Solutions Manual

Second Edition

Discrete-Time Control Systems

Katsuhiko Ogata
University of Minnesota

PRENTICE HALL, UPPER SADDLE RIVER, NJ 07458

A Pearson Education Company
Upper Saddle River, New Jersey 07458

10 9 8 7 6 5

ISBN 0-13-317190-6

Printed in the United States of America

Table of Contents

Preface

This solutions manual for Discrete-Time Control Systems, second edition, contains solutions to all B problems (unsolved problems in the text).

All the materials in the text may be covered in two quarters. In a semester course, the instructor will have some flexibility in choosing the subjects to be covered. If the student has an adequate background in the vector-matrix analysis, then by leaving approximately 10 percent of the text material to the student's self study, most of the important subjects of the text may be covered in one semester. In a quarter course, a good part of the first six chapters may be covered.

Katsuhiko Ogata

CHAPTER 2

B-2-1.

$$X(z) = \mathcal{Z}\left[\frac{1}{a}(1 - e^{-at})\right] = \frac{1}{a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}\right)$$

$$= \frac{1}{a}\frac{z^{-1}(1 - e^{-aT})}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$$

B-2-2.

Method 1: Noting that

$$\mathcal{Z}[k] = \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad \mathcal{Z}[k^2] = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

it can be expected that $\mathcal{Z}[k^3]$ will involve a term $(1 - z^{-1})^4$ in the denominator. Since

$$\mathcal{Z}[k^3] = \sum_{k=0}^{\infty} k^3 z^{-k} = z^{-1} + 2^3 z^{-2} + 3^3 z^{-3} + 4^3 z^{-4} + \cdots$$

$$= z^{-1} + 8z^{-2} + 27z^{-3} + 64z^{-4} + \cdots$$

and

$$(z^{-1} + 8z^{-2} + 27z^{-3} + 64z^{-4} + \cdots)(1 - z^{-1})^4$$

$$= (z^{-1} + 7z^{-2} + 19z^{-3} + 37z^{-4} + \cdots)(1 - z^{-1})^3$$

$$= (z^{-1} + 6z^{-2} + 12z^{-3} + 18z^{-4} + \cdots)(1 - z^{-1})^2$$

$$= (z^{-1} + 5z^{-2} + 6z^{-3} + 6z^{-4} + \cdots)(1 - z^{-1})$$

$$= z^{-1} + 4z^{-2} + z^{-3}$$

we find

$$\mathcal{Z}[k^3] = \frac{z^{-1} + 4z^{-2} + z^{-3}}{(1 - z^{-1})^4} = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$$

Method 2:

$$\mathcal{Z}[k^3] = \mathcal{Z}[k \cdot k^2] = -z\frac{dX(z)}{dz}$$

where

$$X(z) = \mathcal{Z}[k^2] = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

Since

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\right] = -\frac{z^{-2} + 4z^{-3} + z^{-4}}{(1 - z^{-1})^4}$$

we have

$$\mathcal{Z}\left[k^3\right] = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$$

B-2-3.

Method 1: Noting that

$$\mathcal{Z}\left[te^{-at}\right] = \frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$$

we have

$$\mathcal{Z}\left[t^2e^{-at}\right] = \mathcal{Z}\left[-\frac{\partial}{\partial a}\, te^{-at}\right] = \frac{\partial}{\partial a}\left[\frac{-Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}\right]$$

$$= \frac{T^2e^{-aT}(1 + e^{-aT}z^{-1})z^{-1}}{(1 - e^{-aT}z^{-1})^3}$$

Method 2:

$$\mathcal{Z}\left[t^2e^{-at}\right] = \sum_{k=0}^{\infty} (kT)^2e^{-akT}z^{-k}$$

$$= T^2\left[(e^{aT}z)^{-1} + 4(e^{aT}z)^{-2} + 9(e^{aT}z)^{-3} + 16(e^{aT}z)^{-4} + \cdots\right]$$

Referring to Problem A-2-2, we have

$$\mathcal{Z}\left[t^2e^{-at}\right] = \frac{T^2e^{-aT}z^{-1}(1 + e^{-aT}z^{-1})}{(1 - e^{-aT}z^{-1})^3}$$

B-2-4.

$$\mathcal{Z}\left[x(k)\right] = \mathcal{Z}\left[9k(2^{k-1})\right] - \mathcal{Z}\left[2^k\right] + \mathcal{Z}\left[3\right]$$

$$= \frac{9z^{-1}}{(1 - 2z^{-1})^2} - \frac{1}{1 - 2z^{-1}} + \frac{3}{1 - z^{-1}}$$

$$= \frac{2 + z^{-2}}{(1 - 2z^{-1})^2(1 - z^{-1})}$$

B-2-5. Referring to Problem A-2-4, we have

$$\mathcal{Z}\left[\sum_{h=0}^{k} a^h\right] = \frac{1}{1 - z^{-1}}\, X(z)$$

where

$$X(z) = \mathcal{Z}\left[a^h\right] = \frac{1}{1 - az^{-1}}$$

Hence

$$\mathcal{Z}\left[\sum_{h=0}^{k} a^h\right] = \frac{1}{1 - z^{-1}} \cdot \frac{1}{1 - az^{-1}}$$

B-2-6.

$$\mathcal{Z}\left[ka^{k-1}\right] = \mathcal{Z}\left[\frac{\partial}{\partial a} a^k\right] = \frac{\partial}{\partial a}\mathcal{Z}\left[a^k\right] = \frac{\partial}{\partial a}\left(\frac{1}{1 - az^{-1}}\right)$$

$$= \frac{z^{-1}}{(1 - az^{-1})^2} = \frac{z}{(z - a)^2}$$

$$\mathcal{Z}\left[k(k - 1)a^{k-2}\right] = \mathcal{Z}\left[\frac{\partial}{\partial a}(ka^{k-1})\right] = \frac{\partial}{\partial a}\left[\frac{z}{(z - a)^2}\right]$$

$$= \frac{2z}{(z - a)^3} = \frac{(2!)z}{(z - a)^3}$$

$$\mathcal{Z}\left[k(k - 1)\cdots(k - h + 1)a^{k-h}\right]$$

$$= \mathcal{Z}\left[\frac{\partial}{\partial a} k(k - 1)\cdots(k - h + 2)a^{k-h+1}\right]$$

$$= \frac{\partial}{\partial a} X(z, a)$$

where

$$X(z, a) = \frac{(h - 1)!\, z}{(z - a)^h}$$

Hence

$$\mathcal{Z}\left[k(k - 1)\cdots(k - h + 1)a^{k-h}\right] = \frac{\partial}{\partial a} X(z, a)$$

$$= (h - 1)!\, zh(z - a)^{-h-1} = \frac{h!\, z}{(z - a)^{h+1}}$$

B-2-7. From Figure 2-8 we have

$$x(0) = 0, \quad x(1) = 0, \quad x(2) = 0, \quad x(3) = \frac{1}{3}$$

$$x(4) = \frac{2}{3}, \quad x(k) = 1 \quad \text{for } k = 5, 6, 7, \ldots$$

Then

$$X(z) = \sum_{k=0}^{\infty} x(k)\, z^{-k}$$

$$= \frac{1}{3} z^{-3} + \frac{2}{3} z^{-4} + z^{-5} + z^{-6} + z^{-7} + \cdots$$

$$= \frac{1}{3} (z^{-3} + 2z^{-4}) + \frac{z^{-5}}{1 - z^{-1}}$$

$$= \frac{1}{3} \frac{z^{-3} + z^{-4} + z^{-5}}{1 - z^{-1}}$$

B-2-8. By dividing both numerator and denominator by z^4, we have

$$X(z) = 5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

This last equation is already in the form of a power series in z^{-1}. By inspection, we have

$$x(0) = 5$$
$$x(1) = 4$$
$$x(2) = 3$$
$$x(3) = 2$$
$$x(4) = 1$$
$$x(k) = 0 \qquad k \geqslant 5$$

Note that the given X(z) is the z transform of a signal of finite length.

B-2-9.

1. Partial-fraction-expansion method:

$$X(z) = \frac{z^{-1}(0.5 - z^{-1})}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})^2} = \frac{z(0.5z - 1)}{(z - 0.5)(z - 0.8)^2}$$

Hence,

$$\frac{X(z)}{z} = -\frac{8.3333}{z - 0.5} + \frac{8.3333}{z - 0.8} - \frac{2}{(z - 0.8)^2}$$

or

$$X(z) = -\frac{8.3333}{1 - 0.5z^{-1}} + \frac{8.3333}{1 - 0.8z^{-1}} - \frac{2z^{-1}}{(1 - 0.8z^{-1})^2}$$

Thus,

$$x(k) = -8.3333(0.5)^k + 8.3333(0.8)^k - 2k(0.8)^{k-1}, \qquad k = 0, 1, 2, \ldots$$

2. Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-9
»
»% ----- Finding inverse z transform -----
»
»num = [0   0.5   -1   0];
»den = [1   -2.1   1.44   -0.32];
»u = [1   zeros(1,40)];
»x = filter(num,den,u)

x =

 Columns 1 through 7

      0    0.5000    0.0500   -0.6150   -1.2035   -1.6257   -1.8778

 Columns 8 through 14

  -1.9875   -1.9899   -1.9177   -1.7977   -1.6505   -1.4910   -1.3296

 Columns 15 through 21

  -1.1733   -1.0265   -0.8915   -0.7694   -0.6606   -0.5645   -0.4804

 Columns 22 through 28

  -0.4074   -0.3443   -0.2902   -0.2440   -0.2046   -0.1713   -0.1431

 Columns 29 through 35

  -0.1193   -0.0993   -0.0825   -0.0685   -0.0568   -0.0470   -0.0389

 Columns 36 through 41

  -0.0321   -0.0265   -0.0219   -0.0180   -0.0148   -0.0122
```

B-2-10.

$$x(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{z^{-1}}{(1 - z^{-1})(1 + 1.3z^{-1} + 0.4z^{-2})} = 0$$

$$x(\infty) = \lim_{z \to 1} \left[(1 - z^{-1})X(z)\right] = \lim_{z \to 1} \frac{z^{-1}}{1 + 1.3z^{-1} + 0.4z^{-2}} = \frac{1}{2.7}$$

Notice that

$$X(z) = \frac{z^{-1}}{(1 - z^{-1})(1 + 1.3z^{-1} + 0.4z^{-2})}$$

$$= \frac{z^2}{(z - 1)(z + 0.8)(z + 0.5)}$$

$$= \frac{1}{2.7}\left(\frac{z}{z - 1} - \frac{4z}{z + 0.8} + \frac{3z}{z + 0.5}\right)$$

Hence

$$x(k) = \frac{1}{2.7}\left[1 - 4(-0.8)^k + 3(-0.5)^k\right]$$

B-2-11.

1. Inversion integral method:

$$X(z) = \frac{1 + z^{-1} - z^{-2}}{1 - z^{-1}} = \frac{z^2 + z - 1}{(z - 1)z}$$

Hence

$$X(z)z^{k-1} = \frac{(z^2 + z - 1)z^{k-1}}{(z - 1)z}$$

For k = 0:

$$X(z)z^{k-1} = \frac{z^2 + z - 1}{(z - 1)z^2}$$

Thus,

$$x(0) = \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z^2} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z^2} \text{ at double pole } z = 0\right]$$

$$= \lim_{z \to 1} \left[(z - 1)\frac{z^2 + z - 1}{(z - 1)z^2}\right]$$

$$+ \frac{1}{(2 - 1)!} \lim_{z \to 0} \frac{d}{dz}\left[z^2 \frac{z^2 + z - 1}{(z - 1)z^2}\right] = 1 + 0 = 1$$

For k = 1:

$$X(z)z^{k-1} = \frac{z^2 + z - 1}{(z - 1)z}$$

$$x(1) = \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z} \text{ at pole } z = 0\right]$$

$$= \lim_{z \to 1} \left[(z - 1) \frac{z^2 + z - 1}{(z - 1)z}\right] + \lim_{z \to 0} \left[z \frac{z^2 + z - 1}{(z - 1)z}\right]$$

$$= 1 + 1 = 2$$

For k = 2, 3, 4, ...:

$$X(z)z^{k-1} = \frac{(z^2 + z - 1)z^{k-2}}{z - 1}$$

Hence

$$x(k) = \left[\text{residue of } \frac{(z^2 + z - 1)z^{k-2}}{z - 1} \text{ at pole } z = 1\right]$$

$$= \lim_{z \to 1} \left[(z - 1) \frac{(z^2 + z - 1)z^{k-2}}{z - 1}\right] = 1$$

Therefore,

$$x(0) = 1$$

$$x(1) = 2$$

$$x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots$$

2. Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-11
»
»% ----- Finding inverse z transform -----
»
»num = [1  1  -1];
»den = [1  -1  0];
»u = [1  zeros(1,40)];
»x = filter(num,den,u)
```

```
x =

 Columns 1 through 12

   1   2   1   1   1   1   1   1   1   1   1   1

 Columns 13 through 24

   1   1   1   1   1   1   1   1   1   1   1   1

 Columns 25 through 36

   1   1   1   1   1   1   1   1   1   1   1   1

 Columns 37 through 41

   1   1   1   1   1
```

B-2-12.

$$X(z) = \frac{z^{-3}}{(1 - z^{-1})(1 - 0.2z^{-1})} = \frac{1}{z(z - 1)(z - 0.2)}$$

$$= \frac{5}{z} + \frac{1.25}{z - 1} - \frac{6.25}{z - 0.2}$$

Hence

$$x(k) = 5\,\delta_0(k - 1) + 1.25 - 6.25(0.2)^{k-1} \qquad \text{for } k = 1, 2, 3, \ldots$$

That is,

$$x(k) = 0 \qquad \text{for } k = 0, 1, 2$$

$$= 1.25(1 - 0.2^{k-2}) \qquad \text{for } k = 3, 4, 5, \ldots$$

B-2-13.

$$X(z) = \frac{1 + 6z^{-2} + z^{-3}}{(1 - z^{-1})(1 - 0.2z^{-1})} = \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)}$$

$$X(z)z^{k-1} = \frac{(z^3 + 6z + 1)z^{k-1}}{z(z - 1)(z - 0.2)}$$

For k = 0:

$$X(z)z^{k-1} = \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)}$$

Hence

$$x(0) = \left[\text{residue of } \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)} \text{ at double pole } z = 0\right]$$

$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)} \text{ at pole } z = 0.2\right]$$

$$= \frac{1}{(2 - 1)!} \lim_{z \to 0} \frac{d}{dz}\left[\frac{z^3 + 6z + 1}{(z - 1)(z - 0.2)}\right]$$

$$+ \lim_{z \to 1}\left[\frac{z^3 + 6z + 1}{z^2(z - 0.2)}\right] + \lim_{z \to 0.2}\left[\frac{z^3 + 6z + 1}{z^2(z - 1)}\right]$$

$$= 60 + 10 - 69 = 1$$

For k = 1:

$$X(z)z^{k-1} = \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)}$$

Hence

$$x(1) = \left[\text{residue of } \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)} \text{ at pole } z = 0\right]$$

$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)} \text{ at pole } z = 0.2\right]$$

$$= \lim_{z \to 0}\left[\frac{z^3 + 6z + 1}{(z - 1)(z - 0.2)}\right] + \lim_{z \to 1}\left[\frac{z^3 + 6z + 1}{z(z - 0.2)}\right]$$

$$+ \lim_{z \to 0.2}\left[\frac{z^3 + 6z + 1}{z(z - 1)}\right] = 5 + 10 - 13.8 = 1.2$$

For k = 2, 3, 4, ...:

$$X(z)z^{k-1} = \frac{(z^3 + 6z + 1)z^{k-2}}{(z - 1)(z - 0.2)}$$

$$x(k) = \left[\text{residue of } \frac{(z^3 + 6z + 1)z^{k-2}}{(z - 1)(z - 0.2)} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{(z^3 + 6z + 1)z^{k-2}}{(z - 1)(z - 0.2)} \text{ at pole } z = 0.2\right]$$

$$= \lim_{z \to 1} \left[\frac{(z^3 + 6z + 1)z^{k-2}}{z - 0.2} \right] + \lim_{z \to 0.2} \left[\frac{(z^3 + 6z + 1)z^{k-2}}{z - 1} \right]$$

$$= 10 - 2.76(0.2)^{k-2}$$

In summarizing, we have

$$x(0) = 1$$

$$x(1) = 1.2$$

$$x(k) = 10 - 2.76(0.2)^{k-2} \qquad \text{for } k = 2, 3, 4, \ldots$$

B-2-14.

1. Direct division method:

$$X(z) = \frac{z^{-1} - z^{-3}}{1 + 2z^{-2} + z^{-4}} = z^{-1} - 3z^{-3} + 5z^{-5} - 7z^{-7} + \cdots$$

Hence

$$x(0) = 0, \quad x(1) = 1, \quad x(2) = 0, \quad x(3) = -3$$

$$x(4) = 0, \quad x(5) = 5, \quad x(6) = 0, \quad x(7) = -7, \ldots$$

2. Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-14
»
»% ----- Finding inverse z transform -----
»
»num = [0   1   0   -1   0];
»den = [1   0   2   0   1];
»u = [1   zeros(1,40)];
»x = filter(num,den,u)
```

```
x =

 Columns 1 through 12

   0   1   0  -3   0   5   0  -7   0   9   0  -11

 Columns 13 through 24

   0   13   0  -15   0   17   0  -19   0   21   0  -23
```

Columns 25 through 36

0	25	0	-27	0	29	0	-31	0	33	0	-35

Columns 37 through 41

0	37	0	-39	0

B-2-15.

$$X(z) = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^2}$$

$$X(z)z^{k-1} = \frac{(0.368z^2 + 0.478z + 0.154)z^{k-1}}{(z-1)z^2}$$

For k = 0:

$$X(z)z^{k-1} = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^3}$$

$$x(0) = \left[\text{residue of } \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^3} \text{ at triple pole } z = 0\right]$$

$$+ \left[\text{residue of } \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^3} \text{ at pole } z = 1\right]$$

$$= \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^2}{dz^2}\left[\frac{0.368z^2 + 0.478z + 0.154}{z-1}\right]$$

$$+ \lim_{z \to 1}\left[\frac{0.368z^2 + 0.478z + 0.154}{z^3}\right] = -1 + 1 = 0$$

For k = 1:

$$X(z)z^{k-1} = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^2}$$

$$x(1) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left[\frac{0.368z^2 + 0.478z + 0.154}{z-1}\right]$$

$$+ \lim_{z \to 1}\left[\frac{0.368z^2 + 0.478z + 0.154}{z^2}\right] = -0.632 + 1 = 0.368$$

For k = 2:

$$X(z)z^{k-1} = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z}$$

$$x(2) = \lim_{z \to 0} \left[\frac{0.368z^2 + 0.478z + 0.154}{z - 1} \right] + \lim_{z \to 1} \left[\frac{0.368z^2 + 0.478z + 0.154}{z} \right] = -0.154 + 1 = 0.846$$

For $k = 3, 4, 5, \ldots$:

$$X(z)z^{k-1} = \frac{(0.368z^2 + 0.478z + 0.154)z^{k-3}}{z - 1}$$

$$x(k) = \lim_{z \to 1} \left[(0.368z^2 + 0.478z + 0.154)z^{k-3} \right] = 1$$

In summary, we have

$$x(0) = 0$$

$$x(1) = 0.368$$

$$x(2) = 0.846$$

$$x(k) = 1 \qquad \text{for } k = 3, 4, 5, \ldots$$

B-2-16.

Case 1:

$$u(k) = 1 \qquad \text{for } k = 0, 1, 2, \ldots$$
$$= 0 \qquad \text{for } k < 0$$

The z transform of the given difference equation is

$$z^2X(z) - 1.3zX(z) + 0.4X(z) = U(z) = \frac{z}{z - 1}$$

or

$$X(z) = \frac{z}{(z - 0.8)(z - 0.5)(z - 1)} = -\frac{16.6666z}{z - 0.8} + \frac{6.6666z}{z - 0.5} + \frac{10z}{z - 1}$$

Hence

$$x(k) = -16.6666(0.8)^k + 6.6666(0.5)^k + 10 \qquad \text{for } k = 0, 1, 2, \ldots$$

Case 2:

$$u(0) = 1$$

$$u(k) = 0 \qquad \text{for } k \neq 0$$

The z transform of the given difference equation is

$$(z^2 - 1.3z + 0.4)X(z) = U(z) = 1$$

or

$$X(z) = \frac{1}{(z - 0.8)(z - 0.5)} = \frac{3.3333}{z - 0.8} - \frac{3.3333}{z - 0.5}$$

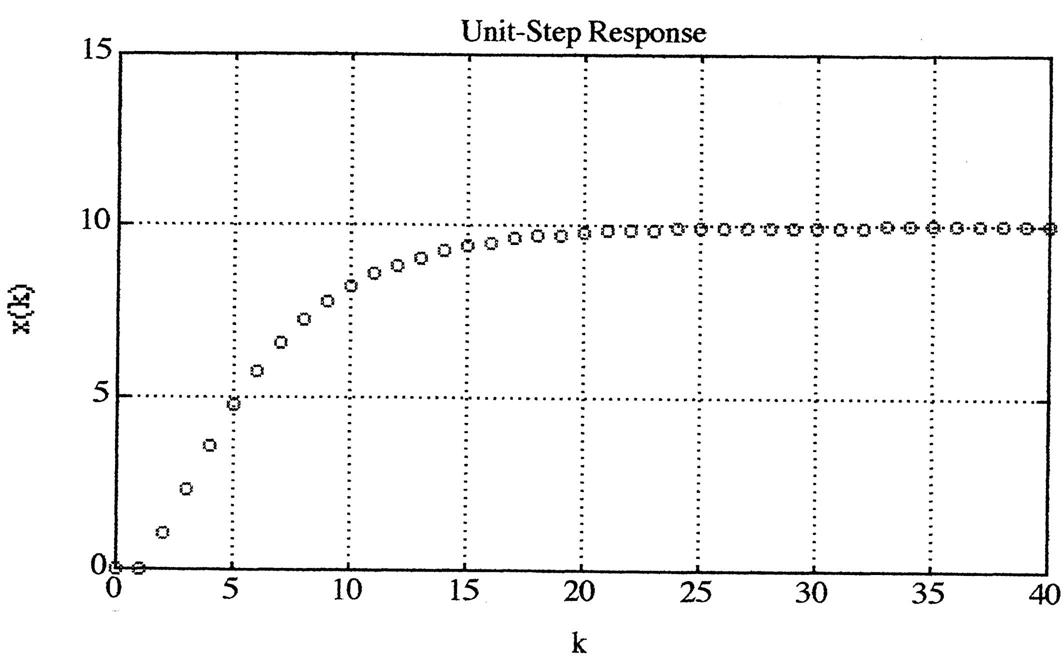
Hence,

$$x(0) = 0$$

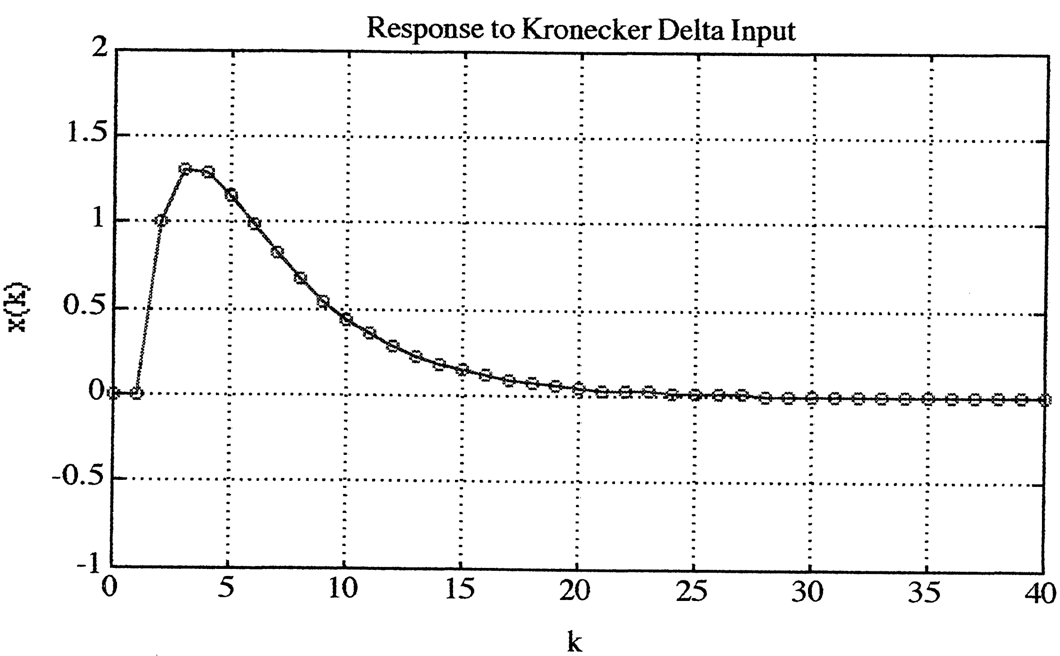
$$x(k) = 3.3333(0.8)^{k-1} - 3.3333(0.5)^{k-1}, \quad \text{for } k = 1, 2, 3, \ldots$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-16 (Part 1)
»
»% ----- Unit-step response -----
»
»num = [0  0  1];
»den = [1  -1.3  0.4];
»u = ones(1,41);
»v = [0  40  0  15];
»axis(v);
»k = 0:40;
»x = filter(num,den,u);
»plot(k,x,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('x(k)')
```



```
»% MATLAB Program for Problem B-2-16 (Part 2)
»
»% ----- Response to Kronecker delta input -----
»
»num =[0  0  1];
»den = [1  -1.3  0.4];
»u = [1  zeros(1,40)];
»v = [0  40  -1  2];
»axis(v);
»k = 0:40;
»x = filter(num,den,u);
»plot(k,x,'o',k,x,'-')
»grid
»title('Response to Kronecker Delta Input')
»xlabel('k')
»ylabel('x(k)')
»
```



B-2-17.

$$x(k + 2) - x(k + 1) + 0.25x(k) = u(k + 2)$$

The z transform of this difference equation is

$$\left[z^2X(z) - z^2x(0) - zx(1)\right] - \left[zX(z) - zx(0)\right] + 0.25X(z)$$
$$= z^2U(z) - z^2u(0) - zu(1)$$

Substituting the initial data into this last equation, we get

$$(z^2 - z + 0.25)X(z) = \frac{z^3}{z - 1}$$

or

$$X(z) = \frac{z^3}{(z - 1)(z^2 - z + 0.25)}$$

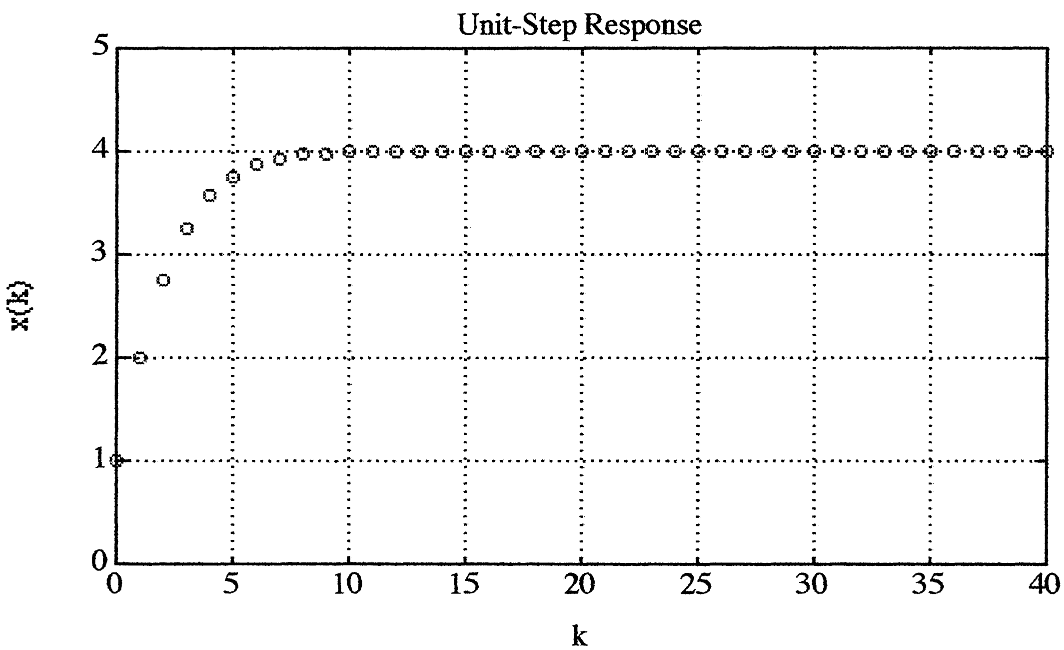
$$= \frac{4z}{z - 1} - \frac{3z}{z - 0.5} - \frac{0.5z}{(z - 0.5)^2}$$

Hence

$$x(k) = 4 - (3 + k)(0.5)^k \qquad \text{for } k = 0, 1, 2, \ldots$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-17
»
»% ----- Unit-step repone -----
»
»num = [1  0  0];
»den = [1  -1  0.25];
»u = ones(1,41);
»v = [0  40  0  5];
»axis(v);
»k = 0:40;
»x = filter(num,den,u);
»plot(k,x,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('x(k)')
```



B-2-18.

$$x(k + 2) - 1.3679x(k + 1) + 0.3679x(k) = 0.3679u(k + 1) + 0.2642u(k)$$

The z transform of this equation is

$$z^2X(z) - z^2x(0) - zx(1) - 1.3679 \left[zX(z) - zx(0)\right] + 0.3679X(z)$$
$$= 0.3679 \left[zU(z) - zu(0)\right] + 0.2642\ U(z)$$

Noting that $x(0) = 0$ and $x(1) = 0.5820$, we have

$$(z^2 - 1.3679z + 0.3679)X(z) = (0.3679z + 0.2642)U(z)$$

or

$$\frac{X(z)}{U(z)} = \frac{0.3679z^{-1} + 0.2642z^{-2}}{1 - 1.3679z^{-1} + 0.3679z^{-2}}$$

Since

$$U(z) = 1.5820 - 0.5820z^{-1}$$

we have

$$X(z) = \frac{0.5820z^{-1} + 0.2038z^{-2} - 0.1538z^{-3}}{1 - 1.3679z^{-1} + 0.3679z^{-2}}$$

$$= 0.5820z^{-1} + z^{-2} + z^{-3} + \cdots$$

Hence

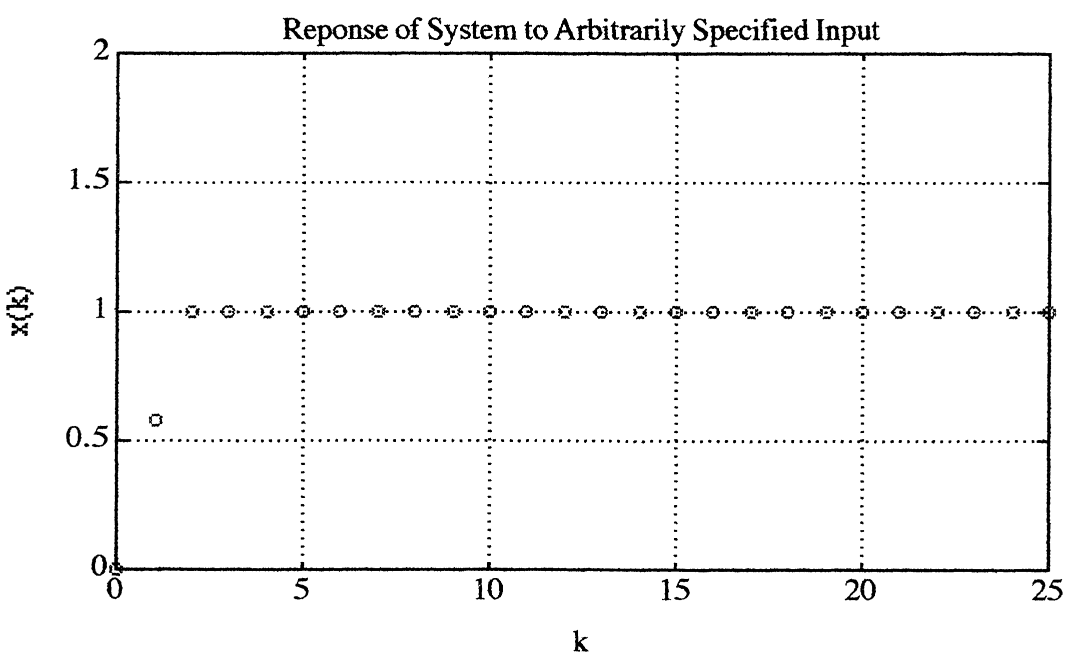
$$x(0) = 0$$

$$x(1) = 0.5820$$

$$x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-18
»
»% ----- Response to arbitrary inpt -----
»
»num = [0   0.3679   0.2642];
»den = [1   -1.3679   0.3679];
»u = [1.582   -0.5820   zeros(1,24)];
»v = [0   25   0   2];
»axis(v);
»k = 0;25;
»x = filter(num,den,u);
»plot(k,x,'o')
»grid
»title('Reponse of System to Arbitrarily Specified Input')
»xlabel('k')
»ylabel('x(k)')
```



CHAPTER 3

B-3-1. When the switch is closed, the input voltage is transmitted to the output. Assume that the switch stays closed for a period $kT \leqslant t < kT + \Delta$, where Δ is very small compared to the sampling period T. The transfer function for this closed period is

$$\frac{M(s)}{E(s)} = \frac{R_0}{R_0(R_iCs + 1) + R_i} \doteqdot \frac{1}{R_iCs + 1} \qquad (R_i \ll R_0)$$

The charge up time constant R_iC must be very small so that the capacitor will be charged to voltage e(kT) almost instantaneously.

When the switch is open, the hold capacitor C will start discharging through the resistor R_0. The output voltage will decay exponentially, but if R_0C is large compared to the sampling period T, the circuit will appear to hold the voltage e(kT) for the time period $kT \leqslant t < (k + 1)T$. Thus,

$$m(t) \doteqdot e(kT) \qquad kT \leqslant t < (k + 1)T$$

Hence the circuit acts as a zero-order hold.

B-3-2. The differential equation for the circuit is

$$RC\dot{x} + x = e$$

For $kT \leqslant t < (k + 1)T$, $e(t) = e(kT) =$ constant. Thus, we have

$$RC\dot{x} + x = e(kT) \qquad kT \leqslant t < (k + 1)T$$

Taking the Laplace transform of this last equation and considering $t = kT$ to be the initial time, we obtain

$$RC\left[sX(s) - x(kT)\right] + X(s) = \frac{e(kT)}{s}$$

or

$$X(s) = \frac{1}{RCs + 1}\left[\frac{e(kT)}{s} + RCx(kT)\right] = \frac{e(kT)}{s} + \frac{RC\left[x(kT) - e(kT)\right]}{RCs + 1}$$

The inverse Laplace transform of this last equation is

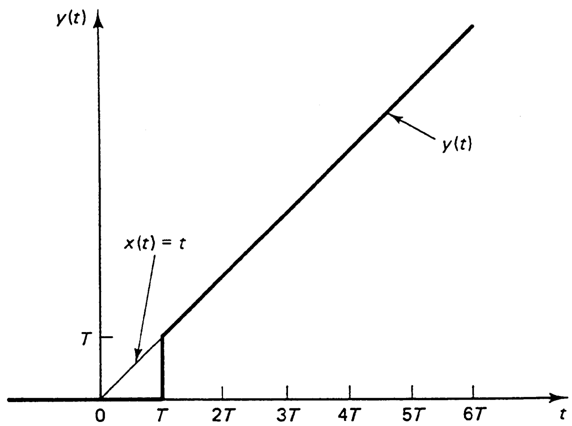
$$x(t) = e(kT) + \left[x(kT) - e(kT)\right] e^{-\frac{1}{RC}(t - kT)} \qquad kT \leqslant t < (k + 1)T$$

Substituting $t = (k + 1)T - 0$ into this last equation, we obtain the desired difference equation as follows:

$$x((k + 1)T-) = x(kT)e^{-\frac{T}{RC}} + (1 - e^{-\frac{T}{RC}})e(kT)$$

B-3-3. For the unit-ramp input $x(t)$, the output $y(t)$ of the first-order hold can be sketched as shown in the figure below. The equation for the curve $y(t)$ is

$$y(t) = (t - T)1(t - T) + T1(t - T)$$



The Laplace transform of this last equation gives

$$Y(s) = \frac{1}{s^2} e^{-Ts} + \frac{1}{s} Te^{-Ts} = e^{-Ts} \frac{1 + Ts}{s^2}$$

The Laplace transform of the pulsed unit-ramp function

$$x^*(t) = \sum_{k=0}^{\infty} kT\delta(t - kT)$$

is

$$X^*(s) = \sum_{k=0}^{\infty} kTe^{-kTs} = Te^{-Ts} + 2Te^{-2Ts} + 3Te^{-3Ts} + \cdots$$

$$= \frac{Te^{-Ts}}{(1 - e^{-Ts})^2}$$

Hence, the transfer function of the first-order hold is

$$G_{h1}(s) = \frac{Y(s)}{X^*(s)} = \frac{e^{-Ts}(Ts + 1)/s^2}{Te^{-Ts}/(1 - e^{-Ts})^2} = \frac{Ts + 1}{T}\left(\frac{1 - e^{-Ts}}{s}\right)^2$$

B-3-4.

1. Residue method:

$$X(z) = \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -1\right]$$

$$+ \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -2\right]$$

$$= \lim_{s \to -1} \left[(s + 1) \frac{s + 3}{(s + 1)(s + 2)} \frac{z}{z - e^{Ts}} \right]$$

$$+ \lim_{s \to -2} \left[(s + 2) \frac{s + 3}{(s + 1)(s + 2)} \frac{z}{z - e^{Ts}} \right]$$

$$= \frac{2z}{z - e^{-T}} - \frac{z}{z - e^{-2T}} = \frac{2}{1 - e^{-T}z^{-1}} - \frac{1}{1 - e^{-2T}z^{-1}}$$

$$= \frac{1 + e^{-T}(1 - 2e^{-T})z^{-1}}{(1 - e^{-T}z^{-1})(1 - e^{-2T}z^{-1})}$$

2. <u>Method based on impulse response function</u>:

$$X(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{2}{s + 1} - \frac{1}{s + 2}$$

The inverse Laplace transform of this equation gives

$$x(t) = 2e^{-t} - e^{-2t}$$

Hence

$$X(z) = \frac{2}{1 - e^{-T}z^{-1}} - \frac{1}{1 - e^{-2T}z^{-1}}$$

$$= \frac{1 + e^{-T}(1 - 2e^{-T})z^{-1}}{(1 - e^{-T}z^{-1})(1 - e^{-2T}z^{-1})}$$

<u>B-3-5</u>.

$$X(s) = \frac{K}{(s + a)(s + b)} = \frac{K}{b - a} \left(\frac{1}{s + a} - \frac{1}{s + b} \right)$$

1. <u>Residue method</u>:

$$X(z) = \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -a \right]$$

$$+ \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -b \right]$$

$$= \lim_{s \to -a} \left[(s + a) \frac{K}{(s + a)(s + b)} \frac{z}{z - e^{Ts}} \right]$$

$$+ \lim_{s \to -b} \left[(s + b) \frac{K}{(s + a)(s + b)} \frac{z}{z - e^{Ts}} \right]$$

$$= \frac{K}{b - a} \frac{z}{z - e^{-aT}} + \frac{K}{a - b} \frac{z}{z - e^{-bT}}$$

$$= \frac{K}{b - a} \left(\frac{1}{1 - e^{-aT}z^{-1}} - \frac{1}{1 - e^{-bT}z^{-1}} \right)$$

2. <u>Method based on impulse response function</u>:

$$x(t) = \frac{K}{b - a} (e^{-at} - e^{-bt})$$

Hence

$$X(z) = \frac{K}{b - a} \left(\frac{1}{1 - e^{-aT}z^{-1}} - \frac{1}{1 - e^{-bT}z^{-1}} \right)$$

<u>B-3-6</u>.

$$X(s) = \frac{1 - e^{-Ts}}{s} \frac{1}{(s + a)^2}$$

$$X(z) = (1 - z^{-1}) \mathcal{Z}\left[\frac{1}{s(s + a)^2}\right]$$

$$= (1 - z^{-1}) \mathcal{Z}\left[\frac{1}{a^2} \frac{1}{s} - \frac{1}{a^2} \frac{1}{s + a} - \frac{1}{a} \frac{1}{(s + a)^2}\right]$$

$$= (1 - z^{-1}) \left[\frac{1}{a^2} \frac{1}{1 - z^{-1}} - \frac{1}{a^2} \frac{1}{1 - e^{-aT}z^{-1}} - \frac{1}{a} \frac{Ta^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}\right]$$

$$= \frac{1}{a^2} \frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} - \frac{1}{a} \frac{(1 - z^{-1})Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$$

<u>B-3-7</u>.

$$y(k + 1) + 0.5y(k) = x(k), \qquad y(0) = 0$$

The z transform of this equation is

$$zY(z) - zy(0) + 0.5Y(z) = X(z) = \frac{z}{z - 1}$$

Solving this equation for $Y(z)$, noting that $y(0) = 0$, we obtain

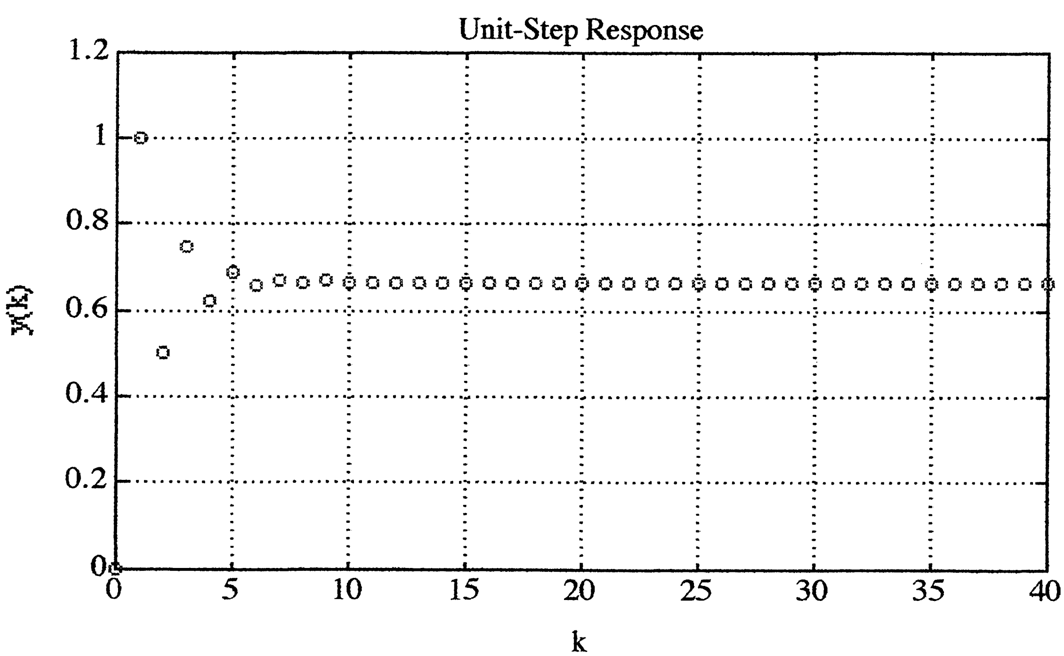
$$Y(z) = \frac{z}{(z + 0.5)(z - 1)} = -\frac{1}{1.5} \frac{z}{z + 0.5} + \frac{1}{1.5} \frac{z}{z - 1}$$

Hence

$$y(k) = -\frac{1}{1.5} (-0.5)^k + \frac{1}{1.5} = \frac{2}{3}\left[1 - (-0.5)^k\right]$$

<u>Computational solution with MATLAB</u>;

```
»% MATLAB Program for Problem B-3-7
»
»% ----- Unit-step response -----
»
»num = [0   1];
»den = [1   0.5];
»x = ones(1,41);
»v = [0   40   0   1.2];
»axis(v);
»k = 0:40;
»y = filter(num,den,x);
»plot(k,y,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('y(k)')
```



B-3-8.

$$y(k + 2) + y(k) = x(k), \qquad y(k) = 0 \text{ for } k < 0$$

Since x(k) is a unit-step sequence, we have

$$X(z) = \frac{z}{z - 1}$$

The initial data are

$$y(0) = x(-2) - y(-2) = 0, \qquad y(1) = x(-1) - y(-1) = 0$$

Hence

$$z^2 Y(z) + Y(z) = X(z) = \frac{z}{z - 1}$$

or

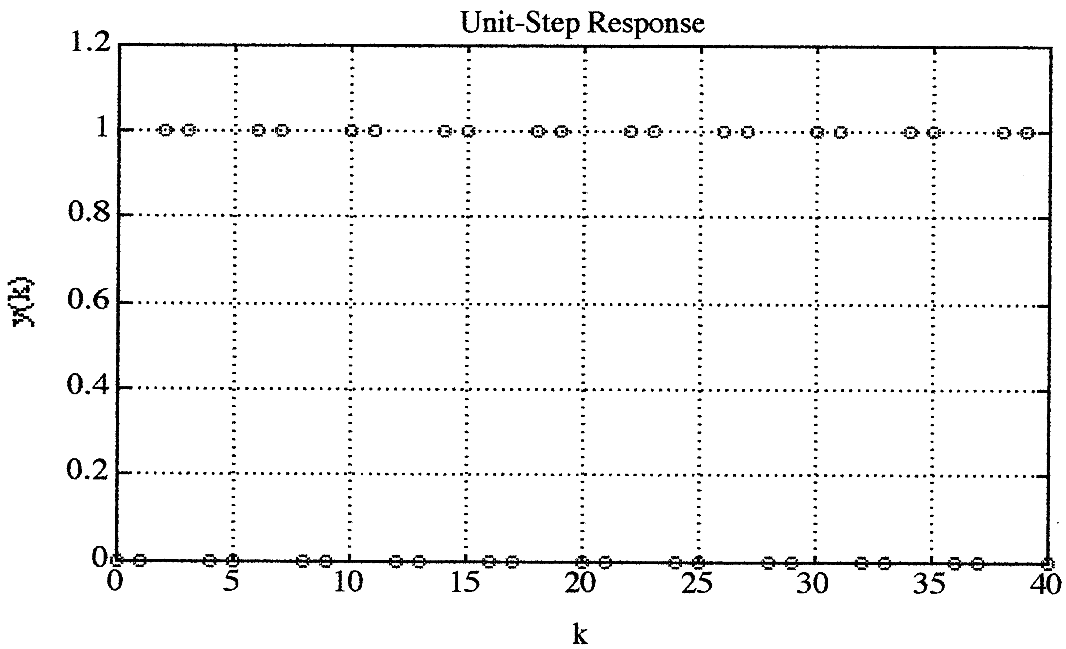
$$Y(z) = \frac{1}{z^2 + 1} \frac{z}{z - 1} = -\frac{1}{2}\left(\frac{1 + z^{-1}}{1 + z^{-2}} - \frac{1}{1 - z^{-1}}\right)$$

from which we obtain

$$y(k) = -\frac{1}{2}\left(\cos\frac{k\pi}{2} + \sin\frac{k\pi}{2} - 1\right) \qquad k = 0, 1, 2, \ldots$$

<u>Computational solution with MATLAB</u>:

```
»% MATLAB Program for Problem B-3-8
»
»% ----- Unit-step response -----
»
»num = [0  0  1];
»den = [1  0  1];
»x = ones(1,41);
»v = [0  40  0  1.2];
»axis(v);
»k = 0:40;
»y = filter(num,den,x);
»plot(k,y,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('y(k)')
```



B-3-9.

$$y(k) - ay(k-1) = x(k) \qquad -1 < a < 1$$

The z transform of this equation is

$$Y(z) - az^{-1}Y(z) = X(z)$$

Hence

$$Y(z) = \frac{X(z)}{1 - az^{-1}}$$

Then

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

and the weighting sequence g(k) is given by

$$g(k) = a^k$$

If two systems which are described by the given difference equation are connected in series, then the weighting sequence $g_0(k)$ of the resulting system can be obtained as the inverse z transform of

$$G_0(z) = \frac{1}{1 - az^{-1}} \frac{1}{1 - az^{-1}} = z \frac{z^{-1}}{(1 - az^{-1})^2}$$

or

$$g_0(k) = (k + 1)a^k$$

B-3-10. The z transform of the given difference equation is

$$Y(z) - z^{-1}Y(z) + 0.24z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

from which we get

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - z^{-1} + 0.24z^{-2}}$$

$$= \frac{8}{1 - 0.6z^{-1}} - \frac{7}{1 - 0.4z^{-1}}$$

Hence, the weighting sequence g(k) is

$$g(k) = 8(0.6)^k - 7(0.4)^k$$

For the unit-step sequence input we have

$$X(z) = \frac{1}{1 - z^{-1}}$$

and the output Y(z) becomes

$$Y(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.24z^{-2}} \frac{1}{1 - z^{-1}}$$

$$= \frac{14}{3} \frac{1}{1 - 0.4z^{-1}} - \frac{12}{1 - 0.6z^{-1}} + \frac{25}{3} \frac{1}{1 - z^{-1}}$$

Thus,

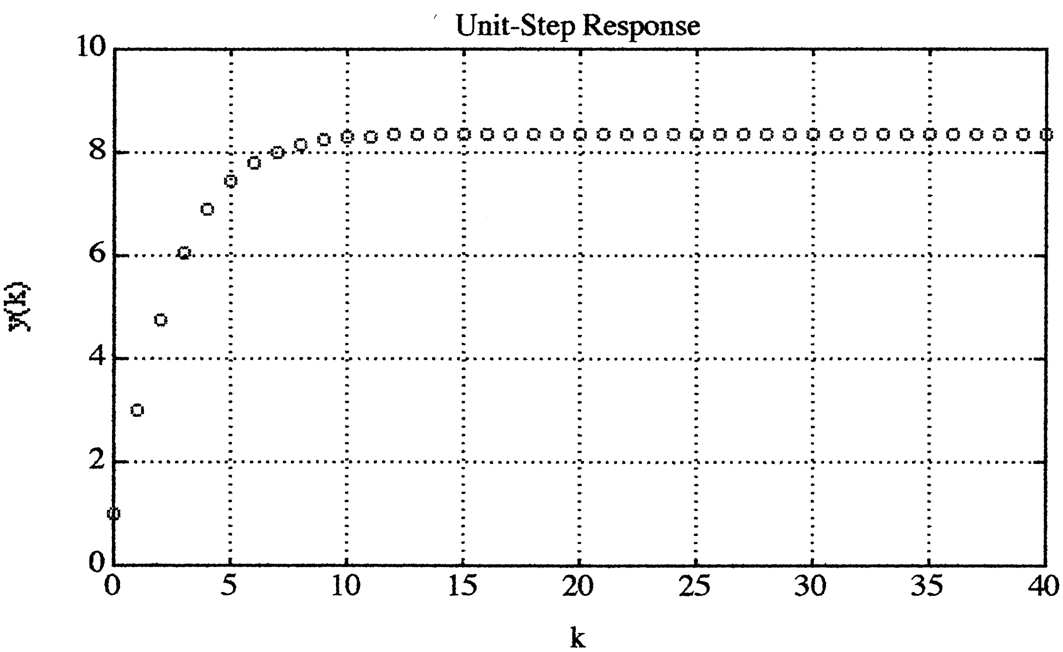
$$y(k) = \frac{14}{3} (0.4)^k - 12(0.6)^k + \frac{25}{3}$$

where k = 0, 1, 2, ...

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-3-10
»
»% ----- Unit-step response -----
»
»num = [1  1  0];
»den = [1  -1  0.24];
»x = ones(1,41);
»v = [0  40  0  10];
»axis(v);
```

```
»k = 0:40;
»y = filter(num,den,x);
»plot(k,y,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('y(k)')
```



<u>B-3-11</u>.

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1 - 0.5z^{-1}}{(1 - 0.3z^{-1})(1 + 0.7z^{-1})}$$

where

$$X(z) = \frac{1}{1 - z^{-1}}$$

The output Y(z) is given by

$$Y(z) = \frac{1 - 0.5z^{-1}}{(1 - 0.3z^{-1})(1 + 0.7z^{-1})} \frac{1}{1 - z^{-1}}$$

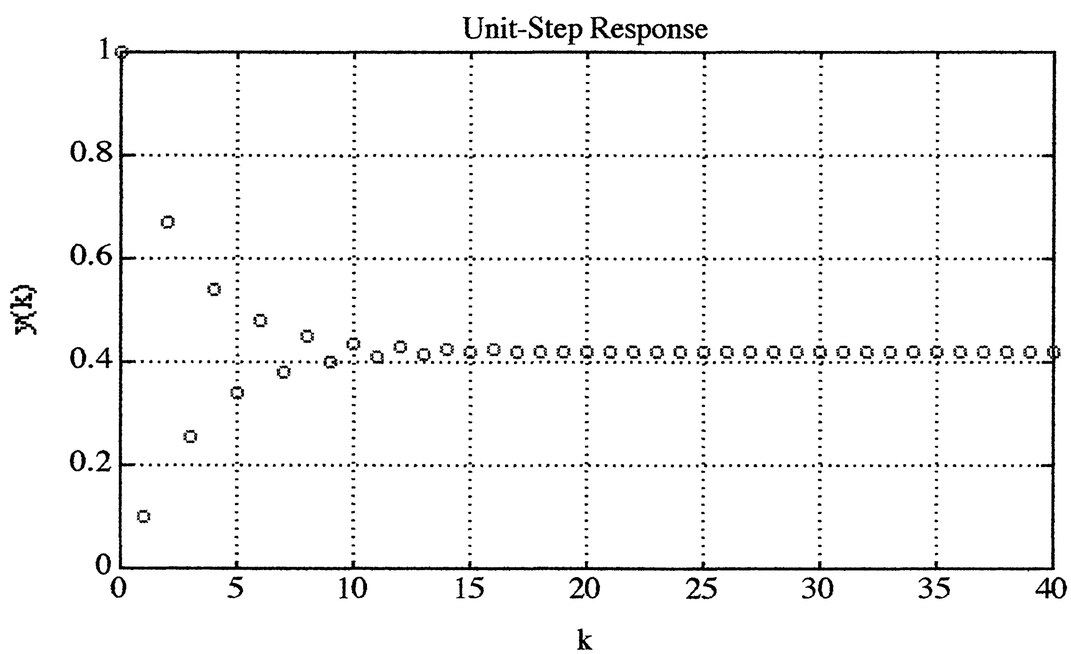
$$= \frac{3}{35} \frac{1}{1 - 0.3z^{-1}} + \frac{42}{85} \frac{1}{1 + 0.7z^{-1}} + \frac{50}{119} \frac{1}{1 - z^{-1}}$$

Hence

$$y(k) = \frac{3}{35}(0.3)^k + \frac{42}{85}(-0.7)^k + \frac{50}{119}$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-3-11
»
»% ----- Unit-step response -----
»
»num = [1  -0.5  0];
»den = [1  0.4  -0.21];
»x = ones(1,41);
»v = [0  40  0  1];
»axis(v);
»k = 0:40;
»y = filter(num,den,x);
»plot(k,y,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('y(k)')
```



B-3-12.

$$Y(s) = \frac{1}{(s+1)(s+2)} X^*(s) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)X^*(s)$$

By taking the starred Laplace transform of this last equation, we obtain

$$Y^*(s) = \left(\frac{1}{s+1}\right)^* X^*(s) - \left(\frac{1}{s+2}\right)^* X^*(s)$$

Hence

$$Y(z) = \mathcal{Z}\left[\frac{1}{s+1}\right] X(z) - \mathcal{Z}\left[\frac{1}{s+2}\right] X(z)$$

$$= \frac{1}{1 - e^{-T}z^{-1}} X(z) - \frac{1}{1 - e^{-2T}z^{-1}} X(z)$$

Since

$$X(z) = \frac{1}{1 - z^{-1}}$$

we have

$$Y(z) = \frac{1}{1 - e^{-T}z^{-1}} \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-2T}z^{-1}} \frac{1}{1 - z^{-1}}$$

$$= \frac{e^{-T}}{e^{-T} - 1} \frac{1}{1 - e^{-T}z^{-1}} + \frac{1}{1 - e^{-T}} \frac{1}{1 - z^{-1}}$$

$$- \frac{e^{-2T}}{e^{-2T} - 1} \frac{1}{1 - e^{-2T}z^{-1}} - \frac{1}{1 - e^{-2T}} \frac{1}{1 - z^{-1}}$$

Hence

$$y(kT) = \frac{e^{-T}}{e^{-T} - 1} (e^{-T})^k + \frac{1}{1 - e^{-T}} - \frac{e^{-2T}}{e^{-2T} - 1} (e^{-2T})^k - \frac{1}{1 - e^{-2T}}$$

For $T = 0.1$ we have

$$y(k) = -9.5083(0.9048)^k + 4.5167(0.8187)^k + 4.9917$$

B-3-13.

$$H(z) = \frac{0.5z^3 + 0.4127z^2 + 0.1747z - 0.0874}{z^3}$$

$$= 0.5 + 0.4127z^{-1} + 0.1747z^{-2} - 0.0874z^{-3}$$

Hence

$$h(0) = 0.5$$

$$h(1) = 0.4127$$

$$h(2) = 0.1747$$

$$h(3) = -0.0874$$

$$h(k) = 0 \qquad \text{for } k = 4, 5, 6, \ldots$$

Noting that

$$u(k) = 1 \qquad \text{for } k = 0, 1, 2, \ldots$$

and

$$y(k) = \sum_{j=0}^{k} h(k - j)u(j)$$

we have

$$y(0) = \sum_{j=0}^{0} h(0 - j)u(j) = h(0)u(0) = 0.5 \times 1 = 0.5$$

$$y(1) = \sum_{j=0}^{1} h(1 - j)u(j) = h(1)u(0) + h(0)u(1)$$

$$= 0.4127 \times 1 + 0.5 \times 1 = 0.9127$$

$$y(2) = \sum_{j=0}^{2} h(2 - j)u(j) = h(2)u(0) + h(1)u(1) + h(0)u(2)$$

$$= 0.1747 \times 1 + 0.4127 \times 1 + 0.5 \times 1 = 1.0874$$

$$y(3) = \sum_{j=0}^{3} h(3 - j)u(j) = h(3)u(0) + h(2)u(1)$$

$$+ h(1)u(2) + h(0)u(3)$$

$$= -0.0874 \times 1 + 0.1747 \times 1 + 0.4127 \times 1 + 0.5 \times 1$$

$$= 1$$

$$y(4) = \sum_{j=0}^{4} h(4 - j)u(j) = h(4)u(0) + h(3)u(1) + h(2)u(2)$$

$$+ h(1)u(3) + h(0)u(4)$$

$$= 0 \times 1 - 0.0874 \times 1 + 0.1747 \times 1 + 0.4127 \times 1$$

$$+ 0.5 \times 1 = 1$$

$$y(k) = 1 \qquad \text{for } k = 5, 6, 7, \ldots$$

B-3-14. Since

$$Y^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + j\omega_s k)X^*(s + j\omega_s k)$$

and

$$X^*(s) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s + j\omega_s h) + \frac{1}{2}x(0+)$$

we have

$$X^*(s + j\omega_s k) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s + j\omega_s h + j\omega_s k) + \frac{1}{2}x(0+)$$

By letting h + k = m, we obtain

$$X^*(s + j\omega_s k) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(s + j\omega_s m) + \frac{1}{2}x(0+) = X^*(s)$$

Substitution of this last equation into the expression for Y*(s) gives

$$Y^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + j\omega_s k)X^*(s)$$

Since G*(s) can be given by

$$G^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + j\omega_s k)$$

we obtain

$$Y^*(s) = G^*(s)X^*(s)$$

B-3-15. From Figure 3-67 we obtain

$$C(s) = G(s)E^*(s)$$

$$E(s) = R(s) - H_2(s)M^*(s)$$

$$M(s) = H_1(s)G(s)E^*(s)$$

By taking the starred Laplace transforms of the preceding equations, we obtain

$$C^*(s) = G^*(s)E^*(s)$$

$$E^*(s) = R^*(s) - H_2^*(s)M^*(s)$$

$$M^*(s) = \left[GH_1(s)\right]^*E^*(s)$$

Hence

$$E^*(s) = R^*(s) - H_2^*(s)\left[GH_1(s)\right]^*E^*(s)$$

or

$$E^*(s) = \frac{R^*(s)}{1 + H_2^*(s)\left[GH_1(s)\right]^*}$$

and

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + H_2^*(s)\left[GH_1(s)\right]^*}$$

Thus,

$$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + H_2^*(s)\left[GH_1(s)\right]^*}$$

or

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + H_2(z)GH_1(z)}$$

<u>B-3-16</u>.

$$G(z) = \mathcal{Z}\left[G_{h0}(s)\frac{K}{s + a}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{K}{s(s + a)}\right]$$

$$= \frac{K}{a}\,\frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} = \frac{K}{a}\,\frac{(1 - e^{-a})z^{-1}}{1 - e^{-a}z^{-1}} \qquad (T = 1)$$

Then

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{K(1 - e^{-a})z^{-1}}{a + \left[K - (K + a)e^{-a}\right]z^{-1}}$$

<u>B-3-17</u>.

$$C(s) = G_2(s)M^*(s)$$

$$M(s) = G_1(s)E^*(s) - G_2(s)M^*(s)$$

$$E(s) = R(s) - H(s)G_2(s)M^*(s)$$

Hence

$$M^*(s) = G_1^*(s)E^*(s) - G_2^*(s)M^*(s)$$

$$E^*(s) = R^*(s) - \left[HG_2(s)\right]^*M^*(s)$$

Thus,

$$M^*(s) = G_1^*(s)\left\{R^*(s) - \left[HG_2(s)\right]^*M^*(s)\right\} - G_2^*(s)M^*(s)$$

or

$$M^*(s) = \frac{G_1^*(s)R^*(s)}{1 + G_1^*(s)\left[HG_2(s)\right]^* + G_2^*(s)}$$

Since

$$C^*(s) = G_2^*(s)M^*(s)$$

we have

$$C^*(s) = \frac{G_1^*(s)G_2^*(s)R^*(s)}{1 + G_1^*(s)\left[HG_2(s)\right]^* + G_2^*(s)}$$

or

$$C(z) = \frac{G_1(z)G_2(z)R(z)}{1 + G_1(z)HG_2(z) + G_2(z)}$$

The continuous-time output $C(s)$ can be given by

$$C(s) = G_2(s)M^*(s)$$

Thus

$$C(s) = \frac{G_2(s)G_1^*(s)R^*(s)}{1 + G_1^*(s)\left[HG_2(s)\right]^* + G_2^*(s)}$$

B-3-18.

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,\frac{K}{s}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{K}{s^2}\right]$$

$$= (1 - z^{-1})\,\frac{KTz^{-1}}{(1 - z^{-1})^2}$$

Since $T = 1$, we have

$$G(z) = \frac{Kz^{-1}}{1 - z^{-1}}$$

Hence

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{Kz^{-1}}{1 + (K - 1)z^{-1}}$$

Since

$$R(z) = \frac{1}{1 - z^{-1}}$$

we have

$$C(z) = \frac{Kz^{-1}}{1 + (K - 1)z^{-1}}\,\frac{1}{1 - z^{-1}}$$

$$= -\frac{1}{1 + (K - 1)z^{-1}} + \frac{1}{1 - z^{-1}}$$

The inverse z transform of $C(z)$ gives

$$c(k) = -(1 - K)^k + 1$$

The output sequence $c(k)$ will converge to 1 if $0 < K < 2$. Otherwise, $c(k)$ approaches infinity as k approaches infinity.

Continuous-time output $c(t)$: From Figure 3-70 we have

$$C(s) = \left(\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}\right) E^*(s)$$

$$E(s) = R(s) - C(s)$$

Thus

$$E^*(s) = R^*(s) - C^*(s)$$

$$= R^*(s) - \left(\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}\right)^* E^*(s)$$

or

$$E^*(s) = \frac{R^*(s)}{1 + \left(\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}\right)^*}$$

Hence

$$C(s) = \frac{\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}}{1 + \left(\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}\right)^*} R^*(s)$$

Since

$$\mathcal{Z}\left[\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{K}{s^2}\right] = \frac{Kz^{-1}}{1 - z^{-1}}$$

by substituting $z = e^{Ts} = e^{s}$ into this last equation, we obtain

$$\left(\frac{K}{s}\,\frac{1 - e^{-Ts}}{s}\right)^* = \frac{Ke^{-s}}{1 - e^{-s}}$$

Therefore,

$$C(s) = \frac{K}{s^2}\,\frac{1 - e^{-s}}{1 + (K - 1)e^{-s}}$$

$$= \frac{K}{s^2}\left[1 - Ke^{-s} + K(K - 1)e^{-2s} - K(K - 1)^2e^{-3s} + \cdots\right]$$

The inverse Laplace transform of this last equation gives

$$c(t) = Kt - K^2(t - 1)1(t - 1) + K^2(K - 1)(t - 2)1(t - 2)$$
$$- K^2(K - 1)^2(t - 3)1(t - 3) + \cdots$$

B-3-19. Since $T = 1$, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,\frac{1}{s(s + 1)}\right]$$

$$= \frac{e^{-1}z^{-1} - 2e^{-1}z^{-2} + z^{-2}}{(1 - z^{-1})(1 - e^{-1}z^{-1})}$$

Hence

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{e^{-1}z^{-1} - 2e^{-1}z^{-2} + z^{-2}}{1 - z^{-1} - e^{-1}z^{-2} + z^{-2}}$$

For a Kronecker delta input

$$R(z) = 1$$

Hence

$$C(z) = \frac{z^{-1}(e^{-1} - 2e^{-1}z^{-1} + z^{-1})}{1 - z^{-1} + (1 - e^{-1})z^{-2}} = \frac{z^{-1}(0.3679 + 0.2642z^{-1})}{1 - z^{-1} + 0.6321\ z^{-2}}$$

$$= 0.3679z^{-1}\left(\frac{1 - 0.5z^{-1}}{1 - z^{-1} + 0.6321z^{-2}}\right)$$

$$+ 0.7250z^{-1}\left(\frac{0.6182z^{-1}}{1 - z^{-1} + 0.6321z^{-2}}\right)$$

The inverse z transform of this last equation gives

$$c(k) = 0.3679\ (0.7951)^{k-1} \cos(0.8907(k - 1))$$

$$+ 0.7250\ (0.7951)^{k-1} \sin(0.8907(k - 1))$$

Notice that

$$c(0) = 0$$

$$c(1) = 0.3679$$

$$c(2) = 0.6321$$

$$c(3) = 0.3996$$

$$\vdots$$

B-3-20. From Figure 3-72 we obtain

$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - B(s)$$

$$B(s) = \frac{1 - e^{-Ts}}{s}\ C^*(s)$$

Thus

$$C(s) = G(s)\left[R(s) - B(s)\right] = G(s)R(s) - G(s)\ \frac{1 - e^{-Ts}}{s}\ C^*(s)$$

Hence

$$C^*(s) = \left[GR(s)\right]^* - \left[G(s)\,\frac{1 - e^{-Ts}}{s}\right]^* C^*(s)$$

or

$$C^*(s) = \frac{\left[GR(s)\right]^*}{1 + \left[G(s)\,\frac{1 - e^{-Ts}}{s}\right]^*}$$

Note that for T = 0.2 we have

$$\mathcal{Z}\left[G(s)\,\frac{1 - e^{-Ts}}{s}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$$= (1 - z^{-1})\,\mathcal{Z}\left[\frac{1}{s(s + 1)}\right] = \frac{0.1813z^{-1}}{1 - 0.8187z^{-1}}$$

and

$$\left[GR(s)\right]^* = \left[\frac{1}{s + 1}\,\frac{1}{s}\right]^*$$

or

$$GR(z) = \frac{0.1813z^{-1}}{(1 - z^{-1})(1 - 0.8187z^{-1})}$$

Hence

$$C(z) = \frac{0.1813z^{-1}}{1 - 1.6375z^{-1} + 0.6375z^{-2}}$$

$$= 0.1813z^{-1} + 0.2968z^{-2} + 0.3705z^{-3} + 0.4174z^{-4} + \cdots$$

The final value $c(\infty)$ can be obtained as follows:

$$c(\infty) = \lim_{z \to 1}\left[(1 - z^{-1})C(z)\right]$$

$$= \lim_{z \to 1}\left[(1 - z^{-1})\,\frac{0.1813z^{-1}}{(1 - 0.6375z^{-1})(1 - z^{-1})}\right] = 0.500$$

B-3-21.

$$M(z) = \frac{K_I}{1 - z^{-1}}\left[R(z) - C(z)\right] - \left[K_P + K_D(1 - z^{-1})\right] C(z)$$

$$C(z) = G(z)M(z)$$

$$= G(z)\,\frac{K_I}{1 - z^{-1}}\,R(z) - G(z)\left[\frac{K_I}{1 - z^{-1}} + K_P + K_D(1 - z^{-1})\right] C(z)$$

Thus

$$\frac{C(z)}{R(z)} = \frac{G(z)\dfrac{K_I}{1 - z^{-1}}}{1 + G(z)\left[\dfrac{K_I}{1 - z^{-1}} + K_P + K_D(1 - z^{-1})\right]}$$

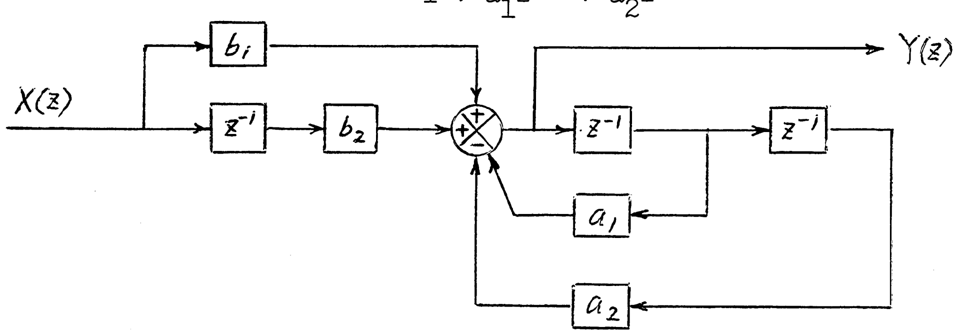
B-3-22.

1. Direct programming: The z transform of the given difference equation becomes

$$Y(z) + a_1 z^{-1}Y(z) + a_2 z^{-2}Y(z) = b_1 X(z) + b_2 z^{-1}X(z)$$

Thus

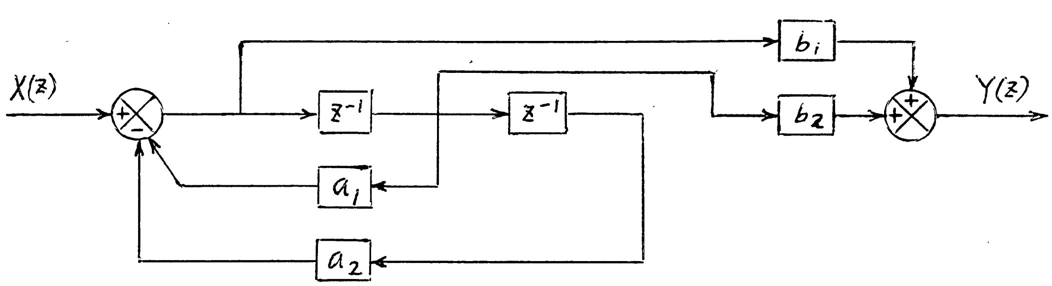
$$\frac{Y(z)}{X(z)} = \frac{b_1 + b_2 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$



2. Standard programming:

$$\frac{Y(z)}{H(z)} = b_1 + b_2 z^{-1}$$

$$\frac{H(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$



3. Ladder programming:

$$\frac{Y(z)}{X(z)} = \frac{b_1 + b_2 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_1 z^2 + b_2 z}{z^2 + a_1 z + a_2}$$

$$= A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z + \cfrac{1}{A_2}}}}$$

where

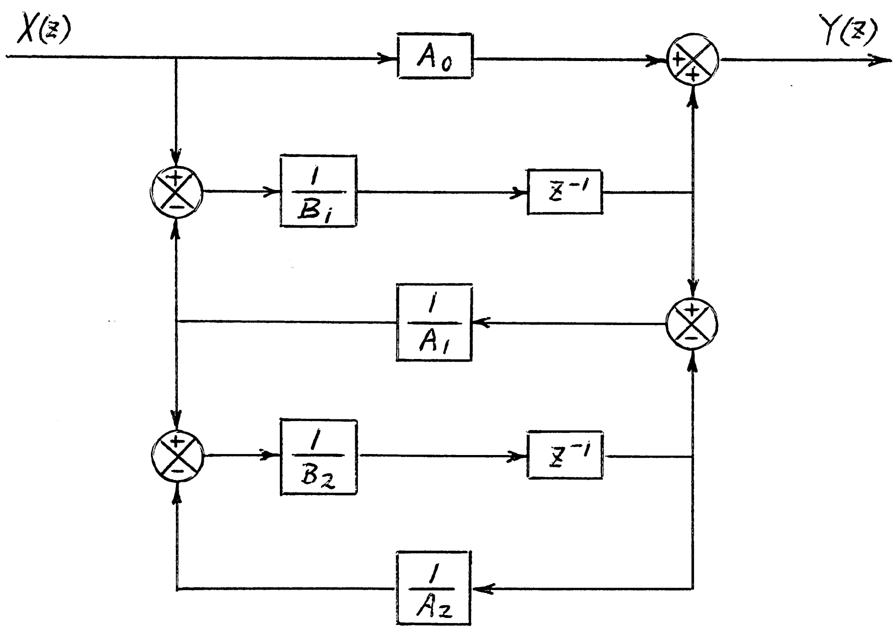
$$A_0 = b_1$$

$$B_1 = \frac{1}{b_2 - a_1 b_1}$$

$$A_1 = \cfrac{b_2 - a_1 b_1}{\cfrac{a_2 b_1}{b_2 - a_1 b_1} + a_1}$$

$$B_2 = - \cfrac{\cfrac{a_2 b_1}{b_2 - a_1 b_1} + a_1}{\cfrac{a_2 (b_2 - a_1 b_1)}{\cfrac{a_2 b_1}{b_2 - a_1 b_1} + a_1} + a_2 b_1}$$

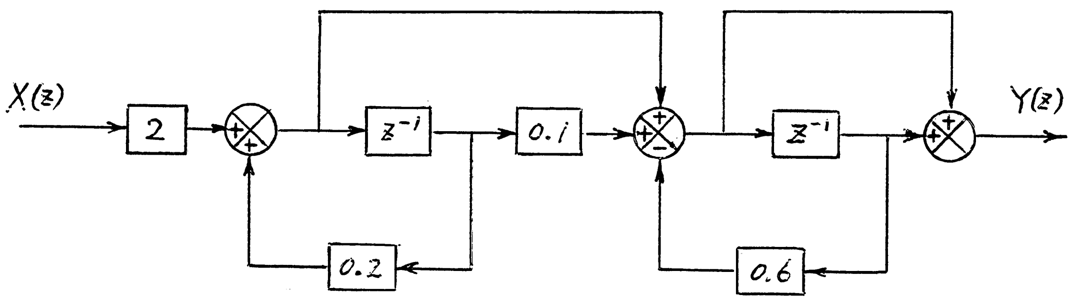
$$\frac{1}{A_2} = \cfrac{-1}{\cfrac{b_2 - a_1 b_1}{\cfrac{a_2 b_1}{b_2 - a_1 b_1} + a_1} + b_1}$$



B-3-23.

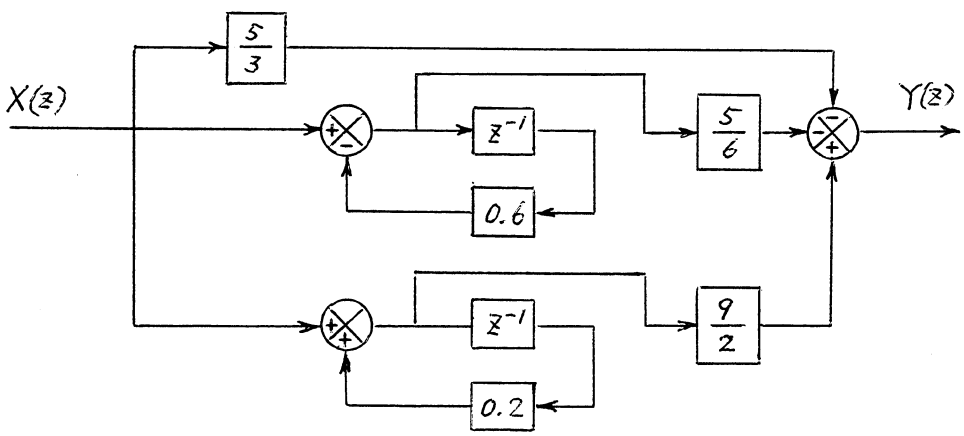
Series scheme:

$$G(z) = 2\,\frac{1 + 0.1z^{-1}}{1 - 0.2z^{-1}}\,\frac{1 + z^{-1}}{1 + 0.6z^{-1}}$$



Parallel scheme:

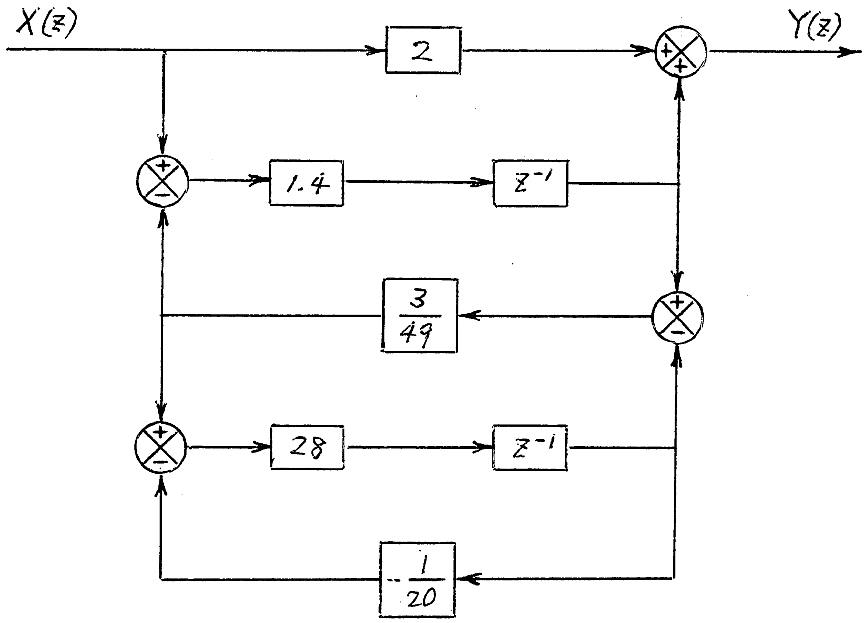
$$G(z) = -\frac{5}{3} - \frac{\frac{5}{6}}{1 + 0.6z^{-1}} + \frac{\frac{9}{2}}{1 - 0.2z^{-1}}$$



Ladder scheme:

$$G(z) = \frac{2z^2 + 2.2z + 0.2}{z^2 + 0.4z - 0.12}$$

$$= 2 + \cfrac{1}{\cfrac{1}{1.4}z + \cfrac{1}{\cfrac{49}{3} + \cfrac{1}{\cfrac{1}{28}z - \cfrac{1}{20}}}}$$



B-3-24. From Figure 3-63, we have

$$V(z) = \frac{1}{T}(1 - z^{-1})X(z)$$

When the input x(k) is a unit-step sequence,

$$X(z) = \frac{1}{1 - z^{-1}}$$

Hence,

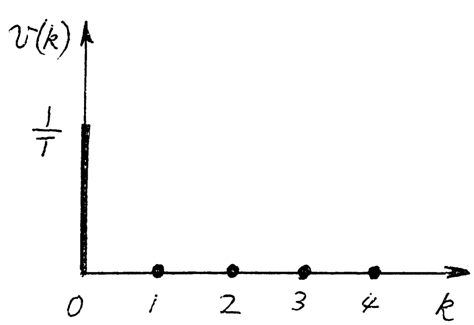
$$V(z) = \frac{1}{T}(1 - z^{-1})\frac{1}{1 - z^{-1}} = \frac{1}{T}$$

The inverse z transform of V(z) gives

$$v(0) = \frac{1}{T}$$

$$v(k) = 0, \quad k = 1, 2, 3, \ldots$$

The output v(k) versus k is shown below.



B-3-25. From Figure 3-73 we have

$$\frac{Y(z)}{X(z)} = \frac{T}{1 - z^{-1}}$$

or

$$Y(z) = \frac{T}{1 - z^{-1}} X(z)$$

Note that in Problem A-2-4 we showed that

$$\mathcal{Z}\left[\sum_{h=0}^{k} x(h)\right] = \frac{1}{1 - z^{-1}} X(z)$$

Hence

$$Y(z) = \frac{T}{1 - z^{-1}} X(z) = T \mathcal{Z}\left[\sum_{h=0}^{k} x(hT)\right]$$

The inverse z transform of this last equation gives

$$y(kT) = T \sum_{h=0}^{k} x(hT)$$

$$= T[x(0) + x(T) + \cdots + x(kT)]$$

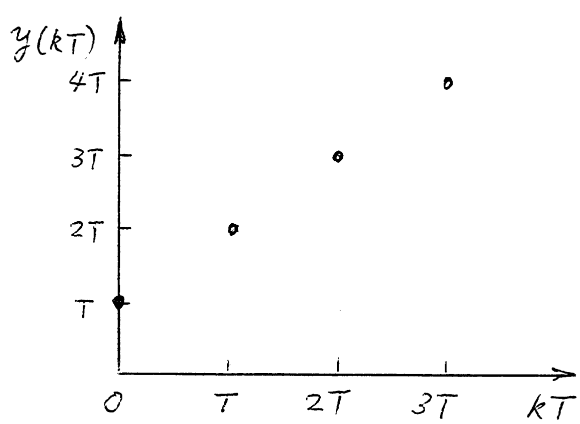
Thus, y(kT) approximates an area made by the input. [Note that y(0) is Tx(0).] The system acts as an integrator.

When the input x(kT) is a unit-step sequence,

$$y(kT) = T[x(0) + x(T) + \cdots + x(kT)]$$

$$= T(1 + 1 + \cdots + 1) = T(k + 1)$$

A plot of the output y(kT) versus kT is shown below.



B-3-26. From Figure 3-74, we obtain

$$\frac{Y(z)}{X(z)} = T\left(\frac{z^{-1}}{1 - z^{-1}}\right)$$

Hence,

$$Y(z) = T\frac{z^{-1}}{1 - z^{-1}}X(z)$$

In Problem A-2-4 we showed that

$$\mathcal{Z}\left[\sum_{h=0}^{k-1} x(h)\right] = \frac{z^{-1}}{1 - z^{-1}}X(z)$$

Hence,

$$Y(z) = T\frac{z^{-1}}{1 - z^{-1}}X(z) = T\mathcal{Z}\left[\sum_{h=0}^{k-1} x(hT)\right]$$

The inverse z transform of this last equation gives

$$y(kT) = T\left[\sum_{h=0}^{k-1} x(hT)\right]$$

$$= T[x(0) + x(T) + \cdots + x((k - 1)T)]$$

or

$$y(0) = 0$$

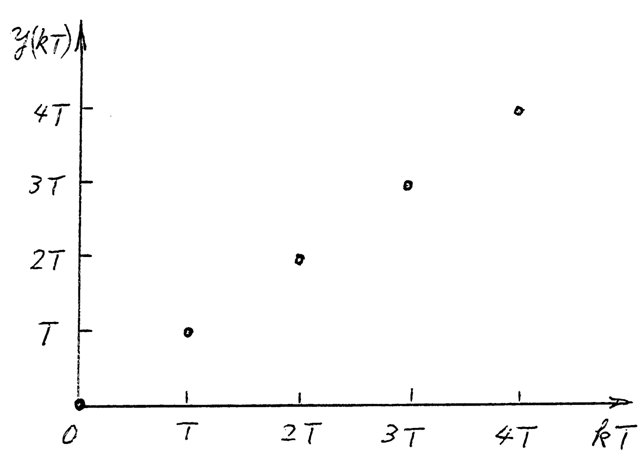
$$y(T) = Tx(0)$$

$$y(2T) = T[x(0) + x(T)]$$

$$y(3T) = T[x(0) + x(T) + x(2T)]$$

$$\vdots$$

The output approximates the area made by the input. The output $y(kT)$ versus kT when the input $x(kT)$ is a unit-step sequence is shown below.



B-3-27. From Figure 3-75 we obtain

$$\frac{Y(z)}{X(z)} = \frac{T}{2}\left(\frac{1}{1 - z^{-1}} + \frac{z^{-1}}{1 - z^{-1}}\right)$$

Hence,

$$Y(z) = \frac{T}{2}\left[\frac{1}{1 - z^{-1}} X(z) + \frac{z^{-1}}{1 - z^{-1}} X(z)\right]$$

Referring to Problem A-2-4, we have

$$Y(z) = \frac{T}{2}\left\{\mathcal{Z}\left[\sum_{h=0}^{k} x(hT)\right] + \mathcal{Z}\left[\sum_{h=0}^{k-1} x(hT)\right]\right\}$$

The inverse z transform of this last equation gives

$$y(kT) = \frac{T}{2}\left[\sum_{h=0}^{k} x(hT) + \sum_{h=0}^{k-1} x(hT)\right]$$

$$= \frac{T}{2}[x(0) + x(T) + \cdots + x(kT) + x(0) + x(T) + \cdots + x((k - 1)T)]$$

Hence,

$$y(0) = \tfrac{1}{2}Tx(0)$$

$$y(T) = T[x(0) + \tfrac{1}{2}x(T)]$$

$$y(2T) = T[x(0) + x(T) + \tfrac{1}{2}x(2T)]$$

$$y(3T) = T[x(0) + x(T) + x(2T) + \tfrac{1}{2}x(3T)]$$

$$\vdots$$

When the input x(kT) is a unit-step sequence, we have

$$y(0) = 0.5T$$

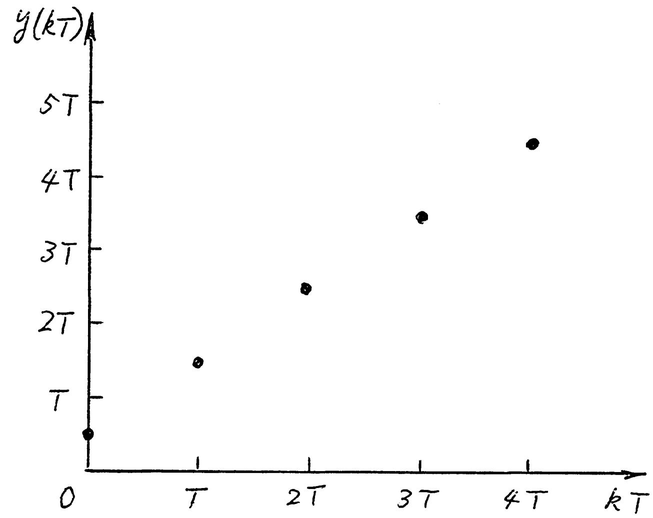
$$y(T) = 1.5T$$

$$y(2T) = 2.5T$$

$$\vdots$$

$$y(kT) = (k + 0.5)T$$

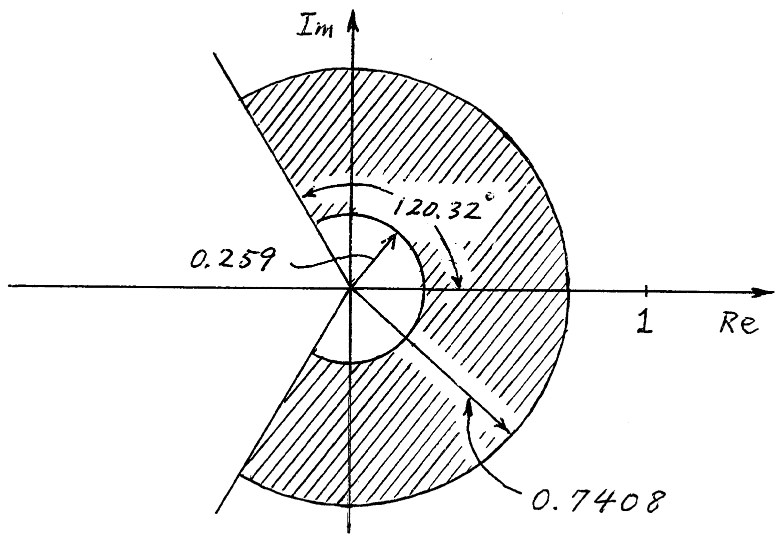
A plot of y(kT) versus kT, when the input x(kT) is a unit-step sequence, is shown on next page.


y(kT)
5T
4T
3T
2T
T
0
T
2T
3T
4T
kT

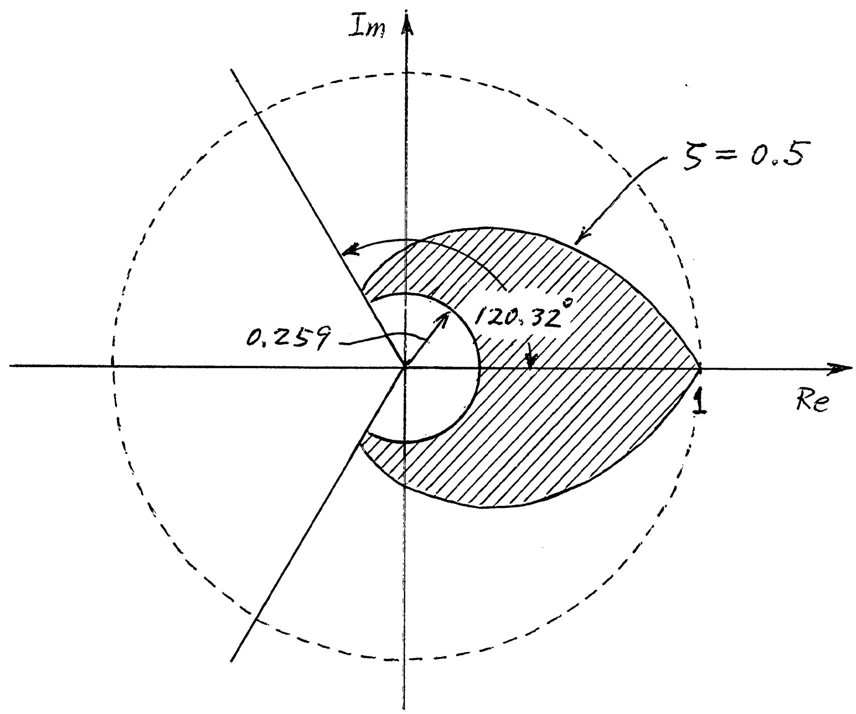
CHAPTER 4

B-4-1.

(a)



(b)



B-4-2. We shall apply the Jury stability test to this problem.

$$P(z) = z^3 + 2.1z^2 + 1.44z + 0.32$$

$$= a_0z^3 + a_1z^2 + a_2z + a_3$$

Thus

$$a_0 = 1, \quad a_1 = 2.1, \quad a_2 = 1.44, \quad a_3 = 0.32$$

The conditions for stability are

1. $|a_3| < a_0$

   This condition is safisfied, since $|0.32| < 1$.

2. $P(1) > 0$

   Since

$$P(1) = 1 + 2.1 + 1.44 + 0.32 = 4.86 > 0$$

   the condition is satisfied.

3. $P(-1) < 0$

   This condition is also satisfied, since

$$P(-1) = -1 + 2.1 - 1.44 + 0.32 = -0.02 < 0$$

4. $|b_2| > |b_0|$

   Since

$$b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix} = \begin{vmatrix} 0.32 & 1 \\ 1 & 0.32 \end{vmatrix} = -0.8976$$

$$b_0 = \begin{vmatrix} a_3 & a_2 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 0.32 & 1.44 \\ 1 & 2.1 \end{vmatrix} = -0.768$$

   the condition is satisfied.

Thus, we see that all conditions for stability are satisfied. Hence, no roots of the characteristic equation lie outside the unit circle centered at the origin of the z plane.

B-4-3.

$$\frac{Y(z)}{X(z)} = \frac{1}{z^3 + 0.5z^2 - 1.34z + 0.24}$$

Define

$$P(z) = z^3 + 0.5z^2 - 1.34z + 0.24$$
$$= a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

Then

$$a_0 = 1$$
$$a_1 = 0.5$$
$$a_2 = -1.34$$

$$a_3 = 0.24$$

The Jury stability conditions are

1. $|a_3| < a_0$

   This condition is satisfied.

2. $P(1) > 0$

   Since

$$P(1) = 1 + 0.5 - 1.34 + 0.24 = 0.4 > 0$$

   the condition is satisfied.

3. $P(-1) < 0$

   Since

$$P(-1) = -1 + 0.5 + 1.34 + 0.24 = 1.08 > 0$$

   the condition is not satisfied.

4. $|b_2| > |b_0|$

   Since condition (3) is not satisfied (the system is unstable), it is not necessary to test condition (4).

The conclusion is that the system is unstable.

B-4-4.

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-s}}{s} \frac{K}{s(s+1)}\right] = (1 - z^{-1})\, \mathcal{Z}\left[\frac{K}{s^2(s+1)}\right]$$

$$= \frac{K\left[e^{-1}z^{-1} + (1 - 2e^{-1})z^{-2}\right]}{(1 - z^{-1})(1 - e^{-1}z^{-1})}$$

Hence

$$\frac{C(z)}{R(z)} = \frac{K\left[e^{-1}z^{-1} + (1 - 2e^{-1})z^{-2}\right]}{(1 - z^{-1})(1 - e^{-1}z^{-1}) + K\left[e^{-1}z^{-1} + (1 - 2e^{-1})z^{-2}\right]}$$

Noting that $e^{-1} = 0.3679$, the characteristic equation of the system becomes

$$z^2 - (1.3679 - 0.3679K)z + 0.3679 + 0.2642K = 0$$

Define

$$P(z) = z^2 - (1.3679 - 0.3679K)z + 0.3679 + 0.2642K$$

$$= a_0 z^2 + a_1 z + a_2$$

Then

$$a_0 = 1$$

$$a_1 = -1.3679 + 0.3679K$$

$$a_2 = 0.3679 + 0.2642K$$

For stability, we must have

$$|a_2| < a_0$$

$$P(1) > 0$$

$$P(-1) > 0$$

Therefore, we require

$$|0.3679 + 0.2642K| < 1$$

which yields

$$-5.1775 < K < 2.3925 \qquad (1)$$

Also, from

$$P(1) = 1 - (1.3679 - 0.3679K) + 0.3679 + 0.2642K$$

$$= 0.6321K > 0$$

we obtain

$$K > 0 \qquad (2)$$

and from

$$P(-1) = 1 + 1.3679 - 0.3679K + 0.3679 + 0.2642K$$

$$= 2.7358 - 0.1037K > 0$$

we have

$$26.38 > K \qquad (3)$$

From Inequalities (1), (2), and (3), we obtain the range of gain K for stability to be

$$0 < K < 2.3925$$

B-4-5.

$$P(z) = z^2 - (1.3679 - 0.3679K)z + 0.3679 + 0.2642K = 0$$

By substituting $z = (w + 1)/(w - 1)$ into this last equation, we obtain

$$\left(\frac{w+1}{w-1}\right)^2 - (1.3679 - 0.3679K)\frac{w+1}{w-1} + 0.3679 + 0.2642K = 0$$

which can be simplified to

$$0.6321Kw^2 + (1.2642 - 0.5284K)w + 2.7358 - 0.1037K = 0$$

The Routh array becomes

$$\begin{array}{lll} w^2 & 0.6321K & 2.7358 - 0.1037K \\ w^1 & 1.2642 - 0.5284K & 0 \\ w^0 & 2.7358 - 0.1037K & \end{array}$$

For stability, we require

$$\begin{array}{lll} 0.6321K > 0 & \text{or} & 0 < K \\ 1.2642 - 0.5284K > 0 & \text{or} & K < 2.3925 \\ 2.7358 - 0.1037K > 0 & \text{or} & K < 26.38 \end{array}$$

Hence

$$0 < K < 2.3925$$

B-4-6.

$$\frac{Y(z)}{X(z)} = G(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Since the output $Y(z)$ is given by

$$Y(z) = G(z)X(z)$$

we have

$$\left|y(k)\right| = \left|\sum_{h=0}^{k} g(k - h)x(h)\right| \leqslant \left|\sum_{h=0}^{k} g(k - h)M_1\right|$$

Since all poles of $G(z)$ lie inside the unit circle in the z plane, the system is stable. For such a stable system

$$\left|g(k)\right| \leqslant a^k < 1$$

where a is a constant, $0 < a < 1$. Hence

$$\left|y(k)\right| \leqslant \left|\left[g(k) + g(k - 1) + \cdots + g(0)\right]M_1\right|$$

$$\leqslant \left|(a^k + a^{k-1} + \cdots + 1)M_1\right|$$

$$= \frac{1 - a^{k+1}}{1 - a} M_1$$

By defining

$$\frac{1}{1-a}M_1 = M_2$$

we have

$$|y(k)| \leqslant \frac{1-a^{k+1}}{1-a}M_1 \leqslant M_2 - \frac{a^{k+1}}{1-a}M_1 \leqslant M_2$$

Thus, the output y(k) is bounded.

B-4-7.

Stability: The system is stable if the weighting sequence g(kT) vanishes for large k.

Instability: The system is unstable if g(kT) grows without bound as k increses indefinitely.

Critical stability: The system is critically stable if g(kT) approaches a constant nonzero value or a bounded oscillation for large values of k.

B-4-8.

$$G(z) = \frac{K(z+1)}{(z-1)(z-0.6065)}$$

The characteristic equation for the system is

$$z^2 + (K - 1.6065)z + 0.6065 + K = 0$$

The critical value of gain K for stability can be determined easily by use of the Jury stability criterion. Define

$$\begin{aligned} P(z) &= z^2 + (K - 1.6065)z + 0.6065 + K \\ &= a_0z^2 + a_1z + a_2 = 0 \end{aligned}$$

Then

$$a_0 = 1, \qquad a_1 = K - 1.6065, \qquad a_2 = 0.6065 + K$$

The conditions for stability are

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0$

Thus we require

$$|0.6065 + K| < 1$$

$$P(1) = 1 + K - 1.6065 + 0.6065 + K = 2K > 0$$

$$P(-1) = 1 - K + 1.6065 + 0.6065 + K = 3.213 > 0$$

Hence

$$0 < K < 0.3935$$

The critical value of gain K for stability is 0.3935.

Since

$$G(z) = \frac{K(z + 1)}{(z - 1)(z - 0.6065)}$$

we have

$$\underline{/G(z)} = \underline{/z + 1} - \underline{/z - 1} - \underline{/z - 0.6065}$$

Define

$$z = \sigma + j\omega$$

The angle condition is

$$\underline{/\sigma + j\omega + 1} - \underline{/\sigma + j\omega - 1} - \underline{/\sigma + j\omega - 0.6065} = 180^\circ$$

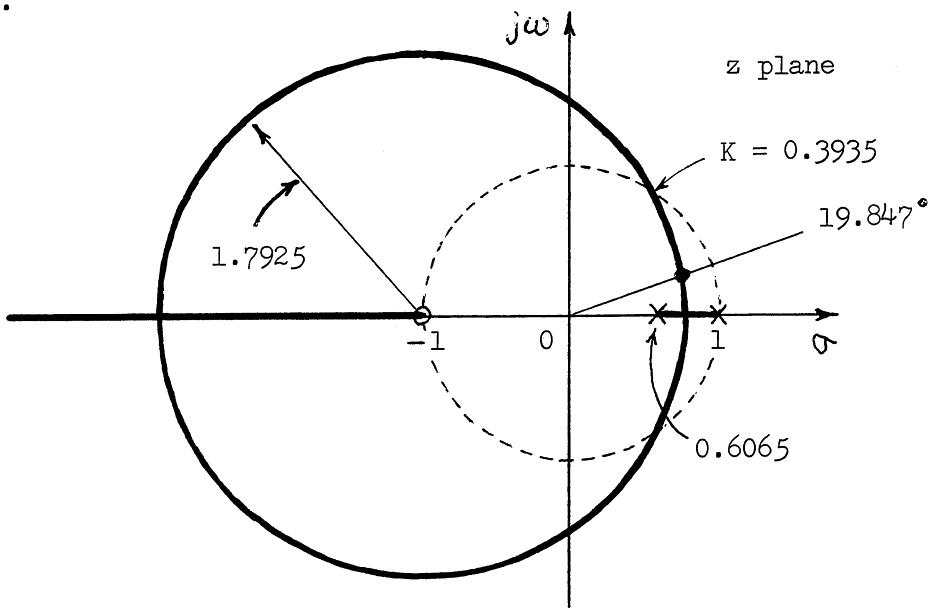
Hence

$$\tan^{-1}\frac{\omega}{\sigma + 1} - \tan^{-1}\frac{\omega}{\sigma - 1} = 180^\circ + \tan^{-1}\frac{\omega}{\sigma - 0.6065}$$

Taking the tangent of both sides of this equation and simplifying, we get

$$\omega = 0 \qquad \text{and} \qquad (\sigma + 1)^2 + \omega^2 = (1.7925)^2$$

Thus, the root loci consist of a part of the real axis (between -1 and $-\infty$) and a circle with center at $\sigma = -1$, $\omega = 0$ and the radius equal to 1.7925, as shown below.



The value of gain K that will yield the damping ratio ζ of the closed-loop poles equal to 0.5 can be determined from Equations (4-29) and (4-30). Since T is given as 0.1 sec, we have

$$|z| = e^{-0.1 \times 0.5\omega_n} = e^{-0.05\omega_n}$$

$$\angle z = 0.1 \times \omega_n \sqrt{1 - 0.5^2} = 0.0866\ \omega_n$$

By trial and error we find that the point that corresponds to $\zeta = 0.5$ and $\omega_n = 4$ rad/sec, that is, the point for which

$$|z| = e^{-0.05 \times 4} = 0.8187$$

$$\angle z = 0.0866 \times 4 = 0.3464 \text{ rad} = 19.847^\circ$$

is on the root locus. This point is

$$z = e^{-0.05 \times 4} \angle 19.847^\circ = 0.8187 \angle 19.847^\circ$$

$$= 0.7701 + j0.2780$$

The value of gain K that corresponds to this closed-loop pole is found from the magnitude condition

$$\left| \frac{K(z + 1)}{(z - 1)(z - 0.6065)} \right|_{z = 0.7701 + j0.2780} = 1$$

as follows:

$$K = \frac{0.3606 \times 0.3226}{1.7918} = 0.0649$$

When gain K is set to 0.0649, or K = 0.0649, the damping ratio ζ of the dominant closed-loop poles is 0.5. With this gain value, the damped natural frequency ω_d is found as

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4\sqrt{1 - 0.5^2} = 3.464$$

The number of samples per cycle of the damped sinusoidal oscillation is

$$\frac{360^\circ}{19.847^\circ} = 18.14$$

or 18.14 samples per cycle.

B-4-9.

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{1}{s(s + 1)}\right] = (1 - z^{-1})\ \mathcal{Z}\left[\frac{1}{s^2(s + 1)}\right]$$

$$= \frac{(T - 1 + e^{-T})z^{-1} + (1 - e^{-T} - Te^{-T})z^{-2}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}$$

Since T = 0.1 sec, G(z) becomes

$$G(z) = \frac{0.004837z^{-1}(1 + 0.9674z^{-1})}{(1 - z^{-1})(1 - 0.9048z^{-1})} = \frac{0.004837(z + 0.9674)}{(z - 1)(z - 0.9048)}$$

Since the number of samples per cycle of damped sinusoidal oscillation is specified as 8, one of the dominant closed-loop poles must be on the line having an angle of 45° and passimg through the origin. Thus, the desired dominant closed-loop pole location in the upper half z plane can be determined as the intersection of the line having an angle of 45° and the $\zeta = 0.5$ locus. The equations for the constant ζ locus are

$$|z| = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right)$$

and

$$\angle z = 2\pi\frac{\omega_d}{\omega_s}$$

Hence for $\zeta = 0.5$ and $\angle z = \frac{1}{4}\pi$, we have

$$\frac{\omega_d}{\omega_s} = \frac{1}{8}$$

and

$$|z| = \exp\left(-\frac{2\pi \times 0.5}{\sqrt{1-0.5^2}}\ \frac{1}{8}\right) = \exp\left(-\frac{\pi}{0.866}\ \frac{1}{8}\right) = 0.6354$$

The desired dominant closed-loop pole in the upper half z plane is located at

$$z = 0.6354\ \angle 45^\circ = 0.4493 + j0.4493$$

In order to have a closed-loop pole at this location, we need to add a phase lead angle of 78.59°. The digital controller must give this necessary phase lead angle.

We shall choose the digital controller $G_D(z)$ to be

$$G_D(z) = K\frac{z+\alpha}{z+\beta} = K\frac{z - 0.9048}{z + \beta}$$

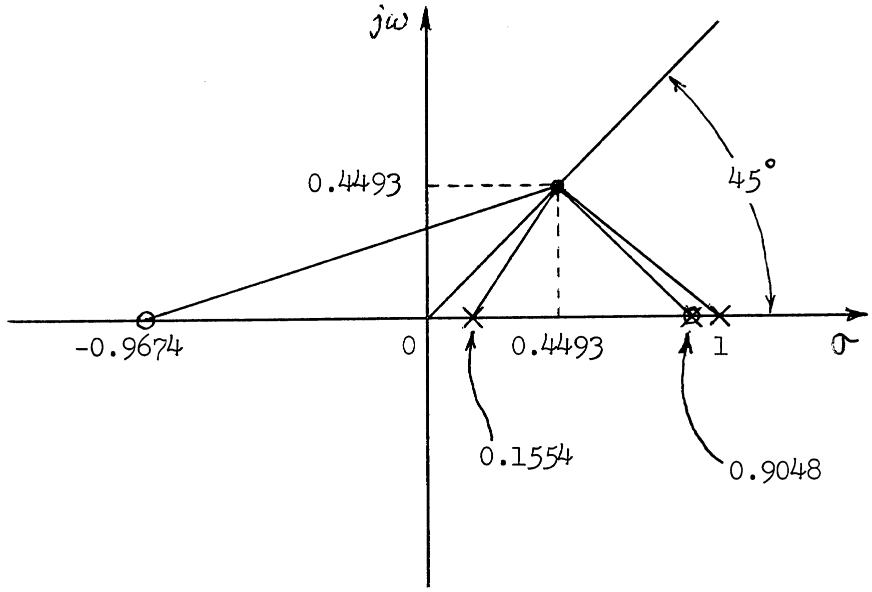
(Here we chose $\alpha = -0.9048$.) Then, from the angle condition we find that the controller pole must be located at $z = 0.1554$, or $\beta = -0.1554$. (See the diagram next page.)

The controller $G_D(z)$ is now given by

$$G_D(z) = K\frac{z - 0.9048}{z - 0.1554}$$

The open-loop pulse transfer function becomes

$$G_D(z)G(z) = K\frac{0.004837(z + 0.9674)}{(z - 0.1554)(z - 1)}$$



Using the magnitude condition, the gain K can be determined as follows:

$$\left| K \frac{0.004837(z + 0.9674)}{(z - 0.1554)(z - 1)} \right|_{z = 0.4493 + j0.4493} = 1$$

or

$$K = 53.08$$

Thus, the digital controller has the following pulse transfer function:

$$G_D(z) = 53.08 \frac{z - 0.9048}{z - 0.1554}$$

The static velocity error constant K_v is determined as follows:

$$K_v = \lim_{z \to 1} \frac{1 - z^{-1}}{0.1} (53.08) \frac{z - 0.9048}{z - 0.1554} \frac{(0.004837)(z + 0.9674)}{(z - 1)(z - 0.9048)}$$

$$= 5.98$$

We shall next obtain the unit-step response sequence. Since the open-loop pulse transfer function is

$$G_D(z)G(z) = 0.2567 \frac{z + 0.9674}{(z - 0.1554)(z - 1)}$$

the closed-loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{0.2567(z + 0.9674)}{(z - 0.1554)(z - 1) + 0.2567(z + 0.9674)}$$

$$= \frac{0.2567z^{-1} + 0.2483z^{-2}}{1 - 0.8987z^{-1} + 0.4037z^{-2}}$$

Since the input is

$$R(z) = \frac{1}{1 - z^{-1}}$$

we obtain

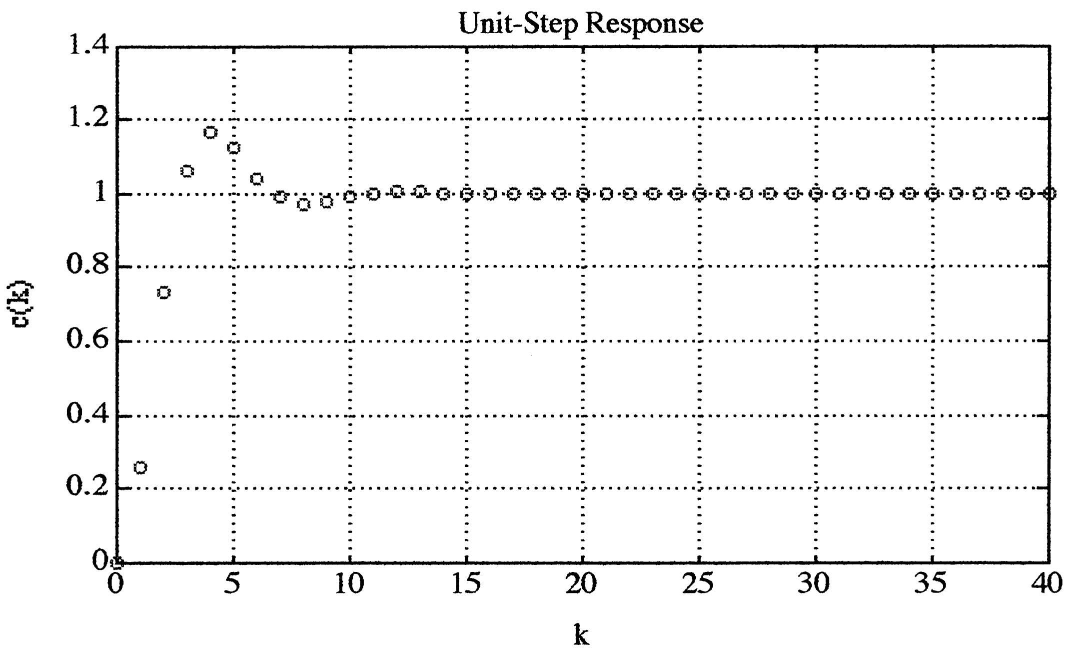
$$C(z) = \frac{0.2567z^{-1} + 0.2483z^{-2}}{(1 - 0.8987z^{-1} + 0.4037z^{-2})(1 - z^{-1})}$$

$$= \frac{0.2567z^{-1} + 0.2483z^{-2}}{1 - 1.8987z^{-1} + 1.3024z^{-2} - 0.4037z^{-3}}$$

$$= 0.2567z^{-1} + 0.7357z^{-2} + 1.0625z^{-3} + 1.1629z^{-4}$$
$$+ 1.1212z^{-5} + 1.0431z^{-6} + 0.9898z^{-7} + 0.9735z^{-8}$$
$$+ 0.9803z^{-9} + 0.9930z^{-10} + 1.0017z^{-11} + 1.0043z^{-12} + \cdots$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-4-9
»
»% ----- Unit-step response -----
»
»num = [0   0.2567   0.2483];
»den = [1   -0.8987   0.4037];
»r = ones(1,41);
»v = [0   40   0   1.4];
»axis(v);
»k = 0:40;
»c = filter(num,den,r);
»plot(k,c,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('c(k)')
```



B-4-10. Assume that the digital controller is of proportional-plus-integral type.

$$G_D(z) = K_P + \frac{K_I}{1 - z^{-1}} = \frac{(K_P + K_I) - K_P z^{-1}}{1 - z^{-1}}$$

$$= \frac{(K_P + K_I)\left(z - \frac{K_P}{K_P + K_I}\right)}{z - 1}$$

Noting that the sampling period T is 0.2 sec, we obtain

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{10}{(s + 1)(s + 5)}\right]$$

$$= (1 - z^{-1}) \mathcal{Z}\left[\frac{10}{s(s + 1)(s + 5)}\right]$$

$$= \frac{0.1372(z + 0.6706)}{(z - 0.8187)(z - 0.3679)}$$

Hence, the open-loop pulse transfer function becomes

$$G_D(z)G(z) = \frac{(K_P + K_I)\left(z - \frac{K_P}{K_P + K_I}\right)}{z - 1} \frac{0.1372(z + 0.6706)}{(z - 0.8187)(z - 0.3679)}$$

Let us choose the controller zero to cancel the plant pole at z = 0.8187, or

$$\frac{K_P}{K_P + K_I} = 0.8187 \qquad (1)$$

Then

$$G_D(z) = (K_P + K_I)\,\frac{z - 0.8187}{z - 1}$$

and

$$G_D(z)G(z) = \frac{(K_P + K_I)(z - 0.8187)}{(z - 1)}\;\frac{0.1372(z + 0.6706)}{(z - 0.8187)(z - 0.3679)}$$

Referring to the root-locus plot for this system shown on next page, the circular locus intersects the $\zeta = 0.5$ locus at point P, where

$$z = 0.7099 \,\angle 34^\circ$$

$$= 0.5885 + j0.3970$$

The number of samples per cycle of sinusoidal oscillation is

$$\frac{360^\circ}{34^\circ} = 10.59$$

Since this number is greater than 8, the requirement is satisfied. Thus, point P is satisfactory as a closed-loop pole location.

The magnitude of gain $K_P + K_I$ can be determined from the magnitude condition.

$$\left|\frac{(K_P + K_I)(0.1372)(z + 0.6706)}{(z - 1)(z - 0.3679)}\right|_{z = 0.5885 + j0.3970} = 1$$

or

$$K_P + K_I = 1.4337 \qquad (2)$$

From Equations (1) and (2), we have

$$K_P = 1.1738 \quad \text{and} \quad K_I = 0.2599$$

Hence the open-loop pulse transfer function becomes

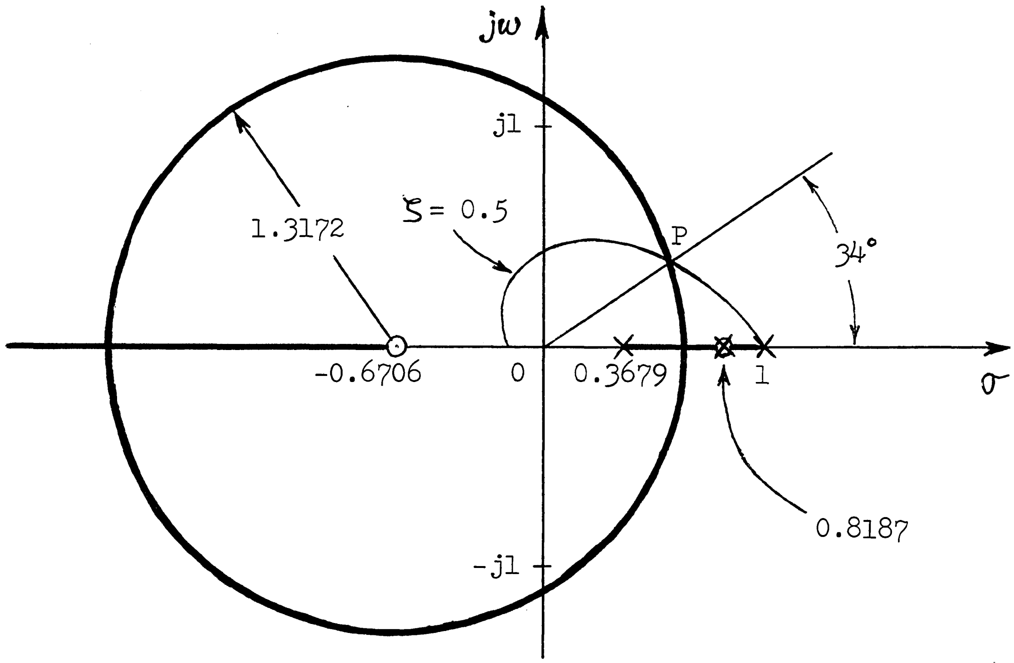
$$G_D(z)G(z) = 1.4337\,\frac{z - 0.8187}{z - 1}\;\frac{0.1372(z + 0.6706)}{(z - 0.8187)(z - 0.3679)}$$

$$= 0.1967\,\frac{z^{-1}(1 + 0.6706z^{-1})}{(1 - z^{-1})(1 - 0.3679z^{-1})}$$

The static velocity error constant K_v is obtained as

$$K_v = \lim_{z \to 1} \frac{1 - z^{-1}}{0.2}\,(0.1967)\,\frac{z^{-1}(1 + 0.6706z^{-1})}{(1 - z^{-1})(1 - 0.3679z^{-1})}$$

$$= 2.599$$



B-4-11. Let us choose the sampling period T to be 5 sec. Then

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{e^{-5s}}{s + 0.4}\right] = (1 - z^{-1}) z^{-1} \mathcal{Z}\left[\frac{1}{s(s + 0.4)}\right]$$

$$= \frac{2.1617}{z(z - 0.1353)}$$

The PI controller has the following pulse transfer function:

$$G_D(z) = K_P + \frac{K_I}{1 - z^{-1}} = (K_P + K_I)\,\frac{z - \dfrac{K_P}{K_P + K_I}}{z - 1}$$

The location of the dominant closed-loop pole in the upper half z plane may be determined from

$$|z| = \exp\left(-\frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \frac{\omega_d}{\omega_s}\right)$$

$$\angle z = 2\pi \frac{\omega_d}{\omega_s}$$

For $\zeta = 0.5$ and $\angle z = 2\pi/10$ we have

$$\frac{\omega_d}{\omega_s} = \frac{1}{10}$$

Hence

$$|z| = \exp\left(-\frac{2\pi \times 0.5}{\sqrt{1 - 0.5^2}}\,\frac{1}{10}\right) = \exp(-0.3628) = 0.6958$$

Thus the desired location for the dominant closed-loop pole in the upper half z plane is

$$z = 0.6958 \angle 36^\circ = 0.5629 + j0.4090$$

The open-loop pulse transfer function is

$$G_D(z)G(z) = (K_P + K_I)\,\frac{z - \dfrac{K_P}{K_P + K_I}}{z - 1}\,\frac{2.1617}{z(z - 0.1353)}$$

The angle contributions of the poles at $z = 0$, $z = 0.1353$, and $z = 1$ at the closed-loop pole at $z = 0.5629 + j0.4090$ are -36.002°, -43.726°, and -136.902°, respectively. Thus, the total angle contribution becomes

$$-36.002^\circ - 43.726^\circ - 136.902^\circ = -216.63^\circ$$

Hence, the angle deficiency is 36.63°. To add the phase lead angle of 36.63° to the system, we need to choose the zero of the controller at $z = 0.0127$. That is, we choose

$$\frac{K_P}{K_P + K_I} = 0.0127 \tag{1}$$

Then, the open-loop pulse transfer function becomes

$$G_D(z)G(z) = (K_P + K_I)\,\frac{z - 0.0127}{z - 1}\,\frac{2.1617}{z(z - 0.1353)}$$

and the magnitude condition becomes

$$\left|(K_P + K_I)\,\frac{z - 0.0127}{z - 1}\,\frac{2.1617}{z(z - 0.1353)}\right|_{z = 0.5629 + j0.4090} = 1$$

or

$$K_P + K_I = 0.1663 \tag{2}$$

From Equations (1) and (2) we find

$$K_P = 0.002 \qquad \text{and} \qquad K_I = 0.1643$$

The open-loop pulse transfer function now becomes

$$G_D(z)G(z) = 0.3595\,\frac{z - 0.0127}{(z - 1)z(z - 0.1353)}$$

and the closed-loop pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{0.3595(z - 0.0127)}{(z - 1)z(z - 0.1353) + 0.3595(z - 0.0127)}$$

$$= \frac{0.3595z - 0.004566}{z^3 - 1.1353z^2 + 0.4948z - 0.004566}$$

$$= \frac{0.3595z^{-2} - 0.004566z^{-3}}{1 - 1.1353z^{-1} + 0.4948z^{-2} - 0.004566z^{-3}}$$

For the unit-step input

$$R(z) = \frac{1}{1 - z^{-1}}$$

we have

$$C(z) = \frac{0.3595z^{-2} - 0.004566z^{-3}}{(1 - 1.1353z^{-1} + 0.4948z^{-2} - 0.004566z^{-3})(1 - z^{-1})}$$

$$= \frac{0.3595z^{-2} - 0.004566z^{-3}}{1 - 2.1353z^{-1} + 1.6301z^{-2} - 0.4994z^{-3} + 0.004566z^{-4}}$$

$$= 0.3595z^{-2} + 0.7631z^{-3} + 1.0434z^{-4} + 1.1636z^{-5}$$
$$+ 1.1631z^{-6} + 1.1045z^{-7} + 1.0386z^{-8} + 0.9929z^{-9}$$
$$+ 0.9733z^{-10} + 0.9734z^{-11} + 0.9830z^{-12} + 0.9937z^{-13}$$
$$+ 1.0012z^{-14} + \cdots$$

<u>B-4-12</u>.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 4\sqrt{1 - 0.5^2} = 3.4641$$

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.1} = 62.832$$

Thus

$$\frac{\omega_d}{\omega_s} = 0.05513$$

The dominant closed-loop pole in the upper half z plane is at

$$|z| = \exp\left(-\frac{2\pi \times 0.5}{\sqrt{1 - 0.5^2}} \times 0.05513\right) = \exp(-0.2000) = 0.8187$$

$$\angle z = 2\pi\frac{\omega_d}{\omega_s} = 2\pi \times 0.05513 = 0.3464 \text{ rad} = 19.847°$$

The PD controller has the following pulse transfer function:

$$G_D(z) = K_P + K_D(1 - z^{-1})$$

$$= (K_P + K_D)\,\frac{z - \dfrac{K_D}{K_P + K_D}}{z}$$

The pulse transfer function of the plant is

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,\frac{1}{s^2}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{1}{s^3}\right]$$

$$= \frac{(0.1)^2 z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^2} = 0.005\,\frac{z + 1}{(z - 1)^2}$$

Thus, the open-loop pulse transfer function is

$$G_D(z)G(z) = (K_P + K_D)\,\frac{z - \dfrac{K_D}{K_P + K_D}}{z}\,(0.005)\,\frac{z + 1}{(z - 1)^2}$$

The location of the desired closed-loop pole in the upper half z plane is

$$z = 0.8187\,\underline{/19.847^\circ} = 0.7701 + j0.2780$$

The total angle contribution from the zero at $z = -1$, the pole at $z = 0$, and the double pole at $z = 1$ is -270.10°. Hence, the controller zero must contribute 90.10°. This requires that the zero be located at $z = 0.7719$. That is,

$$\frac{K_D}{K_P + K_D} = 0.7719 \qquad (1)$$

The open-loop pulse transfer function becomes

$$G_D(z)G(z) = (K_P + K_D)\,\frac{z - 0.7719}{z}\,(0.005)\,\frac{z + 1}{(z - 1)^2}$$

The magnitude condition is

$$\left|(K_P + K_D)\,\frac{z - 0.7719}{z}\,(0.005)\,\frac{z + 1}{(z - 1)^2}\right|_{z = 0.7701 + j0.2780}$$

$$= 1$$

or

$$K_P + K_D = 42.779 \qquad (2)$$

From Equations (1) and (2) we have

$$K_P = 9.758 \quad \text{and} \quad K_D = 33.021$$

and the desired digital controller is given by

$$G_D(z) = 42.779 \frac{z - 0.7719}{z}$$

$$= 42.779(1 - 0.7719z^{-1})$$

The number of samples per cycle of damped sinusoidal oscillation is

$$n = \frac{360°}{19.847°} = 18.14$$

B-4-13. In order to increase the value of static velocity error constant K_v to 12 $\sec^{-1}$, we modify $G_D(z)$ as follows:

$$\hat{G}_D(z) = K \frac{z - 0.8187}{z - 0.1595} \frac{z - 0.96}{z - 0.99}$$

Since the added pole and zero are close together, the closed-loop pole locations will not be changed very much. The open-loop pulse transfer function becomes

$$\hat{G}_D(z)G(z) = K \frac{(z - 0.96)(0.01873)(z + 0.9356)}{(z - 0.1595)(z - 0.99)(z - 1)}$$

The magnitude condition is

$$\left| K \frac{(z - 0.96)(0.01873)(z + 0.9356)}{(z - 0.1595)(z - 0.99)(z - 1)} \right|_{z = 0.4493 + j0.4493} = 1$$

from which we obtain

$$K = 14.40$$

Thus

$$\hat{G}_D(z) = 14.40 \frac{1 - 0.8187z^{-1}}{1 - 0.1595z^{-1}} \frac{1 - 0.96z^{-1}}{1 - 0.99z^{-1}}$$

The static velocity error constant K_v is obtained as

$$K_v = \lim_{z \to 1} \frac{1 - z^{-1}}{T} \hat{G}_D(z)G(z)$$

$$= \lim_{z \to 1} \frac{1 - z^{-1}}{0.2} (14.40) \frac{(1 - 0.96z^{-1})(0.01873)(1 + 0.9356z^{-1})}{(1 - 0.1595z^{-1})(1 - 0.99z^{-1})(1 - z^{-1})}$$

$$= 12.42$$

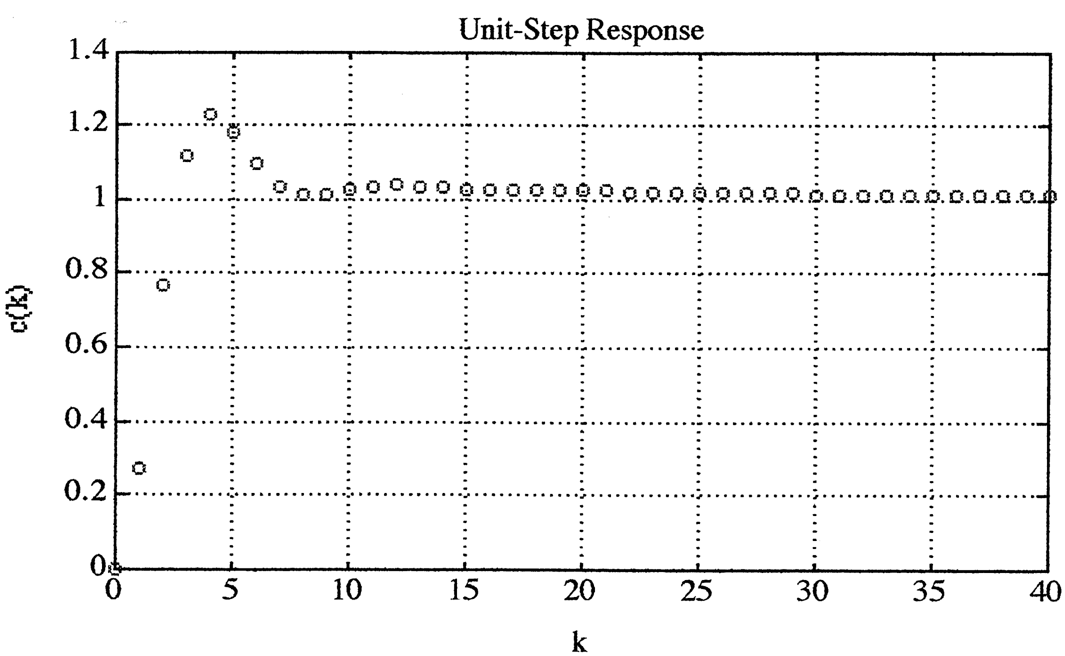
This value of K_v is satisfactory.

The closed-loop pulse transfer function can be obtained as

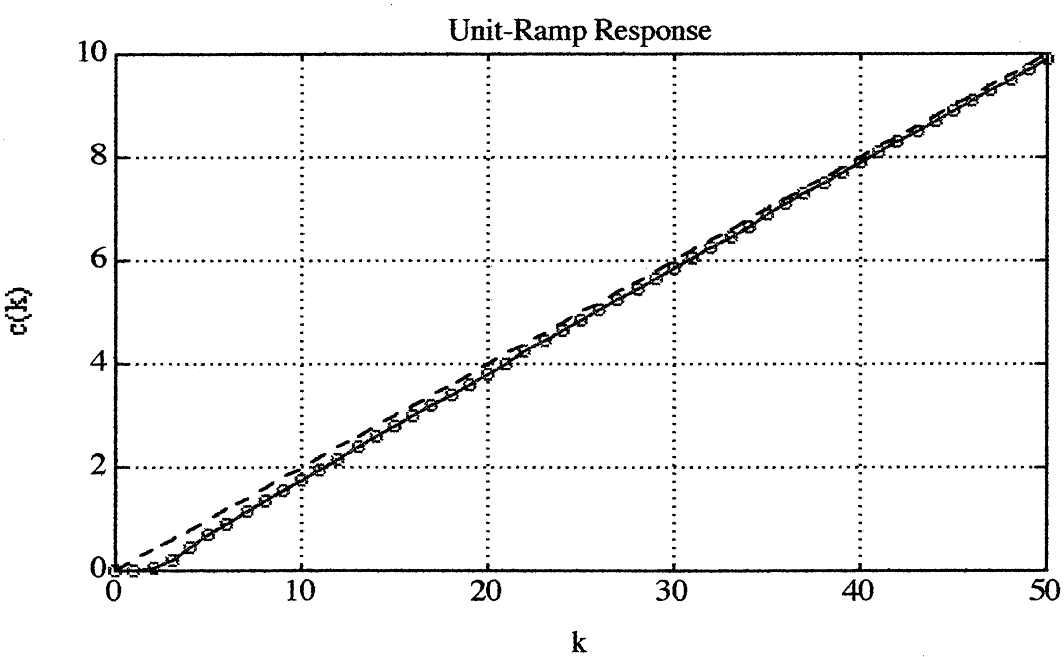
$$\frac{C(z)}{R(z)} = \frac{0.2697z^{-1} - 0.0066z^{-2} - 0.2422z^{-3}}{1 - 1.8798z^{-1} + 1.3008z^{-2} - 0.4001z^{-3}}$$

The unit-step response and unit-ramp response of the system can be obtained by use of MATLAB as shown below.

```
»% MATLAB Program for Problem B-4-13 (Part 1)
»
»% ----- Unit-step response -----
»
»num =[0  0.2697  -0.0066  -0.2422];
»den = [1  -1.8798  1.3008  -0.4001];
»r = ones(1,41);
»v = [0  40  0  1.4];
»axis(v);
»k = 0:40;
»c = filter(num,den,r);
»plot(k,c,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('c(k)')
```



```
»% MATLAB Program for Problem B-4-13 (Part 2)
»
»% ----- Unit-ramp response -----
»
»num = [0   0.2697   -0.0066   -0.2422];
»den = [1   -1.8798   1.3008   -0.4001];
»v = [0   50   0   10];
»axis(v);
»k = 0:50;
»r = [0.2*k];
»c = filter(num,den,r);
»plot(k,c,'o',k,c,'-',k,0.2*k,'--')
»grid
»title('Unit-Ramp Response')
»xlabel('k')
»ylabel('c(k)')
```



B-4-14.

$$\hat{G}(z) = \mathcal{Z}[KG(s)] = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{K}{s(s + 10)}\right] = (1 - z^{-1})\mathcal{Z}\left[\frac{K}{s^2(s + 10)}\right]$$

$$= 0.01\ K \frac{0.2642z^{-2} + 0.3679z^{-1}}{(1 - z^{-1})(1 - 0.3679z^{-1})} = \frac{0.3679\ K(z + 0.7181)}{100(z - 1)(z - 0.3679)}$$

Since T = 0.1, we have

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

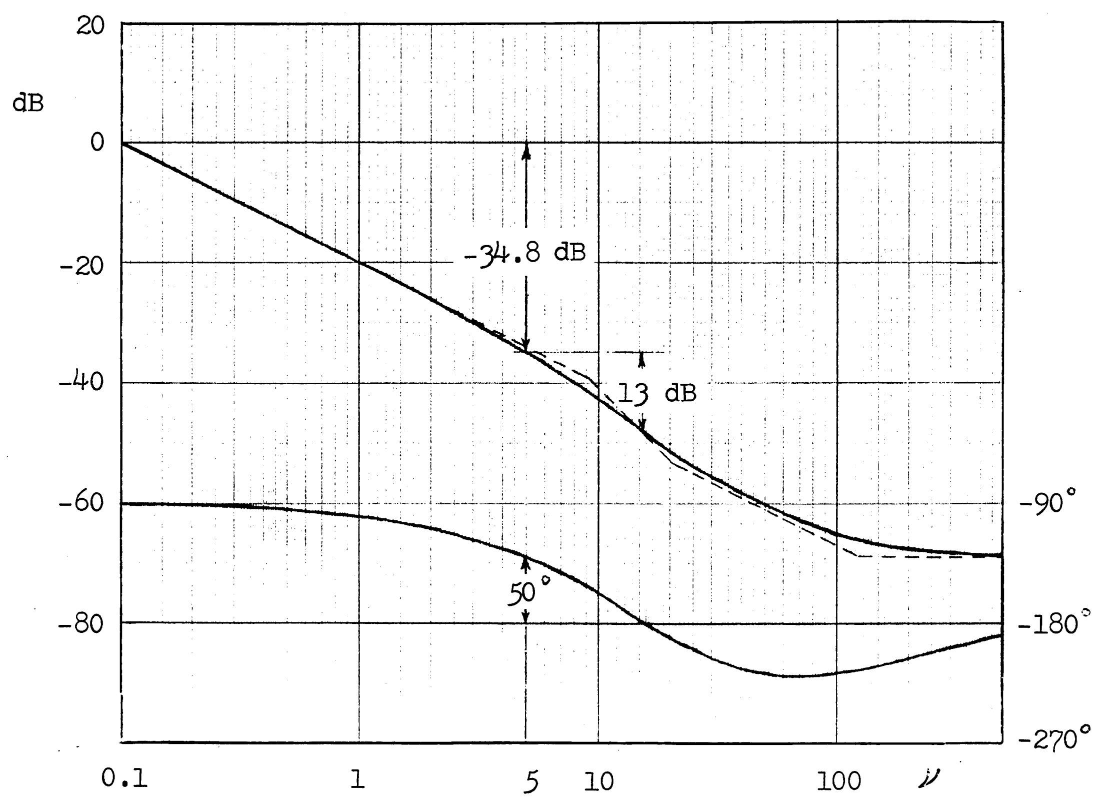
Then, $\hat{G}(w)$ becomes as follows:

$$\hat{G}(w) = \frac{0.3679\ K\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.7181\right)}{100\left(\frac{1 + 0.05w}{1 - 0.05w} - 1\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.3679\right)}$$

$$= \frac{0.1\ K(1 - 0.05w)(0.0082w + 1)}{w(0.1082w + 1)}$$

$$= \frac{0.1\ K\ (1 - \frac{1}{20}\ w)(\frac{1}{121.94}\ w + 1)}{w(\frac{1}{9.2421}\ w + 1)}$$

The Bode diagram of $\hat{G}(j\nu)$ with K = 1 is shown below. At ν = 5 rad/sec the phase angle is $-130°$ and the magnitude $|\hat{G}(j5)|$ is -34.8 dB. Hence, to obtain the phase margin of $50°$, we need to increase the magnitude of $\hat{G}(j5)$ by 34.8 dB. (That is, the entire magnitude curve must be raised by 34.8 dB.)



Thus, we require that the gain K be set such that

$$20 \log K \text{ dB} = 34.8 \text{ dB}$$

or

$$K = 55.0$$

With this gain value, the gain margin is 13 dB. The static velocity error constant K_v is obtained as

$$K_v = \lim_{w \to 0} w\hat{G}(w) = \lim_{w \to 0} w \frac{5.50(1 - 0.05w)(0.0082w + 1)}{w(0.1082w + 1)} = 5.50$$

B-4-15.

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{K}{s(s + 0.5)}\right] = (1 - z^{-1}) \mathcal{Z}\left[\frac{K}{s^2(s + 0.5)}\right]$$

$$= K \frac{0.004918z^{-1} + 0.004836z^{-2}}{(1 - z^{-1})(1 - 0.9512z^{-1})}$$

$$= 0.004918 \ K \frac{z + 0.9835}{(z - 1)(1 - 0.9512)}$$

Noting that T = 0.1 sec, we have

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

Hence

$$G(w) = 0.004918 \frac{K\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.9835\right)}{\left(\frac{1 + 0.05w}{1 - 0.05w} - 1\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9512\right)}$$

$$= \frac{2K(1 - \frac{1}{20} w)(1 + \frac{1}{2404} w)}{w(1 + \frac{1}{0.5002} w)}$$

Assume that the controller $G_D(w)$ has the unity gain at w = 0, or

$$G_D(0) = 1$$

Then, using the requirement that $K_v = 20 \text{ sec}^{-1}$, we determine gain K.

$$K_v = \lim_{w \to 0} wG_D(w)G(w)$$

$$= \lim_{w \to 0} wG_D(w) \frac{2K(1 - \frac{1}{20} w)(1 + \frac{1}{2404} w)}{w(1 + \frac{1}{0.5002} w)} = 2K = 20$$

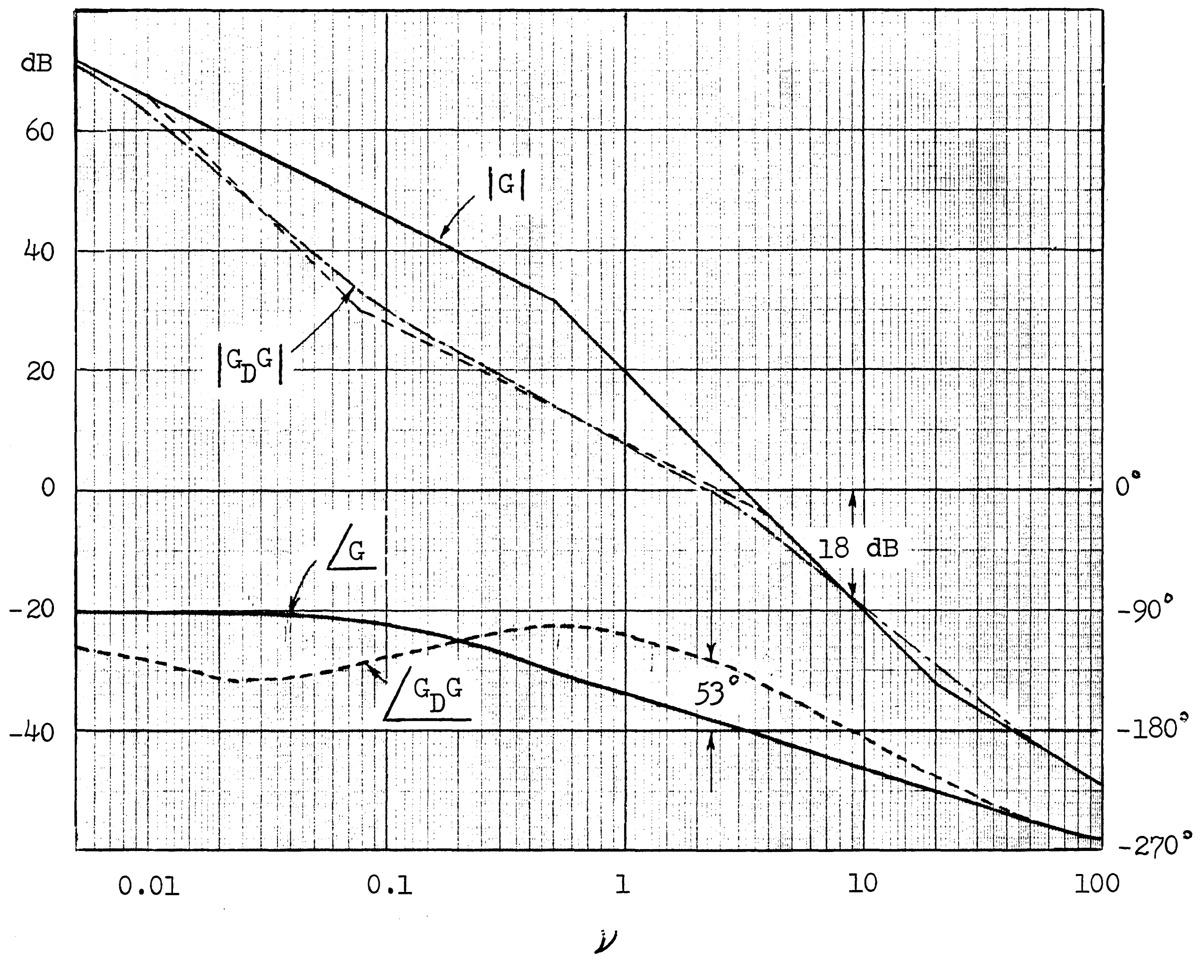
Hence

$$K = 10$$

A Bode diagram of

$$G(w) = \frac{20(1 - \frac{1}{20} w)(1 + \frac{1}{2404} w)}{w(1 + \frac{1}{0.5002} w)}$$

is shown below.



By use of the conventional design technique, we find that the following lag-lead network will satisfy the requirements:

$$G_D(w) = \frac{(1 + \frac{1}{0.08} w)}{(1 + \frac{1}{0.01} w)} \frac{(1 + \frac{1}{0.5} w)}{(1 + \frac{1}{4} w)}$$

The gain crossover frequency is $\nu = 2.3$ rad/sec. The phase margin is approximately 53° and the gain margin is 18 dB.

Next, we transform $G_D(w)$ into $G_D(z)$. Since

$$w = \frac{2}{T} \frac{z - 1}{z + 1} = 20 \frac{z - 1}{z + 1}$$

we have

$$G_D(z) = \frac{(1 + \frac{1}{0.08} 20 \frac{z - 1}{z + 1})(1 + \frac{1}{0.5} 20 \frac{z - 1}{z + 1})}{(1 + \frac{1}{0.01} 20 \frac{z - 1}{z + 1})(1 + \frac{1}{4} 20 \frac{z - 1}{z + 1})}$$

$$= 0.8572 \frac{(z - 0.9920)(z - 0.9512)}{(z - 0.9990)(z - 0.6667)}$$

Noting that

$$G(z) = 0.04918 \frac{z + 0.9833}{(z - 1)(z - 0.9512)}$$

we have

$$G_D(z)G(z) = 0.04216 \frac{(z - 0.9920)(z + 0.9833)}{(z - 0.9990)(z - 0.6667)(z - 1)}$$

The characteristic equation of the closed-loop system is

$$(z - 0.9990)(z - 0.6667)(z - 1) + 0.04216(z - 0.9920)(z + 0.9833)$$
$$= 0$$

This is a third degree equation. One root is located near $z = 0.999$. The other two roots are obtained from

$$z^2 - 1.6245z + 0.7080 = 0$$

Thus, the dominant closed-loop poles, which are the roots of this last equation, are located at

$$z = 0.812 \pm j0.220 = 0.841 \angle 15.2°$$

Hence, the number of samples per cycle of damped sinusoidal oscillations is

$$\frac{360°}{15.2°} = 23.7$$

B-4-16. For $T = 0.1$ sec, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{5}{(s + 1)(s + 2)}\right] = (1 - z^{-1}) \mathcal{Z}\left[\frac{5}{s(s + 1)(s + 2)}\right]$$

$$= \frac{0.02263(z + 0.9061)}{(z - 0.9048)(z - 0.8187)}$$

Using the transformation

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

we have

$$G(w) = \frac{0.02263\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.9061\right)}{\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9048\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.8187\right)}$$

$$= \frac{2.500\left(1 - \frac{1}{20} w\right)\left(1 + \frac{1}{406} w\right)}{(1 + w)\left(1 + \frac{1}{1.994} w\right)}$$

Notice that in order to have the static velocity error constant $K_v = 5 \text{ sec}^{-1}$, we need the controller $G_D(w)$ to include an integrator.

Using the conventional design approach, we find the following $G_D(w)$ will satisfy the requirements that the phase margin be 60°, the gain margin be not less than 12 db, and K_v be equal to 5 sec^{-1}.

$$G_D(w) = \frac{2}{w}\left(\frac{1 + \frac{1}{0.1} w}{1 + \frac{1}{0.01} w}\right)\left(\frac{1 + w}{1 + \frac{1}{10} w}\right)$$

Then the open-loop pulse transfer function becomes

$$G_D(w)G(w) = \frac{5}{w} \frac{\left(1 + \frac{1}{0.1} w\right)\left(1 - \frac{1}{20} w\right)\left(1 + \frac{1}{406} w\right)}{\left(1 + \frac{1}{0.01} w\right)\left(1 + \frac{1}{10} w\right)\left(1 + \frac{1}{1.994} w\right)}$$

From the Bode diagram of $G_D(w)G(w)$ (see next page), we find the phase margin to be approximately 60° and the gain margin to be approximately 22 dB. The gain crossover frequency is $\nu = 0.5$ rad/sec. The phase crossover frequency is $\nu = 3.5$ rad/sec.

Next, using the following transformation:

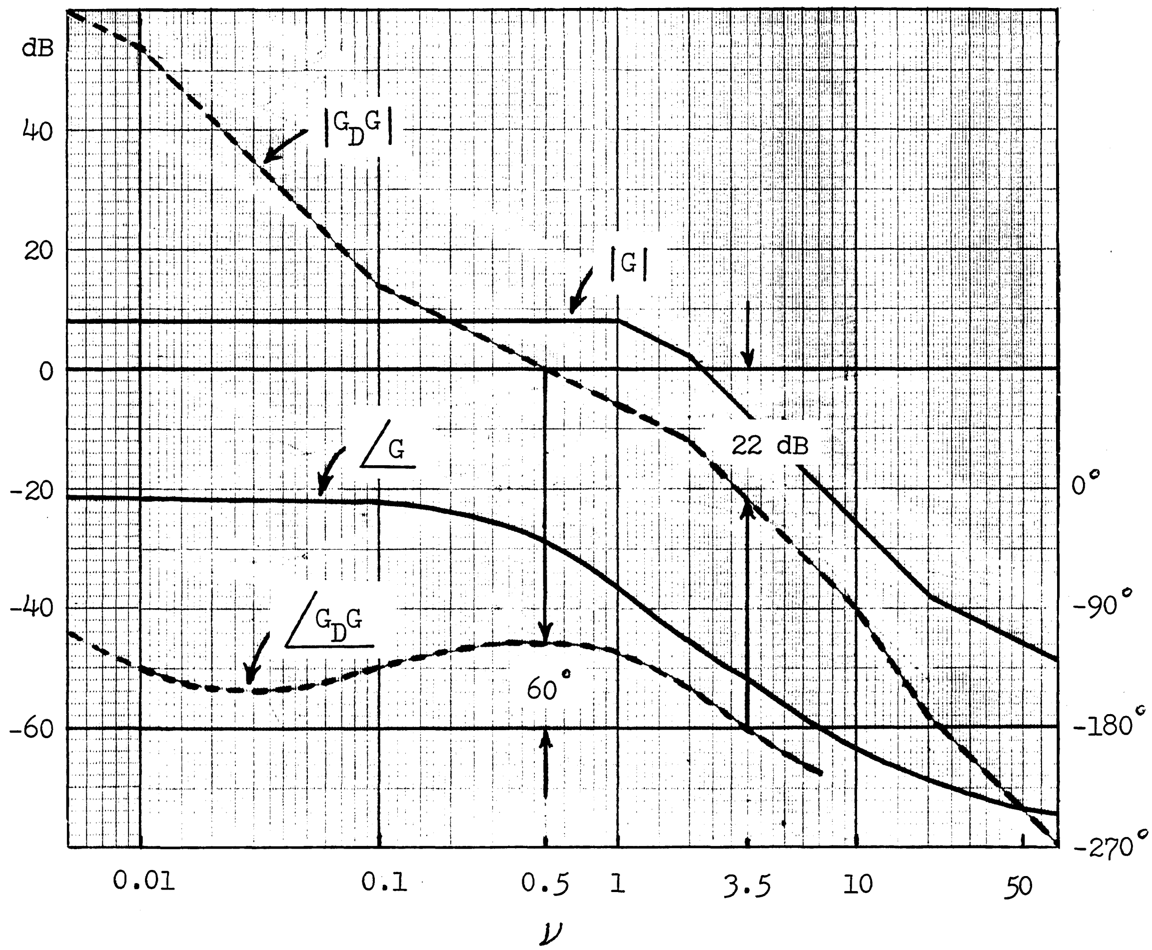
$$w = \frac{2}{0.1} \frac{z - 1}{z + 1} = 20 \frac{z - 1}{z + 1}$$

we obtain $G_D(z)$ as follows:

$$G_D(z) = \frac{2\left(1 + \frac{1}{0.1} 20 \frac{z - 1}{z + 1}\right)\left(1 + 20 \frac{z - 1}{z + 1}\right)}{20\left(\frac{z - 1}{z + 1}\right)\left(1 + \frac{1}{0.01} 20 \frac{z - 1}{z + 1}\right)\left(1 + \frac{1}{10} 20 \frac{z - 1}{z + 1}\right)}$$

$$= 0.07035 \frac{(z + 1)(z - 0.9900)(z - 0.9048)}{(z - 1)(z - 0.9990)(z - 0.3333)}$$

The digital controller $G_D(z)$ defined by this last equation satisfies all the requirements of the problem and is, therefore, satisfactory.



B-4-17. Noting that T = 0.1 sec, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{K(2s + 1)}{s(s + 1)(0.2s + 1)}\right]$$

$$= (1 - z^{-1}) \mathcal{Z}\left[\frac{K(2s + 1)}{s^2(s + 1)(0.2s + 1)}\right]$$

$$= 0.0419\, K \frac{(z - 0.9512)(z + 0.8328)}{(z - 1)(z - 0.9048)(z - 0.6065)}$$

Using the transformation

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

we obtain

$$G(w) = 0.0419 \frac{K\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9512\right)\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.8328\right)}{\left(\frac{1 + 0.05w}{1 - 0.05w} - 1\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9048\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.6065\right)}$$

$$= \frac{K\left(1 - \frac{1}{20}w\right)\left(1 + \frac{1}{0.5}w\right)\left(1 + \frac{1}{219.2}w\right)}{w(1 + w)\left(1 + \frac{1}{4.90}w\right)}$$

Assume that the digital controller $G_D(w)$ to be designed here has the low-frequency gain of unity, or

$$G_D(0) = 1$$

The requirement that the static velocity error constant be 10 sec^{-1} determines the value of gain K.

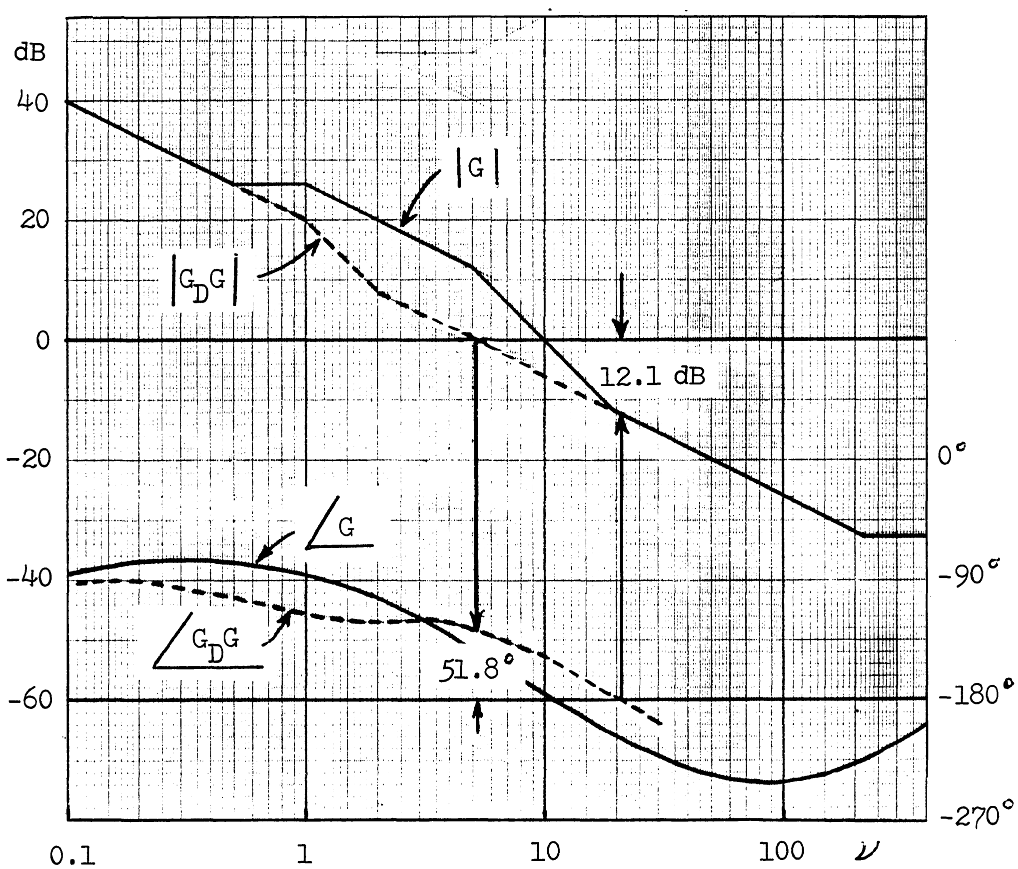
$$K_v = \lim_{w \to 0} wG_D(w)G(w) = K = 10$$

Thus

$$K = 10$$

A Bode diagram for the following G(w) is shown below.

$$G(w) = \frac{10\left(1 + \frac{1}{0.5}w\right)\left(1 - \frac{1}{20}w\right)\left(1 + \frac{1}{219.2}w\right)}{w(1 + w)\left(1 + \frac{1}{4.90}w\right)}$$



Using the conventional design approach we find the following $G_D(w)$ will satisfy all the requirements of the system:

$$G_D(w) = \left(\frac{1 + \frac{1}{2}w}{1 + \frac{1}{0.5}w}\right)\left(\frac{1 + \frac{1}{4.90}w}{1 + \frac{1}{19.6}w}\right)$$

Noting that

$$w = \frac{2}{0.1}\frac{z-1}{z+1} = 20\,\frac{z-1}{z+1}$$

we obtain

$$G_D(z) = \frac{\left(1 + \frac{1}{2}\,20\,\frac{z-1}{z+1}\right)\left(1 + \frac{1}{4.90}\,20\,\frac{z-1}{z+1}\right)}{\left(1 + \frac{1}{0.5}\,20\,\frac{z-1}{z+1}\right)\left(1 + \frac{1}{19.6}\,20\,\frac{z-1}{z+1}\right)}$$

$$= 0.6748\,\frac{(z - 0.8182)(z - 0.6065)}{(z - 0.9512)(z - 0.0101)}$$

The gain crossover frequency is $\nu = 5.2$ rad/sec and the phase margin is 51.8°. The phase crossover frequency is $\nu = 21$ rad/sec and the gain margin is 12.1 dB. Also, $K_v = 10\ \text{sec}^{-1}$. Thus, all requirements are met. Since

$$G_D(z)G(z) = 0.2827\,\frac{(z - 0.8182)(z + 0.8328)}{(z - 0.0101)(z - 1)(z - 0.9048)}$$

the characteristic equation for the closed-loop system is

$$(z - 0.0101)(z - 1)(z - 0.9048) + 0.2827(z - 0.8182)(z + 0.8328) = 0$$

or

$$z^3 - 1.6322z^2 + 0.9282z - 0.2018 = 0$$

which can be factored as follows:

$$(z - 0.761)(z - 0.4361 + j0.2741)(z - 0.4361 - j0.2741) = 0$$

Notice that the real closed-loop pole at $z = 0.761$ is close to a zero at $z = 0.8182$. The zero nearby at the closed-loop pole effectively cancel the effects of this closed-loop pole. Therefore, a pair of complex conjugate poles at $z = 0.4361 \pm j0.2741$ can be considered dominant. The closed-loop pole at

$$z = 0.4361 + j0.2741 = 0.5151\,\underline{/32.15^\circ}$$

is located on a line having an angle of 32.15°. Hence, the number of samples per cycle of damped oscillations is

$$\frac{360^\circ}{32.15^\circ} = 11.2$$

B-4-18. Since T = 1 sec, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} G_p(s)\right] = (1 - z^{-1})\, \mathcal{Z}\left[\frac{1}{s(s + 2)}\right]$$

$$= \frac{0.4323z^{-1}}{1 - 0.1353z^{-1}}$$

Define the closed-loop pulse transfer function as F(z), or

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = F(z) \tag{1}$$

If G(z) is expanded into a series in z^{-1}, then the first term is $0.4323z^{-1}$. Hence, F(z) must begin with a term in z^{-1}, or

$$F(z) = a_1z^{-1} + a_2z^{-2} + \cdots + a_Nz^{-N} \tag{2}$$

where $N \geqslant n$ and n is the order of the system.

Since the input is a unit-step function, from Equation (4-48) we have

$$1 - F(z) = (1 - z^{-1})N(z)$$

Notice that G(z) involves neither zero nor pole outside the unit circle. Therefore, there is no requirement on 1 - F(z) from the stability viewpoint.

Since the system should not exhibit intersampling ripples after steady-state is reached, we require U(z) to be of the following type of series in z^{-1}:

$$U(z) = b_0 + b_1z^{-1} + \cdots + b_{N-1}z^{-N+1} + b(z^{-N} + z^{-N-1} + \cdots)$$

Because the plant transfer function $G_p(s)$ does not involve an integrator, b must not be zero. From Figure 4-75,

$$U(z) = \frac{C(z)}{G(z)} = \frac{C(z)}{R(z)} \frac{R(z)}{G(z)} = F(z) \frac{R(z)}{G(z)}$$

$$= F(z) \frac{1}{1 - z^{-1}} \frac{1 - 0.1353z^{-1}}{0.4323z^{-1}} \tag{3}$$

Since U(z) should be of an infinite series, F(z) should not be divisible by $1 - z^{-1}$.

In the absence of other requirements on F(z), we may choose N(z) = 1, or

$$1 - F(z) = 1 - z^{-1}$$

Then

$$F(z) = z^{-1} \tag{4}$$

Thus, in Equation (2), $a_1 = 1$, $a_2 = a_3 = \cdots = a_N = 0$. Clearly, $F(z)$ is not divisible by the factor $1 - z^{-1}$.

From Equation (1) we obtain

$$G_D(z) = \frac{F(z)}{G(z)\left[1 - F(z)\right]} = \frac{z^{-1}}{\dfrac{0.4323z^{-1}}{1 - 0.1353z^{-1}}(1 - z^{-1})}$$

$$= 2.3132 \frac{1 - 0.1353z^{-1}}{1 - z^{-1}}$$

Note that from Equations (3) and (4) we have

$$U(z) = 2.3132 \frac{1 - 0.1353z^{-1}}{1 - z^{-1}}$$

$$= 2.3132 + 2(z^{-1} + z^{-2} + z^{-3} + \cdots)$$

The sequence $u(k)$ in the unit-step response is constant for $k = 1, 2, 3, \ldots$. The system output stays constant at unity and there is no intersampling ripples after the settling time is reached.

CHAPTER 5

B-5-1.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

B-5-2.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

B-5-3.

$$\frac{Y(z)}{U(z)} = \frac{1 + 6z^{-1} + 8z^{-2}}{1 + 4z^{-1} + 3z^{-2}} = \frac{z^2 + 6z + 8}{z^2 + 4z + 3} = 1 + \frac{1.5}{z+1} + \frac{0.5}{z+3}$$

Referring to Equation (5-112), we have $p_1 = -1$, $p_2 = -3$, $b_0 = 1$, $c_1 = 1.5$, and $c_2 = 0.5$. Hence

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$

B-5-4.

$$\frac{Y(z)}{U(z)} = \frac{z + 2}{z^2 + z + 0.16} = \frac{z^{-1} + 2z^{-2}}{1 + z^{-1} + 0.16z^{-2}}$$

Direct programming:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Nested programming:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.16 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Partial-fraction-expansion programming:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

B-5-5.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_3 + a_3 b_0 & \vdots & b_2 - a_2 b_0 & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + b_0 u(k)$$

B-5-6. From the block diagram we obtain

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = x_3(k)$$

$$x_3(k+1) = -0.2x_1(k) - x_2(k) - 0.5x_3(k) + u(k)$$

and

$$y(k) = 0.6x_1(k) + 2x_2(k) + x_3(k) + 2u(k)$$

Hence,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & -1 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [0.6 \quad 2 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + 2u(k)$$

(This is in a controllable canonical form.)

B-5-7. From the block diagram we obtain

$$y(k) = h(0)u(k) + h(1)x_1(k) + h(2)x_2(k) + \cdots + h(n)x_n(k)$$

and

$$x_1(k+1) = u(k)$$

$$x_2(k+1) = x_1(k)$$

$$x_3(k+1) = x_2(k)$$

$$\vdots$$

$$x_n(k+1) = x_{n-1}(k)$$

Thus, the state-space representation for the system becomes

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [h(1) \quad h(2) \quad \cdots \quad h(n)] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + h(0)u(k)$$

B-5-8. From Figure 5-14 we obtain the following discrete-time state space equations:

$$x_1(k+1) = x_2(k) + u_1(k)$$

$$x_2(k+1) = 3x_1(k) + 2x_3(k)$$

$$x_3(k+1) = -12x_1(k) - 7x_2(k) - 6x_3(k) + u_2(k)$$

$$y_1(k) = 2x_2(k) + 2u_1(k)$$

$$y_2(k) = x_3(k) + u_2(k)$$

Rewriting in the form of vector-matrix equations, we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

These two equations are state-space equations for the system being considered.

To diagonalize the state matrix, let us define

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

Then, the characteristic equation becomes

$$|\lambda\mathbf{I} - \mathbf{G}| = (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The characteristic roots are $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$. The matrix $\mathbf{G}$ can be diagonalized by use of the following transformation matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

(For information on obtaining such a diagonalizing transformation matrix $\mathbf{P}$, see Appendix A.) The inverse of matrix $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \begin{bmatrix} 4.5 & 2.5 & 1 \\ -3 & -2 & -1 \\ 2.5 & 1.5 & 1 \end{bmatrix}$$

Thus,

$$\mathbf{P}^{-1}\mathbf{G}\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Now let us define a new state vector $\hat{\underset{\sim}{x}}$ as follows:

$$\underset{\sim}{x} = P\hat{\underset{\sim}{x}}$$

Then, in terms of the new state vector, state space equations can be written as follows:

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \\ \hat{x}_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{x}_3(k) \end{bmatrix} + \begin{bmatrix} 4.5 & 1 \\ -3 & -1 \\ 2.5 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -2 & -8 & -6 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{x}_3(k) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

Notice that the initial data $\hat{x}_1(0)$, $\hat{x}_2(0)$, and $\hat{x}_3(0)$ are obtained from

$$\begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \\ \hat{x}_3(0) \end{bmatrix} = \underset{\sim}{P}^{-1}\underset{\sim}{x}(0) = \begin{bmatrix} 4.5 & 2.5 & 1 \\ -3 & -2 & -1 \\ 2.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

<u>B-5-9</u>.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.2 \\ 1 & 0 & -1 \\ 0 & 1 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.6 \\ 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + 2u(k)$$

<u>B-5-10</u>.

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,\frac{1}{s^2}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{1}{s^3}\right] = \frac{1}{2}\,\frac{T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}$$

Hence

$$\frac{Y(z)}{U(z)} = \frac{G(z)}{1 + G(z)} = \frac{T^2 z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^2 + T^2 z^{-1}(1 + z^{-1})}$$

$$= \frac{\frac{1}{2}T^2 z^{-1} + \frac{1}{2}T^2 z^{-2}}{1 - (2 - \frac{1}{2}T^2)z^{-1} + (1 + \frac{1}{2}T^2)z^{-2}}$$

Notice that for this system, $b_0 = 0$, $b_1 = \frac{1}{2}T^2$, $b_2 = \frac{1}{2}T^2$, $a_1 = -2 + \frac{1}{2}T^2$, and $a_2 = 1 + \frac{1}{2}T^2$.

Direct programming:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - \frac{1}{2}T^2 & 2 - \frac{1}{2}T^2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \frac{1}{2}T^2 & \frac{1}{2}T^2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Nested programming:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -1 - \frac{1}{2}T^2 \\ 1 & 2 - \frac{1}{2}T^2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}T^2 \\ \frac{1}{2}T^2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

B-5-11.

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + 2z^{-2}}{1 + 0.7z^{-1} + 0.12z^{-2}} = \frac{z + 2}{z^2 + 0.7z + 0.12}$$

$$= \frac{17}{z + 0.3} - \frac{16}{z + 0.4}$$

Hence

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 17 & -16 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

B-5-12.

$$\frac{Y(z)}{U(z)} = \frac{1}{(z+1)(z+2)(z+3)} = \frac{0.5}{z+1} - \frac{1}{z+2} + \frac{0.5}{z+3}$$

Also,

$$\frac{zY(z)}{U(z)} = -\frac{0.5}{z+1} + \frac{2}{z+2} - \frac{1.5}{z+3}$$

$$\frac{z^2Y(z)}{U(z)} = \frac{0.5}{z+1} - \frac{4}{z+2} + \frac{4.5}{z+3}$$

Hence

$$\begin{bmatrix} Y(z)/U(z) \\ zY(z)/U(z) \\ z^2Y(z)/U(z) \end{bmatrix} = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 2 & -1.5 \\ 0.5 & -4 & 4.5 \end{bmatrix} \begin{bmatrix} 1/(z+1) \\ 1/(z+2) \\ 1/(z+3) \end{bmatrix}$$

Define

$$\frac{X_1(z)}{U(z)} = \frac{1}{z+1}, \quad \frac{X_2(z)}{U(z)} = \frac{1}{z+2}, \quad \frac{X_3(z)}{U(z)} = \frac{1}{z+3}$$

Then

$$zX_1(z) = -X_1(z) + U(z)$$
$$zX_2(z) = -2X_2(z) + U(z)$$
$$zX_3(z) = -3X_3(z) + U(z)$$

Therefore,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

The initial data are obtained as follows:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 2 & -1.5 \\ 0.5 & -4 & 4.5 \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix} = \begin{bmatrix} 6 & 5 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}$$

B-5-13. We shall assume that $n = 3$. (It is a simple matter to extend the derivation here to the case of an arbitrary positive integer n.)

$$y(k+3) + a_1(k)y(k+2) + a_2(k)y(k+1) + a_3(k)y(k)$$
$$= b_0(k)u(k+3) + b_1(k)u(k+2) + b_2(k)u(k+1) + b_3(k)u(k)$$

Define

$$x_1(k) = y(k) - h_0(k)u(k)$$

and

$$x_1(k+1) = x_2(k) + h_1(k)u(k)$$
$$x_2(k+1) = x_3(k) + h_2(k)u(k)$$
$$x_3(k+1) = -a_3(k)x_1(k) - a_2(k)x_2(k) - a_1(k)x_3(k) + h_3(k)u(k)$$

where $h_0(k)$, $h_1(k)$, $h_2(k)$, and $h_3(k)$ are undetermined functions at this stage.

Then

$$\begin{aligned}
y(k+1) &= x_1(k+1) + h_0(k+1)u(k+1) \\
&= x_2(k) + h_1(k)u(k) + h_0(k+1)u(k+1) \\
y(k+2) &= x_2(k+1) + h_1(k+1)u(k+1) + h_0(k+2)u(k+2) \\
&= x_3(k) + h_2(k)u(k) + h_1(k+1)u(k+1) \\
&\quad + h_0(k+2)u(k+2) \\
y(k+3) &= x_3(k+1) + h_2(k+1)u(k+1) + h_1(k+2)u(k+2) \\
&\quad + h_0(k+3)u(k+3) \\
&= -a_3(k)x_1(k) - a_2(k)x_2(k) - a_1(k)x_3(k) + h_3(k)u(k) \\
&\quad + h_2(k+1)u(k+1) + h_1(k+2)u(k+2) \\
&\quad + h_0(k+3)u(k+3)
\end{aligned}$$

and we have

$$\begin{aligned}
y(k+3) &+ a_1(k)y(k+2) + a_2(k)y(k+1) + a_3(k)y(k) \\
&= \left[h_3(k) + a_1(k)h_2(k) + a_2(k)h_1(k) + a_3(k)h_0(k)\right]u(k) \\
&\quad + \left[h_2(k+1) + a_1(k)h_1(k+1) + a_2(k)h_0(k+1)\right]u(k+1) \\
&\quad + \left[h_1(k+2) + a_1(k)h_0(k+2)\right]u(k+2) \\
&\quad + h_0(k+3)u(k+3) \\
&= b_3(k)u(k) + b_2(k)u(k+1) + b_1(k)u(k+2) \\
&\quad + b_0(k)u(k+3)
\end{aligned}$$

Hence we obtain

$$\left[h_3(k) + a_1(k)h_2(k) + a_2(k)h_1(k) + a_3(k)h_0(k) - b_3(k)\right]u(k)$$
$$+ \left[h_2(k + 1) + a_1(k)h_1(k + 1) + a_2(k)h_0(k + 1) - b_2(k)\right]u(k + 1)$$
$$+ \left[h_1(k + 2) + a_1(k)h_0(k + 2) - b_1(k)\right]u(k + 2)$$
$$+ \left[h_0(k + 3) - b_0(k)\right]u(k + 3) = 0$$

This last equation must hold for any $u(k)$, $u(k + 1)$, $u(k + 2)$, and $u(k + 3)$. Hence, each of the coefficients of $u(k)$, $u(k + 1)$, $u(k + 2)$, and $u(k + 3)$ must be equal to zero. Thus, we obtain

$$h_0(k + 3) = b_0(k)$$
$$h_1(k + 2) + a_1(k)h_0(k + 2) = b_1(k)$$
$$h_2(k + 1) + a_1(k)h_1(k + 1) + a_2(k)h_0(k + 1) = b_2(k)$$
$$h_3(k) + a_1(k)h_2(k) + a_2(k)h_1(k) + a_3(k)h_0(k) = b_3(k)$$

It follows that

$$h_0(k) = b_0(k - 3)$$
$$h_1(k) = b_1(k - 2) - a_1(k - 2)b_0(k - 3)$$
$$h_2(k) = b_2(k - 1) - a_2(k - 1)b_0(k - 3) - a_1(k - 1)\left[b_1(k - 2) - a_1(k - 2)b_0(k - 3)\right]$$
$$h_3(k) = b_3(k) - a_3(k)b_0(k - 3) - \left[a_2(k) - a_1(k)a_1(k - 1)\right] \cdot \left[b_1(k - 2) - a_1(k - 2)b_0(k - 3)\right] - a_1(k)\left[b_2(k - 1) - a_2(k - 1)b_0(k - 3)\right]$$

The initial conditions are given by

$$x_1(0) = y(0) - h_0(0)u(0)$$
$$x_2(0) = y(1) - h_0(1)u(1) - h_1(0)u(0)$$
$$x_3(0) = y(2) - h_0(2)u(2) - h_1(1)u(1) - h_2(0)u(0)$$

<u>B-5-14</u>. Note first that

$$g_1(\mathbf{G}) = (\mathbf{G} - z_k\mathbf{I})^0 = \mathbf{I}, \qquad g_2(\mathbf{G}) = \mathbf{G} - z_k\mathbf{I}$$
$$g_1(z_1) = (z_1 - z_k)^0 = 1, \qquad g_2(z_1) = z_1 - z_k$$
$$g_1(z_2) = (z_2 - z_k)^0 = 1, \qquad g_2(z_2) = z_2 - z_k$$

Thus,

$$g_1(\underset{\sim}{G}) = g_1(z_1)\underset{\sim}{X}_1 + g_1(z_2)\underset{\sim}{X}_2 = \underset{\sim}{X}_1 + \underset{\sim}{X}_2$$

$$g_2(\underset{\sim}{G}) = g_2(z_1)\underset{\sim}{X}_1 + g_2(z_2)\underset{\sim}{X}_2 = (z_1 - z_k)\underset{\sim}{X}_1 + (z_2 - z_k)\underset{\sim}{X}_2$$

Rewriting, we have

$$\underset{\sim}{I} = \underset{\sim}{X}_1 + \underset{\sim}{X}_2$$

$$\underset{\sim}{G} - z_k\underset{\sim}{I} = (z_1 - z_k)\underset{\sim}{X}_1 + (z_2 - z_k)\underset{\sim}{X}_2$$

The eigenvalues of the given $\underset{\sim}{G}$ are

$$z_1 = 0, \qquad z_2 = -2$$

Now choose $z_k = z_1$. Then

$$\underset{\sim}{I} = \underset{\sim}{X}_1 + \underset{\sim}{X}_2$$

$$\underset{\sim}{G} = z_1\underset{\sim}{X}_1 + z_2\underset{\sim}{X}_2 = -2\underset{\sim}{X}_2$$

from which we obtain

$$\underset{\sim}{X}_1 = \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}, \qquad \underset{\sim}{X}_2 = \begin{bmatrix} 0 & -0.5 \\ 0 & 1 \end{bmatrix}$$

Hence

$$(z\underset{\sim}{I} - \underset{\sim}{G})^{-1} = \sum_{k=1}^{2} \frac{\underset{\sim}{X}_k}{z - z_k} = \frac{\underset{\sim}{X}_1}{z - z_1} + \frac{\underset{\sim}{X}_2}{z - z_2} = \frac{\underset{\sim}{X}_1}{z} + \frac{\underset{\sim}{X}_2}{z + 2}$$

$$= \begin{bmatrix} \frac{1}{z} & \frac{1}{2z} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2(z+2)} \\ 0 & \frac{1}{z+2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{z} & \frac{1}{z(z+2)} \\ 0 & \frac{1}{z+2} \end{bmatrix}$$

<u>B-5-15</u>. Note that

$$(z\underset{\sim}{I} - \underset{\sim}{G})^{-1} = \begin{bmatrix} z + a_1 & a_2 & a_3 \\ -1 & z & 0 \\ 0 & -1 & z \end{bmatrix}^{-1}$$

$$= \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} \begin{bmatrix} z^2 & -(a_2 z + a_3) & -a_3 z \\ z & z^2 + a_1 z & -a_3 \\ 1 & z + a_1 & z^2 + a_1 z + a_2 \end{bmatrix}$$

Hence

$$(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H} = \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} \begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{C}(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}$$
$$= \left[b_1 - a_1 b_0 \;\vdots\; b_2 - a_2 b_0 \;\vdots\; b_3 - a_3 b_0 \right] \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} \begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix}$$

and

$$F(z) = \mathbf{C}(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H} + D$$
$$= \frac{(b_1 - a_1 b_0) z^2 + (b_2 - a_2 b_0) z + (b_3 - a_3 b_0)}{z^3 + a_1 z^2 + a_2 z + a_3} + b_0$$
$$= \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

B-5-16. The pulse transfer function $F(z)$ is given by

$$F(z) = \mathbf{C}(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H} + D$$
$$= [1 \quad 0 \quad 0] \begin{bmatrix} z + a_1 & -1 & 0 \\ a_2 & z & -1 \\ a_3 & 0 & z \end{bmatrix}^{-1} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} + b_0$$

First note that

$$|z\mathbf{I} - \mathbf{G}| = z^3 + a_1 z^2 + a_2 z + a_3$$

and

$$\begin{bmatrix} z + a_1 & -1 & 0 \\ a_2 & z & -1 \\ a_3 & 0 & z \end{bmatrix}^{-1} = \frac{1}{|z\mathbf{I} - \mathbf{G}|} \begin{bmatrix} z^2 & z & 1 \\ -(a_2 z + a_3) & (z + a_1) z & z + a_1 \\ -a_3 z & -a_3 & z(z + a_1) + a_2 \end{bmatrix}$$

Then, F(z) can be written as follows:

$$F(z) = \frac{1}{|z\underset{\sim}{I} - \underset{\sim}{G}|}[z^2 \quad z \quad 1]\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} + b_0$$

$$= \frac{1}{|z\underset{\sim}{I} - \underset{\sim}{G}|}(h_1 z^2 + h_2 z + h_3) + b_0$$

$$= \frac{h_1 z^2 + h_2 z + h_3}{z^3 + a_1 z^2 + a_2 z + a_3} + b_0$$

Thus, the pulse transfer function for the system is

$$F(z) = \frac{b_0 z^3 + (a_1 b_0 + h_1)z^2 + (a_2 b_0 + h_2)z + a_3 b_0 + h_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

$$= \frac{b_0 + (a_1 b_0 + h_1)z^{-1} + (a_2 b_0 + h_2)z^{-2} + (a_3 b_0 + h_3)z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

B-5-17.

$$Y_1(z) = \frac{1}{1 - z^{-1}} U_1(z) + \frac{1 + z^{-1}}{1 - z^{-1}} U_2(z)$$

$$Y_2(z) = \frac{1}{1 + 0.6z^{-1}} U_1(z) + \frac{1 + z^{-1}}{1 + 0.6z^{-1}} U_2(z)$$

Hence

$$Y_1(z) = \frac{z}{z - 1} U_1(z) + \frac{z + 1}{z - 1} U_2(z)$$

$$Y_2(z) = \frac{z}{z + 0.6} U_1(z) + \frac{z + 1}{z + 0.6} U_2(z)$$

which can be modified to

$$Y_1(z) = U_1(z) + \frac{1}{z - 1} U_1(z) + U_2(z) + \frac{2}{z - 1} U_2(z) \qquad (1)$$

$$Y_2(z) = U_1(z) - \frac{0.6}{z + 0.6} U_1(z) + U_2(z) + \frac{0.4}{z + 0.6} U_2(z) \qquad (2)$$

Now define state variables $X_1(z)$ and $X_2(z)$ as follows:

$$X_1(z) = \frac{1}{z - 1}\left[U_1(z) + 2U_2(z)\right]$$

$$X_2(z) = \frac{1}{z + 0.6}\left[-0.6U_1(z) + 0.4U_2(z)\right]$$

Then

$$zX_1(z) = X(z) + U_1(z) + 2U_2(z)$$

$$zX_2(z) = -0.6X_2(z) - 0.6U_1(z) + 0.4U_2(z)$$

In terms of the state variables $X_1(z)$ and $X_2(z)$, Equations (1) and (2) become, respectively, as follows:

$$Y_1(z) = X_1(z) + U_1(z) + U_2(z)$$

$$Y_2(z) = X_2(z) + U_1(z) + U_2(z)$$

Hence, the state equation and the output equation for the given system are

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.6 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -0.6 & 0.4 \end{bmatrix}\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

B-5-18. The state transition matrix is given by

$$\Psi(k) = G^k = \mathcal{Z}^{-1}\left[(zI - G)^{-1}z\right]$$

Since

$$(zI - G)^{-1} = \begin{bmatrix} z & -1 \\ 0.24 & z+1 \end{bmatrix}^{-1} = \frac{1}{(z+0.4)(z+0.6)}\begin{bmatrix} z+1 & 1 \\ -0.24 & z \end{bmatrix}$$

we have

$$\mathcal{Z}^{-1}\left[(zI - G)^{-1}z\right] = \mathcal{Z}^{-1}\begin{bmatrix} \frac{3z}{z+0.4} - \frac{2z}{z+0.6} & \frac{5z}{z+0.4} - \frac{5z}{z+0.6} \\ -\frac{1.2z}{z+0.4} + \frac{1.2z}{z+0.6} & -\frac{2z}{z+0.4} + \frac{3z}{z+0.6} \end{bmatrix}$$

Hence

$$\Psi(k) = G^k = \begin{bmatrix} 3(-0.4)^k - 2(-0.6)^k & 5(-0.4)^k - 5(-0.6)^k \\ -1.2(-0.4)^k + 1.2(-0.6)^k & -2(-0.4)^k + 3(-0.6)^k \end{bmatrix}$$

B-5-19. Define steady-state vectors of $\mathbf{x}(k)$ and $\mathbf{y}(k)$ as $\mathbf{x}_e$ and $\mathbf{y}_e$, respectively. Then, we have

$$\mathbf{x}_e = \mathbf{G}\mathbf{x}_e + \mathbf{H}\mathbf{u}_0$$

$$\mathbf{y}_e = \mathbf{C}\mathbf{x}_e + \mathbf{D}\mathbf{u}_0$$

where $\mathbf{u}_0$ = constant vector. By solving these two equations for $\mathbf{x}_e$ and $\mathbf{y}_e$, we obtain

$$\mathbf{x}_e = (\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{u}_0$$

$$\mathbf{y}_e = \mathbf{C}(\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{u}_0 + \mathbf{D}\mathbf{u}_0$$

B-5-20. Since

$$\mathbf{x}(k) = \mathbf{G}^k\mathbf{x}(0)$$

we have

$$J = \sum_{k=0}^{\infty} \mathbf{x}^*(k)\mathbf{Q}\mathbf{x}(k) = \sum_{k=0}^{\infty} \left[\mathbf{G}^k\mathbf{x}(0)\right]^*\mathbf{Q}\left[\mathbf{G}^k\mathbf{x}(0)\right]$$

$$= \mathbf{x}^*(0)\left[\sum_{k=0}^{\infty} (\mathbf{G}^k)^*\mathbf{Q}\mathbf{G}^k\right]\mathbf{x}(0)$$

Define

$$\sum_{k=0}^{\infty} (\mathbf{G}^*)^k\mathbf{Q}\mathbf{G}^k = \mathbf{P}$$

Although matrix $\mathbf{P}$ is the sum of an infinite series, it is a finite matrix because $\mathbf{G}$ is a stable matrix. (See Problem A-5-19.) The matrix $\mathbf{P}$ can be written as follows:

$$\mathbf{P} = \sum_{k=0}^{\infty} (\mathbf{G}^*)^k\mathbf{Q}\mathbf{G}^k = \mathbf{Q} + \sum_{k=1}^{\infty} (\mathbf{G}^*)^k\mathbf{Q}\mathbf{G}^k = \mathbf{Q} + \sum_{k=0}^{\infty} (\mathbf{G}^*)^{k+1}\mathbf{Q}\mathbf{G}^{k+1}$$

$$= \mathbf{Q} + \mathbf{G}^*\left[\sum_{k=0}^{\infty} (\mathbf{G}^*)^k\mathbf{Q}\mathbf{G}^k\right]\mathbf{G} = \mathbf{Q} + \mathbf{G}^*\mathbf{P}\mathbf{G}$$

Using this matrix $\mathbf{P}$, we have

$$J = \mathbf{x}^*(0)\left[\sum_{k=0}^{\infty} (\mathbf{G}^*)^k\mathbf{Q}\mathbf{G}^k\right]\mathbf{x}(0) = \mathbf{x}^*(0)\mathbf{P}\mathbf{x}(0)$$

B-5-21. In the Liapunov equation

$$\underset{\sim}{G}^*\underset{\sim}{P}\underset{\sim}{G} - \underset{\sim}{P} = -\underset{\sim}{Q}$$

let us choose $\underset{\sim}{Q}$ to be $\underset{\sim}{I}$. Then

$$\begin{bmatrix} 1 & 0.5 \\ -1.2 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & -1.2 \\ 0.5 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} p_{12} + 0.25p_{22} & -1.2p_{11} - 1.6p_{12} \\ -1.2p_{11} - 1.6p_{12} & 1.44p_{11} - p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which yields

$$p_{12} + 0.25p_{22} = -1$$

$$-1.2p_{11} - 1.6p_{12} = 0$$

$$1.44p_{11} - p_{22} = -1$$

Solving these three equations for p_{11}, p_{12}, and p_{22}, we obtain

$$p_{11} = \frac{10}{3.12}, \quad p_{12} = -\frac{5}{2.08}, \quad p_{22} = \frac{17.52}{3.12}$$

Hence, matrix $\underset{\sim}{P}$ is given by

$$\underset{\sim}{P} = \begin{bmatrix} \frac{10}{3.12} & -\frac{5}{2.08} \\ -\frac{5}{2.08} & \frac{17.52}{3.12} \end{bmatrix}$$

Clearly, matrix $\underset{\sim}{P}$ is positive definite.

A Liapunov function $V(\underset{\sim}{x}(k))$ is given by

$$V(\underset{\sim}{x}(k)) = \underset{\sim}{x}^*(k)\underset{\sim}{P}\underset{\sim}{x}(k)$$

B-5-22. Define

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 3 & 0 \\ -3 & -2 & -3 \\ 1 & 0 & 0 \end{bmatrix}$$

Then

$$\left| z\underset{\sim}{I} - \underset{\sim}{G} \right| = z^3 + z^2 + 7z + 9 = 0$$

Define

$$P(z) = z^3 + z^2 + 7z + 9 = a_0z^3 + a_1z^2 + a_2z + a_3 = 0$$

Then

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 7, \qquad a_3 = 9$$

The first requirement of Jury stability test

$$|a_3| < a_0$$

is not satisfied. Therefore, the origin of the system is unstable.

B-5-23. Define

$$\underset{\sim}{G} = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix}$$

Then, the characteristic equation becomes

$$|z\underset{\sim}{I} - \underset{\sim}{G}| = \begin{vmatrix} z - \cos T & -\sin T \\ \sin T & z - \cos T \end{vmatrix} = z^2 - 2z\cos T + 1$$

$$= (z - \cos T + j\sin T)(z - \cos T - j\sin T) = 0$$

Thus, two roots of the characteristic equation lie on the unit circle in the z plane. Thus, the system is stable in the sense of Liapunov, but is not asymptotically stable. [Note that if $T = \pi, 2\pi, 3\pi, \ldots$, then the state variables $x_1(k)$ and $x_2(k)$ are uncoupled.]

B-5-24. Since the system equations are

$$x_1(k + 1) = x_1(k) + 0.2x_2(k) + 0.4$$

$$x_2(k + 1) = 0.5x_1(k) - 0.5$$

the equilibrium state can be determined from

$$x_{1e} = x_{1e} + 0.2x_{2e} + 0.4$$

$$x_{2e} = 0.5x_{1e} - 0.5$$

as follows:

$$x_{1e} = -3, \qquad x_{2e} = -2$$

Define

$$\hat{x}_1(k) = x_1(k) - x_{1e} = x_1(k) + 3$$

$$\hat{x}_2(k) = x_2(k) - x_{2e} = x_2(k) + 2$$

Then, the original system equations are modified into

$$\hat{x}_1(k + 1) = \hat{x}_1(k) + 0.2\hat{x}_2(k)$$

$$\hat{x}_2(k + 1) = 0.5\hat{x}_1(k)$$

or

$$\begin{bmatrix} \hat{x}_1(k + 1) \\ \hat{x}_2(k + 1) \end{bmatrix} = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}$$

Define

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 0 \end{bmatrix}$$

Then, the eigenvalues of $\underset{\sim}{G}$ are found as

$$z_1 = 1.0916, \qquad z_2 = -0.0916$$

Since $|z_1| > 1$, the equilibrium state of the system is unstable.

CHAPTER 6

B-6-1. For complete state controllability, we require

$$\text{rank}\left[\underset{\sim}{H} \,\vdots\, \underset{\sim}{GH}\right] = \text{rank}\begin{bmatrix} 1 & a+b \\ 1 & c+d \end{bmatrix} = 2$$

Thus, the condition is $a + b \neq c + d$.

For complete observability, we require

$$\text{rank}\left[\underset{\sim}{C}^* \,\vdots\, \underset{\sim}{G}^*\underset{\sim}{C}^*\right] = \text{rank}\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} = 2$$

The condition is $b \neq 0$.

B-6-2. Note that $\underset{\sim}{x}(2)$ is given by

$$\underset{\sim}{x}(2) = \underset{\sim}{G}^2\underset{\sim}{x}(0) + \underset{\sim}{GH}u(0) + \underset{\sim}{H}u(1)$$

or

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} -0.16 & -1 \\ 0.16 & 0.84 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0.5 \\ -0.66 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(1)$$

Hence

$$x_1(2) = -1 = -0.16x_1(0) - x_2(0) + 0.5u(0) + u(1)$$

$$x_2(2) = 2 = 0.16x_1(0) + 0.84x_2(0) - 0.66u(0) + 0.5u(1)$$

Since $x_1(0) = 1$ and $x_2(0) = -1$, we find

$$u(0) = -3.9560, \qquad u(1) = 0.1380$$

B-6-3. Notice that

$$\text{rank}\left[\underset{\sim}{H} \,\vdots\, \underset{\sim}{GH}\right] = \text{rank}\begin{bmatrix} 1 & -0.8 \\ -0.8 & 0.64 \end{bmatrix} = 1$$

Since the rank of the controllability matrix is 1, the system is not completely state controllable. However, some state can be controllable. That means some states can be brought to some other states. Notice that

$$\underset{\sim}{x}(2) = \underset{\sim}{G}^2\underset{\sim}{x}(0) + \underset{\sim}{GH}u(0) + \underset{\sim}{H}u(1)$$

or

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} -0.16 & -1 \\ 0.16 & 0.84 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} -0.8 \\ 0.64 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} u(1)$$

1.)

$$\begin{bmatrix} 0 \\ -0.008 \end{bmatrix} = \begin{bmatrix} -0.16 & -1 \\ 0.16 & 0.84 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -0.8 \\ 0.64 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} u(1)$$

or

$$-0.8\, u(0) + u(1) = -\,0.84$$

$$0.64\, u(0) - 0.8\, u(1) = 0.672$$

These two equations can be satisfied by an infinite number of combinations of u(0) and u(1). For example,

$$u(0) = 0, \qquad u(1) = -0.84$$

will satisfy the two equations. Hence it is possible to bring the given initial state to

$$\begin{bmatrix} 0 \\ -0.008 \end{bmatrix}$$

2.)

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.16 & -1 \\ 0.16 & 0.84 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -0.8 \\ 0.64 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} u(1)$$

or

$$-0.8\, u(0) + u(1) = -1.84$$

$$0.64\, u(0) - 0.8\, u(1) = 2.68$$

There is no set of values u(0) and u(1) that satisfies these two equations. Hence, it is not possible to bring the given initial state to

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

B-6-4.

$$\text{rank}\left[\underset{\sim}{H} \,\vdots\, \underset{\sim}{GH} \,\vdots\, \underset{\sim}{G}^2\underset{\sim}{H}\right] = \text{rank}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & b \\ 0 & b & 0 \end{bmatrix} = 2$$

Hence, the system is not completely state controllable. Note that $\underset{\sim}{x}(3)$ is given by

$$\underset{\sim}{x}(3) = \underset{\sim}{G}^3\underset{\sim}{x}(0) + \underset{\sim}{G}^2\underset{\sim}{H}u(0) + \underset{\sim}{GH}u(1) + \underset{\sim}{H}u(2)$$

1.)

$$\begin{bmatrix} x_1(3) \\ x_2(3) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & -\frac{a}{b} \\ -\frac{a^2}{b} & 0 & b + \frac{a^2}{b^2} \\ ab + \frac{a^3}{b^2} & b^2 & -\frac{a}{b}\left(b + \frac{a^2}{b^2}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ b & 0 & 1 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

or

$$0 = a + b - \frac{a}{b} + u(1) \tag{1}$$

$$0 = - \frac{a^2}{b} + b + \frac{a^2}{b^2} + bu(0) + u(2) \tag{2}$$

$$0 = ab + \frac{a^3}{b^2} + b^2 - a - \frac{a^3}{b^3} + bu(1) \tag{3}$$

From Equation (1) we have

$$u(1) = - a - b + \frac{a}{b} \tag{4}$$

By substituting Equation (4) into Equation (3), we have

$$\frac{a^3}{b^2} (1 - \frac{1}{b}) = 0$$

This last equation is satisfied if $a = 0$, or $b = 1$, or $a = 0$ and $b = 1$. If $a = 0$ or $b = 1$, then Equation (2) can be written as

$$bu(0) + u(2) = - b \qquad \text{if } a = 0 \tag{5}$$

$$u(0) + u(2) = - 1 \qquad \text{if } b = 1 \tag{6}$$

There exist infinitely many sets of $u(0)$ and $u(2)$ that satisfy Equation (5) or (6). Thus, if $a = 0$ or $b = 1$ or $a = 0$ and $b = 1$, then it is possible to bring $\underset{\sim}{x}(3)$ to the origin. Otherwise, it is not possible to bring $\underset{\sim}{x}(3)$ to the origin.

2.) If the initial state is $\underset{\sim}{x}(0) = \underset{\sim}{0}$, then

$$\underset{\sim}{x}(3) = \underset{\sim}{G}^2\underset{\sim}{H}u(0) + \underset{\sim}{G}\underset{\sim}{H}u(1) + \underset{\sim}{H}u(2)$$

or

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ b & 0 & 1 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

or

$$1 = u(1)$$

$$1 = bu(0) + u(2)$$

$$1 = bu(1)$$

If $b = 1$, then there are infinitely many sets of $u(0)$ and $u(2)$ that satisfy the last three equations. Hence, if $b = 1$, then it is possible to bring the initial state $\underset{\sim}{x}(0) = \underset{\sim}{0}$ to the given state $\underset{\sim}{x}(3)$; otherwise it is not possible.

B-6-5. Note that

$$y(0) = x_1(0) \quad \text{and} \quad y(1) = x_1(1)$$

Thus,

$$x_1(0) = 1 \quad \text{and} \quad x_1(1) = 2$$

Also, from the state equation,

$$x_1(1) = x_2(0)$$

Thus,

$$x_2(0) = 2$$

Therefore,

$$\underset{\sim}{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

State $\underset{\sim}{x}(1)$ is obtained as follows:

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [2] = \begin{bmatrix} 2 \\ -0.16 \end{bmatrix}$$

and state $\underset{\sim}{x}(2)$ is

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -0.16 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [-1] = \begin{bmatrix} -0.16 \\ -1.16 \end{bmatrix}$$

B-6-6. Define $\underset{\sim}{H} = \underset{\sim}{G}\underset{\sim}{C}^*$. Then the state equation becomes

$$\underset{\sim}{x}(k+1) = \underset{\sim}{G}\underset{\sim}{x}(k) + \underset{\sim}{G}\underset{\sim}{C}^* u(k) = \underset{\sim}{G}\underset{\sim}{x}(k) + \underset{\sim}{H} u(k)$$

Notice that

$$\left[\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}^2\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}^3\underset{\sim}{H}\right] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$\text{rank}\left[\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}^2\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}^3\underset{\sim}{H}\right] = 4$$

Therefore, the system is completely state controllable.

The observability matrix becomes as follows:

$$\left[\underset{\sim}{C}^* \;\vdots\; \underset{\sim}{G}^*\underset{\sim}{C}^* \;\vdots\; (\underset{\sim}{G}^*)^2\underset{\sim}{C}^* \;\vdots\; (\underset{\sim}{G}^*)^3\underset{\sim}{C}^* \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence

$$\text{rank}\left[\underset{\sim}{C}^* \;\vdots\; \underset{\sim}{G}^*\underset{\sim}{C}^* \;\vdots\; (\underset{\sim}{G}^*)^2\underset{\sim}{C}^* \;\vdots\; (\underset{\sim}{G}^*)^3\underset{\sim}{C}^* \right] = 4$$

and the system is completely observable.

Next, we shall show that every initial state can be brought to the origin in at most 4 sampling periods if and only if the control signal is given by

$$u(k) = -\underset{\sim}{C}\underset{\sim}{x}(k) \tag{1}$$

Noting that for the system

$$\underset{\sim}{x}(k+1) = \underset{\sim}{G}\underset{\sim}{x}(k) + \underset{\sim}{G}\underset{\sim}{C}^* u(k)$$

we have

$$\begin{aligned} \underset{\sim}{x}(4) &= \underset{\sim}{G}^4\underset{\sim}{x}(0) + \underset{\sim}{G}^3(\underset{\sim}{G}\underset{\sim}{C}^*)u(0) + \underset{\sim}{G}^2(\underset{\sim}{G}\underset{\sim}{C}^*)u(1) \\ &\quad + \underset{\sim}{G}(\underset{\sim}{G}\underset{\sim}{C}^*)u(2) + \underset{\sim}{G}\underset{\sim}{C}^* u(3) \\ &= \underset{\sim}{G}^4\underset{\sim}{x}(0) + \underset{\sim}{G}^4\underset{\sim}{C}^* u(0) + \underset{\sim}{G}^3\underset{\sim}{C}^* u(1) \\ &\quad + \underset{\sim}{G}^2\underset{\sim}{C}^* u(2) + \underset{\sim}{G}\underset{\sim}{C}^* u(3) \end{aligned}$$

Let us set $\underset{\sim}{x}(4) = \underset{\sim}{0}$. Then, we obtain

$$\underset{\sim}{0} = \underset{\sim}{G}^4\underset{\sim}{x}(0) + \underset{\sim}{G}^4\underset{\sim}{C}^* u(0) + \underset{\sim}{G}^3\underset{\sim}{C}^* u(1) + \underset{\sim}{G}^2\underset{\sim}{C}^* u(2) + \underset{\sim}{G}\underset{\sim}{C}^* u(3)$$

which may be solved for $\underset{\sim}{x}(0)$ as follows:

$$\underset{\sim}{x}(0) = -\underset{\sim}{C}^* u(0) - \underset{\sim}{G}^{-1}\underset{\sim}{C}^* u(1) - \underset{\sim}{G}^{-2}\underset{\sim}{C}^* u(2) - \underset{\sim}{G}^{-3}\underset{\sim}{C}^* u(3)$$

or

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(0) - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(1) - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(2) - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(3)$$

Hence we get

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = - \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} \tag{2}$$

Every initial state can be brought to the origin, that is, $x(4) = 0$, if and only if Equation (2) is satisfied. Equation (2) can be written in the form of Equation (1) as derived below.

From the state equation we have

$$x_1(k + 1) = x_2(k)$$
$$x_2(k + 1) = x_3(k)$$
$$x_3(k + 1) = x_4(k)$$
$$x_4(k + 1) = x_1(k) + u(k)$$

Therefore, Equation (2) can be written as follows:

$$u(0) = - x_1(0)$$
$$u(1) = - x_2(0) = - x_1(1)$$
$$u(2) = - x_3(0) = - x_1(2)$$
$$u(3) = - x_4(0) = - x_1(3)$$

which can be combined into one equation as follows:

$$u(k) = - \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$$= - \underset{\sim}{Cx}(k) \tag{3}$$

The control law given by Equation (3) is the only control law for this system that will bring any given initial state $\underset{\sim}{x}(0)$ to the origin in at most 4 sampling periods.

B-6-7.

$$\underset{\sim}{G}(T) = e^{\underset{\sim}{A}T} = \begin{bmatrix} e^{-3T}(\cos 4T + \frac{3}{4}\sin 4T) & \frac{1}{4}e^{-3T}\sin 4T \\ -\frac{25}{4}e^{-3T}\sin 4T & e^{-3T}(\cos 4T - \frac{3}{4}\sin 4T) \end{bmatrix}$$

$$\underset{\sim}{H}(T) = \int_0^T \begin{bmatrix} \frac{1}{4}e^{-3t}\sin 4t \\ e^{-3t}(\cos 4t - \frac{3}{4}\sin 4t) \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{1}{100}e^{-3T}(-3\sin 4T - 4\cos 4T) + \frac{1}{25} \\ \frac{1}{4}e^{-3T}\sin 4T \end{bmatrix}$$

If $T = \pi n/4$, then

$$G(\frac{\pi n}{4}) = \begin{bmatrix} e^{-\frac{3\pi n}{4}}\cos n\pi & 0 \\ 0 & e^{-\frac{3\pi n}{4}}\cos n\pi \end{bmatrix}$$

and

$$H(\frac{\pi n}{4}) = \begin{bmatrix} \frac{1}{25}(1 - e^{-\frac{3\pi n}{4}}\cos n\pi) \\ 0 \end{bmatrix}$$

Using $G(\pi n/4)$ and $H(\pi n/4)$ thus obtained, we have

$$\text{rank}\left[\underset{\sim}{H} \,\vdots\, \underset{\sim}{GH}\right] < 2$$

and

$$\text{rank}\left[\underset{\sim}{C}^* \,\vdots\, \underset{\sim}{G}^*\underset{\sim}{C}^*\right] < 2$$

Thus, the system is uncontrollable and unobservable if $T = \pi n/4$.

B-6-8.

1. Controllable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

2. Observable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

3. Diagonal canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

B-6-9.

1. Controllable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0.5 & 1.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$

2. Observable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -0.5 \\ 1.8 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$

3. <u>Diagonal canonical form</u>:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 - j0.5 & 0 \\ 0 & 0.5 + j0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \left[0.9 + j0.4 \;\vdots\; 0.9 - j0.4 \right] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$

<u>B-6-10</u>. The system equations in the controllable canonical form are

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + Du(k)$$

By use of the transformation $\mathbf{x}(k) = \mathbf{Q}\hat{\mathbf{x}}(k)$, these two equations become

$$\hat{\mathbf{x}}(k+1) = \mathbf{Q}^{-1}\mathbf{G}\mathbf{Q}\hat{\mathbf{x}}(k) + \mathbf{Q}^{-1}\mathbf{H}u(k)$$

$$y(k) = \mathbf{C}\mathbf{Q}\hat{\mathbf{x}}(k) + Du(k)$$

where

$$\mathbf{Q}^{-1}\mathbf{G}\mathbf{Q} = \hat{\mathbf{G}} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}, \quad \mathbf{Q}^{-1}\mathbf{H} = \hat{\mathbf{H}} = \begin{bmatrix} b_3 - a_3 b_0 \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

Referring to Section 6-4, the transformation matrix $\mathbf{Q}$ that will give this $\hat{\mathbf{G}}$ matrix can be given by

$$\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}$$

where

$$\mathbf{W} = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{N} = \left[\mathbf{C}^* \;\vdots\; \mathbf{G}^*\mathbf{C}^* \;\vdots\; (\mathbf{G}^*)^2\mathbf{C}^* \right]$$

Define

$$\mathbf{C}^* = \begin{bmatrix} b_3 - a_3 b_0 \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

Then

$$\underset{\sim}{W}\underset{\sim}{N}^* = \begin{bmatrix} a_2b_3 - a_3b_2 & a_1b_3 - a_3b_1 & b_3 - a_3b_0 \\ a_1b_3 - a_3b_1 & a_1b_2 - a_2b_1 + b_3 - a_3b_0 & b_2 - a_2b_0 \\ b_3 - a_3b_0 & b_2 - a_2b_0 & b_1 - a_1b_0 \end{bmatrix}$$

The desired transformation matrix $\underset{\sim}{Q}$ is given by

$$\underset{\sim}{Q} = (\underset{\sim}{W}\underset{\sim}{N}^*)^{-1}$$

B-6-11. First note that the rank of the controllability matrix $[\underset{\sim}{H} \,\vdots\, \underset{\sim}{G}\underset{\sim}{H}]$ is two. So, arbitrary pole placement is possible. Define

$$\underset{\sim}{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

Then

$$u(kT) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

and we have

$$\begin{bmatrix} x_1((k+1)T) \\ x_2((k+1)T) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} - \begin{bmatrix} \frac{1}{2}T^2k_1 & \frac{1}{2}T^2k_2 \\ Tk_1 & Tk_2 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

or

$$\begin{bmatrix} x_1((k+1)T) \\ x_2((k+1)T) \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}T^2k_1 & T - \frac{1}{2}T^2k_2 \\ -Tk_1 & 1 - Tk_2 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

The characteristic equation becomes

$$\left| z\underset{\sim}{I} - (\underset{\sim}{G} - \underset{\sim}{H}\underset{\sim}{K}) \right| = \begin{vmatrix} z - 1 + \frac{1}{2}T^2k_1 & -T + \frac{1}{2}T^2k_2 \\ Tk_1 & z - 1 + Tk_2 \end{vmatrix}$$

$$= z^2 + (-2 + \tfrac{1}{2}T^2k_1 + Tk_2)z + \tfrac{1}{2}T^2k_1 - Tk_2 + 1 = 0$$

The desired characteristic equation is

$$(z - \mu_1)(z - \mu_2) = z^2 - (\mu_1 + \mu_2)z + \mu_1\mu_2 = 0$$

By equating the coefficients of the corresponding terms of the two characteristic equations, we obtain

$$\mu_1 + \mu_2 = 2 - \tfrac{1}{2}T^2k_1 - Tk_2$$

$$\mu_1\mu_2 = \tfrac{1}{2}T^2k_1 - Tk_2 + 1$$

from which k_1 and k_2 are determined as follows:

$$k_1 = \frac{1}{T^2}(1 - \mu_1 - \mu_2 + \mu_1\mu_2), \quad k_2 = \frac{1}{2T}(3 - \mu_1 - \mu_2 - \mu_1\mu_2)$$

Thus,

$$K = \left[\frac{1}{T^2} (1 - \mu_1 - \mu_2 + \mu_1 \mu_2) \;\vdots\; \frac{1}{2T} (3 - \mu_1 - \mu_2 - \mu_1 \mu_2) \right]$$

B-6-12. First note that the rank of the controllability matrix is 3 and, therefore, it is possible to determine the necessary state feedback gain matrix $\underset{\sim}{K}$ for deadbeat response.

Referring to Equation (6-65) and noting that for deadbeat response $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we have

$$\underset{\sim}{K} = \left[\alpha_3 - a_3 \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1 \right] \underset{\sim}{T}^{-1}$$

$$= \left[-a_3 \;\vdots\; -a_2 \;\vdots\; -a_1 \right] \underset{\sim}{T}^{-1}$$

where

$$a_1 = 0, \qquad a_2 = -0.84, \qquad a_3 = 0.16$$

and

$$\underset{\sim}{T} = \underset{\sim}{M}\underset{\sim}{W}$$

Matrices $\underset{\sim}{M}$ and $\underset{\sim}{W}$ are given by

$$\underset{\sim}{M} = \left[\underset{\sim}{H} \;\vdots\; \underset{\sim}{G}\underset{\sim}{H} \;\vdots\; \underset{\sim}{G}^2\underset{\sim}{H} \right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0.68 \\ 1 & 0.68 & 0.68 \end{bmatrix}$$

$$\underset{\sim}{W} = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.84 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Hence

$$\underset{\sim}{T} = \underset{\sim}{M}\underset{\sim}{W} = \begin{bmatrix} 0.16 & 1 & 1 \\ -0.16 & 1 & 1 \\ -0.16 & 0.68 & 1 \end{bmatrix}$$

and

$$\underset{\sim}{T}^{-1} = \begin{bmatrix} 3.125 & -3.125 & 0 \\ 0 & 3.125 & -3.125 \\ 0.5 & -2.625 & 3.125 \end{bmatrix}$$

Thus

$$\underset{\sim}{K} = \begin{bmatrix} -0.16 & 0.84 & 0 \end{bmatrix} \begin{bmatrix} 3.125 & -3.125 & 0 \\ 0 & 3.125 & -3.125 \\ 0.5 & -2.625 & 3.125 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 3.125 & -2.625 \end{bmatrix}$$

B-6-13. First, notice that $\underset{\sim}{G}$ is a nonsingular matrix. Then check the observability condition.

$$\text{rank}\left[\underset{\sim}{C}^* \;\vdots\; \underset{\sim}{G}^*\underset{\sim}{C}^*\right] = \text{rank}\begin{bmatrix} 1 & -0.16 \\ 1 & 0 \end{bmatrix} = 2$$

Hence the eigenvalues of $\underset{\sim}{G} - \underset{\sim}{K}_e\underset{\sim}{C}\underset{\sim}{G}$ can be arbitrarily placed in the z plane by a proper choice of $\underset{\sim}{K}_e$. Referring to Equation (6-137), we have

$$\underset{\sim}{K}_e = \phi(\underset{\sim}{G})\begin{bmatrix} \underset{\sim}{C}\underset{\sim}{G} \\ \underset{\sim}{C}\underset{\sim}{G}^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For the deadbeat response, $\phi(z)$, the desired characteristic polynomial of the error dynamics, is z^2. Hence

$$\phi(\underset{\sim}{G}) = \underset{\sim}{G}^2$$

Since

$$CG = \begin{bmatrix} -0.16 & 0 \end{bmatrix}, \qquad CG^2 = \begin{bmatrix} 0 & -0.16 \end{bmatrix}$$

we obtain $\underset{\sim}{K}_e$ as follows:

$$\underset{\sim}{K}_e = \underset{\sim}{G}^2\begin{bmatrix} CG \\ CG^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.16 & -1 \\ 0.16 & 0.84 \end{bmatrix}\begin{bmatrix} -\frac{1}{0.16} & 0 \\ 0 & -\frac{1}{0.16} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6.25 \\ -5.25 \end{bmatrix}$$

The desired current observer is then given by

$$\begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} + \begin{bmatrix} 6.25 \\ -5.25 \end{bmatrix}\left\{ y(k+1) - \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix}\right\}$$

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

B-6-14. Note that for this system

$$G_{aa} = 0, \qquad \underset{\sim}{G}_{ab} = \begin{bmatrix} 0 & -0.25 \end{bmatrix}, \qquad G_{ba} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{G}_{bb} = \begin{bmatrix} 0 & 0 \\ 1 & 0.5 \end{bmatrix}, \qquad H_a = 1, \qquad \underset{\sim}{H}_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Referring to Equation (6-153) we have

$$\mathbf{K}_e = \phi(\mathbf{G}_{bb}) \begin{bmatrix} \mathbf{G}_{ab} \\ \mathbf{G}_{ab}\mathbf{G}_{bb} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where for deadbeat response

$$\phi(\mathbf{G}_{bb}) = \mathbf{G}_{bb}^{\,2}$$

Since

$$\phi(\mathbf{G}_{bb}) = \mathbf{G}_{bb}^{\,2} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.25 \end{bmatrix}, \qquad \mathbf{G}_{ab}\mathbf{G}_{bb} = \begin{bmatrix} -0.25 & -0.125 \end{bmatrix}$$

we have

$$\mathbf{K}_e = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0 & -0.25 \\ -0.25 & -0.125 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

The equations for the minimum-order observer are

$$\begin{bmatrix} \tilde{x}_2(k) \\ \tilde{x}_3(k) \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_2(k) \\ \tilde{\eta}_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y(k)$$

and

$$\tilde{\eta}(k + 1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tilde{\eta}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(k) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u(k)$$

B-6-15.

MATLAB Program for Problem B-6-15

```
% ---------- Pole placement (Deadbeat response) ----------

% ***** This program determines state feedback gain matrix
% K for deadbeat response using Ackermann's formula *****

% ***** Enter matrices G and H *****
G = [0  1;-0.16  -1];
H = [0;1];

% ***** Enter the controllability matrix M and check its
% rank *****

M = [H  G*H];
rank(M)
```

```
ans =

     2

% ***** Since the rank of M is 2, the system is completely
% state controllable and thus arbitrary pole placement is
% possible *****

% ***** For deadbeat response, the desired characteristic
% polynomial becomes Phi = G^2 *****

Phi = G^2;

% ***** State feedback gain matrix K can be given by *****

K = [0  1]*inv(M)*Phi

K =

    -0.1600    -1.0000

k1 = K(1), k2 = K(2)

k1 =

    -0.1600


k2 =

    -1
```

B-6-16.

MATLAB Program for Problem B-6-16

```
% ---------- Design of state observer ----------

% ***** This program determines state observer gain matrix Ke
% by use of Ackermann's formula *****

% ***** Enter matrices G and C *****

G = [0  -0.16;1  -1];
C = [0  1];
```

```
% ***** Enter the observability matrix N and check its rank *****

N = [C'  G'*C'];
rank(N)

ans =

     2

% ***** Since the rank of the observability matrix is 2, design
% of observer is possible *****

% ***** Enter the desired characteristic polynomial by defining
% the following matrix J and entering statement poly(J) *****

J = [0.5+0.5*i    0
     0    0.5-0.5*i];

JJ = poly(J)

JJ =

   1.0000    -1.0000    0.5000

% ***** Enter characteristic polynomial Phi *****

Phi = polyvalm(poly(J),G);

% ***** The observer gain matrix Ke is obtained from *****

Ke = Phi*inv(N')*[0;1]

Ke =

    0.3400
   -2.0000
```

B-6-17. The system equations are

$$x(k + 1) = 0.5\,x(k) + u(k)$$

$$y(k) = x(k)$$

$$v(k + 1) = v(k) + r(k) - y(k)$$

$$u(k) = K_1 v(k) - K_2 x(k)$$

Thus,

$$u(k+1) = -K_2 x(k+1) + K_1 v(k+1)$$
$$= -K_2 \left[0.5\, x(k) + u(k)\right] + K_1 \left[v(k) + r(k) - y(k)\right]$$
$$= (0.5\, K_2 - K_1)x(k) + (1 - K_2)u(k) + K_1 r(k)$$

Hence, the state equation in terms of x and u becomes

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5\, K_2 - K_1 & 1 - K_2 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 \end{bmatrix} r(k) \qquad (1)$$

and the output equation becomes

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

Define

$$x_e(k) = x(k) - x(\infty)$$
$$u_e(k) = u(k) - u(\infty)$$

Then

$$\begin{bmatrix} x_e(k+1) \\ u_e(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5\, K_2 - K_1 & 1 - K_2 \end{bmatrix} \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} z - 0.5 & -1 \\ -0.5\, K_2 + K_1 & z - 1 + K_2 \end{vmatrix}$$
$$= z^2 + (K_2 - 1.5)z + 0.5 - K_2 + K_1 = 0$$

The desired characteristic equation is

$$z^2 = 0$$

Hence we choose $K_1 = 1$ and $K_2 = 1.5$. Thus, the integral gain constant K_1 is

$$K_1 = 1$$

and the state feedback gain constant K_2 is

$$K_2 = 1.5$$

[It is noted that Equation (6-193) must be modified if it is to be applied to this problem, since the configuration of the integral controller is different from that shown in Figure 6-18.]

To determine the output y(k), notice that

$$y(k) = x(k)$$

By substituting $K_1 = 1$, $K_2 = 1.5$, and $r(k) = 1$ into Equation (1), we obtain

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assume that the initial state is

$$\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where a and b are arbitrary. Then

$$\begin{bmatrix} x(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5\,a + b \\ -0.25\,a - 0.5\,b + 1 \end{bmatrix}$$

$$\begin{bmatrix} x(2) \\ x(2) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} 0.5\,a + b \\ -0.25\,a - 0.5\,b + 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

and

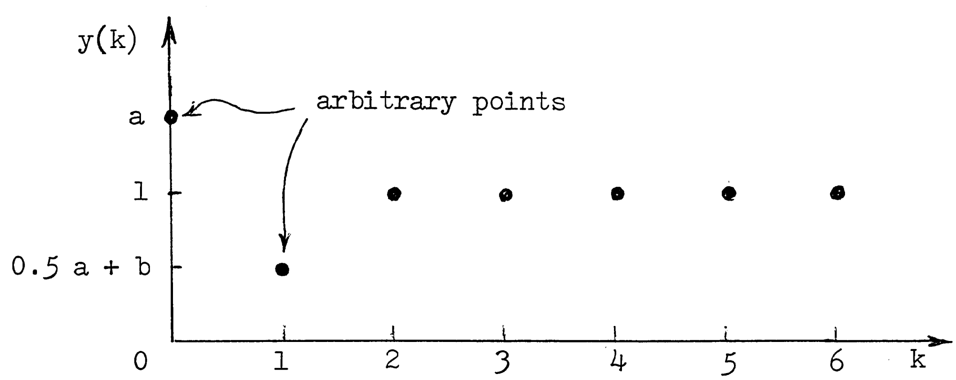
$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad \text{for } k = 3, 4, 5, \ldots$$

Hence

$$y(0) = x(0) = a$$

$$y(1) = x(1) = 0.5\,a + b$$

$$y(k) = x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots.$$



A sample response plot is shown above.

B-6-18. The system equations are

$$x(k+1) = 0.5\,x(k) + u(k)$$

$$y(k) = x(k)$$

$$u(k) = -K_2 x(k) + K_1 v(k)$$

$$v(k) = v(k-1) + r(k) - y(k)$$

Hence

$$\begin{aligned} u(k+1) &= -K_2 x(k+1) + K_1 v(k+1) \\ &= -K_2\left[0.5\,x(k) + u(k)\right] + K_1\left[v(k) - y(k+1) + r(k+1)\right] \\ &= (0.5K_2 - 0.5K_1)x(k) + (1 - K_1 - K_2)u(k) + K_1 r(k+1) \end{aligned}$$

Considering x(k) and u(k) to be new state variables, we can obtain the following state space equations:

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5K_2 - 0.5K_1 & 1 - K_1 - K_2 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 \end{bmatrix} r(k+1) \qquad (1)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

where $r(k) = 1$ for $k = 0, 1, 2, \ldots$. Define

$$x_e(k) = x(k) - x(\infty)$$

$$u_e(k) = u(k) - u(\infty)$$

Then

$$\begin{bmatrix} x_e(k+1) \\ u_e(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5K_2 - 0.5K_1 & 1 - K_1 - K_2 \end{bmatrix} \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix}$$

Define

$$\hat{\mathbf{G}} = \begin{bmatrix} G & H \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \hat{\mathbf{H}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note that Equation (6-193) applies to this system, since the configuration of the integral controller is the same as that shown in Figure 6-18. Matrix $\hat{K}$ can be determined from

$$\hat{\mathbf{K}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{H}} & \hat{\mathbf{G}}\hat{\mathbf{H}} \end{bmatrix}^{-1} \phi(\hat{\mathbf{G}})$$

where

$$\phi(\hat{\mathbf{G}}) = \hat{\mathbf{G}}^2$$

Thus,

$$\hat{\underset{\sim}{K}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0.25 & 0.5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 \end{bmatrix}$$

Then constants K_1 and K_2 are determined from Equation (6-193).

$$\begin{bmatrix} K_2 & K_1 \end{bmatrix} = \begin{bmatrix} 0.25 & 1.5 \end{bmatrix} \begin{bmatrix} -0.5 & 1 \\ 0.5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$$

That is,

$$K_1 = 1 \qquad \text{and} \qquad K_2 = 0.5$$

The output $y(k)$ can be obtained as follows: First note that

$$y(k) = x(k)$$

By substituting $K_1 = 1$, $K_2 = 0.5$, and $r(k) = 1$ into Equation (1), we obtain

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now assume that the initial state is

$$\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where a and b are arbitrary constants. Then

$$\begin{bmatrix} x(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5\,a + b \\ -0.25\,a - 0.5\,b + 1 \end{bmatrix}$$

$$\begin{bmatrix} x(2) \\ u(2) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} 0.5\,a + b \\ -0.25\,a - 0.5\,b + 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

and

$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad \text{for } k = 3, 4, 5, \ldots$$

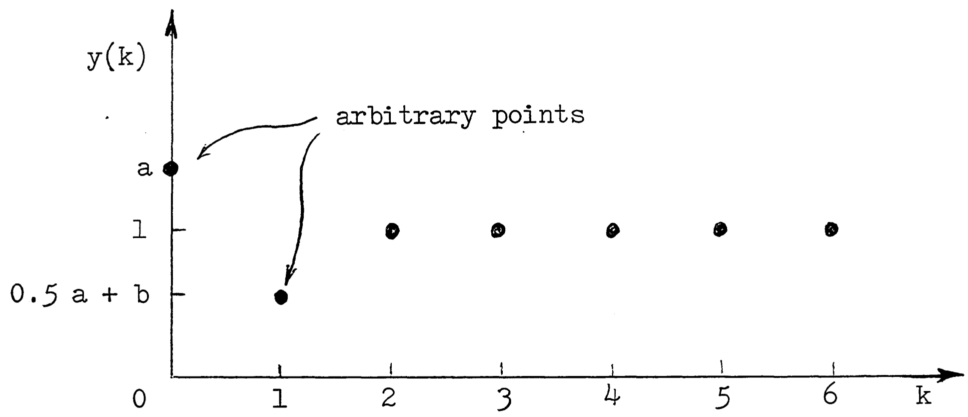
Hence,

$$y(0) = x(0) = a$$

$$y(1) = x(1) = 0.5\,a + b$$

$$y(k) = x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots$$

The output sequence y(k) for a unit step input is shown below.



CHAPTER 7

B-7-1. For this problem

$$a_1 = -(\lambda_1 + \lambda_2), \quad a_2 = \lambda_1 \lambda_2, \quad b_1 = b, \quad b_2 = -b\lambda_3$$

Thus,

$$\mathbf{E} = \begin{bmatrix} \lambda_1 \lambda_2 & 0 & -b\lambda_3 & 0 \\ -(\lambda_1 + \lambda_2) & \lambda_1 \lambda_2 & b & -b\lambda_3 \\ 1 & -(\lambda_1 + \lambda_2) & 0 & b \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The determinant of $\mathbf{E}$ can be expanded as follows:

$$|\mathbf{E}| = \begin{vmatrix} \lambda_1\lambda_2 & 0 \\ -(\lambda_1+\lambda_2) & \lambda_1\lambda_2 \end{vmatrix} \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} - \begin{vmatrix} \lambda_1\lambda_2 & 0 \\ 1 & -(\lambda_1+\lambda_2) \end{vmatrix} \begin{vmatrix} b & -b\lambda_3 \\ 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} \lambda_1\lambda_2 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} b & -b\lambda_3 \\ 0 & b \end{vmatrix} + \begin{vmatrix} -(\lambda_1+\lambda_2) & \lambda_1\lambda_2 \\ 1 & -(\lambda_1+\lambda_2) \end{vmatrix} \begin{vmatrix} -b\lambda_3 & 0 \\ 0 & 0 \end{vmatrix}$$

$$- \begin{vmatrix} -(\lambda_1+\lambda_2) & \lambda_1\lambda_2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -b\lambda_3 & 0 \\ 0 & b \end{vmatrix} + \begin{vmatrix} 1 & -(\lambda_1+\lambda_2) \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -b\lambda_3 & 0 \\ b & -b\lambda_3 \end{vmatrix}$$

$$= 0 - 0 + \lambda_1 \lambda_2 b^2 + 0 - (\lambda_1 + \lambda_2) b^2 \lambda_3 + b^2 \lambda_3^2$$

$$= \lambda_1 \lambda_2 b^2 - b^2(\lambda_1 + \lambda_2)\lambda_3 + b^2 \lambda_3^2$$

$$= b^2(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$$

B-7-2. Sylvester matrix $\mathbf{E}$ is given by

$$\mathbf{E} = \begin{bmatrix} 0.1 & 0 & -0.24 & 0 \\ -0.7 & 0.1 & 0.2 & -0.24 \\ 1 & -0.7 & 1 & 0.2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Define matrices $\mathbf{D}$ and $\mathbf{M}$ as follows:

$$\underset{\sim}{D} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then $\underset{\sim}{M}$ is obtained from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1} \underset{\sim}{D}$$

A MATLAB solution for determining $\underset{\sim}{M}$ is shown below.

```
E =

    0.1000         0   -0.2400         0
   -0.7000    0.1000    0.2000   -0.2400
    1.0000   -0.7000    1.0000    0.2000
         0    1.0000         0    1.0000

/ inv(E)

ans =

  -29.5455  -12.2727   -4.6364   -2.0182
  -51.1364  -19.3182   -8.4091   -2.9545
  -16.4773   -5.1136   -1.9318   -0.8409
   51.1364   19.3182    8.4091    3.9545

/ D = [1;0;0;0];
/ M = (inv(E))*D

M =

  -29.5455
  -51.1364
  -16.4773
   51.1364
```

Thus, $\alpha(z)$ and $\beta(z)$ are determined as follows:

$$\alpha(z) = -51.1364z - 29.5455$$

$$\beta(z) = 51.1364z - 16.4773$$

B-7-3. The following Diophantine equation

$$\gamma(z)A(z) + \beta(z)B(z) = F(z)[H(z) - A(z)]$$

can be written as

$$[\gamma(z) + F(z)]A(z) + \beta(z)B(z) = F(z)H(z)$$

Define

$$\gamma(z) + F(z) = \alpha(z)$$

Then

$$\gamma(z) = \alpha(z) - F(z)$$

and

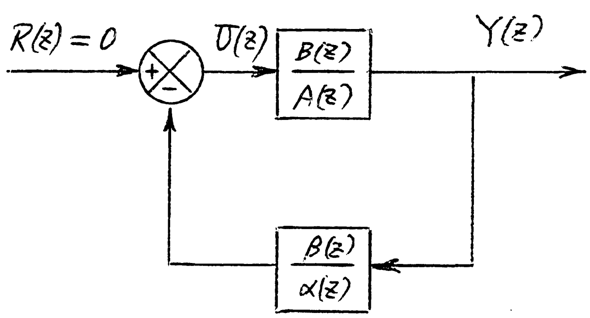
$$U(z) = -\frac{\alpha(z) - F(z)}{F(z)} U(z) - \frac{\beta(z)}{F(z)} Y(z)$$

$$= \left[-\frac{\alpha(z)}{F(z)} + 1\right] U(z) - \frac{\beta(z)}{F(z)} Y(z)$$

which can be simplified to

$$\frac{U(z)}{Y(z)} = -\frac{\beta(z)}{\alpha(z)}$$

The block diagram for this system is shown below.



Note that $\beta(z)/\alpha(z)$ is the observed-state feedback regulator. The block diagram shown above is the same as that shown in Figure 7-3 which represents an observed-state feedback regulator system.

[Note that in Example 6-11 we designed an observed-state feedback regulator system. The block diagram for the system was shown in Figure 6-13. The feed-forward transfer function was the plant transfer function and the feedback transfer function was an observed-state feedback regulator.]

B-7-4. From Figure 7-6(a) we have

$$U(z) = 8R(z) - \left[-U(z) + \frac{24z - 16}{z} Y(z) + \frac{z + 0.32}{z} U(z) \right]$$

Hence

$$U(z) = -0.32z^{-1}U(z) - 24Y(z) + 16z^{-1}Y(z) + 8R(z) \tag{1}$$

By taking the inverse z transform of this last equation, we obtain

$$u(k) = -0.32u(k - 1) - 24y(k) + 16y(k - 1) + 8r(k)$$

$$k = 1, 2, 3, \ldots$$

$$u(0) = -24y(0) + 8r(0)$$

To find u(k) versus k for the unit-step input r(k) = 1, we need to find U(z)/R(z). From Figure 7-6(b), we have

$$\frac{Y(z)}{R(z)} = \frac{8(0.02z + 0.02)}{z^2 - 1.2z + 0.52}$$

Hence,

$$Y(z) = \frac{8(0.02z + 0.02)}{z^2 - 1.2z + 0.52} R(z) \tag{2}$$

Equation (1) can be written as

$$(z + 0.32)U(z) = -(24z - 16)Y(z) + 8zR(z) \tag{3}$$

By substituting Equation (2) into Equation (3), we obtain

$$(z + 0.32)U(z) = -(24z - 16) \frac{8(0.02z + 0.02)}{z^2 - 1.2z + 0.52} R(z) + 8zR(z)$$

$$= \frac{8z^3 - 13.44z^2 + 2.88z + 2.56}{z^2 - 1.2z + 0.52} R(z)$$

By dividing both sides of this last equation by (z + 0.32), we obtain

$$U(z) = \frac{8z^2 - 16z + 8}{z^2 - 1.2z + 0.52} R(z)$$

Hence,

$$\frac{U(z)}{R(z)} = \frac{8z^2 - 16z + 8}{z^2 - 1.2z + 0.52}$$

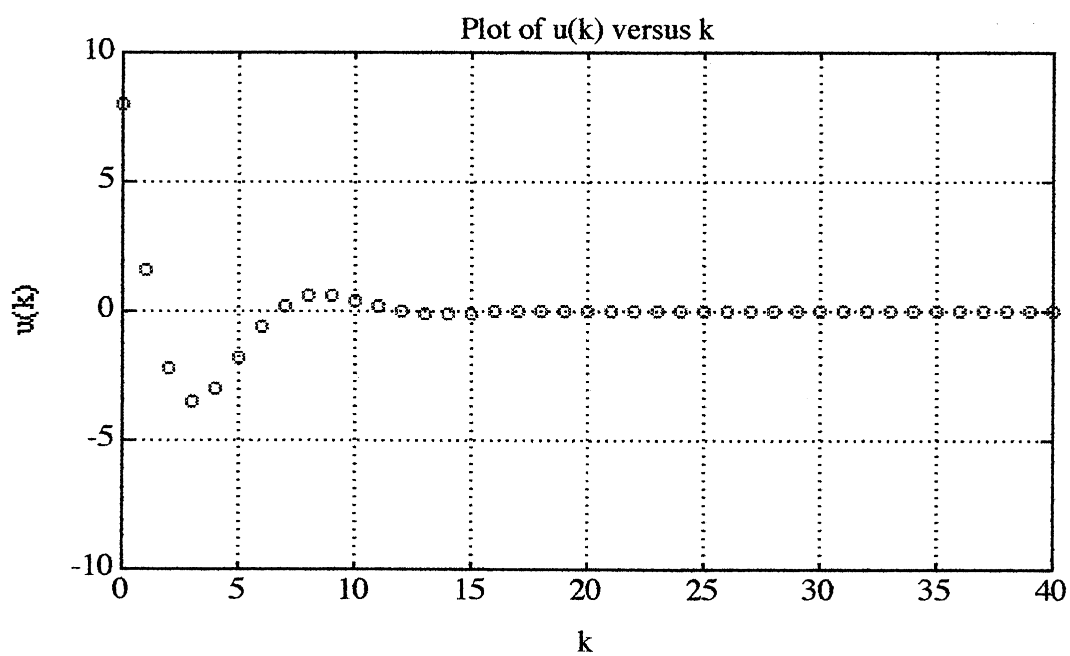
The same equation as above can also be obtained as follows:

$$\frac{U(z)}{R(z)} = \frac{U(z)}{Y(z)} \frac{Y(z)}{R(z)} = \frac{(z - 1)^2}{0.02(z + 1)} \frac{8(0.02z + 0.02)}{z^2 - 1.2z + 0.52}$$

$$= \frac{8(z - 1)^2}{z^2 - 1.2z + 0.52} = \frac{8z^2 - 16z + 8}{z^2 - 1.2z + 0.52}$$

The following MATLAB program will give u(k) versus k for the unit-step sequence, r(k) = 1 (k = 0, 1, 2, ...).

```
»% MATLAB Program for Problem B-7-4
»
»% ----- Plot of u(k) verus k when r(k) is a unit-step function -----
»
»num = [8  -16  8];
»den = [1  -1.2  0.52];
»r = ones(1,41);
»v = [0  40  -10  10];
»axis(v);
»k = 0:40;
»u = filter(num,den,r);
»plot(k,u,'o')
»grid
»title('Plot of u(k) versus k')
»xlabel('k')
»ylabel('u(k)')
```



B-7-5.

$$F(z) = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H} + D$$

$$= [1 \quad 0 \quad 0] \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ 0.16 & -0.84 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{z^3 - 0.84z + 0.16}$$

Hence,

$$\frac{Y(z)}{U(z)} = \frac{1}{z^3 - 0.84z + 0.16} = \frac{B(z)}{A(z)}$$

where

$$A(z) = z^3 - 0.84z + 0.16$$

$$B(z) = 1$$

Next, solving the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = H(z)F(z)$$

or

$$\alpha(z)(z^3 - 0.84z + 0.16) + \beta(z) = z^5$$

we obtain

$$\alpha(z) = z^2 + 0.84$$

$$\beta(z) = -0.16z^2 + 0.7056z - 0.1344$$

Referring to Figure 7-4, we have

$$\frac{Y(z)}{R(z)} = K_0 \frac{\alpha(z)B(z)}{H(z)F(z)} = K_0 \frac{z^2 + 0.84}{z^5}$$

To determine gain K_0, we set $y(\infty) = 1$ for the unit-step input.

$$\lim_{k \to \infty} y(k) = \lim_{z \to 1} (1 - z^{-1})Y(z) = \lim_{z \to 1} \left(\frac{z - 1}{z}\right)\left(K_0 \frac{z^2 + 0.84}{z^5}\right)\left(\frac{z}{z - 1}\right)$$

$$= 1.84K_0 = 1$$

from which we get

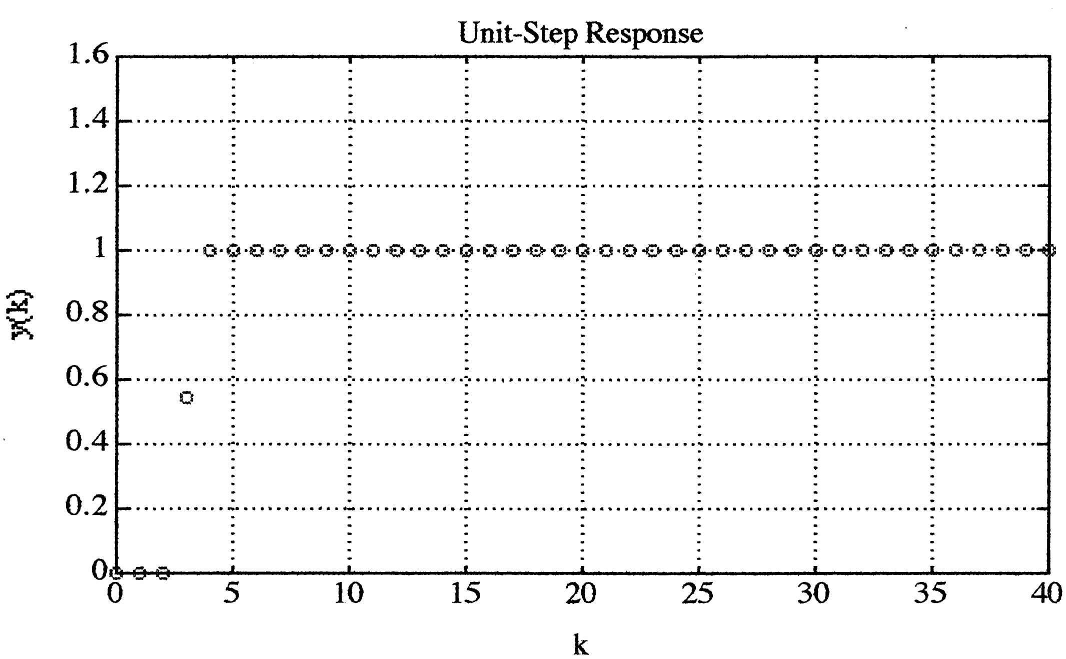
$$K_0 = \frac{1}{1.84} = 0.5435$$

Hence

$$\frac{Y(z)}{R(z)} = \frac{0.5435(z^2 + 0.84)}{z^5}$$

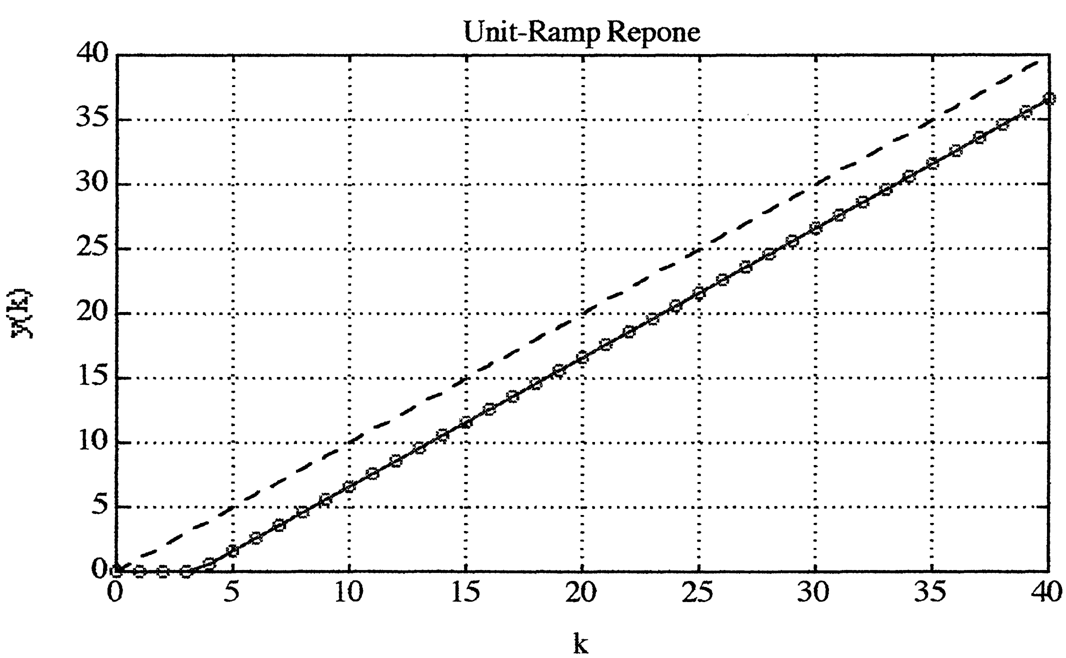
To obtain the unit-step response, we may enter the next MATLAB program into the computer.

```
»% MATLAB Program for Problem B-7-5 (Part 1)
»
»% ----- Unit-step response -----
»
»num = [0  0  0  0.5435  0.4565];
»den = [1  0  0  0  0];
»r = ones(1,41);
»v = [0  40  0  1.6];
»axis(v);
»k = 0:40;
»y = filter(num,den,r);
»plot(k,y,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('y(k)')
```



To obtain the unit-ramp response, the following MATLAB program may be entered into the computer.

```
»% MATLAB Program for Problem B-7-5 (Part 2)
»
»% ----- Unit-ramp response -----
»
»num = [0  0  0  0.5435  0.4565];
»den = [1  0  0  0  0];
»v = [0  40  0  40];
»axis(v);
»k = 0:40;
»r = [k];
»y = filter(num,den,r);
»plot(k,y,'o',k,y,'-',k,k,'--')
»grid
»title('Unit-Ramp Repone')
»xlabel('k')
»ylabel('y(k)')
```



B-7-6. Referring to the solution to Problem B-7-5, $Y(z)/U(z)$ can be obtained as

$$\frac{Y(z)}{U(z)} = \frac{1}{z^3 - 0.84z + 0.16} = \frac{B(z)}{A(z)}$$

where

$$A(z) = z^3 - 0.84z + 0.16$$

$$B(z) = 1$$

The solution to the following Diophantine equation

$$\alpha(z)A(z) + \beta(z)B(z) = H(z)F(z)$$

where

$$H(z) = z^3$$

$$F(z) = z^2$$

was obtained in the solution to Problem B-7-5 as

$$\alpha(z) = z^2 + 0.84$$

$$\beta(z) = -0.16z^2 + 0.7056z - 0.1344$$

Referring to Figure 7-5, we have

$$\frac{Y(z)}{R(z)} = \frac{K_0B(z)}{H(z)} = \frac{K_0}{z^3}$$

To determine constant K_0, we require $y(\infty)$ in the unit-step response to be unity, or

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1} \left(\frac{z-1}{z}\right)\left(\frac{K_0}{z^3}\right)\left(\frac{z}{z-1}\right)$$

$$= K_0 = 1$$

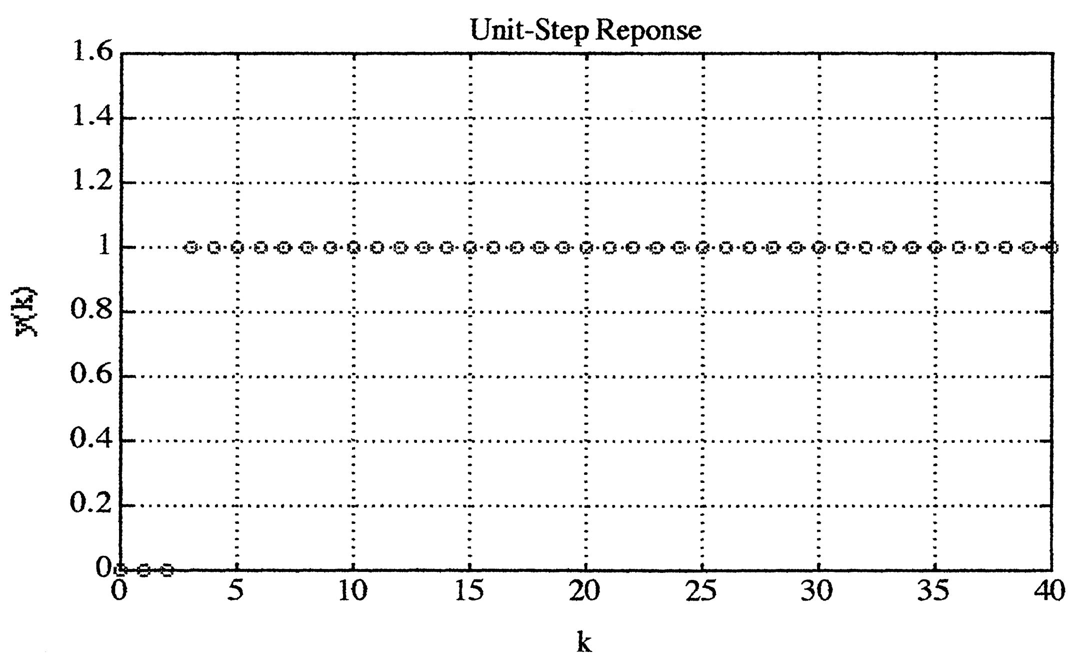
Hence

$$\frac{Y(z)}{R(z)} = \frac{1}{z^3}$$

The system has three closed-loop poles at the origin. Hence, the system is a dead-beat system.

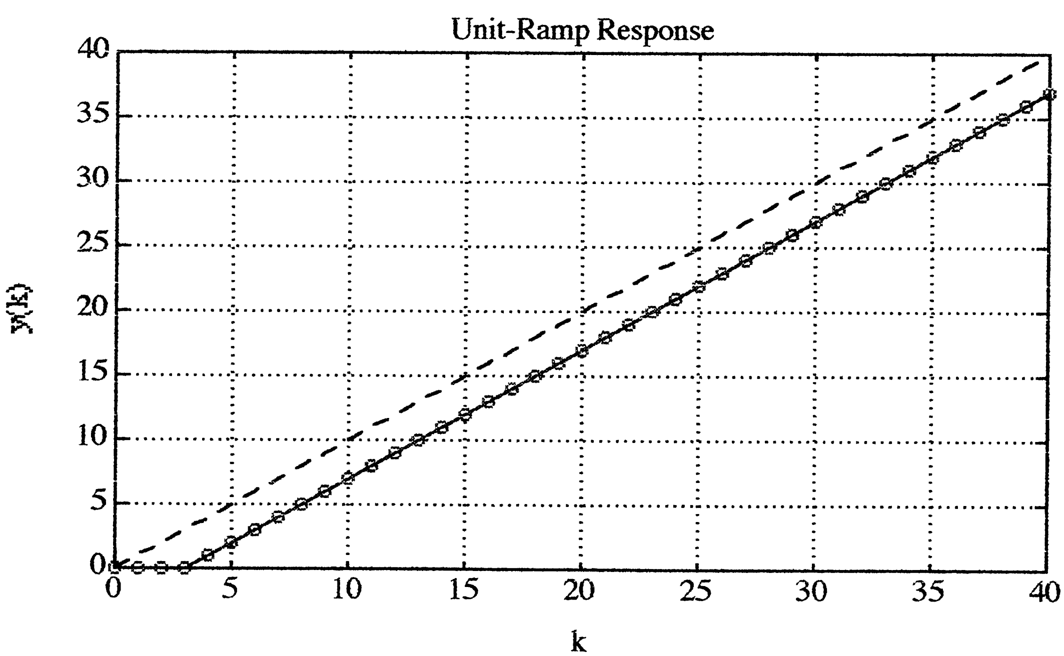
To obtain the unit-step response, we may enter the following MATLAB program into the computer.

```
»% MATLAB Program for Problem B-7-6 (Part 1)
»
»% ----- Unit-step reponse -----
»
»num = [0  0  0  1];
»den = [1  0  0  0];
»r = ones(1,41);
»v = [0  40  0  1.6];
»axis(v);
»k = 0:40;
»y = filter(num,den,r);
»plot(k,y,'o')
»grid
»title('Unit-Step Reponse')
»xlabel('k')
»ylabel('y(k)')
```



To obtain the unit-ramp response, the following MATLAB program may be entered into the computer.

```
»% MATLAB Program for Problem B-7-6 (Part 2)
»
»% ----- Unit-ramp response -----
»
»num = [0  0  0  1];
»den = [1  0  0  0];
»v = [0  40  0  40];
»axis(v);
»k = 0:40;
»r = [k];
»y = filter(num,den,r);
»plot(k,y,'o',k,y,'-',k,k,'--')
»grid
»title('Unit-Ramp Response')
»xlabel('k')
»ylabel('y(k)')
```



B-7-7. First, we determine the transfer function $Y(z)/U(z)$.

$$\frac{Y(z)}{U(z)} = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H}$$

$$= [1 \quad 0 \quad 0] \begin{bmatrix} z & 0 & 0.25 \\ -1 & z & 0 \\ 0 & -1 & z - 0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{z^2 - 0.75z}{z^3 - 0.5z^2 + 0.25} = \frac{B(z)}{A(z)}$$

Hence

$$A(z) = z^3 - 0.5z^2 + 0.25$$

$$B(z) = z^2 - 0.75z$$

Thus,

$$a_1 = -0.5, \quad a_2 = 0, \quad a_3 = 0.25$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = -0.75, \quad b_3 = 0$$

Next, we solve the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = H(z)F(z)$$

or

$$\alpha(z)(z^3 - 0.5z^2 + 0.25) + \beta(z)(z^2 - 0.75z) = z^5$$

where

$$\alpha(z) = \alpha_0 z^2 + \alpha_1 z + \alpha_2$$

$$\beta(z) = \beta_0 z^2 + \beta_1 z + \beta_2$$

To determine $\alpha(z)$ and $\beta(z)$ we define Sylvester matrix $\underset{\sim}{E}$.

$$\underset{\sim}{E} = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & -0.75 & 0 & 0 \\ -0.5 & 0 & 0.25 & 1 & -0.75 & 0 \\ 1 & -0.5 & 0 & 0 & 1 & -0.75 \\ 0 & 1 & -0.5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We also define matrices $\underset{\sim}{D}$ and $\underset{\sim}{M}$ as follows:

$$\underset{\sim}{D} = \begin{bmatrix} d_5 \\ d_4 \\ d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then $\underset{\sim}{M}$ is determined from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1}\,\underset{\sim}{D}$$

A MATLAB solution for determining $\underset{\sim}{M}$ is shown below.

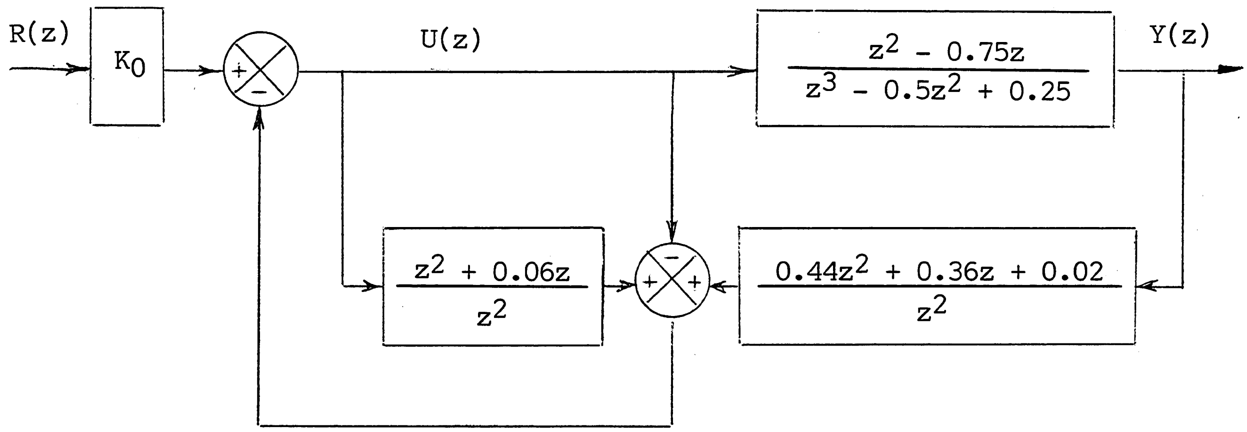
```
E =

    0.2500         0         0         0         0         0
         0    0.2500         0   -0.7500         0         0
   -0.5000         0    0.2500    1.0000   -0.7500         0
    1.0000   -0.5000         0         0    1.0000   -0.7500
         0    1.0000   -0.5000         0         0    1.0000
         0         0    1.0000         0         0         0

/ D = [0;0;0;0;0;1];
/ M = inv(E)*D

M =

         0
    0.0600
    1.0000
    0.0200
    0.3600
    0.4400
```

Thus, $\alpha(z)$ and $\beta(z)$ are determined as follows:

$$\alpha(z) = z^2 + 0.06z$$

$$\beta(z) = 0.44z^2 + 0.36z + 0.02$$

The block diagram for the designed system is shown on next page.



The gain constant K_0 is determined from the requirement that $y(\infty)$ is unity in the unit-step response. Since

$$\frac{Y(z)}{R(z)} = \frac{K_0F(z)B(z)}{H(z)F(z)} = \frac{K_0B(z)}{H(z)} = \frac{K_0(z - 0.75)}{z^2}$$

we have

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}\left(\frac{z - 1}{z}\right)\left[\frac{K_0(z - 0.75)}{z^2}\right]\left(\frac{z}{z - 1}\right)$$

$$= 0.25K_0 = 1$$

Hence K_0 is determined as

$$K_0 = 4$$

The designed system is of second order.

B-7-8. We shall assume that the designed system has the block diagram of Figure 7-9. For the given plant,

$$\frac{Y(z)}{U(z)} = \frac{0.6z + 0.5}{(z - 1)^2} = \frac{B(z)}{A(z)}$$

Thus,

$$A(z) = (z - 1)^2 = z^2 - 2z + 1 = z^2 + a_1z + a_2$$

$$B(z) = 0.6z + 0.5 = b_0z^2 + b_1z + b_2$$

Hence,

$$a_1 = -2, \quad a_2 = 1, \quad b_0 = 0, \quad b_1 = 0.6, \quad b_2 = 0.5$$

[Clearly, there are no common factors between A(z) and B(z) and the numerator

$B(z)$ is a stable polynomial.]

In the design process we choose $H(z)$ as the desired characteristic polynomial of degree 2. Let us choose a stable polynomial of degree 1 as $H_1(z)$, or

$$H_1(z) = z + 0.5$$

[Choice of $H_1(z)$ is, in a sense, arbitrary as long as it is a stable polynomial.] Now define

$$H(z) = B(z)H_1(z) = (0.6z + 0.5)(z + 0.5)$$

(This is the desired characteristic polynomial for this system.) Next, we choose

$$F(z) = z$$

[$F(z)$ can be any stable first-degree polynomial.] Define

$$\begin{aligned} D(z) &= F(z)B(z)H_1(z) \\ &= z(0.6z + 0.5)(z + 0.5) \\ &= 0.6z^3 + 0.8z^2 + 0.25z \end{aligned}$$

Hence,

$$d_0 = 0.6, \quad d_1 = 0.8, \quad d_2 = 0.25, \quad d_3 = 0$$

We determine first-degree polynomials $\alpha(z)$ and $\beta(z)$ by solving the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)B(z)H_1(z) = D(z)$$

or

$$\alpha(z)(z^2 - 2z + 1) + \beta(z)(0.6z + 0.5) = z(0.6z + 0.5)(z + 0.5) \tag{1}$$

The 4 x 4 Sylvester matrix $\underset{\sim}{E}$ for this problem becomes as follows:

$$\underset{\sim}{E} = \begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ 1 & a_1 & b_0 & b_1 \\ 0 & 1 & 0 & b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ -2 & 1 & 0.6 & 0.5 \\ 1 & -2 & 0 & 0.6 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Define

$$\underset{\sim}{D} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \\ 0.8 \\ 0.6 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then matrix $\underset{\sim}{M}$ can be obtained from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1} \underset{\sim}{D}$$

The MATLAB solution for $\underset{\sim}{M}$ is

$$\underset{\sim}{M} = \begin{bmatrix} 0.5 \\ 0.6 \\ -1 \\ 2.5 \end{bmatrix}$$

$\alpha(z)$ and $\beta(z)$ are given by

$$\alpha(z) = \alpha_0 z + \alpha_1 = 0.6z + 0.5$$

$$\beta(z) = \beta_0 z + \beta_1 = 2.5z - 1$$

However, Equation (1) can be solved easily without using MATLAB. Since Equation (1) can be written as

$$\alpha(z)(z^2 - 2z + 1) = (0.6z + 0.5)z^2 + (0.6z + 0.5)[-\beta(z) + 0.5z] \qquad (2)$$

and $\beta(z)$ is a first-degree polynomial, the coefficients of the z^2 terms must be the same. Thus,

$$\alpha(z) = 0.6z + 0.5$$

Then Equation (2) can be simplified to

$$z^2 - 2z + 1 = z^2 - \beta(z) + 0.5z$$

from which we obtain

$$\beta(z) = 2.5z - 1$$

This result is the same as the MATLAB solution.

Using $\alpha(z)$ and $\beta(z)$ thus determined, $Y(z)/V(z)$ becomes as follows:

$$\frac{Y(z)}{V(z)} = \frac{F(z)B(z)}{F(z)B(z)H_1(z)} = \frac{1}{H_1(z)} = \frac{1}{z + 0.5}$$

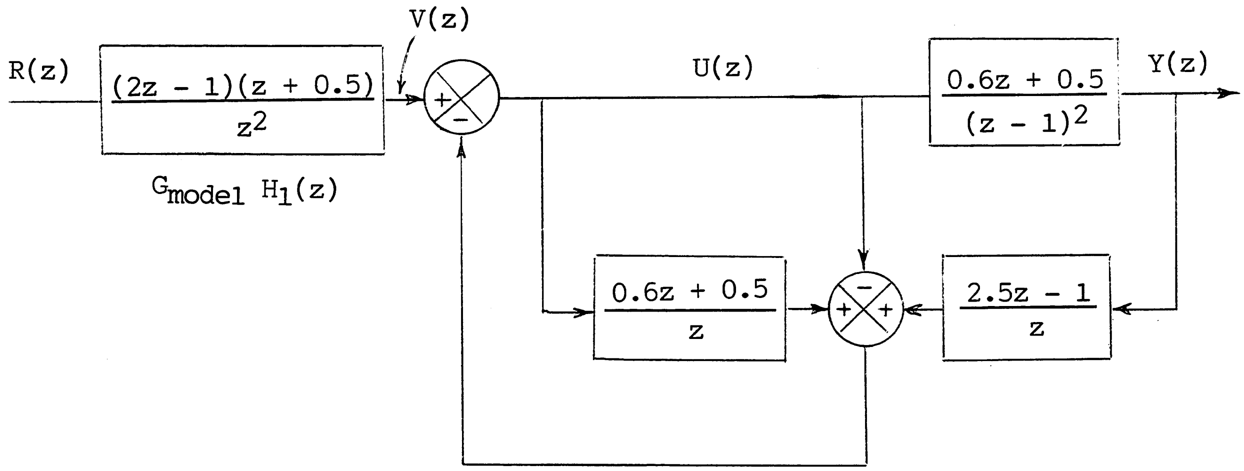
Since $V(z)/R(z)$ is

$$\frac{V(z)}{R(z)} = G_{model}\, H_1(z) = \frac{2z - 1}{z^2}(z + 0.5)$$

the pulse transfer function $Y(z)/R(z)$ becomes

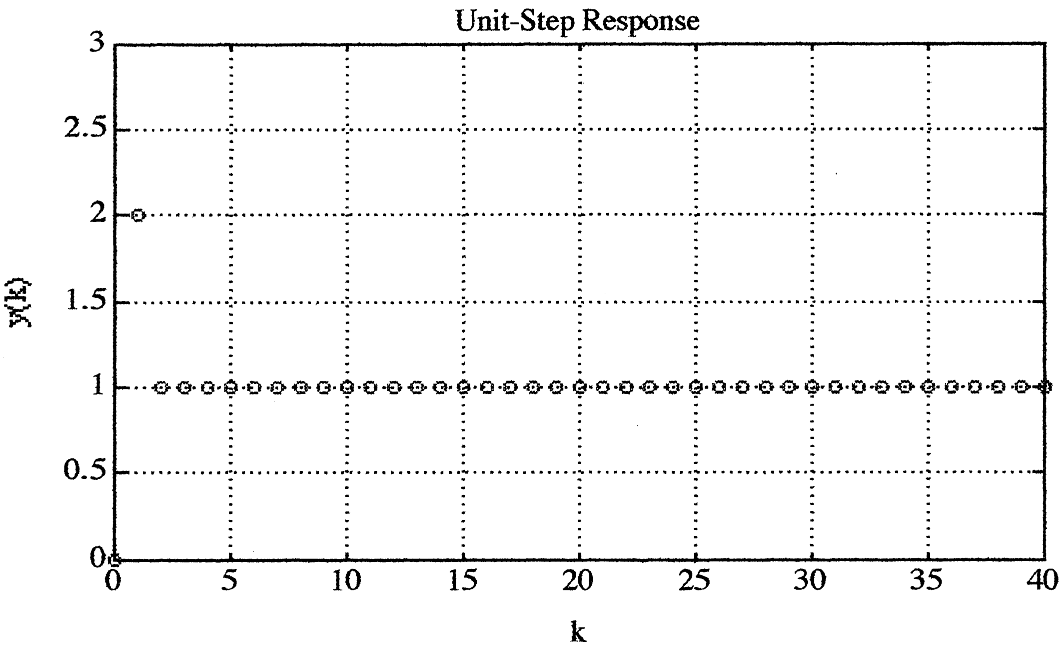
$$\frac{Y(z)}{R(z)} = \frac{Y(z)}{V(z)}\,\frac{V(z)}{R(z)} = \frac{2z - 1}{z^2} = G_{model}$$

The designed model-matching control system has the block diagram as shown in the next page.



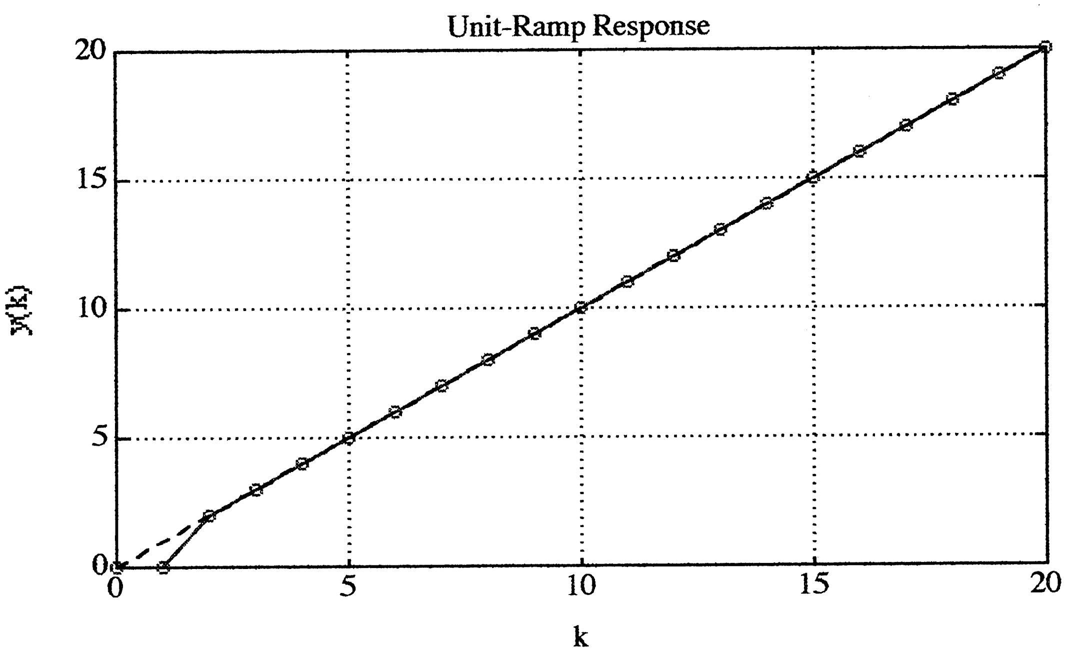
The unit-step response of this system can be obtained by entering the following MATLAB program into the computer.

```
»% MATLAB Program for Problem B-7-8 (Part 1)
»
»% ----- Unit-step response -----
»
»num = [0  2  -1];
»den = [1  0  0];
»r = ones(1,41);
»v = [0  40  0  3];
»axis(v);
»k = 0:40;
»y = filter(num,den,r);
»plot(k,y,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('y(k)')
```



The unit-ramp response of the system can be obtained by entering the following MATLAB program into the computer.

```
»% MATLAB Program for Problem B-7-8 (Part 2)
»
»% ----- Unit-ramp response -----
»
»num = [0  2  -1];
»den = [1  0  0];
»v = [0  20  0  20];
»axis(v);
»k = 0:20;
»r = [k];
»y = filter(num,den,r);
»plot(k,y,'o',k,y,'-',k,k,'--')
»grid
»title('Unit-Ramp Response')
»xlabel('k')
»ylabel('y(k)')
```



B-7-9. We shall assume that the designed system has the same block diagram as that of Figure 7-9. For the given plant,

$$\frac{Y(z)}{U(z)} = \frac{0.01873z + 0.01752}{z^2 - 1.8187z + 0.8187} = \frac{B(z)}{A(z)}$$

Thus,

$$A(z) = z^2 - 1.8187z + 0.8187 = z^2 + a_1 z + a_2$$

$$B(z) = 0.01873z + 0.01752 = b_0 z^2 + b_1 z + b_2$$

Hence,

$$a_1 = -1.8187, \quad a_2 = 0.8187$$

$$b_0 = 0, \quad b_1 = 0.01873, \quad b_2 = 0.01752$$

[Clearly, there are no common factors between A(z) and B(z) and the numerator B(z) is a stable polynomial.]

In the design process we choose H(z) as the desired characteristic polynomial of degree 2. Let us choose a stable polynomial of degree 1 as $H_1(z)$, or

$$H_1(z) = z + 0.5$$

[Choice of $H_1(z)$ is, in a sense, arbitrary as long as it is a stable polynomial.] Now define

$$H(z) = B(z)H_1(z) = (0.01873z + 0.01752)(z + 0.5)$$

(This is the desired characteristic polynomial for this system.) Next, we choose

$$F(z) = z$$

[F(z) can be any stable first-degree polynomial.] Define

$$\begin{aligned} D(z) &= F(z)B(z)H_1(z) \\ &= z(0.01873z + 0.01752)(z + 0.5) \\ &= 0.01873z^3 + 0.026885z^2 + 0.00876z \end{aligned}$$

Hence,

$$d_0 = 0.01873, \quad d_1 = 0.026885, \quad d_2 = 0.00876, \quad d_3 = 0$$

Define

$$\alpha(z) = \alpha_0 z + \alpha_1$$

$$\beta(z) = \beta_0 z + \beta_1$$

We determine $\alpha(z)$ and $\beta(z)$ by solving the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)B(z)H_1(z) = D(z)$$

or

$$\alpha(z)(z^2 - 1.8187z + 0.8187) + \beta(z)(0.01873z + 0.01752)$$
$$= 0.01873z^3 + 0.026885z^2 + 0.00876z$$

The 4 x 4 Sylvester matrix $\underset{\sim}{E}$ for this problem becomes as follows:

$$\underset{\sim}{E} = \begin{bmatrix} 0.8187 & 0 & 0.01752 & 0 \\ -1.8187 & 0.8187 & 0.01873 & 0.01752 \\ 1 & -1.8187 & 0 & 0.01873 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Define

$$\underset{\sim}{D} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.00876 \\ 0.026885 \\ 0.01873 \end{bmatrix}, \qquad M = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then matrix $\underset{\sim}{M}$ can be determined from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1} \underset{\sim}{D}$$

A MATLAB solution for the determination of $\underset{\sim}{M}$ is given below.

```
E =

    0.8187         0    0.0175         0
   -1.8187    0.8187    0.0187    0.0175
    1.0000   -1.8187         0    0.0187
         0    1.0000         0         0

/ D = [0;0.00876;0.026885;0.01873];
/ format long
/ M = inv(E)*D

M =

   0.01752000000000
   0.01873000000000
  -0.81870000000000
   2.31870000000000
```

Hence

$$\alpha(z) = \alpha_0 z + \alpha_1 = 0.01873z + 0.01752$$

$$\beta(z) = \beta_0 z + \beta_1 = 2.3187z - 0.8187$$

Using $\alpha(z)$ and $\beta(z)$ thus determined, $Y(z)/V(z)$ becomes as follows:

$$\frac{Y(z)}{V(z)} = \frac{F(z)B(z)}{F(z)B(z)H_1(z)} = \frac{1}{H_1(z)} = \frac{1}{z + 0.5}$$

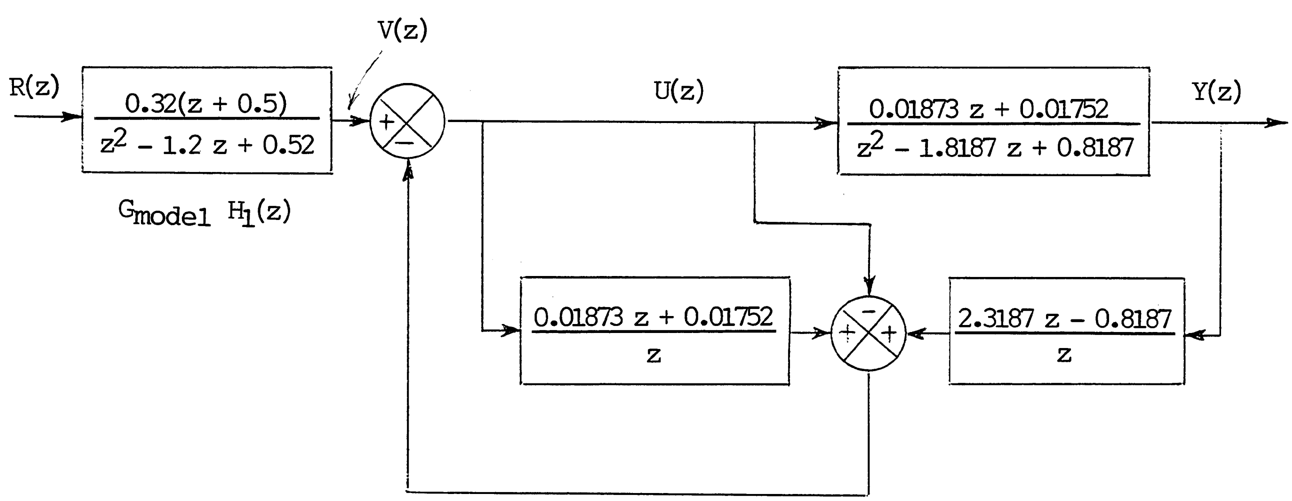
Since $V(z)/R(z)$ is

$$\frac{V(z)}{R(z)} = G_{\text{model}}\, H_1(z) = \frac{0.32}{z^2 - 1.2z + 0.52}\,(z + 0.5)$$

the pulse transfer function $Y(z)/R(z)$ becomes

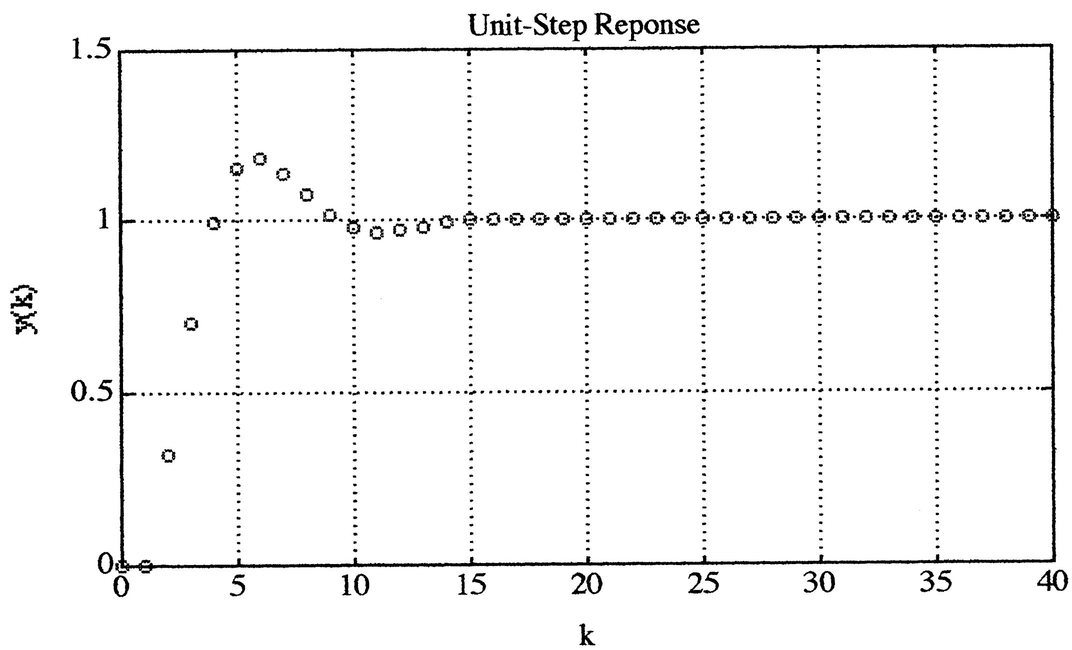
$$\frac{Y(z)}{R(z)} = \frac{Y(z)}{V(z)}\,\frac{V(z)}{R(z)} = \frac{0.32}{z^2 - 1.2z + 0.52} = G_{\text{model}}$$

The designed model-matching control system has the block diagram as shown in the next page.



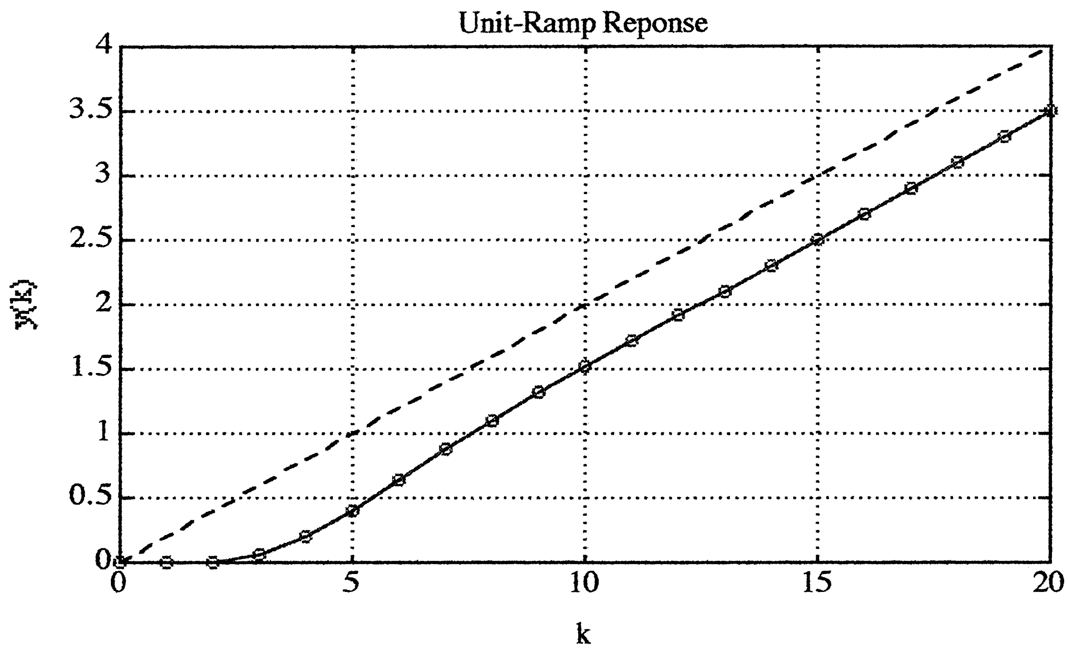
The unit-step response of this system can be obtained by entering the following MATLAB program into the computer.

```
»% MATLAB Program for Problem B-7-9 (Part 1)
»
»% ----- Unit-step repone -----
»
»num = [0  0  0.32];
»den = [1  -1.2  0.52];
»r = ones(1,41);
»v = [0  40  0  1.5];
»axis(v);
»k = 0:40;
»y = filter(num,den,r);
»plot(k,y,'o')
»grid
»title('Unit-Step Reponse')
»xlabel('k')
»ylabel('y(k)')
```



The unit-ramp response of the system can be obtained by entering the following MATLAB program into the computer.

```
»% MATLAB Program for Problem B-7-9 (Part 2)
»
»% ----- Unit-ramp respone -----
»
»num = [0  0  0.32];
»den = [1  -1.2  0.52];
»v = [0  20  0  4];
»axis(v);
»k = 0:20;
»r = [0.2*k];
»y = filter(num,den,r);
»plot(k,y,'o',k,y,'-',k,0.2*k,'--')
»grid
»title('Unit-Ramp Reponse')
»xlabel('k')
»ylabel('y(k)')
```



Unit-Ramp Reponse
4
3.5
3
2.5
2
1.5
1
0.5
0
y(k)
0
5
10
15
20
k

CHAPTER 8

B-8-1. Referring to Equation (8-23), we have

$$\mathbf{P}(k) = \mathbf{Q} + \mathbf{G}^*\mathbf{P}(k+1)\left[\mathbf{I} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}^*\mathbf{P}(k+1)\right]^{-1}\mathbf{G}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_{11}(k+1) & p_{12}(k+1) \\ p_{12}(k+1) & p_{22}(k+1) \end{bmatrix}$$

$$\times\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_{11}(k+1) & p_{12}(k+1) \\ p_{12}(k+1) & p_{22}(k+1) \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}$$

The boundary condition for $\mathbf{P}(k)$ is

$$\mathbf{P}(8) = \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence

$$\mathbf{P}(7) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}^{-1}$$

$$\times\begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}$$

$$\mathbf{P}(6) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}$$

$$\times\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2560 & -0.2143 \\ -0.2143 & 1.7143 \end{bmatrix}$$

Similarly, we can obtain $\mathbf{P}(5)$, $\mathbf{P}(4)$, ..., $\mathbf{P}(0)$ as shown in Table B-8-1.

Next we determine the feedback gain matrix $\mathbf{K}(k)$. Referring to Equation (8-27), we have

$$\mathbf{K}(k) = \mathbf{R}^{-1}\mathbf{H}^*(\mathbf{G}^*)^{-1}\left[\mathbf{P}(k) - \mathbf{Q}\right]$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}^{-1}\left\{\begin{bmatrix} p_{11}(k) & p_{12}(k) \\ p_{12}(k) & p_{22}(k) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} p_{11}(k) - 1 & p_{12}(k) \\ p_{12}(k) & p_{22}(k) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} p_{12}(k) & p_{22}(k) - 1 \end{bmatrix}$$

Hence, we find

$$\underset{\sim}{K}(8) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{K}(7) = \begin{bmatrix} -0.1667 & 0.6667 \end{bmatrix}$$

$$\underset{\sim}{K}(6) = \begin{bmatrix} -0.2143 & 0.7143 \end{bmatrix}$$

Similarly, we get $\underset{\sim}{K}(5)$, $\underset{\sim}{K}(4)$, ..., $\underset{\sim}{K}(0)$ as given in Table B-8-1.

To compute $\underset{\sim}{x}(k)$, first define

$$\underset{\sim}{K}(k) = \begin{bmatrix} K_1(k) & K_2(k) \end{bmatrix}$$

Then

$$u(k) = - \underset{\sim}{K}(k)\underset{\sim}{x}(k) = - \begin{bmatrix} K_1(k) & K_2(k) \end{bmatrix} \underset{\sim}{x}(k)$$

and

$$\underset{\sim}{x}(k + 1) = \underset{\sim}{G}\underset{\sim}{x}(k) + \underset{\sim}{H}u(k) = \begin{bmatrix} \underset{\sim}{G} - \underset{\sim}{H}\underset{\sim}{K}(k) \end{bmatrix} \underset{\sim}{x}(k)$$

$$= \begin{bmatrix} - K_1(k) & 1 - K_2(k) \\ -0.5 - K_1(k) & 1 - K_2(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Hence

$$\underset{\sim}{x}(1) = \begin{bmatrix} 0.2114 & 0.2803 \\ -0.2886 & 0.2803 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix}$$

$$\underset{\sim}{x}(2) = \begin{bmatrix} 0.2114 & 0.2803 \\ -0.2886 & 0.2803 \end{bmatrix} \begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix} = \begin{bmatrix} 0.2032 \\ -0.2885 \end{bmatrix}$$

Similarly, we can obtain $\underset{\sim}{x}(3)$, $\underset{\sim}{x}(4)$, ..., $\underset{\sim}{x}(8)$.

Since the control sequence u(k) is given by

$$u(k) = - \underset{\sim}{K}(k)\underset{\sim}{x}(k)$$

we find

$$u(0) = - \underset{\sim}{K}(0)\underset{\sim}{x}(0) = - \begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = - 1.0166$$

$$u(1) = - \underset{\sim}{K}(1)\underset{\sim}{x}(1) = - \begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix} \begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix} = 0.2198$$

Similarly, we obtain u(2), u(3), ..., u(7).

Finally, the minimum value of J is obtained as follows:

$$J_{min} = \frac{1}{2} \underset{\sim}{x}^*(0)\underset{\sim}{P}(0)\underset{\sim}{x}(0) = \frac{1}{2} \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= 5.1348$$

Table B-8-1

k	$\mathbf{P}(k)$	$\mathbf{K}(k)$	$\mathbf{x}(k)$	$u(k)$
0	$\begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	-1.0166
1	$\begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix}$	0.2198
2	$\begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} 0.2033 \\ -0.2885 \end{bmatrix}$	0.2506
3	$\begin{bmatrix} 1.2704 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} -0.0379 \\ -0.1395 \end{bmatrix}$	0.0924
4	$\begin{bmatrix} 1.2703 & -0.2113 \\ -0.2113 & 1.7194 \end{bmatrix}$	$\begin{bmatrix} -0.2113 & 0.7194 \end{bmatrix}$	$\begin{bmatrix} -0.0471 \\ -0.0282 \end{bmatrix}$	0.0103
5	$\begin{bmatrix} 1.2697 & -0.2118 \\ -0.2118 & 1.7176 \end{bmatrix}$	$\begin{bmatrix} -0.2118 & 0.7176 \end{bmatrix}$	$\begin{bmatrix} -0.0179 \\ 0.0057 \end{bmatrix}$	-0.0079
6	$\begin{bmatrix} 1.2560 & -0.2143 \\ -0.2143 & 1.7143 \end{bmatrix}$	$\begin{bmatrix} -0.2143 & 0.7143 \end{bmatrix}$	$\begin{bmatrix} -0.0022 \\ 0.0068 \end{bmatrix}$	-0.0053
7	$\begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}$	$\begin{bmatrix} -0.1667 & 0.6667 \end{bmatrix}$	$\begin{bmatrix} 0.0015 \\ 0.0026 \end{bmatrix}$	-0.0015
8	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0011 \\ 0.0004 \end{bmatrix}$	0

B-8-2. Referring to Equation (8-74), matrix $\mathbf{P}$ can be determined from

$$\mathbf{P} = \mathbf{Q} + \mathbf{G}^*\mathbf{P}(\mathbf{I} + \mathbf{H}R^{-1}\mathbf{H}^*\mathbf{P})^{-1}\mathbf{G}$$

By substituting the given matrices $\mathbf{G}$, $\mathbf{H}$, $\mathbf{Q}$, and R into this last equation, we obtain

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$\times \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 \\ -0.5 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} -0.5p_{12} & -0.5p_{22} \\ p_{12} & p_{22} \end{bmatrix} \frac{1}{1 + p_{11}} \begin{bmatrix} 1 & -p_{12} \\ 0 & 1 + p_{11} \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 \\ -0.5 & 1 \end{bmatrix}$$

By simplifying this last equation,

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \frac{1}{1 + p_{11}} \cdot \begin{bmatrix} -0.25p_{12}^2 + 0.25p_{22}(1 + p_{11}) & 0.5p_{12}^2 - 0.5p_{22}(1 + p_{11}) \\ 0.5p_{12}^2 - 0.5p_{22}(1 + p_{11}) & -p_{12}^2 + p_{22}(1 + p_{11}) \end{bmatrix}$$

from which we obtain

$$p_{11} = 1 + \frac{1}{1 + p_{11}} [-0.25p_{12}^2 + 0.25p_{22}(1 + p_{11})] \tag{1}$$

$$p_{12} = \frac{1}{1 + p_{11}} [0.5p_{12}^2 - 0.5p_{22}(1 + p_{11})] \tag{2}$$

$$p_{22} = 0.5 + \frac{1}{1 + p_{11}} [-p_{12}^2 + p_{22}(1 + p_{11})] \tag{3}$$

From Equation (3) we get

$$p_{12}^2 = 0.5(1 + p_{11}) \tag{4}$$

By substituting Equation (4) into Equation (1), we obtain

$$p_{11} = 0.875 + 0.25p_{22} \tag{5}$$

Substitution of Equation (4) into Equation (2) yields

$$p_{12} = 0.25 - 0.5p_{22} \tag{6}$$

From Equations (4) and (5), we have

$$p_{12}^2 = 0.9375 + 0.125p_{22} \tag{7}$$

From Equations (6) and (7), we obtain

$$p_{12}^2 + 0.25p_{12} - 1 = 0$$

from which we get

$$p_{12} = -1.1328 \quad \text{or} \quad 0.8828$$

Since $\underset{\sim}{P}$ being a positive definite matrix requires $p_{22} > 0$, we choose

$$p_{12} = -1.1328$$

[See Equation (6).]   Then, from Equation (6),

$$p_{22} = 0.5 - 2p_{12} = 2.7656$$

From Equation (5) we get

$$p_{11} = 0.875 + 0.25 \times 2.7656 = 1.5664$$

The $\underset{\sim}{P}$ matrix is thus determined as

$$\underset{\sim}{P} = \begin{bmatrix} 1.5664 & -1.1328 \\ -1.1328 & 2.7656 \end{bmatrix}$$

Notice that matrix $\underset{\sim}{P}$ is positive definite. Thus, it is the desired $\underset{\sim}{P}$ matrix.

Referring to Equation (8-79), the optimal control law is given by

$$\begin{aligned} u(k) &= - (\underset{\sim}{R} + \underset{\sim}{H}^*\underset{\sim}{PH})^{-1}\underset{\sim}{H}^*\underset{\sim}{PGx}(k) \\ &= - \begin{bmatrix} 0.2207 & -0.4414 \end{bmatrix} \underset{\sim}{x}(k) = - \underset{\sim}{Kx}(k) \end{aligned}$$

where

$$\underset{\sim}{K} = \begin{bmatrix} 0.2207 & -0.4414 \end{bmatrix}$$

The minimum value of J is given by

$$J_{min} = \frac{1}{2} \underset{\sim}{x}^*(0)\underset{\sim}{Px}(0) = 4.1328$$

B-8-3. From Equation (8-87) we have

$$\underset{\sim}{G}^*\underset{\sim}{PG} - \underset{\sim}{P} = - \underset{\sim}{Q}$$

or

$$\begin{bmatrix} 1 & a \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a & -1 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

which can be simplified to

$$\begin{bmatrix} 2ap_{12} + a^2p_{22} & p_{11} + (a-2)p_{12} - ap_{22} \\ p_{11} + (a-2)p_{12} - ap_{22} & p_{11} - 2p_{12} \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

This equation is equivalent to the following three equations:

$$2ap_{12} + a^2p_{22} = -1$$

$$p_{11} + (a-2)p_{12} - ap_{22} = 0$$

$$p_{11} - 2p_{12} = -0.5$$

Solving these three equations for the p_{ij}'s, we obtain

$$\underset{\sim}{P} = \begin{bmatrix} -\dfrac{(2 + 0.5a^2)}{a(a + 2)} & \dfrac{0.5a - 1}{a(a + 2)} \\ \dfrac{0.5a - 1}{a(a + 2)} & -\dfrac{2}{a(a + 2)} \end{bmatrix}$$

Since $-1 \leqslant a < 0$, $\underset{\sim}{P}$ is positive definite.

The performance index J can now be written as

$$J = \frac{1}{2} \underset{\sim}{x}^*(0)\underset{\sim}{P}\underset{\sim}{x}(0) = \frac{1}{2} \frac{-0.5a^2 + a - 6}{a(a + 2)}$$

To obtain the optimal value of a, we set

$$\frac{dJ}{da} = 0$$

This yields

$$a = -0.8730 \quad \text{or} \quad 6.8730$$

Since $-1 \leqslant a < 0$, we discard $a = 6.8730$. Since $d^2J/da^2 > 0$ with $a = -0.8730$, the minimum value of J occurs when $a = -0.8730$. The minimum value of J becomes

$$J_{min} = \frac{1}{2} \frac{-0.5a^2 + a - 6}{a(a + 2)} \bigg|_{a = -0.8730} = 3.6865$$

B-8-4. The optimal control law is given by

$$u(k) = -Kx(k)$$

where K is the undetermined gain constant. Hence

$$x(k + 1) = (0.3679 - 0.6321\,K)x(k)$$

The performance index can be written as

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left[x^2(k) + K^2x^2(k)\right] = \frac{1}{2} \sum_{k=0}^{\infty} (1 + K^2)x^2(k)$$

Referring to Equation (8-94), we have

$$\begin{aligned}(1 + K^2)x^2(k) &= -\left[px^2(k + 1) - px^2(k)\right] \\ &= -\left[p(0.3679 - 0.6321\,K)^2x^2(k) - px^2(k)\right]\end{aligned}$$

Hence

$$\left[1 + K^2 + p(0.3679 - 0.6321\,K)^2 - p\right]x^2(k) = 0$$

This last equation must hold for any x(k). Therefore, we require that

$$1 + K^2 + p(0.3679 - 0.6321\,K)^2 - p = 0$$

or

$$p = \frac{1 + K^2}{1 - (0.3679 - 0.6321\ K)^2}$$

Noting that the performance index J can be given by

$$J = \frac{1}{2}\ px^2(0)$$

to minimize this value of J [for a given x(0)] with respect to K, we set

$$\frac{dp}{dK} = 0$$

or

$$\frac{dp}{dK} = \frac{2K\left[1 - (0.3679 - 0.6321\ K)^2\right] - (1 + K^2)\left[2(0.3679 - 0.6321\ K)0.6321\right]}{\left[1 - (0.3679 - 0.6321\ K)^2\right]^2}$$

$$= 0$$

which can be simplified as

$$K^2 + 5.4363\ K - 1 = 0$$

from which we obtain

$$K = -5.6144 \quad \text{or} \quad 0.1781$$

For K = 0.1781, we have p = 1.1037. For K = -5.6144, p becomes negative. Hence we choose

$$K = 0.1781, \qquad p = 1.1037$$

Thus, the optimal control law is given by

$$u(k) = -\ 0.1781\ x(k)$$

and the minimum value of the performance index is given by

$$J_{min} = \frac{1}{2}\ (1.1037)x^2(0) = 0.5518\ x^2(0)$$

B-8-5. Suppose that we attempt to find a positive definite matrix $\mathbf{P}$ for this system. Let us assume that $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = 1$. Since the given $\mathbf{G}$ matrix is nonsingular, Equation (8-101):

$$\mathbf{P} = \mathbf{Q} + \mathbf{G}^*\mathbf{P}(\mathbf{I} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}^*\mathbf{P})^{-1}\ \mathbf{G}$$

can be rewritten as follows:

$$(\mathbf{G}^*)^{-1}\mathbf{P}\mathbf{G}^{-1} = (\mathbf{G}^*)^{-1}\mathbf{Q}\mathbf{G}^{-1} + \mathbf{P}(\mathbf{I} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}^*\mathbf{P})^{-1} \qquad (1)$$

Noting that

$$(\underset{\sim}{G}^*)^{-1} = \begin{bmatrix} -0.5 & 0 \\ -0.5 & 1.5 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 \\ -2/3 & 2/3 \end{bmatrix}$$

$$\underset{\sim}{G}^{-1} = \begin{bmatrix} -0.5 & -0.5 \\ 0 & 1.5 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -2/3 \\ 0 & 2/3 \end{bmatrix}$$

Equation (1) can be written as

$$\begin{bmatrix} -2 & 0 \\ -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} -2 & -2/3 \\ 0 & 2/3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -2/3 \\ 0 & 2/3 \end{bmatrix}$$

$$+ \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\right)^{-1}$$

which can be simplified to

$$\begin{bmatrix} 4p_{11} & (4/3)p_{11} - (4/3)p_{12} \\ (4/3)p_{11} - (4/3)p_{12} & (4/9)p_{11} - (4/9)p_{12} - (4/9)p_{12} + (4/9)p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4/3 \\ 4/3 & 8/9 \end{bmatrix} + \frac{1}{1 + p_{11}} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & -p_{12}^2 + p_{22}(1 + p_{11}) \end{bmatrix}$$

from which we get the following three equations:

$$4p_{11} = 4 + \frac{p_{11}}{1 + p_{11}}$$

$$\frac{4}{3}p_{11} - \frac{4}{3}p_{12} = \frac{4}{3} + \frac{p_{12}}{1 + p_{11}}$$

$$\frac{4}{9}p_{11} - \frac{8}{9}p_{12} + \frac{4}{9}p_{22} = \frac{8}{9} + \frac{1}{1 + p_{11}}\left[-p_{12}^2 + p_{22}(1 + p_{11})\right]$$

Solving the first of the three equations, we obtain

$$p_{11} = 1.1328$$

Solving the second equation gives

$$p_{12} = 0.09825$$

The last of the three equations yields

$$p_{22} = -0.8428$$

Hence

$$\mathbf{P} = \begin{bmatrix} 1.1328 & 0.09825 \\ 0.09825 & -0.8428 \end{bmatrix}$$

Clearly, $\mathbf{P}$ is not positive definite. For this system no positive-definite matrix $\mathbf{P}$ exists that satisfy the Riccati equation. (This means that the quadratic optimal control approach can not be applied to this system.)

B-8-6. Consider the dual counterpart of the original system

$$\mathbf{x}(k+1) = \mathbf{G}^*\mathbf{x}(k) + \mathbf{C}^*\mathbf{u}(k) \tag{1}$$

Since the original system is completely state controllable and observable, the system defined by Equation (1) is completely state controllable. The Riccati equation considered here is

$$\mathbf{P} = \mathbf{Q} + \mathbf{GPG}^* - \mathbf{GPC}^*(\mathbf{R} + \mathbf{CPC}^*)^{-1}\mathbf{CPG}^* \tag{2}$$

Consider the equation

$$\mathbf{P} = \mathbf{Q} + \hat{\mathbf{K}}^*\mathbf{R}\hat{\mathbf{K}} + (\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C})\mathbf{P}(\mathbf{G}^* - \mathbf{C}^*\hat{\mathbf{K}}) \tag{3}$$

Assume here that $\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C}$ is a stable matrix. (We shall later prove that it is a stable matrix.)

We shall first prove that matrix $\mathbf{P}$ satisfying Equation (3) is unique. Then we show that matrix $\mathbf{P}$ satisfying Equation (2) is unique. Suppose that there exist two matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ that satisfy Equation (3). Then

$$\mathbf{P}_1 = \mathbf{Q} + \hat{\mathbf{K}}^*\mathbf{R}\hat{\mathbf{K}} + (\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C})\mathbf{P}_1(\mathbf{G}^* - \mathbf{C}^*\hat{\mathbf{K}}) \tag{4}$$

$$\mathbf{P}_2 = \mathbf{Q} + \hat{\mathbf{K}}^*\mathbf{R}\hat{\mathbf{K}} + (\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C})\mathbf{P}_2(\mathbf{G}^* - \mathbf{C}^*\hat{\mathbf{K}}) \tag{5}$$

Subtracting Equation (5) from Equation (4), we obtain

$$\mathbf{P}_1 - \mathbf{P}_2 = (\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C})(\mathbf{P}_1 - \mathbf{P}_2)(\mathbf{G}^* - \mathbf{C}^*\hat{\mathbf{K}})$$

Define

$$\mathbf{P}_1 - \mathbf{P}_2 = \hat{\mathbf{P}}$$

Then

$$\hat{\mathbf{P}} = (\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C})\hat{\mathbf{P}}(\mathbf{G}^* - \mathbf{C}^*\hat{\mathbf{K}}) \tag{6}$$

Notice that if $\hat{\mathbf{P}} \neq \mathbf{0}$, then there exists an eigenvector $\mathbf{x}_i$ of matrix $(\mathbf{G} - \hat{\mathbf{K}}^*\mathbf{C})^*$ such that

$$\hat{\mathbf{P}}\mathbf{x}_i \neq \mathbf{0}$$

Let us define the eigenvalue that is associated with the eigenvector x_i to be λ_i. Then

$$(G - \hat{K}^*C)^* x_i = \lambda_i x_i$$

Hence, from Equation (6) we have

$$(G - \hat{K}^*C)\hat{P}(G^* - C^*\hat{K})x_i - \hat{P}x_i = (G - \hat{K}^*C)\hat{P}\,\lambda_i x_i - \hat{P}x_i$$

$$= \left[\lambda_i (G - \hat{K}^*C) - I\right]\hat{P}x_i = 0 \qquad (7)$$

Equation (7) implies that λ_i^{-1} is an eigenvalue of $G - K^*C$. Since $|\lambda_i| < 1$, we have $|\lambda_i^{-1}| > 1$. This contradicts the assumption that $G - K^*C$ is a stable matrix. Hence, $\hat{P}$ must be a zero matrix, or

$$P_1 = P_2$$

Thus, we have proved the uniqueness of matrix P that satisfies Equation (3).

Notice that Equation (3) can be written as follows:

$$P = Q + \hat{K}^*R\hat{K} + (G - \hat{K}^*C)P(G^* - C^*\hat{K})$$

$$= Q + \hat{K}^*R\hat{K} + GPG^* - \hat{K}^*CPG^* - GPC^*\hat{K} + \hat{K}^*CPC^*\hat{K}$$

$$= Q + GPG^* + \left[(R + CPC^*)^{\frac{1}{2}}\hat{K} - (R + CPC^*)^{-\frac{1}{2}}CPG^*\right]^*$$

$$\cdot \left[(R + CPC^*)^{\frac{1}{2}}\hat{K} - (R + CPC^*)^{-\frac{1}{2}}CPG^*\right] - GPC^*(R + CPC^*)^{-1}CPG^* \qquad (8)$$

If we choose

$$\hat{K} = (R + CPC^*)^{-1}CPG^* \qquad (9)$$

then Equations (8) and (2) become identical. (Note that $\hat{K}$ is the optimal gain matrix that minimizes the performance index.) Hence, matrix P satisfying Equation (2) is unique.

Next, we shall prove that matrix P is positive definite. Notice that

$$P = Q + \hat{K}^*R\hat{K} + (G - \hat{K}^*C)P(G^* - C^*K) \qquad (10)$$

may be written as

$$P = \sum_{k=0}^{\infty} (G - \hat{K}^*C)^k (Q + \hat{K}^*R\hat{K})(G^* - C^*\hat{K})^k \qquad (11)$$

because

$$P = (G - \hat{K}^*C)^0 (Q + \hat{K}^*R\hat{K})(G^* - C^*\hat{K})^0$$

$$+ \sum_{k=1}^{\infty} (G - \hat{K}^*C)^k (Q + \hat{K}^*R\hat{K})(G^* - C^*\hat{K})^k$$

$$= Q + \hat{K}^*R\hat{K} + (G - \hat{K}^*C)\left[\sum_{k=0}^{\infty}(G - \hat{K}^*C)^k(Q + \hat{K}^*R\hat{K}) \cdot (G^* - C^*\hat{K})^k\right](G^* - C^*\hat{K})$$

$$= Q + \hat{K}^*R\hat{K} + (G - \hat{K}^*C)P(G^* - C^*\hat{K})$$

Since $Q + \hat{K}^*R\hat{K}$ is a positive definite matrix, from Equation (11), matrix P is also positive definite. Hence, matrix P given by Equation (10) is positive definite. Consequently, we have shown that if $G - \hat{K}^*C$ is a stable matrix and if $\hat{K}$ is given by Equation (9), or

$$\hat{K} = (R + CPC^*)^{-1}CPG^*$$

then matrix P that satisfies Equation (2) is unique and is positive definite.

Finally, we shall prove that if Equation (3) is satisfied by positive definite matrices P and $Q + \hat{K}^*R\hat{K}$, then matrix $G - \hat{K}^*C$ is a stable matrix. Let us define the eigenvector associated with an eigenvalue λ_i of $(G - \hat{K}^*C)^*$ as x_i. Then

$$(G - \hat{K}^*C)^*x_i = \lambda_i x_i$$

By premultiplying both sides of Equation (3) by x_i^* and postmultiplying both sides by x_i, we obtain

$$x_i^*Px_i = x_i^*(Q + \hat{K}^*R\hat{K})x_i + x_i^*(G - \hat{K}^*C)P(G^* - C^*\hat{K})x_i$$

Hence

$$x_i^*Px_i = x_i^*(Q + \hat{K}^*R\hat{K})x_i + \bar{\lambda}_i x_i^*P\ \lambda_i x_i$$

or

$$\left|\lambda_i\right|^2 x_i^*Px_i - x_i^*Px_i = - x_i^*(Q + \hat{K}^*R\hat{K})x_i$$

which can be written as

$$(\left|\lambda_i\right|^2 - 1)x_i^*Px_i = - x_i^*(Q + K^*RK)x_i$$

Since both $x_i^*Px_i$ and $x_i^*(Q + \hat{K}^*R\hat{K})x_i$ are positive definite, we have

$$\left|\lambda_i\right|^2 - 1 < 0$$

or

$$\left|\lambda_i\right| < 1$$

Hence, we have proved that matrix $(G - \hat{K}^*C)^*$ is a stable matrix.

Next, for the original system

$$x(k + 1) = Gx(k) + Hu(k)$$

replace all G's by G^*'s and all C's by H^*'s in the Riccati equation (2) to get

$$P = Q + G^*PG - G^*PH(R + H^*PH)^{-1} H^*PG \tag{12}$$

Then, by exactly the same approach, it can be proved that this Riccati equation [Equation (12)] has a unique positive definite solution and matrix $\underset{\sim}{G} - \underset{\sim}{HK}$ is a stable matrix.

B-8-7.

MATLAB Program for Problem B-8-7

```
% ---------- Minimum-energy control problem ----------

% ***** Solution of minimum-energy control problem, where
% the number of unknown variables is greater than that of
% equations, is obtained by use of the right pseudoinverse *****

% ***** Enter matrices G, H, and x0 *****

G = [1  0.6321;0  0.3679];
H = [0.3679;0.6321];
x0 = [5;-5];

% ***** Enter matrices f1, f2, and f3 *****

f1 = inv(G)*H;
f2 = inv(G)^2*H;
f3 = inv(G)^3*H;

% ***** Enter matrix F = [f1  f2  f3] and compute the right
% pseudoinverse FRM = F'*inv(F*F') *****

F = [f1  f2  f3];
FRM = F'*inv(F*F')
FRM =

     0.7910     0.7191
     0.5000     0.4738
    -0.2910    -0.1929

% ***** Optimal control uopt can be given by *****

uopt = - FRM*x0

uopt =

    -0.3598
    -0.1310
     0.4908

u0 = uopt(1), u1 = uopt(2), u2 = uopt(3)

u0 =

   -0.3598
```

```
u1 =

    -0.1310

u2 =

    0.4908

% ***** The minimum value of performance index J can be
% given by *****

Jmin = 0.5*uopt'*uopt

Jmin =

    0.1937
```

B-8-8. Notice that the rank of $[\mathbf{H} \quad \mathbf{GH}]$ is two and the given system is completely state controllable. We shall solve three cases separately.

Case 1 n = 2: Using the right pseudoinverse we shall determine u(0), u(1), and u(2). Since state $\mathbf{x}(3)$ can be given by

$$\mathbf{x}(3) = \mathbf{G}^3\mathbf{x}(0) + \mathbf{G}^2\mathbf{H}u(0) + \mathbf{GH}u(1) + \mathbf{H}u(2)$$

if we set $\mathbf{x}(3) = \mathbf{0}$, then

$$\mathbf{x}(0) = - \mathbf{G}^{-1}\mathbf{H}u(0) - \mathbf{G}^{-2}\mathbf{H}u(1) - \mathbf{G}^{-3}\mathbf{H}u(2)$$

Define

$$\mathbf{f}_i = \mathbf{G}^{-i}\mathbf{H}$$

Then

$$\mathbf{x}(0) = - \mathbf{f}_1 u(0) - \mathbf{f}_2 u(1) - \mathbf{f}_3 u(2) \tag{1}$$

where

$$\mathbf{f}_1 = \mathbf{G}^{-1}\mathbf{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{f}_2 = \mathbf{G}^{-2}\mathbf{H} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{f}_3 = \mathbf{G}^{-3}\mathbf{H} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence, Equation (1) becomes

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

By use of the right pseudoinverse, we obtain

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = - \underset{\sim}{F}^{RM} \underset{\sim}{x}(0) = - \underset{\sim}{F}^*(\underset{\sim}{F}\underset{\sim}{F}^*)^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

where

$$\underset{\sim}{F} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

and

$$F^{RM} = F^*(FF^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Hence

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \tag{2}$$

The minimum value of J is given by

$$J_{min} = \frac{1}{2} \left[u^2(0) + u^2(1) + u^2(2) \right] = \frac{1}{2} (1 + 1 + 0) = 1$$

Notice that by using the control sequence given by Equation (2) we have

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The state $\underset{\sim}{x}(2)$ becomes $\underset{\sim}{0}$ in two sampling periods, or $\underset{\sim}{x}(2) = \underset{\sim}{0}$. Thus, in this case the minimum energy control is the same as the time optimal control.

<u>Case 2 n = 3</u>: State $\underset{\sim}{x}(4)$ can be given by

$$\underset{\sim}{x}(4) = \underset{\sim}{G}^4\underset{\sim}{x}(0) + \underset{\sim}{G}^3\underset{\sim}{H}u(0) + \underset{\sim}{G}^2\underset{\sim}{H}u(1) + \underset{\sim}{G}\underset{\sim}{H}u(2) + \underset{\sim}{H}u(3)$$

Substituting $\underset{\sim}{0}$ for $\underset{\sim}{x}(4)$ in this equation yields

$$\underset{\sim}{x}(0) = - \underset{\sim}{G}^{-1}\underset{\sim}{H}u(0) - \underset{\sim}{G}^{-2}\underset{\sim}{H}u(1) - \underset{\sim}{G}^{-3}\underset{\sim}{H}u(2) - \underset{\sim}{G}^{-4}\underset{\sim}{H}u(3)$$

Define

$$\underset{\sim}{f}_i = \underset{\sim}{G}^{-i}\underset{\sim}{H}$$

Then

$$\underset{\sim}{x}(0) = - \underset{\sim}{f}_1u(0) - \underset{\sim}{f}_2u(1) - \underset{\sim}{f}_3u(2) - \underset{\sim}{f}_4u(3) \tag{3}$$

where

$$\underset{\sim}{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underset{\sim}{f}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underset{\sim}{f}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \underset{\sim}{f}_4 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Hence, Equation (3) becomes

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix}$$

By use of the right pseudoinverse, we obtain

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} = - \underset{\sim}{F}^{RM} \underset{\sim}{x}(0) = - \underset{\sim}{F}^*(\underset{\sim}{FF}^*)^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

where

$$\underset{\sim}{F} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

and

$$\underset{\sim}{F}^{RM} = \underset{\sim}{F}^*(\underset{\sim}{FF}^*)^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \\ 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Hence

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} = - \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \\ 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Then, the minimum value of J, when $n = 3$, can be obtained as

$$J_{min} = \frac{1}{2} [u^2(0) + u^2(1) + u^2(2) + u^2(3)] = \frac{5}{6}$$

Case 3   n = 4:   State $\mathbf{x}(4)$ can be given by

$$\mathbf{x}(5) = \mathbf{G}^5\mathbf{x}(0) + \mathbf{G}^4\mathbf{H}u(0) + \mathbf{G}^3\mathbf{H}u(1) + \mathbf{G}^2\mathbf{H}u(2) + \mathbf{GH}u(3) + \mathbf{H}u(4)$$

Substituting $\mathbf{0}$ for $\mathbf{x}(5)$ in this last equation yields

$$\mathbf{x}(0) = - \mathbf{G}^{-1}\mathbf{H}u(0) - \mathbf{G}^{-2}\mathbf{H}u(1) - \mathbf{G}^{-3}\mathbf{H}u(2) - \mathbf{G}^{-4}\mathbf{H}u(3) - \mathbf{G}^{-5}\mathbf{H}u(4)$$

Define

$$\mathbf{f}_i = \mathbf{G}^{-i}\mathbf{H}$$

Then

$$\mathbf{x}(0) = - \mathbf{f}_1u(0) - \mathbf{f}_2u(1) - \mathbf{f}_3u(2) - \mathbf{f}_4u(3) - \mathbf{f}_5u(4) \tag{4}$$

where

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{f}_4 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{f}_5 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence Equation (4) becomes

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ u(4) \end{bmatrix}$$

By use of the right pseudoinverse, we obtain

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ u(4) \end{bmatrix} = - \mathbf{F}^{RM}\mathbf{x}(0) = - \mathbf{F}^*(\mathbf{FF}^*)^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

where

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & -3 \end{bmatrix}$$

$$\mathbf{F}^{RM} = \mathbf{F}^*(\mathbf{FF}^*)^{-1} = \begin{bmatrix} 5/8 & 3/8 \\ 3/8 & 7/24 \\ 1/4 & 1/12 \\ 1/8 & 5/24 \\ 1/8 & -1/8 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ u(4) \end{bmatrix} = - \begin{bmatrix} 5/8 & 3/8 \\ 3/8 & 7/24 \\ 1/4 & 1/12 \\ 1/8 & 5/24 \\ 1/8 & -1/8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix}$$

Then, the minimum value of J is obtained as

$$J_{min} = \frac{1}{2} [u^2(0) + u^2(1) + u^2(2) + u^2(3) + u^2(4)] = \frac{5}{6}$$

Note that in this case we have $u(4) = 0$. However, this is not always the case. If the initial state were $x_1(0) \neq x_2(0)$, i.e.,

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

then $u(4)$ would be $1/8$.

B-8-9. From Figure 8-11 we obtain the following equations:

$$x(k + 1) = 0.5x(k) + 2u(k)$$

$$u(k) = k_1v(k) - k_2x(k)$$

$$v(k) = r(k) - y(k) + v(k - 1)$$

$$y(k) = x(k)$$

where k_1 is the integral gain constant and k_2 is the feedback gain constant. In this problem, k_1 and k_2 are variables and must be determined such that the system is stable and will exhibit an acceptable transient response to the unit-step input.

Since

$$\begin{aligned} v(k + 1) &= r(k + 1) - y(k + 1) + v(k) \\ &= -0.5x(k) + v(k) - 2u(k) + r(k + 1) \end{aligned}$$

we obtain

$$\begin{bmatrix} x(k + 1) \\ v(k + 1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k + 1) \qquad (1)$$

For $k = \infty$, we have

$$\begin{bmatrix} x(\infty) \\ v(\infty) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x(\infty) \\ v(\infty) \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u(\infty) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(\infty) \qquad (2)$$

For any step input, $r(k + 1) = r(\infty) = r$. Define

$$x_e(k) = x(k) - x(\infty)$$

$$v_e(k) = v(k) - v(\infty)$$

$$u_e(k) = u(k) - u(\infty)$$

Subtracting Equation (2) from Equation (1), we obtain

$$\begin{bmatrix} x_e(k + 1) \\ v_e(k + 1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_e(k) \\ v_e(k) \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u_e(k) \qquad (3)$$

Note that

$$u_e(k) = k_1 v_e(k) - k_2 x_e(k)$$

Define

$$x_1(k) = x_e(k)$$

$$x_2(k) = v_e(k)$$

$$w(k) = u_e(k)$$

Then, Equation (3) can be written as follows:

$$\begin{bmatrix} x_1(k + 1) \\ x_2(k + 1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} w(k)$$

where

$$w(k) = - [k_2 \quad -k_1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Rewriting, we have

$$\underset{\sim}{x}(k + 1) = \underset{\sim}{G}\underset{\sim}{x}(k) + \underset{\sim}{H}w(k)$$

$$w(k) = -\underset{\sim}{K}\underset{\sim}{x}(k)$$

where

$$\underset{\sim}{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad \underset{\sim}{G} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad \underset{\sim}{H} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \underset{\sim}{K} = [k_2 \quad -k_1]$$

B-8-10.

MATLAB program for Problem B-8-10 (Part 1)

```
% ---------- Design of a servo system based on minimization
% of a quadratic performance index ----------

% ***** The following program solves steady-state Riccati equation and gives
% optimal feedback gain matrix K *****

% ***** Enter matrices G, H, Q, and R *****

G = [0.5  0;-0.5  1];
H = [2;-2];
Q = [100  0;0  1];
R = [1];

% ***** Start with the solution of steady-state Riccati equation
% with P = [0  0;0  0] *****

P = [0  0;0  0];
P = Q + G'*P*G - G'*P*H*inv(R+H'*P*H)*H'*P*G;

% ***** Check solution P every 10 or 20 steps of iteration.
% Stop iteration when P stays constant *****

for i = 1:20,
  P = Q + G'*P*G - G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P

P =

  100.0624    -0.0115
   -0.0115    10.1892

for i = 1:20,
  P = Q + G'*P*G - G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P

P =

  100.0624    -0.0119
   -0.0119    10.5107

for i = 1:20,
  P = Q + G'*P*G - G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P

P =

  100.0624    -0.0119
   -0.0119    10.5167
```

```
for i = 1:20,
  P = Q + G'*P*G - G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P

P =

   100.0624     -0.0119
     -0.0119    10.5168

for i = 1:20,
  P = Q + G'*P*G - G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P

P =

   100.0624     -0.0119
     -0.0119    10.5168

% ***** P matrix stays constant.  Thus steady state has
% been reached.  The steady state P matrix is *****

P

P =

   100.0624     -0.0119
     -0.0119    10.5168

% ***** Optimal feedback gain matrix K is obtained from *****

K = inv(R + H'*P*H)*H'*P*G

K =

      0.2494     -0.0475

k1 = -K(2)

k1 =

      0.0475

k2 = K(1)

k2 =

      0.2494
```

To obtain the unit-step response curve [y(k) versus k], we may proceed as follows: Since

$$x(k+1) = 0.5x(k) + 2u(k)$$
$$= 0.5x(k) + 2[-k_2x(k) + k_1v(k)]$$
$$= (0.5 - 2k_2)x(k) + 2k_1v(k)$$

and

$$v(k+1) = v(k) + r(k+1) - y(k+1)$$
$$= v(k) + r - (0.5 - 2k_2)x(k) - 2k_1v(k)$$
$$= (1 - 2k_1)v(k) + (-0.5 + 2k_2)x(k) + r$$

we get

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 - 2k_2 & 2k_1 \\ -0.5 + 2k_2 & 1 - 2k_1 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \qquad (1)$$

$$y(k) = x(k) = [1 \quad 0] \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + [0]r \qquad (2)$$

For a unit-step input, $r = 1$.

The unit-step response y(k) versus k can be obtained by first converting the state-space equations [Equations (1) and (2)] into the pulse transfer function Y(z)/R(z):

```
[num, den] = ss2tf(GG,HH,CC,DD)
```

where

```
GG = [0.5 - 2k2     2k1;-0.5 + 2k2     1 - 2k1]

HH = [0;1]

CC = [1     0]

DD = [0]
```

and then using the command <u>filter</u> as follows:

```
y = filter(num,den,r)
```

where r is a unit-step function.

To obtain the response v(k), first note that

$$v(k) = [0 \quad 1] \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} = \mathbf{FF} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

where $\mathbf{FF} = [0 \quad 1]$. Then use the following command:

```
[numv,denv] = ss2tf(GG,HH,FF,DD)

v = filter(numv,denv,r)
```

MATLAB program shown next yields the response y(k) versus k and v(k) versus k.

MATLAB Program for Problem B-8-10 (Part 2)

```
% ---------- Unit-step response of designed system ----------

% ***** This program calculates the response of the system
% when subjected to a unit-step input. The values that are
% used for k1 and k2 are computed in MATLAB Program Part 1. The
% response is obtained using the method to convert the discrete-
% time state-space equations into pulse transfer function form. The
% response is then found with the conventional 'filter' command *****

% ***** Enter values of k1 and k2 *****

k1 = 0.0475; k2 = 0.2494;

% ***** Enter matrices GG,HH,CC,FF,DD *****

GG = [0.5-2*k2  2*k1;-0.5+2*k2  1-2*k1];
HH = [0;1];
CC = [1  0];
FF = [0  1];
DD = [0];

% ***** To obtain the response y(k), convert state-space equations
% into pulse transfer function Y(z)/R(z) *****

[num,den] = ss2tf(GG,HH,CC,DD);

% ***** Enter command to obtain unit-step response y(k) *****

r = ones(1,101);
axis([0  100  0  1.2]);
k = 0:100;
y = filter(num,den,r);
plot(k,y,'o',k,y,'-')
grid
title('Output y(k) to Unit-Step Input')
xlabel('k')
ylabel('y(k)')

% ***** To obtain the response v(k), convert state-space equations
% into pulse transfer function V(z)/R(z) *****

[numv,denv] = ss2tf(GG,HH,FF,DD);

% ***** Enter command to obtain v(k) *****

axis([0  100  0  12]);
k = 0:100;
v = filter(numv,denv,r);
plot(k,v,'o',k,v,'-')
grid
title('Output v(k) of Integrator')
xlabel('k')
ylabel('v(k)')
```

The unit-step response y(k) versus k is shown in Figure (a). The response v(k) versus k is shown in Figure (b).

Notice that the system is stable and exhibits nonoscillatory response characteristics. The response characteristics depend on a set of $\underset{\sim}{Q}$ and R chosen in the performance index J.

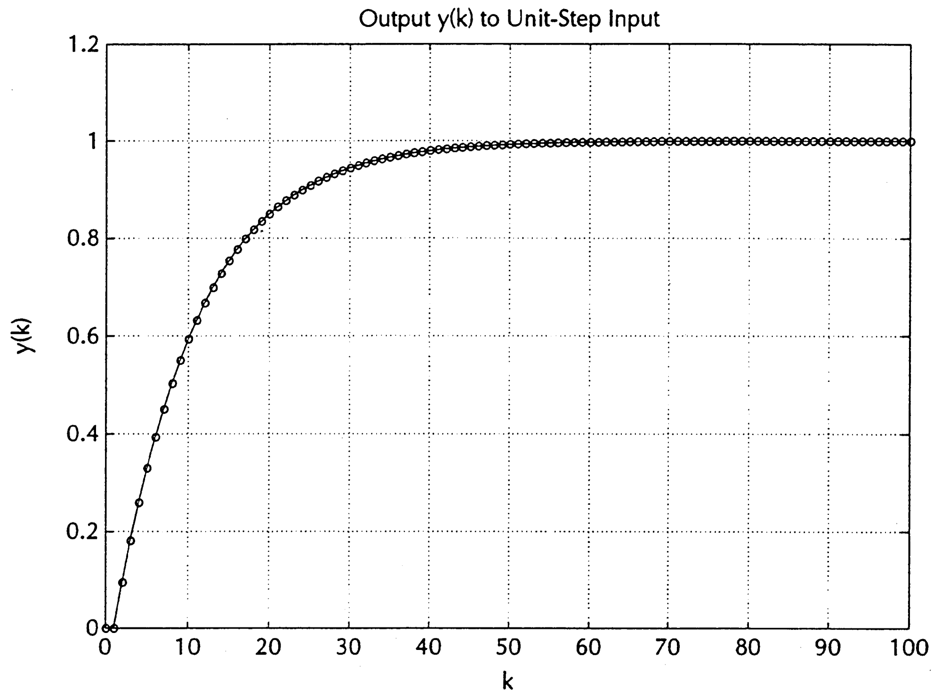


Figure (a)

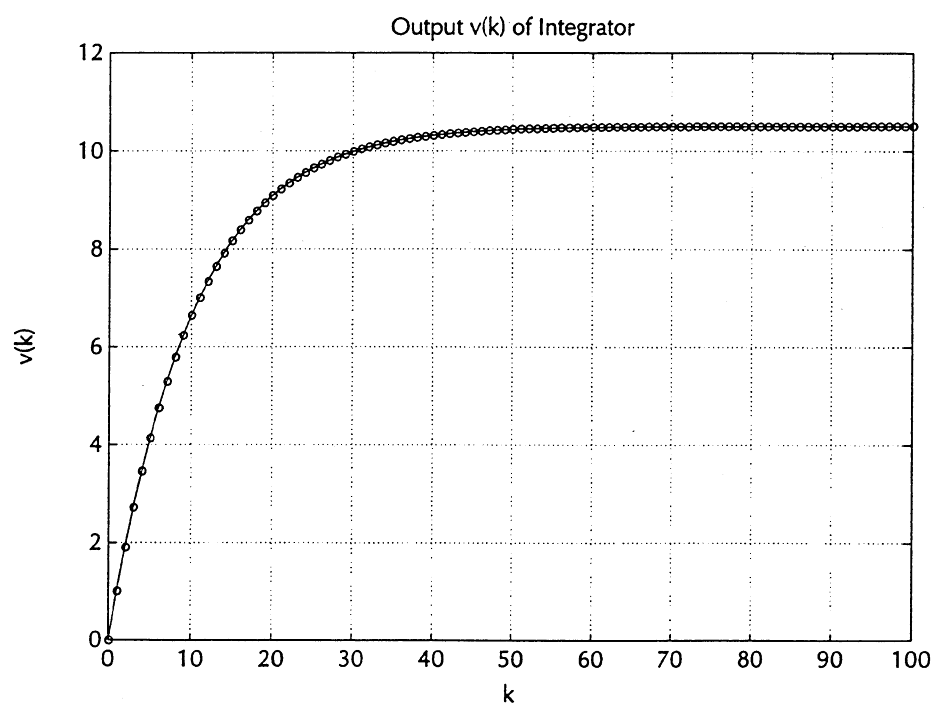


Figure (b)
