Solutions Set 1 ST 525

ST 525 Solutions

Set 1 1.1 — 1.5

1.1 (a) $S(c) = E(Y-c)^2 = EY^2 - 2cEY + c^2$

$$\frac{dS}{dc} = -2EY + 2c = 0 \text{ when } c = E(Y)$$

S is minimum when $c = E(Y)$ since

$$\frac{d^2S}{dc^2} = 2 > 0 \quad \forall c.$$

(b) $E\left[(Y - f(X))^2 \mid X\right]$

$$= E\left[(Y - E(Y|X) + E(Y|X) - f(X))^2 \mid X\right]$$

$$= E\left[(Y - E(Y|X))^2 \mid X\right]$$

$$+ 2E\left[(Y - E(Y|X))(E(Y|X) - f(X)) \mid X\right]$$

$$+ E\left[(E(Y|X) - f(X))^2 \mid X\right]$$

$$= E\left[(Y - E(Y|X))^2 \mid X\right]$$

$$+ \cancel{2(E(Y|X) - f(X))\, E\left[Y - E(Y|X) \mid X\right]}$$

$$+ E\left[(E(Y|X) - f(X))^2 \mid X\right]$$

$$\geq E\left[(Y - E(Y|X))^2 \mid X\right] \quad \text{for all } f(X)$$

i.e. $E\left[(Y - f(X))^2 \mid X\right]$ is minimized when $f(X) = E(Y|X)$.

(c) $E(Y - f(X))^2 = E\left[E\left((Y - f(X))^2 \mid X\right)\right]$

$E\left((Y - E(Y|X))^2 \mid X\right) \leq E\left((Y - f(X))^2 \mid X\right)$ by (b)

Taking expectations of each side gives

$E(Y - E(Y|X))^2 \leq E(Y - f(X))^2$ for all $f(X)$.

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1.2

1.2 (a) $E\left((X_{n+1} - f(\underset{\sim}{X}))^2 \mid \underset{\sim}{X}\right)$ $\quad (\underset{\sim}{X} = (X_1,\ldots,X_n))$

$$= E\left[\left(X_{n+1} - E(X_{n+1}\mid \underset{\sim}{X}) + E(X_{n+1}\mid\underset{\sim}{X}) - f(\underset{\sim}{X})\right)^2 \mid \underset{\sim}{X}\right]$$

$$= E\left[\left(X_{n+1} - E(X_{n+1}\mid\underset{\sim}{X})\right)^2 \mid \underset{\sim}{X}\right]$$

$$+2E\left[\left(X_{n+1} - E(X_{n+1}\mid\underset{\sim}{X})\right)\left(E(X_{n+1}\mid\underset{\sim}{X}) - f(\underset{\sim}{X})\right) \mid \underset{\sim}{X}\right]$$

$$+E\left[\left(E(X_{n+1}\mid\underset{\sim}{X}) - f(\underset{\sim}{X})\right)^2 \mid \underset{\sim}{X}\right]$$

$$\geq E\left[\left(X_{n+1} - E(X_{n+1}\mid\underset{\sim}{X})\right)^2 \mid \underset{\sim}{X}\right]$$

since

$$E\left[\left(X_{n+1} - E(X_{n+1}\mid\underset{\sim}{X})\right)\left(E(X_{n+1}\mid\underset{\sim}{X}) - f(\underset{\sim}{X})\right) \mid \underset{\sim}{X}\right]$$

$$= \left(E(X_{n+1}\mid\underset{\sim}{X}) - f(\underset{\sim}{X})\right) \overbrace{E\left[X_{n+1} - E(X_{n+1}\mid\underset{\sim}{X}) \mid \underset{\sim}{X}\right]}^{0}$$

$$= 0$$

(b) Taking expectations on both sides of (a) gives

$$E\left[\left(X_{n+1} - f(\underset{\sim}{X})\right)^2\right] \geq E\left[\left(X_{n+1} - E(X_{n+1}\mid\underset{\sim}{X})\right)^2\right]$$

(c) By (b) the minimum mean-squared error predictor of X_{n+1} in terms of $X_1,\ldots,X_n$ when $X_1, X_2, \ldots$ are iid with mean μ and $EX_i^2 < \infty$ is

$$E(X_{n+1}\mid\underset{\sim}{X}) = E(X_{n+1}) = \mu.$$

(d) Suppose $\sum_{i=1}^{n} \alpha_i X_i$ is unbiased for μ, i.e. $\sum\alpha_i = 1$.

$$E\left(\sum\alpha_i X_i - \mu\right)^2 = E\left(\sum\alpha_i X_i - \bar{X}\right)^2 + 2E\left(\sum\alpha_i X_i - \bar{X}\right)(\bar{X}-\mu) + E(\bar{X}-\mu)^2$$

$$\geq E(\bar{X}-\mu)^2 \text{ since } 2^{\text{nd}} \text{ term is zero}$$

$$\left[\quad E\left(\sum \alpha_i X_i - \bar{X}\right)(\bar{X}-\mu) = \operatorname{cov}\left(\sum \alpha_i X_i - \bar{X}, \bar{X}-\mu\right)\right.$$

$$= \operatorname{cov}\left(\sum \alpha_i X_i, \sum \tfrac{1}{n} X_i\right) - \operatorname{cov}\left(\sum \tfrac{1}{n} X_i, \sum \tfrac{1}{n} X_i\right)$$

$$\left. = \sum_1^n \frac{\alpha_i}{n}\sigma^2 - \sum_1^n \frac{1}{n^2}\sigma^2 = 0 \quad \right]$$

(e) Suppose $\sum \alpha_i X_i$ is unbiased for μ, i.e. $\sum \alpha_i = 1$.

Then

$$E\left(X_{n+1} - \sum \alpha_i X_i\right)^2 = E\left(X_{n+1} - \bar{X}\right)^2$$

$$+ 2E\left[\left(X_{n+1} - \bar{X}\right)\left(\bar{X} - \sum \alpha_i X_i\right)\right] + E\left(\bar{X} - \sum \alpha_i X_i\right)^2$$

$$\geq E\left(X_{n+1} - \bar{X}\right)^2$$

since the second term is zero

$$\left[\quad \operatorname{Cov}\left(X_{n+1} - \bar{X}, \bar{X} - \sum \alpha_i X_i\right) = -\operatorname{Cov}(\bar{X}, \bar{X}) + \operatorname{cov}\left(\bar{X}, \sum_{i=1}^n \alpha_i X_i\right)\right.$$

$$\left. = 0 \text{ as in (d)} \quad \right]$$

(12)

(f) $E(S_{n+1} \mid S_1, \ldots, S_n) = E(S_n + X_{n+1} \mid S_1, \ldots, S_n)$

$= S_n + E(X_{n+1} \mid S_1, \ldots, S_n)$

$= S_n + \mu$ since X_{n+1} is independent of $S_1, \ldots, S_n$.

1.3 (a) EX_t is independent of t since the dist'n of X_t is independent of t and EX_t exists.

(b) $E(X_{t+h} X_t)$ - - - - - - - - - joint distribution of X_{t+h} and X_t is independent of t and $EX_t^2 < \infty$.

(5)

Hence $\{X_t\}$ is weakly stationary

1.4 (a) $EX_t = a$ indep of t

$$\mathrm{Cov}(X_{t+h}, X_t) = \begin{cases} (b^2+c^2)\sigma^2, & h=0 \\ 0, & h=\pm 1 \\ bc\sigma^2, & h=\pm 2 \\ 0, & \text{otherwise}, \end{cases}$$

indep. of t.

$\therefore \{X_t\}$ is stationary

(b) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$

$EX_t = 0$, indep. of t

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{cov}\big(Z_1 \cos c(t+h) + Z_2 \sin c(t+h),\ Z_1 \cos ct + Z_2 \sin ct\big)$$

$$= \sigma^2 (\cos c(t+h) \cos ct + \sin c(t+h) \sin ct$$

$$= \sigma^2 \cos(ch),$$

indep. of t

$\therefore \{X_t\}$ is stationary

(c) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

$EX_t = 0$, indep of t

$$\mathrm{Cov}(X_{t+1}, X_t) = \sigma^2 \cos c(t+1) \sin ct$$

$\therefore \{X_t\}$ is not stationary (except in the trivial case when c is an integer multiple of 2π).

(d) $X_t = a + bZ_0$

$EX_t = a$, indep of t.

$Cov(X_{t+h}, X_t) = b^2\sigma^2$ indep of t

$\therefore \{X_t\}$ is stationary

(e) $X_t = Z_0 \cos ct$

$EX_t = 0$, indep of t

$$Cov(X_{t+h}, X_t) = \sigma^2 \cos c(t+h) \cos ct$$

$$= \frac{\sigma^2}{2}\left[\cos c(2t+h) + \cos ch\right]$$

$\therefore \{X_t\}$ is non-stationary (except in the trivial case when c is an integer multiple of 2π)

(f) $X_t = Z_t Z_{t-1}$

$EX_t = 0$,

$$Cov(X_{t+h}, X_t) = E(X_{t+h}X_t)$$

$$= E(Z_{t+h}Z_{t+h-1}Z_tZ_{t-1})$$

$$= \begin{cases} \sigma^4 & \text{if } h = 0, \\ 0 & \text{otherwise}, \end{cases}$$

indep of t

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$\therefore \{X_t\}$ is stationary ($WN(0, \sigma^4)$ in fact)

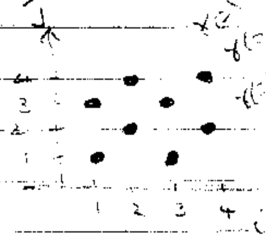
1.5 $X_t = Z_t + \theta Z_{t-2}$, $\{Z_t\} \sim WN(0,1)$

(a) $\gamma(h) = \begin{cases} 1+\theta^2, & h=0 \\ \theta, & h=\pm 2 \\ 0, & \text{otherwise} \end{cases}$, $\rho(h) = \begin{cases} 1, & h=0, \\ \frac{\theta}{1+\theta^2}, & h=\pm 2 \\ 0, & \text{otherwise} \end{cases}$

3 $\gamma(h) \overset{\theta=0.8}{=} \begin{cases} 1.64, & h=0 \\ 0.8, & h=\pm 2 \\ 0, & \text{otherwise} \end{cases}$ $\left(\begin{cases} 1.64 \\ -0.8 \\ 0 \end{cases} \text{if } \theta=-.8 \right)$

(b) $\mathrm{Cov}\left(\frac{1}{4}(X_1+\cdots+X_4), \frac{1}{4}(X_1+\cdots+X_4)\right)$

$= \frac{1}{16} \sum_{j=1}^{4} \sum_{i=1}^{4} \mathrm{Cov}(X_i, X_j)$



$= \frac{1}{16}\left[4\gamma(0) + 4\gamma(2)\right]$

3 $= \frac{1}{16} \times 4 \times 2.44 = \underline{\underline{0.61}}$

(c) $\mathrm{Cov}(\bar{X}_4, \bar{X}_4)$

$\overset{\theta=-.8}{=} \frac{1}{16}\left[4 \times (1.64 - 0.8)\right] = \underline{0.21}$

4 The negative lag-2 correlation in (c) means that positive deviations of X_i from zero tend to be followed two time units later by a compensating negative deviation, resulting in smaller variability in the sample mean than in (b) (and also smaller than if $\{X_t\}$ were iid $(0, 1.64)$ in which case we would have $\mathrm{Var}(\bar{X}_4) = 0.41$).

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SET 2

1.7 $E(X_t + Y_t) = \mu_X + \mu_Y$ is independent of t, and since $\text{Cov}(X_s, Y_t) = 0$ for all t and s, $\text{Cov}(X_{t+h} + Y_{t+h}, X_t + Y_t) = \gamma_X(h) + \gamma_Y(h)$ which is independent of t. 4

1.10 For $m_t = \sum_{k=0}^{p} c_k t^k$, we have

$$\begin{aligned} \nabla m_t &= \sum_{k=0}^{p} c_k t^k - \sum_{k=0}^{p} c_k (t-1)^k \\ &= p c_p t^{p-1} + \sum_{k=0}^{p-2} b_k t^k \end{aligned}$$

since $(t-1)^p = t^p - pt^{p-1} + \ldots$. Consequently, ∇m_t is a polynomial of degree $p-1$ and therefore by successive application of the difference operator ∇ we deduce that $\nabla^{p+1} m_t = 0$. 4

1.12 (a) We first prove that a linear filter $\{a_j\}$ passes a polynomial of degree p if and only if

$$\begin{cases} \sum_j a_j = 1, \\ \sum_j j^r a_j = 0, \quad r = 1, \ldots, p. \end{cases}$$

To prove this, it is enough to show $\sum_j a_j (t+j)^r = t^r$ for $r = 0, \ldots, p$. But, $(t+j)^r = \sum_{k=0}^{r} \binom{r}{k} t^k j^{r-k}$ so that

$$\begin{aligned} \sum_j a_j (t+j)^r &= \sum_{k=0}^{r} \binom{r}{k} t^k \left(\sum_j a_j j^{r-k} \right) \\ &= t^r \end{aligned}$$

for $r = 0, \ldots, p$ if and only if the above conditions hold. 4

(b) For Spencer's 15-point moving average filter, $\{a_j, j = -7, \ldots, 7\}$ it is a simple matter to check that

$$\sum_{-7}^{7} a_j = 1$$

$$\sum_{-7}^{7} j^r a_j = 0, \quad \text{for } r = 1, 2, 3.$$ 2

1.14 $a_0 = \frac{3}{9}$, $a_1 = \frac{4}{9} = a_{-1}$

$$a_2 = -\frac{1}{9} = a_{-2}$$

(i) $\sum a_i = \frac{3}{9} + \frac{8}{9} - \frac{2}{9} = 1$

$\sum i a_i = 0$

$\sum i^2 a_i = \frac{4}{9} + \frac{4}{9} - \frac{4}{9} - \frac{4}{9} = 0$

$\sum i^3 a_i = 0$

$\therefore$ by 1.12(a) the filter passes cubic trend without distortion. **4**

(ii) If $s_t = s_{t-3}$ and $\sum_1^3 s_t = 0$

then $\frac{3}{9} s_t + \frac{4}{9} s_{t-1} - \frac{1}{9} s_{t-2} + \frac{4}{9} s_{t+1} - \frac{1}{9} s_{t+2}$

$$= \frac{3}{9} s_t + \frac{3}{9} s_{t+1} + \frac{3}{9} s_{t+2}$$

$$= 0$$

(since $s_{t-2} = s_{t+1}$ and $s_{t-1} = s_{t+2}$)

$\therefore$ arbitrary seasonal component of period 3 is eliminated. **4**

1.15 (a) Since s_t has period 12,

$$\begin{aligned}\nabla_{12} X_t &= \nabla_{12}(a + bt + s_t + Y_t)\\ &= 12b + Y_t - Y_{t-12}\end{aligned}$$

so that

$$W_t := \nabla \nabla_{12} X_t = Y_t - Y_{t-1} - Y_{t-12} - Y_{t-13}.$$

Then $EW_t = 0$ and

$$\begin{aligned}&\mathrm{Cov}(W_{t+h}, W_t)\\ &\quad= \mathrm{Cov}(Y_{t+h} - Y_{t+h-1} - Y_{t+h-12} + Y_{t+h-13}, Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13})\\ &\quad= 4\gamma(h) - 2\gamma(h-1) - 2\gamma(h+1) + \gamma(h-11) + \gamma(h+11) - 2\gamma(h-12)\\ &\qquad - 2\gamma(h+12) + \gamma(h+13) + \gamma(h-13)\end{aligned}$$

where $\gamma(\cdot)$ is the autocovariance function of $\{Y_t\}$. Since EW_t and $\mathrm{Cov}(W_{t+h}, W_t)$ are independent of t, $\{W_t\}$ is stationary. Also note that $\{\nabla_{12} X_t\}$ is stationary. 4

(b) $X_t = (a + bt)s_t + Y_t$

$\nabla_{12} X_t = bts_t - b(t-12)s_{t-12} + Y_t - Y_{t-12}$

$= 12bs_{t-12} + Y_t - Y_{t-12}.$

Now let $U_t = \nabla_{12}^2 X_t = Y_t - 2Y_{t-12} + Y_{t-24}$. Then $EU_t = 0$ and

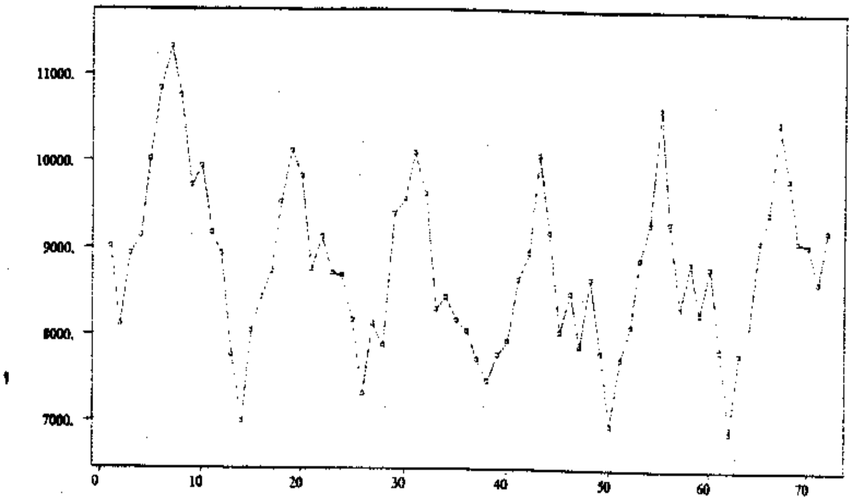
$$\begin{aligned}\mathrm{Cov}(U_{t+h}, U_t) &= \mathrm{Cov}(Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24})\\ &= 6\gamma(h) - 4\gamma(h+12) - 4\gamma(h-12) + \gamma(h+24) + \gamma(h-24)\end{aligned}$$

4

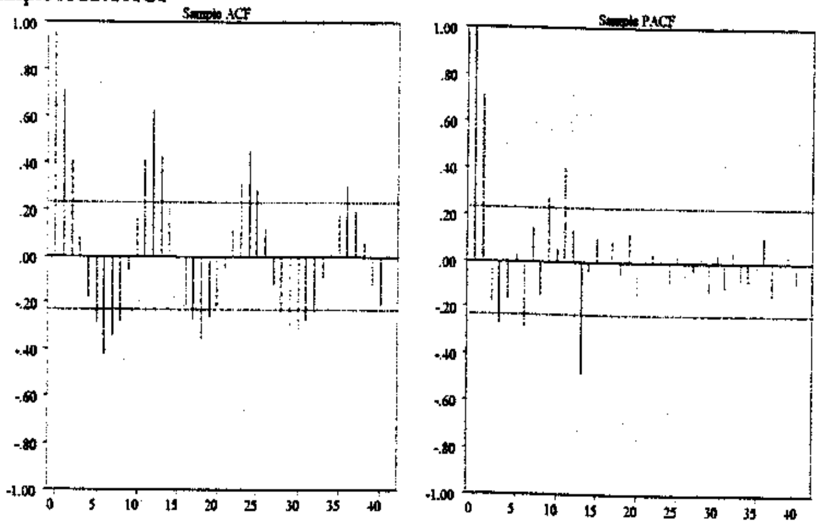
which is independent of t. Hence $\{U_t\}$ is stationary.

2.4

1.18 **Monthly Accidental Deaths, U.S.A., Jan, 73 – Dec, 78**

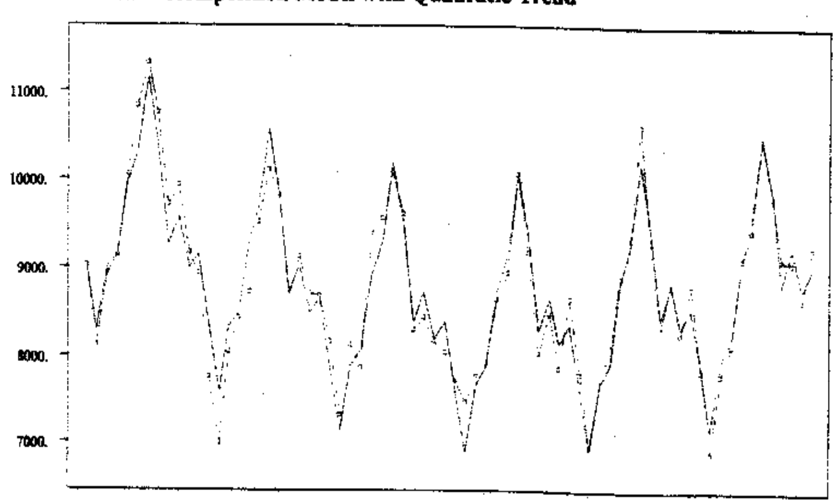


Sample ACF/PACF

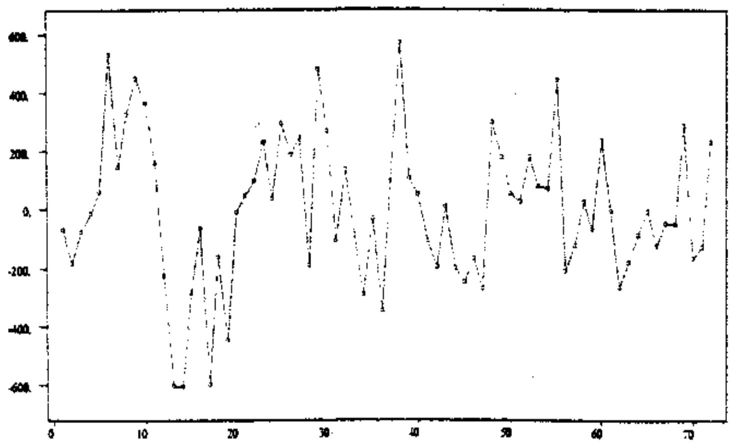


The slowly damped period-12 oscillations suggest trend and seasonality, clearly not iid, > 95% outside bounds

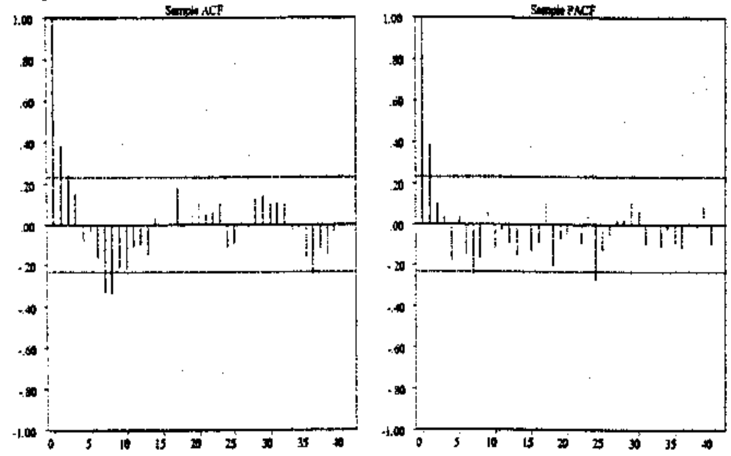
Fit of Classical Decomposition Model with Quadratic Trend



Residuals from Fit



Sample ACF/PACF of Residuals



Four out of 40 sample autocorrelations are outside the 95% bounds and several others are close suggesting that the residuals are not iid.

Further Tests of IID Residual Hypothesis

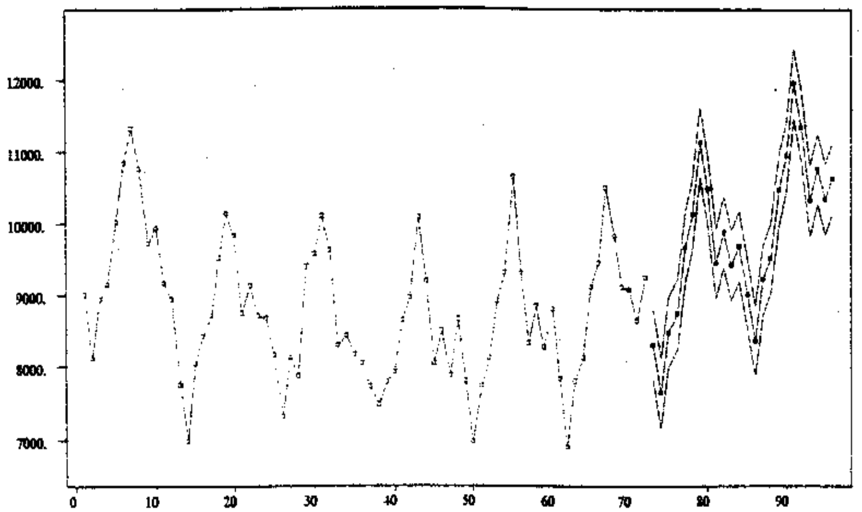
ITSM::Pest(Tests of randomness on residuals)

Ljung - Box statistic = 55.384 Chi-Square (20), p-value = .00004
McLeod - Li statistic = 15.829 Chi-Square (20), p-value = .72716
 # Turning points = 43.000~AN(46.667,sd = 3.5324), p-value = .29926
 # Diff sign points = 35.000~AN(35.500,sd = 2.4664), p-value = .83935
 # Rank points = 1245.~AN(1278.,sd = 102.85), p-value = .74833
Order of Min AICC YW Model for Residuals = 1

The first and last tests suggest rejection, the first at all levels greater than .00004. In view of the sample ACF and the extremely small p-value of the first test, I would reject the null hypothesis of iid residuals.

2.5

Forecasts of 24 future values



The red values are the forecasts, shown with upper and lower 95% prediction bounds. The observed numbers of accidental deaths at times 73 – 78 were 7798, 7406, 8363, 8460, 9217 and 9316, all of which lie between the bounds. Forecasts beyond six months are clearly strongly influenced by the assumption of a quadratic trend and should be treated with some scepticism.

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Set 3

2.1, 2.3, 2.5, 2.7, 2.8, 2.10

2.1 $S(a,b) = E(X_{n+h} - aX_n - b)^2$ to be minimized

$$= E\left((X_{n+h}-\mu) - a(X_n-\mu) - b - a\mu + \mu\right)^2$$

$$= \gamma(0) + a^2\gamma(0) + (b + a\mu - \mu)^2 - 2a\gamma(h)$$

$$\frac{\partial S}{\partial a} = 2a\gamma(0) + 2\mu(b + a\mu - \mu) - 2\gamma(h)$$

$$\frac{\partial S}{\partial b} = 2(b + a\mu - \mu)$$

S is clearly minimized w.r.t. b when
$$b = \mu(1-a)$$

Setting $b = \mu(1-a)$ in $\frac{\partial S}{\partial a}$ and equating to zero $\Rightarrow$

$$a = \frac{\gamma(h)}{\gamma(0)} = \rho(h)$$

6 $\therefore$ $S(a,b)$ is min when
$$a = \rho(h), \quad b = \mu(1-\rho(h))$$

2.3 (a) $X_t = Z_t + .3Z_{t-1} - .4Z_{t-2}$, $\{Z_t\} \sim WN(0,1)$

$$\gamma(0) = 1 + .3^2 + (-.4)^2 = 1.25$$
$$\gamma(1) = .3 - .4 \times .3 = .18$$
$$\gamma(2) = -.4$$
$$\gamma(h) = 0, \quad h > 2$$

$$\gamma(h) = \gamma(-h)$$

(b) $$Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}, \quad \{\tilde{Z}_t\} \sim WN(0, .25)$$

$$\gamma(0) = (.25) + (1.2)^2(.25) + (1.6)^2(.25)$$

$$= 1.25$$

$$\gamma(1) = -1.2(.25) + (1.6)(1.2)(.25)$$

$$= .18$$

$$\gamma(2) = -1.6(.25)$$

$$= -.4$$

$$\gamma(h) = 0, \quad h > 2.$$

$$\gamma(-h) = \gamma(h)$$

6 Same acvf as in (a).

2.5 $\sum_{j=1}^{m} \theta^j X_{n-j}$ converges absolutely (with prob 1)

as $m \to \infty$ since

$$E \sum_1^\infty |\theta|^j |X_{n-j}| \le \sum_1^\infty |\theta|^j E|X_{n-j}|$$

$$\le \sum_1^\infty |\theta|^j \sqrt{\gamma(0)+\mu^2}$$

by Cauchy-Schwarz

$$< \infty \text{ since } |\theta| < 1$$

$\therefore \sum_1^\infty |\theta^j| |X_{n-j}| < \infty$ with probability 1.

$\sum_1^m \theta^j X_{n-j}$ cges in m.s. as $m \to \infty$ since

for $m>k$, $$E|S_m - S_k|^2 = E\left(\sum_{r=k+1}^{m} \theta^r X_{n-r}\right)^2$$

$$= \sum_s \sum_r \theta^{r+s} E(X_{n-r}X_{n-s})$$

$$= \sum\sum \theta^{r+s}\left(\gamma(r-s)+\mu^2\right)$$

$$\leq \sum\sum |\theta|^{r+s}\left(\gamma(0)+\mu^2\right)$$

$$= \left(\gamma(0)+\mu^2\right)\left(\sum_{r=k+1}^{m}|\theta|^{r}\right)^2$$

6 $\rightarrow 0$ as $k, m \rightarrow \infty$ since $\sum_0^\infty |\theta|^r < \infty$.

2.7

$$\frac{1}{1-\phi z} = \frac{-1/\phi z}{1-1/\phi z}$$

$$= -\frac{1}{\phi z}\left(1+\frac{1}{\phi z}+\frac{1}{(\phi z)^2}+\cdots\right)$$

since $|\phi z| > 1$

4 $$= -\sum_{j=1}^{\infty}(\phi z)^{-j}.$$

2.8

$$X_t = \phi X_{t-1} + Z_t$$

$$= Z_t + \phi(Z_{t-1} + \phi X_{t-2})$$

$$= \cdots$$

$$= Z_t + \phi Z_{t-1} + \cdots + \phi^n Z_{t-n} + \phi^{n+1} X_{t-n-1}$$

$\mathrm{Var}(X_t - \phi^{n+1} X_{t-n-1})$

$$= \gamma(0)\left(1+\phi^{2n+2}\right) - 2\phi^{n+1}\gamma(n+1)$$

$$\leq \gamma(0)\left(1+2|\phi|^{n+1}+|\phi|^{2n+2}\right)$$

$$= 4\gamma(0)$$ if $\{X_t\}$ is stationary and $|\phi| = 1$.

$\mathrm{Var}\left(Z_t + \phi Z_{t-1} + \cdots + \phi^n Z_{t-n}\right)$

6 $= n\sigma^2$ if $|\phi| = 1$. Since this is not $\leq 4\gamma(0)$ for all n, $\{X_t\}$ cannot be stat if $|\phi| = 1$.

2.10. $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

$$\phi = .5$$
$$\theta = .5$$

See 2.3 equn (2.3.3)

$$\Rightarrow X_t = \sum_0^\infty \psi_j Z_{t-j}$$

where $\psi_0 = 1$

$$\psi_j = (\phi+\theta)\phi^{j-1}, \quad j \geq 1$$
$$= (.5)^{j-1}, \quad j \geq 1$$

See 2.3 equn (2.3.5)

$$\Rightarrow Z_t = \sum_0^\infty \pi_j X_{t-j}$$

where $\pi_0 = 1$

$$\pi_j = -(\phi+\theta)(-\theta)^{j-1}, \quad j \geq 1$$
$$= -(-.5)^{j-1}, \quad j \geq 1.$$

6

Agrees with ITSM.

ST 525 Solutions Set 4

2.12 MA(1) $X_t = Z_t - .6Z_{t-1}$, $\{Z_t\} \sim WN(0,1)$

$$\bar{x}_{100} = .157$$

95% CI for μ

$$\text{Var}\,\bar{X}_{100} = \frac{1}{n}\sum_{-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma(h)$$

$$= \frac{1}{100}\left[\gamma(0) + 2 \times \frac{99}{100}\gamma(1)\right]$$

$$= \frac{1}{100}\left[1.36 - 1.98 \times .6\right]$$

$$= .00172$$

$\therefore$ 95% conf. bounds for μ are approx

$$\bar{x}_{100} \pm 1.96\sqrt{.00172}$$

.157

$$= .157 \pm 1.96 \times .0415$$

$$= .157 \pm .0813 = .076 , .238$$

Reject $H_0: \mu = 0$ in favour of $H_a: \mu \neq 0$ at significance level .05 since 95% bounds for μ do not include the value 0.

(c)

(NOTE If $\{X_t\}$ were iid $(0, 1.36)$ the conclusion would differ)

2.13 (a) $\hat{\rho}(1) = .438$, $\hat{\rho}(2) = .145$

AR(1) $X_t - \phi X_{t-1} = Z_t$

Bartlett (2.4.12) $\Rightarrow$ $\text{Var}\,\hat{\rho}(1) \simeq \frac{1}{n}(1-\phi^2)$

$$\text{Var}\,\hat{\rho}(2) \simeq \frac{1}{n}\left[(1+\phi^2)^2 - 4\phi^4\right]$$

Approx 95% CB's for $\rho(1)$: $\hat{\rho}(1) \pm \frac{1.96}{\sqrt{n}}(1-\phi^2)^{1/2}$

$\rho(2)$: $\hat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}(1-\phi^2)^{1/2}(1+3\phi^2)^{1/2}$

With $\phi = \hat{\phi} = \hat{\rho}(1)$, $n = 100$, $\hat{\rho}(1) = \cdot 438$, $\hat{\rho}(2) = \cdot 148$, these become

$$\rho(1) : \cdot 438 \pm \cdot 196 \times \cdot 899 = \cdot 262, \cdot 614$$

$$\rho(2) : \cdot 148 \pm \cdot 196 \times 1.128 = -\cdot 073, \cdot 369$$

(4) Not consistent with $\phi = \cdot 8$ since both $\rho(1) = \cdot 8$ and $\rho(2) = \cdot 64$ are outside the bounds

(b) MA(1) : $X_t = Z_t + \theta Z_{t-1}$

Bartlett $\Rightarrow$ $\operatorname{Var} \hat{\rho}(1) \simeq \frac{1}{n}\left(1 - 3\rho(1)^2 + 4\rho(1)^4\right)$

$\operatorname{Var} \hat{\rho}(2) \simeq \frac{1}{n}\left(1 + 2\rho(1)^2\right)$

Approx 95% bounds $\rho(1) \doteq \hat{\rho}(1) \pm \frac{1.96}{\sqrt{n}}\left(1 - 3\rho(1)^2 + 4\rho(1)^4\right)^{1/2}$

$\rho(2) \doteq \hat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}\left(1 + 2\rho(1)^2\right)^{1/2}$

Setting $\rho(1) = \hat{\rho}(1)$, $n = 100$, $\hat{\rho}(1) = \cdot 438$, $\hat{\rho}(2) = \cdot 148$, these become

$$\rho(1) : \cdot 438 \pm \cdot 196 \times \cdot 756 = \cdot 290, \cdot 586$$

$$\rho(2) : \cdot 148 \pm \cdot 196 \times 1.176 = -\cdot 082, \cdot 378$$

$$\left(\theta = \cdot 6 \Rightarrow \rho(1) = \frac{\theta}{1+\theta^2} = \cdot 4412, \ \rho(2) = 0 \right)$$

(4) The bounds are consistent with $\rho(1) = \cdot 4412$, $\rho(2) = 0$ and hence the data are consistent with $X_t = Z_t + \cdot 6 Z_{t-1}$.

2.14 $X_t = A\cos(\omega t) + B\sin\omega t$, A, B uncorrelated $(0, 1)$.

(a) $P_1 X_2 = \phi_{11} X_1$

where $\gamma(0)\phi_{11} = \gamma(1) \Rightarrow \phi_{11} = \rho(1) = \cos\omega$

and $E(X_2 - P_1 X_2)^2 = \gamma(0) - \phi_{11}\gamma(1) = \gamma(0)(1 - \cos^2\omega)$

(2) $= \sin^2\omega$

Note 2.14 is an example in which the matrix Γ_n in the equation $\Gamma_n \underset{\sim}{\phi}_n = \underset{\sim}{\gamma}_n$ is singular for $n \geq 3$. This is because $X_3 = (2\cos\omega)X_2 - X_1$

(b) $$P_2 X_3 = \phi_{21} X_2 + \phi_{22} X_1$$

where $$\begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \underset{\sim}{\phi}_2 = \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix},$$

i.e. $$\begin{bmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \cos\omega \\ \cos 2\omega \end{bmatrix}.$$

$$\therefore \phi_{22}(\cos^2\omega - 1) = \cos^2\omega - \cos 2\omega$$
$$= \cos^2\omega - 2\cos^2\omega + 1$$

$$\therefore \phi_{22} = -1$$

and
$$\phi_{21} = \cos\omega - \phi_{22}\cos\omega = 2\cos\omega$$

$$\therefore P_2 X_3 = 2\cos(\omega) \cdot X_2 - X_1$$

and
$$E(X_3 - P_2 X_3)^2 = \gamma(0) - \underset{\sim}{\phi}_2' \underset{\sim}{\gamma}$$
$$= 1 - \begin{bmatrix} 2\cos\omega & -1 \end{bmatrix} \begin{bmatrix} \cos\omega \\ \cos 2\omega \end{bmatrix}$$
$$= 1 - 2\cos^2\omega + \cos 2\omega$$
$$= 0$$

4

(c) From (b) and stationarity,
$$P(X_{n+1} \mid X_n, X_{n-1}) = (2\cos\omega) X_n - X_{n-1}$$

with MSE = 0

Since $(2\cos\omega) X_{n+1} - X_{n-1}$ is a linear comb'n of X_s, $-\infty < s \leq n$, and since it is impossible to find a predictor of this form with smaller MSE, we conclude that

4

$$\tilde{P}_n X_{n+1} = (2\cos\omega) X_n - X_{n-1} \quad \text{with MSE} = 0$$

==============

ITSM::Pest(ACF/PACF) 2.16 4.4

==============

# of Lags = 40

Sample Autocorrelations:

Sample Variance = 1382.18510000

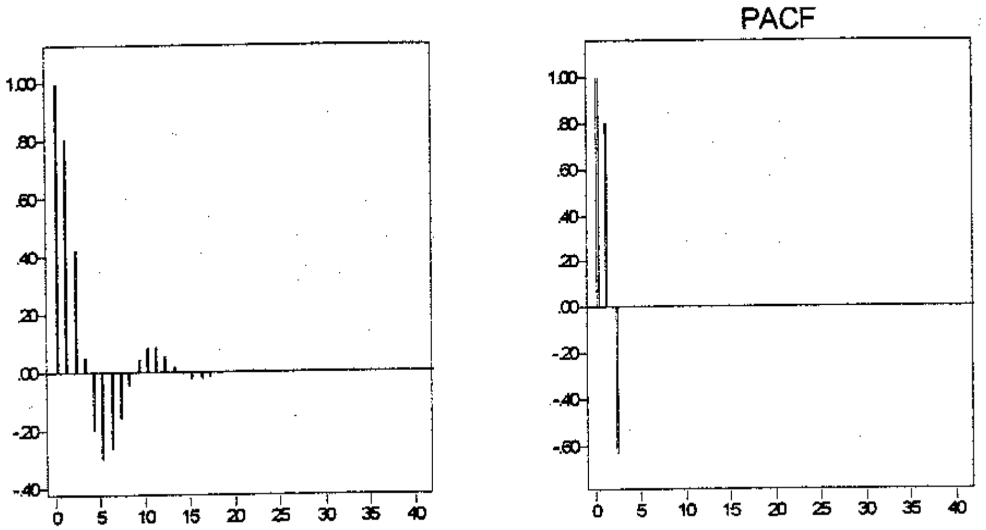
1.0000	.8062	.4281	.0696	-.1694
-.2662	-.2117	-.0437	.1637	.3305
.4099	.3941	.2882	.1431	.0197
-.0548	-.1020	-.1448	-.1770	-.1676
-.1042	-.0186	.0416	.0485	-.0035
-.1001	-.1820	-.2315	-.2505	-.2415
-.2073	-.1500	-.0931	-.0786	-.0974
-.1343	-.1682	-.1857	-.1839	-.1808

Sample Partial Autocorrelations:

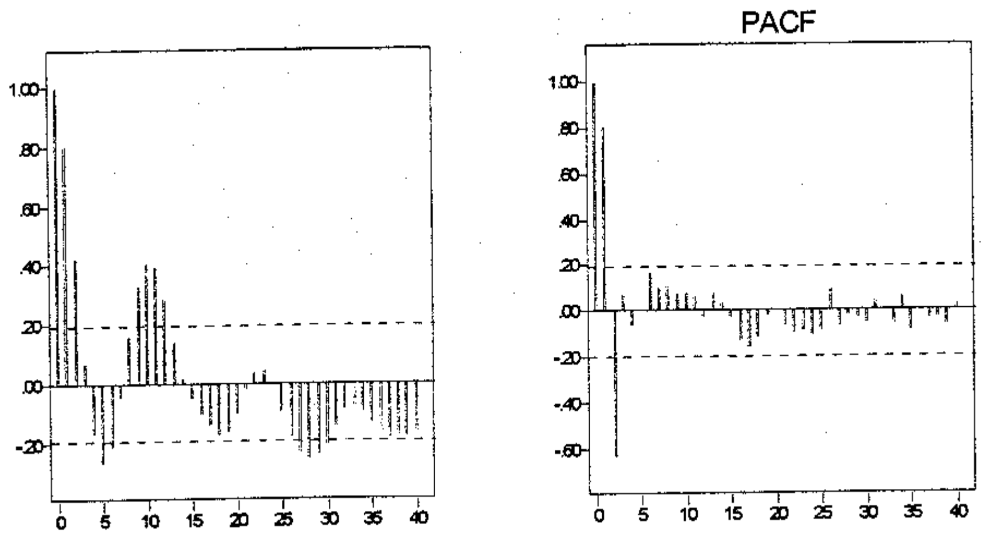
1.0000	.8062	-.6341	.0805	-.0611
.0011	.1698	.1074	.1117	.0800
.0765	.0669	-.0328	.0748	.0369
-.0314	-.1330	-.1571	-.1146	-.0204
.0012	-.0628	-.0988	-.0922	-.1089
-.0901	.0941	-.0735	-.0214	-.0280
-.0599	.0425	-.0017	-.0660	.0638
-.0891	-.0018	-.0373	-.0293	-.0612

```
================================
ITSM::Pest(Preliminary estimates)
================================

Method: Yule-Walker

Fitted Model:
X(t) = 1.318 X(t-1) - .6341 X(t-2)
     + Z(t)

WN Variance = 232.894980

AR Coefficients
      1.317501        -.634121


Ratio of AR coeff. to 1.96 * (standard error)
      8.693289      -4.184136

(Residual SS)/N = 232.895

WN variance estimate (Yule Walker): 289.214

-2Log(Like) = 830.924987
AICC = 837.174987
```

MODEL ACF/PACF



SAMPLE ACF/PACF



ITSM::(ARMA Forecast) 2.17 4.7

Step	Prediction	sqrt(MSE)
1	88.89157	15.26090
2	85.04872	25.24195
3	70.54270	30.32859
4	53.86785	31.75219
5	41.09730	31.79898
6	34.84597	32.01229
7	34.70793	32.56133
8	38.49016	33.02344
9	43.56078	33.20603

2.18 (1) $X_t = Z_t - \theta Z_{t-1}$, $|\theta|<1$, $\{Z_t\} \sim WN(0,\sigma^2)$

(2) $Z_t = X_t + \theta X_{t-1} + \theta^2 X_{t-2} + \cdots$

Setting $t = n+1$ in (2) and applying $\tilde{P}_n$ to each side

$$\Rightarrow \tilde{P}_n X_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j}$$

$$\because \quad \text{--- (1) ---}$$

(3) $$\Rightarrow \tilde{P}_n X_{n+1} = -\theta Z_n$$

(b) Prediction error $= X_{n+1} - \tilde{P}_n X_{n+1} = Z_{n+1}$

$$\therefore \quad MSE = E Z_{n+1}^2 = \sigma^2 .$$

Set 5

CHAPTER 3

3.1. (a) $\phi(z) = 1 + .2z - .48z^2 = (1 + .8z)(1 - .6z) = 0$ when $z = -1/.8$ or $z = 1/.6$. Since both of these zeros are outside the unit circle, the process is causal. Process is obviously invertible since $Z_t = X_t + .2X_{t-1} - .48X_{t-2}$.

(b) $\phi(z) = 1 + 1.9z + .88z^2 = (1 + 1.1z)(1 + .8z) = 0$ when $z = -1/1.1$ or $z = -1/.8$. Since the first of these two zeros lies inside the unit circle, the process is not causal.
$\theta(z) = 1+.2z+.7z^2 = 0$ when $z = \frac{-.2\pm i\sqrt{2.76}}{1.4}$ and $|z|^2 = [(-.2)^2+2.76]/(1.4)^2 = 1.429 > 1$ which implies the process is invertible.

(c) $\phi(z) = 1 + .6z = 0$ when $z = -10/6 \implies$ causal.
$\theta(z) = 1 + 1.2z = 0$ when $z = -5/6 \implies$ not invertible.

(d) $\phi(z) = (1+.9z)^2 = 0$ when $z = -10/9 \implies$ causal. AR(p) processes are always invertible.

(e) $\phi(z) = 1 + 1.6z = 0$ when $z = -10/16 \implies$ not causal.
$\theta(z) = 1 - .4z + .04z^2 = (1 - .2z)^2 = 0$ when $z = 5 \implies$ invertible.

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3.3 $\psi_0 = 1$, $\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$ where $\psi_j := 0$ for $j < 0$

(a)
$\psi_0 = 1$
$\psi_1 = (-.2)\psi_0 = -.2$
$\psi_2 = (-.2)\psi_1 + (.48)\psi_0 = .52$
$\psi_3 = (-.2)\psi_2 + (.48)\psi_1 = -.20$
$\psi_4 = (-.2)\psi_3 + (.48)\psi_2 = .2896$
$\psi_5 = (-.2)\psi_4 + (.48)\psi_3 = -.15392$

(c)
$\psi_0 = 1$
$\psi_1 = 1.2 - (.6)\psi_0 = 0.6$
$\psi_2 = (-.6)\psi_1 = -0.36$
$\psi_3 = (-.6)\psi_2 = 0.216$
$\psi_4 = (-.6)\psi_3 = -0.1296$
$\psi_5 = (-.6)\psi_4 = 0.07776$

(d)
$\psi_0 = 1$
$\psi_1 = (-1.8)\psi_0 = -1.8$
$\psi_2 = (-1.8)\psi_1 + (-.81)\psi_0 = 2.43$
$\psi_3 = (-1.8)\psi_2 + (-.81)\psi_1 = -2.916$
$\psi_4 = (-1.8)\psi_3 + (-.81)\psi_2 = 3.2805$
$\psi_5 = (-1.8)\psi_4 + (-.81)\psi_3 = -3.54294$

These all agree with ITSM.

9

===============
ITSM::Pest(ACF/PACF) 3.2 (a)
===============
# of Lags = 40

5.2

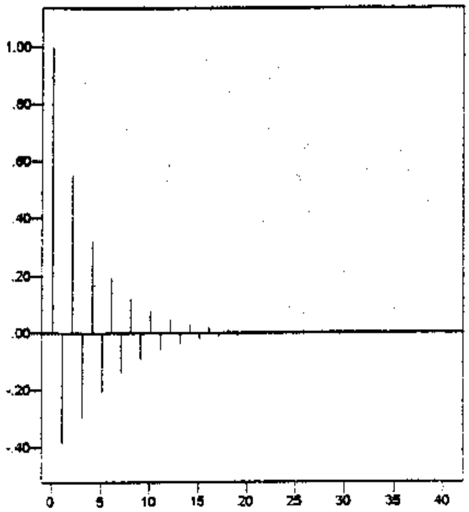
Model Autocorrelations:

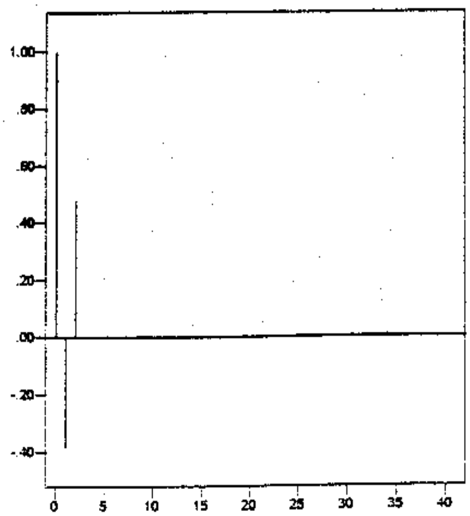
Gamma(0) = 1.52496246

1.0000	-.3846	.5569	-.2960	.3265
-.2074	.1982	-.1392	.1230	-.0914
.0773	-.0593	.0490	-.0383	.0312
-.0246	.0199	-.0158	.0127	-.0101
.0081	-.0065	.0052	-.0041	.0033
-.0027	.0021	-.0017	.0014	-.0011
.0009	-.0007	.0006	-.0004	.0004
-.0003	.0002	-.0002	.0001	-.0001

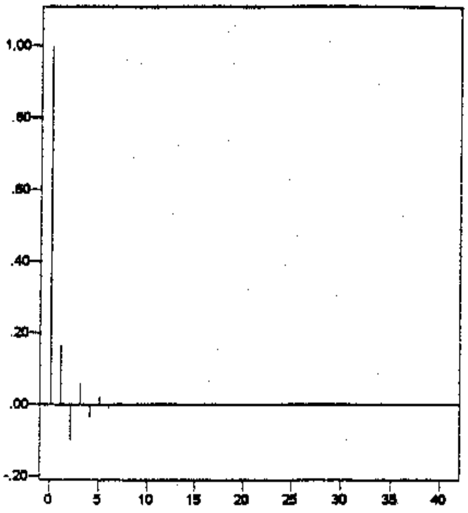
Model Partial Autocorrelations:

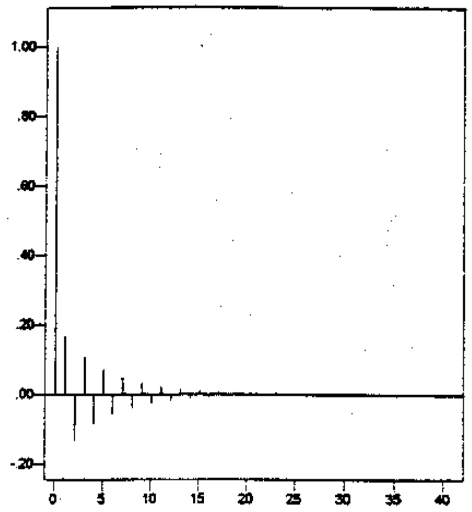
1.0000	-.3846	.4800	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000





3.2 (c)





Gamma(0) = 1.56250000

1.0000	.1680	-.1008	.0605	-.0363
.0218	-.0131	.0078	-.0047	.0028
-.0017	.0010	-.0006	.0004	-.0002
.0001	-.0001	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000

Model Partial Autocorrelations:

1.0000	.1680	-.1328	.1068	-.0869
.0713	-.0587	.0486	-.0403	.0334
-.0278	.0231	-.0192	.0160	-.0133
.0111	-.0093	.0077	-.0064	.0054
-.0045	.0037	-.0031	.0026	-.0022
.0018	-.0015	.0012	-.0010	.0009
-.0007	.0006	-.0005	.0004	-.0003
.0003	-.0002	.0002	-.0002	.0001

ITSM::Pest(ACF/PACF)   3.2 (d)

5.4

# of Lags = 40

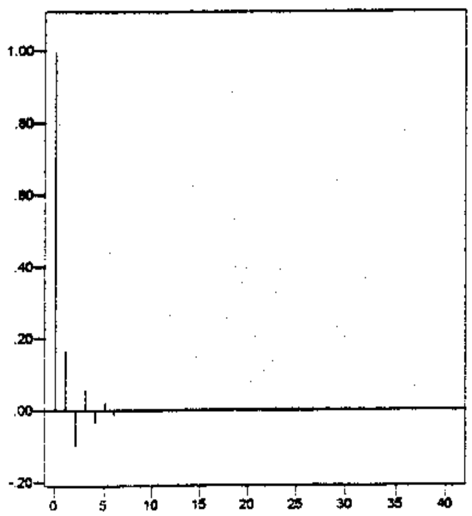
Model Autocorrelations:

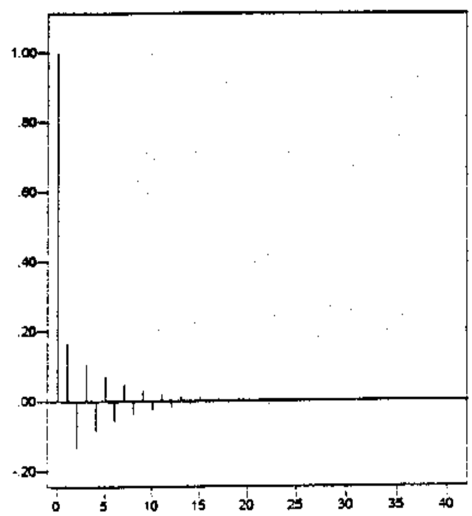
Gamma(0) = 1.56250000

1.0000	.1680	-.1008	.0605	-.0363
.0218	-.0131	.0078	-.0047	.0028
-.0017	.0010	-.0006	.0004	-.0002
.0001	-.0001	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000
.0000	.0000	.0000	.0000	.0000

Model Partial Autocorrelations:

1.0000	.1680	-.1328	.1068	-.0869
.0713	-.0587	.0486	-.0403	.0334
-.0278	.0231	-.0192	.0160	-.0133
.0111	-.0093	.0077	-.0064	.0054
-.0045	.0037	-.0031	.0026	-.0022
.0018	-.0015	.0012	-.0010	.0009
-.0007	.0006	-.0005	.0004	-.0003
.0003	-.0002	.0002	-.0002	.0001





6

3.7 (1) $X_t = Z_t + \theta Z_{t-1}$, $\{Z_t\} \sim WN(0, \sigma^2)$, $|\theta| > 1$.

Show (2) $W_t \doteq \sum_0^\infty (-\theta)^{-j} X_{t-j}$ is $WN(0, \sigma_W^2)$ and find σ_W^2

From (1) $Z_{t-1} = \frac{1}{\theta} X_t - \frac{1}{\theta^2} X_{t+1} + \frac{1}{\theta^3} X_{t+2} - \cdots$

(3) $= \frac{1}{\theta} U_t$

where $U_t = X_t - \frac{1}{\theta} X_{t+1} + \frac{1}{\theta^2} X_{t+2} - \cdots$

Claim $\{U_t\}$ has the same mean and ACVF as $\{W_t\}$

[Pf. $EU_t = 0 = EW_t$

$Cov(U_{t+h}, U_t) = \sum\sum a_j a_k \gamma(h+j-k)$

where $a_j = \begin{cases} \left(-\frac{1}{\theta}\right)^{-j}, & j \leq 0 \\ 0, & j > 0 \end{cases}$

$Cov(W_{t+h}, W_t) = \sum\sum a_j a_k \gamma(h+j-k)$

where $a_j = \begin{cases} \left(-\frac{1}{\theta}\right)^{j}, & j \geq 0 \\ 0, & j < 0 \end{cases}$

$= Cov(U_{t+h}, U_t)$]

But from (3), $\{U_t\} \sim WN(0, \theta^2\sigma^2)$

Hence $\{W_t\} \sim WN(0, \theta^2\sigma^2)$

Applying the filter $(1 + \frac{1}{\theta}B)$ to each side of (2) gives

$$W_t + \frac{1}{\theta} W_{t-1} = \left(1 + \frac{1}{\theta}B\right)\left(\sum_0^\infty (-\theta)^{-j} B^j\right) X_t$$

$$= \left(1 + \frac{1}{\theta}B\right)\left(1 - \frac{1}{\theta}B + \frac{1}{\theta^2}B^2 - \cdots\right) X_t$$

$$= \left(1 + B\left(\frac{1}{\theta} - \frac{1}{\theta}\right) + B^2\left(\frac{1}{\theta^2} - \frac{1}{\theta^2}\right) + \cdots\right) X_t$$

$$= X_t.$$

3.8. By equation (3.1.14), the stationary solution of the difference equations, $X_t = \phi X_{t-1} + Z_t$ with $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $|\phi| > 1$, is given by $X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$. It follows that $EX_t = 0$ and

$$\gamma_X(h) = \frac{\sigma^2}{\phi^2 - 1}\phi^{-|h|}.$$

Now define

$$\tilde{Z}_t = X_t - \phi^{-1}X_{t-1}.$$

Then $E\tilde{Z}_t = EX_t - \phi^{-1}EX_{t-1} = 0$ and

$$\begin{aligned}
\mathrm{Cov}(\tilde{Z}_{t+h}, \tilde{Z}_t) &= \mathrm{Cov}(X_{t+h} - \phi^{-1}X_{t+h-1}, X_t - \phi^{-1}X_{t-1}) \\
&= \gamma_X(h) - \phi^{-1}\gamma_X(h-1) - \phi^{-1}\gamma_X(h+1) + \phi^{-2}\gamma_X(h) \\
&= \begin{cases} \frac{\sigma^2}{\phi^2-1}(1 - 2\phi^{-2} + \phi^{-2}) = \frac{\sigma^2}{\phi^2}, & \text{if } h = 0, \\ \frac{\sigma^2}{\phi^2-1}(\phi^{-h} - \phi^{-h} - \phi^{-h-2} + \phi^{-2-h}) = 0 & \text{if } h > 0. \end{cases}
\end{aligned}$$

Thus $\{\tilde{Z}_t\} \sim \mathrm{WN}(0, \tilde{\sigma}^2)$ with $\tilde{\sigma}^2 = \sigma^2/\phi^2$. Note that the causal representation has a smaller white noise variance than the noncausal representation. 6

3.9. (a) The autocovariance $\gamma(\cdot)$ is given by

$$\gamma(h) = \begin{cases} \sigma^2(1 + \theta_1^2 + \theta_{12}^2), & h = 0, \\ \sigma^2\theta_1, & h = \pm 1, \\ \sigma^2\theta_1\theta_{12}, & h = \pm 11, \\ \sigma^2\theta_{12}, & h = \pm 12, \\ 0, & \text{otherwise.} \end{cases}$$

(b) From ITSM, the sample mean is 28.831 and

h	0	1	2	3	4	5	6
$\hat{\gamma}(h)$	152670	-54327	-15072	14585	-17178	6340	17421

h	7	8	9	10	11	12	13
$\hat{\gamma}(h)$	-31165	-1088	15277	-12435	29802	-50867	13768

h	14	15	16	17	18	19	20
$\hat{\gamma}(h)$	17758	-6206	-9656	27982	-29456	3693	7569

(c) Matching $\gamma(1), \gamma(11), \gamma(12)$ with $\hat{\gamma}(1), \hat{\gamma}(11), \hat{\gamma}(12)$, we have

$$\hat{\theta}_1 = \hat{\gamma}(11)/\hat{\gamma}(12) = -.586,$$

$$\hat{\theta}_{12} = \hat{\gamma}(11)/\hat{\gamma}(1) = -.549,$$

$$\hat{\sigma}^2 = \hat{\gamma}(1)/\hat{\theta}_1 = 92740.$$

so that the model for $\{Y_t = \nabla \nabla_{12} X_t\}$ is

$$Y_t = 28.831 + Z_t - .586Z_{t-1} - .549Z_{t-12}, \qquad \{Z_t\} \sim WN(0, 92740).$$

9

3+3+3

3.10. (a) From ITSM,

$$\hat\gamma(0) = 676789 = \hat\sigma^2/(1-\hat\phi^2)$$
$$\hat\gamma(1) = 495633 = \hat\phi\hat\sigma^2/(1-\hat\phi^2)$$

and solving for $\hat\sigma^2$ and $\hat\phi$, we obtain $\hat\phi = .73233, \hat\sigma^2 = 313822$. The sample mean is $\hat\mu = 4503$ giving us the model

$$Y_t = .73233Y_{t-1} + Z_t, \qquad \{Z_t\} \sim WN(0, 313822)$$

where $Y_t = X_t - \hat\mu = X_t - 4503$.

(b) The best linear predictor of Y_{31} is

$$\hat Y_{31} = \hat\phi Y_{30} = .7323(3885 - 4503) = -452$$

8

and hence $\hat X_{31} = \hat Y_{31} + \hat\mu = 4051$. The mean squared error of prediction is $E(Y_{31} - \hat Y_{31})^2 = EZ_{31}^2 = \sigma^2$ which we estimate by $\hat\sigma^2 = 313822$.

(c) 95% prediction bounds

$$4051 \pm 1.96\sqrt{313822} = 2953,\ 5149$$

1098

Set 6

3.11 The best linear predictor of X_3 in terms of X_2 and X_1 $(X_t = Z_t + \theta Z_{t-1})$

is $$P_2 X_3 = a_1 X_2 + a_2 X_1$$

where $\Gamma \underset{\sim}{a} = \underset{\sim}{\gamma}$, $\Gamma = \mathrm{cov}\left(\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}, \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}\right)$, $\underset{\sim}{\gamma} = \mathrm{cov}\left(X_3, \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}\right)$

i.e. $$\begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \theta \\ 0 \end{bmatrix}$$

$$\Rightarrow a_1 = -\frac{(1+\theta^2)\, a_2}{\theta}$$

$$\Rightarrow \left[-\frac{(1+\theta^2)^2}{\theta} + \theta\right] a_2 = \theta$$

$$\Rightarrow a_2 = -\frac{\theta^2}{1+\theta^2+\theta^4} = \text{PACF at lag 2}$$ 6

(The PACF at lag n turns out to be $-\theta^n \dfrac{1-\theta^2}{1-\theta^{2n+2}}$, $n \geq 1$)

3.13. (a) $r_n = 1 + \theta^2 - \theta^2 / r_{n-1}$ and hence $r_n - 1 = \theta^2 \frac{r_{n-1}-1}{r_{n-1}}$ or

$$\frac{1}{r_n - 1} = \theta^{-2} \frac{r_{n-1}}{r_{n-1}-1} \Longrightarrow -1 + \frac{r_n}{r_n - 1} = \theta^{-2} \frac{r_{n-1}}{r_{n-1}-1}.$$

Defining $y_n = \frac{r_n}{r_n - 1}$, we have $y_0 = \frac{r_0}{r_0 - 1} = \frac{1+2\theta\phi+\theta^2}{(\phi+\theta)^2}$ and $y_n = \theta^{-2} y_{n-1} + 1$.

(b) From the recursion derived in (a)

$$y_n = 1 + \theta^{-2} y_{n-1} = 1 + \theta^{-2}(1 + \theta^{-2} y_{n-2}) = \cdots = 1 + \theta^{-2} + \cdots + \theta^{-2n+2} + \theta^{-2n} y_0.$$

Since $r_n = y_n/(y_n - 1)$, we have for $n \geq 1$

$$r_n = \frac{1 + \theta^{-2} + \cdots + \theta^{-2n+2} + \theta^{-2n} y_0}{\theta^{-2} + \cdots + \theta^{-2n+2} + \theta^{-2n} y_0}$$

$$\theta_{n1} = \frac{\theta}{r_{n-1}}.$$

(c) From (b)

$$\lim_{n\to\infty} r_n = \begin{cases} \frac{(1-\theta^{-2})^{-1}}{\theta^{-2}/(1-\theta^{-2})} = \theta^2, & |\theta| > 1, \\ 1, & |\theta| = 1, \\ \frac{y_0 + \theta^2 + \theta^4 + \cdots}{y_0 + \theta^2 + \theta^4 + \cdots} = 1, & |\theta| < 1 \end{cases}$$

and

$$\lim_{n\to\infty} \theta_{n1} = \begin{cases} \theta^{-1}, & |\theta| > 1, \\ \theta, & |\theta| = 1, \\ \theta, & |\theta| < 1. \end{cases}$$ 9

4.1 $\int_{-\pi}^{\pi} e^{i(k-h)\lambda}\,d\lambda = \int_{-\pi}^{\pi}\left[\cos(k-h)\lambda + i\sin(k-h)\lambda\right]d\lambda$ 6.2

$$= \begin{cases} \left[\dfrac{\sin(k-h)\lambda - i\cos(k-h)\lambda}{k-h}\right]_{-\pi}^{\pi} = 0 & \text{if } k \neq h \\ \int_{-\pi}^{\pi} 1\,d\lambda = 2\pi & \text{if } k = h \end{cases}$$

4

4.4. Since

$$\frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\omega h}\gamma(h) = \frac{1}{2\pi}\left[1 - .5(e^{-2i\omega} + e^{2i\omega}) - .25(e^{-3i\omega} + e^{3i\omega})\right]$$
$$= \frac{1}{2\pi}\left[1 - \cos 2\omega - .5\cos 3\omega\right]$$
$$< 0 \quad \text{at } \omega = 0,$$

6

$\gamma(h)$ cannot be an autocovariance function.

4.5. By Problem 1.10, $\{Z_t := X_t + Y_t\}$ is stationary with autocovariance function $\gamma(h) = \gamma_X(h) + \gamma_Y(h)$ and hence

$$\gamma(h) = \gamma_X(h) + \gamma_Y(h) = \int_{-\pi}^{\pi} e^{i\omega h}\,dF_X(\omega) + \int_{-\pi}^{\pi} e^{i\omega h}\,dF_Y(\omega)$$
$$= \int_{-\pi}^{\pi} e^{i\omega h}\,dF_z(\omega)$$

6

where $F_z(\omega) = F_X(\omega) + F_Y(\omega)$ is the spectral distribution function of $\{Z_t\}$.

4.6. By Problem 1.10, $\gamma_X(h) = \gamma_U(h) + \gamma_Y(h)$, where

$$\gamma_U(h) = \nu^2\cos\frac{\pi h}{3} = \frac{\nu^2}{2}e^{-i\pi h/3} + \frac{\nu^2}{2}e^{i\pi h/3},$$
$$\gamma_Y(h) = \begin{cases} 7.25\sigma^2, & h = 0, \\ 2.5\sigma^2, & h = \pm 1, \\ 0, & |h| > 1. \end{cases}$$

Moreover by Problem 4.7, $F_X(\lambda) = F_U(\lambda) + F_Y(\lambda)$ where

$$F_U(\lambda) = \begin{cases} 0, & \lambda < -\pi/3, \\ \nu^2/2, & -\pi/3 \le \lambda < \pi/3, \\ \nu^2, & \pi/3 \le \lambda, \end{cases}$$
$$F_Y(\lambda) = \int_{-\pi}^{\lambda} \frac{\sigma^2}{2\pi}(7.25 + 5\cos\lambda)\,d\lambda$$
$$= \frac{\sigma^2}{2\pi}\left[7.25(\lambda + \pi) + 5\sin\lambda\right].$$

8

4.8. The spectral density of $\{X_t\}$ is

$$f_X(\lambda) = (2\pi)^{-1}|1 - .99e^{-i3\lambda}|^{-2} = (2\pi)^{-1}(1.9801 - 1.98\cos 3\lambda)^{-1}.$$ 3

This spectral density has sharp peaks at the frequencies $\lambda = 0, \pm 2\pi/3$, which suggests sample paths that are quite smooth and nearly periodic with period 3. The spectral density of the filtered process $Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$ is 2 2

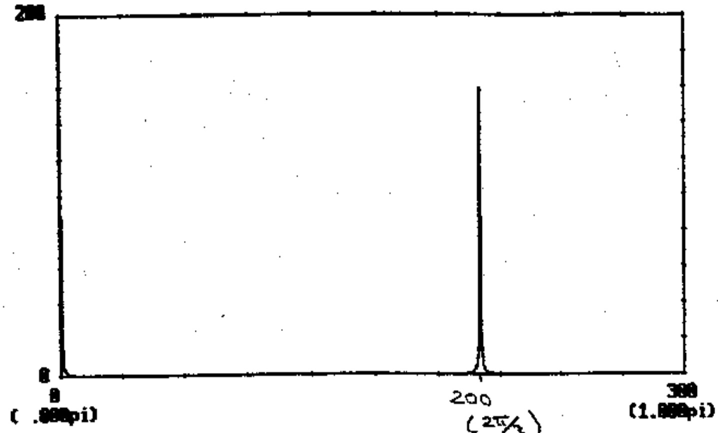
$$f_Y(\lambda) = \frac{1}{9}|e^{-i\lambda} + 1 + e^{i\lambda}|^2 f_X(\lambda)$$
$$= \frac{1}{9}(3 + 4\cos\lambda + 2\cos 2\lambda)f_X(\lambda)$$ 3

and $f_X(2\pi/3) = 10000/(2\pi)$, $f_Y(2\pi/3) = 0$. So this filter effectively eliminates the strong periodic component in the $\{X_t\}$ data. 2

14

2

for smoothing

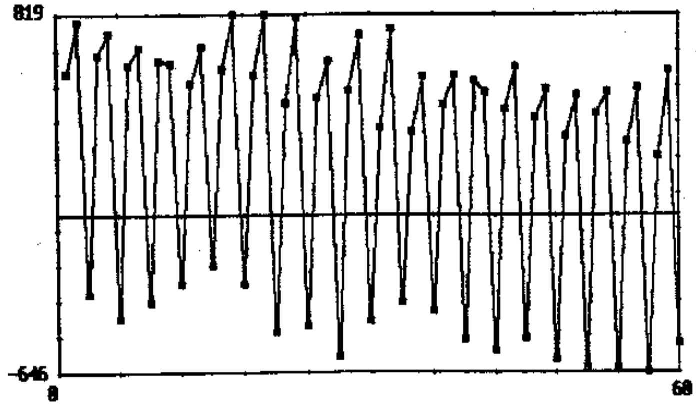


Vertical scale: 1 unit = .100000E+02;
Max. on vertical scale = .200000E+04; Min. = .000000E+00

Spec density

$$X_t - .99 X_{t-3} = Z_t$$

$$\{Z_t\} \sim WN(0,1)$$

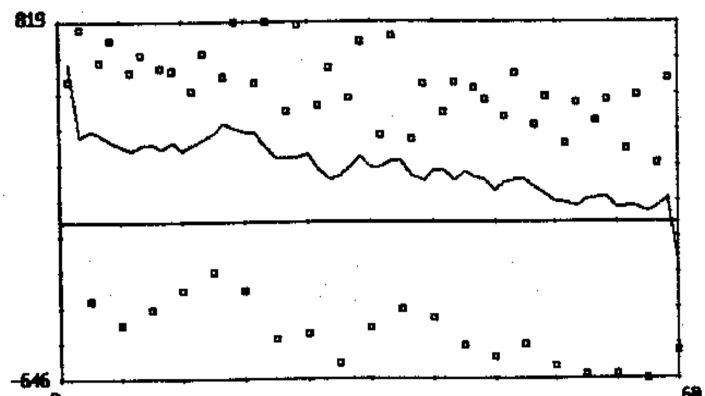


Vertical scale: 1 unit = .100000E-01;
Max. on vertical scale = .818753E+01; Min. = -.645983E+01

Simulated

$X_1, \ldots, X_{60}$

Strong component with period 3.



Vertical scale: 1 unit = .010000

$$Y_t = \frac{1}{3}\left(X_{t-1} + X_t + X_{t+1}\right)$$

Period 3 component essentially eliminated

Set 7 St 525

5.1 $\sqrt{n}\begin{bmatrix}\hat\phi_1-\phi_1\\ \hat\phi_2-\phi_2\end{bmatrix} \xrightarrow{d} N(\underset{\sim}{0}, \sigma^2\Gamma_2^{-1})$

$\hat\Gamma_2 = \begin{bmatrix}1382.2 & 1114.4\\ 1114.4 & 1382.2\end{bmatrix}$

$\hat\gamma(0) = 1382.2$
$\hat\gamma(1) = 1114.4$
$\hat\gamma(2) = 591.73$
$\hat\gamma(3) = \underline{96.216}$

$\hat\Gamma_2^{-1} = \begin{bmatrix}.002067 & -.001667\\ -.001667 & .002067\end{bmatrix}$

$\begin{bmatrix}\hat\phi_1\\ \hat\phi_2\end{bmatrix} = \hat\Gamma_2^{-1}\begin{bmatrix}\hat\gamma(1)\\ \hat\gamma(2)\end{bmatrix} = \begin{bmatrix}1.318\\ -.634\end{bmatrix}$, $\hat\sigma^2 = \hat\gamma(0) - [1.318 \ \ -.634]\begin{bmatrix}1114.4\\ 591.73\end{bmatrix}$

$= 1382.2 - 1093.0 = 289.2$

Estimated SD of $\hat\phi_1$: $\frac{\hat\sigma}{\sqrt n}\sqrt{(\hat\Gamma_2^{-1})_{11}} = .0773$

" " " $\hat\phi_2$: $\frac{\hat\sigma}{\sqrt n}\sqrt{(\hat\Gamma_2^{-1})_{22}} = .0773$

Approx 95% confidence bounds

ϕ_1 : $1.318 \pm 1.96 \times .0773 = 1.166, 1.470$
$(.152)$

ϕ_2 : $-.634 \pm 1.96 \times .0773 = -.786, -.482$

5.2 (a) DL algorithm (cf pp 141, 68)

$\hat\phi_{11} = \hat\rho(1) = \frac{\hat\gamma(1)}{\hat\gamma(0)} = .80625$

$\hat v_1 = \hat\gamma(0)[1-\hat\rho(1)^2] = 483.72$

$\hat\phi_{22} = \frac{\hat\gamma(2) - \hat\phi_{11}\hat\gamma(1)}{\hat v_1} = -.63412$

$\hat\phi_{21} = \hat\phi_{11} - \hat\phi_{11}\hat\phi_{22} = 1.3175$

$\hat v_2 = \hat v_1(1-\hat\phi_{22}^2) = 289.21$

$\hat\phi_{33} = \frac{\hat\gamma(3) - \hat\phi_{21}\hat\gamma(2) - \hat\phi_{22}\hat\gamma(1)}{\hat v_2} = .08047$

$\hat\phi_{32} = \hat\phi_{22} - \hat\phi_{33}\hat\phi_{21} = -.7401$

$\hat\phi_{31} = \hat\phi_{21} - \hat\phi_{33}\hat\phi_{22} = 1.3685$

$\hat v_3 = \hat v_2(1-\hat\phi_{33}^2) = 287.34$

(b) If the data are from an AR(2) process, $\phi_{33}=0$ and $\hat\phi_{33}$ is an observation from $N(0, \frac{1}{100})$

$|\hat\phi_{33}| < 1.96\,\frac{1}{10}$.

Hence we would not reject $H_0: \phi_{33}=0$ at level .05.

5.3 $\quad X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t \quad , \quad \{Z_t\} \sim WN(0, \sigma^2)$

(a) Causal $\iff \left\{ 1 - \phi z - \phi^2 z^2 = 0 \Rightarrow |z| > 1 \right\}$

$$\iff \left\{ z^2 - \phi z - \phi^2 = 0 \Rightarrow |z| < 1 \right\}$$

$$\iff \left| \frac{\phi}{2} \pm \sqrt{\frac{5\phi^2}{4}} \right| < 1$$

$$\iff \frac{\phi^2}{4} + \frac{5\phi^2}{4} \pm \phi\sqrt{\frac{5\phi^2}{4}} < 1$$

$$\iff \left| \phi\sqrt{\frac{5\phi^2}{4}} \right| < 1 - \frac{3\phi^2}{2}$$

$$\iff \phi^2 \frac{\sqrt{5}}{2} < 1 - \frac{3\phi^2}{2}$$

$$\iff \phi^2 < \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2} = .3820$$

4 $\qquad \iff |\phi| < \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{\sqrt{5}-1}{2} = .618$

(b) YW equns

$$\hat{\gamma}(0) - \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^2\hat{\gamma}(2) = \hat{\sigma}^2 \qquad (1)$$
$$\hat{\gamma}(1) - \hat{\phi}\hat{\gamma}(0) - \hat{\phi}^2\hat{\gamma}(1) = 0 \qquad (2)$$
$$\hat{\gamma}(2) - \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^2\hat{\gamma}(0) = 0 \qquad (3)$$
$$\vdots$$

$$(2) \Rightarrow \quad .687 - \hat{\phi} - .687\hat{\phi}^2 = 0$$
$$\Rightarrow \quad \hat{\phi}^2 + 1.456\hat{\phi} - 1 = 0$$
$$\Rightarrow \quad \hat{\phi} = -.728 \pm \sqrt{(.728)^2 + 1}$$
$$= -.728 \pm 1.237$$
$$= .509 \text{ or } -1.965$$

Since the derivation of the YW equns assumes causality it is essential to choose $\hat{\phi} = .509$

$$(3) \Rightarrow \quad \hat{\rho}(2) = \hat{\phi}\hat{\rho}(1) + \hat{\phi}^2 = .609$$

4 $\qquad (1) \Rightarrow \quad \hat{\sigma}^2 = 6.06\left[1 - (.509)(.687) - (.509)^2(.609)\right]$

$$= 2.985$$

5.4 (a) Under H_0: $\{X_t - \mu\}$ is independent WN

$\hat\rho(h)$, $h = 1, 2, 3, \ldots$ are iid $N(0, \frac{1}{n})$ approx for large n

For $n = 200$, $\frac{1.96}{\sqrt{n}} = .1386$

$\hat\rho(1)$, $\hat\rho(2)$ and $\hat\rho(3)$ are all outside
3 the bounds $\pm 1.96/\sqrt{n}$, hence reject H_0
($\hat\rho(1)$, $\hat\rho(2)$ are very far outside)

(b) $\hat\mu = \bar{x} = 3.82$

$$R_2 \hat{\boldsymbol\phi} = \boldsymbol\rho_2 \Rightarrow \begin{bmatrix} 1 & .427 \\ .427 & 1 \end{bmatrix} \begin{bmatrix} \hat\phi_1 \\ \hat\phi_2 \end{bmatrix} = \begin{bmatrix} .427 \\ .475 \end{bmatrix}$$

4 $$\Rightarrow \begin{bmatrix} \hat\phi_1 \\ \hat\phi_2 \end{bmatrix} = \begin{bmatrix} 1.2230 & -.5222 \\ -.5222 & 1.2230 \end{bmatrix} \begin{bmatrix} .427 \\ .475 \end{bmatrix} = \begin{bmatrix} .274 \\ .358 \end{bmatrix}$$

$$\hat\sigma^2 = \hat\gamma(0)\left[1 - (.274)(.427) - (.358)(.475)\right]$$

$$= .820$$

(c) Using the large-sample approx'n

$$\bar{X} - \mu \sim N\left(0, \frac{1}{n}\sum_{-\infty}^{\infty} \gamma(h)\right),$$

we can approximate $\sum_{-\infty}^{\infty} \gamma(h)$ by $2\pi \hat f(0)$
where $\hat f(0)$ is the spectral density at freq. 0
of the model fitted in (b)

3 $$2\pi \hat f(0) = \hat\sigma^2 \frac{1}{|1 - \hat\phi_1 - \hat\phi_2|^2} = 6.055$$

$\therefore \bar{X} - \mu \sim N(0, .0303)$ approx.

Since $\bar{x} = 3.82 \gg 1.96\sqrt{.03}$ we reject
$H_0: \mu = 0$ in favour of $\mu \neq 0$
at significance level .05

(d) $$\begin{bmatrix} \hat{\phi}_1 - \phi_1 \\ \hat{\phi}_2 - \phi_1 \end{bmatrix} \sim N\left(\underset{\sim}{0}, \frac{\sigma^2}{n}\Gamma_p^{-1}\right) = N\left(\underset{\sim}{0}, \frac{\sigma^2}{\gamma(0)n}R_p^{-1}\right)$$

$$= N\left(\underset{\sim}{0}, \frac{1}{n}[1-\phi_1\rho(1) - \cdots - \phi_p\rho(p)]R_p^{-1}\right)$$

$$\simeq N\left(\underset{\sim}{0}, \frac{1}{n}[1-\hat{\phi}_1\hat{\rho}(1) - \cdots - \hat{\phi}_p\hat{\rho}(p)]R_p^{-1}\right)$$

$$= N\left(\underset{\sim}{0}, \frac{1}{200}(.713)\begin{bmatrix} 1.223 & -.5222 \\ -.5222 & 1.223 \end{bmatrix}\right)$$

Est'd SD of $\hat{\phi}_1 = \sqrt{\frac{.713 \times 1.223}{200}} = .0660$

-- -- -- $\hat{\phi}_2 =$ -- $= .0660$

95% c. bounds

ϕ_1 : $.274 \pm 1.96 \times .0660 = .145, .403$

4 $.129$

ϕ_2 : $.358 \pm 1.96 \times .0660 = .229, .487$

(e) $\hat{\phi}_{11} = \hat{\rho}(1) = .427$ = PACF at lag 1

3 $\hat{\phi}_{22} = \hat{\phi}_2 = .358 =$ -- -- -- 2

$\hat{\phi}_{hh} = 0, \quad h > 2$

5.5 3000 observations of a Gaussian MA(1) with coefficient 0.6 and WNV=1.0 were generated using ITSM. The option Transform>Subsequence allows you to pick out the independent subsamples 1-200, 301-500, …, 2701-2900, and get independent estimates for each of these ten samples of length 200. Using the seed 1111111111 gives the results below. Notice that the moment estimator is conveniently calculated in Excel by pasting in the sample ACF at lag 1 and computing the moment estimator from the explicit formula on p. 146.

Simulation	rho(1)hat	thetahat	Moments	Innov. Alg.	MLE
1	0.3901	0.479966	0.48	0.592	0.6686
2	0.4001	0.500208	0.5002	0.5909	0.6135
3	0.4611	0.665026	0.665	0.5649	0.557
4	0.4569	0.649853	0.6499	0.6567	0.61
5	0.5489	#NUM!	1	0.755	0.6771
6	0.464	0.676099	0.6761	0.548	0.4859
7	0.4274	0.562754	0.5628	0.5284	0.5521
8	0.4238	0.553757	0.5538	0.6318	0.6378
9	0.3825	0.46532	0.4653	0.5324	0.5741
10	0.3611	0.426912	0.4269	0.5103	0.5581
	Sample variance =		0.027606	0.005475	0.003502
	Sample mean =		0.598	0.59104	0.59342
	Asymptotic theory:		0.023704	0.005	0.0032
			0.6	0.6	0.6

The MLE is clearly best in terms of sample variance and the moment estimator is worst. The empirical means and variances of the estimators are in good agreement with the asymptotic theory.

Set 8

5.9. We can write the joint (Gaussian) density of the first n observations for $n > p$ as

$$f(x_1,\ldots,x_n) = f(x_1,\ldots,x_p) f_{X_{p+1}|X_t,t\le p}(x_{p+1}|x_t, t \le p) \cdots f_{X_n|X_t,t\le n-1}(x_n|x_t, t \le n-1).$$

The first factor is, by equation (5.2.1),

$$\frac{1}{(\sigma\sqrt{2\pi})^p (\det G_p)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\mathbf{X}_p' G_p^{-1} \mathbf{X}_p\right)\right\}$$

and the remaining factors are

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{1}{2\sigma^2}(X_t - \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p})^2\right\},\ t = p+1,\ldots,n,$$

since, conditional on X_s, $s < t$, X_t has the normal distribution with mean $\phi_1 X_{t-1} + \cdots + \phi_p X_{t-p}$ and variance σ^2. Multiplying these factors together gives the required result. 6

5.11. The reduced likelihood is $l_2(\phi) = \ln \frac{S(\phi)}{2} + \frac{1}{2}(\ln r_0 + \ln r_1)$, where from Problem **5.9**

$$\begin{aligned} S(\phi) &= \frac{x_1^2}{r_0} + \frac{(x_2 - \phi x_1)^2}{r_1} \\ &= x_1^2(1-\phi^2) + (x_2 - \phi x_1)^2 \\ &= x_1^2 + x_2^2 - 2\phi x_1 x_2 \end{aligned}$$

since $r_0 = (1-\phi^2)^{-1}$ and $r_1 = 1$. Therefore

$$\begin{aligned} l_2(\phi) &= -\frac{1}{2}\ln(1-\phi^2) + \ln\left[\frac{1}{2}(x_1^2 + x_2^2 - 2\phi x_1 x_2)\right] \\ \frac{\partial l_2(\phi)}{\partial \phi} &= -\frac{1}{2}\left(\frac{-2\phi}{1-\phi^2}\right) + \frac{-2x_1x_2}{x_1^2 + x_2^2 - 2\phi x_1 x_2} \\ &= 0 \text{ for } \phi = \frac{2x_1x_2}{x_1^2 + x_2^2}, \end{aligned}$$

and hence

$$\begin{aligned} \hat{\phi} &= \frac{2x_1x_2}{x_1^2 + x_2^2} \\ \hat{\sigma}^2 &= \frac{1}{2}S(\hat{\phi}) = \frac{1}{2}\frac{(x_1^2 - x_2^2)^2}{x_1^2 + x_2^2}. \end{aligned}$$

8

Note that if $|x_1| = |x_2|$, then $\hat{\phi} = \mathrm{sgn}(x_1 x_2)$, $\hat{\sigma}^2 = 0$.

5.12 $\quad \ell(\phi) = \log \dfrac{S(\phi)}{n} + \dfrac{1}{n} \sum_{j=0}^{n-1} \log r_j$

where $r_0 = \dfrac{1}{1-\phi^2}$ and $r_j = 1$, $j \geq 1$.

$$\therefore \ell(\phi) = -\log n + \log\left[x_1^2(1-\phi^2) + \sum_{j=2}^{n} (x_j - \phi x_{j-1})^2\right]$$

$$+ \frac{1}{n} \log \frac{1}{1-\phi^2}$$

$$\therefore \frac{d\ell}{d\phi} = \frac{-2\phi x_1^2 - 2\sum_2^n x_{j-1}(x_j - \phi x_{j-1})}{S(\phi)}$$

$$+ \frac{1}{n} \frac{2\phi}{1-\phi^2}$$

Zero when

$$(1-\phi^2)\left(\sum_2^n x_j x_{j-1} - \phi \sum_3^n x_{j-1}^2\right)$$

$$= \frac{\phi}{n}\left(\sum_{j=1}^n x_j^2 + \phi^2 \sum_3^n x_{j-1}^2 - 2\phi \sum_2^n x_j x_{j-1}\right).$$

6 (for large n $\hat{\phi} \simeq \sum_2^n x_j x_{j-1} / \sum_3^n x_{j-1}^2$)

6.1 $\quad \phi(B)(1-B)^d X_t = \Theta(B) Z_t \qquad$ — ①

$\phi(B)(1-B)^d (A_0 + A_1 t + \cdots + A_{d-1} t^{d-1}) = 0 \qquad$ — ②

since $A_0, \ldots, A_{d-1}$ are indep of t and we know from Problem 1.10 that $(1-B)^d$ annihilates polynomials in t of degree $\leq d-1$

Adding ① and ② shows that

4 $\quad X_t + A_0 + \cdots + A_{d-1} t^{d-1}$ satisfies ①.

6.7 (a) Take logs to stabilize the change in variability with the level of the series.

Apply $(1-B^{12})$ to eliminate seasonal component. (Not clear whether $(1-B)$ should then be applied but it's better not to difference more than necessary so try without)

Subtract mean since we'll be fitting a zero-mean model.

Sample ACF suggests AR(13) or MA(13)

Preliminary estimation Burg MIN AICC ⇒ AR(13) AICC = −424.9

IA MA(13) ⇒ AICC = −430.5

Pursuing the MA(13) possibility and setting small coeffs to zero and using maximum likelihood estimation ⇒ MA(12), AICC = −434.96.

In view of the possibility that we could have applied $(1-B)$ to the data, now try ARMA(1,12). Use MLE, successively setting small coeffs to zero to get

$$\left\{\begin{array}{l}\underline{\text{ARMA}(1,12)}\ \text{(see next page)}\quad \underline{\text{AICC} = -449.5}\\ \underline{\text{Residual tests all passed}}.\end{array}\right.$$

4 (Check that this is better than models for $\nabla\nabla_{12}X_t$)

(b) 95% confidence bounds for coefficients

ϕ : $.875 \pm 1.96 \times .0502 = .777, .973$

θ_1 : $-.304 \pm 1.96 \times .1130 = -.525, -.083$

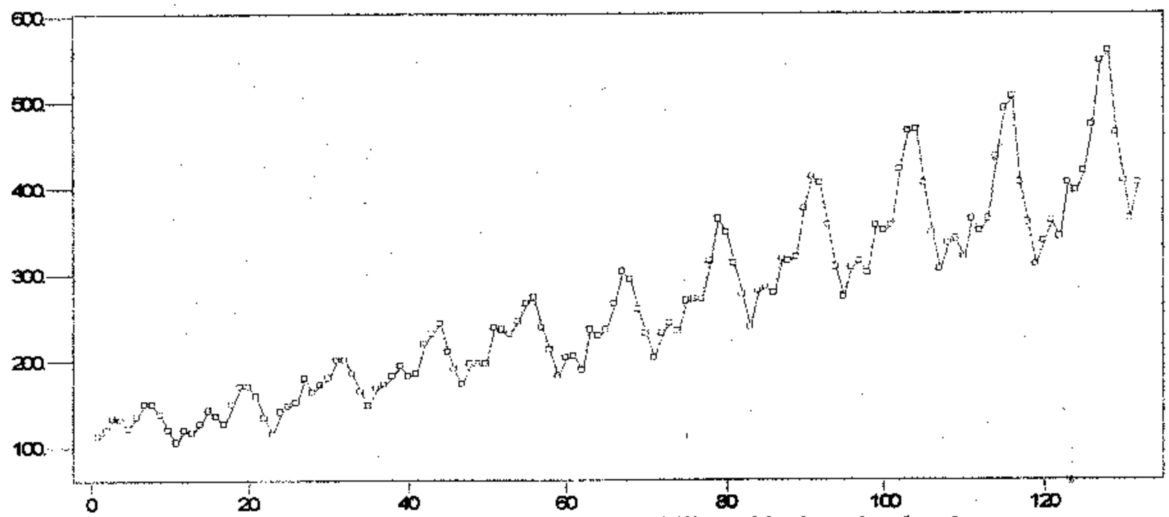
θ_3 : $-.225 \pm 1.96 \times .1020 = -.436, -.025$

θ_4 : $-.236 \pm 1.96 \times .1065 = -.445, -.027$

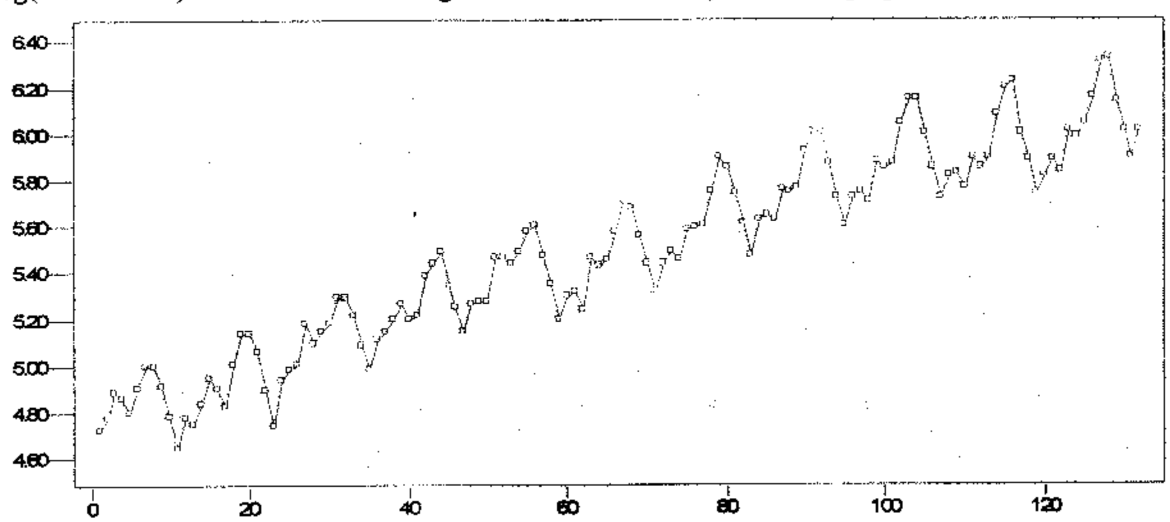
4 θ_{12} : $-.671 \pm 1.96 \times .1006 = -.868, -.474$

Problem 6.7

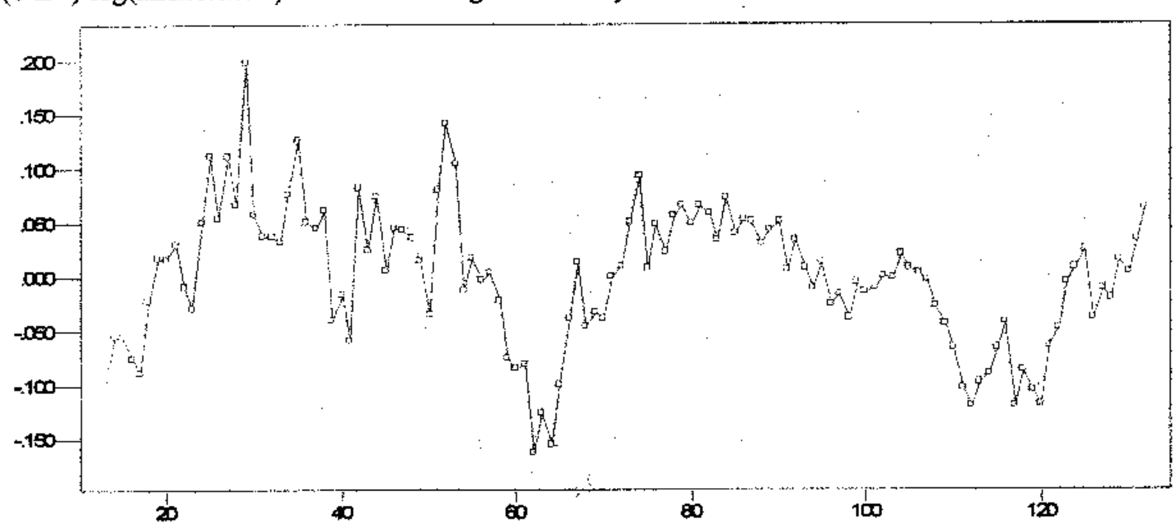
Graph of airshort.tsm



Graph of log(airshort.tsm) Transform used to give constant variability with changing level.



Graph of $(1-B^{12})$ log(airshort.tsm) Difference to get stationary data.



ITSM::Pest(Maximum likelihood estimates)

Method: Maximum Likelihood

$$
\begin{aligned}
X(t) = {} & .8746\, X(t-1) \\
& + Z(t) - .3041\, Z(t-1) + .000\, Z(t-2) - .2255\, Z(t-3) \\
& - .2359\, Z(t-4) + .000\, Z(t-5) + .000\, Z(t-6) + .000\, Z(t-7) \\
& + .000\, Z(t-8) + .000\, Z(t-9) + .000\, Z(t-10) + .000\, Z(t-11) \\
& - .6710\, Z(t-12)
\end{aligned}
$$

WN Variance = .001038

AR Coefficients

.874556

Standard Error of AR Coefficients

.050218

MA Coefficients

-.304085	.000000	-.225482	-.235878
.000000	.000000	.000000	.000000
.000000	.000000	.000000	-.670951

Standard Error of MA Coefficients

.112950	.000000	.101994	.106503
.000000	.000000	.000000	.000000
.000000	.000000	.000000	.100629

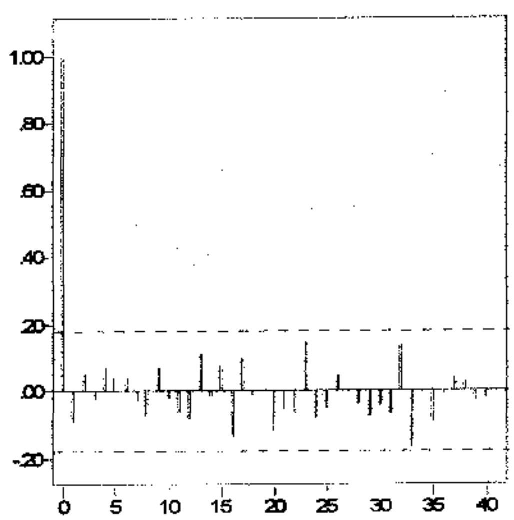
(Residual SS)/N = .00103839

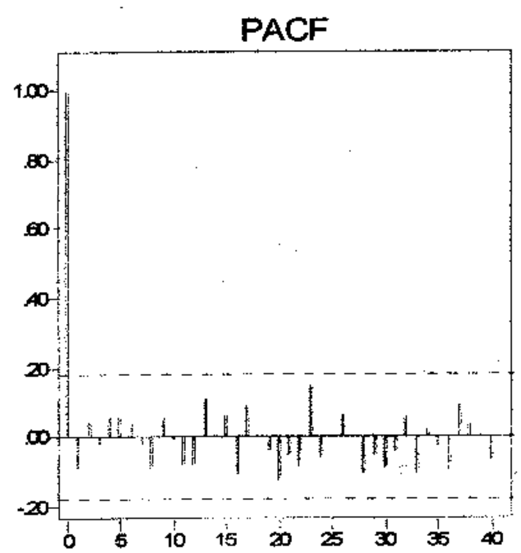
AICC = -449.502764

BIC = -457.835374

-2Log(Likelihood) = -462.246127

(c) Sample ACF/PACF of residuals

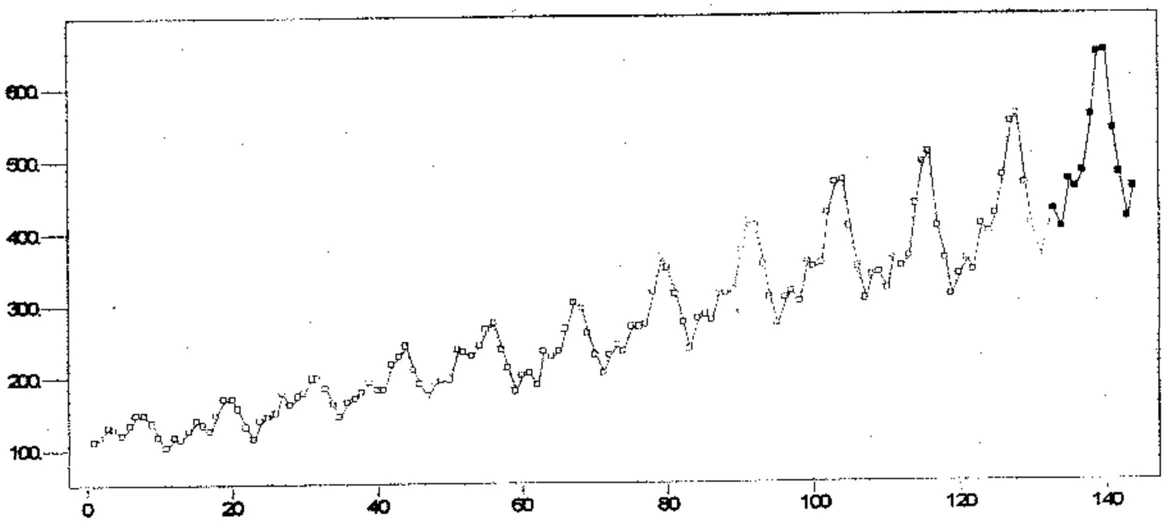




ITSM::Pest(Tests of randomness on residuals)

Ljung - Box statistic = 20.694 Chi-Square (20)
McLeod - Li statistic = 21.201 Chi-Square (25)
# of Turning points = 87.000 ~ AN(78.667, sd = 4.5838)
# of Difference sign points = 58.000 ~ AN(59.500, sd = 3.1754)
# of Rank points = 3388. ~ AN(3570., sd = 220.44)
Order of Min AICC YW Model for Residuals = 0 4

(d) Graph of data and forecasts



2

(e) Forecast values and estimated S.D.'s of the logged data (from ITSM96)

133	.605331E+01	.341946E-01
134	.599448E+01	.414673E-01
135	.614629E+01	.476478E-01
136	.612535E+01	.502547E-01
137	.617129E+01	.509512E-01
138	.632220E+01	.517421E-01
139	.646299E+01	.526304E-01
140	.646905E+01	.536276E-01
141	.628651E+01	.547460E-01
142	.616668E+01	.559880E-01
143	.602568E+01	.573593E-01
144	.612228E+01	.588717E-01

<Press any key to continue>

Prediction bounds (95%) for values of original series at time 144
exp(6.1223-1.96*.05887), exp(6.1223+1.96*.05887) = 406.23, 511.67. **Bounds enclose 432.** 4

(f) Observed forecast errors

Observed values

417	391	419	461	472	535	622	606	508	461	390	432

Forecast values

425.52	401.21	466.98	457.31	478.80	556.80	640.98	644.87	537.28	476.60	413.92	455.91

Errors

8.52	10.21	47.98	-3.69	6.80	21.80	18.98	38.87	29.28	15.60	23.92	23.91

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