Solutions Manual

for

Microwave Engineering
4^{th} edition

David Pozar
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Chapter 1

1.1 This is an open-ended question where the focus of the answer may be largely chosen by the student or the instructor. Some of the relevant historical developments related to the early days of radio are listed here (as cited from T. S. Sarkar, R. J. Mailloux, A. A. Oliner, M. Salazar-Palma, and D. Sengupta, *History of Wireless*, Wiley, N.J., 2006):

1865: James Clerk Maxwell published his work on the unification of electric and magnetic phenomenon, including the introduction of the displacement current and the theoretical prediction of EM wave propagation.

1872: Mahlon Loomis, a dentist, was issued US Patent 129,971 for "aerial telegraphy by employing an 'aerial' used to radiate or receive pulsations caused by producing a disturbance in the electrical equilibrium of the atmosphere". This sounds a lot like radio, but in fact Loomis was not using an RF source, instead relying on static electricity in the atmosphere. Strictly speaking this method does not involve a propagating EM wave. It was not a practical system.

1887-1888: Heinrich Hertz studied Maxwell's equations and experimentally verified EM wave propagation using spark gap sources with dipole and loop antennas.

1893: Nikola Tesla demonstrated a wireless system with tuned circuits in the transmitter and receiver, with a spark gap source.

1895: Marconi transmitted and received a coded message over a distance of 1.75 miles in Italy.

1894: Oliver Lodge demonstrated wireless transmission of Morse code over a distance of 60 m, using coupled induction coils. This method relied on the inductive coupling between the two coils, and did not involve a propagating EM wave.

1897: Marconi was issued a British Patent 12,039 for wireless telegraphy.

1901: Marconi achieved the first trans-Atlantic wireless transmission.

1943: The US Supreme Court invalidated Marconi's 1904 US patent on tuning using resonant circuits as being superseded by prior art of Tesla, Lodge, and Braun.

So it is clear that many workers contributed to the development of wireless technology during this time period, and that Marconi was not the first to develop a wireless system that relied on the propagation of electromagnetic waves. On the other hand, Marconi was very successful at making radio practical and commercially viable, for both shipping and land-based services.

1.2 $E_y = E_o \cos(\omega t - kx)$, $E_o = 5$ V/m, $f = 2.4$ GHz.
$\epsilon_r = 2.54$, $x_1 = 0.1$, $x_2 = 0.15$

a) $\eta = \eta_o/\sqrt{\epsilon_r} = 236.6\,\Omega$

$H_z = E_y/\eta = 0.0211 \cos(\omega t - kz)$

b) $v_p = c/\sqrt{\epsilon_r} = 1.88 \times 10^8$ m/sec

c) $\lambda = v_p/f = 0.0784$ m , $k = 2\pi/\lambda = 80.11\ m^{-1}$

d) $\Delta\phi = k(x_2 - x_1) = 80.11(.15 - .1) = 4.00$ rad $= 229.5°$

1.3 $\bar{E} = E_o(a\hat{x} + b\hat{y})e^{-jk_o z}$; a, b real

Let $\bar{E} = A(\hat{x} - j\hat{y})e^{-jk_o z} + B(\hat{x} + j\hat{y})e^{-jk_o z}$

where A, B are the amplitudes of the RHCP and LHCP components. Equating vector components gives

$\hat{x}$: $A + B = aE_o$

$\hat{y}$: $-jA + jB = bE_o$, or $A - B = jbE_o$

so

$A = E_o(a + jb)/2$

$B = E_o(a - jb)/2$

check: if $a=1$, $b=2$ then $A = (\frac{1}{2} + j)E_o$, $B = (\frac{1}{2} - j)E_o$

(agrees with Problem 1.5 from 3rd ed.)

1.4 From eq. (1.76),

$$\bar{H} = \frac{1}{\eta_0}\hat{n}\times\bar{E} \quad , \quad \bar{E} = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}$$

$$\bar{S} = \bar{E}\times\bar{H}^* = \frac{1}{\eta_0}\bar{E}\times\hat{n}\times\bar{E}^*$$

$$= \frac{1}{\eta_0}\left[(\bar{E}\cdot\bar{E}^*)\hat{n} - (\bar{E}\cdot\hat{n})\bar{E}^*\right] \qquad \text{(from B.5)}$$

Since $\bar{k}\cdot\bar{E}_0 = k_0\hat{n}\cdot\bar{E}_0 = 0$ from (1.69) and (1.74), we have

$$\bar{S} = \frac{\hat{n}}{\eta_0}\bar{E}\cdot\bar{E}^* = \frac{\hat{n}}{\eta_0}|E_0|^2 \quad W/m^2 \quad \checkmark$$

1.5 Writing general plane wave fields in each region:

$\bar{E}^i = \hat{x}e^{-jk_0z}$ $\qquad \bar{H}^i = \frac{\hat{y}}{\eta_0}e^{-jk_0z}$ $\qquad$ for $z<0$

$\bar{E}^r = \hat{x}\Gamma e^{jk_0z}$ $\qquad \bar{H}^r = \frac{-\hat{y}}{\eta_0}\Gamma e^{jk_0z}$ $\qquad$ for $z<0$

$\bar{E}^s = \hat{x}(Ae^{-jkz} + Be^{jkz})$ $\qquad \bar{H}^s = \frac{\hat{y}}{\eta}(Ae^{-jkz} - Be^{jkz})$ $\qquad$ for $0<z<d$

$\bar{E}^t = \hat{x}Te^{-jk_0(z-d)}$ $\qquad \bar{H}^t = \frac{\hat{y}}{\eta_0}Te^{-jk_0(z-d)}$ $\qquad$ for $z>d$

Now match E_x and H_y at $z=0$ and $z=d$ to obtain four equations for Γ, T, A, B:

$z=0$: $\qquad 1+\Gamma = A+B \qquad \frac{1}{\eta_0}(1-\Gamma) = \frac{1}{\eta}(A-B)$

$z=d$: $\qquad j(-A+B) = T \qquad \frac{j}{\eta}(-A-B) = \frac{T}{\eta_0} \qquad (\text{since } d=\lambda_0/4\sqrt{\epsilon_r})$

Solving for Γ gives

$$\Gamma = \frac{\eta^2-\eta_0^2}{\eta^2+\eta_0^2} \quad \checkmark$$

CHECK:
$\lambda/4$ TRANSFORMER $\Rightarrow Z_{in} = \eta^2/\eta_0$, $\Gamma = \frac{\eta^2/\eta_0 - \eta_0}{\eta^2/\eta_0+\eta_0} = \frac{\eta^2-\eta_0^2}{\eta^2+\eta_0^2}$.

1.6 The incident, reflected, and transmitted fields can be written as,

$$\bar{E}^i = E_0(\hat{x} - j\hat{y})e^{-jk_0 z} \qquad \bar{H}^i = j\frac{E_0}{\eta_0}(\hat{x} - j\hat{y})e^{-jk_0 z} \qquad (RHCP)$$

$$\bar{E}^r = E_0\Gamma(\hat{x} - j\hat{y})e^{jk_0 z} \qquad \bar{H}^r = -j\frac{E_0}{\eta_0}\Gamma(\hat{x} - j\hat{y})e^{jk_0 z} \qquad (LHCP)$$

$$\bar{E}^t = E_0 T(\hat{x} - j\hat{y})e^{-\gamma z} \qquad \bar{H}^t = j\frac{E_0}{\eta}T(\hat{x} - j\hat{y})e^{-\gamma z} \qquad (RHCP)$$

Matching fields at $z=0$ gives

$$\Gamma = \frac{\eta - \eta_0}{\eta + \eta_0}, \qquad T = \frac{2\eta}{\eta + \eta_0}$$

The Poynting vectors are: $\quad (\hat{x} - j\hat{y}) \times (\hat{x} - j\hat{y})^* = 2j\hat{z}$

For $z<0$: $\quad \bar{S}^- = (\bar{E}^i + \bar{E}^r) \times (\bar{H}^i + \bar{H}^r)^* = \dfrac{2\hat{z}|E_0|^2}{\eta_0}\left(1 - |\Gamma|^2 + \Gamma e^{2jk_0 z} + \Gamma^* e^{-2jk_0 z}\right)$ ✓

For $z>0$: $\quad \bar{S}^+ = \bar{E}^t \times \bar{H}^{t*} = \dfrac{2\hat{z}|E_0|^2|T|^2}{\eta^*}e^{-2\alpha z}$ ✓

at $z=0$,

$$\bar{S}^- = \frac{2\hat{z}|E_0|^2}{\eta_0}\left(1 - |\Gamma|^2 + \Gamma - \Gamma^*\right) = \frac{2\hat{z}|E_0|^2}{\eta_0}(1+\Gamma)(1-\Gamma^*) \checkmark$$

$$\bar{S}^+ = 2\hat{z}|E_0|^2 \frac{4\eta}{|\eta + \eta_0|^2} \qquad \left(\text{using } T = \frac{2\eta}{\eta + \eta_0}\right)$$

$$= \frac{2\hat{z}|E_0|^2}{\eta_0}\left(\frac{2\eta}{\eta + \eta_0}\right)\left(\frac{2\eta_0}{\eta + \eta_0}\right)^* = \frac{2\hat{z}|E_0|^2}{\eta_0}(1+\Gamma)(1-\Gamma^*) \checkmark$$

Thus $\bar{S}^- = \bar{S}^+$ at $z=0$, and power is conserved.

1.7 From Table 1.1,

$$\gamma = j\omega\sqrt{\mu_0 \epsilon} = 2\pi j f \sqrt{\mu_0 \epsilon_0}\,\sqrt{5-j2} = j\,\frac{2\pi(1000)}{300}\sqrt{5.385\angle{-22^\circ}}$$

$$= 48.5\angle 79^\circ = 9.25 + j\,47.6 = \alpha + j\beta \quad (\text{neper/m, rad/m})$$

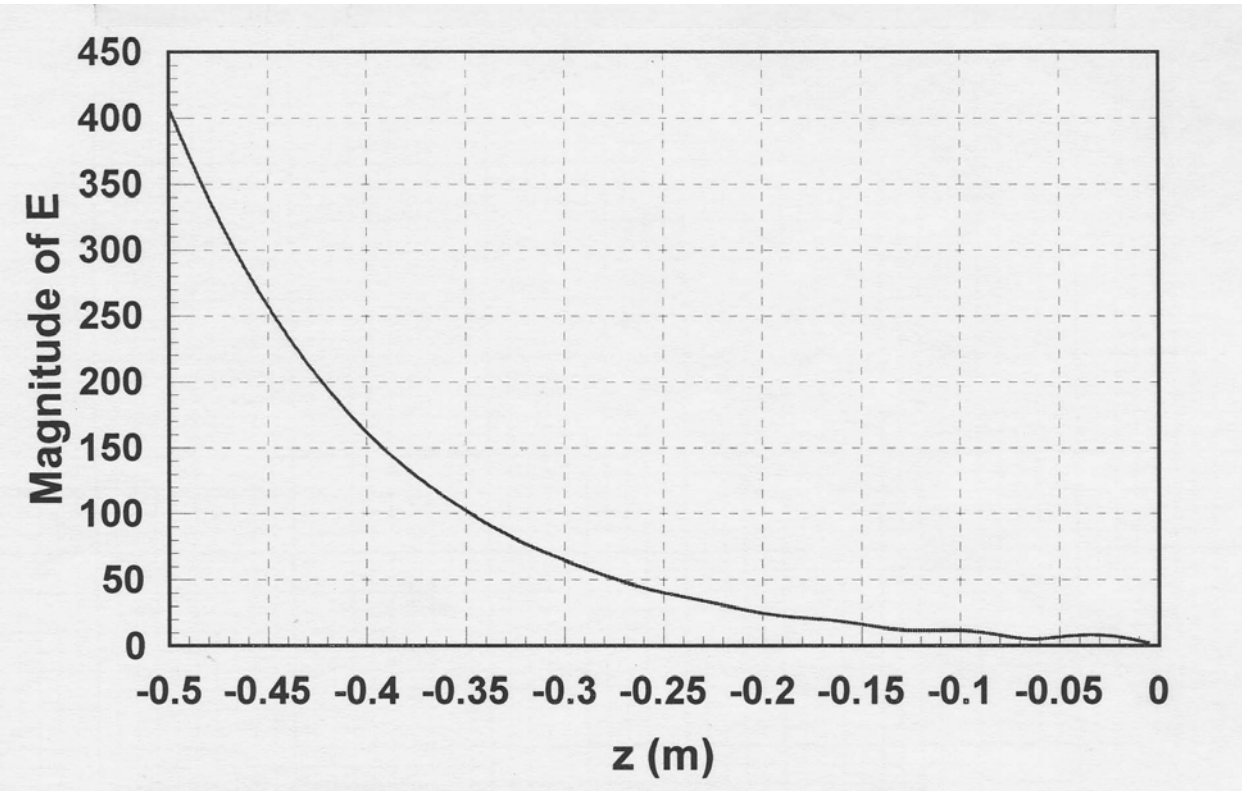
$$\eta = \frac{j\omega\mu}{\gamma} = \frac{\cancel{j\omega\sqrt{\mu_0\epsilon_0}}}{\cancel{j\omega\sqrt{\mu_0\epsilon}}}\sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{\eta_0}{\sqrt{5-j2}} = \frac{377}{2.32\angle{-11^\circ}} = 163\angle 11^\circ \;\Omega$$

$$\Gamma = -1$$

For $z<0$, $\bar{E} = \bar{E}^i + \bar{E}^r = 4\hat{x}\left(e^{-\gamma z} - e^{\gamma z}\right)$

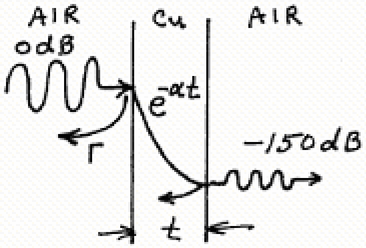
$$|\bar{E}| = 4\left|e^{-\alpha z}e^{-j\beta z} - e^{\alpha z}e^{j\beta z}\right|$$

$|\bar{E}|$ vs z is plotted below.



1.8

The total loss through the sheet is the product of the transmission losses at the air-copper and copper-air interfaces, and the exponential loss through the sheet.



$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}} = 2.09\times10^{-6}\,m = \frac{1}{\alpha}$$

$$\eta_c = \frac{(1+j)}{\sigma\delta_s} = 8.2\times10^{-3}(1+j)\,\Omega$$

a) Power transfer from air into copper is given by,

$$1-|\Gamma|^2 \quad , \quad \Gamma = \frac{\eta_c-\eta_o}{\eta_c+\eta_o} \simeq \frac{8.2\times10^{-3}(1+j)-377}{377} = -0.999956+j\,4.35E{-5}$$

This yields a power transfer of $-40.6\,dB$ into the copper. By symmetry, the same transfer occurs for the copper-air interface.

b) the attenuation within the copper sheet is,

$$\text{copper att.} = 150\,dB - 40.6\,dB - 40.6\,dB = 68.8\,dB$$
$$= -20\log e^{-t/\delta_s} \Rightarrow \underline{t = 0.017\ mm}\ ✓$$

(J. Mead provided this correction on 9/04)

1.9 From Table 1.1,

$$\gamma = j\omega\sqrt{\mu_0\epsilon} = j\frac{2\pi(3000)}{300}\sqrt{3(1-j.1)} = 5.435 + j108.964 = \alpha + j\beta \ \text{m}^{-1}$$

$$\eta = \frac{\eta_0}{\sqrt{\epsilon_r(1-j.1)}} = 217.121 \angle 2.855°$$

a) $$S_i = Re\left\{\frac{|\bar{E}_i(z=0)|^2}{\eta^*}\right\} = 46.000 \ \text{W/m}^2 \ ✓$$

$$\Gamma = -1 \ \text{at} \ z = \ell = 20 \ \text{cm}$$

$$\bar{E}_r = \Gamma\bar{E}_i(z=\ell)e^{\gamma(z-\ell)} = -100\hat{x}\,e^{-2\gamma\ell}e^{\gamma z}$$

$$S_r = Re\left\{\frac{|\bar{E}_r(z=0)|^2}{\eta^*}\right\} = 0.595 \ \text{W/m}^2 \ ✓$$

b) $$\bar{E}_t = \bar{E}_i + \bar{E}_r$$

$$\bar{E}_t(z=0) = 100\hat{x}(1-e^{-2\gamma\ell}), \quad \bar{H}_t(z=0) = \frac{100\hat{y}}{\eta}(1+e^{-2\gamma\ell})$$

$$S_{in} = Re\left\{\bar{E}_t \times \bar{H}_t^* \cdot \hat{z}\right\} = 45.584 \ \text{W/m}^2$$

But $S_i - S_r = 45.405 \ \text{W/m}^2 \neq S_{in}$. This is because S_i and S_r individually are not physically meaningful in a lossy medium.

(The above were computed using a FORTRAN program, with 6 digit precision. The error between $S_i - S_r$ and S_{in} is only about 0.4% - this would be larger if the loss were greater.)

1.10 As in Example 1.3, assume outgoing plane wave fields in each region. To get J_{sx}, we need H_y, since $\hat{n}\times(\bar{H}_2-\bar{H}_1)=\bar{J}_s$ ($\hat{n}=\hat{z}$). Then we must have E_x to get $\bar{S}=\bar{E}\times\bar{H}^*=\pm S\hat{z}$. So the form of the fields must be,

for $z<0$, $\bar{E}_1=\hat{x}Ae^{jk_0z}$ $\qquad$ for $z>0$, $\bar{E}_2=\hat{x}Be^{-jkz}$

$\bar{H}_1=\frac{-\hat{y}}{\eta_0}Ae^{jk_0z}$ $\qquad$ $\bar{H}_2=\hat{y}\frac{B}{\eta}e^{-jkz}$

with $k_0=\omega\sqrt{\mu_0\epsilon_0}$, $k=\omega\sqrt{\mu_0\epsilon_0\epsilon_r}$, $\eta_0=\sqrt{\mu_0/\epsilon_0}$, $\eta=\sqrt{\mu_0/\epsilon_0\epsilon_r}$, and A and B are unknown amplitudes to be determined.

The boundary conditions at $z=0$ are, from (1.36) and (1.37),

$$(\bar{E}_2-\bar{E}_1)\times\hat{n}=0 \quad\Rightarrow\quad A=B$$

$$\hat{z}\times(\bar{H}_2-\bar{H}_1)=\bar{J}_s \quad\Rightarrow\quad -\left(\frac{B}{\eta}+\frac{A}{\eta_0}\right)=J_0$$

$$\therefore\quad A=B=\frac{-J_0\eta\eta_0}{\eta+\eta_0}$$

1.11

This current sheet will generate obliquely propagating plane waves. From (1.132)-(1.133), assume

$$\left.\begin{aligned}\bar{E}_1 &= A(\hat{x}\cos\theta_1 + \hat{z}\sin\theta_1)\, e^{-jk_0(x\sin\theta_1 - z\cos\theta_1)} \\ \bar{H}_1 &= \frac{-A}{\eta_0}\hat{y}\, e^{-jk_0(x\sin\theta_1 - z\cos\theta_1)}\end{aligned}\right\} \text{for } z<0$$

$$\left.\begin{aligned}\bar{E}_2 &= B(\hat{x}\cos\theta_2 - \hat{z}\sin\theta_2)\, e^{-jk(x\sin\theta_2 + z\cos\theta_2)} \\ \bar{H}_2 &= \frac{B}{\eta}\hat{y}\, e^{-jk(x\sin\theta_2 + z\cos\theta_2)}\end{aligned}\right\} \text{for } z>0$$

with $k_0 = \omega\sqrt{\mu_0\epsilon_0}$, $k = \sqrt{\epsilon_r}\, k_0$, $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, $\eta = \eta_0/\sqrt{\epsilon_r}$.

Apply boundary conditions at $z=0$:

$$\hat{z}\times(\bar{E}_2 - \bar{E}_1) = 0 \implies A\cos\theta_1 e^{-jk_0 x\sin\theta_1} - B\cos\theta_2 e^{-jkx\sin\theta_2} = 0$$

$$\hat{z}\times(\bar{H}_2 - \bar{H}_1) = J_s \implies \frac{A}{\eta_0} e^{-jk_0 x\sin\theta_1} + \frac{B}{\eta} e^{-jkx\sin\theta_2} = -J_0 e^{-j\beta x}$$

For phase matching we must have $k_0\sin\theta_1 = k\sin\theta_2 = \beta$

$\therefore\ \theta_1 = \sin^{-1}\beta/k_0$ ✓ $\quad \theta_2 = \sin^{-1}\beta/k$ (must have $\beta < k_0$)

Then, $\quad A\cos\theta_1 = B\cos\theta_2$, $\quad \dfrac{A}{\eta_0} + \dfrac{B}{\eta} = -J_0$

$$A = \frac{-J_0\eta\eta_0\cos\theta_2}{\eta\cos\theta_2 + \eta_0\cos\theta_1}, \quad B = \frac{-J_0\eta\eta_0\cos\theta_1}{\eta\cos\theta_2 + \eta_0\cos\theta_1}$$

Check: If $\beta = 0$, then $\theta_1 = \theta_2 = 0$, and $A = B = \dfrac{-J_0\eta\eta_0}{\eta + \eta_0}$,

which agrees with Problem 1.10 ✓

1.12

This solution is identical to the parallel polarized dielectric case of Section 1.8, except for the definitions of k_1, k_2, η_1, and η_2. Thus,

$$k_o \sin\theta_i = k_o \sin\theta_r = k \sin\theta_t \quad ; \; k = k_o\sqrt{\mu_r}$$

$$\Gamma = \frac{\eta\cos\theta_t - \eta_o\cos\theta_i}{\eta\cos\theta_t + \eta_o\cos\theta_i} \qquad T = \frac{2\eta\cos\theta_i}{\eta\cos\theta_t + \eta_o\cos\theta_i}$$

$$\eta = \eta_o\sqrt{\mu_r}$$

There will be a Brewster angle if $\Gamma = 0$. This requires that,

$$\eta\cos\theta_t = \eta_o\cos\theta_i$$

$$\sqrt{\mu_r}\sqrt{1-\left(\frac{k_o}{k}\right)^2\sin^2\theta_i} = \cos\theta_i = \sqrt{1-\sin^2\theta_i}$$

$$\mu_r\left(1-\frac{1}{\mu_r}\cancel{\sin^2\theta_i}\right) = 1-\cancel{\sin^2\theta_i}$$

or, $\mu_r = 1$. This implies a uniform region, so there is no Brewster angle for $\mu_r \neq 1$. ✓

1.13 Again, this solution is similar to the perpendicular polarized case of Section 1.8, except for the definition of k_1, k_2, η_1, η_2. Thus,

$$\Gamma = \frac{\eta \cos\theta_i - \eta_0 \cos\theta_t}{\eta \cos\theta_i + \eta_0 \cos\theta_t} \quad , \quad T = \frac{2\eta \cos\theta_i}{\eta \cos\theta_i + \eta_0 \cos\theta_t}$$

a Brewster angle exists if

$$\eta \cos\theta_i = \eta_0 \cos\theta_t$$

$$\sqrt{\mu_r}\sqrt{1-\sin^2\theta_i} = \sqrt{1-\frac{1}{\mu_r}\sin^2\theta_i}$$

$$\mu_r^2 - \mu_r^2 \sin^2\theta_i = \mu_r - \sin^2\theta_i$$

$$\mu_r = (\mu_r+1)\sin^2\theta_i$$

$$\sin\theta_i = \sin\theta_b = \sqrt{\frac{\mu_r}{1+\mu_r}} < 1 \quad \checkmark$$

Thus, a Brewster angle does exist for this case.

1.14 $\bar{E} = 3\hat{x} - 2\hat{y} + 5\hat{z}$

$$\bar{D} = [\epsilon]\bar{E} = \begin{bmatrix} 1 & 3j & 0 \\ -3j & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3-6j \\ -4-9j \\ 20 \end{bmatrix} = (3-6j)\hat{x} + (-4-9j)\hat{y} + 20\hat{z}$$

1.15

$$D_x = \epsilon_0(\epsilon_r E_x + j\kappa E_y)$$
$$D_y = \epsilon_0(-j\kappa E_x + \epsilon_r E_y)$$
$$D_z = \epsilon_0 E_z$$

Then,
$$D_+ = D_x - jD_y = \epsilon_0(\epsilon_r - \kappa)E_x - j\epsilon_0(\epsilon_r - \kappa)E_y = \epsilon_0(\epsilon_r - \kappa)E_+$$
$$D_- = D_x + jD_y = \epsilon_0(\epsilon_r + \kappa)E_x + j\epsilon_0(\epsilon_r + \kappa)E_y = \epsilon_0(\epsilon_r + \kappa)E_-$$

OR,
$$\begin{bmatrix} D_+ \\ D_- \\ D_z \end{bmatrix} = \begin{bmatrix} (\epsilon_r - \kappa) & 0 & 0 \\ 0 & (\epsilon_r + \kappa) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \\ E_z \end{bmatrix}$$

From Maxwell's equations,
$$\left.\begin{array}{l} \nabla \times \bar{E} = -j\omega\mu \bar{H} \\ \nabla \times \bar{H} = j\omega[\epsilon]\bar{E} \end{array}\right\} \quad \begin{array}{l} \nabla \times \nabla \times \bar{E} = -j\omega\mu \nabla \times \bar{H} = \omega^2\mu[\epsilon]\bar{E} \\ \nabla^2 \bar{E} + \omega^2\mu[\epsilon]\bar{E} = 0 \quad (\text{CARTESIAN}) \end{array}$$

Expanding this wave equation gives,
$$\nabla^2 E_x + \omega^2\mu\epsilon_0(\epsilon_r E_x + j\kappa E_y) = 0 \qquad (1)$$
$$\nabla^2 E_y + \omega^2\mu\epsilon_0(-j\kappa E_x + \epsilon_r E_y) = 0 \qquad (2)$$
$$\nabla^2 E_z + k_0^2 E_z = 0 \qquad (3)$$

adding (1) $+j$(2) gives $\nabla^2(E_x + jE_y) + \omega^2\mu\epsilon_0[(\epsilon_r + \kappa)E_x + j(\epsilon_r + \kappa)E_y] = 0$
$$\nabla^2 E^+ + \omega^2\mu\epsilon_0(\epsilon_r + \kappa)E^+ = 0$$
$$\therefore \beta_+ = k_0\sqrt{\epsilon_r + \kappa} \quad ✓$$

adding (1) $-j$(2) gives $\nabla^2(E_x - jE_y) + \omega^2\mu\epsilon_0[(\epsilon_r - \kappa)E_x - j(\epsilon_r - \kappa)E_y] = 0$
$$\nabla^2 E^- + \omega^2\mu\epsilon_0(\epsilon_r - \kappa)E^- = 0$$
$$\therefore \beta_- = k_0\sqrt{\epsilon_r - \kappa} \quad ✓$$

Note that the wave equations for E^+, E^- must be satisfied simultaneously. Thus, for E^+ we must have $E^- = 0$. This implies that $E_y = jE_x = jE_0$. The actual electric field is then,
$$\bar{E}^+ = \hat{x}E_x + \hat{y}E_y = E_0(\hat{x} + j\hat{y})e^{-j\beta_+ z} \qquad (\text{LHCP})$$

This is a LHCP wave. Similarly for $\bar{E}^-$ we must have $E^+ = 0$:
$$\bar{E}^- = \hat{x}E_x + \hat{y}E_y = E_0(\hat{x} - j\hat{y})e^{-j\beta_- z} \qquad (\text{RHCP})$$

1.16 Comparing (1.118), (1.125), and (1.129) shows that

$$E_t = \frac{J_t}{\sigma} = \frac{J_s}{\sigma \delta_s} = R_s J_s .$$

Thus $\bar{E}_t = R_s \bar{J}_s = R_s\, \hat{n} \times \bar{H}$ is the desired surface impedance relation. Applying this to the surface integral of (1.155) gives, on S,

$$\left[(\bar{E}_1 \times \bar{H}_2) - (\bar{E}_2 \times \bar{H}_1) \right] \cdot \hat{n} = R_s \left[(\hat{n} \times \bar{H}_{1t}) \times \bar{H}_{2t} - (\hat{n} \times \bar{H}_{2t}) \times \bar{H}_{1t} \right]$$

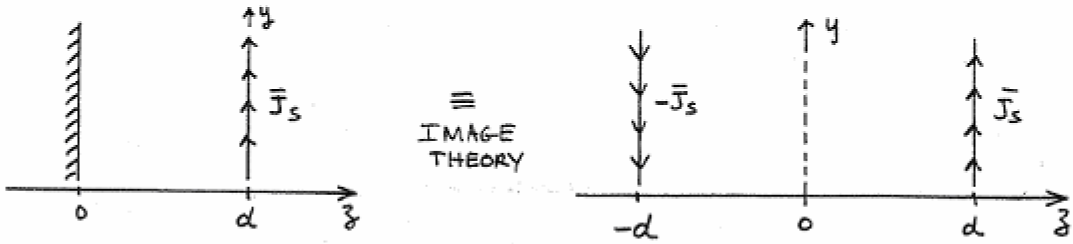
(USING B.5)

$$= R_s \left[\cancelto{0}{(\bar{H}_{2t} \cdot \hat{n})}\, \bar{H}_{1t} - \cancel{(\bar{H}_{2t} \cdot \bar{H}_{1t})}\, \hat{n} - \cancelto{0}{(\bar{H}_{1t} \cdot \hat{n})}\, H_{2t} + \cancel{(\bar{H}_{1t} \cdot \bar{H}_{2t})}\, \hat{n} \right]$$

$$= 0$$

So (1.157) is obtained.

1.17



First find the fields due to the source at $z=d$. From (1.139)–(1.140),

For $z<d$,
$$\bar{E}_1 = A\hat{y}\, e^{-jk_o(x\sin\theta - z\cos\theta)}$$
$$\bar{H}_1 = \frac{A}{\eta_o}(\hat{x}\cos\theta + \hat{z}\sin\theta)\, e^{-jk_o(x\sin\theta - z\cos\theta)}$$

For $z>d$,
$$\bar{E}_2 = B\hat{y}\, e^{-jk_o(x\sin\theta + z\cos\theta)}$$
$$\bar{H}_2 = \frac{B}{\eta_o}(-\hat{x}\cos\theta + \hat{z}\sin\theta)\, e^{-jk_o(x\sin\theta + z\cos\theta)}$$

Apply boundary conditions at $z=d$:

$$\hat{z}\times[\bar{E}(d^+) - \bar{E}(d^-)] = 0 \Rightarrow A e^{jk_od\cos\theta} = Be^{-jk_od\cos\theta}$$

$$\hat{z}\times[\bar{H}(d^+) - \bar{H}(d^-)] = \bar{J}_s \Rightarrow [-B\cos\theta\, e^{-jk_od\cos\theta} - A\cos\theta\, e^{jk_od\cos\theta}]\cdot e^{-jk_ox\sin\theta} = \eta_o J_o e^{-j\beta x}$$

For phase matching, $k_o\sin\theta = \beta$

Then, $A = \dfrac{-\eta_o J_o}{2\cos\theta}e^{-jk_od\cos\theta}$ $\qquad B = \dfrac{-\eta_o J_o}{2\cos\theta}e^{jk_od\cos\theta}$

$$\bar{E} = \frac{-\eta_o J_o \hat{y}}{2\cos\theta}\begin{cases} e^{-jk_o[x\sin\theta - (z-d)\cos\theta]} & z<d \\ e^{-jk_o[x\sin\theta + (z-d)\cos\theta]} & z>d \end{cases}$$

The fields due to the source at $z=-d$ can then be found by replacing d with $-d$, and J_o with $-J_o$:

$$\bar{E} = \frac{\eta_0 J_0 \hat{y}}{2\cos\theta} \begin{cases} e^{-jk_0[x\sin\theta - (z+d)\cos\theta]} & z < -d \\ e^{-jk_0[x\sin\theta + (z+d)\cos\theta]} & z > -d \end{cases}$$

Combining these results gives the total fields:

$$\bar{E} = \frac{-j\eta_0 J_0 \hat{y}}{\cos\theta} \begin{cases} e^{-jk_0 x\sin\theta}\, e^{-jk_0 d} \sin(k_0 z\cos\theta) & 0 < z < d \\ e^{-jk_0 x\sin\theta}\, e^{-jk_0 z} \sin(k_0 d\cos\theta) & z > d \end{cases}$$

CHECK: If $\beta = 0$, then $\theta = 0$ and we have,

$$\bar{E} = -j\eta_0 J_0 \hat{y} \begin{cases} e^{-jk_0 d} \sin k_0 z & \text{for } 0 < z < d \\ e^{-jk_0 z} \sin k_0 d & \text{for } z > d \end{cases}$$

This agrees with the results in (1.161)–(1.162).

1.18

$$\nabla \times \bar{E} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right) + \hat{z}\,\frac{1}{\rho}\left(\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right)$$

$$\nabla \times \nabla \times \bar{E} = \hat{\rho}\left[\frac{-1}{\rho^2}\frac{\partial^2 E_\rho}{\partial \phi^2} - \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial^2 E_z}{\partial \rho \partial z} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial \rho \partial \phi} + \frac{1}{\rho^2}\frac{\partial E_\phi}{\partial \phi}\right]$$

$$+\hat{\phi}\left[-\frac{\partial^2 E_\phi}{\partial z^2} + \frac{1}{\rho}\frac{\partial^2 E_z}{\partial \phi \partial z} - \frac{\partial^2 E_\phi}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \frac{E_\phi}{\rho^2} - \frac{1}{\rho^2}\frac{\partial E_\rho}{\partial \phi} + \frac{1}{\rho}\frac{\partial^2 E_\rho}{\partial \phi \partial \rho}\right]$$

$$+\hat{z}\left[\frac{-\partial^2 E_z}{\partial \rho^2} - \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_\rho}{\partial \rho \partial z} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial \phi \partial z} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial z} - \frac{1}{\rho}\frac{\partial E_z}{\partial \rho}\right]$$

$$\nabla(\nabla \cdot \bar{E}) = \hat{\rho}\left[\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{\partial^2 E_z}{\partial \rho \partial z} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial \rho \partial \phi} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial E_\phi}{\partial \phi} - \frac{E_\rho}{\rho^2}\right]$$

$$+\hat{\phi}\left[\frac{1}{\rho}\frac{\partial^2 E_z}{\partial \phi \partial z} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{1}{\rho}\frac{\partial^2 E_\rho}{\partial \rho \partial \phi} + \frac{1}{\rho^2}\frac{\partial E_\rho}{\partial \phi}\right]$$

$$+\hat{z}\left[\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial \phi \partial z} + \frac{\partial^2 E_\rho}{\partial \rho \partial z} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial z}\right]$$

If we apply ∇^2 to the cylindrical components of $\bar{E}$ we get:

$$\nabla^2 \bar{E} \stackrel{?}{=} \hat{\rho}\,\nabla^2 E_\rho + \hat{\phi}\,\nabla^2 E_\phi + \hat{z}\,\nabla^2 E_z \qquad \text{(THIS IS NOT A VALID STEP!)}$$

$$= \hat{\rho}\left[\frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} + \frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial^2 E_\rho}{\partial z^2}\right]$$

$$+\hat{\phi}\left[\frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial z^2}\right]$$

$$+\hat{z}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right]$$

Note that the $\hat{\rho}$ and $\hat{\phi}$ components of $\nabla \times \nabla \times \bar{E}$ and $\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$ do not agree. This is because $\hat{\rho}$ and $\hat{\phi}$ are not constant vectors, so $\nabla^2 \bar{E} \neq \hat{\rho}\,\nabla^2 E_\rho + \hat{\phi}\,\nabla^2 E_\phi + \hat{z}\,\nabla^2 E_z$. The $\hat{z}$ components are equal.

Chapter 2

2.1

$i(t, z) = 1.8 \cos(3.77 \times 10^9 t - 18.13 z)$ mA

$\omega = 3.77 \times 10^9$ rad/sec, $\beta = 18.13\ m^{-1}$, $Z_0 = 75\ \Omega$

a) $f = \omega/2\pi = 3.77 \times 10^9/2\pi = 600$ MHz

b) $v_p = \omega/\beta = 2.08 \times 10^8$ m/sec

c) $\lambda = 2\pi/\beta = 0.346$ m

d) $\epsilon_r = (c/v_p)^2 = 2.08$ (Teflon)

e) $I(z) = 1.8\, e^{-j\beta z}$ (mA)

f) $v(t, z) = 0.135 \cos(\omega t - \beta z)$ V.

2.2

$R = 4.0\ \Omega/m$, $G = 0.02$ S/m, $L = 0.5\ \mu H/m$, $C = 200$ pF/m

$f = 800$ MHz, $\ell = 30$ cm

$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = 0.540 + j\,50.268$ /m

$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)} = 49.99 + j\,0.46\ \Omega$

with $R = G = 0$, $\beta = \omega\sqrt{LC} = 50.265$ rad/m

$Z_0 = \sqrt{L/C} = 50.0\ \Omega$

Note that β, Z_0 w/o loss are very close to values with loss.

For $\ell = 30$ cm, att $= \alpha \ell = 0.162$ np $\left(\frac{8.686\,dB}{np}\right) = 1.4$ dB

2.3 From Table 2.1:

$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = 0.00825\,\Omega$

$L = \frac{\mu_0}{2\pi}\ln\frac{b}{a} = 2.40\times10^{-7}$ H/m

$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln b/a} = 9.64\times10^{-11}$ Fd/m

$R = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) = 3.76\ \Omega/m$

$G = \frac{2\pi\omega\epsilon_0\epsilon_r\tan\delta}{\ln b/a} = 2.42\times10^{-4}$ S/m

For small loss, $Z_0 = \sqrt{L/C} = 49.9\,\Omega$ ✓

From (2.85a), $\alpha \simeq \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) = 0.044$ np/m $= 0.38$ dB/m ✓

From RG-402U cable data: $Z_0 = 50\,\Omega$, $\alpha = 13$ dB/100ft
$= 0.426$ dB/m

Check using formulas from Example 2.7: ↑ (difference due to braided outer conductor, not solid)

$Z_0 = \sqrt{\frac{\mu}{\epsilon}}\ \frac{\ln b/a}{2\pi} = 49.9\,\Omega$ ✓

$\alpha_c = \frac{R_s}{2\eta\ln b/a}\left(\frac{1}{a}+\frac{1}{b}\right) = 0.0376$ np/m $= 0.326$ dB/m ✓

$\alpha_d = \frac{\omega\epsilon_0\epsilon_r}{2}\eta\tan\delta = 0.00605$ np/m $= 0.052$ dB/m

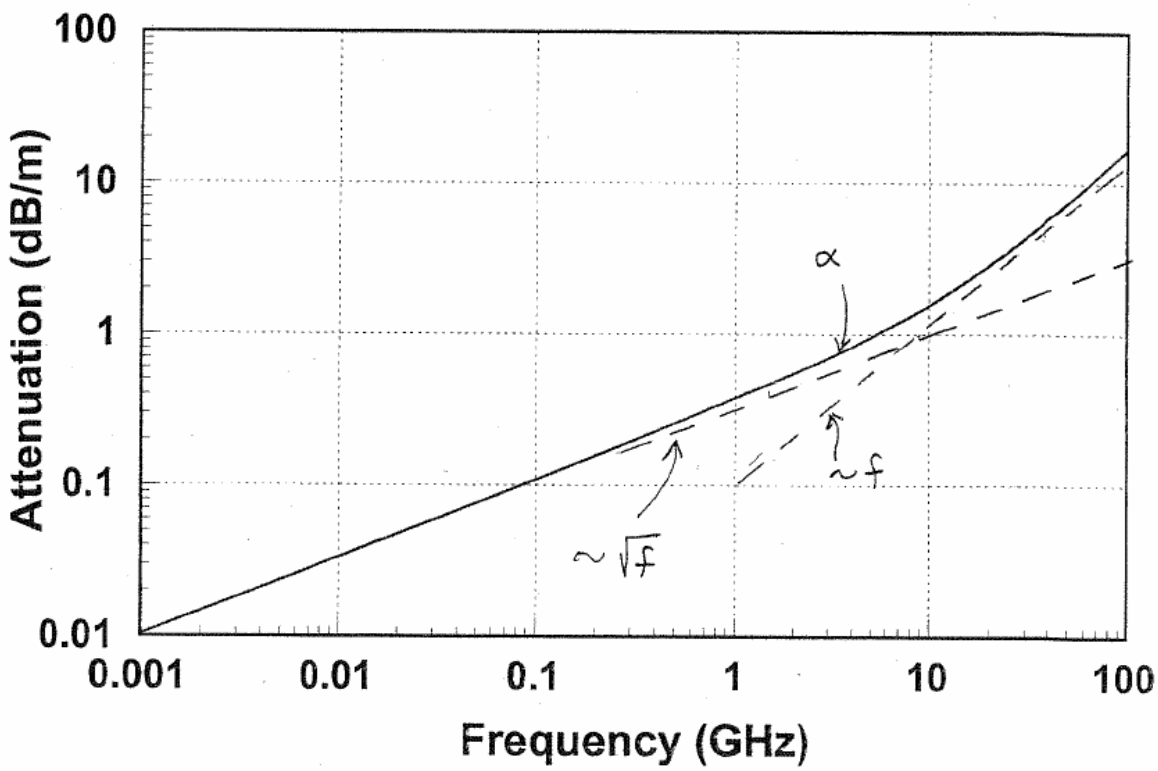
$\alpha_T = 0.378$ dB/m ✓

also verified with Serenade.

2.4 Using the formulas of Problem 2.3, with $\alpha \simeq \frac{1}{2}(R/Z_0 + G Z_0)$:

f	$R_s(\Omega)$	$R(\Omega)$	G (S)	α (Np/m)	α (dB/m)
1 MHz	2.6×10^{-4}	0.118	2.42×10^{-7}	1.19×10^{-3}	0.0103
10 MHz	8.25×10^{-4}	0.376	2.42×10^{-6}	3.82×10^{-3}	0.0332
100 MHz	2.6×10^{-3}	1.18	2.42×10^{-5}	1.24×10^{-3}	0.1078
1 GHz	8.25×10^{-3}	3.76	2.42×10^{-4}	4.365×10^{-2}	0.379
10 GHz	2.6×10^{-2}	11.8	2.42×10^{-3}	1.785×10^{-1}	1.55
100 GHz	8.25×10^{-3}	37.6	2.42×10^{-2}	1.96	17.0

Results are plotted below (with additional data points). Note that the frequency dependence is between $\sqrt{f}$ ($R \sim \sqrt{f}$), and f ($G \sim f$), at low and high frequencies.



2.5 Ignoring fringing fields, $\bar{E}$ and $\bar{H}$ can be assumed as,

$$E_y = \frac{-V_o}{d} \ \text{V/m} \ , \quad H_x = \frac{V_o}{d\eta} = \frac{I_o}{W} \ \text{A/m} \ , \quad \eta = \sqrt{\mu/\epsilon} \ .$$

Then $\bar{E} \times \bar{H}^* = \hat{z}\,|S|$ ✓ and $I_o = V_o\left(\frac{W}{\eta d}\right)$.

From (2.17) - (2.20),

$$L = \frac{\mu_o}{I_o^2}\int_s |\bar{H}|^2 ds = \frac{\mu_o}{I_o^2}\int_{x=0}^{W}\int_{y=0}^{d}\left(\frac{I_o}{W}\right)^2 dx\,dy = \frac{\mu_o d}{W} \quad \text{H/m} \quad ✓$$

$$C = \frac{\epsilon}{V_o^2}\int_s |\bar{E}|^2 ds = \frac{\epsilon}{V_o^2}\int_{x=0}^{W}\int_{y=0}^{d}\left(\frac{-V_o}{d}\right)^2 dx\,dy = \frac{\epsilon W}{d} \quad \text{Fd/m} \quad ✓$$

$$R = \frac{R_s}{I_o^2}\int_{C_1+C_2} |\bar{H}|^2 d\ell = \frac{2R_s}{I_o^2}\int_{x=0}^{W}\left(\frac{I_o}{W}\right)^2 dx = \frac{2R_s}{W} \quad \Omega/\text{m} \quad ✓$$

$$G = \frac{\omega\epsilon''}{V_o^2}\int_s |\bar{E}|^2 ds = \frac{\omega\epsilon''}{V_o^2}\int_{x=0}^{W}\int_{y=0}^{d}\left(\frac{-V_o}{d}\right)^2 dx\,dy = \frac{\omega\epsilon'' W}{d} \quad \text{S/m} \quad ✓$$

These results agree with those in Table 2.1

2.6 Assume $E_z = H_z = 0$, $\partial/\partial x = \partial/\partial y = 0$.

Then Maxwell's curl equations reduce to,

$$-\frac{\partial E_y}{\partial z} = -j\omega\mu H_x \quad (1) \qquad -\frac{\partial H_y}{\partial z} = j\omega\epsilon E_x \quad (3)$$

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y \quad (2) \qquad \frac{\partial H_x}{\partial z} = j\omega\epsilon E_y \quad (4)$$

Since $E_x = 0$ at $y=0$ and $y=d$, and $\partial/\partial y = 0$, we must have $E_x \equiv 0$. Then (3) implies $H_y \equiv 0$. So we have,

$$\frac{\partial E_y}{\partial z} = j\omega\mu H_x \qquad \frac{\partial H_x}{\partial z} = j\omega\epsilon E_y$$

Now let $E_y = \frac{1}{d} V(z)$ and $H_x = \frac{-1}{W} I(z)$.

Then the voltage and current are,

$$V(z) = \int_{y=0}^{d} E_y \, dy \qquad I(z) = \int_{x=0}^{W} (\hat{y} \times \bar{H}) \cdot \hat{z} \, dx = -\int_{x=0}^{W} H_x \, dx$$

Then,

$$\left.\begin{aligned} \frac{\partial V}{\partial z} &= -j\frac{\omega\mu d}{W} I(z) \Rightarrow L = \frac{\mu d}{W} \\ \frac{\partial I(z)}{\partial z} &= -j\frac{\omega\epsilon W}{d} V(z) \Rightarrow C = \frac{\epsilon W}{d} \end{aligned}\right\} \text{agree with Table 2.1}$$

2.7 Using KVL:

$$-v(z)+R\frac{\Delta z}{2}i(z)+L\frac{\Delta z}{2}\frac{\partial i(z)}{\partial t}+R\frac{\Delta z}{2}i(z+\Delta z)+L\frac{\Delta z}{2}\frac{\partial i(z+\Delta z)}{\partial t}+v(z+\Delta z)=0$$

divide by Δz and let $\Delta z\to 0$:

$$\frac{\partial v(z)}{\partial z}=-R\,i(z)-L\frac{\partial i(z)}{\partial t} \quad ✓$$

Using KCL:

$$i(z)-\Delta z\left[G+C\frac{\partial}{\partial t}\right]\left[v(z)-\frac{\Delta z}{2}\left(R+L\frac{\partial}{\partial t}\right)i(z)\right]-i(z+\Delta z)=0$$

divide by Δz and let $\Delta z\to 0$:

$$\frac{\partial i(z)}{\partial z}=-G\,v(z)-C\frac{\partial v(z)}{\partial t} \quad ✓$$

These results agree with (2.2a,b).

2.8 $z_L = Z_L/Z_0 = 0.400 - j0.267$

From Smith chart, $\Gamma_L = 0.461\angle 215^\circ$

$SWR = 2.71$

$\Gamma_{in} = 0.461\angle 359^\circ$ ✓

$Z_{in} = 203 - j5.2\ \Omega$

2.9 $\lambda_g = \lambda_0/\sqrt{\epsilon_r} = \frac{300}{3000\sqrt{2.56}} = 6.25\text{ cm}$

$\ell = \frac{2.0\text{ cm}}{6.25\text{ cm}/\lambda_g} = 0.320\lambda_g \qquad \beta\ell = \frac{2\pi}{\lambda_g}(.32\lambda_g) = 115.2^\circ$

Smith chart solution: $Z_{in} = 18.98 - j20.55\ \Omega$ ✓

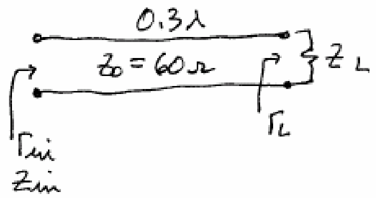
$\Gamma_{in} = 0.62\angle 212^\circ$ ✓

$\Gamma_L = 0.62\angle 83^\circ$ ✓

$SWR = 4.27$ ✓

These results were verified with the analytical formulas.

2.10



$$\Gamma_L = 0.4\angle 60^\circ = 0.2 + j\,0.3464$$

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L} = 60\frac{1.2+j.3464}{.8-j.3464} = \frac{74.94\angle 16.1^\circ}{.8718\angle -23.4^\circ} = 66.3 + j\,54.7\,\Omega \quad ✓$$

$$\Gamma_{in} = \Gamma_L e^{-2j\beta\ell} = .4\angle 60-216 = .4\angle -156^\circ = .4\angle 204^\circ \quad ✓$$

$$Z_{in} = 26.6 - j\,10.3\,\Omega \quad ✓ \quad \text{(Smith Chart)}$$

2.11

C: $Z_{oc} = -j/\omega C = -j\,12.73\,\Omega = -jZ_0 \cot\beta\ell$ $\qquad C = 5\,pF$

$\tan\beta\ell = 100/12.73 \Rightarrow \beta\ell = 82.74^\circ$ ✓

$\lambda_0 = 0.12\,m$, $\beta = 2\pi\sqrt{\epsilon_e}/\lambda_0 = 3854^\circ/m \Rightarrow \ell = \underline{2.147\,cm}$ ✓

L: $Z_{oc} = j\omega L = +j\,78.5\,\Omega = -jZ_0\cot\beta\ell$ $\qquad L = 5\,nH$

$\tan\beta\ell = -100/78.5 \Rightarrow \beta\ell = 128.1^\circ$ ✓ $\Rightarrow \ell = \underline{3.324\,cm}$ ✓

These results were verified with Serenade.

2.12 $|\Gamma| = \dfrac{S-1}{S+1} = \dfrac{0.5}{2.5} = 0.2$

$$|\Gamma| = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = \left|\frac{100 - Z_0}{100 + Z_0}\right| \qquad (Z_0 \text{ real})$$

So either, $\dfrac{100 - Z_0}{100+Z_0} = 0.2 \Rightarrow Z_0 = Z_L\dfrac{1-\Gamma}{1+\Gamma} = 100\left(\dfrac{.8}{1.2}\right) = 66.7\,\Omega$ ✓

or $\dfrac{100-Z_0}{100+Z_0} = -0.2 \Rightarrow Z_0 = Z_L\dfrac{1-\Gamma}{1+\Gamma} = 100\left(\dfrac{1.2}{.8}\right) = 150\,\Omega$ ✓

2.13

$$Z_{sc} = jZ_0 \tan\beta\ell \qquad , \quad Z_{oc} = -jZ_0 \cot\beta\ell$$

$$Z_{sc} \cdot Z_{oc} = Z_0^2 \quad \Rightarrow \quad Z_0 = \sqrt{Z_{sc} Z_{oc}}$$

2.14

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{30 + j40}{130 + j40} = \frac{50\angle 53^\circ}{136\angle 17^\circ} = 0.367\angle 36^\circ \ \checkmark$$

$$P_{LOAD} = P_{INC} - P_{REF} = P_{INC}\left(1 - |\Gamma|^2\right) = 30\left[1 - (.367)^2\right] = 25.9 \text{ W} \ \checkmark$$

2.15

$$RL = -20 \log |\Gamma|$$

$$SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$|\Gamma| = 10^{-RL/20}$$

$$|\Gamma| = \frac{SWR - 1}{SWR + 1}$$

SWR	$\lvert\Gamma\rvert$	RL(dB)
1.00	0.0	∞
1.01	.005	46.0
1.02	.01	40.0
1.05	.024	32.3
1.07	.0316	30.0
1.10	.0476	26.4
1.20	.091	20.8
1.22	.100	20.0
1.50	.200	14.0
1.92	.316	10.0
2.00	.333	9.5
2.50	.429	7.4

2.16 $V_g = 15\text{v RMS}, \; Z_g = 75\Omega, \; Z_0 = 75\Omega, \; Z_L = 60 - j40\Omega, \; \ell = 0.7\lambda.$

a) $$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{-15 - j40}{135 - j40} = \frac{42.7\angle -110.6^\circ}{140.8\angle -16.5^\circ} = 0.303\angle -94^\circ = -0.021 - j0.302$$

$$P_L = \left(\frac{V_g}{2}\right)^2 \frac{1}{Z_0}\left(1 - |\Gamma|^2\right) = 0.681\ \text{W} \checkmark$$

This method is actually based on $P_L = P_{inc}(1-|\Gamma|^2)$. It is the simplest method, but only applies to lossless lines.

b) $$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell} = 75\frac{60 + j190.8}{198.1 + j184.7} = 75\frac{200\angle 72.5^\circ}{270.8\angle 43^\circ}$$

$$= 55.4\angle 29.5^\circ = 48.2 + j27.3\ \Omega$$

$$P_L = \left|\frac{V_g}{Z_g + Z_{in}}\right|^2 Re\,(Z_{in}) = \left|\frac{15}{123.2 + j27.3}\right|^2 (48.2) = 0.681\ \text{W} \checkmark$$

This method computes $P_L = P_{in} = |I_{in}|^2 R_{in}$, and also applies only to lossless lines.

c) $$V(z) = V^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$$

$$V_L = V(0) = V^+(1 + \Gamma) \qquad V^+ = \frac{V_g}{2} = 7.5\text{v}$$

$$= 7.5(1 - .021 - j.302)$$
$$= 7.68\angle -17^\circ$$

$$P_L = \left|\frac{V_L}{Z_L}\right|^2 Re(Z_L) = \left(\frac{7.68}{72.1}\right)^2 (60) = 0.681\ \text{W} \checkmark$$

This method computes $P_L = |I_L|^2 R_L$, and applies to lossy as well as lossless lines. Note the concept that $V^+ = V_g/2$ requires a good understanding of the transmission line equations, and only applies here because $Z_g = Z_0$.

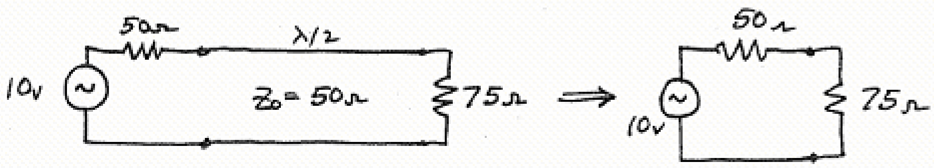
2.17

$$Z_L = jX$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX - Z_0}{jX + Z_0}$$

$$|\Gamma|^2 = \Gamma\Gamma^* = \frac{(jX - Z_0)}{(jX + Z_0)}\,\frac{(-jX - Z_0)}{(-jX + Z_0)} = \frac{X^2 - j\cancel{Z_0}X + j\cancel{Z_0}X + Z_0^2}{X^2 + Z_0^2} = 1 \checkmark$$

2.18



POWER DELIVERED BY SOURCE $= \frac{1}{2}\frac{(10)^2}{50+75} = 0.400\text{ W}$ ✓

POWER DISSIPATED IN 50Ω LOAD $= \frac{1}{2}(50)\left(\frac{10}{50+75}\right)^2 = 0.160\text{ W}$ ✓

POWER TRANSMITTED DOWN LINE $= \frac{1}{2}(75)\left(\frac{10}{50+75}\right)^2 = 0.240\text{ W}$ ✓

INCIDENT POWER $= \frac{1}{2}(50)\left(\frac{10}{50+50}\right)^2 = 0.250\text{ W}$ ✓

REFLECTED POWER $= P_{INC}|\Gamma|^2 = .250\left|\frac{75-50}{75+50}\right|^2 = 0.010\text{ W}$ ✓

—

$P_{INC} - P_{REF} = .250 - .010 = 0.240 = P_{TRANS}$ ✓

$P_{DISS} + P_{TRANS} = .160 + .240 = 0.400 = P_{SOURCE}$ ✓

2.19

$$\Gamma = \frac{-20-j40}{180-j40} = \frac{44.7\angle{-116.6^\circ}}{184.4\angle{-12.5^\circ}} = 0.24\angle{-104^\circ} = -0.058-j\,0.233 \checkmark$$

$$V_L = 10\,\frac{80-j40}{180-j40} = 10\,\frac{89.4\angle{-26^\circ}}{184\angle{-12.5^\circ}} = 4.86\angle{-13.5^\circ}$$

$$V(z) = V^+\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right] \qquad V^+ = 10\,\frac{100}{100+100} = 5\,\text{V} \checkmark$$

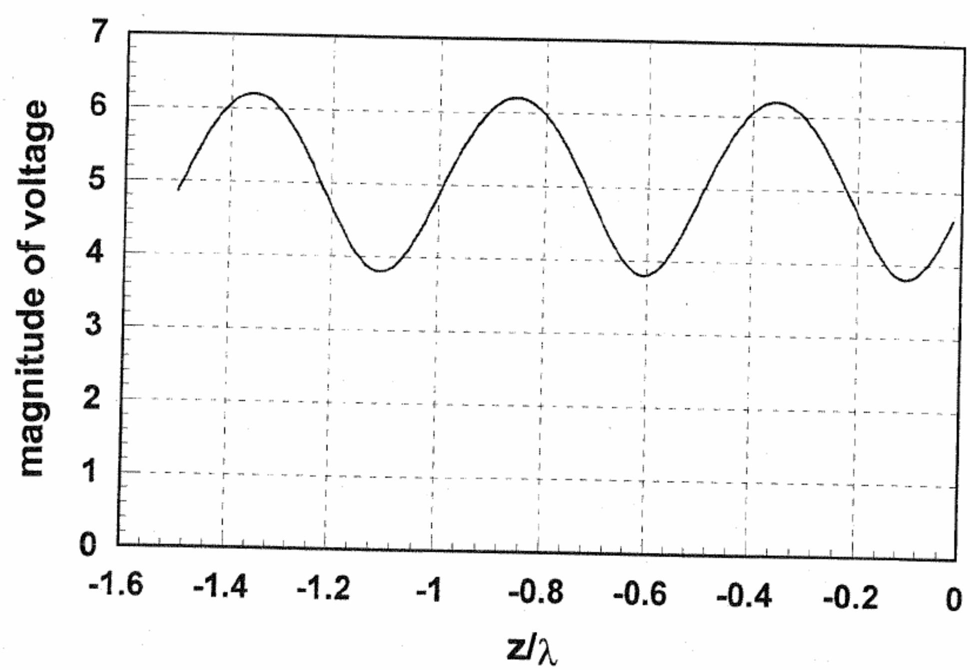
So $\quad V(z) = 5\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]$

$V_{MAX} = 5(1+|\Gamma|) = 5(1.24) = 6.2$ at $z = -0.355\lambda$

$V_{MIN} = 5(1-|\Gamma|) = 5(.76) = 3.8$ at $z = -0.105\lambda$

These results repeat every $\lambda/2$.

$|V(z)|$ is plotted below:



2.20

$Z_0 = 50\,\Omega$, $Z_L = 60 + j50\,\Omega$, $\ell = 0.4\lambda$

From Smith chart, ($z_L = 1.2 + j1.0$)

a) $SWR = 2.46$ ✓

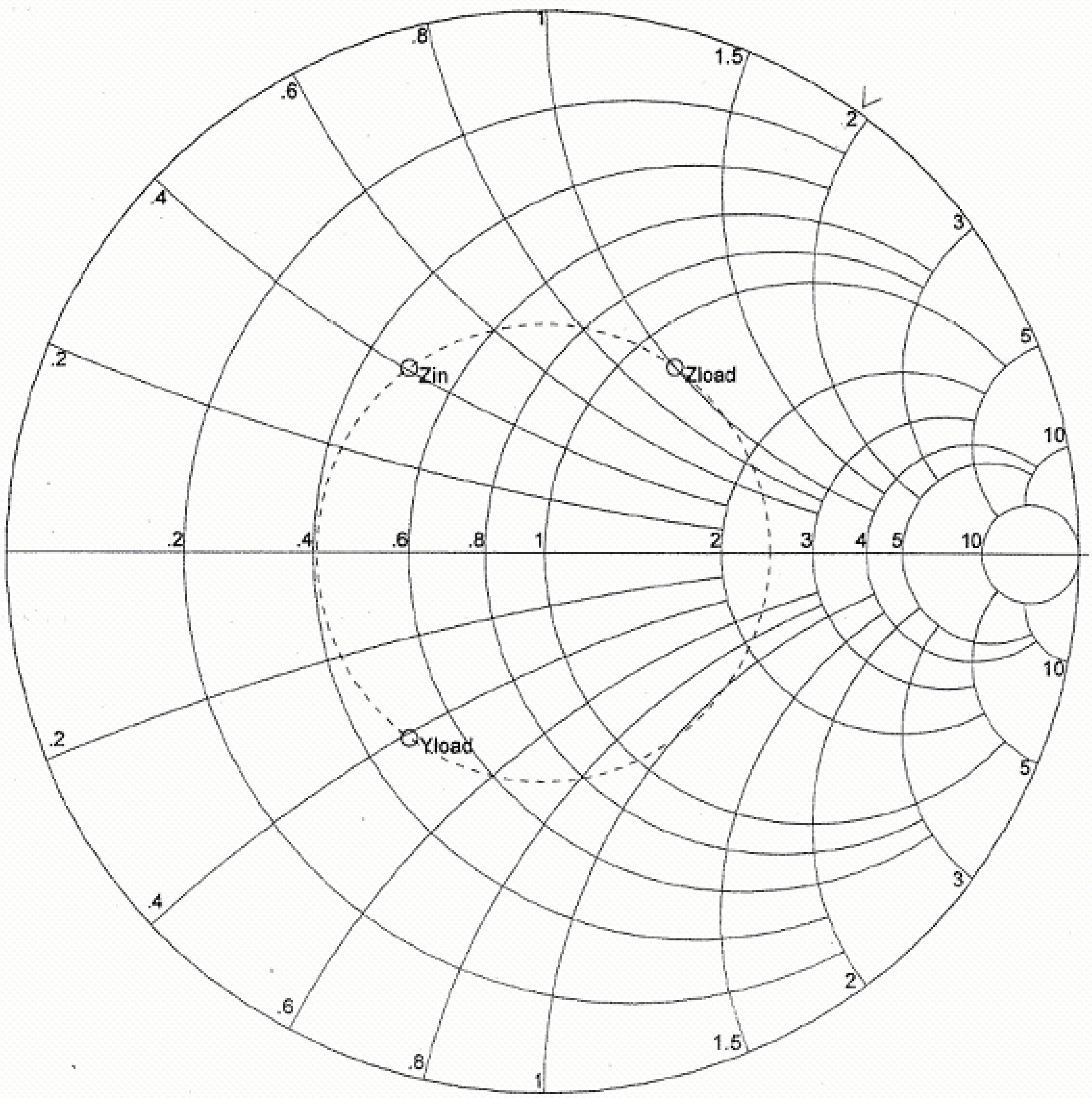
b) $\Gamma = 0.422 \angle 54^\circ$ ✓

c) $Y_L = (.492 - j.410)/50 = 9.84 - j\,8.2$ mS ✓

d) $Z_{in} = 24.5 + j\,20.3\ \Omega$

e) $\ell_{MIN} = 0.325\lambda$

f) $\ell_{max} = 0.075\lambda$



2.21

a) $\ell = 0$ or $\ell = 0.5\lambda$ ✓

b) $\ell = 0.25\lambda$ ✓

c) $\ell = 0.125\lambda$ ✓

d) $\ell = 0.406\lambda$ ✓

e) $\ell = 0.021\lambda$ ✓

These results check with $Z_{in} = jZ_0 \tan \beta\ell$.

2.22

a) $\ell = 0.25\lambda$ ✓

b) $\ell = 0\lambda$ or 0.5λ ✓

c) $\ell = 0.375\lambda$ ✓

d) $\ell = 0.656\lambda - 0.5\lambda = 0.156\lambda$ ✓

e) $\ell = 0.271\lambda$ ✓

(add $\lambda/4$ to results of P.2.22)

(also check with $Z_{in} = -jZ_0 \cot \beta\ell$)

2.23

$\lambda = 4.2$ cm. From the Smith chart, $\ell_{MIN} = .9/4.2 = 0.214\lambda$ from the load, So $z_L = 2 - j.9 \Rightarrow \underline{\underline{Z_L = 100 - j45\,\Omega}}$ ✓

Analytically, using (2.58)-(2.60),

$\Gamma = |\Gamma| e^{j\theta}$, $|\Gamma| = \dfrac{2.5-1}{2.5+1} = 0.428$

$\theta = \pi + 2\beta\ell_{MIN} = 180 + 2(360)(.214) = -26°$ ✓

Then,

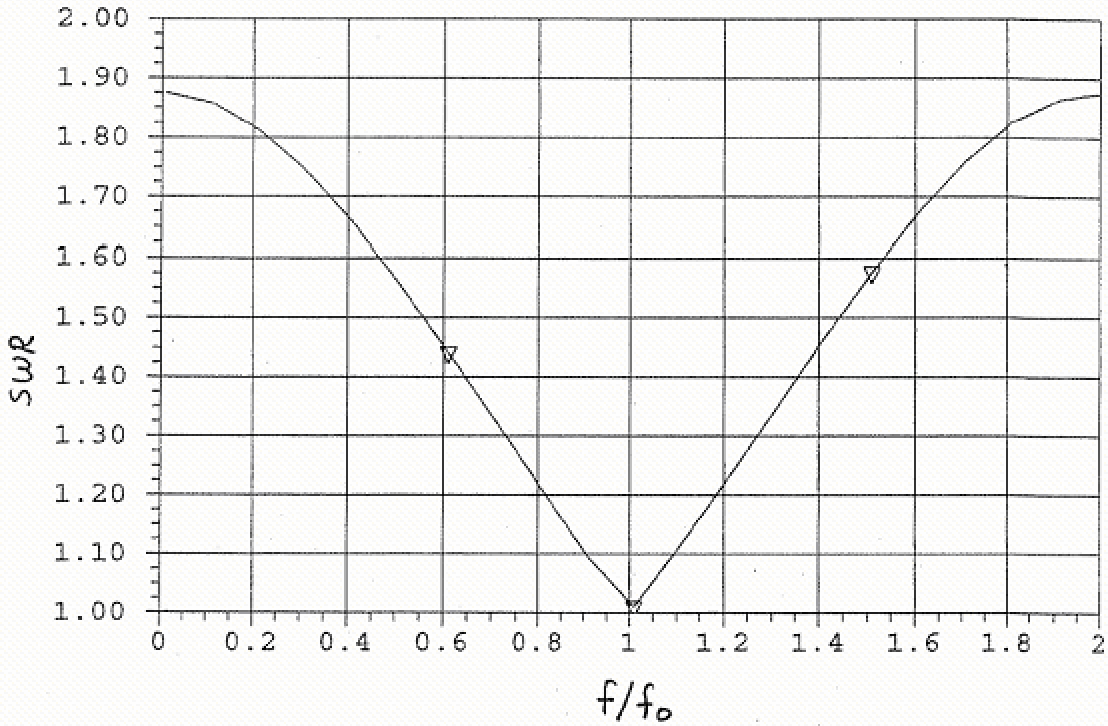
$$Z_L = \frac{1 + .428\angle{-26°}}{1 - .428\angle{-26°}}(50) = 50\,\frac{1.4\angle{-7.7°}}{.643\angle{17°}} = 109\angle{-25°}$$

$$= \underline{\underline{99 - j46\,\Omega}}$$

2.24

$Z_t = \sqrt{40(75)} = 54.77\ \Omega$

The VSWR is plotted vs f/f_o below:



2.25 On the $\lambda/4$ transformer, the voltage can be expressed as,

$$V(z) = V^+ e^{-j\beta z} + \Gamma V^+ e^{j\beta z} \quad , \quad \Gamma = \frac{R_L - \sqrt{Z_o R_L}}{R_L + \sqrt{Z_o R_L}}$$

at $z = -\ell$, $\quad V(-\ell) = V^i = V^+ \left[e^{j\beta\ell} + \Gamma e^{-j\beta\ell} \right]$

$$V^+ = \frac{V^i}{\left[e^{j\beta\ell} + \Gamma e^{-j\beta\ell} \right]} \quad , \quad V^- = \Gamma V^+$$

(assuming V^i with a phase reference at $z = -\ell$.)

2.26 From (2.70),

$$V_0^+ = V_g \frac{Z_{in}}{Z_{in} + Z_g} \frac{1}{(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell})}$$

From (2.67),

$$Z_{in} = Z_0 \frac{1 + \Gamma_\ell e^{-2j\beta\ell}}{1 - \Gamma_\ell e^{-2j\beta\ell}}$$

Then,

$$\frac{Z_{in}}{Z_{in} + Z_g} = \frac{Z_0(1 + \Gamma_\ell e^{-2j\beta\ell})}{Z_0(1 + \Gamma_\ell e^{-2j\beta\ell}) + Z_g(1 - \Gamma_\ell e^{-2j\beta\ell})}$$

$$= \frac{Z_0(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell})\, e^{-j\beta\ell}}{(Z_0 + Z_g) + \Gamma_\ell (Z_0 - Z_g)\, e^{-2j\beta\ell}}$$

$$= \frac{Z_0(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell})\, e^{-j\beta\ell}}{(Z_0 + Z_g)\left[1 + \Gamma_\ell \frac{Z_0 - Z_g}{Z_0 + Z_g} e^{-2j\beta\ell}\right]}$$

Thus,

$$V_0^+ = V_g \frac{Z_0 e^{-j\beta\ell}}{(Z_0 + Z_g)(1 - \Gamma_\ell \Gamma_g e^{-2j\beta\ell})}, \quad \text{since } \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}.$$

2.27

$$\frac{\partial \alpha_c}{\partial a} = \frac{R_s}{2\eta}\left[\frac{1}{a}\left(\frac{1}{\ln b/a}\right)^2\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{\ln b/a}\left(\frac{-1}{a^2}\right)\right] = 0$$

$$a\left(\frac{1}{a}+\frac{1}{b}\right) = \ln b/a$$

$$(1+b/a) = b/a \ \ln b/a$$

If $x = b/a$, then $1+x = x \ln x$.

(If $\frac{\partial \alpha_c}{\partial b}$ is taken, the same result is obtained if $x = a/b$)

Now solve this equation for x:

Using interval-halving method:

x	$x \ln x - x - 1$
1	−2.0
2	−1.6
3	−.704
4	.545
3.5	−.115
3.6	.011
3.55	−.052
→ 3.59	−.01

For $x = \frac{b}{a} = 3.59$,

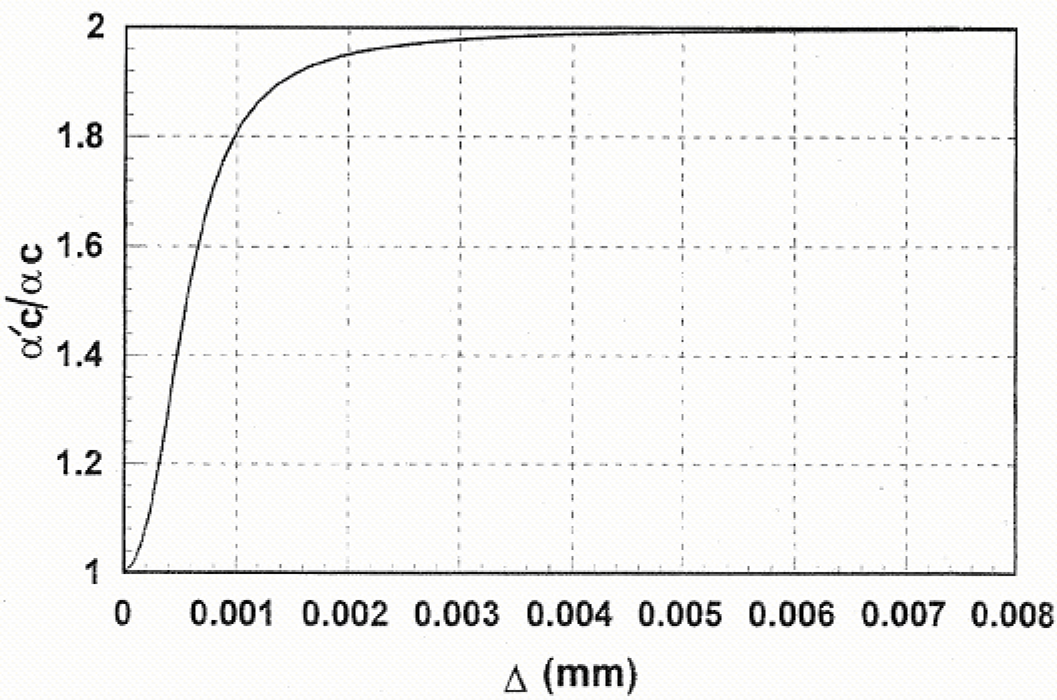
$$Z_0 = \frac{\eta}{2\pi}\ln\frac{b}{a} = \frac{377}{\sqrt{\epsilon_r}}\ln(3.59) = \frac{76.7}{\sqrt{\epsilon_r}} \simeq 77\,\Omega \text{ for } \epsilon_r = 1.$$

Thus, for an air dielectric, minimum attenuation occurs for a characteristic impedance near $77\,\Omega$.

2.28 The skin depth of copper at 10 GHz is $\delta_s = 6.60 \times 10^{-7}$ m.

Then, compute $\frac{\alpha_c'}{\alpha_c} = 1 + \frac{2}{\pi} \tan^{-1} 1.4 \left(\frac{\Delta}{\delta_s}\right)^2$ (2.107)

The results are plotted below.



2.29 Since the generator is matched to the line,

$V_o^+ = \frac{V_g}{2} e^{-\gamma \ell}$ (phase reference at $z=0$)

$\alpha = 0.5\, dB/\lambda = 0.0576$ neper/λ

$\gamma \ell = (\alpha + j\beta)\ell = 0.1325 + j108°$ ✓

Thus $|V_o^+| = \frac{10}{2} e^{-\alpha \ell} = 4.38\,v.$

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-50}{100+50} = 0.333$, $\Gamma(\ell) = \Gamma e^{-2\gamma \ell}$

From (2.92)-(2.94) we then have,

$P_{in} = \frac{|V_o^+|^2}{2Z_0}\left[1 - |\Gamma(\ell)|^2\right] e^{2\alpha \ell} = \frac{(4.38)^2}{100}\left[e^{2(.1325)} - (.333)^2 e^{-2(.1325)}\right]$

$= \underline{0.2337\ w}$ (power delivered to line)

$P_L = \frac{|V_o^+|^2}{2Z_0}(1 - |\Gamma|^2) = \frac{(4.38)^2}{100}\left[1 - (.333)^2\right] = \underline{0.1706\,w}$ (power to load

$P_{Loss} = P_{in} - P_L = 0.2337 - 0.1706 = \underline{0.0631\,w}$

The input impedance is,

$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma \ell}{Z_0 + Z_L \tanh \gamma \ell} = 50 \frac{100 + 50(.845 + j2.19)}{50 + 100(.845 + j2.19)} = 32.5 - j12.4\ \Omega$

The input current is,

$I_{in} = \frac{V_g}{R_g + Z_{in}} = \frac{10}{82.5 - j12.4} = 0.1199\angle 8.5°$ A

The generator power is,

$P_s = \frac{1}{2} V_g |I_{in}| = 5(.1199) = \underline{0.600}\ w$

Power lost in R_g is,

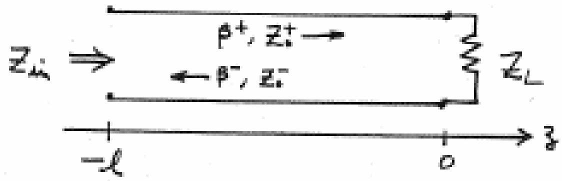
$P_{R_g} = \frac{1}{2}|I_{in}|^2 R_g = \frac{1}{2}(.1199)^2(50) = \underline{0.3594\,w}$

CHECK:

$P_L + P_{Loss} + P_{R_g} = .1706 + .0631 + .3594 = 0.5931\ w \simeq P_s$ ✓

$P_{in} + P_{R_g} = .2337 + .3594 = 0.5931\ w \simeq P_s$ ✓

2.30



$$V(z) = V_0^+ e^{-j\beta^+ z} + V_0^- e^{j\beta^- z}$$

$$I(z) = \frac{V_0^+}{Z_0^+} e^{-j\beta^+ z} - \frac{V_0^-}{Z_0^-} e^{j\beta^- z}$$

at $z=0$ (load), $\quad V(0) = V_0^+ + V_0^-$

$$I(0) = \frac{V_0^+}{Z_0^+} - \frac{V_0^-}{Z_0^-}$$

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+/Z_0^+ - V_0^-/Z_0^-} = \frac{1 + V_0^-/V_0^+}{\frac{1}{Z_0^+} - \frac{V_0^-}{V_0^+}\frac{1}{Z_0^-}}$$

as usual, let $\Gamma(0) = V_0^-/V_0^+$. Then,

$$Z_L\left(\frac{1}{Z_0^+} - \Gamma\frac{1}{Z_0^-}\right) = 1 + \Gamma$$

$$\frac{Z_L}{Z_0^+} - 1 = \Gamma\left(1 + \frac{Z_L}{Z_0^-}\right)$$

$$\Gamma = \Gamma(0) = \frac{Z_L - Z_0^-}{Z_L + Z_0^+} \qquad \text{(at load)}$$

The input impedance is,

$$Z_{in} = \frac{V(-\ell)}{I(-\ell)} = \frac{V_0^+\left[e^{j\beta^+\ell} + \Gamma e^{-j\beta^-\ell}\right]}{V_0^+\left[\frac{1}{Z_0^+} e^{j\beta^+\ell} - \Gamma\frac{1}{Z_0^-} e^{-j\beta^-\ell}\right]}$$

$$= \frac{(Z_L + Z_0^+)\, e^{j\beta^+\ell} + (Z_L - Z_0^-)\, e^{-j\beta^-\ell}}{\frac{1}{Z_0^+}(Z_L + Z_0^+)\, e^{j\beta^+\ell} - \frac{1}{Z_0^-}(Z_L - Z_0^-)\, e^{-j\beta^-\ell}}$$

This result does not simplify much further. From (2.42),

$\Gamma(-\ell) = \Gamma(0)\, e^{-j(\beta^- + \beta^+)\ell}$ (reflection coefficient at the input)

2.31

The incident wave amplitude is $V_1^+ = 10\frac{50}{25+50} = 6.67$ V

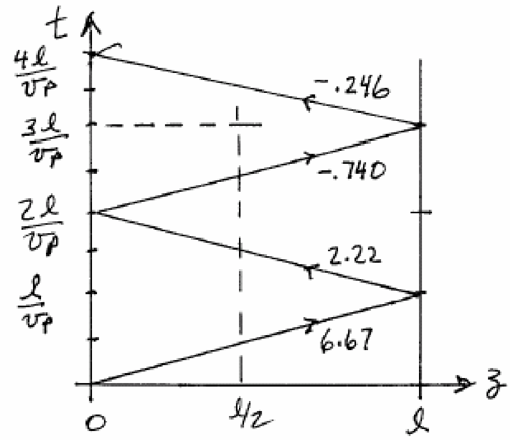
The reflection coefficients are,

$$\Gamma_L = \frac{100-50}{100+50} = 0.333 \quad , \quad \Gamma_g = \frac{25-50}{25+50} = -0.333$$

$V_1^- = \Gamma_L V_1^+ = 2.22$ V

$V_2^+ = V_1^- \Gamma_g = -.740$ V

$V_2^- = V_2^+ \Gamma_L = -.246$ V



at $z = l/2$ and $t = 3l/v_p$,

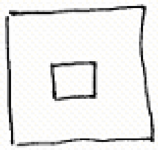
$V_T = 6.67 + 2.22 - 0.740$

$= 8.15$ V

Chapter 3

3.1

<u>Variation on coax:</u>

\#1. 

Square Coax (actually used in micromachined circuits). Easier to fabricate in micromachined form, using deposition and vias. TEM mode, so dispersion is low. Losses can be low if dielectric loss is small and metalization is good. Higher order modes will exist above cutoff frequency.

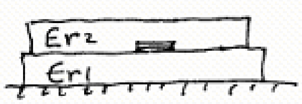
\# 2. 

coax with two dielectric cores. Will not be strictly TEM, thus slightly dispersive. May also allow control of Z_0 without changing a, b, possibly for use as $\lambda/4$ matching section without discontinuity in diameters. More expensive than standard coax.

<u>Variation on microstrip:</u>

\#1 

air-filled microstrip may be supported with foam spacers. No dielectric loss, light weight, pure TEM mode, low dispersion, fewer higher order modes. Lower Cost.

\# 2 

covered microstrip - provides physical protection of metalization. More expensive, higher dielectric loss, heavier. These drawbacks would be minimized if the top layer was very thin

3.2 Let $k_c^2 = k^2 - \beta^2$

H_x: multiply (3.3a) by $\omega\epsilon$, multiply (3.4b) by β, and add:

$$\omega\epsilon \frac{\partial E_z}{\partial y} - j\beta^2 H_x - \beta \frac{\partial H_z}{\partial x} = -j\omega^2\mu\epsilon H_x$$

$$H_x = \frac{j}{k_c^2}\left[\omega\epsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x}\right] \checkmark$$

H_y: multiply (3.3b) by $-\omega\epsilon$, multiply (3.4a) by β, and add:

$$\omega\epsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y} + j\beta^2 H_y = j\omega^2\mu\epsilon H_y$$

$$H_y = \frac{-j}{k_c^2}\left[\omega\epsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y}\right] \checkmark$$

E_x: multiply (3.3b) by $-\beta$, multiply (3.4a) by $\omega\mu$, and add:

$$j\beta^2 E_x + \beta \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y} = j\omega^2\mu\epsilon E_x$$

$$E_x = \frac{-j}{k_c^2}\left[\beta \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right] \checkmark$$

E_y: multiply (3.3a) by β, multiply (3.4b) by $\omega\mu$, and add:

$$\beta \frac{\partial E_z}{\partial y} + j\beta^2 E_y - \omega\mu \frac{\partial H_z}{\partial x} = j\omega^2\mu\epsilon E_y$$

$$E_y = \frac{j}{k_c^2}\left[\beta \frac{\partial E_z}{\partial y} - \omega\mu \frac{\partial H_z}{\partial x}\right] \checkmark$$

3.3 From (3.66)-(3.67),

$$H_z = B_n \cos\frac{n\pi y}{d} e^{-j\beta z}$$

$$H_y = \frac{j\beta}{k_c} B_n \sin\frac{n\pi y}{d} e^{-j\beta z}$$

From (3.71), $$P_o = \frac{\omega\mu d W \beta}{4 k_c^2} |B_n|^2 \quad \text{for } n>0,\ \beta \text{ real}$$

From (2.97), the power lost in both plates is,

$$P_\ell = 2\left(\frac{R_s}{2}\right)\int_s |\bar{H}_t|^2 ds = R_s \int_{z=0}^{l} \int_{x=0}^{w} \left[|H_y(y=0)|^2 + |H_z(y=0)|^2\right] dx\,dz$$

$$= R_s W |B_n|^2$$

Then, $\alpha_c = \dfrac{P_\ell}{2P_o} = \dfrac{2R_s k_c^2}{kd\eta\beta}$. (agrees with (3.72)) ✓

3.4 From Appendix I, $a = 1.07$ cm, $b = 0.43$ cm.

$f_{c_{10}} = \frac{c}{2a} = 14.02$ GHz ✓
$f_{c_{20}} = \frac{c}{a} = 28.04$ GHz ✓ } LOWEST ORDER MODES
$f_{c_{01}} = \frac{c}{2b} = 34.88$ GHz ✓

The fractional BW from $f_{c_{10}}$ to $f_{c_{20}}$ is $\dfrac{(28-14)}{(28+14)/2} = 67\%$

The fractional BW of the recommended operating range of 18.0–26.5 GHz is $\dfrac{(26.5-18.0)}{(26.5+18.0)/2} = 38\%$ (reduction of 29%)

3.5 K-band guide, $\ell = 10$ cm, $\epsilon_r = 2.55$, $\tan\delta = 0.0015$
copper, $f = 15$ GHz. $a = 1.07$ cm, $b = 0.43$ cm, $\sigma = 5.8 \times 10^7$

$$f_{c_{10}} = \frac{c}{2a\sqrt{\epsilon_r}} = 8.78 \text{ GHz} \checkmark \quad , \quad f_{c_{20}} = 17.6 \text{ GHz} \text{ (one prop. mode)}$$

$$k = \sqrt{\epsilon_r}\, k_0 = 501.67 \text{ m}^{-1} \qquad R_S = \sqrt{\frac{\omega\mu}{2\sigma}} = 0.03195\,\Omega$$

$$\beta_{10} = \sqrt{k^2 - (\pi/a)^2} = 406.78 \text{ m}^{-1} \checkmark \qquad \eta = 377/\sqrt{\epsilon_r} = 236.1\,\Omega$$

From (3.29) $$\alpha_d = \frac{k^2 \tan\delta}{2\beta} = 0.464 \text{ np/m} = 4.03 \text{ dB/m} \checkmark$$

From (3.96) $$\alpha_c = \frac{R_s}{a^3 b \beta k \eta}\left(2b\pi^2 + a^3 k^2\right) = 0.0495 \text{ np/m} = 0.430 \text{ dB/m} \checkmark$$

$$\text{Loss} = (\alpha_c + \alpha_d)\ell = 0.446 \text{ dB}$$
$$\Delta\phi = \beta\ell = 2330.7^\circ$$

3.6 In the section of guide of width $a/2$, the TE_{10} mode is below cutoff (evanescent), with an attenuation constant α:

$$k = \frac{2\pi(12{,}000)}{300} = 251.3 \text{ m}^{-1} \checkmark$$

$$\alpha = \sqrt{\left(\frac{\pi}{a/2}\right)^2 - k^2} = \sqrt{\left(\frac{2\pi}{.02286}\right)^2 - (251.3)^2} = 111.3 \text{ nepers/m} \checkmark$$

To obtain 100 dB attenuation (ignoring reflections),

$$-100 \text{ dB} = 20 \log e^{-\alpha\ell}$$
$$10^{-5} = e^{-\alpha\ell}$$
$$\ell = \frac{11.5}{111.3} = 0.103 \text{ m} = \underline{10.3 \text{ cm}} \checkmark$$

3.7 The TE_{10} H-fields from (3.89) are:

$$H_x = \frac{j\beta a A}{\pi} \sin \frac{\pi x}{a} e^{-j\beta x}$$

$$H_y = 0$$

$$H_z = A \cos \frac{\pi x}{a} e^{-j\beta x}$$

$\bar{J}_s = \hat{n} \times \bar{H}$, so the surface currents are,

ON BOTTOM WALL: $\hat{n} = \hat{y}$; $\bar{J}_s = -\hat{z} \frac{j\beta a A}{\pi} \sin \frac{\pi x}{a} e^{-j\beta z} + \hat{x} A \cos \frac{\pi x}{a} e^{-j\beta z}$

ON TOP WALL: $\hat{n} = -\hat{y}$; $\bar{J}_s = \hat{z} \frac{j\beta a A}{\pi} \sin \frac{\pi x}{a} e^{-j\beta z} - \hat{x} A \cos \frac{\pi x}{a} e^{-j\beta z}$

ON LEFT SIDE WALL: $\hat{n} = \hat{x}$, $x=0$; $\bar{J}_s = -\hat{y} A e^{-j\beta z}$

ON RIGHT SIDE WALL: $\hat{n} = -\hat{x}$, $x=a$; $\bar{J}_s = -\hat{y} A e^{-j\beta z}$

Note that the top and bottom currents are the negative of each other.

Along the centerline of the top or bottom (broad) walls, $x = a/2$, so the surface currents can be reduced to,

$$\bar{J}_s = \pm \hat{z} \frac{j\beta a A}{\pi} e^{-j\beta z},$$

which shows that current flow is only in the longitudinal direction. Thus a narrow longitudinal slot will not break any current lines, and will have a negligible effect on the operation of the waveguide.

3.8 From (3.101),

$$\bar{E} \times \bar{H}^* \cdot \hat{z} = E_x H_y^* - E_y H_x^*$$

$$= \frac{\omega \epsilon \beta m^2 \pi^2}{a^2 k_c^4} |B|^2 \cos^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b}$$

$$+ \frac{\omega \epsilon \beta n^2 \pi^2}{b^2 k_c^4} |B|^2 \sin^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b}$$

So the power flow down the guide is,

$$P_o = \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \bar{E} \times \bar{H}^* \cdot \hat{z} \, dx \, dy = \frac{\omega \epsilon \beta \pi^2 |B|^2}{2 k_c^4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \frac{ab}{4} = \frac{\omega \epsilon \beta a b}{8 k_c^2} |B|^2$$

The power loss in the walls is,

$$P_\ell = \frac{R_s}{2} \int_c |\bar{H}_t|^2 \, ds = R_s \left\{ \int_{x=0}^{a} |H_x(y=0)|^2 dx + \int_{y=0}^{b} |H_y(x=0)|^2 dy \right\}$$

$$= R_s \left\{ \frac{\omega^2 \epsilon^2 n^2 \pi^2}{b^2 k_c^4} |B|^2 \frac{a}{2} + \frac{\omega^2 \epsilon^2 m^2 \pi^2}{a^2 k_c^4} |B|^2 \frac{b}{2} \right\}$$

$$= R_s \frac{\omega^2 \epsilon^2 \pi^2}{2 k_c^4} |B|^2 \left(\frac{n^2 a}{b^2} + \frac{m^2 b}{a^2} \right)$$

So the attenuation is,

$$\alpha_c = \frac{P_\ell}{2 P_o} = \frac{R_s \omega^2 \epsilon^2 \pi^2 4 k_c^2}{2 k_c^4 \omega \epsilon \beta a b} \left(\frac{n^2 a}{b^2} + \frac{m^2 b}{a^2} \right)$$

$$= \frac{2 R_s k \pi^2}{k_c^2 \beta \eta} \left(\frac{n^2}{b^3} + \frac{m^2}{a^3} \right) \quad \text{nepers/m} \checkmark$$

3.9 From (3.109), the propagation constant is a solution of,

$$k_a \tan k_d t + k_d \tan k_a (a-t) = 0,$$

where
$$\beta = \sqrt{k_0^2 - k_a^2} = \sqrt{\epsilon_r k_0^2 - k_d^2} \qquad (3.106)$$

Since $\beta = 0$ at cutoff, we have that $k_a = k_0$, and $k_d = \sqrt{\epsilon_r}\, k_0$. Thus we must find the root of the following equation:

$$f(k_0) = k_0 \tan \sqrt{\epsilon_r}\, k_0 t + \sqrt{\epsilon_r}\, k_0 \tan k_0 t = 0 \quad (\text{since } t = a/2)$$

We know that $k_c = k_0$ must be between k_c of the empty guide, and k_c for the completely filled guide:

$k_c(\text{EMPTY}) = \frac{\pi}{a} = 137.\ m^{-1}$

$k_c(\text{FILLED}) = \frac{\pi}{\sqrt{\epsilon_r}\, a} = 92.\ m^{-1}$

k_0	$f(k_0)$
95	−1366
100	−362
105	−44
110	171
106	2.3
105.9	−2.25
✓ → 105.95	.017

This result is accurate to at least four figures, and agrees with a result given in the Waveguide Handbook.

The cutoff frequency is,

$$f_c = \frac{k_c c}{2\pi} = 5.06\ GHz$$

3.10 The lowest order mode will have an H_z component which is even in x, and no variation in y. Thus, h_z can be written as,

$$h_z(x,y) = \begin{cases} A \cos k_d x & \text{for } |x| < W/2 \quad (k_c = k_d) \\ B e^{-k_a |x|} & \text{for } |x| > W/2 \quad (k_c = j k_a) \end{cases}$$

where k_d and k_a are the cutoff wavenumbers in the dielectric and air regions, respectively, satisfying

$$\beta = \sqrt{\epsilon_r k_0^2 - k_d^2} = \sqrt{k_0^2 - k_a^2} \qquad \text{(phase matching)}$$

Next, we need e_y, from (3.19d):

$$e_y(x,y) = \frac{j\omega\mu}{k_c^2} \frac{\partial h_z}{\partial x} = \begin{cases} -j\frac{\omega\mu A}{k_d} \sin k_d x & \text{for } |x| < W/2 \\ \frac{j\omega\mu B}{k_a} e^{-k_a x} & \text{for } x > W/2 \end{cases}$$

Matching h_z and e_y at $x = W/2$ gives,

$$A \cos k_d W/2 = B e^{-k_a W/2}$$

$$\frac{-A}{k_d} \sin k_d W/2 = \frac{B}{k_a} e^{-k_a W/2}$$

Setting the determinant of these equations to zero gives,

$$k_a \tan k_d W/2 + k_d = 0.$$

A TEM mode cannot exist by itself because of the impossibility of phase matching at $x = W/2$. (For a TEM mode, $\beta = k$ in both regions, which is not possible.)

3.11 Maxwell's curl equations are,

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad , \quad \nabla \times \bar{H} = j\omega\epsilon \bar{E}$$

The ρ and ϕ components in cylindrical form are,

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = -j\omega\mu H_\rho \qquad \frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = j\omega\epsilon E_\rho$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\phi \qquad \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = j\omega\epsilon E_\phi$$

Now assume $\bar{E}(\rho,\phi,z) = \bar{e}(\rho,\phi)e^{-j\beta z}$

$$\bar{H}(\rho,\phi,z) = \bar{h}(\rho,\phi)e^{-j\beta z}$$

Then $\partial/\partial z \rightarrow -j\beta$, and the above equations reduce to:

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} + j\beta E_\phi = -j\omega\mu H_\rho \quad (1) \qquad \frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + j\beta H_\phi = j\omega\epsilon E_\rho \quad (3)$$

$$-j\beta E_\rho - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\phi \quad (2) \qquad -j\beta H_\rho - \frac{\partial H_z}{\partial \rho} = j\omega\epsilon E_\phi \quad (4)$$

Multiply (2) by $-\beta$, multiply (3) by $\omega\mu$, and add:

$$j\beta^2 E_\rho + \beta\frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi} = j\omega^2\mu\epsilon E_\rho$$

$$E_\rho = \frac{-j}{k_c^2}\left[\beta\frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right]$$

Multiply (1) by β, multiply (4) by $\omega\mu$, and add:

$$\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} + j\beta^2 E_\phi - \omega\mu\frac{\partial H_z}{\partial \rho} = j\omega^2\mu\epsilon E_\phi$$

$$E_\phi = \frac{-j}{k_c^2}\left[\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial \rho}\right]$$

Multiply (1) by $\omega\epsilon$, multiply (4) by β, and add:

$$\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \phi} - j\beta^2 H_\rho - \beta\frac{\partial H_z}{\partial \rho} = -j\omega^2\mu\epsilon H_\rho$$

$$H_\rho = \frac{j}{k_c^2}\left[\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \phi} - \beta\frac{\partial H_z}{\partial \rho}\right]$$

$$\omega\epsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi} + j\beta^2 H_\phi = j\omega^2\mu\epsilon H_\phi$$

$$H_\phi = \frac{-j}{k_c^2}\left[\omega\epsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi}\right]$$

with $k_c^2 = k^2 - \beta^2$.

These results agree with those of (3.110). ✓

3.12 Let $A=1$, $B=0$ in (3.141). Then the transverse fields are,

$$E_\rho = \frac{-j\beta}{k_c} \sin n\phi \, J_n'(k_c\rho)\, e^{-j\beta z}$$

$$E_\phi = \frac{-j\beta n}{k_c^2 \rho} \cos n\phi \, J_n(k_c\rho)\, e^{-j\beta z}$$

$$H_\rho = \frac{j\omega\epsilon n}{k_c^2 \rho} \cos n\phi \, J_n(k_c\rho)\, e^{-j\beta z}$$

$$H_\phi = \frac{-j\omega\epsilon}{k_c} \sin n\phi \, J_n'(k_c\rho)\, e^{-j\beta z}$$

$$\bar{E}\times\bar{H}^*\cdot\hat{z} = E_\rho H_\phi^* - E_\phi H_\rho^* .$$

The power flow down the guide is, for $n>0$,

$$P_o = \frac{1}{2}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\left[\frac{\beta\omega\epsilon}{k_c^2}\sin^2 n\phi\, J_n'^2(k_c\rho) + \frac{\beta\omega\epsilon n^2}{k_c^4\rho^2}\cos^2 n\phi\, J_n^2(k_c\rho)\right]\rho\, d\phi\, d\rho$$

$$= \frac{\beta\omega\epsilon\pi}{2k_c^2}\int_{\rho=0}^{a}\left[J_n'^2(k_c\rho) + \frac{n^2}{k_c^2\rho^2}J_n^2(k_c\rho)\right]\rho\, d\rho \qquad \begin{aligned} &\text{Let } x = k_c\rho \\ &dx = k_c d\rho \\ &k_c a = p_{nm} \end{aligned}$$

$$= \frac{\beta\omega\epsilon\pi}{2k_c^4}\int_{x=0}^{p_{nm}}\left[J_n'^2(x) + \frac{n^2}{x^2}J_n^2(x)\right]x\, dx = \frac{\beta\omega\epsilon\pi}{4k_c^4}\, p_{nm}^2\, J_n'^2(p_{nm}) \quad \text{(SEE C.16)}$$

The power lost in the conducting wall is,

$$P_\ell = \frac{R_s}{2}\int_{z=0}^{1}\int_{\phi=0}^{2\pi}\left|H_\phi(\rho=a)\right|^2 a\, d\phi\, dz = \frac{aR_s}{2}\frac{\omega^2\epsilon^2}{k_c^2}J_n'^2(p_{nm})\int_{\phi=0}^{2\pi}\sin^2 n\phi\, d\phi$$

$$= \frac{aR_s\omega^2\epsilon^2\pi}{2k_c^2}J_n'^2(p_{nm})$$

The attenuation is then,

$$\alpha_c = \frac{P_\ell}{2P_o} = \frac{aR_s\omega^2\epsilon^2\pi\, 4k_c^4}{4k_c^2\,\beta\omega\epsilon\pi p_{nm}^2} = \frac{aR_s\omega\epsilon k_c^2}{\beta p_{nm}^2} = \frac{kR_s}{\beta\eta a} \text{ neper/m} ✓$$

3.13 $a = 0.4$ cm, $\epsilon_r = 1.50$, copper, $\tan\delta = 0.0002$

TE$_{11}$: $f_c = \dfrac{p'_{11} c}{2\pi a \sqrt{\epsilon_r}} = 17.94$ GHz ✓

TM$_{01}$: $f_c = \dfrac{p_{01} c}{2\pi a \sqrt{\epsilon_r}} = 23.44$ GHz ✓

TE$_{21}$: $f_c = \dfrac{p'_{21} c}{2\pi a \sqrt{\epsilon_r}} = 29.76$ GHz ✓

TE$_{01}$: $f_c = \dfrac{p'_{01} C}{2\pi a \sqrt{\epsilon_r}} = 37.35$ GHz. ✓

at 20 GHz, $k_0 = 418.88\ m^{-1}$, $\beta = \sqrt{\epsilon_r k_0^2 - (p'_{11}/a)^2} = 226.63\ m^{-1}$ ✓

$$\alpha_d = \frac{\epsilon_r k_0^2 \tan\delta}{2\beta} = 0.116 \text{ np/m} = 1.01 \text{ dB/m}$$ ✓

$R_s = 0.0369\,\Omega$

$$\alpha_c = \frac{R_s}{a k_0 \eta_0 \beta}\left(k_c^2 + \frac{k^2}{p'^2_{11} - 1}\right) = 0.083 \text{ np/m} = 0.721 \text{ dB/m}$$ ✓

3.14 From (3.153), $\Phi(\rho,\phi) = \dfrac{V_o \ln b/\rho}{\ln b/a}$

From (3.13) and Appendix,

$$\bar{e}(\rho,\phi) = -\nabla_t \Phi(\rho,\phi) = -\left(+\hat{\rho}\frac{\partial \Phi}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\right) = \frac{V_o \hat{\rho}}{\rho \ln b/a}$$

Then,

$$\bar{E}(\rho,\phi,z) = \bar{e}(\rho,\phi)\, e^{-j\beta z} = \frac{V_o \hat{\rho}\, e^{-j\beta z}}{\rho \ln b/a} \quad \checkmark \qquad (4.155) \text{ of 1st Ed.}$$

From (3.18),

$$\bar{h}(\rho,\phi) = \frac{1}{\eta}\hat{z}\times\bar{e}(\rho,\phi) = \frac{V_o \hat{\phi}}{\eta \rho \ln b/a}$$

Then,

$$\bar{H}(\rho,\phi,z) = \frac{V_o \hat{\phi}\, e^{-j\beta z}}{\eta \rho \ln b/a} \quad \checkmark \qquad (4.157) \text{ of 1st Ed.}$$

The potential between the two conductors is,

$$V_{ab} = \int_{\rho=a}^{b} E_\rho(\rho,\phi,z)\,d\rho = V_o e^{-j\beta z} \quad \checkmark \qquad (4.158) \text{ of 1st Ed.}$$

The current on the inner conductor is,

$$I_a = \int_{\phi=0}^{2\pi} H_\phi(a,\phi,z)\, a\, d\phi = \frac{2\pi V_o e^{-j\beta z}}{\eta \ln b/a} \quad \checkmark \qquad (4.159) \text{ of 1st Ed.}$$

The characteristic impedance is,

$$Z_o = \frac{V_{ab}}{I_o} = \frac{\eta \ln b/a}{2\pi} \quad \checkmark \qquad (4.162) \text{ of 1st Ed.}$$

3.15 The solution is similar to the TE mode case for the coax, but with e_z in place of h_z:

$$e_z(\rho,\phi) = (A \sin n\phi + B \cos n\phi)\left[C J_n(k_c\rho) + D Y_n(k_c\rho)\right]$$

Then the boundary condition that $e_z = 0$ at $\rho = a$ and at $\rho = b$ yields two equations:

$$C J_n(k_c a) + D Y_n(k_c a) = 0$$
$$C J_n(k_c b) + D Y_n(k_c b) = 0$$

or,

$$J_n(k_c a) Y_n(k_c b) = J_n(k_c b) Y_n(k_c a)$$

For the TM_{01} mode, $n=0$. Let $x = k_c a$. Then for $b = 2a$, we have that $k_c b = 2x$, and so the above equation can be written as,

$$f(x) = J_0(x) Y_0(2x) - J_0(2x) Y_0(x) = 0$$

We know that k_c should be greater than k_c for a circular waveguide of radius b, for which $k_{c_{01}} = p_{01}/b = 2.405/2a$, which implies that $x = 1.2$. So we can begin the root search at $x = 1.2$. Using a table of Bessel functions gives the following results in only a few minutes:

x	$J_0(x)$	$Y_0(x)$	$J_0(2x)$	$Y_0(2x)$	$f(x)$
1.2	.671	.228	.003	.510	.342
1.5	.512	.382	−.260	.377	.292
2.0	.224	.510	−.397	−.017	.198
3.1	−.292	.343	.202	−.248	.003
3.2	−.320	.307	.243	−.200	−.011

Linear interpolation between $x = 3.1$ and 3.2 gives a more accurate value for the root:

$$f(x) \simeq .003 + \frac{.003 - (-.011)}{3.1 - 3.2}(x - 3.1)$$

$$\simeq .437 - .14x = 0$$

$$x = \frac{.437}{.14} = \underline{\underline{3.12 = k_c a}}$$

3.16 From (3.175),

$$\bar{E}\times\bar{H}^*\cdot\hat{z} = -E_y H_x^* = \begin{cases} \dfrac{\omega\mu_0\beta|B|^2}{k_c^2}\sin^2 k_c x & \text{for } 0\le x\le d \\ \dfrac{\omega\mu_0\beta|B|^2}{h^2}\cos^2 k_c d\, e^{-2h(x-d)} & \text{for } d\le x<\infty \end{cases}$$

The power flow is,

$$P_o = \frac{1}{2}\int_{x=0}^{\infty}\int_{y=0}^{1}\bar{E}\times\bar{H}^*\cdot\hat{z}\,dy\,dx$$

$$= \frac{\omega\mu_0\beta|B|^2}{2k_c^2}\int_{x=0}^{d}\sin^2 k_c x\,dx + \frac{\omega\mu_0\beta|B|^2}{2h^2}\cos^2 k_c d\int_{x=d}^{\infty}e^{-2h(x-d)}dx$$

$$= \frac{\omega\mu_0\beta|B|^2}{2}\left[\frac{1}{k_c^2}\left(\frac{x}{2}-\frac{\sin 2k_c x}{4k_c}\right)\Big|_0^d + \frac{\cos^2 k_c d}{h^2}\left(\frac{e^{-2h(x-d)}}{-2h}\right)\Big|_d^\infty\right]$$

$$= \frac{\omega\mu_0\beta|B|^2}{2}\left[\frac{1}{k_c^2}\left(\frac{d}{2}-\frac{\sin 2k_c d}{4k_c}\right)+\frac{\cos^2 k_c d}{2h^3}\right]$$

The power loss is,

$$P_\ell = \frac{R_s}{2}\int_S|\bar{H}_t|^2 ds = \frac{R_s}{2}\int_{y=0}^{1}\int_{z=0}^{1}\left[|\overset{0}{H_x(x=0)}|^2+|H_z(x=0)|^2\right]dz\,dy$$

$$= \frac{R_s}{2}|B|^2$$

So the attenuation is,

$$\alpha_c = \frac{P_\ell}{2P_o} = \frac{2R_s}{4\omega\mu_0\beta\left[\frac{1}{k_c^2}\left(\frac{d}{2}-\frac{\sin 2k_c d}{4k_c}\right)+\frac{\cos^2 k_c d}{2h^3}\right]}$$

$$= \frac{R_s}{k_0\eta_0\beta\left[\frac{d}{k_c^2}-\frac{\sin 2k_c d}{2k_c^3}+\frac{\cos^2 k_c d}{h^3}\right]}\quad\checkmark$$

3.17 Following the derivation in Section 3.6 for the TM surface waves of a dielectric slab:

$$k_c^2 = \mu_r k_0^2 - \beta^2 \qquad \text{for } 0 \le y \le d$$
$$h^2 = \beta^2 - k_0^2 \qquad \text{for } y \ge d$$

Then,

$$e_z(x,y) = \begin{cases} A \sin k_c y & \text{for } 0 \le y \le d \\ B e^{-hy} & \text{for } y > d \end{cases}$$

This form of e_z is selected to satisfy $e_z = 0$ at $y=0$, and to have exponential decay for $y \to \infty$ (radiation condition). next, we need H_x ($H_y = E_x = H_z = 0$): From (3.23a),

$$H_x = \frac{j\omega\epsilon_0}{k_c^2}\frac{\partial E_z}{\partial y} = \begin{cases} \frac{j\omega\epsilon_0}{k_c} A \cos k_c y & \text{for } 0 \le y \le d \\ \frac{j\omega\epsilon_0}{h} B e^{-hy} & \text{for } y > d \end{cases}$$

at $y = d$:

E_z continuous $\Rightarrow$ $A \sin k_c d = B e^{-hd}$

H_x continuous $\Rightarrow$ $\frac{A}{k_c} \cos k_c d = \frac{B}{h} e^{-hd}$

or,

$$h \cos k_c d = k_c \sin k_c d$$
$$\underline{h = k_c \tan k_c d} \quad \checkmark$$

and,

$$\underline{h^2 + k_c^2 = (\mu_r - 1) k_0^2} \quad \checkmark$$

These two equations must be solved simultaneously to find h and k_c.

3.18 TM_{om} mode. $H_z = 0$ $\quad E_z(\rho,\phi,z) = e_z(\rho,\phi)e^{-j\beta z}$

(No TEM mode can be supported by this line because of the impossibility of phase matching at $\rho = b$)

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_c^2\right)e_z(\rho,\phi) = 0$$

$\partial/\partial\phi = 0$ for $n=0$ modes $\Rightarrow E_\phi = H_\rho = 0$.

Thus,

$$e_z(\rho,\phi) = \begin{cases} A\,J_0(k_d\rho) + B\,Y_0(k_d\rho) & \text{for } a \le \rho \le b \\ C\,J_0(k_a\rho) + D\,Y_0(k_a\rho) & \text{for } b \le \rho \le c \end{cases}$$

where $\quad \beta^2 = \epsilon_r k_0^2 - k_d^2 = k_0^2 - k_a^2$.

The boundary conditions are:

$e_z = 0$ at $\rho = a$ and $\rho = c$,

e_z and H_ϕ are continuous at $\rho = b$.

From (3.110d),

$$H_\phi = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial\rho},$$

So we get the following four equations:

$$A\,J_0(k_d a) + B\,Y_0(k_d a) = 0$$

$$C\,J_0(k_a c) + D\,Y_0(k_a c) = 0$$

$$A\,J_0(k_d b) + B\,Y_0(k_d b) = C\,J_0(k_a b) + D\,Y_0(k_a b)$$

$$\epsilon_r k_d\left[A\,J_0'(k_d b) + B\,Y_0'(k_d b)\right] = k_a\left[C\,J_0'(k_a b) + D\,Y_0'(k_a b)\right]$$

k_a and k_d can be expressed in terms of β, and β can be found so that the determinant of the above system of equations vanishes. This is as far as we can go without actual values for $a, b, c,$ and ϵ_r.

3.19 STRIPLINE: $100\,\Omega$, $b = 1.02\,mm$, $\epsilon_r = 2.2$, copper, $\tan\delta = 0.001$

$f = 5\ GHz$.

$\lambda_g = \lambda_o/\sqrt{\epsilon_r} = c/\sqrt{\epsilon_r} f = 4.045\ cm$

$\sqrt{\epsilon_r}\, Z_o = 148.3 > 120\,\Omega$

$$x = \frac{30\pi}{\sqrt{\epsilon_r} Z_o} - 0.441 = 0.194$$

$$W/b = .85 - \sqrt{.6 - x} = 0.213 \Rightarrow W = 0.2174\ mm$$

(PCAAD: $W = 0.218\ mm$)

$$\left.\begin{array}{l}\alpha_d = 0.0067\ dB/cm \\ \alpha_c = 0.044\ dB/cm\end{array}\right\}\ \text{PCAAD}$$

3.20 MICROSTRIP: $100\,\Omega$, $d = 0.51\,mm$, $\epsilon_r = 2.2$, copper

$\tan\delta = 0.001$, $f = 5\ GHz$.

First try $W/d < 2$: $$A = \frac{Z_o}{60}\sqrt{\frac{\epsilon_r + 1}{2}} + \frac{\epsilon_r - 1}{\epsilon_r + 1}\left(.23 + .11/\epsilon_r\right) = 2.213$$

$$W/d = \frac{8e^A}{e^{2A} - 2} = .896 \Rightarrow W = 0.457\ mm\ ✓$$

$$\epsilon_e = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2}\frac{1}{\sqrt{1 + 12d/W}} = 1.758\ ✓$$

$$\lambda_g = c/\sqrt{\epsilon_r} f = 4.525\ cm$$

$$\left.\begin{array}{l}\alpha_d = 0.0048\ dB/cm \\ \alpha_c = 0.017\ dB/cm\end{array}\right\}\ \text{PCAAD}$$

3.21 $\lambda_o = 0.12\,m$, $\beta = 2\pi\sqrt{\epsilon_e}/\lambda_o = 3986.7°/m$ $(f = 2.5\,GHz)$

$C = 5\,pF$: From P2.11, $\beta\ell = 82.74° \Rightarrow \ell = 2.0754\,cm$ $(Z_{in} = -j12.73\,\Omega)$

$L = 5\,nH$: From P2.11, $\beta\ell = 128.1° \Rightarrow \ell = 3.2132\,cm$ $(Z_{in} = +j78.5\,\Omega)$

From SERENADE:

LOSSLESS: C: $Z_{in} = -j12.63\,\Omega$ ✓
L: $Z_{in} = +j78.7\,\Omega$ ✓

LOSSY: C: $Z_{in} = 0.27 - j12.82\,\Omega$
$(t = 0.5\,mil)$ L: $Z_{in} = 0.66 + j76.7\,\Omega$

3.22 $k_o = \frac{2\pi f}{c} = 104.7\ m^{-1}$ ✓ ; $R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi(5\times10^9)(4\pi\times10^{-7})}{2(5.813\times10^7)}} = 0.018\Omega$ ✓

MICROSTRIP CASE:

First try $W/d > 2$: From (3.197), $B = \frac{377\pi}{2Z_o\sqrt{\epsilon_r}} = 8.0$

$$W/d = \frac{2}{\pi}\left[B-1-\ln(2B-1)+\frac{\epsilon_r-1}{2\epsilon_r}\left\{\ln(B-1)+.39-\frac{.61}{\epsilon_r}\right\}\right] = 3.09 > 2 \checkmark$$

$W = 3.09(.16\ cm) = \underline{0.494\ cm}$

From (3.195), $\epsilon_e = \frac{\epsilon_r+1}{2}+\frac{\epsilon_r-1}{2}\frac{1}{\sqrt{1+12d/W}} = 1.87$ ✓ $\Rightarrow \lambda_g = \frac{c}{\sqrt{\epsilon_e}f} = \underline{4.38\ cm}$ ✓

From (3.198),

$$\alpha_d = \frac{k_o\epsilon_r(\epsilon_e-1)}{2\sqrt{\epsilon_e}(\epsilon_r-1)}\tan\delta = \underline{0.061}\ \text{neper/m} \checkmark$$

From (3.199),

$$\alpha_c = \frac{R_s}{Z_oW} = \underline{0.073}\ \text{neper/m} \checkmark$$

Total MS Loss:

$$\text{LOSS} = (.061+.073)\left(\frac{np}{m}\right)(16\lambda_g)\left(.0438\frac{m}{\lambda_g}\right)\left(\frac{8.686\ dB}{neper}\right)$$
$$= 0.82\ dB \checkmark$$

STRIPLINE CASE:

From (3.180), $\sqrt{\epsilon_r}Z_o = \sqrt{2.2}(50) = 74 < 120$. $x = \frac{30\pi}{\sqrt{\epsilon_r}Z_o} - .441 = 0.833$

$W/b = x = 0.833 \Rightarrow W = .833(.32\ cm) = \underline{0.267\ cm}$ ✓

$\lambda_g = \frac{c}{\sqrt{\epsilon_r}f} = \underline{4.045\ cm}$ ✓ $A = 1 + \frac{2W}{b-t} + \frac{1}{\pi}\frac{b+t}{b-t}\ln\left(\frac{2b-t}{t}\right) = 4.73$

From (3.181),

$$\alpha_c = \frac{2.7\times10^{-3}R_s\epsilon_rZ_o}{30\pi b}A = \underline{0.084}\ \text{neper/m} \checkmark$$

From (3.30),

$$\alpha_d = \frac{k\tan\delta}{2} = \frac{\sqrt{2.2}(104.7)(.001)}{2} = \underline{0.078}\ \text{neper/m} \checkmark$$

Total S.L. Loss:

$$\text{LOSS} = (.084+.078)\left(\frac{np}{m}\right)(16\lambda_g)\left(.04045\frac{m}{\lambda_g}\right)\left(\frac{8.686\ dB}{neper}\right)$$
$$= \underline{0.91\ dB} \checkmark$$

Thus the microstrip line should be used.

3.23 $H_z(x,y,z) = h_z(x,y)e^{-j\beta z}$; h_z real, β real

From (3.19),

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial h_z}{\partial x}e^{-j\beta z} \qquad H_y = \frac{-j\beta}{k_c^2}\frac{\partial h_z}{\partial y}e^{-j\beta z}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y}e^{-j\beta z} \qquad E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}e^{-j\beta z}$$

$$\bar{E}\times\bar{H}^* = (E_x H_y^* - E_y H_x^*)\hat{z} - E_x H_z^*\hat{y} + E_y H_z^*\hat{x}$$

$$= \frac{\omega\mu\beta}{k_c^4}\left[\left(\frac{\partial h_z}{\partial y}\right)^2 + \left(\frac{\partial h_z}{\partial x}\right)^2\right]\hat{z} + \frac{j\omega\mu}{k_c^2}\left(\frac{\partial h_z}{\partial y}\hat{y} + \frac{\partial h_z}{\partial x}\hat{x}\right)h_z$$

So if h_z is real (or a real function times a complex constant), there is real power flow only in the z-direction.

3.24 Write the incident, reflected, and transmitted TE_{10} fields as follows:

$$E_y^i = E_0 \sin\frac{\pi x}{a}e^{-j\beta_a z} \qquad E_y^r = E_0\Gamma \sin\frac{\pi x}{a}e^{j\beta_a z}$$

$$H_x^i = \frac{-E_0}{Z_a}\sin\frac{\pi x}{a}e^{-j\beta_a z} \qquad H_x^r = \frac{E_0\Gamma}{Z_a}\sin\frac{\pi x}{a}e^{j\beta_a z}$$

$$E_y^t = E_0 T \sin\frac{\pi x}{a}e^{-j\beta_d z}$$

$$H_x^t = \frac{-E_0 T}{Z_d}\sin\frac{\pi x}{a}e^{-j\beta_d z}$$

where $\beta_a = \sqrt{k_0^2 - (\pi/a)^2}$ $\qquad Z_a = k_0\eta_0/\beta_a$

$\beta_d = \sqrt{\epsilon_r k_0^2 - (\pi/a)^2}$ $\qquad Z_d = k_0\eta_0/\beta_d$

Match fields at $z=0$ to obtain:

$1 + \Gamma = T$ (E_y continuous)

$\frac{1}{Z_a}(-1+\Gamma) = \frac{-T}{Z_d}$ (H_x continuous)

Solving for Γ gives,

$$\Gamma = \frac{Z_d - Z_a}{Z_d + Z_a} \checkmark$$

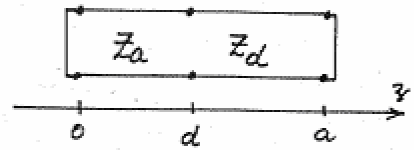
which agrees with the transmission line theory result if Z_{TE} is used as Z_0 in each region. ✓ 50

3.25

$$Z_{TM} = \eta \beta / k = \eta_o \beta / k_o \sqrt{\epsilon_r}$$

for $0 < x < d$, $Z_a = \eta_o k_{xa} / k_o$

for $d < x < a$, $Z_d = \eta_o k_{xd} / \sqrt{\epsilon_r} k_o$



$$\beta = \sqrt{\epsilon_r k_o^2 - k_{xd}^2 - (n\pi/b)^2} = \sqrt{k_o^2 - k_{xa}^2 - (n\pi/b)^2}$$

Applying (3.215):

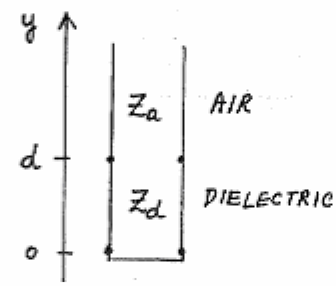
$$Z_a \tan \beta_a d + Z_d \tan \beta_d (a-b) = 0$$

The m-th root of this equation applies to the TM_{mn} mode.

3.26

$$Z_a = k_{ya} \eta_o / k_o = -j h \eta_o / k_o$$

$$Z_d = k_{yd} \eta / k = k_{yd} \eta_o / k_o$$



$$\beta = \sqrt{\mu_r k_o^2 - k_{yd}^2} = \sqrt{k_o^2 - k_{ya}^2} = \sqrt{k_o^2 + h^2}$$

Applying (3.215):

$$Z_a + j Z_d \tan k_{yd} d = 0$$

$$h = k_{yd} \tan k_{yd} = 0$$

This agrees with the solution to Problem 3.17, with $k_c = k_{yd}$. ✓

3.27 For X-band guide, $a = 2.286$ cm.

$$k = \frac{2\pi f\sqrt{\epsilon_r}}{c} = \frac{2\pi(9500)\sqrt{2.08}}{300} = 287.\ \text{m}^{-1} \checkmark$$

$$\beta = \sqrt{k^2 - (\pi/a)^2} = 252.\ \text{m}^{-1} \checkmark$$

speed of light in Teflon $= \frac{c}{\sqrt{\epsilon_r}} = \frac{3\times10^8}{\sqrt{2.08}} = \underline{2.08\times10^8}$ m/sec ✓

phase velocity $= v_p = \frac{\omega}{\beta} = \frac{2\pi(9.5\times10^9)}{252.} = \underline{2.37\times10^8}$ m/sec ✓

From (3.231),

group velocity $= v_g = \left(\frac{d\beta}{d\omega}\right)^{-1} = \left(\frac{d\beta}{dk}\frac{dk}{d\omega}\right)^{-1} = \left(\frac{k}{\beta}\sqrt{\mu\epsilon}\right)^{-1}$

$$= \frac{\beta}{k\sqrt{\mu\epsilon}} = \frac{252(2.08\times10^8)}{287.} = 1.83\times10^8 \text{ m/sec}$$

Note that $v_g < \frac{c}{\sqrt{\epsilon_r}} < v_p$.

3.28

$$P_{MAX} = c\,a^2 \ln\frac{b}{a}$$

$$\frac{dP_{MAX}}{da} = 2a\ln\frac{b}{a} - \frac{a^2}{a} = 0$$

$$2\ln\frac{b}{a} - 1 = 0$$

$$2\ln x = 1$$

$$\ln x = 0.5$$

$$x = 1.65$$

$$Z_0 = \frac{377}{2\pi}\ln\frac{b}{a} = \frac{120\pi}{2\pi}\left(\frac{1}{2}\right) = \underline{30\,\Omega}$$

3.29 alumina, $\epsilon_r = 9.9$, $d = 2.0$ mm, $W = 1.93$ mm
$Z_0 = 50\Omega$. $\epsilon_e = 6.771$

$$f_{T1} = \frac{c}{2\pi d}\sqrt{\frac{2}{\epsilon_r - 1}}\tan^{-1}\epsilon_r = 11.3 \text{ GHz}$$

$$f_{T2} = \frac{c}{4d\sqrt{\epsilon_r - 1}} = 12.5 \text{ GHz}.$$

$$f_{T3} = \frac{c}{\sqrt{\epsilon_r}(2W + d)} = 16.3 \text{ GHz}$$

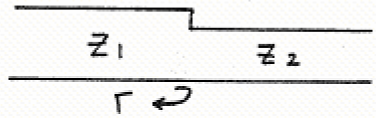
$$f_{T4} = \frac{c}{2d\sqrt{\epsilon_r}} = 23.8 \text{ GHz}.$$

It would be adviseable to keep the operating frequency below 10 GHz for this line.

Chapter 4

4.1 Using a transmission line analogy gives,

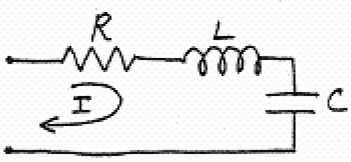
$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$



where $Z_1 = k_o\eta_o/\beta_1$, $Z_2 = k_o\eta_o/\beta_2$.

But $\beta_1 = \beta_2 = \sqrt{k_o^2 - (\pi/a)^2}$ in both regions, since only the height (b) of the guide changes. Thus, $\Gamma = 0$ from above. This is obviously not correct, as E_y should be zero for $b/2 < y < b$. Higher order TE_{1n} modes must be considered, in a mode matching procedure. This will result in a solution where $\Gamma \neq 0$. Consideration of only the dominant mode is not adequate.

4.2



$$P_\ell = \tfrac{1}{2}|I|^2 R \implies R = \frac{P_\ell}{\tfrac{1}{2}|I|^2}$$

$$W_m = \tfrac{1}{4} L |I|^2 \implies L = \frac{2W_m}{\tfrac{1}{2}|I|^2}$$

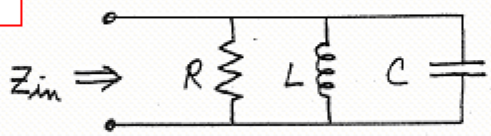
$$W_e = \tfrac{1}{4} C |V_c|^2 = \frac{1}{4\omega^2 C}|I|^2 \implies \frac{1}{\omega^2 C} = \frac{2W_e}{\tfrac{1}{2}|I|^2}$$

The input impedance is,

$$Z_{in} = R + j\left(\omega L + \frac{1}{\omega C}\right) = \frac{P_\ell + 2j\omega(W_m - W_e)}{\tfrac{1}{2}|I|^2} \checkmark$$

In agreement with (4.17)

4.3



$$Z = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} = \frac{1}{\frac{1}{R} + j\omega\left(C - \frac{1}{\omega^2 L}\right)}$$

$$Z(-\omega) = \frac{1}{\frac{1}{R} - j\omega\left(C - \frac{1}{\omega^2 L}\right)} = Z^*(\omega) \quad \checkmark$$

4.4

$$V_1 = 10\angle 90^\circ \qquad I_1 = 0.2\angle 90^\circ \qquad Z_0 = 50\,\Omega$$

$$V_2 = 8\angle 0 \qquad I_2 = 0.16\angle -90^\circ$$

$$V_n^+ = (V_n + Z_0 I_n)/2$$

$$V_n^- = (V_n - Z_0 I_n)/2$$

$$V_1^+ = \tfrac{1}{2}\left[10j + 50(.2j)\right] = 10\angle 90^\circ$$

$$V_1^- = \tfrac{1}{2}\left[10j - 50(.2j)\right] = 0$$

$$V_2^+ = \tfrac{1}{2}\left[8 + 50(-.16j)\right] = 4 - 4j = 5.66\angle -45^\circ$$

$$V_2^- = \tfrac{1}{2}\left[8 - 50(-.16j)\right] = 4 + 4j = 5.66\angle +45^\circ$$

$$Z_{in}^{(1)} = \frac{V_1}{I_1} = \frac{10j}{.2j} = 50\,\Omega$$

$$Z_{in}^{(2)} = \frac{V_2}{I_2} = \frac{8}{-.16j} = 50j = 50\angle 90^\circ\,\Omega$$

4.5

$$P_{in} = \tfrac{1}{2}[V]^t[I]^* = \tfrac{1}{2}[V]^t[Y]^*[V]^*$$
$$= \tfrac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} V_m Y_{mn}^* V_n^*$$

If lossless, $Re\{P_{in}\}=0$. Since the V_n's are independent, we first let all $V_m=0$, except for V_n. Then,

$$P_{in} = \tfrac{1}{2} V_n Y_{nn}^* V_n^* = \tfrac{1}{2}|V_n|^2 Y_{nn}^*$$
$$\therefore \quad Re\{Y_{nn}^*\} = Re\{Y_{nn}\} = 0 \quad \checkmark$$

Now let all port voltages be zero except for V_m and V_n. Then,

$$P_{in} = \tfrac{1}{2} V_m Y_{mn}^* V_n^* + \tfrac{1}{2} V_n Y_{nm}^* V_m^*$$

So,

$$Re\{V_m Y_{mn}^* V_n^* + V_n Y_{nm}^* V_m^*\} = 0$$

If $Y_{mn}=Y_{nm}$ (reciprocal), then

$$Re\{Y_{mn}^*(V_m V_n^* + V_n V_m^*)\} = Re\{Y_{mn}^*[(V_m V_n^* + (V_m V_n^*)^*]\} = 0$$

Since $A+A^*$ is real, we must have $Re\{Y_{mn}\}=0$ ✓

4.6 Let $[Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$, and show that Z_{ij}'s can be found such that $P_{in}=0$, but not all Z_{ij}'s are pure imaginary.

$$P_{in} = \tfrac{1}{2}[I]^t[Z]^t[I]^* = \tfrac{1}{2}(I_1 Z_{11} I_1^* + I_1 Z_{21} I_2^* + I_2 Z_{12} I_1^* + I_2 Z_{22} I_2^*)$$
$$= \tfrac{1}{2}(Z_{11}|I_1|^2 + Z_{22}|I_2|^2 + Z_{21} I_1 I_2^* + Z_{12} I_2 I_1^*)$$

To be lossless, we must have $Re\{Z_{11}\} = Re\{Z_{21}\} = 0$.

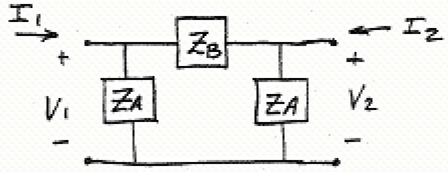
Also, $Re\{Z_{21} I_1 I_2^* + Z_{12} I_2 I_1^*\} = 0$.

This will occur if $Z_{12} = -Z_{21}^*$ (since $Re\{A-A^*\}=0$).

For example, if $Z_{12} = a+jb$, then $Z_{21} = -a+jb$.

Thus, $[Z]$ is not symmetric, and the answer is NO.

4.7



From (4.28),

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0} = \frac{V_1}{V_1\left(\frac{2Z_A+Z_B}{Z_A(Z_A+Z_B)}\right)} = \frac{Z_A(Z_A+Z_B)}{2Z_A+Z_B} \checkmark = Z_{22} \quad \text{(BY SYMMETRY)}$$

$$Z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0} = \frac{I_1 Z_{11}\left(\frac{Z_A}{Z_A+Z_B}\right)}{I_1} = \frac{Z_A^2}{2Z_A+Z_B} \checkmark = Z_{12} \quad \text{(BY RECIPROCITY)}$$

From (4.29),

$$Y_{11} = \frac{I_1}{V_1}\bigg|_{V_2=0} = \frac{I_1}{I_1\left(\frac{Z_A Z_B}{Z_A+Z_B}\right)} = \frac{Z_A+Z_B}{Z_A Z_B} \checkmark = Y_{22} \quad \text{(BY SYMMETRY)}$$

$$Y_{21} = \frac{I_2}{V_1}\bigg|_{V_2=0} = \frac{-V_1/Z_B}{V_1} = \frac{-1}{Z_B} \checkmark = Y_{12} \quad \text{(BY RECIPROCITY)}$$

CHECK: $[Z][Y] = [U]$?

$$Z_{11}Y_{11} + Z_{12}Y_{21} = \frac{(Z_A+Z_B)^2}{Z_B(2Z_A+Z_B)} - \frac{Z_A^2}{Z_B(2Z_A+Z_B)} = \frac{2Z_AZ_B+Z_B^2}{Z_B(2Z_A+Z_B)} = 1 \checkmark$$

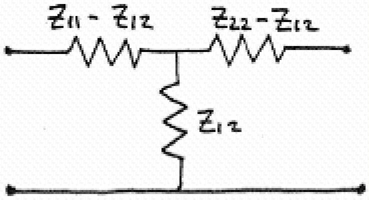
$$Z_{11}Y_{12} + Z_{12}Y_{22} = \frac{-Z_A(Z_A+Z_B)}{Z_B(2Z_A+Z_B)} + \frac{Z_A(Z_A+Z_B)}{Z_B(2Z_A+Z_B)} = 0 \checkmark$$

Similarly for the T-network. The results are,

$$Z_{11} = Z_{22} = \frac{Y_A+Y_B}{Y_AY_B} \checkmark \qquad Z_{12} = Z_{21} = \frac{-1}{Y_B} \checkmark$$

$$Y_{11} = Y_{22} = \frac{Y_A(Y_A+Y_B)}{2Y_A+Y_B} \checkmark \qquad Y_{12} = Y_{21} = \frac{Y_A^2}{2Y_A+Y_B} \checkmark$$

4.8 Model the two-port as below:



Then, $Z_{sc}^{(1)} = Z_{11} - Z_{12} + \dfrac{Z_{12}(Z_{22} - Z_{12})}{Z_{22}} = Z_{11} - Z_{12}^2/Z_{22}$

$Z_{sc}^{(2)} = Z_{22} - Z_{12}^2/Z_{11}$

$Z_{oc}^{(1)} = Z_{11}$ ✓

$Z_{oc}^{(2)} = Z_{22}$ ✓

From the first equation,

$$Z_{12}^2 = -(Z_{sc}^{(1)} - Z_{11})Z_{22} = (Z_{oc}^{(1)} - Z_{sc}^{(1)})Z_{oc}^{(2)}.$$

4.9 From Table 4.1 the ABCD parameters for a transmission line section are,

$A = D = \cos\beta\ell$, $B = jZ_0 \sin\beta\ell$, $C = jY_0 \sin\beta\ell$.

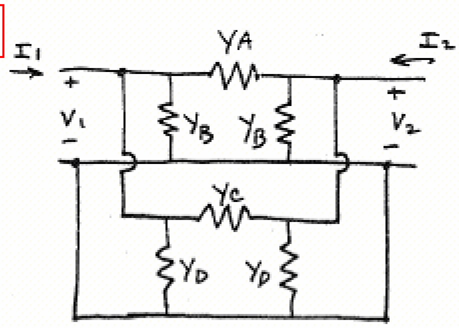
Now use Table 4.2 to convert to Z-parameters:

$Z_{11} = \dfrac{A}{C} = \dfrac{\cos\beta\ell}{jY_0 \sin\beta\ell} = -jZ_0 \cot\beta\ell$ ✓

$Z_{12} = Z_{21} = \dfrac{1}{C} = -jZ_0 \csc\beta\ell$ ✓

$Z_{22} = \dfrac{D}{C} = \dfrac{\cos\beta\ell}{jY_0 \sin\beta\ell} = -jZ_0 \cot\beta\ell$ ✓

4.10



$$[Y_1] = \begin{bmatrix} Y_A+Y_B & -Y_A \\ -Y_A & Y_A+Y_B \end{bmatrix}$$

$$[Y_2] = \begin{bmatrix} Y_C+Y_D & -Y_C \\ -Y_C & Y_C+Y_D \end{bmatrix}$$

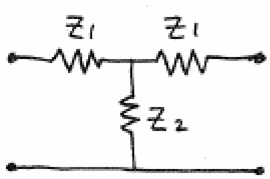
adding [Y] matrices gives:

$$[Y] = [Y_1] + [Y_2] = \begin{bmatrix} Y_A+Y_B+Y_C+Y_D & -Y_A-Y_C \\ -Y_A-Y_C & Y_A+Y_B+Y_C+Y_D \end{bmatrix}$$

By direct calculation, we obtain similar results:

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = Y_A+Y_B+Y_C+Y_D \checkmark \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -(Y_A+Y_C) \checkmark$$

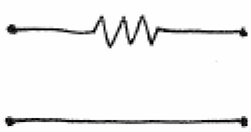
Now apply to bridged-T network (Example 5.7 of 1st edition)



$$[Z_A] = \begin{bmatrix} Z_1+Z_2 & Z_2 \\ Z_2 & Z_1+Z_2 \end{bmatrix}$$

$$[Y_A] = \frac{1}{D}\begin{bmatrix} Z_1+Z_2 & -Z_2 \\ -Z_2 & Z_1+Z_2 \end{bmatrix} \checkmark$$

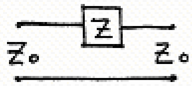
$$D = (Z_1+Z_2)^2 - Z_2^2 = Z_1^2 + 2Z_1Z_2 \checkmark$$



$$[Y_B] = \begin{bmatrix} 1/Z_3 & -1/Z_3 \\ -1/Z_3 & 1/Z_3 \end{bmatrix} \checkmark$$

$$[Y_{TOT}] = [Y_A] + [Y_B] = \begin{bmatrix} \frac{1}{Z_3} + \frac{Z_1+Z_2}{D} & -\left(\frac{1}{Z_3} + \frac{Z_2}{D}\right) \\ -\left(\frac{1}{Z_3} + \frac{Z_2}{D}\right) & \frac{1}{Z_3} + \frac{Z_1+Z_2}{D} \end{bmatrix} \checkmark$$

4.11

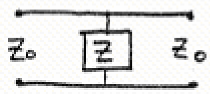


From Table 4.1, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$

convert to [S] using Table 4.2:

$$S_{11} = \frac{1 + Z/Z_0 - 1}{1 + Z/Z_0 + 1} = \frac{Z}{2Z_0 + Z} \quad ; \quad S_{12} = \frac{2}{1 + Z/Z_0 + 1} = \frac{2Z_0}{2Z_0 + Z}$$

$$1 - S_{11} = \frac{2Z_0}{2Z_0 + Z} = S_{12} \checkmark$$



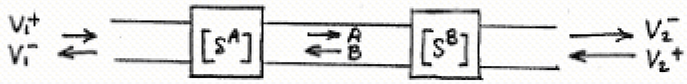
From Table 4.1, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/Z & 1 \end{bmatrix}$

convert to [S]:

$$S_{11} = \frac{1 - Z_0/Z - 1}{1 + Z_0/Z + 1} = \frac{-Z_0}{2Z + Z_0} \quad ; \quad S_{12} = \frac{2}{1 + Z_0/Z + 1} = \frac{2Z}{2Z + Z_0}$$

$$1 + S_{11} = \frac{2Z}{2Z + Z_0} = S_{12} \checkmark$$

4.12 Define wave amplitudes as shown:



Then,

$$\begin{bmatrix} V_1^- \\ A \end{bmatrix} = [S^A] \begin{bmatrix} V_1^+ \\ B \end{bmatrix} \qquad \begin{bmatrix} B \\ V_2^- \end{bmatrix} = [S^B] \begin{bmatrix} A \\ V_2^+ \end{bmatrix} \qquad \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = [S] \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

$S_{21} = \left.\dfrac{V_2^-}{V_1^+}\right|_{V_2^+=0}$. When $V_2^+=0$, we have $B = S_{11}^B A$, $V_2^- = S_{21}^B A$.

Then,

$$A = S_{21}^A V_1^+ + S_{22}^A B = S_{21}^A V_1^+ + S_{22}^A S_{11}^B A$$

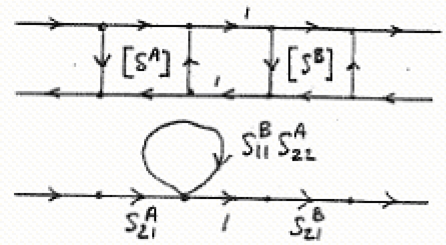
$$\frac{V_2^-}{S_{21}^B} = S_{21}^A V_1^+ + S_{22}^A S_{11}^B \frac{V_2^-}{S_{21}^B}$$

$$V_2^- \left(\frac{1 - S_{22}^A S_{11}^B}{S_{21}^B} \right) = S_{21}^A V_1^+$$

So,

$$S_{21} = \frac{S_{21}^A S_{21}^B}{1 - S_{22}^A S_{11}^B} \checkmark$$

SIGNAL FLOWGRAPH SOLUTION:



$$\therefore\ S_{21} = \frac{S_{21}^A S_{21}^B}{1 - S_{11}^B S_{22}^A} \checkmark$$

4.13

a) $[S] = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix}$ $S_{12} = S_{21}$ since reciprocal

$[S]$ is unitary if lossless, so

1st Row: $|S_{11}|^2 + |S_{21}|^2 = 1$ (or 1st col.)

$|S_{21}|^2 = 1 - |S_{11}|^2$ ✓

b) $[S] = \begin{bmatrix} S_{11} & S_{21} \\ 0 & S_{22} \end{bmatrix}$ $S_{12} \neq S_{21}$ since nonreciprocal

1st Row: $|S_{11}|^2 + |S_{21}|^2 = 1$

1st Col.: $|S_{11}|^2 = 1$

$\therefore |S_{21}| = 0$

4.14

a) to be lossless, [S] must be unitary. From 1st row:

$$|S_{11}|^2+|S_{12}|^2+|S_{13}|^2+|S_{14}|^2=(.178)^2+(.6)^2+(.4)^2=0.552\neq 1$$

so the network is not lossless.

b) The [S] matrix is symmetric, so it is reciprocal.

c) When ports 2, 3, 4 are matched, $\Gamma = S_{11}$.

so $RL = -20\log|\Gamma| = -20\log(.178) = 15.0\text{ dB}$

d) For ports 1 and 3 terminated with Z_0, we have

$V_1^+ = 0,\ V_3^+ = 0$, so $V_4^- = S_{42} V_2^+$

$$IL = -20\log|S_{42}| = -20\log(.3) = 10.5\text{ dB}$$

phase delay $= +45°$

e) For a short at port 3, Z_0 on other ports, we have

$$V_2^+ = V_4^+ = 0$$
$$V_3^+ = -V_3^-$$
$$V_1^- = S_{11}V_1^+ + S_{13}V_3^+ = S_{11}V_1^+ - S_{13}V_3^-$$
$$V_3^- = S_{31}V_1^+$$

Then,

$$\Gamma^{(1)} = \frac{V_1^-}{V_1^+} = S_{11} - S_{13}S_{31} = 0.178j - (.4\angle 45)(.4\angle 45)$$
$$= 0.178j - .16j = 0.018j = 0.018\angle 90°$$

4.15 A matched, reciprocal, 3-port network has an [S] matrix of the following form:

$$[S] = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{bmatrix}$$

If the network is lossless, then [S] must be unitary:

$$|S_{12}|^2 + |S_{13}|^2 = 1 \qquad (1) \qquad\qquad S_{13}S_{23}^* = 0 \qquad (4)$$

$$|S_{12}|^2 + |S_{23}|^2 = 1 \qquad (2) \qquad\qquad S_{12}S_{13}^* = 0 \qquad (5)$$

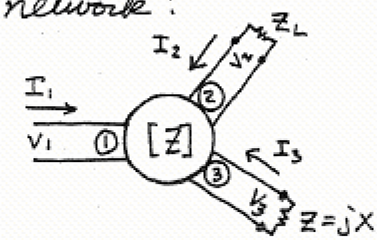
$$|S_{13}|^2 + |S_{23}|^2 = 1 \qquad (3) \qquad\qquad S_{12}S_{23}^* = 0 \qquad (6)$$

To show that a contradiction exists, assume that $S_{12}=0$, in order to satisfy (5) and (6). Then from (1), $|S_{13}|^2=1$, and from (3), $|S_{23}|=0$. But then (2) will be contradicted. Similarly, a contradiction will follow if we let $S_{13}=0$, or $S_{23}=0$.

A circulator is an example of a nonreciprocal, lossless, matched 3-port network.

4.16 For this problem it is easiest to use the z-matrix for a lossless reciprocal 3-port network:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} jX_{11} & jX_{12} & jX_{13} \\ jX_{12} & jX_{22} & jX_{23} \\ jX_{13} & jX_{23} & jX_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$



If we terminate port 3 in a reactance jX,

then $V_3 = -jX I_3$. Then we must find jX so that $V_2 = 0$ for $V_1 \neq 0$. If $V_2 = 0$, then $I_2 = 0$:

$$V_3 = jX_{13} I_1 + jX_{33} I_3 = -jX I_3$$

$$I_3 = \frac{-X_{13} I_1}{X_{33} + X}$$

$$V_2 = jX_{12} I_1 + jX_{23} I_3 = \left(jX_{12} - \frac{jX_{23} X_{13}}{X_{33} + X} \right) I_1 = 0$$

So,

$$X_{12} X_{33} + X X_{12} - X_{13} X_{23} = 0$$

$$X = \frac{X_{13} X_{23} - X_{12} X_{33}}{X_{12}} \checkmark$$

CHECK: The input impedance at Port 1 is,

$$Z_{in}^{(1)} = \frac{V_1}{I_1} = \frac{jX_{11} I_1 + jX_{13} I_3}{I_1} = jX_{11} + jX_{13} \left(\frac{-X_{13}}{X_{33} + X} \right)$$

$$= j \left(X_{11} - \frac{X_{13}^2}{X_{33} + X} \right)$$ which is pure imaginary ✓

4.17

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & S_{13} \\ S_{13} & S_{13} & S_{33} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \end{bmatrix}$$

Assume the network is fed at port 1, so $V_1^+ \neq 0$. Port 2 is terminated in a matched load, so $V_2^+ = 0$. Port 3 is terminated in a reactive load, so $V_3^+ = e^{j\phi} V_3^-$. We must find $e^{j\phi}$ so that $V_1^-/V_1^+ = 0$.

$$V_3^- = S_{13} V_1^+ + S_{33} V_3^+ = e^{-j\phi} V_3^+$$

$$V_3^+ = \frac{S_{13} V_1^+}{e^{-j\phi} - S_{33}}$$

$$V_1^- = S_{11} V_1^+ + S_{13} V_3^+ = S_{11} V_1^+ + \frac{S_{13}^2 V_1^+}{e^{-j\phi} - S_{33}}$$

$$\frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{13}^2}{e^{-j\phi} - S_{33}} = 0 \Rightarrow e^{-j\phi} = S_{33} - \frac{S_{13}^2}{S_{11}} . \checkmark$$

We should also verify that this quantity has unit magnitude:

$$\left| S_{33} - S_{13}^2/S_{11} \right|^2 = \frac{|S_{11}|^2 |S_{33}|^2 - |S_{13}|^4 - S_{11}^* S_{33}^* S_{13}^2 - S_{11} S_{33} S_{13}^{*2}}{|S_{11}|^2}$$

The unitary properties of [S] lead to four equations:

$$|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 = 1 \qquad S_{11} S_{12}^* + S_{12} S_{11}^* + |S_{13}|^2 = 0$$

$$2|S_{13}|^2 + |S_{33}|^2 = 1 \qquad S_{12} S_{13}^* + S_{11} S_{13}^* + S_{13} S_{33}^* = 0$$

Eliminating S_{12} from the two equations on the right yields,

$$-2|S_{11}|^2 - \frac{S_{11} S_{13}^* S_{33}}{S_{13}} - \frac{S_{11}^* S_{13} S_{33}^*}{S_{13}^*} + |S_{13}|^2 = 0$$

or,

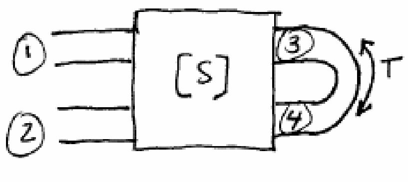
$$-S_{11} S_{13}^{*2} S_{33} - S_{11}^* S_{13}^2 S_{33}^* = 2|S_{11}|^2 |S_{13}|^2 - |S_{13}|^4$$

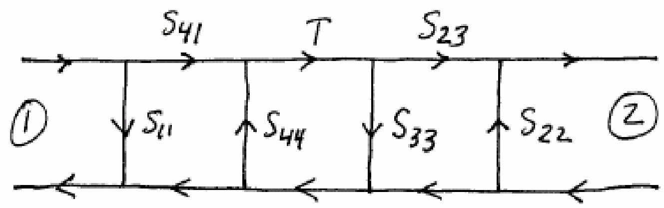
Then,

$$|S_{33} - S_{13}^2/S_{11}|^2 = \frac{|S_{11}|^2 |S_{33}|^2 + \cancel{|S_{13}|^4} + 2|S_{11}|^2 |S_{13}|^2 - \cancel{|S_{13}|^4}}{|S_{11}|^2}$$

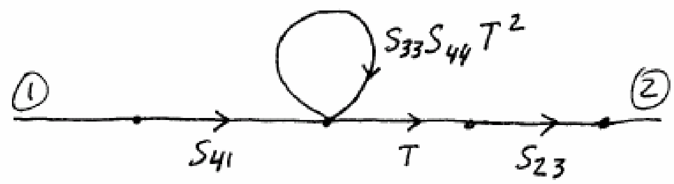
$$= |S_{33}|^2 + 2|S_{13}|^2 = 1 \checkmark$$

4.18 signal flow graph solution:





$T = 1\angle -45^\circ$



$$T_{21} = \frac{S_{41}S_{23}}{1 - S_{33}S_{44}T^2} = \frac{(.4\angle 45)(.7\angle -45)}{1 - (.6\angle 45)(.5\angle 45)(1\angle -90)} = 0.4\angle -90^\circ$$

$IL = -20\log(.4) = 7.96\ dB$

delay $= +90^\circ$.

4.19 From (4.62),

$$S'_{ij} = \frac{\sqrt{Z_{0j}}}{\sqrt{Z_{0i}}} S_{ij}$$

So,

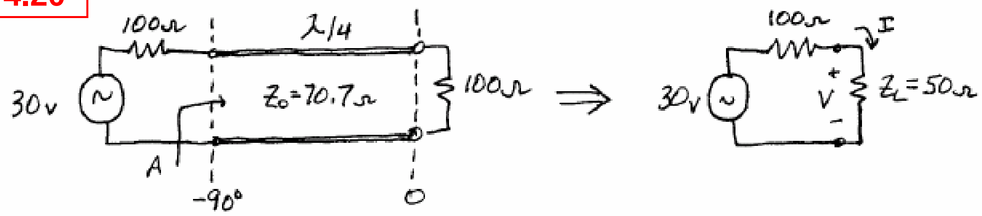
$S'_{11} = S_{11}$ ✓

$S'_{12} = \sqrt{\frac{Z_{02}}{Z_{01}}}\, S_{12}$ ✓

$S'_{21} = \sqrt{\frac{Z_{01}}{Z_{02}}}\, S_{21}$ ✓

$S'_{22} = S_{22}$ ✓

4.20



$Z_{in}(\text{at } A) = (70.7)^2/100 = 50\Omega$

with reference at A: Let $Z_R = Z_L^* = 50\Omega$. Then $\Gamma_p = 0$ (not conj. mat.)

$$V = 30\,\frac{50}{150} = 10\text{ v}$$

$$I = \frac{30}{150} = 0.2\text{ A}$$

$$a = \frac{1}{2\sqrt{R_R}}(V + Z_R I) = \frac{1}{2\sqrt{50}}(10+10) = 1.414$$

$$P_L = \tfrac{1}{2}|a|^2 = 1\text{ W} \checkmark$$

with reference at B: Let $Z_R = Z_L^* = 100\Omega$. Then $\Gamma_p = 0$

$$\Gamma = \frac{100 - Z_0}{100 + Z_0} = 0.1716$$

$$V(z) = V_0^+(e^{-j\beta z} + \Gamma e^{+j\beta z})$$

$$V(-90°) = 10 = V_0^+(j - .1716j)$$

$$V_0^+ = -j\,12.07$$

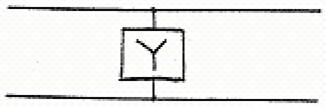
$$V(0) = V_0^+(1+\Gamma) = -j\,14.14$$

$$I(0) = -j\,.1414$$

$$a = \frac{1}{2\sqrt{100}}(-j14.14 - j14.14) = -j\,1.414$$

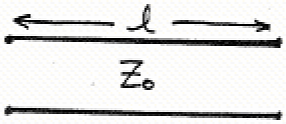
$$P_L = \tfrac{1}{2}|a|^2 = 1\text{ W} \checkmark$$

4.21



$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 1$ ✓ $\quad C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = Y$ ✓

$B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = 0$ ✓ $\quad D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = 1$ ✓



for $I_2=0$, $\quad V_1 = V^+(e^{j\beta\ell} + e^{-j\beta\ell}) = V^+ 2\cos\beta\ell$

$V_2 = 2V^+ = V_1/\cos\beta\ell$

So,

$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \cos\beta\ell$ ✓

$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{V_1}{Z_{in}V_2} = \frac{\cos\beta\ell}{-jZ_0\cot\beta\ell} = jY_0 \sin\beta\ell$ ✓

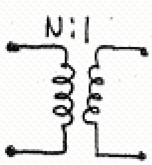
for $V_2=0$, $\quad V_1 = V^+(e^{j\beta\ell} - e^{-j\beta\ell}) = V^+ 2j\sin\beta\ell$

$I_2 = \frac{2V^+}{Z_0}$

So,

$B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = jZ_0\sin\beta\ell$ ✓

$D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{V_1}{Z_{in}I_2} = \frac{B}{Z_{in}} = \frac{jZ_0\sin\beta\ell}{jZ_0\tan\beta\ell} = \cos\beta\ell$ ✓



$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{NV_2}{V_2} = N$ ✓

$B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = 0$ ✓

$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = 0$ ✓

$D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{1}{N}$ ✓

4.22 NOTE: Difference in signs for Z and ABCD.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \left.\frac{V_1}{V_2}\frac{V_2}{I_1}\right|_{I_2=0} = A/C \checkmark$$

for $I_1 = 0$,
$$V_1 = AV_2 - BI_2$$
$$0 = CV_2 - DI_2 \Rightarrow \quad V_2 = DI_2/C$$
$$V_1 = \left(\frac{AD}{C} - B\right) I_2$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{AD-BC}{C} \checkmark$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 1/C \checkmark$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = D/C \checkmark$$

4.23

DIRECT CALCULATION:

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{V_1}{V_1 \frac{1/Y}{Z+1/Y}} = 1 + YZ$$

$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = \frac{V_1}{V_1/Z} = Z$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{I_1}{I_1/Y} = Y$$

$$D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = 1$$

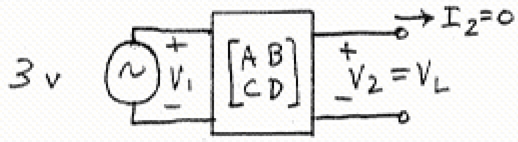
CHECK: $AD - BC = 1 + YZ - ZY = 1 \checkmark$

CALCULATION USING CASCADE: (From Table 4.1)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} = \begin{bmatrix} 1+ZY & Z \\ Y & 1 \end{bmatrix} \checkmark$$

4.24 Using Table 4.1, the ABCD matrix of the cascade of four components (including load) is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 50 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & j50 \\ j/50 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/25 & 1 \end{bmatrix} = \begin{bmatrix} 3j & 25j \\ j/25 & 0 \end{bmatrix}$$

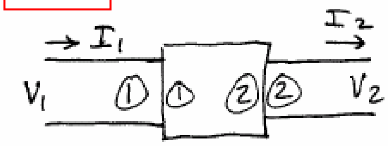


$$V_1 = AV_2 + BI_2 = AV_2 = AV_L$$

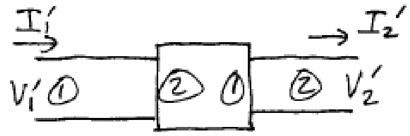
$$V_L = \frac{V_1}{A} = \frac{3}{3j} = 1\angle{-90^\circ} \;\checkmark$$

(verified with Serenade)

4.25



$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$



$\begin{bmatrix} D & B \\ C & A \end{bmatrix}$

$$I_1' = -I_2 \,, \quad I_2' = -I_1$$
$$V_1' = V_2 \,, \quad V_2' = V_1$$

$$\begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1' \\ -I_1' \end{bmatrix}$$

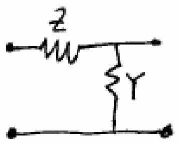
inverting:
$$\begin{bmatrix} V_1' \\ -I_1' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix}$$

since $AD - BC = 1$ if reciprocal.

Rewriting:
$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} D & B \\ C & A \end{bmatrix} \begin{bmatrix} V_2' \\ I_2' \end{bmatrix}$$

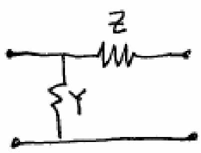
Example:

<u>original</u>



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} = \begin{bmatrix} 1+ZY & Z \\ Y & 1 \end{bmatrix}$$

<u>reversed</u>



$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & Z \\ Y & 1+ZY \end{bmatrix}$$

so $A' = D, \; B' = B, \; C' = C, \; D' = A$ ✓

4.26

$$V_1 = AV_2 - BI_2 \qquad V_n = V_n^+ + V_n^-$$

$$I_1 = CV_2 - DI_2 \qquad I_n = (V_n^+ - V_n^-)/Z_0$$

So,

$$V_1^+ + V_1^- = A(V_2^+ + V_2^-) - B(V_2^+ - V_2^-)/Z_0$$

$$V_1^+ - V_1^- = C(V_2^+ + V_2^-)Z_0 - D(V_2^+ - V_2^-)$$

For $V_2^+ = 0$,

$$V_1^+ + V_1^- = (A + B/Z_0)V_2^-$$

$$V_1^+ - V_1^- = (CZ_0 + D)V_2^-$$

eliminate V_2^+:

$$V_1^+ + V_1^- = \frac{A + B/Z_0}{CZ_0 + D}(V_1^+ - V_1^-)$$

$$V_1^-(CZ_0 + D + A + B/Z_0) = V_1^+(A + B/Z_0 - CZ_0 - D)$$

$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \frac{A + B/Z_0 - CZ_0 - D}{A + B/Z_0 + CZ_0 + D} \quad \checkmark$$

eliminate V_1^-:

$$2V_1^+ = (A + B/Z_0 + CZ_0 + D)V_2^-$$

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{2}{A + B/Z_0 + CZ_0 + D} \quad \checkmark$$

for $V_1^+ = 0$ the above set reduces to,

$$V_1^- = (A - B/Z_0)V_2^+ + (A + B/Z_0)V_2^-$$
$$-V_1^- = (CZ_0 - D)V_2^+ + (CZ_0 + D)V_2^-$$

eliminate V_1^-:

$$(A - B/Z_0 + CZ_0 - D)V_2^+ + (A + B/Z_0 + CZ_0 + D)V_2^- = 0$$

$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_1^+=0} = \frac{-A + B/Z_0 - CZ_0 + D}{A + B/Z_0 + CZ_0 + D} \quad \checkmark$$

eliminate V_2^-:

$$\frac{V_1^-}{A + B/Z_0} - \frac{A - B/Z_0}{A + B/Z_0}V_2^+ = \frac{-V_1^-}{CZ_0 + D} - \frac{CZ_0 - D}{CZ_0 + D}V_2^+$$

$$V_1^-\left(\frac{1}{A + B/Z_0} + \frac{1}{CZ_0 + D}\right) = V_2^+\left(\frac{A - B/Z_0}{A + B/Z_0} - \frac{CZ_0 - D}{CZ_0 + D}\right)$$

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0} = \frac{\dfrac{A - B/Z_0}{A + B/Z_0} - \dfrac{CZ_0 - D}{CZ_0 + D}}{\dfrac{1}{A + B/Z_0} + \dfrac{1}{CZ_0 + D}} = \frac{2(AD - BC)}{A + B/Z_0 + CZ_0 + D} \quad \checkmark$$

These results agree with Table 4.2.

4.27

a) Using the S-parameters, the transmission coefficient from Port 1 to Port 4 is,

$$T = \frac{V_4^-}{V_1^+} = \frac{1}{V_1^+}\left(\frac{-1}{\sqrt{2}}\right)(V_2^+ + jV_3^+) = \frac{1}{V_1^+}\left(\frac{-1}{\sqrt{2}}\right)(\Gamma V_2^- + j\Gamma V_3^-)$$

$$= \frac{1}{V_1^+}\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)(\Gamma)(j+j)V_1^+ = j\Gamma \checkmark$$

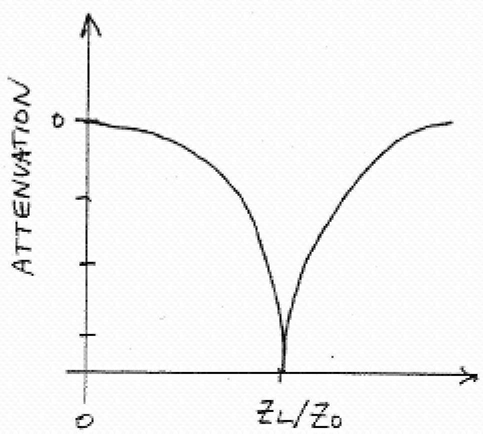
$$\text{attenuation} = |T| = |\Gamma| = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|$$

at port 1 the reflected wave is,

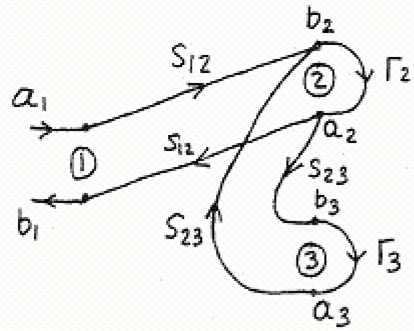
$$V_1^- = \frac{-1}{\sqrt{2}}(jV_2^+ + V_3^+) = \frac{-1}{\sqrt{2}}\Gamma(jV_2^- + V_3^-) = \frac{1}{2}\Gamma(-1+1)V_1^+ = 0 \checkmark$$

b)

Z_L/Z_0	atten. (dB)
0	0
0.172	3
1	∞
5.83	3
∞	0

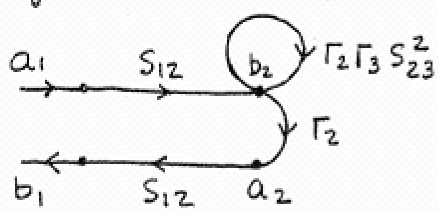


4.28 The signal flowgraph is as follows:



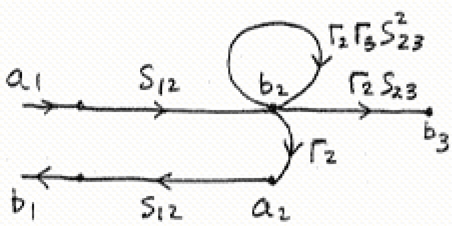
Let $\Gamma_{in} = \dfrac{b_1}{a_1}$

Using the reduction rules:



$$b_2 = a_1 \frac{S_{12}}{1-\Gamma_2\Gamma_3 S_{23}^2} \quad \checkmark$$

$$b_1 = a_1 \frac{S_{12}^2 \Gamma_2}{1-\Gamma_2\Gamma_3 S_{23}^2} \quad \checkmark$$



$$b_3 = b_2 \, \Gamma_2 S_{23}$$

Then,

$$\frac{P_2}{P_1} = \frac{b_2^2 - a_2^2}{a_1^2 - b_1^2} = \frac{b_2^2(1-|\Gamma_2|^2)}{a_1^2(1-|\Gamma_{in}|^2)} = \frac{|S_{12}|^2(1-|\Gamma_2|^2)}{|1-\Gamma_2\Gamma_3 S_{23}^2|^2\left(1-\dfrac{|S_{12}^2|^2|\Gamma_2|^2}{|1-\Gamma_2\Gamma_3 S_{23}^2|^2}\right)}$$

$$= \frac{|S_{12}|^2(1-|\Gamma_2|^2)}{|1-\Gamma_2\Gamma_3 S_{23}^2|^2 - |S_{12}^2\Gamma_2|^2} \quad \checkmark$$

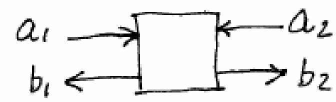
$$\frac{P_3}{P_1} = \frac{b_3^2 - a_3^2}{a_1^2 - b_1^2} = \frac{b_3^2(1-|\Gamma_3|^2)}{a_1^2(1-|\Gamma_{in}|^2)} = \frac{|S_{12}|^2|\Gamma_2 S_{23}|^2(1-|\Gamma_3|^2)}{|1-\Gamma_2\Gamma_3 S_{23}^2|^2\left(1-\dfrac{|S_{12}^2\Gamma_2|^2}{|1-\Gamma_2\Gamma_3 S_{23}^2|^2}\right)}$$

$$= \frac{|S_{12}|^2|S_{23}|^2|\Gamma_2|^2(1-|\Gamma_3|^2)}{|1-\Gamma_2\Gamma_3 S_{23}^2|^2 - |S_{12}^2\Gamma_2|^2} \quad \checkmark$$

(verified by direct calculation using S-parameters)

4.29

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$



$$\left.\begin{aligned} a_1 &= T_{11} b_2 + T_{12} a_2 \\ b_1 &= T_{21} b_2 + T_{22} a_2 \end{aligned}\right\} \text{ T-parameters}$$

$$\left.\begin{aligned} b_1 &= S_{11} a_1 + S_{12} a_2 \\ b_2 &= S_{21} a_1 + S_{22} a_2 \end{aligned}\right\} \text{ S-parameters}$$

$$T_{11} = \left.\frac{a_1}{b_2}\right|_{a_2=0} = 1/S_{21} \quad ✓$$

$$T_{12} = \left.\frac{a_1}{a_2}\right|_{b_2=0} = -S_{22}/S_{21} \quad ✓$$

$$T_{21} = \left.\frac{b_1}{b_2}\right|_{a_2=0} = S_{11}/S_{21} \quad ✓$$

$$T_{22} = \left.\frac{b_1}{a_2}\right|_{b_2=0} = S_{11}\frac{a_1}{a_2} + S_{12} = S_{11}\left(\frac{-S_{22}}{S_{21}}\right) + S_{12} = S_{12} - S_{11}S_{22}/S_{21}$$

$$= \frac{S_{12}S_{21} - S_{11}S_{22}}{S_{21}} \quad ✓$$

4.30

$$Z_{oc} = -jZ_0 \cot \beta\Delta \simeq \frac{-jZ_0}{\beta\Delta} = \frac{-jZ_0 c}{\omega\sqrt{\epsilon_e}\,\Delta} = \frac{-j}{\omega C_f}$$

$$\therefore\ \Delta \simeq \frac{Z_0 c C_f}{\sqrt{\epsilon_e}}$$ (agrees with T. Edwards, p. 123)

For $C_f = 0.075$ pF, $\epsilon_e = 1.894$, $Z_0 = 50\ \Omega$,

this gives $\underline{\Delta = 0.082 \text{ cm}}$

(Using $\epsilon_r = 2.2$, $d = 0.158$ cm, $W = 0.487$ cm)

The Hammerstad & Bekkadal approximation gives

$$\Delta = 0.412d\left(\frac{\epsilon_e + .3}{\epsilon_e - .258}\right)\left(\frac{W + .262d}{W + .813d}\right) = \underline{0.075 \text{ cm}}$$

4.31 The complex reflected power can be computed using (4.88):

$$P_r = \int_s \bar{E}^r \times \bar{H}^{r*} \cdot \hat{z}\, ds = -\int_{x=0}^{a}\int_{y=0}^{b} E_y^r H_x^{r*}\, dx\, dy$$

$$= -b\int_{x=0}^{a}\left[\sum_n A_n \sin\frac{n\pi x}{a} e^{j\beta_n^a z}\right]\left[\sum_m \frac{A_m^*}{Z_m^{a*}} \sin\frac{m\pi x}{a} e^{-j\beta_m^{a*} z}\right]dx$$

$$= \frac{-ab}{2}\sum_{n=1}^{\infty}\frac{|A_n|^2}{Z_n^{a*}} e^{j(\beta_n^a - \beta_n^{a*})z}$$

The only propagating mode is the $n=1$ (TE_{10}) mode, so β_1^a is real, and β_n^a is imaginary for $n>1$. Let $\alpha_n = j\beta_n = \sqrt{(n\pi/a)^2 - k_0^2}$ for $n>1$. Then $Z_1^a = k_0\eta_0/\beta_1^a$, and $Z_n^a = k_0\eta_0/\beta_n^a = jk_0\eta_0/\alpha_n$ for $n>1$.

Then $$P_r = \frac{-ab}{2}\left[\frac{|A_1|^2\beta_1^a}{k_0\eta_0} - j\sum_{n=2}^{\infty}\frac{|A_n|^2\alpha_n}{k_0\eta_0} e^{2\alpha_n z}\right] \text{ for } z<0.$$

So we see that $\text{Im}\{P_r\} > 0$, indicating an inductive load.

4.32 This solution is essentially the same as the analysis in Section 4.6. Let $d = (a-c)/2$

$$E_y^i = \sin\frac{\pi x}{a}\, e^{-j\beta_1^a z} \qquad H_x^i = \frac{-1}{Z_1^a}\sin\frac{\pi x}{a}\, e^{-j\beta_1^a z}$$

$$E_y^r = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} A_n \sin\frac{n\pi x}{a}\, e^{j\beta_n^a z} \qquad H_x^r = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{A_n}{Z_n^a}\sin\frac{n\pi x}{a}\, e^{j\beta_n^a z}$$

$$E_y^t = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} B_n \sin\frac{n\pi}{c}(x-d)\, e^{-j\beta_n^c z} \qquad H_x^t = -\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{B_n}{Z_n^c}\sin\frac{n\pi}{c}(x-d)\, e^{-j\beta_n^c z}$$

where $\beta_n^a = \sqrt{k_0^2 - (n\pi/a)^2}$, $\beta_n^c = \sqrt{k_0^2 - (n\pi/c)^2}$

The solution has the same form as (4.97):

$$\frac{a}{2}A_m + \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{k=1\\ \text{odd}}}^{\infty} \frac{2 Z_k^c I_{km} I_{kn}}{c\, Z_n^a} A_n = \sum_{\substack{k=1\\ \text{odd}}}^{\infty} \frac{2 Z_k^c I_{km} I_{k1}}{c\, Z_1^a} - \frac{a}{2}\delta_{m1}$$

for $m = 1, 3, 5, \ldots$,

and,
$$I_{mn} = \int_{x=d}^{d+c} \sin\frac{m\pi}{c}(x-d)\sin\frac{n\pi x}{a}\, dx$$

$$\delta_{mn} = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

4.33 From (4.110) the source current is,

$$\bar{J}_s = \hat{x}\,\frac{2B^+ m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \hat{y}\,\frac{2B^+ n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

From Table 3.2, the transverse fields for ± traveling TM_{mn} modes are,

$$E_x = \frac{\mp j\beta m\pi}{k_c^2 a} C^{\pm}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{\mp j\beta z}$$

$$E_y = \frac{\mp j\beta n\pi}{k_c^2 b} C^{\pm}\sin\frac{m\pi x}{a}\cos\frac{m\pi y}{b}e^{\mp j\beta z}$$

$$H_x = \frac{j\omega\epsilon n\pi}{k_c^2 b} C^{\pm}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{\mp j\beta z}$$

$$H_y = \frac{-j\omega\epsilon m\pi}{k_c^2 a} C^{\pm}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{\mp j\beta z}$$

where $C^{\pm}$ are the unknown amplitudes. At $z=0$, E_t is continuous, so $C^+ = -C^-$. Also, $\hat{z}\times(\bar{H}^+ - \bar{H}^-) = \bar{J}_s$,

or $H_y^+ - H_y^- = J_{sx}$ and $-H_x^+ + H_x^- = J_{sy}$. So,

$$J_{sx}: \quad \frac{-j\omega\epsilon m\pi}{k_c^2 a}(C^+ - C^-) = 2B^+\frac{m\pi}{a} \Rightarrow C^+ = -C^- = \frac{k_c^2 B^+}{-j\omega\epsilon}$$

$$J_{sy}: \quad \frac{j\omega\epsilon n\pi}{k_c^2 b}(-C^+ + C^-) = 2B^+\frac{n\pi}{a} \Rightarrow C^+ = -C^- = \frac{k_c^2 B^+}{-j\omega\epsilon} \quad \checkmark$$

Since these fields satisfy Maxwell's equations and the boundary conditions, they must form the unique solution.

4.34 Following Example 4.8:

$$\bar{J}(x,y,z) = I(y)\,\delta(x-a/2)\,\delta(z) \quad \text{for } 0<y<d.$$

$$\bar{e}_1 = \hat{y}\sin\frac{\pi x}{a}, \quad \bar{h}_1 = \frac{-\hat{x}}{Z_1}\sin\frac{\pi x}{a}, \quad Z_1 = k_0\eta_0/\beta,$$

From (4.119),

$$P_1 = \frac{ab}{Z_1}$$

From (4.118),

$$A_1^+ = \frac{-1}{P_1}\int_V \sin\frac{\pi x}{a}\, e^{j\beta_1 z}\, I(y)\,\delta(x-a/2)\,\delta(z)\,dx\,dy\,dz$$

$$= \frac{-I_0}{P_1}\int_{y=0}^{d}\frac{\sin k(d-y)}{\sin kd}\,dy = \frac{-I_0}{P_1\sin kd}\int_0^d \sin kw\,dw$$

(let $w = d-y$)

$$= \frac{I_0 Z_1(\cos kd - 1)}{kab\sin kd}$$

The total power flow in the TE_{10} mode is,

$$P = \frac{ab\,|A_1^+|^2}{2Z_1},$$

for both + and − traveling waves, since $|A_1^+| = |A_1^-|$.
Then the radiation resistance is,

$$R_{in} = \frac{2P}{I_0^2} = \frac{ab|A_1^+|^2}{I_0^2 Z_1} = \frac{Z_1}{ab}\,\frac{(1-\cos kd)^2}{k^2\sin^2 kd}.$$

$$= \frac{Z_1}{k^2ab}\,\frac{\left(2\sin^2\frac{kd}{2}\right)^2}{4\sin^2\frac{kd}{2}\cos^2\frac{kd}{2}} = \frac{Z_1}{k^2ab}\tan^2\frac{kd}{2} \quad ✓$$

4.35 Following Example 4.8:

$$\bar{J}(x,y,z) = I\,\delta(z)\left[\delta(x-a/4) - \delta(x-3a/4)\right]\hat{y} \quad \text{for } 0<y<b$$

From Table 3.2,

$$TE_{10}: \quad \bar{e}_1 = \hat{y}\sin\frac{\pi x}{a} \qquad \bar{h}_1 = \frac{-\hat{x}}{Z_1}\sin\frac{\pi x}{a} \qquad Z_1 = k_0\eta_0/\beta_1$$

$$TE_{20}: \quad \bar{e}_2 = \hat{y}\sin\frac{2\pi x}{a} \qquad \bar{h}_2 = \frac{-\hat{x}}{Z_2}\sin\frac{2\pi x}{a} \qquad Z_2 = k_0\eta_0/\beta_2$$

$$P_1 = ab/Z_1 \qquad \beta_1 = \sqrt{k_0^2-(\pi/a)^2}$$

$$P_2 = ab/Z_2 \qquad \beta_2 = \sqrt{k_0^2-(2\pi/a)^2}$$

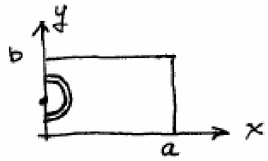
From (4.118):

$$A_1^+ = \frac{-1}{P_1}\int_v \bar{E}_1^-\cdot\bar{J}\,dv = -\frac{Ib}{P_1}\left(\sin\frac{\pi}{4} - \sin\frac{3\pi}{4}\right) = 0 \quad ✓$$

$$A_2^+ = \frac{-1}{P_2}\int_v \bar{E}_2^-\cdot\bar{J}\,dv = \frac{-Ib}{P_2}\left(\sin\frac{\pi}{2} - \sin\frac{3\pi}{2}\right) = \frac{-2Ib}{a}$$

Since the excitation has an odd symmetry about the center of the guide, it will only excite modes that have an electric field with an odd symmetry about $x=a/2$. This implies the TE_{m0} modes, for m even, will be excited. The TE_{10} mode is not excited.

4.36 By image theory, the half-loop on the side wall can be replaced with a full loop without the wall. For a small loop, the equivalent magnetic dipole moment is,



$$\bar{P}_m = \hat{z}\, I_o \pi r_o^2\, \delta(x)\,\delta(y-b/2)\,\delta(z)$$

$$\bar{M} = j\omega\mu_o \bar{P}_m$$

$$= \hat{z}\, j\omega\mu_o I_o \pi r_o^2\, \delta(x)\delta(y-b/2)\,\delta(z) \quad V/m^2$$

For the TE_{10} mode,

$$\bar{e}_1 = \hat{y} \sin \pi x/a$$

$$\bar{h}_1 = \frac{-\hat{x}}{Z_1} \sin \pi x/a$$

$$h_{z1} = \frac{j\pi}{k_o \eta_o a} \cos \pi x/a$$

where $Z_1 = k_o\eta_o/\beta_1$, $P_1 = ab/Z_1$

From (4.128),

$$A_1^+ = \frac{1}{P_1} \int_v (-\bar{h}_1 + \hat{z}\, h_{z1}) \cdot \bar{M}\, e^{j\beta_1 z}\, dv$$

$$= \frac{Z_1}{ab} \int_v h_{z1} M\, dv = \frac{-\pi^2 Z_1 I_o r_o^2}{a^2 b} = A_1^-$$

These results are for a <u>full</u> loop - reduce by $\frac{1}{2}$ for half-loop.

4.37

FIRST SOLUTION: (all fields and currents are TE_{10})

$$E_y = B \sin\frac{\pi x}{a}\left[e^{-j\beta z} - e^{j\beta z}\right] = -2jB\sin\frac{\pi x}{a}\sin\beta z \qquad 0<z<d$$

$$H_x = \frac{B}{Z_1}\sin\frac{\pi x}{a}\left[-e^{-j\beta z} - e^{j\beta z}\right] = \frac{-2B}{Z_1}\sin\frac{\pi x}{a}\cos\beta z \qquad 0<z<d$$

This satisfies $E_y=0$ at $z=0$.

$$E_y = C\sin\frac{\pi x}{a}e^{-j\beta(z-d)} \qquad z>d$$

$$H_x = \frac{C}{Z_1}\sin\frac{\pi x}{a}e^{-j\beta(z-d)} \qquad z>d$$

at $z=d$, E_y is continuous, so

$$-2jB\sin\beta d = C$$

at $z=d$, $\hat{z}\times(\bar{H}^+ - \bar{H}^-) = \bar{J}_s$, or

$$\frac{C}{Z_1} + \frac{2B}{Z_1}\cos\beta d = \frac{2\pi A}{a}$$

Solving for B, C:

$$B = \frac{\pi Z_1 A}{a}e^{-j\beta d}, \qquad C = \frac{\pi Z_1 A}{a}\left(e^{-2j\beta d} - 1\right)$$

SECOND SOLUTION: (using (4.105) and (4.106b)):

E_y due to J_{sy} at $z=d$:

$$E_y^{\pm} = \frac{-\pi Z_1 A}{a}\sin\frac{\pi x}{a}e^{\mp j\beta(z-d)}$$

E_y due to $-J_{sy}$ at $z=-d$:

$$E_y^{\pm} = \frac{\pi Z_1 A}{a}\sin\frac{\pi x}{a}e^{\mp j\beta(z+d)}$$

For $0<z<d$,

$$E_y = \frac{\pi Z_1 A}{a}\sin\frac{\pi x}{a}\left[e^{-j\beta(z+d)} - e^{j\beta(z-d)}\right] = \frac{-2j\pi Z_1 A}{a}e^{-j\beta d}\sin\frac{\pi x}{a}\sin\beta z \ \checkmark$$

For $z>d$,

$$E_y = \frac{\pi Z_1 A}{a}\sin\frac{\pi x}{a}\left[e^{-j\beta(z+d)} - e^{-j\beta(z-d)}\right] = \frac{-2j\pi Z_1 A}{a}\sin\beta d\sin\frac{\pi x}{a}e^{-j\beta z} \ \checkmark$$

These results agree with those from the first solution.

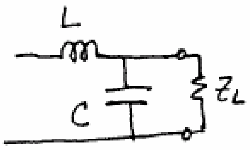
Chapter 5

5.1

a) $Z_L = 150 - j200\,\Omega$

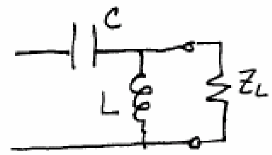
$z_L = 1.5 - j2$ inside $1+jx$ circle

\#1 $b_1 = 0.107 \Rightarrow C = \frac{b}{2\pi f Z_0} = 0.0568\,pF$ ✓



$x_1 = 1.78 \Rightarrow L = \frac{x Z_0}{2\pi f} = 9.44\,nH$ ✓

\#2 $b_2 = -.747 \Rightarrow L = \frac{-Z_0}{2\pi f b} = 7.10\,nH$ ✓

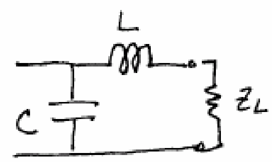


$x_2 = -1.78 \Rightarrow C = \frac{-1}{2\pi f x Z_0} = 0.298\,pF$ ✓

b) $Z_L = 20 - j90\,\Omega$

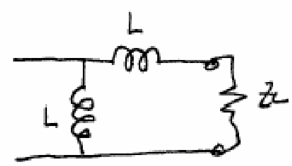
$z_L = 0.2 - j.9$ outside $1+jx$ circle

\#1 $x_1 = 1.30 \Rightarrow L = \frac{x Z_0}{2\pi f} = 6.90\,nH$ ✓



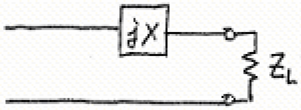
$b_1 = 2.00 \Rightarrow C = \frac{b}{2\pi f Z_0} = 1.06\,pF$ ✓

\#2 $x_2 = 0.5 \Rightarrow L = \frac{-Z_0}{2\pi f} = 2.65\,nH$ ✓

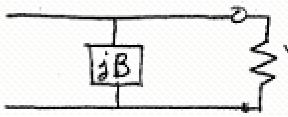


$b_2 = -2.00 \Rightarrow L = \frac{-Z_0}{2\pi f b} = 2.65\,nH$ ✓

verified w/ PCAAD 7.0

5.2 a) 

Z_L matching is possible if $Z_L = Z_0 - jX$ ✓

b) 

Y_L matching is possible if $Y_L = \frac{1}{Z_0} - jB$ ✓

5.3 Analytical Solutions:

From (5.9), $t = \dfrac{80 \pm \sqrt{100[(75-100)^2 + (80)^2]/75}}{100-75} = 3.2 \pm 3.871$

$t_1 = 7.071 \quad , \quad t_2 = -0.671$

From (5.10) the possible stub positions are,

$d_1 = \frac{\lambda}{2\pi} \tan^{-1} t_1 = 0.2276\lambda$ ✓

$d_2 = \frac{\lambda}{2\pi}(\pi + \tan^{-1} t_2) = 0.4059\lambda$ ✓

From (5.8b) the required stub susceptances are,

$$B = \frac{R_L^2 t - (Z_0 - X_L t)(X_L + Z_0 t)}{Z_0[R_L^2 + (X_L + Z_0 t)^2]}$$

$B_1 = 0.0129 \quad , \quad B_2 = -0.0129$

From (5.11a) the o.c. stub lengths are,

$\ell_1 = \frac{-\lambda}{2\pi} \tan^{-1}(B_1 Z_0) = 0.3776\lambda$ ✓ ($\lambda/2$ added to get $\ell_1 > 0$)

$\ell_2 = \frac{-\lambda}{2\pi} \tan^{-1}(B_2 Z_0) = 0.1224\lambda$ ✓

5.4

Use B_1, B_2 from Problem 5.3 with (5.11b):

$$\ell_1 = \frac{\lambda}{2\pi} \tan^{-1} \frac{1}{Z_0 B_1} = 0.1276\lambda \checkmark$$

$$\ell_2 = \frac{\lambda}{2\pi} \tan^{-1} \frac{1}{Z_0 B_2} = 0.3724\lambda \checkmark \quad (\lambda/2 \text{ added to get } \ell_2 > 0)$$

5.5

Smith chart solutions:

The normalized load impedance is $z_L = 1.2 + j0.8$

The stub positions and required reactances are,

$$d_1 = 0.346 - 0.172 = 0.174\lambda \checkmark, \quad x_1 = +j0.753$$

$$d_2 = (.5 - .172) + 0.153 = 0.481\lambda \checkmark, \quad x_2 = -j0.753$$

open ckt stub lengths are,

$$\ell_1 = .25 + .103 = 0.353\lambda \checkmark$$

$$\ell_2 = .397 - .25 = 0.147\lambda \checkmark$$

verified w/ PCAAD 7.0

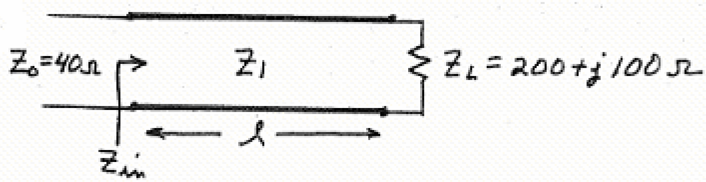
5.6

The required stub lengths for s.c. stubs are $\lambda/4$ longer (or shorter) than the o.c. stub lengths:

$$\ell_1 = .353 - .25 = 0.103\lambda \checkmark$$

$$\ell_2 = .147 + .25 = 0.397\lambda \checkmark$$

5.7



To match this load, we must find Z_1 and ℓ so that $Z_{in} = Z_0 = 40\,\Omega$:

$$Z_{in} = 40 = Z_1 \frac{(200+j100) + jZ_1 t}{Z_1 + j(200+j100)t} \quad , \text{ with } t = \tan\beta\ell.$$

$$(40Z_1 - 4000t) + j\,8000t = 200Z_1 + j(100 + Z_1 t)Z_1$$

Equating real and imaginary parts gives two equations for the two unknowns, Z_1 and t: (if they exist!)

$$\text{Re}: \quad 40Z_1 - 4000t = 200Z_1 \Rightarrow Z_1 = -25t$$

$$\text{Im}: \quad 8000t = Z_1(100 + Z_1 t)$$

$$8000t = -25t(100 - 25t^2)$$

$$t = \pm\sqrt{16.8} = \pm 4.10 \qquad (\text{use } -4.10 \text{ so that } Z_1 > 0) \checkmark$$

Then, $\beta\ell = \tan^{-1}(-4.10) = -76.3^\circ \equiv 104^\circ \Rightarrow \underline{\ell = 0.288\lambda}$

The characteristic impedance is then,

$$Z_1 = -25(-4.10) = 102.5\,\Omega \quad \checkmark$$

(Note: Not all load impedances can be matched in this way – a good exam problem to determine which impedances can be matched using this technique!)

5.8 From (2.91) the impedance of a terminated lossy line is,

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \quad , \quad \gamma l = \alpha l + j\beta l$$

For $Z_L = \infty$ (o.c.), the normalized input admittance is,

$$y_{in} = \tanh \gamma l = \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tanh \alpha l \tan \beta l}$$

The normalized input susceptance is,

$$b_{in} = \frac{\tan \beta l (1 - \tanh^2 \alpha l)}{1 + \tanh^2 \alpha l \tan^2 \beta l}$$ (at this point, we could find max. b_{in} by calculating b_{in} vs. l)

Since maximum susceptance for a lossless line is obtained for $\beta l = \pi/2$, we expect βl to be close to $\pi/2$ for the lossy case. So let $\beta l = \pi/2 + \Delta$, where Δ is small. Also, αl is small, so we have $\tanh \alpha l \simeq \alpha l$, and $\tan \beta l = -\cot \Delta \simeq -1/\Delta$. Then,

$$b_{in} \simeq \frac{\frac{-1}{\Delta}(1 - \alpha^2 l^2)}{1 + \alpha^2 l^2/\Delta^2} \simeq \frac{-1}{\Delta + \alpha^2 l^2/\Delta}$$

To maximize b_{in}, we can minimize $\Delta + \alpha^2 l^2/\Delta$ with respect to l:

$$\frac{d}{dl}(\Delta + \alpha^2 l^2/\Delta) = \frac{d\Delta}{dl} + \frac{2\alpha^2 l}{\Delta} + \alpha^2 l^2 \left(\frac{-1}{\Delta^2}\right)\frac{d\Delta}{dl} = 0$$

or, since $\frac{d\Delta}{dl} = \beta$,

$$\beta + \frac{2\alpha^2 l}{\Delta} - \frac{\alpha^2 l^2}{\Delta^2}\beta = 0$$

since $\Delta = \beta l - \pi/2$, we have,

$$l^2 \beta (\alpha^2 + \beta^2) - \pi l (\alpha^2 + \beta^2) + \beta \frac{\pi^2}{4} = 0$$

Solve for l:

$$\ell = \frac{\pi(\alpha^2+\beta^2) \pm \sqrt{\pi^2(\alpha^2+\beta^2)^2 - \beta^2\pi^2(\alpha^2+\beta^2)}}{2\beta(\alpha^2+\beta^2)}$$

$$= \frac{\pi}{2\beta} \pm \frac{\pi\alpha}{2\beta\sqrt{\alpha^2+\beta^2}} \simeq \frac{\pi}{2\beta} \pm \frac{\pi\alpha}{2\beta^2} \qquad (\text{since } \alpha^2 << \beta^2)$$

Then,

$$\Delta = \beta\ell - \pi/2 \simeq \frac{\pi\alpha}{2\beta} \simeq \alpha\ell \qquad (\text{since } \beta \simeq \pi/2\ell)$$

The corresponding value of b_{in} is,

$$b_{in}^{MAX} = \frac{\pm 1}{\alpha\ell + \alpha\ell} = \frac{\pm 1}{2\alpha\ell} = \frac{\pm 2}{\alpha\lambda} \qquad (\text{since } \ell \simeq \lambda/4)$$

For $\alpha = 0.01$ neper/λ, $b_{in}^{MAX} = \frac{\pm 2}{.01} = \underline{\pm 200}$

(This checks with direct calculation of y_{in} vs. ℓ.)

The reactance of a short-circuited line is the dual case of the above problem, so $x_{in}^{MAX} = \pm 200$.

5.9 Smith chart solution:

1. plot $y_L = 0.4 + j1.2$ on admittance chart
2. plot rotated $1+jb$ circle
3. add a stub susceptance of $j0.6$ or $-j1.0$ to move to rotated $1+jb$ circle
4. move $\lambda/8$ toward generator, to $1+jb$ circle
5. add a stub susceptance of $+j3.0$ or $-j1.0$ to move to center of chart.
6. the o.c. stub lengths are,

$$\ell_1 = 0.086\lambda \checkmark \quad \text{or} \quad \ell_1 = 0.375\lambda \checkmark$$

$$\ell_2 = 0.199\lambda \checkmark \quad \text{or} \quad \ell_2 = 0.375\lambda \checkmark$$

(see attached Smith chart for first solution)

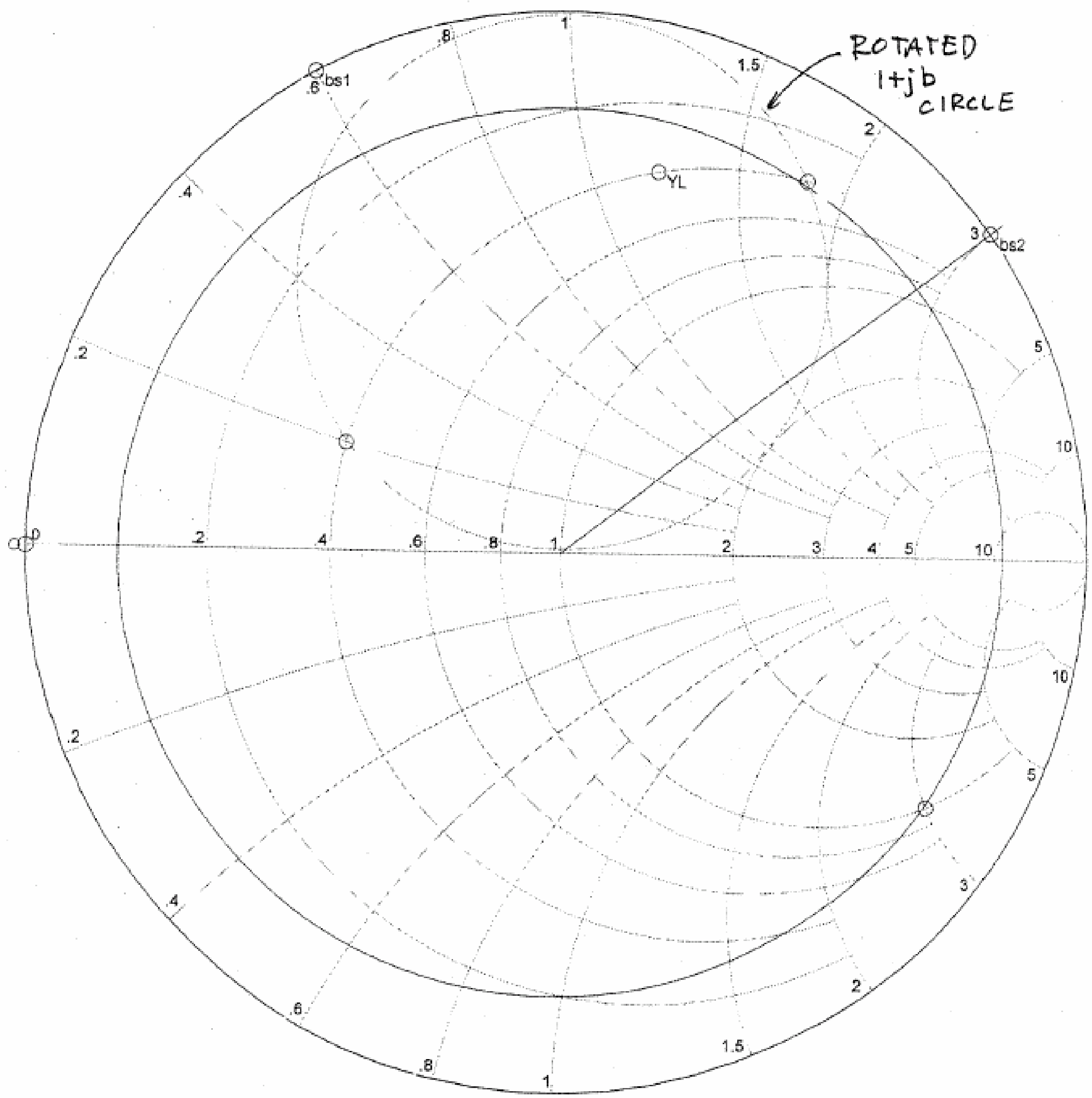
Analytic solution: $\quad t = \tan \beta d = 1$

$$b_1 = -b_L + 1 \pm \sqrt{2g_L - g_L^2} = 0.6,\ -1.0$$

$$b_2 = \frac{1}{g_L}\left[\pm\sqrt{2g_L - g_L^2} + g_L\right] = 3.0,\ -1.0$$

$$\ell_1 = \frac{\lambda}{2\pi}\tan^{-1} b_1 = 0.086\lambda \quad \text{or} \quad \ell_1 = 0.375\lambda$$

$$\ell_2 = \frac{\lambda}{2\pi}\tan^{-1} b_2 = 0.198\lambda \quad \text{or} \quad \ell_2 = 0.375\lambda$$



Smith chart for P. 5.9 (#1)

5.10 Analytic Solution: let $t = \tan\beta d = \tan 135° = -1.0$

From (5.22) the first stub susceptance is

$$b_1 = -b_L + \frac{1 \pm \sqrt{(1+t^2)g_L - g_L^2 t^2}}{t} = -3 \quad \text{or} \quad -1.4 \checkmark$$

From (5.23) the second stub susceptance is

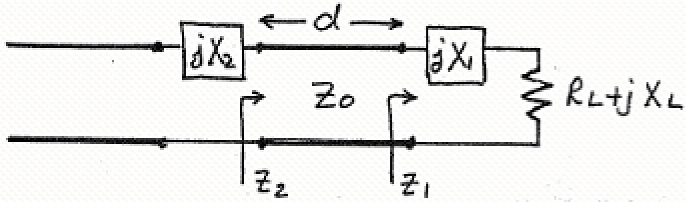
$$b_2 = \frac{\pm\sqrt{(1+t^2)g_L - g_L^2 t^2} + g_L}{g_L t} = -3 \quad \text{or} \quad 1.0 \checkmark$$

The S.C. stub lengths are, from (5.24b),

$$\ell_1 = 0.051\lambda \quad \text{or} \quad 0.0987\lambda$$

$$\ell_2 = 0.051\lambda \quad \text{or} \quad 0.375\lambda$$

5.11



$$Z_1 = R_L + j(X_L + X_1)$$

$$Z_2 = Z_0 \frac{R_L + j(X_L + X_1 + Z_0 t)}{Z_0 + jt(R_L + jX_L + jX_1)} = Z_0 \qquad t = \tan\beta d$$

Solving for R_L:

$$R_L = Z_0 \frac{1+t^2}{2t^2}\left[1 \pm \sqrt{1 - \frac{4t^2(Z_0 - X_L t - X_1 t)^2}{Z_0(1+t^2)^2}}\right]$$

So we must have,

$$0 \le R_L \le Z_0 \frac{1+t^2}{2t^2} = \frac{Z_0}{\sin^2\beta d}$$

The first stub reactance is,

$$X_1 = -X_L + \frac{Z_0 \pm \sqrt{(1+t^2)R_L Z_0 - R_L^2 t^2}}{t}$$

The second stub reactance is,

$$X_2 = \frac{\pm Z_0\sqrt{Z_0 R_L(1+t^2) - R_L^2 t^2} + R_L Z_0}{R_L t}$$

The stub lengths are given by,

$$\ell_{oc} = \frac{1}{2\pi}\tan^{-1}\left(\frac{Z_0}{X}\right), \qquad \ell_{sc} = \frac{1}{2\pi}\tan^{-1}\left(\frac{X}{Z_0}\right)$$

5.12 Using the Smith chart ($Z_0 = 100\,\Omega$)

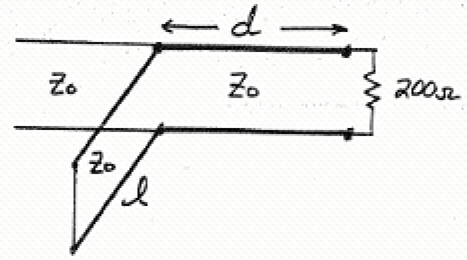
a) a single short-circuited shunt stub:

at f_0, $d_1 = 0.152\lambda$ ✓ $d_2 = 0.348\lambda$ ✓

$b_1 = -0.7$ $b_2 = +0.7$

$\ell_1 = 0.153\lambda$ ✓ $\ell_2 = 0.347$ ✓

$|\Gamma_1| = 0$ $|\Gamma_2| = 0$



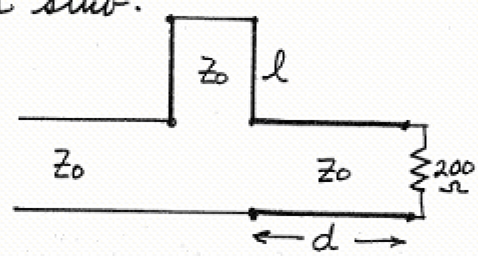
b) a single short-circuited series stub:

at f_0, $d_1 = 0.098\lambda$ ✓ $d_2 = 0.402\lambda$ ✓

$x_1 = 0.7$ $x_2 = -0.7$

$\ell_1 = 0.097\lambda$ ✓ $\ell_2 = 0.403\lambda$ ✓

$|\Gamma_1| = 0$ $|\Gamma_2| = 0$



c) a double short-circuited shunt stub: (let $d = \lambda/8$)

at f_0,

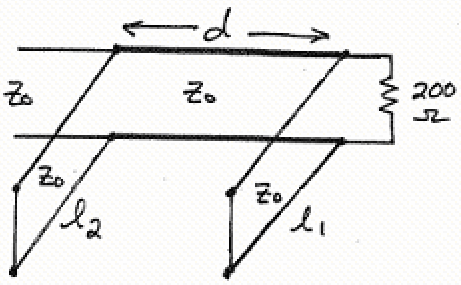
$b_1 = 0.14$ $b_1' = 1.85$

$\ell_1 = 0.272\lambda$ ✓ $\ell_1' = 0.421\lambda$ ✓

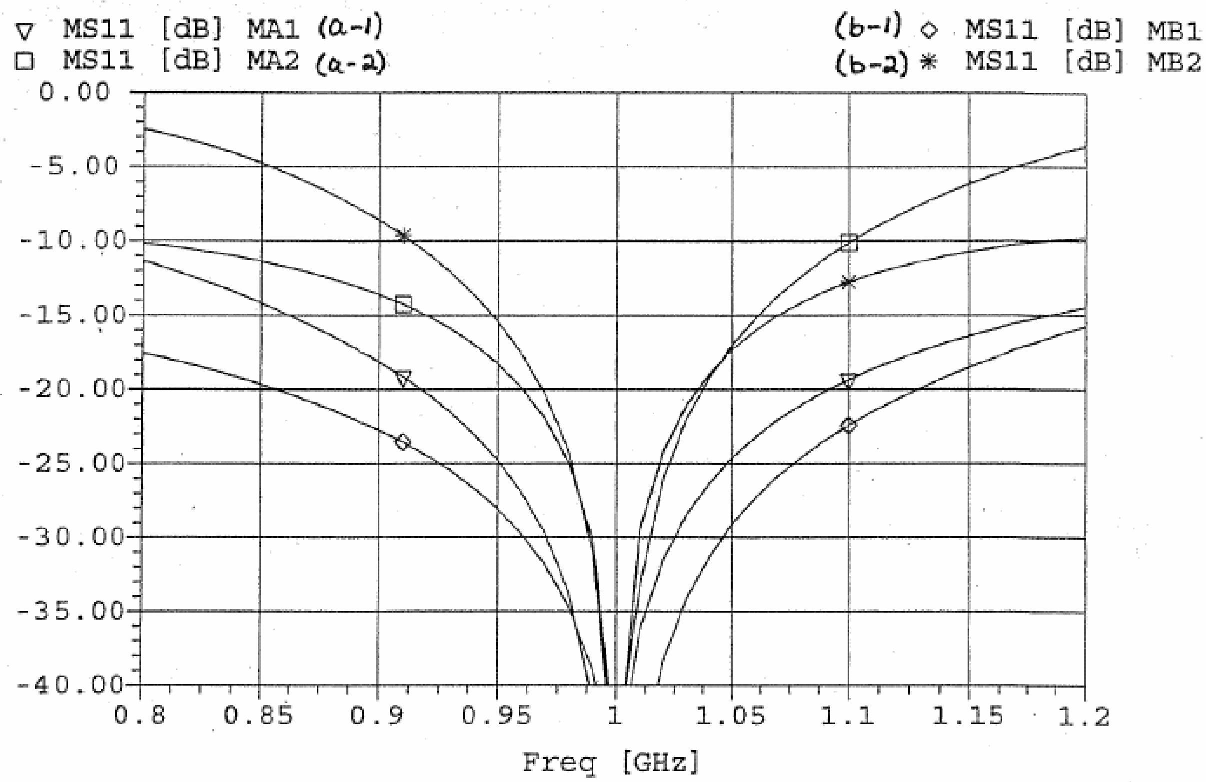
$b_2 = -0.73$ $b_2' = 2.75$

$\ell_2 = 0.15\lambda$ ✓ $\ell_2' = 0.444\lambda$ ✓

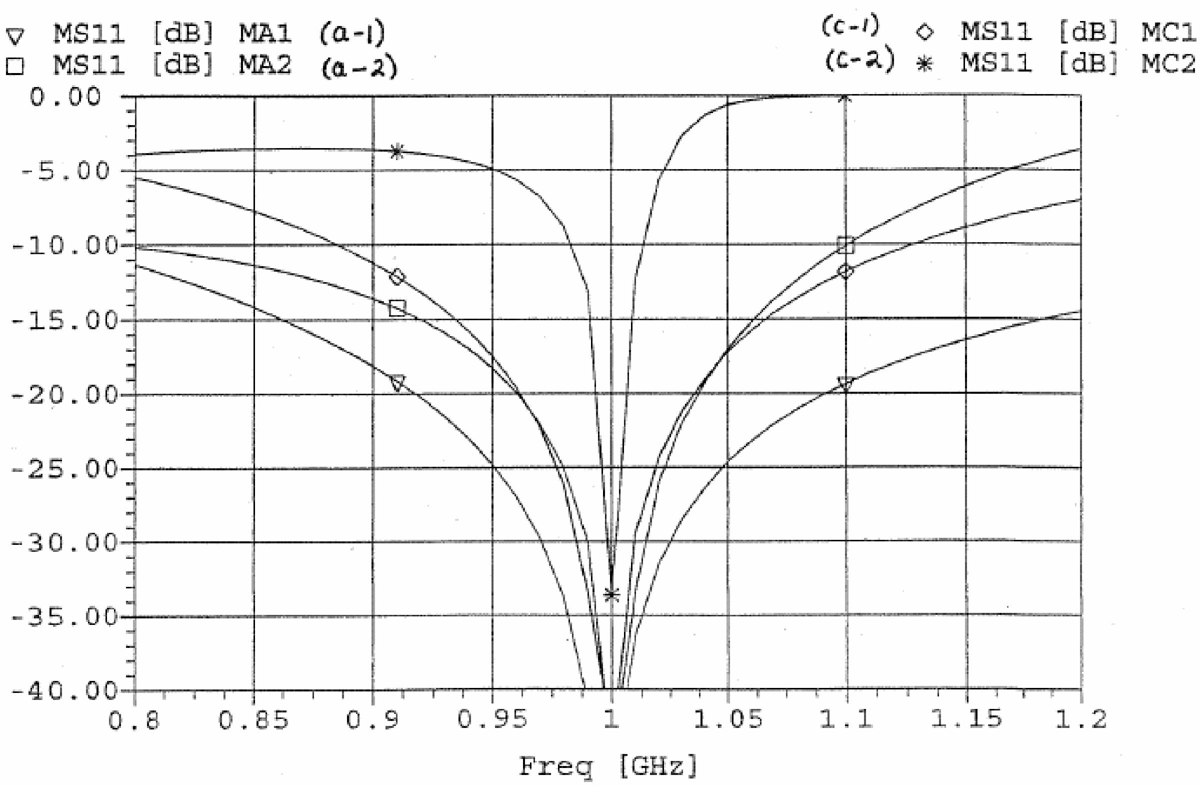
$|\Gamma| = 0$ $|\Gamma'| = 0$



Plots of return loss vs. f/f for these six solutions are shown on the following page. (only 4 curves could be plotted per graph). These results show that the tuner of solution (b-1), the series stub tuner, gives the best bandwidth. This is probably because the stub length and line length are shortest for this case, giving the smallest frequency variation.



▽ MS11 [dB] MA1 (a-1)
□ MS11 [dB] MA2 (a-2)
(b-1) ◇ MS11 [dB] MB1
(b-2) * MS11 [dB] MB2
0.00
-5.00
-10.00
-15.00
-20.00
-25.00
-30.00
-35.00
-40.00
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
Freq [GHz]



▽ MS11 [dB] MA1 (a-1)
□ MS11 [dB] MA2 (a-2)
(c-1) ◇ MS11 [dB] MC1
(c-2) * MS11 [dB] MC2
0.00
-5.00
-10.00
-15.00
-20.00
-25.00
-30.00
-35.00
-40.00
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
Freq [GHz]

5.13

$Z = \sqrt{100(350)} = 187.1\,\Omega$ ✓

An SWR of 2 corresponds to a reflection coefficient magnitude of,

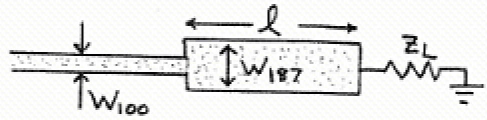
$$\Gamma_m = \frac{S-1}{S+1} = 1/3$$

Then from (5.33) the bandwidth is,

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\,\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right] = 71\%$$

MICROSTRIP LAYOUT:

$\epsilon_r = 2.2$, $d = 0.159$ cm, $f = 4$ GHz



First try $w/d < 2$:

for W_{100}, $A_{100} = 2.213$, $W_{100}/d = 0.896 < 2$ (OK), $W_{100} = 0.142$ cm

for W_{187}, $A_{187} = 4.047$, $W_{187}/d = 0.140 < 2$ (OK), $W_{187} = 0.022$ cm

From (3.195), ϵ_e for W_{187} is $\epsilon_e = 1.66$.

Then the physical length of the $\lambda/4$ transformer is,

$$\ell = \frac{\lambda_g}{4} = \frac{c}{4\sqrt{\epsilon_e}\,f} = 1.455 \text{ cm} \checkmark$$

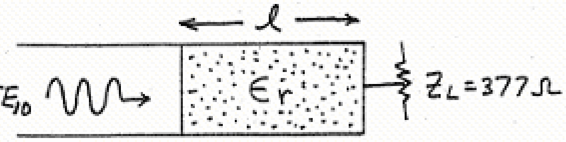
5.14

From (5.34) and (5.36), the partial reflection coefficients are,

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{150-100}{150+100} = 0.2 \quad ; \quad \Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2} = \frac{225-150}{225+150} = 0.2$$

Since the approximate expression for Γ in (5.42) is identical to the numerator for the exact expression in (5.41), the greatest error will occur when the denominator of (5.41) departs from unity to the greatest extent. This occurs for $\theta = 0$ or $180°$. Then (5.41) gives the exact Γ as 0.384, while (5.42) gives the approximate $\Gamma = 0.4$. Thus the error is about 4%.

5.15



$a = 2.286$ cm

$k_o = \frac{2\pi f}{c} = 209.4\ \mathrm{m}^{-1}$

In the air-filled guide,

$$\beta_a = \sqrt{k_o^2 - (\pi/a)^2} = 158.0\ \mathrm{m}^{-1}\ \checkmark$$

$$Z_a = \frac{k_o\eta_o}{\beta_a} = \frac{(209.4)(377)}{158} = 499.6\ \Omega\ \checkmark$$

So the matching section impedance must be,

$$Z_m = \sqrt{Z_a Z_L} = \sqrt{(499.6)(377)} = 434.0\ \Omega$$

$$= \frac{k_m\eta_m}{\beta_m} = \frac{k_o\eta_o}{\beta_m}$$

So the propagation constant of the matching section must be,

$$\beta_m = \frac{k_o\eta_o}{Z_m} = \frac{(209.4)(377)}{434} = 181.9\ \mathrm{m}^{-1}$$

$$= \sqrt{\epsilon_r k_o^2 - (\pi/a)^2}$$

Solving for ϵ_r:

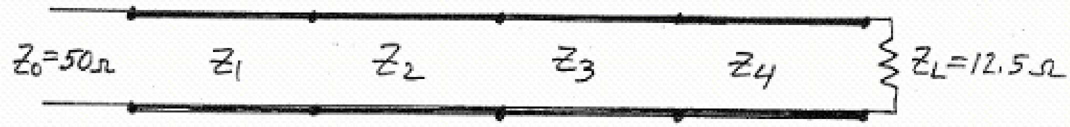
$$\epsilon_r = \frac{\beta_m^2 + (\pi/.02286)^2}{(209.4)^2} = \underline{1.185}\ \checkmark$$

The physical length of the matching section is,

$$\ell = \frac{\lambda_g}{4} = \frac{2\pi}{4\beta_m} = \frac{\pi}{2\beta_m} = 0.86\ \mathrm{cm}\ \checkmark$$

(Note that this type of matching is not possible if $Z_L > Z_a$.)

5.16



a) Using (5.53):

$n=0$: $\ln Z_1/Z_0 = 2^{-4} C_0^4 \ln 12.5/50 \Rightarrow Z_1 = 45.85\Omega$

$n=1$: $\ln Z_2/Z_1 = 2^{-4} C_1^4 \ln 12.5/50 \Rightarrow Z_2 = 32.42\Omega$

$n=2$: $\ln Z_3/Z_2 = 2^{-4} C_2^4 \ln 12.5/50 \Rightarrow Z_3 = 19.28\Omega$

$n=3$: $\ln Z_4/Z_3 = 2^{-4} C_3^4 \ln 12.5/50 \Rightarrow Z_4 = 13.63\Omega$

Check: $n=4$: $\ln Z_5/Z_4 = 2^{-4} C_4^4 \ln 12.5/50 \Rightarrow Z_5 = 12.50\Omega = Z_L$ ✓

Can also check with data in Table 5.1, using $Z_L/Z_0 = 4$, which gives $Z_1 = 13.65\Omega$, $Z_2 = 19.30\Omega$, $Z_3 = 32.38\Omega$, $Z_4 = 45.79\Omega$
(source and load are reversed in this case)

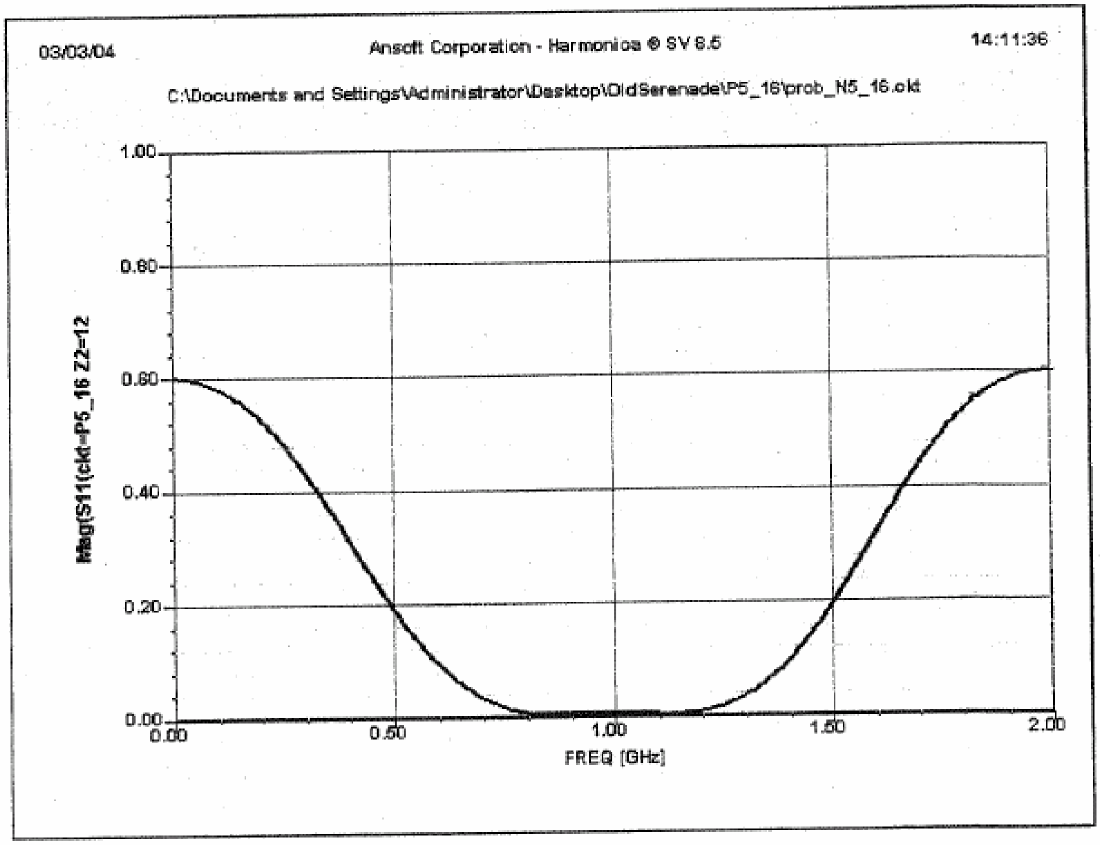
From (5.55), $A \simeq \frac{1}{2^{N+1}} \ln Z_L/Z_0 = -0.0433$

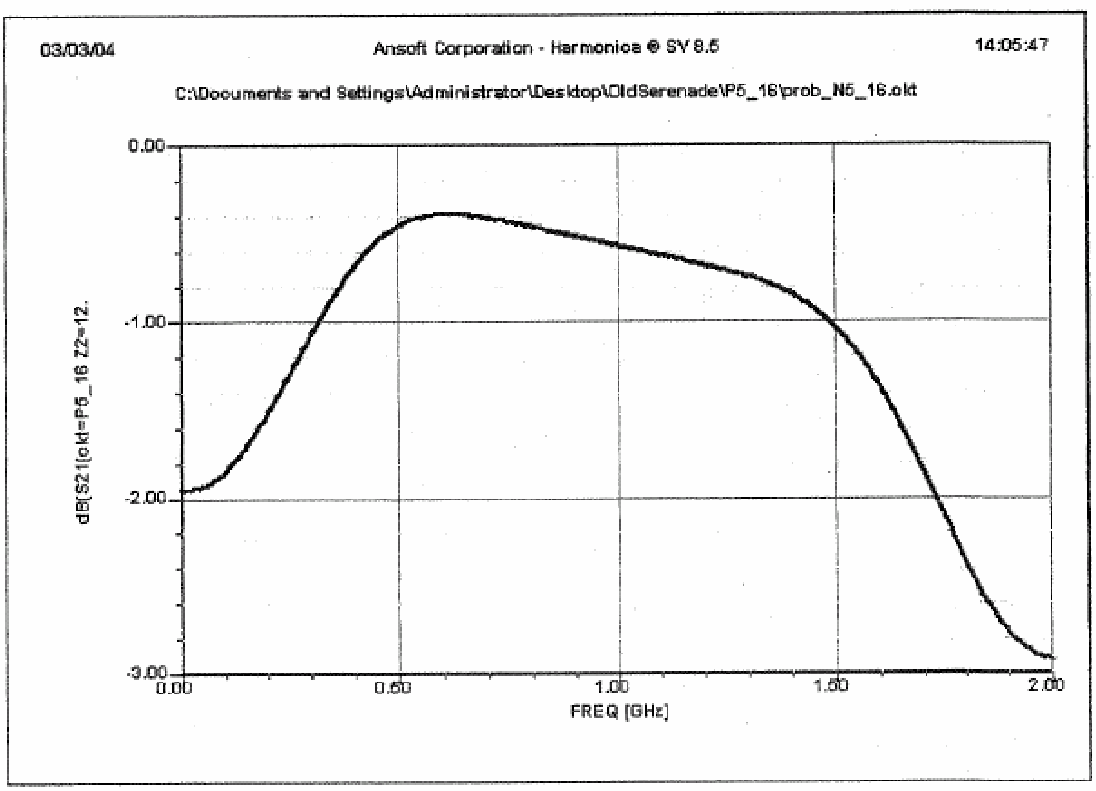
$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{1}{2}\left|\frac{\Gamma_m}{A}\right|^{1/N}\right] = 69\% \quad \text{(agrees with plot)}$$

b) Microstrip line widths and lengths:

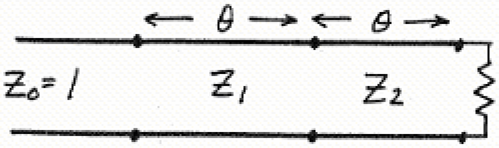
Z_c	W (cm)	ϵ_e	$\lambda_g/4$ (cm)
45.85	0.356	3.239	4.16
32.42	0.597	3.402	4.06
19.28	1.175	3.627	3.94
13.63	1.781	3.756	3.87

Results from Serenade modeling of parts a) and b) are shown on the following page. Note the good match, and the insertion loss of about 0.5dB.


03/03/04
Ansoft Corporation - Harmonica ® SV 8.5
14:11:36
C:\Documents and Settings\Administrator\Desktop\OldSerenade\P5_16\prob_N5_16.ckt
Mag(S11(ckt=P5_16 Z2=12
1.00
0.80
0.60
0.40
0.20
0.00
0.00
0.50
1.00
1.50
2.00
FREQ [GHz]


03/03/04
Ansoft Corporation - Harmonica ® SV 8.5
14:05:47
C:\Documents and Settings\Administrator\Desktop\OldSerenade\P5_16\prob_N5_16.ckt
dB(S21(ckt=P5_16 Z2=12.
0.00
-1.00
-2.00
-3.00
0.00
0.50
1.00
1.50
2.00
FREQ [GHz]

5.17



From (5.50) the desired input reflection coefficient response is ($N=2$):

$$\Gamma(\theta) = 2A(1+\cos 2\theta)$$

From the above circuit, we have that $\Gamma(0) = \frac{R-1}{R+1} = 0.2$, so $A = 0.2/4 = 0.05$.

Now we calculate the input reflection coefficient of the above circuit using ABCD matrices and conversion to S-parameters:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\theta & jZ_1 \sin\theta \\ jY_1 \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & jZ_2 \sin\theta \\ jY_2 \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - Z_1 Y_2 \sin^2\theta & j(Z_1+Z_2)\cos\theta \sin\theta \\ j(Y_1+Y_2)\sin\theta\cos\theta & \cos^2\theta - Y_1 Z_2 \sin^2\theta \end{bmatrix}$$

Using Table 4.2 to convert to S-parameters gives the input reflection coefficient as,

$$\Gamma(\theta) = S_{11} + \frac{S_{12}S_{21}\Gamma_\ell}{1 - S_{22}\Gamma_\ell} = \frac{A+B-C-D}{S} + \frac{4\Gamma_\ell/S^2}{1 - \frac{-A+B-C+D}{S}\Gamma_\ell}$$

$$= \frac{(A+B-C-D)\left[S - \Gamma_\ell(-A+B-C+D)\right] + 4\Gamma_\ell}{S\left[S - \Gamma_\ell(-A+B-C+D)\right]}$$

where $S = A+B+C+D$, $\Gamma_\ell = \frac{R-1}{R+1}$.

This result can be equated to $2A(1+\cos 2\theta)$, and solved for Z_1 and Z_2, but this is a <u>very</u> lengthy procedure. Instead, we will first evaluate both expressions at $\theta = 90°$:

$\Gamma(90°) = 0$, and $\left.\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|_{\theta=90°} = \begin{bmatrix} -Z_1Y_2 & 0 \\ 0 & -Y_1Z_2 \end{bmatrix}$

So $\Gamma(\theta)$ reduces to the following equation:

$$(-Z_1Y_2 + Y_1Z_2)\left[-(Y_1Z_2 + Z_1Y_2) - \Gamma_\ell(Z_1Y_2 - Y_1Z_2)\right] + 4\Gamma_\ell = 0$$

$$(Z_1^2Y_2^2 - Y_1^2Z_2^2) + \Gamma_\ell(Z_1^2Y_2^2 + Y_1^2Z_2^2 + 2) = 0$$

$$(Z_1^4 - Z_2^4) + \Gamma_\ell(Z_1^4 + Z_2^4 + 2Z_1^2Z_2^2) = 0$$

$$(Z_1^2 - Z_2^2) + \Gamma_\ell(Z_1^2 + Z_2^2) = 0$$

$$Z_2^2 = Z_1^2\frac{1+\Gamma_\ell}{1-\Gamma_\ell} = Z_1^2R \Rightarrow \underline{\underline{Z_2 = Z_1\sqrt{R}}} \checkmark \quad (\text{for } Z_0 = 1)$$

another equation is harder to find, so we will make use of the fact that the transformer will be symmetric:

$$\frac{Z_1 - 1}{Z_1 + 1} = \frac{R - Z_2}{R + Z_2} = \frac{R/Z_2 - 1}{R/Z_2 + 1}$$

Thus, $\frac{R}{Z_2} = Z_1$ or $\underline{\underline{Z_1 = R^{1/4}}}$ $(\text{for } Z_0 = 1)$

If $R = 1.5$, these results reduce to,

$$Z_1 = (1.5)^{1/4} = 1.1067 \checkmark$$

$$Z_2 = 1.1067\sqrt{1.5} = 1.3554 \checkmark$$

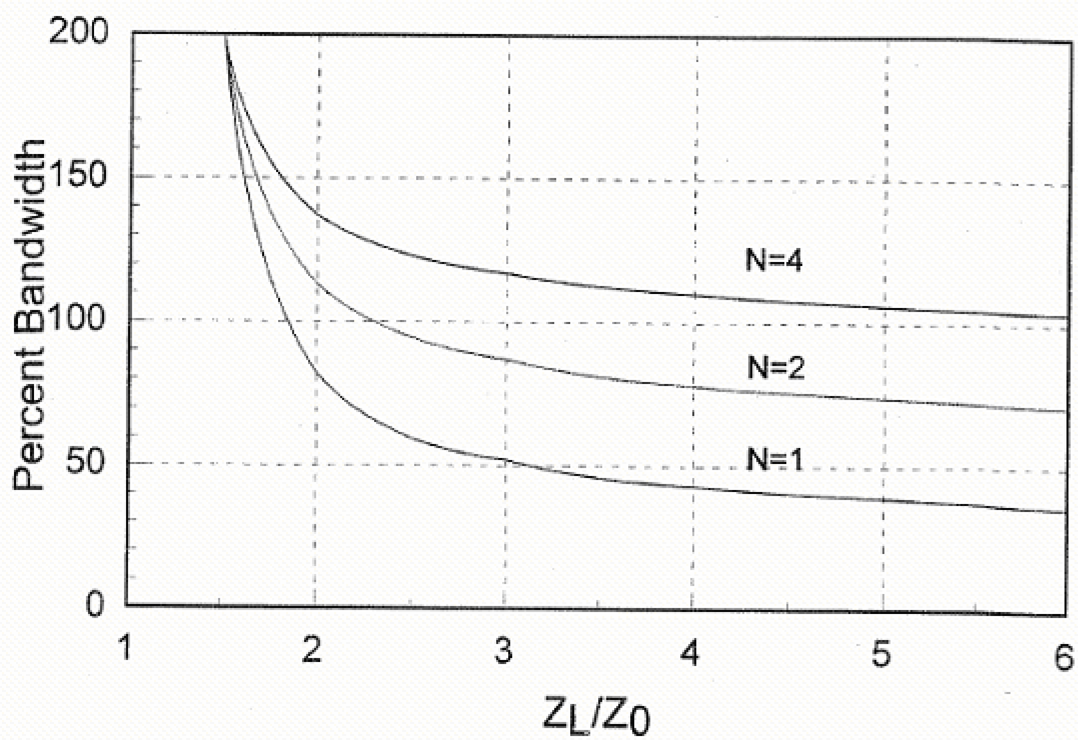
which agree with Table 5.1

5.18 From (5.55), the fractional bandwidth is,

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{A}\right)^{1/N}\right], \text{ with } A = 2^{-N}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|$$

Z_L/Z_0	N=1		N=2		N=4	
	A	$\Delta f/f_0$-%	A	$\Delta f/f_0$-%	A	$\Delta f/f_0$-%
1.5	0.1000	200	0.0500	200	0.0125	200
2.0	0.1667	82	0.0833	113	0.0208	137
3.0	0.2500	52	0.1250	87	0.0313	117
4.0	0.3000	43	0.1500	78	0.0375	110
5.0	0.3333	39	0.1667	74	0.0417	106
6.0	0.3571	36	0.1786	71	0.0446	104

This data is plotted in the graph below.



5.19 $Z_0 = 50\Omega, \quad Z_L = 30\Omega, \quad SWR_{MAX} = 1.25$

$$|\Gamma_m| = \frac{SWR_m - 1}{SWR_m + 1} = 0.111 = A$$

From (5.61), $\Gamma(\theta) = 2e^{-j4\theta}\left[\Gamma_0 \cos 4\theta + \Gamma_1 \cos 2\theta + \frac{1}{2}\Gamma_2\right]$

$$= A e^{-j4\theta} \, T_4(\sec\theta_m \cos\theta)$$

$$= A e^{-j4\theta}\left[\sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3) - 4\sec^2\theta_m \cdot (\cos 2\theta + 1) + 1\right]$$

From (5.63),

$$\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\left|\frac{\ln Z_L/Z_0}{2\Gamma_m}\right|\right)\right]$$

$$= \cosh\left[\frac{1}{4}\cosh^{-1}\left(\frac{1}{2(.111)}\ln 30/50\right)\right] = 1.0687$$

so,

$$\theta_m = \cos^{-1}(1/\sec\theta_m) = 20.66^\circ$$

Equate $\cos 4\theta$ terms:

$$2\Gamma_0 = A\sec^4\theta_m \Rightarrow \Gamma_0 = 0.07246 = \Gamma_4$$

Equate $\cos 2\theta$ terms:

$$2\Gamma_1 = A(4\sec^4\theta_m - 4\sec^2\theta_m) \Rightarrow \Gamma_1 = 0.03607 = \Gamma_3$$

Equate constant terms:

$$\Gamma_2 = A(3\sec^4\theta_m - 4\sec^2\theta_m + 1) = 0.03831$$

Compute Z_n's: (reverse Z_L and Z_0)

$$Z_1 = Z_L \frac{1+\Gamma_0}{1-\Gamma_0} = 34.69\,\Omega$$

$$Z_2 = Z_1 \frac{1+\Gamma_1}{1-\Gamma_1} = 37.29\,\Omega$$

$$Z_3 = Z_2 \frac{1+\Gamma_2}{1-\Gamma_2} = 40.26\,\Omega$$

$$Z_4 = Z_3 \frac{1+\Gamma_3}{1-\Gamma_3} = 43.27\,\Omega$$

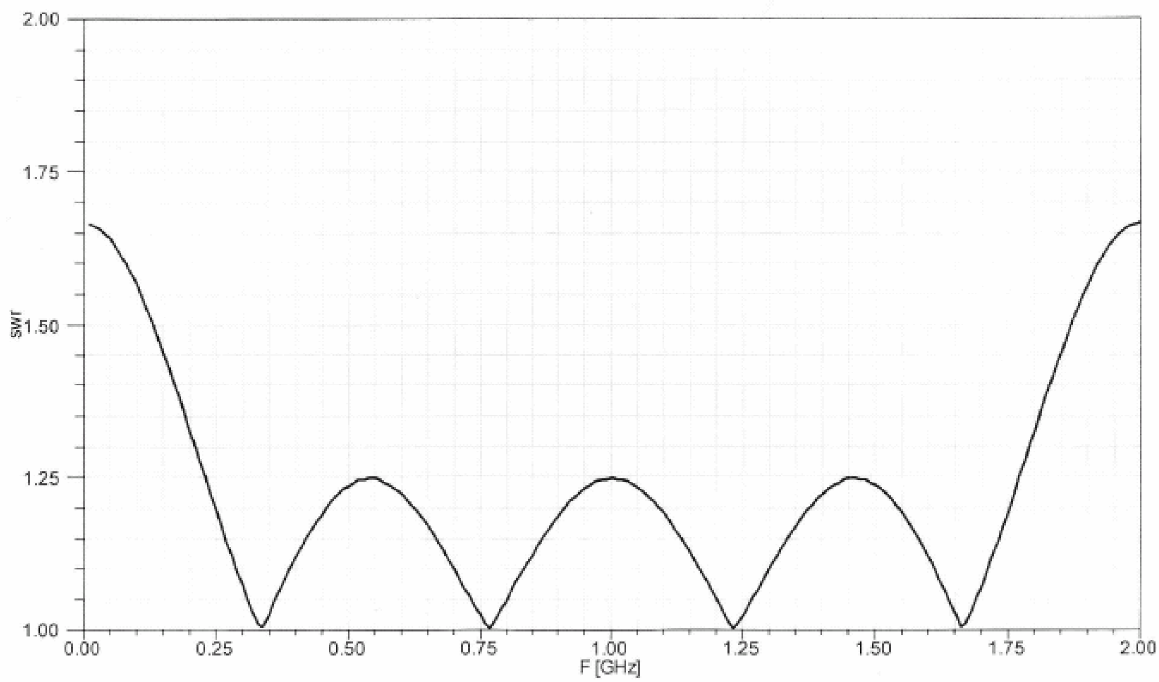
check: $Z_5 = Z_4 \frac{1+\Gamma_4}{1-\Gamma_4} = 50.03\,\Omega \simeq Z_0$ ✓

From (5.64) the bandwidth is,

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi} = 154\,\%$$

From the graph,

$$\frac{\Delta f}{f_0} \simeq \frac{1.77 - .225}{1} = 154.5\,\%$$ ✓



5.20 From (5.61) and (5.60b),

$$\Gamma(\theta) = A e^{-2j\theta} T_2(\sec\theta_m \cos\theta) = A e^{-2j\theta}\left[\sec^2\theta_m(1+\cos 2\theta) - 1\right]$$

$$\Gamma(0) = A T_2(\sec\theta_m) = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = \frac{R-1}{R+1} = 0.2 \quad ; \quad A = \Gamma_m = 0.05$$

As in Problem 5.17, we will evaluate $\Gamma(\theta)$ for $\theta = 90°$. Then $\Gamma(90°) = \Gamma_m$. Also, as in Problem 5.17, from symmetry we have that $Z_1 Z_2 = R$. Then,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -Z_1^2/R & 0 \\ 0 & -R/Z_1^2 \end{bmatrix} \qquad (Z_0 = 1)$$

$$\Gamma(90°) = \Gamma_m = \frac{\left(-Z_1^2/R + R/Z_1^2\right)\left[-\left(Z_1^2/R + R/Z_1^2\right) - \Gamma_\ell\left(Z_1^2/R - R/Z_1^2\right)\right] + 4\Gamma_\ell}{-\left(Z_1^2/R + R/Z_1^2\right)\left[-\left(Z_1^2/R + R/Z_1^2\right) - \Gamma_\ell\left(Z_1^2/R - R/Z_1^2\right)\right]}$$

$$= \frac{(R^2 - Z_1^4)\left[-(Z_1^4 + R^2) - \Gamma_\ell(Z_1^4 - R^2)\right] + 4\Gamma_\ell R^2 Z_1^4}{(Z_1^4 + R^2)\left[(Z_1^4 + R^2) + \Gamma_\ell(Z_1^4 - R^2)\right]}$$

$$\Gamma_m(Z_1^4 + R^2)^2 + \Gamma_m\Gamma_\ell(Z_1^4 + R^2)(Z_1^4 - R^2) = -(R^2 - Z_1^4)(Z_1^4 + R^2) + \Gamma_\ell(Z_1^4 - R^2)^2 + 4\Gamma_\ell R^2 Z_1^4$$

$$Z_1^8(\Gamma_m - 1)(\Gamma_\ell + 1) + 2Z_1^4 R^2(\Gamma_m - \Gamma_\ell) - R^4(\Gamma_m + 1)(\Gamma_\ell - 1) = 0$$

For $\Gamma_m = 0.05$, $\Gamma_\ell = 0.2$, $R = 1.5$:

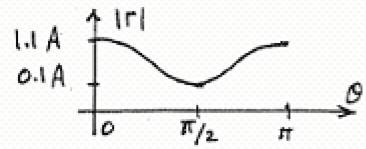
$$-1.140 Z_1^8 - 0.6750 Z_1^4 + 4.2525 = 0$$

$$Z_1^4 = \frac{0.675 \pm 4.455}{-2.280} = 1.65789 \Rightarrow Z_1 = 1.1347 Z_0 \checkmark$$

$$Z_2 = R/Z_1 = 1.3219 Z_0 \checkmark$$

These results agree with Table 5.2.

5.21 $|\Gamma(\theta)| = A(0.1 + \cos^2\theta)\ ,\quad 0 < \theta < \pi$



From (5.46a), for $N=2$,

$$|\Gamma(\theta)| = 2\left(\Gamma_0 \cos 2\theta + \tfrac{1}{2}\Gamma_1\right) = A(0.1 + \cos^2\theta)$$

$$= A(0.6 + 0.5\cos 2\theta)$$

When $\theta = 0$, $|\Gamma(0)| = 1.1A = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{1.5-1}{1.5+1} = 0.2 \implies A = 0.182$

Equating coefficients of $\cos 2\theta$:

$$2\Gamma_0 = 0.5A \implies \Gamma_0 = 0.0455$$

Equating constant terms:

$$\Gamma_1 = 0.6A = 0.109$$

so the characteristic impedances are,

$$Z_1 = Z_0 \frac{1+\Gamma_0}{1-\Gamma_0} = 1.095\, Z_0$$

$$Z_2 = Z_1 \frac{1+\Gamma_1}{1-\Gamma_1} = 1.245\, Z_1 = 1.363\, Z_0$$

CHECK: at $\theta = \pi/2$, the input impedance to the transformer will be,

$$Z_{in} = \frac{Z_1^2}{(Z_2^2/Z_L)} = \frac{Z_L Z_0 Z_1^2}{Z_2^2} = 0.968\, Z_0$$

So the input reflection coefficient is,

$$\Gamma_{in} = \left|\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right| = 0.016$$

which is reasonably close to $|\Gamma(\pi/2)| = 0.1A = 0.018$

5.22

$$\frac{d(\ln Z/Z_0)}{dz} = A \sin\frac{\pi z}{L}$$

$$\ln(Z/Z_0) = B - \frac{LA}{\pi}\cos\frac{\pi z}{L}$$

$$Z(z) = C e^{-\frac{LA}{\pi}\cos\frac{\pi z}{L}}$$

$$Z(0) = Z_0 = C e^{-LA/\pi} \quad , \quad Z(L) = Z_L = C e^{+LA/\pi}$$

Solve for C, A to get,

$$C = \sqrt{Z_0 Z_L}$$

$$A = \frac{-\pi}{2L}\ln(Z_0/Z_L) \checkmark$$

From (5.67),

$$\Gamma(\theta) = \frac{1}{2}\int_{z=0}^{L} e^{-2j\beta z}\frac{d}{dz}(\ln Z/Z_0)\,dz$$

$$= \frac{1}{2}\int_{z=0}^{L} A\sin\frac{\pi z}{L}\, e^{-2j\beta z}\,dz$$

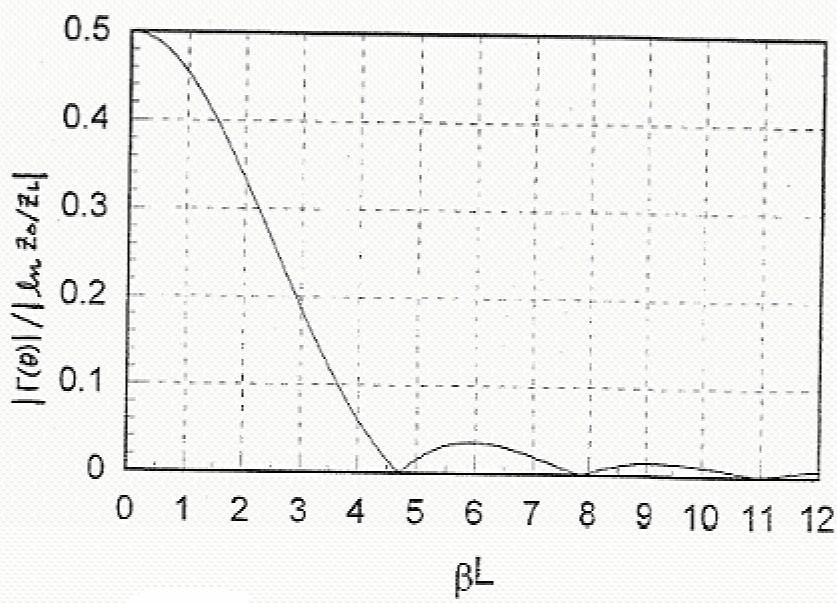
$$= \frac{A}{2}\,\frac{e^{-2j\beta z}\left[-2j\beta\sin\frac{\pi z}{L} - \frac{\pi}{L}\cos\frac{\pi z}{L}\right]}{(\pi/L)^2 - 4\beta^2}\Bigg|_0^L$$

$$= \frac{\pi A}{2L} e^{-j\beta L}\,\frac{(e^{-j\beta L} + e^{j\beta L})}{(\pi/L)^2 - 4\beta^2}$$

So,

$$|\Gamma(\theta)| = \frac{\pi^2}{2}\left|\ln\frac{Z_0}{Z_L}\right|\left|\frac{\cos\beta L}{\pi^2 - (2\beta L)^2}\right| \checkmark$$

This result is plotted as shown:



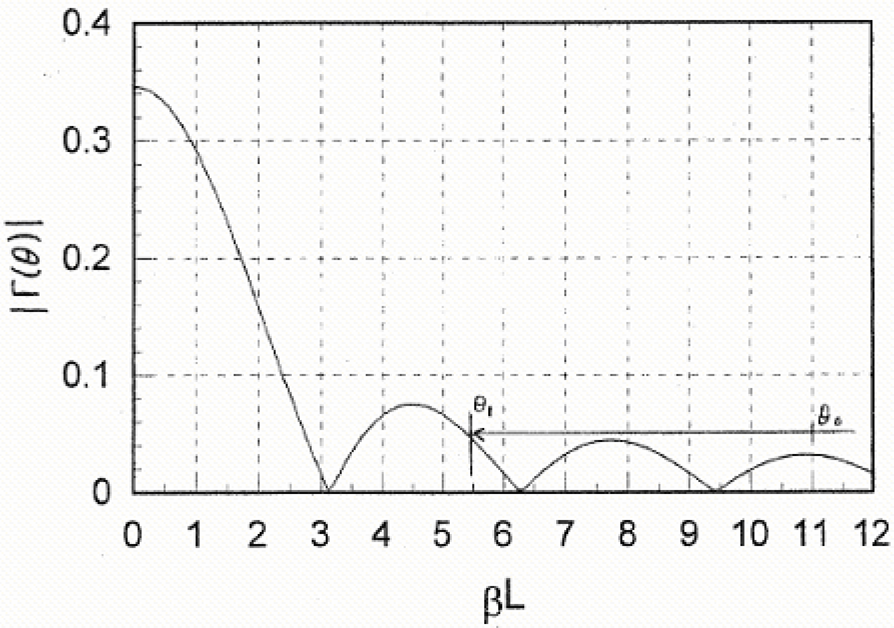
5.23 From (5.68), $Z(z) = Z_0 e^{az}$ for $0 < z < L$.

$$a = \frac{1}{L} \ln \frac{Z_L}{Z_0} = \frac{0.693}{L}$$

From (5.70),

$$|\Gamma(\theta)| = \frac{1}{2} \left| \ln \frac{Z_L}{Z_0} \right| \left| \frac{\sin \beta L}{\beta L} \right| = 0.346 \left| \frac{\sin \beta L}{\beta L} \right| \checkmark$$

This result is plotted in the graph shown below:



We see that the lower frequency limit for $|\Gamma| \leq 0.05$ is $\theta_1 = 5.5$. To obtain 100% bandwidth, we must have,

$$\frac{\theta_2 - \theta_1}{(\theta_1 + \theta_2)/2} = 1 \quad , \text{ or } \quad \theta_2 = 3\theta_1 = 16.5$$

Then at the center frequency,

$$\theta_0 = \frac{\theta_1 + \theta_2}{2} = 11.0 = \beta L$$

So,

$$L = \frac{11 \lambda_0}{2\pi} = 1.75 \lambda_0 \checkmark$$

From (5.64), θ_m for a Chebyshev transformer with 100% bandwidth is,

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi} = 1 \implies \theta_m = \pi/4.$$

Then from (5.63),

$$\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]$$

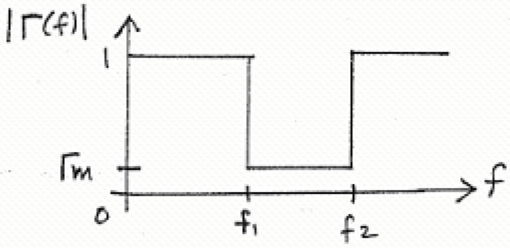
$$1.414 = \cosh\left[\frac{1}{N}(2.5846)\right] \implies N = 2.93 \implies \underline{N=3}$$

So $N=3$ sections would be required, for a length of $3\lambda_0/4$ at the center frequency. ✓
This is much shorter than the exponential taper matching section.

5.24 From Figure 5.22 the Bode-Fano limit for a parallel RC load is,

$$\int_0^\infty \ln\frac{1}{|\Gamma(\omega)|}\, d\omega \le \frac{\pi}{RC}$$

The optimum reflection coefficient magnitude response will be as shown:

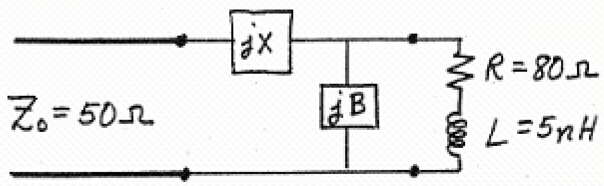


$$\text{Thus, } \ln\frac{1}{\Gamma_m} \le \frac{\pi}{\Delta\omega RC} = \frac{\pi}{2\pi(10.6-3.1)\times10^9(75)(0.6\times10^{-12})}$$

$$\le 1.48$$

$$\Gamma_m > 0.228 \implies RL < \underline{6.4 dB}$$

5.25 L-section matching solution:



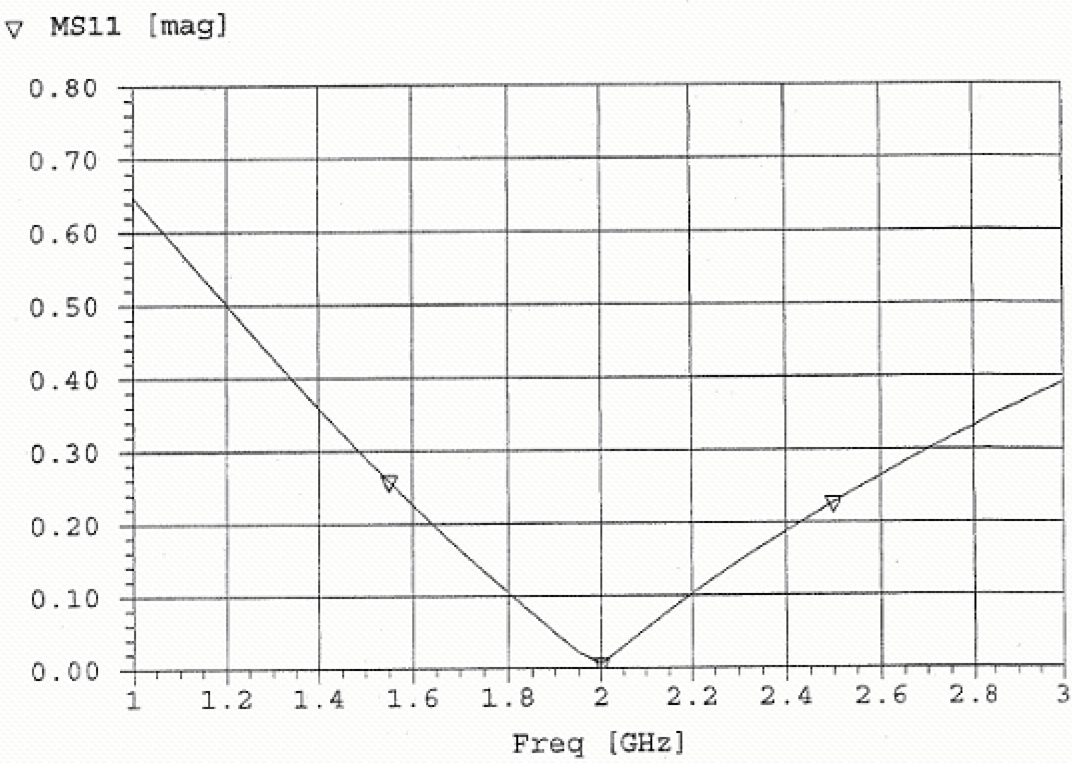
at $f = 2\text{GHz}$, $Z_L = 80 + j63\Omega$, $z_L = 1.6 + j1.26$ (INSIDE $1+jx$)

a Smith chart solution gives,

$jb = -j1.8 \Rightarrow$ INDUCTOR with $L = 22.1nH$. ✓

$jx = -j1.25 \Rightarrow$ CAPACITOR with $C = 1.27pF$. ✓

The input reflection coefficient magnitude is plotted below, where it is seen that the bandwidth for $|\Gamma| < 0.1$ is 20%.



Bode-Fano limit:

From Figure 5.22d, the Bode-Fano criteria gives a bandwidth limit of

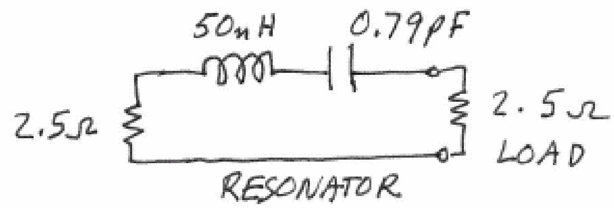
$$\Delta\omega = \frac{\pi R}{L} \frac{1}{\ln 1/\Gamma_m} = 2.18 \times 10^{10} = \omega_2 - \omega_1$$

$$\frac{\Delta f}{f_0} = \frac{f_2 - f_1}{f_0} = \frac{2.18 \times 10^{10}}{2\pi\,(2 \times 10^9)} = 174\%$$

This is considerably more than the bandwidth of the L-section match.

Chapter 6

6.1



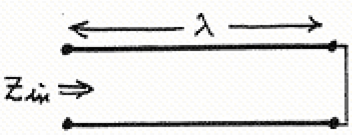
$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 800\ MHz$$

$$Q_0 = \frac{\omega_0 L}{R} = 100$$

$$Q_e = \frac{\omega_0 L}{R_L} = 100$$

$$Q_L = 50$$

6.2



$\ell = \lambda = \frac{2\pi v_p}{\omega_0}$ for $\omega = \omega_0$

This circuit has a series-type resonance, like the short-circuited $\lambda/2$ resonator. Thus, let

$$\beta \ell = \frac{\omega_0 \ell}{v_p} + \frac{\Delta\omega \ell}{v_p} = 2\pi\left(1 + \frac{\Delta\omega}{\omega_0}\right)$$

Then from (6.24) the input impedance is,

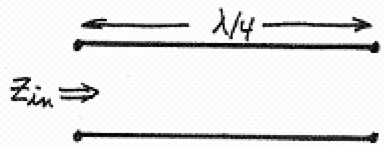
$$Z_{in} \simeq Z_0 \frac{\alpha\ell + j2\pi\frac{\Delta\omega}{\omega_0}}{1 + j2\pi\frac{\Delta\omega}{\omega_0}} \simeq Z_0\left(\alpha\ell + j2\pi\frac{\Delta\omega}{\omega_0}\right) = R + 2jL\Delta\omega$$

Thus, $R = Z_0 \alpha \ell$, $L = \frac{\pi Z_0}{\omega_0}$.

and,

$$Q = \frac{\omega_0 L}{R} = \frac{\pi Z_0}{Z_0 \alpha \ell} = \frac{\pi}{\alpha \ell} = \frac{\beta}{2\alpha} \quad \left(\text{since } \ell = \lambda = \frac{2\pi}{\beta} \text{ at res.}\right)$$

6.3



$$\ell = \frac{\lambda}{4} = \frac{\pi v_p}{2\omega_0} \quad \text{for } \omega = \omega_0$$

This circuit has a series-type resonance, like the short-circuited $\lambda/2$ line. So let,

$$\beta\ell = \frac{\omega_0 \ell}{v_p} + \frac{\Delta\omega \ell}{v_p} = \frac{\pi}{2}\left(1 + \frac{\Delta\omega}{\omega_0}\right)$$

Then,

$$\tan\beta\ell = \tan\frac{\pi}{2}\left(1 + \frac{\Delta\omega}{\omega_0}\right) = -\cot\frac{\Delta\omega\pi}{2\omega_0} \simeq \frac{-2\omega_0}{\pi\Delta\omega}$$

The input impedance is,

$$Z_{in} = Z_0 \frac{1 + j\tan\beta\ell \tanh\alpha\ell}{\tanh\alpha\ell + j\tan\beta\ell} \simeq \frac{1 - j\frac{2\omega_0}{\pi\Delta\omega}\alpha\ell}{\alpha\ell - j\frac{2\omega_0}{\pi\Delta\omega}}$$

$$\simeq Z_0 \frac{\alpha\ell + j\frac{\pi\Delta\omega}{2\omega_0}}{1 + j\frac{\pi\Delta\omega}{2\omega_0}\alpha\ell} \simeq Z_0\left(\alpha\ell + j\frac{\pi\Delta\omega}{2\omega_0}\right) = R + 2jL\Delta\omega$$

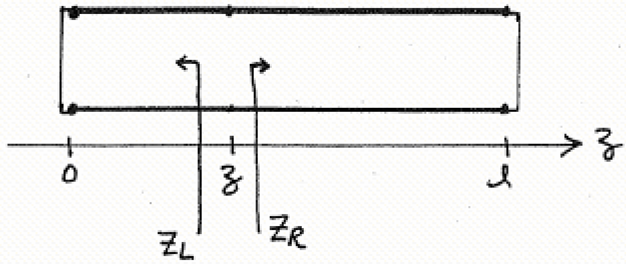
$$\therefore\ R = Z_0\alpha\ell \ \checkmark \quad , \quad L = \frac{\pi Z_0}{4\omega_0} \ \checkmark$$

Then,

$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha} \ \checkmark$$

(since $\ell = \frac{\lambda}{4} = \frac{\pi}{2\beta}$ at resonance)

6.4

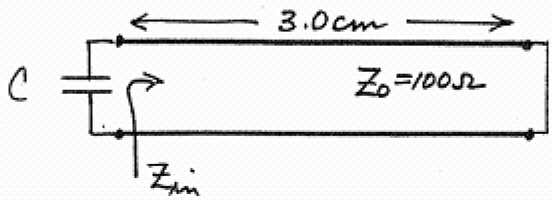


$\beta l = \pi$

$$Z_L = j Z_0 \tan \beta z$$

$$Z_R = j Z_0 \tan \beta(l - z) = j Z_0 \tan(\pi - \beta z) = -j Z_0 \tan \beta z = Z_L^* \checkmark$$

6.5



$f_0 = 6$ GHz

$\beta = \frac{2\pi f}{c} = 125.7 \text{ m}^{-1}$ for an air-filled line

$\beta l = (125.7)(0.03) = 216° \checkmark$

$Z_{in} = j Z_0 \tan \beta l = j(100) \tan 216° = j72.6\Omega = j\omega L \checkmark$

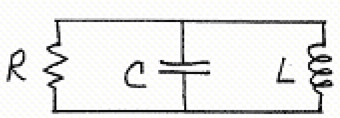
To achieve resonance we must have,

$$Z_{in} = (jX_C)^* = \frac{j}{\omega C}$$

So,

$$C = \frac{1}{\omega X_{in}} = 0.365 \text{ pF} \checkmark$$

The equivalent circuit at 6 GHz, with the shunt resistor, is as follows:



$R = 10,000\ \Omega$

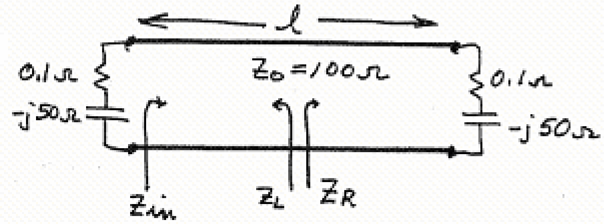
$C = 0.365$ pF

$$L = \frac{X_{in}}{\omega} = \frac{72.6}{2\pi(6 \times 10^9)} = 1.93 \text{ nH} \checkmark$$

So the Q is,

$$Q = \omega RC = 2\pi(6\times 10^9)(10,000)(0.365\times 10^{-12}) = 138. \checkmark$$

6.6



Since the resonator is symmetrical, at the midpoint of the line we must have, $Z_L = Z_R^* = Z_R$, or $\text{Im}\{Z_R\} = 0$:

Let $t = \tan \beta\ell/2$ and $Z_L = R_L + jX_L$. $(R_L = 0.1,\ X_L = -50.)$

$$Z_R = Z_0 \frac{Z_L + jZ_0 t}{Z_0 + jZ_L t} = Z_0 \frac{R_L + j(X_L + Z_0 t)}{(Z_0 - X_L t) + jR_L t}$$

$$= Z_0 \frac{R_L(Z_0 - X_L t) + R_L t(X_L + Z_0 t) + j(X_L + Z_0 t)(Z_0 - X_L t) - jR_L^2 t}{(Z_0 - X_L t)^2 + (R_L t)^2}$$

$$\text{Im}\{Z_R\} = 0 \Rightarrow \quad (X_L + Z_0 t)(Z_0 - X_L t) - R_L^2 t = 0$$

$$-X_L Z_0 t^2 + (Z_0^2 - X_L^2 - R_L^2)t + Z_0 X_L = 0$$

$$5000t^2 + 7500t - 5000 = 0$$

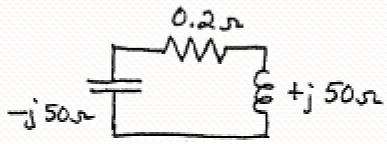
$$t^2 + 1.5t - 1 = 0$$

$$t = \frac{-1.5 \pm \sqrt{(1.5)^2 + 4}}{2} = -0.75 \pm 1.25 = \begin{cases} 0.50 \Rightarrow \beta\ell = 53.1^\circ \\ -2.00 \Rightarrow \beta\ell = -126.9^\circ = 53.1^\circ \end{cases}$$

So,

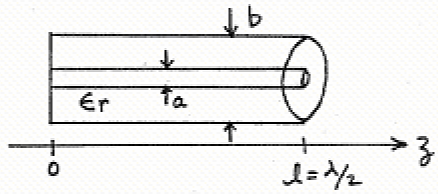
$$\ell = \frac{53.1^\circ}{360^\circ}\lambda = 0.148\lambda \qquad \tan \beta\ell = 1.332$$

CHECK: $$Z_{in} = 100 \frac{(.1 - j50) + j133.2}{100 + j(.1 - j50)(1.332)} = 0.1 + j50\,\Omega \checkmark$$



$$Q = \frac{\omega_0 L}{R} = \frac{X_L}{R} = \frac{50}{0.2} = 250.$$

6.7



$\beta\ell=\pi$

$\beta=\pi/\ell$

$\eta=\sqrt{\mu_0/\epsilon}$

From Section 2.2 the TEM fields of a coaxial line are,

$$\bar{E}^{\pm}=\hat{\rho}\frac{V_0}{\rho\ln b/a}e^{\mp j\beta z}\quad,\qquad \bar{H}^{\pm}=\pm\hat{\phi}\frac{V_0}{\eta\ln b/a}e^{\mp j\beta z}$$

$E_\rho=0$ at $z=0$ in the resonator, so the standing wave fields can be written as,

$$E_\rho=\frac{V_0}{\rho\ln b/a}\left[e^{-j\beta z}-e^{j\beta z}\right]=\frac{-2jV_0}{\rho\ln b/a}\sin\beta z$$

$$H_\phi=\frac{V_0}{\eta\ln b/a}\left[e^{-j\beta z}+e^{j\beta z}\right]=\frac{2V_0}{\eta\ln b/a}\cos\beta z$$

From (1.84) and (1.86) the time-average stored electric and magnetic energies are,

$$W_e=\frac{\epsilon}{4}\int_v|E|^2dv=\frac{\epsilon}{4}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{\ell}\left(\frac{2V_0}{\rho\ln b/a}\right)^2\sin^2\frac{\pi z}{\ell}\,\rho\,dz\,d\phi\,d\rho$$

$$=\frac{\pi\epsilon V_0^2}{\ln b/a}$$

$$W_m=\frac{\mu_0}{4}\int_v|\bar{H}|^2dv=\frac{\mu_0}{4}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{\ell}\left(\frac{2V_0}{\eta\ln b/a}\right)^2\cos^2\frac{\pi z}{\ell}\,\rho\,dz\,d\phi\,d\rho$$

$$=\frac{\pi\mu_0V_0^2}{\eta^2\ln b/a}=\frac{\pi\epsilon V_0^2}{\ln b/a}=W_e\ \checkmark$$

6.8

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{Z_o^2}{R + j(\omega L - \frac{1}{\omega C})} = \frac{1}{\frac{R}{Z_o^2} + j\omega\left(\frac{L}{Z_o^2} - \frac{1}{\omega^2 C Z_o^2}\right)}$$

The input impedance of a parallel RLC circuit is,

$$Z_{in} = \frac{1}{\frac{1}{R'} + \frac{1}{j\omega L'} + j\omega C'} = \frac{1}{\frac{1}{R'} + j\omega\left(C' - \frac{1}{\omega^2 L'}\right)}$$

Thus the original circuit acts as a parallel $R'L'C'$ resonator with $R' = Z_o^2/R$, $C' = L/Z_o^2$, $L' = C Z_o^2$.
(This is the basis for using $\lambda/4$ lines as impedance and admittance inverters.)

6.9

air-filled, aluminium, X-band,
$d = 2.0\text{cm}$, $a = 2.286\text{ cm}$, $b = 1.016\text{ cm}$
$\sigma_{al} = 3.816 \times 10^7 \text{ S/m}$

$$f_{101} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2} = 9.965\text{GHz} \checkmark \quad R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 0.0321\,\Omega$$
$$k = 208.7\text{ m}^{-1}$$

$$f_{202} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{d}\right)^2} = 16.372\text{GHz} \checkmark \quad R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 0.0412\,\Omega$$
$$k = 342.9\text{ m}^{-1}$$

From (6.46), $(2\ell^2 a^3 b + 2bd^3 + \ell^2 a^3 d + ad^3) =$
$= 34.54 + 48.17\,\ell^2 \text{ cm}^4$

$$Q_{101} = \frac{k^3 a^3 d^3 b \eta_0}{2\pi^2 R_s} \frac{1}{(34.54 + 48.17)10^{-8}} = 6349. \checkmark$$

$$Q_{102} = \frac{k^3 a^3 d^3 b \eta_0}{2\pi^2 R_s} \frac{1}{(34.54 + 4 \times 48.17)10^{-8}} = 7987. \checkmark$$

verified w/ RECCAVITY.FOR

6.10 From Table 3.2, the magnetic fields of the TM_{11} waveguide mode are,

$$H_x^{\pm} = \frac{B^{\pm}}{b} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} e^{\mp j\beta z}$$

$$H_y^{\mp} = \frac{B^{\pm}}{a} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} e^{\mp j\beta z}$$

To have current maxima at $z=0,d$ the cavity fields must be,

$$H_x = \frac{A}{b} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} \cos\frac{\pi z}{d}$$

$$H_y = \frac{A}{a} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} \cos\frac{\pi z}{d}$$

The stored magnetic energy is,

$$W_m = \frac{\mu_0}{4}\int_v |\bar{H}|^2 dv = \frac{\mu_0}{4} A^2 \frac{a}{2}\frac{b}{2}\frac{d}{2}\left(\frac{1}{b^2}+\frac{1}{a^2}\right) = \frac{abd\mu_0 A^2}{32}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

The power lost in the walls is,

$$P_\ell = \frac{R_s}{2}\int_s |\bar{H}_t|^2 ds = R_s\left\{\int_{x=0}^{a}\int_{z=0}^{d} |H_x(y=0)|^2 dz\,dx + \int_{y=0}^{b}\int_{z=0}^{d} |H_y(x=0)|^2 dy\,dz + \right.$$

$$\left. + \int_{x=0}^{a}\int_{y=0}^{b}\left[|H_x(z=0)|^2 + |H_y(z=0)|^2\right] dx\,dy\right\}$$

$$= \frac{A^2 R_s}{4}\,\frac{a^3 d + b^3 d + a^3 b + a b^3}{a^2 b^2}$$

Then,

$$Q = \frac{\omega_0 (W_e + W_m)}{P_\ell} = \frac{2\omega_0 W_m}{P_\ell} = \frac{k_0 \eta_0}{4 R_s}\,\frac{abd\,(a^2+b^2)}{(a^3 d + b^3 d + a^3 b + a b^3)} \checkmark$$

6.11 From Section 3.3 the transverse fields of the TE_{10} mode in the two regions can be written as,

$$E_y = \begin{cases} A \sin\frac{\pi x}{a} \sin\beta_a z & \text{for } 0<z<d-t \\ B \sin\frac{\pi x}{a} \sin\beta_d (d-z) & \text{for } d-t<z<d \end{cases}$$

$$H_x = \begin{cases} -j\frac{A}{Z_a} \sin\frac{\pi x}{a} \cos\beta_a z & \text{for } 0<z<d-t \\ -j\frac{B}{Z_d} \sin\frac{\pi x}{a} \cos\beta_d (d-z) & \text{for } d-t<z<d \end{cases}$$

where $\beta_a = \sqrt{k_o^2-(\pi/a)^2}$, $\beta_d = \sqrt{\epsilon_r k_o^2-(\pi/a)^2}$

$Z_a = k_o\eta_o/\beta_a$, $Z_d = k\eta/\beta_d = k_o\eta_o/\beta_d$

Continuity of E_y, H_x at $z=d-t$:

E_y: $\quad A\sin\beta_a(d-t) = B\sin\beta_d t$

H_x: $\quad \frac{A}{Z_a}\cos\beta_a(d-t) = \frac{B}{Z_d}\cos\beta_d t$

Divide to obtain:

$$Z_a \tan\beta_a(d-t) = Z_d \tan\beta_d t$$

$$\beta_d \tan\beta_a(d-t) = \beta_a \tan\beta_d t$$

This equation can be solved for k_o. β_a and β_d are functions of k_o as given above.

6.12 TM modes: $(\nabla^2+k^2)E_z=0$

Let $E_z(x,y,z)=X(x)Y(y)Z(z)$.
Substitute into wave equation and divide by XYZ:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+k^2=0$$

By the separation of variables argument,

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2 \quad \Rightarrow X(x)=A\cos k_x x+B\sin k_x x$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2 \quad \Rightarrow Y(y)=C\cos k_y y+D\sin k_y y$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=-k_z^2 \quad \Rightarrow Z(z)=E\cos k_z z+F\sin k_z z$$

with $k^2=k_x^2+k_y^2+k_z^2$.

Now, $E_z=0$ for $x=0,a$ and $y=0,b$. Therefore, $A=C=0$ and $k_x=\frac{m\pi}{a}$, $k_y=\frac{n\pi}{b}$. To enforce the remaining boundary conditions, we need E_x or E_y: From Maxwell's equations,

$$E_x=\frac{1}{k^2-k_z^2}\frac{\partial^2E_z}{\partial x\partial z}=\frac{1}{k^2-k_z^2}(Bk_x\cos k_x x)(D\sin k_y y)\cdot$$
$$\cdot(-k_z E\sin k_z z+k_z F\cos k_z z)$$

For $E_x=0$ at $z=0,d$ we must have $F=0$, and $k_z=\frac{\ell\pi}{d}$.

Thus,
$$k^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{\ell\pi}{d}\right)^2,$$

which determines the resonant frequencies. The solution for TE modes is similar.

6.13 From Table 3.5 the fields of the TM_{nm0} mode are ($\beta = 0$):

$$E_z = A \sin n\phi \, J_n(k_c \rho)$$

$$H_\rho = \frac{j\omega\epsilon n}{k_c^2 \rho} A \cos n\phi \, J_n(k_c\rho)$$

$$H_\phi = \frac{-j\omega\epsilon}{k_c} A \sin n\phi \, J_n'(k_c\rho) \, , \qquad k_c = p_{nm}/a = k$$

The stored electric energy is,

$$W_e = \frac{\epsilon}{4}\int_v |\bar{E}|^2 dv = \frac{A^2\epsilon}{4}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{d} \sin^2 n\phi \, J_n^2(k_c\rho)\, \rho \, d\rho \, d\phi \, dz$$

$$= \frac{A^2\epsilon}{4}\pi d \frac{a^2}{2} J_n'^2(p_{nm}) = \frac{A^2 a^2 \pi d \epsilon}{8} J_n'^2(p_{nm}) \quad \text{(using C.14}$$

The power loss due to finite conductivity is,

$$P_\ell = \frac{R_s}{2}\int_s |\bar{H}_t|^2 ds$$

$$= \frac{R_s}{2}\left\{\int_{\phi=0}^{2\pi}\int_{z=0}^{d} |H_\phi(\rho=a)|^2 a \, d\phi \, dz + 2\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\left[|H_\rho|^2 + |H_\phi|^2\right]\rho \, d\rho \, d\phi\right\}$$

$$= \frac{A^2 R_s}{2}\left\{\frac{\pi a d}{\eta^2} J_n'^2(p_{nm}) + \frac{2\pi}{\eta^2}\frac{p_{nm}^2}{2k_c^2} J_n'^2(p_{nm})\right\}$$

$$= \frac{A^2 R_s \pi}{2\eta^2}(ad + a^2) J_n'^2(p_{nm})$$

Then, $$Q_c = \frac{2\omega W_e}{P_\ell} = \frac{\omega a^{\cancel{2}}\pi d\epsilon(2\eta^2)}{4R_s\pi \cancel{a}(d+a)} = \frac{adk\eta}{2R_s(d+a)} \quad \checkmark$$

The power lost in the dielectric is,

$$P_d = \frac{\omega\epsilon''}{2}\int_v |\bar{E}|^2 dv = \frac{\omega\epsilon}{2}\tan\delta\int_v|\bar{E}|^2 dv = \frac{2kW_e}{\eta\epsilon}\tan\delta$$

So, $$Q_d = \frac{2\omega W_e}{P_\ell} = \frac{1}{\tan\delta} \quad \checkmark \qquad \text{(as in (6.48))}$$

6.14 From Figure 6.10, maximum Q for the TE_{111} mode occurs for $2a/d \simeq 1.7$. From (6.53a) the resonant frequency is,

$$f_{111} = \frac{c}{2\pi\sqrt{\epsilon_r}}\sqrt{\left(\frac{p'_{11}}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2} = \frac{3\times10^8}{2\pi\sqrt{1.5}}\sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{1.7\pi}{2a}\right)^2}$$

$$= \frac{1.264\times10^8}{a} = 6\times10^9\ Hz \Rightarrow \underline{a = 2.107\ cm}$$

$$d = \frac{2a}{1.7} = \underline{2.479\ cm}$$

$\sigma_{AU} = 4.1\times10^7$ S/m , $R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 0.024\ \Omega$

$k = 2\pi f\sqrt{\epsilon_r}/c = 153.9\ m^{-1}$

$\beta = \frac{\pi}{d} = 126.7\ m^{-1}$

From (6.57) the unloaded Q is, (due to conductor losses)

$$Q_c = \frac{(ka)^3 \eta a d \left[1-\left(\frac{1}{p'_{11}}\right)^2\right]}{4(p'_{11})^2 R_s \left\{\frac{ad}{2}\left[1+\left(\frac{\beta a}{p'^2_{11}}\right)^2\right] + \left(\frac{\beta a^2}{p'_{11}}\right)^2\left(1-\frac{1}{p'^2_{11}}\right)\right\}}$$

$$= 10,985\ \checkmark$$

The unloaded Q due to dielectric loss is

$$Q_d = \frac{1}{\tan\delta} = 2,000\ \checkmark$$

Then the total Q is,

$$Q = \frac{1}{\frac{1}{Q_d} + \frac{1}{Q_c}} = 1,692\ \checkmark$$

(results checked with FORTRAN program CIRCAVITY.FOR)

6.15 Choose coordinate system so that $b < a < d$.
Then the dominant resonant mode is the TE_{101} mode:

$$f_{101} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2} = 5.2 \text{ GHz}$$

or,

$$\frac{1}{a^2} + \frac{1}{d^2} = \left(\frac{2f_{101}}{c}\right)^2 = (34.7)^2$$

The next two higher modes must be either the TM_{110}, TE_{102}, or TE_{011} modes:

$$\left(\frac{2f_{110}}{c}\right)^2 = \frac{1}{a^2} + \frac{1}{b^2} = (34.7)^2 + \frac{1}{b^2} - \frac{1}{d^2}$$

$$\left(\frac{2f_{102}}{c}\right)^2 = \frac{1}{a^2} + \frac{4}{d^2} = (34.7)^2 + \frac{3}{d^2}$$

$$\left(\frac{2f_{011}}{c}\right)^2 = \frac{1}{b^2} + \frac{1}{d^2}$$

Since $d > a$, $f_{011} < f_{110}$

Try $f_{011} = 6.5$ GHz ; $f_{110} = 7.2$ GHz

Then we have, $\frac{1}{b^2} - \frac{1}{d^2} = 1100.$

$$\frac{1}{b^2} + \frac{1}{d^2} = 1878.$$

Solving gives,

$b = 2.60$ cm ✓

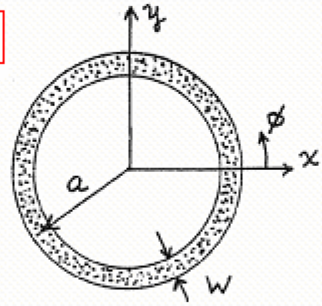
$d = 5.00$ cm ✓

$a = 3.53$ cm ✓

CHECK: $b < a < d$ OK ✓

$f_{102} = 7.35 \text{ GHz} > f_{110} = 7.2 \text{ GHz}$ ✓

6.16



$$e^{\pm j\beta a\phi} = e^{\pm jn\phi} \quad , \; n=1,2,3\ldots$$

FOR PERIODICITY

So, $\quad \beta a = \dfrac{2\pi a}{\lambda_g} = \dfrac{2\pi a\sqrt{\epsilon_e}\,f}{c} = n$

$$f = \frac{nc}{2\pi a\sqrt{\epsilon_e}} \; ; \quad n=1,2,3\ldots$$

(The ring circumference is $2\pi a = n\lambda_g$)

The above result assumes $a \gg W$, so that curvature effects can be neglected. This type of resonator is most often coupled using a gap feed to a microstripline.

6.17 For TM_{nm0} modes we have $H_z=0$ and $\frac{\partial}{\partial z}=0$. The wave equation for E_z is,

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2\right)E_z = 0 \qquad \text{(from 3.134)} \; (\beta = k)$$

The general solution is,

$$E_z = (A_n \cos n\phi + B_n \sin n\phi)\, J_n(k\rho) \qquad \text{(finite at } \rho=0)$$

Since the choice of $\sin n\phi$ or $\cos n\phi$ (or any combination) depends only on the choice of the $\phi=0$ reference, we can let $B_n=0$. Then,

$$E_z = A_n \cos n\phi \, J_n(k\rho)$$

We can find H_ϕ from (3.110d):

$$H_\phi = \frac{-j\omega\epsilon}{k^2}\frac{\partial E_z}{\partial\rho} = \frac{-j\omega\epsilon}{k} A_n \cos n\phi \, J_n'(k\rho)$$

For $H_\phi=0$ at $\rho=a$ we require $J_n'(ka)=0$, or $ka = p'_{nm}$.
So the resonant frequency is,

$$f_{nm0} = \frac{ck}{2\pi\sqrt{\epsilon_r}} = \frac{c\,p'_{nm}}{2\pi a\sqrt{\epsilon_r}}$$

and,

$$f_{110} = \frac{c\,p'_{11}}{2\pi a\sqrt{\epsilon_r}} = \frac{1.841c}{2\pi a\sqrt{\epsilon_r}} \quad \checkmark$$

This solution neglects the effect of fringing fields.

6.18 From (6.70), $\tan \beta L/2 = \alpha/\beta$,

with
$$\alpha = \sqrt{\left(\frac{2.405}{a}\right)^2 - k_o^2}$$
$$\beta = \sqrt{\epsilon_r k_o^2 - (2.405/a)^2}$$

The value of k_o at resonance must lie between $k_o = \frac{2.405}{a} = 602$, and $k_o = \frac{2.405}{a\sqrt{\epsilon_r}} = 100$.

We carry out a trial-and-error numerical search as follows:

	k_o	α	β	$\tan \beta L/2 - \alpha/\beta$
	110	592	275	−1.8
	120	590	399	−1.02
	150	583	672	.008
	145	584	631	−.12
→	149	583	664	−.0018

Thus, the resonant frequency is,
$$f_o = \frac{c k_o}{2\pi} = 7.11 \text{ GHz} \checkmark$$

(measured value is 7.8 GHz)

6.19 Following the analysis of Section 6.5, for $TE_{01\delta}$ mode:

$$H_z = H_0 J_0(k_c\rho)\, e^{\pm j\beta z}$$

$$E_\phi = \frac{j\omega\mu_0 H_0}{k_c} J_0'(k_c\rho)\, e^{\pm j\beta z} = A\, J_0'(k_c\rho)\, e^{\pm j\beta z}$$

$$H_\rho = \frac{\mp j\beta H_0}{k_c} J_0'(k_c\rho)\, e^{\pm j\beta z} = \frac{\mp A}{Z_{TE}} J_0'(k_c\rho)\, e^{\pm j\beta z}$$

for $|z| < L/2$, $\beta = \sqrt{\epsilon_r k_0^2 - k_c^2} = \sqrt{\epsilon_r k_0^2 - (p_{01}/a)^2}$; $Z_{TE} = \frac{\omega\mu_0}{\beta} = Z_d$

for $|z| > L/2$, $j\beta = \alpha = \sqrt{k_c^2 - k_0^2} = \sqrt{(p_{01}/a)^2 - k_0^2}$; $Z_{TE} = \frac{j\omega\mu_0}{\alpha} = Z_a$

So the standing wave fields can be written as,

$$E_\phi = \begin{cases} A\, J_0'(k_c\rho)\left[e^{-j\beta z} - e^{j\beta z}\right] = -2jA\, J_0'(k_c\rho)\sin\beta z & \text{for } |z| < L/2 \\ B\, J_0'(k_c\rho)\, e^{-\alpha z} & \text{for } z > L/2 \end{cases}$$

$$H_\rho = \begin{cases} \frac{A}{Z_d} J_0'(k_c\rho)\left[e^{-j\beta z} + e^{j\beta z}\right] = \frac{2A}{Z_d} J_0'(k_c\rho)\cos\beta z & \text{for } |z| < L/2 \\ \frac{B}{Z_a} J_0'(k_c\rho)\, e^{-\alpha z} & \text{for } z > L/2 \end{cases}$$

Continuity of E_ϕ and H_ρ at $z = L/2$ gives:

$$E_\phi: \quad -2jA \sin\beta L/2 = B e^{-\alpha L/2}$$

$$H_\rho: \quad \frac{2A}{Z_d}\cos\beta L/2 = \frac{B}{Z_a} e^{-\alpha L/2}$$

dividing gives:

$$-j Z_d \tan\beta L/2 = Z_a$$

$$\frac{-j}{\beta}\tan\beta L/2 = j/\alpha$$

$$\tan\beta L/2 + \beta/\alpha = 0 \quad ✓$$

6.20 assume $a>b$

Because of the magnetic wall boundary conditions on the sidewalls, a rectangular dielectric waveguide along the z-axis would support TE modes with an H_z field of the form,

$$H_z = H_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

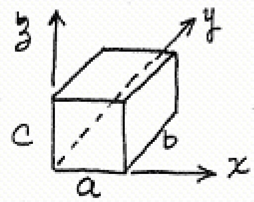
so the lowest order TE mode would be the TE_{11} mode. But $H_z \equiv 0$ for TM modes, so the lowest order TM mode would have,

$$H_x = H_0 \sin\frac{\pi x}{a}, \quad H_y = 0 \quad (\text{if } a>b)$$

So the dominant mode of this resonator must be the $TM_{10\delta}$ mode. Thus we can write,

$$E_y = E_0 \sin\frac{\pi x}{a} e^{\pm j\beta z}$$

$$H_x = \frac{\pm E_0}{Z_{TM}} \sin\frac{\pi x}{a} e^{\pm j\beta z},$$



where,

$$\beta = \sqrt{\epsilon_r k_0^2 - (\pi/a)^2} \qquad \text{for } |z| < c/2$$

$$j\beta = \alpha = \sqrt{(\pi/a)^2 - k_0^2} \qquad \text{for } z > c/2,$$

and $Z_{TM} = Z_d = \beta\eta/k = \beta\eta_0/\epsilon_r k_0 \qquad$ for $|z|<c/2$,

$Z_{TM} = Z_a = -j\alpha\eta_0/k_0 \qquad$ for $z > c/2$

Then the standing wave fields can be written as,

$$E_y = \begin{cases} A\sin\frac{\pi x}{a}\left[e^{-j\beta z} + e^{j\beta z}\right] = 2A\sin\frac{\pi x}{a}\cos\beta z & \text{for } |z|<c/2 \\ B\sin\frac{\pi x}{a} e^{-\alpha z} & \text{for } z > c/2 \end{cases}$$

$$H_x = \begin{cases} \frac{A}{Z_d}\sin\frac{\pi x}{a}\left[-e^{-j\beta z} + e^{j\beta z}\right] = \frac{2jA}{Z_d}\sin\frac{\pi x}{a}\sin\beta z & \text{for } |z|<c/2 \\ \frac{-B}{Z_a}\sin\frac{\pi x}{a} e^{-\alpha z} & \text{for } z>c/2 \end{cases}$$

Continuity of E_y, H_x at $z = c/2$:

$$2A\cos\beta c/2 = Be^{-\alpha c/2}$$

$$\frac{2jA}{Z_d}\sin\beta c/2 = \frac{-B}{Z_a} e^{-\alpha c/2}$$

divide to get:

$$\alpha\epsilon_r \tan\beta c/2 + \beta = 0$$

6.21

a) $d = \ell\lambda_o/2 = \frac{\ell}{2}\frac{c}{f_o} \Rightarrow f_o = \frac{\ell c}{2d}$ ✓

b) $E_x = E_o \sin k_o z$

$H_y = \frac{jE_o}{\eta_o} \cos k_o z$

$$W_e = \frac{\epsilon_o}{4}\int_{z=0}^{d} |E_x|^2 dz = \frac{\epsilon_o |E_o|^2}{4}\int_{z=0}^{d} \sin^2 \frac{\ell\pi z}{d} dz = \frac{\epsilon_o |E_o|^2 d}{8}$$

$$W_m = \frac{\mu_o}{4}\int_{z=0}^{d} |H_y|^2 dz = \frac{\mu_o |E_o|^2}{4\eta_o^2}\int_{z=0}^{d} \cos^2 \frac{\ell\pi z}{d} dz = \frac{\mu_o |E_o|^2 d}{8\eta_o^2} = \frac{\epsilon_o |E_o|^2 d}{8}$$

Thus $W_e = W_m$ at resonance ✓

$$P_c = 2\left(\frac{R_s}{2}\right)|H_y(z=0)|^2 = \frac{R_s |E_o|^2}{\eta_o^2}, \quad R_s = \sqrt{\frac{\omega\mu_o}{2\sigma}}$$

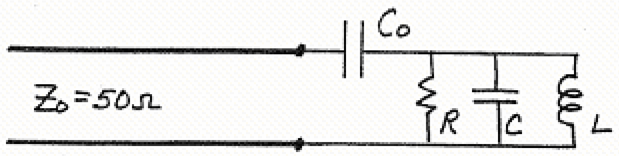
$$Q_c = \omega(W_e + W_m)/P_c = \frac{\omega\epsilon_o d\eta_o^2}{4R_s} = \frac{c\pi\ell\epsilon_o\eta_o^2}{4R_s} = \frac{\pi\ell\eta_o}{4R_s}$$

c) $f_o = \frac{(25)(3\times10^8)}{2(.04)} = \underline{93.8 \text{ GHz}}$

$R_s = 0.08\ \Omega$

$Q_c = \frac{\pi(25)(377)}{4(.08)} = \underline{92{,}500}$

6.22



$R = 1000\ \Omega$
$L = 1.26\ nH$
$C = 0.804\ pF$

The simplest way to solve this problem is graphically, with a Smith chart. The admittance of the resonator at frequencies near resonance is,

$$Y_R = \frac{1}{R} + j\frac{2Q\Delta\omega}{R\omega_o},$$

where $\omega_o = \frac{1}{\sqrt{LC}} = 3.142 \times 10^{10}$ RPS ; $f_o = \frac{\omega_o}{2\pi} = 5.00$ GHz

$$Q = \frac{R}{\omega_o L} = 25.3$$

Normalized to Z_o, we have $y_R = Z_o Y_R = 0.05 + j2.53\frac{\Delta\omega}{\omega_o}$. We can plot y_R on a Smith chart, versus $\Delta\omega/\omega_o$. For $\Delta\omega = 0$, $y_R = 0.05$. For $\Delta\omega = \pm 0.1\omega_o$, $y_R = 0.05 \pm j0.253$.
Next, convert this locus to z_R, an impedance locus. Then we see that a series capacitive reactance of $-jX_{C_o} = -j4.2$ will yield an input impedance of $z_{in} = 1$. This corresponds to a resonator admittance $y_R = 0.05 - j0.22$. So the resonant frequency will be,

$$\Delta\omega = \frac{-0.22\omega_o}{2.53} = -0.0869\omega_o$$

$$\omega_r = \omega_o + \Delta\omega = (1 - .0869)\omega_o = 0.913\omega_o$$

so, $\quad f_r = \frac{\omega_r}{2\pi} = 4.566$ GHz (note lowering from f_o)

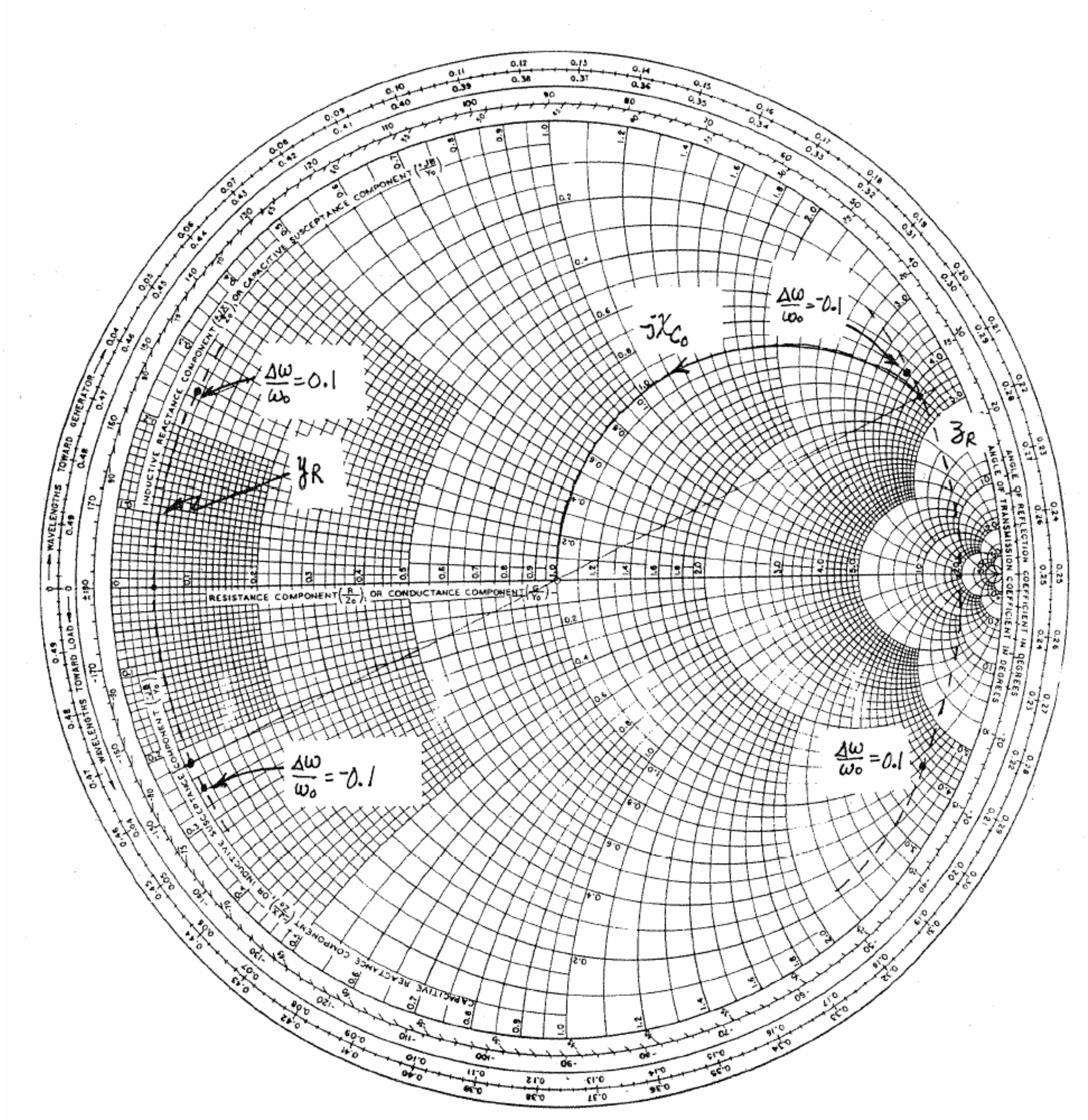
The coupling capacitor value is,

$$C_o = \frac{1}{4.2 Z_o \omega_r} = 0.166\ pF.$$

CHECK: at 4.566 GHz, $Y_R = (1 - j4.39) \times 10^{-3}$ S

$$Z_R = 49.2 + j216.5\ \Omega \simeq 50 + jX_{C_o}$$

$$\frac{1}{j\omega C_o} = -j210.$$



RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE COMPONENT (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE COMPONENT (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
Δω/ω₀ = 0.1
Δω/ω₀ = -0.1
Δω/ω₀ = -0.1
Δω/ω₀ = 0.1
y_R
z_R
-jx_{C_0}

6.23 Assume TE_{101} mode, as in Section 6.6.

At 9 GHz, $k_0 = 188.\ m^{-1}$; $\beta_0 = 140.5\ m^{-1}$; $l = \frac{\lambda_g}{2} = \frac{\pi}{\beta_0} = 2.24$ cm.

$\frac{\omega_0}{2\pi} = f_0 = 9$ GHz is the resonant frequency of the closed cavity, and does not include the effect of the coupling aperture. For a high-Q cavity, the actual resonant frequency, ω_1, will be close to ω_0. So we can approximately compute x_L using ω_0. From (6.89),

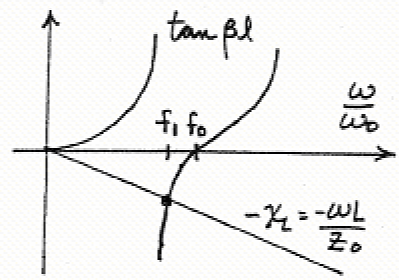
$$x_L = \sqrt{\frac{\pi k_0 \omega_1}{2Q\beta^2 C}} = 0.016 = \frac{\omega L}{Z_0} \Rightarrow \frac{L}{Z_0} = 2.83\times10^{-13}$$

Then solve (6.85) for ω:

$$\tan\beta l + x_L = 0$$

Numerical trial-and-error:

f	β	x_L	$\tan\beta l + x_L$
9	140.	.0160	.01
8.9	137.7	.0158	−.04
8.97	139.65	.0159	.0025



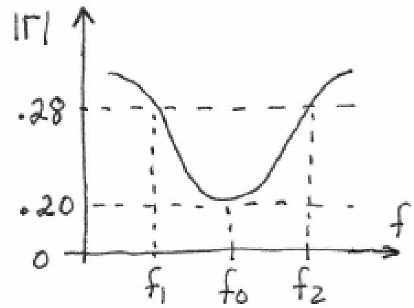
Thus, $f_1 = 8.97$ GHz

6.24 $f_1 = 2.9985$ GHz
$f_2 = 3.0015$ GHz

so $f_0 = 3.0000$ GHz, $BW = 0.1\%$, $Q_L = \frac{1}{BW} = 1000$.

at resonance, $RL = 14$ dB $\Rightarrow \Gamma = 0.200 \Rightarrow r = \frac{1+\Gamma}{1-\Gamma} = 1.5$

at f_1 or f_2, $RL = 11$ dB $\Rightarrow \Gamma = 0.282$



assuming a series resonance, from (6.91), $g = \frac{Z_0}{R} = \frac{1}{r} = 0.667$

$Q_0 = (1+g)Q_L = \underline{\underline{1667}}$

assuming parallel resonator: $g = \frac{R}{Z_0} = r = 1.5$

$Q_0 = (1+g)\,Q_L = \underline{\underline{2500}}$

6.25

f (GHz)	IL (dB)	$\lvert S_{21}\rvert$ (dB)	$\lvert S_{21}\rvert$
3.0000	1.94	−1.94	0.800
2.9925	4.95	−4.95	
3.0075	4.95	−4.95	

$Q_L = \frac{3}{.015} = 200$, $g = \frac{S}{1-S} = \frac{.8}{1-.8} = 4.0$

From (6.91) $Q_0 = Q_L(1+g) = 1000$ ✓

6.26 The unperturbed TE_{101} cavity fields are,

$$E_y = A \sin\frac{\pi x}{a} \sin\frac{\pi z}{d}$$

$$H_x = \frac{-jA}{Z} \sin\frac{\pi x}{a} \cos\frac{\pi z}{d} \qquad ; \; Z = k\eta/\beta$$

$$H_z = \frac{j\pi A}{k\eta a} \cos\frac{\pi x}{a} \sin\frac{\pi z}{d}$$

Then the numerator in (6.95) is,

$$\int_{v_0} \left(\Delta\epsilon |\bar{E}_0|^2 + \Delta\mu |\bar{H}_0|^2\right) dv = (\mu_r - 1)\mu_0 \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{t} \left(|H_x|^2 + |H_z|^2\right) dz\, dy\, dx$$

$$= \mu_0(\mu_r - 1)\frac{ab}{2} A^2 \int_{z=0}^{t} \left(\frac{1}{Z^2} \cos^2\frac{\pi z}{d} + \frac{\pi^2}{k^2\eta^2 a^2} \sin^2\frac{\pi z}{d}\right) dz$$

$$= \mu_0(\mu_r - 1)\frac{ab}{2} A^2 \left[\frac{1}{Z^2}\left(\frac{z}{2} + \frac{\sin\frac{2\pi z}{d}}{4\pi/d}\right)\Bigg|_0^t + \frac{\pi^2}{k^2\eta^2 a^2}\left(\frac{z}{2} - \frac{\sin\frac{2\pi z}{d}}{4\pi/d}\right)\Bigg|_0^t \right]$$

$$= \mu_0(\mu_r - 1)\frac{ab}{2} A^2 \left[\frac{t}{2\eta^2} + \frac{\beta^2 - \pi^2/a^2}{k^2\eta^2} \frac{d}{4\pi} \sin\frac{2\pi t}{d} \right]$$

The denominator in (6.95) is $\dfrac{abd\,\epsilon_0 A^2}{2}$, so

$$\frac{\omega - \omega_0}{\omega_0} = \frac{-(\mu_r - 1)ab\,\eta^2 [\bullet]}{abd}$$

$$= \frac{-(\mu_r - 1)}{d}\left(\frac{t}{2} + \frac{\beta^2 - \pi^2/a^2}{k^2}\frac{d}{4\pi} \sin\frac{2\pi t}{d}\right)$$

For $t \ll d$ this simplifies to,

$$\frac{\omega - \omega_0}{\omega_0} \simeq -(\mu_r - 1)\left(\frac{t}{d}\right)\left(\frac{\beta^2}{k^2}\right)$$

6.27 Following Example 6.8:

at $x = a/2$, $z = 0$: $E_y = 0$

$$H_x = \frac{-jA}{Z} \quad , \; Z = k_0\eta_0/\beta$$

$$H_y = 0$$

Then,

$$\int_{\Delta v} (\mu |\bar{H}_0|^2 - \epsilon |\bar{E}_0|^2)\, dv = \mu_0 \frac{A^2}{Z^2} \Delta V \quad ; \; \Delta V = \pi \ell r_0^2$$

$$\int_{\Delta v} (\mu |\bar{H}_0|^2 + \epsilon |\bar{E}_0|^2)\, dv = \frac{V_0 \epsilon_0 A^2}{2}$$

So (6.102) reduces to,

$$\frac{\omega - \omega_0}{\omega_0} = \frac{2\mu_0 \Delta V}{Z^2 \epsilon_0 V_0} = \frac{2\eta_0^2 \Delta V \beta^2}{k_0^2 \eta_0^2 V_0} = \frac{2\beta^2}{k_0^2} \frac{\Delta V}{V_0}$$

(an increase in resonant frequency)