The relevant appendices can be found under 'Related Documents' for each experiment at this link:
http://sky.campus.mcgill.ca/Exp/exp.a5w#

Fourier Optics

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Warnings: Be careful when using the He-Na laser. Direct eye exposure can severely damage your retina; if need be, wear protective goggles. Specular reflections (such as mirrors) are as dangerous as direct exposure. Diffuse reflections (such as reflection on paper) are not as dangerous, but you should still be careful about highly reflective surfaces, including the optical table.

Contents

1 Introduction

Optical data processing, spatial filtering and holography are amongst a large variety of optical applications that are closely related to a basic concept in modern optics: Fourier optics. The goal of this experiment is to demonstrate how optical image formation can be treated within the mathematical framework of Fourier methods. Once the concept is established through a few fundamental experiments, some image processing and holography will demonstrate the practical applications of the subject.

Part I
Fourier Optics

2 Theory

We will first turn to the theoretical basis of the Fourier methods. You are advised to consult Appendices A and C for a more complete treatment of the subject.

2.1 Fourier Series

You are probably familiar with Fourier series as a mathematical method. Let us restate their basic theorems. A periodic function f(t), of angular frequency ω, can be considered as the sum of harmonic functions whose frequencies are multiples of ω, which is the fundamental frequency – a square wave example is shown in figure 1. We write:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right). \tag{1}$$

Or,

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t}. \tag{2}$$

The amplitude of each term of the series can be calculated using the following relations:

$$a_0 = \frac{1}{T} \int_0^T f(t) \, dt, \tag{3}$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t) \, dt, \tag{4}$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t) \, dt, \tag{5}$$

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-in\omega t} \, dt, \tag{6}$$

where $T = \frac{2\pi}{\omega}$ is the period of f(t).
We define the frequency spectrum as a plot of the coefficients as a function of $n\omega$.

2.2 Fourier Transform

We can get to a more general definition by considering a non-periodic function f(t). In this case, Fourier series need to be replaced by Fourier integrals. A way to think about the difference between those is to view a non-periodic function as a periodic function with infinite period. Thus, to express such functions as Fourier decompositions, it is necessary to sum over a continuous spectrum of frequencies. The sum of (eq.
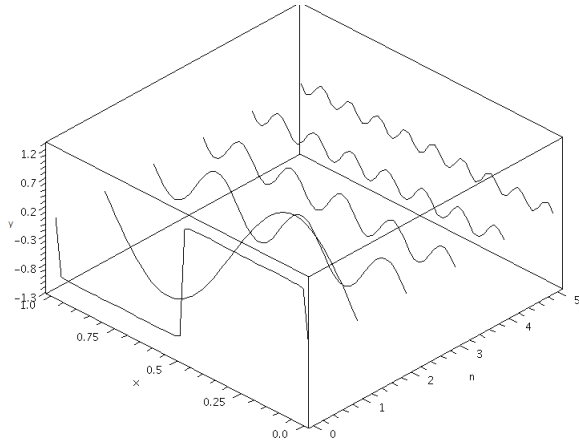
Figure 1: Example of the Fourier decomposition of a square wave. The first few components of the Fourier series are plotted, and the sum of the series is plotted at n=0. The plot of y vs n represent the frequency spectrum of the square wave.

2) can then be changed to an integral over frequencies ν:

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi\nu t} \, d\nu \tag{7}$$

where F(ν) is the equivalent of a Fourier coefficient:

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\nu t} \, dt. \tag{8}$$

The last two equations define the Fourier transform and inverse transform. It is possible to simplify by using the notation:

$$F(\nu) = \mathcal{F}[f(t)]$$
$$f(t) = \mathcal{F}^{-1}[F(\nu)]$$

The mathematically inclined might have noticed that the Fourier transform indicates a correspondence between two spaces: the space of time t and the space of frequencies ν. As we will see later, this experiment is not concerned with time dependent phenomena but rather with position dependent ones. We will study the correspondence between the spatial position x and the spatial frequency ν_x.

As a side note, it is possible to express the Fourier transform in terms of the angular frequency. In such a case, we normalize by introducing a factor of $1/\sqrt{2\pi}$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

2.3   Bandwidth Theorem

You have learned in your quantum mechanics class that momentum and position form a Fourier transform pair. You have also learned that the Heisenberg uncertainty principle states that $\Delta x \Delta p \geq \frac{\hbar}{2}$. In fact, this kind of uncertainty principle can be generalized to any wave phenomenon. In the case of a time-dependent pulse, it can be shown that[1]

$$\Delta\omega\Delta t \approx 2\pi \iff \Delta\nu\Delta t \approx 1 \tag{9}$$

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[1]See H. J. Pain. The Physics of Vibrations and Waves. Wiley, Chichester, UK, 2005, pp.132-135.

where $\Delta\nu$ and $\Delta\omega$ are frequency spreads and Δt is the time interval of a pulse. The relations presented above are often called the *bandwidth theorem*. We define $\Delta\nu$, $\Delta\omega$ and Δt as the full width at half maximum (FWHM) of the peaks they are associated to. Also, we consider positive frequencies only, as shown on figure 2. The theorem is only meaningful for non-periodic functions.
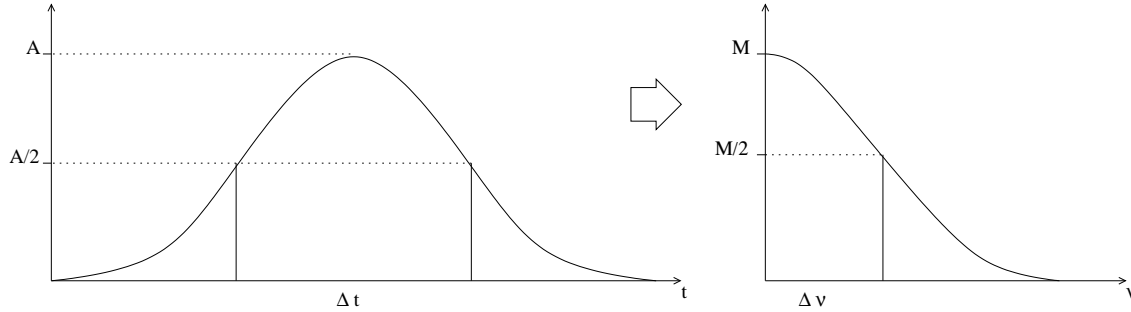

Figure 2: Illustration of the quantities involved in bandwidth theorem.

2.4   Fourier Optics

Interference

This experiment will study optical wave phenomena that are dependent on interference, or the ability of several beams of light to interact with each other. For our studies, the light of the beams must be at the same frequency and must be coherent. The use of a laser is therefore appropriate.

Quantitatively, waves can be described by complex-valued functions, having an amplitude and a phase. A typical detector returns a response proportional to the intensity of the wave, which is the square of the field. In the uni-dimensional case, the combination of two waves $A = a\mathrm{e}^{i\alpha}$ and $B = b\mathrm{e}^{i\beta}$ has an intensity given by:

$$
\begin{aligned}
(A + B)^2 &= (A + B)\overline{(A + B)} & (10)\\
&= \|A\|^2 + \|B\|^2 + A\overline{B} + \overline{A}B & (11)\\
&= a^2 + b^2 + 2ab\cos(\alpha - \beta). & (12)
\end{aligned}
$$

The first two terms of the last line represent the amplitude of the individual waves; the last term introduces a dependence on their respective phase and thus constitutes the interference term.

Given that the waves have the same wavelength λ and are in phase at the source, we can deduce the laws for constructive and destructive interference.[2] Indeed, the intensity is maximal when:

$$\Delta x = m\lambda, \tag{13}$$

and is minimal when:

$$\Delta x = (m + 1/2)\lambda, \tag{14}$$

where Δx is the path length difference between both waves and m is an integer.

Diffraction

In order to understand, or at least gain some intuition about the theory of Fourier optics, it is necessary to become familiar with Huygens-Fresnel principle. According to this principle, each point of a wavefront (light, mechanical wave, etc.) acts as a punctual source; the resulting wave at each position is the sum of

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[2]See Eugene Hecht. Optics. Addison-Wesley, New York, NY, USA, third edition, 1998.

each of the resulting "wavelets" at this position. If one takes a small aperture that allows only a fraction of the wavefront to propagate, the opening acts as a new source. This phenomenon is called diffraction. Figures 3(a) and 3(b) illustrate this idea when dealing with small and large apertures.
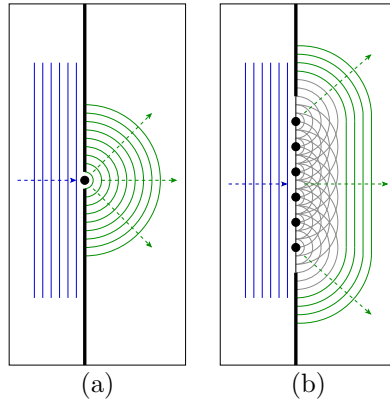

Figure 3: Illustration of diffraction based in Huygens-Fresnel principle in the case of small (a) and large (b) apertures.

We can now turn to a more quantitative formulation of Huygens-Fresnel principle. As you probably know, a wave can be described as having an amplitude and a complex phase. Furthermore, punctual sources emit what are called spherical waves that decrease in amplitude as the distance increases. Also, points of the wavefront that are equidistant from the source all have the same phase since they were emitted simultaneously. The intensity, or the square of the wave function should be inversely proportional to the increase in the area of a sphere it covers, so as r^{-2}. The resulting wave function is then:

$$\psi_{spherical} = \frac{A}{r}e^{ikr}, \tag{15}$$

where k is the spherical wavenumber, i.e. $k = \frac{2\pi}{\lambda}$, r is the distance from the source and A is an amplitude.
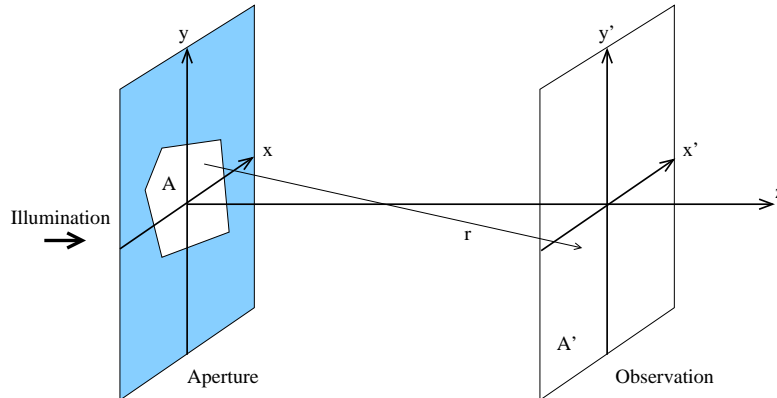
Figure 4: Diffraction of the incident light on a aperture. The observation plane is at a distance z from the aperture.

We are interested in the diffraction pattern produced by an aperture, as in figure 4. $A(x, y)$ is a 2-dimensional function describing that opening. It takes values varying from 0, when the light intensity is entirely blocked, to 1, when it is fully transmitted. The Huygens-Fresnel principle suggests that the resulting diffraction pattern is the sum of the waves produced by punctual sources at every point of the aperture. The

following integral describes the diffracted wavefront observed at a distance z from the aperture:

$$A'(x', y') = \frac{z}{i\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x,y) \frac{e^{ikr}}{r^2} \, dx \, dy, \tag{16}$$

where, $r = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$ is the distance between a given point (x, y) of the aperture and a point (x', y') of the observation plane. The extra factor of z/r corresponds to the cosine of the angle between vectors r and z, since we only sum over the component of the wave in the z direction.[3] The above formula is a special case of the Fresnel-Kirchhoff diffraction formula from scalar diffraction theory.[4] The factor of $1/i\lambda$ comes up in the derivation of this formula.

Fraunhofer Diffraction

A possible approximation is to take the observation plane to be far. Using Taylor series, in such a case, it is possible to approximate r such that:

$$r \approx z\left(1 + \frac{(x-x')^2}{2z^2} + \frac{(y-y')^2}{2z^2}\right) \tag{17}$$

$$\approx z + \frac{x^2 + y^2}{2z} - \frac{xx' + yy'}{z} + \frac{x'^2 + y'^2}{2z}. \tag{18}$$

This is the Fresnel approximation.

Furthermore, it is possible to approximate the observation plane to be at infinity; this idea is called the Fraunhofer approximation. In this case, we assume $(x^2 + y^2)/z \to 0$ as $z \to \infty$. The term $\frac{x^2+y^2}{2z}$ is thus omitted so that eq. (16) takes the form:

$$A'(\nu_x, \nu_y) = \frac{e^{ikz}}{i\lambda z} e^{i\pi\lambda(\nu_x^2 + \nu_y^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x,y) e^{i2\pi(x\nu_x + y\nu_y)} \, dx \, dy, \tag{19}$$

where $\nu_x = \frac{x'}{\lambda z}$ and $\nu_y = \frac{y'}{\lambda z}$. Since we will only observe the relative intensity of the signal, the terms in front of the integrals can be considered to be a constant c:

$$A'(\nu_x, \nu_y) = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x,y) e^{i2\pi(x\nu_x + y\nu_y)} \, dx \, dy. \tag{20}$$

We recognize the 2-dimensional case of the Fourier transform as defined in eq. (8). In our case, however, the "time - frequency" correspondence (t - ν) is replaced by a "position - spatial frequency" correspondence (x - ν_x).

Performing a Fourier Transform Experimentally

In theory, for the Fourier transform of an aperture to be observed, it would be necessary to measure the signal at a near infinite distance from the opening. Luckily, it is possible to use a converging lens in order to focus the diffracted light at a finite distance, as shown on figure 5. It is often asserted that the lens itself performs the Fourier transform; this is not totally accurate. The lens does allow one to observe the pattern but the transform is a consequence of the diffraction from the aperture.

It is also interesting to note the relation between the position on the transform plane (x') and the spatial frequency of the aperture function (ν_x). According to eq. (19), the relation is given by $\nu_x = \frac{x'}{\lambda z}$ where λ is the wavelength of the laser and z is the distance between the object and the plane. The use of a lens as shown above stops the divergence of the wave after a distance f. Thus, the proper formula is:

$$\nu_x = \frac{x'}{\lambda f}. \tag{21}$$

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[3]This situation is analogous to the case you probably encountered in electromagnetism, where one has to calculate the electric potential at a given distance from a charge distribution. In our case, we are not concerned with a point-charge distribution $\frac{\sigma(x,y)}{4\pi\epsilon_0 r}$, but rather with a point-source distribution $\frac{A(x,y)e^{ikr}}{i\lambda r}$.

[4]See Thomas Kreis. Handbook of Holographic Interferometry: Optical and Digital Methods. Wiley-VCH, Weinheim, Germany, 2005.
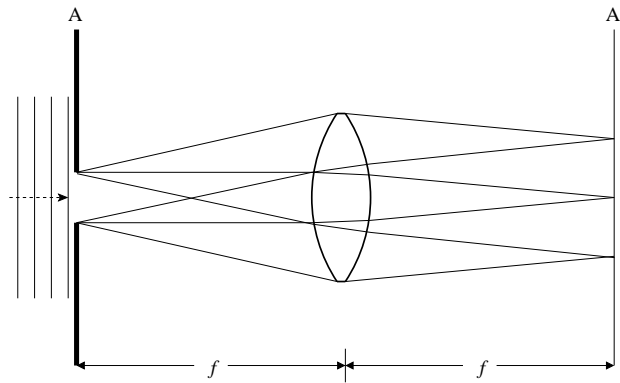
Figure 5: The image of the Fourier transform of an aperture can be observed using a converging lens of focal length f.
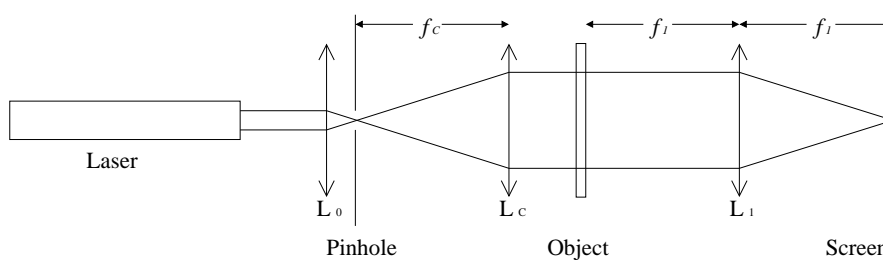
3 Manipulations

Figure 6: Experimental setup for studying the optical Fourier spectrum. L_0 is a 10X microscope objective, L_C and L_1 are converging lenses (focal lengths $f_C \approx 30$cm and $f_1 \approx 50$cm). L_0 and L_C form a beam expander. The object can be any grating you want to study.

First, we study the lens system presented on figure 6. This arrangement constitutes a frequency analyzer. When you set up the apparatus, you will notice that the optical table is too small to have every component in line; you must make use of mirrors. The diffraction patterns (or interference patterns) are measured with a digital camera. You may take the picture of the reflection of the diffraction pattern using plain paper as a screen; it is recommended, however, to observe the diffraction pattern by transmission using the translucent slide that is part of the collection of slides. That way, the camera can be placed on the optical axis, minimizing distortion. If the diffraction pattern is too small in the picture, than the resolution becomes limited by the pixel resolution. To minimize this effect, you should arrange for the picture to cover roughly the size of the translucent screen. This requires that you play with the distance from the camera to the screen, the zoom value and possibly the use of the "macro" option.

Note that for some distance and zoom values, the camera may not be able to focus on the slide, resulting in a blurred picture. It is recommended that you use the fully manual control of exposure time and aperture so as to control possible saturation of some part of the diffraction pattern. You may also want to take overexposed pictures to visualize the faint regions of the diffraction pattern when measuring the diffraction of a single slit. It is recommended that you take equivalent pictures with various exposure times and that you select the best one. Because of the huge contrast difference between the black background and the very intense diffraction pattern, it is difficult to get a good picture using automatic control. The camera provided has automatic focus (manual focus is the preferred option). Because it is often not well defined, the camera may have difficulties to focus on the diffraction pattern. Thus, if you use automatic focus, it is recommended that you turn on some lights in the room so as to light up the frame of the slide on which the camera can focus.

7

You have to transfer the pictures to a computer. You can perform a quantitative analysis of the picture using your preferred photo analysis program. In the case of the diffraction patterns of a grating or of a single slit, you should generate a projection of the diffraction pattern on the axis normal to the diffraction pattern and calculate the various distances (to be compared to theory) by using a distance calibration that you can obtain by including some ruler or calibrated graph paper in your pictures of the diffraction pattern.

3.1 Uni-dimensional Periodic Functions

The aperture used is a simple amplitude grating; you may pick any slides presenting such a periodic pattern. Now, you will measure the spatial frequency of the grating using two methods. Firstly, using a traveling microscope, take a measurement of the spacing between the grating's lines and calculate the spatial frequency in lines/meter (ask a technician for where to find the traveling microscope). Secondly, measure the diffraction grating and use equation (21) to determine the fundamental spatial frequency ν.

Repeat the experiment with gratings of different frequency, and try both slides with a grating in one direction and with a grating in both directions. How does the interference pattern change? You may also try the grating with a spherical pattern.

3.2 Non-Periodic Functions: Bandwidth Theorem

We are now concerned with non-periodic functions such as a simple slit. These are the optical counterparts of pulse functions discussed in the introduction. Thus, we should be able to formulate an analogous bandwidth theorem for "position - spatial frequencies" domain. Find the equivalent of eq. 9 in our case. Show that the theorem is valid by measuring the diffraction pattern of slits with different widths.

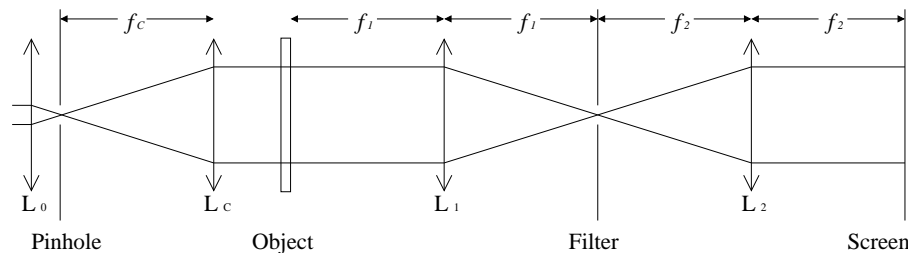## 3.3 Inverse Fourier Transform: Image Reconstruction

Figure 7: Experimental setup for studying image reconstruction. L_C, L_1 and L_2 are converging lenses (focal lengths $f_C \approx 30$cm, $f_1 \approx 50$cm and $f_2 \approx 50$cm). This setup completes the system shown on figure 6.

The Fourier transform has an inverse, which maps a transformed function back to its initial value. That is, if $A'(\nu_x, \nu_y) = \mathcal{F}[A(x, y)]$, then $A(x, y) = \mathcal{F}^{-1}[A'(\nu_x, \nu_y)]$. The optical system presented on figure 7 performs both of these tasks. Experiment with the slide of your choice to verify that the reconstruction is valid. This setup is even longer than the preceding one. You may need to use an extra mirror to make it all fit on the optical table.

L_1 focuses the transform on the plane marked "filter", while L_2 takes the diffracted pattern and reproduces the initial object on the screen. Not that for this section, the "filter" plane is empty. You should find where the interference pattern is the sharpest and use that point as your filter plane to properly set the distance with other instruments. In the next section, you will have to position a low-pass or a high pass filter at that location.

Part II
Spatial Filtering

4 Introduction

So far, you have experimented with the bases of Fourier optics; it is now time to explore some of its applications. Image processing by spatial filtering is the first one that you will be studying. This method is quite simple: the idea is to remove some part of the Fourier transform before transforming the diffraction pattern back to the image. High and low frequencies occupy different positions on the transform plane, which gives the freedom to filter out some components of an image.

The origin of the transform plane, where the zero frequency can be found, is on the axis of the optical system, at the centre of the plane. This fact suggest that frequencies are increasing radially outward from this point. A low pass filter would therefore allow only the light intensity in the neighborhood of the origin, and vice-versa for a high pass filter. Furthermore, the spatial frequencies in different directions are placed in distinct positions in the transform plane. The measurements are taken in similar fashion to the first part of the experiment; use a digital camera.

5 Manipulations

5.1 Uni-dimensional Filtering

You will now study the effect of filters on a one dimensional amplitude grating. You should build this filter using empty slide frames, electric tape or any other material you think is appropriate.

What would you define as a high or a low pass filter in one dimension? Build both of these and study their effect on the image focused on the screen. Relate your results to the Fourier series. A low pass filter keeps only the lower frequency components of the image, which carry the most intensity. The high pass filter keeps only the terms in the Fourier series that "smooth out" the image. Knowing this, what should you expect to obtain in each case?

5.2 Bi-dimensional filtering

Take a slide that presents an image containing different spatial frequencies, such as slide 14, 20 or 21. Where are the regions that present higher and lower spatial frequencies?

What would you define as a high or low pass filter in the 2-dimensional case? Build them and observe their effect. Again, use the decomposition in terms of the Fourier series to interpret your results. Use the image of a grid such as slide 11. Try to allow a diagonal (45°) row of dots to pass. Based on a geometrical argument, you should make a prediction about the relation between the spatial frequencies of this image and the one corresponding to a vertical or horizontal grid. Compare your prediction with experimental results. Again, you should measure the spatial frequencies by using the camera, as discussed above.

5.3 Image Processing

The goal of this section is to modify images in specific ways by using appropriate filters. Use the cloud chamber simulation photograph (slide 22). The straight lines correspond to incoming particles, in the cloud chamber experiment, while the curved one are caused by interactions. In a typical experiment, it would be convenient to remove the large bands. Your goal is to remove these lines leaving only the curved tracks. Based on your observations from the previous section, design a filter that will accomplish this task.

Similarly, spatial filtering makes it possible to view any of different pictures stored on a single slide. Each picture must be encoded using lines at a particular angle. Ask the lab technician for the AB slide, which presents a pattern similar to that of figure 8. Note that the letters are represented by lines going in different directions. Build filters that will enable you to individually reconstruct the letters A and B. (Hint: Since the object is not very visible on the slide, you should probably block the central spot corresponding to the zero spatial frequency.)
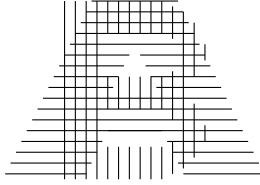

Figure 8: Schematic representation of the AB slide.

A half-tone image represents gray tones by small dots of different sizes. By filtering, it is possible to obtain a continuous-tone image. Use slide 24 and experiment the effect of different filters, such as circular apertures and slits at different angles. For each of them, try to predict what the image should look like. Based on your experimental results, build a filter that removes the dots from the picture.

5.4   Supplemental Activities

At this point, you probably have enough experience to perform experiments that you may have come across in textbooks or that you may design on your own. Here are a few ideas:

- Prove the convolution theorem for the Fourier transform;

- Observe the Fourier transform of different functions as circular apertures, multiple slits, etc. Explain the observed diffraction pattern;

Part III
Holography

6   Introduction

See Appendices B and D for more details on, respectively, the theory and the applications of holography

Holography involves the recording of a wave, including both the amplitude and the phase of the signal. Another name for this technique is wavefront reconstruction. In simpler words, this means that when an hologram is reconstructed, an observer can see the same light signal as he or she would have seen during the recording. Now, a hologram is, by definition, the recording of an interference pattern. As we saw previously, interference implies an interaction between different coherent waves, in this case an object wave and a reference wave. An intuitive example of a hologram is presented on figure 9.

Mathematically, each wave can be represented as a complex function. Let $U(x,y) = A(x,y)\mathrm{e}^{i\phi(x,y)}$ be the object wave and $R_r(x,y) = B(x,y)\mathrm{e}^{i\psi(x,y)}$ be the reference wave. The intensity resulting from the sum of both of these is given by:

$$I(x,y) = |A(x,y)|^2 + |B(x,y)|^2 + 2A(x,y)B(x,y)\cos\left(\psi(x,y) - \phi(x,y)\right) \qquad (22)$$

where the last term corresponds to the interference $U\overline{R_r} + \overline{U}R_r$ and includes both $A(x,y)$ and $\phi(x,y)$. Thus, the interference pattern contains all the information about the object wave, including both its amplitude and its phase.

Figure 9: Graphical illustration of the recording (a) and the reconstruction (b) of a waveform. The hologram is the recording of the interference pattern between the reference wave and the object wave (a). When the reference beam is shone on the processed hologram, the object wave is reconstructed (b).

The interference pattern can be stored on a holographic film. The transmission function of the film is defined as the proportion of incident light that is transmitted as a function of position. In this case, it is given by:

$$t(x, y) = C + \beta \left(|U|^2 + U\overline{R_r} + \overline{U}R_r \right). \tag{23}$$

The resulting waveform, when an illumination beam R is shone on the hologram, can be written as:

$$Rt = U_1 + U_2 + U_3 + U_4,$$

where

$$
\begin{aligned}
U_1 &= CR, \\
U_2 &= \beta |U|^2 R, \\
U_3 &= \beta \overline{R_r} RU, \\
U_4 &= \beta R_r R\overline{U}.
\end{aligned}
$$

Here, U_3 is the reconstruction of the object wave and is equal to U, the object wave, up to a constant factor. We call it the *virtual image* since it can be observed directly. U_4 is proportional to $\overline{U}$, which means that the wave converges, as easily deduced from the functional form of $\overline{U}$. It is possible to focus this image on a screen, hence the name of *real image*. U_3 and U_4 are together called the twin images. U_1 and U_2 are simple constants.

7 Manipulations

This part of the experiment uses holographic film rather than the digital camera.

7.1 Michelson Interferometer

To be successful at making a hologram, one must keep in mind that the interference fringes can be very close to one another. The stability of the system is thus critical. Air vents must be blocked and the use of an air table is preferable in order to isolate the setup from surrounding vibrations. It is possible to study the stability of the optical system by setting up a Michelson interferometer as shown on figure 10. You can observe the effect of vibrations on the interference pattern. During the exposition, avoid instabilities resulting in a movement of greater than a quarter fringe and preferably you want an even better stability. When you are satisfied with the setup, you can proceed to constructing the hologram.
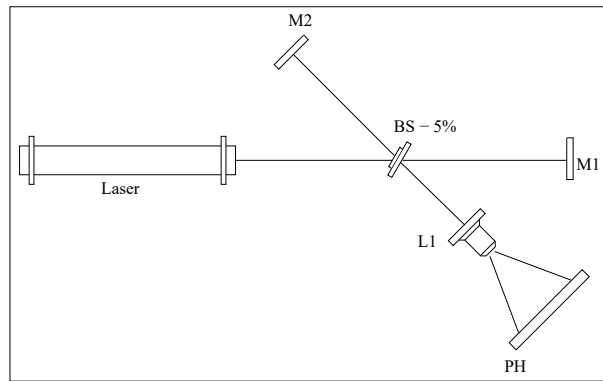
Figure 10: Experimental setup for the Michelson interferometer. M1 and M2 are mirrors, BS -5% is a 5% beam splitter, L1 is a 60X microscope objective and PH is a screen.
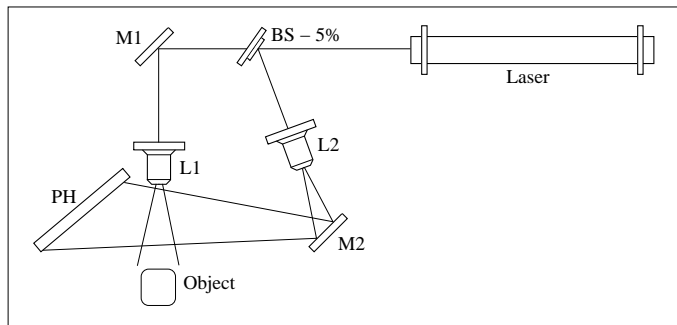
7.2   Transmission Hologram

Figure 11: Experimental system for transmission hologram recording. M1 and M2 are mirrors, BS - 5% is a 5% beam splitter, L1 and L2 are 60X microscope objectives, and PH is the photographic plate.

First, roughly layout the positions of the components as shown in figure 11. When this is done, you can fine tune the position and orientation of each component. Your goal is to obtain a reference beam and an object beam that have approximately the same intensity on the screen. To make sure this is the case, successively block each beam and observe the light on a sheet of white paper placed in front of the photographic plate. You also want to make sure that the whole photographic plate is spanned by the beams.

If you wish to improve the quality of your hologram, you might want to use a pinhole to "clean up" the reference beam. The positioning of this component is not trivial; some practice might be necessary. You need to be careful in the manipulation of the holographic film. It is particularly sensitive to red (632nm) light; manipulate it only under a green safety light. Make sure to turn the laser off when loading the film on the photographic plate. Put the emulsion (sticky) side toward the beams. After you pull out the security plate, wait a few seconds before turning on the laser. This will allow any vibration to die out. The exposure time will vary depending on the intensity of your beams. Proper holograms have been recorded in 30 to 45 seconds but your setup may differ slightly. It is preferable to first try a small piece of film as a test.

When you have recorded the hologram, it still needs to be developed. The general procedure goes as follows: 10 minutes in the D-19 developer, 1 minute in the stopper, 10 minutes in the fixer and about 10 minutes in the wash bath – ask help from a professor or a technician when you first try it. As it is being developed, you can monitor the hologram by indirectly shining the green safety light on it. A proper hologram should be greenish but still mostly translucent. If you see that the hologram becomes very dark

while still in the developer, you can remove it even though 10 minutes have not passed. As soon as the hologram is in the fixer, you can turn the lights back on, it will not ruin the result.
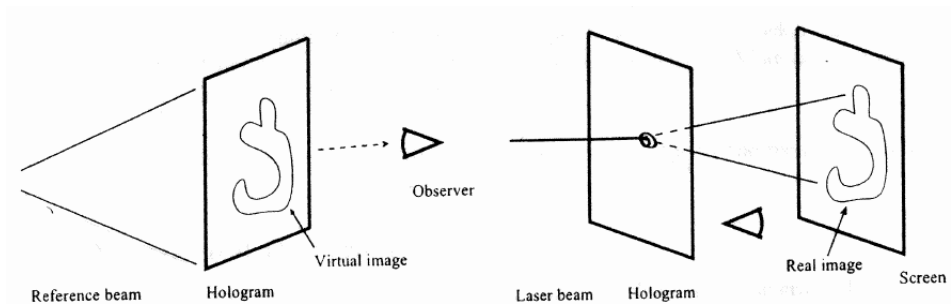

Figure 12: Illustration of the reconstruction procedure for the virtual image (left) and the real image (right).

After the hologram is dry, you have two ways of reconstructing the wave: putting it back in the reference beam or having a non-expanded laser beam go through it. Figure 12 illustrates the procedure. At what angle between the film and the beam are the images easier to observe? Is there a relation between this angle and the experimental setup used to record the hologram? How does this relate to the theory?
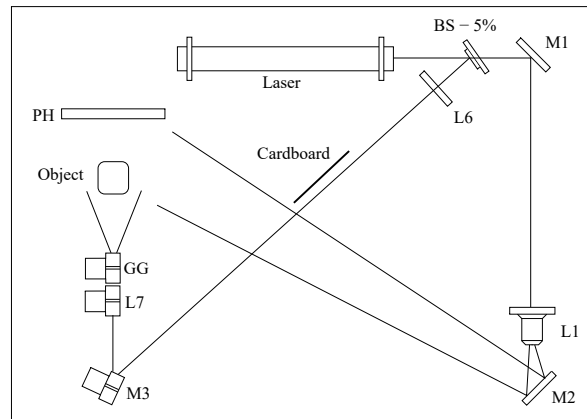
7.3   Optional: Rainbow Hologram

Figure 13: Experimental system for rainbow hologram recording. M1 and M2 are circular mirrors. M3 is a tall mirror. L6 is a small cylindrical lens while L7 is a tall cylindrical lens. GG is a ground glass. PH is the photographic plate. BS-5% is a 5% beam splitter.

You might wonder how common holograms are made. In fact, there exists a plethora of hologram types that can be reconstructed under white light. The setup presented on figure 13 allows to record what is called a rainbow hologram, a type of reflection hologram. You may attempt to record one of those by taking even greater care about stability then in the case of a transmission hologram.

It is important to pick an object that is not too large and that will only block a small portion of light coming from the ground glass. The type of emulsion used for the transmission hologram is appropriate as well in this case. The development procedure is the same for this type of hologram but it is preferable that the film be pale in order to improve reconstruction quality. Hence, a considerably shorter exposure is appropriate. The best reconstruction results have been achieved in direct sunlight, placing the film in front of a dark background. It was also observed that the angle at which the hologram is placed is quite important. You should expect to see the dark silhouette of the object in front of an iridescent background. You will notice that the colors change depending on the angle at which the film is placed.

7.4 Optional: Digital Hologram

Digital holography is a process by which one records a hologram with a digital camera and then reconstructs the object wave numerically. It substitutes the process of putting the hologram back in the reference beam by a clever mathematical trick.

Theory

As we saw previously, the information stored on a hologram corresponds to an interference pattern between a reference beam and an object beam. The intensity is given by:

$$I(x, y) = \left| E_P + E_R^2 \right|^2 = E_P \overline{E_P} + E_R \overline{E_R} + \overline{E_P} E_R + E_P \overline{E_R}, \tag{24}$$

where E_P is the object wave and E_R is the reference wave. Each of those is described by a complex-valued two-dimensional function. Let $h(\xi, \epsilon)$ be the function describing the recorded intensity on the hologram plane $\xi - \eta$ and $b(x, y)$, the complex function corresponding to the object wave. The hologram $(\xi - \eta)$ and the object $(x - y)$ planes are separated by a distance d.

The optical reconstruction of the images would require the illumination of the hologram by the reference beam. It is possible to numerically model this process by multiplying the function $h(\xi, \eta)$ by the conjugate of the reference wave $r^*(\xi, \eta)$. The latter can be approximated to 1 if the reference wave is planar and parallel to the hologram plane. The real image is then determined approximately by the inverse Fresnel Transform of this product:[5]

$$b(x, y) = e^{\frac{i\pi}{d\lambda}(x^2 + y^2)} \int \int h(\xi, \eta) r^*(\xi, \eta) e^{\frac{i\pi}{d\lambda}(\xi^2 + \eta^2)} e^{\frac{-2i\pi}{d\lambda}(x\xi + y\eta)} d\xi d\eta, \tag{25}$$

where λ is the wavelength of the laser used. You might have recognized a 2-dimensional Fourier transform in the previous formula. The only difference are the two following terms:

$$z(x, y) = e^{\frac{i\pi}{d\lambda}(x^2 + y^2)} \tag{26}$$

$$w(\xi, \eta) = e^{\frac{i\pi}{d\lambda}(\xi^2 + \eta^2)} \tag{27}$$

These are called chirp functions. We can thus rewrite (eq. 25) as:

$$b = z \cdot \mathcal{F}^{-1}\{h \cdot r^* \cdot w\}, \tag{28}$$

where $\mathcal{F}^{-1}$ represents inverse the Fourier transform operator. As it is usual in numerical applications, the description we have of $h(\xi, \eta)$ is discrete and is given by an image matrix. This fact enables us to use the Fast Fourier Transform algorithm to compute the Fresnel transform.

Reconstruction Algorithm. *Given h, a M×N matrix; d the distance between the camera recording chip and the object; $\Delta\eta$ and $\Delta\xi$, the vertical and horizontal centre-to-centre distances of pixels; and λ, the wavelength of the laser. Then the following method reconstructs numerically the image stored on hologram h:*

1. *If the reference wave plane is not parallel to the camera, multiply the matrix elements of h by $r^* = \exp\left\{\frac{-2\pi i}{\lambda}\eta \sin\theta\right\}$, where θ is the angle between the two.*

2. *Multiply each element (k,l) of $h \cdot r^*$ by $w(k, l) = \exp\left\{\frac{i\pi}{d\lambda}(k^2\Delta\xi^2 + l^2\Delta\eta^2)\right\}$.*

3. *Compute the Inverse 2D Fast Fourier Transform of the resulting matrix.*

4. *Multiply each elements (k,l) of the transform by $z(k, l) = \exp\left\{-i\pi d\lambda\left(\frac{k^2}{\Delta\xi^2 N^2} + \frac{l^2}{\Delta\eta^2 M^2}\right)\right\}$.*

---

[5]This formula is the inverse version of (eq. 16). The Fresnel approximation is used in place of the Fraunhofer approximation.
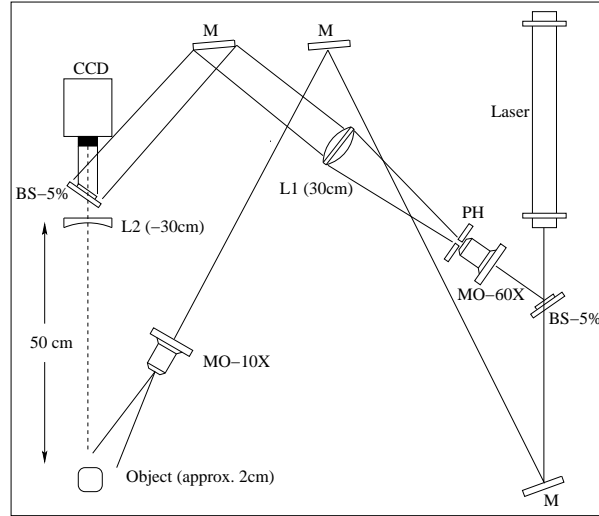
Procedure

Figure 14: Experimental setup for digital holography. BS-5% are 5% beam splitters, MO-60X and MO-10X are microscope objective of 60X and 10X respectively. L1 is a converging lens (f=30cm) and L2 is a diverging lens (f=-30cm). PH is a pinhole. M's are mirrors. CCD is a CCD camera. The object is about 2 cm across.

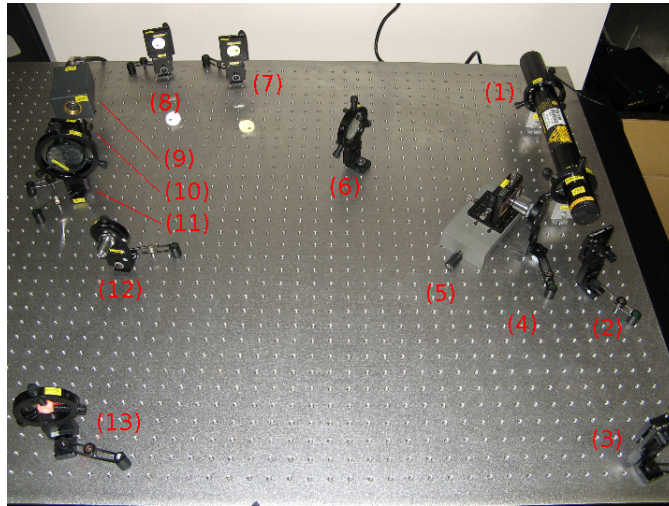Figure 15: Picture of the experimental setup for digital holography. (1) Laser; (2) 5% beam splitter; (3) mirror; (4) 60X microscope objective; (5) pinhole; (6) converging lens (f=30cm); (7) & (8) mirrors; (9) CCD camera; (10) 5% beam splitter; (11) diverging lens (f=-30cm); (12) 10X microscope objective; (13) object.

In order to simplify the reconstruction algorithm, it is necessary to make sure that the reference beam is planar and that the wavefront is parallel to the camera's iris. To obtain a planar beam, use a microscope objective to enlarge the beam. Then, place a converging lens such that the distance that separates it from the objective is equal to its focal length. Fine tuning the second beam splitter will enable you to have an incident beam at right angle with the camera. The size of the camera recording chip being quite small, you need the object to have a reduced angular size. You can achieve this by enlarging the distance between the camera and the object. A diverging lens can also be used to save some room on the table (11 on figure 15). The intensity of the reference and the object beams should be similar, which implies that the illumination must be much more powerful than the reference beam to correct for the distance.

Finally, when you place the object, you should observe what the camera captures at the same time. At one point, the interference pattern will be varying a lot, even though the object is stable; this indicates that the alignment is correct. This method was shown to give some results as presented on figure 16. The object used was a Teflon nut.
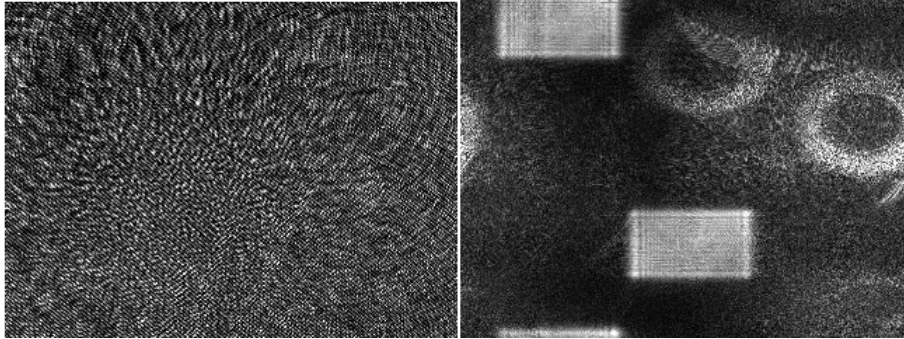

Figure 16: Example of a recorded hologram on the right and of the reconstructed image on the left. The contrast of the reconstructed image was slightly enhanced with a photo edition software. We can distinguish both the real and virtual images.

7.5   Matlab Implementation of the Reconstruction Algorithm

holo.m

```
% Digital Hologram Reconstruction Algorithm
% This algorithm is based on the Fresnel transform, feel free to improve the code.
% h: hologram , d: distance CCD-object

function [ima] = holo(h,d)

[M,N] = size(h);
hd = double(h)/255;
ps = 7.4e-6; % Pixel size
lw = 632.8e-9; % Laser wavelength

mea = mean( mean( hd ) );
for m = 1:M
    for n = 1:N
        % DC term filtering
        hd(m,n) = hd(m,n) - mea;
        % Chirp functions
        chirp(m,n) = exp(i*pi*(ps^2*(m-1)^2 + ps^2*(n-1)^2)/d/lw);
        pre(m,n) = exp(i*pi*(1/ps^2*(m-1)^2/M^2 + 1/ps^2*(n-1)^2/N^2)*d*lw);
    end;
end;

% Reconstruction
mult = (hd.*chirp);
ima = abs(pre.*ifft2(mult));

% Display image
showsc(ima);
```