Style Guide For Writing Mathematical Proofs

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A solution to a math problem is an argument. Therefore, it should be phrased and structured in the same way as an argument, with the intent to convince the reader of a certain idea. This is similar to writing an essay, where both the essay and the mathematical solution may be viewed in terms of *content* and *style*.

The content of a solution consists of the calculations or deductions that you perform to answer the question. In any math course, the lectures are generally devoted to developing content and to teaching you how to perform the kinds of calculations and deductions that will be required to answer the sorts of questions you will encounter.

Style is another matter entirely. Often, style is demonstrated but not discussed specifically. Students are expected to develop their writing style on their own, based on examples seen in class and in textbooks. This guide is intended to help you get started by explicitly discussing aspects of style.

What is style? First, let us consider what it is not:

- Style is not a question of margin size, single- or double-spacing, or any other technical printing issue. This is *formatting*. So long as your work is easy to read, most math instructors do not have formatting preferences.
- Similarly, style is not grammar. Although we will discuss how grammar and style are related, it is possible to use good style even if you have difficulties with English grammar.
- Style is not separate from content, nor is it any less important. If your style is very poor, your content becomes irrelevant, and your work impossible to read. Your work will be graded on content and style *together*.

An essay is an *argument*. In mathematics, essays are often called "proofs". An essay has a *target audience*, for whom it is written. These are the people to whom you are trying to convince that your argument is valid. Your choice of target audience determines how much explanation is required. For a first-year mathematics course, you should write with your peers as your target audience. This means that you should explain each of your deductions; you cannot include too much detail.

In addition to the target audience, every argument has a *thesis*: this is the main point of your argument. Every other statement in your argument is aimed toward convincing the audience that the thesis statement is true. Usually, your thesis statement is determined by the question you are answering.

So what is style? Style is the relationship between the structure of the argument and its content. For example, essays consist of paragraphs, which are composed of sentences which in turn are composed of phrases. It is the phrases which are the essay's content; the style is the way in which the phrases are connected to form sentences, the way in which the sentences are ordered to form paragraphs, and the organization of the paragraphs to form the argument. This is also true for a mathematical argument, although mathematical equations, inequalities, and symbols may also be used alongside the sentences.

Good style involves structuring phrases, sentences and paragraphs so as to make the essay's argument easy to follow. More than this, good style involves organizing the content in such a way that the structure of the essay actually mirrors that of the argument itself.

Consider, for example, the following question: "Prove that $\sqrt{ab} \leq (a+b)/2$ for all $a, b > 0$."

It is tempting to answer with:

$$\begin{aligned}
\sqrt{ab} &\leq \frac{a+b}{2} \\
ab &\leq \left(\frac{a+b}{2}\right)^2 \\
ab &\leq \frac{a^2+2ab+b^2}{4} \\
4ab &\leq a^2+2ab+b^2 \\
0 &\leq a^2-2ab+b^2 \\
0 &\leq (a-b)^2 \text{true}
\end{aligned}$$

An equation (or inequality) is a *phrase*: it has a subject (the left-hand side), a verb ("is", as in "is equal to" or "is less than"), and an object (the right-hand side). The example above is not an essay. It is a list of phrases, but contains no sentences or paragraphs. Try reading it aloud: "The square root of a times b is less than or equal to one half a plus b a times b is less than or equal to..."

An analogous essay would be written in point form without a clear argument, that is to say, with no effort given to style.

To fix the grammar, we can put periods on the end of each complete idea:

$$\begin{aligned}
\sqrt{ab} &\leq \frac{a+b}{2}. \\
ab &\leq \left(\frac{a+b}{2}\right)^2. \\
&\vdots
\end{aligned}$$

We now read, "The square root of a time b is less than or equal to one-half a plus b. The product ab is less than or equal to..." and so on. This makes for better grammar. However, the sentences are still disconnected. We want to create a flow of ideas that connects one line to the next.

It is important to keep in mind that we are trying to convince our audience that $\sqrt{ab}$ is in fact less than or equal to $(a+b)/2$. A good question to ask ourselves then is: Why should the reader believe the first sentence? The answer is: Because the second sentence is true. Why should the reader believe that the second sentence is true? Because the third one is true, and so forth.

To construct an argument, we should join the phrases together so as to make the connection between them clear:

$$\begin{aligned}
\sqrt{ab} &\leq \frac{a+b}{2}, \text{ because} \\
ab &\leq \left(\frac{a+b}{2}\right)^2, \text{ because} \\
ab &\leq \frac{a^2+2ab+b^2}{4}, \text{ because} \\
4ab &\leq a^2+2ab+b^2, \text{ because} \\
0 &\leq a^2-2ab+b^2, \text{ because} \\
0 &\leq (a-b)^2\text{true is always true.}
\end{aligned}$$

We now have a simple essay with a single paragraph. Our paragraph consists of a single, very long sentence. Describing an entire argument in one poorly constructed sentence is acceptable in mathematics, although not in most other types of essays. As long as the phrases are joined together in such a way that the logical connection between them is clear, it is mathematically correct.

This attempt at style is better than no attempt at all. However, we can improve further. In general, we read from top to bottom. The logical flow of this argument, though, starts at the bottom and flows upwards. The reader wants to be convinced of the truth of the first phrase, but has to read the second phrase to find the justification. In fact, the only phrase that is true without prior argumentation is the last phrase. It is for this reason that we write the last phrase first. Then we write the direct result of that phrase, and so on:

$$
\begin{aligned}
0 &\leq x^2 \text{ for any number } x. \\
\text{Thus, } 0 &\leq (a-b)^2 = a^2 - 2ab + b^2. \\
\text{Thus, } 4ab &\leq a^2 + 2ab + b^2. \\
\text{Thus, } ab &\leq \frac{a^2 + 2ab + b^2}{4} = \left(\frac{a+b}{2}\right)^2. \\
\text{Thus, } \sqrt{ab} &\leq \frac{a+b}{2}, \text{ since } a, b > 0 \text{ means } ab > 0.
\end{aligned}
$$

Try reading this out loud. The argument is much clearer now that the logical argument flows in the same direction as the sentences. Each line is a direct result of the previous line. In this format, the argument starts with something that is clearly true, makes a series of deductions, and ends with the conclusion that the thesis statement is true.

Notice that we have included some extra explanation here (e.g., "since $a, b > 0 \ldots$", etc.). Although this is mostly an issue of content (writing for the target audience), there is a stylistic consideration here. The word "thus" means that the following phrase is true *because* the proceeding sentence is true. This is not the case in the last line of the solution, since we require $a, b > 0$ for the phrase $\sqrt{ab} \leq (a+b)/2$ to even make sense.

An alternative way to structure the paragraph is as a single, long sentence. This structure is perhaps more common than the one above, although a bit harder to read:

$$
\begin{aligned}
0 &\leq x^2 \text{ for any number x }, \\
\Rightarrow 0 &\leq (a-b)^2 = a^2 - 2ab + b^2, \\
\Rightarrow 4ab &\leq a^2 + 2ab + b^2, \\
\Rightarrow ab &\leq \frac{a^2 + 2ab + b^2}{4} = \left(\frac{a+b}{2}\right)^2, \\
\Rightarrow \sqrt{ab} &\leq \frac{a+b}{2}, \text{ since } a, b > 0 \text{ means } ab > 0.
\end{aligned}
$$

(The $\Rightarrow$ symbol reads "which implies.")

As a second example, consider the question: Calculate

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-1}. \tag{1}$$

Solutions to simple calculations like this can easily be structured as a single sentence:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+x-2}{x^2-1} &= \lim_{x\to 1} \frac{(x+2)(x-1)}{(x+1)(x-1)} \\
&= \lim_{x\to 1} \frac{(x+2)}{(x+1)} \\
&= \frac{1+2}{1+1} \\
&= 3.
\end{aligned}$$

Is this good style? Yes, but with one problem. The reader expects that each phrase be justified by the one before it. This is the case with the second phrase, since $x^2+x-2=(x+2)(x-1)$ and $x^2-1=(x+1)(x-1)$ for all numbers x is basic arithmetic, with which you may assume that the reader is familiar. However,

$$\frac{(x+2)(x-1)}{(x+1)(x-1)}$$

is *not* equal to

$$\frac{(x+2)}{(x+1)}$$

for all numbers x. In particular the first expression is not defined at $x=1$, while the second one is defined and equal to 3.

Our argument *is* correct, though, since we are taking the limit of these two expressions and not their value at $x=1$. The problem is that the structure of our argument is not clear: we have actually used a theorem in going from the second phrase to the third, but have not explained the deduction. In general, if the justification for a phrase includes anything beyond the truth of the previous phrase, you must explicitly state that justification:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+x-2}{x^2-1} &= \lim_{x\to 1} \frac{(x+2)(x-1)}{(x+1)(x-1)} \\
&= \lim_{x\to 1} \frac{(x+2)}{(x+1)},
\end{aligned}$$

since

$$\lim_{x\to a} f(x) \quad = \quad \lim_{x\to a} g(x)$$

if $f(x) = g(x)$ for all numbers $x \neq a$. Thus,

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-1} = \frac{1+2}{1+1} = 3.$$

We have discussed how to join phrases into sentences, and sentences into paragraphs. What about paragraphs? For short questions such as those in the previous two examples, one paragraph will usually suffice. For longer questions, such as those that ask you to sketch a curve, you may require several paragraphs.

A simple rule is that each paragraph should be about a single topic. So, if you are sketching a curve, for example, you might have one paragraph discussing the function (zeros, asymptotes, etc.), and a second discussing the derivative, a third discussing the second derivative, and a fourth tying these ideas together and presenting the actual sketch.

If your argument is complicated enough to merit more than one paragraph, you should end with a concluding paragraph which summarizes your results. In the curve sketching example, the conclusion is the fourth paragraph where you present the final sketch.

Most essays outside of mathematics have an introductory paragraph. This is not strictly necessary in a math course, as long as you are solving a problem which includes the method in the question. For example, the statement "Use induction to prove the following..." tells you that your argument will follow the format for induction. If the method is not specified and you wish to use induction, you should start your argument with a phrase similar to, "We will prove the following statement using induction." If the method you are using is complicated, you may wish to summarize the steps of your argument so that the reader knows what is going on.

After you have written your solution, re-read it and ask yourself the following questions:

- Is each paragraph about a single topic?
- Is each paragraph made up of complete, properly punctuated sentences?
- Does each phrase follow logically from the one before it?
- Is each logical leap justified?
- Does the progression from one phrase to the next match the argument you are trying to make?

Finally, try reading your solution aloud. It is amazing how many errors and omissions you can catch simply by doing this.

In general, writing style is the set of properties which can make an argument easier for the reader to follow. The details are up to you. But the effect should be the same for every well-written piece of mathematics: the reader understands what you are trying to say and can follow your argument easily.