.$$

Because $\vec{F_2}$ has magnitude 20 pounds, we have $\sqrt{a^2+b^2} = 20$. Thus

$$a = \sqrt{20^2 - b^2} = 19.3 \text{ pounds}$$

where the positive square root is required because the sum $\vec{F_1} + \vec{F_2}$ points east and not west. Thus

$$\vec{F_2} = 19.3\vec{i} - 5.1\vec{j} \text{ pounds}.$$

23. The force exerted on the object from the first rope $\vec{F}_1 = 100\cos(30°)\vec{i} + 100\sin(30°)\vec{j} = 86.60\vec{i} + 50\vec{j}$ and the force exerted from the second rope is $\vec{F}_2 = 70\cos(80°)\vec{i} - 70\sin(80°)\vec{j} = 12.16\vec{i} - 68.94\vec{j}$. The sum of these two forces is $\vec{F}_1 + \vec{F}_2 = 98.76\vec{i} - 18.94\vec{j}$. See Figure 13.19. In order for the object to move vertically, the total force on the object must be in the form $\vec{F} = 0\vec{i} + 0\vec{j} + b\vec{k}$ for some b. Thus the force vector for the crane is

$$\vec{F}_c = -98.76\vec{i} + 18.94\vec{j} + b\vec{k}$$

for some b. To find b, we use the fact that $\|\vec{F}_c\| = 3000$. Thus,

$$\begin{aligned} \|\vec{F}_c\| &= 3000 \\ \sqrt{(98.76)^2 + (18.94)^2 + b^2} &= 3000 \\ b &= \pm 2998.31 \end{aligned}$$

We use the positive value of b since we want the object to go up rather than down. The force exerted by the crane is

$$\vec{F}_c = -98.76\vec{i} + 18.94\vec{j} + 2998.31\vec{k}.$$

The total force acting on the object is $2998.31\vec{k}$, or 2998.31 newtons straight up.

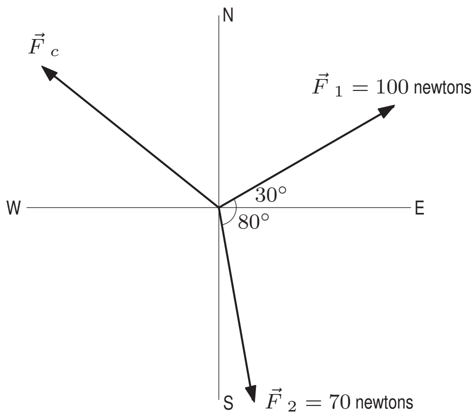


Figure 13.19: Horizontal forces on object

24. (a) The displacement vector of the moon relative to the earth is

$$\vec{r} = 384\vec{i}.$$

The displacement vector of the spaceship relative to the earth is

$$\vec{r_E} = 280\vec{i} + 90\vec{j}.$$

The displacement vector of the spaceship relative to the moon is

$$\vec{r_L} = \vec{r_E} - \vec{r} = -104\vec{i} + 90\vec{j}.$$

See Figure 13.20.

(b) Distance of spaceship from Earth $= \|\vec{r_E}\| = \sqrt{280^2 + 90^2} = \sqrt{86500} = 294.109$ thousand km.
Distance of spaceship from the moon $= \|\vec{r_L}\| = \sqrt{(-104)^2 + 90^2} = \sqrt{18916} = 137.535$ thousand km.

(c) See Figure 13.20. The gravitational force of the earth, $\overrightarrow{F_E}$, is parallel to $\vec{r_E}$ but of length 461 and in the opposite direction:

$$\overrightarrow{F_E} = -\frac{461}{\sqrt{186500}}(280\vec{i} + 90\vec{j}) = -438.885\vec{i} - 141.070\vec{j}.$$

The gravitational force of the moon, $\overrightarrow{F_L}$, is parallel to $\vec{r_L}$ but of length 26 and in the opposite direction:

$$\overrightarrow{F_L} = -\frac{26}{\sqrt{18916}}(-104\vec{i} + 90\vec{j}) = 19.660\vec{i} - 17.041\vec{j}.$$

The resulting force, $\vec{F}$ is

$$\vec{F} = \overrightarrow{F_E} + \overrightarrow{F_L} = 419.225\vec{i} - 158.084\vec{j}.$$

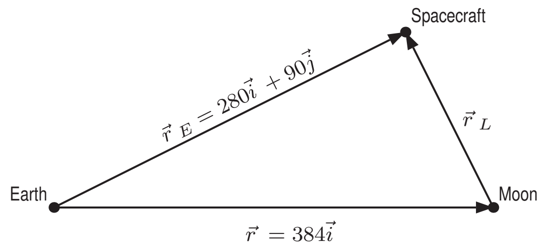


Figure 13.20

25. The speed of the particle before impact is v, so the speed after impact is $0.8v$. If we consider the barrier as being along the x-axis (see Figure 13.21), then the $\vec{i}$-component is $0.8v\cos 60^\circ = 0.8v(0.5) = 0.4v$.
Similarly, the $\vec{j}$-component is $0.8v\sin 60^\circ = 0.8v(0.8660) \approx 0.7v$. Thus

$$\vec{v}_{\text{after}} = 0.4v\vec{i} + 0.7v\vec{j}.$$

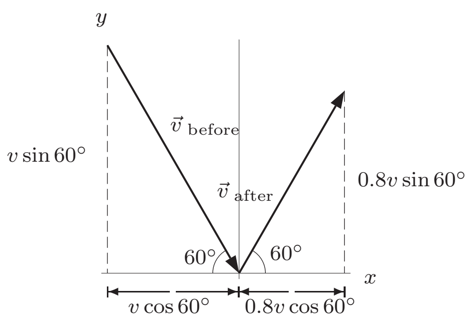


Figure 13.21

26. The total scores are out of 300 and are given by the total score vector $\vec{v} + 2\vec{w}$:

$$\begin{aligned}\vec{v} + 2\vec{w} &= (73, 80, 91, 65, 84) + 2(82, 79, 88, 70, 92)\\ &= (73, 80, 91, 65, 84) + (164, 158, 176, 140, 184)\\ &= (237, 238, 267, 205, 268).\end{aligned}$$

To get the scores as a percentage, we divide by 3, giving

$$\frac{1}{3}(237, 238, 267, 205, 268) \approx (79.00, 79.33, 89.00, 68.33, 89.33).$$

27. Since there are 16 ounces in a pound, we multiply the vector by $1/16$ to get $0.01875\vec{i} + 0.0125\vec{j} + 0.03125\vec{k}$ in dollars per ounce.

28. (a) Since the radius of the circle is 1 meter, the circumference is 2π meters. Thus, the object is moving at 2π meters/minute, or $\pi/30$ meters/second ≈ 0.11 meters/second.

(b) 30 seconds after passing the point $(0, 1)$, the object is at the point $(-1, 0)$. (Since it completes 1 revolution each minute, it will move π radians in 30 seconds.) This is true regardless of whether the point is moving clockwise or counterclockwise. However, since the velocity vector, $\vec{v}$, is tangential to the curve in the direction of motion, it will have an opposite sign if the motion is in the opposite direction. So, moving clockwise $\vec{v} = 2\pi\vec{j}$, and moving counterclockwise $\vec{v} = -2\pi\vec{j}$, if the speed is measured in meters/minute.

29. The speed is a scalar which equals 30 times the circumference of the circle per minute. So it is a constant. The velocity is a vector. Since the direction of the motion changes all the time, the velocity is not constant. This implies that the acceleration is nonzero.

30.

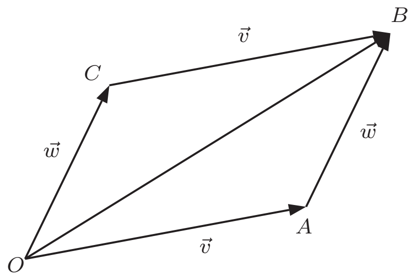


Figure 13.22

The vector $\vec{v} + \vec{w}$ is equivalent to putting the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ end-to-end as shown in Figure 13.22; the vector $\vec{w} + \vec{v}$ is equivalent to putting the vectors $\overrightarrow{OC}$ and $\overrightarrow{CB}$ end-to-end. Since they form a parallelogram, $\vec{v} + \vec{w}$ and $\vec{w} + \vec{v}$ are both equal to the vector $\overrightarrow{OB}$, we have $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

31.

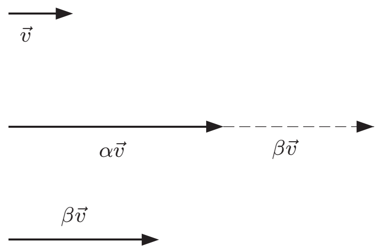


Figure 13.23

The vectors $\vec{v}$, $\alpha\vec{v}$ and $\beta\vec{v}$ are all parallel. Figure 13.23 shows them with $\alpha, \beta > 0$, so all the vectors are in the same direction. Notice that $\alpha\vec{v}$ is a vector α times as long as $\vec{v}$ and $\beta\vec{v}$ is β times as long as $\vec{v}$. Therefore $\alpha\vec{v} + \beta\vec{v}$ is a vector $(\alpha + \beta)$ times as long as $\vec{v}$, and in the same direction. Thus,

$$\alpha\vec{v} + \beta\vec{v} = (\alpha + \beta)\vec{v}.$$

32.

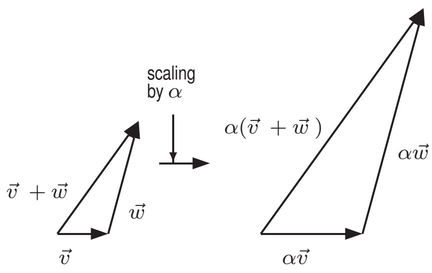


Figure 13.24

The effect of scaling the left-hand picture in Figure 13.24 is to stretch each vector by a factor of α (shown with $\alpha > 1$). Since, after scaling up, the three vectors $\alpha\vec{v}$, $\alpha\vec{w}$, and $\alpha(\vec{v} + \vec{w})$ form a similar triangle, we know that $\alpha(\vec{v} + \vec{w})$ is the sum of the other two: that is

$$\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}.$$

33. Assume $\alpha,\ \beta > 0$. The vector $\beta\vec{v}$ is in the same direction and β times as long as $\vec{v}$. The vector $\alpha(\beta\vec{v})$ is in the same direction and α times as long as $\beta\vec{v}$, and so is $\alpha\beta$ times as long as $\vec{v}$ and in the same direction as $\vec{v}$. Thus,

$$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}.$$

34. Since the zero vector has zero length, adding it to $\vec{v}$ has no effect.

35. According to the definition of scalar multiplication, $1 \cdot \vec{v}$ has the same direction and magnitude as $\vec{v}$, so it is the same as $\vec{v}$.

36. By Figure 13.25, the vectors $\vec{v} + (-1)\vec{w}$ and $\vec{v} - \vec{w}$ are equal.

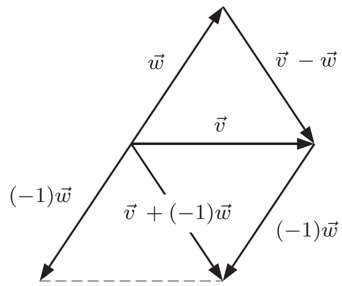


Figure 13.25

37.

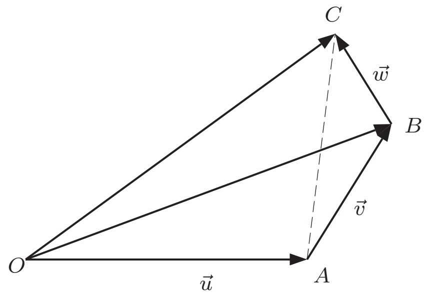


Figure 13.26

The vector $\vec{u} + \vec{v}$ is represented by $\overrightarrow{OB}$. The vector $(\vec{u} + \vec{v}) + \vec{w}$ is represented by $\overrightarrow{OB}$ followed by $\overrightarrow{BC}$, which is therefore $\overrightarrow{OC}$. Now $\vec{v} + \vec{w}$ is represented by $\overrightarrow{AC}$. So $\vec{u} + (\vec{v} + \vec{w})$ is $\overrightarrow{OA}$ followed by $\overrightarrow{AC}$, which is $\overrightarrow{OC}$. Since we get the vector $\overrightarrow{OC}$ by both methods, we know

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

38. (a) Target A is at the point $(30, 0, 3)$; Target B is at the point $(20, 15, 0)$; Target C is the point $(12, 30, 8)$. You fire from the point $P = (0, 0, 5)$. The vectors to each of these targets are $\overrightarrow{PA} = 30\vec{i} - 2\vec{k}$, $\overrightarrow{PB} = 20\vec{i} + 15\vec{j} - 5\vec{k}$, $\overrightarrow{PC} = 12\vec{i} + 30\vec{j} + 3\vec{k}$.

(b) You fire from the point $Q = (0, -1, 3)$, so $\overrightarrow{QA} = 30\vec{i} + \vec{j}$, $\overrightarrow{QB} = 20\vec{i} + 16\vec{j} - 3\vec{k}$, $\overrightarrow{QC} = 12\vec{i} + 31\vec{j} + 5\vec{k}$.

Strengthen Your Understanding

39. The vectors $\vec{v} = 2\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{w} = 3\vec{i} + 2\vec{j} + \vec{k}$ have both magnitude $\sqrt{14}$ and the same $\vec{k}$-component, but different $\vec{i}$-components and $\vec{j}$-components, so they are not the same vector.

40. If the $\vec{j}$-component of $\vec{v}$ is larger than or equal to 2 (or smaller than or equal to -2), then the magnitude of $\vec{v}$ is greater than the magnitude of $\vec{w}$. For example, $\vec{v} = 0.5\vec{i} + 2\vec{j}$.

41. Writing $\vec{F} = a\vec{i} + b\vec{j}$, we have:

$$\vec{R} = \vec{F} + \vec{G} = (a+1)\vec{i} + (b+1)\vec{j}.$$

Choosing $a = 1$ and $b = -2$, results in a positive $\vec{i}$-component and a negative $\vec{j}$-component for $\vec{R}$. So $\vec{F} = \vec{i} - 2\vec{j}$ is a possible vector. There are many possible others.

42. If $\vec{u}$ and $\vec{v}$ are not parallel, then $\vec{u}$, $\vec{v}$ and $\vec{u} + \vec{v}$ form a triangle. In any triangle, the length of any side is less than the sum of the lengths of the other two sides: $\|\vec{u} + \vec{v}\| < \|\vec{u}\| + \|\vec{v}\|$. Instead, choose $\vec{v} = c\vec{u}$, for any $c > 0$. For example, let $\vec{u} = \vec{i}$ and $\vec{v} = 2\vec{i}$. Then

$$\|\vec{i} + 2\vec{i}\| = \|3\vec{i}\| = 3 = 1 + 2 = \|\vec{i}\| + \|2\vec{i}\|.$$

43. Yes. Velocity describes how fast something moves and its direction of travel.

44. No. Speed describes only how fast something moves. It does not specify a direction.

45. Yes. Force describes how hard something is pushed or pulled and the direction of the push or pull.

46. No. Area measures the size of the region occupied by a two-dimension figure or surface, and gives no information about direction.

47. Yes. Acceleration measures the rate of change of velocity and the direction of the change.

48. No. Volume measures the size of the region occupied by a three-dimension object and gives no information about direction.

Solutions for Section 13.3

Exercises

1. $\vec{a}\cdot\vec{y} = (2\vec{j}+\vec{k})\cdot(4\vec{i}-7\vec{j}) = -14.$

2. $\vec{c}\cdot\vec{y} = (\vec{i}+6\vec{j})\cdot(4\vec{i}-7\vec{j}) = (1)(4)+(6)(-7) = 4-42 = -38.$

3. $\vec{a}\cdot\vec{b} = (2\vec{j}+\vec{k})\cdot(-3\vec{i}+5\vec{j}+4\vec{k}) = (0)(-3)+(2)(5)+(1)(4) = 0+10+4 = 14.$

4. $\vec{a}\cdot\vec{z} = (2\vec{j}+\vec{k})\cdot(\vec{i}-3\vec{j}-\vec{k}) = (0)(1)+(2)(-3)+(1)(-1) = 0-6-1 = -7.$

5. $\vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{y} = (\vec{i}+6\vec{j})\cdot(2\vec{j}+\vec{k})+(2\vec{j}+\vec{k})\cdot(4\vec{i}-7\vec{j}) = 12-14 = -2.$

6. $\vec{c}+\vec{y} = (\vec{i}+6\vec{j})+(4\vec{i}-7\vec{j}) = 5\vec{i}-\vec{j}$, so

$$\vec{a}\cdot(\vec{c}+\vec{y}) = (2\vec{j}+\vec{k})\cdot(5\vec{i}-\vec{j}) = -2.$$

7. Since $\vec{a}\cdot\vec{b}$ is a scalar and $\vec{a}$ is a vector, the answer to this equation is a vector parallel to $\vec{a}$. We have

$$\vec{a}\cdot\vec{b} = (2\vec{j}+\vec{k})\cdot(-3\vec{i}+5\vec{j}+4\vec{k}) = 0(-3)+2(5)+1(4) = 14.$$

Thus,

$$(\vec{a}\cdot\vec{b})\cdot\vec{a} = 14\vec{a} = 14(2\vec{j}+\vec{k}) = 28\vec{j}+14\vec{k}$$

8. Since $\vec{a}\cdot\vec{y}$ and $\vec{c}\cdot\vec{z}$ are both scalars, the answer to this equation is the product of two numbers and therefore a number. We have

$$\vec{a}\cdot\vec{y} = (2\vec{j}+\vec{k})\cdot(4\vec{i}-7\vec{j}) = 0(4)+2(-7)+1(0) = -14$$
$$\vec{c}\cdot\vec{z} = (\vec{i}+6\vec{j})\cdot(\vec{i}-3\vec{j}-\vec{k}) = 1(1)+6(-3)+0(-1) = -17$$

Thus,

$$(\vec{a}\cdot\vec{y})(\vec{c}\cdot\vec{z}) = 238$$

9. Since $\vec{c}\cdot\vec{c}$ is a scalar and $(\vec{c}\cdot\vec{c})\vec{a}$ is a vector, the answer to this equation is another scalar. We could calculate $\vec{c}\cdot\vec{c}$, then $(\vec{c}\cdot\vec{c})\vec{a}$, and then take the dot product $((\vec{c}\cdot\vec{c})\vec{a})\cdot\vec{a}$. Alternatively, we can use the fact that

$$((\vec{c}\cdot\vec{c})\vec{a})\cdot\vec{a} = (\vec{c}\cdot\vec{c})(\vec{a}\cdot\vec{a}).$$

Since

$$\vec{c}\cdot\vec{c} = (\vec{i}+6\vec{j})\cdot(\vec{i}+6\vec{j}) = 1^2+6^2 = 37$$
$$\vec{a}\cdot\vec{a} = (2\vec{j}+\vec{k})\cdot(2\vec{j}+\vec{k}) = 2^2+1^2 = 5,$$

we have,

$$(\vec{c}\cdot\vec{c})(\vec{a}\cdot\vec{a}) = 37(5) = 185$$

10. A normal vector can be obtained from the coefficients: $\vec{n} = 2\vec{i}+\vec{j}-\vec{k}$.

11. Rewriting the equation as

$$2x - 2z = 3x + 3y$$

or

$$x + 3y + 2z = 0$$

tells us that a normal vector is

$$\vec{n} = \vec{i} + 3\vec{j} + 2\vec{k}.$$

12. A normal vector can be obtained from the coefficients of x, y, z in the equation of the plane and is: $\vec{n} = 1.5\vec{i} + 3.2\vec{j} + \vec{k}$.

13. Writing the equation in the form

$$3x + 4y - z = 7$$

shows that a normal vector is

$$\vec{n} = 3\vec{i} + 4\vec{j} - \vec{k}$$

14. Rewriting the equation as

$$\pi x - \pi = (1 - \pi)y - (1 - \pi)z + \pi$$

gives

$$\pi x + (\pi - 1)y + (1 - \pi)z = 2\pi$$

so a normal vector is

$$\vec{n} = \pi\vec{i} + (\pi - 1)\vec{j} + (1 - \pi)\vec{k}.$$

15. The plane is

$$3x - y + 4z = 3 \cdot 1 - 1 \cdot 5 + 4 \cdot 2$$
$$3x - y + 4z = 6.$$

16. The plane is

$$5x + 4y - z = 5 \cdot 2 + 4(-1) - 1 \cdot 3$$
$$5x + 4y - z = 3.$$

17. The equation of the plane is

$$(x - 1) - (y - 3) + (z - 5) = 0,$$

which can be written as $x - y + z = 3$.

18. Since the plane is normal to the vector $5\vec{i} + \vec{j} - 2\vec{k}$ and passes through the point $(0, 1, -1)$, an equation for the plane is

$$5x + y - 2z = 5 \cdot 0 + 1 \cdot 1 + (-2) \cdot (-1) = 3$$
$$5x + y - 2z = 3.$$

19. Two planes are parallel if their normal vectors are parallel. Since the plane $2x + 4y - 3z = 1$ has normal vector $\vec{n} = 2\vec{i} + 4\vec{j} - 3\vec{k}$, the plane we are looking for has the same normal vector and passes through the point $(1, 0, -1)$. Thus the plane we want has equation:

$$2x + 4y - 3z = 2 \cdot 1 + 4 \cdot 0 + (-3) \cdot (-1) = 5$$

20. Two planes are parallel if their normal vectors are parallel. Since the plane $3x + y + z = 4$ has normal vector $\vec{n} = 3\vec{i} + \vec{j} + \vec{k}$, the plane we are looking for has the same normal vector and passes through the point $(-2, 3, 2)$. Thus, it has the equation

$$3x + y + z = 3 \cdot (-2) + 3 + 2 = -1.$$

21. The normal vector to the plane is $\vec{n} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, so, for some d, the equation of the plane is

$$2x - 3y + 5z = d.$$

Substituting the point $(4, 5, -2)$ we see that $d = -17$. Thus the equation of the plane is

$$2x - 3y + 5z = -17.$$

22.

$$\begin{aligned}\cos\theta &= \frac{(\vec{i} + \vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} - \vec{k})}{\|\vec{i} + \vec{j} + \vec{k}\| \|\vec{i} - \vec{j} - \vec{k}\|} \\ &= \frac{(1)(1) + (1)(-1) + (1)(-1)}{\sqrt{1^1 + 1^2 + 1^2}\sqrt{1^2 + (-1)^2 + (-1)^2}} \\ &= -\frac{1}{3}.\end{aligned}$$

So, $\theta = \arccos(-1/3) \approx 1.911$ radians, or about 109.471°.

23.

$$\begin{aligned}\cos\theta &= \frac{(\vec{i} + \vec{k}) \cdot (\vec{j} - \vec{k})}{\|\vec{i} + \vec{k}\| \|\vec{j} - \vec{k}\|} \\ &= \frac{(1)(0) + (0)(1) + (1)(-1)}{\sqrt{1^1 + 0^2 + 1^2}\sqrt{0^2 + (1)^2 + (-1)^2}} \\ &= -\frac{1}{2}.\end{aligned}$$

So, $\theta = \arccos(-1/2) = 2\pi/3$ radians, or 120°.

24.

$$\begin{aligned}\cos\theta &= \frac{(\vec{i} + \vec{j} - \vec{k}) \cdot (2\vec{i} + 3\vec{j} + \vec{k})}{\|\vec{i} + \vec{j} - \vec{k}\| \|2\vec{i} + 3\vec{j} + \vec{k}\|} \\ &= \frac{(1)(2) + (1)(3) + (-1)(1)}{\sqrt{1^2 + 1^2 + (-1)^2}\sqrt{2^2 + 3^2 + 1^2}} \\ &= \frac{4}{\sqrt{42}}.\end{aligned}$$

So, $\theta = \arccos(4/\sqrt{42}) \approx 0.906$ radians, or $\approx 51.887^\circ$.

25.

$$\begin{aligned}\cos\theta &= \frac{(\vec{i} + \vec{j}) \cdot (\vec{i} + 2\vec{j} - \vec{k})}{\|\vec{j} + \vec{j}\| \|\vec{i} + 2\vec{j} - \vec{k}\|} \\ &= \frac{(1)(1) + (1)(2) + (0)(-1)}{\sqrt{1^2 + 1^2 + 0^2}\sqrt{1^2 + 2^2 + (-1)^2}} \\ &= \frac{3}{\sqrt{2}\sqrt{6}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.\end{aligned}$$

So, $\theta = \arccos(\sqrt{3}/2) = \pi/6$ radians, or 30°.

26.

$$\begin{aligned}\cos\theta &= \frac{(\vec{i}) \cdot (2\vec{i} + 3\vec{j} - \vec{k})}{\|\vec{i}\| \|2\vec{i} + 3\vec{j} - \vec{k}\|} \\ &= \frac{(1)(2) + (0)(3) + (0)(-1)}{\sqrt{1^2 + 0^2 + 0^2}\sqrt{2^2 + 3^2 + (-1)^2}} \\ &= \frac{2}{\sqrt{14}}.\end{aligned}$$

So, $\theta = \arccos(2/\sqrt{14}) = 1.007$ radians, or 57.688°.

Problems

27. (a) Dividing $\vec{v}$ by its magnitude produces a unit vector $\vec{u}$ in the same direction as $\vec{v}$:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{2^2+3^2}}(2\vec{i} + 3\vec{j}) = \frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j}.$$

(b) Any vector $\vec{w} = a\vec{i} + b\vec{j}$ such that $\vec{v} \cdot \vec{w} = 2a + 3b = 0$ is perpendicular to $\vec{v}$. For example, $\vec{w} = 3\vec{i} - 2\vec{j}$ has this property, as do all scalar multiples of $3\vec{i} - 2\vec{j}$.

28. (a) The plane can be written as $5x - 2y - z + 7 = 0$, so the vector $5\vec{i} - 2\vec{j} - \vec{k}$ is normal to the plane. The vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ is parallel to $5\vec{i} - 2\vec{j} - \vec{k}$ if one is a scalar multiple of the other. This occurs if the coefficients are in proportion:

$$\frac{\lambda}{5} = \frac{1}{-2} = \frac{0.5}{-1}.$$

Solving gives $\lambda = -2.5$.

(b) Substituting $x = a + 1$, $y = a$, $z = a - 1$ into the equation of the plane gives

$$\begin{aligned} a - 1 &= 5(a+1) - 2a + 7 \\ a - 1 &= 5a + 5 - 2a + 7 \\ -13 &= 2a \\ a &= -6.5. \end{aligned}$$

29. (a) On the x-axis, $y = z = 0$, so $5x = 21$, giving $x = \frac{21}{5}$. So the only such point is $(\frac{21}{5}, 0, 0)$.

(b) Other points are $(0, -21, 0)$, and $(0, 0, 3)$. There are many other possible answers.

(c) $\vec{n} = 5\vec{i} - \vec{j} + 7\vec{k}$. It is the normal vector.

(d) The vector between two points in the plane is parallel to the plane. Using the points from part (b), the vector $3\vec{k} - (-21\vec{j}) = 21\vec{j} + 3\vec{k}$ is parallel to the plane.

30. (a) The vector $\vec{n} = 3\vec{i} - \vec{j} - \vec{k}$ is perpendicular to the plane since the plane can be written in the form $(3\vec{i} - \vec{j} - \vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) = -2$.

(b) Find two points in the plane by putting x and y-values into the equation and calculating the corresponding z-values. If $x = 1$ and $y = 0$, then $z = 2 + 3(1) - 0 = 5$, so the point $(1, 0, 5)$ is on the plane. So is the point $(0, 1, 1)$. Therefore the vector $(\vec{i} + 5\vec{k}) - (\vec{j} + \vec{k}) = \vec{i} - \vec{j} + 4\vec{k}$ is parallel to the plane. (To check, we take the dot product with $\vec{n} = 3\vec{i} - \vec{j} - \vec{k}$ and see if we get zero: $(3\vec{i} - \vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} + 4\vec{k}) = 3 + 1 - 4 = 0$. Therefore $\vec{i} - \vec{j} + 4\vec{k}$ is parallel to the plane.)

31. (a) Writing the plane in the form $2x + 3y - z = 0$ shows that a normal vector is

$$\vec{n} = 2\vec{i} + 3\vec{j} - \vec{k}.$$

Any multiple of this vector is also a correct answer.

(b) Any vector perpendicular to $\vec{n}$ is parallel to the plane, so one possible answer is

$$\vec{v} = 3\vec{i} - 2\vec{j}.$$

Many other answers are possible.

32. (a) goes with (I).

(b) goes with (III), (IV).

(c) goes with (II), (III).

(d) goes with (II).

33. (a) Perpendicular vectors have a dot product of 0. Since $\vec{a} \cdot \vec{c} = 1(-2) - 3(-1) - 1 \cdot 1 = 0$, and $\vec{b} \cdot \vec{d} = 1(-1) + 1(-1) + 2 \cdot 1 = 0$, the pairs we want are $\vec{a}, \vec{c}$ and $\vec{b}, \vec{d}$.

(b) Parallel vectors are multiples of one another, so there are no parallel vectors in this set.

(c) Since $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta$, the dot product of the vectors we want is positive. We have

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 1 \cdot 1 - 3 \cdot 1 - 1 \cdot 2 = -4 \\ \vec{a} \cdot \vec{d} &= 1(-1) - 3(-1) - 1 \cdot 1 = 1 \\ \vec{b} \cdot \vec{c} &= 1(-2) + 3(-1) + 2 \cdot 1 = -1 \\ \vec{c} \cdot \vec{d} &= -2(-1) - 1(-1) + 1 \cdot 1 = 4, \end{aligned}$$

and we already know $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{d}=0$. Thus, the pairs of vectors with an angle of less than $\pi/2$ between them are $\vec{a},\vec{d}$ and $\vec{c},\vec{d}$.

(d) Vectors with an angle of more than $\pi/2$ between them have a negative dot product, so pairs are $\vec{a},\vec{b}$ and $\vec{b},\vec{c}$.

34. Vectors $\vec{v}_1$, $\vec{v}_4$, and $\vec{v}_8$ are all parallel to each other. Vectors $\vec{v}_3$, $\vec{v}_5$, and $\vec{v}_7$ are all parallel to each other, and are all perpendicular to the vectors in the previous sentence. Vectors $\vec{v}_2$ and $\vec{v}_9$ are perpendicular.

35. (a) Any multiple of $\vec{v}$ will work, for example, $8\vec{i}+6\vec{j}$.
(b) Any vector $\vec{w}$ such that $\vec{v}\cdot\vec{w}=0$ will work, such as $-3\vec{i}+4\vec{j}$.

36. In general, $\vec{u}$ and $\vec{v}$ are perpendicular when $\vec{u}\cdot\vec{v}=0$.
In this case, $\vec{u}\cdot\vec{v}=(t\vec{i}-\vec{j}+\vec{k})\cdot(t\vec{i}+t\vec{j}-2\vec{k})=t^2-t-2$.
This is zero when $t^2-t-2=0$, i.e. when $(t-2)(t+1)=0$, so $t=2$ or -1.
In general, $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{v}=\alpha\vec{u}$ for some real number α.
Thus we need $\alpha t\vec{i}-\alpha\vec{j}+\alpha\vec{k}=t\vec{i}+t\vec{j}-2\vec{k}$, so we need $\alpha t=t$, and $-\alpha=t$, and $\alpha=-2$. But if $\alpha=-2$, we can't have $\alpha t=t$ unless $t=0$, and if $t=0$, we can't have $-\alpha=t$, so there are no values of t for which $\vec{u}$ and $\vec{v}$ are parallel.

37. (a) Increasing $\|\vec{v}\|$ increases $\vec{v}\cdot\vec{w}$ because $\vec{v}\cdot\vec{w}=\|\vec{v}\|\,\|\vec{w}\|\cos\theta$, and $\cos\theta$ is positive.
(b) Increasing θ decreases $\vec{v}\cdot\vec{w}$ because $\cos\theta$ is a decreasing function.

38. Let

$$\vec{a}=\vec{a}_{\text{parallel}}+\vec{a}_{\text{perp}}$$

where $\vec{a}_{\text{parallel}}$ is parallel to $\vec{d}$, and $\vec{a}_{\text{perp}}$ is perpendicular to $\vec{d}$. Then $\vec{a}_{\text{parallel}}$ is the projection of $\vec{a}$ in the direction of $\vec{d}$:

$$\begin{aligned}\vec{a}_{\text{parallel}}&=\left(\vec{a}\cdot\frac{\vec{d}}{\|\vec{d}\|}\right)\frac{\vec{d}}{\|\vec{d}\|}\\&=\left((3\vec{i}+2\vec{j}-6\vec{k})\cdot\frac{(2\vec{i}-4\vec{j}+\vec{k})}{\sqrt{2^2+4^2+1^2}}\right)\frac{(2\vec{i}-4\vec{j}+\vec{k})}{\sqrt{2^2+4^2+1^2}}\\&=-\frac{8}{21}(2\vec{i}-4\vec{j}+\vec{k})\\&=-\frac{8}{21}\vec{d}\end{aligned}$$

Since we now know $\vec{a}$ and $\vec{a}_{\text{parallel}}$, we can solve for $\vec{a}_{\text{perp}}$:

$$\begin{aligned}\vec{a}_{\text{perp}}&=\vec{a}-\vec{a}_{\text{parallel}}\\&=(3\vec{i}+2\vec{j}-6\vec{k})-\left(-\frac{8}{21}\right)(2\vec{i}-4\vec{j}+\vec{k})\\&=\frac{79}{21}\vec{i}+\frac{10}{21}\vec{j}-\frac{118}{21}\vec{k}.\end{aligned}$$

Thus we can now write $\vec{a}$ as the sum of two vectors, one parallel to $\vec{d}$, the other perpendicular to $\vec{d}$:

$$\vec{a}=-\frac{8}{21}\vec{d}+\left(\frac{79}{21}\vec{i}+\frac{10}{21}\vec{j}-\frac{118}{21}\vec{k}\right)$$

39. We first find displacement vectors $\overrightarrow{AB}=(4-2)\vec{i}+(2-2)\vec{j}+(1-2)\vec{k}=2\vec{i}-\vec{k}$ and $\overrightarrow{AC}=(2-2)\vec{i}+(3-2)\vec{j}+(1-2)\vec{k}=\vec{j}-\vec{k}$. Then

$$\begin{aligned}\cos(\angle BAC)&=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|}\\&=\frac{1}{\sqrt{5}\sqrt{2}}\\&=0.3162.\end{aligned}$$

Thus angle BAC is 71.57° (or 1.25 radians.)

40. If $P = (5, 0, 0)$, $Q = (0, -3, 0)$, $R = (0, 0, 2)$, then vectors along the three sides of the triangle are

$$\overrightarrow{QP} = (5-0)\vec{i} + (0-(-3))\vec{j} = 5\vec{i} + 3\vec{j}$$
$$\overrightarrow{RP} = (5-0)\vec{i} + (0-2)\vec{k} = 5\vec{i} - 2\vec{k}$$
$$\overrightarrow{QR} = (0-(-3))\vec{j} + (2-0)\vec{k} = 3\vec{j} + 2\vec{k}$$

Thus, the lengths of the sides of the triangle are

$$\|\overrightarrow{QP}\| = \|5\vec{i} + 3\vec{j}\| = \sqrt{25+9} = \sqrt{34}$$
$$\|\overrightarrow{RP}\| = \|5\vec{i} - 2\vec{k}\| = \sqrt{25+4} = \sqrt{29}$$
$$\|\overrightarrow{QR}\| = \|3\vec{j} + 2\vec{k}\| = \sqrt{9+4} = \sqrt{13}.$$

The angle between the vectors $\vec{v}$ and $\vec{w}$ is given by

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} \quad \text{so} \quad \theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

Thus,

$$\begin{aligned}\text{Angle at } P &= \arccos\left(\frac{\overrightarrow{QP}\cdot\overrightarrow{RP}}{\|\overrightarrow{QP}\|\|\overrightarrow{RP}\|}\right)\\ &= \arccos\left(\frac{(5\vec{i} + 3\vec{j})\cdot(5\vec{i} - 2\vec{k})}{\sqrt{34}\sqrt{29}}\right)\\ &= \arccos\left(\frac{25}{\sqrt{34}\sqrt{29}}\right)\\ &= 37.235^\circ.\end{aligned}$$

$$\begin{aligned}\text{Angle at } Q &= \arccos\left(\frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\|\overrightarrow{QP}\|\|\overrightarrow{QR}\|}\right)\\ &= \arccos\left(\frac{(5\vec{i} + 3\vec{j})\cdot(3\vec{j} + 2\vec{k})}{\sqrt{34}\sqrt{13}}\right)\\ &= \arccos\left(\frac{9}{\sqrt{34}\sqrt{13}}\right)\\ &= 64.654^\circ.\end{aligned}$$

Now we use the fact that the angles must add up to 180°. Thus

$$\text{Angle at } R = 180^\circ - (37.235^\circ + 64.654^\circ) = 78.111^\circ.$$

41. (a) The points A, B and C are shown in Figure 13.27.

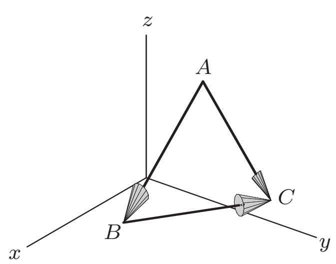


Figure 13.27

First, we calculate the vectors which form the sides of this triangle:
$\overrightarrow{AB} = (4\vec{i} + 2\vec{j} + \vec{k}) - (2\vec{i} + 2\vec{j} + 2\vec{k}) = 2\vec{i} - \vec{k}$
$\overrightarrow{BC} = (2\vec{i} + 3\vec{j} + \vec{k}) - (4\vec{i} + 2\vec{j} + \vec{k}) = -2\vec{i} + \vec{j}$
$\overrightarrow{AC} = (2\vec{i} + 3\vec{j} + \vec{k}) - (2\vec{i} + 2\vec{j} + 2\vec{k}) = \vec{j} - \vec{k}$

Now we calculate the lengths of each of the sides of the triangles:
$\|\overrightarrow{AB}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$
$\|\overrightarrow{BC}\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$
$\|\overrightarrow{AC}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
Thus the length of the shortest side of S is $\sqrt{2}$.

(b) $\cos \angle BAC = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\|} = \frac{2\cdot 0 + 0\cdot 1 + (-1)\cdot(-1)}{\sqrt{5}\cdot\sqrt{2}} \approx 0.32$

42. (a) We first find the unit vector in direction $\vec{v}$. Since $\|\vec{v}\| = \sqrt{3^2 + 4^2} = 5$, the unit vector in direction of $\vec{v}$ is $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$. Then

$$\begin{aligned} \vec{F}_{\text{parallel}} &= (\vec{F} \cdot \vec{u})\vec{u} \\ &= (4 \cdot 0.6 + 1 \cdot 0.8)\vec{u} \\ &= 3.2\vec{u} \\ &= 1.92\vec{i} + 2.56\vec{j}. \end{aligned}$$

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = (4\vec{i} + \vec{j}) - (1.92\vec{i} + 2.56\vec{j}) = 2.08\vec{i} - 1.56\vec{j}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = 4 \cdot 3 + 1 \cdot 4 = 16.$$

43. (a) We first find the unit vector in direction $\vec{v}$. Since $\|\vec{v}\| = \sqrt{3^2 + 4^2} = 5$, the unit vector in direction of $\vec{v}$ is $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$. Then

$$\begin{aligned} \vec{F}_{\text{parallel}} &= (\vec{F} \cdot \vec{u})\vec{u} \\ &= (0.2 \cdot 0.6 - 0.5 \cdot 0.8)\vec{u} \\ &= -0.28\vec{u} \\ &= -0.168\vec{i} - 0.224\vec{j}. \end{aligned}$$

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = (0.2\vec{i} - 0.5\vec{j}) - (-0.168\vec{i} - 0.224\vec{j}) = 0.368\vec{i} - 0.276\vec{j}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = 0.2 \cdot 3 - 0.5 \cdot 4 = -1.4.$$

44. (a) We first find the unit vector in direction $\vec{v}$. Since $\|\vec{v}\| = \sqrt{3^2 + 4^2} = 5$, the unit vector in direction of $\vec{v}$ is $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$. Then

$$\begin{aligned} \vec{F}_{\text{parallel}} &= (\vec{F} \cdot \vec{u})\vec{u} \\ &= (9 \cdot 0.6 + 12 \cdot 0.8)\vec{u} \\ &= 15\vec{u} \\ &= 9\vec{i} + 12\vec{j}. \end{aligned}$$

Notice that the component of $\vec{F}$ in direction $\vec{v}$ is equal to $\vec{F}$. This makes sense (and could have been predicted) since $\vec{F}$ is parallel to $\vec{v}$.

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \vec{0}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = 9 \cdot 3 + 12 \cdot 4 = 75.$$

45. (a) We first find the unit vector in direction $\vec{v}$. Since $||\vec{v}|| = \sqrt{3^2+4^2} = 5$, the unit vector in direction of $\vec{v}$ is $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$. Then

$$\begin{aligned}\vec{F}_{\text{parallel}} &= (\vec{F}\cdot\vec{u})\vec{u}\\ &= (-0.4\cdot 0.6 + 0.3\cdot 0.8)\vec{u}\\ &= \vec{0}.\end{aligned}$$

Notice that the component of $\vec{F}$ in direction $\vec{v}$ is equal to $\vec{0}$. This makes sense (and could have been predicted) since $\vec{F}$ is perpendicular to $\vec{v}$.

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \vec{F}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F}\cdot\vec{v} = -0.4\cdot 3 + 0.3\cdot 4 = 0.$$

Notice that since the force is perpendicular to the displacement, the work done is zero.

46. (a) We first find the unit vector in direction $\vec{v}$. Since $||\vec{v}|| = \sqrt{3^2+4^2} = 5$, the unit vector in direction of $\vec{v}$ is $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$. Then

$$\begin{aligned}\vec{F}_{\text{parallel}} &= (\vec{F}\cdot\vec{u})\vec{u}\\ &= (-3\cdot 0.6 - 5\cdot 0.8)\vec{u}\\ &= -5.8\vec{u}\\ &= -3.48\vec{i} - 4.64\vec{j}.\end{aligned}$$

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = (-3\vec{i} - 5\vec{j}) - (-3.48\vec{i} - 4.64\vec{j}) = 0.48\vec{i} - 0.36\vec{j}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F}\cdot\vec{v} = -3\cdot 3 - 5\cdot 4 = -29.$$

47. (a) We first find the unit vector in direction $\vec{v}$. Since $||\vec{v}|| = \sqrt{3^2+4^2} = 5$, the unit vector in direction of $\vec{v}$ is $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$. Then

$$\begin{aligned}\vec{F}_{\text{parallel}} &= (\vec{F}\cdot\vec{u})\vec{u}\\ &= (-6\cdot 0.6 - 8\cdot 0.8)\vec{u}\\ &= -10\vec{u}\\ &= -6\vec{i} - 8\vec{j}\\ &= \vec{F}.\end{aligned}$$

Notice that $\vec{F}$ is in the opposite direction from $\vec{v}$ so it makes sense (and could have been predicted) that $\vec{F}_{\text{parallel}} = \vec{F}$.

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \vec{0}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F}\cdot\vec{v} = -6\cdot 3 - 8\cdot 4 = -50.$$

48. (a) We first find the unit vector in direction $\vec{v}$. Since $||\vec{v}|| = \sqrt{2^2+3^2} = \sqrt{13}$, the unit vector in direction of $\vec{v}$ is $\vec{u} = \vec{v}/\sqrt{13}$. Then

$$\begin{aligned}\vec{F}_{\text{parallel}} &= (\vec{F}\cdot\vec{u})\vec{u}\\ &= (-60/\sqrt{13})\vec{u}\\ &= \frac{-120}{13}\vec{i} + \frac{-180}{13}\vec{j}\\ &= -9.231\vec{i} - 13.846\vec{j}.\end{aligned}$$

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = (-20\vec{j}) - (-9.231\vec{i} - 13.846\vec{j}) = 9.231\vec{i} - 6.154\vec{j}).$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = -60.$$

49. (a) We first find the unit vector in direction $\vec{v}$. Since $||\vec{v}|| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$, the unit vector in direction of $\vec{v}$ is $\vec{u} = \vec{v}/\sqrt{26}$. Then

$$\begin{aligned}\vec{F}_{\text{parallel}} &= (\vec{F} \cdot \vec{u})\vec{u} \\ &= (20/\sqrt{26})\vec{u} \\ &= \frac{100}{26}\vec{i} + \frac{-20}{26}\vec{j} \\ &= 3.846\vec{i} - 0.769\vec{j}.\end{aligned}$$

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = (-20\vec{j}) - (3.846\vec{i} - 0.769\vec{j}) = -3.846\vec{i} - 19.231\vec{j}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = 20.$$

50. (a) Notice that $\vec{v}$ is parallel to $\vec{F}$ (although in the opposite direction.) Therefore, $\vec{F}_{\text{parallel}} = \vec{F}$.

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \vec{0}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = -60.$$

51. (a) Notice that $\vec{v}$ is perpendicular to $\vec{F}$. Therefore, $\vec{F}_{\text{parallel}} = \vec{0}$.

(b) We have

$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \vec{F}.$$

(c) Since work is the dot product of the force and displacement vectors, we have

$$W = \vec{F} \cdot \vec{v} = 0.$$

The work is zero since the direction of the force is perpendicular to the direction of the displacement.

52. Let the room be put in the coordinate system as shown in Figure 13.28.

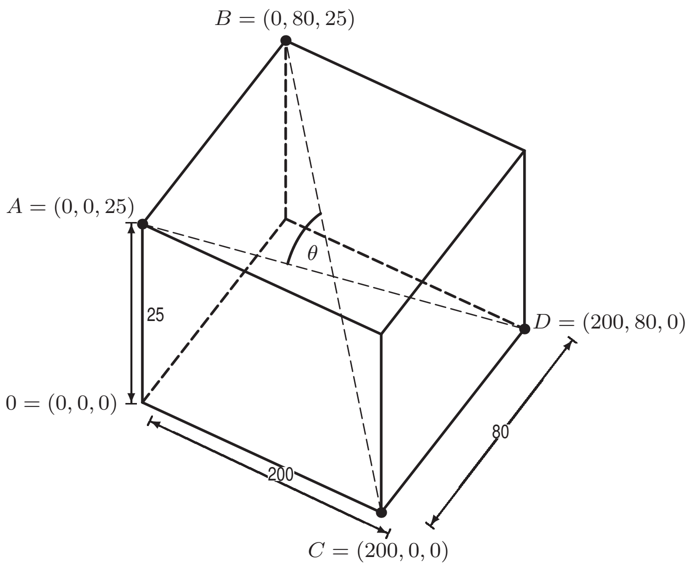


Figure 13.28

Then the vectors of the two strings are given by:
$\overrightarrow{AD} = (200\vec{i} + 80\vec{j} + 0\vec{k}) - (0\vec{i} + 0\vec{j} + 25\vec{k}) = 200\vec{i} + 80\vec{j} - 25\vec{k}$
$\overrightarrow{BC} = (200\vec{i} + 0\vec{j} + 0\vec{k}) - (0\vec{i} + 80\vec{j} + 25\vec{k}) = 200\vec{i} - 80\vec{j} - 25\vec{k}$.
Let the angle between $\overrightarrow{AD}$ and $\overrightarrow{BC}$ be θ. Then we have

$$\begin{aligned}\cos\theta &= \frac{\overrightarrow{AD}\cdot\overrightarrow{BC}}{\|\overrightarrow{AD}\|\,\|\overrightarrow{BC}\|}\\ &= \frac{200(200)+(80)(-80)+(-25)(-25)}{\sqrt{200^2+80^2+(-25)^2}\sqrt{(200)^2+(-80)^2+(-25)^2}}\\ &= \frac{34225}{47025}\\ &= 0.727804\end{aligned}$$

53. We need to find the speed of the wind in the direction of the track. Looking at Figure 13.29, we see that we want the component of $\vec{w}$ in the direction of $\vec{v}$. We calculate

$$\begin{aligned}\|\vec{w}_{\text{parallel}}\| = \|\vec{w}\|\cos\theta = \frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|} &= \frac{(5\vec{i}+\vec{j})\cdot(2\vec{i}+6\vec{j})}{\|2\vec{i}+6\vec{j}\|}\\ &= \frac{16}{\sqrt{40}} \approx 2.53\\ &< 5\end{aligned}$$

Therefore, the race results will not be disqualified.

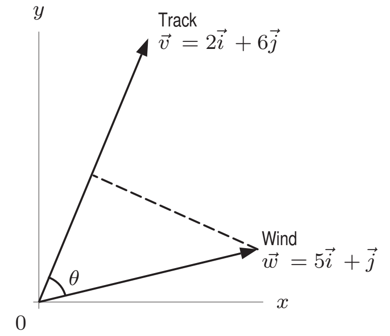


Figure 13.29

54. We find the component of the wind $\vec{w}$ in the direction of the airplane. A direction vector for the airplane is $\vec{v} = \vec{i} - \vec{j}$. The component of $\vec{w}_1$ in the direction of $\vec{v}$ is given by

$$\frac{\vec{w}_1\cdot\vec{v}}{\|\vec{v}\|} = \frac{-4\cdot 1 + (-1)\cdot(-1)}{\sqrt{2}} = -2.12.$$

Similarly, we find the component for each wind vector in the direction of the airplane. We see that, in the direction of the airplane, the component of $\vec{w}_1$ is -2.12, of $\vec{w}_2$ is 2.12, of $\vec{w}_3$ is -6.36, of $\vec{w}_4$ is 5.66, and of $\vec{w}_5$ is 4.95. The vector $\vec{w}_4$ increases the plane's speed the most and the vector $\vec{w}_3$ slows the plane down the most.

55. (a) The speed of the current is $\|\vec{c}\| = \sqrt{5} = 2.24$ m/sec.
(b) The speed of the current in the direction of the canoe's motion is the component of $\vec{c}$ in the direction of $\vec{v}$. This is given by:

$$\begin{aligned}\text{Speed of current in direction of canoe's motion} = \frac{\vec{c}\cdot\vec{v}}{\|\vec{v}\|} &= \frac{(1)(5)+(2)(3)}{\sqrt{5^2+3^2}}\\ &= \frac{11}{\sqrt{34}}\\ &= 1.89 \text{ m/sec.}\end{aligned}$$

Notice that the speed of the current in the direction of the canoe is less than the speed of the current in the direction in which the current is moving.

56. Let $\vec{u} = 3\vec{i} + 4\vec{j}$ and $\vec{v} = 5\vec{i} - 12\vec{j}$. We seek a vector $\vec{w} = x\vec{i} + y\vec{j}$ such that the cosine of the angle between $\vec{u}$ and $\vec{w}$ equals the cosine of the angle between $\vec{v}$ and $\vec{w}$. Thus

$$\frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\|\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$$

or

$$\frac{3x + 4y}{5\sqrt{x^2 + y^2}} = \frac{5x - 12y}{13\sqrt{x^2 + y^2}}.$$

Simplifying, we have $x = -8y$. The vector we want is of the form $\vec{w} = -8y\vec{i} + y\vec{j}$, but should we take $y > 0$ or $y < 0$? The smaller of the two angles formed by $\vec{u}$ and $\vec{v}$ is between 0° and 180^0, and so $\vec{w}$ must make an acute angle with $\vec{u}$ and $\vec{v}$. If $y > 0$ then $\vec{u} \cdot \vec{w} = -20y < 0$ indicating an obtuse angle and if $y < 0$ then $\vec{u} \cdot \vec{w} = -20y > 0$ indicating an acute angle. We have $\vec{w} = -8y\vec{i} + y\vec{j}$ with $y < 0$. Thus $\vec{w}$ can be any positive multiple of the vector $8\vec{i} - \vec{j}$.

57. The planes are parallel, with normal vectors $\vec{n} = 2\vec{i} - 5\vec{j} + \vec{k}$. Pick any point on $2x - 5y + z = 10$, say $(5, 0, 0)$, and any point on $z = 5y - 2x$, say $(0, 0, 0)$. The vector between them is $\vec{d} = 5\vec{i}$, so we want to find the magnitude of the component of $\vec{d}$ in the direction of $\vec{n}$. A unit vector in the direction of $\vec{n}$ is

$$\vec{v} = \frac{1}{\sqrt{2^2 + (-5)^2 + 1^2}}\left(2\vec{i} - 5\vec{j} + \vec{k}\right),$$

so

$$\text{Distance} = |\vec{d} \cdot \vec{v}| = \frac{5\vec{i} \cdot (2\vec{i} - 5\vec{j} + \vec{k})}{\sqrt{30}} = \frac{10}{\sqrt{30}}.$$

58. We have

$$\begin{aligned}\vec{p} \cdot \vec{q} &= (1.00)(43) + (3.50)(57) + (4.00)(12) + (2.75)(78) + (5.00)(20) + (3.00)(35)\\ &= 710 \text{ dollars.}\end{aligned}$$

The vendor took in \$710 in from sales. The quantity $\vec{p} \cdot \vec{q}$ represents the total revenue earned.

59. The vector $\vec{a}$ represents the averages of the exams, written as decimals. The vector $\vec{w}$ represents the weightings.

$$\vec{w} \cdot \vec{a} = 0.1 \cdot 0.75 + 0.15 \cdot 0.91 + 0.25 \cdot 0.84 + 0.5 \cdot 0.87 = 0.8565 = 85.65\%$$

The dot product, 86.65%, represents the class average of the four exams in the course.

60. If $\vec{x}$ and $\vec{y}$ are two consumption vectors corresponding to points satisfying the same budget constraint, then

$$\vec{p} \cdot \vec{x} = k = \vec{p} \cdot \vec{y}.$$

Therefore we have

$$\vec{p} \cdot (\vec{x} - \vec{y}) = \vec{p} \cdot \vec{x} - \vec{p} \cdot \vec{y} = 0.$$

Thus $\vec{p}$ and $\vec{x} - \vec{y}$ are perpendicular; that is, the difference between two consumption vectors on the same budget constraint is perpendicular to the price vector.

61. Property 2 says that multiplying one of the vectors by a scalar simply multiplies the dot product by the same scalar. If $\lambda > 0$, then when one vector is multiplied by λ, the angle between the vectors does not change, but the length of one vector, and hence the dot product, is multiplied by λ. The result remains true when $\lambda < 0$. For a justification in the case when $\lambda < 0$, see Problem 67 on page 1235.

62. We want to show that $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ and $\vec{c}$ are perpendicular. We do this by taking their dot product:

$$((\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}) \cdot \vec{c} = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{c}) = 0.$$

Since the dot product is 0, the vectors $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ and $\vec{c}$ are perpendicular.

63. Suppose $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

- Property 1:

  We calculate both $\vec{v} \cdot \vec{w}$ and $\vec{w} \cdot \vec{v}$ using the algebraic definition of the dot product:

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

$$\vec{w} \cdot \vec{v} = w_1v_1 + w_2v_2 + w_3v_3$$

  But since ordinary multiplication of scalars is commutative, $v_1w_1 = w_1v_1$ and so on. Therefore

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}.$$

- Property 2:

  First we observe that

$$\lambda\vec{w} = \lambda(w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = (\lambda w_1)\vec{i} + (\lambda w_2)\vec{j} + (\lambda w_3)\vec{k}$$

$$\lambda\vec{v} = \lambda(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) = (\lambda v_1)\vec{i} + (\lambda v_2)\vec{j} + (\lambda v_3)\vec{k}.$$

  Now we calculate the three quantities $\vec{v}\cdot(\lambda\vec{w})$ and $\lambda(\vec{v}\cdot\vec{w})$ and $(\lambda\vec{v})\cdot\vec{w}$

$$\begin{aligned}\vec{v}\cdot(\lambda\vec{w}) &= v_1(\lambda w_1) + v_2(\lambda w_2) + v_3(\lambda w_3)\\ \lambda(\vec{v}\cdot\vec{w}) &= \lambda(v_1w_1 + v_2w_2 + v_3w_3)\\ (\lambda\vec{v})\cdot\vec{w} &= (\lambda v_1)w_1 + (\lambda v_2)w_2 + (\lambda v_3)w_3\end{aligned}$$

  Since ordinary multiplication is associative and commutative, we know that $v_1(\lambda w_1) = \lambda v_1 w_1 = (\lambda v_1)w_1$ and so on. Thus, we have $\vec{v}\cdot(\lambda\vec{w}) = (\lambda\vec{v})\cdot\vec{w}$.

  In addition, the distributive property of ordinary multiplication tells us that

$$\lambda(v_1w_1 + v_2w_2 + v_3w_3) = \lambda v_1w_1 + \lambda v_2w_2 + \lambda v_3w_3$$

  Thus, we know that all three quantities are equal

$$\vec{v}\cdot(\lambda\vec{w}) = \lambda(\vec{v}\cdot\vec{w}) = (\lambda\vec{v})\cdot\vec{w}$$

- Property 3:

  First we observe that

$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{i} + (v_2 + w_2)\vec{j} + (v_3 + w_3)\vec{k}.$$

  Next we calculate the quantities $((\vec{v} + \vec{w})\cdot\vec{u})$ and $(\vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u})$.

$$\begin{aligned}(\vec{v} + \vec{w})\cdot\vec{u} &= (v_1 + w_1)u_1 + (v_2 + w_2)u_2 + (v_3 + w_3)u_3\\ \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u} &= (v_1u_1 + v_2u_2 + v_3u_3) + (w_1u_1 + w_2u_2 + w_3u_3).\end{aligned}$$

  The distributive law of ordinary multiplication shows that $(v_1 + w_1)u_1 = v_1u_1 + w_1u_1$, and so on. Thus, the dot product is distributive also:

$$(\vec{v} + \vec{w})\cdot\vec{u} = \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u}$$

64. Since $\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$, $(\vec{u} - \vec{v})\cdot\vec{w} = 0$. This equality holds for any $\vec{w}$, so we can take $\vec{w} = \vec{u} - \vec{v}$. This gives

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v})\cdot(\vec{u} - \vec{v}) = 0,$$

that is,

$$\|\vec{u} - \vec{v}\| = 0.$$

This implies $\vec{u} - \vec{v} = 0$, that is, $\vec{u} = \vec{v}$.

65. Since the dot product of a vector with itself is the square of the vector's magnitude, we can check that two vectors have the same magnitude by computing dot products. We have

$$\begin{aligned}\left(\frac{\vec{u}}{\|\vec{u}\|^2} - \frac{\vec{v}}{\|\vec{v}\|^2}\right)\cdot\left(\frac{\vec{u}}{\|\vec{u}\|^2} - \frac{\vec{v}}{\|\vec{v}\|^2}\right) &= \frac{\vec{u}\cdot\vec{u}}{\|\vec{u}\|^4} - \frac{\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u}}{\|\vec{u}\|^2\|\vec{v}\|^2} + \frac{\vec{v}\cdot\vec{v}}{\|\vec{v}\|^4}\\ &= \frac{1}{\|\vec{u}\|^2} - \frac{2\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2\|\vec{v}\|^2} + \frac{1}{\|\vec{v}\|^2}.\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\vec{u}}{\|\vec{u}\|\|\vec{v}\|} - \frac{\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)\cdot\left(\frac{\vec{u}}{\|\vec{u}\|\|\vec{v}\|} - \frac{\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) &= \frac{\vec{u}\cdot\vec{u}}{\|\vec{u}\|^2\|\vec{v}\|^2} - \frac{\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u}}{\|\vec{u}\|^2\|\vec{v}\|^2} + \frac{\vec{v}\cdot\vec{v}}{\|\vec{u}\|^2\|\vec{v}\|^2}\\ &= \frac{1}{\|\vec{v}\|^2} - \frac{2\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2\|\vec{v}\|^2} + \frac{1}{\|\vec{u}\|^2}\end{aligned}$$

The two dot products are equal, which shows that the vectors $\vec{u}/\|\vec{u}\|^2 - \vec{v}/\|\vec{v}\|^2$ and $\vec{u}/(\|\vec{u}\|\|\vec{v}\|) - \vec{v}/(\|\vec{u}\|\|\vec{v}\|)$ have the same magnitude.

66. If $\vec{u} = \vec{0}$, then both sides of the equation are zero. If $\vec{u} \neq \vec{0}$, write $\vec{v}_{\text{parallel}}$, $\vec{w}_{\text{parallel}}$, and $(\vec{v} + \vec{w})_{\text{parallel}}$ for the components of $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ in the direction of $\vec{u}$. Then Figure 13.34 shows that

$$\vec{v}_{\text{parallel}} + \vec{w}_{\text{parallel}} = (\vec{v} + \vec{w})_{\text{parallel}}.$$

So

$$\left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} + \left(\frac{\vec{w} \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} = \left(\frac{(\vec{v} + \vec{w}) \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u}.$$

Thus, since $\vec{u} \neq \vec{0}$, we deduce that

$$\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} + \frac{\vec{w} \cdot \vec{u}}{\|\vec{u}\|^2} - \frac{(\vec{v} + \vec{w}) \cdot \vec{u}}{\|\vec{u}\|^2} = 0,$$

so

$$\vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = (\vec{v} + \vec{w}) \cdot \vec{u}.$$

67. Suppose θ is the angle between $\vec{u}$ and $\vec{v}$.

(a) By the definition of scalar multiplication, we know that $-\vec{v}$ is in the opposite direction of $\vec{v}$, so the angle between $\vec{u}$ and $-\vec{v}$ is $\pi - \theta$. (See Figure 13.30.) Hence,

$$\begin{aligned}\vec{u} \cdot (-\vec{v}) &= \|\vec{u}\|\|-\vec{v}\|\cos(\pi - \theta)\\ &= \|\vec{u}\|\|\vec{v}\|(-\cos\theta)\\ &= -(\vec{u} \cdot \vec{v})\end{aligned}$$

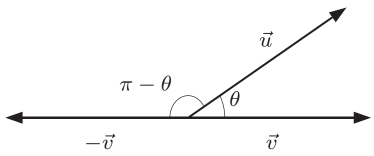


Figure 13.30

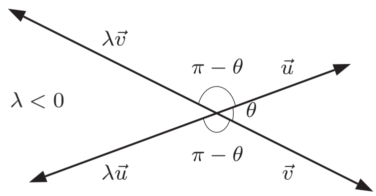


Figure 13.31

(b) If $\lambda < 0$, the angle between $\vec{u}$ and $\lambda\vec{v}$ is $\pi - \theta$, and so is the angle between $\lambda\vec{u}$ and $\vec{v}$. (See Figure 13.31.) So we have,

$$\begin{aligned}\vec{u} \cdot (\lambda\vec{v}) &= \|\vec{u}\|\|\lambda\vec{v}\|\cos(\pi - \theta)\\ &= |\lambda|\|\vec{u}\|\|\vec{v}\|(-\cos\theta)\\ &= -\lambda\|\vec{u}\|\|\vec{v}\|(-\cos\theta) \quad \text{since } |\lambda| = -\lambda\\ &= \lambda\|\vec{u}\|\|\vec{v}\|\cos\theta\\ &= \lambda(\vec{u} \cdot \vec{v})\end{aligned}$$

By a similar argument, we have

$$\begin{aligned}(\lambda\vec{u}) \cdot \vec{v} &= \|\lambda\vec{u}\|\|\vec{v}\|\cos(\pi - \theta)\\ &= -\lambda\|\vec{u}\|\|\vec{v}\|(-\cos\theta)\\ &= \lambda(\vec{u} \cdot \vec{v})\end{aligned}$$

68. Let $\vec{u}$ and $\vec{v}$ be the displacement vectors from C to the other two vertices. Then

$$\begin{aligned}c^2 &= \|\vec{u} - \vec{v}\|^2\\ &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})\\ &= \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}\\ &= \|\vec{u}\|^2 - 2\|u\|\|v\|\cos C + \|\vec{v}\|^2\\ &= a^2 - 2ab\cos C + b^2\end{aligned}$$

69. We substitute $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and by the result of Problem 66, we expand as follows:

$$\begin{aligned}(\vec{u}\cdot\vec{v})_{\text{geom}} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k})\cdot\vec{v} \\ &= (u_1\vec{i})\cdot\vec{v} + (u_2\vec{j})\cdot\vec{v} + (u_3\vec{k})\cdot\vec{v}\end{aligned}$$

where all the dot products are defined geometrically By the result of Problem 67 we can write

$$(\vec{u}\cdot\vec{v})_{\text{geom}} = u_1(\vec{i}\cdot\vec{v})_{\text{geom}} + u_2(\vec{j}\cdot\vec{v})_{\text{geom}} + u_3(\vec{k}\cdot\vec{v})_{\text{geom}}.$$

Now substitute $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and expand, again using Problem 66 and the geometric definition of the dot product:

$$\begin{aligned}(\vec{u}\cdot\vec{v})_{\text{geom}} &= u_1\left(\vec{i}\cdot(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})\right)_{\text{geom}} \\ &\quad + u_2\left(\vec{j}\cdot(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})\right)_{\text{geom}} \\ &\quad + u_3\left(\vec{k}\cdot(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})\right)_{\text{geom}} \\ &= u_1v_1(\vec{i}\cdot\vec{i})_{\text{geom}} + u_1v_2(\vec{i}\cdot\vec{j})_{\text{geom}} + u_1v_3(\vec{i}\cdot\vec{k})_{\text{geom}} \\ &\quad + u_2v_1(\vec{i}\cdot\vec{i})_{\text{geom}} + u_2v_2(\vec{i}\cdot\vec{j})_{\text{geom}} + u_2v_3(\vec{i}\cdot\vec{k})_{\text{geom}} \\ &\quad + u_3v_1(\vec{i}\cdot\vec{i})_{\text{geom}} + u_3v_2(\vec{i}\cdot\vec{j})_{\text{geom}} + u_3v_3(\vec{i}\cdot\vec{k})_{\text{geom}}\end{aligned}$$

The geometric definition of the dot product shows that

$$\begin{aligned}\vec{i}\cdot\vec{i} &= \|\vec{i}\|\,\|\vec{i}\|\cos 0 = 1 \\ \vec{i}\cdot\vec{j} &= \|\vec{i}\|\,\|\vec{j}\|\cos\frac{\pi}{2} = 0.\end{aligned}$$

Similarly $\vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ and $\vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$. Thus, the expression for $(\vec{u}\cdot\vec{v})_{\text{geom}}$ becomes

$$\begin{aligned}(\vec{u}\cdot\vec{v})_{\text{geom}} &= u_1v_1(1) + u_1v_2(0) + u_1v_3(0) \\ &\quad + u_2v_1(0) + u_2v_2(1) + u_2v_3(0) \\ &\quad + u_3v_1(0) + u_3v_2(0) + u_3v_3(1) \\ &= u_1v_1 + u_2v_2 + u_3v_3.\end{aligned}$$

70. (a) Since $q(t) = (\vec{v} + t\vec{w})\cdot(\vec{v} + t\vec{w}) = \|\vec{v} + t\vec{w}\|^2$ and since the length of any vector is nonnegative, we must have

$$q(t) = \|\vec{v} + t\vec{w}\|^2 \geq 0$$

for all real t.

(b) Using the distributive law

$$\begin{aligned}q(t) &= (\vec{v} + t\vec{w})\cdot(\vec{v} + t\vec{w}) = \vec{v}\cdot\vec{v} + t\vec{w}\cdot\vec{v} + \vec{v}\cdot t\vec{w} + t^2\vec{w}\cdot\vec{w} \\ &= \|\vec{v}\|^2 + 2(\vec{v}\cdot\vec{w})t + \|\vec{w}\|^2t^2.\end{aligned}$$

If $\vec{w} \neq 0$, then $\|\vec{w}\| \neq 0$ and $q(t)$ is quadratic in t.

(c) Since $q(t) \geq 0$, the quadratic has one repeated root or no roots, so the discriminant must be less than or equal to zero. Thus,

$$(2\vec{v}\cdot\vec{w})^2 - 4\|\vec{v}\|^2\|\vec{w}\|^2 \leq 0.$$

Taking square roots, we have

$$|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

If $\vec{w} = 0$, then $q(t)$ is no longer a quadratic. However, in that case,

$$|\vec{v}\cdot\vec{w}| = 0 = \|\vec{v}\|\|\vec{w}\|$$

so the inequality still holds.

Strengthen Your Understanding

71. The expression $(\vec{u}\cdot\vec{v})\cdot\vec{w}$ does not make sense. You cannot take the dot product of the scalar $\vec{u}\cdot\vec{v}$ and the vector $\vec{w}$.

72. The formula $\vec{v}_{\text{parallel}} = (\vec{u} \cdot \vec{v})\vec{u}$ works only when $\vec{u}$ is a unit vector. We can see that $\vec{v}_{\text{parallel}} \neq 3(\vec{i} + \vec{j})$ because

$$\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}} = (2\vec{i} + \vec{j}) - (3\vec{i} + 3\vec{j}) = -\vec{i} - 2\vec{j},$$

which is not perpendicular to $\vec{u} = \vec{i} + \vec{j}$.

73. Given an equation for a plane, to find the normal vector, we put x, y, z all on the same side of the equation. This gives $2x + 3y - z = 0$. Then a normal vector is $2\vec{i} + 3\vec{j} - \vec{k}$.

74. The displacement vector from $(1, 1)$ to (a, b) is $(a-1)\vec{i} + (b-1)\vec{j}$, so we want

$$((a-1)\vec{i} + (b-1)\vec{j}) \cdot (\vec{i} + 2\vec{j}) = (a-1) + 2(b-1) = a + 2b - 3 = 0.$$

For example, if $b = 3$, then $a = 3 - 2(3) = -3$. In general, any (a, b) with $a = 3 - 2b$ is a possible example.

75. A plane perpendicular to $\vec{i} + 2\vec{j} + 3\vec{k}$ is $x + 2y + 3z = 0$. Solving for z, we get $z = (-1/3)x + (-2/3)y$, so we can let $f(x, y) = (-1/3)x + (-2/3)y$.

76. False. The dot product is a scalar.

77. True. Components of a normal vector can be read directly from coefficients of x, y and z in the equation for a plane.

78. True. The cosine of the angle between the vectors is negative when the angle is between $\pi/2$ and π.

79. False. The equation $z = x + y$ has normal $\vec{i} + \vec{j} - \vec{k}$, which is not parallel to $\vec{i} + \vec{j} + \vec{k}$. An equation satisfying the given conditions is $x + y + z = 6$.

80. True. The vector from $(0, 1, 0)$ to $(1, 1, 0)$ is $\vec{i}$, while the vector from $(0, 1, 0)$ to $(0, 1, 1)$ is $\vec{k}$, and $\vec{i} \cdot \vec{k} = 0$.

81. True. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$, which cannot be negative.

82. False. If the vectors are nonzero and perpendicular, the dot product will be zero (e.g. $\vec{i} \cdot \vec{j} = 0$).

83. False. If $\vec{v}$ and $\vec{w}$ are different vectors, but both are perpendicular to $\vec{u}$, then both $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$ are zero, yet $\vec{v} \neq \vec{w}$. For example, take $\vec{u} = \vec{i}$, $\vec{v} = \vec{j}$ and $\vec{w} = \vec{k}$.

84. True. Using the distributive property, and the fact that $\vec{u} \cdot (-\vec{v}) = -\vec{u} \cdot \vec{v}$, we have

$$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{u} \cdot (-\vec{v}) + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 - \|\vec{v}\|^2.$$

85. True. This vector is $\vec{v}_{\text{Perp}}$, the component of $\vec{v}$ perpendicular to the unit vector $\vec{u}$. To check, calculate the dot product

$$\vec{u} \cdot (\vec{v} - (\vec{v} \cdot \vec{u})\vec{u}) = \vec{u} \cdot \vec{v} - (\vec{v} \cdot \vec{u})(\vec{u} \cdot \vec{u}) = \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} = 0,$$

since $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1$.

Solutions for Section 13.4

Exercises

1. $\vec{v} \times \vec{w} = \vec{k} \times \vec{j} = -\vec{i}$ (remember $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the axes, and you must use the right hand rule.)

2. $\vec{v} = -\vec{i}$, and $\vec{w} = \vec{j} + \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{j} - \vec{k}$$

3. $\vec{v} = \vec{i} + \vec{k}$, and $\vec{w} = \vec{i} + \vec{j}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} + \vec{k}$$

4. $\vec{v} = \vec{i} + \vec{j} + \vec{k}$, and $\vec{w} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\vec{i} + 2\vec{j}$$

5. $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$, and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} + 3\vec{j} + 7\vec{k}$$

6. $\vec{v} = 2\vec{i} - \vec{j} - \vec{k}$, and $\vec{w} = -6\vec{i} + 3\vec{j} + 3\vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ -6 & 3 & 3 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}.$$

7. Since $\vec{v} = -3\vec{i} + 5\vec{j} + 4\vec{k}$ and $\vec{w} = \vec{i} - 3\vec{j} - \vec{k}$,

$$\begin{aligned} \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 5 & 4 \\ 1 & -3 & -1 \end{vmatrix} &= (5(-1) - 4(-3))\vec{i} - ((-3)(-1) - 4(1))\vec{j} + ((-3)(-3) - 5(1))\vec{k} \\ &= (-5 + 12)\vec{i} - (3 - 4)\vec{j} + (9 - 5)\vec{k} \\ &= 7\vec{i} + \vec{j} + 4\vec{k}. \end{aligned}$$

8.

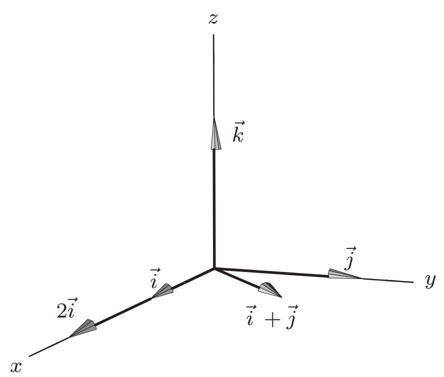


Figure 13.32

By the definition of cross product, $2\vec{i} \times (\vec{i} + \vec{j})$ is in the direction of $\vec{k}$. The magnitude of it equals to the area of the parallelogram which is

$$\|2\vec{i}\| \cdot \|\vec{i} + \vec{j}\| \sin\frac{\pi}{4} = 2\sqrt{2}\sin\frac{\pi}{4} = 2\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2.$$

So $2\vec{i} \times (\vec{i} + \vec{j}) = 2\vec{k}$. See Figure 13.32.

9.

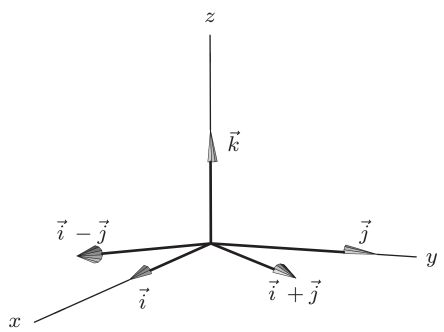


Figure 13.33

By definition, $(\vec{i}+\vec{j})\times(\vec{i}-\vec{j})$ is in the direction of $-\vec{k}$. The magnitude is

$$\|\vec{i}+\vec{j}\|\cdot\|\vec{i}-\vec{j}\|\sin\frac{\pi}{2}=\sqrt{2}\cdot\sqrt{2}\cdot 1=2.$$

So $(\vec{i}+\vec{j})\times(\vec{i}-\vec{j})=-2\vec{k}$. See Figure 13.33.

10.

$$\begin{aligned}[(\vec{i}+\vec{j})\times\vec{i}]\times\vec{j} &= (\vec{i}\times\vec{i}+\vec{j}\times\vec{i})\times\vec{j}\\ &= (\vec{0}-\vec{k})\times\vec{j}\\ &= -\vec{k}\times\vec{j}\\ &= \vec{j}\times\vec{k}=\vec{i}.\end{aligned}$$

11.

$$\begin{aligned}(\vec{i}+\vec{j})\times(\vec{i}\times\vec{j}) &= (\vec{i}+\vec{j})\times\vec{k}\\ &= (\vec{i}\times\vec{k})+(\vec{j}\times\vec{k})\\ &= -\vec{j}+\vec{i}=\vec{i}-\vec{j}.\end{aligned}$$

12.

$$\begin{aligned}\vec{a}\times\vec{b} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 1 & -1\\ 1 & -4 & 2\end{vmatrix}\\ &= \begin{vmatrix}1 & -1\\ -4 & 2\end{vmatrix}\vec{i}-\begin{vmatrix}3 & -1\\ 1 & 2\end{vmatrix}\vec{j}+\begin{vmatrix}3 & 1\\ 1 & -4\end{vmatrix}\vec{k}\\ &= -2\vec{i}-7\vec{j}-13\vec{k}.\end{aligned}$$

Since

$$\vec{a}\cdot(\vec{a}\times\vec{b})=3(-2)+(-7)-(-13)=0$$

and

$$\vec{b}\cdot(\vec{a}\times\vec{b})=1(-2)-4(-7)+2(-13)=0,$$

$\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

13. We find that $\vec{v}\times\vec{w}=-6\vec{i}+7\vec{j}+8\vec{k}$ and $\vec{w}\times\vec{v}=6\vec{i}-7\vec{j}-8\vec{k}$. Notice that

$$\vec{v}\times\vec{w}=-(\vec{w}\times\vec{v}).$$

14. We can form the displacement vectors $\vec{a} = -\vec{i} + \vec{j} + 0\vec{k}$ from $(1,0,0)$ to $(0,1,0)$ and $\vec{b} = -\vec{i} + 0\vec{j} + \vec{k}$ from $(1,0,0)$ to $(0,0,1)$. A normal vector to the plane is $\vec{a} \times \vec{b} = \vec{i} + \vec{j} + \vec{k}$. Using the point $(1,0,0)$, the plane can be written as $(x-1)+y+z=0$ or $x+y+z=1$.

15. The displacement vector from $(3,4,2)$ to $(-2,1,0)$ is:

$$\vec{a} = -5\vec{i} - 3\vec{j} - 2\vec{k}.$$

The displacement vector from $(3,4,2)$ to $(0,2,1)$ is:

$$\vec{b} = -3\vec{i} - 2\vec{j} - \vec{k}.$$

Therefore the vector normal to the plane is:

$$\vec{n} = \vec{a} \times \vec{b} = -\vec{i} + \vec{j} + \vec{k}.$$

Using the first point, the equation of the plane can be written as:

$$-(x-3)+(y-4)+(z-2)=0.$$

The equation of the plane is thus:

$$-x+y+z=3.$$

16. We first calculate the cross product:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 4 & 3 \\ 1 & 1 & 1 \end{vmatrix} = (4-3)\vec{i} - (5-3)\vec{j} + (5-4)\vec{k} = \vec{i} - \vec{j} + \vec{k}.$$

Then

$$\text{Volume} = |(\vec{b} \times \vec{c}) \cdot \vec{a}| = |(\vec{i} - \vec{j} + \vec{k}) \cdot (3\vec{i} + 4\vec{k} + 5\vec{j})| = |3-4+5| = 4.$$

17. We first calculate the cross product:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (1-1)\vec{i} - (-1-1)\vec{j} + (1+1)\vec{k} = 0\vec{i} + 2\vec{j} + 2\vec{k}.$$

Then

$$\text{Volume} = |(\vec{b} \times \vec{c}) \cdot \vec{a}| = |(0\vec{i} + 2\vec{j} + 2\vec{k}) \cdot (-\vec{i} + \vec{k} + \vec{k})| = |0+2+2| = 4.$$

18. We first calculate the cross product:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{vmatrix} = (6-0)\vec{i} - (0-0)\vec{j} + (0+0)\vec{k} = 6\vec{i}.$$

Then

$$\text{Volume} = |(\vec{b} \times \vec{c}) \cdot \vec{a}| = |(6\vec{i}) \cdot (\vec{i} + 8\vec{j} + 7\vec{k})| = |6+0+0| = 6.$$

19. We first calculate the cross product:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (0-1)\vec{i} - (1-0)\vec{j} + (1-0)\vec{k} = -\vec{i} - \vec{j} + \vec{k}.$$

Then

$$\text{Volume} = |(\vec{b} \times \vec{c}) \cdot \vec{a}| = |(-\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} + \vec{k} + 2\vec{k})| = |-1-1+2| = 0.$$

The volume is zero because $\vec{a} = \vec{b} + \vec{c}$, so the three vectors lie in a plane; they do not make a 3-dimensional parallelepiped

Problems

20. Normal vectors of the planes are $\vec{n}_1 = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{n}_2 = 4\vec{i} + \vec{j} - 3\vec{k}$ respectively. The line of intersection is perpendicular to both normals. (Picture the pages in a partially open book.) We can use $\vec{n}_1 \times \vec{n}_2 = 4\vec{i} + 26\vec{j} + 14\vec{k}$.

21. We use the same normal $\vec{n} = 4\vec{i} + 26\vec{j} + 14\vec{k}$ and the point $(0,0,0)$ to get $4(x-0) + 26(y-0) + 14(z-0) = 0$, or $4x + 26y + 14z = 0$.

22. We use the same normal $\vec{n} = 4\vec{i} + 26\vec{j} + 14\vec{k}$ and the point $(4,5,6)$ to get $4(x-4) + 26(y-5) + 14(z-6) = 0$, or $4x + 26y + 14z = 230$.

23. The origin $(0,0,0)$ is on the plane. A vector normal to the plane is given by the cross product of the vectors from the origin to the given points: $(\vec{i} + 3\vec{j}) \times (2\vec{i} + 4\vec{j} + \vec{k}) = 3\vec{i} - \vec{j} - 2\vec{k}$. Thus, an equation for the plane is $3x - y - 2z = 0$.

24. The normal vectors to the two planes are $\vec{n}_1 = 4\vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{n}_2 = \vec{i} + 5\vec{j} - \vec{k}$. A vector parallel to the line of intersection of the two planes is perpendicular to both these normal vectors, so

$$\text{Vector parallel to line} = \vec{n}_1 \times \vec{n}_2 = -7\vec{i} + 6\vec{j} + 23\vec{k}.$$

25. The normal vectors to the planes are $\vec{n_1} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{n_2} = 4\vec{i} + \vec{j} - 3\vec{k}$. The line of intersection is perpendicular to both normal vectors (picture the pages in a partially open book). Hence the vector we need is $\vec{n_1} \times \vec{n_2} = 4\vec{i} + 26\vec{j} + 14\vec{k}$.

26. The vector parallel to the line of intersection is $4\vec{i} + 26\vec{j} + 14\vec{k}$ and this is normal to the desired plane. Therefore, $4x + 26y + 14z = 0$ is the equation of the plane.

27. We use the same normal vector $\vec{n} = 4\vec{i} + 26\vec{j} + 14\vec{k}$ and the point $(4,5,6)$ to get $4(x-4) + 26(y-5) + 14(z-6) = 0$.

28. Normal vectors to the planes are

$$\vec{n}_1 = \vec{i} - \vec{j} + \vec{k} \quad \text{and} \quad \vec{n}_2 = 2\vec{i} + \vec{j} - 2\vec{k}.$$

The vector $\vec{n}_1 \times \vec{n}_2$ is perpendicular to both planes and is normal to the plane we want:

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \vec{i} + 4\vec{j} + 3\vec{k}.$$

The plane through the origin with normal $\vec{n}_1 \times \vec{n}_2$ is

$$x + 4y + 3z = 0.$$

29. (a) Since

$$\overrightarrow{PQ} = (3\vec{i} + 5\vec{j} + 7\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 2\vec{i} + 3\vec{j} + 4\vec{k},$$

and

$$\overrightarrow{PR} = (2\vec{i} + 5\vec{j} + 3\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = \vec{i} + 3\vec{j},$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 1 & 3 & 0 \end{vmatrix} = -12\vec{i} + 4\vec{j} + 3\vec{k},$$

which is a vector perpendicular to the plane containing P, Q and R. Since

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{(-12)^2 + 4^2 + 3^2} = 13,$$

the unit vectors which are perpendicular to a plane containing P, Q, and R are

$$-\frac{12}{13}\vec{i} + \frac{4}{13}\vec{j} + \frac{3}{13}\vec{k},$$

or the unit vector pointing to the opposite direction,

$$\frac{12}{13}\vec{i} - \frac{4}{13}\vec{j} - \frac{3}{13}\vec{k}.$$

(b) The angle between PQ and PR is θ for which

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\| \cdot \|\overrightarrow{PR}\|} = \frac{2 \cdot 1 + 3 \cdot 3 + 4 \cdot 0}{\sqrt{2^2 + 3^2 + 4^2} \cdot \sqrt{1^2 + 3^2 + 0^2}} = \frac{11}{\sqrt{290}},$$

so

$$\theta = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.76^\circ.$$

(c) The area of triangle $PQR = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{13}{2}$.

(d) Let d be the distance from R to the line through P and Q (see Figure 13.34), then

$$\frac{1}{2}d \cdot \|\overrightarrow{PQ}\| = \text{the area of } \triangle PQR = \frac{13}{2}.$$

Therefore,

$$d = \frac{13}{\|\overrightarrow{PQ}\|} = \frac{13}{\sqrt{2^2+3^2+4^2}} = \frac{13}{\sqrt{29}}.$$

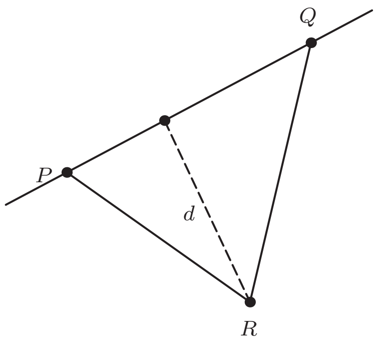


Figure 13.34

30. (a) We first find two displacement vectors: $\overrightarrow{AB} = (3-(-1))\vec{i} + (2-3)\vec{j} + (4-0)\vec{k} = 4\vec{i} - \vec{j} + 4\vec{k}$ and $\overrightarrow{AC} = 2\vec{i} - 4\vec{j} + 5\vec{k}$. The normal vector, $\vec{n}$, to the plane is perpendicular to these two vectors, so we have

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = 11\vec{i} - 12\vec{j} - 14\vec{k}.$$

Using the normal vector, we see that the equation of the plane is $11x-12y-14z = d$ for some number d. Substituting one of the points gives $d = -47$. Therefore, an equation for the plane is

$$11x - 12y - 14z = -47.$$

(b) The area of the triangle is given by

$$\text{Area} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{461} = 10.74.$$

31. Since $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$, we can conclude that $\vec{v} \times \vec{w}$ is parallel to the z-axis.

32. (a) Since $\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta$ and $||\vec{v} \times \vec{w}|| = ||\vec{v}||\,||\vec{w}||\sin\theta$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{||\vec{v} \times \vec{w}||}{\vec{v} \cdot \vec{w}} = \frac{3}{5} = 0.6.$$

(b) Then $\theta = \tan^{-1}(0.6) = 0.540$.

33. Since

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \sin\theta,$$

and

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta,$$

so

$$\frac{\|\vec{v} \times \vec{w}\|}{\vec{v} \cdot \vec{w}} = \frac{\|\vec{v}\| \cdot \|\vec{w}\| \sin\theta}{\|\vec{v}\| \cdot \|\vec{w}\| \cos\theta} = \tan\theta,$$

so

$$\tan\theta = \frac{\|2\vec{i} - 3\vec{j} + 5\vec{k}\|}{3} = \frac{\sqrt{38}}{3} = 2.055.$$

34. (a) Since $\vec{v}\cdot\vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta$ and $||\vec{v}\times\vec{w}|| = ||\vec{v}||\,||\vec{w}||\sin\theta$, we find

$$||\vec{v}\times\vec{w}|| = ||12\vec{i} - 3\vec{j} + 4\vec{k}|| = \sqrt{12^2 + (-3)^2 + 4^2} = 13.$$

Then

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{||\vec{v}\times\vec{w}||}{\vec{v}\cdot\vec{w}} = \frac{13}{8} = 1.625.$$

(b) Then $\theta = \tan^{-1}(1.625) = 1.019$.

35. (a) Increases the force.

(b) The Magnus force is perpendicular to the direction the ball is travelling (which is into the page) and the axis of spin. Thus $\vec{F}_M$ points down and to the left. The ball curves to the left and drops faster than a ball that is not spinning.

36. (a) The vector $\vec{k}\times\vec{r}$ is in the xy-plane because it is perpendicular to $\vec{k}$. It is also perpendicular to $\vec{r}$, so it points in either the same or opposite direction as $\vec{v}$. The right hand rule shows that it is in the same direction. Finally, $\vec{k}\times\vec{r}$ has magnitude $||\vec{k}\times\vec{r}|| = ||\vec{k}||||\vec{r}||\sin(90^\circ) = ||\vec{r}|| = ||\vec{v}||$. Thus $\vec{k}\times\vec{r}$ and $\vec{v}$ have the same directions and magnitudes, and so they are equal.

(b) Since $\vec{k}\times\vec{r} = \vec{k}\times(x\vec{i} + y\vec{j}) = -y\vec{i} + x\vec{j}$ is the position vector of P, we have $P = (-y, x)$.

37. (a) We have

$$\overrightarrow{P_1P_2} = 2\vec{i} + 4\vec{j} + 2\vec{k},$$

and

$$\overrightarrow{P_3P_4} = 2\vec{i} + 4\vec{j} + 2\vec{k}.$$

so these two displacement vectors are equal. Also,

$$\overrightarrow{P_1P_3} = 3\vec{i} \qquad \text{and} \qquad \overrightarrow{P_2P_4} = 3\vec{i},$$

so these two vectors are also equal. These points form a parallelogram.

(b) Vectors along adjacent sides of the parallelogram are $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. Since

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 2 \\ 3 & 0 & 0 \end{vmatrix} = 6\vec{j} - 12\vec{k},$$

we have

$$\text{Area of parallelogram } = ||\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}|| = ||6\vec{j} - 12\vec{k}|| = \sqrt{180}.$$

38. The three vectors $\vec{a} = \overrightarrow{P_1P_2} = 2\vec{i} + 4\vec{j} + 2\vec{k}$ and $\vec{b} = \overrightarrow{P_1P_3} = 3\vec{i}$ and $\vec{c} = \overrightarrow{P_1P_5} = \vec{i} + 4\vec{k}$ determine the parallelepiped. To find the volume of the parallelepiped, we first compute the cross product $\vec{b}\times\vec{c}$:

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -12\vec{j}.$$

We have

$$\text{Volume of parallelepiped } = \left|(\vec{b}\times\vec{c})\cdot\vec{a}\right| = \left|(-12\vec{j})\cdot\vec{a}\right| = |-48| = 48.$$

39. First let

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} \quad \vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

so $\vec{b} + \vec{c} = (b_1 + c_1)\vec{i} + (b_2 + c_2)\vec{j} + (b_3 + c_3)\vec{k}$. Now, using the general formula for cross products, we have:

$$\begin{aligned}
&\vec{a}\times(\vec{b} + \vec{c}) \\
&= [a_2(b_3 + c_3) - a_3(b_2 + c_2)]\vec{i} + [a_3(b_1 + c_1) - a_1(b_3 + c_3)]\vec{j} + [a_1(b_2 + c_2) - a_2(b_1 + c_1)]\vec{k} \\
&= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\vec{i} + (a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3)\vec{j} \\
&\quad +(a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\vec{k} \\
&= (a_2b_3 - a_3b_2)\vec{i} + (a_2c_3 - a_3c_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_3c_1 - a_1c_3)\vec{j} \\
&\quad +(a_1b_2 - a_2b_1)\vec{k} + (a_1c_2 - a_2c_1)\vec{k} \\
&= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} + (a_2c_3 - a_3c_2)\vec{i} + (a_3c_1 - a_1c_3)\vec{j} \\
&\quad +(a_1c_2 - a_2c_1)\vec{k} \\
&= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})
\end{aligned}$$

Thus, $\vec a \times (\vec b + \vec c) = \vec a \times \vec b + \vec a \times \vec c$.

40. If $\lambda = 0$, then all three cross products are $\vec 0$, since the cross product of the zero vector with any other vector is always 0.

If $\lambda > 0$, then $\lambda\vec v$ and $\vec v$ are in the same direction and $\vec w$ and $\lambda\vec w$ are in the same direction. Therefore the unit normal vector $\vec n$ is the same in all three cases. In addition, the angles between $\lambda\vec v$ and $\vec w$, and between $\vec v$ and $\vec w$, and between $\vec v$ and $\lambda\vec w$ are all θ. Thus,

$$\begin{aligned}(\lambda\vec v) \times \vec w &= \|\lambda\vec v\|\|\vec w\| \sin\theta\vec n\\ &= \lambda\|\vec v\|\|\vec w\| \sin\theta\vec n\\ &= \lambda(\vec v \times \vec w)\\ &= \|\vec v\|\|\lambda\vec w\| \sin\theta\vec n\\ &= \vec v \times (\lambda\vec w)\end{aligned}$$

If $\lambda < 0$, then $\lambda\vec v$ and $\vec v$ are in opposite directions, as are $\vec w$ and $\lambda\vec w$ in opposite directions. Therefore if $\vec n$ is the normal vector in the definition of $\vec v \times \vec w$, then the right-hand rule gives $-\vec n$ for $(\lambda\vec v) \times \vec w$ and $\vec v \times (\lambda\vec w)$. In addition, if the angle between $\vec v$ and $\vec w$ is θ, then the angle between $\lambda\vec v$ and $\vec w$ and between $\vec v$ and $\lambda\vec w$ is $(\pi - \theta)$. Since if $\lambda < 0$, we have $|\lambda| = -\lambda$, so

$$\begin{aligned}(\lambda\vec v) \times \vec w &= \|\lambda\vec v\|\|\vec w\| \sin(\pi-\theta)(-\vec n)\\ &= |\lambda|\,\|\vec v\|\|\vec w\| \sin(\pi-\theta)(-\vec n)\\ &= -\lambda\|\vec v\|\|\vec w\| \sin\theta(-\vec n)\\ &= \lambda\|\vec v\|\|\vec w\| \sin\theta\vec n\\ &= \lambda(\vec v \times \vec w).\end{aligned}$$

Similarly,

$$\begin{aligned}\vec v \times (\lambda\vec w) &= \|\vec v\|\|\lambda\vec w\| \sin(\pi-\theta)(-\vec n)\\ &= -\lambda\|\vec v\|\|\vec w\| \sin\theta(-\vec n)\\ &= \lambda(\vec v \times \vec w).\end{aligned}$$

41. The quantities $\left|\vec a \cdot (\vec b \times \vec c)\right|$ and $\left|(\vec a \times \vec b) \cdot \vec c\right|$ both represent the volume of the same parallelepiped, namely that defined by the three vectors $\vec a$, $\vec b$, and $\vec c$, and therefore must be equal. Thus, the two triple products $\vec a \cdot (\vec b \times \vec c)$ and $(\vec a \times \vec b) \cdot \vec c$ must be equal except perhaps for their sign. In fact, both are positive if $\vec a$, $\vec b$, $\vec c$ are right-handed and negative if $\vec a$, $\vec b$, $\vec c$ are left-handed. This can be shown by drawing a picture:

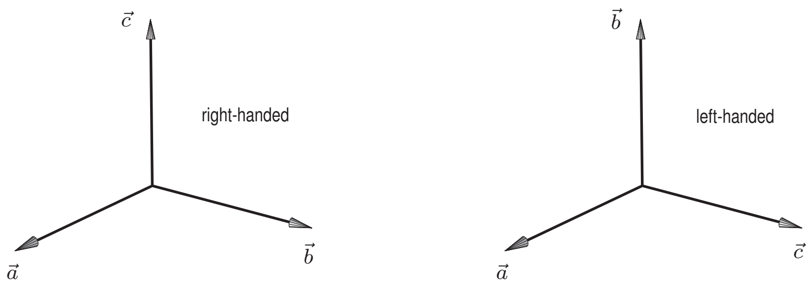


Figure 13.35

42. If θ is the angle between $\vec a$ and $\vec b$, then

$$\begin{aligned}\|\vec a \times \vec b\|^2 &= (\|\vec a\|\|\vec b\| \sin\theta)^2\\ &= \|\vec a\|^2\|\vec b\|^2 \sin^2\theta\\ &= \|\vec a\|^2\|\vec b\|^2(1-\cos^2\theta)\\ &= \|\vec a\|^2\|\vec b\|^2 - \|\vec a\|^2\|\vec b\|^2\cos^2\theta\\ &= \|\vec a\|^2\|\vec b\|^2 - (\vec a \cdot \vec b)^2.\end{aligned}$$

43. Solve for $\vec{c}$ to get $\vec{c} = -(\vec{a} + \vec{b})$. So

$$\begin{aligned}\vec{b} \times \vec{c} &= \vec{b} \times (-(\vec{a} + \vec{b}))\\ &= -(\vec{b} \times (\vec{a} + \vec{b}))\\ &= -(\vec{b} \times \vec{a} + \vec{b} \times \vec{b})\\ &= -(\vec{b} \times \vec{a} + \vec{0})\\ &= -(\vec{b} \times \vec{a})\\ &= \vec{a} \times \vec{b}.\end{aligned}$$

Also,

$$\begin{aligned}\vec{c} \times \vec{a} &= -(\vec{a} + \vec{b}) \times \vec{a}\\ &= -((\vec{a} + \vec{b}) \times \vec{a})\\ &= -(\vec{a} \times \vec{a} + \vec{b} \times \vec{a})\\ &= -(\vec{0} + \vec{b} \times \vec{a})\\ &= \vec{a} \times \vec{b}.\end{aligned}$$

Therefore, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

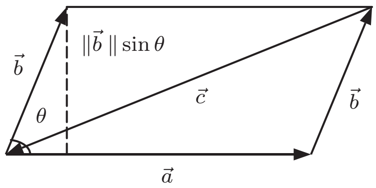


Figure 13.36

Geometrically, the magnitude of the cross product of two vectors is equal to the area of the parallelogram formed by the vectors. If $\vec{a} + \vec{b} + \vec{c} = 0$, then we can think of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as forming a triangle. (See Figure 13.36.) So by showing that

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a},$$

we are showing that the areas of the parallelograms formed by any two sides of the same triangle are equal.

44. The cross product is given by

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \vec{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \vec{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \vec{k}$$

so

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

45. Write $\vec{v}$ and $\vec{w}$ in components and expand using the distributive property of the cross product.

$$\begin{aligned}\vec{v} \times \vec{w} &= (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \times (w_1\vec{i} + w_2\vec{j} + w_3\vec{k})\\ &= v_1w_1\vec{i} \times \vec{i} + v_1w_2\vec{i} \times \vec{j} + v_1w_3\vec{i} \times \vec{k}\\ &\quad + v_2w_1\vec{j} \times \vec{i} + v_2w_2\vec{j} \times \vec{j} + v_2w_3\vec{j} \times \vec{k}\\ &\quad + v_3w_1\vec{k} \times \vec{i} + v_3w_2\vec{k} \times \vec{j} + v_3w_3\vec{k} \times \vec{k}\end{aligned}$$

Now we use the fact that $\vec{i} \times \vec{i} = \vec{0}, \vec{i} \times \vec{j} = \vec{k}, \vec{i} \times \vec{k} = -\vec{j}, \vec{j} \times \vec{i} = -\vec{k}, \vec{j} \times \vec{j} = \vec{0}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}, \vec{k} \times \vec{j} = -\vec{i}, \vec{k} \times \vec{k} = \vec{0}$. Thus we have

$$\begin{aligned}\vec{v} \times \vec{w} &= \vec{0} + v_1w_2\vec{k} + v_1w_3(-\vec{j}) + v_2w_1(-\vec{k}) + \vec{0} + v_2w_3\vec{i} + v_3w_1\vec{j} + v_3w_2(-\vec{i}) + \vec{0}\\ &= (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}.\end{aligned}$$

46. Any vector $\vec{v}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$ will have the property that its dot product with $\vec{a}$ and $\vec{b}$ is 0, that is

$$\begin{aligned}\vec{a}\cdot\vec{v} &= a_1x + a_2y + a_3z = 0,\\ \vec{b}\cdot\vec{v} &= b_1x + b_2y + b_3z = 0.\end{aligned}$$

Multiply the first equation by b_1 and the second by a_1 and subtract to get

$$(b_1a_2 - a_1b_2)y + (b_1a_3 - a_1b_3)z = 0 \quad \text{or} \quad y = \frac{-(b_1a_3 - a_1b_3)z}{(b_1a_2 - a_1b_2)} \quad (\text{for } b_1a_2 \neq a_1b_2)$$

Multiply the second equation by a_2 and the first by b_2 and subtract to get

$$(b_2a_1 - a_2b_1)x + (b_2a_3 - a_2b_3)z = 0 \quad \text{or} \quad x = \frac{-(b_2a_3 - a_2b_3)z}{(b_2a_1 - a_2b_1)}.$$

So

$$\vec{v} = \frac{-(b_2a_3 - a_2b_3)z}{(b_2a_1 - a_2b_1)}\vec{i} - \frac{(b_1a_3 - a_1b_3)z}{(b_1a_2 - a_1b_2)}\vec{j} + z\vec{k}.$$

Pick $z = b_2a_1 - b_1a_2$ and multiply out, and we see that the algebraic method of finding a cross product yields the same result as our standard method.

47. (a) Since $\vec{c}$ is perpendicular to $\vec{a}\times\vec{b}$, and since $\vec{a}\times\vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{b}$, it follows that $\vec{c}$ must be in the plane containing $\vec{a}$ and $\vec{b}$.

(b) Using the expression given in the problem for $\vec{c}$, we get

$$\begin{aligned}\vec{a}\cdot\vec{c} &= \vec{a}\cdot(\vec{a}\times(\vec{b}\times\vec{a}))\\ &= (\vec{a}\times\vec{a})\cdot(\vec{b}\times\vec{a})\\ &= \vec{0}\cdot(\vec{b}\times\vec{a}) = 0.\end{aligned}$$

and

$$\begin{aligned}\vec{b}\cdot\vec{c} &= \vec{b}\cdot(\vec{a}\times(\vec{b}\times\vec{a}))\\ &= (\vec{b}\times\vec{a})\cdot(\vec{b}\times\vec{a})\\ &= \|\vec{b}\times\vec{a}\|^2\\ &= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2.\end{aligned}$$

(c) Since $\vec{c}$ lies in the plane containing $\vec{a}$ and $\vec{b}$, it is of the form $\vec{c} = x\vec{a} + y\vec{b}$ for some scalars x and y. Thus, using the fact that $\vec{a}\cdot\vec{c} = 0$ from part (b), we have

$$\vec{a}\cdot\vec{c} = \vec{a}\cdot(x\vec{a} + y\vec{b}) = x\|\vec{a}\|^2 + y(\vec{a}\cdot\vec{b}) = 0.$$

Similarly, using the fact that $\vec{b}\cdot\vec{c} = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$ from part (b), we have

$$\vec{b}\cdot\vec{c} = \vec{b}\cdot(x\vec{a} + y\vec{b}) = x(\vec{a}\cdot\vec{b}) + y\|\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2.$$

Solving these two linear equations in x and y, we find $x = -\vec{a}\cdot\vec{b}$ and $y = \|\vec{a}\|^2$.

48. Problem 41 tells us that $(\vec{u}\times\vec{v})\cdot\vec{w} = \vec{u}\cdot(\vec{v}\times\vec{w})$. Using this result on the triple product of $(\vec{a} + \vec{b})\times\vec{c}$ with any vector $\vec{d}$ together with the fact that the dot product distributes over addition gives us:

$$\begin{aligned}[(\vec{a} + \vec{b})\times\vec{c}]\cdot\vec{d} &= (\vec{a} + \vec{b})\cdot(\vec{c}\times\vec{d})\\ &= \vec{a}\cdot(\vec{c}\times\vec{d}) + \vec{b}\cdot(\vec{c}\times\vec{d}) \qquad \text{(dot product is distributive)}\\ &= (\vec{a}\times\vec{c})\cdot\vec{d} + (\vec{b}\times\vec{c})\cdot\vec{d} \qquad \text{(using Problem 41 again)}\\ &= [(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})]\cdot\vec{d}. \qquad \text{(dot product is distributive)}\end{aligned}$$

So, since $[(\vec{a} + \vec{b})\times\vec{c}]\cdot\vec{d} = [(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})]\cdot\vec{d}$, then

$$[(\vec{a} + \vec{b})\times\vec{c}]\cdot\vec{d} - [(\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})]\cdot\vec{d} = 0,$$

Since the dot product is distributive, we have

$$[((\vec{a}+\vec{b})\times\vec{c})-(\vec{a}\times\vec{c})-(\vec{b}\times\vec{c})]\cdot\vec{d}=0.$$

Since this equation is true for all vectors $\vec{d}$, by letting

$$\vec{d}=((\vec{a}+\vec{b})\times\vec{c})-(\vec{a}\times\vec{c})-(\vec{b}\times\vec{c}),$$

we get

$$\|(\vec{a}+\vec{b})\times\vec{c}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}\|^2=0$$

and hence

$$(\vec{a}+\vec{b})\times\vec{c}-(\vec{a}\times\vec{c})-(\vec{b}\times\vec{c})=\vec{0}.$$

Thus

$$(\vec{a}+\vec{b})\times\vec{c}=(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c}).$$

49. The area vector for face $OAB=\frac{1}{2}\vec{b}\times\vec{a}$.
The area vector for face $OBC=\frac{1}{2}\vec{a}\times\vec{c}$.
The area vector for face $OAC=\frac{1}{2}\vec{b}\times\vec{c}$.
The area vector for face $ABC=\frac{1}{2}(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$.

$$\frac{1}{2}\vec{b}\times\vec{a}+\frac{1}{2}\vec{c}\times\vec{b}+\frac{1}{2}\vec{a}\times\vec{c}+\frac{1}{2}(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=$$
$$\frac{1}{2}\vec{b}\times\vec{a}+\frac{1}{2}\vec{c}\times\vec{b}+\frac{1}{2}\vec{a}\times\vec{c}+\frac{1}{2}(\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}-\vec{a}\times\vec{a})=0.$$

50. First let two adjoining sides of the rectangle be our vectors $\vec{a}$ and $\vec{b}$. See Figure 13.37. So we have

$$\vec{a}=\vec{j}\quad\text{and}\quad\vec{b}=2\vec{i}$$

Since it faces downward (that is, in the negative z direction), according to the right hand rule $\vec{A}=\vec{a}\times\vec{b}$. So, by the formula for cross products

$$\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & 0\\ 2 & 0 & 0\end{vmatrix}=-2\vec{k}.$$

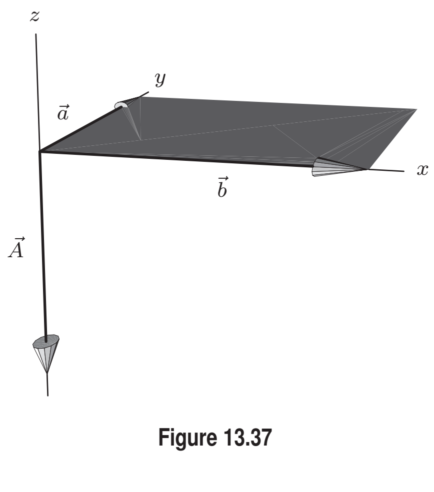


Figure 13.37

51. The area vector for a circle has a magnitude πr^2 and direction normal to the plane of the circle. See Figure 13.38. Thus, since our circle is facing in the positive x direction, and $r = 2$,

$$\vec{A} = \pi(2)^2\vec{i} = 4\pi\vec{i}.$$

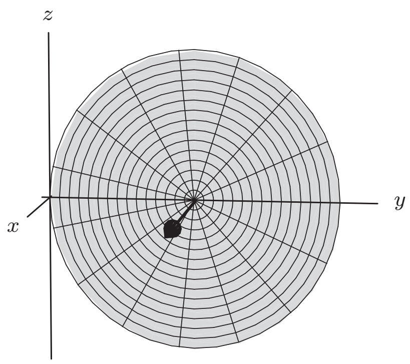


Figure 13.38

52. First choose vectors $\vec{a}$ and $\vec{b}$ along the sides of the triangle. Let's choose

$$\vec{a} = \overrightarrow{AB} = 2\vec{i} - \vec{j} - \vec{k}, \quad \vec{b} = \overrightarrow{AC} = \vec{i} - \vec{j}.$$

These vectors then form two sides of a parallelogram whose area is $\|\vec{a} \times \vec{b}\|$. Our triangle forms half of this parallelogram, so the area of triangle $ABC = \frac{1}{2}\|\vec{a} \times \vec{b}\|$. Since there are two possible orientations, the area of the triangle ABC is represented by one of the following two vectors:

$$\pm\frac{1}{2}(\vec{a} \times \vec{b}) = \pm\frac{1}{2}(2\vec{i} - \vec{j} - \vec{k}) \times (\vec{i} - \vec{j}) = \pm\left(-\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} - \frac{1}{2}\vec{k}\right)$$

We want the upward orientation, so we pick the negative sign, giving

$$\vec{A} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \frac{1}{2}\vec{k}.$$

53. (a) Since

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k},$$

we have

$$\text{Area of } S = \|\vec{u} \times \vec{v}\| = \left((u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2\right)^{1/2}.$$

(b) The two edges of R are given by the projections of $\vec{u}$ and $\vec{v}$ onto the xy-plane. These are the vectors $\vec{U}$ and $\vec{V}$, obtained by omitting the $\vec{k}$-components of $\vec{u}$ and $\vec{v}$: we have $\vec{U} = u_1\vec{i} + u_2\vec{j}$ and $\vec{V} = v_1\vec{i} + v_2\vec{j}$, Thus

$$\text{Area of } R = \|\vec{U} \times \vec{V}\| = \|(u_1v_2 - u_2v_1)\vec{k}\| = |u_1v_2 - u_2v_1|.$$

(c) The vector $m\vec{i} + n\vec{j} - \vec{k}$ is normal to the plane $z = mx + ny + c$. Since the vectors $\vec{u}$ and $\vec{v}$ are in the plane (they're the sides of $\vec{S}$), the vector $\vec{u} \times \vec{v}$ is also normal to the plane. Thus, these two vectors are scalar multiples of one another. Suppose

$$\vec{u} \times \vec{v} = \lambda(m\vec{i} + n\vec{j} - \vec{k})$$

Since the $\vec{k}$ component of $\vec{u} \times \vec{v}$ is $(u_1v_2 - u_2v_1)\vec{k}$, comparing the $\vec{k}$-components tells us that

$$\lambda = -(u_1v_2 - u_2v_1).$$

Thus,

$$-(u_1v_2 - u_2v_1)(m\vec{i} + n\vec{j} - \vec{k}) = \vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k},$$

so

$$m = \frac{u_2v_3 - u_3v_2}{u_2v_1 - u_1v_2}$$
$$n = \frac{u_3v_1 - u_1v_3}{u_2v_1 - u_1v_2}.$$

(d) We have

$$\begin{aligned}(1+m^2+n^2)\cdot(\text{Area of } R)^2 &= \left(1+\left(\frac{u_2v_3-u_3v_2}{u_2v_1-u_1v_2}\right)^2+\left(\frac{u_3v_1-u_1v_3}{u_2v_1-u_1v_2}\right)^2\right)(u_1v_2-u_2v_1)^2\\ &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\ &= (\text{Area of } S)^2.\end{aligned}$$

Strengthen Your Understanding

54. If $\vec{n}$ is unit vector perpendicular to $\vec{u}$ and $\vec{v}$, then so is $-\vec{n}$. There are exactly two possible unit vectors perpendicular to two given nonparallel vectors. The right-hand rule gives one of the two.

55. $\vec{u}\times\vec{v}=\vec{0}$ when $\vec{u}$ and $\vec{v}$ are parallel, not perpendicular.

56. Since $\vec{u}\times\vec{v}$ is perpendicular to the plane containing $\vec{u}$ and $\vec{v}$, we want $\vec{k}$ to be perpendicular to the plane containing $\vec{u}$ and $\vec{v}$. That is, we want $\vec{u}$ and $\vec{v}$ to both lie in the xy-plane. Thus choose $\vec{u}$ to be any nonzero vector in the xy-plane not parallel to $\vec{v}$. For example, let $\vec{u}=2\vec{i}+\vec{j}$.

57. We could let $\vec{v}=a\vec{i}+b\vec{j}+c\vec{k}$, compute $\|\vec{u}\times\vec{v}\|$ in terms of a,b,c, set the length equal to 10, and solve to find values of a,b,c. It is much easier to think geometrically. Let $\vec{v}$ be any vector perpendicular to $\vec{u}$, forming a rectangle of area 10. Since $\|\vec{u}\|=5$, that means we can choose v to be any vector perpendicular to $\vec{u}$ of length 2. So first find a unit vector perpendicular to $\vec{u}$, say $4\vec{i}-3\vec{j}$, unitize it to get $(4/5)\vec{i}-(3/5)\vec{j}$ and then scale it by 2. We get $(8/5)\vec{i}-(6/5)\vec{j}$.

58. True. The cross product yields a vector.

59. False. $\vec{u}\times\vec{v}$ has direction *perpendicular* to both $\vec{u}$ and $\vec{v}$.

60. False. This is only true when $\vec{u}$ and $\vec{v}$ are perpendicular. In general, $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$. The value of $\|\vec{u}\times\vec{v}\|$ is the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$.

61. True. The left-hand side evaluates to $\vec{k}\cdot\vec{k}=1$, while the right-hand side evaluates to $\vec{i}\cdot\vec{i}=1$.

62. False. If $\vec{u}$ and $\vec{w}$ are two different vectors both of which are parallel to $\vec{v}$, then $\vec{v}\times\vec{u}=\vec{v}\times\vec{w}=\vec{0}$, but $\vec{u}\neq\vec{w}$. A counterexample is $\vec{v}=\vec{i}$, $\vec{u}=2\vec{i}$ and $\vec{w}=3\vec{i}$.

63. True. Since $(\vec{v}\times\vec{w})$ is perpendicular to $\vec{v}$, the dot product with $\vec{v}$ is zero.

64. True. The cross product is a vector in 3-space, while the dot product is a scalar, so they cannot be equal.

65. True. The cross product $(\vec{i}+\vec{j})\times(\vec{j}+2\vec{k})=2\vec{i}-2\vec{j}+\vec{k}$, which has magnitude $\sqrt{2^2+(-2)^2+1^2}=3$. Since the triangle has area of $1/2$ the parallelogram with the given vectors as sides, the triangle has area $3/2$.

66. True. Any vector $\vec{w}$ that is parallel to $\vec{v}$ will give $\vec{v}\times\vec{w}=\vec{0}$.

67. False. It is not true in general, but there are special cases when $\vec{v}\times\vec{w}=\vec{w}\times\vec{v}$. For example, when $\vec{v}$ is parallel to $\vec{w}$, or when one of the vectors is $\vec{0}$. In either case the cross products $\vec{v}\times\vec{w}$ and $\vec{w}\times\vec{v}$ are both the zero vector.

Solutions for Chapter 13 Review

Exercises

1. Scalar. $\vec{u}\cdot\vec{v}=(2\vec{i}-3\vec{j}-4\vec{k})\cdot(\vec{k}-\vec{j})=2\cdot0-3(-1)-4\cdot1=-1$.

2. Vector. We calculate

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -3 & -4\\ 3 & -1 & 1\end{vmatrix}=-7\vec{i}-14\vec{j}+7\vec{k}.$$

3. $4-9+4=-1$

4. $\vec{i}\cdot(-\vec{i})=-1$

5. $\vec{a}=-2\vec{j}$, $\vec{b}=3\vec{i}$, $\vec{c}=\vec{i}+\vec{j}$, $\vec{d}=2\vec{j}$, $\vec{e}=\vec{i}-2\vec{j}$, $\vec{f}=-3\vec{i}-\vec{j}$.

6. Resolving $\vec{v}$ into components gives $\vec{v} = 8\cos(40°)\vec{i} - 8\sin(40°)\vec{j} = 6.13\vec{i} - 5.14\vec{j}$. Notice that the component in the $\vec{j}$ direction must be negative.

7. $5\vec{c} = 5\vec{i} + 30\vec{j}$

8. $\vec{c} + \vec{x} + \vec{y} = \vec{i} + 6\vec{j} - 2\vec{i} + 9\vec{j} + 4\vec{i} - 7\vec{j} = 3\vec{i} + 8\vec{j}$.

9. $||\vec{x} - \vec{c}|| = || - 2\vec{i} + 9\vec{j} - (\vec{i} + 6\vec{j})|| = || - 3\vec{i} + 3\vec{j}|| = 3\sqrt{2}$.

10. $\vec{v} + 2\vec{w} = 2\vec{i} + 3\vec{j} - \vec{k} + 2(\vec{i} - \vec{j} + 2\vec{k}) = 4\vec{i} + \vec{j} + 3\vec{k}$.

11. $3\vec{v} - \vec{w} - \vec{v} = 2\vec{v} - \vec{w} = 2(2\vec{i} + 3\vec{j} - \vec{k}) - (\vec{i} - \vec{j} + 2\vec{k}) = 3\vec{i} + 7\vec{j} - 4\vec{k}$.

12. $||\vec{v} + \vec{w}|| = ||3\vec{i} + 2\vec{j} + \vec{k}|| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$.

13. $\vec{v} \cdot \vec{w} = (2\vec{i} + 3\vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} + 2\vec{k}) = 2 \cdot 1 + 3 \cdot (-1) + (-1) \cdot 2 = -3$.

14. $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix} = (6-1)\vec{i} - (4+1)\vec{j} + (-2-3)\vec{k} = 5\vec{i} - 5\vec{j} - 5\vec{k}$.

15. For any vector $\vec{v}$, we have $\vec{v} \times \vec{v} = \vec{0}$.

16. Since $\vec{v} \cdot \vec{w} = 2 \cdot 1 + 3(-1) + (-1)2 = -3$, we have $(\vec{v} \cdot \vec{w})\vec{v} = -6\vec{i} - 9\vec{j} + 3\vec{k}$.

17. Since $\vec{v} \times \vec{w}$ is perpendicular to $\vec{w}$, we have $(\vec{v} \times \vec{w}) \cdot \vec{w} = 0$.

18. We have $\vec{v} \times \vec{w} = 5\vec{i} - 5\vec{j} - 5\vec{k}$, so

$$(\vec{v} \times \vec{w}) \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -5 & -5 \\ 1 & -1 & 2 \end{vmatrix} = (-10-5)\vec{i} - (10+5)\vec{j} + (-5+5)\vec{k} = -15\vec{i} - 15\vec{j}.$$

19. The cross product of two parallel vectors is $\vec{0}$, so the cross product of any vector with itself is $\vec{0}$.

20. A normal vector can be obtained from the coefficients of x, y, z in the equation of the plane and is: $\vec{n} = 2\vec{i} + \vec{j} - \vec{k}$.

21. The equation can be rewritten as

$$z - 5x + 10 = 15 - 3y$$
$$-5x + 3y + z = 5$$

so $\vec{n} = -5\vec{i} + 3\vec{j} + \vec{k}$.

22. If the planes are parallel, they have a common normal vector $\vec{n}$. Rewrite the equation of the plane as $4x - 3y - z = -8$ so that $\vec{n} = 4\vec{i} - 3\vec{j} - \vec{k}$ and the desired plane is $4(x-0) - 3(y-0) - (z-0) = 0$ or $4x - 3y - z = 0$.

23. (a) We have $\vec{v} \cdot \vec{w} = 3 \cdot 4 + 2 \cdot (-3) + (-2) \cdot 1 = 4$.

(b) We have $\vec{v} \times \vec{w} = -4\vec{i} - 11\vec{j} - 17\vec{k}$.

(c) A vector of length 5 parallel to $\vec{v}$ is

$$\frac{5}{||\vec{v}||}\vec{v} = \frac{5}{\sqrt{17}}(3\vec{i} + 2\vec{j} - 2\vec{k}) = 3.64\vec{i} + 2.43\vec{j} - 2.43\vec{k}.$$

(d) The angle between vectors $\vec{v}$ and $\vec{w}$ is found using

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||} = \frac{4}{\sqrt{17}\sqrt{26}} = 0.190,$$

so $\theta = 79.0°$.

(e) The component of vector $\vec{v}$ in the direction of vector $\vec{w}$ is

$$\frac{\vec{v} \cdot \vec{w}}{||\vec{w}||} = \frac{4}{\sqrt{26}} = 0.784.$$

(f) The answer is any vector $\vec{a}$ such that $\vec{a} \cdot \vec{v} = 0$. One possible answer is $2\vec{i} - 2\vec{j} + \vec{k}$.

(g) A vector perpendicular to both is the cross product:

$$\vec{v} \times \vec{w} = -4\vec{i} - 11\vec{j} - 17\vec{k}.$$

24. Since $||2\vec{i} + 3\vec{j} - \vec{k}|| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$, vectors of length 10 are

$$\pm\frac{10}{\sqrt{14}}(2\vec{i} + 3\vec{j} - \vec{k}).$$

25. We take the cross product of $\vec{i} + \vec{j}$ and $\vec{i} - \vec{j} - \vec{k}$ and then make a unit vector parallel to the cross product.

$$(\vec{i} + \vec{j}) \times (\vec{i} - \vec{j} - \vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -\vec{i} + \vec{j} - 2\vec{k}.$$

Since $||-\vec{i} + \vec{j} - 2\vec{k}|| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = 6$, unit vectors are

$$\pm\frac{-\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{6}}.$$

26. We want a unit vector of the form $a\vec{i} + b\vec{j}$ such that

$$(a\vec{i} + b\vec{j}) \cdot (3\vec{i} - 2\vec{j}) = 3a - 2b = 0.$$

Let's take $a = 2$ and $b = 3$. Then the vector $2\vec{i} + 3\vec{j}$ is perpendicular to $3\vec{i} - 2\vec{j}$, but $2\vec{i} + 3\vec{j}$ is not a unit vector. Since $||2\vec{i} + 3\vec{j}|| = \sqrt{13}$, unit vectors are

$$\pm\frac{2\vec{i} + 3\vec{j}}{\sqrt{13}}.$$

27. $\vec{n} = 4\vec{i} + 6\vec{k}$ (the coefficients of x, y, z are the same as the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$.)

28. First, we rewrite the equation of the plane as

$$x - y - z = 1.$$

In this form, the coefficients of $x, y,$ and z are the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$ in a vector that is perpendicular to the plane. So any scalar multiple of $\vec{i} - \vec{j} - \vec{k}$ is perpendicular to the plane.

29. The vector $\vec{w}$ we want is shown in Figure 13.39, where the given vector is $\vec{v} = 4\vec{i} + 3\vec{j}$. The vectors $\vec{v}$ and $\vec{w}$ are the same length and the two angles marked α are equal, so the two right triangles shown are congruent. Thus

$$a = -3 \qquad \text{and} \qquad b = 4.$$

Therefore

$$\vec{w} = -3\vec{i} + 4\vec{j}.$$

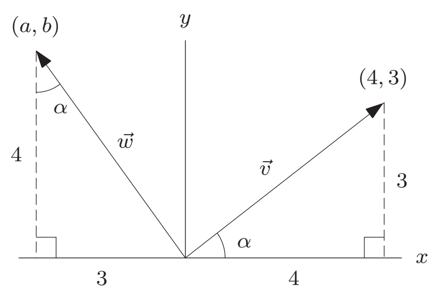


Figure 13.39

30. The cross product of two vectors is perpendicular to both of them, so a possible answer is

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \vec{i} - 2\vec{j} - 5\vec{k}.$$

31. To determine if two vectors are parallel, we need to see if one vector is a scalar multiple of the other one. Since $\vec{u} = -2\vec{w}$, and $\vec{v} = \frac{1}{4}\vec{q}$ and no other pairs have this property, only $\vec{u}$ and $\vec{w}$, and $\vec{v}$ and $\vec{q}$ are parallel.

32. Since $\vec{F} = 2\vec{d}$, the two vectors are parallel in the same direction, so

$$\vec{F}_{\text{parallel}} = \vec{F} \text{ and } \vec{F}_{\text{perp}} = \vec{0}.$$

The work done is

$$W = \vec{F} \cdot \vec{d} = 2 + 8 = 10.$$

Notice that this is the same as the magnitude of the force, $\|\vec{F}\| = \sqrt{20}$, times the distance traveled, $\|\vec{d}\| = \sqrt{5}$, since the force is the same direction as the displacement.

33. Since $\vec{F} = -2\vec{d}$, the two vectors are parallel in opposite directions, so

$$\vec{F}_{\text{parallel}} = \vec{F} \text{ and } \vec{F}_{\text{perp}} = \vec{0}.$$

The work done is

$$W = \vec{F} \cdot \vec{d} = -10.$$

Note that work done is negative since the force is in the opposite direction to the displacement.

34. Since $\vec{F} \cdot \vec{d} = 0$, the two vectors are perpendicular, so

$$\vec{F}_{\text{parallel}} = \vec{0} \text{ and } \vec{F}_{\text{perp}} = \vec{F}.$$

The work done is

$$W = \vec{F} \cdot \vec{d} = 0.$$

No work is done since the force is perpendicular to the displacement.

35. The unit vector in the direction of $\vec{d}$ is $\vec{u} = (3/5)\vec{i} - (4/5)\vec{j}$. Thus

$$\vec{F}_{\text{parallel}} = \left(\vec{F} \cdot \vec{u}\right)\vec{u} = -\frac{10}{5}\vec{u} = -\frac{6}{5}\vec{i} + \frac{8}{5}\vec{j},$$
$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \frac{16}{5}\vec{i} + \frac{12}{5}\vec{j}.$$

Notice that $\vec{F}_{\text{perp}} \cdot \vec{u} = 0$, as we expect. The work done is

$$W = \vec{F} \cdot \vec{d} = 6 - 16 = -10.$$

The work is negative since $\vec{F}_{\text{parallel}}$ is in the opposite direction of the displacement vector $\vec{d}$.

36. The unit vector in the direction of $\vec{d}$ is $\vec{u} = (1/\sqrt{2})(\vec{i} + \vec{j})$. Thus

$$\vec{F}_{\text{parallel}} = \left(\vec{F} \cdot \vec{u}\right)\vec{u} = \frac{2}{\sqrt{2}}\vec{u} = \vec{i} + \vec{j},$$
$$\vec{F}_{\text{perp}} = \vec{F} - \vec{F}_{\text{parallel}} = \vec{i} - \vec{j}.$$

Notice that $\vec{F}_{\text{perp}} \cdot \vec{u} = 0$, as we expect. The work done is

$$W = \vec{F} \cdot \vec{d} = 2 - 0 = 2.$$

37. The unit vector in the direction of $\vec{d} = 3\vec{j}$ is $\vec{u} = \vec{j}$. Thus, the parallel component of $\vec{F}$ is just its j component, and the perpendicular component is its i component:

$$\vec{F}_{\text{parallel}} = 2\vec{j} \text{ and } \vec{F}_{\text{perp}} = 5\vec{i}.$$

The work done is

$$W = \vec{F} \cdot \vec{d} = 6.$$

38. The area of the triangle is half the area of the parallelogram created by these two vectors, and the area of the parallelogram is the magnitude of the cross product. We first calculate the cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 4 & -2 & 1 \end{vmatrix} = 0\vec{i} - 5\vec{j} - 10\vec{k}.$$

The area of the triangle is given by:

$$\text{Area of triangle} = \frac{1}{2}\text{ Area of parallelogram} = \frac{1}{2}\|\vec{a} \times \vec{b}\| = \frac{1}{2}\| - 5\vec{j} - 10\vec{k}\| = \frac{1}{2}\sqrt{125} = 5.590.$$

Problems

39. (a) True, since vectors $\vec{c}$ and $\vec{f}$ point in the same direction and have the same length.
(b) False, since vectors $\vec{a}$ and $\vec{d}$ point in opposite directions. We have $\vec{a} = -\vec{d}$.
(c) False, since $-\vec{b}$ points in the opposite direction to $\vec{b}$, the vectors $-\vec{b}$ and $\vec{a}$ are perpendicular.
(d) True. The vector $\vec{f}$ can be "moved" to point directly up the z-axis.
(e) True. We move in the positive x-direction following vector $\vec{a}$ and then in the positive y-direction following vector $-\vec{b}$. The resulting sum is the vector $\vec{e}$.
(f) False, vector $\vec{d}$ is the negative of the vector $\vec{g} - \vec{c}$. It is true that $\vec{d} = \vec{c} - \vec{g}$.

40. Let the velocity vector of the airplane be $\vec{V} = x\vec{i} + y\vec{j} + z\vec{k}$ in km/hr. We know that $x = -y$ because the plane is traveling northwest. Also, $\|\vec{V}\| = \sqrt{x^2 + y^2 + z^2} = 200$ km/hr and $z = 300$ m/min $= 18$ km/hr. We have $\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + x^2 + 18^2} = 200$, so $x = -140.8$, $y = 140.8$, $z = 18$. (The value of x is negative and y is positive because the plane is heading northwest.) Thus,

$$\vec{v} = -140.8\vec{i} + 140.8\vec{j} + 18\vec{k}.$$

41. The velocity vector of the plane with respect to the air has the form

$$\vec{v} = a\vec{i} + 80\vec{k} \text{ where } \|\vec{v}\| = 480.$$

(See Figure 13.40.) Therefore $\sqrt{a^2 + 80^2} = 480$ so $a = \sqrt{480^2 - 80^2} \approx 473.3$ km/hr. We conclude that $\vec{v} \approx 473.3\vec{i} + 80\vec{k}$.

The wind vector is

$$\begin{aligned}\vec{w} &= 100(\cos 45^\circ)\vec{i} + 100(\sin 45^\circ)\vec{j} \\ &\approx 70.7\vec{i} + 70.7\vec{j}\end{aligned}$$

The velocity vector of the plane with respect to the ground is then

$$\begin{aligned}\vec{v} + \vec{w} &= (473.3\vec{i} + 80\vec{k}) + (70.7\vec{i} + 70.7\vec{j}) \\ &= 544\vec{i} + 70.7\vec{j} + 80\vec{k}\end{aligned}$$

From Figure 13.41, we see that the velocity relative to the ground is

$$544\vec{i} + 70.7\vec{j}.$$

The ground speed is therefore $\sqrt{544^2 + 70.7^2} \approx 548.6$ km/hr.

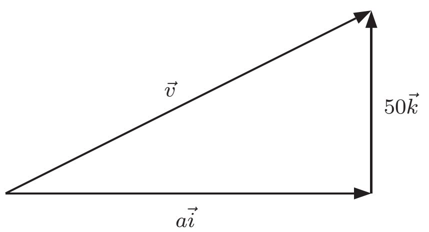


Figure 13.40: Side view

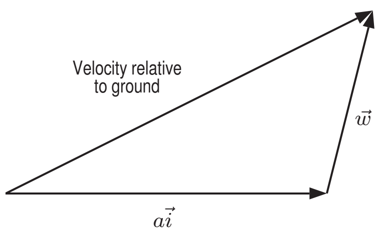


Figure 13.41: Top view

42. (a) See Figure 13.42. Notice that the velocity vectors are tangent to the curve, they point in the direction of motion, and they are longer when the rocket is moving faster.

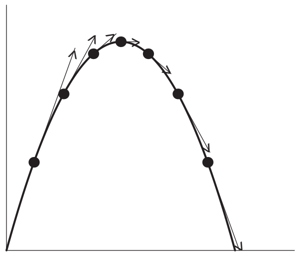


Figure 13.42

(b) If the rocket has a parachute, it comes down more slowly. The velocity vectors on the downward part of the graph are shorter for this rocket.

43. See Figure 13.43.

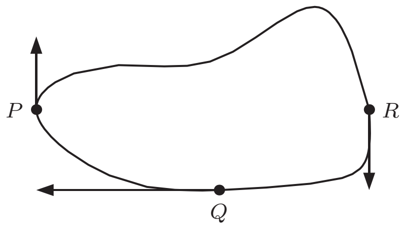


Figure 13.43

44. At the point P, the velocity of the car is changing the quickest; not in magnitude, but in direction only. The acceleration vector is therefore the longest at this point. The direction of the vector is directed in toward the center of the track because the difference in velocity vectors at nearby points is a vector pointing toward the center.

45. Since $3\vec{i} + \sqrt{3}\vec{j} = \sqrt{3}(\sqrt{3}\vec{i} + \vec{j})$, we know that $3\vec{i} + \sqrt{3}\vec{j}$ and $\sqrt{3}\vec{i} + \vec{j}$ are scalar multiples of one another, and therefore parallel.

Since $(\sqrt{3}\vec{i} + \vec{j}) \cdot (\vec{i} - \sqrt{3}\vec{j}) = \sqrt{3} - \sqrt{3} = 0$, we know that $\sqrt{3}\vec{i} + \vec{j}$ and $\vec{i} - \sqrt{3}\vec{j}$ are perpendicular.
Since $3\vec{i} + \sqrt{3}\vec{j}$ and $\sqrt{3}\vec{i} + \vec{j}$ are parallel, $3\vec{i} + \sqrt{3}\vec{j}$ and $\vec{i} - \sqrt{3}\vec{j}$ are perpendicular, too.

46. Let the x-axis point east and the y-axis point north. We resolve the forces into components. Since the first force points 50° south of east with a force of 25 newtons, we have

$$\vec{F_1} = 25\cos(50^\circ)\vec{i} - 25\sin 50^\circ\vec{j} = 16.070\vec{i} - 19.151\vec{j}.$$

Since $\vec{F_1}$ lies in the fourth quadrant, the coefficient of $\vec{i}$ is positive and the coefficient of $\vec{j}$ is negative.

The second force points 70° north of west with a force of 60 newtons, so we have

$$\vec{F_2} = -60\cos(70^\circ)\vec{i} + 60\sin 70^\circ\vec{j} = -20.521\vec{i} + 56.382\vec{j}.$$

Since $\vec{F_2}$ lies in the second quadrant, the coefficient of $\vec{i}$ is negative and the coefficient of $\vec{j}$ is positive.

The third force must make the total force equal to zero, so we have

$$\begin{aligned}
\vec{F_1} + \vec{F_2} + \vec{F_3} &= \vec{0} \\
\vec{F_3} &= -(\vec{F_1} + \vec{F_2}) \\
&= -((16.070\vec{i} - 19.151\vec{j}) + (-20.521\vec{i} + 56.382\vec{j})) \\
&= -(-4.451\vec{i} + 37.231\vec{j}) \\
&= 4.451\vec{i} - 37.231\vec{j}.
\end{aligned}$$

The magnitude of this force is $\|\vec{F_3}\| = \sqrt{4.451^2 + 37.231^2} = 37.50$ newtons. The direction is $\arctan(37.231/4.451) = 83.20^\circ$ south of east.

47. If the vectors are perpendicular, we need

$$\vec{v} \cdot \vec{w} = (2a\vec{i} - a\vec{j} + 16\vec{k}) \cdot (5\vec{i} + a\vec{j} - \vec{k}) = 10a - a^2 - 16 = 0.$$

Solving $10a - a^2 - 16 = -(a-2)(a-8) = 0$ gives $a = 2, 8$.

48. Since a normal vector of the plane is $\vec{n} = -\vec{i} + 2\vec{j} + \vec{k}$, an equation for the plane is

$$-x + 2y + z = -1 + 2 \cdot 0 + 2 = 1$$
$$-x + 2y + z = 1.$$

49. Since the plane is normal to the vector $2\vec{i} - 3\vec{j} + 7\vec{k}$ and passes through the point $(1, -1, 2)$, an equation for the plane is

$$2x - 3y + 7z = 2 \cdot 1 - 3 \cdot (-1) + 7 \cdot 2 = 19$$
$$2x - 3y + 7z = 19.$$

50. See Figure 13.44. One way to find the angle at A is to find the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Since $\overrightarrow{AB} = -1\vec{i} - 7\vec{j}$ and $\overrightarrow{AC} = -5\vec{i} - 3\vec{j}$, we have

$$\begin{aligned}\cos(\angle BAC) &= \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} \\ &= \frac{(-1)(-5) + (-7)(-3)}{\sqrt{50}\sqrt{34}} \\ &= 0.6306.\end{aligned}$$

Thus the angle at vertex A is 50.91°. Similarly, we see that the angle at vertex B is 53.13° and (since the angles of a triangle add up to 180°) the angle at vertex C is 75.96°.

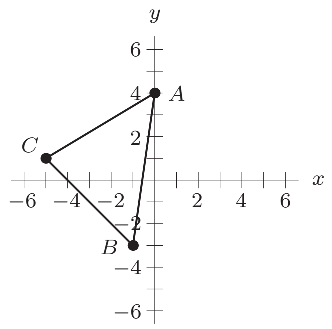


Figure 13.44

51. Let $\vec{r}_1$ be the displacement vector $\overrightarrow{PQ}$ and let $\vec{r}_2$ be the displacement vector $\overrightarrow{PR}$. Then

$$\vec{r}_1 = (1+2)\vec{i} + (3-2)\vec{j} + (-1-0)\vec{k} = 3\vec{i} + \vec{j} - \vec{k},$$
$$\vec{r}_2 = (-4+2)\vec{i} + (2-2)\vec{j} + (1-0)\vec{k} = -2\vec{i} + \vec{k},$$
$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = \vec{i} - (3-2)\vec{j} + 2\vec{k} = \vec{i} - \vec{j} + 2\vec{k}.$$

The area of the triangle $= \frac{1}{2}\|\vec{r}_1 \times \vec{r}_2\| = \frac{1}{2}\sqrt{1^2 + 1^2 + 2^2} = \frac{\sqrt{6}}{2}$.

52. (a) The displacement vector $\overrightarrow{AB}$ lies in the plane and is given by

$$\overrightarrow{AB} = (0-2)\vec{i} + (1-1)\vec{j} + (3-0)\vec{k} = -2\vec{i} + 3\vec{k}.$$

Similarly, the displacement vector $\overrightarrow{AC}$ also lies in the plane,

$$\overrightarrow{AC} = (1-2)\vec{i} + (0-1)\vec{j} + (1-0)\vec{k} = -\vec{i} - \vec{j} + \vec{k}.$$

(b) The vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and is therefore perpendicular to the plane.

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-2\vec{i} + 3\vec{k}) \times (-\vec{i} - \vec{j} + \vec{k}) = 3\vec{i} - \vec{j} + 2\vec{k}.$$

(c) The normal vector to the plane is $\vec{n} = 3\vec{i} - \vec{j} + 2\vec{k}$, so the equation is of the form

$$3x - y + 2z = d.$$

Substituting, for example, $x = 1$, $y = 0$, $z = 1$ gives $d = 5$:

$$3x - y + 2z = 5.$$

53. (a) If we let $\overrightarrow{PQ}$ in Figure 13.45 be the vector from point P to point Q and $\overrightarrow{PR}$ be the vector from P to R, then

$$\overrightarrow{PQ} = -\vec{i} + 2\vec{k}$$
$$\overrightarrow{PR} = 2\vec{i} - \vec{k},$$

then the area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is:

$$\text{Area of parallelogram} = \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} \right\| = \|3\vec{j}\| = 3.$$

Thus, the area of the triangle PQR is

$$\begin{pmatrix} \text{Area of} \\ \text{triangle} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \text{Area of} \\ \text{parallelogram} \end{pmatrix} = \frac{3}{2} = 1.5.$$

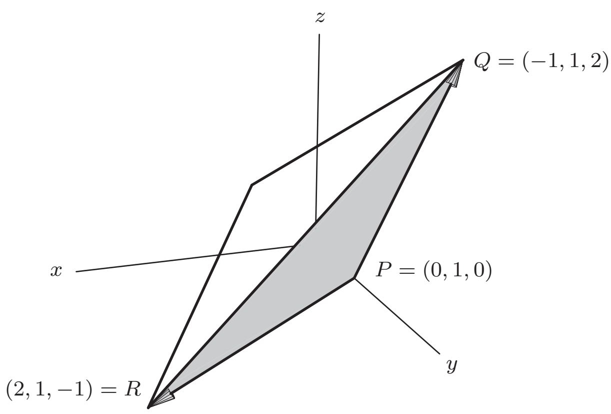


Figure 13.45

(b) Since $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane PQR, and from above, we have $\vec{n} = 3\vec{j}$, the equation of the plane has the form $3y = C$. At the point $(0, 1, 0)$ we get $3 = C$, therefore $3y = 3$, i.e., $y = 1$.

54. Find an arbitrary point on the plane $2x + 4y - z = -1$, say $A = (0, 0, 1)$. The normal $\vec{n}$ to the plane at B is $\vec{n} = 2\vec{i} + 4\vec{j} - \vec{k}$ and $\overrightarrow{PA} = -2\vec{i} + \vec{j} - 2\vec{k}$. See Figure 13.46.

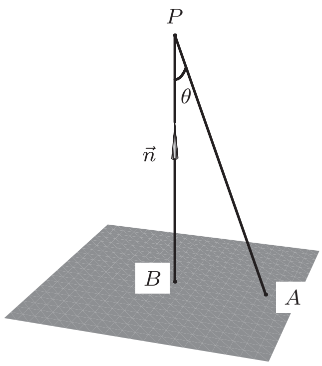


Figure 13.46

So the distance d from the point P to the plane is

$$\begin{aligned} d &= \|\overrightarrow{PB}\| = \|\overrightarrow{PA}\|\cos\theta \\ &= \frac{\overrightarrow{PA}\cdot\vec{n}}{\|\vec{n}\,\|} \quad \text{since } \overrightarrow{PA}\cdot\vec{n} = \|\overrightarrow{PA}\|\|\vec{n}\,\|\cos\theta) \\ &= \frac{(-2\vec{i}+\vec{j}-2\vec{k}\,)\cdot(2\vec{i}+4\vec{j}-\vec{k}\,)}{\sqrt{2^2+4^2+(-1)^2}} \\ &= \frac{2}{\sqrt{21}}. \end{aligned}$$

55. The displacement from $(1,1,1)$ to $(1,4,5)$ is

$$\vec{r_1} = (1-1)\vec{i} + (4-1)\vec{j} + (5-1)\vec{k} = 3\vec{j} + 4\vec{k}\,.$$

The displacement from $(-3,-2,0)$ to $(1,4,5)$ is

$$\vec{r_2} = (1+3)\vec{i} + (4+2)\vec{j} + (5-0)\vec{k} = 4\vec{i} + 6\vec{j} + 5\vec{k}\,.$$

A normal vector is

$$\vec{n} = \vec{r_1}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 4 \\ 4 & 6 & 5 \end{vmatrix} = (15-24)\vec{i} - (-16)\vec{j} + (-12)\vec{k} = -9\vec{i} + 16\vec{j} - 12\vec{k}\,.$$

The equation of the plane is

$$\begin{aligned} -9x + 16y - 12z &= -9\cdot 1 + 16\cdot 1 - 12\cdot 1 = -5 \\ 9x - 16y + 12z &= 5. \end{aligned}$$

We pick a point A on the plane, $A = (\frac{5}{9}, 0, 0)$ and let $P = (0,0,0)$. (See Figure 13.47.) Then $\vec{PA} = (5/9)\vec{i}\,$.

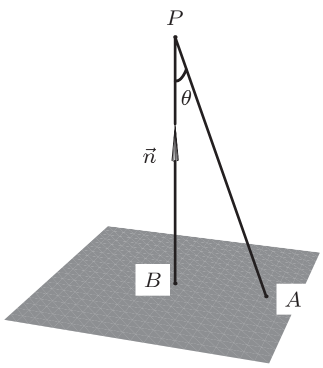


Figure 13.47

So the distance d from the point P to the plane is

$$\begin{aligned} d &= \|\overrightarrow{PB}\| = \|\overrightarrow{PA}\|\cos\theta \\ &= \frac{\overrightarrow{PA}\cdot\vec{n}}{\|\vec{n}\,\|} \quad \text{since } \overrightarrow{PA}\cdot\vec{n} = \|\overrightarrow{PA}\|\|\vec{n}\,\|\cos\theta) \\ &= \left|\frac{(\frac{5}{9}\vec{i}\,)\cdot(-9\vec{i}+16\vec{j}-12\vec{k}\,)}{\sqrt{9^2+16^2+12^2}}\right| \\ &= \frac{5}{\sqrt{481}} = 0.23. \end{aligned}$$

56. (a) 500 km/hr in the west direction, so $\vec{v} = -500\vec{i}$.

(b) While traveling at constant altitude, the plane travels 250 km westward. Thus the coordinates of the point where the plane begins to descend are $(550, 60, 4) - (250, 0, 0) = (300, 60, 4)$.

(c) The vector from the plane to the airport at the time it begins its descent is $(200\vec{i} + 10\vec{j}) - (300\vec{i} + 60\vec{j} + 4\vec{k}) = -100\vec{i} - 50\vec{j} - 4\vec{k}$. Velocity is a vector of length 200 km/hr in the direction of $-100\vec{i} - 50\vec{j} - 4\vec{k}$. Since $\sqrt{(-100)^2 + (-50)^2 + (-4)^2} \approx 111.9$, a unit vector in the direction of descent is $-\frac{100}{111.9}\vec{i} - \frac{50}{111.9}\vec{j} - \frac{4}{111.9}\vec{k}$. Thus

$$\text{Velocity vector} = 200\left(-\frac{100}{111.9}\vec{i} - \frac{50}{111.9}\vec{j} - \frac{4}{111.9}\vec{k}\right) = -178.7\vec{i} - 89.4\vec{j} - 7.2\vec{k}.$$

57. Let $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ be the vector. We will use the properties given in the problem to find v_x, v_y, and v_z. If $\vec{v}$ has magnitude 10, then $\|\vec{v}\| = 10$.

If $\vec{v}$ makes an angle of 45° with the x-axis, then its x-component, v_x, is given by:

$$v_x = \vec{v} \cdot \vec{i} = \|\vec{v}\| \cos 45^\circ = 10\left(\frac{\sqrt{2}}{2}\right) = 7.0710.$$

Similarly, if $\vec{v}$ makes a 75° angle with the y-axis, then its y-component, v_y, is given by:

$$v_y = \vec{v} \cdot \vec{j} = \|\vec{v}\| \cos 75^\circ = 10(0.25882) = 2.5882.$$

We now have two components of $\vec{v}$:

$$\vec{v} = 7.0710\vec{i} + 2.5882\vec{j} + v_z\vec{k}.$$

We only need to find v_z. To do this we use the fact that $\sqrt{\vec{v} \cdot \vec{v}} = \|\vec{v}\| = 10$.

$$\begin{aligned}
\vec{v} \cdot \vec{v} &= 100 \\
v_x^2 + v_y^2 + v_z^2 &= 100 \\
v_z^2 &= 100 - v_x^2 - v_y^2 \\
v_z^2 &= \pm\sqrt{100 - v_x^2 - v_y^2} \\
v_z &= \pm 6.580
\end{aligned}$$

Since the problem tells us that the $\vec{k}$-component is positive, $v_z = +6.580$. Thus

$$\vec{v} = 7.0710\vec{i} + 2.5882\vec{j} + 6.580\vec{k}.$$

58. (a) Suppose $\vec{v} = \overrightarrow{OP}$ as in Figure 13.48. The $\vec{i}$ component of $\overrightarrow{OP}$ is the projection of $\overrightarrow{OP}$ on the x-axis:

$$\overrightarrow{OT} = v \cos\alpha\vec{i}.$$

Similarly, the $\vec{j}$ and $\vec{k}$ components of $\overrightarrow{OP}$ are the projections of $\overrightarrow{OP}$ on the y-axis and the z-axis respectively. So:

$$\begin{aligned}
\overrightarrow{OS} &= v\cos\beta\vec{j} \\
\overrightarrow{OQ} &= v\cos\gamma\vec{k}
\end{aligned}$$

Since $\vec{v} = \overrightarrow{OT} + \overrightarrow{OS} + \overrightarrow{OQ}$, we have

$$\vec{v} = v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}.$$

(b) Since

$$\begin{aligned}
v^2 = \vec{v} \cdot \vec{v} &= (v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}) \cdot \\
&\quad (v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}) \\
&= v^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)
\end{aligned}$$

so

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

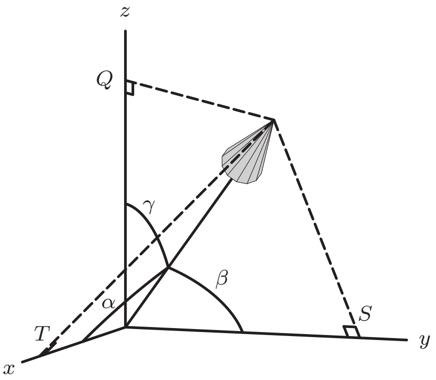


Figure 13.48

59. Let the x-axis point east and the y-axis point north. Denote the forces exerted by Charlie, Sam and Alice by $\vec{F}_C, \vec{F}_S$ and $\vec{F}_A$ (see Figure 13.49).

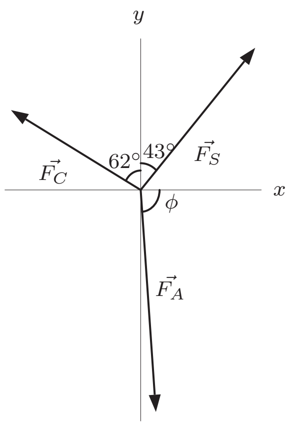


Figure 13.49

Since $\|\vec{F}_C\| = 175$ newtons and the angle θ from the x-axis to $\vec{F}_C$ is $90° + 62° = 152°$, we have

$$\vec{F}_C = 175\cos 152°\vec{i} + 175\sin 152°\vec{j} \approx -154.52\vec{i} + 82.16\vec{j}.$$

Similarly,

$$\vec{F}_S = 200\cos 47°\vec{i} + 200\sin 47°\vec{j} \approx 136.4\vec{i} + 146.27\vec{j}.$$

Now Alice is to counterbalance Sam and Charlie, so the resultant force of the three forces $\vec{F}_C, \vec{F}_S$ and $\vec{F}_A$ must be 0, that is,

$$\vec{F}_C + \vec{F}_S + \vec{F}_A = 0.$$

Thus, we have

$$\begin{aligned}\vec{F}_A &= -\vec{F}_C - \vec{F}_S \\ &\approx -(-154.52\vec{i} + 82.16\vec{j}) - (136.4\vec{i} + 146.27\vec{j}) \\ &= 18.12\vec{i} - 228.43\vec{j}\end{aligned}$$

and, $\|\vec{F}_A\| = \sqrt{18.12^2 + (-228.43)^2} \approx 229.15$ newtons.

If ϕ is the angle from the x-axis to $\vec{F}_A$, then

$$\phi = \arctan\frac{-228.43}{18.12} \approx -85.5°.$$

CAS Challenge Problems

60. $(\vec a \times \vec b) \cdot \vec c = 0$, $(\vec a \times \vec b) \times (\vec a \times \vec c) = \vec 0$

Since $\vec c$ is the sum of a scalar multiple of $\vec a$ and a scalar multiple of $\vec b$, it lies in the plane containing $\vec a$ and $\vec b$. On the other hand, $\vec a \times \vec b$ is perpendicular to this plane, so $\vec a \times \vec b$ is perpendicular to $\vec c$. Therefore, $(\vec a \times \vec b) \cdot \vec c = 0$. Also, $\vec a \times \vec c$ is also perpendicular to the plane, thus parallel to $\vec a \times \vec b$, and thus $(\vec a \times \vec b) \times (\vec a \times \vec c) = \vec 0$.

61. The first parallelepiped has volume

$$|(\vec a \times \vec b) \cdot \vec c| = |ywr - vzr + zus - xws + xvt - yut|.$$

The second has volume $|(\vec a \times \vec b) \cdot (2\vec a - \vec b + \vec c)|$, which also simplifies to $|ywr - vzr + zus - xws + xvt - yut|$. Both parallelepipeds have base with edges $\vec a$ and $\vec b$. The third edge of the first one is $\vec c$ and the third edge of the second one is $\vec c + 2\vec a - \vec b$. Thus the top face of the second parallelepiped is obtained by shifting the top face of the first by $2\vec a - \vec b$. Since this is parallel to the base, the second parallelepiped has the same altitude as the first. Since the volume of a parallelepiped is product of the area of its base with its height, the two parallelepipeds have the same volume.

62. (a) From the geometric definition of the dot product, we have

$$\cos\theta = \frac{|\vec a \cdot \vec b|}{\|\vec a\|\|\vec b\|} = \frac{10}{\sqrt{14}\sqrt{9}}.$$

Using $\sin^2\theta = 1 - \cos^2\theta$, we get

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + y + 2z &= 0 \\ x^2 + y^2 + z^2 &= \|\vec a\|^2\|\vec b\|^2(1 - \cos^2\theta) = (14)(9)\left(1 - \frac{100}{(14)(9)}\right) \end{aligned}$$

Solving these equations we get $x = -1$, $y = -4$, $z = 3$ or $x = 1$, $y = 4$, and $z = -3$. Thus $\vec c = -\vec i - 4\vec j + 3\vec k$ or $\vec c = \vec i + 4\vec j - 3\vec k$.

(b) $\vec a \times \vec b = \vec i + 4\vec j - 3\vec k$. This is the same as one of the answers in part (a). The conditions in part (a) ensured that $\vec c$ is perpendicular to $\vec a$ and $\vec b$ and that it has magnitude $\|\vec a\|\|\vec b\||\sin\theta|$. The cross product is the solution that, in addition, satisfies the right-hand rule.

63. (a) We have

$$\begin{aligned} \|\overrightarrow{AB}\| &= \|2\vec i\| = 2 \\ \|\overrightarrow{AC}\| &= \|\vec i + \sqrt{3}\vec j\| = \sqrt{1+3} = 2 \\ \|\overrightarrow{AD}\| &= \|\vec i + (1/\sqrt{3})\vec j + 2\sqrt{2/3}\vec k\| = \sqrt{1 + (1/3) + (8/3)} = \sqrt{4} = 2 \\ \|\overrightarrow{BC}\| &= \|-\vec i + \sqrt{3}\vec j\| = \sqrt{1+3} = 2 \\ \|\overrightarrow{BD}\| &= \|-\vec i + (1/\sqrt{3})\vec j + 2\sqrt{2/3}\vec k\| = \sqrt{1 + (1/3) + (8/3)} = \sqrt{4} = 2 \\ \|\overrightarrow{CD}\| &= \|(1/\sqrt{3} - \sqrt{3})\vec j + 2\sqrt{2/3}\vec k)\| = \sqrt{(1/3 - 2 + 3) + 8/3} = \sqrt{4} = 2 \end{aligned}$$

Thus all the points are 2 units apart.

(b) By solving the equations

$$\begin{aligned} x^2 + y^2 + z^2 &= (x-2)^2 + y^2 + z^2 \\ x^2 + y^2 + z^2 &= (x-1)^2 + (y - \sqrt{3})^2 + z^2 \\ x^2 + y^2 + z^2 &= (x-1)^2 + (y - 1/\sqrt{3})^2 + (z - 2\sqrt{2/3})^2 \end{aligned}$$

we get $P = (1, 1/\sqrt{3}, \sqrt{6}/6)$.

(c) The cosine of the angle APB is $1/3$ and the angle is 109.471°.

64. (a) $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane containing P, Q, R, and therefore parallel to the normal vector $a\vec i + b\vec j + c\vec k$.

(b)

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= (tv - sw - ty + wy + sz - vz)\vec i + \\ &(-tu + rw + tx - wx - rz + uz)\vec j + (su - rv - sx + vx + ry - uy)\vec k \end{aligned}$$

(c) After substituting $z = (d - ax - by)/c$, $w = (d - au - bv)/c$, $t = (d - ar - bs)/c$ into the result of part (a), and simplifying the expression, we obtain:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \frac{a(s(u-x)+vx-uy+r(-v+y))}{c}\vec{i} + \\ &\frac{b(s(u-x)+vx-uy+r(-v+y))}{c}\vec{j} + (s(u-x)+vx-uy+r(-v+y))\vec{k} \\ &= \frac{(s(u-x)+vx-uy+r(-v+y))}{c}(a\vec{i} + b\vec{j} + c\vec{k}).\end{aligned}$$

Thus $\overrightarrow{PQ} \times \overrightarrow{PR}$ is a scalar multiple of $a\vec{i} + b\vec{j} + c\vec{k}$, and hence parallel to it.

PROJECTS FOR CHAPTER THIRTEEN

1. (a) Let $r = \|\vec{a}\,\|$ and $s = \|\vec{b}\,\|$, and let α, β, be the angles between $\vec{a}$, $\vec{b}$, and the x-axis as shown in the figure. Suppose θ is the angle between $\vec{a}$ and $\vec{b}$. We drew the figure with $\alpha < \beta$ and thus $\beta - \alpha = \theta$. If $\alpha > \beta$, then $\alpha - \beta = \theta$. In both cases we know that

$$\text{Area of parallelogram} = \|\vec{a}\,\|\|\vec{b}\,\|\sin\theta.$$

Using the formula

$$\sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha,$$

and the fact that $a_1 = r\cos\alpha$, $a_2 = r\sin\alpha$, $b_1 = s\cos\beta$, and $b_2 = s\sin\beta$, we get

$$\begin{aligned}a_1b_2 - a_2b_1 &= (r\cos\alpha)(s\sin\beta) - (r\sin\alpha)(s\cos\beta) \\ &= rs(\cos\alpha\sin\beta - \sin\alpha\cos\beta) \\ &= rs\sin(\beta-\alpha) \qquad \text{(from } \sin(\beta-\alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha) \\ &= \|\vec{a}\,\|\|\vec{b}\,\|\sin(\beta-\alpha)\end{aligned}$$

If $\beta > \alpha$, we have $\beta - \alpha = \theta$, so

$$a_1b_2 - a_2b_1 = \|\vec{a}\,\|\|\vec{b}\,\|\sin\theta = \text{Area of parallelogram.}$$

If $\beta < \alpha$, we have $\alpha - \beta = \theta$, so

$$|a_1b_2 - a_2b_1| = \|\vec{a}\,\|\|\vec{b}\,\||\sin(\beta-\alpha)| = \|\vec{a}\,\|\|\vec{b}\,\|\sin\theta = \text{Area of parallelogram.}$$

(b) The sign of $a_1b_2 - a_2b_1$ is the same as the sign of $\beta - \alpha$, so the sign of $a_1b_2 - a_2b_1$ tells us whether the rotation from $\vec{a}$ to $\vec{b}$ is counterclockwise (then $a_1b_2 - a_2b_1$ is positive) or clockwise (then $a_1b_2 - a_2b_1$ is negative).

(c) Part (a) tells us that

$$\text{Area of the parallelogram} = |a_1b_2 - a_2b_1|.$$

The algebraic definition of the cross product is

$$\vec{a} \times \vec{b} = (a_1b_2 - a_2b_1)\vec{k}.$$

The geometric definition has magnitude given by $\|\vec{a} \times \vec{b}\,\| =$ Area of parallelogram. So the magnitude of the algebraic definition agrees with the magnitude of the geometric definition. To check agreement of the direction of $\vec{a} \times \vec{b}$ for the two definitions, we notice that $(a_1b_2 - a_2b_1)\vec{k}$ is perpendicular to $\vec{a}$ and $\vec{b}$ since $\vec{a}$ and $\vec{b}$ are in the $\vec{i}, \vec{j}$-plane. Also, part (b) says $(a_1b_2 - a_2b_1)\vec{k}$ will point up (down) when the rotation from $\vec{a}$ to $\vec{b}$ is counterclockwise (clockwise). So the direction of the algebraic definition obeys the right-hand rule.

2. (a) Since the vectors $\vec{a}_1, \ldots, \vec{a}_4$ show the square roots of the relative frequencies of the alleles, and the relative frequencies in a population add to 1, we have

$$\|\vec{a}_2\| = \sqrt{\sqrt{0.10}^2 + \sqrt{0.09}^2 + \sqrt{0.12}^2 + \sqrt{0.69}^2} = 1$$
$$\|\vec{a}_3\| = \sqrt{\sqrt{0.21}^2 + \sqrt{0.07}^2 + \sqrt{0.06}^2 + \sqrt{0.66}^2} = 1$$
$$\|\vec{a}_4\| = \sqrt{\sqrt{0.22}^2 + \sqrt{0.00}^2 + \sqrt{0.21}^2 + \sqrt{0.57}^2} = 1$$

$$\vec{a}_2 \cdot \vec{a}_3 = \sqrt{0.10}\cdot\sqrt{0.21} + \sqrt{0.09}\cdot\sqrt{0.07} + \sqrt{0.12}\cdot\sqrt{0.06} + \sqrt{0.69}\cdot\sqrt{0.66} = 0.9840$$
$$\vec{a}_3 \cdot \vec{a}_4 = \sqrt{0.21}\cdot\sqrt{0.22} + \sqrt{0.07}\cdot\sqrt{0.00} + \sqrt{0.06}\cdot\sqrt{0.21} + \sqrt{0.66}\cdot\sqrt{0.57} = 0.9405.$$

The distance between the English and the Bantus is given by θ where

$$\cos\theta = \frac{\vec{a}_2 \cdot \vec{a}_3}{\|\vec{a}_2\|\|\vec{a}_3\|} = \frac{0.9840}{1\cdot 1} = 0.9840$$

so $\theta = 10.3^\circ$.
The distance between the English and the Koreans is given by ϕ where

$$\cos\phi = \frac{\vec{a}_3 \cdot \vec{a}_4}{\|\vec{a}_3\|\|\vec{a}_4\|} = \frac{0.9405}{1\cdot 1} = 0.9405$$

so $\phi \approx 19.9^\circ$. Hence the English are genetically closer to the Bantus than to the Koreans.

(b) Let $\vec{f}_1$ be the 4-vector showing the relative frequencies of the alleles in the Eskimo population. Let $\vec{f}_2$, $\vec{f}_3$, $\vec{f}_4$ be the corresponding vectors for the Bantu, English, and Korean populations, respectively. Let $\vec{f}_5$ be the 4-vector for the relative frequencies for the half Eskimo, half Bantu population, and let $\vec{a}_5$ be the 4-vector for the square roots of the relative frequencies . So

$$\vec{f}_5 = \frac{1}{2}\vec{f}_1 + \frac{1}{2}\vec{f}_2 = (0.195, 0.045, 0.075, 0.685)$$
$$\vec{a}_5 = (\sqrt{0.195}, \sqrt{0.045}, \sqrt{0.075}, \sqrt{0.685}).$$

Then

$$\|\vec{a}_5\| = \sqrt{\sqrt{0.195}^2 + \sqrt{0.045}^2 + \sqrt{0.075}^2 + \sqrt{0.685}^2} = 1$$
$$\vec{a}_3 \cdot \vec{a}_5 = \sqrt{0.21}\cdot\sqrt{0.195} + \sqrt{0.07}\cdot\sqrt{0.045} + \sqrt{0.06}\cdot\sqrt{0.075} + \sqrt{0.66}\cdot\sqrt{0.685} = 0.9980.$$

So the distance between the English population and the half Eskimo, half Bantu population is

$$\theta = \arccos\frac{\vec{a}_3 \cdot \vec{a}_5}{\|\vec{a}_3\|\|\vec{a}_5\|} = \arccos\frac{0.9980}{1\cdot 1}$$
$$= \arccos 0.9980 = 3.6^\circ.$$

Since $3.6 < 10.3$, the English are closer to the Bantu/Eskimo mix than to the Bantu alone.

(c) Suppose that x is the fraction of the population that is Eskimo, where $0 \le x \le 1$. Then $(1-x)$ is the fraction that is Bantu. (For example, $x = 0.5$, in part (b).) Let $\vec{f}_6$ be the 4-vector of relative frequencies and $\vec{a}_6$ the 4-vector of square roots of relative frequencies for a population that is x Eskimo and $(1-x)$ Bantu. We have

$$\vec{f}_6 = x\vec{f}_1 + (1-x)\vec{f}_2 = \vec{f}_2 + x(\vec{f}_1 - \vec{f}_2)$$
$$= (0.10 + 0.19x, 0.09 - 0.09x, 0.12 - 0.09x, 0.69 - 0.01x).$$

Then, as before

$$\|\vec{a}_6\| = 1$$

and

$$\vec{a}_3 \cdot \vec{a}_6 = \sqrt{0.21} \cdot \sqrt{0.10 + 0.19x} + \sqrt{0.07} \cdot \sqrt{0.09 - 0.09x} + \sqrt{0.06} \cdot \sqrt{0.12 - 0.09x} + \sqrt{0.66} \cdot \sqrt{0.69 - 0.01x}.$$

Since $\cos\theta$ is a decreasing function of θ for $0 \le \theta \le \pi$, to minimize the angle $\theta = \arccos \dfrac{\vec{a}_3 \cdot \vec{a}_6}{\|\vec{a}_3\|\|\vec{a}_6\|}$, we must maximize

$$f(x) = \frac{\vec{a}_3 \cdot \vec{a}_6}{\|\vec{a}_3\|\|\vec{a}_6\|} = \vec{a}_3 \cdot \vec{a}_6.$$

Using a calculator or computer, we find that the maximum of this function for $0 \le x \le 1$ is

$$f(0.4788) = 0.9980.$$

So the minimum distance of $\theta = \arccos(0.9980) = 3.6^{\circ}$ from the English occurs at a mix of about 47.88% Eskimo and 52.12% Bantu.

3. (a) Let the forces $\vec{F}_1$ from bar AB on joint A and $\vec{F}_2$ from AE on A be given by

$$\begin{aligned} \vec{F}_1 &= a\vec{i} \\ \vec{F}_2 &= f(\cos 65.38^{\circ}\, \vec{i} + \sin 65.38^{\circ}\, \vec{j}\,). \end{aligned}$$

The sum of $\vec{F}_1$, $\vec{F}_2$, and the upward supporting force at A must be the zero vector. Hence

$$\begin{aligned} \vec{F}_1 + \vec{F}_2 + 12500\vec{j} &= \vec{0} \\ (a + f\cos 65.38^{\circ})\vec{i} + (f \sin 65.38^{\circ} + 12500)\vec{j} &= \vec{0} \\ a + f\cos 65.38^{\circ} &= 0 \\ f\sin 65.38^{\circ} + 12500 &= 0. \end{aligned}$$

Solving the last two equations for f and a gives

$$\begin{aligned} f &= \frac{-12500}{\sin 65.38^{\circ}} = -13750 \text{ lb} \\ a &= -f\cos 65.38^{\circ} = 5730 \text{ lb}. \end{aligned}$$

There is a 5730 lb force from AB acting to the right on joint A. Since the bar is pulling the joint, AB is under 5730 lb tension.

There is a 13750 lb force from AE acting downward on joint A. Since the bar is pushing the joint, AE is under 13750 lb compression.

(b) Let the forces $\vec{G}_1$ from bar BC on joint C and $\vec{G}_2$ from CD on C be given by

$$\begin{aligned} \vec{G}_1 &= b\vec{i} \\ \vec{G}_2 &= g(\cos 114.62^{\circ}\, \vec{i} + \sin 114.62^{\circ}\, \vec{j}\,). \end{aligned}$$

The sum of $\vec{G}_1$, $\vec{G}_2$, and the upward supporting force at C must be the zero vector. Hence

$$\begin{aligned} \vec{G}_1 + \vec{G}_2 + 17500\vec{j} &= \vec{0} \\ (b + g\cos 114.62^{\circ})\vec{i} + (g \sin 114.62^{\circ} + 17500)\vec{j} &= \vec{0} \\ b + g\cos 114.62^{\circ} &= 0 \\ g\sin 114.62^{\circ} + 17500 &= 0. \end{aligned}$$

Solving the last two equations for g and b gives

$$\begin{aligned} g &= \frac{-17500}{\sin 114.62^{\circ}} = -19250 \text{ lb} \\ b &= -g\cos 114.62^{\circ} = -8020 \text{ lb}. \end{aligned}$$

There is an 8020 lb force from BC acting to the left on joint C. Since the bar is pulling the joint, BC is under 8020 lb tension.

There is a 19250 lb force from CD acting downward on joint C. Since the bar is pushing the joint, CD is under 19250 lb compression.

(c) Let the forces $\vec{H}_1$ from bar DE on joint D and $\vec{H}_2$ from BD on D be given by

$$\begin{aligned}\vec{H}_1 &= c\vec{i}\\ \vec{H}_2 &= h(\cos 65.38°\,\vec{i} + \sin 65.38°\,\vec{j}\,).\end{aligned}$$

The force $\vec{H}_3$ from CD on D is the opposite of force $\vec{G}_2$ of CD on C computed in part (c). The sum of the forces $\vec{H}_1$, $\vec{H}_2$, $\vec{H}_3$ and the downward force from the weight at D must be the zero vector. Hence

$$\begin{aligned}\vec{H}_1 + \vec{H}_2 + \vec{H}_2 - 20000\vec{j} &= \vec{0}\\ (c + h\cos 65.38° - g\cos 114.62°)\vec{i} + (h\sin 63.38° - g\sin 114.62° - 20000)\vec{j} &= \vec{0}\\ c + h\cos 65.38° - g\cos 114.62° &= 0\\ h\sin 63.38° - g\sin 114.62° - 20000 &= 0.\end{aligned}$$

Since we found g in part (b) we can solve the last two equations for h and c. We have

$$\begin{aligned}h &= \frac{g\sin 114.62° + 20000}{\sin 65.38°} = 2750 \text{ lb}\\ c &= g\cos 114.62° - h\cos 65.38 = 6880 \text{ lb}.\end{aligned}$$

There is an 6880 lb force from DE acting to the right on joint D. Since the bar is pushing the joint, DE is under 6880 lb compression.

There is a 2750 lb force from BD acting upward on joint D. Since the bar is pushing the joint, BD is under 2750 lb compression.

(d) Let the force $\vec{P}$ from bar BE on joint E be given by

$$\vec{P} = p(\cos 114.62°\,\vec{i} + \sin 114.62°\,\vec{j}\,).$$

The other forces acting on E are $-\vec{F}_2$ from AE computed in part (a), $-\vec{H}_1$ from DE computed in part (c), and the downward force from the weight at E. The sum of these four forces must be the zero vector. Hence

$$\begin{aligned}\vec{P} - \vec{F}_2 - \vec{H}_1 - 10000\vec{j} &= \vec{0}\\ (p\cos 114.62° - f\cos 65.38° - c)\vec{i} + (p\sin 114.62° - f\sin 65.38° - 10000)\vec{j} &= \vec{0}\\ p\cos 114.62° - f\cos 65.38° - c &= 0\\ p\sin 114.62° - f\sin 65.38° - 10000 &= 0.\end{aligned}$$

Since we found f in part (a) and c in part (c) we can solve either of the last two equations for p. Using the last equation, we have

$$p = \frac{f\sin 65.38° + 10000}{\sin 114.62°} = -2750 \text{ lb}$$

There is an 2750 lb force from BE acting downward right on joint E. Since the bar is pulling the joint, BE is under 2750 lb tension.

CHAPTER FOURTEEN

Solutions for Section 14.1

Exercises

1. Using difference quotients to approximate the partial derivatives

$$f_x(3,2) \approx \frac{\Delta z}{\Delta x} = \frac{0-2}{6-1} = -\frac{2}{5}$$
$$f_y(3,2) \approx \frac{\Delta z}{\Delta y} = \frac{2-(-1)}{5-0} = \frac{3}{5}.$$

2. If h is small, then

$$f_x(3,2) \approx \frac{f(3+h,2)-f(3,2)}{h}.$$

With $h = 0.01$, we find

$$f_x(3,2) \approx \frac{f(3.01,2)-f(3,2)}{0.01} = \frac{\frac{3.01^2}{(2+1)} - \frac{3^2}{(2+1)}}{0.01} = 2.00333.$$

With $h = 0.0001$, we get

$$f_x(3,2) \approx \frac{f(3.0001,2)-f(3,2)}{0.0001} = \frac{\frac{3.0001^2}{(2+1)} - \frac{3^2}{(2+1)}}{0.0001} = 2.0000333.$$

Since the difference quotient seems to be approaching 2 as h gets smaller, we conclude

$$f_x(3,2) \approx 2.$$

To estimate $f_y(3,2)$, we use

$$f_y(3,2) \approx \frac{f(3,2+h)-f(3,2)}{h}.$$

With $h = 0.01$, we get

$$f_y(3,2) \approx \frac{f(3,2.01)-f(3,2)}{0.01} = \frac{\frac{3^2}{(2.01+1)} - \frac{3^2}{(2+1)}}{0.01} = -0.99668.$$

With $h = 0.0001$, we get

$$f_y(3,2) \approx \frac{f(3,2.0001)-f(3,2)}{0.0001} = \frac{\frac{3^2}{(2.0001+1)} - \frac{3^2}{(2+1)}}{0.0001} = -0.9999667.$$

Thus, it seems that the difference quotient is approaching -1, so we estimate

$$f_y(3,2) \approx -1.$$

3. Using first $\Delta x = 0.1$ and $\Delta y = 0.1$, we have the estimates:

$$f_x(1,3) \approx \frac{f(1.1,3)-f(1,3)}{0.1}$$
$$= \frac{0.0470-0.0519}{0.1} = -0.0493,$$

and

$$f_y(1,3) \approx \frac{f(1,3.1)-f(1,3)}{0.1}$$
$$= \frac{0.0153-0.0519}{0.1} = -0.3660.$$

Now, using $\Delta x = 0.01$ and $\Delta y = 0.01$, we have the estimates:

$$f_x(1,3) \approx \frac{f(1.01,3)-f(1,3)}{0.01}$$
$$= \frac{0.0514-0.0519}{0.01} = -0.0501,$$

and

$$f_y(1,3) \approx \frac{f(1,3.01)-f(1,3)}{0.01}$$
$$= \frac{0.0483-0.0519}{0.01} = -0.3629.$$

4. (a) Dollars/Year.
(b) Negative. You expect to pay less for an older car.
(c) Dollars/Dollar
(d) Positive. You expect to pay more for a car that was more expensive new.

5. $\partial P/\partial t$: The unit is dollars per month. This is the rate at which payments change as the number of months it takes to pay off the loan changes. The sign is negative because payments decrease as the pay-off time increases.

$\partial P/\partial r$: The unit is dollars per percentage point. This is the rate at which payments change as the interest rate changes. The sign is positive because payments increase as the interest rate increases.

6. (a) The units of $\partial c/\partial x$ are units of concentration/distance. (For example, (gm/cm^3)/cm.) The practical interpretation of $\partial c/\partial x$ is the rate of change of concentration with distance as you move down the blood vessel at a fixed time. We expect $\partial c/\partial x < 0$ because the further away you get from the point of injection, the less of the drug you would expect to find (at a fixed time).
(b) The units of $\partial c/\partial t$ are units of concentration/time. (For example, (gm/cm^3)/sec.) The practical interpretation of $\partial c/\partial t$ is the rate of change of concentration with time, as you look at a particular point in the blood vessel. We would expect the concentration to first increase (as the drug reaches the point) and then decrease as the drug dies away. Thus, we expect $\partial c/\partial t > 0$ for small t and $\partial c/\partial t < 0$ for large t.

7. (a) If you borrow \$8000 at an interest rate of 1% per month and pay it off in 24 months, your monthly payments are \$376.59.
(b) The increase in your monthly payments for borrowing an extra dollar under the same terms as in (a) is about 4.7 cents.
(c) If you borrow the same amount of money for the same time period as in (a), but if the interest rate increases by 1%, the increase in your monthly payments is about \$44.83.

8. (a) We expect f_p to be negative because if the price of the product increases, the sales usually decrease.
(b) If the price of the product is \$8 per unit and if \$12000 has been spent on advertising, sales increase by approximately 150 units if an additional \$1000 is spent on advertising.

9. (a) Negative. As the price of beef goes up, we expect people to buy less beef and so the quantity of beef sold goes down. If b increases, we expect Q to decrease.
(b) Positive. As the price of chicken goes up, we expect people to buy more beef. As c increases, we expect Q to increase.
(c) We estimate that

$$\frac{\Delta Q}{\Delta b} \approx \frac{-213}{1} \text{ kg/dollar}.$$

If the price of beef rose by one dollar, the store would sell approximately 213 fewer kilograms of beef.

10. Moving right from P in the direction of increasing x increases f, so $f_x(P) > 0$.
Moving up from P in the direction of increasing y increases f, so $f_y(P) > 0$.

11. Moving right from Q in the direction of increasing x increases f, so $f_x(Q) > 0$.
Moving up from Q in the direction of increasing y decreases f, so $f_y(Q) < 0$.

12. Moving right from R in the direction of increasing x decreases f, so $f_x(R) < 0$.
Moving up from R in the direction of increasing y decreases f, so $f_y(R) < 0$.

13. Moving right from S in the direction of increasing x decreases f, so $f_x(S) < 0$.
Moving up from S in the direction of increasing y increases f, so $f_y(S) > 0$.

14. For $f_w(10, 25)$ we get

$$f_w(10, 25) \approx \frac{f(10+h, 25) - f(10, 25)}{h}.$$

Choosing $h = 5$ and reading values from Table 12.2 on page 672 of the text, we get

$$f_w(10, 25) \approx \frac{f(15, 25) - f(10, 25)}{5} = \frac{13 - 15}{5} = -0.4^\circ\text{F/mph}$$

This means that when the wind speed is 10 mph and the true temperature is 25°F, as the wind speed increases from 10 mph by 1 mph we feel an approximately 0.4°F drop in temperature. This rate is negative because the temperature you feel drops as the wind speed increases.

15. Using a difference quotient with $h = 5$, we get

$$f_T(5, 20) \approx \frac{f(5, 20+5) - f(5, 20)}{5} = \frac{19 - 13}{5} = 1.2^\circ\text{F}/^\circ\text{F}.$$

This means that when the wind speed is 5 mph and the true temperature is 20°F, the apparent temperature increases by approximately 1.2°F for every increase of 1°F in the true temperature. This rate is positive because the true temperature you feel increases as true temperature increases.

16. Since the average rate of change of the temperature adjusted for wind-chill is about -0.8 (drops by 0.8°F), with every 1 mph increase in wind speed from 5 mph to 10 mph, when the true temperature stays constant at 20°F, we know that

$$f_w(5, 20) \approx -0.8.$$

Problems

17. The values of z increase as we move in the direction of increasing x-values, so f_x is positive. The values of z decrease as we move in the direction of increasing y-values, so f_y is negative. We see in the contour diagram that $f(2, 1) = 10$. We estimate the partial derivatives:

$$f_x(2, 1) \approx \frac{\Delta z}{\Delta x} = \frac{14 - 10}{4 - 2} = 2,$$
$$f_y(2, 1) \approx \frac{\Delta z}{\Delta y} = \frac{6 - 10}{2 - 1} = -4.$$

18. From the contour diagram, approximate values of f at nearby x-values are $f(3, 5) = 10$, $f(6.3, 5) = 8$, $f(0.4, 5) = 12$. Difference quotient approximations are

$$f_x(3, 5) \approx \frac{f(6.3) - f(3)}{6.3 - 3} = -0.61 \qquad f_x(3, 5) \approx \frac{f(0.4) - f(3)}{0.4 - 3} = -0.77.$$

Another reasonable approximation is obtained by averaging the two difference quotients:

$$f_x(3, 5) \approx \text{ Average } = \frac{-0.61 - 0.77}{2} = -0.7.$$

19. The partial derivative, $\partial Q/\partial b$ is the rate of change of the quantity of beef purchased with respect to the price of beef, when the price of chicken stays constant. If the price of beef increases and the price of chicken stays the same, we expect consumers to buy less beef and more chicken. Thus when b increases, we expect Q to decrease, so $\partial Q/\partial b < 0$.

On the other hand, $\partial Q/\partial c$ is the rate of change of the quantity of beef purchased with respect to the price of chicken, when the price of beef stays constant. An increase in the price of chicken is likely to cause consumers to buy less chicken and more beef. Thus when c increases, we expect Q to increase, so $\partial Q/\partial c > 0$.

20. (a) An increase in the price of a new car will decrease the number of cars bought annually. Thus $\dfrac{\partial q_1}{\partial x} < 0$. Similarly, an increase in the price of gasoline will decrease the amount of gas sold, implying $\dfrac{\partial q_2}{\partial y} < 0$.

(b) Since the demands for a car and gas complement each other, an increase in the price of gasoline will decrease the total number of cars bought. Thus $\dfrac{\partial q_1}{\partial y} < 0$. Similarly, we may expect $\dfrac{\partial q_2}{\partial x} < 0$.

21. We have

$$f_t(18,6) \approx \frac{\Delta P}{\Delta t} = \frac{90-93}{20-18} = -1.5 \text{ percent/month}.$$

Eighteen months after rats are exposed to a formaldehyde concentration of 6 ppm, the percent of rats surviving is decreasing at a rate of about 1.5 per month. In other words, during the eighteenth month, an additional 1.5% of the rats die.

We have

$$f_c(18,6) \approx \frac{\Delta P}{\Delta c} = \frac{82-93}{15-6} = -1.22 \text{ percent/ppm}.$$

If the original concentration increases by 1 ppm, the percent surviving after 18 months decreases by about 1.22.

22. The fact that $f_x(P) > 0$ tells us that the values of the function on the contours increase as we move to the right in Figure 14.3 past the point P. Thus, the values of the function on the contours

(a) Decrease as we move upward past P. Thus $f_y(P) < 0$.
(b) Decrease as we move upward past Q (since Q and P are on the same contour line.) Thus $f_y(Q) < 0$.
(c) Decrease as we move to the right past Q. Thus $f_x(Q) < 0$

23. (a) The partial derivative $g_x(x,y)$ is zero at points where a contour has a horizontal tangent line. See Figure 14.1.
(b) The partial derivative $g_y(x,y)$ is zero at points where a contour has a vertical tangent line. See Figure 14.2.

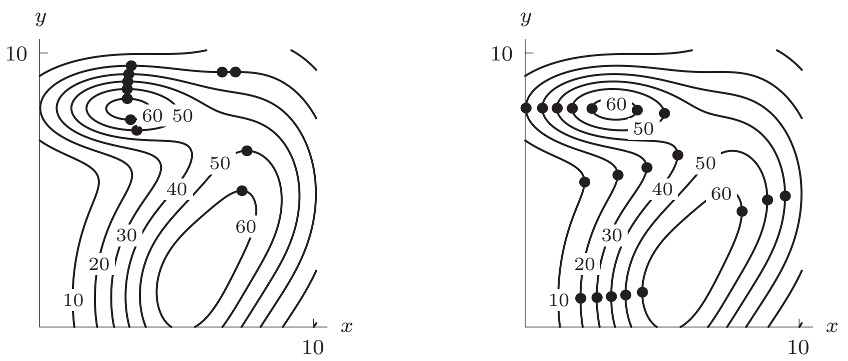


Figure 14.1 **Figure 14.2**

24. (a) (i) Near A, the value of z increases as x increases, so $f_x(A) > 0$.
(ii) Near A, the value of z decreases as y increases, so $f_y(A) < 0$.

(b) $f_x(P)$ changes from positive to negative as P moves from A to B along a straight line, because after P crosses the y-axis, z decreases as x increases near P.
$f_y(P)$ does not change sign as P moves from A to B along a straight line; it is negative along AB.

25. (a) For points near the point $(0,5,3)$, moving in the positive x direction, the surface is sloping down and the function is decreasing. Thus, $f_x(0,5) < 0$.
(b) Moving in the positive y direction near this point the surface slopes up as the function increases, so $f_y(0,5) > 0$.

26. Locating the points $(3, 2, f(3, 2))$ and $(1, 2, f(1, 2))$ on the graph in Figure 14.3 we see that $f_x(1, 2)$ and $f_x(3, 2)$ are both negative with $f_x(1, 2) < f_x(3, 2)$. Similarly, $f_y(3, 2)$ and $f_y(1, 2)$ are both positive with $f_y(3, 2) < f_y(1, 2)$. Therefore,

$$f_x(1, 2) < f_x(3, 2) < 0 < f_y(3, 2) < f_y(1, 2).$$

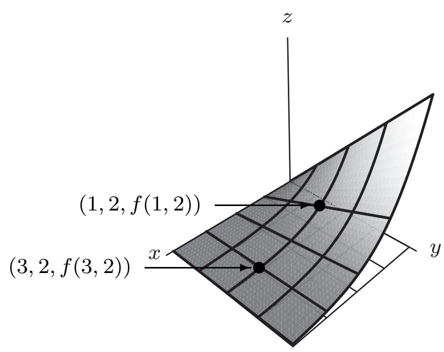


Figure 14.3

27. (a) Estimate $\partial P/\partial r$ and $\partial P/\partial L$ by using difference quotients and reading values of P from the graph:

$$\frac{\partial P}{\partial r}(8, 4000) \approx \frac{P(16, 4000) - P(8, 4000)}{16 - 8} = \frac{100 - 80}{8} = 2.5,$$

and

$$\frac{\partial P}{\partial L} \approx \frac{P(8, 5000) - P(8, 4000)}{5000 - 4000} = \frac{100 - 80}{1000} = 0.02.$$

$P_r(8, 4000) \approx 2.5$ means that at an interest rate of 8% and a loan amount of \$4000 the monthly payment increases by approximately \$2.50 for every one percent increase of the interest rate. $P_L(8, 4000) \approx 0.02$ means the monthly payment increases by approximately \$0.02 for every \$1 increase in the loan amount at an 8% rate and a loan amount of \$4000.

(b) Using difference quotients and reading from the graph

$$\frac{\partial P}{\partial r}(8, 6000) \approx \frac{P(14, 6000) - P(8, 6000)}{14 - 8} = \frac{140 - 120}{6} = 3.33,$$

and

$$\frac{\partial P}{\partial L}(8, 6000) \approx \frac{P(8, 7000) - P(8, 6000)}{7000 - 6000} = \frac{140 - 120}{1000} = 0.02.$$

Again, we see that the monthly payment increases with increases in interest rate and loan amount. The interest rate is $r = 8\%$ as in part (a), but here the loan amount is $L = \$6000$. Since $P_L(8, 4000) \approx P_L(8, 6000)$, the increase in monthly payment per unit increase in loan amount remains the same as in part a). However, in this case, the effect of the interest rate is different: here the monthly payment increases by approximately \$3.33 for every one percent increase of interest rate at $r = 8\%$ and loan amount of \$6000.

(c)

$$\frac{\partial P}{\partial r}(13, 7000) \approx \frac{P(19, 7000) - P(13, 7000)}{19 - 13} = \frac{180 - 160}{6} = 3.33,$$

and

$$\frac{\partial P}{\partial L}(13,7000) \approx \frac{P(13,8000)-P(13,7000)}{8000-7000} = \frac{180-160}{1000} = 0.02.$$

The figures show that the rates of change of the monthly payment with respect to the interest rate and loan amount are roughly the same for $(r, L) = (8, 6000)$ and $(r, L) = (13, 7000)$.

28. The sign of $\partial f/\partial P_1$ tells you whether f (the number of people who ride the bus) increases or decreases when P_1 is increased. Since P_1 is the price of taking the bus, as it increases, f should decrease. This is because fewer people will be willing to pay the higher price, and more people will choose to ride the train. On the other hand, the sign of $\dfrac{\partial f}{\partial P_2}$ tells you the change in f as P_2 increases. Since P_2 is the cost of riding the train, as it increases, f should increase. This is because fewer people will be willing to pay the higher fares for the train, and more people will choose to ride the bus.

Therefore, $\dfrac{\partial f}{\partial P_1} < 0$ and $\dfrac{\partial f}{\partial P_2} > 0$.

29. (i) Statement (i) indicates that, as v increases at a constant temperature, W will decrease. Therefore, $f_v(T, v) < 0$, and so statement (i) matches formula (c).

(ii) Statement (ii) indicates that, as T increases at a constant riding speed, W also increases. Therefore, $f_T(T, v) > 0$, and so statement (ii) matches formula (a).

We now see that formula (c) does not match either given statement. In words, the statement "$f(0, v) \leq 0$" is saying that, if the air temperature is held constant at $0°$F then, no matter what speed you are biking at, you will feel at least as cold as $0°$F.

30. (a) Estimating $T(x, t)$ from the figure in the text at $x = 15$, $t = 20$ gives

$$\left.\frac{\partial T}{\partial x}\right|_{(15,20)} \approx \frac{T(23,20)-T(15,20)}{23-15} = \frac{20-23}{8} = -\frac{3}{8}\ °\text{C per m},$$

$$\left.\frac{\partial T}{\partial t}\right|_{(15,20)} \approx \frac{T(15,25)-T(15,20)}{25-20} = \frac{25-23}{5} = \frac{2}{5}\ °\text{C per min}.$$

At 15 m from heater at time $t = 20$ min, the room temperature decreases by approximately $3/8°$C per meter and increases by approximately $2/5°$C per minute.

(b) We have the estimates,

$$\left.\frac{\partial T}{\partial x}\right|_{(5,12)} \approx \frac{T(7,12)-T(5,12)}{7-5} = \frac{25-27}{2} = -1\ °\text{C per m},$$

$$\left.\frac{\partial T}{\partial t}\right|_{(5,12)} \approx \frac{T(5,40)-T(5,12)}{40-12} = \frac{30-27}{28} = \frac{3}{28}\ °\text{C per min}.$$

At $x = 5$, $t = 12$ the temperature decreases by approximately $1°$C per meter and increases by approximately $3/28°$C per minute.

31. The quantity $H_T(10, 0.1)$ is approximated by a difference quotient. The first partial derivative with respect to T is approximated by

$$H_T(10,0.1) \approx \frac{H(10+\Delta T,0.1)-H(10,0.1)}{\Delta T} \quad \text{for small } \Delta T.$$

We are free to choose ΔT. If we take $H(10,\ 0.1) = 110$ and $H(20,\ 0.1) = 100$, we get the approximation

$$H_T(10,0.1) \approx \frac{H(20,0.1)-H(10,0.1)}{10} = \frac{95-120}{10} = -2.5.$$

(Note that you may get a different answer if you read different values from the graph.) The geometric meaning of the partial derivative $H_T(10, 0.01)$ that we just approximated is the slope of the curve shown in the figure in the text corresponding to $w = 0.1$ at the point where $T = 10$. In practical terms, we have found that for fog at $10°$ C containing 0.1 g water per m^3 of fog, a $1°$ C increase in temperature will reduce the heat requirement for dissipating the fog by about 1 calories per cubic meter of fog.

32. Reading values of H from the graph gives Table 14.1. In order to compute $H_T(T, w)$ at $T = 30$, it is useful to have values of $H(T, w)$ for $T = 40°$C. The column corresponding to $w = 0.4$ is not used in this problem.

Table 14.1 *Estimated values of $H(T, w)$ (in calories/meter^3)*

T (°C) \ w (gm/m^3)	0.1	0.2	0.3	0.4
10	110	240	330	450
20	100	180	260	350
30	70	150	220	300
40	65	140	200	270

Table 14.2 *Estimated values of $H_T(T, w)$ (in calories/meter^3/°C)*

T (°C) \ w (gm/m^3)	0.1	0.2	0.3
10	−1.0	−6.0	−7.0
20	−3.0	−3.0	−4.0
30	−0.5	−1.0	−2.0

The estimates for $H_T(T, w)$ in Table 14.2 are now computed using the formula

$$H_T(T, w) \approx \frac{H(T + 10, w) - H(T, w)}{10}.$$

33. Values of H from the graph are given in Table 14.3. In order to compute $H_w(T, w)$ for $w = 0.3$, it is useful to have the column corresponding to $w = 0.4$. The row corresponding to $T = 40$ is not used in this problem. The partial derivative $H_w(T, w)$ can be approximated by

$$H_w(10, 0.1) \approx \frac{H(10, 0.1 + h) - H(10, 0.1)}{h} \quad \text{for small } h.$$

We choose $h = 0.1$ because we can read off a value for $H(10, 0.2)$ from the graph. If we take $H(10,\ 0.2) = 240$, we get the approximation

$$H_w(10, 0.1) \approx \frac{H(10, 0.2) - H(10, 0.1)}{0.1} = \frac{240 - 110}{0.1} = 1300.$$

In practical terms, we have found that for fog at 10° C containing 0.1 g/m^3 of water, an increase in the water content of the fog will increase the heat requirement for dissipating the fog at the rate given by $H_w(10, 0.1)$. Specifically, a 1 g/m^3 increase in the water content will increase the heat required to dissipate the fog by about 1300 calories per cubic meter of fog.

Wetter fog is harder to dissipate. Other values of $H_w(T, w)$ in Table 14.4 are computed using the formula

$$H_w(T, w) \approx \frac{H(T, w + 0.1) - H(T, w)}{0.1},$$

where we have used Table 14.3 to evaluate H.

Table 14.3 *Estimated values of $H(T, w)$ (in calories/meter^3)*

T (°C) \ w (gm/m^3)	0.1	0.2	0.3	0.4
10	110	240	330	450
20	100	180	260	350
30	70	150	220	300
40	65	140	200	270

Table 14.4 *Table of values of $H_w(T, w)$ (in cal/gm)*

T (°C) \ w (gm/m^3)	0.1	0.2	0.3
10	1300	900	1200
20	800	800	900
30	800	700	800

34. (a) $\dfrac{\partial p}{\partial c} = f_c(c, s) =$ rate of change in blood pressure as cardiac output increases while systemic vascular resistance remains constant.

(b) Suppose that $p = kcs$. Note that c (cardiac output), a volume, s (SVR), a resistance, and p, a pressure, must all be positive. Thus k must be positive, and our level curves should be confined to the first quadrant. Several level curves are shown in Figure 14.4. Each level curve represents a different blood pressure level. Each point on a given curve is a combination of cardiac output and SVR that results in the blood pressure associated with that curve.

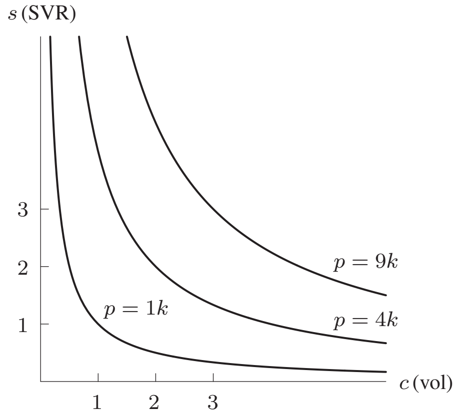


Figure 14.4

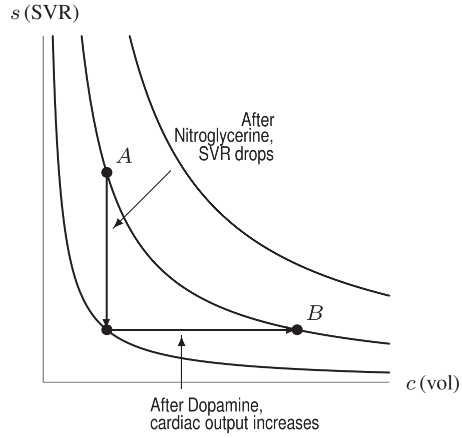


Figure 14.5

(c) Point B in Figure 14.5 shows that if the two doses are correct, the changes in pressure will cancel. The patient's cardiac output will have increased and his SVR will have decreased, but his blood pressure won't have changed.
(d) At point F in Figure 14.6, the patient's blood pressure is normalized, but his/her cardiac output has dropped and his SVR is up.

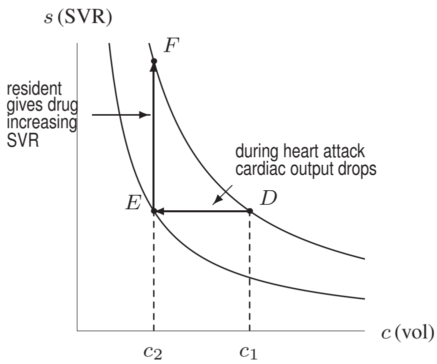


Figure 14.6

Note: c_1 and c_2 are the cardiac outputs before and after the heart attack, respectively.

35. (a) Since $f_x > 0$, the values on the contours increase as you move to the right. Since $f_y > 0$, the values on the contours increase as you move upward. See Figure 14.7.

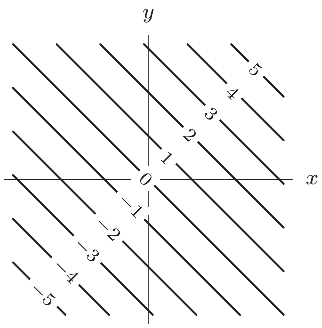


Figure 14.7: $f_x > 0$ and $f_y > 0$

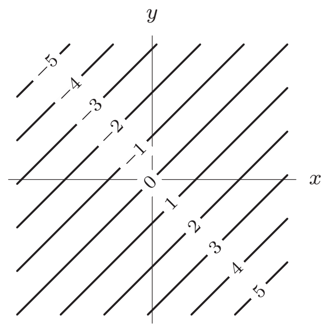


Figure 14.8: $f_x > 0$ and $f_y < 0$

(b) Since $f_x > 0$, the values on the contours increase as you move to the right. Since $f_y < 0$, the values on the contours decrease as you move upward. See Figure 14.8.

(c) Since $f_x < 0$, the values on the contours decrease as you move to the right. Since $f_y > 0$, the values on the contours increase as you move upward. See Figure 14.9.

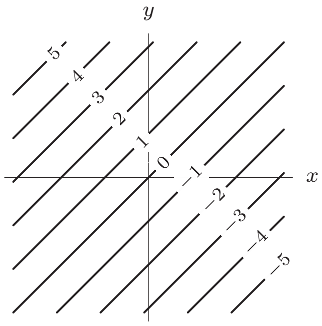


Figure 14.9: $f_x < 0$ and $f_y > 0$

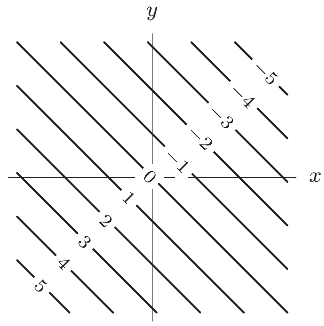


Figure 14.10: $f_x < 0$ and $f_y < 0$

(d) Since $f_x < 0$, the values on the contours decrease as you move to the right. Since $f_y < 0$, the values on the contours decrease as you move upward. See Figure 14.10.

Strengthen Your Understanding

36. The units of $\partial f/\partial x$ and $\partial f/\partial y$ are the same if x and y have the same units. They are different if x and y have different units.

37. If a formula for $f(x, y)$ does not contain y explicitly, then all difference quotients are zero:

$$\frac{f(x, y+h) - f(x, y)}{h} = 0 \text{ for } h \neq 0.$$

Hence the partial derivative with respect to y is defined and $f_y(x, y) = 0$.

38. Since $f_x < 0$, the slope of f in the x-direction is negative. Since $f_y > 0$, the slope of f in the y-direction is positive. A possible table of a linear function satisfying these conditions is:

x \ y	0	5	10
0	50	52	54
10	40	42	44
20	30	32	34

39. One type of function that would satisfy the given condition would be one with $f_x = 4$ and $f_y = -1$. A function with these partial derivatives is $f(x, y) = 4x - y$.

40. True. This is the instantaneous rate of change of f in the x-direction at the point $(10, 20)$.

41. False. The graph of f is a paraboloid or bowl-shape, with lowest point at the origin and opening in the positive z-direction. At $(1, 1)$ the function is increasing in the y-direction, so $f_y(1, 1) > 0$.

42. True. Slicing the graph of f in the xz-plane yields a semicircle, and at the point $x = 0$ this semicircle has a horizontal tangent line. Thus $f_x(0, 0) = 0$. A similar argument shows that $f_y(0, 0) = 0$.

43. False. The units of $\partial P/\partial V$ are the units of P divided by the units of V, which gives $(\text{grams/cm}^3)/\text{cm}^3$, or grams/cm^6.

44. True. The property $f_x(a, b) > 0$ means that f increases in the positive x-direction near (a, b), so f must decrease in the negative x-direction near (a, b).

45. True. When we fix $s = k$, then $g(r, k) = r^2 + k$, whose graph is a parabola. Since this parabola is concave up, we know its slope increases as r increases.

46. True. Using difference quotients to approximate $g_u(1,1)$, we find that

$$\frac{g(1+0.01,1)-g(1,1)}{0.01} = \frac{(2.01)^{1.01}-2^1}{0.01} \approx 2.40816$$
$$\frac{g(1+0.001,1)-g(1,1)}{0.001} = \frac{(2.001)^{1.001}-2^1}{0.001} \approx 2.38847$$
$$\frac{g(1+0.0001,1)-g(1,1)}{0.0001} = \frac{(2.0001)^{1.0001}-2^1}{0.0001} \approx 2.38651.$$

47. False. Increasing m, the number of miles on the engine, for fixed d generally lowers the price of the car, so $\partial P/\partial m < 0$. A higher original cost, d, with fixed m usually gives an increased sales price, so $\partial P/\partial d > 0$.

48. True. A function with constant $f_x(x,y)$ and $f_y(x,y)$ has constant x-slope and constant y-slope, and therefore has a graph which is a plane.

49. False. Having zero for the x and y partial derivatives at a single point means only that cross-sections through the graph have horizontal tangents at that point. For example, $f(x,y) = x^2+y^2$ has $f_x(0,0) = f_y(0,0) = 0$, but f is not constant.

Solutions for Section 14.2

Exercises

1. (a) Make a difference quotient using the two points $(3,2)$ and $(3,2.01)$ that have the same x-coordinate 3 but whose y-coordinates differ by 0.01. We have

$$f_y(3,2) \approx \frac{f(3,2.01)-f(3,2)}{2.01-2} = \frac{28.0701-28}{0.01} = 7.01.$$

(b) Differentiating gives $f_y(x,y) = x+2y$, so $f_y(3,2) = 3+2\cdot 2 = 7$.

2. $f_x(x,y) = 10xy^3+8y^2-6x$ and $f_y(x,y) = 15x^2y^2+16xy$.

3. We have

$$f_x(x,y) = 3x^2+6xy \quad \text{and} \quad f_y(x,y) = 3x^2-4y,$$

so

$$f_x(1,2) = 15 \quad \text{and} \quad f_y(1,2) = -5.$$

4. $\dfrac{\partial}{\partial y}(3x^5y^7-32x^4y^3+5xy) = 21x^5y^6-96x^4y^2+5x$

5. $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}\left[(x^2+x-y)^7\right] = 7(x^2+x-y)^6(2x+1) = (14x+7)(x^2+x-y)^6.$
$\dfrac{\partial z}{\partial y} = \dfrac{\partial}{\partial y}\left[(x^2+x-y)^7\right] = -7(x^2+x-y)^6.$

6. Thinking of A, α, β as constants,

$$f_x = (\alpha+\beta)A^\alpha x^{\alpha+\beta-1}y^{1-\alpha-\beta}$$
$$f_y = (1-\alpha-\beta)A^\alpha x^{\alpha+\beta}y^{-\alpha-\beta}.$$

7. The differentiation is easier if we rewrite f using the rules of logarithms:

$$f(x,y) = 0.6\ln x + 0.4\ln y.$$

Then

$$f_x = \frac{0.6}{x}$$
$$f_y = \frac{0.4}{y}.$$

8. $z_x = \dfrac{1}{2ay}(-2)\dfrac{1}{x^3} + \dfrac{15x^4abc}{y} = -\dfrac{1}{ax^3y} + \dfrac{15abcx^4}{y} = \dfrac{15a^2bcx^7 - 1}{ax^3y}$

9. $z_x = 2xy + 10x^4y$

10. $V_r = \frac{2}{3}\pi rh$

11. $\dfrac{\partial}{\partial T}\left(\dfrac{2\pi r}{T}\right) = -\dfrac{2\pi r}{T^2}$

12. $\dfrac{\partial}{\partial x}(a\sqrt{x}) = a \cdot \dfrac{1}{2}x^{-1/2} = \dfrac{a}{2\sqrt{x}}$

13. $\dfrac{\partial}{\partial x}(xe^{\sqrt{xy}}) = e^{\sqrt{xy}} + xe^{\sqrt{xy}} \cdot \dfrac{1}{2}(xy)^{-1/2}y = e^{\sqrt{xy}}(1 + \dfrac{xy}{2\sqrt{xy}}) = e^{\sqrt{xy}}(1 + \dfrac{\sqrt{xy}}{2})$

14. We have

$$\frac{\partial}{\partial t}e^{\sin(x+ct)} = e^{\sin(x+ct)} \cdot \cos(x+ct) \cdot c.$$

15. $F_m = g$

16. $a_v = \dfrac{2v}{r}$

17. $\dfrac{\partial A}{\partial h} = \dfrac{1}{2}(a+b)$

18. $\dfrac{\partial}{\partial m}\left(\dfrac{1}{2}mv^2\right) = \dfrac{1}{2}v^2$

19. $\dfrac{\partial}{\partial B}\left(\dfrac{1}{u_0}B^2\right) = \dfrac{2B}{u_0}$

20. $\dfrac{\partial}{\partial r}\left(\dfrac{2\pi r}{v}\right) = \dfrac{2\pi}{v}$

21. $F_v = \dfrac{2mv}{r}$

22. $\dfrac{\partial}{\partial v_0}(v_0 + at) = 1 + 0 = 1$

23. $\dfrac{\partial F}{\partial m_2} = \dfrac{Gm_1}{r^2}$

24. With a constant, we have

$$\frac{\partial}{\partial x}\left(\frac{1}{a}e^{-x^2/a^2}\right) = \frac{1}{a}e^{-x^2/a^2}\left(-\frac{2x}{a^2}\right) = -\frac{2x}{a^3}e^{-x^2/a^2}.$$

25. With x constant, we use the product and chain rules

$$\frac{\partial}{\partial a}\left(\frac{1}{a}e^{-x^2/a^2}\right) = -\frac{1}{a^2}e^{-x^2/a^2} + \frac{1}{a}e^{-x^2/a^2}\left(\frac{2x^2}{a^3}\right) = -\frac{e^{-x^2/a^2}}{a^4}\left(a^2 - 2x^2\right).$$

26. $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left[e^{xy}(\ln y)\right] = \dfrac{\partial e^{xy}}{\partial x}\ln y = ye^{xy}(\ln y).$

27. $\dfrac{\partial}{\partial t}(v_0t + \dfrac{1}{2}at^2) = v_0 + \dfrac{1}{2} \cdot 2at = v_0 + at$

28. Since we take the partial derivative with respect to θ, we think of π and ϕ as constant. Thus,

$$\frac{\partial}{\partial\theta}(\cos(\pi\theta\phi) + \ln(\theta^2 + \phi)) = \cos(\pi\theta\phi) \cdot \pi\phi + \frac{1}{\theta^2 + \phi} \cdot 2\theta.$$

29. $\dfrac{\partial}{\partial M}\left(\dfrac{2\pi r^{3/2}}{\sqrt{GM}}\right) = 2\pi r^{3/2}(-\dfrac{1}{2})(GM)^{-3/2}(G) = -\pi r^{3/2} \cdot \dfrac{G}{GM\sqrt{GM}} = -\dfrac{\pi r^{3/2}}{M\sqrt{GM}}$

30. $f_a = e^a\sin(a+b) + e^a\cos(a+b)$

31. $z_x = \cos(5x^3y - 3xy^2) \cdot (15x^2y - 3y^2) = (15x^2y - 3y^2)\cos(5x^3y - 3xy^2)$

32. $g_x(x,y) = \dfrac{\partial}{\partial x}\ln(ye^{xy}) = (ye^{xy})^{-1}\dfrac{\partial}{\partial x}(ye^{xy}) = (ye^{xy})^{-1} \cdot y\dfrac{\partial}{\partial x}(e^{xy}) = (ye^{xy})^{-1} \cdot y \cdot y \cdot e^{xy} = y$

33. $\dfrac{\partial F}{\partial L} = \dfrac{\partial}{\partial L}\left[3\sqrt{LK}\right] = 3\dfrac{\partial}{\partial L}\left[(LK)^{1/2}\right] = \dfrac{3K}{2\sqrt{LK}} = \dfrac{3}{2}\sqrt{\dfrac{K}{L}}.$

34. $\dfrac{\partial V}{\partial r} = \dfrac{8}{3}\pi rh$ and $\dfrac{\partial V}{\partial h} = \dfrac{4}{3}\pi r^2$.

35. $u_E = \frac{1}{2}\epsilon_0 \cdot 2E + 0 = \epsilon_0 E$

36. $\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}\right) = \dfrac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)} \cdot \left(\dfrac{-2(x-\mu)}{2\sigma^2}\right) = \dfrac{-(x-\mu)}{\sqrt{2\pi}\sigma^3}e^{-(x-\mu)^2/(2\sigma^2)}.$

37. $\dfrac{\partial Q}{\partial K} = c\gamma(a_1K^{b_1} + a_2L^{b_2})^{\gamma-1} \cdot a_1b_1K^{b_1-1}$

38. $z_x = 7x^6 + yx^{y-1}$, and $z_y = 2^y \ln 2 + x^y \ln x$

39. We regard x as constant and differentiate with respect to y using the product rule:

$$\frac{\partial z}{\partial y} = 2e^{x+2y}\sin y + e^{x+2y}\cos y$$

Substituting $x = 1$, $y = 0.5$ gives

$$\left.\frac{\partial z}{\partial y}\right|_{(1,0.5)} = 2e^2\sin(0.5) + e^2\cos(0.5) = 13.6.$$

40.

$$\frac{\partial f}{\partial x} = \ln(y\cos x) + x\frac{1}{y\cos x}(-y\sin x) = \ln(y\cos x) - x\tan x$$
$$\left.\frac{\partial f}{\partial x}\right|_{(\pi/3,1)} = \ln(\cos\pi/3) - \pi/3\tan\pi/3 \approx -2.51$$

Problems

41. (a) From contour diagram,

$$\begin{aligned} f_x(2,1) &\approx \frac{f(2.3,1) - f(2,1)}{2.3-2} \\ &= \frac{6-5}{0.3} = 3.3, \\ f_y(2,1) &\approx \frac{f(2,1.4) - f(2,1)}{1.4-1} \\ &= \frac{6-5}{0.4} = 2.5. \end{aligned}$$

(b) A table of values for f is given in Table 14.5.

Table 14.5

		y		
		0.9	1.0	1.1
x	1.9	4.42	4.61	4.82
	2.0	4.81	5.00	5.21
	2.1	5.22	5.41	5.62

From Table 14.5 we estimate $f_x(2,1)$ and $f_y(2,1)$ using difference quotients:

$$f_x(2,1) \approx \frac{5.41 - 5.00}{2.1-2} = 4.1$$
$$f_y(2,1) \approx \frac{5.21 - 5.00}{1.1-1} = 2.1.$$

We obtain better estimates by finer data in the table.

(c) $f_x(x,y) = 2x$, $f_y(x,y) = 2y$. So the true values are $f_x(2,1) = 4$, $f_y(2,1) = 2$.

42. (a) The difference quotient for evaluating $f_w(2,2)$ is

$$f_w(2,2) \approx \frac{f(2+0.01,2)-f(2,2)}{h} = \frac{e^{(2.01)\ln 2}-e^{2\ln 2}}{0.01} = \frac{e^{\ln(2^{2.01})}-e^{\ln(2^2)}}{0.01}$$
$$= \frac{2^{(2.01)}-2^2}{0.01} \approx 2.78$$

The difference quotient for evaluating $f_z(2,2)$ is

$$f_z(2,2) \approx \frac{f(2,2+0.01)-f(2,2)}{h}$$
$$= \frac{e^{2\ln(2.01)}-e^{2\ln 2}}{0.01} = \frac{(2.01)^2-2^2}{0.01} = 4.01$$

(b) Using the derivative formulas we get

$$f_w = \frac{\partial f}{\partial w} = \ln z \cdot e^{w\ln z} = z^w \cdot \ln z$$
$$f_z = \frac{\partial f}{\partial z} = e^{w\ln z} \cdot \frac{w}{z} = w \cdot z^{w-1}$$

so

$$f_w(2,2) = 2^2 \cdot \ln 2 \approx 2.773$$
$$f_z(2,2) = 2 \cdot 2^{2-1} = 4.$$

43. (a) Since $\partial z/\partial x = 6x - ay$ if the surface is sloping up at $(1,2)$ we have $6-2a > 0$, so $a < 3$.
(b) Since $\partial z/\partial y = 8y - ax$, at $(1,2)$ we have $\partial z/\partial y = 16 - a > 0$ if $a < 3$. Thus the surface is sloping upward in the y-direction at $(1,2)$.

44. (a) To calculate $\partial B/\partial t$, we hold P constant and differentiate B with respect to t:

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}(Pe^{rt}) = Pre^{rt}.$$

In financial terms, $\partial B/\partial t$ represents the change in the amount of money in the bank as one unit of time passes by.
(b) To calculate $\partial B/\partial P$, we hold t constant and differentiate B with respect to P:

$$\frac{\partial B}{\partial P} = \frac{\partial}{\partial P}(Pe^{rt}) = e^{rt}.$$

In financial terms, $\partial B/\partial P$ represents the change in the amount of money in the bank at time t as you increase the amount of money that was initially deposited by one unit.

45. (a) $\dfrac{\partial g}{\partial m} = \dfrac{G}{r^2}$ and $\dfrac{\partial g}{\partial r} = -\dfrac{2Gm}{r^3}$
(b) For constant r, the graph of g against m is a straight line through the origin with slope $\dfrac{\partial g}{\partial m} = \dfrac{G}{r^2}$. Thus g increases as m increases, while r is constant. See Figure 14.11.

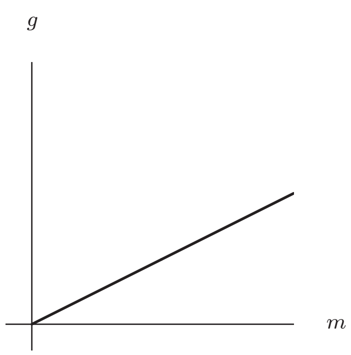


Figure 14.11

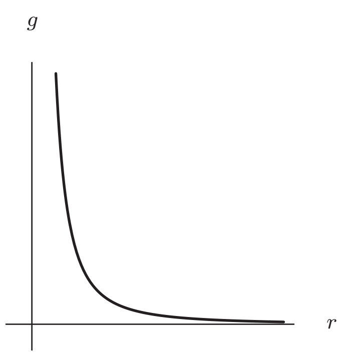


Figure 14.12

For constant m, the graph of g against r has the shape shown in Figure 14.12, (the same shape as the graph of $y = \frac{1}{x^2}$). The slope of the graph in Figure 14.12 is $\frac{\partial g}{\partial r} = -\frac{2Gm}{r^3}$. So as r increases, g decreases, since the slope is negative.

46. Substituting $w = 65$ and $h = 160$, we have

$$f(65, 160) = 0.01(65^{0.25})(160^{0.75}) = 1.277 \text{ m}^2.$$

This tells us that a person who weighs 65 kg and is 160 cm tall has a surface area of about 1.277 m^2. Since

$$f_w(w, h) = 0.01(0.25w^{-0.75})h^{0.75} \text{ m}^2\text{/kg},$$

we have $f_w(65, 160) = 0.005$ m^2/kg. Thus, an increase of 1 kg in weight increases surface area by about 0.005 m^2. Since

$$f_h(w, h) = 0.01w^{0.25}(0.75h^{-0.25}) \text{ m}^2\text{/cm},$$

we have $f_h(65, 160) = 0.006$ m^2/cm. Thus, an increase of 1 cm in height increases surface area by about 0.006 m^2.

47. (a) $\frac{\partial E}{\partial m} = c^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)$. We expect this to be positive because energy increases with mass.

(b) $\frac{\partial E}{\partial v} = mc^2 \cdot \left(-\frac{1}{2}\right)(1 - v^2/c^2)^{-3/2}\left(-\frac{2v}{c^2}\right) = \frac{mv}{(1 - v^2/c^2)^{3/2}}$. We expect this to be positive because energy increases with velocity.

48. The function in $h(x, t)$ tells us the depth of the water in cm at position x meters and time t seconds. Thus, $h_x(2, 5)$ is in cm per meter and $h_t(2, 5)$ is in cm per second. So

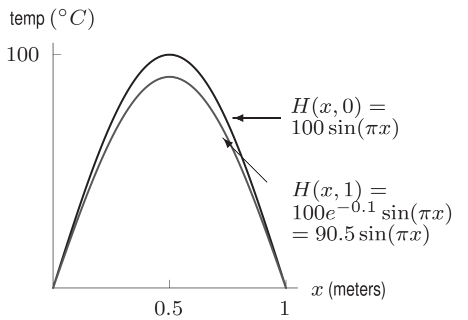
$$\begin{aligned} h_x(x, t) &= -0.5\sin(0.5x - t) \\ h_x(2, 5) &= -0.5\sin(0.5(2) - 5) = -0.378 \text{ cm/meter.} \\ h_t(x, t) &= \sin(0.5x - t) \\ h_t(2, 5) &= \sin(0.5(2) - 5) = 0.757 \text{ cm/second.} \end{aligned}$$

At position $x = 2$ and time $t = 5$, the value of $h_x(2, 5)$ is the rate of change of depth of the water with distance—the slope of the surface of the wave. The value of $h_t(2, 5)$ is the vertical velocity of the surface of the water of the $x = 2$ at time $t = 5$.

49. (a) Substituting $t = 0$ and $t = 1$ into the formula for H gives:

$$\begin{aligned} H(x, 0) &= 100\sin(\pi x) \\ H(x, 1) &= 100e^{-0.1}\sin(\pi x) = 90.5\sin(\pi x). \end{aligned}$$

The graphs of $H(x, 0)$ and $H(x, 1)$ are shown below.



(b) To calculate $H_x(x, t)$, we hold t constant and differentiate with respect to x:

$$\begin{aligned} H_x(x, t) &= \frac{\partial}{\partial x}H(x, t) = \frac{\partial}{\partial x}(100e^{-0.1t}\sin(\pi x)) = 100\pi e^{-0.1t}\cos(\pi x) \\ H_x(0.2, t) &= 100\pi e^{-0.1t}\cos(0.2\pi) = 254.2e^{-0.1t} \text{ °C/meter} \\ H_x(0.8, t) &= 100\pi e^{-0.1t}\cos(0.8\pi) = -254.2e^{-0.1t} \text{ °C/meter.} \end{aligned}$$

The practical interpretation of these partial derivatives is the rate of change in temperature at $x = 0.2$ and $x = 0.8$ as we increase the distance from the end $x = 0$. Notice that $e^{-0.1t}$ is positive for all t. Given the formula for $H(x,t)$, we see that the closer the position to the center of the rod, the hotter the temperature. The partial derivative $H_x(0.2,t)$ has a positive sign because, at $x = 0.2$ as we increase x, we get closer to the center of the rod which is hottest. The partial derivative $H_x(0.8,t)$ has a negative sign because, at $x = 0.8$ as we increase x, we get further away from the center of the rod which is hottest.

(c) To calculate $H_t(x,t)$, we hold x constant and differentiate with respect to t:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(100e^{-0.1t}\sin(\pi x)) = -10e^{-0.1t}\sin(\pi x)\ ^\circ\text{C/second.}$$

For all t, and for $0 < x < 1$ (that is, for all t and all x inside the rod), the partial derivative $H_t(x,t)$ is negative. In terms of heat, $H_t(x,t)$ represents the rate at which the temperature of the rod is changing as time passes at position x and time t. Thus, the temperature inside the rod is always decreasing.

50. We compute the partial derivatives:

$$\frac{\partial Q}{\partial K} = b\alpha K^{\alpha-1}L^{1-\alpha} \quad \text{so} \quad K\frac{\partial Q}{\partial K} = b\alpha K^{\alpha}L^{1-\alpha}$$
$$\frac{\partial Q}{\partial L} = b(1-\alpha)K^{\alpha}L^{-\alpha} \quad \text{so} \quad L\frac{\partial Q}{\partial L} = b(1-\alpha)K^{\alpha}L^{1-\alpha}$$

Adding these two results, we have:

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = b(\alpha + 1 - \alpha)K^{\alpha}L^{1-\alpha} = Q.$$

51. Since $f_x(x,y) = 4x^3y^2 - 3y^4$, we could have

$$f(x,y) = x^4y^2 - 3xy^4.$$

In that case,

$$f_y(x,y) = \frac{\partial}{\partial y}(x^4y^2 - 3xy^4) = 2x^4y - 12xy^3$$

as expected. More generally, we could have $f(x,y) = x^4y^2 - 3xy^4 + C$, where C is any constant.

Strengthen Your Understanding

52. The variable with respect to which the partial derivative is taken needs to be specified. The function $f(x,y) = x^2y^3$ has two partial derivatives, $f_x = 2xy^3$ and $f_y = 3y^2x^2$.

53. The partial derivative $f_x(0,0)$ is defined as a limit of difference quotients:

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h}.$$

Evaluating a difference quotient for a specific value of h, such as $h = 0.01$, does not tell us the value or the sign of the partial derivative.

For example, if $f(x,y) = x^2 + y^2$, then $f(0.01,0) - f(0,0) = 0.01^2 > 0$, but $f_x(x,y) = 2x$, so $f_x(0,0) = 0$.

54. A linear function with the required partial derivatives is $g(x,y) = 2x + 3y$. Adding to g a nonlinear function with both partial derivatives equal to zero at the origin solves the problem. For example, $f(x,y) = 2x + 3y + x^2$ has $f_x(0,0) = 2$ and $f_y(0,0) = 3$.

55. If $f(x,y) = xy$ and $g(x,y) = xy + y^2$, then

$$\begin{aligned} f_x &= g_x = y \\ f_y &= x \\ g_y &= x + 2y \neq f_y. \end{aligned}$$

56. If $f(x,y)$ is defined by a formula that does not contain x, then $f_x = 0$ everywhere. For example, if $f(x,y) = y^2 + 1$, then $f_x = 0$ and f is not constant.

57. True. Any function of the form $f(x, y) = xy + c$, where c is a constant satisfies $f_x = y$ and $f_y = x$.

58. True. Substituting $\partial z/\partial u = \cos v$ and $\partial z/\partial v = -u \sin v$ into the equation yields

$$\cos v \frac{\partial z}{\partial u} - \frac{\sin v}{u}\frac{\partial z}{\partial v} = \cos^2 v + \sin^2 v = 1.$$

59. True. Since g is a function of x only, it can be treated like a constant when taking the y partial derivative.

60. False. Differentiating with respect to s gives

$$k_s = rse^s + re^s.$$

At the point $(-1, 2)$ we have $k_s(-1, 2) = -3e^2$, which is negative, so k is decreasing in the s-direction.

61. False. In order to have $f_x(x, y) = y^2$, we would need f to have the form $f(x, y) = xy^2 + h(y)$, where $h(y)$ is a function of y only. Then the y-partial derivative of f would be $f_y(x, y) = 2xy + h'(y)$, where $h'(y)$ is the derivative of $h(y)$. No matter what function $h(y)$ is, it will contain no x's, so it is impossible for f_y to equal x^2.

62. False. The function f could be any function of x only. For example, $f(x, y) = x^2$ has $f_y(x, y) = 0$.

63. False. Treating y constant and taking the derivative with respect to x yields $f_x(x, y) = ye^{g(x)}g'(x)$, using the chain rule.

64. False. For example, consider $f(x, y) = xy$. This function is symmetric, since $f(y, x) = yx = xy = f(x, y)$, but $f_x(x, y) = y$ and $f_y(x, y) = x$ are not equal.

65. Calculate the partial derivatives and check. The answer is (d), and

$$x\frac{\partial}{\partial x}(x^{0.4}y^{0.6}) = x0.4x^{-0.6}y^{0.6} = 0.4x^{0.4}y^{0.6}$$
$$y\frac{\partial}{\partial y}(x^{0.4}y^{0.6}) = yx^{0.4}0.6y^{-0.4} = 0.6x^{0.4}y^{0.6}.$$

Thus

$$xf_x + yf_y = 0.4x^{0.4}y^{0.6} + 0.6x^{0.4}y^{0.6} = x^{0.4}y^{0.6} = f.$$

Solutions for Section 14.3

Exercises

1. The partial derivatives are

$$z_x = e^{x/y} \quad \text{and} \quad z_y = \frac{-x}{y}e^{x/y} + e^{x/y},$$

so

$$z_x(1, 1) = e \quad \text{and} \quad z_y(1, 1) = -e^1 + e^1 = 0.$$

The tangent plane to $z = ye^{x/y}$ at $(x, y) = (1, 1)$ has equation

$$\begin{aligned} z &= z(1, 1) + z_x(1, 1)(x - 1) + z_y(1, 1)(y - 1) \\ &= e + e(x - 1) + 0(y - 1) \\ &= ex. \end{aligned}$$

2. Since

$$z_x = y\cos xy, \text{ we have } z_x\left(2, \frac{3\pi}{4}\right) = \frac{3\pi}{4}\cos\left(2 \cdot \frac{3\pi}{4}\right) = 0.$$
$$z_y = x\cos xy, \text{ we have } z_y\left(2, \frac{3\pi}{4}\right) = 2\cos\left(2 \cdot \frac{3\pi}{4}\right) = 0.$$

Since

$$z\left(2, \frac{3\pi}{4}\right) = \sin\left(2 \cdot \frac{3\pi}{4}\right) = -1,$$

the tangent plane is

$$\begin{aligned} &z = z\left(2, \frac{3\pi}{4}\right) + z_x\left(2, \frac{3\pi}{4}\right)(x - 2) + z_y\left(2, \frac{3\pi}{4}\right)\left(y - \frac{3\pi}{4}\right) \\ &z = -1. \end{aligned}$$

3. Differentiating gives

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+1}, \quad \text{so} \quad \left.\frac{\partial z}{\partial x}\right|_{(0,3,9)} = 0$$

$$\frac{\partial z}{\partial y} = 2y, \quad \text{so} \quad \left.\frac{\partial z}{\partial y}\right|_{(0,3,9)} = 6.$$

Thus, the tangent plane is

$$z = 9 + 0(x-0) + 6(y-3)$$
$$z = 6y - 9.$$

4. We have

$$z = e^y + x + x^2 + 6.$$

The partial derivatives are

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(1,0)} = (2x+1)\Big|_{(x,y)=(1,0)} = 3$$
$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,0)} = e^y\Big|_{(x,y)=(1,0)} = 1.$$

So the equation of the tangent plane is

$$z = 9 + 3(x-1) + y = 6 + 3x + y.$$

5. The partial derivatives are

$$z_x = x \quad \text{and} \quad z_y = 4y,$$

so

$$z(2,1) = 4, \quad z_x(2,1) = 2 \quad \text{and} \quad z_y(2,1) = 4.$$

The tangent plane to $z = \frac{1}{2}(x^2 + 4y^2)$ at $(x,y) = (2,1)$ has equation

$$\begin{aligned} z &= z(2,1) + z_x(2,1)(x-2) + z_y(2,1)(y-1) \\ &= 4 + 2(x-2) + 4(y-1) \\ &= -4 + 2x + 4y. \end{aligned}$$

6. The surface is given by

$$z = f(x,y) = x^2 + y^2 - 1,$$

where $f(1,3) = 9$ and

$$f_x = 2x \quad \text{and } f_y = 2y$$

and

$$f_x(1,3) = 2 \quad \text{and } f_y(1,3) = 6.$$

Thus, the tangent plane is

$$z = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$$
$$z = 9 + 2(x-1) + 6(y-3)$$
$$z = 2x + 6y - 11.$$

7. The surface is given by

$$z = f(x,y) = 40 - x^2y^2,$$

and $f(2,3) = 40 - 2^2 \cdot 3^2 = 4$. We have

$$f_x(x,y) = -2xy^2, \quad \text{so} \quad f_x(2,3) = -36.$$

and

$$f_y(x,y) = -2x^2y, \quad \text{so} \quad f_y(2,3) = -24.$$

Thus, the tangent plane is

$$\begin{aligned} z &= f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) \\ z &= 4 - 36(x-2) - 24(y-3) \\ z &= -36x - 24y + 148. \end{aligned}$$

8. The surface is given by

$$z = f(x,y) = 6 - x^2y - \ln(xy),$$

where $f(4, 0.25) = 2$ and

$$f_x = -2xy - \frac{1}{xy}\cdot y = -2xy - \frac{1}{x}$$

and

$$f_y = -x^2 - \frac{1}{xy}\cdot x = -x^2 - \frac{1}{y}.$$

Thus

$$f_x(4, 0.25) = -2\cdot 4(0.25) - \frac{1}{4} = -2.25,$$

and

$$f_y(4, 0.25) = -4^2 - \frac{1}{0.25} = -20,$$

so the tangent plane is

$$\begin{aligned} z &= f(4, 0.25) + f_x(4, 0.25)(x-4) + f_y(4, 0.25)(y-0.25) \\ z &= 2 - 2.25(x-4) + 20(y-0.25) \\ z &= -2.25x - 20y + 16. \end{aligned}$$

9. $df = y\cos(xy)\,dx + x\cos(xy)\,dy$

10. Since $g_u = 2u + v$ and $g_v = u$, we have

$$dg = (2u+v)\,du + u\,dv$$

11. Since $z_x = -e^{-x}\cos(y)$ and $z_y = -e^{-x}\sin(y)$, we have

$$dz = -e^{-x}\cos(y)dx - e^{-x}\sin(y)dy.$$

12. $dh = e^{-3t}\cos(x+5t)\,dx + (-3e^{-3t}\sin(x+5t) + 5e^{-3t}\cos(x+5t))\,dt$

13. We have $dg = g_x\,dx + g_t\,dt$. Finding the partial derivatives, we have $g_x = 2x\sin(2t)$ so $g_x(2, \frac{\pi}{4}) = 4\sin(\pi/2) = 4$, and $g_t = 2x^2\cos(2t)$ so $g_t(2, \frac{\pi}{4}) = 8\cos(\frac{\pi}{2}) = 0$. Thus $dg = 4\,dx$.

14. We have $df = f_x\,dx + f_y\,dy$. Finding the partial derivatives, we have $f_x = e^{-y}$ so $f_x(1,0) = e^{-0} = 1$, and $f_y = -xe^{-y}$ so $f_y(1,0) = -1e^{-0} = -1$. Thus, $df = dx - dy$.

15. We have $dP = P_L dL + P_K dK$.
Now

$$\begin{aligned} P_K &= (1.01)(0.75)K^{-0.25}L^{0.25} \\ P_K(100, 1) &\approx 2.395, \end{aligned}$$

and

$$\begin{aligned} P_L &= (1.01)(0.25)K^{0.75}L^{-0.75} \\ P_L(100, 1) &\approx 0.008 \end{aligned}$$

Thus

$$dP \approx 2.395\,dK + 0.008\,dL.$$

16. We have $dF = F_m\,dm + F_r\,dr$.

$$F_m = \frac{G}{r^2},\ F_m(100,10) = \frac{G}{(10)^2} = \frac{G}{100} = 0.01G$$

$$F_r = \frac{-2Gm}{r^3},\ F_r(100,10) = \frac{-2G100}{(10)^3} = \frac{-G}{5} = -0.2G.$$

Thus,

$$dF = 0.01G\,dm - 0.2G\,dr.$$

Problems

17. (a) The units are dollars/square foot.
(b) The price of land 300 feet from the beach and of area near 1000 square feet is greater for larger plots by about \$3 per square foot.
(c) The units are dollars/foot.
(d) The price of a 1000 square foot plot about 300 feet from the beach is less for plots farther from the beach by about \$2 per extra foot from the beach.
(e) Compared to the 998 ft^2 plot at 295 ft from the beach, the other plot costs about $7 \times 3 = \$21$ more for the extra 7 square feet but about $10 \times 2 = \$20$ less for the extra 10 feet you have to walk to the beach. The net difference is about a dollar, and the smaller plot nearer the beach is cheaper.

18. (a) Since the equation of a tangent plane should be linear, this answer is wrong.
(b) The student did not substitute the values $x = 2$, $y = 3$ into the formulas for the partial derivatives used in the formula of a tangent plane.
(c) Let $f(x,y) = z = x^3 - y^2$. Since $f_x(x,y) = 3x^2$ and $f_y(x,y) = -2y$, substituting $x = 2$, $y = 3$ gives $f_x(2,3) = 12$ and $f_y(2,3) = -6$. Then the equation of the tangent plane is

$$z = 12(x-2) - 6(y-3) - 1, \quad \text{or} \quad z = 12x - 6y - 7.$$

19. (a) The two tables of values are Table 14.6 and 14.7.

Table 14.6

		y		
		1.9	2.0	2.1
x	0.9	0.3847	0.3697	0.3510
	1.0	0.3481	0.3345	0.3176
	1.1	0.3150	0.3027	0.2873

Table 14.7

		y		
		1.99	2.00	2.01
x	0.99	0.3394	0.3379	0.3363
	1.00	0.3360	0.3345	0.3330
	1.01	0.3327	0.3312	0.3297

Both tables are nearly linear. To check this, observe that the increments in each row (column) are equal, or nearly so. Table 14.7 is more linear due to finer data.

(b) Table 14.7 shows $f(1,2) \approx 0.3345$. Also

$$\begin{aligned} f_x(1,2) &\approx \frac{f(1.01,2) - f(1,2)}{1.01 - 1} \\ &= \frac{0.3312 - 0.3345}{0.01} = -0.3300, \\ f_y(1,2) &\approx \frac{f(1,2.01) - f(1,2)}{2.01 - 2} \\ &= \frac{0.3330 - 0.3345}{0.01} = -0.1500. \end{aligned}$$

Using the estimates for the partial derivatives that we just made from the Table 14.7, we get that the local linearization of f around $(1,2)$ is

$$\begin{aligned} f(x,y) &\approx f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 0.3345 - 0.33(x-1) - 0.15(y-2). \end{aligned}$$

Now we use $f_x = -e^{-x}\sin(y)$ and $f_y = e^{-x}\cos(y)$, giving

$$f_x(1,2) = -0.3345,$$
$$f_y(1,2) = -0.1531.$$

These values of the partial derivatives tell us that the local linearization of f around $(1,2)$ is

$$f(x,y) \approx 0.3345 - 0.3345(x-1) - 0.1531(y-2).$$

Notice that the two linearizations agree up to two decimal places.

20. We have

$$f_x(3,1) = \left.\frac{\partial f}{\partial x}\right|_{(3,1)} = 2xy|_{(3,1)} = 6,$$

and

$$f_y(3,1) = \left.\frac{\partial f}{\partial y}\right|_{(3,1)} = x^2|_{(3,1)} = 9.$$

Also $f(3,1) = 9$. So the local linearization is,

$$z = 9 + 6(x-3) + 9(y-1).$$

21. The tangent plane approximation gives

$$\Delta h \approx h_x \Delta x + h_y \Delta y,$$

or

$$h(x,y) \approx h(a,b) + h_x \Delta x + h_y \Delta y.$$

With $(a,b) = (600,100)$ and $(x,y) = (605,98)$, we see that $\Delta x = 5$ and and $\Delta x = -2$. Thus

$$h(605,98) \approx 300 + 12(5) - 8(-2) = 376.$$

Using the information given, this is our best estimate for $h(605,98)$. However, it may not be a good estimate if the derivatives are changing rapidly near the point $(600,100)$.

22. We need the partial derivatives, $f_x(1,0)$ and $f_y(1,0)$. We have

$$f_x(x,y) = 2xe^{xy} + x^2ye^{xy}, \qquad \text{so } f_x(1,0) = 2$$
$$f_y(x,y) = x^3e^{xy}, \qquad \text{so } f_y(1,0) = 1.$$

(a) Since $f(1,0) = 1$, the tangent plane is

$$z = f(1,0) + 2(x-1) + 1(y-0) = 1 + 2(x-1) + y.$$

(b) The linear approximation can be obtained from the equation of the tangent plane:

$$f(x,y) \approx 1 + 2(x-1) + y.$$

(c) At $(1,0)$, the differential is

$$df = f_x dx + f_y dy = 2dx + dy.$$

23. Since $f_x(x,y) = \frac{x}{\sqrt{x^2+y^3}}$ and $f_y(x,y) = \frac{3y^2}{2\sqrt{x^2+y^3}}$,
$f_x(1,2) = \frac{1}{\sqrt{1^2+2^3}} = \frac{1}{3}$ and $f_y(1,2) = \frac{3\cdot 2^2}{2\sqrt{1^2+2^3}} = 2$.
Thus the differential at the point $(1,2)$ is

$$df = df(1,2) = f_x(1,2)dx + f_y(1,2)dy = \frac{1}{3}dx + 2dy.$$

Using the differential at the point $(1,2)$, we can estimate $f(1.04, 1.98)$. Since

$$\triangle f \approx f_x(1,2)\triangle x + f_y(1,2)\triangle y$$

where $\triangle f = f(1.04,1.98) - f(1,2)$ and $\triangle x = 1.04 - 1$ and $\triangle y = 1.98 - 2$, we have

$$\begin{aligned} f(1.04,1.98) &\approx f(1,2) + f_x(1,2)(1.04-1) + f_y(1,2)(1.98-2) \\ &= \sqrt{1^2+2^3} + \frac{0.04}{3} - 2(0.02) \approx 2.973. \end{aligned}$$

24. (a) Calculating partial derivatives, we obtain $g_u(u,v) = 2u+v$ and $g_v(u,v) = u$. Therefore, $dg = g_u(u,v)\,du + g_v(u,v)\,dv = (2u+v)\,du + u\,dv$, and so our final answer is $dg = (2u+v)\,du + u\,dv$.

(b) Let $(u,v) = (1,2)$. In moving from $(1,2)$ to $(1.2,\ 2.1)$, we see that $\Delta u = du = 0.2$ and $\Delta v = dv = 0.1$. Therefore, we have

$$\begin{aligned} dg &= (2(1)+2)(0.2) + (1)(0.1) \\ &= 0.8 + 0.1 \\ &= 0.9, \end{aligned}$$

which is our final answer.

25. Local linearization gives us the approximation

$$\begin{aligned} T(x,y) &\approx T(2,1) + T_x(2,1)(x-2) + T_y(2,1)(y-1) \\ T(x,y) &\approx 135 + 16(x-2) - 15(y-1). \end{aligned}$$

Thus,

$$T(2.04, 0.97) \approx 135 + 16(2.04-2) - 15(0.97-1) = 136.09^\circ\text{C}.$$

26. We first recall that the formula for a volume of a right circular cylinder is $V = \pi r^2 h$, where r is the radius of the cylinder and h is the height of the cylinder. In our case, we are being asked to calculate dV, the approximate change in V, when r changes from 50 to 51 cm and h changes from 100 to 101 cm. We have

$$\begin{aligned} dV &= V_r(r,h)\,dr + V_h(r,h)\,dh \\ &= 2\pi r h\,dr + \pi r^2\,dh \\ &= 2\pi(50)(100)\cdot 1 + \pi(50)^2 \cdot 1 \\ &= 10000\pi + 2500\pi, \end{aligned}$$

so we conclude that the volume changes by approximately 12500π or 39270 cm^3.

27. Making use of the values of P_r and P_L from the solution to Problem 27 on page 1270, we have the local linearizations:

For $(r,L) = (8,4000)$,

$$P(r,L) \approx 80 + 2.5(r-8) + 0.02(L-4000),$$

For $(r,L) = (8,6000)$,

$$P(r,L) \approx 120 + 3.33(r-8) + 0.02(L-6000),$$

For $(r,L) = (13,7000)$,

$$P(r,L) \approx 160 + 3.33(r-13) + 0.02(L-7000).$$

28. A linear approximation near $(480,20)$ is given by

$$f(T,P) \approx f(480,20) + f_T(480,20)(T-480) + f_P(480,20)(P-20).$$

Directly from the table on page 774, we have

$$\begin{aligned} f_T(480,20) &\approx \frac{f(480,20) - f(500,20)}{-20} = \frac{27.85 - 28.46}{-20} = 0.0305 \\ f_P(480,20) &\approx \frac{f(480,20) - f(480,22)}{-2} = \frac{27.85 - 25.31}{-2} = -1.27 \end{aligned}$$

This yields the linear approximation near $(480,20)$

$$\begin{aligned} f(T,P) &\approx 27.85 + 0.0305(T-480) - 1.27(P-20) \\ &= 0.0305T - 1.27P + 38.61. \end{aligned}$$

29. (a) The linear approximation gives

$$f(520,24) \approx 24.20, \quad f(480,24) \approx 23.18,$$
$$f(500,22) \approx 25.52, \quad f(500,26) \approx 21.86.$$

The approximations for $f(520,24)$ and $f(500,26)$ agree exactly with the values in the table; the other two do not. The reason for this is that the partial derivatives were estimated using difference quotients with these values.

(b) We could get a more balanced estimate by using a difference quotient that uses the values on both sides. Thus, we could estimate the partial derivatives as follows:

$$\begin{aligned} f_T(500,24) &\approx \frac{f(520,24) - f(480,24)}{40} \\ &= \frac{(24.20 - 23.19)}{40} = 0.02525, \end{aligned}$$

and

$$\begin{aligned} f_p(500,24) &\approx \frac{f(500,26) - f(500,22)}{4} \\ &= \frac{(21.86 - 25.86)}{4} = -1. \end{aligned}$$

This yields the linear approximation

$$V = f(T,p) \approx 23.69 + 0.02525(T - 500) - (p - 24) \text{ ft}^3.$$

This approximation yields values

$$f(520,24) \approx 24.195, \quad f(480,24) \approx 23.185,$$
$$f(500,22) \approx 25.69, \quad f(500,26) \approx 21.69.$$

Although none of these predictions are accurate, the error in the predictions that were wrong before has been reduced. This new linearization is a better all-round approximation for values near $(500,24)$.

30. Letting ΔT denote the change in temperature between these two points, we have

$$\begin{aligned} \Delta T &\approx f_x(3,5)\Delta x + f_t(3,5)\Delta t \\ &= \left(-2\,\frac{^\circ\text{C}}{\text{m}}\right)(-0.5 \text{ m}) + \left(1.2\,\frac{^\circ\text{C}}{\text{min}}\right)(1 \text{ min}) \\ &= 1^\circ\text{C} + 1.2^\circ\text{C} \\ &= 2.2^\circ\text{C}. \end{aligned}$$

We therefore see that the temperature would be about 2.2°C warmer at our destination.

31. (a) Solving for P gives

$$P = f(T,V) = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}.$$

(b) Since

$$\begin{aligned} \frac{\partial P}{\partial T} &= f_T(T,V) = \frac{nR}{V - nb} \\ \frac{\partial P}{\partial V} &= f_V(T,V) = \frac{-nRT}{(V - nb)^2} + \frac{2n^2 a}{V^3}, \end{aligned}$$

we have

$$\begin{aligned} \Delta P &\approx f_T(T_0,V_0)\Delta T + f_V(T_0,V_0)\Delta V \\ \Delta P &\approx \frac{nR}{V_0 - nb}\Delta T + \left(-\frac{nRT_0}{(V_0 - nb)^2} + \frac{2n^2 a}{V_0^3}\right)\Delta V. \end{aligned}$$

32. (a) When $V = 25$ and $P = 1$, we have $T = 304.9$. The differential dT is

$$dT = \frac{\partial T}{\partial V}\,dV + \frac{\partial T}{\partial P}\,dP = (-16.574\frac{1}{V^2} + 1.06\frac{1}{V^3} + 12.187P)\,dV + (-0.3879 + 12.187V)\,dP.$$

When $V = 25$ and $P = 1$ this is

$$dT = 12.16\,dV + 304.29\,dP.$$

(b) If $\Delta P = 0.1$ and $\Delta T = 0$, then

$$0 \approx (12.16)\Delta V + (304.29)(0.1),$$

so

$$\Delta V \approx -\frac{30.429}{12.16} \approx -2.5.$$

Thus the volume would have to decrease by about 2.5 dm^3, or about 10%.

33. (a) For a mass m of liquid, we have $\rho = m/V$, so

$$d\rho = \frac{-m}{V^2}\,dV = \frac{-m}{V^2}\beta V\,dT = -\beta\frac{m}{V}\,dT = -\beta\rho\,dT.$$

(b) From part (a), we have $\Delta\rho \approx -\beta\rho\Delta T$, so

$$\beta \approx -\frac{1}{\rho}\cdot\frac{\Delta\rho}{\Delta T}.$$

Thus, in the limit as ΔT and $\Delta\rho$ become very small, we have

$$\beta = -\frac{1}{\rho}\frac{d\rho}{dT} = -\frac{1}{\rho}\left(\begin{array}{c}\text{Slope of tangent line}\\ \text{in Figure 14.13}\end{array}\right).$$

We use Figure 14.13 to estimate ρ, $\Delta\rho$, and ΔT. We use these values to approximate β.

From Figure 14.13 we see that $\rho \approx 997$ when $T = 20$. In addition, we see that $\rho \approx 1000$ when $T = 0$. Between these points, the temperature change is $\Delta T = 20 - 0 = 20$, and the density change is $\Delta\rho = 997 - 1000 = -3$. Thus, $\Delta\rho \approx -\beta\rho\Delta T$

$$\beta \approx -\frac{1}{\rho}\cdot\frac{\Delta\rho}{\Delta T} = -\frac{1}{997}\cdot\frac{(-3)}{20} \approx 0.00015.$$

At $T = 80$ we have $\rho \approx 973$ and at $T = 60$, we have $\rho \approx 983$. Thus $\Delta T = 80 - 60 = 20$, and $\Delta\rho = 973 - 983 = -10$, so when $T = 80$ we have,

$$\beta \approx -\frac{1}{\rho}\cdot\frac{\Delta\rho}{\Delta T} = -\frac{1}{973}\cdot\frac{(-10)}{20} \approx 0.0005.$$

As you can see from Figure 14.13, using $\Delta T = 20$ may not give a very good approximation. To get a better approximation, use a smaller value of ΔT.

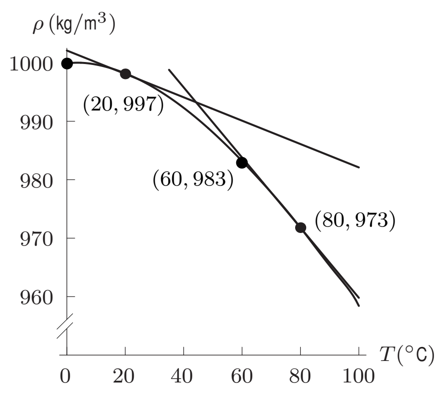


Figure 14.13

34. The error in η is approximated by $d\eta$, where

$$d\eta = \frac{\partial \eta}{\partial r}dr + \frac{\partial \eta}{\partial p}dp.$$

We need to find

$$\frac{\partial \eta}{\partial r} = \frac{\pi}{8}\frac{p4r^3}{v}$$

and

$$\frac{\partial \eta}{\partial p} = \frac{\pi}{8}\frac{r^4}{v}.$$

For $r = 0.005$ and $p = 10^5$ we get

$$\frac{\partial \eta}{\partial r}(0.005, 10^5) = 3.14159 \cdot 10^7, \quad \frac{\partial \eta}{\partial p}(0.005, 10^5) = 0.39270,$$

so that

$$d\eta = \frac{\partial \eta}{\partial r}dr + \frac{\partial \eta}{\partial p}dp.$$

is largest when we take all positive values to give

$$d\eta = 3.14159 \cdot 10^7 \cdot 0.00025 + 0.39270 \cdot 1000 = 8246.68.$$

This seems quite large but $\eta(0.005, 10^5) = 39269.9$ so the maximum error represents about 20% of any value computed by the given formula. Notice also the relative error in r is $\pm 5\%$, which means the relative error in r^4 is $\pm 20\%$.

35. At temperature t_0, the length is l_0 and the period is

$$T_0 = 2\pi\sqrt{\frac{l_0}{g}}.$$

Now suppose the temperature changes by Δt, causing a change in length, Δl. Since $l = l_0(1 + \alpha(t - t_0))$, we have

$$\Delta l \approx \frac{dl}{dt}\Delta t = l_0 \alpha \Delta t.$$

The change in period, ΔT, caused by this change in length is given by

$$\Delta T \approx \frac{dT}{dl}\Delta l = \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2}l^{-1/2}\Delta l = \frac{\pi}{\sqrt{gl}}\Delta l.$$

At $t = t_0$, we have $l = l_0$, so substituting for Δl and using the fact that $T_0 = 2\pi\sqrt{l_0/g}$, we have

$$\Delta T \approx \frac{\pi}{\sqrt{gl_0}}\Delta l = \frac{\pi}{\sqrt{gl_0}}l_0\alpha\Delta t = \pi\alpha\sqrt{\frac{l_0}{g}}\Delta t = \alpha\frac{T_0}{2}\Delta t.$$

Now we have to figure out how many seconds a day the clock gains or loses as a result of this change in period. We assume the units of l and g are chosen to given T in seconds. When the period is T, the pendulum executes N oscillations per day, where

$$N = \frac{\text{Number of seconds in a day}}{\text{Period in seconds}} = \frac{24 \cdot 60 \cdot 60}{T} = \frac{86400}{T}.$$

Thus, when the period changes from T_0 to $T_0 + \Delta T$, we have

$$\Delta N \approx \frac{dN}{dT}\Delta T = -\frac{86400}{T_0^2}\Delta T.$$

Substituting for ΔT gives

$$\Delta N \approx -\frac{86400}{T_0^2} \cdot \frac{\alpha T_0}{2}\Delta t = -\frac{43200\alpha}{T_0}\Delta t.$$

The clock records T_0 seconds as having passed for each oscillation executed. Therefore the number of seconds gained or lost per day when N changes by ΔN is given by

$$\text{Number of seconds gained/lost per day} = T_0\Delta N = -43200\Delta t.$$

The number of seconds is negative and the clock is slow if $\Delta t > 0$; the number of seconds is positive and the clock is fast if $\Delta t < 0$. The answer $-43200\Delta t$ is independent of l_0.

36. (a) The local linearization of $f(x,y)$ at (a,b) is

$$g(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The contour of g through (a,b) is $g(a,b) = c$, where

$$c = f(a,b) + f_x(a,b)(a-a) + f_y(a,b)(b-b) = f(a,b).$$

Thus, the contour of g through (a,b) is

$$\begin{aligned} f(a,b) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ 0 &= f_x(a,b)(x-a) + f_y(a,b)(y-b). \end{aligned}$$

Since the local linearization $g(x,y)$ gives the best linear approximation to f at (a,b), the contour of g through (a,b) must be the tangent line to the contour of f through (a,b). Thus, the tangent line is

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0.$$

(b) Putting the equation in part (a) in slope-intercept form we have

$$y = -\frac{f_x(a,b)}{f_y(a,b)}x + a\frac{f_x(a,b)}{f_y(a,b)} + b.$$

The slope is the coefficient of x, which is $-f_x(a,b)/f_y(a,b)$.

(c) We have $f_x(3,4) = 10$ and $f_y(3,4) = 3$. The tangent line has the equation $10(x-3) + 3(y-4) = 0$.

Strengthen Your Understanding

37. A correct equation for the tangent plane at the point $(3,4)$ is

$$z = f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4).$$

38. The equations of the tangent planes are

$$\begin{aligned} z &= f(0,0) + f_x(0,0)x + f_y(0,0)y \\ z &= g(0,0) + g_x(0,0)x + g_y(0,0)y. \end{aligned}$$

If the function values at the origin are not equal, $f(0,0) \neq g(0,0)$, then the two planes are parallel but not identical. For example, if $f(x,y) = x^2 + y^2$ and $g(x,y) = x^2 + y^2 + 2$, then the tangent planes to the graphs of these functions at the origin are $z = 0$ and $z = 2$, respectively.

39. The given equation is not linear, so it is not the equation of a plane.

Let $f(x,y) = x^2y$. The tangent plane to the surface $z = f(x,y)$ at the point $(1,2)$ has equation

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2).$$

The derivatives $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$ must be evaluated at the point $(1,2)$. The correct tangent plane equation is

$$z = 2 + 4(x-1) + (y-2).$$

40. For any function $f(x,y)$ and any constant C, the two functions $f(x,y)$ and $g(x,y) = f(x,y) + C$ have the same differential. For example, $f(x,y) = xy^2$ and $g(x,y) = xy^2 + 10$ have the same differential:

$$df = y^2 dx + 2xy\,dy$$

and

$$dg = y^2 dx + 2xy\,dy.$$

41. The plane $z = 3$ is a horizontal plane parallel to the xy-plane. This plane is tangent to the sphere of radius 3 centered at the origin. Other examples are possible, the graph of the paraboloid $z = 3 - x^2 - y^2$ also has tangent plane $z = 3$ at $(0,0,3)$.

42. True. The partials are $f_x = 2xye^{x^2}$ and $f_y = e^{x^2}$, so $f_x(0,1) = 0$ and $f_y(0,1) = 1$. Then the tangent plane has equation $z = f(0,1) + 0(x-0) + 1(y-1) = 1 + y - 1 = y$.

43. True. The change in f is approximately $df = 2 \cdot 2 \cdot (-0.1) + \sin(2) \cdot 0.0002$. Since $-1 \le \sin x \le 1$, the term $\sin(2) \cdot 0.0002$ is small in comparison to the first term, which is -0.4.

In fact, $df = 2 \cdot 2(-0.1) + \sin(2) \cdot 0.0002 \approx -0.3998$.

44. False. The graph of f is a paraboloid, opening upward. The tangent plane to this surface at any point lies completely under the surface (except at the point of tangency). So the local linearization *underestimates* the value of f at nearby points.

45. False. As a counterexample, consider $f(x,y) = x^2 + y^2$ and $g(x,y) = 2x + y^2$ at the point $(1,1)$. Then both functions have differential $2\,dx + 2\,dy$ at the point $(1,1)$.

46. False. As a counterexample, consider $f(x,y) = x^2 + y^2$ and $g(x,y) = 2x + y^2 - 1$ at the point $(1,1)$. Then both functions have tangent plane $z = 2 + 2(x-1) + 2(y-1)$ at the point $(1,1)$.

47. True. If f is a constant function, there is no change in f between any two points. Alternatively, $f_x = f_y = 0$, so $df = 0\,dx + 0\,dy = 0$.

48. True. If f is linear, then $f(x,y) = mx + ny + c$ for some m, n and c. So $f_x = m$ and $f_y = n$ giving $df = m\,dx + n\,dy$, which is linear in the variables dx and dy.

49. False. The property of local linearity is that most functions behave *approximately* linearly close to a point. While the contours may *appear* parallel and equally spaced, unless the function is already linear they won't be exactly parallel or evenly spaced, no matter how closely we zoom.

Solutions for Section 14.4

Exercises

1. Since the partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{15}{2}x^4 - 0 = \frac{15}{2}x^4$$
$$\frac{\partial f}{\partial y} = 0 - \frac{24}{7}y^5 = -\frac{24}{7}y^5$$

we have

$$\text{grad}\, f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left(\frac{15}{2}x^4\right)\vec{i} - \left(\frac{24}{7}y^5\right)\vec{j}\,.$$

2. Since the partial derivatives are

$$\frac{\partial Q}{\partial K} = 50 \quad \text{and} \quad \frac{\partial Q}{\partial L} = 100$$

we have

$$\nabla f = 50\vec{i} + 100\vec{j}\,.$$

3. Since the partial derivatives are

$$\frac{\partial f}{\partial m} = 2m + 0 = 2m$$
$$\frac{\partial f}{\partial n} = 0 + 2n = 2n$$

we have

$$\text{grad}\, f = \frac{\partial f}{\partial m}\vec{i} + \frac{\partial f}{\partial n}\vec{j} = 2m\vec{i} + 2n\vec{j}\,.$$

4. Since the partial derivatives are

$$z_x = e^y \quad \text{and} \quad z_y = xe^y$$

we have

$$\nabla z = e^y\vec{i} + xe^y\vec{j}\,.$$

5. Since the partial derivatives are

$$\frac{\partial f}{\partial \alpha} = \frac{1}{2}(5\alpha^2+\beta)^{-1/2}(10\alpha+0) = \frac{5\alpha}{\sqrt{5\alpha^2+\beta}}$$
$$\frac{\partial f}{\partial \beta} = \frac{1}{2}(5\alpha^2+\beta)^{-1/2}(0+1) = \frac{1}{2\sqrt{5\alpha^2+\beta}},$$

we have

$$\text{grad } f = \frac{\partial f}{\partial \alpha}\vec{i} + \frac{\partial f}{\partial \beta}\vec{j} = \left(\frac{5\alpha}{\sqrt{5\alpha^2+\beta}}\right)\vec{i} + \left(\frac{1}{2\sqrt{5\alpha^2+\beta}}\right)\vec{j}\,.$$

6. Since the partial derivatives are

$$f_r = 2\pi rh \quad \text{and} \quad f_h = \pi r^2,$$

we have

$$\nabla f = 2\pi rh\vec{i} + \pi r^2\vec{j}\,.$$

7. Since the partial derivatives are

$$z_x = e^y \quad \text{and} \quad z_y = xe^y + e^y + ye^y,$$

we have

$$\nabla z = e^y\vec{i} + e^y(1+x+y)\vec{j}\,.$$

8. Since the partial derivatives are

$$f_K = 0.3K^{-0.7}L^{0.7} \quad \text{and} \quad f_L = 0.7K^{0.3}L^{-0.3},$$

we have

$$\nabla f = 0.3\left(\frac{L}{K}\right)^{0.7}\vec{i} + 0.7\left(\frac{K}{L}\right)^{0.3}\vec{j}\,.$$

9. Since the partial derivatives are

$$f_r = \sin\theta \quad \text{and} \quad f_\theta = r\cos\theta,$$

we have

$$\nabla f = \sin\theta\vec{i} + r\cos\theta\vec{j}\,.$$

10. Since the partial derivatives are

$$f_x = \frac{2x}{x^2+y^2} \quad \text{and} \quad f_y = \frac{2y}{x^2+y^2},$$

we have

$$\nabla f = \frac{2}{x^2+y^2}(x\vec{i} + y\vec{j}\,).$$

11. Since the partial derivatives are

$$z_x = \left(\frac{1}{y}\right)\cos\left(\frac{x}{y}\right), \quad z_y = \left(\frac{-x}{y^2}\right)\cos\left(\frac{x}{y}\right)$$

we have

$$\nabla z = \frac{1}{y}\cos\left(\frac{x}{y}\right)\vec{i} - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)\vec{j}\,.$$

12. Since the partial derivatives are

$$z_x = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = \frac{y}{y^2+x^2}, \quad \text{and} \quad z_y = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2+x^2},$$

we have

$$\nabla z = \frac{y}{y^2+x^2}\vec{i} - \frac{x}{y^2+x^2}\vec{j}\,.$$

13. Since the partial derivatives are

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= \frac{(2\alpha-3\beta)(2+0)-(2-0)(2\alpha+3\beta)}{(2\alpha-3\beta)^2} \\
&= \frac{4\alpha-6\beta-(4\alpha+6\beta)}{(2\alpha-3\beta)^2} \\
&= -\frac{12\beta}{(2\alpha-3\beta)^2} \\
\frac{\partial f}{\partial \beta} &= \frac{(2\alpha-3\beta)(0+3)-(0-3)(2\alpha+3\beta)}{(2\alpha-3\beta)^2} \\
&= \frac{(6\alpha-9\beta)+(6\alpha+9\beta)}{(2\alpha-3\beta)^2} \\
&= \frac{12\alpha}{(2\alpha-3\beta)^2}
\end{aligned}$$

we have

$$\text{grad}\, f = \frac{\partial f}{\partial \alpha}\vec{i} + \frac{\partial f}{\partial \beta}\vec{j} = \left(-\frac{12\beta}{(2\alpha-3\beta)^2}\right)\vec{i} + \left(\frac{12\alpha}{(2\alpha-3\beta)^2}\right)\vec{j}\,.$$

14. Since the partial derivatives are

$$\begin{aligned}
z_x &= \frac{e^y(x+y)-xe^y}{(x+y)^2} = \frac{ye^y}{(x+y)^2} \\
z_y &= \frac{xe^y(x+y)-xe^y}{(x+y)^2} = \frac{e^y(x^2+xy-x)}{(x+y)^2}
\end{aligned}$$

we have

$$\nabla z = \frac{ye^y}{(x+y)^2}\vec{i} + \frac{e^y(x^2+xy-x)}{(x+y)^2}\vec{j}$$

15. Since the partial derivatives are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xy+7y^3 \\
\frac{\partial f}{\partial y} &= x^2+21xy^2
\end{aligned}$$

we have

$$\begin{aligned}
\text{grad}\, f\Big|_{(1,2)} &= \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)\Bigg|_{(1,2)} \\
&= \left((2xy+7y^3)\vec{i} + (x^2+21xy^2)\vec{j}\,\right)\Bigg|_{(1,2)} \\
&= \left(2(1)(2)+7(2)^3\right)\vec{i} + \left((1)^2+21(1)(2)^2\right)\vec{j} \\
&= (4+56)\vec{i} + (1+84)\vec{j} \\
&= 60\vec{i} + 85\vec{j}
\end{aligned}$$

16. Since the partial derivatives are

$$\frac{\partial f}{\partial m} = 10m + 0 = 10m$$
$$\frac{\partial f}{\partial n} = 0 + 12n^3 = 12n^3$$

we have

$$\begin{aligned} \operatorname{grad} f\Big|_{(5,2)} &= \frac{\partial f}{\partial m}\vec{i} + \frac{\partial f}{\partial n}\vec{j}\Big|_{(5,2)} \\ &= 10m\vec{i} + 12n^3\vec{j}\Big|_{(5,2)} \\ &= 10(5)\vec{i} + 12(2)^3\vec{j} \\ &= 50\vec{i} + 96\vec{j} \end{aligned}$$

17. Since the partial derivatives are

$$f_r = 2\pi(h+r) \quad \text{and} \quad f_h = 2\pi r,$$

we have

$$\nabla f(2,3) = 10\pi\vec{i} + 4\pi\vec{j}.$$

18. The gradient of $e^{\sin y}$ at $x = 0$, $y = \pi$ is given by

$$\operatorname{grad}(e^{\sin y})\Big|_{(0,\pi)} = 0\vec{i} + (\cos y e^{\sin y})\vec{j}\Big|_{(0,\pi)} = (-1)e^0\vec{j} = -\vec{j}.$$

19. Since the partial derivatives are

$$\frac{\partial f}{\partial x} = \cos(x^2)\cdot(2x) + 0 = 2x\cos(x^2)$$
$$\frac{\partial f}{\partial y} = 0 - \sin y = -\sin y$$

we have

$$\begin{aligned} \operatorname{grad} f\Big|_{(\frac{\sqrt{\pi}}{2},0)} &= \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)\Big|_{(\frac{\sqrt{\pi}}{2},0)} \\ &= \left(\left(2x\cos(x^2)\right)\vec{i} + (-\sin y)\vec{j}\right)\Big|_{(\frac{\sqrt{\pi}}{2},0)} = \left(2(\frac{\sqrt{\pi}}{2})\cos(\frac{\pi}{4})\right)\vec{i} + (-\sin 0)\vec{j} \\ &= \left(\sqrt{\pi}(\frac{1}{\sqrt{2}})\right)\vec{i} + 0 \\ &= \left(\sqrt{\frac{\pi}{2}}\right)\vec{i} \end{aligned}$$

20. Since the partial derivatives are

$$f_x = \frac{2x+y}{x^2+xy}$$
$$f_y = \frac{x}{x^2+xy}$$

at the point $(4,1)$, we have

$$f_x(4,1) = \frac{9}{20} = 0.45$$
$$f_y(4,1) = \frac{4}{20} = 0.2.$$

Then

$$\text{grad} f(4,1) = 0.45\vec{i} + 0.2\vec{j}\,.$$

21. Since the partial derivatives are

$$f_x = \frac{-2x}{(x^2+y^2)^2} \quad \text{and} \quad f_y = \frac{-2y}{(x^2+y^2)^2},$$

we have

$$\nabla f = \frac{1}{100}(2\vec{i} - 6\vec{j}\,).$$

22. Since the partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{2}(\tan x + y)^{-1/2}\left(\frac{1}{\cos^2 x} + 0\right) = \frac{1}{2\cos^2 x\sqrt{\tan x + y}},$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{2}(\tan x + y)^{-1/2}(0+1) = \frac{1}{2\sqrt{\tan x + y}},$$

then

$$\text{grad}\, f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left(\frac{1}{2\cos^2 x\sqrt{\tan x + y}}\right)\vec{i} + \left(\frac{1}{2\sqrt{\tan x + y}}\right)\vec{j}\,.$$

Hence we have

$$\begin{aligned}
\text{grad}\, f\bigg|_{(0,1)} &= \left(\frac{1}{2(\cos(0))^2\sqrt{\tan(0)+1}}\right)\vec{i} + \left(\frac{1}{2\sqrt{\tan(0)+1)}}\right)\vec{j} \\
&= \left(\frac{1}{2(1)^2\sqrt{0+1}}\right)\vec{i} + \left(\frac{1}{2\sqrt{0+1}}\right)\vec{j} \\
&= \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}
\end{aligned}$$

23. Since $f_x = y$ and $f_y = x + 3y^2$, at $(1,2)$ we have $\text{grad}\, f = 2\vec{i} + 13\vec{j}$. Thus

$$f_{\vec{u}}\,(1,2) = \text{grad}\, f \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right) = \frac{2\cdot 3 + 13(-4)}{5} = -\frac{46}{5}.$$

24. Since $f_x = 3$ and $f_y = -4$, we have $\text{grad}\, f = 3\vec{i} - 4\vec{j}$. Thus

$$f_{\vec{u}}\,(1,2) = \text{grad}\, f \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right) = \frac{3\cdot 3 + (-4)\cdot(-4)}{5} = 5.$$

25. Since $f_x = 2x$ and $f_y = -2y$, at $(1,2)$ we have $\text{grad}\, f = 2\vec{i} - 4\vec{j}$. Thus

$$f_{\vec{u}}\,(1,2) = \text{grad}\, f \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right) = \frac{2\cdot 3 - 4(-4)}{5} = \frac{22}{5}.$$

26. Since $f_x = 2\cos(2x-y)$ and $f_y = -\cos(2x-y)$, at $(1,2)$ we have $\text{grad}\, f = 2\cos(0)\vec{i} - \cos(0)\vec{j} = 2\vec{i} - \vec{j}$. Thus

$$f_{\vec{u}}\,(1,2) = \text{grad}\, f \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right) = \frac{2\cdot 3 - 1(-4)}{5} = \frac{10}{5} = 2.$$

27. Since $\|\vec v\| = 5$, we see that $\vec v$ is not a unit vector. The unit vector $\vec u$ in the direction of $\vec v$ is

$$\vec u = \frac{\vec v}{\|\vec v\|} = \frac{4}{5}\vec i - \frac{3}{5}\vec j\,.$$

The partial derivatives are $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. So

$$f_{\vec u}\,(2,6) = f_x(2,6)\cdot\left(\frac{4}{5}\right) + f_y(2,6)\cdot\left(-\frac{3}{5}\right) = 24\left(\frac{4}{5}\right) + 4\left(-\frac{3}{5}\right) = \frac{84}{5}.$$

28. Since grad $f = f_x\vec i + f_y\vec j$ and $df = f_x dx + f_y dy$ we have

$$df = ydx + xdy.$$

29. Since grad $f = f_x\vec i + f_y\vec j$ and $df = f_x dx + f_y dy$ we have

$$df = (2x + 3e^y)dx + 3xe^y dy.$$

30. Since $df = f_x dx + f_y dy$ and grad $f = f_x\vec i + f_y\vec j$ we have

$$\text{grad}\, f = 2x\vec i + 10y\vec j\,.$$

31. Since $df = f_x dx + f_y dy$ and grad $f = f_x\vec i + f_y\vec j$ we have

$$\text{grad}\, f = (x+1)ye^x\vec i + xe^x\vec j\,.$$

32. Since the values of z decrease as we move in direction $\vec i$ from the point $(-2,2)$, the directional derivative is negative.

33. Since the values of z decrease as we move in direction $\vec j$ from the point $(0,-2)$, the directional derivative is negative.

34. Since the values of z decrease as we move in direction $\vec i + 2\vec j$ from the point $(0,-2)$, the directional derivative is negative.

35. Since the values of z increase as we move in direction $\vec i - 2\vec j$ from the point $(0,-2)$, the directional derivative is positive.

36. Since the values of z stay approximately the same (since the direction is tangent to the contour) as we move in direction $\vec i + \vec j$ from the point $(-1,1)$, the directional derivative is approximately zero.

37. Since the values of z increase as we move in direction $-\vec i + \vec j$ from the point $(-1,1)$, the directional derivative is positive.

38. The approximate direction of the gradient vector at point $(-2,0)$ is $-\vec i$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

39. The approximate direction of the gradient vector at point $(0,-2)$ is $-\vec j$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

40. The approximate direction of the gradient vector at point $(2,0)$ is $\vec i$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

41. The approximate direction of the gradient vector at point $(0,2)$ is $\vec j$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

42. The approximate direction of the gradient vector at point $(-2,2)$ is $-\vec i + \vec j$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

43. The approximate direction of the gradient vector at point $(-2,-2)$ is $-\vec i - \vec j$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

44. The approximate direction of the gradient vector at point $(2,2)$ is $\vec{i}+\vec{j}$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

45. The approximate direction of the gradient vector at point $(2,-2)$ is $\vec{i}-\vec{j}$, since the gradient vector is perpendicular to the contour and points in the direction of increasing z-values. Answers may vary since answers are approximate and any positive multiple of the vector given is also correct.

Problems

46. The distance from P to Q is $\sqrt{(3.03-3)^2+(3.96-4)^2}=0.05$, so the directional derivative is approximately

$$f_{\vec{u}}(P)\approx\frac{f(Q)-f(P)}{0.05}=\frac{20-15}{0.05}=100.$$

47. (a) The unit vector $\vec{u}$ in the same direction as $\vec{v}$ is

$$\vec{u}=\frac{1}{\|\vec{v}\|}\vec{v}=\frac{1}{\sqrt{(-1)^2+3^2}}\vec{v}=\frac{-1}{\sqrt{10}}\vec{i}+\frac{3}{\sqrt{10}}\vec{j}=-0.316228\vec{i}+0.948683\vec{j}.$$

The vector $\vec{w}$ of length 0.1 in the direction of $\vec{v}$ is

$$\vec{w}=0.1\vec{u}=-0.0316228\vec{i}+0.0948683\vec{j}.$$

The displacement vector from P to Q is $\vec{w}$. Hence

$$Q=(4-0.0316228,5+0.0948683)=(3.96838,5.09487).$$

(b) Since the distance from P to Q is 0.1, the directional derivative of f at P in the direction of Q is approximately

$$f_{\vec{u}}\approx\frac{f(Q)-f(P)}{0.1}=\frac{3.01052-3}{0.1}=0.1052.$$

(c) We have

$$\begin{aligned}\text{grad}\,f(x,y)&=\frac{1}{2\sqrt{x+y}}\vec{i}+\frac{1}{2\sqrt{x+y}}\vec{j}\\ \text{grad}\,f(4,5)&=\frac{1}{6}\vec{i}+\frac{1}{6}\vec{j}.\end{aligned}$$

The directional derivative at $P=(4,5)$ in the direction of $\vec{u}$ is

$$f_{\vec{u}}(4,5)=\text{grad}\,f(4,5)\cdot\vec{u}=\frac{1}{6}\frac{-1}{\sqrt{10}}+\frac{1}{6}\frac{3}{\sqrt{10}}=\frac{1}{3\sqrt{10}}=0.1054.$$

48. (a) The partial derivatives are given by

$$f_x=e^x(\tan y)+4xy,\quad f_y=e^x(\sec^2 y)+2x^2.$$

Thus

$$f_x(0,\frac{\pi}{4})=1\qquad\text{and}\qquad f_y(0,\frac{\pi}{4})=2,$$

and so

$$\text{grad}\,f(0,\frac{\pi}{4})=\vec{i}+2\vec{j}.$$

The unit vector $\vec{u_1}$ in the direction of $\vec{i}-\vec{j}$ is $\frac{1}{\sqrt{2}}(\vec{i}-\vec{j})$. Then the directional derivative of f at $(0,\frac{\pi}{4})$ in the direction of $\vec{i}-\vec{j}$ is

$$\begin{aligned}f_{\vec{u_1}}(0,\frac{\pi}{4})&=\text{grad}\,f(0,\frac{\pi}{4})\cdot\vec{u_1}\\ &=(\vec{i}+2\vec{j})\cdot(\frac{1}{\sqrt{2}}\vec{i}-\frac{1}{\sqrt{2}}\vec{j})\\ &=\frac{1}{\sqrt{2}}-\sqrt{2}=-\frac{\sqrt{2}}{2}.\end{aligned}$$

(b) The unit vector $\vec{u_2}$ in the direction of $\vec{i}+\sqrt{3}\vec{j}$ is $\vec{u_2}=\frac{1}{2}(\vec{i}+\sqrt{3}\vec{j})$. From part (a),

$$\text{grad}\, f(0,\frac{\pi}{4})=\vec{i}+2\vec{j}.$$

Then the directional derivative of f at $(0,\frac{\pi}{4})$ in the direction of $\vec{i}+\sqrt{3}\vec{j}$ is

$$\begin{aligned} f_{\vec{u_2}}(0,\frac{\pi}{4}) &= \text{grad}\, f(0,\frac{\pi}{4})\cdot\vec{u_2}=(\vec{i}+2\vec{j})\cdot(\frac{1}{2}\vec{i}+\frac{\sqrt{3}}{2}\vec{j}) \\ &= \frac{1}{2}+\sqrt{3}. \end{aligned}$$

49. We want the directional derivative in the direction of $\vec{u}$ at $(1,2)$, so we want to calculate $f_{\vec{u}}(1,2)$. Since $f_x=2x$ and $f_y=2y$, at the point $(1,2)$, we have grad $f(1,2)=2\vec{i}+4\vec{j}$. Since $\vec{u}$ is a unit vector, we obtain

$$\begin{aligned} f_{\vec{u}}(1,2) &= \text{grad}\, f(1,2)\cdot\vec{u} \\ &= (2\vec{i}+4\vec{j})\cdot(0.6\vec{i}+0.8\vec{j}) \\ &= 2(0.6)+4(0.8) \\ &= 4.4. \end{aligned}$$

50. (a) First we will find a unit vector in the same direction as the vector $\vec{v}=3\vec{i}-2\vec{j}$. Since this vector has magnitude $\sqrt{13}$, a unit vector is

$$\vec{u}_1=\frac{1}{\|\vec{v}\|}\vec{v}=\frac{3}{\sqrt{13}}\vec{i}-\frac{2}{\sqrt{13}}\vec{j}.$$

The partial derivatives are

$$\begin{aligned} f_x(x,y) &= \frac{(1+x^2)-(x+y)\cdot 2x}{(1+x^2)^2}=\frac{1-x^2-2xy}{(1+x^2)^2}, \\ \text{and}\quad f_y(x,y) &= \frac{1}{1+x^2}, \end{aligned}$$

then, at the point P, we have

$$\begin{aligned} f_x(P) &= f_x(1,-2)=\frac{1-1^2-2\cdot1\cdot(-2)}{(1+1^2)^2}=1, \\ f_y(P) &= f_y(1,-2)=f_y(1,-2)=\frac{1}{1+1^2}=\frac{1}{2}. \end{aligned}$$

Thus

$$\begin{aligned} f_{\vec{u}_1}(P) &= \text{grad}\, f(P)\cdot\vec{u}_1 \\ &= (\vec{i}+\frac{1}{2}\vec{j})\cdot(\frac{3}{\sqrt{13}}\vec{i}-\frac{2}{\sqrt{13}}\vec{j}) \\ &= \frac{3}{\sqrt{13}}-\frac{1}{\sqrt{13}}=\frac{2}{\sqrt{13}}. \end{aligned}$$

(b) The unit vector in the same direction as the vector $\vec{v}=-\vec{i}+4\vec{j}$ is

$$\begin{aligned} \vec{u}_2 &= \frac{1}{\|\vec{v}\|}\vec{v}=\frac{1}{\sqrt{(-1)^2+4^2}}(-\vec{i}+4\vec{j}) \\ &= -\frac{1}{\sqrt{17}}\vec{i}+\frac{4}{\sqrt{17}}\vec{j}. \end{aligned}$$

Since we have calculated from part (a) that $f_x(P)=1$ and $f_y(P)=1/2$,

$$\begin{aligned} f_{\vec{u}_2}(P) &= \text{grad}\, f(P)\cdot\vec{u}_2 \\ &= (\vec{i}+\frac{1}{2}\vec{j})\cdot(-\frac{1}{\sqrt{17}}\vec{i}+\frac{4}{\sqrt{17}}\vec{j}) \\ &= -\frac{1}{\sqrt{17}}+\frac{2}{\sqrt{17}}=\frac{1}{\sqrt{17}}. \end{aligned}$$

(c) The direction of greatest increase is $\operatorname{grad} f$ at P. By part (a) we have found that

$$f_x(P) = 1 \quad \text{and} \quad f_y(P) = \frac{1}{2}.$$

Therefore the direction of greatest increase is

$$\operatorname{grad} f(P) = \vec{i} + \frac{1}{2}\vec{j}\,.$$

51. We have

$$\nabla f = (2xy^3)\vec{i} + (3x^2y^2)\vec{j}\,.$$

(a) A vector in the direction of maximum rate of change is $\nabla f(-1,2) = -16\vec{i} + 12\vec{j}$.
(b) A vector in the direction of minimum rate of change is $-\nabla f(-1,2) = 16\vec{i} - 12\vec{j}$.
(c) Any vector perpendicular to ∇f gives a direction in which the rate of change is zero. One possible answer is $12\vec{i} + 16\vec{j}$.

52. $\operatorname{grad} f(\pi/4, 1) = (\imath + \jmath)/\sqrt{2}$. The displacement is $\imath + \jmath$, so the direction is $\imath + j)/\sqrt{2}$ and the directional derivative is 1.

53. (a) The average rate of change of the function is the "rise over the run", or the change in the z-values divided by the horizontal distance. The vector from the point $(3,1)$ to the point $(1,2)$ is $\vec{v} = -2\vec{i} + \vec{j}$. We have

$$\text{Average rate of change} = \frac{f(1,2) - f(3,1)}{\| -2\vec{i} + \vec{j}\,\|} = \frac{1.6931 - 9}{\sqrt{5}} = -3.268.$$

(b) The instantaneous rate of change is given by the directional derivative in the direction $\vec{v} = -2\vec{i} + \vec{j}$. We first find the gradient vector:

$$\operatorname{grad} f = (2x)\vec{i} + \frac{1}{y}\vec{j}\,,$$

so $\operatorname{grad} f(3,1) = 6\vec{i} + \vec{j}$. In the direction of $\vec{v} = -2\vec{i} + \vec{j}$,

$$\text{Directional derivative } = \operatorname{grad} f \cdot \frac{\vec{v}}{\|\vec{v}\,\|} = \frac{-12+1}{\sqrt{5}} = -4.919.$$

54. We see that

$$\operatorname{grad} f = (3y)\vec{i} + (3x+2y)\vec{j}\,,$$

so at the point $(2,3)$, we have

$$\operatorname{grad} f = 9\vec{i} + 12\vec{j}\,.$$

(a) The directional derivative is $\nabla f \cdot \dfrac{\vec{v}}{\|\vec{v}\,\|} = \dfrac{(9)(3) + (12)(-1)}{\sqrt{10}} = \dfrac{15}{\sqrt{10}} \approx 4.74.$
(b) The direction of maximum rate of change is $\nabla f(2,3) = 9\vec{i} + 12\vec{j}$.
(c) The maximum rate of change is $\|\nabla f\| = \sqrt{225} = 15$.

55. (a) The directional derivative should be a number, not a vector.
(b) Since the partial derivatives are

$$f_x(x,y) = 2xe^y, \quad f_y(x,y) = x^2e^y, \quad f_x(1,0) = 2, \quad f_y(1,0) = 1.$$

we have

$$\operatorname{grad} f(1,0) = 2\vec{i} + \vec{j}\,.$$

The unit vector in the direction of $\vec{v}$ is $\vec{u} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$. Thus, the correct answer is

$$f_{\vec{u}}(1,0) = \operatorname{grad} f(1,0) \cdot \vec{u} = 2 \cdot \frac{4}{5} + 1 \cdot \frac{3}{5} = \frac{8}{5} + \frac{3}{5} = \frac{11}{5}.$$

56. $f_{\vec{i}}\,(4,1)$ means the rate of change of f in the x direction at $(4,1)$. Thus,

$$f_{\vec{i}}\,(4,1) \approx \frac{f(5,1)-f(4,1)}{1}.$$

The point $(5,1)$ is about 2/3 of the way from the contour for $f=3$ to the contour for $f=4$, so we estimate $f(5,1)=3.7$. Thus

$$f_{\vec{i}}\,(4,1) \approx \frac{3.7-2}{1} = 1.7.$$

We can get a better estimate if we average this result with the difference quotient obtained by going the other way, that is

$$f_{\vec{i}}\,(4,1) \approx \frac{f(4,1)-f(3,1)}{1} = \frac{2-1}{1} = 1.$$

Averaging the two estimates gives $f_{\vec{i}}\,(4,1) \approx 1.35$

57. $f_{\vec{j}}\,(4,1)$ means the rate of change of f in the y direction at $(4,1)$. Thus

$$f_{\vec{j}}\,(4,1) \approx \frac{f(4,2)-f(4,1)}{1}.$$

The point $(4,2)$ is about 1/3 of the way from the contour for $f=2$ to the contour for $f=3$, so we estimate $f(4.2)=1.3$. Thus

$$f_{\vec{j}}\,(4,1) \approx \frac{1.3-2}{1} = -0.7.$$

We can get a better estimate by averaging this with the difference quotient obtained by going the other way, that is,

$$f_{\vec{j}}\,(4,1) \approx \frac{f(4,1)-f(4,0)}{1} = \frac{2-4}{1} = -2.$$

Averaging the two estimates we get $f_{\vec{j}}\,(4,1) \approx -1.35$.

58. Since $\vec{u} = (\vec{i} - \vec{j})/\sqrt{2}$, we head away from the point $(4,1)$ toward the point $(5,0)$. Since the points $(4,1)$ and $(5,0)$ are a distance $\sqrt{2}$ apart,

$$f_{\vec{u}}\,(4,1) \approx \frac{f(5,0)-f(4,1)}{\sqrt{2}}.$$

The point $(5,0)$ is about half way between the contour for $f=5$ and the contour for $f=6$, so we estimate $f(5,0)=5.5$. Thus

$$f_{\vec{u}}\,(4,1) \approx \frac{5.5-2}{\sqrt{2}} = 2.5.$$

We can get a better estimate by averaging this with the difference quotient obtained by going the other way, that is,

$$f_{\vec{u}}\,(4,1) \approx \frac{f(4,1)-f(3,2)}{\sqrt{2}} = \frac{2-0.5}{\sqrt{2}} = 1.1$$

Averaging these two results gives an estimate $f_{\vec{u}}\,(4,1) \approx 1.8$.

59. Since $\vec{u} = (-\vec{i} + \vec{j})/\sqrt{2}$, we head away from the point $(4,1)$ toward the point $(3,2)$. Since the points $(4,1)$ and $(3,2)$ are a distance $\sqrt{2}$ apart,

$$f_{\vec{u}}\,(4,1) \approx \frac{f(3,2)-f(4,1)}{\sqrt{2}}.$$

The point $(3,2)$ is about half way between the point where $f=$ and the contour for $f=1$, so we estimate $f(3,2)=0.5$. Thus

$$f_{\vec{u}}\,(4,1) \approx \frac{0.5-2}{\sqrt{2}} = -1.1$$

We can get a better estimate by averaging this with the difference quotient obtained by going the other way, that is,

$$f_{\vec{u}}\,(4,1) \approx \frac{f(4,1)-f(5,0)}{\sqrt{2}} = \frac{2-5.5}{\sqrt{2}} = -2.5.$$

Averaging these two results gives an estimate $f_{\vec{u}}\,(4,1) \approx -1.8$.

60. Since $\vec{u} = (-2\vec{i} + \vec{j})/\sqrt{5}$, we head away from the point $(4,1)$ in the direction $(-2\vec{i} + \vec{j})/\sqrt{5}$, that is, toward the point $(2,2)$. From the graph, we see that $f(4,1) = 2$ and $f(2,2) = 0$. Since the points $(4,1)$ and $(2,2)$ are a distance $\sqrt{5}$ apart, we have

$$f_{\vec{u}}(4,1) \approx \frac{f(2,2) - f(4,1)}{\sqrt{5}} = \frac{0-2}{\sqrt{5}} = -0.9$$

We can get a better estimate by averaging this with the difference quotient obtained by going the other way, that is,

$$f_{\vec{u}}(4,1) \approx \frac{f(4,1) - f(6,0)}{\sqrt{5}} = \frac{2-8}{\sqrt{5}} = -2.7.$$

Averaging these two results gives an estimate $f_{\vec{u}}(4,1) \approx -1.8$.

61. First, we check that $2^2 + 3^2 = 13$. Then let $f(x,y) = x^2 + y^2$ so that the given curve is the contour $f(x,y) = 13$. Since $f_x = 2x$ and $f_y = 2y$, we have $\operatorname{grad} f(2,3) = 4\vec{i} + 6\vec{j}$. Since gradients are perpendicular to contours, a vector normal to the curve at $(2,3)$ is $\vec{n} = 4\vec{i} + 6\vec{j}$. Using the normal vector to a line the same way we use the normal vector to a plane, we get that an equation of the tangent line is $4(x-2) + 6(y-3) = 0$.

62. First, we check that $(2)(3) = 6$. Then let $f(x,y) = xy$ so that the given curve is the contour $f(x,y) = 6$. Since $f_x = y$ and $f_y = x$, we have $\operatorname{grad} f(2,3) = 3\vec{i} + 2\vec{j}$. Since gradients are perpendicular to contours, a vector normal to the curve at $(2,3)$ is $\vec{n} = 3\vec{i} + 2\vec{j}$. Using the normal vector to a line the same way we use the normal vector to a plane, we get that the equation of the tangent line is $3(x-2) + 2(y-3) = 0$.

63. First, we check that $3 = 2^2 - 1$. Then let $f(x,y) = y - x^2 + 1$ so that the given curve is the contour $f(x,y) = 0$. Since $f_x = -2x$ and $f_y = 1$, we have $\operatorname{grad} f(2,3) = -4\vec{i} + \vec{j}$. Since gradients are perpendicular to contours, a vector normal to the curve at $(2,3)$ is $\vec{n} = -4\vec{i} + 1\vec{j}$. Using the normal vector to a line the same way we use the normal vector to a plane, we get that an equation of the tangent line is $-4(x-2) + (y-3) = 0$. Notice, if we had instead found the slope of the tangent line using $dy/dx = 2x$, we get $(y-3) = 4(x-2)$, which agrees with the equation we got using the gradient.

64. First, we check that $(3-2)^2 + 2 = (2)(3) - 3$. Then let $f(x,y) = (y-x)^2 - xy + 5$ so that the given curve is the contour $f(x,y) = 0$. Since $f_x = -2(y-x) - y$ and $f_y = 2(y-x) - x$, we have $\operatorname{grad} f(2,3) = -5\vec{i} + 0\vec{j}$. Since gradients are perpendicular to contours, a vector normal to the curve at $(2,3)$ is $\vec{n} = -5\vec{i}$; in other words, the tangent line is a vertical line. Thus the equation of the tangent line is $x = 2$.

65. (a) In the $\vec{i} - \vec{j}$ direction the function is decreasing, so the value of $g_{\vec{u}}(2,5)$ is negative.
(b) In the $\vec{i} + \vec{j}$ direction the function is decreasing, so the value of $g_{\vec{u}}(2,5)$ negative as well.

66. At the point $(1.2, 0)$, the value of the function is 4.2. Nearby, the largest value is 8.9 at the point $(1.4, -1)$. Since the gradient vector points in the direction of maximum increase, it points into the fourth quadrant.

67. (a) Negative. ∇f is perpendicular to the level curve at the point P, so its x-component which is $\nabla f \cdot \vec{i}$ is negative.
(b) Positive. The y-component of ∇f is in the same direction as $\vec{j}$ at P and hence the dot product will be positive.
(c) Positive. The partial derivative with respect to x at Q is positive because the value of f is increasing in the positive x direction at Q. (Note that Q lies between the level curves with values 3 and 4 and that the one with value 4 is further in the positive x direction from Q.)
(d) Negative. Again, Q lies between the level curves with values 3 and 4 and the one with value 3 is further from Q in the positive y direction, so the partial derivative with respect to y at Q is negative.

68. $\|\nabla f\|$ at P is larger because the level curves are closer there.

69. At points (x,y) where the gradients are defined and are not the zero vector, the level curves of f and g intersect at right angles if and only if $\operatorname{grad} f \cdot \operatorname{grad} g = 0$.

We have $\operatorname{grad} f \cdot \operatorname{grad} g = (\vec{i} + \vec{j}) \cdot (\vec{i} - \vec{j}) = 0$. The level curves of f and g are straight lines that cross at right angles. See Figure 14.14.

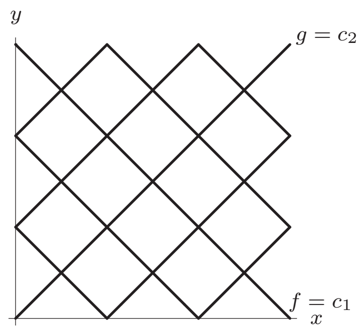


Figure 14.14

70. At points (x, y) where the gradients are defined and are not the zero vector, the level curves of f and g intersect at right angles if and only if $\text{grad } f \cdot \text{grad } g = 0$.

We have $\text{grad } f \cdot \text{grad } g = (2\vec{i} + 3\vec{j}\,) \cdot (2\vec{i} - 3\vec{j}\,) = -5 \neq 0$. The level curves of f and g are straight lines that do not cross at right angles. See Figure 14.15.

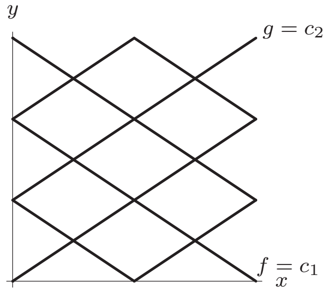


Figure 14.15

71. At points (x, y) where the gradients are defined and are not the zero vector, the level curves of f and g intersect at right angles if and only if $\text{grad } f \cdot \text{grad } g = 0$.

We have $\text{grad } f \cdot \text{grad } g = (2x\vec{i} - \vec{j}\,) \cdot ((1/x)\vec{i} + 2\vec{j}\,) = 0$ at all points where both functions are defined. The level curves of f are parabolas that intersect the level curves of g in right angles. See Figure 14.16.

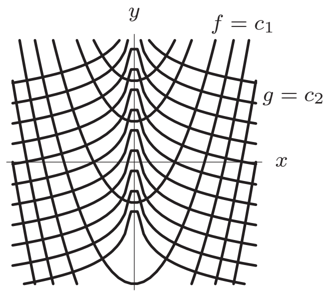


Figure 14.16

72. At points (x, y) where the gradients are defined and are not the zero vector, the level curves of f and g intersect at right angles if and only if $\text{grad } f \cdot \text{grad } g = 0$.

We have $\text{grad } f \cdot \text{grad } g = (2x\vec{i} - 2y\vec{j}\,) \cdot (y\vec{i} + x\vec{j}\,) = 0$. The level curves of f and g are hyperbolas that cross at right angles except at the point $(0, 0)$. See Figure 14.17.

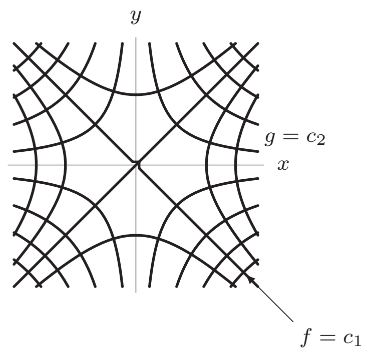


Figure 14.17

73. (a) The graph of $z = y^2$ is in Figure 14.18.
(b) If $z = c$, then $y^2 = c$, so the level curves are $y = \pm\sqrt{c}$. See Figure 14.19.
(c) The level curves in part (b) show that the direction of the greatest increase is in the y direction if the point is in the upper half xy-plane (where $y > 0$). Since the point $(2, 3, 9)$ is in the upper half xy-plane, we climb fastest in the direction $\vec{j}$, parallel to the y-axis.

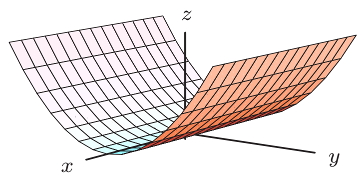


Figure 14.18

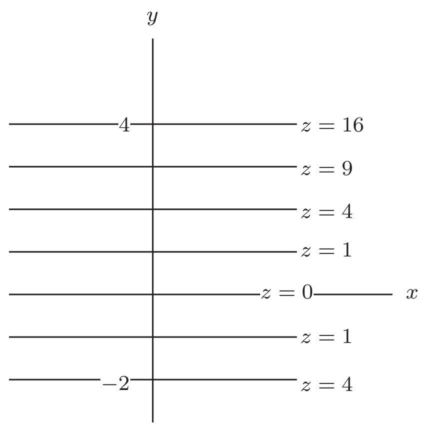


Figure 14.19

74. (a) The fastest descent is in the direction of $-\nabla f$, so

$$-\nabla f(1,3) = (4x\vec{i} + 2y\vec{j})\Big|_{(1,3)} = 4\vec{i} + 6\vec{j}.$$

Any positive multiple of this vector points in the same direction.
(b) If you start to move in this direction, the slope of the path is the rate of change in your height with distance, which is $-||\nabla f|| = -\sqrt{4^2+6^2} = -\sqrt{52}$.

75. (a) We have $f(x, y) = C$ for points (x, y) at a distance C from P, points on the circle of radius C centered at P. The level curves of f are circles centered at P
(b) The gradient of f at the point (x, y) points in the direction that you should move that point to increase its distance from P most rapidly, away from P on the line from P through (x, y). The vector grad f points directly away from P at every point.
(c) The magnitude $\|\operatorname{grad} f(x, y)\|$ is the rate change of f as you go in the direction of $\operatorname{grad} g(x, y)$, which is directly away from P. Every unit farther away from P increases f by 1 because f is the distance from P, so $\|\operatorname{grad} f(x, y)\| = 1$.

76. The vector from $(2, 1)$ to $(1, 3)$ is $\vec{v_1} = (1-2)\vec{i} + (3-1)\vec{j} = -\vec{i} + 2\vec{j}$. A unit vector in this direction is $\vec{u_1} = -\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$.
A vector from $(2, 1)$ to $(5, 5)$ is $\vec{v_2} = (5-2)\vec{i} + (5-1)\vec{j} = 3\vec{i} + 4\vec{j}$. A unit vector in this direction is $\vec{u_2} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.
The directional derivative along $\vec{u_1}$ is

$$z_{\vec{u_1}}(1,2) = \nabla z \cdot \left(-\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}\right) = -\frac{1}{\sqrt{5}}\frac{\partial z}{\partial x} + \frac{2}{\sqrt{5}}\frac{\partial z}{\partial y}.$$

So

$$-\frac{1}{\sqrt{5}}\frac{\partial z}{\partial x} + \frac{2}{\sqrt{5}}\frac{\partial z}{\partial y} = -\frac{2}{\sqrt{5}},$$

that is,
(1)

$$-\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = -2.$$

The directional derivative along $\vec{u_2}$ is

$$z_{\vec{u_2}}(1,2) = \nabla z \cdot \left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right) = \frac{3}{5}\frac{\partial z}{\partial x} + \frac{4}{5}\frac{\partial z}{\partial y},$$

so

$$\frac{3}{5}\frac{\partial z}{\partial x} + \frac{4}{5}\frac{\partial z}{\partial y} = 1,$$

that is,
(2)

$$3\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = 5.$$

Now we solve the system of equations (1) and (2). Multiplying equation (1) by 3 gives
(3)

$$-3\frac{\partial z}{\partial x} + 6\frac{\partial}{\partial y} = -6.$$

Adding (2) and (3) we get:

$$10\frac{\partial z}{\partial y} = -1.$$

So

$$\frac{\partial z}{\partial y} = -0.1$$

and from equation (1)

$$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + 2 = 2(-\frac{1}{10}) + 2 = 1.8.$$

77. Directional derivative = $\nabla f \cdot \vec{u}$, where $\vec{u}$ = unit vector. If we move from $(4,5)$ to $(5,6)$, we move in the direction $\vec{i} + \vec{j}$ so $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$. So,

$$\nabla f \cdot \vec{u} = f_x\left(\frac{1}{\sqrt{2}}\right) + f_y\left(\frac{1}{\sqrt{2}}\right) = 2.$$

Similarly, if we move from $(4,5)$ to $(6,6)$, the direction is $2\vec{i} + \vec{j}$ so $\vec{u} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$. So

$$\nabla f \cdot \vec{u} = f_x\left(\frac{2}{\sqrt{5}}\right) + f_y\left(\frac{1}{\sqrt{5}}\right) = 3.$$

Solving the system of equations for f_x and f_y

$$f_x + f_y = 2\sqrt{2}$$
$$2f_x + f_y = 3\sqrt{5}$$

gives

$$f_x = 3\sqrt{5} - 2\sqrt{2}$$
$$f_y = 4\sqrt{2} - 3\sqrt{5}.$$

Thus at $(4,5)$,

$$\nabla f = (3\sqrt{5} - 2\sqrt{2})\vec{i} + (4\sqrt{2} - 3\sqrt{5})\vec{j}.$$

78. (a) We have

$$g_x = \frac{x}{\sqrt{x^2+3y+3}}, \quad \text{so } g_x(1,4) = \frac{1}{4}$$
$$g_y = \frac{3/2}{\sqrt{x^2+3y+3}}, \quad \text{so } g_y(1,4) = \frac{3}{8}$$
$$\text{grad}\, g(1,4) = \frac{1}{4}\vec{i} + \frac{3}{8}\vec{j}.$$

(b) Using the value $g(1,4) = 4$ and the partial derivatives in part (a) we have

$$g(x,y) = \sqrt{x^2+3y+3} \approx 4 + \frac{1}{4}(x-1) + \frac{3}{8}(y-4).$$

(c) Using the linearization in part (b), we have

$$g(1.01, 3.98) \approx 4 + \frac{1}{4}(0.01) + \frac{3}{8}(-0.02) = 3.995.$$

79. $f_x = 2x$, $f_y = -2y$, so grad $f(3,-1) = 6\vec{i} + 2\vec{j}$. For the direction $\theta = \pi/4$, the direction is $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, so $f_{\vec{u}}(3,-1) = (6\vec{i} + 2\vec{j}) \cdot (\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}) = \frac{8}{\sqrt{2}} = 4\sqrt{2}$.
The directional derivative is largest in the direction of the gradient vector grad $f(3,-1) = 6\vec{i} + 2\vec{j}$.

80. The temperature (Fahrenheit) as a function of position y in miles and time t in hours is given by

$$H = f(y,t) = 30 - 0.05y - 5t.$$

At time t the moose is at position $y = 20t + C$ where C is an unknown constant that depends on where the moose was at $t = 0$. At time t the moose perceives temperature $H = g(t) = f(20t + C, t)$. The rate of change, $g'(t)$, of temperature with time, can be evaluated with the chain rule:

$$g'(t) = \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial t} = -0.05(20) - 5 = -6\ ^\circ/\text{hour}.$$

81. Assume that the x-axis points east and the y-axis points north. We are given that $\|\nabla f\| = 5$ and that ∇f is in the direction $\vec{i} + \vec{j}$. Since $\|\vec{i} + \vec{j}\| = \sqrt{2}$ and ∇f is a multiple of $\vec{i} + \vec{j}$, we have

$$\nabla f = \frac{5}{\sqrt{2}}(\vec{i} + \vec{j}).$$

The rate of change toward the north is the directional derivative in direction $\vec{j}$, which is

$$\nabla f \cdot \vec{j} = \frac{5}{\sqrt{2}}(\vec{i} + \vec{j}) \cdot \vec{j} = \frac{5}{\sqrt{2}}.$$

82. Let's put a coordinate plane on the area you are hiking, with your trail along the x-axis and the second trail branching off at the origin as in Figure 14.20. You are moving in the positive x direction. Let $h(x,y)$ be the elevation at the point (x,y) on the mountain.

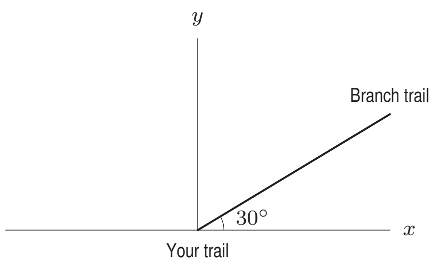


Figure 14.20: Two trails

Since the trail along the x-axis ascends at a 20° angle, we have $h_x(0,0) = \tan 20^\circ$. Since the trail is the steepest path, grad h must point along your trail in the positive x direction. Thus

$$\text{grad}\, h = h_x\vec{i} + 0\vec{j} = \tan 20^\circ \vec{i}.$$

We must compute the rate of change of elevation in the direction of the branch trail. The unit vector in this direction is $\vec{u} = \cos 30^\circ \vec{i} + \sin 30^\circ \vec{j}$, and thus the directional derivative is

$$h_{\vec{u}} = (\text{grad}\, h) \cdot \vec{u} = (\tan 20^\circ \vec{i}) \cdot (\cos 30^\circ \vec{i} + \sin 30^\circ \vec{j}) = (\tan 20^\circ)(\cos 30^\circ) = 0.3152.$$

The angle of ascent of the branch trail is thus $\tan^{-1}(0.3152) = 17.5^\circ$.

83. (a) P corresponds to greatest rate of increase of f and Q corresponds to greatest rate of decrease of f. See Figure 14.21.
(b) The points are marked in Figure 14.22.
(c) Amplitude is $\|$ grad $f\|$. The equation is

$$f_{\vec{u}} = \|\,\text{grad}\, f\| \cos\theta.$$

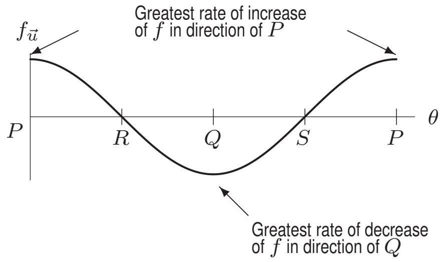


Figure 14.21

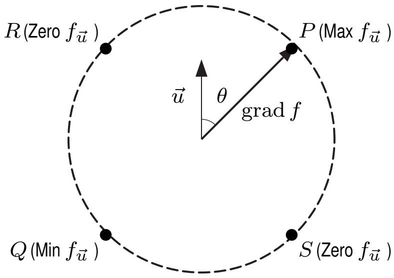


Figure 14.22

84. (a) Since $f(x,y) = 5y - x^2 - y^2$, we have $\nabla f(x,y) = -2x\vec i + (5-2y)\vec j$, so $\nabla f(1,1) = -2\vec i + 3\vec j$, which is the direction of steepest ascent. Therefore, the initial rate of steepest ascent is $\|\nabla f(1,1)\| = \sqrt{13}$ meters ascended for each horizontal meter covered.

(b) In order to go straight northwest, we want to travel along the vector $\vec v = -\vec i + \vec j$. A unit vector that points in the same direction as $\vec v$ is therefore given by $\vec u = \vec v/\|\vec v\| = -(1/\sqrt{2})\vec i + (1/\sqrt{2})\vec j$, so

$$\begin{aligned} f_{\vec u}(1,1) &= \nabla f(1,1)\cdot\vec u \\ &= (-2)(-1/\sqrt{2}) + 3(1/\sqrt{2}) = 3.54 \end{aligned}$$

meters ascended for each horizontal meter traveled.

(c) Here, we are being asked to determine vectors $\vec u$ such that $f_{\vec u}(1,1) = 0$. Since $f_{\vec u}(1,1) = 0$ if and only if $(-2\vec i + 3\vec j)\cdot(u_1\vec i + u_2\vec j) = 0$ or $-2u_1 + 3u_2 = 0$, we see by inspection that $\vec u = 3\vec i + 2\vec j$ and $\vec u = -3\vec i - 2\vec j$ give the two possible directions.

85. (a) To estimate the change in f, we use the gradient vector to estimate the change in f in moving from P to Q. Because the contours are approximately parallel, moving from P to Q takes you to the same contour as moving from P to R. (See Figure 14.23.) If θ is the angle between $\vec u$ and grad $f(a,b)$, then

$$\begin{aligned} \begin{matrix}\text{Change in } f\\ \text{between } P \text{ and } Q\end{matrix} &= \text{Change in } f \\ &= \begin{pmatrix}\text{Rate of change}\\ \text{in direction } PR\end{pmatrix}\begin{pmatrix}\text{Distance traveled}\\ \text{between } P \text{ and } R\end{pmatrix} \\ &\approx \|\operatorname{grad} f\|(h\cos\theta). \end{aligned}$$

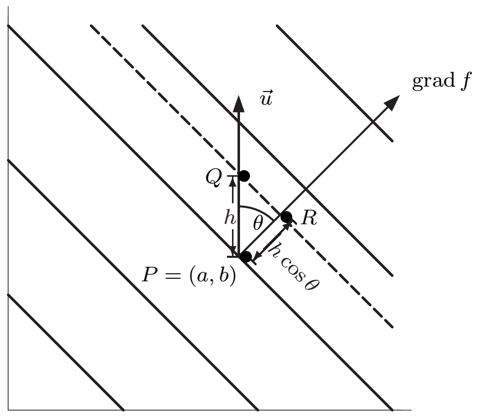


Figure 14.23

(b) Since $\vec u$ is a unit vector, we use the definition of $f_{\vec u}(a,b)$ to estimate

$$\begin{aligned} f_{\vec u}(a,b) &\approx \frac{\text{Change in } f}{h} \approx \frac{\|\operatorname{grad} f(a,b)\|h\cos\theta}{h} \\ &= \|\operatorname{grad} f(a,b)\|\cos\theta = \|\operatorname{grad} f\|\|\vec u\|\cos\theta = \operatorname{grad} f(a,b)\cdot\vec u. \end{aligned}$$

This approximation gets better as we choose h smaller and smaller, and in the limit we get the formula:

$$f_{\vec{u}}(a,b) = \text{grad}\, f(a,b) \cdot \vec{u}\,.$$

86. (a) The ellipse is a level curve of the function $f(x,y) = x^2/2 + y^2$. The vector $\text{grad}\, f = x\vec{i} + 2y\vec{j}$ is perpendicular to the contours of f at the point (x,y). Thus, the vector $\vec{w} = a\vec{i} + 2b\vec{j}$ is perpendicular to the line L tangent to the ellipse at the point (a,b).

(b) Let $\vec{u}$ be the vector from $P = (-1,0)$ to (a,b), so $\vec{u} = (a+1)\vec{i} + b\vec{j}$. The distance from P to the line is

$$p = \frac{|\vec{u}\cdot\vec{w}|}{\|\vec{w}\|} = \frac{\left|a(a+1)+2b^2\right|}{\sqrt{a^2+4b^2}} = \frac{|a(a+1)+2(1-a^2/2)|}{\sqrt{a^2+4(1-a^2/2)}} = \frac{|2+a|}{\sqrt{4-a^2}} = \sqrt{\frac{2+a}{2-a}}$$

where the last equality follows from the substitution $b^2 = 1 - a^2/2$ and the fact that $|a| < 2$.

(c) Let $\vec{v}$ be the vector from $Q = (1,0)$ to (a,b), so $\vec{v} = (a-1)\vec{i} + b\vec{j}$. The distance from Q to L is

$$q = \frac{|\vec{v}\cdot\vec{w}|}{\|\vec{w}\|} = \frac{\left|a(a-1)+2b^2\right|}{\sqrt{a^2+4b^2}} = \frac{|a(a-1)+2b^2|}{\sqrt{a^2+4(1-a^2/2)}} = \frac{|2-a|}{\sqrt{4-a^2}} = \sqrt{\frac{2-a}{2+a}}.$$

(d) We have

$$pq = \sqrt{\frac{2+a}{2-a}}\sqrt{\frac{2-a}{2+a}} = 1.$$

87. (a) Let $\vec{v} = -f_y(a,b)\vec{i} + f_x(a,b)\vec{j}$. Then $\vec{v} \neq 0$ because $\text{grad}\, f \neq 0$. Since $\vec{v}\cdot\text{grad}\, f(a,b) = (-f_y(a,b)\vec{i} + f_x(a,b)\vec{j}\,)\cdot(f_x(a,b)\vec{i} + f_y(a,b)\vec{j}\,) = 0$ we see that the vector $\vec{v}$ is perpendicular to $\text{grad}\, f(a,b)$. The contour C is perpendicular to $\text{grad}\, f(a,b)$ at the point (a,b) by the geometric property of the gradient vector. Since the vector $\vec{v}$ and the contour C at (a,b) are both perpendicular to the same vector they are parallel to each other, which is another way of saying that $\vec{v}$ is tangent to C at (a,b).

(b) A line parallel to a vector $r\vec{i} + s\vec{j}$ with $r \neq 0$ has slope s/r. By part (a) the line tangent to C at (a,b) is parallel to $\vec{v} = r\vec{i} + s\vec{j}$ where $r = -f_y(a,b)$ and $s = f_x(a,b)$. If the tangent line is not vertical, then $f_y(a,b) \neq 0$. Thus the tangent line has slope $s/r = -f_x(a,b)/f_y(a,b)$.

88. The direction of most rapid increase of the sum $f+g$ is given by the vector $\text{grad}(f(x,y)+g(x,y)) = \text{grad}\, f(x,y) + \text{grad}\, g(x,y)$.

(a) A unit vector in the direction of $\vec{w}$ is the vector $\vec{u} = \vec{w}/\|\vec{w}\|$. The rates of change of f and g in the direction of $\vec{w}$ are the directional derivatives

$$f_{\vec{u}}(x,y) = \text{grad}\, f(x,y)\cdot\vec{u} = \text{grad}\, f(x,y)\cdot\vec{w}/\|\vec{w}\|$$
$$g_{\vec{u}}(x,y) = \text{grad}\, g(x,y)\cdot\vec{u} = \text{grad}\, g(x,y)\cdot\vec{w}/\|\vec{w}\|.$$

The two directional derivatives are equal because

$$\begin{aligned}
\text{grad}\, f(x,y)\cdot\vec{w} &= \text{grad}\, f(x,y)\cdot(\text{grad}\, f(x,y) + \text{grad}\, g(x,y)) \\
&= \|\text{grad}\, f(x,y)\|^2 + \text{grad}\, f(x,y)\cdot\text{grad}\, g(x,y) \\
&= \|\text{grad}\, g(x,y)\|^2 + \text{grad}\, f(x,y)\cdot\text{grad}\, g(x,y) \\
&= \text{grad}\, g(x,y)\cdot(\text{grad}\, f(x,y) + \text{grad}\, g(x,y)) \\
&= \text{grad}\, g(x,y)\cdot\vec{w}
\end{aligned}$$

(b) From the calculation in the solution to part (a), we have $\text{grad}\, f(x,y)\cdot\vec{w} = \text{grad}\, g(x,y)\cdot\vec{w}$. Since $\|\text{grad}\, f(x,y)\| = \|\text{grad}\, g(x,y)\|$ we have

$$\frac{\text{grad}\, f(x,y)\cdot\vec{w}}{\|\text{grad}\, f(x,y)\|\|\vec{w}\|} = \frac{\text{grad}\, g(x,y)\cdot\vec{w}}{\|\text{grad}\, g(x,y)\|\|\vec{w}\|}$$

which shows that the angle between $\vec{w}$ and $\text{grad}\, f(x,y)$ is the same as the angle between $\vec{w}$ and $\text{grad}\, g(x,y)$. Since $\text{grad}\, f(x,y)$ is perpendicular to the contour of f through P and $\text{grad}\, g(x,y)$ is perpendicular to the contour of g through P, this shows the $\vec{w}$ makes equal angles with the two contours. Thus $\vec{w}$ bisects the angle between the two contours.

Strengthen Your Understanding

89. Directional derivatives are scalars, not vectors.

90. Gradients are vectors, not scalars.

91. The closer together the contours, the longer the gradient vector.

92. We have

$$f_{\vec{u}}(0,0) = \vec{u} \cdot \text{grad} f(0,0) = \vec{u} \cdot \left(2\vec{i} + 3\vec{j}\right).$$

Many unit vectors make this dot product negative, for example $\vec{u} = -i$ or $\vec{u} = -j$.

93. A possible answer is in Figure 14.24, where the gradient at P is shorter than the gradient at Q because the contours are closer at Q than at P.

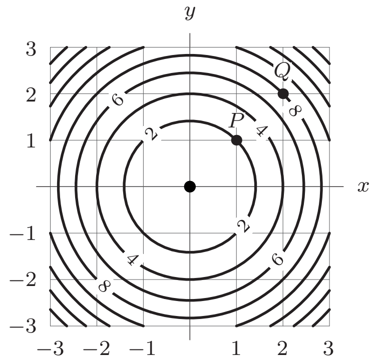


Figure 14.24

94. (a) The gradient vector $\nabla f(P)$ is perpendicular to the contour of f that goes through the point P. It points in the direction of maximal positive rate of change of f at the point P.

(b) The magnitude $\|\nabla f(P)\|$ is the directional derivative of f at the point P in the direction of the gradient vector $\nabla f(P)$ itself. Thus the magnitude equals the maximum directional rate of change of f at the point P.

(c) Given a unit vector $\vec{u}$, the dot product $\nabla f(P) \cdot \vec{u}$ equals the directional derivative $f_{\vec{u}}(P)$ of f at the point P in the direction of $\vec{u}$.

95. False. For example, suppose f is the linear function $f(x,y) = x + y$. Then $f_x = f_y = 1$ at all points. Consider the contour $f = 1$, which is the line $x + y = 1$ of slope -1. There is no point on this contour where the slope is $f_y/f_x = 1$.

96. False. The gradient vector of a function of two variables, $f(x,y)$, is a vector in 2-space given by $f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$.

97. False. Left side is a vector, right side is a scalar.

98. False. The gradient is *perpendicular* to the contour of f at (a,b).

99. True. If $\vec{u}$ is a unit vector, then the directional derivative is given by the formula $f_{\vec{u}}(a,b) = \text{grad}\, f(a,b) \cdot \vec{u}$.

100. True. The components of the gradient vector are the x and y partial derivatives, $f_x(a,b)$ and $f_y(a,b)$. These are the directional derivatives of f in the $\vec{i}$ and $\vec{j}$ directions, respectively.

101. False. The directional derivative is a scalar, not a vector.

102. False. The gradient vector at $(3,4)$ has no relation to the direction of the vector $3\vec{i} + 4\vec{j}$. For example, if $f(x,y) = x+2y$, then $\text{grad}\, f = \vec{i} + 2\vec{j}$, which is not perpendicular to $3\vec{i} + 4\vec{j}$. The gradient vector $\text{grad}\, f(3,4)$ *is* perpendicular to the contour of f passing through the point $(3,4)$.

103. True. The gradient points in the direction of maximal increase of f, and the opposite direction gives the direction of maximum decrease for f.

104. False. The gradient points in the direction of greatest *local* maximum increase. This means that f increases in the $\vec{i}$ direction only *near* the point $(1,2)$. We cannot conclude that f keeps increasing in that direction as far away from $(1,2)$ as $(10,2)$.

105. True. The length of the gradient gives the maximal rate of increase. We have $\text{grad}\, g(3,0) = 6\vec{i}$ and $\text{grad}\, h(3,0) = 6\vec{j}$, so $\|\text{grad}\, g(3,0)\| = \|\text{grad}\, h(3,0)\| = 6$.

106. True. The length of the gradient gives the maximal directional derivative in any direction. The gradient vector is $\text{grad}\, f(0,0) = \vec{i} + \vec{j}$, which has length $\sqrt{2}$.

107. True. It is the rate of change of f in the direction of $\vec{u}$ at the point (x_0, y_0).

108. False. $f_{\vec{u}}\,(a,b) = \|\nabla f(a,b)\|\cos\theta$, where θ is the angle between grad f and $\vec{u}$.

109. Must be true, because at any point grad f is perpendicular to level curves through that point.

110. True. Take the direction perpendicular to grad f at that point. If grad $f = 0$, any direction will do.

111. Is never true. If $\|\operatorname{grad} f\| = 0$, then grad $f = 0$, so grad $f \cdot \vec{u} = 0$ for any unit vector $\vec{u}$. Thus the directional derivative must be zero.

Solutions for Section 14.5

Exercises

1. Since $f_x = 2x$, $f_y = 0$ and $f_z = 0$, we have

$$\operatorname{grad} f = 2x\vec{i}\,.$$

2. We have $f_x = 2x$, $f_y = 3y^2$, and $f_z = -4z^3$. Thus

$$\operatorname{grad} f = 2x\vec{i} + 3y^2\vec{j} - 4z^3\vec{k}\,.$$

3. Since $f(x, y, z) = e^{x+y+z} = e^x e^y e^z$, we have $f_x = e^x e^y e^z$, $f_y = e^x e^y e^z$ and $f_z = e^x e^y e^z$, so

$$\operatorname{grad} f = e^x e^y e^z(\vec{i} + \vec{j} + \vec{k}\,).$$

4. Since $f_x = -\sin(x + y)$, $f_y = -\sin(x + y) + \cos(y + z)$, $f_z = \cos(y + z)$, we have

$$\operatorname{grad} f = -\sin(x + y)\vec{i} + (\cos(y + z) - \sin(x + y))\vec{j} + \cos(y + z)\vec{k}\,.$$

5. Since $f_x = -2xyz^2/(1 + x^2)^2$, $f_y = z^2/(1 + x^2)$, and $f_z = 2yz/(1 + x^2)$,

$$\operatorname{grad} f = \frac{-2xyz^2}{(1 + x^2)^2}\vec{i} + \frac{z^2}{1 + x^2}\vec{j} + \frac{2yz}{1 + x^2}\vec{k}\,.$$

6. We have $f_x = -2x/(x^2 + y^2 + z^2)^2$, $f_y = -2y/(x^2 + y^2 + z^2)^2$, and $f_z = -2z/(x^2 + y^2 + z^2)^2$. Thus

$$\operatorname{grad} f = \frac{-2}{(x^2 + y^2 + z^2)^2}(x\vec{i} + y\vec{j} + z\vec{k}\,).$$

7. We have $f_x = x/\sqrt{x^2 + y^2 + z^2}$, $f_y = y/\sqrt{x^2 + y^2 + z^2}$, and $f_z = z/\sqrt{x^2 + y^2 + z^2}$. Thus

$$\operatorname{grad} f = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\vec{i} + y\vec{j} + z\vec{k}\,).$$

8. We have $f_x = e^y \sin z$, $f_y = xe^y \sin z$, and $f_z = xe^y \cos z$. Thus

$$\operatorname{grad} f = e^y \sin z\vec{i} + xe^y \sin z\vec{j} + xe^y \cos z\vec{k}\,.$$

9. Since $f_x = y$, $f_y = x$, and $f_z = e^z \cos(e^z)$,

$$\operatorname{grad} f = y\vec{i} + x\vec{j} + e^z \cos(e^z)\,\vec{k}\,.$$

10. Since $f_{x_1} = 2x_1x_2^3x_3^4$, $f_{x_2} = 3x_1^2x_2^2x_3^4$, $f_{x_3} = 4x_1^2x_2^3x_3^3$, we have

$$\operatorname{grad} f = (2x_1x_2^3x_3^4)\vec{i} + (3x_1^2x_2^2x_3^4)\vec{j} + (4x_1^2x_2^3x_3^3)\vec{k}\,.$$

11. Since $f_p = e^p$, $f_q = 1/q$, $f_r = 2re^{r^2}$, we have

$$\text{grad}\, f = e^p \vec{i} + \frac{1}{q}\vec{j} + 2re^{r^2}\vec{k}.$$

12. We have

$$\text{grad}(e^{z^2} + y\ln(x^2+5)) = y\frac{2x}{x^2+5}\vec{i} + \ln(x^2+5)\vec{j} + 2ze^{z^2}\vec{k}.$$

13. We have $f_x = 0$, $f_y = 2yz$ and $f_z = y^2$. Thus $\text{grad}\, f = 2yz\vec{j} + y^2\vec{k}$ and $\text{grad}\, f(1,0,1) = \vec{0}$.

14. We have $f_x = 2$, $f_y = 3$ and $f_z = 4$ so $\text{grad}\, f = 2\vec{i} + 3\vec{j} + 4\vec{k}$ at all points.

15. We have

$$\text{grad}(x^2+y^2-z^4)\bigg|_{(3,2,1)} = 2x\vec{i} + 2y\vec{j} - 4z^3\vec{k}\bigg|_{(3,2,1)} = 6\vec{i} + 4\vec{j} - 4\vec{k}.$$

16. We have $f_x = yz$, $f_y = xz$, and $f_z = yz$. Thus $f_x(1,2,3) = 6$, $f_y(1,2,3) = 3$, and $f_z(1,2,3) = 2$, so

$$\text{grad}\, f = 6\vec{i} + 3\vec{j} + 2\vec{k}.$$

17. We have

$$\begin{aligned}\text{grad}(\sin(xy)+\sin(yz))\bigg|_{(1,\pi,-1)} &= y\cos(xy)\vec{i} + (x\cos(xy) + z\cos(yz))\vec{j} + y\cos(yz)\vec{k}\bigg|_{(1,\pi,-1)} \\ &= -\pi\vec{i} - \pi\vec{k}.\end{aligned}$$

18. We have

$$\text{grad}(x\ln(yz))\bigg|_{(2,1,e)} = \ln(yz)\vec{i} + \frac{x}{y}\vec{j} + \frac{x}{z}\vec{k}\bigg|_{(2,1,e)} = \vec{i} + 2\vec{j} + \frac{2}{e}\vec{k}.$$

19. We have $\text{grad}\, f = y\vec{i} + x\vec{j} + 2z\vec{k}$, so $\text{grad}\, f(1,2,3) = 2\vec{i} + \vec{j} + 6\vec{k}$. A unit vector in the direction we want is $u = (1/\sqrt{3})(\vec{i} + \vec{j} + \vec{k})$. Therefore, the directional derivative is

$$\text{grad}\, f(1,2,3)\cdot\vec{u} = \frac{2\cdot 1 + 1\cdot 1 + 6\cdot 1}{\sqrt{3}} = \frac{9}{\sqrt{3}}.$$

20. We have $\text{grad}\, f = y\vec{i} + x\vec{j} + 2z\vec{k}$, so $\text{grad}\, f(1,1,1) = \vec{i} + \vec{j} + 2\vec{k}$. A unit vector in the direction we want is $u = (1/\sqrt{14})(\vec{i} + 2\vec{j} + 3\vec{k})$. Therefore, the directional derivative is

$$\text{grad}\, f(1,1,1)\cdot\vec{u} = \frac{1\cdot 1 + 1\cdot 2 + 2\cdot 3}{\sqrt{14}} = \frac{9}{\sqrt{14}}.$$

21. We have $\text{grad}\, f = y\vec{i} + x\vec{j} + 2z\vec{k}$, so $\text{grad}\, f(1,1,0) = \vec{i} + \vec{j}$. A unit vector in the direction we want is $u = (1/\sqrt{2})(-\vec{i} + \vec{k})$. Therefore, the directional derivative is

$$\text{grad}\, f(1,1,0)\cdot\vec{u} = \frac{1(-1) + 1\cdot 0 + 0\cdot 1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}.$$

22. We have $\text{grad}\, f = y\vec{i} + x\vec{j} + 2z\vec{k}$, so $\text{grad}\, f(0,1,1) = \vec{i} + 2\vec{k}$. A unit vector in the direction we want is $u = (1/\sqrt{2})(-\vec{i} + \vec{k})$. Therefore, the directional derivative is

$$\text{grad}\, f(0,1,1)\cdot\vec{u} = \frac{1(-1) + 0\cdot 0 + 2\cdot 1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

23. We have grad $f = y\vec{i} + x\vec{j} + 2z\vec{k}$, so grad $f(2,3,4) = 3\vec{i} + 2\vec{j} + 8\vec{k}$. Let $\vec{u}$ be a unit vector making an angle of $3\pi/4$ with grad $f(2,3,4)$. Then, the directional derivative is

$$\text{grad}\, f(2,3,4)\cdot\vec{u} = \|\,\text{grad}\, f(2,3,4)\|\|\vec{u}\,\|\cos\left(\frac{3\pi}{4}\right) = \sqrt{77}(1)\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{\frac{77}{2}}.$$

24. We have grad $f = y\vec{i} + x\vec{j} + 2z\vec{k}$, so grad $f(2,3,4) = 3\vec{i} + 2\vec{j} + 8\vec{k}$. The maximum rate of change of f at $(2,3,4)$ is in the direction of grad $f(2,3,4)$ and the directional derivative in that direction is the maximum rate of change, namely $\|\,\text{grad}\, f(2,3,4)\| = \sqrt{77}$.

25. First, we check that $(-1)^2 - (1)^2 + 2^2 = 4$. Then let $f(x,y,z) = x^2 - y^2 + z^2$ so that the given surface is the level surface $f(x,y,z) = 4$. Since $f_x = 2x$, $f_y = -2y$, and $f_z = 2z$, we have grad $f(-1,1,2) = -2\vec{i} - 2\vec{j} + 4\vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = -2\vec{i} - 2\vec{j} + 4\vec{k}$. Thus an equation for the tangent plane is

$$-2(x+1) - 2(y-1) + 4(z-2) = 0.$$

26. First, we check that $2 = (-1)^2 + (1)^2$. Then let $f(x,y,z) = z - x^2 - y^2$ so that the given surface is the level surface $f(x,y,z) = 0$. Since $f_x = -2x$, $f_y = -2y$, and $f_z = 1$, we have grad $f(-1,1,2) = 2\vec{i} - 2\vec{j} + \vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = 2\vec{i} - 2\vec{j} + \vec{k}$. Thus an equation for the tangent plane is

$$2(x+1) - 2(y-1) + (z-2) = 0.$$

Note that you could also view the surface as the graph of the function $z = g(x,y) = x^2 + y^2$ and get the equation of the tangent plane using the local linearization of g.

27. First, we check that $1 = 2^2 - 3$. Then we let $f(x,y,z) = y^2 - z^2 + 3$, so that the given surface is the level surface $f(x,y,z) = 0$. Since $f_x = 0$, $f_y = 2y$, and $f_z = -2z$, we have grad $f(-1,1,2) = 2\vec{j} - 4\vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = 2\vec{j} - 4\vec{k}$. Thus an equation for the tangent plane is

$$2(y-1) - 4(z-2) = 0.$$

28. First, we check that $(-1)^2 - (-1)(1)(2) = 3$. Then let $f(x,y,z) = x^2 - xyz$ so that the given surface is the level surface $f(x,y,z) = 3$. Since $f_x = 2x - yz$, $f_y = -xz$, and $f_z = -xy$, we have grad $f(-1,1,2) = -4\vec{i} + 2\vec{j} + \vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = -4\vec{i} + 2\vec{j} + \vec{k}$. Thus an equation for the tangent plane is

$$-4(x+1) + 2(y-1) + (z-2) = 0.$$

29. First, we check that $\cos(-1+1) = e^{-2+2}$. Then we let $f(x,y,z) = \cos(x+y) - e^{xz+2}$, so that the given surface is the level surface $f(x,y,z) = 0$. Since $f_x = -\sin(x+y) - ze^{xz+2}$, $f_y = -\sin(x+y)$, and $f_z = -xe^{xz+2}$, we have grad $f(-1,1,2) = -2\vec{i} + \vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = -2\vec{i} + \vec{k}$. Thus an equation for the tangent plane is

$$-2(x+1) + (z-2) = 0.$$

30. First, we check that $1 = 4/(2(-1)+3(2))$. We could let $f(x,y,z) = y - 4/(2x+3y)$, but instead let's try $f(x,y,z) = y(2x+3z)$ so that the given surface is the level surface $f(x,y,z) = 4$. Since $f_x = 2y$, $f_y = 2x+3z$, and $f_z = 3y$, we have grad $f(-1,1,2) = 2\vec{i} + 4\vec{j} + 3\vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = 2\vec{i} + 4\vec{j} + 3\vec{k}$. Thus an equation for the tangent plane is

$$2(x+1) + 4(y-1) + 3(z-2) = 0.$$

31. (a) The unit vector $\vec{u}_1$ in the direction of $\vec{v}_1 = \vec{i} - \vec{k}$ is $\vec{u}_1 = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}$. We have

$$\begin{aligned} f_x(x,y,z) &= 6xy^2, && \text{and } f_x(-1,0,4) = 0 \\ f_y(x,y,z) &= 6x^2y + 2z, && \text{and } f_y(-1,0,4) = 8 \\ f_z(x,y,z) &= 2y, && \text{and } f_z(-1,0,4) = 0. \end{aligned}$$

So,

$$\begin{aligned} f_{\vec{u}_1}(-1,0,4) &= f_x(-1,0,4)\left(\frac{1}{\sqrt{2}}\right) + f_y(-1,0,4)(0) + f_z(-1,0,4)\left(-\frac{1}{\sqrt{2}}\right) \\ &= 0\left(\frac{1}{\sqrt{2}}\right) + 8(0) + 0\left(-\frac{1}{\sqrt{2}}\right) \\ &= 0. \end{aligned}$$

(b) The unit vector $\vec{u}_2 = -\frac{1}{\sqrt{19}}\vec{i} + \frac{3}{\sqrt{19}}\vec{j} + \frac{3}{\sqrt{19}}\vec{k}$ is in the direction of $\vec{v}_2 = -\vec{i} + 3\vec{j} + 3\vec{k}$. Using the partial derivatives from part (a),

$$f_{\vec{u}_2}(-1,0,4) = 0\left(-\frac{1}{\sqrt{19}}\right) + 8\left(\frac{3}{\sqrt{19}}\right) + 0\left(\frac{3}{\sqrt{19}}\right) = \frac{24}{\sqrt{19}}.$$

32. The unit vector $\vec{u}$ in the direction of $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{2}{3}\right)\vec{i} + \left(\frac{1}{3}\right)\vec{j} - \left(\frac{2}{3}\right)\vec{k}.$$

The partial derivatives are

$$\begin{aligned} f_x(x,y,z) &= 2x + 3y, \\ f_y(x,y,z) &= 3x, \\ f_z(x,y,z) &= 2. \end{aligned}$$

Thus, we have

$$\begin{aligned} f_{\vec{u}}(2,0,-1) &= f_x(2,0,-1)\left(\frac{2}{3}\right) + f_y(2,0,-1)\left(\frac{1}{3}\right) + f_z(2,0,-1)\left(-\frac{2}{3}\right) \\ &= 4\left(\frac{2}{3}\right) + 6\left(\frac{1}{3}\right) + 2\left(-\frac{2}{3}\right) = \frac{10}{3} \end{aligned}$$

33. (a) We have $f_x = 2x - yz$, $f_y = 2y - xz$ and $f_z = -xy$ so

$$\text{grad } f = (2x - yz)\vec{i} + (2y - xz)\vec{j} - xy\vec{k}.$$

(b) At the point $(2,3,1)$ we have

$$\text{grad } f(2,3,1) = \vec{i} + 4\vec{j} + 6\vec{k}.$$

Thus an equation of the tangent plane to the level surface at the point $(2,3,1)$ is

$$(x-2) + 4(y-3) + 6(z-1) = 0$$

or

$$x + 4y + 6z = 20.$$

34. Since

$$z_x = \frac{-x}{\sqrt{17 - x^2 - y^2}} \quad \text{and} \quad z_y = \frac{-y}{\sqrt{17 - x^2 - y^2}},$$

we have

$$z(3,2) = \sqrt{17 - 9 - 4} = 2, \quad z_x(3,2) = -\frac{3}{2} \quad z_y(3,2) = \frac{-2}{2} = -1.$$

The tangent plane to $z = \sqrt{17 - x^2 - y^2}$ at $(x, y) = (3, 2)$ is

$$z = z(3,2) + z_x(3,2)(x-3) + z_y(3,2)(y-2) = 2 + (\frac{-3}{2})(x-3) + (-1)(y-2) = \frac{17}{2} - \frac{3}{2}x - y,$$

or

$$2z + 3x + 2y = 17.$$

35. Since

$$z_x = -8/yx^2 \quad \text{and} \quad z_y = -8/xy^2,$$

we have

$$z(1,2) = 8/(1)(2) = 4, \quad z_x(1,2) = -8/(2)(1)^2 = -4, \quad z_y(1,2) = -8/(1)(2)^2 = -2.$$

The tangent plane to $z = 8/xy$ at $(x, y) = (1, 2)$ is

$$z = z(1,2) + z_x(1,2)(x-1) + z_y(1,2)(y-2) = 4 - 4(x-1) - 2(y-2) = 12 - 4x - 2y.$$

36. The surface is given by $F(x, y, z) = 0$ where $F(x, y, z) = x - y^3z^7$. The normal direction is

$$\text{grad}\, F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} = \vec{i} - 3y^2z^7\vec{j} - 7y^3z^6\vec{k}.$$

Thus, at $(1, -1, -1)$ a normal vector is $\vec{i} + 3\vec{j} + 7\vec{k}$. The tangent plane has the equation

$$1(x-1) + 3(y-(-1)) + 7(z-(-1)) = 0$$
$$x + 3y + 7z = -9.$$

37. At the point $P = (1, 2, 3)$ we have $\text{grad}\, f = 2 \cdot 3\vec{i} + 1 \cdot 3\vec{j} + 1 \cdot 2\vec{k} = 6\vec{i} + 3\vec{j} + 2\vec{k}$. Hence the tangent plane to the level surface $f(x, y, z) = 0$ at the point P is given by the equation

$$6(x-1) + 3(y-2) + 2(z-1) = 0.$$

38. At the point $P = (10, -10, 30)$ we have $\text{grad}\, f = 2 \cdot 10\vec{i} + 30^2\vec{j} + 2(-10)30\vec{k} = 20\vec{i} + 900\vec{j} - 600\vec{k}$. Hence the tangent plane to the level surface $f(x, y, z) = 0$ at the point P is given by the equation

$$20(x-10) + 900(y+10) - 600(z-30) = 0.$$

39. A normal to the surface is $2x\vec{i} + 2y\vec{j} + 2z\vec{k}$; a normal to the tangent plane at $(2, 3, 2)$ is

$$\vec{n} = 4\vec{i} + 6\vec{j} + 4\vec{k}.$$

The tangent plane can be written as

$$4(x-2) + 6(y-3) + 4(z-2) = 0$$

or

$$4x + 6y + 4z = 4 \cdot 2 + 6 \cdot 3 + 4 \cdot 2 = 34$$
$$2x + 3y + 2z = 17.$$

40. Let $f(x, y, z) = x^2 + y^2$ so that the surface is the level surface $f(x, y, z) = 1$. Since

$$\text{grad } f = 2x\vec{i} + 2y\vec{j}$$

we have

$$\text{grad } f(1, 0, 1) = 2\vec{i}.$$

Thus an equation of the tangent plane at the point $(1, 0, 1)$ is

$$2(x - 1) + 0(y - 0) + 0(z - 1) = 0$$

or

$$2(x - 1) = 0.$$

The tangent plane is given by the equation

$$x = 1.$$

41. Since $z = 2x + y + 3$ is a plane it is its own tangent plane.

42. Let $f(x, y, z) = 3x^2 - 4xy + z^2$ so that the surface is the level surface $f(x, y, z) = 0$. Since

$$\text{grad } f = (6x - 4y)\vec{i} - 4x\vec{j} + 2z\vec{k},$$

we have

$$\text{grad } f(a, a, a) = 2a\vec{i} - 4a\vec{j} + 2a\vec{k}.$$

Thus an equation of the tangent plane at (a, a, a) is

$$2a(x - a) - 4a(y - a) + 2a(z - a) = 0$$

or

$$2ax - 4ay + 2az = 0.$$

Since $a \neq 0$, we can write the plane more simply as

$$2x - 4y + 2z = 0.$$

Notice that it is the same plane for every a.

43. The point on the surface $z = 9/(x + 4y)$ where $x = 1$ and $y = 2$ has third coordinate $z = 9/(1 + 8) = 1$. We want the tangent plane to the surface at the point $(1, 2, 1)$.

Let $f(x, y, z) = z(x + 4y)$ so that the given surface is the level surface $f(x, y, z) = 9$. Since

$$\text{grad } f = z\vec{i} + 4z\vec{j} + (x + 4y)\vec{k}$$

we have

$$\text{grad } f(1, 2, 1) = \vec{i} + 4\vec{j} + 10\vec{k}.$$

Thus an equation of the tangent plane at $(1, 2, 1)$ is

$$(x - 1) + 4(y - 2) + 9(z - 1) = 0$$

so

$$x + 4y + 9z = 18.$$

Problems

44. The point $(-1, -1, 3)$ lies above the point $(-1, -1)$. The vector grad $g(-1, -1)$ points horizontally in the direction in which g increases most rapidly and lies directly under the path of steepest ascent. (See Figures 14.25 and 14.26.)

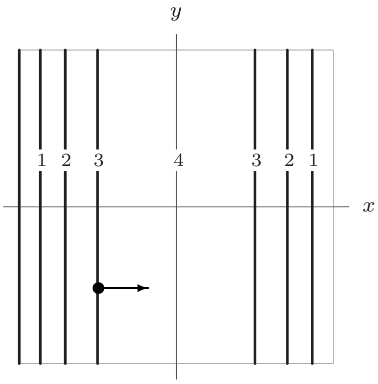


Figure 14.25: Contour diagram for $z = g(x, y) = 4 - x^2$ showing direction of grad $g(-1, -1)$

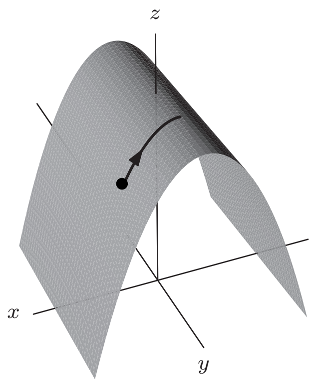


Figure 14.26: Graph of $g(x, y) = 4 - x^2$ showing path of steepest ascent from the point $(-1, -1, 3)$

45. The gradient of (a) is $2x\vec{i} + 2y\vec{j} + 2z\vec{k}$, which points radially outward from the origin, so (a) goes with (III).
The gradient of (c) is parallel to the gradient of (a) but pointing inward, so (c) goes with (IV).
The gradient of (b) is $2x\vec{i} + 2y\vec{j}$, which points radially outward from the z-axis, so (b) goes with (I).
The gradient of (d) is parallel to the gradient of (b) but pointing inward, so (d) goes with (II).

46. (a) The surface is the level surface $F(x, y, z) = 7$, where $F(x, y, z) = x^2 + y^2 - xyz$. Thus the normal vector to the tangent plane is grad $F = (2x - yz)\vec{i} + (2y - xz)\vec{j} + (-xy)\vec{k}$. Evaluated at $(2, 3, 1)$, we get the normal to the plane

$$\vec{n} = \vec{i} + 4\vec{j} - 6\vec{k}.$$

Thus the equation of the plane is

$$(x - 2) + 4(y - 3) - 6(z - 1) = 0.$$

(b) Solving $x^2 + y^2 - xyz = 7$ for z, we get

$$z = \frac{x^2 + y^2 - 7}{xy}.$$

Thus, we have

$$f(x, y) = \frac{x^2 + y^2 - 7}{xy} = \frac{x}{y} + \frac{y}{x} - \frac{7}{xy}.$$

We have

$$f_x(x, y) = \frac{1}{y} - \frac{y}{x^2} + \frac{7}{x^2 y}$$
$$f_y(x, y) = -\frac{x}{y^2} + \frac{1}{x} + \frac{7}{xy^2}.$$

Thus $f_x(2, 3) = 1/3 - 3/4 + 7/12 = 1/6$ and $f_y(2, 3) = -2/9 + 1/2 + 7/18 = 2/3$. Thus the equation of the tangent plane is

$$z = 1 + (1/6)(x - 2) + (2/3)(y - 3).$$

This is the same as the answer to part (a) when that equation is solved for z.

47. At the point $(1, 2, 1)$, we have

$$\text{grad}\, f(1, 2, 1) = 2 \cdot 1\vec{i} + 2 \cdot 2\vec{j} + 2 \cdot 1\vec{k} = 2\vec{i} + 4\vec{j} + 2\vec{k}.$$

The normal to the plane pointing away from the origin is

$$\vec{n} = \vec{i} + 2\vec{j} + 3\vec{k}.$$

Thus we find the rate of change of f in the direction of the unit vector

$$\vec{u} = \frac{\vec{n}}{||\vec{n}||} = \frac{\vec{n}}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k}).$$

The rate of change is given by the directional derivative

$$f_{\vec{u}} = \text{grad}\, f \cdot \vec{u} = 2\vec{i} + 4\vec{j} + 4\vec{k} \cdot \frac{(\vec{i} + 2\vec{j} + 3\vec{k})}{\sqrt{14}} = \frac{22}{\sqrt{14}}.$$

48. (a) To get a normal vector to the surface $z = \cos x \sin y$ at the point $(0, \pi/2, 1)$, we first represent the surface S by the equation $F(x, y, z) = z - \cos x \sin y = 0$. Then we calculate the gradient of F which is normal to S

$$\text{grad } F = \sin x \sin y\vec{i} - \cos x \cos y\vec{j} + \vec{k}.$$

At the point $(0, \pi/2, 1)$,

$$\text{grad } F(0, \pi/2, 1) = \vec{k}.$$

(b) The plane with normal $\vec{k}$ and through the point $(0, \pi/2, 1)$ is

$$z = 1.$$

So $z = 1$ is the equation of the tangent plane at the point $(0, \pi/2, 1)$

49. (a) Since $f(x, y, z) = \sin(x^2 + y^2 + z^2)$, the level surfaces of f are spheres centered at the origin.

(b) Since $f_x = 2x \sin(x^2 + y^2 + z^2)$ and $f_y = 2y \sin(x^2 + y^2 + z^2)$ and $f_z = 2z \sin(x^2 + y^2 + z^2)$, we have

$$\text{grad } f = 2x \sin(x^2 + y^2 + z^2)\vec{i} + 2y \sin(x^2 + y^2 + z^2)\vec{j} + 2z \sin(x^2 + y^2 + z^2)\vec{k}.$$

(c) We can write the formula for grad f as

$$\text{grad } f = 2\vec{r} \sin(x^2 + y^2 + z^2),$$

so grad f is parallel to $\vec{r}$ at the point (x, y, z). Thus, the angle between grad f and $\vec{r}$ is $0°$ or $180°$. The angle is $0°$ if $\sin(x^2 + y^2 + z^2)$ is positive at that point and $180°$ if $\sin(x^2 + y^2 + z^2)$ is negative at that point. (We are assuming that $\sin(x^2 + y^2 + z^2) \neq 0$.)

50. (a) To do this you need to imagine the surfaces with the normal vector $\vec{n}$ at P.

For (I),	For (II),	For (III),	For (IV),
$\vec{n}$ is $(-, -, +)$	$\vec{n}$ is $(+, +, +)$	$\vec{n}$ is $(+, -, +)$	$\vec{n}$ is $(-, +, +)$
or $(+, +, -)$	or $(-, -, -)$	or $(-, +, -)$	or $(+, -, -)$
so (E)	so (F)	so (G)	so (H)

(b) Since the equation of the tangent plane is of the form

$$n_1x + n_2y + n_3z = k,$$

the coefficients of x, y, z in the plane must have the same sign as the components of the normal. Hence (I)-(E)-(L); (II)-(F)-(J); (III)-(G)-(M); (IV)-(H)-(K).

51. (a) A normal vector at (x, y, z) is given by

$$\begin{aligned}\nabla F(x, y, z) &= \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} \\ &= 2x\vec{i} - \frac{1}{z^2}\vec{j} + \frac{2y}{z^3}\vec{k}\end{aligned}$$

The direction of maximum increase of F is ∇F. At $(0, 0, 1)$ this is $-\vec{j}$, which is a unit vector. So $\vec{u_1} = -\vec{j}$.
At $(1, 1, 1)$ this is $2\vec{i} - \vec{j} + 2\vec{k}$, so the unit vector $\vec{u_2} = \frac{2\vec{i} - \vec{j} + 2\vec{k}}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$.

(b) A normal vector at $(0, 0, 1)$ is $\nabla f(0, 0, 1) = (2 \cdot 0)\vec{i} - \frac{1}{1^2}\vec{j} + \frac{2 \cdot 0}{1^3}\vec{k} = -\vec{j}$
The equation of the tangent plane at the point $(0, 0, 1)$ is

$$((x - 0)\vec{i} + (y - 0)\vec{j} + (z - 1)\vec{k}) \cdot (-\vec{j}) = 0,$$

i.e.,

$$-y = 0, \quad \text{so} \quad y = 0.$$

A normal vector at $(1, 1, 1)$ is $\nabla f(1, 1, 1) = (2 \cdot 1)\vec{i} - (\frac{1}{1^2})\vec{j} + (\frac{2 \cdot 1}{1^3})\vec{k} = 2\vec{i} - \vec{j} + 2\vec{k}$.
The equation of the tangent plane at $(1, 1, 1)$ is

$$((x - 1)\vec{i} + (y - 1)\vec{j} + (z - 1)\vec{k}) \cdot (2\vec{i} - \vec{j} + 2\vec{k}) = 0,$$

i.e.,

$$2x - 2 - y + 1 + 2z - 2 = 0,$$

so

$$2x - y + 2z = 3.$$

(c) The vector ∇F is parallel to the xy-plane when it is perpendicular to $\vec{k}$, i.e. when

$$(2x\vec{i} - \frac{1}{z^2}\vec{j} + \frac{2y}{z^3}\vec{k}) \cdot \vec{k} = 0,$$

that is,

$$\frac{2y}{z^3} = 0 \quad \text{so} \quad y = 0.$$

Points that meet these requirements are those (x, y, z) such that

$$x^2 - \frac{y}{z^2} = 0 \quad \text{and} \quad y = 0,$$

i.e., points such that $x = y = 0$ and $z \neq 0$, that is, the z-axis minus the origin. (Observe that we must exclude the points where $z = 0$ because the surface is not defined there: the expression $x^2 - (y/z^2)$ is undefined when $z = 0$.)

52. (a) The vector grad $f(x, y)$ is perpendicular to the level curve of f through (x, y):

$$\text{grad}\, f(x, y) = (e^x - 1)\cos y\vec{i} - (e^x - x)\sin y\vec{j}$$

Thus, at the point $(2, 3)$,

$$\text{grad}\, f(2, 3) = (e^2 - 1)\cos 3\vec{i} - (e^2 - 2)\sin 3\vec{j}$$

The vector grad f points in the direction of greatest increase in f, so the vector we want is

$$\begin{aligned} -\text{grad}\, f(2, 3) &= -(e^2 - 1)\cos 3\vec{i} + (e^2 - 2)\sin 3\vec{j} \\ &= 6.33\vec{i} + 0.76\vec{j} \end{aligned}$$

(b) To find a vector normal to the surface, we write the surface in the form

$$F(x, y, z) = (e^x - x)\cos y - z = 0.$$

Then

$$\text{grad}\, F = (e^x - 1)\cos y\vec{i} - (e^x - x)\sin y\vec{j} - \vec{k}.$$

So, at P,

$$\begin{aligned} \text{grad}\, F &= (e^2 - 1)\cos 3\vec{i} - (e^2 - 2)\sin y\vec{j} - \vec{k} \\ &= -6.33\vec{i} - 0.76\vec{j} - \vec{k}. \end{aligned}$$

The vector $\vec{v}$ is perpendicular to grad F, so

$$\vec{v} \cdot \text{grad}\, F = (5\vec{i} + 4\vec{j} + a\vec{k}) \cdot (-6.33\vec{i} - 0.76\vec{j} - \vec{k}) = 0.$$

This gives

$$-5(6.33) - 4(0.76) - a = 0$$

so

$$a = -34.69$$

53. (a) A normal to the surface is given by the gradient of the function $f(x, y, z) = x^2 + y^2 + 3z^2$,

$$\text{grad}\, f = 2x\vec{i} + 2y\vec{j} + 6z\vec{k}.$$

At the point $(0.6, 0.8, 1)$, a normal to the surface and to the tangent plane is

$$\vec{n} = 1.2\vec{i} + 1.6\vec{i} + 6\vec{k}.$$

Since the plane goes through the point $(0.6, 0.8, 1)$, its equation is

$$1.2(x - 0.6) + 1.6(y - 0.8) + 6(z - 1) = 0$$

$$1.2x + 1.6y + 6z = 8.$$

(b) We want to know if there are x, y, z values such that a normal to the surface is parallel to the normal to the plane. That is, is $2x\vec{i} + 2y\vec{j} + 6z\vec{k}$ parallel to $8\vec{i} + 6\vec{j} + 30\vec{k}$? Yes, if $2x = 8t, \quad 2y = 6t, \quad 6z = 30t$ for some value of t. That is, if

$$x = 4t, \quad y = 3t, \quad z = 5t.$$

Substituting these equations into the equation for the surface and solving for t, we get

$$\begin{aligned}(4t)^2 + (3t)^2 + 3\,(5t)^2 &= 4\\ 100t^2 &= 4\\ t = \pm\sqrt{\frac{4}{100}} &= \pm 0.2.\end{aligned}$$

So there are two points on the surface at which the tangent plane is parallel to the plane $8x + 6y + 30z = 1$. They are

$$\pm(0.8, 0.6, 1).$$

54. (a) The height is given by $f(4, 3) = 2(4^2) - (3^2) = 23$. Your house is 23 units above the xy-plane.

(b) We want a directional derivative so we start by computing the gradient. Since the partial derivatives are $f_x = 4x$ and $f_y = -2y$, we have $f_x(4, 3) = 16$ and $f_y(4, 3) = -6$. The gradient vector is grad $f = 16\vec{i} - 6\vec{j}$. Letting $\vec{v} = -4\vec{i} - 3\vec{j}$, we have $\|\vec{v}\| = 5$, so a unit vector in the same direction is $\vec{u} = (-4/5)\vec{i} - (3/5)\vec{j} = -0.8\vec{i} - 0.6\vec{j}$. The directional derivative in the direction of $\vec{u}$ at the point $(4, 3)$ is

$$\begin{aligned}f_{\vec{u}}\,(4, 3) &= \text{ grad } f(4, 3) \cdot \vec{u}\\ &= (16\vec{i} - 6\vec{j}) \cdot (-0.8\vec{i} - 0.6\vec{j})\\ &= -9.2.\end{aligned}$$

The ground slopes down in that direction quite steeply, going down at a rate of 9.2 units for every one horizontal unit moved.

(c) The water runs off in the direction of minimum slope, which is the direction of the negative of the gradient vector at that point. We already saw that grad $f(4, 3) = 16\vec{i} - 6\vec{j}$, so the water runs off in the direction of $-16\vec{i} + 6\vec{j}$.

(d) We can write the surface $z = 2x^2 - y^2$ as $F(x, y, z) = 2x^2 - y^2 - z = 0$. Then $F_x = 4x$ and $F_y = -2y$ and $F_z = -1$. At the point $(4, 3, 23)$, we have $F_x(4, 3, 23) = 16$ and $F_y(4, 3, 23) = -6$ and $F_z(4, 3, 23) = -1$. The equation of the tangent plane is

$$\begin{aligned}16(x - 4) - 6(y - 3) - 1(z - 23) &= 0\\ 16x - 6y - z &= 23\end{aligned}$$

55. (a) See Figure 14.27.

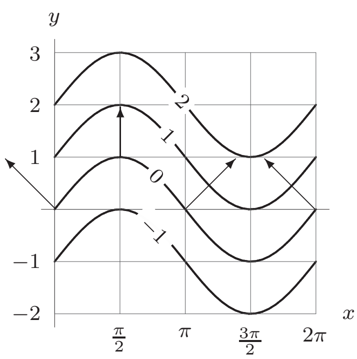


Figure 14.27

(b) The bug is walking parallel to the y-axis. Looking to the right or left, the bug sees higher contours — thus it is in a valley.

(c) See Figure 14.27.

56. (a) The plane $z = 5$ is horizontal. The surface $z = 1 + x^2 + y^2$ is bowl-shaped, with its lowest point at $(0, 0, 1)$. At this point its tangent plane is horizontal and therefore parallel to the plane $z = 5$.

(b) The tangent plane to the surface $z = f(x, y)$ at the point where $(x, y) = (a, b)$ has equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Thus, for $z = 1 + x^2 + y^2$, the tangent plane is

$$\begin{aligned} z &= (1 + a^2 + b^2) + 2a(x - a) + 2b(y - b) \\ &= (1 - a^2 - b^2) + 2ax + 2by. \end{aligned}$$

This is parallel to $z = 5 + 6x - 10y$ when $6 = 2a$, and $-10 = 2b$ so $a = 3$, $b = -5$. Then $z = f(3, -5) = 1 + 3^2 + (-5)^2 = 35$, so the point on the surface whose tangent plane is parallel to $z = 5 + 6x - 10y$ is $(3, -5, 35)$.

57. (a) In the direction of grad F:

$$\operatorname{grad} F\Big|_{(-1,1,1)} = \left((2x + 2xz^2)\vec{i} + (4y^3)\vec{j} + (2x^2z)\vec{k}\right)\Big|_{(-1,1,1)} = -4\vec{i} + 4\vec{j} + 2\vec{k}.$$

(b) The rate of change in the direction of grad F with respect to distance $= \|\nabla F\| = \sqrt{16 + 16 + 4} = 6$. Now we want rate of change with respect to time. If we move at 4 units/sec:

$$\begin{aligned} \text{Rate of change of } \frac{\text{Conc}}{\text{Time}} &= \text{Rate of change of } \frac{\text{Conc}}{\text{Dist}} \times \text{Rate of change of } \frac{\text{Dist}}{\text{Time}} \\ &= 6 \times 4 = 24\text{mg/cm}^3\text{/sec}. \end{aligned}$$

58. The tangent plane is given by

$$3(x - 2) + 10(y - 1) - 5(z - 7) = 0$$

so

$$3x + 10y - 5z + 19 = 0.$$

59. Since $f_{\vec{u}} = \nabla f \cdot \vec{u}$, we see that $f_{\vec{u}_1}$ and $f_{\vec{u}_4}$ are both positive because the angles between $\vec{u}_1$ and ∇f and between $\vec{u}_4$ and ∇f are both between 0 and $\pi/2$. In addition, $f_{\vec{u}_1}$ is larger because the angle between $\vec{u}_1$ and ∇f is smaller than the angle between $\vec{u}_4$ and $\vec{f}$. Similarly, $f_{\vec{u}_2}$ and $f_{\vec{u}_3}$ are both negative, and $f_{\vec{u}_3}$ is more negative. Thus,

$$f_{\vec{u}_3} < f_{\vec{u}_2} < 0 < f_{\vec{u}_4} < f_{\vec{u}_1}$$

60. (a) We have $\nabla G = (2x - 5y)\vec{i} + (-5x + 2yz)\vec{j} + (y^2)\vec{k}$, so $\nabla G(1, 2, 3) = -8\vec{i} + 7\vec{j} + 4\vec{k}$. The rate of change is given by the directional derivative in the direction $\vec{v}$:

$$\begin{aligned} \text{Rate of change in density} &= \nabla G \cdot \frac{\vec{v}}{\|\vec{v}\|} = (-8\vec{i} + 7\vec{j} + 4\vec{k}) \cdot \frac{(2\vec{i} + \vec{j} - 4\vec{k})}{\sqrt{21}} \\ &= \frac{-16 + 7 - 16}{\sqrt{21}} = \frac{-25}{\sqrt{21}} \approx -5.455. \end{aligned}$$

(b) The direction of maximum rate of change is $\nabla G(1, 2, 3) = -8\vec{i} + 7\vec{j} + 4\vec{k}$.

(c) The maximum rate of change is $\|\nabla G(1, 2, 3)\| = \sqrt{(-8)^2 + 7^2 + 4^2} = \sqrt{129} \approx 11.36$.

61. (a) The function $T(x, y, z) =$ constant where $x^2 + y^2 + z^2 =$ constant. These surfaces are spheres centered at the origin.

(b) Calculating the partial derivative with respect to x gives

$$\frac{\partial T}{\partial x} = -2xe^{-(x^2+y^2+z^2)}.$$

Similar calculations for the other variables shows that

$$\operatorname{grad} T = (-2x\vec{i} - 2y\vec{j} - 2z\vec{k})e^{-(x^2+y^2+z^2)}.$$

(c) At the point $(1,0,0)$

$$\operatorname{grad} T(1,0,0) = -2e^{-1}\vec{i}.$$

Moving from the point $(1,0,0)$ to $(2,1,0)$, you move in the direction

$$(2-1)\vec{i} + (1-0)\vec{j} = \vec{i} + \vec{j}.$$

A unit vector in this direction is

$$\vec{u} = \frac{\vec{i} + \vec{j}}{\sqrt{2}}.$$

The directional derivative of $T(x,y,z)$ in this direction at the point $(1,0,0)$ is

$$T_{\vec{u}}(1,0,0) = -2e^{-1}\vec{i} \cdot \frac{\vec{i} + \vec{j}}{\sqrt{2}} = -\sqrt{2}e^{-1}.$$

Since you are moving at a speed of 3 units per second,

$$\text{Rate of change of temperature } = -\sqrt{2}e^{-1} \cdot 3 = -3\sqrt{2}e^{-1} \text{ degrees/second.}$$

62. We have

$$\frac{\partial P}{\partial x} = 5e^{-0.1\sqrt{x^2+y^2+z^2}}(-0.1)(2x)\frac{1}{2}(x^2+y^2+z^2)^{-1/2} = -0.5e^{-0.1\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2+z^2}},$$

and similarly

$$\frac{\partial P}{\partial y} = -0.5e^{-0.1\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \text{and} \quad \frac{\partial P}{\partial z} = -0.5e^{-0.1\sqrt{x^2+y^2+z^2}}\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Thus the gradient of P at $(0,0,1)$ is

$$\operatorname{grad} P = \left.\frac{\partial P}{\partial x}\right|_{(x,y,z)=(0,0,1)}\vec{i} + \left.\frac{\partial P}{\partial y}\right|_{(x,y,z)=(0,0,1)}\vec{j} + \left.\frac{\partial P}{\partial z}\right|_{(x,y,z)=(0,0,1)}\vec{k} = -0.5e^{-0.1}\vec{k}.$$

Let $\vec{u}$ be a unit vector in the direction of $\vec{v}$, so

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Then

$$\begin{aligned}\text{Rate of change of pressure in atm/sec} &= (\text{Directional derivative in direction } \vec{u} \text{ in atm/mi})(\text{Speed of spacecraft in mi/sec})\\ &= P_{\vec{u}}\|\vec{v}\| = (\operatorname{grad} P \cdot \vec{u})\|\vec{v}\| = \operatorname{grad} P \cdot \frac{\vec{v}}{\|\vec{v}\|}\|\vec{v}\| = \operatorname{grad} P \cdot \vec{v}\\ &= -0.5e^{-0.1}\vec{k} \cdot (\vec{i} - 2.5\vec{k}) = 0.5 \cdot 2.5e^{-0.1} = 1.131 \text{ atm/sec.}\end{aligned}$$

63. (a) is (V) since $\vec{r} + \vec{a}$ is a vector not a scalar.
(b) is (IV) since $\operatorname{grad}(\vec{r} \cdot \vec{a}) = \operatorname{grad}(a_1x + a_2y + a_3z) = \vec{a}$.
(c) is (V) since $\vec{r} \times \vec{a}$ is a vector not a scalar.

64. We must calculate the gradient of φ.

$$\begin{aligned}\operatorname{grad}\varphi(x,y,z) &= \operatorname{grad}\frac{GmM}{\|\vec{r}\|}\\ &= GMm \operatorname{grad}\frac{1}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

Now

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{-1/2}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}.$$

The partial derivatives with respect to y and z are similar, so

$$\text{grad}\,\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Thus,

$$\begin{aligned}\text{grad}\,\varphi &= -GMm\frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}\\ &= -GMm\frac{\vec{r}}{\|\vec{r}\|^3}\\ &= \vec{F}\end{aligned}$$

65. The tangent plane to $z=\sqrt{2x^2+2y^2-25}$ at $(x,y)=(4,3)$ is

$$\begin{aligned}z &= z(4,3)+z_x(4,3)(x-4)+z_y(4,3)(y-3)\\ &= \sqrt{2(4)^2+2(3)^2-25}+\frac{2(4)}{\sqrt{2(4)^2+2(3)^2-25}}(x-4)+\frac{2(3)}{\sqrt{2(4)^2+2(3)^2-25}}(y-3)\\ &= 5+\frac{8}{5}(x-4)+\frac{6}{5}(y-3).\end{aligned}$$

The tangent plane to $z=\frac{1}{5}(x^2+y^2)$ at $(x,y)=(4,3)$ is

$$\begin{aligned}z &= z(4,3)+z_x(4,3)(x-4)+z_y(4,3)(y-3)\\ &= \frac{1}{5}(4^2+3^2)+\frac{2}{5}(4)(x-4)+\frac{2}{5}(3)(y-3)\\ &= 5+\frac{8}{5}(x-4)+\frac{6}{5}(y-3).\end{aligned}$$

Thus the two surfaces are tangential at the point $(4,3,5)$.

66. The points of intersection are $(x,y)=(a,b)$ such that

$$\begin{aligned}\frac{1}{2}(a^2+b^2-1) &= \frac{1}{2}(1-a^2-b^2)\\ a^2+b^2-1 &= 1-(a^2+b^2)\\ 2(a^2+b^2) &= 2\\ a^2+b^2 &= 1,\end{aligned}$$

which are points on the unit circle.

The tangent plane to $z=\frac{1}{2}(x^2+y^2-1)$ at $(x,y)=(a,b)$ such that $a^2+b^2=1$ is

$$\begin{aligned}z &= z(a,b)+z_x(a,b)(x-a)+z_y(a,b)(y-b)\\ &= 0+a(x-a)+b(y-b)\\ &= ax+by-(a^2+b^2)\\ &= ax+by-1, \text{ or } ax+by-z=1.\end{aligned}$$

Similarly, the tangent to $z=\frac{1}{2}(1-x^2-y^2)$ at $(x,y)=(a,b)$ such that $a^2+b^2=1$ is $z=-ax-by+1$ or $ax+by+z=1$. The normal vector to the plane $ax+by-z=1$ is $\vec{n_1}=a\vec{i}+b\vec{j}-\vec{k}$ and the normal vector to the plane $ax+by+z=1$ is $\vec{n_2}=a\vec{i}+b\vec{j}+\vec{k}$.

Since $\vec{n_1}\cdot\vec{n_2}=a^2+b^2-1=1-1=0$, $\vec{n_1}$ and $\vec{n_2}$ are perpendicular, hence the two surfaces are orthogonal at all points of intersection.

67.

$$\begin{aligned}\text{grad}(\vec{\mu}\cdot\vec{r}) &= \text{grad}(\mu_1x+\mu_2y+\mu_3z)\\ &= \mu_1\vec{i}+\mu_2\vec{j}+\mu_3\vec{k}=\vec{\mu}.\end{aligned}$$

68.

$$\begin{aligned}\text{grad}(\|\vec{r}\|^a) &= \text{grad}((x^2+y^2+z^2)^{a/2}) \\ &= \frac{a}{2}(x^2+y^2+z^2)^{(a/2)-1}(2x)\vec{i} + \frac{a}{2}(x^2+y^2+z^2)^{(a/2)-1}(2y)\vec{j} \\ &\quad + \frac{a}{2}(x^2+y^2+z^2)^{(a/2)-1}(2z)\vec{k} \\ &= a(x^2+y^2+z^2)^{(a-2)/2}(x\vec{i}+y\vec{j}+z\vec{k}) \\ &= a\|\vec{r}\|^{a-2}\vec{r}.\end{aligned}$$

69. If write $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then we know

$$\text{grad}\, f(x,y,z) = g(x,y,z)(x\vec{i}+y\vec{j}+z\vec{k}) = g(x,y,z)\vec{r}$$

so grad f is everywhere radially outward, and therefore perpendicular to a sphere centered at the origin. If f were not constant on such a sphere, then grad f would have a component tangent to the sphere. Thus, f must be constant on any sphere centered at the origin.

Strengthen Your Understanding

70. The gradient vector grad $f(x,y)$ points in the direction perpendicular to the level curves $f(x,y) = C$ in the xy-plane.

71. The correct equation of the tangent plane is

$$f_x(0,0,0)x + f_y(0,0,0)y + f_z(0,0,0)z = 0.$$

72. The surface $z = f(x,y)$ can be rewritten $f(x,y) - z = 0$ with normal at $(0,0)$ given by

$$\vec{n} = f_x(0,0)\vec{i} + f_y(0,0)\vec{j} - \vec{k}.$$

To have normal vector $\vec{n} = \vec{i} - 2\vec{j} - \vec{k}$, we take $f_x(0,0) = 1$ and $f_y(0,0) = -2$. One example is $f(x,y) = x - 2y$.

73. Any function $f(x,y,z) = 2x + 3y + 4z + C$ where C is a constant has grad $f = 2\vec{i} + 3\vec{j} + 4\vec{k}$. For example, we can take

$$f(x,y,z) = 2x + 3y + 4z + 100.$$

74. We have

$$\text{grad}\, f = 2\vec{i} - 3\vec{j}.$$

We want vectors $\vec{u}$ and $\vec{v}$ which are perpendicular to grad f. Two possibilities are $\vec{k}$ and $3\vec{i} + 2\vec{j}$. Creating unit vectors gives

$$\vec{u} = \vec{k} \qquad \text{and} \qquad \vec{v} = \frac{1}{\sqrt{3^2+2^2}}\left(3\vec{i} + 2\vec{j}\right).$$

Then grad $f \cdot \vec{u} = $ grad $f \cdot \vec{v} = 0$, so $f_{\vec{u}} = f_{\vec{v}} = 0$.

75. False. The equation $z = 2 + 2x(x-1) + 3y^2(y-1)$ is not linear. The correct equation is $z = 2 + 2(x-1) + 3(y-1)$, which is obtained by evaluating the partial derivatives at the point $(1,1)$.

76. True. For example, the function $f(x,y) = x^2 + y^2$ has a horizontal tangent plane at $(0,0)$.

77. This is never true, because, if θ is the angle between grad f and the z-axis at any point, then $f_{\vec{k}} = \|\text{grad}\, f\| \cos\theta \le \|\text{grad}\, f\|$.

78. Can be true, For example for $f(x,y,z) = 4x - 3z$, we have grad $f = 4\vec{i} - 3\vec{k}$ then $\|\text{grad}\, f\| = \sqrt{4^2+3^2} = 5$ and $f_{\vec{k}} = f_z = -3$.

79. (a) The units of $||\text{grad}\, f||$ are °C per meter. It represents the rate of change of temperature with distance as you move in the direction of grad f.

(b) The units of grad $f \cdot \vec{v}$ are °C per second. It represents the rate of change of temperature with time as you move with velocity $\vec{v}$.

(c) The units of $||\text{grad}\, f|| \cdot ||\vec{v}\,||$ are °C per second. It represents the rate of change of temperature with time if you move in the direction of grad f with speed $||\vec{v}\,||$.

Solutions for Section 14.6

Exercises

1. Using the chain rule we see:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= -y^2 e^{-t} + 2xy\cos t\\ &= -(\sin t)^2 e^{-t} + 2e^{-t}\sin t\cos t\\ &= \sin(t)e^{-t}(2\cos t - \sin t)\end{aligned}$$

We can also solve the problem using one variable methods:

$$\begin{aligned} z &= e^{-t}(\sin t)^2\\ \frac{dz}{dt} &= \frac{d}{dt}(e^{-t}(\sin t)^2)\\ &= \frac{de^{-t}}{dt}(\sin t)^2 + e^{-t}\frac{d(\sin t)^2}{dt}\\ &= -e^{-t}(\sin t)^2 + 2e^{-t}\sin t\cos t\\ &= e^{-t}\sin t(2\cos t - \sin t)\end{aligned}$$

2. Using the chain rule we see:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 2t(\sin y + y\cos x) + \frac{1}{t}(x\cos y + \sin x)\\ &= 2t\sin(\ln t) + 2t\ln(t)\cos(t^2) + t\cos(\ln t) + \frac{\sin t^2}{t}\end{aligned}$$

This problem can also be solved using one variable methods. Attempting to solve the problem that way will demonstrate the advantage of using the chain rule.

3. Substituting into the chain rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)(2) + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)(-2t)\\ &= \cos\left(\frac{x}{y}\right)\left(\frac{2y+2xt}{y^2}\right) = 2\cos\left(\frac{2t}{1-t^2}\right)\frac{1+t^2}{(1-t^2)^2}\end{aligned}$$

4. This is a case where substituting is easier:

$$\begin{aligned} z &= \ln(t^{-2} + t)\\ \frac{dz}{dt} &= \frac{1-2t^{-3}}{t^{-2}+t}\\ &= \frac{t^3-2}{t+t^4}\end{aligned}$$

If you use the chain rule the solution is:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= \frac{-2x}{t^2(x^2+y^2)} + \frac{y}{\sqrt{t}(x^2+y^2)}\end{aligned}$$

$$= \frac{-2}{t^3((1/t)^2+t)} + \frac{1}{(1/t)^2+t}$$
$$= \frac{-2}{t+t^4} + \frac{t^2}{1+t^3}$$
$$= \frac{t^3-2}{t+t^4}$$

5. Substituting into the chain rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^y(2) + xe^y(-2t)$$
$$= 2e^y(1-xt) = 2e^{1-t^2}(1-2t^2).$$

6. Substituting into the chain rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^y(2) + (xe^y + e^y + ye^y)(-2t)$$
$$= 2e^y(1-xt-t-yt) = 2e^{1-t^2}(1-2t^2-2t+t^3).$$

7. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{1}{y}\right)\frac{1}{u} + \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{-x}{y^2}\right)\cdot 0$$
$$= \frac{1}{vu}\cos\left(\frac{\ln u}{v}\right).$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{1}{y}\right)\cdot 0 + \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{-x}{y^2}\right)\cdot 1 = -\frac{\ln u}{v^2}\cos\left(\frac{\ln u}{v}\right).$$

8. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$= \frac{1}{x}\cdot 2(u^2+v^2)(2u) + \frac{1}{y}\cdot 2(u^3+v^3)(3u^2)$$
$$= \frac{4u(u^2+v^2)}{(u^2+v^2)^2} + \frac{6u^2(u^3+v^3)}{(u^3+v^3)^2}$$
$$= \frac{4u}{u^2+v^2} + \frac{6u^2}{u^3+v^3}.$$

Similarly, we have

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$
$$= \frac{1}{x}\cdot 2(u^2+v^2)(2v) + \frac{1}{y}\cdot 2(u^3+v^3)(3v^2)$$
$$= \frac{4v(u^2+v^2)}{(u^2+v^2)^2} + \frac{6v^2(u^3+v^3)}{(u^3+v^3)^2}$$
$$= \frac{4v}{u^2+v^2} + \frac{6v^2}{u^3+v^3}.$$

9. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^y\left(\frac{1}{u}\right) + xe^y \cdot 0 = \frac{e^v}{u}.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = e^y(0) + xe^y \cdot 1 = e^v \ln u.$$

10. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^y\left(\frac{1}{u}\right) + e^y(1+x+y) \cdot 0 = \frac{e^v}{u}.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = e^y(0) + e^y(1+x+y) \cdot 1 = (1 + \ln u + v)e^v.$$

11. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^y(2u) + xe^y(2u)\\ &= 2ue^y(1+x) = 2ue^{(u^2-v^2)}(1+u^2+v^2).\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = e^y(2v) + xe^y(-2v)\\ &= 2ve^y(1-x) = 2ve^{(u^2-v^2)}(1-u^2-v^2).\end{aligned}$$

12. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^y(2u) + (xe^y + e^y + ye^y)(2u)\\ &= 2ue^y(1+x+1+y) = 2ue^y(x+y+2)\\ &= 2ue^{(u^2-v^2)}(u^2+v^2+u^2-v^2+2) = 2ue^{(u^2-v^2)}(2u^2+2)\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = e^y(2v) + (xe^y + e^y + ye^y)(-2v)\\ &= 2ve^y(1-x-1-y) = -2ve^y(x+y)\\ &= -2ve^{(u^2-v^2)}(u^2+v^2+u^2-v^2) = -4u^2ve^{(u^2-v^2)}.\end{aligned}$$

13. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

First to find $\partial z/\partial u$

$$\begin{aligned}\frac{\partial z}{\partial u} &= (e^{-y} - ye^{-x})\sin v + (-xe^{-y} + e^{-x})(-v\sin u)\\ &= (e^{-v\cos u} - v(\cos u)e^{-u\sin v})\sin v - (-u(\sin v)e^{-v\cos u} + e^{-u\sin v})v\sin u\end{aligned}$$

Now we find $\partial z/\partial v$ using the same method.

$$\begin{aligned}\frac{\partial z}{\partial v} &= (e^{-y} - ye^{-x})u\cos v + (-xe^{-y} + e^{-x})\cos u\\ &= (e^{-v\cos u} - v(\cos u)e^{-u\sin v})u\cos v + (-u(\sin v)e^{-v\cos u} + e^{-u\sin v})\cos u\end{aligned}$$

14. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

This problem is most easily solved by substitution:

$$\begin{aligned} z &= \cos(u^2((\cos v)^2 + (\sin v)^2)) \\ &= \cos u^2 \\ \frac{\partial z}{\partial u} &= -2u \sin u^2 \\ \frac{\partial z}{\partial v} &= 0 \end{aligned}$$

This problem can also be solved using the chain rule but it is more difficult.

15. Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{1}{1+(\frac{x}{y})^2}(\frac{1}{y})(2u) + \frac{1}{1+(\frac{x}{y})^2}(\frac{-x}{y^2})(2u) \\ &= 2u(\frac{y-x}{y^2+x^2}) = \frac{-2uv^2}{u^4+v^4} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{1}{1+(\frac{x}{y})^2}(\frac{1}{y})(2v) + \frac{1}{1+(\frac{x}{y})^2}(\frac{-x}{y^2})(-2v) \\ &= 2v(\frac{y+x}{y^2+x^2}) = \frac{2vu^2}{u^4+v^4}. \end{aligned}$$

Problems

16. By the chain rule

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (3x^2y^2)(3t^2) + (2x^3y)(2t) \\ &= (3t^6t^4)3t^2 + (2t^9t^2)2t \\ &= 9t^{12} + 4t^{12} \\ &= 13t^{12} \end{aligned}$$

Directly, we have $z = t^9 \cdot t^4 = t^{13}$, so $dz/dt = 13t^{12}$.

17. Using the chain rule we have

$$\begin{aligned} \frac{\partial w}{\partial \rho} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho} \\ &= (2x)\sin\phi\cos\theta + (2y)\sin\phi\sin\theta - (2z)\cos\phi \\ &= 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi, \\ &= 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi \\ &= 2\rho\sin^2\phi - 2\rho\cos^2\phi \\ &= -2\rho\cos 2\phi. \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} \\ &= (2x)(-\rho\sin\phi\sin\theta) + (2y)\rho\sin\phi\cos\theta \\ &= -2\rho^2\sin^2\phi\cos\theta\sin\theta + 2\rho^2\sin^2\phi\sin\theta\cos\theta \\ &= 0 \end{aligned}$$

18. (a) $\partial f/\partial t$
(b) $(\partial f/\partial x)(dx/dt)$
(c) $(\partial f/\partial y)(dy/dt)$

19. When $t = 1$,

$$x = g(1) = 3 \quad \text{and} \quad y = h(1) = 10,$$

so $z = f(3, 10) = 7$. Thus

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t)$$

gives

$$\begin{aligned} \left.\frac{\partial z}{\partial t}\right|_{t=1} &= f_x(3, 10) \cdot g'(1) + f_y(3, 10) \cdot h'(1) \\ &= 100 \cdot 4 + 0.1 \cdot 11 \\ &= 400 + 1.1 = 401.1. \end{aligned}$$

20. The voltage at any time t is given by $V = IR$ where R is the resistance for the whole circuit. (In this case $R = R_1R_2/(R_1 + R_2)$.) So the rate at which the voltage is changing is

$$\begin{aligned} \frac{dV}{dt} &= \frac{dI}{dt}R + I\frac{dR}{dt} \\ &= \frac{dI}{dt}R + I\left(\frac{\partial R}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial R}{\partial R_2}\frac{dR_2}{dt}\right) \\ &= \frac{dI}{dt}R + I\left(\frac{R_2^2}{(R_1 + R^2)^2}\frac{dR_1}{dt} + \frac{R_1^2}{(R_1 + R_2)^2}\frac{dR_2}{dt}\right) \\ &= 0.01\left(\frac{15}{8}\right) + 2\left(\frac{25}{64}(0.5) + \frac{9}{64}(-0.1)\right) \\ &= 0.3812. \end{aligned}$$

So the voltage is increasing by 0.3812 volts/sec.

21. Let $p(x, t)$ be the air pressure in pascals at x km east of the island at time t hours after the ship passes the island. We want to compute $\partial p/\partial t$.

Let $S(t)$ be the air pressure on the ship at time t, so that $S(t) = p(10t, t)$. By the chain rule we have

$$\frac{dS}{dt} = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial t}\frac{dt}{dt} = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial t}$$

Since $dS/dt = (-50 \text{ pascal})/(2 \text{ hour}) = -25$ pascal/hour, and $\partial p/\partial x = -2$ pascal/km, and $dx/dt = 10$ km/hour, solving for $\partial p/\partial t$, we have

$$\frac{\partial p}{\partial t} = -25 \text{ pascal/hour} - (-2 \text{ pascal/km})(10 \text{ km/hour}) = -5 \text{ pascal/hour}$$

22. Let the square cross section have side length $x(T)$ and the bar have length $L(T)$ at temperature T°C; then

$$\text{Volume of the bar} = V = x^2L.$$

Using the chain rule we have:

$$\begin{aligned} \frac{dV}{dT} &= \frac{\partial V}{\partial x}\frac{dx}{dT} + \frac{\partial V}{\partial L}\frac{dL}{dT} \\ &= 2xL\frac{dx}{dT} + x^2\frac{dL}{dT} \\ &= (2xL + x^2) \cdot 13 \cdot 10^{-6}, \end{aligned}$$

since $dx/dT = dL/dT = 13 \cdot 10^{-6}$. When $L = 3$ and $x = 0.05$, the rate of increase of volume is

$$\frac{dV}{dT} = (2 \cdot (0.05) \cdot 3 + 0.05^2) \cdot 13 \times 10^{-6} = 3.93 \cdot 10^{-6}\text{m}^3/°C.$$

23.

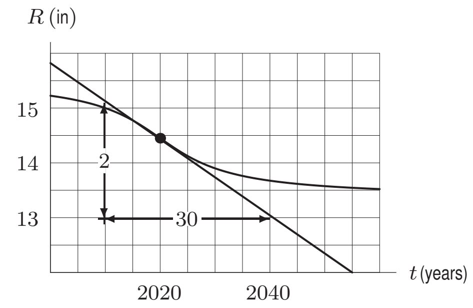


Figure 14.28: Global warming predictions: Rainfall as a function of time

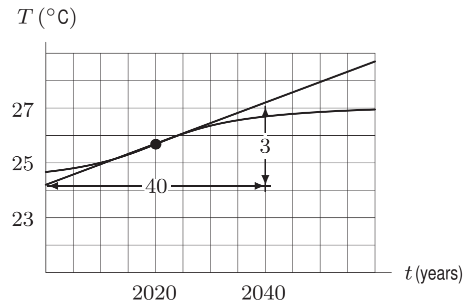


Figure 14.29: Global warming predictions: Temperature as a function of time

We know that, as long as the temperature and rainfall stay close to their current values of $R = 15$ inches and $T = 30°$C, a change, ΔR, in rainfall and a change, ΔT, in temperature produces a change, ΔC, in corn production given by

$$\Delta C \approx 3.3\Delta R - 5\Delta T.$$

Now both R and T are functions of time t (in years), and we want to find the effect of a small change in time, Δt, on R and T. Figure 14.28 shows that the slope of the graph for R versus t is about $-2/30 \approx -0.07$ in/year when $t = 2020$. Similarly, Figure 14.29 shows the slope of the graph of T versus t is about $3/40 \approx 0.08°$C/year when $t = 2020$. Thus, around the year 2020,

$$\Delta R \approx -0.07\Delta t \quad \text{and} \quad \Delta T \approx 0.08\Delta t.$$

Substituting these into the equation for ΔC, we get

$$\Delta C \approx (3.3)(-0.07)\Delta t - (5)(0.08)\Delta t \approx -0.6\Delta t.$$

Since at present $C = 100$, corn production will decline by about 0.6 % between the years 2020 and 2021. Now $\Delta C \approx -0.6\Delta t$ tells us that when $t = 2020$,

$$\frac{\Delta C}{\Delta t} \approx -0.6, \quad \text{and therefore, that} \quad \frac{dC}{dt} \approx -0.6.$$

24. (a) The level surfaces of f are concentric spheres centered at the origin. This is because if $f(x, y, z)$ is fixed, then $g(\rho)$ is fixed, which means ρ must be fixed. Fixed ρ gives a sphere.

(b) By the chain rule, we find

$$\frac{\partial f}{\partial x} = \frac{dg}{d\rho}\frac{\partial \rho}{\partial x} = g'(\rho)\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x = g'(\rho)\frac{x}{\sqrt{x^2 + y^2 + z^2}}.$$

Thus

$$\vec{F} = \text{grad}\, f = \frac{dg}{d\rho} \cdot \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\rho},$$

so $\vec{F}(x, y, z)$ is parallel to $x\vec{i} + y\vec{j} + z\vec{k}$. A unit vector in the direction of $\vec{F}$ at $(1, 2, 2)$ is

$$\vec{u} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

(c) We have

$$||\vec{F}|| = \left|\frac{dg}{d\rho}\right| \cdot \left|\left|\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\rho}\right|\right| = \left|\frac{dg}{d\rho}\right|.$$

Since $\rho = \sqrt{1^2 + 2^2 + 2^2} = 3$ at $(1, 2, 3)$, we estimate $g'(3)$ from the graph, obtaining $g'(\rho(1, 2, 2)) = g'(3) \approx 1/3$. Thus

$$||\vec{F}(1, 2, 2)|| \approx \frac{1}{3}.$$

(d) From the answers to parts (b) and (c), we know that $\vec{F}(1, 2, 2)$ is a vector of length about $1/3$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$. Thus, we estimate

$$\vec{F}(1, 2, 2) \approx \frac{1}{9}(\vec{i} + 2\vec{j} + 2\vec{k}).$$

(e) We estimate $\vec{F}(3,0,0) \approx \frac{1}{3}\vec{i}$, because $||\vec{F}(3,0,0)|| = g'(\rho(3,0,0)) = g'(3) \approx 1/3$, and the direction of $\vec{F}(x,y,z)$ is parallel to $x\vec{i} + y\vec{j} + z\vec{k} = 3\vec{i}$.

(f) (i) We have $||\vec{F}(P)|| = ||\vec{F}(Q)||$ because $||\vec{F}(P)|| = |g'(\rho(P))| = |g'(\rho(Q))| = ||\vec{F}(Q)||$.

(ii) At each point, $\vec{F}(\vec{r})$ points in the same direction as $\vec{r}$, where $x\vec{i} + y\vec{j} + z\vec{k}$.

25. By the chain rule,

$$\frac{\partial z}{\partial t} = g_u u_t + g_v v_t + g_w w_t.$$

Thus, there are three terms.

26.

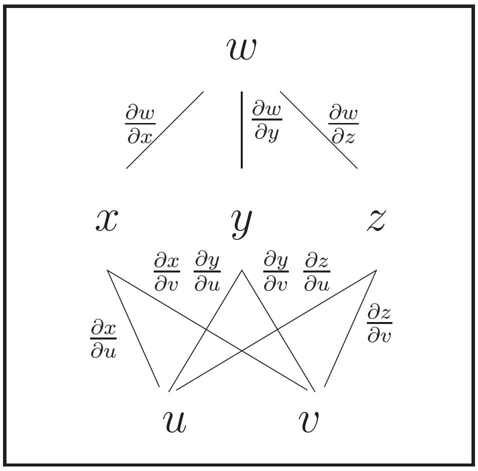


Figure 14.30

The tree diagram in Figure 14.30 tells us that

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

27.

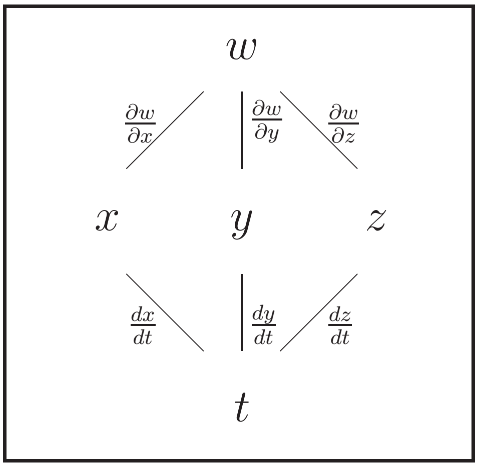


Figure 14.31

From the tree diagram in Figure 14.31, we get

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

28. All are done using the chain rule.

(a) We have $u = x$, $v = 3$. Thus $du/dx = 1$ and $dv/dx = 0$ so

$$f'(x) = F_u(x,3)(1) + F_v(x,3)(0) = F_u(x,3).$$

(b) We have $u = 3$, $v = x$. Thus $du/dx = 0$ and $dv/dx = 1$ so

$$f'(x) = F_u(3,x)(0) + F_v(3,x)(1) = F_v(3,x).$$

(c) We have $u = x$, $v = x$. Thus $du/dx = dv/dx = 1$ so

$$f'(x) = F_u(x,x)(1) + F_v(x,x)(1) = F_u(x,x) + F_v(x,x).$$

(d) We have $u = 5x$, $v = x^2$. Thus $du/dx = 5$ and $dv/dx = 2x$ so

$$f'(x) = F_u(5x,x^2)(5) + F_v(5x,x^2)(2x).$$

29. Using the chain rule,

$$z_u(u,v) = f_x(x,y)\cdot x_u(u,v) + f_y(x,y)\cdot y_u(u,v).$$

Since $x(1,2) = 5$ and $y(1,2) = 3$, substituting gives

$$z_u(1,2) = f_x(5,3)\cdot x_u(1,2) + f_y(5,3)\cdot y_u(1,2) = b\cdot e + d\cdot p.$$

30. Using the chain rule,

$$z_v(u,v) = f_x(x,y)\cdot x_v(u,v) + f_y(x,y)\cdot y_v(u,v).$$

Since $x(1,2) = 5$ and $y(1,2) = 3$, substituting gives

$$z_v(1,2) = f_x(5,3)\cdot x_v(1,2) + f_y(5,3)\cdot y_v(1,2) = b\cdot k + d\cdot q.$$

31. Implicit differentiation with respect to x of the equation $f(x,y) = f(a,b)$ gives $f_x(x,y)dx/dx + f_y(x,y)dy/dx = 0$. Thus $f_y(x,y)dy/dx = -f_x(x,y)$. If $f_y(a,b) \neq 0$ we can solve for the slope of the level curve at the point (a,b): $dy/dx = -f_x(a,b)/f_y(a,b)$.

32. We have $z = h(x,y)$ where $h(x,y) = f(x)g(y)$, $x = t$, and $y = t$. The chain rule gives $dz/dt = (\partial h/\partial x)dx/dt + (\partial h/\partial y)dy/dt = \partial h/\partial x + \partial h/\partial y$. Since $\partial h/\partial x = f'(x)g(y)$ and $\partial h/\partial y = f(x)g'(y)$ we have $dz/dt = f'(x)g(y) + f(x)g'(y) = f'(t)g(t) + f(t)g'(t)$.

33. Let $g(t) = f(tx,ty)$. We use the chain rule, with $u = tx$ and $v = ty$ as our variables. Then we have

$$\begin{aligned} g'(t) &= \frac{\partial f(u,v)}{\partial u}\frac{du}{dt} + \frac{\partial f(u,v)}{\partial v}\frac{dv}{dt} \\ &= f_u(u,v)\,x + f_v(u,v)\,y. \end{aligned}$$

At $t = 1$, we have $u = x$ and $v = y$. So

$$g'(1) = x\,f_x(x,y) + y\,f_y(x,y).$$

On the other hand, since $f(x,y)$ is homogeneous of degree p we also have $g(t) = t^p f(x,y)$. Thus we have

$$g'(t) = p\,t^{p-1} f(x,y)$$

and

$$g'(1) = p\,f(x,y).$$

Thus,

$$x\,f_x(x,y) + y\,f_y(x,y) = p\,f(x,y).$$

34. Use chain rule for the equation $0 = F(x, y, f(x, y))$. Differentiating both sides with respect to x, remembering $z = f(x, y)$ and regarding y as a constant gives:

$$0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx}.$$

Since $dx/dx = 1$, we get

$$-\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial x},$$

so

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}.$$

Similarly, differentiating both sides of the equation $0 = F(x, y, f(x, y))$ with respect to y gives:

$$0 = \frac{\partial F}{\partial y}\frac{dy}{dy} + \frac{\partial F}{\partial z}\frac{dz}{dy}.$$

Since $dy/dy = 1$, we get

$$-\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial y},$$

so

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

35. Using the chain rule,

$$z_u(u, v) = f_x(x, y) \cdot x_u(u, v) + f_y(x, y) \cdot y_u(u, v).$$

Since $x(4, 5) = 2$ and $y(4, 5) = 3$, substituting gives

$$z_u(4, 5) = f_x(2, 3) \cdot x_u(4, 5) + f_y(2, 3) \cdot y_u(4, 5) = b \cdot e + d \cdot p.$$

36. Using the chain rule,

$$z_v(u, v) = f_x(x, y) \cdot x_v(u, v) + f_y(x, y) \cdot y_v(u, v).$$

Since $x(4, 5) = 2$ and $y(4, 5) = 3$, substituting gives

$$z_v(4, 5) = f_x(2, 3) \cdot x_v(4, 5) + f_y(2, 3) \cdot y_v(4, 5) = b \cdot k + d \cdot q.$$

37. (a) We will use the chain rule identities,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}.$$

These equations are to be in terms of $\partial z/\partial x$ and $\partial z/\partial y$, so we may calculate the other terms, switching from Cartesian to polar coordinates. Recall polar coordinates :

$$x = r\cos\theta, \quad y = r\sin\theta$$

Thus we have

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \frac{\partial(r\cos\theta)}{\partial r} = \cos\theta \\
\frac{\partial y}{\partial r} &= \frac{\partial(r\sin\theta)}{\partial r} = \sin\theta \\
\frac{\partial x}{\partial \theta} &= \frac{\partial(r\cos\theta)}{\partial \theta} = -r\sin\theta \\
\frac{\partial y}{\partial \theta} &= \frac{\partial(r\sin\theta)}{\partial \theta} = r\cos\theta
\end{aligned}$$

Now, substituting into the equations for $\partial z/\partial r$ and $\partial z/\partial \theta$, we get

$$(1) \qquad \frac{\partial z}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$

$$(2) \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}.$$

We will call these equations (1) and (2).

(b) Now we solve for $\partial z/\partial x$ and $\partial z/\partial y$. From (2) we get:

$$(3)\qquad \frac{\partial z}{\partial x} = \left(\frac{\partial z}{\partial \theta} - r\cos\theta\frac{\partial z}{\partial y}\right)\left(\frac{-1}{r\sin\theta}\right),$$

Now substitute (3) into (1):

$$\begin{aligned}\frac{\partial z}{\partial r} &= \cos\theta\left(\frac{\partial z}{\partial \theta} - r\cos\theta\frac{\partial z}{\partial y}\right)\left(\frac{-1}{r\sin\theta}\right) + \sin\theta\frac{\partial z}{\partial y}\\ &= -\frac{\cos\theta}{r\sin\theta}\frac{\partial z}{\partial \theta} + \frac{\cos^2\theta}{\sin\theta}\frac{\partial z}{\partial y} + \sin\theta\frac{\partial z}{\partial y}\end{aligned}$$

Now solve for $\partial z/\partial y$:

$$\begin{aligned}\frac{\partial z}{\partial y}\left(\frac{\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta}{\sin\theta}\right) &= \frac{\partial z}{\partial r} + \frac{\cos\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}\\ \frac{\partial z}{\partial y}(\frac{1}{\sin\theta}) &= \frac{\partial z}{\partial r} + \frac{\cos\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}\\ \frac{\partial z}{\partial y} &= \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}.\end{aligned}$$

Now, substitute $\partial z/\partial y$ into equation (3) and solve for $\partial z/\partial x$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \left(\frac{\partial z}{\partial \theta} - r\cos\theta\frac{\partial z}{\partial y}\right)\frac{-1}{r\sin\theta}\\ &= \frac{-1}{r\sin\theta}\frac{\partial z}{\partial \theta} + \frac{\cos\theta}{\sin\theta}\left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right)\\ &= \cos\theta\frac{\partial z}{\partial r} + \frac{\cos^2\theta - 1}{r\sin\theta}\frac{\partial z}{\partial \theta}\\ &= \cos\theta\frac{\partial z}{\partial r} - \frac{\sin^2\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}\\ &= \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}.\end{aligned}$$

(c) Now we use the chain rule to get $\partial z/\partial x$ and $\partial z/\partial y$.

$$(4)\qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}, \qquad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

We will call this equation (4).
As before, we will calculate some of these partials using $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$

$$\begin{aligned}\frac{\partial r}{\partial y} &= \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta\\ \frac{\partial \theta}{\partial y} &= \frac{\partial\arctan(y/x)}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}(x^{-1}) = \frac{x}{(x^2+y^2)} = \frac{\cos\theta}{r}\\ \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta\\ \frac{\partial \theta}{\partial x} &= \frac{1}{1+(\frac{y}{x})^2}(-\frac{y}{x^2}) = -\frac{y}{(x^2+y^2)} = -\frac{\sin\theta}{r}\end{aligned}$$

Now, substituting these into (4), we get:

$$\begin{aligned}\frac{\partial z}{\partial y} &= \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\\ \frac{\partial z}{\partial x} &= \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\end{aligned}$$

Note that these equations match with those found in part (b).

38. Using $x = r\cos\theta$ and $y = r\sin\theta$ we compute $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$$

So we have

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

In addition we have,

$$\frac{1}{r}\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}(-\sin\theta) + \frac{\partial z}{\partial y}\cos\theta$$

thus,

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

Adding we get

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

39. Since $\left(\frac{\partial U}{\partial P}\right)_V$ involves the variables P and V, we are viewing U as a function of these two variables, so $U = U_3(P, V)$. Then

$$\left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U_3(P,V)}{\partial P}.$$

40. To calculate $\left(\frac{\partial U}{\partial P}\right)_T$, we think of U as a function of P and T, as in $U_1(T, P)$. Thus

$$\left(\frac{\partial U}{\partial P}\right)_T = \frac{\partial U_1}{\partial P}.$$

41. From the example, we know that for this gas

$$dU = \left(\frac{\partial U}{\partial P}\right)_V dP + \left(\frac{\partial U}{\partial V}\right)_P dV = 7dP + 8dV.$$

In addition, we have

$$PdV + VdP = 3dV + 4dP = 2dT.$$

We substitute for $dP = (2dT - 3dV)/4$ into the expression for dU, giving

$$dU = 7\left(\frac{2dT - 3dV}{4}\right) + 8dV$$
$$dU = \frac{7}{2}dT + \frac{11}{4}dV.$$

Comparing with the formula for dU obtained from the function U_2, where $U = U_2(T, V)$:

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV,$$

we have

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{7}{2} \quad \text{and} \quad \left(\frac{\partial U}{\partial V}\right)_T = \frac{11}{4}.$$

42. The partial derivative on the left side of the equation is obtained by thinking of T as a function of V and P. The partial derivative on the right side is obtained by thinking of V as a function of T and P.

Thinking of T as a function of V and P, we have

$$dT = \left(\frac{\partial T}{\partial V}\right)_P dV + \left(\frac{\partial T}{\partial P}\right)_V dP.$$

Thinking of V as a function of T and P. we also have

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dP.$$

Solving for dT in terms of dV and dP gives

$$dT = \frac{1}{\left(\frac{\partial V}{\partial T}\right)_P}\left(dV - \left(\frac{\partial V}{\partial P}\right)_P dP\right).$$

Comparing coefficients of dV in the two expressions for dT gives

$$\left(\frac{\partial T}{\partial V}\right)_P = 1\Big/\left(\frac{\partial V}{\partial T}\right)_P.$$

43. From Example 7, we know that

$$\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_V + \left(\frac{\partial U}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P.$$

Interchanging the roles of T and V, we get

$$\left(\frac{\partial U}{\partial V}\right)_P = \left(\frac{\partial U}{\partial V}\right)_T + \left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P.$$

Using the result of Problem 42, namely $\left(\frac{\partial T}{\partial V}\right)_P = 1\Big/\left(\frac{\partial V}{\partial T}\right)_P$, gives

$$\left(\frac{\partial U}{\partial V}\right)_P = \left(\frac{\partial U}{\partial V}\right)_T + \frac{\left(\frac{\partial U}{\partial T}\right)_V}{\left(\frac{\partial V}{\partial T}\right)_P}.$$

44. (a) Thinking of V as a function of P and T gives

$$dV = \left(\frac{\partial V}{\partial P}\right)_T dP + \left(\frac{\partial V}{\partial T}\right)_P dT.$$

(b) Substituting for dV in the following expression for dU,

$$dU = \left(\frac{\partial U}{\partial P}\right)_V dP + \left(\frac{\partial U}{\partial V}\right)_P dV,$$

we get

$$dU = \left(\frac{\partial U}{\partial P}\right)_V dP + \left(\frac{\partial U}{\partial V}\right)_P\left(\left(\frac{\partial V}{\partial P}\right)_T dP + \left(\frac{\partial V}{\partial T}\right)_P dT\right).$$

Rearranging terms gives

$$dU = \left(\left(\frac{\partial U}{\partial P}\right)_V + \left(\frac{\partial U}{\partial V}\right)_P\cdot\left(\frac{\partial V}{\partial P}\right)_T\right)dP + \left(\frac{\partial U}{\partial V}\right)_P\cdot\left(\frac{\partial V}{\partial T}\right)_P dT.$$

(c) The formula for dU obtained by thinking of U as a function of P and T is

$$dU = \left(\frac{\partial U}{\partial T}\right)_P dT + \left(\frac{\partial U}{\partial P}\right)_T dP.$$

(d) Comparing coefficients of dP and dT in the two formulas gives

$$\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial V}\right)_P\cdot\left(\frac{\partial V}{\partial T}\right)_P$$
$$\left(\frac{\partial U}{\partial P}\right)_T = \left(\frac{\partial U}{\partial P}\right)_V + \left(\frac{\partial U}{\partial V}\right)_P\cdot\left(\frac{\partial V}{\partial P}\right)_T.$$

45. As the introduction to this problem indicates, we can differentiate with respect to x inside the integral:

$$f'(x) = \int_0^b F_u(x, y)\, dy.$$

46. By the Fundamental Theorem of Calculus, we substitute $y = x$ in the integrand:

$$f'(x) = F(b, x).$$

47. (a) By Problem 45, treating w as the constant b:

$$G_u(u, w) = \int_0^w F_u(u, y)\, dy.$$

By Problem 46, treating u as the constant b:

$$G_w(u, w) = F(u, w).$$

(b) To differentiate $f(x) = G(x, x)$, we apply the chain rule to $G(u, w)$ with $u = x, w = x$. Since $\partial u/\partial x = \partial w/\partial x = 1$, we get

$$f'(x) = G_u(x, x)(1) + G_w(x, x)(1).$$

By part (a),

$$f'(x) = \int_0^x F_x(x, y)\, dy + F(x, x).$$

Strengthen Your Understanding

48. Writing $z = f(x, y)$, with $x = g(t)$ and $y = h(t)$, the chain rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t).\end{aligned}$$

49. According to the chain rule, we evaluate C_x, R_x and T_x at points (x, y) and we evaluate C_R and T_R at points (R, T). Since we know $R(0, 2) = 5$ and $T(0, 2) = 1$, the chain rule gives:

$$C_x(0, 2) = C_R(5, 1)R_x(0, 2) + C_T(5, 1)T_x(0, 2).$$

50. The partial derivatives f_x and f_y should be evaluated at $(2, 3)$:

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = f_x(2, 3)g'(0) + f_y(2, 3)h'(0).$$

51. We have

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} \\ &= 2g(t)h(t)g'(t) + g(t)^2h'(t).\end{aligned}$$

If $h(0) = 0$, $h'(0) = 1$, and $g(0) = 3$, then $(dz/dt)|_{t=0} = 9$. For example, we can take $h(t) = t$ and $g(t) = 3 + t$.

52. We have

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= f_x(x,y)2e^{2t} + f_y(x,y)\cos t\\ \frac{dz}{dt}\Big|_{t=0} &= f_x(1,0)\cdot 2 + f_y(1,0)\cdot 1 = 2f_x(1,0) + f_y(1,0).\end{aligned}$$

If $f_x(1,0) = 4$ and $f_y(1,0) = 2$, then $dz/dt|_{t=0} = 10$. For example, we can take $f(x,y) = 4x + 2y$.

53. A possible answer is $z = x + y$, $x = e^t$ and $y = t^2$.

54. A possible answer is $w = uv$, $u = 2s^2 + t$ and $v = e^{st}$.

55. A possible answer is $z = x + y$, $x = t$ and $y = t$.

56. By the chain rule,

$$\frac{dz}{dt} = g_u u_x x' + g_u u_y y' + g_u u_t + g_v v_x x' + g_v v_y y' + g_v v_t.$$

Thus, the answer is (c).

Solutions for Section 14.7

Exercises

1. Calculating the partial derivatives:

$$\frac{\partial f}{\partial x} = 2(x+y), \qquad \frac{\partial^2 f}{\partial x^2} = 2.$$

Therefore, we get

$$\frac{\partial f}{\partial y} = 2(x+y), \qquad \frac{\partial^2 f}{\partial y^2} = 2, \qquad \frac{\partial^2 f}{\partial y \partial x} = 2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 2.$$

2. Calculating the partial derivatives:

$$\frac{\partial f}{\partial x} = 3(x+y)^2, \quad \frac{\partial^2 f}{\partial x^2} = 6(x+y).$$
$$\frac{\partial f}{\partial y} = 3(x+y)^2, \quad \frac{\partial^2 f}{\partial y^2} = 6(x+y).$$

Consequently, we get

$$\frac{\partial^2 f}{\partial x \partial y} = 6(x+y), \quad \frac{\partial^2 f}{\partial y \partial x} = 6(x+y).$$

3. We have $f_x = 6xy + 5y^3$ and $f_y = 3x^2 + 15xy^2$, so $f_{xx} = 6y$, $f_{xy} = 6x + 15y^2$, $f_{yx} = 6x + 15y^2$, and $f_{yy} = 30xy$.

4. We have $f_x = 2ye^{2xy}$ and $f_y = 2xe^{2xy}$, so $f_{xx} = 4y^2e^{2xy}$, $f_{xy} = 4xye^{2xy} + 2e^{2xy}$, $f_{yx} = 4xye^{2xy} + 2e^{2xy}$, and $f_{yy} = 4x^2e^{2xy}$.

5. Since $f = (x+y)e^y$, the partial derivatives are

$$\begin{aligned}&f_x = e^y, \quad f_y = e^y(x+1+y)\\ &f_{xx} = 0, \quad f_{yx} = e^y = f_{xy}\\ &f_{yy} = xe^y + e^y + e^y + ye^y = e^y(x+2+y).\end{aligned}$$

6. Since $f(x,y) = xe^y$, the partial derivatives are

$$\begin{aligned}&f_x = e^y, \quad f_y = xe^y\\ &f_{xx} = 0, \quad f_{xy} = e^y = f_{yx}, \quad f_{yy} = xe^y.\end{aligned}$$

7. Since $f(x,y) = \sin(x/y)$, the first partial derivatives are:

$$f_x = (\cos(\frac{x}{y}))\frac{1}{y}, \quad f_y = (\cos(\frac{x}{y}))(\frac{-x}{y^2}).$$

Thus, the second partial derivatives are

$$\begin{aligned} f_{xx} &= -(\sin(\frac{x}{y}))(\frac{1}{y^2}) \\ f_{xy} &= -(\sin(\frac{x}{y}))(\frac{-x}{y^2})(\frac{1}{y}) + (\cos(\frac{x}{y}))(\frac{-1}{y^2}) = f_{yx} \\ f_{yy} &= -(\sin(\frac{x}{y}))(\frac{-x}{y^2})^2 + (\cos(\frac{x}{y}))(\frac{2x}{y^3}). \end{aligned}$$

8. Since $f(x,y) = \sqrt{x^2+y^2}$, we have

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2}} \\ f_{xx} &= \frac{\sqrt{x^2+y^2} - x(\frac{x}{\sqrt{x^2+y^2}})}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{3/2}} \\ f_{xy} &= -\frac{1}{2}\frac{x(2y)}{(x^2+y^2)^{3/2}} = \frac{-xy}{(x^2+y^2)^{3/2}} = f_{yx} \\ f_{yy} &= \frac{\sqrt{x^2+y^2} - y\frac{y}{\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{x^2}{(x^2+y^2)^{3/2}}. \end{aligned}$$

9. We have

$$f_x = 15x^2y^2 - 7y^3 + 18x \qquad \text{and} \qquad f_y = 10x^3y - 21xy^2,$$

so

$$f_{xx} = 30xy^2 + 18 \quad \text{and} \quad f_{xy} = 30x^2y - 21y^2 \quad \text{and} \quad f_{yx} = 30x^2y - 21y^2 \quad \text{and} \quad f_{yy} = 10x^3 - 42xy.$$

10. Since $f(x,y) = \sin(x^2+y^2)$, we have

$$\begin{aligned} f_x &= (\cos(x^2+y^2))2x \quad , f_y = (\cos(x^2+y^2))2y \\ f_{xx} &= -(\sin(x^2+y^2))4x^2 + 2\cos(x^2+y^2) \\ f_{xy} &= -(\sin(x^2+y^2))4xy = f_{yx} \\ f_{yy} &= -(\sin(x^2+y^2))4y^2 + 2\cos(x^2+y^2). \end{aligned}$$

11. We have $f_x = 6\cos 2x \cos 5y$ and $f_y = -15\sin 2x \sin 5y$, so $f_{xx} = -12\sin 2x \cos 5y$, $f_{xy} = -30\cos 2x \sin 5y$, $f_{yx} = -30\cos 2x \sin 5y$, and $f_{yy} = -75\sin 2x \cos 5y$.

12. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= (y-1)(x+1)^2 \\ f_x(x,y) &= 2(y-1)(x+1) \\ f_y(x,y) &= (x+1)^2 \\ f_{xx}(x,y) &= 2(y-1) \\ f_{yy}(x,y) &= 0 \\ f_{xy}(x,y) &= 2(x+1) \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = -1 - 2x + y - x^2 + 2xy$$

Notice this is the same as what you get if you expand $(y-1)(x+1)^2$ and then keep only the terms of degree 2 or less.

13. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= (x-y+1)^2 \\ f_x(x,y) &= 2(x-y+1) \\ f_y(x,y) &= -2(x-y+1) \\ f_{xx}(x,y) &= 2 \\ f_{yy}(x,y) &= 2 \\ f_{xy}(x,y) &= -2 \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 + 2x - 2y + x^2 - 2xy + y^2$$

Notice this is the same as what you get if you expand $(x-y+1)^2$.

14. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= e^{-2x^2-y^2} \\ f_x(x,y) &= -4xe^{-2x^2-y^2} \\ f_y(x,y) &= -2ye^{-2x^2-y^2} \\ f_{xx}(x,y) &= -4e^{-2x^2-y^2} + 16x^2e^{-2x^2-y^2} \\ f_{yy}(x,y) &= -2e^{-2x^2-y^2} + 4y^2e^{-2x^2-y^2} \\ f_{xy}(x,y) &= 8xye^{-2x^2-y^2} \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 - 2x^2 - y^2$$

15. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= e^x \cos y \\ f_x(x,y) &= e^x \cos y \\ f_y(x,y) &= -e^x \sin y \\ f_{xx}(x,y) &= e^x \cos y \\ f_{yy}(x,y) &= -e^x \cos y \\ f_{xy}(x,y) &= -e^x \sin y \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$$

Notice this is the same as what you get if you multiply the quadratic approximations for e^x and $\cos y$, that is $(1 + x + x^2/2)(1 - y^2/2)$, and then keep only the terms of degree 2 or less.

16. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= (1+2x-y)^{-1} \\ f_x(x,y) &= -2(1+2x-y)^{-2} \\ f_y(x,y) &= (1+2x-y)^{-2} \\ f_{xx}(x,y) &= 8(1+2x-y)^{-3} \\ f_{yy}(x,y) &= 2(1+2x-y)^{-3} \\ f_{xy}(x,y) &= -4(1+2x-y)^{-3} \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 - 2x + y + 4x^2 - 4xy + y^2$$

Notice this is the same as what you get if you substitute $u = y - 2x$ in the quadratic approximation $Q(u) = 1 + u + u^2$ for $1/(1-u)$.

17. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= \cos(x+3y) \\ f_x(x,y) &= -\sin(x+3y) \\ f_y(x,y) &= -3\sin(x+3y) \\ f_{xx}(x,y) &= -\cos(x+3y) \\ f_{yy}(x,y) &= -9\cos(x+3y) \\ f_{xy}(x,y) &= -3\cos(x+3y) \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 - \frac{1}{2}x^2 - 3xy - \frac{9}{2}y^2$$

Notice this is the same as what you get if you substitute $x + 3y$ for u in the single variable quadratic approximation $Q(u) = 1 - u^2/2$ for $\cos u$.

18. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2$$

So first we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= \sin 2x + \cos y \\ f_x(x,y) &= 2\cos 2x \\ f_y(x,y) &= -\sin y \\ f_{xx}(x,y) &= -4\sin 2x \\ f_{yy}(x,y) &= -\cos y \\ f_{xy}(x,y) &= 0 \end{aligned}$$

We substitute into the formula to get for our answer:

$$Q(x,y) = 1 + 2x - \frac{1}{2}y^2$$

19. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

So first we find all the relevant derivatives:

$$\begin{aligned}
f(x,y) &= \ln(1+x^2-y)\\
f_x(x,y) &= \frac{2x}{1+x^2-y}\\
f_y(x,y) &= \frac{-1}{1+x^2-y}\\
f_{xx}(x,y) &= \frac{2(1+x^2-y)-4x^2}{(1+x^2-y)^2}\\
f_{yy}(x,y) &= \frac{-1}{(1+x^2-y)^2}\\
f_{xy}(x,y) &= \frac{2x}{(1+x^2-y)^2}
\end{aligned}$$

Substituting into the formula we get as our answer:

$$Q(x,y) = -y + x^2 - \frac{y^2}{2}$$

20. Since $f(x,y) = \ln(1+x-2y)$, the first and second derivatives are

$$\begin{aligned}
f_x &= \frac{1}{1+x-2y}\\
f_y &= \frac{-2}{1+x-2y}\\
f_{xx} &= \frac{-1}{(1+x-2y)^2}\\
f_{xy} &= \frac{2}{(1+x-2y)^2}\\
f_{yy} &= \frac{-4}{(1+x-2y)^2},
\end{aligned}$$

so we find that

$$\begin{aligned}
f(0,0) &= 0\\
f_x(0,0) &= 1\\
f_y(0,0) &= -2\\
f_{xx}(0,0) &= -1\\
f_{xy}(0,0) &= 2\\
f_{yy}(0,0) &= -4.
\end{aligned}$$

The best quadratic approximation for $f(x,y)$ for (x,y) near $(0,0)$ is

$$f(x,y) \approx x - 2y - \frac{1}{2}x^2 + 2xy - 2y^2.$$

21. Since $f(x,y) = \sqrt{1+2x-y}$, the first and second derivatives are

$$\begin{aligned}
f_x &= \frac{1}{\sqrt{1+2x-y}}\\
f_y &= \frac{-1}{2\sqrt{1+2x-y}}\\
f_{xx} &= \frac{-1}{(1+2x-y)^{3/2}}
\end{aligned}$$

$$f_{xy} = \frac{1}{2(1+2x-y)^{3/2}}$$
$$f_{yy} = \frac{-1}{4(1+x-2y)^{3/2}},$$

so we find that

$$\begin{aligned} f(0,0) &= 1 \\ f_x(0,0) &= 1 \\ f_y(0,0) &= -1/2 \\ f_{xx}(0,0) &= -1 \\ f_{xy}(0,0) &= 1/2 \\ f_{yy}(0,0) &= -1/4. \end{aligned}$$

The best quadratic approximation for $f(x,y)$ for (x,y) near $(0,0)$ is

$$f(x,y) \approx 1 + x - \frac{1}{2}y - \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{8}y^2.$$

22. (a) $f_x(P) < 0$ because f decreases as you go to the right.
(b) $f_y(P) = 0$ because f does not change as you go up.
(c) $f_{xx}(P) > 0$ because f_x increases as you go to the right (f_x changes from a large negative number to a small negative number).
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

23. (a) $f_x(P) < 0$ because f decreases as you go to the right.
(b) $f_y(P) = 0$ because f does not change as you go up.
(c) $f_{xx}(P) < 0$ because f_x decreases as you go to the right (f_x changes from a small negative number to a large negative number).
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

24. (a) $f_x(P) > 0$ because f increases as you go to the right.
(b) $f_y(P) = 0$ because f does not change as you go up.
(c) $f_{xx}(P) < 0$ because f_x decreases as you go to the right. (Since the level curves are further apart as you go to the right, the rate of change of f decreases. Thus, f_x changes from a large positive to a small positive number.)
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

25. (a) $f_x(P) > 0$ because f increases as you go to the right.
(b) $f_y(P) = 0$ because f does not change as you go up.
(c) $f_{xx}(P) > 0$ because f_x increases as you go to the right. (The rate of change of f is larger when you go to the right since the level curves are closer together. Thus f_x changes from a small positive to a large positive number.)
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

26. (a) $f_x(P) = 0$ because f does not change as you go to the right.
(b) $f_y(P) > 0$ because f increases as you go up.
(c) $f_{xx}(P) = 0$ because f_x does not change as you go to the right.
(d) $f_{yy}(P) < 0$ because f_y decreases as you go up (Since the level curves are further apart as you move up, the rate of change of f is slower, that is, f_y decreases as you move up.)
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

27. (a) $f_x(P) = 0$ because f does not change as you go to the right.
(b) $f_y(P) < 0$ because f decreases as you go up.
(c) $f_{xx}(P) = 0$ because f_x does not change as you go to the right.
(d) $f_{yy}(P) < 0$ because f_y decreases as you go up. (f_y changes from a negative number with smaller magnitude to a negative number with larger magnitude.)
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

28. (a) $f_x(P) < 0$ because f decreases as you go to the right.
(b) $f_y(P) < 0$ because f decreases as you go up.
(c) $f_{xx}(P) = 0$ because f_x does not change as you go to the right. (Notice that the level curves are equidistant and parallel, so the partial derivatives of f do not change if you move horizontally or vertically.)
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

29. (a) $f_x(P) > 0$ because f increases as you go to the right.
(b) $f_y(P) > 0$ because f increases as you go up.
(c) $f_{xx}(P) = 0$ because f_x does not change as you go to the right. (Notice that the level curves are equidistant and parallel, so the partial derivatives of f do not change if you move horizontally or vertically.)
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

30. (a) $f_x(P) < 0$ because f decreases as you go to the right.
(b) $f_y(P) > 0$ because f increases as you go up.
(c) $f_{xx}(P) > 0$ because f_x increases as you move right (f_x changes from negative numbers with larger magnitude to negative numbers with smaller magnitude).
(d) $f_{yy}(P) > 0$ because the level curves are closer together as you move up, so f_y increases as you go up.
(e) $f_{xy}(P) < 0$ because the rate of change of f with respect to x is a negative number at P and a negative number with larger magnitude higher up. Therefore f_x decreases as the point moves up.

31. (a) $f_x(P) > 0$ because f increases as you go to the right.
(b) $f_y(P) < 0$ because f decreases as you go up.
(c) $f_{xx}(P) < 0$ because the level curves are further apart as you go to the right, so the rate of increase of f is slower as you move to the right. Therefore, f_x decreases as you go to the right.
(d) $f_{yy}(P) < 0$ because f_y decreases as you move up (f_y changes from a negative number with smaller magnitude to a negative number with larger magnitude).
(e) $f_{xy}(P) > 0$ because the rate of change of f with respect to x at P is lower than at points above P. Therefore f_x increases as you move up.

Problems

32. We have $f(1,0) = 1$ and the relevant derivatives are:

$$\begin{aligned} f_x &= \frac{1}{2}(x+2y)^{-1/2} \quad \text{so} \quad f_x(1,0) = \frac{1}{2} \\ f_y &= (x+2y)^{-1/2} \quad \text{so} \quad f_y(1,0) = 1 \\ f_{xx} &= -\frac{1}{4}(x+2y)^{-3/2} \quad \text{so} \quad f_{xx}(1,0) = -\frac{1}{4} \\ f_{xy} &= -\frac{1}{2}(x+2y)^{-3/2} \quad \text{so} \quad f_{xy}(1,0) = -\frac{1}{2} \\ f_{yy} &= -(x+2y)^{-3/2} \quad \text{so} \quad f_{yy}(1,0) = -1\,. \end{aligned}$$

Thus the linear approximation, $L(x,y)$ to $f(x,y)$ at $(1,0)$, is given by:

$$\begin{aligned} f(x,y) \approx L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + \frac{1}{2}(x-1) + y\,. \end{aligned}$$

The quadratic approximation, $Q(x,y)$ to $f(x,y)$ near $(1,0)$, is given by:

$$\begin{aligned} f(x,y) \approx Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\quad + f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= 1 + \frac{1}{2}(x-1) + y - \frac{1}{8}(x-1)^2 - \frac{1}{2}(x-1)y - \frac{1}{2}y^2\,. \end{aligned}$$

The values of the approximations are

$$\begin{aligned} L(0.9, 0.2) &= 1 - 0.05 + 0.2 = 1.15 \\ Q(0.9, 0.2) &= 1 - 0.05 + 0.2 - 0.00125 + 0.01 - 0.02 = 1.13875 \end{aligned}$$

and the exact value is

$$f(0.9, 0.2) = \sqrt{1.3} \approx 1.14018.$$

Observe that the quadratic approximation is closer to the exact value.

33. We have $f(1,0) = 0$ and the relevant derivatives are:

$$\begin{aligned} f_x &= 2xy \quad \text{so} \quad f_x(1,0) = 0 \\ f_y &= x^2 \quad \text{so} \quad f_y(1,0) = 1 \\ f_{xx} &= 2y \quad \text{so} \quad f_{xx}(1,0) = 0 \\ f_{xy} &= 2x \quad \text{so} \quad f_{xy}(1,0) = 2 \\ f_{yy} &= 0 \quad \text{so} \quad f_{yy}(1,0) = 0\,. \end{aligned}$$

Thus the linear approximation, $L(x, y)$ to $f(x, y)$ at $(1, 0)$, is given by:

$$\begin{aligned} f(x,y) \approx L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= y\,. \end{aligned}$$

The quadratic approximation, $Q(x, y)$ to $f(x, y)$ near $(1, 0)$, is given by:

$$\begin{aligned} f(x,y) \approx Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\quad + f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= y + 2(x-1)y\,. \end{aligned}$$

The values of the approximations are

$$\begin{aligned} L(0.9, 0.2) &= 0.2 \\ Q(0.9, 0.2) &= 0.2 + 2(-0.1)(0.2) = 0.16 \end{aligned}$$

and the exact value is

$$f(0.9, 0.2) = (0.81)(0.2) = 0.162.$$

Observe that the quadratic approximation is closer to the exact value.

34. We have $f(1,0) = 1$ and the relevant derivatives are:

$$\begin{aligned} f_x &= e^{-y} \quad \text{so} \quad f_x(1,0) = 1 \\ f_y &= -xe^{-y} \quad \text{so} \quad f_y(1,0) = -1 \\ f_{xx} &= 0 \quad \text{so} \quad f_{xx}(1,0) = 0 \\ f_{xy} &= -e^{-y} \quad \text{so} \quad f_{xy}(1,0) = -1 \\ f_{yy} &= xe^{-y} \quad \text{so} \quad f_{yy}(1,0) = 1\,. \end{aligned}$$

Thus the linear approximation, $L(x, y)$ to $f(x, y)$ at $(1, 0)$, is given by:

$$\begin{aligned} f(x,y) \approx L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + (x-1) - y\,. \end{aligned}$$

The quadratic approximation, $Q(x, y)$ to $f(x, y)$ near $(1, 0)$, is given by:

$$\begin{aligned} f(x,y) \approx Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\quad + f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= 1 + (x-1) - y - (x-1)y + \frac{1}{2}y^2\,. \end{aligned}$$

The values of the approximations are

$$\begin{aligned} L(0.9, 0.2) &= 1 - 0.1 - 0.2 = 0.7 \\ Q(0.9, 0.2) &= 1 - 0.1 - 0.2 + 0.02 + 0.02 = 0.74 \end{aligned}$$

and the exact value is

$$f(0.9, 0.2) = (0.9)e^{-0.2} \approx 0.737$$

Observe that the quadratic approximation is closer to the exact value.

35. Differentiating, we get

$$F_x = e^x \sin y + e^y \cos x \quad F_y = e^x \cos y + e^y \sin x$$
$$F_{xx} = e^x \sin y - e^y \sin x \quad F_{yy} = -e^x \sin y + e^y \sin x = -F_{xx}$$

Thus, $F_{xx} + F_{yy} = 0$.

36. We have $f(1,0) = 0$ and the relevant derivatives are:

$$\begin{aligned} f_x &= \cos(x-1)\cos y \quad \text{so} \quad f_x(1,0) = 1 \\ f_y &= -\sin(x-1)\sin y \quad \text{so} \quad f_y(1,0) = 0 \\ f_{xx} &= -\sin(x-1)\cos y \quad \text{so} \quad f_{xx}(1,0) = 0 \\ f_{xy} &= -\cos(x-1)\sin y \quad \text{so} \quad f_{xy}(1,0) = 0 \\ f_{yy} &= -\sin(x-1)\cos y \quad \text{so} \quad f_{yy}(1,0) = 0\,. \end{aligned}$$

Thus the linear approximation, $L(x,y)$ to $f(x,y)$ at $(1,0)$, is given by:

$$\begin{aligned} f(x,y) \approx L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= x - 1. \end{aligned}$$

The quadratic approximation, $Q(x,y)$ to $f(x,y)$ near $(1,0)$, is given by:

$$\begin{aligned} f(x,y) \approx Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\quad + f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= x - 1. \end{aligned}$$

Thus the linear and quadratic approximations are the same. The values of the approximations are

$$L(0.9, 0.2) = Q(0.9, 0.2) = -0.1,$$

and the exact value is

$$f(0.9, 0.2) = \sin(-0.1)\cos(0.2) \approx -0.098$$

37. Differentiating, we get

$$F_x = -e^{-x}\sin y,\ F_y = e^{-x}\cos y,\ F_{xx} = e^{-x}\sin y,\ F_{yy} = -e^{-x}\sin y = -F_{xx}.$$

Thus, $F_{xx} + F_{yy} = 0$.

38. Differentiating, we get

$$\begin{aligned} F_x &= \frac{1}{1+(\frac{y}{x})}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} \\ F_y &= \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \\ F_{xx} &= (-y)\frac{-1}{(x^2+y^2)^2}(2x) = \frac{2xy}{(x^2+y^2)^2} \\ F_{yy} &= \frac{-x}{(x^2+y^2)^2}(2y) = \frac{-2xy}{(x^2+y^2)^2} = -F_{xx} \end{aligned}$$

Thus, $F_{xx} + F_{yy} = 0$.

39. First let us take the partial derivatives:

$$\begin{aligned} u_t &= ae^{at}\sin(bx) \\ u_x &= be^{at}\cos(bx) \\ u_{xx} &= -b^2e^{at}\sin(bx) \end{aligned}$$

Substituting into the equation, we have

$$ae^{at}\sin(bx) = u_t = u_{xx} = -b^2e^{at}\sin(bx)$$

So, $a = -b^2$.

40. (a) Taking partial derivatives of u, we get

$$\begin{aligned}
u_t &= -\frac{\pi}{4}(\pi t)^{-\frac{3}{2}}e^{-x^2/(4t)} + \frac{x^2}{4t^2}e^{-x^2/(4t)}\frac{1}{2\sqrt{\pi t}} \\
&= -\frac{\pi}{4(\pi t)^{\frac{3}{2}}}e^{-x^2/(4t)} + \frac{x^2}{8t^2\sqrt{\pi t}}e^{-x^2/(4t)} \\
&= -\frac{1}{4t\sqrt{\pi t}}e^{-x^2/(4t)} + \frac{x^2}{8t^2\sqrt{\pi t}}e^{-x^2/(4t)} \\
u_x &= \frac{1}{2\sqrt{\pi t}}\left(\frac{-2x}{4t}\right)e^{-x^2/(4t)} \\
&= -\frac{x}{4t\sqrt{\pi t}}e^{-x^2/(4t)} \\
u_{xx} &= -\frac{1}{4t\sqrt{\pi t}}e^{-x^2/(4t)} - \frac{x}{4t\sqrt{\pi t}}\left(\frac{-2x}{4t}\right)e^{-x^2/(4t)} \\
&= -\frac{1}{4t\sqrt{\pi t}}e^{-x^2/(4t)} + \frac{x^2}{8t^2\sqrt{\pi t}}e^{-x^2/(4t)}
\end{aligned}$$

So we have

$$u_t = u_{xx},$$

showing that u satisfies the heat equation.

(b) See Figure 14.32. Note that as time progresses the heat at the origin decreases and flows out toward the ends of the rod, until at $t = 10$ the temperature appears to be leveling out toward being constant throughout the rod.

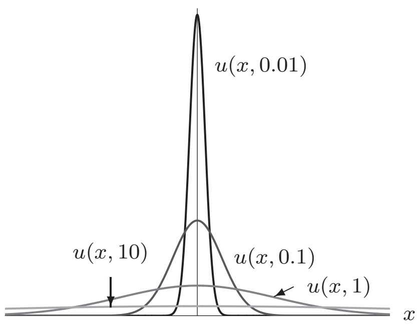


Figure 14.32

41. The graph of f is concave up as we move parallel to the x-axis from the point $(0,0)$, so $f_{xx}(0,0)$ is positive. The graph of f is concave down as we move parallel to the y-axis from the point $(0,0)$, so $f_{yy}(0,0)$ is negative.

42. Since $z_y = g(x), z_{yy} = 0$, because g is a function of x only.

43. (a) $z_{yx} = z_{xy} = 4y$

(b) $z_{xyx} = \dfrac{\partial}{\partial x}(z_{xy}) = \dfrac{\partial}{\partial x}(4y) = 0$

(c) $z_{xyy} = z_{yxy} = \dfrac{\partial}{\partial y}(4y) = 4$

44. (a) Moving parallel to the x-axis means that the z-labels on the contours increase, so z is an increasing function of x. Moving parallel to the y-axis, the z-labels decrease, so z is a decreasing function of y.

(b) Since z is an increasing function of x, we have $f_x > 0$. Similarly, $f_y < 0$.

(c) Since the contours get closer together as we move parallel to the x-axis, we have $f_{xx} > 0$. This means that z is increasing faster and faster as x increases. Similar reasoning shows that $f_{yy} < 0$.

(d) The vector $\text{grad} f$ is perpendicular to the level curves and points in the direction of increasing f values. See Figure 14.33.

(e) The vector $\text{grad} f$ is longer at P because the contours are closer together at P than at Q.

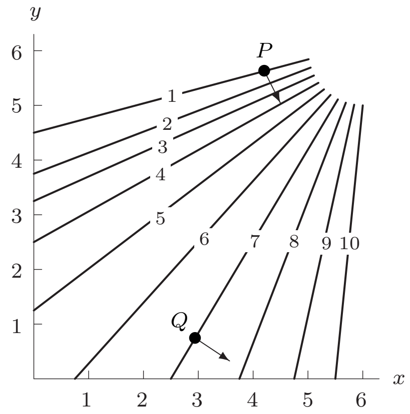


Figure 14.33

45. The table of values is linear, because all the rows are linear and have the same slope, 1, and all the columns are linear and have the same slope, 2. A linear function has no quadratic terms, so the coefficients of x^2, xy, and y^2 are all zero. We have $d = e = f = 0$.

46. The rows of the table are not linear, so there is at least one nonzero quadratic term in the polynomial $P(x, y)$. The coefficients of the quadratic terms are closely related to the second order partial derivatives.

Since the x-slope increases as we move along a row in the direction of increasing x, we have $\partial^2 P/\partial x^2 > 0$. Hence $d = (\partial^2 P/\partial x^2)/2 > 0$.

Since the y-slope is constant as we move across the table in the direction of increasing x, we have $\partial/\partial x(\partial P/\partial y) = 0$. Hence $e = \partial^2 P/\partial x \partial y = 0$.

Since the y-slope is constant as we move down a column in the direction of increasing y, we have $\partial^2 P/\partial y^2 = 0$. Hence $f = (\partial^2 P/\partial y^2)/2 = 0$.

47. The rows of the table are linear but they do not all have the same slope, so the polynomial $P(x, y)$ is not linear. Alternatively, since the columns of the table are not linear, $P(x, y)$ is not linear. There is at least one nonzero quadratic term in P. The coefficients of the quadratic terms are closely related to the second order partial derivatives.

Since the x-slope is constant as we move across a row in the direction of increasing x, we have $\partial^2 P/\partial x^2 = 0$. Hence $d = (\partial^2 P/\partial x^2)/2 = 0$.

Since the x-slope increases as we move down the table in the direction of increasing y, we have $\partial/\partial y(\partial P/\partial x) > 0$. Hence $e = \partial^2 P/\partial y \partial x > 0$.

Since the y-slope decreases as we move down a column in the direction of increasing y, we have $\partial^2 P/\partial y^2 < 0$. Hence $f = (\partial^2 P/\partial y^2)/2 < 0$.

48. Neither the rows nor the columns of the table are linear, so the polynomial $P(x, y)$ is not linear. There is at least one nonzero quadratic term in P. The coefficients of the quadratic terms are closely related to the second order partial derivatives.

Since the x-slope increases as we move across a row in the direction of increasing x, we have $\partial^2 P/\partial x^2 > 0$. Hence $d = (\partial^2 P/\partial x^2)/2 > 0$.

Since the x-slope decreases as we move down the table in the direction of increasing y, we have $\partial/\partial y(\partial P/\partial x) < 0$. Hence $e = \partial^2 P/\partial y \partial x < 0$.

Since the y-slope increases as we move down a column in the direction of increasing y, we have $\partial^2 P/\partial y^2 > 0$. Hence $f = (\partial^2 P/\partial y^2)/2 > 0$.

49. (a) The vertical spacing between the contours just north and just south of the trail increases as you move eastward along the trail.

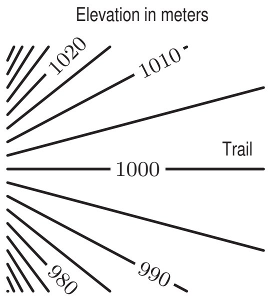


Figure 14.34

(b) Let $h = f(x, y)$, where h is elevation in meters, and x and y are distances in meters east and north of the start of the trail. Hence the trail begins at $(0, 0)$ and lies along the line $y = 0$. We also know:

- The trail is level. Hence $\partial h/\partial x = 0$ at all points of the trail.
- There is a mountain to the left. Hence $\partial h/\partial y > 0$ at all points of the trail.
- The slope up to the left is getting more gentle as you hike east. Thus $\partial h/\partial y$ is a decreasing function of x. Hence $\partial/\partial x(\partial h/\partial y) = (\partial^2 h)/(\partial x \partial y) < 0$.

(c) The decision to delay turning off the trail was based on the second mixed partial, $(\partial^2 h)/(\partial x \partial y)$.

50. (a) Increasing L causes production Y to increase, so $\partial Y/\partial L > 0$.

(b) The increase in production resulting from hiring an additional worker is approximately $\partial Y/\partial L$. This number is larger for larger values of K, so $\partial Y/\partial L$ is an increasing function of K. Its derivative with respect to K is positive. Hence

$$\frac{\partial}{\partial K}\left(\frac{\partial Y}{\partial L}\right) = \frac{\partial^2 Y}{\partial K \partial L} > 0.$$

51. (a) Person A. Weight gain of 1 pound results in an approximate surface area increase of $\partial S/\partial w$. We know that $\partial S/\partial w$ is an increasing function of height h, because

$$\frac{\partial}{\partial h}\left(\frac{\partial S}{\partial w}\right) > 0.$$

Thus $\partial S/\partial w$ is larger for the taller person, A.

(b) Person B. Weight gain of 1 pound results in an approximate surface area increase of $\partial S/\partial w$. We know that $\partial S/\partial w$ is a decreasing function of weight w, because

$$\frac{\partial}{\partial w}\left(\frac{\partial S}{\partial w}\right) < 0.$$

Thus $\partial S/\partial w$ is greater for the lighter person, B.

52. (a) Since P and Q lie on the same level curve, we have $a = k$.

(b) We have $b = f_x$ and $c = f_y$. Since the gradient of f at P (respectively Q) points toward M or away from M, from the figure, we see $f_x(P)$ and $f_y(P)$ have opposite signs, while $f_x(Q)$ and $f_y(Q)$ have the same signs. Thus Q is the point (x_1, y_1), so P is (x_2, y_2).

(c) Since $b = f_x(Q) > 0$ and $c = f_y(Q) > 0$, the value of f must increase as we go away from M. Thus, M must be a minimum (the surface is a valley).

(d) Since M is a minimum, $m = f_x(P) < 0$ and $n = f_y(P) > 0$.

53. (a) Calculate the partial derivatives:

$$\begin{aligned}
f(x, y) &= \sin x \sin y & f(0, 0) &= 0 & f(\tfrac{\pi}{2}, \tfrac{\pi}{2}) &= 1\\
f_x(x, y) &= \cos x \sin y & f_x(0, 0) &= 0 & f_x(\tfrac{\pi}{2}, \tfrac{\pi}{2}) &= 0\\
f_y(x, y) &= \sin x \cos y & f_y(0, 0) &= 0 & f_y(\tfrac{\pi}{2}, \tfrac{\pi}{2}) &= 0\\
f_{xx}(x, y) &= -\sin x \sin y & f_{xx}(0, 0) &= 0 & f_{xx}(\tfrac{\pi}{2}, \tfrac{\pi}{2}) &= -1\\
f_{xy}(x, y) &= \cos x \cos y & f_{xy}(0, 0) &= 1 & f_{xy}(\tfrac{\pi}{2}, \tfrac{\pi}{2}) &= 0\\
f_{yy}(x, y) &= -\sin x \sin y & f_{yy}(0, 0) &= 0 & f_{yy}(\tfrac{\pi}{2}, \tfrac{\pi}{2}) &= -1
\end{aligned}$$

Thus, the Taylor polynomial about $(0,0)$ is

$$f(x,y) \approx Q_1(x,y) = xy.$$

The Taylor polynomial about $(\frac{\pi}{2}, \frac{\pi}{2})$ is

$$f(x,y) \approx Q_2(x,y) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.$$

(b)

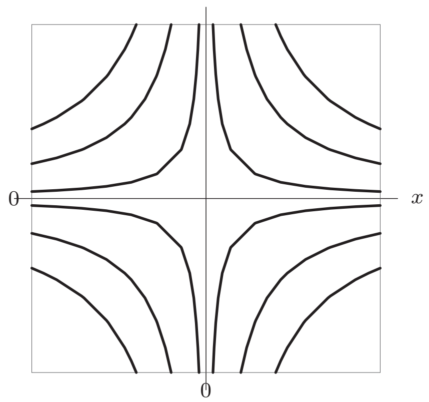


Figure 14.35: $f(x,y) \approx xy$: Quadratic approximation about $(0,0)$

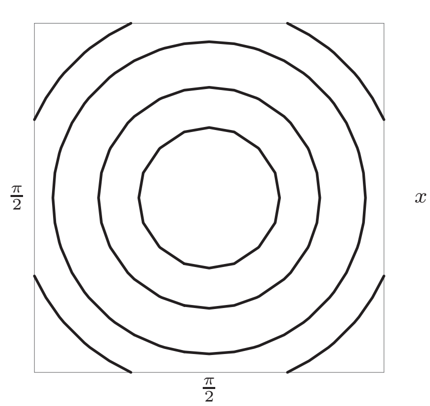


Figure 14.36: $f(x,y) \approx 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 - \frac{1}{2}(y - \frac{\pi}{2})^2$: Quadratic approximation about $(\frac{\pi}{2}, \frac{\pi}{2})$

54. (a) The definition of f_x is:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

(b) We define $f_{xy} = (f_x)_y$ as follows:

$$f_{xy}(a,b) = (f_x)_y(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}.$$

(c) Substituting the expression for f_x into the definition f_{xy}:

$$\begin{aligned} f_{xy}(a,b) &= \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k} \\ &= \lim_{k\to 0} \frac{1}{k}\left(\lim_{h\to 0} \frac{f(a+h,b+k) - f(a,b+k)}{h} - \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}\right) \\ &= \lim_{k\to 0}\lim_{h\to 0} \frac{f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b)}{hk}. \end{aligned}$$

(d) Similarly,

$$f_{yx}(a,b) = \lim_{h\to 0}\lim_{k\to 0} \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk}.$$

(e) The numerators in the two expressions in part (c) and (d) are the same (just swap the middle terms), so the only difference between them is the order in which the limits are taken. To be sure f_{xy} and f_{yx} are equal, we have to assume we can swap the order of the limits. Swapping limits can be a tricky business, but it can be done in this case if f_{xy} and f_{yx} are continuous.

55. (a) (i) Dollars/Year. Negative. The value of the car decreases with age.

(ii) Dollars/Dollar. Positive. For two cars of the same age, the one that had the highest value when new costs more now.

(b) The experts say that $\partial P/\partial C$ is a decreasing function of age. This means that $\partial/\partial A(\partial P/\partial C) = \partial^2 P/(\partial A \partial C) < 0$.

(c) The term eCA, because $\partial^2 P/(\partial A \partial C) = e$.

56. (a) (II) At first $\partial T/\partial S > 0$, but after increasing S, we have $\partial T/\partial S < 0$. Hence $\partial T/\partial S$ is a decreasing function of S. This means that $\partial^2 T/\partial S^2 < 0$.

(b) (I) At first $\partial T/\partial V > 0$, but after increasing V, we have $\partial T/\partial V < 0$. Hence $\partial T/\partial V$ is a decreasing function of V. This means that $\partial^2 T/\partial V^2 < 0$.

(c) (III) At first $\partial T/\partial S < 0$, but after increasing V, we have $\partial T/\partial S > 0$. Hence $\partial T/\partial S$ is an increasing function of V. This means that

$$\frac{\partial}{\partial V}\left(\frac{\partial T}{\partial S}\right) = \frac{\partial^2 T}{\partial V \partial S} > 0.$$

57. Let us first calculate the values of all the partial derivatives at $(0,0)$ that we need:

$$\begin{aligned}
f(x,y) &= (x+2y+1)^{1/2}, & f(0,0) &= 1,\\
f_x(x,y) &= \tfrac{1}{2}(x+2y+1)^{-1/2}, & f_x(0,0) &= 1/2,\\
f_y(x,y) &= (x+2y+1)^{-1/2}, & f_y(0,0) &= 1,\\
f_{xx}(x,y) &= -\tfrac{1}{4}(x+2y+1)^{-3/2}, & f_{xx}(0,0) &= -1/4,\\
f_{xy}(x,y) &= -\tfrac{1}{2}(x+2y+1)^{-3/2}, & f_{xy}(0,0) &= -1/2,\\
f_{yy}(x,y) &= -(x+2y+1)^{-3/2}, & f_{yy}(0,0) &= -1.
\end{aligned}$$

(a) The local linearization $L(x,y)$ of f at $(0,0)$ is given by

$$f(x,y) \approx L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = 1 + \frac{1}{2}x + y.$$

(b) The second-order Taylor polynomial, $Q(x,y)$, for f at $(0,0)$ is given by

$$\begin{aligned}
f(x,y) &\approx Q(x,y)\\
&= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2}y^2\\
&= 1 + \frac{1}{2}x + y - \frac{1}{8}x^2 - \frac{1}{2}xy - \frac{1}{2}y^2.
\end{aligned}$$

Notice that the local linearization of f is the same as the linear part of the Taylor polynomial of degree 2 for f. The extra terms in the Taylor polynomial of degree 2 can be thought of as "correction terms" to the linear approximation.

(c) Table 14.8 records the values of $f(x,y)$, $L(x,y)$, and $Q(x,y)$. Observe that the quadratic approximations $Q(x,y)$ are closer to the true values $f(x,y)$ than are the linear approximations $L(x,y)$. Of course both approximations are exact at $(0,0)$.

Table 14.8 *Linear and quadratic approximations to f near $(0,0)$*

Point	Linear	Quadratic	True
(x,y)	$L(x,y)$	$Q(x,y)$	$f(x,y)$
(0, 0)	1	1	1
(0.1, 0.1)	1.15	1.13875	1.140175
(−0.1, 0.1)	1.05	1.04875	1.048809
(0.1, −0.1)	0.95	0.94875	0.948683
(−0.1, −0.1)	0.85	0.83875	0.836660

58. The contour diagrams in Figures 14.37–14.42 use the fact that

$$\begin{aligned}
f(x,y) &= \sqrt{x+2y+1},\\
L(x,y) &= 1 + \frac{1}{2}x + y,\\
Q(x,y) &= 1 + \frac{1}{2}x + y - \frac{1}{8}x^2 - \frac{1}{2}xy - \frac{1}{2}y^2.
\end{aligned}$$

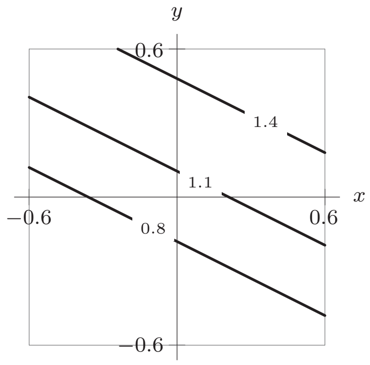


Figure 14.37: $f(x,y)$

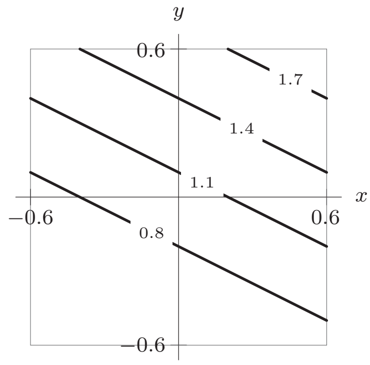


Figure 14.38: $L(x,y)$

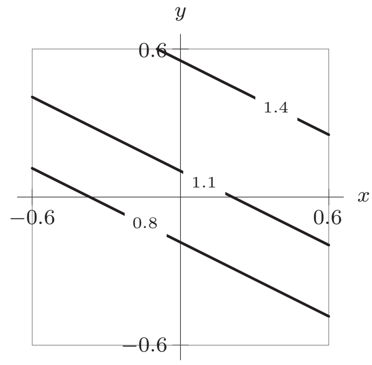


Figure 14.39: $Q(x,y)$

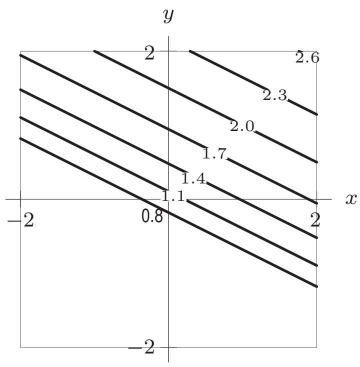


Figure 14.40: $f(x,y)$

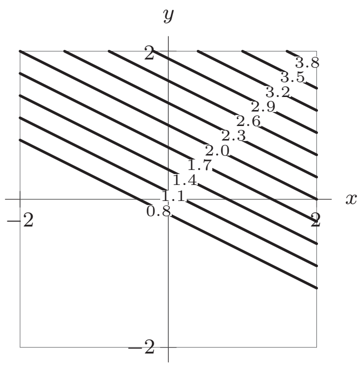


Figure 14.41: $L(x,y)$

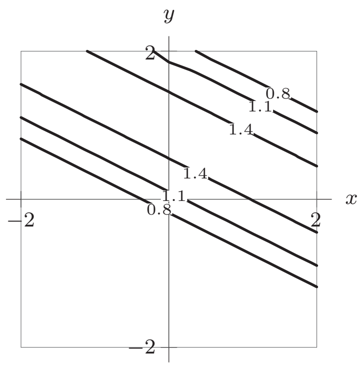


Figure 14.42: $Q(x,y)$

The contours for $f(x,y)$ and $L(x,y)$ are straight lines; those for $L(x,y)$ are equally spaced because $L(x,y)$ is a linear function. The contours for $f(x,y)$ are straight lines because if we set

$$f(x,y) = \sqrt{x+2y+1} = \text{constant}$$

then

$$x + 2y + 1 = \text{constant}.$$

However, the contours of $f(x,y)$ are not equally spaced because $f(x,y)$ is not linear.

In the "close up" diagram $[-0.6, 0.6] \times [-0.6, 0.6]$, the contours of $Q(x,y)$ look like lines (though they are not). The contour diagram of $Q(x,y)$ is more similar to the contour diagram of $f(x,y)$ than is $L(x,y)$. This is because $Q(x,y)$ is a better approximation to $f(x,y)$ than is $L(x,y)$.

In the $[-2,2] \times [-2,2]$ diagram, the values on the level curves of $L(x,y)$ and $Q(x,y)$ show that neither of them is a good approximation to $f(x,y)$ away from the origin.

59. Letting $G(t) = f(t,0)$ so that $G'(t) = f_x(t,0)$, the Fundamental Theorem tells us that $\int_{t=0}^{a} G'(t)dt = G(a) - G(0)$. Thus

$$\int_{t=0}^{a} f_x(t,0)dt = f(a,0) - f(0,0)$$

Letting $H(t) = f(a,t)$ so that $H'(t) = f_y(a,t)$, the Fundamental Theorem tells us that $\int_{t=0}^{b} H'(t)dt = H(b) - H(0)$. Thus

$$\int_{t=0}^{b} f_y(a,t)dt = f(a,b) - f(a,0)$$

Thus

$$\begin{aligned} f(0,0) &+ \int_{t=0}^{a} f_x(t,0)dt + \int_{t=0}^{b} f_y(a,t)dt \\ &= f(0,0) + (f(a,0) - f(0,0)) + (f(a,b) - f(a,0)) \\ &= f(a,b). \end{aligned}$$

60. We have

$$\begin{aligned}
|f(a,b)| &= \left| f(0,0) + \int_{t=0}^{a} f_x(t,0)dt + \int_{t=0}^{b} f_y(a,t)dt \right| \\
&\le |f(0,0)| + \left| \int_{t=0}^{a} f_x(t,0)dt \right| + \left| \int_{t=0}^{b} f_y(a,t)dt \right| \\
&\le 0 + \left| \int_{t=0}^{a} |f_x(t,0)|\, dt \right| + \left| \int_{t=0}^{b} |f_y(a,t)|\, dt \right| \\
&\le \left| \int_{t=0}^{a} A dt \right| + \left| \int_{t=0}^{b} B dt \right| \\
&= A\,|a| + B\,|b|
\end{aligned}$$

Strengthen Your Understanding

61. The function $f(x,y) = x^3 + y^4$ is not the zero function. Nevertheless, the values of f, f_x, f_y, f_{xx}, f_{xy}, and f_{yy} at $(0,0)$ are all zero. Therefore, the quadratic approximation of f near $(0,0)$ is the zero function.

62. If $f_x = xy$ and $f_y = y^2$, then

$$\begin{aligned}
f_{xy} &= \frac{\partial}{\partial y} xy = x \\
f_{yx} &= \frac{\partial}{\partial x} y^2 = 0.
\end{aligned}$$

Since f_{xy} and f_{yx} are continuous, we expect $f_{xy} = f_{yx}$. Thus there is no function f with the given partial derivatives.

63. One example is $f(x,y) = x^2 + y^2$. More generally, functions of the form $f(x,y) = g(x) + h(y)$ with $g'' \neq 0$ and $h'' \neq 0$ all have $f_{xx} \neq 0$, $f_{yy} \neq 0$, and $f_{xy} = 0$.

64. Since the quadratic approximation of a function near $(0,0)$ depends only on the values of the function and its first and second order partial derivatives at $(0,0)$, changing a function by adding another function for which all these values are zero does not change the quadratic approximation. For example, adding x^3 to a function $f(x,y)$ does not change the quadratic approximation near $(0,0)$.

For example, $f(x,y) = 2x + y^2$ and $g(x,y) = 2x + y^2 + x^3$ both have quadratic approximation $Q(x,y) = 2x + y^2$ near $(0,0)$.

65. If f and g have exactly the same contour diagrams inside a circle about the origin, then any partial derivative of f of any order has the same value at the point $(0,0)$ as the corresponding partial derivative of g. Since the Taylor polynomials of degree 2 for f and g near $(0,0)$ are constructed from the values of the partial derivatives at $(0,0)$, the two Taylor polynomials are identical. It makes no difference what the contour diagrams look like outside the circle. See Figure 14.43 for one possible solution.

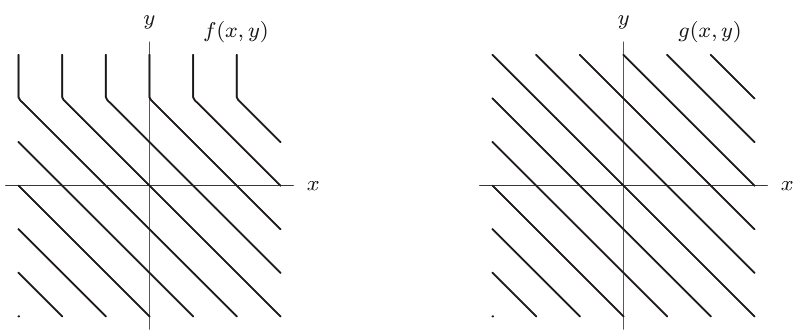


Figure 14.43

Solutions for Section 14.8

Exercises

1. Not differentiable at $(0,0)$.
2. Not differentiable at $(-1,0)$.
3. Not differentiable at all points on the x or y axes.
4. Not differentiable where $x=-2$ or where $y=3$; that is not differentiable at all points on the lines $x=-2$ and $y=3$.
5. Differentiable at all points.
6. Not differentiable where $x=0$; that is, not differentiable on y-axis.
7. Differentiable everywhere, since $|x-3|^2=(x-3)^2$.
8. Differentiable at all points, since $\cos|y|=\cos y$.
9. Not differentiable at $(1,2)$.
10. Differentiable at all points.

Problems

11. (a) The contour diagram for $f(x,y)=\frac{x}{y}+\frac{y}{x}$ is shown in Figure 14.44.

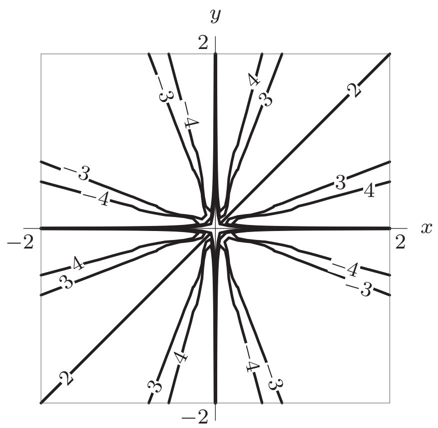


Figure 14.44

(b) If $x\neq 0$ and $y\neq 0$ then f is differentiable at (x,y). Now we need to look at points of the form $(x,0)$, where $x\neq 0$ and $(0,y)$, where $y\neq 0$. The function f is not differentiable at these points as it is not continuous.

(c) For $x\neq 0$ and $y\neq 0$,

$$f_x(x,y)=\frac{1}{y}-\frac{y}{x^2}.$$

So f_x exists for $x\neq 0$, $y\neq 0$, and it is continuous.

For all points $(x_0,0)$ on the x-axis we have:

$$f_x(x_0,0)=\lim_{x\to x_0}\frac{f(x,0)-f(x_0,0)}{x-x_0}=0.$$

Thus, f_x exists but is not continuous at these points.

For points $(0,y_0)$ on the y-axis we have:

$$\lim_{x\to 0}\frac{f(x,y_0)-f(0,y_0)}{x}=\lim_{x\to 0}\left(\frac{1}{y_0}+\frac{y_0}{x^2}\right).$$

This limit does not exist, so the partial derivative $f_x(0,y_0)$ does not exist.

Similarly, for $x \neq 0$ and $y \neq 0$,

$$f_y(x, y) = -\frac{x}{y^2} + \frac{1}{x}.$$

For points $(0, y_0)$ on the y-axis we have $f_y(0, y_0) = 0$, while $f_y(x_0, 0)$ does not exist for $x_0 \neq 0$.

Both $f_x(x, y)$ and $f_y(x, y)$ are continuous at (x, y) only for $x \neq 0$ and $y \neq 0$.

(d) We claim f is not continuous at $(0, 0)$. Let $x = t$ and $y = t$, where $t \to 0$, $t \neq 0$. Then

$$f(x, y) = f(t, t) = 2, \qquad \text{for} \quad t \neq 0.$$

So,

$$\lim_{t\to 0} f(t, t) = 2 \neq f(0, 0) = 0,$$

and therefore

$$\lim_{(x,y)\to(0,0)} f(x, y) \neq f(0, 0).$$

Thus, f is not differentiable at $(0, 0)$ since f is not continuous at $(0, 0)$.

(e) From part (c) we have $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. The functions f_x and f_y are not continuous at $(0, 0)$.

12. (a) The contour diagram for $f(x, y) = 2xy/(x^2 + y^2)^2$ is shown in Figure 14.45.

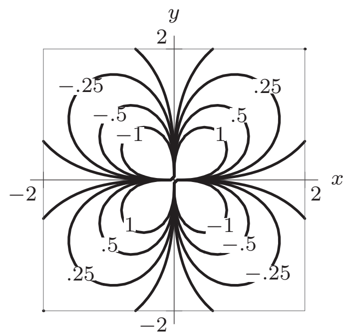


Figure 14.45

(b) The function f is differentiable at all points $(x, y) \neq (0, 0)$, as it is a rational fraction with denominator $(x^2 + y^2)^2 = 0$ only when $(x, y) = (0, 0)$.

(c) The partial derivatives of f at points $(x, y) \neq (0, 0)$ are given by

$$f_x(x, y) = \frac{2y(y^2 - 3x^2)}{(x^2 + y^2)^3},$$

$$f_y(x, y) = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3}.$$

Both f_x and f_y are continuous for $(x, y) \neq (0, 0)$.

(d) The function f is not continuous at $(0, 0)$. To see this, let $x = y = t$ for $t \neq 0$. Then,

$$f(x, y) = f(t, t) = \frac{2t^2}{4t^4} = \frac{1}{2t^2},$$

and so $\lim_{t\to 0} f(t, t)$ does not exist. Hence, f is not differentiable at $(0, 0)$.

(e) At $(0, 0)$, the partial derivatives of f are given by

$$f_x(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x\to 0} \frac{0 - 0}{x} = 0,$$

$$f_y(0, 0) = \lim_{y\to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y\to 0} \frac{0 - 0}{y} = 0.$$

We claim that $\lim_{(x,y)\to(0,0)} f_x(x, y)$ does not exist. To see this, let $x = y = t$ for $t \neq 0$. Then,

$$f_x(x, y) = f_x(t, t) = \frac{2t(t^2 - 3t^2)}{(2t^2)^3} = \frac{-4t^3}{8t^6} = -\frac{1}{2t^3}$$

and so the limit

$$\lim_{t\to 0} f_x(t, t) = \lim_{t\to 0} \frac{-1}{2t^3}$$

does not exist. Hence, f_x is not continuous at $(0, 0)$. Similarly, f_y is not continuous at $(0, 0)$.

13. (a) The contour diagram for $f(x,y) = x^2y/(x^4+y^2)$ is shown in Figure 14.46.

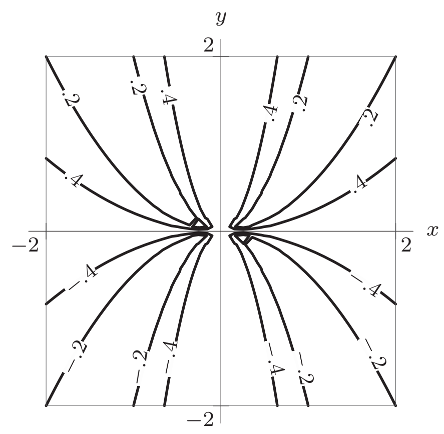


Figure 14.46

(b) The function f is differentiable at all $(x,y) \neq (0,0)$ as it is a rational fraction with denominator which is zero only when $(x,y) = (0,0)$.

(c) The partial derivatives of f are given by

$$f_x(x,y) = \frac{2xy(y^2 - x^4)}{(x^4+y^2)^2}, \qquad \text{for} \quad (x,y) \neq (0,0),$$

$$f_y(x,y) = \frac{x^2(x^4 - y^2)}{(x^4+y^2)^2}, \qquad \text{for} \quad (x,y) \neq (0,0).$$

Both f_x and f_y are continuous at $(x,y) \neq (0,0)$.

(d) We use the definition of differentiability. If f were differentiable at $(0,0)$, then the linear approximation of f at $(0,0)$ would be $L(x,y) = mx + ny$, where $m = f_x(0,0)$ and $n = f_y(0,0)$. We have

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = 0,$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0.$$

So, we need to compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^4+y^2)\sqrt{x^2+y^2}}.$$

This limit is not zero since if we choose $x = y = t$, $t > 0$, we have

$$\frac{x^2y}{(x^4+y^2)\sqrt{x^2+y^2}} = \frac{t}{\sqrt{2}\cdot|t|\cdot(t^2+1)} = \frac{1}{\sqrt{2}(t^2+1)},$$

which converges to $1/\sqrt{2} \neq 0$, as $t \to 0, t > 0$. Hence, f is not differentiable at $(0,0)$.

(e) The partial derivative f_x is not continuous at $(0,0)$ since if we choose $x = y = t \neq 0$, we have

$$f_x(x,y) = f_x(t,t) = \frac{2t^2(t^2-t^4)}{(t^4+t^2)^2} = \frac{2(1-t^2)}{(t^2+1)^2},$$

and so,

$$\lim_{t\to 0} f_x(t,t) = 2 \neq f_x(0,0) = 0.$$

Similarly f_y is not continuous at $(0,0)$.

14. (a) The contour diagram for $f(x, y) = xy/\sqrt{x^2 + y^2}$ is shown in Figure 14.47.

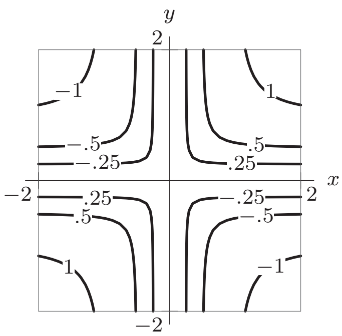


Figure 14.47

(b) By the chain rule, f is differentiable at all points (x, y) where $x^2 + y^2 \neq 0$, and so at all points $(x, y) \neq (0, 0)$.

(c) The partial derivatives of f are given by

$$f_x(x, y) = \frac{y^3}{(x^2 + y^2)^{3/2}}, \qquad \text{for} \quad (x, y) \neq (0, 0),$$

and

$$f_y(x, y) = \frac{x^3}{(x^2 + y^2)^{3/2}}, \qquad \text{for} \quad (x, y) \neq (0, 0).$$

Both f_x and f_y are continuous at $(x, y) \neq (0, 0)$.

(d) If f were differentiable at $(0, 0)$, the chain rule would imply that the function

$$g(t) = \begin{cases} f(t, t), & t \neq 0 \\ 0, & t = 0 \end{cases}$$

would be differentiable at $t = 0$. But

$$g(t) = \frac{t^2}{\sqrt{2t^2}} = \frac{1}{\sqrt{2}} \cdot \frac{t^2}{|t|} = \frac{1}{\sqrt{2}} \cdot |t|,$$

which is not differentiable at $t = 0$. Hence, f is not differentiable at $(0, 0)$.

(e) The partial derivatives of f at $(0, 0)$ are given by

$$f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0} \frac{\frac{x \cdot 0}{\sqrt{x^2 + 0^2}} - 0}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0,$$

$$f_y(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \to 0} \frac{\frac{0 \cdot y}{\sqrt{0^2 + y^2}} - 0}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0.$$

The limit $\lim\limits_{(x,y) \to (0,0)} f_x(x, y)$ does not exist since if we choose $x = y = t$, $t \neq 0$, then

$$f_x(x, y) = f_x(t, t) = \frac{t^3}{(2t^2)^{3/2}} = \frac{t^3}{2\sqrt{2} \cdot |t|^3} = \begin{cases} \frac{1}{2\sqrt{2}}, & t > 0, \\ -\frac{1}{2\sqrt{2}}, & t < 0. \end{cases}$$

Thus, f_x is not continuous at $(0, 0)$. Similarly, f_y is not continuous at $(0, 0)$.

15. (a)

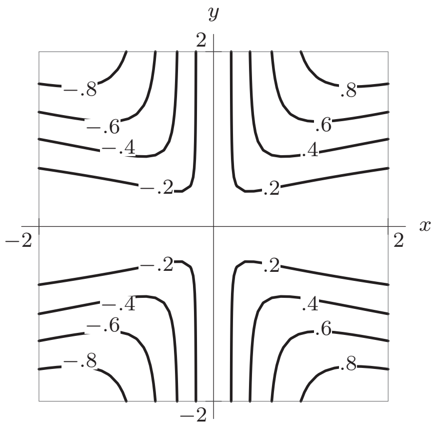


Figure 14.48

(b) f is differentiable at all $(x, y) \neq (0, 0)$ as it is a rational function with nonvanishing denominator.

(c)

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = 0$$

$$f_y(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = 0$$

(d) Let us use the definition. If f were differentiable, the linear approximation of f would be $L(x, y) = mx + ny$, where $m = f_x(0, 0) = 0$ and $n = f_y(0, 0) = 0$. So let's compute

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}.$$

This limit is not zero as, for $x = y = t \to 0$, $t > 0$,

$$\frac{xy^2}{(x^2+y^2)^{3/2}} = \frac{t^3}{2\sqrt{2}|t|^3} \overset{\substack{t \to 0 \\ t > 0}}{\longrightarrow} \frac{1}{2\sqrt{2}}.$$

Hence f is not differentiable at $(0, 0)$.

(e)

$$g(t) = f(x(t), y(t)) = \frac{ab^2t^3}{(a^2+b^2)t^2} = \frac{ab^2}{a^2+b^2}t$$

So

$$g'(0) = \frac{ab^2}{a^2+b^2}.$$

(f) $f_x(0, 0) \cdot x'(0) + f_y(0, 0) \cdot y'(0) = 0$, as $f_x(0, 0) = f_y(0, 0) = 0$. Suppose the chain rule holds, then

$$g'(t) = f_x(x(t), y(t)) \cdot x'(t) + f_x(x(t), y(t))y'(t).$$

But $g'(0) = \frac{ab^2}{a^2+b^2}$ from part (e) and $g'(0) \neq 0$ since $a \neq 0$ and $b \neq 0$. Hence the chain rule does not hold. This happens because f was not differentiable at $(0, 0)$.

(g) If $\vec{u} = a\vec{i} + b\vec{j}$, then $a^2 + b^2 = 1$ as $\vec{u}$ is a unit vector. Thus,

$$f_{\vec{u}}(0,0) = \lim_{t \to 0} \frac{f(at, bt)}{t} = \lim_{t \to 0} \frac{g(t)}{t} = g'(0) = \frac{a^2 b}{a^2+b^2} = a^2 b.$$

16. (a) The contour diagram of $f(x, y) = \sqrt{|xy|}$ is shown in Figure 14.49.

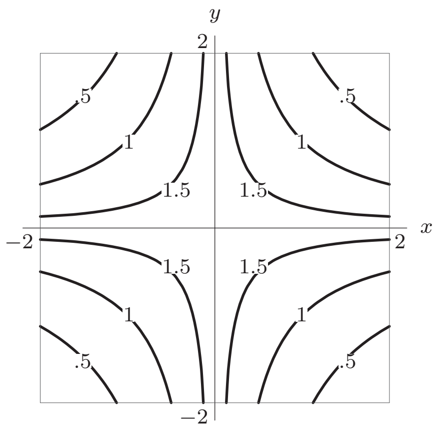


Figure 14.49

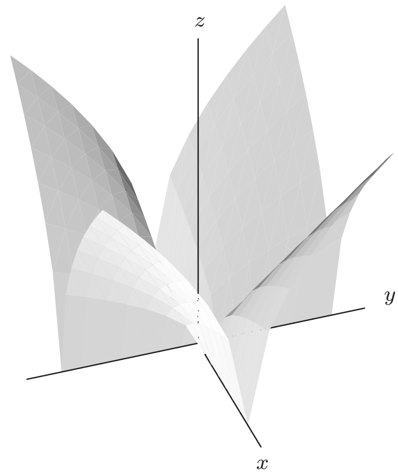


Figure 14.50

(b) The graph of $f(x, y) = \sqrt{|xy|}$ is shown in Figure 14.50.

(c) f is clearly differentiable at (x, y) where $x \neq 0$ and $y \neq 0$. So we need to look at points $(x_0, 0)$, $x_0 \neq 0$ and $(0, y_0)$, $y_0 \neq 0$. At $(x_0, 0)$:

$$f_x(x_0, 0) = \lim_{x \to x_0} \frac{f(x, 0) - f(x_0, 0)}{x - x_0} = 0$$

$$f_y(x_0, 0) = \lim_{y \to 0} \frac{f(x_0, y) - f(x_0, 0)}{y} = \lim_{y \to 0} \frac{\sqrt{|x_0 y|}}{y}$$

which does not exist. So f is not differentiable at the points $(x_0, 0)$, $x_0 \neq 0$. Similarly, f is not differentiable at the points $(0, y_0)$, $y_0 \neq 0$.

(d)

$$f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = 0$$

$$f_y(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = 0$$

(e) Let $\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$:

$$f_{\vec{u}}(0, 0) = \lim_{t \to 0^+} \frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) - f(0, 0)}{t} = \lim_{t \to 0^+} \frac{\sqrt{\frac{t^2}{2}}}{t} = \frac{1}{\sqrt{2}}.$$

We know that $\nabla f(0, 0) = \vec{0}$ because both partial derivatives are 0. But if f were differentiable, $f_{\vec{u}}(0, 0) = \nabla f(0, 0) \cdot \vec{u} = f_x(0, 0) \cdot \frac{1}{\sqrt{2}} + f_y(0, 0) \cdot \frac{1}{\sqrt{2}} = 0$. But since, in fact, $f_{\vec{u}}(0, 0) = 1/\sqrt{2}$, we conclude that f is not differentiable.

17. (a) The contour diagram of $f(x, y) = xy^2/(x^2 + y^4)$ is shown in Figure 14.51.

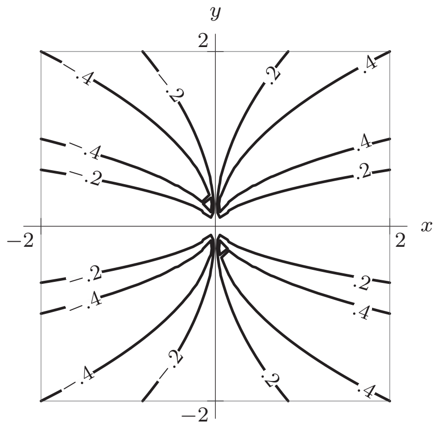


Figure 14.51

(b) Let $\vec{u} = a\vec{i} + b\vec{j}$ be the unit vector. Then

$$f_{\vec{u}}(0,0) = \lim_{t\to 0} \frac{f(at,bt)}{t} = \lim_{t\to 0} \frac{ab^2t^3}{t(a^2t^2+b^4t^4)} = \frac{ab^2}{a^2} = \frac{b^2}{a} \quad \text{if} \quad a \neq 0$$

and

$$f_{\vec{u}}(0,0) = 0 \qquad \text{if} \quad a = 0.$$

(c) f is not continuous at $(0,0)$. To see this let $x = t^2$, $y = t$, $t \to 0$, $t \neq 0$. Then

$$f(x,y) = \frac{t^4}{t^4+t^4} = \frac{1}{2} \overset{t\to 0}{\longrightarrow} \frac{1}{2} \neq f(0,0) = 0.$$

So f is not differentiable at $(0,0)$ either.

18. (a) If f were differentiable at $(0,0)$, then

$$f_{\vec{u}}(0,0) = \text{grad}\, f(0,0)\cdot\vec{u} = f_x(0,0)\cdot\frac{1}{\sqrt{2}} + f_y(0,0)\cdot\frac{1}{\sqrt{2}} = 0$$

which contradict the information that $f_{\vec{u}}(0,0) = 3$.

(b) Let

$$f(x,y) = \begin{cases} \frac{3}{\sqrt{2}}\left(\frac{x^2}{y} + \frac{y^2}{x}\right), & x \neq 0 \text{ and } y \neq 0, \\ 0, & x = 0 \text{ or } y = 0. \end{cases}$$

Then $f_x(0,0) = 0$, $f_y(0,0) = 0$:

$$f_{\vec{u}}(0,0) = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) - 0}{t} = \lim_{t\to 0} \frac{3}{\sqrt{2}t}\left(\frac{t^2}{2}\cdot\frac{\sqrt{2}}{t} + \frac{t^2}{2}\cdot\frac{\sqrt{2}}{t}\right) = 3.$$

19. (a) Differentiating gives

$$f_x(x,y) = \frac{(3x^2y^3 - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

similarly,

$$f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \qquad \text{for} \quad (x,y) \neq (0,0).$$

(b) We find the partial derivatives at the origin by using the limit definition:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = 0$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0$$

(c) Let's compute:

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}.$$

Let's switch to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then $(x,y)\to(0,0)$ is equivalent to $r\to 0$. Therefore:

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{r\to 0} \frac{r^5(\cos^4\theta\sin\theta + 4\cos^4\sin^3\theta - \sin^5\theta)}{r^4} = 0 = f_x(0,0).$$

Similarly, $\lim_{(x,y)\to(0,0)} f_y(x,y) = 0 = f_y(0,0)$. So f_x and f_y are continuous.

(d) From part (c) f is differentiable at $(0,0)$.

20. The fact that f is differentiable says that its graph is well approximated by a plane near (a,b). Since a plane is a graph of continuous function, it is reasonable to expect f to be continuous too. To prove this, we have to show that

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \quad \text{or equivalently} \quad \lim_{\substack{h\to 0\\k\to 0}} f(a+h,b+k) = f(a,b).$$

Suppose f is differentiable at (a,b). Then there is a linear function

$$L(x,y) = f(a,b) + m(x-a) + n(y-b)$$

such that

$$\lim_{\substack{h\to 0\\k\to 0}} \frac{f(a+h,b+k) - L(a+h,b+k)}{\sqrt{h^2+k^2}} = 0.$$

If this limit is 0, then we also have

$$\lim_{\substack{h\to 0\\k\to 0}} f(a+h,b+k) - L(a+h,b+k) = 0.$$

Substituting for $L(a+h,b+k)$, this gives

$$\lim_{\substack{h\to 0\\k\to 0}} (f(a+h,b+k) - f(a,b) - mh - nk) = 0.$$

So,

$$\lim_{\substack{h\to 0\\k\to 0}} f(a+h,b+k) - f(a,b) = 0.$$

Therefore f is continuous at (a,b).

Strengthen Your Understanding

21. The converse of this statement is true. If a function is differentiable at the origin, then it is continuous at the origin. However, a function can be continuous and not differentiable at a point; for example, $f(x,y) = \sqrt{x^2+y^2}$ is continuous but not differentiable at $(0,0)$.

22. A counterexample is provided by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

We have $f_x(0,0) = f_y(0,0)$, but $f(x,y)$ is not continuous or differentiable at $(0,0)$.

23. The function $f(x,y) = \sqrt{x^2+y^2}$ whose graph is a cone with vertex at the origin is not differentiable at the origin.

24. Taking $f(x,y) = |x-1|$ gives a function that is not differentiable on the line $x = 1$.

25. (a) Differentiable; point is at top of hemisphere.
(b) Not differentiable; hemisphere vertical at this point.
(c) Not differentiable; top point of cone with ellipse-shaped cross-section.
(d) Differentiable; point on side of cone.

Solutions for Chapter 14 Review

Exercises

1. Vector. Taking the gradient and substituting $x = 1, y = 2$ gives

$$\operatorname{grad}(x^3 e^{-y/2})\Big|_{(1,2)} = \left(3x^2 e^{-y/2}\vec{i} - \frac{x^3}{2}e^{-y/2}\vec{j}\right)\Big|_{(1,2)} = 3e^{-1}\vec{i} - \frac{1}{2}e^{-1}\vec{j}.$$

2. Scalar. The gradient of $f(x, y)$ at $x = 1, y = 1$ is given by

$$\operatorname{grad}(x^2y^3)\Big|_{(1,1)} = \left(2xy^3\vec{i} + 3x^2y^2\vec{j}\right)\Big|_{(1,1)} = 2\vec{i} + 3\vec{j}.$$

Thus the directional derivative is

$$f_{\vec{u}} = (2\vec{i} + 3\vec{j}) \cdot \frac{\vec{i} + \vec{j}}{\sqrt{2}} = \frac{2+3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

3. Vector. $\operatorname{grad}((\cos x)e^y + z) = -(\sin x)e^y\vec{i} + (\cos x)e^y\vec{j} + \vec{k}$.

4. Scalar. Differentiating with respect to x using the chain rule gives

$$\frac{\partial f}{\partial x} = 2xy^2 e^{x^2y}$$

so, differentiating again using the product rule, we have

$$\frac{\partial^2 f}{\partial x^2} = 2y^2 e^{x^2y} + 4x^2y^3 e^{x^2y}.$$

5. Taking the derivative of f with respect to x and treating y as a constant we get $f_x = 2xy + 3x^2 - 7y^6$. To find f_y we take the derivative of f with respect to y and treat x as a constant. Thus, $f_y = x^2 - 42xy^5$.

6. First we distribute to get $w = 6400\pi gh^2 - 320\pi gh^3$. Taking the partial with respect to h we get $\dfrac{\partial w}{\partial h} = 12{,}800\pi gh - 960\pi gh^2$.

7. $\dfrac{\partial T}{\partial l} = \dfrac{2\pi}{\sqrt{g}} \cdot \dfrac{1}{2}l^{-1/2} = \dfrac{\pi}{\sqrt{lg}}$.

8. To find the derivative of B with respect to t, note that the variable is only in the exponent. Thus,

$$\frac{\partial B}{\partial t} = P(1 + r)^t \ln(1 + r).$$

To find the derivative of B with respect to r, note that this time the exponent is a constant, so we can use the power rule to obtain

$$\frac{\partial B}{\partial r} = tP(1 + r)^{t-1}.$$

9. For both partial derivatives we use the quotient rule. Thus,

$$f_x = \frac{2xy(x^2 + y^2) - x^2y2x}{(x^2 + y^2)^2} = \frac{2x^3y + 2xy^3 - 2x^3y}{(x^2 + y^2)^2} = \frac{2xy^3}{(x^2 + y^2)^2},$$

and

$$f_y = \frac{x^2(x^2 + y^2) - x^2y2y}{(x^2 + y^2)^2} = \frac{x^4 + x^2y^2 - 2x^2y^2}{(x^2 + y^2)^2} = \frac{x^4 - x^2y^2}{(x^2 + y^2)^2}.$$

10. To find the derivative of F with respect to r we first rewrite $F = G\mu m y(r^2+y^2)^{-3/2}$ and then use the chain rule. Thus,

$$\frac{\partial F}{\partial r} = G\mu m y\left(-\frac{3}{2}\right)(r^2+y^2)^{-5/2}\cdot 2r.$$

To find the derivative of F with respect to y we use the quotient rule to obtain

$$\frac{\partial F}{\partial y} = \frac{G\mu m(r^2+y^2)^{3/2} - G\mu m y(\frac{3}{2})(r^2+y^2)^{1/2}\cdot 2y}{(r^2+y^2)^3}.$$

11. $\dfrac{\partial f}{\partial p} = \dfrac{\partial}{\partial p}\left[e^{p/q}\right] = \dfrac{1}{q}e^{p/q}.$
$\dfrac{\partial f}{\partial q} = \dfrac{\partial}{\partial q}\left[e^{p/q}\right] = \dfrac{\partial}{\partial q}\left[e^{pq^{-1}}\right] = (-pq^{-2})e^{pq^{-1}} = -\dfrac{p}{q^2}e^{p/q}.$

12. $z_x = -\sin x, \quad z_x(2,3) = -\sin 2 \approx -0.9$

13. Since we take the derivative with respect to N, we use the power rule to obtain $f_N = c\alpha N^{\alpha-1}V^{\beta}$.

14. This function is symmetric with respect to x and y. Therefore the answers look very similar. Using the chain rule we have

$$f_x = \frac{2(x-a)}{2\sqrt{(x-a)^2+(y-b)^2}} = \frac{(x-a)}{\sqrt{(x-a)^2+(y-b)^2}}$$

and

$$f_y = \frac{2(y-b)}{2\sqrt{(x-a)^2+(y-b)^2}} = \frac{(y-b)}{\sqrt{(x-a)^2+(y-b)^2}}.$$

15. Using the chain rule

$$\frac{\partial}{\partial \omega}\left(\tan\sqrt{\omega x}\right) = \frac{1}{\cos^2\left(\sqrt{\omega x}\right)}\cdot\frac{1}{2}(\omega x)^{-1/2}\cdot x = \frac{x}{2\sqrt{\omega x}\cos^2\left(\sqrt{\omega x}\right)}.$$

16. $\dfrac{\partial y}{\partial t} = \cos(ct-5x)c = c\cos(ct-5x)$

17. Using the quotient rule

$$\begin{aligned} z_y &= \frac{(15xy-8)(21x^2y^6-2y) - 15x(3x^2y^7-y^2)}{(15xy-8)^2} \\ &= \frac{315x^3y^7 - 168x^2y^6 - 30xy^2 + 16y - 45x^3y^7 + 15xy^2}{(15xy-8)^2} \\ &= \frac{270x^3y^7 - 168x^2y^6 - 15xy^2 + 16y}{(15xy-8)^2} \end{aligned}$$

18. Using the quotient rule,

$$\frac{\partial \alpha}{\partial \beta} = \frac{(2y\beta+5)e^{x\beta-3}x - 2ye^{x\beta-3}}{(2y\beta+5)^2} = \frac{[(2y\beta+5)x-2y]e^{x\beta-3}}{(2y\beta+5)^2} = \frac{(5x-2y+2xy\beta)e^{x\beta-3}}{(2y\beta+5)^2}$$

19. Using the chain rule,

$$\begin{aligned} \frac{\partial}{\partial w}(\sqrt{2\pi xyw - 13x^7y^3v}) &= \frac{1}{2}(2\pi xyw - 13x^7y^3v)^{-1/2}(2\pi xy - 0) \\ &= \frac{\pi xy}{\sqrt{2\pi xyw - 13x^7y^3v}}. \end{aligned}$$

20. Using the product and chain rules,

$$\begin{aligned}
&\frac{\partial}{\partial \lambda}\left(\frac{x^2y\lambda - 3\lambda^5}{\sqrt{\lambda^2-3\lambda+5}}\right)\\
&= \left[(x^2y-15\lambda^4)\sqrt{\lambda^2-3\lambda+5} - \frac{(2\lambda-3)(x^2y\lambda-3\lambda^5)}{2\sqrt{\lambda^2-3\lambda+5}}\right]\frac{1}{\lambda^2-3\lambda+5}\\
&= \frac{(x^2y-15\lambda^4)\cdot 2(\lambda^2-3\lambda+5)-(2\lambda-3)(x^2y\lambda-3\lambda^5)}{2(\lambda^2-3\lambda+5)\sqrt{\lambda^2-3\lambda+5}}\\
&= \frac{x^2y[2(\lambda^2-3\lambda+5)-(2\lambda-3)\lambda]-15\lambda^4\cdot 2(\lambda^2-3\lambda+5)+(2\lambda-3)\cdot 3\lambda^5}{2(\lambda^2-3\lambda+5)\sqrt{\lambda^2-3\lambda+5}}\\
&= \frac{x^2y(-3\lambda+10)-3\lambda^4(8\lambda^2-27\lambda+50)}{2(\lambda^2-3\lambda+5)\sqrt{\lambda^2-3\lambda+5}}
\end{aligned}$$

21. Using the chain rule and the quotient rule,

$$\begin{aligned}
&\frac{\partial}{\partial w}\left(\frac{x^2yw-xy^3w^7}{w-1}\right)^{-7/2}\\
&= -\frac{7}{2}\left(\frac{x^2yw-xy^3w^7}{w-1}\right)^{-9/2}\left(\frac{(w-1)(x^2y-7xy^3w^6)-(x^2yw-xy^3w^7)(1)}{(w-1)^2}\right)\\
&= -\frac{7}{2}\left(\frac{w-1}{x^2yw-xy^3w^7}\right)^{-9/2}\left(\frac{(w-1)(x^2y-7xy^3w^6)-(x^2yw-xy^3w^7)}{(w-1)^2}\right)\\
&= \frac{7}{2}\left(\frac{w-1}{x^2yw-xy^3w^7}\right)^{-9/2}\cdot\frac{x^2y+6xy^3w^7-7xy^3w^6}{(w-1)^2}
\end{aligned}$$

22. To find the partial derivative with respect to x we treat y and a as constants: $\frac{\partial}{\partial x}(e^x\cos(xy)+ay^2) = e^x\cos(xy)+e^x\cdot(-\sin(xy))\cdot y$.

To find the partial derivative with respect to y we treat x and a as constants: $\frac{\partial}{\partial y}(e^x\cos(xy)+ay^2) = e^x\cdot(-\sin(xy))\cdot x+2ay$.

To find the partial derivative with respect to a we treat x and y as constants: $\frac{\partial}{\partial a}(e^x\cos(xy)+ay^2) = y^2$.

23. $\dfrac{\partial f_0}{\partial L} = \dfrac{1}{2\pi}(-\dfrac{1}{2})(LC)^{-3/2}C = -\dfrac{1}{4\pi}\dfrac{C}{LC\sqrt{LC}} = -\dfrac{1}{4\pi L\sqrt{LC}}$

24. The first order partial derivative f_x is

$$f_x = -x(x^2+y^2)^{-3/2}$$

Thus the second order partials are

$$f_{xx} = -(x^2+y^2)^{-3/2}+3x^2(x^2+y^2)^{-5/2} = (2x^2-y^2)(x^2+y^2)^{-5/2}$$

and

$$f_{xy} = 3xy(x^2+y^2)^{-5/2}.$$

25. The first order partial derivatives are

$$u_x = e^x\sin y, \quad u_y = e^x\cos y.$$

Thus the second order partials are

$$u_{xx} = e^x\sin y, \quad u_{yy} = -e^x\sin y.$$

26. The first order partial derivative V_r is

$$V_r = 2\pi rh.$$

Thus the second order partials are

$$V_{rr} = 2\pi h, \quad V_{rh} = 2\pi r.$$

27. The first order partials are

$$f_x = \cos(x - 2y), \quad f_y = -2\cos(x - 2y).$$

The second order partials f_{xx} and f_{yx} are

$$f_{xx} = -\sin(x - 2y), \quad f_{yx} = 2\sin(x - 2y).$$

The third order partials are then

$$f_{xxy} = 2\cos(x - 2y), \quad f_{yxx} = 2\cos(x - 2y).$$

Note that these are equal as we would expect, since the order does not matter for higher order partial derivatives of smooth functions.

28. We have

$$\frac{\partial^2}{\partial x^2}(e^{ax-bt}) + \frac{\partial^2}{\partial t^2}(e^{ax-bt}) = (a^2 + b^2)e^{ax-bt}.$$

29. Since $f_x = 2x$, $f_y = 2y + 3y^2$ and $f_z = 0$, we have

$$\text{grad } f = 2x\vec{i} + (2y + 3y^2)\vec{j}.$$

30. Since the partial derivatives are

$$f_x = 3x^2 - yz, \quad f_y = -xz, \quad f_z = 3z^2 - xy,$$

we have

$$\nabla f = (3x^2 - yz)\vec{i} - xz\vec{j} + (3z^2 - xy)\vec{k}.$$

31. Since $f(x, y, z) = \dfrac{1}{xyz}$, we have

$$f_x = -\frac{1}{x^2yz}, \quad f_y = -\frac{1}{xy^2z} \quad f_z = -\frac{1}{xyz^2},$$

we have

$$\text{grad } f = -\frac{1}{xyz}\left(\frac{1}{x}\vec{i} + \frac{1}{y}\vec{j} + \frac{1}{z}\vec{k}\right).$$

32. Since the partial derivatives are

$$f_x = -y\cos(y^2 - xy) \quad \text{and} \quad f_y = (2y - x)\cos(y^2 - xy),$$

we have

$$\nabla f = \cos(y^2 - xy)(-y\vec{i} + (2y - x)\vec{j}).$$

33. Since the partial derivatives are

$$z_x = (2x)\cos(x^2 + y^2), \quad \text{and} \quad z_y = (2y)\cos(x^2 + y^2),$$

$$\nabla z = 2x\cos(x^2 + y^2)\vec{i} + 2y\cos(x^2 + y^2)\vec{j}.$$

34. Since $f(x, y, z) = xe^y + \ln x + \ln z$, we have $f_x = e^y + 1/x$, $f_y = xe^y$, $f_z = 1/z$, so

$$\text{grad } f = \left(e^y + \frac{1}{x}\right)\vec{i} + xe^y\vec{j} + \frac{1}{z}\vec{k}.$$

35. Since $f_x = 2x\cos(x^2 + y^2 + z^2)$, $f_y = 2y\cos(x^2 + y^2 + z^2)$ and $f_z = 2z\cos(x^2 + y^2 + z^2)$, we have

$$\text{grad } f = 2x\cos(x^2 + y^2 + z^2)\vec{i} + 2y\cos(x^2 + y^2 + z^2)\vec{j} + 2z\cos(x^2 + y^2 + z^2)\vec{k}.$$

36. Since the partial derivatives are

$$f_\rho = \sin\phi\cos\theta, \quad f_\phi = \rho\cos\phi\cos\theta, \quad f_\theta = -\rho\sin\phi\sin\theta,$$

we have

$$\nabla f = \sin\phi\cos\theta\vec{i} + \rho\cos\phi\cos\theta\vec{j} - \rho\sin\phi\sin\theta\vec{k}.$$

37. Since the partial derivatives are

$$\frac{\partial f}{\partial s} = (t^2-2t+4)(-\frac{1}{2})s^{-3/2} = -\frac{(t^2-2t+4)}{2s\sqrt{s}}$$
$$\frac{\partial f}{\partial t} = \frac{1}{\sqrt{s}}(2t-2)$$

we have

$$\text{grad } f = \frac{\partial f}{\partial s}\vec{i} + \frac{\partial f}{\partial t}\vec{j} = \left(-\frac{(t^2-2t+4)}{2s\sqrt{s}}\right)\vec{i} + \left(\frac{1}{\sqrt{s}}(2t-2)\right)\vec{j}.$$

38. Since the partial derivatives are

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad f_y = \frac{y}{\sqrt{x^2+y^2}},$$

we have

$$\nabla f = \frac{1}{\sqrt{x^2+y^2}}(x\vec{i} + y\vec{j}).$$

39. Since the partial derivatives are

$$\frac{\partial f}{\partial x} = \cos(xy)\cdot(y) - \sin(xy)\cdot(y) = y[\cos(xy) - \sin(xy)]$$
$$\frac{\partial f}{\partial y} = \cos(xy)\cdot(x) - \sin(xy)\cdot(x) = x[\cos(xy) - \sin(xy)]$$

we have

$$\begin{aligned}\text{grad } f &= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\\ &= y[\cos(xy) - \sin(xy)]\vec{i} + x[\cos(xy) - \sin(xy)]\vec{j}.\end{aligned}$$

40. We have $f_x = 2x$, $f_y = 0$ and $f_z = 0$. Thus grad $f = 2x\vec{i}$ and grad $f(0,0,0) = \vec{0}$.

41. We have $f_x = 2xz$, $f_y = 0$ and $f_z = x^2$. Thus grad $f = 2xz\vec{i} + x^2\vec{k}$ and grad $f(1,1,1) = 2\vec{i} + \vec{k}$.

42. We have grad $f = 3x^2\vec{i} - 3y^2\vec{j}$, so grad $f(2,-1) = 12\vec{i} - 3\vec{j}$. A unit vector in the direction we want is $\vec{u} = (1/\sqrt{2})(\vec{i} - \vec{j})$. Therefore, the directional derivative is

$$\text{grad } f(-2,1)\cdot\vec{u} = \frac{12\cdot 1 - 3(-1)}{\sqrt{2}} = \frac{15}{\sqrt{2}}.$$

43. We have grad $f = e^y\vec{i} + xe^y\vec{j}$, so grad $f(3,0) = \vec{i} + 3\vec{j}$. A unit vector in the direction we want is $\vec{u} = (1/5)(4\vec{i} - 3\vec{j})$. Therefore, the directional derivative is

$$\text{grad } f(3,0)\cdot\vec{u} = \frac{1\cdot 4 + 3(-3)}{5} = -1.$$

44. We have grad $f = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$, so grad $f(2,3,4) = 4\vec{i} + 6\vec{j} - 8\vec{k}$. A unit vector in the direction we want is $\vec{u} = (1/3)(2\vec{i} - 2\vec{j} + \vec{k})$. Therefore, the directional derivative is

$$\text{grad } f(2,3,4)\cdot\vec{u} = \frac{4\cdot 2 + 6(-2) - 8\cdot 1}{3} = -4.$$

45. The unit vector $\vec{u}$ in the direction of $\vec{v} = \vec{i} - \vec{k}$ is $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}$. We have

$$\begin{aligned} f_x(x,y,z) &= 6xy^2, && \text{and } f_x(-1,0,4) = 0 \\ f_y(x,y,z) &= 6x^2y + 2z, && \text{and } f_y(-1,0,4) = 8 \\ f_z(x,y,z) &= 2y, && \text{and } f_z(-1,0,4) = 0. \end{aligned}$$

So,

$$\begin{aligned} f_{\vec{u}}(-1,0,4) &= f_x(-1,0,4)\left(\frac{1}{\sqrt{2}}\right) + f_y(-1,0,4)(0) + f_z(-1,0,4)\left(-\frac{1}{\sqrt{2}}\right) \\ &= 0\left(\frac{1}{\sqrt{2}}\right) + 8(0) + 0\left(-\frac{1}{\sqrt{2}}\right) \\ &= 0. \end{aligned}$$

46. The unit vector $\vec{u} = -\frac{1}{\sqrt{19}}\vec{i} + \frac{3}{\sqrt{19}}\vec{j} + \frac{3}{\sqrt{19}}\vec{k}$ is in the direction of $\vec{v} = -\vec{i} + 3\vec{j} + 3\vec{k}$. We have

$$\begin{aligned} f_x(x,y,z) &= 6xy^2, && \text{and } f_x(-1,0,4) = 0 \\ f_y(x,y,z) &= 6x^2y + 2z, && \text{and } f_y(-1,0,4) = 8 \\ f_z(x,y,z) &= 2y, && \text{and } f_z(-1,0,4) = 0. \end{aligned}$$

So,

$$f_{\vec{u}}(-1,0,4) = 0\left(-\frac{1}{\sqrt{19}}\right) + 8\left(\frac{3}{\sqrt{19}}\right) + 0\left(\frac{3}{\sqrt{19}}\right) = \frac{24}{\sqrt{19}}.$$

47. We have $\text{grad}\, f = (e^{x+z}\cos y)\vec{i} - (e^{x+z}\sin y)\vec{j} + (e^{x+z}\cos y)\vec{k}$, so $\text{grad}\, f(1,0,-1) = \vec{i} + \vec{k}$. A unit vector in the direction we want is $\vec{u} = (1/\sqrt{3})(\vec{i} + \vec{j} + \vec{k})$. Therefore, the directional derivative is

$$\text{grad}\, f(1,0,-1) \cdot \vec{u} = \frac{1 \cdot 1 + 1 \cdot 1}{\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

48. The given curve is the contour for $f(x,y) = x^2 - y^2$ passing through the point $(2,1)$. Thus a normal vector is $\text{grad}\, f(2,1)$. We have $\text{grad}\, f = 2x\vec{i} - 2y\vec{j}$, so a normal vector is $\text{grad}\, f(2,1) = 4\vec{i} - 2\vec{j}$.

49. The given surface is the level surface for $f(x,y,z) = xy + xz + yz$ passing through the point $(1,2,3)$. Thus a normal vector is $\text{grad}\, f(1,2,3)$. We have $\text{grad}\, f = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$, so a normal vector is $\text{grad}\, f(1,2,3) = 5\vec{i} + 4\vec{j} + 3\vec{k}$.

50. The given surface is the level surface for $f(x,y,z) = z^2 - 2xyz - x^2 - y^2$ passing through the point $(1,2,-1)$. Thus a normal vector is $\text{grad}\, f(1,2,-1)$. We have

$$\text{grad}\, f = (-2x - 2yz)\vec{i} + (-2y - 2xz)\vec{j} + (2z - 2xy)\vec{k},$$

so a normal vector is $\text{grad}\, f(1,2,-1) = 2\vec{i} - 2\vec{j} - 6\vec{k}$.

51. Let $f(x,y,z) = z^2 - 4x^2 - 3y^2$ so that the surface (a hyperboloid) is the level surface $f(x,y,x) = 9$. Since

$$\text{grad}\, f = -8x\vec{i} - 6y\vec{j} + 2z\vec{k}$$

we have

$$\text{grad}\, f(1,1,4) = -8\vec{i} - 6\vec{j} + 8\vec{k}.$$

Thus an equation of the tangent plane at $(1,1,4)$ is

$$-8(x-1) - 6(y+6) + 8(z-4) = 0$$

so

$$-4x - 3y + 4z = 9.$$

52. Let $f(x, y, z) = x^3 - 2y^2 + z$ so that the given surface is the level surface $f(x, y, z) = 0$. Since

$$\text{grad } f = 3x^2\vec{i} - 4y\vec{j} + \vec{k}$$

we have

$$\text{grad } f(1, 0, -1) = 3\vec{i} + \vec{k}.$$

Thus an equation of the tangent plane at $(1, 0, -1)$ is

$$3(x - 1) + 0(y - 0) + 1(z - (-1)) = 0$$

or

$$3x + z = 2.$$

53. Let $f(x, y, z) = z - 1/(xy)$. We have

$$\text{grad } f = \frac{1}{x^2y}\vec{i} + \frac{1}{xy^2}\vec{j} + \vec{k}.$$

Thus

$$\text{grad } f(1, 1, 1) = \vec{i} + \vec{j} + \vec{k}.$$

The tangent plane to the level surface at $(1, 1, 1)$ is

$$(x - 1) + (y - 1) + (z - 1) = 0$$

or

$$x + y + z = 3.$$

54. The first and second derivatives are

$$\begin{aligned}
f_x &= 2xy^2 - 5y^3 \\
f_y &= 2x^2y - 15xy^2 \\
f_{xx} &= 2y^2 \\
f_{xy} &= 4xy - 15y^2 \\
f_{yx} &= 4xy - 15y^2 \\
f_{yy} &= 2x^2 - 30xy.
\end{aligned}$$

55. Using the chain rule we see:

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \sin y\,(\cos t) + x(\cos y)\,(-\sin t) \\
&= \sin(\cos t)\cos t - \sin t\cos(\cos t)\sin t \\
&= \sin(\cos t)\cos t - \sin^2 t\cos(\cos t).
\end{aligned}$$

We can also solve the problem using one variable methods:

$$\begin{aligned}
z &= \sin t\ \sin(\cos t) \\
\frac{dz}{dt} &= \frac{d}{dt}(\sin t\ \sin(\cos t)) \\
&= \frac{d\sin t}{dt}\sin(\cos t) + \sin t\frac{d(\sin(\cos t))}{dt} \\
&= \cos t\sin(\cos t) + \sin t\cos(\cos t)(-\sin t) \\
&= \cos t\sin(\cos t) - \sin^2 t\cos(\cos t).
\end{aligned}$$

56. This is a case where substituting is easier:

$$\begin{aligned} z &= \sin((2t)^2 + (t^2)^2) \\ &= \sin(4t^2 + t^4) \\ \frac{dz}{dt} &= (8t + 4t^3)\cos(4t^2 + t^4). \end{aligned}$$

If you use the chain rule the solution is:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= 2x\cos(x^2 + y^2)(2) + 2y\cos(x^2 + y^2)(2t) \\ &= 2(2t)\cos(4t^2 + t^4)(2) + 2(t^2)\cos(4t^2 + t^4)(2t) \\ &= (2 \cdot 2t \cdot 2 + 2 \cdot t^2 \cdot 2t)\cos(4t^2 + t^4) \\ &= (8t + 4t^3)\cos(4t^2 + t^4). \end{aligned}$$

57. Substituting into the chain rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= 2(x^2 + y) \cdot 2x \cdot 2 + 2(x^2 + y) \cdot 2t \\ &= (x^2 + y)(8x + 4t) \\ &= ((2t)^2 + t^2)(8 \cdot 2t + 4t) \\ &= (4t^2 + t^2)(20t) \\ &= 5t^2(20t) \\ &= 100t^3. \end{aligned}$$

58. Substituting into the chain rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= ((x + y)e^x + e^x) \cdot (2t) + e^x \cdot (-2t) \\ &= (t^2 + (1 - t^2))e^{t^2} \cdot (2t) + e^{t^2} \cdot (2t) + e^{t^2} \cdot (-2t) \\ &= 2te^{t^2}(t^2 + 1 - t^2 + 2t - 2t) \\ &= 2te^{t^2}. \end{aligned}$$

This is a case where substituting is much easier:

$$\begin{aligned} z &= (t^2 + (1 - t^2))e^{t^2} \\ &= e^{t^2} \\ \frac{dz}{dt} &= 2te^{t^2}. \end{aligned}$$

59. Using the chain rule we see:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= \frac{1}{x} \cdot 3t^2 + \frac{1}{y} \cdot 2t \\ &= \frac{1}{t^3} \cdot 3t^2 + \frac{1}{t^2 + 1} \cdot 2t \\ &= \frac{3}{t} + \frac{2t}{t^2 + 1}. \end{aligned}$$

This problem can also be solved using one variable methods.

$$\begin{aligned} z &= \ln(t^2+1) + \ln t^3 \\ &= \ln(t^2+1) + 3\ln t \\ \frac{dz}{dt} &= \frac{2t}{t^2+1} + \frac{3}{t}. \end{aligned}$$

60. Substituting into the chain rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial p}\frac{dp}{dt} + \frac{\partial z}{\partial q}\frac{dq}{dt} \\ &= q\cos(pq)\cdot\cos t + p\cos(pq)\cdot(-\sin(t^2)\cdot 2t) \\ &= \cos(pq)\left(\cos t^2\,\cos t - 2t\sin t\sin(t^2)\right) \\ &= \cos(\sin t\,\cos t^2)\left(\cos t^2\,\cos t - 2t\sin t\sin(t^2)\right). \end{aligned}$$

61. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= (x+1)^3(y+2) \\ f_x(x,y) &= 3(x+1)^2(y+2) \\ f_y(x,y) &= (x+1)^3 \\ f_{xx}(x,y) &= 6(x+1)(y+2) \\ f_{yy}(x,y) &= 0 \\ f_{xy}(x,y) &= 3(x+1)^2 \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 2 + 6x + y + 6x^2 + 3xy$$

Notice this is the same as what you get if you expand $(x+1)^3(y+2)$ and keep only the terms of degree 2 or less.

62. The quadratic Taylor expansion about $(0,0)$ is given by

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= \cos x\cos 3y \\ f_x(x,y) &= -\sin x\cos 3y \\ f_y(x,y) &= -3\cos x\sin 3y \\ f_{xx}(x,y) &= -\cos x\cos 3y \\ f_{yy}(x,y) &= -9\cos x\cos 3y \\ f_{xy}(x,y) &= 3\sin x\sin 3y \end{aligned}$$

Now we evaluate each of these derivatives at $(0,0)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 - \frac{1}{2}x^2 - \frac{9}{2}y^2$$

Notice this is the same as what you get if you multiply together the quadratic approximations for $\cos x$ and $\cos 3y$ and then keep only the terms of degree 2 or less.

63. The quadratic Taylor expansion about $(3,5)$ is given by

$$f(x,y) \approx Q(x,y) = f(3,5) + f_x(3,5)(x-3) + f_y(3,5)(y-5) + \frac{1}{2}f_{xx}(3,5)(x-3)^2 +$$

$$f_{xy}(3,5)(x-3)(y-5) + \frac{1}{2}f_{yy}(3,5)(y-5)^2.$$

First we find all the relevant derivatives

$$\begin{aligned} f(x,y) &= (2x-y)^{1/2} \\ f_x(x,y) &= (2x-y)^{-1/2} \\ f_y(x,y) &= -\frac{1}{2}(2x-y)^{-1/2} \\ f_{xx}(x,y) &= -(2x-y)^{-3/2} \\ f_{yy}(x,y) &= -\frac{1}{4}(2x-y)^{-3/2} \\ f_{xy}(x,y) &= \frac{1}{2}(2x-y)^{-3/2} \end{aligned}$$

Now we evaluate each of these derivatives at $(3,5)$ and substitute into the formula to get as our final answer:

$$Q(x,y) = 1 + (x-3) - \frac{1}{2}(y-5) - \frac{1}{2}(x-3)^2 + \frac{1}{2}(x-3)(y-5) - \frac{1}{8}(y-5)^2.$$

Problems

64. The gradient of (a) is $2x\vec{i} + 2y\vec{j} + 2z\vec{k}$, which points radially outward from the origin and increases in magnitude, so (a) goes with (IV).

The gradient of (b) is

$$\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}},$$

which points radially outward and has magnitude 1. Thus (b) goes with (V).

The gradient of (c) is $3\vec{i} + 4\vec{j}$, which is constant and parallel to the xy-plane, so (c) goes with (I).

The gradient of (d) is $3\vec{i} + 4\vec{k}$, which is constant and parallel to xz-plane, so (d) goes with (II).

65. (a) Let $f(x,y,z) = 2x^2 - 2xy^2 + az$ so that the given surface is the level surface $f(x,y,z) = a$. Since

$$\text{grad}\, f = (4x - 2y^2)\vec{i} - 4xy\vec{j} + a\vec{k}$$

we have

$$\text{grad}\, f(1,1,1) = 2\vec{i} - 4\vec{j} + a\vec{k}.$$

Thus an equation of the tangent plane at $(1,1,1)$ is

$$2(x-1) - 4(y-1) + a(z-1) = 0$$

or

$$2x - 4y + az = a - 2.$$

(b) Substituting $x = 0$, $y = 0$, $z = 0$ into the equation for the tangent plane in part (a) we have $0 = a - 2$. The tangent plane passes through the origin if $a = 2$.

66. (a) This means you must pay a mortgage payment of \$1090.08/month if you have borrowed a total of \$92,000 at an interest rate of 14%, on a 30-year mortgage.

(b) This means that the rate of change of the monthly payment with respect to the interest rate is \$72.82; i.e., your monthly payment will go up by approximately \$72.82 for one percentage point increase in the interest rate for the \$92,000 borrowed under a 30-year mortgage.

(c) It should be *positive*, because the monthly payments will increase if the total amount borrowed is increased.

(d) It should be *negative*, because as you increase the number of years in which to pay the mortgage, you should have to pay less each month.

67. (a) At Q, R, we have $f_x < 0$ because f decreases as we move in the x-direction.
(b) At Q, P, we have $f_y > 0$ because f increases as we move in the y-direction.
(c) At all four points, P, Q, R, S, we have $f_{xx} > 0$, because f_x is increasing as we move in the x-direction. (At P, S, we see that f_x is positive and getting larger; at Q, R, we see that f_x is negative and getting less negative.)
(d) At all four points, P, Q, R, S, we have $f_{yy} > 0$, so there are none with $f_{yy} < 0$. The reasoning is similar to part (c).

68. Estimating from the contour diagram, using positive increments for Δx and Δy, we have, for point A,

$$\left.\frac{\partial n}{\partial x}\right|_{(A)} \approx \frac{1.5-1}{67-59} = \frac{1/2}{8} = \frac{1}{16} \approx 0.06 \quad \frac{\text{foxes/km}^2}{\text{km}}$$

$$\left.\frac{\partial n}{\partial y}\right|_{(A)} \approx \frac{0.5-1}{60-51} = -\frac{1/2}{9} = -\frac{1}{18} \approx -0.06 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

So, from point A the fox population density increases as we move eastward. The population density decreases as we move north from A.

At point B,

$$\left.\frac{\partial n}{\partial x}\right|_{(B)} \approx \frac{0.75-1}{135-115} = -\frac{1/4}{20} = -\frac{1}{80} \approx -0.01 \quad \frac{\text{foxes/km}^2}{\text{km}}$$

$$\left.\frac{\partial n}{\partial y}\right|_{(B)} \approx \frac{0.5-1}{120-110} = -\frac{1/2}{10} = -\frac{1}{20} \approx -0.05 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

So, fox population density decreases as we move both east and north of B. However, notice that the partial derivative $\partial n/\partial x$ at B is smaller in magnitude than the others. Indeed if we had taken a negative Δx we would have obtained an estimate of the opposite sign. This suggests that better estimates for$B$ are

$$\left.\frac{\partial n}{\partial x}\right|_{(B)} \approx 0 \quad \frac{\text{foxes/km}^2}{\text{km}}$$

$$\left.\frac{\partial n}{\partial y}\right|_{(B)} \approx -0.05 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

At point C,

$$\left.\frac{\partial n}{\partial x}\right|_{(C)} \approx \frac{2-1.5}{135-115} = \frac{1/2}{20} = \frac{1}{40} \approx 0.02 \quad \frac{\text{foxes/km}^2}{\text{km}}$$

$$\left.\frac{\partial n}{\partial y}\right|_{(C)} \approx \frac{2-1.5}{80-55} = \frac{1/2}{25} = \frac{1}{50} \approx 0.02 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

So, the fox population density increases as we move east and north of C. Again, if these estimates were made using negative values for Δx and Δy we would have had estimates of the opposite sign. Thus, better estimates are

$$\left.\frac{\partial n}{\partial x}\right|_{(C)} \approx 0 \quad \frac{\text{foxes/km}^2}{\text{km}}$$

$$\left.\frac{\partial n}{\partial y}\right|_{(C)} \approx 0 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

69. The derivative $\partial c/\partial x = b$ is the rate of change of the cost of producing one unit of the product with respect to the amount of labor used (in man hours) when the amount of raw material used stays the same. Thus $\partial c/\partial x = b$ represents the hourly wage.

70. (a) The difference quotient for approximating $f_u(u, v)$ is given by

$$f_u(u,v) \approx \frac{f(u+h,v)-f(u,v)}{h}.$$

Putting $(u, v) = (1, 3)$ and $h = 0.001$, the difference quotient is

$$f_u(1,3) \approx \frac{1.001(1.001^2+3^2)^{3/2} - 1(1^2+3^2)^{3/2}}{0.001}$$
$$\approx \frac{31.6639 - 31.6228}{0.001} \approx \frac{0.0411}{0.001} \approx 41.1$$

(b) Using the derivative formulas

$$f_u = \frac{\partial f}{\partial u} = (u^2+v^2)^{3/2} + 3u^2(u^2+v^2)^{1/2} = (u^2+v^2)^{1/2}(4u^2+v^2)$$

so

$$f_u(1,3) = (1^2+3^2)^{1/2} \cdot (4 \cdot 1^2 + 3^2) \approx 41.11$$

We see that the approximation in part (a) was reasonable.

71. Substituting for G, M, m, and r, we have

$$F = \frac{GMm}{r^2} = \frac{(6.67 \cdot 10^{-11})(6 \cdot 10^{24})(70)}{(6.4 \cdot 10^6)^2} = 684 \text{ newtons.}$$

The gravitational force on this person is about 684 newtons.

Differentiating gives

$$\frac{\partial F}{\partial m} = \frac{GM}{r^2} = 9.77 \text{ newtons/kg}$$

and

$$\frac{\partial F}{\partial r} = \frac{-2GMm}{r^3} = -0.000214 \text{ newtons/meter.}$$

These partial derivatives tell us that the gravitational force increases by about 9.77 newtons for an increase of 1 kg in the mass, and the gravitational force decreases by about 0.000214 newtons if the distance from the center of the earth increases by 1 meter.

72. (a) The area of a circle of radius r is given by

$$A = \pi r^2$$

and the perimeter is

$$L = 2\pi r.$$

Thus we get

$$r = \frac{L}{2\pi}$$

and

$$A = \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2\pi}{4\pi^2} = \frac{L^2}{4\pi}.$$

Thus we get

$$\pi = \frac{L^2}{4A}.$$

(b) We will treat π as a function of L and A.

$$d\pi = \frac{\partial \pi}{\partial L}dL + \frac{\partial \pi}{\partial A}dA = \frac{2L}{4A}dL - \frac{L^2}{4A^2}dA.$$

If L is in error by a factor λ, then $\Delta L = \lambda L$, and if A is in error by a factor μ, then $\Delta A = \mu A$. Therefore,

$$\Delta\pi \approx \frac{2L}{4A}\Delta L - \frac{L^2}{4A^2}\Delta A$$
$$= \frac{2L}{4A}\lambda L - \frac{L^2}{4A^2}\mu A$$
$$= \frac{2\lambda L^2}{4A} - \frac{\mu L^2}{4A} = (2\lambda - \mu)\frac{L^2}{4A} = (2\lambda - \mu)\pi,$$

so π is in error by a factor of $2\lambda - \mu$.

73. (a) The linear approximation

$$\Delta F \approx F_x \Delta x + F_y \Delta y$$

gives, for $x = 400$ and $y = 50$,

$$\Delta F \approx 40x^{-1/3}y^{1/3}\Delta x + 20x^{2/3}y^{-2/3}\Delta y = 20\Delta x + 80\Delta y.$$

The company increases its skilled labor by 5 hours, so $\Delta y = 5$. Since output is to remain constant, we have $\Delta F = 0$. Making these substitutions in the linear approximation we get

$$20\Delta x + 80 \cdot 5 = 0, \quad \text{so } \Delta x = -20.$$

With 5 additional hours per day of skilled labor and 20 fewer hours per day of unskilled labor, the company can keep its output at the current level.

(b) When $x = 400$ and $y = 50$, output is given by

$$F(400, 50) = 60 \cdot 400^{2/3} \cdot 50^{1/3} = 12{,}000$$

When y is increased to 55, if output remains constant, we have

$$12{,}000 = 60 \cdot x^{2/3} \cdot 55^{1/3}.$$

Solving for x gives

$$x = \left(\frac{12000}{60 \cdot 55^{1/3}}\right)^{3/2} = 381.385 \text{ hours per day.}$$

Thus, the number of hours of unskilled labor is reduced by $400 - 381.385 = 18.615$ hours per day.

74. (a) If the volume is held constant, $\Delta V = 0$, so $\Delta U \approx 27.32\Delta T$. Thus the energy increases if the temperature increases.

(b) If the temperature is held constant, then $\Delta T = 0$, so $\Delta U \approx 840\Delta V$. Thus the energy increases if the volume increases (yes, it sounds bizarre, but remember the temperature is being held constant).

(c) First, we convert 100 cm^3 to 0.0001 m^3. Now, using the differential approximation,

$$\begin{aligned}\Delta U &\approx 840\,\Delta V + 27.32\,\Delta T \\ &= (840)(-0.0001) + (27.32)(2) \\ &= -0.084 + 54.64 \approx 55 \text{ joules.}\end{aligned}$$

75. Since $\text{grad}\, f = f_x\vec{i} + f_y\vec{j}$, we see that f_x is given by the $\vec{i}$-component of $\text{grad}\, f$ and f_y is given by the $\vec{j}$-component of $\text{grad}\, f$. Also, f_{xx} is the rate of change of f_x in the x-direction and f_{yy} is the rate of change of f_y in the y-direction.

(a) At P, S, we have $f_x > 0$.

(b) At R, S, we have $f_y < 0$.

(c) At all four points, P, Q, R, S, we have $f_{xx} > 0$ because f_x increases as we move in the x-direction. (At P, S, we see f_x is positive and gets larger; at Q, R, we see f_x is negative and getting less negative.)

(d) No points. At P, Q, the value of f_y is positive and increasing as y increases. At R, S, the value of f_y is negative and increasing (getting less negative) as y increases.

76. The directional derivative is approximately the change in z (as we move in direction $\vec{v}$) divided by the horizontal change in position. The directional derivative is $f_{\vec{i}} \approx \dfrac{2-1}{4-1} = \dfrac{1}{3} \approx 0.3$.

77. The directional derivative is approximately the change in z (as we move in direction $\vec{v}$) divided by the horizontal change in position. The directional derivative is $f_{\vec{j}} \approx \dfrac{2-1}{4-1} = \dfrac{1}{3} \approx 0.3$.

78. The directional derivative is approximately the change in z (as we move in direction $\vec{v}$) divided by the horizontal change in position. In the direction of $\vec{v}$, the directional derivative $\approx \dfrac{2-1}{\|\vec{i} + \vec{j}\|} = \dfrac{1}{\sqrt{2}} \approx 0.7$.

79. The directional derivative is approximately the change in z (as we move in direction $\vec{v}$) divided by the horizontal change in position. In the direction of $\vec{v}$, the directional derivative $\approx \dfrac{3.2-2}{\|\vec{i} + \vec{j}\|} = \dfrac{1.2}{\sqrt{2}} \approx 0.8$.

80. The directional derivative is approximately the change in z (as we move in direction $\vec{v}$) divided by the horizontal change in position. In the direction of $\vec{v}$ we go from point $(3, 3)$ to point $(1, 4)$. We have the directional derivative $\approx \dfrac{2-3}{\|-2\vec{i} + \vec{j}\|} = \dfrac{-1}{\sqrt{5}} \approx -0.4$.

81. The directional derivative is approximately the change in z (as we move in direction $\vec{v}$) divided by the horizontal change in position. In the direction of $\vec{v}$ we go from point $(4,1)$ to point $(2,2)$, and the value of z remains unchanged. We have the directional derivative ≈ 0.

82. The gradient vectors are perpendicular to the level curves. To determine the length of the gradient vector, we estimate the rate of change of the function from the contour diagram. At $(1,1)$, the value of f changes from 1 to 2 in a distance of $\sqrt{2}$ (as it moves from $(1,1)$ to $(2,2)$), so the length of grad f is $\frac{1}{\sqrt{2}} \approx 0.7$. At $(1,4)$, the value of f changes slightly faster (the lines are closer together), so grad f is slightly longer here. In fact, the value of f changes from 2 to 3 as we move from $(1,4)$ to $(2.2, 4.2)$ a distance of about 1.2. So the new length is $1/1.2 = 0.8$.

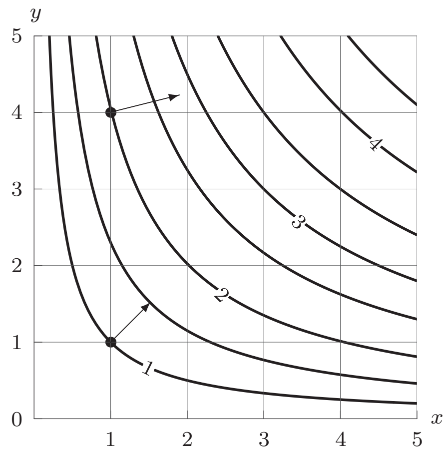


Figure 14.52: Gradient Vectors

83. (a) The gradient vector at the point $x = 1, y = 2$ is

$$\nabla z = \nabla(x^2 y) = 2xy\vec{i} + x^2\vec{j} = 4\vec{i} + \vec{j}.$$

The unit vector making an angle of $5\pi/4$ with the x-axis is

$$\vec{u} = \cos\frac{5\pi}{4}\vec{i} + \sin\frac{5\pi}{4}\vec{j} = -\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}.$$

The directional derivative in this direction is

$$f_{\vec{u}}(1,2) = \nabla z \cdot \vec{u} = (4\vec{i} + \vec{j}) \cdot \frac{\sqrt{2}}{2}(-\vec{i} - \vec{j}) = -\frac{\sqrt{2}}{2}(4+1) = -\frac{5\sqrt{2}}{2}.$$

(b) The directional derivative is a maximum in the direction of the gradient vector $\nabla z = 4\vec{i} + \vec{j}$.

84. (a) Incorrect. $\|\text{grad}\, H\|$ is not the rate of change of H. In fact, there's no such thing as the rate of change of H, although directional derivatives can give its rate of change in a particular direction. For example, this expression would give the wrong answer if the ant was crawling along a contour of H, since then the rate of change of the temperature it experiences is zero even though $\|\text{grad}\, H\|$ and $\vec{v}$ might not be zero.

(b) Correct. If $\vec{u} = \vec{v}/\|\vec{v}\|$, then

$$\begin{aligned}\text{grad}\, H \cdot \vec{v} &= \text{grad}\, H \cdot \frac{\vec{v}}{\|\vec{v}\|}\|\vec{v}\| = (\text{grad}\, H \cdot \vec{u})\|\vec{v}\| = H_{\vec{u}}\|\vec{v}\| \\ &= \text{(Rate of change of } H \text{ in direction } \vec{u} \text{ in deg/cm)(Speed of ant in cm/sec)} \\ &= \text{Rate of change of } H \text{ in deg/sec.}\end{aligned}$$

(c) Incorrect, this is the directional derivative, which gives the rate of change with respect to distance, not time.

85. (a) We choose a coordinate system with the origin at the buoy, the x-axis pointing east and the y-axis pointing north. Then the boat is at the point with position vector $\vec{i} - 2\vec{j}$. So a vector pointing in the direction of the buoy from the boat's position is $\vec{0} - (\vec{i} - 2\vec{j}) = -\vec{i} + 2\vec{j}$, and a unit vector in this direction is $\vec{u} = (1/\sqrt{5})(-\vec{i} + 2\vec{j})$. We have

$$\text{grad}\, h = -30 \cdot 2x\vec{i} - 20 \cdot 2y\vec{j} = -60x\vec{i} - 40y\vec{j},$$

so at the boat's position, $x = 1$ and $y = -2$, we have

$$\text{grad}\, h = -60 \cdot 1\vec{i} - 40(-2)\vec{j} = -60\vec{i} + 80\vec{j}.$$

So the directional derivative of the depth in the direction of the buoy is

$$\text{grad}\, h \cdot \vec{u} = (-60\vec{i} + 80\vec{j}) \cdot \frac{1}{\sqrt{5}}(-\vec{i} + 2\vec{j}) = \frac{60 + 160}{\sqrt{5}} = 98.387 \text{ ft/mile}.$$

So the depth is increasing at a rate of 98.387 ft/mile.

(b) The boat is moving at 3 mph and from part (a) we know that the depth is changing at 98.387 ft/mi. So

$$\begin{aligned}\text{Rate of change of depth with respect to time} &= \text{(Speed)(Rate of change of depth with respect to distance)} \\ &= 3\frac{\text{miles}}{\text{hour}} 98.387 \frac{\text{ft}}{\text{mile}} = 295.161 \text{ ft/hour}.\end{aligned}$$

86. (a) The vector $\text{grad}\, f = 2\vec{i} - 5\vec{j}$ is perpendicular to the level curve at the point $(1, 3)$. Now the vector $5\vec{i} + 2\vec{j}$ is perpendicular to the vector $2\vec{i} - 5\vec{j}$. Thus, the vector $5\vec{i} + 2\vec{j}$ is tangent to curve. (There are many other vectors with this property, such as $-5\vec{i} - 2\vec{j}$, $10\vec{i} + 4\vec{j}$, etc.) The slope of the tangent line is therefore $2/5$. Since the tangent line goes through the point $(1, 3)$, its equation is

$$y - 3 = \frac{2}{5}(x - 1)$$

or

$$y = \frac{2}{5}x + \frac{13}{5}.$$

(b) The surface $z = f(x, y)$ can be written in the form

$$F(x, y, z) = f(x, y) - z = 0.$$

The normal the this surface is

$$\text{grad}\, F = f_x\vec{i} + f_y\vec{j} - \vec{k}.$$

Thus, at the point $(1, 3, 7)$, the normal is

$$\text{grad}\, F(1, 3, 7) = 2\vec{i} - 5\vec{j} - \vec{k}.$$

Thus, the equation of the tangent plane is

$$2x - 5y - z = 2(1) - 5(3) - 7 = -20$$

$$2x - 5y - z + 20 = 0.$$

87. (a) We want the partial derivative with respect to x at the point $(2, 1, 5)$, so

$$\left.\frac{\partial H}{\partial x}\right|_{(2,1,5)} = -2xe^{-(x^2+2y^2+3z^2)}\Big|_{(2,1,5)} = -4e^{-81}\ {}^\circ\text{C/meter}.$$

(b) By the chain rule,

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial t} = -4e^{-81} \cdot 10 = -40e^{-81}\ {}^\circ\text{C/sec}.$$

(c) The magnitude of the gradient gives the maximum rate of change, so

$$\text{grad}\, H\Big|_{(2,1,5)} = -e^{-(x^2+2y^2+3z^2)}(2x\vec{i} + 4y\vec{j} + 6z\vec{k})\Big|_{(2,1,5)} = -e^{-81}(4\vec{i} + 4\vec{j} + 30\vec{k}).$$

Thus

$$||\,\text{grad}\, H|| = e^{-81}\sqrt{4^2 + 4^2 + 30^2} = \sqrt{932}e^{-81}\ {}^\circ\text{C/meter}.$$

88. The point $(4,1,3)$ lies on the surface. The surface is the level surface of the function

$$F(x,y,z) = f(x,y) - z = 0.$$

The normal to the surface at the point $(4,1,3)$ is

$$\text{grad}\, F(4,1,3) = f_x(4,1)\vec{i} + f_y(4,1)\vec{j} - \vec{k} = 2\vec{i} - \vec{j} - \vec{k}.$$

Thus the equation of the tangent plane is

$$2x - y - z = 2(4) - 1 - 3 = 4$$
$$2x - y - z = 4$$

89. The temperature increases fastest in the direction of the gradient vector, namely $\text{grad}\, T = -2x\vec{i} - 2y\vec{j}$. Figure 14.53 shows that the gradient vector points radially inward.

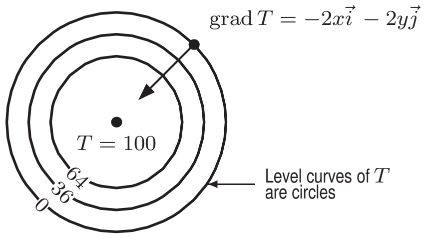


Figure 14.53

90. (a) If $\vec{j}$ points north and $\vec{i}$ points east, then the direction the car is driving is $\vec{j} - \vec{i}$. A unit vector in this direction is

$$\vec{u} = \frac{1}{\sqrt{2}}(\vec{j} - \vec{i}).$$

The gradient of the height function is

$$\text{grad}\, h = \frac{\partial h}{\partial E}\vec{i} + \frac{\partial h}{\partial N}\vec{j} = 50\vec{i} + 100\vec{j}.$$

So the directional derivative is

$$h_{\vec{u}} = \text{grad}\, g \cdot \vec{h} = (50\vec{i} + 100\vec{j}) \cdot \frac{1}{\sqrt{2}}(\vec{j} - \vec{i}) = \frac{100 - 50}{\sqrt{2}} = 35.355 \text{ ft/mi}.$$

(b) The car is traveling at v mi/hr, so

$$\text{Rate of change of with respect to time} = v\frac{\text{miles}}{\text{hour}} 35.355 \frac{\text{ft}}{\text{mile}} = 35.355v \text{ ft/hour}.$$

91. At points (x,y) where the gradients are defined and are not the zero vector, the level curves of f and g intersect at right angles if and only $\text{grad}\, f \cdot \text{grad}\, g = 0$.

We have

$$\begin{aligned}\text{grad}\, f \cdot \text{grad}\, g &= \left(\left(\frac{x}{\sqrt{x^2+y^2}} + 1\right)\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j}\right) \cdot \left(\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right)\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j}\right) \\ &= \left(\frac{x^2}{x^2+y^2} - 1\right) + \frac{y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} - 1 = 0.\end{aligned}$$

The level curves of f and g are parabolas that cross at right angles except at the point $(0,0)$. See Figure 14.54.

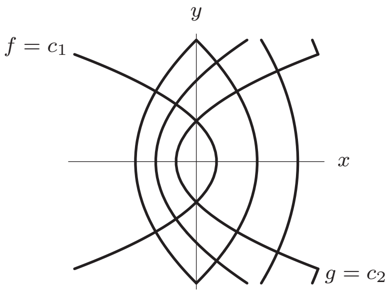


Figure 14.54

92. Since $V = m/r$, where r is the distance from (x, y, z) to (x_0, y_0, z_0), we have $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, so

$$\begin{aligned}
V &= \frac{m}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}, \\
\frac{\partial V}{\partial x} &= \frac{-m\,(x-x_0)}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}}, \\
\frac{\partial^2 V}{\partial x^2} &= \frac{3m\,(x-x_0)^2}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{5/2}} - \frac{m}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}}
\end{aligned}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \frac{3m\,(y-y_0)^2}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{5/2}} - \frac{m}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}}$$

and,

$$\frac{\partial^2 V}{\partial z^2} = \frac{3m\,(z-z_0)^2}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{5/2}} - \frac{m}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}},$$

So

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= \frac{3m\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{5/2}} - 3\frac{m}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\
&= 0
\end{aligned}$$

93. Write $V = f(u)$ where $u = x + ct$, then using the chain rule

$$\frac{\partial V}{\partial x} = \frac{df}{du} \cdot \frac{\partial u}{\partial x} = f'(u)(1).$$

Similarly,

$$\frac{\partial V}{\partial t} = \frac{df}{du} \cdot \frac{\partial u}{\partial t} = f'(u)(c) = cf'(u).$$

Thus

$$\frac{\partial V}{\partial t} = cf'(u) = c\frac{\partial V}{\partial x}.$$

94. (a) The chain rule gives

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (2u - e^v)1 + (-ue^v)2.$$

At $(x, y) = (1, 2)$, we have $u = 1 + 2 \cdot 2 = 5$ and $v = 2 \cdot 1 - 2 = 0$, so

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,2)} = (2 \cdot 5 - e^0)1 - 5e^0 \cdot 2 = -1.$$

(b) The chain rule gives

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = (2u - e^v)2 + (-ue^v)(-1).$$

At $(x, y) = (1, 2)$, we have $(u, v) = (5, 0)$, so

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,2)} = (2 \cdot 5 - e^0)2 + 5e^0 = 23.$$

95. All are done using the chain rule.

(a) We have $u = x$, $v = y$, $w = 3$. Thus $\partial u/\partial x = 1$, $\partial v/\partial x = 0$, $\partial w/\partial x = 0$, so

$$G_x(x, y) = F_u(x, y, 3)(1) + F_v(x, y, 3)(0) + F_w(x, y, 3)(0) = F_u(x, y, 3).$$

(b) We have $u = 3$, $v = y$, $w = x$. Thus $\partial u/\partial x = 0$, $\partial v/\partial x = 0$, $\partial w/\partial x = 1$, so

$$G_x(x, y) = F_u(3, y, x)(0) + F_v(3, y, x)(0) + F_w(3, y, x)(1) = F_w(3, y, x).$$

(c) We have $u = x$, $v = y$, $w = x$. Thus $\partial u/\partial x = 1$, $\partial v/\partial x = 0$, $\partial w/\partial x = 1$, so

$$G_x(x, y) = F_u(x, y, x)(1) + F_v(x, y, x)(0) + F_w(x, y, x)(1) = F_u(x, y, x) + F_w(x, y, x).$$

(d) We have $u = x$, $v = y$, $w = xy$. Thus $\partial u/\partial x = 1$, $\partial v/\partial x = 0$, $\partial w/\partial x = y$, so

$$\begin{aligned} G_x(x, y) &= F_u(x, y, xy)(1) + F_v(x, y, xy)(0) + F_w(x, y, xy)(y) \\ &= F_u(x, y, xy) + yF_w(x, y, xy). \end{aligned}$$

96. Average productivity increases as x_1 increases if $\frac{\partial}{\partial x_1}(\text{average productivity}) > 0$. Now

$$\begin{aligned} \frac{\partial}{\partial x_1}(\text{average productivity}) &= \frac{\partial}{\partial x_1}\left(\frac{P}{x_1}\right) \\ &= \frac{1}{x_1}\frac{\partial P}{\partial x_1} + P\frac{\partial}{\partial x_1}\left(\frac{1}{x_1}\right) \\ &= \frac{1}{x_1}\frac{\partial P}{\partial x_1} - \frac{P}{x_1^2} = \frac{1}{x_1}\left(\frac{\partial P}{\partial x_1} - \frac{P}{x_1}\right) \end{aligned}$$

So $\frac{\partial}{\partial x_1}(\text{average productivity}) > 0$ means that $\left(\frac{\partial P}{\partial x_1} - \frac{P}{x_1}\right) > 0$, i.e.,

$$\frac{\partial P}{\partial x_1} > \frac{P}{x_1}.$$

97. The differential is

$$dP = \frac{\partial P}{\partial L}\,dL + \frac{\partial P}{\partial K}\,dK = 10L^{-0.75}K^{0.75}\,dL + 30L^{0.25}K^{-0.25}dK.$$

When $L = 2$ and $K = 16$, this is

$$dP \approx 47.6\,dL + 17.8\,dK.$$

98. The error, dT, in the period T is given by

$$dT = \frac{\partial T}{\partial l}dl + \frac{\partial T}{\partial g}dg,$$

where

$$\frac{\partial T}{\partial l} = \frac{\sqrt{\frac{l}{g}}\,\pi}{l}$$

and

$$\frac{\partial T}{\partial g} = -\frac{\sqrt{\frac{l}{g}}\,\pi}{g},$$

so that

$$T_l(2, 9.8) = 0.7096, \quad T_g(2, 9.8) = -0.1448.$$

We also have that

$$dl = -0.01, \quad dg = 0.01.$$

The maximum discrepancy in the period is then given by

$$dT = 0.7096(-0.01) - 0.1448(0.01) = -0.008544.$$

99. Looking at the contour diagram, we see that the contours are almost straight lines that are reasonably evenly spaced, so treating the function as linear is not a bad approximation. The monthly payment m is a function of two variables, P, the dollars you borrow, and r, the interest rate. We are going to use the formula

$$m(P, r) \approx m(P_0, r_0) + m_p(P_0, r_0)(P - P_0) + m_r(P_0, r_0)(r - r_0)$$

to approximate the payment $m(P, r)$. The amount of \$6000 and the interest rate 11% would probably be reasonable choices for our P_0 and r_0.
Directly from the Figure 12.8 on page 691, we have

$$m(6000, 0.11) = \$130$$

Next we approximate $m_p(6000, 0.11)$ by a difference quotient.

$$m_p(6000, 0.11) \approx \frac{m(6000 + h, 0.11) - m(6000, 0.11)}{h}$$

Here we choose $h = 500$, and we get $m(6500, 0.11) = 140$ from the figure, so

$$m_p(6000, 0.11) \approx \frac{140 - 130}{500} = 0.02$$

Next we approximate $m_r(6000, 0.11)$ by taking $h = 0.03$ and $m(6000, 0.11 + 0.03) = m(6000, 0.14) = 140$ from the figure,

$$m_r(6000, 0.11) \approx \frac{m(6000, 0.11 + h) - m(6000, 0.11)}{h} = \frac{140 - 130}{0.03} \approx 333.33$$

Thus we have:

$$\begin{aligned} m(P, r) &\approx 130 + 0.02(P - 6000) + 333.33(r - 0.11) \\ &= -26.67 + 0.02P + 333.33r \quad \text{in dollars} \end{aligned}$$

The constants in the answer tell us several useful things. The slope of 0.02 along the P axis tells us how much our monthly payment will increase if we decide to borrow more money; we can expect to pay about 2 cents on each extra dollar every month. The slope in the r direction tells us how much our payment will change if the interest rate changes. We can expect to pay an extra \$333.33 each month for each point the interest rate goes up. The constant c is a gauge of the non-linearity of this function. We know that if we borrow no money at zero percent interest, we should expect to not have a monthly payment ($m = 0$). The constant we calculated is negative which implies that the bank will pay us \$26.67 each month if we do not borrow any money! So, we should bear in mind that the function we have calculated is only useful close to the point at which we made the approximation, in this case $P_0 = 6000$ and $r_0 = 11\%$.

100. (a) The local linearization of f near $(2, 1)$ is $7 - 3(x - 2) + 4(y - 1)$, so the equation of the tangent plane is:

$$z = 7 - 3(x - 2) + 4(y - 1).$$

(b) We use local linearity. The contour is $f(x, y) = 7$, so to find the tangent line to the contour, we set the local linearization for f equal to 7:

$$7 - 3(x - 2) + 4(y - 1) = 7$$

Thus, an equation for the tangent line is

$$-3(x - 2) + 4(y - 1) = 0.$$

101. (a) Rewrite the equation of the tangent plane:

$$-3(x-2)+4(y-1)-(z-7)=0.$$

Then we see a normal vector to the plane is $-3\vec{i}+4\vec{j}-\vec{k}$.

(b) Since the gradient is perpendicular to contours, a normal vector to the tangent line is $-3\vec{i}+4\vec{j}$. Or, setting the local linearization for f equal to 7, we get that the equation for the tangent line is $-3(x-2)+4(y-1)=0$.Thus a normal vector to the tangent line is $-3\vec{i}+4\vec{j}$.

102. The directional derivative, or slope, of f at $(2,1)$ in the direction perpendicular to these contours is the length of the gradient. At the point $(2,1)$, we have grad $f=-3\vec{i}+4\vec{j}$, so $\|\text{grad } f\|=5$. Thus if d is the distance between the contours, we have

$$\frac{7.3-7}{d}=\text{Slope in direction perpendicular to contours}=5.$$

Thus $d=0.3/5=0.06$.

103. Using local linearity with the slope in the x direction of -3 and slope in the y direction of 4, we get the values in Table 14.9.

Table 14.9

		y		
		0.9	1.0	1.1
x	1.8	7.2	7.6	8.0
	2.0	6.6	7.0	7.4
	2.2	6.0	6.4	6.8

104. See Figure 14.55.

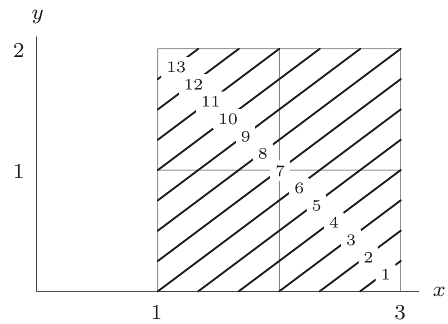


Figure 14.55

105. The bug is heading in the direction of − grad f. In that direction, the rate of cooling in degrees per *centimeter* is given by the magnitude of the gradient, which is 5. Since the bug is moving 3 centimeters per minute, in degrees per *minute*, we have:

$$\text{Rate of cooling}=5\frac{\text{degrees }^\circ\text{C}}{\text{cm}}\cdot 3\frac{\text{cm}}{\text{min}}=15\frac{\text{degrees }^\circ\text{C}}{\text{min}}.$$

106. By the chain rule,

$$f_r(2,1)=f_x(2,1)\cos\theta+f_y(2,1)\sin\theta$$
$$f_\theta(2,1)=f_x(2,1)(-r\sin\theta)+f_y(2,1)(r\cos\theta).$$

Since $x^2+y^2=r^2$, we have $r=\sqrt{2^2+1^2}=\sqrt{5}$, and $\cos\theta=x/r=2/\sqrt{5}$, and $\sin\theta=y/r=1/\sqrt{5}$. Thus

$$f_r(2,1)=(-3)\frac{2}{\sqrt{5}}+(4)\frac{1}{\sqrt{5}}=-\frac{2}{\sqrt{5}}$$
$$f_\theta(2,1)=(-3)(-1)+(4)(2)=11.$$

For $\vec{u}=(1/\sqrt{5})(2\vec{i}+\vec{j})$, we have $f_{\vec{u}}(2,1)=-2/\sqrt{5}$. Since $\vec{u}$ is pointing radially out from the origin to the point $(2,1)$, we expect that $f_{\vec{u}}(2,1)=f_r(2,1)$.

107. To increase f as much as possible, we should head in the direction of the gradient from the point $(2,1)$. The rate of increase of f in the direction of the gradient is the magnitude of the gradient. Since at the point $(2,1)$, we know $\text{grad}\, f = -3\vec{i} + 4\vec{j}$, the magnitude is 5. The furthest we can go from $(2,1)$, inside the circle, is 0.1 units, so the most we can increase f is $(0.1)(5) = 0.5$. Thus

$$\text{Largest value of function} \approx f(2,1) + 0.5 = 7.5.$$

This value is achieved at the point obtained from $(2,1)$ by a displacement of 0.1 units in the direction of $\text{grad}\, f$, that is, a displacement by the vector

$$(0.1)\frac{\text{grad}\, f}{\|\,\text{grad}\, f\|} = (0.1)((-3/5)\vec{i} + (4/5)\vec{j}\,) = -0.06\vec{i} + 0.08\vec{j}\,.$$

Thus, the largest value of f is achieved at the point $(2-0.06, 1+0.08) = (1.94, 1.08)$.

108. (a) Fix $y = 3$. When x changes from 2.00 to 2.01, $f(x,3)$ decreases from 7.56 to 7.42. So

$$\left.\frac{\partial f}{\partial x}\right|_{(2,3)} \approx \left.\frac{\Delta f}{\Delta x}\right|_{(2,3)} = \frac{7.42 - 7.56}{2.01 - 2.00} = \frac{-0.14}{0.01} = -14.$$

Fix $x = 2$, when y changes from 3.00 to 3.02, $f(2,y)$ increases from 7.56 to 7.61. So

$$\left.\frac{\partial f}{\partial y}\right|_{(2,3)} \approx \left.\frac{\Delta f}{\Delta y}\right|_{(2,3)} = \frac{7.61 - 7.56}{3.02 - 3.00} = \frac{0.05}{0.02} = 2.5.$$

(b) Since the unit vector $\vec{u}$ of the direction $\vec{i} + 3\vec{j}$ is

$$\vec{u} = \frac{\vec{i} + 3\vec{j}}{\|\vec{i} + 3\vec{j}\,\|} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}\,,$$

$$\begin{aligned} f_{\vec{u}}\,(2,3) &= \text{grad}\, f(2,3)\cdot\vec{u} \approx \left(\left.\frac{\Delta f}{\Delta x}\right|_{(2,3)}\vec{i} + \left.\frac{\Delta f}{\Delta y}\right|_{(2,3)}\vec{j}\right)\cdot\vec{u} \\ &= (-14\vec{i} + 2.5\vec{j}\,)\cdot\left(\frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}\right) = -\frac{6.5}{\sqrt{10}} \approx -2.055. \end{aligned}$$

(c) Maximum rate equals $\|\,\text{grad}\, f\| \approx \sqrt{(-14)^2 + (2.5)^2} \approx 14.221$ in the direction of the gradient which is approximately equal to $-14\vec{i} + 2.5\vec{j}$.

(d) The equation of the level curve is

$$f(x,y) = f(2,3) = 7.56.$$

(e) The vector must be perpendicular to $\text{grad}\, f$, so $\vec{v} = 2.5\vec{i} + 14\vec{j}$ is a possible answer. (There are many others.).

(f) The differential at the point $(2,3)$ is

$$df = -14\,dx + 2.5\,dy.$$

If $dx = 0.03$, $dy = 0.04$, we get

$$df = -14(0.03) + 2.5(0.04) = -0.32.$$

The df approximates the change in f when (x,y) changes from $(2,3)$ to $(2.03, 3.04)$.

109. Let us first collect the computations that we will need.

$$\begin{aligned} f(x,y) &= \cos(x+2y)\sin(x-y), \\ f_x(x,y) &= \cos(x+2y)\cos(x-y) - \sin(x-y)\sin(x+2y) \\ &= \cos(x+2y+x-y) = \cos(2x+y), \\ f_y(x,y) &= -\cos(x+2y)\cos(x-y) - 2\sin(x-y)\sin(x+2y) \\ &= -\cos(x+2y-(x-y)) - \sin(x+2y)\sin(x-y) \\ &= -\cos(3y) + \frac{1}{2}\left[\cos(2x+y) - \cos(3y)\right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\cos(2x+y) - \frac{3}{2}\cos(3y),\\ f_{xx}(x,y) &= -2\sin(2x+y),\\ f_{xy}(x,y) &= -\sin(2x+y),\\ f_{yy}(x,y) &= -\frac{1}{2}\sin(2x+y) + \frac{9}{2}\sin(3y). \end{aligned}$$

Then

$$\begin{aligned} f(0,0) &= 0,\\ f_x(0,0) &= 1,\\ f_y(0,0) &= -1,\\ f_{xx}(0,0) &= 0,\\ f_{xy}(0,0) &= 0,\\ f_{yy}(0,0) &= 0. \end{aligned}$$

Hence the quadratic Taylor polynomial $P(x,y)$ of $f(x,y)$ at $(0,0)$ is

$$\begin{aligned} P(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y\\ &\quad + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2\\ &= x - y. \end{aligned}$$

110. (a) The first-order Taylor polynomial of a function f about a point (a,b) is equal to

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Computing the partial derivatives, we get:

$$\begin{aligned} f_x &= 2(x-1)e^{(x-1)^2+(y-3)^2}\\ f_y &= 2(y-3)e^{(x-1)^2+(y-3)^2}\\ f_x(0,0) &= 2(-1)e^{(-1)^2+(-3)^2}\\ &= -2e^{10}\\ f_y(0,0) &= 2(-3)e^{(-1)^2+(-3)^2}\\ &= -6e^{10} \end{aligned}$$

Thus,

$$f(x,y) \approx e^{10} - 2e^{10}x - 6e^{10}y$$

(b) The second-order Taylor polynomial of a function f about the point $(1,3)$ is given by

$$\begin{aligned} &f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)\\ &\quad + \frac{1}{2}f_{xx}(1,3)(x-1)^2 + f_{xy}(1,3)(x-1)(y-3) + \frac{1}{2}f_{yy}(1,3)(y-3)^2. \end{aligned}$$

Computing the partial derivatives, we get:

$$\begin{aligned} f_x &= 2(x-1)e^{(x-1)^2+(y-3)^2}\\ f_y &= 2(y-3)e^{(x-1)^2+(y-3)^2}\\ f_{xx} &= (4(x-1)^2+2)e^{(x-1)^2+(y-3)^2}\\ f_{xy} &= 4(x-1)(y-3)e^{(x-1)^2+(y-3)^2}\\ f_{yy} &= (4(y-3)^2+2)e^{(x-1)^2+(y-3)^2} \end{aligned}$$

Substituting in the point $(1,3)$ to these partial derivatives, we get:

$$f_x(1,3) = 0$$

$$\begin{aligned} f_y(1,3) &= 0 \\ f_{xy}(1,3) &= 0 \\ f_{xx}(1,3) &= (4(0)^2+2)e^{0^2+0^2} = 2 \\ f_{yy}(1,3) &= (4(0)^2+2)e^{0^2+0^2} = 2 \end{aligned}$$

Thus,

$$f(x,y) \approx e^0 + 0(x-1) + 0(y-3) + \frac{2}{2}(x-2)^2 + 0(x-1)(y-3) + \frac{2}{2}(y-3)^2$$

$$f(x,y) \approx 1 + (x-1)^2 + (y-3)^2.$$

(c) A vector perpendicular to the level curve is grad f. At the point $(0,0)$, we have

$$\text{grad } f = f_x(0,0)\vec{i} + f_y(0,0)\vec{j}$$

Computing partial derivatives, we have

$$\begin{aligned} f_x &= 2(x-1)e^{(x-1)^2+(y-3)^2} \\ f_y &= 2(y-3)e^{(x-1)^2+(y-3)^2} \\ f_x(0,0) &= 2(-1)e^{(-1)^2+(-3)^2} \\ &= -2e^{10} \\ f_y(0,0) &= 2(-3)e^{(-1)^2+(-3)^2} \\ &= -6e^{10} \end{aligned}$$

Therefore, a perpendicular vector is grad $f = -2e^{10}\vec{i} - 6e^{10}\vec{j}$. Any multiple of grad f, say $-2\vec{i} - 6\vec{j}$, will do.

(d) Since the surface can be represented by the level surface

$$F(x,y,z) = f(x,y) - z = 0,$$

a vector perpendicular to the surface at $(0,0)$ is given by

$$\text{grad } F = f_x(0,0)\vec{i} + f_y(0,0)\vec{j} - \vec{k} = -2e^{10}\vec{i} - 6e^{10}\vec{j} - \vec{k}$$

111. (a) $f_t(\frac{\pi}{2}, \frac{\pi}{2})$ is negative since we move from zero contour of f to negative contours as t slightly increases and $x = \frac{\pi}{2}$. $f_t(\frac{\pi}{2}, \pi)$ is positive since f increases with time t from $t = \pi$ and $x = \frac{\pi}{2}$.

When $f_t(\frac{\pi}{2}, b)$ is positive, the point on the string at $x = \frac{\pi}{2}$ is moving upward at $t = b$. It moves downward for $f_t(\frac{\pi}{2}, b)$ negative at $t = b$.

(b) $f_t(\frac{\pi}{2}, t)$ is positive for $\pi < t < 2\pi$. For these t, f moves to larger and larger contours at $x = \frac{\pi}{2}$.

(c) For any fixed values of t between 0 and $\frac{\pi}{2}$, f increases with x for x between 0 and $\frac{3\pi}{2}$. Also for any fixed t between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$, f increases with x for x between 0 and $\frac{3\pi}{2}$. Hence, $f_x(x,t)$ is positive for $0 < x < 3\pi/2$ and t either satisfying $0 < t < \pi/2$ or $3\pi/2 < t < 5\pi/2$.

CAS Challenge Problems

112. (a) Using a CAS to calculate the partial derivatives, we find

$$f(0,0) = \tfrac{1}{6} \quad f_x(0,0) = \tfrac{1}{9} \quad f_y(0,0) = 1$$
$$f_{xx}(0,0) = \tfrac{-1}{54} \quad f_{xy}(0,0) = \tfrac{2}{3} \quad f_{yy}(0,0) = 3$$

Thus the quadratic approximation is

$$Q(x,y) = \frac{1}{6} + \frac{x}{9} + y - \frac{x^2}{108} + \frac{2xy}{3} + \frac{3y^2}{2}.$$

(b) The CAS gives

$$\begin{aligned} \frac{e^x}{5+e^{2x}} &= \frac{1}{6} + \frac{x}{9} - \frac{x^2}{108} + \cdots \\ (1+\sin 3y)^2 &= 1 + 6y + 9y^2 + \cdots \end{aligned}$$

Multiplying the two together, we get

$$\left(\frac{1}{6}+\frac{x}{9}-\frac{x^2}{108}+\cdots\right)(1+6y+9y^2+\cdots)=\frac{1}{6}+\frac{x}{9}+y-\frac{x^2}{108}+\frac{2xy}{3}+\frac{3y^2}{2}+\cdots$$

which is the quadratic approximation we obtained in part (a).

(c) We have

$$g(x)=g(0)+g'(0)x+\frac{g''(0)}{2}x^2+\cdots$$
$$h(y)=h(0)+h'(0)y+\frac{h''(0)}{2}y^2+\cdots$$

Multiplying these together, we get

$$g(0)h(0)+g'(0)h(0)x+g(0)h'(0)y+\frac{g''(0)}{2}h(0)x^2+g'(0)h'(0)xy+g(0)\frac{h''(0)}{2}y^2+\cdots$$

On the other hand,

$$f(0,0)=g(0)h(0),\quad f_x(0,0)=g'(0)h(0),\quad f_y(0,0)=g(0)h'(0),$$
$$f_{xx}(0,0)=g''(0)h(0),\quad f_{xy}(0,0)=g'(0)h'(0),\quad f_{yy}(0,0)=g(0)h''(0)$$

Thus the quadratic approximation to f is the same as the product of the approximations to g and h.

113. (a) We have $f(1,2)=A_0+A_1+2A_2+A_3+2A_4+4A_5$, $f_x(1,2)=A_1+2A_3+2A_4$, and $f_y(1,2)=A_2+A_4+4A_5$, so the linear approximation is

$$L(x,y)=A_0+A_1+2A_2+A_3+2A_4+4A_5+(A_1+2A_3+2A_4)(x-1)+(A_2+A_4+4A_5)(y-2).$$

Also, $m(t)=1+B_1t$ and $n(t)=2+C_1t$.

(b) Using a CAS to compute the derivatives, we find that they are both the same:

$$\frac{d}{dt}f(x(t),y(t))|_{t=0}=\frac{d}{dt}l(m(t),n(t))|_{t=0}=A_1B_1+2A_3B_1+2A_4B_1+A_2C_1+A_4C_1+4A_5C_1.$$

This can be explained using the chain rule and the fact that the derivative of a function at a point is the same as the derivative of its linear approximation there:

$$\begin{aligned}\frac{d}{dt}f(x(t),y(t))|_{t=0}&=\frac{\partial}{\partial x}f(x(0),y(0))x'(0)+\frac{\partial}{\partial y}f(x(0),y(0))y'(0)\\&=\frac{\partial}{\partial x}l(x(0),y(0))m'(0)+\frac{\partial}{\partial y}l(x(0),y(0))n'(0)=\frac{d}{dt}l(m(t),n(t))|_{t=0}\end{aligned}$$

114. (a) We have

$$\begin{aligned}f(1,2)&=A_0+A_1+2A_2+A_3+2A_4+4A_5,\\f_x(1,2)&=A_1+2A_3+2A_4,\\f_y(1,2)&=A_2+A_4+4A_5,\\f_{xx}(1,2)&=2A_3,\\f_{xy}(1,2)&=A_4,\\f_{yy}(1,2)&=2A_5.\end{aligned}$$

Thus $Q(x,y)=(A_0+A_1+2A_2+A_3+2A_4+4_A5)+(A_1+2A_3+2A_4)(x-1)+(A_2+A_4+4A_5)(y-2)+A_3(x-1)^2+A_4(x-1)(y-2)+A_5(y-2)^2$. Expanding this expression in powers of x and y, we find

$$Q(x,y)=A_0+A_1x+A_2y+A_3x^2+A_4xy+A_5y^2=f(x,y)$$

.

(b) The quadratic expansion for f about $(1,2)$ is equal to f itself. This is because f is already a quadratic function, and is true about any point (a,b).

(c) The linear approximation of $f(x,y)$ at $(1,2)$ is:

$$A_0+A_1+2A_2+A_3+2A_4+4A_5+(A_1+2A_3+2A_4)(x-1)+(A_2+A_4+4A_5)(y-2).$$

When we expand this we get

$$A_0-A_3+A_4-4A_5+(A_1+2A_3+2A_4)x+(A_2+A_4+4A_5)y.$$

This is not the same as $f(x,y)$, and it is not even the same as the linear part of $f(x,y)$, namely $A_0+A_1x+A_2y$.

115. The computer algebra system gives:

$$\frac{dw}{dt} = \frac{d}{dt} f\left(x(u(t),v(t)), y(u(t),v(t)), z(t)\right) = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

This agrees with the result obtained using the chain rule and the tree diagram in Figure 14.56. The diagram shows four paths from w to t, each corresponding to one of the terms in the expression for dw/dt.

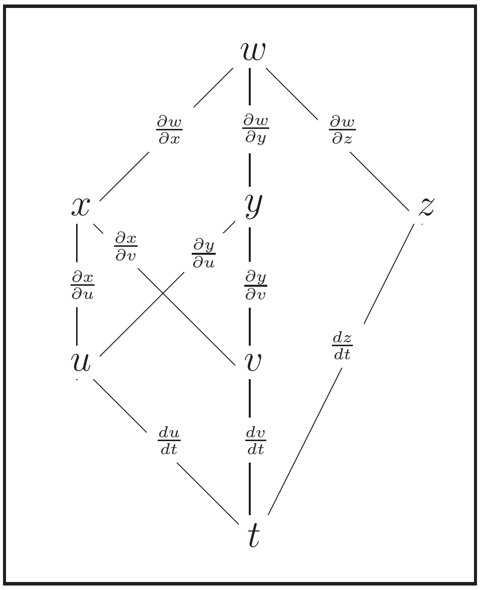


Figure 14.56

PROJECTS FOR CHAPTER FOURTEEN

1. (a) Calculating the necessary partial derivatives:

$$T(x,y,z,t) = \frac{1}{(4\pi Kt)^{3/2}} e^{-(x^2+y^2+z^2)/4Kt}$$
$$\frac{\partial T}{\partial t} = \frac{1}{(4\pi Kt)^{3/2}} e^{-(x^2+y^2+z^2)/4Kt}\left[\frac{x^2+y^2+z^2}{4Kt^2} - \frac{3}{2}\frac{1}{t}\right]$$
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{(4\pi Kt)^{3/2}} e^{-(x^2+y^2+z^2)/4Kt}\left[\frac{x^2}{4K^2t^2} - \frac{1}{2Kt}\right]$$

Thus,

$$K(T_{xx}+T_{yy}+T_{zz}) = \frac{1}{(4\pi Kt)^{3/2}} e^{-(x^2+y^2+z^2)/4Kt}\left[\frac{x^2+y^2+z^2}{4Kt^2} - \frac{3}{2t}\right] = T_t$$

(b) For $t =$ constant, the level surfaces are given by

$$x^2+y^2+z^2 = c^2, \qquad c = \text{constant}$$

which is an equation of a sphere centered at origin.

(c) With $t =$ constant, $\operatorname{grad} T = T_x\vec{i} + T_y\vec{j} + T_z\vec{k}$. Since

$$\frac{\partial T}{\partial x} = -\frac{2x}{4Kt}T = -\frac{x}{2Kt}T$$

and similarly for T_y and T_z. Thus we have

$$\begin{aligned}\operatorname{grad} T(x,y,z) &= -(x\vec{i} + y\vec{j} + z\vec{k})\frac{1}{2Kt}\frac{1}{(4\pi Kt)^{3/2}}e^{-(x^2+y^2+z^2)/4Kt} \\ &= (-\vec{r})\frac{1}{2Kt}\frac{1}{(4\pi Kt)^{3/2}}e^{-r^2/4Kt}, \quad \text{where} \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\end{aligned}$$

The heat flows toward lower temperatures, that is in the direction of $-\operatorname{grad} T$. Since $\operatorname{grad} T$ is in the direction of $-\vec{r}$, heat is flowing outward and with exponentially decreasing magnitude.

2. (a) First, note that m, n, and $(1-q)$ are all positive.

Suppose we increase m, the number of students. Then there are more birthdays to match in a fixed size year, and so q increases. Thus we expect $\partial q/\partial m$ to be positive, which is the case. On the other hand, if we increase n, the number of days in a year, then there are more slots for birthdays in a year, and so less chance of matching. This means q decreases, and so we expect $\partial q/\partial n$ to be negative, which is the case.

(b) Since we do not have an exact formula for q, we find approximate values for q using $q(23,365) = 0.5073$ and local linearization. The local linearization of q at $(23,365)$ is

$$\begin{aligned}\Delta q &\approx \frac{\partial q}{\partial m}\Delta m + \frac{\partial q}{\partial n}\Delta n \\ &= \frac{23}{365}(1-0.5073)(m-23) - \frac{23^2}{(2)(365^2)}(1-0.5073)(n-365).\end{aligned}$$

When $m = 21$ and $n = 365$ we have

$$\Delta q \approx \frac{23}{365}(1-0.5073)(21-23) - \frac{23^2}{(2)(365^2)}(1-0.5073)(365-365) = -0.0621.$$

Thus

$$\begin{aligned}q(21,365) &= q(23,365) + \Delta q \\ &\approx 0.5073 - 0.0621 \\ &= 0.4452.\end{aligned}$$

(c) When $m = 24$ and $n = 358$, we have

$$\Delta q \approx \frac{23}{365}(1-0.5073)(24-23) - \frac{23^2}{(2)(365^2)}(1-0.5073)(358-365) = 0.0379.$$

Thus

$$\begin{aligned}q(24,358) &= q(23,365) + \Delta q \\ &\approx 0.5073 + 0.0379 \\ &= 0.5452.\end{aligned}$$

(d) We want to compare the values of q when $m = 25$ and $n = 365$ and when $m = 23$ and $n = 365 - 31 = 334$. When $m = 25$ and $n = 365$ we have

$$\Delta q \approx \frac{23}{365}(1-0.5073)(25-23) - \frac{23^2}{(2)(365^2)}(1-0.5073)(365-365) = 0.0621.$$

On the other hand, when $m = 23$ and $n = 334$ we have

$$\Delta q \approx \frac{23}{365}(1-0.5073)(23-23) - \frac{23^2}{(2)(365^2)}(1-0.5073)(334-365) = 0.0312.$$

So adding 2 more students gives you a greater increase in probability than no birthdays in December.

(e) The number of possible choices of birthdays for all the students is n^m since each of the m students could choose (have a birthday on) any of the n days. We count the number of these choices for which there are no matching birthdays as follows. The first student can choose any of n days for his/her birthday. The second student can choose any birthday that does not match the first student's, that is, any of $n-1$ days. The third student can choose any birthday that does not match the first two students', that is, any of $n-2$ days; and so on for all m students. So the number of choices of birthdays for all the students for which there is no matching is $n(n-1)(n-2)\cdots(n-(m-1))$. Thus, the fraction of possible choices for which there is no matching is

$$\frac{n(n-1)(n-2)\cdots(n-(m-1))}{n^m}.$$

Then q is one minus the chance of no matching birthdays:

$$q(m,n) = 1 - \frac{n(n-1)(n-2)\cdots(n-(m-1))}{n^m}.$$

CHAPTER FIFTEEN

Solutions for Section 15.1

Exercises

1. The contour diagrams in (I) and (V) show that the value of the function decreases as we move in any direction from the critical point. Therefore, the function has a local maximum at P. The contour diagrams in (II) and in (VI) show that the value of the function increases as we move in any direction from the critical point, and therefore P is a local minimum. The contour lines in (III) and (IV) are hyperbolas, and therefore, the critical point, P, is a saddle point.

2. The point A is not a critical point and the contour lines look like parallel lines. The point B is a critical point and is a local maximum; the point C is a saddle point.

3. (a) None.
(b) Points E and G.
(c) Points D and F.

4. At the origin $g(0,0) = 0$. Since $y^3 \geq 0$ for $y > 0$ and $y^3 < 0$ for $y < 0$, the function g takes on both positive and negative values near the origin, which must therefore be a saddle point. The second derivative test does not tell you anything since $D = 0$.

5. At the origin $f(0,0) = 0$. Since $x^6 \geq 0$ and $y^6 \geq 0$, the point $(0,0)$ is a local (and global) minimum. The second derivative test does not tell you anything since $D = 0$.

6. At the origin, the second derivative test gives

$$\begin{aligned} D = k_{xx}k_{yy} - (k_{xy})^2 &= \left((-\sin x \sin y)(-\sin x \sin y) - (\cos x \cos y)^2\right)\Big|_{x=0,y=0} \\ &= \sin^2 0 \sin^2 0 - \cos^2 0 \cos^2 0 \\ &= -1 < 0. \end{aligned}$$

Thus $k(0,0)$ is a saddle point.

7. At the origin $h(0,0) = 1$. Since $\cos x$ and $\cos y$ are never above 1, the origin must be a local (and global) maximum. The second derivative test

$$\begin{aligned} D = h_{xx}h_{yy} - (h_{xy})^2 &= \left((-\cos x \cos y)(-\cos x \cos y) - (\sin x \sin y)^2\right)\Big|_{x=0,y=0} \\ &= \left(\cos^2 x \cos^2 y - \sin^2 x \sin^2 y\right)\Big|_{x=0,y=0} \\ &= 1 > 0 \end{aligned}$$

and $h_{xx}(0,0) < 0$, so $(0,0)$ is a local maximum.

8. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$\begin{aligned} f_x &= 2x - 2y = 0, \\ f_y &= -2x + 6y - 8 = 0. \end{aligned}$$

We see from the first equation that $x = y$. Substituting this into the second equation shows that $y = 2$. The only critical point is $(2,2)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (2)(6) - (-2)^2 = 8.$$

Since $D > 0$ and $f_{xx} = 2 > 0$, the function f has a local minimum at the point $(2,2)$.

9. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$\begin{aligned} f_x &= 6 - 2x + y = 0, \\ f_y &= x - 2y = 0. \end{aligned}$$

We see from the second equation that $x = 2y$. Substituting this into the first equation shows that $y = 2$. The only critical point is $(4, 2)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (-2)(-2) - 1^2 = 3.$$

Since $D > 0$ and $f_{xx} = -2 < 0$, the function f has a local maximum at $(4, 2)$.

10. The partial derivatives give

$$f_x = 2x + 4 = 0 \quad \text{and} \quad f_y = -2y + 2 = 0.$$

Thus we have $x = -2$ and $y = 1$, so $(-2, 1)$ is a critical point. We use the discriminant:

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = 2(-2) - 0 = -4 < 0.$$

Since $D < 0$, we have a saddle point.

11. Setting $f_x = 0$ and $f_y = 0$ to find the critical point, we have

$$\begin{aligned} f_x = -6x - 4 + 2y = 0 &\quad \text{and} \quad f_y = 2x - 10y + 48 = 0, \text{ or} \\ 2y - 6x = 4 &\quad \text{and} \quad 10y - 2x = 48. \end{aligned}$$

Solving these equations simultaneously gives $x = 1$ and $y = 5$.

Since $f_{xx} = -6$, $f_{yy} = -10$ and $f_{xy} = 2$ for all (x, y), at $(1, 5)$ the discriminant

$$D = (-6)(-10) - (2)^2 = 56 > 0, \quad \text{and} \quad f_{xx} < 0.$$

Thus $f(x, y)$ has a local maximum value at $(1, 5)$.

12. The partial derivatives give

$$f_x = -2x + 6 = 0 \quad \text{and} \quad f_y = 4y - 8 = 0.$$

Thus we have $x = 3$ and $y = 2$, so $(3, 2)$ is a critical point. We use the discriminant:

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (-2)2 = -4 < 0.$$

Since $D < 0$, we have a saddle point.

13. At a critical point, $f_x = 0$ and $f_y = 0$.

$$\begin{aligned} f_x &= 2xy - 2y = 2y(x - 1) = 0, \\ f_y &= x^2 + 4y - 2x = 0. \end{aligned}$$

In order for f_x to be zero, we need $y = 0$ or $x = 1$. If $y = 0$, then $f_y = x^2 - 2x = x(x - 2) = 0$ so $x = 0$ or $x = 2$. Therefore $(0, 0)$ and $(2, 0)$ are both critical points. If $x = 1$, then $f_y = 4y - 1 = 0$ so $y = 0.25$. The three critical points are $(0, 0)$, $(2, 0)$, and $(1, 0.25)$.

We use the discriminant to classify the three points. We have

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - f_{xy}^2 \\ &= (2y)(4) - (2x - 2)^2 \\ &= 8y - (2x - 2)^2. \end{aligned}$$

Checking each critical point, we see that
$D(0, 0) = -4 < 0$, so $(0, 0)$ is a saddle point.
$D(2, 0) = -4 < 0$, so $(2, 0)$ is also a saddle point.
$D(1, 0.25) = 2 > 0$ and $f_{xx}(1, 0.25) = 0.5 > 0$, so $(1, 0.25)$ is a local minimum.

14. At a critical point

$$\begin{aligned} f_x(x, y) &= 6x^2 - 6xy + 12x = 0 \\ f_y(x, y) &= -3x^2 - 12y = 0 \end{aligned}$$

From the second equation, we conclude that $-3(x^2 + 4y) = 0$, so $y = -\frac{1}{4}x^2$. Substituting for y in the first equation gives

$$6x^2 - 6x\left(-\frac{1}{4}x^2\right) + 12x = 0$$

or

$$x^2 + \frac{1}{4}x^3 + 2x = \frac{x}{4}(4x + x^2 + 8) = 0.$$

Thus $x = 0$ or $x^2 + 4x + 8 = 0$. The quadratic has no real solutions, so the only one critical point is $(0, 0)$.

At $(0, 0)$, we have

$$D(0, 0) = f_{xx}f_{yy} - (f_{xy})^2 = (12)(-12) - 0^2 = -144 < 0,$$

so $(0, 0)$ is a saddle point.

15. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 3 = 0$$
$$f_y = 3y^2 - 3 = 0$$

shows that $x = \pm 1$ and $y = \pm 1$. There are four critical points: $(1, 1)$, $(-1, 1)$, $(1, -1)$ and $(-1, -1)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x)(6y) - (0)^2 = 36xy.$$

At the points $(1, -1)$ and $(-1, 1)$, we have $D = -36 < 0$, so f has saddle points at $(1, -1)$ and $(-1, 1)$. At $(1, 1)$, we have $D = 36 > 0$ and $f_{xx} = 6 > 0$, so f has a local minimum at $(1, 1)$. At $(-1, -1)$, we have $D = 36 > 0$ and $f_{xx} = -6 < 0$, so f has a local maximum at $(-1, -1)$.

16. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 6x = 0$$
$$f_y = 3y^2 - 3 = 0$$

shows that $x = 0$ or $x = 2$ and $y = -1$ or $y = 1$. There are four critical points: $(0, -1)$, $(0, 1)$, $(2, -1)$, and $(2, 1)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x - 6)(6y) - (0)^2 = (6x - 6)(6y).$$

At the point $(0, -1)$, we have $D > 0$ and $f_{xx} < 0$, so f has a local maximum.
At the point $(0, 1)$, we have $D < 0$, so f has a saddle point.
At the point $(2, -1)$, we have $D < 0$, so f has a saddle point.
At the point $(2, 1)$, we have $D > 0$ and $f_{xx} > 0$, so f has a local minimum.

17. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 3 = 0$$
$$f_y = 3y^2 - 12y = 0$$

shows that $x = -1$ or $x = 1$ and $y = 0$ or $y = 4$. There are four critical points: $(-1, 0)$, $(1, 0)$, $(-1, 4)$, and $(1, 4)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x)(6y - 12) - (0)^2 = (6x)(6y - 12).$$

At critical point $(-1, 0)$, we have $D > 0$ and $f_{xx} < 0$, so f has a local maximum at $(-1, 0)$.
At critical point $(1, 0)$, we have $D < 0$, so f has a saddle point at $(1, 0)$.
At critical point $(-1, 4)$, we have $D < 0$, so f has a saddle point at $(-1, 4)$.
At critical point $(1, 4)$, we have $D > 0$ and $f_{xx} > 0$, so f has a local minimum at $(1, 4)$.

18. Find the critical point(s) by setting

$$f_x = (xy + 1) + (x + y) \cdot y = y^2 + 2xy + 1 = 0,$$
$$f_y = (xy + 1) + (x + y) \cdot x = x^2 + 2xy + 1 = 0,$$

then we get $x^2 = y^2$, and so $x = y$ or $x = -y$.

If $x = y$, then $x^2 + 2x^2 + 1 = 0$, that is, $3x^2 = -1$, and there is no real solution. If $x = -y$, then $x^2 - 2x^2 + 1 = 0$, which gives $x^2 = 1$. Solving it we get $x = 1$ or $x = -1$, then $y = -1$ or $y = 1$, respectively. Hence, $(1, -1)$ and $(-1, 1)$ are critical points.

Since

$$f_{xx}(x, y) = 2y,$$
$$f_{xy}(x, y) = 2y + 2x \quad \text{and}$$
$$f_{yy}(x, y) = 2x,$$

the discriminant is

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= 2y \cdot 2x - (2y+2x)^2 \\ &= -4(x^2+xy+y^2). \end{aligned}$$

thus

$$\begin{aligned} D(1,-1) &= -4(1^2 + 1\cdot(-1) + (-1)^2) = -4 < 0, \\ D(-1,1) &= -4((-1)^2 + (-1)\cdot 1 + 1^2) = -4 < 0. \end{aligned}$$

Therefore $(1,-1)$ and $(-1,1)$ are saddle points.

19. At a critical point, $f_x = 0$, $f_y = 0$.

$$f_x = 8y - (x+y)^3 = 0, \text{ we know } 8y = (x+y)^3.$$

$$f_y = 8x - (x+y)^3 = 0, \text{ we know } 8x = (x+y)^3.$$

Therefore we must have $x = y$. Since $(x+y)^3 = (2y)^3 = 8y^3$, this tells us that $8y - 8y^3 = 0$. Solving gives $y = 0, \pm 1$. Thus the critical points are $(0,0)$, $(1,1)$, $(-1,-1)$.

$f_{yy} = f_{xx} = -3(x+y)^2$, and $f_{xy} = 8 - 3(x+y)^2$.

The discriminant is

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= 9(x+y)^4 - \left(64 - 48(x+y)^2 + 9(x+y)^4\right) \\ &= -64 + 48(x+y)^2. \end{aligned}$$

$D(0,0) = -64 < 0$, so $(0,0)$ is a saddle point.
$D(1,1) = -64 + 192 > 0$ and $f_{xx}(1,1) = -12 < 0$, so $(1,1)$ is a local maximum.
$D(-1,-1) = -64 + 192 > 0$ and $f_{xx}(-1,-1) = -12 < 0$, so $(-1,-1)$ is a local maximum.

20. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$\begin{aligned} f_x &= e^{2x^2+y^2}(4x) = 0, \\ f_y &= e^{2x^2+y^2}(2y) = 0, \end{aligned}$$

shows that the only critical point is $(0,0)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = e^{2x^2+y^2}(4+(4x)^2)\cdot e^{2x^2+y^2}(2+(2y)^2) - (e^{2x^2+y^2}(4x\cdot 2y))^2.$$

At $(0,0)$, we have $D = 4\cdot 2 - 0^2 > 0$ and $f_{xx} = 4 > 0$, so the function has a local minimum at the point $(0,0)$.

Problems

21. At the critical point $x = 1$, $y = 0$,

$$\begin{aligned} f_x &= 2x + A = 0, \quad \text{ so } 2 + A = 0 \text{ or } A = -2 \\ f_y &= 2y = 0. \end{aligned}$$

Thus, $f(x,y) = x^2 - 2x + y^2 + B$ has a critical point at $(1,0)$. Since $f_{xx} = 2$ and $f_{yy} = 2$ and $f_{xy} = 0$ at $(1,0)$,

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2\cdot 2 - 0^2 = 4,$$

so the second derivative test shows the critical point at $(1,0)$ is a local minimum. The value of the minimum is

$$f(1,0) = 1^2 - 2\cdot 1 + 0^2 + B = 20, \quad \text{ so } B = 21.$$

22. At a local minimum,

$$f_x = 2x + y + a = 0$$
$$f_y = x + 2y + b = 0.$$

Since these equations must be satisfied by $x = 2$, $y = 5$, we have

$$a = -2.2 - 5 = -9$$
$$b = -2 - 2 \cdot 5 = -12.$$

The point $(2, 5)$ gives a local minimum since

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2 \cdot 2 - 1^2 > 0 \text{ and } f_{xx} = 2 > 0.$$

To ensure that $f(2, 5) = 11$, we substitute

$$f(2,5) = 2^2 + 2 \cdot 5 + 5^2 - 9 \cdot 2 - 12 \cdot 5 + c = 11$$
$$c = 50.$$

23. (a) This function has only one critical point, a local maximum, where $(x-a)^2 + (y-b)^2 = 0$; that is, at (a, b). Taking partial derivatives gives the same result:

$$f_x(x,y) = e^{-(x-a)^2-(y-b)^2}(-2(x-a))$$
$$f_y(x,y) = e^{-(x-a)^2-(y-b)^2}(-2(y-b)).$$

Since $e^{-(x-a)^2-(y-b)^2}$ is never zero, f_x and f_y are only 0 at $x = a$, $y = b$.

(b) We have $a = -1$, $b = 5$.

(c) The point $(-1, 5)$ is a local maximum because $e^{-(x-a)^2-(y-b)^2}$ is largest where $-(x-a)^2 - (y-b)^2$ is zero.

24. We have $f_x = 2kx - 4y$ and $f_y = 2y - 4x$, so $f_{xx} = 2k$, $f_{xy} = -4$, and $f_{yy} = 2$. The discriminant is

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (2k)(2) - (-4)^2 = 4k - 16.$$

Since $D = 4k - 16$, we see that $D < 0$ when $k < 4$. The function has a saddle point at the point $(0, 0)$ when $k < 4$. When $k > 4$, we have $D > 0$ and $f_{xx} > 0$, so the function has a local minimum at the point $(0, 0)$. When $k = 4$, the discriminant is zero, and we get no information about this critical point. By looking at the values of the function in Table 15.1, it appears that f has a local minimum at the point $(0, 0)$ when $k = 4$.

Table 15.1

y \ x	-0.1	0	0.1
-0.1	0.01	0.01	0.09
0	0.04	0	0.04
0.1	0.09	0.01	0.01

(a) The function $f(x, y)$ has a saddle point at $(0, 0)$ if $k < 4$.

(b) There are no values of k for which this function has a local maximum at the point $(0, 0)$.

(c) The function $f(x, y)$ has a local minimum at $(0, 0)$ if $k \geq 4$.

25. (a) P is a local maximum.

(b) Q is a saddle point.

(c) R is a local minimum.

(d) S is none of these.

26. Figure 15.1 shows the gradient vectors around P and Q pointing perpendicular to the contours and in the direction of increasing values of the function.

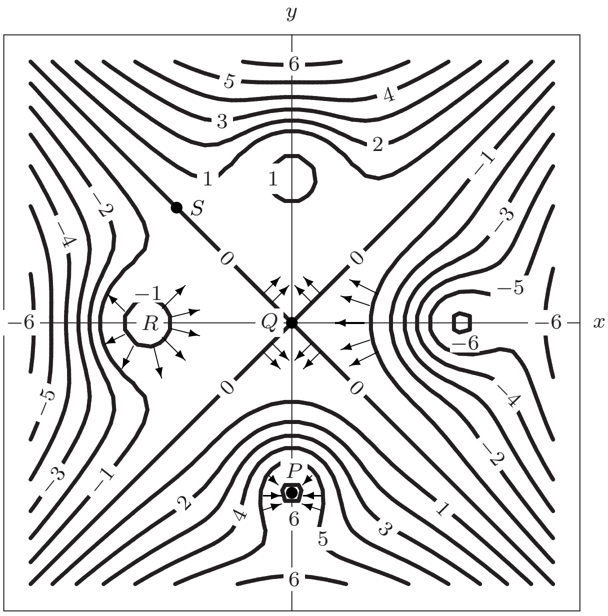


Figure 15.1

27. Figure 15.2 shows the direction of ∇f at the points where $\|\nabla f\|$ is largest, since at those points the contours are closest together.

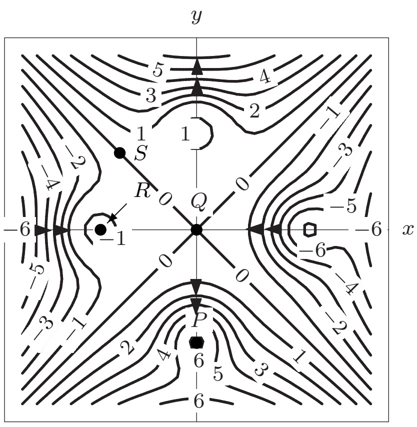


Figure 15.2

28. First, we identify the critical points. The partials are $f_x(x,y) = 3x^2$ and $f_y(x,y) = -2ye^{-y^2}$. These will vanish simultaneously when $x = 0$ and $y = 0$, so our only critical point is $(0,0)$. The discriminant is

$$D = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y) = (6x)(4y^2e^{-y^2} - 2e^{-y^2}) - 0 = 12xe^{-y^2}(2y^2 - 1).$$

Unfortunately, the discriminant is zero at the origin so the second derivative test can tell us nothing about our critical point. We can, however, see that we are at a saddle point by looking at the behavior of $f(x,y)$ along the line $y = 0$. Here we have $f(x,0) = x^3 + 1$, so for positive x, we have $f(x,0) > 1 = f(0,0)$ and for negative x, we have $f(x,0) < 1 = f(0,0)$. So $f(x,y)$ has neither a maximum nor a minimum at $(0,0)$.

29. At a critical point,

$$f_x = \cos x \sin y = 0 \quad \text{so} \quad \cos x = 0 \text{ or } \sin y = 0;$$

and

$$f_y = \sin x \cos y = 0 \quad \text{so} \quad \sin x = 0 \text{ or } \cos y = 0.$$

Case 1: Assume $\cos x = 0$. This gives

$$x = \cdots - \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$$

(This can be written more compactly as: $x = k\pi + \pi/2$, for $k = 0, \pm1, \pm2, \cdots$.)
If $\cos x = 0$, then $\sin x = \pm 1 \neq 0$. Thus in order to have $f_y = 0$ we need $\cos y = 0$, giving

$$y = \cdots - \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$$

(More compactly, $y = l\pi + \pi/2$, for $l = 0, \pm1, \pm2, \cdots$)
Case 2: Assume $\sin y = 0$. This gives

$$y = \cdots - 2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

(More compactly, $y = l\pi$, for $l = 0, \pm1, \pm2, \cdots$)
If $\sin y = 0$, then $\cos y = \pm1 \neq 0$, so to get $f_y = 0$ we need $\sin x = 0$, giving

$$x = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

(More compactly, $x = k\pi$ for $k = 0, \pm0, \pm1, \pm2, \cdots$)
Hence we get all the critical points of $f(x, y)$. Those from Case 1 are as follows:

$$\cdots\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)\cdots$$
$$\cdots\left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\cdots$$
$$\cdots\left(\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)\cdots$$

Those from Case 2 are as follows:

$$\cdots(-\pi, -\pi), (-\pi, 0), (-\pi, \pi), (-\pi, 2\pi)\cdots$$
$$\cdots(0, -\pi), (0, 0), (0, \pi), (0, 2\pi)\cdots$$
$$\cdots(\pi, -\pi), (\pi, 0), (\pi, \pi), (\pi, 2\pi)\cdots$$

More compactly these points can be written as,

$$(k\pi, l\pi), \text{ for } k = 0, \pm1, \pm2, \cdots, l = 0, \pm1, \pm2, \cdots$$
$$\text{and } (k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2}), \text{ for } k = 0, \pm1, \pm2, \cdots, l = 0, \pm1, \pm2, \cdots$$

To classify the critical points, we find the discriminant. We have

$$f_{xx} = -\sin x \sin y, \qquad f_{yy} = -\sin x \sin y, \qquad \text{and} \qquad f_{xy} = \cos x \cos y.$$

Thus the discriminant is

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - f_{xy}^2 \\ &= (-\sin x \sin y)(-\sin x \sin y) - (\cos x \cos y)^2 \\ &= \sin^2 x \sin^2 y - \cos^2 x \cos^2 y \\ &= \sin^2 y - \cos^2 x. \text{ (Use: } \sin^2 x = 1 - \cos^2 x \text{ and factor.)} \end{aligned}$$

At points of the form $(k\pi, l\pi)$ where $k = 0, \pm1, \pm2, \cdots; l = 0, \pm1, \pm2, \cdots$, we have
$D(x, y) = -1 < 0$ so $(k\pi, l\pi)$ are saddle points.
At points of the form $(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2})$ where $k = 0, \pm1, \pm2, \cdots; l = 0, \pm1, \pm2, \cdots$
$D(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2}) = 1 > 0$, so we have two cases:
If k and l are both even or k and l are both odd, then
$f_{xx} = -\sin x \sin y = -1 < 0$, so $(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2})$ are local maximum points.
If k is even but l is odd or k is odd but l is even, then
$f_{xx} = 1 > 0$ so $(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2})$ are local minimum points.

30. To find critical points, set partial derivatives equal to zero:

$$f_x = \sin x = 0 \quad \text{when} \quad x = 0,\ \pm\pi,\ \pm 2\pi, \cdots$$

$$f_y = y = 0 \quad \text{when} \quad y = 0.$$

The critical points are

$$\cdots(-2\pi,0),(-\pi,0),(0,0),(\pi,0),(2\pi,0),(3\pi,0)\cdots$$

To classify, calculate $D = f_{xx}f_{yy} - (f_{xy})^2 = \cos x$.
At the points $(0,0),(\pm 2\pi,0),(\pm 4\pi,0),(\pm 6\pi,0),\cdots$

$$D = (1) > 0 \quad \text{and} \quad f_{xx} > 0 \quad (\text{Since} f_{xx}(0,2k\pi) = \cos(2k\pi) = 1).$$

Therefore $(0,0),(\pm 2\pi,0),(\pm 4\pi,0),(\pm 6\pi,0),\cdots$ are local minima.
At the points $(\pm\pi,0),(\pm 3\pi,0),(\pm 5\pi,0),(\pm 7\pi,0),\cdots$, we have $\cos(2k+1)\pi = -1$, so

$$D = (-1) < 0.$$

Therefore $(\pm\pi,0),(\pm 3\pi,0),(\pm 5\pi,0),(\pm 7\pi,0),\cdots$ are saddle points.

31. To find the critical points, we must solve the equations

$$\frac{\partial f}{\partial x} = e^x(1-\cos y) = 0$$
$$\frac{\partial f}{\partial y} = e^x(\sin y) = 0.$$

The first equation has solution

$$y = 0, \pm 2\pi, \pm 4\pi, \ldots.$$

The second equation has solution

$$y = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Since x can be anything, the lines

$$y = 0, \pm 2\pi, \pm 4\pi, \ldots$$

are lines of critical points.

We calculate

$$\begin{aligned} D = (f_{xx})(f_{yy}) - (f_{xy})^2 &= e^x(1-\cos y)e^x\cos y - (e^x\sin y)^2 \\ &= e^{2x}(\cos y - \cos^2 y - \sin^2 y) \\ &= e^{2x}(\cos y - 1) \end{aligned}$$

At any critical point on one of the lines $y = 0$, $y = \pm 2\pi$, $y = \pm 4\pi, \ldots$,

$$D = e^{2x}(1-1) = 0.$$

Thus, D tells us nothing. However, all along these critical lines, the value of the function, f, is zero. Since the function f is never negative, the critical points are all both local and global minima.

32. (a) $(1,3)$ is a critical point. Since $f_{xx} > 0$ and the discriminant

$$D = f_{xx}f_{yy} - f_{xy}^2 = f_{xx}f_{yy} - 0^2 = f_{xx}f_{yy} > 0,$$

the point $(1,3)$ is a minimum.

(b) See Figure 15.3.

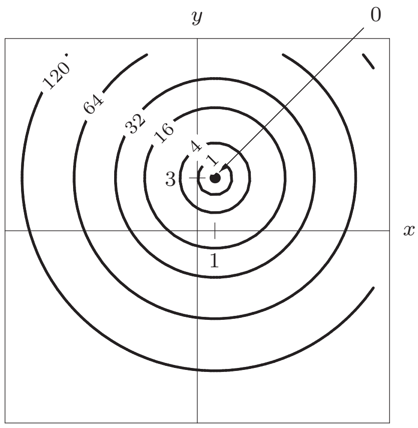


Figure 15.3

33. (a) (a, b) is a critical point. Since the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = -f_{xy}^2 < 0$, (a, b) is a saddle point.
(b) See Figure 15.4.

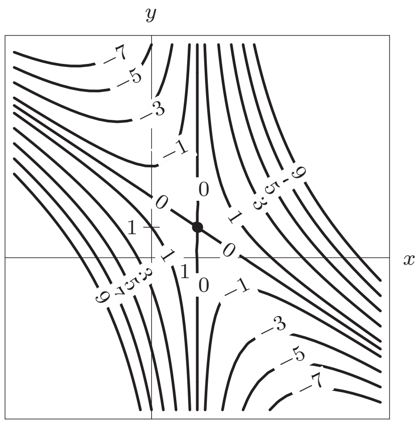


Figure 15.4

34. Begin by constructing little pieces of the contour diagram around each of the points $(-1, 0)$, $(3, 3)$, and $(3, -3)$ where some information about f is given. The general shape will be as in Figure 15.5, and the directions of increasing contour values are indicated for each part. Then complete the diagram in any way. One possible solution is given in Figure 15.6.

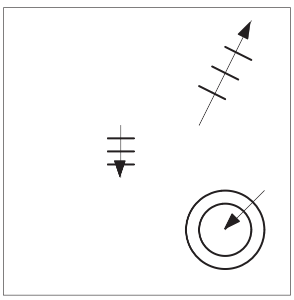


Figure 15.5: Part of contour diagram with arrows showing direction of increasing function values

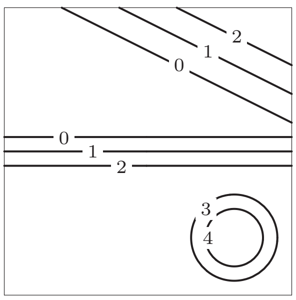


Figure 15.6: Contour diagram of $f(x, y)$

35. Since $(2, 4)$ is a local minimum, the contours around $(2, 4)$ are closed curves with increasing values as we go away from the point $(2, 4)$. We assume that the function values continue to increase as we move parallel to the y-axis to the point $(2, 1)$. Since $(2, 1)$ is a saddle point, we draw the contours so that the values go down as we move up or down from this point, and up as we move to the left or right. One possible contour diagram is shown in Figure 15.7.

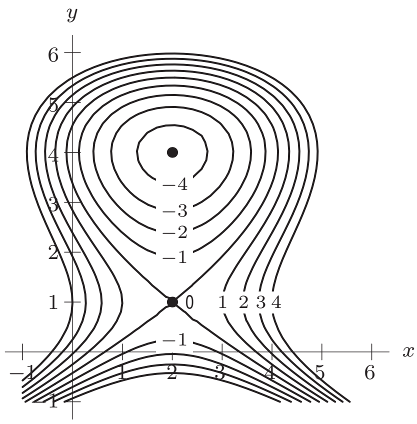


Figure 15.7

36. (a) We set partial derivatives to zero:

$$f_x = 2ax - 2y - 4 = 0$$
$$f_y = 2by - 2x - 6 = 0.$$

So we have $ax - y = 2$ and $-x + by = 3$, leading to

$$x = \frac{2b+3}{ab-1} \quad \text{and} \quad y = \frac{3a+2}{ab-1}.$$

(b) The discriminant is given by

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (2a)(2b) - 2^2 = 4(ab-1)$$

If $a = b = 2$, then $D = 4(2 \cdot 2 - 1) = 12$. Since $f_{xx} = 4 > 0$, the critical point is a local minimum.

(c) We use the fact that $D = 4(ab - 1)$ and $f_{xx} = 2a$.
If $a > 0$ and $ab > 1$, then $D > 0$ and $f_{xx} > 0$ so we have a local minimum.
If $a < 0$ and $ab > 1$, then $D > 0$ and $f_{xx} > 0$ so we have a local maximum.
If $ab < 1$, then $D < 0$ so we have a saddle point.

37. (a) Setting the partial derivatives equal to 0, we have

$$f_x(x, y) = 2x(x^2 + y) + 2x(x^2 - y) = 4x^3 = 0$$
$$f_y(x, y) = -(x^2 + y) + (x^2 - y) = -2y = 0.$$

Thus, $(0, 0)$ is the only critical point.

(b) Calculating D gives

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (12x^2)(-2) - 0^2 = -24x^2.$$

At $x = 0, y = 0$, we have

$$D(0, 0) = 0.$$

Thus the second derivative test tells us nothing about the nature of the critical point.

(c) Since $f(0, 0) = 0$, we sketch contours with values near 0. The contour $f = 0$ is given by

$$(x^2 - y)(x^2 + y) = 0,$$

that is, the two parabolas

$$y = x^2 \quad \text{and} \quad y = -x^2$$

We also sketch the contours $f = 1$ and $f = -1$. See Figure 15.8.

Since there are values of the function which are both positive (above $f(0,0)$) and negative (below ($f(0,0)$), near the critical point $(0,0)$, the origin is neither a local maximum nor a local minimum; it is a saddle point.

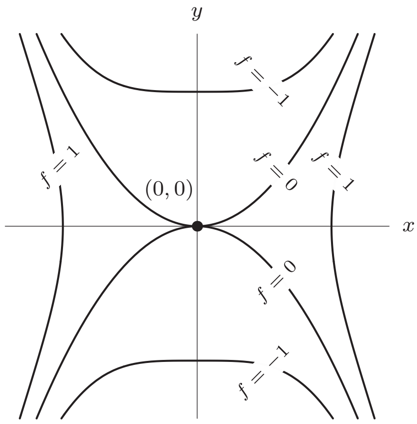


Figure 15.8

38. The first order partial derivatives are

$$f_x(x,y) = 2kx - 2y \qquad \text{and} \qquad f_y(x,y) = 2ky - 2x.$$

And the second order partial derivatives are

$$f_{xx}(x,y) = 2k \qquad f_{xy}(x,y) = -2 \qquad f_{yy}(x,y) = 2k$$

Since $f_x(0,0) = f_y(0,0) = 0$, the point $(0,0)$ is a critical point. The discriminant is

$$D = (2k)(2k) - 4 = 4(k^2 - 1).$$

For $k = \pm 2$, the discriminant is positive, $D = 12$. When $k = 2$, $f_{xx}(0,0) = 4$ which is positive so we have a local minimum at the origin. When $k = -2$, $f_{xx}(0,0) = -4$ so we have a local maximum at the origin. In the case $k = 0$, $D = -4$ so the origin is a saddle point.

Lastly, when $k = \pm 1$ the discriminant is zero, so the second derivative test can tell us nothing. Luckily, we can factor $f(x,y)$ when $k = \pm 1$. When $k = 1$,

$$f(x,y) = x^2 - 2xy + y^2 = (x-y)^2.$$

This is always greater than or equal to zero. So $f(0,0) = 0$ is a minimum and the surface is a trough-shaped parabolic cylinder with its base along the line $x = y$.

When $k = -1$,

$$f(x,y) = -x^2 - 2xy - y^2 = -(x+y)^2.$$

This is always less than or equal to zero. So $f(0,0) = 0$ is a maximum. The surface is a parabolic cylinder, with its top ridge along the line $x = -y$.

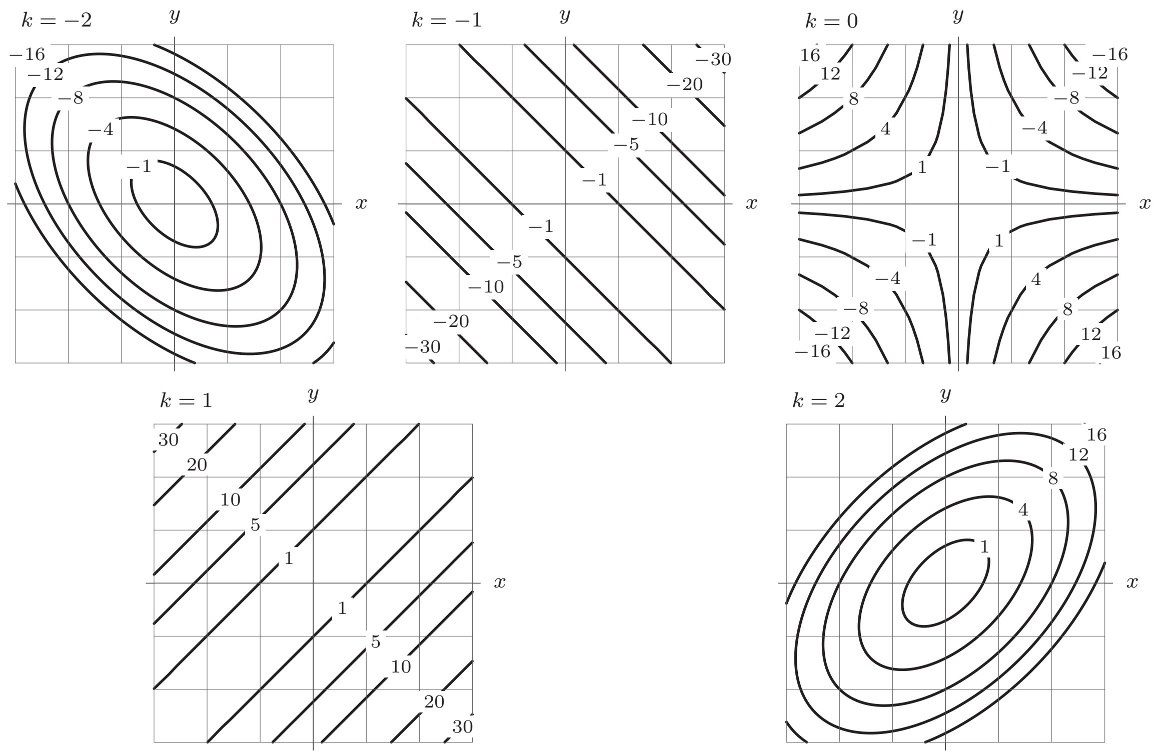


Figure 15.9

39. The partial derivatives are

$$f_x(x,y) = 3x^2 - 3y^2 \qquad \text{and} \qquad f_y(x,y) = -6xy.$$

Now $f_x(x,y)$ will vanish if $x = \pm y$ and $f_y(x,y)$ will vanish if either $x = 0$ or $y = 0$. Since the partial derivatives are defined everywhere, the only critical points are where $f_x(x,y)$ and $f_y(x,y)$ vanish simultaneously. $(0,0)$ is the only critical point.

To find the contour for $f(x,y) = 0$, we solve the equation $x^3 - 3xy^2 = 0$. This can be factored into

$$f(x,y) = x(x - \sqrt{3}\,y)(x + \sqrt{3}\,y) = 0$$

whose roots are $x = 0$, $x = \sqrt{3}\,y$ and $x = -\sqrt{3}\,y$. Each of these roots describes a line through the origin; the three of them divide the plane into six regions. Crossing any one of these lines will change the sign of only one of the three factors of $f(x,y)$, which will change the sign of $f(x,y)$.

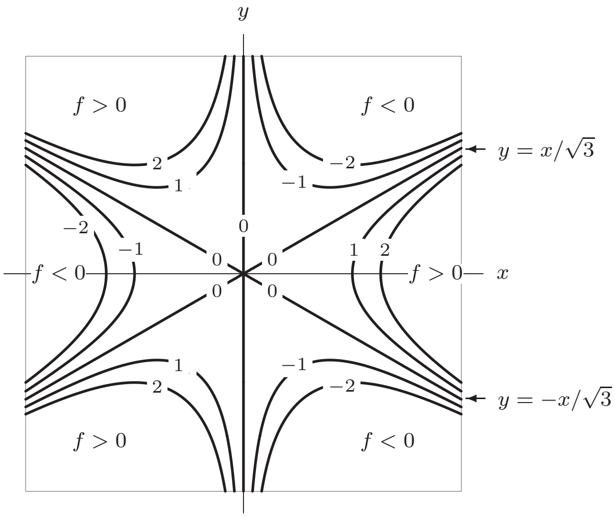


Figure 15.10

40. Contours near a local extremum are approximate ellipses and contours near a saddle point are approximate hyperbolas. The functions in the contour plots (a) and (b) have a local extremum at the origin; those in (c) and (d) have a saddle point at the origin.

All four functions have value zero at the critical point at the origin. Graph (I) corresponds to a local maximum, because the values near the origin are all below zero. Graph (III) corresponds to a local minimum, because the values near the origin are all above zero. Graphs (II) and (IV) correspond to saddle points because there are both positive and negative values near the critical point at the origin.

Contour plot (a) shows that on circles centered at the origin, z takes values farther from zero at $\theta = \pi/2$ and $\theta = 3\pi/2$ than at $\theta = 0$ and $\theta = \pi$. Thus (a) matches (III), and the critical point at the origin is a local minimum.

The function represented by Contour plot (b) takes values farther from zero at $\theta = 0$ and $\theta = \pi$ than at $\theta = \pi/2$ and $\theta = 3\pi/2$, so corresponds to (I) and has a local maximum at the origin.

The function in Contour plot (c) has value zero on the asymptotes of the hyperbolas, which are at $\theta = \pi/4, 3\pi/4, 5\pi/4$ and $7\pi/4$. This matches Graph (II), which represents a saddle point.

The function in Contour plot (d) has value zero on the asymptotes of the hyperbolas, which are at $\theta = 0, \pi/2, \pi$ and $3\pi/2$. This matches Graph (IV),which represents a saddle point.

Strengthen Your Understanding

41. If $f_x = f_y = 0$ at $(1, 3)$, then $(1, 3)$ is a critical point of f. However, $(1, 3)$ may be a saddle point and not a local maximum or minimum.

42. If $D = 0$ at (a, b), the point (a, b) can be a maximum, a minimum, or a saddle point. We are only sure that (a, b) is a saddle point if $D < 0$.

43. If both cross-sections are concave up, then $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$, but we could still have $D < 0$ when $f_{xy}(a, b)$ is large enough, giving a saddle point.

For example, let $f(x, y) = x^2 + y^2 + 7xy$. Then $(0, 0)$ is a critical point. We have $f_{xx}(0, 0) = f_{yy}(0, 0) = 2 > 0$, but $f_{xy}(0, 0) = 7$, so $D = (2)(2) - 7^2 < 0$.

44. Let $f(x, y) = e^x$ Then $f_x = e^x$, which is never 0.

45. The function $f(x, y) = 4 - (x - 2)^2 - (y + 3)^2$ has a local maximum at $(2, -3, 4)$.

46. True. By definition, a critical point is either where the gradient of f is zero or does not exist.

47. False. The point P_0 could be a saddle point of f.

48. False. The point P_0 could be a saddle point of f.

49. True. If P_0 were not a critical point of f, then $\text{grad}\, f(P_0)$ would point in the direction of maximum increase of f, which contradicts the fact that P_0 is a local maximum or minimum.

50. True. The graph of this function is a cone that opens upward with its vertex at the origin.

51. False. The graph of this function is a saddle shape, with a saddle point at the origin. The function increases in the $\vec{i}$ direction and decreases in the $\vec{j}$ direction.

52. True. Adding 5 to the function shifts the graph 5 units vertically, which leaves the (x, y) coordinates of the local extrema intact.

53. True. Multiplying by -1 turns the graph of f upside down, so local maxima become local minima and vice-versa.

54. False. For example, the linear function $f(x, y) = x + y$ has no local extrema at all.

55. False. The statement is only true for points sufficiently close to P_0.

56. False. Local maxima are only high points for f when compared to nearby values; the global maximum is the largest of *any* values of f over its entire domain.

Solutions for Section 15.2

Exercises

1. Mississippi lies entirely within a region designated as 80s so we expect both the maximum and minimum daily high temperatures within the state to be in the 80s. The southwestern-most corner of the state is close to a region designated as 90s, so we would expect the temperature here to be in the high 80s, say 87-88. The northern-most portion of the state is located near the center of the 80s region. We might expect the high temperature there to be between 83-87.

Alabama also lies completely within a region designated as 80s so both the high and low daily high temperatures within the state are in the 80s. The southeastern tip of the state is close to a 90s region so we would expect the temperature here to be about 88-89 degrees. The northern-most part of the state is near the center of the 80s region so the temperature there is 83-87 degrees.

Pennsylvania is also in the 80s region, but it is touched by the boundary line between the 80s and a 70s region. Thus we expect the low daily high temperature to occur there and be about 80 degrees. The state is also touched by a boundary line of a 90s region so the high will occur there and be 89-90 degrees.

New York is split by a boundary between an 80s and a 70s region, so the northern portion of the state is likely to be about 74-76 while the southern portion is likely to be in the low 80s, maybe 81-84 or so.

California contains many different zones. The northern coastal areas will probably have the daily high as low as 65-68, although without another contour on that side, it is difficult to judge how quickly the temperature is dropping off to the west. The tip of Southern California is in a 100s region, so there we expect the daily high to be 100-101.

Arizona will have a low daily high around 85-87 in the northwest corner and a high in the 100s, perhaps 102-107 in its southern regions.

Massachusetts will probably have a high daily high around 81-84 and a low daily high of 70.

2. The maximum value, which is about 11, occurs at $(5.1, 4.9)$. The minimum value, which is about -1, occurs at $(1, 3.9)$.

3. The maximum value, which is slightly above 30, say 30.5, occurs approximately at the origin. The minimum value, which is about 20.5, occurs at $(2.5, 5)$.

4. The maxima occur at about $(\pi/2, 0)$ and $(\pi/2, 2\pi)$. The minimum occurs at $(\pi/2, \pi)$. The maximum value is about 1, the minimum value is about -1.

5. To maximize $z = x^2 + y^2$, it suffices to maximize x^2 and y^2. We can maximize both of these at the same time by taking the point $(1, 1)$, where $z = 2$. It occurs on the boundary of the square. (Note: We also have maxima at the points $(-1, -1)$, $(-1, 1)$ and $(1, -1)$ which are on the boundary of the square.)

To minimize $z = x^2 + y^2$, we choose the point $(0, 0)$, where $z = 0$. It does not occur on the boundary of the square.

6. To maximize $z = -x^2 - y^2$ it suffices to minimize x^2 and y^2. Thus, the maximum is at $(0, 0)$, where $z = 0$. It does not occur on the boundary of the square.

To minimize $z = -x^2 - y^2$, it suffices to maximize x^2 and y^2. Do this by taking the point $(1, 1)$, $(-1, -1)$, $(-1, 1)$, or $(1, -1)$ where $z = -2$. These occur on the boundary of the square.

7. To maximize this function, it suffices to maximize x^2 and minimize y^2. We can do this by choosing the point $(1, 0)$, or $(-1, 0)$ where $z = 1$. These occur on the boundary of the square.

To minimize $z = x^2 - y^2$, it suffices to maximize y^2 and minimize x^2. We can do this by taking the point $(0, 1)$, or $(0, -1)$ where $z = -1$. These occur on the boundary of the square.

8. The function f has no global maximum or global minimum.

9. The function g has a global minimum (it is 0) but no global maximum.

10. The function h has no global maximum or minimum.

11. Since $f(x,y) \le 0$ for all x, y and since $f(0,0) = 0$, the function has a global maximum (it is 0) and no global minimum.

12. Suppose x is fixed. Then for large values of y the sign of f is determined by the highest power of y, namely y^3. Thus,

$$f(x,y) \to \infty \quad \text{as} \quad y \to \infty$$
$$f(x,y) \to -\infty \quad \text{as} \quad y \to -\infty.$$

So f does not have a global maximum or minimum.

Problems

13. (a) The critical points of f are the point(s) at which the partial derivatives, f_x and f_y, are zero. We have

$$f_x = 4x - 3y + 1$$
$$f_y = -3x + 16y - 1.$$

Solving the linear system $f_x = 0$, $f_y = 0$, we find $(x,y) = (-13/55, 1/55)$. To classify this point, we have to find the sign of $f_{xx} \cdot f_{yy} - f_{xy}^2$ there. We calculate

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 4 \cdot 16 - 9 = 55 > 0.$$

Thus, $(-13/55, 1/55)$ is a local minimum.

(b) We complete the square in the following way:

$$\begin{aligned} 2x^2 - 3xy + 8y^2 + x - y &= 2x^2 - x(3y-1) + 8y^2 - y \\ &= 2(x - \frac{1}{4}(3y-1))^2 - \frac{1}{8}(3y-1)^2 + 8y^2 - y \\ &= 2(x - \frac{1}{4}(3y-1))^2 + \frac{1}{8}(55y^2 - 2y - 1) \\ &= 2(x - \frac{1}{4}(3y-1))^2 + \frac{55}{8}\left(y - \frac{1}{55}\right)^2 - \frac{7}{55}. \end{aligned}$$

Therefore the function has a global minimum located at the point (x,y) where both the two squares vanish. The coordinates of that point satisfy:

$$x - \frac{1}{4}(3y-1) = 0 \qquad \text{and} \qquad y - \frac{1}{55} = 0.$$

The two conditions again give the point $(x,y) = (-13/55, 1/55)$. The contour diagram for f is shown in Figure 15.11. The fact that f can be written in this way as the sum of two squares shows that the point $(x,y) = (-13/55, 1/55)$ is a global minimum.

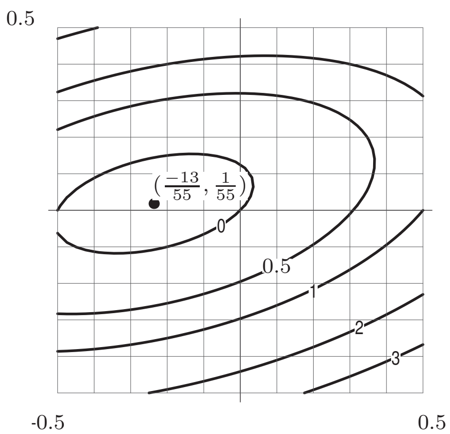


Figure 15.11: The contour diagram for $f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$

14. We calculate the partial derivatives and set them to zero.

$$\frac{\partial\,(\text{range})}{\partial t} = -10t - 6h + 400 = 0$$
$$\frac{\partial\,(\text{range})}{\partial h} = -6t - 6h + 300 = 0.$$

$$10t + 6h = 400$$
$$6t + 6h = 300$$

Solving, we obtain

$$4t = 100$$

so

$$t = 25$$

Solving for h, we obtain $6h = 150$, yielding $h = 25$. Since the range is quadratic in h and t, the second derivative test tells us this is a local and global maximum. So the optimal conditions are $h = 25\%$ humidity and $t = 25°\text{C}$.

15. Let the sides be x, y, z cm. Then the volume is given by $V = xyz = 32$.

The surface area S is given by

$$S = 2xy + 2xz + 2yz.$$

Substituting $z = 32/(xy)$ gives

$$S = 2xy + \frac{64}{y} + \frac{64}{x}.$$

At a critical point,

$$\frac{\partial S}{\partial x} = 2y - \frac{64}{x^2} = 0$$
$$\frac{\partial S}{\partial y} = 2x - \frac{64}{y^2} = 0,$$

The symmetry of the equations (or by dividing the equations) tells us that $x = y$ and

$$2x - \frac{64}{x^2} = 0$$
$$x^3 = 32$$
$$x = 32^{1/3} = 3.17 \text{ cm}.$$

Thus the only critical point is $x = y = (32)^{1/3}$ cm and $z = 32/\left((32)^{1/3}\cdot(32)^{1/3}\right) = (32)^{1/3}$ cm. At the critical point

$$S_{xx}S_{yy} - (S_{xy})^2 = \frac{128}{x^3}\cdot\frac{128}{y^3} - 2^2 = \frac{(128)^2}{x^3y^3} - 4.$$

Since $D > 0$ and $S_{xx} > 0$ at this critical point, the critical point $x = y = z = (32)^{1/3}$ is a local minimum. Since $S \to \infty$ as $x, y \to \infty$, the local minimum is a global minimum.

16. If the coordinates of the corner on the plane are (x, y, z), the volume of the box is $V = xyz$. Since $z = 1 - 3x - 2y$ on the plane, the volume is given by

$$V = xy(1 - 3x - 2y) = xy - 3x^2y - 2xy^2.$$

The domain is the triangular region $0 \le x \le \frac{1}{3}, 0 \le y \le (1 - 3x)/2$. At a critical point,

$$\frac{\partial V}{\partial x} = y - 6xy - 2y^2 = y(1 - 6x - 2y) = 0$$
$$\frac{\partial V}{\partial y} = x - 3x^2 - 4xy = x(1 - 3x - 4y) = 0,$$

One solution is $x = y = 0$. Another is $x = 0, y = \frac{1}{2}$; another is $y = 0, x = \frac{1}{3}$. Another is the solution of

$$1 - 6x - 2y = 0$$
$$1 - 3x - 4y = 0,$$

namely $x = \frac{1}{9}, y = \frac{1}{6}$.

If either $x = 0$ or $y = 0$, then $V = 0$, so these solutions do not give the maximum volume. Since

$$D = V_{xx}V_{yy} - (V_{xy})^2 = (-6y)(-4x) - (1 - 6x - 4y)^2$$

$$D\left(\frac{1}{9}, \frac{1}{6}\right) = \left(-6 \cdot \frac{1}{6}\right)\left(-4 \cdot \frac{1}{9}\right) - \left(1 - 6 \cdot \frac{1}{9} - 4 \cdot \frac{1}{6}\right)^2 = \frac{4}{9} - \frac{1}{9} = \frac{1}{3} > 0,$$

and $V_{xx}(\frac{1}{9}, \frac{1}{6}) = -1 < 0$, the point $x = \frac{1}{9}$, $y = \frac{1}{6}$, is a local maximum at which $V = (1/9)(1/6) - 3(1/9)^2(1/6) - 2(1/9)(1/6)^2 = 1/162$.

Since all points on the boundary of the domain give $V = 0$, the local maximum is a global maximum.

17.

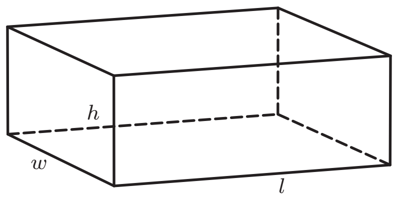


Figure 15.12

Let w, h and l be width, height and length of the suitcase in cm. Then its volume $V = lwh$, and $w + h + l \leq 135$. To maximize the volume V, choose $w + h + l = 135$, and thus $l = 135 - w - h$,

$$\begin{aligned} V &= wh(135 - w - h) \\ &= 135wh - w^2h - wh^2 \end{aligned}$$

Differentiating gives

$$\begin{aligned} V_w &= 135h - 2wh - h^2, \\ V_h &= 135w - w^2 - 2wh. \end{aligned}$$

Find the critical points by solving $V_w = 0$ and $V_h = 0$:

$$\begin{aligned} V_w = 0 \quad &\text{gives} \quad 135h - h^2 = 2wh, \\ V_h = 0 \quad &\text{gives} \quad 135w - w^2 = 2wh. \end{aligned}$$

As $hw \neq 0$, we cancel h (and w respectively) in the above equations and get

$$\begin{aligned} 135 - h &= 2w \\ 135 - w &= 2h \end{aligned}$$

Subtracting gives

$$w - h = 2(w - h)$$

hence $w = h$. Therefore, substituting into the equation $V_w = 0$

$$135h - h^2 = 2h^2$$

and therefore

$$3h^2 = 135h.$$

Since $h \neq 0$, we have

$$h = \frac{135}{3} = 45.$$

So $w = h = 45$ cm. Thus, $l = 135 - w - h = 45$ cm. To check that this critical point is a maximum, we find

$$\begin{aligned} V_{ww} &= -2h, \quad V_{hh} = -2w, \\ V_{wh} &= 135 - 2w - 2h, \end{aligned}$$

so

$$D = V_{ww}V_{hh} - V_{wh}^2 = 4hw - (135 - 2w - 2h)^2.$$

At $w = h = 45$, we have $V_{ww} = -2(45) < 0$ and $D = 4(45)^2 - (135 - 90 - 90)^2 = 6075 > 0$, hence V is maximum at $w = h = l = 45$.

Therefore, the suitcase with maximum volume is a cube with dimensions width $=$ height $=$ length $= 45$ cm.

18. The box is shown in Figure 15.13. Cost of four sides $= (2hl + 2wh)(1)$¢. Cost of two bottoms $= (2wl)(2)$¢. Thus the total cost C (in cents) of the box is

$$C = 2(hl + wh) + 4wl.$$

But volume $wlh = 512$, so $l = 512/(wh)$, thus

$$C = \frac{1024}{w} + 2wh + \frac{2048}{h}.$$

To minimize C, find the critical points of C by solving

$$C_h = 2w - \frac{2048}{h^2} = 0,$$
$$C_w = 2h - \frac{1024}{w^2} = 0.$$

We get

$$2wh^2 = 2048$$
$$2hw^2 = 1024.$$

Since $w, h \neq 0$, we can divide the first equation by the second giving

$$\frac{2wh^2}{2hw^2} = \frac{2048}{1024},$$

so

$$\frac{h}{w} = 2,$$

thus

$$h = 2w.$$

Substituting this in $C_h = 0$, we obtain $h^3 = 2048$, so $h = 12.7$ cm. Thus $w = h/2 = 6.35$ cm, and $l = 512/(wh) = 6.35$ cm. Now we check that these dimensions minimize the cost C. We find that

$$D = C_{hh}C_{ww} - C_{hw}^2 = (\frac{4096}{h^3})(\frac{2048}{w^3}) - 2^2,$$

and at $h = 12.7$, $w = 6.35$, $C_{hh} > 0$ and $D = 16 - 4 > 0$, thus C has a local minimum at $h = 12.7$ and $w = 6.35$. Since C increases without bound as $w, h \to 0$ or ∞, this local minimum must be a global minimum.

Therefore, the dimensions of the box that minimize the cost are $w = 6.35$ cm, $l = 6.35$ cm and $h = 12.7$ cm.

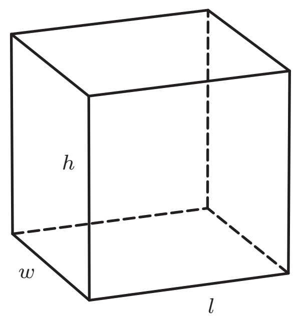


Figure 15.13

19. The square of the distance from the point (x, y, z) to the origin is

$$S = x^2 + y^2 + z^2.$$

If the point is on the plane, $z = 1 - 3x - 2y$, we have

$$S = x^2 + y^2 + (1 - 3x - 2y)^2.$$

At the critical point

$$\frac{\partial S}{\partial x} = 2x + 2(1-3x-2y)(-3) = 2(10x+6y-3) = 0$$
$$\frac{\partial S}{\partial y} = 2y + 2(1-3x-2y)(-2) = 2(6x+5y-2) = 0.$$

Simplifying gives

$$10x + 6y = 3$$
$$6x + 5y = 2,$$

with solution $x = 3/14, y = 1/7$. At this point $z = 1/14$. We have

$$D = S_{xx}S_{yy} - (S_{xy})^2 = (20)(10) - 12^2 = 56,$$

so $D > 0$ and $S_{xx} > 0$. Thus, the point $x = 3/14, y = 1/7$ is a local minimum. Since $S \to \infty$ as $x, y \to \pm\infty$, the local minimum is a global minimum. Thus, $x = 3/14, y = 1/7, z = 1/14$ is the closest point to the origin on the plane.

20. We minimize the square of the distance from the point (x, y, z) to the origin:

$$S = x^2 + y^2 + z^2.$$

Since $z^2 = 9 - xy - 3x$, we have

$$S = x^2 + y^2 + 9 - xy - 3x.$$

At a critical point

$$\frac{\partial S}{\partial x} = 2x - y - 3 = 0$$
$$\frac{\partial S}{\partial y} = 2y - x = 0,$$

so $x = 2y$, and

$$2(2y) - y - 3 = 0$$

giving $y = 1$, so $x = 2$ and $z^2 = 9 - 2\cdot 1 - 3\cdot 2 = 1$, so $z = \pm 1$. We have

$$D = S_{xx}S_{yy} - (S_{xy})^2 = 2\cdot 2 - (-1)^2 = 4 - 1 > 0,$$

so, since $D > 0$ and $S_{xx} > 0$, the critical points are local minima. Since $S \to \infty$ as $x, y \to \pm\infty$, the local minima are global minima.

If $x = 2, y = 1, z = \pm 1$, we have $S = 2^2 + 1^2 + 1^2 = 6$, so the shortest distance to the origin is $\sqrt{6}$.

21. (a) We draw the level curves (parallel straight lines) of $f(x, y) = ax + by + c$. We can see that the level lines with the maximum and minimum f-values which intersect with the disk are the level lines that are tangent to the boundary of the disk. Therefore, the maximum and minimum occur at the boundary of the disk. See Figure 15.14.

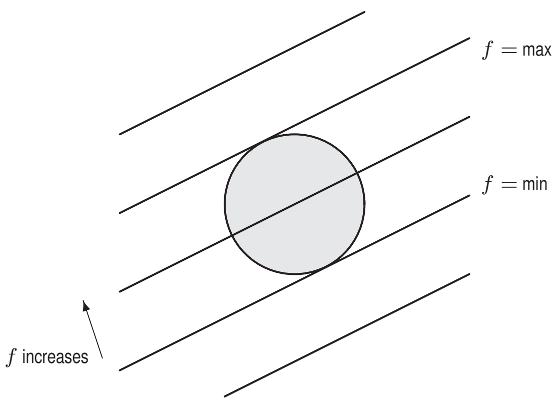


Figure 15.14

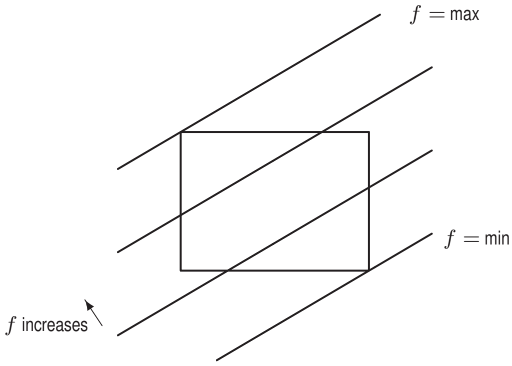


Figure 15.15

Figure 15.16

(b) Similar to part (a), we see the level lines with the largest and smallest f-values which intersect with the rectangle must pass the corner of the rectangle. So the maximum and minimum occur at the corners of rectangle. See Figure 15.15. When the level curves are parallel to a pair of the sides, then the points on the sides are all maximum or minimum, as shown below in Figure 15.16.

(c) The graph of f is a plane. The part of the graph lying above a disk R is either a flat disk, in which case every point is a maximum, or is a tilted ellipse, in which case you can see that the maximum will be on the edge. Similarly, the part lying above a rectangle is either a rectangle or a tilted parallelogram, in which case the maximum will be at a corner.

22. (a) The revenue $R = p_1q_1 + p_2q_2$. Profit $= P = R - C = p_1q_1 + p_2q_2 - 2q_1^2 - 2q_2^2 - 10$.

$$\frac{\partial P}{\partial q_1} = p_1 - 4q_1 = 0 \quad \text{gives } q_1 = \frac{p_1}{4}$$
$$\frac{\partial P}{\partial q_2} = p_2 - 4q_2 = 0 \quad \text{gives } q_2 = \frac{p_2}{4}$$

Since $\frac{\partial^2 P}{\partial q_1^2} = -4$, $\frac{\partial^2 P}{\partial q_2^2} = -4$ and $\frac{\partial^2 P}{\partial q_1 \partial q_2} = 0$, at $(p_1/4, p_2/4)$ we have that the discriminant, $D = (-4)(-4) > 0$ and $\frac{\partial^2 P}{\partial q_1^2} < 0$, thus P has a local maximum value at $(q_1, q_2) = (p_1/4, p_2/4)$. Since P is quadratic in q_1 and q_2, this is a global maximum. So the maximum profit is

$$P = \frac{p_1^2}{4} + \frac{p_2^2}{4} - 2\frac{p_1^2}{16} - 2\frac{p_2^2}{16} - 10 = \frac{p_1^2}{8} + \frac{p_2^2}{8} - 10.$$

(b) The rate of change of the maximum profit as p_1 increases is

$$\frac{\partial}{\partial p_1}(\max\ P) = \frac{2p_1}{8} = \frac{p_1}{4}.$$

23. The total revenue is

$$R = pq = (60 - 0.04q)q = 60q - 0.04q^2,$$

and as $q = q_1 + q_2$, this gives

$$R = 60q_1 + 60q_2 - 0.04q_1^2 - 0.08q_1q_2 - 0.04q_2^2.$$

Therefore, the profit is

$$\begin{aligned} P(q_1, q_2) &= R - C_1 - C_2 \\ &= -13.7 + 60q_1 + 60q_2 - 0.07q_1^2 - 0.08q_2^2 - 0.08q_1q_2. \end{aligned}$$

At a local maximum point, we have grad $P = \vec{0}$:

$$\frac{\partial P}{\partial q_1} = 60 - 0.14q_1 - 0.08q_2 = 0,$$
$$\frac{\partial P}{\partial q_2} = 60 - 0.16q_2 - 0.08q_1 = 0.$$

Solving these equations, we find that

$$q_1 = 300 \quad \text{and} \quad q_2 = 225.$$

To see whether or not we have found a local maximum, we compute the second-order partial derivatives:

$$\frac{\partial^2 P}{\partial q_1^2} = -0.14, \quad \frac{\partial^2 P}{\partial q_2^2} = -0.16, \quad \frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.08.$$

Therefore,

$$D = \frac{\partial^2 P}{\partial q_1^2}\frac{\partial^2 P}{\partial q_2^2} - \frac{\partial^2 P}{\partial q_1 \partial q_2} = (-0.14)(-0.16) - (-0.08)^2 = 0.016,$$

and so we have found a local maximum point. The graph of $P(q_1, q_2)$ has the shape of an upside down paraboloid since P is quadratic in q_1 and q_2, hence $(300, 225)$ is a global maximum point.

24. (a) This tells us that an increase in the price of either product causes a decrease in the quantity demanded of both products. An example of products with this relationship is tennis rackets and tennis balls. An increase in the price of either product is likely to lead to a decrease in the quantity demanded of both products as they are used together. In economics, it is rare for the quantity demanded of a product to increase if its price increases, so for q_1, the coefficient of p_1 is negative as expected. The coefficient of p_2 in the expression could be either negative or positive. In this case, it is negative showing that the two products are complementary in use. If it were positive, however, it would indicate that the two products are competitive in use, for example Coke and Pepsi.

(b) The revenue from the first product would be $q_1p_1 = 150p_1 - 2p_1^2 - p_1p_2$, and the revenue from the second product would be $q_2p_2 = 200p_2 - p_1p_2 - 3p_2^2$. The total sales revenue of both products, R, would be

$$R(p_1, p_2) = 150p_1 + 200p_2 - 2p_1p_2 - 2p_1^2 - 3p_2^2.$$

Note that R is a function of p_1 and p_2. To find the critical points of R, set $\nabla R = 0$, i.e.,

$$\frac{\partial R}{\partial p_1} = \frac{\partial R}{\partial p_2} = 0.$$

This gives

$$\frac{\partial R}{\partial p_1} = 150 - 2p_2 - 4p_1 = 0$$

and

$$\frac{\partial R}{\partial p_2} = 200 - 2p_1 - 6p_2 = 0$$

Solving simultaneously, we have $p_1 = 25$ and $p_2 = 25$. Therefore the point $(25, 25)$ is a critical point for R. Further,

$$\frac{\partial^2 R}{\partial p_1^2} = -4, \frac{\partial^2 R}{\partial p_2^2} = -6, \frac{\partial^2 R}{\partial p_1 \partial p_2} = -2,$$

so the discriminant at this critical point is

$$D = (-4)(-6) - (-2)^2 = 20.$$

Since $D > 0$ and $\partial^2 R/\partial p_1^2 < 0$, this critical point is a local maximum. Since R is quadratic in p_1 and p_2, this is a global maximum. Therefore the maximum possible revenue is

$$\begin{aligned} R &= 150(25) + 200(25) - 2(25)(25) - 2(25)^2 - 3(25)^2 \\ &= (6)(25)^2 + 8(25)^2 - 7(25)^2 \\ &= 4375. \end{aligned}$$

This is obtained when $p_1 = p_2 = 25$. Note that at these prices, $q_1 = 75$ units, and $q_2 = 100$ units.

25. Let $P(K, L)$ be the profit obtained using K units of capital and L units of labor. The cost of production is given by

$$C(K, L) = kK + \ell L,$$

and the revenue function is given by

$$R(K, L) = pQ = pAK^aL^b.$$

Hence, the profit is

$$P = R - C = pAK^aL^b - (kK + \ell L).$$

In order to find local maxima of P, we calculate the partial derivatives and see where they are zero. We have:

$$\frac{\partial P}{\partial K} = apAK^{a-1}L^b - k,$$
$$\frac{\partial P}{\partial L} = bpAK^aL^{b-1} - \ell.$$

The critical points of the function $P(K, L)$ are solutions (K, L) of the simultaneous equations:

$$\frac{k}{a} = pAK^{a-1}L^b,$$
$$\frac{\ell}{b} = pAK^aL^{b-1}.$$

Multiplying the first equation by K and the second by L, we get

$$\frac{kK}{a} = \frac{\ell L}{b},$$

and so

$$K = \frac{\ell a}{kb}L.$$

Substituting for K in the equation $k/a = pAK^{a-1}L^b$, we get:

$$\frac{k}{a} = pA\left(\frac{\ell a}{kb}\right)^{a-1} L^{a-1}L^b.$$

We must therefore have

$$L^{1-a-b} = pA\left(\frac{a}{k}\right)^a\left(\frac{\ell}{b}\right)^{a-1}.$$

Hence, if $a + b \neq 1$,

$$L = \left[pA\left(\frac{a}{k}\right)^a\left(\frac{\ell}{b}\right)^{(a-1)}\right]^{1/(1-a-b)},$$

and

$$K = \frac{\ell a}{kb}L = \frac{\ell a}{kb}\left[pA\left(\frac{a}{k}\right)^a\left(\frac{\ell}{b}\right)^{(a-1)}\right]^{1/(1-a-b)}.$$

To see if this is really a local maximum, we apply the second derivative test. We have:

$$\frac{\partial^2 P}{\partial K^2} = a(a-1)pAK^{a-2}L^b,$$
$$\frac{\partial^2 P}{\partial L^2} = b(b-1)pAK^aL^{b-2},$$
$$\frac{\partial^2 P}{\partial K \partial L} = abpAK^{a-1}L^{b-1}.$$

Hence,

$$\begin{aligned} D &= \frac{\partial^2 P}{\partial K^2}\frac{\partial^2 P}{\partial L^2} - \left(\frac{\partial^2 P}{\partial K \partial L}\right)^2 \\ &= ab(a-1)(b-1)p^2A^2K^{2a-2}L^{2b-2} - a^2b^2p^2A^2K^{2a-2}L^{2b-2} \\ &= ab((a-1)(b-1) - ab)p^2A^2K^{2a-2}L^{2b-2} \\ &= ab(1-a-b)p^2A^2K^{2a-2}L^{2b-2}. \end{aligned}$$

Now a, b, p, A, K, and L are positive numbers. So, the sign of this last expression is determined by the sign of $1 - a - b$.

(a) We assumed that $a + b < 1$, so $D > 0$, and as $0 < a < 1$, then $\partial^2 P/\partial K^2 < 0$ and so we have a unique local maximum. To verify that the local maximum is a global maximum, we focus on the cost. Let $C = kK + \ell L$. Since $K \geq 0$ and $L \geq 0$, $K \leq C/k$ and $L \leq C/\ell$. Therefore the profit satisfies:

$$\begin{aligned} P &= pAK^aL^b - (kK + \ell L) \\ &\leq pA\left(\frac{C}{k}\right)^a\left(\frac{C}{\ell}\right)^b - C \\ &= mC^{a+b} - C \end{aligned}$$

where $m = pA(1/k)^a(1/\ell)^b$. Since $a + b < 1$, the profit is negative for large costs C, say $C \geq C_0$ ($C_0 = m^{1-a-b}$ will do). Therefore, in the KL-plane for $K \geq 0$ and $L \geq 0$, the profit is less than or equal to zero everywhere on or above the line $kK + \ell L = C_o$. Thus the global maximum must occur inside the triangle bounded by this line and the K and L axes. Since $P \leq 0$ on the K and L axes as well, the global maximum must be in the interior of the triangle at the unique local maximum we found.

In the case $a + b < 1$, we have decreasing returns to scale. That is, if the amount of capital and labor used is multiplied by a constant $\lambda > 0$, we get less than λ times the production.

(b) Now suppose $a + b \geq 1$. If we multiply K and L by λ for some $\lambda > 0$, then

$$Q(\lambda K, \lambda L) = A(\lambda K)^a(\lambda L)^b = \lambda^{a+b}Q(K, L).$$

We also see that

$$C(\lambda K, \lambda L) = \lambda C(K, L).$$

So if $a + b = 1$, we have

$$P(\lambda K, \lambda L) = \lambda P(K, L).$$

Thus, if $\lambda = 2$, so we are doubling the inputs K and L, then the profit P is doubled and hence there can be no maximum profit.

If $a + b > 1$, we have increasing returns to scale and there can again be no maximum profit: doubling the inputs will more than double the profit. In this case, the profit increases without bound as K, L go toward infinity.

26. We have

$$\begin{aligned} f_x &= 2x(y+1)^3 = 0 \quad \text{only when } x = 0 \text{ or } y = -1 \\ f_y &= 3x^2(y+1)^2 + 2y = 0 \quad \text{never when } y = -1 \text{ and only for } y = 0 \text{ when } x = 0 \end{aligned}$$

We conclude that $f_x = 0$ and $f_y = 0$ only when $x = 0, y = 0$, so f has only one critical point, namely $(0, 0)$.

The second derivative test at $(0, 0)$ gives

$$\begin{aligned} D = f_{xx}f_{yy} - (f_{xy})^2 &= 2(y+1)^3(6x^2(y+1) + 2) - (6x(y+1)^2)^2 \\ &= 2(1)(2) - 0 > 0 \quad \text{when } x = 0, y = 0 \end{aligned}$$

Since $f_{xx} > 0$ at $(0, 0)$, this means f has a local minimum at $(0, 0)$.

[Alternatively, if we expand $(y + 1)^3$, then we can view $f(x, y)$ as $x^2 + y^2 +$ (terms of degree 3 or greater in x and y), which means that f behaves likes $x^2 + y^2$ near $(0, 0)$.]

Although $(0, 0)$ is a local minimum, it cannot be a global minimum since for fixed x, say $x = 1$, the function $f(x, y)$ is a cubic polynomial in y and cubics take on arbitrarily large positive and negative values.

In the single-variable case, suppose a function f defined on the real line is differentiable and its derivative is continuous. Then if f has only one critical point, say $x = 0$, and if that critical point is a local minimum, it must also be a global minimum. This is because f' cannot change sign without $f' = 0$ so we must have $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$. Thus f is decreasing for all $x < 0$ and increasing for all $x > 0$, so $x = 0$ is the global minimum for f.

27. The variables are a and b, so we set

$$\begin{aligned} \frac{\partial S}{\partial a} &= 2(a + b) + 8(4a + b - 2) + 18(9a + b - 4) = 0 \\ \frac{\partial S}{\partial b} &= 2(a + b) + 2(4a + b - 2) + 2(9a + b - 4) = 0, \end{aligned}$$

so, collecting terms and dividing by 4 and 2 respectively,

$$\begin{aligned} 49a + 7b - 22 &= 0 \\ 14a + 3b - 6 &= 0. \end{aligned}$$

Solving gives $a = 24/49$, $b = -2/7$.

Since there is only one critical point and S is unbounded as $a, b \to \infty$, this critical point is the global minimum. Therefore, the best fitting parabola is

$$y = \frac{24}{49}x^2 - \frac{2}{7}.$$

28. Let the line be in the form $y = b + mx$. Then, when x equals 0, 1, and 2, y equals b, $b + m$, and $b + 2m$ respectively. The sum of the squares of the vertical distances, which is what we want to minimize, is

$f(m, b) = (4 - b)^2 + (3 - (b + m))^2 + (1 - (b + 2m))^2$

To find critical points, set each partial derivative equal to zero.

$$\begin{aligned} f_m &= 0 + 2(3 - (b + m))(-1) + 2(1 - (b + 2m))(-2) \\ &= 6b + 10m - 10 \\ f_b &= 2(4 - b)(-1) + 2(3 - (b + m))(-1) + 2(1 - (b + 2m))(-1) \\ &= 6b + 6m - 16 \end{aligned}$$

Setting both partial derivatives equal to zero and dividing by 2, we get a system of equations:

$$\begin{aligned} 3b + 5m &= 5 \\ 3b + 3m &= 8 \end{aligned}$$

with solutions $m = -\frac{3}{2}$ and $b = \frac{25}{6}$. Thus, the line is $y = \frac{25}{6} - \frac{3}{2}x$.

29. (a) We have $f(2, 1) = 120$.

(i) If $x > 20$ then $f(x, y) > 10x > 200 > f(2, 1)$.

(ii) If $y > 20$ then $f(x, y) > 20y > 400 > f(2, 1)$.

(iii) If $x < 0.01$ and $y \le 20$ then $f(x, y) > 80/(xy) > 80/((0.01)(20)) = 400 > f(2, 1)$.

(iv) If $y < 0.01$ and $x \le 20$ then $f(x, y) > 80/(xy) > 80/((20)(0.01)) = 400 > f(2, 1)$.

(b) The continuous function f must achieve a minimum at some point (x_0, y_0) in the closed and bounded region R' : $0.01 \le x \le 20, 0.01 \le y \le 20$. Since $(2, 1)$ is in R', we must have $f(x_0, y_0) \le f(2, 1)$. By part (a), $f(x_0, y_0)$ is less than all values of f in the part of R that is outside R', so $f(x_0, y_0)$ is a minimum for f on all of R. Since (x_0, y_0) is not on the boundary of R, it must be a critical point of f.

(c) The only critical point of f in R is the point $(2, 1)$, so by part (b) f has a global minimum there.

30. (a) The function f is continuous in the region R, but R is not closed and bounded so a special analysis is required.

Notice that $f(x, y)$ tends to ∞ as (x, y) tends farther and farther from the origin or tends toward any point on the x or y axis. This suggests that a minimum for f, if it exists, can not be too far from the origin or too close to the axes. For example, if $x > 10$ then $f(x, y) > 4x > 40$, and if $y > 10$ then $f(x, y) > 5y > 50$. If $0 < x < 0.1$ then $f(x, y) > 2/x > 20$, and if $0 < y < 0.1$ then $f(x, y) > 3/y > 30$.

Since $f(1, 1) = 14$, a global minimum for f if it exists must be in the smaller region R' : $0.1 \le x \le 10$, $0.1 \le y \le 10$. The region R' is closed and bounded and so f does have a minimum value at some point in R', and since that value is at most 14, it is also a global minimum for all of R.

(b) Since the region R has no boundary, the minimum value must occur at a critical point of f. At a critical point we have

$$f_x = -\frac{2}{x^2} + 4 = 0 \qquad f_y = -\frac{3}{y^2} + 5 = 0.$$

The only critical point is $(\sqrt{1/2}, \sqrt{3/5}) \approx (0.7071, 0.7746)$, at which f achieves the minimum value $f(\sqrt{1/2}, \sqrt{3/5}) = 4\sqrt{2} + 2\sqrt{15} \approx 13.403$.

31. (a) There is one variable, p, in this problem; P_0 and P_F are constants. At the minimum energy,

$$\begin{aligned} \frac{dE}{dp} = \frac{2p}{P_0^2} - \frac{2P_F^2}{p^3} &= 0 \\ p^4 &= P_0^2 P_F^2 \\ p &= \sqrt{P_0 P_F}. \end{aligned}$$

This value of p gives a minimum for E because it is the only critical point and the value of E grows toward ∞ as $p \to \infty$.

(b) There are now two variables, p_1 and p_2, so at the minimum energy,

$$\frac{\partial E}{\partial p_1} = \frac{2p_1}{P_0^2} - \frac{2p_2^2}{p_1^3} = 0$$
$$\frac{\partial E}{\partial p_2} = \frac{2p_2}{p_1^2} - \frac{2P_F^2}{p_2^3} = 0.$$

Solving these equations simultaneously:

$$\frac{p_1}{P_0^2} = \frac{p_2^2}{p_1^3}, \quad \text{so } p_1^4 = P_0^2 p_2^2 \quad \text{and taking square roots gives} \quad p_1^2 = P_0 p_2.$$
$$\frac{p_2}{p_1^2} = \frac{P_F^2}{p_2^3}, \quad \text{so } p_2^4 = P_F^2 p_1^2 \quad \text{and taking square roots gives} \quad p_2^2 = P_F p_1.$$

Substituting $p_2^2 = P_F p_1$ into $p_1^4 = P_0^2 p_2^2$ gives

$$p_1^4 = P_0^2 P_F p_1$$
$$p_1^3 = P_0^2 P_F \quad \text{so} \quad p_1 = \sqrt[3]{P_0^2 P_F}.$$

Substituting $p_1^2 = P_0 p_2$ into $p_2^4 = P_F^2 p_1^2$ gives

$$p_2^4 = P_F^2 P_0 p_2 \quad \text{so} \quad p_2 = \sqrt[3]{P_0 P_F^2}.$$

The critical value of E is again a minimum because the critical point is unique and the value of E tends to ∞ as $p_1, p_2 \to \infty$.

32. (a) Look at the formula $q = Kp^{-E}a^{\theta}$. Since price, p, has a negative exponent, when the price is increased, less is sold. Since advertising, a, has a positive exponent, when the amount of advertising is increased, more is sold.

(b) Since $q = Kp^{-E}a^{\theta}$, we have

$$\frac{\partial q}{\partial p} = Ka^{\theta}\frac{\partial}{\partial p}(p^{-E}) = Ka^{\theta}(-Ep^{-E-1}) = -\frac{EKp^{-E}a^{\theta}}{p} = -\frac{Eq}{p}$$

and

$$\frac{\partial q}{\partial a} = Kp^{-E}\frac{\partial}{\partial a}(a^{\theta}) = Kp^{-E}\theta a^{\theta-1} = \frac{\theta K p^{-E}a^{\theta}}{a} = \frac{\theta q}{a}.$$

(c) Profit is revenue minus cost. Revenue is pq. Cost is the sum of the cost cq of producing the items and the cost $p_a a$ of advertising. Hence

$$\pi = \text{Profit} = \text{Revenue} - \text{Cost} = pq - cq - p_a a.$$

(d) At maximum profit, both partial derivatives are zero:

$$\frac{\partial \pi}{\partial p} = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial a} = 0.$$

(e) Using $\pi = pq - cq - p_a a$, we have

$$\frac{\partial \pi}{\partial p} = q + p\frac{\partial q}{\partial p} - c\frac{\partial q}{\partial p} = q + (p-c)\frac{\partial q}{\partial p}$$
$$\frac{\partial \pi}{\partial p} = q + (p-c)\left(-\frac{E}{p}q\right) = q\left(1 - (p-c)\frac{E}{p}\right).$$

In addition

$$\frac{\partial \pi}{\partial a} = p\frac{\partial q}{\partial a} - c\frac{\partial q}{\partial a} - p_a = q(p-c)\frac{\theta}{a} - p_a.$$

(f) At maximum profit, $\partial \pi/\partial p = 0$, so solving for p gives

$$1 - (p-c)\frac{E}{p} = 0$$
$$\frac{(p-c)E}{p} = 1$$
$$\frac{p-c}{p} = \frac{1}{E}$$

In addition, at maximum profit, $\partial\pi/\partial a = 0$, so we have

$$q(p-c)\frac{\theta}{a} = p_a$$
$$\frac{p-c}{p_a} = \frac{a}{q\theta}.$$

(g) By parts (e) and (f), at maximum profit we have

$$\frac{p-c}{p}\cdot\frac{p_a}{p-c} = \frac{1}{E}\cdot\frac{q\theta}{a}$$

and hence

$$\frac{p_a a}{pq} = \frac{\theta}{E}.$$

The numerator, $p_a a$, is the amount the company spends on advertising. The denominator, pq, is the company's revenue. The monopoly spends a fixed fraction, θ/E of its revenue for advertising, no matter how the price of advertising might change.

Strengthen Your Understanding

33. If the region is closed and bounded and the function is continuous, then it must have a global maximum, even if it has no critical points.

34. The Extreme Value Theorem does not say what happens when R is unbounded. For example, suppose the region R is the whole xy-plane. Some functions may have a global minimum, such as $f(x,y) = x^2 + y^2$, and others may not, such as $f(x,y) = x + y$.

35. The local maximum is not necessarily the global maximum. For example, let $f(x,y)$ be a function with the contour diagram in Figure 15.17.

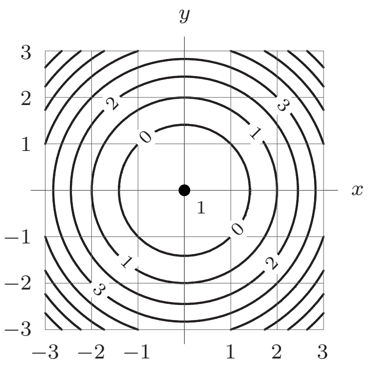


Figure 15.17

This function has a local maximum value of 1 at the origin. However, 1 is definitely not the global maximum value since the function attains higher values at other input values.

36. Let $f(x,y) = x + y$, which tends to $+\infty$ as $x \to \infty$ and tends to $-\infty$ as $x \to -\infty$.

37. Let $f(x,y) = x^2 + y^2$ and let R be the square $0 \le x \le 1$, $0 \le y \le 1$. Then the maximum value of f occurs at $(1,1)$.

38. True. For unconstrained optimization, global extrema occur at one (or more) of the local extrema.

39. False. For example, the linear function $f(x,y) = x + y$ has neither a global minimum or global maximum on all of 2-space.

40. True. The region is the unit disk without its boundary (the unit circle), and the distance between any two points in this region is less than 2—it does not stretch off to infinity in any direction.

41. False. The region is the unit disk without its boundary (the unit circle), so it is not closed (in fact, it is open).

42. True. The global minimum is 0, which occurs at the origin. This is clear since the function $f(x,y) = x^2 + y^2$ is greater than or equal to zero everywhere, and is only zero at the origin.

43. False. On the given region the function f is always less than one. By picking points closer and closer to the circle $x^2 + y^2 = 1$ we can make f larger and larger (although never larger than one). There is no point in the open disk that gives f its largest value.

44. False. While f can only have (at most) one largest *value*, it may attain this value at more than one point. For example, the function $f(x, y) = \sin(x + y)$ has a global maximum of 1 at both $(\pi/2, 0)$ and $(0, \pi/2)$.

45. True. The region is both closed and bounded, guaranteeing both a global maximum and minimum.

46. True. The global minimum could occur on the boundary of the region.

Solutions for Section 15.3

Exercises

1. Our objective function is $f(x, y) = x + y$ and our equation of constraint is $g(x, y) = x^2 + y^2 = 1$. To optimize $f(x, y)$ with Lagrange multipliers, we solve $\nabla f(x, y) = \lambda \nabla g(x, y)$ subject to $g(x, y) = 1$. The gradients of f and g are

$$\nabla f(x, y) = \vec{i} + \vec{j},$$
$$\nabla g(x, y) = 2x\vec{i} + 2y\vec{j}.$$

So the equation $\nabla f = \lambda \nabla g$ becomes

$$\vec{i} + \vec{j} = \lambda(2x\vec{i} + 2y\vec{j})$$

Solving for λ gives

$$\lambda = \frac{1}{2x} = \frac{1}{2y},$$

which tells us that $x = y$. Going back to our equation of constraint, we use the substitution $x = y$ to solve for y:

$$\begin{aligned} g(y, y) = y^2 + y^2 &= 1 \\ 2y^2 &= 1 \\ y^2 &= \frac{1}{2} \\ y &= \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}. \end{aligned}$$

Since $x = y$, our critical points are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points we find that the maximum value is $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \sqrt{2}$ and the minimum value is $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = -\sqrt{2}$.

2. The Lagrange conditions give:

$$1 = \lambda 2x, \quad 3 = \lambda 2y.$$

Thus $2\lambda = 1/x = 3/y$, so $y = 3x$. Substituting this into the constraint, we get $x^2 + (3x)^2 = 10$, so $x = \pm 1$. Since $y = 3x$, the points satisfying the Lagrange conditions are $(1, 3)$ and $(-1, -3)$. Since $f(1, 3) = 12$ and $f(-1, -3) = -8$, the maximum value is 12 at $(1, 3)$ and the minimum value is -8 at $(-1, -3)$.

3. The Lagrange conditions give:

$$2(x - 1) = \lambda 2x, \quad 2(y + 2) = \lambda 2y.$$

We can't have $x = 0$, since then the first equation becomes $-2 = 0$. Similarly, $y \neq 0$. Thus we can divide by x and y. Solving both equations for λ and setting the expressions equal, we get

$$\frac{x - 1}{x} = \frac{y + 2}{y}.$$

Thus, we have $y(x - 1) = x(y + 2)$, so $y = -2x$. Substituting this into the constraint, we get $x^2 + (-2x)^2 = 5$, so $x = \pm 1$. Since $y = -2x$, the points satisfying the Lagrange conditions are $(1, -2)$ and $(-1, 2)$. Since $f(1, -2) = 0$ and $f(-1, 2) = 20$, the maximum value is 20 at $(-1, 2)$ and the minimum value is 0 at $(1, -2)$.

4. The Lagrange conditions give:

$$3x^2 = \lambda 6x, \quad 1 = \lambda 2y.$$

If $x = 0$ in the first equation, then from the constraint, $y = \pm 2$. Thus $(0, \pm 2)$ are two points satisfying the Lagrange conditions. If $x \neq 0$, we can solve for λ from both equations and setting the expressions equal, we get

$$\frac{1}{2}x = \frac{1}{2y},$$

so $y = 1/x$. Substituting this into the constraint, we get $3x^2 + (1/x)^2 = 4$. Therefore $3x^4 + 1 = 4x^2$, so

$$3x^4 - 4x^2 + 1 = (x^2 - 1)(3x^2 - 1) = 0.$$

Therefore $x = \pm 1, x = \pm 1/\sqrt{3}$. Since $y = 1/x$, we get

$$(1, 1), \quad (-1, -1), \quad (1/\sqrt{3}, \sqrt{3}), \quad (-1/\sqrt{3}, -\sqrt{3}),$$

as points satisfying the Lagrange conditions. The corresponding values of $f(x, y)$ are $2, -2, 1.92, -1.92$. There are also the points $(0, \pm 2)$ we found for the case $x = 0$, and the values for f there are ± 2. Thus the maximum value is 2 at $(1, 1)$ and $(0, 2)$ and the minimum value is -2 at $(-1, -1)$ and $(0, -2)$.

5. Our objective function is $f(x, y) = 3x - 2y$ and our equation of constraint is $g(x, y) = x^2 + 2y^2 = 44$. Their gradients are

$$\nabla f(x, y) = 3\vec{i} - 2\vec{j},$$
$$\nabla g(x, y) = 2x\vec{i} + 4y\vec{j}.$$

So the equation $\nabla f = \lambda \nabla g$ becomes $3\vec{i} - 2\vec{j} = \lambda(2x\vec{i} + 4y\vec{j})$. Solving for λ gives us

$$\lambda = \frac{3}{2x} = \frac{-2}{4y},$$

which we can use to find x in terms of y:

$$\begin{aligned} \frac{3}{2x} &= \frac{-2}{4y} \\ -4x &= 12y \\ x &= -3y. \end{aligned}$$

Using this relation in our equation of constraint, we can solve for y:

$$\begin{aligned} x^2 + 2y^2 &= 44 \\ (-3y)^2 + 2y^2 &= 44 \\ 9y^2 + 2y^2 &= 44 \\ 11y^2 &= 44 \\ y^2 &= 4 \\ y &= \pm 2. \end{aligned}$$

Thus, the critical points are $(-6, 2)$ and $(6, -2)$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points, we find that the maximum is $f(6, -2) = 18 + 4 = 22$ and the minimum value is $f(-6, 2) = -18 - 4 = -22$.

6. The objective function is $f(x, y) = 2xy$ and the constraint equation is $g(x, y) = 5x + 4y = 100$, so $\operatorname{grad} f = (2y)\vec{i} + (2x)\vec{j}$ and $\operatorname{grad} g = 5\vec{i} + 4\vec{j}$. Setting $\operatorname{grad} f = \lambda \operatorname{grad} g$ gives

$$2y = 5\lambda,$$
$$2x = 4\lambda.$$

From the first equation we have $\lambda = 2y/5$, and from the second equation we have $\lambda = x/2$. Setting these equal gives

$$y = 1.25x.$$

Substituting this into the constraint equation $5x + 4y = 100$ gives $x = 10$ and $y = 12.5$. A maximum or minimum value for f subject to the constraint can occur only at $(10, 12.5)$.

We have $f(10, 12.5) = 250$. From Figure 15.18, we see that the point $(10, 12.5)$ gives a maximum.

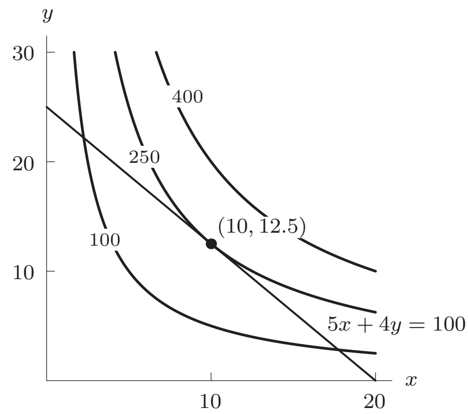


Figure 15.18

7. Let $f(x_1, x_2) = {x_1}^2 + {x_2}^2$ and $g(x_1, x_2) = x_1 + x_2$. Then grad $f = \lambda$ grad g gives

$$2x_1 = \lambda$$
$$2x_2 = \lambda,$$

so $x_1 + x_2 = 1$ gives

$$\frac{\lambda}{2} + \frac{\lambda}{2} = 1 \quad \text{or} \quad \lambda = 1.$$

Thus

$$x_1 = x_2 = \frac{1}{2}.$$

Since $f(x_1, x_2)$ becomes arbitrarily large as $x_1, x_2 \to \infty$, there is no global maximum. The global minimum is given by

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

8. Our objective function is $f(x, y) = x^2 + y$ and our equation of constraint is $g(x, y) = x^2 - y^2 = 1$. Their gradients are

$$\nabla f(x, y) = 2x\vec{i} + \vec{j},$$
$$\nabla g(x, y) = 2x\vec{i} - 2y\vec{j}.$$

Thus $\nabla f = \lambda \nabla g$ gives

$$2x = \lambda 2x$$
$$1 = -\lambda 2y$$

But x cannot be zero, since the constraint equation, $-y^2 = 1$, would then have no real solution for y. So the equation $\nabla f = \lambda \nabla g$ becomes

$$\lambda = \frac{2x}{2x} = \frac{1}{-2y}$$
$$1 = \frac{1}{-2y}$$
$$-2y = 1$$
$$y = -\frac{1}{2}.$$

Substituting this into our equation of constraint we find

$$g(x, -\frac{1}{2}) = x^2 - \left(-\frac{1}{2}\right)^2 = 1$$
$$x^2 = \frac{5}{4}$$
$$x = \pm\frac{\sqrt{5}}{2}.$$

So the critical points are $(\frac{\sqrt{5}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{5}}{2}, -\frac{1}{2})$. Evaluating f at these points we find $f(\frac{\sqrt{5}}{2}, -\frac{1}{2}) = f(-\frac{\sqrt{5}}{2}, -\frac{1}{2}) = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$. This is the minimum value for $f(x, y)$ constrained to $g(x, y) = 1$. To see this, note that for $x^2 = y^2 + 1$, $f(x, y) = y^2 + 1 + y = (y + 1/2)^2 + 3/4 \geq 3/4$. Alternatively, see Figure 15.19. To see that f has no maximum on $g(x, y) = 1$, note that $f \to \infty$ as $x \to \infty$ and $y \to \infty$ on the part of the graph of $g(x, y) = 1$ in quadrant I.

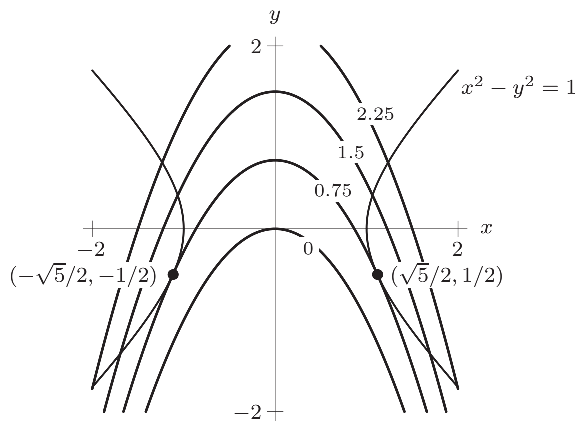


Figure 15.19: Graph of $x^2 - y^2 = 1$

9. The objective function is $f(x, y, z) = x + 3y + 5z$ and the equation of constraint is $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Their gradients are

$$\nabla f(x, y, z) = \vec{i} + 3\vec{j} + 5\vec{k},$$
$$\nabla g(x, y, z) = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

So the equation $\nabla f = \lambda \nabla g$ becomes $\vec{i} + 3\vec{j} + 5\vec{k} = \lambda(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$. Solving for λ we find

$$\lambda = \frac{1}{2x} = \frac{3}{2y} = \frac{5}{2z}.$$

Which provides us with the equations

$$2y = 6x$$
$$10x = 2z.$$

Solving the first equation for y gives us $y = 3x$. Solving the second equation for z gives us $z = 5x$. Substituting these into the equation of constraint, we can find x:

$$x^2 + (3x)^2 + (5x)^2 = 1$$
$$x^2 + 9x^2 + 25x^2 = 1$$
$$35x^2 = 1$$
$$x^2 = \frac{1}{35}$$
$$x = \pm\sqrt{\frac{1}{35}} = \pm\frac{\sqrt{35}}{35}.$$

Since $y = 3x$ and $z = 5x$, the critical points are at $\pm(\frac{\sqrt{35}}{35}, 3\frac{\sqrt{35}}{35}, \frac{\sqrt{35}}{7})$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points, we find the maximum is $f(\frac{\sqrt{35}}{35}, 3\frac{\sqrt{35}}{35}, \frac{\sqrt{35}}{7}) = \sqrt{35}\frac{35}{35} = \sqrt{35}$, and the minimum value is $f(-\frac{\sqrt{35}}{35}, -3\frac{\sqrt{35}}{35}, -\frac{\sqrt{35}}{7}) = -\sqrt{35}$.

10. Our objective function is $f(x, y, z) = x^2 - y^2 - 2z$ and our equation of constraint is $g(x, y, z) = x^2 + y^2 - z = 0$. To optimize $f(x, y, z)$ with Lagrange multipliers, we solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ subject to $g(x, y, z) = 0$. The gradients of f and g are

$$\nabla f(x, y, z) = 2x\vec{i} - 2y\vec{j} - 2\vec{k},$$
$$\nabla g(x, y, z) = 2x\vec{i} + 2y\vec{j} - \vec{k}.$$

We get

$$\begin{aligned} 2x &= 2\lambda x \\ -2y &= 2\lambda y \\ -2 &= -\lambda \\ x^2 + y^2 &= z. \end{aligned}$$

The third equation gives $\lambda = 2$ and from the first $x = 0$, from the second $y = 0$ and from the fourth $z = 0$. So the only solution is $(0, 0, 0)$, and $f(0, 0, 0) = 0$.

To see what kind of extreme point is $(0, 0, 0)$, let (a, b, c) be a point which satisfies the constraint, i.e. $a^2 + b^2 = c$. Then $f(a, b, c) = a^2 - b^2 - 2c = -a^2 - 3b^2 \leq 0$. The conclusion is that 0 is the maximum value of f and that there is no minimum.

11. The Lagrange conditions give:

$$yz = \lambda 2x, \quad xz = \lambda 2y, \quad xy = \lambda 8z.$$

We note that if $x = 0$, then the objective function $f(x, y, z) = xyz$ has value 0 and this cannot be the maximum or minimum value since xyz can take on both positive and negative values. Similarly, we can assume that $y \neq 0$ and $z \neq 0$. Solving for λ and setting expressions equal, we get:

$$\frac{yz}{x} = \frac{xz}{y} = \frac{xy}{4z}.$$

Thus $y^2z = x^2z$ so $y^2 = x^2$, and $4xz^2 = xy^2$, so $y^2 = 4z^2$. Therefore

$$x^2 + y^2 + 4z^2 = y^2 + y^2 + y^2 = 12,$$

so $y = \pm 2$. Since $x = \pm y$ and $z = \pm y/2$, there are eight points satisfying the Lagrange conditions, each of the form $(\pm 2, \pm 2, \pm 1)$. Thus the maximum value of the objective function $f(x, y, z) = xyz$ is 4 at $(2, 2, 1)$,$(2, -2, -1)$, $(-2, 2, -1)$,$(-2, -2, 1)$, and the minimum value is -4 at $(-2, -2, -1)$,$(2, 2, -1)$,$(2, -2, 1)$,$(-2, 2, 1)$.

12. The region $x^2 + y^2 \leq 4$ is the shaded disk of radius 2 centered at the origin (including the circle $x^2 + y^2 = 4$) shown in Figure 15.20.

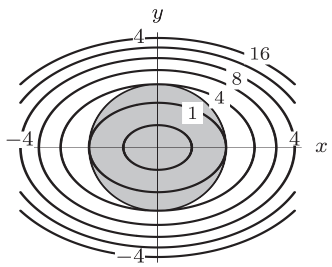


Figure 15.20

We will first find the local maxima and minima in the interior of the disk. So we need to find the extrema of

$$f(x, y) = x^2 + 2y^2 \quad \text{in the region} \quad x^2 + y^2 < 4.$$

For this we compute the critical points:

$$\begin{aligned} f_x &= 2x = 0 \\ f_y &= 4y = 0 \end{aligned}$$

So the critical point is $(0, 0)$. As $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 4$ and $f_{xy}(0, 0) = 0$ we have

$$D = f_{xx}(0, 0) \cdot f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 8 > 0 \quad \text{and} \quad f_{xx}(0, 0) = 2 > 0.$$

Therefore $(0, 0)$ is a minimum point and $f(0, 0) = 0$.

Now let's find the local extrema of f on the boundary of the disk, hence this time we have to solve a constraint problem. We want the extrema of $f(x,y) = x^2 + 2y^2$ subject to $g(x,y) = x^2 + y^2 - 4 = 0$. We use Lagrange multipliers:

$$\text{grad}\, f = \lambda\, \text{grad}\, g \quad \text{and} \quad x^2 + y^2 = 4,$$

which give

$$\begin{aligned} 2x &= 2\lambda x \\ 4y &= 2\lambda y \\ x^2 + y^2 &= 4. \end{aligned}$$

From the first equation we have $x = 0$ or $\lambda = 1$. If $x = 0$, from the last equation $y^2 = 4$ and therefore $(0,2)$ and $(0,-2)$ are solutions.

If $x \neq 0$ then $\lambda = 1$ and from the second equation $y = 0$. Substituting this into the third equation we get $x^2 = 4$ so $(2,0)$ and $(-2,0)$ are the other two solutions.

The region $x^2 + y^2 \leq 4$ is closed and bounded, so maximum and minimum values of f in the the region exist. Therefore, as $f(0,2) = f(0,-2) = 8$ and $f(2,0) = f(-2,0) = 4$, $(0,2)$ and $(0,-2)$ are global maxima and $(0,0)$ is the global minimum on the whole region. The maximum value of f is 8 and the minimum value of f is 0.

13. The region $x^2 + y^2 \leq 2$ is the shaded disk of radius $\sqrt{2}$ centered at the origin (including the circle $x^2 + y^2 = 2$) as shown in Figure 15.21.

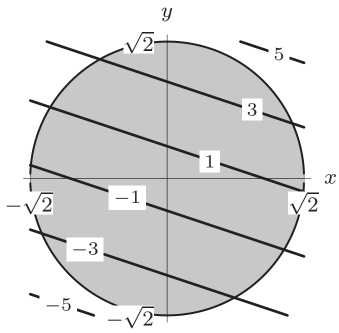


Figure 15.21

We first find the local maxima and minima of f in the interior of our disk. So we need to find the extrema of

$$f(x,y) = x + 3y, \quad \text{in the region} \quad x^2 + y^2 < 2.$$

As

$$\begin{aligned} f_x &= 1 \\ f_y &= 3 \end{aligned}$$

f does not have critical points. Now let's find the local extrema of f on the boundary of the disk. We want to find the extrema of $f(x,y) = x + 3y$ subject to the constraint $g(x,y) = x^2 + y^2 - 2 = 0$. We use Lagrange multipliers

$$\text{grad}\, f = \lambda\, \text{grad}\, g \quad \text{and} \quad x^2 + y^2 = 2,$$

which give

$$\begin{aligned} 1 &= 2\lambda x \\ 3 &= 2\lambda y \\ x^2 + y^2 &= 2. \end{aligned}$$

As λ cannot be zero, we solve for x and y in the first two equations and get $x = \frac{1}{2\lambda}$ and $y = \frac{3}{2\lambda}$. Plugging into the third equation gives

$$8\lambda^2 = 10$$

so $\lambda = \pm\frac{\sqrt{5}}{2}$ and we get the solutions $(\frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}})$ and $(-\frac{1}{\sqrt{5}}, -\frac{3}{\sqrt{5}})$. Evaluating f at these points gives

$$f(\frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}}) = 2\sqrt{5} \quad \text{and}$$
$$f(-\frac{1}{\sqrt{5}}, -\frac{3}{\sqrt{5}}) = -2\sqrt{5}$$

The region $x^2 + y^2 \leq 2$ is closed and bounded, so maximum and minimum values of f in the region exist. Therefore $(\frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}})$ is a global maximum of f and $(-\frac{1}{\sqrt{5}}, -\frac{3}{\sqrt{5}})$ is a global minimum of f on the whole region $x^2 + y^2 \leq 2$.

14. The domain $x^2+2y^2 \leq 1$ is the shaded interior of the ellipse $x^2+2y^2 = 1$ including the boundary, shown in Figure 15.22.

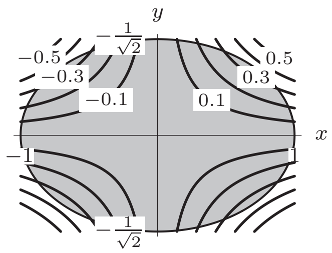


Figure 15.22

First we want to find the local maxima and minima of f in the interior of the ellipse. So we need to find the extrema of

$$f(x, y) = xy, \quad \text{in the region} \quad x^2 + 2y^2 < 1.$$

For this we compute the critical points:

$$f_x = y = 0 \quad \text{and} \quad f_y = x = 0.$$

So there is one critical point, $(0, 0)$. As $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$ and $f_{xy}(0, 0) = 1$ we have

$$D = f_{xx}(0, 0) \cdot f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = -1 < 0$$

so $(0, 0)$ is a saddle and f does not have local extrema in the interior of the ellipse.

Now let's find the local extrema of f on the boundary, hence this time we'll have a constraint problem. We want the extrema of $f(x, y) = xy$ subject to $g(x, y) = x^2 + 2y^2 - 1 = 0$. We use Lagrange multipliers:

$$\text{grad}\, f = \lambda\, \text{grad}\, g \quad \text{and} \quad x^2 + 2y^2 = 1$$

which give

$$y = 2\lambda x$$
$$x = 4\lambda y$$
$$x^2 + 2y^2 = 1$$

From the first two equations we get

$$xy = 8\lambda^2 xy.$$

So $x = 0$ or $y = 0$ or $8\lambda^2 = 1$.

If $x = 0$, from the last equation $2y^2 = 1$ so $y = \pm\frac{\sqrt{2}}{2}$ and we get the solutions $(0, \frac{\sqrt{2}}{2})$ and $(0, -\frac{\sqrt{2}}{2})$.
If $y = 0$, from the last equation we get $x^2 = 1$ and so the solutions are $(1, 0)$ and $(-1, 0)$.
If $x \neq 0$ and $y \neq 0$ then $8\lambda^2 = 1$, hence $\lambda = \pm\frac{1}{2\sqrt{2}}$. For $\lambda = \frac{1}{2\sqrt{2}}$

$$x = \sqrt{2}y$$

and plugging into the third equation gives $4y^2 = 1$ so we get the solutions $(\frac{\sqrt{2}}{2}, \frac{1}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{1}{2})$.

For $\lambda = -\frac{1}{2\sqrt{2}}$ we get

$$x = -\sqrt{2}y$$

and plugging into the third equation gives $4y^2 = 1$, and the solutions $(\frac{\sqrt{2}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{2}}{2}, \frac{1}{2})$. So finally we have the solutions: $(1,0)$, $(-1,0)$, $(\frac{\sqrt{2}}{2}, \frac{1}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{2}}{2}, -\frac{1}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{1}{2})$.

Evaluating f at these points gives:

$$f(0, \frac{\sqrt{2}}{2}) = f(0, -\frac{\sqrt{2}}{2}) = f(1,0) = f(-1,0) = 0$$
$$f(\frac{\sqrt{2}}{2}, \frac{1}{2}) = f(-\frac{\sqrt{2}}{2}, -\frac{1}{2}) = \frac{\sqrt{2}}{4}$$
$$f(\frac{\sqrt{2}}{2}, -\frac{1}{2}) = f(-\frac{\sqrt{2}}{2}, \frac{1}{2}) = -\frac{\sqrt{2}}{4}.$$

The region $x^2 + 2y^2 \le 1$ is closed and bounded, so the maximum and minimum values of f in the region exist. Hence the maximum value of f is $\frac{\sqrt{2}}{4}$ and the minimum value of f is $-\frac{\sqrt{2}}{4}$.

15. The region $x + y \ge 1$ is the shaded half plane (including the line $x + y = 1$) shown in Figure 15.23.

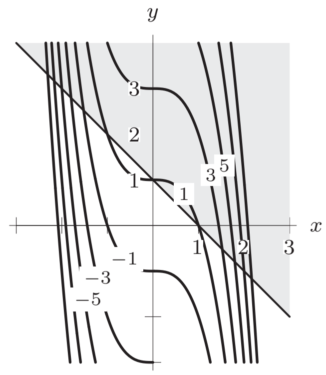


Figure 15.23

Let's look for the critical points of f in the interior of the region. As

$$f_x = 3x^2$$
$$f_y = 1$$

there are no critical points inside the shaded region. Now let's find the extrema of f on the boundary of our region. We want the extrema of $f(x,y) = x^3 + y$ subject to the constraint $g(x,y) = x + y - 1 = 0$. We use Lagrange multipliers

$$\text{grad}\, f = \lambda\, \text{grad}\, g \quad \text{and} \quad x + y = 1,$$

which give

$$3x^2 = \lambda$$
$$1 = \lambda$$
$$x + y = 1.$$

From the first two equations we get $3x^2 = 1$, so the solutions are

$$(\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}) \quad \text{and} \quad (-\frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}).$$

Evaluating f at these points we get

$$f(\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}) = 1 - \frac{2}{3\sqrt{3}}$$
$$f(-\frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}) = 1 + \frac{2}{3\sqrt{3}}.$$

From the contour diagram in Figure 15.23, we see that $(\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}})$ is a local minimum and $(-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}})$ is a local maximum of f on $x+y=1$. Are they global extrema as well?

If we take x very big and $y=1-x$ then $f(x,y)=x^3+y=x^3-x+1$ which can be made as big as we want (if we choose x big enough). So there will be no global maximum.

Similarly, taking x negative with big absolute value and $y=1-x$, $f(x,y)=x^3+y=x^3-x+1$ can be made as small as we want (if we choose x small enough). So there is no global minimum. This can also be seen from Figure 15.23.

16. First, we look for critical points for f:

$$2(x+3)=0, \qquad 2(y-3)=0.$$

Thus the only critical point for f is $(-3,3)$, but this point does not satisfy the constraint $x^2+y^2\le 2$, so we do not use it. The Lagrange conditions are

$$2(x+3)=\lambda 2x, \quad 2(y-3)=\lambda 2y.$$

If $x=0$, the first equation becomes $3=0$, so $x\neq 0$. Similarly, $y\neq 0$. Solving for λ and setting expressions equal, we have

$$\frac{x+3}{x}=\frac{y-3}{y}.$$

Thus, $(x+3)y=(y-3)x$, so $y=-x$. Substituting this into the constraint equation, we get $2x^2=2$, so $x=\pm 1$. Since $y=-x$, the points satisfying the Lagrange conditions are $(1,-1)$ and $(-1,1)$. Since $f(1,-1)=32$ and $f(-1,1)=8$, the maximum value is 32 at $(1,-1)$ and the minimum value is 8 at $(-1,1)$.

17. We first find the critical points of f:

$$f_x=2xy=0, \quad f_y=x^2+6y-1=0.$$

From the first equation, we get either $x=0$ or $y=0$. If $x=0$, from the second equation we get $6y-1=0$ so $y=1/6$. If instead $y=0$, then from the second equation $x=\pm 1$. We conclude that the critical points are $(0,1/6)$, $(1,0)$, and $(-1,0)$. All three critical points satisfy the constraint $x^2+y^2\le 10$.

The Lagrange conditions, $\operatorname{grad} f=\lambda \operatorname{grad} g$, are:

$$2xy=\lambda 2x, \qquad x^2+6y-1=\lambda 2y$$

From the first equation, when $x\neq 0$, we divide by x to get $\lambda=y$. Substituting into the second equation, we get

$$x^2+6y-1=2y^2.$$

Then using $x^2=10-y^2$ from the constraint, we have

$$10-y^2+6y-1=2y^2,$$

so $3y^2-6y-9=0$. Factoring, we get $3(y-3)(y+1)=0$. From the constraint, we get $x=\pm 1$ when $y=3$ and $x=\pm 3$ for $y=-1$. If instead $x=0$, so that we cannot divide by x in the first Lagrange equation, then from the constraint, $y=\pm\sqrt{10}$. Summarizing, the following points are either critical points or satisfy the Lagrange conditions:

$$(1,0),(-1,0),(0,1/6),(\pm 1,3),(\pm 3,-1),(0,\pm\sqrt{10}).$$

These are the candidates for global maximum or minimum points. The corresponding values for $f(x,y)=x^2y+3y^2-y$ are:

$$0,0,-1/12,27,-5,30\mp\sqrt{10}.$$

The largest value is $30+\sqrt{10}$ at the point $(0,-\sqrt{10})$ and the smallest value is -5 at $(\pm 3,-1)$.

18. (a) Minimum. The minimum value of f on the constraint is $f(P)=20$.
(b) Neither. We have $f(Q)=30$, which is neither a minimum nor a maximum of f on the constraint because $f(P)=20<f(Q)<f(R)=40$. The fact that a contour of f and the constraint curve are tangent at Q is not enough to conclude that $f(Q)$ is either a maximum or minimum of f subject to the constraint.
(c) Neither. We have $f(R)=40$, which is neither a minimum nor a maximum of f on the constraint because $f(P)=20<f(R)<f(S)=50$.
(d) Maximum. The maximum value of f on the constraint is $f(S)=50$.

Problems

19. The function $f(x, y) = x + y - (x - y)^2$ attains a maximum value at a critical point inside the triangle or somewhere on its boundary.

At a critical point we have

$$f_x = 1 - 2(x - y) = 0, \quad \text{so } 2(x - y) = 1$$
$$f_y = 1 + 2(x - y) = 0, \quad \text{so } 2(x - y) = -1.$$

The equations have no solution, so f has no critical points.

We next find the maximum of f on each edge of the triangle separately. Each edge is a constraint and the maximum on an edge can be found by the method of Lagrange multipliers. It is also very easy to use the equation of an edge to change the 2-variable constrained maximum problem into a 1-variable maximum problem, which we do for the two edges lying on the x and y-axes.

On the boundary segment $x = 0$, $0 \le y \le 1$, we have $f(0, y) = y - y^2$ which attains a maximum at the point $(0, 1/2)$.

On the boundary segment $0 \le x \le 1$, $y = 0$, we have $f(x, 0) = x - x^2$ which attains a maximum at the point $(1/2, 0)$.

On the boundary segment $x + y = 1$, $0 \le x \le 1$, $0 \le y \le 1$, we use the method of Lagrange multipliers with constraint $g(x, y) = x + y = 1$. The equations

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g = 1$$

give

$$1 - 2(x - y) = \lambda \quad 1 + 2(x - y) = \lambda \quad x + y = 1$$

with solution $(x, y) = (1/2, 1/2)$. On the edge we have $f(x, y) = x + y - (x - y)^2 = 1 - (x - y)^2$ so we see that $(1/2, 1/2)$ gives a constrained maximum. Since

$$f\left(0, \frac{1}{2}\right) = \frac{1}{4} \qquad f\left(\frac{1}{2}, 0\right) = \frac{1}{4} \qquad f\left(\frac{1}{2}, \frac{1}{2}\right) = 1$$

we learn that f attains a maximum value of 1 at the point $(x, y) = (1/2, 1/2)$ on the boundary of the triangle.

20. (a) The contour for $z = 1$ is the line $1 = 2x + y$, or $y = -2x + 1$. The contour for $z = 3$ is the line $3 = 2x + y$, or $y = -2x + 3$. The contours are all lines with slope -2. See Figure 15.24.

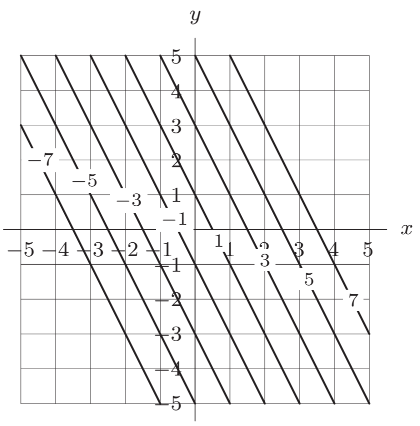


Figure 15.24

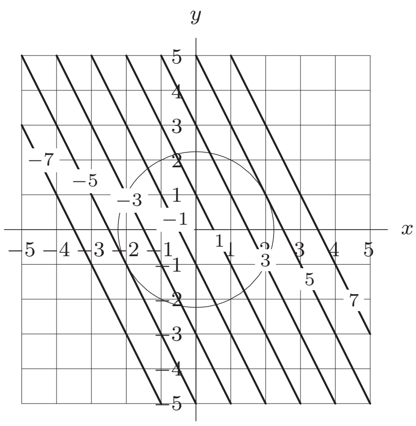


Figure 15.25

(b) The graph of $x^2 + y^2 = 5$ is a circle of radius $\sqrt{5} = 2.236$ centered at the origin. See Figure 15.25.

(c) The circle representing the constraint equation in Figure 15.25 appears to be tangent to the contour close to $z = 5$ at the point $(2, 1)$, and this is the contour with the highest z-value that the circle intersects. The circle is tangent to the contour $z = -5$ approximately at the point $(-2, -1)$, and this is the contour with the lowest z-value that the circle intersects. Therefore, subject to the constraint $x^2 + y^2 = 5$, the function f has a maximum value of about 5 at the point $(2, 1)$ and a minimum value of about -5 at the point $(-2, -1)$.

Since the radius vector, $2\vec{i} + \vec{j}$, at the point $(2, 1)$ is perpendicular to the line $2x + y = 5$, the maximum is exactly 5 and occurs at $(2, 1)$. Similarly, the minimum is exactly -5 and occurs at $(-2, -1)$.

(d) The objective function is $f(x, y) = 2x + y$ and the constraint equation is $g(x, y) = x^2 + y^2 = 5$, and so grad $f = 2\vec{i} + \vec{j}$ and grad $g = (2x)\vec{i} + (2y)\vec{j}$. Setting grad $f = \lambda$ grad g gives

$$2 = \lambda(2x),$$
$$1 = \lambda(2y).$$

On the constraint, $x \neq 0$ and $y \neq 0$. Thus, from the first equation, we have $\lambda = 1/x$, and from the second equation we have $\lambda = 1/(2y)$. Setting these equal gives

$$x = 2y.$$

Substituting this into the constraint equation $x^2 + y^2 = 5$ gives $(2y)^2 + y^2 = 5$ so $y = -1$ and $y = 1$. Since $x = 2y$, the maximum or minimum values occur at $(2, 1)$ or $(-2, -1)$. Since $f(2, 1) = 5$ and $f(-2, -1) = -5$, the function $f(x, y) = 2x + y$ subject to the constraint $x^2 + y^2 = 5$ has a maximum value of 5 at the point $(2, 1)$ and a minimum value of -5 at the point $(-2, -1)$. This confirms algebraically what we observed graphically in part (c).

21. Let $g(x, y) = 2x + 3y$, so the line is $g(x, y) = 6$. At the maximum on the line grad $f = \lambda$ grad g, so

$$f_x = \alpha x^{\alpha-1} y^{1-\alpha} = \lambda \cdot 2$$
$$f_y = (1 - \alpha) x^{\alpha} y^{-\alpha} = \lambda \cdot 3.$$

Dividing to eliminate λ we have

$$\frac{\alpha}{1-\alpha} x^{-1} y^1 = \frac{2}{3}.$$

Since $(1.5, 1)$ is a critical point, we have

$$\frac{\alpha}{1-\alpha}(1.5)^{-1} 1^1 = \frac{2}{3}$$
$$3\alpha = 2(1.5)(1 - \alpha) = 3(1 - \alpha)$$
$$3\alpha = 3 - 3\alpha$$
$$\alpha = 0.5.$$

22. We know that a maximum or minimum value of f subject to the constraint equation $g(x, y) = c$ occurs where grad f is parallel to grad g, or at the endpoints of the constraint. The vectors grad f and grad g are parallel where the graph of $g(x, y) = c$ is tangent to the contours of f, which occurs at approximately $x = 6$ and $y = 6$. At the point $(6, 6)$, we have $f = 400$. The graph of $g(x, y) = c$ crosses the contours $f = 300$, $f = 200$, $f = 100$ but does not cross any contours with f-values greater than 400. We see that the maximum of f subject to the constraint is 400 at the point $(6, 6)$. It appears that f takes on its minimum value (less than 100) at one of the endpoints, which are approximately $(10.5, 0)$ and $(0, 13.5)$.

23. (a) The curves are shown in Figure 15.26.

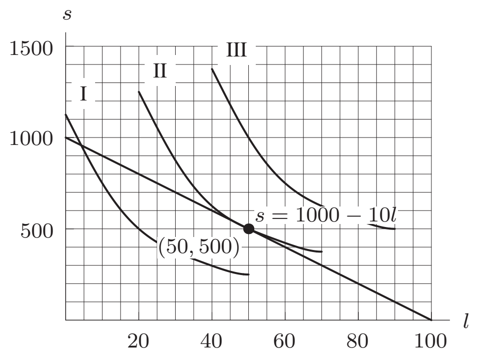


Figure 15.26

(b) The income equals \$10/hour times the number of hours of work:

$$s = 10(100 - l) = 1000 - 10l.$$

(c) The graph of this constraint is the straight line in Figure 15.26.

(d) For any given salary, curve III allows for the most leisure time, curve I the least. Similarly, for any amount of leisure time, curve III also has the greatest salary, and curve I the least. Thus, any point on curve III is preferable to any point on curve II, which is preferable to any point on curve I. We prefer to be on the outermost curve that our constraint allows. We want to choose the point on $s = 1000 - 10l$ which is on the most preferable curve. Since all the curves are concave up, this occurs at the point where $s = 1000 - 10l$ is *tangent* to curve II. So we choose $l = 50$, $s = 500$, and work 50 hours a week.

24. (a) The gradient vectors, ∇f, point inward around a local maximum. See the two points marked A in Figure 15.27.

(b) Some of the gradient vectors around a saddle are pointing inward toward the point; some are pointing outward away from the point. See the point marked B in Figure 15.27.

(c) The critical points on $g = 1$ are at points where ∇f is perpendicular to the curve $g = 1$. There are four of them, all marked with a dot in Figure 15.27. Imagine the level surfaces of f sketched in everywhere perpendicular to ∇f; the maximum value of f is at the point marked C in Figure 15.27

(d) Again imagine level curves of f. The minimum value of f is at the point marked D.

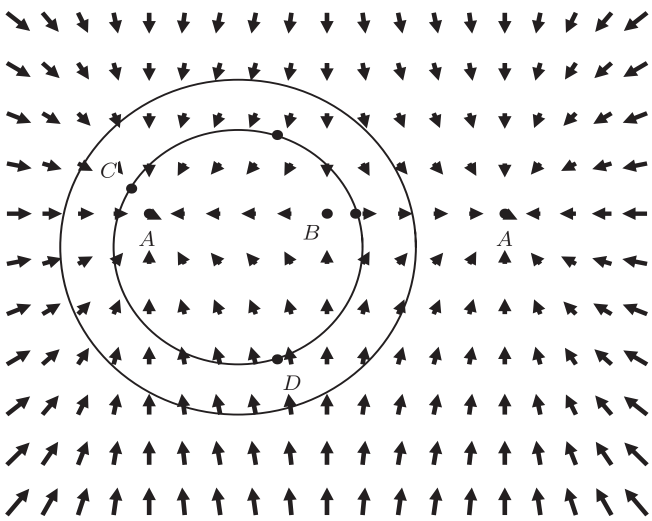


Figure 15.27

(e) At C, the maximum on $g = 1$, the vector ∇g points outward (because it points toward g = 2), while ∇f points inward. The Lagrange multiplier, λ, is defined so that $\nabla f = \lambda \nabla g$, so λ must be negative.

25. (a) The point P gives a minimum; the maximum is at one of the end points of the line segment (either the x- or the y-intercept). The value of λ is negative, since f decreases in the direction in which g increases.

(b) The point P gives a maximum; the minimum is at the x- or y-intercept. The value of λ is positive, since f and g increase in the same direction.

26. Since λ is the additional quantity of f that is obtained by relaxing the constraint by 1 unit, λ is larger if the level curves of f are close together near the optimal point. The answer is I < II < III.

27. The maximum and minimum values change by approximately $\lambda \Delta c$. The Lagrange conditions give:

$$3 = \lambda 2x, \quad -2 = \lambda 4y.$$

Solving for λ and setting the expressions equal, we get $x = -3y$. Substituting into the constraint, we get $y = \pm 2$, so the points satisfying the Lagrange conditions are $(-6, 2)$ and $(6, -2)$. The corresponding values of $f(x, y) = 3x - 2y$ are -22 and 22. From the first equation, we have $\lambda = 3/(2x)$. Thus the minimum value changes by $3/(-12)\Delta c = -\Delta c/4$ and the maximum changes by $3/(12)\Delta c = \Delta c/4$.

28. The maximum and minimum values change by approximately $\lambda \Delta c$. The Lagrange conditions give:

$$y = \lambda 8x, \quad x = \lambda 2y.$$

Solving for λ and setting the expressions equal, we get $4x^2 = y^2$. Substituting into the constraint, we get $x = \pm 1$. Since $y = \pm 2x$, the points satisfying the Lagrange conditions are $(1, 2), (-1, 2), (1, -2), (-1, -2)$. Since $f(x, y) = xy$, we get a maximum value of 2 at $(1, 2), (-1, -2)$ and a minimum value of -2 at $(1, -2), (-1, 2)$. Since $\lambda = y/(8x)$, the maximum value changes by $(2/8)\Delta c = \Delta c/4$ and the minimum changes by $-(2/8)\Delta c/) = -\Delta c/4$.

29. (a) The company wishes to maximize $P(x,y)$ given the constraint $C(x,y) = 50,000$. The objective function is $P(x,y)$ and the constraint equation is $C(x,y) = 50,000$. The Lagrange multiplier λ is approximately equal to the change in $P(x,y)$ given a one unit increase in the budget constraint. In other words, if we increase the budget by \$1, we can produce about λ more units of the good.

(b) The company wishes to minimize $C(x,y)$ given the constraint equation $P(x,y) = 2000$. The objective function is $C(x,y)$ and the constraint equation is $P(x,y) = 2000$. The Lagrange multiplier λ is approximately equal to the change in $C(x,y)$ given a one unit increase in the production constraint. In other words, it costs about λ dollars to produce one more unit of the good.

30.

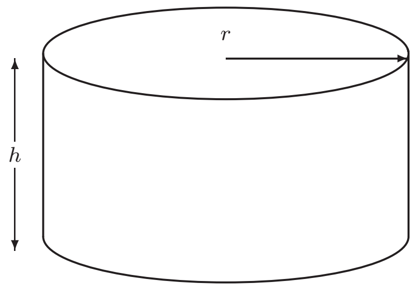


Figure 15.28

Let V be the volume and S be the surface area of the container. Then

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi rh + 2\pi r^2$$

where h is the height and r is the radius as shown in Figure 15.28. We have $V = 100\,\text{cm}^3$ as our constraint. Since

$$\begin{aligned} \nabla S &= (2\pi h + 4\pi r)\vec{i} + 2\pi r\vec{j} = \pi((2h+4r)\vec{i} + 2r\vec{j}\,) \\ \text{and} \qquad \nabla V &= 2\pi rh\vec{i} + \pi r^2\vec{j} = \pi(2rh\vec{i} + r^2\vec{j}\,), \end{aligned}$$

at the optimum

$$\begin{gathered} \nabla S = \lambda \nabla V, \text{ we have} \\ \pi((2h+4r)\vec{i} + 2r\vec{j}\,) = \pi\lambda(2rh\vec{i} + r^2\vec{j}\,), \\ \text{that is} \qquad 2h + 4r = 2\lambda rh \quad \text{and} \quad 2r = \lambda r^2, \qquad \text{hence} \quad \lambda = \frac{2}{r}. \end{gathered}$$

We assume $r \neq 0$ or else we have a very awkward cylinder. Then, plug $\lambda = 2/r$ into the first equation to obtain:

$$\begin{aligned} 2h + 4r &= 2\left(\frac{2}{r}\right)rh \\ 2h + 4r &= 4h \\ h &= 2r. \end{aligned}$$

Finally, solve for r and h using the constraint:

$$\begin{aligned} V = \pi r^2 h &= 100 \\ \pi r^2(2r) &= 100 \\ r^3 &= \frac{50}{\pi} \\ r &= \sqrt[3]{\frac{50}{\pi}}. \end{aligned}$$

Solving for h, we obtain $h = 2r = 2\sqrt[3]{\frac{50}{\pi}}$.

31. (a) We want to minimize C subject to $g = x + y = 39$. Solving $\nabla C = \lambda \nabla g$ gives

$$\begin{aligned} 10x + 2y &= \lambda \\ 2x + 6y &= \lambda \end{aligned}$$

so $y = 2x$. Solving with $x + y = 39$ gives $x = 13, y = 26, \lambda = 182$. Therefore $C = \$4349$.

(b) Since $\lambda = 182$, increasing production by 1 will cause costs to increase by approximately \$182. (because $\lambda = \frac{\|\nabla C\|}{\|\nabla g\|}$ = rate of change of C with g). Similarly, decreasing production by 1 will save approximately \$182.

32. Using Lagrange multipliers, let $G = 2000 - 5x - 10y = 0$ be the constraint.
$\nabla P = \left(1 + \frac{2xy^2}{2 \cdot 10^8}\right)\vec{i} + \left(2 + \frac{2yx^2}{2 \cdot 10^8}\right)\vec{j} = \left(1 + \frac{xy^2}{10^8}\right)\vec{i} + \left(2 + \frac{yx^2}{10^8}\right)\vec{j}$.
$\nabla G = -5\vec{i} - 10\vec{j}$.
Now, $\nabla P = \lambda \nabla G$, so

$$1 + \frac{xy^2}{10^8} = -5\lambda \quad \text{and} \quad 2 + \frac{yx^2}{10^8} = -10\lambda.$$

Thus

$$2 + \frac{2xy^2}{10^8} = 2 + \frac{yx^2}{10^8}.$$

Solving, we get $2y = x$ or $x = 0$ or $y = 0$.

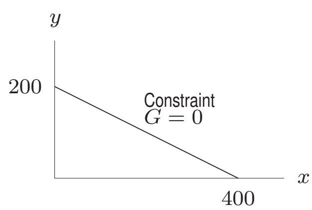


Figure 15.29

From $G = 0$ we have: when $x = 0, y = 200$, when $y = 0, x = 400$, and when $x = 2y, x = 200, y = 100$. So (0,200), (400,0) and (200,100) are the critical points and they include the end points.
Substitute into P: $P(0, 200) = 400$, $P(400, 0) = 400$, $P(200, 100) = 402$ so the organization should buy 200 sacks of rice and 100 sacks of beans.

33. (a) Let c be the cost of producing the product. Then $c = 10W + 20K = 3000$. At optimum production,

$$\nabla q = \lambda \nabla c.$$

$\nabla q = \left(\frac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}}\right)\vec{i} + \left(\frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}}\right)\vec{j}$, and $\nabla c = 10\vec{i} + 20\vec{j}$. Equating we get

$$\tfrac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}} = \lambda 10, \quad \text{and} \quad \tfrac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}} = \lambda 20.$$

Dividing yields $K = \frac{1}{6}W$, so substituting into c gives

$$10W + 20\left(\frac{1}{6}W\right) = \frac{40}{3}W = 3000.$$

Thus $W = 225$ and $K = 37.5$. Substituting both answers to find λ gives

$$\lambda = \frac{\frac{9}{2}(225)^{-\frac{1}{4}}(37.5)^{\frac{1}{4}}}{10} = 0.2875.$$

We also find the optimum quantity produced, $q = 6(225)^{\frac{3}{4}}(37.5)^{\frac{1}{4}} = 862.57$.

(b) At the optimum values found above, marginal productivity of labor is given by

$$\frac{\partial q}{\partial W}\bigg|_{(225,37.5)} = \frac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}}\bigg|_{(225,37.5)} = 2.875,$$

and marginal productivity of capital is given by

$$\frac{\partial q}{\partial K}\bigg|_{(225,37.5)} = \frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}}\bigg|_{(225,37.5)} = 5.750.$$

The ratio of marginal productivity of labor to that of capital is

$$\frac{\frac{\partial q}{\partial W}}{\frac{\partial q}{\partial K}} = \frac{1}{2} = \frac{10}{20} = \frac{\text{cost of a unit of L}}{\text{cost of a unit of K}}.$$

(c) When the budget is increased by one dollar, we substitute the relation $K_1 = \frac{1}{6}W_1$ into $10W_1 + 20K_1 = 3001$ which gives $10W_1 + 20(\frac{1}{6}W_1) = \frac{40}{3}W_1 = 3001$. Solving yields $W_1 = 225.075$ and $K_1 = 37.513$, so $q_1 = 862.86 = q + 0.29$. Thus production has increased by $0.29 \approx \lambda$, the Lagrange multiplier.

34. (a) The problem is to maximize

$$V = 1000D^{0.6}N^{0.3}$$

subject to the budget constraint in dollars

$$40000D + 10000N \le 600000$$

or (in thousand dollars)

$$40D + 10N \le 600$$

(b) Let $B = 40D + 10N = 600$ (thousand dollars) be the budget constraint. At the optimum

$$\nabla V = \lambda \nabla B,$$

so

$$\frac{\partial V}{\partial D} = \lambda\frac{\partial B}{\partial D} = 40\lambda$$

$$\frac{\partial V}{\partial N} = \lambda\frac{\partial B}{\partial N} = 10\lambda.$$

Thus

$$\frac{\partial V/\partial D}{\partial V/\partial N} = 4.$$

Therefore, at the optimum point, the rate of increase in the number of visits with respect to an increase in the number of doctors is four times the corresponding rate for nurses. This factor of four is the same as the ratio of the salaries.

(c) Differentiating and setting $\nabla V = \lambda \nabla B$ yields

$$600D^{-0.4}N^{0.3} = 40\lambda$$
$$300D^{0.6}N^{-0.7} = 10\lambda$$

Thus, we get

$$\frac{600D^{-0.4}N^{0.3}}{40} = \lambda = \frac{300D^{0.6}N^{-0.7}}{10}$$

So

$$N = 2D.$$

To solve for D and N, substitute in the budget constraint:

$$600 - 40D - 10N = 0$$

$$600 - 40D - 10\cdot(2D) = 0$$

So $D = 10$ and $N = 20$.

$$\lambda = \frac{600(10^{-0.4})(20^{0.3})}{40} \approx 14.67$$

Thus the clinic should hire 10 doctors and 20 nurses. With that staff, the clinic can provide

$$V = 1000(10^{0.6})(20^{0.3}) \approx 9{,}779 \text{ visits per year.}$$

(d) From part c), the Lagrange multiplier is $\lambda = 14.67$. At the optimum, the Lagrange multiplier tells us that about 14.67 extra visits can be generated through an increase of $\$1{,}000$ in the budget. (If we had written out the constraint in dollars instead of thousands of dollars, the Lagrange multiplier would tell us the number of extra visits per dollar.)

(e) The marginal cost, MC, is the cost of an additional visit. Thus, at the optimum point, we need the reciprocal of the Lagrange multiplier:

$$\text{MC} = \frac{1}{\lambda} \approx \frac{1}{14.67} \approx 0.068 \text{ (thousand dollars)},$$

that is, at the optimum point, an extra visit costs the clinic 0.068 thousand dollars, or $\$68$.

This production function exhibits declining returns to scale (e.g. doubling both inputs less than doubles output, because the two exponents add up to less than one). This means that for large V, increasing V will require increasing D and N by more than when V is small. Thus the cost of an additional visit is greater for large V than for small. In other words, the marginal cost will rise with the number of visits.

35. (a) The solution to Problem 33 gives $\lambda = 0.29$. We recalculate λ with a budget of $\$4000$.

The condition that $\text{grad}\, q = \lambda\, \text{grad}(\text{budget})$ in Problem 33 gives

$$\frac{9}{2}W^{-1/4}K^{1/4} = \lambda(10) \quad \text{and} \quad \frac{3}{2}W^{3/4}K^{-3/4} = \lambda(20),$$

so $K = \frac{1}{6}W$. Substituting into the budget constraint after replacing the budget of \$3000 by \$4000 gives

$$10W + 20(\frac{1}{6}W) = \frac{40}{3}W = 4000.$$

Thus, $W = 300$ and $K = 50$ and $q = 1150.098$.

Multiplying the first equation by W and the second by K and adding gives

$$W(\frac{9}{2}W^{-1/4}K^{1/4}) + K(\frac{3}{2}W^{3/4}K^{-3/4}) = W(10\lambda) + K(20\lambda).$$

So

$$\left(\frac{9}{2} + \frac{3}{2}\right) W^{3/4}K^{1/4} = \lambda(10W + 20K)$$
$$6W^{3/4}K^{1/4} = \lambda(4000)$$

Thus,

$$\lambda = \frac{6W^{3/4}K^{1/4}}{4000} = \frac{1150.098}{4000} = 0.29$$

Thus, the value of λ remains unchanged.

(b) The solution to Problem 34 shows that $\lambda = 14.67$. We solve the problem again with a budget of \$700,000.

The condition that $\text{grad}\, V = \lambda\, \text{grad}\, B$ in Problem 34 gives

$$600D^{-0.4}N^{0.3} = 40\lambda$$
$$300D^{0.6}N^{-0.7} = 10\lambda$$

Thus, $N = 2D$. Substituting in the budget constraint after replacing the budget of 600 by 700 (the budget in measured in thousands of dollars) gives

$$40D + 10(2D) = 700$$

so $D = 11.667$ and $N = 23.337$ and $V = 11234.705$. As in part a), we multiply the first equation by D and the second by N and add:

$$D(600D^{-0.4}N^{0.3}) + N(300D^{0.6}N^{-0.7}) = D(40\lambda) + N(10\lambda),$$

so

$$(600 + 300)D^{0.6}N^{0.3} = \lambda(400 + 10N)$$
$$900D^{0.6}N^{0.3} = \lambda(700)$$

Since $V = 1000D^{0.6}N^{0.3} = 11234.705$, we have

$$\lambda = \frac{900D^{0.6}N^{0.3}}{700} = \frac{9}{7}(\frac{V}{1000}) = 14.44.$$

Thus, the value of λ has changed with the budget.

(c) We are interested in the marginal increase of production with budget (that is, the value of λ) and whether it is affected by the budget.

Suppose $\$B$ is the budget. In part (a) we found

$$\lambda = \frac{6W^{3/4}K^{1/4}}{B}$$

and in part (b) we found

$$\lambda = \frac{900D^{0.6}N^{0.3}}{B}.$$

In part (a), both W and K are proportional to B. Thus, $W = c_1B$ and $K = c_2B$, so

$$\begin{aligned}\lambda &= \frac{6(c_1B)^{3/4}(c_2B)^{1/4}}{B}\\ &= \frac{6c_1^{3/4}C_2^{1/4}B^{3/4}B^{1/4}}{B}\\ &= 6c_1^{3/4}c_2^{1/4}.\end{aligned}$$

So we see λ is independent of B.

In part (b), both D and N are proportional to B, so $D = c_3B$ and $N = c_4B$. Thus,

$$\begin{aligned}\lambda &= \frac{900(c_3B)^{0.6}(c_4B)^{0.3}}{B}\\ &= \frac{900c_3^{0.6}C_4^{0.3}B^{0.6}B^{0.3}}{B}\\ &= 900c_3^{0.6}c_4^{0.3}\frac{1}{B^{0.1}}.\end{aligned}$$

So we see λ is not independent of B.

The crucial difference is that the exponents in Problem 33 add to 1, that is $3/4+1/4 = 1$, whereas the exponents in Problem 34 do not add to 1, since $0.6 + 0.3 = 0.9$.

Thus, the condition that must be satisfied by the Cobb-Douglas production function

$$Q = cK^aL^b$$

to ensure that the value of λ is not affected by production is that

$$a + b = 1.$$

This is called constant returns to scale.

36. (a) For a given budget, maximum production is achieved at the point on the budget line that is tangent to a production contour. Approximate points are shown in Figure 15.30

(b) Estimating production quantities from the contours gives the following values, or close to them:

Budget, B (in dollars)	2000	4000	6000	8000	10000
Max production, M (in pairs of skis per week)	17	33	50	67	83

(c) We estimate the derivative dM/dB at $B = \$6000$ with a difference quotient. Taking values from the table for $B = 6000$ and $B = 8000$ we have

$$\lambda = \frac{dM}{dB} \approx \frac{67-50}{8000-6000} = 0.0085 \text{ pairs of skis per week per dollar.}$$

For example, increasing the budget from $\$6000$ to $\$7000$ would increase maximum production by approximately $0.0085(1000) = 8.5$ pairs of skis per week.

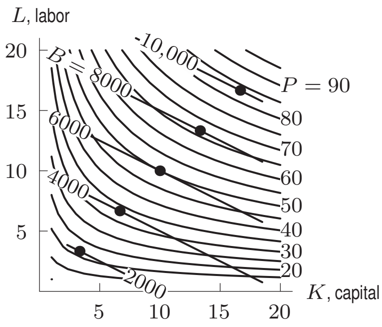


Figure 15.30

37. Since patient 1 has a visit every x_1 weeks, this patient has $1/x_1$ visits per week. Similarly, patient 2 has $1/x_2$ visits per week. Thus, the constraint is

$$g(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} = m$$

To minimize

$$f(x_1, x_2) = \frac{v_1}{v_1 + v_2} \cdot \frac{x_1}{2} + \frac{v_2}{v_1 + v_2} \cdot \frac{x_2}{2}$$

subject to $g(x_1, x_2)$, we solve the equations

$$\begin{aligned} \operatorname{grad} f &= \lambda \operatorname{grad} g \\ g(x_1, x_2) &= m. \end{aligned}$$

This gives us the equations

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{v_1}{v_1 + v_2} \cdot \frac{1}{2} = \lambda \left(-\frac{1}{x_1^2}\right) = \lambda \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2} &= \frac{v_2}{v_1 + v_2} \cdot \frac{1}{2} = \lambda \left(-\frac{1}{x_2^2}\right) = \lambda \frac{\partial g}{\partial x_2} \\ \frac{1}{x_1} + \frac{1}{x_2} &= m. \end{aligned}$$

Dividing the first equation by the second gives

$$\frac{v_1}{v_2} = \frac{x_2^2}{x_1^2}.$$

As v_1, v_2, x_1, x_2, m are strictly positive we have

$$\frac{x_2}{x_1} = \left(\frac{v_1}{v_2}\right)^{\frac{1}{2}}.$$

Substituting for x_2 in the constraint occasion gives

$$\frac{1}{x_1} + \left(\frac{v_2}{v_1}\right)^{\frac{1}{2}} \cdot \frac{1}{x_1} = m$$

solving for x_1 gives

$$\begin{aligned} \frac{1}{x_1}\left(1 + (\frac{v_1}{v_2})^{\frac{1}{2}}\right) &= m \\ x_1 &= \frac{(v_1)^{\frac{1}{2}} + (v_2)^{\frac{1}{2}}}{m \cdot (v_1)^{\frac{1}{2}}}, \end{aligned}$$

and similarly

$$x_2 = \frac{(v_1)^{\frac{1}{2}} + (v_2)^{\frac{1}{2}}}{m \cdot (v_2)^{\frac{1}{2}}}.$$

38. We want to optimize

$$f(x_1, x_2) = \frac{v_1}{v_1+v_2} \cdot \frac{x_1}{2} + \frac{v_2}{v_1+v_2} \cdot \frac{x_2}{2}$$

subject to

$$g(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} = m.$$

At the optimum point, x_1, x_2, and the Lagrange multiplier λ must satisfy the equations

$$\begin{aligned}\frac{v_1}{v_1+v_2} \cdot \frac{1}{2} &= -\frac{\lambda}{x_1^2} \\ \frac{v_2}{v_1+v_2} \cdot \frac{1}{2} &= -\frac{\lambda}{x_2^2} \\ \frac{1}{x_1} + \frac{1}{x_2} &= m.\end{aligned}$$

Solving the first and second equations for $1/x_1$ and $1/x_2$, respectively, gives

$$\begin{aligned}\frac{1}{x_1^2} &= -\frac{1}{2\lambda} \cdot \frac{v_1}{(v_1+v_2)} \\ \frac{1}{x_2^2} &= -\frac{1}{2\lambda} \cdot \frac{v_2}{(v_1+v_2)}\end{aligned}$$

substituting into the constraint gives (note that $\lambda < 0$):

$$\left(-\frac{1}{2\lambda} \cdot \frac{v_1}{(v_1+v_2)}\right)^{1/2} + \left(\frac{-1}{2\lambda} \cdot \frac{v_2}{(v_1+v_2)}\right)^{1/2} = \left(-\frac{1}{2\lambda}\right)^{1/2} \cdot \frac{(v_1)^{1/2}+(v_2)^{1/2}}{(v_1+v_2)^{1/2}} = m.$$

So

$$-\frac{1}{2\lambda} \cdot \frac{v_1+v_2+2(v_1v_2)^{1/2}}{v_1+v_2} = m^2.$$

and thus

$$\lambda = -\frac{1}{2m^2}\left(1 + \frac{2(v_1v_2)^{1/2}}{v_1+v_2}\right).$$

The units of λ are weeks^2 (since the units of m are 1/weeks). The Lagrange multiplier measures df/dm, which represents the rate of change in the expected delay in tumor detection as the available number of visits per week increases. The negative sign represents the fact that as the number of visits per week increases, the delay decreases.

39. (a) The objective function is the complementary energy, $\dfrac{f_1^2}{2k_1} + \dfrac{f_2^2}{2k_2}$, and the constraint is $f_1 + f_2 = mg$. The Lagrangian function is

$$\mathcal{L}(f_1, f_2, \lambda) = \frac{f_1^2}{2k_1} + \frac{f_2^2}{2k_2} - \lambda(f_1 + f_2 - mg).$$

We look for solutions to the system of equations we get from $\operatorname{grad} \mathcal{L} = \vec{0}$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial f_1} &= \frac{f_1}{k_1} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial f_2} &= \frac{f_2}{k_2} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -(f_1 + f_2 - mg) = 0.\end{aligned}$$

Combining $\dfrac{\partial \mathcal{L}}{\partial f_1} - \dfrac{\partial \mathcal{L}}{\partial f_2} = \dfrac{f_1}{k_1} - \dfrac{f_2}{k_2} = 0$ with $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0$ gives the two equation system

$$\begin{aligned}\frac{f_1}{k_1} - \frac{f_2}{k_2} &= 0 \\ f_1 + f_2 &= mg.\end{aligned}$$

Substituting $f_2 = mg - f_1$ into the first equation leads to

$$f_1 = \frac{k_1}{k_1 + k_2} mg$$
$$f_2 = \frac{k_2}{k_1 + k_2} mg.$$

(b) Hooke's Law states that for a spring

Force of spring = Spring constant · Distance stretched or compressed from equilibrium.

Since $f_1 = k_1 \cdot \lambda$ and $f_2 = k_2 \cdot \lambda$, the Lagrange multiplier λ equals the distance the mass stretches the top spring and compresses the lower spring.

40. (a) Let $f(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i{}^2 = x_1{}^2 + x_2{}^2 + x_3{}^2$ and $g(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i = 1$. Then $\text{grad}\, f = \lambda\, \text{grad}\, g$ gives

$$2x_1 = \lambda \quad \text{and} \quad 2x_2 = \lambda \quad \text{and} \quad 2x_3 = \lambda.$$

so

$$x_1 = x_2 = x_3 = \frac{\lambda}{2}.$$

Then $x_1 + x_2 + x_3 = 1$ gives

$$3\frac{\lambda}{2} = 1 \quad \text{so} \quad \lambda = \frac{2}{3} \quad \text{so } x_1 = x_2 = x_3 = \frac{1}{3}.$$

These values of x_1, x_2, x_3 give the minimum (rather than maximum) because the value of f increases without bound as $x_2, x_2, x_3 \to \infty$.

(b) A similar argument shows that $\sum_{i=1}^{n} x_i$ has its minimum value subject to $\sum_{i=1}^{n} x_i = 1$ when

$$x_1 = x_2 = \cdots = x_n = \frac{1}{n}.$$

41. The maximum of $f(x, y) = ax^2 + bxy + cy^2$ subject to the constraint $g(x, y) = 1$ where $g(x, y) = x^2 + y^2$ occurs where $\text{grad}\, f = \lambda\, \text{grad}\, g$. Since $\text{grad}\, f = (2ax + by)\vec{i} + (bx + 2cy)\vec{j}$ and $\text{grad}\, g = 2x\vec{i} + 2y\vec{j}$ we have

$$2ax + by = 2x\lambda$$
$$bx + 2cy = 2y\lambda$$
$$x^2 + y^2 = 1$$

Adding x times the first equation to y times the second gives $x(2ax + by) + y(bx + 2cy) = (2x^2 + 2y^2)\lambda$. Dividing by 2 and using the constraint equation gives $f(x, y) = ax^2 + bxy + cy^2 = (x^2 + y^2)\lambda = \lambda$. This equation holds for all solutions (x, y, λ) of the three equations, including the solution that corresponds to the maximum value of f subject to the constraint. Thus the maximum value is $f(x, y) = \lambda$.

42. Let (x, y, z) be a point on the paraboloid. The square of the distance from (x, y, z) to the point $(1, 2, 10)$ is given by

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 10)^2,$$

and so we wish to minimize $f(x, y, z)$ subject to the constraint

$$g(x, y, z) = x^2 + y^2 - z = 0.$$

We look for solutions to the equations $\text{grad}\, f = \lambda\, \text{grad}\, g$ and $g = 0$:

$$2(x - 1) = 2\lambda x,$$
$$2(y - 2) = 2\lambda y,$$
$$2(z - 10) = -\lambda,$$
$$x^2 + y^2 - z = 0.$$

If $\lambda = 0$, the first three equations would imply that $(x, y, z) = (1, 2, 10)$, which does not satisfy the fourth equation and so $\lambda \neq 0$. The first equation then implies that $x \neq 0$ and the second equation implies that $y \neq 0$, so we can eliminate λ from the first three equations to get:

$$\frac{x-1}{x} = \frac{y-2}{y} \quad \text{and} \quad \frac{y-2}{y} = -2(z-10).$$

These give

$$y = 2x \quad \text{and} \quad z = \frac{2-y}{2y} + 10.$$

Substituting for x and z in the equation $z = x^2 + y^2$, we obtain

$$\frac{2-y}{2y} + 10 = \frac{y^2}{4} + y^2,$$

which simplifies to give

$$5y^3 - 38y - 4 = 0.$$

Let $h(y) = 5y^3 - 38y - 4$. We find that $h(-3) < 0$ and $h(-1) > 0$, and so the cubic $h(y)$ has a root between -3 and -1. Similarly, since $h(-1) > 0$ and $h(0) < 0$, then $h(y)$ has one a between -1 and 0. Finally, as $h(0) < 0$ and $h(3) > 0$, we see that $h(y)$ has a root between 0 and 3. Let's find the root lying between 0 and 3. Using a calculator, we find that this root is approximately given by

$$y_1 \approx 2.808,$$

and so, using $y = 2x$, the corresponding point (x, y, z) on the paraboloid $z = x^2 + y^2$ is given by

$$(x_1, y_1, z_1) \approx (1.404, 2.808, 9.856).$$

The remaining roots of $h(y)$ are given by $y_2 \approx -0.1055$ and $y_3 \approx -2.7026$. The corresponding points on the paraboloid $z = x^2 + y^2$ which satisfy $y = 2x$ are then

$$(x_3, y_3, z_3) \approx (-1.3513, -2.7026, 9.1301),$$
$$(x_2, y_2, z_2) \approx (-0.0528, -0.1055, 0.0139),$$

and so we easily see that the point $(1.404, 2.808, 9.856)$ will be closest to $(1, 2, 10)$. Therefore, the minimum distance is

$$d = \sqrt{(1.404-1)^2 + (2.808-2)^2 + (9.856-10)^2} \approx 0.9148.$$

43. (a) The objective function $f(x, y) = px + qy$ gives the cost to buy x units of input 1 at unit price p and y units of input 2 at unit price q.

The constraint $g(x, y) = u$ tells us that we are only considering the cost of inputs x and y that can be used to produce quantity u of the product.

Thus the number $C(p, q, u)$ gives the minimum cost to the company of producing quantity u if the inputs it needs have unit prices p and q.

(b) The Lagrangian function is

$$\mathcal{L}(x, y, \lambda) = px + qy - \lambda(xy - u).$$

We look for solutions to the system of equations we get from $\operatorname{grad} \mathcal{L} = \vec{0}$:

$$\frac{\partial \mathcal{L}}{\partial x} = p - \lambda y = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = q - \lambda x = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(xy - u) = 0.$$

We see that $\lambda = p/y = q/x$ so $y = px/q$. Substituting for y in the constraint $xy = u$ leads to $x = \sqrt{qu/p}$, $y = \sqrt{pu/q}$ and $\lambda = \sqrt{pq/u}$. The minimum cost is thus

$$C(p, q, u) = p\sqrt{\frac{qu}{p}} + q\sqrt{\frac{pu}{q}} = 2\sqrt{pqu}.$$

44. (a) The objective function $U(x, y)$ gives the utility to the consumer of x units of item 1 and y units of item 2.

Since $px + qy$ gives the cost to buy x units of item 1 at unit price p and y units of item 2 at unit price q, the constraint $px + qy = I$ tells us that we are only considering the utility of inputs x and y that can be purchased with budget I.

Thus the number $V(p, q, I)$ gives the maximum utility the consumer can get with a budget of I if the two items have unit prices p and q.

The indirect utility function tells how much utility the consumer can buy, depending on his budget and the prices of the two items.

(b) The value of the Lagrange multiplier λ is the rate of change of the maximum utility V the consumer can get with his budget as the budget increases. This means that for small changes ΔI in the budget, smart buying will result in a change $\Delta V \approx \lambda\Delta I$ in the utility to the consumer of his purchases.

(c) The Lagrangian function is

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(px + qy - I).$$

We look for solutions to the system of equations we get from grad $\mathcal{L} = \vec{0}$:

$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda p = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = x - \lambda q = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(px + qy - I) = 0.$$

We see that $\lambda = y/p = x/q$ so $y = px/q$. Substituting for y in the constraint $px + qy = I$ leads to $x = I/(2p)$, $y = I/(2q)$ and $\lambda = I/(2pq)$. The maximum utility is thus

$$V(p, q, I) = U(\frac{I}{2p}, \frac{I}{2q}) = \frac{I}{2p} \cdot \frac{I}{2q} = \frac{I^2}{4pq}.$$

The marginal utility of money is

$$\lambda(p, q, I) = \lambda = \frac{I}{2pq}.$$

45. (a) The critical points of $h(x, y)$ occur where

$$h_x(x, y) = 2x - 2\lambda = 0$$
$$h_y(x, y) = 2y - 4\lambda = 0.$$

The only critical point is $(x, y) = (\lambda, 2\lambda)$ and it gives a minimum value for $h(x, y)$. That minimum value is $m(\lambda) = h(\lambda, 2\lambda) = \lambda^2 + (2\lambda)^2 - \lambda(2\lambda + 4(2\lambda) - 15) = -5\lambda^2 + 15\lambda$.

(b) The maximum value of $m(\lambda) = -5\lambda^2 + 15\lambda$ occurs at a critical point, where $m'(\lambda) = -10\lambda + 15 = 0$. At this point, $\lambda = 1.5$ and $m(\lambda) = -5 \cdot 1.5^2 + 15 \cdot 1.5 = 11.25$.

(c) We want to minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 15$, where $g(x, y) = 2x + 4y$. The Lagrangian function is $\mathcal{L}(x, y, \lambda) = x^2 + y^2 - \lambda(2x + 4y - 15)$ so we solve the system of equations

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y - 4\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(2x + 4y - 15) = 0.$$

The first two equations give $x = \lambda$ and $y = 2\lambda$. Substitution into the third equation gives $2\lambda + 4(2\lambda) - 15 = 0$ or $\lambda = 1.5$. Thus $x = 1.5$ and $y = 3$. The minimum value of $f(x, y)$ subject to the constraint is $f(1.5, 3) = 1.5^3 + 3^2 = 11.25$.

(d) The two question have the same answer.

46. (a) By the method of Lagrange multipliers, the point $(2, 1)$ is a candidate when the gradient for f at $(2, 1)$ is a multiple of the gradient of the constraint function at $(2, 1)$. The constraint function is $g(x, y) = x^2 + y^2$, so grad $g = 2x\vec{i} + 2y\vec{j}$. We have grad $g(2, 1) = 4\vec{i} + 2\vec{j}$. This is not a multiple of grad $f(2, 1) = -3\vec{i} + 4\vec{j}$, so $(2, 1)$ is not a candidate.

(b) The constraint function is $g(x,y) = (x-5)^2+(y+3)^2$, so $\text{grad}\, g = 2(x-5)\vec{i} + 2(y+3)\vec{j}$. We have $\text{grad}\, g(2,1) = -6\vec{i} + 8\vec{j}$. This is a multiple of $\text{grad}\, f(2,1) = -3\vec{i} + 4\vec{j}$, so $(2,1)$ is a candidate.

The contours near $(2,1)$ are parallel straight lines with increasing f-values as we move in the direction of $-3\vec{i} + 4\vec{j}$ (approximately toward the northwest). The center of the constraint circle is at $(5,-3)$, approximately southeast of the point $(2,1)$. Thus the point $(2,1)$ is a candidate for a maximum.

In general, if the constraint is a circle and $\text{grad}\, f$ points outside the circle, then the point is a candidate for a maximum.

(c) The constraint function is $g(x,y) = (x+1)^2 + (y-5)^2$. Thus $\text{grad}\, g(2,1) = 6\vec{i} - 8\vec{j}$, which is again a multiple of $\text{grad}\, f$, so $(2,1)$ is a candidate.

This time the center of the constraint circle, $(-1,5)$ is approximately northwest of $(2,1)$, the same general direction in which $\text{grad}\, f$ is pointing. This means the point $(2,1)$ is a candidate for a minimum.

In general, if the constraint is a circle and $\text{grad}\, f$ points inside the circle, then the point is a candidate for a minimum.

47. (a) If the prices are p_1 and p_2 and the budget is b, the quantities consumed are constrained by

$$p_1x_1 + p_2x_2 = b.$$

We want to maximize

$$u(x_1,x_2) = a\ln x_1 + (1-a)\ln x_2$$

subject to the constraint

$$p_1x_1 + p_2x_2 = b.$$

Using Lagrange multipliers, we solve

$$\frac{\partial u}{\partial x_1} = \frac{a}{x_1} = \lambda p_1$$
$$\frac{\partial u}{\partial x_2} = \frac{1-a}{x_2} = \lambda p_2,$$

giving $x_1 = a/(\lambda p_1)$ and $x_2 = (1-a)/(\lambda p_2)$. Substituting into the constraint, we get

$$\frac{a}{\lambda} + \frac{(1-a)}{\lambda} = b$$

so

$$\lambda = \frac{1}{b}.$$

Thus

$$x_1 = \frac{ab}{p_1} \qquad x_2 = \frac{(1-a)b}{p_2}$$

so the maximum satisfaction is given by

$$\begin{aligned} S = u(x_1,x_2) = u\left(\frac{ab}{p_1}, \frac{(1-a)b}{p_2}\right) &= a\ln\left(\frac{ab}{p_1}\right) + (1-a)\ln\left(\frac{(1-a)b}{p_2}\right) \\ &= a\ln a + a\ln b - a\ln p_1 + (1-a)\ln(1-a) + (1-a)\ln b - (1-a)\ln p_2 \\ &= a\ln a + (1-a)\ln(1-a) + \ln b - a\ln p_1 - (1-a)\ln p_2. \end{aligned}$$

(b) We want to calculate the value of b needed to achieve $u(x_1,x_2) = c$. Thus, we solve for b in the equation

$$c = a\ln a + (1-a)\ln(1-a) + \ln b - a\ln p_1 - (1-a)\ln p_2.$$

Since

$$\ln b = c - a\ln a - (1-a)\ln(1-a) + a\ln p_1 + (1-a)\ln p_2,$$

we have

$$b = \frac{e^c \cdot e^{a\ln p_1} \cdot e^{(1-a)\ln p_2}}{e^{a\ln a} \cdot e^{(1-a)\ln(1-a)}} = \frac{e^c p_1^a p_2^{1-a}}{a^a(1-a)^{(1-a)}}.$$

48. (a) The constraints $x = c$ are the vertical lines on the contour diagram of f. Maxima occur where a vertical line is tangent to a contour. Four contours with vertical tangents are shown and we can imagine others. The curve we want is shown in Figure 15.31.

(b) The constraints $x = c$ are the vertical lines on the graph of cross-sections of f. Since f is on the vertical axis, the maximum value of f on the constraint occurs on the cross section that crosses the constraint at the highest point. Not all cross-sections are shown, so we imagine the others to find the highest point. The curve we want goes across the top of all the cross-sections, as shown in Figure 15.32.

(c) The curve in part (b) shows the maximum value of f as a function of the constraining value $x = c$. Since the Lagrange multiplier λ is the rate of change of the maximum value of f with respect to c, the value of λ is the slope of that envelope curve.

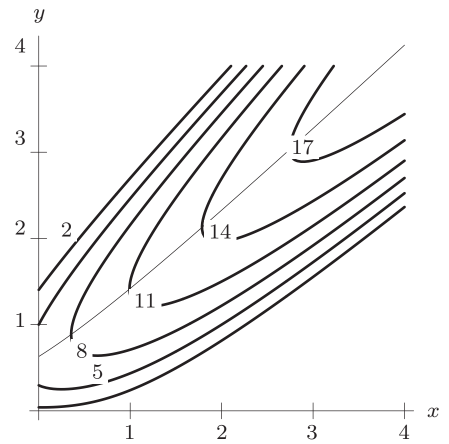


Figure 15.31

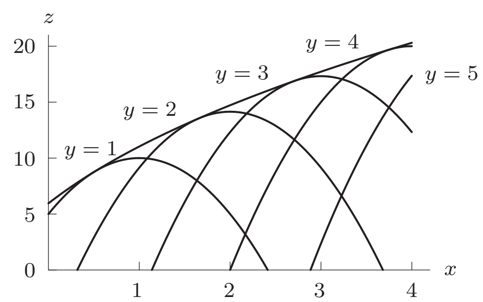


Figure 15.32

Strengthen Your Understanding

49. Let $g(x, y) = x + y$. The critical points occur where

$$\frac{\partial f}{\partial x} = y = \lambda \quad \text{and} \quad \frac{\partial f}{\partial y} = x = \lambda.$$

Thus $x = y$. Since the critical point must lie on the line $x + y = 2$, we have $x = y = 1$.

The value of f decreases as we move away from $(1, 1)$, so $(1, 1)$ is the maximum on the constraint. The maximum value of f is $f(1, 1) = 1$.

50. The maximum value of f can occur at the endpoints. For example, the level curves of f in Figure 15.33 show that f has a maximum at $(0, 3)$ even though the level curves of f are nowhere tangent to the constraint.

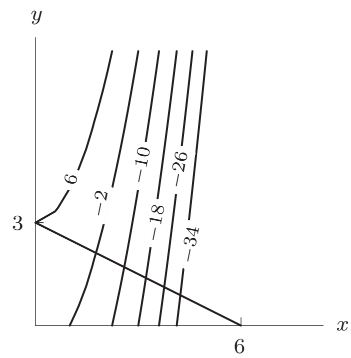


Figure 15.33

51. It is helpful to think of a contour diagram for f. For example, if the contours are all parallel with the highest contour tangent to $x^2+y^2=5$ at $(3,4)$, that would work. Since the tangent at $(3,4)$ has normal $3\vec{i}+4\vec{j}$, this suggests the linear function $f(x,y)=3x+4y$

52. Let f be any function having a global minimum at $(0,0)$, for example $f(x,y)=x^2+y^2$.

53. Let $f(x,y)=x^2+y^2$. Then

$$\text{grad}\, f = 2x\vec{i}+2y\vec{j}\,.$$

So at $(1,1)$,

$$\text{grad}\, f = 2\vec{i}+2\vec{j}\,.$$

If $g(x,y)=x+y$,

$$\text{grad}\, f = 2\,\text{grad}\, g.$$

The point $(1,1)$ is a critical point. It is a minimum on the line $x+y=2$ since the value of f increases without bound as we move away from $(1,1)$.

54. Let $f(x,y)=10-x^2-y^2$. Then

$$\text{grad}\, f = -2x\vec{i}-2y\vec{j}\,.$$

If $g(x,y)=x+y$, then

$$\text{grad}\, g = \vec{i}+\vec{j}\,.$$

Then grad $f=\lambda\,$grad g gives

$$-2x=\lambda$$
$$-2y=\lambda.$$

So, $x=y$ and since $x+y=4$, we have $x=y=2$.

Thus $f(x,y)=10-x^2-y^2$ has a critical point at $(2,2)$. This point gives a maximum. There is no minimum value as f decreases as we go away from $(2,2)$.

55. The constraint is the line segment in the first quadrant joining the points $(0,3)$ and $(6,0)$. A possible contour diagram is shown in Figure 15.34. The maximum value of f is 6 at the point $(0,3)$, an endpoint of the constraint.

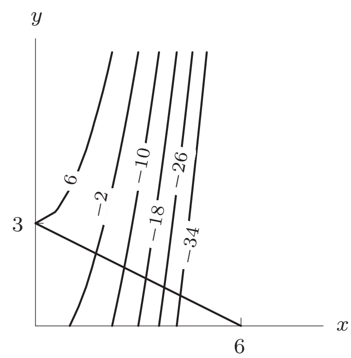


Figure 15.34

56. The maximum is $f=4$ and occurs at $(0,4)$. The minimum is $f=2$ and occurs at about $(4,2)$.

57. The maximum is $f=4$ and occurs at $(0,4)$. The minimum is $f=0$ and occurs at the origin.

58. True. The point (a,b) must lie on the constraint $g(x,y)=c$, so $g(a,b)=c$.

59. False. The point (a,b) is not necessarily a critical point of f, since it is a constrained extremum.

60. True. The constraint is the same as $x=y$, so along the constraint $f=2x$, which grows without bound as $x\to\infty$.

61. False. The condition grad $f=\lambda$grad g yields the two equations $1=\lambda 2x$ and $2=\lambda 2y$. Substituting $x=2$ in the first equation gives $\lambda=1/4$, while setting $y=-1$ in the second gives $\lambda=-1$, so the point $(2,-1)$ is not a local extremum of f constrained to $x+2y=0$.

62. False. Since grad f and grad g point in opposite directions, they are parallel. Therefore (a, b) could be a local maximum or local minimum of f constrained to $g = c$. However the information given is not enough to determine that it is a minimum. If the contours of g near (a, b) increase in the opposite direction as the contours of f, then at a point with grad $f(a, b) = \lambda$grad $g(a, b)$ we have $\lambda \leq 0$, but this can be a local maximum or minimum.

For example, $f(x, y) = 4 - x^2 - y^2$ has a local maximum at $(1, 1)$ on the constraint $g(x, y) = x + y = 2$. Yet at this point, grad $f = -2\vec{i} - 2\vec{j}$ and grad $g = \vec{i} + \vec{j}$, so grad f and grad g point in opposite directions.

63. False. A maximum for f subject to a constraint need not be a critical point of f

64. False. The condition for the Lagrange multiplier λ is grad $f(a, b) = \lambda$ grad $g(a, b)$.

65. False. Just as a critical point need not be a maximum or minimum for unconstrained optimization, a point satisfying the Lagrange condition need not be a maximum or minimum for a constrained optimization.

66. True. Since $f(a, b) = M$, we must satisfy the Lagrange conditions that $f_x(a, b) = \lambda g(a, b)$ and $f_y(a, b) = \lambda g_y(a, b)$, for some λ. Thus $f_x(a, b)/f_y(a, b) = g_x(a, b)/g_y(a, b)$.

67. True. Since $f(a, b) = m$, the point (a, b) must satisfy the Lagrange condition that $f_x(a, b) = \lambda g(a, b)$, for some λ. In particular, if $g_x(a, b) = 0$, then $f_x(a, b) = 0$.

68. False. Whether increasing c will increase M depends on the sign of λ at a point (a, b) where $f(a, b) = M$

69. True. The value of λ at a maximum point gives the proportional change in M for a change in c.

70. False. The value of λ at a minimum point gives the proportional change in m for a change in c. If $\lambda > 0$ and the change in c is positive, the change in m will also be positive.

Solutions for Chapter 15 Review

Exercises

1. At a critical point,

$$f_x = 2x + 2y - 4 = 0$$
$$f_y = 2x - 2y - 8 = 0$$

Solving these equations gives, the critical point $x = 3$, $y = -1$. To classify the critical point, we find

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 2(-2) - 2^2 = -8.$$

Since $D < 0$, we have a saddle point at $(3, -1)$.

2. The critical points of f are obtained by solving $f_x = f_y = 0$, that is

$$f_x(x, y) = 2y^2 - 2x = 0 \quad \text{and} \quad f_y(x, y) = 4xy - 4y = 0,$$

so

$$2(y^2 - x) = 0 \quad \text{and} \quad 4y(x - 1) = 0$$

The second equation gives either $y = 0$ or $x = 1$. If $y = 0$ then $x = 0$ by the first equation, so $(0, 0)$ is a critical point. If $x = 1$ then $y^2 = 1$ from which $y = 1$ or $y = -1$, so two further critical points are $(1, -1)$, and $(1, 1)$.

Since

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(4x - 4) - (4y)^2 = 8 - 8x - 16y^2,$$

we have

$$D(0, 0) = 8 > 0, \quad D(1, 1) = D(1, -1) = -16 < 0,$$

and $f_{xx} = -2 < 0$. Thus, $(0, 0)$ is a local maximum; $(1, 1)$ and $(1, -1)$ are saddle points.

3. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 6x = 0$$
$$f_y = 2y + 10 = 0$$

shows that $x = 0$ or $x = 2$ and $y = -5$. There are two critical points: $(0, -5)$ and $(2, -5)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x - 6)(2) - (0)^2 = 12x - 12.$$

When $x = 0$, we have $D = -12 < 0$, so f has a saddle point at $(0, -5)$. When $x = 2$, we have $D = 12 > 0$ and $f_{xx} = 6 > 0$, so f has a local minimum at $(2, -5)$.

4. At a critical point

$$f_x(x,y) = 2xy - 2y = 0$$
$$f_y(x,y) = x^2 + 4y - 2x = 0.$$

From the first equation, $2y(x-1) = 0$, so either $y = 0$ or $x = 1$. If $y = 0$, then $x^2 - 2x = 0$, so $x = 0$ or $x = 2$. Thus $(0,0)$ and $(2,0)$ are critical points. If $x = 1$, then $1^2 + 4y - 2 = 0$, so $y = 1/4$. Thus $(1, 1/4)$ is a critical point. Now

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2y \cdot 4 - (2x-2)^2 = 8y - 4(x-1)^2,$$

so

$$D(0,0) = -4, \quad D(2,0) = -4, \quad D(1, \tfrac{1}{4}) = 2$$

so $(0,0)$ and $(2,0)$ are saddle points. Since $f_{yy} = 4 > 0$, we see that $(1, 1/4)$ is a local minimum.

5. Critical points occur where $f_x = f_y = 0$:

$$f_x(x,y) = \frac{-80}{x^2y} + 10 + 10y.$$

$$f_y(x,y) = \frac{-80}{xy^2} + 10x + 20.$$

Substituting $x = 2$, $y = 1$ gives

$$f_x(2,1) = \frac{-80}{2^2 \cdot 1} + 10 + 10.1 = 0$$
$$f_y(2,1) = \frac{-80}{2 \cdot 1^2} + 10.2 + 20 = 0.$$

So $(2,1)$ is a critical point.

To determine if this critical point is a minimum we use the second derivative test.

$$f_{xx} = \frac{160}{x^3y}, \quad f_{xx}(2,1) = 20,$$
$$f_{yy} = \frac{160}{xy^3}, \quad f_{yy}(2,1) = 80,$$
$$f_{xy} = \frac{80}{x^2y^2} + 10, \quad f_{xy}(2,1) = 30.$$

So $D = 20 \cdot 80 - 30^2 = 700 > 0$ and $f_{xx}(2,1) > 0$, therefore the point $(2,1)$ is a local minimum.

6. The partial derivatives are

$$f_x = \cos x + \cos(x+y).$$
$$f_y = \cos y + \cos(x+y).$$

Setting $f_x = 0$ and $f_y = 0$ gives

$$\cos x = \cos y$$

For $0 < x < \pi$ and $0 < y < \pi$, $\cos x = \cos y$ only if $x = y$. Then, setting $f_x = f_y = 0$:

$$\cos x + \cos 2x = 0,$$
$$\cos x + 2\cos^2 x - 1 = 0,$$
$$(2\cos x - 1)(\cos x + 1) = 0.$$

So $\cos x = 1/2$ or $\cos x = -1$, that is $x = \pi/3$ or $x = \pi$. For the given domain $0 < x < \pi$, $0 < y < \pi$, we only consider the solution when $x = \pi/3$ then $y = x = \pi/3$. Therefore, the critical point is $(\frac{\pi}{3}, \frac{\pi}{3})$.

Since

$$f_{xx}(x,y) = -\sin x - \sin(x+y) \quad f_{xx}(\tfrac{\pi}{3}, \tfrac{\pi}{3}) = -\sin\tfrac{\pi}{3} - \sin\tfrac{2\pi}{3} = -\sqrt{3}$$
$$f_{xy}(x,y) = -\sin(x+y) \quad f_{xy}(\tfrac{\pi}{3}, \tfrac{\pi}{3}) = -\sin\tfrac{2\pi}{3} = -\tfrac{\sqrt{3}}{2}$$
$$f_{yy}(x,y) = -\sin y - \sin(x+y) \quad f_{yy}(\tfrac{\pi}{3}, \tfrac{\pi}{3}) = -\sin\tfrac{\pi}{3} - \sin\tfrac{2\pi}{3} = -\sqrt{3}$$

the discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2$$
$$= (-\sqrt{3})(-\sqrt{3}) - (-\tfrac{\sqrt{3}}{2})^2 = \tfrac{9}{4} > 0.$$

Since $f_{xx}(\frac{\pi}{3}, \frac{\pi}{3}) = -\sqrt{3} < 0$, $(\frac{\pi}{3}, \frac{\pi}{3})$ is a local maximum.

7. We find critical points:

$$f_x(x,y) = 12 - 6x = 0$$
$$f_y(x,y) = 6 - 2y = 0$$

so $(2,3)$ is the only critical point. At this point

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (-6)(-2) = 12 > 0,$$

and $f_{xx} < 0$, so $(2,3)$ is a local maximum. Since this is a quadratic, the local maximum is a global maximum.

Alternatively, we complete the square, giving

$$f(x,y) = 10 - 3(x^2 - 4x) - (y^2 - 6y) = 31 - 3(x-2)^2 - (y-3)^2.$$

This expression for f shows that its maximum value (which is 31) occurs where $x = 2, y = 3$.

8. The partial derivatives are $f_x = 2x - 3y, f_y = 3y^2 - 3x$. For critical points, solve $f_x = 0$ and $f_y = 0$ simultaneously. From $2x - 3y = 0$ we get $x = \frac{3}{2}y$. Substituting it into $3y^2 - 3x = 0$, we have that

$$3y^2 - 3(\frac{3}{2}y) = 3y^2 - \frac{9}{2}y = y(3y - \frac{9}{2}) = 0.$$

So $y = 0$ or $3y - \frac{9}{2} = 0$, that is, $y = 0$ or $y = 3/2$. Therefore the critical points are $(0,0)$ and $(\frac{9}{4}, \frac{3}{2})$.
The contour diagram for f in Figure 15.35 (drawn by a computer), shows that $(0,0)$ is a saddle point and that $(\frac{9}{4}, \frac{3}{2})$ is a local minimum.

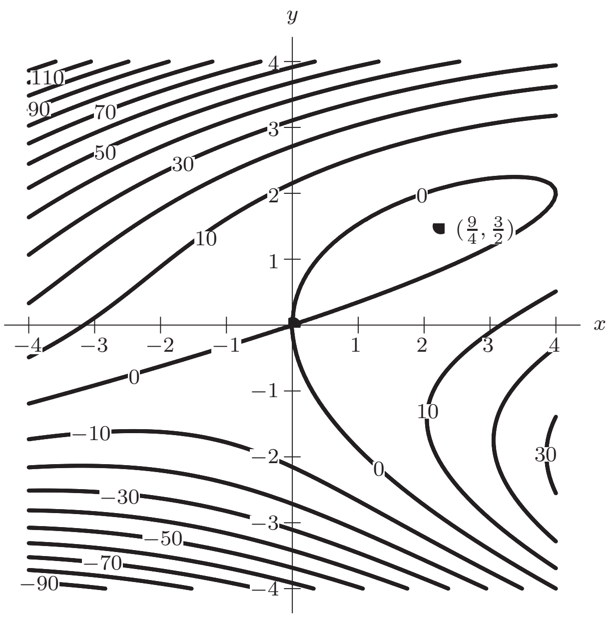


Figure 15.35: Contour map of $f(x,y) = x^2 + y^3 - 3xy$

We can also see that $(0,0)$ is a saddle point and $(\frac{9}{4}, \frac{3}{2})$ is a local minimum analytically. Since $f_{xx} = 2, f_{yy} = 6y, f_{xy} = -3$, the discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 12y - (-3)^2 = 12y - 9.$$

$D(0,0) = -9 < 0$, so $(0,0)$ is a saddle point.
$D(\frac{9}{4}, \frac{3}{2}) = 9 > 0$ and $f_{xx} = 2 > 0$, we know that $(\frac{9}{4}, \frac{3}{2})$ is a local minimum. The point $(\frac{9}{4}, \frac{3}{2})$ is not a global minimum since $f(\frac{9}{4}, \frac{3}{2}) = -1.6875$, whereas $f(0,-2) = -8$.

9. Note that the x-axis and the y-axis are not in the domain of f. Since $x \neq 0$ and $y \neq 0$, by setting $f_x = 0$ and $f_y = 0$ we get

$$f_x = 1 - \frac{1}{x^2} = 0 \text{ when } x = \pm 1$$
$$f_y = 1 - \frac{4}{y^2} = 0 \text{ when } y = \pm 2$$

So the critical points are $(1,2), (-1,2), (1,-2), (-1,-2)$. Since $f_{xx} = 2/x^3$ and $f_{yy} = 8/y^3$ and $f_{xy} = 0$, the discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = \left(\frac{2}{x^3}\right)\left(\frac{8}{y^3}\right) - 0^2 = \frac{16}{(xy)^3}.$$

Since $D < 0$ at the points $(-1,2)$ and $(1,-2)$, these points are saddle points. Since $D > 0$ at $(1,2)$ and $(-1,-2)$ and $f_{xx}(1,2) > 0$ and $f_{xx}(-1,-2) < 0$, the point $(1,2)$ is a local minimum and the point $(-1,-2)$ is a local maximum. No global maximum or minimum, since $f(x,y)$ increases without bound if x and y increase in the first quadrant; $f(x,y)$ decreases without bound if x and y decrease in the third quadrant.

10. The partial derivatives are

$$f_x = y + \frac{1}{x},\ f_y = x + 2y.$$

For critical points, solve $f_x = 0$ and $f_y = 0$ simultaneously. From $f_y = x + 2y = 0$ we get that $x = -2y$. Substituting into $f_x = 0$, we have

$$y + \frac{1}{x} = y - \frac{1}{2y} = \frac{1}{y}(y^2 - \frac{1}{2}) = 0$$

Since $\frac{1}{y} \neq 0$, $y^2 - \frac{1}{2} = 0$, therefore

$$y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2},$$

and $x = \mp\sqrt{2}$. So the critical points are $\left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$. But x must be greater than 0, so $\left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right)$ is not in the domain.

The contour diagram for f in Figure 15.36 (drawn by computer), shows that $\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$ is a saddle point of $f(x,y)$.

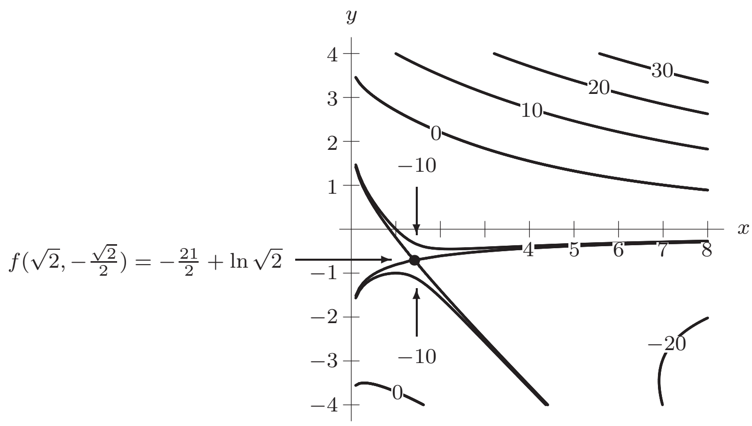


Figure 15.36: Contour map of $f(x,y) = xy + \ln x + y^2 - 10$

We can also see that $\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$ is a saddle point analytically.

Since $f_{xx} = -\frac{1}{x^2}$, $f_{yy} = 2$, $f_{xy} = 1$, the discriminant is:

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2$$
$$= -\frac{2}{x^2} - 1.$$

$D\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right) = -2 < 0$, so $\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$ is a saddle point.

11. The objective function is $f(x,y) = 3x - 4y$ and the constraint equation is $g(x,y) = x^2 + y^2 = 5$, so grad $f = 3\vec{i} - 4\vec{j}$ and grad $g = (2x)\vec{i} + (2y)\vec{j}$. Setting grad $f = \lambda$ grad g gives

$$3 = \lambda(2x),$$
$$-4 = \lambda(2y).$$

From the first equation we have $\lambda = 3/(2x)$, and from the second equation we have $\lambda = -2/y$. Setting these equal gives

$$x = -\frac{3}{4}y.$$

Substituting this into the constraint equation $x^2 + y^2 = 5$ gives $y^2 = 16/5$, so $y = 4/\sqrt{5}$ and $y = -4/\sqrt{5}$. Since $x = -\frac{3}{4}y$, there are two points where a maximum or a minimum might occur:

$$(-3/\sqrt{5}, 4/\sqrt{5}) \quad \text{and} \quad (3/\sqrt{5}, -4/\sqrt{5}).$$

Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Since $f(-3/\sqrt{5}, 4/\sqrt{5}) = -5\sqrt{5}$ and $f(3/\sqrt{5}, -4/\sqrt{5}) = 5\sqrt{5}$, we see that f has a minimum value at $(-3/\sqrt{5}, 4/\sqrt{5})$ and a maximum value at $(3/\sqrt{5}, -4/\sqrt{5})$.

12. The objective function is $f(x,y) = x^2 + y^2$ and the equation of constraint is $g(x,y) = x^4 + y^4 = 2$. Their gradients are

$$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j},$$
$$\nabla g(x,y) = 4x^3\vec{i} + 4y^3\vec{j}.$$

So the equation $\nabla f = \lambda \nabla g$ becomes $2x\vec{i} + 2y\vec{j} = \lambda(4x^3\vec{i} + 4y^3\vec{j})$. This tells us that

$$2x = 4\lambda x^3,$$
$$2y = 4\lambda y^3.$$

Now if $x = 0$, the first equation is true for any value of λ. In particular, we can choose λ which satisfies the second equation. Similarly, $y = 0$ is solution.

Assuming both $x \neq 0$ and $y \neq 0$, we can divide to solve for λ and find

$$\lambda = \frac{2x}{4x^3} = \frac{2y}{4y^3}$$
$$\frac{1}{2x^2} = \frac{1}{2y^2}$$
$$y^2 = x^2$$
$$y = \pm x.$$

Going back to our equation of constraint, we find

$$g(0,y) = 0^4 + y^4 = 2, \quad \text{so } y = \pm\sqrt[4]{2}$$
$$g(x,0) = x^4 + 0^4 = 2, \quad \text{so } x = \pm\sqrt[4]{2}$$
$$g(x,\pm x) = x^4 + (\pm x)^4 = 2, \quad \text{so } x = \pm 1.$$

Thus, the critical points are $(0, \pm\sqrt[4]{2})$, $(\pm\sqrt[4]{2}, 0)$, $(1, \pm 1)$ and $(-1, \pm 1)$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points, we find

$$f(1,1) = f(1,-1) = f(-1,1) = f(-1,-1) = 2,$$
$$f(0,\sqrt[4]{2}) = f(0,-\sqrt[4]{2}) = f(\sqrt[4]{2},0) = f(-\sqrt[4]{2},0) = \sqrt{2}.$$

Thus, the minimum value of $f(x,y)$ on $g(x,y) = 2$ is $\sqrt{2}$ and the maximum value is 2.

13. The objective function is $f(x,y) = x^2 + y^2$ and the constraint equation is $g(x,y) = 4x - 2y = 15$, so grad $f = (2x)\vec{i} + (2y)\vec{j}$ and grad $g = 4\vec{i} - 2\vec{j}$. Setting grad $f = \lambda$ grad g gives

$$2x = 4\lambda,$$
$$2y = -2\lambda.$$

From the first equation we have $\lambda = x/2$, and from the second equation we have $\lambda = -y$. Setting these equal gives

$$y = -0.5x.$$

Substituting this into the constraint equation $4x - 2y = 15$ gives $x = 3$. The only critical point is $(3, -1.5)$.

We have $f(3, -1.5) = (3)^2 + (1.5)^2 = 11.25$. One way to determine if this point gives a maximum or minimum value or neither for the given constraint is to examine the contour diagram of f with the constraint sketched in, Figure 15.37. It appears that moving away from the point $P = (3, -1.5)$ in either direction along the constraint increases the value of f, so $(3, -1.5)$ is a point of minimum value.

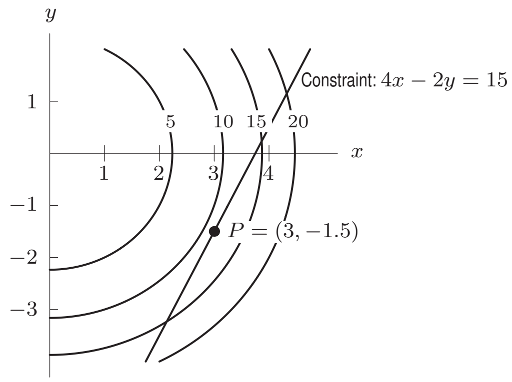


Figure 15.37

14. The objective function is $f(x, y) = x^2 - xy + y^2$ and the equation of constraint is $g(x, y) = x^2 - y^2 = 1$. The gradients of f and g are

$$\nabla f(x, y) = (2x - y)\vec{i} + (-x + 2y)\vec{j},$$
$$\nabla g(x, y) = 2x\vec{i} - 2y\vec{j}.$$

Therefore the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ gives

$$2x - y = 2\lambda x$$
$$-x + 2y = -2\lambda y$$
$$x^2 - y^2 = 1.$$

Let us suppose that $\lambda = 0$. Then $2x = y$ and $2y = x$ give $x = y = 0$. But $(0, 0)$ is not a solution of the third equation, so we conclude that $\lambda \neq 0$. Now let's multiply the first two equations

$$-2\lambda y(2x - y) = 2\lambda x(-x + 2y).$$

As $\lambda \neq 0$, we can cancel it in the equation above and after doing the algebra we get

$$x^2 - 4xy + y^2 = 0$$

which gives $x = (2 + \sqrt{3})y$ or $x = (2 - \sqrt{3})y$.

If $x = (2 + \sqrt{3})y$, the third equation gives

$$(2 + \sqrt{3})^2 y^2 - y^2 = 1$$

so $y \approx \pm 0.278$ and $x \approx \pm 1.038$. These give the critical points $(1.038, 0.278)$, $(-1.038, -0.278)$.

If $x = (2 - \sqrt{3})y$, from the third equation we get

$$(2 - \sqrt{3})^2 y^2 - y^2 = 1.$$

But $(2 - \sqrt{3})^2 - 1 \approx -0.928 < 0$ so the equation has no solution. Evaluating f gives

$$f(1.038, 0.278) = f(-1.038, -0.278) \approx 0.866$$

Since $y \to \infty$ on the constraint, rewriting f as

$$f(x, y) = \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2$$

shows that f has no maximum on the constraint. The minimum value of f is 0.866. See Figure 15.38.

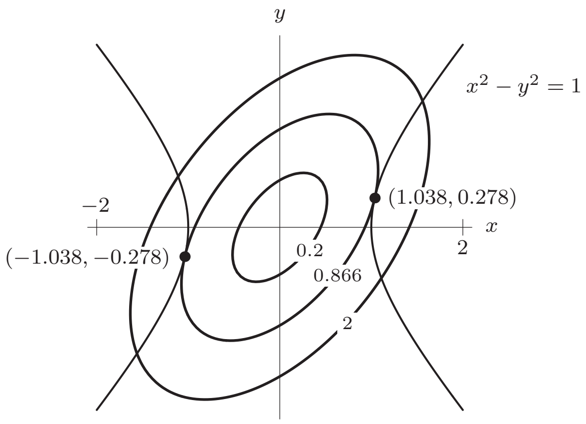


Figure 15.38

15. The objective function is $f(x, y) = x^2 + 2y^2$ and the constraint equation is $g(x, y) = 3x + 5y = 200$, so grad $f = (2x)\vec{i} + (4y)\vec{j}$ and grad $g = 3\vec{i} + 5\vec{j}$. Setting grad $f = \lambda$ grad g gives

$$2x = 3\lambda,$$
$$4y = 5\lambda.$$

From the first equation, we have $\lambda = 2x/3$, and from the second equation we have $\lambda = 4y/5$. Setting these equal gives

$$x = 1.2y.$$

Substituting this into the constraint equation $3x + 5y = 200$ gives $y = 23.256$. Since $x = 1.2y$, we have $x = 27.907$. A maximum or minimum value of f can occur only at $(27.907, 23.256)$.

We have $f(27.907, 23.256) = 1860.484$. From Figure 15.39, we see that the point $(27.907, 23.256)$ is a minimum value of f subject to the given constraint.

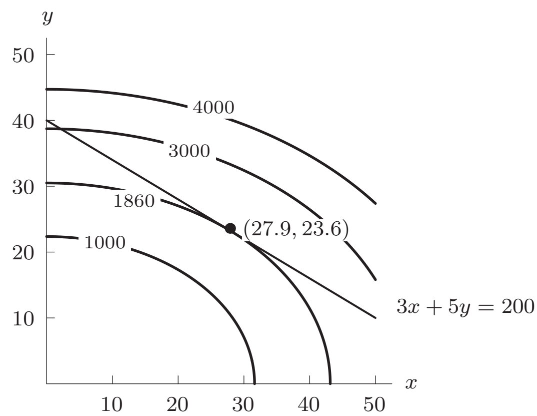


Figure 15.39

16. Our objective function is $f(x,y)=xy$ and our equation of constraint is $g(x,y)=4x^2+y^2=8$. Their gradients are

$$\nabla f(x,y)=y\vec{i}+x\vec{j},$$
$$\nabla g(x,y)=8x\vec{i}+2y\vec{j}.$$

So the equation $\nabla f=\lambda\nabla g$ becomes $y\vec{i}+x\vec{j}=\lambda(8x\vec{i}+2y\vec{j})$. This gives

$$8x\lambda=y \quad \text{and} \quad 2y\lambda=x.$$

Multiplying, we get

$$8x^2\lambda=2y^2\lambda.$$

If $\lambda=0$, then $x=y=0$, which does not satisfy the constraint equation. So $\lambda\neq 0$ and we get

$$\begin{aligned} 2y^2&=8x^2\\ y^2&=4x^2\\ y&=\pm 2x. \end{aligned}$$

To find x, we substitute for y in our equation of constraint.

$$\begin{aligned} 4x^2+y^2&=8\\ 4x^2+4x^2&=8\\ x^2&=1\\ x&=\pm 1 \end{aligned}$$

So our critical points are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating $f(x,y)$ at the critical points, we have

$$\begin{aligned} f(1,2)=f(-1,-2)&=2\\ f(1,-2)=f(1,-2)&=-2. \end{aligned}$$

Thus, the maximum value of f on $g(x,y)=8$ is 2, and the minimum value is -2.

17. We will use the Lagrange multipliers with:
Objective function: $f(x,y)=-3x^2-2y^2+20xy$
Constraint: $g(x,y)=x+y-100$
We first find

$$\begin{aligned} \nabla f&=(-6x+20y)\vec{i}+(-4y+20x)\vec{j}\\ \nabla g&=\vec{i}+\vec{j}. \end{aligned}$$

To optimize f, we must solve the equations

$$\begin{aligned} \nabla f&=\lambda\nabla g\\ (-6x+20y)\vec{i}+(-4y+20x)\vec{j}&=\lambda(\vec{i}+\vec{j})=\lambda\vec{i}+\lambda\vec{j} \end{aligned}$$

We have a vector equation, so we equate the coordinates:

$$\begin{aligned} -6x+20y&=\lambda\\ 20x-4y&=\lambda.\\ \text{So}\quad -6x+20y&=20x-4y\\ 24y&=26x\\ y&=\frac{13}{12}x \end{aligned}$$

Substituting into the constraint equation $x+y=100$, we obtain:

$$\begin{aligned} x+\frac{13}{12}x&=100\\ \frac{25}{12}x&=100\\ x&=48. \end{aligned}$$

Consequently, $y=52$, and $f(48,52)=37{,}600$. The point $(48,52)$ leads to the extreme value of $f(x,y)$, given that $x+y=100$. Note that f has no minimum on the line $x+y=100$ since $f(x,100-x)=-3x^2-2(100-x)^2+20x(100-x)=-25x^2+2400x-20000$ which goes to $-\infty$ as x goes to $\pm\infty$. Therefore, the point $(48,52)$ gives the maximum value for f on the line $x+y=100$.

18. Our objective function is $f(x,y,z) = x^2 - 2y + 2z^2$ and our equation of constraint is $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. To optimize $f(x,y,z)$ with Lagrange multipliers, we solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ subject to $g(x,y,z) = 0$. The gradients of f and g are

$$\begin{aligned}\nabla f(x,y,z) &= 2x\vec{i} - 2\vec{j} + 4z\vec{k}\,,\\ \nabla g(x,y) &= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}\,.\end{aligned}$$

We get,

$$\begin{aligned}x &= \lambda x\\ -1 &= \lambda y\\ 2z &= \lambda z\\ x^2 + y^2 + z^2 &= 1.\end{aligned}$$

From the first equation we get $x = 0$ or $\lambda = 1$.

If $x = 0$ we have

$$\begin{aligned}-1 &= \lambda y\\ 2z &= \lambda z\\ y^2 + z^2 &= 1.\end{aligned}$$

From the second equation $z = 0$ or $\lambda = 2$. So if $z = 0$, we have $y = \pm 1$ and we get the solutions $(0,1,0)$,$(0,-1,0)$. If $z \neq 0$ then $\lambda = 2$ and $y = -\frac{1}{2}$. So $z^2 = \frac{3}{4}$ which gives the solutions $(0, -\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(0, -\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

If $x \neq 0$, then $\lambda = 1$, so $y = -1$, which implies, from the equation $x^2 + y^2 + z^2 = 1$, that $x = 0$, which contradicts the assumption.

Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Therefore, evaluating f at the critical points, we get $f(0,1,0) = -2$, $f(0,-1,0) = 2$ and $f(0, -\frac{1}{2}, \frac{\sqrt{3}}{2}) = f(0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{5}{2}$. So the maximum value of f is $\frac{5}{2}$ and the minimum is -2.

19. Our objective function is $f(x,y,z) = 2x + y + 4z$ and our equation of constraint is $g(x,y,z) = x^2 + y + z^2 = 16$. Their gradients are

$$\begin{aligned}\nabla f(x,y,z) &= 2\vec{i} + 1\vec{j} + 4\vec{k}\,,\\ \nabla g(x,y,z) &= 2x\vec{i} + 1\vec{j} + 2z\vec{k}\,.\end{aligned}$$

So the equation $\nabla f = \lambda \nabla g$ becomes $2\vec{i} + 1\vec{j} + 4\vec{k} = \lambda(2x\vec{i} + 1\vec{j} + 2z\vec{k})$. Solving for λ we find

$$\begin{aligned}\lambda &= \frac{2}{2x} = \frac{1}{1} = \frac{4}{2z}\\ \lambda &= \frac{1}{x} = 1 = \frac{2}{z}.\end{aligned}$$

Which tells us that $x = 1$ and $z = 2$. Going back to our equation of constraint, we can solve for y.

$$\begin{aligned}g(1,y,2) &= 16\\ 1^2 + y + 2^2 &= 16\\ y &= 11.\end{aligned}$$

So our one critical point is at $(1,11,2)$. The value of f at this point is $f(1,11,2) = 2 + 11 + 8 = 21$. This is the maximum value of $f(x,y,z)$ on $g(x,y,z) = 16$. To see this, note that for $y = 16 - x^2 - z^2$,

$$f(x,y,z) = 2x + 16 - x^2 - z^2 + 4z = 21 - (x-1)^2 - (z-2)^2 \leq 21.$$

As $y \to -\infty$, the point $(-\sqrt{16-y}, y, 0)$ is on the constraint and $f(-\sqrt{16-y}, y, 0) \to -\infty$, so there is no minimum value for $f(x,y,z)$ on $g(x,y,z) = 16$.

20. We first find the critical points in the disk

$$\nabla z = (8x - y)\vec{i} + (8y - x)\vec{j}$$

Setting $\nabla z = 0$ gives $8x - y = 0$ and $8y - x = 0$. The only solution is $x = y = 0$. So $(0, 0)$ is the only critical point in the disk.

Next we find the extremal values on the boundary using Lagrange multipliers. We have objective function $z = 4x^2 - xy + 4y^2$ and constraint $G = x^2 + y^2 - 2 = 0$.

$$\begin{aligned}\nabla z &= (8x - y)\vec{i} + (8y - x)\vec{j}\\ \nabla G &= 2x\vec{i} + 2y\vec{j}\end{aligned}$$

$\nabla z = \lambda \nabla G$ gives

$$\begin{aligned}8x - y &= 2\lambda x\\ 8y - x &= 2\lambda y\end{aligned}$$

If $\lambda = 0$ we get

$$\begin{aligned}8x - y &= 0\\ 8y - x &= 0\end{aligned}$$

with only solutions $x = y = 0$, which does not satisfy the constraint: $x^2 + y^2 - 2 = 0$. Therefore $\lambda \neq 0$ and we get:

$$2\lambda y(8x - y) = 2\lambda x(8y - x)$$

and

$$y(8x - y) = x(8y - x).$$

So $x^2 = y^2$, $x = \pm y$.

Substitute into $G = 0$, we get $2x^2 - 2 = 0$ so $x = \pm 1$. The extremal points on the boundary are therefore $(1, 1), (1, -1), (-1, 1), (-1, -1)$. The region $x^2 + y^2 \leq 2$ is closed and bounded, so minimum values of f in the region exist. We check the values of z at these points :

$$z(1, 1) = 7, \quad z(-1, -1) = 7, \quad z(1, -1) = 9, \quad z(-1, 1) = 9, \quad z(0, 0) = 0$$

Thus $(-1, 1)$ and $(1, -1)$ give the maxima over the closed disk and $(0, 0)$ gives the minimum.

21. The region $x^2 \geq y$ is the shaded region in Figure 15.40 which includes the parabola $y = x^2$.

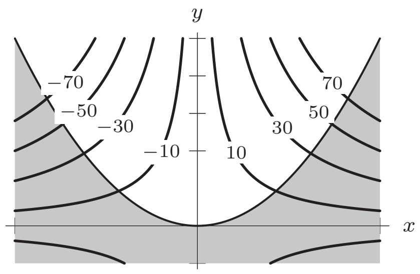


Figure 15.40

We first want to find the local maxima and minima of f in the interior of our region. So we need to find the extrema of

$$f(x, y) = x^2 - y^2, \quad \text{in the region} \quad x^2 > y.$$

For this we compute the critical points:

$$\begin{aligned}f_x &= 2x = 0\\ f_y &= -2y = 0.\end{aligned}$$

As $(0,0)$ does not belong to the region $x^2 > y$, we have no critical points. Now let's find the local extrema of f on the boundary of our region, hence this time we have to solve a constraint problem. We want to find the extrema of $f(x,y) = x^2 - y^2$ subject to $g(x,y) = x^2 - y = 0$. We use Lagrange multipliers:

$$\text{grad}\, f = \lambda\, \text{grad}\, g \quad \text{and} \quad x^2 = y.$$

This gives

$$\begin{aligned} 2x &= 2\lambda x \\ 2y &= \lambda \\ x^2 &= y. \end{aligned}$$

From the first equation we get $x = 0$ or $\lambda = 1$.

If $x = 0$, from the third equation we get $y = 0$, so one solution is $(0,0)$. If $x \neq 0$, then $\lambda = 1$ and from the second equation we get $y = \frac{1}{2}$. This gives $x^2 = \frac{1}{2}$ so the solutions $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$.

So $f(0,0) = 0$ and $f(\frac{1}{\sqrt{2}}, \frac{1}{2}) = f(-\frac{1}{\sqrt{2}}, \frac{1}{2}) = \frac{1}{4}$. From Figure 15.40 showing the level curves of f and the region $x^2 \geq y$, we see that $(0,0)$ is a local minimum of f on $x^2 = y$, but not a global minimum and that $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$ are global maxima of f on $x^2 = y$ but *not* global maxima of f on the whole region $x^2 \geq y$.

So there are no global extrema of f in the region $x^2 \geq y$.

22. The region $x^2 + y^2 \leq 1$ is the shaded disk of radius 1 centered at the origin (including the circle $x^2 + y^2 = 1$) shown in Figure 15.41.

Let's first compute the critical points of f in the interior of the disk. We have

$$\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= -2y = 0, \end{aligned}$$

whose solution is $x = y = 0$. So the only one critical point is $(0,0)$. As $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = -2$ and $f_{xy}(0,0) = 0$,

$$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - (f_{xy}(0,0))^2 = 0$$

which does not tell us anything about the nature of the critical point $(0,0)$.

But, if we choose x,y very small in absolute value and such that $x^3 > y^2$, then $f(x,y) > 0$. If we choose x,y very small in absolute value and such that $x^3 < y^2$, then $f(x,y) < 0$. As $f(0,0) = 0$, we conclude that $(0,0)$ is a saddle point.

We can get the same conclusion looking at the level curves of f around $(0,0)$, as shown in Figure 15.42.

So, f does not have extrema in the interior of the disk.

Now, let's find the local extrema of f on the circle $x^2 + y^2 = 1$. So we want the extrema of $f(x,y) = x^3 - y^2$ subject to the constraint $g(x,y) = x^2 + y^2 - 1 = 0$. Using Lagrange multipliers we get

$$\text{grad}\, f = \lambda\, \text{grad}\, g \quad \text{and} \quad x^2 + y^2 = 1,$$

which gives

$$\begin{aligned} 3x^2 &= 2\lambda x \\ -2y &= 2\lambda y \\ x^2 + y^2 &= 1. \end{aligned}$$

From the second equation $y = 0$ or $\lambda = -1$.

If $y = 0$, from the third equation we get $x^2 = 1$, which gives the solutions $(1,0)$, $(-1,0)$.

If $y \neq 0$ then $\lambda = -1$ and from the first equation we get $3x^2 = -2x$, hence $x = 0$ or $x = -\frac{2}{3}$. If $x = 0$, from the third equation we get $y^2 = 1$, so the solutions $(0,1)$,$(0,-1)$. If $x = -\frac{2}{3}$, from the third equation we get $y^2 = \frac{5}{9}$, so the solutions $(-\frac{2}{3}, \frac{\sqrt{5}}{3})$, $(-\frac{2}{3}, -\frac{\sqrt{5}}{3})$.

Evaluating f at these points we get

$$f(1,0) = 1, \quad f(-1,0) = f(0,1) = f(0,-1) = -1$$

and

$$f\left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\right) = f\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right) = -\frac{23}{27}.$$

The region $x^2+y^2 \le 1$ is closed and bounded, so maximum and minimum values of f in the region exist. Therefore the maximum value of f is 1 and the minimum value is -1.

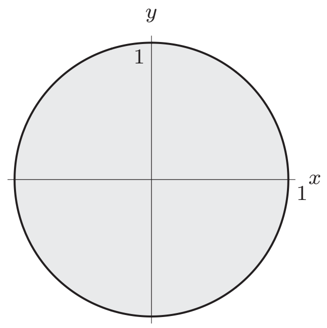


Figure 15.41

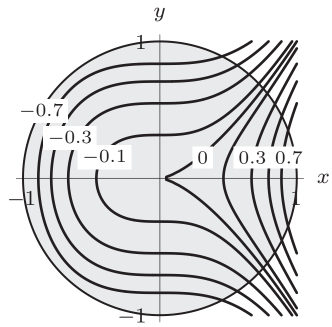


Figure 15.42: Level curves of f

23. If $x = 10$ then $f(x, y) = 100 - y^2$ is a parabola opening downward, so it has a maximum but no minimum.

24. If $y = 10$ then $f(x, y) = x^2 - 100$ is a parabola opening upward, so it has a minimum but no maximum.

25. If $x^2+y^2 = 10$, then $f(x, y) = x^2 - y^2 = 2x^2 - 10$ and x has values in the interval $-\sqrt{10} \le x \le \sqrt{10}$. Hence $f(x, y)$ has a maximum on the constraint at $(x, y) = (\pm\sqrt{10}, 0)$ and a minimum at $(x, y) = (0, \pm\sqrt{10})$.

26. If $xy = 10$, then $f(x, y) = x^2 - y^2 = x^2 - 100/x^2$ and x can take any nonzero value. Since

$$\lim_{x\to\infty}\left(x^2 - \frac{100}{x^2}\right) = \infty,$$

we see f has no maximum on the constraint. Since

$$\lim_{x\to 0}\left(x^2 - \frac{100}{x^2}\right) = -\infty,$$

we see f has no minimum on the constraint.

Problems

27. The function $f(x, y) = 0.3\ln x + 0.7\ln y$ has domain $x > 0, y > 0$. (It is not defined for $x \le 0$ or $y \le 0$.)

At a critical point grad $f = \lambda$ grad g, so

$$\frac{0.3}{x} = 2\lambda$$
$$\frac{0.7}{y} = 3\lambda.$$

Dividing gives

$$\frac{0.3}{x}\cdot\frac{y}{0.7} = \frac{2}{3}$$

$$y = \frac{2}{3}\cdot\frac{0.7}{0.3}x = \frac{14}{9}x.$$

Substituting into $2x + 3y = 6$ gives

$$2x + 3\left(\frac{14}{9}x\right) = 6$$
$$\frac{20}{3}x = 6 \qquad \text{so} \quad x = 0.9.$$

Thus, $y = 14(0.9)/9 = 1.4$.

This is the only critical point on the constraint, and value of f decreases toward $-\infty$ as x or $y \to 0$. Thus $(0.9, 1.4)$ gives the maximum:

$$f(0.9, 1.4) = 0.3\ln(0.9) + 0.7\ln(1.4) = 0.204.$$

28. (a) The distance is

$$D = \sqrt{(x-3)^2 + (y-4)^2}.$$

(b) We want to minimize D subject to the constraint $x^2 + y^2 = 1$. The same values of x and y which minimize D also minimize $D^2 = (x-3)^2 + (y-4)^2$, so we work with D^2. Thus, at a critical point

$$\begin{aligned} \text{grad}(D^2) &= \lambda\, \text{grad}(x^2 + y^2) \\ 2(x-3) &= 2\lambda x \\ 2(y-4) &= 2\lambda y \end{aligned}$$

so we see $\lambda \neq 1$ and

$$x = \frac{3}{1-\lambda} \quad \text{and} \quad y = \frac{4}{1-\lambda},$$

giving

$$y = \frac{4}{3}x.$$

Substituting into $x^2 + y^2 = 1$, we have

$$\begin{aligned} x^2 + \left(\frac{4}{3}x\right)^2 &= 1 \\ \frac{25}{9}x^2 &= 1 \\ x &= \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}. \end{aligned}$$

Since $y = 4x/3$, the critical points are

$$\left(\frac{3}{5}, \frac{4}{5}\right) \quad \text{and} \quad \left(-\frac{3}{5}, -\frac{4}{5}\right),$$

that is

$$(0.6, 0.8) \quad \text{and} \quad (-0.6, -0.8),$$

The distances between $(3, 4)$ and these points are

$$\begin{aligned} D(0.6, 0.8) &= \sqrt{(3-0.6)^2 + (4-0.8)^2} = 4 \\ D(-0.6, -0.8) &= \sqrt{(3+0.6)^2 + (4+0.8)^2} = 6. \end{aligned}$$

Thus, the minimum distance is 4 and occurs at $(0.6, 0.8)$.

(c) The maximum distance is 6 and occurs at $(-0.6, -0.8)$.

29. The maximum and minimum values change by approximately $\lambda\Delta c$. The Lagrange conditions give:

$$2x = \lambda 4x^3, \quad 2y = \lambda 4y^3.$$

If $x = 0$, then $y = \pm 2^{1/4}$ from the constraint. If $y = 0$, then $x = \pm 2^{1/4}$. If $x \neq 0$ and $y \neq 0$, we can solve for λ and set the expressions equal to get $x^2 = y^2$, so $y = \pm x$.

Thus, there are eight points satisfying the Lagrange conditions: four of the form $(0, \pm 2^{1/4})$ or $(\pm 2^{1/4}, 0)$, and four of the form $(\pm 1, \pm 1)$. Since $f(x, y) = x^2 + y^2$, we get a maximum value of 2 at the four points of the form $(\pm 1, \pm 1)$ and a minimum value of $2^{2/4} = \sqrt{2}$ at the other four points. For the maximum value, we use $\lambda = 1/(2x^2) = 1/2$, so the change is approximately $\Delta c/2$. For the minimum value at $(0, \pm 2^{1/4})$, we use $\lambda = 1/(2y^2)$, so there the change is approximately $\Delta c/(2\sqrt{2})$. Similarly, the change at $\pm 2^{1/4}, 0)$ is also $\Delta c/(2\sqrt{2})$.

30. Let the line be in the form $y = b + mx$. When x equals -1, 0 and 1, then y equals $b - m$, b, and $b + m$, respectively. The sum of the squares of the vertical distances, which is what we want to minimize, is

$$f(m, b) = (2 - (b - m))^2 + (-1 - b)^2 + (1 - (b + m))^2.$$

To find the critical points, we compute the partial derivatives with respect to m and b,

$$\begin{aligned} f_m &= 2(2 - b + m) + 0 + 2(1 - b - m)(-1) \\ &= 4 - 2b + 2m - 2 + 2b + 2m \\ &= 2 + 4m, \\ f_b &= 2(2 - b + m)(-1) + 2(-1 - b)(-1) + 2(1 - b - m)(-1) \\ &= -4 + 2b - 2m + 2 + 2b - 2 + 2b + 2m \\ &= -4 + 6b. \end{aligned}$$

Setting both partial derivatives equal to zero, we get a system of equations:

$$2 + 4m = 0,$$
$$-4 + 6b = 0.$$

The solution is $m = -1/2$ and $b = 2/3$. One can check that it is a minimum. Hence, the regression line is $y = \frac{2}{3} - \frac{1}{2}x$.

31. Since $f_{xx} < 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, the point $(1, 3)$ is a maximum. See Figure 15.43.

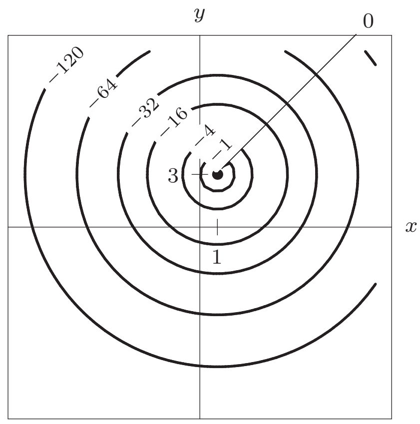


Figure 15.43

32. At a local maximum value of f,

$$\frac{\partial f}{\partial x} = -2x - B = 0.$$

We are told that this is satisfied by $x = -2$. So $-2(-2) - B = 0$ and $B = 4$. In addition,

$$\frac{\partial f}{\partial y} = -2y - C = 0$$

and we know this holds for $y = 1$, so $-2(1) - C = 0$, giving $C = -2$. We are also told that the value of f is 15 at the point $(-2, 1)$, so

$$15 = f(-2, 1) = A - ((-2)^2 + 4(-2) + 1^2 - 2(1)) = A - (-5), \text{ so } A = 10.$$

Now we check that these values of A, B, and C give $f(x, y)$ a local maximum at the point $(-2, 1)$. Since

$$f_{xx}(-2, 1) = -2,$$
$$f_{yy}(-2, 1) = -2$$

and

$$f_{xy}(-2, 1) = 0,$$

we have that $f_{xx}(-2, 1)f_{yy}(-2, 1) - f_{xy}^2(-2, 1) = (-2)(-2) - 0 > 0$ and $f_{xx}(-2, 1) < 0$. Thus, f has a local maximum value 15 at $(-2, 1)$.

33. (a) (i) Suppose $N = kA^p$. Then the rule of thumb tells us that if A is multiplied by 10, the value of N doubles. Thus

$$2N = k(10A)^p = k10^pA^p.$$

Thus, dividing by $N = kA^p$, we have

$$2 = 10^p$$

so taking logs to base 10 we have

$$p = \log 2 = 0.3010.$$

(where $\log 2$ means $\log_{10} 2$). Thus,

$$N = kA^{0.3010}.$$

(ii) Taking natural logs gives

$$\ln N = \ln(kA^p)$$
$$\ln N = \ln k + p \ln A$$
$$\ln N \approx \ln k + 0.301 \ln A$$

Thus, $\ln N$ is a linear function of $\ln A$.

(b) Table 15.2 contains the natural logarithms of the data:

Table 15.2 $\ln N$ *and* $\ln A$

Island	$\ln A$	$\ln N$
Redonda	1.1	1.6
Saba	3.0	2.2
Montserrat	2.3	2.7
Puerto Rico	9.1	4.3
Jamaica	9.3	4.2
Hispaniola	11.2	4.8
Cuba	11.6	4.8

Using a least squares fit we find the line:

$$\ln N = 1.20 + 0.32 \ln A$$

This yields the power function:

$$N = e^{1.20} A^{0.32} = 3.32 A^{0.32}$$

Since 0.32 is pretty close to $\log 2 \approx 0.301$, the answer does agree with the biological rule.

34. We want to minimize

$$C = f(q_1, q_2) = 2q_1^2 + q_1 q_2 + q_2^2 + 500$$

subject to the constraint $q_1 + q_2 = 200$ or $g(q_1, q_2) = q_1 + q_2 - 200 = 0$.

Since $\nabla f = (4q_1 + q_2)\vec{i} + (2q_2 + q_1)\vec{j}$ and $\nabla g = \vec{i} + \vec{j}$, $\nabla f = \lambda \nabla g$ gives

$$4q_1 + q_2 = \lambda$$
$$2q_2 + q_1 = \lambda.$$

Solving we get

$$4q_1 + q_2 = 2q_2 + q_1$$

so

$$3q_1 = q_2.$$

We want

$$q_1 + q_2 = 200$$
$$q_1 + 3q_1 = 4q_1 = 200.$$

Therefore

$$q_1 = 50 \text{ units}, \quad q_2 = 150 \text{ units}.$$

35. (a) Let $g(x, y) = x + y$. We are minimizing $f(x, y) = x^2 + 2y^2$ subject to the constraint $g(x, y) = c$.

The method of Lagrange multipliers is to solve the equations

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g = c,$$

which are

$$2x = \lambda \qquad 4y = \lambda \qquad x + y = c.$$

We have

$$x = \frac{2c}{3} \qquad y = \frac{c}{3} \quad \lambda = \frac{4c}{3},$$

so there is a critical point at $(2c/3, c/3)$. Since moving away from the origin increases values of f in Figure 15.44, we see that f has a minimum on the constraint. The minimum value is

$$m(c) = f\left(\frac{2c}{3}, \frac{c}{3}\right) = \frac{2c^2}{3}.$$

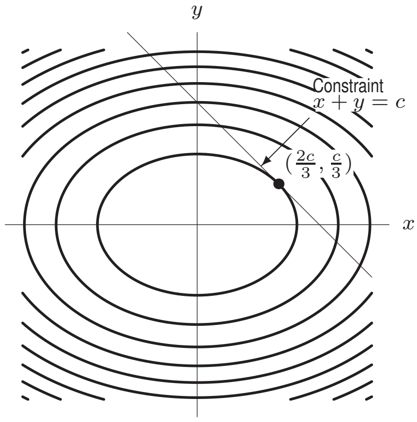


Figure 15.44

(b) Calculations in part (a) showed that $\lambda = 4c/3$.
(c) The multiplier λ is the rate of change of $m(c)$ as c increases and the constraint moves: that is, $\lambda = m'(c)$.

36. We are comparing the maxima of $f(x, y)$ subject to the two different constraints $g(x, y) = 240$ and $g(x, y) = 242$. As c changes in the constraint $g(x, y) = c$, we have

$$\text{Change in maximum of } f \approx \lambda \times \text{Change in } c = 20 \cdot 2 = 40.$$

Since the maximum with $c = 240$ is 6300, we have

$$\text{Maximum of } f \text{ constrained by } g(x, y) = 242 \approx 6300 + 40 = 6340.$$

37. Constraint is $G = P_1x + P_2y - K = 0$.
Since $\nabla Q = \lambda \nabla G$, we have

$$cax^{a-1}y^b = \lambda P_1 \quad \text{and} \quad cbx^a y^{b-1} = \lambda P_2.$$

Dividing the two equations yields $\dfrac{cax^{a-1}y^b}{cbx^a y^{b-1}} = \dfrac{\lambda P_1}{\lambda P_2}$, or simplifying, $\dfrac{ay}{bx} = \dfrac{P_1}{P_2}$. Hence, $y = \dfrac{bP_1}{aP_2}x$.

Substitute into the constraint to obtain $P_1x + P_2\dfrac{bP_1}{aP_2}x = P_1\left(\dfrac{a+b}{a}\right)x = K$, giving

$$x = \frac{aK}{(a+b)P_1} \quad \text{and} \quad y = \frac{bK}{(a+b)P_2}.$$

We now check that this is indeed the maximization point. Since $x, y \geq 0$, possible maximization points are $(0, \dfrac{K}{P_2})$, $(\dfrac{K}{P_1}, 0)$, and $(\dfrac{aK}{(a+b)P_1}, \dfrac{bK}{(a+b)P_2})$. Since $Q = 0$ for the first two points and Q is positive for the last point, it follows that $(\dfrac{aK}{(a+b)P_1}, \dfrac{bK}{(a+b)P_2})$ gives the maximal value.

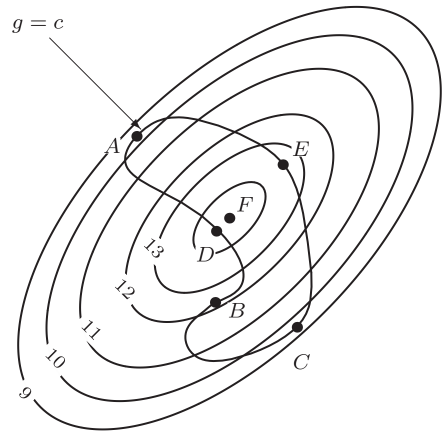
38. (a) To be producing the maximum quantity Q under the cost constraint given, the firm should be using K and L values given by

$$\frac{\partial Q}{\partial K} = 0.6aK^{-0.4}L^{0.4} = 20\lambda$$
$$\frac{\partial Q}{\partial L} = 0.4aK^{0.6}L^{-0.6} = 10\lambda$$
$$20K + 10L = 150.$$

Hence $\dfrac{0.6aK^{-0.4}L^{0.4}}{0.4aK^{0.6}L^{-0.6}} = 1.5\dfrac{L}{K} = \dfrac{20\lambda}{10\lambda} = 2$, so $L = \dfrac{4}{3}K$. Substituting in $20K + 10L = 150$, we obtain $20K + 10\left(\dfrac{4}{3}\right)K = 150$. Then $K = \dfrac{9}{2}$ and $L = 6$, so capital should be reduced by $\dfrac{1}{2}$ unit, and labor should be increased by 1 unit.

(b) $\dfrac{\text{New production}}{\text{Old production}} = \dfrac{a4.5^{0.6}6^{0.4}}{a5^{0.6}5^{0.4}} \approx 1.01$, so tell the board of directors, "Reducing the quantity of capital by $1/2$ unit and increasing the quantity of labor by 1 unit will increase production by 1% while holding costs to \$150."

39. (a) Points A, B, C, D, E; that is, where a level curve of f and the constraint curve are parallel.
(b) Point F since the value of f is greatest at this point.
(c) Point D has the greatest f value of the points A, B, C, D, E.



40. We want to minimize the function $h(x, y)$ subject to the constraint that

$$g(x, y) = x^2 + y^2 = 1{,}000^2 = 1{,}000{,}000.$$

Using the method of Lagrange multipliers, we obtain the following system of equations:

$$h_x = -\frac{10x + 4y}{10{,}000} = 2\lambda x,$$
$$h_y = -\frac{4x + 4y}{10{,}000} = 2\lambda y,$$
$$x^2 + y^2 = 1{,}000{,}000.$$

Multiplying the first equation by y and the second by x we get

$$\frac{-y(10x + 4y)}{10{,}000} = \frac{-x(4x + 4y)}{10{,}000}.$$

Hence:

$$2y^2 + 3xy - 2x^2 = (2y - x)(y + 2x) = 0,$$

and so the climber either moves along the line $x = 2y$ or $y = -2x$.

We must now choose one of these lines and the direction along that line which will lead to the point of minimum height on the circle. To do this we find the points of intersection of these lines with the circle $x^2 + y^2 = 1{,}000{,}000$, compute the corresponding heights, and then select the minimum point.

If $x = 2y$, the third equation gives

$$5y^2 = 1{,}000^2,$$

so that $y = \pm 1{,}000/\sqrt{5} \approx \pm 447.21$ and $x = \pm 894.43$. The corresponding height is $h(\pm 894.43, \pm 447.21) = 2400$ m. If $y = -2x$, we find that $x = \pm 447.21$ and $y = \mp 894.43$. The corresponding height is $h(\pm 447.21, \mp 894.43) = 2900$ m. Therefore, she should travel along the line $x = 2y$, in either of the two possible directions.

41. The objective function is

$$f(x, y, z) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2},$$

and the constraint is

$$g(x, y, z) = Ax + By + Cz + D = 0.$$

Partial derivatives of f and g are

$$\begin{aligned} f_x &= \frac{\frac{1}{2}\cdot 2\cdot (x-a)}{f(x,y,z)} = \frac{x-a}{f(x,y,z)}, \\ f_y &= \frac{\frac{1}{2}\cdot 2\cdot (y-b)}{f(x,y,z)} = \frac{y-b}{f(x,y,z)}, \\ f_z &= \frac{\frac{1}{2}\cdot 2\cdot (z-c)}{f(x,y,z)} = \frac{z-c}{f(x,y,z)}, \end{aligned}$$

$$g_x = A, \quad g_y = B, \quad \text{and} \quad g_z = C.$$

Using Lagrange multipliers, we need to solve the equations

$$\operatorname{grad} f = \lambda \operatorname{grad} g$$

where $\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$ and $\operatorname{grad} g = g_x\vec{i} + g_y\vec{j} + g_z\vec{k}$. This gives a system of equations:

$$\begin{aligned} &\frac{x-a}{f(x,y,z)} = \lambda A \\ &\frac{y-b}{f(x,y,z)} = \lambda B \\ &\frac{z-c}{f(x,y,z)} = \lambda C \\ &Ax + By + Cz + D = 0. \end{aligned}$$

Now $\frac{x-a}{A} = \frac{y-b}{B} = \frac{z-c}{C} = \lambda f(x,y,z)$ gives

$$\begin{aligned} x &= \frac{A}{B}(y-b) + a, \\ z &= \frac{C}{B}(y-b) + c, \end{aligned}$$

Substitute into the constraint,

$$\begin{gathered} A\left(\frac{A}{B}(y-b) + a\right) + By + C\left(\frac{C}{B}(y-b) + c\right) + D = 0, \\ \left(\frac{A^2}{B} + B + \frac{C^2}{B}\right) y = \frac{A^2}{B} b - Aa + \frac{C^2}{B} b - Cc - D. \end{gathered}$$

Hence

$$\begin{aligned} y &= \frac{(A^2 + C^2)b - B(Aa + Cc + D)}{A^2 + B^2 + C^2}, \\ y - b &= \frac{-B(Aa + Bb + Cc + D)}{A^2 + B^2 + C^2} \end{aligned}$$

$$\begin{aligned} x - a &= \frac{A}{B}(y-b) \\ &= \frac{-A(Aa+Bb+Cc+D)}{A^2+B^2+C^2} \\ z - c &= \frac{C}{B}(y-b) \\ &= \frac{-C(Aa+Bb+Cc+D)}{A^2+B^2+C^2} \end{aligned}$$

Thus the minimum $f(x,y,z)$ is

$$\begin{aligned} f(x,y,z) &= \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} \\ &= \Big[\left(\frac{-A(Aa+Bb+Cc+D)}{A^2+B^2+C^2}\right)^2 + \left(\frac{-B(Aa+Bb+Cc+D)}{A^2+B^2+C^2}\right)^2 \\ &\quad + \left(\frac{-C(Aa+Bb+Cc+D)}{A^2+B^2+C^2}\right)^2 \Big]^{1/2} \\ &= \frac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}. \end{aligned}$$

The geometric meaning is finding the shortest distance from a point (a,b,c) to the plane $Ax+By+Cz+D=0$.

42. We first express the revenue R in terms of the prices p_1 and p_2:

$$\begin{aligned} R(p_1,p_2) &= p_1q_1 + p_2q_2 \\ &= p_1(517 - 3.5p_1 + 0.8p_2) + p_2(770 - 4.4p_2 + 1.4p_1) \\ &= 517p_1 - 3.5p_1^2 + 770p_2 - 4.4p_2^2 + 2.2p_1p_2. \end{aligned}$$

At a local maximum we have grad $R = 0$, and so:

$$\begin{aligned} \frac{\partial R}{\partial p_1} &= 517 - 7p_1 + 2.2p_2 = 0, \\ \frac{\partial R}{\partial p_2} &= 770 - 8.8p_2 + 2.2p_1 = 0. \end{aligned}$$

Solving these equations, we find that

$$p_1 = 110 \quad \text{and} \quad p_2 = 115.$$

To see whether or not we have a found a local maximum, we compute the second-order partial derivatives:

$$\frac{\partial^2 R}{\partial p_1^2} = -7, \quad \frac{\partial^2 R}{\partial p_2^2} = -8.8, \quad \frac{\partial^2 R}{\partial p_1 \partial p_2} = 2.2.$$

Therefore,

$$D = \frac{\partial^2 R}{\partial p_1^2}\frac{\partial^2 R}{\partial p_2^2} - \frac{\partial^2 R}{\partial p_1 \partial p_2} = (-7)(-8.8) - (2.2)^2 = 56.76,$$

and so we have found a local maximum point. The graph of $P(p_1,p_2)$ has the shape of an upside down paraboloid. Since P is quadratic in q_1 and q_2, $(110, 115)$ is a global maximum point.

43. We want to minimize cost $C = 100L + 200K$ subject to $Q = 900L^{1/2}K^{2/3} = 36000$. Using Lagrange multipliers, we get

$$\begin{aligned} \nabla Q &= \left(450L^{-1/2}K^{2/3}\right)\vec{i} + \left(600L^{1/2}K^{-1/3}\right)\vec{j}\,. \\ \nabla C &= 100\vec{i} + 200\vec{j} \end{aligned}$$

$\nabla C = \lambda \nabla Q$ gives

$$100 = \lambda 450 L^{-1/2}K^{2/3} \quad \text{and} \quad 200 = \lambda 600 L^{1/2}K^{-1/3}.$$

Since $\lambda \neq 0$ this gives

$$450L^{-1/2}K^{2/3} = 300L^{1/2}K^{-1/3}.$$

Solving, we get $L = (3/2)K$. Substituting into $Q = 36{,}000$ gives

$$900\left(\frac{3}{2}K\right)^{1/2}K^{2/3} = 36{,}000.$$

Solving yields $K = \left[40 \cdot \left(\frac{2}{3}\right)^{1/2}\right]^{6/7} \approx 19.85$, so $L \approx \frac{3}{2}(19.85) = 29.78$. We can thus calculate cost using $K = 20$ and $L = 30$ which gives $C = \$7,000$.

44. We wish to minimize the objective function

$$C(x, y, z) = 20x + 10y + 5z$$

subject to the budget constraint

$$Q(x, y, z) = 20x^{1/2}y^{1/4}z^{2/5} = 1,200.$$

Therefore, we solve the equations $\operatorname{grad} C = \lambda \operatorname{grad} Q$ and $Q = 1,200$:

$$\begin{aligned} 20 &= 10\lambda x^{-1/2}y^{1/4}z^{2/5} \quad \text{or} \quad \lambda = 2x^{1/2}y^{-1/4}z^{-2/5}, \\ 10 &= 5\lambda x^{1/2}y^{-3/4}z^{2/5}, \quad \text{or} \quad \lambda = 2x^{-1/2}y^{3/4}z^{-2/5}, \\ 5 &= 8\lambda x^{1/2}y^{1/4}z^{-3/5}, \quad \text{or} \quad \lambda = 0.625x^{-1/2}y^{-1/4}z^{3/5}, \\ 20x^{1/2}y^{1/4}z^{2/5} &= 1,200. \end{aligned}$$

The first and second equations imply that

$$x = y,$$

while the second and third equations imply that

$$3.2y = z.$$

Substituting for x and z in the constraint equation gives

$$20y^{1/2}y^{1/4}(3.2y)^{2/5} = 1200$$

$$y \approx 23.47,$$

and so

$$x \approx 23.47 \quad \text{and} \quad z \approx 75.1.$$

45. Cost of production, C, is given by $C = p_1W + p_2K = b$. At the optimal point, $\nabla q = \lambda \nabla C$.
Since $\nabla q = \left(c(1-a)W^{-a}K^a\right)\vec{i} + \left(caW^{1-a}K^{a-1}\right)\vec{j}$ and $\nabla C = p_1\vec{i} + p_2\vec{j}$, we get

$$c(1-a)W^{-a}K^a = \lambda p_1 \text{ and } caW^{1-a}K^{a-1} = \lambda p_2.$$

Now, marginal productivity of labor is given by $\frac{\partial q}{\partial W} = c(1-a)W^{-a}K^a$ and marginal productivity of capital is given by $\frac{\partial q}{\partial K} = caW^{1-a}K^{a-1}$, so their ratio is given by

$$\frac{\frac{\partial q}{\partial W}}{\frac{\partial q}{\partial K}} = \frac{c(1-a)W^{-a}K^a}{caW^{1-a}K^{a-1}} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}$$

which is the ratio of the cost of one unit of labor to the cost of one unit of capital.

46. (a) The objective function is the energy loss, $i_1^2R_1 + i_2^2R_2$, and the constraint is $i_1 + i_2 = I$, where I is a constant. The Lagrangian function is

$$\mathcal{L}(i_1, i_2, \lambda) = i_1^2R_1 + i_2^2R_2 - \lambda(i_1 + i_2 - I).$$

We look for solutions to the system of equations we get from $\operatorname{grad} \mathcal{L} = \vec{0}$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial i_1} &= 2i_1R_1 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial i_2} &= 2i_2R_2 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -(i_1 + i_2 - I) = 0. \end{aligned}$$

Combining $\dfrac{\partial \mathcal{L}}{\partial i_1} - \dfrac{\partial \mathcal{L}}{\partial i_2} = 2(i_1R_1 - i_2R_2) = 0$ with $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0$ gives the two equation system

$$\begin{aligned} i_1R_1 - i_2R_2 &= 0 \\ i_1 + i_2 &= I. \end{aligned}$$

Substituting $i_2 = I - i_1$ into the first equation leads to

$$i_1 = \frac{R_2}{R_1 + R_2} I$$
$$i_2 = \frac{R_1}{R_1 + R_2} I.$$

(b) Ohm's Law states that across a resistor

$$\text{Voltage} = \text{Current} \cdot \text{Resistance}.$$

Since $\lambda/2 = i_1 \cdot R_1 = i_2 \cdot R_2$, the Lagrange multiplier λ equals twice the voltage across the resistors.

47. Let the sides of the base be x and y cm. Let the height be z cm. Then the volume is given by $xyz = 32$ and the surface area, S, is given by

$$S = xy + 2xz + 2yz.$$

Substituting $z = 32/(xy)$ gives

$$S = xy + \frac{64}{y} + \frac{64}{x}.$$

At a critical point

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0$$
$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0.$$

The symmetry of the equations tells us that $x = y$ and

$$x - \frac{64}{x^2} = 0$$
$$x^3 = 64$$
$$x = 4 \text{ cm}.$$

Thus the only critical point is $x = y = 4$ cm and $z = 32/(4 \cdot 4) = 2$ cm. At the critical point

$$D = S_{xx}S_{yy} - (S_{xy})^2 = \frac{128}{x^3} \cdot \frac{128}{y^3} - 1^2 = \frac{(128)^2}{x^3y^3} - 1.$$

Since $D > 0$ and $S_{xx} > 0$ at this critical point, the critical point $x = y = 4, z = 2$ is a local minimum. Since $S \to \infty$ as $x, y \to \infty$, the local minimum is a global minimum.

48. The point P is the solution to the constraint optimization problem of maximizing the square of the distance function.

$$D = x^2 + y^2 + x^2$$

subject to the constraint

$$g(x, y, z) = f(x, y) - z = 0.$$

(We take the square of the distance between the point (x, y, z) and the origin, which is

$$\text{Distance } = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2},$$

because it makes the calculations easier.) Therefore, at point P, we have $\nabla D = \lambda \nabla g$, so ∇D is parallel to ∇g.

We know that ∇g is perpendicular to the surface $g(x, y, z) = 0$; that is, perpendicular to the surface $z = f(x, y)$. Also

$$\nabla D = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

At point P, whose position vector is $\vec{p} = a\vec{i} + b\vec{j} + c\vec{k}$, we have

$$\nabla D = 2(a\vec{i} + b\vec{j} + c\vec{k}) = 2\vec{p}.$$

Thus, $\vec{p}$ is parallel to ∇D and therefore $\vec{p}$ is also perpendicular to the surface.

49. You should try to anticipate your opponent's choice. After you choose a value λ, your opponent will use calculus to find the point (x, y) that maximizes the function $f(x, y) = 10 - x^2 - y^2 - 2x - \lambda(2x + 2y)$. At that point, we have $f_x = -2x - 2 - 2\lambda = 0$ and $f_y = -2y - 2\lambda = 0$, so your opponent will choose $x = -1 - \lambda$ and $y = -\lambda$. This gives a value $\mathcal{L}(-1-\lambda, -\lambda, \lambda) = 10 - (-1-\lambda)^2 - (-\lambda)^2 - 2(-1-\lambda) - \lambda(2(-1-\lambda) + 2(-\lambda)) = 11 + 2\lambda + 2\lambda^2$ which you want to make as small as possible. You should choose λ to minimize the function $h(\lambda) = 11 + 2\lambda + 2\lambda^2$. You choose λ so that $h'(\lambda) = 2 + 4\lambda = 0$, or $\lambda = -1/2$. Your opponent then chooses $(x, y) = (-1 - \lambda, -\lambda) = (-1/2, 1/2)$, giving a final score of $\mathcal{L}(-1/2, 1/2, -1/2) = 10.5$. No choice of λ that you can make can force the value of $\mathcal{L}$ below 10.5. But your choice of $\lambda = -1/2$ makes it impossible for your opponent to force the value of $\mathcal{L}$ above 10.5.

50. The wetted perimeter of the trapezoid is given by the sum of the lengths of the three walls, so

$$p = w + \frac{2d}{\sin\theta}$$

We want to minimize p subject to the constraint that the area is fixed at 50 m^2. A trapezoid of height h and with parallel sides of lengths b_1 and b_2 has

$$A = \text{Area} = h\frac{(b_1 + b_2)}{2}.$$

In this case, d corresponds to h and b_1 corresponds to w. The b_2 term corresponds to the width of the exposed surface of the canal. We find that $b_2 = w + (2d)/(\tan\theta)$. Substituting into our original equation for the area along with the fact that the area is fixed at 50 m^2, we arrive at the formula:

$$\text{Area} = \frac{d}{2}\left(w + w + \frac{2d}{\tan\theta}\right) = d\left(w + \frac{d}{\tan\theta}\right) = 50$$

We now solve the constraint equation for one of the variables; we will choose w to give

$$w = \frac{50}{d} - \frac{d}{\tan\theta}.$$

Substituting into the expression for p gives

$$p = w + \frac{2d}{\sin\theta} = \frac{50}{d} - \frac{d}{\tan\theta} + \frac{2d}{\sin\theta}.$$

We now take partial derivatives:

$$\frac{\partial p}{\partial d} = -\frac{50}{d^2} - \frac{1}{\tan\theta} + \frac{2}{\sin\theta}$$
$$\frac{\partial p}{\partial \theta} = \frac{d}{\tan^2\theta} \cdot \frac{1}{\cos^2\theta} - \frac{2d}{\sin^2\theta} \cdot \cos\theta$$

From $\partial p/\partial\theta = 0$, we get

$$\frac{d \cdot \cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos^2\theta} = \frac{2d}{\sin^2\theta} \cdot \cos\theta.$$

Since $\sin\theta \neq 0$ and $\cos\theta \neq 0$, canceling gives

$$1 = 2\cos\theta$$

so

$$\cos\theta = \frac{1}{2}.$$

$$\text{Since} \quad 0 < \theta < \frac{\pi}{2}, \quad \text{we get} \quad \theta = \frac{\pi}{3}.$$

Substituting into the equation $\partial p/\partial d = 0$ and solving for d gives:

$$\frac{-50}{d^2} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}/2} = 0$$

which leads to

$$d = \sqrt{\frac{50}{\sqrt{3}}} \approx 5.37\text{m}.$$

Then

$$w = \frac{50}{d} - \frac{d}{\tan\theta} \approx \frac{50}{5.37} - \frac{5.37}{\sqrt{3}} \approx 6.21 \text{ m}.$$

When $\theta = \pi/3$, $w \approx 6.21$ m and $d \approx 5.37$ m, we have $p \approx 18.61$ m.

Since there is only one critical point, and since p increases without limit as d or θ shrink to zero, the critical point must give the global minimum for p.

CAS Challenge Problems

51. (a) The partial derivatives of f are

$$\frac{\partial f}{\partial x} = \frac{\sqrt{a+x} + \left(-1+\sqrt{a+x}\right)\, y}{2\sqrt{a+x}\,\sqrt{a+x+y}\left(1+\sqrt{a+x}+y\right)^2}$$
$$\frac{\partial f}{\partial y} = \frac{1-2\,a-2\,x+\sqrt{a+x}-y}{2\sqrt{a+x+y}\left(1+\sqrt{a+x}+y\right)^2}$$

Solving $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, we get $x = 1/4 - a, y = 1$. The discriminant at this point is $D = -16/625$. Thus, by the second derivative test, the point is a saddle point.

(b) The y coordinate of the critical points stays the same and the x coordinate is a units to the left of its position when $a = 0$. The type is always a saddle point. This is because f is obtained from $\dfrac{\sqrt{x+y}}{1+y+\sqrt{x}}$ by substituting $x + a$ for x, so that the graph is shifted a units in the negative x-direction but its shape remains the same.

52. (a) We have $\operatorname{grad} f = 2x\vec{i} + \vec{j}$ and $\operatorname{grad} g = (2x + 2y)\vec{i} + (2x + 2y)\vec{j}$. So the equations to be solved in the method of Lagrange multipliers are

$$2x = \lambda(2x + 2y)$$
$$1 = \lambda(2x + 2y)$$
$$x^2 + 2xy + y^2 - 9 = 0$$

Solving these with a CAS, we get two solutions:

$$x = 1/2, y = -7/2, \lambda = -1/6, \quad \text{or} \quad x = 1/2, y = 5/2, \lambda = 1/6$$

Student A reasons that since $f(1/2, -7/2) = -13/4$ and $f(1/2, 5/2) = 11/4$, the (global) maximum and minimum values are 11/4 and 13/4, respectively. Student B graphs the constraint curve $g = 0$ and a contour diagram of f. The constraint curve turns out to be two straight lines, since the constraint $x^2 + 2xy + y^2 - 9 = 0$, which can be rewritten as $(x + y)^2 = 9$, or $x + y = \pm 3$. The value of f goes to infinity on each of these straight lines. On the line $y = -x + 3$, $f(x, y) = x^2 + y = x^2 - x + 3$, and on the line $y = -x - 3$, $f(x, y) = x^2 + y = x^2 - x - 3$. Thus Student B is correct. The points Student A found are actually local maximum and local minimum values, not global. Since the constraint is not bounded, there is no guarantee that there is a local maximum or minimum. See Figure 15.45.

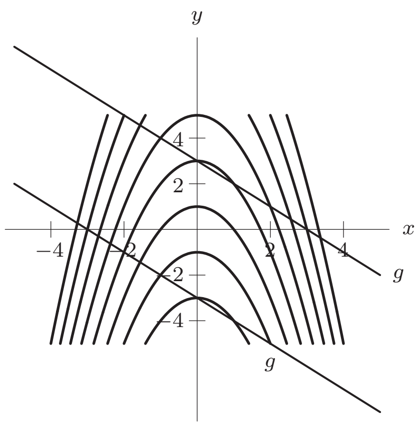


Figure 15.45: Contours of f and two straight lines giving constraint $g = 0$

53. (a) We have $\operatorname{grad} f = 3\vec{i} + 2\vec{j}$ and $\operatorname{grad} g = (4x - 4y)\vec{i} + (-4x + 10y)\vec{j}$, so the Lagrange multiplier equations are

$$3 = \lambda(4x - 4y)$$
$$2 = \lambda(-4x + 10y)$$
$$2x^2 - 4xy + 5y^2 = 20$$

Solving these with a CAS we get $\lambda = -0.4005, x = -3.9532, y = -2.0806$ and $\lambda = 0.4005, x = 3.9532, y = 2.0806$. We have $f(-3.9532, -2.0806) = -11.0208$, and $f(3, 9532, 2.0806) = 21.0208$. The constraint equation is $2x^2 - 4xy + 5y^2 = 20$, or, completing the square, $2(x-y)^2 + 3y^2 = 20$. This has the shape of a skewed ellipse, so the constraint curve is bounded, and therefore the local maximum is a global maximum. Thus the maximum value is 21.0208.

(b) The maximum value on $g = 20.5$ is $\approx 21.0208 + 0.5(0.4005) = 21.2211$. The maximum value on $g = 20.2$ is $\approx 21.0208 + 0.2(0.4005) = 21.1008$.

(c) We use the same commands in the CAS from part (a), with 20 replaced by 20.5 and 20.2, and get the maximum values 21.2198 for $g = 20.5$ and 21.1007 for $g = 20.2$. These agree with the approximations we found in part (b) to 2 decimal places.

PROJECTS FOR CHAPTER FIFTEEN

1. (a) The price of p/q units of B at unit price q is $p/q \cdot q = p$, the same as the price of one unit of A. On a fixed budget, p/q units of B can substitute for one unit of A. Thus the *ERS* is p/q.

(b) The *TRS* measures the rate at which y increases with respect to x as a point (x, y) slides in the direction of decreasing x along a fixed contour $f(x, y) = Q$ of the production function. Thus *TRS* $= -dy/dx$, the negative of the slope of the contour. On the contour the differential $df = f_x dx + f_y dy$ is zero because f has constant value there. Thus $f_x dx + f_y dy = 0$ which gives *TRS* $= -dy/dx = f_x/f_y$.

(c) We want to maximize the production function $f(x, y)$ subject to a budget constraint $px + qy = C$, where C is the fixed budget. At a maximum we have $\operatorname{grad} f = \lambda \operatorname{grad} g$ or

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y.$$

Dividing the first equation by the second gives $f_x/f_y = g_x/g_y$ or *TRS* = *ERS*.

(d) We are asked to minimize the cost function $g(x, y)$ subject to a production constraint $f(x, y) = Q$, where Q is the fixed quantity to be produced. At a minimum we have $\operatorname{grad} g = \lambda \operatorname{grad} f$ or

$$g_x = \lambda f_x$$
$$g_y = \lambda f_y.$$

Dividing the first equation by the second gives $g_x/g_y = f_x/f_y$ or *ERS* = *TRS*.

2.

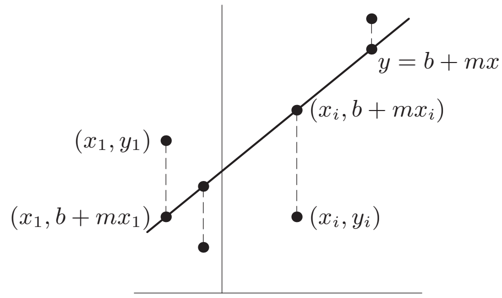


Figure 15.46

(a) Points which are directly above or below each other share the same x coordinate, therefore, the point on the least squares line which is directly above or below the point in question will have x coordinate x_i and from the formula for the least squares line, it will have y coordinate $b + mx_i$. (See Figure 14.1.)

(b) The general distance formula in two dimensions is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, so $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Since the x coordinates are identical for the two points in question, the first term in the square root is zero. This yields $d^2 = (y_i - (b + mx_i))^2$.

(c) In both cases we use the chain rule and our knowledge of summations to show the relationship.

$$\frac{\partial f}{\partial b} = \frac{\partial}{\partial b}\left(\sum_{i=1}^{n}(y_i - (b + mx_i))^2\right) = \sum_{i=1}^{n}\frac{\partial}{\partial b}(y_i - (b + mx_i))^2$$

$$= \sum_{i=1}^{n} 2(y_i - (b + mx_i)) \cdot \frac{\partial}{\partial b}(y_i - (b + mx_i))$$
$$= \sum_{i=1}^{n} 2(y_i - (b + mx_i)) \cdot (-1)$$
$$= -2 \sum_{i=1}^{n} (y_i - (b + mx_i))$$
$$\frac{\partial f}{\partial m} = \frac{\partial}{\partial m}\left(\sum_{i=1}^{n} (y_i - (b + mx_i))^2\right) = \sum_{i=1}^{n} \frac{\partial}{\partial m}(y_i - (b + mx_i))^2$$
$$= \sum_{i=1}^{n} 2(y_i - (b + mx_i)) \cdot \frac{\partial}{\partial m}(y_i - (b + mx_i))$$
$$= \sum_{i=1}^{n} 2(y_i - (b + mx_i)) \cdot (-x_i)$$
$$= -2 \sum_{i=1}^{n} (y_i - (b + mx_i)) \cdot x_i$$

(d) We can separate $\frac{\partial f}{\partial b}$ into three sums as shown:

$$\frac{\partial f}{\partial b} = -2\left(\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} 1 - m\sum_{i=1}^{n} x_i\right)$$

Similarly we can separate $\frac{\partial f}{\partial m}$ after multiplying through by x_i:

$$\frac{\partial f}{\partial m} = -2\left(\sum_{i=1}^{n} y_i x_i - b\sum_{i=1}^{n} x_i - m\sum_{i=1}^{n} {x_i}^2\right)$$

Setting $\frac{\partial f}{\partial b}$ and $\frac{\partial f}{\partial m}$ equal to zero we have:

$$bn + m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$
$$b\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

(e) To solve this pair of linear equations, we multiply the first equation by $\sum_{i=1}^{n} x_i^2$, multiply the second one by $\sum_{i=1}^{n} x_i$, and subtract; we get

$$bn\sum_{i=1}^{n} x_i^2 - b(\sum_{i=1}^{n} x_i)^2 = \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i,$$

So,

$$b = \left(\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i\right) \Big/ \left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right)$$

Similarly,

$$m = \left(n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i\right) \Big/ \left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right)$$

(f) Applying the formulas to the given data, we have $b = -\frac{1}{3}$, $m = 1$ which gives $y = -(1/3) + x$, in agreement with the example.

3. (a) If $p = e^{-x}$ where $x \to \infty$ then $p \to 0$ with $p > 0$ and

$$\lim_{\substack{p\to 0\\ p>0}} (p \ln p) = \lim_{x\to\infty} (-xe^{-x}) = 0,$$

since the exponential decreases faster than any power of x. Alternatively, use l'Hopital's rule:

$$\lim_{\substack{p\to 0\\ p>0}} (p \ln p) = \lim_{\substack{p\to 0\\ p>0}} \frac{\ln p}{1/p} = \lim_{\substack{p\to 0\\ p>0}} \frac{1/p}{-1/p^2} = 0$$

(b) We apply the method of Lagrange multipliers to find the critical points of $S(p_1, \cdots, p_{30})$. The constraint function is $g(p_1, \cdots, p_{30}) = p_1 + \cdots + p_{30}$. We have

$$\frac{\partial S}{\partial p_j} = \frac{\partial}{\partial p_j}\left(-\sum_{i=1}^{30} p_i \frac{\ln p_i}{\ln 2}\right) = -\frac{1}{\ln 2}(\ln p_j + 1),$$

therefore

$$\operatorname{grad} S = -\frac{1}{\ln 2}\sum_{j=1}^{30}(\ln p_j + 1)\vec{k_j}$$

where $\vec{k_1}, \cdots, \vec{k_{30}}$ are the unit vectors corresponding 30 independent directions of the p_j-axes. Also,

$$\operatorname{grad} g = \sum_{j=1}^{30} \vec{k_j}$$

so the condition $\operatorname{grad} S = \lambda \operatorname{grad} g$ becomes

$$-\frac{1}{\ln 2}(\ln p_j + 1) = \lambda, \qquad \text{for} \quad i = 1, \cdots, 30.$$

Thus,

$$\ln p_j = -\lambda \ln 2 - 1$$

and, in particular, all the $p_j s$ must be equal. Since the $p_j s$ have to satisfy the constraint $g(p_1, \cdots, p_{30}) = 1$, we see that $p_j = \frac{1}{30}$ and that the point $(\frac{1}{30}, \frac{1}{30}, \cdots, \frac{1}{30})$ is the only critical point of S. We have

$$S\left(\frac{1}{30}, \frac{1}{30}, \cdots, \frac{1}{30}\right) = -30 \cdot \frac{1}{30}\frac{(-\ln 30)}{\ln 2} = \frac{\ln 30}{\ln 2}.$$

We will not prove that this is indeed the maximum value of S (this requires a higher-dimensional analogue of the second derivative test). Since in part (c) we show that the minimum value of S is 0, the critical point we have found here is not a global minimum; the maximum of S has to be attained somewhere and it is reasonable to believe that it is attained at the unique critical point. The maximum entropy corresponds to maximum uncertainty in the outcome of the competition.

(c) We already know that $S \geq 0$. However, S can be zero: For example, if $p_1 = 1$ and $p_2 = \cdots = p_{30} = 0$, we have $S(1, 0, \cdots, 0) = 0$. Therefore the minimum value of S is 0. Now we determine all the values of $p_i s$ for which $S(p_1, \cdots, p_{30}) = 0$. The condition

$$S = -\sum_{i=1}^{30} \frac{p_i \ln p_i}{2} = 0$$

together with the restrictions $-\ln p_i \geq 0$ shows that, for S to vanish, each individual term in the above sum has to vanish. This means $p_i \ln p_i = 0$ for all $i = 1, \cdots, 30$, that is, $p_i = 0$ or $p_i = 1$ for $i = 1, \cdots, 30$. Since $\sum_{i=1}^{30} p_i = 1$, only one of the $p_i s$ is 1 whereas the other 29 are 0. This corresponds to the case where one of the teams is certain to win, that is, there is no uncertainty. The result can be interpreted by saying that zero entropy implies zero uncertainty.

CHAPTER SIXTEEN

Solutions for Section 16.1

Exercises

1. Using $\Delta x = 3$ and $\Delta y = 0.5$:

$$\text{Lower estimate } = (4+5+3+4)\Delta x\,\Delta y = 16\cdot 3\cdot 0.5 = 24,$$

$$\text{Upper estimate } = (7+10+5+7)\Delta x\,\Delta y = 29\cdot 3\cdot 0.5 = 43.5.$$

2. Mark the values of the function on the plane, as shown in Figure 16.1, so that you can guess respectively at the smallest and largest values the function takes on each small rectangle.

$$\begin{aligned}\text{Lower sum} &= \sum f(x_i,y_i)\Delta x\Delta y\\ &= 4\Delta x\Delta y + 6\Delta x\Delta y + 3\Delta x\Delta y + 4\Delta x\Delta y\\ &= 17\Delta x\Delta y\\ &= 17(0.1)(0.2) = 0.34.\end{aligned}$$

$$\begin{aligned}\text{Upper sum} &= \sum f(x_i,y_i)\Delta x\Delta y\\ &= 7\Delta x\Delta y + 10\Delta x\Delta y + 6\Delta x\Delta y + 8\Delta x\Delta y\\ &= 31\Delta x\Delta y\\ &= 31(0.1)(0.2) = 0.62.\end{aligned}$$

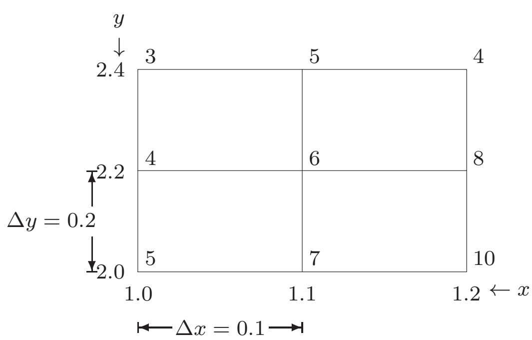


Figure 16.1

3. There are nine squares. Using the largest value of g in each square for the overestimate and the smallest value for the underestimate, we have

$$\text{Overestimate } \approx (2+2.8+3.5+2.8+4+4.9+3.4+4.9+6)\Delta x\Delta y = 34.3\cdot 2\cdot 2 = 137.2.$$

$$\text{Underestimate } \approx (0+0.8+0.9+0.8+2+2.8+0.9+2.8+4)\Delta x\Delta y = 15\cdot 2\cdot 2 = 60.$$

4. In the subrectangle in the top left of the figure given, it appears that $f(x, y)$ has a maximum value of about 9. In the subrectangle in the top middle, $f(x, y)$ has a maximum value of 10. Continuing in this way, and multiplying by Δx and Δy, we have

$$\text{Overestimate} = (9 + 10 + 12 + 7 + 8 + 10 + 5 + 7 + 8) \cdot 10 \cdot 5 = 3800.$$

Similarly, we find

$$\text{Underestimate} = (7 + 7 + 8 + 4 + 5 + 7 + 1 + 3 + 6) \cdot 10 \cdot 5 = 2400.$$

Thus, we expect that

$$2400 \le \int_R f(x, y) dA \le 3800.$$

5. Let R be the rectangle, and let $f(x, y)$ be the population density at the point (x, y) in the rectangle. The population is given by $\int_R f(x, y)\, dA$. We approximate the integral with a 6 term Riemann sum using six 1 km $\times$ 1 km squares in the region. To make a Riemann sum, choose a population density value at one point of each of the squares. For accuracy, we choose the midpoint of each square, but other choices are possible.

The densities at the midpoints can only be estimated from the contour diagram, and different people may make different judgments. The value at the point $(2.5, 1.5)$ is particularly difficult, because the diagram tells us only that the density is between 0 and 200. The table gives one reasonable set of values.

y \ x	0.5	1.5	2.5
1.5	500	450	100
0.5	650	200	400

Each square has area 1 km^2. We have

$$\int_R f(x, y)\, dA \approx 500 \cdot 1 + 450 \cdot 1 + 100 \cdot 1 + 650 \cdot 1 + 200 \cdot 1 + 400 \cdot 1 = 2300 \text{ people.}$$

The population is approximately 2300 people.

Problems

6. The function being integrated is $f(x, y) = 1$, which is positive everywhere. Thus, its integral over any region is positive.

7. The function being integrated is $f(x, y) = 5x$. Since $x > 0$ in R, f is positive in R and thus the integral is positive.

8. The function being integrated is $f(x, y) = 5x$, which is an odd function in x. Since B is symmetric with respect to x, the contributions to the integral cancel out, as $f(x, y) = -f(-x, y)$. Thus, the integral is zero.

9. The function being integrated, $f(x, y) = y^3 + y^5$, is an odd function in y while D is symmetric with respect to y. Then, by symmetry, the positive and negative contributions of f will cancel out and thus its integral is zero.

10. In a region such as B in which $y < 0$, the quantity $y^3 + y^5$ is less than zero. Thus, its integral is negative.

11. The function being integrated, $f(x, y) = y - y^3$, is an odd function in y while D is symmetric with respect to y. By symmetry, the integral is zero.

12. The function being integrated, $f(x, y) = y - y^3$ is always negative in the region B since in that region $-1 < y < 0$ and $|y^3| < |y|$. Thus, the integral is negative.

13. The total area of the square R is $(1.5)(1.5) = 2.25$. See Figure 16.2. On a disk of radius ≈ 0.5 the function has a value of 3 or more, giving a total contribution to the integral of at least $(3)\cdot(\pi \cdot 0.5^2) \approx 2.3$. On less than half of the rest of the square the function has a value between -2 and 0, giving a contribution to the integral of between $(1/2 \cdot 2.25)(-2) = -2.25$ and 0. Since the positive contribution to the integral is therefore greater in magnitude than the negative contribution, $\int_R f\, dA$ is positive.

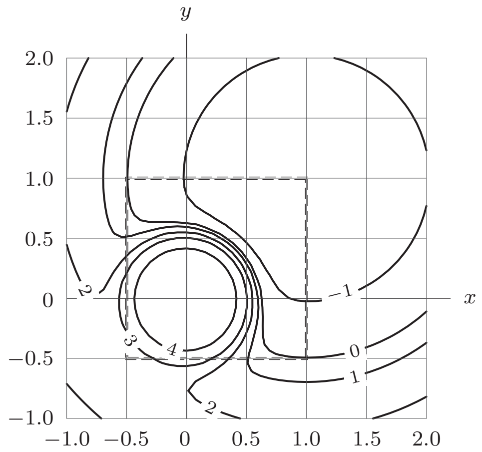


Figure 16.2

14. We use four subrectangles to find an overestimate and underestimate of the integral:

$$\text{Overestimate} = (15 + 9 + 9 + 5)(4)(3) = 456,$$

$$\text{Underestimate} = (5 + 2 + 3 + 1)(4)(3) = 132.$$

A better estimate of the integral is the average of the two:

$$\int_R f(x, y)dA \approx \frac{456 + 132}{2} = 294.$$

The units of the integral are milligrams, and the integral represents the total number of mg of mosquito larvae in this 8 meter by 6 meter section of swamp.

15. Let's break up the room into 25 sections, each of which is 1 meter by 1 meter and has area $\Delta A = 1$.

We shall begin our sum as an upper estimate starting with the lower left corner of the room and continue across the bottom and moving upward using the highest temperature, T_i, in each case. So the upper Riemann sum becomes

$$\begin{aligned}\textstyle\sum_{i=1}^{25} T_i\Delta A &= T_1\Delta A + T_2\Delta A + T_3\Delta A + \cdots + T_{25}\Delta A\\ &= \Delta A(T_1 + T_2 + T_3 + \cdots + T_{25})\\ &= (1)(\;31 + 29 + 28 + 27 + 27 +\\ &\qquad 29 + 28 + 27 + 27 + 26 +\\ &\qquad 27 + 27 + 26 + 26 + 26 +\\ &\qquad 26 + 26 + 25 + 25 + 25 +\\ &\qquad 25 + 24 + 24 + 24 + 24)\\ &= (1)(659) = 659.\end{aligned}$$

In the same way, the lower Riemann sum is formed by taking the lowest temperature, t_i, in each case:

$$\begin{aligned}\textstyle\sum_{i=1}^{25} t_i\Delta A &= t_1\Delta A + t_2\Delta A + t_3\Delta A + \cdots + t_{25}\Delta A\\ &= \Delta A(t_1 + t_2 + t_3 + \cdots + t_{25})\\ &= (1)(\;27 + 27 + 26 + 26 + 25 +\\ &\qquad 26 + 26 + 25 + 25 + 25 +\\ &\qquad 25 + 24 + 24 + 24 + 24 +\\ &\qquad 24 + 23 + 23 + 23 + 23 +\\ &\qquad 23 + 21 + 20 + 21 + 22)\\ &= (1)(602) = 602.\end{aligned}$$

So, averaging the upper and lower sums we get: 630.5.

To compute the average temperature, we divide by the area of the room, giving

$$\text{Average temperature} = \frac{630.5}{(5)(5)} \approx 25.2^\circ\text{C}.$$

Alternatively we can use the temperature at the central point of each section ΔA. Then the sum becomes

$$\begin{aligned}\sum_{i=1}^{25} T_i'\Delta A &= \Delta A \sum_{i=1}^{25} T_i' \\ &= (1)(\ 29 + 28 + 27 + 26.5 + 26 + \\ &\quad\ 27 + 27 + 26 + 26 + 25.5 + \\ &\quad\ 26 + 25.5 + 25 + 25 + 25 + \\ &\quad\ 25 + 24 + 24 + 24 + 24 + \\ &\quad\ 24 + 23 + 22 + 22.5 + 23) \\ &= (1)(630) = 630.\end{aligned}$$

Then we get

$$\text{Average temperature} = \frac{\sum_{i=1}^{25} T_i'\Delta A}{\text{Area}} = \frac{630}{(5)(5)} \approx 25.2^\circ\text{C}.$$

16. We divide the base region into four subrectangles as shown in Figure 16.3. The height of the object at each point (x, y) is given by $f(x, y) = x + y$, we label each corner of the subrectangles with the value of the function at that point. (See Figure 16.3.) Since Volume = Height × Length × Width, and $\Delta x = 2$ and $\Delta y = 3$, we have

$$\text{Overestimate} = (8 + 10 + 5 + 7)(2)(3) = 360,$$

and

$$\text{Underestimate} = (3 + 5 + 0 + 2)(2)(3) = 60.$$

We average these to obtain

$$\text{Volume} \approx \frac{360 + 60}{2} = 210.$$

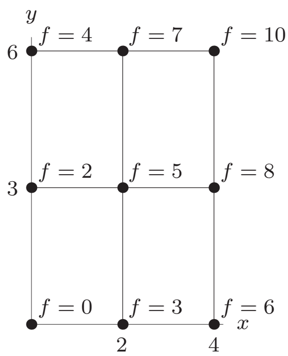


Figure 16.3

Strengthen Your Understanding

17. This is true only if f is nonnegative everywhere. Integrals can be negative, but volumes never can be negative.

18. The sign of an integral depends on the values of f on the region R. If f is positive everywhere on the region R, then $\int_R f(x, y)\, dA$ is positive.

19. We want the value of f at the lower left-hand corner of each subrectangle to be the largest value in the subrectangle. For example, let $f(x, y)$ be a linear function with negative slopes, say $f(x, y) = 5 - x - y$. Let R be the square with vertices $(\pm 1, \pm 1)$.

20. A function whose values are negative everywhere has a negative average value on any region. For example, $f(x,y) = -1$ has average value -1 on the square.

21. False. For example, if $f(x,y) < 0$ for all (x,y) in the region R, then $\int_R f\,dA$ is negative.

22. True. The double integral is the limit of the sum

$$\sum f(x,y)\Delta A = \sum k\Delta A = k\sum \Delta A$$

over rectangles that lie inside the region R. As the area $\Delta A \to 0$, this sum approaches $k \cdot \text{Area}(R)$.

23. False. The function $f(x,y) = e^{xy}$ is largest at the $(1,1)$ corner of R, so for any (x,y) in R we have $e^{xy} \le e^{1\cdot 1} = e$. Then

$$\int_R e^{xy}\,dA = \lim_{\Delta A\to 0}\sum e^{xy}\Delta A \le \lim_{\Delta A\to 0}\sum e\Delta A = e\lim_{\Delta A\to 0}\sum \Delta A = e\cdot \text{Area}(R) = e.$$

So $\int_R e^{xy}\,dA \le e \approx 2.7$.

24. False. For example, if $f = 1$, then $\int_R 1\,dA = \text{Area}(R) = 6$ and $\int_S 1\,dA = \text{Area}(S) = 6$.

25. True. The double integral is the limit of the sum $\sum_{\Delta A\to 0}\rho(x,y)\Delta A$. Each of the terms $\rho(x,y)\Delta A$ is an approximation of the total population inside a small rectangle of area ΔA. Thus the limit of the sum of all of these numbers as $\Delta A \to \infty$ gives the total population of the region R.

26. False. If the graph of f has equal volumes above and below the xy-plane over the region R, the double integral is zero without having $f(x,y) = 0$ everywhere.

27. True. Writing the definition of the integral of g, we have

$$\int_R g\,dA = \lim_{\Delta A\to 0}\sum g(x,y)\Delta A = \lim_{\Delta A\to 0}\sum kf(x,y)\Delta A = k\lim_{\Delta A\to 0}\sum f(x,y)\Delta A = k\int_R f\,dA.$$

28. False. As a counterexample, let R be a rectangle with area 2 and take $f(x,y) = g(x,y) = 1$. Then $\int_R f\cdot g\,dA = \int_R 1\,dA = \text{Area}(R) = 2$, but $\int_R f\,dA = \int_R g\,dA = \text{Area}(R)\cdot\text{Area}(R) = 4$.

29. False. There is no reason to expect this to be true, since the behavior of f on one half of R can be completely unrelated to the behavior of f on the other half. As a counterexample, suppose that f is defined so that $f(x,y) = 0$ for points (x,y) lying in S, and $f(x,y) = 1$ for points (x,y) lying in the part of R that is not in S. Then $\int_S f\,dA = 0$, since $f = 0$ on all of S. To evaluate $\int_R f\,dA$, note that $f = 1$ on the square S_1 which is $0 \le x \le 1, 1 \le y \le 2$. Then $\int_R f\,dA = \int_{S_1} f\,dA = \text{Area}(S_1) = 1$, since $f = 0$ on S.

30. True. Since all points in the region R satisfy $x < y$, it is true that at every point in R, $f(x,y) = x + x < x + y = g(x,y)$. Since all of the values of f in R are less than those of g, the average of the values of f is less than the average of the values of g.

Solutions for Section 16.2

Exercises

1. See Figure 16.4.

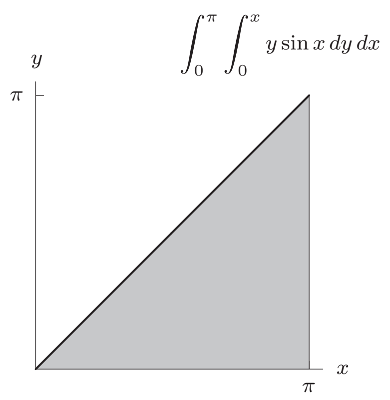


Figure 16.4

2. See Figure 16.5.

3. See Figure 16.6.

4. See Figure 16.7.

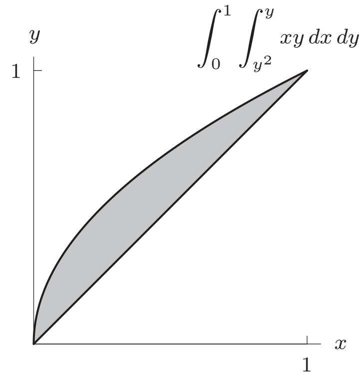


Figure 16.5

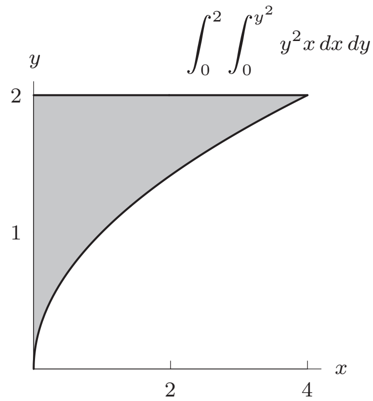


Figure 16.6

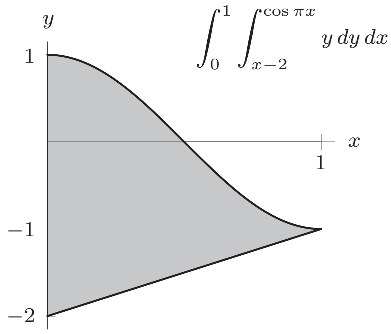


Figure 16.7

5. We evaluate the inside integral first:

$$\int_0^4 (4x+3y)\,dx = (2x^2+3yx)\Big|_0^4 = 32+12y.$$

Therefore, we have

$$\int_0^3 \int_0^4 (4x+3y)\,dxdy = \int_0^3 (32+12y)\,dy = (32y+6y^2)\Big|_0^3 = 150.$$

6. We evaluate the inside integral first:

$$\int_0^3 (x^2+y^2)\,dy = \left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=3} = 3x^2+9.$$

Therefore, we have

$$\int_0^2 \int_0^3 (x^2+y^2)\,dydx = \int_0^2 (3x^2+9)\,dx = (x^3+9x)\Big|_0^2 = 26.$$

7. We evaluate the inside integral first:

$$\int_0^2 (6xy)\,dy = (3xy^2)\Big|_0^2 = 12x.$$

Therefore, we have

$$\int_0^3 \int_0^2 (6xy)\,dydx = \int_0^3 (12x)\,dx = (6x^2)\Big|_0^3 = 54.$$

8. We evaluate the inside integral first:

$$\int_0^2 (x^2y)\,dy = \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=0}^{y=2} = 2x^2.$$

Therefore, we have

$$\int_0^1 \int_0^2 (x^2y)\,dydx = \int_0^1 (2x^2)\,dx = \left(\frac{2x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3}.$$

9. Calculating the inner integral first, we have

$$\int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left(e^{xy}\Big|_0^1 \right) dy = \int_0^1 \left(e^y - e^0\right)\,dy = \int_0^1 (e^y - 1)\,dy = (e^y - y)\Big|_0^1 = e^1 - 1 - (e^0 - 0) = e - 2.$$

10. Calculating the inner integral first, we have

$$\int_0^2 \int_0^y y\,dx\,dy = \int_0^2 yx\Big|_0^y dy = \int_0^2 y^2\,dy = \frac{y^3}{3}\Big|_0^2 = \frac{8}{3}.$$

11. Calculating the inner integral first, we have

$$\int_0^3 \int_0^y \sin x\,dx\,dy = \int_0^3 \left(-\cos x\Big|_0^y \right) dy = \int_0^3 (-\cos y + 1)\;dy = (-\sin y + y)\Big|_0^3 = -\sin 3 + 3.$$

12. Calculating from the inside and using integration by parts, we have

$$\int_0^{\pi/2} \int_0^{\sin x} x\,dy\,dx = \int_0^{\pi/2} xy\Big|_0^{\sin x} dx = \int_0^{\pi/2} x \sin x\,dx = (-x\cos x + \sin x)\Big|_0^{\pi/2} = 1.$$

13. $\displaystyle\int_1^3 \int_0^4 e^{x+y}\,dxdy = \int_1^3 e^x e^y\Big|_0^4 dx = \int_1^3 e^x(e^4 - 1)\,dx = (e^4 - 1)(e^2 - 1)e$. See Figure 16.8.

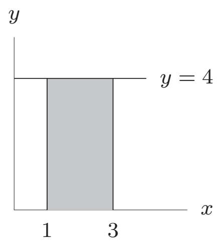


Figure 16.8

14. $\displaystyle\int_0^2 \int_0^x e^{x^2}\,dydx = \int_0^2 e^{x^2} y\Big|_0^x dx = \int_0^2 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_0^2 = \frac{1}{2}(e^4 - 1)$. See Figure 16.9.

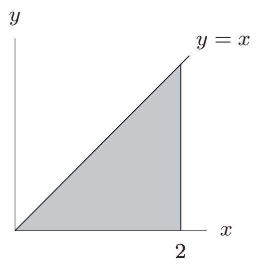


Figure 16.9

15.

$$\begin{aligned}\int_1^5 \int_x^{2x} \sin x\,dy\,dx &= \int_1^5 \sin x \cdot y\Big|_x^{2x} dx \\ &= \int_1^5 \sin x \cdot x\,dx \\ &= (\sin x - x\cos x)\Big|_1^5 \\ &= (\sin 5 - 5\cos 5) - (\sin 1 - \cos 1) \approx -2.68.\end{aligned}$$

See Figure 16.10.

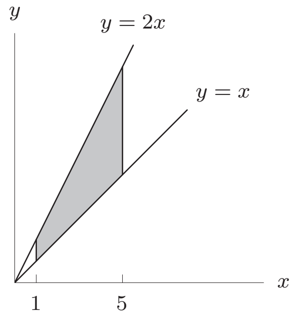


Figure 16.10

16.

$$\int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 \, dx \, dy = \int_1^4 y^3 \frac{x^3}{3}\bigg|_{\sqrt{y}}^{y} dy$$
$$= \frac{1}{3}\int_1^4 (y^6 - y^{\frac{9}{2}})\, dy$$
$$= \frac{1}{3}\left(\frac{y^7}{7} - \frac{y^{11/2}}{11/2}\right)\bigg|_1^4$$
$$= \frac{1}{3}\left[\left(\frac{4^7}{7} - \frac{4^{11/2}\times 2}{11}\right) - \left(\frac{1}{7} - \frac{2}{11}\right)\right] \approx 656.082$$

See Figure 16.11.

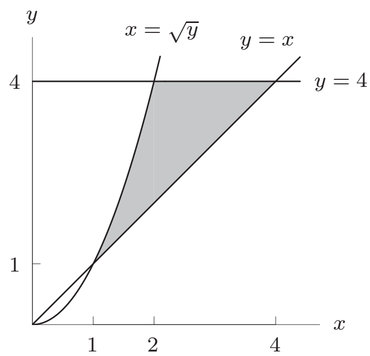


Figure 16.11

17. $\int_1^4 \int_1^2 f \, dy \, dx$ or $\int_1^2 \int_1^4 f \, dx \, dy$

18. This region lies between $x = 0$ and $x = 4$ and between the lines $y = 3x$ and $y = 12$, and so the iterated integral is

$$\int_0^4 \int_{3x}^{12} f(x, y)\, dy dx.$$

Alternatively, we could have set up the integral as follows:

$$\int_0^{12} \int_0^{y/3} f(x, y)\, dx dy.$$

19. The line connecting $(-1,1)$ and $(3,-2)$ is

$$3x+4y=1$$

or

$$y=\frac{1-3x}{4}$$

So the integral becomes

$$\int_{-1}^{3}\int_{-2}^{(1-3x)/4} f\,dy\,dx \quad \text{or} \quad \int_{-2}^{1}\int_{-1}^{(1-4y)/3} f\,dx\,dy$$

20. The line on the left (through points $(0,0)$ and $(3,6)$) is the line $y=2x$; the line on the right (through points $(3,6)$ and $(5,0)$) is the line $y=-3x+15$. See Figure 16.12. One way to set up this iterated integral is:

$$\int_{0}^{6}\int_{y/2}^{(15-y)/3} f(x,y)\,dxdy.$$

The other option for setting up this integral requires two separate integrals, as follows:

$$\int_{0}^{3}\int_{0}^{2x} f(x,y)\,dydx+\int_{3}^{5}\int_{0}^{-3x+15} f(x,y)\,dydx.$$

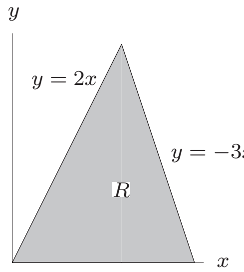


Figure 16.12

21. Two of the sides of the triangle have equations $x=\dfrac{y-1}{2}$ and $x=\dfrac{y-5}{-2}$. So the integral is

$$\int_{1}^{3}\int_{\frac{1}{2}(y-1)}^{-\frac{1}{2}(y-5)} f\,dx\,dy$$

22. The line connecting $(1,0)$ and $(4,1)$ is

$$y=\frac{1}{3}(x-1)$$

So the integral is

$$\int_{1}^{4}\int_{(x-1)/3}^{2} f\,dy\,dx$$

23.

$$\begin{aligned}
\int_R \sqrt{x+y}\,dA &= \int_0^2\int_0^1 \sqrt{x+y}\,dx\,dy \\
&= \int_0^2 \frac{2}{3}(x+y)^{\frac{3}{2}}\Big|_0^1\,dy \\
&= \frac{2}{3}\int_0^2 ((1+y)^{\frac{3}{2}}-y^{\frac{3}{2}})\,dy \\
&= \frac{2}{3}\cdot\frac{2}{5}[(1+y)^{\frac{5}{2}}-y^{\frac{5}{2}}]\Big|_0^2 \\
&= \frac{4}{15}((3^{\frac{5}{2}}-2^{\frac{5}{2}})-(1-0)) \\
&= \frac{4}{15}(9\sqrt{3}-4\sqrt{2}-1)=2.38176
\end{aligned}$$

24. In the other order, the integral is

$$\int_0^1 \int_0^2 \sqrt{x+y}\, dy\, dx.$$

First we keep x fixed and calculate the inside integral with respect to y:

$$\int_0^2 \sqrt{x+y}\, dy = \frac{2}{3}(x+y)^{3/2}\Big|_{y=0}^{y=2} = \frac{2}{3}\left[(x+2)^{3/2} - x^{3/2}\right].$$

Then the outside integral becomes

$$\int_0^1 \frac{2}{3}\left[(x+2)^{3/2} - x^{3/2}\right]\, dx = \frac{2}{3}\left[\frac{2}{5}(x+2)^{5/2} - \frac{2}{5}x^{5/2}\right]\Big|_0^1 = \frac{2}{3}\cdot\frac{2}{5}\left[3^{5/2} - 1 - 2^{5/2}\right] = 2.38176$$

Note that the answer is the same as the one we got in Exercise 23.

25.

$$\begin{aligned}\int_R (5x^2+1)\sin 3y\, dA &= \int_{-1}^1 \int_0^{\pi/3} (5x^2+1)\sin 3y\, dy\, dx \\ &= \int_{-1}^1 (5x^2+1)\left(-\frac{1}{3}\cos 3y\Big|_0^{\pi/3}\right) dx \\ &= \frac{2}{3}\int_{-1}^1 (5x^2+1)\, dx \\ &= \frac{2}{3}(\frac{5}{3}x^3 + x)\Big|_{-1}^1 \\ &= \frac{2}{3}(\frac{10}{3}+2) = \frac{32}{9}\end{aligned}$$

26. The region of integration, R, is shown in Figure 16.13.

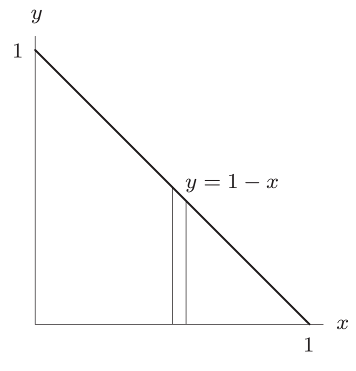


Figure 16.13

Integrating first over y, as shown in the diagram, we obtain

$$\int_R xy\, dA = \int_0^1 \left(\int_0^{1-x} xy\, dy\right) dx = \int_0^1 \frac{xy^2}{2}\Big|_0^{1-x} dx = \int_0^1 \frac{1}{2}x(1-x)^2\, dx$$

Now integrating with respect to x gives

$$\int_R xy\, dA = \left(\frac{1}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4\right)\Big|_0^1 = \frac{1}{24}.$$

27. It would be easier to integrate first in the x direction from $x = y - 1$ to $x = -y + 1$, because integrating first in the y direction would involve two separate integrals.

$$
\begin{aligned}
\int_R (2x+3y)^2\, dA &= \int_0^1 \int_{y-1}^{-y+1} (2x+3y)^2\, dx\, dy \\
&= \int_0^1 \int_{y-1}^{-y+1} (4x^2 + 12xy + 9y^2)\, dx\, dy \\
&= \int_0^1 \left[\frac{4}{3}x^3 + 6x^2y + 9xy^2\right]_{y-1}^{-y+1} dy \\
&= \int_0^1 [\frac{8}{3}(-y+1)^3 + 9y^2(-2y+2)]\, dy \\
&= \left[-\frac{2}{3}(-y+1)^4 - \frac{9}{2}y^4 + 6y^3\right]_0^1 \\
&= -\frac{2}{3}(-1) - \frac{9}{2} + 6 = \frac{13}{6}
\end{aligned}
$$

See Figure 16.14.

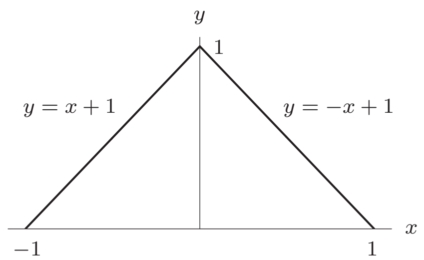


Figure 16.14

Problems

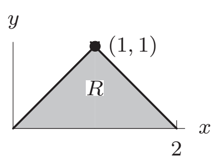
28. The diagonal line has equation $y = 2x$. Integrating with respect to y first gives

$$\int_0^1 \int_0^{2x} xy\, dy\, dx = \int_0^1 x\frac{y^2}{2}\Big|_0^{2x} dx = \int_0^1 2x^3 dx = \frac{x^4}{2}\Big|_0^1 = \frac{1}{2}.$$

29. The right half of the circle is $x = \sqrt{1-y^2}$. Integrating with respect to x first, we have

$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy\, dx\, dy = \int_{-1}^1 \frac{x^2y}{2}\Big|_0^{\sqrt{1-y^2}} dy = \frac{1}{2}\int_{-1}^1 (1-y^2)y\, dy = \frac{y^2}{4} - \frac{y^4}{8}\Big|_{-1}^1 = 0.$$

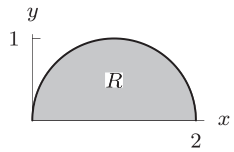
30.



The edges of the triangle are the lines $y = x$ on the left and $y = 2 - x$ on the right. We integrate with respect to x first, giving

$$\int_0^1 \int_y^{2-y} xy\, dx\, dy = \int_0^1 \frac{x^2y}{2}\Big|_y^{2-y} dy = \frac{1}{2}\int_0^1 ((2-y)^2y - y^3)\, dy = \frac{1}{2}\int_0^1 (4y - 4y^2 + y^3 - y^3) dy = y^2 - \frac{2y^3}{3}\Big|_0^1 = \frac{1}{3}.$$

31.



The circle has equation $(x-1)^2+y^2=1$; thus the top half has equation $y=\sqrt{1-(x-1)^2}$. Integrating with respect to y first gives

$$\int_0^2\int_0^{\sqrt{1-(x-1)^2}} xy\,dy\,dx = \int_0^2 \frac{xy^2}{2}\bigg|_0^{\sqrt{1-(x-1)^2}} dx = \int_0^2 \frac{x}{2}(1-(x-1)^2)dx$$
$$= \int_0^2 \left(x^2-\frac{x^3}{2}\right)dx = \left(\frac{x^3}{3}\right)-\left(\frac{x^4}{8}\right)\bigg|_0^2 = \frac{2}{3}.$$

32. (a) We divide the base region into four subrectangles as shown in Figure 16.15. The height of the object at each point (x, y) is given by $f(x, y) = xy$, we label each corner of the subrectangles with the value of the function at that point. (See Figure 16.15.) Since Volume = Height × Length × Width, and $\Delta x = 2$ and $\Delta y = 3$, we have

$$\text{Overestimate} = (12+24+6+12)(2)(3) = 324,$$

and

$$\text{Underestimate} = (0+6+0+0)(2)(3) = 36.$$

We average these to obtain

$$\text{Volume} \approx \frac{324+36}{2} = 180.$$

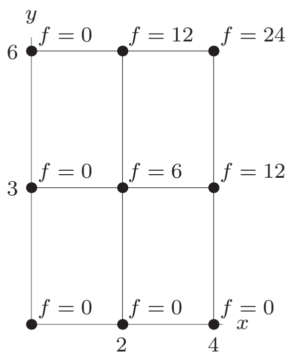


Figure 16.15

(b) We have $f(x, y) = xy$, so

$$\text{Volume} = \int_0^4\int_0^6 xy\,dydx = \int_0^4 \left(\frac{xy^2}{2}\right)\bigg|_{y=0}^{y=6} dx = \int_0^4 18x\,dx = 9x^2\bigg|_0^4 = 144.$$

The volume of this object is 144. Notice that 144 is between the over- and underestimates, 324 and 36, found in part (a).

33. As given, the region of integration is as shown in Figure 16.16.

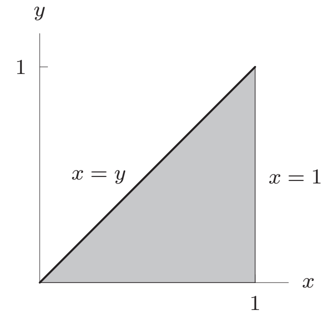


Figure 16.16

Reversing the limits gives

$$\int_0^1 \int_0^x e^{x^2}\, dydx = \int_0^1 \left(ye^{x^2} \Big|_0^x \right) dx = \int_0^1 xe^{x^2}\, dx$$
$$= \frac{e^{x^2}}{2}\Big|_0^1 = \frac{e-1}{2}.$$

34. The function $\sin(x^2)$ has no elementary antiderivative, so we try integrating with respect to y first. The region of integration is shown in Figure 16.17. Changing the order of integration, we get

$$\begin{aligned}\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy &= \int_0^1 \int_0^x \sin(x^2)\, dy\, dx \\ &= \int_0^1 \sin(x^2) \cdot y\Big|_0^x dx \\ &= \int_0^1 \sin(x^2) \cdot x\, dx \\ &= -\frac{\cos(x^2)}{2}\Big|_0^1 \\ &= -\frac{\cos 1}{2} + \frac{1}{2} = \frac{1}{2}(1 - \cos 1) = 0.23.\end{aligned}$$

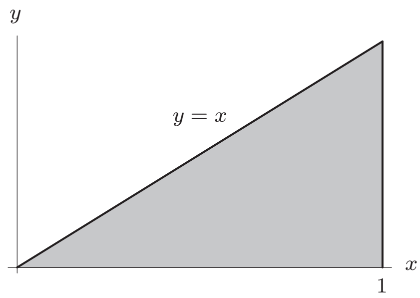


Figure 16.17

35. As given, the region of integration is as shown in Figure 16.18.

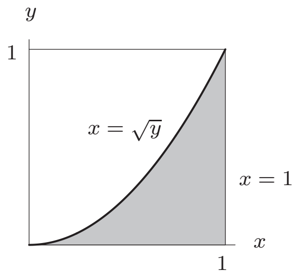


Figure 16.18

Reversing the limits gives

$$\begin{aligned}\int_0^1 \int_0^{x^2} \sqrt{2+x^3}\, dydx &= \int_0^1 \left(y\sqrt{2+x^3}\Big|_0^{x^2}\right) dx \\ &= \int_0^1 x^2\sqrt{2+x^3}\, dx \\ &= \frac{2}{9}(2+x^3)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{9}(3\sqrt{3}-2\sqrt{2}).\end{aligned}$$

36. As given, the region of integration is as shown in Figure 16.19.

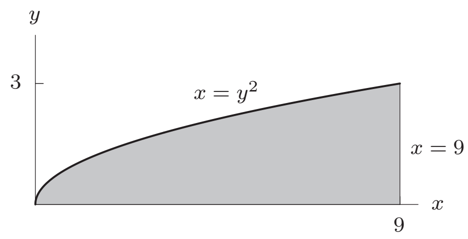


Figure 16.19

Reversing the limits gives

$$\begin{aligned}\int_0^9 \int_0^{\sqrt{x}} y\sin(x^2)\, dydx &= \int_0^9 \left(\frac{y^2\sin(x^2)}{2}\Big|_0^{\sqrt{x}}\right) dx \\ &= \frac{1}{2}\int_0^9 x\sin(x^2)\, dx \\ &= -\frac{\cos(x^2)}{4}\Big|_0^9 \\ &= \frac{1}{4} - \frac{\cos(81)}{4} = 0.056.\end{aligned}$$

37. The region of the integration is shown in Figure 16.20. To make the integration easier, we want to change the order of the integration and get

$$\begin{aligned}\int_0^1 \int_{e^y}^{e} \frac{x}{\ln x}\, dx\, dy &= \int_1^e \int_0^{\ln x} \frac{x}{\ln x}\, dy\, dx \\ &= \int_1^e \frac{x}{\ln x}\cdot y\Big|_0^{\ln x} dx \\ &= \int_1^e x\, dx = \frac{x^2}{2}\Big|_1^e = \frac{1}{2}(e^2-1).\end{aligned}$$

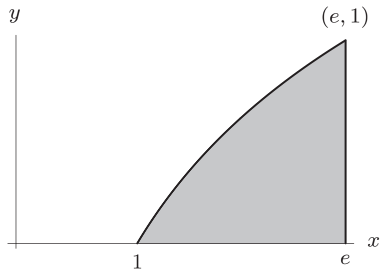


Figure 16.20

38. The region is bounded by $x = 1$, $x = 4$, $y = 2$, and $y = 2x$. Thus

$$\text{Volume} = \int_1^4 \int_2^{2x} (6x^2y)\, dydx.$$

To evaluate this integral, we evaluate the inside integral first:

$$\int_2^{2x} (6x^2y)\, dy = (3x^2y^2)\Big|_2^{2x} = 3x^2(2x)^2 - 3x^2(2^2) = 12x^4 - 12x^2.$$

Therefore, we have

$$\int_1^4 \int_2^{2x} (6x^2y)\, dydx = \int_1^4 (12x^4 - 12x^2)\, dx = \left(\frac{12}{5}x^5 - 4x^3\right)\Big|_1^4 = 2203.2.$$

The volume of this object is 2203.2.

39. (a) The volume is given by

$$V = \int_{-1}^{1}\int_{-1}^{1} x^2 + y^2\, dy\, dx = \int_{-1}^{1} x^2y + \frac{y^3}{3}\Big|_{-1}^{1} dx = 2\int_{-1}^{1} \left(x^2 + \frac{1}{3}\right) dx = 2\left(\frac{x^3}{3} + \frac{x}{3}\right)\Big|_{-1}^{1} = \frac{8}{3}.$$

(b) The region above the surface, together with the region whose volume we found in part (a), make up a box with a square base of side 2 and height 2. The volume of the box is $2 \cdot 2 \cdot 2 = 8$, the volume under the surface is

$$V = 8 - \frac{8}{3} = \frac{16}{3}.$$

40. The region is bounded by $x = 1 - y$, $x = y - 1$, $y = 1$, and $y = 3$. See Figure 16.21. To evaluate this integral we integrate with respect to x first, giving

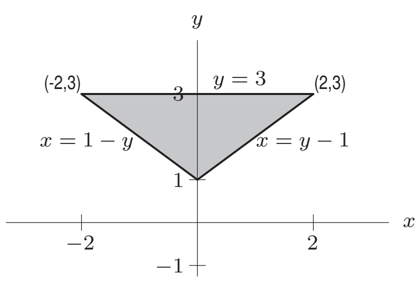


Figure 16.21

$$
\begin{aligned}
\int\int_R (2x^2+y)\,dA &= \int_1^3 \int_{1-y}^{y-1} (2x^2+y)\,dx\,dy \\
&= \int_1^3 \frac{2x^3}{3} + yx\bigg|_{1-y}^{y-1} dy = 2\int_1^3 \frac{2(y-1)^3}{3} + y(y-1)\,dy \\
&= 2\int_1^3 \frac{2y^3}{3} - y^2 + y - \frac{2}{3}\,dy \\
&= \frac{44}{3}
\end{aligned}
$$

41. (a) See Figure 16.22.

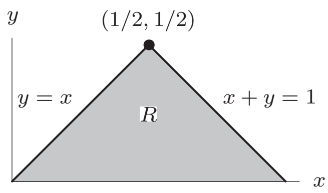


Figure 16.22

(b) If we integrate with respect to x first, we have

$$\int_R f(x,y)\,dA = \int_0^{1/2} \int_y^{1-y} f(x,y)\,dx\,dy.$$

If we integrate with respect to y first, the integral must be split into two parts, so

$$\int_R f(x,y)\,dA = \int_0^{1/2} \int_0^{x} f(x,y)\,dy\,dx + \int_{1/2}^{1} \int_0^{1-x} f(x,y)\,dy\,dx.$$

(c) If $f(x,y) = x$,

$$
\begin{aligned}
\int_R x\,dA &= \int_0^{1/2} \int_y^{1-y} x\,dx\,dy = \int_0^{1/2} \frac{x^2}{2}\bigg|_y^{1-y} dy \\
&= \frac{1}{2}\int_0^{1/2} (1-y)^2 - y^2\,dy = \frac{1}{2}\int_0^{1/2} 1 - 2y\,dy \\
&= \frac{1}{2}(y - y^2)\bigg|_0^{1/2} = \frac{1}{8}.
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\int_R x\,dA &= \int_0^{1/2} \int_0^{x} x\,dy\,dx + \int_{1/2}^{1} \int_0^{1-x} x\,dy\,dx \\
&= \int_0^{1/2} xy\bigg|_0^{x} dx + \int_{1/2}^{1} xy\bigg|_0^{1-x} dx \\
&= \int_0^{1/2} x^2\,dx + \int_{1/2}^{1} x(1-x)\,dx \\
&= \frac{x^3}{3}\bigg|_0^{1/2} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\bigg|_{1/2}^{1} \\
&= \frac{1}{24} + \frac{1}{2} - \frac{1}{3} - \frac{1}{8} + \frac{1}{24} = \frac{1}{8}.
\end{aligned}
$$

42. (a) The line $x = y/2$ is the line $y = 2x$, and $y = x$ and $y = 2x$ intersect at $x = 0$. Thus, R is the shaded region in Figure 16.23. One expression for the integral is

$$\int_R f\, dA = \int_0^3 \int_x^{2x} x^2 e^{x^2}\, dy\, dx.$$

Another expression is obtained by reversing the order of integration. When we do this, it is necessary to split R into two regions on a line parallel to the x-axis along the point of intersection of $y = x$ and $x = 3$; this line is $y = 3$. Then we obtain

$$\int_R f\, dA = \int_0^3 \int_{y/2}^{y} x^2 e^{x^2}\, dx\, dy + \int_3^6 \int_{y/2}^{3} x^2 e^{x^2}\, dx\, dy.$$

(b) We evaluate the first integral. Integrating with respect to y first:

$$\int_0^3 \int_x^{2x} x^2 e^{x^2}\, dy\, dx = \int_0^3 (x^2 e^{x^2} y)\Big|_x^{2x} dx = \int_0^3 x^3 e^{x^2}\, dx.$$

We use integration by parts with $u = x^2$, $v' = xe^{x^2}$. Then $u' = 2x$ and $v = \frac{1}{2}e^{x^2}$, so

$$\int_0^3 x^3 e^{x^2}\, dx = \frac{1}{2}x^2 e^{x^2}\Big|_0^3 - \int_0^3 xe^{x^2}\, dx = \frac{1}{2}x^2 e^{x^2}\Big|_0^3 - \frac{1}{2}e^{x^2}\Big|_0^3$$
$$= \left(\frac{1}{2}(9)e^9 - 0\right) - \left(\frac{1}{2}e^9 - \frac{1}{2}\right) = \frac{1}{2} + 4e^9.$$

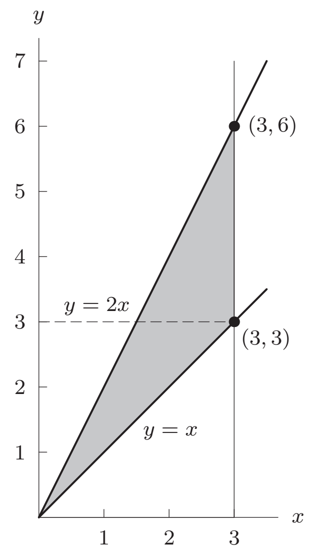


Figure 16.23

43. To find the average value, we evaluate the integral

$$\int_0^3 \int_0^6 (x^2 + 4y)\, dy dx,$$

and then divide by the area of the base region.

To evaluate this integral, we evaluate the inside integral first:

$$\int_0^6 (x^2 + 4y)\, dy = (x^2 y + 2y^2)\Big|_{y=0}^{y=6} = 6x^2 + 72.$$

Therefore, we have

$$\int_0^3 \int_0^6 (x^2+4y)dydx = \int_0^3 (6x^2+72)dx = (2x^3+72x)\Big|_0^3 = 270.$$

The value of the integral is 270. The area of the base region is $3 \cdot 6 = 18$. To find the average value of the function, we divide the value of the integral by the area of the base region:

$$\text{Average value} = \frac{1}{\text{Area}} \int_0^3 \int_0^6 (x^2+4y)\,dydx = \frac{1}{18} \cdot 270 = 15.$$

The average value is 15. This is reasonable, since the smallest value of $f(x, y)$ on this region is 0, and the largest value is $3^2 + 4 \cdot 6 = 33$.

44. To find the average value, we first find the value of the integral

$$\int_0^4 \int_0^3 (xy^2)\,dydx.$$

We evaluate the inside integral first:

$$\int_0^3 (xy^2)\,dy = \left(\frac{xy^3}{3}\right)\Big|_{y=0}^{y=3} = 9x.$$

Therefore, we have

$$\int_0^4 \int_0^3 (xy^2)\,dydx = \int_0^4 (9x)\,dx = \left(\frac{9x^2}{2}\right)\Big|_0^4 = 72.$$

The value of the integral is 72. To find the average value, we divide the value of the integral by the area of the region:

$$\text{Average value} = \frac{1}{\text{Area}} \int_0^4 \int_0^3 (xy^2)\,dydx = \frac{72}{3 \cdot 4} = 6.$$

The average value of $f(x, y)$ on this rectangle is 6. This is reasonable since the smallest value of xy^2 on this region is 0 and the largest value is $4 \cdot 3^2 = 36$.

45. The intersection of the graph of $f(x, y) = 25 - x^2 - y^2$ and xy-plane is a circle $x^2 + y^2 = 25$. The given solid is shown in Figure 16.24.

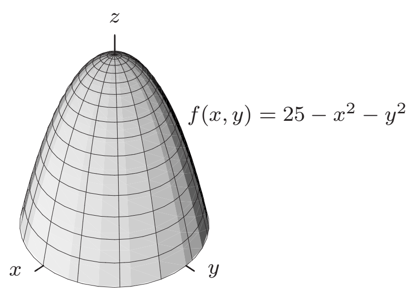


Figure 16.24

Thus the volume of the solid is

$$\begin{aligned} V &= \int_R f(x,y)\,dA \\ &= \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} (25 - x^2 - y^2)\,dx\,dy. \end{aligned}$$

46. The intersection of the graph of $f(x, y) = 25 - x^2 - y^2$ and the plane $z = 16$ is a circle, $x^2 + y^2 = 3^2$. The given solid is shown in Figure 16.25.

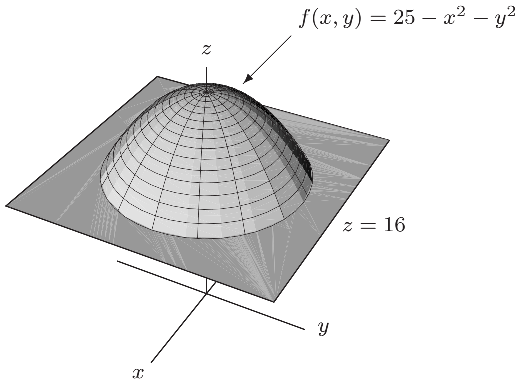


Figure 16.25

Thus, the volume of the solid is

$$\begin{aligned} V &= \int_R (f(x,y) - 16)\, dA \\ &= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (9 - x^2 - y^2)\, dx\, dy. \end{aligned}$$

47. The solid is shown in Figure 16.26, and the base of the integral is the triangle as shown in Figure 16.27.

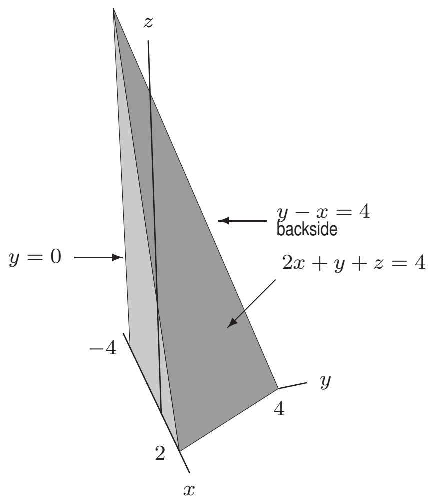


Figure 16.26

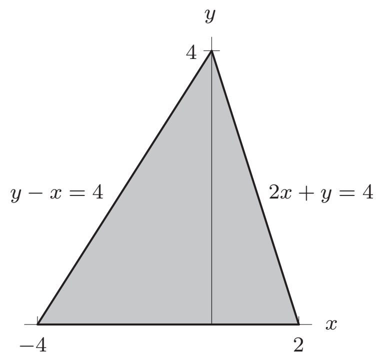


Figure 16.27

Thus, the volume of the solid is

$$\begin{aligned} V &= \int_R z\, dA \\ &= \int_R (4 - 2x - y)\, dA \\ &= \int_0^4 \int_{y-4}^{(4-y)/2} (4 - 2x - y)\, dx\, dy. \end{aligned}$$

48.

$$\begin{aligned}\text{Volume} &= \int_0^2 \int_0^2 xy\,dy\,dx = \int_0^2 \frac{1}{2}xy^2\bigg|_0^2\,dx \\ &= \int_0^2 2x\,dx \\ &= x^2\bigg|_0^2 \\ &= 4\end{aligned}$$

49. The region of integration is shown in Figure 16.28. Thus

$$\text{Volume } = \int_0^1 \int_0^x (x^2+y^2)\,dy\,dx = \int_0^1 \left(x^2y + \frac{y^3}{3}\right)\bigg|_{y=0}^{y=x} dx = \int_0^1 \frac{4}{3}x^3\,dx = \frac{x^4}{3}\bigg|_0^1 = \frac{1}{3}.$$

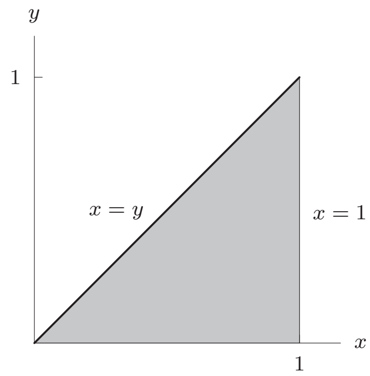


Figure 16.28

50. The region of integration is shown in Figure 16.29. Thus,

$$\begin{aligned}\text{Volume } &= \int_0^9 \int_0^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^9 \left(xy + \frac{y^2}{2}\right)\bigg|_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^9 \left(x^{3/2} + \frac{x}{2}\right) dx = \left(\frac{2}{5}x^{5/2} + \frac{x^2}{4}\right)\bigg|_0^9 = \frac{2349}{20} = 117.45.\end{aligned}$$

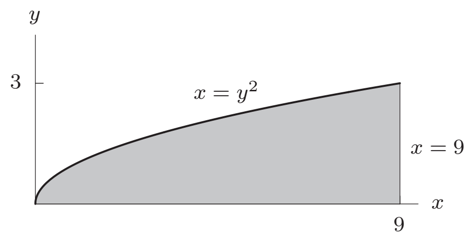


Figure 16.29

51. The plane $2x + y + z = 4$ cuts the xy-plane in the line $2x + y = 4$, so the region of integration is the triangle shown in Figure 16.30. We want to find the volume under the graph of $z = 4 - 2x - y$. Thus,

$$\begin{aligned}\text{Volume } &= \int_0^2 \int_0^{-2x+4} (4 - 2x - y)\,dy\,dx = \int_0^2 \left(4y - 2xy - \frac{y^2}{2}\right)\Big|_0^{-2x+4} dx \\ &= \int_0^2 \left(4(-2x+4) - 2x(-2x+4) - \frac{(-2x+4)^2}{2}\right) dx \\ &= \int_0^2 (2x^2 - 8x + 8)\,dx = \left(\frac{2}{3}x^3 - 4x^2 + 8x\right)\Big|_0^2 = \frac{16}{3}.\end{aligned}$$

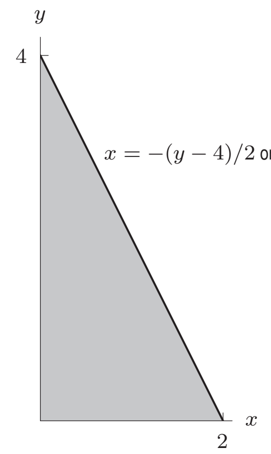


Figure 16.30

52. Let R be the triangle with vertices $(1, 0)$, $(2, 2)$ and $(0, 1)$. Note that $(3x + 2y + 1) - (x + y) = 2x + y + 1 > 0$ for $x, y > 0$, so $z = 3x + 2y + 1$ is above $z = x + y$ on R. We want to find

$$\text{Volume} = \int_R ((3x + 2y + 1) - (x + y))\,dA = \int_R (2x + y + 1)\,dA.$$

We need to express this in terms of double integrals.

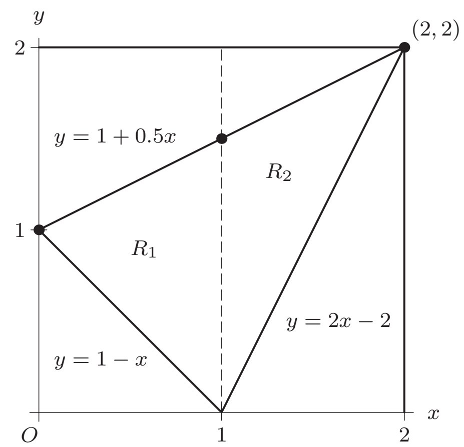


Figure 16.31

To do this, divide R into two regions with the line $x = 1$ to make regions R_1 for $x \leq 1$ and R_2 for $x \geq 1$. See Figure 16.31. We want to find

$$\int_R (2x + y + 1)\, dA = \int_{R_1} (2x + y + 1)\, dA + \int_{R_2} (2x + y + 1)\, dA.$$

Note that the line connecting $(0, 1)$ and $(1, 0)$ is $y = 1 - x$, and the line connecting $(0, 1)$ and $(2, 2)$ is $y = 1 + 0.5x$. So

$$\int_{R_1} (2x + y + 1)\, dA = \int_0^1 \int_{1-x}^{1+0.5x} (2x + y + 1)\, dy\, dx.$$

The line between $(1, 0)$ and $(2, 2)$ is $y = 2x - 2$, so

$$\int_{R_2} (2x + y + 1)\, dA = \int_1^2 \int_{2x-2}^{1+0.5x} (2x + y + 1)\, dy\, dx.$$

We can now compute the double integral for R_1:

$$\begin{aligned}\int_0^1 \int_{1-x}^{1+0.5x} (2x + y + 1)\, dy\, dx &= \int_0^1 \left(2xy + \frac{y^2}{2} + y\right)\Bigg|_{1-x}^{1+0.5x} dx \\ &= \int_0^1 \left(\frac{21}{8}x^2 + 3x\right) dx \\ &= \left(\frac{7}{8}x^3 + \frac{3}{2}x^2\right)\Bigg|_0^1 dx \\ &= \frac{19}{8},\end{aligned}$$

and the double integral for R_2:

$$\begin{aligned}\int_1^2 \int_{2x-2}^{1+0.5x} (2x + y + 1)\, dy\, dx &= \int_1^2 (2xy + y^2/2 + y)\Big|_{2x-2}^{1+0.5x} dx \\ &= \int_0^1 \left(-\frac{39}{8}x^2 + 9x + \frac{3}{2}\right) dx \\ &= \left(-\frac{13}{8}x^3 + \frac{9}{2}x^2 + \frac{3}{2}x\right)\Bigg|_1^2 \\ &= \frac{29}{8}.\end{aligned}$$

So, Volume $= \dfrac{19}{8} + \dfrac{29}{8} = \dfrac{48}{8} = 6$.

53. We want to calculate the volume of the tetrahedron shown in Figure 16.32.

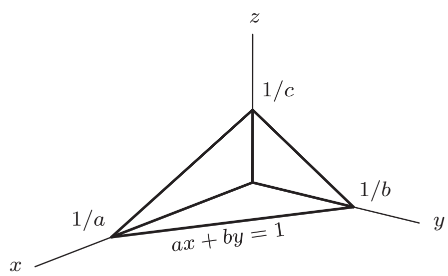


Figure 16.32

We first find the region in the xy-plane where the graph of $ax + by + cz = 1$ is above the xy-plane. When $z = 0$ we have $ax + by = 1$. So the region over which we want to integrate is bounded by $x = 0, y = 0$ and $ax + by = 1$. Integrating with respect to y first, we have

$$\text{Volume} = \int_0^{1/a} \int_0^{(1-ax)/b} z\, dy\, dx = \int_0^{1/a} \int_0^{(1-ax)/b} \frac{1 - by - ax}{c}\, dy\, dx$$

$$
\begin{aligned}
&= \int_0^{1/a} \left(\frac{y}{c} - \frac{by^2}{2c} - \frac{axy}{c}\right)\Bigg|_{y=0}^{y=(1-ax)/b} dx \\
&= \int_0^{1/a} \frac{1}{2bc}(1 - 2ax + a^2x^2)\,dx \\
&= \frac{1}{6abc}.
\end{aligned}
$$

54. The region R is shaded in Figure 16.33. The integral is

$$
\begin{aligned}
\int_R xy\,dA &= \int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} xy\,dx\,dy \\
&= \int_0^a \frac{1}{2}x^2y\Bigg|_{a-y}^{\sqrt{a^2-y^2}} dy = \frac{1}{2}\int_0^a ((a^2-y^2)y - (a-y)^2y)y\,dy \\
&= \frac{1}{2}\int_0^a (2ay^2 - 2y^3)\,dy = \left(\frac{ay^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^a = \frac{a^4}{12}.
\end{aligned}
$$

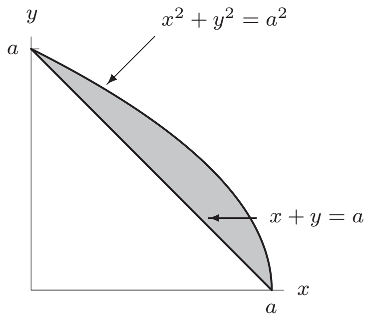


Figure 16.33

55. (a) The contour $f(x, y) = 1$ lies in the xy-plane and has equation

$$2e^{-(x-1)^2-y^2} = 1,$$

so

$$
\begin{aligned}
-(x-1)^2 - y^2 &= \ln(1/2) \\
(x-1)^2 + y^2 &= \ln 2 = 0.69.
\end{aligned}
$$

This is the equation of a circle centered at $(1, 0)$ in the xy-plane.

Other contours are of the form

$$
\begin{aligned}
2e^{-(x-1)^2-y^2} &= c \\
-(x-1)^2 - y^2 &= \ln(c/2).
\end{aligned}
$$

Thus, all the contours are circles centered at the point $(1, 0)$.

(b) The cross-section has equation $z = f(1, y) = e^{-y^2}$. If $x = 1$, the base region in the xy-plane extends from $y = -\sqrt{3}$ to $y = \sqrt{3}$. See Figure 16.34, which shows the circular region below W in the xy-plane. So

$$\text{Area } = \int_{-\sqrt{3}}^{\sqrt{3}} e^{-y^2}\,dy.$$

(c) Slicing parallel to the y-axis, we get

$$\text{Volume } = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-(x-1)^2-y^2}\, dy\, dx.$$

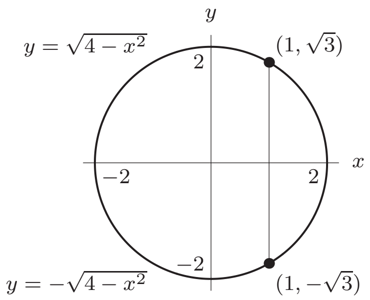


Figure 16.34: Region beneath W in the xy-plane

56. The region bounded by the x-axis and the graph of $y = x - x^2$ is shown in Figure 16.35. The area of this region is

$$A = \int_0^1 (x - x^2)dx = (\frac{x^2}{2} - \frac{x^3}{3})\Big|_0^1$$
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

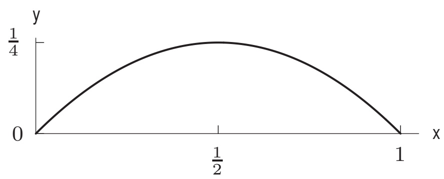


Figure 16.35

So the average distance to the x-axis for points in the region is

$$\text{Average distance} = \frac{\int_R y\, dA}{\text{area}(R)}$$

$$\int_R y\, dA = \int_0^1 \left(\int_0^{x-x^2} y\, dy\right) dx$$
$$= \int_0^1 \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2}\right) dx = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}.$$

Therefore the average distance is $\frac{1/60}{1/6} = 1/10$.

57. (a) One solution would be to arrange that the minimum values of f on the square occur at the corners, so that the corner values give an underestimate of the average. See Figure 16.36.

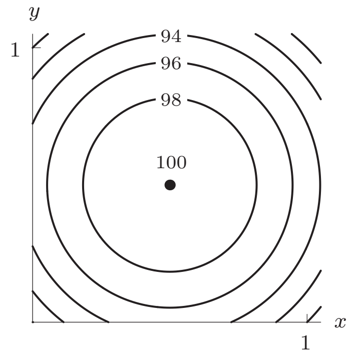


Figure 16.36

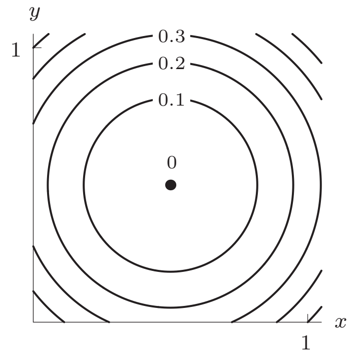


Figure 16.37

(b) One solution would be to arrange that the maximum values of f on the square occur at the corners, so that the corner values give an overestimate of the average. See Figure 16.37.

58. (a) We have

$$\begin{aligned}\text{Average value of } f &= \frac{1}{\text{Area of Rectangle}} \int_{\text{Rectangle}} f\, dA \\ &= \frac{1}{6}\int_{x=0}^{2}\int_{y=0}^{3} (ax+by)\, dydx = \frac{1}{6}\int_0^2 \left(axy + b\frac{y^2}{2}\right)\Big|_{y=0}^{y=3} dx \\ &= \frac{1}{6}\int_0^2 \left(3ax + \frac{9}{2}b\right) dx = \frac{1}{6}\left(\frac{3}{2}ax^2 + \frac{9}{2}bx\right)\Big|_0^2 \\ &= \frac{1}{6}(6a+9b) \\ &= a + \frac{3}{2}b.\end{aligned}$$

The average value will be 20 if and only if $a + (3/2)b = 20$.

This equation can also be expressed as $2a + 3b = 40$, which shows that $f(x,y) = ax + by$ has average value of 20 on the rectangle $0 \le x \le 2$, $0 \le y \le 3$ if and only if $f(2,3) = 40$.

(b) Since $2a + 3b = 40$, we must have $b = (40/3) - (2/3)a$. Any function $f(x,y) = ax + ((40/3) - (2/3)a)y$ where a is any real number is a correct solution. For example, $a = 1$ leads to the function $f(x,y) = x + (38/3)y$, and $a = -3$ leads to the function $f(x,y) = -3x + (46/3)y$, both of which have average value 20 on the given rectangle. See Figure 16.38 and 16.39.

Figure 16.38: $f(x,y) = x + \frac{38}{3}y$

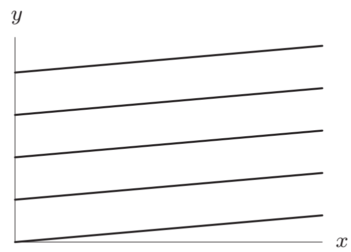


Figure 16.39: $f(x,y) = -3x + \frac{46}{3}y$

59. (a) We have

$$\begin{aligned}\text{Average value of } f &= \frac{1}{\text{Area of Square}} \int_{\text{Square}} f\, dA \\ &= \frac{1}{4}\int_0^2\int_0^2 (ax^2 + bxy + cy^2)\, dydx = \frac{1}{4}\int_0^2 \left(ax^2y + bx\frac{y^2}{2} + c\frac{y^3}{3}\right)\Big|_{y=0}^{y=2} dx\end{aligned}$$

$$= \frac{1}{4}\int_0^2 \left(2ax^2 + 2bx + \frac{8}{3}c\right)\,dx = \frac{1}{4}\left(\frac{2}{3}ax^3 + bx^2 + \frac{8}{3}cx\right)\bigg|_0^2$$
$$= \frac{1}{4}\left(\frac{16}{3}a + 4b + \frac{16}{3}c\right)$$
$$= \frac{4}{3}a + b + \frac{4}{3}c$$

The average value will be 20 if and only if $(4/3)a + b + (4/3)c = 20$.

(b) Since $(4/3)a + b + (4/3)c = 20$, we must have $b = 20 - (4/3)a - (4/3)c$. Any function $f(x, y) = ax^2 + (20 - (4/3)a - (4/3)c)xy + cy^2$ where a and c are any real numbers is a correct solution. For example, $a = 1$, $c = 3$ leads to the function $f(x, y) = x^2 + (44/3)xy + 3y^2$, and $a = -3$, $c = 0$ leads to the function $f(x, y) = -3x^2 + 24xy$, both of which have average value 20 on the given square. See Figures 16.40 and 16.41.

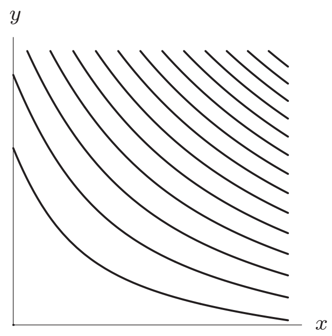


Figure 16.40: $f(x, y) = x^2 + \frac{44}{3}xy + 3y^2$

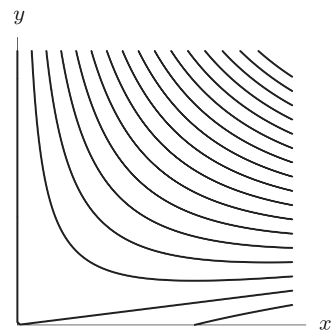


Figure 16.41: $f(x, y) = -3x^2 + 24xy$

60. Assume the length of the two legs of the right triangle are a and b, respectively. See Figure 16.42. The line through $(a, 0)$ and $(0, b)$ is given by $\frac{y}{b} + \frac{x}{a} = 1$. So the area of this triangle is

$$A = \frac{1}{2}ab.$$

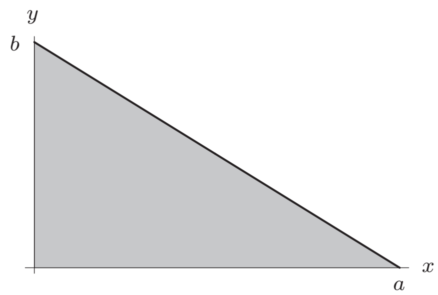


Figure 16.42

Thus the average distance from the points in the triangle to the y-axis (one of the legs) is

$$\text{Average distance} = \frac{1}{A}\int_0^a \int_0^{-\frac{b}{a}x+b} x\,dy\,dx$$
$$= \frac{2}{ab}\int_0^a \left(-\frac{b}{a}x^2 + bx\right)\,dx$$
$$= \frac{2}{ab}\left(-\frac{b}{3a}x^3 + \frac{b}{2}x^2\right)\bigg|_0^a$$
$$= \frac{2}{ab}\left(\frac{a^2b}{6}\right) = \frac{a}{3}.$$

Similarly, the average distance from the points in the triangle to the x-axis (the other leg) is

$$\begin{aligned}\text{Average distance} &= \frac{1}{A}\int_0^b \int_0^{-\frac{a}{b}y+a} y\,dx\,dy \\ &= \frac{2}{ab}\int_0^b \left(-\frac{a}{b}y^2 + ay\right)\,dy \\ &= \frac{2}{ab}\left(\frac{ab^2}{6}\right) = \frac{b}{3}.\end{aligned}$$

61. The force, ΔF, acting on ΔA, a small piece of area, is given by

$$\Delta F \approx p\Delta A,$$

where p is the pressure at that point. Thus, if R is the rectangle, the total force is given by

$$F = \int_R p\,dA.$$

We choose coordinates with the origin at one corner of the plate. See Figure 16.43.

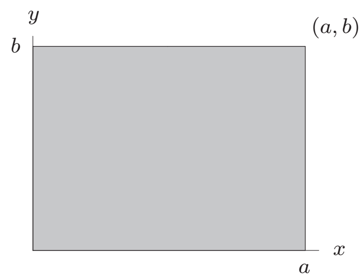


Figure 16.43

Suppose p is proportional to the square of the distance from the corner represented by the origin. Then we have

$$p = k(x^2 + y^2), \quad \text{for some positive constant } k.$$

Thus, we want to compute $\int_R k(x^2+y^2)dA$. Rewriting as an iterated integral, we have

$$\begin{aligned}F = \int_R k(x^2+y^2)\,dA &= \int_0^b \int_0^a k(x^2+y^2)\,dxdy = k\int_0^b \left(\frac{x^3}{3} + xy^2\bigg|_0^a\right)\,dy \\ &= k\int_0^b \left(\frac{a^3}{3} + ay^2\right)\,dy = k\left(\frac{a^3y}{3} + a\,\frac{y^3}{3}\bigg|_0^b\right) \\ &= \frac{k}{3}(a^3b + ab^3).\end{aligned}$$

62. The outer circle is a semicircle of radius 4. This is shown in Figure 16.44, with center at D. Thus, $CE = 2$ and $DC = 2$, while $AD = 4$. Notice that angle ADO is a right angle.

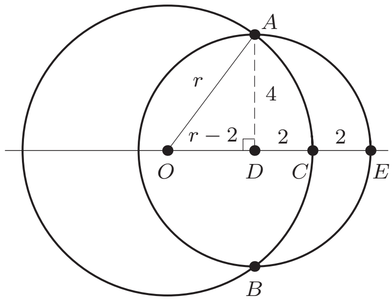


Figure 16.44

Suppose the large circle has center O and radius r. Then $OA = r$ and $OD = OC - DC = r - 2$. Applying Pythagoras' Theorem to triangle OAD gives

$$\begin{aligned} r^2 &= 4^2 + (r-2)^2 \\ r^2 &= 16 + r^2 - 4r + 4 \\ r &= 5. \end{aligned}$$

If we put the origin at O, the equation of the large circle is $x^2 + y^2 = 25$. In the same coordinates, the equation of the small circle, which has center at $D = (3, 0)$, is $(x-3)^2 + y^2 = 16$. The right hand side of the two circles are given by

$$x = \sqrt{25 - y^2} \quad \text{and} \quad x = 3 + \sqrt{16 - y^2}.$$

Since the y-coordinate of A is 4 and the y-coordinate of B is -4, we have

$$\begin{aligned} \text{Area} &= \int_{-4}^{4} \int_{\sqrt{25-y^2}}^{3+\sqrt{16-y^2}} 1\, dx\, dy \\ &= \int_{-4}^{4} (3 + \sqrt{16 - y^2} - \sqrt{25 - y^2})\, dy \\ &= 13.95. \end{aligned}$$

Strengthen Your Understanding

63. The region of integration for the first integral is the triangle bounded by $y = 0, y = x$ and $x = 1$. The region for the second integral is the triangle bounded by $x = 0, x = y$ and $y = 1$. These are not the same triangle.

64. The integral on the right does not make sense. The outside limits of integration contain a variable, but the outside limits should always be constants.

65. Suppose the cylinder has height 2 and the base has boundary $x^2 + y^2 = 1$. Then, integrating with respect to y first gives

$$\text{Volume } = \int_{\text{Base}} 2\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\, dx\, dy.$$

66. The region of integration is bounded by the x-axis and the lines $y = x$ and $x = 1$. Integrating with respect to y first, we have

$$\int_0^1 \int_0^x f(x, y)\, dy\, dx = 4.$$

We start by trying $f(x, y) = x$. Since

$$\int_0^1 \int_0^x x\, dy\, dx = \int_0^1 xy\Big|_{y=0}^{y=x} dx = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3},$$

we take $f(x, y) = 12x$. Many other answers are possible.

67. Since the height of the prism is not specified, we can choose any height we wish, for example 1. If the base triangle has area 6 and the height of the prism is 1 then the volume of the prism is 6. Any right triangle with legs of length 6 and 2 will work. For example, the triangle with vertices $(0,0)$, $(0,2)$ and $(6,0)$ has area 6. Therefore,

$$\int_0^2 \int_0^{6-3y} 1\,dx\,dy$$

represents the volume of a triangular prism whose base has area 6.

68. False. Since the inside integral is performed with respect to x and the outside integral with respect to y, the region of integration is the rectangle $5 \le x \le 12, 0 \le y \le 1$.

69. False. The iterated integral $\int_0^2 \int_0^1 f\,dxdy$ is over a rectangle. The correct limits are $\int_0^1 \int_x^{2-x} f\,dydx$.

70. True. For any point in the region of integration we have $1 \le x \le 2$, and so y is between the positive numbers 1 and 8.

71. False. The sign of $\int_a^b \int_c^d f dydx$ depends on the behavior of the function f on the region of integration. For example, $\int_1^2 \int_1^2 (-x) dydx = -\frac{3}{2}$.

72. True. Since f does not depend on x, the inside integral (which is with respect to x) evaluates to $\int_0^1 f dx = xf\Big|_{x=0}^{x=1} = (f-0) = f$. Thus

$$\int_a^b \int_0^1 f\,dx\,dy = \int_a^b f\,dy.$$

73. False. The given limits describe only the upper half disk where $y \ge 0$. The correct limits are $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f dydx$.

74. True. In the inner integral with respect to y, the function $g(x)$ can be treated as a constant, so

$$\int_a^b \int_c^d g(x)h(y)\,dxdy = \int_a^b g(x)\left(\int_c^d h(y)\,dy\right)\,dx.$$

The result of the integral $\int_c^d h(y)\,dy$ is a constant, so may be factored out of the integral with respect to x. Thus we have

$$\int_a^b g(x)\left(\int_c^d h(y)\,dy\right)\,dx = \left(\int_c^d h(y)\,dy\right)\cdot\left(\int_a^b g(x)\,dx\right).$$

75. False. As a counterexample, consider $\int_0^2 \int_0^2 (x+y)\,dxdy$. We have

$$\int_0^2 \int_0^2 (x+y)\,dxdy = \int_0^2 \left(\frac{x^2}{2} + yx\right)\Bigg|_{x=0}^{x=2} dy = \int_0^2 (2+2y)\,dy = 8$$

and

$$\int_0^2 x\,dx + \int_0^2 y\,dy = \frac{x^2}{2}\Bigg|_{x=0}^{x=2} + \frac{y^2}{2}\Bigg|_{y=0}^{y=2} = 2+2 = 4.$$

Solutions for Section 16.3

Exercises

1.

$$\begin{aligned}\int_W f\,dV &= \int_0^2 \int_{-1}^1 \int_2^3 (x^2+5y^2-z)\,dz\,dy\,dx \\ &= \int_0^2 \int_{-1}^1 (x^2z + 5y^2z - \frac{1}{2}z^2)\Bigg|_2^3 dy\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_0^2 \int_{-1}^1 (x^2 + 5y^2 - \frac{5}{2})\, dy\, dx \\
&= \int_0^2 (x^2 y + \frac{5}{3}y^3 - \frac{5}{2}y)\Big|_{-1}^{1}\, dx \\
&= \int_0^2 (2x^2 + \frac{10}{3} - 5)\, dx \\
&= (\frac{2}{3}x^3 - \frac{5}{3}x)\Big|_0^2 \\
&= \frac{16}{3} - \frac{10}{3} = 2
\end{aligned}
$$

2.

$$
\begin{aligned}
\int_W f\, dV &= \int_0^1 \int_0^1 \int_0^2 (ax + by + cz)\, dz\, dy\, dx \\
&= \int_0^1 \int_0^1 (2ax + 2by + 2c)\, dy\, dx \\
&= \int_0^1 (2ax + b + 2c)\, dx \\
&= a + b + 2c
\end{aligned}
$$

3.

$$
\begin{aligned}
\int_W f\, dV &= \int_0^\pi \int_0^\pi \int_0^\pi \sin x \cos(y+z)\, dz\, dy\, dx \\
&= \int_0^\pi \int_0^\pi \sin x \sin(y+z)\Big|_0^\pi\, dy\, dx \\
&= \int_0^\pi \int_0^\pi \sin x[\sin(y+\pi) - \sin y]\, dy\, dx \\
&= \int_0^\pi \int_0^\pi \sin x(-2\sin y)\, dy\, dx \\
&= -2\int_0^\pi \sin x(-\cos y)\Big|_0^\pi\, dx \\
&= -2\int_0^\pi 2\sin x\, dx \\
&= -4(-\cos x)\Big|_0^\pi \\
&= (-4)(2) = -8
\end{aligned}
$$

4.

$$
\begin{aligned}
\int_W f\, dV &= \int_0^a \int_0^b \int_0^c e^{-x-y-z}\, dz\, dy\, dx \\
&= \int_0^a \int_0^b \int_0^c e^{-x}e^{-y}e^{-z}\, dz\, dy\, dx \\
&= \int_0^a \int_0^b e^{-x}e^{-y}(-e^{-z})\Big|_0^c\, dy\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^a \int_0^b e^{-x}e^{-y}(-e^{-c}+1)\,dy\,dx \\
&= (1-e^{-c})\int_0^a e^{-x}(-e^{-y})\Big|_0^b\,dx \\
&= (1-e^{-b})(1-e^{-c})\int_0^a e^{-x}\,dx \\
&= (1-e^{-a})(1-e^{-b})(1-e^{-c})
\end{aligned}
$$

5. The region is the half cylinder in Figure 16.45.

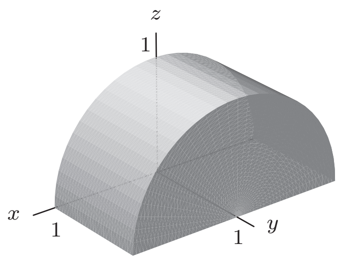


Figure 16.45

6. The region is the half cylinder in Figure 16.46.

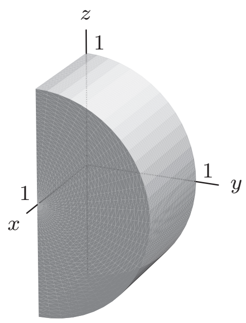


Figure 16.46

7. The region is the cylinder in Figure 16.47.

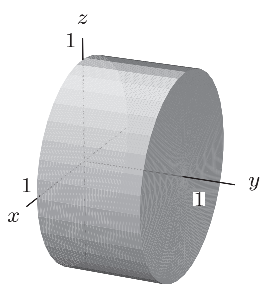


Figure 16.47

8. The region is the half cylinder in Figure 16.48.

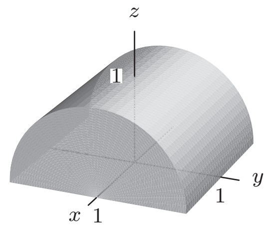


Figure 16.48

9. The region is the hemisphere in Figure 16.49.

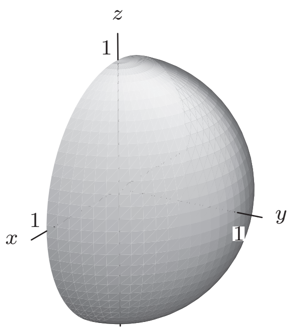


Figure 16.49

10. The region is the quarter sphere in Figure 16.50.

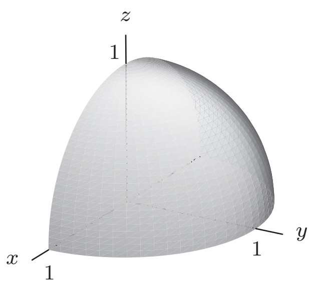


Figure 16.50

11. The region is the quarter sphere in Figure 16.51.

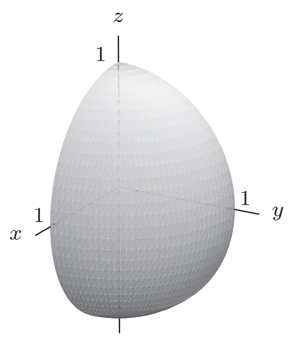


Figure 16.51

12. The region is the hemisphere in Figure 16.52.

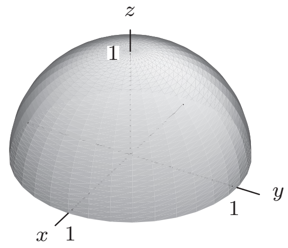


Figure 16.52

13. The region is the quarter sphere in Figure 16.53.

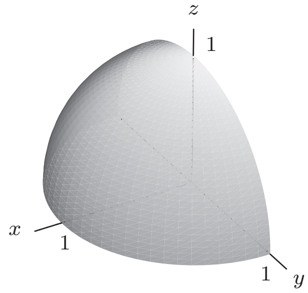


Figure 16.53

Problems

14. Positive. Since e^z is positive on T, its integral is positive.

15. Positive. Since e^z is positive on B, its integral is positive.

16. Zero. Since $\sin z$ is positive on T, and negative (and of equal absolute value) on B the integral is zero because the integrals over the two halves of the sphere cancel.

17. Positive. Since $\sin z$ is positive on T, its integral is positive.

18. Zero. Since $\sin z$ is positive on the upper half of R and negative (and of equal absolute value) on the lower half of R, the integral of $\sin z$ is zero because the integrals over the two halves cancel.

19. Zero. The value of y is positive on the half of the cone above the second and third quadrants and negative (of equal absolute value) on the half of the cone above the third and fourth quadrants. The integral of y over the entire solid cone is zero because the integrals over the four quadrants cancel.

20. Zero. The value of x is positive above the first and fourth quadrants in the xy-plane, and negative (and of equal absolute value) above the second and third quadrants. The integral of x over the entire solid cone is zero because the integrals over the two halves of the cone cancel.

21. Positive. Since z is positive on W, its integral is positive.

22. Zero. You can see this in several ways. One way is to observe that xy is positive on part of the cone above the first and third quadrants (where x and y are of the same sign) and negative (of equal absolute value) on the part of the cone above the second and fourth quadrants (where x and y have opposite signs). These add up to zero in the integral of xy over all of W.

Another way to see that the integral is zero is to write the triple integral as an iterated integral, say integrating first with respect to x. For fixed y and z, the x-integral is over an interval symmetric about 0. The integral of x over such an interval is zero. If any of the inner integrals in an iterated integral is zero, then the triple integral is zero.

23. Zero. Write the triple integral as an iterated integral, say integrating first with respect to x. For fixed y and z, the x-integral is over an interval symmetric about 0. The integral of x over such an interval is zero. If any of the inner integrals in an iterated integral is zero, then the triple integral is zero.

24. Negative. If (x, y, z) is any point inside the cone then $z < 2$. Hence the function $z - 2$ is negative on W and so is its integral.

25. Positive. The function $\sqrt{x^2 + y^2}$ is positive, so its integral over the solid W is positive.

26. Positive. The function e^{-xyz} is a positive function everywhere so its integral over W is positive.

27. Positive. If (x, y, z) is any point inside the solid W then $\sqrt{x^2 + y^2} < z$. Thus the integrand $z - \sqrt{x^2 + y^2} > 0$, and so its integral over the solid W is positive.

28. Figure 16.54 shows a slice through the region for a fixed x. The required volume, V, is given by

$$\begin{aligned} V &= \int_1^2 \int_0^1 \int_y^{3y} dz\, dy\, dx = \int_1^2 \int_0^1 z\Big|_y^{3y} dy\, dx = \int_1^2 \int_0^1 2y\, dy\, dx \\ &= \int_1^2 y^2\Big|_0^1 dx = \int_1^2 dx = 1. \end{aligned}$$

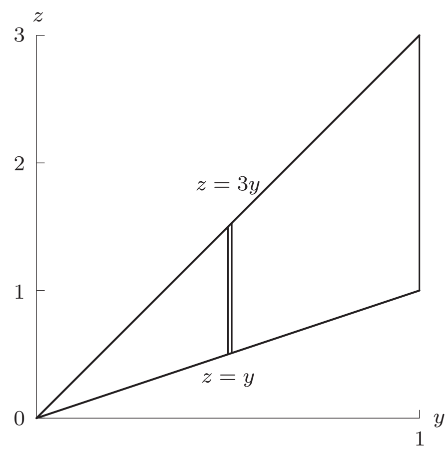


Figure 16.54

29. The required volume, V, is given by

$$\begin{aligned} V &= \int_0^5 \int_0^3 \int_0^{x^2} dz\, dy\, dx \\ &= \int_0^5 \int_0^3 x^2\, dy\, dx \\ &= \int_0^5 x^2 y\Big|_{y=0}^{y=3} dx \\ &= \int_0^5 3x^2\, dx \\ &= 125. \end{aligned}$$

30. We have $z = 2 - x - y$, and the region under the surface in the xy-plane is bounded by the axes and the line $x + y = 2$ or $y = 2 - x$. Thus the volume is given by

$$V = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx = \int_0^2 \int_0^{2-x} (2 - x - y)\, dy\, dx$$

$$= \int_0^2 2y - xy - \frac{y^2}{2}\bigg|_0^{2-x} dx$$
$$= \int_0^2 2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\, dx$$
$$= \int_0^2 \left(\frac{x^2}{2} - 2x + 2\right) dx = \frac{x^3}{6} - x^2 + 2x\bigg|_0^2 = \frac{4}{3}$$

31. (a) Since the trough is symmetric about the xz-plane, we find the mass of sludge in one half and double it. Thus the total mass is given by

$$\text{Mass of sludge } = 2\int_0^{10}\int_0^1\int_y^1 e^{-3x} dz\, dy\, dx.$$

Other orders of integration are possible.

(b) Evaluating gives

$$\text{Mass } = 2\int_0^{10}\int_0^1\int_y^1 e^{-3x} dz\, dy\, dx = 2\int_0^{10}\int_0^1 \left(ze^{-3x}\right)\bigg|_y^1 dy\, dx$$
$$= 2\int_0^{10}\int_0^1 \left(e^{-3x} - ye^{-3x}\right) dy\, dx = 2\int_0^{10}\left(ye^{-3x} - \frac{y^2}{2}e^{-3x}\right)\bigg|_0^1 dx$$
$$= 2\int_0^{10} \frac{e^{-3x}}{2} dx = 2\left(-\frac{e^{-3x}}{6}\right)\bigg|_0^{10} = \frac{1-e^{-30}}{3}.$$

32. The required volume, V, is given by

$$V = \int_0^{10}\int_0^{10-x}\int_{x+y}^{10} dz dy dx$$
$$= \int_0^{10}\int_0^{10-x} (10 - (x+y))\, dy dx$$
$$= \int_0^{10} \left[10y - xy - \frac{1}{2}y^2\right]_{y=0}^{y=10-x} dx$$
$$= \int_0^{10} \frac{1}{2}(10-x)^2\, dx$$
$$= \frac{500}{3}$$

33. Since $z = x + y$ is below $z = 1 + 2x + 2y$ for $x, y \geq 0$, we have

$$V = \int_0^1\int_0^2\int_{x+y}^{1+2x+2y} 1\, dz\, dy\, dx.$$

The order of integration of x and y can be reversed.

34. For $x^2 + y^2 \leq 1$, the paraboloid $z = x^2 + y^2$ is below the sphere $x^2 + y^2 + z^2 = 4$, so

$$V = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{4-x^2-y^2}} 1\, dz\, dy\, dx.$$

The order of integration of x and y can be reversed.

35. The two surfaces are planes given by

$$z = 6 - 2x - 2y$$
$$z = 6 - 3x - 4y.$$

For $x, y \geq 0$, the plane $z = 6 - 2x - 2y$ is above $z = 6 - 3x - 4y$. The region in the xy-plane is shown in Figure 16.55. Thus

$$V = \int_0^1 \int_0^{1-x} \int_{6-3x-4y}^{6-2x-2y} 1\, dz\, dy\, dx.$$

The order of integration of x and y can be reversed.

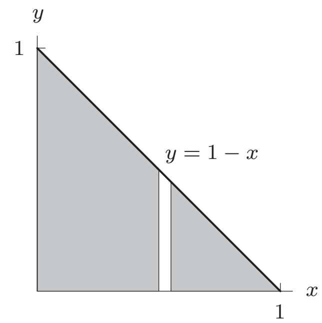


Figure 16.55

36. The top half of the sphere is given by

$$z = \sqrt{9 - x^2 - y^2}.$$

The region in the xy-plane is shown in Figure 16.56. If we integrate with respect to y first, we have to break the region in two pieces. Thus it is easier to integrate with respect to x first, giving

$$V = \int_0^2 \int_y^{(y+2)/2} \int_0^{\sqrt{9-x^2-y^2}} 1\, dz\, dx\, dy.$$

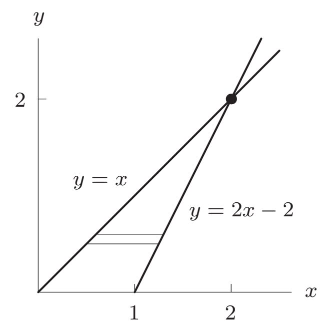


Figure 16.56

37. The sphere $x^2 + y^2 + z^2 = 9$ intersects the plane $z = 2$ in the circle

$$x^2 + y^2 + 2^2 = 9$$
$$x^2 + y^2 = 5.$$

The upper half of the sphere is given by $z = \sqrt{9 - x^2 - y^2}$. Thus, using the limits from Figure 16.57 gives

$$V = \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} \int_2^{\sqrt{9-x^2-y^2}} 1\, dz\, dy\, dx.$$

The order of integration of x and y can be reversed.

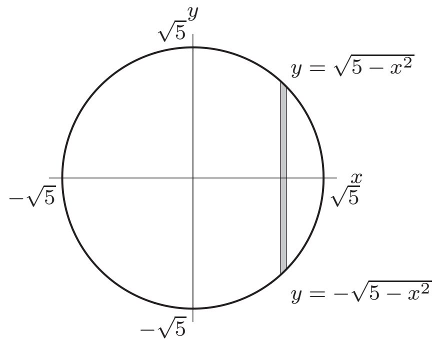


Figure 16.57

38. The top half of the sphere is given by

$$z = \sqrt{4 - x^2 - y^2}.$$

The region in the xy-plane is shown in Figure 16.58. If we integrate with respect to x first, we have to break the region into two pieces. Thus, it is easier to integrate with respect to y first, giving

$$V = \int_0^1 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} 1\, dz\, dy\, dx.$$

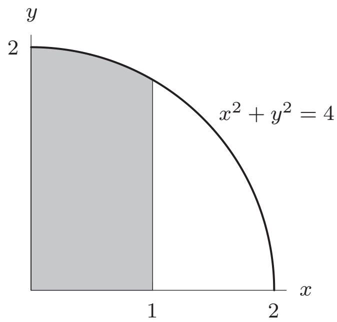


Figure 16.58

39. A slice through W for a fixed value of y is a semi-circle the boundary of which is $z^2 = r^2 - x^2$, for $z \geq 0$, so the inner integral is

$$\int_0^{\sqrt{r^2-x^2}} f(x, y, z)\, dz.$$

Lining up these stacks parallel to x-axis gives a slice from $x = -r$ to $x = r$ giving

$$\int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} f(x, y, z)\, dz\, dx.$$

Finally, there is a slice for each y between 0 and 1, so the integral we want is

$$\int_0^1 \int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} f(x, y, z)\, dz\, dx\, dy.$$

40. A slice through W for a fixed value of x is a semi-circle the boundary of which is $y^2 = 4 - z^2$, for $y \geq 0$, so the inner integral is

$$\int_0^{\sqrt{4-z^2}} f(x, y, z)\, dy.$$

Lining up these stacks parallel to z-axis gives a slice from $z = -2$ to $z = 2$ giving

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-z^2}} f(x, y, z)\, dy\, dz.$$

Finally, there is a slice for each x between 0 and 1, so the integral we want is

$$\int_{0}^{1} \int_{-2}^{2} \int_{0}^{\sqrt{4-z^2}} f(x, y, z)\, dy\, dz\, dx.$$

41. A slice through W for a fixed value of x is a semi-circle the boundary of which is $y^2 = r^2 - x^2 - z^2$, for $y \geq 0$, so the inner integral is

$$\int_{0}^{\sqrt{r^2-x^2-z^2}} f(x, y, z)\, dy.$$

Lining up these stacks parallel to z-axis gives a slice from $z = -\sqrt{r^2 - x^2}$ to $z = \sqrt{r^2 - x^2}$ giving

$$\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{0}^{\sqrt{r^2-x^2-z^2}} f(x, y, z)\, dy\, dz.$$

Finally, there is a slice for each x between $-r$ and r, so the integral we want is

$$\int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{0}^{\sqrt{r^2-x^2-z^2}} f(x, y, z)\, dy\, dz\, dx.$$

42. A slice through W for a fixed value of x is a semi-circle the boundary of which is $z^2 = r^2 - x^2 - y^2$, for $z \geq 0$, so the inner integral is

$$\int_{0}^{\sqrt{r^2-x^2-y^2}} f(x, y, z)\, dz.$$

Lining up these stacks parallel to y-axis gives a slice from $y = -\sqrt{r^2 - x^2}$ to $y = \sqrt{r^2 - x^2}$ giving

$$\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{0}^{\sqrt{r^2-x^2-y^2}} f(x, y, z)\, dz\, dy.$$

Finally, there is a slice for each x between 0 and r, so the integral we want is

$$\int_{0}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{0}^{\sqrt{r^2-x^2-y^2}} f(x, y, z)\, dz\, dy\, dx.$$

43. Figure 16.59 shows a slice through the region for a fixed value of y.

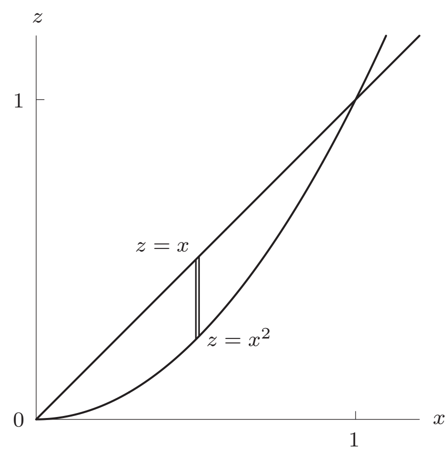


Figure 16.59

We break the region into small cubes of volume $\Delta V = \Delta x \Delta y \Delta z$. A stack of cubes vertically above the point (x, z) in the xz-plane gives the strip shown in Figure 16.59 and so the inner integral is

$$\int_{x^2}^{x} dz$$

The plane and the surface meet when $x = x^2$, giving $x(1-x) = 0$, so $x = 0$ or $x = 1$. Lining up the stacks parallel to the z-axis gives a slice from $x = 0$ to $x = 1$. Thus, the limits on the middle integral are

$$\int_0^1 \int_{x^2}^{x} dz\, dx.$$

Finally, there is a slice for each y between 0 and 3, so the integral we want is

$$\int_0^3 \int_0^1 \int_{x^2}^{x} dz\, dx\, dy.$$

The required volume, V, is given by

$$\begin{aligned} V &= \int_0^3 \int_0^1 \int_{x^2}^{x} dz\, dx\, dy \\ &= \int_0^3 \int_0^1 \left(x - x^2\right) dx\, dy \\ &= \int_0^3 \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 dy \\ &= \int_0^3 \frac{1}{6}\, dy \\ &= \frac{1}{6}\int_0^3 dy \\ &= \frac{1}{6}\cdot 3 \\ &= \frac{1}{2}. \end{aligned}$$

44. The required volume, V, is given by

$$\begin{aligned} V &= \int_0^5 \int_0^{5-x} \int_0^{x+y} dz\, dy\, dx \\ &= \int_0^5 \int_0^{5-x} (x+y)\, dy\, dx \\ &= \int_0^5 xy + \frac{1}{2}y^2\bigg|_{y=0}^{y=5-x} dx \\ &= \int_0^5 \left(x(5-x) + \frac{1}{2}(5-x)^2\right) dx \\ &= \frac{125}{3}. \end{aligned}$$

45. The pyramid is shown in Figure 16.60. The planes $y = 0$, and $y - x = 4$, and $2x + y + z = 4$ intersect the plane $z = -6$ in the lines $y = 0$, $y - x = 4$, $2x + y = 10$ on the $z = -6$ plane as shown in Figure 16.61.

These three lines intersect at the points $(-4, 0, -6)$, $(5, 0, -6)$, and $(2, 6, -6)$. Let R be the triangle in the planes $z = -6$ with the above three points as vertices. Then, the volume of the solid is

$$V = \int_0^6 \int_{y-4}^{(10-y)/2} \int_{-6}^{4-2x-y} dz\, dx\, dy$$

$$
\begin{aligned}
&= \int_0^6 \int_{y-4}^{(10-y)/2} (10 - 2x - y)\, dx\, dy = 162 \\
&= \int_0^6 (10x - x^2 - xy)\Big|_{y-4}^{(10-y)/2} dy \\
&= \int_0^6 (\frac{9y^2}{4} - 27y + 81)\, dy \\
&= 162
\end{aligned}
$$

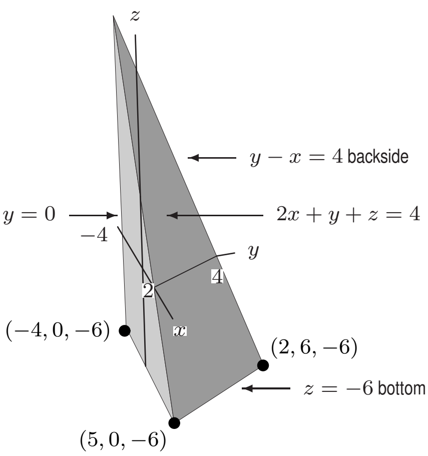


Figure 16.60

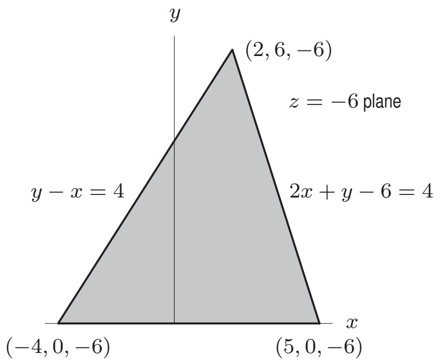


Figure 16.61

46. Since $x + y + z = 1$ can be written as

$$z = 1 - x - y,$$

the plane $z = 1 + x + y$ is above the plane $z = 1 - x - y$ for $x \geq 0, y \geq 0$. The region of integration in the xy-plane is the triangle shown in Figure 16.62. Thus

$$
\begin{aligned}
\text{Volume } &= \int_0^1 \int_0^{1-x} \int_{1-x-y}^{1+x+y} 1\, dz\, dy\, dx = \int_0^1 \int_1^{1-x} z\Big|_{1-x-y}^{1+x+y} dy\, dx \\
&= \int_0^1 \int_0^{1-x} ((1 + x + y) - (1 - x - y))\, dy\, dx = \int_0^1 \int_0^{1-x} (2x + 2y)\, dy\, dx = \int_0^1 2xy + y^2\Big|_0^{1-x} dx \\
&= \int_0^1 (2x(1-x) + (1-x)^2)\, dx = \int_0^1 (1 - x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3}.
\end{aligned}
$$

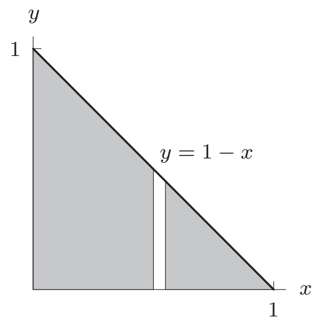


Figure 16.62

47. The plane $x+y+z=1$ cuts the xy-plane in the line $x+y=1$. For $x+y<1$, the plane $x+y+z=1$ is above the xy-plane. For $1<x+y\le 2$, the plane $x+y+z=1$ is below the xy-plane. Therefore, $z=1-x-y$ is positive for $x+y<1$ and negative for $1<x+y<2$. Thus

$$\text{Volume } = \int_{x+y\le 1}(1-x-y)\,dA - \int_{1\le x+y\le 2}(1-x-y)\,dA$$

From Figure 16.63, we see that the region $1\le x+y\le 2$ must be split into two, for example as shown. Integrating with respect to x first, we have

$$\begin{aligned}
\text{Volume } &= \int_0^1\int_0^{1-y}(1-x-y)\,dx\,dy - \left(\int_0^1\int_{1-y}^{2-y}(1-x-y)\,dx\,dy + \int_1^2\int_0^{2-y}(1-x-y)\,dx\,dy\right)\\
&= \int_0^1\left(x-\frac{x^2}{2}-xy\right)\Big|_0^{1-y}dy - \int_0^1\left(x-\frac{x^2}{2}-xy\right)\Big|_{1-y}^{2-y}dy - \int_1^2\left(x-\frac{x^2}{2}-xy\right)\Big|_0^{2-y}dy\\
&= \int_0^1\frac{1-y}{2}(2-(1-y)-2y)\,dy - \int_0^1\left(\frac{2-y}{2}(2-(2-y)-2y)-\frac{1-y}{2}(2-(1-y)-2y)\right)dy\\
&\qquad - \int_1^2\frac{2-y}{2}(2-(2-y)-2y)\,dy\\
&= \int_0^1\frac{(1-y)^2}{2}\,dy + \int_0^1\frac{1}{2}\,dy - \int_1^2\frac{y(y-2)}{2}\,dy\\
&= -\frac{(1-y)^3}{6}\Big|_0^1 + \frac{y}{2}\Big|_0^1 - \frac{1}{2}\left(\frac{y^3}{3}-y^2\right)\Big|_1^2 = \frac{1}{6}+\frac{1}{2}+\frac{1}{3}=1.
\end{aligned}$$

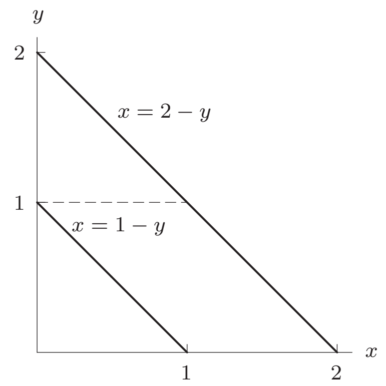


Figure 16.63

48. (a) The solid is shown in Figure 16.64. The density, δ, is given by $\delta=4z$, so

$$\text{Mass} = \int_W 4z\,dV.$$

The top of the solid has equation

$$z = 2-x-y.$$

The base in the xy-plane is shown in Figure 16.65, so the iterated integral must be split into two pieces.

$$\text{Mass} = \int_W 4z\,dV = \int_0^{1/2}\int_0^x\int_0^{2-x-y}4z\,dz\,dy\,dx + \int_{1/2}^1\int_0^{1-x}\int_0^{2-x-y}4z\,dz\,dy\,dx.$$

(b) Evaluating gives

$$\begin{aligned}
\text{Mass} &= \int_0^{1/2}\int_0^x 2z^2\Big|_0^{2-x-y} dy\,dx + \int_{1/2}^1\int_0^{1-x} 2z^2\Big|_0^{2-x-y} dy\,dx \\
&= 2\int_0^{1/2}\int_0^x (2-x-y)^2\,dy\,dx + 2\int_{1/2}^1\int_0^{1-x}(2-x-y)^2\,dy\,dx \\
&= 2\int_0^{1/2} -\frac{(2-x-y)^3}{3}\Big|_0^x dx + 2\int_{1/2}^1 -\frac{(2-x-y)^3}{3}\Big|_0^{1-x} dx \\
&= \frac{2}{3}\int_0^{1/2}\left((2-x)^3-(2-x-x)^3\right)dx + \frac{2}{3}\int_{1/2}^1\left((2-x)^3-1^3\right)dx \\
&= \frac{2}{3}\left(-\frac{(2-x)^4}{4}+\frac{8(1-x)^4}{4}\right)\Big|_0^{1/2} + \frac{2}{3}\left(-\frac{(2-x)^4}{4}-x\right)\Big|_{1/2}^1 \\
&= \frac{2}{3}\cdot\frac{55}{64}+\frac{2}{3}\frac{33}{64}=\frac{88}{96}.
\end{aligned}$$

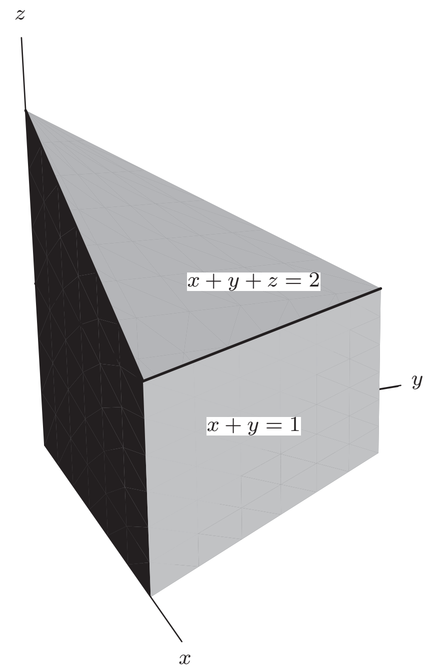


Figure 16.64: The solid

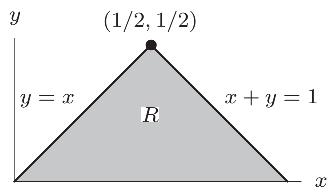


Figure 16.65: Base of solid

49. The region looks like an upside-down trough, with cross section in the xz-plane shown in Figure 16.66, extending a distance of 3 cm in the y-direction.

Since the region and the density function are symmetric about the z-axis, we find the mass of the right side and double it. In grams

$$\begin{aligned}
\text{Mass} &= 2\int_0^3\int_0^1\int_0^{1-x}(10-z)\,dz\,dx\,dy \\
&= 2\int_0^3\int_0^1\left(10z-\frac{z^2}{2}\right)\Big|_0^{1-x} dx\,dy \\
&= 2\int_0^3\int_0^1\left(10(1-x)-\frac{(1-x)^2}{2}\right)dx\,dy
\end{aligned}$$

$$= \int_0^3 \int_0^1 (19 - 18x - x^2)\, dx\, dy$$

$$= 3 \cdot \left(19x - 9x^2 - \frac{x^3}{3}\right)\Bigg|_0^1 = 3\left(19 - 9 - \frac{1}{3}\right) = 29 \text{ gm.}$$

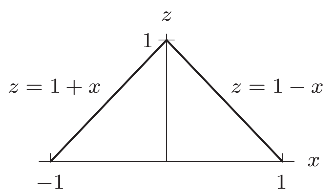


Figure 16.66

50. The region of integration is shown in Figure 16.67, and the mass of the given solid is given by

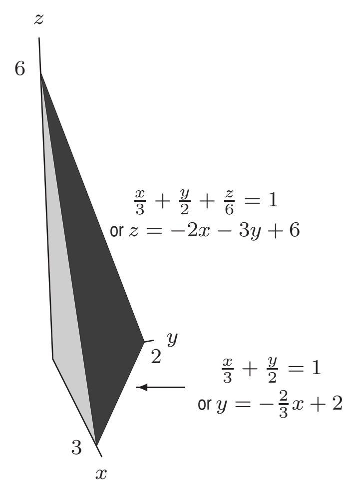


Figure 16.67

$$\text{mass} = \int_R \delta\, dV$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{-2x-3y+6} (x+y)\, dz\, dy\, dx$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} (x+y)z\Big|_0^{-2x-3y+6} dydx$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} (x+y)(-2x-3y+6)\, dydx$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} (-2x^2 - 3y^2 - 5xy + 6x + 6y)\, dydx$$

$$= \int_0^3 \left(-2x^2y - y^3 - \frac{5}{2}xy^2 + 6xy + 3y^2\right)\Bigg|_0^{-\frac{2}{3}x+2} dx$$

$$= \int_0^3 \left(\frac{14}{27}x^3 - \frac{8}{3}x^2 + 2x + 4\right)\, dx$$

$$= \left(\frac{7}{54}x^4 - \frac{8}{9}x^3 + x^2 + 4x\right)\Bigg|_0^3$$
$$= \frac{7}{54}\cdot 3^4 - \frac{8}{9}\cdot 3^3 + 3^2 + 12 = \frac{21}{2} - 3 = \frac{15}{2}.$$

51. The pyramid is shown in Figure 16.68. The planes $y = 0$, and $y - x = 4$, and $2x + y + z = 4$ intersect the plane $z = -6$ in the lines $y = 0$, $y - x = 4$, $2x + y = 10$ as shown in Figure 16.69.

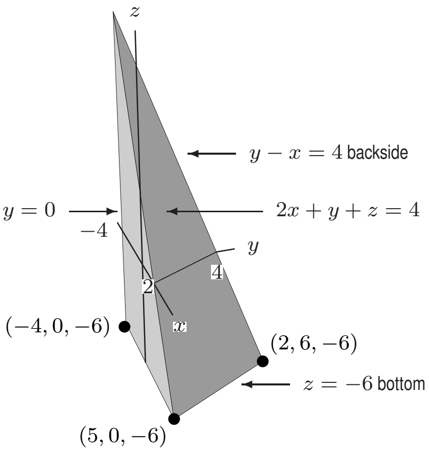


Figure 16.68

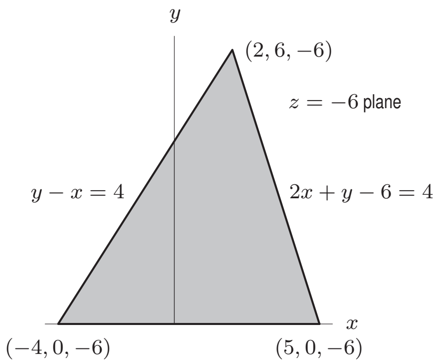


Figure 16.69

These three lines (the edges of the pyramid) intersect the plane $z = -6$ at the points $(-4, 0, -6)$, $(5, 0, -6)$, and $(2, 6, -6)$. Let R be the triangle in the plane $z = -6$ with these three points as vertices. Then, the mass of the solid is

$$\begin{aligned}
\text{Mass} &= \int_0^6 \int_{y-4}^{(10-y)/2} \int_{-6}^{4-2x-y} \delta(x, y, z)\, dz\, dx\, dy \\
&= \int_0^6 \int_{y-4}^{(10-y)/2} \int_{-6}^{4-2x-y} y\, dz\, dx\, dy \\
&= \int_0^6 \int_{y-4}^{(10-y)/2} y(10 - 2x - y)\, dx\, dy \\
&= \int_0^6 y(10x - x^2 - xy)\Bigg|_{x=y-4}^{x=(10-y)/2} dy \\
&= \int_0^6 \left(\frac{9y^3}{4} - 27y^2 + 81y\right) dy \\
&= 243.
\end{aligned}$$

52. (a) Since $\delta(x, y, z) = z$ is measured in grams per cm^3, and since volume is measured in cm^3, the integral represents the mass of the pyramid, in grams.

(b) The pyramid is bounded below by the plane $z = 0$ and is bounded above by the four planes $z = 6 - x$, $z = 6 - y$, $z = x$, and $z = y$. Therefore, since there are four different surfaces bounding the pyramid on the top, we would need four separate triple integrals.

(c) We integrate in the y-direction first. The pyramid is bounded on the left and right sides by the planes $y - z = 0$ and $y + z = 6$, so our y limits of integration are $y = z$ and $y = 6 - z$. After projecting into the xz-plane, we obtain the region bounded by the lines $z = x$, $z = 6 - x$, and $z = 0$, as shown in Figure 16.70:

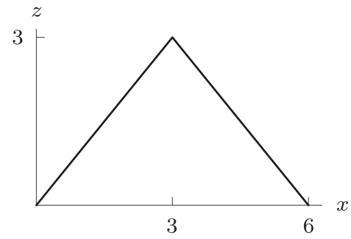


Figure 16.70

Therefore, if we integrate in the x direction next, our limits are $x = z$ and $x = 6 - z$, so we have

$$\begin{aligned}\int_E z dV &= \int_0^3 \int_z^{6-z} \int_z^{6-z} z\, dy\, dx\, dz = \int_0^3 \int_z^{6-z} zy\Big|_z^{6-z} dx\, dz \\ &= \int_0^3 \int_z^{6-z} z(6-2z) dx dz = \int_0^3 z(6-2z)x\Big|_z^{6-z} dz = \int_0^3 z(6-2z)^2\, dz \\ &= \int_0^3 (36z - 24z^2 + 4z^3)\, dz = (18z^2 - 8z^3 + z^4)\Big|_0^3 = 27,\end{aligned}$$

and we conclude that the mass of the pyramid is 27 grams.

53. (a) The vectors $\vec{u} = \vec{i} - \vec{j}$ and $\vec{v} = \vec{i} - \vec{k}$ lie in the required plane so $\vec{p} = \vec{u} \times \vec{v} = \vec{i} + \vec{j} + \vec{k}$ is perpendicular to this plane. Let (x, y, z) be a point in the plane, then $(x-1)\vec{i} + y\vec{j} + z\vec{k}$ is perpendicular to $\vec{p}$, so $((x-1)\vec{i} + y\vec{j} + z\vec{k}) \cdot (\vec{i} + \vec{j} + \vec{j}) = 0$ and so

$$(x-1) + y + z = 0.$$

Therefore, the equation of the required plane is $x + y + z = 1$.

(b) The required volume, V, is given by

$$\begin{aligned}V &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx \\ &= \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx \\ &= \int_0^1 y - xy - \frac{1}{2}y^2\Big|_0^{1-x} dx \\ &= \int_0^1 \left(1 - x - x(1-x) - \frac{1}{2}(1-x)^2\right) dx \\ &= \int_0^1 \frac{1}{2}(1-x)^2 dx \\ &= \frac{1}{6}.\end{aligned}$$

54. We will integrate in the z-direction first. Observe that the bottom of E is given by the plane $z = 0$, and the top of E is given by the plane $z = 4 - 2x - 4y$. After projecting E into the xy-plane, we obtain the region bounded by the lines $3x - 2y = 0$, $2x + 4y = 4$, and $y = 0$ (see Figure 16.71).

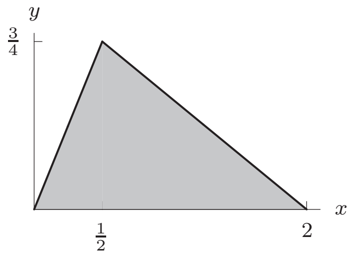


Figure 16.71

We can see from the picture that, if we integrate in the x-direction next, our limits of integration will be $x = (2/3)y$ and $x = 2 - 2y$. Therefore, our final answer is

$$\int_0^{3/4} \int_{\frac{2y}{3}}^{2-2y} \int_0^{4-2x-4y} f(x, y, z)\, dz\, dx\, dy.$$

55. We will integrate in the x-direction first. Observe that the "back" side of E is given by the plane $x = 0$, and the "front" side of E is given by the plane $x = 2 - 2y - (1/2)z$. After projecting E into the yz-plane, we obtain the region bounded by the lines $y = 0$, $z = 0$, $z = 2$, and $4y + z = 4$ (see Figure 16.72).

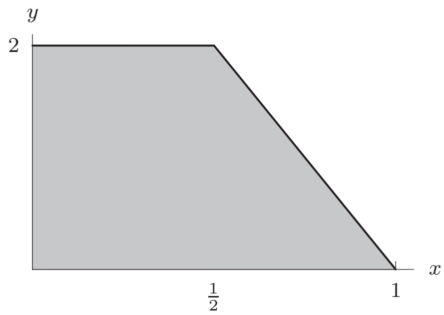


Figure 16.72

We can see from the picture that, if we integrate in the y-direction next, our limits of integration will be $y = 0$ and $x = 1 - (1/4)z$. Therefore, our final answer is

$$\int_0^{2} \int_0^{1-\frac{z}{4}} \int_0^{2-2y-\frac{z}{2}} f(x, y, z)\, dx\, dy\, dz.$$

56. We will integrate in the y-direction first. Observe that the "left" side of E is given by the plane $3x - 2y = 0$, and the "right" side of E is given by the plane $2x+4y+z = 4$. Solving these equations for y gives us a lower limit of $y = (3/2)x$ and an upper limit of $y = 1 - (1/2)x - (1/4)z$. After projecting E into the xz-plane, we obtain the region shown in Figure 16.73.

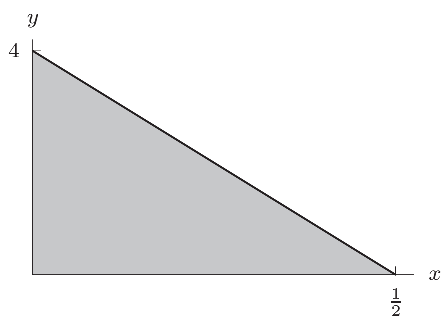


Figure 16.73

We note that the x and the z intercepts in the picture can be determined by observing that the slanted boundary of the projected region is given by the line of intersection of the planes $y = (3/2)x$ and $2x + 4y + z = 4$. Substituting $y = (3/2)x$ into $2x + 4y + z = 4$ yields the following:

$$\begin{aligned} 2x + 4\left(\frac{3x}{2}\right) + z &= 4 \\ 2x + 6x + z &= 4 \\ 8x + z &= 4 \end{aligned}$$

Therefore, the projected region is bounded by the lines $x = 0$, $z = 0$, and $8x + z = 4$ in the xz-plane, and so our final answer is

$$\int_0^{\frac{1}{2}} \int_0^{4-8x} \int_{\frac{3x}{2}}^{1-\frac{x}{2}-\frac{z}{4}} f(x, y, z)\, dy\, dz\, dx.$$

57. Orient the region as shown in Figure 16.74 and use Cartesian coordinates with origin at the center of the sphere. The equation of the sphere is $x^2 + y^2 + z^2 = 25$, and we want the volume between the planes $z = 3$ and $z = 5$. The plane $z = 3$ cuts the sphere in the circle $x^2 + y^2 + 3^2 = 25$, or $x^2 + y^2 = 16$.

$$\text{Volume} = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_3^{\sqrt{25-x^2-y^2}} dz\, dy\, dx.$$

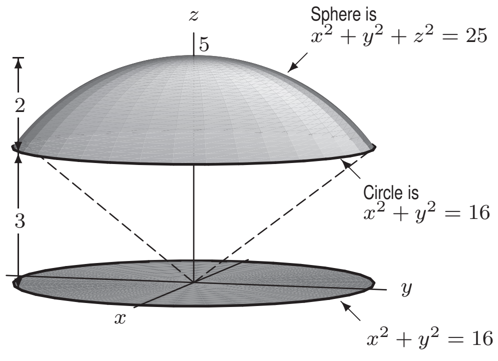


Figure 16.74

58. (a) The equation of the surface of the whole cylinder along the y-axis is $x^2 + z^2 = 1$. The part we want is

$$z = \sqrt{1 - x^2} \qquad 0 \le y \le 10.$$

See Figure 16.75.

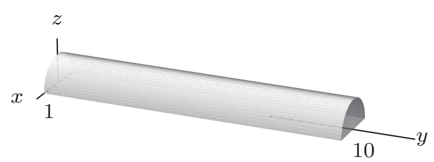


Figure 16.75

(b) The integral is

$$\int_D f(x, y, z)\, dV = \int_0^{10} \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} f(x, y, z)\, dz\, dx\, dy.$$

59. The intersection of two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is shown in Figure 16.76. This region is bounded by four surfaces:

$$z = -\sqrt{1 - x^2}, \quad z = \sqrt{1 - x^2}, \quad y = -\sqrt{1 - z^2}, \quad \text{and} \quad y = \sqrt{1 - z^2}$$

So the volume of the given solid is

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dy\, dz\, dx$$

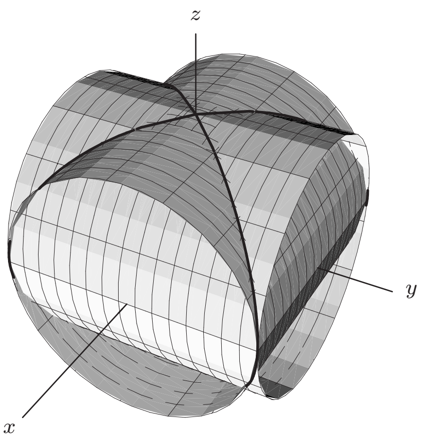


Figure 16.76

60. Integrating in the z-direction first, our lower limit is the plane $z = 0$, and the upper limit is the surface $z = 6y^2$. After projecting E into the xy-plane, we obtain the region in the first quadrant that lies inside the ellipse $x^2 + 3y^2 = 12$ (see Figure 16.77).

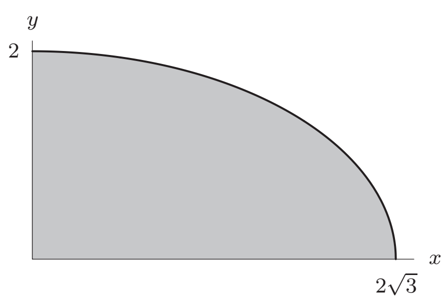


Figure 16.77

Therefore, our x-limits of integration are the line $x = 0$ and the ellipse $x = \sqrt{12 - 3y^2}$, so our answer is

$$\int_0^2 \int_0^{\sqrt{12-3y^2}} \int_0^{6y^2} f(x, y, z)\, dz\, dx\, dy.$$

61. Integrating in the x-direction first, our lower limit of integration is the plane $x = 0$, and the upper limit is the cylinder $x = \sqrt{12 - 3y^2}$. After projecting E into the yz-plane, we obtain the two-dimensional region bounded between the curves $z = 0, y = 2$, and $z = 6y^2$. Substituting $y = 2$ into $z = 6y^2$ gives $z = 24$, which is the z coordinate of the point of intersection of the two curves (see Figure 16.78).

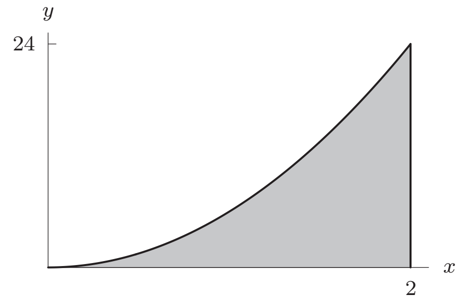


Figure 16.78

Therefore, our final answer is

$$\int_0^2 \int_0^{6y^2} \int_0^{\sqrt{12-3y^2}} f(x,y,z)\,dx\,dz\,dy.$$

62. Integrating in the y-direction first, one bounding surface is$z = 6y^2$, which, after solving for y, becomes $y = \sqrt{z/6}$, our lower limit of integration. Similarly, the other bounding surface is$x^2 + 3y^2 = 12$, which becomes $y = \sqrt{(12 - x^2)/3}$, our upper limit. After projecting E into the xz-plane, we obtain a region bounded by two line segments and a curve (see Figure 16.79).

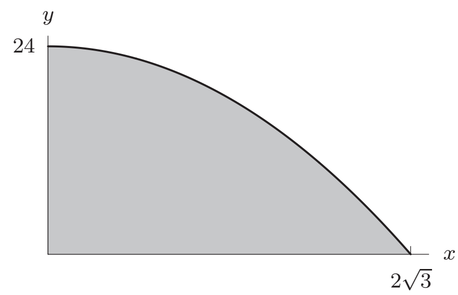


Figure 16.79

We can find the equation of the curve by finding the intersection of the cylinders $y = \sqrt{z/6}$ and $x^2 + 3y^2 = 12$ and projecting into the xz-plane:

$$\begin{aligned} 12 &= x^2 + 3y^2 \\ 12 &= x^2 + 3\left(\frac{z}{6}\right) \\ 24 &= 2x^2 + z \\ z &= 24 - 2x^2 \end{aligned}$$

Therefore, our z-limits of integration are the line $z = 0$ and the parabola $z = 24 - 2x^2$, and so our final answer is

$$\int_0^{\sqrt{12}} \int_0^{24-2x^2} \int_{\sqrt{\frac{z}{6}}}^{\sqrt{\frac{12-x^2}{3}}} f(x,y,z)\,dy\,dz\,dx.$$

63. From the problem, we know that (x, y, z) is in the cube which is bounded by the three coordinate planes, $x = 0$, $y = 0$, $z = 0$ and the planes $x = 2$, $y = 2$, $z = 2$. We can regard the value $x^2 + y^2 + z^2$ as the density of the cube. The average value of $x^2 + y^2 + z^2$ is given by

$$\begin{aligned}
\text{average value} &= \frac{\int_V (x^2+y^2+z^2)\,dV}{\text{volume}(V)} \\
&= \frac{\int_0^2\int_0^2\int_0^2 (x^2+y^2+z^2)\,dx\,dy\,dz}{8} \\
&= \frac{\int_0^2\int_0^2 \left(\frac{x^3}{3} + (y^2+z^2)x\right)\Big|_0^2 dydz}{8} \\
&= \frac{\int_0^2\int_0^2 \left(\frac{8}{3} + 2y^2 + 2z^2\right)\,dydz}{8} \\
&= \frac{\int_0^2 \left(\frac{8}{3}y + \frac{2}{3}y^3 + 2z^2y\right)\Big|_0^2 dz}{8} \\
&= \frac{\int_0^2 \left(\frac{16}{3} + \frac{16}{3} + 4z^2\right)\,dz}{8} \\
&= \frac{\left(\frac{32}{3}z + \frac{4}{3}z^3\right)\Big|_0^2}{8} \\
&= \frac{\left(\frac{64}{3} + \frac{32}{3}\right)}{8} = 4.
\end{aligned}$$

64. As an aid in finding limits of integration, we begin by finding the coordinates of the four points highlighted on the diagram. Clearly, one of them is the origin $(0,0,0)$. The point at the top of E is formed by the intersection of the plane $x+2y+z=4$ with the z-axis, so we substitute $x=0$ and $y=0$ into the equation of the plane to obtain $z=4$, yielding a point with coordinates $(0,0,4)$. Similarly, the intersection of the plane $x+2y+z=4$ with the y-axis produces a point with coordinates $(0,2,0)$. Finally, the last point highlighted on the diagram occurs at the intersection of the surfaces $x+2y+z=4$, $x=2y^2$, and $z=0$. Substituting $z=0$ and $x=2y^2$ into the equation $x+2y+z=4$ yields

$$\begin{aligned}
2y^2 + 2y &= 4 \\
y^2 + y - 2 &= 0 \\
(y+2)(y-1) &= 0
\end{aligned}$$

Since the desired point is in the first octant, its y-coordinate must be $y=1$, which yields $x=2y^2=2$, and we see that the coordinates of the point are given by $(2,1,0)$.

(a) Integrating in the z-direction first, our region is bounded by the plane $z=0$ on the bottom and the plane $x+2y+z=4$ on the top. Solving for z therefore yields a lower limit of $z=0$ and an upper limit of $z=4-x-2y$. After projecting E into the xy-plane, we obtain the picture in Figure 16.80.

The curve at the top of this projected region is formed by the intersection of the plane $x+2y+z=4$ and $z=0$, yielding an equation of $x+2y=4$, or $y=(4-x)/2$. The curve at the bottom of the projected region is simply the portion of the cylinder $x=2y^2$ that lies in the xy-plane. Therefore, our y-limits of integration are $y=(4-x)/2$ and $y=\sqrt{x/2}$, yielding a final answer of

$$\int_0^2 \int_{\sqrt{\frac{x}{2}}}^{\frac{4-x}{2}} \int_0^{4-x-2y} f(x,y,z)\,dz\,dy\,dx.$$

(b) Integrating in the y-direction first, our region is bounded by the parabolic cylinder $x=2y^2$ on one side and the plane $x+2y+z=4$ on the other side. Solving these equations for y yields $y=\sqrt{x/2}$ and $y=(4-x-z)/2$ as our y-limits of integration. After projecting E into the xz-plane, we obtain the picture in Figure 16.81.

We can find the equation of the curve bounding the top of the projected region by finding the intersection of the surfaces $x=2y^2$ and $x+2y+z=4$ and projecting into the xz-plane:

$$\begin{aligned}
x + 2y + z &= 4 \\
x + 2\sqrt{\frac{x}{2}} + z &= 4 \\
x + \sqrt{2x} + z &= 4 \\
z &= 4 - x - \sqrt{2x}
\end{aligned}$$

Therefore, our z-limits of integration are the line $z = 0$ and the curve $z = 4 - x - \sqrt{2x}$, and so our final answer is

$$\int_0^2 \int_0^{4-x-\sqrt{2x}} \int_{\sqrt{\frac{x}{2}}}^{\frac{4-x-z}{2}} f(x,y,z)\, dy\, dz\, dx.$$

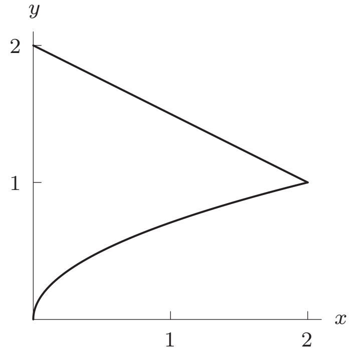


Figure 16.80

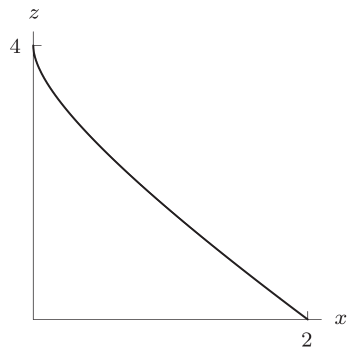


Figure 16.81

65. The mass m is given by

$$\begin{aligned} m &= \int_W 1\, dV = \int_0^1 \int_0^1 \int_0^{x+y+1} 1\, dz\, dy\, dx \\ &= \int_0^1 \int_0^1 (x+y+1)\, dy\, dx \\ &= \int_0^1 \left(xy + y^2/2 + y\right)\Big|_0^1\, dx \\ &= \int_0^1 (x + 3/2)\, dx = 2 \text{ gm.} \end{aligned}$$

Then the x-coordinate of the center of mass is given by

$$\begin{aligned} \bar{x} &= \frac{1}{2}\int_W x\, dV = \frac{1}{2}\int_0^1 \int_0^1 \int_0^{x+y+1} x\, dz\, dy\, dx \\ &= \frac{1}{2}\int_0^1 \int_0^1 x(x+y+1)\, dy\, dx \\ &= \frac{1}{2}\int_0^1 \left(x^2 y + xy^2/2 + xy\right)\Big|_0^1\, dx \\ &= \frac{1}{2}\int_0^1 (x^2 + 3/2x)\, dx = 13/24 \text{ cm.} \end{aligned}$$

An essentially identical calculation (since the region is symmetric in x and y) gives $\bar{y} = 13/24$ cm. Finally, we compute $\bar{z}$:

$$\begin{aligned} \bar{z} &= \frac{1}{2}\int_W z\, dV = \frac{1}{2}\int_0^1 \int_0^1 \int_0^{x+y+1} z\, dz\, dy\, dx \\ &= \frac{1}{2}\int_0^1 \int_0^1 (x+y+1)^2/2\, dy\, dx \\ &= \frac{1}{2}\int_0^1 (x+y+1)^3/6\Big|_0^1\, dx \\ &= \frac{1}{12}\int_0^1 \left((x+2)^3 - (x+1)^3\right) dx = 25/24 \text{ cm.} \end{aligned}$$

So $(\bar{x}, \bar{y}, \bar{z}) = (13/24, 13/24, 25/24)$.

66. The mass m is given by

$$\begin{aligned}
m = \int_W 1\,dV &= \int_0^1 \int_0^{(1-x)/2} \int_0^{(1-x-2y)/3} 1\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{(1-x)/2} \frac{1-x-2y}{3}\,dy\,dx \\
&= \frac{1}{3}\int_0^1 (y - xy - y^2)\Big|_0^{(1-x)/2}\,dx \\
&= \frac{1}{3}\left(\int_0^1 \frac{1-x}{2} - x\frac{1-x}{2} - \left(\frac{1-x}{2}\right)^2\right)\,dx \\
&= \frac{1}{3}\int_0^1 \left(\frac{(1-x)^2}{2} - \frac{(1-x)^2}{4}\right)\,dx \\
&= \frac{1}{3}\left(\frac{(1-x)^3}{12}\right)\Bigg|_0^1 = 1/36 \text{ gm.}
\end{aligned}$$

Then the coordinates of the center of mass are given by

$$\bar{x} = 36\int_W x\,dV = 36\int_0^1 \int_0^{(1-x)/2} \int_0^{(1-x-2y)/3} x\,dz\,dy\,dx = 1/4 \text{ cm.}$$

and

$$\bar{y} = 36\int_W y\,dV = 36\int_0^1 \int_0^{(1-x)/2} \int_0^{(1-x-2y)/3} y\,dz\,dy\,dx = 1/8 \text{ cm.}$$

and

$$\bar{z} = 36\int_W z\,dV = 36\int_0^1 \int_0^{(1-x)/2} \int_0^{(1-x-2y)/3} z\,dz\,dy\,dx = 1/12 \text{ cm.}$$

67. The volume V of the solid is $1 \cdot 2 \cdot 3 = 6$. We need to compute

$$\begin{aligned}
\frac{m}{6}\int_W x^2 + y^2\,dV &= \frac{m}{6}\int_0^1 \int_0^2 \int_0^3 x^2 + y^2\,dz\,dy\,dx \\
&= \frac{m}{6}\int_0^1 \int_0^2 3(x^2 + y^2)\,dy\,dx \\
&= \frac{m}{2}\int_0^1 (x^2 y + y^3/3)\Big|_0^2\,dx \\
&= \frac{m}{2}\int_0^1 (2x^2 + 8/3)\,dx = 5m/3
\end{aligned}$$

68. The volume of the solid is $8abc$, so we need to evaluate

$$\begin{aligned}
\frac{m}{8abc}\int_W (y^2 + z^2)\,dV &= \frac{m}{8abc}\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (y^2 + z^2)\,dx\,dy\,dz \\
&= \frac{m}{8abc}\int_{-c}^{c} \int_{-b}^{b} 2a(y^2 + z^2)\,dy\,dz \\
&= \frac{m}{4bc}\int_{-c}^{c} (y^3/3 + yz^2)\Big|_{-b}^{b}\,dz \\
&= \frac{m}{2c}\int_{-c}^{c} (b^2/3 + z^2)\,dz \\
&= m(b^2 + c^2)/3
\end{aligned}$$

69. By the definition, we have that

$$\begin{aligned} a+b &= \frac{m}{V}\int_W (y^2+z^2)\,dV + \frac{m}{V}\int_W (x^2+z^2)\,dV \\ &= \frac{m}{V}\int_W (x^2+y^2+2z^2)\,dV \\ &= \frac{m}{V}\int_W (x^2+y^2)\,dV + \frac{m}{V}\int_W (2z^2)\,dV \\ &= c + \frac{m}{V}\int_W (2z^2)\,dV \end{aligned}$$

Since z^2 is always positive, the integral $\int_W (2z^2)\,dV$ will be positive, thus $a+b>c$.

Strengthen Your Understanding

70. This would be true if the function f were even in z; that is, if $f(x,y,-z)=f(x,y,z)$, so that the integral over the lower half of the sphere and the upper half of the sphere were equal. But if $f(x,y,z)=z$, then $\int_S f(x,y,z)\,dV=0$ while $\int_U f(x,y,z)\,dV$ is positive.

71. The limits for the innermost integrals should be the same since both integrate first with respect to z.

72. The volume of R is $\pi\cdot 2^2\cdot 3=12\pi$, so we choose f to be the constant function $f(x,y,z)=7/(12\pi)$.

73. Since f is not constant, the integral over different parts of the spherical region must cancel. For example, if we take $f(x,y,z)=z$, the integral over the top and bottom halves of the region cancel. Many other answers are possible.

74. False. The integral gives the total mass of the material contained in W.

75. True. The region lies above the square $0\le x\le 1, 0\le y\le 1$ and below the plane $z=x$.

76. False. The given limits only cover the part of the unit ball in the first octant where $x\ge 0, y\ge 0$, and $z\ge 0$. To cover the entire unit ball the limits are

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f\,dzdydx.$$

77. True. Both sets of limits describe the solid region lying above the triangle $x+y\le 1, x\ge 0, y\ge 0, z=0$ and below the plane $x+y+z=1$.

78. True. Both sets of limits describe the solid region lying above the rectangle $-1\le x\le 1, 0\le y\le 1, z=0$ and below the parabolic cylinder $z=1-x^2$.

79. False. The iterated integral is of the form like $\int_a^b\int_c^d\int_e^k f\,dz\,dy\,dx$ only if the rectangular region has faces parallel to the coordinate axes. More general rectangular regions, such as a cube with one corner at the origin and the opposite corner at $(0,0,1)$ will need to be written as the sum of iterated integrals where the limits are not constant.

80. False. As a counterexample, consider $f(x,y,z)=\frac{1}{2}-x$. Then f is positive on half the cube and negative on the other half. Symmetry can be used to show that $\int_0^1\int_0^1\int_0^1(\frac{1}{2}-x)dz\,dy\,dx=0$.

81. True. Since $\int_W f\,dV=\lim\sum_{i,j,k} f(x_i,y_j,z_k)\Delta V$, where (x_i,y_j,z_k) is a point inside the ijk-th sub-box of volume ΔV, and since $f>g$, we have

$$\lim\sum_{i,j,k} f(x_i,y_j,z_k)\Delta V > \lim\sum_{i,j,k} g(x_i,y_j,z_k)\Delta V = \int_W g\,dV.$$

82. False. As a counterexample, let W_1 be the solid cube $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$, and let W_2 be the solid cube $-\frac{1}{2}\le x\le 0, -\frac{1}{2}\le y\le 0, -\frac{1}{2}\le z\le 0$. Then volume$(W_1)=1$ and volume$(W_2)=\frac{1}{8}$. Now if $f(x,y,z)=-1$, then $\int_{W_1} f\,dV=1\cdot -1$ which is less than $\int_{W_2} f\,dV=\frac{1}{8}\cdot -1$.

83. True. If W is the solid region lying under the graph of f and above the region R in the xy-plane, we can compute the volume of W either using the double integral $\int_R f\,dA$, or using the triple integral $\int_W 1\,dV$.

Solutions for Section 16.4

Exercises

1. $\int_0^{\pi/2}\int_0^{1/2} f\, r dr\, d\theta$

2. $\int_0^{2\pi}\int_0^{\sqrt{2}} f\, r dr\, d\theta$

3. $\int_{\pi/4}^{3\pi/4}\int_0^{2} f\, r dr\, d\theta$

4. $\int_{\pi/2}^{3\pi/2}\int_1^{2} f\, r dr\, d\theta$

5. Since this is a rectangular region, we use Cartesian coordinates. The rectangle is described by the inequalities $1 \le x \le 5$ and $2 \le y \le 4$, so the integral is
$$\int_1^5\int_2^4 f(x,y)\, dy\, dx.$$

6. A circle is best described in polar coordinates. The radius is 5, so r goes from 0 to 5. To include the entire circle, we need θ to go from 0 to 2π. The integral is
$$\int_0^{2\pi}\int_0^5 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

7. This is a portion of a circle so it is best described in polar coordinates. The region is a piece of a ring in which r goes from 2 to 4. Since we include only the portion of the ring below the x-axis, we need θ to go from π to 2π. The integral is
$$\int_{\pi}^{2\pi}\int_2^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

8. Since this is a triangular region we can use Cartesian coordinates. The bottom boundary of the triangle is the line $y = x+1$ and the top boundary is the line $y = 5 - x$. The x limits are 0 to 2. The integral is
$$\int_0^2\int_{x+1}^{5-x} f(x,y)\, dy\, dx.$$

9. See Figure 16.82.

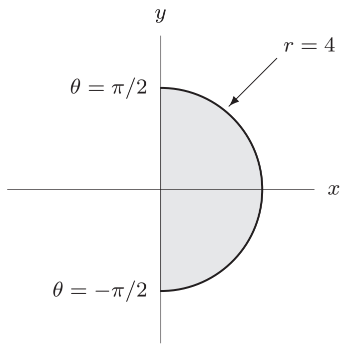


Figure 16.82

10. See Figure 16.83.

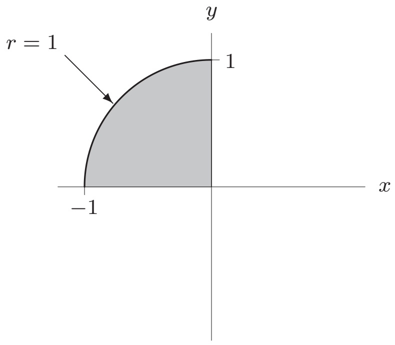


Figure 16.83

11. See Figure 16.84.

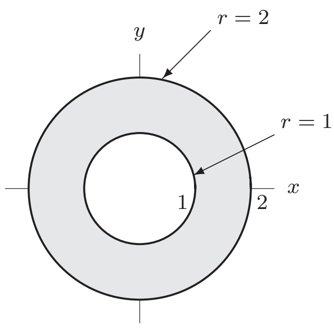


Figure 16.84

12. See Figure 16.85.

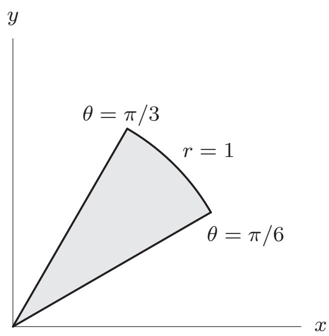


Figure 16.85

13. See Figure 16.86.

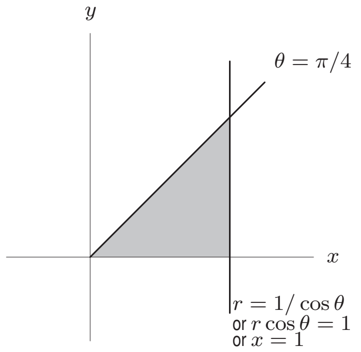


Figure 16.86

14. See Figure 16.87.

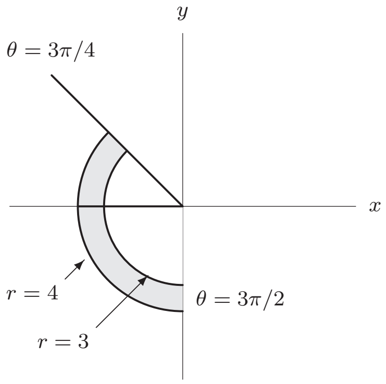


Figure 16.87

15. See Figure 16.88.

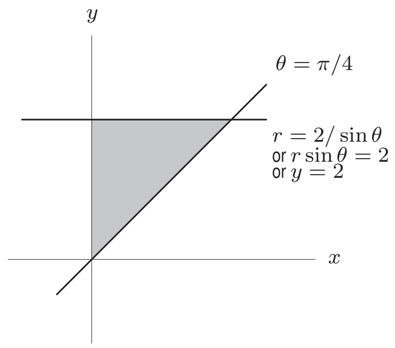


Figure 16.88

Problems

16. The presence of the term x^2+y^2 suggests that we should convert the integral into polar coordinates. Since $\sqrt{x^2+y^2}=r$, the integral becomes

$$\int_R \sqrt{x^2+y^2}\,dxdy = \int_0^{2\pi}\int_2^3 r^2\,drd\theta = \int_0^{2\pi} \left.\frac{r^3}{3}\right|_2^3 d\theta = \int_0^{2\pi} \frac{19}{3}\,d\theta = \frac{38\pi}{3}.$$

17. By using polar coordinates, we get

$$\begin{aligned}\int_R \sin(x^2+y^2)dA &= \int_0^{2\pi}\int_0^2 \sin(r^2)r\,dr\,d\theta \\ &= \int_0^{2\pi} -\frac{1}{2}\cos(r^2)\Big|_0^2\,d\theta \\ &= -\frac{1}{2}\int_0^{2\pi}(\cos 4-\cos 0)\,d\theta \\ &= -\frac{1}{2}(\cos 4-1)\cdot 2\pi = \pi(1-\cos 4)\end{aligned}$$

18. The region is pictured in Figure 16.89.

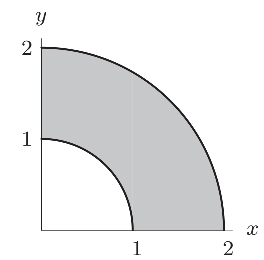


Figure 16.89

Using polar coordinates, we get

$$\int_R (x^2-y^2)dA = \int_0^{\pi/2}\int_1^2 r^2(\cos^2\theta-\sin^2\theta)r dr\,d\theta = \int_0^{\pi/2}(\cos^2\theta-\sin^2\theta)\cdot\frac{1}{4}r^4\Big|_1^2\,d\theta$$

$$\begin{aligned} &= \frac{15}{4}\int_0^{\pi/2}(\cos^2\theta-\sin^2\theta)\,d\theta \\ &= \frac{15}{4}\int_0^{\pi/2}\cos 2\theta\,d\theta \\ &= \frac{15}{4}\cdot\frac{1}{2}\sin 2\theta\Big|_0^{\pi/2} = 0.\end{aligned}$$

19. By the given limits $0 \le x \le -1$, and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, the region of integration is in Figure 16.90.

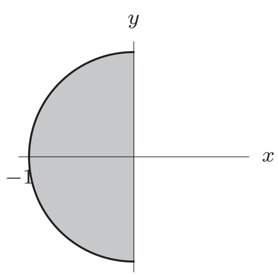


Figure 16.90

In polar coordinates, we have

$$\int_{\pi/2}^{3\pi/2}\int_0^1 r\,\cos\theta\, r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2} \cos\theta\left(\frac{1}{3}r^3\right)\bigg|_0^1 \quad d\theta$$
$$= \frac{1}{3}\int_{\pi/2}^{3\pi/2}\cos\theta \quad d\theta$$
$$= \frac{1}{3}\sin\theta\bigg|_{\pi/2}^{3\pi/2} = \frac{1}{3}(-1-1) = -\frac{2}{3}$$

20. From the given limits, the region of integration is in Figure 16.91.

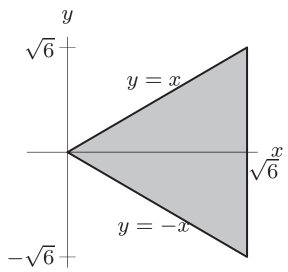


Figure 16.91

In polar coordinates, $-\pi/4 \le \theta \le \pi/4$. Also, $\sqrt{6} = x = r\cos\theta$. Hence, $0 \le r \le \sqrt{6}/\cos\theta$. The integral becomes

$$\int_0^{\sqrt{6}}\int_{-x}^{x} dy\,dx = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{6}/\cos\theta} r\,dr\,d\theta$$
$$= \int_{-\pi/4}^{\pi/4}\left(\frac{r^2}{2}\bigg|_0^{\sqrt{6}/\cos\theta}\right)d\theta = \int_{-\pi/4}^{\pi/4}\frac{6}{2\cos^2\theta}\,d\theta$$
$$= 3\tan\theta\bigg|_{-\pi/4}^{\pi/4} = 3\cdot(1-(-1)) = 6.$$

Notice that we can check this answer because the integral gives the area of the shaded triangular region which is $\frac{1}{2}\cdot\sqrt{6}\cdot(2\sqrt{6}) = 6$.

21. From the given limits, the region of integration is in Figure 16.92.

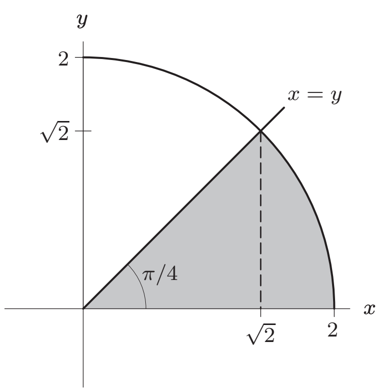


Figure 16.92

So, in polar coordinates, we have,

$$\int_0^{\pi/4}\int_0^2 (r^2\cos\theta\sin\theta)r\,dr\,d\theta = \int_0^{\pi/4}\cos\theta\sin\theta\left(\frac{1}{4}r^4\right)\Big|_0^2\,d\theta$$
$$= 4\int_0^{\pi/4}\frac{\sin(2\theta)}{2}\quad d\theta$$
$$= -\cos(2\theta)\Big|_0^{\pi/4} = 0-(-1)=1.$$

22. (a)

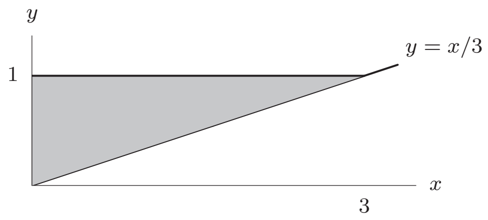


Figure 16.93

(b) $\int_0^1\int_0^{3y} f(x,y)\,dx\,dy.$

(c) For polar coordinates, on the line $y=x/3$, $\tan\theta = y/x = 1/3$, so $\theta=\tan^{-1}(1/3)$. On the y-axis, $\theta=\pi/2$. The quantity r goes from 0 to the line $y=1$, or $r\sin\theta=1$, giving $r=1/\sin\theta$ and $f(x,y)=f(r\cos\theta, r\sin\theta)$. Thus the integral is

$$\int_{\tan^{-1}(1/3)}^{\pi/2}\int_0^{1/\sin\theta} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

23. (a) (i) The two planes contain the z-axis and form the sides of the orange wedge. Since $x\ge 0$, the plane $y=x/\sqrt{3}$ forms an angle of $\pi/6$ radians with the $y=0$ plane, hence $0\le\theta\le\pi/6$. We also have $0\le\rho\le 5$ and $0\le\phi\le\pi$. Hence:

$$\text{Volume} = \int_0^{\pi/6}\int_0^{\pi}\int_0^5 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Evaluating this integral, we get:

$$\text{Volume} = \int_0^{\pi/6}\int_0^{\pi}\left(\sin\phi\left(\frac{5^3}{3}-0\right)\right)d\phi\,d\theta = -\frac{125}{3}\int_0^{\pi/6}(\cos\pi-\cos 0)\;d\theta$$
$$= \frac{250}{3}\frac{\pi}{6} = \frac{125\pi}{9}.$$

(ii) In cylindrical coordinates, the sphere is given by $r^2+z^2=25$.

If we integrate first with respect to z, then z varies between the top and bottom halves of the sphere, or $-\sqrt{25-r^2}\le z\le\sqrt{25-r^2}$. Then, the shadow of the wedge over the xy-plane gives as a circular sector where $0\le r\le 5$, and $0\le\theta\le\pi/6$. Hence:

$$\text{Volume} = \int_0^{\pi/6}\int_0^5\int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta.$$

Evaluating this integral, we get:

$$\text{Volume} = \int_0^{\pi/6}\int_0^5\left(r\left(\sqrt{25-r^2}+\sqrt{25-r^2}\right)\right)dr\,d\theta = \int_0^{\pi/6}\int_0^5\left(2r\sqrt{25-r^2}\right)dr\,d\theta$$
$$= \int_0^{\pi/6}\left(-\frac{(25-r^2)^{3/2}}{3/2}\right)\Big|_0^5\,d\theta$$
$$= \frac{250}{3}\int_0^{\pi/6}d\theta = \frac{250}{3}\frac{\pi}{6} = \frac{125\pi}{9}.$$

If we integrate first with respect to r and consider only the half of the wedge above the xy-plane then, for each value of z, r varies between zero and the radius of the horizontal cross section of the sphere at height z. Hence: $0 \le r \le \sqrt{25-z^2}$. We also have $0 \le z \le 5$ and $0 \le \theta \le \pi/6$. This gives:

$$\text{Volume half wedge } = \int_0^{\pi/6}\int_0^5\int_0^{\sqrt{25-z^2}} r\,dr\,dz\,d\theta,$$

for the top portion of the wedge, or for the full wedge

$$\text{Volume } = 2\int_0^{\pi/6}\int_0^5\int_0^{\sqrt{25-z^2}} r\,dr\,dz\,d\theta.$$

Evaluating the last integral, we get:

$$\text{Volume } = 2\int_0^{\pi/6}\int_0^5\left(\frac{25-z^2}{2}-0\right)dz\,d\theta = \int_0^{\pi/6}\left(\frac{z^3}{3}-25z\right)\Bigg|_0^5 d\theta$$
$$= \frac{250}{3}\int_0^{\pi/6} d\theta = \frac{250}{3}\frac{\pi}{6} = \frac{125\pi}{9}.$$

Spherical coordinates provide the most efficient integration method for calculating the volume of the wedge. This makes sense because the wedge is cut out by spherical fundamental surfaces, and hence, in these coordinates, all integration endpoints are constant.

(b) Since the interior angle of the wedge is $\pi/6$, we need a total of 12 wedges to recover a full sphere of radius 5. Hence:

$$\text{Volume of wedge } = \frac{1}{12}\text{Volume of sphere of radius } 5 = \frac{1}{12}\left(\frac{4}{3}\pi 5^3\right) = \frac{125\pi}{9}.$$

24. Since $r = 2/\cos\theta$ we have $x = r\cos\theta = 2$. Since θ ranges from 0 to $\pi/6$, y ranges from 0 to $y = x/\sqrt{3}$. Converting to Cartesian coordinates we have

$$\int_0^2\int_0^{x/\sqrt{3}} dy\,dx = \int_0^2 y\Big|_0^{x/\sqrt{3}} dx = \int_0^2 \frac{x}{\sqrt{3}}\,dx = \frac{x^2}{2\sqrt{3}}\Bigg|_0^2 = \frac{2}{\sqrt{3}}.$$

25. (a) The region (shaded) is between the circles $x^2+y^2=1$ and $x^2+y^2=4$; see Figure 16.94. The first integral is to the left of the dashed line $x=1$; the second integral is to the right of the dashed line.

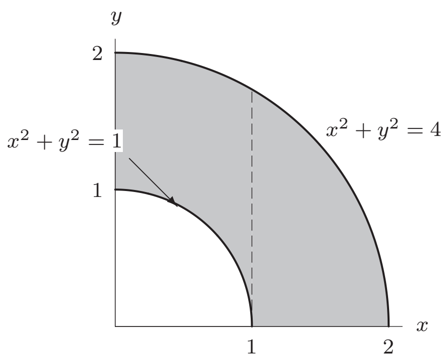


Figure 16.94

(b) Converting to polar coordinates, we find the quantity in part (a) is given by

$$\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx + \int_1^2\int_0^{\sqrt{4-x^2}} x\,dy\,dx = \int_0^{\pi/2}\int_1^2 r\cos\theta\, r\,dr\,d\theta$$
$$= \frac{r^3}{3}\Bigg|_1^2\cdot\sin\theta\Bigg|_0^{\pi/2} = \frac{7}{3}\cdot 1 = \frac{7}{3}.$$

26. The graph of $f(x, y) = 25 - x^2 - y^2$ is an upside down bowl, and the region whose volume we want is contained between the bowl (above) and the xy-plane (below). We must first find the region in the xy-plane where $f(x, y)$ is positive. To do that, we set $f(x, y) \geq 0$ and get $x^2 + y^2 \leq 25$. The disk $x^2 + y^2 \leq 25$ is the region R over which we integrate.

$$\begin{aligned}\text{Volume} = \int_R (25 - x^2 - y^2)\, dA &= \int_0^{2\pi} \int_0^5 (25 - r^2)\, r dr\, d\theta \\ &= \int_0^{2\pi} \left(\frac{25}{2} r^2 - \frac{1}{4} r^4 \right) \Big|_0^5 \, d\theta \\ &= \frac{625}{4} \int_0^{2\pi} d\theta \\ &= \frac{625\pi}{2}\end{aligned}$$

27. First, let's find where the two surfaces intersect.

$$\begin{aligned}\sqrt{8 - x^2 - y^2} &= \sqrt{x^2 + y^2} \\ 8 - x^2 - y^2 &= x^2 + y^2 \\ x^2 + y^2 &= 4\end{aligned}$$

So $z = 2$ at the intersection. See Figure 16.95.

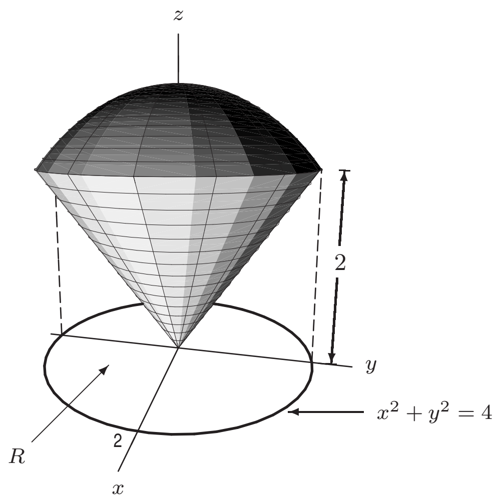


Figure 16.95

The volume of the ice cream cone has two parts. The first part (which is the volume of the cone) is the volume of the solid bounded by the plane $z = 2$ and the cone $z = \sqrt{x^2 + y^2}$. Hence, this volume is given by $\int_R (2 - \sqrt{x^2 + y^2})\, dA$, where R is the disk of radius 2 centered at the origin, in the xy-plane. Using polar coordinates, we have:

$$\begin{aligned}\int_R \left(2 - \sqrt{x^2 + y^2}\right) dA &= \int_0^{2\pi} \int_0^2 (2 - r) \cdot r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\left(r^2 - \frac{r^3}{3} \right) \Big|_0^2 \right] d\theta \\ &= \frac{4}{3} \int_0^{2\pi} d\theta \\ &= 8\pi/3\end{aligned}$$

The second part is the volume of the region above the plane $z = 2$ but inside the sphere $x^2 + y^2 + z^2 = 8$, which is given by $\int_R (\sqrt{8 - x^2 - y^2} - 2)\, dA$ where R is the same disk as before. Now

$$\begin{aligned}
\int_R (\sqrt{8 - x^2 - y^2} - 2)\, dA &= \int_0^{2\pi} \int_0^2 (\sqrt{8 - r^2} - 2) r dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r\sqrt{8 - r^2}\, dr\, d\theta - \int_0^{2\pi} \int_0^2 2r\, dr\, d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{3}(8 - r^2)^{3/2} \Big|_0^2 \right) d\theta - \int_0^{2\pi} r^2 \Big|_0^2 d\theta \\
&= -\frac{1}{3} \int_0^{2\pi} (4^{3/2} - 8^{3/2})\, d\theta - \int_0^{2\pi} 4\, d\theta \\
&= -\frac{1}{3} \cdot 2\pi(8 - 16\sqrt{2}) - 8\pi \\
&= \frac{2\pi}{3}(16\sqrt{2} - 8) - 8\pi \\
&= \frac{8\pi(4\sqrt{2} - 5)}{3}
\end{aligned}$$

Thus, the total volume is the sum of the two volumes, which is $32\pi(\sqrt{2} - 1)/3$.

28. (a) The volume, V, is given by

$$V = \int_{x^2+y^2 \le a^2} e^{-(x^2+y^2)}\, dA.$$

Converting to polar coordinates gives

$$V = \int_0^{2\pi} \int_0^a e^{-r^2}\, r\, dr\, d\theta = \theta \Big|_0^{2\pi} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^a = 2\pi \left(\frac{1}{2} - \frac{1}{2} e^{-a^2} \right) = \pi(1 - e^{-a^2}).$$

(b) As $a \to \infty$, the value of $e^{-a^2} \to 0$, so the volume tends to π.

29. (a) Using polar coordinates, we have

$$\text{Mass } = \int_0^{2\pi} \int_0^3 \frac{1}{r^2 + 1} r\, dr\, d\theta.$$

(b) Integrating with respect to θ first

$$\text{Mass } = 2\pi \int_0^3 \frac{r}{r^2 + 1}\, dr = 2\pi \frac{1}{2} \ln |r^2 + 1| \Big|_0^3 = \pi(\ln 10 - \ln 1) = \pi \ln 10.$$

30. (a)

$$\text{Total Population } = \int_{\pi/2}^{3\pi/2} \int_1^4 \delta(r, \theta)\, r dr\, d\theta.$$

(b) We know that $\delta(r, \theta)$ decreases as r increases, so that eliminates (iii). We also know that $\delta(r, \theta)$ decreases as the x-coordinate decreases, but $x = r\cos\theta$. With a fixed r, x is proportional to $\cos\theta$. So as the x-coordinate decreases, $\cos\theta$ decreases and (i) $\delta(r, \theta) = (4 - r)(2 + \cos\theta)$ best describes this situation.

(c)

$$\begin{aligned}
\int_{\pi/2}^{3\pi/2} \int_1^4 (4 - r)(2 + \cos\theta)\, r dr\, d\theta &= \int_{\pi/2}^{3\pi/2} (2 + \cos\theta)(2r^2 - \frac{1}{3} r^3) \Big|_1^4 d\theta \\
&= 9 \int_{\pi/2}^{3\pi/2} (2 + \cos\theta)\, d\theta \\
&= 9 \Big[2\theta + \sin\theta \Big]_{\pi/2}^{3\pi/2} \\
&= 18(\pi - 1) \approx 39
\end{aligned}$$

Thus, the population is around 39,000.

31. The density function is given by

$$\rho(r) = 10 - 2r$$

where r is the distance from the center of the disk. So the mass of the disk in grams is

$$\begin{aligned}\int_R \rho(r)\, dA &= \int_0^{2\pi}\int_0^5 (10-2r) r dr\, d\theta \\ &= \int_0^{2\pi} \left[5r^2 - \frac{2}{3}r^3\right]_0^5 d\theta \\ &= \int_0^{2\pi} \frac{125}{3}\, d\theta = \frac{250\pi}{3} (\text{grams})\end{aligned}$$

32. The charge density is $\delta = k/r$, where k is a constant.

$$\text{Total charge} = \int_{\text{Disk}} \delta\, dA = \int_0^R \int_0^{2\pi} \frac{k}{r} r\, d\theta\, dr = k\int_0^R\int_0^{2\pi} d\theta\, dr = k\int_0^R 2\pi\, dr = 2k\pi R.$$

Thus the total charge is proportional to R with constant of proportionality $2k\pi$.

33. (a) The curve $r = 1/(2\cos\theta)$ or $r\cos\theta = 1/2$ is the line $x = 1/2$. The curve $r = 1$ is the circle of radius 1 centered at the origin. See Figure 16.96.

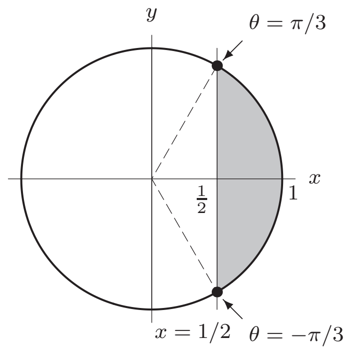


Figure 16.96

(b) The line intersects the circle where $2\cos\theta = 1$, so $\theta = \pm\pi/3$. From Figure 16.96 we see that

$$\text{Area } = \int_{-\pi/3}^{\pi/3}\int_{1/(2\cos\theta)}^{1} r\, dr\, d\theta.$$

Evaluating gives

$$\begin{aligned}\text{Area } &= \int_{-\pi/3}^{\pi/3} \left(\frac{r^2}{2}\Bigg|_{1/(2\cos\theta)}^{1}\right) d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(1 - \frac{1}{4\cos^2\theta}\right) d\theta \\ &= \frac{1}{2}\left(\theta - \frac{\tan\theta}{4}\right)\Bigg|_{-\pi/3}^{\pi/3} = \frac{1}{2}\left(2\frac{\pi}{3} - 2\frac{\sqrt{3}}{4}\right) = \frac{4\pi - 3\sqrt{3}}{12}.\end{aligned}$$

34. (a) The circle $r = 2\cos\theta$ has radius 1 and is centered at $(1, 0)$; the circle $r = 1$ has radius 1 and is centered at the origin. See Figure 16.97.

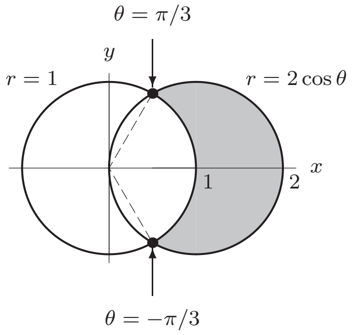


Figure 16.97

(b) The circles intersect where $2\cos\theta = 1$, so $\theta = \pm\pi/3$. From Figure 16.97 we see that

$$\text{Area } = \int_{-\pi/3}^{\pi/3} \int_{1}^{2\cos\theta} r\,dr\,d\theta.$$

Evaluating gives

$$\begin{aligned}\text{Area } &= \int_{-\pi/3}^{\pi/3} \left(\frac{r^2}{2}\bigg|_1^{2\cos\theta} \right) d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta \\ &= \frac{1}{2}\left(2\cos\theta\sin\theta + 2\theta - \theta\right)\bigg|_{-\pi/3}^{\pi/3} = \frac{1}{2}\left(4\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + 2\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}.\end{aligned}$$

35. (a) We must first decide where to put the origin. We locate the origin at the center of one disk and locate the center of the second disk at the point $(1, 0)$. See Figure 16.98. (Other choices of origin are possible.)

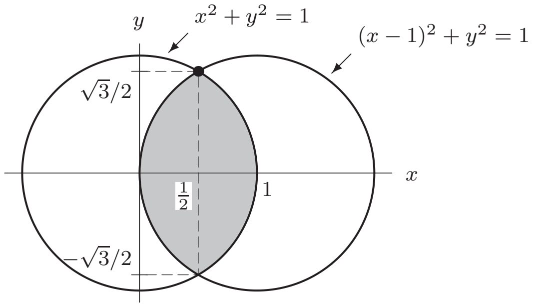


Figure 16.98

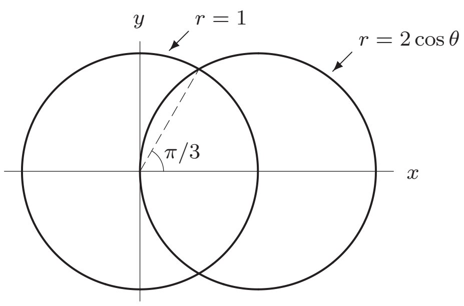


Figure 16.99

By symmetry, the points of intersection of the circles are half-way between the centers, at $x = 1/2$. The y-values at these points are given by

$$y = \pm\sqrt{1 - x^2} = \pm\sqrt{1 - \left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}.$$

We integrate in the x-direction first, so that it is not necessary to set up two integrals. The right-side of the circle $x^2 + y^2 = 1$ is given by

$$x = \sqrt{1 - y^2}.$$

The left side of the circle $(x - 1)^2 + y^2 = 1$ is given by

$$x = 1 - \sqrt{1 - y^2}.$$

Thus the area of overlap is given by

$$\text{Area} = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{1-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx dy.$$

(b) In polar coordinates, the circle centered at the origin has equation $r = 1$. See Figure 16.99. The other circle, $(x-1)^2 + y^2 = 1$, can be written as

$$x^2 - 2x + 1 + y^2 = 1$$
$$x^2 + y^2 = 2x,$$

so its equation in polar coordinates is

$$r^2 = 2r\cos\theta,$$

and, since $r \neq 0$,

$$r = 2\cos\theta.$$

At the top point of intersection of the two circles, $x = 1/2$, $y = \sqrt{3}/2$, so $\tan\theta = \sqrt{3}$, giving $\theta = \pi/3$.

Figure 16.99 shows that if we integrate with respect to r first, we have to write the integral as the sum of two integrals. Thus, we integrate with respect to θ first. To do this, we rewrite

$$r = 2\cos\theta \qquad \text{as} \qquad \theta = \arccos\left(\frac{r}{2}\right).$$

This gives the top half of the circle; the bottom half is given by

$$\theta = -\arccos\left(\frac{r}{2}\right).$$

Thus the area is given by

$$\text{Area} = \int_0^1 \int_{-\arccos(r/2)}^{\arccos(r/2)} r\, d\theta dr.$$

36. The required region is shaded in Figure 16.100. The limits of integration will correspond to the two points where the two curves intersect. These points are given by solving the equation $2+3\cos\theta = 2$. Simplifying this equation gives $\cos\theta = 0$ so the required solutions are $\theta = \pi/2$ and $\theta = -\pi/2$ as shown in Figure 16.100.

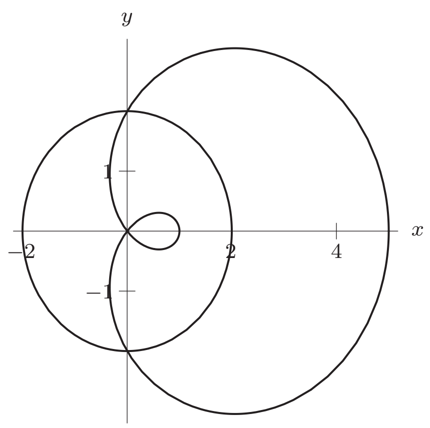


Figure 16.100

The required area is given by

$$\int_R 1\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_2^{2+3\cos\theta} r\, dr\, d\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2+3\cos\theta)^2 - 2^2\, d\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12\cos\theta + 9\cos^2\theta\, d\theta$$

$$= \frac{3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta + 3\cos^2\theta)\,d\theta$$

$$= \frac{3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos\theta + \frac{3}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{3}{2}\left(4\sin\theta + \frac{3}{2}\theta + \frac{3}{4}\sin 2\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 12 + \frac{9\pi}{4}.$$

Strengthen Your Understanding

37. The part of the boundary of R corresponding to $x = 1$ in terms of polar coordinates is not $r = 1$. Rather, it is $r\cos\theta = 1$, which gives $r = 1/\cos\theta$. The angle between $y = 0$ and $y = x$ is $\pi/4$, so we have

$$\int_R x\,dA = \int_0^{\pi/4}\int_0^{1/\cos\theta} (r\cos\theta) r\,dr\,d\theta.$$

38. When converting to polar coordinates, we need an extra factor of r, because $dA = r\,dr\,d\theta$. Thus, we should have:

$$\int_R (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

39. Any region that is a sector of a circle centered at the origin suggests polar coordinates. An example is the quarter disk $0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$.

40. Any integrand that is a function of $\sqrt{x^2+y^2}$ suggests polar coordinates. For example, let $f(x,y) = 1/\sqrt{x^2+y^2}$.

41. (a), (c), (e)

42. The region lies in the first quadrant and is bounded by four lines. The equations $r = 1/\sin\theta$ and $r = 4/\sin\theta$ are the horizontal lines $y = r\sin\theta = 1$ and $y = r\sin\theta = 4$. The equation $\theta = \pi/4$ gives the line $y = x$, and the equation $\theta = \pi/2$ gives the y-axis.

Solutions for Section 16.5

Exercises

1. (a) is (IV); (b) is (II); (c) is (VII); (d) is (VI); (e) is (III); (f) is (V).

2. The plane has equation $\theta = \pi/4$.

3. The top half of the sphere has equation $z = \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$.

4. The cone has equation $z = r$.

5. The cone has equation $\phi = \pi/4$.

6. The plane has equation $\rho\cos\phi = 10$ or $\rho = 10/\cos\phi$.

7. The plane has equation $\rho\cos\phi = 4$ or $\rho = 4/\cos\phi$.

8.

$$\int_W f\,dV = \int_{-1}^{3}\int_0^{2\pi}\int_0^1 (\sin(r^2))\,r dr\,d\theta\,dz$$

$$= \int_{-1}^{3}\int_0^{2\pi} (-\frac{1}{2}\cos r^2)\Big|_0^1\,d\theta\,dz$$

$$= -\frac{1}{2}\int_{-1}^{3}\int_{0}^{2\pi}(\cos 1 - \cos 0)\,d\theta\,dz$$
$$= -\pi\int_{-1}^{3}(\cos 1 - 1)\,dz = -4\pi(\cos 1 - 1) = 4\pi(1-\cos 1)$$

9.

$$\begin{aligned}\int_W f\,dV &= \int_{-1}^{1}\int_{\pi/4}^{3\pi/4}\int_{0}^{4}(r^2+z^2)\,rdr\,d\theta\,dz\\ &= \int_{-1}^{1}\int_{\pi/4}^{3\pi/4}(64+8z^2)\,d\theta\,dz\\ &= \int_{-1}^{1}\frac{\pi}{2}(64+8z^2)\,dz\\ &= 64\pi + \frac{8}{3}\pi = \frac{200}{3}\pi\end{aligned}$$

10.

$$\begin{aligned}\int_W f\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_1^2(\sin\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi/4}\int_1^2\rho^2\sin^2\phi\,d\rho\,d\phi\,d\theta\\ &= \frac{7}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin^2\phi\,d\phi\,d\theta\\ &= \frac{7}{3}\int_0^{2\pi}\int_0^{\pi/4}\frac{1-\cos 2\phi}{2}\,d\phi\,d\theta\\ &= \frac{7}{6}\int_0^{2\pi}(\phi - \frac{1}{2}\sin 2\phi)\Big|_0^{\pi/4}\,d\theta\\ &= \frac{7}{6}\int_0^{2\pi}(\frac{\pi}{4}-\frac{1}{2})\,d\theta\\ &= \frac{7}{6}\cdot 2\pi(\frac{\pi}{4}-\frac{1}{2}) = \frac{7\pi(\pi-2)}{12}\end{aligned}$$

11. We have:

$$\begin{aligned}\int_W f\,dV &= \int_0^5\int_0^{2\pi}\int_{\pi/2}^{\pi}\frac{1}{\rho}\cdot\rho^2\sin\phi\,d\phi\,d\theta\,d\rho\\ &= \int_0^5\int_0^{2\pi}\int_{\pi/2}^{\pi}\rho\sin\phi\,d\phi\,d\theta\,d\rho\\ &= \int_0^5\int_0^{2\pi}\rho\,d\theta\,d\rho\\ &= 2\pi\int_0^5\rho\,d\rho = 25\pi\end{aligned}$$

Note that the integral is improper, but it can be shown that the result is correct.

12. Using Cartesian coordinates, we get:

$$\int_0^3\int_0^1\int_0^5 f\,dz\,dy\,dx$$

13. Using cylindrical coordinates, we get:

$$\int_0^1 \int_0^{2\pi} \int_0^4 f \cdot r dr\, d\theta\, dz$$

14. Using cylindrical coordinates, we get:

$$\int_0^4 \int_0^{\pi/2} \int_0^2 f \cdot r dr\, d\theta\, dz$$

15. Using spherical coordinates, we get:

$$\int_0^{\pi} \int_0^{\pi} \int_2^3 f \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

16. Using spherical coordinates, we get:

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^3 f \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

17. We use Cartesian coordinates, oriented as shown in Figure 16.101. The slanted top has equation $z = mx$, where m is the slope in the x-direction, so $m = 1/5$. Then if f is an arbitrary function, the triple integral is

$$\int_0^5 \int_0^2 \int_0^{x/5} f\, dz\, dy\, dx.$$

Other answers are possible.

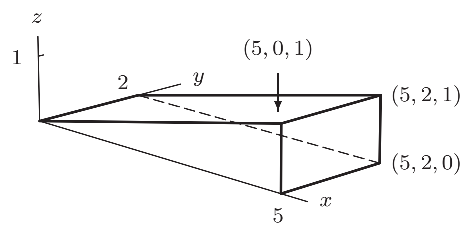


Figure 16.101

18. We choose cylindrical coordinates oriented as in Figure 16.102. The cone has equation $z = r$. Since we have a half cone scooped out of a half cylinder, θ varies between 0 and π. Thus, if f is an arbitrary function, the integral is

$$\int_0^{\pi} \int_0^2 \int_0^r fr\, dzdrd\theta.$$

Other answers are possible.

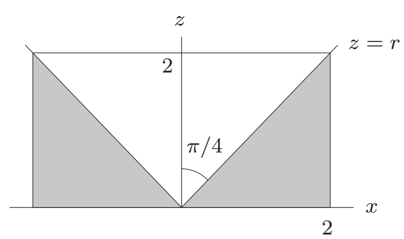


Figure 16.102

Problems

19. In cylindrical coordinates, the sphere has equation $r^2 + z^2 = K^2$. Thus

$$V = \int_0^{2\pi} \int_0^K \int_{-\sqrt{K^2-r^2}}^{\sqrt{K^2-r^2}} r\,dz\,dr\,d\theta.$$

20. In spherical coordinates, the sphere has equation $\rho = K$. Thus

$$V = \int_0^{\pi} \int_0^K \int_0^{2\pi} \rho^2 \sin\phi\,d\theta\,d\rho\,d\phi.$$

21. We use cylindrical coordinates. The cone has radius $r = 2$ when $z = 4$, so its equation is $z = 2r$. Thus, the integral is

$$\int_0^{2\pi} \int_0^2 \int_{2r}^4 f(r,\theta,z) r\,dz\,dr\,d\theta.$$

22. We use spherical coordinates. The cone has radius 2 when $z = 4$, so $\rho \sin\phi = 2$ when $\rho\cos\phi = 4$. Thus $\tan\phi = 1/2$, so $\phi = \arctan(1/2)$. The top of the cone, $z = 4$, is given by $\rho\cos\phi = 4$. Thus, the integral is

$$\int_0^{2\pi} \int_0^{\arctan(1/2)} \int_0^{4/\cos\phi} g(\rho,\phi,\theta)\rho^2 \sin\theta\,d\rho\,d\phi\,d\theta.$$

23. In rectangular coordinates, a cone has equation $z = k\sqrt{x^2+y^2}$ for some constant k. Since $z = 4$ when $\sqrt{x^2+y^2} = \sqrt{2^2} = 2$, we have $k = 2$. Thus, the integral is

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2\sqrt{x^2+y^2}}^4 h(x,y,z)\,dz\,dy\,dx.$$

24. (a) In Cartesian coordinates, the bottom half of the sphere $x^2 + y^2 + z^2 = 1$ is given by $z = -\sqrt{1-x^2-y^2}$. Thus

$$\int_W dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^0 dz\,dy\,dx.$$

(b) In cylindrical coordinates, the sphere is $r^2 + z^2 = 1$ and the bottom half is given by $z = -\sqrt{1-r^2}$. Thus

$$\int_W dV = \int_0^{\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^0 r\,dz\,dr\,d\theta.$$

(c) In spherical coordinates, the sphere is $\rho = 1$. Thus,

$$\int_W dV = \int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

25. (a) Since the cone has a right angle at its vertex, it has equation

$$z = \sqrt{x^2+y^2}.$$

The sphere has equation $x^2 + y^2 + z^2 = 1$, so the top half is given by

$$z = \sqrt{1-x^2-y^2}.$$

The cone and the sphere intersect in the circle

$$x^2 + y^2 = \frac{1}{2}, \qquad z = \frac{1}{\sqrt{2}}.$$

See Figure 16.103. Thus

$$\int_W dV = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{(1/2)-x^2}}^{\sqrt{(1/2)-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

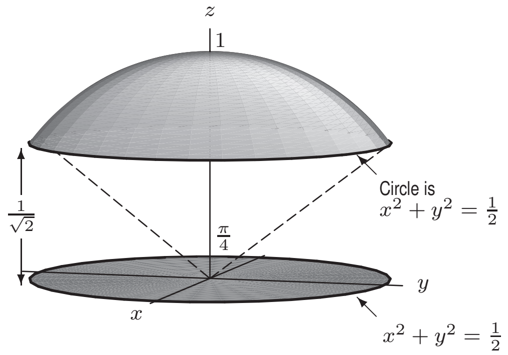


Figure 16.103

(b) In cylindrical coordinates, the cone has equation $z = r$ and the sphere has equation $z = \sqrt{1-r^2}$. Thus

$$\int_W dV = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \int_r^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta.$$

(c) In spherical coordinates, the cone has equation $\phi = \pi/4$ and the sphere is $\rho = 1$. Thus

$$\int_W dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

26. (a) Since the cone has a right angle at its vertex, it has equation

$$z = \sqrt{x^2+y^2}.$$

Figure 16.104 shows the plane with equation $z = 1/\sqrt{2}$. The plane and the cone intersect in the circle $x^2+y^2 = 1/2$. Thus,

$$\int_W dV = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{(1/2)-x^2}}^{\sqrt{(1/2)-x^2}} \int_{\sqrt{x^2+y^2}}^{1/\sqrt{2}} dz\, dy\, dx.$$

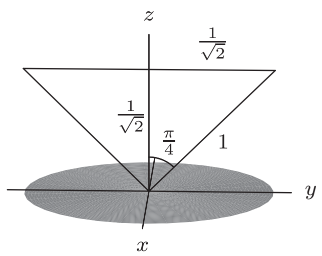


Figure 16.104

(b) In cylindrical coordinates the cone has equation $z = r$, so

$$\int_W dV = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \int_r^{1/\sqrt{2}} r\, dz\, dr\, d\theta.$$

(c) In spherical coordinates, the cone has equation $\phi = \pi/4$ and the plane $z = 1/\sqrt{2}$ has equation $\rho\cos\phi = 1/\sqrt{2}$. Thus

$$\int_W dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{1/(\sqrt{2}\cos\phi)} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

27. (a) In cylindrical coordinates, the cone is $z = r$ and the sphere is $r^2 + z^2 = 4$. The surfaces intersect where $z^2 + z^2 = 2z^2 = 4$. So $z = \sqrt{2}$ and $r = \sqrt{2}$.

$$\text{Volume} = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r\, dzdrd\theta.$$

(b) In spherical coordinates, the cone is $\phi = \pi/4$ and the sphere is $\rho = 2$.

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} \rho^2 \sin\phi\, d\rho d\phi d\theta.$$

28. (a) $\int_0^{2\pi} \int_0^{\pi} \int_1^{2} \rho^2 \sin\phi\, d\rho d\phi d\theta.$

(b) $\int_0^{2\pi} \int_0^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dzdrd\theta - \int_0^{2\pi} \int_0^{1} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dzdrd\theta.$

29. We want the volume of the region above the cone $\phi = \pi/3$ and below the sphere $\rho = 3$:

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{3} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

The order of integration can be altered and other coordinates can be used.

30. We want the volume of the region between the sphere $\rho = 3$ and the cone $z = r$. The sphere can also be written $x^2 + y^2 + z^2 = 3^2$ or $r^2 + z^2 = 9$. The cone can also be written as $\phi = \pi/4$.

The sphere cuts the cone $z = r$ in the circle $2r^2 = 9$, or $r = 3/\sqrt{2}$, lying in the plane $z = 3/\sqrt{2}$.

In cylindrical coordinates,

$$V = \int_0^{2\pi} \int_0^{3/\sqrt{2}} \int_r^{\sqrt{9-r^2}} r\, dz\, dr\, d\theta.$$

In spherical coordinates

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{3} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

The order of integration can be altered and other coordinates can be used.

31. In cylindrical coordinates, the region is the half cylinder given by $5 \le z \le 10$, $\sqrt{2} \le r \le \sqrt{3}$, $0 \le \theta \le \pi$. Thus

$$V = \int_0^{\pi} \int_{\sqrt{2}}^{\sqrt{3}} \int_5^{10} r\, dz\, dr\, d\theta.$$

The order of integration can be altered and other coordinates can be used.

32. The cone can be written $z = r$, and the first quadrant of the xy-plane is given by $0 \le \theta \le \pi/2$. The region $x^2 + y^2 \le 7$ is given by $r \le \sqrt{7}$. Thus

$$V = \int_0^{\pi/2} \int_0^{\sqrt{7}} \int_0^{r} r\, dz\, dr\, d\theta.$$

The order of integration can be altered and other coordinates can be used.

33. We use cylindrical coordinates since the sphere $x^2 + y^2 + z^2 = 10$, or $r^2 + z^2 = 10$, and the plane $z = 1$ can both be simply expressed. The plane cuts the sphere in the circle $r^2 + 1^2 = 10$, or $r = 3$. Thus

$$V = \int_0^{2\pi} \int_0^{3} \int_1^{\sqrt{10-r^2}} r\, dz\, dr\, d\theta,$$

or

$$V = \int_0^{2\pi} \int_1^{\sqrt{10}} \int_0^{\sqrt{10-z^2}} r\, dr\, dz\, d\theta.$$

Order of integration can be altered and other coordinates can be used.

34. In spherical coordinates, the cone $z = r$ is given by $\phi = \pi/4$ and the sphere $x^2 + y^2 + z^2 = 8$ is given by $\rho = \sqrt{8}$. Since ϕ is measured from the positive z-axis, the region we are interested in has $\pi/4 \le \phi \le \pi/2$. Thus

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

The order of integration can be altered and other coordinates can be used.

35. (a) In cylindrical coordinates, the cone has equation $z = \sqrt{3}r$. When $z = 1$, we have $r = 1/\sqrt{3}$, so

$$\text{Volume } = \int_0^{2\pi} \int_0^{1/\sqrt{3}} \int_{\sqrt{3}r}^{1} r \, dz \, dr \, d\theta.$$

(b) Evaluating gives

$$\begin{aligned} \text{Volume } &= 2\pi \int_0^{1\sqrt{3}} z\Big|_{\sqrt{3}r}^{1} r \, dr = 2\pi \int_0^{1\sqrt{3}} (1 - \sqrt{3}r) r \, dr \\ &= 2\pi \left(\frac{r^2}{2} - \frac{\sqrt{3}}{3} r^3 \right) \Big|_0^{1/\sqrt{3}} = 2\pi \left(\frac{1}{6} - \frac{1}{9} \right) = \frac{\pi}{9}. \end{aligned}$$

36. For $x^2 + y^2 \le 1$, the cone is below the plane $z = 10 + x$. In cylindrical coordinates, the plane is $z = 10 + r\cos\theta$, and the cone is $z = r$. Thus

$$\begin{aligned} \text{Volume } &= \int_0^{2\pi} \int_0^1 \int_r^{10+r\cos\theta} r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 rz \Big|_r^{10+r\cos\theta} dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (10r + r^2\cos\theta - r^2) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(5r^2 + \frac{r^3}{3}\cos\theta - \frac{r^3}{3} \right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \left(5 + \frac{1}{3}\cos\theta - \frac{1}{3} \right) d\theta \\ &= \left(\frac{14}{3}\theta + \frac{1}{3}\sin\theta \right) \Big|_0^{2\pi} = \frac{28\pi}{3}. \end{aligned}$$

37. The cone is centered along the positive x-axis and intersects the sphere in the circle

$$\begin{aligned} (y^2 + z^2) + y^2 + z^2 &= 4 \\ y^2 + z^2 &= 2. \end{aligned}$$

We use spherical coordinates with ϕ measured from the x-axis and θ measured in the yz-plane. (Alternatively, the volume we want is equal to the volume between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 4$.) The cone is given by $\phi = \pi/4$. The sphere has equation $\rho = 2$. Thus

$$\begin{aligned} \text{Volume } &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3} \sin\phi \Big|_0^2 d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{8}{3} \sin\phi \, d\phi \, d\theta \end{aligned}$$

$$= \int_0^{2\pi} -\frac{8}{3}\cos\phi\Big|_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{8}{3}\left(1-\frac{1}{\sqrt{2}}\right) d\theta$$
$$= \frac{16\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right).$$

38. Using cylindrical coordinates, the equation of the sphere is $r^2+z^2=4$. The top of the sphere has equation $z=\sqrt{4-r^2}$. When $z=1$ we have $r=\sqrt{3}$. Figure 16.105 shows the limits of integration on the integral.

$$\text{Volume} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^{\sqrt{3}} rz\Big|_1^{\sqrt{4-r^2}} dr = 2\pi\int_0^{\sqrt{3}} (r\sqrt{4-r^2}-r)\,dr$$
$$= 2\pi\left(\frac{(4-r^2)^{3/2}}{-3} - \frac{r^2}{2}\right)\Big|_0^{\sqrt{3}} = \frac{5}{3}\pi.$$

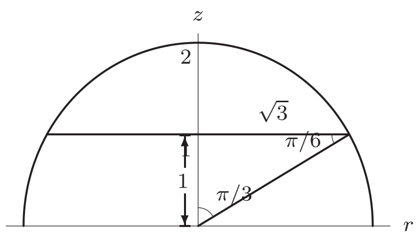


Figure 16.105

39. Use cylindrical coordinates: when $r^2=x^2+y^2=1$, then $x^2+y^2+z^2=1+z^2=2$ so $z=\pm1$. The region W is shown in Figure 16.106.

$$\int_W (x^2+y^2)\,dV = \int_{-1}^1\int_0^{2\pi}\int_1^{\sqrt{2-z^2}} r^2\cdot r\,drd\theta dz$$
$$= \int_{-1}^1\int_0^{2\pi} \frac{r^4}{4}\Big|_1^{\sqrt{2-z^2}} d\theta dz = \frac{1}{4}\int_{-1}^1\int_0^{2\pi}\left((2-z^2)^2-1\right) d\theta dz$$
$$= \frac{2\pi}{4}\int_{-1}^1 \left(3-4z^2+z^4\right) dz$$
$$= \frac{\pi}{2}\left(3z-\frac{4}{3}z^3+\frac{z^5}{5}\right)\Big|_{-1}^1 = \frac{28\pi}{15}.$$

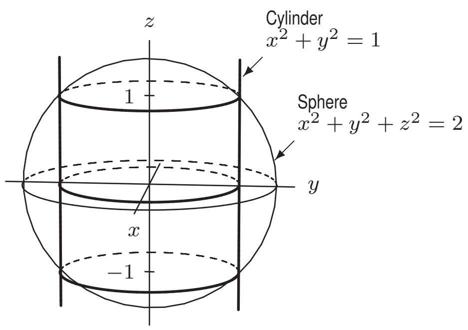


Figure 16.106

40. The region whose volume we want is shown in Figure 16.107:

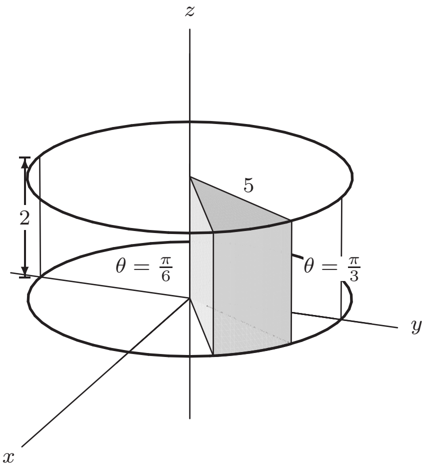


Figure 16.107

Using cylindrical coordinates, the volume is given by the integral:

$$\begin{aligned} V &= \int_0^2 \int_{\pi/6}^{\pi/3} \int_0^5 r\,dr\,d\theta\,dz \\ &= \int_0^2 \int_{\pi/6}^{\pi/3} \frac{r^2}{2}\bigg|_0^5 \,d\theta\,dz \\ &= \frac{25}{2}\int_0^2 \int_{\pi/6}^{\pi/3} d\theta\,dz \\ &= \frac{25}{2}\int_0^2 \left(\frac{\pi}{3}-\frac{\pi}{6}\right) dz \\ &= \frac{25}{2}\cdot\frac{\pi}{6}\cdot 2 = \frac{25\pi}{6}. \end{aligned}$$

41. Orient the cone as shown in Figure 16.108 and use cylindrical coordinates with the origin at the vertex of the cone. Since the angle at the vertex of the cone is a right angle, the angles AOB and COB are both $\pi/4$. Thus, $OB = 5\cos\pi/4 = 5/\sqrt{2}$. The curved surface of the cone has equation $z = r$, so

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi}\int_0^{5/\sqrt{2}}\int_r^{5/\sqrt{2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{5/\sqrt{2}} rz\bigg|_{z=r}^{z=5/\sqrt{2}} dr\,d\theta = \int_0^{2\pi}\int_0^{5/\sqrt{2}} r\left(\frac{5}{\sqrt{2}} - r\right) dr\,d\theta \\ &= \theta\bigg|_0^{2\pi}\left(\frac{5}{\sqrt{2}}\frac{r^2}{2} - \frac{r^3}{3}\right)\bigg|_0^{5/\sqrt{2}} = 2\pi\left(\frac{5}{\sqrt{2}}\cdot\frac{5^2}{2^2} - \frac{5^3}{2\cdot 3\cdot\sqrt{2}}\right) \\ &= 2\pi\cdot\frac{5^3}{2\sqrt{2}}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{5^3\pi}{6\sqrt{2}} = 46.28\text{ cm}^3. \end{aligned}$$

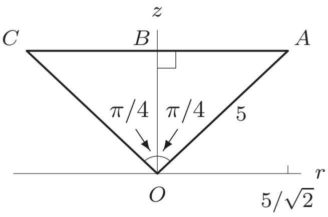


Figure 16.108

42. (a) The angle ϕ takes on values in the range $0 \le \phi \le \pi$. Thus, $\sin\phi$ is nonnegative everywhere in W_1, and so its integral is positive.

(b) The function ϕ is symmetric across the xy plane, such that for any point (x, y, z) in W_1, with $z \neq 0$, the point $(x, y, -z)$ has a $\cos\phi$ value with the same magnitude but opposite sign of the $\cos\phi$ value for (x, y, z). Furthermore, if $z = 0$, then (x, y, z) has a $\cos\phi$ value of 0. Thus, with $\cos\phi$ positive on the top half of the sphere and negative on the bottom half, the integral will cancel out and be equal to zero.

43. (a) The integral is negative. In W_2, we have $0 < z < 1$. Thus, $z^2 - z$ is negative throughout W_2 and thus its integral is negative.

(b) On the top half of the sphere, z is nonnegative, but x can be both positive and negative. Thus, since W_2 is symmetric with respect to the yz plane, the contribution of a point (x, y, z) will be canceled out by its reflection $(-x, y, z)$. Thus, the integral is zero.

44. We must first decide on coordinates. We pick cylindrical coordinates with the z-axis along the axis of the cylinders. The insulation stretches from $z = 0$ to $z = l$. See Figure 16.109. The volume is given by the integral

$$\text{Volume} = \int_0^{2\pi}\int_0^l\int_a^{a+h} r\,dr dz d\theta.$$

Evaluating the integral gives

$$\text{Volume} = \int_0^{2\pi}\int_0^l \left.\frac{r^2}{2}\right|_a^{a+h} dz d\theta = 2\pi z\Big|_0^l\left(\frac{(a+h)^2}{2} - \frac{a^2}{2}\right) = \pi l((a+h)^2 - a^2).$$

To check our answer, notice that the volume is the difference between the volume of two cylinders of radius a and $a + h$. These cylinders have volumes $\pi l(a + h)^2$ and $\pi l a^2$, respectively.

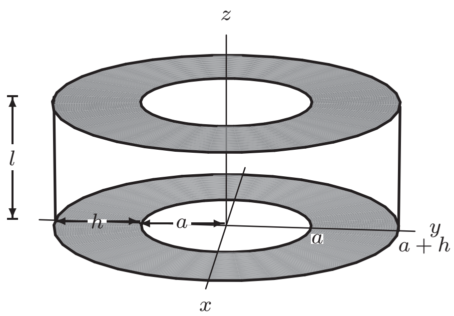


Figure 16.109

45. The plane $(x/p) + (y/q) + (z/r) = 1$ cuts the axes at the points $(p, 0, 0)$; $(0, q, 0)$; $(0, 0, r)$. Since p, q, r are positive, the region between this plane and the coordinate planes is a pyramid in the first octant. Solving for z gives

$$z = r\left(1 - \frac{x}{p} - \frac{y}{q}\right) = r - \frac{rx}{p} - \frac{ry}{q}.$$

The volume, V, is given by the double integral

$$V = \int_R \left(r - \frac{rx}{p} - \frac{ry}{q}\right) dA,$$

where R is the region shown in Figure 16.110. Thus

$$\begin{aligned}
V &= \int_0^p \int_0^{q-qx/p} \left(r - \frac{rx}{p} - \frac{ry}{q}\right) dydx \\
&= \int_0^p \left(\left(ry - \frac{rxy}{p} - \frac{ry^2}{2q}\right)\Bigg|_{y=0}^{y=q-qx/p} \right) dx \\
&= \int_0^p \left(r\left(q - \frac{qx}{p}\right) - \frac{r}{p}x\left(q - \frac{qx}{p}\right) - \frac{r}{2q}\left(q - \frac{qx}{p}\right)^2 \right) dx \\
&= \int_0^p rq - \frac{2rqx}{p} + \frac{rqx^2}{p^2} - \frac{rq^2}{2q}\left(1 - \frac{2x}{p} + \frac{x^2}{p^2}\right) dx \\
&= \left(rqx - \frac{rqx^2}{p} + \frac{rqx^3}{p^2 3} - \frac{rqx}{2} + \frac{rqx^2}{p2} - \frac{rqx^3}{2p^2 3} \right)\Bigg|_0^p \\
&= pqr - pqr + \frac{pqr}{3} - \frac{pqr}{2} + \frac{pqr}{2} - \frac{pqr}{6} = \frac{pqr}{6}.
\end{aligned}$$

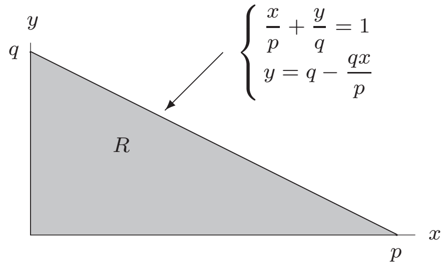


Figure 16.110

46. We must first decide on coordinates. We imagine the vertex of the cone downward, at the origin, with the flat base in the plane $z = h$, as in Figure 16.111. Then, using cylindrical coordinates as in Figure 16.112, we see that the curved surface of the cone has equation $z = hr/a$. Thus the volume is given by

$$\text{Volume} = \int_0^{2\pi} \int_0^a \int_{hr/a}^h r\, dzdrd\theta.$$

Evaluating gives

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^a rz\Bigg|_{z=hr/a}^{z=h} drd\theta = \int_0^{2\pi} \int_0^a \left(hr - \frac{hr^2}{a}\right) drd\theta \\
&= 2\pi\left(\frac{hr^2}{2} - \frac{hr^3}{3a}\right)\Bigg|_0^a = 2\pi h\left(\frac{a^2}{2} - \frac{a^2}{3}\right) = \frac{\pi h a^2}{3}.
\end{aligned}$$

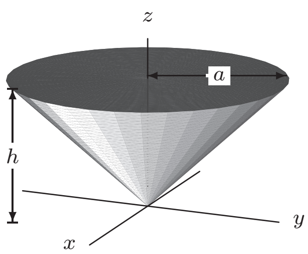


Figure 16.111

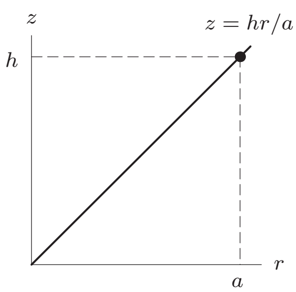


Figure 16.112

47. We must first decide on coordinates. We pick spherical coordinates with the common center of the two spheres as the origin. We imagine the half-melon with the flat side horizontal and the positive z-axis going through the curved surface. See Figure 16.113. The volume is given by the integral

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

Evaluating gives

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \left. \frac{\rho^3}{3} \right|_{p=a}^{p=b} d\phi d\theta = 2\pi(-\cos\phi)\Big|_0^{\pi/2} \left(\frac{b^3}{3} - \frac{a^3}{3} \right) = \frac{2\pi}{3}(b^3 - a^3).$$

To check our answer, notice that the volume is the difference between the volumes of two half spheres of radius a and b. These half spheres have volumes $2\pi b^3/3$ and $2\pi a^3/3$, respectively.

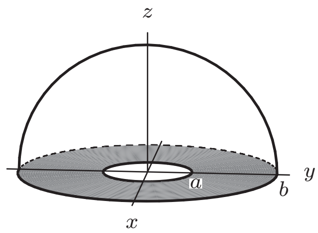


Figure 16.113

48. (a) We use the axes shown in Figure 16.114. Then the sphere is given by $r^2 + z^2 = 25$, so

$$\text{Volume} = \int_0^{2\pi} \int_1^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r \, dz dr d\theta.$$

(b) Evaluating gives

$$\begin{aligned} \text{Volume} = 2\pi \int_1^5 rz \Big|_{z=-\sqrt{25-r^2}}^{z=\sqrt{25-r^2}} dr &= 2\pi \int_1^5 2r\sqrt{25-r^2}\, dr \\ &= 2\pi \left(-\frac{2}{3}\right) (25-r^2)^{3/2} \Big|_1^5 \\ &= \frac{4\pi}{3}(24)^{3/2} = 64\sqrt{6}\pi = 492.5 \text{ mm}^3. \end{aligned}$$

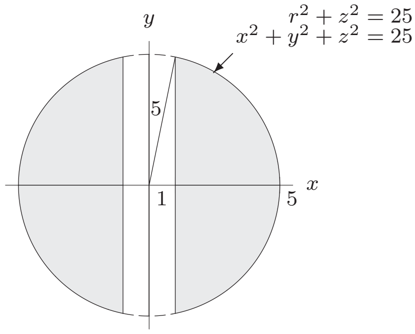


Figure 16.114

49. (a) To find the mass, we integrate the density over the region, W. Converting to cylindrical coordinates, the surface of the pile is $z = 2 - r^2$, so we have

$$\text{Mass } = \int_W (2-z)\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{2-r^2} (2-z)r\,dz\,dr\,d\theta.$$

(b) Evaluating gives

$$\begin{aligned}\text{Mass } &= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(2z - \frac{z^2}{2}\right)\Bigg|_0^{2-r^2} r\,dr\,d\theta \\ &= 2\pi \int_0^{\sqrt{2}} \left(2(2-r^2) - \frac{(2-r^2)^2}{2}\right) r\,dr \\ &= 2\pi \int_0^{\sqrt{2}} 4 - 2r^2 - 2r + 2r^3 - \frac{r^5}{2}\,dr = 2\pi\left(4r - \frac{2r^3}{3} - r^2 + \frac{r^4}{2} - \frac{r^6}{12}\Bigg|_0^{\sqrt{2}}\right) = \left(\frac{16\sqrt{2}}{3} - \frac{4}{3}\right)\pi.\end{aligned}$$

50. The density function can be rewritten as $\delta(\rho,\phi,\theta) = \rho$. So the mass is

$$\begin{aligned}\int_W \delta(P)\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4} \frac{81}{4}\sin\phi\,d\phi\,d\theta \\ &= \frac{81}{4}\int_0^{2\pi} (-\frac{\sqrt{2}}{2}+1)\,d\theta \\ &= \frac{81}{4}\cdot 2\pi\cdot(-\frac{\sqrt{2}}{2}+1) = \frac{81}{4}\pi(-\sqrt{2}+2)\end{aligned}$$

51. We use spherical coordinates because we are integrating over a sphere and the density has spherical symmetry. $D = 2\rho$.

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^3 (2\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

52. Using cylindrical coordinates, the density is given by $\delta = kr^2$ gm/cm^3, where k is a constant. Since $\delta = 2$ when $r = 2$, we have

$$2 = k2^2 \quad \text{so} \quad k = 0.5.$$

The equation of the sphere is $x^2 + y^2 + z^2 = 3^2$, and in cylindrical coordinates,

$$r^2 + z^2 = 9.$$

Thus $r = \sqrt{9 - z^2}$ on the sphere, so

$$\begin{aligned}\text{Mass } &= \int_0^{2\pi}\int_{-3}^{3}\int_0^{\sqrt{9-z^2}} (0.5r^2)r\,dr\,dz\,d\theta \\ &= \int_0^{2\pi}\int_{-3}^{3} 0.5\frac{r^4}{4}\Bigg|_0^{\sqrt{9-z^2}} dz\,d\theta \\ &= \int_0^{2\pi}\int_{-3}^{3} \frac{1}{8}(9-z^2)^2\,dz\,d\theta \\ &= \frac{1}{8}\int_0^{2\pi}\int_{-3}^{3} (81 - 18z^2 + z^4)\,dz\,d\theta \\ &= \frac{2\pi}{8}\left(81z - \frac{18z^3}{3} + \frac{z^5}{5}\right)\Bigg|_{-3}^{3} = \frac{324\pi}{5}\text{ gm}.\end{aligned}$$

53. We use spherical coordinates. The density, δ, of the sphere at a distance ρ from the center is

$$\delta = k\rho^2 \quad \text{for } k \text{ a positive constant.}$$

Thus, for a sphere of radius 1,

$$\begin{aligned}\text{Mass } &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 k\rho^2\cdot\rho^2\sin\theta\,d\rho\,d\phi\,d\theta \\ &= 2\pi\int_0^{\pi} k\frac{\rho^5}{5}\sin\phi\bigg|_0^1\,d\phi = \frac{2\pi k}{5}(-\cos\phi)\bigg|_0^{\pi} = \frac{4\pi k}{5}.\end{aligned}$$

For a sphere of radius 2, a similar calculation gives

$$\begin{aligned}\text{Mass } &= \int_0^{2\pi}\int_0^{\pi}\int_0^2 k\rho^2\cdot\rho^2\sin\theta\,d\rho\,d\phi\,d\theta \\ &= 2\pi k\frac{2^5}{5}(-\cos\phi)\bigg|_0^{\pi} = \frac{128\pi k}{5}.\end{aligned}$$

Therefore

$$\text{Ratio of masses } = \frac{4\pi k/5}{128\pi k/5} = \frac{1}{32}.$$

54. (a) We use spherical coordinates. Since $\delta = 9$ where $\rho = 6$ and $\delta = 11$ where $\rho = 7$, the density increases at a rate of 2 gm/cm^3 for each cm increase in radius. Thus, since density is a linear function of radius, the slope of the linear function is 2. Its equation is

$$\delta - 11 = 2(\rho - 7) \quad \text{so} \quad \delta = 2\rho - 3.$$

(b) Thus,

$$\text{Mass } = \int_0^{2\pi}\int_0^{\pi}\int_6^7 (2\rho-3)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

(c) Evaluating the integral, we have

$$\text{Mass } = 2\pi\left(-\cos\phi\bigg|_0^{\pi}\right)\left(\frac{2\rho^4}{4} - \frac{3\rho^3}{3}\bigg|_6^7\right) = 2\pi\cdot 2(425.5) = 1702\pi \text{ gm } = 5346.991 \text{ gm}.$$

55. (a) First we must choose a coordinate system, since none is given. We pick the xy-plane to be the fixed plane and the z-axis to be the line perpendicular to the plane. Then the distance from a point to the plane is $|z|$, so the density at a point is given by

$$\text{Density} = \rho = k|z|.$$

Using cylindrical coordinates for the integral, we find

$$\text{Mass} = \int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} k|z|r\,dzdrd\theta.$$

(b) By symmetry, we can evaluate this integral over the top half of the sphere, where $|z| = z$. Then

$$\begin{aligned}\text{Mass} &= 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} kzr\,dzdrd\theta = 2k\int_0^{2\pi}\int_0^{a}\frac{z^2}{2}r\bigg|_{z=0}^{z=\sqrt{a^2-r^2}}\,drd\theta \\ &= k\int_0^{2\pi}\int_0^{a} r(a^2-r^2)\,drd\theta = k2\pi\left(\frac{r^2}{2}a^2 - \frac{r^4}{4}\right)\bigg|_0^{a} \\ &= 2\pi k\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi k a^4}{2}.\end{aligned}$$

56. The distance from a point (x, y, z) to the origin is given by $\sqrt{x^2 + y^2 + z^2}$. Thus we want to evaluate

$$\frac{\int_R \sqrt{x^2 + y^2 + z^2}\, dV}{\text{Vol}(R)}$$

where R is the region bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$. See Figure 16.115. We will use spherical coordinates.

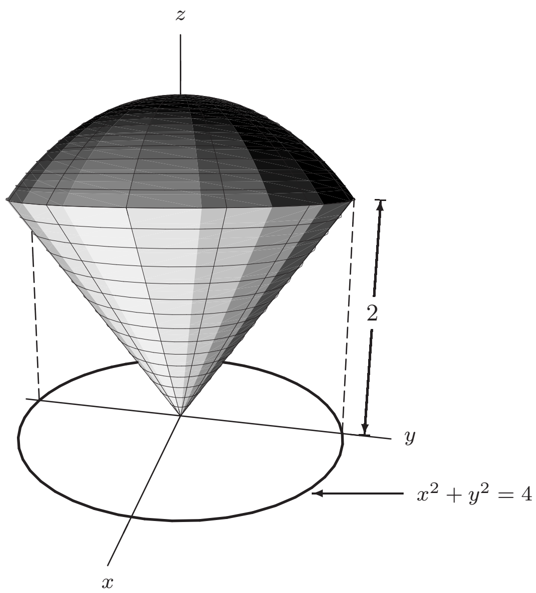


Figure 16.115

In spherical coordinates, the quantity ρ goes from 0 to $\sqrt{8}$, and θ goes from 0 to 2π, and ϕ goes from 0 to $\pi/4$ (because the angle of the cone is $\pi/4$). Thus we have

$$\begin{aligned}
\int_R \sqrt{x^2 + y^2 + z^2}\, dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho(\rho^2 \sin\phi)\, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \cdot \left.\frac{\rho^4}{4}\right|_0^{\sqrt{8}} d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} 16 \sin\phi\, d\phi d\theta \\
&= \int_0^{2\pi} 16(-\cos\phi)\Big|_0^{\pi/4} d\theta \\
&= \int_0^{2\pi} 16\left(1 - \frac{\sqrt{2}}{2}\right) d\theta \\
&= 32\left(1 - \frac{\sqrt{2}}{2}\right)\pi
\end{aligned}$$

From Problem 27 of Section 16.4 we know that $\text{Vol}(R) = 32\pi(\sqrt{2} - 1)/3$, therefore

$$\begin{aligned}
\text{Average distance} &= \frac{\int_R \sqrt{x^2 + y^2 + z^2}\, dV}{\text{Vol}(R)} \\
&= \frac{32\left(1 - \frac{\sqrt{2}}{2}\right)\pi}{[32(\sqrt{2} - 1)\pi/3]} = \frac{3}{\sqrt{2}}.
\end{aligned}$$

57. The total volume of the cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 1^2 \cdot 1 = \frac{1}{3}\pi$, so the total mass is $\frac{1}{3}\pi$ (since the density is always 1). The center of mass z-coordinate is given by

$$\bar{z} = \frac{3}{\pi}\int_C z\,dV$$

Using cylindrical coordinates to evaluate this integral gives

$$\begin{aligned}\bar{z} &= \frac{3}{\pi}\int_0^{2\pi}\int_0^1\int_0^z zr\,dr\,dz\,d\theta\\ &= \frac{3}{\pi}\int_0^{2\pi}\int_0^1 \frac{z^3}{2}\,dz\,d\theta\\ &= \frac{3}{\pi}\int_0^{2\pi}\frac{1}{8}\,d\theta = \frac{3}{4}\end{aligned}$$

58. (a) The mass m of the cone is given by $\int_C \delta\,dV$. In cylindrical coordinates this is

$$\begin{aligned}m &= \int_0^{2\pi}\int_0^1\int_0^z z^2 r\,dr\,dz\,d\theta\\ &= \int_0^{2\pi}\int_0^1 \frac{z^4}{2}\,dz\,d\theta\\ &= \int_0^{2\pi}\frac{1}{10}\,d\theta = \frac{\pi}{5}\end{aligned}$$

(b) The center of mass z-coordinate is given by

$$\bar{z} = \frac{5}{\pi}\int_C z\cdot z^2\,dV$$

Using cylindrical coordinates to evaluate this integral gives

$$\begin{aligned}\bar{z} &= \frac{5}{\pi}\int_0^{2\pi}\int_0^1\int_0^z z^3 r\,dr\,dz\,d\theta\\ &= \frac{5}{\pi}\int_0^{2\pi}\int_0^1 \frac{z^5}{2}\,dz\,d\theta\\ &= \frac{5}{\pi}\int_0^{2\pi}\frac{1}{12}\,d\theta = \frac{5}{6}\end{aligned}$$

Comparing this answer with the center of mass in Problem 57, where the density was constant, it makes sense that the center of mass would be higher in this problem, since more mass is concentrated near the top of the cone.

59. We first need to find the mass of the solid, using cylindrical coordinates:

$$\begin{aligned}m &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{z/a}} r\,dr\,dz\,d\theta\\ &= \int_0^{2\pi}\int_0^1 \frac{z}{2a}\,dz\,d\theta\\ &= \int_0^{2\pi}\frac{1}{4a}\,d\theta = \frac{\pi}{2a}\end{aligned}$$

It makes sense that the mass would vary inversely with a, since increasing a makes the paraboloid skinnier. Now for the z-coordinate of the center of mass, again using cylindrical coordinates:

$$\bar{z} = \frac{2a}{\pi}\int_0^{2\pi}\int_0^1\int_0^{\sqrt{z/a}} zr\,dr\,dz\,d\theta$$

$$= \frac{2a}{\pi} \int_0^{2\pi} \int_0^1 \frac{z^2}{2a}\, dz\, d\theta$$
$$= \frac{2a}{\pi} \int_0^{2\pi} \frac{1}{6a}\, d\theta = \frac{2}{3}$$

60. The volume of the hemisphere is $\frac{2}{3}\pi a^3$ so its mass is $\frac{2}{3}\pi a^3 b$. To find the location of the center of mass, we place the base of the hemisphere on the xy-plane with the origin at its center, so we can describe it in spherical coordinates by $0 \le \rho \le a$, $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le 2\pi$. Then the x-coordinate of the center of mass is, integrating using spherical coordinates:

$$\bar{x} = \frac{3}{2\pi a^3 b} \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho \sin(\phi)\cos(\theta) \cdot \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho = 0$$

since the first integral $\int_0^{2\pi} \cos(\theta)\, d\theta$ is zero. A similar computation shows that $\bar{y} = 0$. Now for the z-coordinate:

$$\bar{z} = \frac{3}{2\pi a^3 b} \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho\cos(\phi) \cdot \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho$$
$$= \frac{3}{2\pi a^3 b} \cdot 2\pi \int_0^a \int_0^{\frac{\pi}{2}} \rho^3 \cos(\phi)\sin(\phi)\, d\phi\, d\rho$$
$$= \frac{3}{a^3 b} \int_0^a \rho^3 \left. \frac{\sin^2(\phi)}{2} \right|_0^{\frac{\pi}{2}} d\rho$$
$$= \frac{3}{2a^3 b} \int_0^a \rho^3\, d\rho = \frac{3a}{8b}$$

So the x and y-coordinates are located at the center of the base, while the z-coordinate is located $\frac{3a}{8b}$ above the center of the base.

61. The sum of the three moments of inertia I for the ball B will be

$$3I = \frac{3}{4\pi a^3} \int_B (y^2+z^2)\, dV + \frac{3}{4\pi a^3} \int_B (x^2+z^2)\, dV + \frac{3}{4\pi a^3} \int_B (x^2+y^2)\, dV$$
$$= \frac{3}{4\pi a^3} \int_B (2x^2+2y^2+2z^2)\, dV,$$

which, in spherical coordinates is

$$\frac{3}{2\pi a^3} \int_B (x^2+y^2+z^2)\, dV = \frac{3}{2\pi a^3} \int_0^a \int_0^{\pi} \int_0^{2\pi} \rho^2 \cdot \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho$$
$$= \frac{3}{a^3} \int_0^a \int_0^{\pi} \rho^4 \sin(\phi)\, d\phi\, d\rho$$
$$= \frac{6}{a^3} \int_0^a \rho^4\, d\rho = \frac{6}{5}a^2.$$

Thus $3I = \frac{6}{5}a^2$, so $I = \frac{2}{5}a^2$.

62. First we need to find the volume of the cone. In spherical coordinates we find:

$$V = \int_0^a \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho = \frac{\pi a^3}{3}$$

Now, to find the moment of inertia about the z-axis we need to compute the integral $\frac{3}{\pi a^3} \int_W x^2 + y^2\, dV$. We can do this in spherical coordinates as

$$\frac{3}{\pi a^3} \int_W x^2 + y^2\, dV = \frac{3}{\pi a^3} \int_0^a \int_0^{\frac{\pi}{3}} \int_0^{2\pi} (\rho^2 \sin^2(\phi)\cos^2(\theta) + \rho^2 \sin^2(\phi)\sin^2(\theta)) \cdot \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho$$

$$= \frac{3}{\pi a^3} \int_0^a \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \rho^4 \sin^3(\phi)\, d\theta\, d\phi\, d\rho$$
$$= \frac{6}{a^3} \int_0^a \int_0^{\frac{\pi}{3}} \rho^4 \sin^3(\phi)\, d\phi\, d\rho$$
$$= \frac{6}{a^3} \frac{5}{24} \int_0^a \rho^4\, d\rho = \frac{a^2}{4}.$$

63. Using spherical coordinates,

$$\text{Stored energy} = \frac{1}{2} \int_a^b \int_0^{\pi} \int_0^{2\pi} \epsilon E^2 \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho = \frac{q^2}{32\pi^2\epsilon} \int_a^b \int_0^{\pi} \int_0^{2\pi} \frac{1}{\rho^2} \sin\phi\, d\theta\, d\phi\, d\rho$$
$$= \frac{q^2}{8\pi\epsilon} \int_a^b \frac{1}{\rho^2}\, d\rho = \frac{q^2}{8\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right).$$

64. Use cylindrical coordinates, with the z-axis being the axis of the cable. Consider a piece of cable of length 1. Then

$$\text{Stored energy} = \frac{1}{2} \int_a^b \int_0^1 \int_0^{2\pi} \epsilon E^2 r\, d\theta\, dz\, dr = \frac{q^2}{8\pi^2\epsilon} \int_a^b \int_0^1 \int_0^{2\pi} \frac{1}{r}\, d\theta\, dz\, dr$$
$$= \frac{q^2}{4\pi\epsilon} \int_a^b \frac{1}{r}\, dr = \frac{q^2}{4\pi\epsilon}(\ln b - \ln a) = \frac{q^2}{4\pi\epsilon} \ln \frac{b}{a}.$$

So the stored energy is proportional to $\ln(b/a)$ with constant of proportionality $q^2/4\pi\epsilon$.

65. The surface $z = 4 - x^2 - y^2$ cuts the xy-plane in the circle $x^2 + y^2 = 4$. Thus

$$\text{Mass } = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} e^{-x-y}\, dz\, dy\, dx \text{ gm.}$$

66. The cylinder has radius 2. Using cylindrical coordinates to find the mass and integrating with respect to r first, we have

$$\text{Mass } = \int_0^{2\pi} \int_0^3 \int_0^2 (1+r) r\, dr\, dz\, d\theta = \int_0^{2\pi} \int_0^3 \left(\frac{r^2}{2} + \frac{r^3}{3}\right)\Bigg|_0^2 dz\, d\theta = 2\pi \cdot 3 \cdot \left(\frac{4}{2} + \frac{8}{3}\right) = 28\pi \text{ gm.}$$

67. In cylindrical coordinates, the density, δ is given by $\delta = kr$ for some positive constant k.

For the smaller cylinder, $x^2 + y^2 \le 1$, $0 \le z \le 2$, whose radius is 1,

$$\text{Mass } = \int_0^{2\pi} \int_0^2 \int_0^1 kr \cdot r\, dr\, dz\, d\theta = 2\pi \cdot 2 \cdot \frac{kr^3}{3}\Bigg|_0^1 = \frac{4\pi k}{3}.$$

For the larger cylinder, $x^2 + y^2 \le 9$, $0 \le z \le 2$, whose radius is 3,

$$\text{Mass } = \int_0^{2\pi} \int_0^2 \int_0^3 kr \cdot r\, dr\, dz\, d\theta = 2\pi \cdot 2 \cdot \frac{kr^3}{3}\Bigg|_0^3 = 36\pi k.$$

Thus, the ratio of the masses is $\dfrac{4\pi k/3}{36\pi k} = \dfrac{1}{27}$.

68. Integrating with respect to z first, we have

$$W = \int_0^1 \int_0^{2\pi} \int_{\sqrt{1-r^2}}^{(\sqrt{9-r^2})-1} r\,dz\,d\theta dr + \int_1^{2\sqrt{2}} \int_0^{2\pi} \int_0^{(\sqrt{9-r^2})-1} r\,dz\,d\theta\,dr$$

or integrating with respect to r first, we have

$$\int_0^1 \int_0^{2\pi} \int_{\sqrt{1-z^2}}^{\sqrt{9-(z+1)^2}} r\,dr\,d\theta\,dz + \int_1^2 \int_0^{2\pi} \int_0^{\sqrt{9-(z+1)^2}} r\,dr\,d\theta\,dz.$$

69. Assume the base of the cylinder sits on the xy-plane with center at the origin. Because the cylinder is symmetric about the z-axis, the force in the horizontal x or y direction is 0. Thus we need only compute the vertical z component of the force. We are going to use cylindrical coordinates; since the force is $G \cdot \text{mass}/(\text{distance})^2$, a piece of the cylinder of volume dV located at (r, θ, z) exerts on the unit mass a force with magnitude $G(\delta\,dV)/(r^2+z^2)$. See Figure 16.116.

$$\text{Vertical component of force} = \frac{G(\delta\,dV)}{r^2+z^2} \cdot \cos\phi = \frac{G\delta\,dV}{r^2+z^2} \cdot \frac{z}{\sqrt{r^2+z^2}} = \frac{G\delta z\,dV}{(r^2+z^2)^{3/2}}.$$

Adding up all the contributions of all the dV's, we obtain

$$\begin{aligned}
\text{Vertical force} &= \int_0^H \int_0^{2\pi} \int_0^R \frac{G\delta zr}{(r^2+z^2)^{3/2}}\,drd\theta dz \\
&= \int_0^H \int_0^{2\pi} (G\delta z) \left(-\frac{1}{\sqrt{r^2+z^2}}\right)\Bigg|_0^R d\theta dz \\
&= \int_0^H \int_0^{2\pi} (G\delta z) \cdot \left(-\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z}\right) d\theta dz \\
&= \int_0^H 2\pi G\delta \left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) dz \\
&= 2\pi G\delta(z - \sqrt{R^2+z^2})\Big|_0^H \\
&= 2\pi G\delta(H - \sqrt{R^2+H^2} + R) = 2\pi G\delta(H + R - \sqrt{R^2+H^2})
\end{aligned}$$

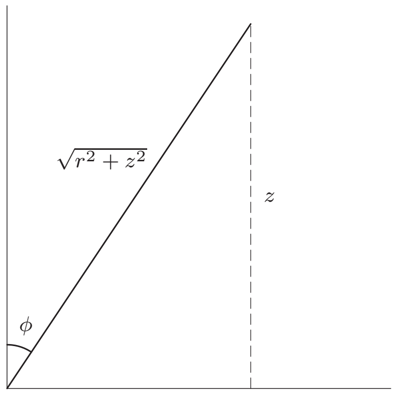


Figure 16.116

70. The charge density is $\delta = kz$, where k is a constant. In cylindrical coordinates,

$$\begin{aligned}
\text{Total charge} &= \int_{\text{Cylinder}} \delta\,dV = \int_0^h \int_0^R \int_0^{2\pi} kzr\,d\theta\,dr\,dz = k \int_0^h \int_0^R 2\pi zr\,dr\,dz \\
&= k\pi \int_0^h R^2 z\,dz = k(\pi R^2)\frac{h^2}{2} = \frac{k\pi}{2} R^2 h^2.
\end{aligned}$$

Thus, the total charge is proportional to R^2h^2 with constant of proportionality $k\pi/2$.

71. The charge density is $\delta = k/\rho$. Integrating in spherical coordinates,

$$\text{Total charge} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{k}{\rho}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = k\int_0^{2\pi}\int_0^{\pi} \frac{R^2}{2}\sin\phi \, d\phi \, d\theta$$
$$= 4\pi k \frac{R^2}{2} = 2\pi k R^2.$$

Thus, the total charge is proportional to R^2 with constant of proportionality $2\pi k$.

72. In the system used in this book the volume element is $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. In the system shown in the problem, ϕ and θ have been interchanged and ρ changed to r. So the volume element is $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

Strengthen Your Understanding

73. (c) $\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

74. The integral is missing part of the volume element in spherical coordinates. The integral $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ gives the volume inside the sphere of radius 1.

75. We cannot switch the order of integration without rewriting the limits of integration if doing so produces limits that are not constant on the outer integral.

76. In spherical coordinates, the upper half of a sphere of radius r is given by $0 \le \rho \le r$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$. Therefore, for a hemisphere of radius 5, we have

$$\text{Volume } = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{5} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

77. Let W be the unit ball $x^2 + y^2 + z^2 \le 1$. It is not easy to integrate

$$\int_W \sqrt{x^2+y^2+z^2} \, dx \, dy \, dz.$$

In spherical coordinates, this integral becomes easy to integrate:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Solutions for Section 16.6

Exercises

1. Yes, p is a joint density function. The values of $p(x,y)$ are nonnegative, since $p(x,y) = 1/2$ for all points inside R and $p(x,y) = 0$ for all other points. The volume under the graph of p over the region R is $(1/2)(5-4)(0-(-2)) = 1$.

2. No, p is not a joint density function. Since $p(x,y) = 0$ outside the region R, the volume under the graph of p is the same as the volume under the graph of p over the region R, which is 2 not 1.

3. No, p is a not joint density function, because $p(x,y) < 0$ for some points (x,y) in the region R. For example, $p(-0.7, 0.1) = -0.6$.

4. Yes, p is a joint density function. Since $x \le y$ everywhere in the region R, we have $p(x,y) = 6(y-x) \ge 0$ for all x and y in R, and $p(x,y) = 0$ for all other (x,y). To check that p is a joint density function, we check that the total volume under the graph of p over the region R is 1:

$$\int_R p(x,y)\, dA = \int_0^1\int_0^y 6(y-x)\, dx\, dy = \int_0^1 6\left(yx - \frac{x^2}{2}\right)\Bigg|_0^y dy = \int_0^1 3y^2\, dx = y^3\Bigg|_0^1 = 1.$$

5. Yes, p is a joint density function. In the region R we have $1 \geq x^2 + y^2$, so $p(x,y) = (2/\pi)(1 - x^2 - y^2) \geq 0$ for all x and y in R, and $p(x,y) = 0$ for all other (x,y). To check that p is a joint density function, we check that the total volume under the graph of p over the region R is 1. Using polar coordinates, we get:

$$\int_R p(x,y)dA = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = \frac{2}{\pi}\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_0^1 d\theta = \frac{2}{\pi}\int_0^{2\pi}\frac{1}{4}\,d\theta = 1.$$

6. Yes, p is a joint density function. Since e^{-x-y} is always positive, $p(x,y) = xye^{-x-y} \geq 0$ for all x and y in R, and hence for all x and y. To check that p is a joint density function, we check that the total volume under the graph of p over the region R is 1. Since $e^{-x-y} = e^{-x}e^{-y}$, we have

$$\int_R xye^{-x-y}dA = \int_0^\infty\int_0^\infty xye^{-x-y}dx\,dy = \int_0^\infty ye^{-y}\left(\int_0^\infty xe^{-x}dx\right)dy.$$

Using integration by parts:

$$\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}(-xe^{-x}-e^{-x})\Big|_0^b = (0-0)-(0-1) = 1.$$

Thus

$$\int_R xye^{-x-y}dA = \int_0^\infty ye^{-y}\left(\int_0^\infty xe^{-x}dx\right)dy = \int_0^\infty ye^{-y}dy = 1.$$

7. We have $p(x,y) = 0$ for all points (x,y) satisfying $x \geq 3$, since all such points lie outside the region R. Therefore the fraction of the population satisfying $x \geq 3$ is 0.

8. The fraction is 0, since $\int_1^1 xy\,dx = 0$, so $\int_{-\infty}^{\infty}\int_1^1 p(x,y)\,dx\,dy = \int_0^1\int_1^1 xy\,dx\,dy$=0.

9. Since $x + y \leq 3$ for all points (x,y) in the region R, the fraction of the population satisfying $x + y \leq 3$ is 1.

10. Since $p(x,y) = 0$ for any (x,y) with $x < 0$ and also $p(x,y) = 0$ for any (x,y) with $y > 1$ or $y < 0$, the fraction of the population is given by the double integral:

$$\int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 \frac{x^2y}{2}\Bigg|_0^1 dy = \int_0^1 \frac{y}{2}dy = \frac{y^2}{4}\Bigg|_0^1 = \frac{1}{4}.$$

11. Since $p(x,y) = 0$ for all (x,y) outside the rectangle R, the population is given by the volume under the graph of p over the region inside the rectangle R and to the right of the line $x = y$. Therefore the fraction of the population is given by the double integral:

$$\int_0^1\int_y^2 xy\,dx\,dy = \int_0^1 \frac{x^2y}{2}\Bigg|_y^2 dy = \int_0^1\left(2y-\frac{y^3}{2}\right)dy = \left(y^2-\frac{y^4}{8}\right)\Bigg|_0^1 = \frac{7}{8}.$$

12. Since $p(x,y) = 0$ for all (x,y) outside the rectangle R, the population is given by the volume under the graph of p over the region inside the rectangle R and below the line $x + y = 1$. This is the same as the region bounded by the x-axis, the y-axis, and the line $x + y = 1$. Therefore the fraction of the population is given by the double integral:

$$\int_0^1\int_0^{1-y} xy\,dx\,dy = \int_0^1 \frac{x^2y}{2}\Bigg|_0^{1-y} dy = \int_0^1 \frac{(1-y)^2y}{2}dy = \left(\frac{y^2}{4}-\frac{y^3}{3}+\frac{y^4}{8}\right)\Bigg|_0^1 = \frac{1}{24}.$$

13. The fraction of the population is given by the double integral:

$$\int_0^{1/2}\int_0^1 xy\,dx\,dy = \int_0^{1/2}\frac{x^2y}{2}\Bigg|_0^1 dy = \int_0^{1/2}\frac{y}{2}dy = \frac{y^2}{4}\Bigg|_0^{1/2} = \frac{1}{16}.$$

14. We are looking for points inside the circle $x^2 + y^2 = 1$ and inside the rectangle R. In the first quadrant, all of the circle and its interior lies inside the rectangle R. Thus the fraction of the population we want is given by the volume under the graph of p over the region inside the circle $x^2 + y^2 = 1$ in the first quadrant. We evaluate this double integral using polar coordinates:

$$\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2} \frac{r^4}{4}\cos\theta\sin\theta\Big|_0^1 d\theta = \frac{1}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta.$$

Making the substitution $w = \sin\theta$, we get:

$$\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = \int_0^1 w\,dw = \frac{1}{2}.$$

Thus the fraction is $(1/4)(1/2) = 1/8$.

Problems

15. (a)

$$\begin{aligned}\int_0^1\int_{1/3}^1 \frac{2}{3}(x+2y)\,dx\,dy &= \int_0^1 \frac{2}{3}(\frac{1}{2}x^2 + 2xy)\Big|_{1/3}^1 dy\\ &= \int_0^1 \frac{2}{3}\left[(\frac{1}{2}+2y) - (\frac{1}{18}+\frac{2}{3}y)\right] dy\\ &= \frac{2}{3}\int_0^1 (\frac{4}{9}+\frac{4}{3}y)\,dy\\ &= \frac{2}{3}\left(\frac{4}{9}y + \frac{2}{3}y^2\right)\Big|_0^1\\ &= \frac{2}{3}\left(\frac{10}{9}\right) = \frac{20}{27}.\end{aligned}$$

(b) It is easier to calculate the probability that $x < (1/3)+y$ does not happen, that is, the probability that $x \geq (1/3)+y$, and subtract it from 1. The probability that $x \geq (1/3)+y$ is

$$\begin{aligned}\int_{1/3}^1\int_0^{x-(1/3)} \frac{2}{3}(x+2y)\,dy\,dx &= \int_{1/3}^1 \frac{2}{3}(xy+y^2)\Big|_0^{x-(1/3)} dx\\ &= \frac{2}{3}\int_{1/3}^1 (x(x-\frac{1}{3}) + (x-\frac{1}{3})^2)\,dx\\ &= \frac{2}{3}\int_{1/3}^1 (2x^2 - x + \frac{1}{9})\,dx\\ &= \frac{2}{3}(\frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{9}x)\Big|_{1/3}^1\\ &= \frac{2}{3}\left[(\frac{2}{3}-\frac{1}{2}+\frac{1}{9}) - (\frac{2}{81}-\frac{1}{18}+\frac{1}{27})\right]\\ &= 44/243.\end{aligned}$$

Thus, the probability that $x < (1/3)+y$ is $1 - (44/243) = 199/243$.

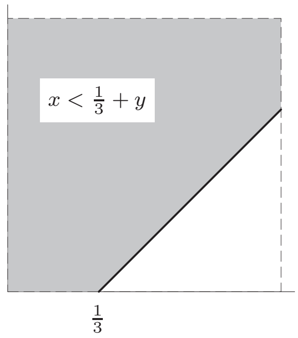


Figure 16.117

16. (a) We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$ for a joint density function. So,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^1\int_x^1 kxydydx$$
$$= \frac{1}{8}k$$

hence $k = 8$.

(b) The region where $x < y < \sqrt{x}$ is sketched in Figure 16.118

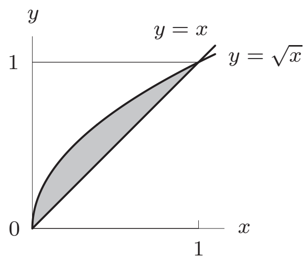


Figure 16.118

So the probability that (x,y) satisfies $x < y < \sqrt{x}$ is given by:

$$\int_0^1\int_x^{\sqrt{x}} 8xy\,dy\,dx = \int_0^1 4x(y^2)\Big|_x^{\sqrt{x}}dx$$
$$= \int_0^1 4x(x - x^2)dx$$
$$= 4\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^1$$
$$= 4\left(\frac{1}{3} - \frac{1}{4}\right)$$
$$= \frac{1}{3}$$

This tells us that in choosing points from the region defined by $0 \le x \le y \le 1$, that $1/3$ of the time we would pick a point from the region defined by $x < y < \sqrt{x}$. These regions are shown in Figure 16.118.

17. (a) For a density function,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^2\int_0^1 kx^2\,dy\,dx$$

$$= \int_0^2 kx^2\,dx$$
$$= \frac{kx^3}{3}\Big|_0^2 = \frac{8k}{3}.$$

So $k = 3/8$.

(b)
$$\int_0^1 \int_0^{2-y} \frac{3}{8}x^2\,dx\,dy = \int_0^1 \frac{1}{8}(2-y)^3\,dy = \frac{-1}{32}(2-y)^4\Big|_0^1 = \frac{15}{32}$$

(c)
$$\int_0^{1/2}\int_0^1 \frac{3}{8}x^2\,dx\,dy = \int_0^{1/2} \frac{1}{8}x^3\Big|_0^1 dy = \int_0^{1/2} \frac{1}{8}\,dy = \frac{1}{16}.$$

18. (a) The area of S is $(2)(4) = 8$. Because the density function $p(x,y)$ is constant on S and the total volume under a density function above the xy-plane is 1, $p(x,y) = 1/8$ for (x,y) in S, and $p(x,y) = 0$ for (x,y) outside S.

(b) The probability that (x,y) is in T is
$$\int_T f(x,y)\,dy\,dx = \frac{1}{8}\int_T dy\,dx = \frac{\text{area}(T)}{8} = \frac{\alpha}{8}.$$

19. Since
$$\sum_x\sum_y f(x,y)\,\Delta x\,\Delta y \approx \int_R f(x,y)\,dx\,dy$$
and since x never exceeds 1, and we can assume that no one lives to be over 100, so y does not exceed 100, we have
$$\text{Fraction of policies} = \int_R f(x,y)\,dx\,dy = \int_{65}^{100}\int_{0.8}^{1} f(x,y)\,dx\,dy,$$
where R is the rectangle: $0.8 \le x \le 1$, $65 \le y \le 100$.

20. (a) Since the exponential function is always positive and λ is positive, $p(t) \ge 0$ for all t, and
$$\int_0^\infty p(t)dt = \lim_{b\to\infty} -e^{-\lambda t}\Big|_0^b = \lim_{b\to\infty} -e^{-bt} + 1 = 1.$$

(b) The density function for the probability that the first substance decays at time t and the second decays at time s is
$$p(t,s) = \lambda e^{-\lambda t}\mu e^{-\mu s} = \lambda\mu e^{-\lambda t - \mu s},$$
for $s \ge 0$ and $t \ge 0$, and is zero otherwise.

(c) We want the probability that the decay time t of the first substance is less than or equal to the decay time s of the second, so we want to integrate the density function over the region $0 \le t \le s$. Thus, we compute
$$\int_0^\infty\int_t^\infty \lambda\mu e^{-\lambda t}e^{-\mu s}\,ds\,dt = \int_0^\infty \lambda e^{-\lambda t}(-e^{-\mu s})\Big|_t^\infty dt$$
$$= \int_0^\infty \lambda e^{-\lambda t}e^{-\mu t}\,dt$$
$$= \int_0^\infty \lambda e^{(-\lambda+\mu)t}\,dt$$
$$= \frac{-\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}\Big|_0^\infty = \frac{\lambda}{\lambda+\mu}.$$

So for example, if $\lambda = 1$ and $\mu = 4$, then the probability that the first substance decays first is $1/5$.

21. (a)
$$\int_{\theta=0}^{\frac{\pi}{6}}\int_{r=\frac{1}{\cos\theta}}^{4} p(r,\theta)r\,dr\,d\theta$$

(b)
$$\int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{6}+\frac{\pi}{12}}\int_{r=\frac{1}{\cos\theta}}^{4} p(r,\theta)r\,dr\,d\theta + \int_{\theta=\frac{\pi}{6}+\frac{\pi}{12}}^{\frac{2\pi}{6}}\int_{r=\frac{1}{\sin\theta}}^{4} p(r,\theta)r\,dr\,d\theta$$

22. (a) If $t \le 0$, then $F(t) = 0$ because the average of two positive numbers can not be negative. If $1 < t$ then $F(t) = 1$ because the average of two numbers each at most 1 is certain to be less than or equal to 1. For any t, we have $F(t) = \int_R p(x, y)dA$ where R is the region of the plane defined by $(x + y)/2 \le t$. Since $p(x, y) = 0$ outside the unit square, we need integrate only over the part of R that lies inside the square, and since $p(x, y) = 1$ inside the square, the integral equals the area of that part of the square. Thus, we can calculate the area using area formulas. For $0 \le t \le 1$, we draw the line $(x + y)/2 = t$, which has x- and y-intercepts of $2t$. Figure 16.119 shows that for $0 < t \le 1/2$,

$$F(t) = \text{ Area of triangle } = \frac{1}{2} \cdot 2t \cdot 2t = 2t^2.$$

In Figure 16.120, when $x = 1$, we have $y = 2t - 1$. Thus, the vertical side of the unshaded triangle is $1 - (2t - 1) = 2 - 2t$. The horizontal side is the same length, so for $1/2 < t \le 1$,

$$F(t) = \text{ Area of Square } - \text{ Area of triangle } = 1^2 - \frac{1}{2}(2 - 2t)^2 - 1 - 2(1 - t)^2.$$

The final result is:

$$F(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 2t^2 & \text{if } 0 < t \le 1/2 \\ 1 - 2(1-t)^2 & \text{if } 1/2 < t \le 1 \\ 1 & \text{if } 1 < t \end{cases}.$$

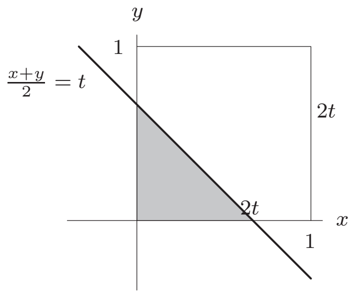


Figure 16.119: For $0 < t \le \frac{1}{2}$

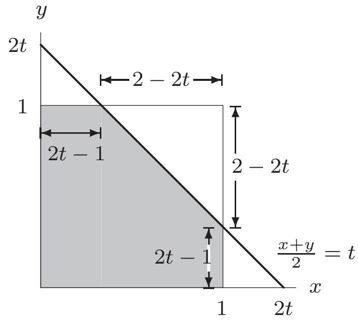


Figure 16.120: For $\frac{1}{2} < t \le 1$

(b) The probability density function $p(t)$ of z is the derivative of its cumulative distribution function. We have

$$p(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 4t & \text{if } 0 < t \le 1/2 \\ 4 - 4t & \text{if } 1/2 < t \le 1 \\ 0 & \text{if } 1 < t \end{cases}.$$

See Figure 16.121.

(c) The values of x and y and equally likely to be near $0, 1/2$, and 1. Notice from the graph of the density function in Figure 16.121 that even though x and y separately are equally likely to be anywhere between 0 and 1, their average $z = (x + y)/2$ is more likely to be near $1/2$ than to be near 0 or 1.

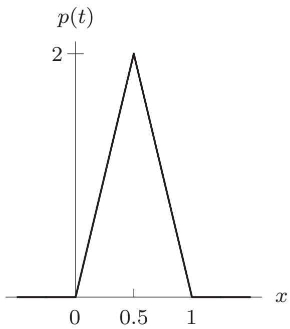


Figure 16.121

Strengthen Your Understanding

23. Since $p_1(x,y)$ and $p_2(x,y)$ are joint density functions, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_1(x,y)\,dx\,dy = 1 \quad \text{and} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_2(x,y)\,dx\,dy = 1.$$

Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (p_1(x,y)+p_2(x,y))\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_1(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_2(x,y)\,dx\,dy = 1+1=2.$$

So $p_1(x,y)+p_2(x,y)$ is not a joint density function.

24. The value of $p(60,170)$ cannot be interpreted as a probability. The probability that w falls in the interval of width Δw around 60 and h falls in the interval of width Δh around 170 is approximately $p(60,170)\Delta w \Delta h$.

25. In order for $f(x,y)$ to be a joint density function, we need

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{c}^{d}\int_{a}^{b} 1\,dx\,dy = (b-a)(d-c),$$

one possibility is $a=0,\ b=1,\ c=0,\ d=1$.

26. In order for $f(x,y)$ to be a joint density function, we need

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{0}^{1}\int_{0}^{2} g(y)\,dx\,dy = 2\int_{0}^{1} g(y)\,dy.$$

Since

$$\int_{0}^{1} y\,dy = \frac{1}{2},$$

one possibility is $g(y)=y$.

27. True. Since y ranges from $-\infty$ to ∞ the double integral gives the probability that $a \le x \le b$ by the definition of joint density function.

28. False. For $p(x,y)$ to be a joint density function one of the restrictions we have is that $p(x,y) \ge 0$ for all x and y.

29. False. The double integral $\int_a^b \int_{-\infty}^{\infty} p(x,y)\,dy\,dx$ is the probability that $a \le x \le b$.

30. True. This follows by the definition of the joint density function.

Solutions for Chapter 16 Review

Exercises

1. The region of integration ranges from $x=0$ to $x=3$ and from $y=0$ to $y=2x$, as shown in Figure 16.122. To evaluate the integral, we evaluate the inside integral first:

$$\int_0^{2x} (x^2+y^2)\,dy = \left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=2x} = x^2(2x)+\frac{(2x)^3}{3} = 2x^3+\frac{8x^3}{3} = \frac{14}{3}x^3.$$

Therefore, we have

$$\int_0^3 \int_0^{2x} (x^2+y^2)\,dydx = \int_0^3 \left(\frac{14}{3}x^3\right)\,dx = \left(\frac{14}{12}x^4\right)\Big|_0^3 = 94.5.$$

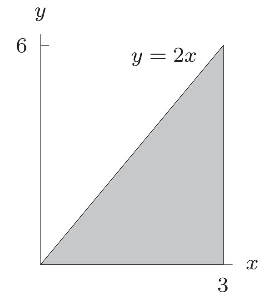


Figure 16.122

2. The region of integration ranges from $x=0$ to $x=\pi$ and from $y=0$ to $y=x$, as shown in Figure 16.123.

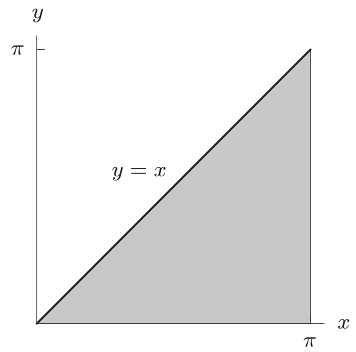


Figure 16.123

Integrating with respect to y first

$$\int_0^\pi \int_0^x \sin x\,dy\,dx = \int_0^\pi y\sin x\Big|_0^x\,dx = \int_0^\pi x\sin x\,dx.$$

Using integration by parts with $u=x$, $v'=\sin x$, so $u'=1$, $v=-\cos x$, we have

$$\begin{aligned}\int_0^\pi \int_0^x \sin x\,dy\,dx &= \int_0^\pi x\sin x\,dx = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx \\ &= -\pi(-1)+\sin x\Big|_0^\pi = \pi.\end{aligned}$$

3. See Figure 16.124.

$$\begin{aligned}\int_{-2}^0 \int_{-\sqrt{9-x^2}}^0 2xy\,dydx &= \int_{-2}^0 x\,y^2\Big|_{-\sqrt{9-x^2}}^0 dx \\ &= -\int_{-2}^0 x(9-x^2)\,dx\end{aligned}$$

$$= \int_{-2}^{0} (x^3 - 9x)\,dx$$
$$= \left(\frac{x^4}{4} - \frac{9}{2}x^2\right)\Bigg|_{-2}^{0}$$
$$= -4 + 18 = 14$$

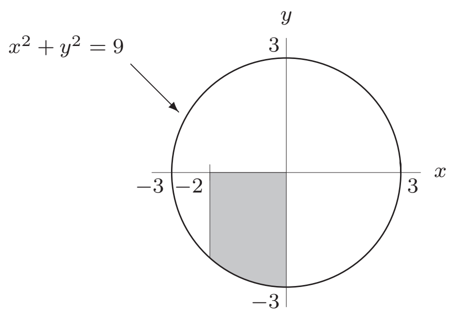


Figure 16.124

4. See Figure 16.125.

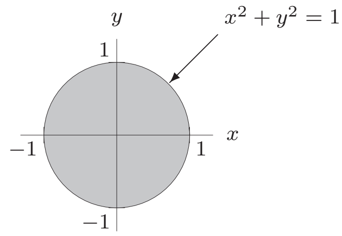


Figure 16.125

5. See Figure 16.126.

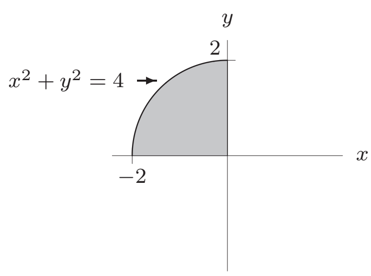


Figure 16.126

6. See Figure 16.127.

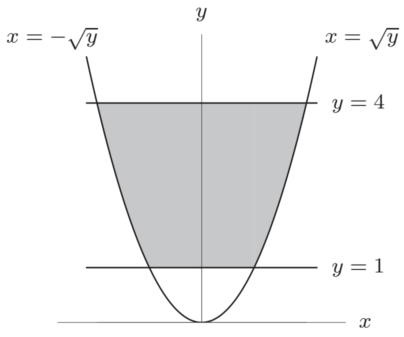


Figure 16.127

7. See Figure 16.128.

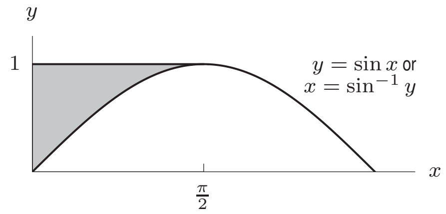


Figure 16.128

8. The region is the half cylinder in Figure 16.129.

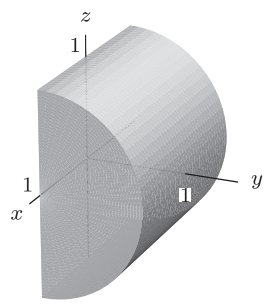


Figure 16.129

9. The region of integration shown in Figure 16.130 is a three-sided pyramid bounded by the xy-plane and the planes $y = 1$, $z - x = 0$ and $y - x = 0$.

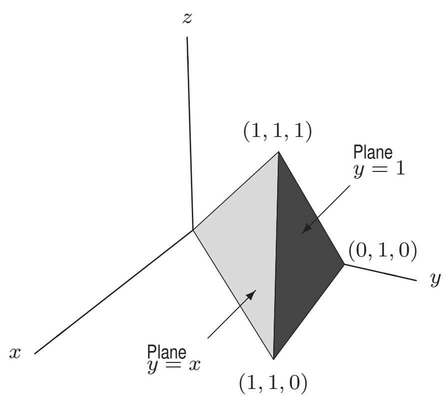


Figure 16.130

10. We use Cartesian coordinates, oriented so that the cube is in the first quadrant. See Figure 16.131. Then, if f is an arbitrary function, the integral is

$$\int_0^2 \int_0^3 \int_0^5 f\, dx\, dy\, dz.$$

Other answers are possible. In particular, the order of integration can be changed.

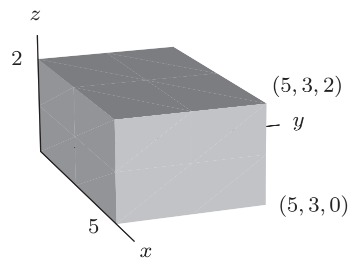


Figure 16.131

11. If we imagine the disk lying horizontally, as in Figure 16.132, we can use cylindrical coordinates with the origin at the center of the flat base. Then, if f is an arbitrary function, the triple integral is

$$\int_0^{2\pi} \int_0^2 \int_0^3 f\, r\, dr dz d\theta.$$

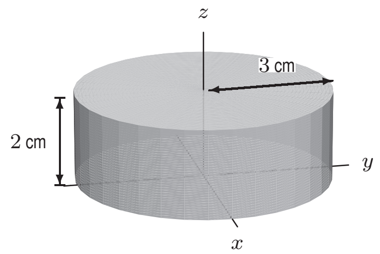


Figure 16.132

12. We use spherical coordinates as in Figure 16.133. Then if f is an arbitrary function, the triple integral is

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_2^5 f\rho^2 \sin\phi\, d\rho d\phi d\theta.$$

Other answers are possible.

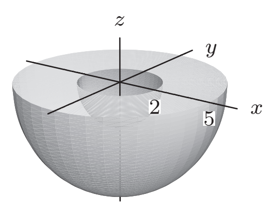


Figure 16.133

13. We use spherical coordinates, as in Figure 16.134. Then if f is an arbitrary function, the triple integral is

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^5 f\rho^2\sin\phi\,d\rho d\phi d\theta.$$

Other answers are possible.

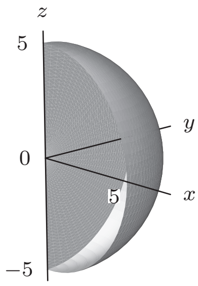


Figure 16.134

14. Integrating with respect to x first we get

$$\int_0^4\int_{\frac{y}{2}-2}^{-y+4} f(x,y)\,dx\,dy$$

Integrating with respect to y first we get

$$\int_{-2}^0\int_0^{2x+4} f(x,y)\,dy\,dx + \int_0^4\int_0^{-x+4} f(x,y)\,dy\,dx.$$

See Figure 16.135

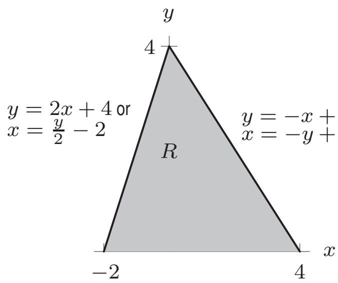


Figure 16.135

15. (a) See Figure 16.136.

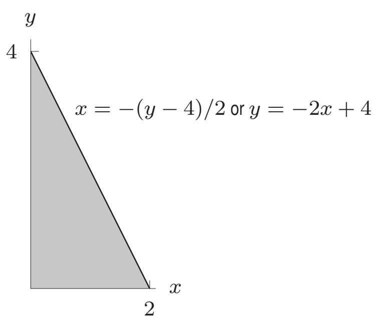


Figure 16.136

(b) $\int_0^2\int_0^{-2x+4} g(x,y)\,dy\,dx.$

16. Compute in polar coordinates:

$$\begin{aligned}\int_R \sqrt{x^2+y^2}\,dA &= \int_0^{\pi}\int_1^2 r\cdot r\,dr\,d\theta\\ &= \int_0^{\pi}\left[\frac{r^3}{3}\right]_1^2 d\theta\\ &= \int_0^{\pi}\left(\frac{8}{3}-\frac{1}{3}\right)d\theta = \frac{7\pi}{3}.\end{aligned}$$

17.

$$\begin{aligned}\int_0^{10}\int_0^{0.1} xe^{xy}\,dy\,dx &= \int_0^{10} e^{xy}\big|_0^{0.1}\,dx\\ &= \int_0^{10}(e^{0.1x}-e^0)\,dx\\ &= \left(\left(\frac{e^{0.1x}}{0.1}\right)-x\right)\Bigg|_0^{10}\\ &= (10e^1-10-10e^0)\\ &= 10e-20 = 10(e-2).\end{aligned}$$

18.

$$\begin{aligned}\int_0^1\int_3^4 (\sin(2-y))\cos(3x-7)\,dx\,dy &= \int_0^1 (\sin(2-y))\left[\frac{\sin(3x-7)}{3}\right]\Bigg|_3^4 dy\\ &= \frac{1}{3}(\sin 5-\sin 2)\int_0^1 \sin(2-y)\,dy\\ &= \frac{1}{3}(\sin 5-\sin 2)\left[\cos(2-y)\right]_0^1\\ &= \frac{1}{3}(\sin 5-\sin 2)(\cos 1-\cos 2).\end{aligned}$$

19.

$$\begin{aligned}\int_0^1\int_0^y (\sin^3 x)(\cos x)(\cos y)\,dx\,dy &= \int_0^1 (\cos y)\left[\frac{\sin^4 x}{4}\right]\Bigg|_0^y dy\\ &= \frac{1}{4}\int_0^1 (\sin^4 y)(\cos y)\,dy\\ &= \frac{\sin^5 y}{20}\Bigg|_0^1\\ &= \frac{\sin^5 1}{20}.\end{aligned}$$

20. First use integration by parts, with y as the variable, $u=x^2y$, $u'=x^2$, $v=\frac{\sin(xy)}{x}$, $v'=\cos(xy)$. Then,

$$\begin{aligned}\int_3^4\int_0^1 x^2y\cos(xy)\,dy\,dx &= \int_3^4\left([xy\sin(xy)]_0^1-\int_0^1 x\sin(xy)\,dy\right)dx\\ &= \int_3^4\left(x\sin x+[\cos(xy)]_0^1\right)dx\\ &= \int_3^4 (x\sin x+\cos x-1)\,dx.\end{aligned}$$

Now use integration by parts again, with $u = x$, $u' = 1$, $v = -\cos x$, $v' = \sin x$. Then,

$$\begin{aligned}\int_3^4 (x\sin x + \cos x - 1)\,dx &= [-x\cos x]_3^4 + \int_3^4 \cos x\,dx + \int_3^4 (\cos x - 1)\,dx \\ &= (-x\cos x + 2\sin x - x)\big|_3^4 \\ &= -4\cos 4 + 2\sin 4 + 3\cos 3 - 2\sin 3 - 1.\end{aligned}$$

Thus,

$$\int_3^4 \int_0^1 x^2 y\cos(xy)\,dy\,dx = -4\cos 4 + 2\sin 4 + 3\cos 3 - 2\sin 3 - 1.$$

21. The region is shown in Figure 16.137.

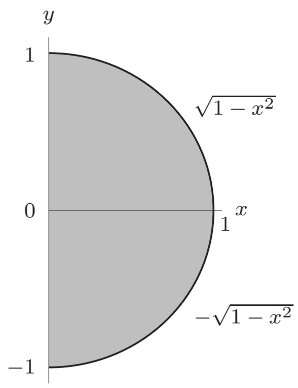


Figure 16.137

The integral has the same values in the upper and lower quarter circles, so we integrate over just the upper circle and multiply by 2. We convert the integral to polar coordinates.

$$\begin{aligned}\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx = 2\int_0^{\pi/2}\int_0^1 e^{-r^2} r\,dr\,d\theta &= \int_0^{\pi/2} (-e^{-r^2})\Big|_0^1\,d\theta \\ &= \int_0^{\pi/2} 1 - e^{-1}\,d\theta \\ &= \frac{\pi}{2}(1 - e^{-1}).\end{aligned}$$

22.

$$\begin{aligned}\int_0^1\int_0^z\int_0^2 (y+z)^7\,dx\,dy\,dz &= \int_0^1\int_0^z 2(y+z)^7\,dy\,dz \\ &= \int_0^1 \frac{2(y+z)^8}{8}\Bigg|_0^z\,dz \\ &= \int_0^1 \frac{(2z)^8 - z^8}{4}\,dz \\ &= \frac{255}{4}\int_0^1 z^8\,dz \\ &= \frac{255}{4}\left[\frac{z^9}{9}\right]_0^1 \\ &= \frac{255}{4}\cdot\frac{1}{9} \\ &= \frac{85}{12}.\end{aligned}$$

23. Evaluating gives

$$\int_0^1\int_0^z\int_0^y xyz\,dx\,dy\,dz = \int_0^1\int_0^z \frac{x^2yz}{2}\bigg|_0^y dy\,dz = \int_0^1\int_0^z \frac{y^3z}{2}\,dy\,dz = \int_0^1 \frac{y^4z}{8}\bigg|_0^z dz = \int_0^1 \frac{z^5}{8}\,dz = \frac{z^6}{48}\bigg|_0^1 = \frac{1}{48}.$$

24. (a) A vertical plane perpendicular to the x-axis: $x = 2$.
(b) A cylinder: $r = 3$.
(c) A sphere: $\rho = \sqrt{3}$.
(d) A cone: $\phi = \pi/4$.
(e) A horizontal plane: $z = -5$.
(f) A vertical half-plane: $\theta = \pi/4$.

25. In spherical coordinates, the spherical cap is part of the surface $\rho = \sqrt{2}$. If α is the angle at the vertex of the cone, we have $\tan(\alpha/2) = 2/2 = 1$, so $\alpha/2 = \pi/4$. Since the cone is below the xy-plane, the angle ϕ ranges from $3\pi/4$ to π. Thus, the integral is given by

$$\int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^{\sqrt{2}} f(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

26. In cylindrical coordinates, the spherical cap has equation $z = -\sqrt{2-r^2}$. If α is the angle at the vertex of the cone, we have $\tan(\alpha/2) = 2/2 = 1$, so $\alpha/2 = \pi/4$. The cone has equation $z = -r$. Thus, the integral is

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{2-r^2}}^{-r} g(r,\theta,z)r\,dz\,dr\,d\theta.$$

27. In rectangular coordinates, the spherical cap has equation $z = -\sqrt{2-x^2-y^2}$. If α is the angle at the vertex of the cone, we have $\tan(\alpha/2) = 2/2 = 1$, so $\alpha/2 = \pi/4$. The cone has equation $z = -\sqrt{x^2+y^2}$. Thus, the integral is

$$\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{2-x^2-y^2}}^{-\sqrt{x^2+y^2}} h(x,y,z)\,dz\,dy\,dx.$$

28. From Figure 16.138, we have the following iterated integrals:

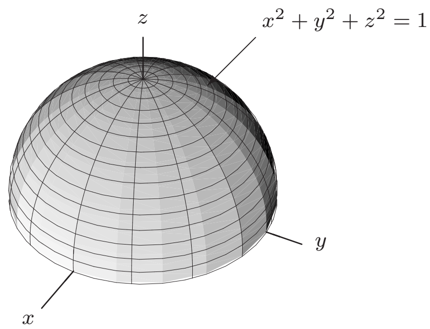


Figure 16.138

(a) $\displaystyle\int_R f\,dV = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

(b) $\displaystyle\int_R f\,dV = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$

(c) $\displaystyle\int_R f\,dV = \int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dz\,dy$

(d) $\displaystyle\int_R f\,dV = \int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$

(e) $$\int_R f\,dV = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$$

(f) $$\int_R f\,dV = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$$

Problems

29. Positive. The value of x is positive on the half-cone, so its integral is positive.

30. Positive. Since z is positive on W, its integral is positive.

31. Positive. If (x, y, z) is any point inside the solid W then $\sqrt{x^2+y^2} < z$. Thus $z - \sqrt{x^2+y^2} > 0$, and so its integral over the solid W is positive.

32. Positive. The function $\sqrt{x^2+y^2}$ is positive, so its integral over the solid W is positive.

33. Negative. If (x, y, z) is any point inside the cone then $z < 2$. Hence the function $z - 2$ is negative on W and so is its integral.

34. Zero. y is positive on the half of the half-cone above the first quadrant in the xy-plane and negative (of equal absolute value) on the half of the half-cone above the fourth quadrant. The integral of y over W is zero because the integrals over each half add up to zero.

35. Zero. You can see this in several ways. One way is to observe that xy is positive on part of the cone above the first quadrant (where x and y are of the same sign) and negative (of equal absolute value) on the part of the cone above the fourth quadrant (where x and y have opposite signs). These add up to zero in the integral of xy over all of W.

Another way to see that the integral is zero is to write the triple integral as an iterated integral, say integrating first with respect to y. For fixed x and z, the y-integral is over an interval symmetric about 0. The integral of y over such an interval is zero. If any of the inner integrals in an iterated integral is zero, then the triple integral is zero.

36. Zero. Write the triple integral as an iterated integral, say integrating first with respect to y. For fixed x and z, the y-integral is over an interval symmetric about 0. The integral of y over such an interval is zero. If any of the inner integrals in an iterated integral is zero, then the triple integral is zero.

37. Positive. The function e^{-xyz} is a positive function everywhere so its integral over W is positive.

38. (a) The top of the tetrahedron is $z = x - y + 2$, and its triangular base is in the second quadrant of the xy-plane bounded by the x- and y-axes, and the line $0 = x - y + 2$, or $y = 2 + x$. Thus the volume is given by

$$V = \int_{-2}^{0} \int_0^{2+x} \int_0^{x-y+2} dz\,dy\,dx.$$

Other orders of integration are possible.

(b) Evaluating gives

$$\begin{aligned} V &= \int_{-2}^{0} \int_0^{2+x} (x-y+2)\,dy\,dx = \int_{-2}^{0} xy - \frac{y^2}{2} + 2y\Big|_0^{2+x} dx \\ &= \int_{-2}^{0} x(2+x) - \frac{1}{2}(2+x)^2 + 2(2+x)\,dx \\ &= \int_{-2}^{0} \left(\frac{x^2}{2} + 2x + 2\right) dx = \frac{x^3}{6} + x^2 + 2x\Big|_{-2}^{0} = \frac{4}{3}. \end{aligned}$$

39. (a) Since z is positive above the xy-plane and negative below the xy-plane, the contributions to $\int_B dV$ and $\int_R zdV$ cancel; both these integrals are zero. Hence (i) and (iii) are zero.
The integral $\int_T z$ is positive.

(b) The equation of the sphere is $x^2+y^2+z^2 = 1$, so the top half is $z = \sqrt{1-x^2-y^2}$. Thus, integrating with respect to z first, we have

$$\int_T z dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{z^2}{2}\bigg|_0^{\sqrt{1-x^2-y^2}} dy\,dx$$
$$= \frac{1}{2}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx$$
$$= \frac{1}{2}\int_{-1}^{1} y - x^2y - \frac{y^3}{3}\bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$
$$= \frac{1}{2}\int_{-1}^{1} \left(2\sqrt{1-x^2} - 2x^2\sqrt{1-x^2} + \frac{2}{3}(1-x^2)^{3/2}\right) dx$$
$$= 1.571$$

40. R is one eighth of a sphere of radius 1, below the xy-plane and under the first quadrant. See Figure 16.139.

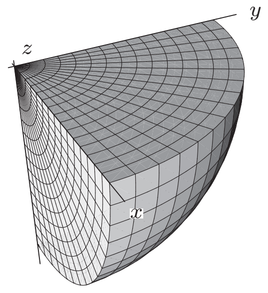


Figure 16.139

41. (a) The region of integration is the region between the cone $z = r$, the xy-plane and the cylinder $r = 3$. In spherical coordinates, $r = 3$ becomes $\rho\sin\phi = 3$, so $\rho = 3/\sin\phi$. The cone is $\phi = \pi/4$ and the xy-plane is $\phi = \pi/2$. See Figure 16.140. Thus, the integral becomes

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{3/\sin\phi} \rho^2\sin\phi\,d\rho d\phi d\theta.$$

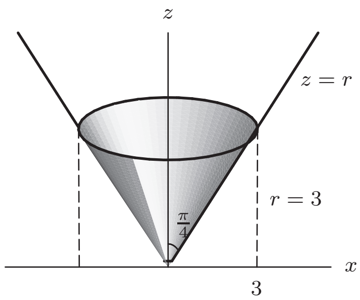


Figure 16.140: Region of integration is between the cone and the xy-plane

(b) The original integral is easier to evaluate, so

$$\int_0^{2\pi}\int_0^{3}\int_0^{r} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{3} zr\bigg|_{z=0}^{z=r} drd\theta = \int_0^{2\pi}\int_0^{3} r^2\,drd\theta = 2\pi\cdot\frac{r^3}{3}\bigg|_0^3 = 18\pi.$$

42. The region stands on a rectangular base in the xy-plane, with vertical sides and a slanting top, the plane $z = 1 + x$. See Figure 16.141. The integral is

$$\int_0^2 \int_0^1 \int_0^{1+x} f(x, y, z)\, dz\, dy\, dx.$$

The order of integration can be altered.

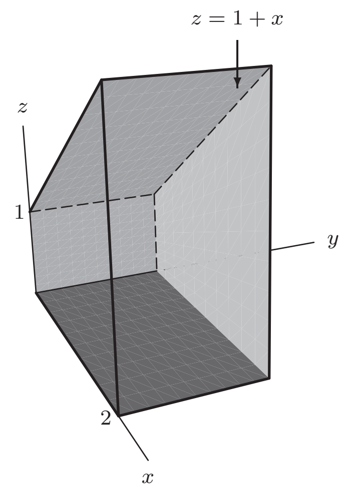


Figure 16.141

43. The region is a solid ring between the planes $z = 2$ and $z = 3$, with inner radius $r = \sqrt{5}$ and outer radius $r = \sqrt{6}$. See Figure 16.142. In cylindrical coordinates, the integral is

$$\int_0^{2\pi} \int_2^3 \int_{\sqrt{5}}^{\sqrt{6}} r\, dr\, dz\, d\theta.$$

The order of integration can be changed.

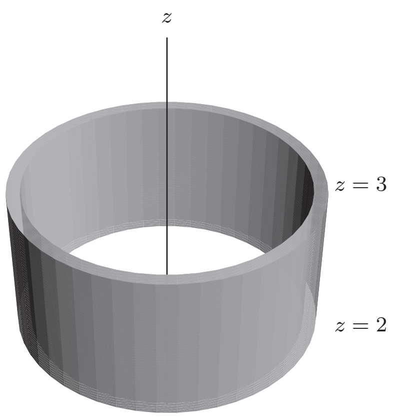


Figure 16.142

44. The region is a hollow half-sphere, with inner radius $\sqrt{3}$ and outer radius $\sqrt{4} = 2$. See Figure 16.143. In spherical coordinates, the integral is

$$\int_0^{\pi} \int_0^{\pi} \int_{\sqrt{3}}^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

The order of integration can be altered.

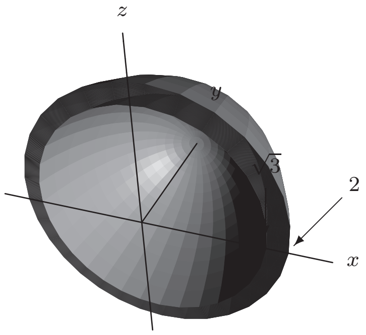


Figure 16.143

45. The region stands on a circular base of radius 1 in the xy-plane and has cylindrical sides. The top is part of a sphere of radius 3. The cylinder meets the sphere where $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 9$, so $1 + z^2 = 9$, $z = \sqrt{8}$. See Figure 16.144. In cylindrical coordinates, the integral is

$$\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{9-r^2}} r\, dz\, dr\, d\theta.$$

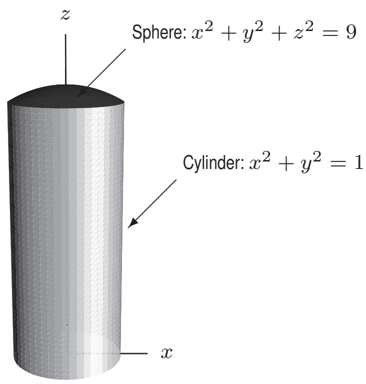


Figure 16.144

46. Positive, since e^{-x} is always positive.
47. Negative, since y^3 is negative on B, where $y < 0$.
48. Positive, since $(x + y^2)$ is positive on R, where $x > 0$.
49. Can't tell, since y^3 is both positive and negative for $x < 0$.
50. Can't tell, since $x < 0$ and $y^2 \geq 0$ on L, where $x < 0$.
51. Zero. The solid sphere is symmetric and z is positive on the top half and negative (of equal absolute value) on the bottom half. The integral of z over the entire solid is zero because the integrals over each half add up to zero.
52. Zero. x is positive on the hemisphere $x^2+y^2+z^2 \leq 1$, $x > 0$ and negative (of equal absolute value) on $x^2+y^2+z^2 \leq 1$, $x < 0$. The integral of x over the entire solid is zero because the integrals over each half add up to zero.
53. Zero. You can see this in several ways. One way is to observe that xy is positive on the part of the sphere above and below the first and third quadrants (where x and y are of the same sign) and negative (of equal absolute value) on the part of the sphere above and below the second and fourth quadrants (where x and y have opposite signs). These add up to zero in the integral of xy over all of W.

    Another way to see that the integral is zero is to write the triple integral as an iterated integral, say integrating first with respect to x. For fixed y and z, the x-integral is over an interval symmetric about 0. The integral of x over such an interval is zero. If any of the inner integrals in an iterated integral is zero, then the triple integral is zero.

54. Zero. Write the triple integral as an iterated integral, say integrating first with respect to x. Then $\sin(\frac{\pi}{2}xy)$ is integrated over an integral symmetric about the origin, and this integral is zero because $\sin(\frac{\pi}{2}xy)$ is an odd function. Since the innermost integral is zero so is the triple integral.

55. Zero. Write the triple integral as an iterated integral, say integrating first with respect to x. For fixed y and z, the x-integral is over an interval symmetric about 0. The integral of x over such an interval is zero. If any of the inner integrals in an iterated integral is zero, then the triple integral is zero.

56. Positive. The function e^{-xyz} is a positive function everywhere so its integral over W is positive.

57. Negative. Since $z^2 - 1 \leq 0$ in the sphere, its integral is negative.

58. Positive. $\sqrt{x^2 + y^2 + z^2}$ is positive on W, so its integral is positive.

59. The integral is over the region $0 \leq x^2 + y^2 \leq 3,\ 1 \leq z \leq 4 - x^2 - y^2$. Using cylindrical coordinates, we get

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{4-r^2} \frac{1}{z^2} r dz\, dr\, d\theta &= \int_0^{2\pi}\int_0^{\sqrt{3}} (-\frac{r}{z})\Big|_1^{4-r^2} dr\, d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} (-\frac{r}{4-r^2} + \frac{r}{1}) dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2}\ln(4-r^2) + \frac{1}{2}r^2\right]_0^{\sqrt{3}} d\theta \\
&= \int_0^{2\pi} (\frac{1}{2}\ln 1 + \frac{3}{2} - \frac{1}{2}\ln 4 - 0) d\theta \\
&= 2\pi(\frac{3}{2} - \ln 2) = \pi(3 - 2\ln 2)
\end{aligned}$$

60. The integral is over the region $x, y \geq 0,\ x^2 + y^2 \leq 1,\ 0 \leq z \leq \sqrt{x^2 + y^2}$. Using cylindrical coordinates, we get

$$\begin{aligned}
\int_0^{\pi/2}\int_0^1\int_0^r (z + r)\, r dz\, dr\, d\theta &= \int_0^{\pi/2}\int_0^1\int_0^r (rz + r^2)\, dz\, dr\, d\theta \\
&= \int_0^{\pi/2}\int_0^1 (\frac{1}{2}r^3 + r^3)\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{3}{8}r^4\Big|_0^1 d\theta \\
&= \frac{3}{8}\cdot\frac{\pi}{2} = \frac{3\pi}{16}
\end{aligned}$$

61. The region is a hemisphere $0 \leq x^2+y^2+z^2 \leq 3^2,\ z \geq 0$, so spherical coordinates are appropriate. Recall the conversion formula $x = \rho\sin\phi\cos\theta$. Then the integral in spherical coordinates becomes

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho^4\sin^3\phi\cos^2\theta\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \frac{243}{5}\sin^3\phi\cos^2\theta\, d\phi\, d\theta \\
&= \frac{243}{5}\int_0^{2\pi}\int_0^{\pi/2} \cos^2\theta\cdot\sin\phi(1-\cos^2\phi)\, d\phi\, d\theta \\
&= \frac{243}{5}\int_0^{2\pi} \cos^2\theta\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{2}} d\theta \\
&= \frac{243}{5}\int_0^{2\pi} \cos^2\theta[-(-1) + \frac{1}{3}(-1)]\, d\theta
\end{aligned}$$

$$= \frac{243}{5} \cdot \frac{2}{3} \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{81}{5}(\theta + \frac{1}{2}\sin 2\theta)\Big|_0^{2\pi} = \frac{81}{5}(2\pi + 0) = \frac{162\pi}{5}$$

62. W is a cylindrical shell, so cylindrical coordinates should be used. See Figure 16.145.

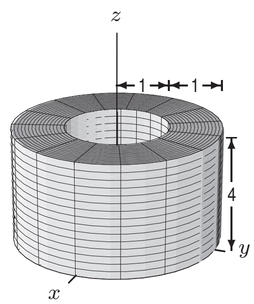


Figure 16.145

$$\begin{aligned}\int_W \frac{z}{(x^2+y^2)^{3/2}}\, dV &= \int_0^4 \int_0^{2\pi} \int_1^2 \frac{z}{r^3}\, r dr\, d\theta\, dz \\ &= \int_0^4 \int_0^{2\pi} \int_1^2 \frac{z}{r^2}\, dr\, d\theta\, dz \\ &= \int_0^4 \int_0^{2\pi} (-\frac{z}{r})\Big|_1^2\, d\theta\, dz \\ &= \int_0^4 \int_0^{2\pi} \frac{z}{2}\, d\theta\, dz \\ &= \int_0^4 \frac{z}{2} \cdot 2\pi\, dz = \frac{1}{2}\pi \cdot z^2\Big|_0^4 = 8\pi\end{aligned}$$

63. (a) The region (shaded) is one eighth of the circle $x^2 + y^2 = 8$; see Figure 16.146. The first integral is above the dashed line $y = 2$; the second integral is below the dashed line.

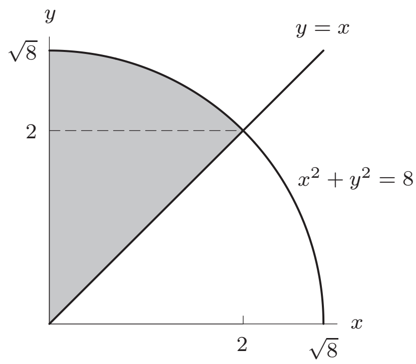


Figure 16.146

(b) Converting to polar coordinates, we find the quantity in part (a) is given by

$$\int_2^{\sqrt{8}} \int_0^{\sqrt{8-y^2}} e^{-x^2-y^2}\, dx\, dy + \int_0^2 \int_0^y e^{-x^2-y^2}\, dx\, dy$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} e^{-r^2} r\,dr\,d\theta$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{\sqrt{8}} = \frac{\pi}{4}\left(\frac{1}{2}e^0 - \frac{1}{2}e^{-8}\right) = \frac{\pi}{8}\left(1 - e^{-8}\right).$$

64. The depth of the lake is given in meters and the diameter in kilometers. We should work with a single unit of length. In this solution we work with kilometers, but meters would work just as well.

The shape of the lake suggests integration in polar coordinates, with r km measured from the center of the island. Thus $t = r - 1$ is the distance in kilometers from the island when r varies between 1 and 5. The depth of the lake r km from the center of the island is

$$\text{Depth} = \frac{100(r-1)(4-(r-1))}{1000} = -\frac{1}{2} + \frac{3r}{5} - \frac{r^2}{10} \text{ km}.$$

$$\text{Volume of the lake } = \int_0^{2\pi} \int_1^5 \left(-\frac{1}{2} + \frac{3r}{5} - \frac{r^2}{10}\right) r\,dr\,d\theta = \frac{32\pi}{5} = 20.1 \text{ km}^3.$$

65. (a) The equation of the curved surface of this half cylinder along the x-axis is $(y-1)^2 + z^2 = 1$. The part we want is

$$z = \sqrt{1-(y-1)^2} \quad 0 \le y \le 2 \quad 0 \le x \le 10.$$

(b) The integral

$$\int_D f(x,y,z)\,dV = \int_0^{10} \int_0^2 \int_0^{\sqrt{1-(y-1)^2}} f(x,y,z)\,dz\,dy\,dx.$$

66. Figure 16.147 shows a slice through the region for a fixed x. Breaking the volume into small cubes each of volume $\Delta V = \Delta x\,\Delta y\,\Delta z$ and stacking the cubes above $(x, y, 0)$, starting at $z = 0$ and going up to $z = x + y$, tells us the inner integral is

$$\int_0^{x+y} dz.$$

Lining up the stacks parallel to the z axis gives a slice, for each fixed value of x, from $y = 0$ to $x + y = 5$, thus the middle integral is

$$\int_0^{5-x} \int_0^{x+y} dz\,dy.$$

Finally, adding up the contributions for $x = 0$ to $x = 5$ gives the volume, V, as

$$V = \int_0^5 \int_0^{5-x} \int_0^{x+y} dz\,dy\,dx = \int_0^5 \int_0^{5-x} (x+y)\,dy\,dx$$

$$= \int_0^5 \left(xy + \frac{1}{2}y^2\right)\Bigg|_{y=0}^{y=5-x} dx = \int_0^5 \left(x(5-x) + \frac{1}{2}(5-x)^2\right) dx = \frac{125}{3}.$$

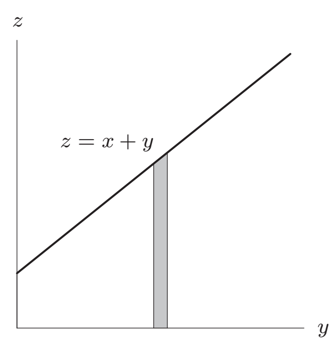


Figure 16.147

67. (a) The region is the half cylinder in Figure 16.148.

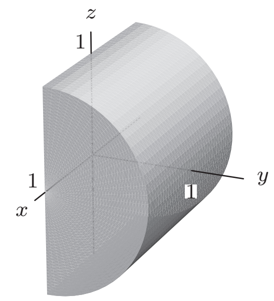


Figure 16.148

(b) Use cylindrical coordinates with x replacing z and y in place of x and z in place of y. Then

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} f(x,y,z)\,dy\,dz\,dx = \int_{-1}^{1}\int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^3\cdot r\,dr\,d\theta\,dx = x\Big|_{-1}^{1}\ \theta\Big|_{-\pi/2}^{\pi/2}\ \frac{r^5}{5}\Big|_{0}^{1} = \frac{2\pi}{5}.$$

68. (a) The density increases at a rate of $(25-1)/12 = 2$ gm/cm^2 for each cm of radius. Thus, at radius r,

$$\text{Density } = 1 + 2r \text{ gm/cm}^2.$$

Thus

$$\text{Mass } = \int_0^{2\pi}\int_0^{12} (1+2r)\,r\,dr\,d\theta \text{ gm}.$$

(b) Evaluating gives

$$\text{Mass } = \int_0^{2\pi}\left(\frac{r^2}{2} + \frac{2}{3}r^3\right)\Big|_0^{12} d\theta = 2\pi\cdot 1224 = 2448\pi \text{ gm}.$$

69. Orient the region as shown in Figure 16.149 and use cylindrical coordinates with the origin at the center of the sphere. The equation of the sphere is $x^2+y^2+z^2 = 25$, or $r^2+z^2 = 25$. If $z = 3$, then $r^2+3^2 = 25$, so $r^2 = 16$ and $r = 4$.

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi}\int_0^{4}\int_3^{\sqrt{25-r^2}} r\,dz dr d\theta = \int_0^{2\pi}\int_0^{4} rz\Big|_{z=3}^{z=\sqrt{25-r^2}} dr d\theta \\ &= \int_0^{2\pi}\int_0^{4} (r\sqrt{25-r^2} - 3r)\,dr d\theta = \theta\Big|_0^{2\pi}\left(-\frac{(25-r^2)^{3/2}}{3} - \frac{3r^2}{2}\right)\Big|_0^{4} \\ &= 2\pi\left(\left(\frac{125}{3} - \frac{27}{3}\right) - 24\right) = \frac{52\pi}{3} = 54.45 \text{ cm}^3.\end{aligned}$$

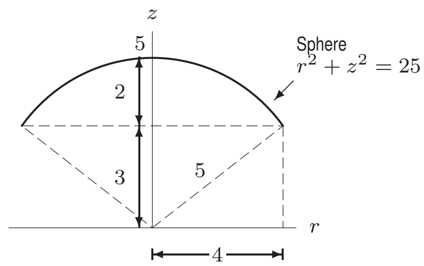


Figure 16.149

70. The region of integration is shown in Figure 16.150, and the mass of the given solid is given by

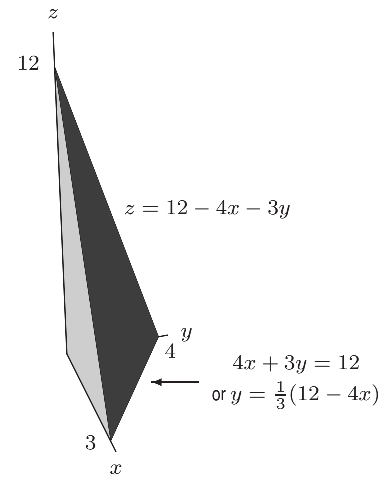


Figure 16.150

$$
\begin{aligned}
\text{Mass} &= \int_R \delta\, dV \\
&= \int_0^3 \int_0^{\frac{1}{3}(12-4x)} \int_0^{12-4x-3y} x^2\, dz\, dy\, dx \\
&= \int_0^3 \int_0^{\frac{1}{3}(12-4x)} x^2 z \Big|_{z=0}^{z=12-4x-3y} dydx \\
&= \int_0^3 \int_0^{\frac{1}{3}(12-4x)} x^2(12 - 4x - 3y)\, dydx \\
&= \int_0^3 x^2(12y - 4xy - \frac{3}{2}y^2)\Big|_0^{y=\frac{1}{3}(12-4x)} dx \\
&= \left(8x^3 - 4x^4 + \frac{8}{15}x^5\right)\Big|_0^3 \\
&= \frac{108}{5}.
\end{aligned}
$$

71. Orient the region as shown in Figure 16.151 and use spherical coordinates with the origin at the center of the sphere. The equation of the sphere is $x^2 + y^2 + z^2 = 25$, or $\rho = 5$. The plane $z = 3$ is the plane $\rho\cos\phi = 3$, so $\rho = 3/\cos\phi$. In Figure 16.151, angle AOB is given by

$$\cos\phi = \frac{3}{5}, \qquad \text{so} \qquad \phi = \arccos(3/5).$$

The volume is given by

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\arccos(3/5)} \int_{3/\cos\phi}^{5} \rho^2 \sin\phi\, d\rho d\phi d\theta = \theta\Big|_0^{2\pi} \int_0^{\arccos(3/5)} \sin\phi \frac{\rho^3}{3}\Big|_{p=3/\cos\phi}^{p=5} d\phi \\
&= 2\pi \int_0^{\arccos(3/5)} \left(\frac{125}{3} - \frac{9}{\cos^3\phi}\right) \sin\phi\, d\phi
\end{aligned}
$$

$$= 2\pi \left(\int_0^{\arccos(3/5)} \frac{125}{3} \sin\phi \, d\phi - \int_0^{\arccos(3/5)} \frac{9}{\cos^3\phi} \sin\phi \, d\phi \right)$$
$$= 2\pi \left(\left(-\frac{125}{3}\cos\phi \right) \Bigg|_0^{\arccos(3/5)} - 9\left(\frac{1}{2\cos^2}\phi \right) \Bigg|_0^{\arccos(3/5)} \right)$$
$$= 2\pi \left(-\frac{125}{3}\left(\frac{3}{5} - 1\right) - \frac{9}{2}\left(\frac{1}{(3/5)^2} - 1 \right) \right)$$
$$= 2\pi \left(-\frac{125}{3}\left(-\frac{2}{5}\right) - \frac{9}{2}\left(\frac{16}{9}\right) \right)$$
$$= 2\pi \left(\frac{50}{3} - 8 \right) = \frac{52\pi}{3} \approx 54.45 \text{ cm}^3.$$

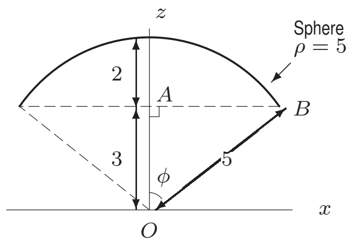


Figure 16.151

72. Let the lower left part of the forest be at $(0, 0)$. Then the other corners have coordinates as shown. The population density function is then given by

$$\rho(x, y) = 10 - 2y$$

The equations of the two diagonal lines are $x = -2y/5$ and $x = 6 + 2y/5$. So the total rabbit population in the forest is

$$\int_0^5 \int_{-\frac{2}{5}y}^{6+\frac{2}{5}y} (10 - 2y)\, dx\, dy = \int_0^5 (10 - 2y)(6 + \frac{4}{5}y)\, dy$$
$$= \int_0^5 (60 - 4y - \frac{8}{5}y^2)\, dy$$
$$= (60y - 2y^2 - \frac{8}{15}y^3) \Bigg|_0^5$$
$$= 300 - 50 - \frac{8}{15} \cdot 125$$
$$= \frac{2750}{15} = \frac{550}{3} \approx 183$$

73. We use spherical coordinates. Since the density, δ, is equal to the distance from the point to the origin, we have

$$\delta = \rho \text{ gm/cm}^3.$$

Therefore the mass of the hemisphere is given by

$$\text{Mass} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho \cdot \rho^2 \sin\phi \, d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^4}{4} \Bigg|_0^2 \sin\phi \, d\phi d\theta$$
$$= 4 \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \, d\phi d\theta = 4 \int_0^{2\pi} (-\cos\phi) \Bigg|_0^{\pi/2} d\theta = 4 \cdot 2\pi \cdot 1 = 8\pi \text{ gm}.$$

74. Since the hole resembles a cylinder, we will use cylindrical coordinates. Let the center of the sphere be at the origin, and let the center of the hole be the z-axis (see Figure 16.152).

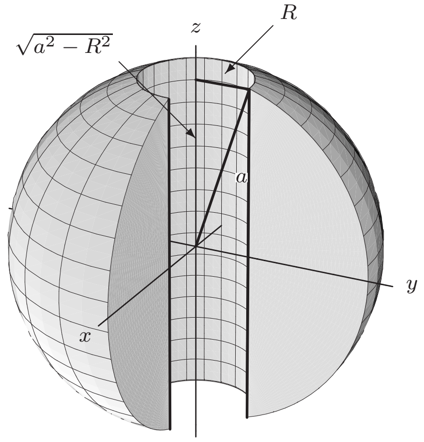


Figure 16.152

Then we will integrate from $z = -\sqrt{a^2 - R^2}$ to $z = \sqrt{a^2 - R^2}$, and each cross-section will be an annulus. So the volume is

$$\begin{aligned}\int_{-\sqrt{a^2-R^2}}^{\sqrt{a^2-R^2}} \int_0^{2\pi} \int_R^{\sqrt{a^2-z^2}} r\,dr\,d\theta\,dz &= \int_{-\sqrt{a^2-R^2}}^{\sqrt{a^2-R^2}} \int_0^{2\pi} \frac{1}{2}(a^2 - z^2 - R^2)\,d\theta\,dz \\ &= \pi \int_{-\sqrt{a^2-R^2}}^{\sqrt{a^2-R^2}} (a^2 - z^2 - R^2)\,dz \\ &= \pi\left[(a^2 - R^2)(2\sqrt{a^2 - R^2}) - \frac{1}{3}(2(a^2 - R^2)^{\frac{3}{2}})\right] \\ &= \frac{4\pi}{3}(a^2 - R^2)^{\frac{3}{2}}\end{aligned}$$

75. We must first decide on coordinates. We pick Cartesian coordinates with the smaller sphere centered at the origin, the larger one centered at $(0, 0, -1)$. A vertical cross-section of the region in the xz-plane is shown in Figure 16.153. The smaller sphere has equation $x^2 + y^2 + z^2 = 1$. The larger sphere has equation $x^2 + y^2 + (z + 1)^2 = 2$.

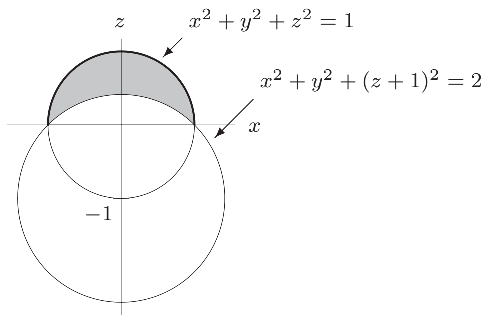


Figure 16.153

Let R represent the region in the xy-plane which lies directly underneath (or above) the region whose volume we want. The curve bounding this region is a circle, and we find its equation by solving the system:

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + y^2 + (z + 1)^2 &= 2 \end{aligned}$$

Subtracting the equations gives

$$\begin{aligned}(z+1)^2 - z^2 &= 1\\ 2z+1 &= 1\\ z &= 0.\end{aligned}$$

Since $z = 0$, the two surfaces intersect in the xy-plane in the circle $x^2 + y^2 = 1$. Thus R is $x^2 + y^2 \le 1$.

The top half of the small sphere is represented by $z = \sqrt{1 - x^2 - y^2}$; the top half of the large sphere is represented by $z = -1 + \sqrt{2 - x^2 - y^2}$. Thus the volume is given by

$$\text{Volume} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-1+\sqrt{2-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

Starting to evaluate the integral, we get

$$\text{Volume} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (\sqrt{1-x^2-y^2} + 1 - \sqrt{2-x^2-y^2})\, dydx.$$

We simplify the integral by converting to polar coordinates

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi} \int_0^1 \left(\sqrt{1-r^2} + 1 - \sqrt{2-r^2}\right) r\, drd\theta\\ &= \int_0^{2\pi} \left(-\frac{(1-r^2)^{3/2}}{3} + \frac{r^2}{2} + \frac{(2-r^2)^{3/2}}{3}\right)\Bigg|_0^1 d\theta\\ &= 2\pi\left(\frac{1}{2} + \frac{1}{3} - \left(-\frac{1}{3} + \frac{2^{3/2}}{3}\right)\right) = 2\pi\left(\frac{7}{6} - \frac{2\sqrt{2}}{3}\right) = 1.41.\end{aligned}$$

76. Suppose the brick is set up as shown in Figure 16.154.

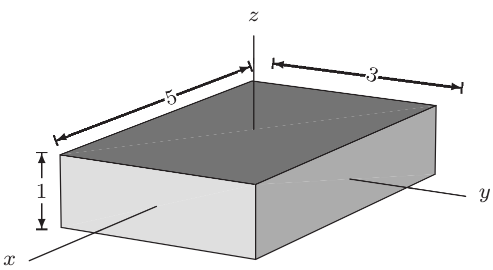


Figure 16.154

The brick has m/v = density = 1. The moment of inertia about the z-axis is

$$\begin{aligned}I_z &= \int_{-5/2}^{5/2} \int_{-3/2}^{3/2} \int_{-1/2}^{1/2} 1(x^2+y^2)\, dz\, dy\, dx\\ &= \int_{-5/2}^{5/2} \int_{-3/2}^{3/2} (x^2+y^2)\, dy\, dx\\ &= \int_{-5/2}^{5/2} (3x^2 + \frac{9}{4})\, dx\\ &= \frac{125}{4} + \frac{45}{4} = \frac{85}{2}\end{aligned}$$

The moment of inertia about the y-axis is

$$\begin{aligned}
I_y &= \int_{-5/2}^{5/2}\int_{-3/2}^{3/2}\int_{-1/2}^{1/2} 1(x^2+z^2)\,dz\,dy\,dx \\
&= \int_{-5/2}^{5/2}\int_{-3/2}^{3/2} (x^2+\frac{1}{12})\,dy\,dx \\
&= \int_{-5/2}^{5/2} (3x^2+\frac{1}{4})\,dx \\
&= \frac{125}{4}+\frac{5}{4}=\frac{65}{2}
\end{aligned}$$

The moment of inertia about the x-axis is

$$\begin{aligned}
I_x &= \int_{-5/2}^{5/2}\int_{-3/2}^{3/2}\int_{-1/2}^{1/2} 1(y^2+z^2)\,dz\,dy\,dx \\
&= \int_{-5/2}^{5/2}\int_{-3/2}^{3/2} (y^2+\frac{1}{12})\,dy\,dx \\
&= \int_{-5/2}^{5/2} (\frac{9}{4}+\frac{1}{4})\,dx \\
&= 5\cdot\frac{10}{4}=\frac{25}{2}
\end{aligned}$$

77. Let the ball be centered at the origin. Since a ball looks the same from all directions, we can choose the axis of rotation; in this case, let it be the z-axis. It is best to use spherical coordinates, so then

$$\begin{aligned}
x^2+y^2 &= (\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2 \\
&= \rho^2\sin^2\phi
\end{aligned}$$

Then $m/v = \text{Density} = 1$, so the moment of inertia is

$$\begin{aligned}
I_z &= \int_0^R\int_0^{2\pi}\int_0^{\pi} 1(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\phi\,d\theta\,d\rho \\
&= \int_0^R\int_0^{2\pi}\int_0^{\pi} \rho^4(\sin\phi)(1-\cos^2\phi)\,d\phi\,d\theta\,d\rho \\
&= \int_0^R\int_0^{2\pi} \rho^4(-\cos\phi+\frac{1}{3}\cos^3\phi)\Big|_0^{\pi}\,d\theta\,d\rho \\
&= \int_0^R\int_0^{2\pi} \frac{4}{3}\rho^4\,d\theta\,d\rho \\
&= \int_0^R \frac{8\pi}{3}\rho^4\,d\rho = \frac{8}{15}\pi R^5
\end{aligned}$$

78. Set up the cylinder with the base centered at the origin on the xy plane, facing up. (See Figure 16.155.) Newton's Law of Gravitation states that the force exerted between two particles is

$$F = G\frac{m_1m_2}{\rho^2}$$

where G is the gravitational constant, m_1 and m_2 are the masses, and ρ is the distance between the particles. We take a small volume element, so $m_1 = m$, and $m_2 = \delta dV$. In cylindrical coordinates, if m is at (0,0,0) and δdV is at (r,θ,z), (see Figure 16.155), then the distance from m to δdV is given by $\rho = \sqrt{r^2+z^2}$ for $r_1 \le r \le r_2$ and $0 \le z \le h$.

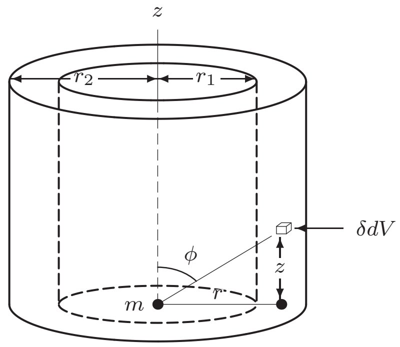


Figure 16.155

Due to the symmetry of the cylinder the sum of all the horizontal forces is zero; the net force on m is vertical. The force acting on the particle as a result of the small piece dV makes an angle ϕ with the vertical and therefore has vertical component

$$\begin{array}{l}\text{Vertical force on}\\ \text{particle from small}\\ \text{piece of cylinder}\end{array} = \frac{Gm\delta dV}{(\sqrt{r^2+z^2})^2}\cdot\cos\phi = \frac{Gm\delta dV}{r^2+z^2}\cdot\frac{z}{\sqrt{r^2+z^2}} = \frac{Gm\delta z}{(r^2+z^2)^{\frac{3}{2}}}\,dV.$$

Thus, since $dV = rdzdrd\theta$,

$$\begin{aligned}\text{Total force} &= \int_0^{2\pi}\int_{r_1}^{r_2}\int_0^h \frac{Gm\delta z}{(r^2+z^2)^{3/2}} r\,dzdrd\theta\\ &= 2\pi Gm\delta\int_{r_1}^{r_2}\int_0^h \frac{zr}{(r^2+z^2)^{3/2}}drdz\\ &= 2\pi Gm\delta\int_{r_1}^{r_2}\left(1-\frac{r}{(r^2+h^2)^{\frac{1}{2}}}\right)dr\\ &= 2\pi Gm\delta(r-(r^2+h^2)^{\frac{1}{2}})\Big|_{r_1}^{r_2}\\ &= 2\pi Gm\delta(r_2-r_1-\sqrt{r_2^2+h^2}+\sqrt{r_1^2+h^2}).\end{aligned}$$

79. (a) The constant k is determined by the condition that $\int_R k(x+y)dA = 1$ where the region R is the quarter disk with radius 10

$$x^2+y^2\le 100 \qquad x\ge 0 \qquad y\ge 0.$$

Using polar coordinates gives the integral

$$\begin{aligned}\int_R k(x+y)dA &= \int_0^{\pi/2}\int_0^{10} k(r\cos\theta+r\sin\theta)r\,dr\,d\theta\\ &= k\int_0^{\pi/2}\int_0^{10} r^2(\cos\theta+\sin\theta)dr\,d\theta\\ &= k\int_0^{\pi/2}\frac{1000}{3}(\cos\theta+\sin\theta)d\theta\\ &= \frac{1000k}{3}(\sin\theta-\cos\theta)\Big|_0^{\pi_2} = \frac{1000k}{3}2 = \frac{2000k}{3}.\end{aligned}$$

Since $2000k/3 = 1$, we have $k = 3/2000$.

(b) Evaluate the integral $\int_S f dA$ where S is the region $0\le r\le 7$, $0\le\theta\le\pi/2$. We have

$$\begin{aligned}\int_S fdA &= \frac{3}{2000}\int_0^{\pi/2}\int_0^7 r^2(\cos\theta+\sin\theta)dr\,d\theta = \frac{3}{2000}\int_0^{\pi/2}\frac{7^3}{3}(\cos\theta+\sin\theta)d\theta\\ &= \frac{7^3}{2000}(\sin\theta-\cos\theta)\Big|_0^{\pi/2} = \frac{7^3}{2000}2 = \frac{343}{1000}.\end{aligned}$$

The probability that the point is closer than 7 units from the origin is $343/1000$.

CAS Challenge Problems

80. The region is the triangle to the right of the y-axis, below the line $y = 1$, and above the line $y = x$. Thus the integral can be written as $\int_0^1 \int_x^1 e^{y^2}\, dydx$ or as $\int_0^1 \int_0^y e^{y^2}\, dxdy$. The second of these integrals can be evaluated easily by hand:

$$\int_0^1 \int_0^y e^{y^2}\, dxdy = \int_0^1 \left(e^{y^2} x \Big|_{x=0}^{x=y} \right) dy = \int_0^1 y e^{y^2}\, dy$$
$$= \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2}(e-1)$$

The other integral cannot be done by hand with the methods you have learned, but some computer algebra systems will compute it and give the same answer.

81. In Cartesian coordinates the integral is

$$\int_D \sqrt[3]{x^2+y^2}\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt[3]{x^2+y^2}\, dydx.$$

In polar coordinates it is

$$\int_D \sqrt[3]{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt[3]{r^2}\, r dr d\theta = \int_0^{2\pi} \int_0^1 r^{5/3}\, dr d\theta$$
$$= \int_0^{2\pi} \frac{3}{8}\, d\theta = \frac{6\pi}{7}$$

The Cartesian coordinate version requires the use of a computer algebra system. Some CASs may be able to handle it and may give the answer in terms of functions called hypergeometric functions. To compare the answers are the same you may need to ask the CAS to give a numerical value for the answer. It's possible your CAS will not be able to handle the integral at all.

82. $\displaystyle\int_0^1 \int_{-1}^0 \frac{x+y}{(x-y)^3}\, dydx = 1/2$ and $\int_{-1}^0 \int_0^1 \frac{x+y}{(x-y)^3}\, dxdy = -1/2$. This does not contradict the theorem because the function is not continuous everywhere inside the region of integration. In fact, it is not even defined at the origin.

83.

$$\text{Average value for } F = \frac{1}{\text{Area}} \int_{-h}^h \int_{-h}^h (a + bx^4 + cy^4 + dx^2y^2 + ex^3y^3)\, dx\, dy$$
$$= \frac{1}{4h^2} \left(4ah^2 + \frac{4bh^6}{5} + \frac{4ch^6}{5} + \frac{4dh^6}{9} \right)$$
$$= a + \frac{1}{45} (9b + 9c + 5d)\, h^4$$

The limit is

$$\lim_{h\to 0} (a + \frac{1}{45} (9b + 9c + 5d)\, h^4) = a.$$

Notice that $F(0,0) = a$.

$$\text{Average value for } G = \frac{1}{4h^2} \int_{-h}^h \int_{-h}^h (a\sin(kx) + b\cos(ky) + c)\, dx\, dy$$
$$= \frac{1}{4h^2} \left(\frac{4\left(ch^2k + bh\sin(hk)\right)}{k} \right)$$
$$= c + \frac{b\sin(hk)}{hk}.$$

The limit is

$$\lim_{h\to 0} \left(c + \frac{b\sin(hk)}{hk} \right) = c + b \lim_{h\to 0} \frac{\sin(hk)}{hk} = b + c.$$

Notice that $G(0,0) = b + c$.

Finally,

$$\text{Average value for } H = \frac{(a+b)\left(-1+e^{2h}\right)\left(-2-2h-h^2+e^{2h}\left(2-2h+h^2\right)\right)}{4e^{2h}h^2}.$$

You may need to simplify the answer given by your CAS to get this form. The limit of this as $h \to 0$ (calculated with a CAS) is 0. This is equal to $H(0,0)$.

In each case the limit of the average values over smaller and smaller squares centered at the origin is equal to the value of the function at the origin. We conjecture that this is true in general for a continuous function. This makes sense because when the square is small, the function is approximately constant on the square with value equal to its value at the origin. Therefore the integral is approximately the area times the value of the function, so the average value is approximately the value of the function. This approximation gets better and better as $h \to 0$.

PROJECTS FOR CHAPTER SIXTEEN

1. (a) We are integrating over the whole plane, so converting to polar coordinates gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\bigg|_0^{\infty} d\theta = \int_0^{2\pi}\frac{1}{2}d\theta = \pi.$$

(b) Rewriting the integrand as a product gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}dxdy.$$

Now e^{-y^2} is a constant as far as the integral with respect to x is concerned, so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}dxdy = \int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)dy.$$

We assume that the integral with respect to x converges, and so is a constant as far as the integral with respect to y is concerned. Thus, we have

$$\int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right).$$

But $\int_{-\infty}^{\infty} e^{-x^2}dx$ and $\int_{-\infty}^{\infty} e^{-y^2}dy$ are the same number, so we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2.$$

(c) Using the results of parts (a) and (b), we have

$$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \pi.$$

Taking square roots and observing that the integral we are looking for is positive, we have

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

2. (a) We want to find the average value of $|x-y|$ over the square $0 \le x \le 1, 0 \le y \le 1$:

$$\text{Average distance between gates} = \int_0^1 \int_0^1 |x-y|\,dy\,dx.$$

Let's fix x, with $0 \le x \le 1$. Then $|x-y| = \begin{cases} y-x & \text{for } y \ge x \\ x-y & \text{for } y \le x \end{cases}$. Therefore

$$\begin{aligned}\int_0^1 |x-y|\,dy &= \int_0^x (x-y)\,dy + \int_x^1 (y-x)\,dy \\ &= \left(xy - \frac{y^2}{2}\right)\Bigg|_0^x + \left(\frac{y^2}{2} - xy\right)\Bigg|_x^1 = x^2 - \frac{x^2}{2} + \frac{1}{2} - x - \frac{x^2}{2} + x^2 \\ &= x^2 - x + \frac{1}{2}.\end{aligned}$$

So,

$$\begin{aligned}\text{Average distance between gates} &= \int_0^1 \int_0^1 |x-y|\,dy\,dx \\ &= \int_0^1 \left(\int_0^1 |x-y|\,dy\right)\,dx = \int_0^1 (x^2 - x + \frac{1}{2})\,dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\Bigg|_0^1 = \frac{1}{3}.\end{aligned}$$

(b) There are $(n+1)^2$ possible pairs (i,j) of gates, $i = 0,\ldots,n, j = 0,\ldots,n$, so the sum given represents the average distances apart of all such gates. The Riemann sum with $\Delta x = \Delta y = 1/n$, if we choose the least x and y-values in each subdivision is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left|\frac{i}{n} - \frac{j}{n}\right|\frac{1}{n^2},$$

which for large n is just about the same as the other sum. For $n = 5$ the sum is about 0.389; for $n = 10$ the sum is about 0.364.

CHAPTER SEVENTEEN

Solutions for Section 17.1

Exercises

1. Since we are moving on the y-axis, $x = 0$, and y goes from -2 to 1. Thus a possible parameterization is

$$x = 0, \quad y = t, \qquad -2 \leq t \leq 1.$$

2. We want a quarter-circle of radius 2 starting at $(2, 0)$ and ending at $(0, 2)$. The equations $x = 2\cos t$, $y = 2\sin t$ describe counterclockwise motion in a circle of radius 2 centered at the origin, passing $(2, 0)$ when $t = 0$ and $(0, 2)$ when $t = \pi/2$. So a possible parameterization is

$$x = 2\cos t, \quad y = 2\sin t, \quad 0 \leq t \leq \pi/2.$$

3. As we move down the straight line from $(0, 3)$ to $(1, 0)$, x increases linearly from 0 to 1 and y decreases linearly from 3 to 0. Thus a possible parameterization is

$$x = t, \quad y = 3 - 3t, \quad 0 \leq t \leq 1.$$

4. We want the bottom half of a semicircle of radius 1 centered at $(0, 1)$. The equations $x = \cos t$, $\quad y = 1 + \sin t$ describe clockwise motion in this circle, passing $(-1, 1)$ when $t = \pi$ and $(1, 1)$ when $t = 2\pi$. So a possible parameterization is

$$x = \cos t, \quad y = 1 + \sin t, \quad \pi \leq t \leq 2\pi.$$

5. We want the straight line segment from $(1, 1)$ to $(3, 2)$. The position vector of $(1, 1)$ is $\vec{i} + \vec{j}$ and the displacement vector from $(1, 1)$ to $(3, 2)$ is $2\vec{i} + \vec{j}$, so the line has equation

$$\vec{r} = \vec{i} + \vec{j} + t(2\vec{i} + \vec{j}),$$

or

$$x = 1 + 2t, \quad y = 1 + t.$$

This passes $(1, 1)$ when $t = 0$ and $(3, 2)$ when $t = 1$, so a possible parameterization is

$$x = 1 + 2t, \quad y = 1 + t, \quad 0 \leq t \leq 1.$$

6. The curve is a segment of a parabola $y = ax^2$ starting at $(0, 0)$ and ending up at $(2, 2)$. Thus the parabola has equation $y = \frac{1}{2}x^2$. Since x goes from 0 to 2, a possible parameterization is

$$x = t, \quad y = \frac{1}{2}t^2, \quad 0 \leq t \leq 2.$$

7. One possible parameterization is

$$x = t, \quad y = 1, \quad z = -t.$$

8. One possible parameterization is

$$x = 3 + t, \quad y = 2t, \quad z = -4 - t.$$

9. One possible parameterization is

$$x = 1, \quad y = 0, \quad z = t.$$

10. One possible parameterization is

$$x = 5, \quad y = -1 + 5t, \quad z = 1 + 2t.$$

11. One possible parameterization is

$$x = 1 + 3t, \quad y = 2 - 3t, \quad z = 3 + t.$$

12. One possible parameterization is

$$x = -3 + 2t, \quad y = 4 + 2t, \quad z = -2 - 3t.$$

13. The displacement vector from the first point to the second is $\vec{v} = (-1 - (-3))\vec{i} + (-3 - (-2))\vec{j} + (-1 - 1)\vec{k} = 2\vec{i} - \vec{j} - 2\vec{k}$. The line through point $(-3, -2, 1)$ and with direction vector $\vec{v} = 2\vec{i} - \vec{j} - 2\vec{k}$ is given by parametric equations

$$x = -3 + 2t,$$
$$y = -2 - t,$$
$$z = 1 - 2t.$$

Other parameterizations of the same line are also possible.

14. The displacement vector from the first point to the second is $\vec{v} = 4\vec{i} - 5\vec{j} - 3\vec{k}$. The line through point $(1, 5, 2)$ and with direction vector $\vec{v} = 4\vec{i} - 5\vec{j} - 3\vec{k}$ is given by parametric equations

$$x = 1 + 4t,$$
$$y = 5 - 5t,$$
$$z = 2 - 3t.$$

Other parameterizations of the same line are also possible.

15. The vector connecting the two points is $3\vec{i} - \vec{j} + \vec{k}$. So a possible parameterization is

$$x = 2 + 3t, \quad y = 3 - t, \quad z = -1 + t.$$

16. The line passes through $(3, -2, 2)$ and $(0, 2, 0)$. The displacement vector from the first of these points to the second is $\vec{v} = (0 - 3)\vec{i} + (2 - (-2))\vec{j} + (0 - 2)\vec{k} = -3\vec{i} + 4\vec{j} - 2\vec{k}$. The line through point $(3, -2, 2)$ and with direction vector $\vec{v} = -3\vec{i} + 4\vec{j} - 2\vec{k}$ is given by parametric equations

$$x = 3 - 3t,$$
$$y = -2 + 4t,$$
$$z = 2 - 2t.$$

Other parameterizations of the same line are also possible.

17. The line passes through $(3, 0, 0)$ and $(0, 0, -5)$. The displacement vector from the first of these points to the second is $\vec{v} = -3\vec{i} - 5\vec{k}$. The line through point $(3, 0, 0)$ and with direction vector $\vec{v} = -3\vec{i} - 5\vec{k}$ is given by parametric equations

$$x = 3 - 3t,$$
$$y = 0,$$
$$z = -5t.$$

Other parameterizations of the same line are also possible.

18. A vector along the line through these points is $\vec{v} = 2\vec{i} + 2\vec{j} - \vec{k}$. Since the line goes through the point $(2, 1, 3)$, a parametric equation for the line segment is

$$x = 2 + 2t, y = 1 + 2t, z = 3 - t \text{ with } 0 \le t \le 1.$$

19. Since the radius is 3, the center is $(0, 0, 5)$, and the circle lies in the $z = 5$ plane, a parametric equation is

$$x = 3\cos t, y = 3\sin t, z = 5, \text{ for } 0 \le t < 2\pi.$$

20. The vector $\vec{n} = 2\vec{i} - 3\vec{j} - \vec{k}$ is normal to the plane, so the line is in the direction of $\vec{n}$ and through the point $(1, 1, 6)$. A possible equation is

$$x = 1 + 2t, y = 1 - 3t, z = 6 - t.$$

21. The xy-plane is where $z = 0$, and to make the particle go in the clockwise direction we start at $(2, 0, 0)$ and head in the negative y-direction. Thus one possible answer is

$$x = 2\cos t, \quad y = -2\sin t, \quad z = 0.$$

22. Since the radius is 2, the circle must be of the form $x = 2\cos t, y = 2\sin t, z = 1$. But this parameterization traces out the circle clockwise when viewed from below. Therefore, the parameterization we want is $x = 2\cos t, y = -2\sin t, z = 1$.

23. The xz-plane is $y = 0$, so one possible answer is

$$x = 2\cos t, \quad y = 0, \quad z = 2\sin t.$$

24. The circle lies in the plane $z = 2$, so one possible answer is

$$x = 3\cos t, \quad y = 3\sin t, \quad z = 2.$$

25. The yz-plane is $x = 0$, so the circle of radius 3 in the yz-plane centered at the origin would have equations

$$x = 0, \quad y = 3\cos t, \quad z = 3\sin t.$$

To move the center to $(0, 0, 2)$ we add 2 to the equation for z, so one possible answer is

$$x = 0, \quad y = 3\cos t, \quad z = 2 + 3\sin t.$$

26. The circle of radius 5 in the yz-plane centered at the origin has equations

$$x = 0, \quad y = 5\cos t, \quad z = 5\sin t.$$

To move the center to $(-1, 0, -2)$, we add -1 to the equation for x and -2 to the equation for z, so one possible answer is

$$x = -1, \quad y = 5\cos t, \quad z = -2 + 5\sin t.$$

27. The xy-plane is $z = 0$, so a possible answer is

$$x = t^2, \quad y = t, \quad z = 0.$$

28. The xy-plane is $z = 0$, so a possible answer is

$$x = t, \quad y = t^3, \quad z = 0.$$

29. The xz-plane is $y = 0$, so a possible answer is

$$x = -3t^2, \quad y = 0, \quad z = t.$$

30. The plane $z = 2$ cuts the cone $z = \sqrt{x^2 + y^2}$ in the circle

$$2 = \sqrt{x^2 + y^2}.$$

This circle is centered on the z-axis, lies in the plane $z = 2$, and has radius 2. A parametric equation is

$$x = 2\cos t, y = 2\sin t, z = 2, \quad \text{for } 0 \le t < 2\pi.$$

31. Since the curve is parallel to the xy-plane, z is constant, and since it passes through $(0, 4, 4)$, we have $z = 4$. One possible answer is

$$x = t, \quad y = 4 - 5t^4, \quad z = 4.$$

32. Since its diameters are parallel to the y and z-axes and its center is in the yz-plane, the ellipse must lie in the yz-plane, $x = 0$. The ellipse with the same diameters centered at the origin would have its y-coordinate range between $-5/2$ and $5/2$ and its z-coordinate range between -1 and 1. Thus this ellipse has equation

$$x = 0, \quad y = \frac{5}{2}\cos t, \quad z = \sin t.$$

To move the center to $(0, 1, -2)$, we add 1 to the equation for y and -2 to the equation for z, so one possible answer for our ellipse is

$$x = 0, \quad y = 1 + \frac{5}{2}\cos t, \quad z = -2 + \sin t.$$

33. Since its diameters lie along the x and y-axes and its center is the origin, the ellipse must lie in the xy-plane, hence at $z = 0$. The x-coordinate ranges between -3 and 3 and the y-coordinate between -2 and 2. One possible answer is

$$x = 3\cos t, \quad y = 2\sin t, \quad z = 0.$$

34. Since its diameters are parallel to the x and z-axes, the ellipse must be parallel to the xz-plane. The ellipse with the same diameters, but centered at the origin, would have its x-coordinate range between $-3/2$ and $3/2$ and its z-coordinate range between -1 and 1. Thus this ellipse has equation

$$x = \frac{3}{2}\cos t, \quad y = 0, \quad z = \sin t.$$

Since our ellipse has center $(0, 1, -2)$, it must be in the plane $y = 1$. To move the center to $(0, 1, -2)$, we add 1 to the equation for y and -2 to the equation for z, so one possible answer for our ellipse is

$$x = \frac{3}{2}\cos t, \quad y = 1, \quad z = -2 + \sin t.$$

35. The displacement vector between the points is $\vec{u} = 3\vec{i} + 5\vec{k}$, so a possible parameterization of the line is

$$x = -1 + 3t, \quad y = 2, \quad z = -3 + 5t.$$

36. The vector from P_0 to P_1 is $\vec{v} = (5+1)\vec{i} + (2+3)\vec{j} = 6\vec{i} + 5\vec{j}$. Since $P_0 = -\vec{i} - 3\vec{j}$, the line is

$$\vec{r}(t) = -\vec{i} - 3\vec{j} + t(6\vec{i} + 5\vec{j}) \quad \text{for } 0 \le t \le 1.$$

In coordinate form, the equations are $x = -1 + 6t$, $y = -3 + 5t$, $0 \le t \le 1$

37. The vector from P_0 to P_1 is $\vec{v} = (4-1)\vec{i} + (1+3)\vec{j} + (-3-2)\vec{k} = 3\vec{i} + 4\vec{j} - 5\vec{k}$. Since P_0 has position vector $\vec{i} - 3\vec{j} + 2\vec{k}$, the line is

$$\vec{r}(t) = \vec{i} - 3\vec{j} + 2\vec{k} + t(3\vec{i} + 4\vec{j} - 5\vec{k}) \quad \text{for } 0 \le t \le 1.$$

In coordinate form the equations are $x = 1 + 3t$, $y = -3 + 4t$, $z = 2 - 5t$.

38. Since the semicircle is in the yz-plane we have $x = 0$. A circle of radius 5 in the yz-plane, centered at the origin and parameterized in the clockwise direction (from the positive z-axis toward the positive y-axis), goes from $(0, 0, 5)$ to $(0, 0, -5)$. It has equations $y = 5\cos t$ and $z = -5\sin t$. The semicircle where $y \ge 0$ is the obtained by restricting t to $-\pi/2 \le t \le \pi/2$. Thus a possible answer is

$$x = 0, \quad y = 5\cos t, \quad z = -5\sin t, \qquad -\pi/2 \le t \le \pi/2.$$

39. Since the semicircle is in the xy-plane we have $z = 0$. A circle of radius 1 in the xy-plane, centered at the origin and parameterized in the counterclockwise direction, goes from $(1, 0, 0)$ to $(-1, 0, 0)$. It has equations $x = \cos t$ and $y = \sin t$. The semicircle where $y \ge 0$ is the obtained by restricting t to $0 \le t \le \pi$. Thus a possible answer is

$$x = \cos t, \quad y = \sin t, \quad z = 0, \qquad 0 \le t \le \pi.$$

40. The graph is parameterized by $x = t$, $y = \sqrt{t}$. To obtain the segment, we restrict t to $1 \le t \le 16$. Thus one possible answer is

$$x = t, \quad y = \sqrt{t}, \qquad 1 \le t \le 16.$$

41. The line segment PQ has length 10, so it must be a diameter of the circle. The center of the circle is therefore the midpoint of PQ, which is the point $(5, 0)$. The upper arc of the circle between P and Q can be parameterized as follows:

$$\vec{r}(t) = 5\vec{i} + 5(-\cos t\vec{i} + \sin t\vec{j}), \quad 0 \le t \le \pi.$$

The lower arc can be parameterized as follows:

$$\vec{r}(t) = 5\vec{i} + 5(\cos t\vec{i} + \sin t\vec{j}), \quad \pi \le t \le 2\pi.$$

42. The equation for z is $z = 3$. The x-coordinate goes from 4 to 0 and the y-coordinate from 0 to -3, so possible equations for x and y are $x = 4\cos t$ and $y = -3\sin t$, with t from 0 to $\pi/2$. Thus one possible answer is

$$x = 4\cos t, \quad y = -3\sin t, \quad z = 3 \qquad 0 \le t \le \pi/2.$$

Problems

43. We find the parameterization in terms of the displacement vector $\overrightarrow{OP} = 2\vec{i} + 5\vec{j}$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = 10\vec{i} + 4\vec{j}$ from P to Q.

$\vec{r}(t) = \overrightarrow{OP} + t\overrightarrow{PQ}$ or, expressed in coordinates, $\vec{r}(t) = (2 + 10t)\vec{i} + (5 + 4t)\vec{j}$. To see that this is correct, note that the equation parameterizes a line because it is linear, that $t = 0$ corresponds to $\overrightarrow{OP} + 0\overrightarrow{PQ} = \overrightarrow{OP}$, the vector from the origin to P, and that $t = 1$ corresponds to $\overrightarrow{OP} + 1\overrightarrow{PQ} = \overrightarrow{OQ}$, the vector from the origin to Q.

44. We find the parameterization in terms of the displacement vector $\overrightarrow{OP} = 2\vec{i} + 5\vec{j}$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = 10\vec{i} + 4\vec{j}$ from P to Q.

$\vec{r}(t) = \overrightarrow{OP} + (t/5)\overrightarrow{PQ}$ or $\vec{r}(t) = (2 + (t/5)10)\vec{i} + (5 + (t/5)4)\vec{j}$

45. We find the parameterization in terms of the displacement vector $\overrightarrow{OP} = 2\vec{i} + 5\vec{j}$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = 10\vec{i} + 4\vec{j}$ from P to Q.

$$\vec{r}(t) = \overrightarrow{OP} + \left(\frac{t-20}{10}\right)\overrightarrow{PQ} = \left(2 + \left(\frac{t-20}{10}\right)10\right)\vec{i} + \left(5 + \left(\frac{t-20}{10}\right)4\right)\vec{j}.$$

46. We find the parameterization in terms of the displacement vector $\overrightarrow{OP} = 2\vec{i} + 5\vec{j}$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = 10\vec{i} + 4\vec{j}$ from P to Q.

$\vec{r}(t) = \overrightarrow{OP} + (t-10)\overrightarrow{PQ}$ or $\vec{r}(t) = (2 + (t-10)10)\vec{i} + (5 + (t-10)4)\vec{j}$

47. We find the parameterization in terms of the displacement vector $\overrightarrow{OP} = 2\vec{i} + 5\vec{j}$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = 10\vec{i} + 4\vec{j}$ from P to Q.

$\vec{r}(t) = \overrightarrow{OP} - t\overrightarrow{PQ}$ or $\vec{r}(t) = (2 - 10t)\vec{i} + (5 - 4t)\vec{j}$

48. Substituting $t = -1$ into the parametric equations tells us that the plane passes through the point

$$(x, y, z) = (8, -12, -6).$$

A vector parallel to the line is $\vec{v} = -3\vec{i} + 5\vec{j} + 6\vec{k}$. This vector is normal to the plane, so an equation for the plane is

$$-3(x-8) + 5(y+12) + 6(z+6) = 0, \quad \text{or}$$
$$-3x + 5y + 6z = -120.$$

49. The line is in the direction of the vector $\vec{v} = 7\vec{i} + 3\vec{j} - 2\vec{k}$ and the vector $\vec{n} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ is normal to the given plane. If the vectors $\vec{v}$ and $\vec{n}$ are perpendicular then the line and the plane are parallel. Since

$$\vec{v} \cdot \vec{n} = (7)(2) + (3)(-3) + (-2)(5) = -5,$$

the line and the plane are not parallel.

50. These equations parameterize a line. Since $(3+t)+(2t)+3(1-t)=6$, we have $x+y+3z=6$. Similarly, $x-y-z=(3+t)-2t-(1-t)=2$. That is, the curve lies entirely in the plane $x+y+3z=6$ and in the plane $x-y-z=2$. Since the normals to the two planes, $\vec{n_1}=\vec{i}+\vec{j}+3\vec{k}$ and $\vec{n_2}=\vec{i}-\vec{j}-\vec{k}$ are not parallel, the line is the intersection of two nonparallel planes, which is a straight line in 3-dimensional space.

51. (a) A vector on the line will lie in both planes and will therefore be orthogonal to both normal vectors. To produce a vector orthogonal to two given vectors, you can take their cross product.

(b) The vector $(\vec{i}+2\vec{j}-3\vec{k})\times(3\vec{i}-\vec{j}+\vec{k})=-\vec{i}-10\vec{j}-7\vec{k}$ is parallel to the line.

(c) We need a point on the line and a vector parallel to the line. We found a vector in part (b). To find a point, we set $z=0$ and solve for x and y in the equations for the planes. We have $x+2y=7$ and $3x-y=0$ from which $x=1$ and $y=3$. Hence, the point $(1,3,0)$ is on the line. Finally, a parametric equation for the line is $\vec{r}=(1-t)\vec{i}+(3-10t)\vec{j}-7t\vec{k}$. Other answers are possible.

52. The vector $2\vec{i}-\vec{j}+\vec{k}$ is in the direction of the line and therefore parallel to the plane. The line (and thus the plane) contains the point $(1,3,4)$. The displacement vector between $(1,3,4)$ and $(2,3,4)$ is the vector $\vec{i}$, and this vector is also parallel to the plane. A normal, $\vec{n}$, to the plane is the cross product

$$\vec{n}=(2\vec{i}-\vec{j}+\vec{k})\times\vec{i}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 1\\ 1 & 0 & 0\end{vmatrix}=\vec{j}+\vec{k}.$$

The equation of the plane has the form $y+z=d$. Substituting either of the points gives

$$y+z=7.$$

53. (a) The line is parallel to a normal vector, $\vec{n}$, to the plane

$$\vec{n}=2\vec{i}-3\vec{j}-\vec{k}.$$

Since the line goes through the point $(1,3,7)$, its equation is

$$\vec{r}=(\vec{i}+3\vec{j}+7\vec{k})+t(2\vec{i}-3\vec{j}-\vec{k}).$$

(b) Rewriting the equation of the line as

$$x=1+2t,\quad y=3-3t,\quad y=7-t,$$

we substitute into the equation of the plane $2x-3y=z$ to get

$$\begin{aligned}2(1+2t)-3(3-3t)&=7-t\\ 4t+9t+t&=7-2+9\\ t&=1.\end{aligned}$$

Thus, the point of intersection is $x=3$, $y=0$, $z=6$.

(c) The distance between $(1,3,7)$ and the plane is measured along the line perpendicular to the plane. Thus, it is the distance from $(1,3,7)$ to $(3,0,6)$:

$$\text{Distance }=\sqrt{(1-3)^2+(3-0)^2+(7-6)^2}=\sqrt{14}.$$

54. (a) The line segment starting at P_0 and ending at P_1 is parametrized by

$$\vec{r}(t)=\overrightarrow{OP_0}+t\overrightarrow{P_0P_1},\quad 0\le t\le 1.$$

We write this in coordinates: Let $P_0=(x_0,y_0,z_0)$ and $P_1=(x_1,y_1,z_1)$. Then a vector between the points is $\overrightarrow{P_0P_1}=(x_1-x_0)\vec{i}+(y_1-y_0)\vec{j}+(z_1-z_0)\vec{k}$, so

$$\vec{r}(t)=(x_0\vec{i}+y_0\vec{j}+z_0\vec{k})+t((x_1-x_0)\vec{i}+(y_1-y_0)\vec{j}+(z_1-z_0)\vec{k}),$$

or

$$\begin{aligned}x(t)&=x_0+t(x_1-x_0)=(1-t)x_0+tx_1\\ y(t)&=y_0+t(y_1-y_0)=(1-t)y_0+ty_1\\ z(t)&=z_0+t(z_1-z_0)=(1-t)z_0+tz_1\end{aligned}$$

Thus we have

$$\vec{r}(t) = (1-t)\overrightarrow{OP_0} + t\overrightarrow{OP_1}.$$

(b) The parametric equation $\vec{r}(t) = t\overrightarrow{OP_0} + (1-t)\overrightarrow{OP_1}$, $0 \le t \le 1$ is the line segment from P_1 to P_0, the same line segment as in part (a), but traversed in the opposite direction.

55. (a) Normal vectors to the two planes are

$$\vec{n}_1 = 2\vec{i} - \vec{j} - 3\vec{k} \quad \text{and} \quad \vec{n}_2 = \vec{i} + \vec{j} + \vec{k}.$$

The vector $\vec{n}_1 \times \vec{n}_2$ is perpendicular to both planes and parallel to the line of intersection:

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 2\vec{i} - 5\vec{j} + 3\vec{k}.$$

(b) To check that the point $(1, -1, 1)$ lies on the planes, substitute into each equation.

$$2x - y - 3z = 2 \cdot 1 - (-1) - 3 \cdot 1 = 0$$
$$x + y + z = 1 - 1 + 1 = 1.$$

Thus, the point lies on both planes.

(c) Parametric equations of the line are

$$x = 1 + 2t, \quad y = -1 - 5t, \quad z = 1 + 3t.$$

56. Since the point of intersection is on the plane $2x - 3y + 5z = -7$ and on the line

$$x = 5 + 7t, \quad y = 4 + 3t, \quad z = -3 - 2t$$

for some t, substituting the equations of the line into the plane equation gives

$$2(5 + 7t) - 3(4 + 3t) + 5(-3 - 2t) = -7.$$

Solving for t gives $t = -2$ and so the point of intersection is $x = 5 + 7(-2) = -9$, $y = 4 + 3(-2) = -2$, $z = -3 - 2(-2) = 1$.

57. The coefficients of t in the parameterizations show that line $\vec{r}_1$ is parallel to the vector $-3\vec{i} + 2\vec{j} + \vec{k}$ and line $\vec{r}_2$ is parallel to $-6\vec{i} + 4\vec{j} + 3\vec{k}$. Since these vectors are not parallel, the lines are not parallel, so the lines are different.

58. The coefficients of t in the parameterizations show that line $\vec{r}_1$ is parallel to the vector $-3\vec{i} + \vec{j} + 2\vec{k}$ and line $\vec{r}_2$ is parallel to $6\vec{i} - 2\vec{j} - 4\vec{k}$. Since these vectors are parallel, the lines are parallel. To see if they are the same line, check whether they have a common point. If so, every point is common. Pick any point on $\vec{r}_1$, say where $t = 0$, which shows that the point $(5, 1, 0)$ is on line $\vec{r}_1$. To determine whether this point is on line $\vec{r}_2$, search for a solution of the simultaneous equations

$$2 + 6t = 5 \quad 2 - 2t = 1 \quad 2 - 4t = 0.$$

The solution of the first equation is $t = 1/2$, which also solves the other two equations, which shows that the point $(5, 1, 0)$ is on line $\vec{r}_2$, corresponding to $t = 1/2$.

Since the two lines are parallel and go through a common point, they are the same line.

59. The coefficients of t in the parameterizations show that line $\vec{r}_1$ is parallel to the vector $-3\vec{i} + \vec{j} + 2\vec{k}$ and line $\vec{r}_2$ is parallel to $6\vec{i} - 2\vec{j} - 4\vec{k}$. Since these vectors are parallel, the lines are parallel. To see if they are the same line, check whether they have a common point. If so, every point is common., Pick any point on $\vec{r}_1$, say where $t = 0$, which shows that the point $(5, 1, 0)$ is on line $\vec{r}_1$. To determine whether this point is on line $\vec{r}_2$, search for a solution of the simultaneous equations

$$2 + 6t = 5 \quad 2 - 2t = 1 \quad 3 - 4t = 0.$$

The solution of the first equation is $t = 1/2$, which is not a solution of the third equation, so there is no common solution.

The two lines are parallel but they are different, because one line contains the point $(5, 1, 0)$ and the other does not.

60. The lines intersect if

$$c + t = s$$
$$1 + t = 1 - s$$
$$5 + t = 3 + s.$$

Solving the last two equations gives $t = -1$ and $s = 1$. Substituting into the first equation gives $c = 2$.

61. (a) The line can be written as

$$x = 2 + 3t, \quad y = 5 + t, \quad z = 2t.$$

We substitute into $x + y + z = 1$ and solve

$$\begin{aligned} (2 + 3t) + (5 + t) + 2t &= 1 \\ 6t + 7 &= 1 \\ t &= -1. \end{aligned}$$

Thus, the point is $(x, y, z) = (2 + 3(-1), 5 - 1, 2(-1)) = (-1, 4, -2)$.

(b) The vector $\vec{v} = 3\vec{i} + \vec{j} + 2\vec{k}$ is parallel to the line; the normal vector $\vec{n} = \vec{i} + \vec{j} + \vec{k}$ is perpendicular to the plane. Thus

$$\vec{v} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} - \vec{j} + 2\vec{k}$$

is perpendicular to the line and lies in the plane. Other answers are possible.

(c) The line passes through $(-1, 4, -2)$ and is parallel to $-\vec{i} - \vec{j} + 2\vec{k}$. Its equation can be written

$$\vec{r} = -\vec{i} + 4\vec{j} - 2\vec{k} + t(-\vec{i} - \vec{j} + 2\vec{k}).$$

62. Add the two equations to get $3x = 8$, or $x = \frac{8}{3}$. Then we have

$$-y + z = \frac{1}{3}.$$

So a possible parameterization is

$$x = \frac{8}{3}, \quad y = t, \quad z = \frac{1}{3} + t.$$

63. Add the two equations to get $2x+3z = 5$, or $x = -\frac{3}{2}z + \frac{5}{2}$. Subtract the two equations to get $2y - z = 1$, or $y = \frac{1}{2}z + \frac{1}{2}$. So a possible parameterization is

$$x = -\frac{3}{2}t + \frac{5}{2}, \quad y = \frac{1}{2}t + \frac{1}{2}, \quad z = t.$$

64. Let $f(x, y, z) = x^2 + y^2 - z$. Then the surface $z = x^2 + y^2$ is a level surface of f at the value 0. The gradient of f is perpendicular to the level surface.

$$\text{grad}\, f = 2x\vec{i} + 2y\vec{j} - \vec{k} = 2\vec{i} + 4\vec{j} - \vec{k}.$$

So a possible parameterization is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 5 - t.$$

65. Parametric equations for a line in 3-space are in the form

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

where (x_0, y_0, z_0) is a point on the line and the direction vector is $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$. We are given the point as $(-4, 2, 3)$. The vector $\vec{j} + \vec{k}$ is parallel to the yz-plane and at an angle of 45° to both positive y- and z-axes. Thus, the direction vector for this line is $\vec{v} = \vec{j} + \vec{k}$. Parametric equations for this line are

$$\begin{aligned} x &= -4, \\ y &= 2 + t, \\ z &= 3 + t. \end{aligned}$$

66. The question is equivalent to asking if the line through $(-3,-4,2)$ and $(4,5,0)$ enters the sphere $x^2+y^2+z^2=1$. A parameterization for this line is given by

$$x=-3+7t,\quad y=-4+9t,\quad z=2-2t.$$

We want to see whether the line intersects the sphere $x^2+y^2+z^2=1$. Substituting we have

$$(-3+7t)^2+(-4+9t)^2+(2-2t)^2=1$$
$$29-122t+134t^2=0$$

Since $(122)^2-4(29)134<0$, this equation has no real solutions. Thus, the line does not enter the sphere and the point is visible.

67. (a) Both paths are straight lines, the first passes through the point $(-1,4,-1)$ in the direction of the vector $\vec{i}-\vec{j}+2\vec{k}$ and the second passes through $(-7,-6,-1)$ in the direction of the vector $2\vec{i}+2\vec{j}+\vec{k}$. The two paths are not parallel.

(b) Is there a time t when the two particles are at the same place at the same time? If so, then their coordinates will be the same, so equating coordinates we get

$$\begin{aligned}-1+t&=-7+2t\\4-t&=-6+2t\\-1+2t&=-1+t.\end{aligned}$$

Since the first equation is solved by $t=6$, the second by $t=10/3$, and the third by $t=0$, no value of t solves all three equations. The two particles never arrive at the same place at the same time, and so they do not collide.

(c) Are there any times t_1 and t_2 such that the position of the first particle at time t_1 is the same as the position of the second particle at time t_2? If so then

$$\begin{aligned}-1+t_1&=-7+2t_2\\4-t_1&=-6+2t_2\\-1+2t_1&=-1+t_2.\end{aligned}$$

We solve the first two equations and get $t_1=2$ and $t_2=4$. This is a solution for the third equation as well, so the three equations are satisfied by $t_1=2$ and $t_2=4$. At time $t=2$ the first particle is at the point $(1,2,3)$, and at time $t=4$ the second is at the same point. The paths cross at the point $(1,2,3)$, and the first particle gets there first.

68. (a) The particle moves clockwise around a circle with center (a,a) and radius b, starting at $(a,a+b)$. The motion has period $2\pi/k$.

(b) (i) Increasing b increases the radius.
(ii) Increasing a moves the center away from the origin along the line $y=x$.
(iii) Increasing k makes the particle move faster and reduces the period.
(iv) If $a=b$, the circle touches both the x- and y-axes at the points $(a,0)$ and $(0,a)$, respectively.

69. (a) The curve is a loop because temperature and salinity go through the same changes every year.
(b) At $t=8$, mid-August.
(c) At $t=4$, mid-April.
(d) From the graph, we have (approximately) $T(5)=15.9$, $T(6)=17.8$, and $T(7)=20.4$. Using a difference quotient, we have

$$\left.\frac{dT}{dt}\right|_{t=6}\approx\frac{T(7)-T(5)}{2}=2.25\ ^\circ\text{C/month}.$$

The seawater temperature in mid-June is increasing at a rate of about 2°C per month.

70. (a) The particle is moving in the direction of the vector $\vec{v}=(-2-0)\vec{i}+(2-1)\vec{j}+(-2-0)\vec{k}=-2\vec{i}+\vec{j}-2\vec{k}$. Since at the point $(0,1,0)$

$$\text{grad}\, c\Big|_{(0,1,0)}=\left(-2x\vec{i}-2y\vec{j}-2z\vec{k}\right)e^{-(x^2+y^2+z^2)}\Big|_{(0,1,0)}=-2e^{-1}\vec{j},$$

the directional derivative, which gives the rate of change we want, is given by

$$c_{\vec{v}}=(-2e^{-1}\vec{j})\cdot\frac{-2\vec{i}+\vec{j}-2\vec{k}}{\sqrt{(-2)^2+1^2+(-2)^2}}=-\frac{2}{3}e^{-1}\ \text{microgr/m}^3\ \text{per meter}.$$

(b) Substituting the parametric equation of the curve into the concentration, we have the concentration as a function of t:

$$c(t) = e^{-\left(\left(1-t^2\right)^2+t^2+\left(1-t^2\right)^2\right)} = e^{-\left(2(1-t^2)^2+t^2\right)}.$$

Differentiating gives

$$c'(t) = -\left(4\left(1-t^2\right)(-2t)+2t\right)e^{-\left(2(1-t^2)^2+t^2\right)}.$$

Setting $c'(t) = 0$ and observing that $e^{-\left(2(1-t^2)^2+t^2\right)} \neq 0$, we have

$$\begin{aligned} -\left(4\left(1-t^2\right)(-2t)+2t\right) &= 0 \\ t\left(-4\left(1-t^2\right)+1\right) &= 0 \\ t\left(4t^2-3\right) &= 0 \\ t &= 0, \pm\frac{\sqrt{3}}{2}. \end{aligned}$$

Since $c(t) \to 0$ as $t \to \infty$, one of the critical points gives the global maximum. Substituting gives

$$\begin{aligned} c(0) &= e^{-2} = 0.135 \\ c\left(\pm\frac{\sqrt{3}}{2}\right) &= e^{-\left(2(1-3/4)^2+3/4\right)} = e^{-7/8} = 0.417. \end{aligned}$$

Thus, the maxima occur at $t = \pm\sqrt{3}/2$ seconds.

71. The three shadows appear as a circle, a cosine wave and a sine wave, respectively.

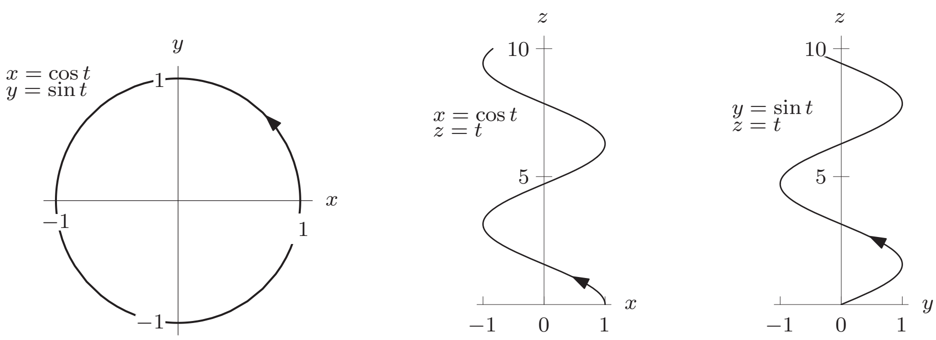


Figure 17.1

72. (a) Equations II represent the line $y = x$.
(b) Equations IV represent the line $x + y = a$. Since $a > 0$, this line is not through the origin.
(c) Equations V give the hyperbola $x^2 - y^2 = a^2$.
(d) Equations I represent the circle $x^2 + y^2 = a^2$, traversed clockwise starting at $(0, a)$.
(e) Equations III represent the circle $x^2 + y^2 = a^2$, traversed counterclockwise starting at $(a, 0)$.

73. (a) Parametric equations are

$$x = 2 + at, \quad y = 1 + bt, \quad z = 3 + ct.$$

(b) The line goes through the origin if the position vector $2\vec{i} + \vec{j} + 3\vec{k}$ is parallel to the vector $a\vec{i} + b\vec{j} + c\vec{k}$. This occurs if a, b, c are in the ratio $2 : 1 : 3$; that is if

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{3}.$$

74. (a) The vector $-2\vec{i} + 7\vec{j} + 4\vec{k}$ is parallel to the line. A normal to the plane is $a\vec{i} + b\vec{j} + c\vec{k}$. We want the normal to the plane to be parallel to the line, so we take $a = -2, b = 7, c = 4$. Any value of d will do, for example $d = 0$.
(b) The same values of a, b, c as in part (a) work, though now we need to choose d so that the point $(5, 3, 0)$ lies on the plane. So $a = -2, b = 7, c = 4$ and

$$d = -2(5) + 7(3) + 4(0) = 11.$$

(c) The normal $a\vec{i} + b\vec{j} + c\vec{k}$ must be perpendicular to the vector $-2\vec{i} + 7\vec{j} + 4\vec{k}$, so

$$-2a + 7b + 4c = 0$$

We can choose any values of a, b, c which satisfy this equation, so $a = 7, b = 2, c = 0$ work. To ensure that the point $(5, 3, 0)$, which lies on the line, also lies on the plane, substitute the coordinates of the point into the plane, giving

$$d = 7x + 2y + 0z = 7(5) + 2(3) = 41.$$

75. The helices wind around a cylinder of radius α, which explains the significance of α. As t increases from 0 to 2π, the helix winds once around the cylinder, climbing upward a distance of $2\pi\beta$. Thus β controls how stretched out the helix is in the vertical direction. See Figure 17.2 and Figure 17.3.

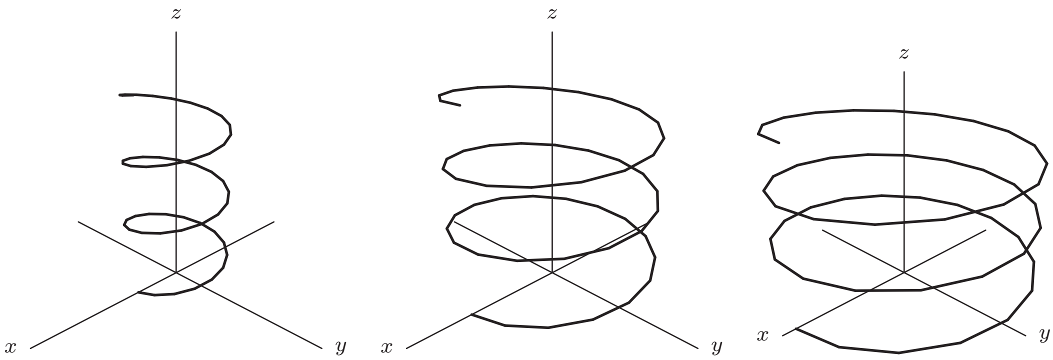


Figure 17.2: Three values of α with the same β

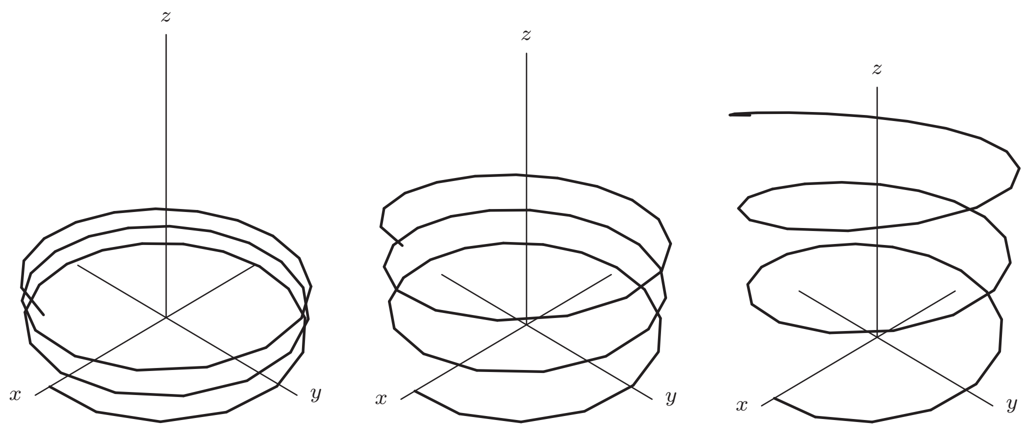


Figure 17.3: Three values of β with the same α

76. The displacement from the point $(1, 2, 3)$ to the point $(3, 5, 7)$ is $3\vec{i} + 5\vec{j} + 7\vec{k} - (\vec{i} + 2\vec{j} + 3\vec{k}) = 2\vec{i} + 3\vec{j} + 4\vec{k}$. So the equation of the line is

$$x\vec{i} + y\vec{j} + z\vec{k} = 1\vec{i} + 2\vec{j} + 3\vec{k} + t(2\vec{i} + 3\vec{j} + 4\vec{k})$$

or

$$x\vec{i} + y\vec{j} + z\vec{k} = (1 + 2t)\vec{i} + (2 + 3t)\vec{j} + (3 + 4t)\vec{k}.$$

The square of the distance from a point (x, y, z) on the line to the origin, denoted by $D(t)$ is

$$\begin{aligned} D(t) &= (x - 0)^2 + (y - 0)^2 + (z - 0)^2 \\ &= (1 + 2t)^2 + (2 + 3t)^2 + (3 + 4t)^2 \\ &= 1 + 4t + 4t^2 + 4 + 12t + 9t^2 + 9 + 24t + 16t^2 \end{aligned}$$

$$\begin{aligned}&= 14 + 40t + 29t^2\\ &= 29\left(t^2 + \frac{40}{29}t + \frac{14}{29}\right)\\ &= 29\left(\left(t + \frac{20}{29}\right)^2 - \left(\frac{20}{29}\right)^2 + \frac{14}{29}\right).\end{aligned}$$

Since $D(t)$ is minimum when $t = -20/29$ and

$$D(-20/29) = 29\left(-\left(\frac{20}{29}\right)^2 + \frac{14}{29}\right) = \frac{6}{29},$$

the shortest distance is $\sqrt{6/29}$.

77. The line $\vec{r} = \vec{a} + t\vec{b}$ is parallel to the vector $\vec{b}$ and through the point with position vector $\vec{a}$.

(a) is (vii). The equation $\vec{b} \cdot \vec{r} = 0$ is a plane perpendicular to $\vec{b}$ and satisfied by $(0, 0, 0)$.

(b) is (ii). For any constant k, the equation $\vec{b} \cdot \vec{r} = k$ is a plane perpendicular to $\vec{b}$. If $k = ||\vec{a}|| \neq 0$, the plane does not contain the origin.

(c) is (iv). The equation $(\vec{a} \times \vec{b}) \cdot (\vec{r} - \vec{a}) = 0$ is the equation of a plane which is satisfied by $\vec{r} = \vec{a}$, so the point with position vector $\vec{a}$ lies on the plane. Since $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$, the plane is parallel to the line, and therefore it contains the line.

78. (a) Parametric equations are

$$x = 1 + 2t, \quad y = 5 + 3t, \quad z = 2 - t.$$

(b) We want to minimize D, the square of the distance of a point to the origin, where

$$D = (x-0)^2 + (y-0)^2 + (z-0)^2 = (1+2t)^2 + (5+3t)^2 + (2-t)^2.$$

Differentiating to find the critical points gives

$$\begin{aligned}\frac{dD}{dt} = 2(1+2t)2 + 2(5+3t)3 + 2(2-t)(-1) &= 0\\ 2 + 4t + 15 + 9t - 2 + t &= 0\\ t &= \frac{-15}{14}.\end{aligned}$$

Thus

$$\begin{aligned}x &= 1 + 2\left(\frac{-15}{14}\right) = \frac{-8}{7}\\ y &= 5 + 3\left(\frac{-15}{14}\right) = \frac{25}{14}\\ z &= 2 - \left(\frac{-15}{14}\right) = \frac{43}{14}.\end{aligned}$$

Since the distance of the point on the line from the origin increases without bound as the magnitude of x, y, z increase, the only critical point of D must be a global minimum. Therefore, the point $(-8/7, 25/14, 43/14)$ is the point on the line closest to the origin.

79. Since the origin is beneath Denver and 1650 meters = 1.65 km, Denver's coordinates, in kilometers, are $(0, 0, 1.65)$. From Figure 17.4, we see the x and y coordinates of Bismark are given by

$$x = 850\cos 60^\circ = 425 \text{ km} \quad \text{and} \quad y = 850\sin 60^\circ = 736 \text{ km}.$$

Since 550 meters = 0.55 km, the coordinates of Bismark in kilometers are (425, 736, 0.55).

The velocity vector, $\vec{v}$, of the plane is parallel to the vector $\overrightarrow{DB}$ joining Denver to Bismark, where $\overrightarrow{DB} = 425\vec{i} + 736\vec{j} + (0.55 - 1.65)\vec{k} = 425\vec{i} + 736\vec{j} - 1.1\vec{k}$.

Since $||\overrightarrow{DB}|| = \sqrt{425^2 + 736^2 + 1.1^2} \approx 850$ km and the plane is moving at 650 km/hr, the velocity vector is given by

$$\vec{v} = \frac{650}{850}(425\vec{i} + 736\vec{j} - 1.1\vec{k}) = 325\vec{i} + 563\vec{j} - 0.84\vec{k}.$$

Since the plane is 8000 m $=$ 8 km above Denver, it passes through the point $(0, 0, 9.65)$. Therefore the parametric equation is

$$\vec{r} = 9.65\vec{k} + t(325\vec{i} + 563\vec{j} - 0.84\vec{k}).$$

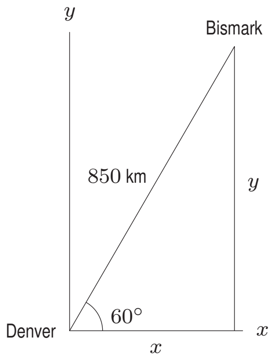


Figure 17.4

80. (a) We look along the line that passes through $P = (1, -2, -1)$ and is parallel to $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$. The question is which plane, the blue or the yellow, this line first meets.

Parametric equations for the line are

$$x = 1 + t, \quad y = -2 + 2t, \quad z = -1 + t.$$

We substitute these into the equations of the respective planes and solve for t in each case:

$$\begin{array}{cc} (1+t) + 3(-2+2t) - 2(-1+t) = 6 & 2(1+t) + (-2+2t) + (-1+t) = 3 \\ 5t - 3 = 6 & 5t - 1 = 3 \\ t = \frac{9}{5} & t = \frac{4}{5} \end{array}$$

From this we see that the line first intersects the yellow plane $2x + y + z = 3$, when $t = 4/5$. So you see the yellow plane. (Note that we did not need to find the points of intersection of the line with the planes.)

(b) A vector from P to a point on the green line gives a direction looking directly at the line. If we get a parametric equation for the green line then we can write down a vector from P to any variable point on the line.

To get a parametric equation we need a vector parallel to the green line and a point that lies on the green line. We take the cross product of the normal of the blue plane, $\vec{n_b} = \vec{i} + 3\vec{j} - 2\vec{k}$, and the normal of the yellow plane, $\vec{n_y} = 2\vec{i} + \vec{j} + \vec{k}$. This gives a vector $5\vec{i} - 5\vec{j} - 5\vec{k}$, so we take $\vec{u} = \vec{i} - \vec{j} - \vec{k}$ as a vector parallel to the green line.

We also need one point on the line. For that, we can choose a value of z, and find the corresponding values of x and y on both the blue and yellow planes. Taking $z = 0$, say, gives the equations $x + 3y = 6$ and $2x + y = 3$, which have $x = 3/5$ and $y = 9/5$ as solutions. So a point on the green line is $Q = (3/5, 9/5, 0)$. Therefore a parametric equation for the green line is

$$x = \frac{3}{5} + t, \quad y = \frac{9}{5} - t, \quad z = -t.$$

A vector from $P = (1, -2, -1)$ to a variable point on the line is then, for $-\infty < t < \infty$,

$$\vec{v} = \left(1 - \left(\frac{3}{5} + t\right)\right)\vec{i} + \left(-2 - \left(\frac{9}{5} - t\right)\right)\vec{j} + (-1 - (-t))\vec{k} = \left(-\frac{2}{5} + t\right)\vec{i} + \left(-\frac{19}{5} + t\right)\vec{j} + (-1 + t)\vec{k}.$$

Thus, if we look in the direction of $\vec{v}$, for any value of t, we look at the line.

(c) Consider the plane that contains the point P and the green line; let's call it the green plane. The green plane divides 3-space into two half-spaces. From P, if we look in a direction pointing into one of the half-spaces we see the yellow plane (as in part (a)) and if we look in a direction pointing into the other half-space we see the blue plane. We have to figure out which half-space is which.

We need a normal vector to the green plane. We know the point $P = (1, -2, -1)$ on the plane and the equation of the green line. We find that a normal vector to the green plane is $\vec{n} = 2\vec{i} + \vec{j} + \vec{k}$.

From part (a) we know that the vector $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$ points from P into the half-space where we see the yellow plane. The dot product of $\vec{n}$ and $\vec{v}$ is

$$\vec{n} \cdot \vec{v} = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 = 5 > 0.$$

This means that any vector pointing into this half-space has a positive dot product with $\vec{n}$. Thus the condition on a general vector $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$ to point into this half-space is

$$2a + b + c > 0;$$

Similarly, $\vec{w}$ points into the half-space where we see the blue plane if

$$2a + b + c < 0.$$

81. (a) If $\vec{n} \cdot \vec{v} = 0$, then $\vec{n}$ and $\vec{v}$ are perpendicular. Since P_1 is perpendicular to $\vec{n}$ and L is parallel to $\vec{v}$, we see that P_1 and L are parallel. In fact, L may lie in the plane.

(b) Since $\vec{n} \times \vec{v}$ is perpendicular to $\vec{n}$ and to $\vec{v}$, the vector $\vec{n} \times \vec{v}$ is parallel to P_1 and perpendicular to L. Thus, P_2, which is perpendicular to $\vec{n} \times \vec{v}$, is

(i) Perpendicular to P_1.

(ii) Parallel to L.

82. (a) (i) is the original graph reflected in both the x- and y-axes, so (C).

(ii) is the original graph reflected in the y-axis, so (A).

(iii) is the original graph shifted right by 1, so (D).

(iv) is the original graph shifted right by 1 and up by 1, so (G).

(b) (i) Not possible; all points would lie on a circle

(ii) Not possible; spiral would be equally spaced.

(iii) Possible; spirals increase in diameter as t increases.

(iv) Not possible; spiral would be equally spaced.

(v) Not possible; all points would lie on a circle

Strengthen Your Understanding

83. The two parameterizations are different, but the curves they describe are the same. A shift of the curve in space by two units in the $\vec{i}$-direction can be parameterized by $\vec{r}_2(t) = \vec{r}(t) + 2\vec{i}$.

84. The curve is a helix centered on the z-axis. All its points are at distance $|R|$ from the z-axis. The distance from the origin to the point on the curve with position vector $\vec{r}(t)$ is given by

$$\text{Distance} = \sqrt{\vec{r}(t) \cdot \vec{r}(t)} = \sqrt{R^2 + t^2}.$$

85. The curves $x = \cos t, y = \sin t, z = 0$ and $x = 0, y = \cos t, z = \sin t$ are both unit circles centered at the origin. The first is in the xy-plane, and the second is in the yz-plane.

86. Examples of two different lines through the point $(1, 2, 3)$ are given by

$$\vec{r}(t) = \vec{i} + 2\vec{j} + 3\vec{k} + t\left(\vec{i} + 2\vec{j}\right)$$

and

$$\vec{r}(t) = \vec{i} + 2\vec{j} + 3\vec{k} + t\left(\vec{i} - \vec{k}\right).$$

87. The line

$$x = t, y = 2t, z = 3 + 4t$$

can also be parameterized by

$$x = t^3, y = 2t^3, z = 3 + 4t^3,$$

and the functions $x = t^3$, $y = 2t^3$, and $z = 3 + 4t^3$ are not linear functions of t.

88. False. The y coordinate is zero when $t = 0$, but when $t = 0$ we have $x = 2$ so the curve never passes through $(0, 0)$.

89. True. Every point (x, y) on this curve satisfies $y = (t^2)^2 = x^2$.

90. False. For example, the graph of $x = \cos t, y = \sin t$ for $0 \le t \le 2\pi$ is a circle. A circle is not the graph of a function, since for some values of x there are two values of y.

91. True. Every y-coordinate is one less than every x-coordinate, so the equation of the line is $y = x - 1$.

92. False. When $t = 0$, we have $(x, y) = (0, -1)$. When $t = \pi/2$, we have $(x, y) = (-1, 0)$. Thus the circle is being traced out clockwise.

93. True. The functions e^t and $\ln t$ are inverses, so $\ln e^t = t$. Thus if $x = e^t, y = t$, we have $y = t = \ln e^t = \ln x$.

94. True. Taking two values for t, say $t = 0$ and $t = 1$ give the points $(1, 0)$ and $(0, 2)$, which lie on a line with equation $y = -2x+2$. The second parameterization describes the same set of points, since $y = -4s+2 = -2(2s)+2 = -2x+2$.

95. True. Adding the equations $z = x + y$ and $z = 1 - x - y$ gives $2z = 1$ or $z = \frac{1}{2}$. Thus the line of intersection is parallel to the xy-plane at height $z = \frac{1}{2}$. Letting x be the parameter t and $z = \frac{1}{2}$ in the first plane's equation gives $\frac{1}{2} = t + y$ or $y = \frac{1}{2} - t$. The same result is obtained by setting $x = t$ and $z = \frac{1}{2}$ in the second plane's equation.

96. True. To find an intersection point, we look for values of s and t that make the coordinates in the first line the same as the coordinates in the second. Setting $x = t$ and $x = 2s$ equal, we see that $t = 2s$. Setting $y = 2 + t$ equal to $y = 1 - s$, we see that $t = -1 - s$. Solving both $t = 2s$ and $t = -1 - s$ yields $t = -\frac{2}{3}$, $s = -\frac{1}{3}$. These values of s and t will give equal x and y coordinates on both lines. We need to check if the z coordinates are equal also. In the first line, setting $t = -\frac{2}{3}$ gives $z = \frac{7}{3}$. In the second line, setting $s = -\frac{1}{3}$ gives $z = -\frac{1}{3}$. As these are not the same, the lines do not intersect.

97. False. All points on this line lie in the plane $x = 1$, so the line is parallel to the yz-plane.

98. True. The $\vec{j}$ component of $\vec{r}$ is always one more than twice the $\vec{i}$ component, so the line is $y = 2x + 1$.

99. False. The line $\vec{r}_1(t)$ is in the direction of the vector $\vec{i} - 2\vec{j}$, while the line $\vec{r}_2(t)$ is in the direction of the vector $\vec{2}\,i - \vec{j}$. Since these vectors are not parallel (they are not scalar multiples of one another) the lines are not parallel.

Solutions for Section 17.2

Exercises

1. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(2+3t)\vec{i} + \frac{d}{dt}(4+t)\vec{j} + \frac{d}{dt}(1-t)\vec{k} = 3\vec{i} + \vec{j} - \vec{k}.$$

The acceleration vector $\vec{a}$ is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(3)}{dt}\vec{i} + \frac{d(1)}{dt}\vec{j} - \frac{d(1)}{dt}\vec{k} = \vec{0}$$

2. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(2+3t^2)\vec{i} + \frac{d}{dt}(4+t^2)\vec{j} + \frac{d}{dt}(1-t^2)\vec{k} = 6t\vec{i} + 2t\vec{j} - 2t\vec{k}.$$

The acceleration vector $\vec{a}$ is given by:

$$\vec{a} = \frac{d(6t)}{dt}\vec{i} + \frac{d(2t)}{dt}\vec{j} - \frac{d(2t)}{dt}\vec{k} = 6\vec{i} + 2\vec{j} - 2\vec{k}.$$

3. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}t\vec{i} + \frac{d}{dt}t^2\vec{j} + \frac{d}{dt}t^3\vec{k} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}.$$

The acceleration vector $\vec{a}$ is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(1)}{dt}t\vec{i} + \frac{d(2t)}{dt}\vec{j} + \frac{d(3t^2)}{d}\vec{k} = 2\vec{j} + 6t\vec{k}.$$

4. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d(t)}{dt}\vec{i} + \left(\frac{d}{dt}(t^3 - t)\right)\vec{j} = \vec{i} + (3t^2 - 1)\vec{j}.$$

The acceleration vector $\vec{a}$ is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(1)}{dt}\vec{i} + \left(\frac{d}{dt}(3t^2 - 1)\right)\vec{j} = 6t\vec{j}.$$

5. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(3\cos t)\vec{i} + \frac{d}{dt}(4\sin t)\vec{j} = -3\sin t\vec{i} + 4\cos t\vec{j}.$$

The acceleration vector $\vec{a}$ is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-3\sin t)\vec{i} + \frac{d}{dt}(4\cos t)\vec{j} = -3\cos t\vec{i} - 4\sin t\vec{j}.$$

6. Since $\vec{r}(t) = 3\cos(t^2)\vec{i} + 3\sin(t^2)\vec{j} + t^2\vec{k}$, we have

$$\vec{v}(t) = -6t\sin(t^2)\vec{i} + 6t\cos(t^2)\vec{j} + 2t\vec{k},$$
$$\vec{a}(t) = (-6\sin(t^2) - 12t^2\cos(t^2))\vec{i} + (6\cos(t^2) - 12t^2\sin(t^2))\vec{j} + 2\vec{k}.$$

7. The velocity vector $\vec{v}$ is given by:

$$\begin{aligned}\vec{v} &= \frac{d}{dt}(t)\vec{i} + \frac{d}{dt}(t^2)\vec{j} + \frac{d}{dt}(t^3)\vec{k} \\ &= \vec{i} + 2t\vec{j} + 3t^2\vec{k}.\end{aligned}$$

The speed is given by:

$$\|\vec{v}\| = \sqrt{1 + 4t^2 + 9t^4}.$$

Now $\|\vec{v}\|$ is never zero since $1 + 4t^2 + 9t^4 \geq 1$ for all t. Thus, the particle never stops.

8. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(\cos 3t)\vec{i} + \frac{d}{dt}(\sin 5t)\vec{j} = -3\sin 3t\vec{i} + 5\cos 5t\vec{j}.$$

The speed is given by

$$\|\vec{v}\| = \sqrt{9\sin^2(3t) + 25\cos^2(5t)}.$$

Thus, $\|\vec{v}\| = 0$ when $\sin(3t) = \cos(5t) = 0$ but there are no values of t for which this is true, so the particle never stops.

9. To find $\vec{v}(t)$ we first find $dx/dt = 6t$ and $dy/dt = 3t^2$. Therefore, the velocity vector is $\vec{v} = 6t\vec{i} + 3t^2\vec{j}$. The speed of the particle is given by the magnitude of the vector,

$$\|\vec{v}\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(6t)^2 + (3t^2)^2} = 3|t| \cdot \sqrt{4 + t^2}.$$

The particle stops when $\vec{v} = \vec{0}$, so when $6t = 3t^2 = 0$. Therefore, the particle stops when $t = 0$.

10. The velocity vector $\vec{v}$ is given by:

$$\begin{aligned}\vec{v} &= \frac{d}{dt}((t-1)^2)\vec{i} + \frac{d}{dt}(2)\vec{j} + \frac{d}{dt}(2t^3 - 3t^2)\vec{k} \\ &= 2(t-1)\vec{i} + (6t^2 - 6t)\vec{k}.\end{aligned}$$

The speed is given by:

$$\|\vec{v}\| = \sqrt{(2(t-1))^2 + (6t^2 - 6t)^2} = 2|t-1|\sqrt{1 + 9t^2}.$$

The particle stops when $\vec{v} = \vec{0}$, so when $2(t-1) = (6t^2 - 6t) = 0$. Since these are all satisfied only by $t = 1$, this is the only time that the particle stops.

11. To find $\vec{v}(t)$ we first find $dx/dt = 6t\cos(t^2)$ and $dy/dt = -6t\sin(t^2)$. Therefore, the velocity is $\vec{v} = 6t\cos(t^2)\vec{i} - 6t\sin(t^2)\vec{j}$. The speed of the particle is given by

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(6t\cos(t^2))^2 + (-6t\sin(t^2))^2} \\ &= \sqrt{36t^2(\cos(t^2))^2 + 36t^2(\sin(t^2))^2} \\ &= 6|t|\sqrt{\cos^2(t^2) + \sin^2(t^2)} \\ &= 6|t|.\end{aligned}$$

The particle comes to a complete stop when speed is 0, that is, if $6|t| = 0$, and so when $t = 0$.

12. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(3\sin^2 t)\vec{i} + \frac{d}{dt}(\cos t - 1)\vec{j} + \frac{d}{dt}(t^2)\vec{k} = 6\sin t\cos t\vec{i} - \sin t\vec{j} + 2t\vec{k}.$$

The speed is given by:

$$\|\vec{v}\| = \sqrt{36\sin^2 t\cos^2 t + \sin^2 t + 4t^2}.$$

The particle comes to a stop when $\vec{v} = \vec{0}$, so when when $6\sin t\cos t = -\sin t = 2t = 0$, and so the particle stops when $t = 0$.

13. We have

$$\text{Length} = \int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt = \int_1^2 \sqrt{5^2 + 4^2 + (-1)^2}\,dt = \sqrt{42}.$$

This is the length of a straight line from the point $(8, 5, 2)$ to $(13, 9, 1)$.

14. We have

$$\text{Length} = \int_0^{2\pi} \sqrt{(-3\sin 3t)^2 + (5\cos 5t)^2}\, dt.$$

We cannot find this integral symbolically, but numerical methods show Length ≈ 24.6.

15. We have

$$\begin{aligned}\text{Length} &= \int_0^1 \sqrt{(-e^t\sin(e^t))^2 + (e^t\cos(e^t))^2}\, dt \\ &= \int_0^1 \sqrt{e^{2t}}\, dt = \int_0^1 e^t\, dt \\ &= e - 1.\end{aligned}$$

This is the length of the arc of a unit circle from the point $(\cos 1, \sin 1)$ to $(\cos e, \sin e)$—in other words between the angles $\theta = 1$ and $\theta = e$. The length of this arc is $(e-1)$.

16. The velocity vector is

$$\vec{v} = \vec{r}\,'(t) = 2\vec{i} + \frac{1}{t}\vec{j} + 2t\vec{k},$$

so

$$\begin{aligned}\text{Length of curve} &= \int_1^2 ||\vec{v}\,||\, dt = \int_1^2 \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt = \int_1^2 \sqrt{\frac{4t^2+1+4t^4}{t^2}}\, dt \\ &= \int_1^2 \sqrt{\frac{(1+2t^2)^2}{t^2}}\, dt = \int_1^2 \frac{1+2t^2}{t}\, dt = \int_1^2 \left(\frac{1}{t} + 2t\right) dt = 3 + \ln 2.\end{aligned}$$

Note that when we took the square root, we used the fact that $(1+2t^2)/t$ is positive for $1 \le t \le 2$.

17. The velocity vector $\vec{v}$ is

$$\begin{aligned}\vec{v} &= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = 3(2\pi)(-\sin(2\pi t))\vec{i} + 3(2\pi)\cos(2\pi t)\vec{j} + 0\vec{k} \\ &= -6\pi\sin(2\pi t)\vec{i} + 6\pi\cos(2\pi t)\vec{j}.\end{aligned}$$

The acceleration vector $\vec{a}$ is

$$\begin{aligned}\vec{a} &= \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = -6\pi(2\pi)\cos(2\pi t)\vec{i} + 6\pi(2\pi)(-\sin(2\pi t))\vec{j} \\ &= -12\pi^2\cos(2\pi t)\vec{i} - 12\pi^2\sin(2\pi t)\vec{j}.\end{aligned}$$

To check that $\vec{v}$ and $\vec{a}$ are perpendicular, we check that the dot product is zero:

$$\begin{aligned}\vec{v}\cdot\vec{a} &= (-6\pi\sin(2\pi t)\vec{i} + 6\pi\cos(2\pi t)\vec{j}\,)\cdot(-12\pi^2\cos(2\pi t)\vec{i} - 12\pi^2\sin(2\pi t)\vec{j}\,) \\ &= 72\pi^3\sin(2\pi t)\cos(2\pi t) - 72\pi^3\cos(2\pi t)\sin(2\pi t) = 0\end{aligned}$$

The speed is

$$\|\vec{v}\,\| = \| -6\pi\sin(2\pi t)\vec{i} + 6\pi\cos(2\pi t)\vec{j}\,\| = 6\pi\sqrt{\sin^2(2\pi t) + \cos^2(2\pi t)} = 6\pi,$$

and so is constant. The magnitude of the acceleration is

$$\|\vec{a}\,\| = \| -12\pi^2\cos(2\pi t)\vec{i} - 12\pi^2\sin(2\pi t)\vec{j}\,\| = 12\pi^2\sqrt{\cos^2(2\pi t) + \sin^2(2\pi t)} = 12\pi^2,$$

which is also constant.

18. The velocity vector $\vec{v}$ is

$$\begin{aligned}\vec{v} &= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = 0\vec{i} + 2(3)\cos(3t)\vec{j} + 2(3)(-\sin(3t))\vec{k} \\ &= 6\cos(3t)\vec{j} - 6\sin(3t)\vec{k}.\end{aligned}$$

The acceleration vector $\vec{a}$ is

$$\begin{aligned}\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} &= 6(3)(-\sin(3t))\vec{j} - 6(3)\cos(3t)\vec{k} \\ &= -18\sin(3t)\vec{j} - 18\cos(3t)\vec{k}.\end{aligned}$$

To check that $\vec{v}$ and $\vec{a}$ are perpendicular, we check that the dot product is zero:

$$\begin{aligned}\vec{v}\cdot\vec{a} &= (6\cos(3t)\vec{j} - 6\sin(3t)\vec{k})\cdot(-18\sin(3t)\vec{j} - 18\cos(3t)\vec{k}) \\ &= -108\cos(3t)\sin(3t) + 108\sin(3t)\cos(3t) = 0.\end{aligned}$$

The speed is

$$\|\vec{v}\| = \|6\cos(3t)\vec{j} - 6\sin(3t)\vec{k}\| = 6\sqrt{\sin^2(3t)+\cos^2(3t)} = 6,$$

and so is constant. The magnitude of the acceleration is

$$\|\vec{a}\| = \|-18\sin(3t)\vec{j} - 18\cos(3t)\vec{k}\| = 18\sqrt{\sin^2(3t)+\cos^2(3t)} = 18,$$

which is also constant.

19. In vector form the parameterization is

$$\vec{r} = 2\vec{i} + 3\vec{j} + 5\vec{k} + t^2(\vec{i} - 2\vec{j} - \vec{k}).$$

Thus the motion is along the straight line through $(2,3,5)$ in the direction of $\vec{i} - 2\vec{j} - \vec{k}$. The velocity vector $\vec{v}$ is

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = 2t(\vec{i} - 2\vec{j} - \vec{k})$$

The acceleration vector $\vec{a}$ is

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = 2(\vec{i} - 2\vec{j} - \vec{k}).$$

The speed is

$$\|\vec{v}\| = 2|t|\|\vec{i} - 2\vec{j} - \vec{k}\| = 2\sqrt{6}|t|.$$

The acceleration vector is constant and points in the direction of $\vec{i} - 2\vec{j} - \vec{k}$. When $t < 0$ the absolute value $|t|$ is decreasing, hence the speed is decreasing. Also, when $t < 0$ the velocity vector $2t(\vec{i} - 2\vec{j} - \vec{k})$ points in the direction opposite to $\vec{i} - 2\vec{j} - \vec{k}$. When $t > 0$ the absolute value $|t|$ is increasing and hence the speed is increasing. Also, when $t > 0$ the velocity vector points in the same direction as $\vec{i} - 2\vec{j} - \vec{k}$.

20. In vector form the parameterization is

$$\vec{r} = \vec{i} + -5\vec{j} - 2\vec{k} + (2t^3+3t)(-\vec{i} + 2\vec{j} + 3\vec{k}).$$

Thus the motion is along the straight line through $(1,-5,-2)$ in the direction of $-\vec{i} + 2\vec{j} + 3\vec{k}$. The velocity vector $\vec{v}$ is

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = (6t^2+3)(-\vec{i} + 2\vec{j} + 3\vec{k})$$

The acceleration vector $\vec{a}$ is

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = 12t(-\vec{i} + 2\vec{j} + 3\vec{k}).$$

The speed is

$$\|\vec{v}\| = |6t^2+3|\|-\vec{i} + 2\vec{j} + 3\vec{k}\| = 3\sqrt{14}|2t^2+1| = 3\sqrt{14}(2t^2+1).$$

The graph of the speed is a parabola opening upward with vertex at $t = 0$. Thus the speed is decreasing when $t < 0$ and increasing when $t > 0$. The velocity vector always points in the same direction $-\vec{i} + 2\vec{j} + 3\vec{k}$, since $6t^2 + 3$ is always positive. The acceleration vector points in the opposite direction to $-\vec{i} + 2\vec{j} + 3\vec{k}$ when $t < 0$ and in the same direction when $t > 0$. Thus the acceleration vector points in the opposite direction to the speed when $t < 0$ and in the same direction when $t > 0$.

21. At $t = 2$, the position and velocity vectors are

$$\vec{r}(2) = (2-1)^2\vec{i} + 2\vec{j} + (2\cdot 2^3 - 3\cdot 2^2)\vec{k} = \vec{i} + 2\vec{j} + 4\vec{k},$$
$$\vec{v}(2) = 2\cdot(2-1)\vec{i} + (6\cdot 2^2 - 6\cdot 2)\vec{k} = 2\vec{i} + 12\vec{k}.$$

So we want the line going through the point $(1, 2, 4)$ at the time $t = 2$, in the direction $2\vec{i} + 12\vec{k}$:

$$x = 1 + 2(t-2), \quad y = 2 \quad z = 4 + 12(t-2).$$

Problems

22. A parameterization is

$$\vec{r}(t) = 5\vec{i} + 4\vec{j} - 2\vec{k} + (t-4)(2\vec{i} - 3\vec{j} + \vec{k})$$

or equivalently

$$x = 5 + 2(t-4), \quad y = 4 - 3(t-4), \quad z = -2 + (t-4).$$

23. The velocity vector for this motion is

$$\vec{v} = (2t-6)\vec{i} + (3t^2-3)\vec{j}.$$

The motion is vertical when the component in the $\vec{i}$ direction is 0 and motion in $\vec{j}$ direction is not 0. Motion in $\vec{i}$ direction is 0 when

$$2t - 6 = 0,$$
$$t = 3.$$

At that time, motion in $\vec{j}$ direction is not 0. The motion is horizontal when the component in the $\vec{j}$ direction is 0 and motion in $\vec{i}$ direction is not 0. Motion in $\vec{j}$ direction is 0 when

$$3t^2 - 3 = 0,$$
$$t = 1, -1.$$

At these times, motion in $\vec{i}$ direction is not 0. To determine the end behavior, recall that a polynomial is approximated by its highest powered term for large values (positive or negative) of the independent variable. Thus, as $t \to \pm\infty$, we have $x \approx t^2$ and $y \approx t^3$. The end behavior, and the x and y coordinates when the motion is vertical or horizontal, are shown in Table 17.1. The graph is shown in Figure 17.5.

Table 17.1

t	x	y
$-\infty$	$+\infty$	$-\infty$
-1	7	2
1	-5	-2
3	-9	18
$+\infty$	$+\infty$	$+\infty$

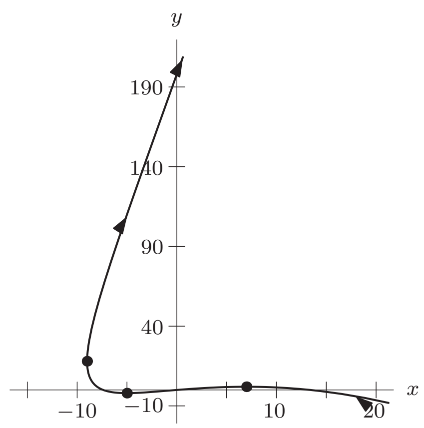


Figure 17.5

24. The velocity vector for this motion is

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} = (3t^2 - 12)\vec{i} + (2t + 10)\vec{j}\,.$$

The motion is vertical when the component in the $\vec{i}$ direction is 0 and motion in $\vec{j}$ direction is not 0. Motion in $\vec{i}$ direction is 0 when

$$\begin{aligned} 3t^2 - 12 &= 0, \\ t &= 2, -2. \end{aligned}$$

At these times, motion in $\vec{j}$ direction is not 0. The motion is horizontal when the component in the $\vec{j}$ direction is 0 and motion in $\vec{i}$ direction is not 0. Motion in $\vec{j}$ direction is 0 when

$$\begin{aligned} 2t + 10 &= 0, \\ t &= -5. \end{aligned}$$

At this time, the motion in $\vec{i}$ direction is not 0. To determine the end behavior, recall that a polynomial is approximated by its highest powered term for large values (positive or negative) of the independent variable. Thus, as $t \to \pm\infty$, we have $x \approx t^3$ and $y \approx t^2$. The end behavior, and the x and y coordinates when the motion is vertical or horizontal, are shown in Table 17.2. The graph is shown in Figure 17.6.

Table 17.2

t	x	y
$-\infty$	$-\infty$	$+\infty$
-5	-65	-25
-2	16	-16
2	-16	24
$+\infty$	$+\infty$	$+\infty$

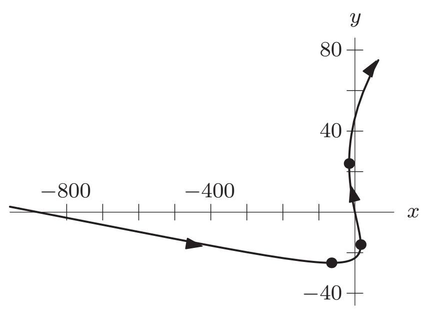


Figure 17.6

25. Plotting the positions on the xy plane and noting their times gives the graph shown in Figure 17.7.

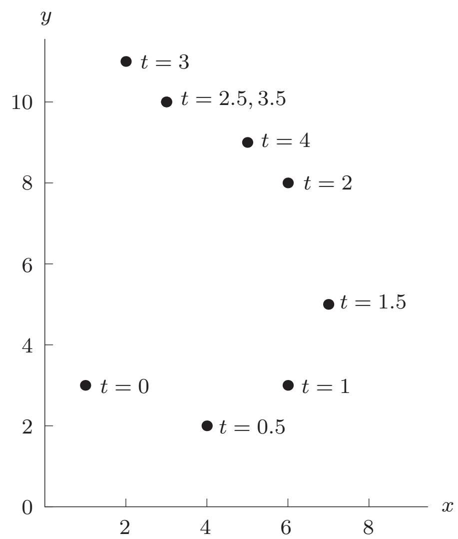


Figure 17.7

(a) We approximate dx/dt by $\Delta x/\Delta t$ calculated between $t = 1.5$ and $t = 2.5$:

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{3-7}{2.5-1.5} = \frac{-4}{1} = -4.$$

Similarly,

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = \frac{10-5}{2.5-1.5} = \frac{5}{1} = 5.$$

So,

$$\vec{v}\,(2) \approx -4\vec{i} + 5\vec{j} \quad \text{and} \quad \text{Speed} = \|\vec{v}\,\| = \sqrt{41}.$$

(b) The particle is moving vertically at about time $t = 1.5$. Note that the particle is momentarily stopped at about $t = 3$; however it is not moving parallel to the y-axis at this instant.

(c) The particle stops at about time $t = 3$ and reverses course.

26. (a) The vector $\overrightarrow{PQ}$ between the points is given by

$$\overrightarrow{PQ} = 2\vec{i} + 5\vec{j} + 3\vec{k}.$$

Since $||\overrightarrow{PQ}|| = \sqrt{2^2+5^2+3^2} = \sqrt{38}$, the velocity vector of the motion is

$$\vec{v} = \frac{5}{\sqrt{38}}(2\vec{i} + 5\vec{j} + 3\vec{k}).$$

(b) The motion is along a line starting at the point $(3, 2, -5)$ and with the velocity vector from part (a). The equation of the line is

$$\vec{r} = 3\vec{i} + 2\vec{j} - 5\vec{k} + t\vec{v} = 3\vec{i} + 2\vec{j} - 5\vec{k} + \frac{5}{\sqrt{38}}(2\vec{i} + 5\vec{j} + 3\vec{k})t,$$

so

$$x = 3 + \frac{10}{\sqrt{38}}t, \qquad y = 2 + \frac{25}{\sqrt{38}}t, \qquad z = -5 + \frac{15}{\sqrt{38}}t.$$

27. (a) The particle starts at $(2, -1, 5)$ so $\vec{r}_0 = 2\vec{i} - \vec{j} + 5\vec{k}$. In 5 seconds, the particle moves through a displacement given by $\overrightarrow{PQ} = 3\vec{i} + 4\vec{j} - 6\vec{k}$. Its velocity, $\vec{v}$, is given by

$$\vec{v} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} - \frac{6}{5}\vec{k} = 0.6\vec{i} + 0.8\vec{j} - 1.2\vec{k}.$$

Thus, the equation of the motion is

$$\vec{r} = 2\vec{i} - \vec{j} + 5\vec{k} + t(0.6\vec{i} + 0.8\vec{j} - 1.2\vec{k})$$

or

$$x = 2 + 0.6t, \quad y = -1 + 0.8t, \quad z = 5 - 1.2t$$

where $0 \le t \le 5$.

(b) The velocity vector in part (a), $\vec{v} = 0.6\vec{i} + 0.8\vec{j} - 1.2\vec{k}$, means that the particle is moving with

$$\text{Speed} = ||\vec{v}\,|| = \sqrt{(0.6)^2 + (0.8)^2 + (1.2)^2} = 1.562.$$

To make the speed equal 5, take a new velocity vector given by

$$\vec{v} = \frac{5}{1.562}(0.6\vec{i} + 0.8\vec{j} - 1.2\vec{k}) = 1.92\vec{i} + 2.56\vec{j} - 3.84\vec{k}.$$

Thus, the equation of the motion is

$$\vec{r} = 2\vec{i} - \vec{j} + 5\vec{k} + t(1.92\vec{i} + 2.56\vec{j} - 3.84\vec{k})$$

or

$$x = 2 + 1.92t, \quad y = -1 + 2.56\,t, \quad z = 5 - 3.84t.$$

The particle reaches $Q = (5, 3, -1)$ when

$$2 + 1.92t = 5$$

or $t = 1.56$ seconds. The parametric equations describe the motion from P to Q when $0 \le t \le 1.56$.

28. (a) We substitute $x = 1 + t, y = 5 + 2t, z = -7 + t$ into $x + y + z = 1$ and solve for t:

$$\begin{aligned}(1+t) + (5+2t) + (t-7) &= 1\\ 4t - 1 &= 1\\ t &= 0.5 \text{ sec.}\end{aligned}$$

When $t = 0.5$, the particle is at the point $(x, y, z) = (1 + (0.5), 5 + 2(0.5), (0.5) - 7) = (1.5, 6, -6.5)$.

(b) The particle's velocity is

$$\vec{v} = \vec{i} + 2\vec{j} + \vec{k},$$

so

$$\text{Speed } = ||1^2 + 2^2 + 1^2|| = \sqrt{6} \text{ meters/sec.}$$

29. (a) At $t = 0$, we have $\vec{r}(0) = 0\vec{i} + 0\vec{j} + 6.4\vec{k}$, so the stone's initial position is $(0, 0, 6.4)$. Thus the rooftop is 6.4 meters above the ground.

(b) The stone hits the ground when the height above the ground is 0; that is, when its z coordinate is 0:

$$\begin{aligned}6.4 - 4.9t^2 &= 0\\ t &= \pm\sqrt{\frac{6.4}{4.9}} = \pm 1.14.\end{aligned}$$

Since t must be positive, the stone hits the ground about 1.14 seconds after it is thrown.

(c) The velocity of the stone at time t is given by

$$\vec{v}(t) = \vec{r}\,'(t) = 10\vec{i} - 5\vec{j} - 9.8t\vec{k},$$

so when the stone hits the ground at $t = 1.14$ seconds,

$$\vec{v}(1.14) = 10\vec{i} - 5\vec{j} - 9.8(1.14)\vec{k} = 10\vec{i} - 5\vec{j} - 11.172\vec{k}.$$

The stone's speed is given by $||\vec{v}(1.14)|| = \sqrt{10^2 + 5^2 + 11.172^2} = 15.81$ meters/sec.

(d) The stone hits the ground at the point with position vector

$$\vec{r}(1.14) = 10(1.14)\vec{i} - 5(1.14)\vec{j} + (6.4 - 4.9(1.14)^2)\vec{k},$$

which is the point $(11.4, -5.7, 0)$.

(e) The acceleration of the stone at time t is given by

$$\vec{a}(t) = \vec{v}\,'(t) = -9.8\vec{k}.$$

Thus, the acceleration is constant; the stone hits the ground at an acceleration of -9.8 meters/sec^2; that is 9.8 meters/sec^2 downward.

30. (a) Since $z = 90$ feet when $t = 0$, the tower is 90 feet high.

(b) The child reaches the bottom when $z = 0$, so $t = 90/5 = 18$ minutes.

(c) Her velocity is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = -(10\sin t)\vec{i} + (10\cos t)\vec{j} - 5\vec{k},$$

so

$$\text{Speed} = ||\vec{v}|| = \sqrt{(-10\sin t)^2 + (10\cos t)^2 + (-5)^2} = \sqrt{10^2 + 5^2} = \sqrt{125} \text{ ft/min.}$$

(d) Her acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = -(10\cos t)\vec{i} - (10\sin t)\vec{j} \text{ ft/min}^2.$$

31. (a) The height at time t is given by $z = 100 - (t-5)^2$, so the maximum height occurs at $t = 5$ secs, when $r = 10\vec{i} + 15\vec{j} + 100\vec{k}$, so the point is $(10, 15, 100)$.

(b) The velocity of the particle at time t is given by

$$\vec{v} = 2\vec{i} + 3\vec{j} - 2(t-5)\vec{k},$$

so the speed is

$$||\vec{v}|| = \sqrt{2^2 + 3^2 + 2^2(t-5)^2} = \sqrt{13 + 4(t-5)^2} \text{ cm/sec.}$$

Thus, the maximum speed occurs when $t = 0$ secs and when $t = 10$ secs and is given by $||\vec{v}|| = \sqrt{13 + 4(5^2)} = \sqrt{113} = 10.630$ cm/sec.

(c) The minimum speed occurs when $t = 5$ and is given by $||\vec{v}|| = \sqrt{13} = 3.606$ cm/sec.

32. We want dw/dt at $t = 0$. The chain rule gives

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Since $\vec{r}\,'(0) = 2\vec{i} + 3\vec{j} + 6\vec{k}$ and

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k},$$

we have

$$x'(0) = 2, \quad y'(0) = 3, \quad z'(0) = 6.$$

Since $\text{grad}\, f = 4\vec{i} - 3\vec{j} + \vec{k}$ at the point $(7, 2, 5)$ which the particle reaches at time $t = 0$, we have

$$\left.\frac{\partial w}{\partial x}\right|_{(7,2,5)} = 4, \quad \left.\frac{\partial w}{\partial y}\right|_{(7,2,5)} = -3, \quad \left.\frac{\partial w}{\partial z}\right|_{(7,2,5)} = 1.$$

Thus

$$\left.\frac{dw}{dt}\right|_{t=0} = 4 \cdot 2 - 3 \cdot 3 + 1 \cdot 6 = 5$$

33. (a) The ball hits the ground when $y = 0$, so

$$2 + 25t - 4.9t^2 = 0.$$

The quadratic formula gives

$$t = \frac{-25 \pm \sqrt{25^2 - 4 \cdot 2(-4.9)}}{-2(4.9)} = -0.079 \text{ or } 5.181 \text{ sec.}$$

We need the positive answer, $t = 5.181$ sec.

(b) At the time the ball hits the ground, $x = 20(5.181) = 103.616$ meters. Thus, the ball hits the ground after 5.181 seconds at a point 103.616 meters horizontally from where it was thrown.

(c) $h = 2$ meters.

(d) $g = 9.8$ meters/sec^2.

(e) Since $v\cos\theta = 20$ and $v\sin\theta = 25$, we have

$$\tan\theta = \frac{v\sin\theta}{v\cos\theta} = \frac{25}{20} = 1.25,$$

so

$$\theta = \arctan(1.25) = 0.896.$$

Then

$$v = \frac{20}{\cos 0.896} = 32.016 \text{ meters/sec.}$$

34. (a) To eliminate t, substitute $t = x/20$ into the equation for z. This gives

$$z = 5\left(\frac{x}{20}\right) - 0.5\left(\frac{x}{20}\right)^2 = \frac{x}{4} - \frac{x^2}{800}.$$

(b) To decide when the particle is at ground level, set the equation for z equal to 0 and solve for t:

$$5t - 0.5t^2 = 0$$
$$-0.5t(t - 10) = 0,$$

so $t = 0$ and $t = 10$ seconds.

(c) The particle's velocity is $\vec{v} = x'(t)\vec{i} + z'(t)\vec{k}$, so

$$\vec{v} = 20\vec{i} + (5 - t)\vec{k}.$$

(d) The particle's speed is

$$||\vec{v}|| = \sqrt{20^2 + (5-t)^2} = \sqrt{400 + (5-t)^2} \text{ m/s.}$$

(e) No. The quantity $\sqrt{400 + (5-t)^2}$ is never 0.

(f) The particle is at its highest point halfway between the times when it is at ground level, or when $t = 5$. Alternatively, the highest point occurs when $z' = 0$, that is

$$z'(t) = 5 - t = 0 \quad \text{so} \quad t = 5 \text{ seconds.}$$

35. (a) The top of the tower is at the point $(0, 0, 20)$, so we want $\vec{r}(0) = 20\vec{k}$. This is (I) and (IV). Only (IV) is going downward.

Projectile (IV) hits the ground when $z = 0$, which occurs when $20 - t^2 = 0$, so $t = \sqrt{20} = 4.5$. (We take the positive root since the projectile is launched when $t = 0$.) At this time, $\vec{r}(\sqrt{20}) = 8.9\vec{j}$, so the projectile hits the ground at the point $(0, 8.9, 0)$, which is 8.9 meters from the base of the tower in the direction of the tree.

(b) To hit the top of the tree, the projectile must go through the point $(0, 20, 20)$. This is (II).

The projectile reaches the top of the tree when $2t^2 = 20$, so (taking the positive root) $t = \sqrt{10} = 3.2$ sec. The projectile is launched from $\vec{r}(0) = \vec{0}$, the base of the tower.

(c) Projectiles launched from somewhere on the tower have $x(0) = y(0) = 0$ and $0 \le z(0) \le 20$. Only (III) and (V) have nonzero $x(0)$ and $y(0)$.

To hit the tree, there must be a time for which the projectile is at a point $(0, 20, z)$ for some $0 \le z \le 20$.

Since (III) has $x(t) = 20$ for all t, it does not hit the tree. So (V) is the answer.

For (V), we have $2t = 20$, when $t = 10$ sec. Then $\vec{r}(10) = 20\vec{j} + 10\vec{k}$, so the projectile hits the tree at $(0, 20, 10)$, which is half way up.

36. (a) For any positive constant k, the parameterization

$$x = -5\sin(kt) \quad y = 5\cos(kt)$$

moves counterclockwise on a circle of radius 5 starting at the point $(0, 5)$. We choose k to make the period 8 seconds. If $k \cdot 8 = 2\pi$, then $k = \pi/4$ and the parameterization is

$$x = -5\sin\left(\frac{\pi t}{4}\right) \qquad y = 5\cos\left(\frac{\pi t}{4}\right).$$

(b) Since it takes 8 seconds for the particle to go around the circle

$$\text{Speed} = \frac{\text{Circumference of circle}}{8} = \frac{2\pi(5)}{8} = \frac{5\pi}{4} \text{ cm/sec.}$$

37. Since the acceleration due to gravity is -9.8 m/sec^2, we have $\vec{r}''(t) = -9.8\vec{k}$. Integrating gives

$$\vec{r}'(t) = C_1\vec{i} + C_2\vec{j} + (-9.8t + C_3)\vec{k},$$
$$\vec{r}(t) = (C_1t + C_4)\vec{i} + (C_2t + C_5)\vec{j} + (-4.9t^2 + C_3t + C_6)\vec{k}.$$

The initial condition, $\vec{r}(0) = \vec{0}$, implies that $C_4 = C_5 = C_6 = 0$, thus

$$\vec{r}(t) = C_1t\vec{i} + C_2t\vec{j} + (-4.9t^2 + C_3t)\vec{k}.$$

To find the position vector, we need to find the values of C_1, C_2, and C_3. This we do using the coordinates of the highest point. When the rocket reaches its peak, the vertical component of the velocity is zero, so $-9.8t + C_3 = 0$. Thus, at the highest point, $t = C_3/9.8$. At that time

$$\vec{r}(t) = 1000\vec{i} + 3000\vec{j} + 10000\vec{k},$$

so, for the same value of t:

$$C_1t = 1000,$$
$$C_2t = 3000,$$
$$-4.9t^2 + C_3t = 10,000,$$

Substituting $t = C_3/9.8$ into the third equation gives

$$-4.9\left(\frac{C_3}{9.8}\right)^2 + \frac{C_3^2}{9.8} = 10{,}000$$
$$C_3^2 = 2(9.8)10{,}000$$
$$C_3 = 442.7$$

Then $C_1 = \frac{1000}{C_3/9.8} = 22.1$ and $C_2 = \frac{3000}{C_3/9.8} = 66.4$. Thus,

$$\vec{r}(t) = 22.1t\vec{i} + 66.4t\vec{j} + (442.7t - 4.9t^2)\vec{k}.$$

38. (a) The parametric equation describing Emily's motion is

$$x = 10\cos\left(\frac{2\pi}{20}t\right) = 10\cos\left(\frac{\pi}{10}t\right), \quad y = 10\sin\left(\frac{2\pi}{20}t\right) = 10\sin\left(\frac{\pi}{10}t\right) \quad z = \text{constant}.$$

Her velocity vector is

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = -\pi\sin\left(\frac{\pi}{10}t\right)\vec{i} + \pi\cos\left(\frac{\pi}{10}t\right)\vec{j}.$$

Her speed is given by:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{\left(-\pi\sin\left(\frac{\pi}{10}t\right)\right)^2 + \left(\pi\cos\left(\frac{\pi}{10}t\right)\right)^2 + 0^2} \\ &= \pi\sqrt{\sin^2\left(\frac{\pi}{10}t\right) + \cos^2\left(\frac{\pi}{10}t\right)} \\ &= \pi\sqrt{1} = \pi \text{ m/sec},\end{aligned}$$

which is independent of time (as we expected). This is certainly the long way to solve this problem though, since we could have simply divided the circumference of the circle (20π) by the time taken for a single rotation (20 seconds) to arrive at the same answer.

(b) When Emily drops the ball, it initially has Emily's velocity vector, but it immediately begins accelerating in the z-direction due to the force of gravity. The motion of the ball will then be tangential to the merry-go-round, curving down to the ground. In order to find the tangential component of the ball's motion, we must know Emily's velocity at the moment she dropped the ball. Then we can integrate the velocity and obtain the position of the ball. Assuming Emily drops the ball at time $t = 0$, her position and velocity vector are

$$\vec{r}(0) = 10\vec{i} + 3\vec{k} \text{ and } \vec{v}(0) = \pi\vec{j}.$$

Thus, the ball has velocity only in the y-direction when it is dropped. In the z-direction, we have

$$\text{Acceleration} = \frac{d^2z}{dt^2} = -9.8 \text{ m/sec}^2.$$

Since the initial velocity 0 and initial height 3, we have

$$z = 3 - 4.9t^2.$$

The ball touches the ground when $z = 0$, that is, when $t = 0.78$ sec. In that time, the ball also travels $\pi(0.78) = 2.45$ meters in the y-direction. So, the final position is $(10, 2.45, 0)$. The distance between this point and $P = (10, 0, 0)$ is 2.45 meters.

(c) The distance of the ball from Emily when it hits the ground is found by finding Emily's position at $t = 0.78$ sec and using the distance formula. Emily's position when the ball hits the ground is $(10\cos(0.078\pi), 10\sin(0.078\pi), 3) = (9.70, 2.43, 3)$. The distance between this point and the point where the ball struck the ground is:

$$d \approx \sqrt{(10 - 9.70)^2 + (2.45 - 2.43)^2 + (0 - 3)^2} = 3.01 \text{ meters}.$$

Note that the merry-go-round does not rotate very much in the 0.78 sec needed for the ball to reach the ground, so our answer makes sense.

39. Since the particle moves in a circle of radius a we have $||\vec{r}(t)|| = a$ so

$$\vec{r}(t) \cdot \vec{r}(t) = a^2.$$

Differentiating with respect to t gives

$$\frac{d}{dt}\vec{r}(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \frac{d}{dt}\vec{r}(t) = 0$$

so

$$\vec{r}\,'(t) \cdot \vec{r}(t) = 0.$$

Thus, the position vector, $\vec{r}(t)$, and the velocity vector, $\vec{v} = \vec{r}\,'(t)$, are perpendicular at all times t.

40. (a) The center of the wheel moves horizontally, so its y-coordinate will never change; it will equal 1 at all times. In one second, the wheel rotates 1 radian, which corresponds to 1 meter on the rim of a wheel of radius 1 meter, and so the rolling wheel advances at a rate of 1 meter/sec. Thus the x-coordinate of the center, which equals 0 at $t = 0$, will equal t at time t. At time t the center will be at the point $(x, y) = (t, 1)$.

(b) By time t the spot on the rim will have rotated t radians clockwise, putting it at angle $-t$ as in Figure 17.8. The coordinates of the spot with respect to the center of the wheel are $(\cos(-t), \sin(-t))$. Adding these to the coordinates $(t, 1)$ of the center gives the location of the spot as $(x, y) = (t + \cos t, 1 - \sin t)$. See Figure 17.9.

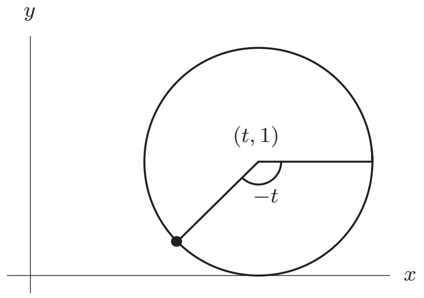


Figure 17.8

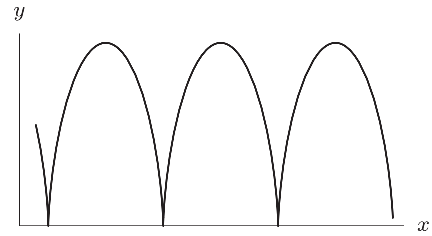


Figure 17.9

41. (a) No. The height of the particle is given by $2t$; the vertical velocity is the derivative $d(2t)/dt = 2$. Because this is a positive constant, the vertical component of the velocity vector is upward at a constant speed of 2.

(b) When $2t = 10$, so $t = 5$.

(c) The velocity vector is given by

$$\begin{aligned}\vec{v}\,(t) &= \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \\ &= -(\sin t)\vec{i} + (\cos t)\vec{j} + 2\vec{k}\,.\end{aligned}$$

From (b), the particle is at 10 units above the ground when $t = 5$, so at $t = 5$,

$$\vec{v}\,(5) = 0.959\vec{i} + 0.284\vec{j} + 2\vec{k}\,.$$

Therefore, $\vec{v}\,(5) = -\sin(5)\vec{i} + \cos(5)\vec{j} + 2\vec{k}$.

(d) At this point, $t = 5$, the particle is located at

$$\vec{r}\,(5) = (\cos(5), \sin(5), 10) = (0.284, -0.959, 10).$$

The tangent vector to the helix at this point is given by the velocity vector found in part (c), that is, $\vec{v}\,(5) = 0.959\vec{i} + 0.284\vec{j} + 2\vec{k}$. So, the equation of the tangent line is

$$\vec{r}\,(t) = 0.284\vec{i} - 0.959\vec{j} + 10\vec{k} + (t - 5)(0.959\vec{i} + 0.284\vec{j} + 2\vec{k}\,).$$

42. We have velocity vector $\vec{v}\,(t) = -\alpha \sin t\vec{i} + \alpha \cos t\vec{j} + \beta\vec{k}$. For the speed we compute

$$\text{Speed } = (\alpha^2 \cos^2 t + \alpha^2 \sin^2 t + \beta^2)^{1/2} = \sqrt{\alpha^2 + \beta^2}$$

which does not depend on t.

43. (a) Let the ant begin the trip at time $t = 0$, and let's place the origin of our coordinate system at the center of the disk. We align the axes so that at time $t = 0$ the radius along which the ant crawls falls on the positive x-axis. At time t seconds, the ant is at a distance of $r = t$ cm from the origin and at angle $\theta = 2\pi t$ radians from the positive x-axis. The Cartesian coordinates of this point are $(x, y) = (r\cos\theta, r\sin\theta) = (t\cos(2\pi t), t\sin(2\pi t))$. We can write the parametric equations of the ant's motion in vector form as

$$\vec{r}\,(t) = t\cos(2\pi t)\vec{i} + t\sin(2\pi t)\vec{j}\,, 0 \le t \le 100.$$

(b) The velocity vector of the ant is the derivative

$$\vec{v}\,(t) = \vec{r}\,'(t) = (\cos(2\pi t) - 2\pi t\sin(2\pi t))\vec{i} + (\sin(2\pi t) + 2\pi t\cos(2\pi t))\vec{j}\,.$$

The speed is the magnitude of the velocity vector

$$\begin{aligned}\|\vec{v}\,\| &= ((\cos(2\pi t) - 2\pi t\sin(2\pi t))^2 + (\sin(2\pi t) + 2\pi t\cos(2\pi t))^2)^{1/2}\\ &= (1 + 4\pi^2 t^2)^{1/2}\ \text{cm/sec}.\end{aligned}$$

Observe that the speed of the ant is increasing. Even though the ant is crawling at constant rate on the disk, the turning of the disk moves the ant faster and faster as it gets closer to the edge.

(c) The acceleration vector is

$$\vec{a}\, = \vec{v}\,'(t) = (-4\pi\sin(2\pi t) - 4\pi^2 t\cos(2\pi t))\vec{i} + (4\pi\cos(2\pi t) - 4\pi^2 t\sin(2\pi t))\vec{j}\,.$$

The magnitude of the acceleration is

$$\begin{aligned}\|\vec{a}\,\| &= ((-4\pi\sin(2\pi t) - 4\pi^2 t\cos(2\pi t))^2 + (4\pi\cos(2\pi t) - 4\pi^2 t\sin(2\pi t))^2)^{1/2}\\ &= 4\pi(1 + \pi^2 t^2)^{1/2}\ \text{cm/sec}^2.\end{aligned}$$

44. (a) Since $x = R\cos(\omega t)$ and $y = R\sin(\omega t)$, and $x^2 + y^2 = R^2\cos^2(\omega t) + R^2\sin^2(\omega t) = R^2$, we have motion around a circle of radius R centered at the origin. The particle moves counterclockwise, completing one revolution in time $2\pi/\omega$. Thus, the period $= 2\pi/\omega$.

(b) The velocity vector is

$$\vec{v}\, = \frac{d\vec{r}}{dt} = -\omega R\sin(\omega t)\vec{i} + \omega R\cos(\omega t)\vec{j}\,.$$

We expect the velocity, $\vec{v}$, to be tangent to the circle. To verify that this, we compute

$$\begin{aligned}\vec{v}\cdot\vec{r}\, &= (-\omega R\sin(\omega t)\vec{i} + \omega R\cos(\omega t)\vec{j}\,)\cdot(R\cos(\omega t)\vec{i} + R\sin(\omega t)\vec{j}\,)\\ &= -\omega R^2\sin(\omega t)\cos(\omega t) + \omega R^2\cos(\omega t)\sin(\omega t) = 0.\end{aligned}$$

This shows that the velocity vector is perpendicular to the radius from the center of the circle to the particle, which moves counterclockwise.

The speed is $\|\vec{v}\,\| = \omega R$, which is constant. Notice that this makes sense, because in time $2\pi/\omega$, the particle travels a distance of $2\pi R$, giving a speed of $2\pi R/(2\pi/\omega) = \omega R$.

(c) The acceleration vector is

$$\vec{a}\, = \frac{d\vec{v}}{dt} = -\omega^2 R\cos(\omega t)\vec{i} - \omega^2 R\sin(\omega t)\vec{j}\, = -\omega^2\vec{r}\,.$$

The acceleration vector points in the direction opposite to the position vector $\vec{r}$, and thus points toward the center of the circle. It has constant magnitude $\|\vec{a}\,\| = \omega^2 R = \|v\|^2/R$.

45. (a) Let x represent horizontal displacement (in cm) from some starting point and y the distance (in cm) above the ground. Since

$$25\ \text{km/hr} = \frac{25\cdot 10^5}{60^2} = 694.444\ \text{cm/sec},$$

if t is in seconds, the motion of the center of the pedal is given by

$$x\vec{i} + y\vec{j}\, = 694.444t\vec{i} + 30\vec{j}\,.$$

The circular motion of your foot relative to the center is described by

$$h\vec{i} + k\vec{j}\, = 20\cos(2\pi t)\vec{i} + 20\sin(2\pi t)\vec{j}\,,$$

so the motion of the light on your foot relative to the ground is described by

$$x\vec{i} + y\vec{j}\, = (694.444t + 20\cos(2\pi t))\vec{i} + (30 + 20\sin(2\pi t))\vec{j}\,.$$

(b) See Figure 17.10.

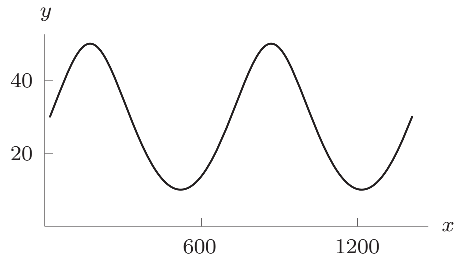


Figure 17.10

(c) Suppose your pedal is rotating with angular velocity ω radians/sec, so that the motion is described by

$$x\vec{i} + y\vec{j} = (694.444t + 20\cos\omega t)\vec{i} + (30 + 20\sin\omega t)\vec{j}.$$

The light moves backward if dx/dt is negative. Since

$$\frac{dx}{dt} = 694.444 - 20\omega\sin\omega t,$$

the minimum value of dx/dt occurs when $\omega t = \pi/2$, and then

$$\frac{dx}{dt} = 694.444 - 20\omega < 0$$

giving

$$\omega \geq 34.722 \text{ radians/sec.}$$

Since there are 2π radians in a complete revolution, an angular velocity of 34.722 radians/sec means $34.722/2\pi \approx 5.526$ revolutions/sec.

46. At time t object B is at the point with position vector $\vec{r}_B(t) = \vec{r}_A(2t)$, which is exactly where object A is at time $2t$. Thus B visits the same points as A, but does so at different times; A gets there later. While B covers the same path as A, it moves twice as fast. To see this, note for example that between $t = 1$ and $t = 3$, object B moves along the path from $\vec{r}_B(1) = \vec{r}_A(2)$ to $\vec{r}_B(3) = \vec{r}_A(6)$ which is traversed by object A during the time interval from $t = 2$ to $t = 6$. It takes A twice as long to cover the same ground.

In the case where $\vec{r}_A(t) = t\vec{i} + t^2\vec{j}$, both objects move on the parabola $y = x^2$. Both A and B are at the origin at time $t = 0$, but B arrives at the point $(2, 4)$ at time $t = 1$, whereas A does does not get there until $t = 2$.

47. In uniform circular motion the velocity vector is tangent to the circle of motion and the acceleration vector is directed toward the center of the circle. At all times the velocity $\vec{v}$ and acceleration $\vec{a}$ are perpendicular. Since $\vec{v}\cdot\vec{a} = (2\vec{i} + \vec{j})\cdot(\vec{i} + \vec{j}) = 3 \neq 0$, $\vec{v}$ and $\vec{a}$ are not perpendicular, and so the object can not be in uniform circular motion.

48. The acceleration vector points from the object to the center of the orbit, and the velocity vector points from the object tangent to the circle in the direction of motion. From Figure 17.11 we see that the movement is counterclockwise.

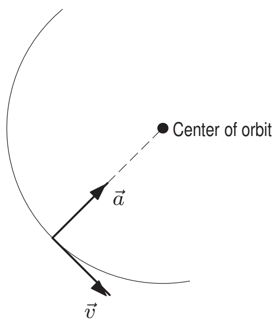


Figure 17.11

49. (a) Using the product rule for differentiation we get

$$\frac{d}{dt}(\vec r\cdot\vec r\,) = \vec r\cdot\frac{d\vec r}{dt} + \frac{d\vec r}{dt}\cdot\vec r\, = 2\vec r\cdot\frac{d\vec r}{dt}.$$

(b) Since $\vec a$ is a constant, $d\vec a\,/dt = 0$ so the product rule gives

$$\frac{d}{dt}(\vec a\times\vec r\,) = \vec a\times\frac{d\vec r}{dt}.$$

(c) The product rule gives

$$\frac{d}{dt}(r^3\vec r\,) = r^3\frac{d\vec r}{dt} + \frac{d}{dt}(r^3)\vec r\, = r^3\frac{d\vec r}{dt} + 3r^2\vec r\,.$$

50. (a) Since $\operatorname{grad} f(1,7,2) = \vec i - (\sqrt6)\vec j + \vec k$ is normal to the tangent plane, and since the plane goes through the point $(1,7,2)$, an equation is

$$x-\sqrt6 y+z = 1\cdot 1-\sqrt6\cdot 7+1\cdot 2 = 3-7\sqrt6$$
$$x-\sqrt6 y+z = 3-7\sqrt6.$$

(b) A vector normal to the level surface is $\vec n = \vec i - (\sqrt6)\vec j + \vec k$. The curve C pass through the point $(1,7,2)$ at $t=0$, so we find the tangent to the curve at this point:

$$\vec r\,'(t) = 2(t+1)\vec i - 7\sin t\vec j + 2e^t\vec k\,,$$

$$\vec r\,'(0) = 2\vec i - 7\cdot 0\vec j + 2e^0\vec k = 2\vec i + 2\vec k\,.$$

The angle, θ, between $\vec n$ and $\vec r\,'(0)$ is given by

$$\cos\theta = \frac{\vec n\cdot\vec r\,'(0)}{\|\vec n\,\|\cdot\|\vec r\,'(0)\|} = \frac{(\vec i-(\sqrt6)\vec j+\vec k\,)\cdot(2\vec i+2\vec k\,)}{\|\vec i-(\sqrt6)\vec j+\vec k\,\|\cdot\|2\vec i+2\vec k\,\|} = \frac{1\cdot 2+1\cdot 2}{\sqrt{1^2+(\sqrt6)^2+1^2}\cdot\sqrt{2^2+2^2}} = \frac{4}{\sqrt8\sqrt8} = \frac12.$$

Thus, $\theta = \arccos(1/2) = \pi/3$.

(c) By the chain rule, the rate at which the concentration, c, is changing is

$$\frac{dc}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \operatorname{grad} f\cdot\vec r\,'(t).$$

Thus, at $t=0$,

$$\frac{dc}{dt} = \operatorname{grad} f\cdot\vec r\,'(0) = (\vec i-(\sqrt6)\vec j+\vec k\,)\cdot(2\vec i+2\vec k\,) = 4 \text{ ppm per second.}$$

51. Since $\vec v = s\cos\theta\,\vec i + s\sin\theta\,\vec j$, the unit vector in the direction of $\vec v$ is

$$\vec T = \cos\theta\,\vec i + \sin\theta\,\vec j$$

and

$$\vec N = \vec k\times\vec T = -\sin\theta\,\vec i + \cos\theta\,\vec j\,.$$

Using the chain rule to differentiate $\vec v$ with respect to t, we have

$$\begin{aligned}\vec a &= (s'\cos\theta - s(\sin\theta)\theta')\vec i + (s'\sin\theta + s(\cos\theta)\theta')\vec j\\ &= s'(\cos\theta\,\vec i+\sin\theta\,\vec j\,) + s\theta'(-\sin\theta\,\vec i+\cos\theta\,\vec j\,)\\ &= s'\vec T + s\theta'\vec N\,.\end{aligned}$$

52. Let $\vec F$ be the force towards O. Then $\vec F$ is parallel to $\vec r\,(t)$ but in the opposite direction, so $\vec F = -k\vec r\,(t)$ for some constant k. However, by Newton's second law, $\vec F = m\vec a$, where $\vec a\,(t) = \vec r\,''(t)$ is the acceleration of the particle. Now consider

$$\begin{aligned}\frac{d}{dt}(\vec r\,(t)\times\vec v\,(t)) &= \frac{d}{dt}(\vec r\,(t))\times\vec v\,(t) + \vec r\,(t)\times\frac{d}{dt}(\vec v\,(t))\\ &= \vec v\,(t)\times\vec v\,(t) + \vec r\,(t)\times\vec a\,(t) = \vec 0\,.\end{aligned}$$

since $\vec r\,(t)$ and $\vec a\,(t)$ are parallel and $\vec v\,(t)\times\vec v\,(t) = \vec 0$. Thus, $\vec r\,(t)\times\vec v\,(t) = \vec c$, a constant.

Since $\vec r\,(t)\times\vec v\,(t) = \vec c$, the particle moves so that $\vec r\,(t)$ is perpendicular to $\vec c$. Thus, $\vec r\,(t)$ is always in the plane perpendicular to $\vec c$ that contains the fixed point O.

53. (a) If $\Delta t = t_{i+1} - t_i$ is small enough so that C_i is approximately a straight line, then we can make the linear approximations

$$x(t_{i+1}) \approx x(t_i) + x'(t_i)\Delta t,$$
$$y(t_{i+1}) \approx y(t_i) + y'(t_i)\Delta t,$$
$$z(t_{i+1}) \approx z(t_i) + z'(t_i)\Delta t,$$

and so

$$\begin{aligned}\text{Length of } C_i &\approx \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2} \\ &\approx \sqrt{x'(t_i)^2(\Delta t)^2 + y'(t_i)^2(\Delta t)^2 + z'(t_i)^2(\Delta t)^2} \\ &= \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\Delta t.\end{aligned}$$

(b) From point (a) we obtain the approximation

$$\begin{aligned}\text{Length of } C &= \sum \text{length of } C_i \\ &\approx \sum \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\Delta t.\end{aligned}$$

The approximation gets better and better as Δt approaches zero, and in the limit the sum becomes a definite integral:

$$\begin{aligned}\text{Length of } C &= \lim_{\Delta t \to 0} \sum \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\,\Delta t \\ &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.\end{aligned}$$

Strengthen Your Understanding

54. Velocity and acceleration are orthogonal for uniform circular motion, which is motion at constant speed. If the speed is not constant, then velocity and acceleration are not orthogonal. For example, if

$$\vec{r}(t) = (\cos t^2)\vec{i} + (\sin t^2)\vec{j}$$

then

$$\vec{v}(t) \cdot \vec{a}(t) = 4t \neq 0.$$

55. Acceleration is a vector, not a scalar.

56. The length of the parameter interval is not the same as the length of the curve parameterized. The length of the curve is given by

$$\text{Length } = \int_A^B \|\vec{v}(t)\|\,dt.$$

57. The particle with position

$$\vec{r}(t) = \left(t + 2t^2\right)\vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

has velocity $\vec{v}(0) = \vec{i} + 2\vec{j}$ and acceleration $\vec{a}(0) = 4\vec{i} + 6\vec{k}$.

58. The curve

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}, \quad a \le t \le b$$

with velocity

$$\vec{v}(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$$

has

$$\text{Length} = \int_a^b \sqrt{\vec{v}(t) \cdot \vec{v}(t)}\,dt = \int_a^b \sqrt{2}\,dt = (b-a)\sqrt{2}.$$

If $b - a = 10/\sqrt{2}$, the length is 10. For example, the interval $0 \le t \le 10/\sqrt{2}$ corresponds to a piece of the helix of length 10.

59. False. The velocity vector is $\vec{v}(t) = \vec{r}\,'(t) = 2t\vec{i} - \vec{j}$. Then $\vec{v}(-1) = -2\vec{i} - \vec{j}$ and $\vec{v}(1) = 2\vec{i} - \vec{j}$, which are not equal.

60. True. The velocity vector is $\vec{v}(t) = \vec{r}\,'(t) = 2t\vec{i} - \vec{j}$, so the speed is $s(t) = \sqrt{4t^2+1}$. Then $s(-1) = s(1) = \sqrt{5}$.

61. False. While this is true for motion in a circle with constant speed, it is not true in general. For a counterexample, consider motion along a parabola $\vec{r}(t) = t\vec{i} + t^2\vec{j}$. Then $\vec{v}(t) = \vec{i} + 2t\vec{j}$ and $\vec{a}(t) = 2\vec{j}$. Taking the dot product gives $\vec{v}(t)\cdot\vec{a}(t) = 4t$, which is not zero for all t. Thus the velocity and acceleration vectors are not always perpendicular.

62. False. If a particle is moving along a line with nonconstant speed, then the acceleration and velocity vectors are parallel. For a counterexample, consider motion along the line $\vec{r}(t) = t^2\vec{i} + t^2\vec{j}$. Then $\vec{v}(t) = 2t\vec{i} + 2t\vec{j}$ and $\vec{a}(t) = 2\vec{i} + 2\vec{j}$, so $\vec{v}(t) = t\vec{a}(t)$. Thus in this case the velocity vector and acceleration vectors are parallel at all points.

63. False. As a counterexample, consider the curve $\vec{r}(t) = t^2\vec{i} + t^2\vec{j}$ for $0 \le t \le 1$. In this case, when t is replaced by $-t$, the parameterization is the same, and is not reversed.

64. True. The length of the curve C is given by $\int_a^b ||\vec{v}(t)||\,dt = \int_a^b 1\,dt = b - a$.

65. False. The velocity of the particle is given by $\vec{v}(t) = \vec{r}\,'(t) = 3\vec{i} + 2\vec{j} + \vec{k}$, so speed is constant: $||\vec{v}(t)|| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$. So the particle never stops.

66. False. As a counterexample, consider motion along the helix $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$. In this case the speed is $||\vec{v}(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$. Thus the particle has constant speed, but is traveling along a helix, not a line.

67. True. We find the tangent vector to a parametrically defined curve by differentiating $\vec{r}(t)$.

68. False. Suppose $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, so $\vec{r}(t)$ traces out the unit circle. Then $\vec{r}(t)$ lies along the radius of the circle and

$$\vec{r}\,'(t) = (-\sin t)\vec{i} + (\cos t)\vec{j}$$

and $\vec{r}\,'(t)$ is tangent to the circle. Thus $\vec{r}(t)$ and $\vec{r}\,'(t)$ are perpendicular, so their cross product is not zero.

69. False. Suppose $\vec{r}(t) = t\vec{i} + t\vec{j}$. Then $\vec{r}\,'(t) = \vec{i} + \vec{j}$ and

$$\vec{r}\,'(t)\cdot\vec{r}(t) = (t\vec{i} + t\vec{j})\cdot(\vec{i} + \vec{j}) = 2t.$$

So $\vec{r}\,'(t)\cdot\vec{r}(t) \neq 0$ for $t \neq 0$.

70. False. This result is true for uniform circular motion, but is not true in general.

Solutions for Section 17.3

Exercises

1. $\vec{V} = x\vec{i}$

2. $\vec{V} = -y\vec{i}$

3. $\vec{V} = x\vec{i} + y\vec{j} = \vec{r}$

4. $\vec{V} = -y\vec{i} + x\vec{j}$

5. $\vec{V} = -x\vec{i} - y\vec{j} = -\vec{r}$

6. $\vec{V} = \dfrac{\vec{r}}{||\vec{r}||}$: vectors are of unit length and point outward.

7. (a) Parallel to y-axis.
(b) Length increasing in x-direction.
(c) Length not dependent on y.

8. (a) Not parallel to either axis.
(b) Length does not change as x increases.
(c) The length increases as y increases.

9. (a) Parallel to x-axis
(b) Length increases as x increases
(c) Length decreases as y increases.

10. (a) Since $\text{grad}(x^4 + e^{3y}) = 4x^3\vec{i} + 3e^{3y}\vec{j}$, the vector field is parallel to neither axis.
(b) The length increases as x increases.
(c) The length increases as y increases.

11. See Figure 17.12.

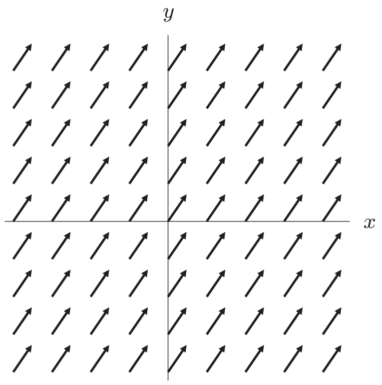


Figure 17.12: $\vec{F}(x, y) = 2\vec{i} + 3\vec{j}$

12. See Figure 17.13.

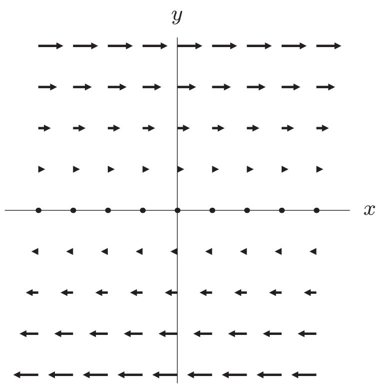


Figure 17.13: $\vec{F}(x, y) = y\vec{i}$

13. See Figure 17.14.

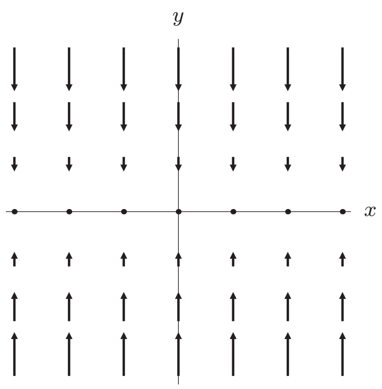


Figure 17.14: $\vec{F}(x, y) = -y\vec{j}$

14. See Figure 17.15.

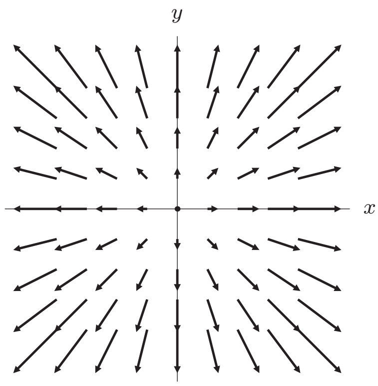


Figure 17.15: $\vec{F}(\vec{r}) = 2\vec{r}$

15. See Figure 17.16.

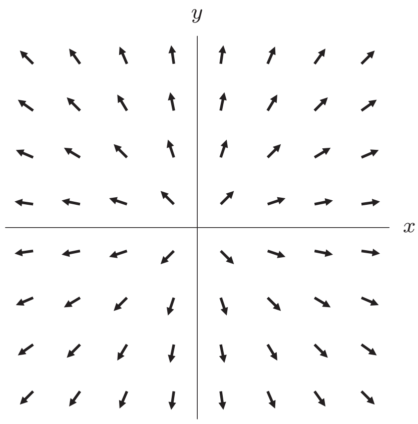


Figure 17.16: $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{\|\vec{r}\|}$

16. See Figure 17.17.

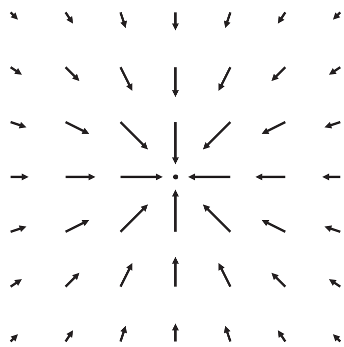


Figure 17.17: $\vec{F}(\vec{r}) = -\vec{r}/\|\vec{r}\|^3$

17. The vector field points in a clockwise direction around the origin. Since $\|y\vec{i} - x\vec{j}\| = \sqrt{y^2 + x^2}$, the vectors get longer as you go away from the origin. See Figure 17.18.

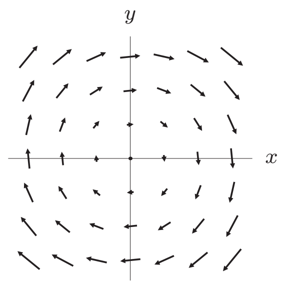


Figure 17.18

18. See Figure 17.19.

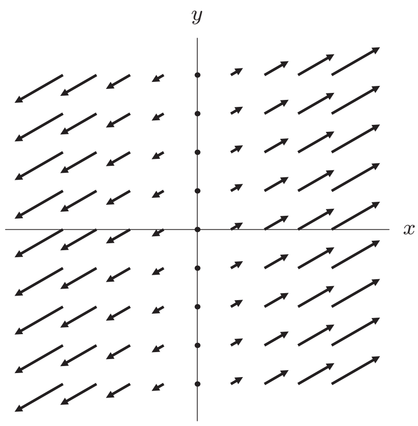


Figure 17.19: $\vec{F}(x, y) = 2x\vec{i} + x\vec{j}$

19.

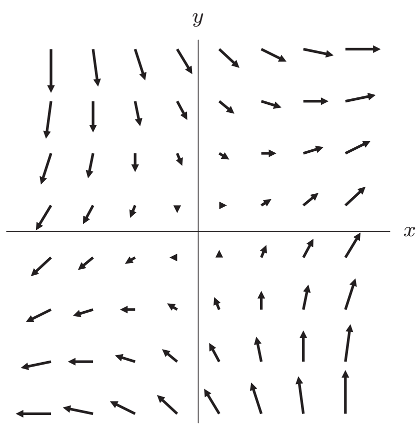


Figure 17.20: $\vec{F}(x, y) = (x+y)\vec{i} + (x-y)\vec{j}$

20. (a) III. The vector field $\vec{r}$ points outward and increases in length farther from the origin.
(b) II. The vector field $-y\vec{i} + x\vec{j}$ is tangent to a circle centered at the origin. Since $|| - y\vec{i} + x\vec{j}|| = \sqrt{x^2 + y^2}$, the vector field given has length 1 everywhere.
(c) IV. The vector field $-\vec{r}$ points toward the origin and increases in length farther from the origin.
(d) VI. The vector field $y\vec{i} - x\vec{j}$ is tangent to a circle centered at the origin and points clockwise.

21. Notice that for a repulsive force, the vectors point outward, away from the particle at the origin, for an attractive force, the vectors point toward the particle. So we can match up the vector field with the description as follows:

(a) IV
(b) III
(c) I
(d) II

Problems

22. $\vec{F}(x,y) = a\vec{i} + b\vec{j}$ for any real numbers a and b is a constant vector field. For example, $\vec{F}(x,y) = 3\vec{i} - 4\vec{j}$.

23. If $\vec{F}(x,y) = f(x,y)\vec{v}$, where $f(x,y)$ is any positive nonconstant function and $\vec{v}$ is any nonzero vector, then $\vec{F}$ has nonconstant magnitude $\|\vec{F}\| = f\|\vec{v}\|$. It has constant direction because all its vectors are in the same direction as $\vec{v}$. For example $\vec{F}(x,y) = (1+x^2)(3\vec{i} + 2\vec{j})$.

24. If

$$\vec{F}(x,y) = \frac{1}{\sqrt{f(x,y)^2 + g(x,y)^2}}(f(x,y)\vec{i} + g(x,y)\vec{j}),$$

then $\vec{f}$ has constant magnitude $\|\vec{F}\| = 1$. For many choices of f and g the vector field $\vec{F}$ is nonconstant because its direction is nonconstant. For example,

$$\vec{F}(x,y) = \frac{1}{\sqrt{1+x^2}}(\vec{i} - x\vec{j}).$$

25. Many answers are possible. For example, $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ has nonconstant magnitude $\|\vec{F}\| = \sqrt{x^2+y^2}$. Its direction is also nonconstant, since $\vec{F}(1,0) = \vec{i}$ and $\vec{F}(0,1) = \vec{j}$.

26. If $\vec{F}(x,y) = f(x,y)((1+y^2)\vec{i} - (x+y)\vec{j})$ where $f(x,y)$ is any function, then $\vec{F} \cdot \vec{G} = 0$, which shows that $\vec{F}$ is perpendicular to $\vec{G}$.

For example $\vec{F}(x,y) = (y + \cos x)((1+y^2)\vec{i} - (x+y)\vec{j})$.

27. Vector fields (B) and (C) both appear to be constant, and therefore correspond to the equally spaced level curves in (I) and (II). Since the gradient points toward increasing values of the function, (B) corresponds to (II) and (C) corresponds to (I).

Vector field (A) points away from the center, so it corresponds to (IV), which has a minimum in the center.

Vector field (D) points toward the center, so it corresponds to (III) which has a maximum at the center.

28. The sketches show that the vector fields point in different directions on the y-axis, so we examine the formulas for the vector fields on the y-axis. On the y-axis, where $x = 0$, we have:

$\vec{F}(0,y) = y\vec{j}$, a vector pointing up if $y > 0$ and down if $y < 0$, as in I
$\vec{G}(0,y) = -y\vec{i}$, a vector pointing left if $y > 0$ and right if $y < 0$, as in II
$\vec{H}(0,y) = -y\vec{j}$, a vector pointing down if $y > 0$ and up if $y < 0$, as in III
So $\vec{F}$ is I, $\vec{G}$ is II, and $\vec{H}$ is III.

29. The sketches show that the vector fields can be distinguished by the directions they point on the coordinate axes, so we examine the formulas for the vector fields on the axes.

(a) $\vec{F}(0,y) + \vec{G}(0,y) = -y\vec{i} + y\vec{j} = y(-\vec{i} + \vec{j})$, a vector pointing up to the left if $y > 0$ and down to the right if $y < 0$, as in II.
(b) $\vec{F}(0,y) + \vec{H}(0,y) = \vec{0}$, the zero vector as in III.
(c) $\vec{G}(0,y) + \vec{H}(0,y) = -y\vec{i} - y\vec{j} = -y(\vec{i} + \vec{j})$, a vector pointing down to the left if $y > 0$ and up to the right if $y < 0$, as in I and IV. $\vec{G}(x,0) + \vec{H}(x,0) = x\vec{i} + x\vec{j} = x(\vec{i} + \vec{j})$, a vector pointing up to the right if $x > 0$ and down to the left if $x < 0$, as in II and IV, so $\vec{G} + \vec{H}$ is IV.
(d) $-\vec{F}(x,0) + \vec{G}(x,0) = -x\vec{i} + x\vec{j} = x(-\vec{i} + \vec{j})$, a vector pointing up to the left if $x > 0$ and down to the right if $x < 0$, as in I.

30. One possible solution is $\vec{F}(x, y) = x\vec{i}$. See Figure 17.21.

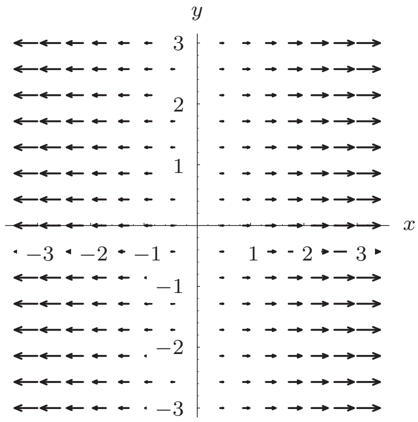


Figure 17.21

31. If we let $\vec{F}(x, y) = \dfrac{-x\vec{i} - y\vec{j}}{\sqrt{x^2 + y^2}}$, then all vectors will be of unit length and will point toward the origin. See Figure 17.22.

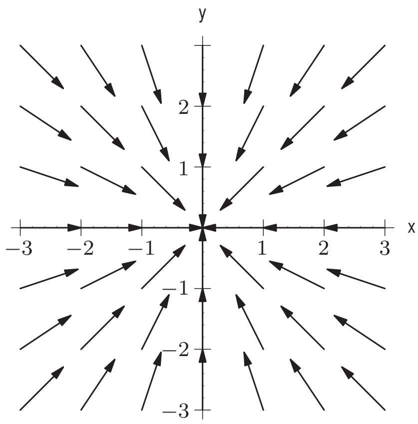
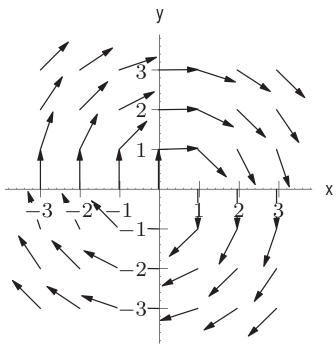


Figure 17.22

32. The position vector at each point is $\vec{r} = x\vec{i} + y\vec{j}$. We want to find $\vec{F}(x, y) = A\vec{i} + B\vec{j}$ such that $\vec{F} \cdot \vec{r} = Ax + By = 0$. One possible answer is let $A = y$ and $B = -x$. So $\vec{F}(x, y) = y\vec{i} - x\vec{j}$. Since the vectors are of unit length, we get $\vec{F}(x, y) = \dfrac{y\vec{i} - x\vec{j}}{\sqrt{x^2 + y^2}}$.



33. (a) The line l is parallel to the vector $\vec{v} = \vec{i} - 2\vec{j} - 3\vec{k}$. The vector field $\vec{F}$ is parallel to the line when $\vec{F}$ is a multiple of $\vec{v}$. Taking the multiple to be 1 and solving for x, y, z we find a point at which this occurs:

$$\begin{aligned} x &= 1 \\ x + y &= -2 \\ x - y + z &= -3 \end{aligned}$$

gives $x = 1$, $y = -3$, $z = -7$, so a point is $(1, -3, -7)$. Other answers are possible.

(b) The line and vector field are perpendicular if $\vec{F} \cdot \vec{v} = 0$, that is

$$\begin{aligned} (x\vec{i} + (x+y)\vec{j} + (x-y+z)\vec{k}) \cdot (\vec{i} - 2\vec{j} - 3\vec{k}) &= 0 \\ x - 2x - 2y - 3x + 3y - 3z &= 0 \\ -4x + y - 3z &= 0. \end{aligned}$$

One point which satisfies this equation is $(0, 0, 0)$. There are many others.

(c) The equation for this set of points is $-4x + y - 3z = 0$. This is a plane through the origin.

34. (a) The vector field $\vec{L} = 0\vec{F} + \vec{G} = -y\vec{i} + x\vec{j}$ is shown in Figure 17.23.
(b) The vector field $\vec{L} = a\vec{F} + \vec{G} = (ax - y)\vec{i} + (ay + x)\vec{j}$ where $a > 0$ is shown in Figure 17.24.
(c) The vector field $\vec{L} = a\vec{F} + \vec{G} = (ax - y)\vec{i} + (ay + x)\vec{j}$ where $a < 0$ is shown in Figure 17.25.

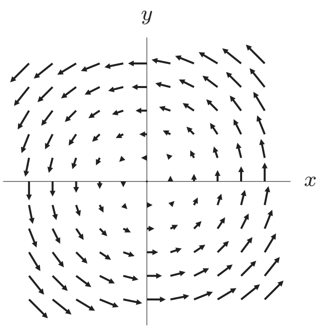


Figure 17.23

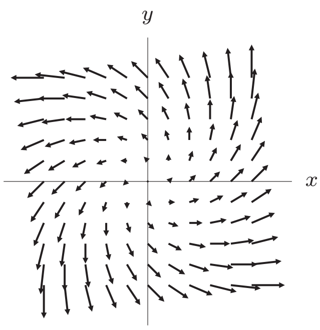


Figure 17.24

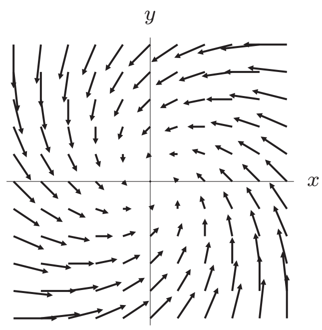


Figure 17.25

35. (a) The vector field $\vec{L} = \vec{F} + 0\vec{G} = x\vec{i} + y\vec{j}$ is shown in Figure 17.26.
(b) The vector field $\vec{L} = \vec{F} + b\vec{G} = (x - by)\vec{i} + (y + bx)\vec{j}$ where $b > 0$ is shown in Figure 17.27.
(c) The vector field $\vec{L} = \vec{F} + b\vec{G} = (x - by)\vec{i} + (y + bx)\vec{j}$ where $b < 0$ is shown in Figure 17.28.

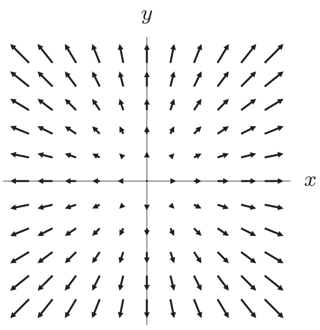


Figure 17.26

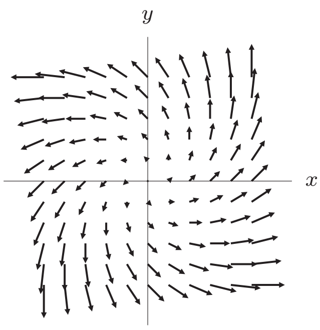


Figure 17.27

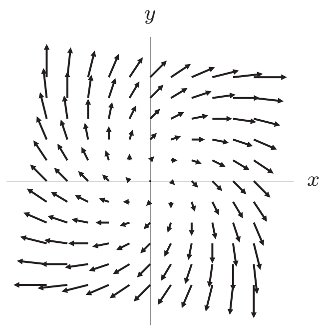


Figure 17.28

36. (a) Since the velocity of the water is the sum of the velocities of the individual fields, then the total field should be

$$\vec{v} = \vec{v}_{\text{stream}} + \vec{v}_{\text{fountain}}.$$

It is reasonable to represent $\vec{v}_{\text{stream}}$ by the vector field $\vec{v}_{\text{stream}} = A\vec{i}$, since $A\vec{i}$ is a constant vector field flowing in the i-direction (provided $A > 0$). It is reasonable to represent $\vec{v}_{\text{fountain}}$ by

$$\vec{v}_{\text{fountain}} = K\vec{r}_r/r^2 = K(x^2+y^2)^{-1}(x\vec{i}+y\vec{j}),$$

since this is a vector field flowing radially outward (provided $K > 0$), with decreasing velocity as r gets larger. We would expect the velocity to decrease as the water from the fountain spreads out. Adding the two vector fields together, we get

$$\vec{v} = A\vec{i} + K(x^2+y^2)^{-1}(x\vec{i}+y\vec{j}), \quad A > 0, K > 0.$$

(b) The constants A and K signify the strength of the individual components of the field. A is the strength of the flow of the stream alone (in fact it is the speed of the stream), and K is the strength of the fountain acting alone.

(c)

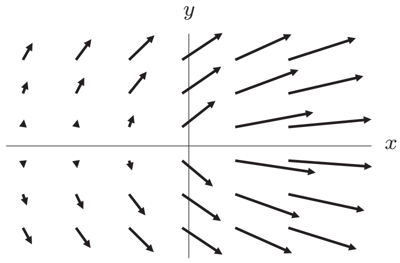


Figure 17.29: $A = 1, K = 1$

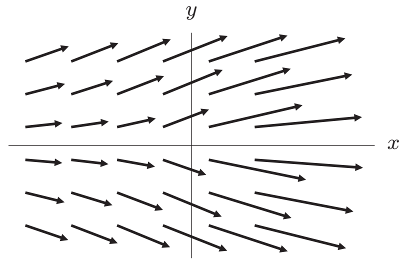


Figure 17.30: $A = 2, K = 1$

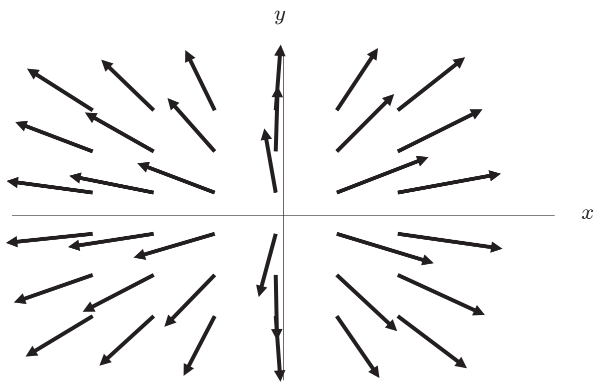


Figure 17.31: $A = 0.2, K = 2$

37. (a) The gradient is perpendicular to the level curves. See Figures 17.33 and 17.32. A function always increases in the direction of its gradient; this is why the values on the level curves of f and g increase as we approach the origin.

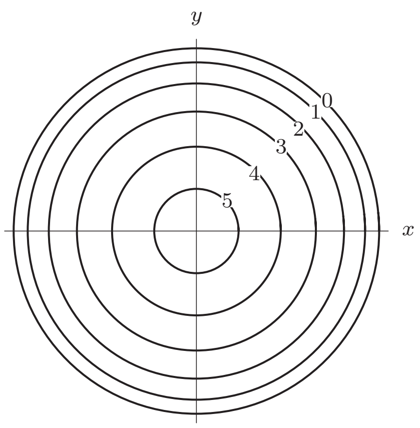


Figure 17.32: Level curves $z = f(x, y)$

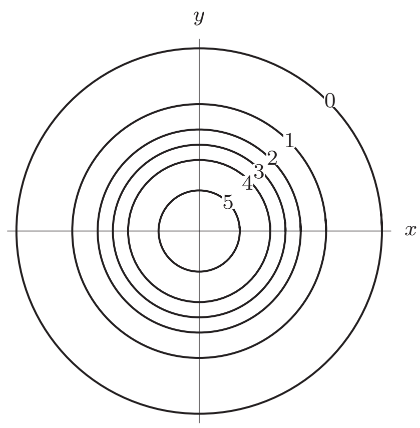


Figure 17.33: Level curves $z = g(x, y)$

(b) f climbs faster at outside, slower at center; g climbs slower at outside, faster at center:

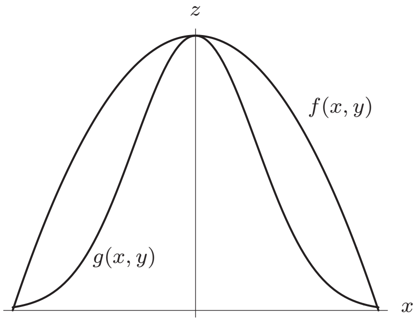


Figure 17.34

This can be understood if we notice that the magnitude of the gradient of f decreases as one approaches the origin whereas the magnitude of the gradient of g increases (at least for a while - what happens very close to the origin depends on the behavior of $\operatorname{grad} g$ in the region. One possibility for g is shown in Figure 17.34; the graph of g could also have a sharp peak at 0 or even blow up.)

38. (a) Dividing a vector $\vec{F}$ by its magnitude always produces the unit vector in the same direction as $\vec{F}$.
(b) Since
$$\|\vec{N}\| = \|(1/F)(-v\vec{i} + u\vec{j})\| = (1/F)\sqrt{v^2 + u^2} = (1/F)F = 1,$$
then $\vec{N}$ is a unit vector. We check that $\vec{N}$ is perpendicular to $\vec{F}$ using the dot product of $\vec{N}$ and $\vec{F}$:
$$\vec{N} \cdot \vec{F} = (1/F)(-v\vec{i} + u\vec{j}) \cdot (u\vec{i} + v\vec{j}) = 0.$$
Which side of $\vec{F}$ does $\vec{N}$ point? The vector $\vec{k}$ is pointing out of the diagram. Since the cross product $\vec{k} \times \vec{F}$ is perpendicular to both $\vec{k}$ and $\vec{F}$, then $\vec{N}$ lies in the xy-plane and points at a right angle to the direction of $\vec{F}$. By the right-hand rule, $\vec{N}$ points to the left as shown in the figure.

Strengthen Your Understanding

39. To obtain a plot of the vector field $\vec{G}(x, y, z) = \vec{F}(2x, 2y, 2z)$, move the vectors in a plot of $\vec{F}(x, y, z)$ halfway to the origin, without changing their directions or magnitudes. For example, to plot the vector $\vec{G}(1, 2, 3)$, move the vector $\vec{F}(2, 4, 6)$ to the point $(1, 2, 3)$.

Doubling the lengths of the arrows in the vector field $\vec{F}(x, y, z)$ gives a plot of the vector field $\vec{H}(x, y, z) = 2\vec{F}(x, y, z)$.

40. The values of a vector field are vectors, not scalars. The formula $\vec{F}(x, y, z) = x^2 - yz$ is not a vector field because the expression $x^2 - yz$ is not a vector.

41. The vector field
$$\vec{F}(x, y, z) = (x^2 + 1)\left(\vec{i} + \vec{j} + \vec{k}\right)$$
is nonconstant and is parallel to $\vec{i} + \vec{j} + \vec{k}$ at every point.

42. Any vector field that is not the zero vector at any point and that has nonconstant direction can be divided by its magnitude to produce a nonconstant vector field of unit vectors. For example, we have
$$\vec{F}(x, y, z) = \frac{1}{\sqrt{(x^2 + 1)^2 + z^2 + y^2}}\left((x^2 + 1)\vec{i} + z\vec{j} + y\vec{k}\right).$$

Solutions for Section 17.4

Exercises

1. Since $x'(t) = 0$ and $y'(t) = 2$, we have $x = x_0$ and $y = 2t + y_0$. Thus, the solution curves are $x = $ constant.

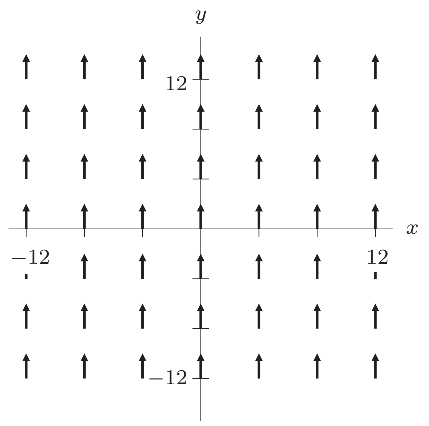


Figure 17.35: The field $\vec{v} = 2\vec{j}$

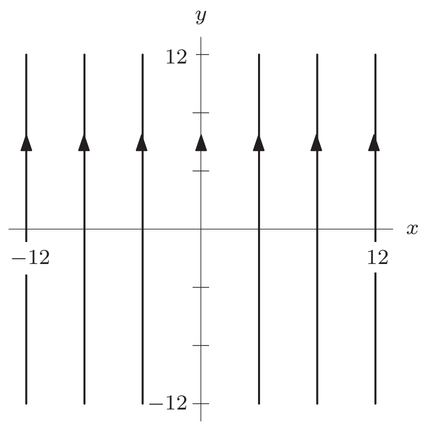


Figure 17.36: The flow $x = $ constant

2. Since $x'(t) = 3$ and $y'(t) = 0$, we have $x = 3t + x_0$ and $y = y_0$. Thus, the solution curves are $y = $ constant.

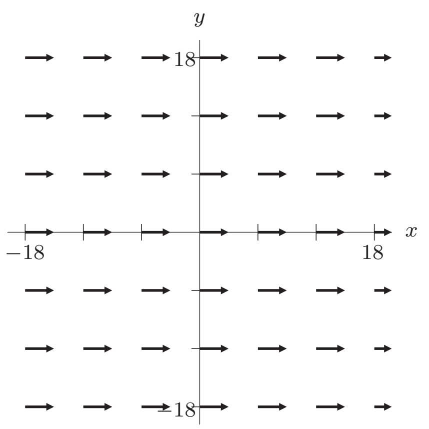


Figure 17.37: The field $\vec{v} = 3\vec{i}$

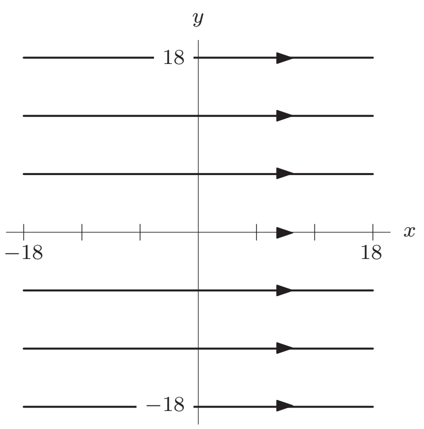


Figure 17.38: The flow $y = $constant

3. Since $x'(t) = 3$ and $y'(t) = -2$, we have $x = 3t + x_0$ and $y = -2t + y_0$. Thus the flow lines are straight lines parallel to the vector $3\vec{i} - 2\vec{j}$. Alternatively, we have $\frac{dy}{dx} = -\frac{2}{3}$. Thus, $y = -\frac{2}{3}x + c$, where c is a constant.

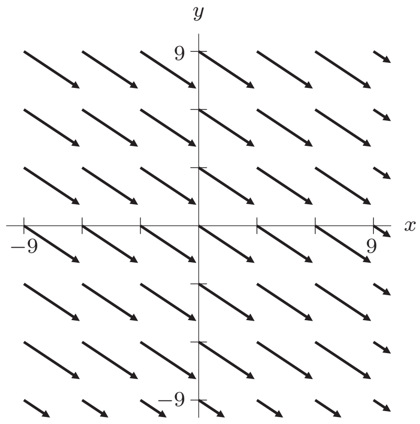


Figure 17.39: The field $\vec{v} = 3\vec{i} - 2\vec{j}$

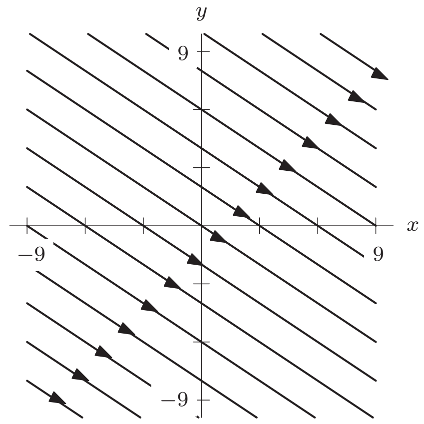


Figure 17.40: The flow $y = -\frac{2}{3}x + c$

4. As

$$\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j},$$

the system of differential equations is

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = 0. \end{cases}$$

Since

$$\frac{d}{dt}(x(t)) = \frac{d}{dt}(ae^t) = x$$

and

$$\frac{d}{dt}(y(t)) = \frac{d}{dt}(b) = 0,$$

the given flow satisfies the system. The solution curves are the horizontal lines $y = b$. See Figures 17.41 and 17.42.

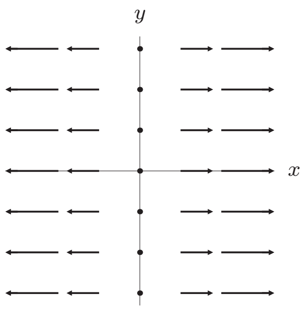


Figure 17.41: $\vec{v}(t) = x\vec{i}$

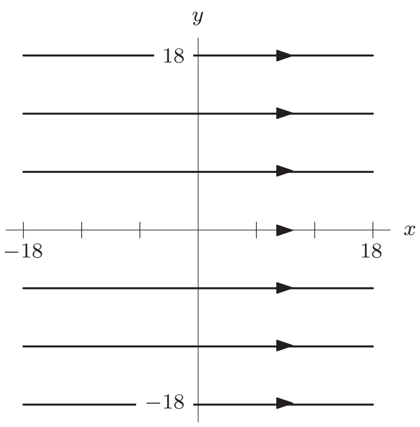


Figure 17.42: The flow $x(t) = ae^t,\ y(t) = b$

5. As

$$\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j},$$

the system of differential equations is

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = x. \end{cases}$$

Since

$$\frac{d(x(t))}{dt} = \frac{d}{dt}(a) = 0$$

and

$$\frac{d(y(t))}{dt} = \frac{d}{dt}(at+b) = a = x,$$

the given flow satisfies the system. The solution curves are the vertical lines $x = a$. See Figures 17.43 and 17.44.

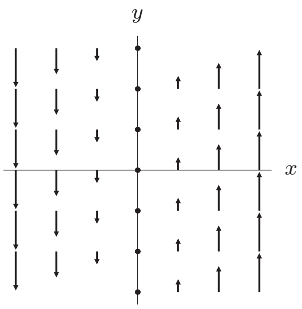


Figure 17.43: $\vec{v}(t) = x\vec{j}$

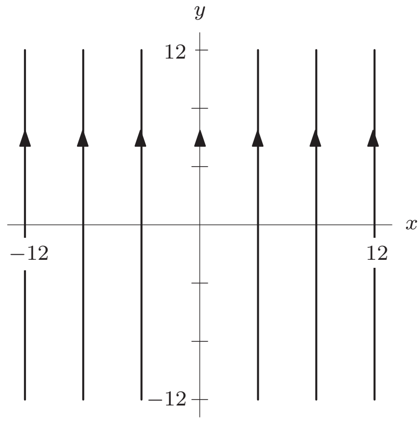


Figure 17.44: The flow $x(t) = a,\ y(t) = at + b$

6.

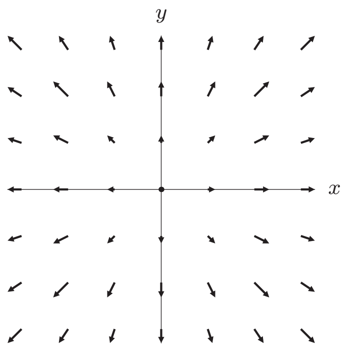


Figure 17.45: $\vec{v}(t) = x\vec{i} + y\vec{j}$

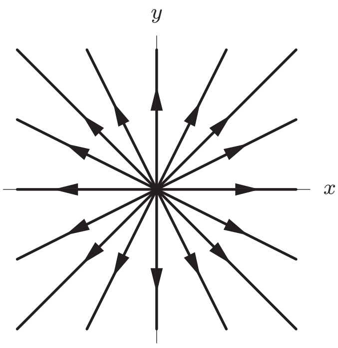


Figure 17.46: The flow $x = ae^t$, $y = be^t$.

As

$$\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j},$$

the system of differential equations is

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = y. \end{cases}$$

Since

$$\frac{dx(t)}{dt} = \frac{d}{dt}[ae^t] = ae^t = x(t)$$

and

$$\frac{dy(t)}{dt} = \frac{d}{dt}[be^t] = be^t = y(t),$$

the given flow satisfies the system. By eliminating the parameter t in $x(t)$ and $y(t)$, the solution curves obtained are $y = \frac{b}{a}x$.

7. As

$$\vec{v}\,(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\,,$$

the system of differential equations is

$$\begin{cases} \frac{dx}{dt} = & x \\ \frac{dy}{dt} = & -y. \end{cases}$$

Since

$$\frac{dx(t)}{dt} = \frac{d}{dt}[ae^t] = ae^t = x(t)$$

and

$$\frac{dy(t)}{dt} = \frac{d}{dt}[be^{-t}] = -be^{-t} = y(t),$$

the given flow satisfies the system. By eliminating the parameter t in $x(t)$ and $y(t)$, the solution curves obtained are $xy = ab$. See Figures 17.47 and 17.48.

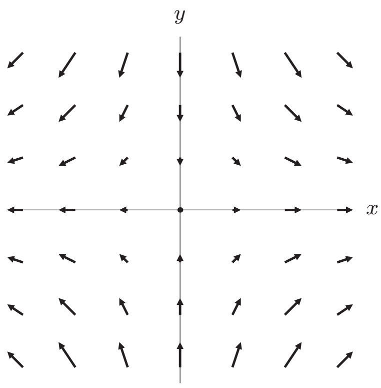


Figure 17.47: $\vec{v}\,(t) = x\vec{i} - y\vec{j}$

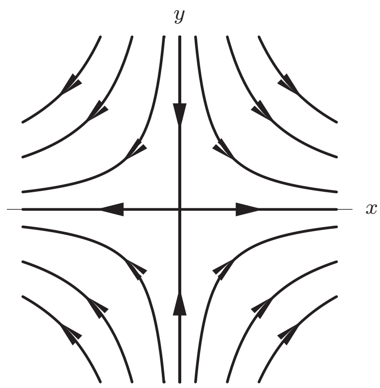


Figure 17.48: The flow $x = ae^t$, $y = be^{-t}$

8. As

$$\vec{v}\,(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\,,$$

the system of differential equations is

$$\begin{cases} \frac{dx}{dt} = & y \\ \frac{dy}{dt} = & -x. \end{cases}$$

Since

$$\frac{dx(t)}{dt} = \frac{d}{dt}[a\sin t] = a\cos t = y(t)$$

and

$$\frac{dy(t)}{dt} = \frac{d}{dt}[a\cos t] = -a\sin t = -x(t),$$

the given flow satisfies the system. By eliminating the parameter t in $x(t)$ and $y(t)$, the solution curves obtained are $x^2 + y^2 = a^2$. See Figures 17.49 and 17.50.

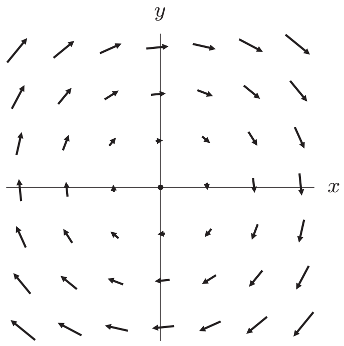


Figure 17.49: $\vec{v}(t) = y\vec{i} - x\vec{j}$

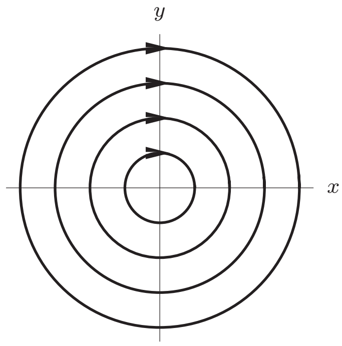


Figure 17.50: The flow $x = a\sin t$, $y = a\cos t$

9. As

$$\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j},$$

the system of differential equations is

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x. \end{cases}$$

Since

$$\frac{dx(t)}{dt} = \frac{d}{dt}[a(e^t + e^{-t})] = a(e^t - e^{-t}) = y(t)$$

and

$$\frac{dy(t)}{dt} = \frac{d}{dt}[a(e^t - e^{-t})] = a(e^t + e^{-t}) = x(t),$$

the given flow satisfies the system. By eliminating the parameter t in $x(t)$ and $y(t)$, the solution curves obtained are $x^2 - y^2 = 4a^2$. See Figures 17.51 and 17.52.

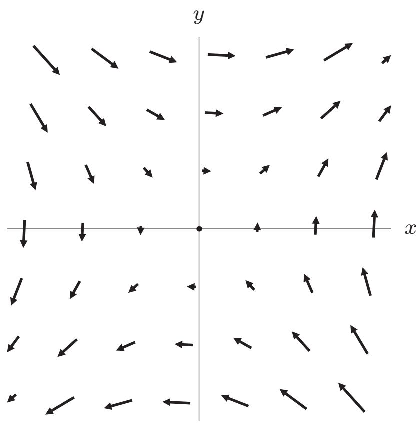


Figure 17.51: $\vec{v}(t) = y\vec{i} + x\vec{j}$

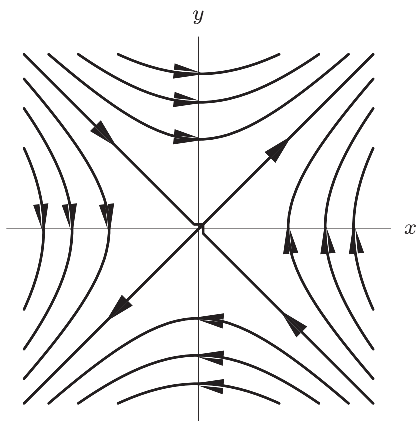


Figure 17.52: The flow $x(t) = a(e^t + e^{-t}),\ y(t) = a(e^t - e^{-t})$

10. The vector field is given by $\vec{v} = y^2\vec{i} + 2x^2\vec{j}$, that is, the flow line $(x(t), (y(t))$ satisfies

$$x'(t) = y^2$$
$$y'(t) = 2x^2$$

We'll use Euler's method with $\Delta t = 0.1$ to find the parameterized curve $(x(t), y(t))$ through $(1, 2)$. So

$$x_{n+1} = x_n + 0.1y_n^2$$
$$y_{n+1} = y_n + (0.1)2x_n^2.$$

Initially, that is when $t = 0$, we have $(x_0, y_0) = (1, 2)$. Then

$$x_1 = x_0 + 0.1y_0^2 = 1 + 0.1 \cdot 2^2 = 1.4$$
$$y_1 = y_0 + 0.1 \cdot 2x_0^2 = 2 + 0.1 \cdot 2 \cdot 1^2 = 2.2.$$

Thus, we see that after one step, $x_1 = 1.4$ and $y_1 = 2.2$. Further values are given in the Table 17.3.

Table 17.3

x	1.4	1.884	2.556	3.646	5.770
y	2.2	2.592	3.302	4.609	7.268

Problems

11. This corresponds to area A in Figure 17.53.

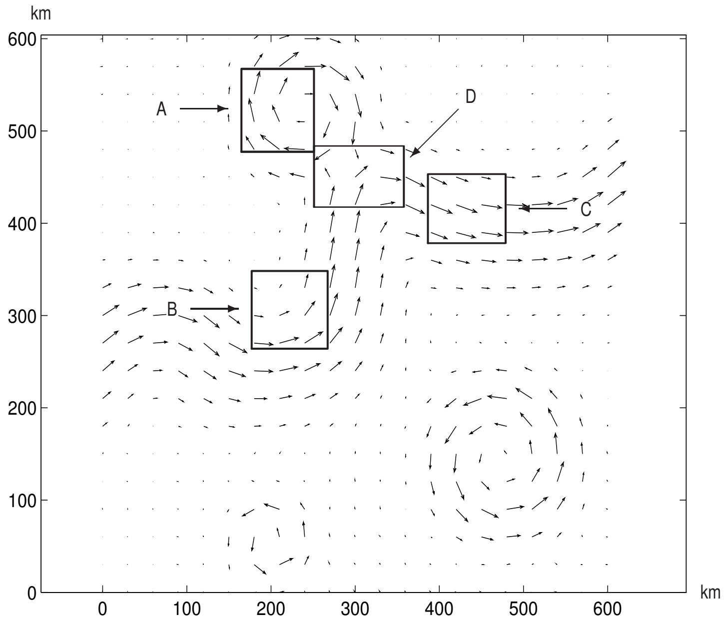


Figure 17.53

12. This corresponds to area B in Figure 17.53 in Problem 11.

13. This corresponds to area C in Figure 17.53 in Problem 11.

14. This corresponds to area D in Figure 17.53 in Problem 11.

15. (a) At every point (x, y) in the plane, the vector $\vec{G}(x, y)$ has the same direction as $\vec{F}(x, y)$ but $\vec{G}(x, y)$ is twice as long. For the case where $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$ see Figures 17.54 and 17.55.

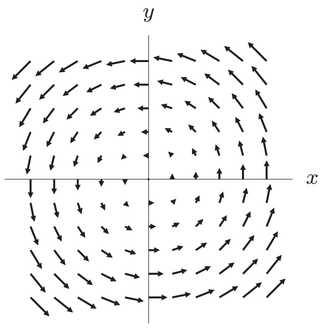


Figure 17.54: $\vec{F} = -y\vec{i} + x\vec{j}$

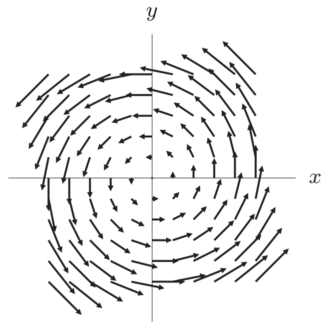


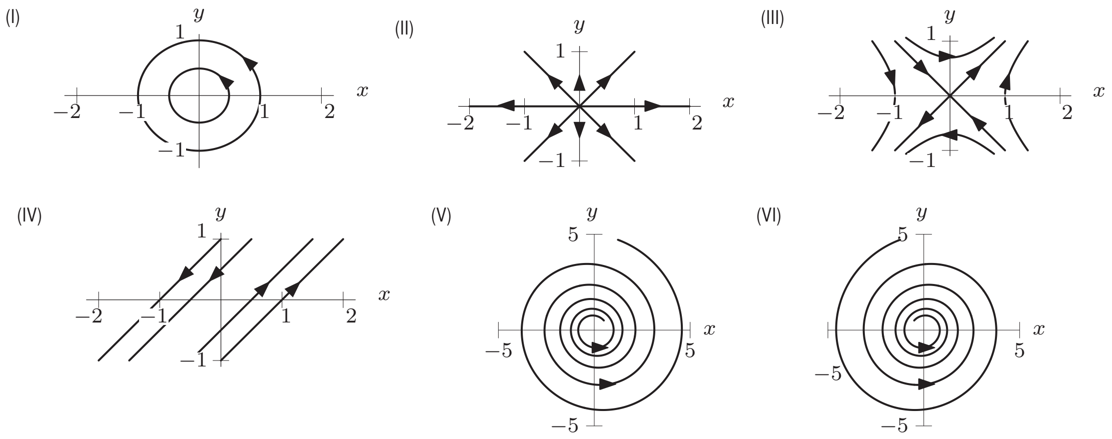
Figure 17.55: $G(x, y) = 2(-y\vec{i} + x\vec{j})$

(b) At every point in the plane the two vector fields $\vec{F}$ and $\vec{G}$ have the same direction. Therefore the flow lines of the two vector fields have the same slopes at every point. By the uniqueness of solutions of differential equations with initial conditions, the flow lines of the two vector fields through any given point must be the same. This means that if two objects are placed at the same point, one into the flow of $\vec{F}$ and the other into the flow of $\vec{G}$, they will move on exactly the same paths. However, they will move at different speeds. The two flows will have different parameterizations.

For the case where $\vec{F}(t) = -y\vec{i} + x\vec{j}$ both flows are circular about the origin, but the flow of $\vec{G}$ is twice as fast as the flow of $\vec{F}$.

16. The directions of the flow lines are as shown.

(a) III
(b) I
(c) II
(d) V
(e) VI
(f) IV



17. The object's motion is described by a function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, where $\frac{d\vec{r}}{dt}(t) = \vec{F}(x(t), y(t)) = u(x(t), y(t))\vec{i} + v(x(t), y(t))\vec{j}$. Using the chain rule to differentiate, we have

$$\begin{aligned}
\vec{a}(t) &= \frac{d^2\vec{r}}{dt^2} \\
&= \frac{du}{dt}\vec{i} + \frac{dv}{dt}\vec{j} \\
&= (u_x\frac{dx}{dt} + u_y\frac{dy}{dt})\vec{i} + (v_x\frac{dx}{dt} + v_y\frac{dy}{dt})\vec{j} \\
&= (u_x u + u_y v)\vec{i} + (v_x u + v_y v)\vec{j}
\end{aligned}$$

18. (a) Perpendicularity is indicated by zero dot product. We have $\vec{v} \cdot \operatorname{grad} H = (-H_y\vec{i} + H_x\vec{j}) \cdot (H_x\vec{i} + H_y\vec{j}) = 0$.
(b) If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ is a flow line we have, using the chain rule,

$$\frac{d}{dt}H(x(t), y(t)) = H_x\frac{dx}{dt} + H_y\frac{dy}{dt} = H_x(-H_y) + H_y(H_x) = 0.$$

Thus $H(x(t), y(t))$ is constant which shows that a flow line stays on a single level curve of H.

For a different solution, use geometric reasoning. The vector field $\vec{v}$ is tangent to the level curves of H because, by part (a), $\vec{v}$ and the level curves are both perpendicular to the same vector field $\operatorname{grad} H$. Thus the level curves of H and the flowlines of $\vec{v}$ run in the same direction.

19. Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ be a flow line of $\vec{v}$. If $f(x, y)$ has the same value at all points $(x(t), y(t))$ then the flow line lies on a level curve of f. We can check whether

$$g(t) = f(x(t), y(t)) = x(t)y(t)$$

is constant by computing the derivative $g'(t)$. Since $\vec{v} = x\vec{i} - y\vec{j}$, we have $dx/dt = x$ and $dy/dt = -y$. Thus,

$$g'(t) = x\frac{dy}{dt} + y\frac{dx}{dt} = -xy + yx = 0$$

and $g(t)$ is constant. This means that the flow line lies on a level curve of f. The flow lines are parameterized hyperbolas with equation $xy = c$.

20. Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ be a flow line of $\vec{v}$. If $f(x, y)$ has the same value at all points $(x(t), y(t))$ then the flow line lies on a level curve of f. We can check whether

$$g(t) = f(x(t), y(t)) = x(t)^2 - y(t)^2$$

is constant by computing the derivative $g'(t)$. Since $\vec{v} = y\vec{i} + x\vec{j}$, we have $dx/dt = y$ and $dy/dt = x$. Thus,

$$g'(t) = 2x\frac{dx}{dt} - 2y\frac{dy}{dt} = 2xy - 2yx = 0$$

and g is constant. This means that the flow line lies on a level curve of f. The flow lines are parameterized hyperbolas with equation $x^2 - y^2 = c$.

21. Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ be a flow line of $\vec{v}$. If $f(x, y)$ has the same value at all points $(x(t), y(t))$ then the flow line lies on a level curve of f. We can check whether

$$g(t) = f(x(t), y(t)) = bx(t)^2 - ay(t)^2$$

is constant by computing the derivative $g'(t)$. Since $\vec{v} = ay\vec{i} + bx\vec{j}$, we have $dx/dt = ay$ and $dy/dt = bx$. Thus,

$$g'(t) = 2bx\frac{dx}{dt} - 2ay\frac{dy}{dt} = 2abxy - 2abyx = 0$$

and g is constant. This means that the flow line lies on a level curve of f. The flow lines are parameterized conic sections with equation $bx^2 - ay^2 = c$, hyperbolas if a and b are both positive or both negative, and ellipses if a and b have opposite sign.

22. (a) Each vector in the vector field $\vec{v}$ is horizontal, tangent to a circle whose center is on the z-axis, and pointing counterclockwise when viewed from above. Thus, $\vec{v}$ is parallel to $-y\vec{i} + x\vec{j}$. The point (x, y, z) is moving on a circle of radius $r = \sqrt{x^2 + y^2}$ and has

$$\text{Speed} = \frac{2\pi r}{24} = \frac{\pi r}{12}.$$

Since the vector at the point (x, y, z) has magnitude $\pi r/12$ and is parallel to the unit vector $(-y\vec{i} + x\vec{j})/\sqrt{x^2 + y^2}$, we have

$$\vec{v} = \frac{\pi r}{12}\left(\frac{-y\vec{i} + x\vec{j}}{\sqrt{x^2 + y^2}}\right) = \frac{\pi}{12}(-y\vec{i} + x\vec{j}) \text{ meters/hr.}$$

(b) A point moves in a horizontal circle, centered on the z-axis, and oriented counter-clockwise when viewed from above. These circles are the flow lines.

23. (a) The flow line $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ is the path with $\vec{r}\,'(t) = \vec{v} = (ax - y)\vec{i} + (x + ay)\vec{j}$. Thus

$$x' = ax - y \quad y' = x + ay.$$

We show $h(t)$ is constant on a flow line by showing that $h'(t) = 0$. Using the chain rule, we have

$$\begin{aligned} h'(t) &= e^{-2at}(2xx' + 2yy') - 2ae^{-2at}(x^2 + y^2) \\ &= e^{-2at}(2x(ax - y) + 2y(x + ay)) - 2ae^{-2at}(x^2 + y^2) = 0. \end{aligned}$$

The function $h(t) = e^{-at}(x^2 + y^2)$ is constant because its derivative is zero.

(b) We have $h(0) = e^{a\cdot 0}(x^2 + y^2) = 1$ for points on the unit circle $x^2 + y^2 = 1$ at $t = 0$. Hence, along the same flow line, $h(t) = e^{-2at}(x^2 + y^2) = 1$ for all t. Thus at time t, the particle's coordinates satisfy

$$x^2 + y^2 = e^{2at},$$

which is the equation of the circle of radius e^{at} centered at the origin.

This result matches what can be seen in the vector field plot in Figure 17.56. If $a < 0$, the flow lines move toward the origin. If $a > 0$, the flow lines move away from the origin, and if $a = 0$ they circle the origin.

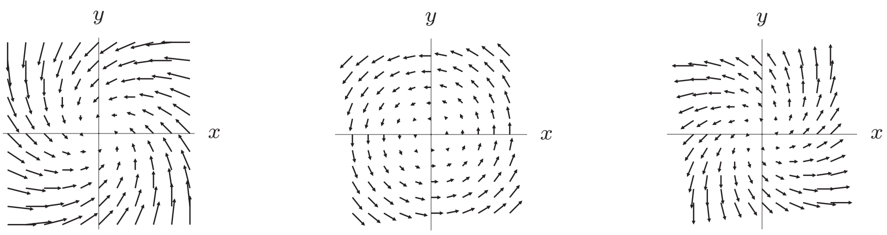


Figure 17.56

Strengthen Your Understanding

24. The vector field $\vec{F} = -y\vec{i} + x\vec{j}$ has linear components and circular flow lines.

25. Although the vectors in the vector field all point in the same direction, their lengths could vary. For example $\vec{F}(x, y, z) = (x^2 + 1)\vec{i}$ has flow lines that are all lines parallel to the x-axis pointing in the positive direction, but it is not constant.

26. Since $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ is a flow line of the vector field $\vec{F}(x, y, z)$, we have

$$\vec{F}(t, t^2, t^3) = \vec{v}(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

where $\vec{v}(t)$ is the velocity of $\vec{r}(t)$. Let

$$\vec{F}(x, y, z) = F_1(x, y, z)\vec{i} + F_2(x, y, z)\vec{j} + F_3(x, y, z)\vec{k}.$$

We must determine three functions, F_1, F_2, and F_3 such that

$$F_1(t, t^2, t^3) = 1 \quad F_2(t, t^2, t^3) = 2t \quad F_3(t, t^2, t^3) = 3t^2.$$

One solution is given by

$$F_1(x, y, z) = 1 \quad F_2(x, y, z) = 2x \quad F_3(x, y, z) = 3y.$$

Thus, $\vec{r}(t)$ is a flow line of the vector field

$$\vec{F}(x, y, z) = \vec{i} + 2x\vec{j} + 3y\vec{k}.$$

27. The vector field $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ points away from the origin at every point other than the origin, so its flow lines are rays pointing from the origin.

28. True, since the vectors $x\vec{j}$ are parallel to the y-axis.

29. False. The flow lines are circles centered at the origin.

30. False. The flow lines are lines parallel to the x-axis.

31. False. Each flow line stays in the quadrant in which it originates.

32. True. Any flow line which stays in the first quadrant has $x, y \to \infty$.

33. False. The flow lines for $\vec{F}$ are perpendicular to the contours for f, since the flow lines follow ∇f, which is perpendicular to the contours of f.

34. True. Since a vector tangent to a circle centered at the origin is perpendicular to the radius vector $\vec{r}$ and $\vec{F}(\vec{r})$ is tangent to its flow lines, $\vec{F}(\vec{r})$ must be perpendicular to $\vec{r}$. Thus $\vec{F}(\vec{r}) \cdot \vec{r} = 0$ for all $\vec{r}$.

35. False. If the flow lines are all straight lines parallel to $\vec{v}$, we need $\vec{F}(x, y)$ to be parallel to $\vec{v}$ for all x and y. That does not mean that $\vec{F}(x, y)$ must be equal to $\vec{v}$; it only needs to be a scalar multiple $\vec{v}$. For example, the vector field $\vec{F}(x, y) = 6\vec{i} + 10\vec{j}$ has all its flow lines parallel to $\vec{v}$. Another example is $\vec{F}(x, y) = 3e^{x-y}\vec{i} + 5e^{x-y}\vec{j} = e^{x-y}\vec{v}$, where the scalar multiplied times $\vec{v}$ varies as x and y vary.

36. True. If (x, y) were a point where the y-coordinate along a flow line reached a relative maximum, then the tangent vector to the flow line, namely $\vec{F}(x, y)$, there would have to be horizontal (or $\vec{0}$), that is its $\vec{j}$ component would have to be 0. But the $\vec{j}$ component of $\vec{F}$ is always 2.

37. False. At all points on the x-axis, $y = 0$, so the vector field is a horizontal vector, $\vec{F}(x, 0) = e^x\vec{i}$, pointing to the right, since it is a positive multiple of $\vec{i}$. Thus the x-axis itself is a flow line for $\vec{F}$. Since there can be only one flow line through any point, no flow line can cross the x-axis.

Solutions for Chapter 17 Review

Exercises

1. The line has equation

$$\vec{r} = 2\vec{i} - \vec{j} + 3\vec{k} + t(5\vec{i} + 4\vec{j} - \vec{k}),$$

or, equivalently

$$\begin{aligned} x &= 2 + 5t \\ y &= -1 + 4t \\ z &= 3 - t. \end{aligned}$$

2. The displacement vector from the point $(1, 2, 3)$ to the point $(3, 5, 7)$ is:

$$3\vec{i} + 5\vec{j} + 7\vec{k} - (\vec{i} + 2\vec{j} + 3\vec{k}) = 2\vec{i} + 3\vec{j} + 4\vec{k}.$$

So the equations are

$$\begin{aligned} x &= 1 + 2t, \\ y &= 2 + 3t, \\ z &= 3 + 4t. \end{aligned}$$

3. $x = t, y = 5$.

4. The parameterization $x\vec{i} + y\vec{j} = 2\cos t\vec{i} + 2\sin t\vec{j}$ has the right radius but starts at the point $(2, 0)$. To start at $(0, 2)$, we need $x\vec{i} + y\vec{j} = 2\cos(t + \frac{\pi}{2})\vec{i} + 2\sin(t + \frac{\pi}{2})\vec{j} = 2\sin t\vec{i} + 2\cos t\vec{j}$.

5. The parameterization $x\vec{i} + y\vec{j} = (4 + 4\cos t)\vec{i} + (4 + 4\sin t)\vec{j}$ gives the correct circle, but starts at $(8, 4)$. To start on the x-axis we need

$$x\vec{i} + y\vec{j} = (4 + 4\cos(t - \frac{\pi}{2}))\vec{i} + (4 + 4\sin(t - \frac{\pi}{2}))\vec{j} = (4 + 4\sin t)\vec{i} + (4 - 4\cos t)\vec{j}.$$

6. The parametric equation of a circle is

$$x = \cos t, y = \sin t.$$

When $t = 0$, $x = 1, y = 0$, and when $t = \frac{\pi}{2}$, $x = 0, y = 1$. This shows a counterclockwise movement, so our original equation is correct.

7. The vector $(\vec{i} + 2\vec{j} + 5\vec{k}) - (2\vec{i} - \vec{j} + 4\vec{k}) = -\vec{i} + 3\vec{j} + \vec{k}$ is parallel to the line, so a possible parameterization is

$$x = 2 - t, \quad y = -1 + 3t, \quad z = 4 + t.$$

8. A line perpendicular to the xz-plane will have $x =$ constant, $z =$ constant, $y =$ anything: This is given by $x = 1, y = t, z = 2$.

9. Since the vector $\vec{n} = \text{grad}(2x - 3y + 5z) = 2\vec{i} - 3\vec{j} + 5\vec{k}$ is perpendicular to the plane, this vector is parallel to the line. Thus the equation of the line is

$$x = 1 + 2t, \quad y = 1 - 3t, \quad z = 1 + 5t.$$

10. The xy-plane is where $z = 0$, so one possible answer is

$$x = 3\cos t, \quad y = 3\sin t, \quad z = 0.$$

This goes in the counterclockwise direction because it starts at $(3, 0, 0)$ and heads in the positive y-direction.

11. Since the circle has radius 3, the equation must be of the form $x = 3\cos t, y = 5, z = 3\sin t$. But since the circle is being viewed from farther out on the y-axis, the circle we have now would be seen going clockwise. To correct this, we add a negative to the third component, giving us the equation $x = 3\cos t, y = 5, z = -3\sin t$.

12. We can find this equation in two ways. First we could find two points on the line of intersection and then proceed as in Example 7 on page 921. To find two points just substitute two different values for z and solve for x and y for each value of z. Alternatively, assuming the line is not horizontal (which it turns out not to be), we could take z to be the parameter t, so $z = t$. To find x and y as functions of t we solve the two equations for x and y in terms of t. We have

$$t = 4 + 2x + 5y$$
$$t = 3 + x + 3y.$$

Eliminating x we get

$$-t = -2 - y \quad \text{and} \quad y = -2 + t.$$

Substituting $-2 + t$ for y in the second equation and solving for x, we get

$$x = 3 - 2t.$$

Our equations are therefore

$$x = 3 - 2t, \; y = -2 + t, \; z = t.$$

or

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = 3\vec{i} - 2\vec{j} + t(-2\vec{i} + \vec{j} + \vec{k}).$$

13. See Figure 17.57. The parameterization is

$$\vec{r} = 10\cos\left(\frac{2\pi t}{30}\right)\vec{i} - 10\sin\left(\frac{2\pi t}{30}\right)\vec{j} + 7\vec{k}.$$

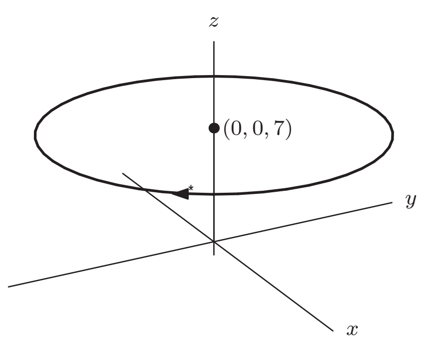


Figure 17.57

14. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(3\cos t)\vec{i} + \frac{d}{dt}(4\sin t)\vec{j} = -3\sin t\vec{i} + 4\cos t\vec{j}.$$

15. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}t\vec{i} + \frac{d}{dt}(t^3 - t)\vec{j} = \vec{i} + (3t^2 - 1)\vec{j}.$$

16. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(2+3t)\vec{i} + \frac{d}{dt}(4+t)\vec{j} + \frac{d}{dt}(1-t)\vec{k} = 3\vec{i} + \vec{j} - \vec{k}.$$

17. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}(2+3t^2)\vec{i} + \frac{d}{dt}(4+t^2)\vec{j} + \frac{d}{dt}(1-t^2)\vec{k} = 6t\vec{i} + 2t\vec{j} - 2t\vec{k}.$$

18. The velocity vector $\vec{v}$ is given by:

$$\vec{v} = \frac{d}{dt}t\vec{i} + \frac{d}{dt}t^2\vec{j} + \frac{d}{dt}t^3\vec{k} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}.$$

19. Vector. Differentiating using the chain rule gives

$$\begin{aligned}\text{Velocity} &= \left(3\cos\sqrt{2t+1}\cdot\frac{1}{2\sqrt{2t+1}}\cdot 2\right)\vec{i} - \left(3\sin\sqrt{2t+1}\cdot\frac{1}{2\sqrt{2t+1}}\cdot 2\right)\vec{j} + \left(\frac{1}{2\sqrt{2t+1}}\cdot 2\right)\vec{k} \\ &= \frac{3\cos\sqrt{2t+1}}{\sqrt{2t+1}}\vec{i} - \frac{3\sin\sqrt{2t+1}}{\sqrt{2t+1}}\vec{j} + \frac{1}{\sqrt{2t+1}}\vec{k}.\end{aligned}$$

20. Scalar. The velocity vector is

$$\vec{v} = 2t\vec{i} + e^t\vec{j},$$

and

$$\text{Speed} = \|\vec{v}\| = \sqrt{(2t)^2 + (e^t)^2} = \sqrt{4t + e^{2t}}.$$

21. Vector. Differentiating using the chain rule gives

$$\text{Velocity} = \left(\frac{-\cos t}{2\sqrt{3+\sin t}}\right)\vec{i} + \left(\frac{-\sin t}{2\sqrt{3+\cos t}}\right)\vec{j}.$$

22. Vector. Differentiating using the product and chain rule gives

$$\begin{aligned}\text{Velocity} &= (e^t + te^t)\vec{i} + 2e^{2t}\vec{j} \\ \text{Acceleration} &= (2e^t + te^t)\vec{i} + 4e^{2t}\vec{j}.\end{aligned}$$

23. No. The first is parallel to the vector $2\vec{i} - \vec{j} + 3\vec{k}$ and the second is parallel to $\vec{i} + 2\vec{j} + 2\vec{k}$.

24. Yes. They are both parallel to the vector $2\vec{i} - \vec{j} + 3\vec{k}$.

25. The direction vectors of the lines, $-\vec{i} + 4\vec{j} - 2\vec{k}$ and $2\vec{i} - 8\vec{j} + 4\vec{k}$, are multiplies of each other (the second is -2 times the first). Thus the lines are parallel. To see if they are the same line, we take the point corresponding to $t = 0$ on the first line, which has position vector $3\vec{i} + 3\vec{j} - \vec{k}$, and see if it is on the second line. So we solve

$$(1+2t)\vec{i} + (11-8t)\vec{j} + (4t-5)\vec{k} = 3\vec{i} + 3\vec{j} - \vec{k}.$$

This has solution $t = 1$, so the two lines have a point in common and must be the same line, parameterized in two different ways.

26. (a) We get the part of the line with $x < 0$ and $y < 0$ and $z < 10$.
(b) We get the part of the line between the points $(0, 0, 10)$ and $(1, 2, 13)$.

27. See Figure 17.58.

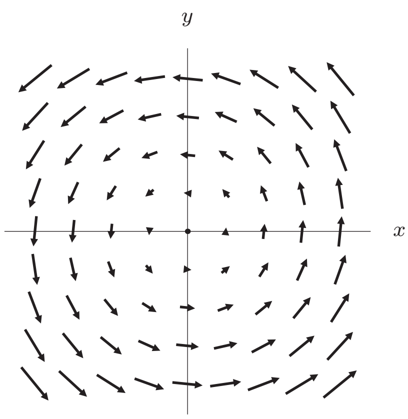


Figure 17.58: $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$

28. At each point, all these vector fields point in the same direction (rotating clockwise around the origin). Since $\|\vec{F}\| = \frac{1}{x^2+y^2}\|y\vec{i} - x\vec{j}\| = \frac{\sqrt{y^2+x^2}}{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}$, the vectors in the field shrink as you go away from the origin. See Figure 17.59.

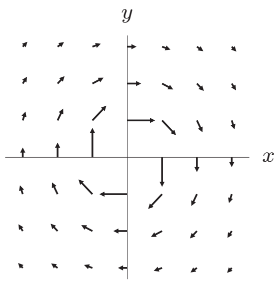


Figure 17.59

29. The vector field points in a clockwise direction around the origin. Since

$$\| \left(\frac{y}{\sqrt{x^2+y^2}}\right)\vec{i} - \left(\frac{x}{\sqrt{x^2+y^2}}\right)\vec{j} \| = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1,$$

the length of the vectors is constant everywhere.

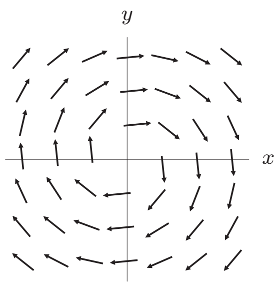


Figure 17.60

Problems

30. Substitute $x = 2t + 1, y = 3t - 2, z = -t + 3$ into the equation of the sphere:

$$\begin{aligned}(x-1)^2 + (y-1)^2 + (z-2)^2 &= 2\\ (2t)^2 + (3t-1)^2 + (-t+1)^2 &= 2\\ 4t^2 + 9t^2 - 6t + 1 + t^2 - 2t + 1 &= 2\\ 14t^2 - 8t &= 0\end{aligned}$$

Thus, $t = 0$, giving $x = 1, y = -2, z = 3$, and when $t = 4/7$, $x = 15/7, y = -2/7, z = 17/7$.

31. (a) To find where the particle is at time equal to 0, we simply substitute 0 in for all t in the equation. Therefore, the particle is at the point with position vector

$$\begin{aligned}\vec{r}(0) &= [2 + 5(0)]\,\vec{i} + (3+0)\vec{j} + 2(0)\vec{k}\\ &= 2\vec{i} + 3\vec{j} + 0\vec{k}\,.\end{aligned}$$

Thus, the particle is at the point $(2, 3, 0)$.

(b) To find the time at which the particle is at the point $(12, 5, 4)$, we solve for t for each component, and the t should be the same, if the curve goes through this point. For the x-component, we get

$$\begin{aligned}2 + 5t &= 12\\ t &= 2.\end{aligned}$$

For the y-component, we get

$$\begin{aligned}3 + t &= 5\\ t &= 2.\end{aligned}$$

And for the z-component, we get

$$\begin{aligned}2t &= 4\\ t &= 2.\end{aligned}$$

Therefore, at $t = 2$ the particle reaches $(12, 5, 4)$.

(c) The particle never reaches $(12, 4, 4)$, because the equation

$$\vec{r} = (2+5t)\vec{i} + (3+t)\vec{j} + 2t\vec{k} = 12\vec{i} + 4\vec{j} + 4\vec{k}$$

has no solution. Thus, the point does not lie on the line.

32. (a) (I) has radius 1 and traces out a complete circle, so $I = C_4$.

(II) has radius 2 and traces out the top half of a circle, so $II = C_1$

(III) has radius 1 and traces out a quarter circle, so $III = C_2$.

(IV) has radius 2 and traces out the bottom half of a circle, so $IV = C_6$.

(b) C_3 has radius $1/2$ and traces out a half circle below the x-axis, so

$$\vec{r} = 0.5\cos t\vec{i} - 0.5\sin t\vec{j}\,.$$

C_5 has radius 2 and traces out a quarter circle below the x-axis starting at the point $(-2, 0)$. Thus we have

$$\vec{r} = -2\cos(t/2)\vec{i} - 2\sin(t/2)\vec{j}\,.$$

33. (a) A vector field associates a vector to every point in a region of the space. In other words, a vector field is a vector-valued function of position given by $\vec{v} = \vec{f}(\vec{r}) = \vec{f}(x, y, z)$

(b) (i) Yes, $\vec{r} + \vec{a} = (x + a_1)\vec{i} + (y + a_2)\vec{j} + (z + a_3)\vec{k}$ is a vector-valued function of position.

(ii) No, $\vec{r} \cdot \vec{a}$ is a scalar.

(iii) Yes.

(iv) $x^2 + y^2 + z^2$ is a scalar.

34. Vector fields (A) and (D) both point radially outward, so they correspond to (I) and (II). Since (A) has vectors that are of constant length, it corresponds to (II), where the level curves are equally spaced. (D) corresponds to (I).

Vector field (B) corresponds to (III) since the vectors in (B) point away from the origin on the x-axis, and the function in (III) increases in this direction. To confirm, the vectors in (B) point toward the origin on the y-axis, and the function decreases away from the origin on the y-axis.

In vector field (C), vectors point away toward the origin on the x-axis and away from the origin on the y-axis. This corresponds to (IV), in which the function decreases away from the origin on the x-axis and increases on the y-axis.

35. Sketches of the vector fields in Figure 17.61 show that $\vec{E}$ is tangent to (IV), $\vec{F}$ is tangent to (I), $\vec{G}$ is tangent to (II), and $\vec{H}$ is tangent to (III).

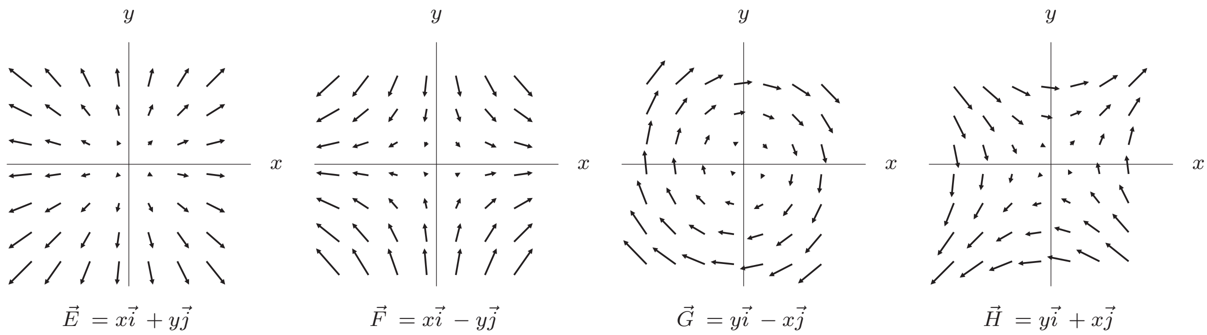


Figure 17.61

36. At time t the particle is $s = t - 7$ seconds from P, so the displacement vector from the point P to the particle is $\vec{d} = s\vec{v}$. To find the position vector of the particle at time t, we add this to the position vector $\vec{r}_0 = 5\vec{i} + 4\vec{j} + 3\vec{k}$ for the point P. Thus a vector equation for the motion is:

$$\begin{aligned}\vec{r} &= \vec{r}_0 + s\vec{v} \\ &= (5\vec{i} + 4\vec{j} + 3\vec{k}) + (t-7)(3\vec{i} + \vec{j} + 2\vec{k}),\end{aligned}$$

or equivalently,

$$x = 5 + 3(t-7), \quad y = 4 + 1(t-7), \quad z = 3 + 2(t-7).$$

Notice that these equations are linear. They describe motion on a straight line through the point $(5, 4, 3)$ that is parallel to the velocity vector $\vec{v} = 3\vec{i} + \vec{j} + 2\vec{k}$.

37. The displacement vector from $(1, 1, 1)$ to $(2, -1, 3)$ is $\vec{d} = (2\vec{i} - \vec{j} + 3\vec{k}) - (\vec{i} + \vec{j} + \vec{k}) = \vec{i} - 2\vec{j} + 2\vec{k}$ meters. The velocity vector has the same direction as $\vec{d}$ and is given by

$$\vec{v} = \frac{\vec{d}}{5} = 0.2\vec{i} - 0.4\vec{j} + 0.4\vec{k} \text{ meters/sec.}$$

Since $\vec{v}$ is constant, the acceleration $\vec{a} = \vec{0}$.

38. Parametric equations for a line in 2-space are

$$\begin{aligned}x &= x_0 + at \\ y &= y_0 + bt\end{aligned}$$

where (x_0, y_0) is a point on the line and $\vec{v} = a\vec{i} + b\vec{j}$ is the direction of motion. Notice that the slope of the line is equal to $\Delta y/\Delta x = b/a$, so in this case we have

$$\begin{aligned}\frac{b}{a} &= \text{Slope} = -2, \\ b &= -2a.\end{aligned}$$

In addition, the speed is 3, so we have

$$\begin{aligned}\|\vec{v}\| &= 3 \\ \sqrt{a^2 + b^2} &= 3 \\ a^2 + b^2 &= 9.\end{aligned}$$

Substituting $b = -2a$ gives

$$\begin{aligned} a^2 + (-2a)^2 &= 9 \\ 5a^2 &= 9 \\ a &= \frac{3}{\sqrt{5}}, -\frac{3}{\sqrt{5}}. \end{aligned}$$

If we use $a = 3/\sqrt{5}$, then $b = -2a = -6/\sqrt{5}$. The point (x_0, y_0) can be any point on the line: we use $(0, 5)$. The parametric equations are

$$x = \frac{3}{\sqrt{5}}t, \quad y = 5 - \frac{6}{\sqrt{5}}t.$$

Alternatively, we can use $a = -3/\sqrt{5}$ giving $b = 6/\sqrt{5}$. An alternative answer, which represents the particle moving in the opposite direction is

$$x = -\frac{3}{\sqrt{5}}t, \quad y = 5 + \frac{6}{\sqrt{5}}t.$$

39. (a) The quantity $\|\operatorname{grad} f\|$ represents the maximum rate of change of temperature with distance at each point. Its units are $^\circ$C per cm.

(b) The speed of the particle is $\sqrt{(g'(t))^2 + (k'(t))^2}$. Its units are cm/sec.

(c) The rate of change of the particle's temperature with time is given by the chain rule

$$\frac{dH}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x \cdot g'(t) + f_y \cdot k'(t).$$

Its units are $^\circ$C/sec.

40. We should have $x = x_0 - 2t$. Since no initial position at time $t = 0$ is given, we can use any point on the line $y = 3x + 7$ as (x_0, y_0). We choose the y-intercept $(0, 7)$. Then $x = 0 - 2t$ and $y = 7 + bt$. Since the slope of the line is 3 and the x-coordinate decreases by 2 units for each unit of time, we know that the y-coordinate decreases by 6 units for each unit of time. Therefore, $b = -6$. Our equations are $x = -2t,\ y = 7 - 6t$.

41. (a) In order for the particle to stop, its velocity $\vec{v} = (dx/dt)\vec{i} + (dy/dt)\vec{j}$ must be zero, so we solve for t such that $dx/dt = 0$ and $dy/dt = 0$, that is

$$\begin{aligned} \frac{dx}{dt} &= 3t^2 - 3 = 3(t-1)(t+1) = 0, \\ \frac{dy}{dt} &= 2t - 2 = 2(t-1) = 0. \end{aligned}$$

The value $t = 1$ is the only solution. Therefore, the particle stops when $t = 1$ at the point $(t^3 - 3t,\ t^2 - 2t)|_{t=1} = (-2, -1)$.

(b) In order for the particle to be traveling straight up or down, the x-component of the velocity vector must be 0. Thus, we solve $dx/dt = 3t^2 - 3 = 0$ and obtain $t = \pm 1$. However, at $t = 1$ the particle has no vertical motion, as we saw in part (a). Thus, the particle is moving straight up or down only when $t = -1$. Since the velocity at time $t = -1$ is

$$\vec{v}(-1) = \left.\frac{dx}{dt}\right|_{t=-1} \vec{i} + \left.\frac{dy}{dt}\right|_{t=-1} \vec{j} = -4\vec{j},$$

the motion is straight down. The position at that time is $(t^3 - 3t,\ t^2 - 2t)|_{t=-1} = (2, 3)$.

(c) For horizontal motion we need $dy/dt = 0$. That happens when $dy/dt = 2t - 2 = 0$, and so $t = 1$. But from part (a) we also have $dx/dt = 0$ also at $t = 1$, so the particle is not moving at all when $t = 1$. Thus, there is no time when the motion is horizontal.

42. (a) The velocity vector is given by differentiating $\vec{r}(t)$ component by component to give

$$\vec{v} = \frac{d\vec{r}}{dt} = (-4\sin 4t\vec{i} + 4\cos 4t\vec{j} + 3\vec{k}).$$

Similarly, differentiating the velocity vector gives the acceleration vector to be

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = (-16\cos 4t\vec{i} - 16\sin 4t\vec{j}).$$

(b) The speed of the particle is

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{4^2(\sin^2 4t + \cos^2 4t) + 3^2} = \sqrt{25} = 5.$$

(c) Since $\|\vec{v}\| = 5$ the particle's speed is constant.

(d) The angle between the position and acceleration vectors, θ, is given

$$\cos(\theta) = \frac{\vec{r}\cdot\vec{a}}{\|\vec{r}\|\|\vec{a}\|}$$

$$\begin{aligned}\vec{r}(0) &= \cos(0)\vec{i} + \sin(0)\vec{j} + 0\vec{k} = \vec{i}\\ \vec{a}(0) &= (-16\cos(0)\vec{i} - 16\sin(0)\vec{j} = -16\vec{i}\\ \vec{r}\cdot\vec{a} &= -16, \quad \|\vec{r}\|| = 1, \quad \|\vec{a}\| = 16\end{aligned}$$

so that

$$\cos(\theta) = -1, \quad \theta = 180^\circ.$$

43. (a) With the center at $(0, 0, 8)$ and a point of the circle at $(0, 5, 8)$, we know that the radius is 5. When $t = 0$, we have $x = 0$ and $y = 5$. Since the stone is rotating horizontally, $z = 0$ for all t. The period is 2π. Thus, the parameterization is:

$$\begin{aligned}x(t) &= 5\sin t\\ y(t) &= 5\cos t\\ z(t) &= 8\end{aligned}$$

This parameterization has the correct period (if t is in seconds) and satisfies the initial conditions.

(b) From our parameterization with t in seconds, we can see that the stone reaches $(5, 0, 8)$ at time $\pi/2$. Thus at $t = \pi/2$,

$$\begin{aligned}\vec{v} &= x_t(\pi/2)\vec{i} + y_t(\pi/2)\vec{j}\\ &= 5\cos(\pi/2)\vec{i} - 5\sin(\pi/2)\vec{j}\\ &= -5\vec{j}.\end{aligned}$$

The acceleration of an object is the second derivative of its position. Thus, at $t = \pi/2$,

$$\begin{aligned}\vec{a} &= x_{tt}(\pi/2)\vec{i} + y_{tt}(\pi/2)\vec{j}\\ &= -5\sin(\pi/2)\vec{i} - 5\cos(\pi/2)\vec{j}\\ &= -5\vec{i}\end{aligned}$$

(c) At the moment in which the stone has left the circle, the only acceleration that acts on the stone is that of gravity. From that, assuming a gravity vector field oriented in the $-z$ direction, we get the differential equations

$$\begin{aligned}z_{tt}(t) &= -g\\ x_{tt}(t) &= y_{tt}(t) = 0.\end{aligned}$$

If we now measure t from the instant the string breaks, then the initial conditions are the velocity and position of the stone at $t = 0$. Since the velocity at the moment of release is $\vec{v} = -5\vec{j}$, we have

$$x_t(0) = 0, \quad y_t(0) = -5, \quad z_t(0) = 0.$$

The initial position at $t = 0$ is:

$$x(0) = 5, \quad y(0) = 0, \quad z(0) = 8.$$

44. (a) Differentiating we have

$$\begin{aligned}x'(t) &= 5, \quad y'(t) = 3, \quad z'(t) = -2t + 2,\\ x''(t) &= 0, \quad y''(t) = 0, \quad z''(t) = -2.\end{aligned}$$

Thus at time $t = 0$,

$$\begin{aligned}\text{Position } &= (x(0), y(0), z(0)) = (0, 0, 15)\\ \text{Velocity } &= x'(0)\vec{i} + y'(0)\vec{j} + z'(0)\vec{k} = 5\vec{i} + 3\vec{j} + 2\vec{k}\\ \text{Acceleration } &= x''(0)\vec{i} + y''(0)\vec{j} + z''(0)\vec{k} = -2\vec{k}.\end{aligned}$$

(b) The particle hits the ground when $z(t) = 0$, so

$$\begin{aligned} 15 - t^2 + 2t &= 0 \\ -(t-5)(t+3) &= 0 \\ t &= -3, 5. \end{aligned}$$

Since $t \geq 0$, the particle hits the ground when $t = 5$. At that time

$$\begin{aligned} \text{Position } &= (x(5), y(5), z(5)) = (25, 15, 0) \\ \text{Velocity } &= x'(5)\vec{i} + y'(5)\vec{j} + z'(5)\vec{k} = 5\vec{i} + 3\vec{j} - 8\vec{k} \end{aligned}$$

so

$$\text{Speed } = \|\text{Velocity}\| = \sqrt{5^2 + 3^2 + (-8)^2} = \sqrt{98}.$$

45. (a) $f_x = \dfrac{\left[2x(x^2+y^2) - 2x(x^2-y^2)\right]}{(x^2+y^2)^2} = \dfrac{4xy^2}{(x^2+y^2)^2}$.

$f_y = \dfrac{\left[-2y(x^2+y^2) - 2y(x^2-y^2)\right]}{(x^2+y^2)^2} = \dfrac{-4yx^2}{(x^2+y^2)^2}$.

$\nabla f(1,1) = \vec{i} - \vec{j}$, i.e., south-east.

(b) We need a vector $\vec{u}$ such that $\nabla f(1,1) \cdot \vec{u} = 0$, i.e., such that $(\vec{i} - \vec{j}) \cdot \vec{u} = 0$. The vector $\vec{u} = \vec{i} + \vec{j}$ clearly works; so does $\vec{u} = -\vec{i} - \vec{j}$. Dividing by the length to get a unit vector, we have $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$ or $\vec{u} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$.

(c) f is a function of x and y, which in turn are functions of t. Thus, the chain rule can be used to show how f changed with t.

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{4xy^2}{(x^2+y^2)^2} \cdot 2e^{2t} - \frac{4x^2y}{(x^2+y^2)^2} \cdot (6t^2+6).$$

At $t = 0, x = 1, y = 1$; so, $\dfrac{df}{dt} = \dfrac{4}{4} \cdot 2 - \dfrac{4}{4} \cdot 6 = -4$.

46. (a) Separate the ant's path into three parts: from $(0,0)$ to $(1,0)$ along the x-axis; from $(1,0)$ to $(0,1)$ via the circle; and from $(0,1)$ to $(0,0)$ along the y-axis. (See Figure 17.62.) The lengths of the paths are 1, $\frac{2\pi}{4} = \frac{\pi}{2}$, and 1 respectively. Thus, the time it takes for the ant to travel the three paths are (using the formula $t = \frac{d}{v}$) $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{2}$ seconds.

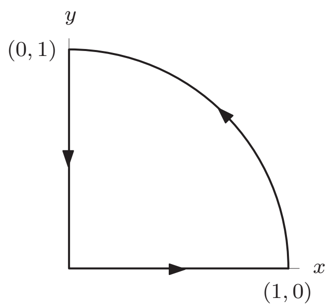


Figure 17.62

From $t = 0$ to $t = \frac{1}{2}$, the ant is heading toward $(1,0)$ so its coordinate is $(2t, 0)$. From $t = \frac{1}{2}$ to $t = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, the ant is veering to the left and heading toward $(0,1)$. At $t = \frac{1}{2}$, it is at $(1,0)$ and at $t = \frac{5}{6}$, it is at $(0,1)$. Thus its position is $(\cos[\frac{3\pi}{2}(t - \frac{1}{2})], \sin[\frac{3\pi}{2}(t - \frac{1}{2})])$. Finally, from $t = \frac{5}{6}$ to $t = \frac{5}{6} + \frac{1}{2} = \frac{4}{3}$, the ant is headed home. Its coordinates are $(0, -2(t - \frac{4}{3}))$.

In summary, the function expressing the ant's coordinates is

$$(x(t), y(t)) = \begin{cases} (2t, 0) & \text{when } 0 \leq t \leq \frac{1}{2} \\ \left(\cos(\frac{3\pi}{2}(t - \frac{1}{2})), \sin(\frac{3\pi}{2}(t - \frac{1}{2}))\right) & \text{when } \frac{1}{2} < t \leq \frac{5}{6} \\ (0, -2(t - \frac{4}{3})) & \text{when } \frac{5}{6} \leq t \leq \frac{4}{3}. \end{cases}$$

(b) To do the reverse path, observe that we can reverse the ant's path by interchanging the x and y coordinates (flipping it with respect to the line $y = x$), so the function is

$$(x(t), y(t)) = \begin{cases} (0, 2t) & \text{when } 0 \leq t \leq \frac{1}{2} \\ \left(\sin(\frac{3\pi}{2}(t - \frac{1}{2})), \cos(\frac{3\pi}{2}(t - \frac{1}{2}))\right) & \text{when } \frac{1}{2} < t \leq \frac{5}{6} \\ (-2(t - \frac{4}{3}), 0) & \text{when } \frac{5}{6} < t \leq \frac{4}{3}. \end{cases}$$

47. Substituting $x = t$, and $y = t^2$ into the temperature function gives $F = 1/(t^2 + t^4)$. To find the rate of change of temperature at time t, we then take the derivative with respect to t.

$$\text{Rate of change of temperature} = \frac{dF}{dt} = -(t^2+t^4)^{-2}(2t+4t^3)$$
$$= -\frac{(2+4t^2)}{t^3(1+t^2)^2}.$$

The chain rule for a function of two variables says:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt},$$

where z is a function of x and y and these are both functions of t. For our case, the temperature F at any point is a function of x and y, and the values of x and y are specified by the parameterized curve on which the bug travels, which is given in terms of t. Thus we may say that:

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt},$$

where dF/dt the rate of change of the temperature of the bug at the time t. We may rewrite this expression to get:

$$\frac{dF}{dt} = \left(\frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j}\right)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right)$$

The first vector here is ∇F, while the second vector is the velocity vector of the bug. Thus, we have

$$\frac{dF}{dt} = \nabla F(x,y)\cdot\vec{v}\,.$$

48. The velocity vector is tangent to the path. At $t = -2$, we have

$$\vec{v} = (3t^2-3)\vec{i} + (2t-2)\vec{j}\,\Big|_{t=-2} = 9\vec{i} - 6\vec{j}\,.$$

Thus, the tangent line has parametric equations where the x-value changes by 9 for each unit change in time, and the y-value changes by -6 for each unit change in time. Also, the tangent line must pass through the point where the particle is at time $t = -2$

$$(t^3-3t,\ t^2-2t)\Big|_{t=-2} = (-2, 8).$$

Therefore, parametric equations for the tangent line are

$$x = -2 + 9(t+2)$$
$$y = 8 - 6(t+2).$$

(Other parameterizations of the line are possible.) See Figure 17.63.

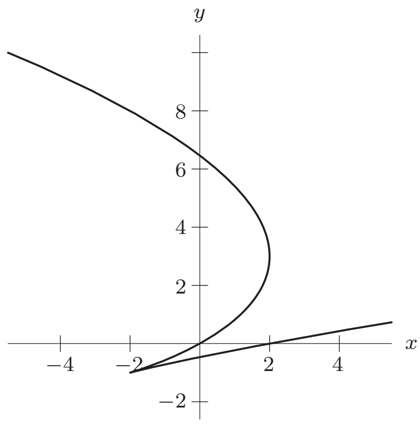


Figure 17.63: The motion given by $x = t^3 - 3t, y = t^2 - 2t$

49. Note that uniform circular motion is possible with the given conditions, since $\vec{v} \cdot \vec{a} = 0$ shows that the velocity and acceleration vectors are perpendicular.

For uniform circular motion in an orbit of radius R, we have $\|\vec{a}\| = \|\vec{v}\|^2/R$. Thus $R = \|\vec{v}\|^2/\|\vec{a}\| = 52/\sqrt{13}$ for both parts (a) and (b).

The center of the orbit is at distance R in the direction of the acceleration vector from the point P on the orbit. The vector

$$R\frac{\vec{a}}{\|\vec{a}\|} = \frac{52}{\sqrt{13}}\frac{2\vec{i} + 3\vec{j}}{\sqrt{13}} = 8\vec{i} + 12\vec{j}$$

thus extends from the point P to the center of the orbit.

(a) The center of the orbit is at the point $(0+8, 0+12) = (8, 12)$
(b) The center of the orbit is at the point $(10+8, 50+12) = (18, 62)$

50. The displacement from the point $(1, 2, 3)$ to the point $(3, 5, 7)$ is $3\vec{i} + 5\vec{j} + 7\vec{k} - (\vec{i} + 2\vec{j} + 3\vec{k}) = 2\vec{i} + 3\vec{j} + 4\vec{k}$. So the equation of the line is

$$x\vec{i} + y\vec{j} + z\vec{k} = 1\vec{i} + 2\vec{j} + 3\vec{k} + t(2\vec{i} + 3\vec{j} + 4\vec{k})$$

or

$$x\vec{i} + y\vec{j} + z\vec{k} = (1+2t)\vec{i} + (2+3t)\vec{j} + (3+4t)\vec{k}.$$

The square of the distance from a point (x, y, z) on the line to the origin, denoted by $D(t)$ is

$$\begin{aligned}
D(t) &= (x-0)^2 + (y-0)^2 + (z-0)^2 \\
&= (1+2t)^2 + (2+3t)^2 + (3+4t)^2 \\
&= 1 + 4t + 4t^2 + 4 + 12t + 9t^2 + 9 + 24t + 16t^2 \\
&= 14 + 40t + 29t^2 \\
&= 29\left(t^2 + \frac{40}{29}t + \frac{14}{29}\right) \\
&= 29\left(\left(t + \frac{20}{29}\right)^2 - \left(\frac{20}{29}\right)^2 + \frac{14}{29}\right).
\end{aligned}$$

Clearly, $D(t)$ is minimum when $t = -20/29$, and

$$D(-20/29) = 29\left(-\left(\frac{20}{29}\right)^2 + \frac{14}{29}\right) = \frac{6}{29}.$$

So the shortest distance is $\sqrt{\frac{6}{29}} = \frac{\sqrt{174}}{29}$.

51. All of the points lie on the unit circle. (You can check this since $x^2 + y^2 = 1$.) The problem is that there is no value of t that gives the point $x = 0$, $y = 1$. This is because

$$y = \frac{t^2 - 1}{t^2 + 1} = 1$$

has no real solution. Only when t approaches positive or negative infinity does the point get close to $(0, 1)$. Technically, it is not a circle.

52. (a) Since there is no horizontal acceleration, if x measures horizontal displacement of the center in meters

$$\frac{d^2x}{dt^2} = 0.$$

Since the initial velocity is 8 m/sec, integrating gives:

$$\frac{dx}{dt} = 8,$$

and since $x = 0$ when $t = 0$,

$$x = 8t.$$

The vertical acceleration is due to gravity. So, if y is vertical displacement of the center in meters:

$$\frac{d^2y}{dt^2} = -g.$$

So
$$\frac{dy}{dt} = -gt + 10,$$
and
$$y = -\frac{gt^2}{2} + 10t + 1.5.$$
Thus, the parametric equations for the center of the baton are
$$x = 8t, \quad y = -\frac{gt^2}{2} + 10t + 1.5.$$

(b) We put a new origin at the center of the baton. Suppose (h, k) are the coordinates of the end of the baton relative to the center. Since the radius of the circular motion is 0.2 m and the angular velocity is $2(2\pi) = 4\pi$ radians/sec and since $x = 0.2$ and $y = 0$ when $t = 0$, we have
$$h = 0.2\cos(4\pi t) \qquad k = 0.2\sin(4\pi t).$$

(c) To find the coordinates of the end of the baton, we add the results from parts (a) and (b), so if x and y represent the position of the end of the baton relative to the ground, we have
$$x = 8t + 0.2\cos(4\pi t) \quad y = -\frac{gt^2}{2} + 10t + 1.5 + 0.2\sin(4\pi t).$$

(d) To sketch this, use $g = 9.8$ meters/sec ^{2}.

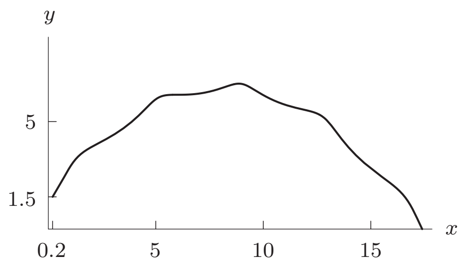


Figure 17.64

53. At time t the particle has polar coordinates $r = \|\vec{r}(t)\| = at$ and $\theta = \omega t$. At time t, the ray from the origin to the particle is at angle ωt radians from the positive x-axis. The ray is therefore rotating at a rate of ω radians per unit time. The parameter ω is the rate of change of the polar angle θ of the particle measured in radians per unit time. The larger ω is, the quicker the particle completes a complete revolution (a $360°$ trip) around the origin. At time t, the particle is at distance at from the origin. Thus a equals the rate of change of the particle's distance from the origin. The larger a is, the faster the particle moves away from the origin.

54. Suppose that the line goes through the point $P = (a, b, c)$ and is parallel to the vector $\vec{u}$. The position vector of the moving object at time t is then given by the formula $\vec{r}(t) = a\vec{i} + b\vec{j} + c\vec{k} + f(t)\vec{u}$ where $f(t)$ is a function so that $f(t)\vec{u}$ is the displacement vector from the point P to the object at time t.

(a) The velocity vector is given by the derivative $\vec{r}\,'(t) = f'(t)\vec{u}$, which is parallel to the line because it is a multiple of the vector $\vec{u}$.

(b) The acceleration vector is given by the second derivative $\vec{r}\,''(t) = f''(t)\vec{u}$ which is parallel to the line because it is a multiple of the vector $\vec{u}$.

55. (a) Since $\vec{F} = \frac{\vec{r}}{\|\vec{r}\|^3}$, the magnitude of $\vec{F}$ is given by
$$\|\vec{F}\| = \frac{\|\vec{r}\|}{\|\vec{r}\|^3} = \frac{1}{\|\vec{r}\|^2}.$$
Now $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, so the magnitude of $\vec{r}$ is given by
$$\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}.$$
Thus,
$$\|\vec{F}\| = \frac{1}{\|\vec{r}\|^2} = \frac{1}{x^2 + y^2 + z^2}.$$

(b) $\vec{F}\cdot\vec{r} = \frac{\vec{r}}{\|\vec{r}\|^3}\cdot\vec{r} = \frac{\|\vec{r}\|^2}{\|\vec{r}\|^3} = \frac{1}{\|\vec{r}\|} = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

(c) A unit vector parallel to $\vec{F}$ and pointing in the same direction is given by $\vec{U} = \frac{\vec{F}}{\|\vec{F}\|}$.
$\vec{F} = \dfrac{\vec{r}}{\|\vec{r}\|^3}$, and $\|\vec{F}\| = \frac{1}{\|\vec{r}\|^2}$. Putting these into the expression for $\vec{U}$ we have

$$\begin{aligned}\vec{U} &= \frac{\vec{F}}{\|\vec{F}\|} = \frac{\frac{\vec{r}}{\|\vec{r}\|^3}}{\frac{1}{\|\vec{r}\|^2}} = \frac{\vec{r}}{\|\vec{r}\|}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}.\end{aligned}$$

(d) A unit vector parallel to $\vec{F}$ and pointing in the opposite direction is given by:

$$\begin{aligned}\vec{V} &= -\frac{\vec{F}}{\|\vec{F}\|} = -\frac{\vec{r}}{\|\vec{r}\|}\\ &= \frac{-x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{-y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{-z}{\sqrt{x^2+y^2+z^2}}\vec{k}.\end{aligned}$$

(e) If $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + \vec{k}$, then $\|\vec{r}\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$.
So, $\vec{F} = \frac{\vec{r}}{\|\vec{r}\|^3} = \frac{\cos t}{\sqrt{8}}\vec{i} + \frac{\sin t}{\sqrt{8}}\vec{j} + \frac{1}{\sqrt{8}}\vec{k} = \frac{\cos t}{2\sqrt{2}}\vec{i} + \frac{\sin t}{2\sqrt{2}}\vec{j} + \frac{1}{2\sqrt{2}}\vec{k}$.

(f) We know that $\vec{F}\cdot\vec{r} = \frac{1}{\|\vec{r}\|}$, so if $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + \vec{k}$, $\vec{F}\cdot\vec{r} = \frac{1}{\sqrt{2}}$.

56. (a) The current, and path that the iceberg would travel, would look like Figure 17.65.

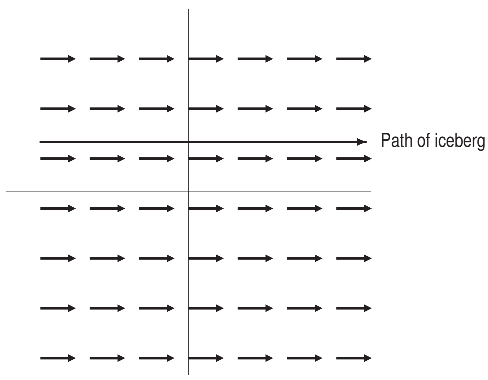


Figure 17.65

To determine the location of the iceberg at time $t = 7$, we must first determine the velocity in the x and y direction. In this current, $V_x = 1$, and $V_y = 0$. To obtain the position, we must integrate the velocity in terms of t. For this current we get

$$\frac{dx}{dt} = 1$$

Hence

$$\begin{aligned}x(7) &= x(0) + \int_0^7 1\cdot dt\\ &= 1 + 7\\ &= 8.\end{aligned}$$

Since $V_y = dy/dt = 0$, y is a constant. Thus at $t = 7$, x has moved from $x = 1$ to $x = 8$ and y has stayed at $y = 3$. Therefore the location at $t = 7$ is $(8, 3)$.

(b) The current and path that the iceberg would travel, would look like Figure 17.66.

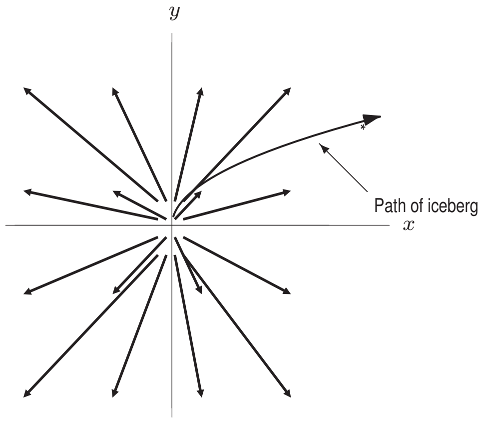


Figure 17.66

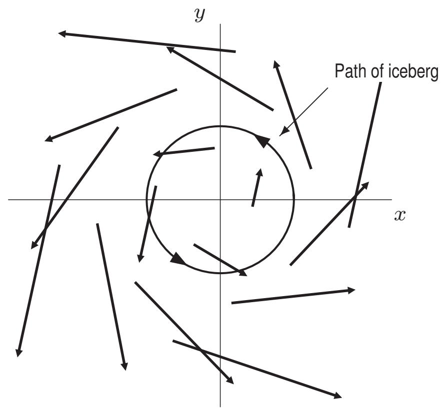


Figure 17.67

Assuming that the iceberg follows the current exactly, we find the position of the iceberg at time $t = 7$ by recognizing that the velocity must be equal to the given vector field.

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = y$$

These are separable equations that are solved for x and y as follows:

$$\begin{aligned}\frac{dx}{dt} &= 2x\\ \int \frac{dx}{2x} &= \int 1\,dt\\ \frac{\ln x}{2} &= t + C\\ x &= k_x e^{2t}\end{aligned}$$

and for y

$$\begin{aligned}\frac{dy}{dt} &= y\\ \int \frac{dy}{y} &= \int 1\,dt\\ \ln y &= t + C\\ y &= k_y e^{t}\end{aligned}$$

We can solve for k_x and k_y, the arbitrary constants, because we know the position of the iceberg at $t = 0$.

$$\begin{aligned}1 &= x(0) = k_x\\ 3 &= y(0) = k_y\end{aligned}$$

so

$$x = e^{2t},\ y = 3e^{t}.$$

We now substitute $t = 7$:

$$x = e^{2\cdot 7} = e^{14} \text{ and } y = 3e^{7}$$

(c) The current, and path that the iceberg would travel, would look like Figure 17.67.

Since

$$\vec{v} = -y\vec{i} + x\vec{j},$$

the system of differential equations satisfied by $x(t)$ and $y(t)$ is

$$\frac{dx}{dt} = -y, \frac{dy}{dt} = x.$$

We differentiate one of the equations and substitute into the other, giving a second order equation

$$\begin{aligned}\frac{dx}{dt} &= -y\\ \frac{d^2x}{dt^2} &= -\frac{dy}{dt}\\ \frac{d^2x}{dt^2} &= -x\\ \frac{d^2x}{dt^2} - x &= 0\end{aligned}$$

This differential equation has a solution of this form :

$$x = A\cos t + B\sin t$$

By taking the derivative and using the fact that $y = -dx/dt$, we get:

$$y = A\sin t - B\cos t$$

We know the initial position of the iceberg, so we can find the constants A and B with the simultaneous equations:

$$\begin{aligned}1 &= x(0) = A\cos 0 + B\sin 0\\ 3 &= y(0) = A\sin 0 - B\cos 0\end{aligned}$$

Thus, $A = 1$ and $B = -3$. Now we evaluate the two expressions for $t = 7$:

$$\begin{aligned}x &= \cos 7 - 3\sin 7 = -1.217\\ y &= \sin 7 + 3\cos 7 = 2.919\end{aligned}$$

and find the position of the iceberg, $(-1.217, 2.919)$.

57. Consider the lines in the directions $\overrightarrow{PQ} = -9\vec{i} - 15\vec{j} + 12\vec{k}$ and $\overrightarrow{RS} = 57\vec{i} - 15\vec{j} + 24\vec{k}$, respectively. If the distance between these lines is always greater than 16 then the beads can always pass without touching. If the distance is less than 16, and if that occurs at a point along the segment from P to Q and from R to S, then the beads will touch at that point.

Vectors in the directions of the two lines are $\vec{u} = -3\vec{i} - 5\vec{j} + 4\vec{k}$ and $\vec{v} = 19\vec{i} - 5\vec{j} + 8\vec{k}$. Parametric equations for the lines are:

$$x(t) = 7 - 3t, \quad y(t) = 12 - 5t, \quad z(t) = -10 + 4t,$$

and

$$x(s) = -20 + 19s, \quad y(s) = 17 - 5s, \quad z(s) = 1 + 8s.$$

(You'll see why we used different parameters for the two lines in a moment.)

The distance between variable points on the lines is a function of s and t; we want the minimum of this function. It is easier to work with the square of the distance. Thus we want to find the minimum of

$$D(s,t) = (-27 + 19s + 3t)^2 + (5 - 5s + 5t)^2 + (11 + 8s - 4t)^2.$$

Computing $\partial D/\partial s$ and $\partial D/\partial t$ and simplifying we find that

$$\frac{\partial D}{\partial s} = 900(-1 + s), \quad \frac{\partial D}{\partial t} = 100(-2 + t).$$

The unique critical point of $D(s,t)$ is $(s,t) = (1,2)$ and the value of D at that point is $D(1,2) = 225$. This must be the minimum value of $D(s,t)$, because $D(s,t)$ is a paraboloid opening upward. (We can also check that this is the minimum with the test for local max and local min.) The distance between the lines is therefore $\sqrt{225} = 15$.

The points on the two lines where $s = 1$ and $t = 2$ are $A = (-1, 12, 9)$ and $B = (1, 2, -2)$; these are the points where the lines are closest. The only question now is whether A and B are along the segments from R to S and from P to Q. In the parameterizations of the lines, R and P correspond to $s = 0$ and $t = 0$, respectively, and S and Q correspond to to $s = 3$ and $t = 3$. So A and B do lie on the given segments. If the beads are centered at these points they will hit because they each have diameter 8 cm, whereas the lines are only 15 cm apart there.

(The solution also shows why we needed different parameters, s and t, for the two lines. The points where the lines are closest together occur at different values of the two parameters: $t = 1$ for one line and $s = 2$ for the other.)

58. Let $\vec{v}$ be the velocity and let $\vec{a}$ be the acceleration. Since the speed $v = |\vec{v}\,|$ is constant, so is $v^2 = \vec{v}\cdot\vec{v}$, and therefore the derivative of v^2 is zero. Differentiating v^2 gives

$$0 = \frac{d}{dt}(v^2) = \frac{d}{dt}(\vec{v}\cdot\vec{v}\,) = \vec{v}\cdot\frac{d}{dt}\vec{v} + \frac{d}{dt}\vec{v}\cdot\vec{v} = \vec{v}\cdot\vec{a} + \vec{a}\cdot\vec{v} = 2\vec{v}\cdot\vec{a}\,,$$

so that

$$\vec{v}\cdot\vec{a} = 0$$

and therefore $\vec{v}$ is perpendicular to $\vec{a}$.

CAS Challenge Problems

59. (a) Since $\vec{e}_1$ and $\vec{e}_2$ are perpendicular, we have $\vec{e}_1\cdot\vec{e}_2 = 0$. The normal vector to the plane is $\vec{i}+\vec{j}+\vec{k}$, and since $\vec{e}_1$ and $\vec{e}_2$ are parallel to the plane, we have $\vec{e}_1\cdot(\vec{i}+\vec{j}+\vec{k}\,) = 0$ and $\vec{e}_2\cdot(\vec{i}+\vec{j}+\vec{k}\,) = 0$. Also, since $\vec{e}_1$ and $\vec{e}_2$ are unit vectors, we have $\vec{e_1}\cdot\vec{e_1} = 1$ and $\vec{e_2}\cdot\vec{e_2} = 1$.

(b) We have

$$ac + bd = 0, a + b = 0, c + d + e = 0, a^2 + b^2 = 1, c^2 + d^2 + e^2 = 1$$

By solving these equations, we can choose, for example,

$$\vec{e_1} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}\,, \vec{e_2} = \frac{1}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} - \frac{\sqrt{2}}{\sqrt{3}}\vec{k}\,.$$

The equations of the circle will then follow from the given formula, with $\vec{r}_0 = \vec{i} + 2\vec{j} + 3\vec{k}$ and $R = 5$.

60. (a)

$$\begin{aligned}\vec{r}\cdot\vec{F} &= (x\vec{i}+y\vec{j}\,)\cdot(-y(1-y^2)\vec{i}+x(1-y^2)\vec{j}\,)\\ &= -xy(1-y^2)+yx(1-y^2) = 0\end{aligned}$$

This means that the tangent line to the flow line at a point is always perpendicular to the vector from the origin to that point. Hence the flow lines are circles centered at the origin.

(b) The circle $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}$ has velocity vector $\vec{v}(t) = -\sin t\vec{i} + \cos t\vec{j} = -y\vec{i} + x\vec{j} = (1-y^2)\vec{F}$. Thus the velocity vector is a scalar multiple of $\vec{F}$, and hence parallel to $\vec{F}$. However, since $\vec{v}(t)$ is not equal to $\vec{F}(\vec{r}(t))$, it is not a flow line.

(c) Using a CAS, we find

$$\vec{v}(t) = -\frac{t}{(1+t^2)^{3/2}}\vec{i} + \left(-\frac{t^2}{(1+t^2)^{3/2}} + \frac{1}{\sqrt{1+t^2}}\right)\vec{j} = -\frac{t}{(1+t^2)^{3/2}}\vec{i} + \frac{1}{(1+t^2)^{3/2}}\vec{j}$$

and

$$\vec{F}(\vec{r}(t)) = -\left(\frac{t\left(1-\frac{t^2}{1+t^2}\right)}{\sqrt{1+t^2}}\right)\vec{i} + \frac{1-\frac{t^2}{1+t^2}}{\sqrt{1+t^2}}\vec{j} = -\frac{t}{(1+t^2)^{3/2}}\vec{i} + \frac{1}{(1+t^2)^{3/2}}\vec{j} = \vec{v}(t).$$

Although the circle parameterized in part (b) has velocity vectors parallel to $\vec{F}$ at each point of the circle, its speed is not equal to the magnitude of the vector field. The circle in part (c) is parameterized at the correct speed to be the flow line.

61. (a) We have

$$\vec{r}\,'(t) = (3ae^{3t} - be^{-t})\vec{i} + (6ae^{3t} + 2be^{-t})\vec{j}$$

and

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= ((ae^{3t}+be^{-t}) + (2ae^{3t}-2be^{-t}))\vec{i} + ((4(ae^{3t}+be^{-t}) + (2ae^{3t}-2be^{-t}))\vec{j}\\ &= (3ae^{3t}-be^{-t})\vec{i} + (6ae^{3t}-be^{-t})\vec{j} = \vec{r}\,'(t).\end{aligned}$$

(b) We want $\vec{r}(0) = \vec{i} - 2\vec{j}$, so we solve

$$a + b = 1$$
$$2a - 2b = -2$$

to get $a = 0$, $b = 1$. So the flow line is $\vec{r}(t) = e^{-t}\vec{i} - 2e^{-t}\vec{j}$, which approaches $(0, 0)$ as $t \to \infty$. For the second flow line, we solve

$$a + b = 1$$
$$2a - 2b = -1.99$$

to get $a = 0.0025$, $b = 0.9975$, so the flow line is

$$\vec{r}(t) = (0.0025e^{3t} + 0.9975e^{-t})\vec{i} + (0.005e^{3t} - 1.995e^{-t})\vec{j},$$

which approaches (∞, ∞) as $t \to \infty$. For the third flow line we solve

$$a + b = 1$$
$$2a - 2b = -2.01$$

which gives $a = -0.0025$, $b = 1.0025$, so the flow line is

$$\vec{r}(t) = (-0.0025e^{3t} + 1.0025e^{-t})\vec{i} + (-0.005e^{3t} + 2.005e^{-t})\vec{j},$$

which approaches $(-\infty, -\infty)$ as $t \to \infty$.

62. Answers may differ depending on the method and CAS used.

(a) Using a CAS to solve for x and y in terms of z and letting $z = t$, we get $x = \frac{20}{13} - \frac{6t}{13}, y = \frac{-1}{13} - \frac{t}{13}, z = t$.

(b) Using a CAS to solve for y and z in terms of x and letting $x = t$, we get $x = t, y = \frac{1}{6}(-2 - 2t + 3t^2), z = \frac{1}{6}(20 - 10t - 3t^2)$.

(c) Using a CAS to solve for x and z in terms of y, we get two solutions

$$x = \sqrt{2 - t^2}, \quad y = t, \quad z = 5 + 5t - 3\sqrt{2 - t^2}$$

and

$$x = -\sqrt{2 - t^2}, \quad y = t, \quad z = 5 + 5t + 3\sqrt{2 - t^2}$$

Each of these is a parameterization of one half of the intersection curve.

PROJECTS FOR CHAPTER SEVENTEEN

1. Set up the coordinate system as shown in Figure 17.68.

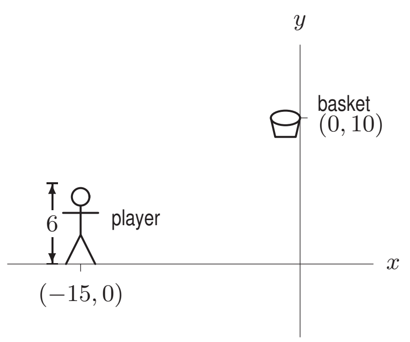


Figure 17.68

(a) We separate the initial velocity vector into its x and y components.

$$V_x = V \cos A$$

$$V_y = V \sin A.$$

Since there is no force acting in the x direction, the x-coordinate of the basketball is just

$$x = (V \cos A)t - 15.$$

For the y-coordinate, we know that

$$y''(t) = -32,$$

so

$$y'(t) = -32t + C_1$$

and

$$y(t) = -16t^2 + C_1 t + C_2.$$

We also know that $y'(0) = V \sin A$ and $y(0) = 6$. Substituting these values in, we get $C_1 = V \sin A$, $C_2 = 6$ and thus

$$y = -16t^2 + (V \sin A)t + 6.$$

(b) Use a graphing calculator or computer to plot the path of the basketball for various values of V and A. Many pairs of V and A put the shot in the basket. For example, $V = 26, A = 60°$, $V = 32, A = 30°$.

(c) Now that we have the equations, we need to find a relationship between V and A that ensures that the basketball goes through the hoop (i.e., the curve passes through $(0, 10)$). So we set

$$x = (V \cos A)t - 15 = 0$$

$$y = -16t^2 + (V \sin A)t + 6 = 10.$$

From the first equation, we get $t = \frac{15}{V \cos A}$. Then we substitute that into the second equation:

$$-16\left(\frac{15}{V \cos A}\right)^2 + (V \sin A)\left(\frac{15}{V \cos A}\right) = 4$$

$$-\frac{3600}{V^2 \cos^2 A} + 15 \tan A = 4$$

$$V^2 = \frac{3600}{\cos^2 A(15 \tan A - 4)}$$

Keeping in mind that $\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta}$, we have:

$$V^2 = \frac{3600(1 + \tan^2 A)}{15 \tan A - 4}.$$

We can minimize V by minimizing V^2 (since $V > 0$).

$$\frac{d(V^2)}{dA} = \frac{2 \tan A(15 \tan A - 4) - 15(\tan^2 A + 1)}{(15 \tan A - 4)^2} \cdot \frac{3600}{\cos^2 A} = 0$$

$$\frac{3600}{\cos^2 A}\left[\frac{15 \tan^2 A - 8 \tan A - 15}{(15 \tan A - 4)^2}\right] = 0$$

$$15 \tan^2 A - 8 \tan A - 15 = 0$$

$$\tan A = \frac{8 + \sqrt{964}}{30}$$

$$\approx 1.30$$

$$A \approx 52°.$$

2. (a) The product rule gives

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$
$$= \vec{v} \times \vec{v} + \vec{r} \times \vec{a}\,.$$

But the cross product of any vector with itself is $\vec{0}$. So $\vec{v} \times \vec{v} = \vec{0}$. Hence $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{a}$.

(b) The area swept out by the planet is approximately a triangle, with sides $\vec{r}, \vec{r} + \Delta\vec{r}$, and $\Delta\vec{r}$. Since $\|\Delta\vec{r} \times \vec{r}\|$ is the area of the parallelogram formed by $\Delta\vec{r}$ and $\vec{r}$, and since the triangle is half the size of the parallelogram, we have $\Delta A \approx \frac{1}{2}\|\Delta\vec{r} \times \vec{r}\|$.

(c) Dividing by Δt gives

$$\frac{\Delta A}{\Delta t} \approx \frac{1}{2}\left\|\frac{\Delta\vec{r}}{\Delta t} \times \vec{r}\right\|.$$

Taking the limit as $\Delta t \to 0$ and recalling that $\vec{L} = \vec{r} \times \vec{v}$, we get

$$\frac{dA}{dt} = \frac{1}{2}\|\vec{L}\|.$$

(d) Since $\vec{a}$ is directed from the earth to the sun, and $\vec{r}$ from the sun to the earth, we see that $\vec{r}$ and $\vec{a}$ are parallel. So $\vec{r} \times \vec{a} = \vec{0}$, as the cross product of parallel vectors is $\vec{0}$. By part (a), this means $d\vec{L}/dt = \vec{0}$. So $\vec{L}$ must be a constant.

(e) Since $\|\vec{L}\|$ is a constant, part (c) implies that

$$\begin{array}{c}\text{area swept out between} \\ t = t_0 \text{ and } t = t_1\end{array} = \int_{t_0}^{t_1} \frac{dA}{dt}dt = \frac{1}{2}\|\vec{L}\| \int_{t_0}^{t_1} dt = \frac{1}{2}\|\vec{L}\|(t_1 - t_0).$$

So the area swept out over a time interval $t_1 - t_0$ only depends on $t_1 - t_0$, not t_0 and t_1 individually.

(f) Let's compare the triangles of are swept out by the planet when it is closest to and furthest from the sun, for a given size time interval. Since the $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ sides are shorter when the planet is closest to the sun, the central angle and the third side must be larger then. So $\Delta\vec{r}$, and hence $\vec{v} = \Delta\vec{r}/\Delta t$, are larger when the planet is closest to the sun, compared to when the planet is furthest from the sun (for a fixed Δt).

3. **(a)**

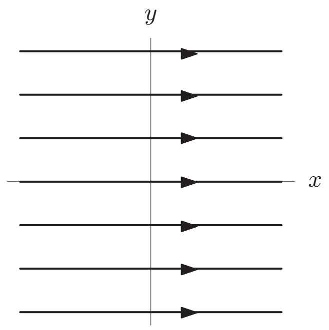


Figure 17.69

(b)

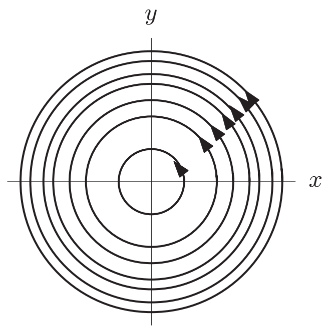


Figure 17.70

(c)

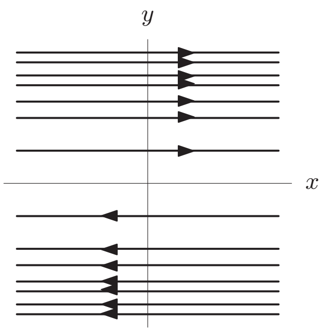


Figure 17.71

(d)

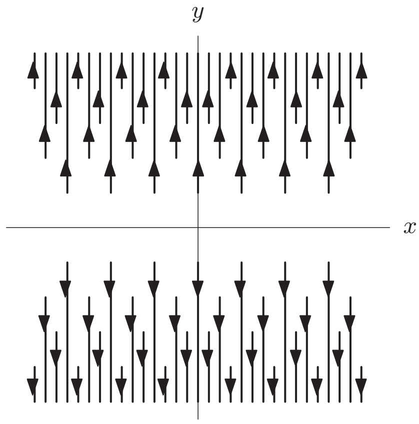


Figure 17.72

CHAPTER EIGHTEEN

Solutions for Section 18.1

Exercises

1. Negative because the vector field points in the opposite direction to the path.

2. Positive, because the vectors are longer on the portion of the path that goes in the same direction as the vector field.

3. Zero, because, by symmetry, the positive integral along the left half of the path cancels the negative integral along the right half.

4. Positive, because the vector field points in the same direction as the path.

5. Zero, because the positive contributions on the upper half of the path cancel the negative contributions on the lower half of the path.

6. Negative, because the vector field points in the opposite direction to the path.

7. Since $\vec{F}$ is perpendicular to the curve at every point along it,

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

8. At every point along the curve, $\vec{F} = 2\vec{j}$ and is parallel to the curve. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = 2 \cdot \text{Length of curve} = 2 \cdot 5 = 10.$$

9. Since $\vec{F}$ is perpendicular to the line, the line integral is 0.

10. Only the $\vec{i}$-component contributes to the integral, so

$$\int_C \vec{F} \cdot d\vec{r} = 6 \cdot \text{ Length of path } = 6 \cdot (7-3) = 24.$$

11. Since $\vec{F}$ is a constant vector field and the curve is a line, $\int_C \vec{F} \cdot d\vec{r} = \vec{F} \cdot \Delta\vec{r}$, where $\Delta\vec{r} = 7\vec{j}$. Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = (3\vec{i} + 4\vec{j}) \cdot 7\vec{j} = 28$$

12. At every point, the vector field is parallel to segments $\Delta\vec{r} = \Delta x\vec{i}$ of the curve. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_2^6 x\vec{i} \cdot dx\vec{i} = \int_2^6 x dx = \left.\frac{x^2}{2}\right|_2^6 = 16.$$

13. The $\vec{j}$-component of $\vec{F}$ does not contribute to the line integral. Since $\Delta\vec{r} = \Delta x\vec{i}$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_2^6 (x\vec{i} + y\vec{j}) \cdot dx\vec{i} = \int_2^6 x dx = \left.\frac{x^2}{2}\right|_2^6 = 16.$$

14. At every point on the path, $\vec{F}$ is parallel to $\Delta\vec{r}$. Suppose r is the distance from the point (x, y) to the origin, so $\|\vec{r}\| = r$. Then $\vec{F} \cdot \Delta\vec{r} = \|\vec{F}\|\|\Delta\vec{r}\| = r\Delta r$. At the start of the path, $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and at the end $r = 6\sqrt{2}$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{2\sqrt{2}}^{6\sqrt{2}} r dr = \left.\frac{r^2}{2}\right|_{2\sqrt{2}}^{6\sqrt{2}} = 32.$$

15. The path is along the y-axis, so only the $\vec{j}$-component contributes to the line integral. Since C is oriented in the $-\vec{j}$ direction, we have

$$\int_C (x\vec{i} + 6\vec{j} - \vec{k}) \cdot d\vec{r} = -6 \cdot \text{Length of path} = -6 \cdot 8 = -48.$$

16. Only the $\vec{i}$-component of the vector field contributes to the integral. This component, $5\vec{i}$, points in the opposite direction to the orientation of the path, which has length 8. Thus,

$$\int_C (5\vec{i} + 7\vec{j}) \cdot d\vec{r} = -5 \cdot \text{ Length of path } = -5 \cdot 8 = -40.$$

17. Only the $\vec{i}$-component contributes to the integral, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_2^3 x^2\vec{i} \cdot \vec{i}\, dx = \int_2^3 x^2\, dx = \frac{x^3}{3}\bigg|_2^3 = \frac{19}{3}.$$

18. Only the $\vec{j}$ component contributes to the integral. On the y-axis, $x = 0$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_3^5 y^2\vec{j} \cdot \vec{j}\, dy = \frac{y^3}{3}\bigg|_3^5 = \frac{98}{3}.$$

19. Since the curve is along the y-axis, only the $\vec{j}$ component of the vector field contributes to the integral:

$$\int_C (2\vec{j} + 3\vec{k}) \cdot d\vec{r} = \int_C 2\vec{j} \cdot d\vec{r} = 2 \cdot \text{ Length of } C = 2 \cdot 10 = 20.$$

20. The path is parallel to the z-axis, so the vector field is perpendicular to the path at every point. Thus, the line integral is 0.

21. Only the $\vec{i}$-component contributes to the line integral, so $d\vec{r} = \vec{i}\, dx$. On C we have $y = 0$, so

$$\int_C ((2y + 7)\vec{i} + 3x\vec{j}) \cdot d\vec{r} = \int_{(1,0,0)}^{(5,0,0)} (7\vec{i} + 3x\vec{j}) \cdot \vec{i}\, dx = \int_1^5 7\, dx = 7 \cdot 4 = 28.$$

22. The vector field $x\vec{i} + y\vec{j} + z\vec{k}$ points radially outward and is everywhere perpendicular to the unit circle. Thus, the line integral is 0.

Problems

23. The line integral along C_1 is positive; the line integrals along C_2 and C_3 appear to be zero.

24. The line integral along C_1 appears to be zero, the line integral along C_2 is positive, and the line integral along C_3 is negative.

25. The line integral along C_1 is negative, the line integral along C_2 is negative, and the line integral along C_3 appears to be zero.

26. The line integral along C_1 appears to be 0, the line integral along C_2 is negative, and the line integral along C_3 is positive.

27. Since it appears that C_1 is everywhere perpendicular to the vector field, all of the dot products in the line integral are zero, hence $\int_{C_1} \vec{F} \cdot d\vec{r} \approx 0$. Along the path C_2 the dot products of $\vec{F}$ with $\Delta\vec{r_i}$ are all positive, so their sum is positive and we have $\int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$. For C_3 the vectors $\Delta\vec{r_i}$ are in the opposite direction to the vectors of $\vec{F}$, so the dot products $\vec{F} \cdot \Delta\vec{r_i}$ are all negative; so, $\int_{C_3} \vec{F} \cdot d\vec{r} < 0$. Thus, we have

$$\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$$

28. The integral $\int_C \vec{F} \cdot d\vec{r}$ is a sum of the line integrals of $\vec{F}$ over each of its three straight segments, which we can compute separately:

$$\int_{PQ} \vec{F} \cdot d\vec{r} = \vec{PQ} \cdot \vec{F} = (4\vec{i} + 2\vec{j}) \cdot \vec{i} = 4$$
$$\int_{QR} \vec{F} \cdot d\vec{r} = \vec{QR} \cdot \vec{F} = (-\vec{i} + 2\vec{j}) \cdot (2\vec{i} - \vec{j}) = -4$$
$$\int_{RS} \vec{F} \cdot d\vec{r} = \vec{RS} \cdot \vec{F} = (-2\vec{i} - 2\vec{j}) \cdot (3\vec{i} + \vec{j}) = -8$$
$$\int_{C} \vec{F} \cdot d\vec{r} = 4 - 4 - 8 = -8.$$

29. (a) See Table 18.1.

Table 18.1

(x, y)	$\vec{F}(x, y)$
$(0, -1)$	$-\vec{i}$
$(1, -1)$	$-\vec{i} + \vec{j}$
$(2, -1)$	$-\vec{i} + 4\vec{j}$
$(3, -1)$	$-\vec{i} + 9\vec{j}$
$(4, -1)$	$-\vec{i} + 16\vec{j}$
$(4, 0)$	$16\vec{j}$
$(4, 1)$	$\vec{i} + 16\vec{j}$
$(4, 2)$	$2\vec{i} + 16\vec{j}$
$(4, 3)$	$3\vec{i} + 16\vec{j}$

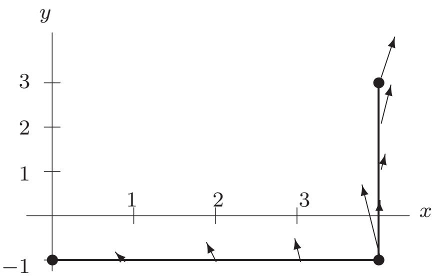


Figure 18.1

(b) See Figure 18.1.

(c) From the point $(0, -1)$ to the point $(4, -1)$, the x-component of the force field is always -1, i.e., it is pushing the object backward with a constant force of 1. Thus, the work done on that part of the path is $-1 \cdot 4 = -4$, because only the horizontal component of the force field contributes to work.

From the point $(4, -1)$ to the point $(4, 3)$, the y-component of the force field is always 16, so it is pushing the object forward with force of 16. Thus, the work done on that part of the path is $16 \cdot 4 = 64$, because only the vertical component of the force field contributes to work.

So the total work done is $-4 + 64 = 60$.

30. The force has no horizontal component. Therefore the (positive) work done in the first half of C_1 will be exactly canceled by the (negative) work done in the second half, so the total work over the path C_1 is zero. The same holds true for C_2, again by virtue of the vertical symmetry of the path and the fact that $\vec{F}$ is constant and because the horizontal part of C_2 contributes zero work. For C_3, the total work will be greater than zero, since the diagonal part of C_3 is in the same general direction as $\vec{F}$ and the horizontal part of C_3 contributes zero work.

31. The dot product of $\vec{F}$ and $10\vec{i}$ is positive if $a > 0$. There are no restrictions on b and c.

32. The $\vec{k}$ component of $\vec{F}$ does not contribute to the line integral. Since the line integral of $y\vec{i}$ around C is negative, for the line integral of $\vec{F}$ to be positive, we need $a < 0$. No restriction on c is needed.

33. The vector field $\vec{F}$ is in the same direction as C if $b > 0$, so we want $b < 0$. No restriction is needed on c.

34. For any value of a, the vector field $ay\vec{i} - ax\vec{j}$ is perpendicular to the vector $\vec{i} + \vec{j} + \vec{k}$ which is in the direction of C. Thus a can take any value. The $\vec{k}$ component of $\vec{F}$ is in the direction of C if the coefficient of $\vec{k}$ is positive, that is, if $c > 1$.

35. The line C is parallel to the z-axis, so $a\vec{i} + b\vec{j}$ does not contribute to the line integral. Thus, there are no restrictions on a and b. The dot product of $\vec{F}$ and $-\vec{k}$ is negative if $c > 3$.

36. (a) See Figure 18.2.

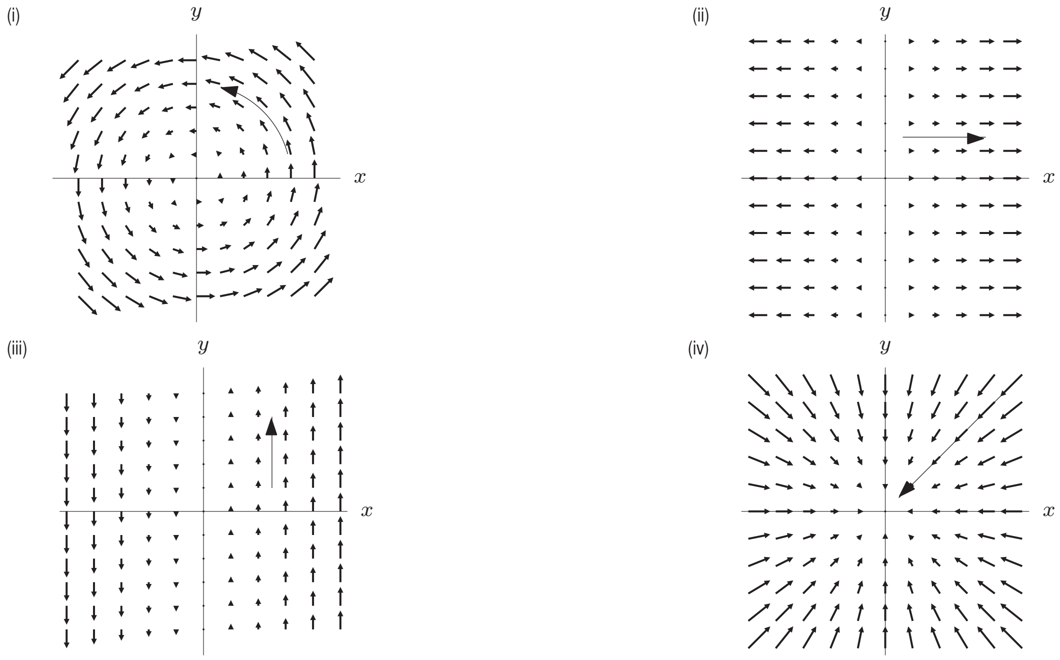


Figure 18.2

(b) For (i) and (iii) a closed curve can be drawn; not for the others.

37. The vector field is $F(\vec{r}) = \vec{r}$. See Figure 18.3. The vector field is perpendicular to the circular arcs at every point, so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r} = 0.$$

Also, since it is radially symmetric,

$$\int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_4} \vec{F} \cdot d\vec{r}.$$

So,

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0.$$

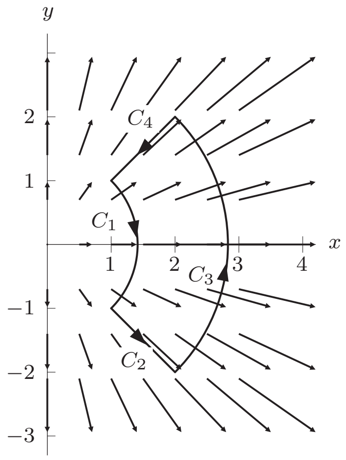


Figure 18.3

38. This vector field is illustrated in Figure 18.4. It is perpendicular to C_2 and C_4 at every point, since $\vec{F}(x,y)\cdot\vec{r}(x,y) = 0$ and C_2 and C_4 are radial line segments, then

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_4} \vec{F}\cdot d\vec{r} = 0.$$

Since C_3 is longer than C_1, and the vector field is larger in magnitude along C_3, the line integral along C_3 has greater absolute value than that along C_1. The line integral along C_3 is positive and the line integral along C_1 is negative, so

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_1} \vec{F}\cdot d\vec{r} > 0.$$

See Figure 18.4.

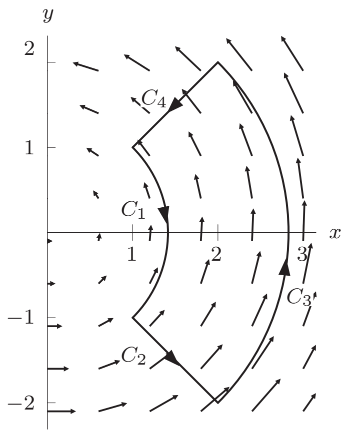


Figure 18.4

39. This vector field is illustrated in Figure 18.5. It is perpendicular to C_2 and C_4 at every point, since $\vec{F}(x,y)\cdot\vec{r}(x,y) = 0$ and C_2 and C_4 are radial line segments, then

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_4} \vec{F}\cdot d\vec{r} = 0.$$

Since C_3 is longer than C_1, and the vector field is larger in magnitude along C_3, the line integral along C_3 has greater absolute value than that along C_1. The line integral along C_1 is positive and the line integral along C_3 is negative, so

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_1} \vec{F}\cdot d\vec{r} < 0.$$

See Figure 18.5.

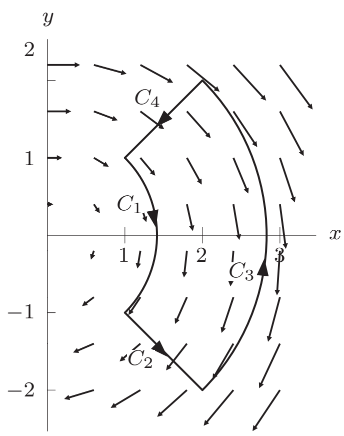


Figure 18.5

40. Since it does not depend on y, this vector field is constant along vertical lines, x = constant. Now let us consider two points P and Q on C_1 which lie on the same vertical line. Because C_1 is symmetric with respect to the x-axis, the tangent vectors at P and Q will be symmetric with respect to the vertical axis so their sum is a vertical vector. But $\vec{F}$ has only horizontal component and thus $\vec{F} \cdot (\Delta\vec{r}(P) + \Delta\vec{r}(Q)) = 0$. As $\vec{F}$ is constant along vertical lines (so $\vec{F}(P) = \vec{F}(Q)$), we obtain

$$\vec{F}(P) \cdot \Delta\vec{r}(P) + \vec{F}(Q) \cdot \Delta\vec{r}(Q) = 0.$$

Summing these products and making $\|\Delta\vec{r}\| \to 0$ gives us

$$\int_{C_1} \vec{F} \cdot d\vec{r} = 0.$$

The same thing happens on C_3, so $\int_{C_3} \vec{F} \cdot d\vec{r} = 0$.

Now let P be on C_2 and Q on C_4 lying on the same vertical line. The respective tangent vectors are symmetric with respect to the vertical axis hence they add up to a vertical vector and a similar argument as before gives

$$\vec{F}(P) \cdot \Delta\vec{r}(P) + \vec{F}(Q) \cdot \Delta\vec{r}(Q) = 0$$

and

$$\int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r} = 0$$

and so

$$\int_{C} \vec{F} \cdot d\vec{r} = 0.$$

See Figure 18.6.

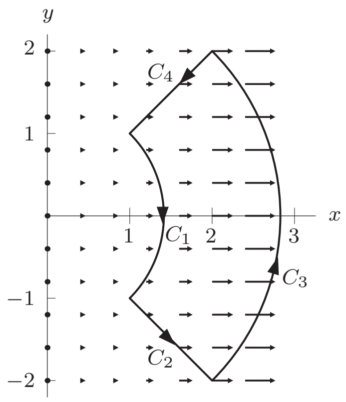


Figure 18.6

41. First of all, $\vec{F}(x, y)$ is perpendicular to the position vector $\vec{r}(x, y) = x\vec{i} + y\vec{j}$ because

$$\vec{F}(x, y) \cdot \vec{r}(x, y) = \frac{-xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2} = 0.$$

Also the magnitude of $\vec{F}$ is inversely proportional to the distance from the origin because

$$\|\vec{F}(x, y)\| = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2} = \frac{1}{\|\vec{r}(x, y)\|}.$$

So $\vec{F}$ is perpendicular to C_2 and C_4 and therefore

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_4} \vec{F} \cdot d\vec{r} = 0.$$

Suppose R is the radius of C_3. On C_3, the vector field $\vec{F}$ has the same direction as the tangent vector which is approximated by $\Delta\vec{r}$, so we have

$$\vec{F}\cdot\Delta\vec{r} = \|\vec{F}\|\cdot\|\Delta\vec{r}\| = \frac{1}{R}\|\Delta\vec{r}\|.$$

When all these products are summed and the limit is taken as $\|\Delta\vec{r}\|\to 0$, we get

$$\begin{aligned}\int_{C_3}\vec{F}\cdot d\vec{r} &= \frac{1}{R}\int_{C_3}\|d\vec{r}\|\\ &= \frac{1}{R}(\text{length of } C_3) = \text{measure of the arc } C_3 \text{ in radians.}\end{aligned}$$

Similarly, suppose r is the radius of C_1. On C_1, the vector field $\vec{F}$ is in the opposite direction to the tangent vector which is approximated by $\Delta\vec{r}$. Hence we have

$$\begin{aligned}\int_{C_1}\vec{F}\cdot d\vec{r} &= -\frac{1}{r}\int_{C_1}\|d\vec{r}\|\\ &= -(\frac{1}{r}(\text{length of } C_1)) = -(\text{measure of } C_1 \text{ in radians}).\end{aligned}$$

Since C_1 and C_3 have the same measure in radians, we have

$$\begin{aligned}\int_C\vec{F}\cdot d\vec{r} &= \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} + \int_{C_4}\vec{F}\cdot d\vec{r}\\ &= -\frac{\pi}{2} + 0 + (+\frac{\pi}{2}) + 0 = 0.\end{aligned}$$

See Figure 18.7.

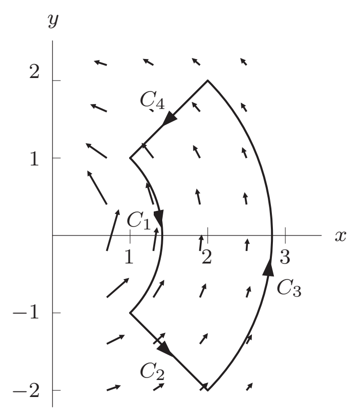


Figure 18.7

42. Figure 18.8 shows the wind velocity vectors on each side of the square, where the speed is v meter/sec on the south side and $(v-12)$ meter/sec on the north side. The circulation is the sum of the line integrals along the four sides of the square. The line integrals along the eastern and western edges are both zero, since the wind velocity is perpendicular to these edges. The integral to the right along the south side equals $(1000\text{ km})(-v\text{ meter/sec}) = -v\times 10^6\text{ meter}^2/\text{sec}$, and the integral to the left along the north side equals $(1000\text{ km})((v-12)\text{ meter/sec}) = (v-12)\times 10^6\text{ meter}^2/\text{sec}$.

$$\text{Total circulation} = -v\times 10^6 + (v-12)\times 10^6 = -1.2\times 10^7\text{ meter}^2/\text{sec}.$$

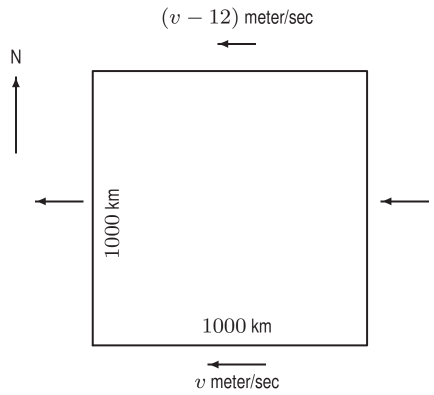


Figure 18.8: Wind velocity across a square

43. Yes, the line integral over C_1 is the negative of the line integral over C_2. One way to see this is to observe that the vector field $x\vec{i} + y\vec{j}$ is symmetric in the y-axis and that C_1 and C_2 are reflections in the y axis (except for orientation). See Figure 18.9. Since the orientation of C_2 is the reverse of the orientation of a mirror image of C_1, the two line integrals are opposite in sign.

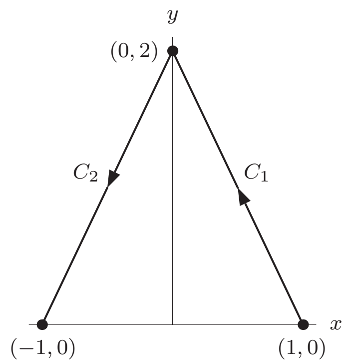


Figure 18.9

44. Since $||\vec{F}|| \leq 7$, the line integral cannot be larger than 7 times the length of the curve. Thus

$$\int_C \vec{F} \cdot d\vec{r} \leq 7 \cdot \text{Circumference of circle} = 7 \cdot 2\pi = 14\pi.$$

The line integral is equal to 14π if $\vec{F}$ is everywhere of magnitude 7, tangent to the curve, and pointing in the direction in which the curve is traversed.

The smallest possible value occurs if the vector field is everywhere of magnitude 7, tangent to the curve and pointing opposite to the direction in which the curve is transversed. Thus

$$\int_C \vec{F} \cdot d\vec{r} \geq -14\pi.$$

45. The line integral is defined by chopping the curve C into little pieces, C_i, and forming the sum

$$\sum_{C_i} \vec{F} \cdot \Delta\vec{r}.$$

When the pieces are small, $\Delta\vec{r}$ is approximately tangent to C_i, and its magnitude is approximately equal to the length of the little piece of curve C_i. This means that $\vec{F}$ and $\Delta\vec{r}$ are almost parallel, the dot product is approximately equal to the product of their magnitudes, i.e.,

$$\vec{F} \cdot \Delta\vec{r} \approx m \cdot (\text{Length of } C_i).$$

When we sum all the dot products, we get

$$\begin{aligned}\sum_{C_i} \vec{F} \cdot \Delta\vec{r} &\approx \sum_{C_i} m \cdot (\text{Length of } C_i) \\ &= m \cdot \sum_{C_i} (\text{Length of } C_i) \\ &= m \cdot (\text{Length of } C)\end{aligned}$$

46. Suppose $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C. Pick any two fixed points P_1, P_2 and curves C_1, C_2 each going from P_1 to P_2. See Figure 18.10. Define $-C_2$ to be the same curve as C_2 except in the opposite direction. Therefore, the curve formed by traversing C_1, followed by C_2 in the opposite direction, written as $C_1 - C_2$, is a closed curve, so by our assumption, $\int_{C_1-C_2} \vec{F} \cdot d\vec{r} = 0$. However, we can write

$$\int_{C_1-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

since C_2 and $-C_2$ are the same except for direction. Therefore,

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0,$$

so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Since C_1 and C_2 are any two curves with the endpoints P_1, P_2, this gives the desired result – namely, that fixing endpoints and direction uniquely determines the value of $\int_C \vec{F} \cdot d\vec{r}$. In other words, the value of the integral $\int_C \vec{F} \cdot d\vec{r}$ does not depend on the path taken. We say the line integral is *path-independent.*

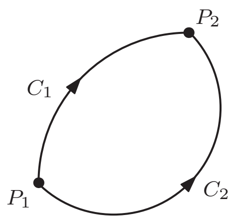


Figure 18.10

47. Pick any closed curve C. Choose two distinct points P_1, P_2 on C. Let C_1, C_2 be the two curves from P_1 to P_2 along C. See Figure 18.11. Let $-C_2$ be the same as C_2, except in the opposite direction. Thus, $C_1 - C_2 = C$. Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

since C_2 and $-C_2$ differ only in direction. But C_1 and C_2 have the same endpoints (P_1 and P_2) and same direction (P_1 to P_2), so by assumption we have $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$. Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0.$$

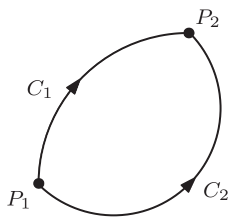


Figure 18.11

48. Let $r = \|\vec{r}\,\|$. Since $\Delta\vec{r}$ points outward, in the opposite direction to $\vec{F}$, we expect the answer to be negative.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C -\frac{GMm\vec{r}}{r^3}\cdot d\vec{r} = \int_{8000}^{10000} -\frac{GMm}{r^2}dr \\ &= \frac{GMm}{r}\bigg|_{8000}^{10000} = GMm\left(\frac{1}{10000}-\frac{1}{8000}\right) \\ &= -2.5\cdot 10^{-5}GMm.\end{aligned}$$

49. Let $r = \|\vec{r}\,\|$. Since $\Delta\vec{r}$ points outward, in the opposite direction to $\vec{F}$, we expect a negative answer. We take the upper limit to be $r=\infty$, so the integral is improper.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C -\frac{GMm\vec{r}}{r^3}\cdot d\vec{r} = \int_{8000}^{\infty} -\frac{GMm}{r^2}dr \\ &= \lim_{b\to\infty}\int_{8000}^{b} -\frac{GMm}{r^2}dr = \lim_{b\to\infty}\frac{GMm}{r}\bigg|_{8000}^{b} = \lim_{b\to\infty} GMm\left(\frac{1}{b}-\frac{1}{8000}\right) \\ &= -\frac{GMm}{8000}\end{aligned}$$

50. The force of the field on the particle at each point is $\vec{E}$, so the force applied in moving the particle against the field is $-\vec{E}$, so

$$\phi(P) = -\int_C \vec{E}\cdot d\vec{r}$$

where C is a path from P_0 to P.

51. Any point P which is a units from the origin can be reached from P_0 by a path C lying on the sphere of radius a. Since $\vec{E}$ is perpendicular to the sphere, $\int_C \vec{E}\cdot d\vec{r} = 0$, so $\phi(P)=0$. On the other hand, if P does not lie on the sphere of radius a, it can be reached by a path consisting of two pieces, C_1 and C_2, one lying on the sphere of radius a and one going straight along a line radiating from the origin (see Figure 18.12). $\int_{C_1}\vec{E}\cdot d\vec{r} = 0$ as before, but $\int_{C_2}\vec{E}\cdot d\vec{r} \neq 0$, since $\vec{E}$ is parallel to C_2 and always points out. Thus, if C is the path consisting of C_1 followed by C_2, we have $\int_C \vec{E}\cdot d\vec{r} = \int_{C_2}\vec{E}\cdot d\vec{r}$. Thus $\int_C \vec{E}\cdot d\vec{r}$ is always positive or always negative along the path C which joins P_0 to P. Hence the set of points with potential zero is the sphere of radius a.

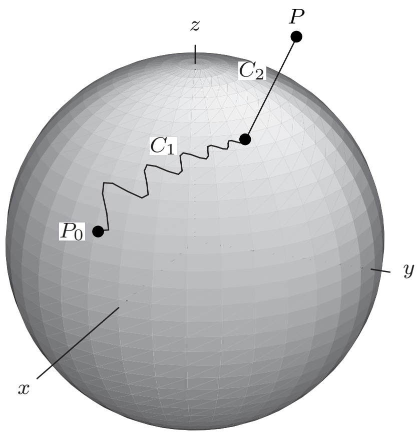


Figure 18.12

52. In Problem 51 we saw that the surface where the potential is zero is a sphere of radius a. Let S be any sphere centered at the origin, and let P_1 be a point on S, and C_1 a path from P_0 to P_1. If P is any point on S, then P can be reached from

P_0 by a path, C, consisting of C_1 followed by C_2, where C_2 is a path from P_1 to P lying entirely on the sphere, S. Then $\int_{C_2} \vec{E} \cdot d\vec{r} = 0$, since $\vec{E}$ is perpendicular to the sphere. So

$$\phi(P) = -\int_C \vec{E} \cdot d\vec{r} = -\int_{C_1} \vec{E} \cdot d\vec{r} - \int_{C_2} \vec{E} \cdot d\vec{r} = -\int_{C_1} \vec{E} \cdot d\vec{r} = \phi(P_1).$$

Thus, ϕ is constant on S. The equipotential surfaces are spheres centered at the origin.

53. (a) Suppose P is b units from the origin. Then P can be reached by a path, C, consisting of two pieces, C_1 and C_2, one lying on the sphere of radius a and one going straight along a line radiating from the origin (see Figure 18.13). We have $\vec{E} \cdot \Delta\vec{r} = 0$ on C_1, and, writing $r = \|\vec{r}\|$, we have $\vec{E} \cdot \Delta\vec{r} = \|\vec{E}\|\Delta r$ on C_2, so

$$\begin{aligned}\phi(P) &= -\int_C \vec{E} \cdot d\vec{r} = -\int_{C_1} \vec{E} \cdot d\vec{r} - \int_{C_2} \vec{E} \cdot d\vec{r} \\ &= 0 - \int_a^b \|\vec{E}\|\, dr = 0 - \int_a^b \frac{Q}{4\pi\epsilon}\frac{1}{r^2}\, dr \\ &= \frac{Q}{4\pi\epsilon}\frac{1}{r}\Big|_a^b = \frac{Q}{4\pi\epsilon}\frac{1}{b} - \frac{Q}{4\pi\epsilon}\frac{1}{a}.\end{aligned}$$

Let P be the point with position vector $\vec{r}$. Then

$$\phi(\vec{r}) = -\frac{Q}{4\pi\epsilon}\frac{1}{a} + \frac{Q}{4\pi\epsilon}\frac{1}{\|\vec{r}\|}.$$

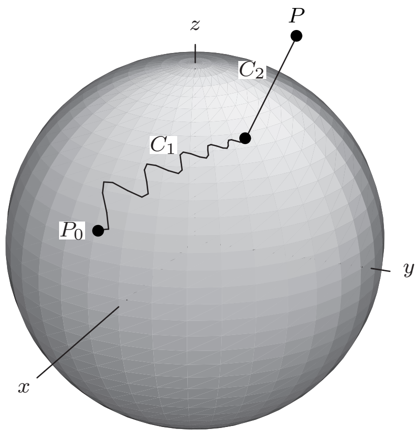


Figure 18.13

(b) If we let $a \to \infty$ in the formula for ϕ, the first term goes to zero and we get the simpler expression

$$\phi(\vec{r}) = \frac{Q}{4\pi\epsilon}\frac{1}{\|\vec{r}\|}.$$

Strengthen Your Understanding

54. This is only true if $\int_C \vec{F} \cdot d\vec{r} > 0$. However, if $\int_C \vec{F} \cdot d\vec{r} < 0$, then $\int_{-C} \vec{F} \cdot d\vec{r} > 0$.

55. The value of a line integral is a scalar, not a vector.

56. The curve C is parallel at every point to the vector $\vec{i} + \vec{j}$. The vector field $\vec{F} = \vec{i} - \vec{j}$ is perpendicular to C, because $\vec{F} \cdot (\vec{i} + \vec{j}) = 0$. We have $\int_C \vec{F} \cdot d\vec{r} = 0$ because the vector field and the curve are orthogonal at every point.

57. Choose for C_1 a curve going in the direction of the vectors in the vector field, and choose for C_2 a curve going in the opposite direction of the vectors in the vector field. See Figure 18.14. A second option for C_2 consists of using C_1 oriented in the downward direction.

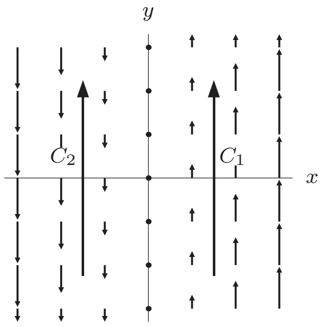


Figure 18.14

58. False. Because $\vec{F} \cdot \Delta\vec{r}$ is a scalar quantity, $\int_C \vec{F} \cdot d\vec{r}$ is also a scalar quantity.

59. True. You can trace out C_2 using the same subdivisions, but each $\Delta\vec{r}$ will have the opposite sign as before and will be traced out twice, so $\int_{C_2} \vec{F} \cdot d\vec{r} = -2\int_{C_1} \vec{F} \cdot d\vec{r} = -6$.

60. True. The vector field $\vec{F} = x\vec{i} + y\vec{j} = \vec{r}$ has radial direction, pointing everywhere perpendicular to the path of integration, so the line integral is zero.

61. True. The line integral is the limit of a sum of dot products, hence is a scalar.

62. False. The relative sizes of the line integrals along C_1 and C_2 depend on the behavior of the vector field $\vec{F}$ along the curves. As a counterexample, take the vector field $\vec{F} = \vec{i}$, and C_1 to be the line from the origin to $(0, 2)$, while C_2 is the line from the origin to $(1, 0)$. Then the length of C_1 is 2, which is greater than the length of C_2, which is 1. However $\int_{C_1} \vec{F} \cdot d\vec{r} = 0$ (since $\vec{F}$ is perpendicular to C_1) while $\int_{C_2} \vec{F} \cdot d\vec{r} > 0$ (since $\vec{F}$ points along C_2).

63. False. For example, the vector field $\vec{F}$ could be perpendicular to C everywhere. For instance, let $\vec{F} = \vec{j}$ and C be the curve $t\vec{i}$, for $0 \le t \le 1$. Alternatively, $\vec{F}$ might point along part of C and in the opposite direction on another part of C and so that the sum cancels out, yielding a zero line integral. For instance, let $\vec{F} = x\vec{i}$ and C be the curve $t\vec{i}$, for $-1 \le t \le 1$.

64. True. All of the dot products $\vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$ in this line integral are positive, since the vector field (the constant $\vec{i}$) points in the same direction as $\Delta\vec{r}_i$.

65. False. All of the dot products $\vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$ in this line integral are zero, since the vector field (the constant $\vec{i}$) points perpendicular to $\Delta\vec{r}_i$.

66. False. The relation between these two line integrals depends on the behavior of the vector field along each of the curves, so there is no reason to expect one to be the negative of the other. As an example, if $\vec{F}(x, y) = y\vec{i}$, then, by symmetry, both line integrals are equal to the same negative number.

67. False. The vector field swirls counterclockwise about the origin, and the path is oriented clockwise, so the line integral is negative.

Solutions for Section 18.2

Exercises

1. If we use the parameterization $x = \sin t, y = \cos t$ for $0 \le t \le \pi$, we have $x' = \cos t$, $y' = -\sin t$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi ((\cos t)\vec{i} + (\sin t)\vec{j}) \cdot ((\cos t)\vec{i} - (\sin t)\vec{j})\, dt = \int_0^\pi (\cos^2 t - \sin^2 t)\, dt.$$

Other answers are possible

2. If we use the parameterization $x = t, y = 1 + t, z = t$ for $0 \le t \le 2$, we have $x' = 1, y' = 1, z' = 1$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 (t\vec{i} + t^2\vec{k}) \cdot (\vec{i} + \vec{j} + \vec{k})dt = \int_0^2 (t + t^2)\, dt.$$

3. If we use the parameterization $x = \cos t, y = \sin t, z = 10$ for $0 \le t \le 2\pi$, we have $x' = -\sin t, y' = \cos t, z' = 0$, so

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (\cos(\cos t)\vec{i} + \cos(\sin t)\vec{j} + (\cos 10)\vec{k}) \cdot (-\sin t)\vec{i} + (\cos t)\vec{j})\, dt \\ &= \int_0^{2\pi} (-\sin t \cos(\cos t) + \cos t \cos(\sin t))\, dt.\end{aligned}$$

4. Only the y-component of the vector field contributes to the line integral. On the curve, $d\vec{r} = \vec{j}\, dy$, so

$$\int_C (3\vec{i} + (y+5)\vec{j}) \cdot d\vec{r} = \int_0^3 (y+5)\, dy = \left(\frac{y^2}{2} + 5y\right)\Bigg|_0^3 = \frac{39}{2}.$$

5. Only the $\vec{i}$-component contributes to the line integral, so $d\vec{r} = \vec{i}\, dx$ and

$$\int_C (2x\vec{i} + 3y\vec{j}) \cdot d\vec{r} = \int_{(1,0,0)}^{(5,0,0)} (2x\vec{i} + 3y\vec{j}) \cdot \vec{i}\, dx = \int_1^5 2x\, dx = x^2\Big|_1^5 = 24.$$

6. We will find the line integral from $(0,0)$ to $(3,1)$ and then take the negative. The line segment is parameterized by

$$x = 3t \quad y = t, \qquad \text{for } 0 \le t \le 1.$$

Then $\vec{r}\,'(t) = 3\vec{i} + \vec{j}$, so

$$\int_C (2y^2\vec{i} + x\vec{j}) \cdot d\vec{r} = -\int_0^1 \left(2t^2\vec{i} + 3t\vec{j}\right) \cdot (3\vec{i} + \vec{j})\, dt = -\int_0^1 (6t^2 + 3t)\, dt = -\left(2t^3 + \frac{3}{2}t^2\right)\Bigg|_0^1 = -\frac{7}{2}.$$

7. The semicircle has radius 1 and is centered at $(2,0)$. It can be parameterized by

$$x = 2 + \cos t \quad y = \sin t, \qquad \text{for } 0 \le t \le \pi.$$

Then $\vec{r}\,'(t) = -\sin t\vec{i} + \cos t\vec{j}$, so

$$\begin{aligned}\int_C (x\vec{i} + y\vec{j}) \cdot d\vec{r} &= \int_0^\pi \left((2+\cos t)\vec{i} + \sin t\vec{j}\right) \cdot (-\sin t\vec{i} + \cos t\vec{j})\, dt \\ &= \int_0^\pi (-2\sin t - \cos t \sin t + \sin t \cos t)\, dt = 2\cos t\Big|_0^\pi = -4.\end{aligned}$$

8. Since $\vec{F} = (x^2+y)\vec{i} + y^3\vec{j}$, the line integral along the third segment, which is parallel to the z-axis, is zero. On the first segment, which is parallel to the y-axis, only the $\vec{j}$-component contributes. On the second segment, which is parallel to the x-axis, only the $\vec{i}$-component contributes. On the first segment $x = 4$ and y varies from 0 to 3; on the second segment $y = 3$ and x varies from 4 to 0. Thus, we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^3 ((4^2+y)\vec{i} + y^3\vec{j}) \cdot \vec{j}\, dy + \int_4^0 ((x^2+3)\vec{i} + 3^3\vec{j}) \cdot \vec{i}\, dx \\ &= \int_0^3 y^3\, dy + \int_4^0 (x^2+3)\, dx = \frac{y^4}{4}\Bigg|_0^3 - \left(\frac{x^3}{3} + 3x\right)\Bigg|_0^4 = \frac{81}{4} - \frac{64}{3} - 12 = -\frac{157}{12}.\end{aligned}$$

9. Only the $\vec{i}$ component of $\vec{F}$ contributes to the line integral. Since C goes a distance of 3 in the $-\vec{i}$ direction, we have

$$\int_C \vec{F} \cdot d\vec{r} = (2\vec{i}) \cdot (-3\vec{i}) = -6.$$

10. Parameterizing C by $x(t) = t, y(t) = t$ for $1 \le t \le 5$, we have $\vec{r}\,'(t) = \vec{i} + \vec{j}$,

$$\int \vec{F} \cdot d\vec{r} = \int_1^5 (3\vec{j} - \vec{i}) \cdot (\vec{i} + \vec{j})\, dt = \int_1^5 2\, dt = 8.$$

11. The curve C is parameterized by $(x, y) = (t, t)$ for $0 \le t \le 3$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 (t\vec{i} + t\vec{j}) \cdot (\vec{i} + \vec{j})dt = \int_0^3 2tdt = t^2\bigg|_0^3 = 9.$$

12. Parameterize the curve: $\vec{r}(t) = \sin t\vec{i} + \cos t\vec{j}$, $0 \le t \le \pi$. Then

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^\pi \left(\cos t\vec{i} - \sin t\vec{j}\right) \cdot \left(\cos t\vec{i} - \sin t\vec{j}\right)\, dt \\ &= \int_0^\pi \left((\cos t)^2 + (-\sin t)^2\right)\, dt = \int_0^\pi 1\, dt = \pi.\end{aligned}$$

13. The line can be parameterized by $(1 + 2t, 2 + 2t)$, for $0 \le t \le 1$, so the integral looks like

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(1 + 2t, 2 + 2t) \cdot (2\vec{i} + 2\vec{j})\, dt \\ &= \int_0^1 [(1 + 2t)^2\vec{i} + (2 + 2t)^2\vec{j}] \cdot (2\vec{i} + 2\vec{j})\, dt \\ &= \int_0^1 2(1 + 4t + 4t^2) + 2(4 + 8t + 4t^2)\, dt \\ &= \int_0^1 (10 + 24t + 16t^2)\, dt \\ &= (10t + 12t^2 + 16t^3/3)\big|_0^1 \\ &= 10 + 12 + 16/3 - (0 + 0 + 0) = 82/3\end{aligned}$$

14. Use $x(t) = t$, $y(t) = t^2$, so $x'(t) = 1$, $y'(t) = 2t$, with $0 \le t \le 2$. Then

$$\begin{aligned}\int \vec{F} \cdot d\vec{r} &= \int_0^2 (-t^2 \sin t\vec{i} + \cos t\vec{j}) \cdot (\vec{i} + 2t\vec{j})\, dt \\ &= \int_0^2 (-t^2 \sin t + 2t \cos t)\, dt = t^2 \cos t\big|_0^2 = 4 \cos 2.\end{aligned}$$

15. Parameterizing C by $x(t) = 3t, y(t) = 2t$ for $0 \le t \le 1$, we have $\vec{r}\,'(t) = 3\vec{i} + 2\vec{j}$, so

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 ((2t)^3\vec{i} + (3t)^2\vec{j}) \cdot (3\vec{i} + 2\vec{j})\, dt \\ &= \int_0^1 (24t^3 + 18t^2)\, dt = 6t^4 + 6t^3\bigg|_0^1 = 12.\end{aligned}$$

16. The curve C is parameterized by

$$\vec{r} = \cos t\vec{i} + \sin t\vec{j}, \qquad \text{for } 0 \le t \le 2\pi,$$

so,

$$\vec{r}\,'(t) = -\sin t\vec{i} + \cos t\vec{j}.$$

Thus,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (2\sin t\vec{i} - \sin(\sin t)\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})dt \\ &= \int_0^{2\pi} (-2\sin^2 t - \sin(\sin t)\cos t)dt \\ &= \sin t \cos t - t + \cos(\sin t)\bigg|_0^{2\pi} \\ &= -2\pi.\end{aligned}$$

17. The parameterization is given, so

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_2^4 \vec{F}(2t,t^3)\cdot(2\vec{i}+3t^2\vec{j})\,dt\\ &= \int_2^4 [(\ln(t^3)\vec{i}+\ln(2t)\vec{j}]\cdot(2\vec{i}+3t^2\vec{j})\,dt\\ &= \int_2^4 (2\ln(t^3)+3t^2\ln(2t))\,dt\\ &= \int_2^4 (6\ln(t)+3t^2\ln(2t))\,dt \qquad \text{since } \ln(t^3)=3\ln(t).\end{aligned}$$

This integral can be computed numerically, or using integration by parts or the integral table, giving

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_2^4 (6\ln(t)+3t^2\ln(2t))\,dt\\ &= (6(t\ln(t)-t)+t^3\ln(2t)-t^3/3)\Big|_2^4\\ &= 240\ln 2-\frac{136}{3}-(28\ln 2-\frac{44}{3})\\ &= 212\ln 2-92/3\approx 116.28.\end{aligned}$$

The expression containing $\ln 2$ was obtained using the properties of the natural log.

18. Parameterizing C by $x(t)=t, y(t)=t, z(t)=t$ for $0\le t\le 2$, we have $\vec{r}\,'(t)=\vec{i}+\vec{j}+\vec{k}$, so

$$\begin{aligned}\int_C (x\vec{i}+6\vec{j}-\vec{k})\cdot d\vec{r} &= \int_0^2 (t\vec{i}+6\vec{j}-\vec{k})\cdot(\vec{i}+\vec{j}+\vec{k})\,dt\\ &= \int_0^2 (t+6-1)\,dt = \frac{t^2}{2}+5t\Big|_0^2 = 12.\end{aligned}$$

19. The triangle C consists of the three paths shown in Figure 18.15.

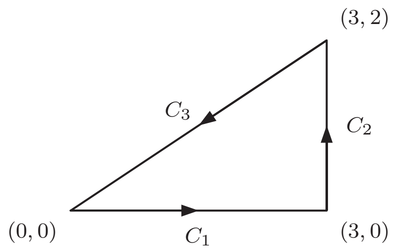


Figure 18.15

Write $C=C_1+C_2+C_3$ where C_1, C_2, and C_3 are parameterized by

$$C_1:(t,0) \text{ for } 0\le t\le 3;\quad C_2:(3,t)\text{ for } 0\le t\le 2;\quad C_3:(3-3t,2-2t)\text{ for } 0\le t\le 1.$$

Then

$$\int_C \vec{F}\cdot d\vec{r}=\int_{C_1}\vec{F}\cdot d\vec{r}+\int_{C_2}\vec{F}\cdot d\vec{r}+\int_{C_3}\vec{F}\cdot d\vec{r}$$

where

$$\int_{C_1}\vec{F}\cdot d\vec{r}=\int_0^3 \vec{F}(t,0)\cdot\vec{i}\,dt=\int_0^3(2t+4)dt=(t^2+4t)\Big|_0^3=21$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^2 \vec{F}(3,t) \cdot \vec{j}\, dt = \int_0^2 (5t+3)dt = (5t^2/2+3t)\Big|_0^2 = 16$$

$$\begin{aligned}\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(3-3t, 2-2t) \cdot (-3\vec{i} - 2\vec{j})dt \\ &= \int_0^1 ((-4t+8)\vec{i} + (-19t+13)\vec{j}) \cdot (-3\vec{i} - 2\vec{j})dt \\ &= 50\int_0^1 (t-1)dt = -25.\end{aligned}$$

So

$$\int_C \vec{F}\, d\vec{r} = 21 + 16 - 25 = 12.$$

20. Since $\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, for $1 \le t \le 2$,
we have $\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$. Then

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_1^2 (t\vec{i} + 2t^3t^2\vec{j} + t\vec{k}) \cdot (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\, dt \\ &= \int_1^2 (t + 4t^6 + 3t^3)\, dt \\ &= \frac{t^2}{2} + \frac{4t^7}{7} + \frac{3t^4}{4}\Big|_1^2 = \frac{2389}{28} \approx 85.32\end{aligned}$$

21. We parameterize C by

$$\vec{r} = 2t\vec{i} + 3t\vec{j} + 4t\vec{k}, \qquad \text{for } 0 \le t \le 1.$$

Then $\vec{r}\,'(t) = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and so

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left((2t)^3\vec{i} + (3t)^2\vec{j} + (4t)\vec{k}\right) \cdot (2\vec{i} + 3\vec{j} + 4\vec{k})dt \\ &= \int_0^1 (16t^3 + 27t^2 + 16t)dt \\ &= 4t^4 + 9t^3 + 8t^2\Big|_0^1 = 21.\end{aligned}$$

22. Since C is given by $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$, we have $\vec{r}\,'(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$. Thus,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{4\pi} (-\sin t\vec{i} + \cos t\vec{j} + 5\vec{k}) \cdot (-\sin t\vec{i} + \cos t\vec{j} + \vec{k})dt \\ &= \int_0^{4\pi} (\sin^2 t + \cos^2 t + 5)dt = \int_0^{4\pi} 6dt = 24\pi.\end{aligned}$$

23. The first step is to parameterize C by

$$(x(t), y(t), z(t)) = (0, 2\cos t, -2\sin t), \quad 0 \le t \le 2\pi.$$

Thus, we have

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} = -2\sin t\vec{j} - 2\cos t\vec{k}.$$

So we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (e^{2\cos t}\vec{i} + \vec{k}) \cdot ((-2\sin t)\vec{j} + (-2\cos t)\vec{k})dt$$

$$= \int_0^{2\pi} -2\cos t dt$$
$$= -2\sin t\Big|_0^{2\pi}$$
$$= 0$$

24. $\int_C 3xdx - y\sin xdy$

25. $\int_C y^2dx + z^2dy + (x^2-5)dz$

26. $\vec{F} = (x+2y)\vec{i} + x^2y\vec{j}$

27. $\vec{F} = e^{-3y}\vec{i} - yz(\sin x)\vec{j} + (y+z)\vec{k}$

28. From $x = t^2$ and $y = t^3$ we get $dx = 2tdt$ and $dy = 3t^2dt$. Hence

$$\int_C ydx + xdy = \int_1^5 t^3(2t)dt + t^2(3t^2)dt = \int_1^5 5t^4dt = 5^5 - 1 = 3124.$$

29. From

$$x = \cos t, \quad y = \sin t, \quad z = 3t$$

we get

$$dx = -\sin t\, dt \qquad dy = \cos t\, dt, \qquad dz = 3dt.$$

Hence

$$\int_C dx + ydy + zdz = \int_0^{2\pi} -\sin t\, dt + \sin t\cos t\, dt + 3t(3dt)$$
$$= \cos t + \frac{1}{2}\sin^2 t + \frac{9}{2}t^2|_0^{2\pi} = 18\pi^2.$$

30. Parameterize C:

$$x = 1+4t, \qquad y = 3+6t, \qquad 0 \le t \le 1$$

so that $dx = 4dt$ and $dy = 6dt$. Hence

$$\int_C 3ydx + 4xdy = \int_0^1 3(3+6t)4dt + 4(1+4t)6dt = \int_0^1 60 + 168tdt = 144.$$

31. Parameterize C:

$$x = 0, \qquad y = 3\cos t, \qquad z = 3\sin t, \qquad 0 \le t \le 2\pi$$

so that

$$dx = 0dt, \qquad dy = -3\sin t\, dt, \qquad dz = 3\cos t\, dt.$$

Hence

$$\int_C xdx + zdy - ydz = \int_0^{2\pi} 0dt + 3\sin t(-3\sin t)dt - 3\cos t(3\cos t)dt = \int_0^{2\pi} -9dt = -18\pi.$$

Problems

32. (a) Since $\vec{r}(t) = t\vec{i} + t^2\vec{j}$, we have $\vec{r}\,'(t) = \vec{i} + 2t\vec{j}$. Thus,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(t,t^2)\cdot(\vec{i} + 2t\vec{j})\, dt$$
$$= \int_0^1 [(3t-t^2)\vec{i} + t\vec{j}]\cdot(\vec{i} + 2t\vec{j})\, dt$$

$$= \int_0^1 (3t + t^2)\, dt$$
$$= (\frac{3t^2}{2} + \frac{t^3}{3})\Big|_0^1 = \frac{3}{2} + \frac{1}{3} - (0+0) = \frac{11}{6}$$

(b) Since $\vec{r}(t) = t^2\vec{i} + t\vec{j}$, we have $\vec{r}\,'(t) = 2t\vec{i} + \vec{j}$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t^2, t) \cdot (2t\vec{i} + \vec{j})\, dt$$
$$= \int_0^1 [(3t^2 - t)\vec{i} + t^2\vec{j}] \cdot (2t\vec{i} + \vec{j})\, dt$$
$$= \int_0^1 (6t^3 - t^2)\, dt$$
$$= (\frac{3t^4}{2} - \frac{t^3}{3})\Big|_0^1$$
$$= \frac{3}{2} - \frac{1}{3} - (0-0) = \frac{7}{6}$$

33. (a) Figure 18.16 shows the curves.

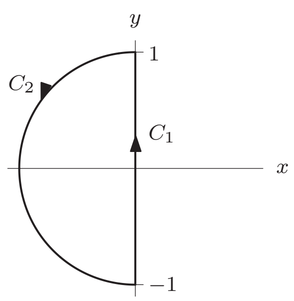


Figure 18.16

(b) On C_1, only the $\vec{j}$-component of $\vec{F}$ contributes to the integral. There $d\vec{r} = \vec{j}\, dy$, so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-1}^1 y\vec{j} \cdot \vec{j}\, dy = \int_{-1}^1 y\, dy = \frac{y^2}{2}\Big|_{-1}^1 = 0.$$

On C_2, we have $\vec{r}\,'(t) = -\sin t\vec{i} + \cos t\vec{j}$, so

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{\pi/2}^{3\pi/2} ((\cos t + 3\sin t)\vec{i} + \sin t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})\, dt$$
$$= \int_{\pi/2}^{3\pi/2} -\cos t \sin t - 3\sin^2 t + \cos t \sin t\, dt = \int_{\pi/2}^{3\pi/2} -3\sin^2 t\, dt$$
$$= -3\left(\frac{t}{2} - \frac{\sin t \cos t}{2}\right)\Big|_{\pi/2}^{3\pi/2} = -\frac{3\pi}{2}.$$

34. First, check that each of these gives a parameterization of L: each has both coordinates equal (as do all points on L) and each begins at $(0,0)$ and ends at $(1,1)$. Now we calculate the line integral of the vector field $\vec{F} = (3x - y)\vec{i} + x\vec{j}$ using each parameterization.

(a) Using $B(t)$ gives

$$\int_L \vec{F} \cdot d\vec{r} = \int_0^{1/2} ((6t - 2t)\vec{i} + 2t\vec{j}) \cdot (2\vec{i} + 2\vec{j})\, dt = \int_0^{1/2} 12t\, dt = 6t^2\Big|_0^{1/2} = \frac{3}{2}.$$

(b) Now we use $C(t)$:

$$\begin{aligned}\int_L \vec{F}\cdot d\vec{r} &= \int_1^2 \left(\left(\frac{3(t^2-1)}{3} - \frac{(t^2-1)}{3}\right)\vec{i} + \frac{t^2-1}{3}\vec{j}\right)\cdot\left(\frac{2t}{3}\vec{i} + \frac{2t}{3}\vec{j}\right)dt \\ &= \int_1^2 \frac{2t}{3}(t^2-1)\,dt = \frac{2}{3}\int_1^2 (t^3-t)\,dt \\ &= \frac{2}{3}\left(\frac{t^4}{4} - \frac{t^2}{2}\right)\Bigg|_1^2 = \frac{3}{2}.\end{aligned}$$

35. (a) The unit circle centered at the origin has equation $x^2+y^2=1$. At any point in the plane, the magnitude of $\vec{F}$ is given by $\|\vec{F}\| = \sqrt{(-y)^2+x^2}$. Along the unit circle, $\|\vec{F}\| = 1$.

(b) Suppose $\vec{r} = x\vec{i} + y\vec{j}$ is a radius vector to a point (x, y) on the unit circle centered at the origin. See Figure 18.17. Then

$$\vec{r}\cdot\vec{F} = (x\vec{i} + y\vec{j})\cdot(-y\vec{i} + x\vec{j}) = -xy + xy = 0.$$

So the vector field is perpendicular to any corresponding radius vector, that is, the vector field is tangent to the circle at every point.

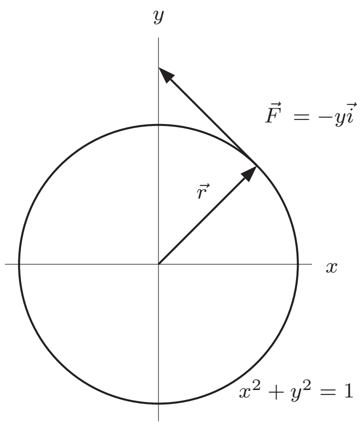


Figure 18.17

(c) We can parameterize C by $(\cos t, \sin t)$, for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\cos t, \sin t)\cdot(-\sin t\vec{i} + \cos t\vec{j})\,dt \\ &= \int_0^{2\pi} (-\sin t\vec{i} + \cos t\vec{j})\cdot(-\sin t\vec{i} + \cos t\vec{j})\,dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt \\ &= \int_0^{2\pi} 1\,dt \\ &= 2\pi\end{aligned}$$

Thus,

$$\int_C \vec{F}\cdot d\vec{r} = 2\pi = \text{Circumference of the unit circle.}$$

36. We parameterize the helical staircase by observing that

$$x = 5\cos t, \quad y = 5\sin t, \quad z = t$$

has the correct radius, but climbs 2π in one revolution. To make it climb 4 meters in one revolution, we write:

$$x = 5\cos t, \quad y = 5\sin t, \quad z = \frac{4t}{2\pi} = \frac{2t}{\pi}.$$

Thus,

$$\vec{r}\,'(t) = -5\sin t\vec{i} + 5\cos t\vec{j} + \frac{2}{\pi}\vec{k}.$$

The gravitational force is given by $\vec{F} = -70g\vec{k}$, and we want to go around 2 turns of the staircase, so we take $0 \le t \le 4\pi$. Thus,

$$\text{Work done by gravity} = \int \vec{F}\cdot d\vec{r} = \int_0^{4\pi} -70g\vec{k}\cdot(-5\sin t\vec{i} + 5\cos t\vec{j} + \frac{2}{\pi}\vec{k})dt$$
$$= \int_0^{4\pi} -\frac{140g}{\pi}dt = -\frac{140g}{\pi}t\Big|_0^{4\pi} = -560g.$$

Notice that the result can also be obtained by multiplying the force by the vertical distance:

$$\text{Gravitational force}\cdot\text{Vertical distance moved} = (-70g)8 = -560g.$$

Now

$$\text{Work done by person} = -\text{Work done by gravity} = 560g.$$

37. The original integral is along the line from $(1,0,0)$ to $(2,2,3)$.

(a) This integral uses the same parameterization, but starting at $(2,2,3)$ and ending at $(1,0,0)$. Thus, the value of the integral is -5.

(b) This integral is along the same line segment from $(1,0,0)$ to $(2,2,3)$ but using the parameterization $x = t^2+1$, $y = 2t^3$, $z = 3t^2$. Thus, the value of the integral is the same, 5.

(c) Using the parameterization in part(b), this integral traverses the line segment twice, from $(2,2,3)$ to $(1,0,0)$ and then back to $(2,2,3)$. Since the segment is traversed once in each direction, the value of the integral is 0.

38. The integral corresponding to $A(t) = (t,t)$ is

$$\int_0^1 3t\,dt.$$

The integral corresponding to $B(t) = (2t,2t)$ is

$$\int_0^{1/2} 12t\,dt.$$

The substitution $s = 2t$ has $ds = 2\,dt$ and $s = 0$ when $t = 0$ and $s = 1$ when $t = 1/2$. Thus, substituting $t = \frac{s}{2}$ into the integral corresponding to $B(t)$ gives

$$\int_0^{1/2} 12t\,dt = \int_0^1 12(\frac{s}{2})(\frac{1}{2}\,ds) = \int_0^1 3s\,ds.$$

The integral on the right-hand side is now the same as the integral corresponding to $A(t)$. Therefore we have

$$\int_0^{1/2} 12t\,dt = \int_0^1 3s\,ds = \int_0^1 3t\,dt.$$

Alternatively, a similar calculation shows that the substitution $t = 2w$ converts the integral corresponding to $A(t)$ into the integral corresponding to $B(t)$.

39. The integral corresponding to $A(t) = (t,t)$ is

$$\int_0^1 3t\,dt.$$

The integral corresponding to $C(t) = (\frac{t^2-1}{3}, \frac{t^2-1}{3})$ is

$$\frac{2}{3}\int_1^2 (t^3 - t)\,dt.$$

The substitution $s = \dfrac{t^2-1}{3}$ has $ds = \dfrac{2}{3}t\,dt$. Also $s=0$ when $t=1$ and $s=1$ when $t=2$. Thus, substituting into the integral corresponding to $C(t)$ gives

$$\frac{2}{3}\int_1^2 (t^3-t)\,dt = \int_1^2 (t^2-1)\frac{2}{3}t\,dt = \int_0^1 3s\,ds.$$

The integral on the right-hand side is the same as the integral corresponding to $A(t)$. Therefore we have

$$\frac{2}{3}\int_1^2 (t^3-t)\,dt = \int_0^1 3s\,ds = \int_0^1 3t\,dt.$$

Alternatively, the substitution $t = \dfrac{w^2-1}{3}$ converts the integral corresponding to $A(t)$ into the integral corresponding to $C(t)$.

40. The integral corresponding to $A(t)=(t,t)$ is

$$\int_0^1 3t\,dt.$$

The integral corresponding to $D(t)=(e^t-1, e^t-1)$ is

$$3\int_0^{\ln 2} (e^{2t}-e^t)\,dt.$$

The substitution $s=e^t-1$ has $ds=e^t\,dt$. Also $s=0$ when $t=0$ and $s=1$ when $t=\ln 2$. Thus, substituting into the integral corresponding to $D(t)$ and using the fact that $e^{2t}=e^t\cdot e^t$ gives

$$3\int_0^{\ln 2} (e^{2t}-e^t)\,dt = 3\int_0^{\ln 2} (e^t-1)e^t\,dt = \int_0^1 3s\,ds.$$

The integral on the right-hand side is the same as the integral corresponding to $A(t)$. Therefore we have

$$3\int_0^{\ln 2} (e^{2t}-e^t)\,dt = \int_0^1 3s\,ds = \int_0^1 3t\,dt.$$

Alternatively, the substitution $t=e^w-1$ converts the integral corresponding to $A(t)$ into the integral corresponding to $B(t)$.

41. (a) The line integral $\int_C (xy\vec{i} + x\vec{j}\,)\cdot d\vec{r}$ is positive. This follows from the fact that all of the vectors of $xy\vec{i} + x\vec{j}$ at points along C point approximately in the same direction as C (meaning the angles between the vectors and the direction of C are less than $\pi/2$).

(b) Using the parameterization $x(t)=t$, $y(t)=3t$, with $x'(t)=1$, $y'(t)=3$, we have

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^4 \vec{F}(t,3t)\cdot(\vec{i}+3\vec{j}\,)\,dt \\ &= \int_0^4 (3t^2\vec{i}+t\vec{j}\,)\cdot(\vec{i}+3\vec{j}\,)\,dt \\ &= \int_0^4 (3t^2+3t)\,dt \\ &= \left(t^3+\frac{3}{2}t^2\right)\Bigg|_0^4 \\ &= 88.\end{aligned}$$

(c) Figure 18.18 shows the oriented path C', with the "turn around" points P and Q. The particle first travels from the origin to the point P (call this path C_1), then backs up from P to Q (call this path C_2), then goes from Q to the point $(4,12)$ in the original direction (call this path C_3). See Figure 18.19. Thus, $C'=C_1+C_2+C_3$. Along the parts of C_1 and C_2 that overlap, the line integrals cancel, so we are left with the line integral over the part of C_1 that does not overlap with C_2, followed by the line integral over C_3. Thus, the line integral over C' is the same as the line integral over the direct route from the point $(0,0)$ to the point $(4,12)$.

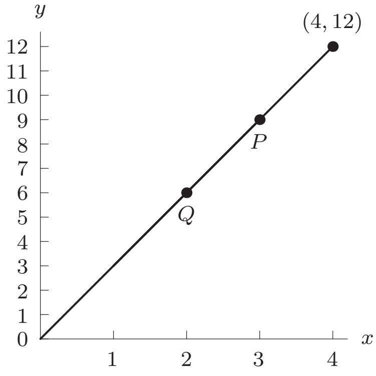


Figure 18.18

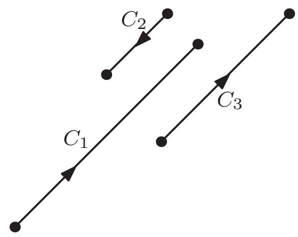


Figure 18.19

(d) The parameterization

$$(x(t), y(t)) = \left(\frac{1}{3}(t^3 - 6t^2 + 11t), (t^3 - 6t^2 + 11t)\right)$$

has $(x(0), y(0)) = (0, 0)$ and $(x(4), y(4)) = (4, 12)$. The form of the parameterization we were given shows that the second coordinate is always three times the first. Thus all points on the parameterized curve lie on the line $y = 3x$.

We have to do a bit more work to guarantee that all points on the curve lie on the line *between* the point $(0, 0)$ and the point $(4, 12)$; it is possible that they might shoot off to, say, $(100, 300)$ before returning to $(4, 12)$. Let's investigate the maximum and minimum values of $f(t) = t^3 - 6t^2 + 11t$ on the interval $0 \le t \le 4$. We can do this on a graphing calculator or computer, or use single-variable calculus. We already know the values of f at the endpoints, namely 0 and 12. We'll look for local extrema:

$$0 = f'(t) = 3t^2 - 12t + 11$$

which has roots at $t = 2 \pm \frac{1}{\sqrt{3}}$. These are the values of t where the particle changes direction: $t = 2 - \frac{1}{\sqrt{3}}$ corresponds to point P and $t = 2 + \frac{1}{\sqrt{3}}$ corresponds to point Q of C'. At these values of t we have $f(2 - \frac{1}{\sqrt{3}}) \approx 6.4$, and $f(2 + \frac{1}{\sqrt{3}}) \approx 5.6$. The fact that these values are between 0 and 12 shows that f takes on its maximum and minimum values at the endpoints of the interval and not in between.

(e) Using the parameterization given in part (d), we have

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} = \frac{1}{3}(3t^2 - 12t + 11)\vec{i} + (3t^2 - 12t + 11)\vec{j}\,.$$

Thus,

$$\begin{aligned}
&\int_{C'} \vec{F} \cdot d\vec{r} \\
&= \int_0^4 \vec{F}\,(\frac{1}{3}(t^3 - 6t^2 + 11t), t^3 - 6t^2 + 11t) \cdot (\frac{1}{3}(3t^2 - 12t + 11)\vec{i} + (3t^2 - 12t + 11)\vec{j}\,)\,dt \\
&= \int_0^4 (\frac{1}{3}(t^3 - 6t^2 + 11t)^2\vec{i} + \frac{1}{3}(t^3 - 6t^2 + 11t)\vec{j}\,) \cdot (\frac{1}{3}(3t^2 - 12t + 11)\vec{i} + (3t^2 - 12t + 11)\vec{j}\,)\,dt \\
&= \int_0^4 \frac{1}{3}(t^3 - 6t^2 + 11t)(3t^2 - 12t + 11)\left\{((t^3 - 6t^2 + 11t)\vec{i} + \vec{j}\,) \cdot (\frac{1}{3}\vec{i} + \vec{j}\,)\right\}\,dt \\
&= \int_0^4 \frac{1}{3}(t^3 - 6t^2 + 11t)(3t^2 - 12t + 11)\left\{\frac{1}{3}(t^3 - 6t^2 + 11t) + 1\right\}\,dt \\
&= \frac{1}{9}\int_0^4 (t^3 - 6t^2 + 11t)(3t^2 - 12t + 11)(t^3 - 6t^2 + 11t + 3)\,dt
\end{aligned}$$

Numerical integration yields an answer of 88, which agrees with the answer found in part b).

Strengthen Your Understanding

42. The substitution of the parameterization into the vector field $\vec{F}$ was done correctly, but instead of computing $d\vec{r} = \vec{r}\,'(t)dt$, the second integral used $\vec{r}(t)\,dt$. Since $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}$, we have $\vec{r}\,'(t)dt = -\sin t\vec{i} + \cos t\vec{j}$, so the correct parameterized line integral is

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} (\cos t\vec{i} - \sin t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})\,dt.$$

43. The sign and value of $\int_C 3\,dx + 4\,dy$ depend on the oriented curve C. For example, if moving along C in the direction of its orientation increases x and y, then the integral is positive, and if it decreases x and y, then the integral is negative.

If the curve is in the direction of the vector $3\vec{i} + 4\vec{j}$, then the integral is positive, but if the curve is in the opposite direction, then the integral is negative.

44. The path is a semicircle of radius 3 centered at the origin, so a vector field that points away from the origin will be perpendicular to the path at all points, and the integral will be zero. For example, if $\vec{F}(x, y) = x\vec{i} + y\vec{j}$, then $\int_C \vec{F} \cdot d\vec{r} = 0$.

45. The vector field points parallel to the x-axis at every point and is constant along any horizontal line, since the magnitude only depends on y. The parameterized path $y = \pi/2$, $x = t$, $0 \le t \le 3$, has length 3, and at every point on the path the vector field points in the direction of the path and has magnitude $\|\sin y\vec{i}\| = \sin(\pi/2) = 1$. So $\int_C \vec{F} \cdot d\vec{r} = 3 \cdot 1 = 3$.

46. False. The relation between these two line integrals depends on the behavior of the vector field along each of the curves, so there is no reason to expect one to be larger than the other. If, for example, the line integral along C_2 is negative, then the line integral along both taken together ($C_1 + C_2$) will be less than the line integral over C_1 by itself. A specific example is given by $\vec{F} = \vec{i}$, with C_1 the line from $(0, 0)$ to $(1, 0)$, and C_2 the line from $(1, 0)$ to $(0, 1)$. Then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{i} \cdot \vec{i}\ dt = 1/2$, and $\int_{C_1+C_2} \vec{F} \cdot d\vec{r} = 1/2 + \int_0^1 \vec{i} \cdot (-\vec{i} + \vec{j})\,dt = 1/2 - 1 = -1/2$.

47. True. The dot product of the integrand $4\vec{i}$ with $\vec{r}\,'(t) = \vec{i} + 2t\vec{j}$ is 4, so the integral has value $\int_0^2 4\,dt = 8$.

48. False. The curves C_1 and C_2 are different. The curve C_1 starts at the point $(1, 0)$ and travels around the unit circle counterclockwise to $(-1, 0)$. The curve C_2 starts at the point $(1, 0)$ and travels around the unit circle *clockwise* to $(-1, 0)$.

49. True. The curves C_1 and C_2 both parameterize the upper unit semicircle with the same orientation (but at different speeds). Since the line integral is independent of parameterization, the integrals over C_1 and C_2 are the same.

50. False. As a counterexample, consider the unit circle C, centered at the origin, oriented counterclockwise and the vector field $\vec{F} = -y\vec{i} + x\vec{j}$. The vector field is always tangent to the circle, and in the same direction as C, so the line integral is positive.

51. False. As a counterexample, consider the vector field $\vec{F} = x\vec{i}$. Then if we parameterize C_1 by $\vec{r}(t) = t\vec{i}$, with $0 \le t \le 1$, we get

$$\int_{C_1} x\vec{i} \cdot d\vec{r} = \int_0^1 t\vec{i} \cdot \vec{i}\,dt = \int_0^1 t\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}.$$

A similar computation for C_2 gives a line integral with value 2.

52. True. If we parameterize C by $\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j}$, with $0 \le t \le 2\pi$, then

$$\int_C (2x\vec{i} + y\vec{j}) \cdot d\vec{r} = \int_0^{2\pi} (2a\cos t\vec{i} + a\sin t\vec{j}) \cdot (-a\sin t\vec{i} + a\cos t\vec{j})\,dt = \int_0^{2\pi} -a^2\cos t\sin t\,dt$$
$$= \left.\frac{a^2\cos^2 t}{2}\right|_0^{2\pi} = 0.$$

53. False. If we parameterize C by $\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j}$, with $0 \le t \le 2\pi$, then

$$\int_C (2y\vec{i} + x\vec{j}) \cdot d\vec{r} = \int_0^{2\pi} (2a\sin t\vec{i} + a\cos t\vec{j}) \cdot (-a\sin t\vec{i} + a\cos t\vec{j})\,dt = \int_0^{2\pi} \left(-2a^2\sin^2 t + a^2\cos^2 t\right)\,dt = -\pi a^2.$$

54. True. The curves C_1 and C_2 are the same (they follow the graph of $y = x^2$ between $(0, 0)$ and $(2, 4)$), except that their orientations are opposite.

55. (a). The two parameterizations give the same path, but one goes in the opposite direction to the other. Since $\vec{F}$ points away from the origin, and C_1 is oriented away from the origin, $\int_{C_1} \vec{F} \cdot d\vec{r}$ is positive, so $\int_{C_2} \vec{F} \cdot d\vec{r}$ is negative.

56. (a). The two parameterizations move along the line $y = x$ in the direction away from the origin, but they have different endpoints. The path C_1 goes from $(0, 0)$ to $(1, 1)$ and the path C_2 goes from $(0, 0)$ to $(\sin 1, \sin 1) \approx (0.84, 0.84)$. Since $\vec{F}$ points away from the origin, and both paths are oriented so that the direction of travel is oriented away from the origin, the integral along the longer path is larger, so $\int_{C_1} \vec{F} \cdot d\vec{r} > \int_{C_2} \vec{F} \cdot d\vec{r}$.

Solutions for Section 18.3

Exercises

1. Since $\vec{F}$ is a gradient field, with $\vec{F} = \text{grad}\, f$ where $f(x, y) = x^2 + y^4$, we use the Fundamental Theorem of Line Integrals. The starting point of the path C is $(2, 0)$ and the end is $(0, 2)$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = f(0, 2) - f(2, 0) = 16 - 4 = 12.$$

2. Since, if $f(x, y, z) = \sin(xy) + e^z$, we have $\text{grad} f = \vec{F}$, we use the Fundamental Theorem of Line Integrals. The starting point of the path is $(0, 0, 0)$ and the end is $(\sqrt{2}, \sqrt{5}, 2)$ so

$$\int_C \vec{F} \cdot d\vec{r} = f(\sqrt{2}, \sqrt{5}, 2) - f(0, 0, 0) = \sin\sqrt{10} + e^2 - 1.$$

Notice that since $\vec{F}$ is a gradient field, the intermediate points on the path do not affect the answer.

3. Negative, not path-independent.

4. Zero, path-independent (a constant vector field is path-independent).

5. Negative, not path-independent.

6. Zero, appears path-independent

7. The field appears to be path-independent. If the path between any two points is a line segment, the line inegral will have a value V. Any detour up or down appears to be perpendicular to the field, thus having no effect. Any detour left or right will be compensated for because the vector field appears to take on the same values for all points along any vertical line. Alternatively, the field could be imagined to be the gradient field of a function like $z = x^2$.

8. The field appears to be path-dependent. If we choose two points lying on a line that passes through the center, it appears that the line integral on a straight path between them has value zero, but a line integral on a circular path clockwise has positive value.

9. The field appears to be path-independent, because the vector field appears constant and could thus be the gradient field of a linear function.

10. The field appears to be path-independent. If the path between any two points is a line segment, the line inegral will have a value V. Any detour left or right appears to be perpendicular to the field, thus having no effect. Any detour up or down will be compensated for because the vector field appears to take on the same values for all points along any horizontal line. Alternatively, the field could be imagined to be the gradient field of a function like $z = y^2$.

11. The field appears to be path-independent. If the path between any two points is a line segment, the line inegral will have a value V. Any detour up-left or down-right appears to be perpendicular to the field, thus having no effect. Any detour up-right or down-left will be compensated for because the vector field appears to take on the same values for all points along any line going from top-left to bottom-right on a 45° angle to the image of the field.

12. The field appears to be path-dependent. If we choose two points lying on a line that passes through the center, it appears that the line integral on a straight path between them has value zero, but a line integral on a circular path counterclockwise has positive value.

13. We know that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2,$$

so, integrating with respect to x, thinking of y as a constant gives

$$f(x, y) = x^2 y + C(y).$$

Differentiating with respect to y gives

$$\frac{\partial f}{\partial y} = x^2 + C'(y),$$

so we take $C(y) = k$ for some constant K^2. Thus

$$f(x,y) = x^2y + K.$$

14. We know that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 8y^3$$

Now think of y as a constant in the equation for $\partial f/\partial x$ and integrate, giving

$$f(x,y) = x^2y + C(y).$$

Since the constant of integration may depend on y, it is written $C(y)$. Differentiating this expression for $f(x,y)$ with respect to y and using the fact that $\partial f/\partial y = x^2 + 8y^3$, we get

$$\frac{\partial f}{\partial y} = x^2 + C'(y) = x^2 + 8y^3.$$

Therefore

$$C'(y) = 8y^3 \quad \text{so} \quad C(y) = 2y^4 + K.$$

for some constant K. Thus,

$$f(x,y) = x^2y + 2y^4 + K.$$

15. Integrating

$$\frac{\partial f}{\partial x} = yze^{xyz} + z^2\cos(xz^2)$$

with respect to x and thinking of y and z as constant gives

$$f(x,y,z) = e^{xyz} + \sin(xz^2) + C(y,z).$$

Differentiating with respect to y and using the fact that $\partial f/\partial y = xze^{xyz}$ gives

$$\frac{\partial f}{\partial y} = xze^{xyz} + \frac{\partial C}{\partial y} = xze^{xyz}.$$

Thus, $\partial C/\partial y = 0$. This means C does not depend on y and can be written $C(z)$, giving:

$$f(x,y,z) = e^{xyz} + \sin(xz^2) + C(z).$$

Differentiating with respect to z, we get

$$\frac{\partial f}{\partial z} = xye^{xyz} + 2zx\cos(xz^2) + C'(z).$$

The expression for $\operatorname{grad} f$ tells us that

$$\frac{\partial f}{\partial z} = xye^{xyz} + 2xz\cos(xz^2).$$

Thus, we have $C'(z) = 0$ so $C =$ constant, giving

$$f(x,y,z) = e^{xyz} + \sin(xz^2) + C.$$

16. Since $\vec{F} = \operatorname{grad} f$ is a gradient vector field, the Fundamental Theorem of Line Integrals give us

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start}) = \left(x^2 + 2y^3 + 3z^4\right)\Big|_{(4,0,0)}^{(0,0,5)} = 3 \cdot 5^4 - 4^2 = 1859.$$

17. Since $\vec{F} = 3x^2\vec{i} + 4y^3\vec{j} = \operatorname{grad}(x^3 + y^4)$, we take $f(x,y) = x^3 + y^4$. Then by the Fundamental Theorem of Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(-1,0) - f(1,0) = (-1)^3 - 1^3 = -2.$$

18. Since $\vec{F} = (x+2)\vec{i} + (2y+3)\vec{j} = \text{grad}\left(\frac{x^2}{2} + 2x + y^2 + 3y\right)$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F} \cdot d\vec{r} = \left(\frac{x^2}{2} + 2x + y^2 + 3y\right)\Bigg|_{(1,0)}^{(3,1)} = \left(\frac{9}{2} + 6 + 1 + 3\right) - \left(\frac{1}{2} + 2\right) = 12.$$

19. Since $\vec{F} = 2\sin(2x+y)\vec{i} + \sin(2x+y)\vec{j} = \text{grad}(-\cos(2x+y))$, we take

$$f(x,y) = -\cos(2x+y).$$

Then, using the Fundamental Theorem of Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(0,5\pi) - f(\pi,0) = -\cos(5\pi) - (-\cos(2\pi)) = -(-1) - (-1) = 2.$$

Notice that only the endpoints of the curve affect the answer.

20. Since $\vec{F} = 2x\vec{i} - 4y\vec{j} + (2z-3)\vec{k} = \text{grad}(x^2 - 2y^2 + z^2 - 3z)$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F} \cdot d\vec{r} = (x^2 - 2y^2 + z^2 - 3z)\Bigg|_{(1,1,1)}^{(2,3,-1)} = (4 - 2\cdot 3^2 + (-1)^2 + 3) - (1^2 - 2\cdot 1^2 + 1^2 - 3) = -7.$$

21. Since $\vec{F} = x^{2/3}\vec{i} + e^{7y}\vec{j} = \text{grad}\left(\frac{3}{5}x^{5/3} + \frac{1}{7}e^{7y}\right)$, we see $\vec{F}$ is a gradient vector field. Therefore,

$$\int_C (x^{2/3}\vec{i} + e^{7y}\vec{j}) \cdot d\vec{r} = 0.$$

22. Since $\vec{F} = x^{2/3}\vec{i} + e^{7y}\vec{j} = \text{grad}\left(\frac{3}{5}x^{5/3} + \frac{1}{7}e^{7y}\right)$, we have

$$\begin{aligned}\int_C (x^{2/3}\vec{i} + e^{7y}\vec{j}) \cdot d\vec{r} &= \frac{3}{5}x^{5/3} + \frac{1}{7}e^{7y}\Bigg|_{(1,0)}^{(0,1)} \\ &= \frac{3}{5}\cdot 0^{5/3} + \frac{1}{7}e^{7\cdot 1} - \frac{3}{5}\cdot 1^{5/3} - \frac{1}{7}e^{7\cdot 0} \\ &= \frac{1}{7}(e^7 - 1) - \frac{3}{5}.\end{aligned}$$

23. Since $\vec{F} = \text{grad}(e^{xy} + \sin z)$, we take $f(x,y,z) = e^{xy} + \sin z$ and use the Fundamental Theorem of Line Integrals

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(3,1,9\pi) - f(0,0,\pi) = e^3 + \sin(9\pi) - e^0 - \sin\pi \\ &= e^3 - 1.\end{aligned}$$

24. Since $\vec{F} = y\sin(xy)\vec{i} + x\sin(xy)\vec{j} = \text{grad}(-\cos(xy))$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F} \cdot d\vec{r} = -\cos(xy)\Bigg|_{(1,2)}^{(3,18)} = -\cos(54) + \cos(2) = \cos(2) - \cos(54).$$

25. Since $\vec{F} = 2xy^2ze^{x^2y^2z}\vec{i} + 2x^2yze^{x^2y^2z}\vec{j} + x^2y^2e^{x^2y^2z}\vec{k} = \text{grad}(e^{x^2y^2z})$ and the curve C is closed, the Fundamental Theorem of Line Integrals tells us that $\int_C \vec{F} \cdot d\vec{r} = 0$, since

$$\int_C \vec{F} \cdot d\vec{r} = e^{x^2y^2z}\Bigg|_{(1,0,1)}^{(1,0,1)} = e^0 - e^0 = 0.$$

Problems

26. Since $\vec{v}$ is a gradient field, the end point should be $(5,4)$, the point with largest value of $f = x^2 + y^2$, and the starting point should be $(0,0)$, the point with smallest f.

27. Since $\vec{F}$ is a gradient field, to maximize $\int_C \vec{v} \cdot d\vec{r}$ we need to choose the path with the end point that makes $2x^2+3y^2$ as large as possible and with the starting point that makes $2x^2+3y^2$ as small as possible. Path PQ satisfies these conditions.

28. The vector field is a gradient field since

$$\cos(xy)e^{\sin(xy)}(y\vec{i}+x\vec{j})+\vec{k} = \text{grad}(e^{\sin(xy)}+z),$$

so we use the Fundamental Theorem of Line Integrals:

$$\begin{aligned}\int_C \left(\cos(xy)e^{\sin(xy)}(y\vec{i}+x\vec{j})+\vec{k}\right)\cdot d\vec{r} &= e^{\sin(xy)}+z\Big|_{(\pi,2,5)}^{(0.5,\pi,7)} \\ &= e^{\sin(\pi/2)}+7-e^{\sin 2\pi}-5 \\ &= e+7-1-5 \\ &= e+1.\end{aligned}$$

29. The vector field $\vec{F}$ points radially outward, and so is everywhere perpendicular to A; thus, $\int_A \vec{F}\cdot d\vec{r} = 0$.

Along the first half of B, the terms $\vec{F}\cdot\Delta\vec{r}$ are negative; along the second half the terms $\vec{F}\cdot\Delta\vec{r}$ are positive. By symmetry the positive and negative contributions cancel out, giving a Riemann sum and a line integral of 0.

The line integral is also 0 along C, by cancellation. Here the values of $\vec{F}$ along the x-axis have the same magnitude as those along the y-axis. On the first half of C the path is traversed in the opposite direction to $\vec{F}$; on the second half of C the path is traversed in the same direction as $\vec{F}$. So the two halves cancel.

30. We parameterize A by $x=t, y=t$, where $0\le t\le 1$. Then

$$\begin{aligned}\int_A \vec{F}\cdot d\vec{r} &= \int_0^1 (t\vec{i}+t\vec{j})\cdot(\vec{i}+\vec{j})\,dt \\ &= \int_0^1 2t\,dt = t^2\Big|_0^1 = 1.\end{aligned}$$

The path B has the parameterization $x=t, y=t^2$, where $0\le t\le 1$. Then we have

$$\begin{aligned}\int_B \vec{F}\cdot d\vec{r} &= \int_0^1 (t\vec{i}+t^2\vec{j})\cdot(\vec{i}+2t\vec{j})\,dt \\ &= \int_0^1 (t+2t^3)\,dt = \frac{t^2}{2}+\frac{2t^4}{4}\Big|_0^1 = 1.\end{aligned}$$

We have to break the path C into two separate parameterizations: $x=t, y=0$, where $0\le t\le 1$ and $x=1, y=t$, where $0\le t\le 1$. Then

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 (t\vec{i}\cdot\vec{i})\,dt + \int_0^1 (\vec{i}+t\vec{j})\cdot\vec{j}\,dt \\ &= \int_0^1 t\,dt + \int_0^1 t\,dt = \frac{1}{2}+\frac{1}{2} = 1.\end{aligned}$$

31. Yes. If $f(x,y)=\frac{1}{2}x^2$, then grad $f = x\vec{i}$.

32. Yes. If $f(x,y)=\frac{1}{3}x^3-xy^2$, then grad $f=(x^2-y^2)\vec{i}-2xy\vec{j}$.

33. Yes. Let

$$f(\vec{r}) = -\frac{1}{r} = -(x^2+y^2+z^2)^{-1/2}$$

Then

$$\begin{aligned}\frac{\partial f}{\partial x} &= x(x^2+y^2+z^2)^{-3/2} \\ \frac{\partial f}{\partial y} &= y(x^2+y^2+z^2)^{-3/2} \\ \frac{\partial f}{\partial z} &= z(x^2+y^2+z^2)^{-3/2}\end{aligned}$$

So grad $f = (x^2+y^2+z^2)^{-3/2}(x\vec{i}+y\vec{j}+z\vec{k}) = \vec{r}/r^3$

34. No. Suppose there were a function f such that $\operatorname{grad} f = \vec{F}$. Then we would have

$$\frac{\partial f}{\partial x} = \frac{-z}{\sqrt{x^2+z^2}}.$$

Hence we would have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{-z}{\sqrt{x^2+z^2}}) = 0.$$

In addition, since $\operatorname{grad} f = \vec{F}$, we have that

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+z^2}}.$$

Thus we also know that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = -xy(x^2+z^2)^{-3/2}.$$

Notice that

$$\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}.$$

Since we expect $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, we have got a contradiction. The only way out of this contradiction is to conclude there is no function f with $\operatorname{grad} f = \vec{F}$. Thus $\vec{F}$ is not a gradient vector field.

35. Since $\vec{F}(x,y,z) = \operatorname{grad}(e^{x^2+yz})$, we use the Fundamental Theorem of line integrals

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \operatorname{grad}\left(e^{x^2+yz}\right) \cdot d\vec{r} = e^{x^2+yz}\Big|_{(0,0,0)}^{(3,0,0)} = e^9 - e^0 = e^9 - 1.$$

36. (a) To find the change in f by computing a line integral, we first choose a path C between the points; the simplest is a line. We parameterize the line by $(x(t), y(t)) = (t, \pi t/2)$, with $0 \le t \le 1$. Then $(x'(t), y'(t)) = (1, \pi/2)$, so the Fundamental Theorem of Line Integrals tells us that

$$\begin{aligned}
f(1,\frac{\pi}{2}) - f(0,0) &= \int_C \operatorname{grad} f \cdot d\vec{r} \\
&= \int_0^1 \operatorname{grad} f\left(t, \frac{\pi t}{2}\right) \cdot \left(\vec{i} + \frac{\pi}{2}\vec{j}\right) dt \\
&= \int_0^1 \left(2te^{t^2} \sin\left(\frac{\pi t}{2}\right)\vec{i} + e^{t^2}\cos\left(\frac{\pi t}{2}\right)\vec{j}\right) \cdot \left(\vec{i} + \frac{\pi}{2}\vec{j}\right) dt \\
&= \int_0^1 \left(2te^{t^2} \sin\left(\frac{\pi t}{2}\right) + \frac{\pi e^{t^2}}{2}\cos\left(\frac{\pi t}{2}\right)\right) dt \\
&= \int_0^1 \frac{d}{dt}\left(e^{t^2}\sin\left(\frac{\pi t}{2}\right)\right) dt \\
&= e^{t^2}\sin\left(\frac{\pi t}{2}\right)\Big|_0^1 = e = 2.718.
\end{aligned}$$

This integral can also be approximated numerically.

(b) The other way to find the change in f between these two points is to first find f. To do this, observe that

$$2xe^{x^2}\sin y\vec{i} + e^{x^2}\cos y\vec{j} = \frac{\partial}{\partial x}\left(e^{x^2}\sin y\right)\vec{i} + \frac{\partial}{\partial y}\left(e^{x^2}\sin y\right)\vec{j} = \operatorname{grad}\left(e^{x^2}\sin y\right).$$

So one possibility for f is $f(x,y) = e^{x^2}\sin y$. Thus,

$$\text{Change in } f\Big|_{(0,0)}^{(1,\pi/2)} = e^{x^2}\sin y\Big|_{(0,0)}^{(1,\pi/2)} = e^1 \sin\left(\frac{\pi}{2}\right) - e^0 \sin 0 = e.$$

The exact answer confirms our calculations in part (a) which show that the answer is e.

37. The unit circle cuts the negative x-axis at $(-1,0,0)$, and it cuts the negative y-axis at $(0,-1,0)$. There is a quarter circle between these points if the circle is traversed counterclockwise.

(a) Since $2\pi x\vec{i} + y^2\vec{j} = \text{grad}(\pi x^2 + y^3/3)$, we use the Fundamental Theorem of Line Integrals:

$$\int_C (2\pi x\vec{i} + y^2\vec{j}) \cdot d\vec{r} = \left(\pi x^2 + \frac{y^3}{3}\right)\Bigg|_{(-1,0,0)}^{(0,-1,0)} = \left(\pi(0^2) + \frac{(-1)^3}{3}\right) - \left(\pi(-1)^2 + \frac{0^3}{3}\right) = -\frac{1}{3} - \pi.$$

(b) Since $\vec{F}$ is not a gradient field, we parameterize c. If $x = \cos t$ and $y = \sin t$, then $\pi \le t \le 3\pi/2$ parametrizes C. Thus

$$\begin{aligned}\int_C (-2y\vec{i} + x\vec{j}) \cdot d\vec{r} &= \int_\pi^{3\pi/2} (-2\sin t\vec{i} + \cos t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})\,dt \\ &= \int_\pi^{3\pi/2} (2\sin^2 t + \cos^2 t)\,dt = \int_\pi^{3\pi/2} (1 + \sin^2 t)\,dt \\ &= \left(t + \frac{t}{2} - \frac{\sin t\cos t}{2}\right)\Bigg|_\pi^{3\pi/2} = \frac{3\pi}{4}.\end{aligned}$$

38. Since $\vec{F} = \text{grad}\left(\frac{x^2+y^2}{2}\right)$, the line integral can be calculated using the Fundamental Theorem of Line Integrals:

$$\int_c \vec{F} \cdot d\vec{r} = \frac{x^2+y^2}{2}\Bigg|_{(0,0)}^{(3/\sqrt{2},3/\sqrt{2})} = \frac{9}{2}.$$

39. This vector field is not a gradient field, so we evaluate the line integral directly. Let C_1 be the path along the x-axis from $(0,0)$ to $(3,0)$ and let C_2 be the path from $(3,0)$ to $(3/\sqrt{2}, 3/\sqrt{2})$ along $x^2 + y^2 = 9$. Then

$$\int_C \vec{H} \cdot d\vec{r} = \int_{C_1} \vec{H} \cdot d\vec{r} + \int_{C_2} \vec{H} \cdot d\vec{r}.$$

On C_1, the vector field has only a $\vec{j}$ component (since $y = 0$), and $\vec{H}$ is therefore perpendicular to the path. Thus,

$$\int_{C_1} \vec{H} \cdot d\vec{r} = 0.$$

On C_2, the vector field is tangent to the path. The path is one eighth of a circle of radius 3 and so has length $2\pi(3/8) = 3\pi/4$.

$$\int_{C_2} \vec{H} \cdot d\vec{r} = \|\vec{H}\| \cdot \text{Length of path} = 3 \cdot \left(\frac{3\pi}{4}\right) = \frac{9\pi}{4}.$$

Thus,

$$\int_C \vec{H} \cdot d\vec{r} = \frac{9\pi}{4}.$$

40. Since $\vec{F} = \text{grad}(y\ln(x+1))$, we evaluate the line integral using the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = y\ln(x+1)\Bigg|_{(0,0)}^{(3/\sqrt{2},3/\sqrt{2})} = \frac{3}{\sqrt{2}}\ln\left(\frac{3}{\sqrt{2}} + 1\right) - 0\ln 1 = \frac{3}{\sqrt{2}}\ln\left(\frac{3}{\sqrt{2}} + 1\right).$$

41. Since $\vec{G} = \text{grad}(e^{xy} + \sin(x+y))$, the line integral can be calculated using the Fundamental Theorem of Line Integrals:

$$\int_c \vec{F} \cdot d\vec{r} = e^{xy} + \sin(x+y)\Bigg|_{(0.0)}^{(3/\sqrt{2},3/\sqrt{2})} = e^{9/2} + \sin\left(\frac{6}{\sqrt{2}}\right) - e^0 = e^{9/2} + \sin(3\sqrt{2}) - 1.$$

42. Since $\vec{F} = yz^2e^{xyz^2}\vec{i} + xz^2e^{xyz^2}\vec{j} + 2xyze^{xyz^2}\vec{k} = \text{grad}(e^{xyz^2})$, we can use the Fundamental Theorem of Line Integrals. The start of the path, where $t = 0$, is $(1, 0, 0)$. The end of the path is $(\cos(1.25\pi), \sin(1.25\pi), 1.25\pi) = (-1/\sqrt{2}, -1/\sqrt{2}, 1.25\pi)$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = e^{xyz^2}\Big|_{(1,0,0)}^{(-1/\sqrt{2},-1/\sqrt{2},1.25\pi)} = e^{(1.25\pi)^2/2} - 1.$$

43. Since $\vec{F} = \text{grad}(x^2 + y^2 + z^2)$, by the Fundamental Theorem of Line Integrals we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \text{grad}(x^2 + y^2 + z^2) \cdot d\vec{r} = (x^2 + y^2 + z^2)\Big|_{(0,0,0)}^{(1,5,9)} = 1^2 + 5^2 + 9^2 = 107.$$

Since $\vec{G} = \vec{F} + y\vec{i}$, we have

$$\int_C \vec{G} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C y\vec{i} \cdot d\vec{r} = 107 + \int_C y\vec{i} \cdot d\vec{r}.$$

To find $\int_C y\vec{i} \cdot d\vec{r}$, we parameterize the line by $\vec{r} = t\vec{i} + 5t\vec{j} + 9t\vec{k}$, for $0 \le t \le 1$. We have $\vec{r}\,'(t) = \vec{i} + 5\vec{j} + 9\vec{k}$, so

$$\int_C y\vec{i} \cdot d\vec{r} = \int_0^1 5t\vec{i} \cdot (\vec{i} + 5\vec{j} + 9\vec{k})\,dt = \int_0^1 5t\,dt = \frac{5t^2}{2}\Big|_0^1 = \frac{5}{2}.$$

Thus,

$$\int_C \vec{G} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C y\vec{i} \cdot d\vec{r} = 107 + \frac{5}{2} = 109.5.$$

44. (a) Since $\vec{F} = \text{grad}(ye^x)$, we can use the Fundamental Theorem which says that

$$\int_C \vec{F} \cdot d\vec{r} = ye^x\Big|_{(1,2)}^{(3,7)} = 7e^3 - 2e^1.$$

It does not matter how the curve goes because the Fundamental Theorem gives the line integral in terms of the values of the function $f(x, y) = ye^x$ at the end points only.

(b) The line is given by $\vec{r} = \vec{i} + 2\vec{j} + t(2\vec{i} + 5\vec{j}) = (1 + 2t)\vec{i} + (2 + 5t)\vec{j}$. Thus, $r'(t) = 2\vec{i} + 5\vec{j}$, so

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left((2 + 5t)e^{1+2t}\vec{i} + e^{1+2t}\vec{j}\right) \cdot (2\vec{i} + 5\vec{j})\,dt = \int_0^1 (4 + 10t + 5)e^{1+2t}dt \\ &= \frac{9e^{1+2t}}{2}\Big|_0^1 + 10\int_0^1 te^{1+2t}dt.\end{aligned}$$

Using integration by parts with $u = t, u' = 1, v' = e^{1+2t}, v = e^{1+2t}/2$ for the second integral, we get

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \frac{9}{2}(e^3 - e^1) + 10\left(\frac{te^{1+2t}}{2}\Big|_0^1 - \int_0^1 \frac{e^{1+2t}}{2}dt\right) \\ &= \frac{9}{2}(e^3 - e^1) + 5e^3 - \frac{5e^{1+2t}}{2}\Big|_0^1 = \frac{9}{2}(e^3 - e^1) + 5e^3 - \frac{5}{2}(e^3 - e^1) \\ &= 7e^3 - 2e.\end{aligned}$$

45. Although this curve is complicated, the vector field is a gradient field since

$$\vec{F} = \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\vec{i} - \cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\vec{j} = \text{grad}\left(-2\cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right).$$

Thus, only the endpoints of the curve, P and Q, are needed. Since $P = (-3\pi/2, 3\pi/2)$ and $Q = (-3\pi/2, -3\pi/2)$ and $\vec{F} = \text{grad}(-2\cos(x/2)\sin(y/2))$, we have

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= -2\cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\Bigg|_{P=(-3\pi/2,3\pi/2)}^{Q=(-3\pi/2,-3\pi/2)} \\ &= -2\cos\left(-\frac{3\pi}{4}\right)\sin\left(-\frac{3\pi}{4}\right) + 2\cos\left(-\frac{3\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right) \\ &= 2\cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right) + 2\cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right) \\ &= -2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = -2.\end{aligned}$$

46. (a) By the Fundamental Theorem of Line Integrals

$$\int_{(0,2)}^{(3,4)} \text{grad}\, f\cdot d\vec{r} = f(3,4) - f(0,2) = 66 - 57 = 9.$$

(b) By the Fundamental Theorem of Line Integrals, since C is a closed path, $\int_C \text{grad}\, f\cdot d\vec{r} = 0$.

47. (a) Three possible paths are shown in Figure 18.20.

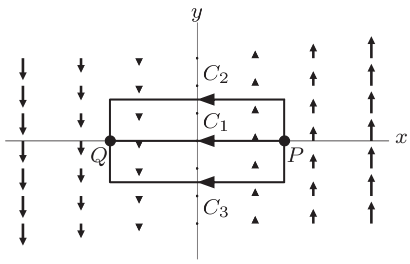


Figure 18.20

Since $\vec{F}$ is perpendicular to the horizontal axis everywhere, $\vec{F}\cdot d\vec{r} = 0$ along C_1.

Since C_2 starts out in the direction of $\vec{F}$, the first leg of C_2 will have a positive line integral. The second horizontal part of C_2 will have a 0 line integral, and the third leg that ends at Q will have a positive line integral. Thus the line integral along C_2 is positive.

A similar argument shows that the line integral along $C_3 < 0$.

(b) No, $\vec{F}$ is not a gradient field, since the line integrals along these three paths joining P and Q do not have the same value.

48. (a) See Figure 18.21.

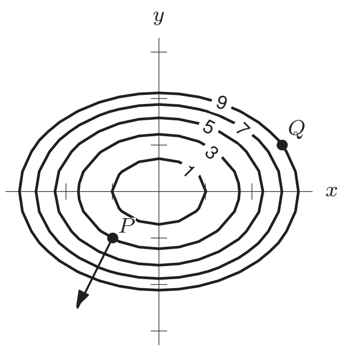


Figure 18.21

(b) Vector at P is shorter than vector at Q.
(c) By the Fundamental Theorem of Line Integrals

$$\int_C \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P) = 9 - 3 = 6.$$

49. Since $\vec{F}$ is a gradient vector field, we use the Fundamental Theorem of line integrals, giving

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \operatorname{grad} f \cdot d\vec{r} = f(\text{end}) - f(\text{start}).$$

(a) The line integral $\int_{C_2} \vec{F} \cdot d\vec{r}$ is 0, since the curve begins and ends on the same contour, so $f(\text{end}) = f(\text{start})$.
(b) Since C_1 crosses more contours than C_4, and since both curves are oriented in the direction of increasing f,

$$0 < \int_{C_4} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r}.$$

Since C_3 goes from higher to lower values of f,

$$\int_{C_3} \vec{F} \cdot d\vec{r} < 0 = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Thus, we have

$$\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r} < \int_{C_4} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r}.$$

(c) Since C_3 and C_4 have endpoints on the same contours, but with start and finish reversed,

$$\int_{C_3} \vec{F} \cdot d\vec{r} = -\int_{C_4} \vec{F} \cdot d\vec{r}.$$

The line integral $\int_{C_3} \vec{F} \cdot d\vec{r}$ is negative because Q_3 is on a contour of lower value than P_3.

50. (a) The integral is positive, because the portion of the path that goes with the vector field is longer than the portion of the path that goes against it, and in addition the vectors are larger in magnitude along the former and smaller in magnitude along the latter.
(b) If it were true that $\vec{F} = \operatorname{grad} f$ for some function f, then the integral around every closed path would be zero. But in part (a) we saw that the integral around one closed path was not zero, so $\vec{F}$ cannot be a gradient vector field.
(c) The region shown is in the first quadrant. In that quadrant, the vectors of $\vec{F_1}$ point away from the origin, so $\vec{F_1}$ does not fit. The vectors of both $\vec{F_2}$ and $\vec{F_3}$ point up and to the left, so they are both possibilities; of these, $\vec{F_2}$ fits best because its vectors get larger in magnitude as you move away from the origin, which fits the diagram. The vectors in $\vec{F}_3$ shrink as you move away from the origin.

51. Since the vector field is path independent, the line integral around the closed curve $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$ is zero. Thus

$$\int_{(0,1)}^{(0,0)} \vec{F} \cdot d\vec{r} = -\left(\int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r} + \int_{(1,1)}^{(0,1)} \vec{F} \cdot d\vec{r}\right) = -(5.1 + 3.2 - 4.7) = -3.6.$$

52. (a) To maximize the line integral, we choose C to be parallel to $\operatorname{grad} f = 3\vec{i} + 4\vec{j}$. Thus C has parametric equation $\vec{r} = (2\vec{i} + \vec{j}) + t\vec{v}$ where $\vec{v} = 3\vec{i} + 4\vec{j}$, so

$$x = 2 + 3t \quad y = 1 + 4t.$$

If the other end of C is at (x_1, y_1), since the length of C is 10, we have

$$\begin{aligned}
\sqrt{(x_1-2)^2 + (y_1-1)^2} &= 10 \\
\sqrt{(3t)^2 + (4t)^2} &= 10 \\
t\sqrt{3^2 + 4^2} &= 10 \\
5t &= 10 \\
t &= 2.
\end{aligned}$$

Thus $t = 2$ at (x_1, y_1), so

$$x_1 = 2 + 2 \cdot 3 = 8 \quad \text{and} \quad y_1 = 1 + 2 \cdot 4 = 9.$$

Thus C ends at the point $(8, 9)$.

(b) By the Fundamental Theorem of Line Integrals,

$$\int_C \operatorname{grad} f \cdot d\vec{r} = f(8,9) - f(2,1) = (3 \cdot 8 + 4 \cdot 9) - (3 \cdot 2 + 4 \cdot 1) = 50.$$

Alternately, since grad f and C are parallel,

$$\int_C \operatorname{grad} f \cdot d\vec{r} = \|\operatorname{grad} f\| \cdot \text{Length of } C = 5 \cdot 10 = 50.$$

53. (a) Since $\vec{r} \cdot \vec{a} = a_1 x + a_2 y + a_3 z$, we have

$$\operatorname{grad}(\vec{r} \cdot \vec{a}) = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} = \vec{a}.$$

(b) By the Fundamental Theorem of Line Integrals, if $\vec{r_0} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$, we have

$$\int_C \operatorname{grad}(\vec{r} \cdot \vec{a}) \cdot d\vec{r} = \vec{r} \cdot \vec{a} \Big|_{(0,0,0)}^{(x_0,y_0,z_0)} = \vec{r}_0 \cdot \vec{a}.$$

(c) Since $\vec{r}_0 \cdot \vec{a} = ||\vec{r}_0|| ||\vec{a}|| \cos\theta = 10||\vec{a}|| \cos\theta$, where the angle between $\vec{r}_0$ and $\vec{a}$, the maximum value of $\vec{r}_0 \cdot \vec{a}$ occurs if $\vec{r}_0$ is parallel to $\vec{a}$. Then $\theta = 0$ and

$$\int_C \operatorname{grad}(\vec{r} \cdot \vec{a}) \cdot d\vec{r} = 10||\vec{a}|| \cos 0 = 10||\vec{a}||.$$

54. (a) Work done by the force is the line integral, so

$$\text{Work done against force} = -\int_C \vec{F} \cdot d\vec{r} = -\int_C (-mg\vec{k}) \cdot d\vec{r}.$$

Since $\vec{r} = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$, we have $\vec{r}\,' = -(\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k}$,

$$\begin{aligned}\text{Work done against force} &= \int_0^{2\pi} mg\vec{k} \cdot (-\sin t\vec{i} + \cos t\vec{j} + \vec{k})dt \\ &= \int_0^{2\pi} mg\, dt = 2\pi\, mg.\end{aligned}$$

(b) We know from physical principles that the force is conservative. (Because the work done depends only on the vertical distance moved, not on the path taken.) Alternatively, we see that

$$\vec{F} = -mg\vec{k} = \operatorname{grad}(-mgz),$$

so $\vec{F}$ is a gradient field and therefore path independent, or conservative.

55. (a) We parameterize the path by $(\cos t, \sin t)$ for $\pi/2 \le t \le \pi$. Since $t = \pi/2$ gives the end point, $(0, 1)$ and $t = \pi$ gives the starting point $(-1, 0)$, we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_\pi^{\pi/2} \vec{F}(\cos t, \sin t) \cdot (-\sin t\vec{i} + \cos t\vec{j})\, dt \\ &= -\int_{\pi/2}^{\pi} (\sin t\vec{i} - \cos t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})\, dt \\ &= -\int_{\pi/2}^{\pi} (-\sin^2 t - \cos^2 t)\, dt \\ &= \int_{\pi/2}^{\pi} 1\, dt = t\Big|_{\pi/2}^{\pi} = \pi/2.\end{aligned}$$

The work done by the force is $+\pi/2$. The work is positive since the force is always in the direction of the path (in fact it is always tangent to C since $\vec{F} \cdot \vec{r} = 0$).

(b) If we redo our computations using the entire unit circle, the only change will be the limits of integration: they'll change to 0 to 2π. This yields an answer of 2π (or -2π, depending on orientation). Since the work around a closed path is not zero, the force is not path-independent.

56. (a) Since $\vec{F}(x,y) - \vec{G}(x,y)$ is parallel to grad $h(x,y)$, it is perpendicular to the level curves of h. Since the oriented path C is on a level curve of h, $\vec{F}(x,y) - \vec{G}(x,y)$ is perpendicular to C at every point of C. Hence

$$\int_C (\vec{F}(x,y) - \vec{G}(x,y)) \cdot d\vec{r} = 0.$$

Therefore

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}.$$

(b) By the Fundamental Theorem of Calculus for Line Integrals we have

$$\int_C \vec{G} \cdot d\vec{r} = \int_C \operatorname{grad}\phi \cdot d\vec{r} = \phi(Q) - \phi(P).$$

Using part (a) we have

$$\int_C \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P).$$

57. (a) We have

$$\begin{aligned}\operatorname{grad} h &= \vec{j}\\ \operatorname{grad}\phi &= y\vec{i} + x\vec{j}\\ \vec{F} - \operatorname{grad}\phi &= x\vec{j} = x \operatorname{grad} h.\end{aligned}$$

Thus, $\vec{F} - \operatorname{grad}\phi$ is a multiple of grad h.

(b) By part (a) the vector fields $\vec{F}$ and grad ϕ have the same components perpendicular to grad h, which is to say the same components in the direction of the level curve C of h. Thus, the line integrals of $\vec{F}$ and grad ϕ along C are equal. Using the Fundamental Theorem of Calculus for Line Integrals, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \operatorname{grad}\phi \cdot d\vec{r} = \phi(Q) - \phi(P) = 80 - 30 = 50.$$

58. (a) We have

$$\begin{aligned}\operatorname{grad} h &= \vec{i}\\ \operatorname{grad}\phi &= 2y\vec{i} + 2x\vec{j}\\ \vec{F} - \operatorname{grad}\phi &= -y\vec{i} = -y \operatorname{grad} h.\end{aligned}$$

Thus, $\vec{F} - \operatorname{grad}\phi$ is a multiple of grad h.

(b) By part (a) the vector fields $\vec{F}$ and grad ϕ have the same components perpendicular to grad h, which is to say the same components in the direction of the level curve C of h. Thus, the line integrals of $\vec{F}$ and grad ϕ along C are equal. Using the Fundamental Theorem of Calculus for Line Integrals, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \operatorname{grad}\phi \cdot d\vec{r} = \phi(Q) - \phi(P) = 60 - 30 = 30.$$

59. (a) We have

$$\begin{aligned}\operatorname{grad} h &= -2x\vec{i} + \vec{j}\\ \operatorname{grad}\phi &= (2x^2 + y)\vec{i} + x\vec{j}\\ \vec{F} - \operatorname{grad}\phi &= -2x^2\vec{i} + x\vec{j} = x \operatorname{grad} h.\end{aligned}$$

Thus, $\vec{F} - \operatorname{grad}\phi$ is a multiple of grad h.

(b) By part (a) the vector fields $\vec{F}$ and grad ϕ have the same components perpendicular to grad h, which is to say the same components in the direction of the level curve C of h. Thus, the line integrals of $\vec{F}$ and grad ϕ along C are equal. Using the Fundamental Theorem of Calculus for Line Integrals, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \operatorname{grad}\phi \cdot d\vec{r} = \phi(Q) - \phi(P) = 384 - 0 = 384.$$

60. (a) We have

$$\begin{aligned}\text{grad}\, h &= \vec{i} + \vec{j} \\ \text{grad}\, \phi &= (x+3y)\vec{i} + (3x+2y)\vec{j} \\ \vec{F} - \text{grad}\, \phi &= (-x-2y)\vec{i} + (-x-2y)\vec{j} = -(x+2y)\,\text{grad}\, h.\end{aligned}$$

Thus, $\vec{F} - \text{grad}\,\phi$ is a multiple of grad h.

(b) By part (a) the vector fields $\vec{F}$ and grad ϕ have the same components perpendicular to grad h, which is to say the same components in the direction of the level curve C of h. Thus, the line integrals of $\vec{F}$ and grad ϕ along C are equal. Using the Fundamental Theorem of Calculus for Line Integrals, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \text{grad}\,\phi \cdot d\vec{r} = \phi(Q) - \phi(P) = 1800 - 1850 = -50.$$

61. (a) By the chain rule

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x x'(t) + f_y y'(t),$$

which is the result we want.

(b) Using the parameterization of C that we were given,

$$\begin{aligned}\int_C \text{grad}\, f \cdot d\vec{r} &= \int_a^b (f_x(x(t),y(t))\vec{i} + f_y(x(t),y(t))\vec{j}\,) \cdot (x'(t)\vec{i} + y'(t)\vec{j}\,)dt \\ &= \int_a^b (f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t))dt.\end{aligned}$$

Using the result of part (a), this gives us

$$\begin{aligned}\int_C \text{grad}\, f \cdot d\vec{r} &= \int_a^b h'(t)dt \\ &= h(b) - h(a) = f(Q) - f(P).\end{aligned}$$

62. (a) The level surfaces are horizontal planes given by $gz = c$, so $z = c/g$. The potential energy increases with the height above the earth. This means that more energy is stored as "potential to fall" as height increases.

(b) The gradient of ϕ points upward (in the direction of increasing potential energy), so $\nabla\phi = g\vec{k}$. The gravitational force acts toward the earth in the direction of $-\vec{k}$. So, $\vec{F} = -g\vec{k}$. The negative sign represents the fact that the gravitational force acts in the direction of the decreasing potential energy.

63. (a) We have

$$\varphi(\vec{r}\,) = \frac{p_1x + p_2y + p_3z}{(x^2+y^2+z^2)^{3/2}}.$$

Taking partial derivatives gives

$$\begin{aligned}\varphi_x(\vec{r}\,) &= \frac{p_1(x^2+y^2+z^2)^{3/2} - (3/2)(p_1x+p_2y+p_3z)(2x)(x^2+y^3+z^2)^{1/2}}{(x^2+y^2+z^3)^3} \\ &= \frac{p_1}{\|\vec{r}\,\|^3} - 3\frac{(\vec{p}\cdot\vec{r}\,)x}{\|\vec{r}\,\|^5} \\ &= -D_1(\vec{r}\,).\end{aligned}$$

Similar computations give $\varphi_y = -D_2$ and $\varphi_z = -D_3$, so grad $\varphi = -\vec{D}$.

(b) The field $\vec{D}$ is necessarily path-independent since it is a gradient vector field.

Strengthen Your Understanding

64. The statement cannot be correct, because $\int_C \vec{F} \cdot d\vec{r}$ is a scalar and $\vec{F}(Q) - \vec{F}(P)$ is a vector. The correct statement of the Fundamental Theorem of Calculus for Line Integrals is as follows: if $\vec{F} = \text{grad}\, f$, then $\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P)$. Evaluate the potential function, f, for $\vec{F}$, not $\vec{F}$ itself, at the endpoints of C.

65. This is only true if $\vec{F}$ is path-independent; if it is not, then f is not well-defined.

66. If a vector field $\vec{F}$ is not a gradient vector field, then $\int_C \vec{F} \cdot d\vec{r}$ can't be evaluated using the Fundamental Theorem of Calculus for Line Integrals. However, other methods such as parameterizing C might work to evaluate the integral.

67. The integral is required to have the same value for all oriented paths from $(0,0)$ to $(1,2)$. This is achieved by choosing $\vec{F} = \operatorname{grad} f$ to be a gradient vector field. Since $\int_C \operatorname{grad} f \cdot d\vec{r} = f(1,2) - f(0,0)$, we seek a function $f(x,y)$ such that $f(1,2) - f(0,0) = 100$. For example, we can take $f(x,y) = 50xy$. With

$$\vec{F} = \operatorname{grad} f = 50y\vec{i} + 50x\vec{j}$$

we have

$$\int_C \vec{F} \cdot d\vec{r} = 100.$$

68. Every gradient field is path-independent. For example,

$$\vec{F} = \operatorname{grad} xy = y\vec{i} + x\vec{j}$$

is path-independent.

69. A path-independent vector field must have zero circulation around all closed paths. Consider a vector field like $\vec{F}(x,y) = |x|\vec{j}$, shown in Figure 18.22.

A rectangular path that is symmetric about the y-axis will have zero circulation: on the horizontal sides, the field is perpendicular, so the line integral is zero. The line integrals on the vertical sides are equal in magnitude and opposite in sign, so they cancel out, giving a line integral of zero. However, this field is not path-independent, because it is possible to find two paths with the same endpoints but different values of the line integral of $\vec{F}$. For example, consider the two points $(0,0)$ and $(0,1)$. The path C_1 in Figure 18.23 along the y axis gives zero for the line integral, because the field is 0 along the y axis, whereas a path like C_2 will have a nonzero line integral. Thus the line integral depends on the path between the points, so $\vec{F}$ is not path-independent.

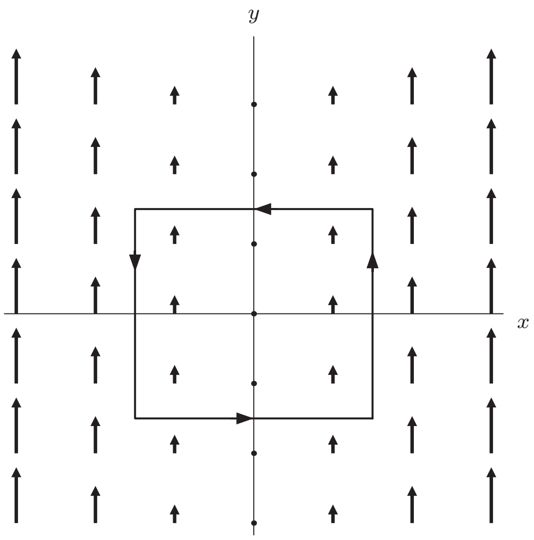


Figure 18.22

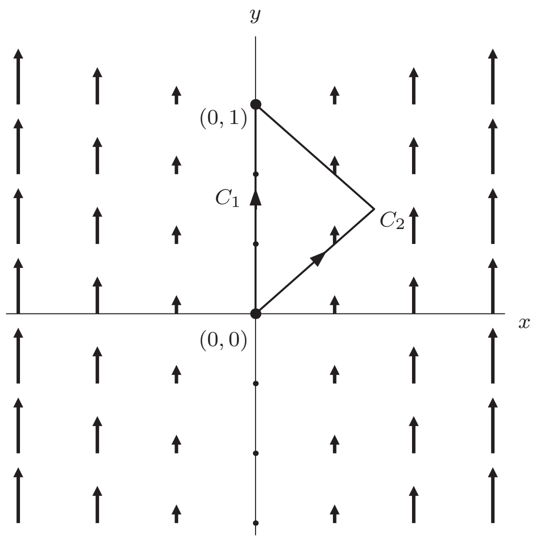


Figure 18.23

70. This is false, because the line integral yields a scalar whereas the total change in $\vec{F}$ would be a vector. In the *special* case when $\vec{F}$ happens to be the gradient of a scalar function f, the line integral does give the total change of the scalar function f along the path—but not of the vector function $\vec{F}$.

71. You can easily come up with counterexamples: suppose that $\vec{F} \neq \vec{G}$ but that both are gradient fields. For example, $\vec{F} = \vec{i}$ and $\vec{G} = \vec{j}$. Then, if C is a closed curve, the line integral around C of both $\vec{F}$ and $\vec{G}$ will equal to zero. But this does not mean that $\vec{F} = \vec{G}$.

72. The total change of f along C depends only on the endpoints of C. If f has the same values at each endpoint (or if C is closed, so that the endpoints coincide) then the total change will be zero. This in no way restricts the shape of the curve C. For example, take $f(x,y) = x^2 + y^2$ and C to be the straight line from the point $(1, 0)$ to the point $(0, 1)$. Then $f(1,0) = f(0,1) = 1$ so the change in f along C is zero, but C is not a contour of f.

73. False. The line integral of a gradient vector field around this circle would be 0, but the converse is not necessarily true. That is, the fact that the line integral around this one circle is zero does not mean $\vec{F}$ is necessarily a gradient field.

74. True. Since $x\vec{i} + y\vec{j} = \text{grad}\left(\frac{1}{2}(x^2+y^2)\right)$, the Fundamental Theorem of Line Integrals gives

$$\int_C (x\vec{i} + y\vec{j}) \cdot d\vec{r} = \frac{1}{2}(x^2+y^2)\Big|_{(0,0)}^{(a,b)} \frac{1}{2}(a^2+b^2).$$

75. False. The line integrals of many vector fields (so called *path independent* or *conservative fields*) are zero around closed curves, but this is not true of all fields. For example, a vector field that is flowing in the same direction as the curve C all along the curve has a positive line integral. A specific example is given by $\vec{F} = -y\vec{i} + x\vec{j}$, where C is the unit circle centered at the origin, oriented counterclockwise.

76. True. By the Fundamental Theorem for Line Integrals, if C is a path from P to Q, then $\int_C \text{grad}\, f \cdot d\vec{r} = f(Q) - f(P)$, so the value of the line integral $\int_C \text{grad}\, f \cdot d\vec{r}$ depends only on the endpoints and not the path.

77. False. The statement is true if C_1 and C_2 have the same initial and final points. For example, $\vec{F}(x,y) = \vec{i}$ is path-independent (since it is the gradient of $f(x,y) = x$), but the line integral of $\vec{F}$ over a path from $(0,0)$ to $(1,0)$ is $f(1,0) - f(0,0) = 1$, but the line integral over a path from $(0,0)$ to $(0,1)$ is $f(0,1) - f(0,0) = 0$.

78. False. However, if $\vec{F}$ is a gradient field, the line integral gives the total change in the potential function f, where $\vec{F} = \text{grad}\, f$.

79. True. The construction at the end of Section 18.3 shows how to make a potential function from a path-independent vector field.

80. True. Since a gradient field is path-independent, and C_1 and C_2 have the same initial and final points, the two line integrals are equal.

81. True. Since the curve can be thought of as having the same initial and final points, if f is a potential function for $\vec{F}$ we have $\int_C \vec{F} \cdot d\vec{r} = \int_C \text{grad}\, f \cdot d\vec{r} = f(P) - f(P) = 0$.

82. False. If there were a potential function f, then $f_x = y^2$, so $f(x,y) = xy^2 + g(y)$, where $g(y)$ is a function of y only. Then $f_y = 2xy + g'(y)$, which depends on x no matter what $g'(y)$ is. Thus f_y cannot be equal to a constant k, and so there is no potential function f such that $\vec{F} = \text{grad}\, f$.

83. False. For example, take $\vec{F} = y\vec{i}$. By symmetry, the line integral of $\vec{F}$ over any circle centered at the origin is zero. But the curve consisting of the upper semicircle connecting $(-a,0)$ to $(a,0)$ has a positive line integral, while the line connecting these points along the x-axis has a zero line integral, so the field cannot be path-independent.

Solutions for Section 18.4

Exercises

1. We have $\dfrac{\partial F_i}{\partial y} = 1$ and $\dfrac{\partial F_2}{\partial x} = -1$, so $\dfrac{\partial F_1}{\partial y} \neq \dfrac{\partial F_2}{\partial x}$ and this cannot be a gradient vector field.

2. Yes, since $\vec{F} = 2xy\vec{i} + x^2\vec{j} = \text{grad}(x^2y)$.

3. The domain of the vector field $\vec{F}(x,y) = y\vec{i} + y\vec{j}$ is the whole xy-plane. In order to see if $\vec{F}$ is a gradient let us apply the curl test:

$$\frac{\partial F_1}{\partial y} = 1$$

and

$$\frac{\partial F_2}{\partial x} = 0$$

So $\vec{F}$ is not the gradient of any function.

4. No, since if $\vec{F} = 2xy\vec{i} + 2xy\vec{j}$,

$$\frac{\partial F_2}{\partial x} = 2y \quad \text{and} \quad \frac{\partial F_1}{\partial y} = 2x,$$

so

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \neq 0.$$

5. The domain of the vector field $\vec{F}(x,y) = (x^2+y^2)\vec{i} + 2xy\vec{j}$ is the whole xy-plane. Let us apply the curl test:

$$\frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}$$

so $\vec{F}$ is the gradient of some function f. In order to compute f we first integrate

$$\frac{\partial f}{\partial x} = x^2 + y^2$$

with respect to x, thinking of y as a constant.

We get

$$f(x,y) = \frac{x^3}{3} + xy^2 + C(y)$$

Differentiating with respect to y and using the fact that $\partial f/\partial y = 2xy$ gives

$$\frac{\partial f}{\partial y} = 2xy + C'(y) = 2xy$$

Thus $C'(y) = 0$ so C is a constant and

$$f(x,y) = \frac{x^3}{3} + xy^2 + C.$$

6. The domain of the vector field $\vec{F} = (2xy^3 + y)\vec{i} + (3x^2y^2 + x)\vec{j}$ is the whole xy-plane. We apply the curl test:

$$\frac{\partial F_1}{\partial y} = 6xy^2 + 1 = \frac{\partial F_2}{\partial x}$$

so $\vec{F}$ is the gradient of a function f. In order to compute f we first integrate

$$\frac{\partial f}{\partial x} = 2xy^3 + y$$

with respect to x thinking of y as a constant. We get

$$f(x,y) = x^2y^3 + xy + C(y)$$

Differentiating with respect to y and using the fact that $\partial f/\partial y = 3x^2y^2 + x$ gives

$$\frac{\partial f}{\partial y} = 3x^2y^2 + x + C'(y) = 3x^2y^2 + x$$

Thus $C'(y) = 0$ so C is constant and

$$f(x,y) = x^2y^3 + xy + C.$$

7. The domain of the vector field $\vec{F} = \frac{\vec{i}}{x} + \frac{\vec{j}}{y} + \frac{\vec{k}}{z}$ is the set of points (x,y,z) in the three space such that $x \neq 0$, $y \neq 0$ and $z \neq 0$. This is what is left in the three space after removing the coordinate planes. This domain has the property that every closed curve is the boundary of a surface entirely contained in it, hence we can apply the curl test.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix}$$

So $\operatorname{curl}\vec{F} = \vec{0}$ and thus $\vec{F}$ is the gradient of a function f. In order to compute f we first integrate

$$\frac{\partial f}{\partial x} = \frac{1}{x}$$

with respect to x, thinking of y and z as constants. We get

$$f(x,y,z) = \ln|x| + C(y,z)$$

Differentiating with respect to y and using the fact that $\partial f/\partial y = 1/y$ gives

$$\frac{\partial f}{\partial y} = \frac{\partial C}{\partial y} = \frac{1}{y}$$

We integrate this relation with respect to y thinking of z as a constant. We get

$$f(x,y,z) = \ln|xy| + K(z)$$

Differentiating with respect to z and using the fact that $\partial f/\partial z = 1/z$ gives

$$\frac{\partial f}{\partial z} = K'(z) = \frac{1}{z}$$

Now we integrate with respect to z and get

$$f(x,y,z) = \ln A|xyz|$$

where A is a positive constant.

8. The domain of the vector field $\vec{F} = \dfrac{\vec{i}}{x} + \dfrac{\vec{j}}{y} + \dfrac{\vec{k}}{xy}$ is the set of points in the three space, (x,y,z) such that $x \neq 0$ and $y \neq 0$. This is the set of points in the three space left after removing the planes $x = 0$ and $y = 0$. This domain has the property that every closed curve is the boundary of a surface entirely contained in it, hence we can apply the curl test.

$$\begin{aligned}
\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{xy} \end{vmatrix} \\
&= \vec{i}\left(\frac{\partial}{\partial y}(\frac{1}{xy}) - \frac{\partial}{\partial z}(\frac{1}{y})\right) - \vec{j}\left(\frac{\partial}{\partial x}(\frac{1}{xy}) - \frac{\partial}{\partial z}(\frac{1}{x})\right) + \vec{k}\left(\frac{\partial}{\partial x}(\frac{1}{y}) - \frac{\partial}{\partial y}(\frac{1}{x})\right) \\
&= -\frac{1}{xy^2}\vec{i} + \frac{1}{x^2y}\vec{j} \neq 0
\end{aligned}$$

Therefore $\vec{F}$ is not the gradient of any function.

9. The domain of the vector field $\vec{F} = 2x\cos(x^2+z^2)\vec{i} + \sin(x^2+z^2)\vec{j} + 2z\cos(x^2+z^2)\vec{k}$ is the whole three space so we can apply the curl test.

$$\begin{aligned}
\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\cos(x^2+y^2) & \sin(x^2+y^2) & 2z\cos(x^2+y^2) \end{vmatrix} \\
&= -4yz\sin(x^2+y^2)\vec{i} + 4xz\sin(x^2+y^2)\vec{j} + (2x\cos(x^2+y^2) + 4xy\sin(x^2+y^2))\vec{k} \neq 0
\end{aligned}$$

As $\operatorname{curl}\vec{F} \neq \vec{0}$, $\vec{F}$ is not the gradient of any function.

10. We have

$$\begin{aligned}
\frac{\partial F_1}{\partial y} &= \frac{(x^2+y^2)1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \\
\frac{\partial F_2}{\partial x} &= -\frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = -\frac{y^2-x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.
\end{aligned}$$

Thus $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$. However, the domain of the vector field contains a "hole" at the origin, so the curl test does not apply. This is not a gradient field. See Example 7 on page 992 of the text.

11. We have $F_1 = y^2$ and $F_2 = x$. By Green's Theorem

$$\int_C \left(y^2\vec{i} + x\vec{j}\right)\cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \int_0^3\int_0^2 (1-2y)\,dx\,dy = -12.$$

12. By Green's Theorem, with R representing the interior of the circle,

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right) dA = -2\int_R dA$$
$$= -2\cdot \text{ Area of circle } = -2\pi(1^2) = -2\pi.$$

13. By Green's Theorem, with R representing the interior of the square,

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(0)\right) dA = \int_R y\,dA$$
$$= \int_0^1\int_0^1 y\,dy\,dx = \int_0^1 \left.\frac{y^2}{2}\right|_0^1 dx = \frac{1}{2}.$$

14. By Green's Theorem, with R representing the interior of the triangle,

$$\int_C \vec{F}\cdot\vec{r} = \int_R \left(\frac{\partial}{\partial x}(2x+3y^2) - \frac{\partial}{\partial y}(2x^2+3y)\right) dA = \int_R (2-3)\,dA = -\int_R dA$$
$$= -\text{ Area of triangle } = -\frac{1}{2}\cdot 4\cdot 3 = -6.$$

15. By Green's Theorem, with R representing the interior of the circle,

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(3y)\right) dA = \int_R (y-3)\,dA.$$

The integral of y over the interior of the circle is 0, by symmetry, because positive contributions of y from the top half of the circle cancel those from the bottom half. Thus

$$\int_R y\,dA = 0.$$

So

$$\int_C \vec{F}\cdot d\vec{r} = \int_R (y-3)\,dA = \int_R -3\,dA = -3\cdot \text{ Area of circle } = -3\cdot\pi(1)^2 = -3\pi.$$

16. Green's theorem gives

$$\int_C ((3x+5y)\vec{i} + (2x+7y)\vec{j}\,)\cdot d\vec{r} = \int\int_R \left(-\frac{\partial}{\partial y}(3x+5y) + \frac{\partial}{\partial x}(2x+5y)\right) dA$$
$$= \int\int_R -3dA = -3\cdot \text{Area of } R = -3\pi m^2.$$

17. (a) The vector field points in the opposite direction to the orientation of the curve, hence the circulation is negative. See Figure 18.24.

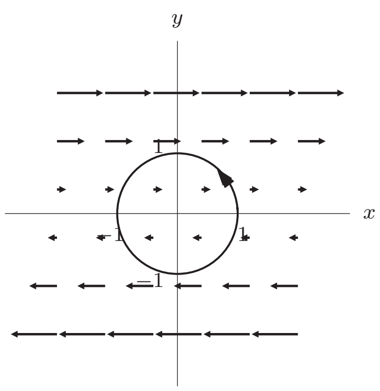


Figure 18.24

(b) Since $\vec{F} = y\vec{i}$, we have $\partial F_1/\partial y = 1$ and $\partial F_2/\partial x = 0$ and $\partial F_1/\partial y = 1$. Thus, using Green's Theorem if R is the region enclosed by the closed curve C (the unit circle centered at the origin and traversed counterclockwise), we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \int_R (-1)\,dx\,dy = -\text{Area of } R = -\pi.$$

Problems

18. The perimeter of the rectangle is a closed curve, C, so we can use Green's Theorem. See Figure 18.25. The curve is traversed in the correct direction to apply Green's Theorem directly. Let R be the interior of the rectangle,

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial (x+y)}{\partial x} - \frac{\partial (\sin x)}{\partial y} \right) dxdy = \int_R 1\,dxdy = \text{Area of rectangle} = 4 \cdot 5 = 20.$$

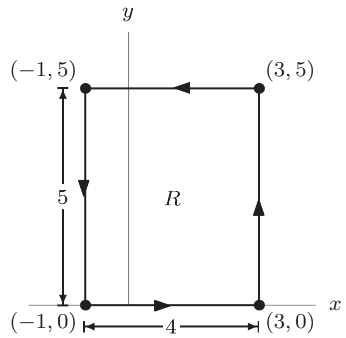


Figure 18.25

19. By Green's Theorem, with R as the interior of the square, we have

$$\begin{aligned}\int_C (\sin(x^2)\cos y)\vec{i} + (\sin(y^2) + e^x)\vec{j} \cdot d\vec{r} &= \int_R \left(\frac{\partial}{\partial x}(\sin(y^2) + e^x) - \frac{\partial}{\partial y}(\sin(x^2) + \cos y) \right) dA \\ &= \int_0^1 \int_0^1 (e^x + \sin y)\,dx\,dy \\ &= \int_0^1 (e^x + x\sin y)\Big|_0^1 dy \\ &= \int_0^1 (e + \sin y) - 1)\,dy \\ &= (ey - \cos y - y)\Big|_0^1 = e - \cos 1 - 1 + \cos 0 = e - \cos 1.\end{aligned}$$

20. The curve is closed, so we can use Green's Theorem. If R represents the interior of the region

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \int_R \left(\frac{\partial (x)}{\partial x} - \frac{\partial (x-y)}{\partial y} \right) dA \\ &= \int_R (1 - (-1))\,dA = \int_R 2\,dA = 2 \cdot \text{Area of sector.}\end{aligned}$$

Since R is $1/8$ of a circle, R has area $\pi(3^2)/8$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = 2 \cdot \frac{9\pi}{8} = \frac{9\pi}{4}.$$

21. The curve is closed, so we can use Green's Theorem. If R represents the interior of the region

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_R \left(\frac{\partial(\sin y)}{\partial x} - \frac{\partial(x+y)}{\partial y}\right) dA$$
$$= \int_R (-1)\, dA = (-1)\cdot \text{Area of sector.}$$

Since R is $1/8$ of a circle, R has area $\pi(3^2)/8$. Thus,

$$\int_C \vec{F}\cdot d\vec{r} = (-1)\cdot\frac{9\pi}{8} = -\frac{9\pi}{8}.$$

22. (a) Since $\vec{F} = \operatorname{grad}(x^2 e^y)$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F}\cdot d\vec{r} = x^2 e^y \Big|_{(0,0)}^{(2,4)} = 4e^4.$$

(b) Since

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y) = 2,$$

we know that $\vec{G}$ is not a gradient field. Parameterizing C by $x(t) = t, y(t) = 2t$ for $0 \le t \le 2$, we have $\vec{r}\,'(t) = \vec{i} + 2\vec{j}$, so

$$\int_C \vec{G}\cdot d\vec{r} = \int_0^2 ((t-2t)\vec{i} + (t+2t)\vec{j})\cdot(\vec{i} + 2\vec{j})\, dt$$
$$= \int_0^2 ((t-2t) + (2t+4t))\, dt = \int_0^2 5t\, dt = \frac{5}{2}t^2\Big|_0^2 = 10.$$

23. (a) By symmetry between quadrants II and IV, this integral is zero. To confirm, we can parameterize. This line is given by

$$\vec{r} = (-\vec{i} + \vec{j}) + t(2\vec{i} - 2\vec{j}) = (-1+2t)\vec{i} + (1-2t)\vec{j} \text{ for } 0 \le t \le 1.$$

Then $\vec{r}\,'(t) = 2\vec{i} - 2\vec{j}$, so

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left((1-2t)\vec{i} + (-1+2t)\vec{j}\right)\cdot(2\vec{i} - 2\vec{j})\, dt = \int_0^1 (4-8t)\, dt = 0.$$

(b) The curve C is closed, so we use Green's Theorem:

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y)\right) dA = \int_R 0\, dA = 0.$$

(c) Since $C = C_2 - C_1$, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r}$$
$$0 = \int_{C_2} \vec{F}\cdot d\vec{r} - 0.$$

So

$$\int_{C_2} \vec{F}\cdot d\vec{r} = 0.$$

(d) The magnitude of $\vec{G}$ is constant on C_2:

$$\|\vec{G}\| = \sqrt{(3y)^2 + (-3x)^2} = 3\sqrt{y^2 + x^2} = 3\sqrt{2}.$$

Thus, since G is everywhere tangent to C_2 and points in the opposite direction to C_2, we have

$$\int_{C_2} \vec{G}\cdot d\vec{r} = -\|\vec{G}\|\cdot \text{ Length of } C_2 = -(3\sqrt{2})\pi\sqrt{2} = -6\pi.$$

(e) The curve C is closed, so we use Green's Theorem:

$$\int_C \vec{G}\cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(-3x) - \frac{\partial}{\partial y}(3y)\right) dA = \int_R (-6)\,dA$$
$$= -6\cdot \text{ Area of } R = -6\frac{\pi(\sqrt{2})^2}{2} = -6\pi.$$

(f) Since $C = C_2 - C_1$, we have

$$\int_C \vec{G}\cdot d\vec{r} = \int_{C_2} \vec{G}\cdot d\vec{r} - \int_{C_1} \vec{G}\cdot d\vec{r}$$
$$-6\pi = -6\pi - \int_{C_1} \vec{G}\cdot d\vec{r}.$$

Hence,

$$\int_{C_1} \vec{G}\cdot d\vec{r} = 0.$$

(g) The curve C is closed, so we use Green's Theorem:

$$\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C (4y\vec{i} - 2x\vec{j})\cdot d\vec{r} = \int_R \frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(4y)\,dA = \int_R -6\,dA$$
$$= -6\cdot \text{ Area of } R = -6\cdot \frac{\pi(\sqrt{2})^2}{2} = -6\pi.$$

Alternately,

$$\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r} = 0 - 6\pi = -6\pi.$$

24. (a) The curve, C, is closed and oriented in the correct direction for Green's Theorem. See Figure 18.26. Writing R for the interior of the circle, we have

$$\int_C \left((x^2-y)\vec{i} + (y^2+x)\vec{j}\right)\cdot d\vec{r} = \int_R \left(\frac{\partial(y^2+x)}{\partial x} - \frac{\partial(x^2-y)}{\partial y}\right) dxdy$$
$$= \int_R (1-(-1))\,dxdy = 2\int_R dxdy$$
$$= 2\cdot \text{Area of circle} = 2(\pi\cdot 3^2) = 18\pi.$$

(b) The circle given has radius R and center (a,b). The argument in part (a) works for any circle of radius R, oriented counterclockwise. So the line integral has the value $2\pi R^2$.

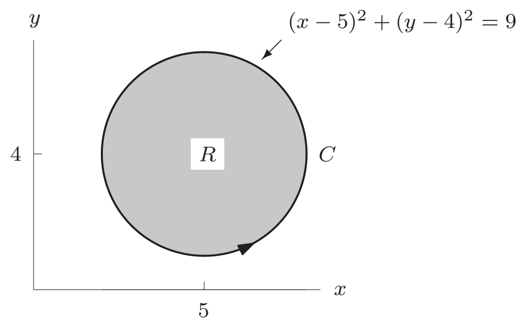


Figure 18.26

25. (a) The particle moving along C_1 starts at the origin and moves simultaneously radially outward and counterclockwise in the xz-plane, describing a spiral. The spiral completes two full revolutions around the origin, starting at $(0, 0, 0)$ and ending at the point $(2, 0, 0)$. The particle moving along C_2 starts at the origin and moves simultaneously outward, counterclockwise, and forward along the positive y-axis, describing a 3-dimensional spiral coil growing in diameter as it wraps around the y-axis. The coil completes two full revolutions around the y-axis, starting at $(0, 0, 0)$ and ending at the point $(2, 2, 0)$.

(b) We suspect $\vec{F}$ is a gradient field and look for a potential function f. We have:

$$f(x, y, z) = \int f_x\, dx = \int yz\, dx = yzx + g(y, z).$$

We now have two expressions for f_y, which we set equal to each other:

$$z(x + 1) = f_y = zx + g_y.$$

This means

$$zx + z = zx + g_y \quad , \text{ or } \quad g_y = z.$$

Thus,

$$g(y, z) = \int g_y\, dy = \int z\, dy = zy + h(z),$$

and also

$$f(x, y, z) = yzx + zy + h(z).$$

Lastly, we set both expressions for f_z equal to each other

$$xy + y + 1 = f_z = yx + y + h'(z).$$

This means $h(z) = z$ which gives us

$$f(x, y, z) = xyz + zy + z.$$

Thus, $\vec{F} = \nabla f$ and we can calculate the line integral using the Fundamental Theorem of Calculus for Line Integrals:

$$\int_{C_2} \vec{F} \cdot dr = f(2, 2, 0) - f(0, 0, 0) = 0 - 0 = 0.$$

(c) The beginning points and the endpoints of C_1 and C_2, respectively, have identical z-values. Since this common z-value is zero, we may look for a gradient field $\vec{G}$ whose potential function has a factor of z, then use the Fundamental Theorem of Calculus for Line Integrals to calculate both integrals. Note that

$$\vec{G} = \vec{F} = \nabla(xyz + zy + z) \quad \text{and}$$
$$\vec{G} = \nabla(ze^{-xyz})$$

both work, for example. In each case we have:

$$\int_{C_1} \vec{G} \cdot dr = \int_{C_2} \vec{G} \cdot dr = 0.$$

There are many other possible answers.

(d) Reasoning as in (c) above, we choose

$$\vec{H}_1 = \vec{F} = \nabla(xyz + zy + z) \quad \text{and}$$
$$\vec{H}_2 = \nabla(ze^{-xyz}),$$

for example. These fields work because, using the Fundamental Theorem of Calculus for Line Integrals, we get

$$\int_{C_1} \vec{H}_1 \cdot d\vec{r} = \int_{C_1} \vec{H}_2 \cdot d\vec{r} = 0.$$

The fields

$$\vec{H}_1 = \nabla(yx^2) \quad \text{and}$$
$$\vec{H}_2 = \nabla(y(x + z))$$

also work, since $y = 0$ at the beginning and endpoints of C_1 and the Fundamental Theorem of Calculus for Line Integrals yields:

$$\int_{C_1} \vec{H}_1 \cdot d\vec{r} = \int_{C_1} \vec{H}_2 \cdot d\vec{r} = 0.$$

There are many other possible answers, and not all of them yield a zero value for the integrals.

26. Since $\vec{F} = x\vec{j}$, we have $\partial F_2/\partial x = 1$ and $\partial F_1/\partial y = 0$. Thus, using Green's Theorem if R is the region enclosed by the closed curve C, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \int_R 1\, dx\, dy = \text{Area of } R$$

27. Using $\vec{F} = x\vec{j} = a\cos t\vec{j}$ and $\vec{r}\,'(t) = -a\sin t\vec{i} + b\cos t\vec{j}$, we have

$$\begin{aligned} A = \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (a\cos t)(b\cos t)\, dt \\ &= ab \int_0^{2\pi} \cos^2 t\, dt \\ &= ab \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt \\ &= \pi ab + \frac{ab}{4}\sin 2t\Big|_0^{2\pi} = \pi ab \end{aligned}$$

The ellipse is shown in Figure 18.27.

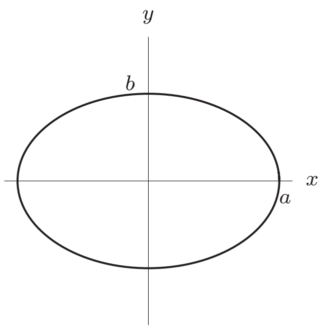


Figure 18.27: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

28. Using $\vec{F} = x\vec{j} = a\cos^3 t$ and $\vec{r}\,'(t) = -3a\cos^2 t\sin t\vec{i} + 3a\sin^2 t\cos t\vec{j}$, we have

$$\begin{aligned} A &= \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (a\cos^3 t)(3a\sin^2 t\cos t)\, dt \\ &= 3a^2 \int_0^{2\pi} \cos^4 t\sin^2 t\, dt = 3a^2 \int_0^{2\pi} \cos^2 t(\sin t\cos t)^2\, dt = 3a^2 \int_0^{2\pi} \cos^2 t \frac{\sin^2 2t}{4}\, dt \\ &= \frac{3a^2}{16} \int_0^{2\pi} (1+\cos 2t)(1-\cos 4t)\, dt \\ &= \frac{3a^2}{16} \int_0^{2\pi} (1+\cos 2t - \cos 4t - \cos 2t\cos 4t)\, dt \\ &= \frac{3a^2}{16} \int_0^{2\pi} (1+\cos 2t - \cos 4t - \frac{1}{2}\cos 6t - \frac{1}{2}\cos 2t)\, dt \\ &= \frac{3a^2}{16} (t - \frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t + \frac{1}{12}\sin 6t + \frac{1}{4}\sin 2t)\Big|_0^{2\pi} = \frac{3\pi a^2}{8} \end{aligned}$$

For the last integral we use the trigonometric formula $\cos 2t\cos 4t = \frac{1}{2}(\cos 6t + \cos 2t)$.

The hypocycloid is shown in Figure 18.28.

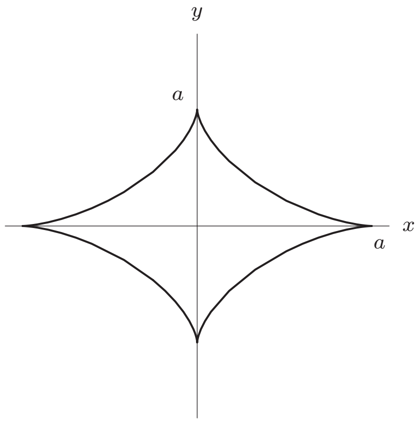


Figure 18.28: $x^{2/3} + y^{2/3} = a^{2/3}$

29. Using $\vec{F} = x\vec{j} = \dfrac{3t^2}{1+t^3}\vec{j}$ and $\vec{r}\,'(t) = \dfrac{1-2t^3}{(1+t^3)^2}\vec{i} + \dfrac{3t(2-t^3)}{(1+t^3)^2}\vec{j}$, we have

$$A = \int_C \vec{F} \cdot d\vec{r} = \int_0^\infty \frac{3t}{1+t^3} \cdot \frac{3t(2-t^3)}{(1+t^3)^2}\, dt$$
$$= 9\int_0^\infty \frac{t^2(2-t^3)}{(1+t^3)^3}\, dt$$

We make the change of variables $u = 1+t^3$ so $du = 3t^2 dt$ and $2 - t^3 = 3 - u$. So

$$A = 3\int_1^\infty \frac{3-u}{u^3} du.$$

This is an improper integral, so it can be computed as follows

$$A = 3\int_1^\infty \frac{3-u}{u^3}\, du = \lim_{b\to\infty} 3\int_1^b \left(\frac{3}{u^3} - \frac{1}{u^2}\right) du$$
$$= \lim_{b\to\infty}\left[9\left(-\frac{1}{2}\right)u^{-2}\Big|_1^b + 3\frac{1}{u}\Big|_1^b\right]$$
$$= \lim_{b\to\infty}\left[-\frac{9}{2}\left(\frac{1}{b^2} - 1\right) + 3\left(\frac{1}{b} - 1\right)\right]$$
$$= -\frac{9}{2}(0-1) + 3(0-1) = \frac{3}{2}.$$

The Folium of Descartes is shown in Figure 18.29.

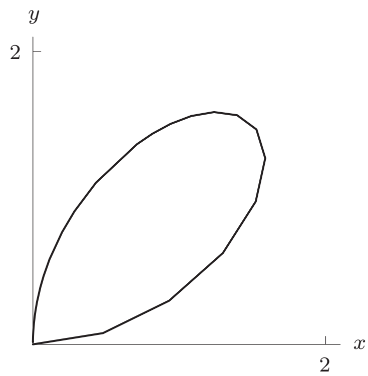


Figure 18.29: $x^3 + y^3 = 3xy$

30. Suppose C encloses a region R. Then, using Green's Theorem, we have

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \\ &= \int_R \frac{\partial}{\partial x}(4x(1-y^2) + x\sin(xy)) - \frac{\partial}{\partial y}(-y^3 + y\sin(xy))\, dA \\ &= \int_R 4(1-y^2) + \sin(xy) + xy\cos(xy) + 3y^2 - \sin(xy) - xy\cos(xy)\, dA \\ &= \int_R (4-y^2)\, dA\end{aligned}$$

This integral over R is largest if C encloses the maximum possible region where $4 - y^2 > 0$, that is, where $-2 \le y \le 2$. Therefore C should be the curve with two sides along the lines $y = -2$ and $y = 2$, as well as two arcs of the circle $x^2 + y^2 = 25$. See Figure 18.30.

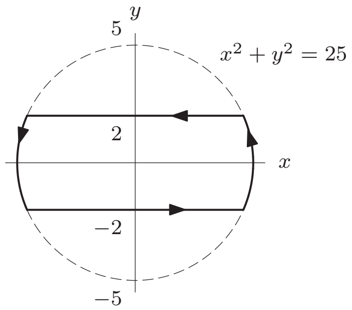


Figure 18.30

31. Since the level curves must be perpendicular to the gradient vectors, if there were a contour diagram fitting this gradient field, it would have to look like Figure 18.31. However, this diagram could not be the contour diagram because the origin is on all contours. This means that $f(0, 0)$ would have to take on more than one value, which is impossible. At a point P other than the origin, we have the same problem. The values on the contours increase as you go counterclockwise around, since the gradient vector points in the direction of greatest increase of a function. But, starting at P, and going all the way around the origin, you would eventually get back to P again, and with a larger value of f, which is impossible.

An additional problem arises from the fact that the vectors in the original vector field are longer as you go away from the origin. This means that if there were a potential function f then $||\,\text{grad}\, f\,||$ would increase as you went away from the origin. This would mean that the level curves of f would get closer together as you go outward which does not happen in the contour diagram in Figure 18.31.

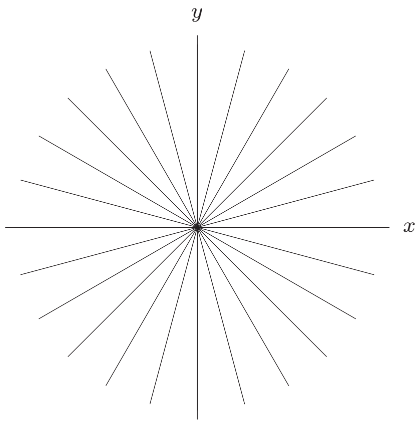


Figure 18.31

32. The drawing of the contour diagrams in Figure 18.32 fitting this gradient field would look like Figure 18.32. The values on the contours would increase both as y increases (for positive x) and as y decreases (for negative x), following the rule that the gradient vector points in the direction of greatest increase of a function. Therefore, it is impossible for this to be a contour diagram.

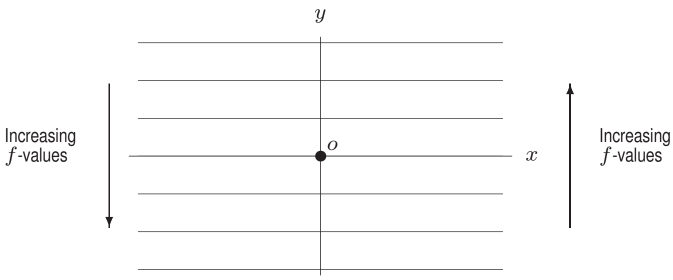


Figure 18.32

33. (a) We see that $\vec{F}, \vec{G}, \vec{H}$ are all gradient vector fields, since

$$\begin{aligned}\text{grad}(xy) &= \vec{F} \quad \text{for all } x, y\\ \text{grad}(\arctan(x/y)) &= \vec{G} \quad \text{except where } y = 0\\ \text{grad}\left((x^2+y^2)^{1/2}\right) &= \vec{H} \quad \text{except at } (0,0).\end{aligned}$$

Other answer are possible. For example $\text{grad}(-\arctan(y/x)) = \vec{G}$ for $x \neq 0$.

(b) Parameterizing the unit circle, C, by $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, we have $\vec{r}\,'(t) = -\sin t\vec{i} + \cos t\vec{j}$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} ((\sin t)\vec{i} + (\cos t)\vec{j}) \cdot ((-\sin t)\vec{i} + (\cos t)\vec{j})\, dt = \int_0^{2\pi} \cos(2t)\, dt = 0.$$

The vector field $\vec{G}$ is tangent to the circle, pointing in the opposite direction to the parameterization, and of length 1 everywhere. Thus

$$\int_C \vec{G} \cdot d\vec{r} = -1 \cdot \text{ Length of circle } = -2\pi.$$

The vector field $\vec{H}$ points radially outward, so it is perpendicular to the circle everywhere. Thus

$$\int_C \vec{H} \cdot d\vec{r} = 0.$$

(c) Green's Theorem does not apply to the computation of the line integrals for $\vec{G}$ and $\vec{H}$ because their domains do not include the origin, which is in the interior, R, of the circle. Green's Theorem does apply to $\vec{F} = y\vec{i} + x\vec{j}$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \int 0\, dx\, dy = 0.$$

34. (a)
I Green's Theorem can be used. The curve is closed and the vector field is smooth throughout the interior of the region enclosed
II Green's Theorem cannot be used. The vector field is not defined at the origin which is inside the curve.
III Green's Theorem cannot be used. The curve is not closed.

(b) For the integral in [I], let R be the region enclosed by C. See Figure 18.33. Green's Theorem gives

$$\begin{aligned}\int_C (x^2+y^2)\, dx + (x^2+y^2)\, dy &= \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_R \left(\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2+y^2)\right) dA\\ &= \int_R (2x-2y)\, dA = \int_0^1 \int_{x^2}^{x} (2x-2y)\, dy\, dx\\ &= \int_0^1 (2xy - y^2)\Big|_{x^2}^{x} dx = \int_0^1 (2x^2 - x^2 - (2x^3 - x^4))\, dx\\ &= \int_0^1 (x^2 - 2x^3 + x^4)\, dx = \left(\frac{x^3}{3} - \frac{2}{4}x^4 + \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{30}.\end{aligned}$$

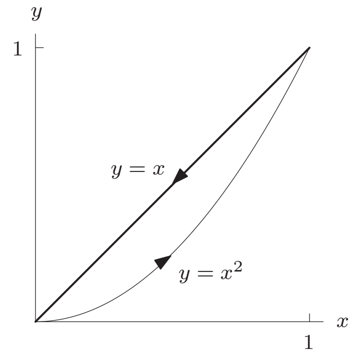


Figure 18.33

35. Green's theorem says that for a closed curve C oriented counterclockwise, bounding region R,

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_R (x^2 + y^2 - 1)\, dA.$$

If R is a region contained strictly inside the unit circle, then $x^2 + y^2 < 1$ for any point (x, y) in R, so $x^2 + y^2 - 1 < 0$, which gives

$$\int_R (x^2 + y^2 - 1)\, dA < 0, \text{ which implies that } \int_C \vec{F} \cdot d\vec{r} < 0.$$

Now, let C be the curve $C_1 - C_2$. Since

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = L_1 - L_2 < 0,$$

we have $L_1 < L_2$. Similarly, if we let $C = C_2 - C_3$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_3} \vec{F} \cdot d\vec{r} = L_2 - L_3 < 0,$$

which gives $L_2 < L_3$. Thus

$$L_1 < L_2 < L_3.$$

36. (a) Writing R_1 for the interior of the circle, Green's Theorem gives

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{R_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_S 3\, dA = 3 \cdot \text{ Area of disk } = 3 \cdot \pi 1^2 = 3\pi.$$

(b) Writing R_2 as the interior of the rectangle, Green's Theorem gives

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{R_2} 3\, dA = 3 \cdot \text{ Area of rectangle } = 3 \cdot 3 \cdot 2 = 18.$$

(c) In parts (a) and (b), we see that the line integral is three times the area enclosed in the curve. Since C_3 encloses a disk of radius 7 and area $\pi \cdot 7^2 = 153.9$, and C_4 encloses a disk of radius 8 and area $\pi \cdot 8^2 = 201.1$, and C_5 encloses a square of side 14 and area $14^2 = 196$, we have

$$\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_5} \vec{F} \cdot d\vec{r} < \int_{C_4} \vec{F} \cdot d\vec{r}.$$

37. (a) We use Green's Theorem. Let R be the region enclosed by the circle C. Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_R \left(\frac{\partial}{\partial x}(e^{y^2} + 12x) - \frac{\partial}{\partial y}(3x^2y + y^3 + e^x)\right) dA$$
$$= \int_R (12 - (3x^2 + 3y^2)\, dA = \int_R (12 - 3(x^2 + y^2))\, dA.$$

Converting to polar coordinates, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\int_0^1 (12 - 3r^2) r\, dr\, d\theta = 2\pi\left(6r^2 - \frac{3}{4}r^4\right)\bigg|_0^1 = 2\pi\left(6 - \frac{3}{4}\right) = \frac{21\pi}{2}.$$

(b) The integrand of the integral over the disk R is $12 - 3(x^2 + y^2)$. Since the integrand is positive for $x^2 + y^2 < 4$ and negative for $x^2 + y^2 > 4$, the integrand is positive inside the circle of radius 2 and negative outside that circle. Thus, the integral over R increases with a until $a = 2$ and then decreases. The maximum value of the line integral occurs when $a = 2$.

38. (a) Taking partial derivatives using the product rule, we have

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{1}{x^2 + y^2} - \frac{x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Similarly,

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{-1}{x^2 + y^2} + \frac{y \cdot 2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus,

$$\text{curl of } \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

(b) Let C be the clock-wise oriented closed path consisting of four pieces

- C_1
- BD, the straight line path from B to D
- $-C_2$
- CA, the straight line path from C to A.

By Green's Theorem and part (a),

$$\int_C \vec{F}\cdot d\vec{r} = 0.$$

Hence

$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{BD} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r} = 0.$$

The key observation is that the vector field $\vec{F}$ is perpendicular to each of the radial paths BD and CA. To see this, consider the radial vector field $\vec{r} = x\vec{i} + y\vec{j}$ which is tangent to the paths BD and CA. Since

$$\vec{r}\cdot\vec{F} = 0$$

the vector fields $\vec{F}$ and $\vec{r}$ are orthogonal.

Therefore, $\int_{BD} \vec{F}\cdot d\vec{r} = \int_{CA} \vec{F}\cdot d\vec{r} = 0$ and it follows that

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

(c) Compute the integral over C_2, which is on the unit circle. On the unit circle, $\vec{F}$ is tangent to the circle and $||\vec{F}|| = 1$. Thus,

$$\int_C \vec{F}\cdot d\vec{r} = ||\vec{F}|| \cdot \text{Length of curve} = 1 \cdot \theta = \theta.$$

39. (a) We can show

$$\operatorname{curl}\vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{qx}{\|\vec{r}\|^3} & \frac{qy}{\|\vec{r}\|^3} & \frac{qz}{\|\vec{r}\|^3} \end{vmatrix} = \vec{0}\,.$$

Let's check, for instance, the $\vec{i}$ component of curl $\vec{E}$:

$$\frac{\partial}{\partial y}\frac{qz}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial z}\frac{qy}{(x^2+y^2+z^2)^{3/2}} = \frac{(-3/2)2qyz-(-3/2)2qzy}{(x^2+y^2+z^2)^{5/2}} = 0.$$

The vector field $\vec{E}$ is a gradient vector field, as curl $\vec{E} = 0$ and $\vec{E}$ is defined everywhere in 3-space except at the origin. This domain satisfies the criteria for the curl test in 3-space. Every closed curve in 3-space which does not pass through 0 bounds a surface not containing the origin.

(b) The function $\varphi(\vec{r}) = q/\|\vec{r}\|$ is a potential for $\vec{E}$, since

$$\frac{\partial\varphi}{\partial x} = q\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -qx(x^2+y^2+z^2)^{-3/2} = -E_1$$

and similarly for $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$; hence $\vec{E} = -\operatorname{grad}\varphi$.

Strengthen Your Understanding

40. To conclude that $\vec{F}$ is path-independent, we must know that $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path in the domain of $\vec{F}$. It is not sufficient to check the equality for a single specific closed path.

41. Since C is not a closed curve, Green's Theorem does not apply. Calculating the line integral requires more information about $\vec{F}$.

42. If $\vec{F} = xy\vec{i} + Q(x,y)\vec{j}$ is a gradient field, then by the curl test we see that

$$\frac{\partial}{\partial y}(xy) = \frac{\partial Q}{\partial x}.$$

Therefore, $\partial Q/\partial x = x$. If $Q(x,y) = x^2/2$, then

$$\vec{F} = xy\vec{i} + \frac{x^2}{2}\vec{j} = \operatorname{grad}\left(\frac{x^2y}{2}\right).$$

43. Because the scalar curl of $\vec{F}$ is zero everywhere except at the origin, the vector field $\vec{F}$ is path independent in any region that does not contain or encircle the origin. Let C_1 be the counterclockwise path on the unit circle from $(1,0)$ to $(0,1)$ and let C_2 be the clockwise path on the unit circle from $(1,0)$ to $(0,1)$. See Figure 18.34.

The paths C_1 and C_2 are not both contained in a single region of the plane that does not contain or encircle the origin, so it is possible that the line integrals of $\vec{F}$ over C_1 and C_2 are not equal. Since $\vec{F}$ is tangent to the unit circle, points counterclockwise, and $\|\vec{F}\| = 1$, we have

$$\int_{C_1}\vec{F}\cdot d\vec{r} = 1\cdot \text{ Length } C_1 = \frac{\pi}{2} \quad\text{and}\quad \int_{C_2}\vec{F}\cdot d\vec{r} = -1\cdot \text{ Length } C_2 = -\frac{3\pi}{2}.$$

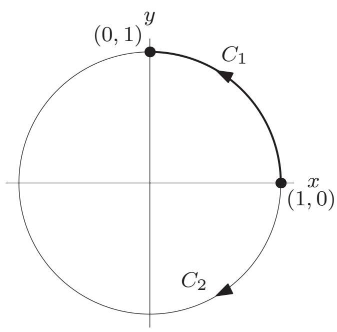


Figure 18.34

44. The vector field $\vec{F} = x\vec{i} + x^2\vec{j}$ is not a gradient field because its scalar curl is not zero:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}x = 2x \neq 0.$$

45. True. The value of $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is $0 - 0 = 0$, so the field is path-independent.

46. False. The fact that $\vec{F} = \operatorname{grad} f$ means that $\vec{F}$ is a potential field, hence $\vec{F}$ is path-independent. Thus $\int_C \vec{F} \cdot d\vec{r} = 0$ since C is closed.

47. True. Since $\vec{F}$ and $\vec{G}$ are both path-independent, we know $\vec{F} = \operatorname{grad} f$ and $\vec{G} = \operatorname{grad} g$ for some scalar functions f and g. Then $\operatorname{grad}(f+g) = \operatorname{grad} f + \operatorname{grad} g = \vec{F} + \vec{G}$, so $\vec{F} + \vec{G}$ is a gradient field, hence path-independent.

48. False. As a counterexample, consider $\vec{F} = x\vec{j}$ and $\vec{G} = y\vec{i}$. Then both of these are path-dependent (they each have nonzero curl), but the curl of $\vec{F} + \vec{G} = y\vec{i} + x\vec{j}$ is zero everywhere, so $\vec{F} + \vec{G}$ is path-independent.

49. True. This vector field has components $F_1 = x, F_2 = y$, and $F_3 = z$. Using the 3-space curl test gives zero for all of the components of $\operatorname{curl}\vec{F}$, so the field is path-independent.

50. True. The value of $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is $0 - 0 = 0$, so the field is path-independent.

51. True. Since $\vec{F}$ is path-independent, we know $\vec{F} = \operatorname{grad} f$ for some scalar function f. Then $\operatorname{grad}(kf) = k \operatorname{grad} f = k\vec{F}$, so $k\vec{F}$ is a gradient field, hence path-independent.

52. False. As a counterexample, consider the vector field $\vec{F} = 2x\vec{i}$, which is path-independent, since it is the gradient of $f(x,y) = x^2$. Multiplying $\vec{F}$ by the function $h(x,y) = y$ gives the field $y\vec{F} = 2xy\vec{i}$. The curl of this vector field is $-2x \neq 0$, so $y\vec{F}$ is path-dependent.

Solutions for Chapter 18 Review

Exercises

1. On the top half of the circle, the angle between the vector field and the curve is less than 90°, so the line integral is positive. On the bottom half of the circle, the angle is more than 90°, so the line integral is negative. However the magnitude of the vector field is larger on the top half of the curve, so the positive contribution to the line integral is larger than the negative. Thus the line integral $\int_C \vec{F} \cdot d\vec{r}$ is positive.

2. The angle between the vector field and the curve is more than 90° at all points on C, so the line integral is negative.

3. (a) The line integral around A is zero, because the curve is perpendicular to the field everywhere.

(b) The line integral along C_1 or C_3 is zero because the curves are everywhere perpendicular to the vector field. Along C_2, the line integral is negative, since $\vec{F}$ points along the opposite direction to the curve. Along C_4, the line integral is positive, since $\vec{F}$ points in the same direction as the curve.

(c) The line integral around C is zero because C_1 and C_3 are perpendicular to the field and the contributions from C_2 and C_4 cancel out.

4. (a) The line integral around A is negative, because the vectors of the field are all pointing in the opposite direction to the direction of the path.

(b) Along C_1, the line integral is positive, since $\vec{F}$ points in the same direction as the curve. Along C_2 or C_4, the line integral is zero, since $\vec{F}$ is perpendicular to the curve everywhere. Along C_3, the line integral is negative, since $\vec{F}$ points in the opposite direction to the curve.

(c) The line integral around C is negative because C_3 is longer than C_1 and the magnitude of the field is bigger along C_3 than C_1.

5. Scalar. The displacement along the line from $(5,2)$ to $(1,8)$ is given by $-4\vec{i} + 6\vec{j}$, so

$$\int_C (3\vec{i} + 4\vec{j}) \cdot d\vec{r} = (3\vec{i} + 4\vec{j}) \cdot (-4\vec{i} + 6\vec{j}) = 12.$$

6. Scalar. Since the path is along the y-axis, the $\vec{i}$ component does not contribute to the line integral. On the path, $d\vec{r} = \vec{j}\, dy$, so

$$\int_C (x\vec{i} + x\vec{j}) \cdot d\vec{r} = \int_2^6 (x\vec{i} + y\vec{j}) \cdot \vec{j}\; dy = \int_2^6 y\, dy = \left.\frac{y^2}{2}\right|_2^6 = 16.$$

7. Since $\vec{F} = 6\vec{i} - 7\vec{j}$, consider the function f

$$f(x, y) = 6x - 7y.$$

Then we see that grad $f = 6\vec{i} - 7\vec{j}$, so we use the Fundamental Theorem of Calculus for Line Integrals:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C \text{grad}\, f \cdot d\vec{r} \\ &= f(4, 4) - f(2, -6) = (-4) - (54) = -58.\end{aligned}$$

8. We know that if $f(x, y) = \dfrac{x^2}{2} + \dfrac{y^2}{2}$, then grad $f = x\vec{i} + y\vec{j} = \vec{F}$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

9. Since $\vec{F}$ is a gradient field, $\vec{F} = \text{grad}\left(\dfrac{x^2}{2} + \dfrac{y^2}{2}\right)$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \left.\left(\frac{x^2}{2} + \frac{y^2}{2}\right)\right|_{(0,0)}^{(0,10)} = \frac{100}{2} - 0 = 50.$$

10. We can parameterize the curve C by $(t, t^2 + 1)$, for $0 \le t \le 1$. Then

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(t, t^2 + 1) \cdot (\vec{i} + 2t\vec{j})dt = \int_0^1 ((-1)\vec{i} + (t^4 + 2t^2 + t + 1)\vec{j}) \cdot (\vec{i} + 2t\vec{j})dt \\ &= \int_0^1 (-1 + 2t(t^4 + 2t^2 + t + 1))dt = \int_0^1 (-1 + 2t^5 + 4t^3 + 2t^2 + 2t)dt \\ &= \left.\left(-t + \frac{2t^6}{6} + \frac{4t^4}{4} + \frac{2t^3}{3} + t^2\right)\right|_0^1 = 2\end{aligned}$$

11. Since $\vec{F} = \text{grad}\left(\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2}\right)$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F} \cdot d\vec{r} = \left.\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right)\right|_{(2,3,0)}^{(0,0,7)} = \frac{7^2}{2} - \left(\frac{2^2}{2} + \frac{3^2}{2}\right) = 18.$$

12. The path can be broken into three line segments: C_1, from $(1, 0)$ to $(-1, 0)$, and C_2, from $(-1, 0)$ to $(0, 1)$, and C_3, from $(0, 1)$ to $(1, 0)$. (See Figure 18.35.)

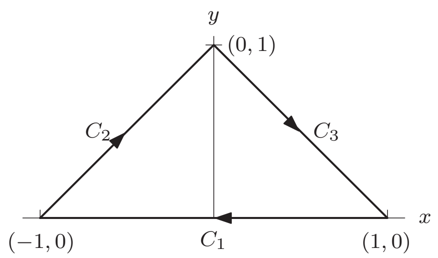


Figure 18.35

Along C_1 we have $y = 0$ so the vector field $xy\vec{i} + (x-y)\vec{j}$ is perpendicular to C_1; Thus, the line integral along C_1 is 0.

C_2 can be parameterized by $(-1+t, t)$, for $0 \le t \le 1$ so the integral is

$$\begin{aligned}\int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(-1+t, t) \cdot (\vec{i} + \vec{j})\, dt \\ &= \int_0^1 [t(-1+t)\vec{i} + (-1)\vec{j}] \cdot (\vec{i} + \vec{j})\, dt \\ &= \int_0^1 (-t + t^2 - 1)\, dt \\ &= (-t^2/2 + t^3/3 - t)\Big|_0^1 \\ &= -1/2 + 1/3 - 1 - (0+0+0) = -7/6\end{aligned}$$

C_3 can be parameterized by $(t, 1-t)$, for $0 \le t \le 1$ so the integral is

$$\begin{aligned}\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(t, 1-t) \cdot (\vec{i} - \vec{j})\, dt \\ &= \int_0^1 (t(1-t)\vec{i} + (2t-1)\vec{j}) \cdot (\vec{i} - \vec{j})\, dt \\ &= \int_0^1 (-t^2 - t + 1)\, dt \\ &= (-t^3/3 - t^2/2 + t)\Big|_0^1 \\ &= -1/3 - 1/2 + 1 - (0+0+0) = 1/6\end{aligned}$$

So the total line integral is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} = 0 + (-\frac{7}{6}) + \frac{1}{6} = -1$$

13. Using x as the parameter we have $dy = 2xdx$. Thus

$$\int_C 3x^2dx + 4ydy = \int_1^5 3x^2dx + 4x^2(2xdx) = \int_1^5 x^2 + 8x^3\, dx = x^3 + 2x^4\Big|_1^5 = 1372.$$

14. Using x as the parameter we have $dy = \cos x\, dx$. Thus

$$\begin{aligned}\int_C ydx + xdy &= \int_0^{\pi/2} \sin x\, dx + x(\cos x\, dx) = \int_0^{\pi/2} \sin x + x\cos x\, dx \\ &= -\cos x + \cos x + x\sin x\Big|_0^{\pi/2} = \frac{\pi}{2}.\end{aligned}$$

15. The domain is all 3-space. Since $F_1 = y$,

$$\operatorname{curl} y\vec{i} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = -\vec{k} \ne \vec{0},$$

so $\vec{F}$ is not path-independent.

16. The domain is all 3-space. Since $F_2 = y$,

$$\operatorname{curl} y\vec{j} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \vec{0},$$

so $\vec{F}$ is path-independent.

17. The domain is all 3-space. Since $F_3 = z$,

$$\text{curl}\, z\vec{k} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \vec{0},$$

so $\vec{F}$ is path-independent.

18. Since $F_2 = F_3 = z$,

$$\text{curl}\,(z\vec{j} + z\vec{k}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = -\vec{i} \neq \vec{0},$$

so $\vec{F}$ is not path-independent.

19. The domain is all 3-space. Since $F_1 = y$, $F_2 = x$,

$$\text{curl}\, y\vec{i} + x\vec{j} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \vec{0},$$

so $\vec{F}$ is path-independent

20. The domain is all 3-space. Since $F_1 = x + y$,

$$\text{curl}\,(x+y)\vec{i} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = -\vec{k} \neq \vec{0},$$

so $\vec{F}$ is not path-independent.

21. The domain is all 3-space. Since $F_1 = yz$, $F_2 = zx$, $F_3 = xy$,

$$\begin{aligned}\text{curl}\,(yz\vec{i} + zx\vec{j} + xy\vec{k}) &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\\ &= (x-x)\vec{i} + (y-y)\vec{j} + (z-z)\vec{k} = \vec{0},\end{aligned}$$

so $\vec{F}$ is path-independent.

22. Since the line is parallel to the y-axis, only the $\vec{j}$-component contributes to the line integral. On C, we have $x = 2$, so $\vec{F} = 10\vec{i} + 6\vec{j}$ and $d\vec{r} = \vec{j}\, dy$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_3^8 6\vec{j} \cdot \vec{j}\, dy = 6 \cdot 5 = 30.$$

23. Since the line is parallel to the x-axis, only the $\vec{i}$-component contributes to the line integral. On C, we have $d\vec{r} = \vec{i}\, dx$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_2^{12} 5x\vec{i} \cdot \vec{i}\, dx = \frac{5}{2}x^2\bigg|_2^{12} = 350.$$

24. Parameterizing C by $x(t) = t$, $y(t) = t^2$, with $1 \leq t \leq 2$, we have $\vec{r}\,'(t) = \vec{i} + 2t\vec{j}$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 (5t\vec{i} + 3t\vec{j}) \cdot (\vec{i} + 2t\vec{j})\, dt = \int_1^2 (5t + 6t^2)\, dt = \left(\frac{5t^2}{2} + 2t^3\right)\bigg|_1^2 = \frac{43}{2}.$$

25. Parameterizing C by $x(t) = 3\cos t$, $y = 3\sin t$, with $0 \leq t \leq \pi$, we have $\vec{r}\,'(t) = -(3\sin t)\vec{i} + (3\cos t)\vec{j}$. Thus,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^\pi (15\cos t\vec{i} + 9\cos t\vec{j}) \cdot (-3\sin t\vec{i} + 3\cos t\vec{j})\, dt\\ &= 9\int_0^\pi (-5\cos t\sin t + 3\cos^2 t)\, dt\\ &= 9\left(\frac{5}{2}\cos^2 t + \frac{3}{2}(\cos t\sin t + t)\right)\bigg|_0^\pi\\ &= \frac{27\pi}{2}.\end{aligned}$$

The integral $\int \cos^2 t$ was calculated using Formula IV-18.

26. We use Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (5x\vec{i} + 3x\vec{j}) \cdot d\vec{r} = \int_R \left(\frac{\partial(3x)}{\partial x} - \frac{\partial(5x)}{\partial y} \right) dx\, dy$$

$$= \int_R 3\, dx\, dy = 3 \cdot \text{ Area of region} = 3(3 \cdot 2 + \frac{1}{2} 3 \cdot 3) = \frac{63}{2}.$$

27. We can calculate this line integral either by calculating a separate line integral for each side, or by adding a line segment, C_1, from $(1, 4)$ to $(1, 1)$ to form the closed curve $C + C_1$. Since we now have a closed curve, we can use Green's Theorem:

$$\int_{C+C_1} \vec{F} \cdot d\vec{r} = \int_{C+C_1} (5x\vec{i} + 3x\vec{j}) \cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(5y) \right) dx\, dy$$

$$= \int_R 3\, dx\, dy = 3 \cdot \text{ Area of region} = 3\left(2 \cdot 3 + \frac{1}{2} 3 \cdot 4\right) = 36.$$

Since $d\vec{r} = -\vec{j}\, dy$ on C_1, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_4^1 3 \cdot 1\vec{j} \cdot (-\vec{j}\ dy) = -3 \cdot 3 = -9.$$

Since

$$\int_{C+C_1} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} - 9 = 36$$

we have

$$\int_C \vec{F} \cdot d\vec{r} = 45.$$

28. Since $\vec{F} = \text{grad}(5x + 4y)$, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = (5x + 4y) \Big|_{(3,4)}^{(7,4)} = (5 \cdot 7 + 4 \cdot 4) - (5 \cdot 3 + 4 \cdot 4) = 20.$$

29. Since $\vec{F} = \text{grad}\left(\frac{5}{2}x^2 + 2y^2\right)$, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \left(\frac{5}{2}x^2 + 2y^2\right)\Big|_{(3,4)}^{(7,4)} = \left(\frac{5}{2} \cdot 7^2 + 2 \cdot 4^2\right) - \left(\frac{5}{2} \cdot 3^2 + 2 \cdot 4^2\right) = 100.$$

30. Since $\vec{F}$ is not a gradient vector field, and

$$\frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(5y) = 4 - 5 = -1,$$

we find the line integral around C_1 by Green's Theorem. The path C_1 is oriented clockwise, so with R_1 as the disk inside C_1, we have

$$\int_{C_1} \vec{F} \cdot d\vec{x} = -\int_{R_1} -1\, dx\, dy = \text{ Area of disk } = 9\pi.$$

For C_2, we parameterize the curve

$$x = 5 - 2\cos t, \quad y = 4 + 2\sin t, \quad 0 \le t \le \pi.$$

Then $\vec{r}\,'(t) = (2\sin t)\vec{i} + (2\cos t)\vec{j}$, so

$$\begin{aligned}
\int_{C_2} \vec{F} \cdot d\vec{r} &= \int_{C_2} (5y\vec{i} + 4x\vec{j}) \cdot d\vec{r} \\
&= \int_0^{\pi} (5(4+2\sin t)\vec{i} + 4(5-2\cos t)\vec{j}) \cdot (2\sin t\vec{i} + 2\cos t\vec{j})\, dt \\
&= \int_0^{\pi} (40\cos t - 16\cos^2 t + 40\sin t + 20\sin^2 t)\, dt \\
&= 88.283.
\end{aligned}$$

The integral has been computed numerically. The exact answer, obtained using the Table of Integrals, is $80 + 2\pi$.

Problems

31. If a vector field is a gradient vector field, it has zero circulation around every closed curve. Vector fields (i) and (iii) do not have this property. Therefore, (ii) and (iv) could represent gradient vector fields.

32. (a) The curves C_1 and C_3 give line integrals which we expect to be zero because at every point, the curve looks perpendicular to the vector field.

(b) The curve C_4 gives a negative line integral because the path is traversed in the direction opposite to the vector field.

(c) The line integrals along C_2, C_5, C_6 and C_7 are all positive. The vector field is path-independent; it is the gradient of a function f whose contours appear to be equally spaced circles centered at the origin; the value of f increases going outward. By the Fundamental Theorem of Line Integrals, the value of a line integral is the difference between the values of f at the two endpoints. The difference between the radii of the circles containing the endpoints of C_2 and the difference between the radii of the circles containing the endpoints of C_6 look about the same, so the line integrals along C_2 and C_6 are approximately equal. Since C_6 and C_7 have the same endpoints, their line integrals are also equal. The difference between the radii of the circles containing the endpoints of C_2 is less than the difference between the radii of the circles containing the endpoints of C_5, so the line integral along C_2 is smaller than the line integral along C_5. Thus

$$C_2 = C_6 = C_7 < C_5.$$

33. The original integral is around the unit circle, oriented counterclockwise.

(a) This integral uses the same parameterization, but goes twice around the circle. The value of the integral is 24.

(b) This integral uses the same parameterization, but with the limits reversed and with $-d\vec{r}$ instead of $d\vec{r}$. Thus, the value of the integral does not change; it is 12.

(c) This integral uses a different parameterization of the circle $x = \sin t, y = \cos t$, which goes once around the circle clockwise. The value of the integral is -12.

34. (a) The path C is a line segment, tangent to $\vec{T} = \vec{i} + \vec{j}$ at every point. Because the path C is on the line $y = x$ we have $\vec{F}(x,y) = 2\vec{i} + 2\vec{j} = 2\vec{T}$ on C. Thus $\vec{F}$ is tangent to C at every point and points in the direction of the orientation of C. The angle between C and $\vec{F}$ is 0.

(b) On C we have $\|\vec{F}\| = \|2\vec{i} + 2\vec{j}\| = 2\sqrt{2}$.

(c) The path C has length $5\sqrt{2}$. Since the vector field $\vec{F}$ is everywhere tangent to C in the direction of the orientation and of constant magnitude $2\sqrt{2}$ we have

$$\int_C \vec{F} \cdot d\vec{r} = \|\vec{F}\| \cdot \text{Length of } C = 2\sqrt{2} \cdot 5\sqrt{2} = 20.$$

35. (a) For path (i), we have $x(t) = t, y(t) = t^2$, so $x'(t) = 1, y'(t) = 2t$. Thus,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(t,t^2) \cdot (\vec{i} + 2t\vec{j})\, dt \\
&= \int_0^1 [(t+t^2)\vec{i} + t\vec{j}] \cdot (\vec{i} + 2t\vec{j})\, dt \\
&= \int_0^1 (t + 3t^2)\, dt
\end{aligned}$$

$$= (\frac{t^2}{2} + t^3)\Big|_0^1$$
$$= \frac{1}{2} + 1 - (0+0) = \frac{3}{2}.$$

For path (ii), we have $x(t) = t^2, y(t) = t$, so $x'(t) = 2t, y'(t) = 1$. Thus,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(t^2, t) \cdot (2t\vec{i} + \vec{j})\,dt \\
&= \int_0^1 [(t^2+t)\vec{i} + t^2\vec{j}] \cdot (2t\vec{i} + \vec{j})\,dt \\
&= \int_0^1 (2t^3 + 3t^2)\,dt \\
&= (\frac{t^4}{2} + t^3)\Big|_0^1 \\
&= \frac{1}{2} + 1 - (0+0) = \frac{3}{2}.
\end{aligned}$$

For path (iii), we have $x(t) = t, y(t) = t^n$, so $x'(t) = 1, y'(t) = nt^{n-1}$. Thus,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(t, t^n) \cdot (\vec{i} + nt^{n-1}\vec{j})\,dt \\
&= \int_0^1 [(t+t^n)\vec{i} + t\vec{j}] \cdot (\vec{i} + nt^{n-1}\vec{j})\,dt \\
&= \int_0^1 (t + t^n + nt^n)\,dt \\
&= \int_0^1 (t + (n+1)t^n)\,dt \\
&= \left(\frac{t^2}{2} + t^{n+1}\right)\Big|_0^1 \\
&= \frac{1}{2} + 1 - (0+0) = \frac{3}{2}.
\end{aligned}$$

(b) If $f(x, y) = xy + x^2/2$, we have $\vec{F} = \operatorname{grad} f$. Each path goes from $(0, 0)$ to $(1, 1)$. Thus in each case

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = \frac{3}{2}.$$

36. The path C is the displacement $\vec{v} = (4-5)\vec{i} + (7-5)\vec{j} = -\vec{i} + 2\vec{j}$. Since the vector field is constant, the line integral is

$$\int_C (2\vec{i} + 13\vec{j}) \cdot d\vec{r} = (2\vec{i} + 13\vec{j}) \cdot (-\vec{i} + 2\vec{j}) = -2 + 26 = 24$$

37. This is a gradient vector field,

$$4x\vec{i} + 3y\vec{j} = \operatorname{grad}\left(2x^2 + \frac{3}{2}y^2\right).$$

Thus,

$$\int_C (4x\vec{i} + 3y\vec{j}) \cdot d\vec{r} = (2x^2 + \frac{3}{2}y^2)\Big|_{(5,5)}^{(4,7)} = 2 \cdot 4^2 + \frac{3}{2} \cdot 7^2 - 2 \cdot 5^2 - \frac{3}{2} \cdot 5^2 = 18.$$

38. This is not a gradient field, so we parameterize the line segment. The vector from $(5, 5)$ to $(4, 7)$ is $\vec{v} = -\vec{i} + 2\vec{j}$, so the parameterization is

$$x = 5 - t, \quad y = 5 + 2t, \quad 0 \le t \le 1,$$

and $d\vec{r} = (-\vec{i} + 2\vec{j})\,dt$. The integral is

$$\int_C ((4x+5y)\vec{i} + (2x+3y)\vec{j}) \cdot d\vec{r}$$

$$= \int_0^1 (4(5-t)+5(5+2t))\vec{i} + (2(5-t)+3(5+2t)\vec{j}) \cdot (-\vec{i} + 2\vec{j})\,dt$$
$$= \int_0^1 -(20-4t+25+10t) + 2(10-2t+15+6t)\,dt$$
$$= \int_0^1 (5+2t)dt = 5t + t^2\Big|_0^1 = 6.$$

39. Since the curve is closed, we use Green's Theorem:

$$\int_C ((4x+5y)\vec{i} + (2x+3y)\vec{j}) \cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(2x+3y) - \frac{\partial}{\partial y}(4x+5y)\right) dA = \int_R -3\,dA = -3 \cdot \text{Area of star.}$$

Each of the four points of the star has area $\frac{1}{2}\cdot 2\cdot 2 = 2$; the square center has area $2^2 = 4$, so area of star is $4\cdot 2+4 = 12$. Thus,

$$\int_C ((4x+5y)\vec{i} + (2x+3y)\vec{j}) \cdot d\vec{r} = -3\cdot 12 = -36.$$

40. (a) By Green's Theorem, if R is the interior of C,

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y)\right) dA = \int_R 3dA = 3\cdot \text{Area of region} = 3\left(2\cdot\frac{1}{2}10(9-1)\right) = 240.$$

(b) On C_1, we have $y = 1$, so $\vec{F} = 2\vec{i} + 5x\vec{j}$. Only the $\vec{i}$ component contributes to the line integral, so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = 2\cdot \text{Length of } C_1 = 2\cdot 20 = 40.$$

(c) Since $C = C_1 + C_2$, we have

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} = 240 - 40 = 200.$$

41. (a) Since $\vec{F} = \text{grad}(x^2e^y)$ is a gradient vector field and C is a closed curve, $\int_C \vec{F} \cdot d\vec{r} = 0$.
(b) Since

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y) = 2,$$

by Green's Theorem,

$$\int_C \vec{G} \cdot d\vec{r} = \int_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) dA = 2\cdot \text{ Area of triangle } = \frac{2\cdot 3\cdot 8}{2} = 24.$$

42. (a) Since $\vec{F} = \text{grad}\left(x^3/3 + x^3y^4\right)$, the Fundamental Theorem of Line Integrals gives

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \frac{x^3}{3} + x^3y^4\Big|_{(2,0)}^{(-2,0)} = -\frac{8}{3} + 0 - \left(\frac{8}{3}+0\right) = -\frac{16}{3}.$$

(b) Since a gradient field is path-independent, and the endpoints of C_1 and C_2 are the same:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = -\frac{16}{3}.$$

(c) The vector field $\vec{G}$ is not a gradient vector field, so we parameterize C_1. Using $x(t) = t$, $y(t) = 0$, from $t = 2$ to $t = -2$ gives

$$\int_{C_1} \vec{G} \cdot d\vec{r} = \int_2^{-2} (t^4 + 0)\vec{i} + (0\vec{j}) \cdot \vec{i}\, dt = \int_2^{-2} t^4\, dt = \frac{t^5}{5}\bigg|_2^{-2} = -\frac{64}{5}.$$

(d) Parameterizing C_2 by $x(t) = 2\cos t, y(t) = 2\sin t$ for $0 \le t \le \pi$ gives

$$\int_{C_2} \vec{G} \cdot d\vec{r} = \int_0^{\pi} \left(\left((2\cos t)^4 + (2\cos t)^3(2\sin t)^2\right)\vec{i} + (2\cos t)^2(2\sin t)^3\vec{j}\right) \cdot (-2\sin t\vec{i} + 2\cos t\vec{j})\, dt$$
$$= 32\int_0^{\pi} (-\cos^4 t\sin t - 2\cos^3 t\sin^3 t + 2\cos^3 t\sin^3 t)\, dt$$
$$= -32\int_0^{\pi} \cos^4 t\sin t\, dt = 32\frac{\cos^5 t}{5}\bigg|_0^{\pi} = -\frac{64}{5}.$$

43. (a) The vector field is everywhere perpendicular to the radial line from the origin to $(2, 3)$, so the line integral is 0.

(b) Since the path is parallel to the x-axis, only the $\vec{i}$ component of the vector field contributes to the line integral. The $\vec{i}$ component is $-3\vec{i}$ on this line, and the displacement along this line is $-2\vec{i}$, so

$$\text{Line integral } = (-3\vec{i}) \cdot (-2\vec{i}) = 6.$$

(c) The circle of radius 5 has equation $x^2 + y^2 = 25$. On this curve, $||\vec{F}|| = \sqrt{(-y^2) + x^2} = \sqrt{25} = 5$. In addition, $\vec{F}$ is everywhere tangent to the circle, and the path is $3/4$ of the circle. Thus

$$\text{Line integral} = ||\vec{F}|| \cdot \text{ Length of curve } = 5 \cdot \frac{3}{4} \cdot 2\pi(5) = \frac{75}{2}\pi.$$

(d) Use Green's Theorem. Writing C for the curve around the boundary of the triangle, we have

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2,$$

so

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\text{Triangle}} 2\, dA = 2 \cdot \text{Area of triangle } = 2 \cdot 7 = 14.$$

44. (a) Since $\vec{F} = (6x + y^2)\vec{i} + 2xy\vec{j} = \text{grad}(3x^2 + xy^2)$, the vector field $\vec{F}$ is path independent, so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = 0.$$

(b) Since C_1 is closed, we use Green's Theorem, so

$$\int_{C_1} \vec{G} \cdot d\vec{r} = \int_{\text{Interior of } C_1} \left(\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial y}(x - y)\right) dA$$
$$= 2\int_{C_1} dA = 2 \cdot \text{Area inside } C_1 = 2 \cdot \frac{1}{2} \cdot 2 \cdot 2 = 4.$$

(c) Since $\vec{F} = \text{grad}(3x^2 + xy^2)$, using the Fundamental Theorem of Line Integrals gives

$$\int_{C_2} \vec{F} \cdot d\vec{r} = (3x^2 + xy^2)\bigg|_{(2,0)}^{(0,-2)} = 0 - 3 \cdot 2^2 = -12.$$

(d) Parameterizing the circle by

$$x = 2\cos t \qquad y = 2\sin t \qquad 0 \le t \le \frac{3\pi}{2},$$

gives

$$x' = -2\sin t \qquad y' = 2\cos t,$$

so the integral is

$$\begin{aligned}\int_{C_2} \vec{G} \cdot d\vec{r} &= \int_0^{3\pi/2} \left((2\cos t - 2\sin t)\vec{i} + (2\cos t + 2\sin t)\vec{j}\right) \cdot (-2\sin t\vec{i} + 2\cos t\vec{j})\, dt \\ &= \int_0^{3\pi/2} 4(-\cos t \sin t + \sin^2 t + \cos^2 t + \sin t \cos t)\, dt \\ &= 4\int_0^{3\pi/2} dt = 4 \cdot \frac{3\pi}{2} = 6\pi.\end{aligned}$$

45. (a) Since $\vec{F} = x\vec{i} + y\vec{j} = \operatorname{grad}\left(\dfrac{x^2+y^2}{2}\right)$, we know that $\vec{F}$ is a gradient vector field. Thus, by the Fundamental Theorem of Line Integrals,

$$\int_{OA} \vec{F} \cdot d\vec{r} = \left.\frac{x^2+y^2}{2}\right|_{(0,0)}^{(3,0)} = \frac{9}{2}.$$

(b) We know that $\vec{F}$ is path independent. If C is the closed curve consisting of the line in part (a) followed by the two-part curve in part (b), then

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

Thus, if ABO is the two-part curve of part (b) and OA is the line in part (a),

$$\int_{ABO} \vec{F} \cdot d\vec{r} = -\int_{OA} \vec{F} \cdot d\vec{r} = -\frac{9}{2}.$$

46. (a) C_1 is a line along the vertical axis; C_2 is a half circle from the positive y to the negative y-axis. See Figure 18.36.

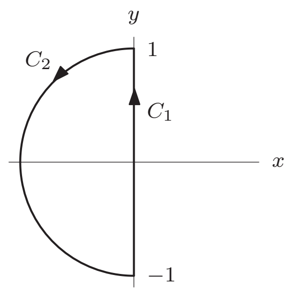


Figure 18.36

(b) Either use Green's Theorem or calculate directly. Using Green's Theorem, with R as the region inside C, we get

$$\begin{aligned}\int_{C_1+C_2} \vec{F} \cdot d\vec{r} &= \int_R \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x+3y)\right) dA \\ &= \int_R -3\, dA = -3(\text{Area of region}) = -3\frac{\pi \cdot 1^2}{2} = -\frac{3\pi}{2}.\end{aligned}$$

47. See Figure 18.37. The example chosen is the vector field $\vec{F}(x,y) = y\vec{j}$ and the path C is the line from $(0,-1)$ to $(0,1)$. Since the vectors are symmetric about the x-axis, the dot products $\vec{F} \cdot \Delta\vec{r}$ cancel out along C to give 0 for the line integral. Many other answers are possible.

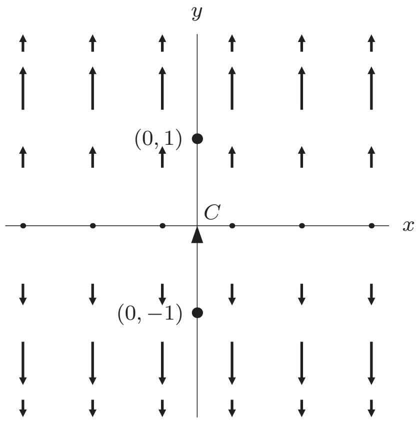


Figure 18.37

48. (a) See Figure 18.38. Notice that $C = C_1 + C_2 + C_3$ is a closed curve.
(b) See Figure 18.39.
(c) (i) Since the component of $\vec{F}$ in the direction of C_1 is $-\vec{i}$,

$$\int_{C_1} \vec{F} \cdot d\vec{r} = -\text{Length of } C_1 = -1.$$

(ii) Since $\vec{F}$ is parallel to C_2 and in the same direction, and $||\vec{F}|| = \sqrt{2}$,

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \sqrt{2} \cdot \text{Length of } C_2 = \sqrt{2} \cdot \sqrt{2} = 2.$$

(iii) Since the component of $\vec{F}$ in the direction of C_3 is $\vec{j}$, and a vector in the direction of C_3 is $-\vec{j}$,

$$\int_{C_3} \vec{F} \cdot d\vec{r} = -\text{Length of } C_3 = -1.$$

(iv) Since $\vec{F}$ is constant, it is a gradient field and C is closed,

$$\int_{C} \vec{F} \cdot d\vec{r} = 0.$$

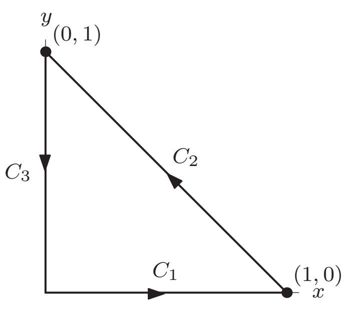


Figure 18.38

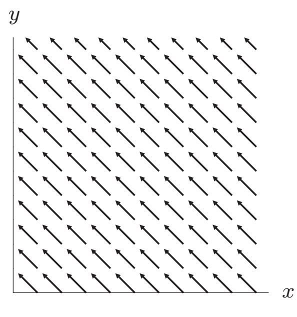


Figure 18.39

49. (a) Since $\dfrac{\partial}{\partial x}(y^5+x) - \dfrac{\partial}{\partial y}(x^3-y) = 1+1 = 2$, any closed curve oriented counterclockwise will do. See Figure 18.40.

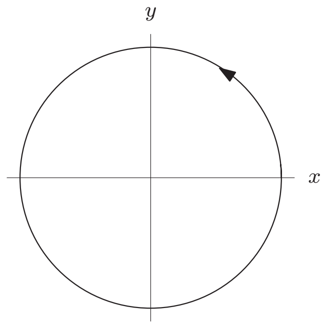


Figure 18.40

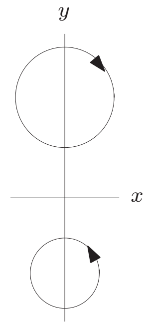


Figure 18.41

(b) Since $\frac{\partial}{\partial x}(y^5 - xy) - \frac{\partial}{\partial y}(x^3) = -y$, any closed curve in the lower half-plane oriented counterclockwise or any closed curve in the upper half-plane oriented clockwise will do. See Figure 18.41. Other answers are possible.

50. A contour of f is a set on which f does not change, so the total change of f from P to Q, $f(P) - f(Q)$, is zero. If C is a part of a contour of f, we know that $\operatorname{grad} f$ is perpendicular to C. This means that the line integral of $\operatorname{grad} f$ along C, which also computes the total change in f between its endpoints, must be zero, since the dot products in its definition are all zero.

51. (a) The vector field ∇f is perpendicular to the level curves, in direction of increasing f. The length of ∇f is the rate of change of f in that direction. See Figure 18.42

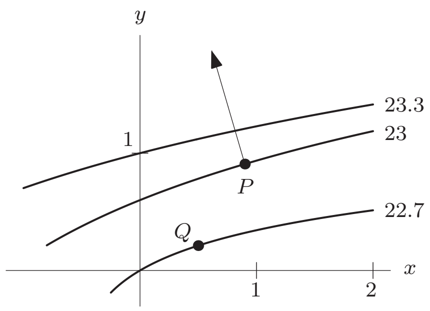


Figure 18.42

(b) Longer.

(c) Using the Fundamental Theorem of Calculus for Line Integrals, we have

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P) = 22.7 - 23 = -0.3.$$

52. (a) (i) The curve C is the line given by $\vec{r} = x\vec{i} + y\vec{j}$, which we can parameterize by $x = t, y = -t+1$ for $0 \le t \le 1$. Then $\vec{r}\,'(t) = \vec{i} - \vec{j}$ so

$$\int_C \vec{v} \cdot d\vec{r} = \int_0^1 ((1-t)\vec{i} + 2t\vec{j}) \cdot (\vec{i} - \vec{j})dt = \int_0^1 (1-3t)dt = -\frac{1}{2}.$$

(ii) The curve C is the circle given by $\vec{r} = x\vec{i} + y\vec{j}$ where $x = \sin t, y = \cos t$ for $0 \le t \le \frac{\pi}{2}$. Thus $\vec{r}\,'(t) = \cos t\vec{i} - \sin t\vec{j}$ and

$$\int_C \vec{v} \cdot d\vec{r} = \int_0^{\pi/2} (\cos t\vec{i} + 2\sin t\vec{j}) \cdot (\cos t\vec{i} - \sin t\vec{j})dt = \int_0^{\pi/2} (\cos^2 t - 2\sin^2 t)dt = -\frac{\pi}{4}.$$

(b) Since the value of the integral along two paths gives different results, $\vec{v}$ is not path independent.

53. (a) The path C is a line segment, tangent to $\vec{T} = \vec{i} + \vec{j}$ at every point. Because the path C is on the line $y = x$ we have $\vec{F}(x, y) = 2\vec{i} - 2\vec{j}$ on C. Hence $\vec{T} \cdot \vec{F} = 0$, which shows that C and $\vec{F}$ are perpendicular at every point of C. The angle between them in $\pi/2$.

(b) $\int_C \vec{F} \cdot d\vec{r} = 0$ because $\vec{F}$ and C are perpendicular at every point of C.

54. By Green's Theorem, if R_a is the interior of C_a

$$\int_{C_a} \vec{F} \cdot d\vec{r} = \int_{R_a} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

The quantity $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is positive for points (x, y) near the origin and negative farther away. This quantity changes sign where

$$\begin{aligned} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= 3(x^2 + y^2) - (x^2 + y^2)^{3/2} = 0 \\ 3(x^2 + y^2) &= (x^2 + y^2)^{3/2} \\ (x^2 + y^2)^{1/2} &= 3. \end{aligned}$$

Thus $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is positive within C_3, the circle of radius 3, and negative outside. The maximum value of the line integral occurs when $a = 3$. Converting to polars,

$$\begin{aligned} \int_{C_3} \vec{F} \cdot d\vec{r} &= \int_{R_3} \left(3(x^2 + y^2) - (x^2 + y^2)^{3/2} \right) dA \\ &= \int_0^{2\pi} \int_0^3 (3r^2 - r^3) r \, dr \, d\theta \\ &= 2\pi \left(\frac{3r^4}{4} - \frac{r^5}{5} \right) \Big|_0^3 = \frac{3^5 \pi}{10}. \end{aligned}$$

55. We'll assume that the rod is positioned along the z-axis, and look at the magnetic field $\vec{B}$ in the xy-plane. If C is a circle of radius r in the plane, centered at the origin, then we are told that the magnetic field is tangent to the circle and has constant magnitude $\|\vec{B}\|$. We divide the curve C into little pieces C_i and then we sum $\vec{B} \cdot \Delta\vec{r}$ computed on each piece C_i. But $\Delta\vec{r}$ points nearly in the same direction as $\vec{B}$, that is, tangent to C, and has magnitude nearly equal to the length of C_i. So the dot product is nearly equal to $\|\vec{B}\| \times$ length of C_i. When all of these dot products are summed and the limit is taken as $\|\Delta\vec{r}\| \to 0$, we get

$$\int_C \vec{B} \cdot d\vec{r} = \|\vec{B}\| \times \text{length of } C = \|\vec{B}\| \times 2\pi r$$

Now Ampère's Law also tells us that

$$\int_C \vec{B} \cdot d\vec{r} = kI$$

Setting these expressions for the line integral equal to each other and solving for $\|\vec{B}\|$ gives $kI = 2\pi r \|\vec{B}\|$, so

$$\|\vec{B}\| = \frac{kI}{2\pi r}.$$

56. (a) An example of a central field is in Figure 18.43.

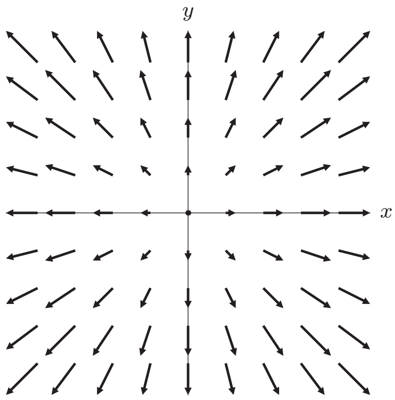


Figure 18.43

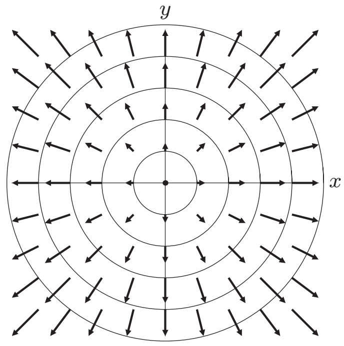


Figure 18.44

(b) The vectors of $\vec{F}$ are radial and the contours of f must be perpendicular to the vectors. Therefore, every contour must be a circle centered at the origin. Sketching some contours results in a diagram like that in Figure 18.44.

(c) No, not every gradient field is a central field, because there are gradient fields which are not perpendicular to circles. An example is the gradient of $f(x, y) = y$, where $\text{grad}\, f = \vec{j}$, so the gradient is parallel to the y axis. Thus, $\vec{F} = \vec{j}$ is an example of a gradient field which is not a central field.

(d) When a particle moves around a circle centered at O, no work is done, because $\vec{F}$ is tangent to the circle. Thus the only work done in moving from P to Q is in moving between the circles. Since $\vec{F}$ is central, the work done on any radial line between C_1 and C_3, for example, depends on only the radii of C_1 and C_3 ($\vec{F}$ is parallel to this path and its magnitude is a function of the distance to the center of the circle only). For that reason, on a path which goes from C_1 to C_2 and then from C_2 to C_3, the same amount of work will be done as on a path direct from C_1 to C_3.

(e) Pick any two points P and Q. Any path between them can be well-approximated by a path which is partly radial and partly around a circle centered at O. By the answer to part d), the work along any such path depends only on the radii of the circles on which P and Q sit, not on the path. Thus, the work done is independent of the path. Hence, $\vec{F}$ must be path-independent and therefore a gradient field.

57. (a) Since $-y\vec{i} + x\vec{j}$ is a counterclockwise rotation, both ω and K must be positive. In order to find the values of ω and K, we must look at the velocity field where we know the magnitude. At a radius of 100 m from the center, we know that $\sqrt{x^2+y^2} = 100$, and that $\|\vec{v}\| = 3 \cdot 10^5$. Thus, using $\vec{v} = \omega(-y\vec{i} + x\vec{j})$ we get

$$\|\vec{v}\| = \omega\sqrt{(-y)^2 + x^2} = 100\omega = 3 \cdot 10^5 \text{ meters/hr},$$

so

$$\omega = 3000 \, \text{rad/hr}.$$

Using $\vec{v} = K(x^2+y^2)^{-1}(-y\vec{i} + x\vec{j})$ gives

$$\|\vec{v}\| = |K|(x^2+y^2)^{-1}\sqrt{(-y)^2 + x^2} = \frac{K100}{100^2} = 3 \cdot 10^5 \text{ meters/hr},$$

so $K = 3 \cdot 10^7$ meters^2·rad/hr.

(b)

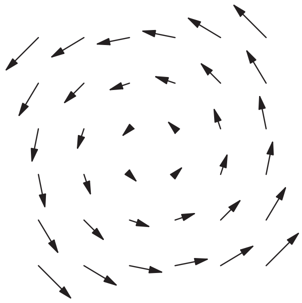


Figure 18.45

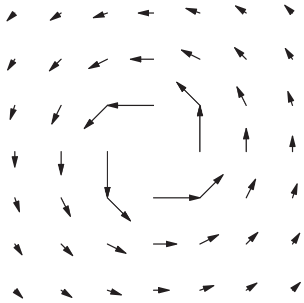


Figure 18.46

The vector field in Figure 18.45 shows the velocity vectors inside the tornado, (i.e. $r < 100$ meters). The vector field in Figure 18.46 shows the velocity vectors as seen from a great distance (i.e. $r >> 100$ meters) with the tornado at the origin.

(c) Let C be the circle of radius r around the origin. If $r < 100$ meters, the velocity vectors at distance r from the origin have magnitude ωr. Since they are tangent, and point counterclockwise, the circulation is

$$\int_C \vec{v}\cdot d\vec{r} = \|\vec{v}\|\cdot \text{Length of } C = 2\omega\pi r^2.$$

If $r \geq 100$ meters, the vectors at distance r from the origin have magnitude K/r and are again tangent to the circle. The circulation here is

$$\int_C \vec{v}\cdot d\vec{r} = \|v\|\cdot \text{ Length of } C = (\frac{K}{r})2\pi r = 2K\pi.$$

58. The free vortex appears to starts at about $r = 200$ meters (that's where the graph changes its behavior) and the tangential velocity at this point is about 200 km/hr $= 2\cdot 10^5$ meters/hr.

Since $\vec{v} = \omega(-y\vec{i} + x\vec{j})$ for $\sqrt{x^2+y^2} \leq 200$, at $r = 200$ we have

$$\|\vec{v}\| = \omega\sqrt{(-y)^2+x^2} = \omega(200) = 2\cdot 10^5 \text{ meters/hr},$$

so

$$\omega = 10^3 \text{ rad/hr}.$$

Since $\vec{v} = K(x^2+y^2)^{-1}(-y\vec{i} + x\vec{j})$ for $\sqrt{x^2+y^2} \geq 200$, at $r = 200$ we have

$$\|\vec{v}\| = K(200^2)^{-1}(200) = \frac{K}{200} = 2\cdot 10^5 \text{ meters/hr}$$

so

$$K = 4\cdot 10^7 \text{ m}^2\text{·rad/hr}.$$

CAS Challenge Problems

59. (a) We parameterize C_a by $\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j}$. Then, using a CAS, we find

$$\begin{aligned}\int_{C_a} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt &= \int_0^{2\pi} a\cos t\left(2a\cos t - \frac{a^3\cos t^3}{3} + a^3\cos t\sin t^2\right)\\ &\quad -a\sin t\left(-(a\sin t) + \frac{2a^3\sin t^3}{3}\right)dt\\ &= -\frac{\pi}{2}(-6a^2+a^4)\end{aligned}$$

The derivative of the expression on the right with respect to a is $-(2\pi)(-3a+a^3)$, which is zero at $a = 0, \pm\sqrt{3}$. Checking at $a = 0$ and as $a \to \infty$, we find the maximum is at $a = \sqrt{3}$.

(b) We have

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = (2 - x^2 + y^2) - (-1 + 2y^2) = 3 - x^2 - y^2.$$

So, by Green's theorem,

$$\int_{C_a} \vec{F}\cdot d\vec{r} = \int\int_{D_a} (3 - x^2 - y^2)\,dA,$$

where D_a is the disk of radius a centered at the origin. The integrand is positive for $x^2 + y^2 < 3$, so it is positive inside the disk of radius $\sqrt{3}$ and negative outside it. Thus the integral has its maximum value when $a = \sqrt{3}$.

60. We parameterize the line from $(0,0)$ to (x,y) by $\vec{r}(t) = t(x\vec{i} + y\vec{j})$. Using a CAS to compute the integral, we get

(a)

$$f(x,y) = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^1 2axyt\,dt = axy + \text{ Constant}$$

(b)

$$f(x,y) = \int_0^1 \vec{F}\,(\vec{r}\,(t)) \cdot \vec{r}\,'(t)dt$$
$$= \int_0^1 (abe^{bt^2xy}txy + (c + abe^{bt^2xy}tx)y)\,dt = ae^{bxy} + cy + \text{ Constant}$$

61. We have

$$\int_{C_1} \vec{F}\,\cdot d\vec{r}\, = \int_0^3 \vec{F}\,(\vec{r}\,(t)) \cdot \vec{r}\,'(t)dt$$
$$= \int_0^3 (2\left(2at + bt^2\right) + 2t\left(2ct + dt^2\right))\,dt = 18a + 18b + 36c + (81d/2)$$

and

$$\int_{C_2} \vec{F}\,\cdot d\vec{r}\, = \int_0^3 \vec{F}\,(\vec{r}\,(t)) \cdot \vec{r}\,'(t)dt$$
$$= \int_0^3 \left(-2\left(2a\,(3-t) + b(3-t)^2\right) - 2\left(2c\,(3-t) + d(3-t)^2\right)(3-t)\right)\,dt$$
$$= -18a - 18b - 36c - (81d/2)$$

The second integral is the negative of the first. This is because C_2 is the same curve as C_1 but traveling in the opposite direction.

PROJECTS FOR CHAPTER EIGHTEEN

1. (a) Since $\|\vec{v}\,(t)\|^2 = \vec{v}\,(t) \cdot \vec{v}\,(t)$ and since $\vec{v}\,(t) = \vec{r}\,'(t) = x'(t)\vec{i}\, + y'(t)\vec{j}\, + z'(t)\vec{k}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\vec{v}\,(t)\|^2 = \frac{1}{2}\frac{d}{dt}(\vec{v}\,(t) \cdot \vec{v}\,(t))$$
$$= \frac{1}{2}\frac{d}{dt}(x'(t)^2 + y'(t)^2 + z'(t)^2)$$
$$= \frac{1}{2}(2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t))$$
$$= x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)$$
$$= (x'(t)\vec{i}\, + y'(t)\vec{j}\, + z'(t)\vec{k}\,) \cdot (x''(t)\vec{i}\, + y''(t)\vec{j}\, + z''(t)\vec{k}\,)$$
$$= \vec{v}\,(t) \cdot \vec{a}\,(t) \quad \text{(Since } \vec{a}\,(t) = x''(t)\vec{i}\, + y''(t)\vec{j}\, + z''(t)\vec{k}\,.\text{)}$$
$$= \vec{a}\,(t) \cdot \vec{v}\,(t).$$

(b) (i) We use $\vec{F}\, = m\vec{a}$ and the parameterization of C given by $r(t)$ for $t_0 \le t \le t_1$. In addition, we need the fact that $\frac{1}{2}\frac{d}{dt}\|\vec{v}\,(t)\|^2 = \vec{a}\,\cdot\vec{v}$:

$$\int_C \vec{F}\,\cdot d\vec{r}\, = \int_C m\vec{a}\,\cdot d\vec{r}\, = \int_{t_0}^{t_1} m\vec{a}\,\cdot \vec{r}\,'dt$$
$$= \int_{t_0}^{t_1} m\,(\vec{a}\,\cdot\vec{v}\,)\,dt$$
$$= \int_{t_0}^{t_1} \frac{m}{2}\left(\frac{d}{dt}\|\vec{v}\,(t)\|^2\right)dt$$
$$= \frac{m}{2}\|\vec{v}\,(t)\|^2\Big|_{t_0}^{t_1}$$
$$= \frac{m}{2}\|\vec{v}\,(t_1)\|^2 - \frac{m}{2}\|\vec{v}\,(t_0)\|^2$$
$$= \text{Kinetic energy at } Q - \text{Kinetic energy at } P.$$

(ii) Since $\vec{F} = -\nabla f$ we use the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C -\nabla f \cdot d\vec{r} = -(f(\vec{r}(t_1)) - f(\vec{r}(t_0)))$$
$$= -(\text{Potential energy at } Q - \text{Potential energy at } P)$$
$$= \text{Potential energy at } P - \text{Potential energy at } Q.$$

(iii) In parts (a) and (b) we derived two expressions for the work done by $\vec{F}$ as the particle moves from P to Q. These two expressions must be equal, so

$$\text{Kinetic energy at } Q - \text{Kinetic energy at } P = \text{Potential energy at } P - \text{Potential energy at } Q.$$

Rewriting this equation we have,

$$(\text{Potential energy } + \text{Kinetic energy}) \text{ at } P = (\text{Potential energy } + \text{Kinetic energy}) \text{ at } Q.$$

This shows that the total energy is the same at P as at Q. Since P and Q are arbitrary points in space, the total energy of a particle moving in a force vector field $\vec{F} = -\nabla f$ is a constant.

2. (a) We have

$$\vec{m} = (x-a)\vec{i} + y\vec{j}.$$

Since $\vec{m}$ has magnitude L we have

$$(x-a)^2 + y^2 = L^2$$

and so

$$x - a = \sqrt{L^2 - y^2}$$

where we have used the fact that $a < x$. Thus

$$\vec{k} \times \vec{m} = -y\vec{i} + (x-a)\vec{j}.$$

Using once more the fact that $\vec{m}$ has magnitude L, we see that the unit vector $\vec{F}$ in the direction of $\vec{k} \times \vec{m}$ is

$$\vec{F} = \frac{1}{L}(\vec{k} \times \vec{m}) = \frac{-y}{L}\vec{i} + \frac{1}{L}\sqrt{L^2 - y^2}\vec{j}.$$

(b) We have

$$\text{curl}\,\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$
$$= \frac{1}{L}\frac{\partial \sqrt{L^2 - y^2}}{\partial x} - \frac{1}{L}\frac{\partial(-y)}{\partial y} = \frac{1}{L}.$$

(c) By Green's Theorem and part (b), we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \text{curl}\,\vec{F}\, dA$$
$$= \int_R \frac{1}{L} dA = \frac{1}{L} \cdot (\text{Area of } R).$$

Since $\vec{F}$ is by definition in part (a) a vector field of unit vectors in the direction of the wheel vector along C, we have

$$(\text{Area of } R) = L\int_C \vec{F} \cdot d\vec{r} = L \cdot (\text{Total roll of planimeter wheel}).$$

3. (a) We take the surface to be a disk of radius r, parallel to the xy-plane and centered on the z-axis. The boundary of the disk is the circle C. We know that the magnetic field, $\vec{B}$, is tangent to the circle and has constant magnitude, $\|\vec{B}\|$, along each circle. Thus, for any such circle,

$$\int_C \vec{B} \cdot d\vec{r} = \|\vec{B}\| \cdot \text{Length of } C = 2\pi r\|\vec{B}\|.$$

If $r \geq r_0$ (where r_0 is the radius of the wire) then the current through the surface is I. Therefore

$$2\pi r\|\vec{B}\| = \int_C \vec{B} \cdot d\vec{r} = kI,$$

so

$$\|\vec{B}\| = \frac{kI}{2\pi r}.$$

If $r < r_0$, then the current flowing through the surface is not I, but the amount of current passing through a disk of radius r. Such a disk has an area which is $(\pi r^2)/(\pi r_0^2)$ of the cross-sectional area of the wire. So the current to use in Ampère's Law is $(\pi r^2)/(\pi r_0^2)\, I$. Thus,

$$2\pi r\|\vec{B}\| = \int_C \vec{B} \cdot d\vec{r} = k\frac{\pi r^2}{\pi r_0^2}I.$$

Solving for $\|\vec{B}\|$ gives

$$\|\vec{B}\| = \frac{kIr}{2\pi r_0^2}.$$

(b) We again use Ampère's Law on a disk of radius r, lying perpendicular to, and centered on, the z-axis (See Figure 18.47). If the boundary, C, of this disk lies inside the torus, then the wire goes through the disk N times, and so the net current through the disk is NI. Thus,

$$2\pi r\|\vec{B}\| = \int_C \vec{B} \cdot d\vec{r} = kNI,$$

which gives

$$\|\vec{B}\| = \frac{kNI}{2\pi r}.$$

On the other hand, if the boundary, C, lies outside the torus, then the net current through the disk is 0. (The wire goes into the disk N times and out of the disk N times, and so the currents cancel.) Hence we have

$$2\pi r\|\vec{B}\| = \int_C \vec{B} \cdot d\vec{r} = 0.$$

So $\|\vec{B}\| = 0$.

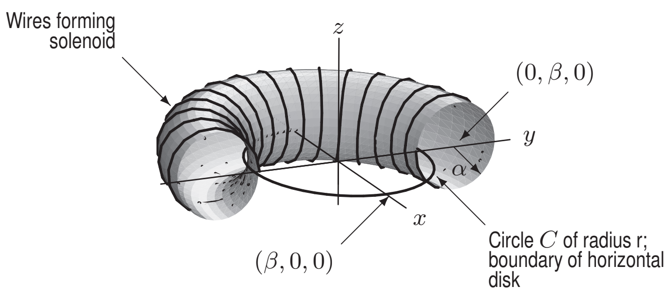


Figure 18.47

4. (a) (i) For each plane $\vec{F} = -c||\vec{v}\,||\vec{v}$ and since $\vec{r}\,'(t) = \vec{v}\,(t)$, the work done is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} (-c||\vec{v}\,(t)||\vec{v}\,(t)) \cdot \vec{v}\,(t)\,dt = -c\int_0^{\pi} ||\vec{v}\,(t)||^3\,dt.$$

The velocity of the first airplane is

$$\vec{v}_1(t) = (-2\sin t)\vec{i} + (2\cos t)\vec{j} + 3\vec{k}\,.$$

Since

$$||\vec{v}_1(t)|| = \sqrt{(-2\cos t)^2 + (2\sin t)^2 + 3^2} = \sqrt{13},$$

the work done by the force $\vec{F}_1$ is

$$W_1 = \int_C \vec{F}_1 \cdot d\vec{r}_1 = -c\int_0^{\pi} ||\vec{v}_1(t)||^3 dt = -c\int_0^{\pi} 13^{3/2} dt = -13^{3/2}\pi c.$$

For the second plane

$$\vec{v}_2(t) = -\frac{4}{\pi}\vec{i} + 3\vec{k}\,,$$

so

$$||\vec{v}_2(t)|| = \sqrt{\frac{16}{\pi^2} + 9},$$

and the work done by the force $\vec{F}_2$ is

$$W_2 = \int_C \vec{F}_2 \cdot d\vec{r}_2 = -c\int_0^{\pi} ||\vec{v}_2(t)||^3 dt = -c\int_0^{\pi} \left(\frac{16}{\pi^2} + 9\right)^{3/2} dt = -\left(\frac{16}{\pi^2} + 9\right)^{3/2} \pi c.$$

(ii) At time $t = 0$, both airplanes are at the point $(2, 0, 0)$. At $t = \pi$, both airplanes are at the point $(-2, 0, 3\pi)$; therefore, their trajectories have the same endpoints. The fact that $W_1 \neq W_2$ shows that the drag force is not conservative, since the work done by a conservative force is the same on all paths with the same endpoints.

(b) (i) The first plane's path is given by

$$\vec{r}_1(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + 3t\vec{k}\,,$$

so

$$||\vec{r}_1(t)|| = \sqrt{(2\cos t)^2 + (2\sin t)^2 + (3t)^2} = \sqrt{4 + 9t^2},$$

and

$$\vec{v}_1(t) = \vec{r}\,'_1(t) = (-2\sin t)\vec{i} + (2\cos t)\vec{j} + 3\vec{k}\,.$$

Thus, the gravitational force acting on the first airplane at time t is

$$\vec{F}_1(t) = -\frac{GMm}{(4 + 9t^2)^{3/2}}((2\cos t)\vec{i} + (2\sin t)\vec{j} + 3t\vec{k}\,),$$

so the total work done by the gravitational force on the first plane is

$$W_1 = \int_0^{\pi} \vec{F}_1(t) \cdot \vec{v}_1(t)\,dt = \int_0^{\pi} -\frac{GMm}{(4 + 9t^2)^{3/2}}(2\cos t\vec{i} + 2\sin t\vec{j} + 3\vec{k}\,) \cdot (-2\sin t\vec{i} + 2\cos t\vec{j} + 3t\vec{k}\,)\,dt$$
$$= \int_0^{\pi} -\frac{GMm}{(4 + 9t^2)^{3/2}} \cdot 9t\,dt.$$

Using the substitution $x = 4 + 9t^2$, we get

$$W_1 = GMm(4 + 9t^2)^{-1/2}\Big|_0^{\pi} = GMm\left(\frac{1}{\sqrt{4 + 9\pi^2}} - \frac{1}{\sqrt{2}}\right).$$

Similarly, for the second plane, we have

$$\vec{r}_2(t) = \left(2 - \frac{4}{\pi}t\right)\vec{i} + 3t\vec{k},$$

so

$$||\vec{r}_2(t)|| = \sqrt{\left(2 - \frac{4}{\pi}t\right)^2 + (3t)^2} = \sqrt{\left(2 - \frac{4}{\pi}t\right)^2 + 9t^2},$$

and

$$\vec{v}_2(t) = \vec{r}\,'_2(t) = -\frac{4}{\pi}\vec{i} + 3\vec{k}.$$

Thus, the gravitational force on the second plane is

$$\vec{F}_2(t) = -\frac{GMm}{((2-4t/\pi)^2 + 9t^2)^{3/2}}\left(\left(2 - \frac{4}{\pi}t\right)\vec{i} + 3t\vec{k}\right),$$

so the work done by the gravitational force on the second plane is

$$\begin{aligned} W_2 &= \int_0^{\pi} \vec{F}_2(t)\cdot\vec{v}_2(t)\,dt \\ &= \int_0^{\pi} -\frac{GMm}{((2-4t/\pi)^2 + 9t^2)^{3/2}}\left(\left(2 - \frac{4}{\pi}t\right)\vec{i} + 3t\vec{k}\right)\cdot\left(-\frac{4}{\pi}\vec{i} + 3\vec{k}\right)\,dt \\ &= \int_0^{\pi} -\frac{GMm}{((2-4t/\pi)^2 + 9t^2)^{3/2}}\left(-\frac{8}{\pi} + \frac{16}{\pi^2}t + 9t\right)\,dt. \end{aligned}$$

Using the substitution $x = (2 - 4t/\pi)^2 + 9t^2$, we get

$$W_2 = GMm((2-(4t/\pi))^2 + 9t^2)^{-1/2}\Big|_0^{\pi} = GMm\left(\frac{1}{\sqrt{4+9\pi^2}} - \frac{1}{\sqrt{2}}\right).$$

(ii) The fact that $W_1 = W_2$ suggests that $\vec{F}$ might be conservative, since a conservative force does the same work along two paths with the same endpoints. That this is indeed the case can be seen from the fact that $\vec{F}$ is a gradient vector field with potential function

$$\varphi(\vec{r}\,) = -\frac{GMm}{||\vec{r}\,||},$$

(so that $\vec{F} = -\operatorname{grad}\varphi$).

CHAPTER NINETEEN

Solutions for Section 19.1

Exercises

1. The surface is a right triangle in the yz-plane with sides 2 and 3, so its area is $(1/2) \cdot 2 \cdot 3 = 3$. The area vector $\vec{A}$ is perpendicular to the plane yz-plane, so $\vec{A}$ is a multiple of $\vec{i}$ with magnitude 3. There are two such vectors, $3\vec{i}$ and $-3\vec{i}$. Since $\vec{A}$ points in the negative x direction, we have $\vec{A} = -3\vec{i}$.

2. The disc has area $\pi R^2 = 25\pi$. The area vector $\vec{A}$ has magnitude 25π and is perpendicular to the xy-plane. Since it points upward, the area vector is $\vec{A} = 25\pi\vec{k}$.

3. The surface is a 5×3 rectangle of area 15. The area vector $\vec{A}$ is perpendicular to the plane $y = 10$, so $\vec{A}$ is a multiple of $\vec{j}$ with magnitude 15. There are two such vectors, $15\vec{j}$ and $-15\vec{j}$. Since $y > 0$ and $\vec{A}$ points away from the xz-plane, we have $\vec{A} = 15\vec{j}$.

4. The surface is a 5×3 rectangle of area 15. The area vector $\vec{A}$ is perpendicular to the plane $y = -10$, so $\vec{A}$ is a multiple of $\vec{j}$ with magnitude 15. There are two such vectors, $15\vec{j}$ and $-15\vec{j}$. Since $y < 0$ and $\vec{A}$ points away from the xz-plane, we have $\vec{A} = -15\vec{j}$.

5. We need a flat surface with area 150 that is perpendicular to the vector $\vec{j}$. There are many examples. For instance, we can take any 3×50 rectangle in the xz-plane oriented in the positive y direction.

6. Scalar. Only the $\vec{j}$-component of the vector field contributes to the flux and $d\vec{A} = \vec{j}\, dA$, and on the disk $y = 6$, so

$$\int_S (x\vec{i} + 6\vec{j}) \cdot d\vec{A} = 6 \cdot \text{Area of disk } = 6 \cdot \pi 3^2 = 54\pi.$$

7. (a) Only the $\vec{k}$-component of the vector field contributes to the flux and $d\vec{A} = \vec{k}\, dA$, so

$$\int_S (4\vec{i} + 5\vec{k}) \cdot d\vec{A} = 5 \cdot \text{Area of square } = 5 \cdot 3^2 = 45.$$

(b) This flux is the negative of the flux in part (a) because the surface is oriented in the opposite direction, so its value is -45.

Calculating directly, here $d\vec{A} = -\vec{k}\, dA$, so

$$\int_S (4\vec{i} + 5\vec{k}) \cdot d\vec{A} = -5 \cdot \text{Area of square } = -5 \cdot 3^2 = -45.$$

8. (a) Only the $\vec{i}$-component of the vector field contributes to the flux and $d\vec{A} = -\vec{i}\, dA$, so

$$\int_S (2\vec{i} + 3\vec{j}) \cdot d\vec{A} = -2 \cdot \text{Area of disk } = -2 \cdot \pi 4^2 = -32\pi.$$

(b) This flux is the negative of the flux in part (a) because the surface is oriented in the opposite direction, so its value is 32π.

Calculating directly, here $d\vec{A} = \vec{i}\, dA$, so

$$\int_S (2\vec{i} + 3\vec{j}) \cdot d\vec{A} = 2 \cdot \text{Area of disk } = 2 \cdot \pi 4^2 = 32\pi.$$

9. (a) The flux is positive, since $\vec{F}$ points in direction of positive x-axis, the same direction as the normal vector.
(b) The flux is negative, since below the xy-plane $\vec{F}$ points toward negative x-axis, which is opposite the orientation of the surface.
(c) The flux is zero. Since $\vec{F}$ has only an x-component, there is no flow across the surface.
(d) The flux is zero. Since $\vec{F}$ has only an x-component, there is no flow across the surface.
(e) The flux is zero. Since $\vec{F}$ has only an x-component, there is no flow across the surface.

10. The vector field $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k} = -z\vec{i} + x\vec{k}$ is a field parallel to the xz-plane that suggests swirling around the origin from the positive x-axis to the positive z-axis.

(a) The flux going through this surface is negative, because $\vec{F} \cdot \vec{n} = (-z\vec{i} + x\vec{k}) \cdot \vec{i} = -z$, z is positive here.
(b) The flux going through this surface is positive, because $\vec{F} \cdot \vec{n} = -z$, z is negative here.
(c) The flux through this surface is negative, because $\vec{F} \cdot \vec{n} = (-z\vec{i} + x\vec{k}) \cdot (-\vec{k}) = -x$, x is positive.
(d) The flux through this surface is negative, because $\vec{F} \cdot \vec{n} = -x$, x is positive.
(e) The flux through this surface is zero, because it is in the xz-plane, which is parallel to the vector field.

11. The vector field $\vec{r}$ is a field that always points away from the origin.

(a) The flux through this surface is zero, because the plane is parallel to the field.
(b) The flux through this surface is zero also, for the same reason.
(c) The flux through this surface is zero also, for the same reason.
(d) The flux through this surface is negative, because the field in that quadrant is going up and away from the origin, and since the orientation is downward, the flux is negative.
(e) The flux through this surface is zero also.

12. The area of the rectangular region is 4, and the orientation vector is $\vec{k}$. So, $\vec{A} = 4\vec{k}$, and the flux is $\vec{v} \cdot \vec{A} = (\vec{i} + 2\vec{j} - 3\vec{k}) \cdot 4\vec{k} = -12$.

13. The rectangular region is parallel to the yz-plane and has area 8. The orientation vector is $\vec{i}$, so $\vec{A} = 8\vec{i}$ and the flux is $\vec{v} \cdot \vec{A} = (\vec{i} + 2\vec{j} - 3\vec{k}) \cdot 8\vec{i} = 8$.

14. The rectangle lies in the plane $z + 2y = 4$. So a normal vector is $2\vec{j} + \vec{k}$ and a unit normal vector is $\frac{1}{\sqrt{5}}(2\vec{j} + \vec{k})$. Since this points in the positive z-direction it is indeed an orientation for the rectangle. Since the area of this rectangle is $4\sqrt{5}$ we have $\vec{A} = 8\vec{j} + 4\vec{k}$,
$\vec{v} \cdot \vec{A} = (\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (8\vec{j} + 4\vec{k}) = 16 - 12 = 4$.

15. The rectangle lies in the plane $3x + 2z = 6$. So $3\vec{i} + 2\vec{k}$ is a normal vector and $\frac{1}{\sqrt{13}}(3\vec{i} + 2\vec{k})$ is a unit normal vector. Since this points in both the positive x-direction and the positive z-direction, it is an orientation for this surface. Since the area of the rectangle is $2\sqrt{13}$, we have $\vec{A} = 6\vec{i} + 4\vec{k}$ and $\vec{v} \cdot \vec{A} = (\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (6\vec{i} + 4\vec{k}) = 6 - 12 = -6$.

16. On the surface, $d\vec{A} = \vec{k}\, dA$, so only the $\vec{k}$ component of $\vec{v}$ contributes to the flux:

$$\text{Flux} = \int_S \vec{v} \cdot d\vec{A} = \int_S (\vec{i} - \vec{j} + 3\vec{k}) \cdot \vec{k}\, dA = 3 \cdot \text{Area of disk} = 3 \cdot \pi 2^2 = 12\pi.$$

17. On the surface, $d\vec{A} = \vec{i}\, dA$, so only the $\vec{i}$ component of $\vec{v}$ contributes to the flux:

$$\text{Flux} = \int_S \vec{v} \cdot d\vec{A} = \int_S (\vec{i} - \vec{j} + 3\vec{k}) \cdot \vec{i}\, dA = \text{Area of triangle} = 4.$$

18. On the surface, $d\vec{A} = \vec{i}\, dA$, so only the $\vec{i}$ component of $\vec{v}$ contributes to the flux:

$$\text{Flux} = \int_S \vec{v} \cdot d\vec{A} = \int_S (\vec{i} - \vec{j} + 3\vec{k}) \cdot \vec{i}\, dA = \text{Area of square} = 4.$$

19. The triangle lies in the plane $x + y + z = 1$, with normal $\vec{i} + \vec{j} + \vec{k}$. A unit normal is $\vec{n} = (\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$, so

$$\text{Flux} = \int_S \vec{v} \cdot d\vec{A} = \int_S (\vec{i} - \vec{j} + 3\vec{k}) \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} dA = \frac{3}{\sqrt{3}} \text{Area of triangle}.$$

The base of the triangle in the xy-plane has length $\sqrt{2}$; the height is $\sqrt{3/2}$, so the area is $\sqrt{3}/2$. Thus

$$\text{Flux} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{3}{2}.$$

20. Since the surface is closed and the vector field is constant, the flux is zero.

21. Since the surface is in the plane $x = 2$, only the $\vec{i}$-component contributes to the flux. The area vector of the surface is $\pi 1^2 \vec{i} = \pi\vec{i}$. Thus,

$$\int_S (2\vec{i} + 3\vec{j} + 5\vec{k}) \cdot d\vec{A} = 2\vec{i} \cdot \pi\vec{i} = 2\pi.$$

22. Since the surface is in the plane $x + y + x = 1$, whose normal vector is $\vec{i} + \vec{j} + \vec{k}$, a unit normal in the direction of the orientation is $(\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$. Thus, the area vector of the surface is $\pi 1^2(\vec{i} + \vec{j} + \vec{k})/\sqrt{3} = \pi(\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$. The flux is given by

$$\int_S (2\vec{i} + 3\vec{j} + 5\vec{k}) \cdot d\vec{A} = (2\vec{i} + 3\vec{j} + 5\vec{k}) \cdot \pi \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} = \frac{\pi(2 + 3 + 5)}{\sqrt{3}} = \frac{10\pi}{\sqrt{3}}.$$

23. Since the surface is closed, the flux of a constant vector field out of it is 0.

24. The only contribution to the flux is from the face $x = 1$, since the vector field is zero or parallel to the other faces. On this face, $\vec{G} = \vec{i}$. This face has area 6, so its area vector $\vec{A} = 6\vec{i}$. Thus

$$\text{Flux} = \vec{i} \cdot \vec{A} = 6.$$

25. The only contribution to the flux is from the face $z = 3$, since the vector field is zero or parallel to the other faces. On this face, $\vec{H} = 3x\vec{k}$. The vector field is everywhere perpendicular to the face $z = 3$ but varies in magnitude from point to point. On this surface, $d\vec{A} = \vec{k}\,dx\,dy$. Thus

$$\text{Flux} = \int_0^2 \int_0^1 3x\vec{k} \cdot \vec{k}\,dx\,dy = \int_0^2 \int_0^1 3x\,dx\,dy = \left.\frac{3x^2}{2}\right|_0^1 \cdot \left. y \right|_0^2 = 3.$$

26. Only the x-component of the vector field contributes to the flux. On the surface, $\vec{F} = 3\vec{i} + 4\vec{j}$ and $d\vec{A} = -\vec{i}\,dA$, so

$$\int_S (3\vec{i} + 4\vec{j}) \cdot d\vec{A} = -3 \cdot \text{Area of disk} = -3 \cdot \pi 5^2 = -75\pi.$$

27. Since $\vec{r}$ is perpendicular to S and $\|\vec{r}\| = 3$ on S, we have

$$\int_S \vec{r} \cdot d\vec{A} = 3 \cdot \text{Area of surface} = 3 \cdot 4\pi 3^2 = 108\pi.$$

28. Only the $\vec{i}$ component contributes to the flux. On S, we have $d\vec{A} = \vec{i}\,dA$ and $x = 3\pi/2$, so

$$\int_S (\sin x\vec{i} + (y^2 + z^2)\vec{j} + y^2\vec{k}) \cdot d\vec{A} = \sin(3\pi/2) \cdot \text{Area of disk} = -1 \cdot \pi(\pi^2) = -\pi^3.$$

29. Since $5\vec{i} + 5\vec{j} + 5\vec{k}$ is perpendicular to S and in the same direction as the orientation, and since $\|5\vec{i} + 5\vec{j} + 5\vec{k}\| = \sqrt{5^2 + 5^2 + 5^2} = \sqrt{75}$ on S, we have

$$\int_S (5\vec{i} + 5\vec{j} + 5\vec{k}) \cdot d\vec{A} = \sqrt{75} \cdot \text{Area of circle} = \sqrt{75} \cdot \pi 3^2 = 9\pi\sqrt{75}.$$

30. Since the vector field $\vec{F}$ lies in the plane of the square, the flux is 0.

31. Since the disk is in the yz-plane, $d\vec{A} = \vec{i}\,dA$ and only the $\vec{i}$ component of $\vec{F}$ contributes to the flux:

$$\text{Flux} = \vec{F} \cdot \vec{A} = 2 \cdot \text{Area of disk} = 2\pi.$$

32. The square is in the plane $y = 5$, so only the $\vec{j}$ component of $\vec{F}$ contributes to the flux. This component is $5\vec{j}$ on the square and the area vector $\vec{A}$ is parallel to $\vec{j}$, so

$$\text{Flux} = \vec{F} \cdot \vec{A} = 5 \cdot \text{Area of square} = 5(1.6)^2 = 12.8.$$

33. The plane is $z = -2$. Since the vector field is $-2\vec{k}$ on the plane, it is parallel to the normal and in the same direction, and of length 2 there. Thus,

$$\text{Flux} = \int_S z\vec{k} \cdot d\vec{A} = 2 \cdot \text{Area of square} = 2 \cdot (\sqrt{14})^2 = 28.$$

34. See Figure 19.1. The vector field is a vortex going around the z-axis, and the square is centered on the x-axis, so the flux going across one half of the square is balanced by the flux coming back across the other half. Thus, the net flux is zero, so

$$\int_S \vec{F} \cdot d\vec{A} = 0.$$

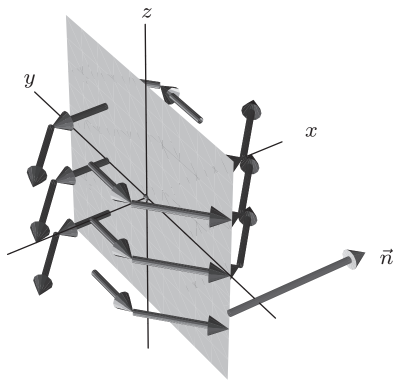


Figure 19.1

35. Only the $\vec{i}$-component contributes to the flux, so

$$\int_S \vec{F} \cdot d\vec{A} = 7 \cdot \text{ Area of disk } = 7 \cdot \pi 2^2 = 28\pi.$$

36. In the plane $y = 3$, we have $\vec{F} = x\vec{i} + 6\vec{j} + 3z\vec{k}$. Only the $\vec{j}$-component contributes to the flux, so

$$\int_S \vec{F} \cdot d\vec{A} = 6 \cdot \text{ Area of square } = 6 \cdot 2^2 = 24.$$

37. Since the vector field is constant, the flux is zero.

38. On the sphere of radius 2, the vector field has $||\vec{F}|| = 10$ and points inward everywhere (opposite to the orientation of the surface). So

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = -||\vec{F}|| \cdot \text{ Area of sphere } = -10 \cdot 4\pi 2^2 = -160\,\pi.$$

39. We have $d\vec{A} = \vec{k}\,dA$, and $z = 4$, so,

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (x\vec{i} + y\vec{j} + (4^2+3)\vec{k}) \cdot \vec{k}\, dA = \int_S 19\,dA$$
$$= 19(\text{Area of rectangle}) = 19(6) = 114.$$

40. The normal to the plane is $\vec{n} = \vec{i} + \vec{j}$; a unit vector in this direction and in the direction of orientation is $(\vec{i} + \vec{j})/\sqrt{2}$. Thus,

$$\text{Flux } = (6\vec{i} + 7\vec{j}) \cdot \frac{\vec{n}}{||\vec{n}||} \cdot \text{Area of triangle } = \frac{6+7}{\sqrt{2}} \cdot 10 = \frac{130}{\sqrt{2}}.$$

41. The only contribution to the flux is from the $\vec{j}$-component, and since $d\vec{A} = \vec{j}\,dx\,dz$ on the square, S, we have

$$\text{Flux } = \int_S (6\vec{i} + x^2\vec{j} - \vec{k}) \cdot d\vec{A} = \int_{-2}^{2}\int_{-2}^{2} x^2\vec{j} \cdot \vec{j}\,dx\,dz = \int_{-2}^{2} \left.\frac{x^3}{3}\right|_{-2}^{2} dz = \frac{16}{3} \cdot 4 = \frac{64}{3}.$$

42. We have $d\vec{A} = \vec{i}\,dA$, and $x = 4$, so,

$$\int_S \vec{F} \cdot d\vec{A} = \int_S ((4+3)\vec{i} + (y+5)\vec{j} + (z+7)\vec{k}) \cdot \vec{i}\, dA = \int_S 7\,dA$$
$$= 7 \cdot \text{Area of rectangle} = 7 \cdot 6 = 42.$$

43. On the sphere of radius 3, the vector field has $||\vec{F}|| = 21$ and points outward everywhere. So

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = ||\vec{F}|| \cdot \text{ Area of sphere } = 21 \cdot 4\pi 3^2 = 756\,\pi.$$

44. The vector field $\vec{F}$ and the area vector on the surface of the sphere are parallel, but in opposite directions. Since $\vec{F}$ is pointing inward and $||\vec{F}|| = 6$ on the surface

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = \int_S ||\vec{F}|| \cos \pi \, dA = -6 \cdot \text{Area of sphere} = -6 \cdot 4\pi 2^2 = -96\pi.$$

45. We have $d\vec{A} = \vec{i}\, dA$, so

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (2z\vec{i} + x\vec{j} + x\vec{k}) \cdot \vec{i}\, dA = \int_S 2z\, dA$$
$$= \int_0^2 \int_0^3 2z\, dz dy = 18.$$

46. Since the vector field is constant, if $\vec{A}$ is the area vector of the square

$$\text{Flux} = \vec{F} \cdot \vec{A}.$$

An upward normal to the plane is $\vec{i} + \vec{j} + \vec{k}$, so a unit vector in this direction is $(\vec{i} + \vec{j} + \vec{k})\sqrt{3}$. The area vector has magnitude 4, so $\vec{A} = 4(\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$. Thus

$$\text{Flux} = (\vec{i} + 2\vec{j}) \cdot \frac{4(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}} = \frac{4(1+2)}{\sqrt{3}} = 4\sqrt{3}.$$

47. Since the disk is in the xy-plane and oriented upward, $d\vec{A} = \vec{k}\, dxdy$ and

$$\int_{\text{Disk}} \vec{F} \cdot d\vec{A} = \int_{\text{Disk}} (x^2 + y^2)\vec{k} \cdot \vec{k}\, dxdy = \int_{\text{Disk}} (x^2 + y^2)\, dxdy.$$

Using polar coordinates

$$\int_{\text{Disk}} \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_0^3 r^2 \cdot r\, dr d\theta = 2\pi \left. \frac{r^4}{4} \right|_0^3 = \frac{81\pi}{2}.$$

48. Since the disk is horizontal and oriented upward, $d\vec{A} = \vec{k}\, dxdy$, so

$$\int_{\text{Disk}} \vec{F} \cdot d\vec{A} = \int_{\text{Disk}} \cos(x^2 + y^2)\vec{k} \cdot \vec{k}\, dxdy = \int_{\text{Disk}} \cos(x^2 + y^2)\, dxdy.$$

Using polar coordinates, since the disk has radius 3, we have

$$\int_{\text{Disk}} \vec{F} \cdot d\vec{A} = \int_{\text{Disk}} \cos(x^2 + y^2) dxdy = \int_0^{2\pi} \int_0^3 \cos(r^2) r dr d\theta$$
$$= \int_0^{2\pi} \left. \frac{1}{2} \sin(r^2) \right|_0^3 d\theta = 2\pi \cdot \left. \frac{1}{2} \sin(r^2) \right|_0^3$$
$$= 2\pi \left(\frac{1}{2} \sin(3^2) - \frac{1}{2} \sin(0^2) \right) = \pi \sin 9.$$

49. Since the disk is oriented in the positive x-direction, $d\vec{A} = \vec{i}\, dydz$, so we have

$$\text{Flux} = \int_{\text{Disk}} \vec{F} \cdot d\vec{A} = \int_{\text{Disk}} e^{y^2+z^2}\vec{i} \cdot \vec{i}\, dydz = \int_{\text{Disk}} e^{y^2+z^2} dydz.$$

To calculate this integral, we use polar coordinates with $y = r\cos\theta$ and $z = r\sin\theta$. Then $r^2 = y^2 + z^2$ and

$$\int_{\text{Disk}} \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_0^2 e^{r^2} \cdot r dr d\theta = 2\pi \cdot \left. \frac{e^{r^2}}{2} \right|_0^2 = \pi(e^4 - 1).$$

50. See Figure 19.2. Since $\vec{F}$ is parallel to the xy plane, there is no flux across the surface, so

$$\int_S \vec{F} \cdot d\vec{A} = 0.$$

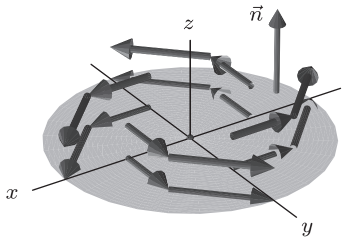


Figure 19.2

51. See Figure 19.3. The area vector of a small area element $\Delta\vec{A}$ is the vector pointing in the direction normal to the surface with magnitude ΔA. The unit vector normal to the surface is $\vec{k}$, so $\Delta\vec{A} = \vec{k}\,\Delta A$. Thus,

$$\int_S \vec{F} \cdot d\vec{A} = \lim_{\|\Delta\vec{A}\|\to 0} \sum \vec{r} \cdot \Delta\vec{A} = \lim_{\|\Delta A\|\to 0} \sum \vec{r} \cdot \vec{k}\ \Delta\vec{A} = \int_S \vec{r} \cdot \vec{k}\ dA.$$

Now $\vec{r} \cdot \vec{k} = 2$ for all the points on S because all such points have z-coordinate equal to 2. Thus, we have

$$\int_S \vec{F} \cdot d\vec{A} = \int_S \vec{r} \cdot \vec{k}\ dA = \int_S 2\,dA = 2 \cdot \text{Area of } S = 8\pi.$$

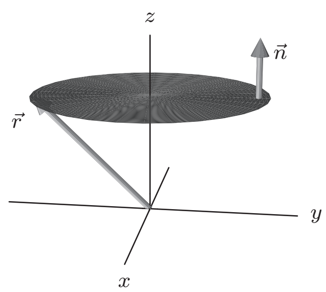


Figure 19.3

52. See Figure 19.4. Since the vector field is parallel to the x-axis, only the two sides perpendicular to the x-axis contribute to the flux integral. On the side where $x = 0$, the vector field is $2\vec{i}$, and hence the flux through that side is $-(2)(3^2) = -18$ (negative because the flow is inward and the normal vector is pointing out). The flow out the other side is at $x = 3$, so $\vec{F} = -\vec{i}$, so the flux out that side is $(-\vec{i}) \cdot (3^2\vec{i}) = -9$. So the net flux is $-18 - 9 = -27$. So

$$\int_S \vec{F} \cdot d\vec{A} = -27.$$

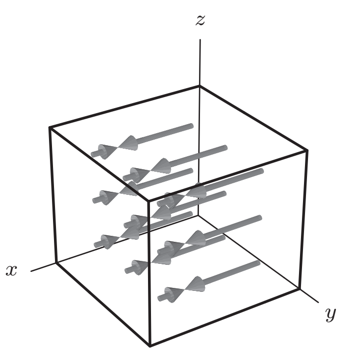


Figure 19.4

Problems

53. (a) (i) The flux $\int_B x\vec{i} \cdot d\vec{A}$ is positive. The vectors $x\vec{i}$ all point out of the box.

(ii) The flux $\int_B y\vec{i} \cdot d\vec{A}$ is zero. The vector field is parallel to the x-axis. For each y-value, the flux entering the box at one end cancels the flux leaving at the other end.

(iii) The flux $\int_S |x|\vec{i} \cdot d\vec{A}$ is zero. The flux entering the sphere where $x < 0$ cancels the flux leaving where $x > 0$.

(iv) Since $\int_S (y-x)\vec{i} \cdot d\vec{A} = \int y\vec{i} \cdot d\vec{A} - \int x\vec{i} \cdot d\vec{A} =$ Zero $-$ Positive , this flux is negative.

(b) From the Divergence Theorem, $\int_B x\vec{i} \cdot d\vec{A}$ is greater. Since $\operatorname{div}(x\vec{i}) = 1$,

$$\int_S x\vec{i} \cdot d\vec{A} = \int_{\text{Inside sphere}} 1 \cdot dV = \text{ Volume of sphere } = \frac{4}{3}\pi.$$

$$\int_B x\vec{i} \cdot d\vec{A} = \int_{\text{Inside box}} 1 \cdot dV = \text{ Volume of box } = 8 > \frac{4}{3}\pi.$$

54. (a) Consider two opposite faces of the cube, S_1 and S_2. The corresponding area vectors are $\vec{A}_1 = 4\vec{i}$ and $\vec{A}_2 = -4\vec{i}$ (since the side of the cube has length 2). Since $\vec{E}$ is constant, we find the flux by taking the dot product, giving

$$\text{Flux through } S_1 = \vec{E} \cdot \vec{A}_1 = (a\vec{i} + b\vec{j} + c\vec{k}) \cdot 4\vec{i} = 4a.$$

$$\text{Flux through } S_2 = \vec{E} \cdot \vec{A}_2 = (a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-4\vec{i}) = -4a.$$

Thus the fluxes through S_1 and S_2 cancel. Arguing similarly, we conclude that, for any pair of opposite faces, the sum of the fluxes of $\vec{E}$ through these faces is zero. Hence, by addition, $\int_S \vec{E} \cdot d\vec{A} = 0$.

(b) The basic idea is the same as in part (a), except that we now need to use Riemann sums. First divide S into two hemispheres H_1 and H_2 by the equator C located in a plane perpendicular to $\vec{E}$. For a tiny patch S_1 in the hemisphere H_1, consider the patch S_2 in the opposite hemisphere which is symmetric to S_1 with respect to the center O of the sphere. The area vectors $\Delta\vec{A}_1$ and $\Delta\vec{A}_2$ satisfy $\Delta\vec{A}_2 = -\Delta\vec{A}_1$, so if we consider S_1 and S_2 to be approximately flat, then $\vec{E} \cdot \Delta\vec{A}_1 = -\vec{E} \cdot \Delta\vec{A}_2$. By decomposing H_1 and H_2 into small patches as above and using Riemann sums, we get

$$\int_{H_1} \vec{E} \cdot d\vec{A} = -\int_{H_2} \vec{E} \cdot d\vec{A}, \quad \text{so} \quad \int_S \vec{E} \cdot d\vec{A} = 0.$$

(c) The reasoning in part (b) can be used to prove that the flux of $\vec{E}$ through any surface with a center of symmetry is zero. For instance, in the case of the cylinder, cut it in half with a plane $z = 1$ and denote the two halves by H_1 and H_2. Just as before, take patches in H_1 and H_2 with $\Delta A_1 = -\Delta A_2$, so that $\vec{E} \cdot \Delta A_1 = -\vec{E} \cdot \Delta\vec{A}_2$. Thus, we get

$$\int_{H_1} \vec{E} \cdot d\vec{A} = -\int_{H_2} \vec{E} \cdot d\vec{A},$$

which shows that

$$\int_S \vec{E} \cdot d\vec{A} = 0.$$

55. Notice that the speed is 3 cm/sec at the center of the pipe and 0 cm/sec at the sides. Suppose $\vec{i}$ is the unit vector parallel to the direction of flow. Then, at a distance r from the center of the pipe, the velocity is given by

$$\vec{v} = \left(3 - \frac{3}{4}r^2\right)\vec{i} \text{ cm/sec.}$$

Divide the circular cross-section into concentric rings of width Δr, so that the velocity is approximately constant on each one. The area of a typical ring is $\Delta A \approx 2\pi r\Delta r$. Then since $\vec{v}$ and $\Delta\vec{A}$ are parallel (see Figure 19.5), we have

$$\text{Flux through ring} \approx \vec{v}\,\Delta\vec{A} = \|\vec{v}\|\|\Delta\vec{A}\| \approx \left(3 - \frac{3}{4}r^2\right)\frac{\text{cm}}{\text{sec}} \cdot (2\pi r\Delta r)\,\text{cm}^2.$$

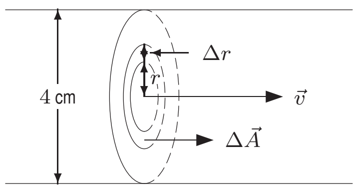


Figure 19.5: Flux through pipe when velocity varies with distance from the center

Thus, the flux through the circular cross-section of the pipe is given by

$$\begin{aligned}\text{Flux} &= \lim_{\|\Delta\vec{A}\|\to 0}\sum \vec{v}\cdot\Delta\vec{A}\\ &= \lim_{\Delta r\to 0}\sum\left(3-\frac{3}{4}r^2\right)2\pi r\Delta r\\ &= \int_{r=0}^{r=2}\left(3-\frac{3}{4}r^2\right)2\pi r\,dr = 6\pi\int_0^2\left(r-\frac{r^3}{4}\right)dr = 6\pi\text{ cm}^3/\text{sec}.\end{aligned}$$

56. (a) The net electric flux through this surface is zero, because the surface is placed so that it is always parallel with the electric field, and there is no flow through the surface.
(b) The net flux is zero, because the flow in through one half of the cylinder is canceled by the flow out through the other half.

57. The $\vec{j}$ components of $\vec{G}$ and $\vec{H}$ do not have flux through S because they are parallel to the cylinder that S lies on, and are therefore tangent to S. The three vector fields, $\vec{F}$, $\vec{G}$, and $\vec{H}$ have the same $\vec{i}$ and $\vec{k}$ components, so they have the same flux through S.

58. Since this vector field points radially out from the origin, it is everywhere parallel to the vector representing the surface area, $d\vec{A}$. Thus since $\|\vec{F}(\vec{r})\| = 1/R^2$ on the surface, S,

$$\vec{F}(\vec{r})\cdot d\vec{A} = \frac{1}{R^2}\,dA,$$

so

$$\int_S \vec{F}(\vec{r})\cdot d\vec{A} = \frac{1}{R^2}\cdot\text{Surface area of sphere} = \frac{1}{R^2}(4\pi R^2) = 4\pi.$$

59. Since this vector field points radially out from the origin, it is everywhere parallel to the area vector, $\Delta\vec{A}$. Thus since $\|\vec{F}(\vec{r})\| = 1/R$ on the surface, S,

$$\vec{F}(\vec{r})\cdot\Delta\vec{A} = \frac{1}{R}\Delta A$$

so

$$\int_S \vec{F}(\vec{r})\cdot d\vec{A} = \frac{1}{R}\lim_{\Delta A\to 0}\sum\Delta A = \frac{1}{R}\cdot\text{Surface area of sphere} = \frac{1}{R}(4\pi R^2) = 4\pi R.$$

60. (a) The vector field is perpendicular to the surface of a sphere centered at the origin. Thus the magnitude of the flux depends on the magnitude of the vector field on the surface. Since for fixed $\vec{r}$, the value of $||\vec{F}||$ decreases as p increases, the maximum flux occurs when $p = 0$.
(b) For a sphere of radius 2 with $p = 0$, we have $||\vec{F}|| = 2$ on the surface. Thus

$$\text{Flux} = \int_S \vec{F}\cdot d\vec{A} = 2\cdot\text{Area of surface} = 2\cdot 4\pi 2^2 = 32\pi.$$

61. (a) For a flat surface, flux through $\vec{A}$ is $\vec{v}\cdot\vec{A}$. Therefore, the flux through each face of the cube is equal to $(-\vec{i}+2\vec{j}+\vec{k})\cdot(\vec{A}$ of the face).

First we shall find the flux through the two faces parallel to the xy-plane, beginning with the one with negative z. The unit vector normal to this face and pointing outward is $-\vec{k}$. The area of the face equals 4, so $\vec{A} = -4\vec{k}$. The flux through the face with negative z equals

$$(-\vec{i}+2\vec{j}+\vec{k})\cdot(-4\vec{k}) = 0+0-4 = -4$$

For the face with positive z, the unit normal vector that points outward is $\vec{k}$. Therefore $\vec{A} = 4\vec{k}$. The flux through this face is given by

$$(-\vec{i}+2\vec{j}+\vec{k})\cdot 4\vec{k} = 0+0+4 = 4$$

Next, we will find the flux through the two faces parallel to the xz-plane, beginning with the one with negative y. A unit vector normal to this face pointing outward is $-\vec{j}$. Therefore $\vec{A} = -4\vec{j}$. The flux then equals

$$(-\vec{i}+2\vec{j}+\vec{k})\cdot(-4\vec{j}) = 0-8+0 = -8$$

For the face with positive y, the unit normal vector pointing outward is $\vec{j}$. Therefore $\vec{A} = 4\vec{j}$. The flux then equals

$$(-\vec{i}+2\vec{j}+\vec{k})\cdot(4\vec{j}) = 0+8+0 = 8$$

Next, we will find the flux through the two faces parallel to the yz plane, beginning with the one with negative x. A unit vector normal to this plane pointing outward is $-\vec{i}$. Therefore $\vec{A} = -4\vec{i}$. The flux then equals

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot (-4\vec{i}) = 4 + 0 + 0 = 4$$

For the face with positive x, the unit normal vector pointing outward is $\vec{i}$. Therefore $\vec{A} = 4\vec{i}$. The flux then equals

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot (4\vec{i}) = -4 + 0 + 0 = -4$$

Adding up all of these fluxes to get the flux out of the entire cube, we get

$$\text{Total flux} = -4 + 4 - 8 + 8 + 4 - 4 = 0$$

(b) For any constant vector field $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, we can calculate the flux out of the cube by the same method.

First we shall find the flux out of the two faces parallel to the xy plane, beginning with the one with negative z. A unit vector normal to this plane, that points negative (because of the orientation of the face) is $-\vec{k}$. The area of the face equals 4, therefore $\vec{A} = -4\vec{k}$. The flux through $\vec{A}$ then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-4\vec{k}) = 0 + 0 - 4c = -4c$$

For the face with positive z, the unit normal vector pointing outward is $\vec{k}$. Therefore $\vec{A} = 4\vec{k}$. The flux then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (4\vec{k}) = 0 + 0 + 4c = 4c.$$

Next, we will find the flux through the two faces parallel to the xz plane, beginning with the one with negative y. A unit vector normal to this plane pointing outward is $-\vec{j}$. Therefore $\vec{A} = -4\vec{j}$. The flux then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-4\vec{j}) = 0 - 4b + 0 = -4b$$

For the face with positive y, the unit normal vector pointing outward is $\vec{j}$. Therefore $\vec{A} == 4\vec{j}$. The flux then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (4\vec{j}) = 0 + 4b + 0 = 4b$$

Next, we will find the flux through the two faces parallel to the yz plane, beginning with the one with negative x. A unit vector normal to this plane pointing outward is $-\vec{i}$. Therefore $\vec{A} = -4\vec{i}$. The flux then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-4\vec{i}) = -4a + 0 + 0 = -4a$$

For the face in the positive x, the unit normal vector pointing outward is $\vec{i}$. Therefore $\vec{A} = 4\vec{i}$. The flux then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (4\vec{i}) = 4a + 0 + 0 = 4a$$

Adding up all of these fluxes to get the flux out of the entire cube, we get

$$\text{Total flux} = -4c + 4c - 4b + 4b + 4a - 4a = 0$$

(c) The answers in parts (a) and (b) make sense because the vector field is constant, and so it does not change as it comes in the one side of the cube, and exits the other side. Therefore the two fluxes cancel each other out, making the total flux zero.

62. (a) Let $\vec{A}$ be the area vector of any face of the tetrahedron in Figure 19.6. The flux through the face equals $\vec{v} \cdot \vec{A}$ because the vector field is constant. Therefore, the flux through each face of the tetrahedron is equal to $(-\vec{i} + 2\vec{j} + \vec{k}) \cdot \vec{A}$, where $\vec{A}$ is the area of that face.

First we shall find the flux out of the triangle in the xy plane. A unit vector normal to that plane, that points negative (because of the orientation of the face), is equal to $-\vec{k}$. The area of the face equals 0.5, therefore $\vec{A} = -0.5\vec{k}$. The flux through $\vec{A}$ then equals

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot (-0.5\vec{k}) = 0 + 0 - 0.5 = -0.5.$$

Next, we will find the flux out of the triangle in the xz plane. A unit vector normal to that plane, that points negative, is equal to $-\vec{j}$. The area of the face equals 0.5, therefore $\vec{A} = -0.5\vec{j}$. The flux through $\vec{A}$ then equals

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot (-0.5\vec{j}) = 0 - 1 + 0 = -1.$$

Next, we will find the flux out of the triangle in the yz plane. A unit vector normal to that plane, that points negative, is equal to $-\vec{i}$. The area of the face equals 0.5, therefore $\vec{A} = -0.5\vec{i}$. The flux through $\vec{A}$ then equals

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot (-0.5\vec{i}) = 0.5 + 0 + 0 = 0.5.$$

Last, we will find the flux out of the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. A unit vector normal to that plane, that points positive, is equal to $\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$. The area of the face equals $\sqrt{3}/2$, since it is an equilateral triangle with side $\sqrt{2}$. Therefore:

$$\vec{A} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})(\sqrt{3}/2) = 0.5(\vec{i} + \vec{j} + \vec{k}).$$

The flux through $\vec{A}$ then equals

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot (0.5\vec{i} + 0.5\vec{j} + 0.5\vec{k}) = -0.5 + 1 + 0.5 = 1.$$

The total flux out of the tetrahedron is $-0.5 - 1 + 0.5 + 1 = 0$. Therefore the flux equals zero.

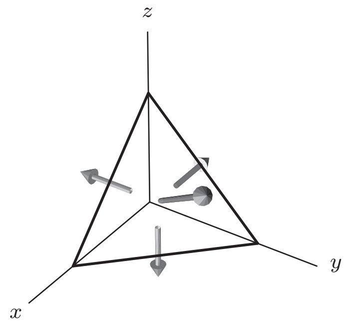


Figure 19.6

(b) For any constant vector field $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, we can find the flux out of the tetrahedron.

First we shall find the flux out of the triangle in the xy plane. A unit vector normal to that plane, that points negative (because of the orientation of the face), is equal to $-\vec{k}$. The area of the face equals 0.5, therefore $\vec{A} = -0.5\vec{k}$. The flux through $\vec{A}$ then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-0.5\vec{k}) = 0 + 0 - 0.5 = -0.5c.$$

Next, we will find the flux out of the triangle in the xz plane. A unit vector normal to that plane, that points negative, is equal to $-\vec{j}$. The area of the face equals 0.5, therefore: $\vec{A} = -0.5\vec{j}$. The flux through $\vec{A}$ then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-0.5\vec{j}) = 0 - 0.5b + 0 = -0.5b.$$

Next, we will find the flux out of the triangle in the yz plane. A unit vector normal to that plane, that points negative, is equal to $(-\vec{i})$. The area of the face equals 0.5, therefore $\vec{A} = -0.5\vec{i}$. The flux through $\vec{A}$ then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (-0.5\vec{i}) = -0.5a + 0 + 0 = -0.5a.$$

Last, we will find the flux out of the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. A unit vector normal to that plane, that points positive, is equal to $\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$. The area of the face equals $\sqrt{3}/2$, therefore:

$$\vec{A} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})(\sqrt{3}/2) = 0.5(\vec{i} + \vec{j} + \vec{k}).$$

The flux through $\vec{A}$ then equals

$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (0.5\vec{i} + 0.5\vec{j} + 0.5\vec{k}) = 0.5a + 0.5b + 0.5c.$$

The total flux out of the tetrahedron is $-0.5c - 0.5b - 0.5a + 0.5a + 0.5b + 0.5c = 0$. Therefore, the flux is equal to zero.

(c) The answers in (a) and (b) make sense because the vector field is constant, so it does not change as it enters through one side of the tetrahedron, and exits the other side. Therefore the two cancel each other out,causing the flux to be equal to zero.

63. Since Pressure = Force/Area, we have

$$\text{Force on a small patch with area } \Delta\vec{A} \text{ at point } (x, y, z) \approx P(x, y, z)\|\Delta\vec{A}\|.$$

This force is directed inward and normal to the surface, so the force is $P(x, y, z)\Delta\vec{A}$ (if S is oriented with the inward normal). For buoyancy, take the upward component of this force, so

$$\text{Buoyancy force} = P(x, y, z)\Delta\vec{A} \cdot \vec{k}.$$

Then:

$$\begin{aligned}
\text{Total buoyancy} &= \lim_{\|\Delta\vec{A}\|\to 0} \sum_s P(x, y, z)\Delta\vec{A} \cdot \vec{k} \\
&= \int_S P(x, y, z)\vec{k} \cdot d\vec{A} \\
&= \int_S \vec{F} \cdot d\vec{A}
\end{aligned}$$

64. (a) From Newton's law of cooling, we know that the temperature gradient will be proportional to the heat flow. If the constant of proportionality is k then we have the equation $\vec{F} = k \operatorname{grad} T$. Since $\operatorname{grad} T$ points in the direction of increasing T, but heat flows toward lower temperatures, the constant k must be negative.

(b) This form of Newton's law of cooling is saying that heat will be flowing in the direction in which temperature is decreasing most rapidly, in other words, in the direction exactly opposite to $\operatorname{grad} T$. This agrees with our intuition which tells us that a difference in temperature causes heat to flow from the higher temperature to the lower temperature, and the rate at which it flows depends on the temperature gradient.

(c) The rate of heat loss from W is given by the flux of the heat flow vector field through the surface of the body. Thus,

$$\begin{matrix}\text{Rate of heat} \\ \text{loss from } W\end{matrix} = \begin{matrix}\text{Flux of } \vec{F} \\ \text{out of } S\end{matrix} = \int_S \vec{F} \cdot d\vec{A} = k \int_S (\operatorname{grad} T) \cdot d\vec{A}$$

65. (a) Figure 19.7 shows the electric field $\vec{E}$. Note that $\vec{E}$ points radially outward from the z-axis.

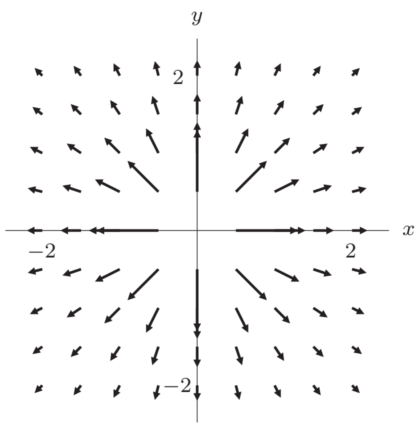


Figure 19.7: The electric field in the xy-plane due to a line of positive charge uniformly distributed along the z-axis: $\vec{E}(x, y, 0) = 2\lambda \dfrac{x\vec{i} + y\vec{j}}{x^2 + y^2}$

(b) On the cylinder $x^2 + y^2 = R^2$, the electric field $\vec{E}$ points in the same direction as the outward normal $\vec{n}$, and

$$\|\vec{E}\| = \frac{2\lambda}{R^2}\|x\vec{i} + y\vec{j}\| = \frac{2\lambda}{R}.$$

So

$$\int_S \vec{E} \cdot d\vec{A} = \int_S \vec{E} \cdot \vec{n}\, dA = \int_S \|\vec{E}\|\, dA = \int_S \frac{2\lambda}{R}\, dA$$
$$= \frac{2\lambda}{R}\int_S dA = \frac{2\lambda}{R} \cdot \text{Area of } S = \frac{2\lambda}{R} \cdot 2\pi R h = 4\pi\lambda h,$$

which is positive, as we expected.

66. (a) The vector field $\vec{B}$ is sketched in Figure 19.8 for $I > 0$.

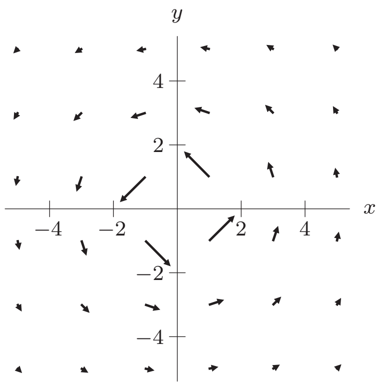


Figure 19.8

(b) The disk S can be parameterized as $z = h$ (viewed as a constant function of x and y), for x, y in the region $\{x^2 + y^2 \le a^2\}$. Hence

$$\int_S \vec{B} \cdot d\vec{A} = \int_{\{x^2+y^2\le a^2\}} \frac{I}{2\pi} \cdot \frac{-y\vec{i} + x\vec{j}}{x^2+y^2} \cdot \vec{k}\, dx\, dy = 0,$$

since $\vec{i} \cdot \vec{k} = 0$ and $\vec{j} \cdot \vec{k} = 0$. The answer is as we would expect, since the vector field $\vec{B}$ is tangent to the surface S, hence there is no flux through S.

(c) The flux of $\vec{B}$ through S_2 is given by $\int_{S_2} \vec{B} \cdot d\vec{A}$. On S_2 we have

$$\vec{B}(x, y, z) = \frac{I}{2\pi} \cdot \frac{-y\vec{i}}{y^2} = -\frac{I}{2\pi y}\vec{i},$$

and

$$d\vec{A} = \vec{n}\, dA = -\vec{i}\, dy\, dz$$

Hence,

$$\int_{S_2} \vec{B} \cdot d\vec{A} = \int_0^h \int_a^b \frac{I}{2\pi} \cdot \frac{(-\vec{i})}{y} \cdot (-\vec{i})\, dy\, dz$$
$$= \frac{I}{2\pi} \int_0^h \int_a^b \frac{1}{y}\, dy\, dz$$
$$= \frac{I}{2\pi} \int_0^h [\ln|y|]_a^b\, dz$$
$$= \frac{I}{2\pi} \int_0^h (\ln|b| - \ln|a|)\, dz$$
$$= \frac{I}{2\pi} h \left(\ln\left|\frac{b}{a}\right|\right).$$

This time we get a non-zero flux since the direction of $\vec{B}$ is everywhere parallel to the orientation of S_2. For $0 < a < b$ the flux is positive since $|b/a| > 1$ and increases as the area S_2 increases. This is as Figure 19.8 would lead us to expect. For $a < b < 0$, the flux is negative since $|b/a| < 1$. If $a < 0 < b$, the flux can be either positive or negative.

67. (a) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of a point on the sphere, then

$$\vec{D}(\vec{r}) = \frac{3zp}{a^5}\vec{r} - \frac{p}{a^3}\vec{k}.$$

The second term is a constant vector field. Hence, by symmetry,

$$\int_S \left(-\frac{p}{a^3}\vec{k}\right) \cdot d\vec{A} = 0.$$

(See the solution to Problem 54 on page 1727). Let us also apply a symmetry argument to

$$\int_S \left(\frac{3zp}{a^5}\vec{r}\right) \cdot d\vec{A}.$$

We will show that the flux of $\vec{D}$ through the upper hemisphere H_1 equals minus the flux of $\vec{D}$ through the lower hemisphere H_2. The flux of $\vec{D}$ through H_1 and H_2 will be computed as limits of Riemann sums.

Consider a small patch P_1 in H_1 and call its reflection about the xy-plane P_2. The contribution of P_1 to the flux of $\vec{D}$ through S is

$$\frac{3z_1p}{a^5}\vec{r} \cdot d\vec{A}_1 = \frac{3z_1p}{a^5}\vec{r} \cdot \text{Area}(P_1)\frac{\vec{r}}{\|\vec{r}\|} = \frac{3z_1p}{a^4} \cdot \text{Area}(P_1),$$

whereas the contribution from P_2 is

$$\frac{3z_2p}{a^5}\vec{r} \cdot d\vec{A}_1 = \frac{3z_2p}{a^4}\text{Area}(P_2).$$

But $\text{Area}(P_1) = \text{Area}(P_2)$ and $z_2 = -z_1$, so the contributions from P_1 and P_2 cancel each other. Dividing H_1 and H_2 into symmetric patches as above, and taking the limit as the areas of the patches become smaller and smaller, one gets

$$\int_{H_1} \left(\frac{3zp}{a^5}\vec{r}\right) \cdot d\vec{A} = -\int_{H_2} \left(\frac{3zp}{a^5}\vec{r}\right) \cdot d\vec{A}.$$

i.e.

$$\int_S \left(\frac{3zp}{a^5}\vec{r}\right) \cdot d\vec{A} = 0.$$

Since we also know that

$$\int_S \left(-\frac{p}{a^3}\vec{k}\right) \cdot d\vec{A} = 0,$$

we can conclude that

$$\int_S \vec{D} \cdot d\vec{A} = 0.$$

(b) By Gauss's Law,

$$\int_S \vec{E} \cdot d\vec{A} = 4\pi(q - q) = 0,$$

which is the same as the flux of $\vec{D}$ through S.

Strengthen Your Understanding

68. The value of a flux integral is a scalar, not a vector.

69. The sign of a flux integral $\int_S \vec{F} \cdot d\vec{A}$ is determined by the interaction between the vector field $\vec{F}$ and the oriented surface S. It can not be determined from either one alone. For example, if S is oriented in the $+\vec{k}$ direction, then the flux through S of $\vec{F} = \vec{k}$ is positive and the flux through S of $\vec{F} = -k$ is negative.

70. The integral $\int_S \vec{F} \cdot d\vec{A}$ is zero if the vector field $\vec{F}$ is tangent to the surface S at every point of S. The displacement vector $\vec{F} = -\vec{i} + \vec{j}$ between the two points $(1, 0, 0)$ and $(0, 1, 0)$ in S is tangent to S. Hence, with the constant vector field $\vec{F} = -\vec{i} + \vec{j}$ we have $\int_S \vec{F} \cdot d\vec{A} = 0$.

71. Let $\vec{F} = z\vec{k}$, and let S be the square $0 \le x \le 1, 0 \le y \le 1, z = 1$, oriented upwards. The vector field is normal to the surface, in the direction of the orientation, and has constant magnitude 1 on the surface, even though it has varying magnitude elsewhere.

$$\int_S \vec{F}\, d\vec{A} = \|\vec{F}\| \cdot \text{Area of } S = 1 \cdot 1 = 1.$$

72. True. By definition, the flux integral is the limit of a sum of dot products, hence is a scalar.

73. False. $\vec{A}$ is *perpendicular* to the flat surface.

74. False. The flux integral measures the *net* flow through the surface. There could be as much flow into the sphere as out, which would give a flux integral of zero. As an example, the constant field $\vec{F} = \vec{i}$ has zero flux integral over the entire sphere, yet is not the zero vector field.

75. True. The flow of this field is in the same direction as the orientation of the surface everywhere on the surface, so the flux is positive.

76. True. The flow of this field is in the same direction as the orientation of the surface everywhere on the surface, so the flux is positive.

77. True. Since the vector field is constant, the negative flux into the bottom of the cube is equal in magnitude to the positive flux out of the top, so these cancel in the sum defining the flux integral. The other four faces of the cube each have zero flux from the field, since the field is parallel to each of them.

78. True. Reversing the orientation on S replaces all of the area vectors $\Delta\vec{A}$ in the sum defining the flux integral with their negatives, so that the flux integral over $-S$ is the negative of the flux integral over S.

79. False. There is no reason to expect a relationship between the flux integrals over S_1 and S_2 simply based on their relative areas. The value of the flux integral over a surface depends both on the shape of the surface and the behavior of the vector field at points on the surface. For example, let S_1 be the square $0 \le x \le 1, 0 \le y \le 1, z = 0$, oriented upward, and let S_2 be the rectangle $0 \le y \le 1, 0 \le z \le 2, x = 0$ with positive orientation in the $\vec{i}$ direction. The area of $S_1 = 1$ and the area of $S_2 = 2$. Then if $\vec{F} = \vec{i}$ we have $\int_{S_1} \vec{F} \cdot d\vec{A} = 0$ (since $\vec{F}$ is parallel to S_1) and $\int_{S_2} \vec{F} \cdot d\vec{A} = 2$. These values do not satisfy $2\int_{S_1} \vec{F} \cdot d\vec{A} = \int_{S_2} \vec{F} \cdot d\vec{A}$.

80. True. In the sum defining the flux integral for $\vec{F}$, we have terms like $\vec{F} \cdot \Delta\vec{A} = (2\vec{G}) \cdot \Delta\vec{A} = 2(\vec{G} \cdot \Delta\vec{A})$. So each term in the sum approximating the flux of $\vec{F}$ is twice the corresponding term in the sum approximating the flux of $\vec{G}$, making the sum for $\vec{F}$ twice that of the sum for $\vec{G}$. Thus the flux of $\vec{F}$ is twice the flux of $\vec{G}$.

81. False. The flux integral measures the net flow *through* the surface S. The vector field $\vec{G}$ could be large in magnitude on S (larger than $||\vec{F}||$), but be parallel to the surface S, and so contribute nothing to the flux. Put another way, a "small" vector field, flowing directly across S, can have greater flux than a much "larger" field flowing parallel to S.

For example, take S to be the square $0 \le x \le 1, 0 \le y \le 1, z = 0$, oriented upward. Then if $\vec{F} = \vec{k}$ and $\vec{G} = 5\vec{i}$, the flux integrals have values $\int_S \vec{F} \cdot d\vec{A} = 1$ and $\int_S \vec{G} \cdot d\vec{A} = 0$ (since $\vec{G}$ is parallel to S). Thus $\int_S \vec{F} \cdot d\vec{A} > \int_S \vec{G} \cdot d\vec{A}$, but $||\vec{F}|| = 1 < 5 = ||\vec{G}||$.

82. For (a), we want the vector field with the largest $\vec{i}$ component so $\vec{F}_1$.
For (b), we want the vector field with the largest $\vec{i}$ component so $\vec{F}_1$.
For (c), we want the vector field with the most negative $\vec{k}$ component, so $\vec{F}_4$.
For (d), we want the vector field with the most negative $\vec{k}$ component, so $\vec{F}_4$.
For (e), we want the vector field with the largest $\vec{j}$ component so $\vec{F}_3$.

Solutions for Section 19.2

Exercises

1. We have
$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\,dy = \left(-3\vec{i} + 5\vec{j} + \vec{k}\right) dx\,dy.$$

2. We have
$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\,dy = \left(-8\vec{i} - 7\vec{j} + \vec{k}\right) dx\,dy.$$

3. We have
$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\,dy = \left(-4x\vec{i} + 6y\vec{j} + \vec{k}\right) dx\,dy.$$

4. We have
$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\,dy = \left(-y\vec{i} + (-x - 2y)\vec{j} + \vec{k}\right) dx\,dy.$$

5. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\, dy = \left(-2\vec{i} + 3\vec{j} + \vec{k}\right) dx\, dy.$$

We have

$$\vec{F} \cdot d\vec{A} = (-20 + 60 + 30) dx\, dy = 70\, dx\, dy.$$

The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_{-2}^{3} \int_0^5 70\, dy\, dx.$$

6. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\, dy = \left(4\vec{i} - 10\vec{j} + \vec{k}\right) dx\, dy.$$

We have

$$\vec{F} \cdot d\vec{A} = (4z - 10x + y)\, dx\, dy.$$

The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^8 \int_0^4 (4z - 10x + y)\, dx\, dy.$$

7. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\, dy = \left(\sin x\vec{i} - 2\cos 2y\vec{j} + \vec{k}\right) dx\, dy.$$

We have

$$\vec{F} \cdot d\vec{A} = (yz\sin x - 2xy\cos 2y + xy) dx\, dy.$$

The region R is a triangle bounded by the x-axis, the y-axis, and the line $y = 5 - x$. Thus, R is described by the inequalities $0 \le y \le 5 - x$, $0 \le x \le 5$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^5 \int_0^{5-x} (yz\sin x - 2xy\cos 2y + xy)\, dy\, dx.$$

8. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\, dy = \left(-e^{3y}\vec{i} - 3xe^{3y}\vec{j} + \vec{k}\right) dx\, dy.$$

We have

$$\vec{F} \cdot d\vec{A} = -3xe^{3y}\cos(x + 2y)\, dx\, dy.$$

The region R is the part of the disk $x^2 + y^2 \le 5^2$ where $x \ge 0$ and $y \ge 0$. Thus, R is described by the inequalities $0 \le y \le \sqrt{5^2 - x^2}$, $0 \le x \le 5$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^5 \int_0^{\sqrt{25-x^2}} -3xe^{3y}\cos(x + 2y)\, dy\, dx.$$

9. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\, dy = \left(-4\vec{i} + 2\vec{j} + \vec{k}\right) dx\, dy.$$

Hence

$$\vec{F} \cdot d\vec{A} = (-12 - 4 + 6) dx\, dy = -10\, dx\, dy.$$

The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^5 \int_0^{10} -10\, dy\, dx = -500.$$

10. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right) dx\, dy = \left(-y\vec{i} - x\vec{j} + \vec{k}\right) dx\, dy.$$

Since $z = xy$ we have

$$\vec{F} \cdot d\vec{A} = (-y + 2x + xy)\, dx\, dy.$$

The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^{10} \int_0^{10} (-y + 2x + xy)\, dx\, dy$$

$$= \int_0^{10} 100 + 40y\, dy = 3000.$$

11. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right)\, dx\, dy = \left(-2x\vec{i} - 2\vec{j} + \vec{k}\right)\, dx\, dy.$$

Since $z = x^2 + 2y$ have

$$\vec{F} \cdot d\vec{A} = (-2x\cos y - 2x^2 - 4y + 1)\, dx\, dy.$$

The flux of $\vec{F}$ through the surface S is given by

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_0^1 \int_0^1 (-2x\cos y - 2x^2 - 4y + 1) dx\, dy\\ &= \int_0^1 \frac{1}{3} - 4y - \cos y\, dy = -\frac{5}{3} - \sin 1 = -2.508.\end{aligned}$$

12. We have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right)\, dx\, dy = \left(-\vec{i} - \vec{j} + \vec{k}\right)\, dx\, dy.$$

Since $z = x + y + 2$ have

$$\vec{F} \cdot d\vec{A} = (y + 2)\, dx\, dy.$$

The (x, y) values for points on S are in the triangle with vertices $(-1, 0)$, $(1, 0)$, $(0, 1)$. The edges of the triangle are given by the x-axis, $y = 1 + x$ and $y = 1 - x$. For this region, it makes sense to integrate on x first, so we describe the region by the inequalities $y - 1 \le x \le 1 - y$, $0 \le y \le 1$. The flux of $\vec{F}$ through the surface S is

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_0^1 \int_{y-1}^{1-y} (y + 2)\, dx\, dy\\ &= \int_0^1 4 - 2y - 2y^2\, dy = \frac{7}{3}.\end{aligned}$$

13. Since the cylinder radius is 10, we have

$$d\vec{A} = 10\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\, dz\, d\theta.$$

Hence

$$\vec{F} \cdot d\vec{A} = 10\left(\cos\theta + 2\sin\theta\right)\, dz\, d\theta.$$

The θz-region corresponding to S is given by $0 \le \theta \le \pi/2$, $0 \le z \le 5$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^{\pi/2} \int_0^5 10\left(\cos\theta + 2\sin\theta\right)\, dz\, d\theta.$$

14. Since the cylinder radius is 10, we have

$$d\vec{A} = 10\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\, dz\, d\theta.$$

Using $x = 10\cos\theta$ and $y = 10\sin\theta$ we have

$$\vec{F} = 10\cos\theta\vec{i} + 20\sin\theta\vec{j} + 3z\vec{k}.$$

Hence

$$\vec{F} \cdot d\vec{A} = 10\left(10\cos^2\theta + 20\sin^2\theta\right)\, dz\, d\theta.$$

The θz-region corresponding to S is given by $0 \le \theta \le 2\pi$, $0 \le z \le 5$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_0^5 10\left(10\cos^2\theta + 20\sin^2\theta\right)\, dz\, d\theta.$$

15. Since the cylinder radius is 6, we have

$$d\vec{A} = 6\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\,dz\,d\theta.$$

Using $x = 6\cos\theta$ and $y = 6\sin\theta$ we have

$$\vec{F} = z^2\vec{i} + e^{6\cos\theta}\vec{j} + \vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = \left(6z^2\cos\theta + 6\sin\theta e^{6\cos\theta}\right)\,dz\,d\theta.$$

The surface S is the part of the cylinder $x^2+y^2=6^2$ that is inside the sphere $x^2+y^2+z^2=10^2$. The cylinder and the sphere intersect at points where $z^2 = 10^2 - 6^2 = 64$, where $z = 8$ or $z = -8$. Therefore, the θz-region corresponding to S is given by $0 \le \theta \le 2\pi$, $-8 \le z \le 8$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_{-8}^{8}\left(6z^2\cos\theta + 6\sin\theta e^{6\cos\theta}\right)\,dz\,d\theta.$$

16. Since the cylinder radius is 2, we have

$$d\vec{A} = 2\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\,dz\,d\theta.$$

Using $x = 2\cos\theta$ and $y = 2\sin\theta$ we have

$$\vec{F} = 8z\cos^2\theta\sin\theta\vec{j} + z^3\vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 16z\cos^2\theta\sin^2\theta\,dz\,d\theta.$$

The surface S is the part of the cylinder $x^2+y^2=2^2$ that is between the xy-plane $z=0$ and the paraboloid $z = x^2+y^2$. The cylinder and the paraboloid intersect at points where $z = 2^2$. Therefore, the θz-region corresponding to S is given by $0 \le \theta \le 2\pi$, $0 \le z \le 4$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^4 16z\cos^2\theta\sin^2\theta\,dz\,d\theta.$$

17. Since the cylinder radius is 5, we have

$$d\vec{A} = 5\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\,dz\,d\theta.$$

Hence

$$\vec{F}\cdot d\vec{A} = 5z\sin\theta\,dz\,d\theta.$$

The θz-region corresponding to S, the part of the cylinder with $y \ge 0$ and $0 \le z \le 20$, is given by $0 \le \theta \le \pi$, $0 \le z \le 20$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{\pi}\int_0^{20} 5z\sin\theta\,dz\,d\theta = \int_0^{\pi} 1000\sin\theta\,d\theta = 2000.$$

18. Since the cylinder radius is 10, we have

$$d\vec{A} = 10\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\,dz\,d\theta.$$

Using $x = 10\cos\theta$ and $y = 10\sin\theta$ we have

$$\vec{F} = 10\sin\theta\vec{i} + 10z\cos\theta\vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 100\cos\theta\sin\theta\,dz\,d\theta.$$

The surface S is the part of the cylinder with $x \ge 0$, $y \ge 0$ and $0 \le z \le 3$, so the θz-region corresponding to S is given by $0 \le \theta \le \pi/2$, $0 \le z \le 3$. The flux of $\vec{F}$ through the surface S is given by

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{\pi/2}\int_0^3 100\cos\theta\sin\theta\,dz\,d\theta \\ &= \int_0^{\pi/2} 300\cos\theta\sin\theta\,d\theta \\ &= 150\sin^2\theta\Big|_{\theta=0}^{\pi/2} = 150.\end{aligned}$$

19. Since the cylinder radius is 2, we have

$$d\vec{A} = 2\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\, dz\, d\theta.$$

Using $x = 2\cos\theta$ and $y = 2\sin\theta$ we have

$$\vec{F} = 4z\cos\theta\sin\theta\vec{j} + 2e^z\cos\theta\vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 8z\cos\theta\sin^2\theta\, dz\, d\theta.$$

The surface S is the part of the cylinder with $0 \le y \le x$ and $0 \le z \le 10$, so the θz-region corresponding to S is given by $0 \le \theta \le \pi/4$, $0 \le z \le 10$. The flux of $\vec{F}$ through the surface S is given by

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{\pi/4}\int_0^{10} 8z\cos\theta\sin^2\theta\, dz\, d\theta \\ &= \int_0^{\pi/4} 400\cos\theta\sin^2\theta\, d\theta \\ &= \frac{400}{3}\sin^3\theta\Big|_{\theta=0}^{\pi/4} = \frac{100\sqrt{2}}{3}.\end{aligned}$$

20. Since the cylinder radius is 1, we have

$$d\vec{A} = \left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)\, dz\, d\theta.$$

Using $x = \cos\theta$ and $y = \sin\theta$ we have

$$\vec{F} = \cos\theta\sin\theta\vec{i} + 2z\vec{j}.$$

Hence

$$\vec{F}\cdot d\vec{A} = \left(\cos^2\theta\sin\theta + 2z\sin\theta\right)\, dz\, d\theta.$$

The surface S is the part of the cylinder with $x \ge 0$, $0 \le y \le 1/2$ and $0 \le z \le 2$, so the θz-region corresponding to S is given by $0 \le \theta \le \pi/6$, $0 \le z \le 2$. The flux of $\vec{F}$ through the surface S is given by

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{\pi/6}\int_0^{2}\left(\cos^2\theta\sin\theta + 2z\sin\theta\right)\, dz\, d\theta \\ &= \int_0^{\pi/6} 2\cos^2\theta\sin\theta + 4\sin\theta\, d\theta \\ &= -\frac{2}{3}\cos^3\theta - 4\cos\theta\Big|_{\theta=0}^{\pi/6} = \frac{14}{3} - \frac{9}{4}\sqrt{3} = 0.770.\end{aligned}$$

21. Since the sphere radius is 10, we have

$$d\vec{A} = 10^2\left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Hence

$$\vec{F}\cdot d\vec{A} = 100\left(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi\right)\sin\phi\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to S, the upper hemisphere $z \ge 0$, is given by $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^{\pi/2} 100\left(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi\right)\sin\phi\, d\phi\, d\theta.$$

22. Since the sphere radius is 5, we have

$$d\vec{A} = 5^2\left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Using $x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$ and $z = 5\cos\phi$ we have

$$\vec{F} = 5\sin\phi\cos\theta\vec{i} + 10\sin\phi\sin\theta\vec{j} + 15\cos\phi\vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 5^2\left(5\sin^2\phi\cos^2\theta + 10\sin^2\phi\sin^2\theta + 15\cos^2\phi\right)\sin\phi\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to S, the entire sphere, is given by $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^{\pi} 25\left(5\sin^2\phi\cos^2\theta + 10\sin^2\phi\sin^2\theta + 15\cos^2\phi\right)\sin\phi\, d\phi\, d\theta.$$

23. Since the sphere radius is 2, we have

$$d\vec{A} = 2^2\left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Using $z = 2\cos\phi$ we have

$$\vec{F} = 4\cos^2\phi\vec{i}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 16\cos^2\phi\sin^2\phi\cos\theta\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to S, the part of the sphere with $x \ge 0$, is given by $-\pi/2 \le \theta \le \pi/2$, $0 \le \phi \le \pi$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_{-\pi/2}^{\pi/2}\int_0^{\pi} 16\cos^2\phi\sin^2\phi\cos\theta\, d\phi\, d\theta.$$

24. Since the sphere radius is 3, we have

$$d\vec{A} = 3^2\left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Using $x = 3\sin\phi\cos\theta$ we have

$$\vec{F} = e^{3\sin\phi\cos\theta}\vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 9\cos\phi\sin\phi e^{3\sin\phi\cos\theta}\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to S, the part of the sphere with $y \ge 0$ and $z \le 0$, is given by $0 \le \theta \le \pi$, $\pi/2 \le \phi \le \pi$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{\pi}\int_{\pi/2}^{\pi} 9\cos\phi\sin\phi e^{3\sin\phi\cos\theta}\, d\phi\, d\theta.$$

25. Since the sphere radius is 20, we have

$$d\vec{A} = 20^2\left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Using $z = 20\cos\phi$ we have

$$\vec{F} = 20\cos\phi\vec{i}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 8000\sin^2\phi\cos\phi\cos\theta\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to S, the part of the sphere where $x \ge 0$, $y \ge 0$ and $z \ge 0$, is given by $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$. The flux of $\vec{F}$ through the surface S is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{\pi/2}\int_0^{\pi/2} 8000\sin^2\phi\cos\phi\cos\theta\, d\phi\, d\theta$$

$$= \int_0^{\pi/2}\frac{8000}{3}\cos\theta\, d\theta = \frac{8000}{3}.$$

26. Since the sphere radius is 4, we have

$$d\vec{A} = 4^2\left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Using $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$ and $z = 4\cos\phi$ we have

$$\vec{F} = 4\sin\phi\sin\theta\vec{i} - 4\sin\phi\cos\theta\vec{j} + 4\cos\phi\vec{k}.$$

Hence

$$\vec{F}\cdot d\vec{A} = 64\cos^2\phi\sin\phi\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to the entire sphere is given by $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. The flux of $\vec{F}$ through the surface S is given by

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{2\pi}\int_0^{\pi} 64\cos^2\phi\sin\phi\, d\phi\, d\theta\\ &= \int_0^{2\pi} -\frac{64}{3}\cos^3\phi\Big|_{\phi=0}^{\pi}\, d\theta\\ &= \int_0^{2\pi}\frac{128}{3}\, d\theta = \frac{256}{3}\pi.\end{aligned}$$

27. Since the sphere radius is 1, we have

$$d\vec{A} = \left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\right)\sin\phi\, d\phi\, d\theta.$$

Using $x = \sin\phi\cos\theta$ and $y = \sin\phi\sin\theta$ we have

$$\vec{F} = \sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j}.$$

Hence

$$\vec{F}\cdot d\vec{A} = \left(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta\right)\, d\phi\, d\theta = \sin^3\phi\, d\phi\, d\theta.$$

The $\theta\phi$-region corresponding to region above the cone $\phi = \pi/4$ is given by $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$. The flux of $\vec{F}$ through the surface S is given by

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{\pi/4}\int_0^{2\pi}\sin^3\phi\, d\theta\, d\phi = 2\pi\int_0^{\pi/4}\sin^3\phi\, d\phi\\ &= 2\pi\int_0^{\pi/4}\sin\phi(1-\cos^2\phi)\, d\phi\\ &= 2\pi(-\cos\phi + \frac{1}{3}\cos^3\phi)\Big|_{\phi=0}^{\pi/4}\\ &= \frac{8-5\sqrt{2}}{6}\pi = 0.486.\end{aligned}$$

28. We find the equation for the plane S in the form $z = f(x,y)$. The rectangle lies in the plane $z + 2y = 4$ we have

$$z = f(x,y) = 4 - 2y.$$

Thus, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})\, dx\, dy = (0\vec{i} + 2\vec{j} + \vec{k})\, dx\, dy = (2\vec{j} + \vec{k})\, dx\, dy.$$

The flux integral is therefore

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^2\int_0^2 (4-2y)\vec{k}\cdot(2\vec{j} + \vec{k})\, dx\, dy = \int_0^2\int_0^2 4 - 2y\, dx\, dy = 8.$$

29. We find the equation for the plane S in the form $z = f(x, y)$. The rectangle lies in the plane $3x + 2z = 6$ so we have

$$z = f(x, y) = 3 - \frac{3}{2}x.$$

Thus, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dx\,dy = \left(\frac{3}{2}\vec{i} + 0\vec{j} + \vec{k}\right)\,dx\,dy = \left(\frac{3}{2}\vec{i} + \vec{k}\right)\,dx\,dy.$$

The flux integral is therefore

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^2\int_0^2 \left(3 - \frac{3}{2}x\right)\vec{k} \cdot \left(\frac{3}{2}\vec{i} + \vec{k}\right)\,dx\,dy = \int_0^2\int_0^2 3 - \frac{3}{2}x\,dx\,dy = 6.$$

Problems

30. We find the equation for the plane S in the form $z = f(x, y)$. We have

$$z = f(x, y) = 1 - x - y.$$

Thus, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dy\,dx = (-(-1)\vec{i} - (-1)\vec{j} + \vec{k})\,dy\,dx = (\vec{i} + \vec{j} + \vec{k})\,dy\,dx.$$

The surface S intersects the xy-plane in the line $x + y = 1$, so the region R in the xy-plane below S is bounded by the x-axis, y-axis, and the line $y = 1 - x$. The flux integral is, therefore,

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_0^1\int_0^{1-x} (1 - x - y)\vec{k} \cdot (\vec{i} + \vec{j} + \vec{k})\,dy\,dx \\ &= \int_0^1\int_0^{1-x} (1 - x - y)\,dy\,dx = \int_0^1 y - xy - \frac{y^2}{2}\bigg|_0^{1-x} dx = \int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2\,dx = \frac{1}{6}.\end{aligned}$$

31. Using $z = f(x, y) = x + y$, we have $d\vec{A} = (-\vec{i} - \vec{j} + \vec{k})\,dx\,dy$. As S is oriented upward, we have

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_0^3\int_0^2 ((x - y)\vec{i} + (x + y)\vec{j} + 3x\vec{k}) \cdot (-\vec{i} - \vec{j} + \vec{k})\,dxdy \\ &= \int_0^3\int_0^2 (-x + y - x - y + 3x)\,dxdy = \int_0^3\int_0^2 x\,dxdy = 6.\end{aligned}$$

32. Writing the surface S as $z = f(x, y) = -y + 1$, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})dxdy.$$

Thus,

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_R \vec{F}(x, y, f(x, y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy \\ &= \int_0^1\int_0^1 (2x\vec{j} + y\vec{k}) \cdot (\vec{j} + \vec{k})\,dxdy \\ &= \int_0^1\int_0^1 (2x + y)\,dxdy = \int_0^1 (x^2 + xy)\bigg|_0^1 dy \\ &= \int_0^1 (1 + y)\,dy = (y + \frac{y^2}{2})\bigg|_0^1 = \frac{3}{2}.\end{aligned}$$

33. Writing the surface S as $z = f(x, y) = y^2 + 5$, we have

$$d\vec{A} = (-f_x\vec{i} + -f_y\vec{j} + \vec{k})dxdy.$$

Thus,

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_R \vec{F}(x, y, f(x, y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\, dxdy \\
&= \int_R (-y\vec{j} + (y^2 + 5)\vec{k}) \cdot (-2y\vec{j} + \vec{k})\, dxdy \\
&= \int_0^1 \int_{-2}^1 (3y^2 + 5)\, dxdy = \int_0^1 (9y^2 + 15)\, dy \\
&= (3y^3 + 15y)\bigg|_0^1 = 18.
\end{aligned}$$

34. One the surface S we have $z = -y + 1$ and $d\vec{A} = (-z_x\vec{i} - z_y\vec{j} + \vec{k})dx\,dy = (\vec{j} + \vec{k})dx\,dy$. The flux is

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_0^1 \int_0^1 ((\ln(x^2)\vec{i} + e^x\vec{j} + \cos(1 - (-y + 1))\vec{k}) \cdot (\vec{j} + \vec{k})\, dx\, dy \\
&= \int_0^1 \int_0^1 (e^x + \cos y)\, dx\, dy = \int_0^1 (e^x + x\cos y)\bigg|_0^1 dy \\
&= \int_0^1 (e + \cos y - 1)\, dy = (ye + \sin y - y)\bigg|_0^1 = e + \sin 1 - 1.
\end{aligned}$$

35. On the curved sides of the cylinder, the $\vec{k}$ component of $\vec{F}$ does not contribute to the flux. Since the $\vec{i}$ and $\vec{j}$ components are constant, these components contribute 0 to the flux on the entire cylinder. Therefore the only nonzero contribution to the flux results from the $\vec{k}$ component through the top, where $z = 2$ and $d\vec{A} = \vec{k}\,dA$, and from the $\vec{k}$ component through the bottom, where $z = -2$ and $d\vec{A} = -\vec{k}\,dA$:

$$\begin{aligned}
\text{Flux} &= \int_{\text{Top}} \vec{F} \cdot d\vec{A} + \int_{\text{Bottom}} \vec{F} \cdot d\vec{A} \\
&= \int_{\text{Top}} 2\vec{k} \cdot \vec{k}\, dA + \int_{\text{Bottom}} (-2\vec{k}) \cdot (-\vec{k}\, dA) \\
&= 4\int_{\text{Top}} dA = 4 \cdot \text{Area of top} = 4 \cdot \pi(3^2) = 36\pi.
\end{aligned}$$

36. On the curved side of the cylinder, only the components $x\vec{i} + z\vec{k}$ contribute to the flux. Since $x\vec{i} + z\vec{k}$ is perpendicular to the curved surface and $||x\vec{i} + z\vec{k}|| = 2$ there (because the cylinder has radius 2), we have

$$\text{Flux through sides} = 2 \cdot \text{Area of curved surface} = 2 \cdot 2\pi \cdot 2 \cdot 6 = 48\pi.$$

On the flat ends, only $y\vec{j}$ contributes to the flux. On one end, $y = 3$ and $d\vec{A} = \vec{j}\,dA$; on the other end, $y = -3$ and $d\vec{A} = -\vec{j}\,dA$. Thus

$$\begin{aligned}
\text{Flux through ends} &= \text{Flux through top} + \text{Flux through bottom} \\
&= 3\vec{j} \cdot \vec{j}\,\pi(2^2) + (-3\vec{j}) \cdot (-\vec{j}\,\pi(2^2)) = 24\pi.
\end{aligned}$$

So,

$$\text{Total flux} = 48\pi + 24\pi = 72\pi.$$

37. Writing the surface S as $z = f(x, y) = -2x - 4y + 1$, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})dxdy.$$

With R as shown in Figure 19.9, we have

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_R \vec{F}(x,y,f(x,y))\cdot(-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy\\ &= \int_R (3x\vec{i} + y\vec{j} + (-2x-4y+1)\vec{k})\cdot(2\vec{i} + 4\vec{j} + \vec{k})\,dxdy\\ &= \int_R (4x+1)\,dxdy = \int_0^1\int_0^{-2x+2}(4x+1)\,dydx\\ &= \int_0^1 (4x+1)(-2x+2)\,dx\\ &= \int_0^1(-8x^2+6x+2)\,dx = \left(-\frac{8x^3}{3}+3x^2+2x\right)\Big|_0^1 = \frac{7}{3}.\end{aligned}$$

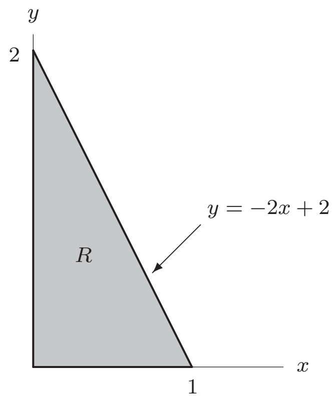


Figure 19.9

38. Writing the surface S as $z = f(x,y) = 25 - x^2 - y^2$, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})dxdy.$$

Thus,

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_R \vec{F}(x,y,f(x,y))\cdot(-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy\\ &= \int_R (x\vec{i} + y\vec{j})\cdot(2x\vec{i} + 2y\vec{j} + \vec{k})\,dxdy\\ &= \int_R 2(x^2+y^2)\,dxdy = \int_0^{2\pi}\int_0^5 2r^2 r\,drd\theta\\ &= \int_0^{2\pi}\frac{r^4}{2}\Big|_0^5 d\theta = \frac{625}{2}(2\pi) = 625\pi.\end{aligned}$$

39. Writing the surface S as $z = f(x,y) = 25 - x^2 - y^2$, we have

$$d\vec{A} = (-f_x\vec{i} - f_y\vec{j} + \vec{k})dxdy = (2x\vec{i} + 2y\vec{j} + \vec{k})dxdy.$$

Thus

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_R \cos(x^2+y^2)\vec{k}\cdot(2x\vec{i} + 2y\vec{j} + \vec{k})\,dxdy\\ &= \int_R \cos(x^2+y^2)\,dxdy = \int_0^{2\pi}\int_0^5 \cos r^2\cdot r\,drd\theta\\ &= \int_0^{2\pi}\frac{\sin r^2}{2}\Big|_0^5 d\theta = \pi\sin 25.\end{aligned}$$

40. Using $z = 1 - x - y$, the upward pointing area element is $d\vec{A} = (\vec{i} + \vec{j} + \vec{k})\,dx\,dy$, so the downward one is $d\vec{A} = (-\vec{i} - \vec{j} - \vec{k})\,dx\,dy$. Since S is oriented downward, we have

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_S (x\vec{i} + y\vec{j} + z\vec{k}) \cdot d\vec{A} \\ &= \int_0^3 \int_0^2 (x\vec{i} + y\vec{j} + (1-x-y)\vec{k}) \cdot (-\vec{i} - \vec{j} - \vec{k})\,dxdy \\ &= \int_0^3 \int_0^2 (-x - y - 1 + x + y)\,dxdy = -6.\end{aligned}$$

41. Using $z = x^2 + y^2$, we find that the upward pointing area element is $d\vec{A} = (-2x\vec{i} - 2y\vec{j} + \vec{k})\,dx\,dy$. Since S is oriented downward, we have $d\vec{A} = (2x\vec{i} + 2y\vec{j} - \vec{k})\,dx\,dy$, so

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_S (x\vec{i} + y\vec{j} + z\vec{k}) \cdot d\vec{A} \\ &= \int_{\text{Disk}} (x\vec{i} + y\vec{j} + (x^2+y^2)\vec{k}) \cdot (2x\vec{i} + 2y\vec{j} - \vec{k})\,dx\,dy \\ &= \int_{\text{Disk}} (2x^2 + 2y^2 - x^2 - y^2)\,dx\,dy = \int_{\text{Disk}} (x^2+y^2)\,dx\,dy \\ &= \int_0^{2\pi} \int_0^1 r^2 r\,dr\,d\theta = 2\pi \cdot \frac{r^4}{4}\bigg|_0^1 = \frac{\pi}{2}.\end{aligned}$$

42. Here $z = \sqrt{9 - x^2 - y^2}$, so

$$z_x = -\frac{x}{\sqrt{9-x^2-y^2}} \quad z_y = -\frac{y}{\sqrt{9-x^2-y^2}}.$$

The flux integral is given by

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_S \left(x\sqrt{9-x^2-y^2}\vec{i} + y\vec{k}\right) \cdot \left(\frac{x}{\sqrt{9-x^2-y^2}}\vec{i} + \frac{y}{\sqrt{9-x^2-y^2}}\vec{j} + \vec{k}\right) dxdy \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2 + y)\,dydx\end{aligned}$$

Changing to polar coordinates gives

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_0^{2\pi} \int_0^3 (r^2\cos^2\theta + r\sin\theta)\,rdrd\theta \\ &= \int_0^{2\pi} \left(\frac{81}{4}\cos^2\theta + \frac{27}{3}\sin\theta\right) d\theta = \frac{81}{4}\pi.\end{aligned}$$

43. We have $0 \le z \le 6$ so $0 \le x^2 + y^2 \le 36$. Let R be the disk of radius 6 in the xy-plane centered at the origin. Because of the cone's point, the flux integral is improper; however, it does converge. We have

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_R \vec{F}(x,y,f(x,y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy \\ &= \int_R (-x\sqrt{x^2+y^2}\vec{i} - y\sqrt{x^2+y^2}\vec{j} + (x^2+y^2)\vec{k}) \\ &\quad \cdot \left(-\frac{x}{\sqrt{x^2+y^2}}\vec{i} - \frac{y}{\sqrt{x^2+y^2}}\vec{j} + \vec{k}\right) dxdy \\ &= \int_R 2(x^2+y^2)\,dxdy\end{aligned}$$

$$= 2\int_0^6 \int_0^{2\pi} r^3 \, d\theta dr$$
$$= 4\pi \int_0^6 r^3 \, dr = 1296\pi.$$

44. Since $y = f(x,z) = x^2 + z^2$, we have

$$d\vec{A} = (-f_x\vec{i} + \vec{j} - f_z\vec{k})\, dxdz = (-2x\vec{i} + \vec{j} - 2z\vec{k})\, dxdz.$$

Thus, substituting $y = x^2 + z^2$ into $\vec{F}$, we have

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_{x^2+z^2\le 1} ((x^2+z^2)\vec{i} + \vec{j} - xz\vec{k}) \cdot (-2x\vec{i} + \vec{j} - 2z\vec{k})\, dxdz \\
&= \int_{x^2+z^2\le 1} (-2x^3 - 2xz^2 + 1 + 2xz^2)\, dxdz \\
&= \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} (1 - 2x^3)\, dxdz \\
&= \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dxdz - \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2x^3\, dxdz \\
&= \text{Area of disk} - \int_{-1}^{1} \left(\frac{x^4}{2}\bigg|_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \right) dz = \pi - 0 = \pi
\end{aligned}$$

45. The plane through the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is given by $x + y + z = 1$, so S is the part of the graph of $z = f(x,y) = 1 - x - y$ above the region R in the xy-plane where $x \ge 0$, $y \ge 0$, and $x + y \le 1$. Thus

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_R \vec{F}(x, y, f(x,y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\, dx\, dy \\
&= \int_R (x^2\vec{i} + y^2\vec{j} + (1-x-y)^2\vec{k}) \cdot (\vec{i} + \vec{j} + \vec{k})\, dx\, dy \\
&= \int_R (x^2 + y^2 + (1-x-y)^2)\, dx\, dy \\
&= \int_0^1 \int_0^{1-x} (1 + 2x^2 + 2y^2 - 2x - 2y + 2xy)\, dy\, dx \\
&= \int_0^1 [(1-x) + 2x^2(1-x) + \frac{2}{3}(1-x)^3 - 2x(1-x) \\
&\quad -(1-x)^2 + x(1-x)^2]\, dx = \frac{1}{4}.
\end{aligned}$$

46. Since the radius of the cylinder is 1, using cylindrical coordinates we have

$$d\vec{A} = (\cos\theta\vec{i} + \sin\theta\vec{j})d\theta dz.$$

Thus,

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_0^6 \int_0^{2\pi} (\cos\theta\vec{i} + \sin\theta\vec{j}) \cdot (\cos\theta\vec{i} + \sin\theta\vec{j})\, d\theta\, dz \\
&= \int_0^6 \int_0^{2\pi} 1\, d\theta\, dz = 12\pi.
\end{aligned}$$

47. Since the radius of the cylinder is 1, using cylindrical coordinates we have $d\vec{A} = (\cos\theta\vec{i} + \sin\theta\vec{j})d\theta dz$. Thus,

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^6\int_0^{2\pi}(z\cos\theta\vec{i} + z\sin\theta\vec{j} + z^3\vec{k})\cdot(\cos\theta\vec{i} + \sin\theta\vec{j})\,d\theta\,dz\\ &= \int_0^6\int_0^{2\pi} z\,d\theta\,dz = 2\pi\left(\frac{z^2}{2}\right)\Big|_0^6 = 36\pi.\end{aligned}$$

48. The flux of $\vec{F}$ through S is given by

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{2\pi}\int_0^{\pi/2}(2\cos\phi\vec{k})\cdot(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})2^2\sin\phi\,d\phi\,d\theta\\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} 8\sin\phi\cos^2\phi d\phi d\theta = 16\pi\left(\frac{-\cos^3\phi}{3}\right)\Big|_{\phi=0}^{\pi/2} = \frac{16\pi}{3}.\end{aligned}$$

49. A parameterization for the surface S is given by $z = \sqrt{1-x^2-y^2}$ over R for $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $-1 \le x \le 1$. Thus

$$z_x = -\frac{x}{\sqrt{1-x^2-y^2}} \quad\text{and}\quad z_y = -\frac{y}{\sqrt{1-x^2-y^2}},$$

so

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_R (y\vec{i} - x\vec{j} + z\vec{k})\cdot(-z_x\vec{i} - z_y\vec{j} + \vec{k})\,dx\,dy\\ &= \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{yx - xy}{\sqrt{1-x^2-y^2}} + z\,dx\,dy = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy\,dx\\ &= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}r\,dr\,d\theta = 2\pi\left[-\frac{1}{2}\cdot\frac{2}{3}(1-r^2)^{3/2}\right]_0^1 = \frac{2\pi}{3}.\end{aligned}$$

50. Since the radius of the sphere is 5, using spherical coordinates we have

$$d\vec{A} = (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})25\sin\phi\,d\theta\,d\phi.$$

Thus,

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(25\cos^2\phi\vec{k})\cdot(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})25\sin\phi\,d\theta\,d\phi\\ &= 625\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cos^3\phi\sin\phi\,d\theta d\phi\\ &= -1250\pi\frac{(\cos\phi)^4}{4}\Big|_0^{\frac{\pi}{2}} = \frac{625}{2}\pi.\end{aligned}$$

51. Since the radius of the sphere is a, using spherical coordinates we have

$$d\vec{A} = (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})a^2\sin\phi\,d\phi\,d\theta.$$

Thus,

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{2\pi}\int_0^{\pi}(a\sin\phi\cos\theta\vec{i} + a\sin\phi\sin\theta\vec{j} + a\cos\phi\vec{k})\cdot\\ &\qquad (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})a^2\sin\phi\,d\phi\,d\theta\\ &= a^3\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta\\ &= 2\pi a^3\int_0^{\pi}\sin\phi\,d\phi = (2\pi a^3)(2) = 4\pi a^3.\end{aligned}$$

52. The $\vec{k}$ -component of $\vec{F}$ does not contribute to the flux as it is perpendicular to the surface. The vector field $x\vec{i}+y\vec{j}$ is everywhere perpendicular to S and has constant magnitude $||x^2+y^2||=1$ on the surface S. Thus

$$\int_S \vec{F}\cdot d\vec{A} = \int_S (x\vec{i}+y\vec{j})\cdot d\vec{A} = 1\cdot \text{Area of } S = 1\frac{\pi}{2} = \frac{\pi}{2}.$$

Alternatively, the flux can be computed by integrating with respect to x and z, treating y as a function of x and z. A parameterization of S is given by $y=\sqrt{1-x^2}$, $0\le x\le 1$,$0\le y\le 1$, $0\le z\le 1$. Thus,

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= \int_0^1\int_0^1 (x\vec{i}+\sqrt{1-x^2}\vec{j}+z\vec{k})\cdot(-y_x\vec{i}+\vec{j}-y_z\vec{k})\,dx\,dz\\
&= \int_0^1\int_0^1 (x\vec{i}+\sqrt{1-x^2}\vec{j}+z\vec{k})\cdot\left(\frac{x}{\sqrt{1-x^2}}\vec{i}+\vec{j}+0\vec{k}\right)dx\,dz\\
&= \int_0^1\int_0^1 \left(\frac{x^2}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right)dx\,dz\\
&= \int_0^1\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx\,dz\\
&= 1\cdot \arcsin x\Big|_0^1 = \frac{\pi}{2}.
\end{aligned}$$

53. We integrate with respect the y and z, thinking of x as a function of y and z. Since $x=\sin y\sin z$, we have $x_y=\cos y\sin z$ and $x_z=\sin y\cos z$. The surface is oriented in the direction of increasing x, so

$$\begin{aligned}
\int_S \vec{F}\cdot\vec{A} &= \int_0^{\pi/2}\int_0^{\pi/2}\vec{F}\cdot(\vec{i}-x_y\vec{j}-x_z\vec{k})\,dy\,dz\\
&= \int_0^{\pi/2}\int_0^{\pi/2}(\sin y\sin z\vec{i}+\vec{j}+\vec{k})\cdot(\vec{i}-\cos y\sin z\vec{j}-\sin y\cos z\vec{k})\,dy\,dz\\
&= \int_0^{\pi/2}\int_0^{\pi/2}(\sin y\sin z-\cos y\sin z-\sin y\cos z)\,dy\,dz\\
&= \int_0^{\pi/2} -\cos y\sin z-\sin y\sin z+\cos y\cos z\Big|_0^{\pi/2}\,dz\\
&= \int_0^{\pi/2}(\sin z-\sin z-\cos z)\,dz = -\sin z\Big|_0^{\pi/2} = -1.
\end{aligned}$$

54. We integrate with respect the x and z, treating y as a function of x and z. Since $y=x^2+z^2$, we have $y_x=2x$ and $y_z=2z$. The region of integration in the xz-plane is given by $x^2+z^2=1$, $x\ge 0$, $z\ge 0$. The orientation is toward the xz-plane, so we have

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= \int_0^1\int_0^{\sqrt{1-z^2}}\vec{F}\cdot(y_x\vec{i}-\vec{j}+y_z\vec{k})\,dx\,dz\\
&= \int_0^1\int_0^{\sqrt{1-z^2}}((x+z)\vec{i}+\vec{j}+z\vec{k})\cdot(2x\vec{i}-\vec{j}+2z\vec{k})\,dx\,dz\\
&= \int_0^1\int_0^{\sqrt{1-z^2}}(2x^2+2xz-1+2z^2)\,dx\,dz.
\end{aligned}$$

Using polar coordinates with $x=r\cos\theta$, $z=r\sin\theta$,

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{\pi/2}\int_0^1 (2r^2+2r^2\cos\theta\sin\theta-1)r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{2} + \frac{r^4}{2}\cos\theta\sin\theta - \frac{r^2}{2}\bigg|_0^1 \right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\cos\theta\sin\theta\, d\theta = \frac{1}{2}\frac{(\sin\theta)^2}{2}\bigg|_0^{\pi/2} = \frac{1}{4}1^2 = \frac{1}{4}.$$

55. On the disk, $z = 0$ and $d\vec{A} = \vec{k}\ dx\,dy$, so

$$\int_S \vec{F}\cdot d\vec{A} = \int_{x^2+y^2\le 1} (xze^{yz}\vec{i} + x\vec{j} + (5 + x^2 + y^2)\vec{k}\,)\cdot\vec{k}\ dx\,dy$$

$$= \int_{x^2+y^2\le 1} (5 + x^2 + y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 (5 + r^2)r\,dr\,d\theta$$

$$= 2\pi\left(\frac{5r^2}{2} + \frac{r^4}{4}\right)\bigg|_0^1 = \frac{11\pi}{2}.$$

56. The plane is $x - z = 0$ over region $0 \le x \le \sqrt{2}$, $0 \le y \le 2$. See Figure 19.10.

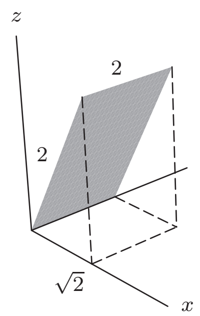


Figure 19.10

$$\text{Flux} = \int_0^2\int_0^{\sqrt{2}} \left((e^{xy} + 3z + 5)\vec{i} + (e^{xy} + 5z + 3)\vec{j} + (3z + e^{xy})\vec{k}\right)\cdot(\vec{i} - \vec{k}\,)\,dx\,dy$$

$$= \int_0^2\int_0^{\sqrt{2}} (e^{xy} + 3z + 5 - 3z - e^{xy})\,dx\,dy = 5(2)(\sqrt{2}) = 10\sqrt{2}$$

Alternatively, since a unit normal to the surface is $\vec{n}/\sqrt{2} = (\vec{i} - \vec{j}\,)/\sqrt{2}$, writing $dA = ||d\vec{A}\,||$, we have

$$\text{Flux} = \int_S \vec{H}\cdot d\vec{A} = \int \vec{H}\cdot\frac{\vec{i} - \vec{k}}{\sqrt{2}}\,dA = \int \frac{5}{\sqrt{2}}\,dA$$

$$= \frac{5}{\sqrt{2}}(\text{Area of slanted square}) = \frac{5}{\sqrt{2}}4 = 10\sqrt{2}.$$

57. (a) The charge is contained in a sphere of radius a centered at the origin, and uniformly distributed through the region enclosed by the sphere.

(b) Since $\vec{e}_\rho$ is the unit vector outward normal to the sphere of radius ρ, we have $\vec{E} = E(\rho)\vec{e}_\rho$. Let S be a sphere of fixed radius ρ, centered at the origin. Then

$$\int_W \delta\,dV = \begin{cases} \frac{4}{3}\pi\rho^3\delta_0 & \rho \le a \\ \frac{4}{3}\pi a^3\delta_0 & \rho > a. \end{cases}$$

On the other hand, since on the sphere $d\vec{A} = \vec{e}_\rho dA$, we have

$$\int_S \vec{E} \cdot d\vec{A} = \int_S E(\rho)\vec{e}_\rho \cdot \vec{e}_\rho dA = E(\rho) \int_S dA = E(\rho)4\pi\rho^2.$$

Therefore, by Gauss's Law,

$$E(\rho)4\pi\rho^2 = \begin{cases} k\frac{4}{3}\pi\rho^3\delta_0 & \rho \le a \\ k\frac{4}{3}\pi a^3\delta_0 & \rho > a. \end{cases}$$

Since $\vec{E} = E(\rho)\vec{e}_\rho$, simplifying gives

$$\vec{E} = \begin{cases} k\dfrac{\delta_0}{3}\rho\vec{e}_\rho & \rho \le a \\ k\dfrac{\delta_0 a^3}{3r^3}\vec{e}_\rho & \rho > a. \end{cases}$$

58. (a) The charge is contained in a cylinder of radius a centered at the origin, and uniformly distributed through the region enclosed by the cylinder.

(b) Since $\vec{e}_r$ is the unit outward pointing normal to the cylinder of radius r, we have $\vec{E} = E(r)\vec{e}_r$. Let S be a cylinder of fixed radius r, height 1, centered along the z-axis. If $r \le a$,

$$\int_W \delta\, dV = \pi r^2\delta_0,$$

and if $r > a$

$$\int_W \delta\, dV = \pi a^2\delta_0.$$

On the other hand, since $d\vec{A} = \vec{e}_r dA$ on S, we can write

$$\int_S \vec{E} \cdot d\vec{A} = \int_S E(r)(\vec{e}_r \cdot \vec{e}_r)dA = E(r)\int_S dA = E(r)2\pi r.$$

(The flux across the top and bottom of the cylinder is zero.) So, by Gauss's Law

$$E(r)2\pi r = \begin{cases} k\pi r^2\delta_0 & \text{if } r \le a \\ \\ k\pi a^2\delta_0 & \text{if } r > a. \end{cases}$$

Since $\vec{E} = E(r)\vec{e}_r$, simplifying gives

$$\vec{E} = \begin{cases} \frac{1}{2}k\delta_0 r\vec{e}_r & \text{if } r \le a \\ \frac{1}{2}k\delta_0\dfrac{a^2}{r}\vec{e}_r & \text{if } r > a. \end{cases}$$

Strengthen Your Understanding

59. The formula $d\vec{A} = \left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) R\, dz\, d\theta$ applies only to surfaces that lie on a cylinder centered on the z-axis of constant radius R. The distance from the cone to the z-axis is not constant, but depends on the point on the cone from which you measure the distance.

60. The orientation vector $\vec{n}$ is a unit vector in the direction of $d\vec{A}$, so

$$\vec{n} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}\left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right)$$

and

$$dA = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

61. For a surface $z = f(x, y)$ oriented upwards, we have

$$d\vec{A} = \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right)\, dx\, dy.$$

If $f(x, y) = -x - y$, then $f_x = -1$, $f_y = -1$, and

$$d\vec{A} = (\vec{i} + \vec{j} + \vec{k})\, dx\, dy.$$

62. The vector field $\vec{F} = x\vec{i} + y\vec{j}$ in polar coordinates is

$$\vec{F}(r, \theta, z) = r\cos\theta\vec{i} + r\sin\theta\vec{j}.$$

For a surface S on the cylinder $r = 10$, oriented outwards, we have

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_T \vec{F}(10, \theta, z) \cdot \left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) 10\, dz\, d\theta \\ &= \int_T \left(10\cos\theta\vec{i} + 10\sin\theta\vec{j}\right) \cdot \left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) 10\, dz\, d\theta \\ &= \int_T 100\, dz\, d\theta = 100 \cdot \int_T dz\, d\theta,\end{aligned}$$

where T is the θz-region corresponding to S. If, for example, $0 \le \theta \le 2$ and $0 \le z \le 3$, then $\int_T dz\, d\theta$ is 6. Hence $\int_S \vec{F} \cdot d\vec{A} = 600$ for the surface S parameterized by $r = 10$, $0 \le \theta \le 2$, $0 \le z \le 3$, oriented outwards.

An intuitive geometric solution is also possible. On the cylinder $x^2 + y^2 = 10^2$ of radius 10, the vector field $\vec{F} = x\vec{i} + y\vec{j}$ has magnitude $\|\vec{F}\| = 10$ and is normal to the cylinder, pointing in the orientation direction. Hence, for a surface S on the cylinder oriented away from the z-axis, we have

$$\int_S \vec{F} \cdot d\vec{A} = 10 \cdot \text{Area of } S.$$

The region S on the cylinder $r = 10$ with $0 \le \theta \le 2$, $0 \le z \le 3$ is a curved rectangle with base $10\Delta\theta = 10 \cdot 2 = 20$ and height 3, so has area 60. Hence $\int_S \vec{F} \cdot d\vec{A} = 600$.

63. True. The area vector for the graph of $f(x, y)$, parametrized in the usual way, is given by $\vec{A} = f_x\vec{i} + f_y\vec{j} + \vec{k}$. The surface area is then the double integral of the magnitude of $\vec{A}$, namely $\sqrt{f_x^2 + f_y^2 + 1}$ over the given rectangle.

64. False. Both surfaces are oriented upward, so $\vec{A}(x, y)$ and $\vec{B}(x, y)$ both point upward. But they could point in different directions, since the graph of $z = -f(x, y)$ is the graph of $z = f(x, y)$ turned upside down.

65. False. The total flux can be 0 without the vector field always being perpendicular to the surface. For example, if $F(x, y, z) = \vec{k}$, then the flux is zero over the sphere, but F is not perpendicular to the sphere except at the north and south poles.

66. (a) The surface can be considered as made up of two rectangles, the original one and another "next door" along the x-axis. Because the vector field is independent of x, the flux through both rectangles are the same. Thus, the flux has doubled.

(b) Because the vector field does not depend on x, the flux is unchanged.

(c) The sign is reversed, but the magnitude of the flux remains the same.

(d) Tripling each side of the rectangle multiplies its area by 9. However, the surface now extends further up the z-axis, where the vector field is not given. If the vector field is larger further up the z-direction (as suggested by the diagram), then the flux has multiplied by a factor of more than 9.

Solutions for Section 19.3

Exercises

1. Scalar. $\operatorname{div}\left((x^2 + y)\vec{i} + (xye^z)\vec{j} - (\ln(x^2 + y^2))\vec{k}\right) = 2x + xe^z$

2. Scalar. $\operatorname{div}((2\sin(xy) + \tan z)\vec{i} + (\tan y)\vec{j} + (e^{x^2+y^2})\vec{k}) = 2y\cos(xy) + 1/\cos^2 y$.

3. The first vector field appears to be diverging more at the origin, since both fields are zero at the origin and the vectors near the origin are larger in field (I) than they are in field (II).

4. (a) Positive. The inflow from the lower left is less than the outflow from the upper right. The net outflow is positive.
(b) Zero. The inflow on the right side is equal to outflow on the left.
(c) Negative. The inflow from above is greater than the outflow below. The net outflow is negative.

5. $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$

6. $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(y) = -1 + 1 = 0$

7. $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(-x+y) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(-z+x) = -1+1-1 = -1$

8. $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x^2-y^2) + \frac{\partial}{\partial y}(2xy) = 2x + 2x = 4x$

9. We have
$$\text{div}(3x^2\vec{i} - \sin(xz)(\vec{i} + \vec{k})) = 6x - z\cos(xz) - x\cos(xz).$$

10. We have
$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}\left(\ln\left(x^2+1\right)\right) = \frac{2x}{x^2+1},\\ \frac{\partial F_2}{\partial y} &= \frac{\partial}{\partial y}(\cos y) = -\sin y,\\ \frac{\partial F_3}{\partial z} &= \frac{\partial}{\partial z}(xye^z) = xye^z.\end{aligned}$$
So,
$$\begin{aligned}\text{div}\,\vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\\ &= \frac{2x}{x^2+1} - \sin y + xye^z.\end{aligned}$$

11. Using the formula for $\vec{a} \times \vec{r}$ in Cartesian coordinates, we get
$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(a_2z - a_3y) + \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_1y - a_2x) = 0$$

12. Taking partial derivatives, we get
$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}\left(\frac{-y}{(x^2+y^2)}\right) + \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)}\right) = \frac{2xy}{(x^2+y^2)^2} - \frac{2yx}{(x^2+y^2)^2} = 0.$$

13. In coordinates, we have
$$\vec{F}(x,y,z) = \frac{(x-x_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\vec{i} + \frac{(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\vec{j}$$
$$+\frac{(z-z_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\vec{k}.$$
So if $(x,y,z) \neq (x_0,y_0,z_0)$, then
$$\begin{aligned}\text{div}\,\vec{F} &= \left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{(x-x_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}}\right)\\ &+\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{(y-y_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}}\right)\\ &+\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{(z-z_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}}\right)\end{aligned}$$

$$= \left(\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}} - \frac{(x-x_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}} \right)$$
$$+ \left(\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}} - \frac{(y-y_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}} \right)$$
$$+ \left(\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}} - \frac{(z-z_0)^2}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}} \right)$$
$$= \frac{3((x-x_0)^2+(y-y_0)^2+(z-z_0)^2) - ((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}}$$
$$= \frac{2}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = \frac{2}{\|\vec{r}-\vec{r}_0\|}.$$

Problems

14. Two vector fields that have positive divergence everywhere are in Figures 19.11 and 19.12.

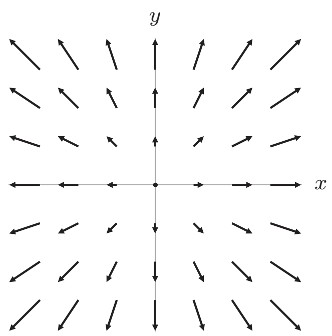


Figure 19.11

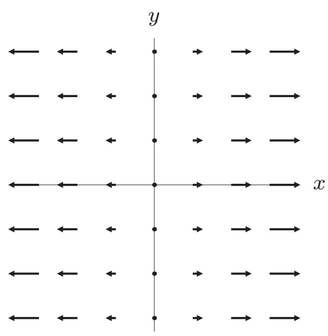


Figure 19.12

15. Two vector fields that have negative divergence everywhere are in Figures 19.13 and 19.14.

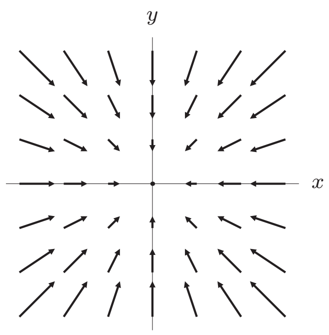


Figure 19.13

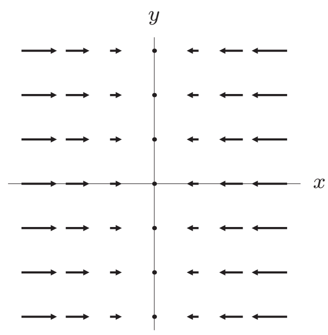


Figure 19.14

16. Two vector fields that have zero divergence everywhere are in Figures 19.15 and 19.16.

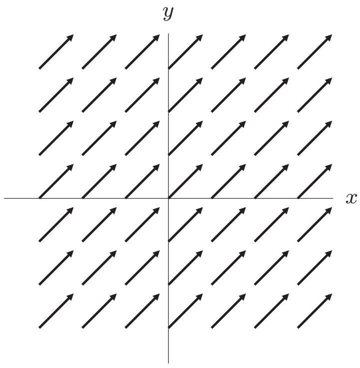


Figure 19.15

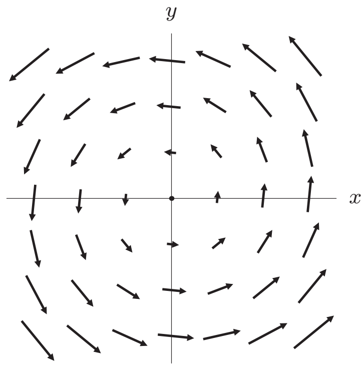


Figure 19.16

17. Since $\operatorname{div}\vec{G}$ at a point is approximately equal to the flux density out of a small region around the point, at $(2,3,-1)$ we have

$$\operatorname{div}\vec{G} \approx \frac{\text{Flux out of small region}}{\text{Volume of region}} = \frac{-0.00004\pi}{4\pi(0.1)^3/3} = -0.030.$$

Note that since the original flux was given into the region, we take the negative to get the flux out of the region.

18. Since $\operatorname{div} F(1,2,3)$ is the flux density out of a small region surrounding the point $(1,2,3)$, we have

$$\operatorname{div}\vec{F}\,(1,2,3) \approx \frac{\text{Flux out of small region around } (1,2,3)}{\text{Volume of region.}}$$

So

$$\begin{aligned}\text{Flux out of region } &\approx (\operatorname{div}\vec{F}\,(1,2,3)) \cdot \text{Volume of region}\\ &= 5 \cdot \frac{4}{3}\pi(0.01)^3\\ &= \frac{0.00002\pi}{3}.\end{aligned}$$

19. (a) (i) By the definition of divergence, if S is a sphere centered at $(2,0,0)$, we have

$$\operatorname{div}\vec{F} = \lim_{\text{Vol}\to 0} \frac{\int_S \vec{F}\cdot d\vec{A}}{\text{Volume of } S}.$$

Thus, if the sphere is small

$$\int_S \vec{F}\cdot d\vec{A} \approx \operatorname{div}\vec{F} \cdot \text{ Volume of } S.$$

Since $\operatorname{div}\vec{F} = x^2+y^2-z$, if S is the sphere given

$$\int_S \vec{F}\cdot d\vec{A} \approx (2^2+0^2-0)\cdot\frac{4}{3}\pi(0.1)^3 = \frac{0.016}{3}\pi.$$

(ii) The relationship between flux, divergence, and volume holds when S is cube. Thus

$$\int_S \vec{F}\cdot d\vec{A} \approx \operatorname{div}\vec{F} \cdot \text{ Volume of } S,$$

gives

$$\int_S \vec{F}\cdot d\vec{A} \approx (0^2+0^2-10)\cdot(0.2)^3 = -0.08.$$

(b) The point $(2,0,0)$ is a source because the flux out of a small region around the point is positive; $(0,0,10)$ is a sink because the flux out of a small region around the point is negative.

20. (a) Since div $\vec{F}$ at a point is approximately equal to the flux density out of a small region around the point, at $(4, 5, 2)$, we have

$$\text{div}\,\vec{F} \approx \frac{\text{Flux out of small region}}{\text{Volume of region}} = \frac{0.0125}{(4/3)\pi(0.1)^3} = 2.984.$$

(b) The flux through the sphere is approximated by

$$\text{Flux through sphere} = \int_S \vec{F} \cdot d\vec{A} \approx \text{div}\,\vec{F} \cdot \text{Volume of sphere} = 2.984 \cdot \frac{4}{3}\pi(0.2)^3 = 0.100.$$

We could also estimate the flux by noticing that it is eight times the original flux, that is, $8(0.0125) = 0.100$.

21. (a) On S_1, $x = a$ and normal is in negative x-direction, so

$$\vec{F} \cdot \Delta\vec{A} = (2a\vec{i} - 3y\vec{j} + 5z\vec{k}) \cdot (-\Delta A\,\vec{i}) = -2a\Delta A.$$

Thus

$$\int_{S_1} \vec{F} \cdot d\vec{A} = \int_{S_1} -2a\,dA = -2a(\text{Area of } S_1) = -2aw^2.$$

On S_2, $x = a + w$ and normal is in positive x-direction, so

$$\vec{F} \cdot \Delta\vec{A} = (2(a+w)\vec{i} - 3y\vec{j} + 5z\vec{k}) \cdot (\Delta A\,\vec{i}) = 2(a+w)\Delta A.$$

Thus

$$\int_{S_2} \vec{F} \cdot d\vec{A} = \int_{S_2} 2(a+w)dA = 2(a+w)(\text{Area of } S_2) = 2(a+w)w^2$$

Calculating the flux through the other sides similarly, we get

$$\begin{aligned}
&\text{Total flux} \\
&= \int_{S_1} \vec{F} \cdot d\vec{A} + \int_{S_2} \vec{F} \cdot d\vec{A} + \int_{S_3} \vec{F} \cdot d\vec{A} + \int_{S_4} \vec{F} \cdot d\vec{A} + \int_{S_5} \vec{F} \cdot d\vec{A} + \int_{S_6} \vec{F} \cdot d\vec{A} \\
&= -2aw^2 + 2(a+w)w^2 + 3bw^2 - 3(b+w)w^2 - 5cw^2 + 5(c+w)w^2 \\
&= (2w - 3w + 5w)w^2 = 4w^3.
\end{aligned}$$

(b) To find div $\vec{F}$ at the point (a, b, c), let the box shrink to the point by letting $w \to 0$. Then

$$\begin{aligned}
\text{div}\,\vec{F} &= \lim_{w\to 0}\left(\frac{\text{Flux through box}}{\text{Volume of box}}\right) \\
&= \lim_{w\to 0}\left(\frac{4w^3}{w^3}\right) = 4.
\end{aligned}$$

(c)

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3y) + \frac{\partial}{\partial z}(5z) = 2 - 3 + 5 = 4.$$

22. Using flux: On $S_1, x = a$ and normal is in negative x-direction, so

$$\vec{F} \cdot \Delta\vec{A} = ((3a+2)\vec{i} + 4a\vec{j} + (5a+1)\vec{k}) \cdot (-\Delta A\,\vec{i}) = -(3a+2)\Delta A$$

Thus

$$\int_{S_1} \vec{F} \cdot d\vec{A} = \int_{S_1} -(3a+2)dA = -(3a+2)(\text{Area of } S_1) = -(3a+2)w^2.$$

On $S_2, x = a + w$ and normal is in the positive x-direction, so

$$\vec{F} \cdot \Delta\vec{A} = [(3(a+w)+2)\,\vec{i} + 4(a+w)\vec{j} + (5(a+w)+1)\,\vec{k}\,] \cdot (\Delta A\,\vec{i}) = (3a+3w+2)\Delta A.$$

Thus

$$\int_{S_2} \vec{F} \cdot d\vec{A} = \int_{S_2} (3a+3w+2)dA = (3a+3w+2)(\text{Area of } S_2) = (3a+3w+2)w^2.$$

Next, we have $\int_{S_3} \vec{F} \cdot d\vec{A} = \int_{S_3} -4xdA$ and $\int_{S_4} \vec{F} \cdot d\vec{A} = \int_{S_4} 4xdA$. Since these two are integrated over the same region in the xz-plane, the two integrals cancel. Similarly, $\int_{S_5} \vec{F} \cdot d\vec{A} = \int_{S_5} -(5x+1)d\vec{A}$ cancels out $\int_{S_6} \vec{F} \cdot d\vec{A} = \int_{S_6} (5x+1)d\vec{A}$. Therefore,

$$\begin{aligned}
&\text{Total flux}\\
&= \int_{S_1} \vec{F} \cdot d\vec{A} + \int_{S_2} \vec{F} \cdot d\vec{A} + \int_{S_3} \vec{F} \cdot d\vec{A} + \int_{S_4} \vec{F} \cdot d\vec{A} + \int_{S_5} \vec{F} \cdot d\vec{A} + \int_{S_6} \vec{F} \cdot d\vec{A}\\
&= -(3a+2)w^2 + (3a+3w+2)w^2 + \int_{S_3} -4xdA + \int_{S_4} 4xdA\\
&\quad + \int_{S_5} -(5x+1)dA + \int_{S_6} (5x+1)dA = 3w^3.
\end{aligned}$$

To find $\operatorname{div} \vec{F}$ at the point (a,b,c), let the box shrink to the point by letting $w \to 0$. Then

$$\begin{aligned}
\operatorname{div} \vec{F} &= \lim_{w\to 0} \left(\frac{\text{Flux through box}}{\text{Volume of box}} \right)\\
&= \lim_{w\to 0} \left(\frac{3w^3}{w^3} \right) = 3.
\end{aligned}$$

Using partial derivatives:

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(3x+2) + \frac{\partial}{\partial y}(4x) + \frac{\partial}{\partial z}(5x+1) = 3$$

23. Figure 19.17 shows a two dimensional cross-section of the vector field $\vec{v} = -2\vec{r}$. The vector field points radially inward, so if we take S to be a sphere of radius R centered at the origin, oriented outward, we have

$$\vec{v} \cdot \Delta\vec{A} = -2R\,\|\Delta\vec{A}\|,$$

for a small area vector $\Delta\vec{A}$ on the sphere. Therefore,

$$\int_S \vec{v} \cdot d\vec{A} = \int_S -2R\,\|d\vec{A}\| = -2R(\text{Surface area of sphere}) = -2R(4\pi R^2) = -8\pi R^3.$$

Thus, we find that

$$\operatorname{div} \vec{v}\,(0,0,0) = \lim_{\text{vol}\to 0} \left(\frac{\int_S \vec{v} \cdot d\vec{A}}{\text{Volume of sphere}} \right) = \lim_{R\to 0} \left(\frac{-8\pi R^3}{\frac{4}{3}\pi R^3} \right) = -6.$$

Notice that the divergence is negative. This is what you would expect, since the vector field represents an inward flow at the origin.

Since $\vec{v} = -2\vec{r} = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$, the coordinate definition give

$$\operatorname{div} \vec{v} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-2z) = -2-2-2 = -6.$$

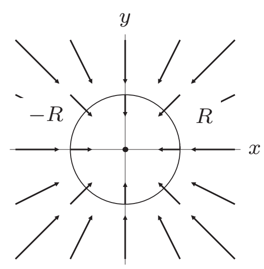


Figure 19.17: The vector field $\vec{v} = -2\vec{r}$

24. (a) Since a is a constant,

$$\text{div}\,\text{grad}\, f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(ay + 2axy) + \frac{\partial}{\partial y}(ax + ax^2 + 3y^2) = 2ay + 6y.$$

(b) Since $\text{div}\,\text{grad}\, f = (2a+6)y$, we have $\text{div}\,\text{grad}\, f = 0$ for all x, y if $a = -3$.

25. (a) Since

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(9a^2x + 10ay^2) + \frac{\partial}{\partial y}(10z^3 - 6ay) + \frac{\partial}{\partial z}(-3z - 10x^2 - 10y^2) = 9a^2 - 6a - 3,$$

we have $\text{div}\,\vec{F} = 0$ when

$$\begin{aligned} 9a^2 - 6a - 3 &= 0 \\ 3(3a+1)(a-1) &= 0 \\ a &= -\frac{1}{3} \quad \text{or} \quad a = 1. \end{aligned}$$

(b) Taking the derivative with respect to a to locate the minimum gives

$$\begin{aligned} \frac{d}{da}(9a^2 - 6a - 3) &= 18a - 6 = 0 \\ a &= \frac{1}{3}. \end{aligned}$$

This value of a gives a minimum because the expression for $\text{div}\,\vec{F}$ is an upward-opening parabola.

26. Take S to be a small sphere of radius R centered at the origin. Then on S we have

$$\|\vec{F}\| = \frac{\|\vec{R}\|}{R^3} = \frac{R}{R^3} = \frac{1}{R^2}.$$

In addition, since $\vec{F}$ points radially outward, $\vec{F}$ is parallel to the outward normal on the surface, so

$$\vec{F} \cdot \Delta\vec{A} = \|\vec{F}\|\Delta A = \frac{1}{R^2}\Delta A.$$

Thus,

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} &= \int_S \frac{1}{R^2} dA = \frac{1}{R^2}\int_S dA \\ &= \frac{1}{R^2} \cdot (\text{Area of sphere}) = \frac{1}{R^2}(4\pi R^2) = 4\pi. \end{aligned}$$

The divergence is therefore given by

$$\text{div}\,\vec{F} = \lim_{\text{Vol}\to 0} \frac{\text{Flux out of sphere}}{\text{Volume inside sphere}} = \lim_{R\to 0}\left(\frac{4\pi}{\frac{4}{3}\pi R^3}\right) = \lim_{R\to 0}\left(\frac{3}{R^3}\right).$$

Since this limit is infinite, or undefined, we say that divergence of this vector field is not defined at the origin.

27. (a) $\text{div}\,\vec{B} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(x+y) = 0$, so this could be a magnetic field.

(b) $\text{div}\,\vec{B} = \frac{\partial}{\partial x}(-z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 0 + 1 + 0 = 1$, so this could not be a magnetic field.

(c) $\text{div}\,\vec{B} = \frac{\partial}{\partial x}(x^2 - y^2 - x) + \frac{\partial}{\partial y}(y - 2xy) + \frac{\partial}{\partial z}(0) = 2x - 1 + 1 - 2x + 0 = 0$, so this could be a magnetic field.

28. (a) We have

$$\text{div}\,\vec{F} = \frac{\partial z}{\partial z} = 1.$$

(b) Above the xy plane, the vector field consists of vectors pointing vertically upward, getting longer as you go up. Below the xy plane, it consists of vectors pointing vertically downward, getting longer as you go down. You can clearly see the divergence on the xy plane, since vectors on either side of it point in opposite directions, but it is not so clear elsewhere. However, the fact that the vectors are getting longer as you go up means that the flux through a cube situated above the xy plane will be non-zero, since the flux out of its top face will be greater than the flux into the bottom face.

29. (a) In Cartesian coordinates,

$$\vec{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\vec{k}.$$

So if $(x,y,z) \neq (0,0,0)$, then

$$\begin{aligned}\operatorname{div}\vec{F}(x,y,z) &= \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right) + \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}\right)\\ &+ \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\right)\\ &= \left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right)\\ &+ \left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}\right)\\ &+ \left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\right)\\ &= \frac{3(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\\ &= 0.\end{aligned}$$

(b)

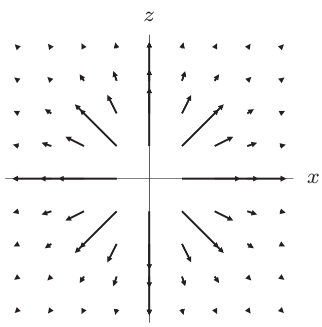


Figure 19.18: The vector field $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{\|\vec{r}\|^3}$ shown in the xz-plane

The vector field is radial (all the arrows point out), so you might think that it is has non-zero divergence. (See Figure 19.18.) However the fact that the divergence is 0 at every point shows that flux density out of any small volume around a point must be 0. This is possible because the arrows also get shorter as you go out.

30. The charges that produce this electric field are concentrated along two vertical lines, one near $x = -1$ and the another one near $x = 1$. This is seen by the change in direction of the field at those lines. Near $x = -1$ the field is being repulsed by the line (seen by the field going away from the line), and the charge is therefore positive. Near $x = 1$ the field is being attracted to the line (seen by the field going toward the line), and the charge is therefore negative.

31. (a) Translating the vector field into rectangular coordinates gives, if $(x,y,z) \neq (0,0,0)$

$$\vec{E}(x,y,z) = \frac{kx}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{ky}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{kz}{(x^2+y^2+z^2)^{3/2}}\vec{k}.$$

We now take the divergence of this to get

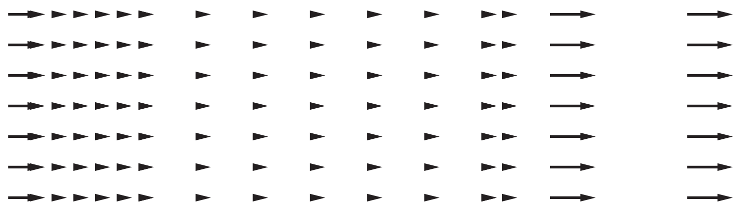
$$\begin{aligned}\operatorname{div}\vec{E} &= k\left(-3\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= 0.\end{aligned}$$

(b) Let S be the surface of a sphere centered at the origin. We have seen that for this field, the flux $\int \vec{E}\cdot d\vec{A}$ is the same for all such spheres, regardless of their radii. So let the constant c stand for $\int \vec{E}\cdot d\vec{A}$. Then

$$\operatorname{div}\vec{E}(0,0,0) = \lim_{\text{vol}\to 0}\frac{\int \vec{E}\cdot d\vec{A}}{\text{Volume inside } S} = \lim_{\text{vol}\to 0}\frac{c}{\text{Volume}}.$$

(c) For a point charge, the charge density is not defined. The charge density is 0 everywhere else.

32. (a) The velocity vector for the traffic flow would look like:



(b) When $0 \le x < 2000$, the velocity is decreasing linearly from 55 to 15, so its formula is $(55 - x/50)\vec{i}$ mph. Then, when $2000 \le x < 7000$, the speed is constant, so $\vec{v}(x) = 15\vec{i}$ mph. Next, when $7000 \le x < 8000$, the velocity is increasing linearly from 15 to 55, so $\vec{v}(x) = (15 + (x - 7000)/25)\vec{i}$ mph. Finally, when $x \ge 8000$, the speed is constant, so $\vec{v}(x) = 55\vec{i}$ mph.

(c) $\text{div}\,\vec{v} = dv(x)/dx$.

At $x = 1000$, $v(x) = 55 - x/50$, so $\text{div}\,\vec{v} = -1/50$.
At $x = 5000$, $v(x) = 15$, so $\text{div}\,\vec{v} = 0$.
At $x = 7500$, $v(x) = 15 + (x - 7000)/25$, so $\text{div}\,\vec{v} = 1/25$.
At $x = 10,000$, $v(x) = 55$, so $\text{div}\,\vec{v} = 0$.
In each case the units of $\text{div}\,\vec{v}$ are $\frac{\text{miles/hour}}{\text{feet}}$.

33. (a) Usually, the distance between cars is more at higher speeds and less at lower speeds. The cars are traveling the fastest at $x = 0$, so at that point, the traffic should be the least dense. Thus,

$$\rho(0) < \rho(1000) < \rho(5000)$$

(b) Since ρ is in cars/mile, $\vec{v}$ is in miles/hour$\vec{v}$ is in km/hour $\rho\vec{v}$ is in cars/hour. The vector quantity $\rho\vec{v}$ gives the number of cars passing through a fixed point in a time interval.

(c) Pick any two points on the highway, $x = a$ and $x = b$ $(a < b)$. We expect $\rho\vec{v}$ to be the same at both places. This is because if more cars pass through a than b, that would mean cars are disappearing (or at least stopping, which we know is not the case since the velocity field is not 0) between a and b. On the other hand, if more cars pass through b than a, that would mean cars are being created between a and b. So we expect $\rho\vec{v}$ to be the same at a and b. Since a and b were chosen arbitrarily, we can say that $\rho\vec{v}$ is constant at all x. This means $\text{div}(\rho\vec{v}) = 0$.

(d) At $x = 0$, $\vec{v}(0) = 55\vec{i}$ and $\rho(0) = 75$. We have $\rho\vec{v}(0) = 4125\vec{i} = \text{constant}$. So $\rho(x) = \|\rho\vec{v}\|/\|\vec{v}(x)\| = 4125/v$.

$$\rho(x) = \frac{4125}{55 - \frac{x}{50}} \text{ if } 0 \le x < 2000$$
$$\rho(x) = \frac{4125}{15} = 275 \text{ if } 2000 \le x < 7000$$
$$\rho(x) = \frac{4125}{15 + \frac{x-7000}{25}} \text{ if } 7000 \le x < 8000$$
$$\rho(x) = \frac{4125}{55} = 75 \text{ if } x \ge 8000$$

(e) We have $\rho(0) = 75$, $\rho(1000) = 118$, $\rho(5000) = 275$, where ρ is given in cars/mile. At $x = 0$, there are 75 cars in a 1-mile stretch of highway. Since there are two lanes, there are about 38 cars in a mile in one lane. A mile is 5280 feet, so that says on average, one car occupies 139 feet. So at $x = 0$, the distance between two cars is 139 feet.

Similarly, we find that at $x = 1000$, the distance is 89 feet, and at $x = 5000$, the distance is 38 feet.

34. (a) We calculate

$$\vec{r} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ c_1 & c_2 & c_3 \end{vmatrix} = (c_3 y - c_2 z)\vec{i} - (c_3 x - c_1 z)\vec{j} + (c_2 x - c_1 y)\vec{k}.$$

Thus

$$\text{div}(\vec{r} \times \vec{c}) = 0.$$

(b) By the Divergence Theorem

$$\int_S (\vec{r} \times \vec{c}) \cdot d\vec{A} = 0.$$

35. Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ with a_1, a_2, and a_3 constant. Then $f\vec{a} = f(x,y,z)(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = f(x,y,z)a_1\vec{i} + f(x,y,z)a_2\vec{j} + f(x,y,z)a_3\vec{k} = fa_1\vec{i} + fa_2\vec{j} + fa_3\vec{k}$. So

$$\begin{aligned}
\operatorname{div}(f\vec{a}) &= \frac{\partial(fa_1)}{\partial x} + \frac{\partial(fa_2)}{\partial y} + \frac{\partial(fa_3)}{\partial z} \\
&= a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z} \quad \text{since } a_1, a_2, a_3 \text{ are constants} \\
&= (\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}) \cdot (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \\
&= (\operatorname{grad} f) \cdot \vec{a}.
\end{aligned}$$

36. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$. Then

$$\begin{aligned}
\operatorname{div}(g\vec{F}) &= \operatorname{div}(gF_1\vec{i} + gF_2\vec{j} + gF_3\vec{k}) \\
&= \frac{\partial}{\partial x}(gF_1) + \frac{\partial}{\partial y}(gF_2) + \frac{\partial}{\partial z}(gF_3) \\
&= \frac{\partial g}{\partial x}F_1 + g\frac{\partial F_1}{\partial x} + \frac{\partial g}{\partial y}F_2 + g\frac{\partial F_2}{\partial y} + \frac{\partial g}{\partial z}F_3 + g\frac{\partial F_3}{\partial z} \\
&= \frac{\partial g}{\partial x}F_1 + \frac{\partial g}{\partial y}F_2 + \frac{\partial g}{\partial z}F_3 + g\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \\
&= (\operatorname{grad} g) \cdot \vec{F} + g\operatorname{div}\vec{F}.
\end{aligned}$$

37. Now $\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$ and $\operatorname{grad} g$ is similar. Thus

$$\operatorname{grad} f \times \operatorname{grad} g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = (f_yg_z - f_zg_y)\vec{i} - (f_xg_z - f_zg_x)\vec{j} + (f_xg_y - f_yg_x)\vec{k}.$$

Therefore

$$\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = \frac{\partial}{\partial x}(f_yg_z - f_zg_y) + \frac{\partial}{\partial y}(f_zg_x - f_xg_z) + \frac{\partial}{\partial z}(f_xg_y - f_yg_x).$$

Expanding using the product rule gives

$$\begin{aligned}
\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = {} & f_{yx}g_z + f_yg_{zx} - f_{zx}g_y - f_zg_{yx} + f_{zy}g_x + f_zg_{xy} \\
& - f_{xy}g_z - f_xg_{zy} + f_{xz}g_y + f_xg_{yz} - f_{yz}g_x - f_yg_{xz}.
\end{aligned}$$

Now consider pairs of terms such as $f_{yx}g_z - f_{xy}g_z$. Since $f_{yx} = f_{xy}$ provided the second derivatives are continuous, these two terms cancel out. All the other terms cancel in pairs, showing that

$$\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = 0.$$

38. Using $\operatorname{div}(g\vec{F}) = (\operatorname{grad} g) \cdot \vec{F} + g\operatorname{div}\vec{F}$, we have

$$\begin{aligned}
\operatorname{div}\vec{F} &= \frac{1}{\|\vec{r}\|^p}\operatorname{div}(\vec{a} \times \vec{r}) + \operatorname{grad}(\frac{1}{\|\vec{r}\|^p}) \cdot \vec{a} \times \vec{r} \\
&= \frac{1}{\|\vec{r}\|^p}0 + \frac{-p}{\|\vec{r}\|^{p+2}}\vec{r} \cdot (\vec{a} \times \vec{r}) \\
&= 0 \quad \text{since } \vec{r} \text{ and } \vec{a} \times \vec{r} \text{ are perpendicular.}
\end{aligned}$$

39. Using $\operatorname{div}(g\vec{F}) = (\operatorname{grad} g) \cdot \vec{F} + g\operatorname{div}\vec{F}$, we have

$$\operatorname{div}\vec{B} = \operatorname{grad}\left(\frac{1}{x^a}\right) \cdot \vec{r} + \frac{1}{x^a}\operatorname{div}\vec{r} = -ax^{-(a+1)}\vec{i} \cdot \vec{r} + x^{-a}(3) = (3-a)x^{-a}.$$

40. Using $\operatorname{div}(g\vec{F}) = (\operatorname{grad} g) \cdot \vec{F} + g\operatorname{div}\vec{F}$, we have

$$\operatorname{div}\vec{G} = \operatorname{grad}(\vec{b} \cdot \vec{r}) \cdot (\vec{a} \times \vec{r}) + \vec{b} \cdot \vec{r}\operatorname{div}(\vec{a} \times \vec{r}) = \vec{b} \cdot (\vec{a} \times \vec{r}) + \vec{v} \cdot \vec{r}0 = \vec{b} \cdot (\vec{a} \times \vec{r}).$$

41. First compute the unit vectors $\vec{T}$ and $\vec{N}$. Since $\vec{T}$ is in the direction of $\vec{F}$ we have

$$\vec{T} = \frac{1}{\|\vec{F}\|}\vec{F} = \frac{1}{F}(u\vec{i} + v\vec{j}).$$

Since $\vec{N}$ is the unit vector in the direction of $\vec{k} \times \vec{F}$ we have

$$\begin{aligned}
\vec{k} \times \vec{F} &= \vec{k} \times (u\vec{i} + v\vec{j}) \\
&= -v\vec{i} + u\vec{j} \\
\vec{N} &= \frac{1}{\| -v\vec{i} + u\vec{j}\|}(-v\vec{i} + u\vec{j}) \\
&= \frac{1}{F}(-v\vec{i} + u\vec{j}).
\end{aligned}$$

The chain rule for partial differentiation of the formulas $u = F\cos\theta$ and $v = F\sin\theta$ gives

$$\begin{aligned}
u_x &= (\cos\theta)F_x - F(\sin\theta)\theta_x \\
v_y &= (\sin\theta)F_y + F(\cos\theta)\theta_y.
\end{aligned}$$

We have

$$\begin{aligned}
\operatorname{div}\vec{F} &= u_x + v_y \\
&= ((\cos\theta)F_x - F(\sin\theta)\theta_x) + ((\sin\theta)F_y + F(\cos\theta)\theta_y) \\
&= (-v\theta_x + u\theta_y) + \frac{1}{F}(uF_x + vF_y) \\
&= (\theta_x\vec{i} + \theta_y\vec{j}) \cdot (-v\vec{i} + u\vec{j}) + \frac{1}{F}(F_x\vec{i} + F_y\vec{j}) \cdot (u\vec{i} + v\vec{j}) \\
&= F\operatorname{grad}\theta \cdot \vec{N} + \operatorname{grad}F \cdot \vec{T}.
\end{aligned}$$

Since the directional derivative of θ in the direction of $\vec{N}$ is $\theta_{\vec{N}} = \operatorname{grad}\theta \cdot \vec{N}$ and the directional derivative of F in the direction of $\vec{T}$ is $F_{\vec{T}} = \operatorname{grad}F \cdot \vec{T}$ we have

$$\operatorname{div}\vec{F} = F\theta_{\vec{N}} + F_{\vec{T}}.$$

42. We have now our temperature a function depending on t, x, y, z, hence $T = T(t, x, y, z)$. For a fixed moment, say t_0, T is a function of only x, y, z. For this moment, $t = t_0$, we have:

$$\begin{matrix}\text{Rate of heat loss} \\ \text{from volume } V\end{matrix} = k\int_S (\operatorname{grad}T) \cdot d\vec{A}.$$

where $\operatorname{grad}T = \left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}\right)\Big|_{t=t_0}$. Now the rate of change, with respect to time, in the average temperature in the region, at $t = t_0$, is proportional to the average rate at which heat is being lost per unit volume at $t = t_0$, so

$$\left.\frac{\partial T_{avg}}{\partial t}\right|_{t=t_0} = -c\left(\frac{\text{Rate heat lost}}{\text{Volume } V}\right)_{t=t_0} = \frac{-ck\int_S(\operatorname{grad}T)\cdot d\vec{A}}{\text{Volume } V}$$

Taking the limit as V shrinks around the point, the average temperature through the region becomes the temperature at that point. Thus using the definition of the divergence (with respect to x, y, z), we have

$$\begin{aligned}
\left.\frac{\partial T}{\partial t}\right|_{t=t_0} &= -ck\lim_{V\to 0}\left(\frac{\int_S(\operatorname{grad}T)\cdot d\vec{A}}{\text{Volume } V}\right) \\
&= (-ck\operatorname{div}\operatorname{grad}T)_{t=t_0}
\end{aligned}$$

As this holds at every moment t_0, one has:

$$\frac{\partial T}{\partial t} = B \cdot \operatorname{div}\operatorname{grad}T,$$

where $B = -ck$ is a function of time only, and the gradient and divergence are taken with respect to the variables x, y, z.

43. (a) At any point $\vec{r} = x\vec{i} + y\vec{j}$, the direction of the vector field $\vec{v}$ is pointing away from the origin, which means it is of the form $\vec{v} = f\vec{r}$ for some positive function f, whose value can vary depending on $\vec{r}$. The magnitude of $\vec{v}$ depends only on the distance r, thus f must be a function depending only on r, which is equivalent to depending only on r^2 since $r \geq 0$. So $\vec{v} = f(r^2)\vec{r} = \left(f(x^2+y^2)\right)(x\vec{i} + y\vec{j})$.

(b) At $(x, y) \neq (0, 0)$ the divergence of $\vec{v}$ is

$$\operatorname{div}\vec{v} = \frac{\partial(K(x^2+y^2)^{-1}x)}{\partial x} + \frac{\partial(K(x^2+y^2)^{-1}y)}{\partial y} = \frac{Ky^2 - Kx^2}{(x^2+y^2)^2} + \frac{Kx^2 - Ky^2}{(x^2+y^2)^2} = 0.$$

Therefore, $\vec{v}$ is a point source at the origin.

(c) The magnitude of $\vec{v}$ is

$$\|\vec{v}\| = K(x^2+y^2)^{-1}|x\vec{i} + y\vec{j}| = K(x^2+y^2)^{-1}(x^2+y^2)^{1/2} = K(x^2+y^2)^{-1/2} = \frac{K}{r}.$$

(d) The vector field looks like that in Figure 19.19:

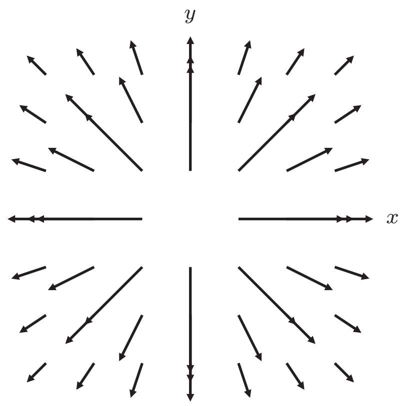


Figure 19.19

(e) We need to show that $\operatorname{grad}\phi = \vec{v}$.

$$\begin{aligned}
\operatorname{grad}\phi &= \frac{\partial}{\partial x}\left(\frac{K}{2}\log(x^2+y^2)\right)\vec{i} + \frac{\partial}{\partial y}\left(\frac{K}{2}\log(x^2+y^2)\right)\vec{j} \\
&= \frac{Kx}{x^2+y^2}\vec{i} + \frac{Ky}{x^2+y^2}\vec{j} \\
&= K(x^2+y^2)^{-1}(x\vec{i} + y\vec{j}) \\
&= \vec{v}
\end{aligned}$$

44. (a) At any point $\vec{r} = x\vec{i} + y\vec{j}$, the direction of the vector field $\vec{v}$ is pointing toward the origin, which means it is of the form $\vec{v} = f\vec{r}$ for some negative function f whose value can vary depending on $\vec{r}$. The magnitude of $\vec{v}$ depends only on the distance r, thus f must be a function depending only on r, which is equivalent to depending only on r^2 since $r \geq 0$. So $\vec{v} = f(r^2)\vec{r} = \left(f(x^2+y^2)\right)(x\vec{i} + y\vec{j})$.

(b) At $(x, y) \neq (0, 0)$ the divergence of $\vec{v}$ is

$$\operatorname{div}\vec{v} = \frac{\partial(K(x^2+y^2)^{-1}x)}{\partial x} + \frac{\partial(K(x^2+y^2)^{-1}y)}{\partial y} = \frac{Ky^2 - Kx^2}{(x^2+y^2)^2} + \frac{Kx^2 - Ky^2}{(x^2+y^2)^2} = 0.$$

Therefore, $\vec{v}$ is a point sink at the origin.

(c) The magnitude of $\vec{v}$ is

$$\|\vec{v}\| = |K|(x^2+y^2)^{-1}|x\vec{i} + y\vec{j}| = |K|(x^2+y^2)^{-1}(x^2+y^2)^{1/2} = |K|(x^2+y^2)^{-1/2} = \frac{|K|}{r}.$$

(remember, $K < 0$)

(d) The vector field looks like the following:

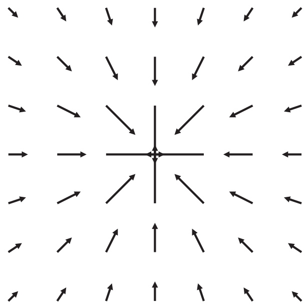


Figure 19.20

(e) We need to show that $\operatorname{grad}\phi = \vec{v}$.

$$\begin{aligned}\operatorname{grad}\phi &= \frac{\partial}{\partial x}(\frac{K}{2}\log(x^2+y^2))\vec{i} + \frac{\partial}{\partial y}(\frac{K}{2}\log(x^2+y^2))\vec{j} \\ &= \frac{Kx}{x^2+y^2}\vec{i} + \frac{Ky}{x^2+y^2}\vec{j} \\ &= K(x^2+y^2)^{-1}(x\vec{i} + y\vec{j}) \\ &= \vec{v}\end{aligned}$$

Strengthen Your Understanding

45. Divergence of a vector field is a scalar not a vector. We have $\operatorname{div}(2x\vec{i}) = 2$.

46. The divergence of a vector field is a scalar function, not a vector field. The correct divergence is

$$\operatorname{div}\vec{F} = 2x + 2 - 2z.$$

47. Only vector fields have a divergence. A scalar function such as $f(x, y, z) = x^2 + yz$ does not have a divergence.

48. If $\vec{F}(x, y, z) = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$. then

$$\operatorname{div}\vec{F}(x, y, z) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z) = 2 + 3 + 4 = 9.$$

More complicated examples work too, such as if $\vec{F}(x, y, z) = (2x + x^2)\vec{i} - 2xy\vec{j}$, giving

$$\operatorname{div}\vec{F}(x, y, z) = \frac{\partial}{\partial x}\left(2x + x^2\right) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}0 = (2 + 2x) - 2x + 0 = 2.$$

49. If

$$\vec{F}(x, y, z) = y^2\vec{i} + xz\vec{j} + x\vec{k}$$

then

$$\operatorname{div}\vec{F}(x, y, z) = \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}x = 0.$$

50. The vector field $\vec{F}(x, y) = 2x\vec{i}$ is not divergence free since $\operatorname{div}\vec{F}(x, y) = 2 \neq 0$.

51. True.

$$\begin{aligned}\operatorname{div}(\vec{F} + \vec{G}) &= \frac{\partial(F_1 + G_1)}{\partial x} + \frac{\partial(F_2 + G_2)}{\partial y} + \frac{\partial(F_3 + G_3)}{\partial z} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} + \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \operatorname{div}\vec{F} + \operatorname{div}\vec{G}.\end{aligned}$$

52. False. Let's compare the x–components of each side of the equation. The x–component of $\text{grad}(\vec F\cdot\vec G)$ is given by

$$\begin{aligned}(\text{grad}(\vec F\cdot\vec G))_1 &= \frac{\partial(F_1G_1+F_2G_2+F_3G_3)}{\partial x}\\ &= \frac{\partial F_1}{\partial x}G_1+F_1\frac{\partial G_1}{\partial x}+\frac{\partial F_2}{\partial x}G_2+F_2\frac{\partial G_2}{\partial x}+\frac{\partial F_3}{\partial x}G_3+F_3\frac{\partial G_3}{\partial x}.\end{aligned}$$

However, the x–component of $\vec F\cdot(\text{div}\,\vec G)+(\text{div}\,\vec F)\cdot\vec G$ is

$$\begin{aligned}(\vec F(\text{div}\,\vec G)+(\text{div}\,\vec F)\vec G)_1 &= F_1(\text{div}\,\vec G)+(\text{div}\,\vec F)G_1\\ &= F_1\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)+\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)G_1.\end{aligned}$$

These two x–components are different and therefore

$$\text{grad}(\vec F\cdot\vec G)\neq\vec F(\text{div}\,\vec G)+(\text{div}\,\vec F)\vec G.$$

53. True. $\text{div}\,\vec F$ is a scalar whose value depends on the point at which it is calculated.

54. False. The divergence is a scalar function that gives flux density at a point.

55. True. The net flow through a small volume is zero since the flow in is canceled by the equal flow out. Alternatively, $\text{div}\vec F=\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial c}{\partial z}=0$.

56. False. As a counterexample, consider $\vec F=x\vec k$. Then $\text{div}\vec F=\frac{\partial x}{\partial z}=0$.

57. False. As a counterexample, consider $\vec F=\vec i$ and $f(x,y,z)=x$. Then $\text{div}(f\vec F)=\text{div}\,x\vec i=1$, and $f\text{div}\vec F=x\cdot 0=0$.

58. False. As a counterexample, consider $\vec F=2x\vec i+2y\vec j+2z\vec k$. Then $\vec F=\text{grad}(x^2+y^2+z^2)$, and $\text{div}\,\vec F=2+2+2\neq 0$.

59. False. As a counterexample, consider $\vec F=x^2\vec i$ Then $\text{div}\,\vec F=2x$, and $\text{grad}\,2x=2\vec i\neq\vec 0$.

60. False. Since $\vec F$ can be written $\vec F(x,y,z)=x\vec i+y\vec j+z\vec k$, the divergence of $\vec F$ is 3.

61. True. Here is a way of constructing a vector field $\vec F$. The idea is to think of f as a function of x (with y and z constant) and take the antiderivative. We define

$$g(x,y,z)=\int_0^x f(t,y,z)\,dt.$$

By the Fundamental Theorem of one-variable calculus, we know $\frac{\partial g}{\partial x}=f$. So, if f is given, the vector field $\vec F=g(x,y,z)\vec i$ has $\text{div}\vec F=f$.

62. False. As a counterexample, note that $\vec F=\vec i$ and $\vec G=\vec j$ both have divergence zero, but are not the same vector fields.

63. False. The left-hand side of the equation, $\text{div}(\text{grad}\,f)$, is a scalar function and the right hand side, $\text{grad}(\text{div}\,\vec F)$, is a vector. There cannot be an equality between a' scalar and a vector.

64. (a), (b), and (e) all depend on the point (x,y,z), so they are vector fields. Since $\text{div}\,\vec r=3$ and $\text{div}\,\vec i=0$, the vectors in (c) and (d) are constant vector fields.

Solutions for Section 19.4

Exercises

1. First directly: On the faces $x=0, y=0, z=0$, the flux is zero. On the face $x=2$, a unit normal is $\vec i$ and $d\vec A=d\text{A}\vec i$. So

$$\int_{S_{x=2}}\vec r\cdot d\vec A=\int_{S_{x=2}}(2\vec i+y\vec j+z\vec k)\cdot(dA\vec i)$$

(since on that face, $x=2$)

$$=\int_{S_{x=2}}2dA=2\cdot(\text{Area of face})=2\cdot 4=8.$$

In exactly the same way, you get

$$\int_{S_{y=2}} \vec{r} \cdot d\vec{A} = \int_{S_{z=2}} \vec{r} \cdot d\vec{A} = 8,$$

so

$$\int_{S} \vec{r} \cdot \overrightarrow{dA} = 3 \cdot 8 = 24.$$

Now using divergence:

$$\operatorname{div} \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,$$

so

$$\text{Flux} = \int_0^2 \int_0^2 \int_0^2 3 \, \text{dx dy dz} = 3 \cdot (\text{Volume of Cube}) = 3 \cdot 8 = 24$$

2. First directly, since the vector field is totally in the $\vec{j}$ direction, there is no flux through the ends. On the side of the cylinder, a normal vector at (x, y, z) is $x\vec{i} + y\vec{j}$. This is in fact a unit normal, since $x^2 + y^2 = 1$ (the cylinder has radius 1). Also, using $x = \cos\theta, y = \sin\theta$, in this case, the element of area dA equals $1 d\theta dz$. So

$$\begin{aligned}\text{Flux} &= \int \vec{F} \cdot d\vec{A} = \int_0^2 \int_0^{2\pi} (y\vec{j}) \cdot (x\vec{i} + y\vec{j}) \, d\theta \, dz \\ &= \int_0^2 \int_0^{2\pi} y^2 \, d\theta \, dz = \int_0^2 \int_0^{2\pi} \sin^2\theta \, d\theta \, dz = \int_0^2 \pi dz = 2\pi.\end{aligned}$$

Now we calculate the flux using the divergence theorem. The divergence of the field is given by the sum of the respective partials of the components, so the divergence is simply $\frac{\partial y}{\partial y} = 1$. Since the divergence is constant, we can simply calculate the volume of the cylinder and multiply by the divergence

$$\text{Flux} = 1\pi r^2 h = 2\pi$$

3. Finding flux directly:
1) On bottom face, $z = 0$ so $\vec{F} = x^2\vec{i} + 2y^2\vec{j}$ is parallel to face so flux is zero.
2) On front face, $y = 0$ so $\vec{F} = x^2\vec{i} + 3z^2\vec{k}$ is parallel to face so flux is zero.
3) On back face, $y = 1$ so $\vec{F} = x^2\vec{i} + 2\vec{j} + 3z^2\vec{k}$ and $\vec{A} = \vec{j}$ so flux is 2.
4) On top face, $z = 1$ so $\vec{F} = x^2\vec{i} + 2y^2\vec{j} + 3\vec{k}$ and $\vec{A} = \vec{k}$ so flux is 3.
5) On side $x = 1$, $\vec{F} = \vec{i} + 2y^2\vec{j} + 3z^2\vec{k}$ and $\vec{A} = -\vec{i}$ so flux is -1.
6) On side $x = 2$, $\vec{F} = 4\vec{i} + 2y^2\vec{j} + 3z^2\vec{k}$ and $\vec{A} = \vec{i}$ so flux is 4.
Total flux is thus 8.

By the Divergence Theorem:

$$\operatorname{div} \vec{F} = 2x + 4y + 6z$$

So, if W is the interior of the box, we have

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_W (2x + 4y + 6z) dV = 2 \int_1^2 \int_0^1 \int_0^1 (x + 2y + 3z) dz \, dy \, dx \\ &= 2 \int_1^2 \int_0^1 \left[xz + 2yz + \frac{3z^2}{2} \right]_0^1 dy \, dx = 2 \int_1^2 \int_0^1 \left(x + 2y + \frac{3}{2} \right) dy \, dx \\ &= 2 \int_1^2 \left[xy + y^2 + \frac{3y}{2} \right]_0^1 dx = 2 \int_1^2 \left(x + 1 + \frac{3}{2} \right) dx \\ &= \int_1^2 (2x + 5) dx = (x^2 + 5x) \Big|_1^2 = 8\end{aligned}$$

4. Since $\operatorname{div} \vec{F} = 1 + 1 + 1 = 3$, the Divergence Theorem gives

$$\int_S \vec{F} \cdot d\vec{A} = \int_W 3 \, dV = 3 \int_W dV = 3 \cdot \text{Volume of the cylinder} = 3\pi.$$

5. The location of the pyramid has not been completely specified. For instance, where is it centered on the xy plane? How is base oriented with respect to the axes? Thus, we cannot compute the flux by direct integration with the information we have. However, we can calculate it using the divergence theorem. First we calculate the divergence of $\vec{F}$.

$$\operatorname{div}\vec{F} = \frac{\partial(-z)}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial x}{\partial z} = 0 + 0 + 0 = 0$$

Thus for any closed surface the flux will be zero, so the flux through our pyramid, regardless of its location or orientation, is zero.

6. Since the surface is closed, the flux of a constant vector field out of it is 0.

7. Since $\operatorname{div}\vec{G} = 1$, if W is the interior of the box, the Divergence Theorem gives

$$\text{Flux} = \int_W 1\,dV = 1 \cdot \text{ Volume of box } = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

8. Since $\operatorname{div}\vec{H} = y$, if W is the interior of the box, the Divergence Theorem gives

$$\text{Flux} = \int_W y\,dV = \int_0^4 \int_0^3 \int_0^2 y\,dx\,dy\,dz = 4 \cdot 2 \cdot \left.\frac{y^2}{2}\right|_0^3 = 36.$$

9. Since $\operatorname{div}\vec{J} = 2xy$, if W is the interior of the box, the Divergence Theorem gives

$$\text{Flux} = \int_W 2xy\,dV = \int_0^4 \int_0^3 \int_0^2 2xy\,dx\,dy\,dz = x^2\Big|_0^2 \left.\frac{y^2}{2}\right|_0^3 z\Big|_0^4 = 72.$$

10. Since $\operatorname{div}\vec{N} = 0$, the flux through the closed surface of the box is 0.

11. We have

$$\operatorname{div}((3x+4y)\vec{i} + (4y+5z)\vec{j} + (5z+3x)\vec{k}) = 3 + 4 + 5 = 12.$$

Let W be the interior of the cube. Then by the divergence theorem,

$$\int_S ((3x+4y)\vec{i} + (4y+5z)\vec{j} + (5z+3x)\vec{k}) \cdot d\vec{A} = \int_W 12\,dV = 12 \cdot \text{Volume of cube} = 12 \cdot (2 \cdot 3 \cdot 4) = 288.$$

12. By the Divergence Theorem,

$$\begin{aligned}\text{Flux} &= \int_S \vec{M} \cdot d\vec{A} = \int_W \operatorname{div}\vec{M}\,dV = \int_W (xy+5)\,dV = \int_0^4 \int_0^3 \int_0^2 (xy+5)\,dx\,dy\,dz \\ &= \int_0^4 \int_0^3 \left.\frac{x^2y}{2} + 5x\right|_0^2 dy\,dz = \int_0^4 \int_0^3 (2y+10)\,dy\,dz = \int_0^4 (y^2+10y)\Big|_0^3 dz = 156.\end{aligned}$$

Problems

13. Since

$$\operatorname{div}\vec{F} = 0 + 1 + 0 = 1,$$

we have

$$\int_S \vec{F} \cdot d\vec{A} = (\operatorname{div}\vec{F}) \cdot \text{Vol} = \frac{4}{3}\pi 3^3 = 36\pi.$$

14. Since $\operatorname{div}\vec{F} = y + z + x$, the flux is given by

$$\int_{\text{Sphere}} \vec{F} \cdot d\vec{A} = \int_{\text{Sphere}} (x+y+z)dV.$$

We calculate the first term of the integral

$$\begin{aligned}\int_{\text{Sphere}} x\,dV &= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} x\,dx\,dy\,dz \\ &= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{x^2}{2}\Big|_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dy\,dz \\ &= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \left(\frac{(\sqrt{1-y^2-z^2})^2}{2} - \frac{(-\sqrt{1-y^2-z^2})^2}{2}\right) dy\,dz \\ &= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 0\,dydz = 0.\end{aligned}$$

The other terms in the integral are zero by a similar calculation.

The same result can be obtained by a symmetry argument, which is much shorter: Since x, y, and z each take equal positive and negative values on half the sphere, the integral of each term is 0. Thus, the flux is zero:

$$\int_{\text{Sphere}} \vec{F}\cdot d\vec{A} = \int_{\text{Sphere}} (x+y+z)dV = 0.$$

15. Since $\operatorname{div}\vec{F} = 3x^2+3y^2+3z^2$, the Divergence Theorem gives

$$\int_S \vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV = \int_W 3(x^2+y^2+z^2)\,dV.$$

Since W is the interior of a cylinder of radius 2 centered on the z-axis, we use cylindrical coordinates, giving

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_W 3(x^2+y^2+z^2)\,dV = 3\int_0^{2\pi}\int_0^2\int_0^5 (r^2+z^2)r\,dz\,dr\,d\theta \\ &= 3\int_0^{2\pi}\int_0^2 \left(r^3z+\frac{rz^3}{3}\right)\Big|_0^5 dr\,d\theta = 3\int_0^{2\pi}\int_0^2 (5r^3+\frac{125}{3}r)\,dr\,d\theta \\ &= 3\cdot 2\pi\left(\frac{5r^4}{4}+\frac{125}{3}\cdot\frac{r^2}{2}\right)\Big|_0^2 = 620\pi.\end{aligned}$$

16. Since $\operatorname{div}\vec{F} = 3x^2+3y^2+3z^2$, the Divergence Theorem gives

$$\text{Flux } = \int_S \vec{F}\cdot d\vec{A} = \int_W (3x^2+3y^2+3z^2)\,dV.$$

In spherical coordinates, the region W lies between the spheres $\rho=2$ and $\rho=3$ and inside the cone $\phi=\pi/4$. Since $3x^2+3y^2+3z^2=3\rho^2$, we have

$$\begin{aligned}\text{Flux } &= \int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^{\pi/4}\int_2^3 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi\cdot\frac{3}{5}\rho^5\Big|_2^3 (-\cos\phi)\Big|_0^{\pi/4} = \frac{633(2-\sqrt{2})}{5}\pi = 232.98.\end{aligned}$$

17. We calculate $\operatorname{div}\vec{F}$ and use the Divergence Theorem:

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x+3e^{yz}) + \frac{\partial}{\partial y}(\ln(x^2z^2+1)+y) + \frac{\partial}{\partial z}(z) = 3.$$

Thus, with S representing the surface of the cylinder and W the region inside, we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_W 3\,dV = 3\cdot\text{Volume of cylinder } = 3\cdot\pi 2^2 4 = 48\pi.$$

18. Since $\operatorname{div}\vec{F} = \operatorname{div}(e^{y^2z^2}\vec{i} + (\tan(0.001x^2z^2)+y^2)\vec{j} + (\ln(1+x^2y^2)+z^2)\vec{k}\,) = 2y+2z$, by the Divergence Theorem, if S is the surface of the box W,

$$\begin{aligned}\text{Flux} = \int_S \vec{F}\cdot d\vec{A} &= \int_W \operatorname{div}\vec{F}\,dV = \int_0^3\int_0^4\int_0^5 (2y+2z)\,dx\,dy\,dz \\ &= \int_0^3\int_0^4 (2xy+2xz)\Big|_0^5\,dy\,dz = \int_0^3\int_0^4 (10y+10z)\,dy\,dz \\ &= \int_0^3 (5y^2+10yz)\Big|_0^4\,dz = \int_0^3 (80+40z)dz \\ &= (80z+20z^2)\Big|_0^3 = 420.\end{aligned}$$

19. We use the Divergence Theorem, with

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y) = 2x.$$

Let W be the interior of the cone. Then

$$\int_S \vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV = \int_W 2x\,dV.$$

The region W is shown in Figure 19.21. Evaluating the integral over W as an iterated integral gives

$$\begin{aligned}\int_W 2x\,dV &= \int_{-1}^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{\sqrt{y^2+z^2}}^1 2x\,dx\,dy\,dz = \int_{-1}^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} x^2\Big|_{\sqrt{y^2+z^2}}^1 dy\,dz \\ &= \int_{-1}^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \left(1-(y^2+z^2)\right)\,dy\,dz = \int_{-1}^1 \left(y - \frac{1}{3}y^3 - z^2y\right)\Big|_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dz \\ &= \int_{-1}^1 \left(2(1-z^2)^{1/2} - \frac{2}{3}(1-z^2)^{3/2} - 2z^2(1-z^2)^{1/2}\right)\,dz = 1.571.\end{aligned}$$

The last integral was computed numerically. It can also be done by trigonometric substitutions; the exact value is $\pi/2$.

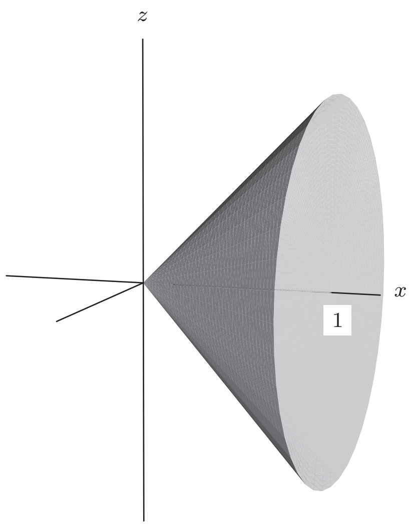


Figure 19.21: The cone $x = \sqrt{y^2+z^2}$, with $0 \le x \le 1$

20. By the Divergence Theorem, if W is the cylinder and S is its surface:

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV = \int_W 10\, dV = 10 \cdot \text{Volume of cylinder} = 10\pi a^3.$$

21. Apply the Divergence Theorem to the solid cone, whose interior we call W. The surface of W consists of S and D. Thus

$$\int_S \vec{F} \cdot d\vec{A} + \int_D \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV.$$

But $\operatorname{div} \vec{F} = 0$ everywhere, since $\vec{F}$ is constant. Thus

$$\int_D \vec{F} \cdot d\vec{A} = -\int_S \vec{F} \cdot d\vec{A} = -3.22.$$

22. Since $\operatorname{div} \vec{r} = \operatorname{div}(x\vec{i} + y\vec{j} + z\vec{k}) = 3$, applying the Divergence Theorem to the vector field $\vec{F} = \vec{r}$ gives

$$\int_S \vec{r} \cdot d\vec{A} = \int_V 3dV = 3\int_V dV = 3V.$$

Thus $\frac{1}{3}\int_S \vec{r} \cdot d\vec{A} = V$.

23. By the Divergence Theorem, $\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV = \int_W 0dV = 0$ for a closed surface S, where W is the region enclosed by S.

24. (a) Since $||\vec{F}|| = ||\vec{r}||/||\vec{r}||^3$, we have $||\vec{F}|| = 1$ on the unit sphere. Thus

$$\text{Flux} = \int_{\text{Sphere}} \vec{F} \cdot d\vec{A} = 1 \cdot \text{ Surface area of sphere } = 4\pi.$$

(b) Since

$$\frac{\vec{r}}{\|\vec{r}\|^3} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\vec{k},$$

we compute the partial derivative of each component

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{(3/2)+1}}$$
$$\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{(3/2)+1}}$$
$$\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{(3/2)+1}}.$$

So

$$\begin{aligned}\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^3}\right) &= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{(3/2)+1}}\\ &= \frac{3(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \frac{0}{(x^2+y^2+z^2)^{5/2}} = 0.\end{aligned}$$

(c) Consider the region W between the box, B, and the sphere, S, with the box oriented outward and the sphere oriented inward. Then the Divergence Theorem says

$$\int_W \operatorname{div} \vec{F}\; dV = \int_B \vec{F} \cdot d\vec{A} + \int_S \vec{F} \cdot d\vec{A}.$$

Since the sphere is now oriented inward,

$$\int_S \vec{F} \cdot d\vec{A} = -4\pi.$$

Since $\operatorname{div}\vec{F} = 0$, we have

$$0 = \int_B \vec{F} \cdot d\vec{A} - 4\pi$$
$$\int_B \vec{F} \cdot d\vec{A} = 4\pi$$

25. (a) Since $\vec{r} = x\vec{i} + y\vec{j}$, for $\vec{r} \neq \vec{0}$, we have

$$\begin{aligned}\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^2}\right) &= \operatorname{div}\left(\frac{x\vec{i} + y\vec{j}}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \\ &= \frac{1}{x^2+y^2} - \frac{x\cdot 2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y\cdot 2y}{(x^2+y^2)^2} \\ &= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2} \\ &= 0.\end{aligned}$$

(b) Since the cylinder contains points where $\vec{r} = \vec{0}$ (on the z-axis) and $\operatorname{div}(\vec{r}/\|\vec{r}\|^2)$ is undefined there, we cannot use the Divergence Theorem.

(c) Since the cylinder is closed, it is oriented outward. Calculating the flux directly, the flux through the ends of the cylinder is 0 since the vector field has no $\vec{k}$-component. (The flux is zero even though the vector field is undefined on the z-axis.) On the curved sides of the cylinder, the vector field is perpendicular to the surface and of length 1. Thus

$$\text{Flux} = \int_S \frac{\vec{r}}{\|\vec{r}\|^2} \cdot d\vec{A} = 1 \cdot \text{Area of curved side} = 1 \cdot 2\pi 1 \cdot 2 = 4\pi.$$

(d) The cylinder, S, in part (c) lies inside this one, S_1. In the space, W, between the cylinders, the divergence is 0. The surface of W consists of S oriented inward and S_1 oriented outward. The Divergence Theorem can be applied to the region W, so

$$\begin{aligned}\int_{S_1-S} \frac{\vec{r}}{\|\vec{r}\|^2} \cdot d\vec{A} &= \int_W \operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^2}\right) dV = 0 \\ \int_{S_1} \frac{\vec{r}}{\|\vec{r}\|^2} \cdot d\vec{A} - \int_S \frac{\vec{r}}{\|\vec{r}\|^2} \cdot d\vec{A} &= 0 \\ \int_{S_1} \frac{\vec{r}}{\|\vec{r}\|^2} \cdot d\vec{A} = \int_S \frac{\vec{r}}{\|\vec{r}\|^2} \cdot d\vec{A} &= 4\pi.\end{aligned}$$

Thus, the flux through this cylinder is the same as the flux through the cylinder in part (c).

26. We use the Divergence Theorem to compare the integrals. We have

$$\begin{aligned}\operatorname{div}\vec{F}_1 &= y^2 + 3z^2 + 3x^2 + 2z^2 + 3y^2 = 3x^2 + 4y^2 + 5z^2 \\ \operatorname{div}\vec{F}_2 &= y^2 + z^2 + x^2 \\ \operatorname{div}\vec{F}_3 &= z^2 + x^2 + z^2 + y^2 + y^2 + z^2 = x^2 + 2y^2 + 3z^2.\end{aligned}$$

For all x, y, z,

$$\operatorname{div}\vec{F}_2 \leq \operatorname{div}\vec{F}_3 \leq \operatorname{div}\vec{F}_1,$$

with equality only at the origin. Since the flux integrals are all through the same surface S, we have

$$\int_S \vec{F}_2 \cdot d\vec{A} < \int_S \vec{F}_3 \cdot d\vec{A} < \int_S \vec{F}_1 \cdot d\vec{A}.$$

27. (a) By the Divergence Theorem, if S is the surface of the rectangular solid W,

$$\begin{aligned}\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV &= \int_0^c \int_0^b \int_0^a 2(6-x)\,dx\,dy\,dz \\ &= \int_0^c \int_0^b 12x - x^2\Big|_0^a dy\,dz = cb(12a - a^2).\end{aligned}$$

(b) To maximize the flux $cb(12a - a^2)$, we take b and c to be as large as possible and choose a to maximize $(12a - a^2)$ on the interval $0 \leq a \leq 10$. Thus we take $a = 6$, $b = c = 10$. Then

$$\text{Flux } = cb(12a - a^2)\bigg|_{a=6,b=10,c=10} = 3600.$$

28. (a) Since $\vec{F}$ is radial, it is everywhere parallel to the area vector, $\Delta\vec{A}$. Also, $||\vec{F}\,|| = 1$ on the surface of the sphere $x^2 + y^2 + z^2 = 1$, so

$$\begin{aligned}\text{Flux through the sphere} &= \int_S \vec{F} \cdot d\vec{A} = \lim_{||\Delta\vec{A}\,||\to 0} \sum \vec{F} \cdot \Delta\vec{A} \\ &= \lim_{||\Delta\vec{A}\,||\to 0} \sum ||\vec{F}\,||\,||\Delta\vec{A}\,|| = \lim_{||\Delta\vec{A}\,||\to 0} \sum ||\Delta\vec{A}\,|| \\ &= \text{Surface area of sphere} = 4\pi \cdot 1^2 = 4\pi.\end{aligned}$$

(b) In Cartesian coordinates,

$$\vec{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\vec{k}.$$

So,

$$\begin{aligned}\operatorname{div}\vec{F}(x,y,z) &= \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &\quad + \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &\quad + \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &= \left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &\quad + \left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &\quad + \left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &= \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} \\ &= 0.\end{aligned}$$

(c) We cannot apply the Divergence Theorem to the whole region within the box, because the vector field $\vec{F}$ is not defined at the origin. However, we can apply the Divergence Theorem to the region, W, between the sphere and the box. Since $\operatorname{div}\vec{F} = 0$ there, the theorem tells us that

$$\int_{\substack{\text{Box}\\\text{(outward)}}} \vec{F} \cdot d\vec{A} + \int_{\substack{\text{Sphere}\\\text{(inward)}}} \vec{F} \cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV = 0.$$

Therefore, the flux through the box and the sphere are equal if both are oriented outward:

$$\int_{\substack{\text{Box}\\\text{(outward)}}} \vec{F} \cdot d\vec{A} = -\int_{\substack{\text{Sphere}\\\text{(inward)}}} \vec{F} \cdot d\vec{A} = \int_{\substack{\text{Sphere}\\\text{(outward)}}} \vec{F} \cdot d\vec{A} = 4\pi.$$

29. We have:

$$\vec{F}(x,y,z) = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$$

Calculating the flux of $\vec{F}$ through the ellipsoid directly would be difficult. However, since $\operatorname{div}\vec{F} = 0$, we can replace the ellipsoid by a sphere. Except at the origin, we have $\operatorname{div}\vec{F} = 0$. Let T be the surface of a sphere centered at the origin inside the ellipsoid S, and let W be the region between S and T. Suppose both S and T are oriented away from the origin. By the Divergence Theorem, we have

$$\text{Flux out of } W = \int_W \operatorname{div}\vec{F}\cdot dV = 0,$$

and therefore

$$\text{Flux out of } W = (\text{Flux out } - \text{ Flux in}) = \int_S \vec{F}\cdot d\vec{A} - \int_T \vec{F}\cdot d\vec{A} = 0.$$

Thus, we have

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}\cdot d\vec{A} &= (\text{Magnitude of } \vec{F} \text{ on sphere})\cdot(\text{Surface area})\\ &= \left(\frac{1}{\text{radius}^2}\right)\cdot(4\pi\cdot\text{radius}^2) = 4\pi.\end{aligned}$$

30. Use the Divergence Theorem. Since

$$\vec{F}(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\vec{i} + y\vec{j} + z\vec{k}),$$

we have $\operatorname{div}\vec{F} = 0$, except at the origin.

Let T be the surface of a sphere inside the cylinder S, and let W be the region between S and T. By the Divergence Theorem,

$$\text{Flux out of } W = \int_{S+T} \vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV = 0.$$

Since S is oriented outward and T is oriented inward,

$$\text{Net flux out of } W = \text{Flux out} - \text{Flux in} = \int_S \vec{F}\cdot d\vec{A} - \int_T \vec{F}\cdot d\vec{A} = 0.$$

so

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}\cdot d\vec{A} &= \text{Magnitude of } \vec{F} \text{ on sphere}\cdot\text{Surface area}\\ &= \left(\frac{1}{\text{Radius}^2}\right)\cdot(4\pi\ \text{Radius}^2) = 4\pi.\end{aligned}$$

31. (a) Let W_1 be the ball inside S_1. By the Divergence Theorem,

$$\int_{S_1} \vec{F}\cdot d\vec{A} = \int_{W_1} (x^2+y^2+z^2+3)\, dV.$$

Using spherical coordinates, we have

$$\begin{aligned}\int_{S_1} \vec{F}\cdot d\vec{A} &= \int_{W_1} (x^2+y^2+z^2+3)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho^2+3)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta\\ &= 2\pi\cdot\left(\frac{\rho^5}{5}+\rho^3\right)\Bigg|_0^1 (-\cos\phi)\Bigg|_0^{\pi} = 2\pi\cdot\frac{6}{5}\cdot 2 = \frac{24}{5}\pi.\end{aligned}$$

(b) If we were to calculate all three integrals using the Divergence Theorem, we would be integrating $\operatorname{div}\vec{F}$ through the interior of each of these regions. Since S_2 lies entirely inside S_3, and S_3 lies entirely inside S_4, and since $\operatorname{div}\vec{F} = x^2+y^2+z^2+3$ is positive everywhere,

$$\int_{S_2} \vec{F}\cdot d\vec{A} < \int_{S_3} \vec{F}\cdot d\vec{A} < \int_{S_4} \vec{F}\cdot d\vec{A}.$$

32. (a) At the point $(1,2,1)$, we have $\operatorname{div}\vec{F} = 1\cdot 2\cdot 1^2 = 2$.

(b) Since the box is small, we use the approximation

$$\operatorname{div}\vec{F} = \text{Flux density} \approx \frac{\text{Flux out of box}}{\text{Volume of box}}.$$

Thus

$$\text{Flux out of box} \approx (\operatorname{div}\vec{F})\cdot(\text{Volume of box}) = 2(0.2)^3 = 0.016.$$

(c) To calculate the flux exactly, we use the Divergence Theorem,

$$\text{Flux out of box} = \int_{\text{Box}} \operatorname{div}\vec{F}\,dV = \int_{\text{Box}} xyz^2\,dV.$$

Since the box has side 0.2, it is given by $0.9 < x < 1.1$, $1.9 < y < 2.1$, $0.9 < z < 1.1$, so

$$\begin{aligned}\text{Flux} &= \int_{0.9}^{1.1}\int_{1.9}^{2.1}\int_{0.9}^{1.1} xyz^2\,dzdydx = \int_{0.9}^{1.1}\int_{1.9}^{2.1} xy\left.\frac{z^3}{3}\right|_{0.9}^{1.1} dydx\\ &= \frac{(1.1)^3-(0.9)^3}{3}\int_{0.9}^{1.1}\left.\frac{xy^2}{2}\right|_{1.9}^{2.1} dx = \frac{(1.1)^3-(0.9)^3}{3}\cdot\frac{(2.1)^2-(1.9)^2}{2}\cdot\left.\frac{x^2}{2}\right|_{0.9}^{1.1}\\ &= \frac{(1.1)^3-(0.9)^3}{3}\cdot\frac{(2.1)^2-(1.9)^2}{2}\cdot\frac{(1.1)^2-(0.9)^2}{2} = 0.016053\ldots.\end{aligned}$$

Notice that you can calculate the flux without knowing the vector field, $\vec{F}$.

33. Any closed surface, S, oriented inward, will work. Then,

$$\int_{S(\text{inward})}\vec{F}\cdot d\vec{A} = -\int_{S(\text{outward})}\vec{F}\cdot d\vec{A},$$

so, by the Divergence Theorem, with W representing the region inside S,

$$\int_{S(\text{inward})}\vec{F}\cdot d\vec{A} = -\int_{W}\operatorname{div}\vec{F}\,dV = -\int_{W}(x^2+y^2+3)dV.$$

The integral on the right is positive because the integrand is positive everywhere. Therefore the flux through S oriented inward is negative.

34. (a) The rate at which heat is generated at any point in the earth is $\operatorname{div}\vec{F}$ at that point. So $\operatorname{div}\vec{F} = 30$ watts/km^3.

(b) Differentiating gives $\operatorname{div}(\alpha(x\vec{i}+y\vec{j}+z\vec{k})) = \alpha(1+1+1) = 3\alpha$ so $\alpha = 30/3 = 10$ watts/km^3. Thus, $\vec{F} = \alpha\vec{r}$ has constant divergence. Note that $\vec{F} = \alpha\vec{r}$ has flow lines going radially outward, and symmetric about the origin.

(c) The vector $\operatorname{grad} T$ gives the direction of greatest increase in temperature. Thus, $-\operatorname{grad} T$ gives the direction of greatest decrease in temperature. The equation $\vec{F} = -k\operatorname{grad} T$ says that heat will flow in the direction of greatest decrease in temperature (i.e. from hot regions to cold), and at a rate proportional to the temperature gradient.

(d) We assume that $\vec{F}$ is given by the answer to part (b). Then, using part (c), we have

$$\vec{F} = 10(x\vec{i}+y\vec{j}+z\vec{k}) = -30{,}000\operatorname{grad} T,$$

so

$$\operatorname{grad} T = -\frac{10}{30{,}000}(x\vec{i}+y\vec{j}+z\vec{k}).$$

Integrating we get

$$T = \frac{-10}{2(30{,}000)}(x^2+y^2+z^2)+C.$$

At the surface of the earth, $x^2+y^2+z^2 = 6400^2$, and $T = 20^\circ$C, so

$$T = \frac{-1}{6000}(6400^2)+C = 20.$$

Thus,

$$C = 20+\frac{6400^2}{6000} = 6847.$$

At the center of the earth, $x^2+y^2+z^2 = 0$, so

$$T = 6847^\circ\text{C}.$$

35. (a) Using the expression given for the force, we have

$$\begin{aligned}\text{Force in } \vec{i} \text{ direction} = \vec{F}\cdot\vec{i} &= -\int_S \delta g z\vec{i}\cdot d\vec{A} \\ &= -\delta g \int_S z\vec{i}\cdot d\vec{A}.\end{aligned}$$

Now apply the Divergence Theorem to this integral. (Notice that in order to do this, you need to orient S outward, hence the minus sign disappears.)

$$\vec{F}\cdot\vec{i} = \delta g \int_V \frac{\partial z}{\partial x} dV = 0.$$

Similarly:

$$\begin{aligned}\text{Force in } \vec{j} \text{ direction} = \vec{F}\cdot\vec{j} &= -\delta g \int_S z\vec{j}\cdot d\vec{A} \\ &= \delta g \int_V \frac{\partial z}{\partial y} dV = 0\end{aligned}$$

(b)

$$\begin{aligned}\text{Force in } \vec{k} \text{ direction} = \vec{F}\cdot\vec{k} &= -\delta g \int_S z\vec{k}\cdot d\vec{A} \\ &= \delta g \int_V \frac{\partial z}{\partial z} dV = \delta g \int_V dV = \delta g V.\end{aligned}$$

36. (a) Taking partial derivatives of $\vec{E}$ gives

$$\begin{aligned}\frac{\partial E_1}{\partial x} = \frac{\partial}{\partial x}[qx(x^2+y^2+z^2)^{-3/2}] &= q[(x^2+y^2+z^2)^{-3/2} + x(-3/2)(2x)(x^2+y^2+z^2)^{-5/2}] \\ &= q(y^2+z^2-2x^2)(x^2+y^2+z^2)^{-5/2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial E_2}{\partial x} &= q(x^2+z^2-2y^2)(x^2+y^2+z^2)^{-5/2} \\ \frac{\partial E_3}{\partial x} &= q(x^2+y^2-2z^2)(x^2+y^2+z^2)^{-5/2}.\end{aligned}$$

Summing, we obtain $\operatorname{div}\vec{E} = 0$.

(b) Since on the surface of the sphere, the vector field $\vec{E}$ and the area vector $\Delta\vec{A}$ are parallel,

$$\vec{E}\cdot\Delta\vec{A} = \|\vec{E}\|\|\Delta\vec{A}\|.$$

Now, on the surface of a sphere of radius a,

$$\|\vec{E}\| = \frac{q\|\vec{r}\|}{\|\vec{r}\|^3} = \frac{q}{a^2}.$$

Thus,

$$\int_{S_a} \vec{E}\cdot d\vec{A} = \int \frac{q}{a^2}\|d\vec{A}\| = \frac{q}{a^2}\cdot \text{Surface area of sphere} = \frac{q}{a^2}\cdot 4\pi a^2 = 4\pi q.$$

(c) It is not possible to apply the Divergence Theorem in part (b) since $\vec{E}$ is not defined at the origin (which lies inside the region of space bounded by S_a), and the Divergence Theorem requires that the vector field be defined everywhere inside S.

(d) Let R be the solid region lying between a small sphere S_a, centered at the origin, and the surface S. Applying the Divergence Theorem and the result of part (a), we get:

$$0 = \int_R \operatorname{div}\vec{E}\, dV = \int_{S_a} \vec{E}\cdot d\vec{A} + \int_S \vec{E}\cdot d\vec{A},$$

where S is oriented with the outward normal vector, and S_a with the inward normal vector (since this is "outward" with respect to the region R). Since

$$\int_{S_a,\text{ inward}} \vec{E}\cdot d\vec{A} = -\int_{S_a,\text{ outward}} \vec{E}\cdot d\vec{A},$$

the result of part (b) yields

$$\int_S \vec{E}\cdot d\vec{A} = 4\pi q.$$

[Note: It is legitimate to apply the Divergence Theorem to the region R since the vector field $\vec{E}$ is defined everywhere in R.]

37. Check that div $\vec{E} = 0$ by taking partial derivatives. For instance,

$$\begin{aligned}\frac{\partial E_1}{\partial x} &= \frac{\partial}{\partial x}[q(x-x_0)[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{-3/2}] \\ &= q[(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2][(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{-5/2}\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\partial E_2}{\partial y} &= q[(x-x_0)^2+(z-z_0)^2-2(y-y_0)^2][(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{-5/2} \\ \frac{\partial E_3}{\partial z} &= q[(x-x_0)^2+(y-y_0)^2-2(z-z_0)^2][(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{-5/2}.\end{aligned}$$

Therefore,

$$\frac{\partial E_1}{\partial x}+\frac{\partial E_2}{\partial y}+\frac{\partial E_3}{\partial z} = 0.$$

The vector field $\vec{E}$ is defined everywhere but at the point with position vector $\vec{r}_0$. If this point lies outside the surface S, the Divergence Theorem can be applied to the region R enclosed by S, yielding:

$$\int_S \vec{E}\cdot d\vec{A} = \int_R \operatorname{div}\vec{E}\, dV = 0.$$

If the charge q is located inside S, consider a small sphere S_a centered at q and contained in R. The Divergence Theorem for the region R' between the two spheres yields:

$$\int_S \vec{E}\cdot d\vec{A} + \int_{S_a} \vec{E}\cdot d\vec{A} = \int_{R'} \operatorname{div}\vec{E}\, dV = 0.$$

In this formula, the Divergence Theorem requires S to be given the outward orientation, and S_a the inward orientation. To compute $\int_{S_a} \vec{E}\cdot d\vec{A}$, we use the fact that on the surface of the sphere, $\vec{E}$ and $\Delta\vec{A}$ are parallel and in opposite directions, so

$$\vec{E}\cdot\Delta\vec{A} = -\|\vec{E}\|\|\Delta\vec{A}\|$$

since on the surface of a sphere of radius a,

$$\|\vec{E}\| = q\frac{\|\vec{r}-\vec{r}_0\|}{\|\vec{r}-\vec{r}_0\|^3} = \frac{q}{a^2}.$$

Then,

$$\int_{S_a} \vec{E}\cdot d\vec{A} = \int -\frac{q}{a^2}\|d\vec{A}\| = \frac{-q}{a^2}\cdot\text{Surface area of sphere} = -\frac{q}{a^2}\cdot 4\pi a^2 = -4\pi q.$$

$$\int_{S_a} \vec{E}\cdot d\vec{A} = -4\pi q.$$

$$\int_S \vec{E}\cdot d\vec{A} - \int_{S_a} \vec{E}\cdot d\vec{A} = 4\pi q.$$

Strengthen Your Understanding

38. The surface S is not the boundary of a solid region, so the Divergence Theorem does not apply.

39. The correct statement of the Divergence Theorem is:

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV.$$

40. Since $\operatorname{div} \vec{F} = 0$ everywhere, the Divergence Theorem shows that

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV = 0$$

for every surface S that bounds a solid region W. For example, we can take S to be a sphere with any center and radius.

41. If S is a sphere of radius 1 centered at the origin and W is the region inside it, then

$$\text{Flux out of } S = \int_S \vec{F}\, dA = \int_W \operatorname{div} \vec{F}\, dV = 3.$$

We pick $\vec{F}$ to be a vector field with constant divergence, $\operatorname{div} \vec{F} = k$, and calculate the constant k so as to get a flux of 3. Let $\vec{F} = kx\vec{i}$. Then $\operatorname{div} \vec{F} = k$ everywhere and

$$\text{Flux out of } S = \int_W k\, dV = k \cdot (\text{Volume of the sphere}) = \frac{4}{3}\pi k.$$

So we have

$$\text{Flux out of } S = \frac{4}{3}\pi k = 3, \quad \text{which gives } k = \frac{9}{4\pi}.$$

Thus, $\vec{F} = (9x/4\pi)\vec{i}$ is one possible answer.

42. False. Since the divergence is positive, $\vec{F}$ has a net outflow per unit volume everywhere.

43. True. The divergence of the field $x\vec{i} + (3y)\vec{j} + (y - 5x)\vec{k}$ is equal to $1 + 3 + 0 = 4$ at all points.

44. False. The divergence of any constant field is zero at all points.

45. False. The vector field could be $\vec{F} = 4x\vec{i}$, which is parallel to the xy-plane and hence has zero flux through the circle. Note that the Divergence Theorem cannot be used to calculate the flux in this case since a circle is not a closed surface enclosing a volume.

46. True. The Divergence Theorem applies in this case and, since $\operatorname{div}\vec{F}$ is constant, the flux of $\vec{F}$ through the cylinder is equal to 4 times the volume of the cylinder, or $4(3\pi)$.

47. False. If this were true, $\operatorname{div} \vec{F}$, which is a function, would always be constant ($\int_S \vec{F} \cdot d\vec{A}$ is a constant). Take $\vec{F} = x^2\vec{i}$, so $\operatorname{div} \vec{F} = 2x$, which is not constant.

48. False. Neither side of this equation makes sense: $\operatorname{div} \vec{F}$ is a scalar, and we cannot take the flux integral of a scalar. On the other side, $\vec{F}$ is a vector field, and we cannot take the triple integral of a vector field.

49. True. By the Divergence theorem, $\int_S \vec{F} \cdot d\vec{A} = -\int_W \operatorname{div} \vec{F}\, dV$, where W is the solid interior of S and the negative sign is due to the inward orientation of S. Since $\operatorname{div} \vec{F} = 0$, we have $\int_S \vec{F} \cdot d\vec{A} = 0$.

50. True. By the Divergence theorem, $\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV$, where W is the solid interior of S. Since $\operatorname{div} \vec{F} = 1$, we have $\int_S \vec{F} \cdot d\vec{A} = \int_W 1\, dV$ which is equal to the volume enclosed by S.

51. True. The Divergence theorem says that $\int_W \operatorname{div} \vec{F}\, dV = \int_S \vec{F} \cdot d\vec{A}$, where S is the outward oriented boundary of W. In this case, the boundary of W consists of the surfaces S_1 and S_2. To give this boundary surface a consistent outward orientation, we use a normal vector on S_1 that points toward the origin, and a normal on S_2 that points away from the origin. Thus $\int_W \operatorname{div} \vec{F}\, dV = \int_{S_2} \vec{F} \cdot d\vec{A} + \int_{S_1} \vec{F} \cdot d\vec{A}$, with S_2 oriented outward and S_1 oriented inward. Reversing the orientation on S_1 so that both spheres are oriented outward yields $\int_W \operatorname{div} \vec{F}\, dV = \int_{S_2} \vec{F} \cdot d\vec{A} - \int_{S_1} \vec{F} \cdot d\vec{A}$.

52. False. The boundary of the cube W consists of six squares, so the Divergence theorem requires adding the flux integrals over the four remaining sides.

53. True. The boundary of the cube W consists of six squares, but four of them are parallel to the xz or yz-planes and so contribute zero flux for this particular vector field. The only two surfaces of the boundary with nonzero flux are S_1 and S_2, which are parallel to the xy-plane.

54. False. The Divergence theorem tells us in this case that $\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV = 0$, where W is the solid ball with boundary S. But this does not necessarily mean that $\operatorname{div} \vec{F} = 0$ at all points in W. The divergence of $\vec{F}$ can be positive at some points in W and negative at other points in W, yielding a triple integral of zero. For example, let $\vec{F} = x^2\vec{i}$. The flux of this vector field through the sphere is 0. (The flux out for $x > 0$ cancels the flux in for $x < 0$.) However, $\operatorname{div} \vec{F} = 2x$, which is not 0.

55. True. Let D be the disk that forms the bottom of the cylinder, $x^2 + y^2 \le 1$, $z = 0$, oriented downward. Then the surface consisting of S_h and D is closed and oriented outward, so the Divergence Theorem says $\int_{S_h+D} \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV$, where W is the solid interior of the cylinder. Since $\operatorname{div} \vec{F} = 0$, we have $\int_{S_h+D} \vec{F} \cdot d\vec{A} = 0$. Writing the flux integral as the sum of integrals over S_h and D gives $\int_{S_h} \vec{F} \cdot d\vec{A} + \int_D \vec{F} \cdot d\vec{A} = 0$, so $\int_{S_h} \vec{F} \cdot d\vec{A} = -\int_D \vec{F} \cdot d\vec{A}$. The flux integral $\int_D \vec{F} \cdot d\vec{A}$ does not depend on the height h, so $\int_{S_h} \vec{F} \cdot d\vec{A}$ is independent of h.

56. Since $\operatorname{div} \vec{F} = 5 + 7 + 9 = 21$, by the Divergence Theorem, if W_i is the region inside S_i, we have

$$Q_i = \int_{W_i} 21\, dV = 21 \cdot \text{ Vol } W_i.$$

Thus, we arrange Q_i by the volume of W_i, so

$$Q_4 < Q_3 < Q_2 < Q_1.$$

Solutions for Chapter 19 Review

Exercises

1. Scalar. Only the $\vec{j}$-component of the vector field contributes to the flux and $d\vec{A} = -\vec{j}\, dA$, so

$$\int_S (3\vec{i} + 4\vec{j}) \cdot d\vec{A} = -4 \cdot \text{Area of disk } = -4 \cdot \pi 5^2 = -100\pi.$$

2. Scalar. Since

$$\operatorname{div}\left(\frac{y\vec{i} - x\vec{j}}{x^2 + y^2}\right) = \frac{-y \cdot 2x}{(x^2 + y^2)^2} + \frac{x \cdot 2y}{(x^2 + y^2)^2} = 0.$$

3. (a) Only the $\vec{i}$-component of the vector field contributes to the flux and $d\vec{A} = \vec{i}\, dA$, so

$$\int_S (\vec{i} + 2\vec{j} + \vec{k}) \cdot d\vec{A} = 1 \cdot \text{Area of square } = 16.$$

(b) Only the $\vec{i}$-component of the vector field contributes to the flux and $d\vec{A} = \vec{i}\, dA$, so

$$\int_S (\vec{i} + 2\vec{j} + \vec{k}) \cdot d\vec{A} = 1 \cdot \text{Area of square } = 16.$$

(c) Only the $\vec{k}$-component of the vector field contributes to the flux and $d\vec{A} = -\vec{k}\, dA$, so

$$\int_S (\vec{i} + 2\vec{j} + \vec{k}) \cdot d\vec{A} = -1 \cdot \text{Area of square } = -16.$$

(d) Only the $\vec{k}$-component of the vector field contributes to the flux and $d\vec{A} = -\vec{k}\, dA$, so

$$\int_S (\vec{i} + 2\vec{j} + \vec{k}) \cdot d\vec{A} = -1 \cdot \text{Area of square } = -16.$$

(e) Only the $\vec{j}$-component of the vector field contributes to the flux and $d\vec{A} = \vec{j}\, dA$, so

$$\int_S (\vec{i} + 2\vec{j} + \vec{k}) \cdot d\vec{A} = 2 \cdot \text{Area of square } = 32.$$

4. Zero, since $d\vec{A}$ is parallel to the x-axis, so $(x\vec{j} + y\vec{k}) \cdot d\vec{A} = 0$.

5. Negative. On S, we have $d\vec{A} = -\vec{i}\, dy\, dz$ and $x = 7$, so $(x\vec{i} + y\vec{k}) \cdot d\vec{A} = -7\, dy\, dz$. Thus, the flux integral is negative.

6. Zero. Since $d\vec{A} = -\vec{i}\, dA$, we have

$$\int_S (y\vec{i} + x\vec{k}) \cdot d\vec{A} = -\int_S y\, dA = 0.$$

The integral is 0 because the disk is centered on the x-axis, so the contributions from the parts of the disk where y is positive and where y is negative cancel. In more detail, since $d\vec{A} = -\vec{i}\, dx\, dy$ on S, we have

$$\int_S (y\vec{i} + x\vec{k}) \cdot d\vec{A} = \int_S -y\, dx\, dy = 0.$$

7. Positive. Since $d\vec{A} = -\vec{i}\, dA$ and $(x - 10) = -3$ on S,

$$\int_S ((x-10)\vec{i} + (x+10)\vec{j}) \cdot d\vec{A} = \int (-3\vec{i} + (x+10)\vec{j}) \cdot (-\vec{i}\, dA) = \int_S 3\, dA = 3 \cdot \text{ Area of } S.$$

8. The disk has area 25π, so its area vector is $25\pi\vec{j}$. Thus

$$\text{Flux} = (2\vec{i} + 3\vec{j}) \cdot 25\pi\vec{j} = 75\pi.$$

9. Since $\vec{F}$ is a constant vector field, the flux through a closed surface is zero. (The flux that enters one side, exits the other side.)

10. The square has area 16, so its area vector is $16\vec{j}$. Since $\vec{F} = 5\vec{j}$ on the square,

$$\text{Flux } = 5\vec{j} \cdot 16\vec{j} = 80.$$

11. The square has area 9, so its area vector is $9\vec{i}$. Since $\vec{F} = -5\vec{i}$ on the square,

$$\text{Flux} = -5\vec{i} \cdot 9\vec{i} = -45.$$

12. Since the square, S, is in the plane $y = 0$ and oriented in the negative y-direction, $d\vec{A} = -\vec{j}\, dxdz$ and

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (0+3)\vec{j} \cdot (-\vec{j}\, dxdz) = -3\int_S dxdz = -3 \cdot \text{Area of square } = -3(2^2) = -12.$$

13. Since the square, S, is oriented upward, $d\vec{A} = \vec{k}\, dxdy$ and

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = \int_S x\vec{k} \cdot \vec{k}\, dxdy = \int_0^3 \int_0^3 x\, dxdy = \int_0^3 \left.\frac{x^2}{2}\right|_0^3 dy = \frac{9}{2}\int_0^3 dy = \frac{27}{2}.$$

14. Since the vector field is constant, Flux $= 0$. The flux through opposite faces of the cube cancel.

15. The only contribution to the flux is from the $\vec{k}$-component, and since the square, S, is oriented upward, we have

$$\text{Flux } = \int_S (6\vec{i} + x^2\vec{j} - \vec{k}) \cdot d\vec{A} = \int_S -\vec{k} \cdot d\vec{A} = -\text{Area of square } = -4.$$

16. All the vectors in the vector field point horizontally (because their z-component is zero), and the surface is horizontal, so there is no flow through the surface and the flux is zero.

17. We have $d\vec{A} = \vec{k}\, dA$, so

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (z\vec{i} + y\vec{j} + 2x\vec{k}) \cdot \vec{k}\; dA = \int_S 2x\, dA$$
$$= \int_0^3 \int_0^2 2x\, dxdy = 12.$$

18. We have $d\vec{A} = \vec{i}\, dA$, so

$$\int_S \vec{F} \cdot d\vec{A} = \int_S ((2+\cos z)\vec{i} + y\vec{j} + 2x\vec{k}) \cdot \vec{i}\, dA = \int_S (2+\cos z)\, dA$$
$$= \int_0^4 \int_0^3 (2+\cos z)\, dy dz = 3(8+\sin 4)$$

19. On the surface S, y is constant, $y=-1$, and $d\vec{A} = -\vec{j}\, dA$, so,

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (x^2\vec{i} + (x+e^{-1})\vec{j} - \vec{k}) \cdot (-\vec{j})\, dA = -\int_S (x+e^{-1})\, dA$$
$$= -\int_0^4 \int_0^2 (x+e^{-1})\, dx\, dz = -4(2+2e^{-1}) = -8(1+e^{-1}).$$

20. Observe that the $\vec{j}$ and $\vec{k}$ components of $\vec{F}$ are parallel to the surface S, so they contribute nothing to the flux integral. On the surface S, the $\vec{i}$ component of $\vec{F}$ equals $5\vec{i}$, because $x=0$ on S. Since $5\vec{i}$ is normal to S and in the direction of the orientation of S, $\int_S \vec{F} \cdot d\vec{A} = \int_S 5\vec{i} \cdot d\vec{A} = \|5\vec{i}\|$(Area of S) $= 20$.

21. There is no flux through the base or top of the cylinder because the vector field is parallel to these faces. For the curved surface, consider a small patch with area $\Delta\vec{A}$. The vector field is pointing radially outward from the z-axis and so is parallel to $\Delta\vec{A}$. Since $\|\vec{F}\| = \sqrt{x^2+y^2} = 2$ on the curved surface of the cylinder, we have $\vec{F} \cdot \Delta\vec{A} = \|\vec{F}\|\|\Delta\vec{A}\| = 2\Delta A$. Replacing ΔA with dA, we get

$$\int_S \vec{F} \cdot d\vec{A} = \int_{\substack{\text{Curved}\\ \text{surface}}} 2\, dA = 2(\text{Area of curved surface}) = 2(2\pi \cdot 2 \cdot 3) = 24\pi.$$

22. The vector field $\vec{F} = -y\vec{i} + x\vec{j} + z\vec{k}$ is tangent to the curved surface of the cylinder. (The area vector is parallel to the vector pointing radially outward from the z-axis, namely $x\vec{i} + y\vec{j}$ and $(-y\vec{i} + x\vec{j} + z\vec{k}) \cdot (x\vec{i} + y\vec{j}) = 0$.) Thus the only contributions to the flux integral are from the top and the bottom. On the top, $z=1$ and $d\vec{A} = dA\vec{k}$, so

$$\vec{F} \cdot d\vec{A} = (-y\vec{i} + x\vec{j} + \vec{k}) \cdot dA\vec{k} = dA.$$

Thus

$$\int_{\text{Top}} \vec{F} \cdot d\vec{A} = \int_{\text{Top}} dA = \text{Area of top} = \pi(1)^2 = \pi.$$

Similarly, on the base, $z=-1$ and $d\vec{A} = (-dA\,\vec{k})$, so

$$\vec{F} \cdot d\vec{A} = (-y\vec{i} + x\vec{j} - \vec{k}) \cdot (-dA\vec{k}) = dA.$$
$$\int_{\text{Base}} \vec{F} \cdot d\vec{A} = \int_{\text{Base}} dA = \text{Area of base} = \pi.$$

Therefore,

$$\text{Total flux through cylinder} = \text{Flux through top} + \text{Flux through base} = 2\pi.$$

23. First we have

$$z_x = \frac{x}{\sqrt{x^2+y^2}} \qquad z_y = \frac{y}{\sqrt{x^2+y^2}}.$$

Although z is not a smooth function of x and y at $(0,0)$, the improper integral that we get converges:

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (x^2\vec{i} + y^2\vec{j} + \sqrt{x^2+y^2}\vec{k}) \cdot (-\frac{x}{\sqrt{x^2+y^2}}\vec{i} - \frac{y}{\sqrt{x^2+y^2}}\vec{j} + \vec{k})\, dA$$
$$= \int_S \left(-\frac{x^3+y^3}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} \right) dA$$

Changing to polar coordinates we have

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_0^{\pi/2} \int_0^1 (-r^2\cos^3\theta - r^2\sin^3\theta + r) r\,dr d\theta \\
&= \int_0^{\pi/2} \left(-\frac{r^4}{4}(\cos^3\theta + \sin^3\theta) + \frac{1}{3}r^3 \Bigg|_{r=0}^{r=1} \right) d\theta \\
&= \int_0^{\pi/2} \left(-\frac{1}{4}(\cos^3\theta + \sin^3\theta) + \frac{1}{3} \right) d\theta \\
&= \int_0^{\pi/2} \left(-\frac{1}{4}(\cos\theta - \cos\theta\sin^2\theta + \sin\theta - \sin\theta\cos^2\theta) + \frac{1}{3} \right) d\theta \\
&= -\frac{1}{4}(\sin\theta - \frac{1}{3}\sin^3\theta - \cos\theta + \frac{1}{3}\cos^3\theta) + \frac{\theta}{3} \Bigg|_0^{\pi/2} \\
&= \frac{\pi}{6} - \frac{1}{3}.
\end{aligned}$$

24. The vector normal to S is $\vec{j}$; the dot product of $\vec{F}$ and $\vec{j}$ is positive if $b > 0$. There are no conditions on a and c.

25. By the symmetry of the sphere, the $\vec{i}$ and $\vec{j}$ components of $\vec{F}$ do not contribute to the flux; only the $\vec{k}$ component contributes. The vector normal to S has a negative $\vec{k}$ component, so we need $c > 0$. There are no conditions on a and b.

26. The sphere is oriented outward. Provided $a > 0$, the vector field points outward, giving positive flux.

27. The normal to the plane is $\vec{i} + \vec{j} + \vec{k}$. Converting this to a unit vector, we see that on the plane $d\vec{A} = (\vec{i} + \vec{j} + \vec{k})/\sqrt{3}\,dA$. The flux integrates the dot product

$$\vec{F} \cdot d\vec{A} = (a\vec{i} + b\vec{j} + c\vec{k}) \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}\,dA = \frac{a+b+c}{\sqrt{3}}\,dA.$$

Thus we need $a + b + c > 0$.

28. (a) By direct calculation, the flux of $\vec{F}$ through the plane $x = 0$ is 0 because the $\vec{i}$-component of the vector field is 0 there. The flux of $\vec{F}$ through S, the square $0 \le y \le 1, 0 \le z \le 1$ in the plane $x = 1$ is

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = \int_0^1 \int_0^1 (1 \cdot y\vec{i} + yz\vec{j} + z \cdot 1\vec{k}) \cdot \vec{i}\,dy\,dz = \int_0^1 \int_0^1 y\,dy\,dz = \int_0^1 \frac{y^2}{2}\Bigg|_0^1 dz = \frac{1}{2}.$$

Similarly, the faces $y = 0$ and $y = 1$ contribute a flux of 0 and $1/2$, respectively, as do the faces $z = 0$ and $z = 1$. Therefore

$$\text{Total flux} = 0 + \frac{1}{2} + 0 + \frac{1}{2} + 0 + \frac{1}{2} = \frac{3}{2}.$$

(b) Since $\operatorname{div}\vec{F} = y + z + x$, the flux is given by

$$\int_{\text{Box}} \vec{F} \cdot d\vec{A} = \int_{\text{Box}} (x + y + z) dV.$$

We calculate the integral

$$\begin{aligned}
\int_{\text{Box}} (x+y+z)\,dV &= \int_0^1 \int_0^1 \int_0^1 (x+y+z)\,dx\,dy\,dz = \int_0^1 \int_0^1 \left(\frac{x^2}{2} + yx + zx \right) \Bigg|_0^1 dy\,dz \\
&= \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z \right) dy\,dz = \int_0^1 \left(\frac{y}{2} + \frac{y^2}{2} + zy \right) \Bigg|_0^1 dz \\
&= \int_0^1 \left(\frac{1}{2} + \frac{1}{2} + z \right) dz = \int_0^1 (1+z)\,dz = \left(z + \frac{z^2}{2} \right) \Bigg|_0^1 = \frac{3}{2}.
\end{aligned}$$

29. (a) We will compute separately the flux of the vector field $\vec{F} = x^3\vec{i} + 2y\vec{j} + 3\vec{k}$ through each of the six faces of the cube.

The face S_I where $x = 1$, which has normal vector $\vec{i}$. Only the $\vec{i}$ component $x^3\vec{i} = \vec{i}$ of $\vec{F}$ has flux through S_I.

$$\int_{S_I} \vec{F} \cdot d\vec{A} = \int_{S_I} \vec{i} \cdot d\vec{A} = \|\vec{i}\|(\text{area of } S_I) = 4.$$

The face S_{II} where $x = -1$, which has normal vector $-\vec{i}$. Only the $\vec{i}$ component $x^3\vec{i} = -\vec{i}$ of $\vec{F}$ has flux through S_{II}.

$$\int_{S_{II}} \vec{F} \cdot d\vec{A} = \int_{S_{II}} -\vec{i} \cdot d\vec{A} = \|-\vec{i}\|(\text{area of } S_{II}) = 4.$$

The face S_{III} where $y = 1$, which has normal vector $\vec{j}$. Only the $\vec{j}$ component $2y\vec{j} = 2\vec{j}$ of $\vec{F}$ has flux through S_{III}.

$$\int_{S_{III}} \vec{F} \cdot d\vec{A} = \int_{S_{III}} 2\vec{j} \cdot d\vec{A} = \|2\vec{j}\|(\text{area of } S_{III}) = 8.$$

The face S_{IV} where $y = -1$, which has normal vector $-\vec{j}$. Only the $\vec{j}$ component $2y\vec{j} = -2\vec{j}$ of $\vec{F}$ has flux through S_{IV}.

$$\int_{S_{IV}} \vec{F} \cdot d\vec{A} = \int_{S_{IV}} -2\vec{j} \cdot d\vec{A} = \|-2\vec{j}\|(\text{area of } S_{IV}) = 8.$$

The face S_V where $z = 1$, which has normal vector $\vec{k}$. Only the $\vec{k}$ component $3\vec{k}$ of $\vec{F}$ has flux through S_V.

$$\int_{S_V} \vec{F} \cdot d\vec{A} = \int_{S_V} 3\vec{k} \cdot d\vec{A} = \|3\vec{k}\|(\text{area of } S_V) = 12.$$

The face S_{VI} where $z = -1$, which has normal vector $-\vec{k}$. Only the $\vec{k}$ component $3\vec{k}$ of $\vec{F}$ has flux through S_{VI}.

$$\int_{S_{VI}} \vec{F} \cdot d\vec{A} = \int_{S_{VI}} 3\vec{k} \cdot d\vec{A} = -\|3\vec{k}\|(\text{area of } S_{VI}) = -12.$$

$$(\text{Total flux through } S) = 4 + 4 + 8 + 8 + 12 - 12 = 24.$$

(b) Since S is a closed surface the Divergence Theorem applies. Since $\operatorname{div}\vec{F} = 3x^2 + 2$,

$$\int_S \vec{F} \cdot d\vec{A} = \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (3x^2 + 2)dzdydx = 24.$$

30. Since $\operatorname{div}\vec{F} = -1 + 2 + 2 = 3$, we use the Divergence Theorem:

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV = 3 \cdot \text{Volume of Sphere} = 3 \cdot \frac{4}{3}\pi 2^3 = 32\pi.$$

31. Since $\operatorname{div}\vec{F} = 2 + 3 + 4 = 9$, the Divergence Theorem gives

$$\int_S \vec{F} \cdot d\vec{A} = \int_{\substack{\text{Interior}\\ \text{of sphere}}} \operatorname{div}\vec{F}\, dV = 9 \cdot \text{Volume of sphere} = 9 \cdot \frac{4}{3}\pi 5^3 = 1500\pi.$$

32. Since $\operatorname{div}\vec{F} = 2x + 2y$, the Divergence Theorem gives

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_{\substack{\text{Interior}\\ \text{of cube}}} (2x + 2y)\, dV \\ &= 2\int_0^3 \int_0^3 \int_0^3 (x + y)\, dx\, dy\, dz\end{aligned}$$

$$= 2\int_0^3\int_0^3 \left(\frac{x^2}{2}+xy\right)\Big|_0^3 dy\,dz$$
$$= 2\int_0^3\int_0^3 \left(\frac{9}{2}+3y\right) dy\,dz$$
$$= 2\int_0^3 \left(\frac{9}{2}y+\frac{3}{2}y^2\right)\Big|_0^3 dz = 2\int_0^3 27\,dz = 162.$$

33. Since $\text{div}\vec{F} = 1+2-1 = 2$, the Divergence Theorem gives

$$\int_S \vec{F}\cdot d\vec{A} = \int_{\substack{\text{Interior}\\ \text{of sphere}}} \text{div}\,\vec{F}\,dV = 2\cdot \text{Volume of sphere} = 2\cdot\frac{4}{3}\pi 1^3 = \frac{8\pi}{3}.$$

34. Since $\text{div}\vec{F} = 3x^2+3y^2$, using cylindrical coordinates to calculate the triple integral gives

$$\int_S \vec{F}\cdot d\vec{A} = \int_{\substack{\text{Interior}\\ \text{of cylinder}}} (3x^2+3y^2)\,dV = 3\int_0^{2\pi}\int_0^5\int_0^2 r^2\cdot r\,dr\,dz\,d\theta = 3\cdot 2\pi\cdot 5\frac{r^4}{4}\Big|_0^2 = 120\pi.$$

35. Since $\text{div}\,\vec{F} = 1+1+1 = 3$, the flux through the closed cylinder, S_1, with interior W, is

$$\int_{S_1} \vec{F}\cdot d\vec{A} = \int_W 3\,dV = 3\cdot \text{ Volume of cylinder } = 3\pi.$$

With the base of the cylinder oriented downward and the top of the cylinder oriented upward,

$$\int_S \vec{F}\cdot d\vec{A} = \int_{S_1} \vec{F}\cdot d\vec{A} - \int_{\text{Base}} \vec{F}\cdot d\vec{A} - \int_{\text{Top}} \vec{F}\cdot d\vec{A}.$$

Since $\vec{F}$ is parallel to the base, the flux through the base is 0. The flux through the top is contributed entirely by the $\vec{k}$ component. Since $z=1$, we have

$$\text{Flux through top } = \int_{\text{Top}} \vec{F}\cdot d\vec{A} = \int_{\text{Top}} (x\vec{i}+y\vec{j}+\vec{k})\cdot d\vec{A} = \int_{\text{Top}} \vec{k}\cdot d\vec{A} = \pi.$$

Thus

$$\int_S \vec{F}\cdot d\vec{A} = 3\pi - \pi = 2\pi.$$

Problems

36.

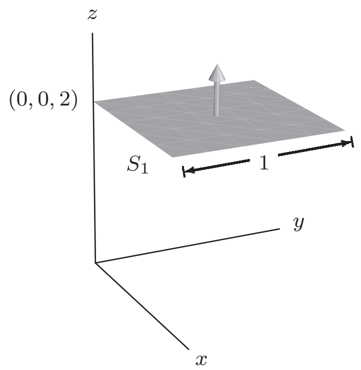


Figure 19.22

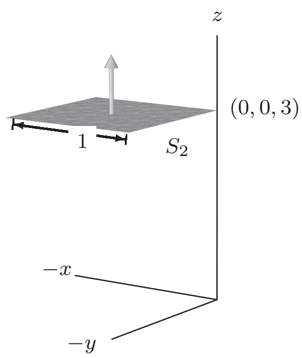


Figure 19.23

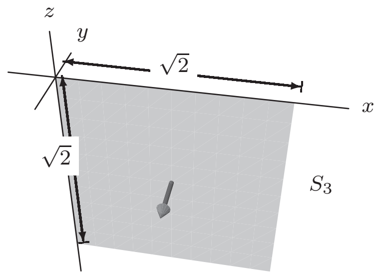


Figure 19.24

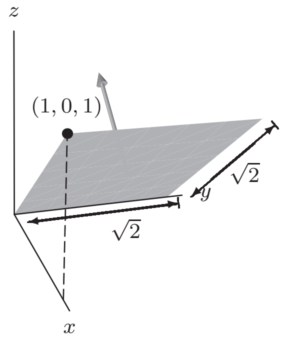


Figure 19.25

$$\text{Flux through } S_1 = \vec{F} \cdot \vec{A} = (-\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{k}) = 1$$
$$\text{Flux through } S_2 = \vec{F} \cdot \vec{A} = (-\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{k}) = 1$$
$$\text{Flux through } S_3 = \vec{F} \cdot \vec{A} = (-\vec{i} - \vec{j} + \vec{k}) \cdot (-2\vec{j}) = 2$$

For S_4, a normal is $-\vec{i} + \vec{k}$ and the area is 2, so $\vec{A} = -\sqrt{2}\vec{i} + \sqrt{2}\vec{k}$

$$\text{Flux through } S_4 = \vec{F} \cdot \vec{A} = (-\vec{i} - \vec{j} + \vec{k}) \cdot (-\sqrt{2}\vec{i} + \sqrt{2}\vec{k}) = 2\sqrt{2}.$$

So,

$$\text{Flux through } S_1 = \text{Flux through } S_2 < \text{Flux through } S_3 < \text{Flux through } S_4.$$

37. (a) We have $\text{grad } f = (y + yze^{xyz})\vec{i} + (x + xze^{xyz})\vec{j} + xye^{xyz}\vec{k}$.

(b) By the Fundamental Theorem of Line Integrals, we have

$$\int_C \text{grad } f \cdot d\vec{r} = (xy + e^{xyz})\Big|_{(1,1,1)}^{(2,3,4)} = (2 \cdot 3 + e^{2\cdot 3\cdot 4} - (1 \cdot 1 + e^{1\cdot 1\cdot 1}) = 5 + e^{24} - e^1.$$

(c) Only the k-component of grad f contributes to the flux integral. On the xy-plane, $d\vec{A} = \vec{k}\,dx\,dy$ and $z = 0$, so

$$\int_S \text{grad } f \cdot d\vec{A} = \int_0^2 \int_0^{\sqrt{4-x^2}} xye^0\,dy\,dx = \int_0^2 \frac{xy^2}{2}\Big|_0^{\sqrt{4-x^2}} = \int_0^2 \frac{x}{2}(4 - x^2)\,dx = \left(x^2 - \frac{x^4}{8}\right)\Big|_0^2 = 2.$$

38. The square of side 2 in the plane $x = 5$, oriented in the positive x-direction, has area vector $\vec{A} = 4\vec{i}$. Since the vector field is constant

$$\text{Flux} = (a\vec{i} + b\vec{j} + c\vec{k}) \cdot 4\vec{i} = 4a = 24.$$

Thus, $a = 6$ and we cannot say anything about the values of b and c.

39. (a) In the first and third integrals, only the $\vec{i}$ component contributes to the flux integral. The orientation is in the positive $\vec{i}$ directions in both cases, the disks are the same size, and the $(x^2 + 4)$ is larger on the surface S_3, so the flux through S_3 is larger than the flux through S_1. In the case of S_2, only the $\vec{j}$-component contributes. Since the disks in S_2 and S_3 are the same size, and S_2 oriented in the positive y-direction, the fact that $(x^2 + 4)$ is larger on S_3 than y is on S_2 tells us that the flux through S_3 is largest.

(b) On S_3, the vector field is $((-3)^2 + 4)\vec{i} + y\vec{j} = 13\vec{i} + y\vec{j}$ and $d\vec{A} = \vec{i}\,dA$, so

$$\int_{S_3} ((x^2 + 4)\vec{i} + y\vec{j}) \cdot d\vec{A} = \int_{S_3} (13\vec{i} + y\vec{j}) \cdot \vec{i}\,dA = \int_{S_3} 13\,dA = 13 \cdot \text{ Area of disk } = 13\pi.$$

40. (a) At the north pole, the area vector of the plate is upward (away from the center of the earth), and so is in the opposite direction to the magnetic field. Thus the magnetic flux is negative.

(b) At the south pole, the area vector of the plate is again away from the center of the earth (because that is upward in the southern hemisphere), and so is in the same direction as the magnetic field. Thus, the magnetic flux is positive.

(c) At the equator the magnetic field is parallel to the plate, so the flux is zero.

41. (a) By the definition of divergence as flux density, we have

$$\text{Flux out of small region} \approx \text{Divergence} \cdot \text{Volume}.$$

(i) At $(2,1,1)$, we have $\operatorname{div}\vec{F} = 2^2 + 1^2 - 1^2 = 4$, so

$$\text{Flux} \approx 4 \cdot \frac{4}{3}\pi(0.1)^3 = \frac{0.016}{3}\pi = 0.0168.$$

(ii) At $(0,0,1)$, we have $\operatorname{div}\vec{F} = 0^2 + 0^2 - 1^2 = -1$, so

$$\text{Flux} \approx -1 \cdot \frac{4}{3}\pi(0.1)^3 = -0.004.$$

(b) The fact that the flux in part (i) is positive tells us that the vector field is pointing, on average, outward (rather than inward) on the sphere around $(2,1,1)$. The fact that the flux in part (ii) is negative tells us that, on average, the vector field is pointing inward (rather than outward) on the sphere around the point $(0,0,1)$.

42. (a) The cube is in Figure 19.26. The vector field is parallel to the x-axis and zero on the yz-plane. Thus the only contribution to the flux is from S_2. On S_2, $x = c$, the normal is outward. Since $\vec{F}$ is constant on S_2, the flux through face S_2 is

$$\begin{aligned}\int_{S_2} \vec{F} \cdot d\vec{A} &= \vec{F} \cdot \vec{A}_{S_2} \\ &= c\vec{i} \cdot c^2\vec{i} \\ &= c^3.\end{aligned}$$

Thus, total flux through box $= c^3$.

(b) Using the geometric definition of divergence

$$\begin{aligned}\operatorname{div}\vec{F} &= \lim_{c\to 0}\left(\frac{\text{Flux through box}}{\text{Volume of box}}\right) \\ &= \lim_{c\to 0}\left(\frac{c^3}{c^3}\right) \\ &= 1\end{aligned}$$

(c) Using partial derivatives,

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 1 + 0 + 0 = 1.$$

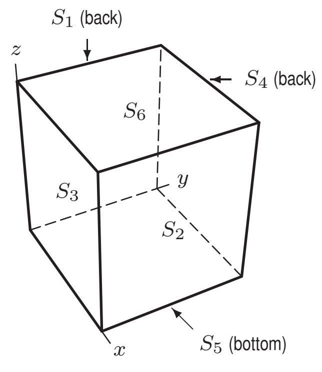


Figure 19.26

43. See Figure 19.26.

(a) Since $2\vec{i}+3\vec{k}$ is a constant field, its contribution to the flux is zero (flux in cancels flux out). Therefore $\int \vec{F}\cdot d\vec{A} = \int (y\vec{j})\cdot d\vec{A} = \int_{S_3} y\vec{j}\cdot d\vec{A} + \int_{S_4} y\vec{j}\cdot d\vec{A}$ since only S_3 and S_4 are perpendicular to $y\vec{j}$. On S_3, $y=0$ so $\int_{S_3} y\vec{j}\cdot d\vec{A} = 0$. On S_4, $y=c$ and normal is in the positive y-direction, so $\int_{S_4} y\vec{j}\cdot d\vec{A} = c(\text{Area of } S_4) = c\cdot c^2 = c^3$. Thus, total flux $= c^3$.

(b) Using the geometric definition of divergence

$$\begin{aligned}\operatorname{div}\vec{F} &= \lim_{c\to 0}\left(\frac{\text{Flux through box}}{\text{Volume of box}}\right)\\ &= \lim_{c\to 0}\left(\frac{c^3}{c^3}\right) = 1.\end{aligned}$$

(c)

$$\frac{\partial}{\partial x}(2)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(3) = 0+1+0 = 1.$$

44. See Figure 19.26.

(a) This vector field points radially outward from the z-axis. Thus, the vector field is parallel to the surface on S_1, S_3, S_5 and S_6, so the only contributions to the flux integral are from S_2 and S_4.
On $S_2, x=c$ and normal is in the positive x-direction, so the flux is

$$\int_{S_2}\vec{F}\cdot d\vec{A} = \int_{S_2}(c\vec{i}+y\vec{j})\cdot(dA\vec{i}) = \int_{S_2} c\,dA = c(\text{Area of } S_2) = c^3.$$

Similarly, the flux through S_4 is

$$\int_{S_4}\vec{F}\cdot d\vec{A} = \int_{S_4}(x\vec{i}+c\vec{j})\cdot(dA\vec{j}) = \int_{S_4} c\,dA = c(\text{Area of } S_4) = c^3.$$

Thus, the total flux through the box $= 2c^3$.

(b) Using the geometric definition of divergence, we have

$$\begin{aligned}\operatorname{div}\vec{F} &= \lim_{c\to 0}\left(\frac{\text{Flux through surface of box}}{\text{Volume of box}}\right)\\ &= \lim_{c\to 0}\left(\frac{2c^3}{c^3}\right) = 2.\end{aligned}$$

(c)

$$\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(0) = 1+1+0 = 2.$$

45. We close the cylinder, S, by adding the circular disk, S_1, at the top, $z=3$. The surface $S+S_1$ is oriented outward, so S_1 is oriented upward. Applying the Divergence Theorem to the closed surface $S+S_1$ enclosing the region W, we have

$$\int_{S+S_1}\vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV$$

$$\int_S \vec{F}\cdot d\vec{A} + \int_{S_1}\vec{F}\cdot d\vec{A} = \int \operatorname{div}\vec{F}\,dV.$$

Since

$$\operatorname{div}\vec{F} = \operatorname{div}(z^2\vec{i}+x^2\vec{j}+5\vec{k}) = 0,$$

we have

$$\int_S \vec{F}\cdot d\vec{A} = -\int_{S_1}\vec{F}\cdot d\vec{A}.$$

Only the $\vec{k}$-component of $\vec{F}$ contributes to the flux through S_1, and $d\vec{A} = \vec{k}\,dx\,dy$ on S_1, so

$$\int_S \vec{F}\cdot d\vec{A} = -\int_{S_1}(z^2\vec{i}+x^2\vec{j}+5\vec{k})\cdot\vec{k}\,dx\,dy = -\int_{S_1} 5\,dx\,dy = -5\cdot\text{ Area of } S_1 = -5\pi(\sqrt{2})^2 = -10\pi.$$

46. We close the cylinder, S, by adding the circular disk, S_1, at the top, $z = 3$. The surface $S + S_1$ is oriented outward, so S_1 is oriented upward. Applying the Divergence Theorem to the closed surface $S + S_1$ enclosing the region W, we have

$$\int_{S+S_1} \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} \, dV$$

$$\int_S \vec{F} \cdot d\vec{A} + \int_{S_1} \vec{F} \cdot d\vec{A} = \int \operatorname{div} \vec{F} \, dV.$$

Since

$$\operatorname{div} \vec{F} = \operatorname{div}(y^2\vec{i} + z^2\vec{j} + (x^2 + y^2)\vec{k}) = 0,$$

we have

$$\int_S \vec{F} \cdot d\vec{A} = -\int_{S_1} \vec{F} \cdot d\vec{A}.$$

Only the $\vec{k}$-component of $\vec{F}$ contributes to the flux through S_1, and $d\vec{A} = \vec{k}\, dx\, dy$ on S_1, so

$$\int_S \vec{F} \cdot d\vec{A} = -\int_{S_1} (y^2\vec{i} + z^2\vec{j} + (x^2 + y^2)\vec{k}) \cdot \vec{k}\, dx\, dy = -\int_{S_1} (x^2 + y^2)\, dx\, dy.$$

Converting to polar coordinates, since the cylinder has radius $\sqrt{2}$, we have

$$\int_S \vec{F} \cdot d\vec{A} = -\int_0^{2\pi} \int_0^{\sqrt{2}} r^2 \cdot r\, dr\, d\theta = -2\pi \left. \frac{r^4}{4} \right|_0^{\sqrt{2}} = -2\pi.$$

47. We close the cylinder, S, by adding the circular disk, S_1, at the top, $z = 3$. The surface $S + S_1$ is oriented outward, so S_1 is oriented upward. Applying the Divergence Theorem to the closed surface $S + S_1$ enclosing the region W, we have

$$\int_{S+S_1} \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} \, dV$$

$$\int_S \vec{F} \cdot d\vec{A} + \int_{S_1} \vec{F} \cdot d\vec{A} = \int \operatorname{div} \vec{F} \, dV.$$

Since

$$\operatorname{div} \vec{F} = \operatorname{div}(z\vec{i} + x\vec{j} + y\vec{k}) = 0,$$

we have

$$\int_S \vec{F} \cdot d\vec{A} = -\int_{S_1} \vec{F} \cdot d\vec{A}.$$

Only the $\vec{k}$-component of $\vec{F}$ contributes to the flux through S_1, and $d\vec{A} = \vec{k}\, dx\, dy$ on S_1, so

$$\int_S \vec{F} \cdot d\vec{A} = -\int_{S_1} (z\vec{i} + x\vec{j} + y\vec{k}) \cdot \vec{k}\, dx\, dy = -\int_{S_1} y\, dx\, dy.$$

Since y is an odd function, by symmetry, its integral over S_1 is zero. Thus,

$$\int_S \vec{F} \cdot d\vec{A} = -\int_S y\, dx\, dy = 0.$$

48. We close the cylinder, S, by adding the circular disk, S_1, at the top, $z = 3$. The surface $S + S_1$ is oriented outward, so S_1 is oriented upward. Applying the Divergence Theorem to the closed surface $S + S_1$ enclosing the region W, we have

$$\int_{S+S_1} \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} \, dV$$

$$\int_S \vec{F} \cdot d\vec{A} + \int_{S_1} \vec{F} \cdot d\vec{A} = \int \operatorname{div} \vec{F} \, dV.$$

Since

$$\operatorname{div} \vec{F} = \operatorname{div}(y^2\vec{i} + x^2\vec{j} + 7z\vec{k}) = 7,$$

we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_W 7\,dV - \int_{S_1} \vec{F}\cdot d\vec{A} = 7\cdot \text{ Volume of cylinder } - \int_{S_1} \vec{F}\cdot d\vec{A}$$

$$= 7\cdot\pi(\sqrt{2})^2 6 - \int_{S_1} \vec{F}\cdot d\vec{A} = 84\pi - \int_{S_1} \vec{F}\cdot d\vec{A}.$$

Only the $\vec{k}$-component of $\vec{F}$ contributes to the flux through S_1, and $d\vec{A} = \vec{k}\,dx\,dy$ and $z = 3$ on S_1, so

$$\int_S \vec{F}\cdot d\vec{A} = 84\pi - \int_{S_1} (y^2\vec{i} + x^2\vec{j} + 7\cdot 3\vec{k})\cdot\vec{k}\,dx\,dy = 84\pi - \int_{S_1} 21\,dx\,dy$$

$$= 84\pi - 21\cdot \text{ Area of } S_1 = 84\pi - 21\cdot\pi(\sqrt{2})^2 = 42\pi.$$

49. We close the cylinder, S, by adding the circular disk, S_1, at the top, $z = 3$. The surface $S + S_1$ is oriented outward, so S_1 is oriented upward. Applying the Divergence Theorem to the closed surface $S + S_1$ enclosing the region W, we have

$$\int_{S+S_1} \vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV$$

$$\int_S \vec{F}\cdot d\vec{A} + \int_{S_1} \vec{F}\cdot d\vec{A} = \int \operatorname{div}\vec{F}\,dV.$$

Since

$$\operatorname{div}\vec{F} = \operatorname{div}(x^3\vec{i} + y^3\vec{j} + \vec{k}) = 3x^2 + 3y^2,$$

we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_W (3x^2 + 3y^2)\,dV - \int_{S_1} \vec{F}\cdot d\vec{A}.$$

To find the integral over W, we use cylindrical coordinates. For the integral over S_1, we use the fact that $d\vec{A} = \vec{k}\,dx\,dy$, so only the k-component of $\vec{F}$ contributes to the flux.

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_{-3}^{3}\int_0^{\sqrt{2}} 3r^2\cdot r\,dr\,dz\,d\theta - \int_{S_1} (x^3\vec{i} + y^3\vec{j} + \vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \theta\Big|_0^{2\pi} z\Big|_{-3}^{3} \frac{3}{4}r^4\Big|_0^{\sqrt{2}} - \int_{S_1} dx\,dy$$

$$= 2\pi\cdot 6\cdot\frac{3}{4}(\sqrt{2})^4 - \text{ Area of } S_1$$

$$= 36\pi - \pi(\sqrt{2})^2 = 34\pi.$$

50. By the Divergence Theorem, since $\operatorname{div}\vec{F} = 0$, the flux through the cone equals the flux upward through the disk $r \le 4$ in the plane $z = 4$. The area vector of the disk is $\vec{A} = \pi 4^2\vec{k}$. Since the flux is negative, $\vec{F} = -c(\vec{i} + \vec{k})$ with $c > 0$. Thus

$$\text{Flux } = \vec{F}\cdot\vec{A} = -c(\vec{i} + \vec{k})\cdot\pi 4^2\vec{k} = -16\pi c = -7.$$

so

$$c = \frac{7}{16\pi} \quad\text{and}\quad \vec{F} = -\frac{7}{16\pi}(\vec{i} + \vec{k}).$$

51. (a) Since $\vec{r} = x\vec{i} + y\vec{j}$, for $\vec{r} \ne \vec{0}$, we have

$$\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^2}\right) = \operatorname{div}\left(\frac{x\vec{i} + y\vec{j}}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right)$$

$$= \frac{1}{x^2 + y^2} - \frac{x\cdot 2x}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} - \frac{y\cdot 2y}{(x^2 + y^2)^2}$$

$$= \frac{x^2 + y^2 - 2x^2 + x^2 + y^2 - 2y^2}{(x^2 + y^2)^2}$$

$$= 0.$$

(b) For $\vec{r} = 0$, which is the z-axis.

(c) Since the cylinder does not include the z-axis, we can use the Divergence Theorem. If S is the cylinder and W is its interior, we have

$$\int_S \frac{\vec{r}}{||\vec{r}\,||^2} \cdot d\vec{A} = \int_W \operatorname{div}\left(\frac{\vec{r}}{||\vec{r}\,||^2}\right) dV = \int_W 0\, dV = 0.$$

(d) Since the vector field $\vec{r}/||\vec{r}\,||^2$ has no component perpendicular to the flux there, it has no flux through the ends. Thus the flux through the sides of this cylinder is also 0.

52. (a) The divergence of $\vec{F}$ must be zero.

(b) We calculate the divergence:

$$\operatorname{div}(a(e^x + y - x)\vec{i} + 12y(1 - e^x)\vec{j}\,) = ae^x - a + 12(1 - e^x) = (a - 12)e^x + 12 - a.$$

Thus, the divergence is 0 if $a = 12$.

53. (a) First we calculate

$$\vec{r} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = (ya_3 - za_2)\vec{i} + (za_1 - xa_3)\vec{j} + (xa_2 - ya_1)\vec{k}\,.$$

Then we calculate

$$\operatorname{div}(\vec{r} \times \vec{a}\,) = \operatorname{div}\left((ya_3 - za_2)\vec{i} + (za_1 - xa_2)\vec{j} + (xa_2 - ya_1)\vec{k}\,\right) = 0.$$

(b) The cube is closed and oriented outward. By the Divergence Theorem,

$$\int_{\text{Box}} (\vec{r} \times \vec{a}\,) \cdot d\vec{A} = \int_{\text{Interior of box}} \operatorname{div}(\vec{r} \times \vec{a}\,) dV = 0.$$

54. We use the Divergence Theorem, where W is the region inside S:

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} F\, dV.$$

Since $W = W_1 + W_2$, where W_1 is the region inside the cube and W_2 is the region outside the cube and inside the sphere S, we find the integrals over W_1 and W_2 separately:

$$\int_{W_1} \operatorname{div} F\, dV = 3 \cdot \text{Volume } W_1 = 3 \cdot 4^3 = 192.$$

$$\int_{W_2} \operatorname{div} F\, dV = 5 \cdot \text{Volume } W_2 = 5 \cdot \left(\frac{4}{3}\pi(10)^3 - 4^3\right) = \frac{20{,}000}{3}\pi - 320.$$

Thus,

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} F\, dV = 192 + \frac{20{,}000}{3}\pi - 320 = \frac{20{,}000}{3}\pi - 128.$$

55. (a) The disk S_2 is in the plane $x = 2$. It is oriented away from the origin because S is closed and oriented outward. See Figure 19.27.

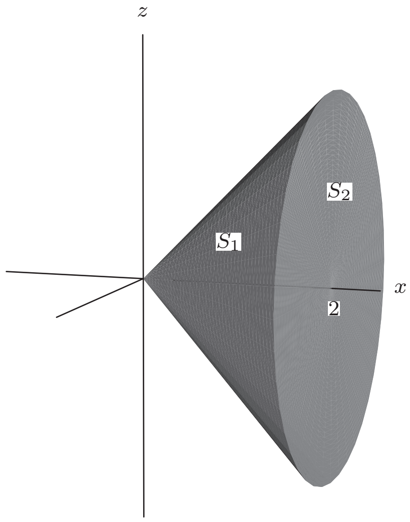


Figure 19.27: The cone $x = \sqrt{y^2+z^2}$ for $0 \le x \le 2$

(b) (i) On S_2, we have $d\vec{A} = \vec{i}\,dA$, so only the $\vec{i}$ component of $\vec{F}$ contributes to the flux through S_2. On S_2, we have $x = 2$ so

$$\int_{S_2} \vec{F}\cdot d\vec{A} = \int_{S_2} 3\cdot 2\vec{i}\cdot\vec{i}\,dA = 6\cdot \text{Area of } S_2 = 6(\pi 2^2) = 24\pi.$$

(ii) Since it is difficult to calculate the flux through S_1 directly, we use the Divergence Theorem, with

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(5z) = 3+4+5 = 12.$$

Let W be the interior of the cone. Then by the Divergence Theorem,

$$\int_S \vec{F}\cdot d\vec{A} = \int_{S_1}\vec{F}\cdot d\vec{A} + \int_{S_2}\vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV,$$

so

$$\int_S \vec{F}\cdot d\vec{A} = \int_W 12\,dV = 12\cdot\text{Volume of cone } = 12\cdot\left(\frac{1}{3}\pi 2^2\cdot 2\right) = 32\pi.$$

Thus

$$\int_{S_1}\vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV - \int_{S_2}\vec{F}\cdot d\vec{A} = 32\pi - 24\pi = 8\pi.$$

56. (a) On the disk, only the $\vec{i}$-component, $5\vec{i}$, contributes to the integral, so

$$\int_{S_1}\vec{r}\cdot d\vec{A} = \int_{S_1} 5\vec{i}\cdot\vec{i}\,dA = 5\cdot\text{ Area of disk } = 5\pi(\sqrt{7})^2 = 35\pi.$$

(b) Using the Divergence Theorem, with W the region within the cylinder and $\operatorname{div}\vec{r} = 3$,

$$\int_{S_2}\vec{r}\cdot d\vec{A} = \int_W 3\,dV = 3\cdot\text{ Volume of cylinder } = 3\cdot\pi(\sqrt{7})^2\cdot 5 = 105\pi.$$

(c) The closed cylinder S_2 has sides consisting of S_1 and S_3, and also a disk, S_4, in the yz-plane $y^2+z^2 \le \sqrt{7}$, $x = 0$. Since the $\vec{i}$ component is 0 on S_4, the flux through S_4 is 0. Thus,

$$\int_{S_2}\vec{r}\cdot d\vec{A} = \int_{S_1}\vec{r}\cdot d\vec{A} + \int_{S_3}\vec{r}\cdot d\vec{A} + \int_{S_4}\vec{r}\cdot d\vec{A}.$$

$$105\pi = 35\pi + \int_{S_3}\vec{r}\cdot d\vec{A} + 0,$$

so

$$\int_{S_3}\vec{r}\cdot d\vec{A} = 105\pi - 35\pi = 70\pi.$$

57. We cannot find $\int_S \vec{F}\cdot d\vec{A}$ directly as we do not know f_1 and f_2. We close the surface by adding D, the disk $x^2+y^2\le 9$ in the xy-plane, oriented upward. If W is the solid region enclosed, the Divergence Theorem tells us

$$\int_{S+D}\vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV = \int_W 5\,dV = 5\cdot \text{ Volume of } W = 5\cdot\frac{1}{2}\cdot\frac{4}{3}\pi 3^3 = 90\pi.$$

Thus,

$$\int_S \vec{F}\cdot d\vec{A} + \int_D \vec{F}\cdot d\vec{A} = 90\pi.$$

Only the $\vec{k}$ component of $\vec{F}$ contributes to the flux through D, where $d\vec{A} = \vec{k}\,dA$, so

$$\int_D \vec{F}\cdot d\vec{A} = \int_D \vec{k}\cdot\vec{k}\,dA = \text{ Area of } D = \pi 3^2 = 9\pi.$$

Thus,

$$\int_S \vec{F}\cdot d\vec{A} = 90\pi - \int_D \vec{F}\cdot d\vec{A} = 90\pi - 9\pi = 81\pi.$$

58. (a) Since $\vec{F} = \dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$,

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}\left(x(x^2+y^2+z^2)^{-3/2}\right)\\ &= 1(x^2+y^2+z^2)^{-3/2} - \frac{3}{2}x(x^2+y^2+z^2)^{-5/2}\cdot 2x\\ &= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

Similar calculations for $\partial F_2/\partial y$ and $\partial F_3/\partial z$ show that

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = \frac{3(x^2+y^2+z^2)-3x^2-3y^2-3z^2}{(x^2+y^2+z^2)^{5/2}} = 0.$$

Thus $\operatorname{div}\vec{F}$ is undefined at the origin, but it is zero everywhere else.

(b) The vector field $\vec{F}$ is everywhere perpendicular to the surface of the sphere, and on the sphere we have

$$\|\vec{F}\| = \frac{10}{10^3} = \frac{1}{10^2}.$$

Thus

$$\int_S \vec{F}\cdot d\vec{A} = \frac{1}{10^2}\cdot\text{Area of sphere} = \frac{1}{10^2}\cdot 4\pi 10^2 = 4\pi.$$

(c) Since $\operatorname{div}\vec{F} = 0$ throughout B_1, by the Divergence Theorem

$$\int_{B_1}\vec{F}\cdot d\vec{A} = \int_{\text{Interior of } B_1}\operatorname{div}\vec{F}\,dV = 0.$$

(d) Since $\operatorname{div}\vec{F}$ is not defined at the origin, we cannot use the Divergence Theorem in the same way as we did in part (c). However, we can apply the Divergence Theorem to W, the space between B_2 and S, with B_2 oriented inward:

$$\int_S \vec{F}\cdot d\vec{A} + \int_{B_2(\text{inward})}\vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\,dV = 0,$$

Thus

$$\int_{B_2(\text{outward})}\vec{F}\cdot d\vec{A} = -\int_{B_2(\text{inward})}\vec{F}\cdot d\vec{A} = \int_S \vec{F}\cdot d\vec{A} = 4\pi.$$

(e) If the origin is outside the surface, the flux through the surface is 0, by the Divergence Theorem. If the origin is inside the surface, then we apply the Divergence Theorem to the region between the surface and a sphere. The sphere is taken large enough to contain the surface entirely within it, or small enough to lie entirely within the surface. We find that the flux is 4π. The only case we do not consider is when the origin actually lies on the surface.

59. We can rewrite $\vec{F}$ in terms of (x, y, z) as

$$\vec{F} = \frac{-Gmx}{\sqrt{(x^2+y^2+z^2)^3}}\vec{i} + \frac{-Gmy}{\sqrt{(x^2+y^2+z^2)^3}}\vec{j} + \frac{-Gmz}{\sqrt{(x^2+y^2+z^2)^3}}\vec{k}$$

Now we find the divergence of $\vec{F}$ in the usual manner

$$\begin{aligned}\operatorname{div}\vec{F} &= \frac{\partial}{\partial x}\frac{-Gmx}{\sqrt{(x^2+y^2+z^2)^3}} + \frac{\partial}{\partial y}\frac{-Gmy}{\sqrt{(x^2+y^2+z^2)^3}} + \frac{\partial}{\partial z}\frac{-Gmz}{\sqrt{(x^2+y^2+z^2)^3}}\\ &= -3Gm\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3Gm}{(x^2+y^2+z^2)^{3/2}}\\ &= 0.\end{aligned}$$

Thus, $\operatorname{div}\vec{F} = 0$ for all points except the origin. We consider a region enclosed by two concentric spheres. Since the divergence of the field is zero at all points except the origin, the volume enclosed contains only points with zero divergence. Consequently, the flux through the surface of the enclosed volume must be zero. Since the field is always inward pointing, this is equivalent to saying that the flux into the outer sphere must equal the flux out of the inner sphere, and so we see that for any two spheres, the flux must be equal, which shows that the flux is independent of the radius of the spheres.

60. Since the divergence is zero at all points not containing the charge, the flux must be zero through any closed surface containing no charge. We imagine a surface composed of two concentric cylinders and their end-caps, where the axis of both cylinders is the z-axis. Then, since no charge is contained in the region enclosed, the flux through the surface must be zero. Now, we know that the field points away from the axis, which means it is parallel to the end-caps. Consequently, there must be no flux through the end-caps. This implies that the flux through the inner cylinder must equal the flux out of the outer cylinder. Since the strength of the field only depends upon the distance from the z axis, the flux through each cylinder is a constant. This implies that the following equation must hold

$$\text{Flux through each cylinder} = E_a 2\pi r_a L = E_b 2\pi r_b L$$

where E_a and E_b are the strengths of the field at r_a and r_b respectively, and L is the length of the cylinders. Dividing through, we can arrive at the following relationship:

$$E_a/E_b = r_b/r_a$$

If we take E_b to be a constant at a fixed value of r_b, then the equation can be simplified to

$$E_a = k/r_a$$

where $k = E_b r_b$. Thus we see that the strength of the field is proportional to $1/r$.

61. (a) If we examine the equation for $\vec{v}$, we see that when $r = 0$, that is, at the center of the pipe, $\vec{v}(0)$ becomes $u\vec{i}$. So u is the speed at the center of the pipe; it is also the maximum speed since $u(1 - r^2/a^2)$ reaches its maximum at $r = 0$.

(b) The flow rate at the wall of the pipe (where $r = a$) is

$$\vec{v}(a) = u(1 - a^2/a^2)\vec{i} = \vec{0}.$$

(c) To find the flux through a circular cross-sectional area, we use polar coordinates in the plane perpendicular to the velocity. In these coordinates, an infinitesimal area, $d\vec{A}$ becomes $r\,dr\,d\theta\vec{i}$. So the flux is given by

$$\begin{aligned}\text{Flux} &= \int_S \vec{v}\cdot d\vec{A} = \int_S u(1-r^2/a^2)\vec{i}\cdot r\,dr\,d\theta\vec{i} = \int_0^{2\pi}\int_0^a u(1-r^2/a^2)r\,dr\,d\theta\\ &= 2\pi u\int_0^a (r - \frac{r^3}{a^2})\,dr = 2\pi u\left(\frac{a^2}{2} - \frac{a^2}{4}\right) = \frac{\pi u a^2}{2}.\end{aligned}$$

62. (a) (i) The integral $\int_W \rho\,dV$ represents the total charge in the volume W.

(ii) The integral $\int_S \vec{J}\cdot d\vec{A}$ represents the total current flowing out of the surface S.

(b) The total current flowing out of the surface S is the rate at which the total charge inside the surface S (i.e., in the volume W) is decreasing. In other words,

$$\text{Rate current flowing out of } S = -\frac{\partial}{\partial t}(\text{charge in } W),$$

so

$$\int_S \vec{J}\cdot d\vec{A} = -\frac{\partial}{\partial t}\left(\int_W \rho\,dV\right).$$

63. (a) Since $\vec{v} = \operatorname{grad}\phi$ we have

$$\vec{v} = \left(1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right)\vec{i} + \frac{-2xy}{(x^2+y^2)^2}\vec{j}$$

(b) Differentiating the components of $\vec{v}$, we have

$$\operatorname{div}\vec{v} = \frac{\partial}{\partial x}\left(1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right) + \frac{\partial}{\partial y}\left(\frac{-2xy}{(x^2+y^2)^2}\right) = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3} + \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} = 0$$

(c) The vector $\vec{v}$ is tangent to the circle $x^2+y^2 = 1$, if and only if the dot product of the field on the circle with any radius vector of that circle is zero. Let (x, y) be a point on the circle. We want to show: $\vec{v}\cdot\vec{r} = \vec{v}(x,y)\cdot(x\vec{i} + y\vec{j}) = 0$. We have:

$$\begin{aligned}
\vec{v}(x,y)\cdot(x\vec{i} + y\vec{j}) &= \left(\left(1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right)\vec{i} + \frac{-2xy}{(x^2+y^2)^2}\vec{j}\right)\cdot(x\vec{i} + y\vec{j}) \\
&= x + x\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{2xy^2}{(x^2+y^2)^2} \\
&= \frac{x(x^2+y^2-1)}{x^2+y^2},
\end{aligned}$$

but we know that for any point on the circle, $x^2 + y^2 = 1$, thus we have $\vec{v}\cdot\vec{r} = 0$. Therefore, the velocity field is tangent to the circle. Consequently, there is no flow through the circle and any water on the outside of the circle must flow around it.

(d) See Figure 19.28.

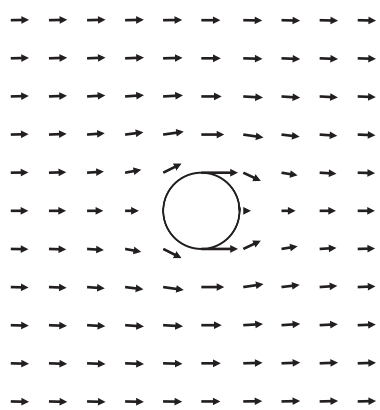


Figure 19.28

64. (a) The distance from any point to the origin is given by $\sqrt{x^2+y^2+z^2}$, so the denominator is simply r^3. The field can then be rewritten in components as $(\frac{kx}{r^3}\vec{i} + \frac{ky}{r^3}\vec{j} + \frac{kz}{r^3}\vec{k})$. Its magnitude is thus:

$$\|\vec{v}\| = \sqrt{K^2\frac{x^2+y^2+z^2}{r^6}} = \sqrt{\frac{K^2}{r^4}} = \frac{K}{r^2}$$

which is only a function of the distance from the origin. It is clear that the vector field points away from the origin for all points (x, y, z), because it is the radius vector $x\vec{i} + y\vec{j} + z\vec{k}$, multiplied by the positive scalar K/r^3. Suppose $(x, y, z) \neq (0, 0, 0)$, then

$$\begin{aligned}
\operatorname{div}\vec{v}(x,y,z) &= \frac{\partial}{\partial x}\left(\frac{Kx}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{Ky}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{Kz}{r^3}\right) \\
&= K\left(\frac{-2x^2+y^2+z^2}{r^5} + \frac{x^2-2y^2+z^2}{r^5} + \frac{x^2+y^2-2z^2}{r^5}\right) = 0.
\end{aligned}$$

Hence, indeed $\vec{v}$ is a point source at the origin.

(b) The dependence of $\vec{v}$ on r, the distance from the origin, is shown in part (a).

(c) The flux through a sphere centered at the origin is calculated as:

$$\text{Flux} = \int_S \vec{v} \cdot d\vec{A}$$

Since the vector field's magnitude is a function only of the distance from the origin, it will be constant over the surface of a sphere centered at the origin. Furthermore, since it is pointed away from the origin, $\vec{v} \cdot d\vec{A}$ will be simply $\|\vec{v}\| \cdot \|d\vec{A}\|$. Thus

$$\text{Total flux out of a sphere of radius } r = \|\vec{v}_r\| \cdot \|\vec{A}_{\text{sphere}}\| = \frac{K}{r^2}\frac{4}{3}\pi r^2 = \frac{4}{3}\pi K.$$

So the flux does not even depend upon r since the rate at which the area of the sphere is increasing is exactly equal to the rate at which the magnitude of the field is decreasing.

(d) This is best handled by observing (as in part (a)) that the divergence of the vector field at any point besides the origin is zero. Since the divergence anywhere but the origin is zero, the net flux through any closed surface not enclosing the origin must also be zero.

CAS Challenge Problems

65. (a) When $x > 0$, the vector $x\vec{i}$ points in the positive x-direction, and when $x < 0$ it points in the negative x-direction. Thus it always points from the inside of the ellipsoid to the outside, so we expect the flux integral to be positive. The upper half of the ellipsoid is the graph of $z = f(x, y) = \frac{1}{\sqrt{2}}(1 - x^2 - y^2)$, so the flux integral is

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} x\vec{i} \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (-xf_x)\,dxdy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{x^2}{\sqrt{1-x^2-y^2}}\,dxdy \\
&= \frac{-\sqrt{2} + 11\,\arcsin(\frac{1}{\sqrt{3}}) + 10\,\arctan(\frac{1}{\sqrt{2}}) - 8\,\arctan(\frac{5}{\sqrt{2}})}{12} = 0.0958.
\end{aligned}$$

Different CASs may give the answer in different forms. Note that we could have predicted the integral was positive without evaluating it, since the integrand is positive everywhere in the region of integration.

(b) For $x > -1$, the quantity $x + 1$ is positive, so the vector field $(x + 1)\vec{i}$ always points in the direction of the positive x-axis. It is pointing into the ellipsoid when $x < 0$ and out of it when $x > 0$. However, its magnitude is smaller when $-1/2 < x < 0$ than it is when $0 < x < 1/2$, so the net flux out of the ellipsoid should be positive. The flux integral is

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (x+1)\vec{i} \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} -(x+1)f_x\,dxdy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{x(1+x)}{\sqrt{1-x^2-y^2}}\,dxdy \\
&= \frac{\sqrt{2} - 11\,\arcsin(\frac{1}{\sqrt{3}}) - 10\,\arctan(\frac{1}{\sqrt{2}}) + 8\,\arctan(\frac{5}{\sqrt{2}})}{12} = 0.0958
\end{aligned}$$

The answer is the same as in part (a). This makes sense because the difference between the integrals in parts (a) and (b) is the integral of $\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(x/\sqrt{1-x^2-y^2})\,dxdy$, which is zero because the integrand is odd with respect to x.

(c) This integral should be positive for the same reason as in part (a). The vector field $y\vec{j}$ points in the positive y-direction when $y > 0$ and in the negative y-direction when $y < 0$, thus it always points out of the ellipsoid. Evaluating the integral we get

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} y\vec{j} \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dxdy \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (-yf_y)\,dxdy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{y^2}{\sqrt{1-x^2-y^2}}\,dxdy
\end{aligned}$$

$$= \frac{\sqrt{2} - 2\ \arcsin(\frac{1}{\sqrt{3}}) - 19\ \arctan(\frac{1}{\sqrt{2}}) + 8\ \arctan(\frac{5}{\sqrt{2}})}{12} = 0.0958.$$

The symbolic answer appears different but has the same numerical value as in parts (a) and (b). In fact the answer is the same because the integral here is the same as in part (a) except that the roles of x and y have been exchanged. Different CASs may give different symbolic forms.

66. (a) The surface has a shape of a flower or trumpet opening in the direction of the positive y-axis. See Figure 19.29. The outer rim is a circle of radius 4, so the surface lies above $z = -4$. Thus $z + 4 > 0$ on the surface, so the vector field $(z + 4)\vec{k}$ points in the positive z direction everywhere on the surface. Thus it crosses the surface in the opposite direction as the orientation when it is below the xy-plane, and in the same direction when it is above the xy-plane. Also, it has smaller magnitude when $-2 \le z \le 2$ than it does when $0 \le z \le 2$, so we expect the negative contribution to the flux integral to be smaller than the positive contribution, so the flux integral should be positive.

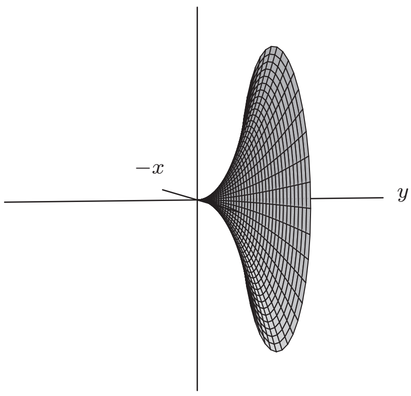


Figure 19.29

(b) The area vector element is

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = s^2 \cos t \vec{i} - (2s^3 \cos(t)^2 + 2s^3 \sin^2 t)\vec{j} + s^2 \sin(t)\vec{k}\ .$$

This points in the direction of the negative y-axis, as required for computing the flux integral. The flux integral is

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_0^2 \vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) dsdt$$
$$= \int_0^{2\pi} \int_0^2 (s^2 \sin t + 2)s^2 \sin t\, dsdt = \frac{32\pi}{5}$$

PROJECTS FOR CHAPTER NINETEEN

1. (a) Taking partial derivatives using the product rule, we have

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$$
$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{x \cdot 2x}{(x^2+y^2+z^2)^{5/2}}$$
$$= \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly,

$$\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$$

and
$$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}.$$
Thus,
$$\text{divergence of } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0.$$

(b) Let S be the closed surface, oriented outward, consisting of three pieces
- S_1
- $-S_2$
- S_3, the surface connecting the boundaries of S_1 and S_2, consisting of line segments on lines through the origin.

By the Divergence Theorem and part (a),
$$\int_S \vec{F} \cdot d\vec{A} = 0.$$
Hence
$$\int_{S_1} \vec{F} \cdot d\vec{A} - \int_{S_2} \vec{F} \cdot d\vec{A} + \int_{S_3} \vec{F} \cdot d\vec{A} = 0.$$

The key observation is that the vector field $\vec{F}$ is tangent to the lateral surface S_3. Therefore, $\int_{S_3} \vec{F} \cdot d\vec{A} = 0$ and it follows that
$$\int_{S_1} \vec{F} \cdot d\vec{A} = \int_{S_2} \vec{F} \cdot d\vec{A}.$$

(c) Compute the integral over S_2, which is on the unit sphere. Notice that $\vec{F}$ is perpendicular to the sphere and that on the sphere, $||\vec{F}|| = 1$. Thus
$$\int_{S_1} \vec{F} \cdot d\vec{A} = \int_{S_2} \vec{F} \cdot d\vec{A} = ||\vec{F}|| \cdot \text{Area of } S_2 = \text{Area of} S_2.$$

2. (a) Since $\vec{e}_\rho$ is a unit vector pointing radially away from the origin, $\vec{e}_\rho = \vec{r}/\|\vec{r}\| = \vec{r}/\rho$. Thus, we have
$$\vec{F} = \frac{f(\rho)}{\rho}\vec{r} = \frac{f(\rho)}{\rho}x\vec{i} + \frac{f(\rho)}{\rho}y\vec{j} + \frac{f(\rho)}{\rho}z\vec{k}.$$

Let $g(\rho) = f(\rho)/\rho$. Since $\rho = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$, and $\partial\rho/\partial x = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x = x/\sqrt{x^2+y^2+z^2}$, using the chain rule, we have
$$\begin{aligned}
\frac{\partial}{\partial x}g(\rho) &= \frac{d}{d\rho}g(\rho) \cdot \frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}g'(\rho)\\
\frac{\partial}{\partial y}g(\rho) &= \frac{y}{\sqrt{x^2+y^2+z^2}}g'(\rho)\\
\frac{\partial}{\partial z}g(\rho) &= \frac{z}{\sqrt{x^2+y^2+z^2}}g'(\rho).
\end{aligned}$$

So
$$\begin{aligned}
\operatorname{div}\vec{F} &= \frac{\partial}{\partial x}(g(\rho)x) + \frac{\partial}{\partial y}(g(\rho)y) + \frac{\partial}{\partial z}(g(\rho)z)\\
&= \left(x\frac{\partial}{\partial x}g(\rho) + g(\rho)\right) + \left(y\frac{\partial}{\partial y}g(\rho) + g(\rho)\right) + \left(z\frac{\partial}{\partial z}g(\rho) + g(\rho)\right)
\end{aligned}$$

$$= \frac{x^2}{\sqrt{x^2+y^2+z^2}}g'(\rho) + \frac{y^2}{\sqrt{x^2+y^2+z^2}}g'(\rho) + \frac{z^2}{\sqrt{x^2+y^2+z^2}}g'(\rho) + 3g(\rho)$$
$$= \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}g'(\rho) + 3g(\rho) = \frac{\rho^2}{\rho}g'(\rho) + 3g(\rho) = \rho g'(\rho) + 3g(\rho)$$
$$= \rho\left(\frac{f'(\rho)}{\rho} - \frac{f(\rho)}{\rho^2}\right) + \frac{3f(\rho)}{\rho} = f'(\rho) + 2\frac{f(\rho)}{\rho}.$$

On the other hand,

$$\frac{1}{\rho^2}\frac{d}{d\rho}(\rho^2 f(\rho)) = \frac{1}{\rho^2}(2\rho f(\rho) + \rho^2 f'(\rho)) = 2\frac{f(\rho)}{\rho} + f'(\rho).$$

Thus,

$$\operatorname{div}\vec{F} = \frac{1}{\rho^2}(\rho^2 f(p)).$$

(b) Applying the formula from part (a), we have

$$\operatorname{div}\vec{F} = \frac{1}{\rho^2}\frac{d}{d\rho}(\rho^2 f(\rho)) = 0.$$

Since $\rho \neq 0$, multiplying by ρ^2 gives us $(d/d\rho)(\rho^2 f(\rho)) = 0$, hence $\rho^2 f(\rho)$ is a constant, say k. So $f(\rho) = k/\rho^2$, and hence

$$\vec{F} = \frac{k}{\rho^2}\vec{e}_\rho.$$

(c) Suppose the sphere, S, has radius R. Since $\vec{F}$ has constant magnitude on S and points perpendicularly to S, we have $\vec{F}\cdot d\vec{A} = f(R)\,dA$, so the flux of $\vec{F}$ out of S is $\int_S \vec{F}\cdot d\vec{A} = \int_S f(R)\,dA = f(R)\int_S dA = f(R)\cdot$ Area of $S = 4\pi R^2 f(R)$. On the other hand, using spherical coordinates, part (a) tells us that:

$$\operatorname{div}\vec{F} = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 f(\rho)\right),$$

so if W is the region enclosed by S then

$$\int_W \operatorname{div}\vec{F}\,dV = \int_0^R\int_0^\pi\int_0^{2\pi}\frac{1}{\rho^2}\frac{d}{d\rho}(\rho^2 f(\rho))\,\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$$
$$= 4\pi\int_0^R \frac{d}{d\rho}\left(\rho^2 f(\rho)\right)\,d\rho = 4\pi\rho^2 f(\rho)\Big|_0^R = 4\pi R^2 f(R).$$

The fact that $\int_W \operatorname{div}\vec{F}\,dV$ equals the flux integral, $\int_S \vec{F}\cdot d\vec{A}$, confirms the Divergence Theorem.

(d) Since we are assuming $\vec{E}$ is spherically symmetric, we can write

$$\vec{E} = E(\rho)\vec{e}_\rho.$$

By part (a) and Gauss's Law,

$$\operatorname{div}\vec{E} = \frac{1}{\rho^2}\frac{d}{d\rho}(\rho^2 E(\rho)) = \begin{cases}\delta_0 & \rho \le a\\ 0 & \rho > a\end{cases}$$

Suppose $\rho \le a$. Then, since $\rho \neq 0$,

$$\frac{d}{d\rho}(\rho^2 E(\rho)) = \delta_0\rho^2.$$

Therefore

$$\rho^2 E(\rho) = \int \delta_0 \rho^2 d\rho = \frac{\delta_0 \rho^3}{3} + C,$$

and so

$$E(\rho) = \frac{\delta_0 \rho}{3} + \frac{C}{\rho^2}.$$

Since $\vec{E}$, and hence E, is assumed to be continuous, we must have $C = 0$, so $E(\rho) = \delta_0 \rho/3$.

Now suppose that $\rho > a$. Then

$$\frac{d}{d\rho}(\rho^2 E(\rho)) = 0,$$

therefore

$$\rho^2 E(\rho) = k,$$

and so

$$E(\rho) = \frac{k}{\rho^2}.$$

Since E is assumed to be continuous,

$$\lim_{\rho \to a^+} E(\rho) = \lim_{\rho \to a^-} E(\rho).$$

Now,

$$\lim_{\rho \to a^+} E(\rho) = \frac{k}{a^2}$$
$$\lim_{\rho \to a^-} E(\rho) = \frac{\delta_0 a}{3}$$

so $k = \delta_0 a^3/3$. Thus

$$E(\rho) = \begin{cases} \dfrac{\delta_0 \rho}{3} & \rho \le a \\ \dfrac{\delta_0 a^3}{3} & \rho > a \end{cases}$$

3. (a) (i) Since the direction of the electric field is perpendicular to the surface of any cylinder with the wire as an axis, it is parallel to the surfaces of the two washers. Consequently, there is no flux through the washers.

(ii) Gauss's Law tells us that the total flux through the surface must be zero, since no charge is contained within it. (Note that the region within the surface S lies between the cylinders.) Since the flux through the washers is zero, the flux into the inner surface must equal the flux out of the outer surface in order for the net flux through the surface to be zero.

(iii) Since the surface area of a cylinder is given by $A = 2\pi RL$ where R is the radius of the cylinder and L is its length, and we know that $E_a A_a = E_b A_b$ (since the fluxes are equal), we have

$$E_b(2\pi bL) = E_a(2\pi aL)$$
$$\frac{E_b}{E_a} = \frac{2\pi aL}{2\pi bL}$$
$$\frac{E_b}{E_a} = \frac{a}{b}.$$

(iv) The equations in part (iii) imply that

$$aE_a = bE_b.$$

Since a, b are arbitrary radii we can say:

$$rE_r = \text{Constant}$$
$$E_r = \text{Constant}\left(\frac{1}{r}\right),$$

for any radius r. This statement tells us that the strength of the electric field at r is proportional to $1/r$.

(b) Since the electric field points perpendicular to the sheet, it is parallel to all sides of the box, except for the two sides parallel to the sheet. Additionally, since there is no charge contained in the box, Gauss's Law tells us the net flux through the surface of the box must be zero. This implies that the flux into the near face must equal the flux out of the far face. Since the faces have the same area, the field must have equal strengths at the two faces in order for their fluxes to be equal. Since we did not use the values of a or b, we see that for all points in space on the same side of the sheet, the field has the same magnitude.

4. (a) (i) In cylindrical coordinates, the position vector of a point (R, θ, z) on the cylinder is given by

$$\vec{r} = R\cos\theta\vec{i} + R\sin\theta\vec{j} + z\vec{k}.$$

So $\|\vec{r}\| = \sqrt{R^2 + z^2}$. For an area element on the cylinder we have

$$d\vec{A} = (\cos\theta\vec{i} + \sin\theta\vec{j})R\,dz\,d\theta,$$

so the flux integral is:

$$\begin{aligned}\int_S \vec{E}\cdot d\vec{A} &= \int_0^{2\pi}\int_{-H}^{H} q\frac{\vec{r}}{\|\vec{r}\|^3}\cdot(\cos\theta\vec{i} + \sin\theta\vec{j})R\,dz\,d\theta \\ &= q\int_0^{2\pi}\int_{-H}^{H}\frac{R\cos^2\theta + R\sin^2\theta}{(R^2+z^2)^{3/2}}R\,dz\,d\theta = 2\pi q\int_{-H}^{H}\frac{R^2\,dz}{(R^2+z^2)^{3/2}}.\end{aligned}$$

To compute this one variable integral, we write:

$$\int_{-H}^{H}\frac{R^2dz}{(R^2+z^2)^{3/2}} = \int_{-H}^{H}\frac{(R^2+z^2)dz}{(R^2+z^2)^{3/2}} - \int_{-H}^{H}\frac{z^2dz}{(R^2+z^2)^{3/2}}$$

We calculate the integral $\displaystyle\int_{-H}^{H}\frac{z^2dz}{(R^2+z^2)^{3/2}} = \int_{-H}^{H} z\frac{zdz}{(R^2+z^2)^{3/2}}$ using integration by parts:

$$\begin{aligned}\int_{-H}^{H}\frac{z^2dz}{(R^2+z^2)^{3/2}} &= \int_{-H}^{H}\frac{dz}{(R^2+z^2)^{1/2}} - \left(\int_{-H}^{H}\frac{dz}{(R^2+z^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}}\Bigg|_{-H}^{H}\right) \\ &= \frac{2H}{\sqrt{R^2+H^2}}.\end{aligned}$$

Note that the integral $\displaystyle\int_{-H}^{H}\frac{dz}{(R^2+z^2)^{1/2}}$ has canceled. Therefore we have

$$\int_S \vec{E}\cdot d\vec{A} = 4\pi q\frac{H}{\sqrt{R^2+H^2}}.$$

(ii) • Let R be fixed. We have

$$\lim_{H\to 0}\int_S \vec{E}\cdot d\vec{A} = \lim_{H\to 0} 4\pi q\frac{H}{\sqrt{H^2+R^2}} = 0.$$
$$\lim_{H\to \infty}\int_S \vec{E}\cdot d\vec{A} = \lim_{H\to \infty} 4\pi q\frac{H}{\sqrt{H^2+R^2}} = 4\pi q.$$

- Now let H be fixed. We have

$$\lim_{R\to 0}\int_S \vec{E}\cdot d\vec{A} = \lim_{R\to 0} 4\pi q\frac{H}{\sqrt{H^2+R^2}} = 4\pi q.$$
$$\lim_{R\to \infty}\int_S \vec{E}\cdot d\vec{A} = \lim_{R\to \infty} 4\pi q\frac{H}{\sqrt{H^2+R^2}} = 0.$$

Each of these results is as we would expect from Gauss's Law.

(b) Let S denote the side of the cylinder. We have

$$\int_T \vec{E}\cdot d\vec{A} = \int_S \vec{E}\cdot d\vec{A} + \int_{\text{Top}} \vec{E}\cdot d\vec{A} + \int_{\text{Bottom}} \vec{E}\cdot d\vec{A}.$$

The result of part (a) shows that

$$\int_S \vec{E}\cdot d\vec{A} = 4\pi q\frac{H}{\sqrt{R^2+H^2}}.$$

Let's compute $\int_{\text{Top}} \vec{E}\cdot d\vec{A}$. The normal at any point on the top is $\vec{n} = \vec{k}$ and so $d\vec{A} = \vec{k}\, r\, dr d\theta$. On the top, $z = H$ so

$$\vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j} + H\vec{k}.$$

Therefore $\|\vec{r}\| = \sqrt{r^2+H^2}$ and we have

$$\begin{aligned}\int_{\text{Top}} \vec{E}\cdot d\vec{A} &= \int_0^{2\pi}\int_0^R q\frac{\vec{r}}{\|\vec{r}\|^3}\cdot\vec{k}\, r\, dr\, d\theta\\ &= q\int_0^{2\pi}\int_0^R \frac{(r\cos\theta\vec{i} + r\sin\theta\vec{j} + H\vec{k})\cdot\vec{k}}{(r^2+H^2)^{3/2}} r\, dr\, d\theta\\ &= qH\int_0^{2\pi}\int_0^R \frac{r\, dr\, d\theta}{(r^2+H^2)^{3/2}} = -2\pi qH\frac{1}{\sqrt{r^2+H^2}}\bigg|_0^R\\ &= -2\pi q\frac{H}{\sqrt{R^2+H^2}} + 2\pi q.\end{aligned}$$

Similarly, or using a symmetry argument, we find that the flux through the bottom is given by

$$\int_{\text{Bottom}} \vec{E}\cdot d\vec{A} = -2\pi q\frac{H}{\sqrt{R^2+H^2}} + 2\pi q.$$

Thus, the total flux is given by

$$\begin{aligned}\int_S \vec{E}\cdot d\vec{A} &= 4\pi q\frac{H}{\sqrt{R^2+H^2}} - 2\pi q\frac{H}{\sqrt{R^2+H^2}} + 2\pi q - 2\pi q\frac{H}{\sqrt{R^2+H^2}} + 2\pi q\\ &= 4\pi q.\end{aligned}$$

CHAPTER TWENTY

Solutions for Section 20.1

Exercises

1. Vector. We have

$$\operatorname{curl}(z\vec{i} - x\vec{j} + y\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -x & y \end{vmatrix} = \vec{i} + \vec{j} - \vec{k}.$$

2. Vector. We have

$$\operatorname{curl}(-2z\vec{i} - z\vec{j} + xy\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2z & -z & xy \end{vmatrix} = (x+1)\vec{i} - (y+2)\vec{j}.$$

3. We have

$$\operatorname{curl}(3x\vec{i} - 5z\vec{j} + y\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x & -5z & y \end{vmatrix} = 6\vec{i}.$$

4. Using the definition in Cartesian coordinates, we have

$$\begin{aligned} \operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right)\vec{i} + \left(-\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}(x^2-y^2)\right)\vec{j} + \left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right)\vec{k} \\ &= 4y\vec{k}. \end{aligned}$$

5. Using the definition in Cartesian coordinates,

$$\begin{aligned} \operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (-x+y) & (y+z) & (-z+x) \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(-z+x) - \frac{\partial}{\partial z}(y+z)\right)\vec{i} + \left(-\frac{\partial}{\partial x}(-z+x) + \frac{\partial}{\partial z}(-x+y)\right)\vec{j} \\ &\quad + \left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial y}(-x+y)\right)\vec{k} \\ &= -\vec{i} - \vec{j} - \vec{k}. \end{aligned}$$

6. Using the definition in Cartesian coordinates,

$$\begin{aligned} \operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & 3xz & 7xy \end{vmatrix} \\ &= (7x-3x)\vec{i} - (7y-2y)\vec{j} + (3z-2z)\vec{k} \\ &= 4x\vec{i} - 5y\vec{j} + z\vec{k}. \end{aligned}$$

7. Using the definition in Cartesian coordinates,

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^3 & z^4 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(z^4) - \frac{\partial}{\partial z}(y^3)\right)\vec{i} + \left(-\frac{\partial}{\partial x}(z^4) + \frac{\partial}{\partial z}(x^2)\right)\vec{j} + \left(\frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial y}(x^2)\right)\vec{k} \\ &= \vec{0}\,.\end{aligned}$$

8. Using the definition in Cartesian coordinates,

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & \cos y & e^{z^2} \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(e^{z^2}) - \frac{\partial}{\partial z}(\cos y)\right)\vec{i} + \left(-\frac{\partial}{\partial x}(e^{z^2}) + \frac{\partial}{\partial z}(e^x)\right)\vec{j} + \left(\frac{\partial}{\partial x}(\cos y) - \frac{\partial}{\partial y}(e^x)\right)\vec{k} \\ &= \vec{0}\,.\end{aligned}$$

9. Using the definition in Cartesian coordinates

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+yz & y^2+xzy & zx^3y^2+x^7y^6 \end{vmatrix} \\ &= (2x^3yz + 6x^7y^5 - xy)\vec{i} + (-3x^2y^2z - 7x^6y^6 + y)\vec{j} + (yz - z)\vec{k}\end{aligned}$$

10. This vector field points radically outward and has unit length everywhere (except the origin). Thus, we would expect its curl to be $\vec{0}$. Computing the curl directly we get

$$\operatorname{curl}\left(\frac{\vec{r}}{\|\vec{r}\|}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{1/2}} & \frac{y}{(x^2+y^2+z^2)^{1/2}} & \frac{z}{(x^2+y^2+z^2)^{1/2}} \end{vmatrix}$$

$$\begin{aligned}\text{The } \vec{i}\text{-component is given by } &= \left(-\frac{1}{2}\cdot\frac{2yz}{(x^2+y^2+z^2)^{3/2}} - \left(-\frac{1}{2}\cdot\frac{2yz}{(x^2+y^2+z^2)^{1/2}}\right)\right)\vec{i} \\ &= 0\end{aligned}$$

Similarly, the $\vec{j}$ and $\vec{k}$ components are also both 0.

11. The circulation of the vector field around the boundary of any square centered at the origin with sides parallel to the axes is positive because the line integrals on the top and bottom sides are positive and the line integrals on the left and right sides are zero. Therefore, we suspect a nonzero curl.

12. The circulation around any square centered on the origin with sides parallel to the axes is zero because the line integrals on all four sides are zero, so we suspect a zero curl.

13. The circulation around any rectangle centered on the origin with sides parallel to the axes is zero because the line integrals on the top and bottom sides add to zero and the line integrals on the left and right sides are zero. We suspect that the vector field has zero curl.

14. The vector field is swirling counterclockwise, so the circulation around any circle centered at the origin is positive. Thus we suspect a nonzero curl.

Problems

15. (a) The vector field always points perpendicularly to both $\vec{k}$ and $\vec{r}$, in the direction determined by the right hand rule, and its magnitude is twice the magnitude of $\vec{r}$. Thus

$$\vec{F}(\vec{r}) = 2\vec{k} \times \vec{r} = -2y\vec{i} + 2x\vec{j}.$$

(b) Using the definition in Cartesian coordinates

$$\begin{aligned}\text{curl}\,\vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & 2x & 0 \end{vmatrix} \\ &= (\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2x))\vec{i} + (\frac{\partial}{\partial z}(-2y) - \frac{\partial}{\partial x}(0))\vec{j} + (\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(-2y))\vec{k} = 4\vec{k}.\end{aligned}$$

This makes sense, because we computed the circulation density of this vector field in the z-direction and found it was 4, and we would expect the z-direction to give the maximum circulation density from the symmetry of the vector field.

16. The part of this vector field in the xy-plane looks like Figure 19.29 on page 1025, and shows no rotational tendency. Thus we expect the curl to be $\vec{0}$. In fact it is, because the circulation around *every* closed curve C is zero, since

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k} = \text{grad}(x^2/2 + y^2/2 + z^2/2),$$

so $\vec{F}$ is a gradient field. Thus the circulation density is zero in any direction, and hence $\text{curl}\,\vec{F}(P) = \vec{0}$ for every point P. Using the formula, we see that

$$\begin{aligned}\text{curl}\,\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\vec{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\vec{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}.\end{aligned}$$

17. The curl is defined in such a way that if $\vec{n}$ is a unit vector and C is a small circle in the plane perpendicular to $\vec{n}$ and with orientation induced by $\vec{n}$, then

$$\begin{aligned}(\text{curl}\,\vec{G})\cdot\vec{n} &= \text{Circulation density} \\ &\approx \frac{\int_C \vec{G}\cdot d\vec{r}}{\text{Area inside } C}\end{aligned}$$

so

$$\text{Circulation} = \int_C \vec{G}\cdot d\vec{r} \approx \left((\text{curl}\,\vec{G})\cdot\vec{n}\right)\cdot \text{Area inside } C.$$

(a) Let C be the circle in the xy-plane, and let $\vec{n} = \vec{k}$. Then

$$\begin{aligned}\text{Circulation} &\approx (2\vec{i} - 3\vec{j} + 5\vec{k})\cdot\vec{k}\cdot\pi(0.01)^2 \\ &= 0.0005\pi.\end{aligned}$$

(b) By a similar argument to part (a), with $\vec{n} = \vec{i}$, we find the circulation around the circle in the yz-plane:

$$\begin{aligned}\text{Circulation} &\approx (2\vec{i} - 3\vec{j} + 5\vec{k})\cdot\vec{i}\cdot\pi(0.01)^2 \\ &= 0.0002\pi.\end{aligned}$$

(c) Similarly for circulations around the circle in the xz-plane,

$$\text{Circulation} \approx -0.0003\pi.$$

18. The vector curl $\vec{F}$ has its component in the x-direction given by

$$(\text{curl } \vec{F})_x \approx \frac{\text{Circulation around small circle around } x\text{-axis}}{\text{Area inside circle}} = \frac{\text{Circulation around } C_2}{\text{Area inside } C_2} = \frac{0.5\pi}{\pi(0.1)^2} = 50.$$

Similar reasoning leads to

$$(\text{curl } \vec{F})_y \approx \frac{\text{Circulation around } C_3}{\text{Area inside } C_3} = \frac{3\pi}{\pi(0.1)^2} = 300,$$
$$(\text{curl } \vec{F})_z \approx \frac{\text{Circulation around } C_1}{\text{Area inside } C_1} = \frac{0.02\pi}{\pi(0.1)^2} = 2.$$

Thus,

$$\text{curl } \vec{F} \approx 50\vec{i} + 300\vec{j} + 2\vec{k}.$$

19. The conjecture is that when the first component of $\vec{F}$ depends only on x, the second component depends only on y, and the third component depends only on z, that is, if

$$\vec{F} = F_1(x)\vec{i} + F_2(y)\vec{j} + F_3(z)\vec{k}$$

then

$$\text{curl } \vec{F} = \vec{0}$$

The reason for this is that if $\vec{F} = F_1(x)\vec{i} + F_2(y)\vec{j} + F_3(z)\vec{k}$, then

$$\begin{aligned}\text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x) & F_2(y) & F_3(z) \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}F_3(z) - \frac{\partial}{\partial z}F_2(y)\right)\vec{i} + \left(-\frac{\partial}{\partial x}F_3(z) + \frac{\partial}{\partial z}F_1(x)\right)\vec{j} + \left(\frac{\partial}{\partial x}F_2(y) - \frac{\partial}{\partial y}F_1(x)\right)\vec{k} \\ &= \vec{0}.\end{aligned}$$

20. (a) We have

$$\text{curl } \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay^3 + be^z & cz + dx^2 & e\sin x + fy \end{vmatrix} = (f-c)\vec{i} + (be^z - e\cos x)\vec{j} + (2dx - 3ay^2)\vec{k}.$$

(b) If curl $\vec{G}$ is parallel to the yz-plane, then it has no $\vec{i}$ component. Thus, $f - c = 0$ or $f = c$.

(c) If curl $\vec{G}$ is parallel to the z-axis, then there are no $\vec{i}$ and $\vec{j}$ components. Thus,

$$f - c = 0 \quad \text{and} \quad be^z - e\cos x = 0.$$

Since the second equation holds for all z and x, we have $b = e = 0$ as well as $f = c$.

21. (a) A thin twig at the origin along the x-axis would only feel the velocity along that axis, and thus go counterclockwise.

(b) Clockwise.

(c) Using the Cartesian coordinate definition, we get

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + (1-1)\vec{k} = \vec{0}.$$

This is as expected, since a paddle-wheel (instead of a twig) placed in the field would not rotate at all.

22. We have

$$\vec{F}(t,x,y,z) = (\cos t\vec{j} + \sin t\vec{k}) \times \vec{r} = (z\cos t - y\sin t)\vec{i} + x\sin t\vec{j} - x\cos t\vec{k},$$

so

$$(\text{curl}\,\vec{F})(t,x,y,z) = 2\cos t\vec{j} + 2\sin t\vec{k}.$$

(a) At $t = 0$ the vector $(\text{curl}\,\vec{F})(0,x,y,z) = 2\vec{j}$ is horizontal, pointing in the y direction.
(b) At $t = \pi/2$ the vector $(\text{curl}\,\vec{F})(\pi/2,x,y,z) = 2\vec{k}$ is vertical, pointing in the z direction.
(c) At $0 < t < \pi/2$ the vector $(\text{curl}\,\vec{F})(t,x,y,z) = 2\cos t\vec{j} + 2\sin t\vec{k}$ is parallel to the yz-plane, making an angle of t radians with the horizontal plane. Thus as t goes from 0 to $\pi/2$, the curl goes steadily from horizontal to vertical.

23. Investigate the velocity vector field of the atmosphere near the fire. If the curl of this vector field is non-zero, there is circulatory motion. Consequently, if the magnitude of the curl of this vector field is large near the fire, a fire storm has probably developed.

24. (a) Figure 20.1 shows a cross-section of the vector field in xy-plane with $\omega = 1$, so $\vec{v} = -y\vec{i} + x\vec{j}$.
Figure 20.2 shows a cross-section of vector field in xy-plane with $\omega = -1$, so $\vec{v} = y\vec{i} - x\vec{j}$.

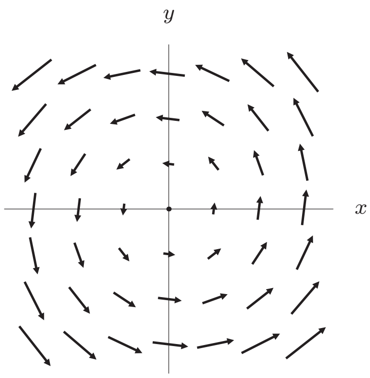


Figure 20.1: $\vec{v} = -y\vec{i} + x\vec{j}$

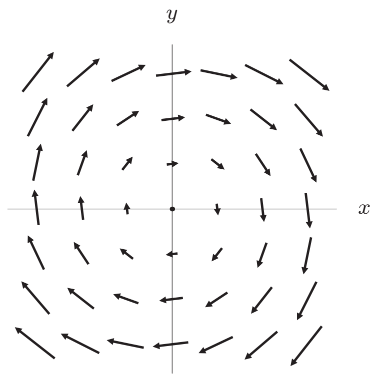


Figure 20.2: $\vec{v} = y\vec{i} - x\vec{j}$

(b) The distance from the center of the vortex is given by $r = \sqrt{x^2+y^2}$. The velocity of the vortex at any point is $-\omega y\vec{i} + \omega x\vec{j}$, and the speed of the vortex at any point is the magnitude of the velocity, or $s = \|\vec{v}\| = \sqrt{(-\omega y)^2 + (\omega x)^2} = |\omega|\sqrt{x^2+y^2} = |\omega|\,r$.
(c) The divergence of the velocity field is given by:

$$\text{div}\,\vec{v} = \frac{\partial(-\omega y)}{\partial x} + \frac{\partial(\omega x)}{\partial y} = 0$$

The curl of the field is:

$$\text{curl}\,\vec{v} = \text{curl}(-\omega y\vec{i} + \omega x\vec{j}) = \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right)\vec{k} = 2\omega\vec{k}$$

(d) We know that $\vec{v}$ has constant magnitude $|\omega|R$ everywhere on the circle and is everywhere tangential to the circle. In addition, if $\omega > 0$, the vector field rotates counterclockwise; if $\omega < 0$, the vector field rotates clockwise. Thus if $\omega > 0$, $\vec{v}$ and $\Delta\vec{r}$ are parallel and in the same direction, so

$$\int_C \vec{v}\cdot d\vec{r} = |\vec{v}|\cdot(\text{Length of } C) = \omega R\cdot 2\pi R = 2\pi\omega R^2$$

If $\omega < 0$, then $|\omega| = -\omega$ and $\vec{v}$ and $\Delta\vec{r}$ are in opposite directions, so

$$\int_C \vec{v}\cdot d\vec{r} = -|\vec{v}|\cdot(\text{Length of } C) = -|\omega|R\cdot(2\pi R) = 2\pi\omega R^2.$$

25. To show that the force field is irrotational we must show that its curl is zero. Let us do this in Cartesian coordinates:

$$\vec{F} = f(r)\vec{r} = f(\sqrt{x^2+y^2+z^2})(x\vec{i} + y\vec{j} + z\vec{k})$$

The third component of curl $\vec{F}$ is

$$\begin{aligned}&\frac{\partial}{\partial x}(f(\sqrt{x^2+y^2+z^2})y) - \frac{\partial}{\partial y}(f(\sqrt{x^2+y^2+z^2})x)\\ &= f'(\sqrt{x^2+y^2+z^2})\cdot\frac{2xy}{2\sqrt{x^2+y^2+z^2}} - f'(\sqrt{x^2+y^2+z^2})\cdot\frac{2xy}{2\sqrt{x^2+y^2+z^2}}\\ &= 0.\end{aligned}$$

A similar computation shows that the other components of curl $\vec{F}$ are 0 too.

26. Let $\vec{C} = a\vec{i} + b\vec{j} + c\vec{k}$. Then

$$\begin{aligned}\operatorname{curl}(\vec{F}+\vec{C}) &= \left(\frac{\partial}{\partial y}(F_3+c) - \frac{\partial}{\partial z}(F_2+b)\right)\vec{i} + \left(\frac{\partial}{\partial z}(F_1+a) - \frac{\partial}{\partial x}(F_3+c)\right)\vec{j}\\ &\quad + \left(\frac{\partial}{\partial x}(F_2+b) - \frac{\partial}{\partial y}(F_1+a)\right)\vec{k}\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\\ &= \operatorname{curl}\vec{F}.\end{aligned}$$

27.

$$\operatorname{curl}\vec{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\vec{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\vec{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\vec{k})$$

$$\begin{aligned}\operatorname{div}\operatorname{curl}\vec{F} &= \frac{\partial}{\partial x}(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\\ &= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}\end{aligned}$$

Since, if $\vec{F}$ has continuous second partial derivatives,

$$\frac{\partial^2 F_3}{\partial x\partial y} = \frac{\partial^2 F_3}{\partial y\partial x},\quad \frac{\partial^2 F_2}{\partial x\partial z} = \frac{\partial^2 F_2}{\partial z\partial x},\quad \text{and}\quad \frac{\partial^2 F_1}{\partial y\partial z} = \frac{\partial^2 F_1}{\partial z\partial y}$$

everything cancels out and we get div curl $\vec{F} = 0$.

28. The Fundamental Theorem of Calculus for Line Integrals states that if C is a path from P to Q, then

$$\int_C \operatorname{grad} f\cdot d\vec{r} = f(Q) - f(P).$$

Since C is a closed path we have

$$\int_c \operatorname{grad} f\cdot d\vec{r} = f(P) - f(P) = 0$$

(a) For any unit vector $\vec{n}$

$$\operatorname{circ}_{\vec{n}}\ \operatorname{grad} f = \lim_{\text{Area}\to 0}\left(\frac{\int \operatorname{grad} f\cdot d\vec{r}}{\text{Area of C}}\right) = \lim_{\text{Area}\to 0}\left(\frac{0}{\text{Area}}\right) = 0$$

where the limit is taken over curves C in a plane perpendicular to $\vec{n}$, and oriented by the right hand rule. Thus the circulation density of grad f is zero in every direction, and hence curl grad $f = \vec{0}$.

(b) Using the Cartesian coordinate definition

$$\begin{aligned}\operatorname{curl}\operatorname{grad} f &= \operatorname{curl}(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k})\\ &= \left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\vec{k}\\ &= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}.\end{aligned}$$

29.

$$\begin{aligned}&\operatorname{curl}(\phi\vec{F})\\ &= \left(\frac{\partial}{\partial y}(\phi F_3) - \frac{\partial}{\partial z}(\phi F_2)\right)\vec{i} + \left(\frac{\partial}{\partial z}(\phi F_1) - \frac{\partial}{\partial x}(\phi F_3)\right)\vec{j} + \left(\frac{\partial}{\partial x}(\phi F_2) - \frac{\partial}{\partial y}(\phi F_1)\right)\vec{k}\\ &= \left(\phi\frac{\partial F_3}{\partial y} + \frac{\partial\phi}{\partial y}F_3 - \phi\frac{\partial F_2}{\partial z} - \frac{\partial\phi}{\partial z}F_2\right)\vec{i} + \left(\phi\frac{\partial F_1}{\partial z} + \frac{\partial\phi}{\partial z}F_1 - \phi\frac{\partial F_3}{\partial x} - \frac{\partial\phi}{\partial x}F_3\right)\vec{j}\\ &\quad + \left(\phi\frac{\partial F_2}{\partial x} + \frac{\partial\phi}{\partial x}F_2 - \phi\frac{\partial F_1}{\partial y} - \frac{\partial\phi}{\partial y}F_1\right)\vec{k}\\ &= \phi\left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\right)\\ &\quad + \left(\left(\frac{\partial\phi}{\partial y}F_3 - \frac{\partial\phi}{\partial z}F_2\right)\vec{i} + \left(\frac{\partial\phi}{\partial z}F_1 - \frac{\partial\phi}{\partial x}F_3\right)\vec{j} + \left(\frac{\partial\phi}{\partial x}F_2 - \frac{\partial\phi}{\partial y}F_1\right)\vec{k}\right)\\ &= \phi\operatorname{curl}\vec{F} + \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right) \times (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\\ &= \phi\operatorname{curl}\vec{F} + (\operatorname{grad}\phi) \times \vec{F}.\end{aligned}$$

30. By Problem 29, $\operatorname{curl}\vec{F} = \operatorname{grad} f \times \operatorname{grad} g + f\operatorname{curl}\operatorname{grad} g = \operatorname{grad} f \times \operatorname{grad} g$, since $\operatorname{curl}\operatorname{grad} g = 0$. Since the cross product of two vectors is perpendicular to both vectors, $\operatorname{curl}\vec{F}$ is perpendicular to $\operatorname{grad} g$. But $\vec{F}$ is a scalar times $\operatorname{grad} g$, so $\operatorname{curl}\vec{F}$ is perpendicular to $\vec{F}$.

31. If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then

$$\begin{aligned}&\operatorname{grad}(\vec{F}\cdot\vec{v})\cdot\vec{u} - \operatorname{grad}(\vec{F}\cdot\vec{u})\cdot\vec{v}\\ &= \operatorname{grad}(F_1v_1 + F_2v_2 + F_3v_3)\cdot(u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) - \operatorname{grad}(F_1u_1 + F_2u_2 + F_3u_3)\cdot(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})\\ &= \frac{\partial F_1}{\partial x}v_1u_1 + \frac{\partial F_2}{\partial x}v_2u_1 + \frac{\partial F_3}{\partial x}v_3u_1 + \frac{\partial F_1}{\partial y}v_1u_2 + \frac{\partial F_2}{\partial y}v_2u_2 + \frac{\partial F_3}{\partial y}v_3u_2 +\\ &\quad \frac{\partial F_1}{\partial z}v_1u_3 + \frac{\partial F_2}{\partial z}v_2u_3 + \frac{\partial F_3}{\partial z}v_3u_3 -\\ &\quad \left(\frac{\partial F_1}{\partial x}u_1v_1 + \frac{\partial F_2}{\partial x}u_2v_1 + \frac{\partial F_3}{\partial x}u_3v_1 + \frac{\partial F_1}{\partial y}u_1v_2 + \frac{\partial F_2}{\partial y}u_2v_2 + \frac{\partial F_3}{\partial y}u_3v_2+\right.\\ &\quad \left.\frac{\partial F_1}{\partial z}u_1v_3 + \frac{\partial F_2}{\partial z}u_2v_3 + \frac{\partial F_3}{\partial z}u_3v_3\right)\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)(u_2v_3 - u_3v_2) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)(u_3v_1 - u_1v_3) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)(u_1v_2 - u_2v_1)\\ &= (\operatorname{curl}\vec{F})\cdot\vec{u}\times\vec{v}.\end{aligned}$$

32. (a) Since $\vec{F}$ is in the xy-plane, $\operatorname{curl}\vec{F}$ is parallel to $\vec{k}$ (because $F_3 = 0$ and F_1, F_2 have no z-dependence). Imagine computing the circulation of $\vec{F}$ counterclockwise around a small rectangle R at the point P with sides of length h parallel to $\vec{F}$ and sides of length t perpendicular to $\vec{F}$ as shown in Figure 20.3. Since $\vec{F}$ is perpendicular to C_2 and C_4, the line integral over these two sides is zero. Assuming that $\vec{F}$ is approximately constant on C_1 and C_3, its value on these sides is $F(Q)\vec{T}$ and $-F(P)\vec{T}$, respectively. Thus, since $\vec{F}$ is parallel to C_1 and C_3, the line integral over C_1 is approximately $F(Q)h$ and the line integral over C_3 is approximately $-F(P)h$. Finally

$$\begin{aligned}\text{curl}\,\vec{F}\,(P) &\approx \frac{\text{Circulation around } R}{\text{Area of } R} \approx \frac{F(Q)h - F(P)h}{ht} = \frac{F(Q)-F(P)}{t} \\ &\approx \text{Directional derivative of } F \text{ in the direction of } \overrightarrow{PQ}.\end{aligned}$$

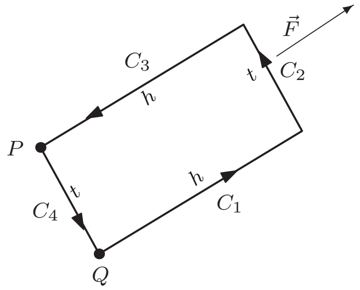


Figure 20.3: Path R used to find curl $\vec{F}$ at P

(b) Since $\vec{F} = F(x,y)\vec{T} = F(x,y)a\vec{i} + F(x,y)b\vec{j}$, with a, b constant, we have

$$\text{curl}\,\vec{F} = (bF_x - aF_y)\vec{k}.$$

Also $\vec{T} \times \vec{k} = (a\vec{i} + b\vec{j}) \times \vec{k} = b\vec{i} - a\vec{j}$, so

$$bF_x - aF_y = (\text{grad}F)\cdot(b\vec{i} - a\vec{j}) = \text{grad}F \cdot ((a\vec{i} + b\vec{j}) \times \vec{k}) = F_{\vec{T}\times\vec{k}},$$

where $F_{\vec{T}\times\vec{k}}$ is the directional derivative of F in the direction of the unit vector $\vec{T} \times \vec{k}$, which is perpendicular to $\vec{F}$. The right-hand rule applied to $\vec{T} \times \vec{k}$ shows that $\vec{T} \times \vec{k}$ is obtained by a *clockwise* rotation of $\vec{T}$ through $90°$.

33. (a) Using $r = (x^2+y^2)^{1/2}$, we calculate $r_x = (1/2)(x^2+y^2)^{-1/2}2x$. Notice that $r_x = x/r$ and, by a similar argument, $r_y = y/r$. We have

$$\begin{aligned}\text{curl}\left(r^A\cdot(-y\vec{i} + x\vec{j})\right) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -r^A y & r^A x & 0 \end{vmatrix} \\ &= \left(\frac{\partial(0)}{\partial y} - \frac{\partial(r^A x)}{\partial z}\right)\vec{i} + \left(\frac{\partial(-r^A y)}{\partial z} - \frac{\partial(0)}{\partial x}\right)\vec{j} + \left(\frac{\partial(r^A x)}{\partial x} - \frac{\partial(-r^A y)}{\partial y}\right)\vec{k} \\ &= 0\vec{i} + 0\vec{j} + c\vec{k} = c\vec{k}\end{aligned}$$

where

$$\begin{aligned}c &= \frac{\partial}{\partial x}(xr^A) - \frac{\partial}{\partial y}(-yr^A) = r^A + Axr^{A-1}r_x + r^A + Ayr^{A-1}r_y \\ &= 2r^A + Ar^{A-1}(xr_x + yr_y) \\ &= 2r^A + Ar^{A-1}\left(\frac{x^2}{r} + \frac{y^2}{r}\right) \\ &= 2r^A + Ar^{A-1}\left(\frac{x^2+y^2}{r}\right) \\ &= 2r^A + Ar^{A-1}\left(\frac{r^2}{r}\right) \\ &= (2+A)r^A.\end{aligned}$$

(b) The curl is in the direction of $\vec{k}$ for $A = -1$, is the zero vector for $A = -2$, and is in the direction of $-\vec{k}$ for $A = -3$.

The counterclockwise flow in the figure suggests the curl is in the $+\vec{k}$ direction in all three cases. However, it is not easy to see all the effects in the picture. The counterclockwise flow indicates that the circulation on any path around $\vec{k}$ is positive, but the curl is the circulation density, not the circulation, in the $\vec{k}$-direction. To understand the curl, we must see how the circulation around the $\vec{k}$-direction changes when the circles get smaller and smaller. We need to take into account the changing length of the vector field as well as the changing direction. This is too subtle an effect to be measured accurately by eye. The three vector fields differ in the rate at which the magnitude of the vector field changes as you move perpendicular to the flow (the shear) and this accounts for the differing directions of their curls. When $A = -1$ the magnitude of the vector field is constant, when $A = -2$ or -3 the magnitude decreases as you move farther from the origin, and the decrease is most rapid for $A = -3$.

(c) The curl has a component only in the $\vec{k}$ direction. Think of the this component as the limit of the circulation density around a small circle in the xy-plane, as the circle shrinks to zero. Thus the sign of the component of the curl tells us that this circulation around the circle centered at $(1, 1, 1)$ is positive for $A = -1$, zero for $A = -2$, and negative for $A = -3$. (This assumes that the circle is small enough that it does not go around the origin, where the vector fields are not defined.)

Since the vector fields and their curls are not defined at $(0, 0, 0)$, the curl calculated in part (b) does not tell us anything about the circulation around a circle centered at $(0, 0, 0)$.

Strengthen Your Understanding

34. The curl of a vector field is another vector field, not a scalar function.

35. The vector field $\vec{F} = y\vec{i}$ is parallel to the x-axis at every point, but $\text{curl}\,\vec{F} = -\vec{k} \neq \vec{0}$.

Though the flow of $\vec{F}$ consists of parallel straight lines, the vector field does exhibit shear, which can produce curl.

36. If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ then

$$\text{curl}\,\vec{F} = \left(\frac{\partial}{\partial y}z^2 - \frac{\partial}{\partial z}y^2\right)\vec{i} + \left(\frac{\partial}{\partial z}x^2 - \frac{\partial}{\partial x}z^2\right)\vec{j} + \left(\frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}x^2\right)\vec{k} = \vec{0}.$$

37. For $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, we have

$$\text{curl}\,\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

If $F_1 = z$ and $F_2 = F_3 = 0$, then $\vec{F} = z\vec{i}$ and $\text{curl}\,\vec{F} = \vec{j}$.

38. True. The circulation density is obtained by dividing the circulation around a circle C (a scalar) by the area enclosed by C (also a scalar), in the limit as the area tends to zero.

39. True.

$$\text{curl}\,\text{grad}\,f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \underbrace{\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)}_{0}\vec{i} + \underbrace{\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)}_{0}\vec{j} + \underbrace{\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)}_{0}\vec{k}$$

40. False. As a counterexample, any constant vector field $\vec{F} = a\vec{i} + b\vec{j} + c\vec{k}$ has $\text{div}\vec{F} = 0$ and $\text{curl}\vec{F} = \vec{0}$.

41. True. Writing $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{G} = G_1\vec{i} + G_2\vec{j} + G_3\vec{k}$, we have $\vec{F} + \vec{G} = (F_1 + G_1)\vec{i} + (F_2 + G_2)\vec{j} + (F_3 + G_3)\vec{k}$. Then the $\vec{i}$ component of $\text{curl}(\vec{F} + \vec{G})$ is

$$\frac{\partial(F_3 + G_3)}{\partial y} - \frac{\partial(F_2 + G_2)}{\partial z} = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} + \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}$$

which is the $\vec{i}$ component of $\text{curl}\vec{F}$ plus the $\vec{i}$ component of $\text{curl}\vec{G}$. The $\vec{j}$ and $\vec{k}$ components work out in a similar manner.

42. False. The left-hand side of the equation does not make sense. The quantity $(\vec{F} \cdot \vec{G})$ is a scalar, so we cannot compute the curl of it.

43. False. For example, take $\vec{F} = z\vec{i}$ and $\vec{G} = x\vec{j}$. Then $\vec{F} \times \vec{G} = xz\vec{k}$, and $\text{curl}(\vec{F} \times \vec{G}) = -z\vec{j}$. However, $(\text{curl}\vec{F}) \times (\text{curl}\vec{G}) = \vec{j} \times \vec{k} = \vec{i}$.

44. True. We calculate the x–components for each side of the equation:

$$\begin{aligned}(\text{curl}(f\vec{G}))_1 &= \frac{\partial(fG_3)}{\partial y} - \frac{\partial(fG_2)}{\partial z}\\ &= \frac{\partial f}{\partial y}G_3 + f\frac{\partial G_3}{\partial y} - \frac{\partial f}{\partial z}G_2 - f\frac{\partial G_2}{\partial z}\\ &= (\frac{\partial f}{\partial y}G_3 - \frac{\partial f}{\partial z}G_2) + f(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z})\\ &= ((\text{grad } f) \times \vec{G})_1 + (f(\text{curl}\,\vec{G}))_1.\end{aligned}$$

Computations for the other two components are similar, so

$$\text{curl}(f\vec{G}) = (\text{grad } f) \times \vec{G} + f \cdot (\text{curl}\,\vec{G}).$$

45. False. For example, take $\vec{F} = z\vec{i} + x\vec{j}$. Then $\text{curl}\vec{F} = \vec{j} + \vec{k}$, which is not perpendicular to $\vec{F}$, since $(z\vec{i} + x\vec{j}) \cdot (\vec{j} + \vec{k}) = x \neq 0$.

46. True. $\vec{F}$ is rotating around the y axis, so by the right hand rule $\text{curl}\,\vec{F}$ has a positive y component. Therefore taking the dot product of $\text{curl}\,\vec{F}$ and $\vec{j}$ will give a positive number.

47. (a)

$$\text{curl}(y\vec{i} - x\vec{j} + z\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{vmatrix} = -2\vec{k}$$

(b)

$$\text{curl}(y\vec{i} + z\vec{j} + x\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\vec{i} - \vec{j} - \vec{k}$$

(c)

$$\text{curl}(-z\vec{i} + y\vec{j} + x\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & y & x \end{vmatrix} = -2\vec{j}$$

(d)

$$\text{curl}(x\vec{i} + z\vec{j} - y\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z & -y \end{vmatrix} = -2\vec{i}$$

(e)

$$\text{curl}(z\vec{i} + x\vec{j} + y\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

So (a), (c), (d) are parallel to the z-, y-, and x-axes, respectively.

Solutions for Section 20.2

Exercises

1. To calculate $\int_C \vec F\cdot d\vec r$ directly, we compute the integral along the paths C_1 and C_2 in Figure 20.4. Now C_1 is parameterized by

$$x(t)=3-t,\quad y(t)=0\quad z(t)=0\quad \text{for } 0\le t\le 6,\quad \text{so } r'(t)=-\vec i,$$

and C_2 is parameterized by

$$x(t)=-3\cos t,\quad y(t)=0,\quad z(t)=3\sin t\quad \text{for } 0\le t\le \pi,\quad \text{so } \vec r\,'(t)=3\sin t\vec i+3\cos t\vec k.$$

Thus

$$\begin{aligned}\int_C \vec F\cdot d\vec r &= \int_{C_1}\vec F\cdot d\vec r+\int_{C_2}\vec F\cdot d\vec r\\ &= \int_0^6 (3-t)(\vec i+\vec j)\cdot(-\vec i)\,dt+\int_0^\pi((-3\cos t+3\sin t)\vec i-3\cos t\vec j)\cdot(3\sin t\vec i+3\cos t\vec k)\,dt\\ &= -\int_0^6(3-t)\,dt+9\int_0^\pi(-\sin t\cos t+\sin^2 t)\,dt\\ &= -3t+\frac{t^2}{2}\bigg|_0^6+9\left(-\frac{\sin^2 t}{2}-\frac12\sin t\cos t+\frac t2\right)\bigg|_0^\pi=\frac{9\pi}{2}.\end{aligned}$$

Formula IV-17 was used to calculate the last integral.

To calculate $\int_C \vec F\cdot d\vec r$ using Stokes' Theorem, we find

$$\operatorname{curl}\vec F=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+z & x & y\end{vmatrix}=\vec i+\vec j+\vec k.$$

For Stokes' Theorem, the semicircular region S in Figure 20.4 is oriented into the page, so $d\vec A=\vec j\;dx\,dz$. Thus,

$$\begin{aligned}\int_C \vec F\cdot d\vec r &= \int_S \operatorname{curl}\vec F\cdot d\vec A=\int_S(\vec i+\vec j+\vec k)\cdot\vec j\;dx\,dz=\int_S dx\,dz\\ &= \text{Area of semicircle }=\frac12\pi 3^2=\frac92\pi.\end{aligned}$$

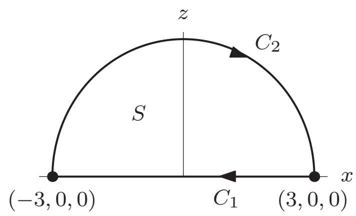


Figure 20.4

2. Since C is the curve $x^2+y^2=4$, oriented counterclockwise, we calculate $\int_C \vec F\cdot d\vec r$ directly using the parameterization

$$x(t)=2\cos t,\quad y(t)=2\sin t,\quad z(t)=0,\quad 0\le t\le 2\pi,\quad \text{so } \vec r\,'(t)=-2\sin t\vec i+2\cos t\vec j.$$

Thus,

$$\begin{aligned}\int_C \vec F\cdot d\vec r &= \int_0^{2\pi}(2\sin t\vec i-2\cos t\vec j)\cdot(-2\sin t\vec i+2\cos t\vec j)\,dt\\ &= -4\int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt=-4\cdot 2\pi=-8\pi.\end{aligned}$$

Since S is given by $z = 4 - x^2 - y^2$, we have

$$\frac{\partial z}{\partial x} = -2x \quad \text{and} \quad \frac{\partial z}{\partial y} = -2y, \quad \text{so} \quad d\vec{A} = (2x\vec{i} + 2y\vec{j} + \vec{k})\,dx\,dy.$$

Also

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\vec{k},$$

so, writing R for the disk below S in the xy-plane, Stokes' Theorem gives

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot d\vec{A} = \int_R -2\vec{k}\cdot(2x\vec{i} + 2y\vec{j} + \vec{k})\,dx\,dy$$
$$= -2\int dx\,dy = -2\cdot\ \text{Area of disk} = -2\cdot\pi 2^2 = -8\pi.$$

3. To calculate $\int_C \vec{F}\cdot d\vec{r}$ directly, we compute the integral along each of the sides C_1, C_2, C_3 in Figure 20.5. Now C_1 is parameterized by

$$x(t) = t, \quad y(t) = 0, \quad z(t) = 0 \quad \text{for } 0 \le t \le 5, \quad \text{so } r'(t) = \vec{i}.$$

Similarly, C_2 is parameterized by

$$x(t) = 5, \quad y(t) = t, \quad z(t) = 0 \quad \text{for } 0 \le t \le 5, \quad \text{so } \vec{r}\,'(t) = \vec{j}.$$

Also, C_3 is parameterized by

$$x(t) = 5 - t, \quad y(t) = 5 - t, \quad z(t) = 0 \quad \text{for } 0 \le t \le 5, \quad \text{so } \vec{r}\,'(t) = -\vec{i} - \vec{j}.$$

Thus

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$
$$= \int_0^5 t(\vec{i} + \vec{j})\cdot\vec{i}\,dt + \int_0^5 (5-t)(\vec{i} + \vec{j})\cdot\vec{j}\,dt + \int_0^5 ((5-t)-(5-t))(\vec{i} + \vec{j})\cdot(-\vec{i} - \vec{j})\,dt$$
$$= \int_0^5 t\,dt + \int_0^5 5 - t\,dt + \int_0^5 0\,dt = \int_0^5 5\,dt = 25.$$

To calculate $\int_C \vec{F}\cdot d\vec{r}$ using Stokes' Theorem, we find

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y + z & x - y + z & 0 \end{vmatrix} = -\vec{i} + \vec{j} + (1 - (-1))\vec{k} = -\vec{i} + \vec{j} + 2\vec{k}.$$

For Stokes' Theorem, the triangular region S in Figure 20.5 is oriented upward, so $d\vec{A} = \vec{k}\ dx\,dy$. Thus

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot d\vec{A} = \int_S (-\vec{i} + \vec{j} + 2\vec{k})\cdot\vec{k}\ dx\,dy$$
$$= \int_S 2\,dx\,dy = 2\cdot\ \text{Area of triangle} = 2\cdot\frac{1}{2}\cdot 5\cdot 5 = 25.$$

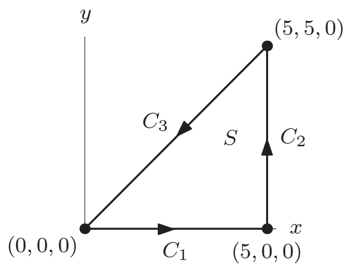


Figure 20.5

4. A sketch of the surface S and curve C which is the union of four curves C_1, C_2, C_3, and C_4, and the region R is shown in Figure 20.6.

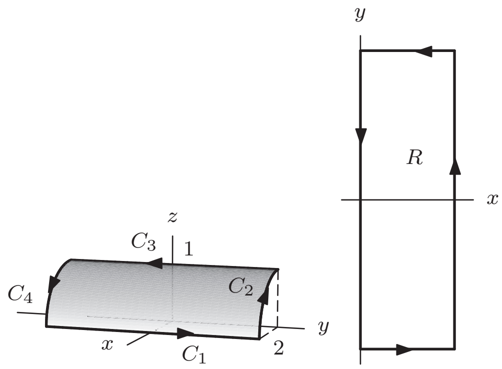


Figure 20.6

To compute the flux integral, we find $d\vec{A}$, oriented upward.

$$d\vec{A} = (2x\vec{i} + \vec{k})dxdy \quad \text{and} \quad \text{curl}\,\vec{F} = -y\vec{i} - z\vec{j} - x\vec{k}.$$

Thus,

$$\begin{aligned}\int_S \text{curl}\,\vec{F} \cdot d\vec{A} &= \int_S (-y\vec{i} - z\vec{j} - x\vec{k}) \cdot (2x\vec{i} + \vec{k})dxdy \\ &= \int_0^1 \int_{-2}^2 (-2xy - x)dydx = -2.\end{aligned}$$

The line integral $\int_S \vec{F} \cdot d\vec{r}$ is the sum of four integrals along C_1, C_2, C_3, and C_4.
On C_1: $x = 1, z = 0, dx = 0, dz = 0$, so

$$\int_S \vec{F} \cdot d\vec{r} = \int_{-2}^2 0\,dy = 0.$$

On C_2: $y = 2, z = 1 - x^2, dy = 0, dz = -2xdx$, so

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} 2xdx + 2(1-x^2)0 + x(1-x^2)(-2x)dx = \int_1^0 (2x - 2x^2 + 2x^4)dx = -\frac{11}{15}.$$

On C_3: $x = 0, z = 1, dx = 0, dz = 0$, so

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{C_3} 0 + ydy + 0 = \int_2^{-2} ydy = 0.$$

On C_4: $y = -2, z = 1 - x^2, dy = 0, dz = -2xdx$, so

$$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{C_4} -2xdx - 2(1-x^2)0 + x(1-x^2)(-2x)dx = \int_0^1 (-2x - 2x^2 + 2x^4)dx = -\frac{19}{15}.$$

Hence

$$\int_C \vec{F} \cdot d\vec{r} = 0 - \frac{11}{15} + 0 - \frac{19}{15} = -2.$$

Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\,\vec{F} \cdot d\vec{A}.$$

5. The boundary of S is C, the circle $x^2+y^2=1,\ z=0$, oriented counterclockwise and parameterized in polar coordinates by
$$\vec r(\theta)=\cos\theta\vec i+\sin\theta\vec j, \quad 0\le\theta\le 2\pi,$$
so,
$$\vec r\,'(\theta)=-\sin\theta\vec i+\cos\theta\vec j.$$
Hence
$$\begin{aligned}\int_C \vec F\cdot d\vec r &= \int_0^{2\pi}(\sin\theta\vec i+0\vec j+\cos\theta\vec k)\cdot(-\sin\theta\vec i+\cos\theta\vec j+0\vec k)d\theta\\ &= \int_0^{2\pi}-\sin^2\theta d\theta=-\pi.\end{aligned}$$
Now consider the integral $\int_S \operatorname{curl}\vec F\cdot d\vec A$. Here $\operatorname{curl}\vec F=-\vec i-\vec j-\vec k$ and the area vector $d\vec A$, oriented upward, is given by
$$d\vec A=2x\vec i+2y\vec j+\vec k\,dxdy.$$
If R is the disk $x^2+y^2\le 1$, then we have
$$\int_S \operatorname{curl}\vec F\cdot d\vec A=\int_R(-\vec i-\vec j-\vec k)\cdot(2x\vec i+2y\vec j+\vec k)dxdy.$$
Converting to polar coordinates gives:
$$\begin{aligned}\int_S \operatorname{curl}\vec F\cdot d\vec A &= \int_0^{2\pi}\int_0^1(-\vec i-\vec j-\vec k)\cdot(2r\cos\theta\vec i+2r\sin\theta\vec j+\vec k)rdrd\theta\\ &= \int_0^{2\pi}\int_0^1(-2r\cos\theta-2r\sin\theta-1)rdrd\theta\\ &= \int_0^{2\pi}\left(\frac{2}{3}(-\cos\theta-\sin\theta)-\frac{1}{2}\right)d\theta\\ &= -\pi.\end{aligned}$$
Thus, we confirm that
$$\int_C \vec F\cdot d\vec r=\int_S \operatorname{curl}\vec F\cdot d\vec A.$$

6. Since
$$\operatorname{curl}\vec F=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & y^2 & z^2\end{vmatrix}=\vec 0$$
the line integral $\int_C \vec F\cdot d\vec r=0$ around any closed curve, including this unit circle.

7. Since
$$\operatorname{curl}\vec F=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y-x & z-y & x-z\end{vmatrix}=-\vec i-\vec j-\vec k,$$
and the area vector of the disk $x^2+y^2\le 5$ is $\vec A=\pi(\sqrt5)^2\vec k=5\pi\vec k$, Stokes' Theorem gives
$$\int_C \vec F\cdot d\vec r=\int_{\text{Disk}}(-\vec i-\vec j-\vec k)\cdot d\vec A=(-\vec i-\vec j-\vec k)\cdot\text{Area vector}=(-\vec i-\vec j-\vec k)\cdot 5\pi\vec k=-5\pi.$$

8. If C is the circle $x^2 + y^2 = 9$ oriented counterclockwise when viewed from above, Stokes' Theorem gives

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}.$$

Only the $\vec{i}$ and $\vec{j}$ components of $\vec{F}$, namely $-y\vec{i} + x\vec{j}$, contribute to the line integral. Since $\left\|-y\vec{i} + x\vec{j}\right\| = 3$ on the circle, and $-y\vec{i} + x\vec{j}$ is tangent to the circle and points in the direction of the orientation of C we have

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r} = \left\|-y\vec{i} + x\vec{j}\right\| \cdot \text{Length of curve} = 3 \cdot 2\pi 3 = 18\pi.$$

Alternatively, the line integral can be evaluated by parameterizing the curve using $x = 3\cos t, y = 3\sin t$ for $0 \le t \le 2\pi$.

9. If C is the rectangular path around the rectangle, traversed counterclockwise when viewed from above, Stokes' Theorem gives

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}.$$

The $\vec{k}$ component of $\vec{F}$ does not contribute to the line integral, and the $\vec{j}$ component contributes to the line integral only along the segments of the curve parallel to the y-axis. Thus, if we break the line integral into four parts

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_{(0,0)}^{(3,0)} \vec{F} \cdot d\vec{r} + \int_{(3,0)}^{(3,2)} \vec{F} \cdot d\vec{r} + \int_{(3,2)}^{(0,2)} \vec{F} \cdot d\vec{r} + \int_{(0,2)}^{(0,0)} \vec{F} \cdot d\vec{r},$$

we see that the first and third integrals are zero, and we can replace $\vec{F}$ by its $\vec{j}$ component in the other two

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_{(3,0)}^{(3,2)} (x+7)\vec{j} \cdot d\vec{r} + \int_{(0,2)}^{(0,0)} (x+7)\vec{j} \cdot d\vec{r}.$$

Now $x = 3$ in the first integral and $x = 0$ in the second integral and the variable of integration is y in both, so

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_0^2 10\,dy + \int_2^0 7\,dy = 20 - 14 = 6.$$

10. (a) A counterclockwise parameterization of the circle is

$$x(t) = \cos t, \quad y(t) = \sin t \quad 0 \le t \le 2\pi.$$

Since $x'(t) = -\sin t,$ and $y'(t) = \cos t$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\sin t\vec{i} - \cos t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})dt$$
$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)dt = -2\pi.$$

(b) $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} - 2\vec{k} = -2\vec{k}.$

(c) Using Stokes' Theorem, where R is the region inside the circle oriented upward

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_R -2\vec{k} \cdot d\vec{A} = -2 \cdot \text{Area of circle} = -2\pi.$$

(d) Stokes' Theorem, which says that if C is a closed curve which is the boundary of a surface R, and $\vec{F}$ is a smooth vector field, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \operatorname{curl} \vec{F} \cdot d\vec{A}.$$

Here, the orientations of C and of R are related by the right-hand rule.

11. (a) We have

$$\operatorname{curl}\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & e^y & x-y-z \end{vmatrix} = -\vec i - \vec j\,.$$

(b) If S is the disk on the plane within the circle C, Stokes' Theorem gives

$$\int_C \vec F\cdot d\vec r = \int_S \operatorname{curl}\vec F\cdot d\vec A\,.$$

For Stokes' Theorem, the disk is oriented upward. Since the unit normal to the plane is $(\vec i + \vec j + \vec k)/\sqrt{3}$ and the disk has radius 3, the area vector of the disk is

$$\vec A = \frac{\vec i + \vec j + \vec k}{\sqrt{3}}\pi(3^2) = 3\sqrt{3}\pi(\vec i + \vec j + \vec k).$$

Thus, using curl $\vec F = -\vec i - \vec j$, we have

$$\int_C \vec F\cdot d\vec r = \left(-\vec i - \vec j\right)\cdot 3\sqrt{3}\pi(\vec i + \vec j + \vec k) = -6\sqrt{3}\pi.$$

12. No, because the curve C over which the integral is taken is not a closed curve, and so it is not the boundary of a surface.

Problems

13. By Stokes' theorem, the circulation of $\vec F$ around C is the flux of curl $\vec F$ through the disk S in the yz-plane enclosed by C. By the right hand rule, a positive normal vector to the disk points in the direction of the negative x-axis, $-\vec i$. Thus curl $\vec F\cdot d\vec A = \operatorname{curl} F\cdot(-\vec i)dA = -dA$, so the flux through S is negative. So the circulation is negative.

14. (a) We use Stokes' theorem, with S the interior of the circle in the xy-plane, oriented upward. Then

$$\int_C \vec F\cdot d\vec r = \int_S \operatorname{curl}\vec F\cdot d\vec A = \int_S ((x^2+z^2)\vec j + 5\vec k)\cdot d\vec A\,.$$

Since $d\vec A = \vec k\ dA$, we have

$$\int_C \vec F\cdot d\vec r = \int_S 5\,dA = 5\cdot \text{ Area of } S\ = 5\cdot\pi 3^2 = 45\pi.$$

(b) Using Stokes' theorem again, this time with S as the interior of the circle in the xz-plane, we have

$$\int_C \vec F\cdot d\vec r = \int_S \operatorname{curl}\vec F\cdot d\vec A = \int_S ((x^2+z^2)\vec j + 5\vec k)\cdot d\vec A\,.$$

Since $d\vec A = \vec j\ dx\,dz$, only the $\vec j$ component of curl $\vec F$ contributes to the flux. Thus, converting to polars gives

$$\int_C \vec F\cdot d\vec r = \int_S (x^2+z^2)\,dx\,dz = \int_0^{2\pi}\int_0^3 r^2 r\,dr\,d\theta = 2\pi\frac{r^4}{4}\bigg|_0^3 = \frac{81\pi}{2}.$$

15. (a) We have

$$\operatorname{curl}(y\vec i + z\vec j + x\vec k) = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\vec i - \vec j - \vec k\,.$$

(b) Let S be the triangular interior of the curve C, oriented upward. Then, by Stokes' Theorem,

$$\int_C (y\vec i + z\vec j + x\vec k)\cdot d\vec r = \int_S \operatorname{curl}(y\vec i + z\vec j + x\vec k)\cdot d\vec A = \int_S -(\vec i + \vec j + \vec k)\cdot d\vec A\,.$$

On the triangle $d\vec A = \vec k\ dA$, so

$$\int_S -(\vec i + \vec j + \vec k)\cdot d\vec A = \int_S -(\vec i + \vec j + \vec k)\cdot\vec k\ dA = -\text{ Area of triangle } = -\frac{1}{2}\cdot 4\cdot 3 = -6.$$

16. (a) We have

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \vec{i}\,(-1) - \vec{j}\,(1) + \vec{k}\,(-1) = -\vec{i} - \vec{j} - \vec{k}\,.$$

(b) (i) Using Stokes' Theorem, with S representing the disk inside the circle, oriented upward, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_S (-\vec{i} - \vec{j} - \vec{k}\,)\cdot\vec{k}\;dA = -\text{ Area of disk } = -4\pi.$$

(ii) This is a right triangle in the plane $x = 2$; it has height 5 and base length 3. Using Stokes' Theorem, with S representing the triangle, oriented toward the origin (in the direction $-\vec{i}\,$), we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_S (-\vec{i} - \vec{j} - \vec{k}\,)\cdot(-\vec{i}\;dA) = \int_S dA = \text{ Area of triangle } = \frac{1}{2}\cdot 3\cdot 5 = \frac{15}{2}.$$

17. (a) We have

$$\operatorname{curl}(z\vec{i} + x\vec{j} + y\vec{k}\,) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}\,.$$

(b) Let S be the square interior of the curve C, oriented toward the origin. Then, by Stokes' Theorem,

$$\int_C (z\vec{i} + x\vec{j} + y\vec{k}\,)\cdot d\vec{r} = \int_S \operatorname{curl}(z\vec{i} + x\vec{j} + y\vec{k}\,)\cdot d\vec{A} = \int_S (\vec{i} + \vec{j} + \vec{k}\,)\cdot d\vec{A}\,.$$

Since the plane has normal vector $\vec{n} = \vec{i} + \vec{j} + \vec{k}\,$, a unit normal in this direction is $(\vec{i} + \vec{j} + \vec{k}\,)/\sqrt{3}$. But S is oriented toward the origin, so $d\vec{A} = -(\vec{i} + \vec{j} + \vec{k}\,)/\sqrt{3}\,dA$. Thus

$$\int_S (\vec{i} + \vec{j} + \vec{k}\,)\cdot d\vec{A} = \int_S (\vec{i} + \vec{j} + \vec{k}\,)\cdot\frac{-(\vec{i} + \vec{j} + \vec{k}\,)}{\sqrt{3}}\,dA = -\frac{3}{\sqrt{3}}\cdot\text{ Area of square } = -\frac{3}{\sqrt{3}}\cdot 2^2 = -4\sqrt{3}.$$

18. Since $\vec{F}$ is constant, $\operatorname{curl}\vec{F} = \vec{0}\,$, so if S is the disk in the plane enclosed by the circle, Stokes' Theorem gives

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = 0.$$

19. Since

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\vec{k} = 2\vec{k}\,,$$

writing S for the disk in the plane enclosed by the circle, Stokes' Theorem gives

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_S 2\vec{k}\cdot d\vec{A}\,.$$

Now $d\vec{A} = \vec{n}\;dA$, where $\vec{n}$ is the unit vector perpendicular to the plane, so

$$\vec{n} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}\,).$$

Thus

$$\int_C \vec{F}\cdot d\vec{r} = \int_S 2\vec{k}\cdot\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}\,)\,dA = \frac{2}{\sqrt{3}}\int_S dA = \frac{2}{\sqrt{3}}\cdot\text{ Area of disk } = \frac{2}{\sqrt{3}}\cdot\pi 2^2 = \frac{8\pi}{\sqrt{3}}.$$

20. Since

$$\operatorname{curl}\vec F=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & -x & y-x\end{vmatrix}=\frac{\partial}{\partial y}(y-x)\vec i-\frac{\partial}{\partial x}(y-x)\vec j+\left(\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial y}(y)\right)\vec k=\vec i+\vec j-2\vec k,$$

writing S for the disk in the plane enclosed by the circle, Stokes' Theorem gives

$$\int_C \vec F\cdot d\vec r=\int_S \operatorname{curl}\vec F\cdot d\vec A=\int_S(\vec i+\vec j-2\vec k)\cdot d\vec A.$$

Now $d\vec A=\vec n\,dA$, where $\vec n$ is the unit vector perpendicular to the plane, so

$$\vec n=\frac{1}{\sqrt3}(\vec i+\vec j+\vec k).$$

Thus

$$\int_C \vec F\cdot d\vec r=\int_S(\vec i+\vec j-2\vec k)\cdot\frac{\vec i+\vec j+\vec k}{\sqrt3}\,dA=\int_S\frac{0}{\sqrt3}\,dA=0.$$

21. Since

$$\operatorname{curl}\vec F=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y+e^x & (\sin y)-x & 2y-x+\cos z^2\end{vmatrix}=2\vec i-(-1)\vec j+(-1-2)\vec k=2\vec i+\vec j-3\vec k,$$

writing S for the disk in the plane enclosed by the circle, Stokes' Theorem gives

$$\int_C \vec F\cdot d\vec r=\int_S \operatorname{curl}\vec F\cdot d\vec A=\int_S(2\vec i+\vec j-3\vec k)\cdot d\vec A.$$

Now $d\vec A=\vec n\,dA$, where $\vec n$ is the unit vector perpendicular to the plane, so

$$\vec n=\frac{1}{\sqrt3}(\vec i+\vec j+\vec k).$$

Thus

$$\int_C \vec F\cdot d\vec r=\int_S(2\vec i+\vec j-3\vec k)\cdot\frac{\vec i+\vec j+\vec k}{\sqrt3}\,dA=\int_S 0\,dA=0.$$

22. Since

$$\operatorname{curl}\vec F=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & -z & y\end{vmatrix}=\left(\frac{\partial}{\partial y}(y)-\frac{\partial}{\partial z}(-z)\right)\vec i-0\vec j+0\vec k=2\vec i,$$

writing S for the disk in the plane enclosed by the circle, Stokes' Theorem gives

$$\int_C \vec F\cdot d\vec r=\int_S \operatorname{curl}\vec F\cdot d\vec A=\int_S 2\vec i\cdot d\vec A.$$

Now $d\vec A=\vec n\,dA$, where $\vec n$ is the unit vector perpendicular to the plane, so

$$\vec n=\frac{1}{\sqrt3}(\vec i+\vec j+\vec k).$$

Thus

$$\int_C \vec F\cdot d\vec r=\int_S 2\vec i\cdot\frac{\vec i+\vec j+\vec k}{\sqrt3}\,dA=\int_S\frac{2}{\sqrt3}\,dA=\frac{2}{\sqrt3}\cdot\text{Area of disk}=\frac{2}{\sqrt3}\pi 2^2=\frac{8\pi}{\sqrt3}.$$

23. Since

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z-y) & (x-z) & (y-x) \end{vmatrix} = 2\vec{i} + 2\vec{j} + 2\vec{k},$$

writing S for the disk in the plane enclosed by the circle, Stokes' Theorem gives

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{curl}\vec{F} \cdot d\vec{A} = \int_S (2\vec{i} + 2\vec{j} + 2\vec{k}) \cdot d\vec{A}.$$

Now $d\vec{A} = \vec{n}\, dA$, where $\vec{n}$ is the unit vector perpendicular to the plane, so

$$\vec{n} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}).$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (2\vec{i} + 2\vec{j} + 2\vec{k}) \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}\, dA = \int_S \frac{6}{\sqrt{3}}\, dA = 2\sqrt{3} \cdot \text{Area of disk} = 2\sqrt{3}\pi 2^2 = 8\sqrt{3}\pi.$$

24. (a) All 3-space, because $(1 + ax^2 + by^2 + cz^2)$ is always positive.

(b) We have

$$\operatorname{grad} f = \operatorname{grad}(\ln(1 + ax^2 + by^2 + cz^2)) = \frac{2ax\vec{i} + 2by\vec{j} + 2cz\vec{k}}{1 + ax^2 + by^2 + cz^2}.$$

(c) We expect $\operatorname{curl}(\operatorname{grad} f) = 0$ because all gradient vector fields satisfy the curl test. Direct calculation gives the same result.

(d) Vector field $\vec{F} = \operatorname{grad} f = \operatorname{grad}(\ln(1 + ax^2 + by^2 + cz^2))$ with $a = 1$, $b = 2$, $c = 3$. So, if P and Q are the start and end points of C:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \operatorname{grad} f \cdot d\vec{r} = f\Big|_P^Q.$$

Since P is the point on C where $t = 0$, we have $P = (\cos 0, \sin 0, 0) = (1, 0, 0)$. Since Q is the point on C with $t = 13\pi/2$, we have $Q = (\cos(13\pi/2), \sin(13\pi/2), 13\pi/2) = (0, 1, 13\pi/2)$. Thus,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \ln(1 + x^2 + 2y^2 + 3z^2)\Big|_{(1,0,0)}^{(0,1,13\pi/2)} \\ &= \ln(3 + 507\pi^2/4) - \ln(2) \end{aligned}$$

25. (a) The equation of the rim, C, is $x^2 + y^2 = 9$, $z = 2$. This is a circle of radius 3 centered on the z-axis, and lying in the plane $z = 2$.

(b) Use Stokes' Theorem, with C oriented clockwise when viewed from above:

$$\int_S \operatorname{curl}(-y\vec{i} + x\vec{j} + z\vec{k}) \cdot d\vec{A} = \int_C (-y\vec{i} + x\vec{j} + z\vec{k}) \cdot d\vec{r}.$$

Since C is horizontal, the $\vec{k}$ component does not contribute to the integral. The remaining vector field, $-y\vec{i} + x\vec{j}$, is tangent to C, of constant magnitude $||-y\vec{i} + x\vec{j}|| = 3$ on C, and points in the opposite direction to the orientation. Thus

$$\int_S \operatorname{curl}(-y\vec{i} + x\vec{j} + z\vec{k}) \cdot d\vec{A} = \int (-y\vec{i} + x\vec{j}) \cdot d\vec{r} = -3 \cdot \text{ Length of curve } = -3 \cdot 2\pi 3 = -18\pi.$$

26. The curl $\vec{F}(x, y, z)$ of this vector field is equal to $-2\vec{j}$. Notice, as a check, that this field rotates in a direction opposite to the direction of C. Therefore we expect a negative line integral. The surface S is the disk parallel to the xz plane with radius 2. The curl points in the opposite direction to the normal vector to the surface with this orientation, so by Stokes' Theorem,

$$\begin{aligned} \text{Circulation around } S &= \int_S \operatorname{curl}\vec{F} \cdot d\vec{A} = \int_S (-2\vec{j}) \cdot d\vec{A} = \int_S -2\, dA \\ &= -2(\text{Area of circle}) = -2\pi 2^2 = -8\pi. \end{aligned}$$

27. We calculate

$$\operatorname{curl}\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & z & 3y \end{vmatrix} = (3-1)\vec{i} - (0-0)\vec{j} + (0-x)\vec{k} = 2\vec{i} - x\vec{k}.$$

Let S be the surface of the square, oriented in the positive x-direction. Then, by Stokes' Theorem

$$\int_C \vec{G}\cdot d\vec{r} = \int_S (\operatorname{curl}\vec{G})\cdot d\vec{A} = \int_S (2\vec{i} - x\vec{k})\cdot d\vec{A}.$$

On the square we have, $d\vec{A} = \vec{i}\,dydz$, so

$$\int_C \vec{G}\cdot d\vec{r} = \int_S (2\vec{i} - x\vec{k})\cdot\vec{i}\,dydz = 2\cdot \text{Area of square} = 2\cdot 36 = 72.$$

28. Since $\vec{F}$ is the curl of a vector field , we can use Stokes' Theorem to replace the spherical surface, S, by the disk, D, of radius 2 centered at the origin and oriented upward. Calculating $\vec{F}$ gives

$$\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3+\cos(z^2) & x+\sin(y^2) & y^2\sin(x^2) \end{vmatrix} = (2y\sin(x^2))\vec{i} - (2xy^2\cos(x^2) + 2z\sin(z^2))\vec{j} + \vec{k}.$$

On the horizontal disk D, only the $\vec{k}$ component of $\vec{F}$ contributes to the flux, so

$$\text{Flux} = \int_S \vec{F}\cdot d\vec{A} = \int_D \vec{F}\cdot d\vec{A} = 1\cdot\text{Area of disk} = 4\pi.$$

29. Use Stokes' theorem, applied to the surface R, oriented upward. Since curl$\vec{F} = \vec{k}$ for $\vec{F} = \frac{1}{2}(-y\vec{i} + x\vec{j})$, we have $\frac{1}{2}\int_C(-y\vec{i} + x\vec{j})\cdot d\vec{r} = \int_R \vec{k}\cdot d\vec{A} = \|\vec{k}\|(\text{area of } R) = \text{area of } R$.

30. The region is shown in Figure 20.7, so $C_2 - C_1$ is the boundary of the region, traversed with the region on the left. Thus, Stokes' Theorem applies with the region oriented upward, so $d\vec{A} = \vec{k}\,dx\,dy$ and

$$\int_{C_2}\vec{F}\cdot d\vec{r} - \int_{C_1}\vec{F}\cdot d\vec{r} = \int_R \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_R 3\vec{k}\cdot\vec{k}\,dx\,dy = 3\cdot\text{ Area of ring } = 3(\pi 5^2 - \pi 2^2) = 63\pi.$$

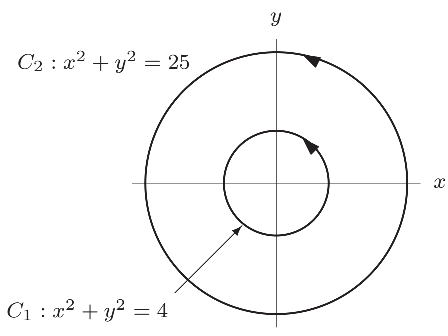


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Figure 20.7

31. Let S be the curved surface of the cylinder. The boundary of S consists of the curves C_1 and C_2, so Stokes' Theorem gives

$$\int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r}.$$

We calculate the flux of curl $\vec{F}$ through S. Only the $\vec{i}$ and $\vec{j}$ components of curl $\vec{F}$ contribute to this flux. The surface S has equation $x^2 + y^2 = 4$, so the component of curl $\vec{F}$ perpendicular to S has constant magnitude on S

$$||\vec{F}|| = ||3x\vec{i} + 3y\vec{j}|| = \sqrt{(3x)^2 + (3y)^2} = 3 \cdot 2 = 6.$$

The vector $d\vec{A}$ has no $\vec{k}$-component on S, and it is parallel to $3x\vec{i} + 3y\vec{j}$ on S. Evaluating the flux integral

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = ||3x\vec{i} + 3y\vec{j}|| \cdot \text{Area of } S = 6 \cdot (2\pi 2 \cdot 5) = 120\pi.$$

Thus,

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 120\pi.$$

32. (a) Let $\vec{F} = x^3\vec{i} + \sin(y^3)\vec{j} + e^{z^3}\vec{k}$. Then

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & \sin(y^3) & e^{z^3} \end{vmatrix} = \vec{0}.$$

(b) Since C is a closed curve, $\int_C \vec{F} \cdot d\vec{r} = 0$ by Stokes' Theorem.
(c) The line integral is 0 for any closed curve C in 3-space.

33. (a) $\vec{F}$ has only $\vec{i}$ and $\vec{j}$ components, and they do not depend on z. Thus $\vec{F}$ is everywhere parallel to the xy-plane, and takes the same values for every value of z.
(b) We have

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}.$$

(c) Since C is in the xy-plane, oriented counterclockwise when viewed from above, for an area element $d\vec{A}$ in S, we have $d\vec{A} = \vec{k}\, dx\, dy$. Thus Stokes' Theorem says

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k} \cdot \vec{k}\, dx\, dy = \int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

(d) Green's Theorem.

34. (a) Notice that the denominators $(z^2+1)^2$ and (z^2+1) are always positive and so affect the magnitude (but not the direction) of the motion of each of the terms.

The $(-y\vec{i} + x\vec{j})$ term represents rotation around the z-axis (counterclockwise when viewed from above). The $-z(x\vec{i} + y\vec{j})$ term represents radial motion (toward the z-axis when $z > 0$ and away when $z < 0$). The $\vec{k}$ term is downward motion. So $\vec{F}$ is a flow rotating inward and downward around the z-axis (for $z > 0$), like an actual bathtub drain.

(b) Let D be the disk representing the drain, oriented downward. Then the rate at which the water is leaving the bathtub is the flux of water flowing out of the drain:

$$\int_D \vec{F} \cdot d\vec{A} = \int_D \vec{F} \cdot (-dA\vec{k}) = \int_D \frac{1}{z^2+1}\, dA.$$

Because D is in the xy-plane, $z = 0$, so

$$\text{Flux out of } D = \int_D dA = \pi \text{ cm}^3\text{/sec}.$$

(c) We have, in units/sec,

$$\operatorname{div} \vec{F} = -\frac{z}{(z^2+1)^2} - \frac{z}{(z^2+1)^2} + \frac{2z}{(z^2+1)^2} = 0.$$

(d) Let W be the closed region bounded by the hemisphere S of radius 1 lying below the xy-plane and the disk, D, in the xy-plane representing the drain. Both S and D are oriented downward, so by the Divergence Theorem, we have:

$$\begin{aligned} 0 &= \int_W \operatorname{div} \vec{F}\, dV = \text{Flux out of } W \;-\; \text{Flux into } W \\ &= \int_S \vec{F} \cdot d\vec{A} - \int_D \vec{F} \cdot d\vec{A} \\ &= \int_S \vec{F} \cdot d\vec{A} - \pi. \end{aligned}$$

Thus,

$$\int_S \vec{F} \cdot d\vec{A} = \pi \text{ cm}^3/\text{sec}.$$

(e) Since C is oriented clockwise, we parameterize the circle by $\vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j}$. In addition, on the drain, $z = 0$. Thus,

$$\begin{aligned} \int_C \vec{G} \cdot d\vec{r} &= \int_C \frac{1}{2}(y\vec{i} - x\vec{j} - (x^2+y^2)\vec{k}) \cdot d\vec{r} \\ &= \frac{1}{2}\int_0^{2\pi} (\cos t\vec{i} - \sin t\vec{j} - \vec{k}) \cdot (\cos t\vec{i} - \sin t\vec{j})\, dt \\ &= \frac{1}{2}\int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \pi. \end{aligned}$$

So,

$$\int_C \vec{G} \cdot d\vec{r} = \pi \text{ cm}^3/\text{sec}.$$

(f) Computing the partial derivatives, we find that

$$\operatorname{curl} \vec{G} = -\frac{y+xz}{(z^2+1)^2}\vec{i} - \frac{yz-x}{(z^2+1)^2}\vec{j} - \frac{1}{z^2+1}\vec{k} = \vec{F}.$$

(g) By Stokes' Theorem, we have:

$$\int_C \vec{G} \cdot d\vec{r} = \int_D \operatorname{curl} \vec{G} \cdot d\vec{A} = \int_D \vec{F} \cdot d\vec{A}.$$

Thus, since curl $\vec{G} = \vec{F}$, Stoke's Theorem tells us that the answers to parts (d) and (e) should be equal.

Strengthen Your Understanding

35. The curve C is not the boundary of a surface, so Stokes' Theorem does not apply.

36. The orientations of the surface S and its boundary curve C are not compatible. For Stokes' Theorem to apply with the downward orientation of the surface, the boundary C must be oriented clockwise in the xy-plane. With the orientations of S and C as given in the problem statement, we have

$$\int_C \vec{F} \cdot d\vec{r} = -\int_S \operatorname{curl} \vec{F} \cdot d\vec{A}.$$

37. Since curl $\vec{F} = 0$ everywhere, Stokes' Theorem shows that

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = 0$$

for every closed curve C that is the boundary of a surface S. For example, we can take C to be any circle with either orientation.

38. The surface S could be the flat disk $x^2 + y^2 \leq 1, z = 0$ oriented upward, since using the right hand rule would orient the boundary circle counterclockwise.

39. True. By Stokes' Theorem, the circulation of $\vec{F}$ around C is the flux of curl $\vec{F}$ through the flat disc S in the xy-plane enclosed by the circle. An area element for S is $d\vec{A} = \pm\vec{k}\,dA$, where the sign depends on the orientation of the circle. Since curl $\vec{F}$ is perpendicular to the z-axis, curl $\vec{F} \cdot d\vec{A} = \pm(\text{curl}\,\vec{F} \cdot \vec{k})dA = 0$, so the flux of curl $\vec{F}$ through S is zero, hence the circulation of $\vec{F}$ around C is zero.

40. True. On a small patch of S that includes the boundary circle, the positive normal is outward. Letting the thumb of the right hand point in this direction makes the fingers curl in the counterclockwise direction.

41. False. Using the right-hand rule gives C_1 oriented clockwise and C_2 oriented counterclockwise when viewed from the positive y-axis.

42. False. The curl needs to be in the flux integral, not the line integral, for a correct statement of Stokes' theorem: $\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\,\vec{F} \cdot d\vec{A}$.

43. True. By Stokes' theorem, both flux integrals are equal to the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the circle $x^2 + y^2 = 1$, oriented counterclockwise when viewed from the positive z-axis.

44. True. By Stokes' theorem, the flux integral is equal to the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the boundary of the closed sphere. Since the sphere has no boundary curve, the line integral is zero. Alternatively, the closed sphere can be divided into two hemispheres S_1 (the top half with upward orientation) and S_2 (the bottom half with downward orientation.) Then S_1 and S_2 both have the circle C ($x^2 + y^2 = 1, z = 0$) as their common boundary, except that for S_1, C is oriented counterclockwise when viewed from above, and as the boundary of S_2, C is oriented clockwise. Then, using Stokes' theorem on S_1 and S_2 gives $\int_S \text{curl}\vec{F} \cdot d\vec{A} = \int_{S_1} \text{curl}\vec{F} \cdot d\vec{A} + \int_{S_2} \text{curl}\vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} = 0$.

The same result can be obtained using the Divergence theorem and the fact that div curl $\vec{F} = 0$.

45. True. Let D be the interior disk of C, oriented by the right hand rule. By Stokes' theorem, $\int_C \vec{F} \cdot d\vec{r} = \int_D \text{curl}\vec{F} \cdot d\vec{A}$, and $\int_C \vec{G} \cdot d\vec{r} = \int_D \text{curl}\vec{G} \cdot d\vec{A}$. Since $\text{curl}\vec{F} = \text{curl}\vec{G}$, we have $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}$.

46. True. Let S be the rectangular region inside C, oriented by the right hand rule. By Stokes' theorem, $\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\vec{F} \cdot d\vec{A} = 0$.

47. False. The condition that $\int_C \vec{F} \cdot d\vec{r} = 0$ implies, by Stokes' theorem, that $\int_S \text{curl}\,\vec{F} \cdot d\vec{A} = 0$. However, curl $\vec{F}$ need not be $\vec{0}$ for this to occur. For example, let $\vec{F} = y\vec{i}$, and let S be the upper unit hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ oriented upward. Then C is the circle $x^2 + y^2 = 1, z = 0$ oriented counterclockwise when viewed from above. The line integral $\int_C y\vec{i} \cdot d\vec{r} = 0$, but curl $\vec{F} = -\vec{k}$.

48. Since

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 3y & x - z & 6y - 7x \end{vmatrix} = (6 - (-1))\vec{i} - (-7 - 2)\vec{j} + (1 - 3)\vec{k} = 7\vec{i} + 9\vec{j} - 2\vec{k},$$

by Stokes' Theorem, if S is the flat interior of the circle, with area vector $\vec{A}$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\,\vec{F} \cdot d\vec{A} = \int_S (7\vec{i} + 9\vec{j} - 2\vec{k}) \cdot d\vec{A} = (7\vec{i} + 9\vec{j} - 2\vec{k}) \cdot \vec{A}.$$

This is a maximum if $\vec{A}$ is parallel to curl $\vec{F} = 7\vec{i} + 9\vec{j} - 2\vec{k}$, so curl $\vec{F}$ is the normal to the plane of the circle. Since the plane containing the circle is through the origin, its equation is

$$7x + 9y - 2z = 0.$$

To give the orientation of S shown by curl $\vec{F}$, notice that curl $\vec{F}$ points downward (because the z-component is negative). Therefore, the circle must be oriented counterclockwise when seen from the negative z-axis, or clockwise from the positive z-axis.

Solutions for Section 20.3

Exercises

1. We have
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & z & y \end{vmatrix} = (1-1)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k} = \vec{0}.$$
Since $\operatorname{curl}\vec{F} = \vec{0}$ and $\vec{F}$ is defined everywhere, we know by the curl test that $\vec{F}$ is a gradient field. In fact, $\vec{F} = \operatorname{grad} f$, where $f(x,y,z) = x^2 + yz$, so f is a potential function for $\vec{F}$.

2. We have
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (0-1)\vec{i} + (0-1)\vec{j} + (0-1)\vec{k} \neq \vec{0}.$$
Since $\operatorname{curl}\vec{F} \neq \vec{0}$, by the curl test $\vec{F}$ is not a gradient field.

3. We have
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+2z & x+z & 2x+y \end{vmatrix} = (1-1)\vec{i} + (2-2)\vec{j} + (1-1)\vec{k} = \vec{0}.$$
Since $\operatorname{curl}\vec{F} = \vec{0}$ and $\vec{F}$ is defined everywhere, we know by the curl test that $\vec{F}$ is a gradient field. In fact, $\vec{F} = \operatorname{grad} f$, where $f(x,y,z) = xy + yz + 2xz$, so f is a potential function for $\vec{F}$.

4. We have
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-2z & x-z & 2x-y \end{vmatrix} = (-1-(-1))\vec{i} + (-2-2)\vec{j} + (1-1)\vec{k} \neq \vec{0}.$$
Since $\operatorname{curl}\vec{F} \neq \vec{0}$, by the curl test $\vec{F}$ is not a gradient field.

5. Since $\operatorname{curl}\vec{G} = 2\vec{k} \neq \vec{0}$, the vector field $\vec{G}$ is not a gradient field.

6. We have
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+z^2 & yx+2yz \end{vmatrix} = ((x+2z)-(x+2z))\vec{i} - (y-y)\vec{j} + (z-z)\vec{k} = \vec{0}.$$
Since $\operatorname{curl}\vec{F} = \vec{0}$ and $\vec{F}$ is defined everywhere, we know by the curl test that $\vec{F}$ is a gradient field. In fact, $\vec{F} = \operatorname{grad} f$, where $f(x,y,z) = xyz + yz^2$, so f is a potential function for $\vec{F}$.

7. We have
$$\operatorname{div}\vec{F} = \frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z} = 0+0+0 = 0.$$
Since $\operatorname{div}\vec{F} = 0$, by the divergence test $\vec{F}$ is a curl field.

8. We have
$$\operatorname{div}\vec{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 0+1+0 = 1.$$
Since $\operatorname{div}\vec{F} \neq 0$, by the divergence test $\vec{F}$ is not a curl field.

9. We have
$$\operatorname{div}\vec{F} = \frac{\partial(2x)}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial(-z)}{\partial z} = 2-1-1 = 0.$$
Since $\operatorname{div}\vec{F} = 0$, by the divergence test $\vec{F}$ is a curl field.

10. We have

$$\operatorname{div}\vec{F} = \frac{\partial(x+y)}{\partial x} + \frac{\partial(y+z)}{\partial y} + \frac{\partial(x+z)}{\partial z} = 1+1+1 = 3.$$

Since $\operatorname{div}\vec{F} \neq 0$, by the divergence test $\vec{F}$ is not a curl field.

11. We have

$$\operatorname{div}\vec{F} = \frac{\partial(-xy)}{\partial x} + \frac{\partial(2yz)}{\partial y} + \frac{\partial(yz-z^2)}{\partial z} = -y + 2z + (y-2z) = 0.$$

Since $\operatorname{div}\vec{F} = 0$, by the divergence test $\vec{F}$ is a curl field.

12. We have

$$\operatorname{div}\vec{F} = \frac{\partial(xy)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial(xy)}{\partial z} = y + x + 0.$$

Since $\operatorname{div}\vec{F} \neq 0$, by the divergence test $\vec{F}$ is not a curl field.

13. The region is all points above the xy-plane. Any curve in that region can be contracted in that region to a point, so the curl test can be used. Also, any surface in that region bounds a solid in that region, so the divergence test can be used.

14. A small circle in the xz-plane centered at the origin surrounds the y-axis and cannot be contracted in the region to a point, so the curl test cannot be applied. A closed surface in the region bounds a solid in the region so the divergence test can be applied.

15. Any closed curve in the region can be contracted in the region to a point; even a small circle around the z-axis can be contracted by pulling it around the end of the positive z-axis. Thus, the curl test can be applied. A closed surface in the region bounds a solid in the region so the divergence test can be applied

16. Any closed curve be contracted in the region to a point. Even a small circle around the missing segment of the x-axis can be contracted by pulling it around either end of the segment. Thus, the curl test can be applied. A sphere of radius 2 centered at the origin contains inside it the missing segment, so it does not bound a solid in the region. Thus, the divergence test cannot be applied.

Problems

17. (a) We calculate the curl of each of these vector fields.

$$\operatorname{curl}\vec{A} = \operatorname{curl}(-by\vec{i}\,) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -by & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial y}(-by)\vec{k} = b\vec{k}\,.$$

(b)

$$\operatorname{curl}\vec{A} = \operatorname{curl}(bx\vec{j}\,) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & bx & 0 \end{vmatrix} = \frac{\partial}{\partial x}(bx)\vec{k} = b\vec{k}\,.$$

(c)

$$-\frac{1}{2}\vec{r} \times \vec{B} = -\frac{1}{2}yb\vec{i} + \frac{1}{2}xb\vec{j}$$

$$\begin{aligned}\operatorname{curl}\vec{A} &= \operatorname{curl}\left(\frac{1}{2}\vec{B} \times \vec{r}\right) \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(1/2)by & (1/2)bx & 0 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial x}\left(\left(\frac{1}{2}\right)bx\right) - \frac{\partial}{\partial y}\left(-\left(\frac{1}{2}\right)by\right)\right)\vec{k} = b\vec{k}\,.\end{aligned}$$

18. Let $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ and try

$$\vec{F} = \vec{v} \times \vec{r} = (a\vec{i} + b\vec{j} + c\vec{k}) \times (x\vec{i} + y\vec{j} + z\vec{k}) = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}.$$

Then

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2a\vec{i} + 2b\vec{j} + 2c\vec{k}.$$

Taking $a = 1$, $b = -\frac{3}{2}$, $c = 2$ gives $\operatorname{curl}\vec{F} = 2\vec{i} - 3\vec{j} + 4\vec{k}$, so the desired vector field is $\vec{F} = (-\frac{3}{2}z - 2y)\vec{i} + (2x - z)\vec{j} + (y + \frac{3}{2}x)\vec{k}$.

19. In Example 3 on page 1051 we showed that $\operatorname{curl}(\vec{b} \times \vec{r}) = 2\vec{b}$. Thus $(1/2)\vec{b} \times \vec{r}$ is a vector potential for $\vec{B}$.

20. Note that

$$\operatorname{div}(2y\vec{i} + 4x\vec{j}) = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(4x) = 0$$

and

$$\operatorname{curl}(3x\vec{i} + 9y\vec{j}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x & 9y & 0 \end{vmatrix} = \vec{0},$$

so $(3x\vec{i} + 9y\vec{j}) + (2y\vec{i} + 4x\vec{j})$ is the required decomposition.

21. Since $\operatorname{div}\vec{G} = 2x + 2y + 2z \neq 0$, there is not a vector potential for $\vec{G}$.

22. Since $\operatorname{div}\vec{F} = 2 + 3 - 5 = 0$, a vector potential does exist. One such is the vector field $\vec{H} = (-xy + 5yz)\vec{i} + (2xy + xz^2)\vec{k}$, but there are many others.

23. (a) Yes. To show this, we use a version of the product rule for curl (Problem 29 on page 1805):

$$\operatorname{curl}(\phi\vec{F}) = \phi \operatorname{curl}\vec{F} + (\operatorname{grad}\phi) \times \vec{F},$$

where ϕ is a scalar function and $\vec{F}$ is a vector field. So

$$\begin{aligned} \operatorname{curl}\left(q\frac{\vec{r}}{\|\vec{r}\|^3}\right) &= \operatorname{curl}\left(\frac{q}{\|\vec{r}\|^3}\vec{r}\right) = \frac{q}{\|\vec{r}\|^3}\operatorname{curl}\vec{r} + \operatorname{grad}\left(\frac{q}{\|\vec{r}\|^3}\right) \times \vec{r} \\ &= \vec{0} + q \operatorname{grad}\left(\frac{1}{\|\vec{r}\|^3}\right) \times \vec{r} \end{aligned}$$

Since the level surfaces of $1/\|\vec{r}\|^3$ are spheres centered at the origin, $\operatorname{grad}(1/\|\vec{r}\|^3)$ is parallel to $\vec{r}$, so $\operatorname{grad}(1/\|\vec{r}\|^3) \times \vec{r} = \vec{0}$. Thus, $\operatorname{curl}\vec{E} = \vec{0}$.

(b) Yes. The domain of $\vec{E}$ is 3-space minus $(0,0,0)$. Any closed curve in this region is the boundary of a surface contained entirely in the region. (If the first surface you pick happens to contain $(0,0,0)$, change its shape slightly to avoid it.)

(c) Yes. Since $\vec{E}$ satisfies both conditions of the curl test, it must be a gradient field. In fact,

$$\vec{E} = \operatorname{grad}\left(-q\frac{1}{\|\vec{r}\|}\right).$$

24. We must show $\operatorname{curl}\vec{A} = \vec{B}$.

$$\begin{aligned} \operatorname{curl}\vec{A} &= \frac{\partial}{\partial y}\left(\frac{-I}{c}\ln(x^2 + y^2)\right)\vec{i} - \frac{\partial}{\partial x}\left(\frac{-I}{c}\ln(x^2 + y^2)\right)\vec{j} \\ &= \frac{-I}{c}\left(\frac{2y}{x^2 + y^2}\right)\vec{i} + \frac{I}{c}\left(\frac{2x}{(x^2 + y^2)}\right)\vec{j} \\ &= \frac{2I}{c}\left(\frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}\right) \\ &= \vec{B}. \end{aligned}$$

25. (a) Yes. This is the case $p = 2$ of Example 5 on page 1052.
(b) No. The domain of $\vec{B}$ is 3-space minus the z-axis. A closed curve C which surrounds the z-axis cannot be contracted to a point without hitting the z-axis, so it cannot remain at all times within the domain.
(c) No. In Example 2 on page 1058 we found that if C is a circle around the origin,

$$\int_C \vec{B} \cdot d\vec{r} = \frac{4\pi I}{c}.$$

Thus $\vec{B}$ has non-zero circulation around C, and hence cannot be a gradient field.

26. (a) Using the product rule from Problem 29 on page 1805, we find

$$\operatorname{curl} \vec{E} = \operatorname{curl}\left(\frac{\vec{r}}{\|\vec{r}\|^p}\right) = \frac{1}{\|\vec{r}\|^p}\operatorname{curl}\vec{r} + \operatorname{grad}\left(\frac{1}{\|\vec{r}\|^p}\right) \times \vec{r}.$$

Now $\operatorname{curl}\vec{r} = \vec{0}$ and $\operatorname{grad}\left(\frac{1}{\|\vec{r}\|^p}\right)$ is parallel to $\vec{r}$, so both terms are zero. Thus $\operatorname{curl}\vec{E} = \vec{0}$.
(b) The domain of $\vec{E}$ is 3-space minus the origin if $p > 0$, and it is all of 3-space if $p \le 0$.
(c) Both domains have the property that any closed curve can be contracted to a point without hitting the origin, so $\vec{E}$ satisfies the curl test for all p. Since $\vec{E}$ has constant magnitude r^{1-p} on the sphere of radius r centered at the origin, and is parallel to the outward normal at every point of the sphere, the sphere must be a level surface of the potential function ϕ, that is, ϕ is a function of r alone. Further, since $\|\vec{E}\| = r^{1-p}$, a good guess is

$$\phi(r) = \int r^{1-p}\,dr,$$

that is,

$$\phi(r) = \begin{cases} \frac{r^{2-p}}{2-p} & \text{if } p \ne 2 \\ \ln r & \text{if } p = 2. \end{cases}$$

You can check that this is indeed a potential function for $\vec{E}$ by checking that $\operatorname{grad}\phi = \vec{E}$.

27. We apply Stokes' Theorem to the vector field $\vec{F}(x,y,z) = u(x,y)\vec{i} + v(x,y)\vec{j} + 0\vec{k}$. Since $\partial u/\partial z = \partial v/\partial z = 0$, it is straightforward to compute that $\operatorname{curl}\vec{F} = (\partial v/\partial x - \partial u/\partial y)\vec{k}$. Therefore, by Stokes' Theorem,

$$\int_C (u\vec{i} + v\vec{j}) \cdot d\vec{r} = \int_R (\partial v/\partial x - \partial u/\partial y)\vec{k} \cdot d\vec{A}$$

Since the surface R is in the xy-plane, oriented upward, $d\vec{A} = dxdy\vec{k}$. The result follows.

28. (a) Although $\operatorname{div}\vec{B} = 0$, the vector field $\vec{B}$ does not satisfy the divergence test because its domain is 3-space minus the origin, which does not have the required property that every closed surface is the boundary of a solid region which is entirely contained within the domain. For example, the solid region inside a sphere centered at the origin contains the origin, hence is not in the domain of $\vec{B}$.
(b) Using the product rule from Problem 29 on page 1805, we find

$$\operatorname{curl}\vec{A} = \left(\frac{1}{\|\vec{r}\|^3}\right)\operatorname{curl}(\mu \times \vec{r}) + \operatorname{grad}\left(\frac{1}{\|\vec{r}\|^3}\right) \times (\vec{\mu} \times \vec{r}).$$

By Example 3 on page 1051,

$$\operatorname{curl}(\vec{\mu} \times \vec{r}) = 2\vec{\mu}$$

and by Problem 68 on page 796

$$\operatorname{grad}\left(\frac{1}{\|\vec{r}\|^3}\right) = -3\frac{1}{\|\vec{r}\|^5}\vec{r}.$$

So

$$\operatorname{curl}\vec{A} = 2\frac{\vec{\mu}}{\|\vec{r}\|^3} - 3\left(\frac{1}{\|\vec{r}\|^5}\right)\vec{r} \times (\vec{\mu} \times \vec{r}).$$

From Problem 47 on page 750, we have

$$\vec{r} \times (\vec{\mu} \times \vec{r}) = \|\vec{r}\|^2\vec{\mu} - (\vec{\mu} \cdot \vec{r})\vec{r}.$$

So

$$\operatorname{curl}\vec{A} = 2\frac{\vec{\mu}}{\|\vec{r}\|^3} - 3\left(\frac{1}{\|\vec{r}\|^5}\right)(\|\vec{r}\|^2\vec{\mu} - (\vec{\mu} \cdot \vec{r})\vec{r}) = -\frac{\vec{\mu}}{\|\vec{r}\|^3} + \frac{3(\vec{\mu} \cdot \vec{r})\vec{r}}{\|\vec{r}\|^5}.$$

(c) No. The divergence test says a vector field must be a curl field if it satisfies the conditions of the test; it does not say the vector field cannot be a curl field if the vector field fails to satisfy the test.

29. (a) Since $\operatorname{curl}\operatorname{grad}\psi = 0$ for any function ψ, $\operatorname{curl}(\vec{A} + \operatorname{grad}\psi) = \operatorname{curl}\vec{A} + \operatorname{curl}\operatorname{grad}\psi = \operatorname{curl}\vec{A} = \vec{B}$.
(b) We have

$$\operatorname{div}(\vec{A} + \operatorname{grad}\psi) = \operatorname{div}\vec{A} + \operatorname{div}\operatorname{grad}\psi = \operatorname{div}\vec{A} + \nabla^2\psi.$$

Thus ψ should be chosen to satisfy the partial differential equation

$$\nabla^2\psi = -\operatorname{div}\vec{A}.$$

Strengthen Your Understanding

30. Since $\operatorname{div}\operatorname{curl}\vec{F} = 0$ for any smooth vector field $\vec{F}$, and $\operatorname{div} x\vec{i} = 1 \neq 0$, there can be no vector field $\vec{F}$ such that $\operatorname{curl}\vec{F} = x\vec{i}$.

31. The expression $\operatorname{curl}\operatorname{div}\vec{F}$ is meaningless, because $\operatorname{div}\vec{F}$ is a scalar function, but curl is only defined for vector fields.

32. If $\vec{F} = x\vec{i} + y\vec{j}$, then $\operatorname{div}\vec{F} \neq 0$, so $\vec{F}$ is not the curl of another vector field.

33. If $f(x, y, z) = x^2$, then $\operatorname{grad} f = 2x\vec{i}$ and $\operatorname{div}\operatorname{grad} f = 2 \neq 0$.

34. True. For example, take $\vec{F} = y\vec{k}$.

35. False. To see why, write $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$. Then $\operatorname{curl}\vec{F} = x\vec{i}$ gives

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = x; \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0; \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

Now take the partial $\frac{\partial}{\partial x}$ of the first equation, $\frac{\partial}{\partial y}$ of the second and $\frac{\partial}{\partial z}$ of the third. This gives

$$\frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} = 1; \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} = 0; \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y} = 0.$$

Assuming the continuity of the second order partials, the equality of mixed partials in the second and third equations shows that $\frac{\partial^2 F_3}{\partial x\partial y} = \frac{\partial^2 F_2}{\partial x\partial z}$, which contradicts the first equation. Thus there cannot be a vector field $\vec{F}$ with $\operatorname{curl}\vec{F} = x\vec{i}$.

36. False. The condition that $\int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = 0$ implies, by Stokes' theorem, that $\int_C \vec{F}\cdot d\vec{r} = 0$. However, $\vec{F}$ need not be a gradient field for this to occur. For example, let $\vec{F} = x\vec{k}$, and let S be the upper unit hemisphere $x^2+y^2+z^2 = 1, z \geq 0$ oriented upward. Then C is the circle $x^2 + y^2 = 1, z = 0$ oriented counterclockwise when viewed from above. The line integral $\int_C x\vec{k}\cdot d\vec{r} = 0$, since the field $\vec{F}$ is everywhere perpendicular to C. The curl of $\vec{F}$ is the constant field $-\vec{j}$, so $\vec{F}$ is not a gradient field. Yet we have $\int_S -\vec{j}\cdot d\vec{A} = 0$, since the constant field $-\vec{j}$ flows in, and then out of the hemisphere S.

37. True. By Stokes' theorem, $\int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_C \vec{F}\cdot d\vec{r}$, where C is one or more closed curves that form the boundary of S. Since $\vec{F}$ is a gradient field, its line integral over any closed curve is zero.

38. (a) Is zero, since, for example, $\dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right) = 0.$

(b) Is not zero. Let $\vec{F} = x\vec{j} + x\vec{k}$, then

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & x \end{vmatrix} = -\vec{j} + \vec{k},$$

so $\vec{F}\times\operatorname{curl}\vec{F} \neq \vec{0}$.

(c) Has components like $\dfrac{\partial}{\partial x}\left(\dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}\right)$, not zero.

(d) Is zero, since $\dfrac{\partial}{\partial x}\left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) = 0.$

(e) Is $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$, not zero.

Solutions for Chapter 20 Review

Exercises

1. We have

$$\operatorname{curl}((x+y)\vec{i} - (y+z)\vec{j} + (x+z)\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & -(y+z) & (x+z) \end{vmatrix} = \vec{i} - \vec{j} - \vec{k}$$

2. We want the $\vec{j}$ component of $\operatorname{curl}\vec{n}$. Since

$$\operatorname{curl}\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & 4y+5z & 6z+7x \end{vmatrix},$$

the $\vec{j}$ component is

$$(\operatorname{curl}\vec{n}) \cdot \vec{j} = -\left(\frac{\partial}{\partial x}(6z+7y) - \frac{\partial}{\partial z}(2x+3y)\right) = -7.$$

3. Fields with zero curl: (c), (d), (f) because these don't appear to be swirling.

4. Fields (a), (c), (e) because they do not appear to be exploding or collapsing.

5. C_2, C_3, C_4, C_6, since line integrals around C_1 and C_5 are clearly nonzero. You can see directly that $\int_{C_2} \vec{F} \cdot d\vec{r}$ and $\int_{C_6} \vec{F} \cdot d\vec{r}$ are zero, because C_2 and C_6 are perpendicular to their fields at every point.

6. Not defined, since we can't take the gradient of a vector function.

7. Defined; scalar because $\vec{F}(\vec{r}) \times \vec{G}(\vec{r})$ is a vector field; the integral represents the flux—a scalar—through the surface S.

8. Defined; scalar because $\operatorname{grad} f$ is a vector field, so $(\operatorname{grad} f) \times \vec{r}$ is a vector field, so we can calculate its divergence, giving a scalar.

9. Defined; vector since $\operatorname{curl}\vec{F}$ is a vector and the cross product of two vectors is a vector.

10. The circulation around any square with sides parallel to the axes and centered at the point is zero because the line integrals on the top and bottom sides add to zero and the line integrals on the left and right sides add to zero. We suspect that the vector field has zero curl.

11. The circulation around the boundary of any square with sides parallel to the axes and enclosing the point is nonzero because the vectors at the bottom are larger than those at the top, so we suspect a nonzero curl at this point.

12. We have

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^4\right) = 2x + 3y^2 + 4z^3,$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^3 & z^4 \end{vmatrix} = \vec{0}.$$

So $\vec{F}$ is not solenoidal, but $\vec{F}$ is irrotational.

13. We have

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx) = y + z + x,$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\vec{i} - z\vec{j} - x\vec{k}.$$

So $\vec{F}$ is not solenoidal and not irrotational.

14. We have

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(\cos x) + \frac{\partial}{\partial y}(e^y) + \frac{\partial}{\partial z}(x+y+z) = -\sin x + e^y + 1$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & e^y & x+y+z \end{vmatrix} = \vec{i} - \vec{j}\,.$$

So $\vec{F}$ is not solenoidal and not irrotational.

15. We have

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(e^{y+z}) + \frac{\partial}{\partial y}(\sin(x+z)) + \frac{\partial}{\partial z}(x^2+y^2) = 0$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y+z} & \sin(x+z) & x^2+y^2 \end{vmatrix} = (2y - \cos(x+z))\vec{i} - \left(2x - e^{y+z}\right)\vec{j} + \left(\cos(x+z) - e^{y+z}\right)\vec{k}\,.$$

So $\vec{F}$ is solenoidal, but $\vec{F}$ is not irrotational.

16. (a) Direct method:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xz & -xy \end{vmatrix} = -2x\vec{i} - (-y)\vec{j} + z\vec{k} = -2x\vec{i} + y\vec{j} + z\vec{k}\,.$$

On the surface, $d\vec{A}$ has no $\vec{i}$-component, so the $\vec{i}$-component of $\text{curl}\,\vec{F}$ does not contribute to the flux. Thus

$$\int_S \text{curl}\,\vec{F}\cdot d\vec{A} = \int_S (y\vec{j} + z\vec{k})\cdot d\vec{A}\,.$$

Since $y\vec{j} + z\vec{k}$ is perpendicular to S and $\|y\vec{j} + z\vec{k}\| = \sqrt{y^2+z^2} = \sqrt{5}$ on S, we have

$$\int_S \text{curl}\,\vec{F}\cdot d\vec{A} = \sqrt{5}\cdot \text{ Area of } S = \sqrt{5}\cdot 2\pi\sqrt{5}\cdot 3 = 30\pi.$$

(b) Using Stokes' theorem, we replace the flux integral by two line integrals around the circular boundaries, C_1 and C_2, of S. See Figure 20.8.

$$\int_S \text{curl}\,\vec{F}\cdot ds = \int_{C_1} \vec{F}\cdot dr + \int_{C_2} \vec{F}\cdot d\vec{r}\,.$$

On C_1, the left boundary, $x = 0$, so $\vec{F} = \vec{0}$, and therefore $\int_{C_1} \vec{F}\cdot d\vec{r} = 0$. On C_2, the right boundary, $x = 3$, so $\vec{F} = 3z\vec{j} - 3y\vec{k}$. This vector field has $\|\vec{F}\| = \sqrt{(3z)^2 + (-3y)^2} = \sqrt{9(z^2+y^2)} = 3\sqrt{5}$. and $\vec{F}$ is tangent to the boundary C_2 and pointing in the same direction as C_2. Thus

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \|\vec{F}\|\cdot \text{ Length of } C_2 = 3\sqrt{5}\cdot 2\pi\sqrt{5} = 30\pi.$$

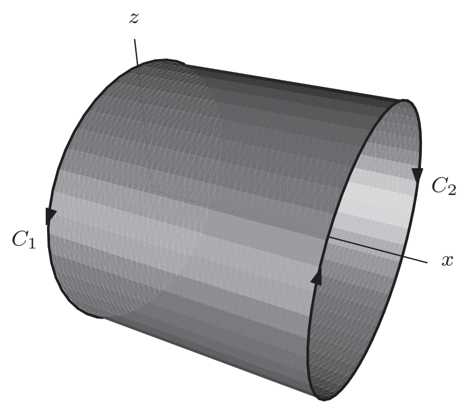

Figure 20.8

17. (a) Let us parameterize the curve C by $\vec{r}(t) = 3\cos t\vec{i} + 3\sin t\vec{j}$, $\quad 0 \le t \le 2\pi$.
Then $d\vec{r} = (-3\sin t\vec{i} + 3\cos t\vec{j})dt$ and so

$$\int_C ((yz^2 - y)\vec{i} + (xz^2 + x)\vec{j} + 2xyz\vec{k}) \cdot d\vec{r} = \int_C (-3\sin t\vec{i} + 3\cos t\vec{j}) \cdot d\vec{r}$$
$$= \int_0^{2\pi} 9dt = 18\pi.$$

(b) Since C is a closed curve, Stokes' Theorem applies. We choose the surface S to be the disk in the xy-plane bounded by C, and it must be oriented upward. Since $\text{curl}\vec{F} = 2\vec{k}$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S 2\vec{k} \cdot d\vec{A} = \|2\vec{k}\|(\text{area of } S) = 2(\pi 3^2) = 18\pi.$$

18. (a) Let C be the boundary of the disc D, given by $y^2 + z^2 \le 1$ and $x = 0$, oriented counterclockwise when viewed from the positive x-axis. Using Stokes' Theorem, we have

$$\int_D \text{curl}\left(e^{x^2}\vec{i} + (x+y)\vec{k}\right) \cdot d\vec{A} = \int_C \left(e^{x^2}\vec{i} + (x+y)\vec{k}\right) \cdot d\vec{r}.$$

Parameterize the curve C by $\vec{r}(t) = \cos t\vec{j} + \sin t\vec{k}$ for $0 \le t \le 2\pi$. Then, $\vec{r}\,'(t) = -\sin t\vec{j} + \cos t\vec{k}$. Thus,

$$\int_C \left(e^{x^2}\vec{i} + (x+y)\vec{k}\right) \cdot d\vec{r} = \int_0^{2\pi} \left(e^0\vec{i} + (0 + \cos t)\vec{k}\right) \cdot (-\sin t\vec{j} + \cos t\vec{k})dt$$
$$= \int_0^{2\pi} \cos^2 t\, dt = \frac{1}{2}(t + \sin t\cos t)\Big|_0^{2\pi} = \pi.$$

(b) We have

$$\text{curl}\left(e^{x^2}\vec{i} + (x+y)\vec{k}\right) = \vec{i} - \vec{j},$$

and $d\vec{A} = \vec{n}\,dA = \vec{i}\,dA$. So,

$$\int_D \text{curl}\left(e^{x^2}\vec{i} + (x+y)\vec{k}\right) \cdot d\vec{A} = \int_D (\vec{i} - \vec{j}) \cdot \vec{i}\,dA = \int_D dA = \text{Area of disc } D = \pi.$$

19. First C is parameterized by

$$\vec{r}(\theta) = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + \vec{k}.$$

Note that C bounds the disk S given by $x^2 + y^2 \le 4$, $z = 1$. Then

$$\vec{r}\,'(\theta) = -2\sin\theta\vec{i} + 2\cos\theta\vec{j}.$$

Now,

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - 2y & 3x - 4y & z + 3y \end{vmatrix} = 3\vec{i} + \vec{j} + 5\vec{k},$$

and $d\vec{A} = \vec{k}\,dA$. Using Stokes' Theorem we get

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\,\vec{F} \cdot d\vec{A}$$
$$= \int_S (3\vec{i} + \vec{j} + 5\vec{k}) \cdot \vec{k}\,dA = \int_S 5dA$$
$$= 5(\text{Area of circle}) = 5(4\pi) = 20\pi.$$

20. First note that $\text{curl} = 2\vec{k}$.

(a) By Stokes' Theorem, $\int_C \vec{F} \cdot d\vec{r} = \int_S 2\vec{k} \cdot d\vec{A}$ where S is the disk of radius 10 in the xy-plane centered at the origin, oriented downward. Since this orientation is opposite to $2\vec{k}$, $\int_S 2\vec{k} \cdot d\vec{A} = -\|2\vec{k}\|(\text{area of } S) = -200\pi$.

(b) By Stokes' Theorem, $\int_C \vec{F} \cdot d\vec{r} = \int_S 2\vec{k} \cdot d\vec{A}$ where S is the disk of radius 10 in the yz-plane centered at the origin, oriented in the negative x direction. Since the vector field $2\vec{k}$ is parallel to the surface S, its flux through the surface is zero.

21. The graph of $\vec{F} = \vec{r}/\|\vec{r}\|^3$ consists of vectors pointing radially outward. There is no swirl, so curl $\vec{F} = \vec{0}$. From Stokes' Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot d\vec{A} = \int_S \vec{0}\cdot d\vec{A} = 0$$

22. The circulation is the line integral $\int_C \vec{F}\cdot d\vec{r}$ which can be evaluated directly by parameterizing the circle, C. Or, since C is the boundary of a flat disk S, we can use Stokes' Theorem:

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot d\vec{A}$$

where S is the disk $x^2+y^2 \le 1$, $z = 2$ and is oriented upward (using the right hand rule). Then curl $\vec{F} = -y\vec{i} - x\vec{j} + \vec{k}$ and the unit normal to S is $\vec{k}$. So

$$\begin{aligned}\int_S \text{curl}\,\vec{F}\cdot d\vec{A} &= \int_S (-y\vec{i} - x\vec{j} + \vec{k})\cdot\vec{k}\;dxdy\\ &= \int_S 1\,dxdy\\ &= \text{Area of } S = \pi\end{aligned}$$

Problems

23. (a) is (I) since $\text{div}(\vec{r} + \vec{a}) = 3$.
(b) is (I) since

$$\text{div}(\vec{r}\times\vec{a}) = \text{div}((a_3y - a_2z)\vec{i} + (a_1z - a_3x)\vec{j} + (a_2x - a_1y)\vec{k}) = 0.$$

(c) is (V) since $\vec{r}\cdot\vec{a}$ is a scalar.
(d) is (III) since $\text{curl}(\vec{r} + \vec{a}) = \vec{0}$.
(e) is (IV) since

$$\text{curl}(\vec{r}\times\vec{a}) = \text{curl}((a_3y - a_2z)\vec{i} + (a_1z - a_3x)\vec{j} + (a_2x - a_1y)\vec{k}) = -2a_1\vec{i} - 2a_2\vec{j} - 2a_3\vec{k} = -2\vec{a}.$$

(f) is (V) since $\vec{r}\cdot\vec{a}$ is a scalar.

24. (a) $\text{grad}(\vec{r}\cdot\vec{a}) = \text{grad}(a_1x + a_2y + a_3z) = \vec{a}$.
(b) Not defined. Can't take the divergence of a scalar.
(c) Not defined. Can't take the curl of a scalar.
(d) Not defined. Can't take the gradient of a vector.
(e) Now

$$\vec{r}\times\vec{a} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ a_1 & a_2 & a_3\end{vmatrix} = (a_3y - a_2z)\vec{i} - (a_3x - a_1z)\vec{j} + (a_2x - a_1y)\vec{k}.$$

Thus,

$$\text{div}(\vec{r}\times\vec{a}) = 0.$$

(f) Using the expression for $\vec{r}\times\vec{a}$ in part (e), we have

$$\text{curl}(\vec{r}\times\vec{a}) = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y\end{vmatrix} = -2a_1\vec{i} - 2a_2\vec{j} - 2a_3\vec{k} = -2\vec{a}.$$

25. (a) Only the y-component of $\vec{F}$ contributes to the flux integral. On S we have $y = 5$ and $d\vec{A} = -\vec{j}\,dA$ (since S is oriented toward the origin), so

$$\int_S \vec{F}\cdot d\vec{A} = \int_S 5^3\vec{j}\cdot(-\vec{j}\,dA) = -125\cdot\text{Area of } S = -125(\pi 3^2) = -1125\pi.$$

(b) Not defined; we cannot integrate a vector field over a solid region in space.

(c) Since

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix} = \vec{0},$$

we have $\int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = 0$.

(d) Not defined; we cannot calculate the gradient of a vector field.

(e) Since

$$\operatorname{div}\vec{F} = \operatorname{div}(x^3\vec{i} + y^3\vec{j} + z^3\vec{k}) = 3x^2 + 3y^2 + 3z^2,$$

we have

$$\int_W \operatorname{div}\vec{F}\,dV = 3\int_W (x^2+y^2+z^2)dV.$$

Converting to spherical coordinates gives

$$\int_W \operatorname{div}\vec{F}\,dV = 3\int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 3\cdot 2\pi\left(-\cos\phi\Big|_0^{\pi}\right)\left(\frac{\rho^5}{5}\Big|_0^2\right) = \frac{384\pi}{5}.$$

(f) Since $\vec{F} = \operatorname{grad}((x^4+y^4+z^4)/4)$, we use the Fundamental Theorem of Line Integrals to get

$$\int_C \vec{F}\cdot d\vec{r} = \frac{x^4+y^4+z^4}{4}\Bigg|_{(0,0,0)}^{(2,3,4)} = \frac{2^4+3^4+4^4}{4} = \frac{353}{4}.$$

Alternatively, we parameterize the line by $x = 2t, y = 3t, z = 4t$ for $0 \le t \le 1$. Then on the line $\vec{F} = 8t^3\vec{i} + 27t^3\vec{j} + 64t^3\vec{k}$ and $r'(t) = 2\vec{i} + 3\vec{j} + 4\vec{k}$, giving

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (8t^3\vec{i} + 27t^3\vec{j} + 64t^3\vec{k})\cdot(2\vec{i}+3\vec{j}+4\vec{k})\,dt = \int_0^1 (16t^3+81t^3+256t^3)\,dt = 353\frac{t^4}{4}\Bigg|_0^1 = \frac{353}{4}.$$

(g) Not defined; $\operatorname{curl}\vec{F}$ is a vector field and we cannot integrate a vector field over a solid region in space.

(h) We have $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ so

$$\int_W \vec{F}\cdot(\vec{i}+\vec{j}+\vec{k})dV = \int_0^3\int_0^2\int_0^1 (x^3+y^3+z^3)\,dx\,dy\,dz = 2\cdot 3\frac{x^4}{4}\Bigg|_0^1 + 1\cdot 3\frac{y^4}{4}\Bigg|_0^2 + 1\cdot 2\frac{z^4}{4}\Bigg|_0^3$$

$$= 2\cdot 3\frac{1^4}{4} + 1\cdot 3\frac{2^4}{4} + 1\cdot 2\frac{3^4}{4} = 54.$$

26. (a) Defined; equal to (e) and (f) by Stokes' Theorem.
(b) Not defined.
(c) Not defined.
(d) Defined; not equal to others in list.
(e) Defined; equal to (a) and (f) by Stokes' Theorem.
(f) Defined; equal to (a) and (e) by Stokes' Theorem.
(g) Defined; not equal to others in list.

27. (a) At P, we have $\operatorname{curl}\vec{F} = 6\vec{i} + 5\vec{j} - 8\vec{k}$, so

$$\operatorname{curl}\vec{F}\cdot(\vec{i}+\vec{j}+\vec{k}) = 6+5-8 = 3.$$

(b) The definition of the curl tells us that if $\vec{n}$ is the unit vector normal to the plane containing the circle and pointing away from the origin, then

$$\operatorname{curl}\vec{F}\cdot\vec{n} \approx \frac{\int_C \vec{F}\cdot d\vec{r}}{\text{Area enclosed by } C}.$$

Since $\vec{n} = (\vec{i}+\vec{j}+\vec{k})/\sqrt{3}$, we have

$$\int_C \vec{F}\cdot d\vec{r} \approx (\operatorname{curl}\vec{F}\cdot\vec{n})\text{Area of circle} = \frac{3}{\sqrt{3}}\pi(0.01)^2 = \frac{0.0003\pi}{\sqrt{3}} = 0.000544.$$

28. .

The vector curl $\vec{G}$ has its component in the x-direction given by

$$(\text{curl } \vec{G})_x \approx \frac{\text{Circulation around small square around } x\text{-axis}}{\text{Area inside square}} = \frac{\text{Circulation around } S_2}{\text{Area inside } S_2} = \frac{6}{(0.1)^2} = 600.$$

Similar reasoning leads to

$$(\text{curl } \vec{G})_y \approx \frac{\text{Circulation around } S_3}{\text{Area inside } S_3} = \frac{-5}{(0.1)^2} = -500.$$

$$(\text{curl } \vec{G})_z \approx \frac{\text{Circulation around } S_1}{\text{Area inside } S_1} = \frac{-0.02}{(0.1)^2} = -2.$$

Thus,

$$\text{curl } \vec{G} \approx 600\vec{i} - 500\vec{j} - 2\vec{k}.$$

29. (a) It appears that div $\vec{F} < 0$, and div $\vec{G} < 0$; div $\vec{G}$ is larger in magnitude (more negative) if the scales are the same.
(b) curl $\vec{F}$ and curl $\vec{G}$ both appear to be zero at the origin (and elsewhere).
(c) Yes, the cylinder with axis along the z-axis will have negative flux through it (ends parallel to xy-plane).
(d) Same as part(c).
(e) No, you cannot draw a closed curve around the origin such that $\vec{F}$ has a non-zero circulation around it because curl is zero. By Stokes' theorem, circulation equals the integral of the curl over the surface bounded by the curve.
(f) Same as part(e)

30. (a) Since the disk is in the plane $z = \sqrt{5}$, only the $\vec{k}$ component of $\vec{F}$ contributes to the integral. Since the disk is oriented downward, $d\vec{A} = -\vec{k}\, dA$ on the disk; in addition, $z = \sqrt{5}$ on the disk. Thus,

$$\int_S \vec{F} \cdot d\vec{A} = \int_S (\sqrt{5}\vec{k}) \cdot (-\vec{k}\; dA) = -\sqrt{5} \cdot \text{ Area of disk } = -\sqrt{5}\pi(\sqrt{10})^2 = -10\sqrt{5}\pi.$$

(b) The line is parallel to the x-axis, so only the $\vec{i}$ component of $\vec{F}$ contributes to the integral. Since $d\vec{r} = \vec{i}\, dx$ and $y = \sqrt{3}$ on the line,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\sqrt{3}\vec{i}) \cdot (\vec{i}\; dx) = \sqrt{3} \cdot \text{ Length of line } = 2\sqrt{3}.$$

(c) Using Stokes' Theorem, if C is the boundary of S with induced orientation,

$$\int_S \text{curl } \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}.$$

The boundary of the rectangle C has four parts. The orientation is counterclockwise when viewed from above. The integral along the x-axis is 0 because the $\vec{i}$ component is 0 there. Similarly, the integral along the y-axis is 0 because the $\vec{j}$ component is 0 there.

The integral along the top of the rectangle uses the reverse orientation to the answer to part (b), so its value is $-2\sqrt{3}$. The integral along the right side depends only on the $\vec{j}$ component at $x = 2$, so the value is of this integral is $\pi \cdot 2 \cdot$ Length of line $= 2\sqrt{3}\pi$. Thus,

$$\int_S \text{curl } \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r} = 0 + 0 - 2\sqrt{3} + 2\sqrt{3}\pi = 2\sqrt{3}(\pi - 1).$$

Alternatively, direct calculations gives

$$\text{curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & \pi x & z \end{vmatrix} = (\pi - 1)\vec{k}.$$

Thus,

$$\int_S \text{curl } \vec{F} \cdot d\vec{A} = \int_S (\pi - 1)\vec{k} \cdot \vec{k}\; dA = (\pi - 1) \cdot \text{ Area of } S = 2\sqrt{3}(\pi - 1).$$

31. We calculate

$$\text{curl}\left(x^2\vec{i}+y^2\vec{j}+(x+y+z)\vec{k}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & y^2 & x+y+z\end{vmatrix}=\vec{i}-\vec{j},$$

and use Stokes' Theorem with D as the surface $(x-1)^2+(y-2)^2\leq 4$ in the xy-plane, oriented in the positive z-direction:

$$\int_C\left(x^2\vec{i}+y^2\vec{j}+(x+y+z)\vec{k}\right)\cdot d\vec{r}=\int_D(\vec{i}-\vec{j})\cdot d\vec{A}.$$

Since $d\vec{A}=\vec{k}\,dA$, we have

$$\int_D(\vec{i}-\vec{j})\cdot d\vec{A}=\int_D 0\,dA=0.$$

32. We use Stokes' Theorem. We have

$$\text{curl}(-y^3\vec{i}+x^3\vec{j}+e^z\vec{k})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -y^3 & x^3 & e^z\end{vmatrix}=(3x^2+3y^2)\vec{k}.$$

So if D is the disk $x^2+y^2\leq 3$, $z=4$, oriented upward, we have

$$\int_C(-y^3\vec{i}+x^3\vec{j}+e^z\vec{k})\cdot d\vec{r}=\int_D(3x^2+3y^2)\vec{k}\cdot d\vec{A}=3\int_D(x^2+y^2)\,dA.$$

Converting to polar coordinates, we have

$$\int_C(-y^3\vec{i}+x^3\vec{j}+e^z\vec{k})\cdot d\vec{r}=3\int_0^{2\pi}\int_0^{\sqrt{3}}r^2\cdot r\,dr\,d\theta=3\cdot 2\pi\frac{r^4}{4}\bigg|_0^{\sqrt{3}}=\frac{27\pi}{2}.$$

33. We calculate

$$\text{curl}\left(\sin x^2\vec{i}+\cos y^2\vec{j}+(x+y)\vec{k}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \sin x^2 & \cos y^2 & x+y\end{vmatrix}=\vec{i}-\vec{j},$$

and use Stokes' Theorem with D as the surface $(y-1)^2+(z-2)^2\leq 4$ in the yz-plane, oriented in the positive x-direction:

$$\int_C\left(\sin x^2\vec{i}+\cos y^2\vec{j}+(x+y)\vec{k}\right)\cdot d\vec{r}=\int_D(\vec{i}-\vec{j})\cdot d\vec{A}.$$

Since $d\vec{A}=\vec{i}\,dA$, we have

$$\int_D(\vec{i}-\vec{j})\cdot d\vec{A}=\int_D 1\,dA=\text{ Area }D=\pi 2^2=4\pi.$$

34. We use Stokes' Theorem. If C is the circle $y^2+z^2=3$, oriented counterclockwise when viewed from the positive x-direction, then Stokes' Theorem gives

$$\int_S\text{curl}\,\vec{F}\cdot d\vec{A}=\int_C\vec{F}\cdot d\vec{r}=\int_C\left((z+y)\vec{i}-(z+x)\vec{j}+(y+x)\vec{k}\right)\cdot d\vec{r}.$$

Since C is in the yz-plane, the $\vec{i}$ component does not contribute to the line integral. On the yz-plane, $x=0$ so

$$\int_S\text{curl}\,\vec{F}\cdot d\vec{A}=\int_C\vec{F}\cdot d\vec{r}=\int_C(-z\vec{j}+y\vec{k})\cdot d\vec{r}.$$

On the circle $y^2 + z^2 = 3$, we have $\left\|-z\vec{j} + y\vec{k}\right\| = \sqrt{3}$ and $-z\vec{j} + y\vec{k}$ is tangent to the curve pointing in the counterclockwise direction, so

$$\int_S \text{curl}\,\vec{F} \cdot d\vec{A} = \int_C (-z\vec{j} + y\vec{k}) \cdot d\vec{r} = \sqrt{3} \cdot \text{Length of curve} = \sqrt{3} \cdot 2\pi\sqrt{3} = 6\pi.$$

35. Since

$$\text{div}(3x\vec{i} + 4y\vec{j} + xy\vec{k}) = 3 + 4 + 0 = 7,$$

we calculate the flux using the Divergence Theorem:

$$\text{Flux} = \int_S (3x\vec{i} + 4y\vec{j} + xy\vec{k}) \cdot d\vec{A} = \int_W 7\,dV = 7 \cdot \text{Volume of box} = 7 \cdot 3 \cdot 5 \cdot 2 = 210.$$

36. We use the Divergence Theorem.

Since $\text{div}\vec{F} = 3 - 1 + 2 = 4$ and a closed surface is oriented outward, the Divergence Theorem gives

$$\int_S \vec{F} \cdot d\vec{A} = \int_{\substack{\text{Interior}\\ \text{of sphere}}} \text{div}\,\vec{F}\;dV = 4 \cdot \text{Volume of sphere} = 4 \cdot \frac{4}{3}\pi 1^3 = \frac{16\pi}{3}.$$

37. We use the Divergence Theorem. Since $\text{div}\vec{F} = 3x^2 + 3y^2 + 3z^2$ and a closed surface is oriented outward, the Divergence Theorem gives

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int_{\substack{\text{Interior}\\ \text{of sphere}}} \text{div}\,\vec{F}\;dV \\
&= \int_{\substack{\text{Interior}\\ \text{of sphere}}} \left(3x^2 + 3y^2 + 3z^2\right) dV \\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 3 \cdot 2\pi(-\cos\phi)\Big|_0^{\pi} \frac{\rho^5}{5}\Big|_0^1 = \frac{12\pi}{5}.
\end{aligned}$$

38. We use Stokes' Theorem. Since

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & y + 2z & z + 3x \end{vmatrix} = -2\vec{i} - 3\vec{j} - \vec{k},$$

if S is the interior of the square, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\;\vec{F} \cdot d\vec{A} = \int_S (-2\vec{i} - 3\vec{j} - \vec{k}) \cdot d\vec{A}$$

Since the area vector of S is $49\vec{j}$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (-2\vec{i} - 3\vec{j} - \vec{k}) \cdot d\vec{A} = -3\vec{j} \cdot 49\vec{j} = -147.$$

39. We use Stokes' Theorem. Since

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y^3 + z & x^3 + y + z & x + y + z^3 \end{vmatrix} = (3x^2 + 3y^2)\vec{k},$$

if S is the disk $x^2 + y^2 \le 10$ oriented upward,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (3x^2 + 3y^2)\vec{k} \cdot d\vec{A} = 3\int_0^{2\pi}\int_0^{\sqrt{10}} r^2 \cdot r\,dr\,d\theta = 3 \cdot 2\pi \cdot \frac{r^4}{4}\Big|_0^{\sqrt{10}} = 150\pi.$$

40. We use the Divergence Theorem. Since $\operatorname{div}\vec{F} = 3y^2 + 3z^2$, we have

$$\int_S \vec{F} \cdot d\vec{A} = \int_{\substack{\text{Interior}\\ \text{of cylinder}}} (3y^2 + 3z^2)\, dV.$$

Converting to cylindrical coordinates with $y = r\cos\theta, z = r\sin\theta$, we have $y^2 + z^2 = r^2$, so

$$\int_S \vec{F} \cdot d\vec{A} = 3\int_{-1}^{1}\int_0^{2\pi}\int_0^4 r^2 \cdot r\, dr\, d\theta\, dx = 3 \cdot 2 \cdot 2\pi \frac{r^4}{4}\bigg|_0^4 = 768\pi.$$

41. We use the fact that if S_1 and S_2 are surfaces which share a common boundary C, and if S_1 and S_2 determine the same orientation of C, then

$$\int_{S_1} \operatorname{curl}\vec{F} \cdot d\vec{A} = \int_{S_2} \operatorname{curl}\vec{F} \cdot d\vec{A}.$$

We replace the surface S by S_0, the base of the cube, oriented upward because S_0 is easier to integrate over. (Both surfaces have the unit square in the xy-plane as the boundary.) Since S_0 is horizontal, we need only find the $\vec{k}$ component of $\operatorname{curl}\vec{F}$:

$$z\text{-component of } \operatorname{curl}\vec{F} = \frac{\partial(ye^x)}{\partial x} - \frac{\partial(-xe^y)}{\partial y} = (ye^x + xe^y)\vec{k}.$$

Thus, we have

$$\int_S \operatorname{curl}\vec{F} \cdot d\vec{A} = \int_{S_0} (ye^x + xe^y)\, dA = \int_0^1\int_0^1 (ye^x + xe^y)\, dxdy = \int_0^1 ye^x + \frac{x^2}{2}e^y\bigg|_0^1 dy$$
$$= \int_0^1 (ey + \frac{e^y}{2} - y)\, dy = (e-1)\frac{y^2}{2} + \frac{e^y}{2}\bigg|_0^1 = e - 1.$$

42. Let S be the disk inside the unit circle, oriented upward. By Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S 4\vec{k} \cdot d\vec{A} = 4 \cdot \text{Area of disk} = 4 \cdot \pi 1^2 = 4\pi.$$

43. (a) Let D be the face of the box which was removed from the yz-plane. We use Stokes' Theorem with D as the surface.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & -z & y \end{vmatrix} = 2\vec{i} - 2\vec{k},$$

so

$$\int_C \vec{F} \cdot d\vec{r} = \int_D \operatorname{curl}\vec{F} \cdot d\vec{A} = \int_D (2\vec{i} - 2\vec{k}) \cdot d\vec{A} = 2(\text{Area of } D) = 2 \cdot 3^2 = 18.$$

(b) To use the Divergence Theorem, we calculate that $\operatorname{div}\vec{F} = 1$. Since the Divergence Theorem requires a closed surface, we close S by adding D, oriented in the direction of the negative x-axis. Then

$$\int_{S+D} \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot d\vec{A} + \int_D \vec{F} \cdot d\vec{A} = \int_{\text{Interior}} 1\, dv = \text{Volume of box} = 27.$$

Now on D, where $x = 0$, we have $\vec{F} = y\vec{i} - z\vec{j} + y\vec{k}$. In addition, on D, we have $d\vec{A} = -\vec{i}\, dy\, dz$, so

$$\int_D \vec{F} \cdot d\vec{A} = \int (y\vec{i} - z\vec{j} + y\vec{k}) \cdot (-i\, dy\, dz) = -\int_0^3\int_0^3 y\, dy\, dz$$
$$= -\int_0^3 \frac{y^2}{2}\bigg|_0^3 dz = \frac{-9}{2}\int_0^3 dz = -\frac{27}{2}.$$

Thus,

$$\int_S \vec{F} \cdot d\vec{A} = 27 - \int_D \vec{F} \cdot d\vec{A} = 27 - \left(-\frac{27}{2}\right) = \frac{81}{2}.$$

44. (a) Can be computed. If W is the interior of the sphere, by the Divergence Theorem, we have

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV = 4 \cdot \text{Volume of sphere} = 4 \cdot \frac{4}{3}\pi \cdot 2^3 = \frac{128\pi}{3}.$$

(b) Cannot be computed.

(c) Can be computed. Use the fact that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$. If W is the inside of the sphere, then by the Divergence Theorem,

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_W \operatorname{div}(\operatorname{curl} \vec{F})dV = \int_W 0\, dV = 0$$

45. (a) $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z & 2xyz & xy \end{vmatrix} = (x - 2xy)\vec{i} - (y - y^2)\vec{j} + (2yz - 2yz)\vec{k} \neq \vec{0}$. Since $\operatorname{curl} \vec{F} \neq 0$, we know $\vec{F}$ is not conservative.

(b) A line integral round the closed path shown in Figure 20.9 is not zero, so $\vec{F}$ is not conservative.

Note: To show that a vector field is not conservative, we only need to find *one* path whose line integral is nonzero. To show that a vector field is conservative, we must show the line integral is zero on *all* paths.

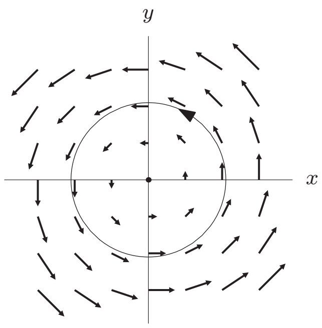

Figure 20.9

46. (a) We have

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right)\vec{k}.$$

Since

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and similarly $\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, we have, provided $x^2 + y^2 \neq 0$,

$$\operatorname{curl} \vec{F} = \vec{0}.$$

The domain of $\operatorname{curl} \vec{F}$ is all points in 3-space except the z-axis.

(b) On C_1, the unit circle $x^2 + y^2 = 1$ in the xy-plane, the vector field $\vec{F}$ is tangent to the circle and $||\vec{F}|| = 1$. Thus

$$\text{Circulation} = \int_{C_1} \vec{F} \cdot d\vec{r} = ||\vec{F}|| \cdot \text{Perimeter of circle} = 2\pi.$$

Note that Stokes' Theorem cannot be used to calculate this circulation since the z-axis pierces any surface which has this circle as boundary.

(c) Consider the disk $(x-3)^2 + y^2 \le 1$ in the plane $z = 4$. This disk has C_2 as boundary and $\text{curl}\,\vec{F} = \vec{0}$ everywhere on this disk. Thus, by Stokes' Theorem $\int_{C_2} \vec{F} \cdot d\vec{r} = 0$.

(d) The square S has an interior region which is pierced by the z-axis, so we cannot use Stokes' Theorem. We consider the region, D, between the circle C_1 and the square S. See Figure 20.10. Stokes' Theorem applies to the region D, provided C_1 is oriented clockwise. Then we have

$$\int_{C_1(\text{clockwise})} \vec{F} \cdot d\vec{r} + \int_S \vec{F} \cdot d\vec{r} = \int_D \text{curl}\,\vec{F} \cdot d\vec{A} = 0.$$

Thus,

$$\int_S \vec{F} \cdot d\vec{r} = -\int_{C_1(\text{clockwise})} \vec{F} \cdot d\vec{r} = \int_{C_1(\text{counterclockwise})} \vec{F} \cdot d\vec{r} = 2\pi.$$

(e) If a simple closed curve goes around the z-axis, then it contains a circle C of the form $x^2 + y^2 = a^2$. The circulation around C is 2π or -2π, depending on its orientation. A calculation similar to that in part (d) then shows that the circulation around the curve is 2π or -2π, again depending on its orientation. If the closed curve does not go around the z-axis, then $\text{curl}\,\vec{F} = \vec{0}$ everywhere on its interior and the circulation is zero.

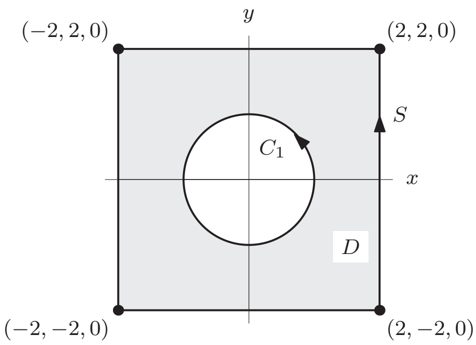

Figure 20.10

47. (a) Notice that $\vec{F}$ is tangent to the circle, because

$$\vec{F} \cdot \vec{r} = \frac{(-y\vec{i} + x\vec{j})}{x^2 + y^2} \cdot (x\vec{i} + y\vec{j}) = 0,$$

and $\vec{F}$ has constant magnitude on the circle $x^2 + y^2 = 9$, since

$$||\vec{F}|| = \left|\left|\frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}\right|\right| = \frac{\sqrt{(-y)^2 + x^2}}{x^2 + y^2} = \frac{1}{3}.$$

Thus, the line integral can be calculated as follows:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = ||\vec{F}|| \cdot \text{ Circumference of circle } = \frac{1}{3} \cdot 2\pi 3 = 2\pi.$$

(b) We have

$$\begin{aligned}
\text{curl}\,\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\vec{k} \\
&= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right)\vec{k} \\
&= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}\vec{k} = \vec{0}.
\end{aligned}$$

Thus, $\text{curl}\,\vec{F} = \vec{0}$ except where $x = y = 0$, on the z-axis, where it is not defined.

(c) Stokes' Theorem cannot be used to calculate $\int_{C_1}$ because the z-axis cuts any surface whose boundary is C_1.
(d) Since the disk D_2 inside C_2 does not intersect the z-axis, Stokes' Theorem can be used:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{D_2} \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_{D_2} \vec{0} \cdot d\vec{A} = 0.$$

(e) The vector field $\vec{F}$ is not a gradient field since $\int_{C_1} \vec{F} \cdot d\vec{r} \neq 0$.

48. Note that planes of the form $mx + ny = d$ are vertical, so their normals have no $\vec{k}$ component.

(a) Since

$$\operatorname{curl} \vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \left(\frac{\partial (x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\vec{k} = 2\vec{k},$$

by Stokes' Theorem, the circulation around curves in the xy-plane and the plane $x + y + z = 0$ can be nonzero. The circulation is 0 for (II), (III), (V).

(b) Since $\vec{G} = \operatorname{grad}(xy)$, the circulation around any closed curve is 0. So the circulation is 0 around (I), (II), (III), (IV), (V).

(c) Since

$$\operatorname{curl} \vec{H} = -\frac{\partial H_2}{\partial z}\vec{i} = -\frac{\partial (z)}{\partial z}\vec{i} = -\vec{i},$$

the circulation can be nonzero around any curve in a plane whose normal has a nonzero $\vec{i}$ component. The circulation is 0 for (I), (III).

49. (a) Only the $\vec{k}$-component of $\vec{F}$ contributes to the line integral, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{10} z\vec{k} \cdot \vec{k}\, dz = \int_0^{10} z\, dz = \left.\frac{z^2}{2}\right|_0^{10} = 50.$$

(b) Only the $\vec{k}$ component of $\vec{F}$ contributes to the flux integral; this component is $10\vec{k}$. Thus,

$$\int_S \vec{F} \cdot d\vec{S} = 10 \cdot \text{ Area of } S = 10\pi(\sqrt{3})^2 = 30\pi.$$

(c) We use the Divergence Theorem, with $\operatorname{div} \vec{F} = 1$. Thus, if W is the region inside the box,

$$\int_S \vec{F} \cdot d\vec{A} = \int_W 1\, dV = 1 \cdot \text{ Volume of box } = 2^3 = 8.$$

(d) Only the $\vec{i}$ and $\vec{j}$ components contribute to the line integral. The horizontal component of the vector field, $y\vec{i} - x\vec{j}$, is tangent to the circle, and points in the clockwise direction. On C, the length of the horizontal component is

$$||y\vec{i} - x\vec{j}|| = \sqrt{y^2 + x^2} = 3.$$

Thus,

$$\int_C \vec{F} \cdot d\vec{r} = -3 \cdot \text{ Circumference of circle } = -3 \cdot 2\pi 3 = -18\pi.$$

CAS Challenge Problems

50. (a) $\operatorname{curl} \vec{F}(1,2,1) = -6\vec{i} - 8\vec{j} + \vec{k}$.

(b) $\int_{C_a} \vec{F} \cdot d\vec{r} = -\frac{3}{2}(4a^2 + a^4)\pi$.

$$\lim_{a \to 0} \frac{\int_{C_a} \vec{F} \cdot d\vec{r}}{\pi a^2} = -6.$$

This is the $\vec{i}$ component of $\operatorname{curl}\vec{F}(1,2,1)$ according to the circulation density formula.

(c) $\int_{D_a} \vec{F} \cdot d\vec{r} = -8a^2\pi$.

$$\lim_{a \to 0} \frac{\int_{C_a} \vec{F} \cdot d\vec{r}}{\pi a^2} = -8.$$

This is the $\vec{j}$ component of $\operatorname{curl}\vec{F}(1,2,1)$.

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(d) $\int_{E_a} \vec{F} \cdot d\vec{r} = -\frac{1}{4}(-4a^2 + 3a^4)\pi$.

$$\lim_{a\to 0} \frac{\int_{C_a} \vec{F} \cdot d\vec{r}}{\pi a^2} = 1.$$

This is the $\vec{k}$ component of curl$\vec{F}$ $(1, 2, 1)$.

(e) The curves C_a are circles of radius a around the point $(1, 2, 1)$ facing in the positive x-direction, so the limit as $a \to 0$ of the circulation around them is the $\vec{i}$-component of the curl, according to the geometric definition. Similarly, the curves D_a are circles facing in the y-direction and the curves E_a are circles facing in the z-direction, so they give the $\vec{j}$ and $\vec{k}$ components of the curl.

51. (a) Let W be the region enclosed by the sphere. We have div $\vec{F} = 2ax+bz+2cy+p+q$ so by the Divergence Theorem $\int_S \vec{F} \cdot d\vec{A} = \int_W (2ax+bz+2cy+p+q)\,dV$. Now $\int_W x\,dV = \int_W y\,dV = \int_W z\,dV = 0$, because W is symmetric about the origin and x, y, and z are odd functions. So $\int_W (2ax+bz+2cy+p+q)\,dV = \int_B (p+q)dV = \frac{4(p+q)\pi R^3}{3}$.

(b) Using spherical coordinates, we calculate the flux integral directly as

$$\int_0^{2\pi}\int_0^{\pi} ((bR^2\cos(\theta)\cos(\phi)\sin(\phi) + aR^2\cos(\theta)^2\sin(\phi)^2)\vec{i}$$
$$+(pR\sin(\theta)\sin(\phi) + cR^2\sin(\theta)^2\sin(\phi)^2)\vec{j} + (qR\cos(\phi)$$
$$+rR^3\cos(\theta)^3\sin(\phi)^3)\vec{k}\,) \cdot (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\,)R^2\sin\phi\,d\phi d\theta = \frac{4(p+q)\pi R^3}{3}.$$

Rather than entering this integral directly into your CAS, it is better to define the vector field and parameterization separately and enter the formula for flux integral through a sphere.

52. (a) div $\vec{F}$ $(1, 1, 1) = 5$.

(b) On a small sphere centered at $(1, 1, 1)$ the divergence is approximately constant and equal to its value at $(1, 1, 1)$, namely 5. Geometrically the divergence is the flux density, so the total flux is approximately the divergence multiplied by the volume, so $\int_S \vec{F} \cdot d\vec{A} \approx 5($ volume of $S) = \frac{20}{3}\pi(0.1)^3 \approx 0.02094...$

(c) The area element on the sphere is $d\vec{A} = (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}\,)a^2\sin\phi\,d\phi d\theta$. Using a CAS to calculate the flux integral we get $\int_S \vec{F} \cdot d\vec{A} = (20\pi a^3/3) + (4\pi a^5/5)$, which is .02097 when $a = .1$. This is close to the approximation we found in part (b). The limit

$$\lim_{a\to 0} \frac{\int_{S_a} \vec{F} \cdot d\vec{A}}{\text{Volume inside } S_a} = \lim_{a\to 0} \frac{(20\pi a^3/3) + (4\pi a^5/5)}{(4/3)\pi a^3} = \lim_{a\to 0}(5 + \frac{3a^2}{5}) = 5.$$

This agrees with the value of the divergence we found in part (a). This makes sense, because the limit is just the geometric definition of divergence.

PROJECTS FOR CHAPTER TWENTY

1. (a) Since S does not touch the wire, it is in a region of space where there is no current. Hence curl $\vec{B} = \vec{0}$ at every point of S. Therefore,

$$\int_S \text{curl}\,\vec{B} \cdot d\vec{A} = 0.$$

The flux of curl $\vec{B}$ through S is 0.

(b) The boundary of S is a curve, C, in two pieces, C_1 and C_2. Given the upward orientation of S, the curve C_1 is oriented clockwise and C_2 is oriented counterclockwise when viewed from above. (See Figure 20.26 in the text.)

The vector field has constant magnitude on each circle and is parallel to the circle. Therefore,

$$\int_{C_2} \vec{B} \cdot d\vec{r} = \|\vec{B}\,(P_2)\| \cdot \text{Length of } C_2 = 2\pi R_2\|\vec{B}\,(P_2)\|.$$

Because C_1 is oriented in the opposite direction

$$\int_{C_1} \vec{B} \cdot d\vec{r} = -\|\vec{B}\,(P_1)\| \cdot \text{Length of } C_1 = -2\pi R_1\|\vec{B}\,(P_1)\|.$$

(c) From Stokes' Theorem,

$$\int_S \operatorname{curl} \vec{B} \cdot d\vec{A} = \int_C \vec{B} \cdot d\vec{r}.$$

Since

$$\int_S \operatorname{curl} \vec{B} \cdot d\vec{A} = 0$$

and

$$\int_C \vec{B} \cdot d\vec{r} = \int_{C_1} \vec{B} \cdot d\vec{r} + \int_{C_2} \vec{B} \cdot d\vec{r}$$

we have

$$\int_{C_2} \vec{B} \cdot d\vec{r} = -\int_{C_1} \vec{B} \cdot d\vec{r}.$$

Hence,

$$2\pi R_2 \|\vec{B}(P_2)\| = -(-2\pi R_1 \|\vec{B}(P_1)\|)$$

and thus

$$\|\vec{B}(P_2)\| = (R_1\|\vec{B}(P_1)\|)\frac{1}{R_2}.$$

Letting $k = R_1\|\vec{B}(P_1)\|$ be constant and thinking of $R_2 = r$ as a variable, we have

$$\|\vec{B}(P_2)\| = k\frac{1}{r}.$$

This relationship shows that $\|\vec{B}(P_2)\|$ is proportional to the reciprocal of the distance r from P_2 to the wire.

(d) If the distance from P to the wire is r, then the distance from Q to the wire is $2r$. Using the proportionality in part (c), we have

$$\|\vec{B}(P)\| = \frac{k}{r}$$

and

$$\|\vec{B}(Q)\| = \frac{k}{2r} = \frac{1}{2}\|\vec{B}(P)\|.$$

Doubling the distance from the wire cuts the magnitude of $\vec{B}$ in half.

(e) Suppose the distance from P to the wire is r, so that

$$\|\vec{B}(P)\| = \frac{k}{r}.$$

If the distance from Q to the wire is R and if $\|\vec{B}(Q)\| = 0.8\|\vec{B}(P)\|$, then

$$\|\vec{B}(Q)\| = \frac{k}{R} = 0.8\frac{k}{r}.$$

Solving gives

$$R = \frac{r}{0.8} = 1.25r,$$

so the distance from the wire must be increased by 25%.

2. First compute the unit vectors $\vec{T}$ and $\vec{N}$. Since $\vec{T}$ is in the direction of $\vec{F}$ we have

$$\vec{T} = \frac{1}{\|\vec{F}\|}\vec{F} = \frac{1}{F}(u\vec{i} + v\vec{j}).$$

Since $\vec{N}$ is the unit vector in the direction of $\vec{k} \times \vec{F}$ we have

$$\begin{aligned}
\vec{k} \times \vec{F} &= \vec{k} \times (u\vec{i} + v\vec{j}) \\
&= -v\vec{i} + u\vec{j} \\
\vec{N} &= \frac{1}{\| - v\vec{i} + u\vec{j}\|}(-v\vec{i} + u\vec{j}) \\
&= \frac{1}{F}(-v\vec{i} + u\vec{j}).
\end{aligned}$$

The chain rule for partial differentiation of the formulas $u = F\cos\theta$ and $v = F\sin\theta$ gives

$$\begin{aligned}
u_y &= (\cos\theta)F_y - F(\sin\theta)\theta_y \\
v_x &= (\sin\theta)F_x + F(\cos\theta)\theta_x.
\end{aligned}$$

Since, for a 2-dimensional vector field, $\text{curl}\vec{F} = (v_x - u_y)\vec{k}$ we have

$$\begin{aligned}
c &= v_x - u_y \\
&= ((\sin\theta)F_x + F(\cos\theta)\theta_x) - ((\cos\theta)F_y - F(\sin\theta)\theta_y) \\
&= (u\theta_x + v\theta_y) + \frac{1}{F}(vF_x - uF_y) \\
&= (\theta_x\vec{i} + \theta_y\vec{j}) \cdot (u\vec{i} + v\vec{j}) - \frac{1}{F}(F_x\vec{i} + F_y\vec{j}) \cdot (-v\vec{i} + u\vec{j}) \\
&= F\,\text{grad}\,\theta \cdot \vec{T} - \text{grad}\,F \cdot \vec{N}.
\end{aligned}$$

Since the directional derivative of θ in the direction of $\vec{T}$ is $\theta_{\vec{T}} = \text{grad}\,\theta \cdot \vec{T}$ and the directional derivative of F in the direction of $\vec{N}$ is $F_{\vec{N}} = \text{grad}\,F \cdot \vec{N}$ we have

$$c = F\theta_{\vec{T}} - F_{\vec{N}}.$$

CHAPTER TWENTY-ONE

Solutions for Section 21.1

Exercises

1. There is just one parameter, s, so the parameterization describes a curve.

2. There are two parameters, s and t, so the parameterization describes a surface.

3. There are two parameters, s and t, so the parameterization describes a surface.

4. There is just one parameter, s, so the parameterization describes a curve.

5. A horizontal disk of radius 5 in the plane $z = 7$.

6. A cylinder of radius 5 centered around the z-axis and stretching around from $z = 0$ to $z = 7$.

7. A helix (curve) of radius 5 which makes one turn about the z-axis, starting at the point $(5, 0, 0)$ and ending at the point $(5, 0, 10\pi)$.

8. Since $z = r = \sqrt{x^2 + y^2}$, we have a cone around the z-axis. Since $0 \le r \le 5$, we have $0 \le z \le 5$, so the cone has height and maximum radius of 5.

9. The top half of the sphere $(z \ge 0)$.

10. The half of the sphere with $y \le 0$.

11. A vertical segment lying between two longitudinal lines ($\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$) and stretching between the poles.

12. Half the horizontal ring around the sphere between two latitude lines ($\phi = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$) in the northern hemisphere.

Problems

13. Two vectors in the plane containing $P = (0, 0, 0)$, $Q = (1, 2, 3)$, and $R = (2, 1, 0)$ are the displacement vectors

$\vec v_1 = \vec{PQ} = \vec i + 2\vec j + 3\vec k$
$\vec v_2 = \vec{PR} = 2\vec i + \vec j$.

Letting $\vec r_0 = 0\vec i + 0\vec j + 0\vec k = \vec 0$ we have the parameterization

$$\begin{aligned}\vec r(s,t) &= \vec r_0 + s\vec v_1 + t\vec v_2\\ &= (s + 2t)\vec i + (2s + t)\vec j + 3s\vec k.\end{aligned}$$

14. Two vectors in the plane containing $P = (1, 2, 3)$, $Q = (2, 5, 8)$, and $R = (5, 2, 0)$ are the displacement vectors

$\vec v_1 = \vec{PQ} = \vec i + 3\vec j + 5\vec k$
$\vec v_2 = \vec{PR} = 4\vec i - 3\vec k$.

Letting $\vec r_0 = \vec i + 2\vec j + 3\vec k$ we have the parameterization

$$\begin{aligned}\vec r(s,t) &= \vec r_0 + s\vec v_1 + t\vec v_2\\ &= (1 + s + 4t)\vec i + (2 + 3s)\vec j + (3 + 5s - 3t)\vec k.\end{aligned}$$

15. To parameterize the plane we need two nonparallel vectors $\vec v_1$ and $\vec v_2$ that are parallel to the plane. Such vectors are perpendicular to the normal vector to the plane, $\vec n = \vec i + \vec j + \vec k$. We can choose any vectors $\vec v_1$ and $\vec v_2$ such that $\vec v_1 \cdot \vec n = \vec v_2 \cdot \vec n = 0$.

One choice is

$$\vec v_1 = \vec i - j \qquad \vec v_2 = \vec i - \vec k.$$

Letting $\vec r_0 = 3\vec i + 5\vec j + 7\vec k$ we have the parameterization

$$\begin{aligned}\vec r(s,t) &= \vec r_0 + s\vec v_1 + t\vec v_2\\ &= (3 + s + t)\vec i + (5 - s)\vec j + (7 - t)\vec k.\end{aligned}$$

16. To parameterize the plane we need two nonparallel vectors $\vec{v}_1$ and $\vec{v}_2$ that are parallel to the plane. Such vectors are perpendicular to the normal vector to the plane, $\vec{n} = \vec{i} + 2\vec{j} + 3\vec{k}$. We can choose any vectors $\vec{v}_1$ and $\vec{v}_2$ such that $\vec{v}_1 \cdot \vec{n} = \vec{v}_2 \cdot \vec{n} = 0$.

One choice is

$$\vec{v}_1 = 2\vec{i} - j \qquad \vec{v}_2 = 3\vec{i} - \vec{k}.$$

Letting $\vec{r}_0 = 5\vec{i} + \vec{j} + 4\vec{k}$ we have the parameterization

$$\begin{aligned}\vec{r}(s,t) &= \vec{r}_0 + s\vec{v}_1 + t\vec{v}_2\\ &= (5+2s+3t)\vec{i} + (1-s)\vec{j} + (4-t)\vec{k}.\end{aligned}$$

17. (a) We want to find s and t so that

$$\begin{aligned}2+s &= 4\\ 3+s+t &= 8\\ 4t &= 12\end{aligned}$$

Since $s = 2$ and $t = 3$ satisfy these equations, the point $(4, 8, 12)$ lies on this plane.

(b) Are there values of s and t corresponding to the point $(1, 2, 3)$? If so, then

$$\begin{aligned}1 &= 2+s\\ 2 &= 3+s+t\\ 3 &= 4t\end{aligned}$$

From the first equation we must have $s = -1$ and from the third we must have $t = 3/4$. But these values of s and t do not satisfy the second equation. Therefore, no value of s and t corresponds to the point $(1, 2, 3)$, and so $(1, 2, 3)$ is not on the plane.

18. If the planes are parallel, then their normal vectors will also be parallel. The equation of the first plane can be written

$$\vec{r} = 2\vec{i} + 4\vec{j} + \vec{k} + s(\vec{i} + \vec{j} + 2\vec{k}) + t(\vec{i} - \vec{j}).$$

A normal vector to the first plane is $\vec{n}_1 = (\vec{i} + \vec{j} + 2\vec{k}) \times (\vec{i} - \vec{j}) = 2\vec{i} + 2\vec{j} - 2\vec{k}$. The second plane can be written

$$\vec{r} = 2\vec{i} + s(\vec{i} + \vec{k}) + t(2\vec{i} + \vec{j} - \vec{k}).$$

A normal vector to the second plane is $\vec{n}_2 = (\vec{i} + \vec{k}) \times (2\vec{i} + \vec{j} - \vec{k}) = -\vec{i} + 3\vec{j} + \vec{k}$. Since $\vec{n}_1$ and $\vec{n}_2$ are not parallel, neither are the two planes.

19. The surface is the plane $z = 1$. The family of parameter curves with s constant and t varying consists of lines in the plane parallel to the y-axis. The family with t constant and s varying consists of lines in the plane parallel to the x-axis.

20. The surface is the cylinder of radius 1 centered on the x-axis with equation $y^2 + z^2 = 1$. The family of parameter curves with s constant and t varying consists of circles on the cylinder, the cross-sections of the cylinder parallel to the yz-plane. The family with t constant and s varying consists of lines on the cylinder parallel to the x-axis.

21. The surface is the graph of the equation $z = x^2 + y^2$, a parabola. The family of parameter curves with s constant and t varying consists of cross-sections of the graph with x fixed. If $s = s_0$, the cross-section has equation $z = s_0^2 + y^2$ in the plane $x = s_0$, so it is a parabola. The family of parameter curves with t constant and s varying consists of cross-sections of the graph with y fixed. If $t = t_0$, the cross-section has equation $z = x^2 + t_0^2$ in the plane $y = t_0$, so it is a parabola.

22. The parametric equations are the spherical coordinate parameterization of the unit sphere centered at the origin, with $s = \theta$ and $t = \phi$. The family of parameter curves with $s = \theta$ constant and $t = \phi$ varying are meridians of constant longitude, semicircles from the north pole to the south pole. The family of parameter curves with $t = \phi$ constant and $s = \theta$ varying consists of latitude circles.

23. Since you walk 5 blocks east and 1 block west, you walk 5 blocks in the direction of $\vec{v}_1$, and 1 block in the opposite direction. Thus,

$$s = 5 - 1 = 4,$$

Similarly,

$$t = 4 - 2 = 2.$$

Hence

$$\begin{aligned} x\vec{i} + y\vec{j} + z\vec{k} &= (x_0\vec{i} + y_0\vec{j} + z_0\vec{k}) + 4\vec{v}_1 + 2\vec{v}_2 \\ &= (x_0\vec{i} + y_0\vec{j} + z_0\vec{k}) + 4(2\vec{i} - 3\vec{j} + 2\vec{k}) + 2(\vec{i} + 4\vec{j} + 5\vec{k}) \\ &= (x_0 + 10)\vec{i} + (y_0 - 4)\vec{j} + (z_0 + 18)\vec{k}. \end{aligned}$$

Thus the coordinates are:

$$x = x_0 + 10, \quad y = y_0 - 4, \quad z = z_0 + 18.$$

24. (a) Nearer to the equator.
(b) Farther from the north pole.
(c) Farther from Greenwich.

25. A horizontal circle in the northern hemisphere at a latitude of 45° north of the equator.

26. A vertical half-circle, going from the north to south poles.

27. Set up the coordinates as in Figure 21.1. The surface is the revolution surface obtained by revolving the curve shown in Figure 21.2 about the z axis. From the measurements given, we obtain the equation of the curve in Figure 21.2:

$$x = \cos\left(\frac{\pi}{3}z\right) + 3, \qquad 0 \le z \le 48$$

(a) Rotating this around the z-axis, and taking $z = t$ as the parameter, we get the parametric equations

$$\begin{aligned} x &= (\cos\left(\frac{\pi}{3}t\right) + 3)\cos\theta \\ y &= (\cos\left(\frac{\pi}{3}t\right) + 3)\sin\theta \\ z &= t \qquad 0 \le \theta \le 2\pi,\ 0 \le t \le 48 \end{aligned}$$

(b) We know that the points in the curve consists of cross-sections of circles parallel to the xy plane and of radius $\cos((\pi/3)z + 3)$. Thus,

$$\text{Area of cross-section} = \pi\left(\cos\left(\frac{\pi}{3}z + 3\right)\right)^2$$

Integrating over z, we get

$$\begin{aligned} \text{Volume } &= \pi\int_0^{48}\left(\cos\frac{\pi}{3}z + 3\right)^2 dz \\ &= \pi\int_0^{48}\left(\cos^2\frac{\pi}{3}z + 6\cos\frac{\pi}{3}z + 9\right) dz \\ &= 456\pi \text{ in}^3. \end{aligned}$$

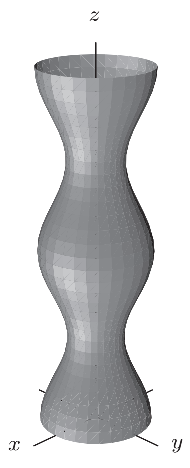


Figure 21.1

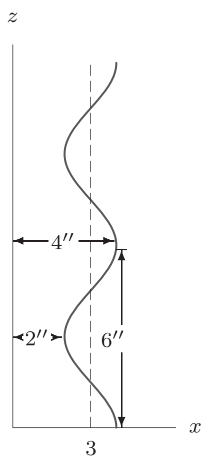


Figure 21.2

28. The sphere $(x-a)^2+(y-b)^2+(z-c)^2=d^2$ has center at the point (a,b,c) and radius d. We use spherical coordinates θ and ϕ as the two parameters. The parameterization of the sphere with center at the origin and radius d is

$$x = d\sin\phi\cos\theta, \quad y = d\sin\phi\sin\theta, \quad z = d\cos\phi.$$

Since the given sphere has center at the point (a,b,c) we add the displacement vector $a\vec{i}+b\vec{j}+c\vec{k}$ to the radial vector corresponding to a parameterization of the sphere with center at the origin and radius d to give

$$\begin{aligned} x &= a + d\sin\phi\cos\theta, \qquad 0 \le \phi \le \pi, \\ y &= b + d\sin\phi\sin\theta, \qquad 0 \le \theta \le 2\pi, \\ z &= c + d\cos\phi. \end{aligned}$$

To check that this is a parameterization for the given sphere, we substitute for x, y, z:

$$\begin{aligned} &(x-a)^2+(y-b)^2+(z-c)^2 \\ &= d^2\sin^2\phi\cos^2\theta + d^2\sin^2\phi\sin^2\theta + d^2\cos^2\phi \\ &= d^2\sin^2\phi + d^2\cos^2\phi = d^2. \end{aligned}$$

29. Let $(\theta, \pi/2)$ be the original coordinates. If $\theta < \pi$, then the new coordinates will be $(\theta+\pi, \pi/4)$. If $\theta \ge \pi$, then the new coordinates will be $(\theta-\pi, \pi/4)$.

30. If we set $z = u$, $x^2+y^2=u^2$ is the equation of a circle with radius $|u|$. Hence a parameterization of the cone is:

$$\begin{aligned} x &= u\cos v, \\ y &= u\sin v, \qquad 0 \le v \le 2\pi, \\ z &= u. \end{aligned}$$

31. Since the parameterization in Example 6 on page 1079 was $r = (1-\frac{z}{h})a$ and since the cone is given by $z = r$, we have $z = (1-\frac{r}{a})h$. The parameterization we want is

$$\begin{aligned} x &= r\cos\theta, \qquad 0 \le r \le a, \\ y &= r\sin\theta, \qquad 0 \le \theta \le 2\pi, \\ z &= \left(1-\frac{r}{a}\right)h. \end{aligned}$$

32. The plane in which the circle lies is parameterized by

$$\vec{r}(p,q) = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + p\vec{u} + q\vec{v}.$$

Because $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, the parameters p and q establish a rectangular coordinate system on this plane exactly analogous to the usual xy-coordinate system, with $(p,q)=(0,0)$ corresponding to the point (x_0,y_0,z_0). Thus the circle we want to describe, which is the circle of radius a centered at $(p,q)=(0,0)$, can be parameterized by

$$p = a\cos t, \qquad q = a\sin t.$$

Substituting into the equation of the plane gives the desired parameterization of the circle in 3-space,

$$\vec{r}(t) = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + a\cos t\vec{u} + a\sin t\vec{v},$$

where $0 \le t \le 2\pi$.

33. (a) Add second and third equations to get $y+z=1+2s$. Thus, $y+z=1+x$ or $-x+y+z=1$, which is the equation of a plane. Now, $s=x/2$, and $t=(y-z+1)/2$, so the conditions $0\le s\le 1$, $0\le t\le 1$ are equivalent to $0\le x\le 2$, $0\le y-z+1\le 2$ or $0\le x\le 2$, $-1\le y-z\le 1$.

(b) The surface is shown in Figure 21.3.

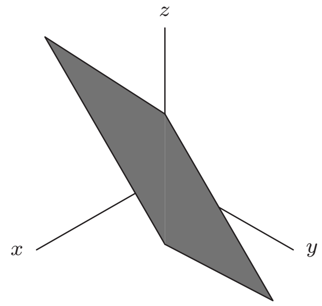


Figure 21.3: The surface $x = 2s$, $y = s + t$, $z = 1 + s - t$, for $0 \leq s \leq 1, 0 \leq t \leq 1$

34. (a) $z^2 = 1 - s^2 - t^2 = 1 - x^2 - y^2$. So $x^2 + y^2 + z^2 = 1$ which is the equation of a sphere. The conditions $s^2 + t^2 \leq 1$, $s, t \geq 0$ are equivalent to $x^2 + y^2 \leq 1$ and $x, y \geq 0$. But if $x^2 + y^2 + z^2 = 1$, then $x^2 + y^2 \leq 1$ is satisfied automatically, so our surface is defined by:

$$x^2 + y^2 + z^2 = 1, \qquad x, y, z \geq 0.$$

(b) The surface $x = s$, $y = t$, $z = \sqrt{1 - s^2 - t^2}$ for $s^2 + t^2 \leq 1$, $s, t \geq 0$ is shown in Figure 21.4.

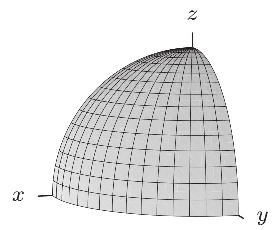


Figure 21.4

35. (a) From the first two equations we get:

$$s = \frac{x+y}{2}, \qquad t = \frac{x-y}{2}.$$

Hence the equation of our surface is:

$$z = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 = \frac{x^2}{2} + \frac{y^2}{2},$$

which is the equation of a paraboloid.

The conditions: $0 \leq s \leq 1$, $0 \leq t \leq 1$ are equivalent to: $0 \leq x + y \leq 2$, $0 \leq x - y \leq 2$. So our surface is defined by:

$$z = \frac{x^2}{2} + \frac{y^2}{2}, \qquad 0 \leq x + y \leq 2 \quad 0 \leq x - y \leq 2$$

(b) The surface is shown in Figure 21.5.

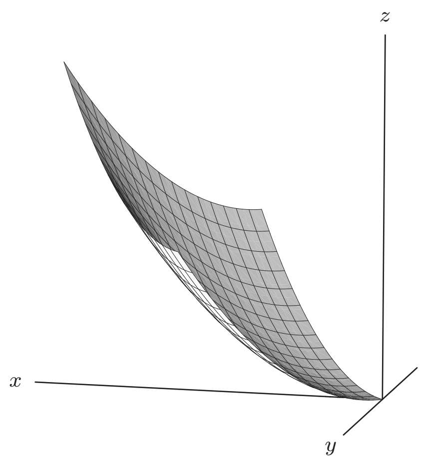


Figure 21.5: The surface $x = s + t$, $y = s - t$, $z = s^2 + t^2$ for $0 \le s \le 1$, $0 \le t \le 1$

Strengthen Your Understanding

36. A counter example is provided by an unusual parameterization of the xy-plane:

$$\vec{r}(s,t) = s\vec{i} + (s+t)\vec{j}.$$

The parameter curves with t constant are lines parallel to $\vec{v}_1 = \vec{i} + \vec{j}$, and the parameter curves with s constant are lines parallel to $\vec{v}_2 = \vec{j}$. Since $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, neither are the parameter curves.

37. A parameter curve for constant ϕ on the sphere is parameterized by

$$\vec{r_1}(\theta) = R\sin\phi\cos\theta\vec{i} + R\sin\phi\sin\theta\vec{j} + R\cos\phi\vec{k}.$$

It is a circle with radius $R\sin\phi$.

38. The point on the unit sphere where $\theta = \pi/4$ and $\phi = \pi/4$ has position vector

$$\vec{r}_0 = \sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}.$$

Vectors perpendicular to the radius vector $\vec{r}_0$ are parallel to the tangent plane. Two such vectors are $\vec{v}_1 = \vec{i} - \vec{j}$ and $\vec{v}_2 = \sqrt{2}\vec{i} - \vec{k}$. A parameterization of the tangent plane can be given by

$$\vec{r}(s,t) = \vec{r}_0 + s\vec{v}_1 + t\vec{v}_2.$$

39. Since

$$x = s+1, y = t+2, z = s+t$$

we have $x + y = z + 3$. Therefore, an equation for the plane is given by

$$f(x,y,z) = x + y - z - 3 = 0.$$

40. To give a parameterized curve on the sphere, we assign a point on the sphere to every parameter value t, by giving values of θ and ϕ. For example, letting $\theta = t$ and $\phi = t^2$ we have the curve

$$\vec{r}_1(t) = \sin t^2\cos t\vec{i} + \sin t^2\sin t\vec{j} + \cos t^2\vec{k}$$

which is not a parameter curve because neither θ nor ϕ is constant.

41. True. The plane passes through the point $(1, -2, 3)$ and contains the vectors $\vec{i}$ and $\vec{j}$.

42. False. There is only one parameter, s. The equations parameterize a line.

43. True. The position vector of a point on the lower hemisphere is the negative of the position vector of the opposite point on the upper hemisphere. As $\vec{r}$ ranges over all points in the upper hemisphere, $-\vec{r}$ ranges over all points in the lower hemisphere.

44. False. For example, if $\vec{r}(s,t) = s\vec{i} + t\vec{j} + \sqrt{1-s^2-t^2}\vec{k}$ with (s,t) inside the unit disk $s^2+t^2 \le 1$ then $\vec{r}(-s,-t)$ parameterizes the same upper hemisphere.

45. True. Adding a constant vector shifts the plane by a corresponding displacement, keeping it parallel to the original plane.

46. True. If the surface is parameterized by $\vec{r}(s,t)$ and the point has parameters (s_0,t_0) then the parameter curves $\vec{r}(s_0,t)$ and $\vec{r}(s,t_0)$ pass through (s_0,t_0).

47. False. For example, the lines of longitude on a sphere correspond to different values of the parameter θ, but all pass through the north and south poles.

48. (I) Part of a plane, since $z = x + y$.

(II) Part of a cylinder of radius 1 with center on the z-axis, since $x^2 + y^2 = 1$.

(III) Part of a sphere of radius 1 centered at the origin, since $x^2+y^2+z^2 = 1$. With $s = \phi$ and $t = \theta$ this is the usual parameterization of the sphere using spherical coordinates.

(IV) Part of a cone with vertex at the origin and central axis on the z-axis, since $z = \sqrt{x^2+y^2}$.

The match-up is: I-b, II-a, III-c, IV-d

Solutions for Section 21.2

Exercises

1. We have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = -1.$$

Therefore,

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 1.$$

2. We have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} 2s & -2t \\ 2t & 2s \end{vmatrix} = 4s^2 + 4t^2.$$

Therefore,

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 4s^2 + 4t^2.$$

Notice we can drop the absolute value signs because in this case the Jacobian is nonnegative for all s and t.

3. We have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} e^s\cos t & -e^s\sin t \\ e^s\sin t & e^s\cos t \end{vmatrix} = (e^s\cos t)^2 + (e^s\sin t)^2.$$

Therefore,

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \left|e^{2s}(\cos^2 t + \sin^2 t)\right| = e^{2s}.$$

Notice we can drop the absolute value signs because the Jacobian in this case is positive for all s and t.

4. We have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} 3s^2-3t^2 & -6st \\ 6st & 3s^2-3t^2 \end{vmatrix} = 9(s^2-t^2)^2 + 36s^2t^2.$$

Therefore, multiplying out and simplifying

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 9\left|s^4 - 2s^2t^2 + t^4 + 4s^2t^2\right| = 9(s^2+t^2)^2.$$

Notice we can drop the absolute value sign since the Jacobian in this case is nonnegative for all s and t.

5. The square T is defined by the inequalities

$$0 \leq s = ax \leq 1 \qquad 0 \leq t = by \leq 1$$

that correspond to the inequalities

$$0 \leq x \leq 1/a = 10 \qquad 0 \leq y \leq 1/b = 1$$

that define R. Thus $a = 1/10$ and $b = 1$.

6. The square T is defined by the inequalities

$$0 \leq s = ax \leq 1 \qquad 0 \leq t = by \leq 1$$

that correspond to the inequalities

$$0 \leq x \leq 1/a = 1 \qquad 0 \leq y \leq 1/b = 1/4$$

that define R. Thus $a = 1$ and $b = 4$.

7. The square T is defined by the inequalities

$$0 \leq s = ax \leq 1 \qquad 0 \leq t = by \leq 1$$

that correspond to the inequalities

$$0 \leq x \leq 1/a = 50 \qquad 0 \leq y \leq 1/b = 10$$

that define R. Thus $a = 1/50$ and $b = 1/10$.

8. Inverting the change of coordinates gives $x = s - at$, $y = t$.

The four edges of R are

$$y = 0, y = 3, y = \frac{1}{4}x, y = \frac{1}{4}(x - 10).$$

The change of coordinates transforms the edges to

$$t = 0, t = 3, t = \frac{1}{4}s - \frac{1}{4}at, t = \frac{1}{4}s - \frac{1}{4}at - \frac{10}{4}.$$

These are equations for the edges of a rectangle in the st-plane if the last two are of the form: $s =$ (Constant). This happens when the t terms drop out, or $a = -4$. With $a = -4$ the change of coordinates gives

$$\int\int_T \left| \frac{\partial(x, y)}{\partial(s, t)} \right| \, ds \, dt$$

over the rectangle

$$T : 0 \leq t \leq 3, 0 \leq s \leq 10.$$

9. Inverting the change of coordinates gives $x = s - at$, $y = t$.

The four edges of R are

$$y = 0, y = 5, y = -\frac{1}{3}x, y = -\frac{1}{3}(x - 10).$$

The change of coordinates transforms the edges to

$$t = 0, t = 5, t = -\frac{1}{3}s + \frac{1}{3}at, t = -\frac{1}{3}s + \frac{1}{3}at + \frac{10}{3}.$$

These are equations for the edges of a rectangle in the st-plane if the last two are of the form: $s =$ (Constant). This happens when the t terms drop out, or $a = 3$. With $a = 3$ the change of coordinates gives

$$\int\int_T \left| \frac{\partial(x, y)}{\partial(s, t)} \right| \, ds \, dt$$

over the rectangle

$$T : 0 \leq t \leq 5, 0 \leq s \leq 10.$$

Problems

10. Given $T = \{(s,t) \mid 0 \le s \le 3,\ 0 \le t \le 2\}$ and

$$\begin{cases} x = 2s - 3t \\ y = \ s - 2t \end{cases}$$

The shaded area in Figure 21.6 is the corresponding region R in the xy-plane.

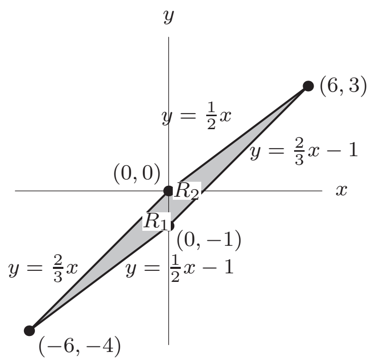


Figure 21.6

Since

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = -1,$$

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 1.$$

Thus we get

$$\int_T \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt = \int_0^3 ds \int_0^2 dt = 6.$$

Since

$$\int_R dx\,dy = \int_{R_1} dx\,dy + \int_{R_2} dx\,dy = \int_{-6}^0 dx \int_{\frac{1}{2}x-1}^{\frac{2}{3}x} dy + \int_0^6 dx \int_{\frac{2}{3}x-1}^{\frac{1}{2}x} dy$$
$$= \int_{-6}^0 \left(\frac{1}{6}x + 1\right) dx + \int_0^6 \left(-\frac{1}{6}x + 1\right) dx = 3 + 3 = 6,$$

thus

$$\int_R dx\,dy = \int_T \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.$$

11. Given

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi, \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$
$$= \cos\phi(\rho^2\cos^2\theta\cos\phi\sin\phi + \rho^2\sin^2\theta\cos\phi\sin\phi)$$
$$+\rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$
$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$$
$$= \rho^2\sin\phi.$$

12. Given

$$\begin{cases} x = 3s - 4t \\ y = 5s + 2t, \end{cases}$$

we have

$$\begin{cases} s = \frac{1}{13}(x + 2y) \\ t = \frac{1}{26}(3y - 5x). \end{cases}$$

Since

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix} = 26,$$

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{13} & \frac{2}{13} \\ -\frac{5}{26} & \frac{3}{26} \end{vmatrix} = \left(\frac{3}{26}\right)\left(\frac{1}{13}\right) + \left(\frac{5}{26}\right)\left(\frac{2}{13}\right) = \frac{1}{26}.$$

So

$$\frac{\partial(x,y)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)} = 26 \cdot \frac{1}{26} = 1.$$

13. Given

$$\begin{cases} x = 2s + t \\ y = s - t, \end{cases}$$

we have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3,$$

hence

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 3.$$

We get

$$\int_R (x+y)\,dA = \int_T 3s \left|\frac{\partial(x,y)}{\partial(s,t)}\right| dsdt = \int_T (3s)(3)\,ds\,dt = 9\int_T s\,ds\,dt,$$

where T is the region in the st-plane corresponding to R.

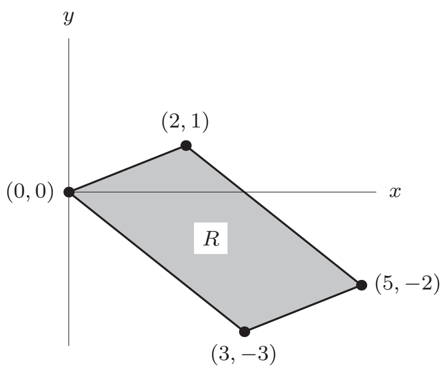


Figure 21.7

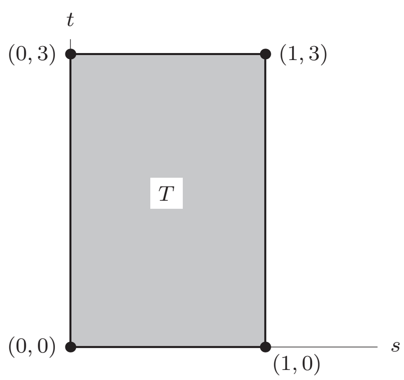


Figure 21.8

Now, we need to find T.

As

$$\begin{cases} x = 2s + t \\ y = s - t \end{cases} \quad \text{or} \quad \begin{cases} s = \frac{1}{3}(x + y) \\ t = \frac{1}{3}(x - 2y), \end{cases}$$

so from the above transformation and Figure 21.7, T is the shaded area in Figure 21.8. Therefore

$$\int_R (x + y)\, dA = 9 \int_0^1 s\, ds \int_0^3 dt = (27)(\frac{1}{2}) = 13.5.$$

14. The area of the ellipse is $\int\int_R dx\, dy$ where R is the region $x^2 + 2xy + 2y^2 \leq 1$. We must change coordinates in both the area element $dA = dx\, dy$ and the region R.

Inverting the coordinate change gives $x = s - t$, $y = t$. Thus

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore

$$dx\, dy = \left| \frac{\partial(x, y)}{\partial(s, t)} \right| ds\, dt = ds\, dt.$$

The region of integration is

$$x^2 + 2xy + 2y^2 = (s - t)^2 + 2(s - t)t + 2t^2 = s^2 + t^2 \leq 1.$$

Let T be the unit disc $s^2 + t^2 \leq 1$. We have

$$\int\int_R dx\, dy = \int\int_T ds\, dt = \text{Area of } T = \pi.$$

15. We must change coordinates in the area element $dA = dx\, dy$, the integrand x and the region R.

Inverting the coordinate change gives $x = \sqrt{s - t}$, $y = s$ where we use the positive square root because the region R is in the first quadrant where $x \geq 0$. Thus

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1/(2\sqrt{s - t}) & -1/(2\sqrt{s - t}) \\ 1 & 0 \end{vmatrix} = \frac{1}{2\sqrt{s - t}}.$$

Therefore

$$dx\, dy = \left| \frac{\partial(x, y)}{\partial(s, t)} \right| ds\, dt = \frac{1}{2\sqrt{s - t}}\, ds\, dt.$$

The integrand is $x = \sqrt{s - t}$.

The region of integration can be transformed by examination of its boundaries. The left and right boundaries of R are given by $y - x^2 = t = 0$ and $y - x^2 = t = -9$. The bottom and top boundaries of R are given by $y = s = 0$ and $y = s = 16$.

Let T be the rectangle $0 \leq s \leq 16$, $-9 \leq t \leq 0$ of area $(9)(16) = 144$. We have

$$\int\int_R x\, dx\, dy = \int\int_T \sqrt{s - t} \frac{1}{2\sqrt{s - t}}\, ds\, dt = \frac{1}{2}(\text{Area of } T) = 72.$$

16. Let

$$\begin{cases} s = x - y \\ t = x + y, \end{cases} \quad \text{that is} \quad \begin{cases} x = \frac{1}{2}(s + t) \\ y = \frac{1}{2}(t - s), \end{cases}$$

we get

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Hence

$$I = \int_R \cos\left(\frac{x - y}{x + y}\right) dx\, dy = \int_T \cos\left(\frac{s}{t}\right) \left|\frac{\partial(x, y)}{\partial(s, t)}\right| ds\, dt = \frac{1}{2} \int_T \cos\left(\frac{s}{t}\right) ds\, dt,$$

where R is the triangle bounded by $x + y = 1$, $x = 0$, $y = 0$ and T is its image which is the triangle bounded by $t = 1$, $s = -t$, $s = t$.
Then

$$I = \frac{1}{2}\int_0^1 \int_{-t}^{t} \cos\left(\frac{s}{t}\right) ds\, dt = \frac{1}{2}\int_0^1 t[\sin(1) - \sin(-1)]\, dt$$
$$= \frac{1}{2}\int_0^1 t \cdot 2\sin 1\, dt = \sin 1 \int_0^1 t\, dt = \frac{\sin 1}{2} = 0.42.$$

17. (a) The probability that $(x + y)/2$ is less than or equal to t is the integral of the joint density function $p(x, y)$ over the infinite region (half plane) where $(x + y)/2 \le t$. Thus

$$F(t) = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{2t-x} e^{-(x^2+y^2)/(2\sigma^2)} dy\, dx.$$

(b) With $u = (x + y)/2$, $v = (x - y)/2$, we have $x = u + v$, $y = u - v$. Thus

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so

$$dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv = 2du\, dv.$$

Also $x^2 + y^2 = 2(u^2 + v^2)$. After writing the limits of integration in the uv-coordinates, we have

$$F(t) = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{t}\int_{-\infty}^{\infty} e^{-(u^2+v^2)/\sigma^2} 2dv\, du = \frac{2}{2\pi\sigma^2}\int_{-\infty}^{t} e^{-u^2/\sigma^2}\int_{-\infty}^{\infty} e^{-v^2/\sigma^2} dv\, du.$$

Continuing using the fact that $\int_{-\infty}^{\infty} e^{-x^2/a^2} dx = a\sqrt{\pi}$ with a replaced by σ, we obtain

$$F(t) = \frac{2}{2\pi\sigma^2}\int_{-\infty}^{t} e^{-u^2/\sigma^2}(\sigma\sqrt{\pi})du = \frac{1}{\sqrt{\pi}\sigma}\int_{-\infty}^{t} e^{-u^2/\sigma^2} du.$$

(c) The probability density function of z is the derivative of its cumulative distribution function $F(t)$. By the Fundamental Theorem of Calculus,

$$p(t) = F'(t) = \frac{1}{\sqrt{\pi}\sigma} e^{-t^2/\sigma^2}.$$

(d) Since the density function for z can be written in the form

$$p(t) = \frac{1}{\sqrt{2\pi}(\sigma/\sqrt{2})} e^{-t^2/(2(\sigma/\sqrt{2})^2)},$$

the distribution of z is normal, with mean 0 and standard deviation $\sigma/\sqrt{2}$.

Notice that the standard deviation of the average $z = (x + y)/2$ is less than the standard deviations of the individual numbers x and y. The average of two random numbers is more likely to be near the mean than are either of the two numbers individually.

18. Let's denote the (x, y) coordinates of the points in the lagoon by L. Since x and y are measured in kilometers and d is measured in meters, and 1 km = 1000 m, the volume of a small piece of the lagoon is given by

$$\Delta V \approx d(x, y)(1000\Delta x)(1000\Delta y)\text{m}^3.$$

Thus, the total volume of the lagoon is given by

$$V = 1000^2 \int_L d(x, y)\, dxdy.$$

Changing coordinates using $u = x/2$ and $v = y - f(x)$ converts the depth function to:

$$d(x(u, v), y(u, v)) = 40 - 160v^2 - 160u^2 = 160(\frac{1}{4} - u^2 - v^2) \text{ meters}.$$

Thus, the points in the lagoon have (u, v) coordinates in the disk, D, given by $u^2 + v^2 \le 1/4$.

The Jacobian of the transformation is:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 2f'(2u) & 1 \end{vmatrix} = 2.$$

Thus, the integral in u, v coordinates is

$$V = 1000^2 \int_L d(x,y)\,dxdy = 10^6 \int_D 160(\frac{1}{4} - u^2 - v^2)2\,dudv = 320 \cdot 10^6 \int_D (\frac{1}{4} - u^2 - v^2)\,dudv.$$

Converting to polar coordinates, we have

$$V = 320 \cdot 10^6 \int_0^{2\pi} \int_0^{1/2} (\frac{1}{4} - r^2) r\,drd\theta = 320 \cdot 10^6 2\pi (\frac{1}{4}\frac{r^2}{2} - \frac{r^4}{4})\Big|_0^{1/2} = 10^7 \pi \text{ m}^3.$$

Strengthen Your Understanding

19. The region R does not correspond to the region T. The region R corresponds separately to both

$$T_1: \ 0 \le s \le 1, \ -2 \le t \le 0 \qquad \text{and to} \qquad T_2: \ 0 \le s \le 1, \ 0 \le t \le 2.$$

In a change of coordinates for the integral over R, we use only one of these two regions. Both the following integrals are correct:

$$\int_R f(x,y)\,dx\,dy = \int_{T_1} f(s,t^2) \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds\,dt$$

and

$$\int_R f(x,y)\,dx\,dy = \int_{T_2} f(s,t^2) \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds\,dt.$$

20. The Jacobian for the change of coordinates is given by

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} 0 & 3t^2 \\ 1 & 0 \end{vmatrix} = -3t^2.$$

The change of coordinates formula requires the absolute value of the Jacobian. The correct formula is

$$\int_R (x+2y)\,dx\,dy = \int_T \left(t^3 + 2s\right)\left(3t^2\right)\,ds\,dt$$

21. The change of coordinates $x = 2s$, $y = 3t$ transforms the region $0 \le s \le 1$, $0 \le t \le 1$ in the st-plane into the rectangle $0 \le x \le 2$, $0 \le y \le 3$ in the xy-plane.

22. Let $x = 2s$, $y = t$. Geometrically, this change of coordinates stretches every region in the horizontal direction by a factor of 2, while leaving the vertical distances the same, which doubles the area.

To see this analytically, let T be a region in the st-plane, corresponding to region R in the xy-plane. We have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2.$$

Hence

$$\text{Area of } R = \int_R dx\,dy = \int_T \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds\,dt = \int_T 2\,ds\,dt = 2 \cdot \text{Area of } T.$$

23. False. The change of variable leaves the value of the integral unchanged. Thus, calculating the value of the s, t-integral gives you the value of the x, y-integral.

24. False. The Jacobian can be negative; we use the absolute value of the Jacobian in the integral.

Solutions for Section 21.3

Exercises

1. Since

$$\vec r\,(s,t) = (s+t)\vec i + (s-t)\vec j + st\vec k\,,$$

we have

$$\frac{\partial \vec r}{\partial s} = \vec i + \vec j + t\vec k \quad \text{and} \quad \frac{\partial \vec r}{\partial t} = \vec i - \vec j + s\vec k\,,$$

so

$$d\vec A = \frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t}\, ds\, dt = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 1 & t \\ 1 & -1 & s \end{vmatrix} ds\, dt = ((s+t)\vec i - (s-t)\vec j - 2\vec k\,)\, ds\, dt.$$

For the opposite orientation, $d\vec A$ has the opposite sign:

$$d\vec A = -((s+t)\vec i - (s-t)\vec j - 2\vec k\,)\, ds\, dt.$$

2. Since

$$\vec r\,(s,t) = \sin t\,\vec i + \cos t\,\vec j + (s+t)\,\vec k\,,$$

we have

$$\frac{\partial \vec r}{\partial s} = 0\,\vec i + 0\,\vec j + \vec k \quad \text{and} \quad \frac{\partial \vec r}{\partial t} = \cos t\,\vec i - \sin t\,\vec j + \vec k\,,$$

so

$$d\vec A = \frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t}\, ds\, dt = \begin{vmatrix} \vec i & \vec j & \vec k \\ 0 & 0 & 1 \\ \cos t & -\sin t & 1 \end{vmatrix} ds\, dt = (\sin t\,\vec i + \cos t\,\vec j\,)\, ds\, dt.$$

For the opposite orientation, $d\vec A$ has the opposite sign:

$$d\vec A = -(\sin t\,\vec i + \cos t\,\vec j\,)\, ds\, dt.$$

3. Since

$$\vec r\,(s,t) = e^s\,\vec i + \cos t\,\vec j + \sin t\,\vec k\,,$$

we have

$$\frac{\partial \vec r}{\partial s} = e^s\,\vec i + 0\,\vec j + 0\,\vec k \quad \text{and} \quad \frac{\partial \vec r}{\partial t} = 0\vec i - \sin t\,\vec j + \cos t\,\vec k\,,$$

so

$$\begin{aligned} d\vec A &= \frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t}\, ds\, dt \\ &= \begin{vmatrix} \vec i & \vec j & \vec k \\ e^s & 0 & 0 \\ 0 & -\sin t & \cos t \end{vmatrix} ds\, dt \\ &= (-e^s \cos t\,\vec j - e^s \sin t\,\vec k\,)\, ds\, dt \\ &= -e^s(\cos t\,\vec j + \sin t\,\vec k\,)\, ds\, dt. \end{aligned}$$

For the opposite orientation, $d\vec A$ has the opposite sign:

$$d\vec A = e^s(\cos t\,\vec j + \sin t\,\vec k\,)\, ds\, dt.$$

4. Since

$$\vec r\,(u,v) = (u+v)\,\vec j\, + (u-v)\,\vec k\,,$$

we have

$$\frac{\partial \vec r}{\partial u} = \vec j\, + \vec k \quad \text{and} \quad \frac{\partial \vec r}{\partial v} = \vec j\, - \vec k\,,$$

so

$$d\vec A\, = \frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\, du\, dv = \begin{vmatrix} \vec i & \vec j & \vec k \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} du\, dv = -2\,\vec i\, du\, dv.$$

For the opposite orientation, $d\vec A$ has the opposite sign:

$$d\vec A\, = 2\,\vec i\, du\, dv.$$

5. Since S is given by

$$\vec r\,(s,t) = (s+t)\vec i\, + (s-t)\vec j\, + (s^2+t^2)\vec k\,,$$

we have

$$\frac{\partial \vec r}{\partial s} = \vec i\, + \vec j\, + 2s\vec k \quad \text{and} \quad \frac{\partial \vec r}{\partial t} = \vec i\, - \vec j\, + 2t\vec k\,,$$

and

$$\frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t} = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 1 & 2s \\ 1 & -1 & 2t \end{vmatrix} = (2s+2t)\vec i\, + (2s-2t)\vec j\, - 2\vec k\,.$$

Since the $\vec k$ component of this vector is negative, it points down, and so has the opposite orientation to the one specified. Thus, we use

$$d\vec A\, = -\frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t}\, ds\, dt,$$

and so we have

$$\begin{aligned}
\int_S \vec F\, \cdot d\vec A\, &= -\int_0^1 \int_0^1 (s^2+t^2)\vec k\, \cdot \left((2s+2t)\vec i\, + (2s-2t)\vec j\, - 2\vec k\,\right)\, ds\, dt \\
&= 2\int_0^1 \int_0^1 (s^2+t^2)\, ds\, dt = 2\int_0^1 \left(\frac{s^3}{3} + st^2\right)\bigg|_{s=0}^{s=1} dt \\
&= 2\int_0^1 (\frac{1}{3} + t^2)\, dt = 2(\frac{1}{3}t + \frac{t^3}{3})\bigg|_0^1 = 2(\frac{1}{3} + \frac{1}{3}) = \frac{4}{3}.
\end{aligned}$$

6. Since S is parameterized by

$$\vec r\,(s,t) = 2s\vec i\, + (s+t)\vec j\, + (1+s-t)\vec k\,,$$

we have

$$\frac{\partial \vec r}{\partial s} = 2\vec i\, + \vec j\, + \vec k \quad \text{and} \quad \frac{\partial \vec r}{\partial t} = \vec j\, - \vec k\,,$$

so

$$\frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t} = \begin{vmatrix} \vec i & \vec j & \vec k \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\vec i\, + 2\vec j\, + 2\vec k\,,$$

which points upward, in the direction opposite to the orientation given. Thus,

$$\begin{aligned}
\int_S \vec F\, \cdot d\vec A\, &= -\int_0^1 \int_0^1 (2s\vec i\, + (s+t)\vec j\,)(-2\vec i\, + 2\vec j\, + 2\vec k\,)\, ds\, dt \\
&= -\int_0^1 \int_0^1 (-4s + 2s + 2t)\, ds dt = \int_0^1 \int_0^1 (2s - 2t)\, ds\, dt \\
&= \int_0^1 \left(s^2 - 2st\bigg|_{s=0}^{s=1}\right) dt = \int_0^1 (1-2t)\, dt = t - t^2\bigg|_0^1 = 0.
\end{aligned}$$

7. The cross-product $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$ is given by

$$\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^s & 0 & 6 \\ 0 & -3\sin(3t) & 0 \end{vmatrix} = 18\sin(3t)\vec{i} - 3e^s\sin(3t)\vec{k}\,.$$

Since the z-component, $-3e^s\sin(3t)$, of $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$ is always negative for $0 \le s \le 4$ and $0 < t < \pi/6$, the vector $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$ points downward and so in the direction of the given orientation of S.

Thus,

$$d\vec{A} = \left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right) ds\,dt$$

and

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^4\int_0^{\pi/6} (e^s\vec{i}\,)\cdot(18\sin(3t)\vec{i} - 3e^s\sin(3t)\vec{j}\,)\,dt\,ds \\ &= \int_0^4\int_0^{\pi/6} 18e^s\sin(3t)\,dt\,ds = -18\int_0^4 \left.\frac{e^s\cos(3t)}{3}\right|_0^{\pi/6} ds \\ &= -6\int_0^4 e^s(0-1)\,ds = 6\int_0^4 e^s\,ds = 6(e^4-1).\end{aligned}$$

8. Since S is parameterized by

$$\vec{r}(s,t) = 3\sin s\vec{i} + 3\cos s\vec{j} + (t+1)\vec{k}\,,$$

we have

$$\frac{\partial\vec{r}}{\partial s} = 3\cos s\vec{i} - 3\sin s\vec{j} \quad\text{and}\quad \frac{\partial\vec{r}}{\partial t} = \vec{k}\,.$$

So

$$\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos s & -3\sin s & 0 \\ 0 & 0 & 1 \end{vmatrix} = -3\sin s\vec{i} - 3\cos s\vec{j}\,,$$

which points toward the z-axis and thus opposite to the orientation we were given. Hence, we use

$$d\vec{A} = -\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\,ds\,dt,$$

and so we have

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= -\int_0^1\int_0^{\pi} (3\cos s\vec{i} + 3\sin s\vec{j}\,)\cdot(-3\sin s\vec{i} - 3\cos s\vec{j}\,)\,ds\,dt \\ &= 9\int_0^1\int_0^{\pi} 2\sin s\cos s\,ds\,dt = 9\int_0^1\int_0^{\pi}\sin 2s\,ds\,dt \\ &= 9\int_0^1 \left(-\left.\frac{\cos 2s}{2}\right|_{s=0}^{s=\pi}\right) dt = 0.\end{aligned}$$

9. Using cylindrical coordinates, we see that the surface S is parameterized by

$$\vec{r}(r,\theta) = r\cos\theta\vec{i} + r\sin\theta\vec{j} + r\vec{k}\,.$$

We have

$$\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\vec{i} - r\sin\theta\vec{j} + r\vec{k}\,.$$

Since the vector $\partial\vec{r}/\partial r \times \partial\vec{r}/\partial\theta$ points upward, in the direction opposite to the specified orientation, we use $d\vec{A} = -\left(\partial\vec{r}/\partial r \times \partial\vec{r}/\partial\theta\right)\,dr\,d\theta$. Hence

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= -\int_0^{2\pi}\int_0^R (r^5\cos^2\theta\sin^2\theta\vec{k}\,)\cdot(-r\cos\theta\vec{i}\,-r\sin\theta\vec{j}\,+r\vec{k}\,)\,dr\,d\theta \\
&= -\int_0^{2\pi}\int_0^R r^6\cos^2\theta\sin^2\theta\,dr\,d\theta \\
&= -\frac{R^7}{7}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta \\
&= -\frac{R^7}{7}\int_0^{2\pi}\sin^2\theta(1-\sin^2\theta)\,d\theta \\
&= -\frac{R^7}{7}\int_0^{2\pi}(\sin^2\theta-\sin^4\theta)\,d\theta \\
&= -(\frac{R^7}{7})(\frac{\pi}{4}) = \frac{-\pi}{28}R^7.
\end{aligned}$$

The cone is not differentiable at the point $(0,0)$. However the flux integral, which is improper, converges.

10. Place the cylinder S of radius a and length L so that its central axis is the z axis between $z = 0$ and $z = L$. A parameterization of S is

$$x = a\cos\theta,\quad y = a\sin\theta,\quad z = t,\quad \text{for } 0\le\theta\le 2\pi,\quad 0\le t\le L.$$

We compute

$$\begin{aligned}
\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial t} &= (-a\sin\theta\vec{i}\,+a\cos\theta\vec{j}\,)\times\vec{k}\, = a\cos\theta\vec{i}\,+a\sin\theta\vec{j} \\
\left\|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial t}\right\| &= a \\
\text{Surface area} &= \int_S dA = \int_R \left\|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial t}\right\| dA = \int_{\theta=0}^{2\pi}\int_{t=0}^{L} a\,dt\,d\theta = 2\pi aL.
\end{aligned}$$

11. A parameterization of S is

$$x = s,\quad y = t,\quad z = 3s+2t,\quad \text{for } 0\le s\le 10,\quad 0\le t\le 20.$$

We compute

$$\begin{aligned}
\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t} &= (\vec{i}\,+3\vec{k}\,)\times(\vec{j}\,+2\vec{k}\,) = -3\vec{i}\,-2\vec{j}\,+\vec{k} \\
\left\|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial t}\right\| &= \sqrt{14} \\
\text{Surface area} &= \int_S dA = \int_R \left\|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial t}\right\| dA = \int_{t=0}^{20}\int_{s=0}^{10}\sqrt{14}\,ds\,dt = 200\sqrt{14}.
\end{aligned}$$

Problems

12. The surface S is parameterized by

$$\vec{r}\,(x,z) = x\vec{i}\,+(x^2+z^2)\vec{j}\,+z\vec{k}\,.$$

The surface S, together with its given orientation $\vec{n}$, is graphed in Figure 21.9. Using the right-hand rule we see that the vector $\vec{r}_x\times\vec{r}_z$ points in the direction of $\vec{n}$. Thus,

$$d\vec{A} = \left(\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial z}\right)dx\,dz = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2x & 0\\ 0 & 2z & 1\end{vmatrix}\,dx\,dz = (2x\vec{i}\,-\vec{j}\,+2z\vec{k}\,)\,dx\,dz.$$

Thus

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_S ((x+z)\vec{i}+\vec{j}+z\vec{k})\cdot(2x\vec{i}-\vec{j}+2z\vec{k})\,dx\,dz\\ &= \int_S (2x^2+2xz-1+2z^2)dx\,dz.\end{aligned}$$

Changing to polar coordinates, $x = r\cos\theta, z = r\sin\theta$, where $1/2 \le r \le 1, 0 \le \theta \le 2\pi$, we obtain

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{2\pi}\int_{1/2}^1 (2r^2+2r^2\sin\theta\cos\theta-1)r\,dr\,d\theta\\ &= \int_0^{2\pi}\left(\frac{r^4}{2}+\frac{r^4}{2}\sin\theta\cos\theta-\frac{r^2}{2}\right)\bigg|_{r=1/2}^{r=1} d\theta\\ &= \int_0^{2\pi}\left(\frac{15}{32}\sin\theta\cos\theta+\frac{3}{32}\right)d\theta\\ &= \frac{15}{64}(\sin\theta)^2+\frac{3}{32}\theta\bigg|_0^{2\pi} = \frac{3\pi}{16}.\end{aligned}$$

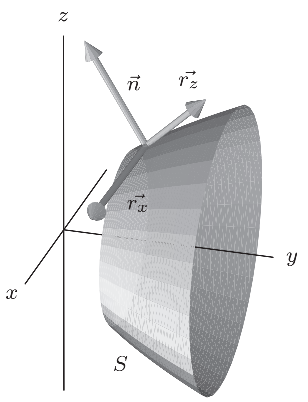


Figure 21.9

13. The plane is parameterized by

$$\vec{r} = x\vec{i}+y\vec{j}+z\vec{k} = x\vec{i}+y\vec{j}+(2-2x-y)\vec{k},$$

where (x, y) is in the disk R lying inside the circle $x^2+y^2 = 2x$. By completing the square, this circle can be rewritten as $(x-1)^2+y^2 = 1$ and so the disk has area π.

We have $dA = \|\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y}\|\,dxdy$, where

$$\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -2\\ 0 & 1 & -1\end{vmatrix} = 2\vec{i}+\vec{j}+\vec{k}$$

and so

$$\left\|\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y}\right\| = \sqrt{6}.$$

Thus, the surface area of the ellipse S is given by

$$\begin{aligned}\text{Surface area} &= \int_S 1\,dA = \int_R \sqrt{6}\,dxdy\\ &= \sqrt{6}\times(\text{Area of disk } x^2+y^2=2x)\\ &= \sqrt{6}\pi.\end{aligned}$$

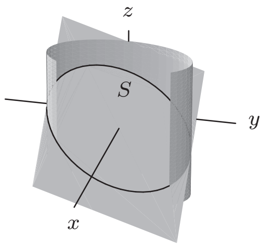


Figure 21.10

14. The surface S is parameterized by

$$\vec{r} = x\vec{i} + f(x)\cos\theta\vec{j} + f(x)\sin\theta\vec{k}, \qquad a \le x \le b, 0 \le \theta \le 2\pi.$$

The area element on A is

$$\begin{aligned} dA &= \left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta} \right\| dx\, d\theta \\ &= \|(\vec{i} + f'(x)\cos\theta\vec{j} + f'(x)\sin\theta\vec{k}) \times (-f(x)\sin\theta\vec{j} + f(x)\cos\theta\vec{k})\|\, dx\, d\theta \\ &= \|f(x)f'(x)\vec{i} - f(x)\cos\theta\vec{j} - f(x)\sin\theta\vec{k}\|\, dx\, d\theta \\ &= f(x)\sqrt{f'(x)^2 + \cos^2\theta + \sin^2\theta}\, dx\, d\theta \\ &= f(x)\sqrt{1 + f'(x)^2}\, dx\, d\theta. \end{aligned}$$

So

$$\text{Surface area} = \int_S dA = \int_0^{2\pi}\int_a^b f(x)\sqrt{1+f'(x)^2}\, dx\, d\theta = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\, dx.$$

15. Let x be the distance d_1. Since w is the total width of the channel, we have $d_2 = w - x$. The flux through a rectangle with dimensions $A = w \times h$, is given by

$$\text{Flux } = \int_A \vec{v} \cdot d\vec{A}.$$

For a thin section of the channel of width dx, we have $d\vec{A} = (h\, dx)\vec{j}$. Thus

$$\begin{aligned} \text{Flux } &= \int_0^w \vec{v} \cdot (h\, dx)\vec{j} = \int_0^w kx(w-x)\vec{j} \cdot (h \cdot dx)\vec{j} = \int_0^w khx(w-x)\, dx \\ &= \int_0^w kh(wx - x^2)\, dx = kh\left(\frac{1}{2}w^3 - \frac{1}{3}w^3\right) = \frac{1}{6}khw^3 \text{meter}^3\text{/sec.} \end{aligned}$$

16. (a) Building on the parameterization $x = \cos u$, $y = \sin u$, $z = 0$ of the circular base of the cone, we get

$$\begin{aligned} x &= (1-v)\cos u + av \\ y &= (1-v)\sin u + bv \\ z &= cv \\ &0 \le u \le 2\pi, \quad 0 \le v \le 1. \end{aligned}$$

Note that $v = 0$ corresponds to the base of the cone and $v = 1$ is its vertex.

(b) Writing $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ we have

$$\begin{aligned} \frac{\partial \vec{r}}{\partial u} &= -(1-v)\sin u\vec{i} + (1-v)\cos u\vec{j} \\ \frac{\partial \vec{r}}{\partial v} &= (a - \cos u)\vec{i} + (b - \sin u)\vec{j} + c\vec{k}. \end{aligned}$$

Thus

$$\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}=\begin{vmatrix}\vec i & \vec j & \vec k\\ -(1-v)\sin u & (1-v)\cos u & 0\\ a-\cos u & b-\sin u & c\end{vmatrix}$$
$$= c(1-v)\cos u\vec i + c(1-v)\sin u\vec j + (1-v)(1-a\cos u - b\sin u)\vec k$$

so

$$\left\|\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}\right\| = (1-v)\sqrt{c^2+(1-a\cos u-b\sin u)^2}.$$

Thus

$$\text{Surface area} = \int_0^{2\pi}\int_0^1 \left\|\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}\right\| du\,dv = \frac{1}{2}\int_0^{2\pi}\sqrt{c^2+(1-a\cos u-b\sin u)^2}du.$$

This is an elliptic integral that can not be evaluated in terms of elementary functions.

(c) We have Surface Area $= (1/2)\int_0^{2\pi}\sqrt{1+(1-2\cos u)^2}du = 5.805$.

17. If S is the part of the graph of $z=f(x,y)$ lying over a region R in the xy-plane, then S is parameterized by

$$\vec r(x,y) = x\vec i + y\vec j + f(x,y)\vec k, \qquad (x,y) \text{ in } R.$$

So

$$\frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y} = (\vec i + f_x\vec k)\times(\vec j + f_y\vec k) = -f_x\vec i - f_y\vec j + \vec k.$$

Since the $\vec k$ component is positive, this points upward, so if S is oriented upward

$$d\vec A = (-f_x\vec i - f_y\vec j + \vec k)\,dx\,dy$$

and therefore we have the expression for the flux integral obtained on page 1018:

$$\int_S \vec F\cdot d\vec A = \int_R \vec F(x,y,f(x,y))\cdot(-f_x\vec i - f_y\vec k + \vec k)\,dx\,dy.$$

18. If S is the part of the cylinder of radius R corresponding to the region T in θz-space, then S is parameterized in cylindrical coordinates by

$$\vec r(\theta,z) = R\cos\theta\vec i + R\sin\theta\vec j + z\vec k, \qquad (\theta,z) \text{ in } T.$$

So

$$\frac{\partial \vec r}{\partial \theta}\times\frac{\partial \vec r}{\partial z} = (-R\sin\theta\vec i + R\cos\theta\vec j)\times\vec k = R\cos\theta\vec i + R\sin\theta\vec j.$$

This points outward, so

$$d\vec A = (R\cos\theta\vec i + R\sin\theta\vec j)\,d\theta\,dz = (\cos\theta\vec i + \sin\theta\vec j)R\,d\theta\,dz$$

and therefore we obtain the expression for the flux integral in cylindrical coordinates on page 1019:

$$\int_S \vec F\cdot d\vec A = \int_T \vec F(R\cos\theta, R\sin\theta, z)\cdot(\cos\theta\vec i + \sin\theta\vec j)R\,d\theta\,dz.$$

19. If S is the part of the sphere of radius R corresponding to the region T in $\theta\phi$-space, then S is parameterized in spherical coordinates by

$$\vec r(\theta,\phi) = R\sin\phi\cos\theta\vec i + R\sin\phi\sin\theta\vec j + R\cos\phi\vec k, \qquad (\theta,\phi) \text{ in } T.$$

So

$$\frac{\partial \vec r}{\partial \theta}\times\frac{\partial \vec r}{\partial \phi} = \begin{vmatrix}\vec i & \vec j & \vec k\\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0\\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi\end{vmatrix}$$
$$= -R^2\sin^2\phi\cos\theta\vec i - R^2\sin^2\phi\sin\theta\vec j - R^2\sin\phi\cos\phi\vec k$$
$$= -R^2\sin\phi(\sin\phi\cos\theta\vec i + \sin\phi\sin\theta\vec j + \cos\phi\vec k).$$

This points inward, so the outward area element is

$$d\vec{A} = (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})R^2\sin\phi\, d\theta\, d\phi,$$

and therefore we obtain the expression for the flux integral in spherical coordinates on page 1020:

$$\begin{aligned}&\int_S \vec{F}\cdot d\vec{A}\\ &= \int_T \vec{F}(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\cdot(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})R^2\sin\phi\, d\theta\, d\phi.\end{aligned}$$

20. In terms of the st-parameterization,

$$d\vec{A} = \frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\, ds\, dt.$$

By the chain rule, we have

$$\begin{aligned}\frac{\partial\vec{r}}{\partial s} &= \frac{\partial\vec{r}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial\vec{r}}{\partial v}\frac{\partial v}{\partial s}\\ \frac{\partial\vec{r}}{\partial t} &= \frac{\partial\vec{r}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\vec{r}}{\partial v}\frac{\partial v}{\partial t}.\end{aligned}$$

So taking the cross product, we get

$$\begin{aligned}\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t} &= \left(\frac{\partial\vec{r}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial\vec{r}}{\partial v}\frac{\partial v}{\partial s}\right)\times\left(\frac{\partial\vec{r}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\vec{r}}{\partial v}\frac{\partial v}{\partial t}\right)\\ &= \left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right)\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}.\end{aligned}$$

Now suppose we are going to change coordinates in a double integral from uv-coordinates to st-coordinates. The Jacobian is

$$\frac{\partial(u,v)}{\partial(s,t)} = \begin{vmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{vmatrix} = \frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}.$$

Since the Jacobian is assumed to be positive, converting from a uv-integral to an st-integral gives:

$$\begin{aligned}\int_T \vec{F}\cdot\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\, dudv &= \int_R \vec{F}\cdot\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\frac{\partial(u,v)}{\partial(s,t)}\, dsdt\\ &= \int_R \vec{F}\cdot\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right) dsdt.\end{aligned}$$

However, we know that this gives us

$$\int_T \vec{F}\cdot\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\, du\, dv = \int_R \vec{F}\cdot\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right) dsdt = \int_R \vec{F}\cdot\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\, dsdt.$$

Thus, the flux integral in uv-coordinates equals the flux integral in st-coordinates.

Strengthen Your Understanding

21. The integral $\int_0^1\int_0^1 f(s,t)\, ds\, dt$ gives the volume of the region between the square $0\le x\le 1, 0\le y\le 1$ in the xy-plane and the surface $z = f(x,y)$.

22. The area of the region R given by $0\le s\le 2, 0\le t\le 3$ in st-space is $\int_0^3\int_0^2 ds\, dt = 6$, but the area of the parameterized surface S in xyz-space depends on its parameterization.

The surface S is parameterized by $\vec{r} = f(s,t)\vec{i} + g(s,t)\vec{j} + h(s,t)\vec{k}$. The surface area of S is given by

$$\text{Area} = \int_S dA = \int_0^3\int_0^2 \left\|\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right\| ds\, dt.$$

23. The region $0 \leq s \leq 1$, $0 \leq t \leq 1$ in the parameter space must correspond to surface area $\int_0^1 \int_0^1 2\,ds\,dt = 2$ on the xy-plane being parameterized. This suggests a parameterization that is a scaling of the usual xy coordinates.

Let $x = 2s$, $y = t$, $z = 0$ so that the parameterization is

$$\vec{r}(s,t) = 2s\vec{i} + t\vec{j}.$$

We have

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = 2\vec{i} \times \vec{j} = 2\vec{k}$$
$$dA = \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| ds\,dt = 2ds\,dt.$$

24. Since $\vec{r} = (s-t)\vec{i} + t^2\vec{j} + (s+t)\vec{k}$ we have

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = -2t\vec{i} - 2\vec{j} + 2t\vec{k}.$$

Therefore, if $\vec{F}(x,y,z) = -\vec{j}$, we have

$$\vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) = 2 > 0$$

so

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^1 \int_0^1 \vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds\,dt = 2 > 0.$$

25. True. At every point on S the vector field $\vec{F}$ is perpendicular to the area vector $d\vec{A}$.

26. True. The vector $2\vec{i} - 4\vec{j} + 6\vec{k}$ is the gradient vector for $f(x,y,z) = x^2 - y^2 + z^2$ at $(1,2,3)$ so it is perpendicular to the surface. Thus it is parallel to the area vector.

27. False. It is true that both $d\vec{A}$ and $3\vec{i} + 4\vec{j} + 5\vec{k}$ are perpendicular to the plane at every point, so they are multiples of each other. However, the ratio between them might not be a constant. For example, $x = s^3$, $y = t^3$, $z = (1/5)(7 - 3s^3 - 4t^3)$ is a parameterization of the plane, but

$$\begin{aligned} d\vec{A} &= (3s^2\vec{i} - (9/5)s^2\vec{k}) \times (3t^2\vec{j} - (12/5)t^2\vec{k})dsdt \\ &= ((27/5)s^2t^2\vec{i} + (36/5)s^2t^2\vec{j} + 9s^2t^2\vec{k})dsdt \\ &= (9/5)s^2t^2(3\vec{i} + 4\vec{j} + 5\vec{k})dsdt. \end{aligned}$$

28. Since the half sphere is $x = -\sqrt{1-y^2-z^2}$, we parameterize in the form $x = x(y,z)$. Thus, the answer is either (e) or (f). We have $\partial\vec{r}/\partial y = (y/\sqrt{1-y^2-z^2})\vec{i} + \vec{j}$ and $\partial\vec{r}/\partial z = (z/\sqrt{1-y^2-z^2})\vec{i} + \vec{k}$, and $\partial\vec{r}/\partial y \times \partial\vec{r}/\partial z = \vec{i} - (y/\sqrt{1-y^2-z^2})\vec{j} - (z/\sqrt{1-y^2-z^2})\vec{k} = -\vec{r}/\sqrt{1-y^2-z^2}$. Our surface is oriented away from the origin, so we want $(\partial\vec{r}/\partial z) \times (\partial\vec{r}/\partial y) = \vec{r}/\sqrt{1-y^2-z^2}$, so the answer is (f).

Solutions for Chapter 21 Review

Exercises

1. Since $x^2 + y^2 = 4z^2$, we have $z = \frac{1}{2}\sqrt{x^2+y^2}$. Thus we have a cone of height 7 and maximum radius 14, centered around the z-axis.

2. This is a parabolic cylinder $y = x^2$, between $x = -5$ and $x = 5$, with its axis along the z-axis, stretching from $z = 0$ to $z = 7$.

3. The disc S is defined by the inequality

$$s^2 + t^2 = a^2x^2 + b^2y^2 \leq 1$$

that corresponds to the inequality $x^2 + y^2 \leq 15^2$ or equivalently

$$\frac{1}{15^2}x^2 + \frac{1}{15^2}y^2 \leq 1$$

that defines R. Thus $a^2 = 1/15^2$ and $b^2 = 1/15^2$. We have $a = 1/15$, $b = 1/15$.

4. The disc S is defined by the inequality

$$s^2 + t^2 = a^2x^2 + b^2y^2 \le 1$$

that corresponds to the inequality $x^2/4 + y^2/9 \le 1$ that defines R. Thus $a^2 = 1/4$ and $b^2 = 1/9$. We have $a = 1/2$, $b = 1/3$.

5. Inverting the change of coordinates gives $x = s - at$, $y = t$.

The four edges of R are

$$y = 15, y = 35, y = 2(x - 10) + 15, y = 2(x - 30) + 15.$$

The change of coordinates transforms the edges to

$$t = 15, t = 35, t = 2s - 2at - 5, t = 2s - 2at - 45.$$

These are equations for the edges of a rectangle in the st-plane if the last two are of the form: $s =$ (Constant). This happens when the t terms drop out, or $a = -1/2$. With $a = -1/2$ the change of coordinates gives

$$\int\int_T \left| \frac{\partial(x, y)}{\partial(s, t)} \right| \, ds\, dt$$

over the rectangle

$$T : 15 \le t \le 35, \frac{5}{2} \le s \le \frac{45}{2}.$$

6. The cross product $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$ is given by

$$\frac{\partial\vec{r}}{\partial s} \times \frac{\partial\vec{r}}{\partial t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2s & 2 & 0 \\ 0 & 2t & 5 \end{vmatrix} = 10\vec{i} - 10s\vec{j} + 4st\vec{k}.$$

Since the z-component, $4st$, of the vector $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$ is positive for $0 < s \le 1, 1 \le t \le 3$, we see that $\partial\vec{r}/\partial s \times \partial\vec{r}/\partial t$ points upward, in the direction of the orientation of S we were given. Thus, we use

$$d\vec{A} = \left(\frac{\partial\vec{r}}{\partial s} \times \frac{\partial\vec{r}}{\partial t} \right) ds\, dt,$$

and so we have

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} &= \int_0^1 \int_1^3 (5t\vec{i} + s^2\vec{j}) \cdot (10\vec{i} - 10s\vec{j} + 4st\vec{k})\, dt\, ds \\ &= \int_0^1 \int_1^3 (50t - 10s^3)\, dt\, ds = \int_0^1 (25t^2 - 10s^3 t)\Big|_{t=1}^{t=3} ds \\ &= \int_0^1 (200 - 20s^3)\, ds = (200s - 5s^4)\Big|_0^1 \\ &= 200 - 5 = 195. \end{aligned}$$

7. Since

$$\vec{r}(a, \theta) = a\cos\theta\vec{i} + a\sin\theta\vec{j} + \sin a^2\vec{k},$$

we have

$$\frac{\partial\vec{r}}{\partial a} \times \frac{\partial\vec{r}}{\partial\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2a\cos a^2 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix} = -2a^2\cos\theta\cos a^2\,\vec{i} - 2a^2\sin\theta\cos a^2\,\vec{j} + a\,\vec{k}.$$

The z-component, a, of the vector $\partial\vec{r}/\partial a \times \partial\vec{r}/\partial\theta$ is positive for $1 \le a \le 3$, $0 \le \theta \le \pi$, so $\partial\vec{r}/\partial a \times \partial\vec{r}/\partial\theta$ points upward, in the direction of the orientation of S we were given. Thus, we use $d\vec{A} = (\partial\vec{r}/\partial a \times \partial\vec{r}/\partial\theta)\, da\, d\theta$, giving

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} &= \int_1^3 \int_0^\pi \left((-\frac{2}{a\cos\theta})\vec{i} + (\frac{2}{a\sin\theta})\vec{j} \right) \cdot \frac{\partial\vec{r}}{\partial a} \times \frac{\partial\vec{r}}{\partial\theta}\, d\theta\, da \\ &= \int_1^3 \int_0^\pi (4a\cos a^2 - 4a\cos a^2)\, d\theta\, da = 0. \end{aligned}$$

8. Two vectors in the plane containing $P = (5, 5, 5)$, $Q = (10, -10, 10)$, and $R = (0, 20, 40)$ are the displacement vectors

$\vec{v}_1 = \vec{PQ} = 5\vec{i} - 15\vec{j} + 5\vec{k}$

$\vec{v}_2 = \vec{PR} = -5\vec{i} + 15\vec{j} + 35\vec{k}$.

Letting $\vec{r}_0 = 5\vec{i} + 5\vec{j} + 5\vec{k}$ we have the parameterization

$$\begin{aligned}\vec{r}(s,t) &= \vec{r}_0 + s\vec{v}_1 + t\vec{v}_2 \\ &= (5 + 5s - 5t)\vec{i} + (5 - 15s + 15t)\vec{j} + (5 + 5s + 35t)\vec{k}.\end{aligned}$$

9. We use spherical coordinates ϕ and θ as the two parameters. The parameterization of the sphere center at the origin and radius 5 is:

$$x = 5\sin\phi\cos\theta, \quad y = 5\sin\phi\sin\theta, \quad z = 5\cos\phi.$$

We have to shift the center of the sphere from the origin to the point $(2, -1, 3)$. This gives

$$x = 2 + 5\sin\phi\cos\theta, \quad y = -1 + 5\sin\phi\sin\theta, \quad z = 3 + 5\cos\phi.$$

Problems

10. (a) The cone of height h, maximum radius a, vertex at the origin and opening upward is shown in Figure 21.11.

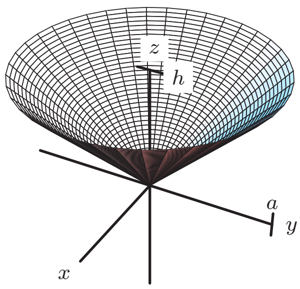


Figure 21.11

By similar triangles, we have

$$\frac{r}{z} = \frac{a}{h},$$

so

$$z = \frac{hr}{a}.$$

Therefore, one parameterization is

$$\begin{aligned}x &= r\cos\theta, && 0 \le r \le a, \\ y &= r\sin\theta, && 0 \le \theta < 2\pi, \\ z &= \frac{hr}{a}.\end{aligned}$$

(b) Since $r = az/h$, we can write the parameterization in part (a) as

$$\begin{aligned}x &= \frac{az}{h}\cos\theta, && 0 \le z \le h, \\ y &= \frac{az}{h}\sin\theta, && 0 \le \theta < 2\pi, \\ z &= z.\end{aligned}$$

11. (a) The surface is the cylinder $x^2 + y^2 = 1$ of radius 1 centered on the z-axis.
(b) The parameter curves with constant s and varying t are helices that wind clockwise around the cylinder as they advance up the cylinder with increasing t. See Figure 21.12.

The parameter curves with constant t and varying s are helices that wind counterclockwise up the cylinder. See Figure 21.13.

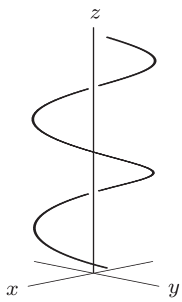


Figure 21.12: Constant s, varying t

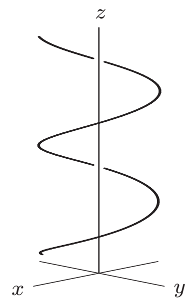


Figure 21.13: Constant t, varying s

12. The plane through $(1, 3, 4)$ and orthogonal to $\vec{n} = 2\vec{i} + \vec{j} - \vec{k}$ is given by $2(x-1) + (y-3) - (z-4) = 0$, that is,

$$2x + y - z - 1 = 0.$$

Thus, thinking of the plane as $z = 2x + y - 1$, one possible parameterization is

$$x = u, \quad y = v, \quad z = 2u + v - 1.$$

13. The parameterization for a sphere of radius a using spherical coordinates is

$$x = a\sin\phi\cos\theta, \quad y = a\sin\phi\sin\theta, \quad z = a\cos\phi.$$

Think of the ellipsoid as a sphere whose radius is different along each axis and you get the parameterization:

$$\begin{cases} x = a\sin\phi\cos\theta, & 0 \le \phi \le \pi, \\ y = b\sin\phi\sin\theta, & 0 \le \theta \le 2\pi, \\ z = c\cos\phi. \end{cases}$$

To check this parameterization, substitute into the equation for the ellipsoid:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \frac{a^2\sin^2\phi\cos^2\theta}{a^2} + \frac{b^2\sin^2\phi\sin^2\theta}{b^2} + \frac{c^2\cos^2\phi}{c^2} \\ &= \sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi = 1. \end{aligned}$$

14. The vase obtained by rotating the curve $z = 10\sqrt{x-1}$, $1 \le x \le 2$, around the z-axis is shown in Figure 21.14.

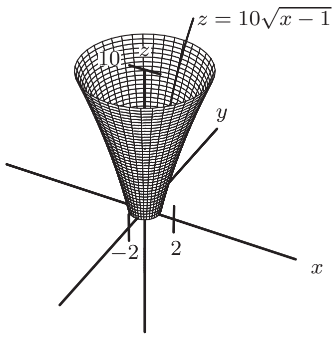


Figure 21.14

At height z, the cross-section is a horizontal circle of radius a. Thus, a point on this horizontal circle is given by

$$\vec{r} = a\cos\theta\vec{i} + a\sin\theta\vec{j} + z\vec{k}.$$

However, the radius a varies, so we need to express it in terms of the other parameters θ and z. If you look at the xz-plane, the radius of this circle is given by x, so solving for x in $z = 10\sqrt{x-1}$ gives

$$a = x = \left(\frac{z}{10}\right)^2 + 1.$$

Thus, a parameterization is

$$\vec{r} = \left(\left(\frac{z}{10}\right)^2 + 1\right)\cos\theta\vec{i} + \left(\left(\frac{z}{10}\right)^2 + 1\right)\sin\theta\vec{j} + z\vec{k}$$

so

$$x = \left(\left(\frac{z}{10}\right)^2 + 1\right)\cos\theta, \quad y = \left(\left(\frac{z}{10}\right)^2 + 1\right)\sin\theta, \quad z = z,$$

where $0 \le \theta \le 2\pi$, $0 \le z \le 10$.

15. (a) As $x^2 + y^2 = 9$ and $s \in [0, \pi]$ is equivalent to $x \ge 0$, and $t \in [0, 1]$ is equivalent to $z \in [1, 2]$. So, $x^2 + y^2 = 9$ is the equation of a cylinder, and our surface is defined by:

$$x^2 + y^2 = 9, \qquad x \ge 0, \quad 1 \le z \le 2.$$

(b) The surface $x = 3\sin s$, $y = 3\cos s$, $z = t + 1$ for $0 \le s \le \pi$, $0 \le t \le 1$ is shown in Figure 21.15.

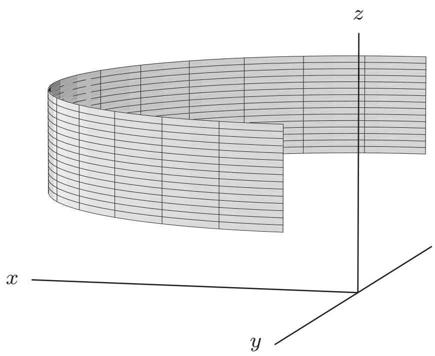


Figure 21.15

16.

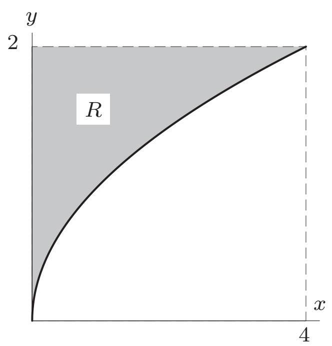


Figure 21.16

Given $T = \{(s,t) \mid 0 \le s \le 2,\ s \le t \le 2\}$ and

$$\begin{cases} x = s^2 \\ y = t, \end{cases}$$

$$R = \{(x,y)\,|\,0 \le x \le 4,\ \sqrt{x} \le y \le 2\}.$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 2s & 0 \\ 0 & 1 \end{vmatrix} = 2s,$$

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 2s \qquad \text{since} \qquad 0 \le s \le 2.$$

$$\int_T \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt = 2\int_0^2 s\,ds \int_s^2 dt = 2\int_0^2 s(2-s)\,ds = 2\left[s^2 - \frac{s^3}{3}\right]_0^2 = \frac{8}{3}.$$

So,

$$\int_R dx\,dy = \int_0^4 dx \int_{\sqrt{x}}^2 dy = \int_0^4 (2-\sqrt{x})\,dx$$
$$= \left[2x - \frac{2}{3}x^{3/2}\right]_0^4 = 8 - \frac{16}{3} = \frac{8}{3}.$$

Thus

$$\int_R dx\,dy = \int_T \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.$$

17. We must change coordinates in the area element $dA = dx\,dy$, the integrand $\sin(x+y)$ and the region R.

Inverting the coordinate change gives $x = (s+t)/2$, $y = (t-s)/2$. Thus

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt = \frac{1}{2}\,ds\,dt.$$

The integrand is $\sin(x+y) = \sin t$.

The region of integration is

$$x^2 + y^2 = \left(\frac{s+t}{2}\right)^2 + \left(\frac{t-s}{2}\right)^2 = \frac{s^2+t^2}{2} \le 1.$$

Let T be the disc $s^2 + t^2 \le 2$ of radius $\sqrt{2}$. We have

$$\int\int_R \sin(x+y)\,dx\,dy = \int\int_T \frac{1}{2}(\sin t)\,ds\,dt = 0.$$

The final integral is zero by symmetry, the integral over the part of the disc where $t < 0$ canceling the integral over the part where $t > 0$.

18. Given

$$\begin{cases} s = xy \\ t = xy^2, \end{cases}$$

we have

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} = xy^2 = t.$$

Since

$$\frac{\partial(s,t)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(s,t)} = 1,$$

$$\frac{\partial(x,y)}{\partial(s,t)} = t \qquad \text{so} \qquad \left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \frac{1}{t}$$

So

$$\int_R xy^2\,dA = \int_T t\left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt = \int_T t(\frac{1}{t})\,ds\,dt = \int_T ds\,dt,$$

where T is the region bounded by $s = 1$, $s = 4$, $t = 1$, $t = 4$.
Then

$$\int_R xy^2\, dA = \int_1^4 ds \int_1^4 dt = 9.$$

19. The elliptic cylindrical surface is parameterized by

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = a\cos\theta\vec{i} + b\sin\theta\vec{j} + z\vec{k} \qquad \text{where } 0 \le \theta \le 2\pi, -c \le z \le c.$$

We have

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\theta & b\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = b\cos\theta\vec{i} + a\sin\theta\vec{j}\,.$$

This vector points away from the z-axis, so we use $d\vec{A} = (b\cos\theta\vec{i} + a\sin\theta\vec{j})\,d\theta dz$, giving

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_{-c}^{c}\int_0^{2\pi} \left(\frac{b}{a}(a\cos\theta)\vec{i} + \frac{a}{b}(b\sin\theta\vec{j}\,)\right)\cdot(b\cos\theta\vec{i} + a\sin\theta\vec{j}\,)\,d\theta\,dz \\ &= \int_{-c}^{c}\int_0^{2\pi} (b^2\cos^2\theta + a^2\sin^2\theta)\,d\theta dz \\ &= 2\pi c(a^2+b^2).\end{aligned}$$

20. The surface of S is parameterized by

$$\vec{r}\,(\theta,\phi) = x\vec{i} + y\vec{j} + z\vec{k}\,,$$

where

$$\begin{cases} x = a + d\sin\phi\cos\theta, \\ y = b + d\sin\phi\sin\theta, \\ z = c + d\cos\phi, \end{cases} \quad \text{for} \quad 0\le\phi\le\pi,\ 0\le\theta\le 2\pi.$$

The vector $\partial\vec{r}/\partial\phi \times \partial\vec{r}/\partial\theta$ points outward by the right-hand rule, so

$$d\vec{A} = \left(\frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta}\right) d\phi d\theta.$$

Thus,

$$\begin{aligned}\vec{F}\cdot d\vec{A} &= \vec{F}\cdot\left(\frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta}\right) d\phi d\theta \\ &= \begin{vmatrix} x^2 & y^2 & z^2 \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \end{vmatrix} d\phi d\theta \\ &= \begin{vmatrix} (a+d\sin\phi\cos\theta)^2 & (b+d\sin\phi\sin\theta)^2 & (c+d\cos\phi)^2 \\ d\cos\phi\cos\theta & d\cos\phi\sin\theta & -d\sin\phi \\ -d\sin\phi\sin\theta & d\sin\phi\cos\theta & 0 \end{vmatrix} d\phi d\theta.\end{aligned}$$

Hence,

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} = d^2\int_0^{2\pi}\int_0^{\pi} (\ & a^2\sin^2\phi\cos\theta + 2ad\sin^3\phi\cos^2\theta + d^2\sin^4\phi\cos^3\theta \\ +\ & b^2\sin^2\phi\sin\theta + 2bd\sin^3\phi\sin^2\theta + d^2\sin^4\phi\sin^3\theta \\ +\ & c^2\sin\phi\cos\phi + 2cd\sin\phi\cos^2\phi + d^2\sin\phi\cos^3\phi)\ d\phi d\theta.\end{aligned}$$

Since

$$\int_0^{2\pi} \cos\theta\, d\theta = \int_0^{2\pi} \sin\theta\, d\theta = \int_0^{2\pi} \cos^3\theta\, d\theta = \int_0^{2\pi} \sin^3\theta\, d\theta = 0,$$

and

$$\int_0^{\pi} \sin\phi\cos\phi\, d\phi = \int_0^{\pi} \sin\phi\cos^3\phi\, d\phi = 0,$$

we have

$$\begin{aligned}\int_S \vec F \cdot \vec A \quad & d^2 \int_0^{2\pi}\int_0^{\pi} (2ad\sin^3\phi\cos^2\theta + 2bd\sin^3\phi\sin^2\theta + 2cd\sin\phi\cos^2\phi)\, d\phi d\theta \\ &= 2\pi d^3 \int_0^{\pi} (a\sin^3\phi + b\sin^3\phi + 2c\sin\phi\cos^2\phi)\, d\phi \\ &= 4\pi d^3 \int_0^{\pi/2} (a\sin^3\phi + b\sin^3\phi + 2c\sin\phi\cos^2\phi)\, d\phi \\ &= \frac{8}{3}\pi d^3 (a+b+c).\end{aligned}$$

PROJECTS FOR CHAPTER TWENTY-ONE

1. The sphere $x^2 + y^2 + z^2 = 1$ is shown in Figure 21.17.

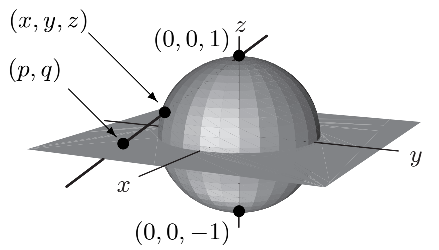


Figure 21.17

(a) The origin corresponds to the south pole.
(b) The circle $x^2 + y^2 = 1$ corresponds to the equator.
(c) We get all the points of the sphere by this parameterization except the north pole itself.
(d) $x^2 + y^2 > 1$ corresponds to the upper hemisphere.
(e) $x^2 + y^2 < 1$ corresponds to the lower hemisphere.

2. We use the angles θ and ϕ shown in the figure in the problem. The angle θ is in the xy-plane measured counterclockwise from the positive x-axis; the angle ϕ is measured perpendicular to the xy-plane.

(a) The circle of radius b in the xy-plane is parameterized by

$$\vec r = b\cos\theta\vec i + b\sin\theta\vec j \qquad 0 \le \theta \le 2\pi.$$

(b) One vector is always $\vec k$. The other vector is in the same direction as $\vec r$ in part (a) but has length 1. Therefore, we take the other vector to be $\vec m = \cos\theta\vec i + \sin\theta\vec j$. Thus, relative to its center, the small circle of radius a can be parameterized by

$$\vec s = a\cos\phi\vec m + a\sin\phi\vec k = a\cos\phi(\cos\theta\vec i + \sin\theta\vec j) + a\sin\phi\vec k \quad 0 \le \phi \le 2\pi.$$

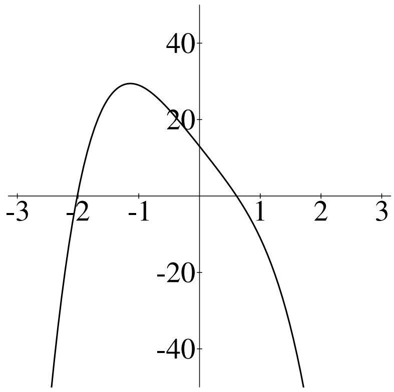
(c) The parameterization of the torus with parameters $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$, is given by

$$\begin{aligned} x\vec i + y\vec j + z\vec k &= \vec r + \vec s \\ &= (b\cos\theta + a\cos\phi\cos\theta)\vec i + (b\sin\theta + a\cos\phi\sin\theta)\vec j + a\sin\phi\vec k.\end{aligned}$$

APPENDIX

Solutions for Section A

1. The graph is



(a) The range appears to be $y \leq 30$.
(b) The function has two zeros.

2. (a) The root is between 0.3 and 0.4, at about 0.35.
(b) The root is between 1.5 and 1.6, at about 1.55.
(c) The root is between -1.8 and -1.9, at about -1.85.

3. The root occurs at about -1.05

4. The root is between -1.7 and -1.8, at about -1.75.

5. The largest root is at about 2.5.

6. There is one root at $x = -1$ and another at about $x = 1.35$.

7. There is one real root at about $x = -1.1$.

8. The root occurs at about 0.9, since the function changes sign between 0.8 and 1.

9. Using a graphing calculator, we see that when x is around 0.45, the graphs intersect.

10. The root occurs between 0.6 and 0.7, at about 0.65.

11. The root occurs between 1.2 and 1.4, at about 1.3.

12. Zoom in on graph: $t = \pm 0.824$. [Note: t must be in radians; one must zoom in two or three times.]

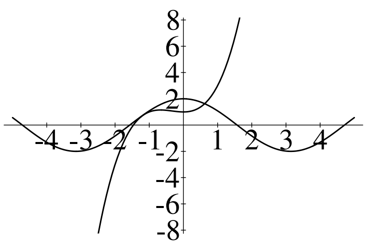
13. (a) Only one real zero, at about $x = -1.15$.
(b) Three real zeros: at $x = 1$, and at about $x = 1.41$ and $x = -1.41$.

14. First, notice that $f(3) \approx 0.5 > 0$ and that $f(4) \approx -0.25 < 0$.
1st iteration: $f(3.5) > 0$, so a zero is between 3.5 and 4.
2nd iteration: $f(3.75) < 0$, so a zero is between 3.5 and 3.75.
3rd iteration: $f(3.625) < 0$, so a zero is between 3.5 and 3.625.
4th iteration: $f(3.588) < 0$, so a zero is between 3.5 and 3.588.
5th iteration: $f(3.545) > 0$, so a zero is between 3.545 and 3.588.
6th iteration: $f(3.578) < 0$, so a zero is between 3.567 and 3.578.
7th iteration: $f(3.572) > 0$, so a zero is between 3.572 and 3.578.
8th iteration: $f(3.575) > 0$, so a zero is between 3.575 and 3.578.

Thus we know that, rounded to two places, the value of the zero must be 3.58. We know that this is the largest zero of $f(x)$ since $f(x)$ approaches -1 for larger values of x.

15. (a) Let $F(x) = \sin x - 2^{-x}$. Then $F(x) = 0$ will have a root where $f(x)$ and $g(x)$ cross. The first positive value of x for which the functions intersect is $x \approx 0.7$.
(b) The functions intersect for $x \approx 0.4$.

16. The graph is



We find one zero at about 0.6. It looks like there might be another one at about -1.2, but zoom in close... closer... closer, and you'll see that though the graphs are very close together, they do not touch, and so there is no zero near -1.2. Thus the zero at about 0.6 is the only one. (How do you know there are no other zeros off the screen?)

17. (a) Since f is continuous, there must be one zero between $\theta = 1.4$ and $\theta = 1.6$, and another between $\theta = 1.6$ and $\theta = 1.8$. These are the only clear cases. We might also want to investigate the interval $0.6 \le \theta \le 0.8$ since $f(\theta)$ takes on values close to zero on at least part of this interval. Now, $\theta = 0.7$ is in this interval, and $f(0.7) = -0.01 < 0$, so f changes sign twice between $\theta = 0.6$ and $\theta = 0.8$ and hence has two zeros on this interval (assuming f is not *really* wiggly here, which it's not). There are a total of 4 zeros.
(b) As an example, we find the zero of f between $\theta = 0.6$ and $\theta = 0.7$. $f(0.65)$ is positive; $f(0.66)$ is negative. So this zero is contained in $[0.65, 0.66]$. The other zeros are contained in the intervals $[0.72, 0.73]$, $[1.43, 1.44]$, and $[1.7, 1.71]$.
(c) You've found all the zeros. A picture will confirm this; see Figure A.1.

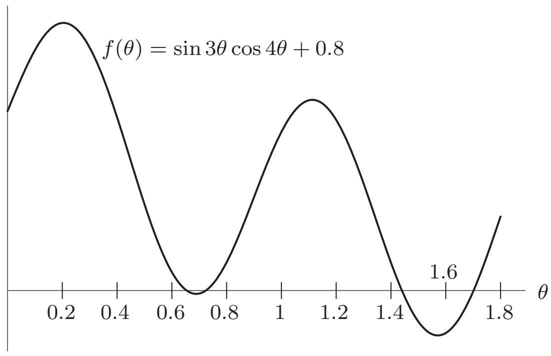


Figure A.1

18. (a) There appear to be two solutions: one on the interval from 1.13 to 1.14 and one on the interval from 1.08 to 1.09. From 1.13 to 1.14, $\frac{x^3}{\pi^3}$ increases from 0.0465 to 0.0478 while $(\sin 3x)(\cos 4x)$ decreases from 0.0470 to 0.0417, so they must cross in between. Similarly, going from 1.08 to 1.09, $\frac{x^3}{\pi^3}$ increases from 0.0406 to 0.0418 while $(\sin 3x)(\cos 4x)$ increases from 0.0376 to 0.0442. Thus the difference between the two changes sign over that interval, so their difference must be zero somewhere in between.
(b) Reasonable estimates are $x = 1.085$ and $x = 1.131$.

19. (a) The first ten results are:

n	0	1	2	3	4	5	6	7	8
1	3.14159	5.05050	5.50129	5.56393	5.57186	5.57285	5.57297	5.57299	5.57299

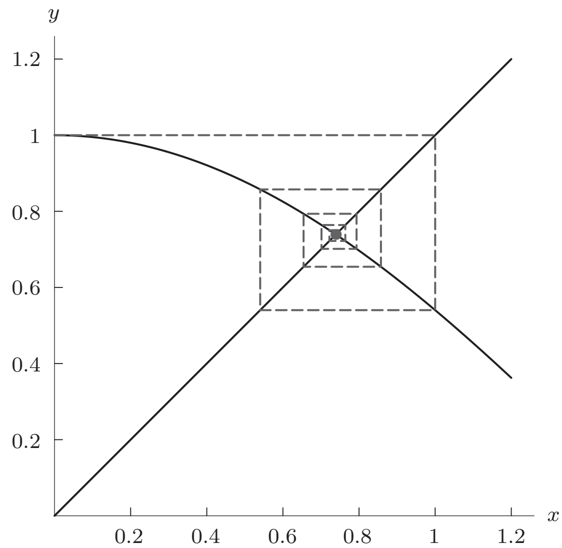
(b) The solution is $x \approx 5.573$. We started with an initial guess of 1, and kept repeating the given procedure until our values converged to a limit at around 5.573. For each number on the table, the procedure was in essence asking the question “Does this number equal 4 times the arctangent of itself?” and then correcting the number by repeating the question for 4 times the arctangent of the number.

(c) P_0 represents our initial guess of $x = 1$ (on the line $y = x$). P_1 is 4 times the arctangent of 1. If we now use take this value for P_1 and slide it horizontally back to the line $y = x$, we can now use this as a new guess, and call it P_2. P_3, of course, represents 4 times the arctangent of P_2, and so on. Another way to make sense of this diagram is to consider the function $F(x) = 4 \arctan x - x$. On the diagram, this difference is represented by the vertical lines connecting P_0 and P_1, P_2 and P_3 and so on. Notice how these lines (and hence the difference between $\arctan x$ and x) get smaller as we approach the intersection point, where $F(x) = 0$.

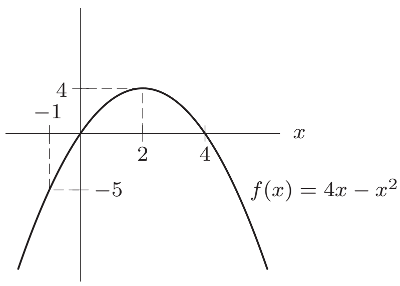
(d) For an initial guess of $x = 10$, the procedure gives a decreasing sequence which converges (more quickly) to the same value of about 5.573. Graphically, our initial guess of P_0 will lie to the right of the intersection on the line $y = x$. The iteration procedure gives us a sequence of $P_1, P_2, \ldots$ that zigzags to the left, toward the intersection point. For an initial guess of $x = -10$, the procedure gives an increasing sequence converging to the other intersection point of these two curves at $x \approx -5.573$. Graphically, we get a sequence which is a reflection through the origin of the sequence we got for an initial guess of $x = 10$. This is so because both $y = x$ and $y = \arctan x$ are odd functions.

20. Starting with $x = 0$, and repeatedly taking the cosine, we get the numbers below. Continuing until the first three decimal places remain fixed under iteration, we have this list and diagram:

x	$\cos x$
0	0.735069
1	0.7401473
0.5403023	0.7356047
0.8575532	0.7414251
0.6542898	0.7375069
0.7934804	0.7401473
0.7013688	0.7383692
0.7639597	0.7395672
0.7221024	0.7387603
0.7504178	0.7393039
0.7314043	0.7389378
0.7442374	etc.

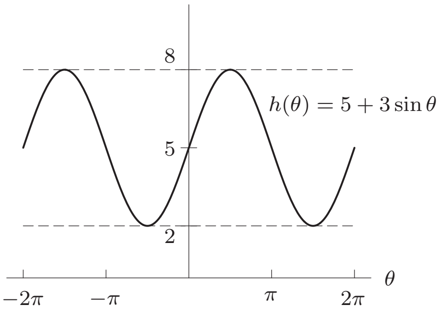


21.



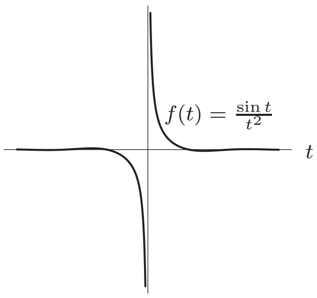
Bounded and $-5 \leq f(x) \leq 4$.

22.



Bounded and $2 \le h(\theta) \le 8$.

23.



Not bounded because $f(t)$ goes to infinity as t goes to 0.

Solutions for Section B

1. $2e^{i\pi/2}$

2. $5e^{i\pi}$

3. $\sqrt{2}e^{i\pi/4}$

4. $5e^{i4.069}$

5. $0e^{i\theta}$, for any θ.

6. $e^{3\pi i/2}$

7. $\sqrt{10}e^{i\theta}$, where $\theta = \arctan(-3) + \pi = 1.893$ is an angle in the second quadrant.

8. $13e^{i\theta}$, where $\theta = \arctan(-\frac{12}{5}) \approx -1.176$ is an angle in the fourth quadrant.

9. $-3 - 4i$

10. $-11 + 29i$

11. $-5 + 12i$

12. $1 + 3i$

13. $\frac{1}{4} - \frac{9i}{8}$

14. $3 - 6i$

15. $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

16. $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$ is one solution.

17. $5^3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -125i$

18. $\sqrt[4]{10}\cos\frac{\pi}{8} + i\sqrt[4]{10}\sin\frac{\pi}{8}$ is one solution.

19. One value of $\sqrt{i}$ is $\sqrt{e^{i\frac{\pi}{2}}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

20. One value of $\sqrt{-i}$ is $\sqrt{e^{i\frac{3\pi}{2}}} = (e^{i\frac{3\pi}{2}})^{\frac{1}{2}} = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

21. One value of $\sqrt[3]{i}$ is $\sqrt[3]{e^{i\frac{\pi}{2}}} = (e^{i\frac{\pi}{2}})^{\frac{1}{3}} = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

22. One value of $\sqrt{7i}$ is $\sqrt{7e^{i\frac{\pi}{2}}} = (7e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \sqrt{7}e^{i\frac{\pi}{4}} = \sqrt{7}\cos\frac{\pi}{4} + i\sqrt{7}\sin\frac{\pi}{4} = \frac{\sqrt{14}}{2} + i\frac{\sqrt{14}}{2}$

23. $(1+i)^{100} = (\sqrt{2}e^{i\frac{\pi}{4}})^{100} = (2^{\frac{1}{2}})^{100}(e^{i\frac{\pi}{4}})^{100} = 2^{50}\cdot e^{i\cdot 25\pi} = 2^{50}\cos 25\pi + i2^{50}\sin 25\pi = -2^{50}$

24. One value of $(1+i)^{2/3}$ is $(\sqrt{2}e^{i\frac{\pi}{4}})^{2/3} = (2^{\frac{1}{2}}e^{i\frac{\pi}{4}})^{\frac{2}{3}} = \sqrt[3]{2}e^{i\frac{\pi}{6}} = \sqrt[3]{2}\cos\frac{\pi}{6} + i\sqrt[3]{2}\sin\frac{\pi}{6} = \sqrt[3]{2}\cdot\frac{\sqrt{3}}{2} + i\sqrt[3]{2}\cdot\frac{1}{2}$

25. One value of $(-4+4i)^{2/3}$ is $[\sqrt{32}e^{(i3\pi/4)}]^{(2/3)} = (\sqrt{32})^{2/3}e^{(i\pi/2)} = 2^{5/3}\cos\frac{\pi}{2} + i2^{5/3}\sin\frac{\pi}{2} = 2i\sqrt[3]{4}$

26. One value of $(\sqrt{3}+i)^{1/2}$ is $(2e^{i\frac{\pi}{6}})^{1/2} = \sqrt{2}e^{i\frac{\pi}{12}} = \sqrt{2}\cos\frac{\pi}{12} + i\sqrt{2}\sin\frac{\pi}{12} \approx 1.366 + 0.366i$

27. One value of $(\sqrt{3}+i)^{-1/2}$ is $(2e^{i\frac{\pi}{6}})^{-1/2} = \frac{1}{\sqrt{2}}e^{i(-\frac{\pi}{12})} = \frac{1}{\sqrt{2}}\cos(-\frac{\pi}{12}) + i\frac{1}{\sqrt{2}}\sin(-\frac{\pi}{12}) \approx 0.683 - 0.183i$

28. Since $\sqrt{5} + 2i = 3e^{i\theta}$, where $\theta = \arctan\frac{2}{\sqrt{5}} \approx 0.730$, one value of $(\sqrt{5}+2i)^{\sqrt{2}}$ is $(3e^{i\theta})^{\sqrt{2}} = 3^{\sqrt{2}}e^{i\sqrt{2}\theta} = 3^{\sqrt{2}}\cos\sqrt{2}\theta + i3^{\sqrt{2}}\sin\sqrt{2}\theta \approx 3^{\sqrt{2}}(0.513) + i3^{\sqrt{2}}(0.859) \approx 2.426 + 4.062i$

29. We have

$$\begin{aligned} i^{-1} &= \frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = -i, \\ i^{-2} &= \frac{1}{i^2} = -1, \\ i^{-3} &= \frac{1}{i^3} = \frac{1}{-i}\cdot\frac{i}{i} = i, \\ i^{-4} &= \frac{1}{i^4} = 1. \end{aligned}$$

The pattern is

$$i^n = \begin{cases} -i & n = -1, -5, -9, \cdots \\ -1 & n = -2, -6, -10, \cdots \\ i & n = -3, -7, -11, \cdots \\ 1 & n = -4, -8, -12, \cdots. \end{cases}$$

Since 36 is a multiple of 4, we know $i^{-36} = 1$.
Since $41 = 4\cdot 10 + 1$, we know $i^{-41} = -i$.

30. Substituting $A_1 = 2 - A_2$ into the second equation gives

$$(1-i)(2-A_2) + (1+i)A_2 = 0$$

so

$$\begin{aligned} 2iA_2 &= -2(1-i) \\ A_2 &= \frac{-(1-i)}{i} = \frac{-i(1-i)}{i^2} = i(1-i) = 1+i \end{aligned}$$

Therefore $A_1 = 2 - (1+i) = 1 - i$.

31. Substituting $A_2 = 2 - A_1$ into the second equation gives

$$\begin{aligned} (i-1)A_1 + (1+i)(2-A_1) &= 0 \\ iA_1 - A_1 - A_1 - iA_1 + 2 + 2i &= 0 \\ -2A_1 &= -2 - 2i \\ A_1 &= 1 + i \end{aligned}$$

Substituting, we have

$$A_2 = 2 - A_1 = 2 - (1+i) = 1 - i.$$

32. (a) To divide complex numbers, multiply top and bottom by the conjugate of $1+2i$, that is, $1-2i$:

$$\frac{3-4i}{1+2i} = \frac{3-4i}{1+2i}\cdot\frac{1-2i}{1-2i} = \frac{3-4i-6i+8i^2}{1^2+2^2} = \frac{-5-10i}{5} = -1-2i.$$

Thus, $a = -1$ and $b = -2$.

(b) Multiplying $(1+2i)(a+bi)$ should give $3-4i$, as the following calculation shows:

$$(1+2i)(a+bi) = (1+2i)(-1-2i) = -1-2i-2i-4i^2 = 3-4i.$$

33. To confirm that $z = \dfrac{a+bi}{c+di}$, we calculate the product

$$\begin{aligned} z(c+di) &= \left(\frac{ac+bd}{c^2+d^2} = \frac{bc-ad}{c^2+d^2}i\right)(c+di) \\ &= \frac{ac^2+bcd-bcd+ad^2+(bc^2-acd+acd+bd^2)i}{c^2+d^2} \\ &= \frac{a(c^2+d^2)+b(c^2+d^2)i}{c^2+d^2} = a+bi. \end{aligned}$$

34. (a)

$$z_1z_2 = (-3-i\sqrt{3})(-1+i\sqrt{3}) = 3+(\sqrt{3})^2+i(\sqrt{3}-3\sqrt{3}) = 6-i2\sqrt{3}.$$
$$\frac{z_1}{z_2} = \frac{-3-i\sqrt{3}}{-1+i\sqrt{3}}\cdot\frac{-1-i\sqrt{3}}{-1-i\sqrt{3}} = \frac{3-(\sqrt{3})^2+i(\sqrt{3}+3\sqrt{3})}{(-1)^2+(\sqrt{3})^2} = \frac{i\cdot 4\sqrt{3}}{4} = i\sqrt{3}.$$

(b) We find (r_1, θ_1) corresponding to $z_1 = -3-i\sqrt{3}$:

$$r_1 = \sqrt{(-3)^2+(\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3};$$
$$\tan\theta_1 = \frac{-\sqrt{3}}{-3} = \frac{\sqrt{3}}{3}, \text{ so } \theta_1 = \frac{7\pi}{6}.$$

(See Figure B.2.) Thus,

$$-3-i\sqrt{3} = r_1e^{i\theta_1} = 2\sqrt{3}\,e^{i\frac{7\pi}{6}}.$$

We find (r_2, θ_2) corresponding to $z_2 = -1+i\sqrt{3}$:

$$r_2 = \sqrt{(-1)^2+(\sqrt{3})^2} = 2;$$
$$\tan\theta_2 = \frac{\sqrt{3}}{-1} = -\sqrt{3}, \text{ so } \theta_2 = \frac{2\pi}{3}.$$

(See Figure B.3.) Thus,

$$-1+i\sqrt{3} = r_2e^{i\theta_2} = 2e^{i\frac{2\pi}{3}}.$$

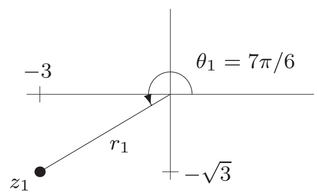


Figure B.2

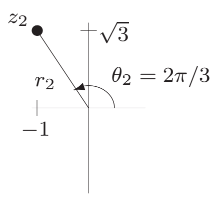


Figure B.3

We now calculate z_1z_2 and $\dfrac{z_1}{z_2}$.

$$\begin{aligned} z_1z_2 &= \left(2\sqrt{3}e^{i\frac{7\pi}{6}}\right)\left(2e^{i\frac{2\pi}{3}}\right) = 4\sqrt{3}e^{i(\frac{7\pi}{6}+\frac{2\pi}{3})} = 4\sqrt{3}e^{i\frac{11\pi}{6}} \\ &= 4\sqrt{3}\left[\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right] = 4\sqrt{3}\left[\frac{\sqrt{3}}{2}-i\frac{1}{2}\right] = 6-i2\sqrt{3}. \\ \frac{z_1}{z_2} &= \frac{2\sqrt{3}e^{i\frac{7\pi}{6}}}{2e^{i\frac{2\pi}{3}}} = \sqrt{3}e^{i(\frac{7\pi}{6}-\frac{2\pi}{3})} = \sqrt{3}e^{i\frac{\pi}{2}} \\ &= \sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = i\sqrt{3}. \end{aligned}$$

These agrees with the values found in (a).

35. First we calculate

$$z_1z_2 = (a_1 + b_1i)(a_2 + b_2i) = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1).$$

Thus, $\overline{z_1z_2} = a_1a_2 - b_1b_2 - i(a_1b_2 + a_2b_1)$.

Since $\bar{z}_1 = a_1 - b_1i$ and $\bar{z}_2 = a_2 - b_2i$,

$$\bar{z}_1\bar{z}_2 = (a_1 - b_1i)(a_2 - b_2i) = a_1a_2 - b_1b_2 - i(a_1b_2 + a_2b_1).$$

Thus, $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$.

36. If the roots are complex numbers, we must have $(2b)^2 - 4c < 0$ so $b^2 - c < 0$. Then the roots are

$$\begin{aligned} x = \frac{-2b \pm \sqrt{(2b)^2 - 4c}}{2} &= -b \pm \sqrt{b^2 - c} \\ &= -b \pm \sqrt{-1(c - b^2)} \\ &= -b \pm i\sqrt{c - b^2}\,. \end{aligned}$$

Thus, $p = -b$ and $q = \sqrt{c - b^2}$.

37. True, since $\sqrt{a}$ is real for all $a \geq 0$.

38. True, since $(x - iy)(x + iy) = x^2 + y^2$ is real.

39. False, since $(1 + i)^2 = 2i$ is not real.

40. False. Let $f(x) = x$. Then $f(i) = i$ but $f(\bar{i}) = \bar{i} = -i$.

41. True. We can write any nonzero complex number z as $re^{i\beta}$, where r and β are real numbers with $r > 0$. Since $r > 0$, we can write $r = e^c$ for some real number c. Therefore, $z = re^{i\beta} = e^ce^{i\beta} = e^{c+i\beta} = e^w$ where $w = c + i\beta$ is a complex number.

42. False, since $(1 + 2i)^2 = -3 + 4i$.

43.

$$\begin{aligned} 1 = e^0 &= e^{i(\theta-\theta)} = e^{i\theta}e^{i(-\theta)} \\ &= (\cos\theta + i\sin\theta)(\cos(-\theta) + i\sin(-\theta)) \\ &= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \end{aligned}$$

44. Using Euler's formula, we have:

$$e^{i(2\theta)} = \cos 2\theta + i\sin 2\theta$$

On the other hand,

$$e^{i(2\theta)} = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

Equating imaginary parts, we find

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

45. Using Euler's formula, we have:

$$e^{i(2\theta)} = \cos 2\theta + i\sin 2\theta$$

On the other hand,

$$e^{i(2\theta)} = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

Equating real parts, we find

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

46. Differentiating Euler's formula gives

$$\frac{d}{d\theta}(e^{i\theta}) = ie^{i\theta} = i(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta$$

Since in addition $\frac{d}{d\theta}(e^{i\theta}) = \frac{d}{d\theta}(\cos\theta + i\sin\theta) = \frac{d}{d\theta}(\cos\theta) + i\frac{d}{d\theta}(\sin\theta)$, by equating imaginary parts, we conclude that $\frac{d}{d\theta}\sin\theta = \cos\theta$.

47. Differentiating Euler's formula twice gives

$$\frac{d^2}{d\theta^2}(e^{i\theta}) = \frac{d^2}{d\theta^2}(\cos\theta + i\sin\theta) = \frac{d^2}{d\theta^2}(\cos\theta) + i\frac{d^2}{d\theta^2}(\sin\theta).$$

But

$$\frac{d^2}{d\theta^2}(e^{i\theta}) = i^2e^{i\theta} = -e^{i\theta} = -\cos\theta - i\sin\theta.$$

Equating real parts, we find

$$\frac{d^2}{d\theta^2}(\cos\theta) = -\cos\theta.$$

48. Replacing θ by $-x$ in the formula for $\sin\theta$:

$$\sin(-x) = \frac{1}{2i}\left(e^{-ix} - e^{ix}\right) = -\frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -\sin x.$$

49. Replacing θ by $(x+y)$ in the formula for $\sin\theta$:

$$\begin{aligned}\sin(x+y) &= \frac{1}{2i}\left(e^{i(x+y)} - e^{-i(x+y)}\right) = \frac{1}{2i}\left(e^{ix}e^{iy} - e^{-ix}e^{-iy}\right)\\ &= \frac{1}{2i}\left((\cos x + i\sin x)(\cos y + i\sin y) - (\cos(-x) + i\sin(-x))(\cos(-y) + i\sin(-y))\right)\\ &= \frac{1}{2i}\left((\cos x + i\sin x)(\cos y + i\sin y) - (\cos x - i\sin x)(\cos y - i\sin y)\right)\\ &= \sin x\cos y + \cos x\sin y.\end{aligned}$$

50. Since x_1, y_1, x_2, y_2 are each functions of the variable t, differentiating the sum gives

$$\begin{aligned}(z_1+z_2)' &= (x_1+iy_1+x_2+iy_2)' = (x_1+x_2+i(y_1+y_2))'\\ &= (x_1+x_2)' + i(y_1+y_2)'\\ &= \left(x_1'+x_2'\right) + i\left(y_1'+y_2'\right)\\ &= (x_1+iy_1)' + (x_2+iy_2)'\\ &= z_1'+z_2'.\end{aligned}$$

Differentiating the product gives

$$\begin{aligned}(z_1z_2)' &= ((x_1+iy_1)(x_2+iy_2))' = (x_1x_2 - y_1y_2 + i(y_1x_2+x_1y_2))'\\ &= (x_1x_2-y_1y_2)' + i(y_1x_2+x_1y_2)'\\ &= \left(x_1'x_2 + x_1x_2' - y_1'y_2 - y_1y_2'\right) + i\left(y_1'x_2 + y_1x_2' + x_1'y_2 + x_1y_2'\right)\\ &= [x_1'x_2 - y_1'y_2 + i(x_1'y_2 + y_1'x_2)] + [x_1x_2' - y_1y_2' + i(y_1x_2' + x_1y_2')]\\ &= \left(x_1'+iy_1'\right)(x_2+iy_2) + (x_1+iy_1)\left(x_2'+iy_2'\right)\\ &= z_1'z_2 + z_1z_2'.\end{aligned}$$

Solutions for Section C

1. (a) $f'(x) = 3x^2 + 6x + 3 = 3(x+1)^2$. Thus $f'(x) > 0$ everywhere except at $x = -1$, so it is increasing everywhere except perhaps at $x = -1$. The function is in fact increasing at $x = -1$ since $f(x) > f(-1)$ for $x > -1$, and $f(x) < f(-1)$ for $x < -1$.

(b) The original equation can have at most one root, since it can only pass through the x-axis once if it never decreases. It must have one root, since $f(0) = -6$ and $f(1) = 1$.

(c) The root is in the interval $[0, 1]$, since $f(0) < 0 < f(1)$.

(d) Let $x_0 = 1$.

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{12} = \frac{11}{12} \approx 0.917 \\ x_2 &= \frac{11}{12} - \frac{f\left(\frac{11}{12}\right)}{f'\left(\frac{11}{12}\right)} \approx 0.913 \\ x_3 &= 0.913 - \frac{f(0.913)}{f'(0.913)} \approx 0.913. \end{aligned}$$

Since the digits repeat, they should be accurate. Thus $x \approx 0.913$.

2. Let $f(x) = x^3 - 50$. Then $f(\sqrt[3]{50}) = 0$, so we can use Newton's method to solve $f(x) = 0$ to obtain $x = \sqrt[3]{50}$. Since $f'(x) = 3x^2$, f' is always positive, and f is therefore increasing. Consequently, f has only one zero. Since $3^3 = 27 < 50 < 64 = 4^3$, let $x_0 = 3.5$. Then

$$\begin{aligned} x_0 &= 3.5 \\ x_1 &= 3.5 - \frac{f(3.5)}{f'(3.5)} \approx 3.694 \end{aligned}$$

Continuing, we find

$$\begin{aligned} x_2 &\approx 3.684 \\ x_3 &\approx 3.684. \end{aligned}$$

Since the digits repeat, x_3 should be correct, as can be confirmed by calculator.

3. Let $f(x) = x^4 - 100$. Then $f(\sqrt[4]{100}) = 0$, so we can use Newton's method to solve $f(x) = 0$ to obtain $x = \sqrt[4]{100}$. $f'(x) = 4x^3$. Since $3^4 = 81 < 100 < 256 = 4^4$, try 3.1 as an initial guess.

$$\begin{aligned} x_0 &= 3.1 \\ x_1 &= 3.1 - \frac{f(3.1)}{f'(3.1)} \approx 3.164 \\ x_2 &= 3.164 - \frac{f(3.164)}{f'(3.164)} \approx 3.162 \\ x_3 &= 3.162 - \frac{f(3.162)}{f'(3.162)} \approx 3.162 \end{aligned}$$

Thus $\sqrt[4]{100} \approx 3.162$.

4. Let $f(x) = x^3 - \frac{1}{10}$. Then $f(10^{-1/3}) = 0$, so we can use Newton's method to solve $f(x) = 0$ to obtain $x = 10^{-1/3}$. $f'(x) = 3x^2$. Since $\sqrt[3]{\frac{1}{27}} < \sqrt[3]{\frac{1}{10}} < \sqrt[3]{\frac{1}{8}}$, try $x_0 = \frac{1}{2}$. Then $x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} \approx 0.467$. Continuing, we find $x_2 \approx 0.464 . x_3 \approx 0.464$. Since $x_2 \approx x_3$, $10^{-1/3} \approx 0.464$.

5. Let $f(x) = \sin x - 1 + x$; we want to find all zeros of f, because $f(x) = 0$ implies $\sin x = 1 - x$.

Graphing $\sin x$ and $1 - x$ in Figure C.4, we see that $f(x)$ has one solution at $x \approx \frac{1}{2}$.

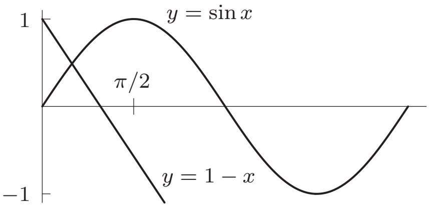


Figure C.4

Letting $x_0 = 0.5$, and using Newton's method, we have $f'(x) = \cos x + 1$, so that

$$x_1 = 0.5 - \frac{\sin(0.5) - 1 + 0.5}{\cos(0.5) + 1} \approx 0.511,$$

$$x_2 = 0.511 - \frac{\sin(0.511) - 1 + 0.511}{\cos(0.511) + 1} \approx 0.511.$$

Thus $\sin x = 1 - x$ has one solution at $x \approx 0.511$.

6. Let $f(x) = \cos x - x$. We want to find all zeros of f, because $f(x) = 0$ implies that $\cos x = x$. Since $f'(x) = -\sin x - 1$, f' is always negative (as $-\sin x$ never exceeds 1). This means f is always decreasing and consequently has at most 1 root. We now use Newton's method. Since $\cos 0 > 0$ and $\cos \frac{\pi}{2} < \frac{\pi}{2}$, $\cos x = x$ for $0 < x < \frac{\pi}{2}$. Thus, try $x_0 = \frac{\pi}{6}$.

$$x_1 = \frac{\pi}{6} - \frac{\cos\frac{\pi}{6} - \frac{\pi}{6}}{-\sin\frac{\pi}{6} - 1} \approx 0.7519,$$
$$x_2 \approx 0.7391,$$
$$x_3 \approx 0.7390.$$

$x_2 \approx x_3 \approx 0.739$. Thus $x \approx 0.739$ is the solution.

7. Let $f(x) = e^{-x} - \ln x$. Then $f'(x) = -e^{-x} - \frac{1}{x}$. We want to find all zeros of f, because $f(x) = 0$ implies that $e^{-x} = \ln x$. Since e^{-x} is always decreasing and $\ln x$ is always increasing, there must be only 1 solution. Since $e^{-1} > \ln 1 = 0$, and $e^{-e} < \ln e = 1$, then $e^{-x} = \ln x$ for some x, $1 < x < e$. Try $x_0 = 1$. We now use Newton's method.

$$x_1 = 1 - \frac{e^{-1} - 0}{-e^{-1} - 1} \approx 1.2689,$$
$$x_2 \approx 1.309,$$
$$x_3 \approx 1.310.$$

Thus $x \approx 1.310$ is the solution.

8. Let $f(x) = e^x \cos x - 1$. Then $f'(x) = -e^x \sin x + e^x \cos x$. Now we use Newton's method, guessing $x_0 = 1$ initially.

$$x_1 = 1 - \frac{f(1)}{f'(1)} \approx 1.5725$$

Continuing: $x_2 \approx 1.364$, $x_3 \approx 1.299$, $x_4 \approx 1.293$, $x_5 \approx 1.293$. Thus $x \approx 1.293$ is a solution. Looking at a graph of $f(x)$ suffices to convince us that there is only one solution.

9. Let $f(x) = \ln x - \frac{1}{x}$, so $f'(x) = \frac{1}{x} + \frac{1}{x^2}$.
Now use Newton's method with an initial guess of $x_0 = 2$.

$$x_1 = 2 - \frac{\ln 2 - \frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} \approx 1.7425,$$
$$x_2 \approx 1.763,$$
$$x_3 \approx 1.763.$$

Thus $x \approx 1.763$ is a solution. Since $f'(x) > 0$ for positive x, f is increasing: it must be the only solution.

10. (a) One zero in the interval $0.6 < x < 0.7$.
(b) Three zeros in the intervals $-1.55 < x < -1.45$, $x = 0$, $1.45 < x < 1.55$.
(c) Two zeros in the intervals $0.1 < x < 0.2$, $3.5 < x < 3.6$.

11. $f'(x) = 3x^2 + 1$. Since f' is always positive, f is everywhere increasing. Thus f has only one zero. Since $f(0) < 0 < f(1)$, $0 < x_0 < 1$. Pick $x_0 = 0.68$.

$$x_0 = 0.68,$$
$$x_1 = 0.6823278,$$
$$x_2 \approx 0.6823278.$$

Thus $x \approx 0.682328$ (rounded up) is a root. Since $x_1 \approx x_2$, the digits should be correct.

12. Let $f(x) = x^2 - a$, so $f'(x) = 2x$.
Then by Newton's method, $x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$
For $a = 2:$
$x_0 = 1$, $x_1 = 1.5$, $x_2 \approx 1.416$, $x_3 \approx 1.414215$, $x_4 \approx 1.414213$ so $\sqrt{2} \approx 1.4142$.
For $a = 10:$
$x_0 = 5$, $x_1 = 3.5$, $x_2 \approx 3.17857$, $x_3 \approx 3.162319$, $x_4 \approx 3.162277$ so $\sqrt{10} \approx 3.1623$.
For $a = 1000:$
$x_0 = 500$, $x_1 = 251$, $x_2 \approx 127.49203$, $x_3 \approx 67.6678$, $x_4 \approx 41.2229$, $x_5 \approx 32.7406$, $x_6 \approx 31.6418$, $x_7 \approx 31.62278$, $x_8 \approx 31.62277$ so $\sqrt{1000} \approx 31.6228$.
For $a = \pi:$
$x_0 = \frac{\pi}{2}$, $x_1 \approx 1.7853$, $x_2 \approx 1.7725$ $x_3 \approx 1.77245$, $x_4 \approx 1.77245$ so $\sqrt{\pi} \approx 1.77245$.

Solutions for Section D

Exercises

1. The magnitude is $\|3\vec{i}\| = \sqrt{3^2 + 0^2} = 3$.
The angle of $3\vec{i}$ is 0 because the vector lies along the positive x-axis.

2. The magnitude is $\|2\vec{i} + \vec{j}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$.
Since $2\vec{i} + \vec{j}$ is in the first quadrant, the angle is $\arctan(1/2) = 0.464$ radians, or about 26.6 degrees.

3. The magnitude is $\| - \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}\,\| = \sqrt{(-\sqrt{2})^2 + \sqrt{2}^2} = 2$.
The direction of the vector $-\sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}$ is given by the angle $\theta = 3\pi/4$ as the vector bisects the second quadrant.

4. $\vec{v} + \vec{w} = (1-2)\vec{i} + (2+3)\vec{j} = (-1)\vec{i} + 5\vec{j} = -\vec{i} + 5\vec{j}$.

5. $2\vec{v} + \vec{w} = (2-2)\vec{i} + (4+3)\vec{j} = 7\vec{j}$.

6. $\vec{w} + (-2)\vec{u} = -2\vec{i} + (3-4)\vec{j} = (-2)\vec{i} + (-1)\vec{j} = -2\vec{i} - \vec{j}$.

7. Calculating magnitudes of each vector yields:

$$\|3\vec{i} + 4\vec{j}\| = \sqrt{3^2 + 4^2} = 5$$
$$\|\vec{i} + \vec{j}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\| - 5\vec{i}\| = \sqrt{(-5)^2 + 0^2} = 5$$
$$\|5\vec{j}\| = \sqrt{0^2 + 5^2} = 5$$
$$\|\sqrt{2}\vec{j}\| = \sqrt{\sqrt{2}^2} = \sqrt{2}$$
$$\|2\vec{i} + 2\vec{j}\| = \sqrt{2^2 + 2^2} = \sqrt{8}$$
$$\| - 6\vec{j}\| = \sqrt{0^2 + (-6)^2} = 6.$$

Thus $3\vec{i} + 4\vec{j}$, $-5\vec{i}$ and $5\vec{j}$ have the same magnitude.
Also $\vec{i} + \vec{j}$ and $\sqrt{2}\vec{j}$ have the same magnitude.

8. Two vectors are in the same direction if one is a positive scalar multiple of the other.

Since
$$2(\vec{i}+\vec{j})=2\vec{i}+2\vec{j}$$
the vectors $\vec{i}+\vec{j}$ and $2\vec{i}+2\vec{j}$ are in the same direction.

Also
$$5\vec{j}=\frac{5}{\sqrt{2}}(\sqrt{2}\vec{j})$$
so $5\vec{j}$ and $\sqrt{2}\vec{j}$ are in the same direction.

9. Two vectors have opposite direction if one is a negative scalar multiple of the other. Since
$$5\vec{j}=\frac{-5}{6}(-6\vec{j})$$
the vectors $5\vec{j}$ and $-6\vec{j}$ have opposite direction. Similarly, $-6\vec{j}$ and $\sqrt{2}\vec{j}$ have opposite direction.

10. We work out the left hand side $\|k\vec{r}\|$ and show that it is the same as the right hand side $|k|\|\vec{r}\|$. We have
$$\|k\vec{r}\|=\|ka\vec{i}+kb\vec{j}\|=\sqrt{(ka)^2+(kb)^2}=\sqrt{k^2(a^2+b^2)}=\sqrt{k^2}\sqrt{a^2+b^2}=|k|\|\vec{r}\|.$$

11. (a) The magnitude of $-3\vec{i}+4\vec{j}$ is $\sqrt{(-3)^2+4^2}=5$, so we want to scale down the magnitude by a factor of 5. Scalar multiplying by $1/5$ does not change the vector's direction, and gives the unit vector $(-3/5)\vec{i}+(4/5)\vec{j}$.

(b) Scalar multiplying by -1 reverses a vector's direction without changing its magnitude. So
$$(-1)\left(\frac{-3}{5}\vec{i}+\frac{4}{5}\vec{j}\right)=\frac{3}{5}\vec{i}-\frac{4}{5}\vec{j}$$
is a unit vector in the direction opposite to the original.

12. A vector making an angle of 90° with the positive x-axis lies along the positive y-axis, so it is of the form $\vec{v}=a\vec{j}$ with a positive. Its magnitude is $\|\vec{v}\|=\sqrt{0^2+a^2}=a$, so $a=5$. The vector is $5\vec{j}$.

13. Scalar multiplication by 2 doubles the magnitude of a vector without changing its direction. Thus, the vector is $2(4\vec{i}-3\vec{j})=8\vec{i}-6\vec{j}$.

14. Scalar multiplication by -1 reverses the direction of a vector without changing its magnitude. Thus, the vector is $(-1)(4\vec{i}-3\vec{j})=-4\vec{i}+3\vec{j}$.

15. The vector is $(4-3)\vec{i}+(4-2)\vec{j}=\vec{i}+2\vec{j}$.

16. In components, the vector from $(6,6)$ to $(-6,-6)$ is $((-6)-6)\vec{i}+((-6)-6)\vec{j}=-12\vec{i}-12\vec{j}$, which is not equal to $6\vec{i}-6\vec{j}$.

17. In components, the vector from $(7,7)$ to $(9,11)$ is $(9-7)\vec{i}+(11-9)\vec{j}=2\vec{i}+2\vec{j}$.

In components, the vector from $(8,10)$ to $(10,12)$ is $(10-8)\vec{i}+(12-10)\vec{j}=2\vec{i}+2\vec{j}$.

The two vectors are equal.

18. In components, the vector of length $\sqrt{2}$ making an angle of $\pi/4$ with the positive x-axis is $\vec{i}+\vec{j}$, which is not the same as $-\vec{i}+\vec{j}$.

19. In components, the vector from $(1,12)$ to $(6,10)$ is $(6-1)\vec{i}+(10-12)\vec{j}=5\vec{i}-2\vec{j}$. The two vectors are equal.

20. The velocity is $\vec{v}(t)=1\vec{i}+2t\vec{j}$. When $t=1$ the velocity vector is $\vec{v}=\vec{i}+2\vec{j}$.

The speed is $\|\vec{v}\|=\sqrt{1^2+2^2}=\sqrt{5}$.

The acceleration is $\vec{a}(t)=2\vec{j}$.

21. The velocity is $\vec{v}(t)=e^t\vec{i}+(1/(1+t))\vec{j}$. When $t=0$, the velocity vector is $\vec{v}=\vec{i}+\vec{j}$.

The speed is $\|\vec{v}\|=\sqrt{1^2+1^2}=\sqrt{2}$.

The acceleration is $\vec{a}(t)=e^t\vec{i}-1/((1+t)^2)\vec{j}$. When $t=0$, the acceleration vector is $\vec{a}=\vec{i}-\vec{j}$.

22. The velocity is $\vec{v}(t)=-5\sin t\vec{i}+5\cos t\vec{j}$. When $t=\pi/2$, the velocity vector is $\vec{v}=-5\vec{i}$.

The speed is $\|\vec{v}\|=\sqrt{(-5)^2+0^2}=5$.

The acceleration is $\vec{a}(t)=-5\cos(t)\vec{i}-5\sin(t)\vec{j}$. For $t=\pi/2$, the acceleration vector is $\vec{a}=-5\vec{j}$.

23. The position vector is
$$\vec{r}(\pi/4)=\cos(\pi/4)\vec{i}+\sin(\pi/4)\vec{j}=(1/\sqrt{2})\vec{i}+(1/\sqrt{2})\vec{j}.$$
The velocity vector is
$$\vec{v}(t)=\frac{d}{dt}\cos t\,\vec{i}+\frac{d}{dt}\sin t\,\vec{j}=-\sin t\,\vec{i}+\cos t\,\vec{j},$$

so

$$\vec{v}\,(\pi/4) = -\sin(\pi/4)\vec{i} + \cos(\pi/4)\vec{j} = (-1/\sqrt{2})\vec{i} + (1/\sqrt{2})\vec{j}\,.$$

The speed is

$$||\vec{v}\,|| = \sqrt{(-1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1.$$

We recognize $\vec{r}\,(t)$ as the parameterization of a unit circle, centered at the origin. Figure D.5 shows the curve together with the position and velocity vectors when $t = \pi/4$. We see that the velocity vector is tangent to the circle.

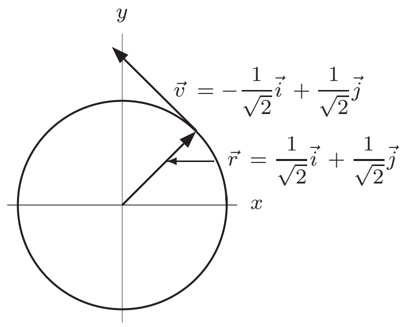


Figure D.5: Position and velocity vectors for motion along a circle