Chapter 1

What is Operations Research?

Set 2.1a

Buy three roundtrip tickets for the first three weeks only—cost = 3× \$400 = \$1200. Though the cost is cheaper, it is not feasible because it covers only three out of the required five weeks.

Given a string of length L:

(1) $h = .3L$, $w = .2L$, Area $= .06L^2$
(2) $h = .1L$, $w = .4L$, Area $= .04L^2$

Solution (2) is better because the area is larger

$L = 2(w + h)$
$w = L/2 - h$

$z = wh = h(L/2 - h) = Lh/2 - h^2$

$\delta z/\delta h = L/2 - 2h = 0$

Thus, $h = L/4$ and $w = L/4$.

Solution is optimal because z is a concave function

(a)
Let T = Total tie to move all four individuals to the other side of the river. the objective is to determine the transfer schedule that minimizes T.

(b)
Let t = crossing time from one side to the other. Use codes 1, 2, 5, and 10 to represent Amy, Jim, John, and Kelly.

continued...

4 cont.

East	Crossing	West
5,10	(1,2)→ **(t = 2)**	1,2
1,5,10	**(t = 1)**←(1)	2
1	(5,10)→ **(t = 10)**	2,5,10
1,2	**(t = 2)**←(2)	5,10
none	(1,2)→ **(t = 2)**	2,5,10
Total = 2 + 1 + 10 + 2 + 2 = 17 minutes		

		Jim	
		Curve	Fast
Joe	Curve	.500	.200
	Fast	.100	.300

(a)
Alternatives:
Joe: Prepare for curve or fast ball.
Jim: Throw curve of fast ball.
(b)
Joe tries to improve his batting score and Jim tries to counter Joe's action by selecting a less favorable strategy. This means that neither player will be satisfied with a single (pure) strategy.

The problem is not an optimization situation in the familiar sense in which the objective is maximized or minimized. Instead, the conflicting situation requires a compromise solution in which neither layer is tempted to change strategy. Game theory (Chapter 14) provides such a solution.

Let L=ops. 1 and 2=20 sec, C=ops. 3 and 4=25 sec, U=op. 5=20 sec

Gant chart: L1+load horse 1, L2=load horse 2, etc.

one joist: 0---L1---**20**---C1---45----U1+L1---**85**----U2+L2----125---U1+L1---**165**---U2+L2---*205*

20-L2-40 45---C2----70 **85**---C1---110 125---C2---140 **165**-C1-190

205---C2---230---U2---250

Total = 250

Loaders utilization=[250-(5+25)]/250=88%

Cutter utilization=[250-(20+15+15+15+15)]/250=68%

two joists: 0---2L1---**40**-----2C1-----**90**----2(U1+L1)---**170**----2C1----220---2U1---260

40---2L2---80 **90**---2C2----140 **170**---2U2---210

Total =260

Loaders utilization=[260-(10+10)]/260=92%

Cutter utilization=[260-(40+30+40)]/250=58%

three joists: 0---3L1---**60**------3C1------**135**------3C2---------210----3U2----270

60---3L2---120 **135**-----3U1-----195

Total =270

Loaders utilization=[270-(15+15)]/270=89%

Cutter utilization=[270-(60+60)]/270=56%

Recommendation: One joist at time gives the smallest time. The problem has other alternatives that combine 1, 2, and 3 joists. Cutter utilization indicates that cutter represents the bottleneck.

CHAPTER 2

Modeling with Linear Programming

Set 2.1a

1

(a) $x_2 - x_1 \geq 1$ or $-x_1 + x_2 \geq 1$

(b) $x_1 + 2x_2 \geq 3$ and $x_1 + 2x_2 \leq 6$

(c) $x_2 \geq x_1$ or $x_1 - x_2 \leq 0$

(d) $x_1 + x_2 \geq 3$

(e) $\frac{x_2}{x_1 + x_2} \leq .5$ or $.5x_1 - .5x_2 \geq 0$

2

(a) $(x_1, x_2) = (1, 4)$

$$\begin{array}{l}(x_1, x_2) \geq 0 \\ 6 \times 1 + 4 \times 4 = 22 < 24 \\ 1 \times 1 + 2 \times 4 = 9 \nleq 6 \quad \text{infeasible}\end{array}$$

(b) $(x, x_2) = (2, 2)$

$$(x_1, x_2) \geq 0$$

$$\left.\begin{array}{rcl}6 \times 2 + 4 \times 2 = 20 & < & 24 \\ 1 \times 2 + 2 \times 2 = 6 & = & 6 \\ -1 \times 2 + 1 \times 2 = 0 & < & 1 \\ 1 \times 2 = 2 & = & 2\end{array}\right\} \text{feasible}$$

$$Z = 5 \times 2 + 4 \times 2 = \$18$$

(c) $(x_1, x_2) = (3, 1.5)$

$$x_1, x_2 \geq 0$$

$$\left.\begin{array}{rcl}6 \times 3 + 4 \times 1.5 = 24 & = & 24 \\ 1 \times 3 + 2 \times 1.5 = 6 & = & 6 \\ -1 \times 3 + 1 \times 1.5 = -1.5 & < & 1 \\ 1 \times 1.5 = 1.5 & < & 2\end{array}\right\} \text{feasible}$$

$$Z = 5 \times 3 + 4 \times 1.5 = \$21$$

(d) $(x_1, x_2) = (2, 1)$

$$x_1, x_2 \geq 0$$

$$\left.\begin{array}{rcl}6 \times 2 + 4 \times 1 = 16 & < & 24 \\ 1 \times 2 + 2 \times 1 = 4 & < & 6 \\ -1 \times 2 + 1 \times 1 = -1 & < & 1 \\ 1 \times 1 = 1 & < & 2\end{array}\right\} \text{feasible}$$

$$Z = 5 \times 2 + 4 \times 1 = \$14$$

(e) $(x_1, x_2) = (2, -1)$

$x_1 \geq 0$, $x_2 < 0$, infeasible

Conclusion: (c) gives the best feasible solution

3

$(x_1, x_2) = (2, 2)$

Let S_1 and S_2 be the unused daily amounts of M1 and M2.

For M1: $S_1 = 24 - (6x_1 + 4x_2) = 4$ tons/day

For M2: $S_2 = 6 - (x_1 + 2x_2)$

$= 6 - (2 + 2 \times 2) = 0$ tons/day

4

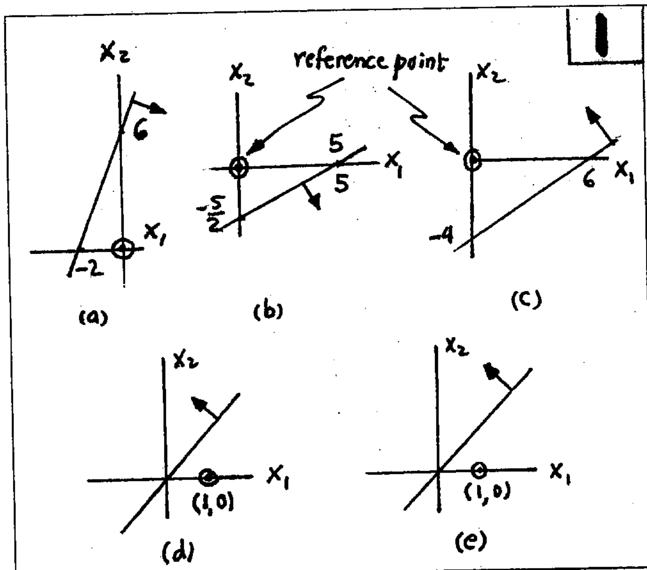
Quantity discount results in the following nonlinear objective function:

$$Z = \begin{cases} 5x_1 + 4x_2, & 0 \leq x_1 \leq 2 \\ 4.5x_1 + 4x_2, & x_1 > 2 \end{cases}$$

The situation cannot be treated as a linear program. Nonlinearity can be accounted for in this case using mixed integer programming (chapter 9).

Set 2.2a

1



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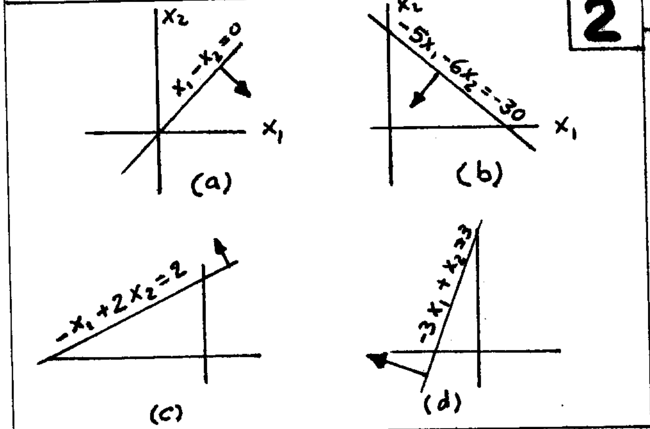
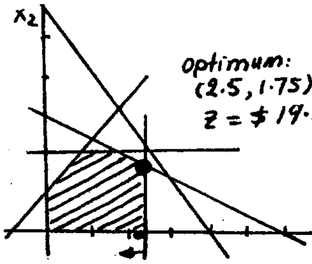


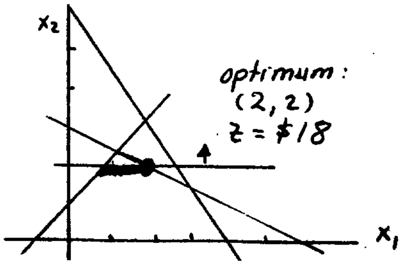
3

(a) $x_1 \le 2.5$



optimum:
$(2.5, 1.75)$
$z = \$19.50$

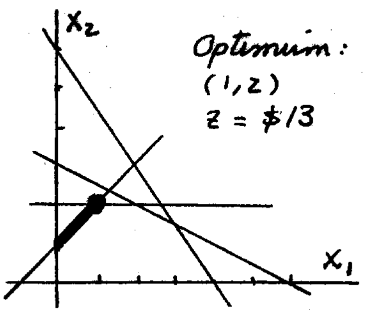
(b) $x_2 \ge 2$



optimum:
$(2, 2)$
$z = \$18$

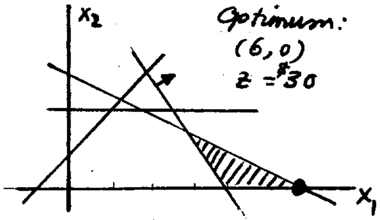
continued...

(c) $-x_1+x_2=1$



Optimum:
$(1,2)$
$z = \$13$

(d) $6x_1+4x_2 \ge 24$



Optimum:
$(6,0)$
$z = \$30$

(e) No feasible space

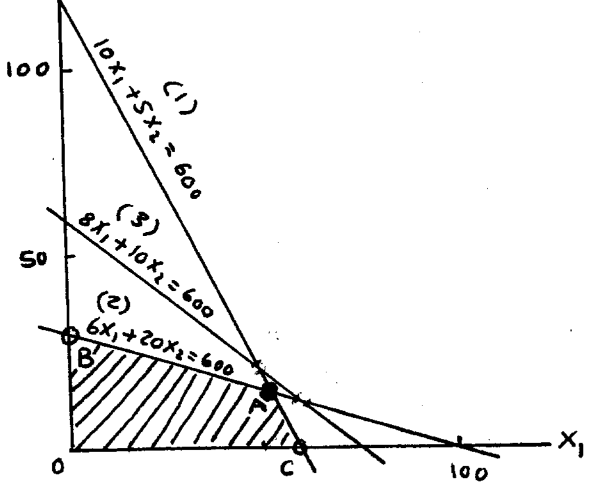
4

x_1 = daily units of product 1
x_2 = daily units of product 2

Maximize $z = 2x_1 + 3x_2$

s.t.

$10x_1 + 5x_2 \le 600$ (1)
$6x_1 + 20x_2 \le 600$ (2)
$8x_1 + 10x_2 \le 600$ (3)



Optimum occurs at A:
$x_1 = 52.94$
$x_2 = 14.12$
$z = \$148.24$

Set 2.2a

5

x_1 = number of units of A

x_2 = number of units of B

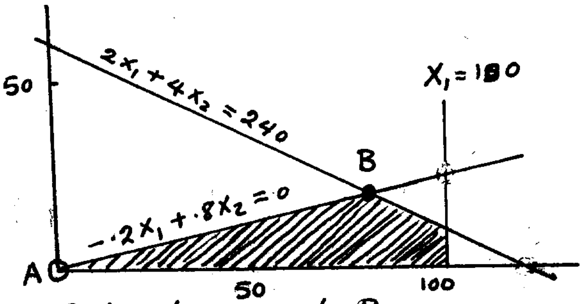
Maximize $z = 20x_1 + 50x_2$

$$\frac{x_1}{x_1+x_2} \geq .8 \quad \text{or} \; -.2x_1 + .8x_2 \leq 0$$

$$x_1 \leq 100$$

$$2x_1 + 4x_2 \leq 240$$

$$x_1, \; x_2 \geq 0$$



Optimal occurs at B:

$x_1 = 80$ units

$x_2 = 20$ units

$z = \$2,600$

6

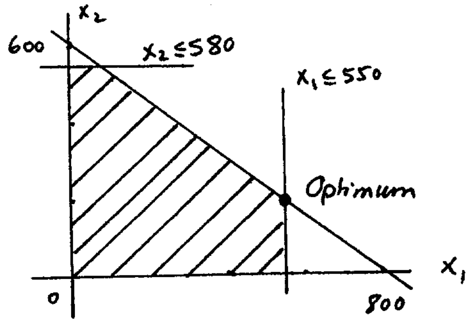
x_1 = number of sheets/day

x_2 = number of bars/day

Maximize $z = 40x_1 + 35x_2$

s.t. $\frac{x_1}{800} + \frac{x_2}{600} \leq 1$

$0 \leq x_1 \leq 550, \quad 0 \leq x_2 \leq 580$



Optimum solution:

$x_1 = 550$ sheets

$x_2 = 187.13$ bars

$z = \$28,549.40$

7

x_1 = \$ invested in A

x_2 = \$ invested in B

Maximize $Z = .05x_1 + .08x_2$

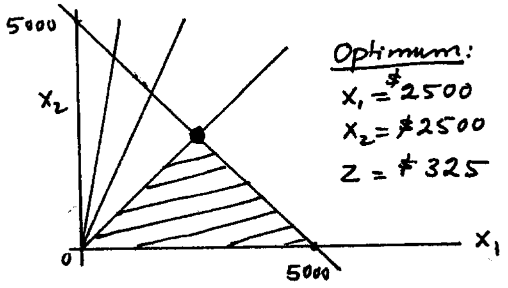
s.t. $x_1 \geq .25(x_1 + x_2)$

$$x_2 \leq .5(x_1 + x_2)$$

$$x_1 \geq .5x_2$$

$$x_1 + x_2 \leq 5000$$

$$x_1, \; x_2 \geq 0$$



Optimum:

$x_1 = \$2500$

$x_2 = \$2500$

$z = \$325$

8

x_1 = number of practical courses

x_2 = number of humanistic courses

Maximize $z = 1500x_1 + 1000x_2$

s.t.

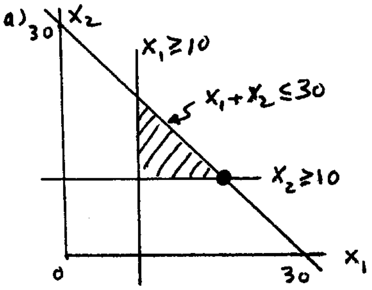
$$x_1 + x_2 \leq 30$$

$$x_1 \geq 10$$

$$x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

(a)



Optimum:

$x_1 = 20$

$x_2 = 10$

$z = \$40,000$

(b) Change $x_1 + x_2 \leq 30$ to $x_1 + x_2 \leq 31$

Optimum $z = \$41,500$

$\Delta z = \$41,500 - 40,000 = \$1500$

Conclusion: Any additional course will be of the practical type.

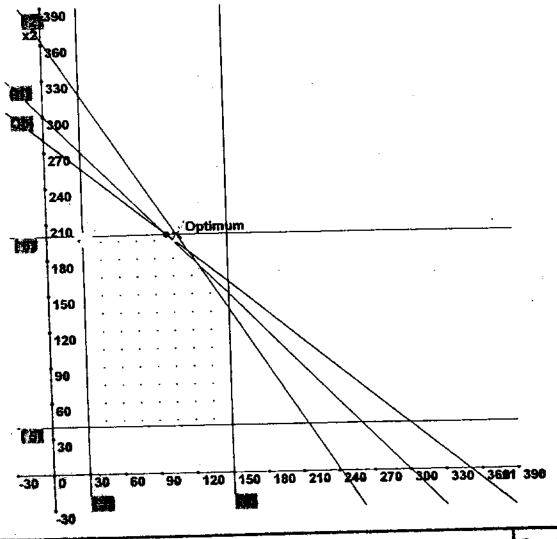
9

x_1 = units of solution A
x_2 = units of solution B

Maximize $z = 8x_1 + 10x_2$
Subject to

$$
\begin{aligned}
.5x_1 + .5x_2 &\le 150\\
.6x_1 + .4x_2 &\le 145\\
x_1 &\ge 30\\
x_1 &\le 150\\
x_2 &\ge 40\\
x_2 &\le 200\\
x_1, x_2 &\ge 0
\end{aligned}
$$

Summary of Optimal Solution:
Objective Value = 2800.00
x1 = 100.00
x2 = 200.00



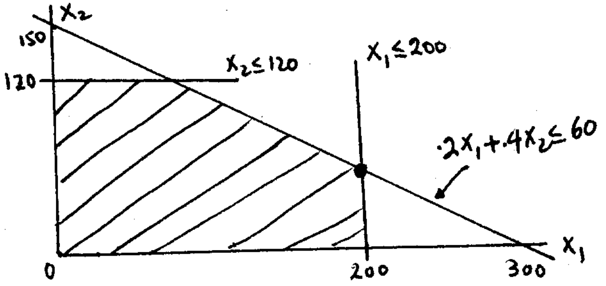
10

x_1 = nbr. of grano boxes
x_2 = nbr. of wheatie boxes

Maximize $Z = x_1 + 1.35x_2$

$$
\begin{aligned}
\text{s.t.}\quad .2x_1 + .4x_2 &\le 60\\
x_1 &\le 200\\
x_2 &\le 120\\
x_1, x_2 &\ge 0
\end{aligned}
$$

continued...



Optimum: $x_1 = 200$, $x_2 = 50$, $Z = \$267.50$

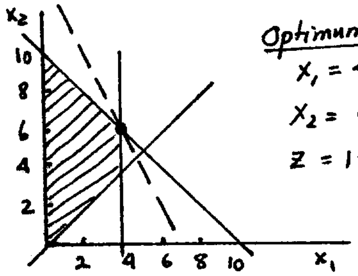
Area allocation: 67% grano, 33% wheatie

11

x_1 = play hours per day
x_2 = work hours per day

Maximize $Z = 2x_1 + x_2$

$$
\begin{aligned}
\text{s.t.}\quad x_1 + x_2 &\le 10\\
x_1 - x_2 &\le 0\\
x_1 &\le 4\\
x_1, x_2 &\ge 0
\end{aligned}
$$



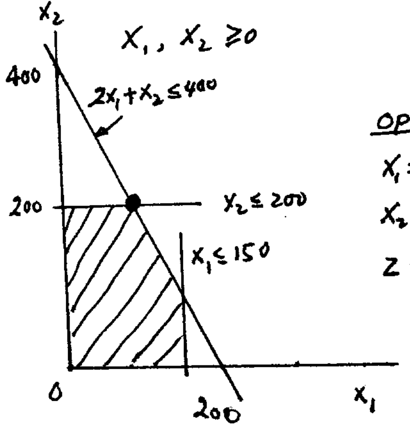
Optimum solution:
$x_1 = 4$ hours
$x_2 = 6$ hours
$z = 14$ "pleasarits"

12

x_1 = Daily nbr. of type 1 hat
x_2 = Daily nbr. of type 2 hat

Maximize $Z = 8x_1 + 5x_2$

$$
\begin{aligned}
\text{s.t.}\quad 2x_1 + x_2 &\le 400\\
x_1 &\le 150\\
x_2 &\le 200\\
x_1, x_2 &\ge 0
\end{aligned}
$$



optimum:
$x_1 = 100$ type 1
$x_2 = 200$ type 2
$Z = \$1800$

continued...

Set 2.2a

13

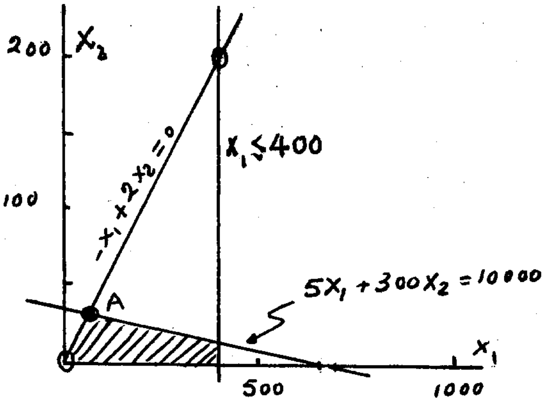
x_1 = radio minutes
x_2 = TV minutes

Maximize $z = x_1 + 25x_2$

s.t. $15x_1 + 300x_2 \le 10{,}000$

$\frac{x_1}{x_2} \ge 2$ or $-x_1 + 2x_2 \le 0$

$x_1 \le 400$, $x_1, x_2 \ge 0$



Optimum occurs at A:
$x_1 =$ 60.61 minutes
$x_2 =$ 30.3 minutes
$z =$ 818.18

14

x_1 = tons of C_1 consumed per hour
x_2 = tons of C_2 consumed per hour

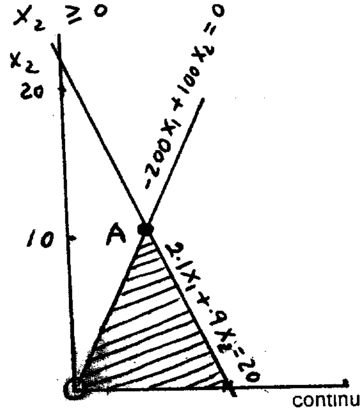
Maximize $z = 12000x_1 + 9000x_2$

s.t.
$1800x_1 + 2100x_2 \le 2000(x_1 + x_2)$

or $-200x_1 + 100x_2 \le 0$

$2.1x_1 + .9x_2 \le 20$

$x_1, x_2 \ge 0$



continued...

(a) Optimum occurs at A:
$x_1 =$ 5.128 tons per hour
$x_2 =$ 10.256 tons per hour
$z =$ 153,846 lb of steam

Optimal ratio $= \frac{5.128}{10.256} = .5$

(b) $2.1x_1 + .9x_2 \le (20+1) = 21$

Optimum $z = 161538$ lb of steam

$\Delta z = 161538 - 153846 = 7692$ lb

15

x_1 = Nbr. of radio commercials beyond the first
x_2 = Nbr. of TV ads beyond the first

Maximize $z = 2000x_1 + 3000x_2 + 5000 + 2000$

s.t. $300(x_1+1) + 2000(x_2+1) \le 20{,}000$

$300(x_1+1) \le .8 \times 20{,}000$

$2000(x_2+1) \le .8 \times 20{,}000$

$x_1, x_2 \ge 0$

or

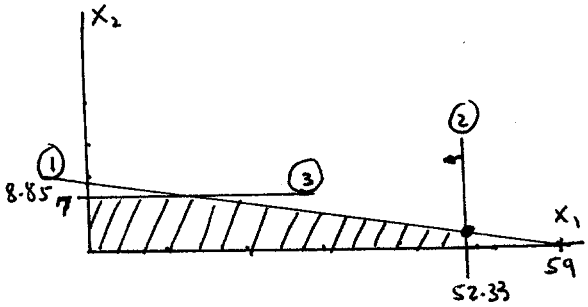
Maximize $z = 2000x_1 + 3000x_2 + 7000$

s.t. $300x_1 + 2000x_2 \le 17700$ ①

$300x_1 \le 15700$ ②

$2000x_2 \le 14000$ ③

$x_1, x_2 \ge 0$



Optimum solution:

Radio Commercials = 52.33+1 = 53.33

TV ads = 1+1 = 2

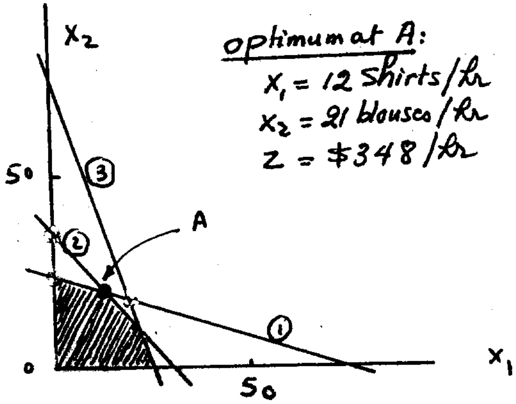
$z = 107666.67 + 7000 = 114666.67$

16

x_1 = number of shirts per hour
x_2 = number of blouses per hour

Maximize $Z = 8x_1 + 12x_2$
s.t.

$$\begin{aligned} 20x_1 + 60x_2 &\le 25\times 60 = 1500 && (1)\\ 70x_1 + 60x_2 &\le 35\times 60 = 2100 && (2)\\ 12x_1 + 4x_2 &\le 5\times 60 = 300 && (3)\\ x_1, x_2 &\ge 0 \end{aligned}$$



optimum at A:
x_1 = 12 shirts/hr
x_2 = 21 blouses/hr
z = \$348/hr

17

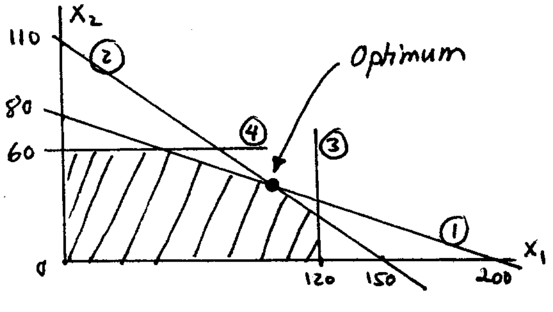
x_1 = Nbr. of desks per day
x_2 = Nbr. of chairs per day

Maximize $Z = 50x_1 + 100x_2$

$$\frac{x_1}{200} + \frac{x_2}{80} \le 1 \qquad (1)$$

$$\frac{x_1}{150} + \frac{x_2}{110} \le 1 \qquad (2)$$

$$x_1 \le 120,\ x_2 \le 60 \qquad (3,4)$$



Optimum:
x_1 = 90 desks
x_2 = 44 chairs
z = \$ 8900

18

x_1 = number of HiFi1 units
x_2 = number of HiFi2 units

Constraints:

$$\begin{aligned} 6x_1 + 4x_2 &\le 480\times .9 = 432\\ 5x_1 + 5x_2 &\le 480\times .86 = 412.8\\ 4x_1 + 6x_2 &\le 480\times .88 = 422.4 \end{aligned}$$

or

$$\begin{aligned} 6x_1 + 4x_2 + S_1 \quad\quad &= 432\\ 5x_1 + 5x_2 \quad + S_2 \quad &= 412.8\\ 4x_1 + 6x_2 \quad\quad + S_3 &= 422.4 \end{aligned}$$

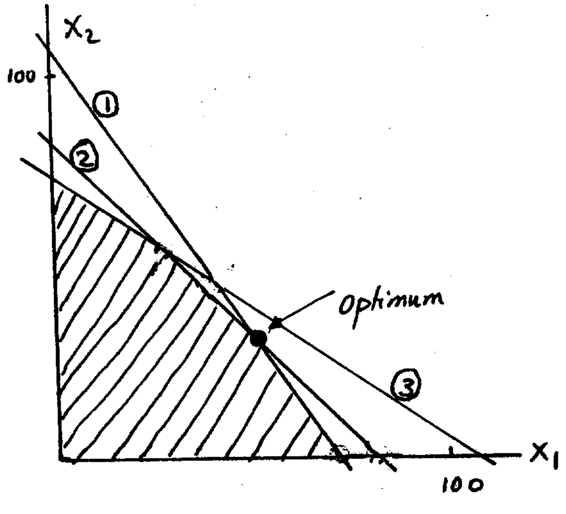
Objective function:

Minimize $S_1 + S_2 + S_3 = 1267.2 - 15x_1 - 15x_2$

Thus, min $S_1 + S_2 + S_3 \equiv$ max $15x_1 + 15x_2$

Maximize $Z = 15x_1 + 15x_2$
s.t.

$$\begin{aligned} 6x_1 + 4x_2 &\le 432 && ①\\ 5x_1 + 5x_2 &\le 412.8 && ②\\ 4x_1 + 6x_2 &\le 422.4 && ③\\ x_1, x_2 &\ge 0 \end{aligned}$$

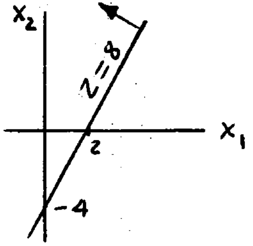


Optimum: (Problem has alternative optima)
x_1 = 50.88 units
x_2 = 31.68 units
z = 1238.4 minutes

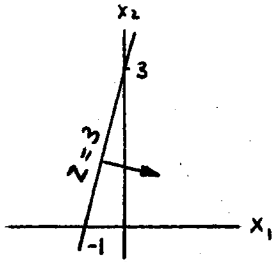
Set 2.2b

1

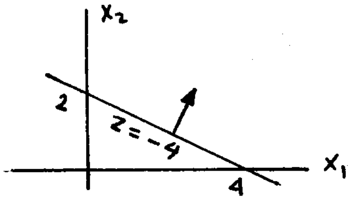
(a)



(b)

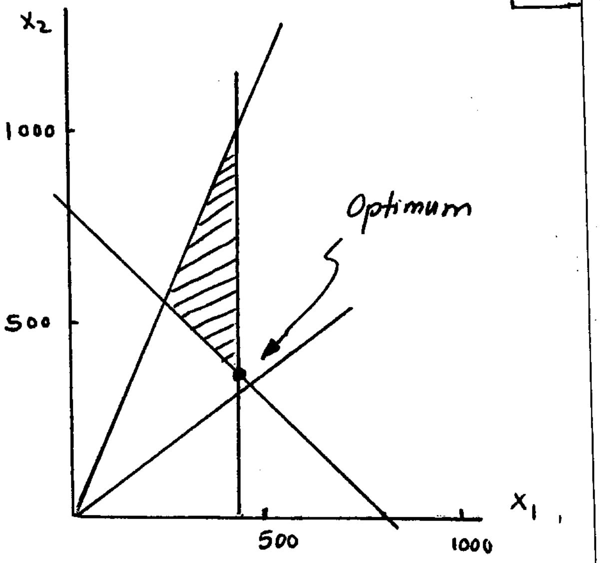


(c)



2

Additional constraint: $x_1 \leq 450$



Optimum Solution:

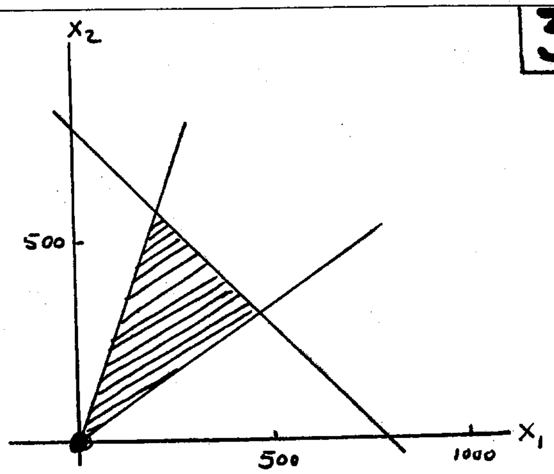
$x_1 = 450$ lb

$x_2 = 350$ lb

$z = \$450$

continued...

3



Optimum: $x_1 = 0$, $x_2 = 0$, $Z = 0$, which is nonsensical

4

x_1 = number of hours/week in store 1

x_2 = number of hours/week in store 2

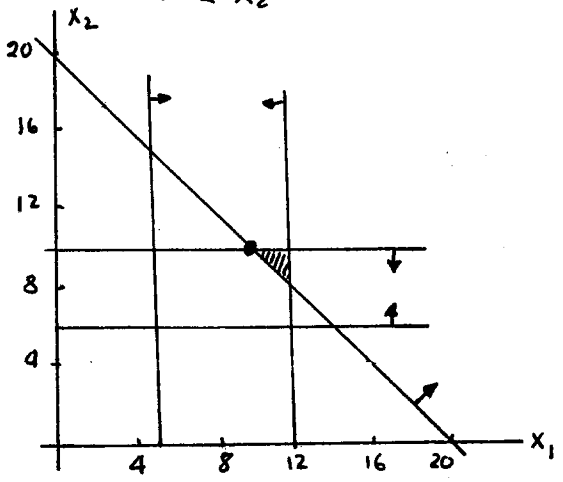
Minimize $Z = 8x_1 + 6x_2$

s.t.

$$x_1 + x_2 \geq 20$$

$$5 \leq x_1 \leq 12$$

$$6 \leq x_2 \leq 10$$



Optimum:

$x_1 = 10$ hours

$x_2 = 10$ hours

$z = 140$ stress index

continued...

5

Let

$x_1 = 10^3$ bbl/day from Iran

$x_2 = 10^3$ bbl/day from Dubai

Refinery capacity $= x_1 + x_2$ 10^3 bbl/day

Minimize $z = x_1 + x_2$

subject to

$$x_1 \geq .4(x_1 + x_2)$$

or

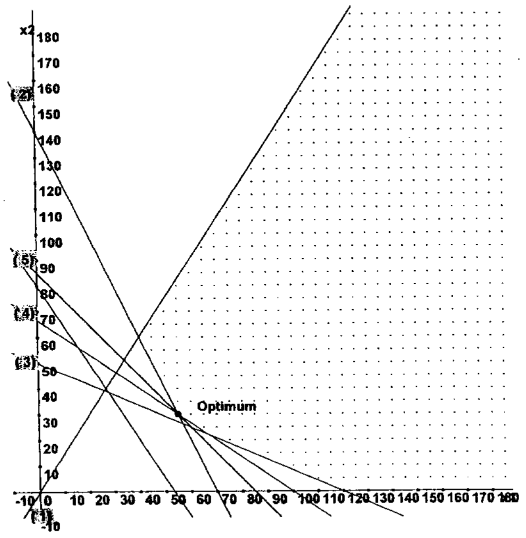
$$-.6x_1 + .4x_2 \leq 0$$
$$.2x_1 + .1x_2 \geq 14$$
$$.25x_1 + .6x_2 \geq 30$$
$$.1x_1 + .15x_2 \geq 10$$
$$.15x_1 + .1x_2 \geq 8$$
$$x_1, x_2 \geq 0$$

Optimum solution from TORA:

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: diet problem

Summary of Optimal Solution:
Objective Value = 85.00
x1 = 55.00
x2 = 30.00



6

Let

$x_1 = 10^3\$$ invested in blue chip stock

$x_2 = 10^3\$$ invested in high-tech stocks

Minimize $z = x_1 + x_2$

Subject to

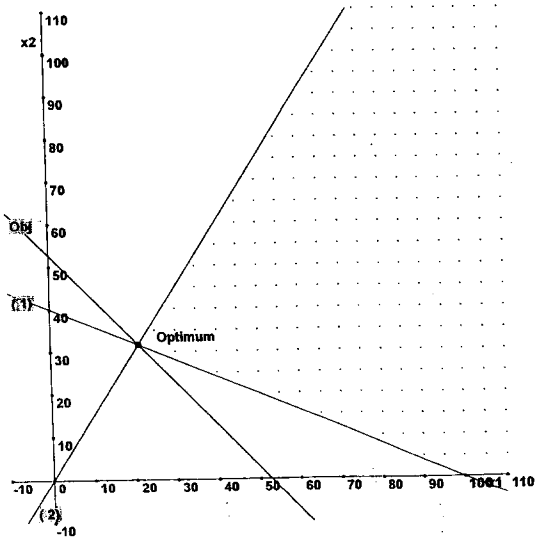
$$.1x_1 + .25x_2 \geq 10$$
$$.6x_1 - .4x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

TORA optimum solution:

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: diet problem

Summary of Optimal Solution:
Objective Value = 52.63
x1 = 21.05
x2 = 31.58

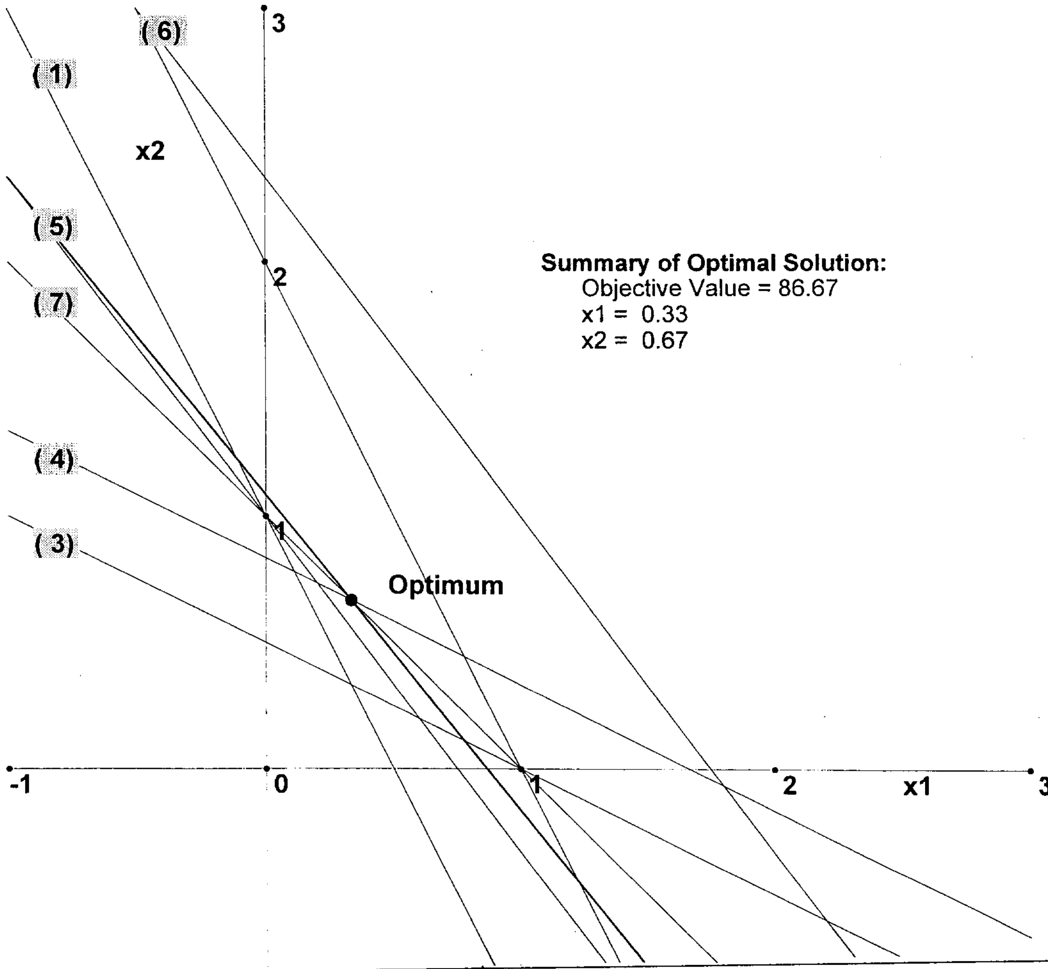


Set 2.2b

7

x_1 = Ratio of scrap A in alloy
x_2 = Ratio of scrap B in alloy

	x1	x2		
Minimize	100.00	80.00		
Subject to				
(1)	0.06	0.03	>=	0.03
(2)	0.06	0.03	<=	0.06
(3)	0.03	0.06	>=	0.03
(4)	0.03	0.06	<=	0.05
(5)	0.04	0.03	>=	0.03
(6)	0.04	0.03	<=	0.07
(7)	1.00	1.00	=	1.00



Summary of Optimal Solution:
Objective Value = 86.67
x1 = 0.33
x2 = 0.67

1

x_e = Nbr. of efficiency apartments
x_d = Nbr. of duplexes
x_s = Nbr. of single-family homes
x_r = Retail space in ft²

Maximize $Z = 600x_e + 750x_d + 1200x_s + 100x_r$

s.t. $x_e \le 500, x_d \le 300, x_s \le 250$

$$x_r \ge 10x_e + 15x_d + 18x_s$$

$$x_r \le 10000$$

$$x_d \ge \frac{x_e + x_s}{2}$$

$$x_e, x_d, x_s, x_r \ge 0$$

Optimal solution:

$Z = 1,595,714.29$
$x_e = 207.14$, $x_d = 228.57$
$x_s = 250$, $x_r = 10,000$

LP does not guarantee integer solution. Use rounded solution or apply integer LP algorithm (Chapter 9).

2

x_i = Acquired portion of property i

Each site is represented by a separate LP. The site that yields the smaller objective value is selected.

Site 1 LP:

Minimize $Z = 25 + x_1 + 2.1x_2 + 2.35x_3 + 1.85x_4 + 2.95x_5$
s.t. $x_4 \ge .75$, all $x_i \ge 0$, $i = 1, 2, \dots, 5$
$20x_1 + 50x_2 + 50x_3 + 30x_4 + 60x_5 \ge 200$

Optimum: $Z = 34.6625$ million \$
$x_1 = .875, x_2 = x_3 = 1, x_4 = .75, x_5 = 1$

Site 2 LP:

Minimize $Z = 27 + 2.8x_1 + 1.9x_2 + 2.8x_3 + 2.5x_4$
s.t. $x_3 \ge .5$, $x_1, x_2, x_3, x_4 \ge 0$
$80x_1 + 60x_2 + 50x_3 + 70x_4 \ge 200$

Optimum: $Z = 34.35$ million \$
$x_1 = x_2 = 1$, $x_3 = x_4 = .5$

Select site 2.

3

x_{ij} = portion of project i completed in year j

Maximize $Z = .05(4x_{11} + 3x_{12} + 2x_{13}) +$
$.07(3x_{22} + 2x_{23} + x_{24}) +$
$.15(4x_{31} + 3x_{32} + 2x_{33} + x_{34}) +$
$.02(2x_{43} + x_{44})$

s.t. $\sum_{j=1}^{3} x_{1j} = 1$, $\sum_{j=3}^{4} x_{4j} = 1$

$.25 \le \sum_{j=2}^{5} x_{2j} \le 1$, $.25 \le \sum_{j=1}^{5} x_{3j} \le 1$

$5x_{11} + 15x_{31} \le 3$
$5x_{12} + 8x_{22} + 15x_{32} \le 6$
$5x_{13} + 8x_{23} + 15x_{33} + 1.2x_{43} \le 7$
$8x_{24} + 15x_{34} + 1.2x_{44} \le 7$
$8x_{25} + 15x_{35} \le 7$

Optimum:

$Z = \$523,750$

$x_{11} = .6$, $x_{12} = .4$
$x_{24} = .225$, $x_{25} = .025$
$x_{32} = .267$, $x_{33} = .387$, $x_{34} = .346$
$x_{43} = 1$

4

x_ℓ = Nbr. of low income units
x_m = Nbr. of middle income units
x_u = Nbr. of upper income units
x_p = Nbr. of public housing units
x_s = Nbr. of school rooms
x_r = Nbr. of retail units
x_c = Nbr. of condemned homes

Maximize $Z = 7x_\ell + 12x_m + 20x_u + 5x_p + 15x_r$
$- 10x_s - 7x_c$

s.t. $100 \le x_\ell \le 200$, $125 \le x_m \le 190$
$75 \le x_u \le 260$, $300 \le x_p \le 600$
$0 \le x_s \le 2/.045$
$.05x_\ell + .07x_m + .03x_u + .025x_p +$
$.045x_s + .1x_r \le .85(50 + .25x_c)$
$x_r \ge .023x_\ell + .034x_m + .046x_u +$
$.023x_p + .034x_s$

continued...

Set 2.3a

$25x_s \geq 1.3x_l + 1.2x_m + .5x_u + 1.4x_p$

Optimum: $z = 8290.30$ thousand \$

$x_l = 100$, $x_m = 125$, $x_u = 227.04$

$x_p = 300$, $x_s = 32.54$, $x_r = 25$

$x_c = 0$

5

x_1 = Nbr. of single-family homes

x_2 = Nbr. of double-family homes

x_3 = Nbr. of triple-family home

x_4 = Nbr. of recreation areas

Maximize $z = 10{,}000x_1 + 12000x_2 + 15000x_3$

s.t.

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq .85 \times 800$$

$$\frac{x_1}{x_1 + x_2 + x_3} \geq .5 \text{ or } .5x_1 - .5x_2 - .5x_3 \geq 0$$

$$x_4 \geq \frac{x_1 + 2x_3 + 3x_3}{200} \text{ or } 200x_4 - x_1 - 2x_2 - 3x_3 \geq 0$$

$$1000x_1 + 1200x_2 + 1400x_3 + 800x_4 \geq 100{,}000$$

$$400x_1 + 600x_2 + 840x_3 + 450x_4 \leq 200{,}000$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Optimum solution:

$x_1 = 339.15$ homes

$x_2 = 0$

$x_3 = 0$

$x_4 = 1.69$ areas

$z = \$3{,}391{,}521.20$

6

New land use constraint:

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq .85(800 + 100)$$

New Optimum solution:

$z = \$3{,}815{,}461.35$

$x_1 = 381.54$ homes

$x_2 = x_3 = 0$

$x_4 = 1.91$ areas

$\Delta z = \$3{,}815{,}461.35 - 3{,}391{,}521.20$

$= \$423{,}940.35$

$\Delta z < \$450{,}000$, the purchasing cost of 100 acres. Hence, the purchase of the new acreage is not recommended.

1

The constraints remain unchanged, but the objective function is changed to

Maximize $z = y - \text{commission}$

where

$$\text{commission} = .001(\text{all transactions in } \$)$$

$$= .001\Big[(x_{12}+x_{13}+x_{14}+x_{15}) + \frac{1}{.769}(x_{21}+x_{23}+x_{24}+x_{25}) + \frac{1}{.625}(x_{31}+x_{32}+x_{34}+x_{35}) + \frac{1}{105}(x_{41}+x_{42}+x_{43}+x_{45}) + \frac{1}{.342}(x_{51}+x_{52}+x_{53}+x_{54})$$

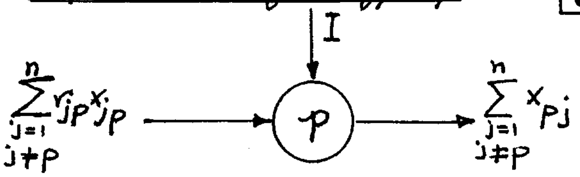
Optimum solution:

	Without	With
z	5.09032	5.06211
y	5.09032	5.08986
Return	1.8064%	1.2421%

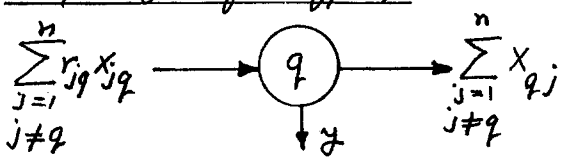
Commission = 5.08986 − 5.06211
= \$ 27,750
or, .555% of the original investment of \$5 million

2

Input I in fund type p:

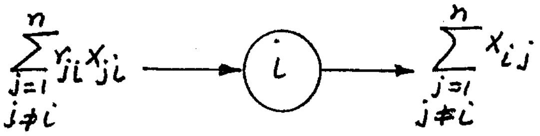


Output y in fund type q:



For specific p and q, the model below can be used to transform any fund to any other fund. In the present problem, p = 1 (\$) and q = 2 (€), 3 (£), 4 (¥), and 5 (KD).

General node i:



Maximize $z = y$

s.t.

$$I + \sum_{\substack{j=1\\ j\neq p}}^{n} r_{jp}x_{jp} = \sum_{\substack{j=1\\ j\neq p}}^{n} x_{pj}$$

$$\sum_{\substack{j=1\\ j\neq q}}^{n} r_{jq}x_{jq} = y + \sum_{\substack{j=1\\ j\neq q}}^{n} x_{qj}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{n} r_{ji}x_{ji} = \sum_{\substack{j=1\\ j\neq i}}^{n} x_{ij}, \quad i \neq p \text{ or } q$$

$$0 \leq x_{ij} \leq cap_i, \text{ all } i \text{ and } j$$

Note: Solver or AMPL is ideal for solving this problem interactively. See files solver2.3b-2.xls and ampl2.3b-2.txt.

Results: (No commission)

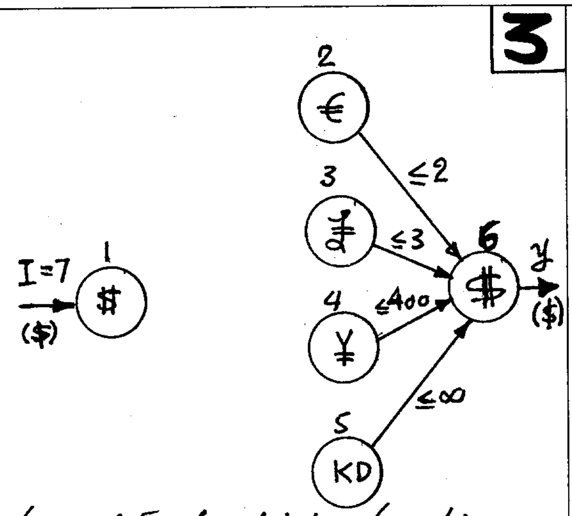
p	q	Rate of return
\$	\$	1.8064%
\$	€	1.7966%
\$	£	1.8287%
\$	¥	2.8515%
\$	KD	1.0471%

Wide discrepancy in ¥ and KD currencies may be attributed to the fact that their exchange rates may not be consistent with the remaining rates. Nevertheless, the problem shows that there may be advantages in targeting accumulation in different currencies.

continued...

Set 2.3b

3



To formulate the objective function correctly, all output currencies are converted to a single currency (arbitrarily chosen to be \$). Thus

$$y = r_{21}x_{26} + r_{31}x_{36} + r_{41}x_{46} + r_{51}x_{56}$$

Maximize $z = y$

s.t. $x_{26} \le 2,\ x_{36} \le 3,\ x_{46} \le 400;\ x_{3j} \le 3.5$

$x_{1j} \le 5,\ x_{2j} \le 3,\ x_{4j} \le 100,\ x_{5j} \le 2.8$, all j

$$I + \sum_{i=2}^{5} x_{i1} r_{i1} = \sum_{j=2}^{5} x_{1j}$$

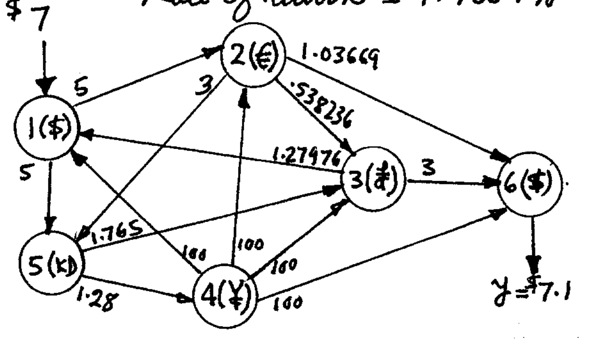
$$y = r_{21}x_{26} + r_{31}x_{36} + r_{41}x_{46} + r_{51}x_{56}$$

$$x_{i6} + \sum_{\substack{j=1\\ j\ne i}}^{5} x_{ij} = \sum_{\substack{j=1\\ j\ne i}}^{5} r_{ji}x_{ji}\ ,\ i = 2,3,4,5$$

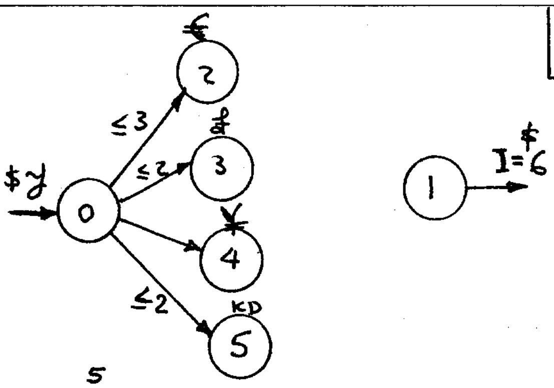
all $x_{ij} \ge 0$, $i \ne j$

Solution: Total accumulation y = \$7.1 million

Rate of return = 1.4354%



4



$$y = \sum_{k=2}^{5} r_{k1}x_{0k} = r_{21}x_{02} + r_{31}x_{03} + r_{41}x_{04} + r_{51}x_{05}$$

Minimize $z = y$

s.t. $y = r_{21}x_{02} + r_{31}x_{03} + r_{41}x_{04} + r_{51}x_{05}$

$x_{02} \le 3,\ x_{03} \le 2,\ x_{05} \le 2$

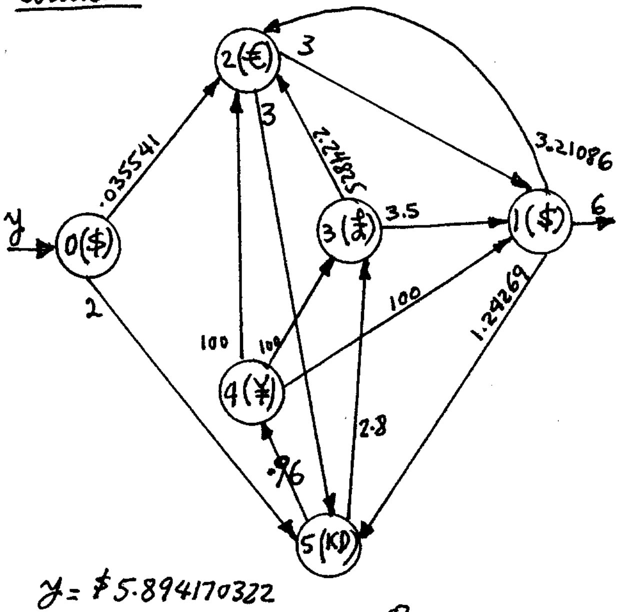
all j: $x_{1j} \le 5,\ x_{2j} \le 3,\ x_{4j} \le 3.5,\ x_{4j} \le 100,\ x_{5j} \le 2.8$

$$x_{0j} + \sum_{\substack{i=1\\ i\ne j}}^{5} r_{ij}x_{ij} = \sum_{\substack{k=1\\ k\ne j}}^{5} x_{jk}\ ,\ j = 2,3,4,5$$

$$\sum_{i=2}^{5} r_{i1}x_{i1} = \sum_{j=2}^{5} x_{1j} + I$$

all $x_{ij} \ge 0$

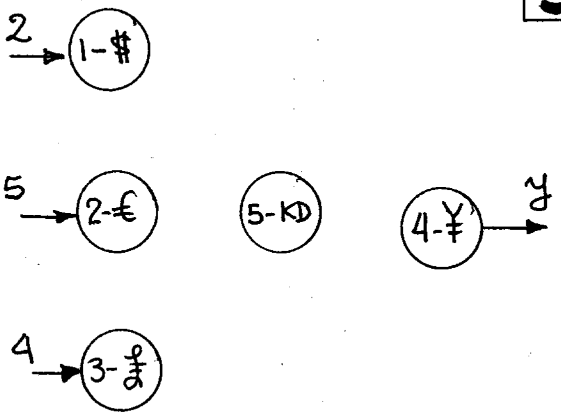
Solution:



y = \$5.894170322

Rate of return = 1.7638%

5



Maximize $z = y$

s.t.

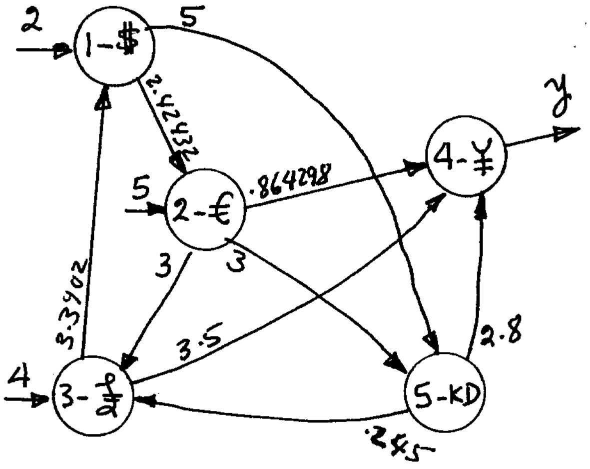
$$y = r_{14}x_{14} + r_{24}x_{24} + r_{34}x_{34} + r_{54}x_{54}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{5} r_{ij}x_{ij} = \sum_{\substack{k=1 \\ k \neq j}}^{5} x_{jk} - \begin{cases} 2, & j=1 \\ 5, & j=2 \\ 4, & j=3 \\ -y, & j=4 \\ 0, & j=5 \end{cases}$$

$$\left.\begin{array}{l} x_{ij} \leq C_i \text{ , for all } i \text{ and } j \\ x_{ij} \geq 0 \text{, for all } i \text{ and } j \end{array}\right\} i \neq j$$

Solution: $y = 1584.91$ million¥

Rate of return = .8853%



Set 2.3c

1

(a) x_i = Undertaken portion of project i

Maximize

$$z = 32.4x_1 + 35.8x_2 + 17.75x_3 + 14.8x_4 + 18.2x_5 + 12.35x_6$$

Subject to

$$10.5x_1 + 8.3x_2 + 10.2x_3 + 7.2x_4 + 12.3x_5 + 9.2x_6 \le 60$$
$$14.4x_1 + 12.6x_2 + 14.2x_3 + 10.5x_4 + 10.1x_5 + 7.8x_6 \le 70$$
$$2.2x_1 + 9.5x_2 + 5.6x_3 + 7.5x_4 + 8.3x_5 + 6.9x_6 \le 35$$
$$2.4x_1 + 3.1x_2 + 4.2x_3 + 5.0x_4 + 6.3x_5 + 5.1x_6 \le 20$$
$$0 \le x_j \le 1, \quad j = 1, 2, \dots, 6$$

TORA optimum Solution:

$x_1 = x_2 = x_3 = x_4 = 1, x_5 = .84, x_6 = 0, z = 116.06$

(b) Add the constraint $x_2 \le x_6$

TORA optimum Solution:

$x_1 = x_2 = x_3 = x_4 = x_6 = 1, x_5 = .03, z = 113.68$

(c) Let S_i be the unused funds at the end of year i and change the right-hand sides of constraints 2, 3, and 4 to $70 + S_1$, $35 + S_2$, and $20 + S_3$, respectively.

TORA optimum solution:

$x_1 = x_2 = x_3 = x_4 = x_5 = 1, \quad x_6 = .71$

$z = \$127.72$ (thousand)

The solution is interpreted as follows:

i	S_i	$S_i - S_{i-1}$	Decision
1	4.96	—	—
2	7.62	+2.66	Don't borrow from yr 1
3	4.62	−3.00	Borrow \$3 from year 2
4	0	−4.62	Borrow \$4.62 from yr 2

The effect of availing excess money for use in later years is that the first five projected are completed and 71% of project 6 is undertaken.

The total revenue increases from \$116,060 to 127,720.

continued...

(d) The slack S_i in period i is treated as an <u>unrestricted</u> variable.

TORA optimum solution: $z = \$131.30$

$S_1 = 2.3, \; S_2 = .4, \; S_3 = -5, \; S_4 = -6.1$

This means that additional funds are needed in years 3 and 4.

Increase in return $= 131.30 - 116.06$
$= \$15.24$

Ignoring the time value of money, the amount borrowed $5 + 6.1 - (2.3 + .4) = \$8.4$. Thus,

$$\text{rate of return} = \frac{15.24 - 8.4}{8.4} \cong 81\%$$

2

x_i = dollar investment in project i, $i = 1, 2, 3, 4$

y_j = dollar investment in bank in year j, $j = 1, 2, 3, 4, 5$

Maximize $z = y_5$

Subject to

$$x_1 + x_2 + x_4 + y_1 \le 10{,}000$$
$$.5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_1 - y_2 = 0$$
$$.3x_1 + .2x_2 + .8x_3 + .6x_4 + 1.065y_2 - y_3 = 0$$
$$1.8x_1 + 1.5x_2 + 1.9x_3 + 1.8x_4 + 1.065y_3 - y_4 = 0$$
$$1.2x_1 + 1.3x_2 + .8x_3 + .95x_4 + 1.065y_4 - y_5 = 0$$

All variables ≥ 0

TORA optimal solution:

$x_1 = 0, x_2 = \$10{,}000, x_3 = \$6000, x_4 = 0$

$y_1 = 0, y_2 = 0, y_3 = \$6800, y_4 = \$33{,}642$

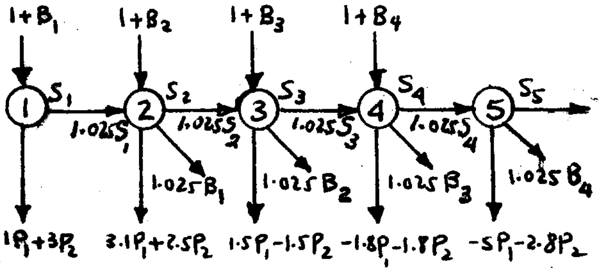
$z = \$53{,}628.73$ at the start of year 5

3

P_i = fraction undertaken of project i, $i = 1, 2$

B_j = million dollars borrowed in quarter j, $j = 1, 2, 3, 4$

S_j = surplus million dollars at the start of quarter j, $j = 1, 2, 3, 4, 5$



(a) Maximize $z = S_5$

subject to

$$P_1 + 3P_2 + S_1 - B_1 = 1$$
$$3.1P_1 + 2.5P_2 - 1.02S_1 + S_2 + 1.025B_1 - B_2 = 1$$
$$1.5P_1 - 1.5P_2 - 1.02\ S_2 + S_3 + 1.025B_2 - B_3 = 1$$
$$-1.8P_1 - 1.8P_2 - 1.02\ S_3 + S_4 + 1.025B_3 - B_4 = 1$$
$$-5P_1 - 2.8P_2 - 1.02\ S_4 + S_5 + 1.025B_4 = 1$$
$$0 \le P_1 \le 1,\quad 0 \le P_2 \le 1$$
$$0 \le B_j \le 1,\quad j = 1, 2, 3, 4$$

<u>Optimum solution</u>:

$P_1 = .7113 \qquad P_2 = 0$

$z = 5.8366$ million dollars

$B_1 = 0$, $B_2 = .9104$ million dollars

$B_3 = 1$ million dollars, $B_4 = 0$

(b) $B_1 = 0$, $S_1 = .2887$ million \$

$B_2 = .9104$, $S_2 = 0$

$B_3 = 1$, $S_3 = 0$

$B_4 = 0$, $S_4 = 1.2553$

The solution shows that $B_i \cdot S_i = 0$, meaning that you can't borrow and also end up with surplus in any quarter. The result makes sense because the cost of borrowing (2.5%) is higher then the return on surplus funds (2%)

4

Assume that the investment program ends at the start of year 11. Thus, the 6-year bond option can be exercised in years 1, 2, 3, 4, and 5 only. Similarly, the 9-year bond can be used in years 1 and 2 only. Hence, from year 6 on, the only option available is insured savings at 7.5%.

Let

I_i = insured savings investments in year i, $i = 1, 2, \cdots, 10$

G_i = 6-year bond investment in year i, $i = 1, 2, \cdots, 5$

M_i = 9-year bond investment in year i, $i = 1, 2$

The objective is to maximize total accumulation at the end of year 10; that is,

$$\text{maximize } z = 1.075I_{10} + 1.079G_5 + 1.085M_2$$

The constraints represent the balance equation for each year's cash flow.

$$I_1 + .98G_1 + 1.02M_1 = 2$$
$$I_2 + .98G_2 + 1.02M_2 = 2 + 1.075I_1 + .079G_1 + .085M_1$$
$$I_3 + .98G_3 = 2.5 + 1.075I_2 + .079(G_1 + G_2) + .085(M_1 + M_2)$$
$$I_4 + .98G_4 = 2.5 + 1.075I_3 + .079(G_1 + G_2 + G_3) + .085(M_1 + M_2)$$
$$I_5 + .98G_5 = 3 + 1.075I_4 + .079(G_1 + G_2 + G_3 + G_4) + .085(M_1 + M_2)$$
$$I_6 = 3.5 + 1.075I_5 + .079(G_1 + G_2 + G_3 + G_4 + G_5) + .085(M_1 + M_2)$$

continued...

Set 2.3c

$$I_7 = 3.5 + 1.075\, I_6 + 1.079\, G_1 + .079\,(G_2 + G_3 + G_4 + G_5) + .085\,(M_1 + M_2)$$

$$I_8 = 4 + 1.075\, I_7 + 1.079\, G_2 + .079\,(G_3 + G_4 + G_5) + .085\,(M_1 + M_2)$$

$$I_9 = 4 + 1.075\, I_8 + 1.079\, G_3 + .079\,(G_4 + G_5) + .085\,(M_1 + M_2)$$

$$I_{10} = 5 + 1.075\, I_9 + 1.079\, G_4 + .079\, G_5 + 1.085\, M_1 + .085\, M_2$$

all variables ≥ 0

```
                        *** OPTIMUM SOLUTION SUMMARY ***
------------------------------------------------------------------------
Title: Problem 26a-14
Final iteration No: 14
Objective value (max) =   46.8500
------------------------------------------------------------------
Variable               Value       Obj Coeff    Obj Val Contrib
------------------------------------------------------------------
x1 I1                 0.0000       0.0000          0.0000
x2 I2                 0.0000       0.0000          0.0000
x3 I3                 0.0000       0.0000          0.0000
x4 I4                 0.0000       0.0000          0.0000
x5 I5                 0.0000       0.0000          0.0000
x6 I6                 4.6331       0.0000          0.0000
x7 I7                 9.6137       0.0000          0.0000
x8 I8                15.4678       0.0000          0.0000
x9 I9                24.6663       0.0000          0.0000
x10 I10              37.5201       1.0750         40.3341
x11 G1                0.0000       0.0000          0.0000
x12 G2                0.0000       0.0000          0.0000
x13 G3                2.9053       0.0000          0.0000
x14 G4                3.1395       0.0000          0.0000
x15 G5                3.9028       1.0790          4.2111
x16 M1                1.9608       0.0000          0.0000
x17 M2                2.1242       1.0850          2.3047
-------------------------------------------------------------
Constraint             RHS          Slack(-)/Surplus(+)
-------------------------------------------------------------
1 (=)                 2.0000           0.0000
2 (=)                 2.0000           0.0000
3 (=)                 2.5000           0.0000
4 (=)                 2.5000           0.0000
5 (=)                 3.0000           0.0000
6 (=)                 3.5000           0.0000
7 (=)                 3.5000           0.0000
8 (=)                 4.0000           0.0000
9 (=)                 4.0000           0.0000
10 (=)                5.0000           0.0000
```

Year	Recommendation
1	Invest all in 9-yr bond
2	Invest all in 9-yr. bond
3	Invest all in 6-yr bond
4	Invest all in 6-yr bond
5	Invest all in 6-yr bond
7	Invest all in insured savings
8	Invest all in insured savings
9	Invest all in insured savings
10	Invest all in insured savings

5

x_{iA} = amount invested in year i, plan A (1000\$)

x_{iB} = amount invested in year i, plan B (1000\$)

$$\text{Maximize } z = 3x_{2B} + 1.7x_{3A}$$

subject to

$$\begin{aligned} x_{1A} + x_{1B} &\leq 100 \\ -1.7x_{1A} + x_{2A} + x_{2B} &= 0 \\ -3x_{1B} - 1.7x_{2A} + x_{3A} &= 0 \end{aligned}$$

$x_{iA}, x_{iB} \geq 0$ for $i = 1, 2, 3$

```
                          *** OPTIMUM SOLUTION SUMMARY ***
---------------------------------------------------------------------------
Title: Problem 2.6a-15
Final iteration No: 4
Objective value (max) =  510.0000
==> ALTERNATIVE solution detected at x2
--------------------------------------------------------------------
Variable               Value       Obj Coeff    Obj Val Contrib
--------------------------------------------------------------------
x1 x1A              100.0000       0.0000          0.0000
x2 x1B                0.0000       0.0000          0.0000
x3 x2A                0.0000       0.0000          0.0000
x4 x2B              170.0000       3.0000        510.0000
x5 x3A                0.0000       1.7000          0.0000
---------------------------------------------------------------
Constraint             RHS          Slack(-)/Surplus(+)
---------------------------------------------------------------
1 (<)               100.0000           0.0000-
2 (<)                 0.0000           0.0000-
3 (<)                 0.0000           0.0000-
```

Optimum solution: Invest \$100,000 in A in yr 1 and \$170,000 in B in yr 2.

Alternative optimum: Invest \$100,000 in B in yr 1 and \$300,000 in A in yr 3.

6

x_i = dollars allocated to choice i, $i = 1, 2, 3, 4$

y = minimum return

$$\text{Maximize } z = \min \begin{cases} -3x_1 + 4x_2 - 7x_3 + 15x_4 \\ 5x_1 - 3x_2 + 9x_3 + 4x_4 \\ 3x_1 - 9x_2 + 10x_3 - 8x_4 \end{cases}$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 500$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The problem can be converted to a linear program as

continued...

Maximize $z = y$

subject to

$$-3x_1 + 4x_2 - 7x_3 + 15x_4 \geq y$$
$$5x_1 - 3x_2 + 9x_3 + 4x_4 \geq y$$
$$3x_1 - 9x_2 + 10x_3 - 8x_4 \geq y$$
$$x_1 + x_2 + x_3 + x_4 \leq 500$$
$$x_1, x_2, x_3, x_4 \geq 0$$

y unrestricted

```
                *** OPTIMUM SOLUTION SUMMARY ***
-----------------------------------------------------------
Title:
Final iteration No: 5
Objective value (max) = 1175.0000
-----------------------------------------------------------
Variable        Value     Obj Coeff   Obj Val Contrib
-----------------------------------------------------------
x1             0.0000      0.0000          0.0000
x2             0.0000      0.0000          0.0000
x3           287.5000      0.0000          0.0000
x4           212.5000      0.0000          0.0000
x5 y        1175.0000      1.0000       1175.0000
-----------------------------------------------------------
Constraint      RHS         Slack(-)/Surplus(+)
-----------------------------------------------------------
1 (>)          0.0000          0.0000+
2 (>)          0.0000       2262.5000+
3 (>)          0.0000          0.0000+
4 (<)        500.0000          0.0000-
```

Allocate \$287.50 to choice 3 and \$212.50 to choice 4. Return = \$1175.00

7

$$i = \begin{cases} 1, & \text{regular savings} \\ 2, & \text{3-month CD} \\ 3, & \text{6-month CD} \end{cases}$$

x_{it} = Deposit in plan i at start of month t

$$t = \begin{cases} 1, 2, \dots, 12 & \text{if } i = 1 \\ 1, 2, \dots, 10 & \text{if } i = 2 \\ 1, 2, \dots, 7 & \text{if } i = 3 \end{cases}$$

y_1 = initial amount on hand to insure a feasible solution

r_i = interest rate for plan $i = 1, 2, 3$

$$\bar{J_i} = \begin{cases} 12, & i = 1 \\ 10, & i = 2 \\ 7, & i = 3 \end{cases}$$

continued...

$$P_i = \begin{cases} 1, & i = 1 \\ 3, & i = 2 \\ 6, & i = 3 \end{cases}$$

d_t = \$ demand for period t

$$\text{Maximize } z = \sum_{\substack{t=1 \\ t-P_i>0}}^{12} \sum_{i=1}^{3} r_i \cdot x_{i,t-P_i} - y_1$$

s.t.

$$y_1 - x_{11} - x_{21} - x_{31} \geq d_1$$
$$1000 + \sum_{\substack{i=1 \\ t-P_i>0}}^{3} (1+r_i) x_{i,t-P_i} - \sum_{\substack{i=1 \\ t \leq J_i}}^{3} x_{it} \geq d_t, \; t=2, \dots, 12$$
$$x_{it}, y_1 \geq 0$$

Solution: (see file ampl2.3c-7.txt)

y_1 = \$1200, $z = -1136.29$

Interest amount = 1200 − 1136.29 = \$63.71

Deposits:

t	x_{1t}	x_{2t}	x_{3t}
1	0	0	0
2	0	200	0
3	286.48	313.53	0
4	0	587.43	0
5	314.37	289.30	0
6	0	734.69	0
7	0	98.20	0
8	0	294.60	
9	0	848.16	
10	0	0	
11	0		
12	0		

Set 2.3d

1

x_{W1} = #wrenches/wk using regular time
x_{W2} = #wrenches/wk using overtime
x_{W3} = #wrenches/wk using subcontracting
x_{C1} = # chisels/wk using regular time
x_{C2} = # chisels/wk using overtime
x_{C3} = # chisels/wk using subcontracting

Minimize $z = 2x_{W1} + 2.8x_{W2} + 3x_{W3} + 2.1x_{C1} + 3.2x_{C2} + 4.2x_{C3}$

Subject to

$x_{W1} \le 550$, $x_{W2} \le 250$

$x_{C1} \le 620$, $x_{C2} \le 280$

$$\frac{x_{C1} + x_{C2} + x_{C3}}{x_{W1} + x_{W2} + x_{W3}} \ge 2$$

or

$$2x_{W1} + 2x_{W2} + 2x_{W3} - x_{C1} - x_{C2} - x_{C3} \le 0$$

$$x_{W1} + x_{W2} + x_{W3} \ge 1500$$

$$x_{C1} + x_{C2} + x_{C3} \ge 1200$$

all variables ≥ 0

(a) Optimum from TORA:

$x_{W1} = 550$, $x_{W2} = 250$, $x_{W3} = 700$

$x_{C1} = 620$, $x_{C2} = 280$, $x_{C3} = 2100$

$z = \$14,918$

(b) Increasing marginal cost ensures that regular time capacity is used before that of overtime, and that overtime capacity is used before that of subcontracting. If the marginal cost function is not monotonically increasing, additional constraints are needed to ensure that the capacity restriction is satisfied.

continued...

2

x_j = number of units produced of product j, $j = 1, 2, 3, 4$

Profit per unit:

Product 1 = $75 - 2\times10 - 3\times5 - 7\times4 = \$12$
Product 2 = $70 - 3\times10 - 2\times5 - 3\times4 = \$18$
Product 3 = $55 - 4\times10 - 1\times5 - 2\times4 = \$2$
Product 4 = $45 - 2\times10 - 2\times5 - 1\times4 = \$11$

Maximize $z = 12x_1 + 18x_2 + 2x_3 + 11x_4$

s.t.

$$2x_1 + 3x_2 + 4x_3 + 2x_4 \le 500$$
$$3x_1 + 2x_2 + x_3 + 2x_4 \le 380$$
$$7x_1 + 3x_2 + 2x_3 + x_4 \le 450$$
$$x_1, x_2, x_3, x_4 \ge 0$$

TORA Solution:

$x_1 = 0$, $x_2 = 133.33$, $x_3 = 0$, $x_4 = 50$

$z = \$2950$

3

x_j = number of units of model j

Maximize $z = 30x_1 + 20x_2 + 50x_3$

Subject to

① $2x_1 + 3x_2 + 5x_3 \le 4000$
② $4x_1 + 2x_2 + 7x_3 \le 6000$
③ $x_1 + .5x_2 + \frac{1}{3}x_3 \le 1500$
④ $\frac{x_1}{3} = \frac{x_2}{2}$, or $2x_1 - 3x_2 = 0$
⑤ $\frac{x_2}{2} = \frac{x_3}{5}$, or $5x_2 - 2x_3 = 0$

$x_1 \ge 200$, $x_2 \ge 200$, $x_3 \ge 150$

```
                    *** OPTIMUM SOLUTION SUMMARY ***
----------------------------------------------------------------------
Title: Problem 2.6a-12
Final iteration No: 4
Objective value (max) =41081.0820
-----------------------------------------------------------------
Variable            Value       Obj Coeff    Obj Val Contrib
-----------------------------------------------------------------
x1                  324.3243    30.0000       9729.7305
x2                  216.2162    20.0000       4324.3242
x3                  540.5405    50.0000      27027.0273
-------------------------------------------------------------
Constraint            RHS            Slack(-)/Surplus(+)
-------------------------------------------------------------
1 (<)               4000.0000          0.0000-
2 (<)               6000.0000        486.4865-
3 (<)               1500.0000        887.3875-
4 (=)                  0.0000          0.0000
5 (=)                  0.0000          0.0000
LB-x1                200.0000        124.3243+
LB-x2                200.0000         16.2162+
LB-x3                150.0000        390.5405+
```

4

x_{ij} = Nbr. cartons in month i from supplier j

I_i = End inventory in period i , $I_0 = 0$

c_{ij} = Price per unit of x_{ij}

h = Holding cost/unit/month

C = Supplier capacity/month

d_i = Demand for month i

$i = 1, 2, 3, \quad j = 1, 2$

$$\text{Minimize } z = \sum_{i=1}^{3}\sum_{j=1}^{2} c_{ij}\, x_{ij} + \frac{h}{2}\left(\sum_{i=1}^{3}\left(\sum_{j=1}^{2} x_{ij} + I_{i-1} + I_i\right)\right.$$

s.t. $x_{ij} \le C$, all i and j

$$\sum_{j=1}^{2} x_{ij} + I_{i-1} - I_i = d_i, \text{ all } i$$

Optimum solution:

i	x_{i1}	x_{i2}	I
1	400	100	0
2	400	400	200
3	200	0	0

Total cost = \$167,450.

5

X_i = Production amount in quarter i

I_i = End inventory for quarter i

$$\text{Minimize } z = 20X_1 + 22X_2 + 24X_3 + 26X_4 + 3.5(I_1 + I_2 + I_3)$$

s.t.

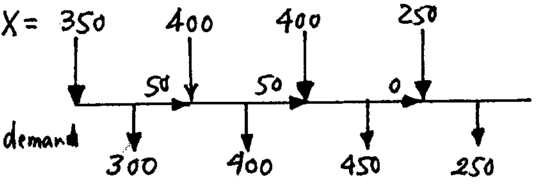
$X_1 = 300 + I_1$ $\qquad X_i \le 400, i = 1,2,3,4$

$I_1 + X_2 = 400 + I_2$ $\qquad I_i \le 100, i = 1,2,3$

$I_2 + X_3 = 450 + I_3$ $\qquad I_0 = I_4 = 0$

$I_3 + X_4 = 250$

Optimum solution:



Total cost = \$32,250

6

x_{ij} = Qty of product i in month j,

$i = 1, 2, \; j = 1, 2, 3$

I_{ij} = End inventory of product i in month j

$$\text{Minimize } z = 30(x_{11}+x_{12}+x_{13}) + 28(x_{21}+x_{22}+x_{23}) + .9(I_{11}+I_{12}+I_{13}) + .75(I_{21}+I_{22}+I_{23})$$

s.t.

$$(x_{1j}/1.25) + x_{2j} \le \begin{cases} 3000, & j=1 \\ 3500, & j=2 \\ 3000, & j=3 \end{cases}$$

$$I_{1,j-1} + x_{1j} - I_{1j} = \begin{cases} 500, & j=1 \\ 5000, & j=2 \\ 750, & j=3 \end{cases}$$

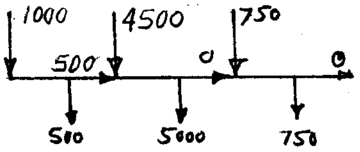
$$I_{2,j-1} + x_{2j} - I_{2j} = \begin{cases} 1000, & j=1 \\ 1200, & j=2 \\ 1200, & j=3 \end{cases}$$

$I_{i0} = 0, i = 1,2$

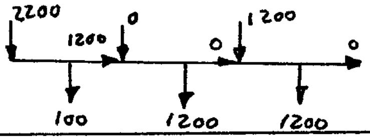
$x_{ij}, I_{ij} \ge 0$

Optimum solution: Cost = \$284,050

Product 1:



Product 2:



7

x_{ij} = Qty by operation i in month j

$i = 1, 2, \; j = 1, 2, 3$

$$\text{Minimize } Z = .2\sum_{j=1}^{3} I_{1j} + .4\sum_{j=1}^{3} I_{2j} + 10x_{11} + 12x_{12} + 11x_{13} + 15x_{21} + 18x_{22} + 16x_{23}$$

$.6x_{11} \le 800, \; .6x_{12} \le 700, \; .6x_{13} \le 550$

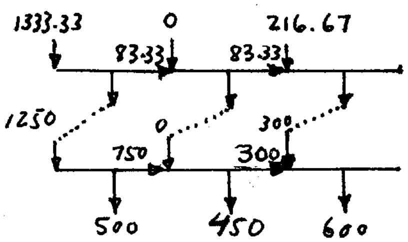
$.8x_{21} \le 1000, \; .8x_{22} \le 850, \; .8x_{23} \le 700$

$$\left.\begin{aligned} x_{1j} + I_{1,j-1} &= x_{2j} + I_{1j} \\ x_{2j} + I_{2,j-1} &= I_{2j} + d_j \end{aligned}\right\} j = 1, 2, 3$$

$I_{i0} = 0, \; i = 1, 2$

I_{ij} = Entering inv. of op. i in month j

Solution: Cost = \$39,720



Set 2.3d

8

x_j = Units of product j, $j=1,2$

y_i^- = Unused hours of machine i
y_i^+ = Overtime hours of machine i } $i=1,2$

Maximize $z = 110x_1 + 118x_2 - 100(y_1^+ + y_2^+)$

s.t.

$$\frac{x_1}{5} + \frac{x_2}{5} + y_1^- - y_1^+ = 8$$

$$\frac{x_1}{8} + \frac{x_2}{4} + y_2^- - y_2^+ = 8$$

$$y_1^+ \le 4, \quad y_2^+ \le 4$$

$$x_1, x_2, y_1^-, y_1^+, y_2^-, y_2^+ \ge 0$$

Solution:

Revenue = \$6,232

$x_1 = 56$, $y_1^+ = 4$ hrs

$x_2 = 4$, $y_2^+ = 0$

$y_1^-, y_2^- = 0$

1

x_S = tons of strawberry/day
x_g = tons of grapes/day
x_a = tons of apples/day

x_A = cans of drink A/day
x_B = cans of drink B/day
x_C = cans of drink C/day
} Each can holds one lb

x_{SA} = lb of strawberry used in drink A/day
x_{SB} = lb of strawberry used in drink B/day
x_{gA} = lb of grapes used in drink A/day
x_{gB} = lb of grapes used in drink B/day
x_{gC} = lb of grapes used in drink C/day
x_{aB} = lb of apples used in drink B/day
x_{aC} = lb of apples used in drink C/day

$$\text{Maximize } z = 1.15x_A + 1.25x_B + 1.2x_C - 200x_S - 100x_g - 90x_a$$

s.t.

$$x_S \le 200,\ x_g \le 100,\ x_a \le 150$$
$$x_{SA} + x_{SB} = 1500x_S$$
$$x_{gA} + x_{gB} + x_{gC} = 1200x_g$$
$$x_{aB} + x_{aC} = 1000x_a$$
$$x_A = x_{SA} + x_{gA}$$
$$x_B = x_{SB} + x_{gB} + x_{aB}$$
$$x_C = x_{gC} + x_{aC}$$
$$x_{SA} = x_{gA},$$
$$x_{SB} = x_{gB},\ x_{gB} = .5x_{aB}$$
$$3x_{gC} = 2x_{aC}$$

all variables ≥ 0

Optimum solution:

x_A = 90,000 cans, x_B = 300,000 cans, x_C = 0

x_{ij}:

	j		
i	A	B	C
S	45,000	75,000	0
g	45,000	75,000	0
a	0	150,000	0
	90,000	300,000	0

x_S = 80 tons, x_g = 100 tons, x_a = 150 tons

z = \$439,000/day

2

x_s = lb of screws per package
x_b = lb of bolts per package
x_n = lb of nuts per package
x_w = lb of washers per package

$$\text{Minimize } z = 1.1x_s + 1.5x_b + \frac{70}{80}x_n + \frac{20}{30}x_w$$

s.t.

$$Y = x_s + x_b + x_n + x_w$$
$$x_s \ge .1Y$$
$$x_b \ge .25Y,\ \frac{x_b}{50} \le x_w,\ \frac{x_b}{10} \le x_n$$
$$x_n \le .15Y$$
$$x_w \le .1Y$$
$$Y \ge 1$$

All variables are nonnegative

Optimum solution:

$Y = 1$, $x_s = .5$, $x_b = .25$, $x_n = .15$, $x_w = .1$

Cost = \$1.12

3

$x_{o,(A,B,C)}$ = lb of oats in cereals A, B, C
$x_r,(A,C)$ = lb of raisins in cereals A, C
$x_{c,(B,C)}$ = lb of coconuts in cereals B, C
$x_{a,(A,B,C)}$ = lb of almond in cereals A, B, C

$$Y_o = x_{oA} + x_{oB} + x_{oC}$$
$$Y_r = x_{rA} + x_{rC}$$
$$Y_c = x_{cB} + x_{cC}$$
$$Y_a = x_{aA} + x_{aB} + x_{aC}$$
$$W_A = x_{oA} + x_{rA} + x_{aA}$$
$$W_B = x_{oB} + x_{cB} + x_{aB}$$
$$W_C = x_{oC} + x_{rC} + x_{cC} + x_{aC}$$

$$\text{Maximize } z = \frac{1}{5}(2W_A + 2.5W_B + 3W_C) - \frac{1}{2000}(100Y_o + 120Y_r + 110Y_c + 200Y_n)$$

s.t.

$$W_A \le 500 \times 5 = 2500$$
$$W_B \le 600 \times 5 = 3000$$
$$W_C \le 500 \times 5 = 4000$$

continued...

Set 2.3e

$Y_0 \le 5 \times 2000 = 10{,}000$

$Y_r \le 2 \times 2000 = 4{,}000$

$Y_c \le 1 \times 2000 = 2{,}000$

$Y_a \le 1 \times 2000 = 2{,}000$

$X_{0A} = \frac{50}{5} X_{rA}$, $X_{0A} = \frac{50}{2} X_{aA}$

$X_{0B} = \frac{60}{2} X_{cB}$, $X_{0B} = \frac{60}{3} X_{aB}$

$X_{0C} = \frac{60}{3} X_{rC}$, $X_{0C} = \frac{60}{4} X_{cC}$, $X_{0C} = \frac{60}{2} X_{aC}$

all variables are nonnegative.

Optimum solution: $Z = \$5384.84$/day

$W_A = 2500$ lb or 500 boxes/day

$W_B = 3000$ lb or 600 boxes

$W_C = 5793.45$ lb or $\simeq 1158$ boxes

$X_0 = 10{,}000$ lb or 5 tons/day

$X_r = 471.19$ lb or .236 ton

$X_c = 428.16$ lb or .214 ton

$X_a = 394.11$ lb or .197 ton

4

$$\left.\begin{aligned} X_{Ai} &= \text{bbl of gasoline A in fuel } i \\ X_{Bi} &= \text{bbl of gasoline B in fuel } i \\ X_{Ci} &= \text{bbl of gasoline C in fuel } i \\ X_{Di} &= \text{bbl of gasoline D in fuel } i \end{aligned}\right\} i = 1, 2$$

$Y_A = X_{A1} + X_{A2}$

$Y_B = X_{B1} + X_{B2}$

$Y_C = X_{C1} + X_{C2}$

$Y_D = X_{D1} + X_{D2}$

$F_1 = X_{A1} + X_{B1} + X_{C1} + X_{D1}$

$F_2 = X_{A2} + X_{B2} + X_{C2} + X_{D2}$

Maximize $Z = 200F1 + 250F2$

$\quad - (120Y_A + 90Y_B + 100Y_C + 150Y_D)$

continued...

s.t.

$X_{A1} = X_{B1}$, $X_{A1} = .5X_{C1}$, $X_{A1} = .25X_{D1}$

$X_{A2} = X_{B2}$, $X_{A2} = 2X_{C2}$, $X_{A2} = \frac{2}{3}X_{D2}$

$Y_A \le 1000$, $Y_B \le 1200$, $Y_C \le 900$, $Y_D \le 1500$

$F_1 \ge 200$, $F_2 \ge 400$

Optimum solution: $Z = \$495{,}416.67$

$Y_A = 958.33$ bbl/day

$Y_B = 958.33$ bbl/day

$Y_C = 516.67$ bbl/day

$Y_D = 1500$ bbl/day

$F1 = 200$ bbl/day

$F_2 = 3733.33$ bbl/day

5

A = bbl of crude A/day

B = bbl of crude B/day

R = bbl of regular gasoline/day

P = bbl of premium gasoline/day

J = bbl of jet gasoline/day

Maximize $Z = 50(R - R^+) + 70(P - P^+)$

$\quad + 120(J - J^+) - (10R^- + 15P^- + 20J^-)$

$\quad - (2R^+ + 3P^+ + 4J^+) - (30A + 40B)$

s.t.

$A \le 2500$, $B \le 3000$

$R = .2A + .25B$, $R + R^- - R^+ = 500$

$P = .1A + .3B$, $P + P^- - P^+ = 700$

$J = .25A + .1B$, $J + J^- - J^+ = 400$

All variables ≥ 0

Optimum solution:

$Z = \$21{,}852.94$

$A = 1176.47$ bbl/day

$B = 1058.82$ bbl/day

$R = 500$ bbl/day

$P = 435.29$ bbl/day

$J = 400$ bbl/day

6

NR = bbl/day of naphta used in regular
NP = bbl/day of naphta used in premium
NJ = bbl/day of naphta used in jet
LR = bbl/day of light used in regular
LP = bbl/day of light used in premium
LJ = bbl/day of light used in jet

Using the other notation in Problem 5,

$$\text{Maximize } z = 50(R-R^+)+70(P-P^+)+120(J-J^+)$$
$$-(10R^-+15P^-+20J^-)-(2R^++3P^++4J^+)$$
$$-(30A+40B)$$

s.t.

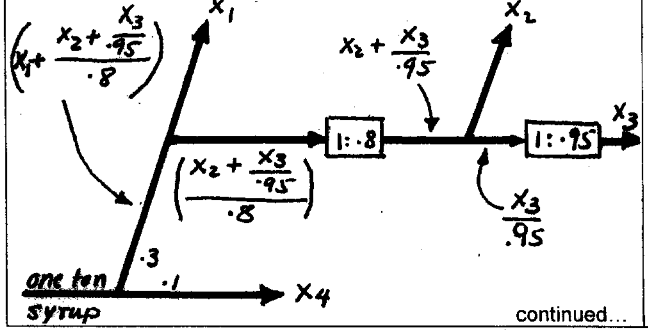
$$A \le 2500, \quad B \le 3000$$
$$R+R^- - R^+ = 500$$
$$P+P^- - P^+ = 700$$
$$J+J^- - J^+ = 400$$
$$.35A+.45B = NR+NP+NJ$$
$$.6A+.5B = LR+LP+LJ$$
$$R = NR+LR$$
$$P = NP+LP$$
$$J = NJ+LJ$$

all variables are nonnegative

<u>Optimum solution</u>: $Z = \$71,473.68$
$A = 1684.21, \ B = 0$
$R = 500, \ P = 700, \ J = 400$

7

x_1 = tons of brown sugar per week
x_2 = tons of white sugar per week
x_3 = tons of powdered sugar per week
x_4 = tons of molasses per week



continued...

$$\text{Maximize } Z = 150x_1+200x_2+230x_3+35x_4$$

s.t.

$$x_4 \le 4000 \times .1$$

or $\quad x_4 \le 400$

$$x_1 + \left(\frac{x_2+\frac{x_3}{.95}}{.8}\right) \le .3 \times 4000$$

or $\quad .76x_1 + .95x_2 + x_3 \le 912$

$$x_1 \ge 25, \ x_2 \ge 25$$
$$x_3 \ge 25, \ x_4 \ge 0$$

Optimum solution from TORA:
x_1 = 25 tons per week
x_2 = 25 tons per week
x_3 = 869.25 tons per week
x_4 = 400 tons per week
$Z = \$222,677.50$

8

A = bbl/hr of stock A
B = bbl/hr of stock B

$$\left.\begin{array}{l} Y_{Ai} = \text{bbl/hr of A used in gasoline } i \\ Y_{Bi} = \text{bbl/hr of B used in gasoline } i \end{array}\right\} i = 1, 2$$

$$\text{Maximize } Z = 7(Y_{A1}+Y_{B1})+10(Y_{A2}+Y_{B2})$$

s.t.

$$A = Y_{A1}+Y_{A2}, \quad A \le 450$$
$$B = Y_{B1}+Y_{B2}, \quad B \le 700$$
$$98Y_{A1}+89Y_{B1} \ge 91(Y_{A1}+Y_{B1})$$
$$98Y_{A2}+89Y_{B2} \ge 93(Y_{A2}+Y_{B2})$$
$$10Y_{A1}+8Y_{B1} \le 12(Y_{A1}+Y_{B1})$$
$$10Y_{A2}+8Y_{B2} \le 12(Y_{A2}+Y_{B2})$$

all variables are nonnegative

<u>Optimum solution</u>:

$Z = \$10,675$
A = 450 bbl/hr
B = 700 bbl/hr
Gasoline 1 production $= Y_{A1}+Y_{B1}$
$= 61.11+213.89 = 275$ bbl/hr
Gasoline 2 production $= Y_{A2}+Y_{B2}$
$= 388.89+486.11 = 875$ bbl/hr

9

S = tons of steel scrap / day
A = tons of alum. scrap / day
C = tons of cast iron scrap / day
Ab = tons of alum. briquettes / day
Sb = tons silicon briquettes / day
a = tons of alum. / day
g = tons of graphite / day
s = tons of silicon / day
aI = tons of alum. in ingot I / day
aII = tons of alum. in ingot II / day
gI = tons of graphite in ingot I / day
gII = tons of graphite in ingot II / day
sI = tons of silicon in ingot I / day
sII = tons of silicon in ingot II / day
I_1 = tons of ingot I / day
I_2 = tons of ingot II / day.

Minimize $Z = 100S + 150A + 75C + 900Ab + 380S_b$

s.t. $S \le 1000,\ A \le 500,\ C \le 2500$

$a = .1S + .95A + Ab$
$g = .05S + .01A + .15C$
$s = .04S + .02A + .08C + S_b$
$I_1 = aI + gI + sI$
$I_2 = aII + gII + sII$
$a_I + a_{II} \le \cancel{s}$, $sI + sII \le s$, $gI + gII \le g$
$.081 I_1 \le aI \le .108 I_1$
$.015 I_1 \le gI \le .03 I_1$
$.025 I_1 \le sI < \infty$
$.062 I_2 \le aII \le .089 I_2$
$.041 I_2 \le gII \le \infty$
$.028 I_2 \le sII \le .041 I_2$
$I_1 \ge 130,\ I_2 \ge 250$

Optimum solution:

$Z = \$117,435.65$
$S = 0,\ A = 38.2,\ C = 1489.41$
$Ab = Sb = 0$
$I_1 = 130,\ I_2 = 250$
$a = 36.29,\ g = 223.79,\ s = 119.92$

10

x_{ij} = tons of ore i allocated to alloy k
w_k = tons of alloy k produced

$$\text{Maximize } z = 200\,w_A + 300\,w_B - 30(x_{1A} + x_{1B}) - 40(x_{2A} + x_{2B}) - 50(x_{3A} + x_{3B})$$

Subject to

Specification constraints:

$.2x_{1A} + .1x_{2A} + .05x_{3A} \le .8W_A$ ①
$.1x_{1A} + .2x_{2A} + .05x_{3A} \le .3W_A$ ②
$.3x_{1A} + .3x_{2A} + .2x_{3A} \ge .5W_A$ ③
$.1x_{1B} + .2x_{2B} + .05x_{3B} \ge .4W_B$ ④
$.1x_{1B} + .2x_{2B} + .05x_{3B} \le .6W_B$ ⑤
$.3x_{1B} + .3x_{2B} + .7x_{3B} \ge .3W_B$ ⑥
$.3x_{1B} + .3x_{2B} + .2x_{3B} \le .7W_B$ ⑦

Ore constraints:

$x_{1A} + x_{1B} \le 1000$
$x_{2A} + x_{2B} \le 2000$
$x_{3A} + x_{3B} \le 3000$

```
                    *** OPTIMUM SOLUTION SUMMARY ***
--------------------------------------------------------------------------
Title: Problem 26a-17
Final iteration No: 12
Objective value (max) =400000.0000
--------------------------------------------------------------------
Variable               Value        Obj Coeff     Obj Val Contrib
--------------------------------------------------------------------
x1 wA              1799.9999        200.0000        359999.9688
x2 wB              1000.0001        300.0000        300000.0312
x3 x1A             1000.0000        -30.0000        -30000.0000
x4 x1B                0.0000        -30.0000           -0.0000
x5 x2A                0.0000        -40.0000           -0.0000
x6 x2B             2000.0001        -40.0000        -80000.0078
x7 x3A             3000.0000        -50.0000       -150000.0000
x8 x3B                0.0000        -50.0000           -0.0000
------------------------------------------------------------------
Constraint               RHS           Slack(-)/Surplus(+)
------------------------------------------------------------------
1 (<)                 0.0000           1090.0000-
2 (<)                 0.0000            290.0000-
3 (>)                 0.0000              0.0000+
4 (>)                 0.0000              0.0000+
5 (<)                 0.0000            200.0000-
6 (>)                 0.0000            300.0002+
7 (<)                 0.0000            100.0000-
8 (<)              1000.0000              0.0000-
9 (<)              2000.0000              0.0000-
10 (<)             3000.0000              0.0000-
```

Solution:

Produce 1800 tons of alloy A and 1000 tons of alloy B.

Set 2.3f

1

h = Regular pay hour

8-hr pay = $8h$

12-hr pay = $12h + \frac{4h}{2} = 14h$

x_i = Nbr 8-hr buses starting in period i

y_i = Nbr. of 12-hr buses starting in period i

Minimize $Z = h\left(8\sum_{i=1}^{6} x_i + 14\sum_{i=1}^{6} y_i\right)$

s.t.

x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	y_5	y_6	
1					1	1				1	1	≥ 4
1	1					1	1				1	≥ 8
	1	1				1	1	1				≥ 10
		1	1				1	1	1			≥ 7
			1	1				1	1	1		≥ 12
				1	1				1	1	1	≥ 4

Solution: $Z = 196h$

$x_1 = 4, x_2 = 4, x_4 = 2, x_5 = 4, x_3 = x_6 = 0$

$y_3 = 6, y_1 = y_2 = y_4 = y_5 = y_6 = 0$

For 8-hr only buses, solution is

$Z = 208h$

$x_1 = x_2 = 4, x_3 = 6, x_4 = 1, x_5 = 11, x_6 = 0$

(8-hr + 12-hr) buses is cheaper.

2

x_i = Nbr. of volunteers starting in hour i

Minimize $z = \sum_{i=1}^{14} x_i$

s.t.

(8:00) $x_1 \geq 4$

(9:00) $x_1 + x_2 \geq 4$

(10:0) $x_1 + x_2 + x_3 \geq 6$

(11:00) $x_2 + x_3 + x_4 \geq 6$

(12:00) $x_3 + x_4 + x_5 \geq 8$

(1:00) $x_4 + x_5 + x_6 \geq 8$

(2:00) $x_5 + x_6 + x_7 \geq 6$

(3:00) $x_6 + x_7 + x_8 \geq 6$

(4:00) $x_7 + x_8 + x_9 \geq 4$

(5:00) $x_8 + x_9 + x_{10} \geq 4$

(6:00) $x_4 + x_{10} + x_{11} \geq 6$

(7:00) $x_{10} + x_{11} + x_{12} \geq 6$

(8:00) $x_{11} + x_{12} + x_{13} \geq 8$

(9:00) $x_{12} + x_{13} \geq 8$

All $x_j \geq 0$

continued...

Solution: $Z = 32$ volunteers

$x_1 = 4, x_3 = 2, x_4 = 6, x_6 = 2, x_7 = 4, x_{10} = 6, x_{12} = 8$

all other $x_i = 0$

3

Same formulation as in Problem 2 with the added constraints $x_5 = 0, x_{11} = 0$

Optimum solution remains the same

4

x_i = Nbr. of casuals starting on day i

($i = 1$: Monday, $i = 7$: Sunday)

Minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

s.t.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
M	1			1	1	1	1	≥ 20
T	1	1			1	1	1	≥ 14
W	1	1	1			1	1	≥ 10
Th	1	1	1	1			1	≥ 15
F	1	1	1	1	1			≥ 18
Sat		1	1	1	1	1		≥ 10
Sun			1	1	1	1	1	≥ 12

Solution: $Z = 20$ workers

$x_1 = 8, x_4 = 6, x_5 = 4, x_6 = 1, x_7 = 1$

5

x_i = Nbr. Students starting at hour i

$i = 1$ (8:01), $i = 9$ (4:01), $x_5 = 0$

Minimize $Z = x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9$

S.t.

	x_1	x_2	x_3	x_4	x_6	x_7	x_8	x_9	
8:01	1								≥ 2
9:01	1	1							≥ 2
10:01	1	1	1						≥ 3
11:01		1	1	1					≥ 4
12:01			1	1					≥ 4
1:01				1	1				≥ 3
2:01					1	1			≥ 3
3:01					1	1	1		≥ 3
4:01						1	1	1	≥ 3

Solution: $Z = 9$ students

$x_1 = 2, x_3 = 1, x_4 = 3, x_7 = 3$

Set 2.3f

6

Let x_i = Nbr. starting on day i and lasting for 7 days

y_{ij}= Nbr. starting shift on day i and *starting* their 2 days off on day j, $i \neq j$

Thus, of the x_1 workers who start on Monday, y_{12} will take T and W off, y_{13} will take W and Th off, and so on, as the following table shows.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	**start on Mon**	y_{12}	$y_{12}+y_{13}$	$y_{13}+y_{14}$	$y_{14}+y_{15}$	$y_{15}+y_{16}$	y_{16}
2	y27	**Tue**	y23	y23+y24	y24+y25	y25+y26	y26+y27
3	y31+y37	y31	**Wed**	y34	y34+y35	y35+y36	y36+y37
4	y41+y47	y41+y42	y42	**Th**	y45	y45+y46	y46+y47
5	y51+y57	y51+y52	y52+y53	y53	**Fri**	y56	y56+y57
6	y61+y67	y61+y62	y62+y63	y63+y64	y64	**Sat**	y67
7	y71	y71+y72	y72+y73	y73+y74	y74+y75	y75	**Su**

Minimize z = $x_1+x_2+x_3+x_4+x_5+x_6+x_7$

Each employee has 2 days off: x_i = sum{j in 1..7,j≠i}y_{ij}

Mon (1) constraint: s -(y27 + y31+ y37 + y41+y47+ y51+ y57 + y61+y67 + y71)>= 12

Tue (2) constraint: s -(y12 + y31 + y41+ y42 + y51+y52 + y61+y62 + y71+y72>= 18

Wed (3) constraint: s -(y12+y13 + y23 + y42 + y52+y53 + y62+y63 + y72+y73>= 20

Th (4) constraint: s -(y13+y14 + y23 +y24+ y24 + y53 + y63+y64 + y73+y74>= 28

Fri (5) constraint: s -(y14+y15 + y24 +y25+ y34 +y35 + y45+ y64 + y74+y75>= 32

Sat(6) constraint: s -(y15+y16+ y25 +y26+ y35 + y36 + y45+y46+ y56+y75>= 40

Sun(7) constraint: s -(y16+ y26 +y27+ y36 + y37 +y46+y47+ y56+y57+y67)>= 40

continued

Solution: 42 employees

Starting		Nbr off						
On	Nbr	M	Tu	Wed	Th	Fri	Sat	Sun
M	16		16	16				
Tu	8				8	8		
Wed	8	8	8					
Th	0							
Fri	6			6	6			
Sat	2	2						2
Sun	2					2	2	
Nbr off		10	24	22	14	10	2	2
Nbr at work		32	18	20	28	32	40	40
Surplus above minimum		20	0	0	0	0	0	0

Set 2.3g

1

	Setting		Number produced	Surplus rolls
	1	3		
5'	0	2	200	50
7'	1	0	200	0
9'	1	1	300	0
Loss/foot	4	1		
No. rolls	200	100		

Trim loss area =

$L(200 \times 4 + 100 \times 1 + 50 \times 5) = 1150L\ ft^2$

(b) 15' standard roll:

	Setting			
	1	2	3	4
5'	3	1	1	0
7'	0	1	0	2
9'	0	0	1	0
trim loss per ft	0	3	1	1

(c) $x_1 + x_2 + 2x_5 \geq 120$

New solution calls for decreasing the number of standard 20'-rolls by 30

(d) $x_1 + x_3 + 2x_6 \geq 240$

New solution calls for increasing the number of standard 20'-rolls by 50

2

x_i = Space (in^2) allocated to cereal i

Maximize $z = 1.1x_1 + 1.3x_2 + 1.08x_3 + 1.25x_4 + 1.2x_5$

s.t.

$16x_1 + 24x_2 + 18x_3 + 22x_4 + 20x_5 \leq 5000$

$x_1 \leq 100, x_2 \leq 85, x_3 \leq 140, x_4 \leq 80, x_5 \leq 90$

$x_i \geq 0$ for all $i = 1, 2, .., 5$

Solution:

$z = \$314/day$

$x_1 = 100, x_3 = 140, x_5 = 44$

$x_2 = x_4 = 0$

3

x_i = Nbr. of ads for issue i, $i = 1, 2, 3, 4$

Minimize $z = S_1^- + S_2^- + S_3^- + S_4^-$

s.t.

$(-30{,}000 + 60{,}000 + 30{,}000)x_1 + S_1^- - S_1^+ = .51 \times 400{,}000$

$(80{,}000 + 30{,}000 - 45{,}000)x_2 + S_2^- - S_2^+ = .51 \times 400{,}000$

$(40{,}000 + 10{,}000)x_3 + S_3^- - S_3^+ = .51 \times 400{,}000$

$(90{,}000 - 25{,}000)x_4 + S_4^- - S_4^+ = .51 \times 400{,}000$

$1500(x_1 + x_2 + x_3 + x_4) \leq 100{,}000$

$x_1, x_2, x_3, x_4 \geq 0$

Solution:

$x_1 = 3.4, x_2 = 3.14, x_3 = 4.08, x_4 = 3.14$

4

x_{ij} = Units of part j produced by department i, $i = 1, 2, 3$, $j = 1, 2$

Maximize $z = \min\{x_{11} + x_{21}, x_{12} + x_{22}, x_{13} + x_{23}\}$

or

Maximize $z = y$

s.t.

$y \leq x_{11} + x_{21}$

$y \leq x_{12} + x_{22}$

$y \leq x_{13} + x_{23}$

$\frac{x_{11}}{8} + \frac{x_{12}}{5} + \frac{x_{13}}{10} \leq 100$

$\frac{x_{21}}{6} + \frac{x_{22}}{12} + \frac{x_{23}}{4} \leq 80$

all $x_{ij} \geq 0$

Solution:

Nbr. of assembly units $= y = 556.2 \simeq 557$

$x_{11} = 354.78, x_{12} = 0$

$x_{21} = 556.52, x_{22} = 201.74$

$x_{31} = 556.52, x_{32} = 0$

5

x_i = tons of coal i, $i = 1, 2, 3$

Minimize $z = 30x_1 + 35x_2 + 33x_3$

s.t.

$2500x_1 + 1500x_2 + 1600x_3 \leq 2000(x_1 + x_2 + x_3)$

$x_1 \leq 30, x_2 \leq 30, x_3 \leq 30$

$x_1 + x_2 + x_3 \geq 50$

Solution: $z = \$1361.11$

$x_1 = 22.22$ tons, $x_2 = 0$, $x_3 = 27.78$ tons.

6

t_i = Green time in secs for highway i,
$i = 1, 2, 3$

Maximize $Z = 3\left(\frac{500}{3600}\right)t_1 + 4\left(\frac{600}{3600}\right)t_2 + 5\left(\frac{400}{3600}\right)t_3$

s.t.

$$\left(\frac{500}{3600}\right)t_1 + \left(\frac{600}{3600}\right)t_2 + \left(\frac{400}{3600}\right)t_3 \le \frac{510}{3600}(2.2\times 60 - 3\times 10)$$

$$t_1 + t_2 + t_3 + 3\times 10 \le 2.2\times 60,\ t_1 \ge 25, t_2 \ge 25, t_3 \ge 25$$

Solution: $Z = \$58.04/\text{hr}$

$t_1 = 25,\ t_2 = 43.6,\ t_3 = 33.4$ Sec

7

y_i = observation i

Define straight line as

$$\hat{y}_i = a \ + b, \quad a, b \text{ unrestricted}$$

$$\text{Minimize } Z = \sum_{i=1}^{10} y_i - \hat{y}_i|$$

$$= \sum_{i=1}^{10} |y_i - ai - b|$$

Let $d_i = |y_i - ai - b|$

Minimize $Z = d_1 + d_2 + \cdots + d_{10}$

s.t.

$y_i - ai - b \le d_i$

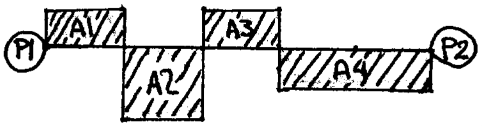
$y_i - ai - b \ge -d_i$

a, b, unrestricted

$d_i \ge 0$

Solution: $\hat{y}_i = 2.85714\,i + 6.42857$

8



$A1 = 2\times 1760\times 10\times 50 = 1760$ (thousand) Yd^3

$A2 = 3520,\ A3 = 1760,\ A4 = 3520$

Distances (center to center) in miles:

$$\begin{array}{c|cc} & A2 & A4 \\ A1 & 2 & 7 \\ A3 & 2 & 3 \\ P1 & 3 & 8 \\ P2 & 7 & 2 \end{array}$$

continued…

Cost (\$) per cubic yd:

$$\begin{array}{c|cc} & (5)\ A2 & (6)\ A4 \\ (1)\ A1 & .2 + 2\times .15 = .50 & .20 + 7\times .15 = 1.25 \\ (2)\ A3 & .20 + 2\times .15 = .50 & .20 + 3\times .15 = .65 \\ (3)\ P1 & 1.70 + 3\times .15 = 2.15 & 1.70 + 8\times .15 = 2.90 \\ (4)\ P3 & 2.10 + 7\times .15 = 3.15 & 2.10 + 2\times .15 = 2.40 \end{array}$$

Using the code $A1 \equiv 1$, $A3 \equiv 2$, $P1 \equiv 3$, $P2 \equiv 4$, $A2 \equiv 5$, $A4 \equiv 6$, let

$x_{ij} = 10^3\, Yd^3$ from source i to destination j
$i = 1,2,3,4,\ j = 5, 6$

$$\text{Minimize } Z = 1000(.5x_{15} + 1.25x_{16} + .5x_{25} + .65x_{26} + 2.15x_{35} + 2.9x_{36} + 3.15x_{45} + 2.4x_{46})$$

s.t.

$x_{15} + x_{16} \le 1760 \qquad x_{35} + x_{36} \le 20{,}000$

$x_{25} + x_{26} \le 1760 \qquad x_{45} + x_{46} \le 15{,}000$

$x_{15} + x_{25} + x_{35} + x_{45} \ge 3520$

$x_{16} + x_{26} + x_{36} + x_{46} \ge 3520$

Solution:

A1→A2: $x_{15} = 1760$ (1000 Cu Yd)
A1→A4: $x_{16} = 0$
A3→A2: $x_{25} = 0$
A3→A4: $x_{26} = 1760$
P1→A2: $x_{35} = 1760$
P1→A4: $x_{36} = 0$
P2→A2: $x_{45} = 0$
P2→A4: $x_{46} = 1760$
Cost = \$10,032,000

9

x_{ij} = Blue regulars on front i in defense line j, $i=$

y_{ij} = Blue reserves on front i in defense line j.

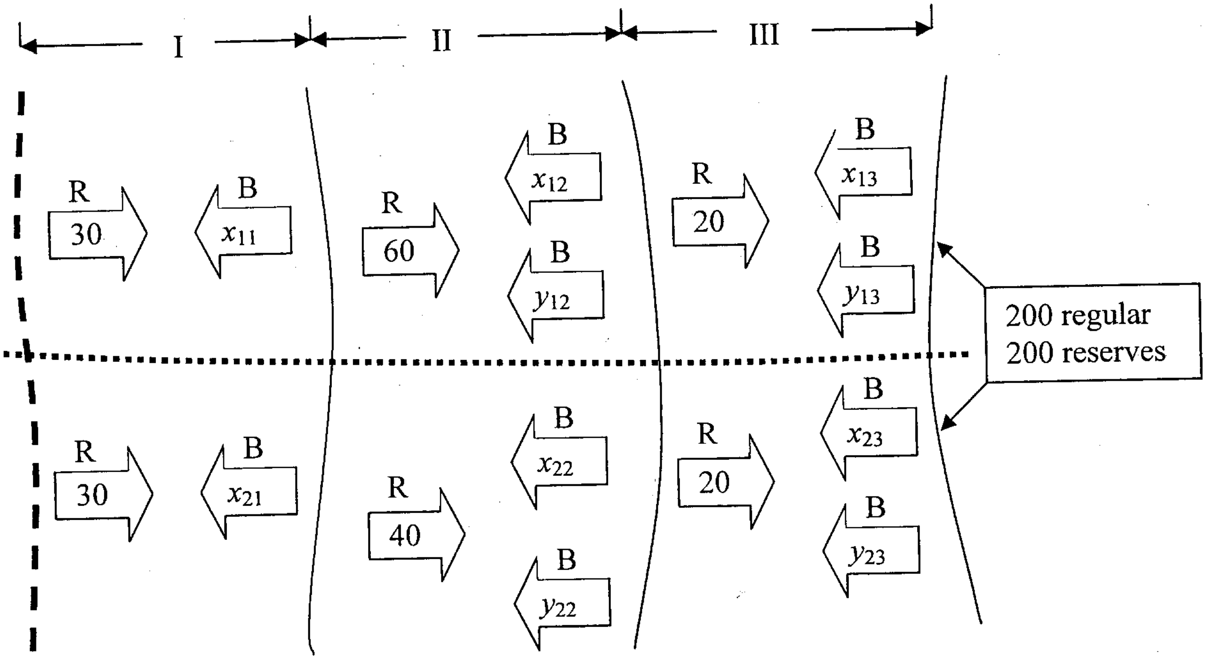
t_{ij} = Delay days on front i in defense line j:

Maximize $Z = \min\{t_{11} + t_{12} + t_{13},\ t_{21} + t_{22} + t_{23}\}$

or

continued…

Set 2.3g



Maximize $Z = T$

s.t.

$$T \le t_{11} + t_{12} + t_{13}$$

$$T \le t_{21} + t_{22} + t_{23}$$

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \le 200$$

$$y_{12} + y_{13} + y_{22} + y_{23} \le 200$$

$$t_{11} = .5 + 8.8\,\frac{x_{11}}{30}$$

$$t_{12} = .75 + 7.9\,\frac{x_{12} + y_{12}}{60}$$

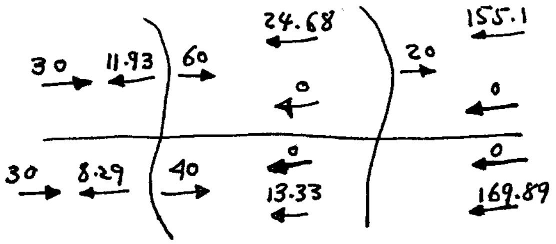
$$t_{13} = .55 + 10.2\,\frac{x_{13} + y_{13}}{20}$$

$$t_{21} = 1.1 + 10.5\,\frac{x_{21}}{30}$$

$$t_{22} = 1.3 + 8.1\,\frac{x_{22} + y_{22}}{40}$$

$$t_{23} = 1.5 + 9.2\,\frac{x_{23} + y_{23}}{20}$$

Solution: Battle duration = 87.65 days



continued...

10

x_i = Efficiency of plant i

Minimize $Z = .2(500)x_1 + .25(3000)x_2 + .15(6000)x_3 + .18(1000)x_4$

s.t.

$$500(1-x_1) \le .00085 \times 215{,}000$$

$$.94(500)(1-x_1) + 3000(1-x_2) \le .0009 \times 220{,}000$$

$$.94^2(500)(1-x_1) + .94(3000)(1-x_2) + 6000(1-x_3) \le .0008 \times 200{,}000$$

$$.94^3(500)(1-x_1) + .94^2(300)(1-x_2) + .94(6000)(1-x_3) + 1000(1-x_4) \le .0008 \times 210{,}000$$

$$0 \le x_1 \le .99$$

$$0 \le x_2 \le .99$$

$$0 \le x_3 \le .99$$

$$0 \le x_4 \le .99$$

Solution

Cost per ton = \$1891.41

Plant 1 efficiency = .99

Plant 2 efficiency = .9661

Plant 3 efficiency = .99

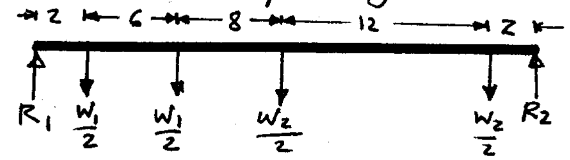
Plant 4 efficiency = .9824

11

W_i = Capacity of yoke i (Kips)

R_1 = Reaction in Kips at left end

R_2 = Reaction in Kips at right end



Maximize $Z = W_1 + W_2$

s.t.

$$R_1 + R_2 = W_1 + W_2$$

$$2\left(\frac{W_1}{2}\right) + 8\left(\frac{W_1}{2}\right) + 16\left(\frac{W_2}{2}\right) + 28\left(\frac{W_2}{2}\right) = 30 R_2$$

$$R_1 \leq 25, \quad R_2 \leq 25$$

$$\frac{W_1}{2} \leq 20, \quad \frac{W_2}{2} \leq 20$$

Solution:

$W_1 = 20.59$ Kips

$W_2 = 29.41$ Kips

12

X_{ij} = Nbr. of aircraft of type i allocated to route j ($i = 1,2,3,4$, $j = 1,2,3,4$)

S_j = Nbr. of passengers not served on route j, $j = 1,2,3,4$

$$\begin{aligned} \text{Minimize } Z = {} & 1000(3x_{11}) + 1100(2x_{12}) \\ & + 1200(2x_{13}) + 1500(x_{14}) \\ & + 800(4x_{21}) + 900(3x_{22}) \\ & + 1000(3x_{23}) + 1000(2x_{24}) \\ & + 600(5x_{31}) + 800(5x_{32}) \\ & + 800(4x_{33}) + 900(2x_{34}) \\ & + 40S_1 + 50S_2 + 45S_3 + 70S_4 \end{aligned}$$

Subject to

$$\sum_{j=1}^{4} x_{1j} \leq 5, \quad \sum_{j=1}^{4} x_{2j} \leq 8, \quad \sum_{j=1}^{4} x_{3j} \leq 10$$

$$50(3x_{11}) + 30(4x_{21}) + 20(5x_{31}) + S_1 = 1000$$

$$50(2x_{12}) + 30(3x_{22}) + 20(5x_{32}) + S_2 = 2000$$

$$50(2x_{13}) + 30(3x_{23}) + 20(4x_{33}) + S_3 = 900$$

$$50(x_{14}) + 30(2x_{24}) + 20(2x_{34}) + S_4 = 1200$$

All x_{ij} and $S_j \geq 0$

continued...

```
                        *** OPTIMUM SOLUTION SUMMARY ***
----------------------------------------------------------------------
Title: Problem 26a-16
Final iteration No: 16
Objective value (min) =221900.0000
==> ALTERNATIVE solution detected at x13
-----------------------------------------------------------------
Variable               Value       Obj Coeff    Obj Val Contrib
-----------------------------------------------------------------
x1 x11                5.0000      3000.0000        14999.9990
x2 x12                0.0000      2200.0000           0.0000
x3 x13                0.0000      2400.0000           0.0000
x4 x14                0.0000      1500.0000           0.0000
x5 x21                0.0000      3200.0000           0.0000
x6 x22                0.0000      2700.0000           0.0000
x7 x23                0.0000      3000.0000           0.0000
x8 x24                8.0000      2000.0000        15999.9990
x9 x31                2.5000      3000.0000         7500.0015
x10 x32               7.5000      4000.0000        29999.9980
x11 x33               0.0000      3200.0000           0.0000
x12 x34               0.0000      1800.0000           0.0000
x13 s1                0.0000        40.0000           0.0000
x14 s2             1250.0000        50.0000        62500.0000
x15 s3              899.9998        45.0000        40499.9922
x16 s4              720.0001        70.0000        50400.0078
-------------------------------------------------------------
Constraint             RHS          Slack(-)/Surplus(+)
-------------------------------------------------------------
1 (<)                 5.0000          0.0000-
2 (<)                 8.0000          0.0000-
3 (<)                10.0000          0.0000-
4 (=)              1000.0000          0.0000
5 (=)              2000.0000          0.0000
6 (=)               900.0000          0.0000
7 (=)              1200.0000          0.0000
```

Solution:

Aircraft Type	Route	Nbr. aircraft
1	1	5
2	4	8
3	1	2.5
3	2	7.5

Fractional solution must be rounded.

Cost = \$221,900

CHAPTER 3

The Simplex Method and Sensitivity Analysis

Set 3.1a

1

$(x_1, x_2) = (3, 1)$

$M1: S_1 = 24 - (6 \times 3 + 4 \times 1) = 2$ tons/day

$M2: S_2 = 6 - (1 \times 3 + 2 \times 1) = 1$ ton/day

2

$$S_1 = x_1 + x_2 - 800$$
$$= 500 + 600 - 800 = 300 \text{ lb}$$

3

$10X_1 - 3X_2 \geq -5 \equiv -10X_1 + 3X_2 \leq 5$

Thus, $-10X_1 + 3X_2 + S_1 = 5$ ①

Also, $10X_1 - 3X_2 \geq -5 \equiv 10X_1 - 3X_2 - S_2 = -5$

Thus, $-10X_1 + 3X_2 + S_2 = 5$ ②

① and ② are the same

4

X_{ij} = number of units of product i manufactured on machine j

LP model

Maximize $Z = 10(X_{11} + X_{12}) + 15(X_{21} + X_{22})$

Subject to

$$|(X_{11} + X_{21}) - (X_{12} + X_{22})| \leq 5$$
$$X_{11} + X_{21} \leq 200$$
$$X_{12} + X_{22} \leq 250$$
$$X_{ij} \geq 0 \text{ for all } i \,\&\, j$$

Equation form:

$|(X_{11} + X_{21}) - (X_{12} + X_{22})| \leq 5$

to

$$X_{11} + X_{21} - X_{12} - X_{22} \leq 5$$
$$X_{11} + X_{21} - X_{12} - X_{22} \geq -5$$

Maximize $z = 10X_{11} + 10X_{12} + 15X_{21} + 15X_{22}$

Subject to

$$\begin{aligned}
X_{11} + X_{21} - X_{12} - X_{22} + S_1 &= 5 \\
-X_{11} - X_{21} + X_{12} + X_{22} + S_2 &= 5 \\
X_{11} + X_{21} + S_3 &= 200 \\
X_{12} + X_{22} + S_4 &= 250
\end{aligned}$$

$X_{ij} \geq 0$ for all i and j

$S_i \geq 0$ for all i

continued...

5

$y = \max\{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$

Hence

$$|x_1 - x_2 + 3x_3| \leq y$$
$$|-x_1 + 3x_2 - x_3| \leq y$$

LP model:

minimize $z = y$

Subject to

$$\begin{aligned}
x_1 - x_2 + 3x_3 &\leq y \\
x_1 - x_2 + 3x_3 &\geq -y \\
-x_1 + 3x_2 - x_3 &\leq y \\
-x_1 + 3x_2 - x_3 &\geq -y
\end{aligned}$$

$$x_1, x_2, x_3, y \geq 0$$

Equation form:

Minimize $z = y$

Subject to

$$\begin{aligned}
-y + x_1 - x_2 + 3x_3 + S_1 &= 0 \\
-y - x_1 + x_2 - 3x_3 + S_2 &= 0 \\
-y - x_1 + 3x_2 - x_3 + S_3 &= 0 \\
-y + x_1 - 3x_2 + x_3 + S_4 &= 0
\end{aligned}$$

$$x_1, x_2, x_3, y, S_1, S_2, S_3, S_4 \geq 0$$

6

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \Leftrightarrow \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & ① \\ \sum_{j=1}^{n} a_{ij} x_j \geq b_i & ② \end{cases}$$

From ②, for $i = 1, 2, \ldots, m$, we have

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \Leftrightarrow \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) \geq \sum_{i=1}^{m} b_i$$
$$\Leftrightarrow \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right) x_j \geq \sum_{i=1}^{m} b_i$$

Thus, ① and ② are equivalent to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m$$
$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right) x_j \geq \sum_{i=1}^{m} b_i$$

1

x_1 = Nbr. $\frac{1}{4}$-lb / day

x_2 = Nbr. cheeseburgers/day

Maximize $z = .2x_1 + .15x_2 - .25x_3^+$

s.t.

$$.25x_1 + .2x_2 + x_3^- - x_3^+ = 200$$

$$x_1 + x_2 \le 900$$

Solution: $z = \$173.35$

$x_1 = 900$, $x_2 = 0$, $x_3^+ = 25$ lb

2

(a) x_j = # units of product j per day, $j=1,2$

x_3^+ = unused minutes of machine time / day

x_3^- = machine overtime per day in minutes

Maximize $z = 6x_1 + 7.5x_2 - .5x_3^-$

Subject to

$$10x_1 + 12x_2 + x_3^+ - x_3^- = 2500$$

$$150 \le x_1 \le 200$$

$$x_2 \le 45$$

$$x_1, x_2 \ge 0$$

$$x_3^+, x_3^- \ge 0$$

TORA optimum solution:

x_1 = 200 units/day

x_2 = 45 units/day

x_3^- = overtime minutes
= 40 minutes/day

z = \$1517.50

(b) Overtime at \$1.50/min yields $x_3^- = 0$, which means no overtime is needed

3

x_j = # of units of products 1, 2, and 3

Maximize $z = 2x_1 + 5x_2 + 3x_3 - 15x_4^+ - 10x_5^+$

Subject to

$$2x_1 + x_2 + 2x_3 + x_4^- - x_4^+ = 80$$

$$x_1 + x_2 + 2x_3 + x_5^- - x_5^+ = 65$$

all variables ≥ 0

Solution: $z = \$325$

$x_2 = 65$ units, $x_4^- = 15$

All other variables = 0

continued…

4

Formulation 1:

Maximize $z = -2x_1 + 3x_2^+ - 3x_2^- - 2x_3^+ + 2x_3^-$

Subject to

$$4x_1 - x_2^+ + x_2^- - 5x_3^+ + 5x_3^- = 10$$

$$2x_1 + 3x_2^+ - 3x_2^- + 2x_3^+ - 2x_3^- = 12$$

all variables ≥ 0

Optimum solution:

$x_1 = 0$

$$\left.\begin{array}{l} x_2^+ = 6.15 \\ x_2^- = 0 \end{array}\right\} \Rightarrow x_2 = 6.15$$

$$\left.\begin{array}{l} x_3^+ = 0 \\ x_3^- = 3.23 \end{array}\right\} \Rightarrow x_3 = -3.23$$

$z = 24.92$

Formulation 2:

Maximize $z = -2x_1 + 3x_2^+ - 2x_3^+ - w$

Subject to

$$4x_1 - x_2^+ - 5x_3^+ + 6w = 10$$

$$2x_1 + 3x_2^+ + 2x_3^+ - 5w = 12$$

all variables ≥ 0

Optimum solution:

$x_1 = 0$

$$\left.\begin{array}{l} x_2^+ = 9.38 \\ w = 3.23 \end{array}\right\} \Rightarrow x_2 = 9.38 - 3.23 = 6.15$$

$$\left.\begin{array}{l} x_3^+ = 0 \\ w = 3.23 \end{array}\right\} \Rightarrow x_3 = 0 - 3.23 = -3.23$$

$z = 24.92$

continued…

Set 3.2a

1

(a)

Equation form:

Maximize $z = 2x_1 + 3x_2$

subject to

$$x_1 + 3x_2 + x_3 = 6$$
$$3x_1 + 2x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(b) Basic (x_1, x_2) (Point B):

$$x_1 + 3x_2 = 6$$
$$3x_1 + 2x_2 = 6$$

Solution: $(x_1, x_2) = (\frac{6}{7}, \frac{12}{7})$, $z = 6\frac{6}{7}$

Basic (x_1, x_3) (Point E):

$$x_1 + x_3 = 6$$
$$3x_1 = 6$$

Solution: $(x_1, x_3) = (2, 4)$, $z = 4$

Basic (x_1, x_4) (Point C):

$$x_1 = 6$$
$$3x_1 + x_4 = 6$$

Solution: $(x_1, x_4) = (6, -12)$

Unique but infeasible

Basic (x_2, x_3) (Point A):

$$3x_2 + x_3 = 6$$
$$2x_2 = 6$$

Solution: $(x_2, x_3) = (3, -3)$

Unique but infeasible

Basic (x_2, x_4) (Point D):

$$3x_2 = 6$$
$$2x_2 + x_4 = 6$$

Solution: $(x_2, x_4) = (2, 2)$, $z = 6$

Basic (x_3, x_4) (Point F):

$$x_3 = 6$$
$$x_4 = 6$$

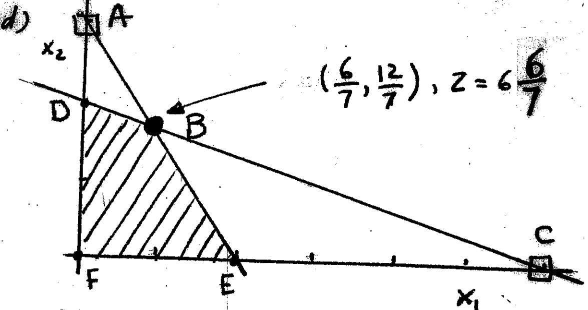
Solution: $(x_3, x_4) = (6, 6)$, $z = 0$

(c) Optimum solution occurs at B:

$(x_1, x_2) = (\frac{6}{7}, \frac{12}{7})$ with $z = 6\frac{6}{7}$

continued...

(d)



(e) From the graph in (d), we have

A: $x_2 = 3$, $x_3 = -3$

C: $x_1 = 6$, $x_4 = -12$

2

(a) Maximize $z = 2x_1 - 4x_2 + 5x_3 - 6x_4$

Subject to

$$x_1 + 4x_2 - 2x_3 + 8x_4 + x_5 = 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 + x_6 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Combination	Solution	Status	z
x_1, x_2	0, 1/2	Feasible	-2
x_1, x_3	8, 3	Feasible	$\boxed{31}$
x_1, x_4	0, 1/4	Feasible	-3/2
x_1, x_5	-1, 3	Infeasible	—
x_1, x_6	2, 3	Feasible	4
x_2, x_3	1/2, 0	Feasible	-2
x_2, x_4	1/2, 0	Feasible	-2
x_2, x_5	1/2, 0	Feasible	-2
x_2, x_6	1/2, 0	Feasible	-2
x_3, x_4	0, 1/4	Feasible	-3/2
x_3, x_5	1/3, 8/3	Feasible	5/3
x_3, x_6	-1, 4	Infeasible	—
x_4, x_5	1/4, 0	Feasible	-3/2
x_4, x_6	1/4, 0	Feasible	-3/2
x_5, x_6	2, 1	Feasible	0

Optimum Solution:

$x_1 = 8$, $x_2 = 0$, $x_3 = 3$, $x_4 = 0$

$z = 31$

continued...

(b) Minimize $z = x_1 + 2x_2 - 3x_3 - 2x_4$

subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Combination	Solution	Status	z
x_1, x_2	infinity of solutions		—
x_1, x_3	4, 0	Feasible	4
x_1, x_4	4, 0	Feasible	4
x_2, x_3	2, 0	Feasible	4
x_2, x_4	2, 0	Feasible	4
x_3, x_4	$-\frac{4}{7}, \frac{16}{7}$	Infeasible	—

Alternative optima:

x_1	x_2	x_3	x_4	z
4	0	0	0	4
0	2	0	0	4

3

maximize $z = x_1 + x_2$

subject to

$$x_1 + 2x_2 + x_3 = 6$$
$$2x_1 + x_2 - x_4 = 16$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Combination	Solution	Status
x_1, x_2	26/3, −4/3	Infeasible
x_1, x_3	8, −2	Infeasible
x_1, x_4	6, −4	Infeasible
x_2, x_3	16, −26	Infeasible
x_2, x_4	3, −13	Infeasible
x_3, x_4	6, −16	Infeasible

continued...

4

Maximize $z = 2x_1 + 3x_2^- - 3x_2^+ + 5x_3$

Subject to

$$-6x_1 + 7x_2^- - 7x_2^+ - 9x_3 - x_4 = 4$$
$$x_1 + x_2^- - x_2^+ + 4x_3 = 10$$
$$x_1, x_2^-, x_2^+, x_3, x_4 \geq 0$$

(x_2, x_2):

$$7x_2^- - 7x_2^+ = 4$$
$$x_2^- - x_2^+ = 10$$

Since $(7x_2^- - 7x_2^+)$ and $(x_2^- - x_2^+)$ are dependent, it is impossible for x_2^- and x_2^+ to be basic simultaneously. This means that at least x_2^- and x_2^+ must be nonbasic at zero level; thus making it impossible for x_2^- and x_2^+ to assume positive values simultaneously in any basic solution.

5

maximize $z = x_1 + 3x_2$

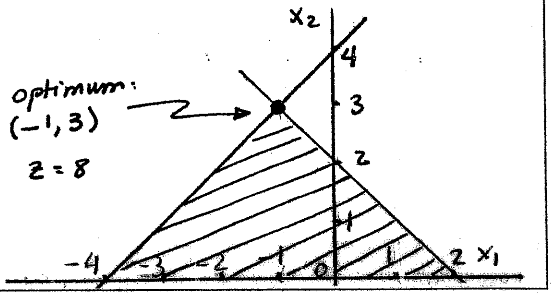
subject to

$$x_1 + x_2 + x_3 = 2$$
$$-x_1 + x_2 + x_4 = 4$$
$$x_1 \text{ unrestricted}$$
$$x_2, x_3 \geq 0$$

Combination	Solution	Status	z
x_1, x_2	−1, 3	Feasible	$\boxed{8}$
x_1, x_3	−4, 6	Feasible	−4
x_1, x_4	2, 6	Feasible	2
x_2, x_3	4, −2	Infeasible	—
x_2, x_4	2, 2	Feasible	6
x_3, x_4	2, 4	Feasible	0

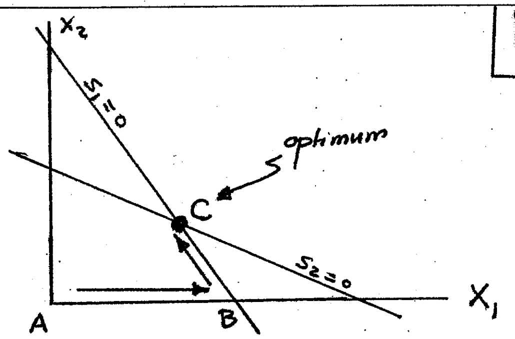
Optimum: $x_1 = -1$, $x_2 = 3$, $z = 8$

(c)



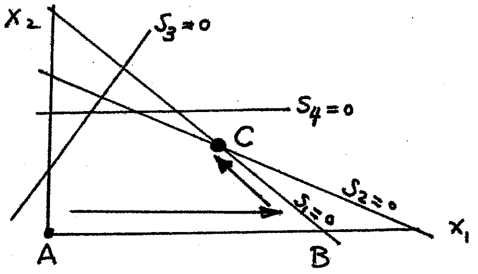
Set 3.3a

1



Extreme point	Basic	Nonbasic
A	S_1, S_2	x_1, x_2
B	X_1, S_2	x_2, S_1
C	X_1, X_2	S_1, S_2

2



Extreme point	Basic	Nonbasic
A	s_1, s_2, s_3, s_4	x_1, x_2
B	x_1, s_2, s_3, s_4	s_1, x_2
C	x_1, x_2, s_3, s_4	s_1, s_2

3

(a) (A, B) adjacent, hence can be on a simplex path. Remaining pairs cannot be on a simplex path because they are not adjacent.

(b) (i) Yes, because connects adjacent extreme points

(ii) No, because C and I are not adjacent.

(iii) No, because the path returns to a previous extreme point.

4

Extreme Point	Basic	Nonbasic
A	S_1, S_2, S_3, S_4	x_1, x_2, x_3
B	S_1, X_1, S_3, S_4	S_2, x_2, S_3
C	X_2, S_2, S_3, S_4	S_1, x_1, x_3
D	S_1, S_2, X_3, S_4	x_1, x_2, S_3
E	x_1, x_2, S_3, S_4	S_1, S_2, x_3
F	x_2, S_2, x_3, S_4	x_1, S_1, S_3
G	S_1, x_1, x_3, S_4	S_2, x_2, S_3
H	S_1, x_1, x_2, x_3	S_2, S_3, S_4
I	x_1, x_2, x_3, S_3	S_1, S_2, S_4
J	x_1, S_2, x_2, x_3	S_1, S_3, S_4

5

(a) x_3 enters at value 1
$z = 0 + 3 \times 1 = 3$

(b) x_1 enters at value 1
$z = 0 + 5 \times 1 = 5$

(c) x_2 enters at value 1
$z = 0 + 7 \times 1 = 7$

(d) Tie broken arbitrarily between x_1, x_2, and x_3. Entering value = 1
$z = 0 + 1 \times 1 = 1$

1

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Sol
z	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2
z	1	0	6	0	5	0	0	30
s_1	0	0	-8	1	-6	0	0	-12
x_1	0	1	2	0	1	0	0	6
s_3	0	0	3	0	1	1	0	7
s_4	0	0	1	0	0	0	1	2

2

(a)

```
Basic |  x1      x2      x3      x4     sx5     sx6     sx7  | Solution
--------------------------------------------------------------------------
z     | -2.00   -1.00    3.00   -5.00    0.00    0.00    0.00 |   0.00
--------------------------------------------------------------------------
 1)sx5|  1.00    2.00    2.00    4.00    1.00    0.00    0.00 |  40.00
 2)sx6|  2.00   -1.00    1.00    2.00    0.00    1.00    0.00 |   8.00
 3)sx7|  4.00   -2.00    1.00   -1.00    0.00    0.00    1.00 |  10.00
--------------------------------------------------------------------------
z     |  3.00   -3.50    5.50    0.00    0.00    2.50    0.00 |  20.00
--------------------------------------------------------------------------
 1)sx5| -3.00    4.00    0.00    0.00    1.00   -2.00    0.00 |  24.00
 2)x4 |  1.00   -0.50    0.50    1.00    0.00    0.50    0.00 |   4.00
 3)sx7|  5.00   -2.50    1.50    0.00    0.00    0.50    1.00 |  14.00
--------------------------------------------------------------------------
z     |  0.38    0.00    5.50    0.00    0.88    0.75    0.00 |  41.00
--------------------------------------------------------------------------
 1)x2 | -0.75    1.00    0.00    0.00    0.25   -0.50    0.00 |   6.00
 2)x4 |  0.62    0.00    0.50    1.00    0.12    0.25    0.00 |   7.00
 3)sx7|  3.12    0.00    1.50    0.00    0.62   -0.75    1.00 |  29.00
--------------------------------------------------------------------------
```

```
(b)
Basic |  x1      x2      x3      x4     sx5     sx6     sx7  | Solution
--------------------------------------------------------------------------
z     | -8.00   -6.00   -3.00    2.00    0.00    0.00    0.00 |   0.00
--------------------------------------------------------------------------
 1)sx5|  1.00    2.00    2.00    4.00    1.00    0.00    0.00 |  40.00
 2)sx6|  2.00   -1.00    1.00    2.00    0.00    1.00    0.00 |   8.00
 3)sx7|  4.00   -2.00    1.00   -1.00    0.00    0.00    1.00 |  10.00
--------------------------------------------------------------------------
z     |  0.00  -10.00   -1.00    0.00    0.00    0.00    2.00 |  20.00
--------------------------------------------------------------------------
 1)sx5|  0.00    2.50    1.75    4.25    1.00    0.00   -0.25 |  37.00
 2)sx6|  0.00    0.00    0.50    2.50    0.00    1.00   -0.50 |   3.00
 3)x1 |  1.00   -0.50    0.25   -0.25    0.00    0.00    0.25 |   2.50
--------------------------------------------------------------------------
z     |  0.00    0.00    6.00   17.00    4.00    0.00    1.00 | 170.00
--------------------------------------------------------------------------
 1)x2 |  0.00    1.00    0.70    1.70    0.40    0.00   -0.10 |  15.00
 2)sx6|  0.00    0.00    0.50    2.50    0.00    1.00   -0.50 |   3.00
 3)x1 |  1.00    0.00    0.60    0.60    0.20    0.00    0.20 |  10.00
--------------------------------------------------------------------------
```

```
(c)
Basic |  x1      x2      x3      x4     sx5     sx6     sx7  | Solution
--------------------------------------------------------------------------
z     | -3.00    1.00   -3.00   -4.00    0.00    0.00    0.00 |   0.00
--------------------------------------------------------------------------
 1)sx5|  1.00    2.00    2.00    4.00    1.00    0.00    0.00 |  40.00
 2)sx6|  2.00   -1.00    1.00    2.00    0.00    1.00    0.00 |   8.00
 3)sx7|  4.00   -2.00    1.00   -1.00    0.00    0.00    1.00 |  10.00
--------------------------------------------------------------------------
z     |  1.00   -1.00   -1.00    0.00    0.00    2.00    0.00 |  16.00
--------------------------------------------------------------------------
 1)sx5| -3.00    4.00    0.00    0.00    1.00   -2.00    0.00 |  24.00
 2)x4 |  1.00   -0.50    0.50    1.00    0.00    0.50    0.00 |   4.00
 3)sx7|  5.00   -2.50    1.50    0.00    0.00    0.50    1.00 |  14.00
--------------------------------------------------------------------------
z     |  0.25    0.00   -1.00    0.00    0.25    1.50    0.00 |  22.00
--------------------------------------------------------------------------
 1)x2 | -0.75    1.00    0.00    0.00    0.25   -0.50    0.00 |   6.00
 2)x4 |  0.62    0.00    0.50    1.00    0.12    0.25    0.00 |   7.00
 3)sx7|  3.12    0.00    1.50    0.00    0.62   -0.75    1.00 |  29.00
--------------------------------------------------------------------------
z     |  1.50    0.00    0.00    2.00    0.50    2.00    0.00 |  36.00
--------------------------------------------------------------------------
 1)x2 | -0.75    1.00    0.00    0.00    0.25   -0.50    0.00 |   6.00
 2)x3 |  1.25    0.00    1.00    2.00    0.25    0.50    0.00 |  14.00
 3)sx7|  1.25    0.00    0.00   -3.00    0.25   -1.50    1.00 |   8.00
--------------------------------------------------------------------------
```

continued...

```
(d)
Basic |  x1      x2      x3      x4     sx5     sx6     sx7  | Solution
--------------------------------------------------------------------------
z     | -5.00    4.00   -6.00    8.00    0.00    0.00    0.00 |   0.00
--------------------------------------------------------------------------
 1)sx5|  1.00    2.00    2.00    4.00    1.00    0.00    0.00 |  40.00
 2)sx6|  2.00   -1.00    1.00    2.00    0.00    1.00    0.00 |   8.00
 3)sx7|  4.00   -2.00    1.00   -1.00    0.00    0.00    1.00 |  10.00
--------------------------------------------------------------------------
z     |-13.00    8.00  -10.00    0.00    0.00   -4.00    0.00 | -32.00
--------------------------------------------------------------------------
 1)sx5| -3.00    4.00    0.00    0.00    1.00   -2.00    0.00 |  24.00
 2)x4 |  1.00   -0.50    0.50    1.00    0.00    0.50    0.00 |   4.00
 3)sx7|  5.00   -2.50    1.50    0.00    0.00    0.50    1.00 |  14.00
--------------------------------------------------------------------------
z     | -7.00    0.00  -10.00    0.00   -2.00    0.00    0.00 | -80.00
--------------------------------------------------------------------------
 1)x2 | -0.75    1.00    0.00    0.00    0.25   -0.50    0.00 |   6.00
 2)x4 |  0.62    0.00    0.50    1.00    0.12    0.25    0.00 |   7.00
 3)sx7|  3.12    0.00    1.50    0.00    0.62   -0.75    1.00 |  29.00
--------------------------------------------------------------------------
```

3

Basic	Ratios			
	x_1	x_2	x_3	x_4
x_5	4/1	4/2	--	4/5
x_6	8/5	--	--	8/6
x_7	3/2	3/3	--	3/3
x_8	--	--	0/1	--
Value	1.5	1	0	0.8
Leaving var	x_7	x_7	x_8	x_5

4

(a) Nonbasic x_1 will improve solution.

Basic	x_1-ratios
x_2	4/5 $\Rightarrow$ x_2 leaves, $x_1 = \frac{4}{5}$
x_3	8/6
x_4	3/3

$x_1 = \frac{4}{5} = .8$, $X_3 = 8 - 6 \times .8 = 3.6$, $X_4 = 3 - 3 \times .8 = .6$
$x_2 = 0$, $Z = .8 \times 1 = .8$

(b) x_1 remains nonbasic at zero. Current solution, $x_2 = 4$, $x_3 = 8$, $x_4 = 3$, $Z = 0$ is optimum

5

Basic solutions consist of <u>one</u> variable each. Thus,

$x_1 = 90/1 = 90$, $\quad z = 5 \times 90 = \boxed{450}$
$x_2 = 90/3 = 30$, $\quad Z = -6 \times 30 = -180$
$x_3 = 90/5 = 18$, $\quad Z = 3 \times 18 = 54$
$x_4 = 90/6 = 15$, $\quad Z = -5 \times 15 = -75$
$x_5 = 90/3 = 30$, $\quad Z = 12 \times 30 = 360$

Optimum solution:

$x_1 = 90$, $x_2 = x_3 = x_4 = x_5 = 0$, $Z = 450$

6

(a) Basic: $(x_8, x_3, x_1) = (12, 6, 0)$, $Z = 620$
Nonbasic: $(x_2, x_4, x_5, x_6, x_7) = (0,0,0,0,0)$

(b) x_2, x_5, x_6 will improve solution.

$\underline{x_2 \text{ enters}}$: $X_2 = \min(\frac{12}{3}, \frac{6}{1}, -) = 4$. Thus,
x_8 leaves, $\Delta Z = 4 \times 5 = 20$

continued...

Set 3.3b

<u>x_5 enters:</u> $x_5 = \min(-, \frac{6}{1}, \frac{0}{6}) = 0$. Thus, $\Delta Z = 1 \times 0 = 0$ (x_1 leaves)

<u>x_6 enters:</u> $x_6 = \min(-, -, -)$. Thus, no leaving variable and x_6 can be increased to ∞. $\Delta Z = +\infty$

(c) x_4 can improve solution.

<u>x_4 enters:</u> $x_4 = \min(-, \frac{6}{3}, -) = 2$. Thus, x_3 leaves. $\Delta Z = -4 \times 2 = -8$

(d) As shown in (b), x_5 cannot change Z because it enters the solution a level zero. x_7 cannot change Z either because its objective equation coefficient $= 0$. $\Delta Z = 0 \times \min(\frac{12}{5}, \frac{6}{3}, -) = 0$

7

(a) Maximize $Z = 3x_1 + 6x_2$:

x_2 is the first entering variable. Resulting path is $A \rightarrow G \rightarrow F \rightarrow E$.

(b) Maximize $Z = 4x_1 + x_2$:

Entering variable $x_1 = \begin{pmatrix}\text{min intercept with} \\ x_1\text{-axis}\end{pmatrix}$

$x_1 = \min(2, 3, 5) = 2$ at B

$\Delta Z = 4 \times 2 = 8$

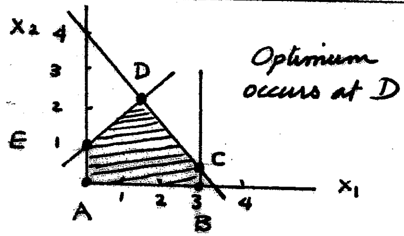
(c) Maximize $z = x_1 + 4x_2$:

Entering variable $x_2 = \begin{pmatrix}\text{min intercept} \\ \text{with } x_2\text{-axis}\end{pmatrix}$

$x_2 = \min(1, 2, 4) = 1$

$\Delta Z = 4 \times 1 = 4$

8



Optimum occurs at D

(a) x_1 will enter first and the iterations will follow the path $A \rightarrow B \rightarrow C \rightarrow D$

(b) x_2 enters first and the iterations will follow the path $A \rightarrow E \rightarrow D$

(c) The most-negative criterion requires more iterations (4 vs. 3). This criterion is only a heuristic, and although it does not guarantee the smallest number of iterations, computational experience demonstrates that, on the average, the most-negative criterion is more efficient.

continued...

(d) Iterations are identical, with the exception of the objective row, which should appear with an opposite sign

9

Optimum tableau:

Basic	x_1	x_2	s_1	s_2	s_3	s_4	
z	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	1	0	¼	-1/2	0	0	3
x_2	0	1	-1/8	¾	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

If s_1 enters, its value $= \min\{\frac{3}{1/4}, -, \frac{5/2}{3/8}, \frac{1/2}{1/8}\} = 4$

New $Z = 21 - 3/4 \times 4 = 18$

If s_2 enters, its value $= \min\{-, \frac{3/2}{3/4}, -, -\} = 2$

New $Z = 21 - 1/2 \times 2 = 20$. The second best Z is associated with s_2 entering the basic solution

10

Not easily extendable because the third best solution may not be an adjacent corner point of the current optimum point.

11

x_1 = number of purses per day

x_2 = number of bags per day

x_3 = number of backpacks per day

Maximize $Z = 24x_1 + 22x_2 + 45x_3$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\le 42 \\ 2x_1 + x_2 + 2x_3 &\le 40 \\ x_1 + .5x_2 + x_3 &\le 45 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

TORA's optimum solution:

$x_1 = 0$, $x_2 = 36$, $x_3 = 2$, $z = \$882$

Status of resources:

<u>Resource</u>	<u>slack</u>	<u>status</u>
Leather	0	scarce
Sewing	0	scarce
Finishing	25	abundant

12

From TORA Iterations module, click All Iterations , then go to the optimal iteration and click any of the associated nonbasic variables (X4, SX6, SX7, SX8). Now, click Next Iteration to produce the new iteration in which the selected variable becomes basic. The associated value of Z will deteriorate.

13

To determine the next-best solution, follow the procedure in Problem 1. First, let X4 enter the basic solution and record the associated value of Z. Next, click View/Modify Input Data and re-solve the problem to produce the same optimum tableau that was used before X4 was entered into the basic solution. Now, enter SX6 into the basic solution and record the associated value of Z. Repeat the procedure for SX7 and SX8. You will get the following results:

Entering variable	Z
X4	2.63
SX6	1.00
SX7	6.40
SX8	1.90

The next-best solution is associated with entering SX7 into the basic solution. Associated values of the variables are

$x_1 = 1.6$

$x_2 = 0$

$x_3 = 1.6$

$x_4 = 0$

$Z = 6.40$

Set 3.4a

1

Iteration	Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
0 (starting)	z	$-4+7M$	$-1+4M$	$-M$	0	0	0	$9M$
x_1 enters	R_1	3	1	0	1	0	0	3
R_1 leaves	R_2	4	3	-1	0	1	0	6
	x_4	1	2	0	0	0	1	4
1	z	0	$\frac{1+5M}{3}$	$-M$	$\frac{4-7M}{3}$	0	0	$4+2M$
x_2 enters	x_1	1	1/3	0	1/3	0	0	1
R_2 leaves	R_2	0	5/3	-1	$-4/3$	1	0	2
	x_4	0	5/3	0	$-1/3$	0	1	3
2	z	0	0	1/5	$8/5-M$	$-1/5-M$	0	18/5
x_3 enters	x_1	1	0	1/5	3/5	$-1/5$	0	3/5
x_4 leaves	x_2	0	1	$-3/5$	$-4/5$	3/5	0	6/5
	x_4	0	0	1	1	-1	1	1
3	z	0	0	0	$7/5-M$	$-M$	$-1/5$	17/5
(optimum)	x_1	1	0	0	2/5	0	$-1/5$	2/5
	x_2	0	1	0	$-1/5$	0	3/5	9/5
	x_3	0	0	1	1	-1	1	1

2

$M=1$:

Optimum solution: $X_1=0$, $X_2=2$, $XR4=1$
$Z=3$

Solution is infeasible because $XR4$ is positive. The reason $M=1$ produces an infeasible solution is that it does not play the role of a penalty relative to the objective coefficients of the real variables, X_1 and X_2. Using $M=1$ makes $XR4$ more attractive than X_1 from the standpoint of minimizat

$M=10$:

Optimum solution: $X_1=.4$, $X_2=1.8$, $Z=3.4$

The solution is feasible because it does not include artificials at positive level. $M=10$ is relatively much larger than the objective coefficients of X_1 and X_2, and hence properly plays the role of a penalty.

$M=1000$:

It produces the optimum solution as with $M=10$. The conclusion is that it suffices to select M reasonably larger than the objective coefficients of the real variables. Actually, $M=1000$ is an "overkill" in this case, and selecting such huge values could result in adverse roundoff error.

3

(a) Minimize $Z = 4X_1 + X_2 + M(R_1+R_2+R_3)$.
subject to

$$\begin{aligned} 3X_1 + X_2 \qquad\qquad + R_1 \qquad\qquad &= 3 \\ 4X_1 + 3X_2 - S_2 \qquad\qquad + R_2 \qquad &= 6 \\ X_1 + 2X_2 \qquad - S_3 \qquad\qquad + R_3 &= 4 \\ X_1, X_2, S_2, S_3, R_1, R_2, R_3 &\geq 0 \end{aligned}$$

Basic	X_1	X_2	S_2	S_3	R_1	R_2	R_3	
Z	-4	-1			$-M$	$-M$	$-M$	0
R_1	3	1			①			3
R_2	4	3	-1			①		6
R_3	1	2		-1			①	4
Z	$-4+8M$	$-1+6M$	$-M$	$-M$	0	0	0	$10M$
R_1	3	1			1			3
R_2	4	3	-1			1		6
R_3	1	2		-1			1	4

(b) Minimize $Z = 4X_1 + X_2 + M\,R_1$
subject to

$$\begin{aligned} 3X_1 + X_2 \qquad + R_1 \qquad\qquad &= 3 \\ 4X_1 + 3X_2 \qquad\qquad + S_2 \qquad &= 6 \\ X_1 + 2X_2 \qquad\qquad\qquad + S_3 &= 4 \end{aligned}$$

Basic	X_1	X_2	R_1	S_2	S_3	
Z	-4	-1	$-M$			0
R_1	3	1	①			3
S_2	4	3		1		6
R_3	1	2			1	4
Z	$-4+3M$	$-1+M$	0	0	0	$3M$
R_1	3	1	1			3
S_2	4	3		1		6
R_3	1	2			1	4

(c) Minimize $Z = 4X_1 + X_2 + M(R_1+R_2)$
subject to

$$\begin{aligned} 3X_1 + X_2 + R_1 \qquad\qquad &= 3 \\ 4X_1 + 3X_2 \qquad + R_2 \qquad &= 6 \\ X_1 + 2X_3 \qquad\qquad + S_3 &= 4 \end{aligned}$$

Basic	X_1	X_2	R_1	R_2	S_3	
Z	-4	-1	$-M$	$-M$	0	0
R_1	3	1	①			3
R_2	4	3		①		6
S_3	1	2			1	4
Z	$-4+7M$	$-1+4M$	0	0	0	$9M$
R_1	3	1	1			3
R_2	4	3		1		6
S_3	1	2			1	4

continued…

(d) Maximize $Z = 4x_1 + x_2 - M(R_1 + R_2)$

subject to

$3x_1 + x_2 + R_1 = 3$

$4x_1 + 3x_2 - S_2 + R_2 = 6$

$x_1 + 2x_2 + S_3 = 4$

Basic	x_1	x_2	S_2	R_1	R_2	S_3	
z	-4	-1	0	Ⓜ	Ⓜ	0	0
R_1	3	1		①			3
R_2	4	3	-1		①		6
S_3	1	2				1	4
z	-4-7M	-1-4M	M	0	0	0	-9M
R_1	3	1		1			3
R_2	4	3	-1		1		6
S_3	1	2				1	4

4

(a) Maximize $z = 5x_1 + 6x_2 - M\textcircled{R_1}$

subject to

$-2x_1 + 3x_2 + \textcircled{R_1} = 3$ (1)

$x_1 + 2x_2 + S_3 = 5$ (3)

$6x_1 + 7x_2 + S_4 = 3$ (4)

$z - (5-2M)x_1 - (6+3M)x_2 = -3M$

(b) Maximize $Z = 2x_1 - 7x_2 - M(R_1 + R_2 + R_5)$

subject to

$-2x_1 + 3x_2 + R_1 = 3$ (1)

$4x_1 + 5x_2 - S_2 + R_2 = 10$ (2)

$6x_1 + 7x_2 + S_4 = 3$ (4)

$4x_1 + 8x_2 - S_5 + R_5 = 5$ (5)

$Z - (2+6M)x_1 - (-7+16M)x_2 + MS_2 + MS_5 = -18M$

(c) Minimize $z = 3x_1 + 6x_2 + MR_5$

subject to

$x_1 + 2x_2 + S_1 = 5$ (3)

$6x_1 + 7x_2 + S_2 = 3$ (4)

$4x_1 + 8x_2 - S_5 + R_5 = 5$ (5)

$Z - (3-4M)x_1 - (6-8M)x_2 - MS_5 = 5M$

(d) Minimize $Z = 4x_1 + 6x_2 + M(R_1 + R_2 + R_5)$

subject to

$-2x_1 + 3x_2 + R_1 = 3$ (1)

$4x_1 + 5x_2 - S_2 + R_2 = 10$ (2)

$4x_1 + 8x_2 - S_5 + R_5 = 5$ (5)

$Z - (4-6M) - (6-16M)x_2 - MS_2 - MS_5 = 18M$

(e) Minimize $z = 3x_1 + 2x_2 + M(R_1 + R_5)$

subject to

$-2x_1 + 3x_2 + R_1 = 3$ (1)

$4x_1 + 8x_2 - S_5 + R_5 = 5$ (5)

$Z - (3-2M)x_1 - (2-11M)x_2 - MS_5 = 8M$

continued...

5

(a)

	Basic	x_1	x_2	x_3	S_2	R_1	R_2	
0	z	-2 -3M	-3 +4M	5 -2M	M	0	0	-17M
	R_1	1	1	1	0	1	0	7
	R_2	$\boxed{2}$	-5	1	-1	0	1	10
I	z	0	$-8-\frac{7M}{2}$	$6-\frac{M}{2}$	$-1-\frac{M}{2}$	0	$1+\frac{3M}{2}$	10 -2M
	R_1	0	$\boxed{7/2}$	1/2	1/2	1	-1/2	2
	x_1	1	-5/2	1/2	-1/2	0	1/2	5
II	z	0	0	50/7	1/7	16/7 +M	-1/7 +M	$\frac{102}{7}$
	x_2	0	1	1/7	1/7	2/7	-1/7	4/7
	x_1	1	0	6/7	-1/7	5/7	1/7	45/7

(b)

	Basic	x_1	x_2	x_3	S_2	R_1	R_2	Sol^n
0	z	-2 +3M	-3 -4M	5 +2M	-M	0	0	17M
	R_1	1	1	1	0	1	0	7
	R_2	2	-5	1	-1	0	1	10
I	z	0	-8 +7M/2	6 +M/2	-1 +M/2	0	1 -3M/2	10 + 2M
	R_1	0	7/2	1/2	1/2	1	-1/2	2
	x_1	1	-5/2	1/2	-1/2	0	1/2	5
II	z	0	0	50/7	1/7	16/7 -M	-1/7 -M	$\frac{102}{7}$
	x_2	0	1	1/7	1/7	2/7	-1/7	4/7
	x_1	1	0	6/7	-1/7	5/7	1/7	45/7
III	z	0	-50	0	-7	-12 -M	7 -M	-14
	x_3	0	7	1	1	2	-1	4
	x_1	1	-6	0	-1	-1	1	3

continued...

Set 3.4a

(c)

	Basic	x_1	x_2	x_3	S_2	R_1	R_2	Sol'n
	z	-1	-2	-1	0	m	m	–
0	R_1	1	1	1	0	1	0	7
	R_2	2	-5	1	-1	0	1	10
I	z	-1 -3m	-2 +4m	-1 -2m	m	0	0	-17M
	R_1	1	1	1	0	1	0	7
	R_2	2	-5	1	-1	0	1	10
II	z	0	-9/2 -7M/2	-1/2 - M/2	-1/2 -M/2	0	1/2 +3M/2	5 -2M
	R_1	0	7/2	1/2	1/2	1	-1/2	2
	x_1	1	-5/2	1/2	-1/2	0	1/2	5
III	z	0	0	1/7	1/7	9/7 +m	-1/7 +M	53/7
	x_2	0	1	1/7	1/7	2/7	-1/7	4/7
	x_1	1	0	6/7	-1/7	5/7	1/7	45/7

(d)

	Basic	x_1	x_2	x_3	S_2	R_1	R_2	Sol'n
	z	-4	8	-3	0	-m	-m	0
0	R_1	1	1	1	0	1	0	7
	R_2	2	-5	1	-1	0	1	10
I	z	-4 +3m	8 -4m	-3 +2m	-m	0	6	17M
	R_1	1	1	1	0	1	0	7
	R_2	2	-5	1	-1	0	1	10
II	z	0	-2 +7M/2	-1 +M/2	-2 +M/2	0	2 -3M/2	20 +2M
	R_1	0	7/2	1/2	1/2	1	-1/2	2
	x_1	1	-5/2	1/2	-1/2	0	1/2	5
III	z	0	0	-5/7	-12/7	4/7 -m	11/7 -M	148/7
	x_2	0	1	1/7	1/7	2/7	-1/7	4/7
	x_1	1	0	6/7	-1/7	-5/7	1/7	45/7

6

In the first iteration, we must substitute out the starting solution variables, x_3 and x_4, in the Z-equation, exactly as we do with the artificial variables

	Basic	x_1	x_2	x_3	x_4	Solution
	Z	-2	-4	-4	3	—
0	x_3	1	1	①	0	4
	x_4	1	4	0	①	8
I	Z	-1	-12	0	0	-8
	x_3	1	1	1	0	4
	x_4	1	$\boxed{4}$	0	1	8
II	Z	2	0	0	3	16
	x_3	3/4	0	1	-1/4	2
	x_2	1/4	1	0	1/4	2

7

After adding surplus S_1 and S_2, Substitute out x_3 in the Z-equation

	Basic	x_1	x_2	S_1	S_2	x_3	x_4	Solution
	Z	-3	-2	0	0	-3	0	—
0	x_3	1	4	-1	0	①	0	7
	x_4	2	1	0	-1	0	1	10
I	Z	0	10	-3	0	0	0	21
	x_3	1	4	-1	0	1	0	7
	x_4	2	1	0	-1	0	1	10
II	Z	-5/2	0	-1/2	0	-5/2	0	7/2
	x_2	1/4	1	-1/4	0	1/4	0	7/4
	x_4	7/4	0	1/4	-1	-1/4	1	33/4

8

Both x_3 and R (the starting solution variables) must be substituted out in the Z-equation

	Basic	x_1	x_2	x_3	R	Solution
	z	-1	-5	-3	M	–
0	x_3	1	2	①	0	3
	R	2	-1	0	①	4
I	z	2-2M	1+M	0	0	9-4M
	x_3	1	2	1	0	3
	R	$\boxed{2}$	-1	0	1	4
II	z	0	2	0	-1+M	5
	x_3	0	5/2	1	-1/2	1
	x_1	1	-1/2	0	1/2	2

9

Maximize $Z = 2x_1 + 5x_2 - MR_1$

subject to

$$\begin{aligned} 3x_1 + 2x_2 - S_1 + R_1 \quad &= 6 \\ 2x_1 + x_2 \quad + S_2 &= 2 \\ x_1, x_2, S_1, R_1, S_2 &\geq 0 \end{aligned}$$

Basic	X_1	X_2	S_1	R_1	S_2	
Z	-2	-5	0	M	0	–
R_1	3	2	-1	1	0	6
S_2	2	1	0	0	1	2
Z	-2-3M	-5-2M	M	0	0	-6M
R_1	3	2	-1	1	0	6
S_2	2	1	0	0	1	2
Z	0	-4-M/2	M	0	1+3M/2	-2 +3M
R_1	0	1/2	-1	1	-3/2	3
x_1	1	1/2	0	0	1/2	1
Z	8+M	0	M	0	5+2M	10 -2M
R_1	-1	0	-1	1	-2	2
x_2	2	1	0	0	1	2

The Z-row shows that the solution is optimal (all nonbasic coefficients in the Z-row are ≥ 0). However, the solution is infeasible because the artificial variable R_1 assumes a positive value. Having a positive value for the artificial variable R_1 is the same as regarding the constraint $3x_1 + 2x_2 \geq 6$ as $3x_1 + 2x_2 \leq 6$, which violates the constraints of the original model.

Set 3.4b

1

In Phase I, we always minimize the sum of the artificial variables because the sum represents a measure of infeasibility in the problem

2

(a) Minimize $r = R_1$
(b) Minimize $r = R_1 + R_2 + R_5$
(c) Minimize $r = R_5$
(d) Minimize $r = R_1 + R_2 + R_5$
(e) Minimize $r = R_1 + R_5$

3

(a) Phase I:

Basic	x_1	x_2	x_3	S_2	R_1	R_2	
r	0	0	0	0	-1	-1	0
R_1	1	1	1	0	1	0	7
R_2	2	-5	1	-1	0	1	10
r	3	-4	2	-1	0	0	17
R_1	1	1	1	0	1	0	7
R_2	2	-5	1	-1	0	1	10
r	0	7/2	1/2	1/2	0	-3/2	2
R_1	0	7/2	1/2	1/2	1	-1/2	2
x_1	1	-5/2	1/2	-1/2	0	1/2	5
r	0	0	0	0	-1	-1	0
x_2	0	1	1/7	1/7	2/7	-1/7	4/7
x_1	1	0	6/7	-1/7	5/7	1/7	45/7

Phase II

Basic	x_1	x_2	x_3	S	Sol^{n}
z	-2	-3	5	0	0
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
z	0	0	50/7	1/7	102/7
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7

(b) Phase I is the same as in (a)

Phase II

Basic	x_1	x_2	x_3	S_2	Sol^{n}
z	-2	-3	5	0	0
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
z	0	0	50/7	1/7	102/7
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
z	0	-50	0	-7	-14
x_3	0	7	1	1	4
x_1	1	-6	0	-1	3

continued...

(c) Phase I is the same as in (a)

Phase II:

Basic	x_1	x_2	x_3	S_2	Sol^{n}
z	-1	-2	-1	0	0
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
z	0	0	1/7	1/7	53/7
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7

(d) Phase I is the same as in (a)

Phase II:

Basic	x_1	x_2	x_3	x_4	Sol^{n}
z	-4	8	-3	0	0
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
z	0	0	-5/7	-12/7	211/7
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7

4

Minimize $r = R_1$
subject to

$$3x_1 + 2x_2 - S_1 + R_1 = 6$$
$$2x_1 + x_2 + S_2 = 2$$
$$x_1, x_2, S_1, R_1, S_2 \geq 0$$

Solution of Phase I by TORA yields $r = 2$, which indicates that the problem has no feasible space

5

Minimize $z = R_2$
subject to

$$2x_1 + x_2 + x_3 + S_1 = 2$$
$$3x_1 + 4x_2 + 2x_3 - S_2 + R_2 = 8$$
$$x_1, x_2, x_3, S_1, S_2, R_2 \geq 0$$

Phase I Optimal solution:

Basic	x_1	x_2	x_3	S_2	S_1	R_2	Sol^{n}
r	-5	0	-2	-1	-4	0	0
x_2	2	1	1	0	1	0	2
R_2	-5	0	-2	-1	-4	1	0

$R_2 = 0$ is basic in the Phase I Solution

continued...

(b)

Phase I (continued): R2 leaves, x1 enters (also x3, s2, and s1 are candidates for the entering variable).

	x1	x2	x3	s2	s1	R2	Sol
r	-5	0	-2	-1	-4	0	0
x2	2	1	1	0	1	0	2
R2	-5	0	-2	-1	-4	1	0
r	0	0	0	0	0	-1	
x2	0	1	1/5	-2/5	-3/5	2/5	2
x1	1	0	2/5	1/5	4/5	-1/5	0

Drop R2-column.

Phase II:

	x1	x2	x3	s2	s1	Sol.
z	-2	-2	-4	0	0	0
x2	0	1	1/5	-2/5	-3/5	2
x1	1	0	2/5	1/5	4/5	0
z	0	0	-14/5	-2/5	2/5	4
x2	0	1	1/5	-2/5	-3/5	2
x1	1	0	2/5	1/5	4/5	0
z	7	0	0	1	6	4
x2	-1/2	1	0	-1/2	-1	2
x3	5/2	0	1	1/2	2	0

Optimum solution:

$x_1 = 0, x_2 = 2, x_3 = 0, z = 4$

Phase I:

	x1	x2	x3	R1	R2	R3	Sol
r	-10	0	-4	-8	0	0	0
x2	2	1	1	1	0	0	2
R2	-5	0	-2	-3	1	0	0
R3	-5	0	-2	-4	0	1	0
r	0	0	1	-2	-2	0	0
x2	0	1	1/5	-1/5	2/5	0	2
x1	1	0	2/5	3/5	-1/5	0	0
R3	0	0	0	-1	-1	1	0

Remove R1- and R2 columns, which gives

	x1	x2	x3	R3	Sol
r	0	0	1	0	0
x2	0	1	1/5	0	2
x1	1	0	2/5	0	0
R3	0	0	0	1	0

The R3-row is R3 = 0, which is redundant. Hence the R3-row and R3-column can be dropped from the tableau with no consequences.

Phase II:

	x1	x2	x3	Sol
z	-3	-2	-3	0
x2	0	1	1/5	2
x1	1	0	2/5	0
z	0	0	-7/5	4
x2	0	1	1/5	2
x1	1	0	2/5	0
z	7/2	0	0	4
x2	-1/2	1	0	2
x1	5/2	0	1	0

Optimum solution:

$x_1 = 0, x_2 = 2, x_3 = 0, z = 4$

Set 3.4b

7

If x_1, x_3, x_4, or x_5 assume a positive value, the value of the objective function at the end of Phase I must necessarily become positive. This follows because these variables have nonzero z-row coefficients in the optimal Phase I tableau. A positive objective value at the end of Phase I means that Phase I solution is infeasible. Since Phase II uses the same constraints as in Phase I, it follows that Phase II must have $x_1 = x_3 = x_4 = x_5 = 0$ as well.

Phase II:

Basic	x_2	R	Sol'n
z	$\textcircled{-2}$	0	0
x_2	$\textcircled{1}$	0	2
R	0	1	0
z	0	0	4
x_2	1	0	2
R	0	1	0

Optimum solution:

$x_1 = 0 \quad x_2 = 2 \quad x_3 = x_4 = x_5 = 0$

$z = 4$

8

$$\begin{aligned} -5x_1 + 6x_2 - 2x_3 + x_4 \quad\quad &= -5 \\ x_1 - 3x_2 - 5x_3 \quad + x_5 \quad &= -8 \\ 2x_1 + 5x_2 - 4x_3 \quad\quad + x_6 &= 9 \end{aligned}$$

x_1	x_2	x_3	x_4	x_5	x_6	R	
0	0	0	0	0	0	-1	
-5	6	-2	1	0	0	-1	-5
1	-3	-5	0	1	0	$\boxed{-1}$	-8
2	5	-4	0	0	1	0	9
-1	3	5	0	-1	0	0	8
-6	9	3	1	-1	0	0	3
-1	3	5	0	-1	0	1	8
2	5	-4	0	0	1	0	9

Phase I problem:

minimize $r = R$

subject to

$$\begin{aligned} -6x_1 + 9x_2 + 3x_3 + x_4 - x_5 \quad\quad &= 3 \\ -x_1 + 3x_2 + 5x_3 \quad - x_5 + R &= 8 \\ 2x_1 + 5x_2 - 4x_3 \quad\quad + x_6 &= 9 \end{aligned}$$

all variables ≥ 0

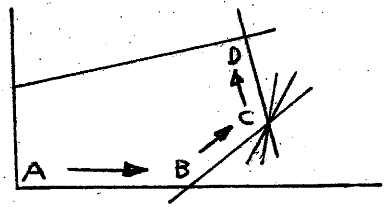
The logic of the procedure is as follows:

In the R-column, enter -1 for any constraint with negative RHS and 0 for all other constraints.

Next, use the R-column as a pivot column and select the pivot element as the one corresponding to the most negative RHS. This procedure will always require one artificial variable regardless of the number of constraints.

1

(a)

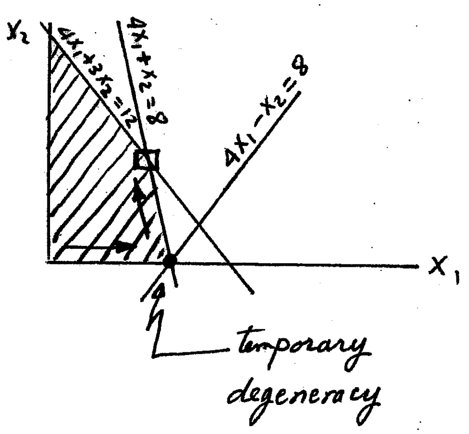


(b) $A: 1$, $B: 1$, $C: \binom{3}{2} = 3$, $D: 1$

2

(a) From TORA, iterations 2 and 3 are degenerate. Degeneracy is removed in iteration 4.

(b)



3

(a) Four iterations

(b) Three iterations: In iteration 2, degeneracy is removed because basic sX5 = 0 corresponds to a negative constraint coefficient in the entering variable column (X2).

(c) In part (a), solution encounters 2 degenerate basic solution at the same corner point. In part (b), only one basic solution was encountered.

Set 3.5b

1

	Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
0	z	-1	-2	-3	0	0	0	0
	s_1	1	2	$\boxed{3}$	1	0	0	10
	s_2	1	1	0	0	1	0	5
	s_3	1	0	0	0	0	1	1
I	z	0	$\boxed{0}$	0	1	0	0	10
	x_3	1/3	2/3	1	1/3	0	0	10/3
	s_2	1	$\boxed{1}$	0	0	1	0	5
	s_3	1	0	0	0	0	1	1
II	z	$\boxed{0}$	0	0	1	0	0	10
	x_3	-1/3	0	1	1/3	-2/3	0	0
	x_2	1	1	0	0	1	0	5
	s_3	$\boxed{1}$	0	0	0	0	1	1
III	z	0	0	0	1	$\boxed{0}$	0	10
	x_3	0	0	1	1/3	-2/3	1/3	1/3
	x_2	0	1	0	0	$\boxed{1}$	-1	4
	x_1	1	0	0	0	0	1	1
IV	z	0	0	0	1	0	0	10
	x_3	0	2/3	1	1/3	0	-1/3	3
	s_2	0	1	0	0	1	-1	4
	x_1	1	0	0	0	0	1	1

Three alternative basic optima:

$$(x_1, x_2, x_3) = \begin{cases} (0, 0, 10/3) \\ (0, 5, 0) \\ (1, 4, 1/3) \end{cases}$$

The associated nonbasic alternative optima are

$$\tilde{x}_1 = \lambda_3$$

$$\tilde{x}_2 = 5\lambda_2 + 4\lambda_3$$

$$\tilde{x}_3 = 10/3\,\lambda_1 + 1/3\,\lambda_3$$

where

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$0 \le \lambda_i \le 1, \quad i = 1, 2, 3$$

2

Basic	x_1	x_2	x_3	s_1	s_2	
z	-2	1	3	0	0	0
s_1	1	-1	$\boxed{5}$	1	0	10
s_2	2	-1	3	0	1	40
z	-7/5	2/5	0	3/5	0	6
x_3	$\boxed{1/5}$	-1/5	1	1/5	0	2
s_2	7/5	-2/5	0	-3/5	1	34
z	0	-1	7	2	0	20
x_1	1	-1	5	1	0	10
s_2	0	$\boxed{1}$	-7	-2	1	20
z	0	0	$\boxed{0}$	$\boxed{0}$	1	40
x_1	1	0	-2	-1	0	30
x_2	0	1	-7	-2	1	20

x_3 and s_1 can yield alternative optima. However, because all their constraint coefficients are negative (in general, ≤ 0), none can yield an alternative (corner point) basic solution.

3

	Basic	x_1	x_2	x_3	s_1	s_2	s_3	soln
0	z	-3	-1	0	0	0	0	0
	s_1	1	2	0	1	0	0	5
	s_2	$\boxed{1}$	1	-1	0	1	0	2
	s_3	7	3	-5	0	0	1	20
I	z	0	2	-3	0	3	0	6
	s_1	0	1	$\boxed{1}$	1	-1	0	3
	x_1	1	1	-1	0	1	0	2
	s_3	0	-4	2	0	-7	1	6
II	z	0	5	0	3	$\boxed{0}$	0	15
	x_3	0	1	1	1	-1	0	3
	x_1	1	2	0	1	0	0	5
	s_3	0	-6	0	-2	-5	1	0

The optimum solution is degenerate because S_3 is basic and equal to zero. Also, it has alternative nonbasic solutions because S_2 has a zero coefficient in the z-row and all its constraint coefficients are ≤ 0.

1

Basic	x_1	x_2	s_1	s_2	
z	-2	-1	0	0	0
s_1	1	-1	1	0	10
s_2	2	0	0	1	40
z	0	-3	2	0	20
x_1	1	-1	1	0	10
s_2	0	2	-2	1	20
z	0	0	-1	3/2	50
x_1	1	0	0	1/2	20
x_2	0	1	-1	1/2	10

unbounded ⟶↑

2

(a)

x_2
-10
-3
0
-1

$\}\Rightarrow$ Solution space unbouned in the direction of x_2

(b) Objective value is unbounded because each unit increase in x_2 increases z by 10

3

If, at any iteration, all the constraint coefficients of a variable are ≤ 0, then the solution space is unbounded in the direction of that variable.

A more "foolproof" way of accomplishing this task is to solve a sequence of LPs in which the objective function is

Maximize $z = x_j$, $j = 1, 2, \cdots, n$

Subject to the constraints of the problem. For the unbounded variables, $z = \infty$.

Set 3.5d

1

x_1 = number of units of T1
x_2 = number of units of T2
x_3 = number of units of T3

Constraints:

$$3x_1 + 5x_2 + 6x_3 \le 1000$$
$$5x_1 + 3x_2 + 4x_3 \le 1200$$
$$x_1 + x_2 + x_3 \ge 500$$
$$x_1, x_2, x_3 \ge 0$$

We can use Phase I to see whether the problem has a feasible solution; that is,

minimize $r = R_3$
subject to

$$\begin{aligned} 3x_1 + 5x_2 + 6x_3 + S_1 \qquad\qquad &= 1000 \\ 5x_1 + 3x_2 + 4x_3 \qquad + S_2 \qquad &= 1200 \\ x_1 + x_2 + x_3 \qquad\qquad - S_3 + R_3 &= 500 \end{aligned}$$
$$x_1, x_2, x_3, S_1, S_2, S_3, R_3 \ge 0$$

Optimum solution from TORA:
$R_3 = r = 225$ units

This is interpreted as a deficiency of 225 units. The most that can be produced is $500 - 225 = 275$ units

2

Basic	x_1	x_2	x_3	S_1	S_2	R_1	Sol$^{\underline{n}}$
z	-3 $-3M$	-2 $-4M$	-3 $-2M$	M	0	0	$-8M$
S_1	2	$\boxed{1}$	1	0	1	0	2
R_1	3	4	2	-1	0	1	8
z	-1 $+5M$	0	-1 $+2M$	M	2 $+4M$	0	4
x_2	2	1	1	0	1	0	2
R_1	-5	0	-2	-1	-4	1	0

Because $R_1 = 0$ in the optimal tableau, the problem has a feasible solution. The optimum solution is

$x_1 = 0,\ x_2 = 2,\ z = 4$

Note that in the first iteration, R_1 could have been used as the leaving variable, in which case it would not be basic in the optimum iteration.

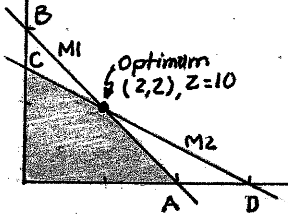
1

x_1 = Nbr. units of product A
x_2 = Nbr. units of product B

Maximize $Z = 2x_1 + 3x_2$

s.t.

$2x_1 + 2x_2 \le 8$ (M1)
$3x_1 + 6x_2 \le 18$ (M2)
$x_1, x_2 \ge 0$



	M1	M2	Z
A=(4,0)		12	8
B=(0,4)		24	12
C=(0,3)	6		9
D=(6,0)	12		12

(a) M1 at C $= 2(0)+2(3) = 6$
M1 at D $= 2(6)+2(0) = 12$
Z at C $= 2(0)+3(3) = 9$
Z at D $= 2(6)+3(0) = 12$
Dual price $= \frac{12-9}{12-6} = \$.50$/unit
Allowable range $= (6 \le M1 \le 12)$
M2 at A $= 3(4)+6(0) = 12$
M2 at B $= 3(0)+6(4) = 24$
Z at A $= 2(4)+3(0) = 8$
Z at B $= 2(0)+3(4) = 12$
Dual price $= \frac{12-8}{24-12} = \$.33$/unit
Range : $12 \le M2 \le 24$

(b) Dual price = \$.50/unit valid in the range $6 \le M1 \le 12$
Increase in revenue $= .5 \times 4 = \$2.00$
Increase in cost $= .3 \times 4 = \$1.20$
Cost < Revenue – purchase recommended

(c) Dual price = \$.33/unit valid in the range $12 \le M2 \le 24$
Purchase price/unit $< \$.33$

(d) Dual price = \$.33/unit valid in the range $12 \le M2 \le 24$. M2 is increased from 18 to 23 units
Increase in revenue
$= 5 \times .33 = \$1.65$

New optimum revenue $= 10 + 1.65 = \$11.65$

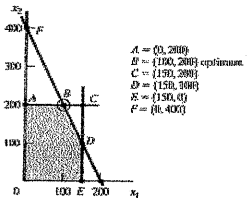
2

x_1 = daily number of type 1 hat
x_2 = daily number of type 2 hat

Maximize $Z = 8x_1 + 5x_2$

$$2x_1 + x_2 \le 400$$
$$x_1 \le 150$$
$$x_2 \le 200$$
$$x_1, x_2 \ge 0$$

(a) Optimum occurs at B:
$x_1 = 100$ type 1 hats
$x_2 = 200$ type 2 hats
$Z = \$1800$



(b) A = (0, 200), C = (150, 200)

	capacity	Z
A	$2\times0+1\times200 = 200$	$8\times0+5\times200 = 1000$
C	$2\times150+1\times200 = 500$	$8\times150+5\times200 = 2200$

Worth/capacity unit $= \frac{2200-1000}{500-200}$
= \$4 per type 2 hat

Range : (200, 500)

(c) Dual price = 0 in the range (100, ∞)
Thus, change from $x_1 \le 150$ to $x_1 \le 120$ has no effect on optimum Z.

(d) Let d = demand limit for type 2 hat

	d	Z
D(150,100)	100	$8(150)+5(100) = \$1700$
F(0,400)	400	$8(0)+5(400) = \$2000$

Dual price $= \frac{2000-1700}{400-100} = \$1.00$

Range (100, 400)

Maximum increase in demand limit for type 2 hat = 400 − 200 = 200 hats

Set 3.6b

1

(a) $\frac{3}{6} \le \frac{c_A}{c_B} \le \frac{2}{2}$, or

$.5 \le \frac{c_A}{c_B} \le 1$ or $1 \le \frac{c_B}{c_A} \le 2$

(b) Maximize $z = 2x_A + 3x_B$

$c_B = 3$: $3 \times .5 \le c_A \le 3 \times 1$

$1.5 \le c_A \le 3$

$c_A = 2$: $2 \times .5 \le c_B \le 2 \times 2$

$1 \le c_B \le 4$

(c) $\frac{c_A}{c_B} = \frac{5}{4} = 1.25$, which falls outside the range $.5 \le \frac{c_A}{c_B} \le 1$. Optimum solution changes and must be computed anew. New solution: $x_A = 4$, $x_B = 0$, $z = \$20$.

(d) Case 1: $z = 5x_A + 3x_B$

$c_A = 5$ falls outside the range (1.5, 3), hence the optimum changes. New Optimum is $x_A = 4$, $x_B = 0$, $z = \$20$.

Case 2: $z = 2x_A + 4x_B$

$c_B = 4$ falls in the range (1, 4), hence optimum is unchanged at $x_A = x_B = 2$, $z = 2(2) + 4(2) = \$12$

2

(a) $\frac{1}{2} \le \frac{c_1}{c_2} \le \frac{6}{4}$, or

$.5 \le \frac{c_1}{c_2} \le 1.5$ or $\frac{2}{3} \le \frac{c_2}{c_1} \le 2$

(b) Given $c_1 = 5$, then

$5(\frac{2}{3}) \le c_2 \le 5(2)$, or $\frac{10}{3} \le c_2 \le 10$

(c) $\frac{c_1}{c_2} = \frac{5}{3} = 1.67$, which falls outside the range $.5 \le \frac{c_1}{c_2} \le 1.5$. Hence the solution changes

3

(a) $\frac{0}{1} \le \frac{c_1}{c_2} \le \frac{2}{1}$, or

$0 \le \frac{c_1}{c_2} \le 2$

(b) $\frac{c_1}{c_2} = 1$, which falls in the range $0 \le \frac{c_1}{c_2} \le 2$. Hence, the solution is unchanged.

1

Feasibility conditions:

$x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2$

$x_3 = 230 + \frac{1}{2}D_2$

$x_6 = 20 - 2D_1 + D_2 + D_3$

(a) $D_1 = 438 - 430 = 8$ min

$D_2 = 500 - 460 = 40$

$D_3 = 410 - 420 = -10$

$x_2 = 100 + \frac{1}{2}(8) - \frac{1}{4}(40) = 94 > 0$

$x_3 = 230 + \frac{1}{2}(40) = 250 > 0$

$x_6 = 20 - 2(8) + 40 - 10 = 34 > 0$

Dual prices:

Resource 1 = \$1/min, $-200 \le D_1 \le 10$

2 = \$2/min, $-20 \le D_2 \le 400$

3 = \$0/min, $-20 \le D_3 < \infty$

New profit = $1350 + D_1 + 2D_2 + 0D_3$

$= 1350 + 8 + 2 \times 40 =$ \$1438

(b) $D_1 = 460 - 430 = 30$ min

$D_2 = 440 - 460 = -20$

$D_3 = 380 - 420 = -40$

$x_2 = 100 + \frac{1}{2}(30) - \frac{1}{4}(-20) = 120 > 0$

$x_3 = 230 + \frac{1}{2}(-20) = 220 > 0$

$x_6 = 20 - 2(30) - 20 - 40 = -100 < 0$

2

(a) Overtime cost $\frac{50}{60} =$ \$.83/min

Revenue (dual price) for operation 1 is \$1/min.

Cost < Revenue ⇒ advantageous

(b) Dual price for operation 2 = \$2/min valid in the range $-20 \le D_2 \le 400$

$D_2 = 120$ minutes

Revenue increase = $120 \times 2 =$ \$240

Cost increase = 2 (\$55) = \$110

Revenue > cost ⇒ accept.

(c) No, resource 3 is already abundant. This is the reason its dual price = 0

(d) Dual price for operation 1 is \$1/min, valid in the range $-200 \le D_1 \le 10$

continued...

$D_1 = 440 - 430 = 10$ min

Cost = $\frac{10}{60} \times 40 =$ \$6.67

New revenue = $1350 + 1 \times 10 =$ \$1360

Net revenue = 1360 − \$6.67 = \$1353.33

(e) Dual price = \$2/min, $-20 \le D_2 \le 400$

$D_2 = -$ min

Decrease in cost = $\frac{15}{60} \times$ \$30 = \$7.50

Lost revenue = 15 × \$2.00 = \$30.00

Lost revenue > Decrease in cost

Not recommended.

3

x_j = units of product $i = 1, 2, 3$

Maximize $z = 20x_1 + 50x_2 + 35x_3$

s.t.

$-.5x_1 + .5x_2 + .5x_3 \le 0$

$x_1 \le 75$

$2x_1 + 4x_2 + 3x_3 \le 240$

$x_1, x_2, x_3 \ge 0$

(a) Solution: $z =$ \$2800

$x_1 = x_2 = 40,\ x_3 = 0$

	x_1	x_2	x_3	S_1	S_2	S_3	
z	0	0	10/3	20/3	0	35/3	2800
x_2	0	0	5/6	2/3	0	1/6	40
S_2	1	0	1/6	4/3	1	-1/6	35
x_1	0	1	-1/6	-4/3	0	1/6	40

(b) $z + 10/3\,x_3 + 20/3\,S_1 + 0\,S_2 + 35/3\,S_3 = 2800$

Dual-price for raw material = \$35/3 /lb

$$\left.\begin{aligned} x_2 &= 40 + D_3/6 \\ S_2 &= 35 - D_3/6 \\ x_1 &= 40 + D_3/6 \end{aligned}\right\} \Rightarrow -240 \le D_3 \le 210$$

$D_3 = 120$ lb falls in the range (−240, 210)

New solution:

$x_1 = 40 + \frac{120}{6} = 60$ units

$x_2 = 40 + \frac{120}{6} = 60$ units

$x_3 = 0$

New revenue = $2800 + (35/3)(120)$

= \$4200

continued...

Set 3.6c

(c) Dual price = 0 , $-35 \le D_2 < \infty$

±10% of 75 = ±7.5 or

Change has no effect on the solution

4

x_j = units of product j, $j = 1, 2, 3$,

Maximize $Z = 4.5x_1 + 5x_2 + 4x_3$

s.t.

$$10x_1 + 5x_2 + 6x_3 \le 600$$
$$6x_1 + 8x_2 + 9x_3 \le 600$$
$$8x_1 + 10x_2 + 12x_3 \le 600$$
$$x_1, x_2, x_3 \ge 0$$

(a) <u>Solution</u>: $Z = \$325$

$x_1 = 50, x_2 = 20, x_3 = 0$

(b) <u>Optimum tableau</u>

	x_1	x_2	x_3	S_1	S_2	S_3	
Z	0	0	2	.083	0	.458	325
x_1	1	0	0	.167	0	-.083	50
S_2	0	0	-.6	.067	1	-.833	140
x_2	0	1	1.2	-.133	0	.167	20

$Z + 2x_3 + .083S_1 + 0S_2 + .458S_3 = 325$

<u>Dual prices</u>:

Process 1: \$.083/min

2 : \$0/min

3 : \$.458/min

Process3 > Process 1

(c) Process 1 : $60 \times .083 = \$4.98$

2 : 0

3 : $60 \times .458 = \$27.48$

5

x_1 = Nbr. of practical courses

x_2 = Nbr. of humanistic courses

Maximize $z = 1500x_1 + 1000x_2$

$$x_1 + x_2 + S_1 = 30 \quad (1)$$
$$x_1 - S_2 = 10 \quad (2)$$
$$x_2 - S_3 = 10 \quad (3)$$
$$x_1, x_2, S_1, S_2, S_3 \ge 0$$

(a) <u>Solution</u>:

$z = \$40,000$

x_1 = 20 courses

x_2 = 10 Courses

continued...

(b) From Tora,

$Z + 1500S_1 + 0S_2 + 500S_3 = 40,000$

S_1 is a slack, S_2 and S_3 are surplus

<u>Dual prices</u>:

Constraint 1 : \$1500 /course

Constraint 2 : \$0/min limit course

Constraint 3 : -\$500/min limit course

Dual price for constraint 1 equals the revenue per practical course. Hence, an additional course must necessarily be of the practical type.

(c) From TORA,

$$\left.\begin{array}{l} S_2 = 10 + D_1 \ge 0 \\ x_1 = 20 + D_1 \ge 0 \end{array}\right\} -10 \le D_1 < \infty$$
$$x_2 = 10$$

Thus, the dual price of \$1500 for constraint 1 is valid for any number of courses ≥ 30-10 = 20.

(d) Dual price = -\$500. To determine the range where it applies, we have from TORA

$$\left.\begin{array}{l} S_1 = 10 - D_3 \ge 0 \\ x_1 = 20 - D_3 \ge 0 \\ x_2 = 10 + D_3 \ge 0 \end{array}\right\} -10 \le D_3 \le 10$$

A unit increase in lower limit on humanistic course offering (i.e. from 10 to 11) decreases revenue by \$500

6

x_1 = Radio minutes

x_2 = TV minutes

x_3 = Newspaper ads

Maximize $Z = x_1 + 50x_2 + 5x_3$

s.t.

$$15x_1 + 300x_2 + 50x_3 \le 10000 \quad (1)$$
$$x_3 \ge 5 \quad (2)$$
$$x_1 \le 400 \quad (3)$$
$$-x_1 + 2x_2 \le 0 \quad (4)$$
$$x_1, x_2, x_3 \ge 0$$

<u>Solution</u>: $Z = 1561.36$

$x_1 = 59.09$ min, $x_2 = 29.55$ min, $x_3 = 5$ ads

continued...

(b) S_1, S_3, S_4 = Slacks associated with constraints 1, 3, and 4

S_2 = Surplus associated with constraint 2

From TORA's optimum tableau:

$$Z + 2.879\,S_2 + .158\,S_1 + 0\,S_2 + 1.364\,S_3 = 1561.36$$

$$59.091 + .006D_1 - .303D_2 \qquad - .909D_4 \geq 0$$

$$5 \qquad + D_2 \qquad \geq 0$$

$$340.909 - .006D_1 + .303D_2 + D_3 + .909D_4 \geq 0$$

$$29.545 + .003D_1 - .152D_2 \qquad + .045D_4 \geq 0$$

Constraint	Dual Price	RHS Range†
1	.158	(250, 66250)
2	−2.879*	(0, 2000)
3	0	(59.09, ∞)
4	1.3636	(−375, 65)

* Negative because S_2 is a surplus variable

† These results are taken from TORA output. They differ from those computed from the given D_i- conditions because of roundoff error

Conclusions:

1. Increasing the lower limit on the number of newspaper ads is not advantageous because the associated dual price is negative (= −2.879)
2. Increasing the upper limit on radio minutes is not warranted because its dual price is zero (the current limit is already abundant).

(c) Dual price = .158/budget \$ valid in the range 250 ≤ \$ ≤ 66250.

50% budget increase = \$5000, or budget will be increased to 15,000.

Increase in Z = .158 × 5000 = 790

7

(a) X_1 = Nbr. Shirts / week

X_2 = Nbr. blouses / week

Maximize $Z = 8X_1 + 12X_2$

s.t.

$$20X_1 + 60X_2 \leq 25 \times 60 \times 40 = 60{,}000$$

$$70X_1 + 60X_2 \leq 35 \times 60 \times 40 = 84{,}000$$

$$12X_1 + 4X_2 \leq 5 \times 60 \times 40 = 12{,}000$$

$$X_1, X_2 \geq 0$$

continued...

Solution: $Z = \$13920$ / week

X_1 = 480 shirts, X_2 = 840 blouses

(b) Let S_1, S_2, and S_3 be the slack variables associated with the cutting, sewing, and packaging constraints. From the optimum TORA tableau, we have

$$Z + .12S_1 + .08S_2 + 0S_3 = 13920$$

Dept.	Worth/hr (Dual price)	
Cutting	\$.12/	= \$7.20/hr
Sewing	\$.08/min	= \$4.80/hr
Packaging	\$0/hr	

(c) Breakeven wages are \$7.20/hr for cutting and \$4.80 for sewing

8

(a) X_1 = units of solution A

X_2 = units of solution B

Maximize $Z = 8X_1 + 10X_2$

s.t.

$$.5X_1 + .5X_2 \leq 150 \qquad (1)$$

$$.6X_1 + .4X_2 \leq 145 \qquad (2)$$

$$30 \leq X_1 \leq 150 \qquad (3)$$

$$40 \leq X_2 \leq 200 \qquad (4)$$

Solution: $Z = \$2800$

X_1 = 100 units, X_2 = 200 units

(b) Define

S_1, S_2, S_3, S_4 = slacks in constraints 1, 2, 3, 4

S_5, S_6 = surplus variables associated with the lower bounds of constraints 3 and 4.

From TORA's optimum tableau:

$$Z + 16S_1 + 0S_2 + 0S_3 + 2S_4 + 0S_5 + 0S_6 = 2800$$

Conditions:

$$S_5 = 70 + 2D_1 - D_4 - D_5 \geq 0$$

$$S_2 = 5 - 1.2D_1 + D_2 + .2D_4 \geq 0$$

$$S_3 = 50 - 2D_1 + D_3 + D_4 \geq 0$$

$$X_1 = 100 + 2D_1 - D_4 \geq 0$$

$$X_2 = 200 + D_4 \geq 0$$

$$S_4 = 160 + D_4 - D_6 \geq 0$$

continued...

Set 3.6c

Constraint	Dual price	RHS-range
1	16	(115, 154.17)
2	0	(140, ∞)
3 (upper)	0	(100, ∞)
3 (lower)	0	(−∞, 100)
4 (upper)	2	(175, 270)
4 (lower)	0	(−∞, 200)

Increase in raw material 1 and in the upper bound on solution B is advantageous because their dual prices (16 and 2) are positive.

(c) Increase in revenue/unit = \$16
Increase in cost/unit = \$20
Not recommended!

(d) Dual price for raw material 2 is zero because it is abundant. No increase is warrented.

9

x_1 = Nbr. DiGi-1
x_2 = Nbr. DiGi-2
S_i = Idle minutes for station i, $i=1,2,3$

The objective is to minimize $S_1+S_2+S_3$. To express the objective function in terms of x_1 and x_2, consider

$$6x_1 + 4x_2 + S_1 = .9\times480 = 432$$
$$5x_1 + 4x_2 + S_2 = .86\times480 = 412.8$$
$$4x_1 + 6x_2 + S_3 = .88\times480 = 422.4$$

Thus, $S_1+S_2+S_3 = 1267.2 - 15x_1 - 14x_2$

(a)
Maximize $z = 15x_1 + 14x_2$
s.t.

$$6x_1 + 4x_2 + S_1 = 432$$
$$5x_1 + 4x_2 + S_2 = 412.8$$
$$4x_1 + 6x_2 + S_3 = 422.4$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

z represents the total used time in the three stations in minutes.

Solution: $z = 1241.28$ minutes
$x_1 = 45.12$ units, $x_2 = 40.32$ units

Utilization $= \frac{1241.28}{1267.20}\times100 = 97.95\%$

continued...

(b) From TORA,

$$z + 1.7S_1 + 0S_2 + 1.2S_3 = 1241.28$$

Conditions:

$$x_1 = .3D_1 - .2D_3 + 45.12 \geq 0$$
$$S_2 = -.7D_1 + D_2 - .2D_3 + 25.92 \geq 0$$
$$x_2 = -.2D_1 + .3D_3 + 40.32 \geq 0$$

Station	Dual Price	RHS Range
1	1.7	281.6, 469.03
2	0	386.88, ∞
3	1.2	288, 552

1% decrease in maintenance time is equivalent to $D_1 = D_2 = D_3 = 4.8$ minutes. This is equivalent to having

Station	Daily minutes
1	436.8
2	417.6
3	427.2

All three daily minutes fall within the allowable ranges. Thus

Station	Increase in utilized time/day
1	4.8×1.7 = 8.16 minutes
2	4.8×0 = 0
3	4.8×1.2 = 5.76

(c) $D_1 = .9(600-480) = 108$ min
$D_2 = .86(600-480) = 103.2$
$D_3 = .88(600-480) = 105.6$

From the conditions in (b)

$$x_1 = .3\times108 - .2\times105.6 + 45.12 = 56.4$$
$$S_2 = -.7\times108 + 103.2 - .2\times105.6 + 25.92 = 32.4$$
$$x_2 = -.2\times108 + .3\times105.6 + 40.32 = 50.4$$

Solution is feasible. Hence dual prices remain applicable and the net utilization is increased by $1.7\times108 + 0\times103.2 + 1.2\times105.6 = 310.32$ minutes. Because station 2 has zero dual price, its capacity need not be increased. The associated cost ~~thus~~ equals $1.5(600-480) + 0 + 1.5(600-480) = \$360$.

The proposal can be improved by recommending that station 2 time remain unchanged.

10

X_1 = Nbr. purses/day
X_2 = Nbr. bags/day
X_3 = Nbr. backpacks/day

Maximize $z = 24X_1 + 22x_2 + 45X_3$
s.t.

$$2X_1 + X_2 + 3X_3 \le 42$$
$$2X_1 + X_2 + 2X_3 \le 40$$
$$X_1 + .5X_2 + X_3 \le 45$$
$$X_1, X_2, X_3 \ge 0$$

Solution: $z = \$882$, $X_1 = 0, X_2 = 2, X_3 = 36$

Letting S_1, S_2, S_3 be the slacks in constraints 1, 2, and 3, we get

$$Z + 20X_1 + S_1 + 21S_2 + 0S_3 = 882$$

Conditions:

$$X_3 = 2 + D_1 - D_2 \ge 0$$
$$X_2 = 36 - 2D_1 + 3D_2 \ge 0$$
$$S_3 = 25 - .5D_2 + D_3 \ge 0$$

Resource	Dual price	RHS Ranges
Leather	1	(40, 60)
Sewing	21	(28, 42)
Finishing	0	(20, ∞)

(a) Available leather = 45 ft² falls in the RHS range. Solution remains feasible. $D_1 = 45 - 42 = 3$. New solution:

$$X_1 = 0$$
$$X_2 = 36 - 2 \times 3 = 30$$
$$X_3 = 2 + 3 = 5$$
$$Z = 882 + 1 \times D_1 = 882 + 1 \times 3 = \$885$$

(b) Available leather = 41 ft² falls in the RHS range and the solution remains feasible. $D_1 = 41 - 42 = -1$

$$X_2 = 36 - (2 \times -1) = 38$$
$$X_3 = 2 - 1 = 1$$
$$Z = 882 + (1 \times -1) = \$881$$

(c) Sewing hours = 38 falls within the RHS range. $D_2 = 38 - 40 = -2$. Dual price = 21

$$X_2 = 36 + 3 \times -2 = 30$$
$$X_3 = 2 - (-2) = 4$$
$$z = 882 + (21 \times -2) = \$840$$

continued....

(d) Sewing hours = 46 hours falls outside the RHS range. Thus, the current optimum basic solution is infeasible. To obtain the new solution, either solve the problem anew or use the algorithms in chapter 4.

(e) Finishing hours = 15, which falls outside the RHS range. Hence, resolve the problem

(f) Sewing hours = 50, which falls in the RHS range. $D_3 = 50 - 45 = 5$. Solution remains unchanged because dual price is zero and D_3 does not appear in the expression for X_2 or X_3.

(g) Dual price = \$21/hr, which is higher than the cost of an additional worker per hour. Hiring is recommended.

11

X_1 = Nbr. model 1 units
X_2 = Nbr. model 2 units

Maximize $z = 3X_1 + 4X_2$
s.t.

$$2X_1 + 3X_2 \le 1200$$
$$2X_1 + X_2 \le 1000$$
$$4X_2 \le 800$$
$$X_1, X_2 \ge 0$$

Solution: $z = \$1750$
$X_1 = 450$, $X_2 = 100$

(a) $S_1 = 0 \Rightarrow$ Resistors scarce
$S_2 = 0 \Rightarrow$ Capacitors scarce
$S_3 = 400 \Rightarrow$ chips abundant

(b) $Z + \frac{5}{4}S_1 + \frac{1}{4}S_2 = 1750$

Resource	Dual price
Resistors	\$1.25/resistor
Capacitors	\$.25/capacitor
Chips	\$0/chip

(c) Conditions:

$$X_1 = 450 - \frac{1}{4}D_1 + \frac{3}{4}D_2 \ge 0$$
$$S_3 = 400 - 2D_1 + 2D_2 + D_3 \ge 0$$
$$X_2 = 100 + \frac{1}{2}D_1 - \frac{1}{2}D_2 \ge 0$$

Feasibility ranges:

$$\left.\begin{array}{l} 450 - .25D_1 \ge 0 \\ 400 - 2D_1 \ge 0 \\ 100 + .5D_1 \ge 0 \end{array}\right\} \Rightarrow -200 \le D_1 \le 200$$

continued...

Set 3.6c

$$\left.\begin{array}{l} 450 + .75 D_2 \ge 0 \\ 400 + 2 D_2 \ge 0 \\ 100 - .5 D_2 \ge 0 \end{array}\right\} \Rightarrow -200 \le D_2 \le 200$$

$400 + D_3 \ge 0 \Rightarrow -400 \le D_3 < \infty$

(d) $D_1 = 1300 - 1200 = 100$ in the allowable range $-200 \le D_1 \le 200$.

$\Delta Z = 100 \times 1.25 = \$125$

$x_1 = 450 - .25 \times 100 = 425$

$x_2 = 100 + .5 \times 100 = 150$

New $Z = 1750 + \Delta Z = \$1875$

(e) $D_3 = 350 - 800 = -450$, which falls outside allowable range $-400 \le D_3$. Thus, basic solution and dual price change and the problem must be solved anew.

(f) $-200 \le D_2 \le 200$, dual price $= .25$. Thus,

$-200 \times .25 \le \Delta Z \le 200 \times .5$

$-50 \le \Delta Z \le 50$

$\$1700 \le Z \le \$1800$

$450 - .75 \times 200 \le x_1 \le 450 + .75 \times 200$

$100 - \frac{1}{2}(-200) \le x_2 \le 100 - \frac{1}{2}(+200)$

(g) Cost of purchasing 500 additional resistors $= 500 \times .40 = \$200$

$D_1 = 500$ resistors

Dual price of \$1.25 is valid in $-200 \le D_1 \le 200$. Thus, for the first 200 resistors alone, HiDec will get an additional revenue of $200 \times 1.25 = \$250$, which is more than the cost of all 500 resistors. Accept.

12

From Example 3.6-2, we have for the TOYCO model

$-200 \le D_1 \le 10$

$-20 \le D_2 \le 400$

$-20 \le D_3 < \infty$

(a) $D_1 = 8$, $D_2 = 40$, $D_3 = -10$

All D_i, $i = 1, 2, 3$ fall within the feasibility ranges. Thus

continued...

$r_1 = \frac{8}{10}$, $r_2 = \frac{40}{400}$, $r_3 = \frac{-10}{-20}$

$r_1 + r_2 + r_3 = .8 + .1 + .5 = 1.4 > 1$

Hence, no conclusion can be made about the feasibility of the new RHS (438, 500, 410). Problem 1(a) shows that these new values do produce a feasible solution.

(b) $D_1 = 30$, $D_2 = -20$, $D_3 = -40$. Because D_1 and D_3 fall outside the given feasibility ranges, the 100% rule cannot be applied in this case

13

(a) From TORA,

$x_1 = 2 + \frac{2}{3} D_1 - \frac{1}{3} D_2 \ge 0$

$x_2 = 2 - \frac{1}{3} D_1 + \frac{2}{3} D_2 \ge 0$

Feasibility ranges:

$-3 \le D_1 \le 6$

$-3 \le D_2 \le 6$

(b) $D_1 = D_2 = \Delta > 0$. Thus

$$\left.\begin{array}{l} x_1 = 2 + \Delta/3 > 0 \\ x_2 = 2 + \Delta/3 > 0 \end{array}\right\} \text{for all } \Delta > 0$$

100% rule for $0 < \Delta \le 3$:

$r_1 = r_2 = \frac{\Delta}{6} \le \frac{3}{6} \Rightarrow r_1 + r_2 < 1$, which confirms feasibility for $0 < D < 3$

100% rule for $3 \le \Delta \le 6$:

$r_1 = r_2 = \frac{\Delta}{6} \Rightarrow \frac{3}{6} \le r_1, r_2 \le \frac{6}{6}$

$r_1 + r_2 \ge 1 \Rightarrow$ cannot confirm feasibility.

100% rule for $\Delta > 6$:

Δ is outside $-3 \le D_1, D_2 \le 6$. Thus, the rule is not applicable.

1

From Section 3.6.3, we have the following optimality conditions for the TOYCO model:

x_1: $4-\frac{1}{4}d_2+\frac{3}{2}d_3-d_1 \geq 0$

x_4: $1+\frac{1}{2}d_2 \geq 0$

x_5: $2-\frac{1}{4}d_2+\frac{1}{2}d_3 \geq 0$

(i) $Z = 2x_1 + x_2 + 4x_3$

$d_1 = 2-3 = -1$, $d_2 = 1-2 = -1$, $d_3 = 4-5 = -1$

x_1: $4-\frac{1}{4}(-1)+\frac{3}{2}(-1)-(-1) = 3.75 > 0$

x_4: $1+\frac{1}{2}(-1) = .5 > 0$

x_5: $2-\frac{1}{4}(-1)+\frac{1}{2}(-1) = 1.75 > 0$

Conclusion: Solution is unchanged

(ii) $Z = 3x_1 + 6x_2 + x_3$

$d_1 = 3-3 = 0$, $d_2 = 6-2 = 4$, $d_3 = 1-5 = -4$

x_1: $4-\frac{1}{4}(4)+\frac{3}{2}(4)-(0) = -3 < 0$

Conclusion: solution changes

(iii) $Z = 8x_1 + 3x_2 + 9x_3$

$d_1 = 8-3 = 5$, $d_2 = 3-2 = 1$, $d_3 = 9-5 = 4$

x_1: $4-\frac{1}{4}(1)+\frac{3}{2}(4)-(5) = 4.75 > 0$

x_4: $1+\frac{1}{2}(1) = 1.5 > 0$

x_5: $2-\frac{1}{4}(1)+\frac{1}{2}(4) = 3.75 > 0$

Conclusion: Solution is unchanged

2

x_1 = Nbr cans of A1

x_2 = Nbr. cans of A2

x_3 = Nbr. cans of BK

Maximize $Z = 80x_1 + 70x_2 + 60x_3$

s.t. $x_1 + x_2 + x_3 \leq 500$ $\Leftarrow S_1$

$x_1 \geq 100$ $\Leftarrow S_2$

$4x_1 - 2x_2 - 2x_3 \leq 0$ $\Leftarrow S_3$

$x_1, x_2, x_3 \geq 0$

TORA optimum tableau:

Basic	x_1	x_2	x_3	s_1	S_2	s_3	Solution
Z	0	0	10	73.33	0	1.67	36666.67
x_2	0	1	1	.67	0	-.17	333.33
x_1	1	0	0	.33	0	.17	166.67
S_2	0	0	0	.33	1	.17	66.67

continued...

(a) $Z = \$366.67$

$x_1 = 166.67$, $x_2 = 333.33$, $x_3 = 0$

(b) Reduced cost for x_3 = 10 cents. Price should be increased by more than 10 cents/can

(c) $d_1 = d_2 = d_3 = -5$ cents

From the optimum tableau, reduced costs:

x_3: $10+d_2-d_3 = 10-5-(-5) = 10 > 0$

S_1: $73.33 + .67d_2 + .33d_1$

$= 73.33 + .67(-5) + .33(-5) = 68.33 > 0$

S_3: $1.67 - .17d_2 + .17d_1 = 1.67 - .17(-5) + .17(-5)$

$= 1.67 > 0$

Conclusion: Solution is unchanged.

3

(a) Available carpenter hours in a 10-day period = 4 × 10 × 8 = 320

x_1 = Nbr. chairs assembled in 10 days

x_2 = Nbr. tables assembled in 10 days

Maximize $Z = 50x_1 + 135x_2$

s.t.

$.5x_1 + 2x_2 \leq 320$

$$4 \leq \frac{x_1}{x_2} \leq 6 \Rightarrow \begin{cases} x_1 - 4x_2 \geq 0 \\ x_1 - 6x_2 \leq 0 \end{cases}$$

$x_1, x_2 \geq 0$

<u>Solution</u>: $Z = \$27,840$, $x_1 = 384$, $x_2 = 64$

(b) Optimum tableau:

	x_1	x_2	s_1	S_2	s_3	solution
Z	0	0	87	0	6.5	27840
x_2	0	1	.2	0	-.1	64
x_1	1	0	1.2	0	.4	384
S_2	0	0	.4	1	.8	128

Optimality conditions:

S_1: $87 + 1.2d_1 + .2d_2 \geq 0$

S_3: $6.5 + .4d_1 - .1d_2 \geq 0$

For $d_1 = -5$, $d_2 = -13.5$:

S_1: $87 + 1.2(-5) + .2(-13.5) = 78.3 > 0$

S_3: $6.5 + .4(-5) - .1(-13.5) = 5.85 > 0$

Solution remains the same

(c) $d_1 = 25-50 = -25$, $d_2 = 120-135 = -15$

S_1: $87 + 1.2(-25) + .2(-15) = 58.5 > 0$

S_3: $6.5 + 4(-25) - .1(-15) = -2 < 0$

Solution changes

Set 3.6d

4

(a) x_1 = Amt. of personal loan (\$)

x_2 = Amt. of car loan (\$)

Maximize $z = .14(x_1 - .03x_1) + .12(x_2 - .02x_2)$
$- .03x_1 - .02x_2$
$= .1058x_1 + .0976x_2$

s.t.

$x_1 + x_2 \le 200{,}000$

$\frac{x_2}{x_1} \ge 2$ or $2x_1 - x_2 \le 0$

$x_1, x_2 \ge 0$

Solution: $z = \$20{,}067$

$x_1 = \$66{,}667$, $x_2 = \$133{,}333$

Rate of return $= \frac{20{,}067}{200{,}000} \times 100 = 10.03\%$

(b) Optimum tableau:

	x_1	x_2	s_1	s_2	Solution
z	0	0	.1003	.0027	20066.67
x_2	0	1	.6667	−.3333	133333.33
x_1	1	0	.3333	.3333	66666.67

Optimality conditions:

s_1: $.1003 + .333d_1 + .6667d_2 \ge 0$

s_2: $.0027 + .3333d_1 - .3333d_2 \ge 0$

New x_1-objective coefficient $= .14(1-.04) - .04$
$= .0944$

New x_2-objective coefficient $= .12(1-.03) - .03$
$= .0864$

$d_1 = .0944 - .1058 = -.0114$

$d_2 = .0864 - .0976 = -.0112$

s_1: $.1003 + .3333(-.0114) + .6667(-.0112)$
$= .08907 > 0$

s_2: $.0027 + .3333(-.0114) - .3333(-.0112)$
$= .00267 > 0$

Solution does not change

5

(a) x_i = Nbr of units of motor i, $i = 1, 2, 3, 4$

Maximize $z = 60x_1 + 40x_2 + 25x_3 + 30x_4$

s.t. $8x_1 + 5x_2 + 4x_3 + 6x_4 \le 8000$

$x_1 \le 500$, $x_2 \le 500$, $x_3 \le 800$, $x_4 \le 750$

$x_1, x_2, x_3, x_4 \ge 0$

Solution: $z = \$59{,}375$, $x_1 = 500$, $x_2 = 500$, $x_3 = 375$
$x_4 = 0$

continued...

(b) Optimality conditions (from TORA):

x_4: $7.5 + 1.5d_3 - d_4 \ge 0$

s_1: $6.25 + .25d_3 \ge 0$

s_2: $10 - 2d_3 + d_1 \ge 0$

s_3: $8.75 - 1.25d_3 + d_2 \ge 0$

From s_3, $8.75 + d_2 \ge 0 \Rightarrow -8.75 \le d_2 < \infty$

Thus, price of type 2 motor can be reduced by at most \$8.75 without causing a solution change.

(c) $d_1 = -15$, $d_2 = -10$, $d_3 = -6.25$, $d_4 = -7.5$

Solution remains the same because

x_4: $7.5 + 1.5(-6.25) - (-7.5) = 5.625 > 0$

s_1: $6.25 + .25(-6.25) = 4.6875 > 0$

s_2: $10 - 2(-6.25) + (-15) = 7.5 > 0$

s_3: $8.75 - 1.25(-6.25) + (-10) = 6.5625 > 0$

(d) Reduced cost for $x_4 = 7.5$. Increase price of type 4 motor by more than \$7.50.

6

(a) x_1 = Cases of juice/day

x_2 = Cases of sauce/day

x_3 = Cases of paste/day

Maximize $z = 21x_1 + 9x_2 + 12x_3$

s.t.

$(1 \times 24)x_1 + (\frac{1}{2} \times 24)x_2 + (\frac{3}{4} \times 24)x_3 \le 60{,}000$

$x_1 \le 2000$, $x_2 \le 5000$, $x_3 \le 6000$

$x_1, x_2, x_3 \ge 0$

Solution: $z = \$51{,}000$

$x_1 = 2000$, $x_2 = 1000$, $x_3 = 0$

(b) From TORA, optimality conditions given d_2:

x_3: $1.5 + 1.5d_2 \ge 0 \Rightarrow d_2 \ge -1$

s_1: $.75 + .083d_2 \ge 0 \Rightarrow d_2 \ge -9$

s_2: $3 - 2d_2 \ge 0 \Rightarrow d_2 \le 1.5$

Thus, $-1 \le d_2 \le 1.5$, or

$9 - 1 \le$ price/case of sauce $\le 9 + 1.5$

Solution mix remains the same if the price per case of sauce remains between \$8 and \$10.50.

7

(a) x_1 = Nbr. regular cabinets/day
x_2 = Nbr. deluxe cabinets/day

Maximize $z = 100x_1 + 140x_2$

s.t.

$$.5x_1 + x_2 \le 180$$
$$x_1 \le 200$$
$$x_2 \le 150$$
$$x_1, x_2 \ge 0$$

Solution: $z = \$31,200$
$x_1 = 200$ regular
$x_2 = 80$ deluxe

(b) From TORA, optimality conditions:

$s_1: 140 + d_2 \ge 0$
$s_2: 30 + d_1 - .5d_2 \ge 0$

$d_1 = 80 - 100 = -20$
$d_2 = 80 - 140 = -60$

$s_1: 140 + (-60) = 80 > 0$
$s_2: 30 + (-20) - .5(-60) = 40 > 0$

Solution remains the same

8

(a) For the original TOYCO model, TORA gives (also see Section 3.6.3)

$$-\infty < d_1 \le 4,\ -2 \le d_2 \le 8,\ -8/3 \le d_3 < \infty$$

(ii) Original $z = 3x_1 + 2x_2 + 5x_3$
New $z = 3x_1 + 6x_2 + x_3$

i	d_i	u_i	v_i	r_i
1	0		4	$0/4 = 0$
2	4		8	$4/8 = 1/2$
3	-4	$-8/3$		$-4/-\frac{8}{3} = 3/2$

$$r_1 + r_2 + r_3 = 0 + 1/2 + 3/2 = 2 > 1$$

The 100% rule is nonconclusive in this case. The solution in Problem 1 (ii) shows that the solution will change

(iii) Original $z = 3x_1 + 2x_2 + 5x_3$
New $z = 8x_1 + 3x_2 + 9x_3$

i	d_i	u_i	v_i	r_i
1	5		4	$5/4$
2	1		8	$1/8$
3	4		∞	$4/\infty = 0$

$$r_1 + r_2 + r_3 = \frac{5}{4} + \frac{1}{8} = \frac{11}{8} > 1$$

continued...

The 100% rule is nonconclusive. Yet Problem 1(iii) shows that the solution remains unchanged.

The two cases demonstrate that the 100% rule is too weak to be effective in decision making, and that it is more reliable to utilize the simultaneous optimality conditions given in Section 3.6.3.

(b) $-30 \le d_1 < \infty,\ -140 \le d_2 \le 60$

New $z = 80x_1 + 80x_2$
Original $z = 100x_1 + 140x_2$

i	d_i	u_i	v_i	r_i
1	-20	-30	∞	$-20/-30 = 2/3$
2	-60	-140	60	$-60/-140 = 3/7$

$$r_1 + r_2 = 2/3 + 3/7 = \frac{23}{21} > 1$$

The 100% rule is nonconclusive. Yet, Problem 7(b) shows that the solution remains unchanged.

Set 3.6e

1

See file solver 3.6e-1.xls in ch3Files

Dual prices for years 1, 2, 3, and 4 are 0, 0, 0, 2.89. Thus, for year 4, one (thousand) additional dollars increases Z by \$2.89 thousand. It is worthwhile to increase the funding for year 4.

2

See file tora 3.6e-2.txt

Constraint	Dual Price	Range
1	5.36	$(0, \infty)$
2	−3.73	$(-\infty, 6000)$
3	−1.13	$(-\infty, 6800)$
4	−1.07	$(-\infty, 33642)$
5	−1.00	$(-\infty, 53628.73)$

(a) Constraint 1: $x_1 + x_2 + x_4 + y_1 \leq 10{,}000$

Dual price = \$5.36/invested \$

Rate of return = 536%

(b) Constraint 2: \$1000 spend on pleasure

$$.5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_1 - y_2 = 1000$$

Dual price = −3.73/pleasure \$

Range = $(-\infty, 6000)$

Spending \$1000 at end of year 1 reduces total return by \$3730.

3

See file tora 3.6e-3.txt in ch3Files

Quarter	Dual price	Range
1	1.2488	.6647, 2.5806
2	1.2443	.6580, 2.6122
3	1.1945	−.2646, 1.1245
4	1.0200	−.2553, ∞
5	1.0000	−4.8366, ∞

(a) An additional \$ available at the start of quarter 1 is worth \$1.24888 at the end of 4 quarters. Similarly, an additional dollar at the start of periods 2, 3, and 4 is worth \$1.2443, \$1.1945, and \$1.02, respectively. The dual price for quarter 4 (= \$1.02) shows that all we can do with the money then is to invest it at 2% for the quarter.

We can use the dual price to determine the rate of return for each quarter — namely,

continued...

quarter 1:

$$1.2488 = 1.2243(1+i_1) \Rightarrow i_1 = .02$$

quarter 2:

$$1.2243 = 1.1945(1+i_2) \Rightarrow i_2 = .025$$

quarter 3:

$$1.1945 = 1.02\,(1+i_3) \Rightarrow i_3 = .171$$

quarter 4:

$$1.02 = 1.0(1+i_4) \Rightarrow i_4 = .02$$

(b) The dual price associated with the upper bound on B_3 (UB-X10) is \$.149. It represents the networth per dollar borrowed in period 3. Also, an extra dollar in period 3 is worth \$1.1945 at the end of the horizon. However, if that dollar is borrowed, it must be repaid as \$1.025 in the next quarter. The repayment is equivalent to forgoing making 2% in interest. Thus, the networth of borrowing in period 3 is

$$1.1945 - 1.025 \times 1.02 = .149$$

This result is consistent with the dual price for the upper bound on B_3

4

```
--------------------------------------------------------------------
ConstraintCurrent RHS        Min RHS        Max RHS        Dual Price
--------------------------------------------------------------------
1 (=)          2.0000         0.0000       infinity            2.1756
2 (=)          2.0000        -0.1667       infinity            2.0173
3 (=)          2.5000        -0.3472       infinity            1.8647
4 (=)          2.5000        -0.5767       infinity            1.7296
5 (=)          3.0000        -0.8248       infinity            1.6044
6 (=)          3.5000        -1.1331       infinity            1.4356
7 (=)          3.5000        -6.1137       infinity            1.3355
8 (=)          4.0000       -11.4678       infinity            1.2423
9 (=)          4.0000       -20.6663       infinity            1.1556
10 (=)         5.0000       -32.5201       infinity            1.0750
```

The dual price provides the worth per additional \$ at the end of year 10.

Annual rate of return:

Period 1: $2.1756 = 2.0173(1+i_1) \Rightarrow i_1 = .0785$

Period 2: $2.0173 = 1.8647(1+i_2) \Rightarrow i_2 = .0818$

Period 3: $1.8647 = 1.7296(1+i_3) \Rightarrow i_3 = .0781$

Period 4: $1.7296 = 1.6044(1+i_4) \Rightarrow i_4 = .0780$

etc...

5

See file tora3.6e-5.txt in ch3Files

The dual price for constraint 1

$$x_{1A} + x_{1B} \le 100,000$$

is \$5.10. Thus, each invested \$ is worth \$5.10 at the end of the investment horizon. Range (0, ∞)

6

Dual price for the constraint

$$x_1 + x_2 + x_3 + x_4 \le 500$$

is \$2.35 per \$ invested, range (0, ∞) The gambler should bet the largest amount possible.

7

See file tora3.6e-7.txt in ch3Files

For, $x_{W1} + x_{W2} + x_{W3} \ge 1500$, the dual price is \$11.4, range (800, ∞)

One extra wrench automatically requires the production of two chisels, thus leading to the following changes:

Cost of one wrench using subcont. = \$3.00
Cost of 2 chisels using subcont. = 2x \$4.20
total = \$11.40

$x_{W1} \le 550$, dual price = −\$1, range (−∞, 1250). If regular time capacity for wrenches is increased by 1 unit, one less wrench will be produced by subcontractor, which saves \$3 − \$2 = \$1.

Similar interpretations can be given for the remaining dual prices

8

See file tora3.6e-8.txt in ch3Files

Machine	Capacity	Dual price	Range
1	500	2	(253.33,570)
2	380	12	(333.33, 750)

The company should pay less than \$2/hr for machine 1 and less than \$12/hr for machine 2.

9

See file tora3.6e-9.txt in ch3Files

(a) Constraint $2x_1 + 3x_2 + 5x_3 \le 4000$ corresponds to raw material A. Its dual price is \$10.27/lb. For a purchase price of \$12/lb, acquisition of additional raw material A is not recommended.

(b) Constraint $4x_1 + 2x_2 + 7x_3 \le 6000$ is associated with raw material B. Its dual price is \$0/lb. Resource B is already abundant. Thus, no additional purchase is recommended.

10

(a) See file tora3.6e-10.txt

Constraint	Dual price
1	0
2	0
3	-400
4	-750
5	0
6	0
7	0

Constraints 3 and 4 have negative dual price. These correspond respectively to the third specification for alloy A and the first specification for alloy B. Changes in these specifications affects profit adversely

(b) For the ore constraints, the dual prices are \$90, \$110, and \$30 per additional ton of ores 1, 2, and 3, respectively. These are the maximum prices the company should pay.

CHAPTER 4

Duality and Post-Optimal Analysis

Set 4.1a

1

Primal:

Minimize $z = 5x_1 + 12x_2 + 4x_3$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + S_1 &= 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3, S_1 &\geq 0 \end{aligned}$$

Dual:

Maximize $w = 10y_1 + 8y_2$

subject to

$$\begin{aligned} y_1 + 2y_2 &\leq 5 \\ 2y_1 - y_2 &\leq 12 \\ y_1 + 3y_2 &\leq 4 \\ y_1 &\leq 0 \end{aligned}$$

y_2 unrestricted

2

Primal:

Minimize $z = 15x_1 + 12x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 - 4x_2 + x_4 &= 5 \\ 3x_1 + x_2 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

Maximize $z = 3y_1 + 5y_2 + 4y_3$

subject to

$$\begin{aligned} y_1 + 2y_2 + 3y_3 &\leq 15 \\ 2y_1 - 4y_2 + y_3 &\leq 12 \\ -y_1 &\leq 0 \Rightarrow y_1 \geq 0 \\ y_2 &\leq 0 \end{aligned}$$

y_3 unrestricted

3

Primal:

Minimize $z = 5x_1^+ - 5x_1^- + 6x_2$

subject to

$$\begin{aligned} x_1^+ - x_1^- + 2x_2 &= 5 \\ -x_1^+ + x_1^- + 5x_2 - x_3 &= 3 \\ 4x_1^+ - 4x_1^- + 7x_2 + x_4 &= 8 \\ x_1^+, x_1^-, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

Maximize $z = 5y_1 + 3y_2 + 8y_3$

Subject to

$$\left.\begin{aligned} y_1 - y_2 + 4y_3 &\leq 5 \\ -y_1 + y_2 - 4y_3 &\leq -5 \end{aligned}\right\} \Rightarrow y_1 - y_2 + 4y_3 = 5$$

$$\begin{aligned} 2y_1 + 5y_2 + 7y_3 &\leq 6 \\ -y_2 &\leq 0 \Rightarrow y_2 \geq 0 \\ y_3 &\leq 0 \end{aligned}$$

y_1 unrestricted

4

(a) *Primal:*

Maximize $z = -5x_1 + 2x_2$

s.t.

$$\begin{aligned} x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

Minimize $w = 2y_1 + 5y_2$

Subject to

$$\begin{aligned} y_1 + 2y_2 &\geq -5 \\ -y_1 + 3y_2 &\geq 2 \\ -y_1 &\geq 0 \Rightarrow y_1 \leq 0 \\ y_2 &\geq 0 \end{aligned}$$

(b) *Primal:*

Minimize $z = 6x_1 + 3x_2$

Subject to

$$\begin{aligned} 6x_1 - 3x_2 + x_3 - x_4 &= 2 \\ 3x_1 + 4x_2 + x_3 - x_5 &= 5 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Dual:

Maximize $w = 2y_1 + 5y_2$

Subject to

$$\begin{aligned} 6y_1 + 3y_2 &\leq 6 \\ -3y_1 + 4y_2 &\leq 3 \\ y_1 + y_2 &\leq 0 \end{aligned}$$

$$\left.\begin{aligned} -y_1 &\leq 0 \\ -y_2 &\leq 0 \end{aligned}\right\} \Rightarrow y_1, y_2 \geq 0$$

(c) *Primal:*

Maximize $z = x_1 + x_2$

Subject to

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ 3x_1 - x_2 &= 6 \end{aligned}$$

x_1, x_2 unrestricted

Dual:

Minimize $w = 5y_1 + 6y_2$

Subject to

$$\begin{aligned} 2y_1 + 3y_2 &= 1 \\ y_1 - y_2 &= 1 \end{aligned}$$

y_1, y_2 unrestricted

5

Primal:

Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR_2$

$$\begin{aligned} x_1 + 2x_2 + x_3 + S_1 \quad &= 10 \\ 2x_1 - x_2 + 3x_3 \quad + R_2 &= 8 \\ x_1, x_2, x_3, S_1, R_2 &\geq 0 \end{aligned}$$

Dual

Minimize $w = 10y_1 + 8y_2$

Subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 12 \\ y_1 + 3y_2 &\geq 4 \\ y_1 \quad &\geq 0 \\ y_2 &\geq -M \\ y_2 \text{ unrestricted} & \end{aligned} \Bigg\} \text{same}$$

6

All parts, (a) through (e), are true

7

(1) max + ($\geq$ constraints):

$$\sum a_{ij} x_j \boxed{-S_i} = b_i \Rightarrow -y_i \geq 0 \Rightarrow y_i \leq 0$$

(2) min + ($\geq$ constraints):

$$\sum a_{ij} x_j \boxed{-S_i} = b_i \Rightarrow -y_i \leq 0 \Rightarrow y_i \geq 0$$

(3) max + ($\leq$ constraints):

$$\sum a_{ij} x_j \boxed{+S_i} = b_i \Rightarrow y_i \geq 0$$

(4) min + ($\leq$ constraints):

$$\sum a_{ij} x_j + S_i = b_i \Rightarrow y_i \leq 0$$

(5) max or min + (= constraint)

$$\sum a_{ij} x_j = b_i \Rightarrow y_i \text{ unrestricted}$$

(6) max + ($x_j \geq 0$):

$$\boxed{\begin{matrix} c_j x_j \\ a_{ij} x_j \end{matrix}} \Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

(7) max + ($x_j \leq 0$):

Let $x_j = -x_j'$, $x_j' \geq 0$

$$\boxed{\begin{matrix} -c_j x_j' \\ -a_{ij} x_j' \end{matrix}} \Rightarrow -\sum_{i=1}^{m} a_{ij} y_i \geq -c_j$$

$$\Rightarrow \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

(8) min + ($x_j \geq 0$):

$$\boxed{\begin{matrix} c_j x_j \\ a_{ij} x_j \end{matrix}} \Rightarrow \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

(9) min + ($x_j \leq 0$):

Let $x_j = -x_j'$, $x_j' \geq 0$

$$\boxed{\begin{matrix} -c_j x_j' \\ -a_{ij} x_j' \end{matrix}} \Rightarrow -\sum_{i=1}^{m} a_{ij} y_i \leq -c_j$$

$$\Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

(10) max or min + (x_j unrestricted)

$$\boxed{\begin{matrix} c_j x_j \\ a_{ij} x_j \end{matrix}} \Rightarrow \sum_{i=1}^{m} a_{ij} y_i = c_j$$

Set 4.2a

$\boxed{1}$

(a) $\underset{3\times\underline{2}}{A}\;\underset{\underline{1}\times 2}{V1}$ undefined

(b) $\underset{3\times\underline{2}}{A}\,\underset{\underline{2}\times 1}{P1} = \begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}9\\ 12\\ 15\end{pmatrix}_{3\times 1}$

(c) $\underset{3\times\underline{2}}{A}\,\underset{\underline{3}\times 1}{P2}$ undefined

(d) $\underset{1\times\underline{2}}{V1}\;\underset{\underline{3}\times 2}{A}$ undefined

(e) $\underset{1\times\underline{3}}{V2}\;\underset{\underline{3}\times 2}{A} = (-1,-2,-3)\begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}$

$= (-14,\ -32)_{1\times 2}$

(f) $\underset{2\times\underline{1}}{P1}\;\underset{\underline{3}\times 1}{P2}$ undefined

(g) $\underset{1\times\underline{2}}{V1}\;\underset{2\times 1}{P1} = (11, 22)\begin{pmatrix}1\\ 2\end{pmatrix}$

$= 55_{1\times 1}$

1

(a)

$$\text{inverse} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{pmatrix}$$

2

(a)

$$\text{inverse} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Set 4.2c

1

Dual: Maximize $w = 50y$

s.t.

$$5y_1 \le 10,\ -7y_1 \le 4,\ 3y_1 \le 5,\ y \ge 0$$

The constraints simplify to

$$0 \le y_1 \le 5/3$$

Thus, max $w = 50 \times \frac{5}{3} = \frac{250}{3} = \min z$

2

Dual:

Maximize $w = 50y_1 + 20y_2 + 30y_3 + 35y_4 + 10y_5 + 90y_6 + 20y_7$

s.t.

$$5y_1 + y_2 + 7y_3 + 5y_4 + 2y_5 + 12y_6 \le 5$$
$$5y_1 + y_2 + 6y_3 + 5y_4 + 4y_5 + 10y_6 + y_7 \le 6$$
$$3y_1 - y_2 - 9y_3 + 5y_4 - 15y_5 - 10y_7 \le 3$$
$$-y_j \le 0 \Rightarrow y_j \ge 0,\ j = 1, 2, \dots, 7$$

From TORA, optimal objective equation is

$$z + 50y_1 + 0y_2 + 90y_3 + 65y_4 + 70y_5 + 110y_6 + 0y_7 + 0S_1 + 20S_2 + 0S_3 = 120$$

(S_1, S_2, S_3) are slack variables.

Thus, $x_1 = 0$, $x_2 = 20$, $x_3 = 0$

Obtaining the solution from the dual is advantgeous computationally because the dual has a smaller number of constraints.

3

Dual: Minimize $w = 30y_1 + 40y_2$

s.t.

$$y_1 + y_2 \ge 5$$
$$5y_1 - 5y_2 \ge 2$$
$$2y_1 - 6y_2 \ge 3$$
$$y_2 \ge 0,\ y_1 \text{ unrestricted}$$

Method 1: $z + 0x_1 + 23x_2 + 7x_3 + 105x_4 + 0x_5 = 150$

Coefficient of $x_4 = 105 \Rightarrow y_1 = 105 + (-100) = 5$

Coefficient of $x_5 = 0 \Rightarrow y_2 = 0$

Method 2:

$$(y_1, y_2) = (5, 0)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$= (5, 0)$$

$$w = 30 \times 5 + 40 \times 0 = 150$$

4

Dual: Maximize $w = 3y_1 + 6y_2 + 4y_3$

s.t.

$$3y_1 + 4y_2 + y_3 \le 4$$
$$y_1 + 3y_2 + 2y_3 \le 1$$
$$-y_2 \le 0 \Rightarrow y_2 \ge 0$$
$$y_3 \le 0$$
$$y_1 \text{ unrestricted}$$

Method 1: $z - 98.6x_4 - 100x_5 - .2x_6 = 3.4$

Coefficient of $x_4 = -98.6 \Rightarrow y_1 = -98.6 + 100 = 1.4$

Coefficient of $x_5 = -100 \Rightarrow y_2 = -100 + 100 = 0$

Coefficient of $x_6 = -.2 \Rightarrow y_3 = -.2$

Method 2:

$$(y_1, y_2, y_3) = (4, 1, 0)\begin{pmatrix} .4 & 0 & -.2 \\ -.2 & 0 & .6 \\ 1 & -1 & 1 \end{pmatrix}$$
$$= (1.4, 0, -.2)$$

$$w = 3 \times 1.4 + 6 \times 0 + 4 \times -.2 = 3.4$$

5

Dual: Minimize $w = 4y_1 + 8y_2$

s.t.

$$y_1 + y_2 \ge 2$$
$$y_1 + 4y_2 \ge 4$$
$$y_1 \ge 4$$
$$y_2 \ge -3$$

Method 1: $z + 2x_1 + 0x_2 + 0x_3 + 3x_4 = 16$

Coefficient of $x_3 = 0 \Rightarrow y_1 = 0 + 4 = 4$

Coefficient of $x_4 = 3 \Rightarrow y_2 = 3 + (-3) = 0$

Method 2:

$$(y_1, y_2) = (4, 4)\begin{pmatrix} 1 & -.25 \\ 0 & .25 \end{pmatrix} = (4, 0)$$

$$w = 4 \times 4 + 8 \times 0 = 16$$

6

Dual: Minimize $w = 3y_1 + 4y_2$

s.t.

$$y_1 + 2y_2 \ge 1$$
$$2y_1 - y_2 \ge 5$$
$$y_1 \ge 3,\ y_2 \text{ unrestricted}$$

Method 1: $z + 2x_2 + 0x_3 + 99x_4 = 5$

Coefficient of $x_3 \neq 0 \Rightarrow y_1 = 0 + 3 = 3$

Coefficient of $x_4 = 99 \Rightarrow y_2 = 99 + (-100) = -1$

Method 2:

$$(y_1, y_2) = (3, 1)\begin{pmatrix} 1 & -.5 \\ 0 & .5 \end{pmatrix} = (3, -1)$$

$$w = 3 \times 3 + 4(-1) = 5$$

7

Maximize $z = x_1 + x_2$

s.t.

$-3x_1 + 3x_2 \le 12$

$-3x_1 + 2x_2 \le -4$

$3x_1 - 5x_2 \le 2$

x_1 unrestricted, $x_2 \ge 0$

TORA solution:

$x_1 = 3.4737,\ x_2 = 1.6842,\ z = 5.1579$

Dual: Minimize $w = 12y_1 - 4y_2 + 2y_3$

s.t.

$y_1 - 3y_2 + 3y_3 = 1$

$3y_1 + 2y_2 - 5y_3 \ge 1$

$y_1, y_2, y_3 \ge 0$

From TORA, the optimal objective row is

$$w - 3.0526y_2 - 1.684y_4 - 96.5263y_5 - 98.3158y_6 = 5.1579$$

(y_5 and y_6 are artificial variables)

Coefficient of $y_5 = -96.5263 \Rightarrow x_1 = -96.5263 + 100 = 3.4737$

Coefficient of $y_6 = -98.3158 \Rightarrow x_2 = -98.3158 + 100 = 1.6842$

8

(a)

Primal	Dual
min $z = 5x_1 + 2x_2$	max $w = 3y_1 + 5y_2$
s.t.	s.t.
$x_1 - x_2 \ge 3$	$y_1 + 2y_2 \le 5$
$2x_1 + 3x_2 \ge 5$	$-y_1 + 3y_2 \le 2$
$x_1, x_2 \ge 0$	$y_1 \ge 0$
	$y_2 \ge 0$
Feasible solutions:	
$x_1 = 3, x_2 = 0, z = 15$	$y_1 = 3, y_2 = 1, w = 14$

Range: $14 \le$ Optimum value ≤ 15

(b)

max $z = x_1 + 5x_2 + 3x_3$	min $w = 3y_1 + 4y_2$
s.t.	s.t.
$x_1 + 2x_2 + x_3 = 3$	$y_1 + 2y_2 \ge 1$
$2x_1 - x_2 = 4$	$2y_1 - y_2 \ge 5$
$x_1, x_2, x_3 \ge 0$	$y_1 \ge 3$
	y_2 unrestricted
Feasible solutions:	
$x_1 = 2, x_2 = 0, x_3 = 1$	$y_1 = 3, y_2 = 0$,
$z = 5$	$w = 9$

Range: $5 \le$ optimum value ≤ 9

continued...

(c)

max $z = 2x_1 + x_2$	min $w = 10y_1 + 40y_2$
s.t.	s.t.
$x_1 - x_2 \le 10$	$y_1 + 2y_2 \ge 2$
$2x_1 \le 40$	$-y_1 \ge 1$
$x_1, x_2 \ge 0$	$y_1 \ge 0$
	$y_2 \ge 0$
Feasible Solution:	
$x_1 = 20,\ x_2 = 20$	No feasible solution.
$z = 60$	

Primal is unbounded because the primal is feasible and the dual has no feasible solution.

(d)

max $z = 3x_1 + 2x_2$	min $w = 3y_1 + 12y_2$
s.t.	s.t.
$2x_1 + x_2 \le 3$	$2y_1 + 3y_2 \ge 3$
$3x_1 + 4x_2 \le 12$	$y_1 + 4y_2 \ge 2$
$x_1, x_2 \ge 0$	$y_1, y_2 \ge 0$
Feasible solutions:	
$x_1 = x_2 = 1$	$y_1 = 2,\ y_2 = 0$
$z = 5$	$w = 6$

Range: $5 \le$ optimum value ≤ 6

9

min $z = 5x_1 + 2x_2$	max $w = 3y_1 + 5y_2$
s.t.	s.t.
$x_1 - x_2 \ge 3$	$y_1 + 2y_2 \le 5$
$2x_1 + 3x_2 \ge 5$	$-y_1 + 3y_2 \le 2$
$x_1, x_2 \ge 0$	$y_1, y_2 \ge 0$

(a) ($x_1 = 3,\ x_2 = 1;\ y_1 = 4,\ y_2 = 1$):

Both primal and dual are infeasible

(b) ($x_1 = 4,\ x_2 = 1;\ y_1 = 1,\ y_2 = 0$):

Primal feasible, $z = 22$

Dual feasible, $w = 3$

Since $z \ne w$, solutions are not optimal.

(c) ($x_1 = 3,\ x_2 = 0;\ y_1 = 5,\ y_2 = 0$):

Primal feasible, $z = 15$

Dual feasible, $w = 15$

Since $z = w$, solutions are optimal

Set 4.2d

1

From TORA using $M = 100$:

	x_1	x_2	x_3	x_4	x_5	
z	-205	88	-304	0	0	-800
x_4	1	2	1	1	0	10
x_5	2	-1	3	0	1	8
z	$-7/3$	$-40/3$	0	0	$304/3$	$32/3$
x_4	$1/3$	$7/3$	0	1	$-1/3$	$22/3$
x_3	$2/3$	$-1/3$	1	0	$1/3$	$8/3$

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Minimize $w = 10y_1 + 8y_2$
s.t. $x_1 + 2x_2 + x_3 \le 10$	s.t. $y_1 + 2y_2 \ge 5$
$2x_1 - x_2 + 3x_3 = 8$	$2y_1 - y_2 \ge 12$
$x_1, x_2, x_3 \geqslant 0$	$y_1 + 3y_2 \ge 4$
	$y_1 \ge 0$
	y_2 unrestricted

Iteration 1: x_5 artificial, $M = 100$

$$\text{Inverse} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix},\ C_B = (0, 4)$$

Constraints:

$$LHS = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 7/3 & 0 & 1 & -1/3 \\ 2/3 & -1/3 & 1 & 0 & 1/3 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} 22/3 \\ 8/3 \end{pmatrix}$$

Objective row:

$$\text{Dual values } (y_1, y_2) = (0,4)\begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = (0, 4/3)$$

Variable	Objective coefficient
x_1	$y_1 + 2y_2 - 5 = 0 + 2(4/3) - 5 = -7/3$
x_2	$2y_1 - y_2 - 12 = 2(0) - (4/3) - 12 = -40/3$
x_3	$y_1 + 3y_2 - 4 = 0 + 3(4/3) - 4 = 0$
x_4	$y_1 - 0 = 0 - 0 = 0$
x_5	$y_2 - (-M) = 4/3 - (-100) = 304/3$

2

Dual:

Minimize $w = 21y_1 + 21y_2$

Subject to

$$2y_1 + 7y_2 \ge 4$$
$$7y_1 + 2y_2 \ge 14$$
$$y_1, y_2 \ge 0$$

(a) $$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix} \Rightarrow \text{feasible}$$

$$(y_1, y_2) = (14, 0)\begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix} = (2, 0)$$

$$\left.\begin{aligned} &\text{obj coeff } x_1 = 2y_1 + 7y_2 - 4 \\ &\qquad = 2\times 2 + 7\times 0 - 4 = 0 \\ &\text{obj coeff of } x_3 = y_1 - 0 = 2 - 0 = 2 \end{aligned}\right\} \Rightarrow \text{optimal}$$

(b) Feasibility:

continued…

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 10.5 \\ -\frac{105}{2} \end{pmatrix} \Rightarrow \text{infeasible}$$

Optimality:

$$(y_1, y_2) = (14, 0)\begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix} = (0, 7)$$

obj coeff of x_1: $2y_1 + 7y_2 - 4 = 2\times0 + 7\times7 - 4 = 45 > 0$

obj coeff of x_4: $y_2 - 0 = 7 - 0 > 0$

Solution is optimal but infeasible

(c) Feasibility:

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 7/45 & -2/45 \\ -2/45 & 7/45 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 7/3 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2) = (14, 4)\begin{pmatrix} 7/45 & -2/45 \\ -2/45 & 7/45 \end{pmatrix} = (2, 0)$$

$$\left.\begin{aligned} &\text{obj coeff of } x_3: y_1 - 0 = 2 - 0 > 0 \\ &\text{obj coeff of } x_4: y_2 - 0 = 0 - 0 = 0 \end{aligned}\right\} \text{optimal}$$

Solution is optimal and feasible

(d) Feasibility:

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -7/2 & 1 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} \frac{21}{2} \\ -\frac{105}{2} \end{pmatrix} \Rightarrow \text{infeasible}$$

Optimality:

$$(y_1, y_2) = (4, 0)\begin{pmatrix} 1/2 & 0 \\ -7/2 & 1 \end{pmatrix} = (2, 0)$$

$$\left.\begin{aligned} &\text{obj coeff of } x_2: 7y_1 + 2y_2 - 14 = 0 \\ &\text{Obj coeff of } x_3: y_1 - 0 = 2 - 0 = 2 \end{aligned}\right\} \text{optimal}$$

Solution optimal but infeasible

3

Dual:

minimize $w = 30y_1 + 60y_2 + 20y_3$

subject to

$$y_1 + 3y_2 + y_3 \ge 3$$
$$2y_1 + 4y_3 \ge 2$$
$$y_1 + 2y_2 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

(a) Feasibility:

$$\begin{pmatrix} x_4 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 50 \\ 20 \end{pmatrix} \text{ feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (0, 5, 0)\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0, 5/2, 0)$$

Obj coeff of x_1: $y_1 + 3y_2 + y_3 - 3 = 0 + 3(\frac{5}{2}) + 0 - 3 = 9/2$

Obj coeff of x_2: $2y_1 + 4y_3 - 2 = 2\times0 + 4\times0 - 2 = -2 < 0$

Solution feasible but not optimal

continued…

b) Feasibility:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 15 \\ 10 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (2, 5, 3) \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix} = (5, 0, -2)$$

obj coeff of x_4 : $y_1 - 0 = 5$

obj. coeff of x_5 : $y_2 - 0 = 0$

obj. coeff of x_6 : $y_3 - 0 = -2 \Rightarrow$ not optimal

(c) Feasibility:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \\ 20 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (2, 5, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} = (1, 2, 0)$$

$$\left.\begin{array}{l} \text{obj coeff of } x_1 : y_1 + 3y_2 + y_3 - 3 = 1 + 6 + 0 - 3 = 4 \\ \text{obj coeff of } x_4 : y_1 - 0 = 1 - 0 = 1 \\ \text{obj coeff of } x_5 : y_2 - 0 = 2 - 0 = 2 \end{array}\right\} \text{optimal}$$

4

Constraints:

$$LHS = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 4 & 3 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -3/5 & 1/5 & 0 \\ 0 & 1 & 4/5 & -3/5 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

Objective coefficients:

$$(y_1, y_2, y_3) = (2, 1, 0) \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= (2/5, 1/5, 0)$$

obj coeff of $x_3 = -y_1 - 0 = -2/5$

obj coeff of $x_4 = -y_2 - 0 = -1/5$

$z = 2 \times 3/5 + 1 \times 6/5 = 12/5$

	x_1	x_2	x_3	x_4	x_5	
z	0	0	$-2/5$	$-1/5$	0	$12/5$
x_1	1	0	$-3/5$	$1/5$	0	$3/5$
x_2	0	1	$4/5$	$-3/5$	0	$6/5$
x_5	0	0	-1	1	1	0

continued…

5

(a)

(i) $$\begin{pmatrix} x_4 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 28/3 \\ 2/3 \end{pmatrix}$$

$z = 4 \times 2/3 = 8/3$

(ii) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 18/5 \\ 14/5 \end{pmatrix}$$

$z = 5 \times \frac{14}{5} + 12 \times \frac{18}{5} = \boxed{57.2}$

(iii) $$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3/7 & -1/7 \\ 1/7 & 2/7 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$z = 12 \times 4 + 4 \times 2 = 56$

Solution in (b) is the best

(b)

$$y_1, y_2 = (12, 5) \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} = \left(\frac{29}{5}, -\frac{2}{5}\right)$$

obj coeff of x_3 : $y_1 + 3y_2 - 4 = \frac{29}{5} + 3\left(-\frac{2}{5}\right) - 4 = \frac{3}{5}$

obj coeff of x_4 : $y_1 - 0 = \frac{29}{5} - 0 = \frac{29}{5}$

Solution is optimal.

6

$$\text{Inverse} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

(a) $$\begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 10 \end{pmatrix}$$

Thus, $b_1 = 30$, $b_2 = 40$

(b) Optimal dual solution:

$$(y_1, y_2) = (5, 0) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (5, 0)$$

(c) $(d, e) = (y_1, y_2) = (5, 0)$

$a = 5y_1 - 5y_2 - 2 = 5 \times 5 - 5 \times 0 - 2 = 23$

$$\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$$

7

objective value:

in dual $= b_1 y_1 + b_2 y_2 + b_3 y_3$

in primal $= c_1 x_1 + c_2 x_2$

$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

Thus, $b_1 = 4$, $b_2 = 6$, $b_3 = 8$

continued…

Set 4.2d

$$(y_1, y_2, y_3) = (0, C_2, C_1)\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= (0, C_2 - C_1, C_1)$$

$$\left.\begin{array}{l} \text{Obj coeff of } x_3 = 0 = y_1 - 0 \\ \text{Obj coeff of } x_4 = 3 = y_2 - 0 \\ \text{Obj coeff of } x_5 = 2 = y_3 - 0 \end{array}\right\} y_1 = 0, y_2 = 3, y_3 = 2$$

Thus, $C_2 - C_1 = 3$ and $C_1 = 2 \Rightarrow C_1 = 2, C_2 = 5$

Now we can determine the objective value as follows:

$$\text{Dual} = b_1 y_1 + b_2 y_2 + b_3 y_3$$
$$= 4 \times 0 + 6 \times 3 + 8 \times 2 = 34$$

$$\text{Primal} = C_1 X_1 + C_2 X_2$$
$$= 2 \times 2 + 5 \times 6 = 34$$

8

<u>Dual</u>:

Minimize $w = 4y_1 + 8y_2$

subject to

$$\begin{array}{l} y_1 + y_2 \geq 2 \\ y_1 + 4y_2 \geq 4 \\ y_1 \qquad\ \ \geq 4 \\ \qquad y_2 \geq -3 \end{array}$$

For basic (x_1, x_2), we have

$$\left.\begin{array}{l} y_1 + y_2 - 2 = 0 \\ y_1 + 4y_2 - 4 = 0 \end{array}\right\} \Rightarrow y_1 = \frac{4}{3}, y_2 = \frac{2}{3}$$

Obj coeff of $x_3 = y_1 - 4 = \frac{4}{3} - 4 = -\frac{8}{3} < 0$

The result shows that the solution is not optimal.

9

For a slack starting basic variable, the dual constraint is of the form

$$y \geq 0$$

(assuming primal maximization).

Thus,

Optimal obj coeff. of basic variable $= y - 0$

For artificial starting basic variable, the dual constraint is $y \geq -M$ if the primal is maximization, and $y \leq M$ if the primal is minimization.

Thus,

$$\text{Optimal obj coeff} = \begin{cases} y + M, & \text{for maximization} \\ y - M, & \text{for minimization} \end{cases}$$

1

From TORA output:

	y_1	y_2	y_3	y_4
	.75	.5	0	0
Range:	(20,36)	(4,6.7)	(-1.5,∞)	(1.5,∞)

(a) $\$750 \times (22-24) = -\$1500$

(b) $\Delta Z = \$500(4.5-6) = -\$750$

(c) $\Delta Z = \$0\,(10-2) = \$0$

2

x_1, x_2, x_3, x_4 = daily units of cables 320, 325, 340, and 370

(a) Maximize $z = 9.4x_1 + 10.8x_2 + 8.75x_3 + 7.8x_4$

subject to

$$10.5x_1 + 9.3x_2 + 11.6x_3 + 8.2x_4 \le 4800$$
$$20.4x_1 + 24.6x_2 + 17.7x_3 + 26.5x_4 \le 9600$$
$$3.2x_1 + 2.5x_2 + 3.6x_3 + 5.5x_4 \le 4700$$
$$5x_1 + 5x_2 + 5x_3 + 5x_4 \le 4500$$
$$x_1 \ge 100,\ x_2 \ge 100,\ x_3 \ge 100,\ x_4 \ge 100$$

```
                        *** OPTIMUM SOLUTION SUMMARY ***

--------------------------------------------------------------------------
Title:
Final iteration No: 3
Objective value (max) = 4011.1582
-----------------------------------------------------------------
Variable               Value       Obj Coeff    Obj Val Contrib
-----------------------------------------------------------------
x1                  100.0000          9.4000           939.9999
x2                  100.0000         10.8000          1080.0000
x3                  138.4181          8.7500          1211.1582
x4                  100.0000          7.8000           780.0000
-------------------------------------------------------------
Constraint              RHS          Slack(-)/Surplus(+)
-------------------------------------------------------------
1 (<)              4800.0000            394.3503-
2 (<)              9600.0000              0.0000-
3 (<)              4700.0000           3081.6948-
4 (<)              4500.0000           2307.9097-
LB-x1               100.0000              0.0000+
LB-x2               100.0000              0.0000+
LB-x3               100.0000             38.4181+
LB-x4               100.0000              0.0000+

                          *** SENSITIVITY ANALYSIS ***
Objective coefficients -- Single Changes:
--------------------------------------------------------------------------
Variable      Current Coeff       Min Coeff       Max Coeff     Reduced Cost
--------------------------------------------------------------------------
x1                   9.4000       -infinity         10.0847           0.6847
x2                  10.8000       -infinity         12.1610           1.3610
x3                   8.7500          8.1559        infinity           0.0000
x4                   7.8000       -infinity         13.1003           5.3003

Right-hand Side -- Single Changes:
--------------------------------------------------------------------------
Constraint       Current RHS         Min RHS         Max RHS       Dual Price
--------------------------------------------------------------------------
1 (<)              4800.0000       4405.6497        infinity           0.0000
2 (<)              9600.0000       8919.9999      10201.7242           0.4944
3 (<)              4700.0000       1618.3052        infinity           0.0000
4 (<)              4500.0000       2192.0903        infinity           0.0000
LB-x1               100.0000          0.0000        133.3333          -0.6847
LB-x2               100.0000         42.1946        127.6423          -1.3610
LB-x3               100.0000       -infinity        138.4181           0.0000
LB-x4               100.0000         56.9826        125.6604          -5.3003
```

continued...

(b) Only soldering capacity can be increased because its dual price is positive.

(c) The fact that the dual prices of the lower bounds on x_1, x_2, and x_4 are negative shows that the lower bounds have adverse effect on profitability. Specifically, one unit decrease in the production of cables SC320, SC325, and SC370 will respectively increase the profit by \$.68, \$1.36, and \$5.30 per cable. These values are valid considering the cables one at a time.

(d) Dual price for soldering is \$.49 per minute, valid in the range (8920, 10201.7) minutes. Hence, the \$.49 additional profit per minute is guaranteed only for up to $\frac{10201-9600}{9600} = 6.26\%$ capacity increase.

3

x_1 = number of jackets per week

x_2 = number of handbags per week

Maximize $z = 350x_1 + 120x_2$

Subject to

$$8x_1 + 2x_2 \le 1200$$
$$12x_1 + 5x_2 \le 1850$$
$$x_1, x_2 \ge 0$$

TORA optimum solution:

$x_1 \cong 144,\ x_2 = 25,\ z = \$53312.50$

Resource	Dual price	Range
Leather	\$19.38/m²	(740,1233.33)
Labor	\$16.25/hr	(1800,3000)

BagCo should not pay more than \$19.38/m² of leather and \$16.25/hr of labor time.

Set 4.3b

1

Dual prices: $y_1=1,\ y_2=2,\ y_3=0$
all in \$/min

$$(1-r_1)\,y_1 + 1.25\,y_2 + y_3 \ge 3$$

Reduced cost of $x_2 = (1-r_1)\times1 + 1.25\times2 + 1\times0 - 3$
$= .5 - r_1$

For x_1 to be just profitable, its reduced cost must be (at least) zero, that is, $.5 - r_1 \le 0$ or $r_1 \ge .5$. This means a reduction of at least 50%

2

Dual constraint for fire trucks:

$$y_2 + 3y_3 \ge 4$$

Reduced cost $= y_2 + 3y_3 - 4$
$= 1\times2 + 3\times0 - 4 = -2 < 0$

New toy is recommended.

3

x_j = number of units of PP_j, $j=1,2,3,4$

Maximize $z = 3x_1 + 6x_2 + 5x_3 + 4x_4$
subject to

$$2x_1 + 5x_2 + 3x_3 + 4x_4 \le 5300$$
$$3x_1 + 4x_2 + 6x_3 + 4x_4 \le 5300$$
$$x_1, x_2, x_3, x_4 \ge 0$$

```
                         *** OPTIMUM SOLUTION SUMMARY ***
--------------------------------------------------------------------------
Title: Problem 4.4b-3
Final iteration No: 4
Objective value (max) = 6814.2856
------------------------------------------------------------
Variable              Value      Obj Coeff    Obj Val Contrib
------------------------------------------------------------
x1                  757.1429      3.0000        2271.4287
x2                  757.1428      6.0000        4542.8569
x3                    0.0000      5.0000           0.0000
x4                    0.0000      4.0000           0.0000
---------------------------------------------------------
Constraint            RHS           Slack(-)/Surplus(+)
---------------------------------------------------------
1 (<)              5300.0000            0.0000-
2 (<)              5300.0000            0.0000-

                          *** SENSITIVITY ANALYSIS ***
Objective coefficients -- Single Changes:
--------------------------------------------------------------------------
Variable       Current Coeff     Min Coeff      Max Coeff      Reduced Cost
--------------------------------------------------------------------------
x1                3.0000          2.9444         4.5000          0.0000
x2                6.0000          4.0000         6.3333          0.0000
x3                5.0000         -infinity       5.1429          0.1429
x4                4.0000         -infinity       5.1429          1.1429

Right-hand Side -- Single Changes:
--------------------------------------------------------------------------
Constraint        Current RHS       Min RHS       Max RHS        Dual Price
--------------------------------------------------------------------------
1 (<)              5300.0000      3533.3334     6625.0000          0.8571
2 (<)              5300.0000      4240.0000     7949.9998          0.4286
```

continued...

From TORA solution:

Variable	Reduced cost
x_3	.1429
x_4	1.1429

Thus,

$$\begin{pmatrix}\text{Rate of deterioration in } z\\ \text{per unit of } x_3\end{pmatrix} = \$.14$$

$$\begin{pmatrix}\text{Rate of deterioration in}\\ z \text{ per unit of } x_4\end{pmatrix} = \$1.14$$

4

Resource	Dual price	Range
Lathe	\$.8571	(5333.33, 6625)
Drill	\$.4286	(4240, 7950)

Reduced cost for x_3
$= .8(3y_1 + 6y_2) - 5$
$= .8(3\times.8571 + 6\times.4286) - 5$
$= -.8857 < 0$

Reduced cost for x_4
$= .8(4y_1 + 4y_2) - 4$
$= .8(4\times.8571 + 4\times.4286) - 4$
$= .1142 > 0$

Only PP_3 will be profitable. PP_4 needs more than

$$1 - \frac{4}{4\times.8571 + 4\times.4286} = 22.2\%$$

improvement to be profitable

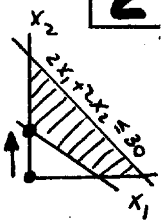
1

(a) No, because A is feasible.
(b) No, because E is feasible. Dual simplex iterations remain infeasible until the last iteration is reached.
(c) L → I → F.

2

(a)
Minimize $z = 2x_1 + 3x_2$
subject to

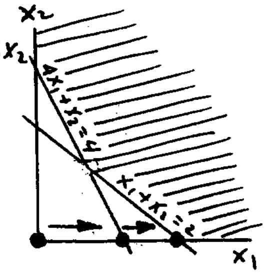
$$2x_1 + 2x_2 \le 30$$
$$-x_1 - 2x_2 \le -10$$
$$x_1,\ x_2 \ge 0$$



Basic	x_1	x_2	x_3	x_4	Sol^n
z	-2	-3	0	0	0
x_3	2	2	1	0	30
x_4	-1	$\boxed{-2}$	0	1	-10
z	-1/2	0	0	-3/2	15
x_3	1	0	1	1	20
x_2	1/2	1	0	-1/2	5

(b)
Minimize $z = 5x_1 + 6x_2$
subject to

$$-x_1 - x_2 \le -2$$
$$-4x_1 - x_2 \le -4$$
$$x_1,\ x_2 \ge 0$$

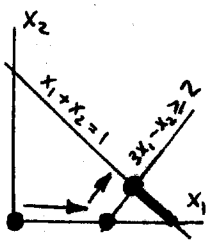


Basic	x_1	x_2	x_3	x_4	Sol^n
z	-5	-6	0	0	0
x_3	-1	-1	1	0	-2
x_4	$\boxed{-4}$	-1	0	1	-4
z	0	-19/4	0	-5/4	5
x_3	0	-3/4	1	-1/4	-1
x_1	1	1/4	0	-1/4	1
z	0	-1	-5	0	10
x_4	0	3	-4	1	4
x_1	1	1	-1	0	2

continued…

(c)
Minimize $z = 4x_1 + 2x_2$
subject to

$$x_1 + x_2 \le 1$$
$$x_1 + x_2 \ge 1$$
$$3x_1 - x_2 \ge 2$$
$$x_1,\ x_2 \ge 0$$

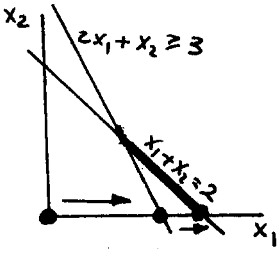


(Convert the equation into two inequalities to fit the dual simplex format.)

Basic	x_1	x_2	x_3	x_4	x_5	Sol^n
z	-4	-2	0	0	0	0
x_3	1	1	1	0	0	1
x_4	-1	-1	0	1	0	-1
x_5	$\boxed{-3}$	1	0	0	1	-2
z	0	-4/3	0	0	-4/3	8/3
x_3	0	4/3	1	0	1/3	1/3
x_4	0	$\boxed{-4/3}$	0	1	-1/3	-1/3
x_1	1	-1/3	0	0	-1/3	2/3
z	0	0	0	-5/2	-1/2	7/2
x_3	0	0	1	1	0	0
x_2	0	1	0	-3/4	1/4	1/4
x_1	1	0	0	-1/4	-1/4	3/4

(d)
Minimize $z = 2x_1 + 3x_2$
Subject to

$$2x_1 + x_2 \ge 3$$
$$x_1 + x_2 \le 2$$
$$x_1 + x_2 \ge 2$$
$$x_1,\ x_2 \ge 0$$



Basic	x_1	x_2	x_3	x_4	x_5	Sol^n
z	-2	-3	0	0	0	0
x_3	$\boxed{-2}$	-1	1	0	0	-3
x_4	1	1	0	1	0	2
x_5	-1	-1	0	0	1	-2
z	0	-2	-1	0	0	3
x_1	1	1/2	-1/2	0	0	3/2
x_4	0	1/2	1/2	1	0	1/2
x_5	0	-1/2	$\boxed{-1/2}$	0	1	-1/2
z	0	-1	0	0	-2	4
x_1	1	1	0	0	-1	2
x_4	0	0	0	1	1	0
x_3	0	1	1	0	-2	1

Set 4.4a

3

Add the constraints $x_1 + x_3 \le M$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
z	-2	1	-1	0	0	0	0	0
s_1	-2	-3	5	1	0	0	0	-4
s_2	1	-9	1	0	1	0	0	-3
s_3	4	6	3	0	0	1	0	8
s_4	$\boxed{1}$	0	1	0	0	0	1	M
z	0	0	1	0	0	0	2	2M
s_1	0	-3	7	1	0	0	2	-4+2M
s_2	0	-9	0	0	1	0	-1	-3-M
s_3	0	6	-1	0	0	1	$\boxed{-4}$	8-4M
x_1	1	0	1	0	0	0	1	M

The second tableau is now optimal but infeasible. We can thus apply the dual simplex to the second tableau. Optimal solution is:

$x_1 = 1.286$, $x_2 = .476$, $x_3 = 0$

$z = 2.095$

4

(a) add the constraint $x_3 \le M$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	0	0	-2	0	0	0	0	0
x_4	1	-2	2	1	0	0	0	-8
x_5	-1	1	1	0	1	0	0	4
x_6	2	-1	4	0	0	1	0	10
x_7	0	0	$\boxed{1}$	0	0	0	1	M
z	0	0	0	0	0	0	2	2M
x_4	1	-2	0	1	0	0	-2	-8-2M
x_5	-1	1	0	0	1	0	-1	4-M
x_6	2	$\boxed{-1}$	0	0	0	1	-4	10-4M
x_7	0	0	1	0	0	0	1	M

Last tableau is optimal but infeasible: application of the dual simplex method yields the solution:

$x_1 = 56/9$, $x_2 = 26/3$, $x_3 = 14/9$

$z = 28/9$

continued...

(b) Add the constraint $x_1 \le M$

	x_1	x_2	s_1	s_2	s_3	s_4	
z	-1	3	0	0	0	0	0
s_1	1	-1	1	0	0	0	2
s_2	-1	-1	0	1	0	0	-4
s_3	-2	2	0	0	1	0	-3
s_4	$\boxed{1}$	0	0	0	0	1	M
z	0	3	0	0	0	1	M
s_1	0	-1	1	0	0	$\boxed{-1}$	2-M
s_2	0	-1	0	1	0	1	-4+M
s_3	0	2	0	0	1	2	-3+2M
x_1	1	0	0	0	0	1	M

Optimum: $x_1 = 3$, $x_2 = 1$ $z = 0$

(c) Add the constraint $x_1 \le M$

	x_1	x_2	s_1	s_2	s_3	s_4	
z	1	-1	0	0	0	0	0
s_1	-1	4	1	0	0	0	-5
s_2	1	-3	0	1	0	0	1
s_3	-2	5	0	0	1	0	-1
s_4	$\boxed{1}$	0	0	0	0	1	M
z	0	-1	0	0	0	-1	-M
s_1	0	4	1	0	0	1	-5+M
s_2	0	$\boxed{-3}$	0	1	0	-1	1-M
s_3	0	5	0	0	1	2	-1+2M
x_1	1	0	0	0	0	1	M

Problem has no feasible solution

(d) Add the constraint $x_3 \le M$

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
z	0	0	-2	0	0	0	0	0
s_1	1	-3	7	1	0	0	0	-5
s_2	-1	1	-1	0	1	0	0	1
s_3	3	1	-10	0	0	1	0	8
s_4	0	0	1	0	0	0	1	M
z	0	0	0	0	0	0	2	2M
s_1	1	$\boxed{-3}$	0	1	0	0	-7	-5-7M
s_2	-1	1	0	0	1	0	1	1+M
s_3	3	1	0	0	0	1	10	8+10M
s_4	0	0	1	0	0	0	1	M

Solution is unbounded

5

Method 1: M-technique (or two-phase method)

Starting tableau:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	R_1	R_2	R_3	Sol^n
z	-6	-7	-3	-5	0	0	0	-M	-M	-M	–
R_1	5	6	-3	4	-1	0	0	1	0	0	12
R_2	0	1	-5	-6	0	-1	0	0	1	0	10
R_3	2	5	1	1	0	0	-1	0	0	1	8

Method 2: Solve the dual problem

Starting tableau:

Basic	y_1	y_2	y_3	s_1	s_2	s_3	s_4	
w	-12	-10	-8	0	0	0	0	0
s_1	5	0	2	1	0	0	0	6
s_2	6	1	5	0	1	0	0	7
s_3	-3	-5	1	0	0	1	0	3
s_4	4	-6	1	0	0	0	1	5

Method 3: Dual simplex

starting tableau:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Sol^n
z	-6	-7	-3	-5	0	0	0	0
s_1	-5	-6	3	-4	1	0	0	-12
s_2	0	-1	5	6	0	1	0	-10
s_3	-2	-5	-1	-1	0	0	1	-8

Optimal solution: $x_1 = 0, x_2 = 10, x_3 = x_4 = 0$
$z = 70$

Method	Number of iterations
1	5
2	3
3	

The dual simplex is the best. It follows because it requires the smallest number of iterations and has the smallest number of constraints.

Set 4.4b

1

Basic	x_1	x_2	x_3	x_4	x_5	
z	1	-1	0	0	0	0
x_3	$\boxed{-1}$	4	1	0	0	-5
x_4	1	-3	0	1	0	1
x_5	-2	5	0	0	1	-1
z						
x_1	1	-4	-1	0	0	5
x_4	0	1	1	1	0	-4
x_5	0	-3	-2	0	1	9

In the second iteration, row 2 has all nonnegative coefficients on the left-hand side. This means that the infeasibility of x_4 cannot be removed, and the problem has no feasible solution.

2

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	0	-2	0	0	0	0
x_4	1	$\boxed{-3}$	7	1	0	0	-5
x_5	-1	1	-1	0	1	0	1
x_6	3	1	-10	0	0	1	8
z	0	0	-2	0	0	0	0
x_2	-1/3	1	-7/3	-1/3	0	0	5/3
x_5	$\boxed{-2/3}$	0	4/3	1/3	1	0	-2/3
x_6	10/3	0	-23/3	1/3	0	1	19/3
z			-2				0
x_1			-4/3				2
x_1			-2				1
x_6			-1				3

Iteration 3 is feasible but nonoptimal. However, x_3 shows that the solution is unbounded.

1

$$\text{new RHS} = \begin{pmatrix} 430 \\ 480 \\ 400 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 430 \\ 480 \\ 400 \end{pmatrix} = \begin{pmatrix} 95 \\ 240 \\ 20 \end{pmatrix}$$

The new solution is feasible with $x_1 = 0$, $x_2 = 95$, $x_3 = 240$. $z = 3 \times 0 + 2 \times 95 + 5 \times 240 = \$1390$, which is better than the current value of z

2

(a)
$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 460 \\ 500 \\ 400 \end{pmatrix} = \begin{pmatrix} 105 \\ 250 \\ -20 \end{pmatrix}$$

Solution is infeasible

	x_1	x_2	x_3	x_4	x_5	x_6	
z	4	0	0	1	2	0	1460
x_2	-1/4	1	0	1/2	-1/4	0	105
x_3	3/2	0	1	0	1/2	0	250
x_6	2	0	0	$\boxed{-2}$	1	1	-20
z	5	0	0	0	5/2	1/2	1450
x_2	1/4	1	0	0	0	1/4	100
x_3	3/2	0	1	0	1/2	0	250
x_4	-1	0	0	1	-1/2	-1/2	10

(b)
$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 500 \\ 400 \\ 600 \end{pmatrix} = \begin{pmatrix} 150 \\ 200 \\ 0 \end{pmatrix}$$

New solution is feasible. $z = \$1300$

(c)
$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 300 \\ 800 \\ 200 \end{pmatrix} = \begin{pmatrix} -50 \\ 400 \\ 400 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
z	4	0	0	1	2	0	1900
x_2	-1/4	1	0	1/2	-1/4	0	-50
x_3	3/2	0	1	0	1/2	0	400
x_6	2	0	0	-2	1	1	400
z	2	8	0	5	0	0	1500
x_5	1	-4	0	-2	1	0	200
x_3	1	2	1	1	0	0	300
x_6	1	4	0	0	0	1	200

continued...

(d)
$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 700 \\ 350 \end{pmatrix} = \begin{pmatrix} 50 \\ 350 \\ 150 \end{pmatrix}$$

Solution is feasible. $z = \$1850$

3

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 28 \\ 8 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5/2 \\ 3/2 \\ -1 \end{pmatrix}$$

	x_1	x_2	s_1	s_2	s_3	s_4	
z	0	0	3/4	1/2	0	0	25
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	5/2
s_3	0	0	3/8	-5/4	1	0	3/2
s_4	0	0	1/8	$\boxed{-3/4}$	0	1	-1
z	0	0	5/6	0	0	2/3	$24\frac{2}{3}$
x_1	1	0	1/6	0	0	-2/3	10/3
x_2	0	1	0	0	0	1	2
s_3	0	0	1/6	0	1	-5/3	7/3
s_2	0	0	-1/6	1	0	-4/3	2/3

4

x_1 = lb limestone in weekly mix
x_2 = lb corn in weekly mix
x_3 = lb soybean meal in weekly mix

Minimize $z = .12x_1 + .45x_2 + 1.6x_3$

s.t.

$$\begin{aligned} x_1 + x_2 + x_3 &\geq Q \\ .38x_1 + .001x_2 + .002x_3 &\geq .008(x_1+x_2+x_3) \\ .38x_1 + .001x_2 + .002x_3 &\leq .012(x_1+x_2+x_3) \\ .09x_2 + .5x_3 &\geq .22(x_1+x_2+x_3) \\ .02x_2 + .08x_3 &\leq .05(x_1+x_2+x_3) \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Q = weekly mix

The constraints simplify to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq Q \\ .372x_1 - .007x_2 - .006x_3 &\geq 0 \\ .368x_1 - .011x_2 - .01x_3 &\leq 0 \\ -.22x_1 - .13x_2 + .28x_3 &\geq 0 \\ -.05x_1 - .03x_2 + .03x_3 &\leq 0 \end{aligned}$$

Week	1	2	3	4	5	6	7	8
Q (lb)	5200	9600	15000	20000	26000	32000	38000	42000

continued...

Set 4.5a

First, we solve the problem using $Q = 5200$ lb, feed requirements for week 1. Then we use sensitivity analysis for the remaining weeks.

<u>Week 1 solution</u> (using TORA)

$$\begin{pmatrix}\text{Basic}\\ \text{vector}\end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ Sx_5 \\ x_3 \\ sx_{11} \end{pmatrix}, \quad Z = \$4224.74$$

$$\text{inverse} = \begin{pmatrix} .649 & 0 & -3.216 & -2.431 & 0 \\ .028 & 0 & 2.637 & -.006 & 0 \\ .004 & -1 & 1.000 & .000 & 0 \\ .323 & 0 & .579 & 2.438 & 0 \\ .011 & 0 & .018 & .146 & 1 \end{pmatrix}$$

Solution given Q:

$$\begin{pmatrix} x_2 \\ x_1 \\ Sx_5 \\ x_3 \\ sx_{11} \end{pmatrix} = (\text{inverse}) \begin{pmatrix} Q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} .649Q \\ .028Q \\ .004Q \\ .323Q \\ .011Q \end{pmatrix}$$

<u>General solution</u>:

$$x_1 = .028Q$$
$$x_2 = .649Q$$
$$x_3 = .323Q$$

$$Z = (.12 \times .028 + .45 \times .649 + 1.6 \times .323)Q$$
$$= .81221\,Q$$

5

B^{-1} = inverse

D_i = change in RHS of constraint i, $i = 1, 2, \dots, m$

<u>Simultaneous</u> feasibility conditions:

$$B^{-1}\begin{pmatrix} b_1 + D_1 \\ \vdots \\ b_m + D_m \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad ①$$

Let $p_i \leq D_i \leq q_i$ be the feasibility range computed from the <u>single</u>-change conditions:

$$B^{-1}\begin{pmatrix} b_1 \\ \vdots \\ b_i + D_i \\ \vdots \\ b_m \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad ②$$

Define
$$\Delta_i = \begin{cases} p_i, & \text{if } D_i < 0 \\ q_i, & \text{if } D_i > 0 \end{cases}$$

Condition ② holds true for $D_i = \Delta_i$ also.

Now, define $r_i \geq 0$, $i = 0, 1, 2, \dots, m$ such that $r_0 + r_1 + \dots + r_m = 1$ Then

$$B^{-1}\left[r_0 \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix} + r_1 \begin{pmatrix} b_1 + \Delta_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix} + \dots + r_m \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m + \Delta_m \end{pmatrix} \right]$$

must also be feasible. The last expression reduces to

$$B^{-1}\left[\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} + \begin{pmatrix} r_1\Delta_1 \\ \vdots \\ r_m\Delta_m \end{pmatrix} \right] \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad ③$$

Next, select $r_i = \frac{D_i}{\Delta_i}$, $i = 1, 2, \dots, m$. Then ③ is the same as condition ①. However, because $r_0 + r_1 + \dots + r_m = 1$, it must be true that $r_1 + r_2 + \dots + r_m \leq 1$. The condition

$$r_1 + r_2 + \dots + r_m \leq 1$$

thus implies that ③, and hence ①, is feasible. The condition is not sufficient because ③ can be satisfied for arbitrary values of $r_0, r_1, \dots,$ and r_m.

6

(a)

$$B^{-1} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

$$Y = (1, 4, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

$$= (-1/4, 5/2, 0, 0)$$

$$X_B = B^{-1} b$$

$$= \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 28 \\ 8 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5/2 \\ 3/2 \\ -1/2 \end{pmatrix}$$

The simplex tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	Soluti
Z	0	0	-1/4	5/2	0	0	13
x_1	1	0	1/4	-1/2	0	6	3
x_2	0	1	-1/8	3/4	0	0	5/2
x_5	0	0	$\boxed{3/8}$	-5/4	1	0	3/2
x_6	0	0	1/8	-3/4	0	1	-1/2

The tableau is both nonoptimal and infeasible.

(b) Apply the primal simplex to the tableau above, disregarding the x_6-row in the ratio test. Thus, x_3 enters the basic solution and x_5 leaves. The resulting tableau is

continued...

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	0	0	5/3	2/3	0	14
x_1	1	0	0	1/3	-2/3	0	2
x_2	0	1	0	1/3	1/3	0	3
x_3	0	0	1	-10/3	8/3	0	4
x_6	0	0	0	-1/3	-1/3	1	-1

The tableau is now optimal but infeasible. Application of the dual simplex method should then lead to feasibility while maintaining the tableau optimal.

continued...

Set 4.5b

1

Current optimum is

$x_1 = 0, \quad x_2 = 100, \quad x_3 = 230$

(a) $4x_1 + x_2 + 2x_3 \leq 570$:

Since $4 \times 0 + 1 \times 100 + 2 \times 230 = 560 < 570$, the additional constraint is redundant and the solution remains unchanged.

(b) $4x_1 + x_2 + 2x_3 \leq 548$:

The current solution violates the new constraints. We use the dual simplex method to determine the new solution.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	4	0	0	1	2	0	0	1350
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	0	100
x_3	$3/2$	0	1	0	$1/2$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	4	$\boxed{1}$	$\boxed{2}$	0	0	0	1	548
z	4	0	0	1	2	0	0	1350
x_1	$-1/4$	1	0	$1/2$	$-1/4$	0	0	100
x_3	$3/2$	0	1	0	$1/2$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	$5/4$	0	0	$\boxed{-1/2}$	$-3/4$	0	1	-12
z	$13/2$	0	0	0	$1/2$	0	2	1326
x_2	$-1/4$	1	0	0	-1	0	1	88
x_3	$3/2$	0	1	0	$1/2$	0	0	230
x_6	-3	0	0	0	4	1	-4	68
x_4	$-5/2$	0	0	1	$3/2$	0	-2	24

Optimum solution:

$x_1 = 0, \quad x_2 = 88, \quad x_3 = 230$

$z = \$1326$

2

Maximize $z = 5x_1 + 6x_2 + 3x_3$

Subject to

$$
\begin{aligned}
5x_1 + 5x_2 + 3x_3 &\leq 50 && (1)\\
x_1 + x_2 - x_3 &\leq 20 && (2)\\
7x_1 + 6x_2 - 9x_3 &\leq 30 && (3)\\
5x_1 + 5x_2 + 5x_3 &\leq 35 && (4)\\
12x_1 + 6x_2 \quad &\leq 90 && (5)\\
x_2 - 9x_3 &\leq 20 && (6)\\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Start with constraints (1), (3), and (4). The associated solution is

$x_1 = 0, \quad x_2 = 6.2, \quad x_3 = .8$

This solution automatically satisfies the remaining constraints (2), (5), and (6). Hence these constraints are discarded as redundant and the optimum solution for the problem is as given above.

1

$\begin{pmatrix}\text{Basic}\\ \text{vector}\end{pmatrix} = \begin{pmatrix}x_2\\ x_3\\ x_6\end{pmatrix}$ Inverse $= \begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$

Nonbasic variables: x_1, x_4, x_5

(a) $z = 2x_1 + x_2 + 4x_3$

$$(y_1, y_2, y_3) = (1, 4, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$$

$$= (1/2, 7/4, 0)$$

Reduced costs:

$x_1: (1/2, 7/4, 0)\begin{pmatrix}1\\3\\1\end{pmatrix} - 2 = 15/4$

$x_4: (1/2, 7/4, 0)\begin{pmatrix}1\\0\\0\end{pmatrix} - 0 = 1/2$

$x_5: (1/2, 7/4, 0\begin{pmatrix}0\\1\\0\end{pmatrix} - 0 = 7/4$

current solution remains optimal

(b) $z = 3x_1 + 6x_2 + x_3$

$$(y_1, y_2, y_3) = (6, 1, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$$

$$= (3, -1, 0)$$

Reduced costs:

$x_1: 1\times3 + 3\times-1 + 1\times0 - 3 = -3 < 0$

$x_4: 1\times3 + 0\times-1 + 0\times0 - 0 = 3$

$x_5: 0\times3 + 1\times-1 + 0\times0 - 0 = -1 < 0$

Solution is not optimal.

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-3	0	0	3	-1	0	830
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	$\boxed{2}$	0	0	-2	1	1	20
z	0	0	0	0	$1/2$	$3/2$	860
x_2	0	1	$1/4$	$1/4$	$-1/4$	$1/8$	$102\frac{1}{2}$
x_3	0	0	0	0	$1/2$	0	215
x_1	1	0	-1	-1	$1/2$	$1/2$	10

Optimum Solution: $x_1 = 10$, $x_2 = 102\frac{1}{2}$, $x_3 = 215$

Problem has alternative optima. $z = \$860$

(c) $z = 8x_1 + 3x_2 + 9x_3$

$$(y_1, y_2, y_3) = (3, 9, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix} = \left(\tfrac{3}{2}, \tfrac{15}{4}, 0\right)$$

Reduced costs:

$x_1: 1\times\frac{3}{2} + 3\times\frac{15}{4} + 1\times0 - 8 = 19/4$

$x_4: 1\times\frac{3}{2} + 3\times0 + 1\times0 - 0 = 3/2$

continued...

$x_5: 0\times3/2 + 1\times15/4 + 0\times0 - 0 = 15/4$

Solution remains optimal

2

Basic vector $= \begin{pmatrix}x_1\\ x_2\\ x_5\\ x_6\end{pmatrix}$, inverse $= \begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix}$

Dual problem:

Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$

Subject to

$$6y_1 + y_2 - y_3 \geq 5$$
$$4y_1 + 2y_2 + y_3 + y_4 \geq 4$$
$$y_1, y_2, y_3, y_4 \geq 0$$

(a) $z = 3x_1 + 2x_2$

$$(y_1, y_2, y_3, y_4) = (3, 2, 0, 0)\begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix}$$

$$= (1/2, 0, 0, 0)$$

Reduced costs:

$x_3: y_1 - 0 = 1/2 - 0 = 1/2$

$x_4: y_2 - 0 = 0 - 0 = 0$

Solution remains optimal.

(b) $z = 8x_1 + 10x_2$

$$(y_1, y_2, y_3, y_4) = (8, 10, 0, 0)\begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix}$$

$$= (3/4, 7/2, 0, 0)$$

Reduced costs:

$x_3: y_1 - 0 = 3/4 - 0 = 3/4$

$x_4: y_2 - 0 = 7/2 - 0 = 7/2$

Solution remains optimal

(c) $z = 2x_1 + 5x_2$

$$(y_1, y_2, y_3, y_4) = (2, 5, 0, 0)\begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix}$$

$$= (-1/8, 11/4, 0, 0)$$

Reduced costs:

$x_3: y_1 - 0 = -1/8 - 0 = -1/8 < 0$

$x_4: y_2 - 0 = 11/4 - 0 = 11/4$

current solution is not optimal.

continued...

Set 4.5c

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	0	$-1/8$	$11/4$	0	0	$27/2$
x_1	1	0	$1/4$	$-1/2$	0	0	3
x_2	0	1	$-1/8$	$3/4$	0	0	$3/2$
x_5	0	0	$3/8$	$-5/4$	1	0	$5/2$
x_6	0	0	$1/8$	$\boxed{-3/4}$	0	1	$1/2$
z	0	0	0	2	0	1	14
x_1	1	0	0	1	0	-2	2
x_2	0	1	0	0	0	1	2
x_5	0	0	0	1	1	-3	1
x_3	0	0	1	-6	0	8	4

Optimum solution:

$x_1 = 2,\ x_2 = 2,\ x_3 = 4,\ z = 14$

3

Let d_j = change in the objective coefficient c_j, $j = 1, 2, \ldots, n$

The simultaneous changes yield the same optimum if (for maximization)

$$(z_j - c_j - d_j) \geq 0,\ j = 1, 2, \ldots, n \quad (1)$$

where

z_j = left-hand of constraint dual$_j$ $= \sum_{i=1}^{m} a_{ij} y_i$

Let $u_j \leq d_j \leq v_j$ be the optimality range computed from the single-change condition

$$z_j - c_j - d_j \geq 0 \quad (2)$$

and define

$$\delta_j = \begin{cases} u_j, & \text{if } d_j < 0 \\ v_j, & \text{if } d_j > 0 \end{cases}$$

Condition (2) holds true also for $d_j = \delta_j$.

Define $r_j \geq 0$, $j = 0, 1, 2, \ldots, n$, such that $r_0 + r_1 + \cdots + r_n = 1$. Then

$r_0(z_1 - c_1, \ldots, z_n - c_n) + r_1(z_1 - c_1 - \delta_1, \ldots, z_n - c_n) + \cdots + r_n(z_1 - c_1, \ldots, z_n - c_n - \delta_n)$

continued...

must be nonnegative. However, the last expression reduces to

$$(z_1 - c_1, \ldots, z_n - c_n) - (r_1\delta_1, \ldots, r_n\delta_n) \geq 0$$

or

$$z_j - c_j - r_j\delta_j \geq 0,\ j = 1, 2, \ldots, n \quad (3)$$

Now, set $r_j = \frac{d_j}{\delta_j}$, then (3) is identical to (1), the desired condition. However, since $r_0 + r_1 + \cdots + r_n = 1$ and $r_0 \geq 0$, then for optimality we must have

$$r_1 + r_2 + \cdots + r_n \leq 1$$

1

Dual constraint for toy trains is

$$y_1+3y_2+y_3 \ge 3$$

where $y_1=1$, $y_2=2$, and $y_3=0$

new reduced cost for x_1 is

$$\frac{P}{100}(y_1+3y_2+y_3)-3.$$

For toy trains to be just profitable, we must have

$$\frac{P}{100}(1+3\times2+1\times0)-3\ge 0$$

or $P \ge 42.86\%$

2

x_1-reduced cost $= .5y_1+y_2+.5y_3-3$

$= .5\times1+1\times2+.5\times0-3 = -.5$

$$x_1\text{-column} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} .5 \\ 1 \\ .5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-1/2	0	0	1	2	0	1350
x_2	0	1	0	1/2	-1/4	0	100
x_3	1/2	0	1	0	1/2	0	230
x_6	$\boxed{1/2}$	0	0	-2	1	1	20
z	0	0	0	-1	3	1	1370
x_2	0	1	0	1/2	-1/4	0	100
x_3	0	0	1	$\boxed{2}$	-1/2	-1	210
x_1	1	0	0	-4	2	2	40
z	0	0	1/2	0	11/4	1/2	1475
x_2	0	1	-1/4	0	-1/8	1/4	$47\frac{1}{2}$
x_4	0	0	1/2	1	-1/4	-1/2	105
x_1	1	0	2	0	1	0	460

3

(a) New dual constraint for fire engines is

$$3y_1+2y_2+4y_3 \ge 5,\ y_1=1,\ y_2=2,\ y_3=0$$

Reduced cost $= 3\times1+2\times2+4\times0-5$

$= 2 > 0$

Fire engines are not profitable

continued...

(b) Reduced cost $= 3\times1+2\times2+4\times0-10 = -3$

$$\begin{pmatrix}\text{tableau}\\ \text{column}\end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

	x_1	x_2	x_3	$\textcircled{x_4}$	x_5	x_6	x_7	
z	4	0	0	-3	1	2	0	1350
x_2	-1/4	1	0	1	1/2	-1/4	0	100
x_3	3/2	0	1	1	0	1/2	0	230
x_7	2	0	0	0	-2	1	1	20
z	13/4	3	0	0	5/2	5/4	0	1650
x_4	-1/4	1	0	1	1/2	-1/4	0	100
x_3	7/4	-1	1	0	-1/2	3/4	0	130
x_7	2	0	0	0	-2	1	1	20

4

x_3 = daily tons of new exterior paint

Maximize $z = 5x_1+4x_2+3.5x_3$

subject to

$$\begin{aligned} 6x_1+4x_2+3/4x_3 &\le 24 \\ x_1+2x_2+3/4x_3 &\le 6 \\ -x_1+x_2+x_3 &\le 1 \\ x_2 &\le 2 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

New dual constraint: $\frac{3}{4}y_1+\frac{3}{4}y_2+y_3 \ge 3.5$

Dual solution: $y_1=3/4$, $y_2=1/2$, $y_3=0$

Reduced cost $= \frac{3}{4}(3/4+1/2)+0-3.5 = -41/16$

$$\begin{pmatrix}\text{Constraint}\\ \text{Column}\end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3/4 \\ 3/4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/16 \\ 15/32 \\ 13/16 \\ -15/32 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	0	0	-41/16	3/4	1/2	0	0	21
x_1	1	0	-3/16	1/4	-1/2	0	0	3
x_2	0	1	15/32	-1/8	3/4	0	0	3/2
x_6	0	0	13/16	3/8	-5/4	1	0	5/2
x_7	0	0	-15/32	1/8	-3/4	0	1	1/2
z	0	5.47	0	.07	4.6	0	0	29.2
x_1	1	.4	0	.2	-.2	0	0	3.6
x_3	0	2.13	1	-.27	1.6	0	0	3.2
x_6	0	-.73	0	.47	-1.8	1	0	1.4
x_7	0	1	0	0	0	0	1	2.0

Optimum solution:

$x_1 = 3.6$ tons, $x_2=0$, $x_3=3.2$ tons

$z = \$29,200$

CHAPTER 5

Transportation Model and its Variants

Set 5.1a

1

(a) False
(b) True
(c) True

2

(a) $\sum a_i = 25$, $\sum b_j = 31$
Add a dummy source whose supply amount is 31 − 25 = 6 units

(b) $\sum a_i = 74$, $\sum b_j = 65$
add a dummy destination whose demand amount is 74 − 65 = 9 units

3

Denver will be 150 cars short. Similarly, Miami will be 50 cars short of satisfying its demand

4

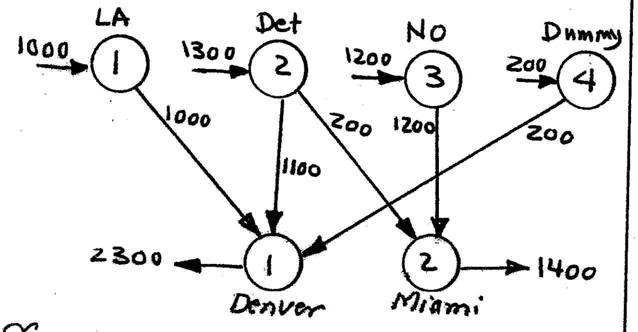
assign a very high cost M to the route from Detroit to Dummy

5

		Den 1	Miami 2	
LA	1	80 1000	M	1000
Det	2	100 1100	108 200	1300
NO	3	102	68 1200	1200
Dummy	4	200 200	300	200
		2300	1900	

use M = $1000 in TORA

Optimum solution from TORA



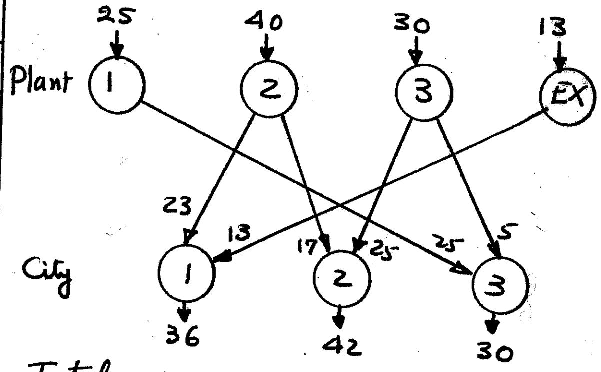
Denver is 200 cars short, Cost = $33,200

6

(a)

	City 1	2	3	
Plant 1	600	700	400 25	25
Plant 2	320 23	300 17	350	40
Plant 3	500	480 25	450 5	30
Excess plant 4	1000 13	1000	M	13
	36	42	30	

(b) M = $10,000 in TORA



Total cost = $49,710

(c) City 1 excess cost = 13 × 1000 = $13,000

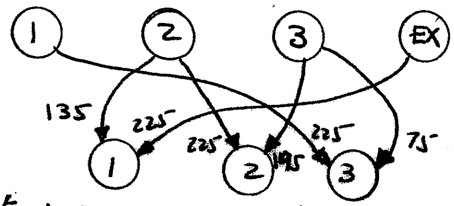
7

Assume units in 100,000 kWh

(a)

	city 1	2	3	
Plant 1	60	70	40 225	225
2	32 135	30 225	35	360
3	50	48 195	45 75	270
EX	100 225	100	M	225
	360	420	300	

(b)



(c) City 1 excess cost = $22,500

Optimum cost = $55,305$

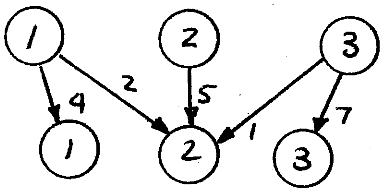
8

Unit transportation cost in thousand \$ per million gallons $= \left(\left(\frac{10¢}{1000} \times 10^6 \times \text{mileage}\right) \times \frac{1}{100}\right) \times \frac{1}{1000}$

$= \frac{\text{mileage}}{10}$

Distribution Area

	1	2	3	
Ref. 1	4 ^{12}	2 ^{18}	^{M}	6
2	^{30}	5 ^{10}	^{8}	5
3	^{20}	1 ^{25}	7 ^{12}	8
	4	8	7	



Total cost = \$243,000

9

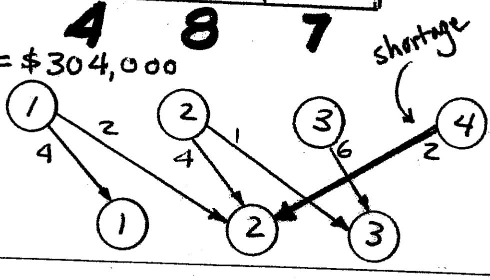
Unit cost in thousand \$ from Dummy source to distribution areas 2 or 3

$= \frac{5¢}{100} \times \frac{10^6}{10^3} = 50$ thousand \$/million gal

Distribution area

	1	2	3	
Ref. 1	4 ^{12}	2 ^{18}	^{M}	6
2	^{30}	4 ^{10}	1 ^{8}	5
3	^{20}	^{25}	6 ^{12}	6
Dummy	^{M}	2 ^{50}	^{50}	2
	4	8	7	

Cost = \$304,000



10

Unit costs in thousand \$ per million gallons:

from refinery 1 to Dummy

$= \frac{\$1.50}{100} \times \frac{10^6}{10^3} = 15$

from refinery 2 to Dummy

$= \frac{\$2.20}{100} \times \frac{10^6}{10^3} = 22$

	1	2	3	Dummy	
Ref. 1	4 ^{12}	2 ^{18}	^{M}	^{15}	6
2	^{30}	5 ^{10}	^{8}	^{22}	5
3	^{20}	1 ^{25}	4 ^{12}	3 ^{0}	8
	4	8	4	3	

Refinery 3 diverts 3 million gallons for use within.

Total cost = \$207,000

11

(a) Total supply = 150+200+250 = 600 crates

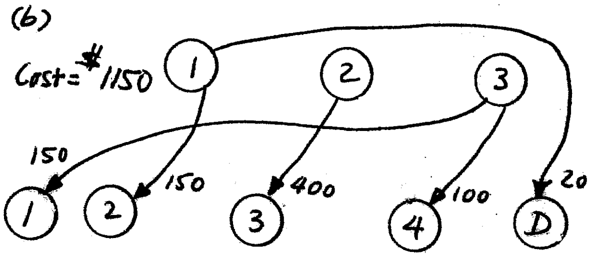
Total demand = 150+150+400+100 = 800 crates

$\begin{pmatrix}\text{Potential overtime supply} \\ \text{by each of orchards 1 \& 2}\end{pmatrix}$ = 800 − 600 = 200 crates

	1	2	3	4	Dummy	
Orch 1	^{1}	150 ^{2}	^{3}	^{2}	200 ^{0}	150+200
2	^{2}	^{4}	400 ^{1}	^{2}	^{0}	200+200
3	150 ^{1}	^{3}	^{5}	100 ^{3}	^{M}	250
	150	150	400	100	200	

(b)

Cost = \$1150



Problem has alternative optima.

(c) Orchad 1 = 0 overtime crates

Orchad 2 = 200 overtime crates

12

Supply/demand quantities are expressed in truckloads, determined by dividing the number of cars by 18 and rounding the result up, if necessary. For example, supply amount at center 1 is $\frac{400}{18} = 22.22$ or 23 truckloads. Expressing unit transportation costs in $1000 per truck load, we get

	1	2	3	4	5	
1	2.5 6	3.75	5	3.5 9	.875 8	23
2	1.25	1.75 3	1.5 9	1.625	2	12
3	1	2.25 9	2.5	3.75	3.25	9
	6	12	9	9	8	

(b) alternative solution exists
Cost = $92,500

13

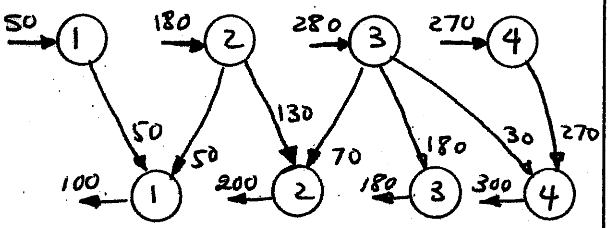
		Denver						Miami						
		M1	M2	M3	M4	M1/2	M3/4	M1	M2	M3	M4	M1/3	M2/4	
L.A.	M3	▨	▨	80 400	▨	▨	80 120	▨	▨	215 180	▨	215	▨	700
	M4	▨	▨	▨	80 300	▨	80	▨	▨	▨	215	▨	215	300
Det	M1	100 500	▨	▨	▨	100	▨	108	▨	▨	▨	108	▨	500
	M2	▨	100 450	▨	▨	100 70	▨	▨	108 75	▨	▨	▨	108 5	600
	M4	▨	▨	▨	100 180	▨	100 100	▨	▨	▨	108 95	▨	108 25	400
N.O.	M1	102 130	▨	▨	▨	102 50	▨	68 590	▨	▨	▨	68 80	▨	800
	M2	▨	102	▨	▨	102	▨	▨	68 400	▨	▨	▨	68	400
		630	450	400	480	120	220	540	475	180	95	80	30	

Optimum solution:

LA-Denver M4 = 300 cars
Det.-Denver M1 = 500 cars
Det.-Denver M2 = 450 cars
Det.-Denver M1/M2 = 70 cars
Det.-Miami M2 = 75 cars
Det.-Miami M2/4 = 5 cars
Det.-Denver M4 = 180 cars
Det.-Denver M3/4 = 100 cars
Det.-Miami M4 = 95 cars
Det.-Miami M2/4 = 25 cars
N.O.-Denver M1 = 130 cars
N.O.-Denver M1/2 = 50 cars
N.O.-Miami M1 = 540 cars
N.O.-Miami M1/3 = 80 cars
N.O.-Miami M2 = 400 cars
Total cost = $343,620

1

	1	2	3	4	
1	40 50	40.4	40.7	41.4	50
2	42 50	40 130	40.3	41	180
3	44	42 70	40 180	40.7 30	280
4	46	44	42	40 270	270
	100	200	180	300	



Cost = $31,461

2

Least-cost starting solution.
(Problem has alternative optima.)

	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Disposal	
New	12 24	12 12	12 2	12	12	12	12	0 86	124
Mon	▨	6	6	3 6	1 18	1	1	0	24
Tue	▨	▨	6 12	6	3	1	1	0	12
Wed	▨	▨	▨	6 14	6	3	1	0	14
Thu	▨	▨	▨	▨	6	6	3 20	0	20
Fri	▨	▨	▨	▨	▨	6 14	6	0 4	18
Sat	▨	▨	▨	▨	▨	▨	6 2	0 12	14
Sun	▨	▨	▨	▨	▨	▨	▨	0 22	22
	24	12	14	20	18	14	22	124	

The given optimum solution is interpreted as summarized below.
Total cost = $804

Day	New	Sharpening Service			Disposal
		Overnite	2-day	3-day	
Mon	24	0	6	18	0
Tue	12	12	0	0	0
Wed	2	14	0	0	0
Thu	0	0	20	0	0
Fri	0	14	0	0	4
Sat	0	2	0	0	12
Sun	0	0	0	0	22

3

	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Disposal	
New	12 24	12	12 8	12	12	12	12	0 92	124
Mon	▨	6	6.5	3 12	3.5	4	4.5	0	24
Tue	▨	▨	6 6	6.5	3 6	3.5	4	0	12
Wed	▨	▨	▨	6 8	6.5	3 6	3.5	0	14
Thu	▨	▨	▨	▨	6 12	6.5	3 8	0	20
Fri	▨	▨	▨	▨	▨	6 8	6.5	0 10	18
Sat	▨	▨	▨	▨	▨	▨	6 14	0	14
Sun	▨	▨	▨	▨	▨	▨	▨	0 22	22
	24	12	14	20	18	14	22	124	

Day	New	Sharpening Service		Disposal
		Overnight	2-day	
Mon	24	12	12	0
Tue	0	6	6	0
Wed	8	8	6	0
Thu	0	12	8	0
Fri	0	8	0	10
Sat	0	14	0	0
Sun	0	0	0	22

Total cost = $840
alternative solution exists

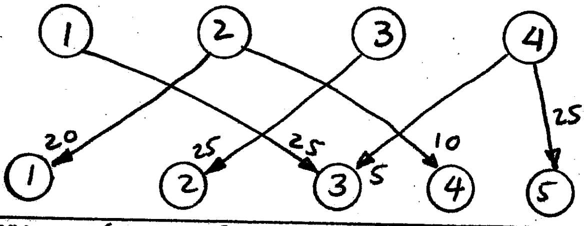
continued...

Set 5.2a

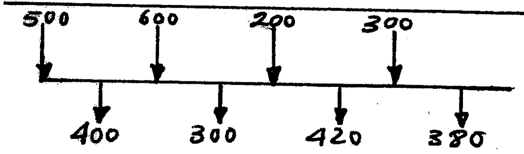
4

Machine \ Task	1	2	3	4	5	
1	10	2	3 25	15	9	25
2	5 20	10	15	2 10	4	30
3	15	5 20	14	7	15	20
4	20	15	13 5	M	8 25	30
	20	20	30	10	25	

Total cost = $560



5



c: $100 $140 $120 $150

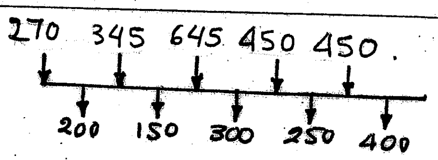
h: $3 $3 $3 $3

	1	2	3	4	Surplus	
1	100 400	103 100	106	M	0	500
2	M	140 200	143 220	146 180	0	600
3	M	M	120 200	123	0	200
4	M	M	M	150 200	0 100	300
	400	300	420	380	100	

Cost = $190,040, Alternative solution exists

Period	Capacity	Amt Prod.	Delivery
1	500	500	400 for 1 100 for 2
2	600	600	200 for 2 220 for 3 180 for 4
3	200	200	200 for 3
4	300	200	200 for 4

6



	1	2	3	4	5	Surplus	
R_1	100 180	104	108	112	116	0	180
O_1	150 20	154	158	162	166	0 70	90
R_2		96 150	100 80	104	108	0	230
O_2		144	148	152	156	0 115	115
R_3			116 220	120	124 50	0 160	430
O_3			174	178	182	0 215	215
R_4				102 250	106 50	0	300
O_4				153	157	0 150	150
R_5					106 300	0	300
O_5					159	0 150	150
	200	150	300	250	400	860	

Cost = $137,720

Alternative solution exists.

Period	Production schedule
1	Regular – 180 engines Overtime – 20 engines
2	Regular: 230 engines
3	Regular 270 engines
4	Regular 300 engines
5	Regular 300 engines

7

	1	2	3	4	5	6	Disposal	
New	200 200	210 180	220.5 140	231.53	243.1	255.26	0 878	1398
1		120	121.5 12	35 188	36.5	38	0	200
2			120 148	121.5	35 32	36.5	0	180
3				120 10	121.5	35 290	0	300
4					120 198	121.5	0	198
5						120	0 230	230
6							0 290	290
	200	180	300	198	230	290	1398	

Cost = $ 170,698

alternative solution exists

		overhaul		
Month	New	1-day	3-day	Disposal
1	200	12	188	0
2	180	148	32	0
3	140	10	290	0
4	0	198	0	0
5	0	0	0	230
6	0	0	0	290

8

(a) Use negative cost values

	Bidder 1	2	3	4	
Loc 1	-520	M	-650 10	-180	10
2	-210	-510 20	M	-430	20
3	-570	-495	-240	-710 30	30
Dummy	0 30	0 10	0 20	0	60
	30	30	30	30	

(b) Bidder 1 = 0 acre

Bidder 2 = 20 acres (location 1)

Bidder 3 = 10 acres (location 2)

Bidder 4 = 30 acres (location 3)

Set 5.3a

(a)

Northwest:

Cost = $42

⑤ 0	① 2	1	6
2	④ 1	③ 5	7
2	4	⑦ 3	7
5	5	10	

Least-cost:

Cost = $37

⑤ 0	2	① 1	6
2	⑤ 1	② 5	7
2	4	⑦ 3	7
5	5	10	

Vogel:

Cost = $37

				Penalty	Penalty
⑤ 0	2	① 1	6	1	1
2	⑤ 1	5	7	1	④
2	4	⑦ 3	7	1	1
5	5	10		step1	Step 2
Penalty ②	1	2	← step1		
Penalty —	1	2	← Step 2		

(b)

Northwest:

Cost = $94

⑦ 1	2	6	7
③ 0	⑨ 4	2	12
3	① 1	⑩ 5	11
10	10	10	

Least-Cost:

Cost = $61

1	2	⑦ 6	7
⑩ 0	4	② 2	12
3	⑩ 1	① 5	11
10	10	10	

continued...

VAM:

Cost = $40

			Penalties		
⑦ 1	2	6	1	1	1
② 0	4	⑩ 2	2	④	—
① 3	⑩ 1	5	2	2	②
			step1	step2	step3

Penalties
{ 1 1 ③ ← step1
2 1 — ← step2
2 1 — ← step3

(c)

Northwest:

Cost = $104

⑨ 5	③ 1	8	12
2	⑦ 4	⑦ 0	14
3	6	④ 7	4
9	10	11	

Least-Cost

Cost = $38

② 5	⑩ 1	8	12
③ 2	4	⑪ 0	14
④ 3	6	7	4
9	10	11	

VAM:

Cost = $38

				Penalties	
② 5	⑩ 1	8	12	4	④
③ 2	4	⑪ 0	14	2	2
④ 3	6	7	4	3	3
9	10	11		Step1	step 2

Penalties
{ 1 3 ⑦ ← Step1
1 3 — ← step 2

continued...

(i)

u \ v	0	2	6	
0	0 ⑤	2 ① −	1 ▨ 5 +	6
−1	2 −3	1 ④ +	5 ⑤ −	9
−3	2 −5	4 −5	3 ⑤	5
	5	5	10	

u \ v	0	−3	1	
0	0 ⑤ −	2 −5	1 ① +	6
4	2 ▨ 2 +	1 ⑤	5 ④ −	9
7	2 0	4 −5	3 ⑤	5
	5	5	10	

u \ v	0	−1	1	
0	0 ①	2 −3	1 ⑤	6
2	2 ④	1 ⑤	5 −2	9
2	2 0	4 −3	3 ⑤	5
	5	5	10	

cost = $33
alternative solution exists

(ii)

u \ v	0	4	2	
0	0 ⑦	4 ①	2 0	8
−1	2 −3	3 ⑤	4 −3	5
−2	1 −3	2 ⓪	0 ⑥	6
	7	6	6	

Problem has alternative optima. Cost = $19
Note: If x_{23} were selected as the zero in place of x_{32}, solution would require one more iteration.

continued...

(iii)

u \ v	M	M−3	M−5	
0	M ④ −	3 ▨ M−6 +	5 M−10	4
7−M	7 ① +	4 ⑥ −	9 −7	7
11−M	1	8 ⓪	6 ⑲	19
	5	6	19	

u \ v	6	3	1	
0	M 6−M	3 ④	5 −4	4
1	7 ⑤ −	4 ② +	9 −7	7
5	1 ▨ 10 +	8 ⓪ −	6 ⑲	19
	5	6	19	

u \ v	6	3	11	
0	M 6−M	3 ④ −	5 ▨ 6 +	4
1	7 ⑤ −	4 ② +	9 3	7
−5	1 ⓪ +	8 −10	6 ⑲ −	19
	5	6	19	

u \ v	0	−3	5	
0	M −M	3 −6	5 ④	4
7	① −	⑥	9 ▨ 3 +	7
1	④ +	−10	6 ⑮ −	19
	5	6	19	

u \ v	0	0	5	
	M −M	3 −3	5 ④	4
	7 −3	4 ⑥	9 ①	7
	1 ⑤	8 −7	6 ⑭	19
	5	6	19	

Cost = $142

continued...

Set 5.3b

(c)

Method	Nbr. of iterations (i)	(ii)	(iii)
NW	3	4	5
Least cost	2	2	2
Vogel	2	1	1

Least-cost starting solution:

2

u \ v	2	1	2	
	5 -3	1 (10)	7 -5	10
	6 (70) −	4 1 +	6 (10)	80
	3 (5) +	2 (10) −	5 -2	15
	5 -3	3 -2	2 (40)	40
	75	20	50	

u \ v	3	1	3	
0	5 -2	1 (10)	7 -4	10
3	6 (60)	4 (10)	6 (10)	80
0	3 (15)	2 -1	5 -2	15
-1	5 -3	3 -3	2 (40)	40
	75	20	50	

Destination 3 will be 40 units short. Optimum cost = $595

Least-cost starting solution:

3

u \ v	2	1	1	
0	5 -3	1 10	7 -6	10
4	6 30	4 (1)	6 50	80
1	3 5	2 10	5 -3	15
-2	0 40	0 -1	M -1-M	40
	75	20	50	

u \ v	3	1	3	
0	5 -2	1 10	7 -4	10
3	6 20	4 10	6 50	80
0	3 15	2 -1	5 -2	15
-3	0 40	0 -2	M -M	40
	75	20	50	

Total cost = $515. Dest. 1 is 40 units Short.

Vogel method:

4

1	2	1	5	0
3	4	5	M	1
2	3	3	3 20	1
1	1	2	2	

1	2	1 20		0
3	4	5		1
2	3	3		1
1	1	2		

3	4	5		1
2	3	3 0		1
1	1	2		

3 20	4 20			1
2 10	3			1
1	1			

u \ v	0	1	1	1	
0	1 -1	2 -1	1 20	3 -2	20
3	3 20	4 20	5 -1	M 4-M	40
2	2 10	3 0	3 0	3 20	30
	30	20	20	20	

Cost = \$240 — alternative solution exists

5

u \ v	2	5	10	
-2	c_{11} 15	c_{12}	c_{13}	15
3	c_{21} 5	c_{22} 25	c_{23}	30
5	c_{31}	c_{32} 5	c_{33} 80	85
	20	30	80	

(a) $c_{ij} = u_i + v_j$ for basic x_{ij}

Thus,

$c_{11} = 2 - 2 = 0$

$c_{21} = 3 + 2 = 5$

$c_{22} = 3 + 5 = 8$

$c_{32} = 5 + 5 = 10$

$c_{33} = 5 + 10 = 15$

Cost $= 15 \times 0 + 5 \times 5 + 25 \times 8 + 5 \times 10 + 80 \times 15 = \$1475$

(b) $u_i + v_j - c_{ij} \leq 0$ for nonbasic x_{ij}

$-2 + 5 - c_{12} \leq 0 \implies c_{12} \geq 3$

$-2 + 10 - c_{13} \leq 0 \implies c_{13} \geq 8$

$3 + 10 - c_{23} \leq 0 \implies c_{23} \geq 13$

$5 + 2 - c_{31} \leq 0 \implies c_{31} \geq 7$

continued...

Problems 6 and 7 on next page

continued...

Set 5.3b

6

(a) For basic x_{ij}, $c_{ij} = u_i + v_j$.

u \ v	2	2	5	
1	$c_{11}=3$ ⑩	$1+2\theta$	$1+3\theta$	10
-1	$2+\theta$	$c_{22}=1$ ⑳	$c_{23}=4$ ⑳	40
	10	20	20	

Cost = 3×10 + 1×20 + 4×20 = \$130

(b) For nonbasic x_{ij}: $u_i + v_j - c_{ij} \le 0$ to satisfy optimality. Hence

$$2+1-(1+2\theta) \le 0 \implies \theta \ge 1$$

$$5+1-(1+3\theta) \le 0 \implies \theta \ge 5/3$$

$$2-1-(2+\theta) \le 0 \implies \theta \ge -1$$

Take $\theta = \frac{5}{3}$ to yield $x_{13} = 0$ as the zero basic variable.

7

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	
min Z =	1	1	2	6	5	1	
s.t.							
	1	1	1				≥5
				1	1	1	≥6
	1			1			≥2
		1			1		≥7
			1			1	≥1

$x_{ij} \ge 0$ for all i and j

Optimum LP solution using TORA:

$Z = 15$, $x_{11} = 2$, $x_{12} = 7$, $x_{23} = 6$

If we replace the first two constraints with equations, we get the optimum solution:

$Z = 27$, $x_{11} = 2$, $x_{12} = 3$,

$x_{22} = 4$, $x_{23} = 2$

The new solution is worse!

1

	u_1	u_2	u_3	v_1	v_2	v_3	v_4	
max	15	25	10	5	15	15	15	
s.t.								
	1			1				≤ 10
	1				1			≤ 2
	1					1		≤ 20
	1						1	≤ 11
		1		1				≤ 12
		1			1			≤ 7
		1				1		≤ 9
		1					1	≤ 20
			1	1				≤ 4
			1		1			≤ 14
			1			1		≤ 16
			1				1	≤ 18

From Table 5-25:

$u_1 = 0,\ u_2 = 5,\ u_3 = 7$

$v_1 = -3,\ v_2 = 2,\ v_3 = 4,\ v_4 = 11$

$$\text{Optimum } w = 15\times 0 + 25\times 5 + 10\times 7 + 5\times -3 + 15\times 2 + 15\times 4 + 15\times 11 = \$435$$

2

$$\text{minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i,\quad i=1,2,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\quad j=1,2,\dots,n$$

Next, consider

$$z' = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}+K)x_{ij}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + K\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + K\sum_{i=1}^{m} a_i$$

continued...

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + K,\ K \text{ is a constant}$$

$$= z + K$$

This result shows that optimization using z and z' yield the same optimum values of x_{ij}.

To show why the dual values associated with a given primal basic solution are not unique, note that, for any constant K,

$$\begin{pmatrix}\text{Dual}\\ \text{Values}\end{pmatrix} = \begin{pmatrix}\text{Original basic}\\ \text{obj. coefficients}\\ + \\ K\end{pmatrix} \times \text{Inverse}$$

This means that even though the optimal primal solution is unique for all K, there are infinity of dual values, each corresponding to a given value of K.

The conclusion is that an arbitrary value assigned to one of the dual variables (e.g., $u_1 = 0$) implies a specific value for the constant K.

Set 5.4a

1

(a-i)

$$\begin{array}{ccccc|c} 3 & 8 & 2 & 10 & 3 & 2 \\ 8 & 7 & 2 & 9 & 7 & 2 \\ 6 & 4 & 2 & 7 & 5 & 2 \\ 8 & 4 & 2 & 3 & 5 & 2 \\ 9 & 10 & 6 & 9 & 10 & 6 \end{array}$$

Row min

$$\begin{array}{ccccc} 1 & 6 & 0 & 8 & 1 \\ 6 & 5 & 0 & 7 & 5 \\ 4 & 2 & 0 & 5 & 3 \\ 6 & 2 & 0 & 1 & 3 \\ 3 & 4 & 0 & 3 & 4 \end{array}$$

Col min → 1 2 0 1 1

$$\begin{array}{ccccc} 0 & 4 & 2 & 7 & \boxed{0} \\ 3 & 1 & \boxed{0} & 4 & 2 \\ 3 & \boxed{0} & 2 & 4 & 2 \\ 5 & 0 & 2 & \boxed{0} & 2 \\ \boxed{0} & 0 & 0 & 0 & 1 \end{array}$$

Assignment:
1-5
2-3
3-2
4-4
5-1
Cost= $21

(a-ii)

$$\begin{array}{ccccc|c} 3 & 9 & 2 & 2 & 7 & 2 \\ 6 & 1 & 5 & 6 & 6 & 1 \\ 9 & 4 & 7 & 10 & 3 & 3 \\ 2 & 5 & 4 & 2 & 1 & 1 \\ 9 & 6 & 2 & 4 & 6 & 2 \end{array}$$

$$\begin{array}{ccccc} 1 & 7 & 0 & 1 & 5 \\ 5 & 0 & 4 & 5 & 5 \\ 6 & 1 & 4 & 7 & 0 \\ 1 & 4 & 3 & 1 & 0 \\ 7 & 4 & 0 & 2 & 4 \end{array}$$

Col min 1 0 0 1 0

continued...

$$\begin{array}{ccccc} \boxed{0} & 7 & 0 & 0 & 5 \\ 4 & \boxed{0} & 4 & 4 & 5 \\ 5 & 1 & 4 & 6 & \boxed{0} \\ 0 & 4 & 3 & \boxed{0} & 0 \\ 6 & 4 & \boxed{0} & 1 & 4 \end{array}$$

Optimum:
1-1
2-2
3-5
4-4
5-3
Cost = $11

2

$$\begin{array}{cccc} 5 & 5 & M & 2 \\ 7 & 4 & 2 & 3 \\ 9 & 3 & 5 & M \\ 7 & 2 & 6 & 7 \end{array} \qquad \begin{array}{cccc} 3 & 3 & M-2 & 0 \\ 5 & 2 & 0 & 1 \\ 6 & 0 & 2 & M-3 \\ 5 & 0 & 4 & 5 \end{array}$$

(All entries are divided by 10 for convenience)

$$\begin{array}{cccc} 0 & 3 & M-2 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 0 & 2 & M-3 \\ 2 & 0 & 4 & 5 \end{array} \qquad \begin{array}{cccc} 0 & 5 & M-2 & \boxed{0} \\ 2 & 4 & \boxed{0} & 1 \\ 1 & \boxed{0} & 0 & M-5 \\ \boxed{0} & 0 & 4 & 5 \end{array}$$

Optimum: 1-4, 2-3, 3-2, 4-1
Cost = $140

3

$$\begin{array}{cc|ccccc} & & & & \text{Job} & & \\ & & 1 & 2 & 3 & 4 & 5 \\ \hline & 1 & 50 & 50 & M & 20 & 0 \\ & 2 & 70 & 40 & 20 & 30 & 0 \\ \text{Worker} & 3 & 90 & 30 & 50 & M & 0 \\ & 4 & 70 & 20 & 60 & 70 & 0 \\ & 5 & 60 & 45 & 30 & 80 & 0 \end{array}$$

Job 5 is dummy

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 30 & M-20 & 0 & 0 \\ 2 & 20 & 20 & 0 & 10 & 0 \\ 3 & 40 & 10 & 30 & M-20 & 0 \\ 4 & 20 & 0 & 40 & 50 & 0 \\ 5 & 10 & 25 & 10 & 60 & 0 \end{array}$$

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 30 & M-20 & \boxed{0} & 10 \\ 2 & 20 & 20 & \boxed{0} & 10 & 10 \\ 3 & 30 & 0 & 20 & M-30 & \boxed{0} \\ 4 & 20 & \boxed{0} & 40 & 50 & 10 \\ 5 & \boxed{0} & 15 & 0 & 50 & 0 \end{array}$$

Optimum:
1-4
2-3
3-5
4-2
5-1

Worker 3 is assigned to dummy job 5.
Thus, worker 5 must replace worker 3.

4

Add a "dummy" operator with zero assignment cost to each job (including the fifth). The optimal solution will show the replacement by indicating which of the current jobs (1 thru 4) is assigned to the dummy operator. If the dummy operator is assigned to the new job, then the new job must assume lower priority to the current four jobs.

(All assignment cost are divided by 10 for convenience.)

		Job					
		1	2	3	4	5	
	1	5	5	M	2	2	
	2	7	4	2	3	1	
Operator	3	9	3	5	M	2	
	4	7	2	6	7	8	
	5	0	0	0	0	0	← Dummy

3	3	M-3	0	0
6	3	1	2	0
7	1	3	M-2	0
5	0	4	5	
0	0	0	0	0

2	2	M-4	$\boxed{0}$	0
5	2	$\boxed{0}$	2	0
6	0	2	M-2	$\boxed{0}$
5	$\boxed{0}$	4	6	7
$\boxed{0}$	0	0	1	0

Optimum:

1 – 4

2 – 3

3 – 5

4 – 2

5 – 1

Since dummy operator is assigned to job 1, new job 5 has higher priority over job 1.

5

Define the following two sets:

Set 1: (DA,3), (DA,10), (DA,17), (DA,25)

Set 2: (AT,7), (AT,12) AT,21), (AT,28).

The idea is to match one element from Set 1 with another element from Set 2. The matching automatically decides the date and location for the purchase of each ticket. For example, consider the following assignment:

(DA,3) – (AT,21)

(DA,10) – (AT,7)

(DA,17) – (AT,28)

(DA,25) – (AT,12)

This assignment can be interpreted as follows:

Ticket 1:	June 3	DA → AT
	June 21	AT → DA
Ticket 2:	June 7	AT → DA
	June 10	DA → AT
Ticket 3:	June 17	DA → AT
	June 28	AT → DA
Ticket 4:	June 12	AT → DA
	June 25	DA → AT

The complete assignment model is given below

	A,7	A,12	A,21	A,28
D,3	400	300	300	280
D,10	300	400	300	300
D,17	300	300	400	300
D,25	300	300	300	400

Optimum:

(D,3) – (A,28) (A,21) – (D,25)

(A,7) – (D,10) (A,12) – (D,17)

Problem has alternative optima.

continued...

Set 5.4a

6

Distance matrix in meters:

candidate areas

$$\begin{array}{ll|cccc|} & & a & b & c & d \\ & 1 & 50 & 50 & 95 & 45 \\ \text{existing} & 2 & 30 & 30 & 55 & 65 \\ \text{centers} & 3 & 70 & 50 & 25 & 55 \\ & 4 & 100 & 60 & 55 & 25 \end{array}$$

A measure of the optimal assignment of new centers to candidate locations must reflect both distance and frequency of trips; that is

existing candidate

$$\begin{array}{ll|cccc|} & & 1 & 2 & 3 & 4 \\ & I & 10 & 7 & 0 & 11 \\ & II & 2 & 1 & 8 & 4 \\ \text{new} & III & 4 & 9 & 6 & 0 \\ & IV & 3 & 5 & 2 & 7 \end{array} \begin{array}{|cccc|} a & b & c & d \\ 50 & 50 & 95 & 45 \\ 30 & 30 & 55 & 65 \\ 70 & 50 & 25 & 55 \\ 100 & 60 & 55 & 25 \end{array}$$

$$\begin{array}{ll|cccc|} & & a & b & c & d \\ & I & 1810 & 1370 & 1940 & \boxed{1180} \\ & II & 1090 & 770 & \boxed{665} & 695 \\ \text{New} & III & \boxed{890} & 770 & 1025 & 1095 \\ & IV & 1140 & \boxed{820} & 995 & 745 \end{array}$$

TORA optimum assignment:

I-d

II-C

III-a

IV-b

7

The ranking of the projects by the different teams can use the following numeric score

1: Highest preference

1: lowest preference

A tie in preference between two or more projects is indicated by assigning the projects the same score. For example, the scores

Project	1	2	3	4	5	6	7	8	9	10
Score	9	9	8	7	3	5	4	1	2	6

indicate that project 8 is the most preferred and projects 1 and 2 tie for the least preferred status.

For the development of the model, we use the following numeric designations for the projects

Project nbr.	Project name
1	Boeing-F15
2	Boeing-F18
3	Boeing Simulation
4	Cargil
5	Cobb-Vantress
6	ConAgra
7	Cooper
8	DaySpring (layout)
9	DaySpring (Materials)
10	JB Hunt
11	Raytheon
12	Tyson South
13	Tyson East
14	WAL-MART
15	Yellow

continued...

The following is a typical summary of preference scores submitted by the 11 teams:

	Team										
	1*	2	3	4	5	6*	7*	8	9*	10	11
1⊗	–	①	2	2	1	–	–	1	–	2	15
2⊗	–	1	3	①	2	–	–	1	–	10	12
3	1	2	5	3	2	13	5	1	4	15	①
4	②	3	6	4	10	5	14	2	1	4	14
5	3	5	4	5	9	4	12	3	3	13	13
6	3	4	2	5	9	8	12	①	2	1	13
7	4	6	①	12	8	9	10	2	5	2	5
8	5	6	7	14	7	9	10	4	6	3	15
9	7	8	9	14	7	1	①	15	1	15	1
10	7	9	12	15	6	3	9	5	4	7	5
11⊗	–	9	13	6	5	–	–	7	–	6	7
12	13	10	14	7	4	②	8	9	15	4	9
13	14	11	1	8	3	13	7	8	①	8	9
14	15	12	5	9	①	14	7	6	2	9	10
15	15	13	7	10	2	15	6	1	3	①	11

* Team does not meet citizenship requirements

⊗ project requiring US citizenship

The problem is modeled as an assignment model. Entries – are replaced by M, a very large value. The model is unbalanced. Thus, 4 artificial teams must be added to balance the model. In the end four projects will not be assigned.

TORA solution:

Project	Team	Score
1	2	1
2	4	1
3	11	1
4	1	1
5	None	–
6	8	1
7	3	1
8	None	–
9	7	1
10	None	–
11	None	–
12	6	2
13	10	1
14	5	1
15	10	1
	Total score	13

continued....

$$\text{Average score} = \frac{13}{11} = 1.18$$

The average score is close to 1, meaning that all preferences are well met.

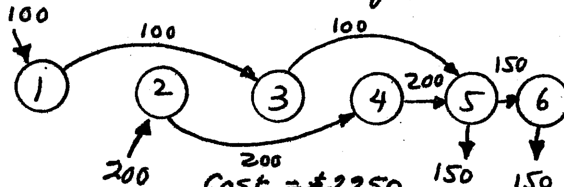
Set 5.5a

1

	3	4	5	6	
1	1 100	4	M	M	100
2	3	2 200	M	M	200
3	0 200	1	6 100	M	B
4	3	0 100	5 200	8	B
5	M	M	0 150	1 150	B
	B	B	150+B	150	

Let B = 300 units

Optimum solution using TORA:



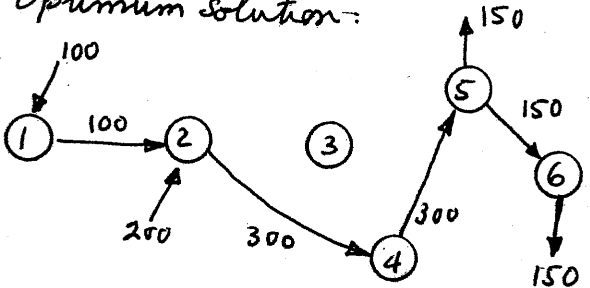
Cost = $2250

2

B = 300 units

	2	3	4	5	6	
1	1 100	5	4	M	M	100
2	0 200	3 0	2 300	M	M	200+B
3	M	0 300	1	6 0	M	B
4	M	3	0	5 300	8	B
5	M	M	M	0 150	1 150	B
	B	B	B	150+B	150	

Optimum solution:

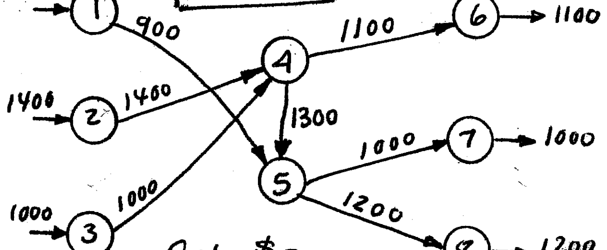


Cost = $2,350

3

	4	5	6	7	8	
1	1	.3 900	M	M	M	900
2	.8 1400	4.3	M	M	M	1400
3	2 1000	4.6	M	M	M	1000
4	0 900	.5 1300	.2 1100	4.5	6	B
5	M	0 1100	3	2.1 1000	1.9 1200	B
	B	B	1100	1000	1200	

B = 3300

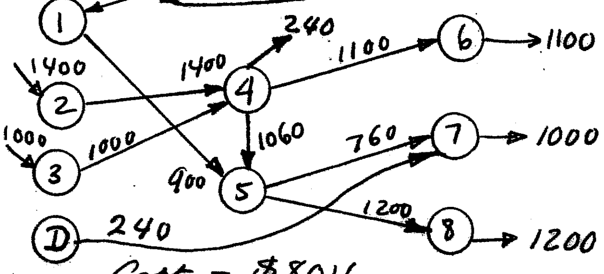


Cost = $8,640

(b)

	4	5	6	7	8	
1	1	.3 900	M	M	M	900
2	.8 1400	4.3	M	M	M	1400
3	2 1000	4.6	M	M	M	1000
4	0 1380	.5 1060	.2 1100	4.5	6	B
5	M	0 1580	3	2.1 760	1.9 1200	B
Dummy	0	0	0	0 240	0	240
	240+B	B	1100	1000	1200	

B = 3540



Cost = $8016

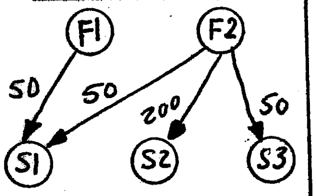
4

	F1	F2	S1	S2	S3	Dum	
F1	0 (500)	6	7 (50)	8	9	0 (150)	200 + B
F2	6	0 (500)	5 (50)	4 (200)	3 (50)	0	300 + B
S1	7	2	0 (500)	5	1	0	B
S2	1	5	1	0 (500)	4	0	B
S3	8	9	7	6	0 (500)	0	B
	B	B	100 + B	200 + B	50 + B	150	

$\boxed{B = 500}$

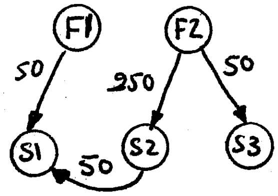
Alternative optima using TORA:

#1



Cost = $1,550

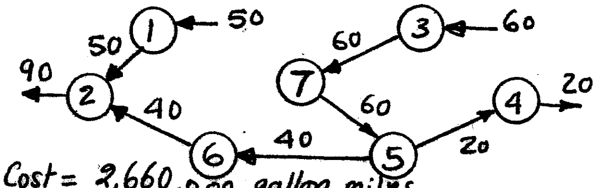
#2



5

assume that units of supply and demand are in thousand gallons. $\boxed{B = 110}$

	2	4	5	6	7	
1	20 (50)	M	M	M	3	50
3	M	30	M	M	9 (60)	60
5	M	2 (20)	0 (50)	4 (40)	10	B
6	8 (40)	M	4	0 (70)	M	B
7	40	M	10 (60)	M	0 (50)	B
	90	20	B	B	B	



Cost = 2,660,000 gallon miles

6

	2	3	4	5	6	7	
1	5	3 (1)	M	M	M	M	1
2	0 (1)	4	1	7	M	M	0+1
3	6	0	5	1	2 (1)	M	0+1
4	M	M	0 (1)	9	M	4	0+1
5	M	M	2	0 (1)	5	8	0+1
6	M	3	M	7	0	3 (1)	0+1
	0+1	0+1	0+1	0+1	0+1	1	

Optimum route using TORA:

$1 \rightarrow 3 \rightarrow 6 \rightarrow 7$

Distance = $3 + 2 + 3 = 8$

7

minimize z.

	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}	x_{35}	x_{36}	x_{46}	x_{47}	x_{56}	x_{67}	
z=	3	4	2	5	7	8	6	4	9	5	3	
①	1	1										= 1000
②			1	1								= 1200
③	-1		-1		1	1	1					= 0
④		-1		-1	-1			1	1			= 0
⑤						-1				1		= -800
⑥							-1	-1		-1	1	= -900
⑦									-1		-1	= -500

Each node yields a constraint. The special characteristics of the model show that each column has exactly +1 and -1, with the remainder of the elements equal to zero.

Set 5.5a

8

x_{ij} = number of laborers hired at the start of period i and terminated at the start of period j.

Define nodes 1, 2, 3, 4, and 5 to correspond to the five months of the horizon. Node 6 is added to allow defining the variables x_{i6} that terminate at the end of the five-month planning horizon. The associated LP is defined below.

	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	
	100	130	180.	220	250	100	130	180	220	100	130	180	100	130	100	min
(1)	1	1	1	1	1											≥ 100
(2)		1	1	1	1	1	1	1	1							≥ 120
(3)			1	1	1		1	1	1	1	1	1				≥ 80
(4)				1	1			1	1		1	1	1	1		≥ 170
(5)					1				1			1		1	1	≥ 50

Let S_1, S_2, S_3, S_4, and S_5 be the surplus variables associated with constraints 1, 2, 3, 4, and 5, respectively. The LP after adding the surplus variables thus appears as

x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	S_1	S_2	S_3	S_4	S_5	
100	130	180	220	250	100	120	180	220	100	130	180	100	130	100						min
1	1	1	1	1											−1					100
	1	1	1	1	1	1	1	1								−1				120
		1	1	1		1	1	1	1	1	1						−1			80
			1	1			1	1		1	1	1	1					−1		170
				1				1			1		1	1					−1	50

Next, perform the following transformation:

1. Leave equation (1) unchanged.
2. Replace equation (2) with (2) − (1).
3. Replace equation (3) with (3) − (2).
4. Replace equation (4) with (4) − (3).
5. Replace equation (5) with (5) − (4)
6. Add a new equation that equals −(5).

These transformations lead to the following LP

x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	S_1	S_2	S_3	S_4	S_5	
100	130	180	220	250	100	130	180	220	100	130	180	100	130	100						min
1	1	1	1	1											−1					100
−1					1	1	1	1							1	−1				20
	−1				−1				1	1	1					1	−1			−40
		−1				−1			−1			1	1				1	−1		90
			−1				−1			−1		−1		1				1	−1	−120
				−1				−1			−1		−1	−1					1	−50

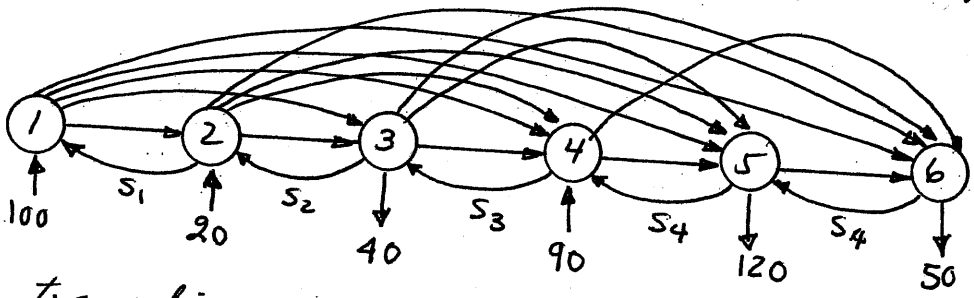
continued...

The last LP has the structure of a transshipment model (See Problem 7). Let

$S_1 = x_{21}$ $S_3 = x_{43}$ $S_5 = x_{65}$

$S_2 = x_{32}$ $S_4 = x_{54}$

Then the LP above can be translated as a network as follows:

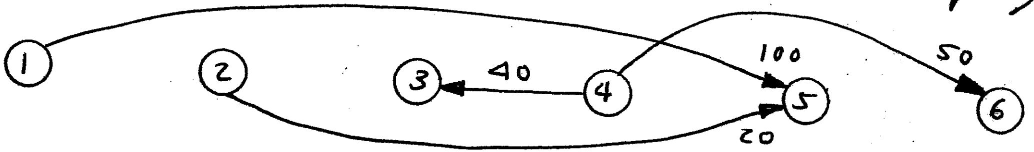


The transshipment model thus appears as

	1	2	3	4	5	6	
1	0	100	130	180	220	250	100 + *B*
2	0	0	100	130	180	220	20 + *B*
3	*M*	0	0	100	130	180	*B*
4	*M*	*M*	0	0	100	130	90 + *B*
5	*M*	*M*	*M*	0	0	100	*B*
6	*M*	*M*	*M*	*M*	0	0	*B*
	B	*B*	40 + *B*	*B*	120 + *B*	50 + *B*	

$B = 550$

The optimum solution from TORA is (Problem has alternative optima)



This solution can be interpreted as follows

1. Hire 100 laborers at the start of period 1 and terminate them at the start of period 5.
2. Hire 20 workers at the start of period 2 and terminate them at the start of period 5.
3. Hire 50 workers at the start of period 4 and terminate them at the start of period 6.

The solution satisfies the labor requirements exactly, except for period 3 where there is a surplus of 40 workers ($x_{43} = 40$).

CHAPTER 6

Network Models

Set 6.1a

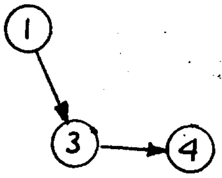
1

(i)

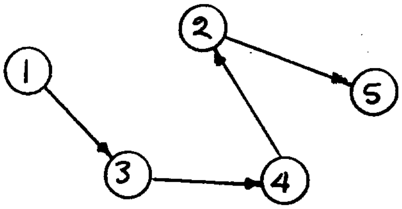
(a) Path: 1-3-4-2

(b) Cycle: 1-3-4-5-1

(c) Tree



(d) Spanning tree:

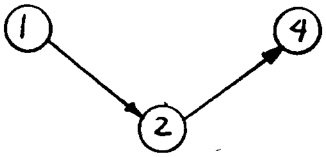


(ii)

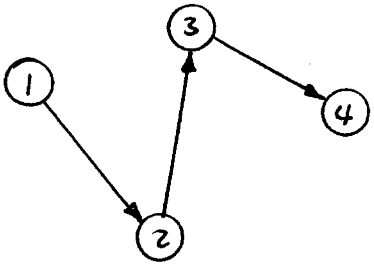
(a) Path: 1-2-3

(b) Cycle: 1-2-3-1

(c) Tree



(d) Spanning Tree:



2

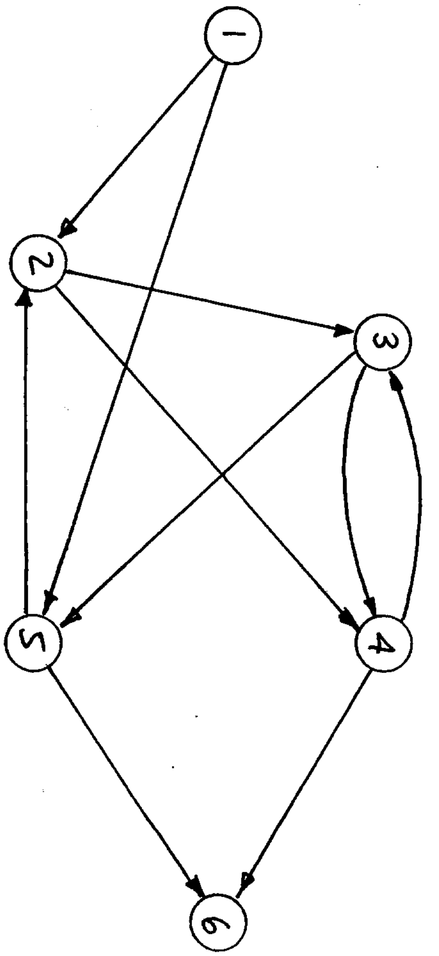
(i) N = {1, 2, 3, 4, 5}

A = {1-2, 1-3, 2-5, 3-4,
3-5, 4-2, 4-5, 5-1}

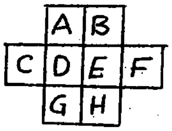
(ii) N = {1, 2, 3, 4}

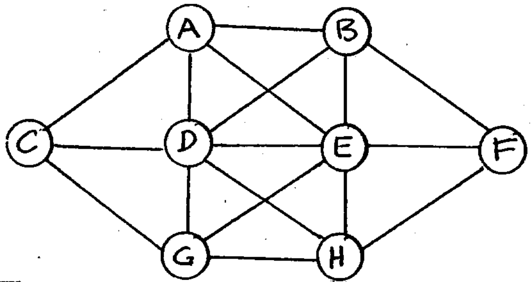
A = {1-2, 1-3, 2-3, 2-4, 3-4}

3

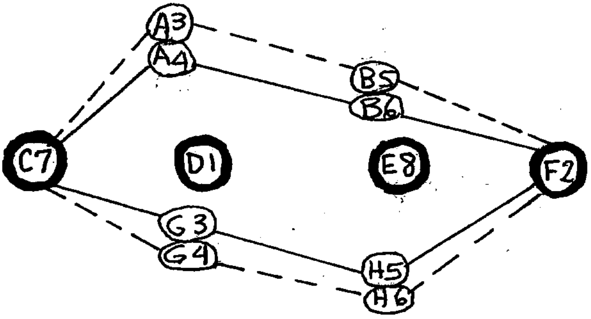


4

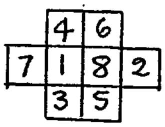


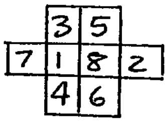


The network shows that nodes connected by an arc cannot hold consecutive numbers. Nodes D and E each has 6 emanating arcs, whereas all the remaining nodes have at most 4 emanating arcs. Because 1 and 8 each can have 6 nonconsecutive neighbors (namely, 1-3, 1-4, 1-5, 1-6, 1-7, 1-8 or 8-6, 8-5, 8-4, 8-3, 8-2, 8-1) and no other number has this property, 1 and 8 *must* be assigned to D and E. Letting D=1 and E=8, we *must* assign C=7 and F=2 because 2 and 7 can't be assigned anywhere else without violating the sequence condition. Next, we have the following possibilities:



Two possible solutions indicated by the solid and dashed arcs:





Switch D=1 and E=8 to two mirror arrangements.

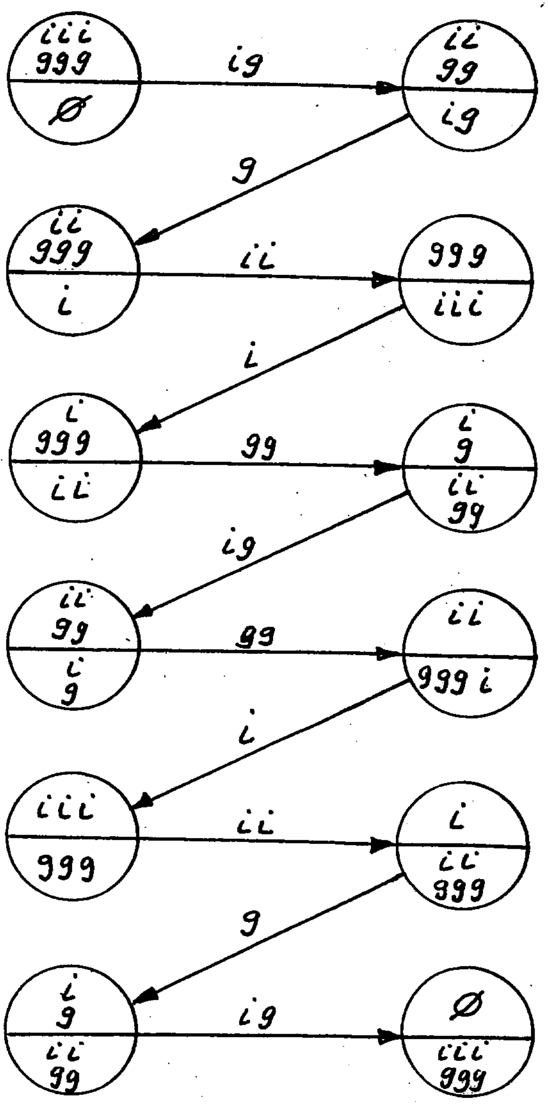
5

Let

i ≡ inmate

g ≡ guard

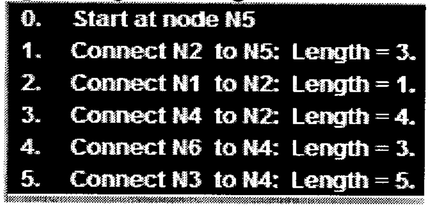
For each node, top half represents the number of i's and g's on the mainland side. The bottom half is that of Alcatraz.



Set 6.2a

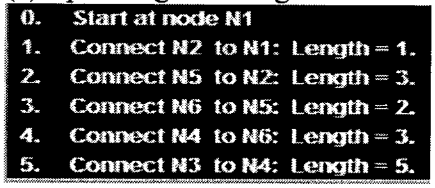
1

Spanning tree length = 16

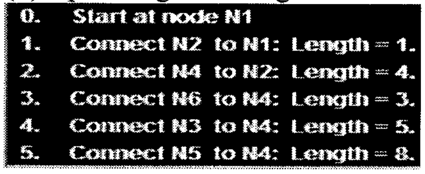

0. Start at node N5
1. Connect N2 to N5: Length = 3.
2. Connect N1 to N2: Length = 1.
3. Connect N4 to N2: Length = 4.
4. Connect N6 to N4: Length = 3.
5. Connect N3 to N4: Length = 5.

2

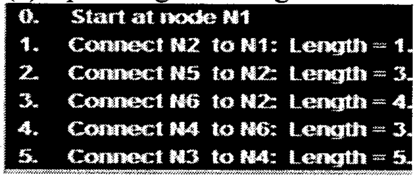
(a) Spanning tree length = 14


0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N5 to N2: Length = 3.
3. Connect N6 to N5: Length = 2.
4. Connect N4 to N6: Length = 3.
5. Connect N3 to N4: Length = 5.

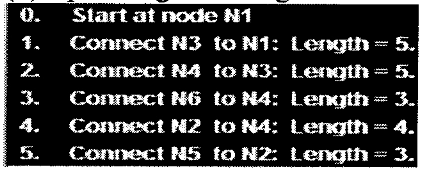
(b) Spanning tree length = 21


0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N4 to N2: Length = 4.
3. Connect N6 to N4: Length = 3.
4. Connect N3 to N4: Length = 5.
5. Connect N5 to N4: Length = 8.

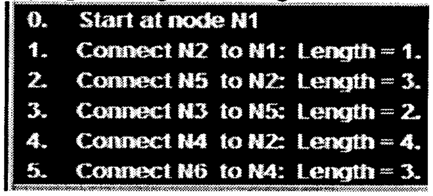
(c) Spanning tree length = 16


0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N5 to N2: Length = 3.
3. Connect N6 to N2: Length = 4.
4. Connect N4 to N6: Length = 3.
5. Connect N3 to N4: Length = 5.

(d) Spanning tree length = 20


0. Start at node N1
1. Connect N3 to N1: Length = 5.
2. Connect N4 to N3: Length = 5.
3. Connect N6 to N4: Length = 3.
4. Connect N2 to N4: Length = 4.
5. Connect N5 to N2: Length = 3.

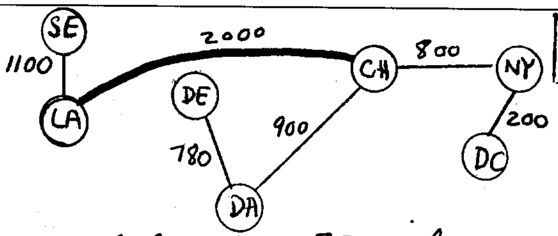
(e) Spanning tree length = 13


0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N5 to N2: Length = 3.
3. Connect N3 to N5: Length = 2.
4. Connect N4 to N2: Length = 4.
5. Connect N6 to N4: Length = 3.

(f) Spanning tree length = 21


0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N4 to N2: Length = 4.
3. Connect N6 to N4: Length = 3.
4. Connect N3 to N4: Length = 5.
5. Connect N5 to N4: Length = 8.

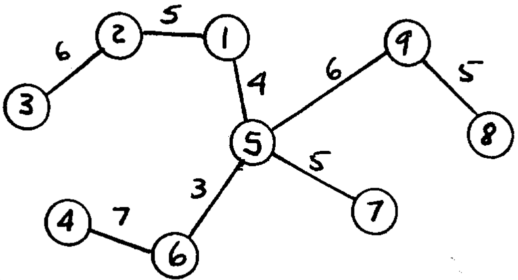
3



Total length = 5780 miles

4

Order of selection: 1-5, 5-6, 1-2, 5-7, 2-3, 5-9, 9-8, 6-4

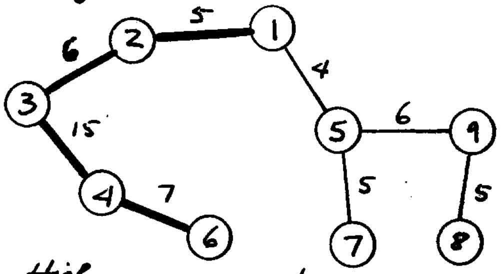


Total length = 41

5

Set length of arcs 3-5, 5-3, 4-5, 5-4, 4-7, 7-4, 5-6, and 6-5 to ∞

Total length = 53



6

(a) $d_{ij} = 1 - \dfrac{n_{ij}}{n_{ij} + m_{ij}}$

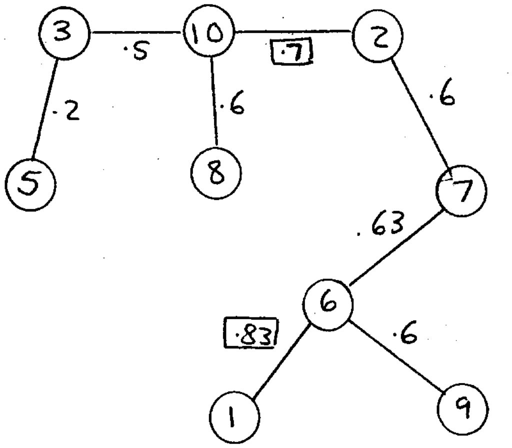
$i-j$	n_{ij}	m_{ij}	d_{ij}
1-2	0	10	1
1-3	0	6	1
1-4	0	8	1
1-5	0	7	1
1-6	1	5	.83
1-7	0	8	1
1-8	0	5	1
1-9	0	4	1
1-10	0	7	1

continued...

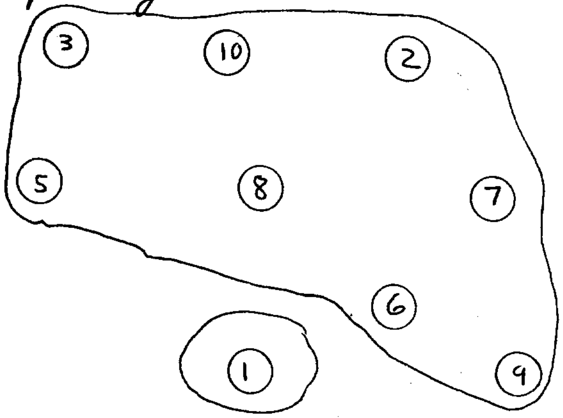
i-j	n_{ij}	m_{ij}	d_{ij}
2-3	1	10	.91
2-4	5	4	.44
2-5	1	11	.92
2-6	1	11	.92
2-7	4	6	.6
2-8	2	7	.78
2-9	0	10	1
2-10	3	7	.7
3-4	0	10	1
3-5	4	1	.2
3-6	2	5	.71
3-7	2	6	.75
3-8	1	5	.83
3-9	1	4	.8
3-10	3	3	.5
4-5	1	9	.9
4-6	0	11	1
4-7	3	6	.67
4-8	0	9	1
4-9	0	8	1
4-10	1	9	.9
5-6	2	6	.75
5-7	2	7	.78
5-8	1	6	.86
5-9	1	5	.83
5-10	3	4	.57
6-7	3	5	.63
6-8	1	6	.86
6-9	2	3	.60
6-10	1	8	.89
7-8	0	9	1
7-9	1	6	.86
7-10	1	9	.9
8-9	1	3	.75
8-10	2	4	.67
9-10	1	5	.83

continued...

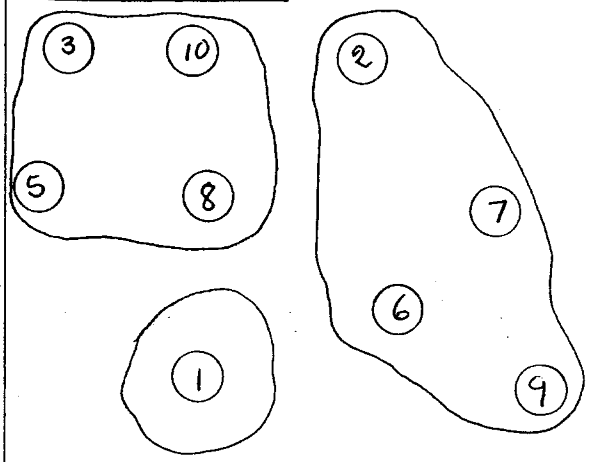
(b) Spanning tree



(c) A 2-cell solution is formed by removing the highest link in the minimal spanning tree.



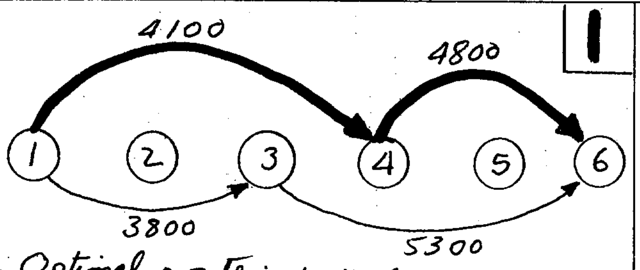
3-cell solution:



continued...

Set 6.3a

1



Optimal route: 1-4-6

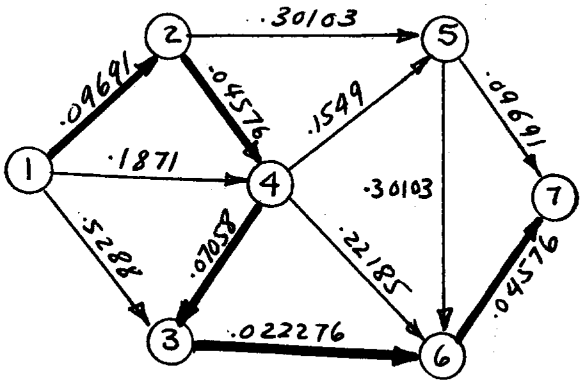
Decision: Buy new cars in 2001 & 2004.

2

$\max(P_1P_2\cdots P_n)$

$\equiv \max(\log P_1 + \log P_2 + \cdots + \log P_n)$

$\equiv \min(-\log P_1 - \log P_2 - \cdots - \log P_n)$



Optimum solution by TORA:

1-2-4-3-6-7

$\sum_{i=1}^{7} -\log P_i = .281286$. Thus,

$\sum_{i=1}^{7} \log P_i = -.281286$.

Hence,

$p = 10^{-.28128} = .52326$

5

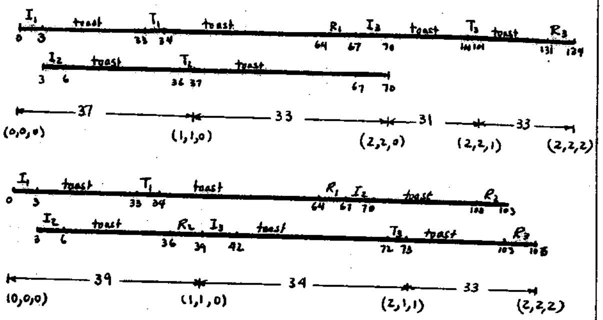
Define

(i, j, k) = number of sides toasted of slices 1, 2, and 3

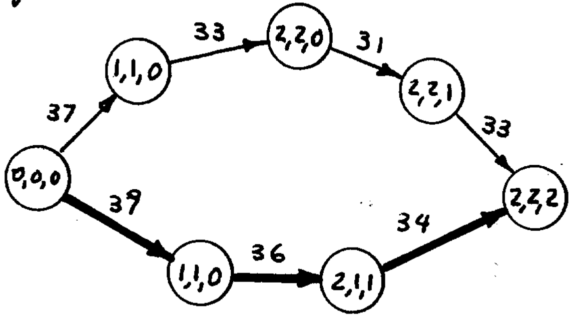
The two time charts below provides a summary of the times between the successive nodes.

Problem 4 on p. 6-7

continued...



The associated network is thus given as



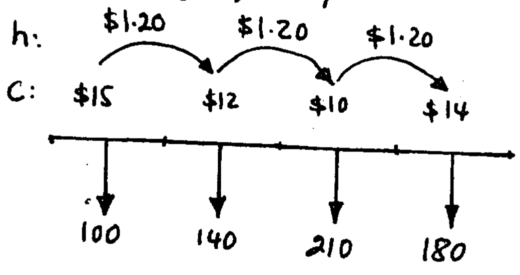
The optimal sequence is (0,0,0) → (1,1,0) → (2,1,1) → (2,2,2). It is interpreted as follows:

- Toast both sides of slice 1 successively (without interruption) in side A.
- Toast side 1 of slice 2 in side B, then remove slice 2.
- Toast both sides of slice 3 in side B
- Toast side 2 of slice 2 in side A after slice 1 is toasted.

Total time = 106 seconds.

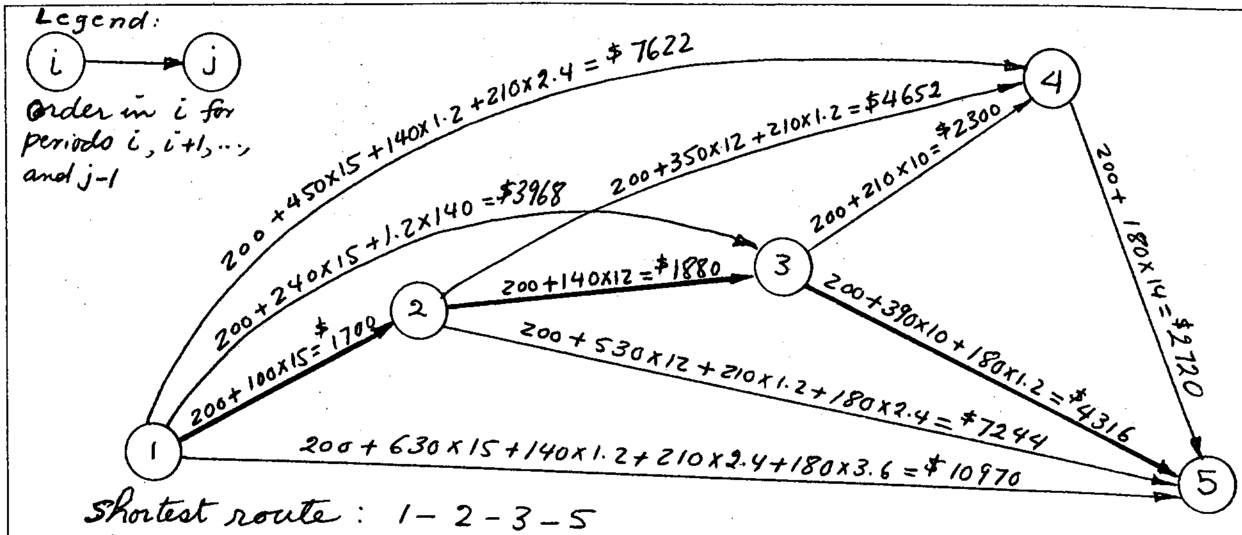
3

Summary of the problem data



Setup cost = $200

continued...



Shortest route: 1-2-3-5

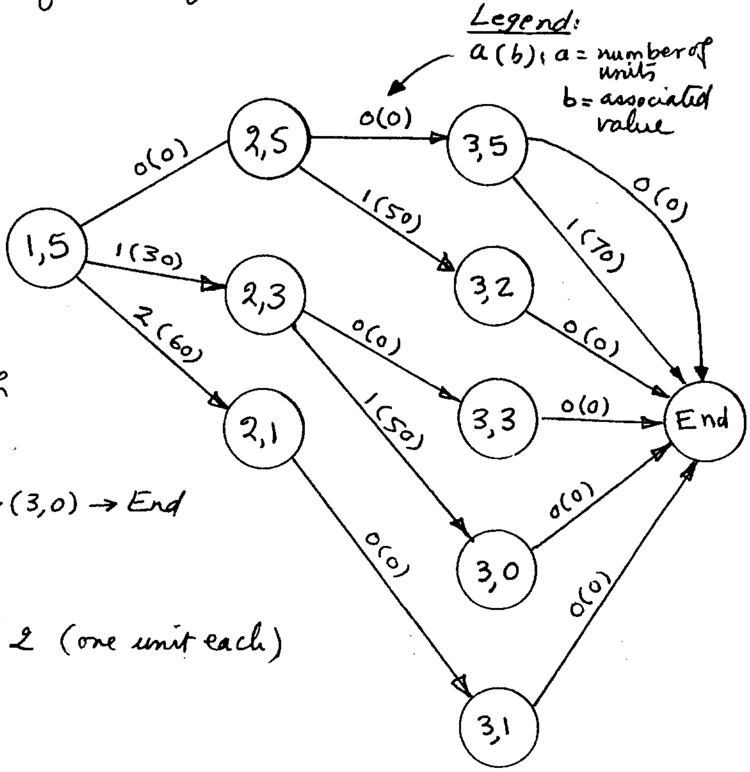
Interpretation of the solution: order 100 units in Period 1, 140 units in Period 2, and 390 units in Period 3. Total cost = $ 7896

4

Define node (i, v), where i is the item number and v is the volume remaining before item i is selected. Each arc represents a feasible value of the number of units of item i.

Item i	1	2	3
Volume/unit	2	3	4
Value/unit	30	50	70

Total available volume = 5 ft^3



The objective is to determine the longest path between (1,5) and (End).

Longest path: (1,5) → (2,3) → (3,0) → End

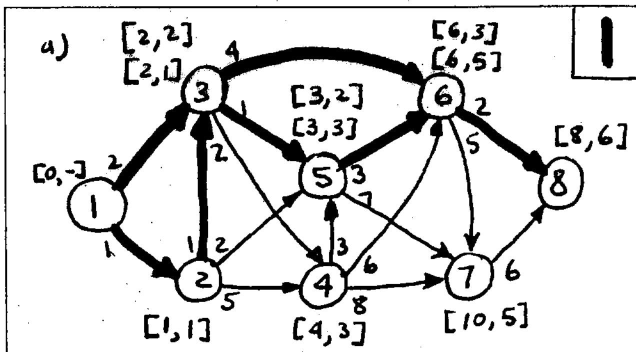
Interpretation of the solution:

Select items 1 and 2 (one unit each)

Total value = 80

1

a)



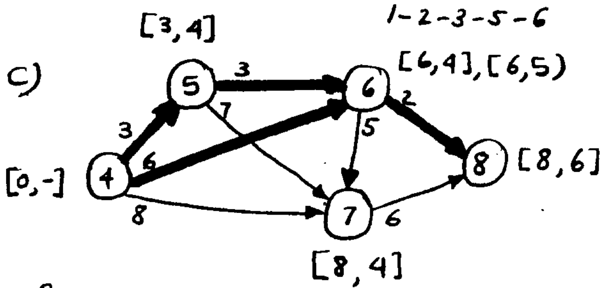
shortest distance = 8:

alternative routes:
1-3-6-8
1-2-3-6-8
1-3-5-6-8
1-2-3-5-6-8

b) From part (a), shortest distance between ① and ⑥ is 6.

alternative routes:
1-3-6
1-3-5-6
1-2-3-6
1-2-3-5-6

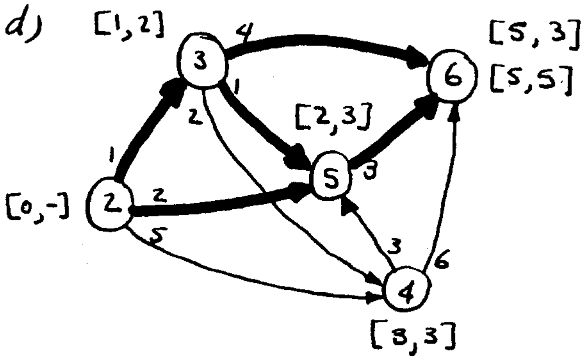
c)



shortest distance = 8

alternative routes:
4-5-6-8
4-6-8

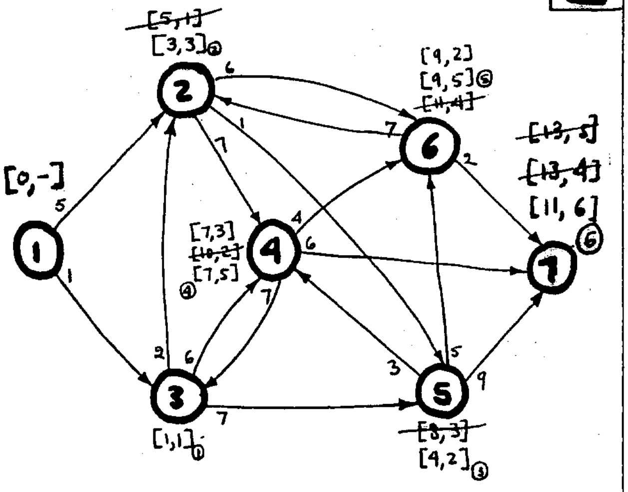
d)



shortest distance = 5

Alternative routes = {
2-3-6
2-3-5-6
2-5-6

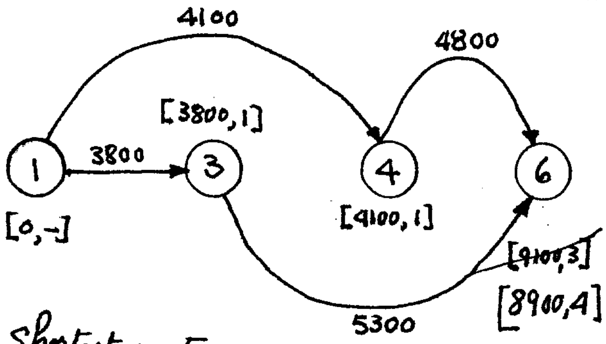
2



Shortest routes:		Length
1-2 :	1-3-2	3
1-3 :	1-3	1
1-4 :	1-3-4 1-3-2-5-4	7
1-5 :	1-3-2-5	4
1-6 :	1-3-2-5-6 1-3-2-6	9
1-7 :	1-3-2-5-6-7 1-3-2-6-7	11

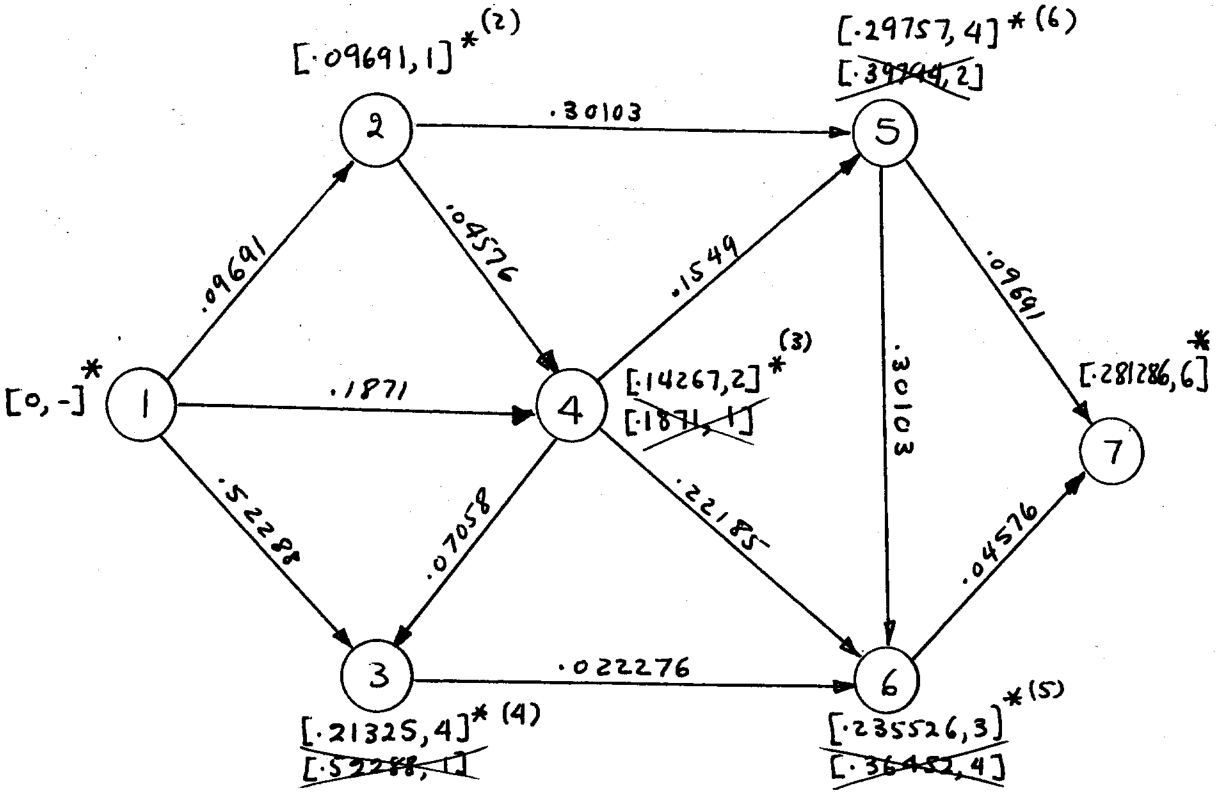
3

(a)



Shortest route: 1-4-6. Cost = $ 8900
Buy in 2001 & 2004

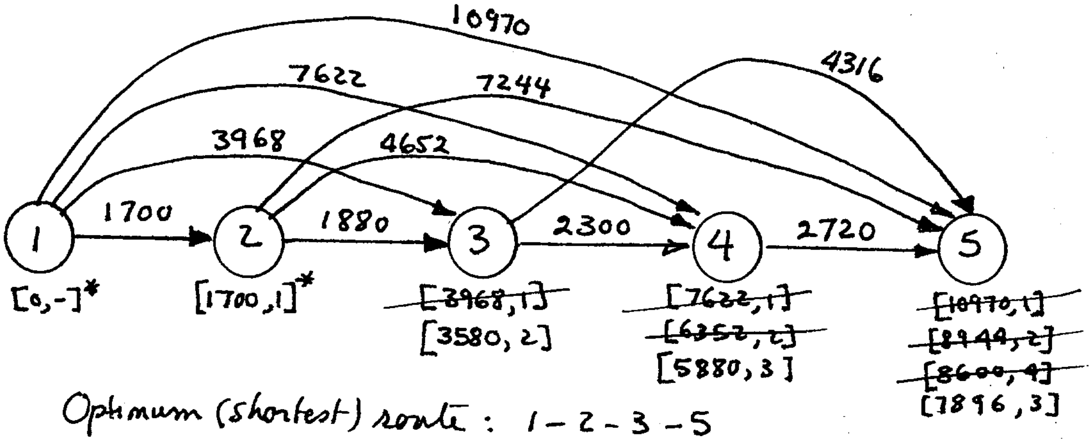
3(b)



Solution: 1-2-4-3-5-6, Route value = .281286

Probability = 10^-.281286 = .52326

3(c)



Optimum (shortest) route: 1-2-3-5

Solution: Order in 1 for 1

Order in 2 for 2

Order in 3 for 3 and 4

Iteration 5

Array D5

	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:		4.00	3.00	9.00	6.00	7.00	12.00
N2:	4.00		1.00	5.00	2.00	3.00	8.00
N3:	3.00	1.00		6.00	3.00	4.00	9.00
N4:	9.00	5.00	6.00		3.00	4.00	3.00
N5:	6.00	2.00	3.00	3.00		1.00	6.00
N6:	7.00	3.00	4.00	1.00	1.00		4.00
N7:	12.00	8.00	9.00	3.00	6.00	4.00	

Array S5

	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:		3	3	3	3	5	4
N2:	3		3	4	5	5	4
N3:	1	2		2	2	5	4
N4:	3	2	2		5	5	7
N5:	3	2	2	4		6	4
N6:	5	5	5	4	5		4
N7:	4	4	4	4	4	6	

Iteration 6

Array D6

	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:		4.00	3.00	8.00	6.00	7.00	11.00
N2:	4.00		1.00	4.00	2.00	3.00	7.00
N3:	3.00	1.00		5.00	3.00	4.00	8.00
N4:	9.00	5.00	6.00		3.00	4.00	3.00
N5:	6.00	2.00	3.00	2.00		1.00	5.00
N6:	7.00	3.00	4.00	1.00	1.00		4.00
N7:	11.00	7.00	8.00	3.00	5.00	4.00	

Array S6

	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:		3	3	6	3	5	6
N2:	3		3	6	5	5	6
N3:	1	2		6	2	5	6
N4:	3	2	2		5	5	7
N5:	3	2	2	6		6	6
N6:	5	5	5	4	5		4
N7:	6	6	6	4	6	6	

(a) $\boxed{1-7}$ distance = 11

1-6-7 ⇒ 1-5-6-7 ⇒ 1-3-5-6-7 ⇒
1-3-2-5-6-7 ⇒ 1-3-2-5-6-4-7

(b) $\boxed{7-1}$ distance = 11

7-6-1
7-6-5-1
7-6-5-3-1
7-6-5-2-3-1

(c) $\boxed{6-7}$ distance = 4

6-4-7

3

Iteration 0

Array D0

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		700.00	200.00	infinity	infinity	infinity
N2:	infinity		300.00	200.00	infinity	400.00
N3:	200.00	300.00		700.00	600.00	infinity
N4:	infinity	200.00	700.00		300.00	100.00
N5:	infinity	infinity	600.00	300.00		500.00
N6:	infinity	400.00	infinity	100.00	500.00	

Array S0

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		2	3	4	5	6
N2:	1		3	4	5	6
N3:	1	2		4	5	6
N4:	1	2	3		5	6
N5:	1	2	3	4		6
N6:	1	2	3	4	5	

Iteration 1

Array D1

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		700.00	200.00	infinity	infinity	infinity
N2:	infinity		300.00	200.00	infinity	400.00
N3:	200.00	300.00		700.00	600.00	infinity
N4:	infinity	200.00	700.00		300.00	100.00
N5:	infinity	infinity	600.00	300.00		500.00
N6:	infinity	400.00	infinity	100.00	500.00	

Array S1

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		2	3	4	5	6
N2:	1		3	4	5	6
N3:	1	2		4	5	6
N4:	1	2	3		5	6
N5:	1	2	3	4		6
N6:	1	2	3	4	5	

Iteration 2

Array D2

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		700.00	200.00	900.00	infinity	1100.00
N2:	infinity		300.00	200.00	infinity	400.00
N3:	200.00	300.00		500.00	600.00	700.00
N4:	infinity	200.00	500.00		300.00	100.00
N5:	infinity	infinity	600.00	300.00		500.00
N6:	infinity	400.00	700.00	100.00	500.00	

Array S2

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		2	3	2	5	2
N2:	1		3	4	5	6
N3:	1	2		2	5	2
N4:	1	2	2		5	6
N5:	1	2	3	4		6
N6:	1	2	2	4	5	

Iteration 3

Array D3

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		500.00	200.00	700.00	800.00	900.00
N2:	500.00		300.00	200.00	900.00	400.00
N3:	200.00	300.00		500.00	600.00	700.00
N4:	700.00	200.00	500.00		300.00	100.00
N5:	800.00	900.00	600.00	300.00		500.00
N6:	900.00	400.00	700.00	100.00	500.00	

Array S3

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		3	3	3	3	3
N2:	3		3	4	3	6
N3:	1	2		2	5	2
N4:	3	2	2		5	6
N5:	3	3	3	4		6
N6:	3	2	2	4	5	

continued...

Set 6.3c

Iteration 4

Array D4

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		500.00	200.00	700.00	800.00	800.00
N2:	500.00		300.00	200.00	500.00	300.00
N3:	200.00	300.00		500.00	600.00	600.00
N4:	700.00	200.00	500.00		300.00	100.00
N5:	800.00	500.00	600.00	300.00		400.00
N6:	800.00	300.00	600.00	100.00	400.00	

Array S4

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		3	3	3	3	4
N2:	3		3	4	4	4
N3:	1	2		2	5	4
N4:	3	2	2		5	6
N5:	3	4	3	4		4
N6:	4	4	4	4	4	

Iteration 5

Array D5

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		500.00	200.00	700.00	800.00	800.00
N2:	500.00		300.00	200.00	500.00	300.00
N3:	200.00	300.00		500.00	600.00	600.00
N4:	700.00	200.00	500.00		300.00	100.00
N5:	800.00	500.00	600.00	300.00		400.00
N6:	800.00	300.00	600.00	100.00	400.00	

Array S5

	N1:	N2:	N3:	N4:	N5:	N6:
N1:		3	3	3	3	4
N2:	3		3	4	4	4
N3:	1	2		2	5	4
N4:	3	2	2		5	6
N5:	3	4	3	4		4
N6:	4	4	4	4	4	

Shortest routes:

From	To	Distance	Route
1	2	500.00	1- 3- 2
1	3	200.00	1- 3
1	4	700.00	1- 3- 2- 4
1	5	800.00	1- 3- 5
1	6	800.00	1- 3- 2- 4- 6
2	1	500.00	2- 3- 1
2	3	300.00	2- 3
2	4	200.00	2- 4
2	5	500.00	2- 4- 5
2	6	300.00	2- 4- 6
3	1	200.00	3- 1
3	2	300.00	3- 2
3	4	500.00	3- 2- 4
3	5	600.00	3- 5
3	6	600.00	3- 2- 4- 6
4	1	700.00	4- 2- 3- 1
4	2	200.00	4- 2
4	3	500.00	4- 2- 3
4	5	300.00	4- 5
4	6	100.00	4- 6
5	1	800.00	5- 3- 1
5	2	500.00	5- 4- 2
5	3	600.00	5- 3
5	4	300.00	5- 4
5	6	400.00	5- 4- 6
6	1	800.00	6- 4- 2- 3- 1
6	2	300.00	6- 4- 2
6	3	600.00	6- 4- 2- 3
6	4	100.00	6- 4
6	5	400.00	6- 4- 5

continued...

4

Iteration 0

Array D0

	N1:joe	N2:bob	N3:kay	N4:jim	N5:rae	N6:kim
N1:joe		1.00	infinity	infinity	infinity	1.00
N2:bob	infinity		1.00	infinity	infinity	infinity
N3:kay	infinity	1.00		1.00	1.00	infinity
N4:jim	infinity	infinity	1.00		infinity	infinity
N5:rae	infinity	infinity	infinity	infinity		1.00
N6:kim	1.00	1.00	infinity	infinity	infinity	

Array S0

	N1:joe	N2:bob	N3:kay	N4:jim	N5:rae	N6:kim
N1:joe		2	3	4	5	6
N2:bob	1		3	4	5	6
N3:kay	1	2		4	5	6
N4:jim	1	2	3		5	6
N5:rae	1	2	3	4		6
N6:kim	1	2	3	4	5	

Iteration 1

Array D1

	N1:joe	N2:bob	N3:kay	N4:jim	N5:rae	N6:kim
N1:joe		1.00	infinity	infinity	infinity	1.00
N2:bob	infinity		1.00	infinity	infinity	infinity
N3:kay	infinity	1.00		1.00	1.00	infinity
N4:jim	infinity	infinity	1.00		infinity	infinity
N5:rae	infinity	infinity	infinity	infinity		1.00
N6:kim	1.00	1.00	infinity	infinity	infinity	

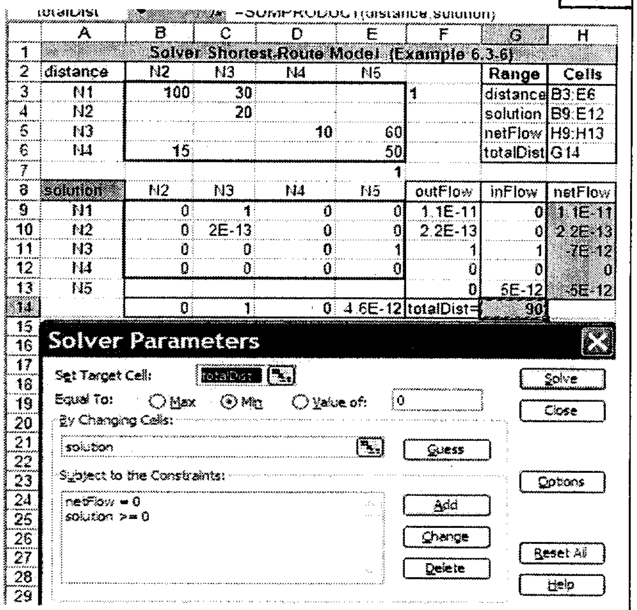
Array S1

	N1:joe	N2:bob	N3:kay	N4:jim	N5:rae	N6:kim
N1:joe		2	3	4	5	6
N2:bob	1		3	4	5	6
N3:kay	1	2		4	5	6
N4:jim	1	2	3		5	6
N5:rae	1	2	3	4		6
N6:kim	1	2	3	4	5	

continued...

1

(a)



totalDist =SUMPRODUCT(distance,solution)

	A	B	C	D	E	F	G	H
1	Solver Shortest-Route Model (Example 6.3-6)							
2	distance	N2	N3	N4	N5		Range	Cells
3	N1	100	30			1	distance	B3:E6
4	N2		20				solution	B9:E12
5	N3			10	60		netFlow	H9:H13
6	N4	15			50		totalDist	G14
7					1			
8	solution	N2	N3	N4	N5	outFlow	inFlow	netFlow
9	N1	0	1	0	0	1.1E-11	0	1.1E-11
10	N2	0	2E-13	0	0	2.2E-13	0	2.2E-13
11	N3	0	0	0	1	1	1	-7E-12
12	N4	0	0	0	0	0	0	0
13	N5					0	5E-12	-5E-12
14		0	1	0	4.6E-12	totalDist=	90	

Set Target Cell: totalDist

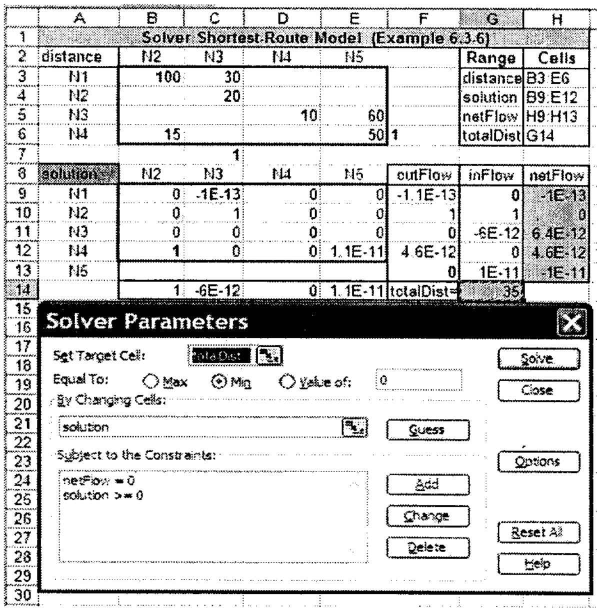
Equal To: ○ Max ● Min ○ Value of: 0

By Changing Cells: solution

Subject to the Constraints:

netFlow = 0
solution >= 0

(b)



	A	B	C	D	E	F	G	H
1	Solver Shortest-Route Model (Example 6.3-6)							
2	distance	N2	N3	N4	N5		Range	Cells
3	N1	100	30				distance	B3:E6
4	N2		20				solution	B9:E12
5	N3			10	60		netFlow	H9:H13
6	N4	15			50	1	totalDist	G14
7			1					
8	solution	N2	N3	N4	N5	outFlow	inFlow	netFlow
9	N1	0	-1E-13	0	0	-1.1E-13	0	-1E-13
10	N2	0	1	0	0	1	1	0
11	N3	0	0	0	0	0	-6E-12	6.4E-12
12	N4	1	0	0	1.1E-11	4.6E-12	0	4.6E-12
13	N5					0	1E-11	-1E-11
14		1	-6E-12	0	1.1E-11	totalDist=	35	

Set Target Cell: totalDist

Equal To: ○ Max ● Min ○ Value of: 0

By Changing Cells: solution

Subject to the Constraints:

netFlow = 0
solution >= 0

2

```
param n;
param start;
param end;
param p{1..n,1..n} default 0;
param rhs{i in 1..n}=if i=start then 1 else (if i=end then -1 else
0);

var x{i in 1..n,j in 1..n:p[i,j]>0}>=0;
var outFlow{i in 1..n}=sum{j in 1..n:p[i,j]>0}x[i,j];
var inFlow{j in 1..n}=sum{i in 1..n:p[i,j]>0}x[i,j];
var logProb=sum{i in 1..n}sum{j in 1..n:p[i,j]>0}-log(p[i,j])*x[i,j];
var prob=2.718^-logProb;

minimize z: sum {i in 1..n, j in 1..n:p[i,j]>0}-log(p[i,j])*x[i,j];
subject to limit {i in 1..n}: outFlow[i]-inFlow[i]=rhs[i];

data;
param n:=7;
param start:=4;
param end:=7;

param p:
  1  2  3  4   5  6  7:=
1 . .8 .3 .65 .  .  .
2 . .  .  .9  .5 .  .
3 . .  .  .   .  .95 .
4 . .  .85 .  .7 .  .
5 . .  .  .   .  .5 .8
6 . .  .  .   .  .  .9;

solve;
display z,logProb,prob, x;
```

1

(a) Surplus capacities:

2-3: 40-0 = 40 units
2-5: 30-20 = 10 units
4-3: 5-0 = 5 units

All other arcs have zero surplus capacities.

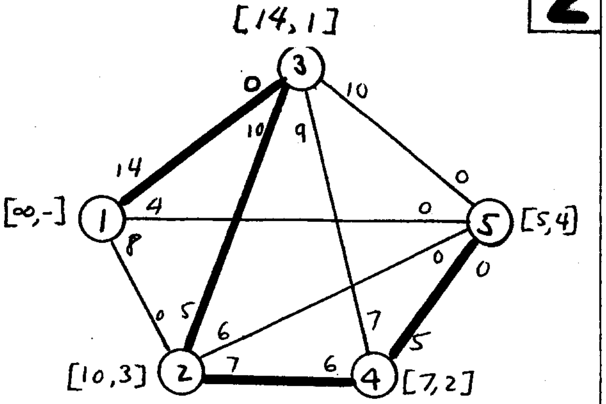
(b)

Flow through node 2 = 20 units
Flow through node 3 = 30 units
Flow through node 4 = 20 units

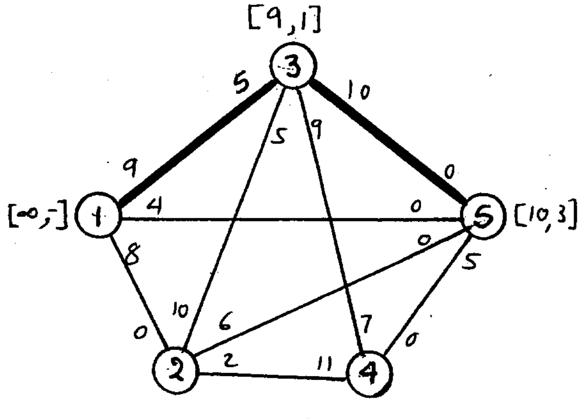
(c)

No, because the arcs out of node 1 have zero surplus capacity

2

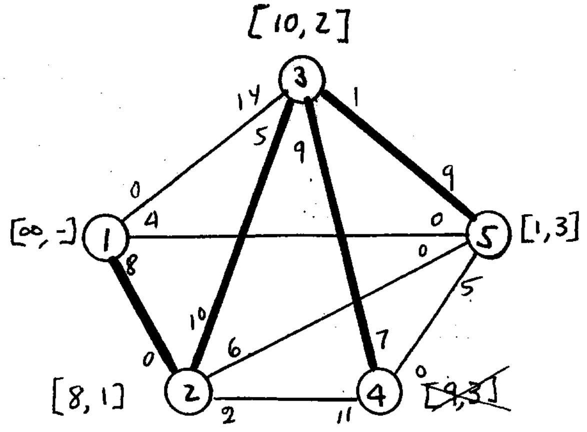


f_1 = 5 units

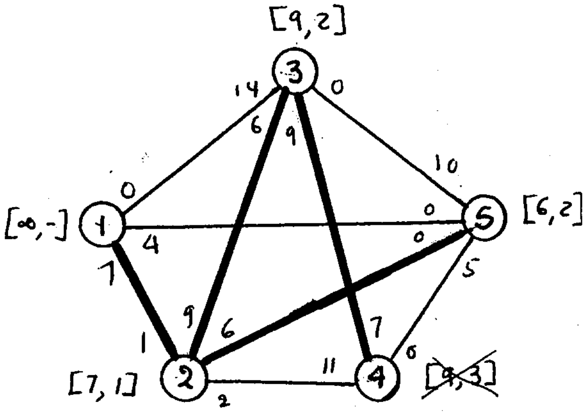


f_2 = 9 units

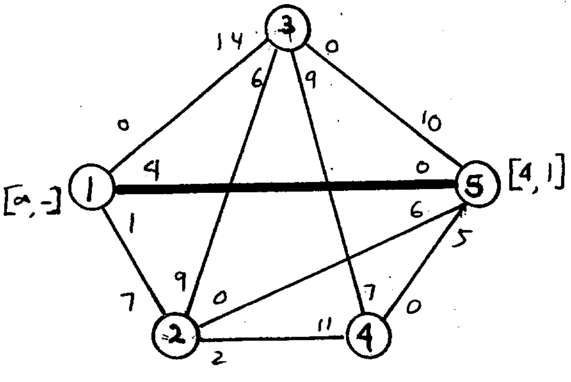
continued...



f_3 = 1 unit



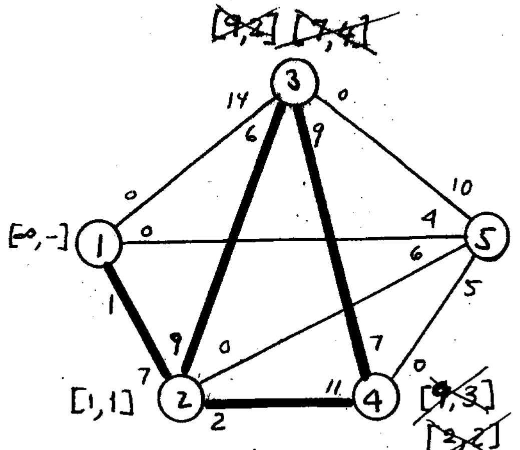
f_4 = 6 units



f_5 = 4 units

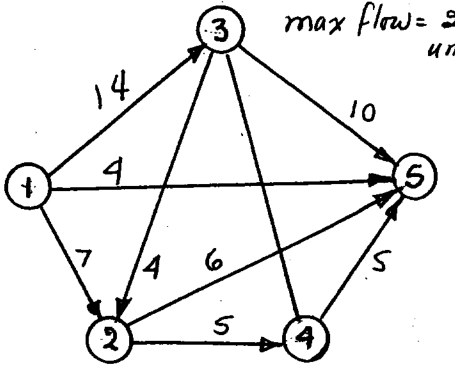
continued...

Set 6.4b



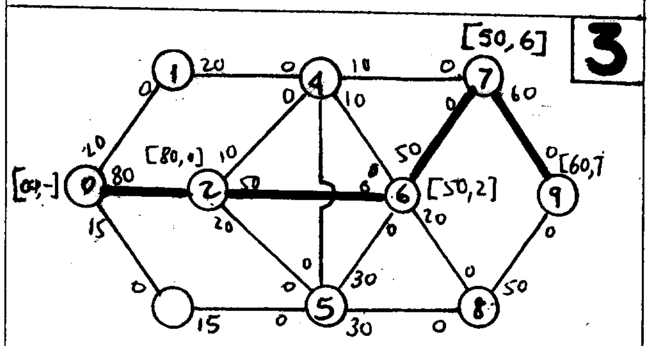
No breakthrough

Solution:

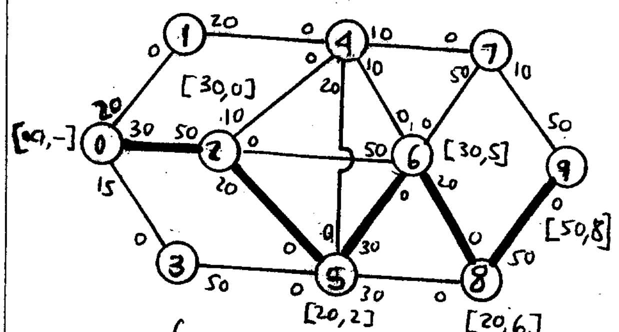


max flow= 25 units

3

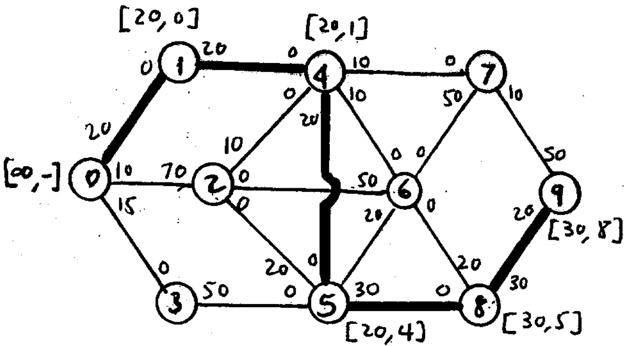


f_1 = 50 units

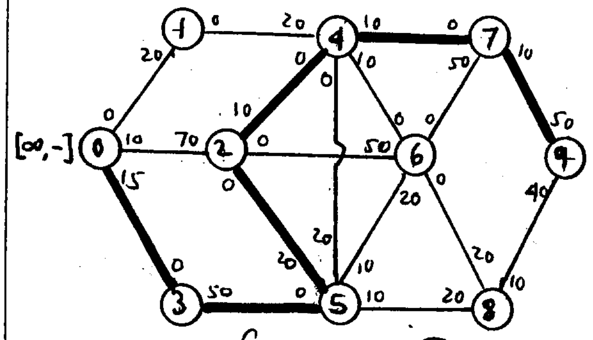


f_2 = 20 units

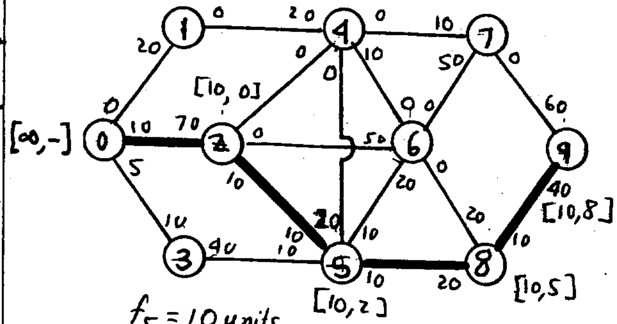
continued...



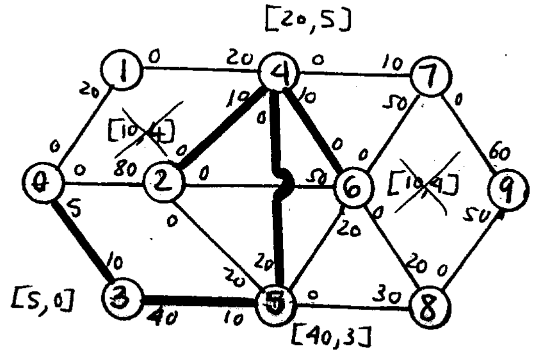
f_3 = 20 units



f_4 = 10 units



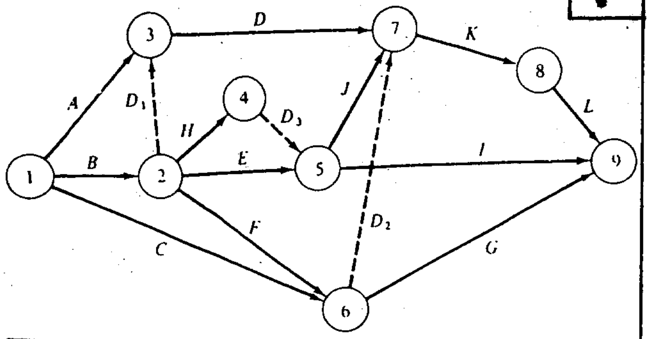
f_5 = 10 units



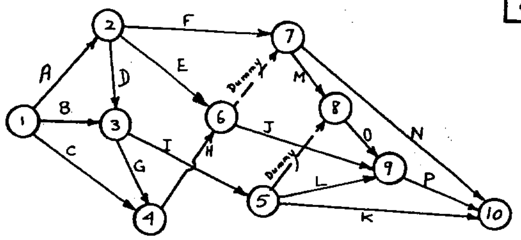
continued...

Set 6.5a

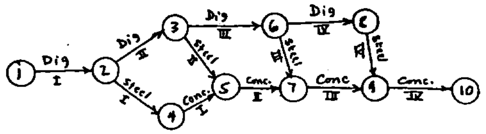
1



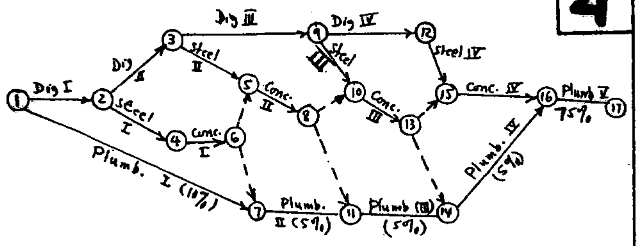
2



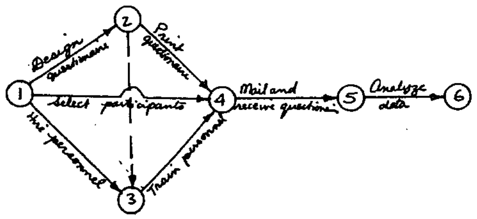
3



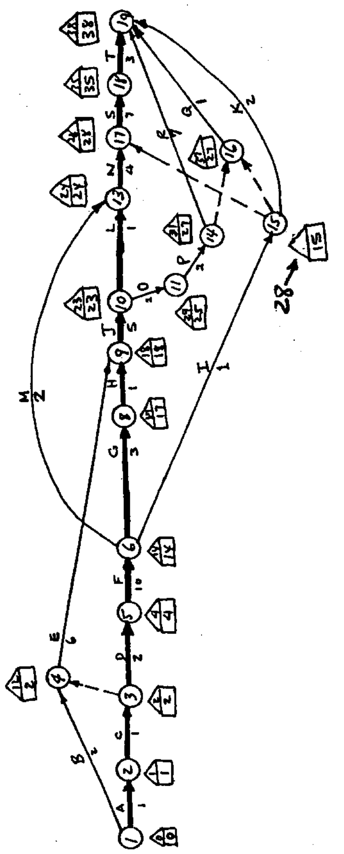
4



5



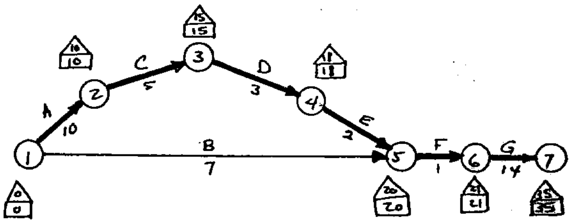
6



Critical path = 1-2-3-5-6-8-10-13-17-18-19

Duration = 38 days

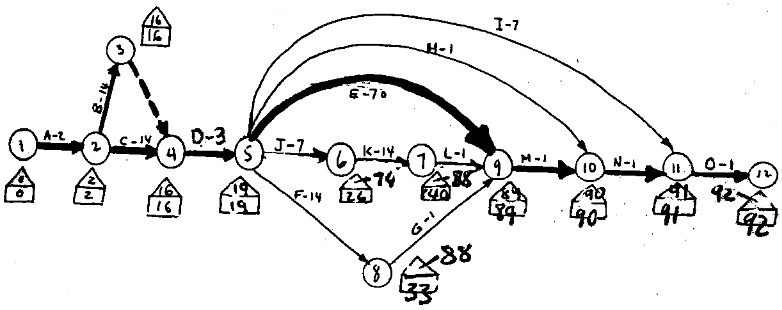
7



Critical path: 1-2-3-4-5-6-7

Duration: 35 days

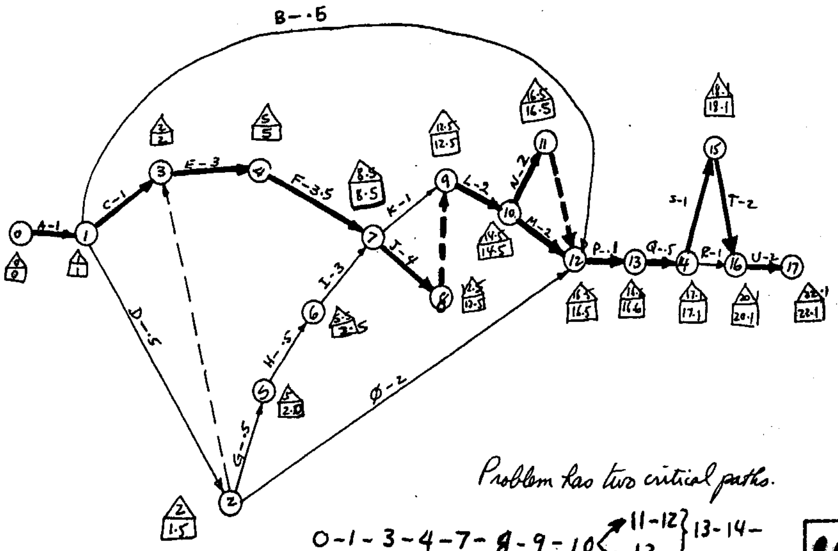
8



critical paths:

1-2-3-4-5-9-10-11-12
1-2-4-5-9-10-11-12

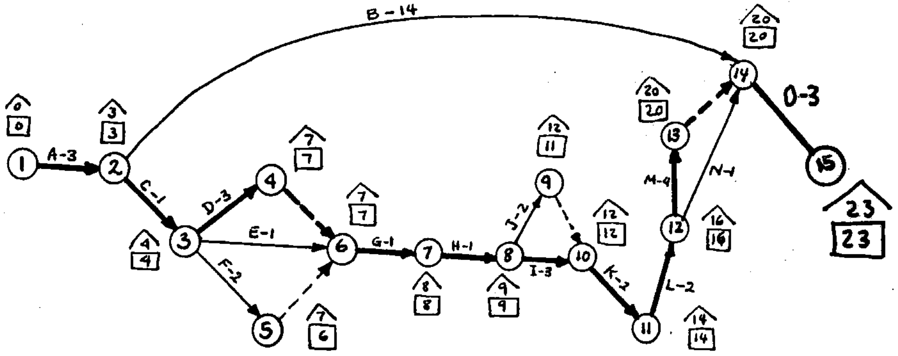
9



Problem has two critical paths.

0-1-3-4-7-8-9-10 ↗ 11-12 } 13-14-
↘ 12 } 15-16-17

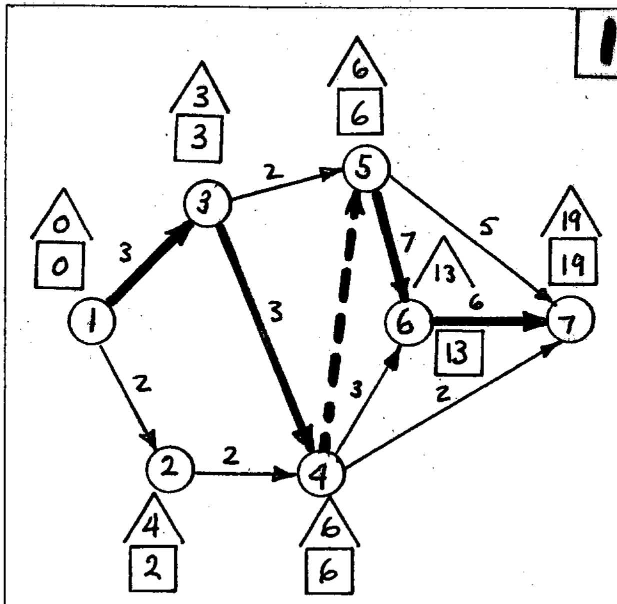
10



Critical path: 1-2-3-4-6-7-8-10-11-12-13-14-15
Duration: 20 days

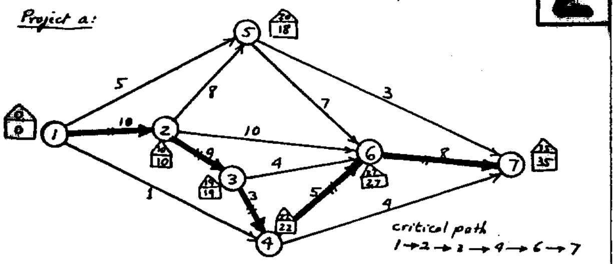
Set 6.5b

1



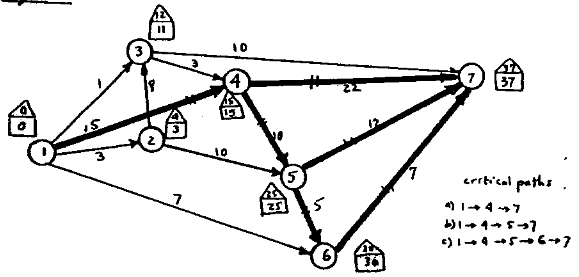
2

Project a:



critical path
1 → 2 → 3 → 4 → 6 → 7

Project b:



critical paths
a) 1 → 4 → 7
b) 1 → 4 → 5 → 7
c) 1 → 4 → 5 → 6 → 7

3

See solution to Problem 6, Set 6.6a

4

See solution to Problem 8, Set 6.6a

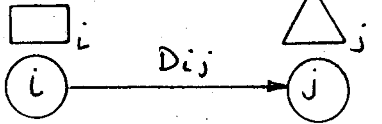
5

See solution to Problem 9, Set 6.6a

6

See solution to Problem 10, Set 6.6a

1



Earliest completion of (i,j) = $\square_i + D_{ij}$

Latest start of (i,j) = $\Delta_j - D_{ij}$

2

Both floats are zero by definition

3

(a) $FF = 10$, $TF = 10$, $D = 4$
maximum delay = 10
(b) $FF = 5$, $TF = 10$, $D = 4$
maximum delay = 5
(c) $FF = 0$, $TF = 10$, $D = 4$
maximum delay = 0

4

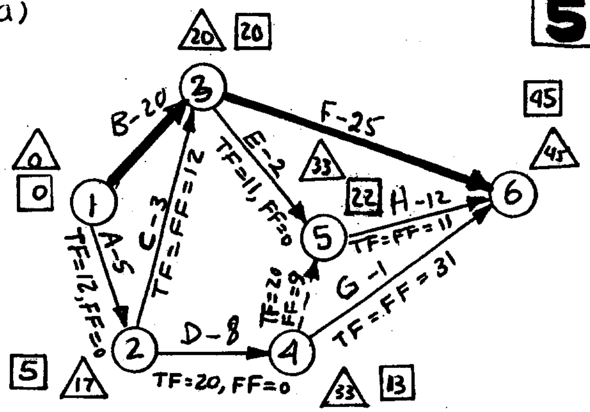
(a) <u>For B</u>: $TF = 5$, $FF = 2$
Because $FF = 2$, a delay of 1 has no effect on succeeding activities.
<u>For C</u>: Starting at time 5 implies no delay. Thus, the earliest start time for E and F is time 8.

(b) <u>For B</u>: Delay = 3, FF = 2. Thus, the start of E and F must be delayed by at least 3 − 2 = 1.
<u>For C</u>: Delay = 7 − 5 = 2, FF = 0. Thus, start of E and F must be delayed by at least 2.
<u>For B & C combined</u>: Start of E and F must be delayed by max(1, 2) = 2.

(c) Delay in B = 6. Because FF of B = 2, the start of E and F must be delayed by 4. Next, a delay of 4 in E will delay critical H by 1 because $FF_E = 3$. Also, a delay of 4 in E will not impact other activities in the project. Thus, the proposed delay in B will delay the entire project by 1 (because of the delay in critical H).

5

(a)



(b) Red-flagged activities are A, D, and E.

(c) $FF_A = 0$: $\text{Delay}_A = 5$ will delay each of C and D by 5.

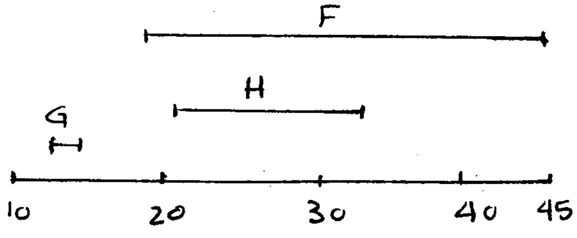
$FF_D = 0$: $\text{Delay}_D = 5$ will delay G by 5.

$FF_C = 12$: $\text{Delay}_C = 5$ does not affect other activities

<u>Conclusion</u>: Start of C, D, and G is delayed by 5.

<u>Note</u>: If you use TORA to experiment with the effect of $\text{Delay}_A = 5$, the chart will only show a delay in C and D, but not in G The effect of C and D on succeeding activities must be done manually. To effect that <u>after</u> $\text{Delay}_A = 5$ is implemented, select C with $\text{delay}_C = 0$ and D with $\text{delay}_D = 0$. $\text{Delay}_C = 0$ produces no action, but $\text{delay}_D = 0$ will delay G and Dummy properly to match $\text{delay}_A = 5$.

(d)



Two unit of equipment are required.

Set 6.5c

6

```
                    *** CPM SOLUTION ***

Title:          .(a)

Size:  7 nodes x 13 activities
-------------------------------------------------------------------------
                        Earliest          Latest        Total    Free
Activity  Duration    start  Compl.    start  compl.    float    float
-------------------------------------------------------------------------
c 1-2        10.0      0.0    10.0      0.0    10.0      0.0      0.0
  1-4         1.0      0.0     1.0     21.0    22.0     21.0     21.0
  1-5         5.0      0.0     5.0     15.0    20.0     15.0     13.0
c 2-3         9.0     10.0    19.0     10.0    19.0      0.0      0.0
  2-5         8.0     10.0    18.0     12.0    20.0      2.0      0.0
  2-6        10.0     10.0    20.0     17.0    27.0      7.0      7.0
c 3-4         3.0     19.0    22.0     19.0    22.0      0.0      0.0
  3-6         4.0     19.0    23.0     23.0    27.0      4.0      4.0
c 4-6         5.0     22.0    27.0     22.0    27.0      0.0      0.0
  4-7         4.0     22.0    26.0     31.0    35.0      9.0      9.0
  5-6         7.0     18.0    25.0     20.0    27.0      2.0      2.0
  5-7         3.0     18.0    21.0     32.0    35.0     14.0     14.0
c 6-7         8.0     27.0    35.0     27.0    35.0      0.0      0.0
```

```
                    *** CPM SOLUTION ***

Title:      (b)

Size:  7 nodes x 13 activities
-------------------------------------------------------------------------
                      Earliest          Latest         Total    Free
Activity Duration   start  Compl.    start  compl.     float    float
-------------------------------------------------------------------------
  1-2        3.0     0.0     3.0      1.0     4.0       1.0      0.0
  1-3        1.0     0.0     1.0     11.0    12.0      11.0     10.0
c 1-4       15.0     0.0    15.0      0.0    15.0       0.0      0.0
  1-6        7.0     0.0     7.0     23.0    30.0      23.0     23.0
  2-3        8.0     3.0    11.0      4.0    12.0       1.0      0.0
  2-5       10.0     3.0    13.0     15.0    25.0      12.0     12.0
  3-4        3.0    11.0    14.0     12.0    15.0       1.0      1.0
  3-7       10.0    11.0    21.0     27.0    37.0      16.0     16.0
c 4-5       10.0    15.0    25.0     15.0    25.0       0.0      0.0
c 4-7       22.0    15.0    37.0     15.0    37.0       0.0      0.0
c 5-6        5.0    25.0    30.0     25.0    30.0       0.0      0.0
c 5-7       12.0    25.0    37.0     25.0    37.0       0.0      0.0
c 6-7        7.0    30.0    37.0     30.0    37.0       0.0      0.0
```

Project (a):

Red flagged activities:

(1-5), TF = 15, FF = 13

(2-5), TF = 2, FF = 0

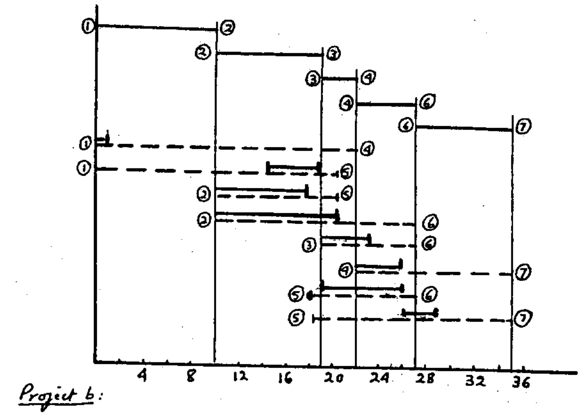
Project (b):

The following activities are red-flagged:

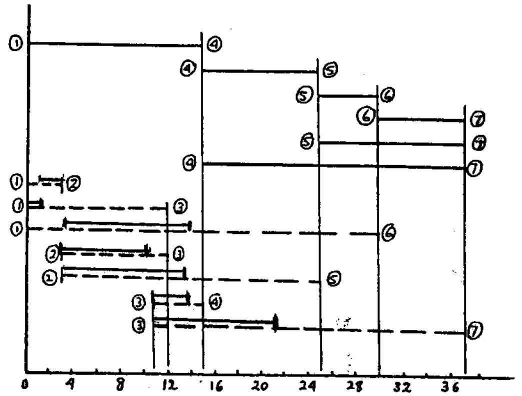
Activity	TF	FF
1 – 2	1	0
1 – 3	11	10
2 – 3	1	0

continued...

Project a:



Project b:



In project (a), note the delay in the start of activity 5-6 to account for the effect of starting (1-5) at time 14.

1

	x_{12}	x_{13}	x_{24}	x_{34}	x_{35}	x_{45}	x_{46}	x_{47}	x_{56}	x_{57}	x_{67}	
Maximize z =	3	3	2	3	2	0	3	2	7	5	6	
Node 1	-1	-1										= -1
Node 2	1		-1									= 0
Node 3		1		-1	-1							= 0
Node 4			1	1		-1	-1	-1				= 0
Node 5					1	1			-1	-1		= 0
Node 6							1		1		-1	= 0
Node 7								1		1	1	= 1

Optimal:

$x_{13} = x_{34}$ x_{45} $x_{56} = x_{67} = 1$

$Z = 19$

2

(a)

	x_{12}	x_{14}	x_{15}	x_{23}	x_{25}	x_{26}	x_{34}	x_{36}	x_{46}	x_{47}	x_{56}	x_{57}	x_{67}	
Maximize z =	10	1	5	9	8	10	3	4	5	4	7	3	8	
Node 1	-1	-1	-1											= -1
Node 2	1			-1	-1	-1								= 0
Node 3				1			-1	-1						= 0
Node 4		1					1		-1	-1				= 0
Node 5			1		-1				1		-1	-1		= 0
Node 6						1		1			1		-1	= 0
Node 7										1		1	1	= 1

Optimum: $x_{12} = x_{23} = x_{34} = x_{46} = x_{67} = 1$, $Z = 35$

(b)

	x_{12}	x_{13}	x_{14}	x_{16}	x_{23}	x_{25}	x_{34}	x_{37}	x_{45}	x_{47}	x_{56}	x_{57}	x_{67}	
Maximize z =	3	1	15	7	8	10	3	10	10	22	5	12	7	
Node 1	-1	-1	-1											= -1
Node 2	1			-1	-1	-1								= 0
Node 3		1			1		-1	-1						= 0
Node 4			1				1		-1	-1				= 0
Node 5						1			1		-1	-1		= 0
Node 6				1						1	1		-1	= 0
Node 7								1				1	1	= 1

Optimum:

$$\left.\begin{array}{l} x_{14} = x_{47} = 1 \\ x_{14} = x_{45} = x_{57} = 1 \\ x_{14} = x_{45} = x_{56} = x_{67} = 1 \end{array}\right\} \text{alternative optima} \quad Z = 37$$

Set 6.5e

I

Project (a)

Title:

Activity	Mean Duration	Variance
1 - 2	4.00	0.11
1 - 4	2.83	0.25
1 - 5	3.83	0.25
2 - 3	5.00	0.11
2 - 5	8.17	0.25
2 - 6	9.50	0.69
3 - 4	10.00	5.44
3 - 6	4.00	0.11
4 - 6	7.67	1.00
4 - 7	6.17	0.25
5 - 6	10.67	1.00
5 - 7	6.00	0.44
6 - 7	4.00	0.11

Title:

Node	Longest Path	Path Mean	Path Std. Dev.
2	1- 2	4.00	0.33
3	1- 2- 3	9.00	0.47
4	1- 2- 3- 4	19.00	2.38
5	1- 2- 5	12.17	0.60
6	1- 2- 3- 4- 6	26.67	2.58
7	1- 2- 3- 4- 6- 7	30.67	2.60

Event	Latest occurrence time, LC	P{occurrence time ≤ LC}
2	4	.5
3	9	.5
4	19	.5
5	16	1.0
6	26.67	.5
7	30.67	.5

LC is determined by carrying out CPM calculations using average duration time

Example of probability calculations:

For node 5:

$$P\{T \le 16\} = P\{z \le \frac{16-12.17}{.6}\}$$
$$= P\{z \le 6.38\} \simeq 1$$

continued...

Project (b)

Title:

Activity	Mean Duration	Variance
1 - 2	2.83	0.25
1 - 3	6.83	0.25
1 - 4	7.17	0.25
1 - 6	2.00	0.11
2 - 3	4.00	0.11
2 - 5	8.00	0.11
3 - 4	15.00	2.78
3 - 7	13.00	0.11
4 - 5	12.17	0.69
4 - 7	10.00	0.44
5 - 6	8.33	0.44
5 - 7	4.33	1.00
6 - 7	6.00	0.11

Title:

Node	Longest Path	Path Mean	Path Std. Dev.
2	1- 2	2.83	0.50
3	1- 3	6.83	0.50
4	1- 3- 4	21.83	1.74
5	1- 3- 4- 5	34.00	1.93
6	1- 3- 4- 5- 6	42.33	2.04
7	1- 3- 4- 5- 6- 7	48.33	2.07

Event	Latest occurrence time, LC	P{Occurrence time ≤ LC}
2	2.83	.5
3	6.83	.5
4	21.83	.5
5	34.00	.5
6	42.33	.5
7	48.33	.5

All events happen to fall on the critical path (using average durations). This is the reason all probabilities = .5

CHAPTER 7

Advanced Linear programming

Set 7.1a

1

$Q = \{x_1, x_2 \mid x_1 + x_2 \le 1,\ x_1 \ge 0,\ x_2 \ge 0$

Let $(\bar{x}_1, \bar{x}_2) \ge 0$ and $(\bar{\bar{x}}_1, \bar{\bar{x}}_2) \ge 0$ be two distinct points in Q and define for $0 \le \lambda \le 1$:

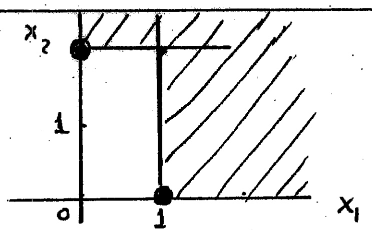
$$(x_1, x_2) = \lambda(\bar{x}_1, \bar{x}_2) + (1-\lambda)(\bar{\bar{x}}_1, \bar{\bar{x}}_2) \ge 0$$

Then,

$$x_1 + x_2 = \lambda\bar{x}_1 + (1-\lambda)\bar{\bar{x}}_1 + \lambda\bar{x}_2 + (1-\lambda)\bar{\bar{x}}_2$$
$$= \lambda(\bar{x}_1 + \bar{x}_2) + (1-\lambda)(\bar{\bar{x}}_1 + \bar{\bar{x}}_2)$$
$$\le \lambda(1) + (1-\lambda)(1) = 1$$

which shows that Q is convex. The result is true even without the nonnegativity restrictions.

2



$Q = \{x_1, x_2 \mid x_1 \ge 1 \text{ or } x_2 \ge 2\}$

Let $(\bar{x}_1, \bar{x}_2) = (1, 0) \in Q$

$(\bar{\bar{x}}_1, \bar{\bar{x}}_2) = (0, 2) \in Q$

Consider

$$(x_1, x_2) = \lambda(1,0) + (1-\lambda)(0,2)$$
$$= (\lambda, 2-2\lambda) \quad 0 \le \lambda \le 1$$

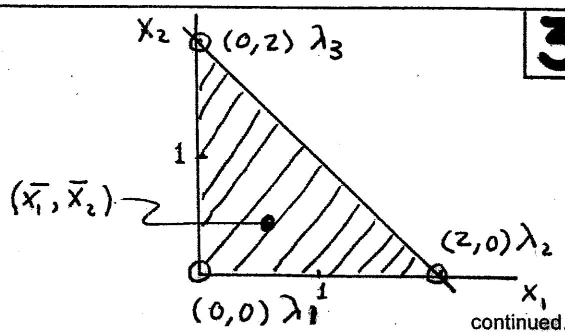
For $0 < \lambda < 1$, we have

$x_1 = \lambda < 1$

$x_2 = 2 - 2\lambda < 2$

Thus, $(x_1, x_2) \notin Q$.

3



continued..

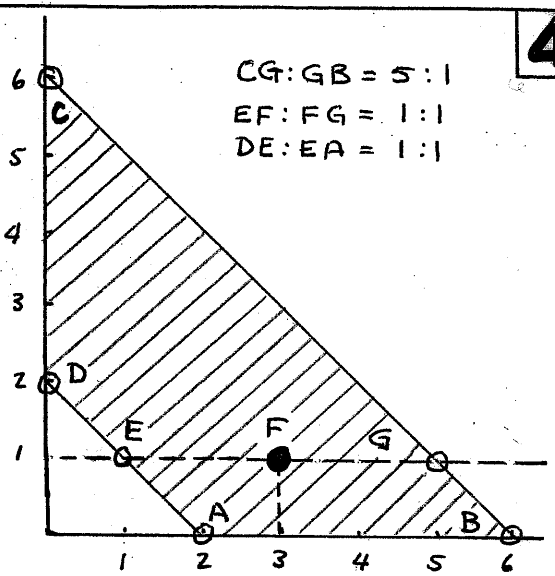
$Q = \{x_1, x_2 \mid x_1 + x_2 \le 2,\ x_1, x_2 \ge 0\}$

$$(\bar{x}_1, \bar{x}_2) = \lambda_1(0,0) + \lambda_2(2,0) + \lambda_3(0,2)$$
$$= (2\lambda_2, 2\lambda_3)$$

where $\lambda_1, \lambda_2, \lambda_3 \ge 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

4



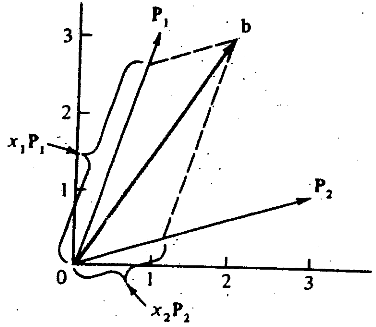
$E = \frac{1}{2}A + \frac{1}{2}D$

$G = \frac{5}{6}B + \frac{1}{6}C$

$$F = \frac{1}{2}E + \frac{1}{2}G$$
$$= \frac{1}{2}\left(\frac{1}{2}A + \frac{1}{2}D\right) + \frac{1}{2}\left(\frac{5}{6}B + \frac{1}{6}C\right)$$
$$= \frac{1}{4}A + \frac{1}{4}D + \frac{5}{12}B + \frac{1}{12}C$$
$$= \frac{1}{4}(2,0) + \frac{1}{4}(0,2) + \frac{5}{12}(6,0) + \frac{1}{12}(0,6)$$
$$= (3,1)$$

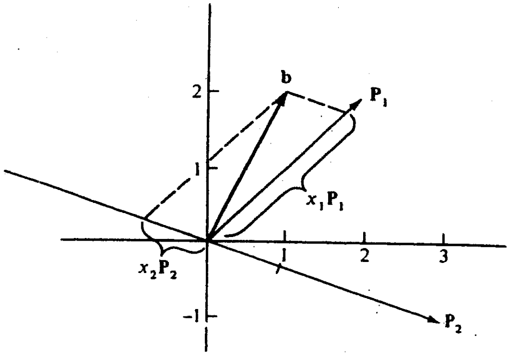
1

(a)



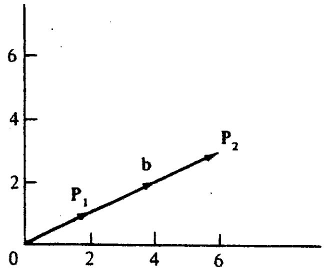
Unique solution:
$(x_1, x_2) = (7/8, 3/8)$,
left-side vectors P_1 and P_2
are independent (basis)

(b)

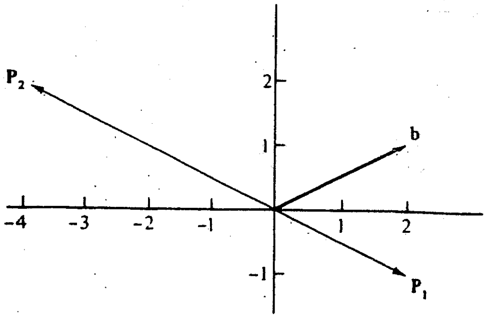


Unique solution:
$(x_1, x_2) = (7/8, -1/4)$,
P_1 and P_2 form a basis

(c)



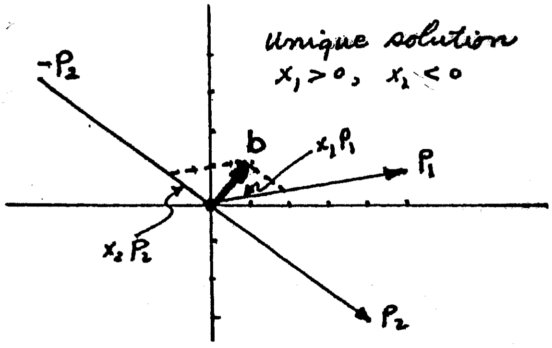
Infinity of solutions:
P_1 and P_2 are dependent
(no basis); b is also
dependent



No solution: P_1 and P_2
are dependent (no basis),
but b is independent

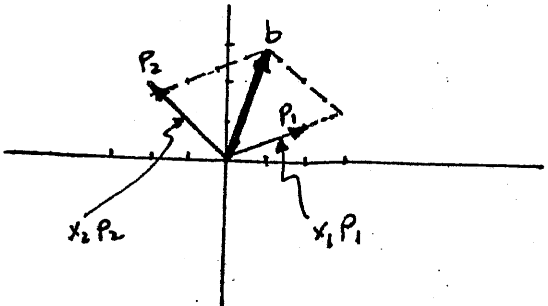
2

(a) $\begin{pmatrix}5 & 4\\ 1 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$



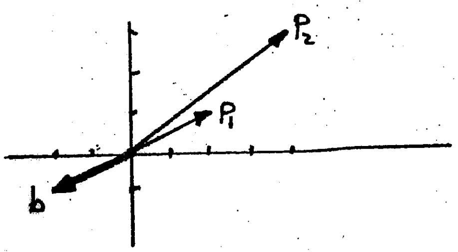
Unique solution
$x_1 > 0, \; x_2 < 0$

b) $\begin{pmatrix}2 & -2\\ 1 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}1\\ 3\end{pmatrix}$



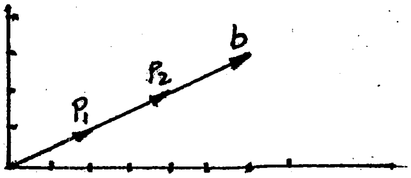
Unique solution: $x_1, x_2 > 0$
$x_1 > 1, \; x_2 < 1$

(c) $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$



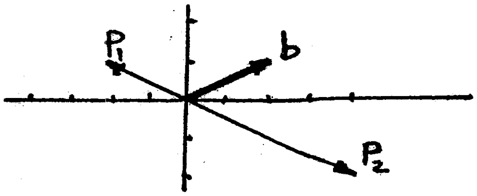
Unique solution: $x_1 < 0$, $x_2 = 0$

(d) $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$



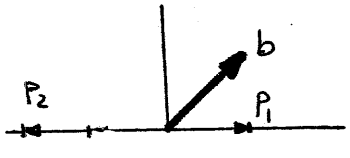
Infinity of solutions

(e) $\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



No solution

(f) $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



No solution

3

(a) $\det(P_1, P_2, P_3) = \det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$

$= -4 \neq 0$, basis

(b) $\det(P_1, P_2, P_4) = \det\begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$

$= -8 \neq 0$, basis

(c) $\det(P_2, P_3, P_4) = \det\begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}$

$= 0$, not a basis

(d) In this problem, a basis must include exactly 3 independent vectors.

4

(a) True

(b) True

(c) True

1

$$B=(P_3,P_4)=\begin{pmatrix}2&4\\-2&6\end{pmatrix}$$

$$B^{-1}=\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix},\quad X_B=\begin{pmatrix}x_3\\x_4\end{pmatrix},\; C_B=(7,5)$$

$$X_B=B^{-1}b=\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}\begin{pmatrix}10\\5\end{pmatrix}=\begin{pmatrix}2\\1.5\end{pmatrix}\geq\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_BB^{-1}=(7,5)\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}=(2.6,-.9)$$

$$\{z_j-c_j\}_{j=1,2}=(2.6,-.9)\begin{pmatrix}2&1\\3&-1\end{pmatrix}-(1,4)$$

$$=(1.5,-.5)$$

$$B^{-1}(P_1,P_2)=\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}\begin{pmatrix}2&1\\3&-1\end{pmatrix}=\begin{pmatrix}0&.5\\.5&0\end{pmatrix}$$

X_B is feasible but not optimal.

Tableau:

	x_1	x_2	x_3	x_4	
z	1.5	-.5	0	0	21.5
x_3	0	.5	1	0	2
x_4	.5	0	0	1	1.5

2

Maximize $z=(5,12,4)\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

Subject to

$$\begin{array}{cccc}\begin{pmatrix}1&2&1&1\\2&-2&-1&0\end{pmatrix}\end{array}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}10\\2\end{pmatrix}$$

$$\begin{array}{cccc}P_1&P_2&P_3&P_4\end{array}$$

$$\det(P_1\,P_2)=\det\begin{pmatrix}1&2\\2&-2\end{pmatrix}$$

$$=-6\neq0\Rightarrow\text{basis}$$

$$\det(P_2\,P_3)=\det\begin{pmatrix}2&1\\-2&-1\end{pmatrix}$$

$$=0\Rightarrow\text{not a basis}$$

$$\det(P_3\,P_4)=\det\begin{pmatrix}1&1\\-1&0\end{pmatrix}$$

$$=1\neq0\Rightarrow\text{basis}$$

3

$X_B=(x_1,x_2,x_5)^T$, $C_B=(2,1,0)$

$$B^{-1}=\begin{pmatrix}3&1&0\\4&3&0\\1&2&1\end{pmatrix}^{-1}=\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}$$

$$C_BB^{-1}=(2,1,0)\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}$$

$$=(2/5,1/5,0)$$

$$(z_3-c_3,\;z_4-c_4)$$

$$=(2/5,1/5,0)\begin{pmatrix}-1&0\\0&-1\\0&0\end{pmatrix}-(0,0)$$

$$=(-2/5,-1/5)\Rightarrow\text{optimal}$$

$$B^{-1}(P_1\;P_2\;P_3\;P_4\;P_5\,|\,b)$$

$$=\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}\left(\begin{array}{ccccc|c}3&1&-1&0&0&3\\4&3&0&-1&0&6\\1&2&0&0&1&3\end{array}\right)$$

$$=\left(\begin{array}{ccccc|c}1&0&-3/5&1/5&0&3/5\\0&1&4/5&-3/5&0&6/5\\0&0&-1&1&1&0\end{array}\right)$$

feasible

$$z=C_B(B^{-1}b)=(2,1,0)\begin{pmatrix}3/5\\6/5\\0\end{pmatrix}=12/5$$

	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	-2/5	-1/5	0	12/5
x_1	1	0	-3/5	1/5	0	3/5
x_2	0	1	4/5	-3/5	0	6/5
x_5	0	0	-1	1	1	0

4

$X_B=(x_3,x_2,x_1)^T$, $C_B=(0,c_2,c_1)$

$$C_BB^{-1}=(0,c_2,c_1)\begin{pmatrix}1&1&-1\\0&1&0\\0&-1&1\end{pmatrix}=(0,c_2-c_1,c_1)$$

For x_3, x_4, and x_5,

$$\{z_j-c_j\}=C_BB^{-1}(P_3,P_4,P_5)-(0,0,0)$$

$$=C_BB^{-1}\quad=C_BB^{-1}=(0,c_2-c_1,c_1)$$

From the tableau, we have

$$(0,c_2-c_1,c_1)=(0,3,2)$$

which gives

$c_1=2$

$c_2=5$

Continued...

Hence,

Optimum $z = c_1x_1 + c_2x_2 + c_3x_3$

$= 2\times 2 + 5\times 6 + 0\times 2 = 34$

To construct the original problem,

$$B^{-1}(P_1\ P_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus,

$$(P_1\ P_2) = B\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Similarly,

$$b = B\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

Original model:

Maximize $z = 2x_1 + 5x_2$

Subject to

$$\begin{aligned} x_1 \quad\quad &\le 4 \\ x_2 &\le 6 \\ x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$

5

All that is needed is to show that the computations lead to the column under x_{II}.

For x_{II}, we have,

$$\{z_j - c_j\} = c_B B^{-1} I - c_{II}$$

$$= c_B B^{-1} - c_{II}$$

Constraint coefficients

$$= B^{-1} I = B^{-1}$$

1

(a) current $B = (P_1, P_2)$
P_1 must leave so that b is enclosed between P_2 and P_3, hence yielding feasible values of x_2 and x_3

(b) $B = (P_2\ P_4)$ is a feasible basis

2

$$z_j - c_j = c_B B^{-1} P_j - c_j$$

Assume for convenience that

$$B = (P_1, P_2, \cdots, P_m)$$

Then, for the basic vectors P_1, $P_2, \cdots$, and P_m, we have

$$\{z_j - c_j\}_{j=1,2,\cdots,m} = c_B B^{-1}(P_1, \cdots, P_m) - (c_1, \cdots, c_m)$$
$$= c_B B^{-1} B - c_B$$
$$= c_B I - c_B = 0$$

3

Let NB represent the set of nonbasic variables at any iteration. Then

$$z = z^* - \sum_{j \in NB} (z_j - c_j) x_j$$

(a) Since

$$z_j - c_j \begin{cases} > 0 & \text{for max} \\ < 0 & \text{for min} \end{cases}$$

it follows that all $x_j = 0$, $j \in NB$ because if any x_j, $j \in NB$ becomes positive $z < z^*$ for max and $z > z^*$ for min, which is not optimal. Thus, $X_B = B^{-1} b$ and $x_j = 0$, $j \in NB$ shows that the solution is unique.

Continued...

(b) If $z_j - c_j = 0$ for at least one $j \in NB$, then x_j can become basic at a value other than zero without changing the optimum value of z. Thus, alternative optima exist.

4

starting tableau (max):

	x_1	$x_2 \cdots x_j \cdots$	x_n	
z	$-c_1$	$-c_2 \cdots -c_j \cdots$	$-c_n$	0

At the starting iteration:

$$B = I, \quad c_B = 0$$

Hence

$$z_j - c_j = c_B B^{-1} P_j - c_j$$
$$= 0(B^{-1} P_j) - c_j$$
$$= -c_j$$

5

Starting tableau (assuming max):

	$\cdots\ x_j\ \cdots$	$R_1\ R_2\ \cdots\ R_m$	
	$\cdots\ -c_j\ \cdots$	$M\ M\ \cdots\ M$	0
R_1 $\vdots$ R_m	$\cdots\ P_j\ \cdots$	I	b

$$B = B^{-1} = I, \quad c_B = (-M, -M, \cdots; -M)$$

$$c_B B^{-1} = (-M, -M, \cdots, -M)$$

$$\{z_j - c_j\} = (-M, -M, \cdots, -M)(P_1, \cdots, P_n \mid I) - (c_1, c_2, \cdots, c_n, -M, \cdots, -M)$$

$$= \big((-M, -M, \cdots, -M)P_1 - c_1, \cdots, (-M, -M, \cdots, -M)P_n - c_n, 0, \cdots, 0\big)$$

which yields the following tableau

$\cdots\ x_j$	$R_1\ \cdots\ R_m$	
$\cdots\ (-M, \cdots, -M)P_j - c_j\ \cdots$	$0\ \cdots\ 0$	$(-M, \cdots, -M)b$

Continued...

Set 7.2a

6

The vectors

$$\begin{pmatrix} c_k \\ p_k \end{pmatrix} \text{ and } \begin{pmatrix} -c_k \\ -p_k \end{pmatrix}$$

correspond to x_k^- and x_k^+, respectively. Assume that both x_k^- and x_k^+ are nonbsic, and let **B** and $\mathbf{c_B}$ correspond to the current solution. Then

$$z_k^- - c_k^- = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{P}_k - c_k$$
$$z_k^+ - c_k^+ = -\mathbf{c}_B\mathbf{B}^{-1}\mathbf{P}_k + c_k = -(z_k^- - c_k^-)$$

Thus, if x_k^- is a candidate for entering the basic solution, then x_k^+ cannot be an entering candidate, and vice versa.

If $z_k^+ - c_k^+ = (z_k^- - c_k^-) = 0$, then possibly one of the two variables may enter the basic solution to provide an alternative optimum. The two variables cannot be basic simultaneously because a basis **B** cannot include two dependent vectors $\mathbf{P}_k$ and $-\mathbf{P}_k$

To show that the two variables cannot replace one another in alternative optima, assume that x_k^- is basic in the optimum solution. Then

$$\mathbf{B}^{-1}\mathbf{P}_k = (0,\ldots,1,\ldots,0)^T$$
$$\mathbf{B}^{-1}(-\mathbf{P}_k) = (0,\ldots,-1,\ldots,0)^T$$

According to the feasibility condition, x_k^+ cannot replace x_k^- because the corresponding pivot element $\mathbf{B}^{-1}(-\mathbf{P}_k)$ is negative, unless $x_k^- = 0$, which is a trivial case.

7

Number of nonbasic variables = $n - m$. In the case of *nondegeneracy*, each entering nonbasic variable will be associated with a *distinct* adjacent extreme point. In the case of *degeneracy*, an entering nonbasic variable can result in a different basic solution without changing the extreme point itself. In this situation, the number of adjacent extreme points in less than $n - m$.

8

Let $x_k = d_k$ (≥ 0) represent the current basic solution. Then, the new basic solution after x_j enters and x_r leaves is

$$x_j = \frac{d_r}{(\mathbf{B}^{-1}\mathbf{P}_j)_r} = \frac{0}{(\mathbf{B}^{-1}\mathbf{P}_j)_r} = 0, \text{ provided } (\mathbf{B}^{-1}\mathbf{P}_j)_r \neq 0$$
$$x'_k = d_k - x_j(\mathbf{B}^{-1}\mathbf{P}_j)_k, \text{ all basic } x_k, k \neq j$$

The last equation is independent of $(\mathbf{B}^{-1}\mathbf{P}_j)_k$ for all k, because $x_j = 0$. Hence, x'_k remains feasible for all k.

9

1. If the minimum ratio corresponds to more than one basic variable, the next iteration is degenerate.
2. If x_j is the entering variable and if the basic variable x_j is zero, the next iteration will continue to be degenerate if $(\mathbf{B}^{-1}\mathbf{P}_j)_k > 0$.
3. Iffor every zero basic variable, *xk*, the pivot element $(\mathbf{B}^{-1}\mathbf{P}j)_k \leq$, then the next iteration will not be degenerate.

10

Under nondegeneracy:

number of extreme points
= number of basic solutions

Under degeneracy:

number of extreme points
< number of basic solutions

11

(a) $x_j = \theta = \dfrac{x_r}{(B^{-1}P_j)_r}$, $(B^{-1}P_j)_r > 0$

For P_j, we have

$$\frac{\text{new } x_j}{\text{old } x_j} = \frac{\dfrac{x_r}{\alpha(B^{-1}P_j)_r}}{\dfrac{x_r}{(B^{-1}P_j)_r}} = \frac{1}{\alpha}$$

(b)

$$\frac{\text{new } x_j}{\text{old } x_j} = \frac{\dfrac{\beta(B^{-1}b)_r}{\alpha(B^{-1}P_j)_r}}{\dfrac{(B^{-1}b)_r}{(B^{-1}P_j)_r}} = \frac{\beta}{\alpha}$$

12

$$\text{New}(z_j - c_j) = C_B\left(\frac{1}{\beta}B^{-1}P_j\right) - \frac{1}{\beta}c_j$$

$$= \frac{1}{\beta}(C_B B^{-1}P_j - c_j)$$

$$= \frac{1}{\beta}(\text{old } z_j - c_j), \quad \beta > 0$$

Conclusion: x_j remains nonbasic

A variable x_j can be made profitable either by increasing c_j or by decreasing z_j (which is the unit usage of resources by activity j). Of course, a combination of the two changes will work as well.

13

$C_B = (c_1, c_2, \cdots, c_m)$

$B = (P_1, P_2, \cdots, P_m)$

For the basic variables

$$z_j - c_j = C_B B^{-1}(P_1, \cdots, P_m) - (c_1, \cdots, c_m)$$

$$= C_B B^{-1}B - C_B$$

$$= C_B I - C_B = \underline{0}$$

Thus, for the basic variable, $z_j - c_j = 0$ regardless of the specific assignment to the vector C_B (e.g., D_B).

This result implies that changes in C_B cannot affect the optimality of the basic variables since these variables are already basic. It may, however, cause a nonbasic variable to become basic.

Set 7.2b

1

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	-2/3	5/6	0	0	0	20
x_1		2/3					4
x_4		4/3					2
x_5		5/3					5
x_6		1					2

2

(a) Starting iteration:

Let x_4 and x_5 be the slacks.

$$X_B = (X_4, X_5)^T,\ C_B = (0,0),\ B = B^{-1} = I$$

First iteration:

$$C_B B^{-1} = (0,0)$$

$$(z_j - c_j)_{j=1,2,3} = (0,0)\begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 4 \end{pmatrix} - (6,-2,3)$$

$$= (-6, 2, -3) \Rightarrow x_1 \text{ enters}$$

$$X_B = B^{-1}b = Ib = b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\alpha^1 = B^{-1}P_1 = P_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\theta = \min_{k=4,5}\{2/2, 4/1\} = 1 \Rightarrow x_4 \text{ leaves}$$

$$B^{-1}_{next} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$$

$$X_B = (X_1, X_5)^T = (1, 3)^T$$

Second iteration:

$$C_B B^{-1} = (6,0)\begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} = (3,0)$$

$$(z_j - c_j)_{j=2,3,4} = (3,0)\begin{pmatrix} -1 & 2 & 1 \\ 0 & 4 & 0 \end{pmatrix} - (-2,3,0)$$

$$= (-1, 3, 3) \Rightarrow x_2 \text{ enters}$$

$$X_B = \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\alpha^2 = B^{-1}P_2 = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

$$\theta = \min_{k=1,5}\left\{—, \frac{3}{1/2}\right\} = 6 \Rightarrow x_6 \text{ leaves}$$

$$B^{-1}_{next} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$X_B = (x_1, x_2)^T = (4, 6)^T,\ C_B = (6,-2)$$

continued...

Third iteration:

$$C_B B^{-1} = (6,-2)\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = (2,2)$$

$$(z_j - c_j)_{j=3,4,5} = (2,2)\begin{pmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} - (3,0,0)$$

$$= (9, 2, 2) \Rightarrow \text{optimal}$$

Optimal solution

$$X_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$z = C_B X_B = 6 \times 4 + (-2)(6) = 12$$

(b)

Starting iteration: Let x_4, x_5, and x_6 be the slack variables.

$$X_B = (x_4, x_5, x_6)^T,\ C_B = (0,0,0),\ B = B^{-1} = I$$

First iteration: $C_B B^{-1} = (0,0,0)$

$$(z_j - c_j)_{j=1,2,3} = (0,0,0)\begin{pmatrix} 4 & 3 & 8 \\ 4 & 1 & 12 \\ 4 & -1 & 3 \end{pmatrix} - (2,1,2)$$

$$= (-2, -1, -2) \Rightarrow x_1 \text{ enters}$$

$$X_B = B^{-1}b = Ib = b = (12, 8, 8)^T$$

$$\alpha^1 = B^{-1}P_1 = P_1 = (4, 4, 4)^T$$

$$\theta = \min_{k=4,5,6}\left\{\frac{12}{4}, \frac{8}{4}, \frac{8}{4}\right\} = 2 \Rightarrow x_5 \text{ leaves}$$

$$B^{-1}_{next} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$X_B = (x_4, x_1, x_6),\ C_B = (0, 2, 0)$$

Second iteration: $C_B B^{-1} = (0, 1/2, 0)$

$$(z_j - c_j)_{j=2,3,5} = (0, 1/2, 0)\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix} - (1,2,0)$$

$$= (-1/2, 4, 1/2) \Rightarrow x_2 \text{ enters}$$

$$X_B = \begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/4 \\ -2 \end{pmatrix}$$

$$\theta = \min_{k=4,1,6} = \left\{\frac{4}{2}, \frac{2}{1/4}, —\right\} = 2, x_4 \text{ leaves}$$

$$B^{-1}_{next} = \begin{pmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -1 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/8 & 3/8 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$$X_B = (x_2, x_1, x_6)^T,\ C_B = (1, 2, 0)$$

continued...

Third iteration: $C_B B^{-1} = (1/4, 1/4, 0)$

$$(z_j - c_j)_{j=3,4,5} = (1/4, 1/4, 0)\begin{pmatrix}8&1&0\\12&0&1\\3&0&0\end{pmatrix} - (2, 0, 0)$$

$$= (3, 1/4, 1/4) \Rightarrow \text{optimal.}$$

Optimal solution:

$$X_B = \begin{pmatrix}x_2\\x_1\\x_6\end{pmatrix} = \begin{pmatrix}1/2 & -1/2 & 0\\ -1/8 & 3/8 & 0\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}12\\8\\8\end{pmatrix} = \begin{pmatrix}2\\3/2\\4\end{pmatrix}$$

$$z = 2\times 3/2 + 1\times 2 + 2\times 0 = 5$$

(c)

Adding artificials, we get

$$\min z = 2x_1 + x_2 + Mx_4 + Mx_5$$

s.t.

$$\left(\begin{array}{ccc|ccc}3&1&0&1&0&0\\4&3&-1&0&1&0\\1&2&0&0&0&1\end{array}\right)\begin{pmatrix}x_1\\x_2\\\vdots\\x_6\end{pmatrix} = \begin{pmatrix}3\\6\\3\end{pmatrix}$$

where x_3 and x_6 are slacks, and x_4 and x_5 are artificials.

Starting solution:

$X_B = (x_4, x_5, x_6)$, $C_B = (M, M, 0)$

$B = B^{-1} = I$

First iteration: $C_B B^{-1} = (M, M, 0)$

$$(z_j - c_j)_{j=1,2,3} = (M, M, 0)\begin{pmatrix}3&1&0\\4&3&-1\\1&2&0\end{pmatrix} - (2, 1, 0)$$

$$= (-2+7M, -1+4M, -M)$$

Thus, x_1 enters.

$$\theta = \min_{k=4,5,6}\left\{\frac{3}{3}, \frac{6}{4}, \frac{3}{1}\right\} = 1 \Rightarrow x_4 \text{ leaves}$$

$$B^{-1}_{next} = \begin{pmatrix}3&0&0\\4&1&0\\1&0&1\end{pmatrix}^{-1} = \begin{pmatrix}1/3&0&0\\-4/3&1&0\\-1/3&0&1\end{pmatrix}$$

$X_B = (x_1, x_5, x_6)^T$, $C_B = (2, M, 0)$

Second iteration: $C_B B^{-1} = \left(\frac{2-4M}{3}, M, 0\right)$

$$(z_j - c_j)_{j=2,3,4} = \left(\frac{2-4M}{3}, M, 0\right)\begin{pmatrix}1&0&1\\3&-1&0\\2&0&0\end{pmatrix} - (1, 0, 0)$$

$$= \left(\frac{5M-1}{3}, -M, \frac{2-4M}{3}\right) \Rightarrow x_2 \text{ enters}$$

$$X_B = \begin{pmatrix}1/3&0&0\\-4/3&1&0\\-1/3&0&1\end{pmatrix}\begin{pmatrix}3\\6\\3\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix}1/2&0&0\\-4/3&1&0\\-1/3&0&1\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}1/3\\5/3\\5/3\end{pmatrix}$$

$$\theta = \min_{k=1,5,6}\left\{\frac{1}{1/3}, \frac{2}{5/3}, \frac{2}{5/3}\right\} \Rightarrow x_5 \text{ leaves}$$

Continued...

$$B^{-1}_{next} = \begin{pmatrix}3&1&0\\4&3&0\\1&2&1\end{pmatrix}^{-1} = \begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}$$

$X_B = (x_1, x_2, x_6)^T$, $C_B = (2, 1, 0)$

Third iteration: $C_B B^{-1} = (2/5, 1/5, 0)$

$$(z_j - c_j)_{j=3,4,5} = (2/5, 1/5, 0)\begin{pmatrix}0&1&0\\-1&0&1\\0&0&0\end{pmatrix} - (0, M, M)$$

$$= (-1/5, 2/5 - M, 1/5 - M)$$

$\Rightarrow$ optimal solution.

Optimal solution:

$$X_B = \begin{pmatrix}x_1\\x_2\\x_6\end{pmatrix} = \begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}\begin{pmatrix}3\\6\\3\end{pmatrix} = \begin{pmatrix}3/5\\6/5\\0\end{pmatrix}$$

$$z = 2\times\frac{3}{5} + 1\times\frac{6}{5} = 12/5$$

(d)

Minimize $z = 5x_1 - 4x_2 + 6x_3 + 8x_4 + Mx_8$

subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + 7x_4 \quad\quad + x_6 \quad\quad &= 46\\ 3x_1 - x_2 + x_3 + 2x_4 \quad\quad\quad + x_7 \quad &= 20\\ 2x_1 + 3x_2 - x_3 + x_4 - x_5 \quad\quad + x_8 &= 18 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

Iteration 0:

$X_B = (x_6, x_7, x_8)$, $C_B = (0, 0, M)$, $B_0 = B_0^{-1} = I$

$$\{z_j - c_j\}_{j=1,2,3,4,5}$$

$$= (0, 0, M)\begin{pmatrix}1&7&3&7&0\\3&-1&1&2&0\\2&3&-1&1&-1\end{pmatrix} - (5, -4, 6, 8, 0)$$

$$= (2M-5, \boxed{3M+4}, -M-6, M-8, -M)$$

x_2 enters

$$B^{-1}P_2 = \begin{pmatrix}7\\-1\\3\end{pmatrix}, \quad B^{-1}b = \begin{pmatrix}46\\20\\18\end{pmatrix}, \quad \theta = \min\left\{\frac{46}{7}, -, \enclose{circle}{\frac{18}{3}}\right\}$$

x_8 leaves

$$B_1 = \begin{pmatrix}1&0&7\\0&1&-1\\0&0&3\end{pmatrix}, \quad B_1^{-1} = \begin{pmatrix}1&0&-7/3\\0&1&1/3\\0&0&1/3\end{pmatrix}$$

$$X_{B_1} = \begin{pmatrix}x_6\\x_7\\x_2\end{pmatrix} = B_1^{-1}b = \begin{pmatrix}4\\26\\6\end{pmatrix}$$

Continued...

Set 7.2b

Iteration 1:

$$X_B = (x_6, x_7, x_2)^T, \quad C_B = (0, 0, -4)$$

$$C_B B^{-1} = (0, 0, -4/3)$$

$$\{z_j - c_j\}_{1,3,4,5}$$

$$= (0, 0, -4/3)\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix} - (5, 6, 8, 0)$$

$$= (-23/3, -30/3, -28/3, \boxed{4/3})$$

x_5 enters

$$B_1^{-1} P_5 = \begin{pmatrix} \boxed{7/3} \\ -1/3 \\ -1/3 \end{pmatrix}, \quad B_1^{-1} b = \begin{pmatrix} 4 \\ 26 \\ 6 \end{pmatrix}$$

x_6 leaves

Iteration 2:

$$X_B = (x_5, x_7, x_2)^T, \quad C_B = (0, 0, -4)$$

$$B_2 = \begin{pmatrix} 0 & 0 & 7 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}, \quad B_2^{-1} = \begin{pmatrix} 3/7 & 0 & 0 \\ 1/7 & 1 & 0 \\ 1/7 & 0 & 1 \end{pmatrix}$$

$$X_{B_2} = \begin{pmatrix} x_5 \\ x_7 \\ x_2 \end{pmatrix} = B^{-1} b = \begin{pmatrix} 12/7 \\ 186/7 \\ 46/7 \end{pmatrix}$$

$$C_B B^{-1} = (-4/7, 0, 0)$$

$$\{z_j - c_j\}_{1,3,4,6}$$

$$= (-4/7, 0, 0)\begin{pmatrix} 1 & 3 & 7 & 1 \\ 3 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix} - (5, 6, 8, 0)$$

$$= (-39/7, -54/7, -12, -4/7) \text{ optimum}$$

$$X_{B_2} = (x_5, x_7, x_2)^T = (12/7, 186/7, 46/7)$$

$$z = -184/7$$

3

Iteration 0:

$$X_{B_0} = (x_2, x_4, x_5)^T, \quad C_B = (7, -10, 0)$$

$$B_0 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_0^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

continued...

$$X_B = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = B_0^{-1} b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

$$C_B B_0^{-1} = (7, -10, 0)\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = (17, 7, -17)$$

$$\{z_j - c_j\}_{j=1,3,6}$$

$$= (17, 7, -17)\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 3 \\ 1 & -3 & 0 \end{pmatrix} - (0, 11, 26)$$

$$= (-17, \boxed{16}, 12) \quad x_3 \text{ enters}$$

$$B_0^{-1} b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, \quad B_0^{-1} P_3 = \begin{pmatrix} \boxed{1} \\ -2 \\ -2 \end{pmatrix} \quad x_2 \text{ leaves}$$

Iteration 1:

$$X_{B_1} = (x_3, x_4, x_5)^T, \quad C_B = (11, -10, 0)$$

$$B_1 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}, \quad B_1^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$X_B = B_1^{-1} b = \begin{pmatrix} 2 \\ 10 \\ 8 \end{pmatrix}$$

$$C_B B_1^{-1} = (11, -10, 0)\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} = (1, -9, -1)$$

$$\{z_j - c_j\}_{1,2,6}$$

$$= (1, -9, -1)\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} - (0, 7, 26)$$

$$= (-1, -16, -52) \Rightarrow \text{optimum}$$

$$X_{B_1} = (x_3, x_4, x_5)^T = (2, 10, 8)^T$$

$$z = -78$$

4

(a) Minimize $z = 2x_1 + x_2 + M(x_4 + x_5)$

subject to

$$\begin{aligned} 3x_1 + x_2 \phantom{{}- x_3} + x_4 \phantom{{}+ x_5 + x_6} &= 3 \\ 4x_1 + 3x_2 - x_3 \phantom{{}+ x_4} + x_5 \phantom{{}+ x_6} &= 6 \\ x_1 + 2x_2 \phantom{{}- x_3 + x_4 + x_5} + x_6 &= 3 \\ x_1, \ldots, x_6 &\geq 0 \end{aligned}$$

Phase I:

Iteration 0:

$$X_B = (x_4, x_5, x_6)^T, \quad C_B = (1, 1, 0)$$

$$B_0^{-1} = I, \quad C_B B^{-1} = (1, 1, 0)$$

continued...

$$\{z_j - c_j\}_{1,2,3} = (1,1,0)\begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix} - (0,0,0)$$

$$= (\boxed{7}, 4, -1), \quad x_1 \text{ enters}$$

$$B_0^{-1}P_1 = B_0^{-1}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \quad B_0^{-1}b = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

$$\theta = \min\left\{\boxed{\frac{3}{3}}, \frac{6}{4}, \frac{3}{1}\right\} \Rightarrow x_4 \text{ leaves}$$

Iteration 1:

$$X_B = (x_1, x_5, x_6)^T, \quad C_B = (0, 1, 0)$$

$$B_1 = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B_1^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$C_B B_1^{-1} = (-4/3, 1, 0)$$

$$\{z_j - c_j\}_{2,3,4} = (-4/3, 1, 0)\begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} - (0,0,1)$$

$$= (\boxed{5/3}, -1, -7/3) \quad x_2 \text{ enters}$$

$$B_1^{-1}P_2 = B_1^{-1}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/3 \\ 5/3 \end{pmatrix},$$

$$B_1^{-1}b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\theta = \min\left\{\frac{1}{1/3}, \frac{2}{5/3}, \frac{2}{5/3}\right\}, \quad x_5 \text{ leaves}$$

Iteration 2:

$$X_B = (x_1, x_2, x_6)^T, \quad C_B = (0, 0, 0)$$

$$B_2 = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad B_2^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Since X_B does not include the artificials x_4 and x_5, we can use to start Phase II.

Continued...

Phase II: objective max $z = 2x_1 + x_2$

Iteration 0:

$$X_B = (x_1, x_2, x_6), \quad C_B = (2, 1, 0)$$

$$B_0^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = B_2^{-1}b = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$C_B B^{-1} = (2,1,0)\begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & 1 & 1 \end{pmatrix} = (2/5, 1/5, 0)$$

$$\{z_j - c_j\}_{j=3} = (2/5, 1/5, 0)\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - 0 = -1/5$$

x_3 enters

$$B_0^{-1}P_3 = \begin{pmatrix} 1/5 \\ -3/5 \\ 1 \end{pmatrix}, \quad B_0^{-1}b = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix} \quad x_6 \text{ leaves}$$

Iteration 1:

$$X_B = (x_1, x_2, x_3), \quad C_B = (2, 1, 0)$$

$$B_1 = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix}, \quad B_1^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\{z_j - c_j\}_{j=6} = (3/5, 0, 1/5)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 = 1/5 > 0$$

optimum!

$$X_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$z = 12/5$$

5

Minimize $z = 3x_1 + 2x_2$

subject to

$$\begin{aligned} -3x_1 - x_2 + x_3 &= -3 \\ -4x_1 - 3x_2 + x_4 &= -6 \\ x_1 + x_2 + x_5 &= 3 \end{aligned}$$

Iteration 0:

$$X_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad B_0 = B_0^{-1} = I$$

Continued...

Set 7.2b

$X_B = \begin{pmatrix}-3\\-6\\3\end{pmatrix} \Longrightarrow x_4$ leaves

$C_B = (0,0,0)$, $C_B B^{-1} = (0,0,0)$

$\{z_j - c_j\}_{1,2}$

$= (0,0,0)\begin{pmatrix}-3 & -1\\-4 & -3\\1 & 1\end{pmatrix} - (3,2) = (-3,-2)$

(row 2 of B_0^{-1})$(P_1\ P_2)$

$= (0,1,0)\begin{pmatrix}-3 & -1\\-4 & -3\\1 & 1\end{pmatrix} = (-4,-3)$

$\theta = \min_{j=1,2}\left\{\left|\frac{-3}{-4}\right|, \left|\frac{-2}{-3}\right|\right\} = 2/3 \Longrightarrow x_2$ enters

Iteration 1:

$X_B = \begin{pmatrix}x_3\\x_2\\x_5\end{pmatrix}$, $B_1 = \begin{pmatrix}1 & -1 & 0\\0 & -3 & 0\\0 & 1 & 1\end{pmatrix}$, $B_1^{-1} = \begin{pmatrix}1 & -1/3 & 0\\0 & -1/3 & 0\\0 & 1/3 & 1\end{pmatrix}$

$X_B = \begin{pmatrix}x_3\\x_2\\x_5\end{pmatrix} = B_1^{-1}b$

$= \begin{pmatrix}1 & -1/3 & 0\\0 & -1/3 & 0\\0 & 1/3 & 1\end{pmatrix}\begin{pmatrix}-3\\-6\\3\end{pmatrix} = \begin{pmatrix}-1\\2\\1\end{pmatrix}$ x_3 leaves

$C_B = (0,2,0)$

$C_B B^{-1} = (0, -2/3, 0)$

$\{z_j - c_j\}_{j=1,4} = (0,-2/3,0)\begin{pmatrix}-3 & 0\\-4 & 1\\1 & 0\end{pmatrix} - (3,0)$

$= (-1/3, -2/3)$

(row 1 of B_1^{-1})$(P_1\ P_4)$

$= (1, -1/3, 0)\begin{pmatrix}-3 & 0\\-4 & 1\\1 & 0\end{pmatrix} = (-5/3, -1/3)$

$\theta = \min_{j=1,4}\left\{\left|\frac{-1/3}{-5/3}\right|, \left|\frac{-2/3}{-1/3}\right|\right\} = 1/5$

x_1 enters

continued...

Iteration 2:

$X_B = \begin{pmatrix}x_1\\x_2\\x_5\end{pmatrix}$

$B_2 = \begin{pmatrix}-3 & -1 & 0\\-4 & -3 & 0\\1 & 1 & 1\end{pmatrix}$

$B_2^{-1} = \begin{pmatrix}-3/5 & 1/5 & 0\\4/5 & -3/5 & 0\\-1/5 & 2/5 & 1\end{pmatrix}$

$X_B = B_2^{-1}b$

$= \begin{pmatrix}-3/5 & 1/5 & 0\\4/5 & -3/5 & 0\\-1/5 & 2/5 & 1\end{pmatrix}\begin{pmatrix}-3\\-6\\3\end{pmatrix}$

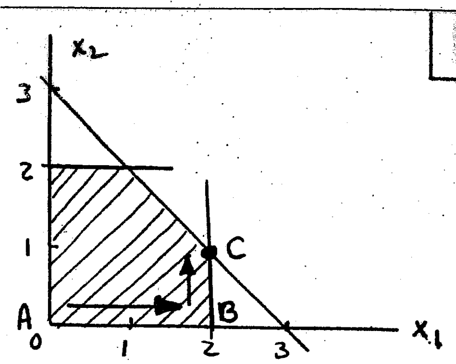
$= \begin{pmatrix}3/5\\6/5\\6/5\end{pmatrix}$

Feasible!

$Z = 3 \times 3/5 + 2 \times 6/5 = 21/5$

1

a)



b)

Iteration 1: x_1 enters

	x_1	x_2	x_3	solution
z	-2	-1	0	0
x_3	1	1	1	3

$\theta = \min\{3/1, -, 2\} = 2$

Substitute x_1 at its upper bound: $x_1 = 2 - x_1'$

	x_1'	x_2	x_3	Solution
z	2	-1	0	2
x_3	-1	1	1	1

This solution ($x_1 = 2$, $x_2 = 0$) coincides with point B in the solution space above. The solution now has $x_1' = 0$, which implies that $x_1 = 2$, thus reducing the solution space to line segment BC.

Iteration 2: x_2 enters

$\theta = \min\{1/1, -, 2\} = 1$

	x_1'	x_2	x_3	Solution
z	1	0	1	3
x_2	-1	1	1	1

Optimum: $x_1' = 0 \Rightarrow x_1 = 2, x_2 = 1$ which is the same as point C.

c) As shown in (b) above, the substitution of the upper-bounding method recognizes the extreme point implicitly by using the substitution

$$x_j = \mu_j - x_j'$$

2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	-6	-2	-8	-4	-2	-10	0	0
x_7	8	1	8	2	2	4	1	13

x_6 enters: $\theta = \min\{13/4, -, 1\} = 1$

$x_6 = 1 - x_6'$

	x_1	x_2	x_3	x_4	x_5	x_6'	x_7	
z	-6	-2	-8	-4	-2	10	0	10
x_7	8	1	8	2	2	-4	1	9

x_3 enters: $\theta = \min\{9/8, -, 1\} = 1$

$x_3 = 1 - x_3'$

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
z	-6	-2	8	-4	-2	10	0	18
x_7	8	1	-8	2	2	-4	1	1

x_1 enters: $\theta = \min\{1/8, -, 1\} = 1/8$, x_7 leaves

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
z	0	-5/4	2	-5/2	-1/2	7	3/4	18 3/4
x_1	1	1/8	-1	1/4	1/4	-1/2	1/8	1/8

x_4 enters: $\theta = \min\{\frac{1/8}{1/4}, -, 1\} = 1/2$, x_1 leaves

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
z	10	0	-8	0	2	2	2	20
x_4	4	1/2	-4	1	1	-2	1/2	1/2

x_3' enters: $\theta = \min\{-, \frac{1/2 - 1}{(-4)}, 1\} = 1/8$

x_4 leaves, $x_4 = 1 - x_4'$

	x_1	x_2	x_3'	x_4'	x_5	x_6'	x_7	
z	2	-1	0	2	0	6	1	21
x_3'	-1	-1/8	1	1/4	-1/4	1/2	-1/8	1/8

x_2 enters: $\theta = \min\{-, \frac{1/8 - 1}{-1/8}, 1\} = 1$

$x_2 = 1 - x_2'$

	x_1	x_2'	x_3'	x_4'	x_5	x_6'	x_7	
z	2	1	0	2	0	6	1	22
x_3'	-1	1/8	1	1/4	-1/4	1/2	-1/8	1/4

Optimum solution:

$x_1 = 0$

$x_2 = 1$

$x_3 = 3/4$

$x_4 = 1$

$x_5 = 0$

$x_6 = 1$

$Z = 22$

Set 7.3a

3

(a) Minimize

	x_1	x_2	x_3	x_4	x_5	
z	-6	2	3	0	0	0
x_4	2	4	2	1	0	8
x_5	1	-2	3	0	1	7

x_3 enters: $\theta = \min\{\frac{7}{3}, —, 1\} = 1$; $x_3 = 1 - x_3'$

	x_1	x_2	x_3'	x_4	x_5	
z	-6	2	-3	0	0	-3
x_4	2	4	-2	1	0	6
x_5	1	-2	-3	0	1	4

x_2 enters: $\theta = \min\{\frac{6}{4}, -, 2\} = 3/2$; x_4 leaves

	x_1	x_2	x_3'	x_4	x_5	
z	-7	0	-2	-1/2	0	-6
x_2	1/2	1	-1/2	1/4	0	3/2
x_5	2	0	-4	1/2	1	7

Optimum: $x_1 = 0$, $x_2 = 3/2$, $x_3 = 1$, $z = -6$

b) Maximize

	x_1	x_2	x_3	x_4	x_5	
z	-3	-5	-2	0	0	0
x_4	1	2	2	1	0	10
x_5	2	4	3	0	1	15

x_2 enters: $\theta = \min\{\frac{15}{4}, —, 3\} = 3$; $x_2 = 3 - x_2'$

	x_1	x_2'	x_3	x_4	x_5	
z	-3	5	-2	0	0	15
x_4	1	-2	2	1	0	4
x_5	2	-4	3	0	1	3

x_1 enters: $\theta = \min\{\frac{3}{2}, -, 4\}$; x_5 leaves

	x_1	x_2'	x_3	x_4	x_5	
z	0	-1	5/2	0	3/2	39/2
x_4	0	0	1/2	1	-1/2	5/2
x_1	1	-2	3/2	0	1/2	3/2

x_2' enters: $\theta = \min\{-, \frac{3/2-2}{-2}, 2\} = 1/4$

x_1 leaves, $x_1 = 4 - x_1'$

	x_1'	x_2'	x_3	x_4	x_5	
z	1/2	0	7/4	0	5/4	83/4
x_4	0	0	1/2	1	-1/2	5/2
x_2'	1/2	1	-3/4	0	-1/4	5/4

Optimum: $x_1 = 4$, $x_2 = 7/4$, $x_3 = 0$, $z = 83/4$

4

(a) Substitute $x_1 = 1 + y_1$, $x_3 = y_3 + 2$

Phase 1: $0 \le y_1 \le 2$, $0 \le x_2 \le 3$, $y_3 \ge 0$

	y_1	x_2	y_3	x_4	x_5	R	
z	1	2	-1	-1	0	0	4
x_5	2	1	1	0	1	0	4
R_1	1	2	-1	-1	0	1	4
z	0	0	0	0	0	-1	0
x_5	3/2	0	3/2	1/2	1	0	2
x_2	1/2	1	-1/2	-1/2	0	1	2

Phase 2:

	y_1	x_2	y_3	x_4	x_5	
z	-2	0	1	-1	0	3
x_5	3/2	0	3/2	1/2	1	2
x_2	1/2	1	-1/2	-1/2	0	2

y_1 enters: $\theta = \min\{\frac{2}{3/2}, —, 2\} = 4/3$; x_5 leaves

	y_1	x_2	y_3	x_4	x_5	
z	0	0	3	-1/3	4/3	17/6
y_1	1	0	1	1/3	2/3	4/3
x_2	0	1	-1	-2/3	-1/3	4/3

x_4 enters: $\theta = \min\{\frac{4/3}{1/3}, \frac{4/3-3}{-2/3}, —\} = 5/2$

x_2 leaves, $x_2 = 1 - x_2'$

	y_1	x_2'	y_3	x_4	x_5	
z	0	1/2	7/2	0	3/2	13/2
y_1	1	-1/2	1/2	0	1/2	1/2
x_4	0	3/2	3/2	1	1/2	5/2

Optimum: $x_1 = 3/2$, $x_2 = 3$, $x_3 = 2$, $z = 13/2$

b) Set $x_1 = 1 + y_1$, $0 \le y_1 \le 2$, $0 \le x_2 \le 1$

Phase 1:

	y_1	x_2	x_3	R	x_4	x_5	
z	-1	2	0	0	0	0	1
R	-1	2	-1	1	0	0	1
x_4	3	2	0	0	1	0	7
x_5	-1	1	0	0	0	1	2
z	-2	0	-1	1	0	0	0
x_2	-1/2	1	-1/2	1/2	0	0	1/2
x_4	4	0	1	-1	1	0	6
x_5	-1/2	0	1/2	-1/2	0	1	3/2

Phase 2:

	y_1	x_2'	x_3	x_4	x_5	
z	0	4	1	0	0	4
y_1	1	2	1	0	0	1
x_4	0	-8	-3	1	0	2
x_5	0	1	1	0	1	2

Optimum: $x_1 = 2$, $x_2 = 1$, $z = 4$

c) Let $x_1 = 1 + y_1$

$0 \le y_1 \le 2,\ 0 \le x_2 \le 5,\ 0 \le x_3 \le 2$

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-4	-2	-6	0	0	0	4
x_4	4	-1	0	1	0	0	5
x_5	-1	1	2	0	1	0	9
x_6	-3	1	4	0	0	1	15

x_3 enters: $\theta = \min\{15/4, -, 2\} = 2$; $x_3 = 2 - x_3'$

	y_1	x_2	x_3'	x_4	x_5	x_6	
z	-4	-2	6	0	0	0	16
x_4	4	-1	0	1	0	0	5
x_5	-1	1	-2	0	1	0	5
x_6	-3	1	-4	0	0	1	7

y_1 enters: $\theta = \min\{\frac{5}{4}, -, 2\} = 5/4$; x_4 leaves

	y_1	x_2	x_3'	x_4	x_5	x_6	
z	0	-3	6	1	0	0	21
y_1	1	-1/4	0	1/4	0	0	5/4
x_5	0	3/4	-2	1/4	1	0	25/4
x_6	0	1/4	-4	3/4	0	1	43/4

x_2 enters: $\theta = \min\{\frac{25}{3}, \frac{5/4-2}{-1/4}, 5\} = 3$

y_1 leaves, $y_1 = 2 - y_1'$

	y_1'	x_2	x_3'	x_4	x_5	x_6	
	12	0	6	-2	0	0	30
x_2	4	1	0	-1	0	0	3
x_5	-3	0	-2	1	1	0	4
x_6	-1	0	-4	1	0	1	10

x_4 enters: $\theta = \min\{4, \frac{3-5}{-1}, -\} = 2$

x_2 leaves, $x_2 = 5 - x_2'$

	y_1'	x_2'	x_3'	x_4	x_5	x_6	
z	4	2	6	0	0	0	34
x_4	-4	1	0	1	0	0	2
x_5	3	-1	-2	0	1	0	2
x_6	1	-1	-4	0	0	1	8

Optimum Solution:

$x_1 = 3$

$x_2 = 5$

$x_3 = 2$

$z = 34$

5

Let $\mathbf{X}_u$ represent the basic *and* nonbasic variables in $\mathbf{X}$ that have been substituted at their upper bound. Also, let $\mathbf{X}_z$ be the remaining basic *and* nonbasic variables. Suppose that the order of the vectors of (**A**, **I**) corresponding to $\mathbf{X}_z$ and $\mathbf{X}_u$ are given by the matrices $\mathbf{D}_z$ and $\mathbf{D}_u$, and let the vector **C** of the objective function be partitioned correspondingly to give $(\mathbf{C}_z, \mathbf{C}_u)$. The equations of the linear programming problem at any iteration then become

$$\begin{pmatrix} 1 & -\mathbf{C}_z & -\mathbf{C}_u \\ 0 & \mathbf{D}_z & \mathbf{D}_u \end{pmatrix}\begin{pmatrix} z \\ \mathbf{X}_z \\ \mathbf{X}_u \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

Instead of dealing with two types of variables, $\mathbf{X}_z$ and $\mathbf{X}_u$, $\mathbf{X}_u$ is put at zero level by using the substitution

$$\mathbf{X}_u = \mathbf{U}_u - \mathbf{X}'_u$$

where $\mathbf{U}_u$ is a subset of **U** representing the upper bounds for the variables in $\mathbf{X}_u$. This gives

$$\begin{pmatrix} 1 & -\mathbf{C}_z & \mathbf{C}_u \\ 0 & \mathbf{D}_z & -\mathbf{D}_u \end{pmatrix}\begin{pmatrix} z \\ \mathbf{X}_z \\ \mathbf{X}'_u \end{pmatrix} = \begin{pmatrix} \mathbf{C}_u\mathbf{U}_u \\ \mathbf{b} - \mathbf{D}_u\mathbf{U}_u \end{pmatrix}$$

The optimality and the feasibility conditions can be developed more easily now, since all nonbasic variables are at zero level. However, it is still necessary to check that no basic or nonbasic variable will exceed its upper bound.

Define $\mathbf{X}_B$ as the basic variables of the current iteration, and let $\mathbf{C}_B$ represent the elements corresponding to $\mathbf{X}_B$ in **C**. Also, let **B** be the basic matrix corresponding to $\mathbf{X}_B$. The current solution is determined from

$$\begin{pmatrix} 1 & -\mathbf{C}_B \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} z \\ \mathbf{X}_B \end{pmatrix} = \begin{pmatrix} \mathbf{C}_u\mathbf{U}_u \\ \mathbf{b} - \mathbf{D}_u\mathbf{U}_u \end{pmatrix}$$

By inverting the partitioned matrix as in Section 4.1.3, the current basic solution is given by

$$\begin{pmatrix} z \\ \mathbf{X}_B \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{C}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{C}_u\mathbf{U}_u \\ \mathbf{b} - \mathbf{D}_u\mathbf{U}_u \end{pmatrix} = \begin{pmatrix} \mathbf{C}_u\mathbf{U}_u + \mathbf{C}_B\mathbf{B}^{-1}(\mathbf{b} - \mathbf{D}_u\mathbf{U}_u) \\ \mathbf{B}^{-1}(\mathbf{b} - \mathbf{D}_u\mathbf{U}_u) \end{pmatrix}$$

By using

$$\mathbf{b}' = \mathbf{b} - \mathbf{D}_u\mathbf{U}_u$$

the complete simplex tableau corresponding to any iteration is

Basic	$\mathbf{X}_z^T$	$\mathbf{X}_u'^T$	Solution
z	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{D}_z - \mathbf{C}_z$	$-\mathbf{C}_B\mathbf{B}^{-1}\mathbf{D}_u + \mathbf{C}_u$	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{b}' + \mathbf{C}_u\mathbf{U}_u$
$\mathbf{X}_B$	$\mathbf{B}^{-1}\mathbf{D}_z$	$-\mathbf{B}^{-1}\mathbf{D}_u$	$\mathbf{B}^{-1}\mathbf{b}'$

6

(a) $b' = b - D_u U_u$

$$= \begin{pmatrix} 7 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix}(3) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_1 \end{pmatrix} = B^{-1}b' = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

(b) $X_B = \begin{pmatrix} x_4 \\ x_2' \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & -4 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & -1/4 \end{pmatrix}$

$b' = b - D_u U_u$

$$= \begin{pmatrix} 7 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_4 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1/4 \\ 0 & -1/4 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 5/4 \end{pmatrix}$$

7

Minimize $z = 6x_1 - 2x_2 - 3x_3$

Subject to

$$2x_1 + 4x_2 + 2x_3 + x_4 = 8$$
$$x_1 - 2x_2 + 3x_3 + x_5 = 7$$
$$0 \le x_1 \le 2, \quad 0 \le x_2 \le 2, \quad 0 \le x_3 \le 1$$

We use the tableau developed in Problem 5 above.

Iteration 0:

$$X_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}, \quad B = B^{-1} = I$$

$$C_B = (0,0), \quad C_B B^{-1} = (0,0)$$

$$\{z_j - c_j\}_{1,2,3}$$
$$= (0,0)\begin{pmatrix} 2 & 4 & 2 \\ 1 & -2 & 3 \end{pmatrix} - (6,-2,-3)$$
$$= (-6, 2, 3), \quad x_3 \text{ enters}$$

$$B^{-1}P_3 = B^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \Longrightarrow \theta_1 = 7/3$$

Since $B^{-1}P_3 > \underline{0}$, $\theta_2 = \infty$

$\theta = \min\{7/3, \infty, 1\} = 1$

Thus, x_3 becomes nonbasic at its upper bound.

New Solution: $X_Z = (x_1, x_2)$, $X_u = X_3$

$U_u = 1$, $C_u = -3$

$$D_Z = \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix}, \quad D_u = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad C_Z = (6,-2)$$

$$b' = \begin{pmatrix} 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}(1) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = B^{-1}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad z = -3$$

Iteration 1: $C_Z = (6,-2)$, $C_u = C_3' = 3$

$P_3' = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, $B = B^{-1} = I$, $C_B = (0,0)$, $C_B B^{-1} = (0,0)$

$$\{z_j - c_j\}_{Z(j=1,2)}$$
$$= (0,0)\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} - (6,-2) = (-6, 2)$$

continued...

$$\{z_j - c_j\}_{u(j=3)}$$
$$= (0,0)\begin{pmatrix} -2 \\ -3 \end{pmatrix} - (3) = -3$$

x_2 enters

$$B^{-1}P_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad X_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$\theta_1 = \frac{6}{4} = 3/2$, $\theta_2 = \infty$ (because $U_5 = \infty$)

$\theta = \min\{3/2, \infty, 2\} = 3/2$

x_4 leaves

Iteration 2: $C_Z = (x_1, x_4)$, $X_u = X_3$

$$X_B = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix}, \quad P_3' = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \quad b' = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix}$$

$$C_B = (-2, 0), \quad C_B B^{-1} = (-1/2, 0)$$

$$\{z_j - c_j\}_{Z(j=1,4)}$$
$$= (-1/2, 0)\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - (6,0) = (-7, 0)$$

$$\{z_j - c_j\}_{u(j=3)}$$
$$= (-1/2, 0)\begin{pmatrix} -2 \\ -3 \end{pmatrix} - 3 = -2$$

Optimum!

$$X_B = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 7 \end{pmatrix}$$

$$x_3 = 1 - 0 = 1$$
$$z = -6$$

8

(a)

To convert the problem into a dual feasible solution, we use the following substitutions:

$x_1 = 2 - x_1'$, $x_2 = 3 - x_2'$

Thus,

minimize $z = 3x_1' + 2x_2' + 2x_3 - 12$

Subject to

$-2x_1' - x_2' + x_3 \leq 1$

$-x_1' + 2x_2' - x_3 \leq -9$

$0 \leq x_1' \leq 2$, $0 \leq x_2' \leq 3$, $0 \leq x_3 \leq 1$

	x_1'	x_2'	x_3	x_4	x_5	
z	-3	-2	-2	0	0	-12
x_4	-2	-1	1	1	0	1
x_5	-1	2	$\boxed{-1}$	0	1	-9

x_5 leaves and x_3 enters

	x_1'	x_2'	x_3	x_4	x_5	
z	-1	-6	0	0	-2	6
x_4	-2	1	0	1	1	-8
x_3	1	-2	1	0	-1	9

x_3 above its upper bound, substitute $x_3 = 1 - x_3'$, then multiply the second row by -1.

	x_1'	x_2'	x_3'	x_4	x_5	
z	-1	-6	0	0	-2	6
x_4	-2	1	0	1	1	-8
x_3'	-1	2	1	0	1	-8

x_3' leaves and x_1' enters

	x_1'	x_2'	x_3'	x_4	x_5	
z	0	-8	-1	0	-3	14
x_4	0	-3	-2	1	-1	8
x_1'	1	-2	-1	0	-1	8

Substitute $x_1' = 2 - x_1$ and multiply second row by -1

Continued...

	x_1	x_2'	x_3'	x_4	x_5	
z	0	-8	-1	0	-3	14
x_4	0	-3	-2	1	-1	8
x_1	1	2	1	0	1	-8

x_1-row shows that the problem has no feasible solution

(b) Let $x_1 = 2 - x_1'$

$x_2 = 3 - x_2'$

This substitution will result in a dual feasible starting solution

	x_1'	x_2'	x_3	x_4	x_5	
z	1	5	2	0	0	17
x_4	-4	-2	2	1	0	12
x_5	1	3	$\boxed{-4}$	0	1	-6
z	3/2	13/2	0	0	1/2	14
x_4	-7/2	-1/2	0	1	1/2	9
x_3	-1/4	-3/4	1	0	-1/4	3/2

Optimum!

$x_1 = 2 - 0 = 2$

$x_2 = 3 - 0 = 3$

$x_3 = 3/2$

$z = 14$

1

Primal:

Maximize $z = CX$

Subject to

$$AX = b \quad \leftarrow Y$$
$$X \geq 0$$

Dual:

Minimize $w = Yb$

Subject to

$$YA \geq C$$

Y unrestricted

Dual in equation form:

Minimize $w = Yb$

subject to

$$YA - IS = C \quad \leftarrow X$$

Y unrestricted

$$S \geq 0$$

Dual of dual:

Maximize $z = CX$

Subject to

$$AX = b$$
$$-X \leq 0 \Rightarrow X \geq 0$$

The first set of constraints is equation because Y is unrestricted

The last problem shows that the dual of the dual is the primal

2

Primal in equation form:

Minimize $z = CX$

Subject to

$$AX - IS = b \quad \leftarrow Y$$
$$X \geq 0$$
$$S \geq 0$$

Dual:

Maximize $w = Yb$

Subject to

$$YA \leq C$$
$$-Y \leq 0 \Rightarrow Y \geq 0$$

Set 7.4b

1

Primal in equation form:

Maximize $z = x_1 + x_2$

Subject to

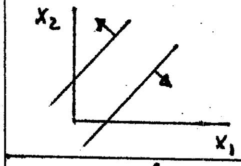
$$x_1 - x_2 + S_1 = -1 \quad \leftarrow y_1$$
$$-x_1 + x_2 + S_2 = -1 \quad \leftarrow y_2$$

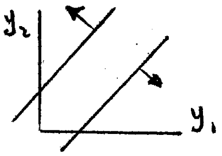
Dual:

Minimize $w = -y_1 - y_2$

Subject to

$$y_1 - y_2 \geq 1$$
$$-y_1 + y_2 \geq 1$$
$$y_1, y_2 \geq 0$$





2

(a) Dual:

Minimize $w = y_1 - 5y_2 + 6y_3$

Subject to

$$2y_1 + 4y_3 \geq 50$$
$$y_1 + 2y_2 \geq 30$$
$$y_3 \geq 10$$

y_1, y_2, y_3 unrestricted.

(b) $2x_1 = -5 \Rightarrow x_1 < 0$, infeasible

(c) Inspection of the second dual constraint shows that y_2 can be increased indefinitely without violating any of the dual constraints. Thus, $w = y_1 \boxed{-5y_2} + 6y_3$ is unbounded.

(d)

$$\text{Primal infeasible} \Rightarrow \begin{cases} \text{dual infeasible} \\ \text{or} \\ \text{dual unbounded} \end{cases}$$

Primal unbounded $\Rightarrow$ dual infeasible

3

(a) Minimize $w = 2y_1 + 5y_2$

Subject to

$$2y_1 + y_2 \geq 5$$
$$-y_1 + 2y_2 \geq 12$$
$$3y_1 + y_2 \geq 4$$
$$y_2 \geq 0$$

y_1 unrestricted

(b)

(i) $B = (P_4\ P_3) = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}, B^{-1} = \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix}$

$X_B = \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13/3 \\ 2/3 \end{pmatrix}$ feasible

$C_B = (0, 4)$

$Y = C_B B^{-1} = (0,4)\begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix} = (4/3, 0)$

Dual feasibility:

$2y_1 + y_2 = 2 \times 4/3 + 1 \times 0 = 8/3 \not\geq 5$

Dual infeasible $\Rightarrow$ primal nonoptimal.

(ii) $B = (P_2 P_3) = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}, B^{-1} = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}$

$X_B = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13/7 \\ 9/7 \end{pmatrix}$ feasible

Dual feasibility:

$Y = C_B B^{-1} = (12, 4)\begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}$

$= (-4/7, 40/7)$

$2y_1 + y_2 = 2(-4/7) + 40/7 = \frac{32}{7} \not\geq 5$

X_B is not optimal

(iii) $B = (P_1\ P_2) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, B^{-1} = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}$

$X_B = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 8/5 \end{pmatrix}$ feasible

Dual feasibility:

$Y = C_B B^{-1} = (5, 12)\begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}$

$= (-2/5, 29/5)$

Y satisfies all dual constraints. Thus X_B is optimal.

continued…

(iv) $B = (P_1 P_4) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$

$X_B = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ feasible

Dual feasibility:

$Y = C_B B^{-1} = (5, 0)\begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} = (5/2, 0)$

Y does not satisfy second dual constraint. X_B is <u>not</u> optimum

4

(a) Dual:

Minimize $w = 4y_1 + 8y_2$

subject to

$$\left.\begin{aligned} y_1 + y_2 &\geq 2 \\ y_1 + 4y_2 &\geq 4 \\ y_1 \quad &\geq 4 \\ y_2 &\geq -3 \end{aligned}\right\} \text{all } y \text{ unrestr.}$$

(b) $X_B = (x_2, x_3)^T$

$B = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 0 & 1/4 \\ 1 & -1/4 \end{pmatrix}$

$C_B = (4, 4)$, $C_B B^{-1} = (4, 0)$

$z_1 - c_1 = C_B B^{-1} P_1 - c_1$

$= (4,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \;=\; 2 > 0$

$z_4 - c_4 = (4,0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - (-3) = 3 > 0$

X_B optimal

(c) x_3 basic $\Rightarrow z_3 - c_3 = 0$, or

$YP_3 - c_3 = (y_1, y_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4 = 0$, or

$y_1 - 4 = 0 \implies y_1 = 4$ ①

x_2 basic $\Rightarrow z_2 - c_2 = 0$, or

$YP_2 - c_2 = (y_1, y_2)\begin{pmatrix} 1 \\ 4 \end{pmatrix} - 4 = 0$, or

$y_1 + 4y_2 = 4$. Given ①, we get $y_2 = 0$.

5

$B^{-1} b = X_B$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} b_1 &= 4 \\ b_2 &= 6 \\ b_3 &= 8 \end{aligned}$$

Dual objective value is

$w = Yb = (0, 3, 2)\begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = 34$

<u>From the dual:</u>

$C_B B^{-1} = Y$

$(c_1, c_2, 0)\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = (0, 3, 2)$

or $\left.\begin{aligned} c_2 - c_1 &= 3 \\ c_1 &= 2 \end{aligned}\right\} \Rightarrow c_1 = 2,\ c_2 = 5$

Primal objective value is

$z = C_B X_B = (2, 5, 0)\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 34$

6

$$\sum_{i=1}^{m} c_i (B^{-1} P_k)_i = (C_B B^{-1}) P_k$$

$$= Y P_k$$

$$= \sum_{i=1}^{m} y_i a_{ik}$$

7

Minimize $w = Yb$

Subject to

$YA = C$

Y unrestricted

8

Dual: Minimize $Y_1 b - Y_2 L + Y_3 U$

subject to

$Y_1 A - Y_2 + Y_3 \geq C$

$Y_1, Y_2, Y_3 \geq 0$

Let $Y = Y_3 - Y_2 \Rightarrow Y$ unrestricted. Hence $Y_1 A + (Y_3 - Y_2) \geq C$ can be written as $Y_1 A + Y \geq C$. Since Y is unrestricted, its value can always be selected such that $Y_1 A + Y \geq C$ is satisfied

Set 7.5a

1

For X_{B_0}:

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (4+14t, 1-t, 2+3t) \geq (0,0,0)$$

The inequalities are satisfied for

$$-2/7 \leq t \leq 1$$

2

(a) $$c_B(t) B_0^{-1} = (2, 5-6t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= (1, 2-3t, 0)$$

$$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (1, 2-3t, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3+3t, 0, 0)$$

$$= (4-12t, 1, 2-3t) \geq (0,0,0)$$

X_{B_0} remains optimal for $t \leq 1/3$

at $t = 1/3$, x_1 enters solution

$$B_0^{-1} P_1 = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 3/2 \\ 2 \end{pmatrix}$$

x_6 leaves.

$$X_{B_1} = (x_2, x_3, x_1)^T$$

$$B_1 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}$$

$$B_1^{-1} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}$$

$$X_{B_1} = B_1^{-1} b = (25/4, 90/4, 5)^T$$

$$c_B(t) B_1^{-1} = (2, 5-6t, 3+3t)\begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}$$

$$= (5-12t, 3t, -2+6t)$$

$$\{z_j - c_j\}_{j=4,5,6}$$

$$= (5-12t, 3t, -2+6t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (0,0,0)$$

$$= (5-12t, 3t, -2+6t)$$

X_{B_1} remains optimal for $1/3 \leq t \leq 5/12$

continued...

at $t = 5/12$, x_4 enters

$$B_1^{-1} P_4 = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 3/2 \\ -1 \end{pmatrix}$$

x_3 leaves

$$X_{B_2} = (x_2, x_4, x_1)^T$$

$$B_2 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 0 & 1 \end{pmatrix}$$

$$B_2^{-1} = \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}$$

$$X_{B_2} = B_2^{-1} b = (5/2, 15, 20)^T$$

$$c_B(t) B_2^{-1} = (2, 0, 3+3t)\, B_2^{-1}$$

$$= (0, 5/6+t, 1/2)$$

$$\{z_j - c_j\}_{j=3,5,6}$$

$$= (0, 5/6+t, 1/2)\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (5-6t, 0, 0)$$

$$= (-10/3+8t, 5/6+t, 1/2)$$

X_{B_2} remains optimal for $5/12 \leq t < \infty$

(b) $$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$c_B(t) B_0^{-1} = (2+t, 5+2t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= (1+t/2, 2+3t/4, 0)$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (1+t/2, 2+3/4t, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3-2t, 0, 0)$$

$$= (4+19t/4, 1+t/2, 2+3t/4) \geq (0,0,0)$$

X_{B_0} is optimal for all $t \geq 0$

(c) $$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$c_B(t) B_0^{-1} = (2+2t, 5-t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= (1+t, 2-t, 0)$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (1+t, 2-t, 0)\begin{pmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3+t, 0, 0)$$

$$= (4-3t, 1+t, 2-t) \geq (0,0,0)$$

continued...

X_{B_0} remains optimal for the range $t \leq 4/3$. At $t = 4/3$, x_1 enters solution. As in Part (a) above, x_6 leaves

$$B_1^{-1} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}, \; X_{B_1} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

$$X_{B_1} = B_1^{-1} b = (25/4, 90/4, 5)^T$$

$$C_B(t) B_1^{-1} = (2+2t, 5-t, 3+t)\, B_1^{-1}$$
$$= (5-2t, t/2, -2 + 3/2\, t)$$

$$\{z_j - c_j\}_{j=4,5,6}$$
$$= (5-2t, t/2, -2+3/2\, t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (0,0,0)$$
$$= (5-2t, \; t/2, \; -2 + 3/2\, t) \geq (0,0,0)$$

X_{B_1} remains optimal for

$$4/3 \leq t \leq 5/2$$

At $t = 5/2$, x_4 enters solution. As in Part (a), we have x_3 leaving and

$$B_2^{-1} = \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}, \; X_{B_2} = \begin{pmatrix} x_2 \\ x_4 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 15 \\ 20 \end{pmatrix}$$

$$C_B(t) B_2^{-1} = (2+2t, 0, 3+t) \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}$$
$$= (0, 5/6 + t/6, 1/2 + t/2)$$

$$\{z_j - c_j\}_{j=3,5,6}$$
$$= (0, 5/6 + t/6, 1/2 + t/2) \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$- (5-t, 0, 0)$$

$$= (-10/3 + 4t/3, \; 5/6 + t/6, \; 1/2 + t/2)$$
$$\geq (0,0,0)$$

X_{B_2} remains optimal for $\frac{5}{2} \leq t < \infty$

3

Minimize $z = (4-t)x_1 + (1-3t)x_2 + (2-2t)x_3$

Subject to

$$\begin{aligned} 3x_1 + x_2 + 2x_3 & & &= 3 \\ 4x_1 + 3x_2 + 2x_3 - x_4 & & &= 6 \\ x_1 + 2x_2 + 5x_3 & & + x_5 &= 4 \end{aligned}$$

$$x_1, x_2, \ldots, x_5 \geq 0$$

Continued...

$$X_{B_0} = (x_1, x_2, x_4)^T = (2/5, 9/5, 1)$$

$$B_0^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$C_B(t) B_0^{-1} = (4-t, 1-3t, 0)\, B_0^{-1}$$
$$= \left(\frac{7+t}{5}, 0, -\frac{1+8t}{5}\right)$$

$$\{z_j - c_j\}_{j=3,5}$$
$$= \left(\frac{7+t}{5}, 0, -\frac{1+8t}{5}\right) \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} - (2-2t, 0)$$
$$= \left(-\frac{1+28t}{5}, -\frac{1+8t}{5}\right) \leq (0,0)$$

B_0 remains optimal for all $t \geq 0$.

4

The dual simplex method requires that the LP problem be put in the form:

Minimize $z = CX$

Subject to

$$-AX \leq -b, \quad X \geq 0$$

Let B_i be the basis associated with critical value t_i in the parametric analysis. To obtain t_{i+1}, we consider

$$\{z_j - c_j\}_{\text{nonbasic } x_j}$$
$$= C_B(t) B_i^{-1}(-P_j) - c_j(t) \leq 0$$

where P_j is the j^{th} column vector of A.

In the present problem, the first two constraints are of the type $\geq$. Hence, only the first two constraints are multiplied by -1.

$$X_{B_0} = (x_3, x_2, x_6)^T = (3/2, 3/2, 0)^T$$

$$B_0^{-1} = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; C_{B_0}(t) = (1, 2+4t, 0)$$

$C_{B_0}(t)\,B_0^{-1} = (-1/2+2t, -1/2-2t, 0)$

$\{z_j - c_j\}_{1,4,5} = C_{B_0} B_0^{-1} P_j' - c_j(t)$

$= (-1/2+2t, -1/2-2t, 0)\begin{pmatrix} -3 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} - (3+t, 0, 0)$

$= (-13t-3, -1/2+2t, 0) \le (0, 0, 0)$

Thus, $t_1 = 1/4 \Rightarrow x_{B_0}$ remains optimal for $0 \le t \le 1/4$.

At $t = 1/4$, x_4 enters and x_6 leaves.

$X_{B_1} = (x_3, x_2, x_4)^T = (3/2, 3/2, 0)^T$

$B_1^{-1} = \begin{pmatrix} 0 & 1/2 & 3/2 \\ 0 & -1/2 & -1/2 \\ 1 & 0 & 1 \end{pmatrix}$, $C_{B_1}(t) = (1, 2+4t, 0)$

$C_{B_1}(t) B_1^{-1} = (0, -1/2-2t, 1/2-2t)$

$\{z_j - c_j\}_{1,5,6} = (0, -1/2-2t, 1/2-2t)\begin{pmatrix} -3 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - (3+t, 0, 0)$

$= (-4-9t, -1/2-2t, 1/2-2t) \le (0, 0, 0)$

Conditions are satisfied for $t \ge 1/4$. Thus, X_{B_1} is optimal for all $t \ge 1/4$.

Summary:

$X_{B_0} = (x_3, x_2, x_6) = (3/2, 3/2, 0)$ is optimal for $0 \le t \le 1/4$

$X_{B_1} = (x_3, x_2, x_4) = (3/2, 3/2, 0)$ is optimal for $t \ge 1/4$

OR

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 = 3/2 \\ x_3 = 3/2 \end{array}\right\} \text{ for all } t \ge 0$$

5

$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$

$C_{B_0}(t) = (2-2t^2, 5-t, 0)$

$B_0^{-1} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

$C_B(t)\, B_0^{-1} = (2-2t^2, 5-t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

$= (1-t^2, t^2/2 - t/2 + 2, 0)$

continued...

$\{z_j - c_j\}_{j=1,4,5}$

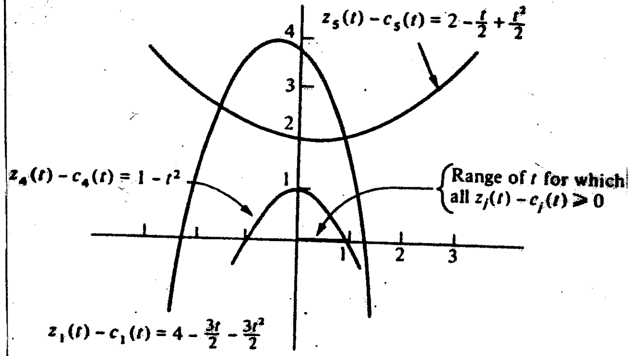
$= (1-t^2, t^2/2 - t/2 + 2, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$- (3+2t^2, 0, 0)$

$= \left(4 - \frac{3t}{2} - \frac{3t^2}{2}, 1-t^2, 2 - \frac{t}{2} + \frac{t^2}{2}\right)$

$\ge (0, 0, 0)$

The graph below summarizes the optimality conditions.



X_{B_0} remains optimal for $0 \le t \le 1$.

1

(a) $X_{B_0} = (X_2, X_3, X_6)^T$

$$= \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40+2t \\ 60-3t \\ 30+6t \end{pmatrix}$$

$$= \begin{pmatrix} 5+t/4 \\ 30-3t/2 \\ 10-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-20 \leq t \leq 10, \quad t_1 = 10$

X_6 leaves at $t = 10$.

(row of B_0^{-1} associated with X_6)$(P_1 \; P_4 \; P_5)$

$$= (-2, 1, 1) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (2, -2, 1)$$

$\{z_j - c_j\}_{j=1,4,5}$

$$= (2,5,0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3,0,0)$$

$= (4, 1, 2)$

	x_1	x_4	x_5
$z_j - c_j$	4	1	2
x_6	2	−2	1

x_4 enters.

$$\text{new } B_1 = (P_2 \; P_3 \; P_4) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

(b) $X_{B_0} = (X_2, X_3, X_6)^T$

$$= \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40-t \\ 60+2t \\ 30-5t \end{pmatrix}$$

$$= \begin{pmatrix} 5-t \\ 30+t \\ 10-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-30 \leq t \leq 5 \qquad t_1 = 5$

X_2 leaves when $t = 5$.

Continued...

(row of B_0^{-1} associated with X_2)$(P_1 P_4 P_5) =$

$$= (1/2, -1/4, 0) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$= (-1/4, 1/2, -1/4)$

$\{z_j - c_j\}_{j=1,4,5}$

$$= (2,5,0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3,0,0)$$

$= (4, 1, 2)$

	x_1	x_4	x_5
$z_j - c_j$	4	1	2
x_6	−1/4	1/2	−1/4

X_5 enters

$$\text{new } B_1 = (P_5 \; P_3 \; P_6) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2

$X_{B_0} = (x_1, x_2, x_4)^T = (2/5, 9/5, 1)$

x_4 = surplus in constraint 2

x_5 = slack in constraint 3

$$B_0^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$X_{B_0}(t) = B_0^{-1} \begin{pmatrix} 3+3t \\ 6+2t \\ 4-t \end{pmatrix} = \begin{pmatrix} 2/5 + 7/5\,t \\ 9/5 - 6/5\,t \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, $0 \leq t \leq 3/2$, $t_1 = 3/2$

At $t = 3/2$, x_2 leaves the solution. To determine the entering variable, we use the dual simplex computations.

(row of B_0^{-1} associated with x_2) (P_3, P_5)

$$= (-1/5, 0, 3/5) \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} = (13/5, 3/5)$$

Because $(13/5, 3/5) \geq 0$, the problem has no feasible solution for $t > 3/2$ (per dual simplex conditions).

<u>Summary:</u>

$x_1 = 2/5, x_2 = 9/5, x_3 = 0$, for $0 \leq t \leq 3/2$

No feasible solution for $t > 3/2$

Set 7.5b

For the dual simplex, the feasibility condition is

$$B^{-1}b'(t) \geq 0$$

where $b'(t)$ is modified such that the element $b_i(t)$ associated with $\geq$ constraint is replaced with $-b_i(t)$.

$$X_{B_0} = (x_3, x_2, x_6)^T = (3/2, 3/2, 0)$$

$$B_0^{-1} = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$b_0'(t) = \begin{pmatrix} -3-2t \\ -6+t \\ 3-4t \end{pmatrix}$$

The top two elements appear with an opposite sign because the first two constraint are of the type ≥ 0, hence reversing their signs in the dual simplex method.

$$B_0^{-1} b_0'(t) = \begin{pmatrix} -3/2 & -1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3-2t \\ -6+t \\ 3-4t \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 + 5/2\,t \\ 3/2 - 3/2\,t \\ -6t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$x_3 = 3/2 + 5/2\,t \geq 0$ gives $t \geq -\frac{3}{5}$

$x_2 = 3/2 - 3/2\,t \geq 0$ gives $t \leq 1$

$x_6 = -6t$ gives $t \leq 0$

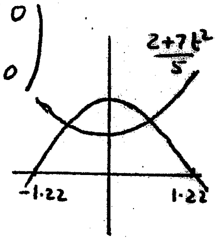
Thus, for $t \geq 0$, the solution X_{B_0} is feasible for $t = 0$ only. Else, the problem has no feasible solution for $t > 0$

4

$$X_{B_0} = (x_1, x_2, x_3)^T$$

$$X_{B_t} = B_0^{-1} b(t) = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3+3t^2 \\ 6+2t^2 \\ 4-t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 + 7/5\,t^2 \\ 9/5 - 6/5\,t^2 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$-1.22 \leq t \leq 1.22$

x_2 leaves at $t = 1.22$

$$(\text{Row 2 of } B_0^{-1})(P_4 \; P_5)$$

$$= (-1/5, 0, 3/5) \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 3/5)$$

$\Rightarrow$ no feasible solution exists for $t > 1.22$

continued…

CHAPTER 8

Goal Programming

Set 8.1a

1

Additional constraint:

$$.075X_g \geq .1(550X_p + 35X_f + 55X_s + .075X_g)$$

The constraint simplifies to

$$55X_p + 3.5X_f + 5.5X_s - .0675X_g \leq 0$$

Thus,

$$55X_p + 3.5X_f + 5.5X_s - .0675X_g + S_5^- - S_5^+ = 0$$

G_5: Minimize S_5^+

2

X_1 = Number of band concerts / yr

X_2 = number of art shows / yr

G_1: Minimize S_1^-

G_2: Minimize S_2^-

G_3: Minimize S_3^-

Constraints:

$$\begin{aligned} 1500X_1 + 3000X_2 &\leq 1500 \\ 200X_1 + S_1^- - S_1^+ &= 1000 \\ 100X_1 + 400X_2 + S_2^- - S_2^+ &= 1200 \\ 250X_2 + S_3^- - S_3^+ &= 800 \end{aligned}$$

all variables are ≥ 0

3

X_1 = in-state freshmen

X_2 = out-of-state freshmen

X_3 = international freshmen

(a) $X_1 + X_2 + X_3 \geq 1200$

(b) $\dfrac{27X_1 + 26X_2 + 23X_3}{X_1 + X_2 + X_3} \geq 25$

(c) $\dfrac{X_3}{X_1 + X_2 + X_3} \geq .1$

(d) $\dfrac{1/2\,X_1 + 2/5\,X_2 + 1/9\,X_3}{1/2\,X_1 + 3/5\,X_2 + 8/9\,X_3} \geq .75$

(e) $\dfrac{X_2}{X_1 + X_2 + X_3} \geq .2$

Goal program:

G_1: minimize S_1^-

G_2: minimize S_2^-

G_3: minimize S_3^-

G_4: minimize S_4^-

G_5: minimize S_5^-

continued...

Constraints:

$$\begin{aligned} X_1 + X_2 + X_3 + S_1^- - S_1^+ &= 1200 \\ 2X_1 + X_2 - 2X_3 + S_2^- - S_2^+ &= 0 \\ -.1X_1 - .1X_2 + .9X_3 + S_3^- - S_3^+ &= 0 \\ 1/8\,X_1 - \tfrac{1}{20}X_2 - 5/9\,X_3 + S_4^- - S_4^+ &= 0 \\ -.2X_1 + .8X_2 - .2X_3 + S_5^- - S_5^+ &= 0 \end{aligned}$$

all variables ≥ 0

4

X_1 = lb of limestone per day

X_2 = lb of corn per day

X_3 = lb of soybean meal per day

$$\begin{aligned} X_1 + X_2 + X_3 &\geq 6000 \\ .38X_1 + .001X_2 + .002X_3 &\leq .012(X_1 + X_2 + X_3) \\ .38X_1 + .001X_2 + .002X_3 &\geq .008(X_1 + X_2 + X_3) \\ .09X_2 + .5X_3 &\geq .22(X_1 + X_2 + X_3) \\ .02X_2 + .08X_3 &\leq .05(X_1 + X_2 + X_3) \end{aligned}$$

Goals:

G_1: minimize S_1^-

G_2: minimize S_2^+

G_3: minimize S_3^-

G_4: minimize S_4^-

G_5: minimize S_5^+

Constraints:

$$\begin{aligned} X_1 + X_2 + X_3 + S_1^- - S_1^+ &= 6000 \\ .368X_1 - .011X_2 - .01X_3 + S_2^- - S_2^+ &= 0 \\ .372X_1 - .007X_2 - .006X_3 + S_3^- - S_3^+ &= 0 \\ -.22X_1 - .13X_2 + .28X_3 + S_4^- - S_4^+ &= 0 \\ -.05X_1 - .03X_2 + .03X_3 + S_5^- - S_5^+ &= 0 \end{aligned}$$

all variables ≥ 0

Goal programming is not suitable for this problem because nutritional requirements <u>must</u> be met. However, goal programming can assist in deciding which nutritional requirements are "demanding" from the standpoint of optimization. The information may then be used to decide if alternative nutritional requirements can be specified in a manner that does not adversely impact cost minimization.

5

x_j = number of production runs in shift j, $j = 1, 2, 3$

$$\frac{500x_1 + 600x_2 + 640x_3}{300x_1 + 280x_2 + 360x_3} = \frac{4}{2}$$

or

$$-100x_1 + 40x_2 - 80x_3 = 0$$

Minimize $Z = S_1^- + S_1^+$

Subject to

$$-100x_1 + 40x_2 - 80x_3 + S_1^- - S_1^+ = 0$$

$$4 \le x_1 \le 5,\ 10 \le x_2 \le 20,\ 3 \le x_3 \le 5$$

6

x_j = number of units of part j, $j = 1, 2, 3, 4$

G_1: minimize S_1^+

G_2: minimize S_2^+

G_3: minimize S_3^+

G_4: minimize S_4^+

G_5: minimize S_5^-

G_6: minimize S_6^-

G_7: minimize S_7^-

G_8: minimize S_8^-

G_9: minimize S_9^+

Constraints:

$$\begin{aligned}
5x_1 + 6x_2 + 4x_3 + 7x_4 + S_1^- - S_1^+ &= 600 \\
3x_1 + 2x_2 + 6x_3 + 4x_4 + S_2^- - S_2^+ &= 600 \\
2x_1 + 4x_2 - 2x_3 + 3x_4 + S_3^- - S_3^+ &= 30 \\
-2x_1 - 4x_2 + 2x_3 - 3x_4 + S_4^- - S_4^+ &= 30 \\
x_1 \qquad\qquad + S_5^- - S_5^+ &= 10 \\
x_2 \qquad\qquad + S_6^- - S_6^+ &= 10 \\
x_3 \qquad\qquad + S_7^- - S_7^+ &= 10 \\
x_4 + S_8^- - S_8^+ &= 10 \\
x_1 - x_2 \qquad\qquad + S_9^- - S_9^+ &= 0
\end{aligned}$$

all variables ≥ 0

7

x_j = units of product j, $j = 1, 2$

G_1: minimize S_1^-

G_2: minimize S_2^-

G_3: minimize S_3^+

G_4: minimize S_4^+

Continued...

Constraints:

$$\begin{aligned}
x_1 + S_1^- - S_1^+ &= 80 \\
x_2 + S_2^- - S_2^+ &= 60 \\
5x_1 + 3x_2 + S_3^- - S_3^+ &= 480 \\
6x_1 + 2x_2 + S_4^- - S_4^+ &= 480
\end{aligned}$$

all variables ≥ 0

8

x_j = number of 1-day stays admitted on day j, $j = 1, 2, 3, 4$

y_j = number of 2-day stays admitted on day j, $j = 1, 2, 3, 4$

w_j = number of 3-day stays admitted on day j, $j = 1, 2, 3, 4$

G_1: minimize S_1^+

G_2: minimize S_2^+

G_3: minimize S_3^+

G_4: minimize S_4^+

Subject to

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 30 \\
y_1 + y_2 + y_3 + y_4 &= 25 \\
w_1 + w_2 + w_3 + w_4 &= 20 \\
x_1 + y_1 + w_1 + S_1^- - S_1^+ &= 20 \\
x_2 + y_1 + y_2 + w_1 + w_2 + S_2^- - S_2^+ &= 30 \\
x_3 + y_2 + y_3 + w_1 + w_2 + w_3 + S_3^- - S_3^+ &= 30 \\
x_4 + y_3 + y_4 + w_2 + w_3 + w_4 + S_4^- - S_4^+ &= 30
\end{aligned}$$

all variables ≥ 0

9

(x, y) = desired home location

G_1: minimize S_1^+

G_2: minimize S_2^-

G_3: minimize S_3^+

Subject to

$$\sqrt{(x-1)^2 + (y-1)^2} + S_1^- - S_1^+ = 25$$

$$\sqrt{(x-20) + (y-15)^2} + S_2^- - S_2^+ = 10$$

$$\sqrt{(x-4)^2 + (y-7)^2} + S_3^- - S_3^+ = 1$$

all variables ≥ 0

10

$\hat{y}$ = estimated value of y given the independent values x_j, $j = 1, 2, \ldots, n$

$= b_0 + b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$

The parameters $b_0, b_1, \ldots, b_n$ are determined by minimizing

$$\sum_{i=1}^{m} | y_i - \hat{y}_i |$$

where m is the number of observed points.

The equivalent goal programming model is given as

$$\text{minimize } z = \sum_{i=1}^{m} (S_i^- + S_i^+)$$

Subject to

$$\hat{y}_i + S_i^- - S_i^+ = y_i, \quad i = 1, 2, \ldots, m$$

$$S_i^-, S_i^+ \geq 0, \quad i = 1, 2, \ldots, m$$

The values of the unknown parameters $b_0, b_1, \ldots, b_n$ are introduced in the optimization problem by using the substitution

$$\hat{y}_i = b_0 + b_1 x_{i1} + b_2 x_{i2} + \cdots + b_n x_{in}$$

Thus, the variables of the model are $S_i^-, S_i^+, b_0, b_1, \ldots, b_n$. Only S_i^- and S_i^+ are required to be nonnegative.

11

$$\text{Minimize} \left[\max_{i=1,2,\ldots,m} \{ |y_i - \hat{y}_i| \} \right]$$

Let

$$d = \max \{ |y_1 - \hat{y}_1|, |y_2 - \hat{y}_2|, \ldots, |y_m - \hat{y}_m| \}$$

continued...

The problem reduces to the following goal program:

minimize $z = d$

Subject to

$$\left.\begin{aligned} \hat{y}_i + d &\geq y_i \\ \hat{y}_i - d &\leq y_i \end{aligned}\right\} i = 1, 2, \ldots, m$$

$$d \geq 0$$

1

Minimize $Z = S_1^- + S_2^- + S_3^- + S_4^+ + S_5^+$

s.t.

$$\begin{aligned}
550x_p + 35x_f + 55x_s + .075x_g + S_1^- - S_1^+ &= 16\\
55x_p - 31.5x_f + 5.5x_s + .0075x_g + S_2^- - S_2^+ &= 0\\
110x_p + 7x_f - 44x_s + .015x_g + S_3^- - S_3^+ &= 0\\
x_g + S_4^- - S_4^+ &= 2\\
55x_p + 3.5x_f + 5.5x_s - .0675x_g + S_5^- - S_5^+ &= 0
\end{aligned}$$

Solution: $x_p = .0201$, $x_f = .0457$, $x_s = .0582$

$x_g = 2$ cents, $S_5^+ = 1.45$, all others $= 0$

Gasoline tax goal is \$1.45 million short of its \$1.6 million

2

Minimize $z = S_1^- + 2S_2^- + S_3^-$

s.t.

$$\begin{aligned}
1500x_1 + 3000x_2 & &\le 15000\\
200x_1 & + S_1^- - S_1^+ &= 1000\\
100x_1 + 400x_2 & + S_2^- - S_2^+ &= 1200\\
250x_2 & + S_3^- - S_3^+ &= 800
\end{aligned}$$

Solution: $z = 175$, $x_1 = 5$, $x_2 = 2.5$.

$S_1^- = S_1^+ = 0$: goal 1 satisfied

$S_2^+ = 300$: goal 2 overachieved by 300 persons

$S_3^- = 175$: goal 3 underachieved by 175 persons

3

(a) Minimize $Z = 2S_2^- + S_3^- + S_4^- + S_5^+$

s.t.

$$\begin{aligned}
x_1 + x_2 + x_3 &\ge 1200\\
2x_1 + x_2 - 2x_3 + S_2^- - S_2^+ &= 0\\
.125x_1 - .05x_2 - .556x_3 + S_3^- - S_3^+ &= 0\\
-.1x_1 - .1x_2 + .9x_3 + S_4^- - S_4^+ &= 0\\
-.2x_1 + .8x_2 - .2x_3 + S_5^- - S_5^+ &= 0
\end{aligned}$$

Solution: $z = 0$: all goals are satisfied

$x_1 \cong 801$, $x_2 \cong 240$, $x_3 = \simeq 159$

$S_2^+ = 15225.6$: ACT score overachieved by 1.27 pts/stud

$S_4^+ = 38.59$: Nbr of international students overachieved by 39 students

(b) Minimize $z = 4S_1^- + 2S_2^- + S_3^- + S_5^-$

$$x_1 + x_2 + x_3 + S_1^- - S_2^+ = 1200$$

Solution in (a) remains the same

4

Minimize $Z = S_1^- + S_2^+ + S_3^- + S_4^- + S_5^+$

s.t.

$$\begin{aligned}
x_1 + x_2 + x_3 + S_1^- - S_1^+ &= 6000\\
.368x_1 - .011x_2 - .01x_3 + S_2^- - S_2^+ &= 0\\
.372x_1 - .007x_2 - .006x_3 + S_3^- - S_3^+ &= 0\\
-.22x_1 - .13x_2 + .28x_3 + S_4^- - S_4^+ &= 0\\
-.05x_1 - .03x_2 + .03x_3 + S_5^- - S_5^+ &= 0
\end{aligned}$$

$Z = 0$: all goals are satisfied

$x_1 = 166.08$ lb, $x_2 = 2778.56$ lb, $x_3 = 3055.36$ lb

$S_3^+ = 24$: G3 overachieved by $\frac{24}{6000} = .004$

$S_4^+ = 457.75$: G4 overachieved by $\frac{457.75}{6000} = .0763$

calcium% = 1.2

Protein% = 22 + 7.63 = 29.63, fiber% = 5

5

Minimize $z = S_1^- + S_1^+$

s.t. $-100x_1 + 40x_2 - 80x_3 + S_1^- - S_1^+ = 0$

$4 \le x_1 \le 5$, $10 \le x_2 \le 20$, $3 \le x_3 \le 5$

Solution: $z = 0$: all goal are satisfied

$x_1 = 4$, $x_2 = 16$, $x_3 = 3$

$S_1^- = S_1^+ = 0$: Production is balanced.

6

Min $Z = S_3^- + S_4^- + 2S_5^- + 2S_6^- + 2S_7^- + 2S_8^- + 2S_9^+$

s.t.

$$\begin{aligned}
5x_1 + 6x_2 + 4x_3 + 7x_4 & &\le 600\\
3x_1 + 2x_2 + 6x_3 + 4x_4 & &\le 600\\
2x_1 + 4x_2 - 2x_3 + 3x_4 & + S_3^- - S_3^+ &= 30\\
-2x_1 - 4x_2 + 2x_3 - 3x_4 & + S_4^- - S_4^+ &= 30\\
x_1 & + S_5^- - S_5^+ &= 10\\
x_2 & + S_6^- - S_6^+ &= 10\\
x_3 & + S_7^- - S_7^+ &= 10\\
x_4 & + S_8^- - S_8^+ &= 10\\
x_1 - x_2 & + S_9^- - S_9^+ &= 0
\end{aligned}$$

$Z = 0$: all goals are satisfied

$x_1 = 10$, $x_2 = 10$, $x_3 = 30$, $x_4 = 10$

7

Assign a relatively large weight to the quota constraint.

Min $z = 100(S_1^- + S_2^-) + (S_3^+ + S_4^+)$

s.t.

$$\begin{aligned}
x_1 + S_1^- - S_1^+ &= 80\\
x_2 + S_2^- - S_2^+ &= 60\\
5x_1 + 3x_2 + S_3^- - S_3^+ &= 480\\
6x_1 + 2x_2 + S_4^- - S_4^+ &= 480
\end{aligned}$$

Solution: $x_1 = 80$, $x_2 = 60$, $S_3^+ = 100$, $S_4^+ = 120$ min

Production quota can be met with 100 min of overtime on machine 1 and 120 min on machine 2

8

Min $Z = S_1^+ + S_2^+ + S_3^+ + S_4^+$

s.t.

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 30\\
y_1 + y_2 + y_3 + y_4 &= 25\\
w_1 + w_2 + w_3 + w_4 &= 20\\
x_1 + y_1 + w_1 + S_1^- - S_1^+ &= 20\\
& = 30\\
x_2 + y_1 + y_2 + w_1 + w_2 + S_2^- - S_2^+ &\\
x_3 + y_2 + y_3 + w_1 + w_2 + w_3 + S_3^- - S_3^+ &= 30\\
x_4 + y_3 + y_4 + w_2 + w_3 + w_4 + S_4^- - S_4^+ &= 30
\end{aligned}$$

Solution: $z = 0$: All goals are met

$x_1 = 5$, $x_2 = 15$, $x_3 = 10$, $x_4 = 0$

Σ 1-day stays = 30

$y_1 = 10$, $y_2 = 0$, $y_3 = 15$, $y_4 = 0$

Σ 2-day stays = 25

$w_1 = 5$, $w_2 = 0$, $w_3 = 0$, $w_4 = 15$

Σ 3-day stays = 20

The solution shows that :

continued...

Set 8.2a

Nbr. beds used on day 1

$= x_1 + y_1 + w_1 = 20$ (= availability 20)

Nbr. beds used on day 2 $= x_2 + y_2 + w_2 = 15$ (< 30)

Nbr. beds used on day 3 $= x_3 + y_3 + w_3 = 25$ (< 30)

Nbr beds used on day 4 $= x_4 + y_4 + w_4 = 15$ (< 30)

Conclusion: All 1-, 2-, and 3-day stays can be met without overbooking

9

$$\hat{y} = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3$$

Minimize $z = \sum_{i=1}^{5} (S_i^- + S_i^+)$

Subject to

$$b_0 + 30b_1 + 4b_2 + 5b_3 + S_1^- - S_1^+ = 40$$
$$b_0 + 39b_1 + 5b_2 + 10b_3 + S_2^- - S_2^+ = 48$$
$$b_0 + 44b_1 + 2b_2 + 14b_3 + S_3^- - S_3^+ = 38$$
$$b_0 + 48b_1 \quad + 18b_3 + S_4^- - S_4^+ = 36$$
$$b_0 + 37b_1 + 3b_2 + 9b_3 + S_5^- - S_5^+ = 41$$

$S_i^+, S_i^- \geq 0, \ i = 1, 2, \ldots, 5$

b_0, b_1, b_2, b_3 unrestricted

TORA Solution:

$b_0 = .8571$

$b_1 = 1.0714$

$b_2 = 2.881$

$b_3 = -.9048$

$S_3^- = 3.0952$

all other S_i^- and $S_i^+ = 0$

Thus, the least-square estimator is given as

$$\hat{y} = .8571 + 1.0714X_1 + 2.881X_2 - .9048X_3$$

10

$$\hat{y} = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3$$

minimize $z = d$

Subject to

$$b_0 + 30b_1 + 4b_2 + 5b_3 + d \geq 40$$
$$b_0 + 39b_1 + 5b_2 + 10b_3 + d \geq 48$$
$$b_0 + 44b_1 + 2b_2 + 14b_3 + d \geq 38$$
$$b_0 + 48b_1 \quad + 18b_3 + d \geq 36$$
$$b_0 + 37b_1 + 3b_2 + 9b_3 + d \geq 41$$

$$b_0 + 30b_1 + 4b_2 + 5b_3 - d \leq 40$$
$$b_0 + 39b_1 + 5b_2 + 10b_3 - d \leq 48$$
$$b_0 + 44b_1 + 2b_2 + 14b_3 - d \leq 38$$
$$b_0 + 48b_1 \quad + 18b_3 - d \leq 36$$
$$b_0 + 37b_1 + 3b_2 + 9b_3 - d \leq 41$$

b_0, b_1, b_2, b_3 unrestricted

$d \geq 0$

TORA Solution:

$b_0 = 27.5536$

$b_1 = -.0893$

$b_2 = 3.2679$

$b_3 = .6429$

$d = 1.1607$

Chebyshev estimator:

$$\hat{y} = 27.5536 - .0893X_1 + 3.2679X_2 + 1.1607X_3$$

1

Minimize $G_1 = S_1^-$
subject to

$$\begin{aligned} 4x_1 + 8x_2 + S_1^- - S_1^+ &= 45 \\ 8x_1 + 24x_2 + S_2^- - S_2^+ &= 110 \\ x_1 + 2x_2 &\leq 10 \\ x_1 &\leq 6 \\ x_1, x_2, S_1^-, S_1^+, S_2^-, S_2^+ &\geq 0 \end{aligned}$$

TORA solution:

$x_1 = 2.5$, $x_2 = 3.75$ $S_1^- = 5$
$S_1^+ = S_2^- = S_2^+ = 0$

Both goals are automatically satisfied.

2

$G_1 \succ G_2 \succ G_3 \succ G4 \succ G5$

G1- Problem Solution:

$x_p = .01745$ $x_f = .0457$, $x_s = .0582$
$x_g = 21.33$

$S_1^- = S_1^+ = S_2^- = S_2^+ = S_3^- = S_3^+ = S_y^-$
$= S_5^- = 0$
$S_4^+ = 19.33$

Goals G1, G2, G3, and G are satisfied.

G4 - Problem:

Minimize $z = S_4^+$
subject to G1-constraints & $S_1^- = S_2^- = S_3^- = 0$
Solution: $x_1 = .0201$, $x_2 = .0457$, $x_3 = .0582$, $x_4 = 2$
$S_5^+ = 1.45$. G_5 is not satisfied

G5-Problem: Minimize $z = S_5^+$ subject to same constraints in G4 & $S_4^+ = 0$
Solution:
Same as in G4, which means that G5 cannot be satisfied.

3

(a) $G1 \succ G2 \succ G3$
G1-Problem:
minimize $G1 = S_1^-$
TORA solution: $S_1^- = 0$, $S_2^- = 0$, $S_3^- = 362.5$
$x_1 = 5$, $x_2 = 1.75$

G2 is satisfied
G3-Problem:
Minimize $G3 = S_3^-$
$S_1^- = 0$, $S_2^- = 0$
TORA solution: $S_3^- = 175$
$x_1 = 5$, $x_2 = 2.5$
G3 remains unsatisfied.
(b) $G3 \succ G2 \succ G1$
G3-Problem: minimize $G3 = S_3^-$
TORA Solution: $S_1^- = 280$, $S_2^- = 0$, $S_3^- = 0$
$x_1 = 3.6$, $x_2 = 3.2$

G2 is satisfied.
G1-Problem: minimize $G1 = S_1^-$
$S_2^- = 0$, $S_3^- = 0$
TORA Solution: $x_1 = 3.6$, $x_2 = 3.2$, $S_1^- = 280$
G1 is not satisfied

4

Problem G1: minimize $G1 = S_2$
TORA Solution: $x_1 = 0$, $x_2 = 1080$, $x_3 = 120$
$S_4^+ = 309.33$, $S_2^- = S_3^- = 0$
G2 (minimize S_3) is satisfied.
G3-Problem: minimize $G3 = S_4^+$
$S_2^- = 0$, $S_3^- = 0$

TORA Solution: $x_1 = 1080$, $x_2 = 0$, $x_3 = 120$
$S_4^+ = 93.33$, $S_5^+ = 240$
G4- Problem: Minimize $G4 = S_5^+$
$S_2^- = 0$, $S_3^- = 0$, $S_4^+ = 9333$

TORA Solution: $x_1 = 1080$, $x_2 = 0$
$x_3 = 120$
$S_5^+ = 240$
G3 and G4 are unsatisfied

CHAPTER 9

Integer Linear Programming

Set 9.1a

1

Max $z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$

subject to

$$\begin{pmatrix} 5 & 4 & 3 & 7 & 8 \\ 1 & 7 & 9 & 4 & 6 \\ 8 & 10 & 2 & 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 25 \\ 25 \\ 25 \end{pmatrix}$$

(a) $x_1 \leq x_5$, $x_3 \leq x_5$, all x_j binary

Solution: $x_2 = x_3 = x_5 = 1$, $Z = 90$

(b) $x_2 + x_3 \leq 1$, all x_j binary

Solution: $x_2 = x_4 = x_5 = 1$, $Z = 85$

Note: When you use TORA, add the upper bound $x_j \leq 1$ for all binary variables.

2

x_i = number of units of item i, $i = 1, 2, \ldots, 5$

Maximize $z = 4x_1 + 7x_2 + 6x_3 + 5x_4 + 4x_5$

subject to

$$\begin{pmatrix} 5 & 8 & 3 & 2 & 7 \\ 1 & 8 & 6 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 112 \\ 109 \end{pmatrix}$$

$x_j \geq 0$ and integer, $j = 1, 2, \ldots, 5$

Solution: $x_1 = 14$, $x_4 = 19$, all others are zero, $z = 151$

3

x_{ij} = number of bottles of type i assigned to individual j

where $i = \begin{cases} 1, & \text{full} \\ 2, & \text{half-full} \\ 3, & \text{empty} \end{cases}$

Total available wine $= 7 + 3\frac{1}{2} = 10\frac{1}{2}$

Share per individual $= \frac{10\frac{1}{2}}{3} = 3\frac{1}{2}$ bottles

Constraints:

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &= 7 \\ x_{21} + x_{22} + x_{23} &= 7 \\ x_{31} + x_{32} + x_{33} &= 7 \end{aligned}\right\} \text{bottle type}$$

$$\left.\begin{aligned} x_{11} + \frac{x_{21}}{2} &= 3.5 \\ x_{12} + \frac{x_{22}}{2} &= 3.5 \\ x_{13} + \frac{x_{23}}{2} &= 3.5 \end{aligned}\right\} \text{amount of wine per individual}$$

continued...

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &= 7 \\ x_{12} + x_{22} + x_{32} &= 7 \\ x_{13} + x_{23} + x_{33} &= 7 \end{aligned}\right\} \text{bottles per individual (redundant)}$$

$x_{ij} \geq 0$ and integer

Use dummy objective function

maximize $z = 0x_{11} + 0x_{12} + \cdots + 0x_{33}$

Feasible solution: (alternative solutions exist)

		individual			
		1	2	3	sum
type	F	3	3	1	7
	H	1	1	5	7
	E	3	3	1	7
	Sum	7	7	7	
	Qty.	3.5	3.5	3.5	

4

x_1 = number of camels to Tarek

x_2 = number of camels to Sharif

x_3 = number of camels to Maisa

x_4 = number of camels to charity (=1)

r = dummy integer variable ≥ 0.

y = total number of camels in the will

Constraints:

$y = x_1 + x_2 + x_3 + 1$

$y = 2r + 1 \Rightarrow y$ is odd

$x_1 \geq 1/2\, y$, $x_2 \geq 1/3\, y$, $x_3 \geq \frac{1}{9} y$

Using a dummy objective function, the problem reduces to

	y	x_1	x_2	x_3	r	
min	0	0	0	0	0	
	1	−1	−1	−1	0	= 1
	1	0	0	0	−2	= 1
	1	−2	0	0	0	≤ 0
	1	0	−3	0	0	≤ 0
	1	0	0	−9	0	≤ 0

continued...

Solution: $y = 27$ camels. Tarek get 14, Sharif gets 9, and maisa get 3.
Note: If you enter the last two constraints in the original fractional form, make sure that 1/3 and 1/9 are accurate to six decimal points (.333333 and .111111). Else, TORA fails to find a solution.

5

x_{ij} = number of apples belonging to child i and sold at price j.

$$i = \begin{cases} 1 \to \text{Jim} \\ 2 \to \text{Bill} \\ 3 \to \text{John} \end{cases} \qquad j = \begin{cases} 1 \to \$1/7 \text{ apples} \\ 2 \to \$3/\text{apple} \end{cases}$$

Allocation of apples to children:

$$x_{11} + x_{12} = 50 \quad (\text{Jim})$$
$$x_{21} + x_{22} = 30 \quad (\text{Bill})$$
$$x_{31} + x_{32} = 10 \quad (\text{John})$$

Allocate same money to each child:

$$\frac{x_{11}}{7} + 3x_{12} = \frac{x_{21}}{7} + 3x_{22}$$
$$\frac{x_{11}}{7} + 3x_{12} = \frac{x_{31}}{7} + 3x_{32}$$

Objective function:

$$\text{maximize } z = \frac{x_{11}}{7} + 3x_{12}$$

ILP:

maximize $z = x_{11} + 21x_{12}$

subject to

$$\begin{aligned} x_{11} + x_{12} &= 50 \\ x_{21} + x_{22} &= 30 \\ x_{31} + x_{32} &= 10 \\ x_{11} + 21x_{12} - x_{21} - 21x_{22} &= 0 \\ x_{11} + 21x_{12} - x_{31} - 21x_{32} &= 0 \\ x_{ij} \geq 0 \text{ and integer} \end{aligned}$$

Solution:

	\$1/7apples	\$3/apple	\$
Jim	42	8	30
Bill	21	9	30
John	0	10	30

Each child returns home with \$30.

6

y = original sum of money
x_1 = amount taken the first night
x_2 = amount taken the second night
x_3 = amount taken the third night
x_4 = amount given by first officer to each mariner

Minimize $z = y$

subject to

$$x_1 = \frac{y-1}{3} + 1$$
$$x_2 = \frac{y - x_1 - 1}{3} + 1$$
$$x_3 = \frac{y, - x_1 - x_2 - 1}{3} + 1$$
$$x_4 = \frac{y, - x_1 - x_2 - x_3 - 1}{3}$$

The ILP is given as

minimize $z = y$

subject to

$$\begin{aligned} 3x_1 \qquad\qquad\qquad - y &= 2 \\ x_1 + 3x_2 \qquad\quad - y &= 2 \\ x_1 + x_2 + 3x_3 \quad - y &= 2 \\ -x_1 - x_2 - x_3 - 3x_4 + y &= 1 \\ x_1, x_2, x_3, x_4, y \geq 0 \text{ and integer} \end{aligned}$$

Solution: $y = 79$ units

Resolve the problem after adding the constraint $y \geq 80$.

Solution: $y = 160$ units

Resolve the problem after adding the constraint $y \geq 161$

Solution: $y = 241$ units

General Solution: $y = 79 + 81n$, $n = 0, 1, 2, \ldots$

Set 9.1a

7

Given A=1 and Z=26, let $x_j = 1$ if word j is selected and 0 if it is not selected.

$x_j = 1$ if word j is selected and 0 if it is not selected.

j	Word	L_{1j}	L_{2j}	L_{3j}	Score
1	AFT	1	6	20	27
2	FAR	6	1	18	25
3	TVA	20	22	1	43
4	ADV	1	4	22	27
5	JOE	10	15	5	30
6	FIN	6	9	14	29
7	OSF	15	19	6	40
8	KEN	11	5	14	30

$\sum_{j=1}^{8} L_{1j}x_j < \sum_{j=1}^{8} L_{2j}x_j$ implies that

$\sum_{j=1}^{8}(L_{2j} - L_{1j}) > 0$, or $\sum_{j=1}^{8}(L_{2j} - L_{1j}) \geq 1$

which translates to

$$5x_1 - 5x_2 + 2x_3 + 3x_4 + 5x_5 + 3x_6 + 4x_7 - 6x_8 \geq 1$$

Similarly, constraint $\sum_{j=1}^{8} L_{2j} < \sum_{j=1}^{8} L_{3j}x_j$ translates to

$$14x_1 + 17x_2 - 21x_3 + 18x_4 - 14x_5 + 5x_6 - 13x_7 + 9x_8 \geq 1$$

<u>ILP</u>:

$$\text{maximize } z = 27x_1 + 25x_2 + 43x_3 + 27x_4 + 30x_5 + 29x_6 + 40x_7 + 30x_8$$

Subject to

$$5x_1 - 5x_2 + 2x_3 + 3x_4 + 5x_5 + 3x_6 + 4x_7 - 6x_8 \geq 1$$

$$14x_1 + 17x_2 - 21x_3 + 18x_4 - 10x_5 + 5x_6 - 13x_7 + 9x_8 \geq 1$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 5$$

$$x_j = (0, 1),\ j = 1, 2, \dots, 8$$

<u>Solution</u>: $x_1 = x_3 = x_4 = x_7 = x_8 = 1$

Selected word	L_{1j}	L_{2j}	L_{3j}	Score
AFT	1	6	20	27
TVA	20	22	1	43
ADV	1	4	22	27
OSF	15	19	6	40
KEN	11	5	14	30
Σ	48	56	63	167

8

Because $\sum_{j=1}^{8} L_{1j}x_j < \sum_{j=1}^{8} L_{2j}x_j < \sum_{j=1}^{8} L_{3j}x_j$, the new objective function

$$\text{Maximize } z = \sum_{j=1}^{8} L_{ij}x_j$$

produces the desired result, including that of Problem 7.

9

C_{ik} = Nbr. of times letter i is repeated in group k, $k = 1, 2$

$$x_{ij} = \begin{cases} 1, & \text{if letter } i \text{ is assigned value } j \\ 0, & \text{therwise} \end{cases}$$

$$\text{Minimize } z = \left| \sum_{i=1}^{9} (C_{i1} - C_{i2}) \sum_{j=1}^{9} j x_{ij} \right|$$

s.t.

$$\sum_{j=1}^{9} x_{ij} = 1, \quad \text{all } i$$

$$\sum_{i=1}^{9} x_{ij} = 1, \quad \text{all } j$$

The objective function is equivalent to

$$\text{Minimize } z = y$$

s.t.

$$-y \leq \sum_{j=1}^{9} (C_{i1} - C_{i2}) \sum_{j=1}^{9} j x_{ij} \leq y$$

<u>Solution</u>: $z = 0$

A=8, E=3, F=7, H=2, O=1, P=4, R=6, S=9, T=5

10

$$x_{ij} = \begin{cases} 1, & \text{if song } i \text{ is on side } j \\ 0, & \text{if song } i \text{ is \underline{not} on side } j \end{cases}$$

$$\text{Minimize } z = |S_1 - S_2|$$

Subject to

$$8x_{11} + 3x_{21} + 5x_{31} + 5x_{41} + 9x_{51} + 6x_{61} + 7x_{71} + 12x_{81} + S_1 = 30$$

$$8x_{21} + 3x_{22} + 5x_{32} + 5x_{42} + 9x_{52} + 6x_{62} + 7x_{72} + 12x_{82} + S_2 = 30$$

$$x_{i1} + x_{i2} = 1, \quad i = 1, 2, \dots, 8$$

Let $y = |S_1 - S_2| \Rightarrow \begin{cases} S_1 - S_2 \leq y \\ S_1 - S_2 \geq -y \end{cases}$

continued...

ILP:

minimize $z = y$

subject to

$$8x_{11}+3x_{21}+5x_{31}+5x_{41}+9x_{51}+6x_{61}+7x_{71}+12x_{81}+S_1=30$$

$$8x_{12}+3x_{22}+5x_{32}+5x_{42}+9x_{52}+6x_{62}+7x_{72}+12x_{82}+S_2=30$$

$x_{i1}+x_{i2}=1,\ i=1,2,\dots,8$

$S_1-S_2-y\le 0$

$S_1-S_2+y\ge 0$

$x_{ij}=(0,1),\ i=1,2,\dots,8;\ j=1,2$

$S_1, S_2, y\ge 0$

Solution:

Side 1: 5-6-8 (27 minutes)

Side 2: 1-2-3-4-7 (28 minutes)

Problem has alternative optima.

Simpler Model:

Minimize $z = y$

subject to

$$8x_{11}+3x_{21}+5x_{31}+5x_{41}+9x_{51}+6x_{61}+7x_{71}+12x_{81}\le y$$

$$8x_{12}+3x_{22}+5x_{32}+5x_{42}+9x_{52}+6x_{62}+7x_{72}+12x_{82}\le y$$

$x_{i1}+x_{i2}=1,\ i=1,2,\dots,8$

$y\ge 0$

Solution:

Side 1: 3-4-6-8, time = 28 minutes

Side 2: 1-2-5-7, time = 27 minutes

11

Add the constraints

$x_{31}+x_{41}=1$

$x_{32}+x_{42}=1$

Use the simpler model in Problem 10; that is,

continued...

minimize $z = y$

subject to

$$8x_{11}+3x_{21}+5x_{31}+5x_{41}+9x_{51}+6x_{61}+7x_{71}+12x_{81}\le y$$

$$8x_{12}+3x_{22}+5x_{32}+5x_{42}+9x_{52}+6x_{62}+7x_{72}+12x_{82}\le y$$

$x_{i1}+x_{i2}=1,\ i=1,2,\dots,8$

$x_{31}+x_{41}=1$

$x_{32}+x_{42}=1$

$x_{ij}=(0,1)$ for all i and j

$y\ge 0$

Solution:

Side 1: 1-2-4-8, $\Sigma = 28$ min

Side 2: 3-5-6-7, $\Sigma = 27$ min

The tape must be at least 28 minutes.

12

$x_{ij}=\begin{cases}1, \text{student } i \text{ selects course } j,\\ 0, \text{otherwise}\end{cases}$

P_{ij} = associated preference score

Maximize $z=\sum_{i=1}^{10}\sum_{j=1}^{6}P_{ij}x_{ij}$

s.t. $\sum_{j=1}^{6}x_{ij}=2,\ i=1,2,\dots,10$

$\sum_{i=1}^{10}x_{ij}\le C_j,\ j=1,2,\dots,6$

Solution: Total score = 1775

Course	Students
1	2,4,9
2	2,8
3	5,6,7,9
4	4,5,7,10
5	1,3,8,10
6	1,3

1

Route	Delivery distance	
1,2,3,4	10+32+14+15+9	= 80
4,3,5	9+15+18+8	= 50
1,2,5	10+32+20+8	= 70
2,3,5	12+14+18+8	= 52
1,4,2	10+17+21+12	= 60
1,3,5	10+8+18+8	= 44

All routes start and end at ABC.

$$x_j = \begin{cases} 1, & \text{if route } j \text{ is selected} \\ 0, & \text{if otherwise} \end{cases}$$

$$\begin{array}{lccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\min z = & 80 & 50 & 70 & 52 & 60 & 44 & \\
\text{subject to} & & & & & & & \\
\text{Customer ①} & 1 & 0 & 1 & 0 & 1 & 1 & \geq 1 \\
② & 1 & 0 & 1 & 1 & 1 & 0 & \geq 1 \\
③ & 1 & 1 & 0 & 1 & 0 & 1 & \geq 1 \\
④ & 1 & 1 & 0 & 0 & 1 & 0 & \geq 1 \\
⑤ & 0 & 1 & 1 & 1 & 0 & 1 & \geq 1
\end{array}$$

$x_j = (0,1),\ j = 1, 2, \ldots, 6$

Solution: $x_5 = x_6 = 1$, all others $= 0$

$z = 104$

Select routes (1,4,2) and (1,3,5). Customer 1 should be visited once using either route

2

Suppose that the 10 individuals are referred to by the code $k = a, b, \ldots, j$. Let

$$x_k = \begin{cases} 1, & \text{individual } k \text{ included} \\ 0, & \text{individual } k \text{ not included.} \end{cases}$$

$k = a, b, c, \ldots, j$.

$$\begin{array}{lcccccccccccl}
 & x_a & x_b & x_c & x_d & x_e & x_f & x_g & x_h & x_i & x_j & & \\
\min z & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
\text{subject to} & & & & & & & & & & & & \\
 & 1 & 1 & 1 & 1 & 1 & & & & & & \geq 1 & \text{(females} \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & \geq 1 & \text{(males} \\
 & 1 & 1 & 1 & & & & & & & 1 & \geq 1 & \text{(Students} \\
 & & & & & 1 & 1 & & & & & \geq 1 & \text{(admin} \\
 & & & & 1 & & & 1 & 1 & 1 & & \geq 1 & \text{(faculty}
\end{array}$$

Solution: Use individuals a, d, and f.

Problem has alternative optima

3

Station	Towns it can serve
1	1, 3, 5
2	2, 4, 6
3	1, 3
4	2, 4
5	1, 5, 6
6	2, 5, 6

$$x_j = \begin{cases} 1, & \text{if station } j \text{ is selected} \\ 0, & \text{if station } j \text{ is } \underline{\text{not}} \text{ selected} \end{cases}$$

Assume that station j can be located in any of the towns its serves.

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Subject to

Station 1: $x_1 + x_3 + x_5 \geq 1$

2: $x_2 + x_4 + x_6 \geq 1$

3: $x_1 + x_3 \geq 1$

4: $x_2 + x_4 \geq 1$

5: $x_1 + x_5 + x_6 \geq 1$

6: $x_2 + x_5 + x_6 \geq 1$

$x_j = (0,1),\ j = 1, 2, \ldots, 6$

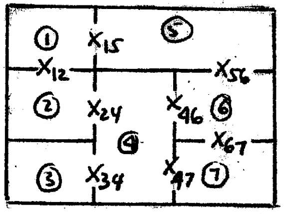
Constraints 3 and 4 are redundant

Solution: Select stations 1 and 2.

4

$x_{ij} = 1$ if guard is posted between rooms i and j; zero otherwise.

One constraint per room.



Minimize $z = x_{12} + x_{15} + x_{24} + x_{34} + x_{46} + x_{47} + x_{56} + x_{67}$

Subject to

Room 1: $x_{12} + x_{15} \geq 1$

2: $x_{12} + x_{24} \geq 1$

3: $x_{34} \geq 1$

4: $x_{24} + x_{34} + x_{46} + x_{47} \geq 1$

5: $x_{15} + x_{56} \geq 1$

6: $x_{46} + x_{56} + x_{67} \geq 1$

7: $x_{47} + x_{67} \geq 1$

$x_{ij} = (0,1)$

Solution: $x_{12} = x_{34} = x_{56} = x_{67} = 1$

Alternative optima exist.

5

$x_j = \begin{cases} 1, \text{ if town } j \text{ is selected} \\ 0, \text{ otherwise} \end{cases}$

I_i = set of cities offering movie i

c_j = cost/show in city j

d_j = miles to city j

n_j = number of movies in city j

$C_j = c_j n_j + d_j \times .75$

$$\text{Minimize } z = \sum_{j=1}^{7} C_j x_j$$

s.t.

$$\sum_{j \in I_i} x_j \geq 1, \quad i = 1, 2, \ldots, 7$$

Note: The formulation assumes that Bill will see all the movies in a visited town regardless of repetitions.

Solution: Cost = \$169.35

Visited town	movies
A	1, 6, 8
C	1, 8, 9
D	2, 4, 7
E	1, 3, 5, 10

Movie 1 will be seen 3 times and movie 8 twice. If Bill wants to see these movies only once, then movie 1 should be seen in city E (Cost \$5.25) and movie 8 should be seen in city A (Cost \$5.50)

Net Cost = 169.35 − (5.50 + 7.00) − 7.00
= \$149.85

6

$x_j = \begin{cases} 1, \text{ if community } j \text{ is selected} \\ 0, \text{ otherwise} \end{cases}$

P_j = population of community j

C_i = set of communities within 25 miles from community i

The idea of the model is that the larger the population of a community, the higher should be its preference for acquiring a new store. At the same

continued...

time, we need to minimize the total number of new stores. Thus, using $1/P_j$ as a weight for x_j is an appropriate way for modeling the objective function

$$\text{Minimize } z = \sum_{j=1}^{10} \frac{1}{P_j} x_j$$

s.t.

$$\sum_{j \in C_i} x_j \geq 1$$

$$x_j = (0, 1), \; j = 1, 2, \ldots, 10$$

Note: The determination of C_i can be automated in AMPL. See ampl9.1b-6.txt

Solution: New stores should be located in Communities 6, 8, and 9

7

$x_t = \begin{cases} 1, \text{ if transmitter } t \text{ is selected} \\ 0, \text{ otherwise} \end{cases}$

C_t = construction cost of transmitter t

$x_c = \begin{cases} 1, \text{ if community } c \text{ is covered by a transmitter} \\ 0, \text{ otherwise} \end{cases}$

S_c = set of transmitters covering community c

P_c = population of community c

$$\text{Maximize } z = \sum_{c=1}^{15} P_c x_c$$

s.t.

$$\sum_{t \in S_c} x_t \geq x_c, \quad c = 1, 2, \ldots, 15$$

$$\sum_{t=1}^{7} C_t x_t \leq 15$$

Examples of the determination of S_c:

$S_1 = \{1, 3\}$, $S_2 = \{1, 2\}$, $S_3 = \{2\}$, $S_4 = \{4\}$
$S_5 = \{2, 6\}$, $S_6 = \{4, 5\}$, $S_7 = \{3, 5, 6\}$

Solution:

Build transmitters 2, 4, 5, 6, and 7. All communities, except community number 1, are covered.

Set 9.1b

8

$$x_j = \begin{cases} 1, & \text{if receiver } j \text{ is installed} \\ 0, & \text{otherwise} \end{cases}, \quad j = 1, 2, \ldots, 8$$

R_i = Set of receivers covering meter i; $i = 1, 2, \ldots, 10$

$R_1 = \{1, 6, 8\}$, $R_2 = \{1, 2\}$, $R_3 = \{1, 2, 5\}$,
$R_4 = \{6, 7, 8\}$, $R_5 = \{3, 7\}$, $R_6 = \{3, 5\}$,
$R_7 = \{3, 4, 6\}$, $R_8 = \{5, 8\}$, $R_9 = \{2, 4, 6, 7\}$
$R_{10} = \{4\}$

Minimize $z = x_1 + x_2 + \cdots + x_8$

s.t.

$$\sum_{j \in R_i} x_j \geq 1, \quad i = 1, 2, \ldots, 10$$

$$x_j = (0, 1), \quad j = 1, 2, \ldots, 8$$

<u>Solution</u>: Install receivers 1, 4, 5, and 7.

9

$$x_{ij} = \begin{cases} 1, & \text{if meter } i \text{ uses receiver } j \\ 0, & \text{otherwise} \end{cases}$$

$y_j = (0, 1)$, $i = 1, 2, \ldots, 10$, $j = 1, 2, \ldots, 8$

Minimize $z = y_1 + y_2 + \cdots + y_8$

s.t.

$$\sum_{i \in S_j} x_{ij} \leq 3 y_j, \quad j = 1, 2, \ldots, 8$$

$$\sum_{i \notin S_j} x_{ij} = 0, \quad j = 1, 2, \ldots, 8$$

$$\sum_{j=1}^{8} x_{ij} \geq 1, \quad i = 1, 2, \ldots, 10$$

where

S_j = Set of meters covered by receiver j

$S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3, 9\}$, etc

continued...

<u>Solution</u>:

Receiver	Covered meters
1	1, 2, 3
3	5, 6
4	7, 9, 10
8	4, 8

Install receivers 1, 3, 4, and 8.

Set 9.1c

1

x_j = Nbr. of units of product j, $j = 1, 2, 3$

$$y_j = \begin{cases} 1, & \text{if } x_j > 0 \\ 0, & \text{if } x_j = 0 \end{cases}$$

$$\text{Maximize } z = (60-30)x_1 + (40-20)x_2 + (120-80)x_3 - 100y_1 - 80y_2 - 150y_3$$

s.t.

$$5x_1 + 3x_2 + 8x_3 \le 3000$$
$$4x_1 + 3x_2 + 5x_3 \le 2500$$
$$x_1 \ge 100,\ x_2 \ge 150,\ x_3 \ge 200$$
$$x_1 \le 5000y_1,\ x_2 \le 5000y_2,\ x_3 \le 5000y_3$$

<u>Solution</u>: $z = \$16670$

$x_1 = 100,\ x_2 = 300,\ x_3 = 200$

2

x_j = number of widget produced on machine j, $j = 1, 2, 3$

$$y_j = \begin{cases} 1, & \text{if machine } j \text{ is used} \\ 0, & \text{if machine } j \text{ is not used} \end{cases}$$

$$\text{Min } z = 2x_1 + 10x_2 + 5x_3 + 300y_1 + 100y_2 + 200y_3$$

Subject to

$$x_1 + x_2 + x_3 \ge 2000$$
$$x_1 - 600y_1 \le 0$$
$$x_2 - 800y_2 \le 0$$
$$x_3 - 1200y_3 \le 0$$
$$x_1, x_2, x_3 \ge 500 \text{ and integer}$$
$$y_1, y_2, y_3 = (0,1)$$

<u>Solution</u>: $x_1 = 600,\ x_2 = 500,\ x_3 = 900$

$z = \$11300$

3

$$x_{ij} = \begin{cases} 1, & \text{if site } i \text{ is assigned to target } j \\ 0, & \text{if otherwise} \end{cases}$$

$$\text{Min } z = 5y_1 + 6y_2 + 2x_{11} + x_{12} + 8x_{13} + 5x_{14} + 4x_{21} + 6x_{22} + 3x_{23} + x_{24}$$

Subject to

$$x_{11} + x_{21} = 1$$
$$x_{12} + x_{22} = 1$$

continued...

$$x_{13} + x_{23} = 1$$
$$x_{14} + x_{24} = 1$$
$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &\le My_1 \\ x_{21} + x_{22} + x_{23} + x_{24} &\le My_2 \end{aligned}\right\} M \ge 4$$
$$y_i = (0,1) \text{ for all } i$$
$$x_{ij} = (0,1) \text{ for all } i \text{ and } j$$

<u>Solution</u>: $z = 18$

site	assigned targets
1	1 and 2
2	3 and 4

4

The problem can be formulated as a regular transportation model. Since total supply = total demand, all three plants must work at full capacity and the setup cost is immaterial in this case. This will not be the case if total supply exceeds total demand.

The ILP formulation is

$$\text{Min } z = 12{,}000y_1 + 11{,}000y_2 + 12{,}000y_3 + 10x_{11} + 15x_{12} + \cdots + 11x_{33}$$

Subject to

$$x_{11} + x_{12} + x_{13} \le 1800y_1$$
$$x_{21} + x_{22} + x_{23} \le 1400y_2$$
$$x_{31} + x_{32} + x_{33} \le 1300y_3$$
$$x_{11} + x_{21} + x_{31} \ge 1200$$
$$x_{12} + x_{22} + x_{32} \ge 1700$$
$$x_{13} + x_{23} + x_{33} \ge 1600$$

$x_{ij} \ge 0$ and integer

$y_i = (0,1)$

<u>Solution</u>: $x_{11} = 1200,\ x_{13} = 600,\ x_{22} = 1400$
$x_{32} = 300,\ x_{33} = 1000.\ y_1 = y_2 = y_3 = 1.$

5

Total supply > Total demand.
Modified constraints:

$$x_{11} + x_{21} + x_{31} \ge 800$$
$$x_{12} + x_{22} + x_{32} \ge 800$$

<u>Solution</u>: $x_{11} = 1000,\ x_{13} = 800,\ x_{21} = 200,\ x_{22} = 800$
$y_1 = y_2 = 1,\ y_3 = 0$. Plant 3 is not used.

Set 9.1c

6

$x_{ijt} = \begin{cases} 1, \text{ if product } i \text{ uses line } j \text{ in period } t \\ 0, \text{ otherwise} \end{cases}$

$v_{ijt} = \begin{cases} 1, \text{ if changeover is made to product } i \text{ on line } j \text{ in period } t \\ 0, \text{ otherwise} \end{cases}$

I_{it} = End inventory of product i in period t

I_{i0} = Initial inventory of product i

D_{it} = Demand of product i in period t

r_{ij} = production rate of product i on line j (units/month)

S_{ij} = Switching cost of product i on line j

C_{ij} = Production cost of product i on line j (\$/unit)

h_i = Holding cost/unit/month of product i

$$\text{Minimize } z = \sum_{i=1}^{3}\sum_{j=1}^{2} c_{ij} r_{ij}\left(\sum_{t=1}^{6} x_{ij}t\right) + \sum_{i=1}^{3}\sum_{j=1}^{2} s_{ij}\left(\sum_{t=1}^{6} v_{ijt}\right) + \sum_{i=1}^{3} h_i\left(\sum_{t=1}^{6} I_{it}\right)$$

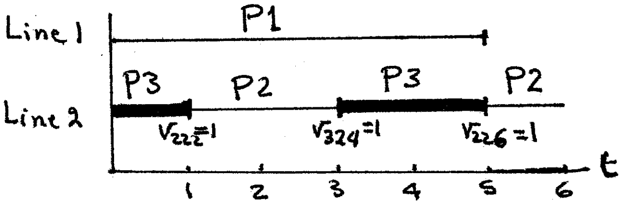
S.t.

$$\sum_{i=1}^{3} x_{ijt} \le 1, \quad =1,2, \quad t=1,2,\dots,6$$

$$v_{ijt} \ge x_{ijt} - x_{ijt-1} \begin{cases} i=1,2,3 \\ j=1,2 \\ t=2,3,\dots,6 \end{cases}$$

$$I_{it} = I_{i0} + \sum_{k=1}^{t}\left(\sum_{j=1}^{2} r_{ij}x_{ijk} - D_{ik}\right), \quad i=1,2,3, \quad t=1,2,\dots,6$$

Solution:



See file ampl9.1c-6.txt.

7

w_{ij} = Line capacity in gal/hr from city i to potential plant j

F_i = Fixed cost for plant located in city i

P_i = Population (in thousands) of city i

$y_i = \begin{cases} 1, \text{ if a plant is constructed in city } i \\ 0, \text{ Otherwise} \end{cases}$

C_{ij} = Construction cost of pipeline between cities i and j in \$/1000 gal/hr

$$\text{Minimize } Z = \sum_{i=1}^{7}\left(\sum_{j=1}^{7} C_{ij}\frac{w_{ij}}{1000} + F_i y_i\right)$$

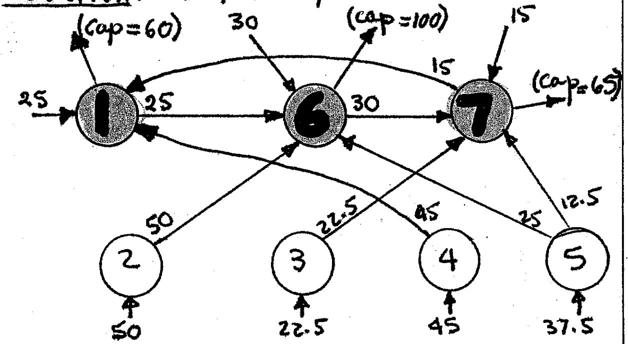
s.t.

$$\sum_{j=1}^{7} w_{ij} \ge 500P_i, \quad i=1,2,\dots,7$$

$$\sum_{i=1}^{7} w_{ij} \le 100{,}000\,y_j, \quad j=1,2,\dots,7$$

$$\sum_{i=1}^{7} y_i \le 4$$

Solution: See file ampl9.1C-7.txt.



Plant 1 capacity = 60,000 gal/hr
6 capacity = 100,000 gal/hr
7 capacity = 65,000 gal/hr
Total cost = \$3,770,875

8

x_{tpc} = gal of product p in compartment c on truck t

$y_{tpc} = \begin{cases} 1, \text{ if compartment } c \text{ on truck } t \text{ is used for product } p \\ 0, \text{ otherwise} \end{cases}$

w_p = Subcontracted gal of product p

continued...

Minimize $z = 5w_1 + 12w_2 + 8w_3 + 10w_4$

s.t.

$$\sum_{t=1}^{4}\sum_{c=1}^{5} x_{tpc} + w_p = \begin{cases} 1,0000, & p=1 \\ 15,000, & p=2 \\ 12,000, & p=3 \\ 8,000, & p=4 \end{cases}$$

$$\sum_{p=1}^{4} y_{tpc} = 1,\ t=1,2,3,4,\ c=1,2,\dots,5$$

$$\left.\begin{aligned} x_{tp1} &\le 500\,y_{tp1} \\ x_{tp2} &\le 750\,y_{tp2} \\ x_{tp3} &\le 1200\,y_{tp3} \\ x_{tp4} &\le 1500\,y_{tp4} \\ x_{tp5} &\le 1750\,y_{tp5} \end{aligned}\right\}\ \begin{aligned} t&=1,2,3,4 \\ p&=1,2,3,4 \end{aligned}$$

Solution: See file ampl9.1C-8.txt

$z = \$148,000$

Truck	Product	500	750	1200	1500	1750
1	2		×		×	×
	4	×		×		
2	2		×			×
	4	×		×	×	
3	2		×		×	×
	4	×		×		
4	2	×	×		×	×
	4			×		

Subcontracting:

Product 1 = 10,000 gal
3 = 12,000 gal
4 = 2,000 gal

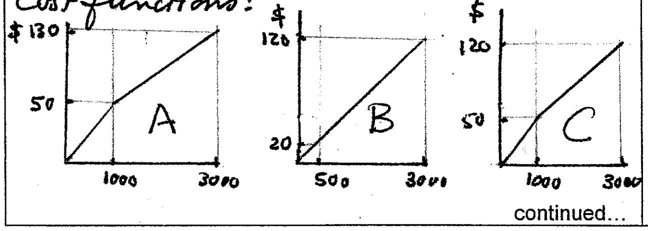
9

r_{ij} = Weight i of cost function j, $i = 0,1,2;\ j = 1,2,3$

$w_{ij} = (0,1)\quad i = 0,1,2,\ j = 1,2,3$

$$y_j = \begin{cases} 1, & \text{if company } j \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

Cost functions:



continued...

Minimize $z = 50r_{11} + 130r_{21} + 20r_{12} + 120r_{22} + 50r_{13} + 120r_{23} + 10y_1 + 20y_2 + 25y_3$

s.t.

$$\left.\begin{aligned} r_{0j} &\le w_{0j} \\ r_{1j} &\le w_{0j} + w_{1j} \\ r_{2j} &\le w_{1j} \end{aligned}\right\}\ j = 1,2,3$$

$$r_{0j} + r_{1j} + r_{2j} = 1,\quad j = 1,2,3$$

$$w_{0j} + w_{1j} = 1,\quad j = 1,2,3$$

$$x_j \le 3000\,y_j,\quad j = 1,2,3$$

$$\sum_{j=1}^{3} x_j \ge 3000$$

Solution: See file ampl9.1c-9

Use company A. Total cost = \$140

10

x_e = Nbr. of Eastern tickets
x_u = Nbr. of USAir tickets
x_c = Nbr. of Continental tickets
$e_1, e_2 = (0,1)$
u, c = nonnegative integers

Maximize $z = 1000(x_e + 1.5x_u + 1.8x_c + 5e_1 + 5e_2 + 10u + 7c)$

s.t.

$$x_e + x_u + x_c = 12$$

$$e_1 \le \frac{x_e}{2}$$

$$e_2 \le \frac{x_e}{6}$$

$$u \le \frac{x_u}{6}$$

$$c \le \frac{x_c}{5}$$

Solution: $z = 39,000$ miles

$x_e = 2$ tickets
$x_u = 0$
$x_c = 10$ tickets

Set 9.1d

1

variables definitions:

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

$1 \le x_{ij} \le 9$ and integer

$$\sum_{j=1}^{3} x_{ij} = 15, \quad i = 1,2,3$$

$$\sum_{i=1}^{3} x_{ij} = 15, \quad j = 1,2,3$$

$$x_{11} + x_{22} + x_{33} = 15$$

$$x_{31} + x_{22} + x_{13} = 15$$

$$x_{11} \ge x_{12}+1 \text{ or } x_{11} \le x_{12}-1$$

$$x_{11} \ge x_{13}+1 \text{ or } x_{11} \le x_{13}-1$$

$$x_{12} \ge x_{13}+1 \text{ or } x_{12} \le x_{13}-1$$

$$x_{11} \ge x_{21}+1 \text{ or } x_{11} \le x_{21}-1$$

$$x_{11} \ge x_{31}+1 \text{ or } x_{11} \le x_{31}-1$$

$$x_{21} \ge x_{31}+1 \text{ or } x_{21} \le x_{31}-1$$

To remove "or" constraints, note that $x_{11} \ge x_{12}+1$ or $x_{11} \le x_{12}-1$ can be replaced with the two simultaneous constraints:

$$\left.\begin{array}{l} -x_{11} + x_{12} + 15y_1 \le 14 \\ -x_{11} + x_{12} + 15y_1 \ge 1 \end{array}\right\} \quad y_1 = (0,1)$$

Using a dummy objective function with all zero coefficients, the following solutions can be found

4	3	8
9	5	1
2	7	6

6	7	2
1	5	9
8	3	4

Other solutions exist.

Note:

If you use TORA to solve the problem, replace $y_j = (0,1)$ with $0 \le y_j \le 1$ for all j

2

x_1 = daily units of product 1

x_2 = daily units of product 2

Maximize $z = 10x_1 + 12x_2$

subject to

$$x_1 + x_2 \le 35$$

$$(x_1 \le 20 \text{ and } x_2 \le 10) \text{ or } (x_1 \le 12 \text{ and } x_2 \le 25)$$

$$x_1, x_2 \ge 0 \text{ and integer}$$

Maximize $z = 10x_1 + 12x_2$

subject to

$$\begin{array}{rcl} x_1 + x_2 & \le & 35 \\ x_1 - 35y & \le & 20 \\ x_2 - 35y & \le & 10 \\ x_1 + 35y & \le & 47 \\ x_2 + 35y & \le & 60 \end{array}$$

$$x_1, x_2, y \ge 0 \text{ and integer}$$

$$y = (0,1) \qquad \boxed{M = 35}$$

Solution: $x_1 = 10$, $x_2 = 25$, $y = 1$, $z = \$400$

Select setting 2.

3

x_j = daily number of units of product j

$$y = \begin{cases} 0, & \text{if location 1 is selected} \\ 1, & \text{if location 2 is selected} \end{cases}$$

Maximize $z = 25x_1 + 30x_2 + 22x_3$

Subject to

$$\begin{pmatrix} 3x_1 + 4x_2 + 5x_3 \le 100 \\ 4x_1 + 3x_2 + 6x_3 \le 100 \end{pmatrix} \text{ or } \begin{pmatrix} 3x_1 + 4x_2 + 5x_3 \le 90 \\ 4x_1 + 3x_2 + 6x_3 \le 120 \end{pmatrix}$$

$$x_1, x_2, x_3 \ge 0 \text{ and integer}$$

Let $M = 1000$, The "or" constraints are equivalent to

$$3x_1 + 4x_2 + 5x_3 \le 100 + My$$

$$4x_1 + 3x_2 + 6x_3 \le 100 + My$$

$$3x_1 + 4x_2 + 5x_3 \le 90 + M(1-y)$$

$$4x_1 + 3x_2 + 6x_3 \le 120 + M(1-y)$$

$$x_1, x_2, x_3 \ge 0 \text{ and integer} \quad y = (0,1)$$

Solution: $x_1 = 26$, $x_2 = 3$, $x_3 = 0$, $y = 1$

Use location 2. $Z = \$740$

4

x_j = Start time of job j, $j = 1, 2, \ldots, 10$

$$y_{ij} = \begin{cases} 1, \text{ if job } i \text{ precedes job } j \\ 0, \text{ otherwise} \end{cases}$$

$w = (0, 1)$

P_j = processing time of job j

d_j = due date of job j

Minimize $Z = S_1^+ + S_2^+ + \cdots + S_{10}^+$

s.t.

$$\left.\begin{aligned} My_{ij} + x_i - x_j &\geq P_j \\ M(1 - y_{ij}) + x_j - x_i &\geq P_i \\ x_j + P_j + S_j^- - S_j^+ &= d_j \end{aligned}\right\} \begin{aligned} i &= 1, 2, \ldots, 10 \\ j &= 1, 2, \ldots, 10 \end{aligned}$$

$$\left.\begin{aligned} x_3 - (x_4 + P_4) &\leq M(1 - w) - \epsilon \\ x_9 + P_9 - x_7 &\leq Mw \end{aligned}\right\} \epsilon <<<$$

<u>Solution</u>: Total delay = 134 (See file ampl9.1d-4.txt)

Job	Start time
1	0
2	85
3	88
4	10
5	47
6	25
7	68
8	101
9	56
10	131

Optimal sequence: 1-4-6-5-9-7-2-3-8-10

5

Remove the last two constraints in Problem 4. Add the following constraints:

$$x_3 + P_3 \leq x_4$$

$$\left.\begin{aligned} x_7 + P_7 &\geq x_8 - Mw \\ x_7 + P_7 &\leq x_8 + Mw \\ x_8 + P_8 &\geq x_7 - M(1 - w) \\ x_8 + P_8 &\leq x_7 + M(1 - w) \end{aligned}\right\}$$

These four constraints translate either $x_7 + P_7 = x_8$ or $x_8 + P_8 = x_7$

<u>Solution</u>: Total delay = 170

optimal sequence: 1-3-4-5-6-9-2-7-8-10

6

x_j = Daily production of product j

Max $Z = 25x_1 + 30x_2 + 45x_3$

<u>Subject to</u>

$$3x_1 + 4x_2 + 5x_3 \leq 100$$

$$4x_1 + 3x_2 + 6x_3 \leq 100$$

$$x_3 \leq 0 \text{ or } x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer}$$

Let $y = (0, 1)$ and $M = 100$. Then,

$$(x_3 \leq 0 \ \underline{\text{or}} \ x_3 \geq 5)$$

is equivalent to

$$(x_3 \leq My \ \underline{\text{and}} \ -x_3 \leq -5 + M(1 - y))$$

which reduces to

$$x_3 - 100y \leq 0 \ \underline{\text{and}} \ -x_3 + 100y \leq 95$$

<u>Solution</u>: $x_1 = 0$, $x_2 = 11$, $x_3 = 11$

$y = 1 \Rightarrow$ produce product 3

$Z = \$825$

Set 9.1d

7

1. Straightforward formulation:

Let x_{it} = 1 if load i is assigned to trailer t, o otherwise
L_i = linear feet of load i
r_i = revenue from load i

$$\text{Maximize } z = \sum_{i=1}^{10} \sum_{t=1}^{2} r_i x_{it} \text{ subject to}$$

$$\sum_{i=1}^{10} L_i x_{it} \le 36, t = 1, 2$$

$$\sum_{t=1}^{2} x_{it} \le 1, i = 1, \ldots, 10, x_{it} = (0,1), i = 1, 2, \ldots 10$$

2. Formulation using if-then:

Let x_{it} = feet in trailer t assigned to load i
$y_i = (0, 1)$, $i = 1, 2, \ldots, 10$, $w_{it} = (0, 1)$, $i = 1, 2, \ldots, 10$, $t = 1, 2$

$$\text{Maximize } z = \sum_{i=1}^{10} \sum_{t=1}^{2} r_i x_{it} \text{ subject to}$$

$$\sum_{i=1}^{10} x_{it} \le 36, t = 1, 2$$

$$x_{i1} \le L_i y_i, \ x_{i2} \le L_i (1 - y_i), i = 1, 2, \ldots, 10$$

(above constraint is not as efficient as $x_{i1} + x_{i2} \le 1, i = 1, 2, \ldots, 10$ in formulation 1)

(if $x_{it} > 0$ then $x_{it} = L_i$) translates to

$$x_{it} \le M(1 - w_{it}), L_i - x_{it} \le M w_{it}, -L_i + x_{it} \le M w_{it}, i = 1, 2, \ldots, 10, t = 1, 2$$

$$x_{it}, w_{it}, y_i = (0,1), i = 1, 2, \ldots, 10, t = 1, 2$$

Solution: z = \$7929. Problem has several alternative optima. (See file ampl9.1d-7.txt.)

	Solution 1		Solution 2	
Trailer	Load	Feet	Load	Feet
1	1	5	1	5
	5	7	2	11
	6	9	6	9
	8	14	9	10
	Total	35 ft	Total	35 ft
2	2	11	4	15
	4	15	5	7
	9	10	8	14
	Total	36 ft	Total	36 ft

8

(a)

Formulation 1:

$$\begin{pmatrix} x_1 \le 1,\ x_2 \le 2 \\ \text{or} \\ x_1 + x_2 \le 3,\ x_1 \ge 2 \end{pmatrix} \equiv \begin{pmatrix} x_1 - My \le 1 \\ x_2 - My \le 2 \\ x_1 + x_2 - M(1-y) \le 3 \\ x_1 + M(1-y) \ge 2 \end{pmatrix} M \ge 3$$

$$y = 0,1,\ x_1, x_2 \ge 0$$

Formulation 2:

$$\begin{pmatrix} x_1 + x_2 \le 3,\ x_2 \le 2 \\ \text{and} \\ (x_1 \le 1 \text{ or } x_1 \ge 2) \end{pmatrix} \equiv \begin{pmatrix} x_1 + x_2 \le 3,\ x_2 \le 2 \\ x_1 - My \le 1 \\ x_1 + M(1-y) \ge 2 \\ y = 0,1,\ x_1, x_2 \ge 0 \end{pmatrix} M \ge 2$$

(b)

$$\begin{pmatrix} x_1 + x_2 \le 3 \\ \text{and} \\ (x_1 \ge 1 \text{ or } x_2 \ge 1) \end{pmatrix} \equiv \begin{pmatrix} x_1 + My \ge 1 \\ x_2 + M(1-y) \ge 1 \\ x_1 + x_2 \le 3 \\ y = 0,1,\ x_1, x_2 \ge 0 \end{pmatrix} M \ge 3$$

(c)

$$\begin{pmatrix} x_1 + x_2 \le 3 \\ \text{and} \\ (x_1 + x_2 \ge 2 \text{ or } x_2 \le 1) \end{pmatrix} \equiv \begin{pmatrix} x_1 + x_2 \le 3 \\ x_1 + x_2 + My \ge 2 \\ x_2 - M(1-y) \le 1 \\ y = 0,1,\ x_1, x_2 \ge 0 \end{pmatrix} M \ge 3$$

9

$$g_i(x_1, x_2, \dots, x_n) \le b_i + My_i$$

$$i = 1, 2, \dots, m$$

$$y_1 + y_2 + \dots + y_m = k$$

$$y_i = (0,1),\quad i = 1, 2, \dots, m$$

10

$$g(x_1, x_2, \dots, x_n) \le b_1 y_1 + b_2 y_2 + \dots + b_m y_m$$

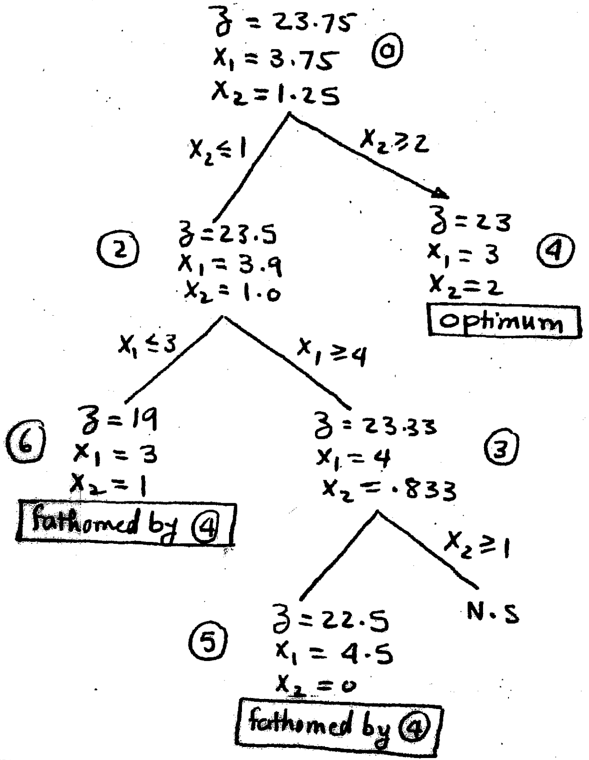
$$y_1 + y_2 + \dots + y_m = 1$$

$$y_i = (0,1),\quad i = 1, 2, \dots, m$$

Set 9.2a

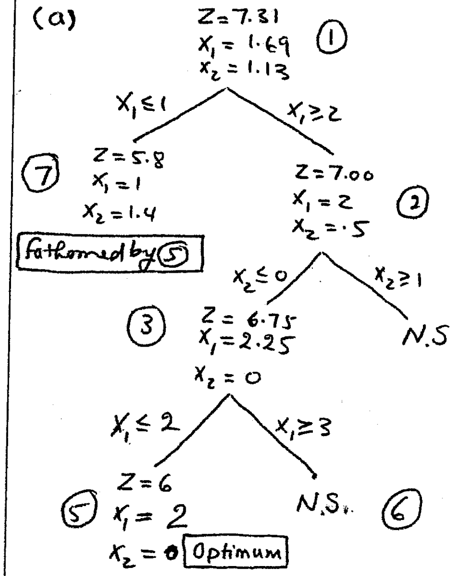
1

Note: all subproblems are solved by TORA using the MODIFY option to create each problem.



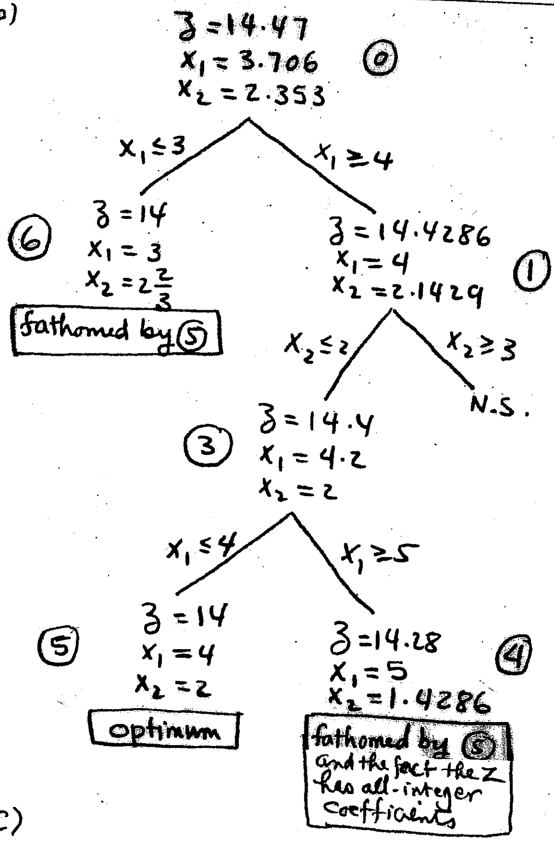
2

(a)

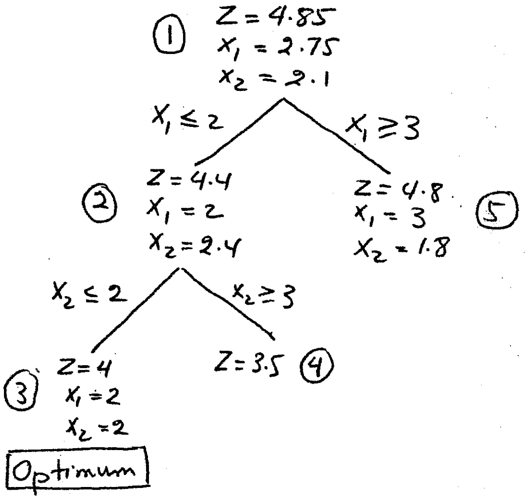


A different tree will result if branch $x_1 \leq 1$ at ① is investigated before $x_1 \geq 2$

(b)



(c)

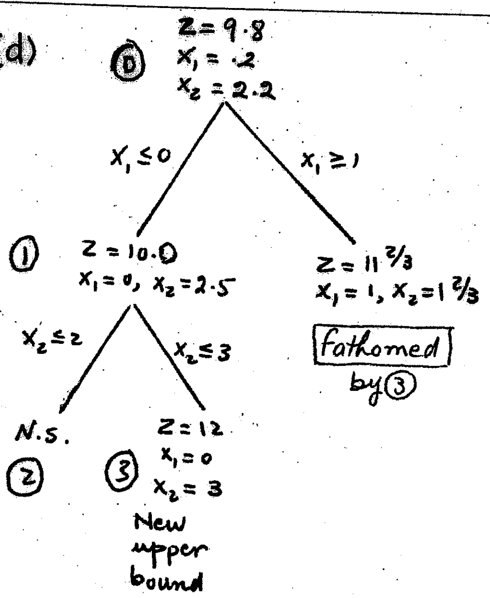


④ and ⑤ are fathomed by ③
Fathoming of ⑤ requires the additional condition that the coefficients of z are all-integer.

continued...

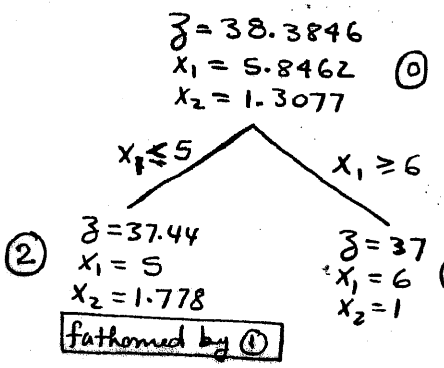
continued...

(d)



Optimum solution: $x_1 = 0,\ x_2 = 3,\ Z = 12$

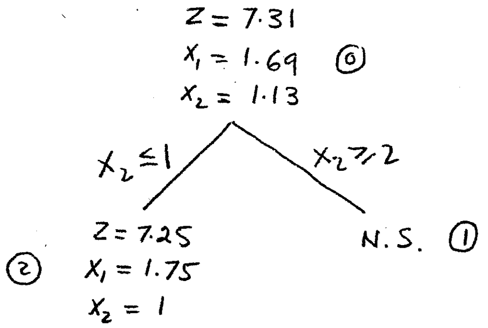
(e)



Optimum: $x_1 = 6,\ x_2 = 1,\ z = 37$

3

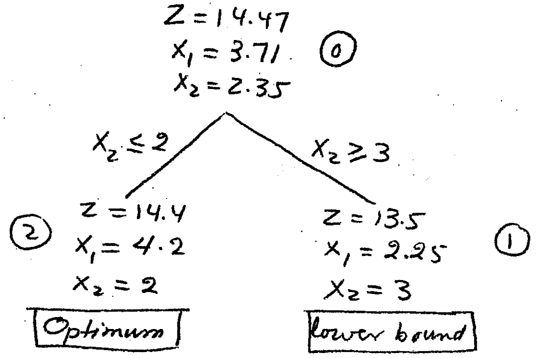
(a)



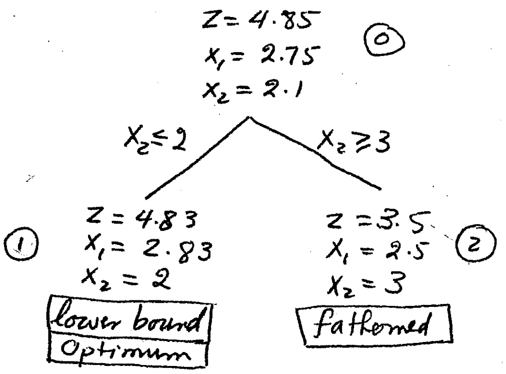
Optimum: $Z = 7.25,\ x_1 = 1.75,\ x_2 = 1$

continued...

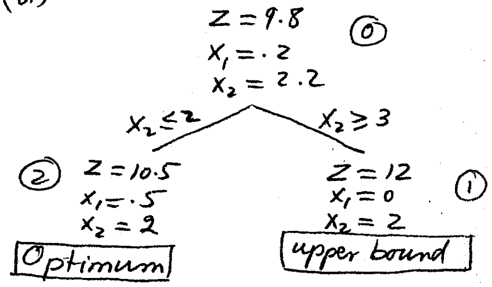
(b)



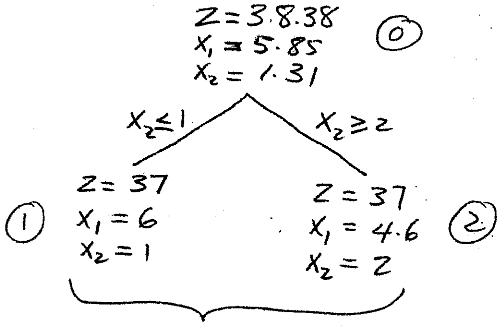
(c)



(d)



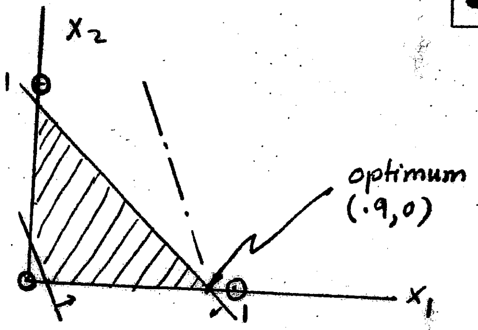
(e)

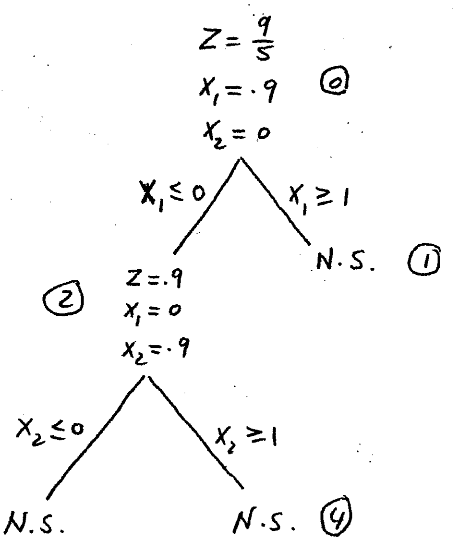


alternative optima.

Set 9.2a

4





Problem has no feasible solution.

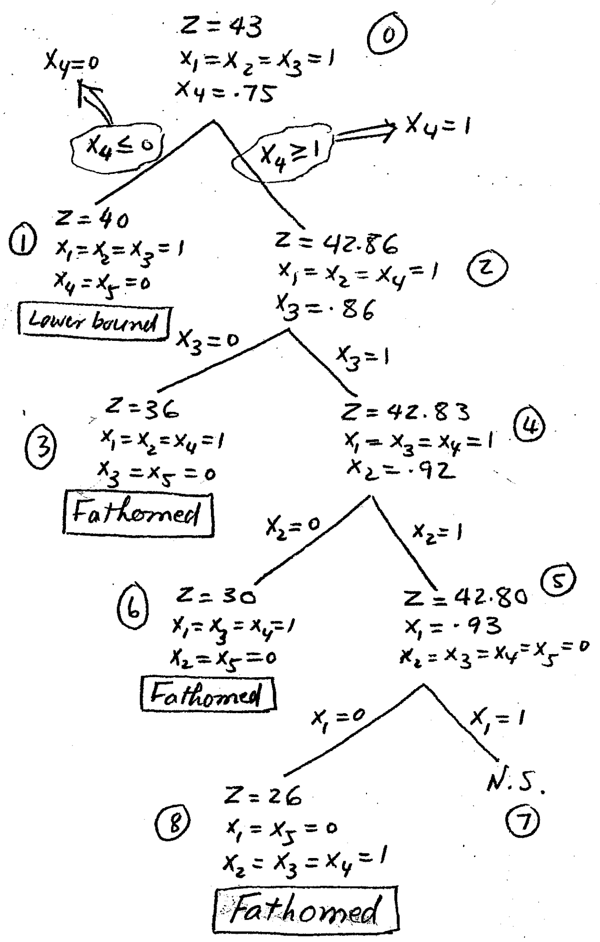
5

Max $z=18x_1+14x_2+8x_3+4x_4$

subject to

$$15x_1+12x_2+7x_3+4x_4+x_5\le 37$$

$$0\le x_j\le 1,\ j=1,2,\dots,5$$



Optimum: $z=40$

$x_1=x_2=x_3=1$

$x_4=x_5=0$

6

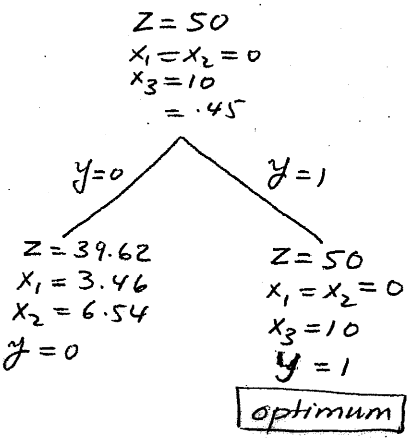
$|-x_1+10x_2-3x_3| \geq 15 \Rightarrow \begin{cases} -x_1+10x_2-3x_3 \geq 15 \\ \text{or} \\ -x_1+10x_2-3x_3 \leq -15 \end{cases}$

The problem is

$\text{Max } z = x_1 + 2x_2 + 5x_3$

Subject to

$$\begin{aligned} -x_1+10x_2-3x_3+My &\geq 15 \qquad (M=100) \\ -x_1+10x_2-3x_3+My &\leq M-15 \\ 2x_1 + x_2 + x_3 &\leq 10 \\ x_1, x_2, x_3 \geq 0, \quad y &= (0,1) \end{aligned}$$



7

(a) Replacing $x_j = (0,1)$ with $0 \leq x_j \leq 1$ and $y = (0,1)$ with $0 \leq y \leq 1$, TORA's ILP automated module determines the optimum in 9 subproblems and verifies optimality after examining 25,739 subproblems.

(b) See file solver9.2a-7b.xls. Solver examined over 25,000 subproblems before verifying optimality.

8

Number of examined subproblems with the objective function bound activated = 29

Number of examined subproblems without the objective bound activated = 35

9

Conversion to binary variables:

$0 \leq x_1 \leq 2 \Rightarrow x_1 = y_{11} + 2y_{12}$

$0 \leq x_2 \leq 3 \Rightarrow x_2 = y_{21} + 2y_{22}$

$0 \leq x_3 \leq 6 \Rightarrow x_3 = y_{31} + 2y_{32} + 4y_{33}$

$\text{Max } z = 18y_{11} + 36y_{12} + 14y_{21} + 28y_{22} + 8y_{31} + 16y_{32} + 32y_{33}$

Subject to

$15y_{11} + 30y_{12} + 12y_{21} + 24y_{22} + 7y_{31} + 14y_{32} + 28y_{33} \leq 43$

all $y_{ij} = (0,1)$

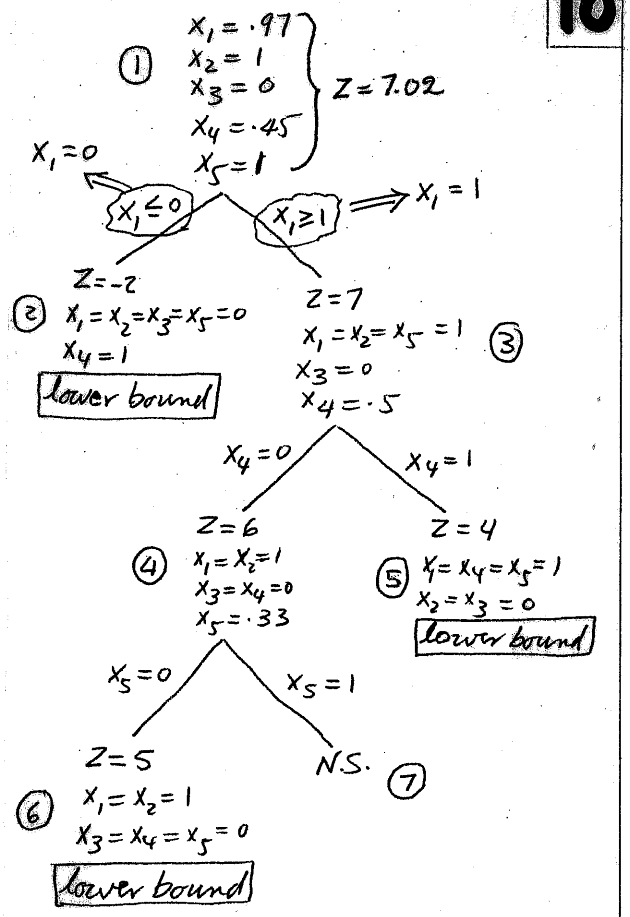
Optimum solution: $z = 50$

$y_{12} = y_{21} = 1 \Rightarrow x_1 = 2, x_2 = 1, x_3 = 0$

The solution takes 6 iterations to find the optimum and 41 to verify it. If the original problem is solved directly, it takes 4 iterations to find the optimum and 29 to verify optimality. The result points to the possibility that binary substitution may not offer any computational advantages.

Set 9.2a

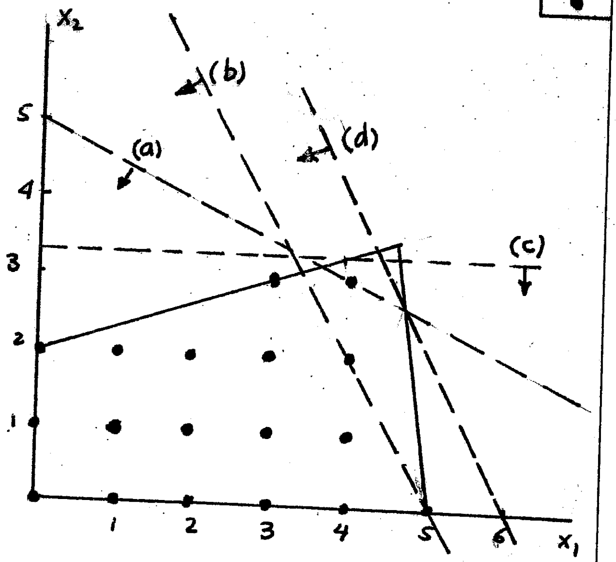
10



If the search sequence is ① → ② → ③ → ④ → ⑤ → ⑥, the lower bound will be successively updated as $z = -2$ at ②, $z = 4$ at ⑤ and $z = 5$ at ⑥. In this case, only node ⑦ is fathomed without being investigated.

If the search sequence is ① → ③ → ④ → ⑥, the first lower bound will be $z = 5$. However, even in this case, the remaining nodes ② and ⑤ must be examined because they have the potential of producing a better solution with $z = 7$ (at ⑤, it could be an alternative solution with $z = 7$). Only node ⑦ need not be examined.

1



(a) $x_1 + 2x_2 \le 10$:

The cut is legitimate because it passes through an integer point and does not eliminate any feasible integer points.

(b) $2x_1 + x_2 \le 10$:

The cut is not legitimate because it eliminates a feasible integer point

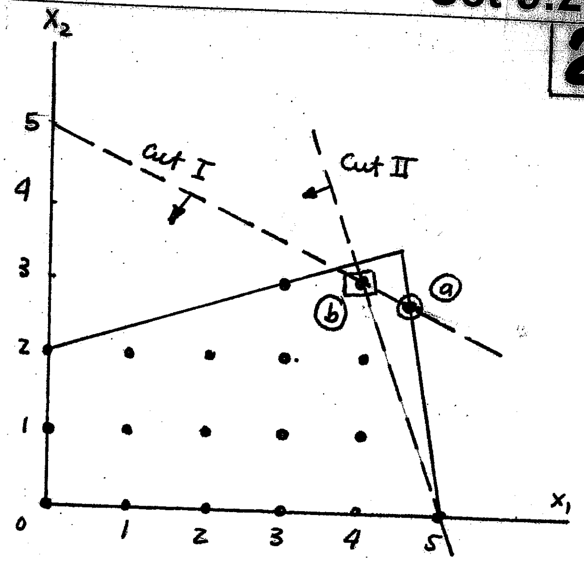
(c) $3x_2 \le 10$:

The cut is not legitimate because it does not pass through an integer point.

(d) $2x_1 + x_2 \le 12$:

The cut is legitimate because it passes through an integer point and does not exclude any feasible integer points. Note that it does not matter that the integer point through which the cut passes is itself infeasible [namely, (6,0)].

2



Cut I produces the continuous optimum at point ⓐ

Cut II (together with I) produces the integer optimum at point ⓑ.

3

Cut I:

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 \le -\frac{1}{2}$$

From the original constraints,

$$x_3 = 6 + x_1 - 3x_2$$

$$x_4 = 35 - 7x_1 - x_2$$

Thus,

$$-\frac{7}{22}(6 + x_1 - 3x_2) - \frac{1}{22}(35 - 7x_1 - x_2) \le -\frac{1}{2}$$

or

$$x_2 \le 3$$

Cut II:

$$-\frac{1}{7}x_4 - \frac{6}{7}S_1 \le -\frac{4}{7}$$

$$S_1 = -\frac{1}{2} + \frac{7}{22}x_3 + \frac{1}{22}x_4$$

or

$$-\frac{1}{7}(35 - 7x_1 - x_2) - \frac{6}{7}\left(-\tfrac{1}{2} + \frac{7}{22}x_3 + \frac{1}{22}x_4\right) \le \frac{-4}{7}$$

or

$$x_1 + x_2 \le 7$$

Set 9.2b

4

From the tableau of cut I, we have

$$x_3 + \frac{1}{7}x_4 - \frac{22}{7}S_1 = 1\frac{4}{7}$$

$$x_3 + \frac{1}{7}x_4 + \left(-4 + \frac{6}{7}\right)S_1 = 1 + \frac{4}{7}$$

Cut: $-\frac{1}{7}x_4 - \frac{6}{7}S_1 \leq -\frac{4}{7}$

This cut happens to be the same as cut II in Example 9.2-2

5

Basic	x_1	x_2	x_3	Solution
z	-1	-2	0	0
x_3	1	1/2	1	13/4
z	3	0	4	13
x_2	2	1	2	13/2

The optimum constraint

$$2x_1 + x_2 + 2x_3 = 6\frac{1}{2}$$

produces the cut $S_1 = -1/2$, which is infeasible.

Next, convert the constraint to

$$4x_1 + 2x_2 \leq 13$$

The associated simplex tableaus are

	Basic	x_1	x_2	x_3	Sol^n
0	z	-1	-2	0	0
	x_3	4	2	1	13
I		3	0	1	13
	x_2	2	1	1/2	$6\frac{1}{2}$

From the optimal constraint

$$2x_1 + x_2 + \tfrac{1}{2}x_3 = 6\tfrac{1}{2},$$

the cut is

$$S_1 - (0)x_1 - \frac{1}{2}x_3 = -\tfrac{1}{2}$$

The dual simplex produces the following iterations:

	Basic	x_1	x_2	x_3	S_1	sol^n
II	z	3	0	1	0	13
	x_2	2	1	1/2	0	$6\frac{1}{2}$
	S_1	0	0	$\boxed{-1/2}$	1	-1/2
III	z	3	0	0	2	12
	x_2	2	1	0	1/2	6
	x_3	0	0	1	-2	1

Optimum: $x_1 = 0$, $x_2 = 6$, $x_3 = 1$, $z = 12$

6

(a) Continuous optimum tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol^n
z	0	0	0	2	2	2	30
x_1	1			3/10	1/5	0	$2\frac{1}{2}$
x_2		1		1/20	1/5	0	$1\frac{1}{4}$
x_3			1	1/4	0	1	$6\frac{1}{4}$

From the x_1-row

$$x_1 + \frac{3}{10}x_4 + \frac{1}{5}x_5 = 2\frac{1}{2},$$

the cut is

$$S_1 - \frac{3}{10}x_4 - \frac{1}{5}x_5 = -\frac{1}{2} \quad \text{(cut I)}$$

adding cut I and solving, we get

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Sol^n
z	0	0	0	0	2/3	2	20/3	80/3
x_1	1				0	0	1	2
x_2		1			1/6	0	1/6	$1\frac{1}{6}$
x_3			1		-1/6	1	5/6	$5\frac{5}{6}$
x_4				1	2/3	0	-10/3	$1\frac{2}{3}$

From the x_3-row

$$x_3 - \frac{1}{6}x_5 + x_6 + \frac{5}{6}S_1 = 5\frac{5}{6},$$

the cut is

$$S_2 - \frac{5}{6}x_5 - \frac{5}{6}S_1 = -\frac{5}{6} \quad \text{(cut I)}$$

continued...

continued...

Cut II produces the following optimum tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Sol^n
	0	0	0	0	0	2	6	4/5	26
x_1	1					0	1	0	2
x_2		1				0	0	1/5	1
x_3			1			1	1	-1/5	6
x_4				1		0	-4	4/5	1
x_5					1	0	1	-6/5	1

which is all optimum and integer

Variable	rounded sol^n	Integer sol^n
x_1	2 (or 3)	2
x_2	1	1
x_3	6	6
z	26 (or 30)	26

If x_1 is rounded to 3, the solution is infeasible

(b)

Continuous optimum tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol^n
z	0	0	0	2	3	5	29
x_3	0	0	1	4/9	1/9	4/9	3 1/3
x_2	0	1	0	1/3	1/3	1/3	3
x_1	1	0	0	1/9	7/9	10/9	5 1/3

From x_3-row, we get cut I:

$$S_1 - \frac{4}{9}x_4 - \frac{1}{9}x_5 - \frac{4}{9}x_6 = -1/3$$

New tableau after cut I:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	
z					5/2	3	9/2	55/2
x_3			1		0	0	1	3
x_2		1			-1	0	3/4	2 3/4
x_1	1				3/4	1	1/4	5 1/4
S_1				1	1/4	1	-9/4	3/4

From x_2-row, we get cut II:

$$S_2 - 3/4 S_1 = -3/4$$

continued...

New tableau after cut II is added:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Sol^n
z	0	0	0	0	5/2	3	0	6	23
x_3	0	0	1	0	0	0	0	4/3	2
x_2	0	1	0	0	-1	0	0	1	2
x_1	1	0	0	0	3/4	1	0	1/3	5
x_4	0	0	0	1	1/4	1	0	-3	3
S_1	0	0	0	0	0	0	1	-4/3	1

Variable	rounded solution	integer sol^n
x_1	5	5
x_2	3	2
x_3	3	2
z	27	23

The rounded solution is infeasible.

Set 9.3a

1

The table gives the number of distinct employees who enter/leave manager's office when switch is made from project i to project j.

	1	2	3	4	5	6
1	–	4	4	6	6	5
2	4	–	6	4	6	3
3	4	6	–	4	8	7
4	6	4	4	–	6	5
5	6	6	8	6	–	5
6	5	3	7	5	5	–

$$x_{ij} = \begin{cases} 1, & \text{if project } j \text{ follows } i \\ 0, & \text{otherwise} \end{cases}$$

minimize $z = Mx_{11} + 4x_{12} + 4x_{13} + \cdots + 5x_{64} + 5x_{65} + Mx_{66}$

Subject to

$$\sum_{j=1}^{6} x_{ij} = 1, \quad i = 1, 2, \ldots, 6$$

$$\sum_{i=1}^{6} x_{ij} = 1, \quad j = 1, 2, \ldots, 6$$

Solution is a tour

$x_{ij} = (0, 1)$

2

Represent Basin, Wald, Bon, and Kiln by nodes 1, 2, 3, 4, and 5, respectively.

$$x_{ij} = \begin{cases} 1, & \text{if city } j \text{ follows city } i \\ 0, & \text{otherwise} \end{cases}$$

Minimize $z = Mx_{11} + 120x_{12} + 220x_{13} + \cdots + 185x_{53} + 190x_{54} + Mx_{55}$

Subject to

$$\sum_{i=1}^{5} x_{ij} = 1, \quad j = 1, 2, \ldots, 5$$

$$\sum_{j=1}^{5} x_{ij} = 1, \quad i = 1, 2, \ldots, 5$$

Solution is a tour

$x_{ij} = (0, 1)$

continued...

3

$$x_{ij} = \begin{cases} 1, & \text{if hole } j \text{ follows hole } i \\ 0, & \text{if otherwise} \end{cases}$$

Minimize $z = Mx_{11} + 1.2x_{12} + \cdots + 1.9x_{65} + Mx_{66}$

Subject to

$$\sum_{i=1}^{6} x_{ij} = 1, \quad j = 1, 2, \ldots, 6$$

$$\sum_{j=1}^{6} x_{ij} = 1, \quad i = 1, 2, \ldots, 6$$

$x_{ij} = (0, 1)$

Solution is a tour

Set 9.3b

(a)

	1	2	3	4
1		10	17	15
2	20		19	18
3	50	44		25
4	45	40	20	

Solution sumamry:

Start city	Tour	Length
1	1-2-4-3-1	98
2	2-4-3-1-2	98
3	3-4-2-1-3	102
4	4-3-2-1-4	99
Reversals		
2-4	1-4-2-3-1	124
4-3	1-2-3-4-1	99
2-4-3	1-3-4-2-1	102

Solution: (1-2-4-3-1).
Min setup cost = $98

continued....

(b)

	1	2	3	4	5	6
1		4	4	6	6	5
2	4		6	4	6	3
3	4	6		4	8	7
4	6	4	4		6	5
5	6	6	8	6		5
6	5	3	7	5	5	

Solution sumamry:

Start city	Tour	Length
1	1-3-4-2-6-5-1	26
2	2-6-5-4-3-1-2	26
3	3-4-2-6-5-1-3	26
4	4-3-1-2-6-5-4	26
5	5-6-2-4-3-1-5	26
6	6-2-4-3-1-5-6	26
Reversals		
3-4	1-4-3-2-6-5-1	30
4-2	1-3-2-4-6-5-1	30
2-6	1-3-4-6-2-5-1	28
6-5	1-3-4-2-5-6-1	28
3-4-2	1-2-4-3-6-5-1	30
4-2-6	1-3-6-2-4-5-1	30
2-6-5	1-3-4-5-6-2-1	26
3-4-2-6	1-6-2-4-3-5-1	30
4-2-6-5	1-3-5-6-2-4-1	30
3-4-2-6-5	1-5-6-2-4-3-1	26

Solution: (1-3-4-2-6-5-1). Alternative solutions
Min traffic = 26 employees

continued....

Set 9.3b

(c)

	1	2	3	4	5
1		120	220	150	210
2	120		80	110	130
3	220	80		160	15
4	150	110	160		190
5	210	130	185	190	

Solution sumamry:

Start city	Tour	Length
1	1-2-3-5-4-1	555
2	2-3-5-4-1-2	555
3	3-5-2-4-1-3	625
4	4-2-3-5-1-4	565
5	5-2-3-4-1-5	730
Reversals		
2-3	1-3-2-5-4-1	770
3-5	1-2-5-3-4-1	745
5-4	1-2-3-4-5-1	760
2-3-5	1-5-3-2-4-1	735
3-5-4	1-2-4-5-3-1	825
2-3-5-4	1-4-5-3-2-1	725

Solutions: 1. (2-3-4-1-5-2)
. 2. (5-2-3-4-1-5)
3. (2-5-1-4-3-2)
Length = 730 miles

Note: Tours 1 and 2 are the same. Tour 3 is the reverse order of tours 1 and 2 because the distance matrix is symmetrical.

(d)

	1	2	3	4	5	6
1		1.2	0.5	2.6	4.1	3.2
2	1.2		3.4	4.6	2.9	5.2
3	0.5	3.4		3.5	4.6	6.2
4	2.6	4.6	3.5		3.8	0.9
5	4.1	2.9	4.6	3.8		1.9
6	3.2	5.2	6.2	0.9	1.9	

Solution sumamry:

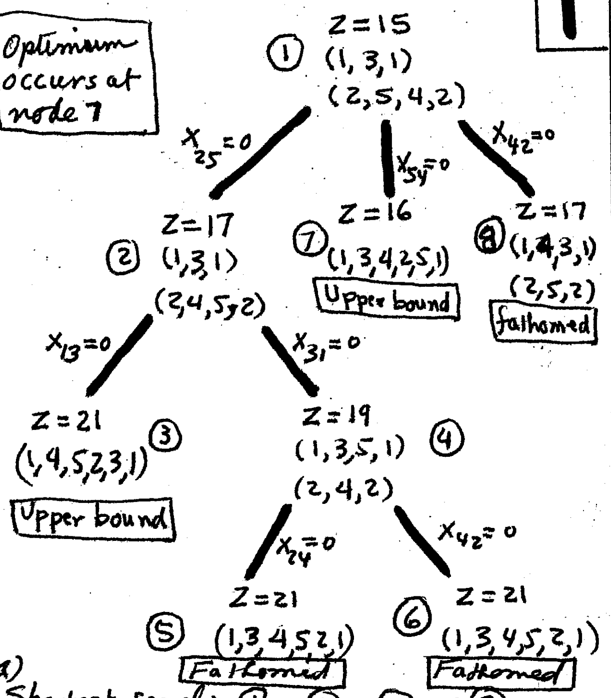
Start city	Tour	Length
1	1-3-2-5-6-4-1	12.2
2	2-1-3-4-6-5-2	10.9
3	3-1-2-5-6-4-3	10.9
4	4-6-5-2-1-3-4	10.9
5	5-6-4-1-3-2-5	12.2
6	6-4-1-3-2-5-6	12.2
Reversals		
1-3	2-3-1-4-6-5-2	12.2
3-4	2-1-4-3-6-5-2	18.3
4-6	2-1-3-6-4-5-2	15.5
6-5	2-1-3-4-5-6-2	16.1
1-3-4	2-4-3-1-6-5-2	16.6
3-4-6	2-1-6-4-3-5-2	16.3
4-6-5	2-1-3-5-6-4-2	13.7
1-3-4-6	2-6-4-3-1-5-2	17.1
3-4-6-5	2-1-5-6-4-3-2	15
1-3-4-6-5	2-5-6-4-3-1-2	10.9

Solutions: 1. (2-1-3-4-6-5-2)
2. (3-1-2-5-6-4-3)
3. (4-6-5-2-1-3-4)
4. (2-5-6-4-3-1-2)
Length =10.9 cm

Note: Tours 1 and 3 are the same. Tours 2 and 4 are the same (also, reverse order of 1 and 3).

1

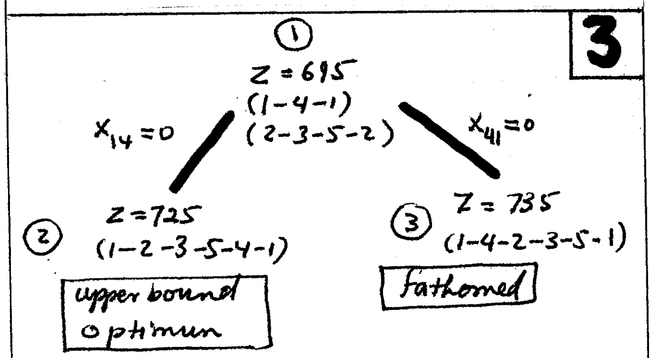
Optimum occurs at node 7



(a) Shortest search: ① → ② → ⑦ → ⑧

(b) longest search: ① → ② → ③ → ④ → ⑤ → ⑥ → ⑦ → ⑧

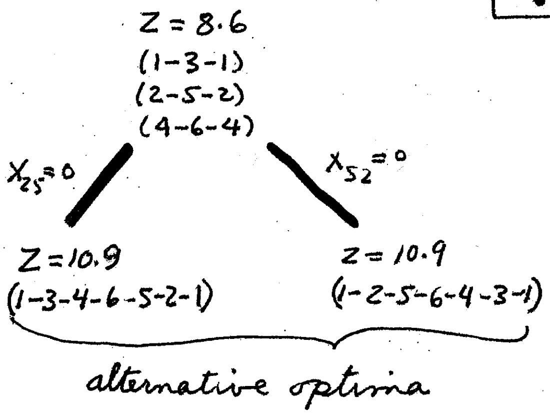
3



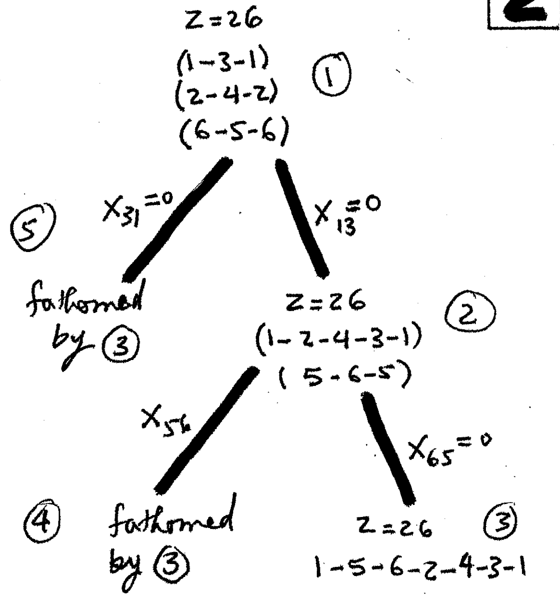
Optimum: 1-2-3-5-4-1

Z = 725

4



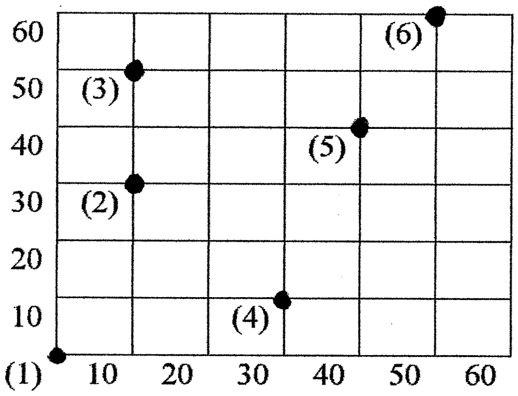
2



Set 9.3d

1

Layout:



Distance matrix:

	1	2	3	4	5	6
1		40	60	40	80	110
2	40		20	40	40	70
3	60	20		60	40	50
4	40	40	60		40	70
5	80	40	40	40		30
6	110	70	50	70	30	

Cuts:

```
6*x[2,3] + u[2] - u[3] <= 5;
6*x[2,4] + u[2] - u[4] <= 5;
6*x[2,5] + u[2] - u[5] <= 5;
6*x[2,6] + u[2] - u[6] <= 5;
6*x[3,2] - u[2] + u[3] <= 5;
6*x[3,4] + u[3] - u[4] <= 5;
6*x[3,5] + u[3] - u[5] <= 5;
6*x[3,6] + u[3] - u[6] <= 5;
6*x[4,2] - u[2] + u[4] <= 5;
6*x[4,3] - u[3] + u[4] <= 5;
6*x[4,5] + u[4] - u[5] <= 5;
6*x[4,6] + u[4] - u[6] <= 5;
6*x[5,2] - u[2] + u[5] <= 5;
6*x[5,3] - u[3] + u[5] <= 5;
6*x[5,4] - u[4] + u[5] <= 5;
6*x[5,6] + u[5] - u[6] <= 5;
6*x[6,2] - u[2] + u[6] <= 5;
6*x[6,3] - u[3] + u[6] <= 5;
6*x[6,4] - u[4] + u[6] <= 5;
6*x[6,5] - u[5] + u[6] <= 5;
```

Solution: See file ampl9.3d-1.txt.
1-2-3-6-5-4-1. Minimum length = 220 meters

2

Cuts:

```
subject to cut[2,3]:  5*X[2,3] + u[2] - u[3] <= 4;

subject to cut[2,4]:  5*X[2,4] + u[2] - u[4] <= 4;

subject to cut[2,5]:  5*X[2,5] + u[2] - u[5] <= 4;

subject to cut[3,2]:  5*X[3,2] - u[2] + u[3] <= 4;

subject to cut[3,4]:  5*X[3,4] + u[3] - u[4] <= 4;

subject to cut[3,5]:  5*X[3,5] + u[3] - u[5] <= 4;

subject to cut[4,2]:  5*X[4,2] - u[2] + u[4] <= 4;

subject to cut[4,3]:  5*X[4,3] - u[3] + u[4] <= 4;

subject to cut[4,5]:  5*X[4,5] + u[4] - u[5] <= 4;

subject to cut[5,2]:  5*X[5,2] - u[2] + u[5] <= 4;

subject to cut[5,3]:  5*X[5,3] - u[3] + u[5] <= 4;

subject to cut[5,4]:  5*X[5,4] - u[4] + u[5] <= 4;
```

Solution: 1-5-2-3-4-1, length = 45.

3

(a) See file ampl9.3d-3a.txt.
(b) See file ampl9.3d-3b.txt

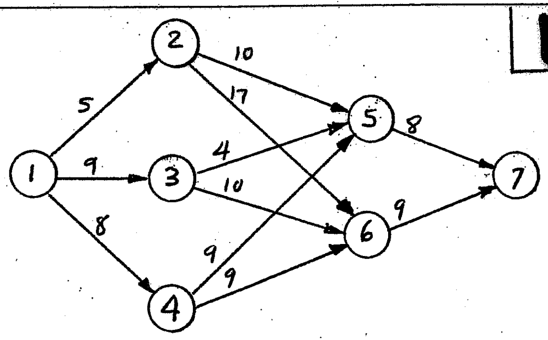
…

CHAPTER 10

Deterministic Dynamic Programming

Set 10.1a

1



Stage 1:

To city	shortest distance	from City
2	5	1
3	9	1
4	8	1

Stage 2:

To city	Shortest distance	from city
5	$\min\{5+10, \underline{9+4}, 8+9\} = 13$	3
6	$\min\{5+17, 9+10, \underline{8+9}\} = 17$	4

stage 3:

To city	Shortest distance	from city
7	$\min\{\underline{13+8}, 17+9\} = 21$	5

Optimum solution: shortest distance = 21 miles

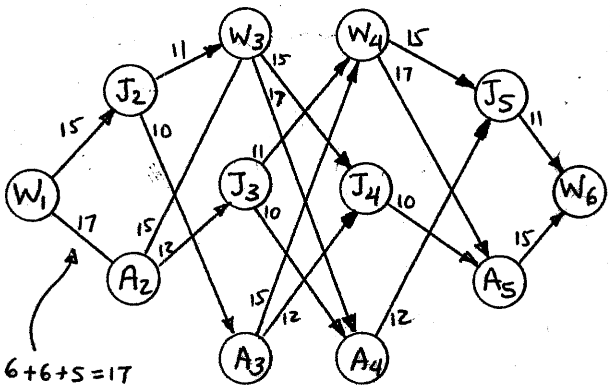
Route: $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$

2

Define node N_i as:

$N \equiv W, J$, and A for Washington, Jefferson, and Adams

i = day on which N is visited



continued...

Stage 1:

To	Longest distance	From
J_2	15	W_1
A_2	17	W_1

Stage 2:

To	Longest distance	From
W_3	$\max\{15+11, \underline{17+15}\} = 32$	A_2
J_3	$17+12 = 29$	A_2
A_3	$15+10 = 25$	J_2

Stage 3:

To	Longest distance	From
W_4	$\max\{\underline{29+11}, \underline{25+15}\} = 40$	J_3 or A_3
J_4	$\max\{\underline{32+15}, 25+12\} = 47$	W_3
A_4	$\max\{\underline{32+17}, 29+10\} = 49$	W_3

Stage 4:

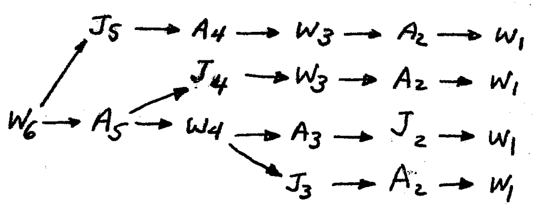
To	Longest distance	From
J_5	$\max\{40+15, \underline{49+12}\} = 61$	A_4
A_5	$\max\{\underline{40+17}, \underline{47+10}\} = 57$	W_4 or J_4

Stage 5:

To	Longest distance	From
W_6	$\max\{\underline{61+11}, \underline{57+15}\} = 72$	J_5 or A_5

Solution: 72 miles or 14.4 miles/day

To determine the optimum routes, start from stage 5.



The routes can be summarized as:

Day	1	2	3	4	5
Route 1	W	A	W	A	J
Route 2	W	A	W	J	A
Route 3	W	J	A	W	A
Route 4	W	A	J	W	A

All routes visit J once and each of W and A twice

1

$$f_i(x_i) = \min_{\substack{\text{feasible}\\(x_i,\,x_{i+1})\\\text{routes}}} \left\{ d(x_i, x_{i+1}) + f_{i+1}(x_{i+1}) \right\},\ i=1,2$$

Stage 3:

$$f_3(x_3) = \min_{\substack{\text{feasible}\\(x_3,\,x_4)}} \left\{ d(x_3, x_4) \right\}$$

x_3	$d(x_3, x_4)$	Optimum sol	
	$x_4 = 7$	$f_3(x_3)$	x_4^*
5	8	8	7
6	9	9	7

Stage 2:

$$f_2(x_2) = \min_{\substack{\text{feasible}\\(x_2,\,x_3)}} \left\{ d(x_2, x_3) + f_3(x_3) \right\}$$

x_2	$d(x_2, x_3) + f_3(x_3)$		Opt. Sol.	
	$x_3 = 5$	$x_3 = 6$	$f_2(x_2)$	x_3^*
2	$10+8=\boxed{18}$	$17+9=26$	18	5
3	$4+8=\boxed{12}$	$10+9=19$	12	5
4	$9+8=\boxed{17}$	$9+9=18$	17	5

Stage 1:

$$f_1(x_1) = \min_{\substack{\text{feasible}\\(x_1,\,x_2)}} \left\{ d(x_1, x_2) + f_2(x_2) \right\}$$

x_1	$d(x_1, x_2) + f_2(x_2)$			Opt. Sol.	
	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_1(x_1)$	x_2^*
1	$5+18=23$	$9+12=\boxed{21}$	$8+17=25$	21	3

Solution: distance = 21

route = 1-3-5-7.

2

$$f_i(x_i) = \max_{\substack{\text{feasible}\\(x_i,\,x_{i+1})\\\text{routes}}} \left\{ d(x_i, x_{i+1}) + f_{i+1}(x_{i+1}) \right\}$$

$i = 1, 2, 3, 4$

Stage 5: $f_5 = \max_{\substack{\text{feasible}\\(x_5,\,x_6)}} \left\{ d(x_5, x_6) \right\}$

x_5	$d(x_5, x_6)$	Opt. Sol.	
	$x_6 = W_6$	$f_5(x_5)$	x_6^*
J_5	11	11	W_6
A_5	15	15	W_6

continued...

Stage 4:

x_4	$d(x_4, x_5) + f_5(x_5)$		Opt. Sol.	
	$x_5 = J_5$	$x_5 = A_5$	$f_4(x_4)$	x_5^*
W_4	$15+11=26$	$17+15=\boxed{32}$	32	A_5
J_4	—	$10+15=\boxed{25}$	25	A_5
A_4	$12+11=\boxed{23}$	—	23	J_5

Stage 3:

x_3	$d(x_3, x_4) + f_4(x_4)$			Opt. Sol.	
	$x_4 = W_4$	$x_4 = J_4$	$x_4 = A_4$	$f_3(x_3)$	x_4^*
W_3	—	$15+25=\boxed{40}$	$17+23=\boxed{40}$	40	J_4, A_4
J_3	$11+32=\boxed{43}$	—	$10+23=33$	43	W_4
A_3	$15+32=\boxed{47}$	$17+25=42$	—	47	W_4

Stage 2:

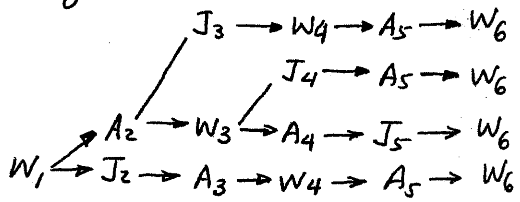
x_2	$d(x_2, x_3) + f_3(x_3)$			Opt. Sol.	
	$x_3 = W_3$	$x_3 = J_3$	$x_3 = A_3$	$f_2(x_2)$	x_3^*
J_2	$11+40=51$	—	$10+47=\boxed{57}$	57	A_3
A_2	$15+40=\boxed{55}$	$12+43=\boxed{55}$	—	55	W_3, J_3

Stage 1:

x_1	$d(x_1, x_2) + f_2(x_2)$		Opt. Sol.	
	$x_2 = J_2$	$x_2 = A_2$	$f_1(x_1)$	x_2^*
W_1	$15+57=\boxed{72}$	$17+55=\boxed{72}$	72	A_2, J_2

Solution:

Longest distance = 72 miles

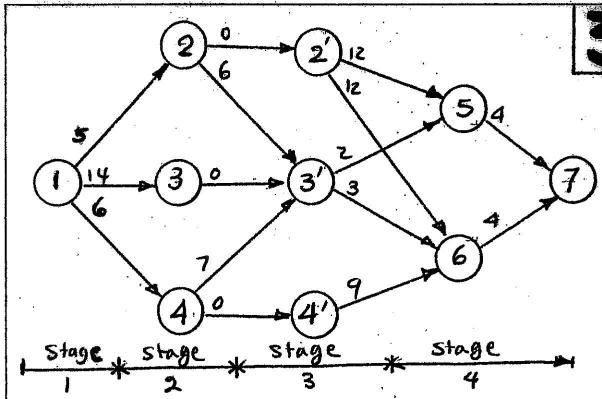


Routes:

	Day				
	1	2	3	4	5
Route 1:	W	A	J	W	A
Route 2:	W	A	W	J	A
Route 4:	W	A	W	A	J
Route 5:	W	J	A	W	A

Set 10.2a

3



$$f_i(x_i) = \min_{\substack{\text{feasible} \\ (x_i, x_{i+1}) \\ \text{routes}}} \left\{ d(x_i, x_{i+1}) + f_{i+1}(x_{i+1}) \right\}$$

$i = 1, 2, 3, 4$

Stage 4:

x_4	$d(x_4, x_5)$	Opt. Sol.	
	$x_5 = 7$	$f_4(x_4)$	x_5^*
5	4	4	7
6	4	4	7

Stage 3:

x_3	$d(x_3, x_4) + f_4(x_4)$		Opt. Sol.	
	$x_4 = 5$	$x_4 = 6$	f_4	x_4^*
2'	12 + 4 = ⑯	12 + 4 = ⑯	16	5, 6
3'	2 + 4 = ⑥	3 + 4 = 7	6	5
4'	—	9 + 4 = ⑬	13	6

Stage 2:

x_2	$d(x_2, x_3) + f_3(x_3)$			Opt. Sol.	
	$x_3 = 2'$	$x_3 = 3'$	$x_3 = 4'$	f_2	x_3^*
2	0 + 16 = 16	6 + 6 = ⑫	—	12	3'
3	—	0 + 6 = ⑥	—	6	3'
4	—	7 + 6 = ⑬	0 + 13 = ⑬	13	3', 4'

Stage 1:

x_1	$d(x_1, x_2) + f_2(x_2)$			Opt. Sol.	
	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_1(x_1)$	x_2^*
1	5 + 12 = ⑰	14 + 6 = 20	6 + 13 = 19	17	2

continued...

Solution:

Distance = 17

Route: 1 - 2 - 3' - 5 - 7

Since ③ is the same as ③', the optimal route is

1 - 2 - 3 - 5 - 7.

1

$(x_1=3) \rightarrow m_1=0 \rightarrow (x_2=3) \rightarrow m_2=1 \rightarrow$
$(x_3 = 3-3=0) \rightarrow m_3=0.$

Solution:
$(m_1, m_2, m_3) = (0, 3, 0)$
Revenue = 47

2

(a)
Stage 3: $\max m_3 = \left[\frac{6}{2}\right] = 3$

	$40m_3$				Opt. Sol.	
x_3	$m_3=0$	$m_3=1$	$m_3=2$	$m_3=3$	f_3	m_3^*
0	⓪	–	–	–	0	0
1	⓪	–	–	–	0	0
2	0	(40)	–	–	40	1
3	0	(40)	–	–	40	1
4	0	40	(80)	–	80	2
5	0	40	(80)	–	80	2
6	0	40	80	(120)	120	3

Stage 2: $\max m_2 = \left[\frac{6}{1}\right] = 6$

	$20m_2 + f_3(x_2 - m_2)$							Opt. Sol.	
x_2	$m_2=0$	1	2	3	4	5	6	f_2	m_2^*
0	⓪	–	–	–	–	–	–	0	0
1	0	(20)	–	–	–	–	–	20	1
2	(40)	20	(40)	–	–	–	–	40	0,2
3	40	(60)	40	(60)	–	–	–	60	1,3
4	(80)	60	(80)	60	(80)	–	–	80	2,4,0
5	80	(100)	80	(100)	80	(100)	–	100	1,3,5
6	(120)	100	(120)	100	(120)	100	(120)	120	0,2, 4,6

Stage 1: $\max m_1 = \left[\frac{6}{4}\right] = 1$

	$70m_1 + f_2(x_1 - 4m_1)$		Opt. Sol.	
x_1	$m_1=0$	$m_1=1$	f_1	m_1^*
6	0+120 = (120)	70+40 = 110	120	0

Optimum solutions:
$(m_1, m_2, m_3) = (0, 0, 3)$
$= (0, 2, 2)$
$= (0, 4, 1)$
$= (0, 6, 0)$
Value = 120

continued...

(b) Stage 3: $\max m_3 = \left[\frac{4}{3}\right] = 1$

	$80m_3$		Opt. Sol.	
x_3	$m_3=0$	$m_3=1$	f_3	m_3^*
0	⓪	–	0	0
1	⓪	–	0	0
2	⓪	–	0	0
3	0	(80)	80	1
4	0	(80)	80	1

Stage 2: $\max m_2 = [4/2] = 2$

	$60m_2 + f_3(x_2 - 2m_2)$			Opt. Sol.	
x_2	$m_2=0$	$m_2=1$	$m_2=2$	f_2	m_2^*
0	⓪	–	–	0	0
1	⓪	–	–	0	0
2	0	(60)	–	60	1
3	(80)	60	–	80	0
4	80	60	(120)	120	2

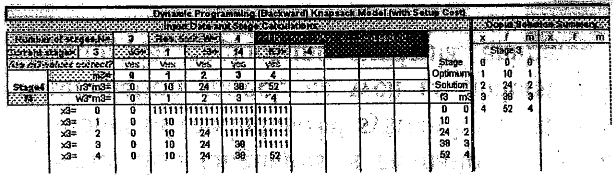
Stage 1: $\max m_1 = [4/1] = 4$

	$30m_1 + f_2(x_1 - m_1)$					Opt. Sol.	
x_1	$m_1=0$	1	2	3	4	f_1	m_1^*
4	(120)	90	(120)	90	(120)	120	0,2,4

Alternative optima:
$(m_1, m_2, m_3) = (0, 2, 0)$
$= (2, 1, 0)$
$= (4, 0, 0)$
$\Big\}$ value $= 120$

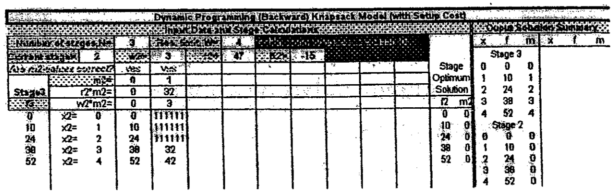
3

Stage 3: $w_3 = 1$, $r_3 = 14$, $K_3 = -4$



```
Dynamic Programming (Backward) Knapsack Model (with Setup Cost)
Input Data and Stage Calculations                                   Optimum Solution Summary
Number of stages,N=  3  Res. limit, W=  4                           x   f   m   x   f   m
Current stage=  3  w3=  1  r3=  14  K3=  -4                         Stage 3
Are m3-values correct?  yes   yes     yes     yes     yes     Stage     0   0   0
              m3=      0      1       2       3       4       Optimum   1  10   1
Stage4      r3*m3=     0     10      24      38      52       Solution  2  24   2
f4          w3*m3=     0      1       2       3       4       f3  m3    3  38   3
        x3=   0        0   111111  111111  111111  111111      0   0    4  52   4
        x3=   1        0     10    111111  111111  111111     10   1
        x3=   2        0     10      24    111111  111111     24   2
        x3=   3        0     10      24      38    111111     38   3
        x3=   4        0     10      24      38      52       52   4
```

Stage 2: $w_2 = 3$, $r_2 = 47$, $K_2 = -15$



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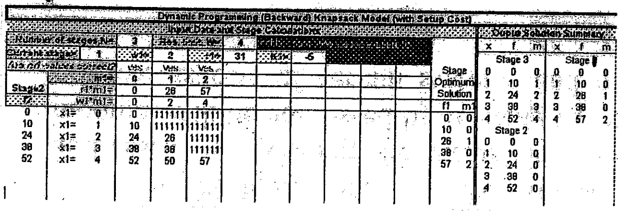
Dynamic Programming (Backward) Knapsack Model (with Setup Cost)
Input Data and Stage Calculations                                   Optimum Solution Summary
Number of stages,N=  3  Res. limit, W=  4                           x   f   m   x   f   m
Current stage=  2  w2=  3  r2=  47  K2=  -15                        Stage 3
Are m2-values correct?  yes   yes                             Stage     0   0   0
              m2=      0      1                               Optimum   1  10   1
Stage3      r2*m2=     0     32                               Solution  2  24   2
f3          w2*m2=     0      3                               f2  m2    3  38   3
  0     x2=   0        0   111111                              0   0    4  52   4
 10     x2=   1       10   111111                             10   0    Stage 2
 24     x2=   2       24   111111                             24   0    0   0   0
 38     x2=   3       38     32                               38   0    1  10   0
 52     x2=   4       52     42                               52   0    2  24   0
                                                                        3  38   0
                                                                        4  52   0
```

continued...

Set 10.3a

Stage 1: $w_1 = 2,\ r_1 = 31,\ K_1 = -5$



Dynamic Programming (Backward) Knapsack Model (with Setup Cost)

Input Data and Stage Calculations

Number of stages N=		3	Res. limit, W=		4			
Current stage=	1	w1=	2	r1=	31	K1=	-5	
Are m1-values correct?		Yes	Yes	Yes				
Stage2	m1=	0	1	2				
	r1*m1=	0	26	57				
f2	W1*m1=	0	2	4				
0	x1=	0	0	111111	111111			
10	x1=	1	10	111111	111111			
24	x1=	2	24	26	111111			
38	x1=	3	38	38	111111			
52	x1=	4	52	50	57			

Stage Optimum Solution

f1	m1
0	0
10	0
26	1
38	0
57	2

Output Solution Summary

x	f	m	x	f	m
Stage 3			Stage 1		
0	0	0	0	0	0
1	10	1	1	10	0
2	24	2	2	26	1
3	38	3	3	38	0
4	52	4	4	57	2
Stage 2					
0	0	0			
1	10	0			
2	24	0			
3	38	0			
4	52	0			

Optimum solution:

$$, = 4 \rightarrow (m_1 = 2) \rightarrow x_2 = (4 - 2\times 2 = 0) \rightarrow$$
$$(m_2 = 0) \rightarrow x_3 = 0 \rightarrow m_3 = 0$$

value = 57

4

x_1 = number of food items
x_2 = number of first-aid items
x_3 = number of cloth pieces

Maximize $z = 3x_1 + 4x_2 + 5x_3$

subject to

$$x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 \le 3$$

$$x_1 \ge 1,\ 1 \le x_2 \le 2,\ x_3 \ge 1$$

Define the state y_i as the volume assigned to items i, $i+1, \dots,$ and n

Recursive equations:

$$f_3(y_3) = \max_{x_3 = 1, \dots, [\frac{y_3}{2}]} \{5x_3\}$$

$$f_2(y_2) = \max_{x_2 = 1, \dots, \min[\frac{y_2}{4}, 2]} \left\{4x_2 + f_3\left(y_2 - \frac{x_2}{4}\right)\right\}$$

$$f_1(y_1) = \max_{x_1 = 1, \dots, y_1} \{3x_1 + f_2(y_1 - x_1)\}$$

Stage 3: (Note: $[a, b) \equiv a \le y < b$)

y_3	$5x_3$						Opt. Sol.	
	$x_3=1$	2	3	4	5	6	f_3	x_3^*
[.5,1)	⑤	—	—	—	—	—	5	1
[1,1.5)	5	⑩	—	—	—	—	10	2
[1.5,2)	5	10	⑮	—	—	—	15	3
[2,2.5)	5	10	15	⑳	—	—	20	4
[2.5,3)	5	10	15	20	㉕	—	25	5
3	5	10	15	20	25	㉚	30	6

continued...

Stage 2:

y_2	$4x_2 + f_3(y_2 - x_2/4)$		Opt. Sol.	
	$x_2 = 1$	$x_2 = 2$	f_2	x_2^*
.25	—	—	—	—
.50	—	—	—	—
.75	4+5 = ⑨	—	9	1
1.00	4+5 = 9	8+5 = ⑬	13	2
1.25	4+10 = ⑭	8+5 = 13	14	1
1.50	4+10 = 14	8+10 = ⑱	18	2
1.75	4+15 = ⑲	8+10 = 18	19	1
2.00	4+15 = 19	8+15 = ㉓	23	2
2.25	4+20 = ㉔	8+15 = 23	24	1
2.50	4+20 = 24	8+20 = ㉘	28	2
2.75	4+25 = ㉙	8+20 = 28	29	1
3.00	4+25 = 29	8+25 = ㉝	33	2

Stage 1:

y_1	$3x_1 + f_2(y_1 - x_1)$		Opt. Sol.	
	$x_1 = 1$	$x_1 = 2$	f_1	x_1^*
3	3 + 23 = ㉖	6+13 = 19	26	1

Solution:

$$(y_1 = 3) \rightarrow x_1 = 1 \rightarrow (y_2 = 3-1 = 2) \rightarrow x_2 = 2 \rightarrow$$
$$(y_1 = 2 - .5 = 1.5) \rightarrow x_3 = 3$$

Revenue = 26

$(x_1, x_2, x_3) = (1, 2, 3)$

5

x_i = number of courses allocated to departments i, $i+1, \dots,$ and n.

$m_i = 1, 2, \dots, 7$, $i = 1, 2, 3, 4$

$x_4 = 1, 2, \dots, 7$ $\quad x_2 = 3, 4, \dots, 9$

$x_3 = 2, 3, \dots, 8$ $\quad x_1 = 4, 5, \dots, 10$

$$f_i(x_i) = \max_{m_i} \{v(m_i) + f_{i+1}(x_i - m_i)\}$$

where $v(m_i)$ = value of m_i courses

continued...

Stage 4:

x_4	$v(m_4)$: $m_4=1$	2	3	4	5	6	7	Opt. Sol. f_4	m_4^*
1	10							10	1
2		20						20	2
3			30					30	3
4				40				40	4
5					50			50	5
6						60		60	6
7							70	70	7

Stage 3:

x_3	$v(m_3)+f_4(x_3-m_3)$: $m_3=1$	2	3	4	5	6	7	Opt. Sol. f_3	m_3^*
2	50	—	—	—	—	—	—	50	1
3	60	70	—	—	—	—	—	70	2
4	70	80	90	—	—	—	—	90	3
5	80	90	100	110	—	—	—	110	4
6	90	100	110	120	110	—	—	120	4
7	100	110	120	130	120	110	—	130	4
8	110	120	130	140	130	120	110	140	4

Stage 2:

x_2	$v(m_2)+f_3(x_2-m_2)$: $m_2=1$	2	3	4	5	6	7	Opt. Sol. f_2	m_2^*
3	70	—	—	—	—	—	—	70	1
4	90	120	—	—	—	—	—	120	2
5	110	140	140	—	—	—	—	140	2,3
6	130	160	160	150	—	—	—	160	2,3
7	140	180	180	170	150	—	—	180	2,3
8	150	190	200	190	170	150	—	200	3
9	160	200	210	210	190	170	150	210	3,4

stage 1:

x_1	$v(m_1)+f_2(x_1-m_1)$: $m_1=1$	2	3	4	5	6	7	Opt. Sol. f_1	m_1^*
10	235	250	240	240	240	220	170	250	3

Solution: $m_1=2$, $m_2=3$, $m_3=4$, $m_4=1$

Total number of points = 250

6

x_1 = number of (2') rows of tomato

x_2 = number of (3') rows of bean

x_3 = number of (2') rows of corn

Maximize $z = 10x_1 + 3x_2 + 7x_3$

Subject to

$$2x_1 + 3x_2 + 2x_3 \le 10$$

$$0 \le x_1 \le 2, \quad x_2 \ge 1, \quad x_3 \ge 0$$

continued...

Define the states as:

y_3 = number of width-feet assigned to corn

y_2 = number of width-feet assigned to corn and bean

y_1 = number of width-feet assigned to corn, bean, and tomato

$y_1 = 10$, $y_2 = 2, 3, \ldots, 10$, $y_3 = 0, 1, \ldots, 7$

Stage 3: $f_3(y_3) = \max\limits_{2x_3 \le y_3} \{7x_3\}$

y_3	$7x_3$: $x_3=0$	1	2	3	4	5	Opt. Sol. f_3	x_3^*
0	0	—	—	—	—	—	0	0
1	0	—	—	—	—	—	0	0
2	0	7	—	—	—	—	7	1
3	0	7	—	—	—	—	7	1
4	0	7	14	—	—	—	14	2
5	0	7	14	—	—	—	14	2
6	0	7	14	21	—	—	21	3
7	0	7	14	21	—	—	21	3

Stage 2: $f_2(y_2) = \max\limits_{\substack{3x_2 \le y_2 \\ x_2 \ge 1}} \{3x_2 + f_3(y_2 - 3x_2)\}$

y_2	$3x_2+f_3(y_2-3x_2)$: $x_2=1$	$x_2=2$	$x_2=3$	Opt. Sol. f_2	x_2^*
3	3 + 0 = 3	—	—	3	1
4	3 + 0 = 3	—	—	3	1
5	3 + 7 = 10	—	—	10	1
6	3 + 7 = 10	6 + 0 = 6	—	10	1
7	3 + 14 = 17	6 + 0 = 6	—	17	1
8	3 + 14 = 17	6 + 7 = 13	—	17	1
9	3 + 21 = 24	6 + 7 = 13	9 + 0 = 9	24	1
10	3 + 21 = 24	6 + 14 = 20	9 + 0 = 9	24	1

Stage 1: $f_1(y_1) = \max\limits_{\substack{2x_1 \le y_1 \\ x_1 \le 2}} \{10x_1 + f_2(y_1 - 2x_1)\}$

y_1	$10x_1+f_2(y_1-2x_1)$: $x_1=0$	$x_1=1$	$x_1=2$	Opt. Sol. f_1	x_1^*
10	0+24=24	10+17=27	20+10=30	30	2

continued...

Set 10.3a

Solution:

$(y_1 = 10) \rightarrow x_1 = 2 \rightarrow (y_2 = 10-4 = 6) \rightarrow x_2 = 1$

$\rightarrow (y_3 = 6-3 = 3) \rightarrow x_1 = 1$

Plant 2 rows of tomatoes, 1 row of beans, and 1 row of corn.

7

$x_j = 1$ if application j is selected, and 0 if otherwise.

maximize $z = 78x_1 + 64x_2 + 68x_3 + 62x_4 + 85x_5$

subject to

$$7x_1 + 4x_2 + 6x_3 + 5x_4 + 8x_5 \le 23$$

$$x_j = (0,1), \quad j = 1, 2, \dots, 5$$

Stage 5: $f_5(y_5) = \max\limits_{8x_5 \le y_5} \{85x_5\}$

y_5	$85x_5$		Opt. Sol.	
	$x_5 = 0$	$x_5 = 1$	f_5	x_5^*
0	0	—	0	0
1	0	—	0	0
⋮	⋮	⋮	⋮	⋮
7	0	—	0	0
8	0	85	85	1
9	0	85	85	1
⋮	⋮	⋮	⋮	⋮
23	0	85	85	1

Stage 4:

$f_4(y_4) = \max\limits_{5x_4 \le y_4} \{62x_4 + f_5(y_4 - 5x_4)\}$

continued...

y_4	$62x_4 + f_5(y_4 - 5x_4)$		Opt. Sol.	
	$x_4 = 0$	$x_4 = 1$	f_4	x_4^*
0	0+0 = 0	—	0	0
1	0+0 = 0	—	0	0
⋮	⋮ ⋮	⋮	⋮	⋮
5	0+0 = 0	62+0 = 62	62	1
6	0+0 = 0	62+0 = 62	62	1
7	0+0 = 0	62+0 = 62	62	1
8	0+85 = 85	62+0 = 62	85	0
⋮	⋮ ⋮	⋮ ⋮	⋮	⋮
12	0+85 = 85	62+0 = 62	85	0
13	0+85 = 85	62+85 = 147	147	1
14	0+85 = 85	62+85 = 147	147	1
⋮	⋮ ⋮	⋮ ⋮	⋮	⋮
23	0+85 = 85	62+85 = 147	147	1

Stage 3: $f_3(y_3) = \max\limits_{6x_3 \le y_3} \{68x_3 + f_4(y_3 - 6x_3)\}$

y_3	$68x_3 + f_4(y_3 - 6x_3)$		Opt. Sol.	
	$x_3 = 0$	$x_3 = 1$	f_3	x_3^*
0	0 + 0 = 0	—	0	0
1	0 + 0 = 0	—	0	0
2	0 + 0 = 0	—	0	0
3	0 + 0 = 0	—	0	0
4	0 + 0 = 0	—	0	0
5	0 + 62 = 62	—	62	0
6	0 + 62 = 62	68 + 0 = 68	68	1
7	0 + 62 = 62	68 + 0 = 68	68	1
8	0 + 85 = 85	68 + 0 = 68	85	0
9	0 + 85 = 85	68 + 0 = 68	85	0
10	0 + 85 = 85	68 + 0 = 68	85	0
11	0 + 85 = 85	68 + 62 = 130	130	1
12	0 + 85 = 85	68 + 62 = 130	130	1
13	0 + 147 = 147	68 + 62 = 130	147	0
14	0 + 147 = 147	68 + 85 = 153	153	1
15	0 + 147 = 147	68 + 85 = 153	153	1
16	0 + 147 = 147	68 + 85 = 153	153	1
17	0 + 147 = 147	68 + 85 = 153	153	1
18	0 + 147 = 147	68 + 85 = 153	153	1
19	0 + 147 = 147	68 + 147 = 215	215	1
20	0 + 147 = 147	68 + 147 = 215	215	1
21	0 + 147 = 147	68 + 147 = 215	215	1
22	0 + 147 = 147	68 + 147 = 215	215	1
23	0 + 147 = 147	68 + 147 = 215	215	1

continued...

Stage 2:

$$f_2(y_2) = \max_{4x_2 \le y_2} \{64x_2 + f_3(y_2 - 4x_2)\}$$

y_2	$64x_2 + f_3(y_2 - 4x_2)$		Opt. Sol.	
	$x_2 = 0$	$x_2 = 1$	f_2	x_2^*
0	0 + 0 = 0	—	0	0
1	0 + 0 = 0	—	0	0
2	0 + 0 = 0	—	0	0
3	0 + 0 = 0	—	0	0
4	0 + 0 = 0	64 + 0 = 64	64	1
5	0 + 62 = 62	64 + 0 = 64	64	1
6	0 + 68 = 68	64 + 0 = 64	68	0
7	0 + 68 = 68	64 + 0 = 64	68	0
8	0 + 85 = 85	64 + 0 = 64	85	0
9	0 + 85 = 85	64 + 62 = 126	126	1
10	0 + 85 = 85	64 + 68 = 132	132	1
11	0 + 130 = 130	64 + 68 = 132	132	1
12	0 + 130 = 130	64 + 85 = 149	149	1
13	0 + 147 = 147	64 + 85 = 149	149	1
14	0 + 153 = 153	64 + 85 = 149	153	0
15	0 + 153 = 153	64 + 130 = 194	194	1
16	0 + 153 = 153	64 + 130 = 194	194	1
17	0 + 153 = 153	64 + 147 = 211	211	1
18	0 + 153 = 153	64 + 153 = 217	217	1
19	0 + 215 = 215	64 + 153 = 217	217	1
20	0 + 215 = 215	64 + 153 = 217	217	1
21	0 + 215 = 215	64 + 153 = 217	217	1
22	0 + 215 = 215	64 + 153 = 217	217	1
23	0 + 215 = 215	64 + 215 = 279	279	1

Stage 1:

$$f_1(y_1) = \max_{7x_1 \le y_1} \{78x_1 + f_2(y_1 - 7x_1)\}$$

y_1	$78x_1 + f_2(y_1 - 7x_1)$		Opt. Sol.	
	$x_1 = 0$	$x_1 = 1$	f_1	x_1^*
23	0 + 279 = 279	78 + 194 = 272	279	0

Solution : $(y_1 = 23) \rightarrow x_1 = 0 \rightarrow (y_2 = 23) \rightarrow$
$x_2 = 1 \rightarrow (y_3 = 23 - 4 = 19) \rightarrow x_3 = 1 \rightarrow$
$(y_4 = 19 - 6 = 13) \rightarrow x_4 = 1 \rightarrow (y_5 = 13 - 5 = 8) \rightarrow x_5 = 1$

All but the first application are accepted.

8

$x_j = 1$ if precinct j is selected, and 0 if otherwise.

Maximize $z = 31x_1 + 26x_2 + 35x_3 + 28x_4 + 24x_5$
subject to

$$3.5x_1 + 2.5x_2 + 4x_3 + 3x_4 + 2x_5 \le 10$$
$$x_j = (0, 1), \quad j = 1, 2, \dots, 5$$

Stage 5: $f_5(y_5) = \max_{\substack{2x_5 \le y_5 \\ x_5 = (0,1)}} \{24x_5\}$

y_5	$24x_5$		Opt. Sol.	
	$x_5 = 0$	$x_5 = 1$	f_5	x_5^*
0	0	—	0	0
.5	0	—	0	0
1.	0	—	0	0
1.5	0	—	0	0
2.	0	24	24	1
2.5	0	24	24	1
↓	↓	↓	↓	↓
10	0	24	24	1

Stage 4:

$$f_4(y_4) = \max_{\substack{3x_4 \le y_4 \\ x_4 = 0,1}} \{28x_4 + f_5(y_4 - 3x_4)\}$$

y_4	$28x_4 + f_5(y_4 - 3x_4)$		Opt. Sol.	
	$x_4 = 0$	$x_4 = 1$	f_4	x_4^*
0	0 + 0 = 0	—	0	0
.5	0 + 0 = 0	—	0	0
1.	0 + 0 = 0	—	0	0
1.5	0 + 0 = 0	—	0	0
2.	0 + 24 = 24	—	24	0
2.5	↓	—	24	0
3.		28 + 0 = 28	28	1
3.5		28 + 0 = 28	28	1
4.		28 + 0 = 28	28	1
4.5		28 + 0 = 28	28	1
5.		28 + 24 = 52	52	1
↓		↓	↓	↓
10	0 + 24 = 24	28 + 24 = 52	52	1

continued...

Stage 3 :

$$f_3(y_3) = \max_{\substack{4x_3 \le y_3 \\ x_3 = 0,1}} \{35x_3 + f_4(y_3 - 4x_3)\}$$

y_3	$35x_3 + f_4(y_3 - 4x_3)$		Opt. Sol.	
	$x_3 = 0$	$x_3 = 1$	f_3	x_3^*
0	0+0 = 0	—	0	0
.5	0+0 = 0	—	0	0
1.	0+0 = 0	—	0	0
1.5	0+0 = 0	—	0	0
2.	0+24 = 24	—	24	0
2.5	0+24 = 24	—	24	0
3.	0+28 = 28	—	28	0
3.5	0+28 = 28	—	28	0
4.	0+28 = 28	35 + 0 = 35	35	0
4.5	0+28 = 28	35 + 0 = 35	35	0
5.	0+52 = 52	35 + 0 = 35	52	0
5.5	$\downarrow$	35 + 0 = 35	52	0
6.		35+24 = 59	59	1
6.5		35+24 = 59	59	1
7.		35+28 = 63	63	1
7.5		35+28 = 63	63	1
8.		35+28 = 63	63	1
8.5		35+28 = 63	63	1
9.		35+52 = 87	87	1
9.5		35+52 = 87	87	1
10.	0+52 = 52	35+52 = 87	87	1

Stage 2 :

$$f_2(y_2) = \max_{\substack{2.5x_2 \le y_2 \\ x_2 = 0,1}} \{26x_2 + f_3(y_2 - 2.5x_2)\}$$

y_2	$26x_2 + f_3(y_2 - 2.5x_2)$		Opt. Sol.	
	$x_2 = 0$	$x_2 = 1$	f_2	x_2^*
0	0 + 0 = 0	—	0	0
.5	0 + 0 = 0	—	0	0
1.	0 + 0 = 0	—	0	0
1.5	0 + 0 = 0	—	0	0
2.	0 + 24 = 24	—	24	0
2.5	0 + 24 = 24	26 + 0 = 26	26	1
3.	0 + 28 = 28	26 + 0 = 26	28	0
35	0 + 28 = 28	26 + 0 = 26	28	0
4.	0 + 35 = 35	26 + 0 = 26	35	0
4.5	0 + 35 = 35	26 + 24 = 50	50	1
5.	0 + 35 = 35	26 + 24 = 50	50	1
5.5	0 + 35 = 35	26 + 28 = 54	54	1
6.	0 + 59 = 59	26 + 28 = 54	59	0
6.5	0 + 59 = 59	26 + 35 = 61	61	1
7.	0 + 63 = 63	26 + 35 = 61	63	0
7.5	0 + 63 = 63	26 + 35 = 61	63	0
8.	0 + 63 = 63	26 + 35 = 61	63	0
8.5	0 + 63 = 63	26 + 59 = 85	85	1
9.	0 + 87 = 87	26 + 59 = 85	87	0
9.5	0 + 87 = 87	26 + 63 = 89	89	1
10.	0 + 87 = 87	26 + 63 = 89	89	1

Stage 1:

$$f_1(y_1) = \max_{\substack{3.5x_1 \le y_1 \\ x_1 = 0,1}} \{31x_1 + f_2(y_1 - 3.5x_1)\}$$

y_1	$31x_1 + f_2(y_1 - 3.5x_1)$			
	$x_1 = 0$	$x_1 = 1$	f_1	x_1^*
10	0+89 = 89	31+61 = 92	92	1

Solution:

$(y_1 = 10) \rightarrow x_1 = 1 \rightarrow (y_2 = 10 - 3.5 = 6.5)$
$\rightarrow x_2 = 1 \rightarrow (y_3 = 6.5 - 2.5 = 4) \rightarrow$
$x_3 = 1 \rightarrow (y_4 = 4 - 4 = 0) \rightarrow x_4 = 0 \rightarrow$
$(y_5 = 0) \rightarrow x_5 = 0.$

Allocate funds to precincts 1, 2, and 3. Total population reached is 3100 + 2600 + 3500 = 9200.

continued...

9

k_j = number of parallel units in component j, $j = 1, 2, 3$

The problem can be written as

maximize $r = r_1(k_1) \cdot r_2(k_1) \cdot r_3(k_3)$

subject to

$c_1(k_1) + c_2(k_2) + c_3(k_3) \le 10$

where

$r_j(k_j)$ = reliability of component j given k_j parallel units

$C_j(k_j)$ = cost of component j given k_j parallel units

Define state as

y_j = capital assigned to components $j, j+1, \dots, 3$

Stage 3: $f_3(y_3) = \max\limits_{k_3 = 1, 2, 3} \{R_3(k_3)\}$

	$R_3(k_3)$			Optimal Solution	
	$k_3 = 1$	$k_3 = 2$	$k_3 = 3$		
y_3	$R = .5, c = 2$	$R = .7, c = 4$	$R = .9, c = 5$	$f_3(y_3)$	k_3^*
2	.5	—	—	.5	1
3	.5	—	—	.5	1
4	.5	.7	—	.7	2
5	.5	.7	.9	.9	3
6	.5	.7	.9	.9	3

Stage 2: $f_2(y_2) = \max\limits_{k_3 = 1, 2, 3} \{R_2(k_2) \cdot f_3[y_2 - c_2(k_2)]\}$

	$R_2(k_2) \cdot f_3[y_2 - c_2(k_2)]$			Optimal Solution	
	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$		
y_2	$R = .7, c = 3$	$R = .8, c = 5$	$R = .9, c = 6$	$f_2(y_2)$	k_2^*
5	.7 × .5 = .35	—	—	.35	1
6	.7 × .5 = .35	—	—	.35	1
7	.7 × .7 = .49	.8 × .5 = .40	—	.49	1
8	.7 × .9 = .63	.8 × .5 = .40	.9 × .5 = .45	.63	1
9	.7 × .9 = .63	.8 × .7 = .56	.9 × .5 = .45	.63	1

Stage 1: $f_1(y_1) = \max\limits_{k_1 = 1, 2, 3} \{R_1(k_1) \cdot f_2[y_1 - c_1(k_1)]\}$

	$R_1(k_1) \cdot f_2[y_1 - c_1(k_1)]$			Optimal Solution	
y_1	$R = .6, c = 1$	$R = .8, c = 2$	$R = .9, c = 3$	$f_1(y_1)$	k_1^*
6	.6 × .35 = .210	—	—	.210	1
7	.6 × .35 = .210	.8 × .35 = .280	—	.280	2
8	.6 × .49 = .294	.8 × .35 = .280	.9 × .35 = .315	.315	3
9	.6 × .63 = .378	.8 × .49 = .392	.9 × .35 = .315	.392	2
10	.6 × .63 = .378	.8 × .63 = .504	.9 × .49 = .441	.504	2

Solution:

$(K_1^*, K_2^*, K_3^*) = (2, 1, 3)$

Composite $r = .504$

continued...

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State y_j = portion of the quantity c allocated to variables j, $j+1, \dots,$ and n.

Stage n: $f_n(y_n) = \max\limits_{x_n \le y_n} \{x_n\}$

State	Opt. Sol.	
	f_n	x_n^*
y_n	y_n	y_n

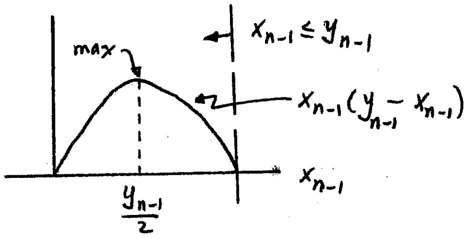
Stage $n-1$: $f_{n-1}(y_{n-1}) = \max\limits_{x_{n-1} \le y_{n-1}} \{x_{n-1} f_n(y_{n-1} - x_{n-1})\}$

Given $f_n(y_n) = y_n$, then

$f_n(y_{n-1} - x_{n-1}) = y_{n-1} - x_{n-1}$

Thus,

$f_{n-1}(y_{n-1}) = \max\limits_{x_{n-1} \le y_{n-1}} \{x_{n-1}(y_{n-1} - x_{n-1})\}$



State	Opt. Sol.	
	f_{n-1}	x_{n-1}^*
y_{n-1}	$(y_{n-1}/2)^2$	$(y_{n-1}/2)$

Stage j

$f_j(y_j) = \max\limits_{x_j \le y_j} \{x_j f_{j+1}(y_j - x_j)\}$

State	Opt. Sol.	
	f_j	x_j^*
y_j	$\left(\frac{y_j}{n-j+1}\right)^{n-j+1}$	$\frac{y_j}{n-j+1}$

Solution: $(y_1 = q) \rightarrow x_1 = \frac{c}{n} \rightarrow (y_2 = \frac{n-1}{n} c) \rightarrow \dots \rightarrow y_j = \frac{n-j+1}{n} c \rightarrow x_j = \frac{c}{n}$

$x_1 = x_2 = \dots = x_n = \frac{c}{n}$, $z = \left(\frac{c}{n}\right)^n$

Set 10.3a

11

$$f_n(y_n) = \min_{x_n = y_n} \{x_n^2\}$$

$$f_i(y_i) = \min_{x_i > 0} \left\{x_i^2 + f_{i+1}\left(\frac{y_i}{x_i}\right)\right\}$$

Stage n:

$$f_n(y_n) = y_n^2, \qquad x_n^* = y_n$$

stage n-1:

$$f_{n-1}(y_{n-1}) = \min_{x_{n-1} > 0} \left\{x_{n-1}^2 + \left(\frac{y_{n-1}}{x_{n-1}}\right)^2\right\}$$

$$\frac{\partial\{\cdot\}}{\partial x_{n-1}} = 2x_{n-1} - 2\frac{y_{n-1}^2}{x_{n-1}^3} = 0$$

or $x_{n-1}^* = \sqrt{y_{n-1}}$, $f_{n-1}(y_{n-1}) = 2y_{n-1}$

stage n-2:

$$f_{n-2}(y_{n-2}) = \min_{x_{n-2} > 0} \left\{x_{n-2}^2 + 2\left(\frac{y_{n-2}}{x_{n-2}}\right)\right\}$$

$$\frac{\partial\{\cdot\}}{\partial x_{n-2}} = 2x_{n-2} - 2\frac{y_{n-2}}{x_{n-2}} = 0$$

or $x_{n-2}^* = (y_{n-2})^{1/3}$, $f_{n-2}(y_{n-2}) = 3y_{n-2}^{2/3}$

Stage i:

Induction yields

$$x_i^* = y_i^{\frac{1}{n-i+1}}, \quad f_i(y_i) = (n-i+1)\,y_i^{\frac{2}{n-i+1}}$$

Stage 1:

$$x_1^* = c^{1/n}, \quad f_1(y_1) = n\,y_1^{2/n}$$

Thus, $y_2 = \frac{y_1}{x_1} = c^{\frac{n-1}{n}} \Rightarrow x_2^* = c^{1/n}$

In general, $y_i = \sqrt[n]{c}$

12

For proper decomposition, let

$x_1 = y_1$, $x_2 = y_4$, $x_3 = y_2$ and $x_4 = y_3$

The problem is then written as

Maximize $z = (x_1+2)^2 + (x_2-5)^2 + x_3x_4$

Subject to

$$x_1 + x_2 + x_3 + x_4 \le 5$$

$x_1, x_2, x_3, x_4 \ge 0$ and integer

Rearrangement of variables allows mixing multiplicative and additive decomposition

continued...

z_j = amount of the resource allocated to variables $j, j+1, \ldots, 4$.

Stage 4: $f_4(z_4) = \max_{x_4 \le y_4} \{x_4\}$

	x_4						Opt. Sol.	
z_4	$x_4 = 0$	1	2	3	4	5	f_4	x_4^*
0	0	—	—	—	—	—	0	0
1	0	1	—	—	—	—	1	1
2	0	1	2	—	—	—	2	2
3	0	1	2	3	—	—	3	3
4	0	1	2	3	4	—	4	4
5	0	1	2	3	4	5	5	5

Stage 3: $f_3(z_3) = \max_{x_3 \le z_3} \{x_3 f_4(z_3 - x_3)\}$

	$x_3 f_4(z_3 - x_3)$						Opt. Sol.	
z_3	$x_3 = 0$	1	2	3	4	5	f_3	x_3^*
0	0x0=0	—	—	—	—	—	0	0
1	0x1=0	1x0=0	—	—	—	—	0	1
2	0x2=0	1x1=1	2x0=0	—	—	—	1	1
3	0x3=0	1x2=2	2x1=2	3x0=0	—	—	2	1,2
4	0x4=0	1x3=3	2x2=4	3x1=3	4x0=0	—	4	2
5	0x5=0	1x4=4	2x3=6	3x2=6	4x1=4	5x0=0	6	2,3

Stage 2: $f_2(z_2) = \max_{x_2 \le z_2} \{(x_2-5)^2 + f_3(z_2 - x_2)\}$

	$(x_2-5)^2 + f_3(z_2 - x_2)$						Opt. Sol.	
z_2	$x_2 = 0$	1	2	3	4	5	f_2	x_2^*
0	25+0=25	—	—	—	—	—	25	0
1	25+0=25	16+0=16	—	—	—	—	25	0
2	25+1=26	16+0=16	9+0=9	—	—	—	26	0
3	25+2=27	16+1=17	9+0=9	4+0=4	—	—	27	0
4	25+4=29	16+2=18	9+1=10	4+0=4	1+0=0	—	29	0
5	25+6=31	16+4=20	9+2=11	4+1=5	1+0=0	0+0=0	31	0

stage 1: $f_1(z_1) = \max_{x_1 \le z_1} \{(x_1+2)^2 + f_2(z_1 - x_1)\}$

	$(x_1+2)^2 + f_2(z_1 - x_1)$						Opt. Sol.	
z_1	$x_1 = 0$	1	2	3	4	5	f_1	x_1^*
5	4+31	9+29	16+27	25+26	36+25	49+25	74	5

$(z_1 = 5) \rightarrow x_1 = 5 \rightarrow (z_2 = 0) \rightarrow x_2 = 0 \rightarrow (z_3 = 0) \rightarrow$
$x_3 = 0 \rightarrow (z_4 = 0) \rightarrow x_4 = 0$

Optimum: $(y_1, y_2, y_3, y_4) = (5, 0, 0, 0)$

$z = 74$

$\boxed{13}$

Define state as

y_i = amount of the resource allocated to variable $i, i+1, \ldots,$ and n

$$g_n(y_n) = \min_{x_3 = y_3} \{f_3(y_3)\}$$

$$g_i(y_i) = \min_{0 \le x_i \le y_i} \{\max[f_i(x_i), g_{i+1}(y_i - x_i)]\}$$

Stage 3: $g_3(y_3) = \min_{x_3 = y_3} \{x_3 - 2\}$

State	$g_3(y_3)$	x_3^*
y_3	$y_3 - 2$	y_3

Stage 2: $\min_{0 \le x_2 \le y_2} \{\max[(5x_2+3), (y_2 - x_2 - 2)]\}$

State	$g_2(y_2)$	x_2^*
$y_2 < 5$	0	3
$y_2 \ge 5$	$\frac{x_2 - 5}{6}$	$\frac{5}{6}x_2 - \frac{7}{6}$

Stage 1: $g_1(y_1) = \min_{x_1 \le y_1} \{\max[x_1 + 5, g_2(y_1 - x_1)]\}$

State	$g_1(y_1)$	x_1^*
$y_1 \le \frac{37}{5}$	0	5
$y_1 > \frac{37}{5}$	$\frac{5y_1 - 37}{11}$	$\frac{5y_1 + 18}{11}$

$$(y_1 = 10) \rightarrow x_1 = \frac{50-37}{11} = \frac{13}{11} \rightarrow$$

$$(y_2 = \frac{97}{11}) \rightarrow x_2 = \frac{97/11 - 5}{6} = \frac{7}{11} \rightarrow$$

$$(y_3 = \frac{90}{11}) \rightarrow x_3 = \frac{90}{11}$$

$$g_1(10) = \frac{5 \times 10 + 18}{11} = \frac{68}{11}$$

Set 10.3b

(a) Stage 5: $b_5=8$

x_4		Opt. Sol.	
	$x_5=8$	f_5	x_5^*
6	0 + 4 + 2(2) = 8	8	8
7	0 + 4 + 2(1) = 6	6	8
8	0 + 0 = 0	0	8

Stage 4: $b_4=6$

x_3				Opt. Sol.	
	$x_4=6$	$x_4=7$	$x_4=8$	f_4	x_4^*
3	0+(4+6)+8	3+(4+8)+6	6+(4+10)+0	18	6
4	0+(4+4)+8	3+(4+6)+6	6+(4+8)+0	16	6
5	0+(4+2)+8	3+(4+4)+6	6+(4+6)+0	14	6
6	0+0+8	3+(4+2)+6	6+(4+4)+0	8	6
7	0+0+8	3+0+6	6+(4+2)+0	8	6
8	0+0+8	3+0+6	6+0+0	6	8

Stage 3: $b_3=3$

x_2							Opt. Sol.	
	$x_3=3$	4	5	6	7	8	f_3	x_3^*
5	0+0 +18	3+0 +16	6+0 +14	9+4 +2+8	12+4 +4+8	15+4 +6+6	18	3
6	0+0 +18	3+0 +16	6+0 +14	9+0 +8	12+4 +2+8	5+4 +4+6	17	6
7	0+0 +18	3+0 +16	6+0 +14	9+0 +8	12+0 +8	15+4 +2+6	17	6
8	0+0 +18	3+0 +16	6+0 +14	9+0 +8	12+0 +8	15+0 +6	17	6

Stage 2: $b_2=5$

x_1					Opt. Sol.	
	$x_2=5$	6	7	8	f_2	x_2^*
6	0+0+18	3+0+17	6+4+2+17	9+4+4+17	18	5
7	0+0+18	3+0+17	6+0+17	9+4+2+17	18	5
8	0+0+18	3+0+17	6+0+17	9+0+17	18	5

Stage 1: $b_1=6$

x_0				Opt. Sol.	
	$x_1=6$	7	8	f_1	x_1^*
0	0+(4+12) +18	3+(4+14) +18	6+(4+16) +18	34	6

Week i	b_i	x_i	
1	6	6	Hire 6
2	5	5	Fire 1
3	3	3	Fire 2
4	6	6	Hire 3
5	8	8	Hire 2

continued...

(b) Stage 5: $b_5=2$

x_4		Opt. Sol.	
	$x_5=2$	f_5	x_5^*
	0+0	0	2

Stage 4: $b_4=8$

x_3		Opt. Sol.	
	$x_4=8$	f_4	x_4^*
7	0+(4+2)+1	6	8
8	0+0+0	0	8

Stage 3: $b_3=7$

x_2			Opt. Sol.	
	$x_3=7$	$x_3=8$	f_3	x_3^*
4	0+4+6+6	3+4+8+0	15	8
5	0+4+4+6	3+4+6+0	13	8
6	0+4+2+6	3+4+4+0	11	8
7	0+0+6	3+4+2+0	6	7
8	0+0+6	3+0+0	6	7

Stage 2: $b_2=4$

x_1						Opt. Sol.	
	$x_2=4$	5	6	7	8	f_2	x_2^*
8	0+0 +15	3+0 +13	6+0 +11	9+0 +6	12+0 +6	15	4,7

Stage 1: $b_1=8$

x_0		Opt. Sol.	
	$x_1=8$	f_1	x_1^*
0	0 + (4+2x8) + 15	35	8

Optimum solution:

Week i	b_i	x_i	
1	8	8	Hire 8
2	4	7	Fire 1
3	7	7	—
4	8	8	Hire 1
5	2	2	Fire 6

Alternative optimum:

Week i	b_i	x_i	
1	8	8	Hire 8
2	4	4	Fire 4
3	7	8	Hire 4
4	8	8	—
5	2	2	Fire 6

2

Let

$C_3(x_{i-1} - x_i) = 100(x_{i-1} - x_i)$

be the severance cost of $x_{i-1} - x_i$ laborers, $x_{i-1} > x_i$

$$f_i(x_i\ ) = \min_{x_i \geq b_i} \{C_1(x_i - b_i) + C_2(x_i - x_{i-1}) + C_3(x_{i-1} - x_i) + f_{i+1}(x_i)\}$$

$i = 1, 2, \ldots, n$

Stage 5 $(b_5 = 6)$:

	$C_1(x_5 - 6) + C_2(x_5 - x_4) + C_3(x_4 - x_5)$	Optimum solution	
x_4	$x_5 = 6$	$f_5(x_4)$	x_5^*
4	3(0) + 4 + 2(2) + 0 = 8	8	6
5	3(0) + 4 + 2(1) + 0 = 6	6	6
6	3(0) + 0 + 0 = 0	0	6

Stage 4 ($b_4 = 4$):

	$C_1(x_4 - 4) + C_2(x_4 - x_3) + C_3(x_4 - x_5) + f_5(x_4)$			Optimum solution	
x_3	$x_4 = 4$	$x_4 = 5$	$x_4 = 6$	$f_4(x_3)$	x_4^*
8	3(0) + 0 + 4 + 8 = 12	3(1) + 0 + 3 + 6 = 12	3(2) + 0 + 2 + 0 = 8	8	6

Stage 3 $(b_3 = 8)$:

	$C_1(x_3 - 8) + C_2(x_3 - x_2) + C_3(x_2 - x_3) + f_4(x_3)$	Optimum solution	
x_2	$x_3 = 8$	$f_3(x_2)$	x_3^*
7	0 + 4 + 2(1) + 0 + 8 = 14	14	8
8	0 + 0 + 0 + 8 = 8	8	8

Stage 2 $(b_2 = 7)$:

	$C_1(x_2 - 7) + C_2(x_3 - x_2) + C_3(x_1 - x_2) + f_3(x_2)$		Optimum solution	
x_1	$x_2 = 7$	$x_2 = 8$	$f_2(x_1)$	x_2^*
5	0 + 4 + 2(2) + 0 + 14 = 22	3(1) + 4 + 2(3) + 0 + 8 = 21	21	8
6	0 + 4 + 2(1) + 0 + 14 = 20	3(1) + 4 + 2(2) + 0 + 8 = 19	19	8
7	0 + 0 + 0 + 14 = 14	3(1) + 4 + 2(1) + 0 + 8 = 17	14	7
8	0 + 0 + 1 + 14 = 15	3(1) + 0 + 0 + 8 = 11	11	8

Stage 1 $(b_1 = 5)$:

	$C_1(x_1 - 5) + C_2(x_1 - x_0) + C_3(x_0 - x_1) + f_2(x_1)$				Optimum solution	
x_0	$x_1 = 5$	$x_1 = 6$	$x_1 = 7$	$x_1 = 8$	$f_1(x_0)$	x_1^*
0	0 + 4 + 2(5) + 0 + 21 = 35	3(1) + 4 + 2(6) + 0 + 19 = 38	3(2) + 4 + 2(7) + 0 + 14 = 38	3(2) + 4 + 2(8) + 0 + 11 = 37	35	5

The optimum solution is determined as

$x_0 = 0 \rightarrow x_1^* = 5 \rightarrow x_2^* = 8 \rightarrow x_3^* = 8 \rightarrow x_4^* = 6 \rightarrow x_5 = 6$

The solution can be translated to the following plan:

Week i	Minimum Labor Force b_i	Actual Labor Force x_i	Decision
1	5	5	Hire 5 workers
2	7	8	Hire 3 workers
3	8	8	No change
4	4	6	Fire 2 workers
5	6	6	No change

3

Let

x_i = number of cars rented in wk i

$C_i(x_i)$ = renal cost in week i

$$= \begin{cases} 220x_i, & \text{if } x_i \leq x_{i-1} \\ 500 + 220x_i, & \text{if } x_i > x_{i-1} \end{cases}$$

$$f_i(x_{i-1}) = \min_{x_i \geq b_i} \{C_i(x_i) + f_{i+1}(x_i)\}$$

$i = 1, 2, 3, 4$

continued...

Stage 4: $b_4 = 8$

		Opt. Sol.	
x_3	$x_4 = 8$	f_4	x_4^*
7	500 + 220×8 = 2260	2260	8
8	220×8 = 1760	1760	8

Stage 3: $b_3 = 7$

			Opt. Sol.	
x_2	$x_3 = 7$	$x_3 = 8$	f_3	x_3^*
4	500 + 220(7) + 2260 = 4300	500 + 220(8) + 1760 = 4020	4020	8
5	500 + 220(7) + 2260 = 4300	500 + 220(8) + 1760 = 4020	4020	8
6	500 + 220(7) + 2260 = 4300	500 + 220(8) + 1760 = 4020	4020	8
7	220×7 + 2260 = 3800	500 + 220(8) + 1760 = 4020	3800	7
8	220×7 + 2260 = 3800	220×8 + 1760 = 3520	3520	8

Stage 2: $b_2 = 4$

						Opt. Sol.	
x_1	$x_2 = 4$	5	6	7	8	f_2	x_2^*
7	220(4) + 4020 = 4900	220(5) + 4020 = 5120	220(6) + 4020 = 5340	220(7) + 3800 = 5340	500 + 220(8) + 3520 = 5780	4900	4
8	220(4) + 4020 = 4900	220(5) + 4020 = 5120	220(6) + 4020 = 5340	220(7) + 3800 = 5340	220(8) + 3520 = 5280	4900	4

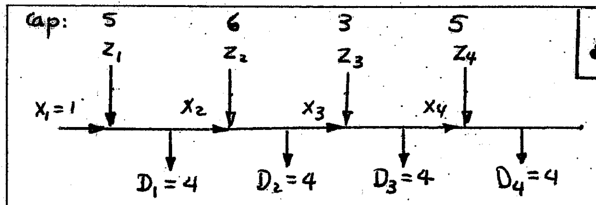
Stage 1: $b_1 = 7$

			Opt. Sol.	
x_0	$x_1 = 7$	$x_1 = 8$	f_1	x_1^*
0	500 + 220(7) + 4900 = 6940	500 + 220(8) + 4900 = 7160	6940	4

Solution:

Week i	b_i	x_i	
1	7	7	Rent 7 cars
2	4	4	Return 3
3	7	8	Rent 4
4	8	8	—

4



c/unit	\$30	33	35	42
h/unit	\$2	3	4	–

z_i = amount produced in period i

$x_1 = 1,\quad 0 \le z_1 \le 5$

$0 \le x_2 \le 2,\quad 0 \le z_2 \le 6$

$0 \le x_3 \le 4,\quad 0 \le z_3 \le 3$

$0 \le x_4 \le 3,\quad 0 \le z_4 \le$

Stage 4: $f_4(x_4) = \min\limits_{\substack{z_4 \ge 0 \\ z_4 + x_4 = 4}} \{42 z_4\}$

						Opt. Sol.	
x_4	$z_4=0$	1	2	3	4	f_4	z_4^*
0	–	–	–	–	42x4	168	4
1	–	–	–	42x3	–	126	3
2	–	–	42x2	–	–	84	2
3	–	42x1	–	–	–	42	1
4	0	–	–	–	–	0	0

Stage 3: $f_3(x_3) = \min\limits_{\substack{z_3 \ge 0 \\ z_3 + x_3 \ge 4}} \{35 z_3 + 4(x_3 + z_3 - 4) + f_4(x_3 + z_3 - 4)\}$

x_3	$z_3=0$	1	2	3	4	f_3	z_3^*
0	–	–	–	–	140+0 +168 =308	308	4
1	–	–	–	105+0 +168 =273	140+4 +126 =270	270	4
2	–	–	70+0 +168 =238	105+4 +126 =235	140+8 +84 =232	232	4
3	–	35+0 +168 =203	70+4 +126 =200	105+8 +84 =193	140+12 +42 =194	193	3
4	0+0 +168 =168	35+4 +126 =165	70+8 +84 =162	105+12 +42 =159	140+16 +0 =156	156	4

continued...

Stage 2:

$f_2(x_2) = \min\limits_{\substack{z_2 \ge 0 \\ z_2 + x_2 \ge 4}} \{33 z_2 + 3(x_2 + z_2 - 4) + f_3(x_2 + z_2 - 4)\}$

								Opt. Sol.	
x_2	$z_2=0$	1	2	3	4	5	6	f_2	z_2
0	–	–	–	–	132 +308 =440	165 +3 270 =438	198 +6 232 =436	436	6
1	–	–	–	99 +308 =407	132 +3 +270 =405	165 +6 +232 =403	198 +9 +193 =400	400	6
2	–	–	66 +308 =374	99 +3 +270 =372	132 +6 +232 =370	165 +9 +193 =367	198 +12 +156 =366	366	6

Stage 1:

$f_1(x_1) = \min\limits_{\substack{z_1 \ge 0 \\ z_1 + x_1 \ge 4}} \{30 z_1 + 2(x_1 + z_1 - 4) + f_2(x_1 + z_1 - 4)\}$

							Opt. Sol.	
x_1	$z_1=0$	1	2	3	4	5	f_1	z_1^*
1	–	–	–	90 +436 =526	120 +2 +400 =522	150 +4 +366 =520	520	5

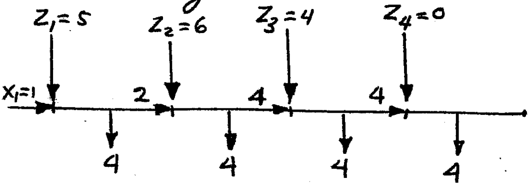
Solution: Cost = \$520

$(x_1 = 1) \rightarrow z_1 = 5 \rightarrow (x_2 = 1+5-4 = 2) \rightarrow$

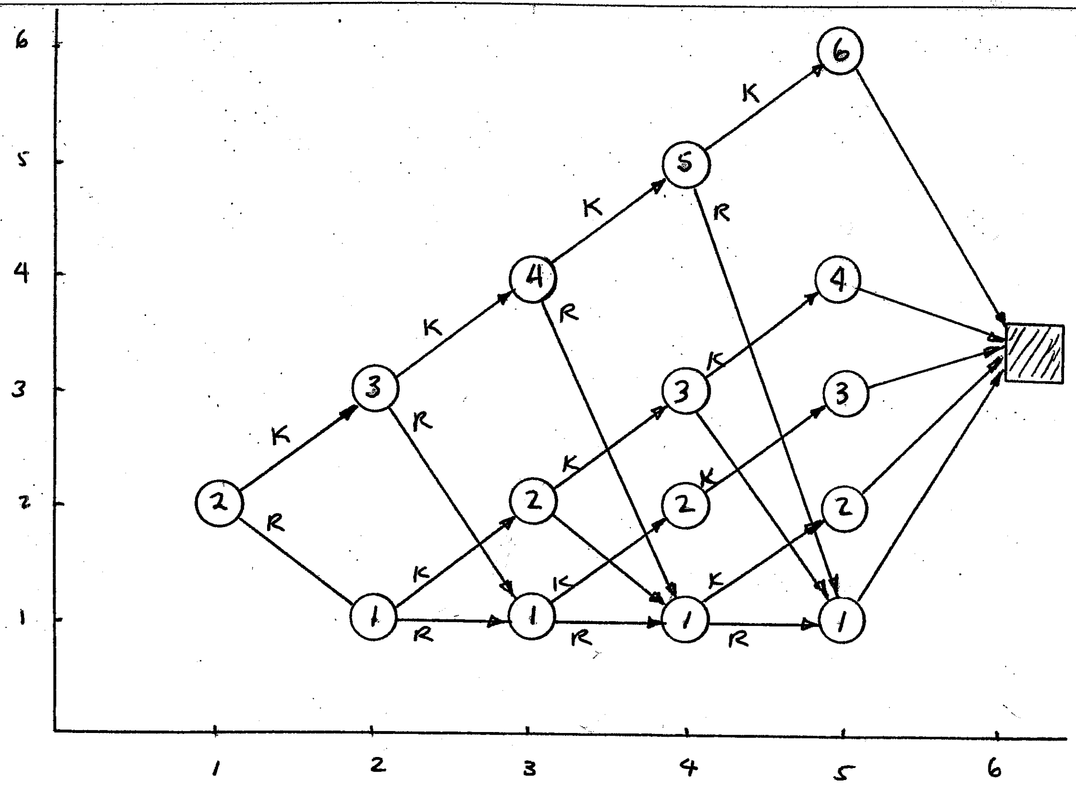
$z_2 = 6 \rightarrow (x_3 = 2+6-4 = 4) \rightarrow z_3 = 4 \rightarrow$

$(x_4 = 4+4-4 = 4) \rightarrow z_4 = 0$

Summary



(a)



Stage 4:

t	K	R	Opt. Sol.	
			f4	Dec.
1	19+60 − .6 = 78.4	20 + 80 + 80 − 100.2 = 79.8	79.8	R
2	18.5 + 50 − 1.2 = 67.3	20 + 60 + 80 − 100.2 = 59.8	67.3	K
3	17.2 + 30 − 1.5 = 45.7	20 + 50 + 80 − 100.2 = 49.8	49.8	R
5	14 + 10 − 1.8 = 22.2	20 + 10 + 80 − 100.2 = 9.8	22.2	K

Stage 3:

t	K	R	Opt. Sol.	
			f_3	Dec.
1	19 − .6 + 67.3 = 85.7	20 + 80 − 100.2 + 79.8 = 79.6	85.7	K
2	18.5 − 1.2 + 49.8 = 67.1	20 + 60 − 100.2 + 79.8 = 59.6	67.1	K
4	15.5 − 1.7 + 22.2 = 36.	20 + 30 − 100.2 + 79.8 = 29.6	36	K

Stage 2:

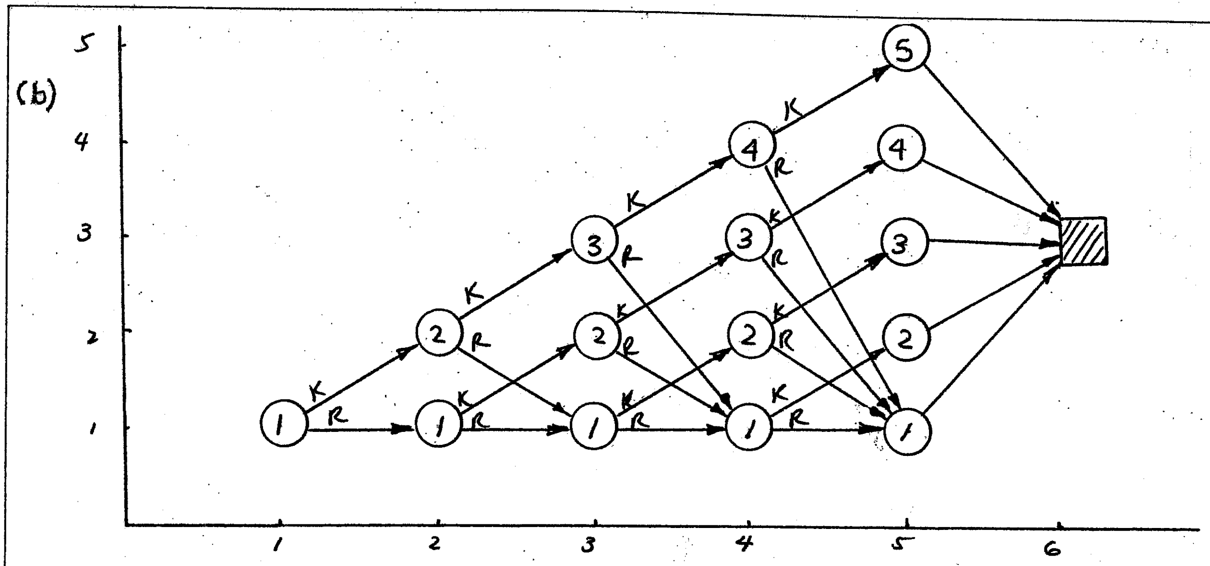
t	K	R	Opt. Sol.	
			f_2	Dec.
1	19 − .6 + 67.1 = 85.5	20 + 80 − 100.2 + 85.7 = 85.5	85.5	K, R
3	17.2 − 1.5 + 36 = 51.7	20 + 50 − 100.2 + 85.7 = 55.5	55.5	R

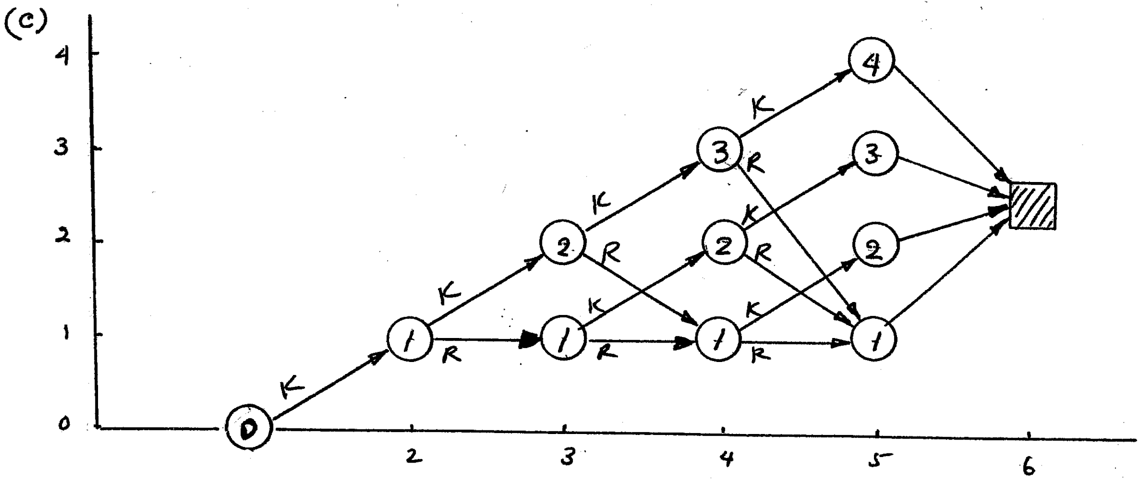
Stage 1:

t	K	R	Opt. Sol.	
			f_1	Dec.
2	18.5 − 1.2 + 55.5 = 72.8	20 + 60 − 100.2 + 85.5 = 65.3	72.8	K

Solution: K → R → K → K, revenue = $72,800

continued...





2

Since income from mowing is constant, it need not be taken into account.

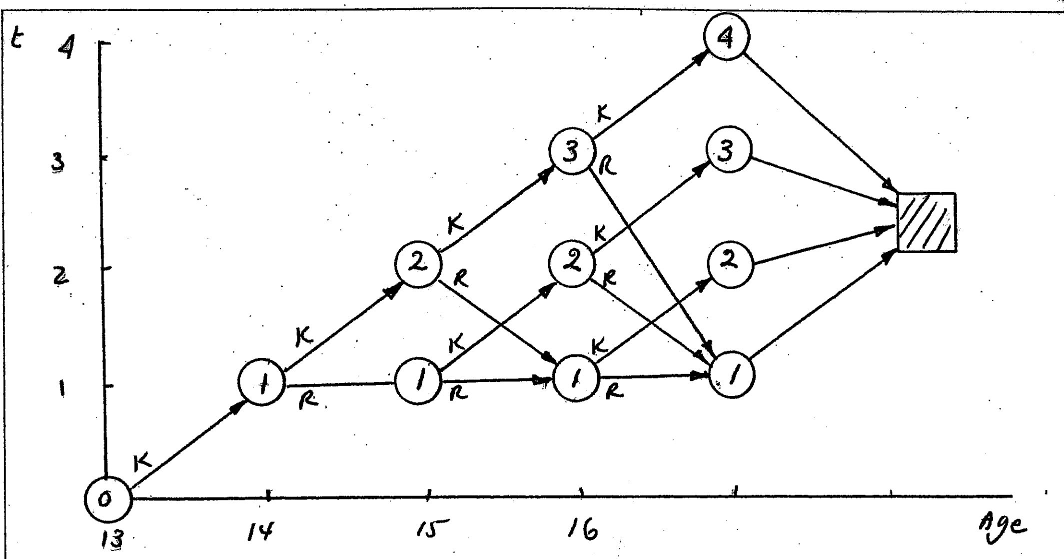
f_4 $\cdot$ $f_4(t) = \min \begin{cases} C(t) - S(t), & K \\ I(t) + c(1) - S(t), & R \end{cases}$

$$f_i(t) = \min \begin{cases} C(t) + f_{i+1}(t+1), & K \\ I(t) + C(1) - S(t) + f_{i+1}(1), & R \end{cases}$$

where,

$C(t)$ = operating cost per year for a t-year old mower
$I(t)$ = cost of a new mower after t years
$S(t)$ = salvage value of a t-year old mower
$f_i(t)$ = minimum cost for periods $i, i+1, \ldots,$ and 4 given t-year mower.

continued...



Stage 4:

t	K	R	Opt. Sol. f_4	Dec.
1	144 − 130 = 14	260 + 120 − 150 − 150 = 80	14	K
2	168 − 110 = 58	260 + 120 − 135 − 150 = 95	58	K
3	192 − 90 = 102	260 + 120 − 120 − 150 = 110	102	K

Stage 3:

t	K	R	Opt. Sol. f_3	Dec.
1	144 + 58 = 202	240 + 120 − 150 + 14 = 224	202	K
2	168 + 102 = 270	240 + 120 − 135 + 14 = 239	239	R

Stage 2:

t	K	R	Opt. Sol. f_2	Dec
1	144 + 239 = 338	220 + 120 − 150 + 202 = 392	338	K

Stage 1: The only option available at the start is K. Cost = 120+338 = 458

Solution: K → K → R → K, total cost = $458

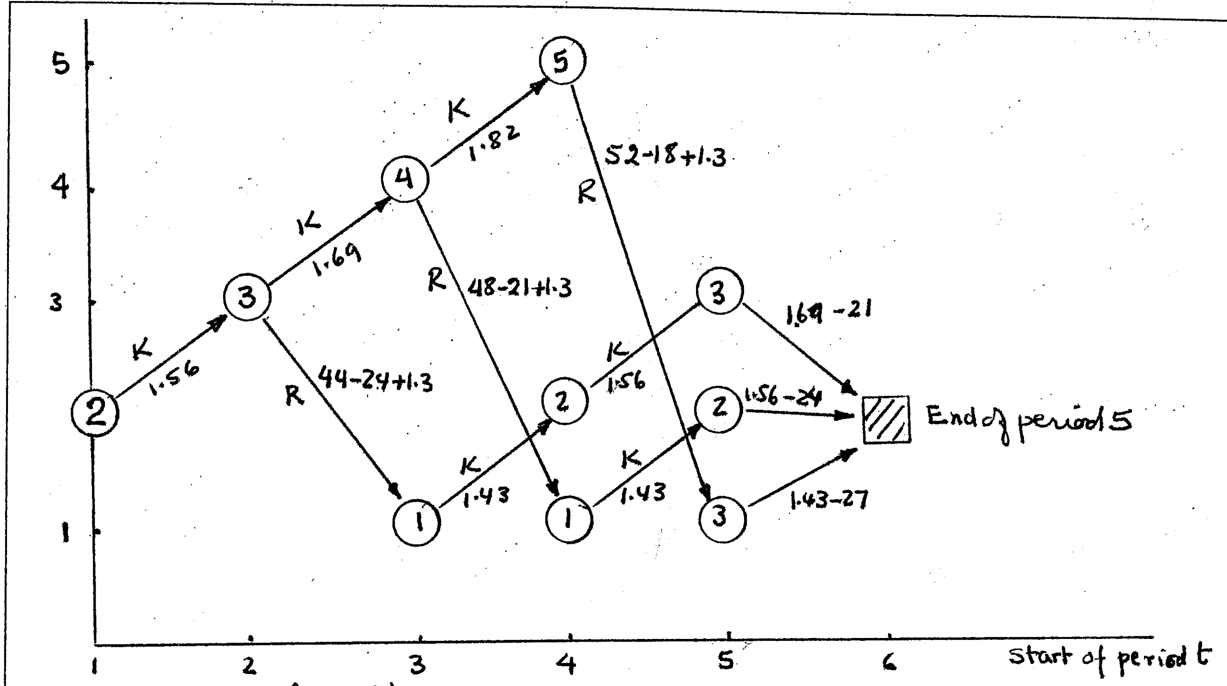
3

$$f_i(t) = \min \begin{cases} c(t) + f_{i+1}(t+1), & K \\ I(t) + c(1) - s(t) + f_{i+1}(1), & R \end{cases} \qquad (2 \le t \le 5)$$

$$f_5(t) = \min \begin{cases} c(t) - s(t), & K \\ I(t) + c(1) - S(t), & R \end{cases}$$

continued...

Set 10.3c



Stage 5: (Start of year 5)

t	K	R	Optimum f_5	Dec
1	1.43 - 27 = -25.57	—	-25.57	K
2	1.56 - 24 = - 22.44	—	-22.44	K
3	1.69 - 21 = -19.31	—	-19.31	K

Stage 4 (Start of year 4):

t	K	R	Optimum f_4	Dec.
1	1.43 + (-22.94) = -21.51	—	-21.51	K
2	1.56 + (-19.31) = -17.75	—	-17.75	K
5	—	52 - 18 + 1.3 + (-25.57) = 9.73	9.73	R

Stage 3 (Start of year 3):

t	K	R	Optimum f_3	Dec.
1	1.43 + (-17.75) = -16.32	—	-16.32	K
4	1.82 + (9.73) = 11.55	48-21+1.3+(-21.51) = 6.79	6.79	R

Stage 2 (Start of year 2):

t	K	R	Optimum f_2	Dec
3	1.69 + 6.79 = 8.48	44-24+1.3-16.32 = 4.98	4.98	R

Stage 1 (Start of year 1): Keep is the only option. Cost = 1.56 + 4.98 = 6.54

Solution:

K → R → K → K → K. Cost = \$6540

4

(a)

$$f_N(T_N) = \max_{T_N \le N} \begin{cases} N^2 - T_N^2 + N - (T_N + 1), & K \\ (N^2 - 0) + N - (0+1) - C + N - T_N, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \le N} \begin{cases} (N^2 - T_i^2) + f_{i+1}(T_i + 1), & K \\ (N^2 - 0) + (N - T_i) - C + f_{i+1}(1), & R \end{cases}$$

For $N = 3$, $C = 10$,

$$f_3(T_3) = \max_{T_3 \le 3} \begin{cases} 11 - T_3 - T_3^2, & K \\ 4 - T_3, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \le 3} \begin{cases} 9 - T_i^2 + f_{i+1}(T_i + 1), & K \\ 2 - T_i + f_{i+1}(1), & R \end{cases}$$

$i = 1, 2$

(b)

Stage 3

T_3	K	R	Optimum f_3	Optimum Dec^n
1	9	3	9	K
2	5	2	5	K
3	-1	1	1	R

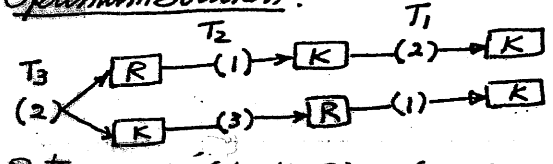
Stage 2:

T_2	K	R	Optimum f_2	Optimum Dec^n
1	8+5 = 13	1+9 = 10	13	K
2	5+1 = 6	0+9 = 9	9	R
3	—	-1+9 = 8	8	R

Stage 1:

T_1	K	R	Optimum f_1	Optimum Dec^n
1	8+9 = 17	1+13 = 14	17	K
2	5+8 = 13	0+13 = 13	13	K,R
3	—	-1+13 = 12	12	R

Optimum Solution:



Return = 13, (K, K, R) or (K, R, K)

5

$$f_4(T_4) = \max_{T_4 \le 4} \begin{cases} \frac{4}{1+T_4} + 4 - (T_4 + 1), & K \\ \frac{4}{1+0} + 4 - (0+1) + 6 + (4 - T_4), & R \end{cases}$$

$$= \max_{T_4 \le 4} \begin{cases} \frac{4}{1+T_4} - T_4 + 3, & K \\ 5 - T_4, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \le 4} \begin{cases} \frac{4}{1+T_i} + f_{i+1}(T_i + 1), & K \\ 2 - T_i + f_{i+1}(1), & R \end{cases}$$

Stage 4

T_4	K	R	Opt. Sol. f_4	Opt. Sol. Dec.
1	4.00	4	4	K,R
2	2.33	3	3	R
3	1.00	2	2	R
4	-0.20	1	1	R

Stage 3

T_3	K	R	Opt. Sol. f_3	Opt. Sol. Dec.
1	2 + 3 = 5	1 + 4 = 5	5	K,R
2	1.33 + 2 = 3.33	0 + 4 = 4	4	R
3	1.00 + 1 = 2.00	-1 + 4 = 3	3	R
4	0.80 + (-) = —	-2 + 4 = 2	2	R

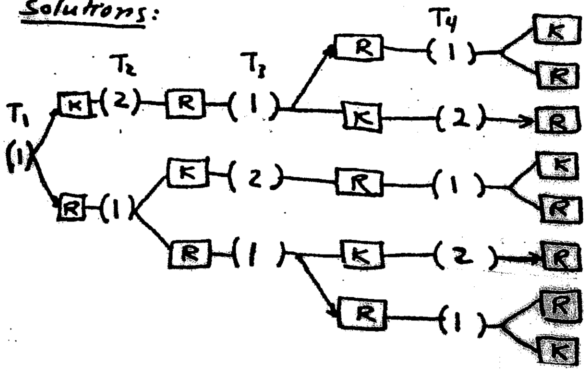
Stage 2:

T_2	K	R	Opt. Sol. f_2	Opt. Sol. Dec.
1	2 + 4 = 6	1 + 5 = 6	6	K,R
2	1.33 + 3 = 4.33	0 + 5 = 5	5	R
3	1.00 + 2 = 3	-1 + 5 = 4	4	R
4	.80 + (-) = —	-2 + 5 = 3	3.	R

Stage 1:

T_1	K	R	Opt. Sol. f_1	Opt. Sol. Dec
1	2 + 5 = 7	1 + 6 = 7	7	K,R
2	1.33 + 4 = 5.33	0 + 6 = 6	6	R
3	1.00 + 3 = 4	-1 + 6 = 5	5	R
4	.8 + (-) = —	-2 + 6 = 4	4	R

Solutions:



continued...

1

$P_1 = 5,\quad P_2 = 4,\quad P_3 = 3,\quad P_4 = 2$

$\alpha_1 = (1+.085)$

$= 1.085$

$\alpha_2 = (1+.08)$

$= 1.08$

$$q_{ij} = \begin{array}{c|c|c} & 1 & 2 \\ \hline 1 & .018 & .023 \\ 2 & .017 & .022 \\ 3 & .021 & .026 \\ 4 & .025 & .030 \end{array}$$

Stage 4: $f_4(x_4) = \max\limits_{0 \le I_4 \le x_4} \{S_4\}$

$S_4 = (1.085 + .025 - 1.08 - .03) I_4 + (1.08 + .03) x_4$

$= 1.11 x_4$

State	Opt. Sol.	
	$f_4(x_4)$	I_4^*
x_4	$1.11\, x_4$	$0 \le I_4 \le x_4$

stage 3: $f_3(x_3) = \max\limits_{0 \le I_3 \le x_3} \{S_3 + f_4(x_4)\}$

$S_3 = (1.085^2 - 1.08^2) I_3 + 1.08^2 x_3$

$= .010825\, I_3 + 1.1664 x_3$

$x_4 = P_4 + (q_{31} - q_{32}) I_3 + q_{32} x_3$

$= 2000 + (.021 - .026) I_3 + .026 x_3$

$= 2000 - .005 I_3 + .026 x_3$

$f_3(x_3) = \max\limits_{0 \le I_3 \le x_3} \{ .010825 I_3 + 1.1664 x_3 + 1.11 (2000 - .005 I_3 + .026 x_3)\}$

$= \max\limits_{0 \le I_3 \le x_3} \{2220 + .005275 I_3 + 1.19526 x_3\}$

State	Opt. Sol.	
	$f_3(x_3)$	I_3^*
x_3	$2220+$ $1.200535 x_3$	x_3

Stage 2: $f_2(x_2) = \max\limits_{0 \le I_2 \le x_2} \{S_2 + f_3(x_3)\}$

$S_2 = (1.085^3 - 1.08^3) I_2 + 1.08^3 x_2$

$= .0175771 I_2 + 1.259712 x_2$

$x_3 = 3000 + (.017 - .022) I_2 + .022 x_2$

$= 3000 - .05 I_2 + .022 x_2$

continued...

1 continued

$f_2(x_2) = \max\limits_{0 \le I_2 \le x_2} \{ .0175771 I_2 + 1.259712 x_2 + 2220 + 1.200535 (3000 - .05 I_2 + .022 x_2)\}$

$= \max\limits_{0 \le I_2 \le x_2} \{5821.61 - .0424496 I_2 + 1.2861238 x_2\}$

State	Opt. Sol.	
	$f_2(x_2)$	I_2^*
	5821.61 $+$ $1.2861238 x_2$	0

stage 1: $f_1(x_1) = \max\limits_{0 \le I_1 \le x_1} \{S_1 + f_2(x_2)\}$

$S_1 = (1.085^4 - 1.08^4) I_1 + 1.08^4 x_1$

$= .0253697 I_1 + 1.360489 x_1$

$x_2 = 4000 - .005 I_1 + .023 x_1$

$f_1(x_1) = \max\limits_{0 \le I_1 \le x_1} \{ .0253697 I_1 + 1.360489 x_1 + 5821.61 + 1.2861238 (4000 - .005 I_1 + .023 x_1)\}$

$= \max\limits_{0 \le I_1 \le x_1} \{10{,}966.11 + .018939 I_1 + 1.3900698 x_1\}$

State	Opt. Sol.	
	$f_1(x_1)$	I_1^*
$x_1 = 5000$	$10{,}966.11+$ $1.4090088 x_1$	5000

$x_2 = 4000 - .005 \times 5000 + .023 \times 5000 = \$4090$

$x_3 = 3000 - .005 \times 0 + .022 \times 4090 \cong \$3090$

$x_4 = 2000 - .005 \times 3090 + .026 \times 3090 = \$2065$

Solution:

$I_1 = x_1 = 5000$: Invest \$5000 in FB

$I_2 = 0$: Invest \$4090 in SB

$I_3 = 3090$: Invest \$3090 in FB

$0 \le I_4 \le \$2065$: Invest \$2065 in FB, SB, or both.

2

x_i = cumulative amount available at the end of period i before a decision is made.

$$f_i(x_i) = \max_{y_i \le x_i} \{ g(y_i) + f_{i+1}(\alpha(x_i - y_i)) \}$$

$$f_n(x_n) = \max_{y_n = x_n} \{ g(y_n) \}$$

where,

$$\alpha = 1.09, \quad g(y) = \sqrt{y}, \quad x_1 = 10{,}000\alpha$$

Stage n:

$$f_n(x_n) = \sqrt{x_n}, \quad y_n^* = x_n$$

Stage $n-1$:

$$f_{n-1}(x_{n-1}) = \max_{y_{n-1} \le x_{n-1}} \{ \sqrt{y_{n-1}} + \sqrt{\alpha(x_{n-1} - y_{n-1})} \}$$

$$\frac{\partial\{\cdot\}}{\partial y_{n-1}} = \frac{1}{2\sqrt{y_{n-1}}} - \frac{\alpha}{2\sqrt{\alpha(x_{n-1} - y_{n-1})}} = 0$$

$$y_{n-1}^* = \frac{x_{n-1}}{1+\alpha}$$

Because $\frac{\partial^2\{\cdot\}}{\partial y_{n-1}^2} < 0$, y_{n-1}^* is a maximum point.

$$f_{n-1}(x_{n-1}) = \sqrt{(1+\alpha)x_{n-1}}$$

Stage $n-2$:

$$f_{n-2}(x_{n-2}) = \max_{y_{n-2} \le x_{n-2}} \{ \sqrt{y_{n-2}} + \sqrt{\alpha(1+\alpha)(x_{n-2} - y_{n-2})} \}$$

$$y_{n-2}^* = \frac{x_{n-2}}{1+\alpha+\alpha^2}$$

$$f_{n-2}(x_{n-2}) = \sqrt{(1+\alpha+\alpha^2)x_{n-2}}$$

Stage i:

By induction, we can show that

$$y_i^* = \frac{x_i}{(1+\alpha+\cdots+\alpha^{n-i})}$$

continued...

$$f_i(x_i) = \sqrt{(1+\alpha+\cdots+\alpha^{n-i})x_i}$$

Hence,

$$x_1 = \alpha C, \quad C = \$10{,}000$$

$$y_1^* = \frac{\alpha C}{(1+\alpha+\cdots+\alpha^{n-1})}$$

$$= \frac{C(1-\alpha)}{(1-\alpha^n)}$$

$$f_1(x_1) = \sqrt{(1+\alpha+\cdots+\alpha^{n-1})x_1}$$

Given $x_1 = \alpha C$,

$$f_1(C) = \sqrt{\alpha(1+\alpha+\cdots+\alpha^{n-1})C}$$

$$= \sqrt{\frac{\alpha(1-\alpha^n)}{(1-\alpha)}C}$$

$$x_2 = \alpha(x_1 - y_1)$$

$$= \alpha^2 C\left(1 - \frac{1}{1+\alpha+\cdots+\alpha^{n-1}}\right)$$

$$= \alpha^3 C\left(\frac{1-\alpha^{n-1}}{1-\alpha^n}\right)$$

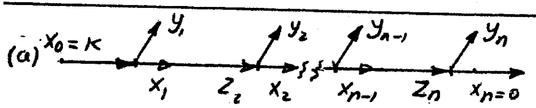
$$y_2^* = \alpha^3 C \frac{(1-\alpha)}{1-\alpha^n}$$

In general, we have

$$y_i^* = \alpha^{i+1} C\left(\frac{1-\alpha}{1-\alpha^{n-i+1}}\right)$$

3

(a)



$$f_n(z_n) = \max_{y_n = z_n \le 2^n K} \{ p_n y_n \}$$

$$f_i(z_i) = \max_{y_i \le z_i \le 2K} \{ p_i y_i + f_{i+1}(2[z_i - y_i]) \}$$

$$i = 1, 2, \cdots, n-1$$

continued...

(b) Stage (year) 3:

z_3	$120y_3$									Optimum	
	$y_3=0$	1	2	3	4	5	6	7	8	f_3	y_3^*
0	0									0	0
1		120								120	1
2			240							240	2
3				360						360	3
4					480					480	4
5						600				600	5
6							720			720	6
7								840		840	7
8									960	960	8

Stage (year) 2:

z_2	$130y_2+f_3(2[z_2-y_2])$						
	$y_2=0$	1	2	3	4	f_2	y_2^*
0	0 + 0 = 0	—				0	0
1	0 + 240 = 240	130 + 0 = 130	—			240	0
2	0 + 480 = 480	130 + 240 = 370	260 + 0 = 260	—		480	0
3	0 + 720 = 720	130 + 480 = 610	260 + 240 = 500	390 + 0 = 390	—	720	0
4	0 + 960 = 960	130 + 720 = 850	260 + 480 = 740	390 + 240 = 630	520 + 0 = 520	960	0

Stage (year) 1:

z_1	$100y_1+f_2(2[z_1-y_1])$			Optimum	
	$y_1=0$	1	2	f_1	y_1^*
0	—	—	—	—	—
1	—	—	—	—	—
2	0 + 960 = 960	100 + 480 = 580	200 + 0 = 200	960	0

Solution:

$z_1=2 \rightarrow y_1=0 \rightarrow z_2=4 \rightarrow y_2=0 \rightarrow z_3=8 \rightarrow y_3=8$

Revenue = $960

$\boxed{1}$

(a)

$$f_2(v_2, w_2) = \max_{\substack{0 \le 7x_2 \le v_2 \\ 0 \le 2x_2 \le w_2}} \{14x_2\}$$

$$= 14 \min\left\{\frac{v_2}{7}, \frac{w_2}{2}\right\}$$

$$x_2^* = \min\left\{\frac{v_2}{7}, \frac{w_2}{2}\right\}$$

$$f_1(v_1, w_1) = \max_{\substack{0 \le 2x_1 \le v_1 \\ 0 \le 7x_1 \le w_1}} \{4x_1 + f_2(v_1 - 2x_1, w_1 - 7x_1)\}$$

$$= \max\left(4x_1 + 14\min\left\{\frac{v_1 - 2x_1}{7}, \frac{w_1 - 7x_1}{2}\right\}\right)$$

For $v_1 = w_1 = 21$, $0 \le x_1 \le 3$,

$$f_1(21,21) = \max \begin{cases} 42, & 0 \le x_1 \le 7/3 \\ 147 - 45x_1, & 7/3 \le x_1 \le 3 \end{cases}$$

$$= 42 \quad \text{for } 0 \le x_1^* \le 7/3$$

Next, $v_2 = v_1 - 2x_1 = 21 - 2x_1^*$

$w_2 = w_2 - 7x_1 = 21 - 7x_1^*$

$$x_2^* = \min\left\{\frac{21 - 2x_1^*}{7}, \frac{21 - 7x_1^*}{2}\right\}$$

$$= 3 - \frac{2}{7}x_1^*, \quad 0 \le x_1^* \le 7/3$$

Problem has infinite alternative solutions.

(b)

$$f_2(v_2, w_2) = \max_{\substack{0 \le x_2 \le v_2 \\ 0 \le 2x_2 \le w_2 \\ x_2 \text{ integer}}} \{7x_2\}$$

$$= 7 \min\left\{[v_2], \left[\frac{w_2}{2}\right]\right\}$$

where $[a]$ = largest integer $\le a$.

$$f_1(v_1, w_1) = \max_{\substack{0 \le 2x_1 \le v_1 \\ 0 \le 5x_1 \le w_1}} \{8x_1 + f_2(v_1 - 2x_1, w_1 - 5x_1)\}$$

$$= \max\left\{8x_1 + 7\min\left([8 - 2x_1], \left[\frac{15 - 5x_1}{2}\right]\right)\right\}$$

$$x_1 \le \min\left\{\left[\frac{v_1}{2}\right], \left[\frac{w_1}{5}\right]\right\} = \min\left\{[8/2], \left[\frac{15}{5}\right]\right\} = 3$$

$$f_1(v_1, w_1) = \max_{x_1 = 0,1,2,3} \left\{8x_1 + 7\min\left([8 - 2x_1], \left[\frac{15 - 5x_1}{2}\right]\right)\right\}$$

continued...

$$= \max_{x_1 = 0,1,2,3} \left\{8x_1 + 7\left[\frac{15 - 5x_1}{2}\right]\right\}$$

$$= 49 \text{ at } x_1^* = 0$$

$v_2 = v_1 - 2x_1 = v_1 = 8$

$w_2 = w_1 - 5x_1 = w_1 = 15$

$$x_2^* = \min\left\{[8], \left[\frac{15}{2}\right]\right\} = 7$$

Optimum: $(x_1, x_2) = (0, 7)$, $Z = 49$

(c)

Forward formulation:

$$f_1(v_1, w_1) = \max_{\substack{0 \le x_1 \le v_1 \\ 0 \le x_1 \le w_1}} (7x_1^2 + 6x_1)$$

$$= \min\{7v_1^2 + 6v_1, 7w_1^2 + 6w_1\}$$

where $x_1^* = \min\{v_1, w_1\}$

$$f_2(v_2, w_2) = \max_{0 \le x_2 \le 5} \{5x_2^2 + \min[7(v_2 - x_2)^2 + 6(v_2 - x_2), 7(w_2 - x_2)^2 + 6(w_2 - x_2)]\}$$

Now, $v_2 = 10$:

$0 \le v_1 = 10 - 2x_2 \Rightarrow 0 \le x_2 \le 5$

$0 \le v_1 - x_1 \Rightarrow 0 \le x_1 \le v_1$

$w_2 = 9$:

$0 \le w_1 = 9 + 3x_2 \Rightarrow x_2 \ge 0$

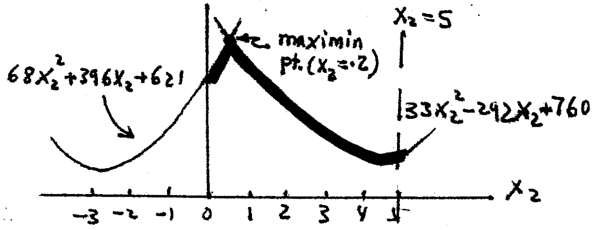
$0 \le w_1 - x_1 \Rightarrow 0 \le x_1 \le w_1$

With $v_2 = 10$ and $w_2 = 9$, we get

$f_2(v_2, w_2)$

$$= \max_{0 \le x_2 \le 5} \{5x_2^2 + \min[28x_2^2 - 292x_2 + 760, 63x_2^2 + 396x_2 + 621]\}$$

$$= \max_{0 \le x_2 \le 5} \{\min[33x_2^2 - 292x_2 + 760, 68x_2^2 + 396x_2 + 621]\}$$



continued...

Optimal solution:

$v_2 = 10,\ w_2 = 9 \Rightarrow x_2^* = .2$

$$\left.\begin{array}{l} v_1 = 10 - 2 \times .2 = 9.6 \\ w_1 = 9 + 3 \times .2 = 9.6 \end{array}\right\} \Rightarrow x_1^* = 9.6$$

Optimal objective value $= 702.92$

2

Maximize $z = r_1 x_1 + r_2 x_2 + \cdots + r_n x_n$

subject to

$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \leq W$

$v_1 x_1 + v_2 x_2 + \cdots + v_n x_n \leq V$

$x_j \geq 0$ and integer

where

x_j = number of units of item j

D.P. backward formulation:

Let

a_j = weight allocated to items $j, j+1, \ldots,$ and n

b_j = volume allocated to items $j, j+1, \ldots,$ and n

$f_j(a_j, b_j)$ = optimum revenue for items $j, j+1,$ and n, given a_j and b_j

$$f_n(a_n, b_n) = \max_{\substack{0 \leq w_n x_n \leq a_n \\ 0 \leq v_n x_n \leq b_n}} \{r_n x_n\}$$

$$f_j(a_j, b_j) = \max_{\substack{0 \leq w_j x_j \leq a_j \\ 0 \leq v_j x_j \leq b_j}} \{r_j x_j + f_{j+1}(a_j - w_j x_j, b_j - v_j x_j)\}$$

Order of computations

$f_n \rightarrow f_{n-1} \rightarrow \cdots \rightarrow f_1$

$a_1 = W$

$b_1 = V$

CHAPTER 11

Deterministic Inventory Models

Set 11.3a

1

$y^* = \sqrt{\frac{2KD}{h}}$, $t_0 = \frac{y^*}{D}$, $TCU(y^*) = \sqrt{2KDh}$

a) $y^* = \sqrt{\frac{2 \times 100 \times 30}{.05}} = 346.4$ units

$t_0 = \frac{346.4}{30} = 11.55$ days

$TCU(y^*) = \frac{100 \times 30}{346.4} + \frac{.05 \times 346.4}{2} = \$17.32$

Policy: order 346.4 units whenever inventory drops to 207.2 units

Effective lead time = 6.91 days

b) $y^* = \sqrt{\frac{2 \times 50 \times 30}{.05}} \cong 245$ units

$t_0^* = \frac{245}{30} = 8.16$ days

$L_e = 5.51$ days

$TCU(y^*) = \frac{50 \times 30}{245} + \frac{.05 \times 245}{2} = \$12.25$

Policy: order 245 units whenever inventory drops to 165.15 units

c) $y^* = \sqrt{\frac{2 \times 100 \times 40}{.01}} = 894.4$ units

$t_0 = \frac{894.4}{40} = 22.36$ days

$L_e = 7.64$ days

$TCU(y^*) = \frac{100 \times 40}{894.4} + \frac{.01 \times 894.4}{2} = \$8.94$

Policy: Order 894.4 units whenever inventory drops to 305.57 units.

d) $y^* = \sqrt{\frac{2 \times 100 \times 20}{.04}} = 316.23$ units

$t_0^* = \frac{316.23}{20} = 15.81$ days

$L_e = 14.19$ days

$TCU(y^*) = \frac{100 \times 20}{316.23} + \frac{.04 \times 316.23}{2} = \$12.65$

Policy: Order 316.23 units whenever inventory drops to 283.8 units.

2

$D = 300$ lb/wk, $K = \$20$, $h = \$.03$/lb/day

(a) $TC/wk = \frac{KD}{y} + \frac{hy}{2}$

$= \frac{20 \times 300}{300} + \frac{7 \times .03 \times 300}{2} = \$51.50$

(b) $y^* = \sqrt{\frac{2 \times 20 \times 300}{(.03 \times 7)}} = 239$ lb

$t_0^* = \frac{239}{300/7} = .8$ wk

$TC/wk = \sqrt{2 \times 20 \times 300 \times .03 \times 7}$

$= \$50.20$

$L_e = 0$ days

Policy: Order 239 lb whenever inventory drops to zero level.

c) Cost difference = 51.50 − 50.20

= \$1.30

3

a) $h = \frac{.35}{7} = \$.05$/unit/day

$D = 50$ units/day, $K = \$20$

$y^* = \sqrt{\frac{2 \times 20 \times 50}{.05}} = 200$ units

$t_0 = \frac{200}{50} = 4$ days

$L = 7$ days, $L_e = 3$ days

$R = 3 \times 50 = 150$ units

Policy: Order 200 units whenever inventory drops to 150 units.

b) Optimum number of orders $= \frac{365}{4}$

$\cong 91$ orders

4

(a) Policy 1: $D = \frac{R}{L_e} = \frac{50}{10} = 5$ units/day

$\text{Cost/day} = \frac{KD}{y} + \frac{hy}{2}$

$= \frac{20 \times 5}{150} + \frac{.02 \times 150}{2} = \$2.17$

Policy 2: $D = \frac{75}{15} = 5$ units/day

$\text{Cost/day} = \frac{20 \times 5}{200} + \frac{.02 \times 200}{2} = \$2.50$

choose policy 1.

(b) $K = \$20$, $D = 5$ units/day

$h = \$.02$, $L = 22$ days

$y^* = \sqrt{\frac{2 \times 20 \times 5}{.02}} = 100$ units

$t_0 = \frac{100}{5} = 20$ days

$L_e = 22 - 20 = 2$ days

Reorder level = $2 \times 5 = 10$ units

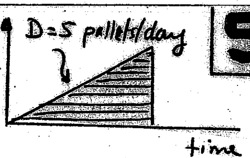
Order 100 units whenever the level drops to 10 units

$\text{Cost/day} = \frac{20 \times 5}{100} + \frac{.02 \times 100}{2} = \$2.00$

continued...

5

$D = 5$ units/day
$h = \$0.10/\text{day}$
$K = \$100$

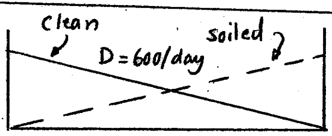


$$y^* = \sqrt{\frac{2 \times 5 \times 100}{.1}} = 100 \text{ pallets}$$

$$t_0 = \frac{100}{5} = 20 \text{ days}$$

Pick up 100 pallets every 20 days.

6



$$TC/day = \frac{K}{y/D} + \frac{h_1 y}{2} + \frac{h_2 y}{2} + .6D$$

$$= \frac{KD}{y} + (h_1 + h_2)\frac{y}{2} + .6D$$

$$y^* = \sqrt{\frac{2KD}{(h_1 + h_2)}} = \sqrt{\frac{2 \times 81 \times 600}{(.01 + .02)}} = 1800 \text{ towels}$$

$$t_0 = 1800/600 = 3 \text{ days}$$

$$Cost/day = \frac{81 \times 600}{1800} + \frac{.03 \times 1800}{2} = \$54$$

Optimal policy: Pick up soiled towels and deliver an equal batch of 1800 towels every 3 days

7

The basic assumption is that the employee will deposit sufficient funds in Europe to take advantage of the higher interest rate and periodically send lump sums to the US to take care of the obligations. This problem in the context of an application of the simple economic lot size formula with no shortages. The idea is that it may be more economical to hold funds longer in European banks to take advantage of their considerably higher interest rate. The cost of wiring funds from overseas (= \$50) may be regarded as the "setup" cost and the lost interest per dollar per year (=.065 - .015 = \$.05) can be treated as the "holding" cost. Using this information, the economic lot size formula will yield

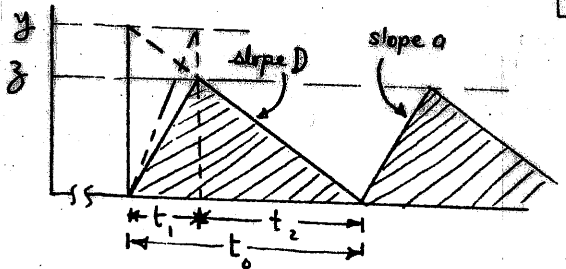
$$\text{Deposit amount} = \sqrt{\frac{2kD}{h}} = \sqrt{\frac{2 \times 50 \times 12000}{.05}} = \$4899$$

$$\text{Time between deposits} = t_0 = \frac{4899}{12000} = .408 \text{ year}$$

$$= 4.9 \text{ months}$$

Optimal policy: Send \$4899 (≈\$5000) every 4.9 (≈ 5) months to the US. The first installment occurs at the start of the year

8



a) From the geometry of the figure,

$$z = t_1(a - D) = \frac{y}{a}(a - D) = y(1 - \frac{D}{a})$$

b)
$$TCU(y) = \frac{K + (z/2)\, t_0 * h}{t_0}$$

$$= \frac{KD}{y} + \frac{h}{2}(1 - \frac{D}{a})y$$

(c) $\frac{\partial TCU(y)}{\partial y} = 0$ gives

$$-\frac{KD}{y^2} + \frac{h}{2}(1 - \frac{D}{a}) = 0$$

$$y^* = \sqrt{\frac{2KD}{h(1 - \frac{D}{a})}}$$

(d)
$$\lim_{a \to \infty} \sqrt{\frac{2KD}{h(1 - \frac{D}{a}}} = \sqrt{\frac{2KD}{h}}$$

9

Alternative 1: Produce

$$y^* = \sqrt{\frac{2KD}{h(1 - \frac{D}{a})}}$$

$$= \sqrt{\frac{2 \times 20 \times \frac{26000}{365}}{.02(1 - \frac{26000/365}{100})}} = 703.7 \text{ units}$$

Total cost/day

$$= \frac{KD}{y^*} + \frac{h}{2}(1 - \frac{D}{a})y^*$$

$$= \frac{200 \times \frac{2600}{365}}{703.7} + \frac{.02}{2}(1 - \frac{26000}{100 \times 365}) \times 703.7$$

$$= \$4.05 \text{ per day}$$

continued...

Set 11.3a

alternative 2: Buy

$$y^* = \sqrt{\frac{2KD}{h}}$$

$$= \sqrt{\frac{2 \times 15 \times \frac{26000}{365}}{.02}}$$

$$= 326.87 \text{ units}$$

Total cost/day

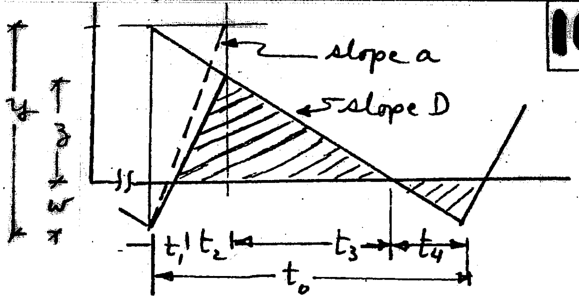
$$= \frac{KD}{y^*} + \frac{h}{2} y^*$$

$$= \frac{15 \times \frac{26000}{365}}{377.45} + \frac{.02}{2} \times 377.45$$

$$= \$6.54/\text{day}$$

The company should produce its own.

10



$$z = y(1 - \frac{D}{a}) - w$$

$$TCU(y,w) = \left[K + \frac{h\{y(1-\frac{D}{a}) - w\}^2 + pw^2}{2D(1-D/a)}\right] / t_0$$

$$= \frac{KD}{y} + \frac{h\{y(1-\frac{D}{a}) - w\}^2 + pw^2}{2y(1-D/a)}$$

Partial derivatives = 0 give

$$-\frac{KD}{y^2} + h\left(\frac{1}{2}(1-\frac{D}{a}) - \frac{w^2}{2y^2(1-D/a)}\right) - \frac{pw^2}{2y^2(1-\frac{D}{a})} = 0$$

$$h\left(\frac{w}{y(1-\frac{D}{a})} - 1\right) + \frac{pw}{y(1-D/a)} = 0$$

This gives,

$$y^* = \sqrt{\frac{2KD(p+h)}{ph(1-D/a)}}, \quad w^* = \sqrt{\frac{2KDh(1-\frac{D}{a})}{p(p+h)}}$$

1

EOQ before quantity dis-count = 1800 towels per Problem 6, Set 11.2a.

Total cost/day given batches of 1800 towels

$$= DC_1 + \frac{KD}{y} + \frac{h_1+h_2}{2} y$$

$$= 600 \times .6 + \frac{81 \times 600}{1800} + \frac{.03 \times 1800}{2} = \$414$$

Total cost/day given batches of 2500 towels

$$= DC_2 + \frac{KD}{y} + \frac{(h_1+h_2)}{2} y$$

$$= 600 \times .5 + \frac{81 \times 600}{2500} + \frac{.03 \times 2500}{2} = \$356.94$$

Take advantage of price discount.

2

$$y_m = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 100 \times 30}{.05}} = 346.41$$

$q = 500$ units

Because $y_m < q$, we need to compute Q.

$$TCU_1(y_m) = DC_1 + \frac{KD}{y_m} + \frac{h y_m}{2}$$

$$= 30 \times 10 + \frac{100 \times 30}{346.41} + \frac{.05 \times 346.41}{2}$$

$$= 317.32$$

The equation for computing Q is

$$Q^2 + \left(\frac{2(8 \times 30 - 317.32)}{.05}\right) Q + \frac{2 \times 100 \times 30}{.05} = 0$$

or

$$Q^2 - 3092.82Q + 120000 = 0$$

This yields $Q = 3053.52$ units.

Because $y_m < q < Q \Rightarrow y^* = q = 500$

$t_0 = \frac{500}{30} = 16.67$ days $\Rightarrow L_e = 4.33$

Order 500 units when inventory drop to 130.

3

$$y_m = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 50 \times 20}{.3}}$$

$$= 81.65 \text{ units}$$

Because $q > y_m$, we need to compute Q.

$$TCU_1(y_m) = 20 \times 25 + \frac{50 \times 20}{81.65} + \frac{.3 \times 81.65}{2}$$

$$= \$524.49$$

Q-equation:

$$Q^2 + \left(\frac{2(22.5 \times 20 - 524.49)}{.3}\right) Q + \frac{2 \times 50 \times 20}{.3} = 0$$

$$Q^2 - 496.63Q + 6666.67 = 0$$

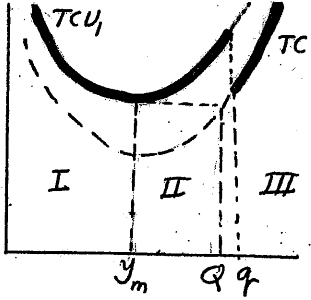
continued…

Thus, $Q = 482.83$

Because $y_m < q < Q \Rightarrow y^* = 150$

Order 150 units when inventory drops to 0

4



From the preceding figure, the discount is not advantageous if

$$TCU_1(y_m) \leq TCU_2(q)$$

or

$$DC_1 + \frac{KD}{y_m} + \frac{h y_m}{2} \leq DC_2 + \frac{KD}{q} + \frac{hq}{2}$$

or

$$20C_1 + \frac{50 \times 20}{81.65} + \frac{.3 \times 81.65}{2}$$

$$\leq 20C_2 + \frac{50 \times 20}{150} + \frac{.3 \times 150}{2}$$

Thus, the condition reduces to

$$C_1 - C_2 \leq .23359$$

Let d = discount factor (<1).

Then $C_2 = (1-d)C_1, \quad 0 < d < 1$

Given $C_1 = 25$, we have

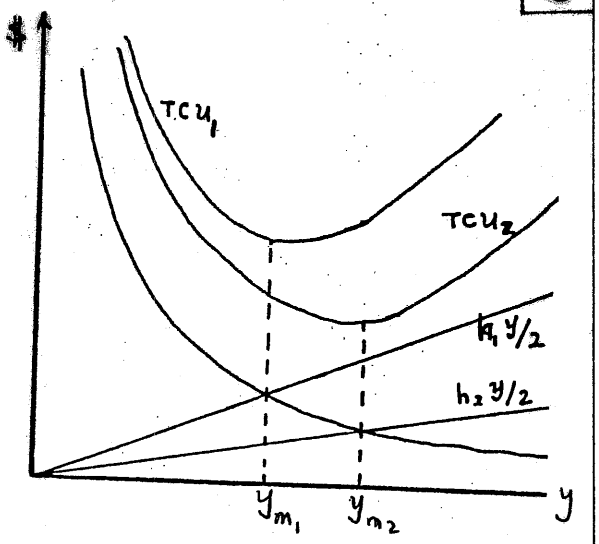
$$25d \leq .233588$$

or

$$d \leq .009344$$

Thus, no advantage if the % discount is $\leq .9344\%$ ($\simeq$ 1%)

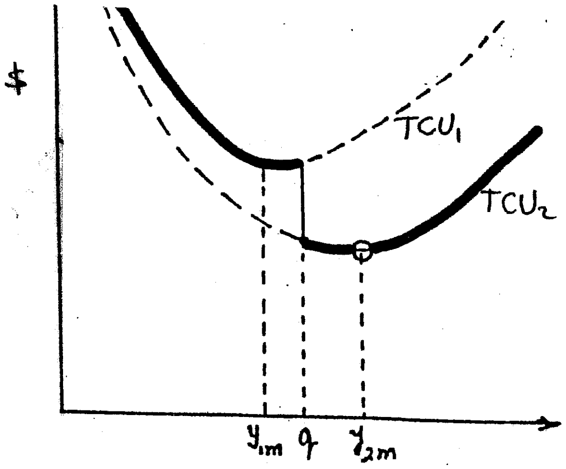
5



$$TCU_1(y) = \frac{KD}{y} + \frac{h_1 y}{2}$$

$$TCU_2(y) = \frac{KD}{y} + \frac{h_2 y}{2}$$

Case 1: $q < y_{2m}$



Solution:

$y^* = y_{2m}$

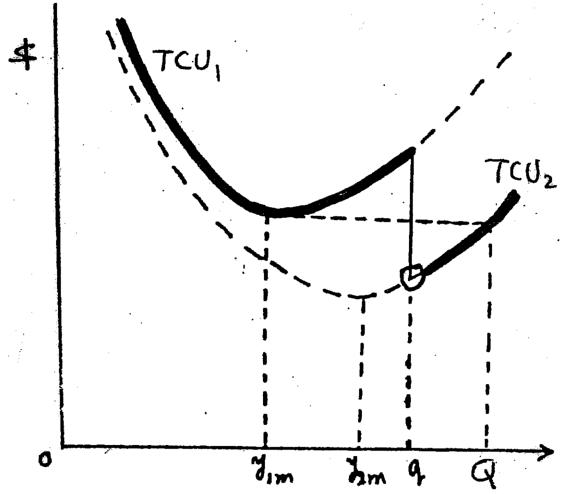
$TCU(y^*) = TCU_2(y_{2m})$

continued...

Case 2: $y_{2m} < q \le Q$

The value of Q is determined from the equation:

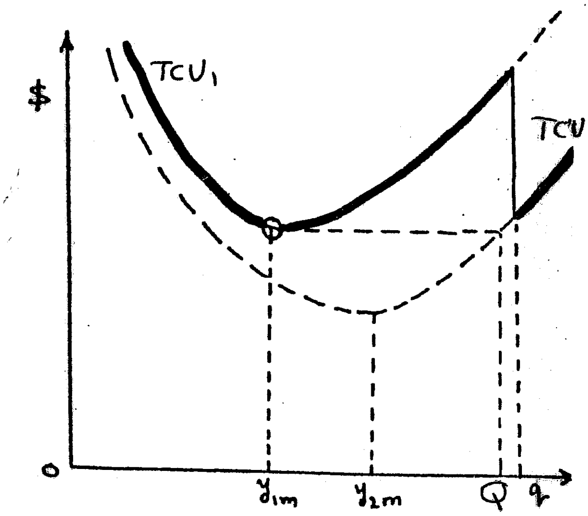
$$TCU_1(y_{1m}) = TCU_2(Q)$$



Solution: $y^* = q$

$TCU(y^*) = TCU_2(q)$

Case 3: $y_{2m} < Q < q$



Solution: $y^* = y_{1m}$, $TCU(y^*) = TCU_1(y_{1m})$

$$TCU(y^*) = \begin{cases} TCU_2(y_{2m}), & q < y_{2m} \\ TCU_2(q), & y_{2m} < q \le Q \\ TCU_1(y_{1m}), & y_{2m} < Q < q \end{cases}$$

1

See file ampl11.3c-1.txt.
AMPL model will not converge unless K_iD_i/y_i is replaced with $K_iD_i/(y_i+\varepsilon)$, where $\varepsilon>0$ and very small.

SOLUTION:
Total cost = 568.11
$y_1 = 4.42$
$y_2 = 6.87$
$y_3 = 4.12$
$y_4 = 7.20$
$y_5 = 5.80$

2

See file ampl11.3c-2.txt.
New constraint:
$(1/2)(y_1 + y_2 + y_3) \leq 25$

SOLUTION:
Total cost = 10.42
$y_1 = 10.83$
$y_2 = 16.85$
$y_3 = 22.32$

3

See file ampl11.3c-3.txt.
New constraint:
Average inventory for item i = $y_i/2$.
$(1/2)(100y_1 + 55y_2 + 100y_3) \leq 1000$

SOLUTION:
Total cost = 14.31
y1 = 5.58
y2 = 7.90
y3 = 10.07

4

See file ampl11.3c-4.txt.
AMPL model will not converge unless K_iD_i/y_i is replaced with $K_iD_i/(y_i+\varepsilon)$, where $\varepsilon>0$ and very small.

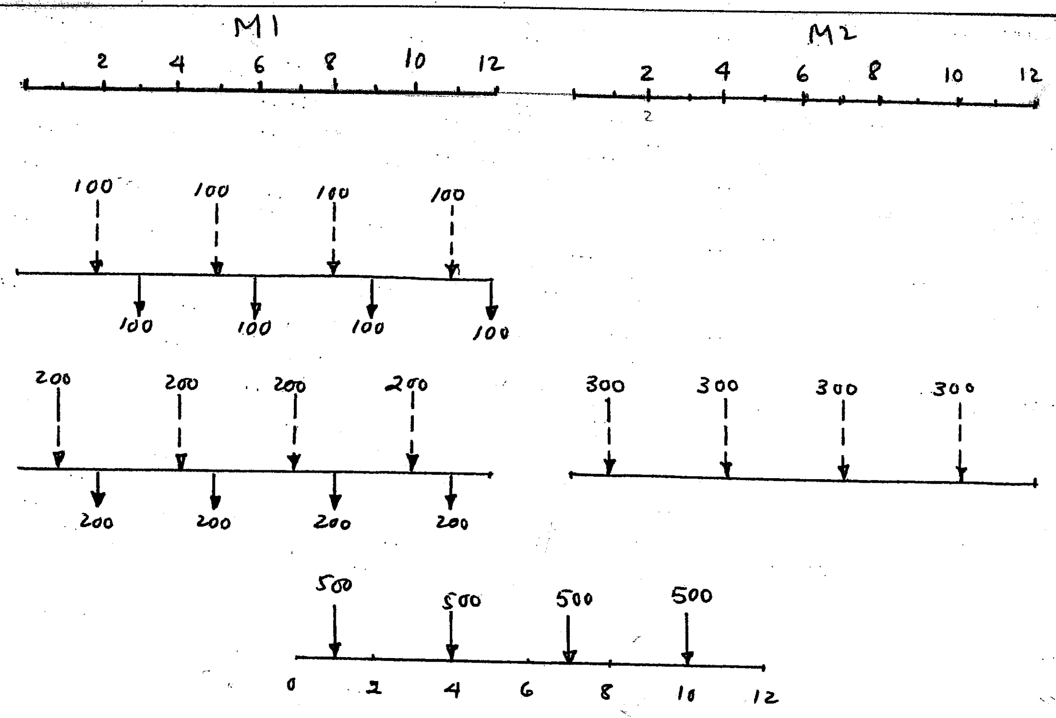
New constraint:
$365(10/y_1 + 20/y_2 + 5/y_3 + 10/y_4) \leq 150$

SOLUTION:
Total cost = 54.71
y1 = 155.30
y2 = 118.81
y3 = 74.36
y4 = 90.09

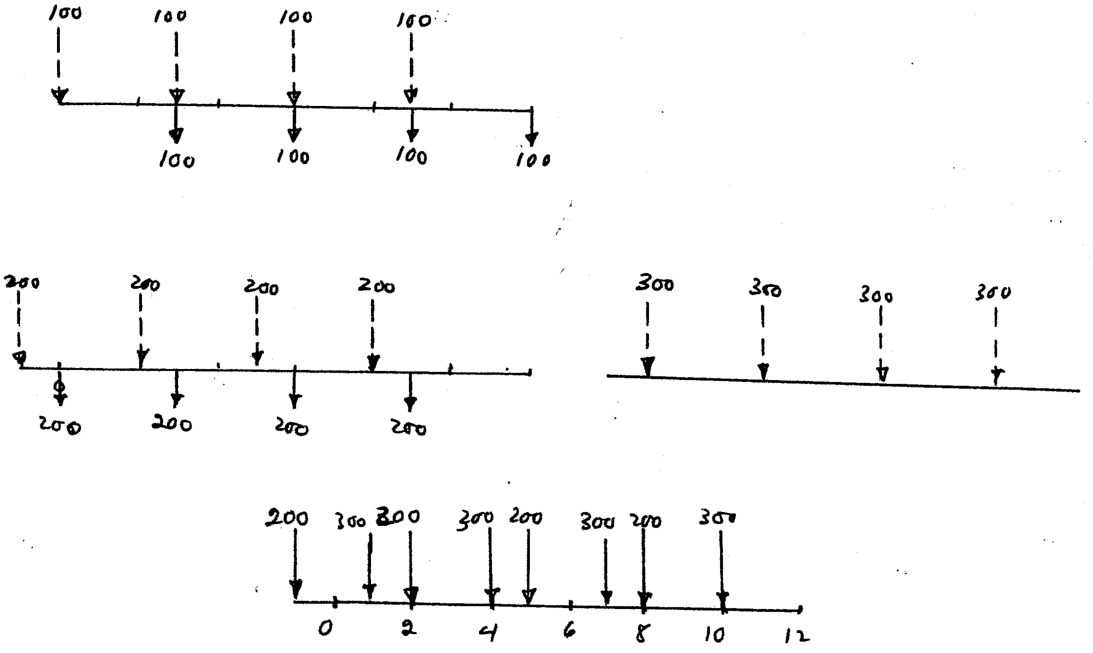
Set 11.4a

1

(a)


M1
2 4 6 8 10 12
M2
2 4 6 8 10 12
100 100 100 100
100 100 100 100
200 200 200 200
200 200 200 200
300 300 300 300
500 500 500 500
0 2 4 6 8 10 12

(b)


100 100 100 100
100 100 100 100
200 200 200 200
200 200 200 200
300 300 300 300
200 300 300 300 200 300 200 300
0 2 4 6 8 10 12

1

	1	2	3	4	surplus	
R_1	5 90	5.1	5.25	5.37	0	90
O_1	7.5 10	7.6 30	7.75 10	7.87	0	50
R_2	▨	3 100	3.15	3.27	0	100
O_2	▨	4.5 60	4.65	4.77	0	60
R_3	▨	▨	4 120	4.12	0	120
O_3	▨	▨	6 80	6.12	0	80
R_4	▨	▨	▨	1 110	0	110
O_4	▨	▨	▨	1.5 50	0 20	70
	100	190	210	160	20	

2

(a)

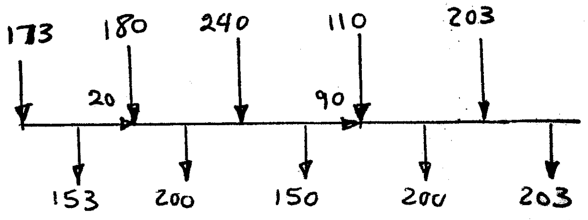
	1	2	3	4	Surplus	
I	11 1	1.3	1.65	1.85	0	11
II	2	1 2.3	3 2.65	2.85	0	4
III	5	5.3	5.65	5 ⑤ 5.85	5 ⓪ 0	10
I	☒	3 2	2.35	2.55	0	3
II	☒	4	11 4.35	1 4.55	0	12
III	☒	6	6.35	⑤ 6.55	10 ⑤ 0	10
I	☒	☒	3 2	2.2	0	3
II	☒	☒	5	8 5.2	0	8
III	☒	☒	7	7.2	4 0	4
IV	☒	☒	10	10.2	10 0	10
I	☒	☒	☒	3 3	0	3
II	☒	☒	☒	8 4	0	8
III	☒	☒	☒	4 5	0	4
IV	☒	☒	☒	7	10 0	10
	11	4	17	29	39	

(b) Additional 10 units are produced as shown by the circled entries in period 4. The problem has alternative solutions.

3

		1	2	3	4	5	Surplus	
1	R	100 4	4.5	5.	5.5	6.	0	100
	O	50 6	6.5	7	7.5	8.	0	50
	S	3 7	20 7.5	8	8.5	9	7 0	30
2	R	☒	40 4	4.5	5.	6	0	40
	O	☒	60 6	6.5	7.	7.5	0	60
	S	☒	80 7	7.5	8	8.5	0	80
3	R	☒	☒	90 4	4.5	5	0	90
	O	☒	☒	60 6	20 6.5	7	0	80
	S	☒	☒	7	70 7.5	8	0	70
4	R	☒	☒	☒	60 4	4.5	0	60
	O	☒	☒	☒	50 6	6.5	0	50
	S	☒	☒	☒	7	7.5	20 0	20
5	R	☒	☒	☒	☒	70 4	0	70
	O	☒	☒	☒	☒	50 6	0	50
	S	☒	☒	☒	☒	83 7	17 0	100
		153	200	150	200	203	44	

Solution summary



Demand

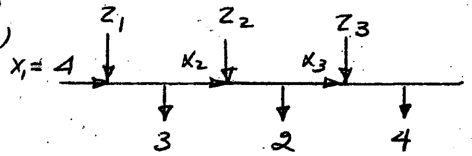
Set 11.4c

1

(a) No, because inventory should not be held needlessly at the end of planning horizon

(b)

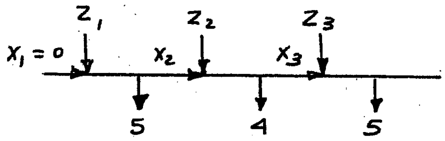
(i)



$0 \le z_1 \le 5$, $1 \le z_2 \le 5$, $0 \le z_3 \le 4$

$x_1 = 4$, $1 \le x_2 \le 6$, $0 \le x_3 \le 4$

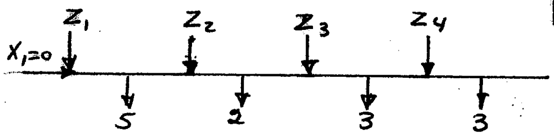
(ii)



$5 \le z_1 \le 14$, $0 \le z_2 \le 9$, $0 \le z_3 \le 5$

$x_1 = 0$, $0 \le x_2 \le 9$, $0 \le x_3 \le 5$

2



Stage 1: $f_1(x_2) = \min_{z_1 = D_1 + x_2} \{K_1 + C_1(z_1) + h_1 x_2\}$

where $C_i(z_i) = \begin{Bmatrix} 1z_i, & 0 \le z_i \le 6 \\ 2z_i, & z_i \ge 7 \end{Bmatrix}$ $i = 1, 2, \ldots, 4$

x_2 \ z_1	$K_1 = 5, h_1 = 1$									Opt. Sol	
	5	6	7	8	9	10	11	12	13	f_1	z_1
0	10									10	5
1		12								12	6
2			15							15	7
3				18						18	8
4					21					21	9
5						24				24	10
6							27			27	11
7								30		30	12
8									33	33	13

continued...

Stage 2:

$f_2(x_3) = \min_{0 \le z_2 \le D_2 + x_3} \{K_2 + C_2(z_2) + h_2 x_3 + f_1(x_3 + D_2 - z_2)\}$

$0 \le z_2 \le 8$, $0 \le x_3 \le 6$, $D_2 = 2$

x_3 \ z_2	$K_2 = 7, h_2 = 1$									Opt. Sol.	
	0	1	2	3	4	5	6	7	8	f_2	z_2
0	15	20	19							15	0
1	19	24	22	21						19	0
2	23	28	26	24	23					23	0,4
3	27	32	30	28	26	25				25	5
4	31	36	34	32	30	28	27			27	6
5	35	40	38	36	34	32	30	30		30	6
6	39	44	42	40	38	36	34	33	33	33	7,8

Stage 3: $0 \le z_3 \le 6$, $0 \le x_4 \le 3$, $D_3 = 3$

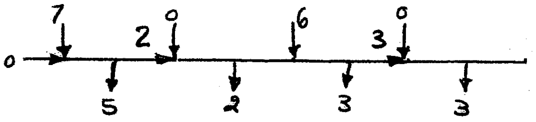
x_4 \ z_3	$K_3 = 9, h_3 = 1$							Opt. Sol.	
	0	1	2	3	4	5	6	f_3	z_3
0	25	33	30	27				25	0
1	28	36	35	32	29			28	0
2	32	39	38	37	34	31		31	5
3	36	43	41	40	39	36	33	33	6

Stage 4: $0 \le z_4 \le 3$, $x_5 = 0$, $D_4 = 3$

x_5 \ z_4	$K_4 = 7, h_4 = 1$				Opt. Sol.	
	0	1	2	3	f_4	z_4
0	33	39	37	35	33	0

Solution:

$(x_5 = 0) \rightarrow z_4 = 0 \rightarrow (x_4 = 3) \rightarrow z_3 = 6 \rightarrow (x_3 = 0) \rightarrow$
$z_2 = 0 \rightarrow (x_2 = 2) \rightarrow z_1 = 7$



Total cost = \$33

3

$$f_1(x_2) = \min_{0 \le z_1 \le D_1 + x_2} \left\{ C_1(z_1) + K_1 + h_1\left(\frac{x_1 + z_1 + x_2}{2}\right) \right\}$$

$$= \min_{0 \le z_1 \le D_1 + x_2} \left\{ K_1 + c_1(z_1) + h_1\left(x_2 + \frac{D_1}{2}\right) \right\}$$

$$f_i(x_{i+1}) = \min_{0 \le z_i \le D_i + x_{i+1}} \left\{ K_i + C_i(z_i) + h_i\left(x_{i+1} + \frac{D_i}{2}\right) + f_{i-1}(x_{i+1} + D_i - z_i) \right\}$$

4

$$f_n(x_n) = \min_{z_n + x_n = D_n} \{ k_n + c_n(z_n) \}$$

$$f_i(x_i) = \min_{D_i \le x_i + z_i \le D_1 + \cdots + D_n} \{ k_i + c_i(z_i) + h_i(x_i + z_i - D_i) + f_{i+1}(x_i + z_i - D_i) \}$$

Stage 3: $D_3 = 4$, $0 \le x_3 \le 4$

						Opt. Sol.	
x_3	$z_3 = 0$	1	2	3	4	f_3	z_3
0					56	56	4
1				36		36	3
2			26			26	2
3		16				16	1
4	0					0	0

Stage 2: $D_2 = 2$

								Opt. Sol.	
x_2	$z_2 = 0$	1	2	3	4	5	6	f_2	z_2
0			83	76	89	102	109	76	3
1		73	66	69	82	89		66	2
2	56	56	59	62	69			56	0,1
3	39	49	52	49				34	0
4	32	42	39					32	0
5	25	29						25	0
6	12							12	0

continued...

Stage 1: $D_1 = 3$

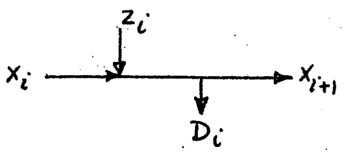
								Opt. Sol.	
x_1	$z_1 = 2$	3	4	5	6	7	8	f_1	z_1
1	99	100	111	115	129	193	151	99	2

Solution:

$(x_1 = 1) \rightarrow z_1 = 2 \rightarrow (x_2 = 0) \rightarrow z_2 = 3 \rightarrow$
$(x_3 = 1) \rightarrow z_3 = 3$

Cost = \$99

5



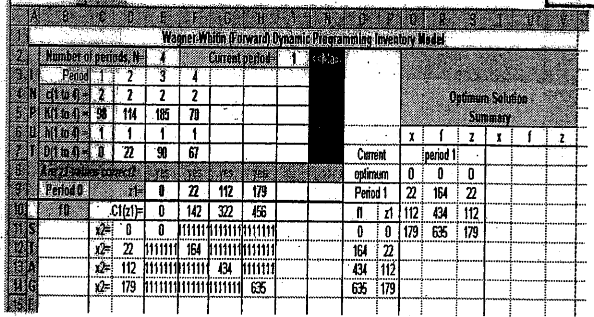
$$\text{Average inventory} = \frac{x_i + z_i + x_{i+1}}{2}$$
$$= \frac{x_i + z_i + x_i + z_i - D_i}{2}$$
$$= x_i + z_i - \frac{D_i}{2}$$

Replace $h_i(x_i + z_i - D_i)$ with $h_i(x_i + z_i - \frac{D_i}{2})$ in the backward formulation of problem 4.

Set 11.4d

1

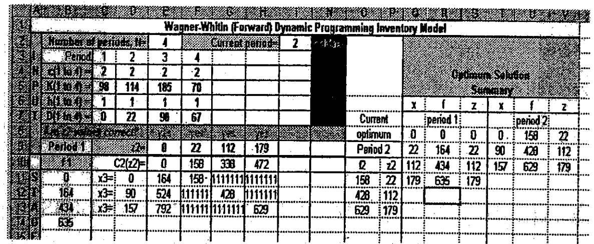
Period 1:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N=			4	Current period=		1								
Period	1	2	3	4										
c(1 to 4) =	2	2	2	2						Optimum Solution Summary				
K(1 to 4) =	98	114	185	70										
h(1 to 4) =	1	1	1	1					x	f	z	x	f	z
D(1 to 4) =	0	22	90	67			Current			period 1				
Are z1 values correct?			yes	yes	yes	yes	optimum		0	0	0			
Period 0		z1=	0	22	112	179	Period 1		22	164	22			
f0		C1(z1)=	0	142	322	456	f1	z1	112	434	112			
	x2=	0	0	1111111	1111111	1111111	0	0	179	635	179			
	x2=	22	1111111	164	1111111	1111111	164	22						
	x2=	112	1111111	1111111	434	1111111	434	112						
	x2=	179	1111111	1111111	1111111	635	635	179						

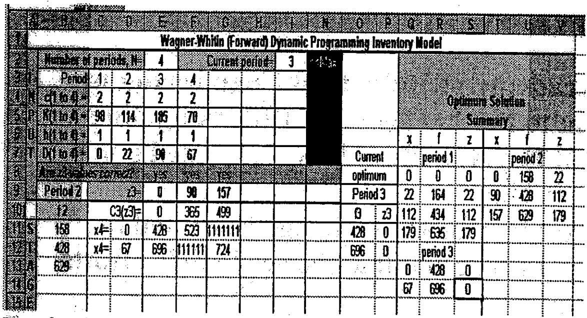
Period 2:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N=			4	Current period=		2								
Period	1	2	3	4										
c(1 to 4) =	2	2	2	2						Optimum Solution Summary				
K(1 to 4) =	98	114	185	70										
h(1 to 4) =	1	1	1	1					x	f	z	x	f	z
D(1 to 4) =	0	22	90	67			Current			period 1			period 2	
Are z2 values correct?			yes	yes	yes	yes	optimum		0	0	0	0	158	22
Period 1		z2=	0	22	112	179	Period 2		22	164	22	90	428	112
f1		C2(z2)=	0	158	338	472	f2	z2	112	434	112	157	629	179
0	x3=	0	164	158	1111111	1111111	158	22	179	635	179			
164	x3=	90	524	111111	428	1111111	428	112						
434	x3=	157	792	111111	1111111	629	629	179						
635														

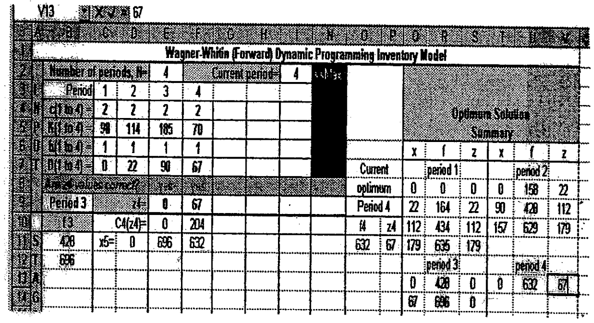
Period 3:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N=			4	Current period=		3								
Period	1	2	3	4										
c(1 to 4) =	2	2	2	2						Optimum Solution Summary				
K(1 to 4) =	98	114	185	70										
h(1 to 4) =	1	1	1	1					x	f	z	x	f	z
D(1 to 4) =	0	22	90	67			Current			period 1			period 2	
Are z3 values correct?			yes	yes	yes		optimum		0	0	0	0	158	22
Period 2		z3=	0	90	157		Period 3		22	164	22	90	428	112
f2		C3(z3)=	0	365	499		f3	z3	112	434	112	157	629	179
158	x4=	0	428	523	1111111		428	0	179	635	179			
428	x4=	67	696	111111	724		696	0		period 3				
629									0	428	0			
									67	696	0			

Period 4:



V13 = 67

Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N=			4	Current period=		4								
Period	1	2	3	4										
c(1 to 4) =	2	2	2	2						Optimum Solution Summary				
K(1 to 4) =	98	114	185	70										
h(1 to 4) =	1	1	1	1					x	f	z	x	f	z
D(1 to 4) =	0	22	90	67			Current			period 1			period 2	
Are z4 values correct?			yes	yes			optimum		0	0	0	0	158	22
Period 3		z4=	0	67			Period 4		22	164	22	90	428	112
f3		C4(z4)=	0	204			f4	z4	112	434	112	157	629	179
428	x5=	0	696	632			632	67	179	635	179			
696										period 3			period 4	
									0	428	0	0	632	67
									67	696	0			

Optimum solution:

Z4	X4	Z3	X3	Z2	X2	Z1
67	0	0	90	112	0	0

Cost = $632

2

Stage 1: $D_1 = 150$, $X_1 = 50$

											Opt. Sol.	
x_2	z_1=100	200	220	260	330	420	550	730	870	920	f_1	z_1^*
0	700										700	100
100		1400									1400	200
120			1540								1540	220
160				1820							1820	160
230					2310						2310	330
320						2940					2940	420
450							3850				3850	550
630								5110			5110	730
770									6090		6090	870
820										6940	6940	920

Stage 2: $D_2 = 100$

											Opt. Sol.	
x_3	z_2=0	100	120	160	230	320	450	630	770	820	f_2	z_2^*
0	1400	1400									1400	100
20	1560		1540								1540	120
60	1880			1820							1820	160
130	2440				2310						2310	230
220	3160					2940					2940	320
350	4200						3850				3850	450
530	5690							5110			5110	630
670	6760								6090		6090	770
720	7160									6940	6940	820

Stage 3: D = 20

										Opt. Sol.	
x_4	z_3=0	20	60	130	220	350	530	670	720	f_3	z_3
0	1540	1580								1540	0
90	1900		1820							1820	60
110	2530			2240						2240	130
200	3340				2780					2780	220
330	4510					3560				3560	350
510	6130						4640			4640	530
650	7390							5480		5480	670
700	7840								5780	5780	720

Stage 4: $D_4 = 40$

									Opt. Sol.	
x_5	z_4=0	40	110	200	330	510	650	750	f_4	z_4
0	1820	1900							1820	0
70	2310		2250						2250	110
160	2940			2700					2700	200
240	3850				3350				3350	330
470	5110					4250			4250	510
610	6090						4950		4950	650
660	6440							5200	5200	700

Stage 5: $D_5 = 70$

								Opt. Sol.	
x_6	z_5=0	70	160	290	470	610	660	f_5	z_5
0	2250	2440						2250	0
90	2880		3160					2880	0
220	3790			4200				3790	0
400	5050				5640			5050	0
540	6030					6760		6030	0
590	6380						7160	6380	0

continued...

Stage 6: $D_6 = 90$

							Opt. Sol.	
x_7	$z_6 = 0$	90	220	400	540	590	f_6	z_6
0	2880	3110					2880	0
130	4180		4600				4180	0
310	5980			6580			5980	0
450	7380				8120		7380	0
500	7880					8670	7880	0

Stage 7: $D_7 = 130$

						Opt. Sol.	
x_8	$z_7 = 0$	130	310	450	500	f_7	z_7
0	4180	3700				3700	130
180	6160		4600			4600	310
320	7700			5300		5300	450
370	8250				5550	5550	500

Stage 8: $D_8 = 180$

					Opt. Sol.	
x_9	$z_8 = 0$	180	320	370	f_8	z_8
0	4600	4720			4600	0
140	5860		5840		5840	320
190	6310			6240	6240	370

Stage 9: $D_9 = 140$

				Opt. Sol.	
	$z_9 = 0$	140	190	f_9	z_9
0	5840	5180		5180	140
50	6340		5380	5380	190

Stage 10: $D_{10} = 50$

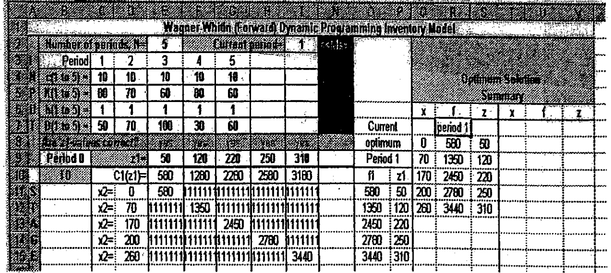
			Opt. Sol.	
x_{11}	$z_{10} = 0$	50	f_{10}	z_{10}
0	5380	5780	5380	0

Solution:

Period	Order amount
1	100
2	120
3	0
4	200
5	0
6	0
7	310
8	0
9	190
10	0

Minimum cost = $5380

Period 1:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N= 5 Current period= 1

Period	1	2	3	4	5
c(1 to 5) =	10	10	10	10	10
K(1 to 5) =	80	70	60	80	60
h(1 to 5) =	1	1	1	1	1
D(1 to 5) =	50	70	100	30	60

Period 0	z1=	50	120	220	250	310
10	C1(z1)=	580	1280	2280	2580	3180
	x2= 0	580	1111111	1111111	1111111	1111111
	x2= 70	1111111	1350	1111111	1111111	1111111
	x2= 170	1111111	1111111	2450	1111111	1111111
	x2= 200	1111111	1111111	1111111	2780	1111111
	x2= 260	1111111	1111111	1111111	1111111	3440

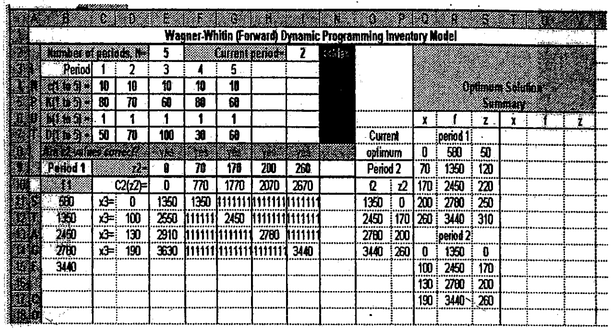
Current optimum Period 1

f1	z1
580	50
1350	120
2450	220
2780	250
3440	310

Optimum Solution Summary

x	f	z	x	f	z
	period 1				
0	580	50			
70	1350	120			
170	2450	220			
200	2780	250			
260	3440	310			

Period 2:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N= 5 Current period= 2

Period	1	2	3	4	5
c(1 to 5) =	10	10	10	10	10
K(1 to 5) =	80	70	60	80	60
h(1 to 5) =	1	1	1	1	1
D(1 to 5) =	50	70	100	30	60

Period 1	z2=	0	70	170	200	260
f1	C2(z2)=	0	770	1770	2070	2670
580	x3= 0	1350	1350	1111111	1111111	1111111
1350	x3= 100	2550	1111111	2450	1111111	1111111
2450	x3= 130	2910	1111111	1111111	2780	1111111
2780	x3= 190	3630	1111111	1111111	1111111	3440
3440						

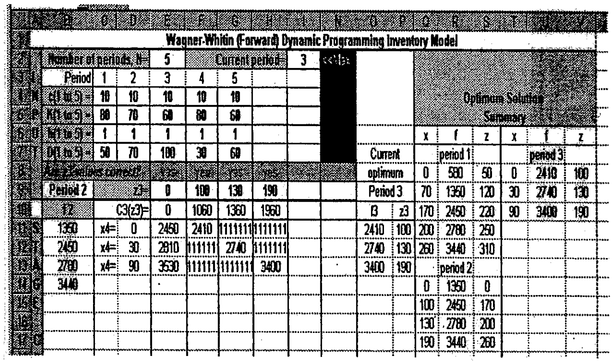
Current optimum Period 2

f2	z2
1350	0
2450	170
2780	200
3440	260

Optimum Solution Summary

x	f	z	x	f	z
	period 1				
0	580	50			
70	1350	120			
170	2450	220			
200	2780	250			
260	3440	310			
	period 2				
0	1350	0			
100	2450	170			
130	2780	200			
190	3440	260			

Period 3:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N= 5 Current period= 3

Period	1	2	3	4	5
c(1 to 5) =	10	10	10	10	10
K(1 to 5) =	80	70	60	80	60
h(1 to 5) =	1	1	1	1	1
D(1 to 5) =	50	70	100	30	60

Period 2	z3=	0	100	130	190
f2	C3(z3)=	0	1060	1360	1960
1350	x4= 0	2450	2410	1111111	1111111
2450	x4= 30	2810	1111111	2740	1111111
2780	x4= 90	3530	1111111	1111111	3400
3440					

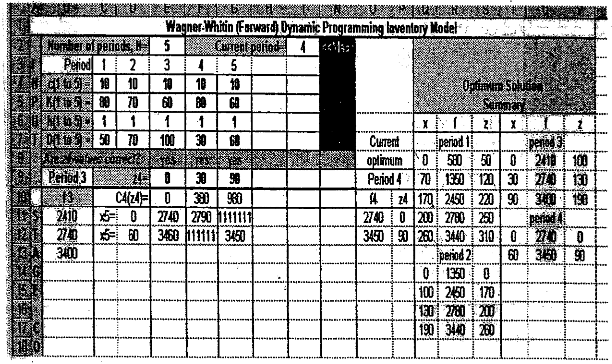
Current optimum Period 3

f3	z3
2410	100
2740	130
3400	190

Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	580	50	0	2410	100
70	1350	120	30	2740	130
170	2450	220	90	3400	190
200	2780	250			
260	3440	310			
	period 2				
0	1350	0			
100	2450	170			
130	2780	200			
190	3440	260			

Period 4:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N= 5 Current period= 4

Period	1	2	3	4	5
c(1 to 5) =	10	10	10	10	10
K(1 to 5) =	80	70	60	80	60
h(1 to 5) =	1	1	1	1	1
D(1 to 5) =	50	70	100	30	60

Period 3	z4=	0	30	90
f3	C4(z4)=	0	380	980
2410	x5= 0	2740	2790	1111111
2740	x5= 60	3460	1111111	3450
3400				

Current optimum Period 4

f4	z4
2740	0
3450	90

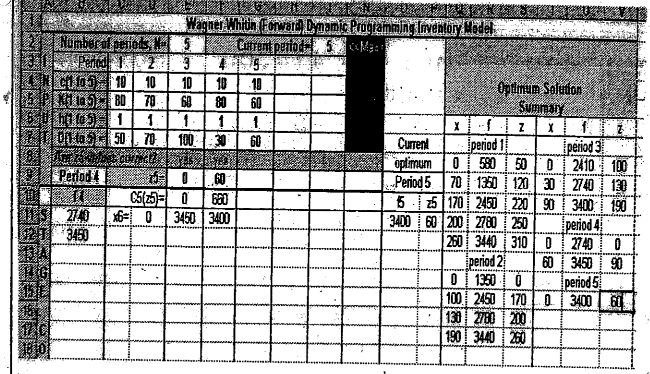
Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	580	50	0	2410	100
70	1350	120	30	2740	130
170	2450	220	90	3400	190
200	2780	250		period 4	
260	3440	310	0	2740	0
	period 2		60	3450	90
0	1350	0			
100	2450	170			
130	2780	200			
190	3440	260			

continued...

Set 11.4d

Period 5:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

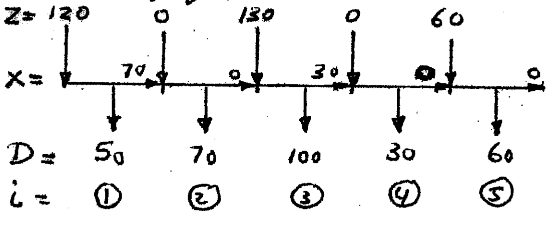
Number of periods, N=			5	Current period=			5	<<Max
Period	1	2	3	4	5			
c(1 to 5) =	10	10	10	10	10			
K(1 to 5) =	80	70	60	80	60			
h(1 to 5) =	1	1	1	1	1			
D(1 to 5) =	50	70	100	30	60			
Are z5-values correct?			yes	yes				
Period 4		z5=	0	60				
f4		C5(z5)=	0	660				
2740	x6=	0	3450	3400				
3450								

Current	
optimum	
Period 5	
f5	z5
3400	60

Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	580	50	0	2410	100
70	1350	120	30	2740	130
170	2450	220	90	3400	190
200	2780	250		period 4	
260	3440	310	0	2740	0
	period 2		60	3450	90
0	1350	0		period 5	
100	2450	170	0	3400	60
130	2780	200			
190	3440	260			

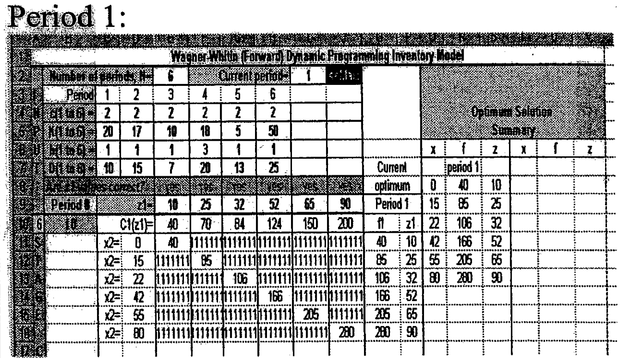
Summary of optimum solution:



Cost = $3400

4

Period 1:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

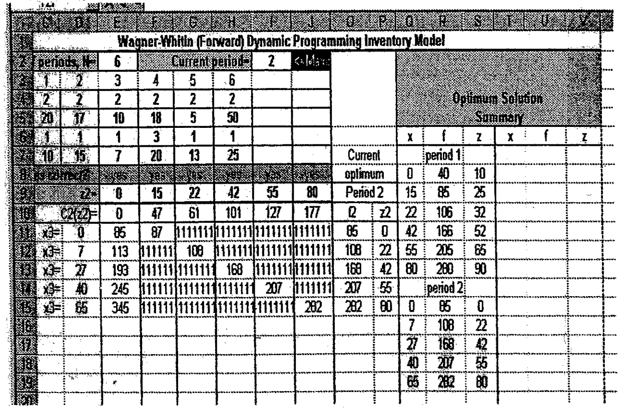
Number of periods, N=			6	Current period=			1	<<Max
Period	1	2	3	4	5	6		
c(1 to 6) =	2	2	2	2	2	2		
K(1 to 6) =	20	17	10	18	5	50		
h(1 to 6) =	1	1	1	3	1	1		
D(1 to 6) =	10	15	7	20	13	25		
Are z1-values correct?			yes	yes	yes	yes	yes	yes
Period 0		z1=	10	25	32	52	65	90
f0		C1(z1)=	40	70	84	124	150	200
	x2=	0	40	1111111	1111111	1111111	1111111	1111111
	x2=	15	1111111	85	1111111	1111111	1111111	1111111
	x2=	22	1111111	1111111	106	1111111	1111111	1111111
	x2=	42	1111111	1111111	1111111	166	1111111	1111111
	x2=	55	1111111	1111111	1111111	1111111	205	1111111
	x2=	80	1111111	1111111	1111111	1111111	1111111	280

Current	
optimum	
Period 1	
f1	z1
40	10
85	25
106	32
166	52
205	65
280	90

Optimum Solution Summary

x	f	z	x	f	z
	period 1				
0	40	10			
15	85	25			
22	106	32			
42	166	52			
55	205	65			
80	280	90			

Period 2



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

periods, N=		6	Current period=			2	<<Max
1	2	3	4	5	6		
2	2	2	2	2	2		
20	17	10	18	5	50		
1	1	1	3	1	1		
10	15	7	20	13	25		
es correct?		yes	yes	yes	yes	yes	yes
	z2=	0	15	22	42	55	80
	C2(z2)=	0	47	61	101	127	177
x3=	0	85	87	1111111	1111111	1111111	1111111
x3=	7	113	1111111	108	1111111	1111111	1111111
x3=	27	193	1111111	1111111	168	1111111	1111111
x3=	40	245	1111111	1111111	1111111	207	1111111
x3=	65	345	1111111	1111111	1111111	1111111	282

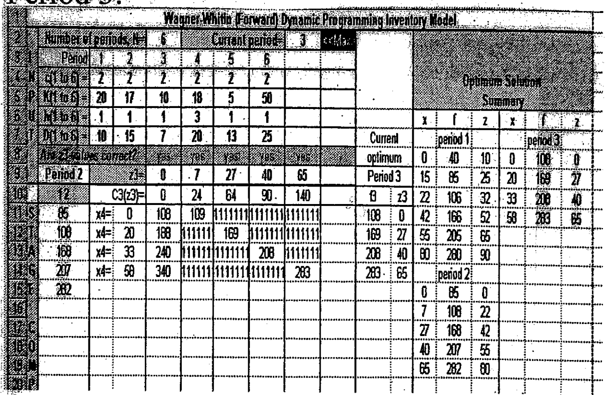
Current	
optimum	
Period 2	
f2	z2
85	0
108	22
168	42
207	55
282	80

Optimum Solution Summary

x	f	z	x	f	z
	period 1				
0	40	10			
15	85	25			
22	106	32			
42	166	52			
55	205	65			
80	280	90			
	period 2				
0	85	0			
7	108	22			
27	168	42			
40	207	55			
65	282	80			

continued...

Period 3:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

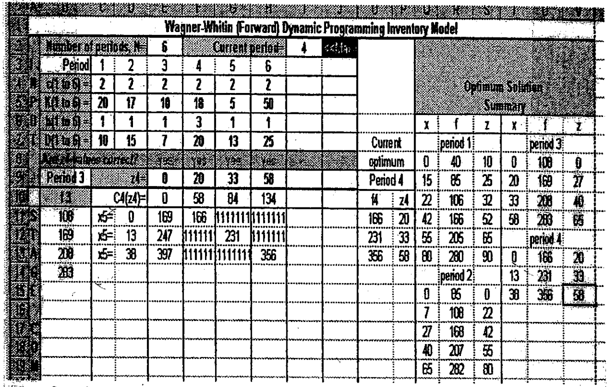
Number of periods, N=			6	Current period=			3	<<Max
Period	1	2	3	4	5	6		
c(1 to 6) =	2	2	2	2	2	2		
K(1 to 6) =	20	17	10	18	5	50		
h(1 to 6) =	1	1	1	3	1	1		
D(1 to 6) =	10	15	7	20	13	25		
Are z3-values correct?			yes	yes	yes	yes	yes	
Period 2		z3=	0	7	27	40	65	
f2		C3(z3)=	0	24	64	90	140	
85	x4=	0	108	109	1111111	1111111	1111111	
108	x4=	20	188	1111111	169	1111111	1111111	
168	x4=	33	240	1111111	1111111	208	1111111	
207	x4=	58	340	1111111	1111111	1111111	283	
282								

Current	
optimum	
Period 3	
f3	z3
108	0
169	27
208	40
283	65

Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	40	10	0	108	0
15	85	25	20	169	27
22	106	32	33	208	40
42	166	52	58	283	65
55	205	65			
80	280	90			
	period 2				
0	85	0			
7	108	22			
27	168	42			
40	207	55			
65	282	80			

Period 4:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

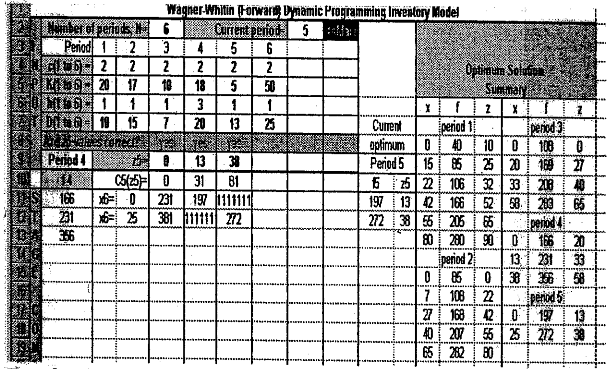
Number of periods, N=			6	Current period=			4	<<Max
Period	1	2	3	4	5	6		
c(1 to 6) =	2	2	2	2	2	2		
K(1 to 6) =	20	17	10	18	5	50		
h(1 to 6) =	1	1	1	3	1	1		
D(1 to 6) =	10	15	7	20	13	25		
Are z4-values correct?			yes	yes	yes	yes		
Period 3		z4=	0	20	33	58		
f3		C4(z4)=	0	58	84	134		
108	x5=	0	169	166	1111111	1111111		
169	x5=	13	247	1111111	231	1111111		
208	x5=	38	397	1111111	1111111	356		
283								

Current	
optimum	
Period 4	
f4	z4
166	20
231	33
356	58

Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	40	10	0	108	0
15	85	25	20	169	27
22	106	32	33	208	40
42	166	52	58	283	65
55	205	65		period 4	
80	280	90	0	166	20
	period 2		13	231	33
0	85	0	38	356	58
7	108	22			
27	168	42			
40	207	55			
65	282	80			

Period 5:



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

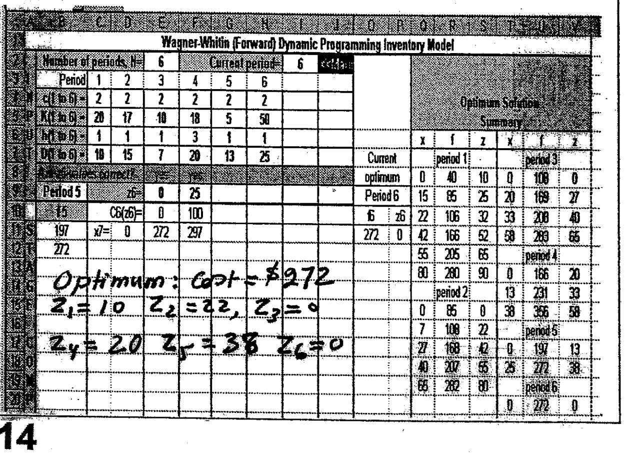
Number of periods, N=			6	Current period=			5	<<Max
Period	1	2	3	4	5	6		
c(1 to 6) =	2	2	2	2	2	2		
K(1 to 6) =	20	17	10	18	5	50		
h(1 to 6) =	1	1	1	3	1	1		
D(1 to 6) =	10	15	7	20	13	25		
Are z5-values correct?			yes	yes	yes			
Period 4		z5=	0	13	38			
f4		C5(z5)=	0	31	81			
166	x6=	0	231	197	1111111			
231	x6=	25	381	1111111	272			
356								

Current	
optimum	
Period 5	
f5	z5
197	13
272	38

Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	40	10	0	108	0
15	85	25	20	169	27
22	106	32	33	208	40
42	166	52	58	283	65
55	205	65		period 4	
80	280	90	0	166	20
	period 2		13	231	33
0	85	0	38	356	58
7	108	22		period 5	
27	168	42	0	197	13
40	207	55	25	272	38
65	282	80			

Period 6



Wagner-Whitin (Forward) Dynamic Programming Inventory Model

Number of periods, N=			6	Current period=			6	<<Max
Period	1	2	3	4	5	6		
c(1 to 6) =	2	2	2	2	2	2		
K(1 to 6) =	20	17	10	18	5	50		
h(1 to 6) =	1	1	1	3	1	1		
D(1 to 6) =	10	15	7	20	13	25		
Are z6-values correct?			yes	yes				
Period 5		z6=	0	25				
f5		C6(z6)=	0	100				
197	x7=	0	272	297				
272								

Current	
optimum	
Period 6	
f6	z6
272	0

Optimum Solution Summary

x	f	z	x	f	z
	period 1			period 3	
0	40	10	0	108	0
15	85	25	20	169	27
22	106	32	33	208	40
42	166	52	58	283	65
55	205	65		period 4	
80	280	90	0	166	20
	period 2		13	231	33
0	85	0	38	356	58
7	108	22		period 5	
27	168	42	0	197	13
40	207	55	25	272	38
65	282	80		period 6	
			0	272	0

Optimum: Cost = $272

$z_1 = 10$ $z_2 = 22$, $z_3 = 0$

$z_4 = 20$ $z_5 = 38$ $z_6 = 0$

1

$i=1,\ K_1=250$:

Period, t	D_t	TC(1, t)	TCU(1, t)
1	60	250	250/1 = 250
2	70	250+1×70 = 320	320/2 = $\underline{160}^*$
3	80	320+2×80 = 480	480/3 = $\underline{160}^*$
4	90	480+3×90 = 750	750/4 = 187.50

Produce 60+70+80 = 210 for 1, 2, and 3

$i=4,\ K_4=300$

Period, t	D_t	TC(4, t)	TCU(4, t)
4	90	300	300/1 = 300
5	85	300+85 = 385	385/2 = 192.5
6	80	385+2×80 = 545	545/3 = $\underline{181.67}$
7	75	545+3×75 = 770	770/4 = 192.5

Produce 90+85+80 = 255 for 4, 5, and 6

$i=7,\ K_7=\$250$:

Period t	D_t	TC(7, t)	TCU(7, t)
7	75	250	250/1 = 250
8	70	250+70 = 320	320/2 = 160
9	65	320+2×65 = 450	450/3 = $\underline{150}$
10	60	480+3×60 = 630	630/4 = 157.50

Produce 75+70+65 = 210 for 7, 8, and 9

$i=10,\ K_{10}=\$250$:

Period t	D_t	TC(10, t)	TCU(10, t)
10	60	250	250/1 = 250
11	55	250+1×55 = 305	305/2 = 152.50
12	50	305+2×50 = 405	405/3 = 135

Produce 60+55+50 = 165 for 10, 11, and 12

2

$i=1,\ K=200$:

t	D_t	TC(1, t)	TCU(1, t)
1	100	200	200/1 = 200
2	120	200+144 = 344	344/2 = 172
3	50	344+2×1.2×50 = 464	464/3 = $\underline{154.67}$
4	70	464+3×1.2×70 = 716	716/4 = 179

Continued...

$i=4,\ K=\$200$:

t	D_t	TC(4, t)	TCU(4, t)
4	70	200	200/1 = 200
5	90	200+1.2×90 = 308	308/2 = $\underline{154}$
6	105	308+2×1.2×105 = 560	560/3 = 186.67

$i=6,\ K=\$200$:

t	D_t	TC(6, t)	TCU(6, t)
6	105	200	200/1 = 200
7	115	200+1.2×115 = 338	338/2 = $\underline{169}$
8	95	338+2×1.2×95 = 566	566/3 = 188.67

$i=8,\ K=\$200$:

t	D_t	TC(8, t)	TCU(8, t)
8	95	200	200/1 = 200
9	80	200+1.2×80 = 296	296/2 = $\underline{148}$
10	85	296+2×1.2×85 = 500	500/3 = 166.67

$i=10,\ K=\$200$:

t	D_t	TC(10, t)	TCU(10, t)
10	85	200	200/1 = 200
11	100	200+1.2×100 = 320	320/2 = $\underline{160}$
12	110	320+2×1.2×110 = 584	548/3 = 194.67

Schedule:

Produce	For periods
270	1, 2, and 3
160	4, and 5
220	6 and 7
175	8 and 9
185	10 and 11
110	12

CHAPTER 12

Review of Probability Theory

Set 12.1a

1

	Eng'g	Non-Eng'g	Sum
Math	150	250	400
Non-math	29	571	600
Sum	179	821	Total=1000

(a) $P\{\text{Eng'g student had math}\} = \frac{150}{1000} = .15$

$P\{\text{Non-eng'g student had math}\} = \frac{250}{1000} = .25$

(b) $P\{\text{Non-eng'g had no math}\} = \frac{571}{1000} = .571$

(c) $P\{\text{Student is non-eng'g}\} = \frac{821}{1000} = .821$

2

Let

n = desired sample size

p_n = prob. n persons have distinct b'days

$$= \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{365-n+1}{365}$$

$1-p_n$ = prob at least two persons out of n have the same b'day

Thus,

$$1-p_n > 1/2$$

means $1-p_n$ is more likely to occur than p_n.

Now,

$$p_n < 1/2$$

or

$$\frac{(365)(364)\cdots(365-n+1)}{(365)^n} < 1/2$$

A spreadsheet solution yields $n \geq 23$

3

$P\{\text{no one shares \underline{your} b'day}\} = \frac{364}{365}$

$P\{\text{no one among } n \text{ persons shares \underline{your} b'day}\}$

$$= \left(\frac{364}{365}\right)^n$$

$P\{\text{at least one person among } n \text{ shares \underline{your} b'day}\}$

$$= 1 - \left(\frac{364}{365}\right)^n$$

Thus, for two or more persons to share *your* b'day with more than 50% chance means

$$1 - \left(\frac{364}{365}\right)^n > 1/2$$

or

$$n \ln\left(\frac{364}{365}\right) < \ln(1/2)$$

or

$$n > \frac{\ln(1/2)}{\ln\left(\frac{364}{365}\right)} \simeq 253$$

The direction of the inequality has been reversed because $\ln x < 0$ for $0 < x < 1$

Set 12.1b

1

E = outcome of first toss

F = outcome of second toss

(a) Sum = 11:

(E&F) = (5&6) or (6&5)

$P\{sum=11\} = 2\left(\frac{1}{6}\times\frac{1}{6}\right) = 1/18$

(b) Sum = even value

(E&F) = (1&[1 or 3 or 5]) or
(2&[2 or 4 or 6]) or
(3&[1 or 3 or 5]) or
(4&[2 or 4 or 6]) or
(5&[1 or 3 or 5]) or
(6&[2 or 4 or 6])

$P\{E\&F\} = 6\times\frac{1}{6}\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right) = 1/2$

(c) Sum = odd value > 3

(E&F) = (1&[4 or 6]) or
(2&[3 or 5]) or
(3&[2 or 4 or 6]) or
(4&[1 or 3 or 5]) or
(5&[2 or 4 or 6]) or
(6&[1 or 3 or 5])

$P\{E\&F\} = 2\times\frac{1}{6}\left(\frac{1}{6}+\frac{1}{6}\right)+4\times\frac{1}{6}\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right) = \frac{4}{9}$

(d) $P\{(2 \text{ or } 4)\&(3 \text{ or } 5)\} = \left(2\times\frac{1}{6}\right)^2 = \frac{1}{9}$

(e) (E&F) = (3&[1 or 2 or 3]) or
(4&[1 or 2 or 3]) or
(5&[1 or 2 or 3]) or
(6&[1 or 2 or 3])

$P\{E\&F\} = 4\times\frac{1}{6}\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right) = \frac{1}{3}$

(f) $P\{4\&[1 \text{ or } 3 \text{ or } 5]) = \frac{1}{6}\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right) = \frac{1}{12}$

2

(a) $\left(P\{2,4,\text{ or } 6\}\right)^2 = (1/2)^2 = 1/4$

(b) $P\{4\&6\}+P\{5\&5\}+P\{6\&4\}$

$= 3\times\left(\frac{1}{6}\times\frac{1}{6}\right)$

$= \frac{1}{12}$

(c) $P\{1\&4\}+P\{1\&5\}+P\{1\&6\}+$
$+P\{2\&5\}+P\{2\&6\}+P\{3\&6\}$
$+P\{4\&1\}+P\{5\&1\}+P\{6\&1\}$
$+P\{5\&2\}+P\{6\&2\}+P\{6\&3\}$

$= 12\times\frac{1}{6}\times\frac{1}{6} = \frac{12}{36} = \frac{1}{3}$

3

Outcome	Probability
TTTH	$(1/2)^4$
HTTTH	$(1/2)^5$
HHTTTH THTTTH	$2\times(1/2)^6$
HTHTTTH THHTTTH TTHTTTH HHHTTTH	$4\times\left(\frac{1}{2}\right)^7$

$$\text{Probability} = \left(\frac{1}{2}\right)^4\left[1+\frac{1}{2}+2\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3\right]$$

$$= \frac{5}{32}$$

4

p = probability Liz wins

we have

P{Liz, Jim, John, or Ann wins}

$= p + 3p + 3p + 6p = 1$

Thus, $p = \frac{1}{13}$

(a) $P\{\text{Jim wins}\} = 3p = \frac{3}{13}$

(b) $P\{\text{Liz or Ann wins}\} = p + 6p$

$= \frac{7}{13}$

(c) P{no woman wins}

$= 1 - \frac{7}{13} = \frac{6}{13}$

Set 12.1c

1

(a) $E = (2 \text{ or } 4)$

$F = (1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5)$

$$P\{E|F\} = \frac{P\{EF\}}{P\{F\}} = \frac{P\{E\}}{P\{F\}} = \frac{2/6}{5/6} = 2/5$$

(b) $E = (3 \text{ or } 5)$

$F = (1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5)$

$$P\{E|F\} = \frac{P\{EF\}}{P\{F\}} = \frac{P\{E\}}{P\{F\}} = \frac{2/6}{5/6} = 2/5$$

2

Joint probabilities:

	WMS up	WMS down	Col. Sum
Dow up	.6	.1	.7
Dow down	.05	.25	.3
Row sum	.65	.35	

(a) $P\{\text{WMS up}\} = .6 + .05 = .65$

(b) $P\{\text{WMS up} \mid \text{Dow up}\} = \frac{.6}{.7} = 6/7$

(c) $P\{\text{WMS down} \mid \text{Dow down}\} = \frac{.25}{.3} = 5/6$

3

$P\{A\} = .4 \quad P\{B\} = .25 \quad P\{AB\} = .15$

(a) $P\{B|A\} = \frac{P\{BA\}}{P\{A\}} = \frac{.15}{.4} = 3/8$

(b) $P\{A|B\} = \frac{P\{AB\}}{P\{B\}} = \frac{.15}{.25} = 3/5$

4

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

If $\frac{P\{AB\}}{P\{B\}} = P\{A\}$ then

$P\{AB\} = P\{A\}P\{B\}$, which shows that A and B must be independent.

5

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

$$= \frac{P\{B|A\}P\{A\}}{P\{B\}}$$

provided $P\{B\} > 0$.

6

(a) $P\{D\} = P\{D, A\} + P\{D, B\}$

$= P\{D|A\}P\{A\} + P\{D|B\}P\{B\}$

$= .1 \times .75 + .2 \times .25$

$= .125$

(b) $$P\{A|D\} = \frac{P\{D|A\}P\{A\}}{P\{D\}}$$

$$= \frac{.1 \times .75}{.125} = .6$$

7

$C \equiv$ cancer

$NC \equiv$ no cancer

$+ \equiv$ test positive

$$P\{C|+\} = \frac{P\{C, +\}}{P\{+\}}$$

$P\{+\} = P\{+, C\} + P\{+, NC\}$

$= P\{+|C\}P\{C\} + P\{+|NC\}P\{NC\}$

$= .9 \times .7 + .1 \times .3$

$= .66$

Thus,

$$P\{C|+\} = \frac{P\{C,+\}}{P\{+\}} = \frac{P\{+|C\}P\{C\}}{P\{+\}}$$

$$= \frac{.9 \times .7}{.66}$$

$$\simeq .9545$$

1

(a) $p(x) = kx$, $x = 1, 2, 3, 4, 5$

$$\sum_{x=1}^{5} p(x) = k(1+2+3+4+5) = 15k = 1$$

Thus, $k = 1/15$, and

$$p(x) = \frac{x}{15}, \quad x = 1, 2, \cdots, 5$$

CDF:

$$P(x) = \sum_{y=1}^{x} \frac{y}{15} = \frac{x(x+1)}{30}, \quad x = 1, 2, \cdots, 5$$

(b) $P\{x = 2 \text{ or } x = 4\} = \frac{2+4}{15} = \frac{2}{5}$

2

(a) $\int_{10}^{20} \frac{k}{x^2} = 1$

$$k\left(\frac{1}{10} - \frac{1}{20}\right) = \frac{k}{20} = 1 \Rightarrow k = 20$$

$$f(x) = \frac{20}{x^2}, \quad 10 \leq x \leq 20$$

(b) $F(x) = \int_{10}^{x} \frac{20}{t^2} dt$

$$= 2 - \frac{20}{x}$$

(i) $P\{x > 12\} = P\{x \geq 12\}$

$$= 1 - \left(2 - \frac{20}{12}\right)$$

$$= \frac{2}{3}$$

(ii) $P\{13 \leq x \leq 15\}$

$$= P\{x \leq 15\} - P\{x \leq 13\}$$

$$= 2 - \frac{20}{15} - \left(2 - \frac{20}{13}\right)$$

$$= .205$$

3

$$P\{Demand = d\} = \frac{1}{500}, \quad 750 \leq d \leq 1250$$

$$P\{d \geq 1100\} = 1 - P\{d \leq 1100\}$$

$$= 1 - \frac{1100 - 750}{500}$$

$$= .3$$

Set 12.3a

1

$$h(x)\begin{cases} x-20, & x=21,22,23,24 \\ 0, & x=10,11,\ldots,20 \end{cases}$$

$$E\{h(x)\} = \sum_{x=10}^{20} 0\left(\frac{1}{15}\right) + \sum_{x=21}^{24}(x-20)\left(\frac{1}{15}\right)$$

$$= \frac{2}{3} \text{ stamp}$$

2

There is no inconsistency because the two cases are mutually exclusive. There can be either surplus or shortage. When surplus occurs, its average value is $3\frac{2}{3}$ stamps. And when shortage occurs, its average value is $\frac{2}{3}$ stamp.

3

(a) $P\{50 \le x \le 70\}$

$$= 1 - P\{35 \le x \le 49\}$$

$$= 1 - \frac{15}{45} = \frac{2}{3}$$

(b) Expected number of unsold copies

$$= \sum_{x=35}^{49}(50-x)p(x) + \sum_{x=50}^{70} 0\,p(x)$$

$$= 50\sum_{x=35}^{49} p(x) - \sum_{x=35}^{49} x\,p(x)$$

$$= 50 \times \frac{15}{45} - \frac{1}{45}(35+\cdots+49)$$

$$= \frac{1}{45}(750-630) = 2.67$$

(c) Expected net profit

$$= (50-2.67)\times 1 - 50 \times .5$$

$$= \$22.33$$

Set 12.3b

1

x:	1	2	3	4	5
$p(x)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$$E\{x\} = \sum_{x=1}^{5} x\,p(x)$$

$$= 1\left(\tfrac{1}{15}\right) + 2\left(\tfrac{2}{15}\right) + 3\left(\tfrac{3}{15}\right) + 4\left(\tfrac{4}{15}\right) + 5\left(\tfrac{5}{15}\right)$$

$$= 3\tfrac{2}{3}$$

$$Var\{x\} = \sum_{x=1}^{5}\left(x - \tfrac{11}{3}\right)^2 p(x)$$

$$= \left(1-\tfrac{11}{3}\right)^2\left(\tfrac{1}{15}\right) + \left(2-\tfrac{11}{3}\right)^2\left(\tfrac{2}{15}\right) + \left(3-\tfrac{11}{3}\right)^2\left(\tfrac{3}{15}\right) + \left(4-\tfrac{11}{3}\right)^2\left(\tfrac{4}{15}\right) + \left(5-\tfrac{11}{3}\right)^2\left(\tfrac{5}{15}\right)$$

$$\simeq 1.556$$

2

$$E\{x\} = \int_{10}^{20} \frac{20x}{x^2}\,dx$$

$$= \left(\ln x \Big|_{10}^{20}\right)(20) = 13.86$$

$$Var\{x\} = 20\int_{10}^{20} \frac{(x-13.86)^2}{x^2}\,dx$$

$$= 20\left[x - 27.72\ln x - \frac{192.10}{x}\right]_{10}^{20}$$

$$= 7.81$$

3

(a) $f(x) = \frac{1}{b-a}, \quad a \le x \le b$

$$E\{x\} = \int_a^b \frac{x}{b-a}\,dx = \left.\frac{x^2}{2(b-a)}\right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

(b) $\int_a^b \frac{(x-\bar{x})^2}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3} - \bar{x}x^2 + x\bar{x}^2\right]_a^b$

$$= \frac{4b^2 + 4a^2 + 4ab - 3b^2 - 3a^2 - 6ab}{12}$$

$$= \frac{(b-a)^2}{12}$$

4

$$Var\{x\} = \int_{-\infty}^{\infty} \{x - E(x)\}^2 dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2E\{x\}\int_{-\infty}^{\infty} x\,f(x)\,dx + (E\{x\})^2\int_{-\infty}^{\infty} f(x)\,dx$$

$$= E\{x^2\} - 2(E\{x\})^2 - (E\{x\})^2$$

$$= E\{x^2\} - (E\{x\})^2$$

5

$$y = cx + d$$

$$E\{y\} = \int (cx+d)\,f(x)\,dx$$

$$= c\int x f(x)\,dx + d\int f(x)\,dx$$

$$= cE\{x\} + d$$

$$Var\{y\} = E\{(cx+d)^2\} - E^2\{cx+d\}$$

$$= E\{c^2x^2 + d^2 + 2cdx\} - [cE\{x\} + d]^2$$

$$= c^2E\{x^2\} + d^2 + 2cd\,E\{x\} - c^2E^2\{x\} - d^2 - 2cd\,E\{x\}$$

$$= c^2\left(E\{x^2\} - E^2\{x\}\right)$$

$$= c^2\ Var\{x\}$$

Set 12.3c

1

(a)

$$p(x_1,x_2) = \begin{array}{c|ccc|c} & 1 & 2 & 3 & P(X_1) \\ \hline 1 & .2 & 0 & .2 & .4 \\ 2 & 0 & .2 & 0 & .2 \\ 3 & .2 & 0 & .2 & .4 \\ \hline P(X_2) & .4 & .2 & .4 & \end{array}$$

X_1	1	2	3
$p(X_1)$	.4	.2	.4

X_2	1	2	3
$p(X_2)$	.4	.2	.4

(b) No, because $p(X_1,X_2) \neq p(X_1)p(X_2)$

(c) $E\{X_1 + X_2\} = E\{X_1\} + E\{X_2\}$

$= 2(1\times.4 + 2\times.2 + 3\times.4)$

$= 4$

(d) $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)\,E(X_2)$

$E(X_1X_2) = 1\times.2 + 2\times0 + 3\times.2 + 2\times0$
$+ 4\times.2 + 6\times0 + 3\times.2 + 6\times0$
$+ 3\times.2 + 6\times0 + 9\times.2$
$= 4.6$

$E\{X_1\} = 2$, $E\{X_2\} = 2$

$Cov(X_1, X_2) = 4.6 - 2\times2 = .6$

(e) $Var\{5X_1 - 6X_2\} = 25\,Var\{X_1\} + 36\,Var\{X_2\}$

$Var\{X_1\} = Var\{X_2\} = E\{X_1^2\} - E^2\{X_1\}$

$= 1\times.4 + 4\times.2 + 9\times.4 - 2^2$

$= .8$

$Var\{5X_1 - 6X_2\} = 25(.8) + 36(.8)$
$+ 2(5)(-6)(.6)$

$= 12.8$

1

$P\{\text{an even number in one throw}\}$

$$= P\{2, 4, \text{ or } 6\}$$

$$= 3\left(\frac{1}{6}\right) = \frac{1}{2}$$

$P\{0 \text{ even number in } 10 \text{ throws}\}$

$$= C_0^{10}\,(1/2)^0(1/2)^{10} = (1/2)^{10}$$

2

$$\text{Probability} = P\{\text{One head in 5 throws}\} + P\{\text{one tail in 5 throws}\}$$

$$= 2\;C_1^5\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4$$

$$= \frac{5}{16}$$

3

Being a fluke is equivalent to a 50-50 chance of being correct.

$$P\{a\text{ fluke}\} = C_8^{10}(1/2)^8\left(\frac{1}{2}\right)^2 + C_9^{10}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + C_{10}^{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0$$

$$= \left(\frac{1}{2}\right)^{10}\left[45+10+1\right]$$

$$= .0547$$

4

Probability of a single match

$$= 6\times\left(\frac{1}{6}\times\frac{1}{6}\right) = \frac{1}{6}$$

$P\{i \text{ matches out of 3 dice}\}$

$$= C_i^3\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{3-i},\quad i=0,1,2,3$$

i	0	1	2	3
P	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

$$\text{Expected payoff} = -1\left(\frac{125}{216}\right) + 1\left(\frac{75}{216}\right) + 2\left(\frac{15}{216}\right) + 3\left(\frac{1}{216}\right) \cong -.08 = -8\text{ cents}$$

5

$$\text{Prob. of a match} = \frac{1}{6}$$

$$\text{Prob. of no match} = \frac{5}{6}$$

$$\text{Expected payoff} = 50\left(\frac{1}{6}\right) - 10\left(\frac{5}{6}\right) = 0$$

6

$$E\{k\} = \sum_{k=}^{n} k\binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=}^{n} k\,\frac{n!}{k!\,(n-k)!}\,p^k q^{n-k}$$

$$= np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}\,p^{k-1}q^{n-k}$$

$$= np\left(\sum_{j=0}^{n-1}\frac{(n-1)!}{j!\,(n-1-j)!}\,p^j q^{n-1-j}\right)$$

$$\text{Var}\{k\} = E\{k^2\} - E^2\{k\}$$

$$E\{k^2\} = \sum_{k=1}^{n} k^2\binom{n}{k}p^k q^{n-k}$$

$$= np\sum_{k=1}^{n-1} k\,\frac{(n-1)!}{(k-1)!(n-k)!}\,p^{k-1}q^{n-k}$$

$$= np\sum_{k=0}^{n-1}(k+1)\,\frac{(n-1)!}{k!\,(n-k-1)!}\,p^k q^{n-1-j}$$

$$= np\big((n-1)p + 1\big)$$

$$= np(np+q)$$

$$\text{Var}\{k\} = np(np+q) - (np)^2$$

$$= npq$$

Set 12.4b

1

$$P\{n \geq 1 | t = 30 \text{ sec}\}$$

$$= \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!}$$

$$= 1 - e^{-\lambda t}$$

$$= 1 - e^{-4 \times .5} = 1 - e^{-2} = .8646$$

2

Case 1: $p = .1$

Binomial:

$$P\{0 \text{ or } 1 \text{ defective}\}$$

$$= C_0^{10}(.01)^0(.99)^{10} + C_1^{10}(.01)^1(.99)^9$$

$$= .99^{10} + 10 \times .01 \times .99^9 = .9957$$

Poisson:

$$\lambda = np = 10 \times .01 = .1$$

$$p_0 + p_1 = \frac{.1^0 e^{-.1}}{0!} + \frac{.1^1 e^{-.1}}{1!}$$

$$= e^{-.1}(1 + .1) \simeq .9953$$

Case 2: $p = .5$

Binomial:

$$P\{0 \text{ or } 1 \text{ defective}\}$$

$$= C_0^{10}(.5)(.5)^{10} + C_1^{10}(.5)^1(.5)^9$$

$$= .5^{10} + 10 \times .5^{10} = .01074$$

Poisson:

$$\lambda = 10 \times .5 = 5$$

$$p_0 + p_1 = \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!}$$

$$= .04043$$

3

$\lambda = 20$ customers/hr

(a) $$p_0 = \frac{20^0 e^{-20}}{0!} \cong 0$$

(b) $$p_{n \geq 3} = 1 - p_0 - p_1 - p_2$$

$$= 1 - \frac{20^0 e^{-20}}{0!} - \frac{20^1 e^{-20}}{1!} - \frac{20^2 e^{-20}}{2!} \simeq 1$$

Note:

$n \geq 3 \Rightarrow$ (1 in service and at least 2 waiting)

4

$$E\{x\} = \sum_{x=1}^{\infty} x \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$= \sum_{x=1}^{\infty} (\lambda t) \frac{(\lambda t)^{x-1}}{(x-1)!} e^{-\lambda t}$$

$$= (\lambda t) \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

$$= \lambda t$$

$$Var\{x\} = E\{x^2\} - E^2\{x\}$$

$$E\{x^2\} = \sum_{x=1}^{\infty} x^2 \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$= \lambda t \sum_{x=1}^{\infty} x \frac{(\lambda t)^{x-1}}{(x-1)!} e^{-\lambda t}$$

$$= \lambda t \sum_{x=0}^{\infty} (x+1) \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

$$= \lambda t \left(\sum_{x=0}^{\infty} x \frac{(\lambda t)^x e^{-\lambda t}}{x!} + \sum_{x=0}^{\infty} \frac{(\lambda t)^x e^{-\lambda t}}{x!} \right)$$

$$= \lambda t (\lambda t + 1)$$

$$Var\{x\} = (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$= \lambda t$$

1

$\lambda_{town} = 5$ customers/min

$\lambda_{rural} = 7$ customers/min

$\lambda = 5+7 = 12$ customers/min.

$$P\{t \le \tfrac{5}{60}\} = 1 - e^{-12 \times \frac{5}{60}}$$

$$= 1 - .368$$

$$= .632$$

2

$$E\{x\} = \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= -\int_0^\infty x d e^{-\lambda x}$$

$$= -\left[x e^{-\lambda x} - \int_0^\infty e^{-\lambda x} dx\right]$$

$$= -\left[x e^{-\lambda x} - \frac{1}{\lambda}\int_0^\infty \lambda e^{-\lambda x} dx\right]$$

$$= -\left[x e^{-\lambda x} - \frac{1}{\lambda}\right]_0^\infty$$

$$= \frac{1}{\lambda}$$

$$Var\{x\} = \int_0^\infty (x - E\{x\})^2 f(x)\, dx$$

$$= \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x} dx - 2\int_0^\infty x e^{-\lambda x} dx + \frac{1}{\lambda^2}\int_0^\infty \lambda e^{-\lambda x} dx$$

$$= -\int_0^\infty x^2 d e^{-\lambda x} - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

$$= -\left[x^2 e^{-\lambda x} - \int_0^\infty e^{-\lambda x} dx^2\right] - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

continued...

$$= \int_0^\infty e^{-\lambda x} dx^2 - x^2 e^{-\lambda x}\Big|_0^\infty - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

$$= 2\int_0^\infty x e^{-\lambda x} dx - x^2 e^{-\lambda x}\Big|_0^\infty - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda} - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

Set 12.4d

1

(a) $P\{x \geq 26\}$

$= 1 - P\{x \leq 26\}$

$= 1 - P\{z \leq \frac{26-22}{2}\}$

$= 1 - P\{z \leq 2\}$

$= 1 - .9772 = .0228$

(b) $P\{x \leq 17\}$

$= P\{z \leq \frac{17-22}{2}\}$

$= P\{z \leq -2.5\}$

$= 1 - .9938$

$= .0062$

2

Distribution of the weight of 5 individuals is normal with

mean = $5 \times 180 = 900$ lb

Standard deviation = $\sqrt{5 \times 15^2} = 33.54$

$P\{x \geq 1000\} = 1 - P\{z \leq \frac{1000-900}{33.54}\}$

$= 1 - P\{z \leq 2.98\}$

$= 1 - .9986$

$= .0014$

3

$x_1 = N(.99, .01)$

$x_2 = N(1, .01)$

$P\{x_1 > x_2\} = P\{x_1 - x_2 \geq 0\}$

mean $\{x_1 - x_2\} = .99 - 1 = -.01$

Standard deviation $\{x_1 - x_2\} = \sqrt{.01^2 + .01^2}$

$= .01414$

$P\{x_1 - x_2 \geq 0\}$

$= P\{z \geq \frac{0-(-.01)}{.01414}\}$

continued...

$= P\{z \geq .7072\}$

$= 1 - P\{z \leq .7072\}$

$\cong 1 - .760283$

$\cong .239717$

1

Step 1: Use ch12SampleMeanVar.xls to compute sample statistics and to prepare for creating the histogram as shown below

Sample Mean and Variance +Histogram

Output:				
Sample size	96		Mean	3.9219
Minimum	0.1000		Variance	6.8809
Maximum	15.9000		Std Dev.	2.6231

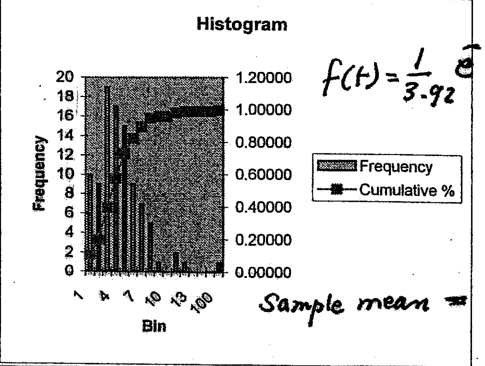
Input:

Enter data in A8:E100

				Bin
4.3	0.9	5.8	2.7	0.5
4.4	4.4	3.4	5.1	1
0.1	4.9	15.9	2.1	1.5
2.5	3.8	2.8	2.1	2
3.4	0.4	0.9	4.5	2.5
6.1	1.1	2.9	7.2	3
2.6	4.9	4.1	11.5	3.5
0.1	4.3	4.3	4.1	4
2.2	5.2	1.1	2.1	4.5
3.5	7.9	5.1	5.8	5
0.5	6.4	2.1	3.2	5.5
3.3	7.1	3.1	2.1	6
3.4	0.7	3.4	7.8	6.5
0.8	1.9	3.1	1.4	7
4.1	4.8	6.7	2.3	7.5
3.3	6.1	5.9	2.8	8
3.1	2.7	2.9	3.8	8.5
3.4	4.2	4.6	5.1	9
0.9	2.4	5.1	2.6	9.5
10.3	5.1	1.1	6.7	10
2.9	8.2	3.3	7.3	10.5
3.1	0.9	6.2	1.4	11
4.5	1.2	10.7	2.3	11.5
3.3	6.9	1.6	1.9	

Step 2: Apply Excel histogram to the sample above. The output below is for bin width of 1. Excel automatically provides the output below, less the columns titled n_i and Chi-value. You can then augment the spreadsheet with formulas to create the right-most columns.

Bin	Oi	Cpi	ni	Chi-value	Revised χ^2
1	10	0.10417	21.60641	6.234669	
2	9	0.19792	16.74353	3.581217	
3	19	0.39583	12.97511	2.797605	
4	17	0.57292	10.05485	4.797204	
5	15	0.72917	7.791835	6.668217	
6	9	0.82292	6.038151	1.452853	25.53176
7	7	0.89583	4.679164	1.15112	1.643731
8	5	0.94792	3.626039	0.520614	
9	1	0.95833	2.809938	1.165818	2.02322
10	0	0.95833	2.177515	2.177515	
11	2	0.97917	1.687428	0.057899	
12	1	0.98958	1.307644	0.072378	2.498492
13	0	0.98958	1.013337	1.013337	
14	0	0.98958	0.785268	0.785268	
15	0	0.98958	0.608531	0.608531	
100	1	1.00000	2.095247	0.572518	
sum	96		96		31.69721



$f(t) = \frac{1}{3.92} e^{-t/3.92}, \ t > 0$

Sample mean = 3.92

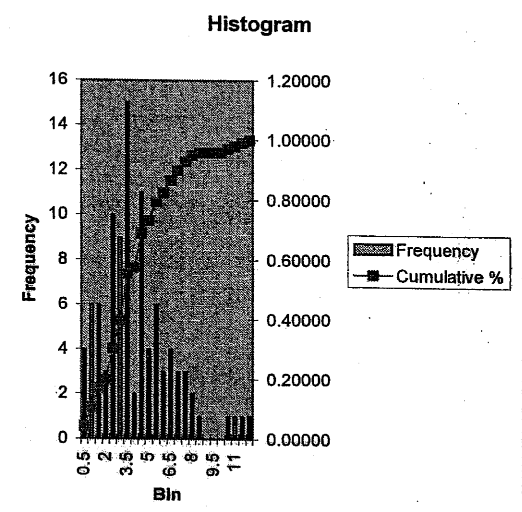
continued...

As can be seen from the output above, the spreadsheet can be modified to compute the χ^2-value. Note that the grouping is necessary to guarantee that $n_i \geq 5$.

χ^2-value = 31.69721, $\chi^2_{9-1-1, .95} = 14.067$, Reject

Binsize = .5:

Bin	Oi	Cpi	ni	Chi-value	Revised χ^2
0.5	4	0.04167	11.49092	4.883325	
1	6	0.10417	10.11549	1.674389	
1.5	6	0.16667	8.904697	0.947507	
2	3	0.19792	7.83883	2.986961	
2.5	10	0.30208	6.900545	1.392154	
3	9	0.39583	6.07457	1.408847	
3.5	15	0.55208	5.347461	17.4235	13.29318
4	2	0.57292	4.707386	1.557114	1.944528
4.5	11	0.68750	4.143925	11.34329	
5	4	0.72917	3.647909	0.033983	1.438188
5.5	6	0.79167	3.211265	2.4218	
6	3	0.82292	2.826886	0.010601	0.533896
6.5	4	0.86458	2.488516	0.918051	
7	3	0.89583	2.190648	0.299021	0.819514
7.5	3	0.92708	1.928434	0.595434	
8	2	0.94792	1.697606	0.053865	
8.5	1	0.95833	1.494407	0.163569	1.541192
9	0	0.95833	1.315531	1.315531	
9.5	0	0.95833	1.158066	1.158066	
10	0	0.95833	1.019449	1.019449	
10.5	1	0.96875	0.897424	0.011725	
11	1	0.97917	0.790005	0.05582	
11.5	1	0.98958	0.695443	0.133375	
100	1	1.00000	5.114583	3.310103	3.310103
Sum	96		96		22.8806



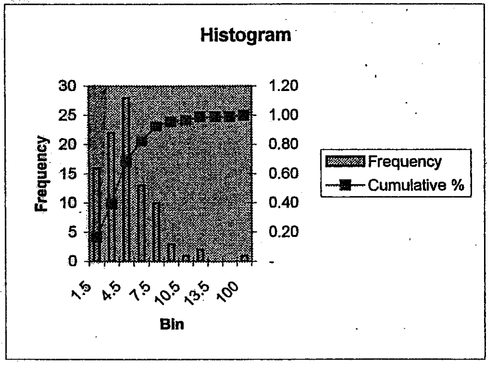
χ^2-value = 22.88

$\chi^2_{12-1-1, .05} = 18.307$ } Reject.

continued...

Set 12.5a

Bin	Oi	Cpi	ni	Chi-value	Revised χ^2
1.5	16	0.17	30.51111	6.901496	
3	22	0.40	20.81395	0.067586	
4.5	28	0.69	14.19877	13.41481	
6	13	0.82	9.686061	1.133814	
7.5	10	0.93	6.607598	1.741691	23.2594
9	3	0.96	4.507544	0.504197	1.692609
10.5	1	0.97	3.074938	1.400148	
12	2	0.99	2.097649	0.004546	1.963661
13.5	0	0.99	1.430966	1.430966	
15	0	0.99	0.97617	0.97617	
100	1	1.00	2.095247	0.572518	
Sum	96		96		26.91567



χ^2-value = 26.92

$\chi^2_{7-1-1,.05} = 11.07$ } Reject.

All three histogram call for rejecting the hypothesis that the sample is drawn from an exponential distribution with an estimated mean value of 3.92.

Note the effect of bin size on the χ^2-value. The larger the bin size, the smaller the number of degrees of freedom for the χ^2, and the tighter are the rejection limits

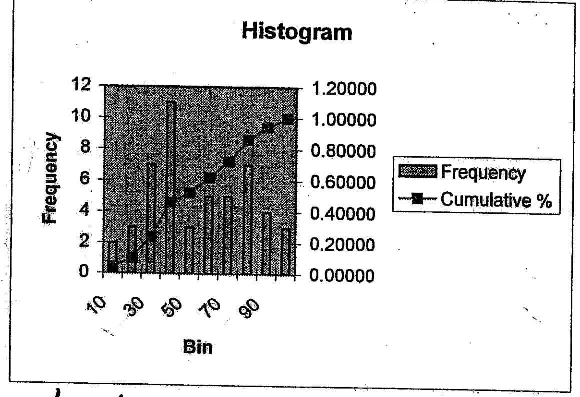
Sample Mean and Variance +Histogram

Output:					
Sample size	50		Mean	50.7620	
Minimum	5.6000		Variance	639.0783	
Maximum	94.8000		Std Dev.	25.2800	
Input:					
Enter data in A8:E100					Bin
25.8	67.3	35.2	36.4	58.7	10
47.9	94.8	61.3	59.3	93.4	20
17.8	34.7	56.4	22.1	48.1	30
48.2	35.8	65.3	30.1	72.5	40
5.8	70.9	88.9	76.4	17.3	50
77.4	66.1	23.9	23.8	36.8	60
5.6	36.4	93.5	36.4	76.7	70
89.3	39.2	78.7	51.9	63.6	80
89.5	58.6	12.8	28.6	82.7	90
38.7	71.3	21.1	35.9	29.2	100

2

(a) $f(x) = \frac{1}{100}, \quad 0 \le x \le 100$

Bin	Oi	Cpi	ni	chi-value	revised chi
10	2	0.04000	5	1.8	
20	3	0.10000	5	0.8	
30	7	0.24000	5	0.8	
40	11	0.46000	5	7.2	
50	3	0.52000	5	0.8	
60	5	0.62000	5	0	
70	5	0.72000	5	0	
80	7	0.86000	5	0.8	
90	4	0.94000	5	0.2	
100	3	1.00000	5	0.8	
sum	50			13.2	



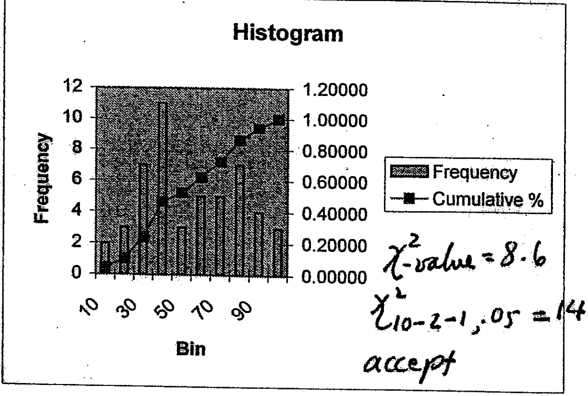
χ^2-value = 13.2, $\chi^2_{10-1,.05} = 16.9$

Conclusion: accept hypothesis

(b) Hypothesis: $f(x) = \frac{1}{94.8-5.6} = \frac{1}{89.2}$

$5.6 \le x \le 94.8$

Bin	Oi	Cpi	ni	chi-value	revised chi
10	2	0.04000	2.466368	0.088186	1.168971
20	3	0.10000	5.605381	1.210981	
30	7	0.24000	5.605381	0.346981	
40	11	0.46000	5.605381	5.191781	
50	3	0.52000	5.605381	1.210981	7.227487
60	5	0.62000	5.605381	0.065381	
70	5	0.72000	5.605381	0.065381	
80	7	0.86000	5.605381	0.346981	
90	4	0.94000	5.605381	0.459781	0.202451
100	3	1.00000	2.690583	0.035583	
sum	50		50	9.022018	8.598909



χ^2-value = 8.6

$\chi^2_{10-2-1,.05} = 14.1$

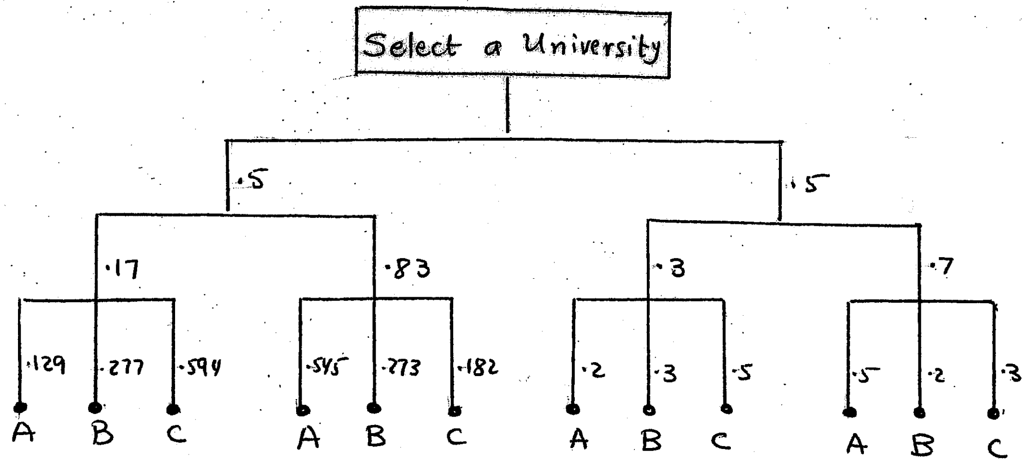
accept

CHAPTER 13

Decision Theory and Games

Set 13.1a

1



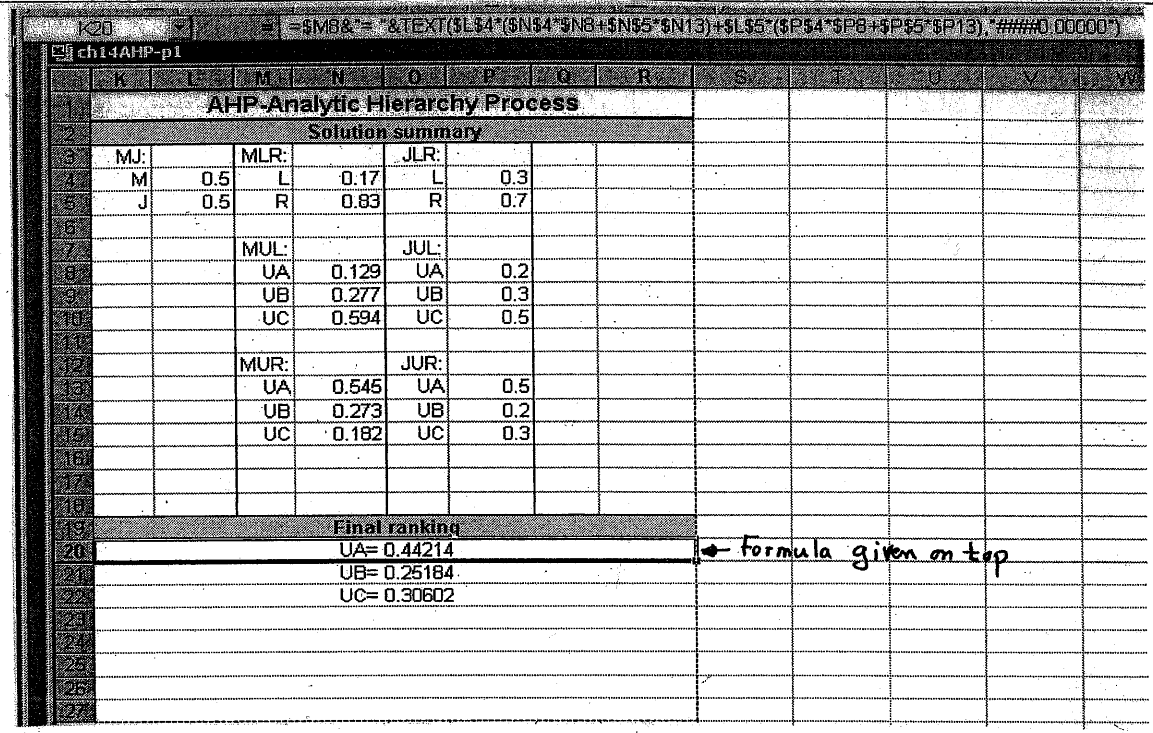
$W_A = .5(.17 \times .129 + .83 \times .545) + .5(.3 \times .2 + .7 \times .5) = \boxed{.44214}$

$W_B = .5(.17 \times .277 + .83 \times .273) + .5(.3 \times .3 + .7 \times .2) = .25184$

$W_C = .5(.17 \times .594 + .83 \times .182) + .5(.3 \times .5 + .7 \times .3) = .30602$

Select A.

1



```
K20  = =$M8&"= "&TEXT($L$4*($N$4*$N8+$N$5*$N13)+$L$5*($P$4*$P8+$P$5*$P13),"####0.00000")
```

ch14AHP-p1

AHP-Analytic Hierarchy Process

Solution summary

	K	L	M	N	O	P
3	MJ:		MLR:		JLR:	
4	M	0.5	L	0.17	L	0.3
5	J	0.5	R	0.83	R	0.7
6						
7			MUL:		JUL:	
8			UA	0.129	UA	0.2
9			UB	0.277	UB	0.3
10			UC	0.594	UC	0.5
11						
12			MUR:		JUR:	
13			UA	0.545	UA	0.5
14			UB	0.273	UB	0.2
15			UC	0.182	UC	0.3

Final ranking

UA= 0.44214 ← Formula given on top

UB= 0.25184

UC= 0.30602

2

$$A = \begin{array}{c|ccc} & I & E & R \\ \hline I & 1 & 2 & .25 \\ E & .5 & 1 & .2 \\ R & 4 & 5 & 1 \end{array}$$

$$N = \begin{bmatrix} .182 & .25 & .172 \\ .091 & .125 & .138 \\ .727 & .625 & .690 \end{bmatrix} \quad \begin{array}{c} \underline{\text{Average}} \\ .201 \\ .118 \\ .681 \end{array}$$

$$A\bar{W} = \begin{bmatrix} 1 & 2 & .25 \\ .5 & 1 & .2 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} .201 \\ .118 \\ .681 \end{bmatrix} = \begin{bmatrix} .60725 \\ .3547 \\ 2.075 \end{bmatrix}$$

$$n_{max} = .60725 + .3547 + 2.075 = 3.037$$

$$CI = \frac{3.073 - 3}{3-1} = .0185$$

$$RI = \frac{1.98(1)}{3} = .66$$

$$CR = \frac{.0185}{3.037} = .028 < .1, \text{ acceptable}$$

Continued...

$$N_I = \begin{bmatrix} .632 & .333 & .769 \\ .211 & .111 & .038 \\ .158 & .556 & .192 \end{bmatrix} \quad \begin{array}{c} \bar{W} \\ .578 \\ .120 \\ .302 \end{array}$$

$$A_I \bar{W} = \begin{bmatrix} 1 & 3 & 4 \\ .33 & 1 & .2 \\ .25 & 5 & 1 \end{bmatrix} \begin{bmatrix} .578 \\ .120 \\ .320 \end{bmatrix} = \begin{bmatrix} 2.146 \\ .373 \\ 1.0465 \end{bmatrix}$$

$$n_{max} = 2.14 + .373 + 1.0465 = 3.5655$$

$$CI = \frac{3.5655 - 3}{2} = .28275$$

$$RI = \frac{1.98(1)}{3} = .66$$

$$CR = \frac{.28275}{.66} = .428 > .1, \text{ not acceptable}$$

$$N_E = \begin{bmatrix} .222 & .100 & .571 \\ .667 & .300 & .143 \\ .111 & .600 & .286 \end{bmatrix} \quad \begin{array}{c} \bar{W} \\ .298 \\ .370 \\ .332 \end{array}$$

Continued...

$$A_E\bar{W} = \begin{bmatrix}1 & .33 & 2\\ 3 & 1 & .5\\ .5 & 2 & 1\end{bmatrix}\begin{bmatrix}.298\\ .370\\ .332\end{bmatrix} = \begin{bmatrix}1.085\\ 1.430\\ 1.221\end{bmatrix}$$

$$n_{max} = 3.736$$

$$CI = \frac{3.736-3}{2} = .368, \quad RI = .66$$

$$CR = \frac{.368}{.66} = .558 > .1, \text{ not acceptable}$$

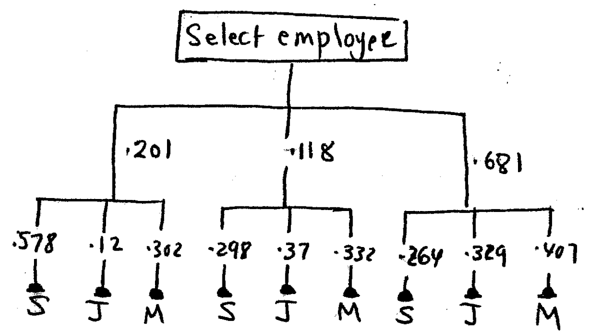
$$N_R = \begin{bmatrix}.25 & .143 & .400\\ .50 & .286 & .200\\ .25 & .571 & .400\end{bmatrix} \qquad \begin{array}{c}\bar{W}\\ .264\\ .329\\ .407\end{array}$$

$$A_R\bar{W} = \begin{bmatrix}1 & .5 & 1\\ 2 & 1 & .5\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}.264\\ .329\\ .407\end{bmatrix} = \begin{bmatrix}.8355\\ 1.0605\\ 1.329\end{bmatrix}$$

$$n_{max} = 3.225$$

$$CI = \frac{3.225-3}{2} = .1125, \quad RI = .66$$

$$CR = \frac{.1125}{.66} = .17 > .1, \text{ not acceptable}$$



$$W_S = .201 \times .578 + .118 \times .298 + .681 \times .264$$
$$= .331$$

$$W_J = .201 \times .12 + .118 \times .37 + .681 \times .329$$
$$= .292$$

$$W_M = .201 \times .302 + .118 \times .332 + .681 \times .407$$
$$= \boxed{.377}$$

Decision:

Select Maisa

3

$$N = \begin{bmatrix}.667 & .667\\ .333 & .333\end{bmatrix} \qquad \begin{array}{c}\bar{W}\\ .667\\ .333\end{array}$$

$$N_k = \begin{bmatrix}.25 & .25\\ .75 & .75\end{bmatrix} \qquad \begin{array}{c}.25\\ .75\end{array}$$

$$N_J = \begin{bmatrix}.8 & .8\\ .2 & .2\end{bmatrix} \qquad \begin{array}{c}.8\\ .2\end{array}$$

$$N_{ky} = \begin{bmatrix}.546 & .571 & .500\\ .272 & .286 & .333\\ .182 & .143 & .167\end{bmatrix} \qquad \begin{array}{c}\bar{W}\\ .539\\ .297\\ .164\end{array}$$

$$A_{ky}\bar{W} = \begin{bmatrix}1 & 2 & 3\\ .5 & 1 & 2\\ .333 & .5 & 1\end{bmatrix}\begin{bmatrix}.539\\ .297\\ .164\end{bmatrix} = \begin{bmatrix}1.625\\ .8945\\ .4922\end{bmatrix}$$

$$n_{max} = 3.01167$$

$$RI = \frac{.01167/2}{.66} = .0088 < .1, \text{ acceptable}$$

$$N_{kW} = \begin{bmatrix}.286 & .333 & .273\\ .143 & .167 & .182\\ .571 & .500 & .545\end{bmatrix} \qquad \begin{array}{c}\bar{W}\\ .297\\ .164\\ .539\end{array}$$

$$A_{kW}\bar{W} = \begin{bmatrix}1 & 2 & .5\\ .5 & 1 & .333\\ 2 & 3 & 1\end{bmatrix}\begin{bmatrix}.297\\ .164\\ .539\end{bmatrix} = \begin{bmatrix}.8945\\ .4922\\ 1.625\end{bmatrix}$$

$$n_{max} = 3.0117$$

$$RI = \frac{.0117/2}{.66} = .008 < .1, \text{ acceptable}$$

$$N_{JY} = \begin{bmatrix}.571 & .750 & .333\\ .143 & .188 & .500\\ .286 & .062 & .167\end{bmatrix} \qquad \begin{array}{c}\bar{W}\\ .551\\ .277\\ .172\end{array}$$

$$A_{JY}\bar{W} = \begin{bmatrix}1 & 4 & 2\\ .25 & 1 & 3\\ .5 & .333 & 1\end{bmatrix}\begin{bmatrix}.551\\ .277\\ .172\end{bmatrix} = \begin{bmatrix}2.003\\ .93075\\ .5398\end{bmatrix}$$

$$n_{max} = 3.476$$

$$RI = \frac{.476/2}{.66} = .3576 > .1, \text{ not acceptable}$$

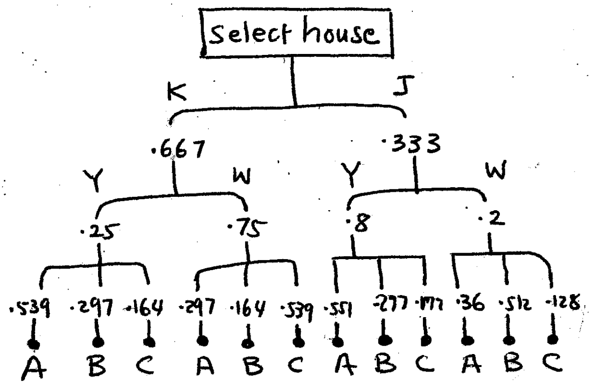
$$N_{JW} = \begin{bmatrix}.308 & .273 & .500\\ .615 & .546 & .375\\ .077 & .182 & .125\end{bmatrix} \qquad \begin{array}{c}\bar{W}\\ .360\\ .512\\ .128\end{array}$$

continued...

$$A_{JW}\bar{W} = \begin{bmatrix} 1 & .5 & 4 \\ 2 & 1 & 3 \\ .25 & .33 & 1 \end{bmatrix} \begin{bmatrix} .360 \\ .512 \\ .128 \end{bmatrix} = \begin{bmatrix} 1.128 \\ 1.616 \\ .3887 \end{bmatrix}$$

$$n_{max} = 3.1333$$

$$RI = \frac{.1333/2}{.66} = .100, \text{ acceptable}$$



$$W_A = .667(.25 \times .539 + .75 \times .297) + .333(.8 \times .551 + .2 \times .36) = \boxed{.4092}$$

$$W_B = .667(.25 \times .297 + .75 \times .164) + .333(.8 \times .277 + .2 \times .512) = .2395$$

$$W_c = .667(.25 \times .164 + .75 \times .539) + .333(.8 \times 172 + .2 \times .128) = .3513$$

Select A.

4

$$N = \begin{bmatrix} .167 & .143 & .172 \\ .167 & .143 & .138 \\ .667 & .714 & .690 \end{bmatrix} \quad \begin{array}{c} \bar{w} \\ .161 \\ .149 \\ .690 \end{array}$$

$$A\bar{w} = \begin{bmatrix} 1 & 1 & .25 \\ 1 & 1 & .20 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} .161 \\ .149 \\ .690 \end{bmatrix} = \begin{bmatrix} .4825 \\ .4480 \\ 2.079 \end{bmatrix}$$

$$n_{max} = 3.0095$$

$$CR = \frac{.0095/2}{.66} = .0072 < .1, \text{ acceptable}$$

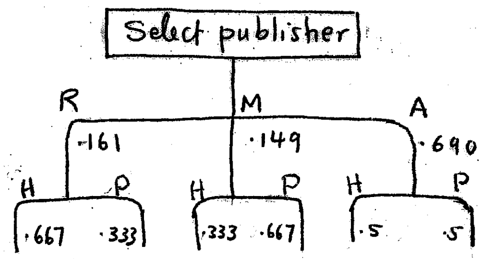
$$N_R = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix} \quad \begin{array}{cc} \bar{w} & \\ .667 & (H) \\ .333 & (P) \end{array}$$

$$N_M = \begin{bmatrix} .333 & .333 \\ .667 & .667 \end{bmatrix} \quad \begin{array}{cc} .333 & (H) \\ .667 & (P) \end{array}$$

$$N_A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \begin{array}{cc} .5 & (H) \\ .5 & (P) \end{array}$$

N_R, N_M, N_A are consistent because they are 2-dimensional.

Continued...



$$W_H = .161 \times .667 + .149 \times .333 + .69 \times .5 = \boxed{.502}$$

$$W_p = .161 \times .333 + .149 \times .667 + .69 \times .5 = .498$$

Choose H.

5

$$N = \begin{bmatrix} .286 & .25 & .294 \\ .143 & .125 & .118 \\ .571 & .625 & .588 \end{bmatrix} \quad \begin{array}{c} \bar{W} \\ .277 \\ .128 \\ .595 \end{array}$$

$$A\bar{W} = \begin{bmatrix} 1 & 2 & .5 \\ .5 & 1 & .2 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} .277 \\ .128 \\ .595 \end{bmatrix} = \begin{bmatrix} .8305 \\ .3855 \\ 1.789 \end{bmatrix}$$

$$n_{max} = 3.005$$

$$RI = \frac{.005/2}{.66} = .0039 < .1, \text{ acceptable}$$

$$N_L = \begin{bmatrix} .3 & .429 & .273 \\ .1 & .142 & .182 \\ .6 & .429 & .546 \end{bmatrix} \quad \begin{array}{c} \bar{W} \\ .334 \\ .141 \\ .525 \end{array}$$

$$A_L\bar{W} = \begin{bmatrix} 1 & 3 & .5 \\ .333 & 1 & .333 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} .334 \\ .141 \\ .525 \end{bmatrix} = \begin{bmatrix} 1.0195 \\ .427 \\ 1.6163 \end{bmatrix}$$

$$n_{max} = 3.06283$$

$$RI = \frac{.06283/2}{.66} = .04 < .1, \text{ acceptable}$$

$$N_C = \begin{bmatrix} .5 & .5 & .5 \\ .25 & .25 & .25 \\ .25 & .25 & .25 \end{bmatrix} \quad \begin{array}{c} \bar{W} \\ .5 \\ .25 \\ .25 \end{array} \quad \text{consistent}$$

$$N_R = \begin{bmatrix} .474 & .471 & .500 \\ .474 & .471 & .444 \\ .052 & .059 & .056 \end{bmatrix} \quad \begin{array}{c} \bar{W} \\ .482 \\ .463 \\ .056 \end{array}$$

Continued...

Set 13.1b

$$A_R\bar{W} = \begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 8 \\ 1/9 & 1/8 & 1 \end{bmatrix} \begin{bmatrix} .482 \\ .463 \\ .056 \end{bmatrix} = \begin{bmatrix} 1.449 \\ 1.393 \\ .167 \end{bmatrix}$$

$$n_{max} = 3.0094$$

$$RI = \frac{.0094/2}{.66} = .0071 < .1, \text{acceptable}$$

$$W_I = .277(.334 \times .1 + .141 \times .2 + .525 \times .3) + .128(.5 \times .3 + .25 \times .5 + .25 \times .2) + .595(.482 \times .7 + .463 \times .1 + .056 \times .3) = .3406$$

$$W_B = .277(.334 \times .5 + .141 \times .4 + .525 \times .2) + .128(.5 \times .4 + .25 \times .2 + .25 \times .4) + .595(.482 \times .1 + .463 \times .4 + .056 \times .2) = .2813$$

$$W_S = .277(.334 \times .4 + .141 \times .4 + .525 \times .5) + .128(.5 \times .3 + .25 \times .3 + .25 \times .4) + .595(.482 \times .2 + .463 \times .5 + .056 \times .5) = \boxed{.3798} \Rightarrow \text{Select Smith}$$

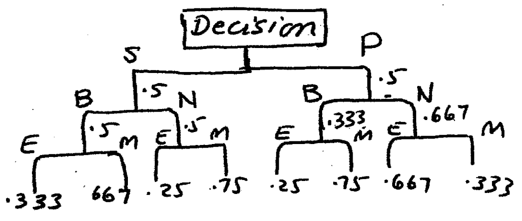
6

$$N_S = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$

$$N_P = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix}$$

$$N_{SB} = \begin{bmatrix} .333 & .333 \\ .667 & .667 \end{bmatrix}, N_{PB} = \begin{bmatrix} .25 & .25 \\ .75 & .75 \end{bmatrix}$$

$$N_{SN} = \begin{bmatrix} .25 & .25 \\ .75 & .75 \end{bmatrix}, N_{PN} = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix}$$



$$W_E = .5(.5 \times .333 + .5 \times .25) + .5(.333 \times .25 + .667 \times .667) = .4097$$

$$W_M = .5(.5 \times .667 + .5 \times .75) + .5(.333 \times .75 + .667 \times .333) = \boxed{.5903}$$

Decision: Keep music program.

7

Car Model	PP/yr	MC	CD	RD
M1	6	1.8	4.5	1.5
M2	8	1.2	2.25	.75
M3	10	.6	1.125	.6
Sum	24	3.6	7.875	2.85

All the comparison matrices are developed based on the average costs

$$A = \begin{array}{c|cccc} & PP & MC & CD & RD \\ \hline PP & 1 & \frac{24}{3.6} & \frac{24}{7.875} & \frac{24}{2.85} \\ MC & \frac{3.6}{24} & 1 & \frac{3.6}{7.875} & \frac{3.6}{2.85} \\ CD & \frac{7.875}{24} & \frac{7.875}{3.6} & 1 & \frac{7.875}{2.85} \\ RD & \frac{2.85}{24} & \frac{2.85}{3.6} & \frac{2.85}{7.875} & 1 \end{array}$$

$$= \begin{pmatrix} 1 & 6.67 & 3.048 & 8.421 \\ .15 & 1 & .457 & 1.263 \\ .328 & 2.188 & 1 & 2.763 \\ .119 & .792 & .362 & 1 \end{pmatrix}$$

$$A_{PP} = \begin{array}{c} M1 \\ M2 \\ M3 \end{array} \begin{bmatrix} 1 & 6/8 & 6/10 \\ 8/6 & 1 & 8/10 \\ 10/6 & 10/8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & .75 & .6 \\ 1.33 & 1 & .8 \\ 1.67 & 1.25 & 1 \end{bmatrix}$$

$$A_{MC} = \begin{array}{c} M1 \\ M2 \\ M3 \end{array} \begin{bmatrix} 1 & 6/4 & 6/2 \\ 4/6 & 1 & 4/2 \\ 2/6 & 2/4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1.5 & 3 \\ .667 & 1 & 2 \\ .333 & .5 & 1 \end{bmatrix}$$

$$A_{CD} = \begin{array}{c} M1 \\ M2 \\ M3 \end{array} \begin{bmatrix} 1 & \frac{4500}{2250} & \frac{4500}{1125} \\ \frac{2250}{4500} & 1 & \frac{2250}{1125} \\ \frac{1125}{4500} & \frac{1125}{2250} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ .5 & 1 & 2 \\ .25 & .5 & 1 \end{bmatrix}$$

$$A_{RD} = \begin{bmatrix} 1 & \frac{1500}{750} & \frac{1500}{600} \\ \frac{7500}{1500} & 1 & \frac{750}{600} \\ \frac{600}{1500} & \frac{600}{750} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2.5 \\ .5 & 1 & 1.25 \\ .4 & .8 & 1 \end{bmatrix}$$

continued...

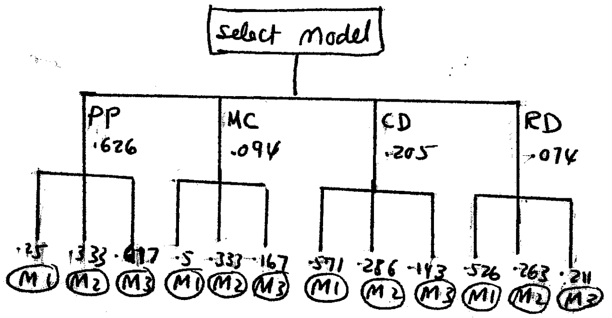
$$N = \begin{bmatrix} .626 & .626 & .626 & .628 \\ .094 & .094 & .094 & .094 \\ .205 & .205 & .205 & .206 \\ .075 & .074 & .074 & .074 \end{bmatrix} \quad \begin{matrix} w \\ .626 \\ .094 \\ .205 \\ .074 \end{matrix}$$

$$N_{PP} = \begin{bmatrix} .250 & .250 & .250 \\ .333 & .333 & .333 \\ .417 & .417 & .417 \end{bmatrix} \quad \begin{matrix} \bar{w} \\ .25 \\ .333 \\ .417 \end{matrix}$$

$$N_{MC} = \begin{bmatrix} .500 & .500 & .500 \\ .333 & .333 & .333 \\ .167 & .167 & .167 \end{bmatrix} \quad \begin{matrix} \bar{w} \\ .5 \\ .333 \\ .167 \end{matrix}$$

$$N_{CD} = \begin{bmatrix} .571 & .571 & .571 \\ .286 & .286 & .286 \\ .143 & .143 & .143 \end{bmatrix} \quad \begin{matrix} \bar{w} \\ .571 \\ .286 \\ .143 \end{matrix}$$

$$N_{RD} = \begin{bmatrix} .526 & .526 & .526 \\ .263 & .263 & .263 \\ .211 & .211 & .211 \end{bmatrix} \quad \begin{matrix} \bar{w} \\ .526 \\ .263 \\ .211 \end{matrix}$$



$$W_{M_1} = .626 \times .25 + .094 \times .5 + .205 \times .571 + .074 \times .526 = .3595$$

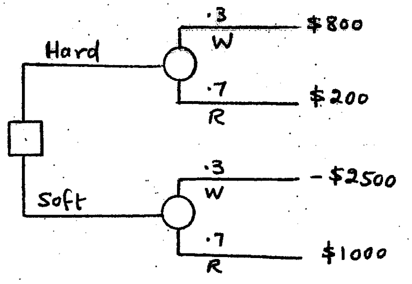
$$W_{M_2} = .626 \times .333 + .094 \times .333 + .205 \times .286 + .074 \times .263 = \boxed{.3185}$$

$$W_{M_3} = .626 \times .417 + .094 \times .167 + .205 \times .143 + .074 \times .211 = .3217$$

Since the comparison matrices are based on costs, the model with the <u>smallest</u> weight is selected. Select M2.

Set 13.2a

1

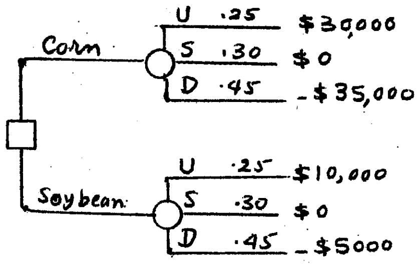


$EV\{hard\} = 800\times.3 + 200\times.7 = \boxed{\$380}$

$EV\{soft\} = -2500\times.3 + 1000\times.7 = -\$50$

Select "hard" button.

2

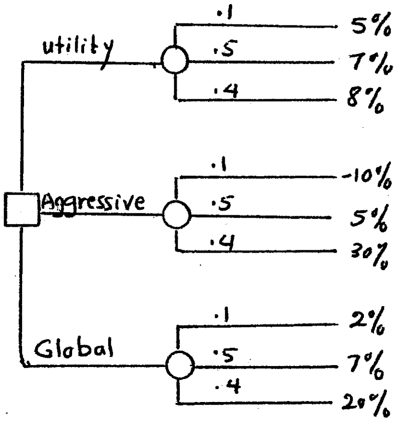


$EV(corn) = 30{,}000\times.25 + 0\times.3 + (-35000)\times.45$
$= -\$8250$

$EV(Soybean) = 10{,}000\times.25 + 0\times.3 + (-5000)\times.45$
$= \boxed{\$250}$

Select soybean

3



$EV(utility) = 5\times.1 + 7\times.5 + 8\times.4 = 7.2\%$

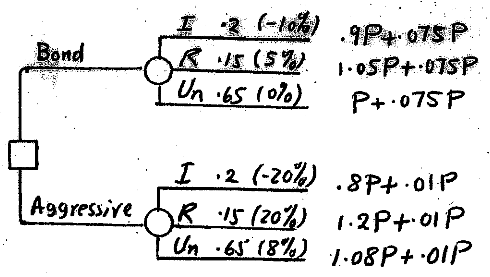
$EV(aggrssive) = -10\times.1 + 5\times.5 + 30\times.4 = \boxed{13.5\%}$

$E(global) = 2\times.1 + 7\times.5 + 20\times.4 = 11.7\%$

Select agressive Stock

4

P = amount invested

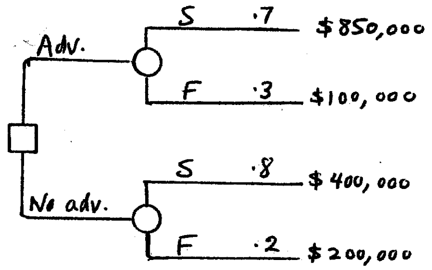


$EV(Bond) = P(.975\times.2 + 1.125\times.15 + 1.075\times.65)$
$= \boxed{1.0625P}$

$EV(Aggressive)$
$= P(.81\times.2 + 1.21\times.15 + 1.09\times.65)$
$= 1.052\ P$

Select Bond

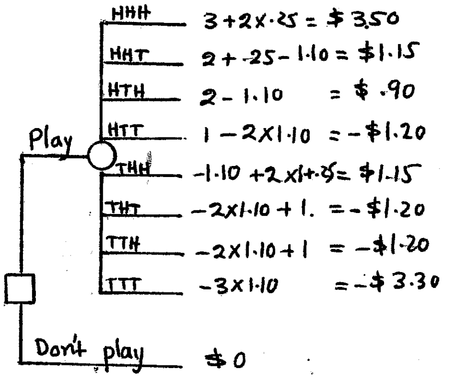
5



$EV(adv.) = 850\times.7 + 100\times.3 = \boxed{\$625{,}000}$

$EV(no\ adv.) = 400\times.8 + 200\times.2 = \$360{,}000$

6



$EV(play) = \frac{1}{8}\{3.5 + 1.15 + .90 - 1.20$
$+ 1.15 - 1.20 - 1.20 - 3.30\}$

$= -\$.025$

$EV(no\ play) = 0$

7

$(even/even) \equiv \{(2,2),(4,4),(6,6)\}$

$(odd/odd) \equiv \{(1,1),(3,3),(5,5)\}$

$(odd/even \text{ or } even/odd)$

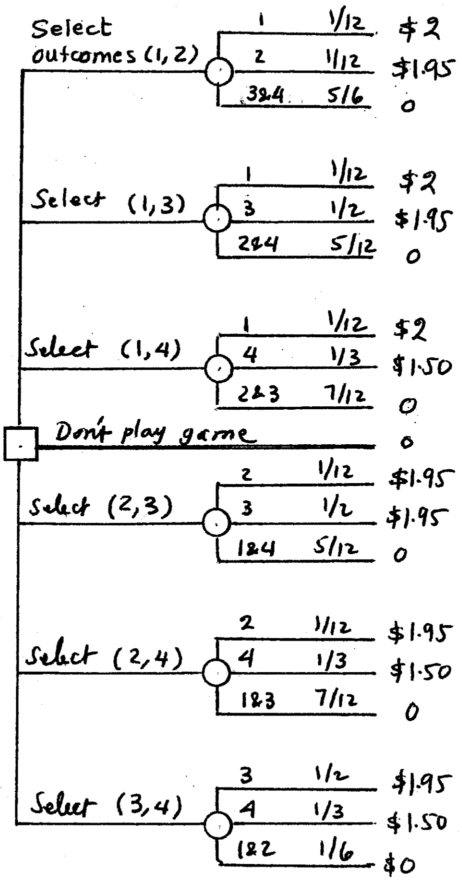
$\equiv \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6)$
$(5,2),(5,4),(5,6),(2,1),(2,3),(2,5)$
$(4,3),(4,5),(4,1),(6,1),(6,3),(6,5)\}$

$P\{e/e\} = 3\times(\frac{1}{6})^2 = 1/12$ (outcome 1)

$P\{0/0\} = 3\times(\frac{1}{6})^2 = 1/12$ (outcome 2)

$P\{e/0 \text{ or } 0/e\} = 18(\frac{1}{6})^2 = 1/2$ (Outcome 3)

$P\{\text{others}\} = 1/3$ (outcome 4)



$EV(1,2) = \frac{1}{12}(2+1.95) - 2 = -\$1.67$

$EV(1,3) = \frac{1}{12}\times 2 + \frac{1}{2}\times 1.95 - 2 = -\$.86$

$EV(1,4) = \frac{1}{12}\times 2 + \frac{1}{3}\times 1.50 - 2 = -\$1.33$

$EV(2,3) = \frac{1}{12}\times 1.95 + \frac{1}{2} 1.95 - 2 = -\$.86$

$EV(2,4) = \frac{1}{12}\times 1.95 + \frac{1}{3}\times 1.50 - 2 = -\$1.34$

$EV(3,4) = \frac{1}{2}\times 1.95 + \frac{1}{3}\times 1.50 - 2 = -\$.53$

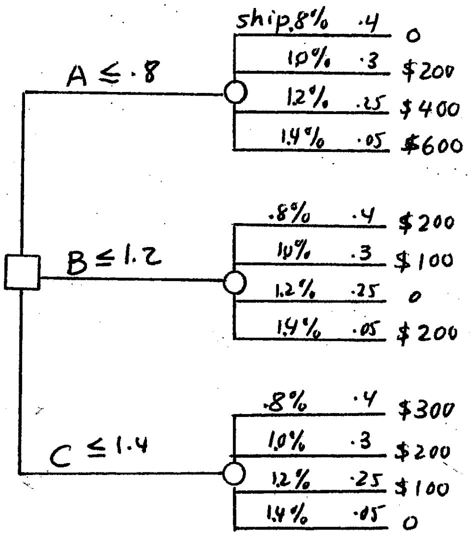
Don't play the game

Continued...

8

Penalty matrix:

$$\begin{array}{c|cccc} & \multicolumn{4}{c}{\text{Lot defective \%}} \\ & .8\% & 1\% & 1.2\% & 1.4\% \\ \hline A(.8\%) & 0 & 200 & 400 & 600 \\ B(1.2\%) & 200 & 100 & 0 & 200 \\ C(1.4\%) & 300 & 200 & 100 & 0 \end{array}$$



$EV(A) = 0\times.4 + 200\times.3 + 400\times.25 + 600\times.05$
$= \$190$

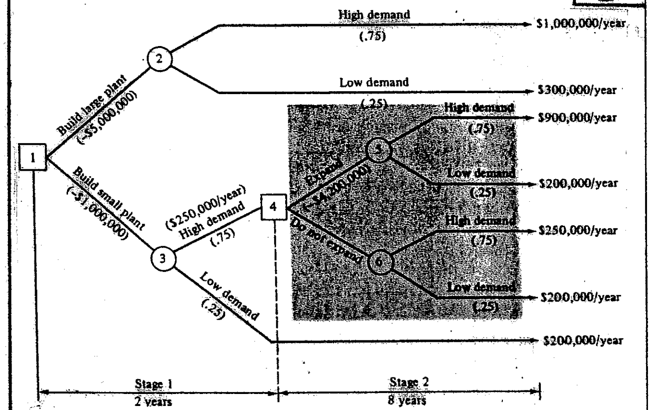
$EV(B) = 200\times.4 + 100\times.3 + 0\times.25 + 200\times.05 = \boxed{\$120}$

$EV(C) = 300\times.4 + 200\times.3 + 100\times.25 + 0\times.05 = \$205$

Select customer B

9

(a)



(b) $E\{\text{profit at node 4} \mid \text{expansion}\}$
$= (900\times.75 + 200\times.25)\times 8 - 4200 = \$1,600,000$

$E\{\text{profit at node 4} \mid \text{no expansion}\}$
$= (250\times.75 + 200\times.25)\times 8$
$= \boxed{\$1,900,000}$

Continued...

9 continued

At node 4, no expansion is recommended.

$E(\text{profit at node1} \mid \text{large plant})$

$= (1000 \times .75 + 300 \times .25) \times 10 - 5000$

$= \boxed{\$3,250,000}$

$E(\text{profit at node 1} \mid \text{small plant})$

$= (1900 + 2 \times 250) \times .75 + 10 \times 200 \times .25 - 1000$

$= \$1,300,000$

Decision: Start with large plant

10

Node 4:

$E(\text{annual profit} \mid \text{expansion})$

$= 900 \times .75 + 200 \times .25 = \$725,000$

$E(\text{annual profit} \mid \text{no expansion})$

$= 250 \times .75 + 200 \times .25 = \$237,500$

$E(\text{profit} \mid \text{expansion}) = 725\,[P|A]_{8}^{10\%} - 4200$

$= 725 \times 5.3349 - 4200$

$= -\$332,198$

$E(\text{profit} \mid \text{no expansion})$

$= 237.5 \times [P|A]_{8}^{10\%}$

$= 237.5 \times 5.3349 = \$1,267,000$

Decision at $\boxed{4}$: no expansion

Node 1:

$E(\text{profit} \mid \text{large plant})$

$= (1000 \times .75 + 300 \times .25)\,[P/A]_{10}^{10\%} - 5000$

$= \$69,295$

$E(\text{profit} \mid \text{small plant})$

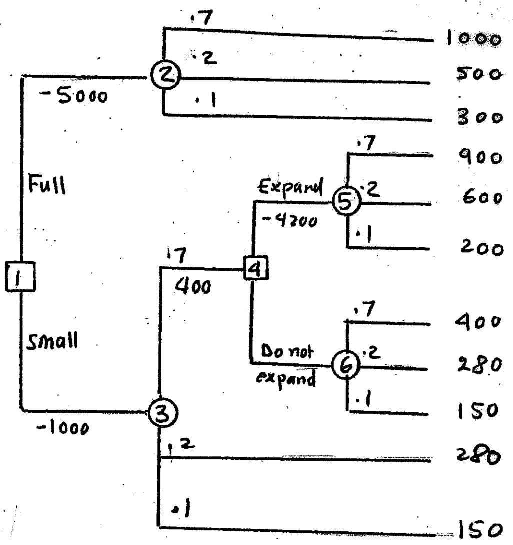
$= \left(1267\,[P|S]_{2}^{10\%} + 250\,[P|A]_{2}^{10\%}\right) \times .75$

$+ 200\,[P|A]_{10}^{10\%} \times .25 - 1000$

$= \$417,970$

Decision: Construct a small plant now and do not expand two years from now.

11



Node 4:

$E(\text{profit} \mid \text{expansion})$

$= (900 \times .7 + 600 \times .2 + 200 \times .1) \times 8 - 4200$

$= \$1,960,000$

$E(\text{profit} \mid \text{no expansion})$

$= (400 \times .7 + 280 \times .2 + 150 \times .1) \times 8$

$= \boxed{\$2,808,000}$

Decision at node 4: Do not expand

Node 1:

$E(\text{profit} \mid \text{large plant})$

$= (1000 \times .7 + 500 \times .2 + 300 \times .1) \times 10 - 5000$

$= \boxed{\$3,300,00}$

$E(\text{profit} \mid \text{small plant})$

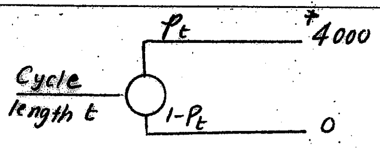
$= (2 \times 400 + 2808) \times .7 + 10 \times 280 \times .2 +$

$10 \times 150 \times .1) - 1000$

$= \$2,235,600$

Choose large plant now.

12



$E(\text{breakdown cost given } t) = 4000P_t$

t=1:

$\text{Cost} = 20 \times 75 = \$1500$

t=2:

$\text{Exp. breakdown cost} = 4000 \times .03 = \$120$

$\text{Av. cost/year} = \frac{1500+120}{2} = \$810$

t=3:

$\text{Exp. breakdown cost} = 120 + 4000 \times .04 = \$280$

$\text{Av. cost/year} = \frac{1500+280}{3} = \$593.33$

t=4:

$\text{Exp. breakdown cost} = 280 + 4000 \times .05 = \$480$

$\text{Av. cost/year} = \frac{1500 * 480}{4} = \$495$

t=5:

$\text{Exp. breakdown cost} = 480 + 4000 \times .06 = \$720$

$\text{Av. cost/year} = \frac{1500+720}{5} = \$444$

t=6:

$\text{Exp. breakdown cost} = 720 + 4000 \times .07 = \$1000$

$\text{Av. cost/year} = \frac{1500+1000}{6} = \$416.67$

t=7:

$\text{Exp. breakdown cost} = 1000 + 4000 \times .08 = \$1320$

$\text{Av. cost/year} = \frac{1500+1320}{7} = \$402.86$

t=8:

$\text{Av. cost/yr} = \frac{1500+1320+4000 \times .09}{8} = \boxed{\$397.50}$

continued...

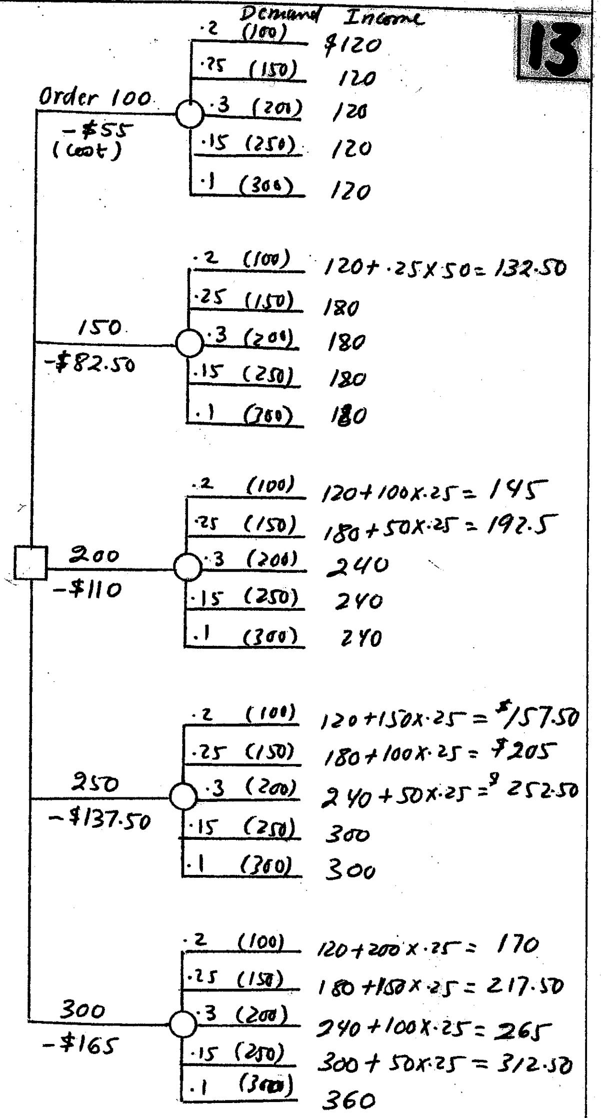
t=9:

$\text{Av. cost/yr} = \frac{1500+1680+4000 \times .1}{9} = \$397.78$

Decision:

Optimum cycle length = 8 , Cost/yr = $397.50

13



$E(\text{profit} \mid 100 \text{ loaves}) = 120 - 55 = \$65$

$E(\text{profit} \mid 150 \text{ loaves}) = 132.50 \times .2 + 180 \times .8 - 82.50 = \$88$

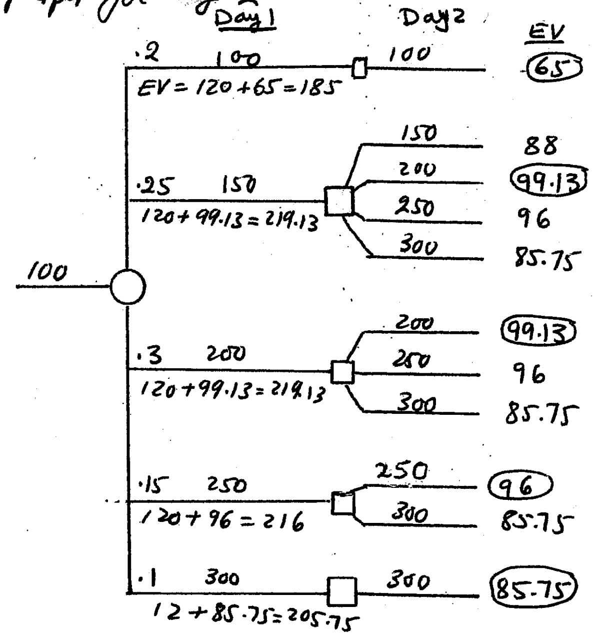
$E(\text{profit} \mid 200 \text{ loaves}) = 145 \times .2 + 192.50 \times .25 + 240 \times .55 - 110 = \boxed{\$99.13}$

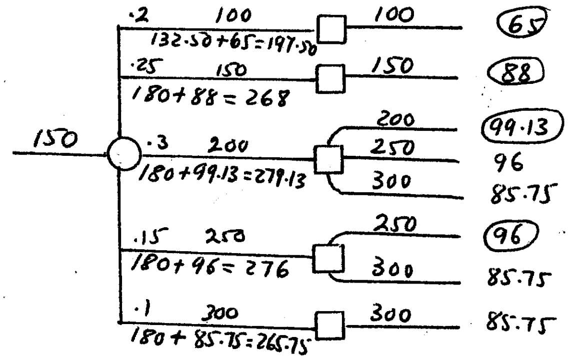
$E(\text{profit} \mid 250 \text{ loaves}) = 157.50 \times .2 + 205 \times .25 + 252.50 \times .3 + 300 \times .25 = \$96$

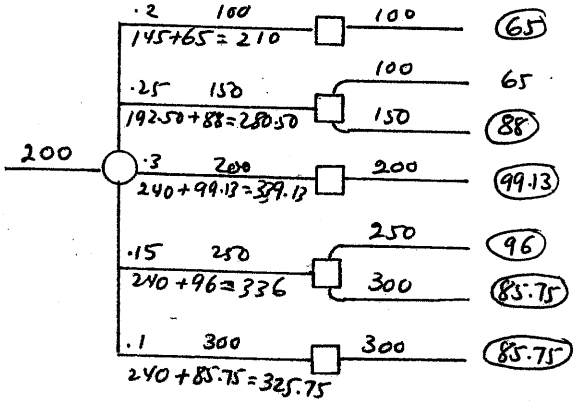
$E(\text{profit} \mid 300 \text{ loaves}) = 170 \times .2 + 217.50 \times .25 + 265 \times .3 + 312.50 \times .15 + 360 \times .1 = \$85.75$

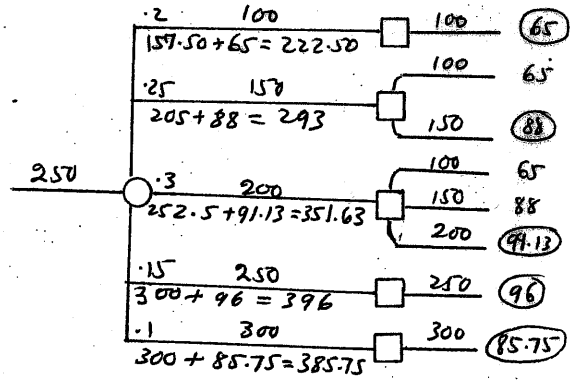
14

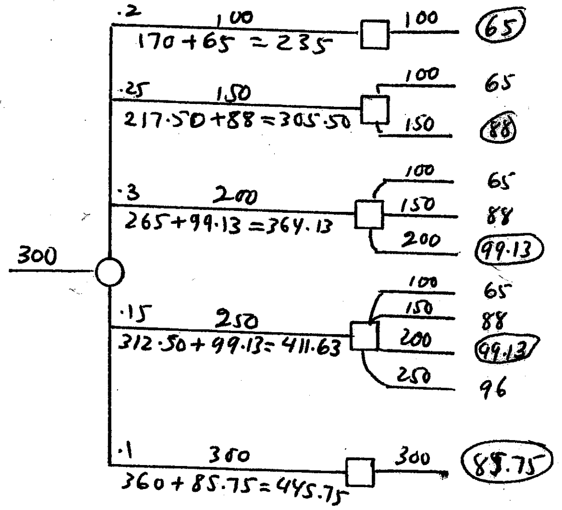
Make use of the results in Problem to determine the expected profit for day 2.











E(profit | 100 loaves) = 185 x .2 + 219.13 x .25 + 216 x .15 + 205.75 x .1 − 55 = $155.50

E(profit | 150 loaves) = 197.50 x .2 + 268 x .25 + 279.13 x .3 + 276 x .15 + 265.75 x .1 − 82.50 = 175.71

E(profit | 200 loaves) = 210 x .2 + 280.5 x .25 + 339.13 x .3 + 336 x .15 + 325.75 x .1 − 110 = 186.84

E(profit | 250 loaves) = 222.5 x .2 + 293 x .25 + 351.63 x .3 + 396 x .15 + 385.75 x .1 − 137.50 = 183.71

E(profit | 300 loaves) = 235 x .2 + 305.5 x .25 + 364.13 x .3 + 411.63 x .15 + 445.75 x .1 − 165 = 173.93

Solution: Revenue = $186.84

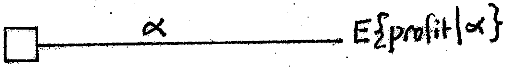
Day 1 : Stock 200 loaves

Day 2 : Stock level = demand

continued...

15

(a) Decision tree



(b) Profit given α

$$= r\alpha(1-p) - c\alpha p$$
$$= \alpha(r - [c+r]p)$$

$c = \$50$ is the loss per defective item
$r = \$5$ is the profit per good item

$$E\{profit|\alpha\} = \alpha\left[r - (c+r)E\{p\}\right\}$$

$$E\{p\} = \int_0^1 p\,\alpha\, p^{\alpha-1}\,dp = \frac{\alpha}{\alpha+1}$$

Hence

$$E\{profit|\alpha\} = \alpha r - (c+r)\frac{\alpha^2}{\alpha+1}$$

$$\frac{\partial E\{profit\}}{\partial \alpha} = r - (c+r)\,\frac{2\alpha(\alpha+1)-\alpha^2}{(\alpha+1)^2}$$
$$= r - (c+r)\,\frac{\alpha(\alpha+2)}{(\alpha+1)^2}$$

Equating the derivative to zero, we get

$$c\alpha^2 + 2c\alpha - r = 0$$

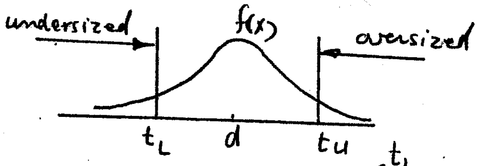
Using $c = \$50$ and $r = \$5$, we get

$$50\alpha^2 + 100\alpha - 5 = 0$$

Thus, $\alpha = .049$ or 49 pieces per day.

16

Let N = number of cylinders



$$E\{cost\} = N\left\{c_1\int_{t_u}^{\infty} f(x)dx + c_2\int_{-\infty}^{t_L} f(x)dx\right\}$$

Let $\Phi(z)$ be the standard normal.

$$E\{cost\} = N\left\{c_1\int_{\frac{t_u-d}{\sigma}}^{\infty}\Phi(z)dz + c_2\int_{-\infty}^{\frac{t_L-d}{\sigma}}\Phi(z)dz\right\}$$

Continued...

(a)

(b)

$$\frac{\partial E\{cost\}}{\partial d} = -\frac{c_2}{\sigma}\phi\left(\frac{t_L-d}{\sigma}\right) + \frac{c_2}{\sigma}\phi\left(\frac{t_u-d}{\sigma}\right)$$
$$= 0$$

Thus,

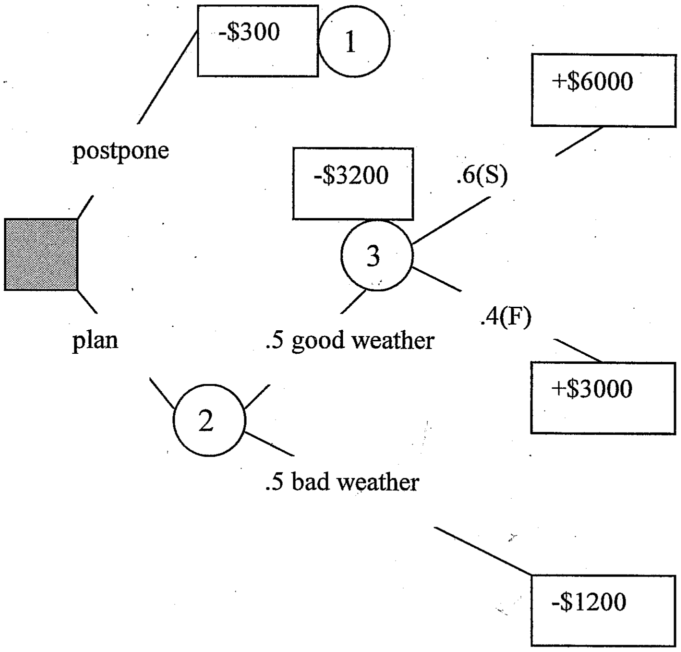
$$\frac{c_2}{c_1} = \frac{\Phi\left(\frac{t_u-d}{\sigma}\right)}{\Phi\left(\frac{t_L-d}{\sigma}\right)}$$
$$= \frac{1}{2}e^{-\frac{1}{2}\left(\frac{t_u-d}{\sigma}\right)^2} + \frac{1}{2}e^{\frac{1}{2}\left(\frac{t_L-d}{\sigma}\right)^2}$$

On simplification, we get

$$d^* = \frac{1}{2}\left(t_L + t_u - \frac{2\sigma^2}{t_L - t_u}\ln\frac{c_2}{c_1}\right)$$

Set 13.2a

17

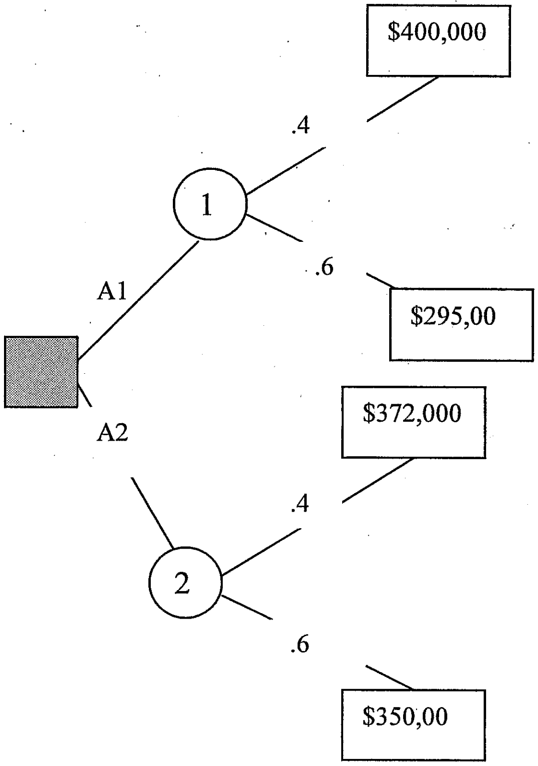


E{Plan}=.5(.6×6000+.4×3000-3200)+.5(-1200)=$200 > -$300
Select "Plan".

	Expected value			
P{good W}	Node 3	Node 2	Node 1	Decision
0	$4,800.00	-$1,200.00	-$300.00	postpone
0.1	$4,800.00	-$920.00	-$300.00	postpone
0.2	$4,800.00	-$640.00	-$300.00	postpone
0.3	$4,800.00	-$360.00	-$300.00	postpone
0.4	$4,800.00	-$80.00	-$300.00	plan
0.5	$4,800.00	$200.00	-$300.00	plan
0.6	$4,800.00	$480.00	-$300.00	plan
0.7	$4,800.00	$760.00	-$300.00	plan
0.8	$4,800.00	$1,040.00	-$300.00	plan
0.9	$4,800.00	$1,320.00	-$300.00	plan
1	$4,800.00	$1,600.00	-$300.00	plan

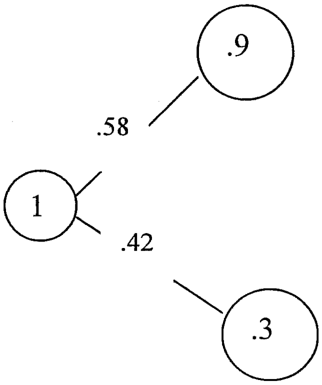
18

(a)



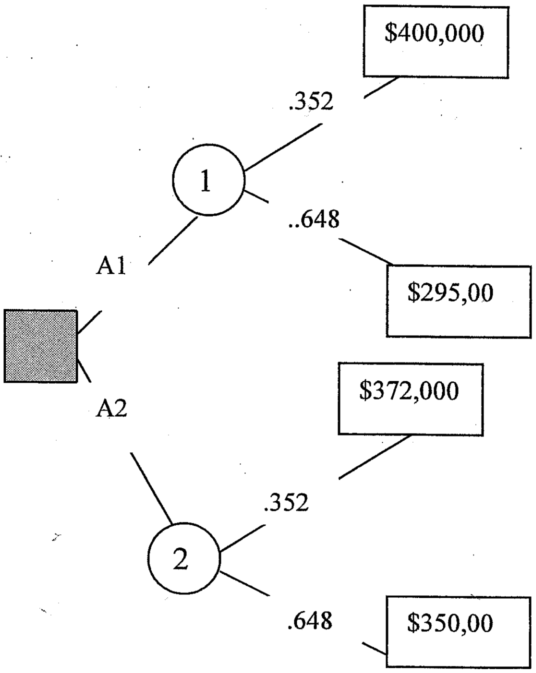
E{A1} = .4×400+.6×295.5=$337,300
E{A2} = .4×372+.6×350=$**358,800**
Use mix A2.

(b)



Expected probability of price increase = .58×.9+.42×.3 = .648

continued...



E{A1} = .352×400+.648×295.5=$332,284
E{A2} = .352×372+.648×350=$**357,744**
Use mix A2.Decision remains the same. Hence, additional cost is not warranted.

19

E{shortage}= $\int_{I}^{200}(x-I)\frac{200}{x^2}dx = 200(\ln\frac{200}{I}+\frac{I}{200}-1) \le 40$

E{surplus}= $\int_{100}^{I}(I-x)\frac{200}{x^2}dx = 200(\ln\frac{100}{I}+\frac{I}{100}-1) \le 20$

Simplifying, we get

$\ln I - \frac{I}{200} \ge 4.098$ (1)

$\ln I - \frac{I}{100} \ge 3.505$ (2)

Using a spreadsheet, the two aspiration levels are satisfied for

$$99 \le I \le 151$$

Set 13.2b

1

States of nature:

m_1 = took calculus

m_2 = didn't take calculus

Outcomes:

v_1: does well

v_2: doesn't do well

P{m}		v_1	v_2
.3	m_1	.75	
.7	m_2	.5	

$P\{v_1\} = .3 \times .75 + .7 \times .5$

$= .575$

2

Prior probabilities:

$P\{A\} = .75, \quad P\{B\} = .25$

Let z represent the event of having one defective in a sample of size five.

$P\{z|A\} = C_1^5(.01)^1(.99)^4 = .04803$

$P\{z|B\} = C_1^5(.02)^1(.98)^4 = .09224$

$P\{z, A\} = .04803 \times .75 = .036022$

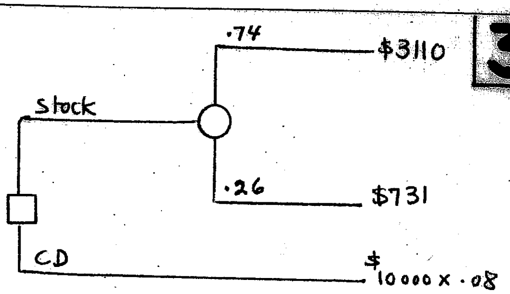
$P\{z, B\} = .09224 \times .25 = .023059$

$P\{z\} = .036022 + .023059 = .059081$

$P\{A|z\} = \frac{.036022}{.059081} = .6097$

$P\{B|z\} = \frac{.023059}{.059081} = .3903$

3



$EV(\text{Stock}) = .74 \times 3110 + .26 \times 731$

$= \$2491.46$

$EV(CD) = 10{,}000 \times .08 = \$800$

Decision: invest in stock

4

(a) $P\{\text{success}\} = .7 \quad P\{\text{failure}\} = .3$

$E\{\text{publisher offer}\} = 20{,}000 + .7(200{,}000 \times 1)$
$+ .3(10{,}000 \times 1)$
$= \$163{,}000$

$E\{\text{revenue if you undertake publishing}\}$
$= -90{,}000 + .7(200{,}000 \times 2) +$
$.3(10{,}000 \times 2) = \$196{,}000$

Decision: Publish it yourself.

(b) Define

m_1 = novel is a success

m_2 = novel is not a success

v_1 = survey predicts success

v_2 = survey does not predict success

$$P\{v_j|m_i\} = \begin{array}{c} \\ m_1 \\ m_2 \end{array} \begin{array}{cc} v_1 & v_2 \\ \left[\begin{array}{c} .8 \\ .15 \end{array}\right. & \left.\begin{array}{c} .2 \\ .85 \end{array}\right] \end{array}$$

Prior probabilities: $P\{m_1\} = .7 \quad P\{m_2\} = .3$

$$P\{m_i, v_j\} = \begin{array}{c} \\ m_1 \\ m_2 \end{array} \begin{array}{cc} v_1 & v_2 \\ \left[\begin{array}{c} .8 \times .7 \\ .15 \times .3 \end{array}\right. & \left.\begin{array}{c} .2 \times .7 \\ .85 \times .3 \end{array}\right] \end{array}$$

$$= \begin{array}{c} \\ m_1 \\ m_2 \end{array} \begin{array}{cc} v_1 & v_2 \\ \left[\begin{array}{c} .56 \\ .045 \end{array}\right. & \left.\begin{array}{c} .14 \\ .255 \end{array}\right] \end{array}$$

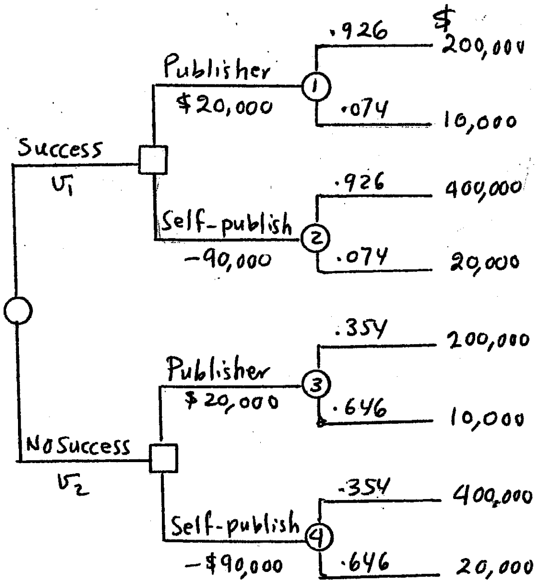
continued...

$P\{v_1\} = .56 + .045 = .605$

$P\{v_2\} = .14 + .255 = .395$

$$P\{m_i|v_j\} = \begin{array}{c} m_1 \\ \\ m_2 \end{array}\begin{bmatrix} \frac{.56}{.605} & \frac{.14}{.395} \\ \\ \frac{.045}{.605} & \frac{.255}{.395} \end{bmatrix}$$

$$= \begin{bmatrix} .926 & .354 \\ .074 & .646 \end{bmatrix}$$



$E\{\text{revenue}|①\} = .926 \times 200 + .074 \times 10 + 20$
$= \$205,940$

$E\{\text{revenue}|②\} = .926 \times 400 + .074 \times 20 - 90$
$= \boxed{\$281,880}$

$E\{\text{revenue}|③\} = .354 \times 200 + .646 \times 10 + 20$
$= \boxed{\$97,260}$

$E\{\text{revenue}|④\} = .354 \times 400 + .646 \times 20 - 90$
$= \$64,520$

Decision: If survey predicts success, publish the book yourself. Otherwise, use the publisher.

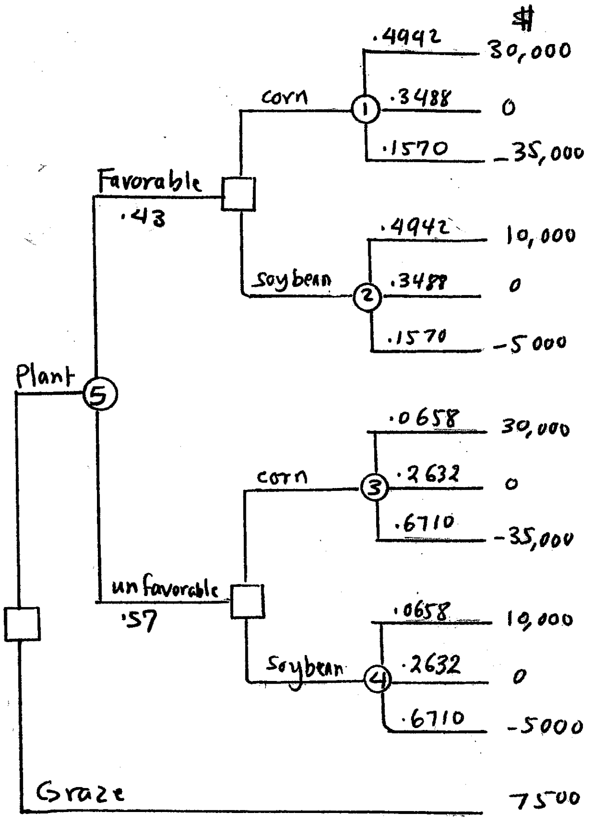
5

$$P\{a|s\} = \begin{array}{c|cc} & a_1 & a_2 \\ \hline s_1 & .85 & .15 \\ s_2 & .5 & .5 \\ s_3 & .15 & .85 \end{array} \qquad P\{s_i\} = \begin{bmatrix} .25 \\ .30 \\ .45 \end{bmatrix}$$

$$P\{s,a\} = \begin{bmatrix} .2125 & .0375 \\ .15 & .15 \\ .0675 & .3825 \end{bmatrix}$$

$P\{a\} = (.43 \quad .57)$

$$P\{s|a\} = \begin{array}{c} a_1 \\ a_2 \end{array}\begin{bmatrix} .4942 & .3488 & .1570 \\ .0658 & .2632 & .6710 \end{bmatrix}$$



$E\{\text{revenue}|①\} = 30 \times .4942 + 0 \times .3488 - 35 \times .1570$
$= \boxed{\$9331}$

$E\{\text{revenue}|②\} = 10 \times .4942 + 0 \times .3488 - 5 \times .1570$
$= \$4157$

$E\{\text{revenue}|③\} = 30 \times .0658 + 0 \times .2632 - 35 \times .6710$
$= -\$21,511$

$E\{\text{revenue}|④\} = 10 \times .0658 + 0 \times .2632 - 5 \times .671$
$= \boxed{-\$2697}$

$E\{\text{revenue}|⑤\} = .43 \times 9331 + (-2697) \times .57 = \$2478$

Decision: Choose grazing

Set 13.2b

6

$$P\{a|v\} = \begin{array}{ccc} & a_1 & a_2 \\ v_1 & .95 & .05 \\ v_2 & .3 & .7 \end{array}, \quad P\{v\} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

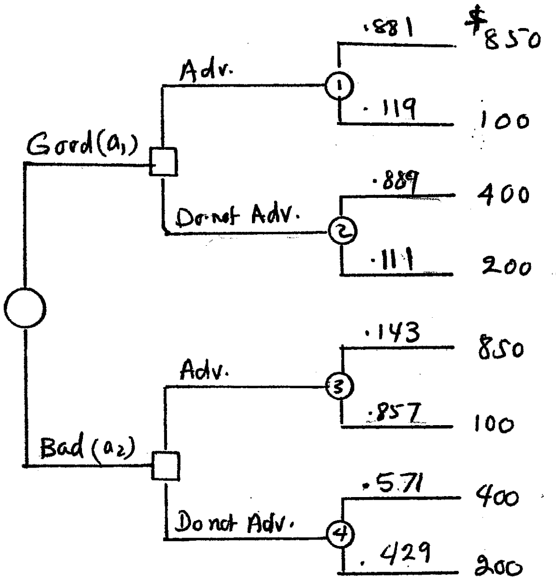
$$P\{v,a\} = \begin{array}{ccc} & a_1 & a_2 \\ v_1 & .665 & .035 \\ v_2 & .090 & .210 \end{array} \quad P\{a\} = (.755, .245)$$

$$P\{v|a\} = \begin{array}{ccc} v_1 & .881 & .143 \\ v_2 & .119 & .857 \end{array}$$

$$P\{a|w\} = \begin{array}{ccc} w_1 & .8 & .2 \\ w_2 & .4 & .6 \end{array}, \quad P\{w\} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$$

$$P\{w,a\} = \begin{array}{ccc} w_1 & .64 & .16 \\ w_2 & .08 & .12 \end{array} \quad P\{a\} = (.72, .28)$$

$$P\{w|a\} = \begin{array}{ccc} w_1 & .889 & .571 \\ w_2 & .111 & .429 \end{array}$$



E{revenue|①} = 850×.881+100×.119 = $\boxed{\$760.75}$

E{revenue|②} = 400×.889+200×.111 = $377.80

E{revenue|③} = 850 × .143+100×.857 = $207.25

E{revenue|④} = 400×.571 +200×.429 = $\boxed{\$314.70}$

Decision:

Advertise if test is good, else do not advertise

7

(a) θ_1 = lot is good (4% defectives)

θ_2 = lot is bad (15% defectives)

z_1 = both items of the sample are good

z_2 = one item is good

z_3 = both items are bad

$P\{\theta_1\} = .95 \quad P\{\theta_2\} = .05$

$P\{z_1|\theta_1\} = C_2^2(.96)^2(.04)^0 = .922$

$P\{z_2|\theta_1\} = C_1^2(.96)^1(.04)^1 = .0768$

$P\{z_3|\theta_1\} = C_0^2(.96)^0(.04)^2 = .0016$

$P\{z_1|\theta_2\} = C_2^2(.85)^2(.15)^0 = .7225$

$P\{z_2|\theta_2\} = C_1^2(.85)^1(.15)^1 = .255$

$P\{z_3|\theta_2\} = C_0^2(.85)^0(.15)^2 = .0225$

$$P\{\theta,z\} = \begin{array}{cccc} & z_1 & z_2 & z_3 \\ \theta_1 & .8759 & .07296 & .00152 \\ \theta_2 & .036125 & .01275 & .001125 \end{array}$$

$$P\{z\} = (\ .912025 \quad .08571 \quad .002645)$$

$$P\{\theta|z\} = \begin{array}{cccc} \theta_1 & .96039 & .85124 & .57467 \\ \theta_2 & .03961 & .14876 & .42533 \end{array}$$

(b)

Case 1: Two good items (z_1)

		G	B
5%	A	$50	$1000
8%	B	$200	$700

E(cost/customer A)

= 50×.96039+1000×.03961 = $\boxed{\$87.63}$

E(cost | customer B)

= 200 ×.96039+700×.03961 = $219.81

Decision: Ship lot to A

Case 2: One good item (z_2)

E(cost | customer A)

= 50× .85124+1000×.14876 = $\boxed{\$191.32}$

E(cost | customer B)

= 200×.85124+700×.14876 = 274.38

Decision: Ship lot to A

Case 3: Both items bad (z_3)

E{cost |A} = 50×.57467+1000×.42533

= $454.06

E{cost | B} = 200×.57467+700×.42533 =

= $\boxed{\$412.67}$

Decision: Ship to B

1

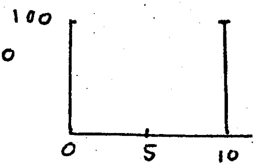
(a) $E\{\text{value of poker game}\}$

$= .5\times 10 + .5\times 0 = \$5$

No advantage

(b)

$$U(x) = \begin{cases} 0, & 0 \le x < 10 \\ 100, & x = 10 \end{cases}$$



(C) Because $U(5) = 0$ and $U(10) = 100$, the decision is to play the poker game

2

Worst condition cost $= 900{,}000 + 350{,}000$
$= \$1{,}250{,}000$

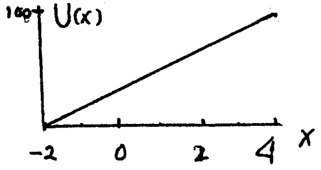
Best condition savings $= 900{,}000$

Lottery:

$U(x) = p\,U(-1{,}250{,}000) + (1-p)U(900{,}000)$
$= p(0) + (1-p)(100)$
$= 100(1-p) = 100 - 100p$

3

(a)



$$\frac{U(0)}{U(4)} = \frac{0-(-2)}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

$U(0) = \frac{1}{3}(100) = 33.33$

Now, $U(0) = p\,U(-2) + (1-p)U(4)$
$= 100(1-p)$

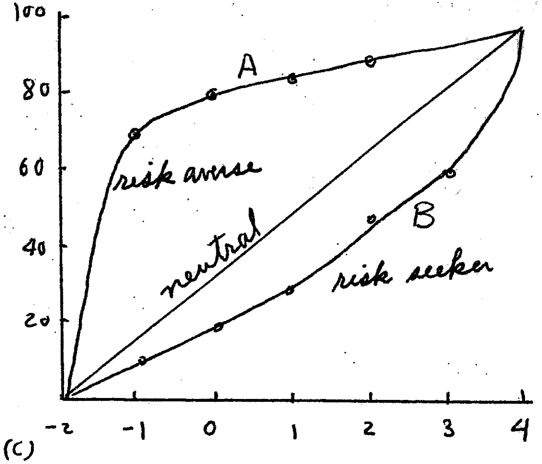
Thus, for $U(0) = 33.33$, $p = .6667$

b)

X	$U(X)_A$	$U(X)_B$
-2	0	0
-1	70	10
0	80	20
1	85	30
2	90	50
3	95	60
4	100	100

Continued...

3 continued



(c) Venture I:

$U_A(3000) = 95, \quad U_A(-1000) = 70$

$EU(I) = .4\times 95 + .6\times 70 = 80$

Venture II:

$U_A(2000) = 90, \quad U_A(0) = 80$

$EU(II) = .4\times 90 + .6\times 80 = \boxed{84}$

Decision: Select II

$$E\{\$\text{ venture II}\} = \frac{84-80}{85-80} = \frac{x-0}{1-0}$$

$\Rightarrow x = .8$ or $\$800$

(d) Venture I:

$U_B(3000) = 60, \quad U_B(-1000) = 10$

$EU(I) = .6\times 60 + .4\times 10 = \boxed{40}$

Venture II:

$U_B(2000) = 50, \quad U_B(0) = 20$

$EU(II) = .6\times 50 + .4\times 20 = 38$

Decision: Select I.

$E\{\$\text{ venture I}\} = \$1500$

Set 13.3a

1

(a)

Laplace:

$E(a_1) = \frac{1}{3}(85+60+40) = 61.67$

$E(a_2) = \frac{1}{3}(92+85+81) = 86$

$E(a_3) = \frac{1}{3}(100+88+82) = \boxed{90}$

Study all night.

Maximin:

Because this is a reward matrix, we use maximin

$$\begin{bmatrix} 85 & 60 & 40 \\ 92 & 85 & 81 \\ 100 & 88 & 82 \end{bmatrix} \begin{array}{l} \text{min} \\ 40 \\ 81 \\ \boxed{82}\ \text{maximin} \end{array}$$

Decision: Study all night

Savage:

$$\text{"Cost" matrix} = \begin{bmatrix} -85 & -60 & -40 \\ -92 & -85 & -81 \\ -100 & -88 & -82 \end{bmatrix}$$

$$\text{Regret matrix} = \begin{bmatrix} 15 & 28 & 42 \\ 8 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Row max} \\ 42 \\ 8 \\ \boxed{0} \leftarrow \text{min} \end{array}$$

Decision: Study all night

Hurwicz:

	Row min	Row max	$\alpha\binom{\text{Row}}{\text{min}}+(1-\alpha)\binom{\text{Row}}{\text{max}}$		At $\alpha = .5$
a_1	-85	-40	$-40-45\alpha$	=	-62.5
a_2	-92	-81	$-81-11\alpha$	=	-86.5
a_3	-100	-82	$-82-18\alpha$	=	$\boxed{-91}$

Decision: Study all night

(b)

$$\text{"Cost" matrix} = \begin{bmatrix} -80 & -60 & 0 \\ -90 & -80 & -80 \\ -90 & -80 & -80 \end{bmatrix}$$

Laplace:

$E(a_1) = \frac{-1}{3}(80+60+0) = -46.67$

$E(a_2) = \frac{-1}{3}(90+80+80) = \boxed{-83.33}$

$E(a_3) = \frac{-1}{3}(90+80+80) = \boxed{-83.33}$

Decision: Select second or third.

continued...

Minimax

$$\begin{bmatrix} -80 & -60 & 0 \\ -90 & -80 & -80 \\ -90 & -80 & -80 \end{bmatrix} \begin{array}{l} 0 \\ \boxed{-80} \leftarrow \\ \boxed{-80} \leftarrow \end{array}$$

Select either the second or the third option

Savage:

$$\begin{bmatrix} 10 & 20 & 80 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} 80 \\ \boxed{0} \leftarrow \\ \boxed{0} \leftarrow \end{array}$$

Select either the second or the third option.

Hurwicz:

	Row min	Row max	$\alpha\binom{\text{Row}}{\text{min}}+(1-\alpha)\binom{\text{Row}}{\text{max}}$	At $\alpha = .5$
a_1	-80	0	-80α	-40
a_2	-90	-80	$-80-10\alpha$	$\boxed{-85}$
a_3	-90	-80	$-80-10\alpha$	$\boxed{-85}$

Select the second or the third option

2

Laplace:

$E(a_1) = \frac{1}{4}(-20+60+30-5) = 16.25$

$E(a_2) = \frac{1}{4}(40+50+35+0) = \boxed{31.25}$

$E(a_3) = \frac{1}{4}(-50+100+45-10) = 21.25$

$E(a_4) = \frac{1}{4}(12+15+15+10) = 13$

Plant wheat

Minimax:

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{bmatrix} 20 & -60 & -30 & 5 \\ -40 & -50 & -35 & 0 \\ 50 & -100 & -45 & 10 \\ -12 & -15 & -15 & -10 \end{bmatrix} \begin{array}{l} \text{Row max} \\ 20 \\ 0 \\ 50 \\ \boxed{-10}\ \text{minimax} \end{array}$$

Recommend grazing.

Savage:

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{bmatrix} 60 & 40 & 15 & 15 \\ 0 & 50 & 10 & 10 \\ 90 & 0 & 0 & 20 \\ 28 & 85 & 30 & 0 \end{bmatrix} \begin{array}{l} \text{Row max} \\ 60 \\ \boxed{50}\ \text{minimax} \\ 90 \\ 85 \end{array}$$

Plant wheat

continued...

2 continued

Hurwicz:

	$\binom{\text{Row}}{\text{min}}$	$\binom{\text{Row}}{\text{max}}$	$\alpha\binom{\text{Row}}{\text{min}}+(1-\alpha)\binom{\text{Row}}{\text{max}}$	at $\alpha=.5$
a_1	-60	20	$20+80$	-20
a_2	-50	0	-50α	-25
a_3	-100	50	$50-150\alpha$	-25
a_4	-15	-10	$-10-5\alpha$	-12.5

Select wheat or soybeans.

3

Laplace:

$$\min_{a_i} \frac{\int_{Q^*}^{Q^{**}} (K_i + c_i Q)\,dQ}{Q^{**} - Q^*}$$

$$= \min_{a_i} \left\{ K_i + \frac{c_i}{2}(Q^{**} - Q^*) \right\}$$

$$E\{a_1\} = 100 + \frac{5}{2}(3000) = \$7600$$

$$E(a_2) = 40 + \frac{12}{2}(3000) = \$18{,}040$$

$$E(a_3) = 150 + \frac{3}{2}(3000) = \$4650$$

$$E(a_4) = 90 + \frac{8}{2}(3000) = \$12{,}090$$

Select machine 3

Minimax:

$$\min_{a_i} \max_{Q^* \le Q \le Q^{**}} \{K_i + c_i Q\}$$

$$= \min_{a_i} \{K_i + c_i Q^{**}\}$$

machine	$\{K_i + c_i Q^{**}\}$
1	$100 + 5 \times 4000 = \$20{,}100$
2	$40 + 12 \times 4000 = \$48{,}040$
3	$150 + 3 \times 4000 = \$12{,}150$
4	$90 + 8 \times 4000 = \$32{,}090$

Select machine 3.

Continued...

Savage:

$$\min_{a_i}\left[\max_{Q^* \le Q \le Q^{**}} \left\{K_i + c_i Q - \min_{a_i}(K_i + c_i Q)\right\}\right]$$

	cost	Regret
a_1	$100+5Q$	$-50+2Q$
a_2	$40+12Q$	$-110+9Q$
a_3	$\boxed{150+3Q}$	0
a_4	$90+8Q$	$-60+5Q$

Select machine 3

Smallest for $1000 \le Q \le 4000$

Hurwicz:

$$\min_{a_i}\{\alpha(K_i + c_i Q^*) + (1-\alpha)(K_i + c_i Q^{**})\}$$

$$= \min_{a_i}\{K_i + c_i(\alpha Q^* + (1-\alpha) Q^{**})\}$$

For $\alpha = 1/2$, we have

a_1 : $100 + 5\left(\frac{1000}{2} + \frac{4000}{2}\right) = \$12{,}600$

a_2 : $40 + 12 \times 2500 = \$30{,}040$

a_3 : $150 + 3 \times 2500 = \$7600$

a_4 : $90 + 8 \times 2500 = \$20{,}090$

Select machine 3.

Set 13.4a

1

(a)

$$\begin{bmatrix} 8 & 6 & 2 & 8 \\ 8 & 9 & \boxed{4} & 5 \\ 7 & 5 & 3 & 5 \end{bmatrix} \begin{matrix} 2 \\ \textcircled{4} \\ 3 \end{matrix}$$

$$\begin{matrix} 8 & 9 & \textcircled{4} & 5 \end{matrix}$$

Saddle point solution at (2,3)

(b)

$$\begin{bmatrix} 4 & -4 & \boxed{-5} & 6 \\ -3 & -4 & -9 & -2 \\ 6 & 7 & -8 & -9 \\ 7 & 3 & -9 & 5 \end{bmatrix} \begin{matrix} \textcircled{-5} \\ -9 \\ -9 \\ -9 \end{matrix}$$

$$\begin{matrix} 7 & 7 & \textcircled{-5} & 6 \end{matrix}$$

Saddle point solution at (1,3)

2

(a) $p \geq 5$, $q \leq 5$

(b) $p \leq 7$, $q \geq 7$

3

(a)

$$\begin{bmatrix} 1 & 9 & 6 & 0 \\ 2 & 3 & 8 & 4 \\ -5 & -2 & 10 & -3 \\ 7 & 4 & -2 & -5 \end{bmatrix} \begin{matrix} 0 \\ \textcircled{2} \\ -5 \\ -5 \end{matrix} \qquad 2 < v < 4$$

$$\begin{matrix} 7 & 9 & 10 & \textcircled{4} \end{matrix}$$

(b)

$$\begin{bmatrix} -1 & 9 & 6 & 8 \\ -2 & 10 & 4 & 6 \\ 5 & 3 & 0 & 7 \\ 7 & -2 & 8 & 4 \end{bmatrix} \begin{matrix} -1 \\ -2 \\ \textcircled{0} \\ -2 \end{matrix} \qquad 0 < v < 7$$

$$\begin{matrix} \textcircled{7} & 10 & 8 & 8 \end{matrix}$$

(c)

$$\begin{bmatrix} 3 & 6 & 1 \\ 5 & 2 & 3 \\ 4 & 2 & -5 \end{bmatrix} \begin{matrix} 1 \\ \textcircled{2} \\ -5 \end{matrix} \qquad 2 < v < 3$$

$$\begin{matrix} 5 & 6 & \textcircled{3} \end{matrix}$$

(d)

$$\begin{bmatrix} 3 & 7 & 1 & 3 \\ 4 & 8 & 0 & -6 \\ 6 & -9 & -2 & 4 \end{bmatrix} \begin{matrix} \textcircled{1} \\ -6 \\ -9 \end{matrix} \qquad 0 < v \leq 1$$

$$\begin{matrix} 6 & 8 & \textcircled{0} & 4 \end{matrix}$$

4

Define the following strategies:

1 - no campaign
2 - TV
3 - Newspaper
4 - Radio
5 - TV + newspaper
6 - TV + radio
7 - Radio + newspaper
8 - TV + radio + newspaper

The payoff is the additional percentage of customers reached by Company A.

	1	2	3	4	5	6	7	8	
1	0	-50	-30	-20	-80	-70	-50	-100	-100
2	50	0	20	30	-30	-20	0	-50	-50
3	30	-20	0	10	-50	-40	-20	-70	-70
4	20	-30	-10	0	-60	-50	-30	-80	-80
5	80	30	50	60	0	10	30	-20	-20
6	70	20	40	50	-10	0	20	-30	-30
7	50	0	20	30	-30	-20	0	-50	-50
8	100	50	70	80	20	30	50	0	⓪
	100	50	70	80	20	30	50	⓪	

The game has a saddle point at (8,8), meaning that both companies should advertise in all three media. The game is fair because its value equals zero.

5

$\min\limits_{j} a_{ij} \leq a_{ij}$, all i & j

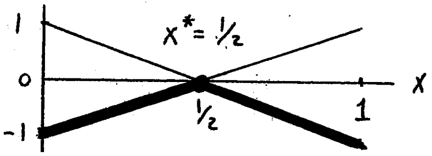
$\max\limits_{i} \min\limits_{j} a_{ij} \leq \max\limits_{i} a_{ij}$, all j

$\leq \min\limits_{j} \max\limits_{i} a_{ij}$

1

$$\begin{array}{ccc|cc} & & & y & 1-y \\ & & & B_H & B_T \\ \hline x & A_H & & 1 & -1 \\ 1-x & A_T & & -1 & 1 \end{array}$$

B's pure strategy	A's expected payoff
B_H	$x + (-1)(1-x) = -1+2x$
B_T	$-x + 1(1-x) = 1-2x$



B's game:

$y - (1-y) = -y + (1-y) \Rightarrow y^* = \frac{1}{2}$

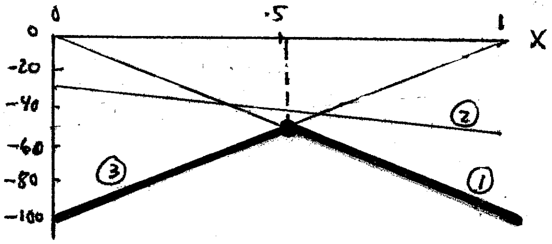
Value of the game $= -1 + 2(\frac{1}{2}) = 0$

2

Robin's Payoff matrix:

$$\begin{array}{cc|ccc} & & 100\text{-}A & 50/50\text{-}A/B & 100\,B \\ \hline x & A & -100 & -50 & 0 \\ (1-x) & B & 0 & -30 & -100 \end{array}$$

Police strategy	Robin's expected payoff
1	$-100x$
2	$-50x + (-30)(1-x) = -30-20x$
3	$-100 + 100x$



Robin's strategy: mix A and B 50-50.
Game cost = \$50

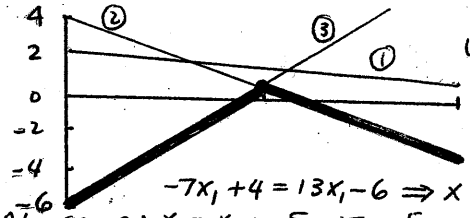
Police strategy:
$-100y_1 = -100(1-y_1) \Rightarrow y_1 = .5$

Solution: $y_1 = .5,\ y_2 = 0,\ y_3 = .5$

3

(a)

B's strategy	A's exp. payoff
1	$-x+2$
2	$-7x+4$
3	$13x-6$



$v = 1/2$

$-7x_1 + 4 = 13x_1 - 6 \Rightarrow x = 1/2$

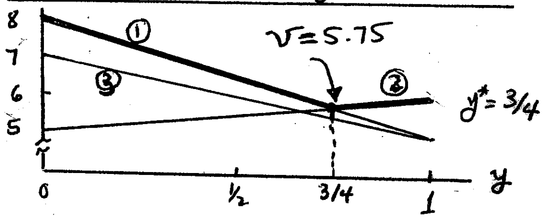
A's game: $x_1 = x_2 = .5,\ v = .5$

B's game: mix B's ② and ③

$-10y_2 + 7 = 10y_2 - 6 \Rightarrow y_2 = 13/20,\ y_3 = 7/20$

(b)

A's pure strategy	B's exp. payoff
1	$-3y+8$
2	$y+5$
3	$-2y+7$



B's game: mix B's ① and ②

$-x_1 + 6 = 3x_1 + 5 \Rightarrow x_1 = 1/4,\ x_2 = 3/4,\ x_3 = 0$

4

(a)

A's strategy	B's exp. payoff
1	$5(\frac{49}{54}) + 50(\frac{5}{54}) + 50(0) = \frac{55}{6}$
2	$1(\frac{49}{54}) + 1(\frac{5}{54}) + .1(0) = 1$
3	$10(\frac{49}{54}) + 1(\frac{5}{54}) + 10(0) = \frac{55}{6}$

$\max\limits_{i}(\text{exp. payoffs}) = \boxed{55/6}$

B's strategy	A's exp. payoff
1	$5(\frac{1}{6}) + 1(0) + 10(5/6) = \frac{55}{6}$
2	$50(\frac{1}{6}) + 1(0) + 1(5/6) = 55/6$
3	$50(1/6) + .1(0) + 10(5/6) = 100/6$

$\min\limits_{j}(\text{exp. payoffs}) = \boxed{55/6}$

value of the game $= 55/6$

(b)

$$v = \left(5(\tfrac{1}{6}) + 1\times 0 + 10\times\tfrac{5}{6}\right)\left(\tfrac{49}{54}\right) + \left(50\times\tfrac{1}{6} + 1\times 0 + 1\times\tfrac{5}{6}\right)(5/54) + (50\times 1/6 + .1\times 0 + 10\times 5/6)\times 0 = \frac{55}{6}$$

Set 13.4c

1

Team 2

$$\begin{array}{cc|cccccc} & & AB & AC & AD & BC & BD & CD \\ \hline & AB & 1 & 0 & 0 & 0 & 0 & -1 \\ & AC & 0 & 1 & 0 & 0 & -1 & 0 \\ \text{Team } 1 & AD & 0 & 0 & 1 & -1 & 0 & 0 \\ & BC & 0 & 0 & -1 & 1 & 0 & 0 \\ & BD & 0 & -1 & 0 & 0 & 1 & 0 \\ & CD & -1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Team 1 LP:

Maximize $z = v$

s.t.

$$\begin{array}{lllllllll} v & -x_1 & & & & & +x_6 & \le 0 \\ v & & -x_2 & & & +x_5 & & \le 0 \\ v & & & -x_3 & +x_4 & & & \le 0 \\ v & & & +x_3 & -x_4 & & & \le 0 \\ v & & +x_2 & & & -x_5 & & \le 0 \\ v & +x_1 & & & & & -x_6 & \le 0 \\ & x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & = 1 \end{array}$$

v unrestricted $x_j \geq 0$

Team 1 Solution: $x_1 = x_6 = .5$, all others $= 0$
Team 2 Solution: $y_1 = y_6 = .5$, all others $= 0$ $\Big\}\; v = 0$

2

(a) Maximize $z = v$

s.t.

$$\begin{array}{lllll} v - 3x_1 & -2x_2 & +x_3 & +x_4 & \le 0 \\ v + 2x_1 & -3x_2 & -2x_3 & +2x_4 & \le 0 \\ v - x_1 & +3x_2 & +2x_3 & -4x_4 & \le 0 \\ v - 2x_1 & & -2x_3 & -x_4 & \le 0 \\ \quad x_1 & +x_2 & +x_3 & +x_4 & = 1 \end{array}$$

v unrestricted, all $x_j \geq 0$

(b) Solution:

value of game $= .5$ in favor of UA

UA strategy: $x_2 = x_4 = .5$, all others $= 0$

DU strategy: $x_2 = .58$, $x_3 = .42$,
all others $= 0$

(c) Expected number of points
$= 60 \times .5 = 30$
in favor of UA

3

$(n_1, n_2) =$ Blotto's allocation between the two posts
$= \{(2,0), (1,1), (0,2)\}$

Enemy's allocation $= \{(3,0), (2,1), (1,2), (0,3)\}$

(a)

$$\begin{array}{c|cccc} & (3,0) & (2,1) & (1,2) & (0,3) \\ \hline (2,0) & -1 & -1 & 0 & 0 \\ (1,1) & 0 & -1 & -1 & 0 \\ (0,2) & 0 & 0 & -1 & -1 \end{array}$$

Maximize $z = v$

s.t.

$$\begin{array}{lllll} v & +x_1 & & & \le 0 \\ v & +x_1 & +x_2 & & \le 0 \\ v & & +x_2 & +x_3 & \le 0 \\ v & & & +x_3 & \le 0 \\ & x_1 & +x_2 & +x_3 & = 1 \end{array}$$

(b) v unrestricted, $x_1, x_2, x_3 \geq 0$

Solution: $v = -\frac{1}{2} \Rightarrow$ enemy wins
$x_1 = .5$, $x_2 = 0$, $x_3 = .5$
$y_1 = .5$, $y_2 = y_3 =$ $y_4 =$

4

$(a, b) =$ (Nbr. shown, Nbr. guessed)

$$\begin{array}{cc|cccc} & & (1,1) & (1,2) & (2,1) & (2,2) \\ \hline & (1,1) & 0 & 2 & -3 & 0 \\ A & (1,2) & -2 & 0 & 0 & 3 \\ & (2,1) & 3 & 0 & 0 & -4 \\ & (2,2) & 0 & -3 & 4 & 0 \end{array}$$

Maximize $z = v$

s.t.

$$\begin{array}{llllll} v & & 2x_2 & -3x_3 & & \le 0 \\ v & -2x_1 & & & +3x_4 & \le 0 \\ v & +3x_1 & & & -4x_4 & \le 0 \\ v & & -3x_2 & +4x_3 & & \le 0 \\ & x_1 & +x_2 & +x_3 & +x_4 & = 1 \end{array}$$

v unrestricted, $x_j \geq 0$

Solution:

Player A:
$x_1 = 0$, $x_2 = .571$, $x_3 = .429$, $x_4 = 0$

Player B:
$y_1 = 0$, $y_2 = .571$, $y_3 = .429$, $y_4 = 0$

value of the game $= 0$

Chapter 14

Probabilistic Inventory Models

Set 14.1a

1

(a) Effective lead time $L = 15-10 = 5$ days

$\mu_L = 100 \times 5 = 500$ units

$\sigma_L = \sqrt{10^2 \times 5} = 22.36$ units

$B \geq 22.36 \times 1.645 \simeq 37$ units

Order 1000 units whenever the inventory level drops to 537 units

(b) Effective lead time $L = 23-20 = 3$ days

$\mu_L = 100 \times 3 = 300$ units

$\sigma_L = \sqrt{10^2 \times 3} = 17.32$ units

$B \geq 17.32 \times 1.645 \simeq 29$ units

Order 1000 units whenever the inventory level drops to 329 units

(c) Effective lead time = 8 days

$\mu_L = 100 \times 8 = 800$ units

$\sigma_L = \sqrt{10^2 \times 8} = 28.28$ units

$B \geq 28.28 \times 1.645 \simeq 47$ units

(d) Effective lead time = 0

$\mu_L = \sigma_L = 0$, $B \geq 0$

Order 1000 units whenever the inventory level drops to 0 unit.

2

Demand/day = N (200, 20)

$h = \$.04$/day/unit, $K = \$100$, $L = 7$ days

order quantity $= \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 100 \times 200}{.04}}$

$= 1000$ units

Cycle length $= \frac{1000}{200} = 5$ days

Effective lead time $= 7-5 = 2$ days

$\mu_L = 200 \times 100 = 200$ units $K_{.02} = 2.06$

$\sigma_L = \sqrt{20^2 \times 2} = 28.28$

$B \geq 28.28 \times 2.06 = 58.27 = 59$ discs

Order 1000 discs whenever the inventory level drops to 459 units.

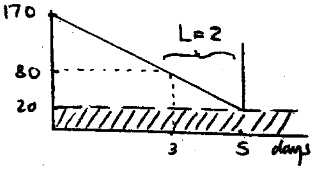
3

Demand/day = N(30,5)

$h = \$.02$/day/unit, $K = \$30$

(a) $L = \frac{80-20}{30}$

$= 2$ days



$\mu_L = 60$ units

$\sigma_L = \sqrt{5^2 \times 2} \simeq 7.07$ units

$P\{\text{demand during } L \geq 80\}$

$= P\{z \geq \frac{80-60}{7.07}\}$

$= P\{z \geq 2.83\}$

$= 1 - .9977 = .0023$

(b) $y = \sqrt{\frac{2 \times 30 \times 30}{.02}} = 300$ rolls

Cycle length $= \frac{300}{30} = 10$ days

Lead time = 2 days

$\mu_L = 2 \times 30 = 60$ units

$\sigma_L = \sqrt{5^2 \times 2} = 7.07$ units

$K_{.1} = 1.28$

$B \geq 7.07 \times 1.28 \simeq 10$

Order 300 rolls whenever the inventory level drops to 70 rolls.

1

(a) $D/y = \frac{1000}{320} = 3.125$ setups

(b) $\frac{KD}{y} = 100 \times 3.125 = \$\,312.50$ /month

(c) $h\left(\frac{y}{2} + R - E\{x\}\right) = 2\left(\frac{320}{2} + 94 - 50\right)$
$= \$\,408$

(d) $pS = 10 \times .20397 \cong \$\,2.04$

(e) $\int_R^\infty f(x)dx = \int_{94}^{100} \frac{1}{100} dx = \frac{100-94}{100} = .06$

2

D = 1000 gallons per month
K = \$100, h = \$2/gal/month
p = \$10/gal.

$f(x) = \frac{1}{50}, \quad 0 \le x \le 50 \quad , E\{x\} = 25$

$\hat{y} = \sqrt{\frac{2 \times 1000(100 + 10 \times 25)}{2}} = 591.6$

$\tilde{y} = \frac{PD}{h} = \frac{10 \times 1000}{2} = 5000$

$\tilde{y} > \hat{y} \Rightarrow$ unique solution exists

$S = \int_R^{50} (x-R)\frac{1}{50}dx = \frac{R^2}{100} - R + 25$

$y_i = \sqrt{\frac{2 \times 1000(100 + 10S)}{2}} = \sqrt{100{,}000 + 10000S}$

$\int_{R_i}^{50} \frac{1}{50} dx = \frac{2y_i}{5000} \Rightarrow R_i = 50 - \frac{y_i}{100}$

Iteration 1:

$S = 0$

$y_1 = \sqrt{100{,}000} = 316.23$ gal

$R_1 = 50 - \frac{316.23}{100} = 46.84$ gal

Iteration 2:

$S = \frac{46.84^2}{100} - 46.84 + 25 = .099856$

$y_2 = \sqrt{100{,}000 + 10{,}000 \times .099856} = 317.80$

$R_2 = 50 - \frac{317.80}{100} = 46.82$

Iteration 3:

$S = \frac{46.82^2}{100} - 46.82 + 25 = .101124$

continued...

2 continued

$y_3 = \sqrt{100{,}000 + 10{,}000 \times .101124} = 317.82$

$R_3 = 50 - \frac{317.82}{100} = 46.82$

Optimum solution:

$y^* \cong 318$ gal, $R^* \cong 47$ gal

3

$f(x) = \frac{1}{20}, \ 40 \le x \le 60, \ E\{x\} = 50$

$\hat{y} = \sqrt{\frac{2 \times 1000(100 + 10 \times 50)}{2}} = 774.6$ gal

$\tilde{y} = \frac{10 \times 1000}{2} = 5000$ gal

$\tilde{y} > \hat{y} \Rightarrow$ unique solution exists

$S = \int_R^{60} (x-R)\frac{1}{20}dx = \frac{1}{20}\left[\frac{x^2}{2} - Rx\right]_R^{60}$
$= \frac{R^2}{40} - 3R + 90$

$y_i = \sqrt{100{,}000 + 10{,}000S}$

$\int_{R_i}^{60} \frac{1}{20} dx = \frac{2y_i}{10 \times 1000} \Rightarrow R_i = 60 - \frac{y_i}{250}$

Iteration 1:

$S = 0$

$y_1 = \sqrt{100{,}000} = 316.23$ gal

$R_1 = 60 - \frac{316.23}{250} = 58.735$

Iteration 2:

$S = \frac{58.7}{40} - 3 \times 58.735 + 90 = .04$

$y_2 = \sqrt{100{,}000 + 10{,}000 \times .04} = 316.823$

$R_2 = 60 - \frac{316.823}{250} = 58.733$ gal

Optimum solution:

$y^* = 316.85 \simeq 317$ gal.
$R^* = 58.73 \simeq 59$ gal.

R^* in the present model is smaller than R^* in Example because f(x) has a smaller variance, and hence less uncertainty.

4

For the normal distribution, it can be shown that the following approximation holds

$$S = \int_R^\infty (x-R)\, f(x)\, dx$$

$$\simeq \sqrt{Var\{x\}}\, L(R_s) \qquad ①$$

where

$Var\{x\}$ = variance of x given $f(x)$

$$R_s = \frac{R - E\{x\}}{\sqrt{Var\{x\}}} \qquad ②$$

$L(R_s)$ = standard normal loss integral

$$= \int_{R_s}^\infty (z - R_s)\, \Phi(z)\, dz$$

$\Phi(z)$ is $N(0,1)$. The values of $L(\cdot)$ can be found in standard statistical tables

$$\int_R^\infty f(x)\,dx = \frac{hy}{pD}$$

or

$$\int_R^\infty \Phi(z)\,dz = \frac{hy}{pD} \qquad ③$$

The steps of the solution algorithm are:

1. Compute first trial

$$y = \sqrt{\frac{2KD}{h}}$$

2. Compute R_s from ③ using the current value of y and the standard normal tables

3. Compute R from ② using the current value of R_s; that is,

$$R = E\{x\} + R_s \sqrt{var\{x\}}$$

4 continued

If two successive values of R are approximately equal, stop. Otherwise, go to step 4

4. Compute S from ① using standard normal loss integral tables. Then find

$$y = \sqrt{\frac{2D(K+pS)}{h}}$$

Go to step 2.

Set 14.2a

1

$$E\{C(y)] = h\sum_{D=0}^{y}(y-D)f(D) + p\sum_{D=y+1}^{\infty}(D-y)f(D)$$

Consider $E\{C(y)\} \le E\{C(y-1)\}$:

$$E\{C(y-1)\} = h\sum_{D=0}^{y-1}(y-1-D)f(D) + p\sum_{D=y}^{\infty}(D-y+1)f(D)$$

$$= h\sum_{D=0}^{y-1}(y-D)f(D) + p\sum_{D=y}^{\infty}(D-y)f(D) - h\sum_{D=0}^{y-1}f(D) + p\sum_{D=y}^{\infty}f(D) - c$$

$$= E\{C(y)\} + p - (h+p)\sum_{D=0}^{y-1}f(D)$$

Thus,

$$E\{C(y-1)\} - E\{C(y)\} = p - (h+p)P\{D\le y-1\} \ge 0$$

Hence

$$P\{D \le y-1\} \le \frac{p}{p+h}$$

Similarly, it can be shown that

$$P\{D \le y\} \ge \frac{p}{p+h}$$

Thus, y^* must satisfy

$$P\{D \le y^*-1\} \le \frac{p}{p+h} \le P\{D \le y^*\}$$

2

$$f(D) = \frac{1}{5}, \quad 10 \le D \le 15$$

$$\int_{10}^{y} f(D)dD \le .1:$$

$$\int_{10}^{y}\frac{1}{5}dD = \frac{y-10}{5} \le .1 \Rightarrow y \le 10.5$$

$$\int_{y}^{15} f(D)dD \le .1:$$

$$\int_{y}^{15}\frac{1}{5}dD = \frac{15-y}{5} \le .1 \Rightarrow y \ge 14.5$$

The two conditions cannot be satisfied simultaneously.

3

$$q = \frac{p}{p+h} = \frac{p}{p+1}$$

y	0	1	2	3	4	5	6
$P\{D\le y\}$	.05	.15	.25	.45	.7	.85	.9

↑ $y = 4$

From the CDF,

$$P\{D \le 4-1\} = .45$$

$$P\{D \le 4\} = .7$$

Thus, $$.45 \le \frac{p}{p+1} \le .7$$

or $.43 \le p \le .82$

4

Maximize expected revenue.

$$E\{revenue\} = -10y + \int_{200}^{y} 25D\,f(D)dD + \int_{y}^{250} 25y\,f(D)dD$$

$$= -10y + \frac{25D^2}{100}\Big]_{200}^{y} + \frac{25y}{50}D\Big]_{y}^{250}$$

$$= -.25y^2 + 115y - 10{,}000$$

$$\frac{\partial E\{revenue\}}{\partial y} = -.5y + 115 = 0$$

$y = 230$ copies

5

$$E\{revenue\} = -7y + \int_{90}^{y}[25D + 5(y-D)]f(D)dD + \int_{y}^{150} 25y\,f(D)dD$$

$$= -\frac{y^2}{6} + 48y - 1350$$

$$\frac{\partial E\{revenue\}}{\partial y} = \frac{-y}{3} + 48$$

$y = 144$ donuts

Decision: Stock 12 dozens

Set 14.2a

6

Use continuous pdf as an approximation

$E\{revenue\} =$

$$-50y + \int_{20}^{y}[110D + 55(y-D)]f(D)\,dD + \int_{y}^{30} 110y\, f(D)\,dD$$

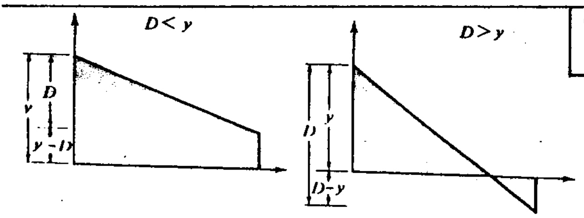
$$= -50y + \frac{1}{10}\left[55yD + \frac{55D^2}{2}\right]_{20}^{y} + 110y\left[\frac{D}{10}\right]_{y}^{30}$$

$$= -2.75y^2 + 175y - 1100$$

$$\frac{\partial E\{revenue\}}{\partial y} = -5.5y + 175 = 0$$

$$y \simeq 32 \text{ coats}$$

7



Average holding inventory = $y - \frac{D}{2}$ Average holding inventory = $\frac{y^2}{2D}$

Average shortage inventory = 0 Average shortage inventory = $\frac{(D-y)^2}{2D}$

$$E\{C(y)\} = c(y-x) + h\left\{\int_0^y \left(y - \frac{D}{2}\right)f(D)\,dD + \int_y^{\infty}\frac{y^2}{2D}f(D)\,dD\right\} + p\int_y^{\infty}\frac{(D-y)^2}{2D}f(D)\,dD$$

$$\frac{\partial E\{\}}{\partial y} = c + \left(\int_0^y f(D)\,dD + \int_y^{\infty}\frac{y}{D}f(D)\,dD\right) - p\int_{,}^{\infty}\left(\frac{D-y}{D}\right)f(D)\,dD = 0$$

$$\int_0^{y^*} f(D)\,dD + y^*\int_{y^*}^{\infty}\frac{f(D)}{D}\,dD = \frac{p-c}{p+h}$$

8

$$f(D) = \frac{1}{100}, \quad 0 \le D \le 100$$

$$\int_0^y f(D)\,dD + y\int_y^{100}\frac{f(D)}{D}\,dD = \frac{p-c}{p+h}$$

$$\int_0^y \frac{1}{100}\,dD + y\int_y^{100}\frac{1}{100D}\,dD = \frac{p-c}{p+h}$$

$$\frac{y}{100} + \frac{y}{100}(\ln 100 - \ln y) = \frac{p-c}{p+h}$$

$$.056y - .01y\ln y = \frac{45-30}{45+25} = .2143$$

Trial and error yield $y^* \simeq 5.5$ units

1

$E\{C(s)\} = K + E\{C(S)\}$

$.25s^2 - 4.5s + 22.5 = 5 + .25S^2 - 4.5S + 22.5$

$.25s^2 - 4.5s + 15.25 = 0$ (for $S = 9$)

Solution: $s = (4.53$ or $13.47)$

Policy: If $x < 4.53$, order $9 - x$

$x \geq 4.53$, do not order

2

$$E\{R(y)\} = -c(y-x) + \int_0^y [rD - h(y-D)]f(D)dD + \int_y^\infty [ry - p(D-y)]f(D)dD$$

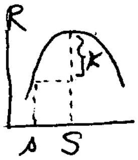
$$\frac{\partial E\{\ \}}{\partial y} = -c - \int_0^y h f(D)dD + ryf(D) + \int_y^\infty (r+p)f(D)dD - ryf(D) = 0$$

Thus,

$$\int_0^{y^*} f(D)dD = \frac{r+p-c}{r+p-h}$$

In the presence of setup cost, we have an s-S policy. Define s such that

$$E\{R(s)\} = E\{R(S)\} - K$$



For the numeric problem,

$E\{R(y)\} = .4y^2 + 5y - 20 - 2x$

$$\int_0^S f(D)dD = \frac{3+4-2}{3+4-1} = .625$$

Thus, $S = 6.25$

Next, $-.4s^2 + 5s - 5.625 = 0$

Thus, $s = 1.25$

Policy:

If $x < 1.25$, order $6.25 - x$

$x \geq 1.25$, do not order

3

$$-\frac{s^2}{6} + 48s - 1350 = -10 - \frac{144^2}{6} + 48 \times 144 - 1350$$

Thus,

$$s^2 - 288s + 20676 = 0$$

$$s = \begin{cases} \boxed{136.25} \\ 151.25 \end{cases}$$

Optimal policy

If $x < 136$, order $144 - x$

$x \geq 136$, do not order

1

$$L(y_i) = \int_0^{y_i} (rD - h(y_i - D)) f(D)\,dD + \int_{y_i}^{\infty} (ry_i + (\alpha r' - p)(D - y_i) f(D)\,dD \qquad i = 1, 2$$

where

$$r' = \begin{cases} r & i = 1 \\ r - c & i = 2 \end{cases}$$

$$g_2(x_2) = \max_{y_2 \geq x_2} \{-c(y_2 - x_2) + L(y_2)\}$$

$$g_1(x_1) = \max_{y_1 \geq x_1} \{-c(y_1 - x_1) + L(y_1) + \alpha E\{g_2(y_1 - D)\}\}$$

For period 2:

$$\frac{\partial f_2(y_2 | x_2)}{\partial y_2} = -c + L'(y_2^*) = 0$$

or

$$\int_0^{y_2^*} f(D)\,dD = \frac{r + p - c - \alpha(r - c)}{r + p + h - \alpha(r - c)}$$

$$g_2(y_1 - D) = \begin{cases} L_2(y_1 - D), & D \leq y_1 - y_2^* \\ -c(y_2^* - y_1 + D) + L(y_2^*), & D \geq y_1 - y_2^* \end{cases}$$

$$E\{\ (y_1 - D)\} = \int_0^{y_1 - y_2^*} L_2(y_1 - D) f(D)\,dD + \int_{y_1 - y_2^*}^{\infty} (-c(y_2^* - y_1 + D) + L(y_2^*)) f(D)\,dD$$

This, when substituted in the expression for $g_1(x_1)$, will yield total expected profit in terms of y_1. Hence, the value of y_1^* can be obtained.

In terms of the given numerical example, we have

$$\frac{1}{10} \int_0^{y_2^*} dD = \frac{2 + 3 - 1 - .8(2 - 1)}{2 + 3 + .1 - .8(2 - 1)} = .75$$

Thus, $y_2^* = 7.5$

$$L(z) = \frac{1}{10} \left\{ \int_0^z (2D - .1(z - D))\,dD + \int_z^{10} (2z + (.8r' - 3)(D - z)\,dD \right\}$$

continued...

1 continued

$$= (.04r' - .255)z^2 + (5 - .8r')z + (4r' - 15)$$

Hence

$$L(y_2) = [.04(2 - 1) - .255]y_2^2 + [5 - .8(2 - 1)]y_2 + [4(2 - 1) - 15]$$

$$= -.215y_2^2 + 4.2y_2 - 11$$

$$L(y_2^*) = L(7.5) = 8.4$$

$$g_2(y_1 - D) = \begin{cases} -.215(y_1 - D)^2 + 4.2(y_1 - D) - 11, & D \leq y_1 - 7.5 \\ .9 - y + D, & D \geq y_1 - 7.5 \end{cases}$$

$$E\{g_2(y_1 - D)\} = \frac{1}{10} \left\{ \int_0^{y_1 - 7.5} [-.215(y_1 - D)^2 + 4.2(y_1 - D) - 11]\,dD + \int_{y_1 - 7.5}^{10} (.9 + y - D)\,dD \right\}$$

$$= \frac{1}{10}(-.072y_1^3 + 2.115y_1^2 - 11y_1 - 5 - y_1^2 - 5.4y_1 - 19.625)$$

$$= \frac{1}{10}(-.072y_1^3 + 1.115y_1^2 - 16.4y_1 - 24.625)$$

$$L(y_1) = (.04 \times 2 - .255)y_1^2 + (5 - .8 \times 2)y_1 + (4 \times 2 - 15)$$

$$= -.175y_1^2 + 3.4y_1 - 7$$

$$g_1(x_1) = \max_{y_1 \geq x_1} \left\{-1(y_1 - x_1) - .175y_1^2 + 3.4y_1 - 7 + \frac{.8}{10}(-.07y_1^3 + 1.115y_1^2 - 16.4y_1 - 24.625)\right\}$$

$$= \max_{y_1 \geq x_1} \{-.00576y_1^3 - .075y_1^2 + .89y_1 - 8.97 + x_1\}$$

$$\frac{\partial\{\cdot\}}{\partial y_1} = -.01728y_1^2 - .15y_1 + .89 = 0$$

$$y_1^* = 9.02$$

continued...

1 continued

Optimal policy:

$$\text{Period } 1\begin{cases}\text{order } 9.02-x_1, & x_1\le 9.02\\ \text{order } 0, & x_1\ge 9.02\end{cases}$$

$$\text{Period } 2\begin{cases}\text{order } 7.5-x_2, & x_2\le 7.5\\ \text{order } 0, & x_2\ge 7.5\end{cases}$$

For the infinite model:

$$\frac{1}{10}\int_0^{y^*}dD=\frac{3+.2(2-1)}{3+.1+.2\times 2}=.915$$

$$y_1^*=9.15>y_2^*>y_1^*$$

2

$$\int_0^{y^*}f(D)\,dD=.08\int_0^{y^*}D\,dD$$

$$=.04\,y^{*2}$$

Thus,

$$.04\,y^{*2}=\frac{p+(1-\alpha)(r-c)}{p+h+(1-\alpha)r}$$

$$=\frac{10+.1\times 2}{10+1+.1\times 10}=.85$$

Thus, $y^*=4.61$

Policy:

order $4.61-x$, if $x\le 4.61$

order 0, if $x\ge 4.61$

3

$$g(x)=\min_{y\ge x}\left\{c(y-x)+h\int_0^y(y-D)^2f(D)\,dD+p\int_y^\infty(D-y)^2f(D)\,dD+\alpha\int_0^\infty g(y-D)f(D)\,dD\right\}$$

$$\frac{\partial\{\cdot\}}{\partial y}=c+2h\int_0^y(y-D)f(D)\,dD-2p\int_y^\infty(D-y)f(D)\,dD+\alpha E\{g'(y-D)\}$$

Continued...

3 continued

Since this is a cost function,

$$g'(y-D)=-c.$$

Now, $\frac{\partial\{\cdot\}}{\partial y}=0$ yields,

$$\left\{(1-\alpha)c+2hy^*\int_0^{y^*}f(D)\,dD-2h\int_0^{y^*}Df(D)\,dD+2py^*\left(1-\int_0^{y^*}f(D)\,dD\right)-2pE\{D\}+2p\int_0^{y}Df(D)\,dD\right\}=0$$

This simplifies to

$$(h-p)\left\{y^*\int_0^{y^*}f(D)\,dD-\int_0^{y^*}Df(D)\,dD\right\}+py^*=\frac{2pE\{D\}-(1-\alpha)c}{2} \quad ①$$

or

$$y^*\left\{\frac{1}{h-p}+\int_0^{y^*}f(D)\,dD-\int_0^{y^*}Df(D)\,dD\right.=\frac{2pE\{D\}-(1-\alpha)c}{2(h-p)}$$

y^* can be determined from the last equation. When $h=p$, ① yields

$$y^*=\frac{2pE\{D\}-(1-\alpha)c}{2p}$$

This result is independent of $f(D)$ except in so far as $E\{D\}$ is concerned.

Chapter 15

Queuing Systems

Set 15.1a

1

(a) Efficiency = 100 − 29 = 71%

(b) For average waiting time ≤ 3 minutes, at least 5 cashiers are needed

For efficiency ≥ 90%, the associated idleness percentage is ≤ 10%. The corresponding number of cashiers is at most 2.

Conclusion:

The two conditions cannot be satisfied simultaneously. At least one of the two conditions must be relaxed.

2

C_A = \$18 per hour

C_B = \$25 per hour

Length of queue A = 4 jobs

Length of queue B = .7 × 4 = 2.8 jobs

Cost of A = \$18 + 4 × \$10 = \$58 per hour

Cost of B = \$25 + 2.8 × \$10 = \$53 per hour

Decision:

Select Model B.

Set 15.2a

1

Situation	Customer	Server
a	Plane	Runway
b	Passenger	Taxi
c	Machinist	Clerk at tool crib
d	Letter	Clerk
e	Student	Registrar's office
f	Cases	Judge
g	Shopper	Cashier
h	Car	Parking space

2

Situation	Calling Source	Customers arrival
a	∞	Individual
b	∞	Individual
c	∞	Individual
d	∞	Bulk
e	∞	Individual
f	∞	Individual
g	∞	Individual
h	∞	Individual

Situation	Interarrival time	Service time
a	Probabilistic	Time to clear runway
b	Probabilistic	Ride time
c	Probabilistic	Time to receive tool
d	Deterministic	Time to process letter
e	Probabilistic	Time to process registr$^{\underline{n}}$
f	Probabilistic	Trial time
g	Probabilistic	check-out time
h	Probabilistic	Parking time.

Situation	Queue Capacity	Queue Discipline
a	∞	FIFO
b	∞	FIFO
c	∞	FIFO
d	∞	Random
e	∞	FIFO
f	∞	FIFO
g	∞	FIFO
h	0	None

3

#	Queueing situation	Customers
1	Arrival of orders	Orders
2	Processing (single machine)	Rush orders
3	Processing (single machine)	Regular jobs
4	Processing (Prod. line)	Rush jobs
5	Processing (Prod. line)	Regular jobs
6	Receipt of completed jobs	Completed orders
7	Tool crib	Tools
8	Machine breakdown	machines

#	Servers	Discipline	Service time	Queue length	Source
1	Foreman	Priority	Sorting time	∞	∞
2	machine	FIFO	Prod. time	∞	∞
3	machine	FIFO	Prod. time	∞	∞
4	Prod. line	FIFO	Prod. time	∞	∞
5	Prod. line	FIFO	Prod. time	∞	∞
6	Shipping facilities	FIFO	Loading time	finite	∞
7	Tool crib	Priority	Exchange time	finite	finite
8	Repair persons	Priority	Repair time	finite	finite

4

(a) T. (b) T. (c) T.

5

(a) None.
(b) None.
(c) None.
(d) None.
(e) Jockey or balk
(f) None
(g) Jockey
(h) None

Set 15.3a

1

(a) Av. interarrival time (in time units)

$$= \frac{1}{\text{arrival rate } \lambda \text{ (in customers/unit time)}}$$

(b) Let $\bar{I}$ = av. interarrival time

(i) $\lambda = \frac{60}{10} = 6$ arrivals/hr

$\bar{I} = 10$ minutes $= \frac{1}{6}$ hour

(ii) $\lambda = \frac{60}{3} = 20$ arrivals/hr

$\bar{I} = \frac{6}{2} = 3$ minutes $= \frac{1}{20}$ hr

(iii) $\lambda = \frac{10}{30} \times 60 = 20$ arrivals/hr

$\bar{I} = \frac{30}{10} = 3$ minutes $= \frac{1}{20}$ hour

(iv) $\lambda = 1/.5 = 2$ arrivals/hour

$\bar{I} = .5$ hour

(c) Let $\bar{S}$ = av. service time

(i) $\mu = \frac{60}{12} = 5$ services/hour

$\bar{S} = 12$ minutes $= .2$ hour

(ii) $\mu = \frac{60}{7.5} = 8$ services/hr

$\bar{S} = 7.5$ min $= .125$ hr

(iii) $\mu = \frac{5}{30} \times 60 = 10$ services/hr

$\bar{S} = \frac{30}{5} = 6$ min $= 1/10$ hr

(iv) $\mu = \frac{1}{.3} = 3.33$ services/hr

$\bar{S} = .3$ hour

2

(a) $\lambda_{hour} = .2$ failures/hr

$\lambda_{week} = .2 \times 24 \times 7 = 33.6$ failures/wk

(b) P{at least one failure in 2 hours}

= P{time betn. failures ≤ 2}

$= P\{t \le 2\} = 1 - e^{-.2 \times 2} \cong .33$

(c) $P\{t > 3 \text{ hrs}\} = 1 - P\{t \le 3\} = e^{-.2 \times 3} \cong .55$

(d) $P\{t \le 1 \text{ hour}\} = 1 - e^{-.2 \times 1} = .18$

3

$\lambda = \frac{1}{.05} = 20$ arrivals/hr

(a) $f(t) = \lambda e^{-\lambda t}$

$= 20 e^{-20t}, \quad t > 0$

(b) $P\{t > \frac{15}{60}\} = P\{t > .25\}$

$= e^{-20 \times .25}$

$= .00674$

(c) $P\{t \le \frac{3}{60}\} = P\{t \le .05\}$

$= 1 - e^{-20 \times .05} = .632$

$P\{t > \frac{5}{60}\} = e^{-\frac{20 \times 5}{60}} = .189$

(d) $t = 45 - 10 = 35$ minutes

Av. # of arrivals in 35 min.

$= 20 \times \frac{35}{60} = 11.67$ arrivals

4

$\lambda = \frac{1}{6}$ arrivals/hr

$P\{t \ge 1\} = e^{-1/6 \times 1} = .846$

$P\{t \le .5\} = 1 - e^{-1/6 \times .5}$

$= 1 - e^{-1/12} = .08$

5

(a) $\lambda = \frac{60}{10} = 6$ arrivals/hr

(b) $P\{t \ge \frac{15}{60}\} = e^{-6 \times \frac{15}{60}} = .223$

(c) $P\{t \le \frac{20}{60}\} = 1 - e^{-6 \times \frac{20}{60}} = .865$

6

(a) $P\{t \le \frac{2}{60}\} = 1 - e^{-35(2/60)}$

$= .6886$

(b) $P\{\frac{2}{60} \le t \le \frac{3}{60}\}$

$= P\{t \le 3/60\} - P\{t \le \frac{2}{60}\}$

$= (1 - e^{-35 \times 3/60}) - (1 - e^{-35 \times 2/60})$

$= e^{-70/60} - e^{-105/60} = .1376$

(c) $P\{t \geq 3/60\} = e^{-35(3/60)}$

$= .1738$

7

$\lambda = \frac{60}{1.5} = 40$ arrivals/hr

Jim's Payoff	−2¢	+2¢
Prob.	$P\{t \ge 1\}$	$P\{t \le 1\}$

$P\{t \ge 1\} = e^{-40(1/60)} = .5134$

$P\{t \le 1\} = 1 - .5134 = .4866$

Jim's exp. payoff/arriving customer

$= -2 \times .5134 + 2 \times .4866$

$= -.0536$ Cent

Jim's exp. payoff/8 hours

$= -.0536(8\lambda)$

$= -.0536 \times 8 \times 40$

$\cong -17.15$ cent

Conclusion: Jim will pay Ann an average of 17 cents every 8 hrs

8

Jim's payoff	2	0	−2
Probability	$P\{t \le 1\}$	$P\{1 \le t \le 1.5\}$	$P\{t \ge 1.5\}$

$P\{t \le 1\} = .4866$

$P\{t \ge 1.5\} = e^{-40(1.5/60)}$

$= .3679$

2	0	−2
.4866	.1455	.3679

Jim's expected payoff/8 hours

$= [2 \times .4866 + 0 \times .1455 - 2 \times .3679] \times 40 \times 8$

$\cong 76$ cents

9

2¢	3¢	−5¢	−6¢
$t \le 1$	$1 \le t \le 1.5$	$1.5 \le t \le 2$	$t \ge 2$

$\lambda = 40$ arrivals/hr

$P\{t \le 1\} = 1 - e^{-40/60} = .4866$

$P\{1 \le t \le 1.5\} = e^{-40(1/60)} - e^{-40(1.5/60)}$

$= .1455$

$P\{1.5 \le t \le 2\} = e^{-40(1.5/60)} - e^{-40(2/60)}$

$= .1043$

$P\{t \ge 2\} = e^{-40(2/60)} = .2636$

Jim's exp. payoff/8 hours

$= 8 \times 40 (2 \times .4866 + 3 \times .1455 - 5 \times .1043 - 6 \times .2636)$

$\simeq -222$ cents

Jim pays Ann an average of $2.22/8 hours.

10

(a) $\lambda = \frac{60}{6} = 10$ customers/hr

$P\{t \le 4 \text{ min}\} = 1 - e^{-10(4/60)}$

$= .4866$

(b)

$$\%\text{discount} = \begin{cases} 10\%, & \text{if } t \le 4 \\ 6\%, & \text{if } 4 < t \le 5 \\ 2\%, & \text{if } t > 5 \end{cases}$$

$P\{t \le 4\} = .4866$

$P\{4 < t \le 5\} = e^{-10(4/60)} - e^{-10(5/60)}$

$= .0788$

$P\{t > 5\} = e^{-10(5/60)} =$

$= .4346$

Expected % discount

$= 10 \times .4866 + 6 \times .0788 + 2 \times .4346$

$= 6.208\%$

11

$$\lambda = \frac{365 \times 24}{9000} = .973 \text{ failure/yr}$$

$$P\{t \le 1\} = 1 - e^{-.973 \times 1}$$

$$= .622$$

12

Lack-of-memory property applies.

(a) The waiting time for the green bus is exponential with mean 10 minutes:

$$f(t) = .1\, e^{-.1t}, \quad t \ge 0$$

(b) The waiting time for the red bus is exponential with mean 7 minutes:

$$f(t) = \frac{1}{7} e^{-t/7}, \quad t \ge 0$$

13

$$E\{t\} = \int_0^\infty t \lambda e^{-\lambda t} dt$$

$$= -\int_0^\infty t\, d\, e^{-\lambda t}$$

$$= -\left(t e^{-\lambda t} - \int_0^\infty e^{-\lambda t} dt\right)$$

$$= -\left(t e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t}\right)\Big]_0^\infty$$

$$= 1/\lambda$$

$$E\{t^2\} = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$$= -\int_0^\infty t^2 d e^{-\lambda t}$$

$$= -\left[t^2 e^{-\lambda t} - \int_0^\infty 2t\, e^{-\lambda t} dt\right]$$

continued...

$$= -\left[t^2 e^{-\lambda t} - \frac{2}{\lambda}\int_0^\infty t \lambda e^{-\lambda t} dt\right]_0^\infty$$

$$= + \frac{2}{\lambda^2}$$

$$Var\{t\} = E\{t^2\} - E^2\{t\}$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

1

TORA input = (5, 0, 0, ∞, ∞)

$$p_{n\geq 5}(t=1hr) = 1-[p_0(1)+\cdots+p_4(1)]$$
$$= 1-e^{-5}\left(1+5+\frac{5^2}{2!}+\frac{5^3}{3!}+\frac{5^4}{4!}\right)$$
$$= 1-.44049 = .55951$$

2

$\lambda = 1$ trip/month

(a) $\lambda t = 3$: TORA input = (3, 0, 0, ∞, ∞)

$$p_0(3) = \frac{(1\times3)^0 e^{-1\times3}}{0!} = .049787$$

(b) $\lambda t = 12$: TORA input = (12, 0, 0, ∞, ∞)

$$p_{n\leq 8}(t=12) = p_0(12)+\cdots+p_8(12)$$
$$= \frac{12^0 e^{-12}}{0!}+\frac{12^1 e^{-12}}{1!}+\cdots+\frac{12^8 e^{-12}}{8!}$$
$$= .15503$$

(c) $p_0(1) = \frac{1^0 e^{-1}}{0!} = e^{-1} = .3679$

TORA input = (1, 0, 0, ∞, ∞)

3

$\lambda = 2$ arrivals/minute

(a) $\lambda t = 2\times5 = 10$ arrivals

(b) $\lambda t = 2\times.5 = 1$

TORA input = (1, 0, 0, ∞, ∞)

$p_0(t=.5) = e^{-2\times.5} = .3679$

(c) $1-p_0(t=.5) = 1-.3679 = .6321$

(d) $\lambda t = 2\times3 = 6$ arrivals

TORA input = (6, 0, 0, ∞, ∞)

$$p_0(t=3) = \frac{(2\times3)^0 e^{-2\times3}}{0!} = .00248$$

4

$\lambda = 1/5 = .2$ arrival/min

(a) $p_2(t=7) = \frac{(.2\times7)^2 e^{-.2\times7}}{2!} = .24167$

TORA input = (1.4, 0, 0, ∞, ∞)

(b) $p_1(t=5) = \frac{(.2\times5)^1 e^{-.2\times5}}{1!} = .36788$

5

$\lambda = 25$ books per day

(a) $\lambda t = 25\times30 = 750$ books = 7.5 shelves

(b) 10 bookcases = $10\times5\times100 = 5000$ books

$$p_{n>5000}(t=30) = 1-[p_0(30)+\cdots+p_{5000}(30)]$$
$$\simeq 0$$

6

(a) $\lambda_{green} = .1$ stop/min, $\lambda_{red} = 1/7$ stop/min

λ combined $= .1+\frac{1}{7} = .24286$ stop/min

$$p_2(5) = \frac{(.24286\times5)^2 e^{-.24286\times5}}{2!} = .219$$

The two buses could be 2 G, 2R or 1G and 1R.

(b) $P\{t\leq2\} = 1-e^{-.243\times2} = .3849$

7

$$E\{n|t\} = \sum_{n=1}^{\infty} n\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
$$= \lambda t e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}$$
$$= \lambda t e^{-\lambda t} e^{+\lambda t} = \lambda t$$

$$E\{n^2|t\} = \sum_{n=0}^{\infty} n^2\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
$$= \sum_{n=1}^{\infty} n^2\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
$$= \lambda t e^{-\lambda t}\sum_{n=1}^{\infty}\frac{n(\lambda t)^{n-1}}{(n-1)!}$$
$$= \lambda t e^{-\lambda t}\frac{\partial}{\partial\lambda t}\left(\lambda t\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\right)$$
$$= \lambda t e^{-\lambda t}\frac{d}{\partial\lambda t}\left(\lambda t e^{\lambda t}\right)$$
$$= \lambda t e^{-\lambda t}\left(\lambda t e^{\lambda t}+e^{\lambda t}\right)$$
$$= (\lambda t)^2+\lambda t$$

Thus,

$$var\{n|t\} = (\lambda t)^2+\lambda t-(\lambda t)^2$$
$$= \lambda t$$

Set 15.4a

8

$$p_0'(t) = -\lambda\, p_0(t) \qquad (1)$$

$$p_n'(t) = -\lambda\, p_n(t) + \lambda\, p_{n-1}(t) \qquad (2)$$

From (1)

$$d\,p_0(t) = -\lambda\, p_0(t)\,dt$$

which yields

$$p_0(t) = A e^{-\lambda t}$$

Because $p_0(0) = 1 \Rightarrow A = 1$, $p_0(t) = e^{-\lambda t}$

For $n = 1$:

$$p_1'(t) = -\lambda\, p_1(t) + \lambda\, p_0(t)$$

$$= -\lambda\, p_1(t) + \lambda e^{-\lambda t}$$

or

$$p_1'(t) + \lambda\, p_1(t) = \lambda e^{-\lambda t}$$

This yields the solution:

$$p_1(t) = e^{-\int \lambda dt}\left\{\int \lambda e^{-\lambda t} e^{-\int \lambda dt}\, dt + C\right\}$$

$$= \lambda t\, e^{-\lambda t} + C$$

Because $p_1(0) = 0$, $C = 0$, and

$$p_1(t) = \frac{\lambda t\, e^{-\lambda t}}{1!}$$

<u>Induction proof</u>:

Given

$$p_i(t) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

then

$$p_{i+1}'(t) + \lambda\, p_{i+1}(t) = \lambda \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

The solution is

continued...

$$p_{i+1}(t) = e^{-\int \lambda dt}\left\{\frac{\lambda (\lambda t)^i e^{-\lambda t} e^{\int \lambda dt}}{i!}\, dt + C\right\}$$

$$= \frac{e^{-\lambda t}(\lambda t)^{i+1}}{(i+1)!} + C$$

Because $p_{i+1}(0) = 0$, $C = 0$, and

$$p_{i+1}(t) = \frac{e^{-\lambda t}(\lambda t)^{i+1}}{(i+1)!}$$

2

$\mu = 3$ dozens/day, $N = 18$

TORA input data $= (0, \mu t, 1, 18, 18)$

(a) $\mu = 3 \times 3 = 9$

$p_0(t=3) = .00532$ (from TORA)

(b) $\mu t = 3 \times 2 = 6$

$\sum_{n=0}^{18} n\, p_n(2) = 11.955$

(c) This part can be solved using Poisson or exponential distributions:

<u>Poisson</u>: $\mu t = 3 \times 1 = 3$

Probability $= p_0(1) + p_1(1) + \cdots + p_{17}(1)$

$= .9502$ (from TORA)

<u>Exponential</u>: mean $= 1/3$ day

P{purchasing at least one dozen in 1 day}

$= P\{\text{time between purchases} \le 1\}$

$= 1 - e^{-3 \times 1} = .9502$

(d) <u>Exponential</u>: $P\{t \le .5\} = 1 - e^{-3 \times .5} = .7769$

<u>Poisson</u>: $p_0(.5) + p_1(.5) + \cdots + p_{17}(.5) = .7769$

(e) $p_0(1) = 0$ $(\mu t = 3 \times 1 = 3)$

3

$N = 40$, $\mu = 10$ calls/hr

TORA input $(0, \mu t, 1, 40, 40)$

(a) $p_{n>0}(t=4) = 1 - p_0(4)$

$= 1 - .521 = .479$

(b) $E\{n|t=4\} = \sum_{n=0}^{40} n\, p_n(4) \simeq 2.5$ blocks

$\cong 25$ tickets

4

$N = 48$, $\mu = \frac{4 \times 10}{8} = 5$ cans/hr

$\mu t = 5 \times 4 = 20$ cans

$p_0(4) \cong .000005$ (from TORA)

$N = 48$, $\mu t = 5 \times 8 = 40$, $p_0(8) = .11958$

5

$\mu = 1/1 = 1$ withdrawl/week

$N = 5$, $\mu t = 4$

$p_0(4) = .37116$

6

$N = 80$ items, $\mu = 5$ items/day

(a) $\mu t = 5 \times 2 = 10$ items

$p_{70}(2) = .1251$

(b) $\mu t = 5 \times 4 = 20$ items

$p_0(4) = .00001$

(c) $\mu t = 5 \times 4 = 20$ items

$E\{n|4\text{ days}\} = \sum_{n=0}^{80} n\, p_n(4) \simeq 60$ items

Av. # of withdrawls $= 80 - 60$

$= 20$ items

7

$\mu = 1/1 = 1$ breakdown/day

$N = 10$, $\mu t = 1 \times 2 = 2$

From TORA, $p_0(2) = .00005$

8

(a) $N = 25$, $\mu = 3$/day

$t = 6$ days, $\mu t = 18$

Av. stock remaining after 6 days

$= E\{n|t=6\} = 7.11$

Av. order size $= 25 - 7.11$

$\simeq 18$ items

(b) $t = 4$, $\mu t = 3 \times 4 = 12$

$p_0(4) = .00069$

(c) $t = 6$, $\mu t = 3 \times 6 = 18$

$p_{n \le 14}(6) = p_0(6) + \cdots + p_{14}(6) = .9696$

9

P{time betn. departures $> T$}

$= P\{\text{no departures during } T\}$

$= P\{N \text{ left after time } T\}$

$= p_N(T)$

$$P\{t > T\} = p_N(T) = \frac{(\mu T)^0 e^{-\mu T}}{0!}$$

$$= e^{-\mu T}$$

Set 15.4b

10

$$p'_N(t) = -\mu\, p_N(t) \qquad (1)$$

$$p'_n(t) = -\mu\, p_n(t) + \mu\, p_{n+1}(t), \quad 0 \leq n < N \qquad (2)$$

From (1), we get

$$p_N(t) = C\, e^{-\mu t}$$

Given $p_N(0) = 1$, then $c = 1$ and

$$p_N(t) = e^{-\mu t}$$

Next, consider (2) for $n = N-1$

$$p'_{N-1}(t) = -\mu\, p_{N-1}(t) + \mu\, p_N(t)$$

$$= -\mu\, p_{N-1}(t) + \mu\, e^{-\mu t}$$

Thus,

$$P_{N-1}(t) = e^{-\int \mu t}\left\{\int \mu e^{-\mu t} e^{\int \mu dt} dt + C\right\}$$

$$= e^{-\mu t}\mu t + C$$

Because $p_{N-1}(0) = 0$, $C = 0$ and $p_{N-1}(t) = (\mu t)e^{-\mu t}$

<u>Induction proof</u>:

Given $p_{n+1}(t) = \dfrac{(\mu t)^{N-n-1} e^{-\mu t}}{(N-n-1)!}$, then

$$p'_n(t) = -\mu\, p_n(t) + \frac{\mu\, (\mu t)^{N-n-1}\, e^{-\mu t}}{(N-n-1)!}$$

Solution gives

$$p_n(t) = \frac{(\mu t)^{N-n}\, e^{-\mu t}}{(N-n)!}$$

1

(a) $P\{0 \text{ counter open}\} = p_0 = \frac{1}{55}$

$P\{1 \text{ counter open}\} = p_1 + p_2 + p_3$

$= \frac{1}{55}(2+ \quad +8) = \frac{14}{55}$

$P\{2 \text{ counters open}\} = p_4 + p_5 + p_6$

$= \frac{1}{55}(8+8+8) = \frac{24}{55}$

$P\{3 \text{ counters open}\} = p_7 + p_8 + \cdots$

$= 1 - (p_0 + \cdots + p_6)$

$= 1 - (\frac{1}{55} + \frac{14}{55} + \frac{24}{55}) = \frac{16}{55}$

(b) Av. # busy counters

$= 0 \times \frac{1}{55} + 1 \times \frac{14}{55} + 2 \times \frac{24}{55} + 3 \times \frac{16}{55}$

= 2 counters

(c) Av. # idle counters = 3 − 2 = 1

2

$\lambda = 1/5 = .2$ arrival/min

= 12 arrivals/hr

(a)

$$\mu_n = \begin{cases} 5 \text{ customers/hr}, & n = 0, 1, 2 \\ 10 \text{ customers/hr}, & n = 3, 4 \\ 15 \text{ customers/hr}, & n = 5, 6 \\ 20 \text{ customers/hr}, & n \geq 7 \end{cases}$$

$p_1 = \frac{12}{5} p_0 = 2.4 p_0$

$p_2 = (\frac{12}{5})^2 p_0 = 5.76 p_0$

$p_3 = (\frac{12}{5})^2 (\frac{12}{10}) p_0 = 6.912 p_0$

$p_4 = (\frac{12}{5})^2 (\frac{12}{10})^2 p_0 = 8.2944 p_0$

$p_5 = (\frac{12}{5})^2 * (\frac{12}{10})^2 (\frac{12}{15}) p_0 = 6.63552 p_0$

$p_6 = (\frac{12}{5})^2 (\frac{12}{10})^2 (\frac{12}{15})^2 p_0 = 5.308416 p_0$

$p_{n \geq 7} = (\frac{12}{5})^2 (\frac{12}{10})^2 (\frac{12}{15})^2 (\frac{12}{20})^{n-6} p_0 = 5.308416 (.6)^{n-6} p_0$

From $\sum_{n=0}^{\infty} p_n = 1$, we get $p_0 = .002587$

$p_1 = .05421$, $p_2 = .13010$, $p_3 = .15612$

$p_4 = .18735$, $p_5 = .14988$, $p_6 = .1199$

$p_{n \geq 7} = .1199(.6)^{n-6}$

(b) $p_{n \geq 7} = 1 - (p_0 + p_1 + \cdots + p_6) = .8$

Continued...

(c) $P\{0 \text{ counter}\} \Rightarrow p_0 = .002587$

$P\{1 \text{ counter}\} = p_1 + p_2 = .18431$

$P\{2 \text{ counters}\} = p_3 + p_4 = .34347$

$P\{3 \text{ counters}\} = p_5 + p_6 = .26978$

$P\{4 \text{ counters}\} = p_7 + p_8 + \cdots = .199853$

Av. # idle counters

$= 4 - (1 \times .18431 + 2 \times .34347 + 3 \times .26978 + 4 \times .199853) \cong 1.52$

3

$$\mu_n = \begin{cases} 5n, & n = 1, 2 \\ 15, & n = 3, 4, \dots \end{cases}$$

$p_1 = (\frac{10}{5}) p_0 = 2 p_0$

$p_2 = (\frac{10}{5})(\frac{10}{10}) p_0 = 2 p_0$

$p_{n \geq 3} = (\frac{10}{5})(\frac{10}{10})(\frac{10}{15})^{n-2} p_0 = 2(\frac{2}{3})^{n-2} p_0$

Thus,

$p_0 + 2p_0 + 2p_0 + [2(\frac{2}{3}) + 2(\frac{2}{3})^2 + \cdots] p_0 = 1$

which gives $p_0 = .1111$

(a) Prob that 3 counters are in use

$= p_{n \geq 3} = 1 - (p_0 + p_1 + p_2)$

$= 1 - (.1111 + .2222 + .2222)$

$= .4445$

(b) $p_{n \leq 2} = p_0 + p_1 + p_2 = .5555$

4

$$\lambda_n = \begin{cases} 12 \text{ cars/hr}, & n = 0, 1, \dots, 10 \\ 0 & n \geq 11 \end{cases}$$

$\mu_n = 60/6 = 10$ cars/hr

$p_n = (\frac{12}{10})^n p_0, \quad n = 1, 2, \dots, 10$

$= 0, \quad n \geq 11$

$p_0 (1 + 1.2 + 1.2^2 + \cdots + 1.2^{10}) = p_0 \frac{1 - 1.2^{11}}{1 - 1.2}$

Thus, $p_0 = .0311$

Continued...

4 continued

(a) $p_{10} = (\frac{12}{10})^{10} p_0 = .19259$

(b) $p_{n \geq 1} = 1 - p_0 = 1 - .0311 = .9689$

(c) Av. length of the lane

$= 0p_0 + 1p_1 + \cdots + 10p_{10}$

$= 1 \times .03732 + 2 \times .04479 + 3 \times .05375 + 4 \times .0645 + 5 \times .0774 + 6 \times .09288 + 7 \times .11145 + 8 \times .13374 + 9 \times .16049 + 10 \times .19259 = 6.71071$

5

$\lambda_n = 6$ arrivals/hr, $n = 0, 1, \ldots, 8$

$= 5$ arrivals/hr, $n = 9, 10, \cdots 11, 12$

$\mu_n = n/.5 = 2n$/hr, $n = 1, 2, 3, 4$

$= 10$/hr, $n \geq 5$

n	P_n
0	0.049526
1	0.148578
2	0.222866
3	0.222866
4	0.16715
5	0.10029
6	0.060174
7	0.036104
8	0.021663
9	0.010831
10	0.005416
11	0.002708
12	0.001354

$p_1 = \frac{6}{2} p_0 = 3p_0$

$p_2 = \frac{6}{2} \cdot \frac{6}{4} p_0 = 4.5 p_0$

$p_3 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot p_0 = 4.5 p_0$

$p_4 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} p_0 = 3.375 p_0$

$p_5 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{6}{10} p_0 = 2.025 p_0$

$p_6 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{6}{10} \cdot \frac{6}{10} p_0 = 1.215 p_0$

$p_7 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{6}{10} \cdot \frac{6}{10} \cdot \frac{6}{10} p_1 = .729 p_0$

$p_8 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot (\frac{6}{10})^4 p_0 = .4374 p_0$

$p_{n \geq 9} = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} (\frac{6}{10})^4 (\frac{5}{10})^{n-8} p_0 = .4374 (.5)^{n-8} p_0$

From $\sum_{n=0}^{12} p_n = 1$, we get $p_0 = .0495$

(a) $p_{12} = .4374 \times .5^4 \times .0495 = .00135$

(b) $p_{n \geq 5} = 1 - (p_0 + p_1 + \cdots + p_4) = .2385$

(c) Av. # busy tables $= 0p_0 + 1p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_{n \geq 5} = 2.9768$

5 continued

(d) $1p_6 + 2p_7 + \cdots + 7p_{12}$

$= 1 \times .0602 + 2 \times .0361 + 3 \times .0217 + 4 \times .0108 + 5 \times .0054 + 6 \times .0027 + 7 \times .00135$

$= .2935$ pair

6

$\lambda = 4$ customers/hr

$$\lambda_n = \begin{cases} 4, & n = 0, 1, \ldots, 4 \\ 0, & n \geq 5 \end{cases}$$

$\mu_n = \frac{60}{15} = 4$ customers/hr

(a) $p_1 = \frac{4}{4} p_0$

$p_2 = (\frac{4}{4})^2 p_0$

$p_3 = (\frac{4}{4})^3 p_0$

$p_4 = (\frac{4}{4})^4 p_0$

$p_0 + p_1 + \cdots + p_4 = 1 \implies p_0 = 1/5$

$p_0 = p_1 \pm p_2 = p_3 = p_4 = 1/5$

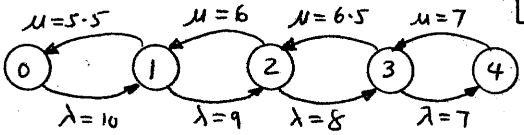
(b) expected # in shop =

$0p_0 + 1p_1 + 2p_2 + 3p_3 + 4p_4$

$= \frac{1}{5}(1 + 2 + 3 + 4) = 2$

(c) $p_4 = .2$

7



(a) $5.5 p_1 = 10 p_0$

$10p_0 + 6p_2 = (5.5 + 9) p_1$

$9p_1 + 6.5p_3 = (6 + 8) p_2$

$8p_2 + 7p_4 = (6.5 + 7) p_3$

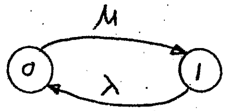
(b) $p_1 = 1.82 p_0$, $p_2 = 2.727 p_0$

$p_3 = 3.3566 p_0$, $p_4 = 3.3566 p_0$

$p_0 + p_1 + \cdots + p_4 = 1 \implies p_0 = .088882$

$p_1 = .1614$, $p_2 = .2422$, $p_3 = .2981$, $p_4 = .$

8



(a) $\mu p_1 = \lambda p_0$

$p_1 = \frac{\lambda}{\mu} p_0$

(b) $p_0 + \frac{\lambda}{\mu} p_0 = 1$

$p_0 = \frac{1}{1+\rho}$, $\rho = \lambda/\mu$

$p_1 = \frac{\rho}{1+\rho}$

(c) $L_s = 0p_0 + 1p_1 = \frac{\rho}{1+\rho}$

(d) $\lambda_{eff} = \lambda p_0 = \frac{\lambda}{1+\rho}$

(e) $W_q = \frac{L_s}{\lambda_{eff}} - \frac{1}{\mu}$

$= \frac{\rho/(1+\rho)}{\lambda/(1+\rho)} - \frac{1}{\mu} = 0$

9

$$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} =$$

$$\lambda_{n-1}\left(\frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-2}}{\mu_{n-1}}\right) +$$

$$\mu_{n+1}\left(\frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_n}{\mu_{n+1}}\right)$$

$$= \mu_n\left(\frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-1}}{\mu_n}\right) +$$

$$\lambda_n\left(\frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-1}}{\mu_n}\right)$$

$$= \mu_n p_n + \lambda_n p_n$$

$$= (\mu_n + \lambda_n) p_n$$

1

(a) $L_q = \sum_{n=6}^{8} (n-5) p_n$

$= 1p_6 + 2p_7 + 3p_8$

$= 1 \times .05847 + 2 \times .03508 + 3 \times .02105$

$= .19177$

(b) $W_q = \frac{L_q}{\lambda_{eff}}$

$= \frac{.1917}{5.8737} = .03265$ hour

$W_s = W_q + \frac{1}{\mu}$

$= .03264 + \frac{1}{2} = .53265$ hour

(c) $\lambda_{lost} = \lambda p_8$

$= 6 \times .02105 = .1263$ car/hr

Number lost/8 hrs $= .1263 \times 8 = 1.01$ cars

(d) Average number of empty spaces

$= c - (L_s - L_q)$

$= c - \sum_{n=0}^{8} n p_n + \sum_{n=c+1}^{8} (n-c) p_n$

$= \left(c \sum_{n=0}^{8} p_n - c \sum_{n=c+1}^{8} p_n\right)$

$- \left(\sum_{n=0}^{8} n p_n - \sum_{n=c+1}^{8} n p_n\right)$

$= c \sum_{n=0}^{c} p_n - \sum_{n=0}^{c} n p_n$

$= \sum_{n=0}^{c-1} (c-n) p_n$

2

(a)

$\lambda_n = 6$ cars/hr, $n = 0, 1, \dots, 6$

$$\mu_n = \begin{cases} \left(\frac{4}{3}\right)n, & n = 1, 2, \dots, 6 \\ 8, & n = 7, 8, 9, 10 \end{cases}$$

$p_n = \left(\frac{6}{4/3}\right)^n \frac{1}{n!} p_0, \ n = 0, 1, \dots, 6$

continued...

2 continued

$$p_n = \frac{\left(\frac{6}{4/3}\right)^n}{6!\, 6^{n-6}} p_0, \quad n = 7, 8, 9, 10$$

$$p_0 \left(1 + \frac{9/2}{1!} + \frac{(9/2)^2}{2!} + \frac{(9/2)^3}{3!} + \frac{(9/2)^4}{4!} + \frac{(9/2)^5}{5!} + \frac{(9/2)^6}{6!} + \frac{(9/2)^7}{6!\,6} + \frac{(9/2)^8}{6!\,6^2} + \frac{(9/2)^9}{6!\,6^3} + \frac{(9/2)^{10}}{6!\,6^4}\right) = 1$$

Thus, $p_0 = .0004$

n	p_n	n	p_n
1	.00304	6	.10027
2	.01141	7	.12534
3	.02852	8	.15667
4	.05348	9	.19584
5	.08022	10	.24480

(b) $\lambda_{eff} = \lambda(1 - p_{10}) = 10(1 - .2448)$

$= 7.552$ cars/hr

(c) $L_s = 0p_0 + 1p_1 + 2p_2 + \dots + 10p_{10}$

$= 7.6941$ cars

(d) $W_s = \frac{L_s}{\lambda_{eff}} = \frac{7.6941}{7.552} = 1.0155$ cars

$W_q = 1.0155 - \frac{1}{4/3} = .2655$

(e) $L_q = \lambda_{eff} W_q$

$= .2655 \times 7.552$

$= 2.005$ cars

Average number of occupied spaces $= L_s - L_q$

$= 7.6941 - 2.005$

$= 5.6891$ spaces

1

(a) % utilization $= 100(1-p_0)$

$= 100\,\frac{\lambda}{\mu}$

$= 100\left(\frac{4}{6}\right) = 66.67\%$

(b) $p_{n\geq 1} = 1 - p_0 = \frac{\lambda}{\mu} = \frac{4}{6} = .6667$

(c) $p_{n\leq 7} = p_0 + p_1 + \cdots + p_7$

$= 1-\left(\frac{\lambda}{\mu}\right)^8 = 1-\left(\frac{4}{6}\right)^8 = .961$

(d) $p_0 + p_1 + \cdots + p_K \geq .99$

From Figure 17-6, $K = 11$

Also, we can determine K from

$1-\rho^{K+1} \geq .99$

$(K+1) \geq \frac{\ln .01}{\ln(4/6)} = 11.$

or $K \geq 11.350 - 1 = 10.358$

Thus, $K \geq 11$

Note that the desired number of parking spaces is almost doubled (from 5 to 11) to accommodate the increase in the acceptance percentage from 90% to 99%.

2

$\lambda = 1/5 = .2$ job/day

$\mu = 1/4 = .25$ job/day

From the TORA output on the next column,

(a) $p_0 = .2$

(b) Av. income/month $= \$50\,\mu t$

$= 50 \times .25 \times 30$

$= \$375$

(c) Av. number of jobs awaiting completion $= L_q = 3.2$ jobs

cost $= 3.2 \times \$40 = \$128$

Continued...

Lambda =	0.20000	Mu =	0.25000
Lambda eff =	0.20000	Rho/c =	0.80000
Ls =	4.00000	Lq =	3.20000
Ws =	20.00000	Wq =	16.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.20000	0.20000	23	0.00118	0.99528
1	0.16000	0.36000	24	0.00094	0.99622
2	0.12800	0.48800	25	0.00076	0.99698
3	0.10240	0.59040	26	0.00060	0.99758
4	0.08192	0.67232	27	0.00048	0.99807
5	0.06554	0.73786	28	0.00039	0.99845
6	0.05243	0.79028	29	0.00031	0.99876
7	0.04194	0.83223	30	0.00025	0.99901
8	0.03355	0.86578	31	0.00020	0.99921
9	0.02684	0.89263	32	0.00016	0.99937
10	0.02147	0.91410	33	0.00013	0.99949
11	0.01718	0.93128	34	0.00010	0.99959
12	0.01374	0.94502	35	0.00008	0.99968
13	0.01100	0.95602	36	0.00006	0.99974
14	0.00880	0.96482	37	0.00005	0.99979
15	0.00704	0.97185	38	0.00004	0.99983
16	0.00563	0.97748	39	0.00003	0.99987
17	0.00450	0.98199	40	0.00003	0.99989
18	0.00360	0.98559	41	0.00002	0.99991
19	0.00288	0.98847	42	0.00002	0.99993
20	0.00231	0.99078	43	0.00001	0.99995
21	0.00184	0.99262	44	0.00001	0.99996
22	0.00148	0.99410			

3

$\lambda = 1/4 = .25$ case/wk

$\mu = 1/1.5 = .66667$ case/wk

M/M/c/GD/N/K Queueing Model

Input Data			
λ =	0.25		0.66667
c =	1		
Sys. Lim., N=	infinity	Source limit, K =	infinity
Output Results			
λ_{eff} =	0.2500		0.3750
Ls =	0.6000	Lq =	0.2250
Ws =	2.4000	Wq =	0.9000

n	Pn	CPn	1-CPn
0	0.625002	0.625002	0.374998
1	0.234375	0.859376	0.140624
2	0.087890	0.947266	0.052734
3	0.032959	0.980225	0.019775
4	0.012359	0.992584	0.007416
5	0.004635	0.997219	0.002781
6	0.001738	0.998957	0.001043
7	0.000652	0.999609	0.000391
8	0.000244	0.999853	0.000147
9	0.000092	0.999945	0.000055
10	0.000034	0.999979	0.000021
11	0.000013	0.999992	0.000008
12	0.000005	0.999997	0.000003
13	0.000002	0.999999	0.000001
14	0.000001	1.000000	0.000000

(a) $L_q = .225$ case

(b) $1-p_0 = 1-.625 = .375$ or 37.5%

(c) $W_s = 2.4$ weeks

4

Present situation:

$\lambda = 90$ cars/hr

$\mu = \frac{3600}{38} = 94.7368$ cars/hr

New situation:

$\lambda = 90$ cars per hour

$\mu = \frac{3600}{30} = 120$ cars per hour

Continued...

Set 15.6b

Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	90.00000	94.73680	90.00000	0.05000	19.00017	18.05017	0.21111	0.20056
2	1	90.00000	120.00000	90.00000	0.25000	3.00000	2.25000	0.03333	0.02500

L_s (present) = 19 cars

% of idle time (new) = p_0(new) × 100

= 100 × .25 = 25%

The device can be justified based on the number of waiting customers, L_s, in the present system, but not on the basis of % idle time in the new one.

5

Scenario 1-- (M/M/1):(GD/infinity/infinity)

```
Lambda =     0.40000          Mu =       0.66667
Lambda eff = 0.40000          Rho/c =     0.60000

Ls =        1.49998           Lq =       0.89998
Ws =         3.74995          Wq =        2.24996
```

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.40000	0.40000	11	0.00145	0.99782
1	0.24000	0.64000	12	0.00087	0.99869
2	0.14400	0.78400	13	0.00052	0.99922
3	0.08640	0.87040	14	0.00031	0.99953
4	0.05184	0.92224	15	0.00019	0.99972
5	0.03110	0.95335	16	0.00011	0.99983
6	0.01866	0.97201	17	0.00007	0.99990
7	0.01120	0.98320	18	0.00004	0.99994
8	0.00672	0.98992	19	0.00002	0.99996
9	0.00403	0.99395	20	0.00001	0.99998
10	0.00242	0.99637			

(a) $p_0 = .4$

(b) $L_q = .9$ car

(c) $W_q = 2.25$ minutes

(d) $P_{n \ge 11} = 1 - CP_{10} = 1 - .99637 = .0036$

6

Scenario 1-- (M/M/1):(GD/infinity/infinity)

```
Lambda =     10.00000         Mu =       12.00000
Lambda eff = 10.00000         Rho/c =     0.83333

Ls =        5.00000           Lq =       4.16667
Ws =         0.50000          Wq =        0.41667
```

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.16667	0.16667	27	0.00121	0.99393
1	0.13889	0.30556	28	0.00101	0.99494
2	0.11574	0.42130	29	0.00084	0.99579
3	0.09645	0.51775	30	0.00070	0.99649
4	0.08038	0.59812	31	0.00059	0.99707
5	0.06698	0.66510	32	0.00049	0.99756
6	0.05582	0.72092	33	0.00041	0.99797
7	0.04651	0.76743	34	0.00034	0.99831
8	0.03876	0.80619	35	0.00028	0.99859
9	0.03230	0.83849	36	0.00024	0.99882
10	0.02692	0.86541	37	0.00020	0.99902
11	0.02243	0.88784	38	0.00016	0.99918
12	0.01869	0.90654	39	0.00014	0.99932
13	0.01558	0.92211	40	0.00011	0.99943
14	0.01298	0.93509	41	0.00009	0.99953
15	0.01082	0.94591	42	0.00008	0.99961
16	0.00901	0.95493	43	0.00007	0.99967
17	0.00751	0.96244	44	0.00005	0.99973
18	0.00626	0.96870	45	0.00005	0.99977
19	0.00522	0.97392	46	0.00004	0.99981
20	0.00435	0.97826	47	0.00003	0.99984
21	0.00362	0.98189	48	0.00003	0.99987
22	0.00302	0.98491	49	0.00002	0.99989
23	0.00252	0.98742	50	0.00002	0.99991
24	0.00210	0.98952	51	0.00002	0.99992
25	0.00175	0.99126	52	0.00001	0.99994
26	0.00146	0.99272	53	0.00001	0.99995

(a) $p_0 + p_1 + p_2 = .4213$

continued...

(b) $1 - CP_2 = 1 - .4213 = .5787$

(c) $W_q = .417$ hour

(d) Let N = spaces (including car being served)

$CP_{N-1} \ge .9$

Because $CP_{11} = .88784$ and $CP_{12} = .90654$,

$N-1 \ge 12 \Rightarrow N \ge 13$.

7

In general, $L_s < L_q + 1$. The reason is that $p_0 > 0$, usually. Consider

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n$$

$$= \sum_{n=1}^{\infty} n p_n - \sum_{n=1}^{\infty} p_n$$

$$= L_s - (1 - p_0)$$

The closer p_0 is to zero, the more likely $L_s \cong L_q + 1$ will hold.

8

Consider

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n$$

$$= \sum_{n=1}^{\infty} (n-1)(1-\rho)\rho^n$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\left(\sum_{n=1}^{\infty} \rho^{n-1}\right)$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \rho^2 (1-\rho) \frac{1}{(1-\rho)^2}$$

$$= \frac{\rho^2}{1-\rho}$$

9

(a) $P\{j \text{ in queue} \mid j \geq 1\}$

$$= p\{n \text{ in system} \mid n \geq 2\}$$

$$= \frac{p_n}{\sum_{j=2}^{\infty} p_j}.$$

Thus,

$$\text{expected number} = \sum_{n=2}^{\infty}(n-1)\frac{p_n}{\sum_{j=2}^{\infty} p_j}$$

$$= \frac{\sum_{n=2}^{\infty} np_n - \sum_{n=2}^{\infty} p_n}{\sum_{n=2}^{\infty} p_n}$$

$$= \frac{\sum_{n=1}^{\infty} np_n - p_1}{\sum_{n=2}^{\infty} p_n} - 1$$

$$= \frac{\frac{\rho}{1-\rho} - \rho(1-\rho)}{1-[(1-\rho)+\rho(1-\rho)]} - 1$$

$$= \frac{1}{1-\rho}$$

(b) Exp. number in queue given the system is not empty

$$= \sum_{n=1}^{\infty}(n-1)\left(\frac{p_n}{\sum_{j=1}^{\infty} p_j}\right)$$

$$= \frac{\sum_{n=1}^{\infty} np_n - \sum_{n=1}^{\infty} p_n}{\sum_{j=1}^{\infty} p_j}$$

$$= \frac{\left(\frac{\rho}{1-\rho}\right) - \rho}{\rho}$$

$$= \frac{\rho}{1-\rho}$$

Continued...

Thus,

Exp. waiting time in queue for those who must wait

$$= \frac{\rho/(1-\rho)}{\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

Set 15.6c

1

$$w(\tau) = (\mu-\lambda)\, e^{-(\mu-\lambda)\tau}, \quad \tau > 0$$

$$\left.\begin{aligned}\lambda &= 1/4 = .25/\text{wk} \\ \mu &= 1/1.5 = .667/\text{wk}\end{aligned}\right\} (\mu-\lambda) = .417$$

$$\rho = \lambda/\mu = \frac{1.5}{4} = .375$$

$$w(\tau) = .417\, e^{-.417\tau}, \quad \tau > 0$$

$$P\{\tau > 1\} = e^{-.417\times 1} = .659$$

2

(a) Standard deviation $= \frac{1}{\mu-\lambda} = \frac{1}{6-4} = .5$

(b) $w(\tau) = (\mu-\lambda)e^{-(\mu-\lambda)\tau}, \quad \tau > 0$

$$P\left\{\frac{1}{2(\mu-\lambda)} \le \tau \le \frac{3}{2(\mu-\lambda)}\right\}$$

$$= (1 - e^{-1.5}) - (1 - e^{-.5})$$

$$= e^{-.5} - e^{-1.5}$$

$$= .3834$$

3

$W_s \le 10$ minutes, $\lambda = 4/\text{hr}$

$$\frac{1}{(\mu-\lambda)} \le \frac{10}{60}\ \text{hr}$$

or $\mu - \lambda \ge 6$

or $\mu \ge 6 + \lambda = 10/\text{hr}$

4

$P\{\tau > \frac{10}{60}\} \le .1$, or

$$e^{-\frac{1}{6}(\mu-4)} \le .1$$

$$\mu - 4 \ge 13.8$$

$$\mu \ge 17.8/\text{hr}$$

5

$P\{\tau > 5\} = e^{-(\mu-\lambda)t} = e^{-.267\times 5} = .2636$

where $\lambda = .4/\text{min}$, $\mu = .667/\text{min}$

Exp. # customers in a 12-hr day

$= \lambda \times 12 \times 60 = .4 \times 12 \times 60 = 288$ cust.

Exp. cost $= 288 \times .2636 \times .5 = \$37.95$

6

Let

$w_{n+1}(t|n)$ = conditional pdf for waiting in queue given there are n customers ahead

= n-fold convolution of the exponential pdf

$$= \frac{\mu(\mu t)^{n-1} e^{-\mu t}}{(n-1)!}$$

$w(t)$ = absolute pdf of waiting time in queue

$g(t,n)$ = joint pdf of t and n

$$= w_{n+1}(t|n)\, p_n$$

$$= \frac{\mu(\mu t)^{n-1} e^{-\mu t}}{(n-1)!} \rho^n (1-\rho)$$

(a) For $t > 0$

$$w(t) = \sum_{n=1}^{\infty} g(t,n)$$

$$= \frac{\mu\rho e^{-\mu t}(1-\rho)}{e^{-\mu\rho t}} \sum_{n=1}^{\infty} \frac{(\mu\rho t)^{n-1} e^{-\mu\rho t}}{(n-1)!}$$

$$= \mu\rho(1-\rho)\, e^{-\mu(1-\rho)t}, \quad t > 0$$

For $t = 0$, $w(0) = p_0 = (1-\rho)$

$$w(t) = \begin{cases} 1-\rho, & t = 0 \\ \mu\rho(1-\rho)e^{-\mu(1-\rho)t}, & t > 0 \end{cases}$$

(b) $W_q = E\{t\}$

$$= \int_0^{\infty} t\, w(t)\, dt$$

$$= 0\, w(0) + \int_{0+}^{\infty} t\, w(t)\, dt$$

$$= \int_{0+}^{\infty} \mu\rho t(1-\rho)\, e^{-\mu(1-\rho)t}\, dt$$

$$= \frac{\rho}{\mu(1-\rho)}$$

1

(a) $p_0 = .3654$

(b) $W_q = .207$ hour

(c) Average number of empty spaces $= 4 - L_q$

$= 4 - .788$

$= 3.212$ spaces

(d) $p_5 = .04812$

(e) $W_s \le 10$ minutes

```
Title: 17.6d-1
Comparative Analysis

Scenario     c   Lambda        Mu   L'da eff         p0        Ls        Lq        Ws        Wq

       1     1   4.00000    6.00000   3.80752   0.36541   1.42256   0.78797   0.37362   0.20695
       2     1   4.00000    7.00000   3.89179   0.44403   1.11691   0.56094   0.28699   0.14413
       3     1   4.00000    8.00000   3.93651   0.50794   0.90476   0.41270   0.22984   0.10484
       4     1   4.00000    9.00000   3.96116   0.55987   0.75340   0.31327   0.19020   0.07908
       5     1   4.00000   10.00000   3.97532   0.60247   0.64199   0.24446   0.16149   0.06149
```

μ (cars/hr)	W_s (hrs)	W_s (min)
6	.3736	22.4
7	.287	17.16
8	.23	13.80
9	.19	11.40
$\boxed{10}$	.16	$\boxed{9.60}$

Desired service rate = 10 cars/hr. Thus, the service time must be reduced from $\frac{60}{6} = 10$ minutes to $\frac{60}{10} = 6$ minutes, a 40% reduction

2

m = number of parking spaces

An arriving car will not find a space if there are $m+1$ cars in the system. Thus, find m such that $p_{m+1} \le .01$

TORA input = $(4, 6, 1, m+1, \infty)$

m	$N = m+1$	p_N
4	5	.04812
5	6	.0311
6	7	.0203
7	8	.01335
$\boxed{8}$	9	$\boxed{.009}$

Select the number of parking spaces $m \ge 8$

Continued...

3

m = number of seats.

The $N = m+1$, and

$\lambda_{eff} = \lambda P_N = 5 P_N$ customers/hr

TORA input = $(6, 5, 1, N, \infty)$

```
Title: 17.6d-3
Comparative Analysis

Scenario     c   Lambda        Mu   L'da eff         p0        Ls        Lq        Ws        Wq

       1     1   6.00000   5.00000   3.62637   0.27473   1.12088   0.39560   0.30909   0.10909
       2     1   6.00000   5.00000   4.06855   0.18629   1.72578   0.91207   0.42418   0.22418
       3     1   6.00000   5.00000   4.32810   0.13438   2.35949   1.49387   0.54516   0.34516
       4     1   6.00000   5.00000   4.49647   0.10071   3.02117   2.12188   0.67190   0.47190
       5     1   6.00000   5.00000   4.61288   0.07742   3.70984   2.78726   0.80423   0.60423
```

m	$N = m+1$	λ_{eff} (customers/hr)
1	2	3.63
2	3	$\boxed{4.07}$

Use two seats or less

4

$\lambda = 10$ generators per hour

$\mu = \frac{60}{15} = 4$ generators per hour

$N = 7 + 1 = 8$

```
Title: 17.6d-4
Scenario 1-- (M/M/1):(GD/8/Infinity)

  Lambda =      10.00000             Mu =        4.00000
  Lambda eff = 3.99843               Rho/c =       2.50000

  Ls =        7.33569                Lq =        6.33609
  Ws =          1.83464              Wq =          1.58464

        n   Probability, pn  Cumulative, Pn          n   Probability, pn  Cumulative, Pn

        0        0.00039          0.00039             5        0.03841          0.06375

        1        0.00098          0.00138             6        0.09603          0.15978
        2        0.00246          0.00383             7        0.24006          0.39984
        3        0.00615          0.00998             8        0.60016          1.00000
        4        0.01536          0.02534
```

(a) $p_8 \cong .6$

(b) $L_q = 6.34$ generators

(c) Let C = belt capacity. Thus, $N = C+1$. The assembly department is kept in operation so long as at least one empty space remains on the belt; that is,

$$P\{\text{empty space on belt}\} = p_0 + p_1 + \cdots + p_C$$

$$= \frac{1-\rho}{1-\rho^{C+2}} \sum_{n=0}^{C} \rho^n$$

$$= \frac{1-\rho}{1-\rho^{C+2}} \cdot \frac{1-\rho^{C+1}}{1-\rho}$$

$$= \frac{1-\rho^{C+1}}{1-\rho^{C+2}}$$

Continued...

Set 15.6d

$$\lim_{c\to\infty} \frac{1-\rho^{c+1}}{1-\rho^{c+2}} = \lim_{c\to\infty} \frac{-(c+1)\rho^{c}}{-(c+2)\rho^{c+1}}$$

$$= \lim_{c\to\infty} \frac{c+1}{(c+2)\rho}$$

$$= \lim_{c\to\infty} \left(\frac{1+1/c}{1+2/c}\right)\frac{1}{\rho}$$

$$= \frac{1}{\rho}$$

In the present example, $\rho = 10/4$ and $1/\rho = .4$. Thus,

$$\lim_{c\to\infty} (p_0+p_1+\cdots+p_c) = 1/\rho = .4$$

This result means that regardless of how large the belt is, the probability of finding an empty space cannot exceed .4. Thus, achieving a 95% utilization for the assembly dept. is impossible.

The result makes sense because the arrival rate λ (=10/hr) is $2\frac{1}{2}$ times larger than the service rate (= 4). The only way we can accomplish the desired result is to reduce λ and/or increase μ.

5

(a) $p_{50} \simeq .00002$

(b) $P\{\text{wish is not fulfilled}\}$

$= P\{48 \text{ or more in resturant}\}$

$= p_{48}+p_{49}+p_{50}$

$= 1-(p_0+p_1+\cdots+p_{47})$

$= 1-.99993$

$= .00007$

continued...

```
Title: 17.6d-5
Scenario 1-- (M/M/1):(GD/50/infinity)
```

TORA input = $(10, 12, 1, 50, \infty)$

```
Lambda =      10.00000          Mu =      12.00000
Lambda eff = 9.99982            Rho/c =    0.83333

Ls =      4.99533               Lq =      4.16201
Ws =       0.49954              Wq =       0.41621

  n  Probability, pn  Cumulative, Pn      n  Probability, pn  Cumulative, Pn

  0      0.16668         0.16668          26      0.00146         0.99281

  1      0.13890         0.30558          27      0.00121         0.99402
  2      0.11575         0.42133          28      0.00101         0.99504
  3      0.09646         0.51779          29      0.00084         0.99588
  4      0.08038         0.59818          30      0.00070         0.99658
  5      0.06699         0.66516          31      0.00059         0.99717

  6      0.05582         0.72098          32      0.00049         0.99765
  7      0.04652         0.76750          33      0.00041         0.99806
  8      0.03876         0.80627          34      0.00034         0.99840
  9      0.03230         0.83857          35      0.00028         0.99868
 10      0.02692         0.86549          36      0.00024         0.99892

 11      0.02243         0.88792          37      0.00020         0.99911
 12      0.01869         0.90662          38      0.00016         0.99928
 13      0.01558         0.92220          39      0.00014         0.99941
 14      0.01298         0.93518          40      0.00011         0.99952
 15      0.01082         0.94600          41      0.00009         0.99962

 16      0.00902         0.95501          42      0.00008         0.99970
 17      0.00751         0.96253          43      0.00007         0.99976
 18      0.00626         0.96879          44      0.00005         0.99982
 19      0.00522         0.97401          45      0.00005         0.99986
 20      0.00435         0.97835          46      0.00004         0.99990

 21      0.00362         0.98198          47      0.00003         0.99993
 22      0.00302         0.98500          48      0.00003         0.99996
 23      0.00252         0.98751          49      0.00002         0.99998
 24      0.00210         0.98961          50      0.00002         1.00000
 25      0.00175         0.99136
```

TORAinput = $(20, 7.5, 1, 15, \infty)$

6

```
Title: 17.6d-6
Scenario 1-- (M/M/1):(GD/15/infinity)

Lambda =      20.00000          Mu =       7.50000
Lambda eff = 7.50000            Rho/c =    2.66667

Ls =      14.40000              Lq =      13.40000
Ws =       1.92000              Wq =       1.78667

  n  Probability, pn  Cumulative, Pn      n  Probability, pn  Cumulative, Pn

  0      0.00000         0.00000           8      0.00065         0.00104

  1      0.00000         0.00000           9      0.00174         0.00278
  2      0.00000         0.00000          10      0.00463         0.00742
  3      0.00000         0.00001          11      0.01236         0.01978
  4      0.00001         0.00002          12      0.03296         0.05273
  5      0.00003         0.00005          13      0.08789         0.14062

  6      0.00009         0.00015          14      0.23438         0.37500
  7      0.00024         0.00039          15      0.62500         1.00000
```

(a) $p_0 \cong 0.$

(b) $p_{n\leq 14} = p_0+\cdots+p_{14} \doteq .375$

(c) $W_s = 1.92$ hours

7

(a) $p_{n\leq 4} = p_0+p_1+\cdots+p_4$

$= .962$

(b) $\lambda_{lost} = \lambda p_5$

$= 5 \times .038 = .19$ cust./hr

(c) $L_s = 0\times .399 + 1\times .249 + 2\times .156$

$+ 3\times .097 + 4\times .061$

$+ 5\times .038$

$= 1.286$

continued...

(d) $W_q = W_s - \frac{1}{\mu}$

$\lambda_{eff} = 5(1-.038) = 4.81$ cust/hr

$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{1.286}{4.81} = .2675 \text{ hour}$$

$$W_q = .2675 - 1/8 = .1424 \text{ hour}$$

8

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}}$$

$$\lim_{\rho\to 1} p_n = \lim_{\rho\to 1} \frac{\rho^n - \rho^{n+1}}{1-\rho^{N+1}} = \lim_{\rho\to 1} \frac{n\rho^{n-1} - (n+1)\rho^n}{-(N+1)\rho^N} = \frac{1}{N+1}$$

Thus,

$$L_s = \sum_{n=0}^{N} np_n = \frac{1}{N+1}\sum_{n=0}^{N} n = \frac{N(N+1)}{2(N+1)} = \frac{N}{2}$$

9

$$W_s = W_q + \frac{1}{\mu}$$

$$\lambda_{eff} W_s = \lambda_{eff} W_q + \frac{\lambda_{eff}}{\mu}$$

Thus,

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

or

$$\lambda_{eff} = \mu(L_s - L_q)$$

Set 15.6e

1

TORA input = (8, 5, 2, ∞, ∞)

```
Title: 17.6e-1
Scenario 1-- (M/M/2):(GD/infinity/infinity)

Lambda =      8.00000            Mu =      5.00000
Lambda eff =  8.00000            Rho/c =   0.80000

Ls =   4.44444                   Lq =   2.84444
Ws =   0.55556                   Wq =   0.35556

  n  Probability, pn  Cumulative, Pn       n  Probability, pn  Cumulative, Pn

  0      0.11111         0.11111          23      0.00131         0.99475

  1      0.17778         0.28889          24      0.00105         0.99580
  2      0.14222         0.43111          25      0.00084         0.99664
  3      0.11378         0.54489          26      0.00067         0.99731
  4      0.09102         0.63591          27      0.00054         0.99785
  5      0.07282         0.70873          28      0.00043         0.99828

  6      0.05825         0.76698          29      0.00034         0.99862
  7      0.04660         0.81359          30      0.00028         0.99890
  8      0.03728         0.85087          31      0.00022         0.99912
  9      0.02983         0.88070          32      0.00018         0.99930
 10      0.02386         0.90456          33      0.00014         0.99944

 11      0.01909         0.92365          34      0.00011         0.99955
 12      0.01527         0.93892          35      0.00009         0.99964
 13      0.01222         0.95113          36      0.00007         0.99971
 14      0.00977         0.96091          37      0.00006         0.99977
 15      0.00782         0.96873          38      0.00005         0.99982

 16      0.00625         0.97498          39      0.00004         0.99985
 17      0.00500         0.97998          40      0.00003         0.99988
 18      0.00400         0.98399          41      0.00002         0.99991
 19      0.00320         0.98719          42      0.00002         0.99992
 20      0.00256         0.98975          43      0.00002         0.99994

 21      0.00205         0.99180          44      0.00001         0.99995
 22      0.00164         0.99344
```

TORA input = (16, 5, 4, ∞, ∞)

```
Title: 17.6e-1
Scenario 2-- (M/M/4):(GD/infinity/infinity)

Lambda =     16.00000            Mu =      5.00000
Lambda eff = 16.00000            Rho/c =   0.80000

Ls =   5.58573                   Lq =   2.38573
Ws =   0.34911                   Wq =   0.14911

  n  Probability, pn  Cumulative, Pn       n  Probability, pn  Cumulative, Pn

  0      0.02730         0.02730          24      0.00138         0.99450

  1      0.08737         0.11467          25      0.00110         0.99560
  2      0.13979         0.25446          26      0.00088         0.99648
  3      0.14911         0.40357          27      0.00070         0.99718
  4      0.11929         0.52285          28      0.00056         0.99775
  5      0.09543         0.61828          29      0.00045         0.99820

  6      0.07634         0.69463          30      0.00036         0.99856
  7      0.06107         0.75570          31      0.00029         0.99885
  8      0.04886         0.80456          32      0.00023         0.99908
  9      0.03909         0.84365          33      0.00018         0.99926
 10      0.03127         0.87492          34      0.00015         0.99941

 11      0.02502         0.89994          35      0.00012         0.99953
 12      0.02001         0.91995          36      0.00009         0.99962
 13      0.01601         0.93596          37      0.00008         0.99970
 14      0.01281         0.94877          38      0.00006         0.99976
 15      0.01025         0.95901          39      0.00005         0.99981

 16      0.00820         0.96721          40      0.00004         0.99985
 17      0.00656         0.97377          41      0.00003         0.99988
 18      0.00525         0.97901          42      0.00002         0.99990
 19      0.00420         0.98321          43      0.00002         0.99992
 20      0.00336         0.98657          44      0.00002         0.99994

 21      0.00269         0.98926          45      0.00001         0.99995
 22      0.00215         0.99140          46      0.00001         0.99996
 23      0.00172         0.99312
```

(a) $\underline{C=2:}$

$P\{\text{all servers are busy}\} = (P_{n\geq 2})^2$

$= (1-.29)^2$

$= .504$

$\underline{C=4:}$

$P\{\text{all servers are busy}\} = 1 - P_{n\leq 3}$

$= 1 - .404$

$= .596$

C = 4 yields a higher probability that all servers are busy.

continued...

(b)

```
Title: 17.6e-1
Comparative Analysis

Scenario      c    Lambda        Mu   L'da eff        p0        Ls        Lq        Ws        Wq
    1         4  16.00000   5.00000   16.00000   0.02730   5.58573   2.38573   0.34911   0.14911
    2         5  16.00000   5.00000   16.00000   0.03715   3.71299   0.51299   0.23208   0.03208
    3         6  16.00000   5.00000   16.00000   0.03977   3.34526   0.14526   0.20908   0.00908
```

For C = 5, Wq = .032 hour ≅ 2 min

C = 4, Wq = .149 hour ≃ 9 min

Select C ≥ 5

2

$\underline{C=2}$: $\lambda = 8$ calls/hr

$\mu = \frac{60}{14.5} = 4.1379$ calls/hr

$\underline{C=4}$: $\lambda = 16$ calls/hr

$\mu = 4.1379$ calls per hour

utilization $= \lambda/\mu c = .967$

```
Title: 6e-2
Comparative Analysis

Scenario      c    Lambda        Mu   L'da eff        p0         Ls         Lq        Ws        Wq
    1         2   8.00000   4.13790    8.00000   0.01695   29.49805   27.56471   3.68726   3.44559
    2         4  16.00000   4.13790   16.00000   0.00332   30.75857   26.89167   1.92241   1.68074
```

$W_q = \begin{cases} 3.446 \text{ hours for } C=2 \\ 1.681 \text{ hours for } C=4 \end{cases}$

Consolidation reduces the waiting time by more than 51%.

3

(a) $\lambda = \frac{60}{5} = 12$ per hour

$\mu = 10$ per hour

$c > \frac{\lambda}{\mu} = 1.2 \Rightarrow c \geq 2$

(b) $\lambda = \frac{60}{2} = 30$ per hour

$\mu = \frac{60}{6} = 10$ per hour

$c > \frac{\lambda}{\mu} = \frac{30}{10} = 3 \Rightarrow c \geq 4$

(c) $\lambda = 30$ per hour, $\mu = 40$ per hr.

$c > \frac{30}{40} = .75 \Rightarrow c \geq 1$

4

$\lambda = 45$ customers/hr

$\mu = \frac{60}{5} = 12$ customers/hr

$c > \frac{45}{12}$ or $c \geq 4$

Desired $W_q \leq 30$ seconds $= .0083$ hr

```
Title: 6e-4
Comparative Analysis

Scenario      c    Lambda        Mu   L'da eff        p0         Ls         Lq        Ws        Wq
    1         4  45.00000  12.00000   45.00000   0.00656   16.72545   12.97545   0.37168   0.28834
    2         5  45.00000  12.00000   45.00000   0.01868    5.13537    1.38537   0.11412   0.03079
    3         6  45.00000  12.00000   45.00000   0.02208    4.12903    0.37903   0.09176   0.00842
    4         7  45.00000  12.00000   45.00000   0.02309    3.86873    0.11873   0.08597   0.00264
```

Select C ≥ 7.

5

TORA input: (20, 12, 3, ∞, ∞)

```
Title: 17.6e-5
Scenario  1-- (M/M/3):(GD/infinity/infinity)

Lambda =     20.00000            Mu =        12.00000
Lambda eff = 20.00000            Rho/c =      0.55556

Ls =      2.04137                Lq =       0.37470
Ws =       0.10207               Wq =        0.01874

   n  Probability, pn  Cumulative, Pn        n  Probability, pn  Cumulative, Pn
   0       0.17266         0.17266          10       0.00218         0.99728

   1       0.28777         0.46043          11       0.00121         0.99849
   2       0.23981         0.70024          12       0.00067         0.99916
   3       0.13323         0.83347          13       0.00037         0.99953
   4       0.07401         0.90748          14       0.00021         0.99974
   5       0.04112         0.94860          15       0.00012         0.99986

   6       0.02284         0.97144          16       0.00006         0.99992
   7       0.01269         0.98414          17       0.00004         0.99996
   8       0.00705         0.99119          18       0.00002         0.99998
   9       0.00392         0.99510          19       0.00001         0.99999
```

m = size of waiting room.

$p_0 + p_1 + \cdots + p_{m+2} \geq .999 \Rightarrow m \geq 10$

6

$C = 2$, $\lambda_{windows} = .8 \times \frac{60}{3} = 16/hr$

$\mu = \frac{60}{5} = 12$ per hour

```
Title: 6e-6
Scenario  1-- (M/M/2):(GD/infinity/infinity)

Lambda =     16.00000            Mu =        12.00000
Lambda eff = 16.00000            Rho/c =      0.66667

Ls =      2.40000                Lq =       1.06667
Ws =       0.15000               Wq =        0.06667

   n  Probability, pn  Cumulative, Pn        n  Probability, pn  Cumulative, Pn
   0       0.20000         0.20000          14       0.00137         0.99726

   1       0.26667         0.46667          15       0.00091         0.99817
   2       0.17778         0.64444          16       0.00061         0.99878
   3       0.11852         0.76296          17       0.00041         0.99919
   4       0.07901         0.84198          18       0.00027         0.99946
   5       0.05267         0.89465          19       0.00018         0.99964

   6       0.03512         0.92977          20       0.00012         0.99976
   7       0.02341         0.95318          21       0.00008         0.99984
   8       0.01561         0.96879          22       0.00005         0.99989
   9       0.01040         0.97919          23       0.00004         0.99993
  10       0.00694         0.98613          24       0.00002         0.99995

  11       0.00462         0.99075          25       0.00002         0.99997
  12       0.00308         0.99383          26       0.00001         0.99998
  13       0.00206         0.99589
```

(a) $P_{n\geq 2} = 1 - (P_0 + P_1)$

$= 1 - .46667$

$= .5333$

(b) $P_0 = .2$

(c) $L_q = 1.067$

(d) NO, because $\lambda > \mu$. The minimum number of windows should $\geq \frac{\lambda}{\mu} = \frac{16}{12} = 1.33$

Number of windows ≥ 2

7

$\lambda = 25 \times \frac{60}{15} = 100$ jobs/hour

$\mu = \frac{60}{2} = 30$ jobs/hour, $C = 4$

```
Title: 6e-7
Scenario  1-- (M/M/4):(GD/infinity/infinity)

Lambda =     100.00000           Mu =        30.00000
Lambda eff = 100.00000           Rho/c =      0.83333

Ls =      6.62194                Lq =       3.28861
Ws =       0.06622               Wq =        0.03289

   n  Probability, pn  Cumulative, Pn        n  Probability, pn  Cumulative, Pn
   0       0.02131         0.02131          28       0.00138         0.99311

   1       0.07103         0.09234          29       0.00115         0.99425
   2       0.11839         0.21073          30       0.00096         0.99521
   3       0.13154         0.34228          31       0.00080         0.99601
   4       0.10962         0.45190          32       0.00066         0.99668
   5       0.09135         0.54325          33       0.00055         0.99723

   6       0.07613         0.61937          34       0.00046         0.99769
   7       0.06344         0.68281          35       0.00038         0.99808
   8       0.05286         0.73568          36       0.00032         0.99840
   9       0.04405         0.77973          37       0.00027         0.99866
  10       0.03671         0.81644          38       0.00022         0.99889

  11       0.03059         0.84703          39       0.00019         0.99907
  12       0.02549         0.87253          40       0.00015         0.99923
  13       0.02125         0.89377          41       0.00013         0.99936
  14       0.01770         0.91148          42       0.00011         0.99946
  15       0.01475         0.92623          43       0.00009         0.99955

  16       0.01229         0.93853          44       0.00007         0.99963
  17       0.01025         0.94877          45       0.00006         0.99969
  18       0.00854         0.95731          46       0.00005         0.99974
  19       0.00711         0.96443          47       0.00004         0.99978
  20       0.00593         0.97035          48       0.00004         0.99982

  21       0.00494         0.97530          49       0.00003         0.99985
  22       0.00412         0.97941          50       0.00002         0.99988
  23       0.00343         0.98284          51       0.00002         0.99990
  24       0.00286         0.98570          52       0.00002         0.99991
  25       0.00238         0.98809          53       0.00001         0.99993

  26       0.00199         0.99007          54       0.00001         0.99994
  27       0.00165         0.99173          55       0.00001         0.99995
```

(a) $P_{n\geq 4} = 1 - CP_3$

$= 1 - .34228 = .65772$

(b) $W_s = .06622$ hour

(c) $L_q = 3.29$ jobs

(d) $P_0 = .021 \Rightarrow 2.1\%$ idleness

(e) Av. # of idle computers $= 4 - (L_s - L_q)$

$= 4 - (6.62 - 3.29) = .67$

8

$\lambda = 15 + 10 + 20 = 45$ customers/hour

$\mu = \frac{60}{6} = 10$ customers/hour

$C > 45/10 = 4.5 \Rightarrow C \geq 5$

```
Title: 6e-8
Comparative Analysis

Scenario     c   Lambda      Mu          L'da eff    p0        Ls         Lq        Ws        Wq
   1         5   45.00000    10.00000    45.00000    0.00496   11.36244   6.86244   0.25250   0.15250
   2         6   45.00000    10.00000    45.00000    0.00914    5.76496   1.26496   0.12811   0.02811
   3         7   45.00000    10.00000    45.00000    0.01046    4.89100   0.39100   0.10869   0.00869
```

(a) $W_s \leq 15/60 = .25$ hour $\Rightarrow C \geq 6$

(b) $\% \text{ idle} = \frac{C - (L_s - L_q)}{C} \times 100$

C	L_s	L_q	$C-(L_s-L_q)$	% idle
5	11.362	6.862	.5	10%
6	5.765	1.265	1.5	25%

Select $C = 5$

(c)

C	5	6	7
P_0	.00496	.00914	.01046

Select $C \leq 6$

Set 15.6e

9

1. Limited space inside a bank or a grocery store
2. Multiple queues appear to offer more corteous service.

10

For c parallel servers:

$$L_q = \frac{\rho}{c-\rho}, \text{ provided } \frac{\rho}{c} \to 1$$

Thus,

$$W_{q_c} = \frac{1}{\lambda_c}\frac{\rho}{c-\rho} = \frac{1}{(c\mu-\lambda_c)}$$

For a single server

$$W_{q_1} = \frac{\lambda_1}{\mu(\mu-\lambda_1)}$$

Because $\lambda_c = c\lambda_1$, we have

$$\frac{W_{q_c}}{W_{q_1}} = \left(\frac{\frac{1}{c(\mu-\lambda_1)}}{\frac{\lambda_1}{\mu(\mu-\lambda_1)}}\right) = \frac{1}{c\left(\frac{\lambda_1}{\mu}\right)}$$

$$= \frac{1}{c\left(\frac{\lambda c/\mu}{c}\right)}$$

$$= \frac{1}{c(\rho/c)}$$

$$\lim_{\frac{\rho}{c}\to 1} \frac{W_{q_c}}{W_{q_1}} = \frac{1}{c}$$

11

Determination of p_0 involves the finite series sum

$$\sum_{n=c}^{\infty}\left(\frac{\rho}{c}\right)^{n-c} = \sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu c}\right)^{j}$$

The series will diverge if $\lambda \geq \mu c$. The condition requires that customers be serviced at a rate faster than the rate at which they arrive at the facility. Else, the queue will build up to infinity.

12

$$L_q = \sum_{n=c}^{\infty}(n-c)p_n$$

$$= \sum_{n=c}^{\infty} n p_n - c\sum_{n=c}^{\infty} p_n + \sum_{n=0}^{c-1} n p_n - \sum_{n=0}^{c-1} n p_n + c\sum_{n=0}^{c-1} p_n - c\sum_{n=0}^{c-1} p_n$$

$$= \sum_{n=0}^{\infty} n p_n - c\sum_{n=0}^{\infty} p_n + \sum_{n=0}^{c-1}(c-n)p_n$$

$$= L_s - c + (\text{number of idle servers})$$

$$= L_s - \bar{c}$$

Now, by definition

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

It follows that $\bar{c} = \frac{\lambda_{eff}}{\mu}$

13

$$p_n = \begin{cases} \frac{\lambda^n}{n!\,\mu^n} p_0, & n \leq c \\ \frac{\lambda^n}{c!\,c^{n-c}\mu^n} p_0, & n \geq c \end{cases}$$

for $c = 1$,

$$p_n = \begin{cases} \frac{\lambda}{\mu} p_0 & n = 1 \\ \left(\frac{\lambda}{\mu}\right)^n p_0 & n \geq 1 \end{cases}$$

Thus,

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \quad n = 1, 2, \ldots$$

14

$$L_q = p_0 \frac{1}{c!}\sum_{n=c+1}^{\infty}(n-c)\frac{(\lambda/\mu)^n}{c^{n-c}}$$

$$= p_0 \frac{(\lambda/\mu)^c}{c!}\sum_{n=c+1}^{\infty}(n-c)\left(\frac{\lambda}{\mu c}\right)^{n-c}$$

$$= p_0 \frac{(\lambda/\mu)^c}{c!}\sum_{j=1}^{\infty} j\left(\frac{\lambda}{\mu c}\right)^{j}$$

$$= p_0 \frac{(\lambda/\mu)^c}{c!}\frac{\lambda}{\mu c}\frac{d}{d\left(\frac{\lambda}{\mu c}\right)}\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu c}\right)^{j}$$

$$= p_0 \frac{(\lambda/\mu)^c}{c!}\left\{\frac{\lambda/\mu c}{(1-\lambda/\mu c)^2}\right\}$$

$$= p_c \frac{\rho/c}{(1-\rho/c)^2} = \frac{\rho}{(c-\rho)^2} p_c$$

15

(a) $P\{\text{a customer is waiting}\}$

$$= P\{\text{at least } c+1 \text{ in system}\}$$

$$= \sum_{n=c+1}^{\infty} p_n$$

$$= \sum_{n=c}^{\infty} p_n - p_c$$

$$= p_0 \frac{\rho^c}{c!} \frac{1}{1-\frac{\rho}{c}} - p_c$$

$$= p_c \left\{\frac{1}{1-\frac{\rho}{c}} - 1\right\}$$

$$= p_c \left(\frac{\rho}{c-\rho}\right)$$

(b) Expected number in queue given the queue is not empty

$$= \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{j=c+1}^{\infty} p_j}$$

$$= \frac{L_q}{\sum_{j=c+1}^{\infty} p_j} = \frac{L_q}{p_c\left(\frac{\rho}{c-\rho}\right)}$$

Now, $L_q = \frac{p_0}{c!} \sum_{n=c+1}^{\infty} (n-c) \frac{\rho^n}{c^{n-c}}$

$$= p_0 \frac{\rho^c}{c!} \sum_{j=1}^{\infty} j \left(\frac{\rho}{c}\right)^j$$

$$= p_0 \frac{\rho^c}{c!} \left(\frac{\rho/c}{(1-\rho/c)^2}\right), \quad \frac{\rho}{c} < 1$$

$$= p_c \left\{\frac{c\rho}{(c-\rho)^2}\right\}, \quad \frac{\rho}{c} < 1$$

Substitution for L_q yield the desired result.

(c) Exp. waiting time for those who must wait = Exp. waiting time given there are c in the system.

$$= \frac{1}{\lambda} \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{n=0}^{\infty} p_n}$$

$$= \frac{L_q/\lambda}{p_c/(1-\rho/c)} = \frac{1}{\mu(c-\rho)}$$

16

First convert the c-channel case into an equivalent single channel. Let the customer just arriving be the jth in queue. Because there are c channels in parallel, the service time, t, of each of the other $j-1$ customers and the (one) customer in service are determined as follows: Let $t_1, t_2, \ldots, t_c$ be the actual service times in the c channels. Then,

$$P\{t > T\} = P\{\min_{1 \le i \le c} t_i > T\}$$

$$= \left(e^{-\mu T}\right)^c = e^{-\mu c T}$$

This is true because if $\min_i > T$, then every t_i must be $> T$.

Now,

$$F_t(T) = 1 - P\{t > T\}$$

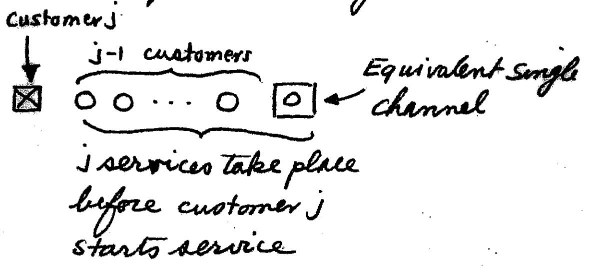
$$= 1 - e^{-\mu c T}, \quad T > 0$$

Thus,

$$f(T) = \frac{\partial F_t(T)}{\partial T} = \mu c e^{-\mu c T}, \quad T > 0$$

which is exponential with mean $\frac{1}{\mu c}$.

The c channels can be converted into an equivalent single channel as



Before customer j starts service, j other customers each with a service time T must be processed first.

Continued...

The assumption here is that all c channels are busy. If there are any idle servers, arriving customer j will have zero waiting time in queue and the special case is treated separately.

Let τ be the waiting time in queue given there are j other customer yet to be serviced. Then

$$\tau = T_1' + T_2 + \cdots + T_j$$

where $T_1', T_2, \ldots, T_j$ are exponential with mean $1/\mu c$. T_1' represents the remaining service time for the customer already in service. The lack of memory property indicate that T_1' is also exponential with mean $1/\mu c$. Thus,

$$W_q(\tau|j) = \frac{\mu c(\mu c\tau)^{j-1} e^{-\mu c\tau}}{(j-1)!}, \quad \tau > 0$$

Let $W_q(\tau)$ be the absolute pdf, then

$$w_q(\tau) = \sum_{j=1}^{\infty} W_q(\tau|j)\, q_j.$$

where

$$q_j = \begin{cases} \sum_{k=0}^{c-1} p_k, & j = 0 \\ p_{c+j-1}, & j > 0 \end{cases}$$

Hence, for $\tau > 0$

$$W_q(\tau) = \sum_{j=1}^{\infty} \frac{\mu c(\mu c\tau)^{j-1} e^{-\mu c\tau}}{(j-1)!} \frac{\rho^{c+j-1}}{c!\, c^{j-1}} p_0$$

$$= \frac{\rho^c \mu c\, e^{-\mu c\tau}}{c!} p_0 \sum_{j=0}^{\infty} \frac{(\rho\mu c\tau/c)^j}{j!}$$

$$= \frac{\rho^c \mu c\, e^{-\mu c\tau}}{c!} p_0\, e^{-\lambda\tau}$$

$$= \frac{\rho^c \mu\, e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0$$

continued...

For $\tau = 0$, the corresponding probability is $\sum_{k=0}^{c-1} p_k$, or

$$1 - \sum_{k=c}^{\infty} p_k = 1 - \sum_{j=0}^{\infty} p_{c+j}$$

$$= 1 - \sum_{j=0}^{\infty} \frac{\rho^{c+j}}{c!\, c^j} p_0$$

$$= 1 - \frac{\rho^c}{c!}\left(\frac{p_0}{1-\frac{\rho}{c}}\right)$$

$$= 1 - \left\{ \frac{\rho^c p_0}{(c-1)!(c-\rho)} \right.$$

Hence,

$$W_q(\tau) = \begin{cases} 1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}, & \tau = 0 \\ \frac{\mu\rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0, & \tau > 0 \end{cases}$$

17

$$P\{\tau > y\} = \int_y^{\infty} w_q(\tau)\, d\tau$$

$$= \frac{c\mu\rho^c p_0}{c!} \int_y^{\infty} e^{-(c\mu-\lambda)\tau}\, d\tau$$

$$= \frac{\rho^c c\mu}{c!(c\mu-\lambda)} e^{-(c\mu-\lambda)y} p_0$$

$$= \frac{\rho^c p_0}{c!\left(1-\frac{\rho}{c}\right)} e^{-(c\mu-\lambda)y}$$

$$= P\{\tau > 0\}\, e^{-(c\mu-\lambda)y}$$

where $P\{\tau > 0\} = 1 - P\{\tau = 0\}$

18

From Problem 16, the waiting time in the system is computed as

$$T = T_1' + T_2 + \cdots + T_j + t_j$$

where

t_j = actual service time for customer j.

t_j is exponential with mean $1/\mu$

Thus, T is the convolution of the waiting time in queue and the actual service time of customer j. This means that $w(T)$ is the convolution of $w_q(\tau)$ and $g(t)$; that is,

$$w(T) = w_q(\tau) * g(t)$$

where

$$g(t) = \mu e^{-\mu t}, \quad t > 0$$

$$w(T) = w_q(0)g(T) + \int_{0+}^{T} w_q(\tau)g(T-\tau)d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}\right)\mu e^{-\mu T} + p_0\int_{0+}^{T}\frac{\mu\rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!}\mu e^{-\mu(T-\tau)}d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}\right)\mu e^{-\mu T} + \frac{\mu\rho^c e^{-\mu T}}{(c-1)!(c-1-\rho)}p_0\left\{1 - e^{-\mu(c-1-\rho)T}\right\}$$

$$= \mu e^{-\mu T} - \frac{\rho^c p_0 \mu e^{-\mu T}}{(c-1)!(c-1-\rho)}\frac{(c-\rho-1)}{(c-\rho)} + \frac{\mu\rho^c e^{-\mu T}p_0}{(c-1)!(c-1-\rho)} - \frac{\mu\rho^c e^{-\mu T}e^{-\mu(c-1-\rho)T}}{(c-1)!(c-1-\rho)}$$

Continued...

$$= \mu e^{-\mu T} + \frac{\rho^c p_0 \mu e^{-\mu T}}{(c-1)!(c-1-\rho)}\left\{\frac{1}{c-\rho} - e^{-\mu(c-1-\rho)T}\right\}$$

$T > 0$

Set 15.6f

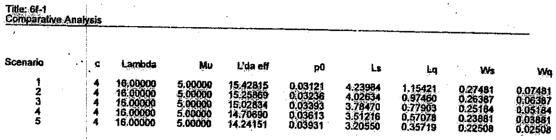
1

(a) $c-(L_s-L_q)=4-(4.24-1.54)$

$= 1.3$ cabs

(b) $p_q = .04468$

(c)



```
Title: 6f-1
Comparative Analysis

Scenario     c   Lambda     Mu        L'da eff   p0        Ls        Lq        Ws        Wq
   1         4   16.00000   5.00000   15.42815   0.03121   4.23984   1.15421   0.27481   0.07481
   2         4   16.00000   5.00000   15.25869   0.03236   4.02634   0.97460   0.26387   0.06387
   3         4   16.00000   5.00000   15.02834   0.03393   3.78470   0.77903   0.25184   0.05184
   4         4   16.00000   5.00000   14.70690   0.03613   3.51216   0.57078   0.23881   0.03881
   5         4   16.00000   5.00000   14.24151   0.03931   3.20550   0.35719   0.22508   0.02508
```

m = length of waiting list

$N = m+4$

m	N	Wq(hr)	Wq(min)
6	10	.075	4.5
5	9	.064	3.83
4	8	.052	3.12
3	7	.039	2.33
2	6	.025	1.5

Select $m \leq 3$

2

$c=2$, $\lambda = 20/hr$, $N=5$

$\mu = 60/6 = 10/hr$

(a) $p_5 = .1818$ or 18.18%

(b) $p_1 = .1818$ or 18.18%

(c) % utilization $= 100\left(\frac{L_s-L_q}{c}\right)$

$= \frac{2.727-1.091}{2} \times 100$

$= 81.8\%$

(d) Probability $= p_2+p_3+p_4 = .54546$

(e) $p_N \leq .1$

N	5	⋯ 8	9	10
p_N	.1818	.1176	.1053	.0952

$N \geq 10$ spaces (including the pumps)

continued...

(f) $p_0 \leq .05$

N	5	⋯ 8	9	10
p_N	.0909	.0588	.0526	.0476

$N \geq 10$

3

$\lambda = 60/10 = 6/hr$

$\mu = 60/30 = 2/hr$, $N=18$

(a) # idle mechanics

$= c-(L_s-L_q)$

$= 3-(9.54-6.71) = .17$

(b) $p_{18} = .0559$

$\lambda_{lost} = .0559 \times 6 = .3354$ job/hr

lost jobs in 10 hrs = 3.354 jobs

(c) $p_{n\leq 17} = p_0+p_1+\cdots+p_{17}$

$= .9441$

(d) $p_{n\leq 2} = p_0+p_1+p_2 = .10559$

(e) $L_q = 6.7081$ mower

(f) $\frac{L_s-L_q}{c} = \frac{9.54-6.71}{3} = .944$

4

$N=40$, $c=30$, $\lambda = 20/hr$

$\mu = 60/60 = 1/hr$

(a) $p_{40} = .00014$

(b) $p_{30}+p_{31}+\cdots+p_{39} = p_{n\leq 39} - p_{n\leq 29}$

$= .99986 - .97533$

$= .02453$

(c) $p_{29} = .01248$

(d) $L_s - L_q = 20.043 - .046 \simeq 20$ spaces

(e) $L_q = .046$

continued...

Set 17.6f

(f) If there are 30 cars or more in the lot, the student will not make it to class. Thus,

$$P\{\text{not finding a parking space}\}$$
$$= p_{30} + p_{31} + \cdots + p_{40} = 1 - P_{n \le 29}$$
$$= 1 - .97533 = .02467$$

No. of students who cannot park during an 8-hr period $= 20 \times .02467 \times 8$
$\cong 4$ students

5

$$1 = p_0\left\{\sum_{n=0}^{c-1}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}\sum_{n=c}^{N}\left(\frac{\rho}{c}\right)^{n-c}\right\}$$
$$= p_0\left\{\sum_{n=0}^{c-1}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}\,\frac{1-(\rho/c)^{N-c+1}}{(1-\rho/c)}\right\}$$
$$p_0 = \left\{\sum_{n=0}^{c-1}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}\left(\frac{1-(\rho/c)^{N-c+1}}{1-\rho/c}\right)\right\}$$

6

$$\bar{c} = L_s - L_q$$
$$= \lambda_{eff}\,(W_s - W_q)$$
$$= \lambda_{eff}\left(\frac{1}{\mu}\right)$$

7

$$1 = \frac{p_0}{c!}\sum_{n=c}^{N}\frac{\rho^n}{c^{n-c}} + p_0\sum_{n=0}^{c-1}\frac{\rho^n}{n!}$$
$$= \frac{p_0\,\rho^c}{c!}\sum_{n=0}^{N-c}\left(\frac{\rho}{c}\right)^n + p_0\sum_{n=0}^{c-1}\frac{\rho^n}{n!}$$
$$= \frac{p_0\,\rho^c}{c!}(N-c+1) + p_0\sum_{n=0}^{c-1}\frac{\rho^n}{n!}$$

Thus,

$$p_0 = \left\{\sum_{n=0}^{c-1}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}(N-c+1)\right\}^{-1}$$
$$L_q = \sum_{n=c}^{N}(n-c)\,p_n$$
$$= \sum_{j=0}^{N-c} j\,p_{j+c}$$
$$= \frac{\rho}{C!}\,\frac{\rho}{c}\sum_{j=0}^{N-c} j\left(\frac{\rho}{c}\right)^{j-1} p_0$$

continued...

$$= \frac{\rho^c}{c!}\sum_{j=0}^{N-c} j\,p_0 \quad \left(\text{because } \frac{\rho}{c} = 1\right)$$
$$= \frac{\rho^c}{c!}\,\frac{(N-c)(N-c+1)}{2}\,p_0$$
$$= \frac{\rho^c(N-c)(N-c+1)}{2c!}\,p_0$$

8

$$\lambda_n = \begin{cases}\lambda, & n = 0, 1, 2, \ldots, c-1\\ 0, & n = c\end{cases}$$

$$\mu_n = n\mu, \quad n = 0, 1, \ldots, c$$

Thus,

$$p_n = \frac{\rho^n}{n!}\,p_0, \quad n = 0, 1, 2, \ldots, c$$
$$\sum_{n=0}^{c} p_n = \sum_{n=0}^{c}\frac{\rho^n}{n!}\,p_0 = 1$$
$$p_0 = \left\{\sum_{n=0}^{c}\frac{\rho^n}{n!}\right\}^{-1}$$

Set 15.6g

1

(a) $p_0 = 0$

(b) $p_{n\geq 10} = 1 - p_{n\leq 9} = 1$

(c) $p_{n\leq 40} - p_{n\leq 29} = .7771 - .13787$

$= .63923$

(d) $L_s = 36$

Net annual equity

$= \$1000 \times 36 \{.1(1-.3)+.9(1+.15)\}$

$= \$39,780$

2

$\lambda = \frac{100}{8} = 12.5/hr$

$\mu = \frac{60}{30} = 2/hr$

(a) $L_s = 6.25 \simeq 7$ seats

(b) $p_{n\geq 8} = 1-(p_0+p_1+\cdots+p_7)$

$= 1 - .7089 = .2911$

(c) $p_0 = .00193$

3

$\rho = .1$

c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
2	1.00000	10.00000	1.00000	0.90476	0.10025	0.00025	0.10025	0.00025
4	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
10	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
20	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
50	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
9999	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000

4

$\rho = 9$

c	λ	μ	λ_{eff}	p_0	L_s	L_q	W_s	W_q
10	9.00000	1.00000	9.00000	0.00007	15.01858	6.01858	1.66873	0.66873
15	9.00000	1.00000	9.00000	0.00012	9.07235	0.07235	1.00804	0.00804
25	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000
50	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000
9999	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000

1. For very small ρ, $(M/M/\infty):(GD/\infty/\infty)$ provides reliable estimates for $(M/M/c):(GD/\infty/\infty)$.
2. For large ρ, $(M/M/\infty)$ gives reliable estimates only if c is large

1

(a) $R=1: \lambda_{eff} = \lambda(22-L_s)$

$= .5(22 - 12.004)$

$= 4.998$

$R=4: \lambda_{eff} = .5(22-2.1) = 9.95$

(b) No. of idle repair persons

$= 4-(L_s-L_q)$

$= 4-(2.1-.11) = 2.01$

(c) $p_0 = .10779$

(d) $R=3:$

$P\{2 \text{ or } 3 \text{ are idle}\} = p_0+p_1$

$= .34492$

```
Title: 6h-1
Scenario  3-- (M/M/3):(GD/22/22)

Lambda =     0.50000            Mu =        5.00000
Lambda eff = 9.76696            Rho/c =      0.03333

Ls =       2.46596              Lq =       0.51257
Ws =        0.25248             Wq =        0.05248

        n  Probability, pn  Cumulative, Pn        n  Probability, pn  Cumulative, Pn

        0          0.10779         0.10779        8          0.00953         0.99244

        1          0.23713         0.34492        9          0.00445         0.99689
        2          0.24899         0.59390       10          0.00193         0.99881
        3          0.16599         0.75990       11          0.00077         0.99959
        4          0.10513         0.86502       12          0.00028         0.99987
        5          0.06308         0.92810       13          0.00009         0.99996

        6          0.03574         0.96384       14          0.00003         0.99999
        7          0.01906         0.98291
```

2

Productivity of repair persons

$$= \frac{\text{Av. \# busy repair persons}}{R}$$

$$= \frac{L_s - L_q}{R}$$

R	Repair prod.	Shop prod.
1	100%	45.44%
2	88.2%	80.15%
3	65.1%	88.7%
4	49.7%	90.45%

R = 2 yield 80.15% shop productivity and also maintain repair productivity at 88.2%

3

Increasing R, in effect, increases the number of machines that remain operative, and hence the chance of additional breakdowns. Stated differently, if all machines remain broken, there will be no new calls for repair service, and $\lambda_{eff} = 0$

4

$\lambda = \frac{60}{45} = 1.33$ machines/hr

$\mu = \frac{60}{8} = 7.5$ machines/hr

$R=1, \quad K=5$

```
Title: 6h-4
Scenario  1-- (M/M/1):(GD/5/5)

Lambda =     1.33333            Mu =        7.50000
Lambda eff = 4.99939            Rho/c =      0.17778

Ls =       1.25045              Lq =       0.58386
Ws =        0.25012             Wq =        0.11679

        n  Probability, pn  Cumulative, Pn        n  Probability, pn  Cumulative, Pn

        0          0.33341         0.33341        3          0.11240         0.95293

        1          0.29637         0.62978        4          0.03996         0.99290
        2          0.21075         0.84053        5          0.00710         1.00000
```

(a) $L_s = 1.25$ machines

(b) $p_0 = .33341$

(c) $W_s = .25$ hour

5

$\lambda = 60/45 = 1.33/\text{hr}$

$\mu = 60/20 = 3/\text{hr}$

$R=4, \quad K=4$

```
Title: 6h-5
Scenario  1-- (M/M/4):(GD/4/4)

Lambda =     1.33333            Mu =        3.00000
Lambda eff = 3.69230            Rho/c =      0.11111

Ls =       1.23077              Lq =       0.00000
Ws =        0.33333             Wq =        0.00000

        n  Probability, pn  Cumulative, Pn        n  Probability, pn  Cumulative, Pn

        0          0.22972         0.22972        3          0.08067         0.99104

        1          0.40839         0.63811        4          0.00896         1.00000
        2          0.27226         0.91037
```

(a) $L_s = 1.23$ worker

(b) $P_0 = .22922$

Set 15.6h

6

$\lambda = \frac{60}{30} = 2$ calls/hr/baby

$\mu = \frac{60}{120} = .5$/hr

$R = 5, \quad K = 5$

```
Title: 6h-6
Scenario  1-- (M/M/5):(GD/5/5)

Lambda =     2.00000          Mu =        0.50000
Lambda eff = 2.00000          Rho/c =      0.80000

Ls =        4.00000           Lq =        0.00000
Ws =         2.00000          Wq =         0.00000

   n   Probability, pn  Cumulative, Pn          n   Probability, pn  Cumulative, Pn
   0        0.00032        0.00032              3        0.20480        0.26272
   1        0.00640        0.00672              4        0.40960        0.67232
   2        0.05120        0.05792              5        0.32768        1.00000
```

(a) No. "awake" babies

$= 5 - L_s = 5 - 4 = 1$ baby

(b) $p_5 = .32768$

(c) $p_{n \le 2} = p_0 + p_1 + p_2 = .05792$

7

$$p_n = \begin{cases} \frac{K\lambda}{\mu} \frac{(K-1)\lambda}{2\mu} \cdots \frac{(K-n)\lambda}{n\mu} p_0, & 0 \le n \le R \\ \frac{K\lambda}{\mu} \frac{(K-1)\lambda}{2\mu} \cdots \frac{(K-R)\lambda}{R\mu} \cdots \frac{K-n}{R\mu} p_0, & R \le n \le K \end{cases}$$

Thus,

$$p_n = \begin{cases} \frac{K(K-1)\cdots(K-n)}{1 \times 2 \times \cdots \times n} \left(\frac{\lambda}{\mu}\right)^n p_0, & 0 \le n \le R \\ \frac{C_n^k \, n!}{R! \, R^{n-R}} \left(\frac{\lambda}{\mu}\right)^n p_0, & R \le n \le K \end{cases}$$

$$= \begin{cases} C_n^K \rho^n p_0, & 0 \le n \le R \\ C_n^k \frac{n! \rho^n}{R! \, R^{n-R}} p_0, & R \le n \le K \end{cases}$$

8

$\bar{R} = L_s - L_q$

$= \lambda_{eff} (W_s - W_q)$

$= \lambda_{eff} \left(\frac{1}{\mu}\right)$

hence $\lambda_{eff} = \mu \bar{R}$

9

$$p_n = \begin{cases} C_n^k \rho^n n! \, p_0, & n = 0, 1 \\ C_n^k n! \rho^n p_0, & n = 1, 2, \cdots, K \end{cases}$$

$$= \frac{K!}{(K-n)!} \rho^n p_0, \quad n = 0, 1, 2, \cdots, K$$

$$L_s = \sum_{n=0}^{K} n p_n = p_0 K! \sum_{n=0}^{K} \frac{n \rho^n}{(K-n)!}$$

$$= K - \left(\frac{1 - p_0}{\rho}\right)$$

1

$$\%\text{idle} = \frac{1-(L_s-L_q)}{1}\times 100$$

$$= [1-(L_s-L_q)]\times 100$$

$$= (1 - 1.333 + .667)\times 100$$

$$= 33.3\%$$

2

(a) $E\{t\} = 14$ min

$Var\{t\} = \frac{(20-8)^2}{12} = 12$ min^2

$\lambda = 4/hr = .0667/min$

$L_s = 7.867$ cars

$W_s = 118$ min $= 1.967$ hours

$L_q = 6.933$ cars

$W_q = 104$ min $= 1.733$ hours

(b) $E\{t\} = 12$ min

$Var\{t\} = 9$ min^2

$\lambda = .0667$ /min

$L_s = 2.5$ cars

$W_s = 37.5$ min $= .625$ hour

$L_q = 1.7$ cars

$W_q = 25.5$ min $= .425$ hr

(c) $E\{t\} = 4\times.2 + 8\times.6 + 15\times.2 = 8.6$ min

$Var\{t\} = (4-8.6)^2(.2) + (8-8.6)^2(.6) + (15-8.6)^2(.2) = 12.64$ min^2

$L_s = 1.0244$ cars

$W_s = 15.3657$ min $= .256$ hr

$L_q = .451$ car

$W_q = 6.765$ min $= .113$ hr

3

$\lambda = .3$ job/day

Service time distribution:

$f(t) = .5, \quad 2 \le t \le 4$ days

$E\{t\} = 3$ days

$Var\{t\} = \frac{4}{12} = .333$ days^2

(a) $L_q = 4.2$ homes

(b) $W_s = 17$ days

(c) $E\{t\} = 1.5$, $Var\{t\} = \frac{1}{12} = .0833$

$L_q = .191$ home

$W_s = 2.14$ days

4

$\lambda = \frac{30}{8\times 60} = .0625$ prescr./min

$E\{t\} = 12 + 3 = 15$ min

$Var\{t\} = 9 + \frac{(4-2)^2}{12} = 9.333$ min^2

(a) $p_0 = .0625$

(b) $L_q = 7.3$ prescriptions

(c) $W_s = 132.17$ min $= 2.2$ hours

5

$\lambda = 1/45$ /min $= .0222$ /min

$E\{t\} = 28 + 4.5 = 32.5$ min

$Var\{t\} = \frac{(6-3)^2}{12} = .75$

(a) $L_q = .9395$ item

(b) $p_0 = .278$

(c) $W_s = 74.78$ min

6

$$L_s = \lambda E\{t\} + \frac{\lambda^2(E^2(t) + Var\{t\})}{2(1-\lambda E\{t\})}$$

$$= \lambda E\{t\} + \frac{(\lambda E\{t\})^2}{2(1-\lambda E\{t\})}$$

$$= \rho + \frac{\rho^2}{2(1-\rho)}$$

Set 15.7a

7

$$L_s = \frac{m\lambda}{\mu} + \frac{\lambda^2\left(\frac{m^2}{\mu^2} + \frac{n}{\mu^2}\right)}{2\left(1 - \frac{m\lambda}{\mu}\right)}$$

$$= m\rho + \frac{m^2\rho^2 + m\rho^2}{2(1-m\rho)}$$

$$= m\rho + \frac{m(m+1)\rho^2}{2(1-m\rho)}$$

8

$$E\{t\} = \frac{1}{\mu}, \quad Var\{t\} = \frac{1}{\mu^2}$$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\frac{1}{\mu^2} + \frac{1}{\mu^2}\right)}{2((1 - \lambda/\mu)}$$

$$= \rho + \frac{\rho^2}{1-\rho}$$

$$= \frac{\rho}{1-\rho}$$

9

(a) Because each server receives every c^{th} customer and the interarrival time at the channel is exponential with mean $1/\lambda$, the interarrival time at each server is the convolution of c exponential distributions each with mean $\frac{1}{\mu}$. This means that the interarrival time is gamma with mean c/λ and variance c/λ^2.

(b) The interarrival time at the i^{th} server is exponential with mean $\frac{1}{\alpha_i \lambda}$. This means that the arrivals at server i is Poisson with mean $\alpha_i \lambda$, $i = 1, 2, \ldots, c$

1

(a) $\mu_2 = \frac{24}{(\frac{1000}{36})\frac{1}{60}} = 5.184$ jobs/day

$\mu_3 = \frac{24}{(\frac{1000}{50})\times\frac{1}{60}} = 7.2$ jobs/day

$\mu_4 = \frac{24}{(\frac{1000}{66})\times\frac{1}{60}} = 9.5$ jobs/day

(b) $ETC_i = 24\, C_{1i} + 80\, Lq_i$

i	λ_i	μ_i	Lq_i	C_{1i}	ETC_i
1	4	4.32	11.57	\$15	\$1285.60
2	4	5.18	2.62	20	689.60
3	4	7.20	.69	24	$\boxed{631.20}$
4	4	9.50	.31	27	672.80

Select model 3.

2

$\lambda = 3/hr$

$\mu_1 = 5/hr, \quad C_1 = \$15$

$\mu_2 = 8/hr, \quad C_2 = \$20$

Cost/Broken machine = \$50/hr

$(M/M/1):(GD/10/10)$:

$\lambda = 3,\ \mu = 5 \Rightarrow L_{s_1} = 8.33$

$(M/M/1):(GD/10/10)$:

$\lambda = 3,\ \mu = 8 \Rightarrow L_{s_2} = 7.33$

$TC_1 = 50L_{s_1} + 15 = 50\times 8.33 + 15$
$= \$431.50/hr$

$TC_2 = 50L_{s_2} + 20 = 50\times 7.33 + 20$
$= \$386.50/hr$

Hire second repair person.

3

$\lambda = 10/hr = .167/min$

Scanner A:

Service time distribution:

$f_A(t) = \frac{1}{(\frac{35}{10})-(\frac{25}{10})} = 1,\ 2.5 \le t \le 3.5$

Continued...

$E_A\{t\} = 3$ min

$Var_A\{t\} = \frac{1}{12}$ min^2

Scanner B:

$f_B(t) = \frac{1}{\frac{35}{15} - \frac{25}{15}} = 1.5,\quad 5/3 \le t \le 7/3$

$E_B\{t\} = 2$ min

$Var_B\{t\} = \frac{(2/3)^2}{12} = 1/27$ min^2

From Excel file PKFormula.xls,

$L_{s_A} = .755$ customer

$L_{s_B} = .419$ customer

$TC_A = .2L_{s_A} + C_A$
$= (.2\times .755 + \frac{\$10}{10\times 60})\times 60 = \$10.06/hr$

$TC_B = .2L_{sB} + C_B$
$= (.2\times .419 + \frac{\$15}{10\times 60})\times 60 = \$6.53/hr$

Select scanner B

4

(a)

μ = number of filled orders/hr

λ = number of requested orders/hr

C_1 = cost/unit increase in production rate

C_2 = cost of waiting/unit waiting time/cust.

$TC(\mu)$ = Total cost/unit waiting time given μ

$= C_1\mu + C_2 L_s$

$= C_1\mu + C_2\frac{\lambda}{\mu-\lambda}$

(b)

$\frac{\partial TC(\mu)}{\partial\mu} = C_1 - C_2\frac{\lambda}{(\mu-\lambda)^2} = 0$

$\mu = \lambda + \sqrt{\frac{C_2}{C_1}\lambda}$

(c) $\lambda = 3$, $C_1 = .1\times 500 = \$50$, $C_2 = \$100$

$\mu = 3 + \sqrt{\frac{100}{50}\times 3} = 5.45$ orders/hr

Optimum production rate

$= 500\times 5.45 \simeq 2725$ pieces/hr

Set 15.9a

5

$\lambda = 80$ jobs/wk

$C_1 = \$250/\text{wk} \qquad C_2 = \$500/\text{job}/\text{wk}$

$$\mu = \lambda + \sqrt{\frac{C_2}{C_1}\lambda}$$

$$= 80 + \sqrt{\frac{500}{250} \times 80} = 92.65 \text{ jobs/wk}$$

6

$\lambda = 25$ groups/hr

Model A: $\mu_A = 26/\text{hr}$, $N = 20$

Operating cost $C_A = \$12000/\text{month}$

From TORA: $p_{20} = .03128$

$L_q = 7.65$ groups

Cost/hr = operating cost/hr + waiting cost/hr + cost of lost customers/hr

$$= \frac{C_A}{30 \times 10} + 10L_q + \lambda p_N \times 15$$

$$= \frac{12000}{30 \times 10} + 10 \times 7.65 + 25 \times .03128 \times 15$$

$$= \$128.23/\text{hr}$$

Model B: $\mu_B = 29/\text{hr}$, $N = 30$

$C_B = \$16000/\text{month}$

From TORA: $p_{30} = .0016$

$L_q = 5.07$ groups

$$\text{Cost/hr} = \frac{\$16000}{30 \times 10} + 10 \times 5.07 + 25 \times .0016 \times 15$$

$$= \$104.63$$

Select model B

7

Let

C_3 = cost/unit time/additional capacity unit.

The cost model in Problem 6 is modified by adding the term $C_3 N$ to the cost equation.

8

p_0 is the probability of running out of stock. Thus,

Cost of lost sales per hour $= C_1 \lambda p_0$

$E\{\text{cost}\}$/unit time

= $E\{\text{lost sales cost}\}$/unit time + $E\{\text{holding cost}\}$/unit time

$$= C_1 \lambda p_0 + C_2 L_s$$

For $(M/M/1):(GD/\infty/\infty)$

$$p_0 = (1-\rho)$$

$$L_s = \frac{\rho}{1-\rho}$$

Thus,

$$E\{\text{cost}\}/\text{unit time} = C_1\lambda(1-\rho) + C_2\frac{\rho}{1-\rho}$$

$$\frac{\partial E\{\text{cost}\}}{\partial \rho} = -C_1\lambda + \frac{C_2}{(1-\rho)^2} = 0$$

Thus,

$$\rho = 1 \pm \sqrt{\frac{C_1\lambda}{C_2}}$$

Under steady state, ρ must be less than 1. Thus,

$$\rho = 1 - \sqrt{\frac{C_1\lambda}{C_2}}$$

The solution requires $\sqrt{\frac{C_1\lambda}{C_2}} < 1$ in order for ρ not to assume a negative value. Note that $\rho = \frac{\lambda}{\mu}$, where λ is a constant. This means that μ is the actual optimization variable.

1

$C_1 = \$20$, $C_2 = \$45$,
$\lambda = 17.5/hr$, $\mu = 10/hr$

Title: 17.9b-1 (M/M/c)(GD/infinity/infinity)
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	17.50000	10.00000	17.50000	0.06667	7.46667	5.71667	0.42667	0.32667
2	3	17.50000	10.00000	17.50000	0.15564	2.21712	0.46712	0.12669	0.02669
3	4	17.50000	10.00000	17.50000	0.17038	1.84206	0.09206	0.10526	0.00526
4	5	17.50000	10.00000	17.50000	0.17314	1.76962	0.01962	0.10112	0.00112

$ETC(c) = 20c + 45L_s$

c	$L_s(c)$	$ETC(c)$
2	7.467	$20\times2+45\times7.467=\$376.02$
→ 3	2.217	$20\times3+45\times2.217=\$159.77$
4	1.842	$20\times4+45\times1.842=\$162.89$
5	1.770	$20\times5+45\times1.770=\$179.65$

Use three clerks

2

$Cost/hr = C_1L_s + C_2c$

$C_1 = \$30$, $C_2 = \$18$

$(M/M/c):(GD/10/10)$: $\lambda = 1/20 = 0.05/hr$
$\mu = 1/3 = 0.333/hr$

Title: 9b-2
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	0.05000	0.33300	0.41603	0.21439	1.67942	0.43010	4.03683	1.03383
2	3	0.05000	0.33300	0.43187	0.24269	1.36246	0.06554	3.15476	0.15175

$(Cost/hr \text{ for } c=2) = 30\times1.68+18\times2 = \$86.40$

$(Cost/hr \text{ for } c=3) = 30\times1.36+18\times3 = \$94.80$

(a) No, because the cost is higher

(b) Schedule loss/breakdown $= C_1W_s$

$c=2$: $W_s = 4.037$ hours
Schedule loss $= 30\times4.037 = \$121.11$

$c=3$: $W_s = 3.155$ hours
Schedule loss $= 30\times3.155 = \$94.65$

3

The problem is similar to the machine repair model. The executives are the "machines" and the WATS line is the "server"

arrival rate/executive = 2 calls/day

$$\text{service rate} = \frac{480}{6} = 80 \text{ calls/day}$$

Continued...

TORA input:

$R=1$: $(2, 80, 1, 100, 100)$

$R=2$: $(2, 80, 2, 100, 100)$

Title: 9b-3
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	2.00000	80.00000	80.00000	0.00000	59.99881	58.99881	0.74999	0.73749
2	2	2.00000	80.00000	159.28000	0.00200	20.35920	18.36820	0.12782	0.11532

(a) No WATS:

Cost/month = (2 calls/8 hrs/exec) × (100 exec) × (6 min/call) × (50¢/min) × (200 hrs/month)
= \$15,000/month

One WATS Line: $L_q = 59$

Cost/month = cost of WATS line + C_1L_q

$= \$2000/month + 59\left(\frac{1¢}{100}\times60\times200\right)$

$= \$9080$

Savings = 15,000 − 9080
= \$5920/month

(b) Two WATS lines: $L_q = 18.4$

Cost/month $= 2\times2000 + 18.4\left(\frac{1¢}{100}\times200\times60\right)$
$= \$6200$

Additional savings
$= 9080 - 6200 = \$2880$

Lease a second WATS line

Set 15.9b

4

Rate of breakdown /machine, λ

$= \frac{57.8}{8 \times 20} = .36125/hr$

$\mu = \frac{60}{6} = 10/hr$

TORA model: (M/M/3):(GD/20/20)

W_s = lost time per breakdown

λ = number of breakdowns/hr/mach

lost time per mach/hr $= \lambda W_s$

From TORA, $W_s = .10118$ hr

Lost revenue/machine/hr

$= 25 \times (.36125 \times .10118) \times \$2$

$= \$1.83$

Lost revenue for all machines

$= 20 \times 1.83 = \$36.60$

Cost of 3 repairpersons/hr

$= 3 \times 20 = \$60.$

5

$TC(c) = cC_1 + C_2 L_s(c)$

$TC(c-1) = (c-1)C_1 + C_2 L_s(c-1)$

$TC(c+1) = (c+1)C_1 + C_2 L_s(c+1)$

$TC(c-1) - TC(c)$

$= -C_1 + C_2\{L_s(c-1) - L_s(c)\}$

$TC(c+1) - TC(c)$

$= C_1 - C_2\{L_s(c) - L_s(c+1)\}$

At a minimum point, we must have

$TC(c-1) \geq TC(c)$

$TC(c+1) \geq TC(c)$

Thus,

$L_s(c-1) - L_s(c) \geq \frac{C_1}{C_2}$

$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2}$

continued...

or

$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2} \leq L_s(c-1) - L_s(c)$

$\frac{C_1}{C_2} = \frac{12}{50} = .24$

c	$L_s(c)$	$L_s(c) - L_s(c+1)$
2	7.467	–
3	2.217	5.25
4	1.842	.375
		$\leftarrow \frac{C_1}{C_2} = .24$
5	1.764	.078

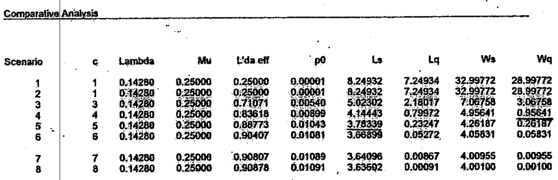
$C^* = 4$

1

$\lambda = 1/7 = .1428$ breakdown/hr

$\mu = .25$ repair per hour

TORA model: $(M/M/R):(GD/10/10)$



```
Comparative Analysis

Scenario    c  Lambda   Mu       L'da eff  p0       Ls       Lq       Ws        Wq

   1        1  0.14280  0.25000  0.25000   0.00001  8.24932  7.24934  32.99772  28.99772
   2        1  0.14280  0.25000  0.25000   0.00001  8.24932  7.24934  32.99772  28.99772
   3        3  0.14280  0.25000  0.71071   0.00540  5.02302  2.18017   7.06758   3.06758
   4        4  0.14280  0.25000  0.83618   0.00899  4.14443  0.79972   4.95641   0.95641
   5        5  0.14280  0.25000  0.88773   0.01043  3.78339  0.23247   4.26187   0.26187
   6        6  0.14280  0.25000  0.90407   0.01081  3.66899  0.05272   4.05831   0.05831

   7        7  0.14280  0.25000  0.90807   0.01089  3.64096  0.00867   4.00955   0.00955
   8        8  0.14280  0.25000  0.90878   0.01091  3.63602  0.00091   4.00100   0.00100
```

(a) From TORA's output

$L_s < 4 \Rightarrow R \geq 5$

(b) From TORA's output

$W_q < 1 \Rightarrow R \geq 4$

2

$C_1 = \$12$

c	L_s
2	7.467
3	2.217
4	1.842

$$2.217 - 1.842 \leq \frac{12}{C_2} \leq 7.467 - 2.217$$

$$.375 \leq \frac{12}{C_2} \leq 5.25$$

or

$$\$2.29 \leq C_2 \leq \$32$$

Chapter 16

Simulation Modeling

Set 16.1a

1

R1	R2	X	Y	(X-1)^2+(Y-2)^2	1=in, 0=out
0.0589	0.6733	-3.411	3.733	22.46021	1
0.4799	0.9486	0.799	6.486	20.164597	1
0.6139	0.5933	2.139	2.933	2.16781	1
0.9341	0.1782	5.341	-1.218	29.199805	0
0.3473	0.5644	-0.527	2.644	2.746465	1
0.3529	0.3646	-0.471	0.646	3.997157	1
0.7676	0.8931	3.676	5.931	22.613737	1
0.3919	0.7876	-0.081	4.876	9.439937	1
0.5199	0.6358	1.199	3.358	1.883765	1
0.7472	0.8954	3.472	5.954	21.7449	1
				Total=	9
				Area estimate=	90

Exact area = 78.54 cm2. Estimate from Figure 18-2 = 78.56 cm2 for a sample size of n=30,000. Current estimate = 90 cm2, which is unreliable because the sample size is too small.

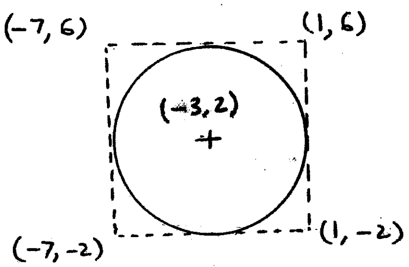
2

(a) $x = -7+8R_1$

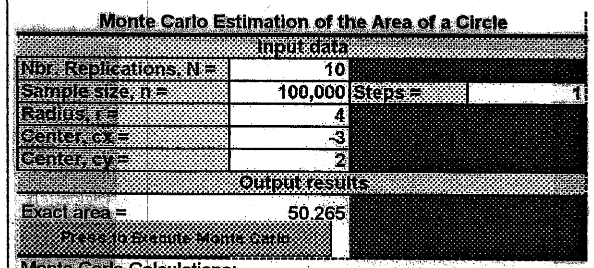
$y = -2+8R_2$

$f(x) = 1/8, \quad -7 \le x \le 1$

$f(y) = 1/8, \quad -2 \le x \le 6$



(b)



Monte Carlo Estimation of the Area of a Circle

Input data

Nbr. Replications, N =	10		
Sample size, n =	100,000	Steps =	1
Radius, r =	4		
Center, cx =	-3		
Center, cy =	2		

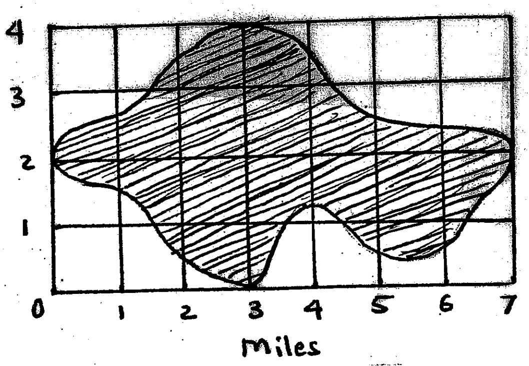
Output results

Exact area = 50.265

Monte Carlo Calculations:

	n=100000
Replication 1	50.223
Replication 2	50.378
Replication 3	50.113
Replication 4	50.260
Replication 5	50.244
Replication 6	50.330
Replication 7	50.327
Replication 8	50.252
Replication 9	50.236
Replication 10	50.467
Mean =	50.283
Std. Deviation =	0.099
95% lower conf. limit =	50.212
95% upper conf. limit =	50.354

3



R_1	R_2	X	Y	in?
.0589	.6733	.4123	2.6932	No
.4799	.9486	3.3593	3.7944	Yes
.6139	.5933	4.2973	2.3732	Yes
.9341	.1782	6.5387	.7128	NO
.3473	.5644	2.4311	2.2576	Yes
.3529	.3646	2.4703	1.4584	Yes
.7676	.8931	5.3732	3.5724	NO
.3919	.7876	2.7433	3.1504	Yes
.5199	.6358	3.6393	2.5432	NO
.7472	.8954	5.2304	3.5816	No

\# points in = 5

Area estimate $= \frac{5}{10} \times (4 \times 7) = 14 \text{ miles}^2$

4

(a) $P\{H\} = .5 \qquad P\{T\} = .5$

If $0 \le R \le .5$, Jim gets \$10

$.5 < R \le 1$, Jan gets \$10

(b)

R	Jan's pay	R	Jan's pay
.0589	-10	.3529	-10
.6733	10	.3646	-10
.4799	-10	.7676	10
.9486	10	.8931	10
.6139	10	.3919	-10
.5933	10	.7876	10
.9341	10	.5199	10
.1782	-10	.6358	10
.3473	-10	.7472	10
.5644	10	.8954	10
$\bar{X}_1 = \$2$		$\bar{X}_2 = \$4$	

continued...

R	Jan's pay	R	Jan's pay	R	Jan's pay
.5861	10	.3455	-10	.7900	10
.1281	-10	.4871	-10	.7698	10
.2867	-10	.8111	10	.2871	-10
.8216	10	.8912	10	.9534	10
.3866	-10	.4291	-10	.1394	-10
.7125	10	.2302	-10	.9025	10
.2108	-10	.5423	10	.1605	-10
.3575	-10	.4208	-10	.3567	-10
.2926	-10	.6975	10	.3070	-10
.8261	10	.5954	10	.5513	10
$\bar{X}_3 = -\$2$		$\bar{X}_4 = \$0$		$\bar{X}_5 = \$0$	

(b) Av. Jan's pay based on 5 repls.

$= 2+4-2+0+0$

$= \$.8$

$$S = \sqrt{\frac{(2-.8)^2+(4-.8)^2+(-2-.8)^2+2(0-.8)^2}{5-1}}$$

$$= \sqrt{\frac{80.8}{4}} = 2.28$$

Confidence interval:

$$.8 - \frac{2.28}{\sqrt{5}}\, t_{.025,4} \le \mu \le .8 + \frac{2.28}{\sqrt{5}}\, t_{.025,4}$$

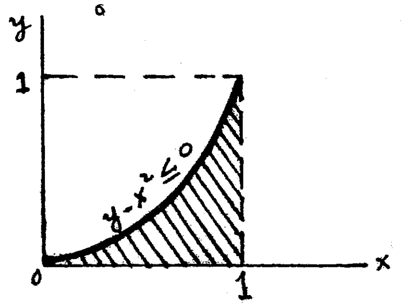
Given $t_{.025,4} = 2.776$, the 95% confidence interval is

$$-2.03 \le \mu \le 3.63$$

(c) Theoretical Jan's payoff = \$0.

5

Estimate $\int_0^1 x^2 dx$



Continued...

(a)

Let x=R1 and y=R2.
Experiment: If R2<R1^2, count point "in".
Estimate of integral = (1x1)(Points "in")/5

(b)

	R1	R2	1=in, 0=out
Rep 1	0.0589	0.6733	0
	0.4799	0.9486	0
	0.6139	0.5933	0
	0.9341	0.1782	1
	0.3473	0.5644	0
	Integral estimate =		0.2
Rep 2	0.3529	0.3646	0
	0.7676	0.8931	0
	0.3919	0.7876	0
	0.5199	0.6358	0
	0.7472	0.8954	0
	Integral estimate =		0
Rep 3	0.5869	0.1281	1
	0.2867	0.8216	0
	0.8261	0.3866	1
	0.7125	0.2108	1
	0.3575	0.2926	0
	Integral estimate =		0.6
Rep 4	0.3455	0.4871	0
	0.8111	0.8912	0
	0.4291	0.2302	0
	0.5954	0.5423	0
	0.4208	0.6975	0
	Integral estimate =		0
overall integral estimate =			0.2
Std. Deviation =			0.244949
95% lower confidence limit =			-0.189714
95% upper confidence limit =			0.5485706
Exact integral value =			0.3333

The given estimate is not "good" when compared with the exact value because sample size (n = 5) is too small.

6

$7 \equiv (6,1), (5,2), (4,3), (3,4), (2,5), (1,6)$

$11 \equiv (6,5), (5,6)$

Monte Carlo experiment:

R	outcome
$0 \le R \le 1/6$	1
$1/6 < R \le 1/3$	2
$1/3 < R \le 1/2$	3
$1/2 < R \le 2/3$	4
$2/3 < R \le 5/6$	5
$5/6 < R \le 1$	6
$0 \le R \le .167$	1
$.167 < R \le .333$	2
$.333 < R \le .5$	3
$.5 < R \le .667$	4
$.667 < R \le .833$	5
$.833 < R \le 1$	6

Continued...

Set 16.1a

R_1	R_2	Sum	Payoff
.0589	.6733	$1+5=\boxed{6}$ point	
.4799	.9486	$3+6=9$	
.6139	.5933	$4+4=8$	
.9341	.1782	$6+2=8$	
.3473	.5644	$3+4=7\rightarrow$	−$10
.3529	.3646	$3+3=\boxed{6}$ point	
.7676	.8931	$5+6=11$	
.3919	.7876	$3+5=8$	
.5199	.6358	$4+4=8$	
.7472	.8954	$5+6=11$	
.5869	.1281	$4+1=5$	
.2867	.8216	$2+5=7\rightarrow$	−$10
.8261	.3866	$5+3=\boxed{8}$ point	
.7125	.2108	$5+2=7\rightarrow$	−$10
.3575	.2926	$3+2=\boxed{5}$ point	
.3455	.4871	$3+3=6$	
.8111	.8912	$5+6=11$	
.4291	.2302	$3+2=5\rightarrow$	$10
.5954	.5423	$4+4=\boxed{8}$ point	
.4208	.6975	$3+5=8\rightarrow$	$10

7

Lead time:

$0 \le R \le .5$, $L = 1$ day

$.5 < R \le 1$, $L = 2$ days

Demand/day:

$0 \le R \le .2$, $d = 0$ unit

$.2 < R \le .9$, $d = 1$ unit

$.9 < R \le 1.$ $d = 2$ units

Let $p(d,L)$ be the joint pdf of demand and lead time. The procedure calls for constructing a frequency table of demand and lead time. The maximum demand during lead time is $2 \times 2 = 4$ units, so that the demand $d = 0, 1, 2, 3, 4$. We will use the random numbers in Table 16-1 in the following manner: First use a random number to generate a lead time. If $L=1$ day, use one

continued...

7 continued

random number to generate the demand in that day. If $L=2$ days, use two random numbers to generate the demands for the two days. For example, $R = .058962$ yields $L=1$. Next, $R = .6733$ gives $d=1$. Thus, we update the frequency table by increasing the frequency of the entry $(d=1, L=1)$ by one. The frequency table using the first two columns of R in Table 16-1 is

d

		0	1	2	3	4
L	1	|	𝍸 ||	||	0	0
	2	||	0	𝍸 ||	||||	0

d

		0	1	2	3	4
L	1	1	7	2	0	0
	2	2	0	7	4	0

Total $n = 23$

Relative frequency table:

		0	1	2	3	4	P(L)
L	1	1/23	7/23	2/23	0	0	10/23
	2	2/23	0	7/23	4/23	0	13/23
p(d)		3/23	7/23	9/23	4/23	0	

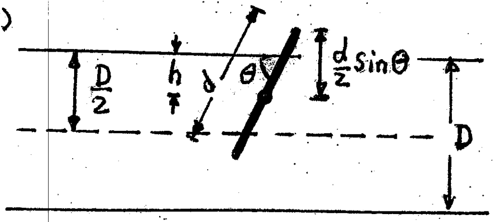
Notice that

$$p(d) = \sum_L p(d,L)$$

$$p(L) = \sum_d p(d,L)$$

8

(a)



From graph, needle will touch line or cross it is

$$h \le \frac{d}{2}\sin\theta$$

(b) Generate $h = R_1 \times D/2$

$\theta = \pi \times R_2$

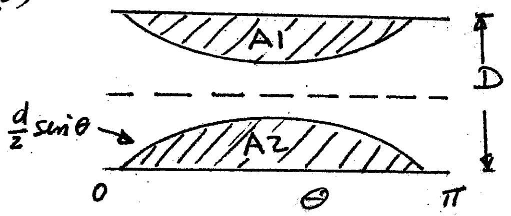
If $h \le \frac{d}{2}\sin\theta$, needle touches. Else it doesn't.

Probability estimate $= \frac{\text{\# touches}}{\text{sample size}}$

(c)

A	B	C	D	E
	D=	20	d=	10
	(RAND()*$C$1)*0.5	RAND()*PI()	$E$1*0.5*SIN(C4)	IF(B4<=D4,1,0)
	h	theta	d*sin(theta)/2	1=touch, 0=else
Rep 1	8.396953573	1.3165558	4.839272983	0
	7.107859045	2.9048959	1.172463622	0
	0.27542965	0.8440783	3.736795168	1
	1.267504547	2.8354706	1.506816139	1
	9.237262421	0.7436482	3.38488765	0
	2.495379696	2.9719552	0.844125326	0
	4.253169953	2.8396976	1.486650397	0
	8.516662244	1.4161445	4.940326141	0
	4.224254495	0.7887632	3.547410981	0
	3.690266876	3.0811599	0.301979787	0
		Estimate of probability=		0.2
Rep 2	0.712918949	1.5238102	4.994481772	1
	9.381794079	2.5979258	2.586388239	0
	1.360072144	2.0189288	4.506289193	1
	8.477675064	1.9724771	4.60202594	0
	0.99443686	1.300734	4.81877136	1
	5.170438974	1.4568612	4.967582038	0
	5.056822846	1.6844549	4.967739087	0
	5.864264693	0.0683356	0.341412027	0
	6.87137267	2.6283793	2.454895584	0
	1.092023022	2.6522347	2.350296303	1
		Estimate of probability=		0.4
Rep3	9.712756211	1.694489	4.961799031	0
	6.686447356	1.2243834	4.702983326	0
	6.436673778	2.4581589	3.157296664	0
	1.324134345	2.2441568	3.908652279	1
	1.775706228	2.255079	3.874363448	1
	0.090587765	2.7080167	2.100592855	1
	4.979938633	2.5138689	2.936520016	0
	8.678634219	2.7348178	1.978247037	0
	2.179672677	1.8339609	4.827857959	1
	9.640572895	1.2431615	4.734030551	0
		Estimate of probability=		0.4
Rep 4	8.227016322	2.6999829	2.136976805	0
	8.757368267	2.1537385	4.174233356	0
	4.203914479	0.1860064	0.92467824	0
	6.098369885	2.1672345	4.13670754	0
	4.960185836	0.7841548	3.531135292	0
	3.899078191	1.8047989	4.863730557	1
	5.840727605	0.727722	3.325852126	0
	6.645324046	0.498725	2.391531067	0
	5.361422671	0.89898	3.91346242	0
	3.223016816	1.6715052	4.974665749	1
		Estimate of probability=		0.2
			Mean value =	0.3
			Std. Deviation =	0.1155
			95% LCL =	0.1163
			95%UCL =	0.4837

(d)



$$\text{Exact probability} = \frac{A_1 + A_2}{\pi D}$$

$$= \frac{2\int_0^{\pi} \frac{d}{2}\sin\theta \, d\theta}{\pi}$$

$$= \frac{2d}{\pi D}$$

(e) From (c),

$$\hat{p} = .3$$

Thus,

$$\frac{2d}{\pi D} = .3$$

or

$$\pi \simeq \frac{2d}{.3D}$$

$$\simeq \frac{2 \times 10}{.3 \times 20}$$

$$\simeq 3.33$$

Set 16.2a

1

(a) Discrete

(b) Continuous

(c) Discrete

2

In discrete simulation, there are two main events: arrivals and departures. An arrival event may experience delay before starting service. When service has been completed, customer leaves the facility.

The description of the discrete simulation situation by arrival and departure events is the reason discrete simulation is associated with queues.

1

Events:

A_1 = rush job arrives
A_2 = regular job arrives
D_1 = rush job departs
D_2 = regular job departs

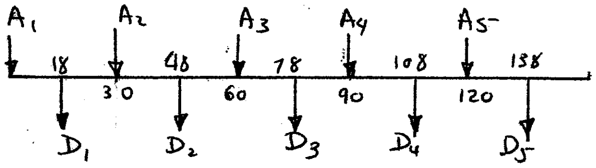
2

A_0 = job arrives at carousel
A_1 = job arrives at station 1
A_2 = job arrives at station 2
A_3 = job arrives at station 3
D_1 = job departs station 1
D_2 = job departs station 2
D_3 = job depart station 3

3

A_1 = car enters lane 1
A_2 = car enters lane 2
A_3 = car goes elsewhere
D_1 = car departs lane 1
D_2 = car departs lane 2.

4



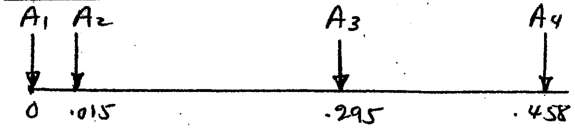
Set 16.3b

1

$t = -\frac{1}{\lambda}\ln(1-R)$

$\lambda =$ 4 customers/hr

Customer	R	t(hrs)	Arrival time
1	—	—	0
2	.0589	.015	0+.015 = .015
3	.6733	.280	.015+.28 = .295
4	.4799	.163	.295+.163 = .458



2

$f(t) = \frac{1}{b-a}, \quad a \le t \le b$

$F(t) = \int_0^t \frac{1}{b-a}dx = \frac{t-a}{b-a}, \ a \le t \le b$

$R = \frac{t-a}{b-a}$

$t = a + (b-a)R$

3

$f_1(t_1) = .5e^{-.5t}, \quad \lambda = 1/2$ arrival/hr

$f_2(t) = \frac{1}{.9}, \quad 1.1 < t < 2$

$R = .0589, \ a_1 = -.2\ln(1-.0589) = .12$hr

$R = .6733, \ d_1 = 1.1 + .9 \times .6733 = 1.71$ hrs

$R = .4799, \ a_2 = -2\ln(1-.4799) = 1.31$ hrs

$R = .9486, \ a_3 = -2\ln(1-.9486) = 5.94$ hrs

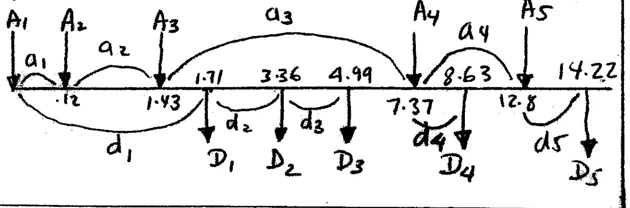
$R = .6139, \ d_2 = 1.1 + .9 \times .6139 = 1.65$ hrs

$R = .5933, \ d_3 = 1.1 + .9 \times .5933 = 1.63$ hrs

$R = .9341, \ a_4 = -2\ln(1-.9341) = 5.44$ hrs

$R = .1782, \ d_4 = 1.1 + .9 \times .1782 = 1.26$ hrs

$R = .3473, \ d_5 = 1.1 + .9 \times .3473 = 1.41$ hrs



4

(a) $0 \le R < .2, \quad d = 0$

$.2 \le R < .5, \quad d = 1$

$.5 \le R < .9, \quad d = 2$

$.9 \le R \le 1., \quad d = 3$

(b)

Day	R	Demand d	Stock level
0	—	—	5
1	.0589	0	5
2	.6733	2	3
3	.4799	1	2

Replenish stock on day 3

5

Repair/.2, Package/.8:

$0 \le R < .2$, go to Repair

$.2 \le R \le 1.$, go to Package

Package/.8, Repair/.2:

$0 \le R < .8$, go to Package

$.8 \le R \le 1$, go to Repair

Example: R = .1 leads to Repair in the first case and to Package in the second case

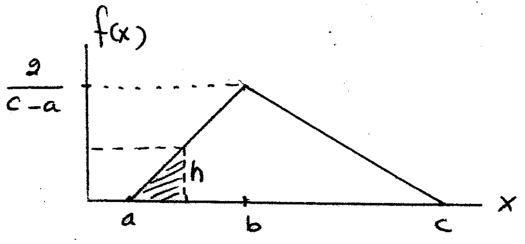
6

$0 \le R < .5$: H

$.5 \le R \le 1.$: T

n	R	outcome	Payoff
1	.0589	H	\$2
1	.6733	T	0
2	.4799	H	$2^2 = \$4$

7



continued...

7 continued

(a)
$$F(x)=\begin{cases}\dfrac{(x-a)^2}{(b-a)(c-a)}, & a\le x\le b\\[2ex] 1-\dfrac{(c-x)^2}{(c-b)(c-a)}, & b\le x\le c\end{cases}$$

For $R=\dfrac{(x-a)^2}{(b-a)(c-a)}$,

$x=a+\sqrt{R(b-a)(c-a)}$, $0\le R\le \dfrac{b-a}{c-a}$

For $R=1-\dfrac{(c-x)^2}{(c-b)(c-a)}$,

$x=c-\sqrt{(c-b)(c-a)(1-R)}$, $\dfrac{b-a}{c-a}\le R\le 1$

(b) $a=1,\ b=3,\ c=7$

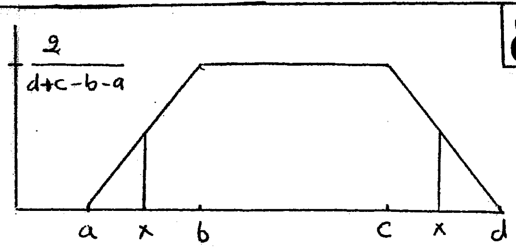
$$\frac{b-a}{c-a}=\frac{3-1}{7-1}=.333$$

Thus,

$$X=\begin{cases}1+\sqrt{(3-1)(7-1)R}\\ \quad =1+\sqrt{12R}, & 0\le R\le .333\\[2ex] 7-\sqrt{(7-3)(7-1)(1-R)}\\ \quad =7-\sqrt{24(1-R)}, & .333\le R\le 1\end{cases}$$

R	X
.0589	1.84
.6733	4.20
.4799	3.47
.9486	5.89
.6139	3.96

8



(a)
$$F(x)=\begin{cases}\dfrac{(x-a)^2}{(b-a)(d+c-b-a)} & a\le x\le b\\[2ex] \dfrac{1}{(b-a)(d+c-b-a)}+\dfrac{2(x-a)}{(d+c-b-a)} & b\le x\le c\\[2ex] 1-\dfrac{(d-x)^2}{(d-c)(d+c-b-a)}. & c\le x\le d\end{cases}$$

Continued...

$R=\dfrac{(x-a)^2}{(b-a)(d+c-b-a)}$ gives

$X=a+\sqrt{(b-a)(d+c-b-a)R}$, $0\le R\le\dfrac{b-a}{(d+c-b-a)}$

$R=\dfrac{1}{(b-a)(d+c-b-a)}+\dfrac{2(x-b)}{(d+c-b-a)}$ gives

$X=\frac{1}{2}\left(R-\dfrac{1}{(b-a)(d+c-b-a)}\right)(d+c-b-a)$,

$$\frac{b-a}{d+c-b-a}\le R\le 1-\frac{d-c}{(d+c-b-a)}$$

$R=1-\dfrac{(d-x)^2}{(d-c)(d+c-b-a)}$

$X=d-\sqrt{(d-c)(d+c-b-a)(1-R)}$,

$$1-\frac{d-c}{(d+c-b-a)}\le R\le 1$$

(b) $a=1,\ b=2,\ c=4,\ d=6$

$1+\sqrt{(2-1)(6+4-2-1)R}=1+\sqrt{7R}$, $0\le R\le .143$

$2+\frac{6+4-2-1}{2}\left(R-\frac{1}{(2-1)(6+4-2-1)}\right.$

$= 2+3.5(R-.143)$,

$.143\le R\le .714$

$6-\sqrt{(6-4)(6+4-2-1)(1-R)}$

$= 6-\sqrt{14(1-R)}$

$.714\le R\le 1$

R	X
.0589	1.64
.6733	3.86
.4799	3.18
.9486	5.15
.6139	3.65

9

$f(x)=pq^x$, $x=0,1,2,\ldots$

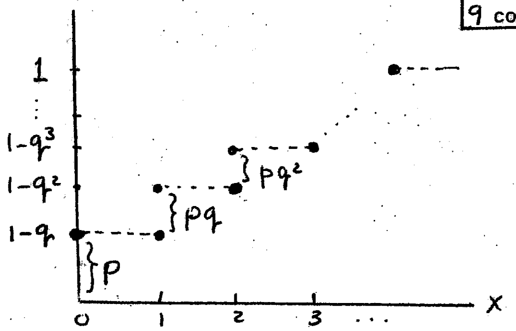
$(p+q)=1$

$$F(x)=p\sum_{t=0}^{x}q^t$$

$$=1-q^{x+1},\quad x=0,1,2,\ldots$$

Continued...

9 continued



Sampling procedure:

if $0 \le R \le p$, then $x = 0$.

For $p < R \le 1$, we have

$$1-q^n \le R \le 1-q^{n+1}$$

or

$$n \le \frac{\ln(1-R)}{\ln q} \le n+1$$

Thus, for $p \le R \le 1$, compute

$$x = \left[\frac{\ln(1-R)}{\ln q}\right]$$

where $[a]$ is the largest integer less than or equal to a.

For $p = .6$, $q = .4$, we have

R	$\frac{\ln(1-R)}{\ln q}$	x
.0589	—	0
.6733	1.22	1
.4799	—	0
.9486	3.24	3
.6139	1.03	1

10

$$f(x) = \alpha\beta^{-\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}, \quad x > 0$$

$$= \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x > 0$$

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x > 0$$

Thus,

$$R = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}$$

or

$$x = \beta\left[-\ln(1-R)\right]^{1/\alpha}$$

1

$$y = -\frac{1}{5}\ln\{(.0589 \times .6733 \times .4799 \times .9486\}$$
$$= .803 \text{ hour}$$

2

$\lambda = 5$ events/hr, $t = 1$

$$e^{-5\times 1} = e^{-5} = .00673$$

i	$R_1 R_2 \cdots R_i$	
1	.0589	
2	.0589 × .6733	= .0397
3	.0397 × .4799	= .0190
4	.0190 × .9486	= .0181
5	.0181 × .6139	= .0111
6	.0111 × .5933	= .00656
7	.00656 × .9341	= .00614

Hence $n = 6$

3

$\mu = 8$, $\sigma = 1$, $N(8,1)$

Convolution method:

$$x = R_1 + R_2 + \cdots + R_{12} = 6.1094$$
$$y = 8 + 1(6.1094 - 6) = 8.1094$$

Box-Miller method:

$$x = \sqrt{-2\ln R_1}\cos(2\pi R_2)$$
$$= \sqrt{-2\ln .0589}\cos(2\pi \times .6733)$$
$$\cong -1.103$$

$$y = 8 + 1(-1.103) = 6.897$$

4

$\lambda = 6$/day $\quad m = 5$

$$y = -\frac{1}{6}\ln(.0589 \times .6733 \times .4799 \times .9486 \times .6139) = .751 \text{ hour}$$

5

$N(27,3)$: $\mu = 27$, $\sigma = 3$

Given R_1 and R_2, we have

$$x_1 = \sqrt{-2\ln R_1}\cos(2\pi R_2)$$
$$x_2 = \sqrt{-2\ln R_1}\sin(2\pi R_2)$$
$$y_1 = \mu + \sigma x_1$$
$$y_2 = \mu + \sigma x_2$$

Continued...

	J	K	L	M	N	O
	Mean = 27		Std. Dev.= 3			
	R1	R2	x1	x2	y1	y2
5	0.0589	0.6733	-1.1030306	-2.108827	23.69091	20.67352
6	0.4799	0.9486	1.149111	-0.384576	30.44733	25.84627
7	0.6139	0.5933	-0.8229152	-0.546495	24.53125	25.36051
				mean y=	25.09163	
				Sy	3.197533	

Formulas:

```
L5=      SQRT(-2*LN(J5))*COS(2*PI()*K5)
M4=      SQRT(-2*LN(J5))*SIN(2*PI()*K5)
N4=      $K$1+L4*$M$1
O4=      $K$1+M4*$M$1
```

6

$$x_i = 10 + (20-10)R_i$$
$$= 10 + 10R_i, \quad i = 1,2,3,4$$
$$t = x_1 + x_2 + x_3 + x_4$$
$$= 40 + 10(R_1 + R_2 + R_3 + R_4)$$

7

	R_1	R_2	R_3	R_4	t (sec)	Σt
1	.0589	.6733	.4799	.9486	61.61	61.60
2	.6139	.5933	.9341	.1782	63.20	124.81
3	.3473	.7676	.8931	.3919	64.00	188.81
4	.7876	.5199	.6358	.7472	66.91	255.72
5	.8954	.5869	.1281	.2867	58.94	314.69

The number of mice that exit the maze in 300 seconds is 4

8

Let $x_1, x_2, \ldots, x_r$ be r successive random deviates obtained from the geometric distribution as given in Problem 9, Set 18.3b. Then

$$x_i = \left[\frac{\ln R_i}{\ln(1-p)}\right], \quad i = 1, 2, \ldots, r$$

Because the negative binomial is the convolution of r independent geometric random variables, it follows that a random negative binomial sample can be determined as

$$x = \sum_{i=1}^{r}\left[\frac{\ln R_i}{\ln(1-p)}\right]$$

Note that $[a]$ represents the largest integer $\leq a$

Set 16.3d

1

Step1: $R = .6139$

$x = .6139$

Step2: $R = .5933$

Step3: $\frac{f(.6139)}{g(.6139)} = .948 > .5933$ Reject x

Step1: $R = .9341$, $x = .9341$

Step2: $R = .1782$

Step3: $\frac{f(.9341)}{g(.9341)} = \frac{.3693}{1.5} = .246 > .1782$ Reject x

Step1: $R = .3473$, $x = .3473$

Step2: $R = .5644$

Step3: $\frac{f(.3473)}{g(.3473)} = .9067 > .5644$ Reject x

Step1: $R = .3529$, $x = .3529$

Step2: $R = .3646$

Step3: $\frac{f(.3529)}{g(.3529)} = .913 > .3646$ Reject x

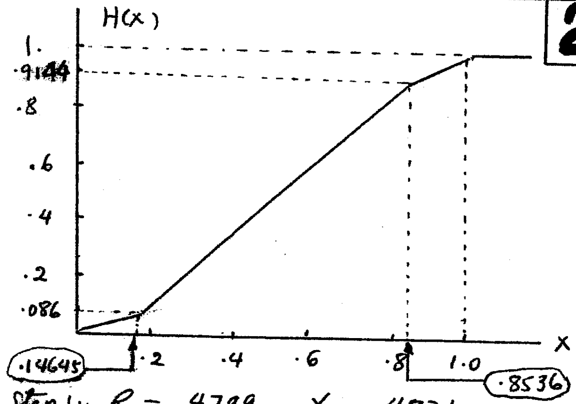
Step1: $R = .7676$, $x = .7676$

Step2: $R = .8931$

Step3: $\frac{f(.7676)}{g(.7676)} = .7135 < .8931$

accept $x = .7676$

2



Step1: $R = .4799$, $X = .4831$

Step2: $R = .9486$

Step3: $\frac{f(.4831)}{g(.4831)} = .9988 > .9486$ Reject X

Step1: $R = .6139$, $X = .5974$

Step2: $R = .5933$

continued...

2 continued

Step3: $\frac{f(.5974)}{g(.5974)} = .962 \geq .5933$ reject X

Step1: $R = .9341$ $X = .8804$

Step2: $R = .1782$

Step3: $\frac{f(.8804)}{g(.8804)} = .842 > .1782$ Reject X

Step1: $R = .3529$, $X = .375$

Step2: $R = .3646$

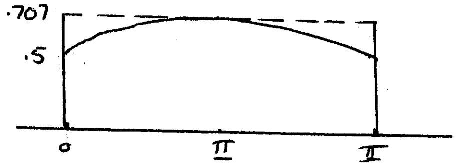
Step 3: $\frac{f(.375)}{g(.375)} = .937 > .3646$ Reject X

Step1: $R = .7676$, $X = .7286$

Step2: $R = .8931$

Step3: $\frac{f(.7286)}{g(.7286)} = \frac{1.186}{1.5} = .791 < .8931$ accept X

3



$f(x) = \frac{\sin(x) + \cos(x)}{2}$ $\quad 0 \le x \le \frac{\pi}{2}$

$\max f(x) = .707$ at $x = \frac{\pi}{4}$

$g(x) = .707$ $\quad 0 \le x \le \pi/2$

$h(x) = \frac{g(x)}{\text{area under } g(x)}$

$= \frac{.707}{.707 \times \frac{\pi}{2}} = .637$ $\quad 0 \le x \le \frac{\pi}{2}$

4

$\int_{12}^{20} \frac{K_1}{t} dt = K_1 \ln \frac{20}{12} = 1$

Thus, $K_1 = 1.96$

$\int_{18}^{22} \frac{K_2}{t^2} dt = K_2 \left(\frac{1}{18} - \frac{1}{22}\right) = 1$

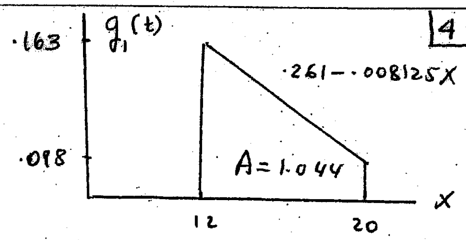
Thus, $K_2 = 99$

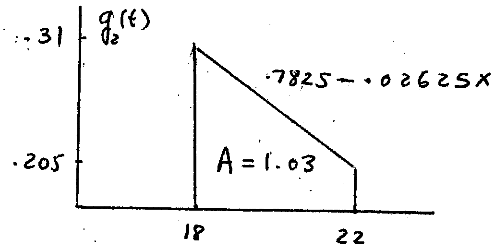
$f_1(t) = \frac{1.96}{t}$, $\quad 12 \le t \le 20$

$f_2(t) = \frac{99}{t^2}$, $\quad 18 \le t \le 22$

continued...

4 continued





$$h_1(t) = \frac{.261 - .008125t}{1.044}$$

$$= .25 - .007783\,t$$

$$H_1(t) = .025x - .00778\frac{x^2}{2}\Big|_{12}^{t}$$

$$= .25t - .003892t^2 - 2.44$$

$$h_2(t) = \frac{.7825 - .02625t}{1.03}$$

$$= .76 - .0255t$$

$$H_2(t) = .76t - .01275t^2 - 9.55$$

Sample computations from $H_2(t)$:

Step 1: $R_1 = .0589$

$$.76t - .01275t^2 - 9.55 = .0589$$

$$t^2 - 59.6t + 753.64 = 0$$

$$t = \frac{59.6 \pm \sqrt{(-59.6)^2 - 4 \times 753.64}}{2}$$

$$= 18.2$$

Step 2: $R = .6733$

Step 3: $$\frac{f_2(18.21)}{g_2(18.21)} = \frac{\left(\frac{99}{18.21^2}\right)}{.7825 - .02625 \times 18.21}$$

$$= .98 > .6733$$

Reject t.

continued...

Set 16.4a

1

Multiplicative Congruential Method

Input data	
b =	17
c =	111
u0 =	7
m =	103
How many numbers?	50

Output results

Press to Generate Sequence

Generated random numbers:

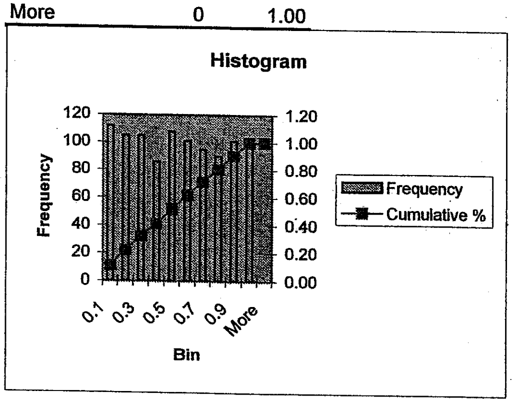
1	0.23301
2	0.03883
3	0.73786
4	0.62136
5	0.64078
6	0.97087
7	0.58252
8	0.98058
9	0.74757
10	0.78641
11	0.44660
12	0.66990
13	0.46602
14	0.00000
15	0.07767
16	0.39806
17	0.84466
18	0.43689
19	0.50485
20	0.66019
21	0.30097
22	0.19417
23	0.37864
24	0.51456
25	0.82524
26	0.10680
27	0.89320
28	0.26214
29	0.53398
30	0.15534
31	0.71845
32	0.29126
33	0.02913
34	0.57282
35	0.81553
36	0.94175
37	0.08738
38	0.56311
39	0.65049
40	0.13592
41	0.38835
42	0.67961
43	0.63107
44	0.80583
45	0.77670
46	0.28155
47	0.86408
48	0.76699
49	0.11650
50	0.05825

2

R=RAND()	Bin
0.813455	0.1
0.21757	0.2
0.937991	0.3
0.840823	0.4
0.19536	0.5
0.681599	0.6
0.829291	0.7
0.377723	0.8
0.149187	0.9
0.965781	1
0.808752	
0.957601	
0.502469	
0.620944	
0.992405	
0.97218	
0.051905	
0.144368	
0.129308	
0.676603	
0.140868	
0.486705	
0.12415	
0.821802	
0.954853	
0.301267	
0.827929	
0.917179	
0.07369	
0.462159	
0.333902	
0.390604	
0.723163	
0.041401	
0.805603	
0.556012	

Bin	*Frequency*	*umulative %*
0.1	112	0.11
0.2	105	0.22
0.3	105	0.32
0.4	86	0.41
0.5	108	0.52
0.6	101	0.62
0.7	95	0.71
0.8	90	0.80
0.9	101	0.90
1	97	1.00
More	0	1.00

Sample size=1000



$\boxed{1}$

$C = 2$ barbers

$f_1(t) = .1 e^{-.1t}, \quad t > 0$

$f_2(t) = \frac{1}{15}, \quad 15 \le t \le 30$

$t_1 = -12 \ln R$

$t_2 = 15 + 15R$

A_1 at $T = 0$:

$T(A_2) = 0 + (-10 \ln .0589) = 28.3$

$T(D_2) = 0 + (15 + 15 \times .6733) = 25.1$

Barber 1 busy

D_2 at $T = 25.1$:

Barber 1 idle

A_2 at $T = 28.3$:

$T(A_3) = 28.3 - 10 \ln .4799 = 35.6$

$T(D_2) = 28.3 + (15 + 15 \times .9486) = 57.5$

Barber 1 busy $\boxed{A_3 \quad D_2}$

A_3 at $T = 35.6$:

$T(A_4) = 35.6 - 10 \ln .6139 = 40.5$

$T(D_3) = 35.6 + (15 + 15 \times .5933) = 59.5$

Barber 2 busy $\boxed{A_4 \quad D_2 \quad D_3}$

A_4 at $T = 40.5$:

$T(A_5) = 40.5 - 10 \ln .9341 = 41.2$

A_4 waits in queue

$\boxed{A_5 \quad D_2 \quad D_3}$ $\boxed{A_4}$ ← queue

A_5 at $T = 41.2$:

$T(A_6) = 41.2 - 10 \ln .1782 = 58.4$

A_5 waits in queue

$\boxed{D_2 \quad A_6 \quad D_3}$ $\boxed{A_4 \quad A_5}$ ← queue

continued...

D_2 at $T = 57.5$:

Barber 1 idle

Take A_4 out of queue

$T(D_4) = 57.5 + 15 + 15 \times .3473 = 77.7$

Barber 1 busy

$\boxed{A_6 \quad D_3 \quad D_4}$ $\boxed{A_5}$ ← queue

A_6 at $T = 58.4$:

$T(A_7) = 58.4 - 10 \ln .5644 = 64.1$

Put A_6 in queue $\boxed{D_3 \quad A_7 \quad D_4}$

D_3 at $T = 59.5$: $\boxed{A_5 \quad A_6}$ ← queue

Barber 2 idle

Take A_5 out of queue

$T(D_5) = 59.5 + 15 + 15 \times .3529 = 79.8$

Barber 2 busy

$\boxed{A_7 \quad D_4 \quad D_5}$ $\boxed{A_6}$ ← queue

A_7 at $T = 64.1$:

$T(A_8) = 64.1 - 10 \ln .3646 = 74.2$

Put A_7 in queue

$\boxed{A_8 \quad D_4 \quad D_5}$ $\boxed{A_6 \quad A_7}$ ← queue

A_8 at $T = 74.2$:

$T(A_9) = 74.2 + (-10 \ln .7676)$

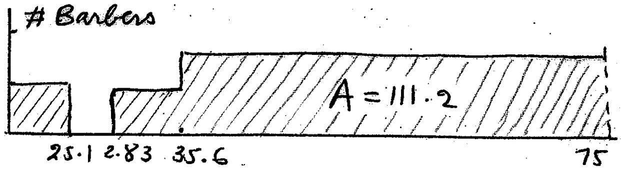
$= 76.8$

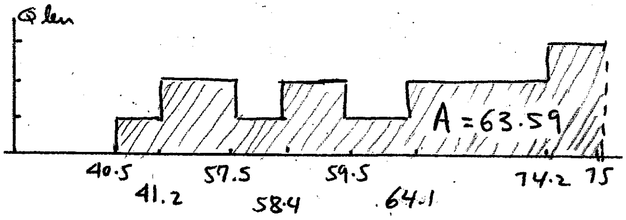
Place A_8 in queue.

$\boxed{A_9 \quad D_4 \quad D_5}$ $\boxed{A_6 \quad A_7 \quad A_8}$ ← queue

continued...

Set 16.5a





Av. facility utilization $= \frac{111.2}{75}$

$= 1.48$ barbers

Av. queue length $= \frac{63.59}{75} = .8$ customer

Av. waiting time in queue $= \frac{63.59}{8}$

$= 7.95$ min

Av. waiting time for those who must wait $= \frac{63.59}{5} = 12.72$ min

2

(a) Observation.

(b) Time.

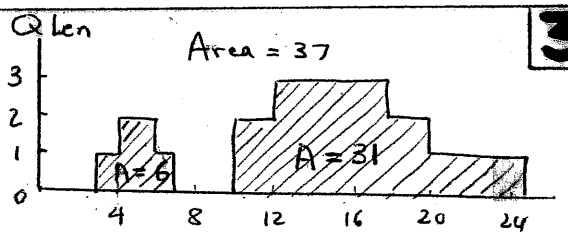
(c) Observation.

(d) Observation.

(e) Observation.

(f) Time.

3

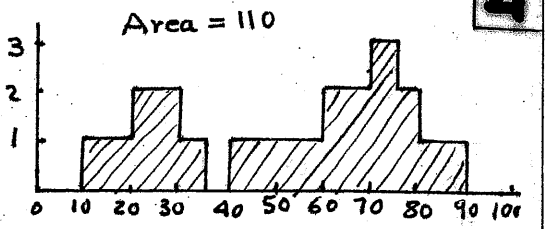


(a) $\bar{Q} = \frac{37}{25} = 1.48$ customers

(b) Number of waiting customers = 5

$\bar{W} = \frac{37}{5} = 7.4$ hours

4



(a) Average utilization

$= \frac{110}{100} = 1.1$ barber

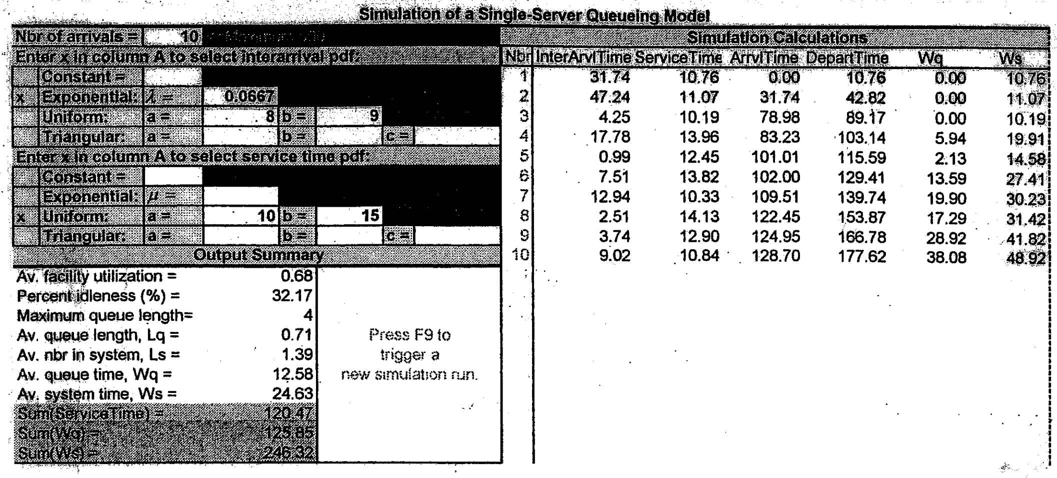
(b) Average idle time

$= \frac{10+(40-35)+(100-90)}{3}$

$= \frac{25}{3}$

$= 8.33$ minutes

1



Simulation of a Single-Server Queueing Model

Nbr of arrivals = 10

Enter x in column A to select interarrival pdf:

	Constant =						
x	Exponential:	λ =	0.0667				
	Uniform:	a =	8	b =	9		
	Triangular:	a =		b =		c =	

Enter x in column A to select service time pdf:

	Constant =						
	Exponential:	μ =					
x	Uniform:	a =	10	b =	15		
	Triangular:	a =		b =		c =	

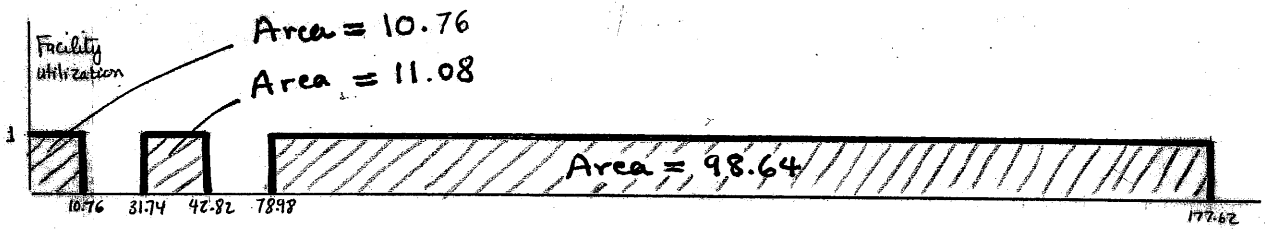
Output Summary

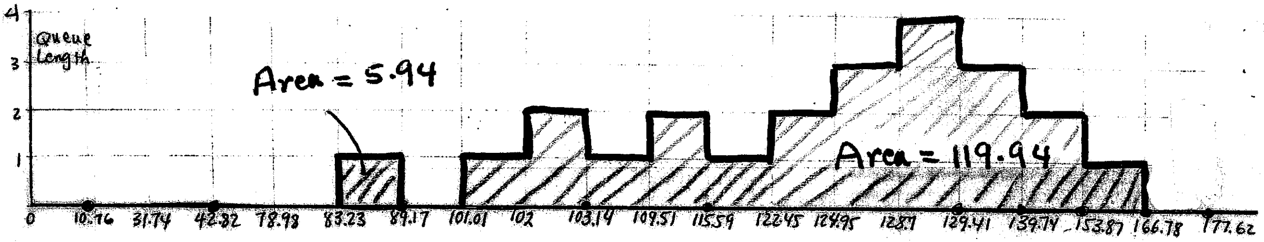
Av. facility utilization =	0.68
Percent idleness (%) =	32.17
Maximum queue length=	4
Av. queue length, Lq =	0.71
Av. nbr in system, Ls =	1.39
Av. queue time, Wq =	12.58
Av. system time, Ws =	24.63
Sum(ServiceTime) =	120.47
Sum(Wq) =	125.85
Sum(Ws) =	246.32

Press F9 to trigger a new simulation run.

Simulation Calculations

Nbr	InterArvlTime	ServiceTime	ArrvlTime	DepartTime	Wq	Ws
1	31.74	10.76	0.00	10.76	0.00	10.76
2	47.24	11.07	31.74	42.82	0.00	11.07
3	4.25	10.19	78.98	89.17	0.00	10.19
4	17.78	13.96	83.23	103.14	5.94	19.91
5	0.99	12.45	101.01	115.59	2.13	14.58
6	7.51	13.82	102.00	129.41	13.59	27.41
7	12.94	10.33	109.51	139.74	19.90	30.23
8	2.51	14.13	122.45	153.87	17.29	31.42
9	3.74	12.90	124.95	166.78	28.92	41.82
10	9.02	10.84	128.70	177.62	38.08	48.92





From the graph:

$$\sum \text{service times} = 10.76 + 11.08 + 98.64 = 120.48$$

$$\sum \text{queue waiting times} = 5.94 + 119.94 = 125.88$$

(The small difference between these answers and the simulation output is because of roundoff error.)

$$\text{Av. facility utilization} = \frac{120.48}{177.62} = .6783$$

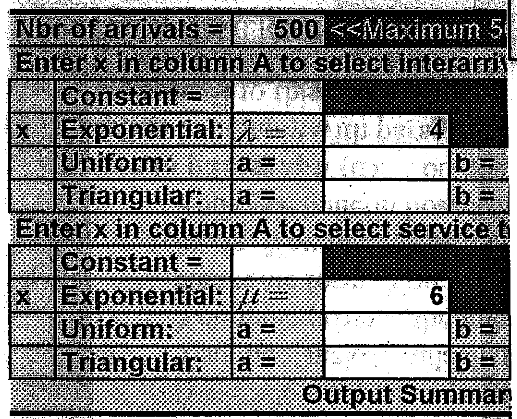
$$\text{Av. queue length} = \frac{125.88}{177.62} = .7087$$

$$\text{Av. waiting time in queue} = \frac{125.88}{10} = 12.588$$

$$\text{Av. waiting time in system} = \frac{120.48 + 125.88}{10} = 24.636$$

Set 16.5b

2



Nbr of arrivals = 500 <<Maximum 5

Enter x in column A to select interarri

Constant =

x Exponential: λ = 4

Uniform: a = b =

Triangular: a = b =

Enter x in column A to select service t

Constant =

x Exponential: μ = 6

Uniform: a = b =

Triangular: a = b =

Output Summar

①

Av. facility utilization =	0.66
Percent idleness (%) =	33.84
Maximum queue length=	0
Av. queue length, Lq =	1.42
Av. nbr in system, Ls =	2.08
Av. queue time, Wq =	0.37
Av. system time, Ws =	0.54

②

Av. facility utilization =	0.61
Percent idleness (%) =	38.65
Maximum queue length=	0
Av. queue length, Lq =	0.91
Av. nbr in system, Ls =	1.52
Av. queue time, Wq =	0.24
Av. system time, Ws =	0.40

③

Av. facility utilization =	0.65
Percent idleness (%) =	35.11
Maximum queue length=	0
Av. queue length, Lq =	0.91
Av. nbr in system, Ls =	1.56
Av. queue time, Wq =	0.22
Av. system time, Ws =	0.38

④

Av. facility utilization =	0.68
Percent idleness (%) =	31.70
Maximum queue length=	0
Av. queue length, Lq =	1.35
Av. nbr in system, Ls =	2.03
Av. queue time, Wq =	0.32
Av. system time, Ws =	0.48

⑤

Av. facility utilization =	0.60
Percent idleness (%) =	39.83
Maximum queue length=	0
Av. queue length, Lq =	1.14
Av. nbr in system, Ls =	1.74
Av. queue time, Wq =	0.30
Av. system time, Ws =	0.46

continued...

Summary:

	utiliz	Lq	Ls	Wq	Wa
mean	.64	1.146	1.786	.29	.452
Std. Dev.	.0339	.2388	.2598	.0608	.0642

95% confidence limits:

$$t_{4,.025} = 2.776$$

$$UCL = \bar{X} + \frac{2.776S}{\sqrt{n}} = \bar{X} + 1.245$$

$$LCL = \bar{X} - 1.245$$

	utiliz	Lq	Ls	Wq	Ws
LCL	.598	.850	1.464	.215	.372
UCL	.682	1.442	2.108	.365	.531

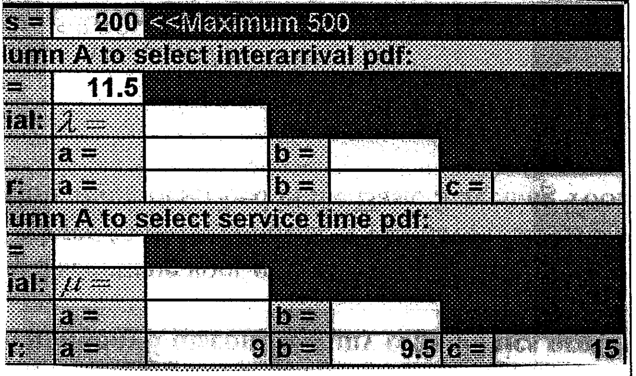
Poisson queue output:

```
Scenario  1-- (M/M/1):(GD/infinity/infinity)


   Lambda =      4.00000                        Mu =       6.00000
   Lambda eff = 4.00000                         Rho/c =     0.66667

   Ls =        2.00000                          Lq =        1.33333
   Ws =         0.50000                         Wq =        0.33333
```

3



s = 200 <<Maximum 500

umn A to select interarrival pdf:

= 11.5

ial: λ =

a = b =

r: a = b = c =

umn A to select service time pdf:

=

ial: μ =

a = b =

r: a = 9 b = 9.5 c = 15

①

Av. facility utilization =	0.96
Percent idleness (%) =	4.20
Maximum queue length=	2
Av. queue length, Lq =	0.12
Av. nbr in system, Ls =	1.08
Av. queue time, Wq =	1.36
Av. system time, Ws =	12.38

continued...

②
Av. facility utilization = 0.96
Percent idleness (%) = 3.85
Maximum queue length= 2
Av. queue length, Lq = 0.12
Av. nbr in system, Ls = 1.08
Av. queue time, Wq = 1.33
Av. system time, Ws = 12.39

③
Av. facility utilization = 0.97
Percent idleness (%) = 2.98
Maximum queue length= 2
Av. queue length, Lq = 0.19
Av. nbr in system, Ls = 1.16
Av. queue time, Wq = 2.14
Av. system time, Ws = 13.33

④
Av. facility utilization = 0.96
Percent idleness (%) = 3.58
Maximum queue length= 2
Av. queue length, Lq = 0.16
Av. nbr in system, Ls = 1.13
Av. queue time, Wq = 1.88
Av. system time, Ws = 12.97

⑤
Av. facility utilization = 0.97
Percent idleness (%) = 3.39
Maximum queue length= 2
Av. queue length, Lq = 0.17
Av. nbr in system, Ls = 1.14
Av. queue time, Wq = 2.00
Av. system time, Ws = 13.12

utilization:

$$\text{mean} = \frac{.96+.96+.97+.96+.97}{5}$$

$$= .964$$

$$\text{St. dev.} = .0311$$

Set 16.6a

1

$W_1 = \frac{14}{3} = 4.67$ (time units)

$W_2 = \frac{10}{4} = 2.5$

$W_3 = \frac{11}{3} = 3.67$

$W_4 = \frac{6}{3} = 2$

$W_5 = \frac{15}{4} = 3.75$

$$\overline{W} = \frac{4.67+2.5+3.67+2+3.75}{5}$$

$= 3.32$ time units

2

Dis-card observations during the transient period (0, 100)

$W_1 = \frac{12+30+10+14+16}{5} = 16.4$ time units

$W_2 = \frac{15+17+20+22}{4} = 18.5$

$W_3 = \frac{10+20+30+15+25+31}{6} = 21.83$

$W_4 = \frac{15+17+20+14+13}{5} = 15.8$

$W_5 = \frac{25+30+15}{3} = 23.33$

$\overline{W} = 19.17 \quad S = 3.3$

Confidence interval

$$\overline{W} \pm t_{.025,4} \frac{S}{\sqrt{n}}$$

$$= 19.17 \pm 2.776 \frac{3.3}{\sqrt{5}}$$

or

$$15.07 \le \mu \le 23.27$$

3

Batch	a_i	b_i	y_i
1	6	7	.869
2	10	7	1.369
3	6	9	.584
	$\bar{a}=7.33$	$\bar{b}=7.67$	$\bar{y}=.941$
			$S_y=.397$

continued...

3 continued

$$y_i = \frac{3\times7.33}{7.67} - \frac{(3-1)(3\times7.33-a_i)}{3\times7.67-b_i}$$

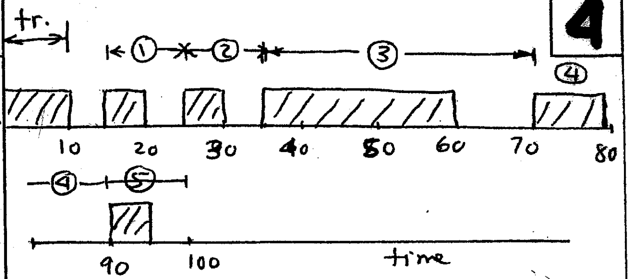
$$= 2.867 - \frac{43.98-2a_i}{23.01-b_i}$$

95% confidence interval:

$$.941-2.776\frac{.397}{\sqrt{3}} \le \mu \le .941+2.776\frac{.397}{\sqrt{3}}$$

$$.305 \le \mu \le 1.577$$

4



(a) Start points are 15, 25, 35, 70, 90

(b)

Batch	a_i	b_i	y_i
1	5	10	.54
2	5	10	.54
3	25	35	.94
4	10	20	.45
5	5	10	.54
	$\bar{a}=10$	17	$\bar{y}=.602$
			$S_y=.193$

$$y_i = \frac{5\times10}{17} - \frac{4(5\times10-a_i)}{5\times17-b_i}$$

$$= 2.94 - \frac{200-4a_i}{85-b_i}$$

$$.602-2.776\frac{.193}{\sqrt{5}} \le \mu \le .602+2.776\frac{.193}{\sqrt{5}}$$

or $.36 \le \mu \le .84$

(c) $t = \frac{90}{5} = 18$

i	1	2	3	4	5
A	8	13	14	10	5
u_i	.44	.72	.78	.56	.28

Mean = .556, Std. Dev. = .2042

Chapter 17

Markov Chains

Set 17.1a

1

States: Models M1, M2, and M3

	M1	M2	M3
M1	0.65	0.2	0.15
M2	0.6	0.15	0.25
M3	0.5	0.1	0.4

2

s1: car on patrol
s2: car responding to a call
s3: car at call scene
s4: apprehension made.
s5 transport to police station

	S1	S2	S3	S4	S5
S1	0.4	0.6	0	0	0
S2	0.1	0.3	0.6	0	0
S3	0.1	0	0.5	0.4	0
S4	0.4	0	0	0	0.6
S5	1	0	0	0	0

3

States: Q0, Q1, Q2, Q3, Q4, paid, bad debt
p{bad,bad}=P{paid,paid}=1
P{Q0,paid}=2000/10000, P{Q0,Q1}=3000/10000,
P{Q0,Q2}=3000/10000, P{Q0,Q3}=2000/10000,
P{Q1,paid}=4000/25000, P{Q1,Q2}=12000/25000,
P{Q1,Q3}=6000/25000, P{Q1,Q4}=3000/25000,

Input Markov chain:

	Q0	Q1	Q2	Q3	Q4	PAID	BAD
Q0	.00	.30	.30	.20	.00	.20	.00
Q1	.00	.00	.48	.24	.12	.16	.00
Q2	.00	.00	.00	.30	.55	.15	.00
Q3	.00	.00	.00	.00	.16	.84	.00
Q4	.00	.00	.00	.00	.00	.50	.50
PAID	.00	.00	.00	.00	.00	1.00	.00
BAD	.00	.00	.00	.00	.00	.00	1.00

4

States: dialysis, cadaver transplant,
living donor transplant, >1year survivors, death

	Dialysis	CTransp	LTransp	>1yrS	Death
Dialysis	0.5	0.3	0.1	0	0.1
CTransp	0.3	0	0	0.5	0.2
LTransp	0.15	0	0	0.75	0.1
>1yrS	0.05	0	0	0.9	0.05
Death	0	0	0	0	1

1

Input Markov chain:

	M1	M2	M3
M1	0.65	0.2	0.15
M2	0.6	0.15	0.25
M3	0.5	0.1	0.4

Output (2-step or 4 yrs.) transition matrix P^2

	M1	M2	M3
M1	0.6175	0.175	0.2075
M2	0.605	0.1675	0.2275
M3	0.585	0.155	0.26

P{M1|M1}=.6175

P{M2|M2}=.1675

P{M3|M3}=.26

2

Initial probabilities:

S1	S2	S3	S4	S5
0	0	1	0	0

Input Markov chain:

	S1	S2	S3	S4	S5
S1	0.4	0.6	0	0	0
S2	0.1	0.3	0.6	0	0
S3	0.1	0	0.5	0.4	0
S4	0.4	0	0	0	0.6
S5	1	0	0	0	0

Output (2-step or 2 patrols) transition matrix P^2

	S1	S2	S3	S4	S5
S1	0.22	0.42	0.36	0	0
S2	0.13	0.15	0.48	0.24	0
S3	0.25	0.06	0.25	0.2	0.24
S4	0.76	0.24	0	0	0
S5	0.4	0.6	0	0	0

Absolute 2-step probabilities =(0 0 1 0 0)P^2

State	Absolute (2-step)
S1	0.25
S2	0.06
S3	0.25
S4	**0.2**
S5	0.24

P{apprehension, S4, in 2 patrols}=.2

3

Initial probabilities:

Q0	Q1	Q2	Q3	Q4	PAID	BAD
0.2	0.1	0.3	0.2	0	0	0.2

Input Markov chain:

	Q0	Q1	Q2	Q3	Q4	PAID	BAD
Q0	.00	.30	.30	.20	.00	0.20	.00
Q1	.00	.00	.48	.24	.12	0.16	.00
Q2	.00	.00	.00	.30	.55	0.15	.00
Q3	.00	.00	.00	.00	.16	0.84	.00
Q4	.00	.00	.00	.00	.00	0.50	.50
PAID	.00	.00	.00	.00	.00	1.00	.00
BAD	.00	.00	.00	.00	.00	0.00	1.

Output (2-step) transition matrix

	Q0	Q1	Q2	Q3	Q4	PAID	BAD
Q0	.00	.00	.14	.16	.23	0.46	0.00
Q1	.00	.00	.00	.14	.30	0.49	0.06
Q2	.00	.00	.00	.00	.05	0.68	0.28
Q3	.00	.00	.00	.00	.00	0.92	0.08
Q4	.00	.00	.00	.00	.00	0.50	0.50
PAID	.00	.00	.00	.00	.00	1.00	0.00
BAD	.00	.00	.00	.00	.00	0.00	1.00

State	Absolute (2-step)	$500,000p
Q0	0	0
Q1	0	0
Q2	0.0288	14400
Q3	0.0468	23400
Q4	0.09124	45620
PAID	0.52866	264330
BAD	0.3045	152250
		$500,000

Set 17.2a

4

(a)

Initial probabilities:

Dialy	CTrans	LTrans	>1yrS	Death
1	0	0	0	0

Input Markov chain:

	Dialy	CTrans	LTrans	>1yrS	Death
Dialy	0.5	0.3	0.1	0	0.1
CTrans	0.3	0	0	0.5	0.2
LTrans	0.15	0	0	0.75	0.1
>1yrS	0.05	0	0	0.9	0.05
Death	0	0	0	0	1

Output (2-step) transition matrix

	Dialy	1stYrC	1stYrL	>1yrS	Death
Dialy	0.355	0.15	0.05	0.225	0.22
CTrans	0.175	0.09	0.03	0.45	0.25
LTrans	0.1125	0.045	0.015	0.675	0.15
>1yrS	0.07	0.015	0.005	0.81	0.1
Death	0	0	0	0	1

State	**Absolute (2-step)**
Dialy	0.355
CTrans	0.15
LTrans	0.05
>1yrS	0.225
Death	0.22

P{transplant}=.15+.05=.2

continued...

(b)

Initial probabilities:

Dialy	CTrans	LTrans	>1yrS	Death
0	0	0	1	0

Input Markov chain:

	Dialy	CTrans	LTrans	>1yrS	Death
Dialy	0.5	0.3	0.1	0	0.1
CTrans	0.3	0	0	0.5	0.2
LTrans	0.15	0	0	0.75	0.1
>1yrS	0.05	0	0	0.9	0.05
Death	0	0	0	0	1

Output (4-step) transition matrix

	Dialy	CTrans	LTrans	>1yrS	Death
Dialy	0.1737	0.0724	0.024	0.363	0.37
CTrans	0.1128	0.0425	0.014	0.465	0.37
LTrans	0.0967	0.0317	0.011	0.602	0.26
>1yrS	0.0847	0.0242	0.008	0.682	0.2
Death	0	0	0	0	1

State	**Absolute (4-step)**
Dialy	0.08474
CTrans	0.02423
LTrans	0.00807
>1yrS	0.68197
Death	0.20099

P{surviving 4 more years} = .68197

1

(a) Using excelMarkovChains.xls, all the states of the chain are periodic with period 3.

$$\mathbf{P}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\mathbf{P}^2=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},$$

$$\mathbf{P}^3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\mathbf{P}^4=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

(b) States 1, 2, and 3 are transient, State 4 is absorbing.

(c) State 1 is transient. States 2 and 3 form a closed set. State 4 is absorbing. States 5 and 6 form a closed set.

(d) All the states communicate and the chain is ergodic.

Set 17.4a

(b)

Population=150,000xP{1-step}

State	Absolute (1-step)	Population
inner	0.106667	*16000*
sub	0.54	*81000*
rural	0.353333	*53000*

Population=150,000xP{2-step}

State	Absolute (2-step)	Population
inner	0.098667	*14800*
sub	0.417667	*62650*
rural	0.483667	*72550*

(c)

Long-run population=150,000xπi

State	Steady state	Population
inner	0.073892	*11084*
sub	0.275862	*41379*
rural	0.650247	*97537*

8

(a)

Initial probabilities:

Equal initial probabilities

Phx	Den	Chi	Atl
0.25	0.25	0.25	0.25

Input Markov chain:

	Phx	Den	Chi	Atl
Phx	0.7	0.06	0.18	0.06
Den	0	0.7	0.18	0.12
Chi	0	0.15	0.7	0.15
Atl	0.03	0.03	0.24	0.7

(b)

State	Absolute (2-step)	No. of cars
Phx	0.1355	*54*
Den	0.2319	*93*
Chi	0.3645	*146*
Atl	0.2681	*107*
	total=	400

continued...

(c)

State	Steady state	No. of cars	
Phx	0.0311	*12*	
Den	0.2442	*98*	
Chi	0.4139	*166*	*>110*
Atl	0.3108	*124*	*>110*
	total=	400	

Chicago and Atlanta will have space availability problem

(d)

State	Mean return time (wks)
Phx	32.17
Den	4.09
Chi	2.42
Atl	3.22

9

(a)

Tally of i followed by j

	0	1	2	3	sum
0	2	2	1	3	8
1	2	1	2	2	7
2	2	3	1	1	7
3	2	0	4	1	7

Input Markov chain:

	0	1	2	3
0	0.25	0.25	0.125	0.375
1	0.28571	0.142857	0.285714	0.28571
2	0.28571	0.428571	0.142857	0.14286
3	0.28571	0	0.571429	0.14286

(b)

Output Results

State	Steady state	Mean return time
0	***0.275862***	***3.6249995***
1	0.215779	4.6343799
2	0.270638	3.6949792
3	0.237722	4.2065916

$\pi_0 =$ ***0.275862***

continued...

Set 17.4a

14

(a) State=(i,j,k)=(# in yr -2,# in yr-1,# in cur yr)
i, j, k = (0 or 1)

Example: (1-0-0) this yr links to (0-0-1) if a contract is secured next yr.

	0-0-0	1-0-0	0-1-0	0-0-1	1-1-0	1-0-1	0-1-1	1-1-1
0-0-0	.1	0	0	.9	0	0	0	0
1-0-0	.2	0	0	.8	0	0	0	0
0-1-0	0	.2	0	0	0	.8	0	0
0-0-1	0	0	.2	0	0	0	.8	0
1-1-0	0	.3	0	0	0	.7	0	0
1-0-1	0	0	.3	0	0	0	.7	0
0-1-1	0	0	0	0	.3	0	0	.7
1-1-1	0	0	0	0	.5	0	0	.5

(b)

State	Steady state
0-0-0	.014859
1-0-0	.066865
0-1-0	.066865
0-0-1	.066865
1-1-0	.178306
1-0-1	.178306
0-1-1	.178306
1-1-1	.249629

Expected # contracts in 3 yrs =
1(.066865+.066865+.066865)+
2(.178306+.178306+.178306)+
3(.249629)= 2.01932

Expected # contracts/yr=2.01932/3=.67311

15

(a) States:0, 1, 2, 3, 4

Input Markov chain

	0	1	2	3	4
0	.5	.5	0	0	0
1	0	.6	.4	0	0
2	0	0	.7	.3	0
3	0	0	0	.8	.2
4	1	0	0	0	0

continued...

(b)

Output Results

State	Steady state	Mean return time
0	.144578	6.9166613
1	.180723	5.5333285
2	.240964	4.1499977
3	.361446	2.7666647
4	.072289	13.833323

Av. # stops bet. suspensions=13.83

(c) P{losing license}=.072289

(d) Fines paid=$400

1

(a) Initial probabilities:

1	2	3	4	5
1	0	0	0	0

Input Markov chain:

1	2	3	4	5
0	.3333	.3333	.3333	0
.3333	0	.3333	0	.3333
.3333	.3333	0	0	.3333
.5	0	0	0	.5
0	.3333	.3333	.3333	0

State	Absolute (3-step)	Steady state
1	.07407	.214286
2	.2963	.214286
3	.2963	.214286
4	.25926	.142857
5	**.07407**	**.214286**

(b) $a_5 = .07407$

(c) $\pi_5 = .214286$

(d)

Matrix I:

	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1

Matrix P:

	1	2	3	4	5
1	0	.3333	.3333	.3333	0
2	.3333	0	.3333	0	.333
3	.3333	.3333	0	0	.333
4	.5	0	0	0	.5
5	0	.3333	.3333	.3333	0

Perform first passage time calculations below:

I-N

i=5	1	2	3	4
1	1	-.333	-.333	-.3333
2	-.333	1	-.333	0
3	-.333	-.333	1	0
4	-.5	0	0	1

continued...

inv(I-N)

	1	2	3	4
1	2	1	1	.6667
2	1	1.625	.875	.3333
3	1	.875	1.625	.3333
4	1	.5	.5	1.3333

Mu

	5
1	***4.6666***
2	3.8333
3	3.8333
4	3.3333

$\mu_{15} = 4.6666$

2

Matrix I:

	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1

Matrix P:

	1	2	3	4	5
1	0	.3333	.3333	.3333	0
2	.3333	0	.3333	0	.333
3	.25	.25	0	.25	.25
4	.3333	0	.3333	0	.333
5	0	.3333	.3333	.3333	0

Perform first passage time calculations below:

I-N

i=5	1	2	3	4
1	1	-.333	-.333	-.3333
2	-.333	1	-.333	0
3	-.25	-.25	1	-.25
4	-.333	0	-.333	1

inv(I-N)

	1	2	3	5
1	2	1	1.3333	***5.3333***
2	1	1.6	1.0667	4.2666
3	1	.8	1.8667	4.4666
4	1	.6	1.0667	4.2666

Mu

5
5.3333
4.2666
4.4666
4.2666

$\mu_{15} = 5.3333$

(as opposed to 4.6666 in Part (d) of Problem 1)

Set 17.5a

3

(a)

Initial probabilities: (Jim-Joe)=(i-j)

3-2	2-3	1-4	4-1	0-5	5-0
1	0	0	0	0	0

Input Markov chain:

	3-2	2-3	1-4	4-1	0-5	5-0
3-2	0	.5	0	.5	0	0
2-3	.5	0	.5	0	0	0
1-4	0	.5	0	0	.5	0
4-1	.5	0	0	0	0	.5
0-5	.3	0	0	0	.7	0
5-0	.3	0	0	0	0	.7

(b)

Output (3-step) transition matrix

	3-2	2-3	1-4	4-1	0-5	5-0
3-2	.075	.375	0	.25	**.125**	**.175**
2-3	.45	0	.25	0	.175	.125
1-4	.105	.325	0	.2	.37	0
4-1	.355	.075	.125	.075	0	.37
0-5	.297	.105	.075	.105	.343	.075
5-0	.297	.105	.075	.105	0	.418

P{Joe wins in 3 tosses}=P{3-2→0-5}=.125

P{Jim wins in 3 tosses}=P{3-2→5-0}=.175

(c)

Output Results

State	Absolute (3-step)	Steady state	Mean return time
3-2	.075	.257143	3.8888891
2-3	.375	.171429	5.8333335
1-4	0	.085714	11.666665
4-1	.25	.128571	7.7777801
0-5	.125	***.142857***	7.0000019
5-0	.175	***.214286***	4.6666665

P\{game ends in Jim's favor\}= $\pi_{5-0} = .214$

P\{game ends in Joe's favor\}= $\pi_{5-0} = .143$

continued...

(d)

Matrix I:

	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Matrix P:

	3-2	2-3	1-4	4-1	0-5	5-0
3-2	0	.5	0	.5	0	0
2-3	.5	0	.5	0	0	0
1-4	0	.5	0	0	.5	0
4-1	.5	0	0	0	0	.5
0-5	.3	0	0	0	.7	0
5-0	.3	0	0	0	0	.7

i=0-5 **I-N**

3-2	1	-1	0	-.5	0
2-3	-.5	1	-.5	0	0
1-4	0	-1	1	0	0
4-1	-.5	0	0	1	-1
5-0	-.3	0	0	0	.3

inv(I-N)

3-2	6	4	2	3	5
2-3	4	4	2	2	3.3
1-4	2	2	2	1	1.7
4-1	6	4	2	4	6.7
5-0	6	4	2	3	8.3

Mu

	0-5	
3-2	***20***	←expected number of tosses till Joe wins
2-3	15.3	
1-4	8.7	
4-1	22.7	
5-0	23.3	

i=5-0 **I-N**

3-2	1	-1	0	-.5	0
2-3	-.5	1	-.5	0	0
1-4	0	-1	1	0	-1
4-1	-.5	0	0	1	0
0-5	-.3	0	0	0	.3

continued...

inv(I-N)

3-2	4	2.7	1.33	2	2.2
2-3	4	4	2	2	3.3
1-4	4	3.3	2.67	2	4.4
4-1	2	1.3	.67	2	1.1
5-0	4	2.7	1.33	2	5.6

Mu

	0-5
3-2	*12.2*
2-3	15.3
1-4	16.4
4-1	7.1
5-0	15.6

←expected number of tosses till Jim wins

4

(a)

Input Markov chain:

	pink	red	orange	white
pink	.6	0	0	.4
red	.5	.4	.1	0
orange	.5	0	.25	.25
white	.5	0	0	.5

(b)

Initial probabilities:

pink	red	orange	white
.25	.25	.25	.25

State	Absolute (5-step)	Steady state
pink	0.55555	0.555556
red	0.00256	0
orange	0.00179	0
white	0.4401	0.444445

After 5 years, 56% pink, 44% white.
Red and orange will vanish. Approximately same result in the long run.

(c)

I-N

j=4(white)	pink	red	orange
pink	.4	0	0
red	-.5	.6	-.1
orange	-.5	0	.75

continued...

inv(I-N)

	pink	red	orange	**Mu** white
pink	2.5	0	0	2.5
red	2.36111	1.66667	.22222	*4.25*
orange	1.66667	0	1.33333	3

It takes 4.25 years from red to white

5

(a)

Input Markov chain:

	A	B	C
A	.75	.1	.15
B	.2	.75	.05
C	.125	.125	.75

(b)

State	Steady state
A	.394737
B	.307018
C	.298246

A: 39.5%, B: 30.7%, C: 29.8%

(c)

I-N

i = 2 (B)	A	C
A	.25	-.15
C	-.125	.25

inv(I-N)

	A	C
A	5.71429	3.42857
C	2.85714	5.71429

Mu

	B
A	*9.14286*
C	8.57143

i = 3 (C)	A	B
A	.25	-.1
B	-.2	.25

	1	2
A	5.88235	2.35294
B	4.70588	5.88235

	C
A	*8.23529*
B	1.5882

A→B: 9.14 years
A→C: 8.23 years

Set 17.6a

1

$$(\mathbf{I}-\mathbf{N})^{-1}=\begin{pmatrix}1.07 & 1.02 & .98 & 0.93\\ 0.07 & 1.07 & 1.03 & 0.98\\ 0 & 0 & 1.07 & 1.02\\ 0 & 0 & 0.07 & 1.07\end{pmatrix}$$

$$(\mathbf{I}-\mathbf{N})^{-1}\mathbf{A}=\begin{pmatrix}.16 & .84\\ .12 & .88\\ .08 & .92\\ .04 & .96\end{pmatrix}$$

Labor cost={\$20×[1.07(30/60)+.98(20/60)] +
\$18[1.02(10/60)+.93(10/600]}/(.84)
=\$27.48

2

(a)
States: 1wk, 2wk, 3wk, Library

Matrix P:

	1	2	3	lib
1	0	0.3	0	0.7
2	0	0	0.1	0.9
3	0	0	0	1
lib	0	0	0	1

(b)

inv(I-N)

	1	2	3
1	1	0.3	.03
2	0	1	.01
3	0	0	1

Mu

	lib
1	1.33
2	1.1
3	1

I keep the book 1.33 wks on the average.

3

(a) **Matrix P:**

	1	2	3	4	5	6	0
1	0	.4	0	0	0	0	.6
2	.6	0	.4	0	0	0	0
3	0	.6	0	.4	0	0	0
4	0	0	.6	0	.4	0	0
5	0	0	0	.6	0	.4	0
6	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

inv(I-N)

	1	2	3	4	5
1	1.5865	0.9774	0.5714	0.3008	0.1203
2	1.4662	2.4436	1.4286	0.7519	0.3008
3	1.2857	2.1429	2.7143	1.4286	0.5714
4	1.0150	1.6917	2.1429	2.4436	0.9774
5	0.6090	1.0150	1.2857	1.4662	1.5865

	MU Absorption
1	3.556391
2	6.390977
3	8.142857
4	8.270677
5	5.962406

	P{i to j} 6	0
1	0.048	0.952
2	0.12	0.88
3	0.229	0.771
4	0.391	0.609
5	0.635	0.365

(b)Average # of bets to termination =8.14286

(c) P{win double}=..229, P{lose all}=.771

4

(a) **Matrix P:**

	1	2	3	4	5(D)	M
1	0.5	0.5	0	0	0	0
2	0	0.5	0.5	0	0	0
3	0	0	0.2	0.5	0	0.3
4	0	0	0	0.5	0.5	0
5(D)	0	0	0	0	1	0
M	0	0	0	0	0	1

(b)

inv(I-N)

	1	2	3	4
1	2	2	1.25	1.25
2	0	2	1.25	1.25
3	0	0	1.25	1.25
4	0	0	0	2

	Mu absorption
1	6.5
2	4.5
3	2.5
4	2

Years as a student = ***6.5 years***

continued...

(c)

P{i to j}=inv(I-N)A

	D	M
1	0.625	0.375
2	0.625	0.375
3	0.625	0.375
4	1	0

P{Master}=***.375***

(d)

Expected pay=
$15,000(5×.625+3×.375)=$63,750

5

(a) States:55, 56, …, 62, quit

Matrix P

	55	56	57	58	59	60	61	62	Q
55	0	.9	0	0	0	0	0	0	.1
56	0	0	.89	0	0	0	0	0	.11
57	0	0	0	.88	0	0	0	0	.12
58	0	0	0	0	.87	0	0	0	.13
59	0	0	0	0	0	.86	0	0	.14
60	0	0	0	0	0	0	.85	0	.15
61	0	0	0	0	0	0	0	1	0
62	0	0	0	0	0	0	0	1	0
Q	0	0	0	0	0	0	0	0	1

(b)

inv(I-N)

	55	56	57	58	59	60	61
55	1	.9	.8	.7	.61	.53	.448
56	0	1	.89	.78	.68	.59	.498
57	0	0	1	.88	.77	.66	.56
58	0	0	0	1	.87	.75	.636
59	0	0	0	0	1	.86	.731
60	0	0	0	0	0	1	.85
61	0	0	0	0	0	0	1

Mu	P{i to j}	
62/Q	62	Q
4.99	***.448***	***.552***
4.44	.498	.502
3.86	.56	***.44***
3.25	.636	.364
2.59	.731	.269
1.85	.85	.15
1	1	0

P{retire at 62}=.448

continued…

(c) P{quit at 57}=.44

(d) P{off payroll}=3.25 years

6

(a)

Matrix P

	Q0	Q1	Q2	Q3	Q4	PAID	BAD
Q0	0	.3	.3	.2	0	.2	0
Q1	0	0	.48	.24	.12	.16	0
Q2	0	0	0	.3	.55	.15	0
Q3	0	0	0	0	.16	.84	0
Q4	0	0	0	0	0	.5	.5
PAID	0	0	0	0	0	1	0
BAD	0	0	0	0	0	0	1

inv(I-N)

	Q0	Q1	Q2	Q3	Q4	**Mu**
Q0	1	.3	.44	.41	.35	**2.49**
Q1	0	1	.48	.38	.45	2.31
Q2	0	0	1	.3	.6	1.9
Q3	0	0	0	1	.16	1.16
Q4	0	0	0	0	1	1

Expected # qrtrs till absorption= 2.49

(b)

P{i to j}

PAID	BAD
.83	***.17***
.78	.22
.7	.3
.92	.08
.5	.5

P{Q0→bad}=.17

P{Q0→Paid}=.83

(c)

	Mu
Q0	2.49
Q1	2.31
Q2	***1.9***
Q3	1.16
Q4	1

Nbr. of qrtrs till settled = 1.9

Set 17.6a

(a) State (i-j)=(Sets won by Andre-Sets won by John)

7

Matrix P:

	0-0	0-1	0-2	1-0	1-1	1-2	2-0	2-1	2-2	2-3	3-0	0-3	1-3	3-1	3-2
0-0	0	**.4**	0	**.6**	0	0	0	0	0	0	0	0	0	0	0
0-1	0	0	**.4**	0	**.6**	0	0	0	0	0	0	0	0	0	0
0-2	0	0	0	0	0	**.6**	0	0	0	0	0	**.4**	0	0	0
1-0	0	0	0	0	**.4**	0	**.6**	0	0	0	0	0	0	0	0
1-1	0	0	0	0	0	**.4**	0	**.6**	0	0	0	0	0	0	0
1-2	0	0	0	0	0	0	0	0	**.6**	0	0	0	**.4**	0	0
2-0	0	0	0	0	0	0	0	**.4**	0	0	**.6**	0	0	0	0
2-1	0	0	0	0	0	0	0	0	**.4**	0	0	0	0	**.6**	0
2-2	0	0	0	0	0	0	0	0	0	**.4**	0	0	0	0	**.6**
2-3	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0
3-0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0
0-3	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0
1-3	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0
3-1	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0
3-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**

(b)

inv(I-N)

	0-0	0-1	0-2	1-0	1-1	1-2	2-0	2-1	2-2
0-0	1	.4	.16	.6	.48	.3	.4	.4	.35
0-1	0	1	.4	0	.6	.5	0	.4	.43
0-2	0	0	1	0	0	.6	0	0	.36
1-0	0	0	0	1	.4	.2	.6	.5	.29
1-1	0	0	0	0	1	.4	0	.6	.48
1-2	0	0	0	0	0	1	0	0	.6
2-0	0	0	0	0	0	0	1	.4	.16
2-1	0	0	0	0	0	0	0	1	.4
2-2	0	0	0	0	0	0	0	0	1

	MU
0-0	***4.07***
0-1	3.27
0-2	1.96
1-0	2.93
1-1	2.48
1-2	***1.6***
2-0	1.56
2-1	1.4
2-2	1

P{i to j}

	2-3	3-0	0-3	1-3	3-1	3-2	P{A}	P{J}
0-0	.1	***.22***	.06	.12	***.26***	***.21***	***.68***	.32
0-1	.2	0	.16	.19	.22	.26	.48	.52
0-2	.1	0	.4	.24	0	.22	.22	.78
1-0	.1	.36	0	.06	.29	.17	.82	.18
1-1	.2	0	0	.16	.36	.29	.65	.35
1-2	.2	0	0	.4	0	.36	***.36***	***.64***
2-0	.1	.6	0	0	.24	.1	.94	.06
2-1	.2	0	0	0	.6	.24	.84	.16
2-2	.4	0	0	0	0	.6	.6	.4

Average # of sets till end of match= 4.07

Probability Andre will win = sum of ($P_{3\text{-}0}+P_{3\text{-}1}+P_{3\text{-}2}$) given 0-0 start= .69

(c)P{Andre wins | current score 1-2}=.36.

(d) The average number of sets till termination is 1.6. In **ONE** set the termination score can be 1-3 (J's favor), or in **TWO** sets it can be 2-3 (J's favor) or 3-2 (A's favor). The average number of sets to termination is thus more than 1 and less then 2 (= 1.6).

8

(a)

Matrix P:

	1	2	3	4	F
1	.2	.8	0	0	0
2	0	.22	.78	0	0
3	0	0	.25	.75	0
4	0	0	0	.3	.7
F	0	0	0	0	1

(b)

inv(I-N)

	1	2	3	4
1	1.25	1.282	1.333	1.429
2	0	1.282	1.333	1.429
3	0	0	1.333	1.429
4	0	0	0	1.429

Mu

	F
1	***5.29***
2	4.04
3	2.76
4	1.43

(c) To be able to take Cal II, the student must finish in 16 weeks (4 transitions) or less. Average number of transitions needed = 5.29. Hence, an average student will not be able to finish Cal I on time.

(d) No!

9

(a)

States: 0, 1, 2, 3, 4, 5, promotion

Matrix P:

	0	1	2	3	4	5	P
0	.2	.7	.1	0	0	0	0
1	0	.2	.7	.1	0	0	0
2	0	0	.2	.7	.1	0	0
3	0	0	0	.2	.7	.1	0
4	0	0	0	0	.2	.7	.1
5	0	0	0	0	0	0	1
P	0	0	0	0	0	0	1

(b)

inv(I-N)

	0	1	2	3	4	5
0	1.25	1.094	1.113	1.11	1.1	.89
1	0	1.25	1.094	1.11	1.1	.89
2	0	0	1.25	1.09	1.1	.89
3	0	0	0	1.25	1.1	.89
4	0	0	0	0	1.3	.88
5	0	0	0	0	0	1

Mu

	P
0	***6.57***
1	5.46
2	4.35
3	3.23
4	2.13
5	1

It takes 6.57 on the averages to be promoted.

10

(a)

Matrix P:

	0	1	2	3	D
0	.5	.5	0	0	0
1	.4	0	.6	0	0
2	.3	0	0	.7	0
3	.2	0	0	0	.8
D	0	0	0	0	1

States: 0, 1, 2, 3, Delete

inv(I-N)

	0	1	2	3
0	5.952	2.976	1.786	1.25
1	3.952	2.976	1.786	1.25
2	2.619	1.31	1.786	1.25
3	1.19	.595	.357	1.25

Mu

	D
0	***12***
1	9.96
2	***6.96***
3	3.39

(b)

A new customer stays 12 years on the list

(c) 6.96 years

11

(a)

states: 108, 109, 110, 111, 112, **107,113**

	108	109	110	111	112	107	113
108	.33	.33	0	0	0	.33	0
109	.33	.33	.33	0	0	0	0
110	0	.33	.33	.33	0	0	0
111	0	0	.33	.33	.333	0	0
112	0	0	0	.33	.333	0	.33
107	0	0	0	0	0	1	0
113	0	0	0	0	0	0	1

(b)

inv(I-N)

	108	109	110	111	112
108	2.5	2	1.5	1	.5
109	2	4	3	2	1
110	1.5	3	4.5	3	1.5
111	1	2	3	4	2
112	.5	1	1.5	2	2.5

continued...

Set 17.6a

	MU absorb
108	7.5
109	12
110	***13.5***
111	12
112	7.5

	P{i to j} 107	 113
108	.83	.17
109	.67	.33
110	.5	.5
111	.33	.67
112	.17	.83

The last two columns (low=107, high=113) provide the answer as a function of the current voltage. For example, if current voltage is 109, P{low}=.67, P{high}=.33

(c)
Setting current voltage at 110 guarantees an average time to failure of 13.5(15)= 202.5 minutes.

12

	Dialysis	1stYrC	1stYrL	>1yrS	Death
Dialysis	.5	.3	.1	0	.1
1stYrC	.3	0	0	.5	.2
1stYrL	.15	0	0	.75	.1
>1yrS	.05	0	0	.9	.05
Death	0	0	0	0	1

	inv(I-N) Dialysis	 1stYrC	 >1yrS	 1stYrL
Dialysis	***3.5398***	1.0619	7.96	.354
1stYrC	1.9469	1.5841	9.38	.1947
1stYrL	1.8584	.5575	11.71	.1858
>1yrS	***1.7699***	.531	***14***	.177

Mu death
12.92
13.11
15.28
16.46

(a) # years on dialysis=3.54 years.
(b) Longevity = 12.92 years.
(c) Life expectancy = 16.46 years
(d) 14 years.
(e) >1yrSurvivor has the highest longevity (= 16.46 years) and the least number of years on dialysis (= 1.7699 years).

Chapter 18

Classical Optimization Theory

Set 18.1a

1

(a) $\frac{\partial f}{\partial x} = 3x^2+1 = 0$

$x = \pm\sqrt{-1/3}$

The necessary condition yields imaginary roots. The problem has no stationary points.

(b) $\frac{\partial f}{\partial x} = 4x^3+2x = 0$

$x = 0, \quad x = \pm\sqrt{-1/2}$

For $x = 0$,

$\frac{\partial^2 f}{\partial x^2} = 12x^2+2 = 2 > 0 \Rightarrow$ min

(c) $\frac{\partial f}{\partial x} = 16x^3 - 2x = 0$

$x = 0, .353, -.353$

$\frac{\partial^2 f}{\partial x^2} = 48x^2 - 2$

$x=0: \frac{\partial^2 f}{\partial x^2} = -2 \Rightarrow$ max

$x = .353: \frac{\partial^2 f}{\partial x^2} = 6 \Rightarrow$ min

$x = -.353: \frac{\partial^2 f}{\partial x^2} = 6 \Rightarrow$ min

(d) $f(x) = (3x-2)^2(2x-3)^2$

$= (6x^2 - 13x + 6)^2$

$\frac{\partial f}{\partial x} = 2(6x^2-13x+6)(12x-13) = 0$

$x = 2/3, 3/2, 13/12$

$\frac{\partial^2 f}{\partial x^2} = 2(216x^2 - 468x + 241)$

$x = 2/3: \frac{\partial^2 f}{\partial x^2} = 50 \Rightarrow$ min

$x = 3/2: \frac{\partial^2 f}{\partial x^2} = 50 \Rightarrow$ min

$x = 13/12: \frac{\partial^2 f}{\partial x^2} = -25 \Rightarrow$ max

(e) $\frac{\partial f}{\partial x} = 30x^4 - 12x^2 = 0 \Rightarrow x = (0, \pm .63)$

$\frac{\partial^2 f}{\partial x^2} = 120x^3 - 24x$

$x = 0: \frac{\partial^2 f}{\partial x^2} = 0, \frac{\partial^3 f}{\partial x^3} = 360x^2 - 24\Big|_{x=0} = -24 \Rightarrow$ inflect

$x = .63: \frac{\partial^2 f}{\partial x^2} = 14.88 \Rightarrow$ min

$x = -.63: \frac{\partial^2 f}{\partial x^2} = -14.88 \Rightarrow$ max

2

(a) $\frac{\partial f}{\partial x_1} = 3x_1^2 - 3x_2 = 0$

$\frac{\partial f}{\partial x_2} = 3x_2^2 - 3x_1 = 0$

$(x_1, x_2) = (0,0), (1,1)$

$$H = \begin{pmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{pmatrix}$$

$\underline{(x_1, x_2) = (0,0)}$:

principal minor determinants $= (0, -9) \Rightarrow$ indefinite

$\Rightarrow (0,0)$ is not an extreme point

$\underline{(x_1, x_2) = (1,1)}$:

Principal minor determinants $= (6, 27) \Rightarrow$ positive definite

$\Rightarrow (1,1)$ is a minimum point.

(b) $\frac{\partial f}{\partial x_1} = 4x_1 + 6 + 2x_2x_3 = 0 \quad (1)$

$\frac{\partial f}{\partial x_2} = 2x_2 + 6 + 2x_1x_3 = 0 \quad (2)$

$\frac{\partial f}{\partial x_3} = 2x_3 + 6 + 2x_1x_2 = 0 \quad (3)$

(3) − (2) yields $(x_3 - x_2) - x_1(x_3 - x_2) = 0$

or $(x_3 - x_2)(1 - x_1) = 0$

Thus, $x_3 = x_2$ or $x_1 = 1$

$\underline{\text{For } x_1 = 1}$:

from (1), $10 + 2x_2x_3 = 0 \quad (4)$

from (2), $2x_2 + 2x_3 + 6 = 0 \quad (5)$

Hence, $x_2 = -(3 + x_3)$. Substituting in (4), then

$10 - 2x_3(3 + x_3) = 0$

or

$x_3^2 + 3x_3 - 5 = 0$

Thus, $x_3 = 1.2$ or $x_3 = -4.2$

$x_2 = -4.2$ or $x_2 = 1.2$

or,

$$(x_1, x_2, x_3) = \begin{cases} (1, -4.2, 1.2) \\ (1, 1.2, -4.2) \end{cases}$$

$\underline{\text{For } x_2 = x_3}$:

from (2), $2x_2 + 6 + 2x_1x_2 = 0$

or, $(1 + x_1) = \frac{-3}{x_2}$

continued...

2 continued

From (1), $2x_1 + 3 + x_2^2 = 0$

Substituting $(1+x_1) = -3/x_2$, then

$$-\frac{3}{x_2} + \frac{1}{2} + \frac{x_2^2}{2} = 0$$

or

$$x_2^3 + x_2 - 6 = 0$$

This gives the solution $x_2 \simeq 1.65$. (The remaining two roots are imaginary.) Thus, $x_1 = \frac{-3}{1.65} - 1 = -2.82$ and $(x_1, x_2, x_3) = (-2.82, 1.65, 1.65)$

$$H = \begin{pmatrix} 4 & 2x_3 & 2x_2 \\ 2x_3 & 2 & 2x_1 \\ 2x_2 & 2x_1 & 2 \end{pmatrix}$$

$\underline{X} = (1, -4.2, 1.2)$:

Principal minor determinants (PMD) $= (4, 2.24, -223) \Rightarrow$ indefinite

$\underline{X} = (1, 1.2, -4.2)$:

PMD $= (4, -62.56, -155.5) \Rightarrow$ indefinite

$X = (-2.82, 1.65, 1.65)$:

PMD $= (4, 2.25, -67.4) \Rightarrow$ indefinite

3

$$\frac{\partial f}{\partial x_1} = 2x_2x_3 - 4x_3 + 2x_1 - 2 = 0$$

$$\frac{\partial f}{\partial x_2} = 2x_1x_3 - 2x_3 + 2x_2 - 4 = 0$$

$$\frac{\partial f}{\partial x_3} = 2x_1x_2 - 4x_1 - 2x_2 + 2x_3 + 4 = 0$$

Solutions: $(0,3,1), (0,1,-1),$
$(2,1,1), (1,2,0), (2,3,-1)$

$$H = \begin{pmatrix} 2 & 2x_3 & 2x_2-4 \\ 2x_3 & 2 & 2x_1-2 \\ 2x_2-4 & 2x_1-2 & 2 \end{pmatrix}$$

$\text{PMD}_{(0,3,1)} = (2, 0, -32)$ indefinite

$\text{PMD}_{(0,1,-1)} = (2, 0, -32)$ indefinite

PMD$(2,1,1) = (2,0,-32)$ indefinite

PMD$(1,2,0) = (2,4,8)$ positive def $\Rightarrow$ min

PMD$(2,3,-1) = (2,0,-32)$ indefinite

4

The problem is equivalent to

Minimize $z = (x_1 - x_1^2)^2 + (x_2 - x_1 - 2)^2$

$$\frac{\partial z}{\partial x_1} = 2(x_2 - x_1^2)(-2x_1) + 2(x_2 - x_1 - 2)(-1) = 0$$

$$\frac{\partial z}{\partial x_2} = 2(x_1 - x_1^2) + 2(x_2 - x_1 - 2) = 0$$

Thus, solve

$$2x_1^3 - 2x_1x_2 + x_1 - x_2 + 2 = 0 \quad ①$$

$$x_1^2 + x_1 - 2x_2 + 2 = 0 \quad ②$$

From (2),

$$x_2 = \frac{x_1^2 + x_1 + 2}{2}$$

From (1), we get

$$2x_1^3 - 3x_1^2 - 3x_1 + 2 = 0$$

Solutions: $(x_1, x_2) = (2, 4)$ and $(-1, 1)$

Note: The given method complicates a simple problem. Nevertheless the idea is interesting

5

From Taylor's theorem

$$f(y_0+h) = f(y_0) + f'(y_0)h + \frac{f''(y_0)h^2}{2!} + \cdots + \frac{f^{(n)}(y_0+\theta h)h^n}{n!}$$

Let $f'(y_0) = f''(y_0) = \cdots = f^{(n-1)}(y_0) = 0$ according to the assumption. Then

$$f(y_0+h) - f(y_0) = \frac{f^{(n)}(y_0+\theta h)h^n}{n!}$$

Because $f^{(n)}(y_0+\theta h)$ has the same sign as $f^{(n)}(y_0)$, then

(1) If n is even: $h^n > 0$ and $f(y_0+h) - f(y_0)$ has the same sign as $f^{(n)}(y_0) \Longrightarrow y_0$ is maximum if $f^{(n)}(y_0) < 0$, and y_0 is min $f^{(n)}(y_0) > 0$.

(2) If n is odd: $h^n < 0$ or > 0, depending on whether $h < 0$ or > 0, respectively. Thus, at y_0, $f(y_0+h) - f(y_0)$ will change sign from negative (positive) to positive (negative) depending on whether $f^{(n)}(y_0) > 0$ (< 0). Thus, y_0 is an inflection point.

Set 18.1b

1

$f(x) = 4x^4 - x^2 + 5$

$\frac{\partial f}{\partial x} = 16x^3 - 2x = 0$

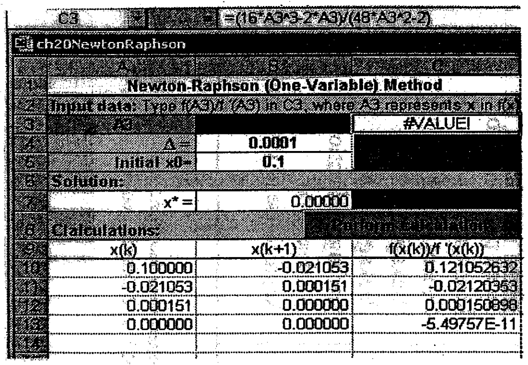
Cell C3 formula : (16*A3^3-2*A3)/((48*A3^2-2)

Solution:

(1) Initial $x_0 = .1 \implies x^* = 0$

(2) Initial $x_0 = 10 \implies x^* = .35355$

(3) Initial $x_0 = -10 \implies x^* = -.35355$



Newton-Raphson (One-Variable) Method

Input data: Type f(A3)/f '(A3) in C3, where A3 represents x in f(x)

A3 #VALUE!

Δ = 0.0001

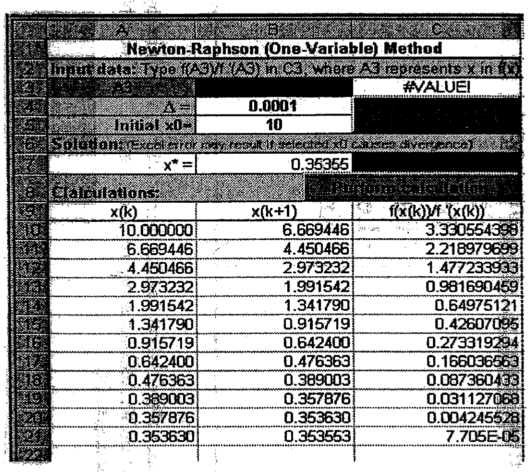
Initial x0= 0.1

Solution:

x* = 0.00000

Calculations:

x(k)	x(k+1)	f(x(k))/f '(x(k))
0.100000	-0.021053	0.121052632
-0.021053	0.000151	-0.02120353
0.000151	0.000000	0.000150898
0.000000	0.000000	-5.49757E-11



Newton-Raphson (One-Variable) Method

Input data: Type f(A3)/f '(A3) in C3, where A3 represents x in f(x)

A3 #VALUE!

Δ = 0.0001

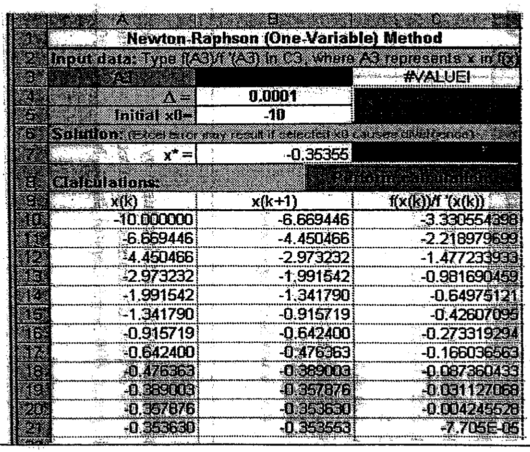
Initial x0= 10

Solution: (Excel error may result if selected x0 causes divergence)

x* = 0.35355

Calculations:

x(k)	x(k+1)	f(x(k))/f '(x(k))
10.000000	6.669446	3.330554398
6.669446	4.450466	2.218979699
4.450466	2.973232	1.477233933
2.973232	1.991542	0.981690459
1.991542	1.341790	0.64975121
1.341790	0.915719	0.42607095
0.915719	0.642400	0.273319294
0.642400	0.476363	0.166036563
0.476363	0.389003	0.087360433
0.389003	0.357876	0.031127068
0.357876	0.353630	0.004245528
0.353630	0.353553	7.705E-05



Newton-Raphson (One-Variable) Method

Input data: Type f(A3)/f '(A3) in C3, where A3 represents x in f(x)

A3 #VALUE!

Δ = 0.0001

Initial x0= -10

Solution: (Excel error may result if selected x0 causes divergence)

x* = -0.35355

Calculations:

x(k)	x(k+1)	f(x(k))/f '(x(k))
-10.000000	-6.669446	-3.330554398
-6.669446	-4.450466	-2.218979699
-4.450466	-2.973232	-1.477233933
-2.973232	-1.991542	-0.981690459
-1.991542	-1.341790	-0.64975121
-1.341790	-0.915719	-0.42607095
-0.915719	-0.642400	-0.273319294
-0.642400	-0.476363	-0.166036563
-0.476363	-0.389003	-0.087360433
-0.389003	-0.357876	-0.031127068
-0.357876	-0.353630	-0.004245528
-0.353630	-0.353553	-7.705E-05

2

$$f(x_1, x_2) = 2x_1^2 + x_2^2 + x_3^2 + 6(x_1 + x_2 + x_3) + 2x_1 x_2 x_3$$

$$\frac{\partial f}{\partial x_1} = 4x_1 + 2x_2 x_3 + 6 = 0 \quad (= F_1)$$

$$\frac{\partial f}{\partial x_2} = 2x_2 + 2x_1 x_3 + 6 = 0 \quad (= F_2)$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + 2x_1 x_2 + 6 = 0 \quad (= F_3)$$

$$\nabla F_1 = (4, 2x_3, 2x_2)$$

$$\nabla F_2 = (2x_3, 2, 2x_1)$$

$$\nabla F_3 = (2x_2, 2x_1, 2)$$

Thus,

$$B = \begin{pmatrix} 4 & 2x_3 & 2x_2 \\ 2x_3 & 2 & 2x_1 \\ 2x_2 & 2x_1 & 2 \end{pmatrix}$$

(note that B is the Hessian matrix)

$$A = \begin{pmatrix} 4x_1 + 2x_2 x_3 + 6 \\ 2x_2 + 2x_1 x_3 + 6 \\ 2x_3 + 2x_1 x_2 + 6 \end{pmatrix}$$

Let $X^0 = (0, 0, 0)$ be the starting point.

$$X^1 = (0,0,0) - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

$$= (-1.5, -3, -3)$$

$$X^2 = (-1.5, -3, -3) \begin{pmatrix} 4 & -6 & -6 \\ -6 & 2 & -3 \\ -6 & -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 18 \\ 9 \\ 9 \end{pmatrix}$$

$$= (-2.68, -4.89, -4.89)$$

We continue in the same manner until $X^k \simeq X^{k+1}$. If the present sequence does not converge, choose another starting point

1

(a) $\partial_c f = -46\partial x_2$

$= -.046$ for $\partial x_2 = .001$

$\begin{pmatrix}\partial x_1\\ \partial x_3\end{pmatrix} = -J^{-1}C\partial x_2$

$= \begin{pmatrix}2.83\\ -2.50\end{pmatrix} \times .001$

$= \begin{pmatrix}.00283\\ -.00250\end{pmatrix}$

$X^0+\partial X = (1-.00283, 2+.001, 3+.0025)$

$= (.99717, 2.001, 3.0025)$

$f(X^0+\partial X) = 57.9538$

$\partial_c f = 58 - 57.9538 = -.04618$

The approximation is better.

(b) $\partial x_1 = 2.83\,\partial x_2$

$\partial x_3 = -2.5\,\partial x_2$

(c) $\nabla_Y f = (6x_2,\ 10x_1x_3)$

$\nabla_Z f = (2x_1 + 5x_3^2)$

$J = \begin{pmatrix}2x_2+2 & x_1\\ 2x_1 & 2x_3\end{pmatrix}$

$C = \begin{pmatrix}x_3\\ 2x_1+2x_2\end{pmatrix}$

At $X^0 = (1, 2, 3)$,

$J^{-1}C = \begin{pmatrix}6 & 1\\ 2 & 6\end{pmatrix}^{-1}\begin{pmatrix}3\\ 6\end{pmatrix}$

$= \begin{pmatrix}6/34 & -1/34\\ -2/34 & 6/34\end{pmatrix}\begin{pmatrix}3\\ 6\end{pmatrix}$

continued...

1 continued

$= \begin{pmatrix}.353\\ .882\end{pmatrix}$

$\partial_c f = \left[47 - (12, 30)\begin{pmatrix}.352\\ .882\end{pmatrix}\right]\partial x_1$

$= 16.316\,\partial x_1$

For $\partial_c f = -.46$, we have

$16.316\,\partial x_1 = -.46$

or $\partial x_1 = -.0282$

Set 18.2b

1

(a) No, the necessary and sufficient conditions are the same in both methods.

(b) The Jacobian method computes the constrained gradient of the objective function directly. The new method computes the constrained objective function from which we can compute the constrained gradient.

2

$Y = (x_2, x_3) \quad Z = (x_1)$

$\nabla f(Y) = (6x_2, 10x_1 x_3)$

$\nabla f(Z) = (2x_1 + 5x_3^2)$

$$J = \begin{pmatrix} 2x_2+2 & x_1 \\ 2x_1 & 2x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 \\ 2 & 6 \end{pmatrix} \text{at } X = (1,2,3)$$

$$C = \begin{pmatrix} x_3 \\ 2x_1+2x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{at } X = (1,2,3)$$

$$J^{-1}C = \begin{pmatrix} 6/34 & -1/34 \\ -2/34 & 6/34 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 6/17 \\ 15/17 \end{pmatrix}$$

$\nabla f(Z) = 47 \quad \nabla f(Y) = (12, 30)$

$$\partial_c f = \left[(47 - (12,30) \begin{pmatrix} 6/17 \\ 15/17 \end{pmatrix} \right] \partial x_1$$

$$= 16.316\, \partial x_1$$

From Example 20.3-1, given $\partial x_2 = .01$, then $\partial x_1 = -.0283$ and

$$\partial_c f = 16.316 \times (-.0283) \simeq -.46$$

3

$Y = x_n$

$Z = (x_1, x_2, \cdots, x_{n-1})$

$\nabla f(Y) = 2x_n$

$\nabla f(Z) = (2x_1, 2x_2, \cdots, 2x_{n-1})$

$$J = \nabla g(Y) = \prod_{i=1}^{n-1} x_i = \frac{C}{x_n},$$

$$C = \nabla g(Z) = \left(\frac{C}{x_1}, \frac{C}{x_2}, \cdots, \frac{C}{x_{n-1}}\right)$$

$$x_i \neq 0, \quad i = 1, 2, \cdots, n$$

$$\nabla_c f = (2x_1, \cdots, 2x_{n-1}) - 2x_n \left(\frac{x_n}{C}\right)\left(\frac{C}{x_1}, \cdots, \frac{C}{x_{n-1}}\right)$$

$$= 0$$

$$i = 1, 2, \ldots, n-1$$

Thus, necessary conditions are

$$2x_i - \frac{2x_n^2}{x_i} = 0, \quad i = 1, 2, \cdots, n-1$$

The solution of these equations yields

$$x_1 = x_2 = \cdots = x_n$$

Hence, from the constraint

$$x_i^* = \sqrt[n]{C}, \quad i = 1, 2, \cdots, n$$

Sufficient conditions:

$$\frac{\partial_c f}{\partial_c x_i} = 2x_i - \frac{2x_n^2}{x_i}, \quad i = 1, 2, \ldots, n-1$$

$$\frac{\partial_c^2 f}{\partial_c^2 x_i^2} = 2 + \frac{2x_n^2}{x_i^2} = 4 \text{ at } x_i^* \text{ for all } i$$

Hence,

$$H = \begin{pmatrix} 4 & & 0 \\ & 4 & \\ 0 & & \ddots\ 4 \end{pmatrix}$$

which is positive definite ⇒ min

$$\frac{\partial f}{\partial g} = \nabla f(Y) J^{-1} \text{ at } X^0$$

$$= 2\sqrt[n]{C} \quad \frac{\sqrt[n]{C}}{2} = 2\sqrt[n]{C^{2-n}}$$

For $\partial g = \delta$,

$$\partial f = 2\delta \sqrt[n]{C^{2-n}} = 2\delta\left(C^{\frac{2-n}{n}}\right)$$

4

$Z = x_1\ ,\quad Y = x_2$

$\nabla f(Z) = 10x_1 + 2x_2$

$\nabla f(Y) = 2x_1 + 2x_2$

$J = \nabla g(Y) = x_1$

$C = \nabla g(Z) = x_2$

$\nabla_c f = (2x_2 + 10x_1) - (2x_1 + 2x_2)\frac{1}{x_1}\, x_2$

$= \frac{-2}{x_1}(x_2^2 - 5x_1^2)$

$\nabla_c f = 0 \Rightarrow x_2 = \pm\sqrt{5}\ x_1$

$g(x) = 0 \Rightarrow x_1^2 = 10/\sqrt{5}$

The stationary points are

$(2.115, 4.729),\ (-2.115, -4.729)$

Sufficiency condition:

$\frac{\partial}{\partial Z}\nabla_c f = 10 + 2\left(\frac{x_2^2}{x_1^2}\right)$

Thus, both stationary points are min

(a) $\partial f = \nabla f(Y)\, J^{-1}\partial g$

$= (2x_1 + 2x_2)(1/x_1)\,\partial g$

$\partial g = -.01$, thus, $\partial f = -.0647$

(b) $\partial f = \nabla f(Y) J^{-1}\partial g + \nabla_c f\, \partial Z$

$= 14\left(\frac{1}{2}\right)(-.01) +$

$[30 - 14)\left(\frac{1}{2}\right)(5)](.01)$

$= -.12$

5

$Y = (x_2, x_3),\ Z = x_1$

at $X^0 = (1, 1, 1)$

$\nabla f(Y^0) = (4x_2 + 5x_1, 20x_3)$

$= (9, 20)$

continued...

5 continued

$\nabla g(Y^0) = \begin{pmatrix} 2x_2+3x_3 & 3x_2 \\ 5x_1 & 2x_3 \end{pmatrix}$

$= \begin{pmatrix} 5 & 3 \\ 5 & 2 \end{pmatrix}$

$\nabla g(Z^0) = \begin{pmatrix} 1 \\ 2x_1 + 5x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

$\partial_c f = \nabla_c f(Y^0) J^{-1}\partial g + \nabla_c f(Y^0, Z^0)\,\partial Z$

$\nabla_c f(Y^0) J^{-1} = (9, 20)\begin{pmatrix} -2/5 & 3/5 \\ 1 & -1 \end{pmatrix}$

$= (82/5, -73/5)$

$\nabla_c f(Y^0, Z^0) = \left[7 - (9,20)\begin{pmatrix} -2/5 & 3/5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix}\right]$

$= 92.8$

$\partial_c f = (82/5, -73/5)\begin{pmatrix} \partial g_1 \\ \partial g_2 \end{pmatrix} + 92.8\,\partial x_1$

For $(\partial g_1\ \ \partial g_2) = (-.01, .02)$, $\partial x_1 = .01$

$\partial_c f = -\frac{.82}{5} - \frac{1.46}{5} + .928 = .472$

6

(a) $Y = (x_1, x_2)\quad Z = (x_3, x_4)$

$J = \nabla g(Y) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, which is singular. We must

(b) select a new set Y and Z

Let

$Y = (x_2, x_4)\ ,\ Z = (x_1, x_3)$

$\nabla f(Z) = (2x_1, 2x_3)$

$\nabla f(Y) = (2x_2, 2x_4)$

$\nabla g(Y) = \begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix},\ J^{-1} = \begin{pmatrix} 3 & -5/2 \\ -1 & 1 \end{pmatrix}$

$\nabla g(Z) = \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix}$

$\nabla_c f = (2x_1, 2x_3) - (2x_2, 2x_4)\begin{pmatrix} 3 & -5/2 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix}$

$= (2x_1 - x_2\ ,\ 2x_3 + 7x_2 - 4x_4)$

$\nabla_c f = 0$ yields

continued...

6 continued

$2x_1 - x_2 = 0$ ①

$2x_3 + 7x_2 - 4x_4 = 0$ ②

$x_1 + 2x_2 + 3x_3 + 5x_4 - 10 = 0$ ③

$x_1 + 2x_2 + 5x_3 + 6x_4 - 15 = 0$ ④

From ①, $2x_1 = x_2$

Substitution in ③ and ④ yields

$$5x_1 + 3x_3 + 5x_4 = 10$$
$$5x_1 + 5x_3 + 6x_4 = 15$$
$$14x_1 + 2x_3 - 4x_4 = 0$$

The solution is

$$(x_1, x_2, x_3, x_4) = \left(\frac{-5}{74}, \frac{-10}{74}, \frac{155}{74}, \frac{60}{74}\right)$$

$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ positive definite

Thus, the stationary point is a minimum point.

For $Y^0 = (-10/74, 60/74)$

$$\nabla f(Y^0) = (-10/37, 60/37)$$

$$\frac{\partial f}{\partial g} = \nabla f(Y^0) J^{-1} = \left(\frac{-10}{37}, \frac{60}{37}\right) \begin{pmatrix} 3 & -5/2 \\ -1 & 1 \end{pmatrix}$$
$$= \left(-\frac{90}{37}, \frac{85}{37}\right)$$

$$\partial_c f = \nabla f(Y_0) J^{-1} \partial g$$
$$= \left(-\frac{90}{37}, \frac{85}{37}\right) \begin{pmatrix} -.01 \\ -.02 \end{pmatrix} \cong -.07$$

7

For the LP problem,

indep. vars = nonbasic variables

dep. vars = basic variables

$$\nabla f(Y) = (c_1, c_2, \dots, c_m) = C_B$$

$$\nabla f(Z) = (c_{m+1}, c_{m+2}, \dots, c_n)$$

$$\nabla g(Y) = J = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & & a_{mm} \end{pmatrix} = B$$

$$\nabla g(Z) = \begin{pmatrix} a_{1,m+1} & \cdots & a_{1n} \\ a_{m,m+1} & \cdots & a_{mn} \end{pmatrix}$$

continued...

7 continued

$$= (P_{m+1}, P_{m+2}, \dots, P_n)$$

$$\nabla_c f = \Big\{ (c_{m+1}, \dots, c_n) - (c_1, \dots, c_m) \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}^{-1} \times (P_{m+1}, \dots, P_n) \Big\}$$

$$= \{ c_j - C_B B^{-1} P_j \}, \; j = m+1, \dots, n$$

$$= \{ c_j - z_j \}, \text{ provided } B^{-1} \text{ exists}$$

The Jacobian method cannot be applied to LP directly without first accounting for the nonnegativity constraints. This is accomplished by using the substitution $x_j = w_j^2$.

1

$$f(\underline{W}) = 5w_1^2 + 3w_2^2$$

s.t.

$$g_1(\underline{W}) = w_1^2 + 2w_2^2 + w_3^2 \quad -6 = 0$$

$$g_2(\underline{W}) = 3w_1^2 + w_2^2 \quad + w_4^2 - 9 = 0$$

$$\underline{Y} = (w_1, w_2), \quad Z = (w_3, w_4)$$

$$\nabla f(\underline{Y}) = (10w_1, 6w_2)$$

$$\nabla f(\underline{Z}) = (0, 0)$$

$$\underline{\nabla} \underline{g}(\underline{Y}) = \begin{pmatrix} 2w_1 & 4w_2 \\ 6w_1 & 2w_1 \end{pmatrix}$$

$$\underline{\nabla} g(\underline{Z}) = \begin{pmatrix} 2w_3 & 0 \\ 0 & 2w_4 \end{pmatrix}$$

$$\underline{J}^{-1} = \frac{1}{-20w_1w_2}\begin{pmatrix} 2w_2 & -4w_2 \\ -6w_1 & 2w_2 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} -1/w_1 & 2/w_1 \\ 3/w_2 & -1/w_1 \end{pmatrix}$$

$$\underline{J}^{-1}\underline{C} = \frac{1}{10}\begin{pmatrix} \frac{-1}{w_1} & \frac{2}{w_1} \\ \frac{3}{w_2} & \frac{-1}{w_2} \end{pmatrix}\begin{pmatrix} 2w_2 & 0 \\ 0 & 2w_4 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} \frac{-2w_3}{w_1} & \frac{4w_4}{w_1} \\ \frac{6w_3}{w_2} & \frac{-2w_4}{w_2} \end{pmatrix}$$

$$\nabla_c f = (0,0) - (10w_1, 6w_2)\begin{pmatrix} -\frac{w_3}{5w_1} & \frac{2w_4}{5w_1} \\ \frac{3w_3}{5w_2} & \frac{-w_4}{5w_2} \end{pmatrix}$$

$$= \left(-\frac{8}{5}w_3, -\frac{14}{5}w_4\right) = \underline{0}$$

$$w_3 = w_4 = 0$$

From the constraints,

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} w_1^2 \\ w_2^2 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} \Rightarrow w_1^2 = \frac{12}{5}, w_2^2 = \frac{9}{5}$$

$$f(\underline{W}_0) = \left(5 \times \frac{12}{5} + 3 \times \frac{9}{5}\right) = 17.4$$

To check if the point is a max, consider

$$H_{\underline{W}_0} = \begin{pmatrix} -8/5 & 0 \\ 0 & -14/5 \end{pmatrix} \Rightarrow \text{negative def.}$$

Thus,

$$X_0 = \left(\frac{12}{5}, \frac{9}{5}, 0, 0\right)$$

is a maximum point.

continued...

Sensitivity coefficients:

$$\underline{\nabla} f(\underline{Y}_0) J^{-1} = (10w_1, 6w_2)\begin{pmatrix} \frac{-1}{10w_1} & \frac{2}{10w_1} \\ \frac{3}{10w_2} & \frac{-1}{10w_2} \end{pmatrix}$$

$$= (.8, 1.4)$$

Dual values:

$$C_B B^{-1} = (5, 3)\begin{pmatrix} -1/5 & 2/5 \\ 3/5 & -1/5 \end{pmatrix} = (.8, 1.4)$$

Dual obj value $= 6 \times .8 + 9 \times 1.4 = 17.4$

Lagrangian Method:

$$L(\underline{W}, \underline{\lambda}) = 5w_1^2 + 3w_2^2 - \lambda_1(w_1^2 + 2w_2^2 + w_3^2 - 6) - \lambda_2(3w_1^2 + w_2^2 + w_4^2 - 9)$$

$$\frac{\partial L}{\partial w_1} = 10w_1 - 2\lambda_1 w_1 - 6\lambda_2 w_2 = 0$$

$$\frac{\partial L}{\partial w_2} = 6w_1 - 4\lambda_1 w_2 - 2\lambda_2 w_2 = 0$$

$$\frac{\partial L}{\partial w_3} = -2\lambda_1 w_3 = 0$$

$$\frac{\partial L}{\partial w_4} = -2\lambda_2 w_4 = 0$$

$$g_1(\underline{W}) = 0$$

$$g_2(\underline{W}) = 0$$

The solution is

$$(W^0, \lambda^0) = \left(\frac{12}{5}, \frac{9}{5}, 0, 0, .8, 1.4\right)$$

Sufficiency condition:

$$H^B = \left[\begin{array}{cc|cccc} 0 & 0 & 2w_1 & 2w_2 & 2w_3 & 0 \\ 0 & 0 & 6w_1 & 2w_2 & 0 & 2w_4 \\ \hline 2w_1 & 6w_1 & 10-2\lambda-6\lambda_2 & 0 & 0 & 0 \\ 2w_2 & 2w_2 & 0 & 6-4\lambda_1-2\lambda_2 & 0 & 0 \\ 2w_3 & 0 & 0 & 0 & -2\lambda_1 & 0 \\ 0 & 2w_4 & 0 & 0 & 0 & -2\lambda_2 \end{array}\right]_{\underline{W}^0, \underline{\lambda}^0}$$

$$= \begin{bmatrix} 0 & 0 & 3 & 2.64 & 0 & 0 \\ 0 & 0 & 9 & 2.64 & 0 & 0 \\ 3 & 9 & 0 & 0 & 0 & 0 \\ 2.64 & 2.64 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.6 & 0 \\ 0 & 0 & 0 & 0 & -1.6 & -2.8 \end{bmatrix}$$

continued...

1 continued

$2m+1 = 5$

The value of the 5th principal minor determinant $= -427$ and that of the 6th principal minor determinant is 1130, following the signs of $(-1)^{m+1}$ and $(-1)^{m+2}$ (−, +, respectively) Hence W^0, λ^0 is a maximum point

2

$$\frac{\partial}{\partial x_1} = 2x_1 - \lambda_1 - \lambda_2 = 0 \quad ①$$

$$\frac{\partial}{\partial x_2} = 4x_2 - 2\lambda_1 x_2 - 5\lambda_2 = 0 \quad ②$$

$$\frac{\partial}{\partial x_3} = 20x_3 - \lambda_1 - \lambda_2 = 0 \quad ③$$

$$\frac{\partial}{\partial \lambda_1} = -(x_1 + x_2^2 + x_3 - 5) = 0 \quad ④$$

$$\frac{\partial}{\partial \lambda_2} = -(x_1 + 5x_2 + x_3 - 7) = 0 \quad ⑤$$

From ① and ③, $x_1 = 10x_3$.

Substitution in ④ and ⑤ yields

$$x_2^2 + 11x_3 = 5 \quad ⑥$$

$$5x_2 + 11x_3 = 7 \quad ⑦$$

⑥ and ⑦ give

$$x_2^2 - 5x_2 + 2 = 0$$

Solution:

$$X_1^0 = (-14.4, 4.56, -1.44)$$

$$X_2^0 = (4.4, .44, .44)$$

For X_1^0, from ② and ③

$$\lambda_1^1 = 38.5, \quad \lambda_2^1 = -67.3$$

For X_2^0, from ② and ③

$$\lambda_1^2 = 10.2, \quad \lambda_2^2 = -1.4$$

Stationary point:

$(X_1^0, \lambda_1^0) = (-14.4, 4.65, -1.44, 38.5, -67.3)$

$(X_2^0, \lambda_2^0) = (4.4, .44, .44, 10.2, -1.4)$

Both points are minima

3

$$L(X,\lambda) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$
$$-\lambda_1(x_1 + 2x_2 + 3x_3 + 5x_4 - 10)$$
$$-\lambda_2(x_1 + 2x_2 + 5x_3 + 6x_4 - 15)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2\lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - 5\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_4} = 2x_4 - 5\lambda_1 - 6\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 + 3x_3 + 5x_4 - 10) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(x_1 + 2x_2 + 5x_3 + 6x_4 - 15) = 0$$

Solution:

$$(X^0, \lambda^0) = \left(\frac{-5}{74}, \frac{-10}{74}, \frac{60}{74}, \frac{-90}{37}, \frac{85}{37}\right)$$

The values of λ^0 are the same as the sensitivity coefficients obtained in Problem 20.2b-6.

1

By definition

$$\lambda = \frac{\partial f}{\partial g}$$

If the right-hand side of $g(X) \geq 0$ is changed to $\partial g \geq 0$, the constraints become more restrictive. This means that the value of $f(X)$ can never improve. Thus,

$$\frac{\partial f}{\partial g} \leq 0 \quad \text{or} \quad \lambda \leq 0$$

2

Replace $g(X) = 0$ with

$$g(X) \leq 0$$
$$-g(X) \leq 0$$

Thus,

$$L(X, \lambda_1, \lambda_2) = f(X) - \lambda_1 (g(X) + S_1^2) - \lambda_2 (-g(X) + S_2^2)$$

The K-T conditions are then given by,

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0 \qquad ①$$

$$\frac{\partial L}{\partial X} = \nabla f(X) - (\lambda_1 - \lambda_2) \nabla g(X) = 0 \qquad ②$$

$$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0 \qquad ③$$

$$\frac{\partial L}{\partial S_2} = -2\lambda_2 S_2 = 0 \qquad ④$$

$$\frac{\partial L}{\partial \lambda_1} = g(X) + S_1^2 = 0 \qquad ⑤$$

$$\frac{\partial L}{\partial \lambda_2} = -g(X) + S_2^2 = 0 \qquad ⑥$$

From ⑤ and ⑥, $S_1^2 + S_2^2 = 0$

Because $S_1^2, S_2^2 \geq 0$, then $S^2 = S_2^2 = 0$

continued...

as should be expected. This means that conditions ③ and ④ are trivial and conditions ⑤ and ⑥ reduce to $g(X) = 0$.

Let $\lambda = \lambda_1 - \lambda_2$

Because $\lambda_1, \lambda_2 \geq 0$, λ is unrestricted in sign.

The K-T conditions become

(i) λ unrestricted in sign

(ii) $\nabla f(X) - \lambda \nabla g(X) = 0$

(iii) $g(X) = 0$

3

(a) $\max f(X) = x_1^3 - x_2^2 + x_1 x_3^2$

s.t.

$$x_1 + x_2^2 + x_3 = 5$$
$$-5x_1^2 + x_2^2 + x_3 \leq -2$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$
$$-x_3 \leq 0$$

$$L(X, \lambda) = f(X) - \lambda_1 (x_1 + x_2^2 + x_3 - 5) - \lambda_2 (-5x_1^2 + x_2^2 + x_3 + S_1^2 + 2) - \lambda_3 (-x_1 + S_2^2) - \lambda_4 (-x_2 + S_3^2) - \lambda_5 (-x_3 + S_4^2)$$

The K-T conditions are

① λ_1 unrestricted

② $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$

③ $$(3x_1^2 + x_3^2, -2x_2, 2x_1 x_3) - (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \begin{pmatrix} 1 & 2x_2 & 1 \\ -10x_1 & 2x_2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (0, 0, 0, 0, 0)$$

④ $$(\lambda_2, \lambda_3, \lambda_4, \lambda_5) \begin{pmatrix} -5x_1 + x_2^2 + x_3 + 2 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = 0$$

continued...

3 continued

⑤ $g(X) = 0$

(b) max $-f(X) = -x_1^4 - x_2^2 - 5x_1x_2x_3$

s.t.

$$x_1 - x_2^2 + x_3^3 - 10 \le 0$$
$$-x_1^3 - x_2^2 - 4x_3^2 + 20 \le 0$$

① $\lambda_1, \lambda_2 \ge 0$

② $(-4x_1^3 - 5x_2x_3, -2x_2 - 5x_1x_3, -5x_1x_2) - (\lambda_1, \lambda_2)\begin{pmatrix} 1 & -2x_2 & 3x_3^2 \\ -3x_1^2 & -2x_2 & -8x_3 \end{pmatrix} = (0,0)$

③ $(\lambda_1, \lambda_2)\begin{pmatrix} x_1 - x_2^2 + x_2^3 - 10 \\ -x_1^3 - x_2^2 - 4x_3^2 + 20 \end{pmatrix} = 0$

④ $x_1 - x_2^2 + x_3^3 - 10 \le 0$

$-x_1^3 - x_2^2 - 4x_3^2 + 20 \le 0$

4

Consider

$L(X,\lambda) = f(X) - \lambda g(X)$

Because all the constraints are equations, the elements of λ are unrestricted. However, because $g(X)$ is a linear function, $g(X)$ can be either convex or concave. Thus, for $\lambda_i > 0$, we take $g_i(X)$ as a convex function so that $-\lambda_i g_i(X)$ is concave. Similarly, if $\lambda_i < 0$, $g_i(X)$ is assumed concave, in which case $-\lambda_i g_i(X)$ is also concave. Given $f(X)$ is concave, hence $L(X,\lambda)$ is concave. If $g(X)$ is nonlinear, it cannot be both convex and concave, a central argument in the case of linear $g(X)$.

5

Maximize $f(X)$

s.t. $g_1(X) \ge 0$

$g_2(X) = 0$

$g_3(X) \le 0$

continued...

5 continued

$$L(X, \lambda_1, \lambda_2, \lambda_3) = f(X) - \lambda_1(-g_1(X) + S_1^2) - \lambda_2(g_2(X)) - \lambda_3(g_3(X) + S_3^2)$$

K-T conditions:

① $\lambda_1 \ge 0$, λ_2 unrestricted, $\lambda_3 \ge 0$

② $\frac{\partial L}{\partial X} = \nabla f(X) + \lambda_1 \nabla g_1(X) - \lambda_2 \nabla g_2(X) - \lambda_3 \nabla g_3(X)$

③ $\frac{\partial L}{\partial S_1} = 2\lambda_1 S_1 = 0$

④ $\frac{\partial L}{\partial S_3} = -2\lambda_3 S_3 = 0$

⑤ $\frac{\partial L}{\partial \lambda_1} = g_1(X) - S_1^2 = 0$

⑥ $\frac{\partial L}{\partial \lambda_2} = -g_2(X) = 0$

⑦ $\frac{\partial L}{\partial \lambda_3} = -(g_3(X) + S_3^2) = 0$

Sufficient conditions:

$f(X)$ concave

$g_1(X)$ concave

$g_2(X)$ linear or $\lambda_2 g_2(X)$ convex

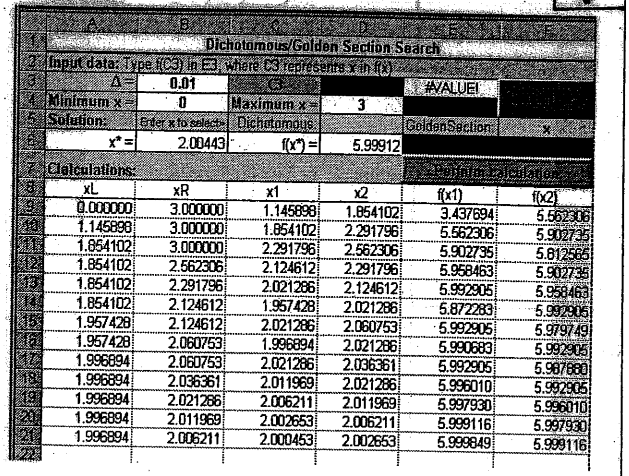
$g_3(X)$ convex

Chapter 19

Nonlinear Programming Algorithms

Set 19.1a

1



Dichotomous/Golden Section Search

Input data: Type f(C3) in E3, where C3 represents x in f(x)

Δ = 0.01 C3 #VALUE!

Minimum x = 0 Maximum x = 3

Solution: Enter x to select> Dichotomous GoldenSection x

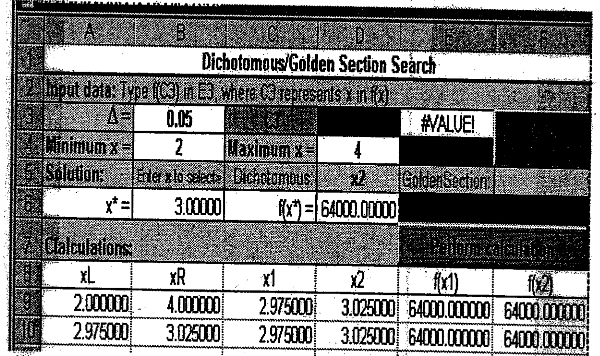
x* = 2.00443 f(x*) = 5.99912

Calculations: Perform calculation

xL	xR	x1	x2	f(x1)	f(x2)
0.000000	3.000000	1.145898	1.854102	3.437694	5.562306
1.145898	3.000000	1.854102	2.291796	5.562306	5.902735
1.854102	3.000000	2.291796	2.562306	5.902735	5.812565
1.854102	2.562306	2.124612	2.291796	5.958463	5.902735
1.854102	2.291796	2.021286	2.124612	5.992905	5.958463
1.854102	2.124612	1.957428	2.021286	5.872283	5.992905
1.957428	2.124612	2.021286	2.060753	5.992905	5.979749
1.957428	2.060753	1.996894	2.021286	5.990683	5.992905
1.996894	2.060753	2.021286	2.036361	5.992905	5.987880
1.996894	2.036361	2.011969	2.021286	5.996010	5.992905
1.996894	2.021286	2.006211	2.011969	5.997930	5.996010
1.996894	2.011969	2.002653	2.006211	5.999116	5.997930
1.996894	2.006211	2.000453	2.002653	5.999849	5.999116

(a) 2

Dichotomous:



Dichotomous/Golden Section Search

Input data: Type f(C3) in E3, where C3 represents x in f(x)

Δ = 0.05 C3 #VALUE!

Minimum x = 2 Maximum x = 4

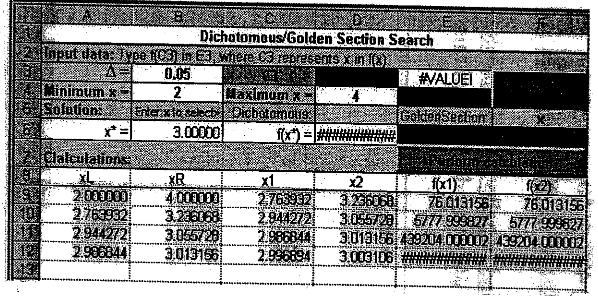
Solution: Enter x to select> Dichotomous: x2 GoldenSection:

x* = 3.00000 f(x*) = 64000.00000

Calculations: Perform calculation

xL	xR	x1	x2	f(x1)	f(x2)
2.000000	4.000000	2.975000	3.025000	64000.000000	64000.000000
2.975000	3.025000	2.975000	3.025000	64000.000000	64000.000000

Golden section:



Dichotomous/Golden Section Search

Input data: Type f(C3) in E3, where C3 represents x in f(x)

Δ = 0.05 C3 #VALUE!

Minimum x = 2 Maximum x = 4

Solution: Enter x to select> Dichotomous GoldenSection x

x* = 3.00000 f(x*) = ############

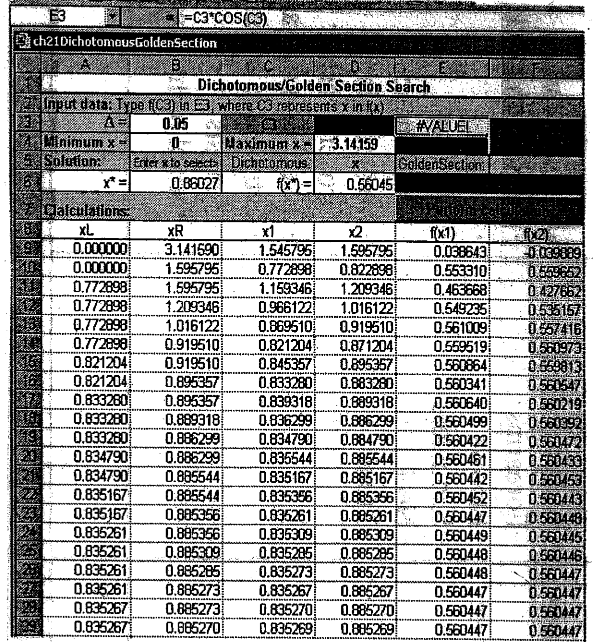
Calculations: Perform calculation

xL	xR	x1	x2	f(x1)	f(x2)
2.000000	4.000000	2.763932	3.236068	76.013156	76.013156
2.763932	3.236068	2.944272	3.055728	5777.999827	5777.999827
2.944272	3.055728	2.986844	3.013156	439204.000002	439204.000002
2.986844	3.013156	2.996894	3.003106	############	############

continued...

(b)

Dichotomous:



E3 = =C3*COS(C3)

ch21DichotomousGoldenSection

Dichotomous/Golden Section Search

Input data: Type f(C3) in E3, where C3 represents x in f(x)

Δ = 0.05 C3 #VALUE!

Minimum x = 0 Maximum x = 3.14159

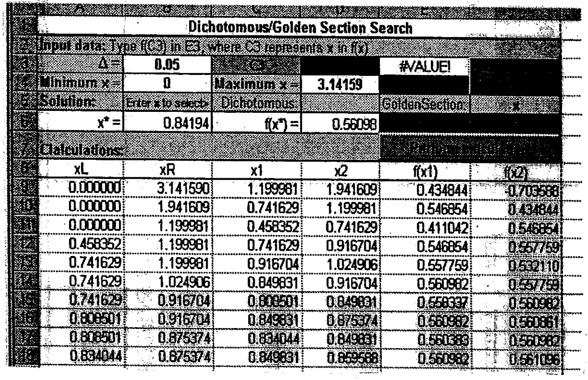
Solution: Enter x to select> Dichotomous x GoldenSection

x* = 0.86027 f(x*) = 0.56045

Calculations: Perform calculation

xL	xR	x1	x2	f(x1)	f(x2)
0.000000	3.141590	1.545795	1.595795	0.038643	-0.039889
0.000000	1.595795	0.772898	0.822898	0.553310	0.559652
0.772898	1.595795	1.159346	1.209346	0.463668	0.427662
0.772898	1.209346	0.966122	1.016122	0.549235	0.535157
0.772898	1.016122	0.869510	0.919510	0.561009	0.557416
0.772898	0.919510	0.821204	0.871204	0.559519	0.560973
0.821204	0.919510	0.845357	0.895357	0.560864	0.559813
0.821204	0.895357	0.833280	0.883280	0.560341	0.560547
0.833280	0.895357	0.839318	0.889318	0.560640	0.560219
0.833280	0.889318	0.836299	0.886299	0.560499	0.560392
0.833280	0.886299	0.834790	0.884790	0.560422	0.560472
0.834790	0.886299	0.835544	0.885544	0.560461	0.560433
0.834790	0.885544	0.835167	0.885167	0.560442	0.560453
0.835167	0.885544	0.835356	0.885356	0.560452	0.560443
0.835167	0.885356	0.835261	0.885261	0.560447	0.560448
0.835261	0.885356	0.835309	0.885309	0.560449	0.560445
0.835261	0.885309	0.835285	0.885285	0.560448	0.560446
0.835261	0.885285	0.835273	0.885273	0.560448	0.560447
0.835261	0.885273	0.835267	0.885267	0.560447	0.560447
0.835267	0.885273	0.835270	0.885270	0.560447	0.560447
0.835267	0.885270	0.835269	0.885269	0.560447	0.560447

Golden Section



Dichotomous/Golden Section Search

Input data: Type f(C3) in E3, where C3 represents x in f(x)

Δ = 0.05 C3 #VALUE!

Minimum x = 0 Maximum x = 3.14159

Solution: Enter x to select> Dichotomous GoldenSection x

x* = 0.84194 f(x*) = 0.56098

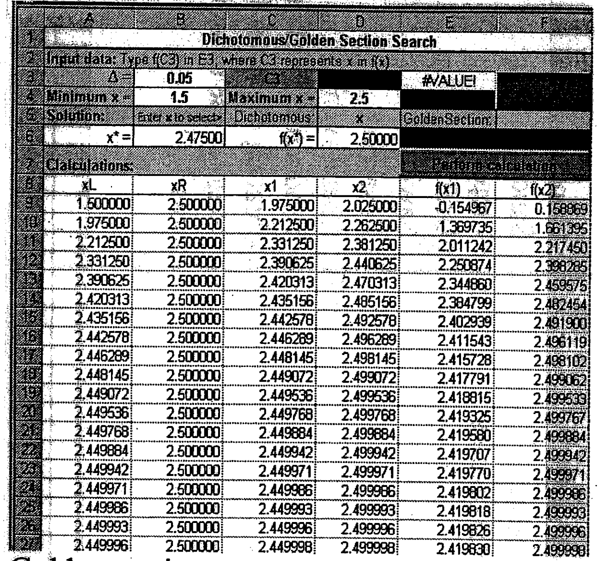
Calculations: Perform calculation

xL	xR	x1	x2	f(x1)	f(x2)
0.000000	3.141590	1.199981	1.941609	0.434844	-0.703588
0.000000	1.941609	0.741629	1.199981	0.546854	0.434844
0.000000	1.199981	0.458352	0.741629	0.411042	0.546854
0.458352	1.199981	0.741629	0.916704	0.546854	0.557759
0.741629	1.199981	0.916704	1.024906	0.557759	0.532110
0.741629	1.024906	0.849831	0.916704	0.560982	0.557759
0.741629	0.916704	0.808501	0.849831	0.558337	0.560982
0.808501	0.916704	0.849831	0.875374	0.560982	0.560861
0.808501	0.875374	0.834044	0.849831	0.560383	0.560982
0.834044	0.875374	0.849831	0.859568	0.560982	0.561096

continued...

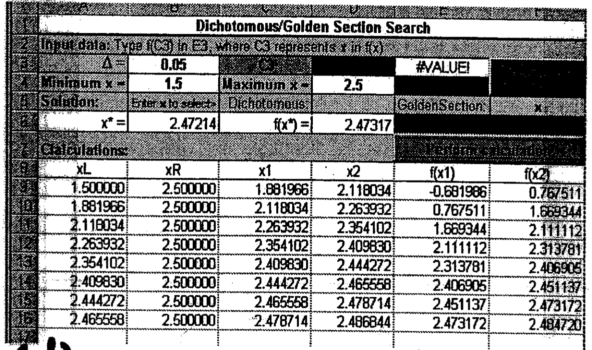
(c)

Dichotomous:



	A	B	C	D	E	F
1	Dichotomous/Golden Section Search					
2	Input data: Type f(C3) in E3, where C3 represents x in f(x)					
3	Δ =	0.05	C3		#VALUE!	
4	Minimum x =	1.5	Maximum x =	2.5		
5	Solution:	Enter x to select>	Dichotomous:	x	GoldenSection:	
6	x* =	2.47500	f(x*) =	2.50000		
7	Clalculations:				Perform calculation	
8	xL	xR	x1	x2	f(x1)	f(x2)
9	1.500000	2.500000	1.975000	2.025000	-0.154967	0.158869
10	1.975000	2.500000	2.212500	2.262500	1.369735	1.661395
11	2.212500	2.500000	2.331250	2.381250	2.011242	2.217450
12	2.331250	2.500000	2.390625	2.440625	2.250874	2.398285
13	2.390625	2.500000	2.420313	2.470313	2.344860	2.459575
14	2.420313	2.500000	2.435156	2.485156	2.384799	2.482454
15	2.435156	2.500000	2.442578	2.492578	2.402939	2.491900
16	2.442578	2.500000	2.446289	2.496289	2.411543	2.496119
17	2.446289	2.500000	2.448145	2.498145	2.415728	2.498102
18	2.448145	2.500000	2.449072	2.499072	2.417791	2.499062
19	2.449072	2.500000	2.449536	2.499536	2.418815	2.499539
20	2.449536	2.500000	2.449768	2.499768	2.419325	2.499767
21	2.449768	2.500000	2.449884	2.499884	2.419580	2.499884
22	2.449884	2.500000	2.449942	2.499942	2.419707	2.499942
23	2.449942	2.500000	2.449971	2.499971	2.419770	2.499971
24	2.449971	2.500000	2.449986	2.499986	2.419802	2.499986
25	2.449986	2.500000	2.449993	2.499993	2.419818	2.499993
26	2.449993	2.500000	2.449996	2.499996	2.419826	2.499996
27	2.449996	2.500000	2.449998	2.499998	2.419830	2.499998

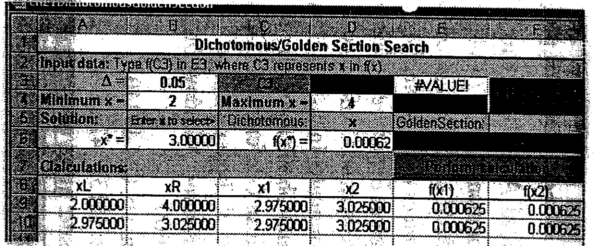
Golden section:



	A	B	C	D	E	F
1	Dichotomous/Golden Section Search					
2	Input data: Type f(C3) in E3, where C3 represents x in f(x)					
3	Δ =	0.05	C3		#VALUE!	
4	Minimum x =	1.5	Maximum x =	2.5		
5	Solution:	Enter x to select>	Dichotomous:		GoldenSection:	x
6	x* =	2.47214	f(x*) =	2.47317		
7	Clalculations:				Perform calculation	
8	xL	xR	x1	x2	f(x1)	f(x2)
9	1.500000	2.500000	1.881966	2.118034	-0.681986	0.767511
10	1.881966	2.500000	2.118034	2.263932	0.767511	1.669344
11	2.118034	2.500000	2.263932	2.354102	1.669344	2.111112
12	2.263932	2.500000	2.354102	2.409830	2.111112	2.313781
13	2.354102	2.500000	2.409830	2.444272	2.313781	2.406905
14	2.409830	2.500000	2.444272	2.465558	2.406905	2.451137
15	2.444272	2.500000	2.465558	2.478714	2.451137	2.473172
16	2.465558	2.500000	2.478714	2.486844	2.473172	2.484720
17						

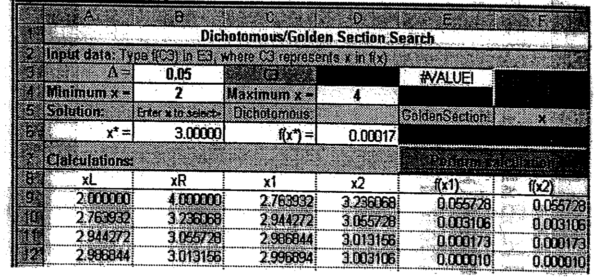
(d)

Dichotomus:



	A	B	C	D	E	F
1	Dichotomous/Golden Section Search					
2	Input data: Type f(C3) in E3, where C3 represents x in f(x)					
3	Δ =	0.05	C3		#VALUE!	
4	Minimum x =	2	Maximum x =	4		
5	Solution:	Enter x to select>	Dichotomous:	x	GoldenSection:	
6	x* =	3.00000	f(x*) =	0.00062		
7	Clalculations:				Perform calculation	
8	xL	xR	x1	x2	f(x1)	f(x2)
9	2.000000	4.000000	2.975000	3.025000	0.000625	0.000625
10	2.975000	3.025000	2.975000	3.025000	0.000625	0.000625

Golden section:

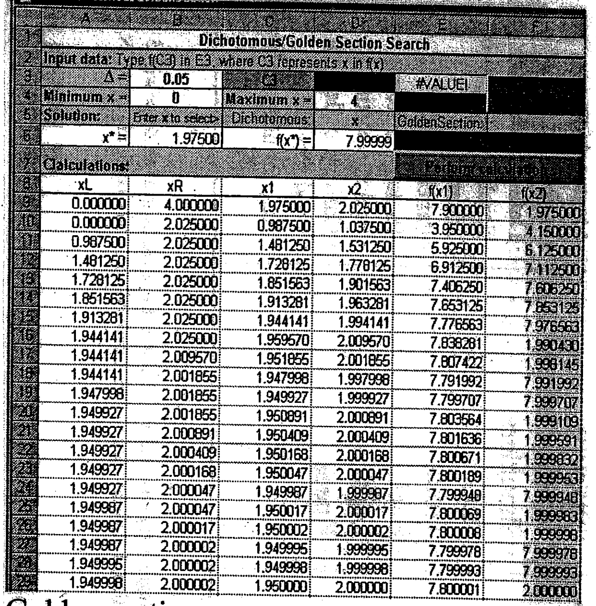


	A	B	C	D	E	F
1	Dichotomous/Golden Section Search					
2	Input data: Type f(C3) in E3, where C3 represents x in f(x)					
3	Δ =	0.05	C3		#VALUE!	
4	Minimum x =	2	Maximum x =	4		
5	Solution:	Enter x to select>	Dichotomous:		GoldenSection:	x
6	x* =	3.00000	f(x*) =	0.00017		
7	Clalculations:				Perform calculation	
8	xL	xR	x1	x2	f(x1)	f(x2)
9	2.000000	4.000000	2.763932	3.236068	0.055728	0.055728
10	2.763932	3.236068	2.944272	3.055728	0.003106	0.003106
11	2.944272	3.055728	2.986844	3.013156	0.000173	0.000173
12	2.986844	3.013156	2.996894	3.003106	0.000010	0.000010

Continued...

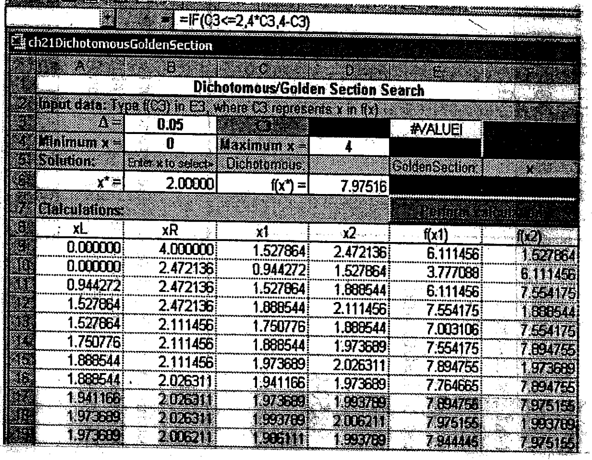
(e)

Dichotomous:



	A	B	C	D	E	F
1	Dichotomous/Golden Section Search					
2	Input data: Type f(C3) in E3, where C3 represents x in f(x)					
3	Δ =	0.05	C3		#VALUE!	
4	Minimum x =	0	Maximum x =	4		
5	Solution:	Enter x to select>	Dichotomous:	x	GoldenSection:	
6	x* =	1.97500	f(x*) =	7.99999		
7	Clalculations:				Perform calculation	
8	xL	xR	x1	x2	f(x1)	f(x2)
9	0.000000	4.000000	1.975000	2.025000	7.900000	1.975000
10	0.000000	2.025000	0.987500	1.037500	3.950000	4.150000
11	0.987500	2.025000	1.481250	1.531250	5.925000	6.125000
12	1.481250	2.025000	1.728125	1.778125	6.912500	7.112500
13	1.728125	2.025000	1.851563	1.901563	7.406250	7.606250
14	1.851563	2.025000	1.913281	1.963281	7.653125	7.853125
15	1.913281	2.025000	1.944141	1.994141	7.776563	7.976563
16	1.944141	2.025000	1.959570	2.009570	7.838281	1.990430
17	1.944141	2.009570	1.951855	2.001855	7.807422	1.998145
18	1.944141	2.001855	1.947998	1.997998	7.791992	7.991992
19	1.947998	2.001855	1.949927	1.999927	7.799707	7.999707
20	1.949927	2.001855	1.950891	2.000891	7.803564	1.999109
21	1.949927	2.000891	1.950409	2.000409	7.801636	1.999591
22	1.949927	2.000409	1.950168	2.000168	7.800671	1.999832
23	1.949927	2.000168	1.950047	2.000047	7.800189	1.999953
24	1.949927	2.000047	1.949987	1.999987	7.799948	7.999948
25	1.949987	2.000047	1.950017	2.000017	7.800069	1.999983
26	1.949987	2.000017	1.950002	2.000002	7.800008	1.999998
27	1.949987	2.000002	1.949995	1.999995	7.799978	7.999978
28	1.949995	2.000002	1.949998	1.999998	7.799993	7.999993
29	1.949998	2.000002	1.950000	2.000000	7.800001	2.000000

Golden section:



=IF(C3<=2,4*C3,4-C3)

ch21DichotomousGoldenSection

	A	B	C	D	E	F
1	Dichotomous/Golden Section Search					
2	Input data: Type f(C3) in E3, where C3 represents x in f(x)					
3	Δ =	0.05	C3		#VALUE!	
4	Minimum x =	0	Maximum x =	4		
5	Solution:	Enter x to select>	Dichotomous:		GoldenSection:	x
6	x* =	2.00000	f(x*) =	7.97516		
7	Clalculations:				Perform calculation	
8	xL	xR	x1	x2	f(x1)	f(x2)
9	0.000000	4.000000	1.527864	2.472136	6.111456	1.527864
10	0.000000	2.472136	0.944272	1.527864	3.777088	6.111456
11	0.944272	2.472136	1.527864	1.888544	6.111456	7.554175
12	1.527864	2.472136	1.888544	2.111456	7.554175	1.888544
13	1.527864	2.111456	1.750776	1.888544	7.003106	7.554175
14	1.750776	2.111456	1.888544	1.973689	7.554175	7.894755
15	1.888544	2.111456	1.973689	2.026311	7.894755	1.973689
16	1.888544	2.026311	1.941166	1.973689	7.764665	7.894755
17	1.941166	2.026311	1.973689	1.993789	7.894756	7.975155
18	1.973689	2.026311	1.993789	2.006211	7.975155	1.993789
19	1.973689	2.006211	1.986111	1.993789	7.944445	7.975155

Set 19.1b

1

Because $f(X)$ is strictly concave, a sufficient condition for optimality is $\nabla f(X) = 0$.

To solve $\nabla f(X) = 0$ by the Newton-Raphson method, consider Taylor's expansion about an initial X^0,

$$\nabla f(X) = \nabla f(X^0) + H(X - X^0)$$

The Hessian matrix H is independent of X because $f(X)$ is quadratic. The given expansion is exact because higher-order derivatives are zero.

Given $\nabla f(X) = 0$, we get

$$X = X^0 - H^{-1} \nabla f(X^0)$$

Because X satisfies $\nabla f(X) = 0$, X must be optimum regardless of the choice of initial X^0

$$\nabla f(X) = (4 - 4X_1 - 2X_2, 6 - 2X_1 - 4X_2)$$

Let

$$X^0 = (5, 5) \Longrightarrow \nabla f(X^0) = (-26, -24)^T$$

$$H = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} -1/3 & 1/6 \\ 1/6 & -1/3 \end{pmatrix}$$

Thus, the optimum is

$$X = \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} -1/3 & 1/6 \\ 1/6 & -1/3 \end{pmatrix} \begin{pmatrix} -26 \\ -24 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix}$$

2

(a) $f(X) = (X_2 - X_1^2)^2 + (1 - X_1)^2$

$$\nabla f(X) = [4(X_1^3 - X_1X_2) + 2(X_1 - 1), 2(X_2 - X_1^2)]$$

$$X^0 = (0, 0)$$

$$\nabla f(X^0) = (-2, 0)^T$$

$$X = (-2r, 0)^T$$

$$h(r) = 16r^4 + 4r^2 + 4r + 1$$

$$r^* = -.2949$$

$$X^1 = (0,0) + (-.2949)(-2,0) = (.5898, 0)$$

continued...

2 continued

(b)

$$\nabla f(X) = C + 2X^TA$$

$$= (1 - 10X_1 - 6X_2 - X_3, \; 3 - 6X_1 - 4X_2, \; 5 - X_1 - X_3)$$

$$X^0 = (0, 0, 0)^T$$

$$\nabla f(X^0) = (1, 3, 5)$$

$$X = (1, 3, 5)r$$

$$h(r) = 35r + r^2(1, 3, 5)A\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

Optimal $r = .299145$

$$X^1 = (.299145, .897436, 1.495726)$$

$$\nabla f(X^1) = (-15.88, 2.84614, 3.205129)$$

$$X = X^1 + r \nabla f(X^1)$$

1

$f_1(x_1) = e^{-x_1} + x_1,\ g^1(x_1) = x_1^2$

$f_2(x_2) = (x_2+1)^2,\ g^2(x_2) = x_2$

k_1	$a_1^{k_1}$	$f_1(a_1^{k_1})$	$g_1(a_1^{k_1})$	Var
1	0	1	0	t_1^1
2	.5	1.1	.25	t_1^2
3	1.	1.37	1.	t_1^3
4	1.5	1.72	2.25	t_1^4
5	1.732	1.91	3.00	t_1^5

k_2	$a_2^{k_2}$	$f_2(a_2^{k_2})$	$g_2(a_2^{k_1})$	Var
1	0	1.	0	t_2^1
2	.5	2.25	.5	t_2^2
3	1.	4.	1.	t_2^3
4	1.5	6.25	1.5	t_2^4
5	2.	9.	2.	t_2^5
6	2.5	12.25	2.5	t_2^6
7	3.	16.	3.	t_2^7

$$\begin{aligned}\text{maximize } z \cong\ & t_1^1 + 1.1t_1^2 + 1.37t_1^3 + \\ & 1.72t_1^4 + 1.91t_1^5 + \\ & t_2^1 + 2.25t_2^2 + 4t_2^3 + \\ & 6.25t_2^4 + 9t_2^5 + 12.25t_2^6 + \\ & 16t_2^7\end{aligned}$$

Subject to

$$\begin{aligned}& .25t_1^2 + t_1^3 + 2.25t_1^4 + 3t_1^5 + \\ & .5t_2^2 + t_2^3 + 1.5t_2^4 + 2t_2^5 + 2.5t_2^6 + \\ & +3t_2^7 \le 3\end{aligned}$$

$0 \le t_1^1 \le y_1^1$

$0 \le t_1^2 \le y_1^1 + y_1^2$

$0 \le t_1^3 \le y_1^2 + y_1^3$

$0 \le t_1^4 \le y_1^3 + y_1^4$

$0 \le t_1^5 \le y_1^4$

$0 \le t_2^1 \le y_2^1$

$0 \le t_2^2 \le y_2^1 + y_2^2$

$0 \le t_2^3 \le y_2^2 + y_2^3$

$0 \le t_2^4 \le y_2^3 + y_2^4$

$0 \le t_2^5 \le y_2^4 + y_2^5$

$0 \le t_2^6 \le y_2^5 + y_2^6$

$0 \le t_2^7 \le y_2^6$

$t_2^1 + t_2^2 + t_2^3 + t_2^4 + t_2^5 + t_2^6 + t_2^7 = 1$

$t_1^1 + t_1^2 + t_1^3 + t_1^4 + t_1^5 = 1$

$y_1^i = (0,1) \quad i = 1, 2 \cdots, 5$

$y_2^i = (0,1) \quad i = 1, 2, \ldots, 7$

2

Use the formulation in Problem 1, less all the constraints in y_i^j. We use S_1, t_1^1, and t_2^1 as the starting basic solution mainly for simplicity and to avoid using artificial starting basic variables! This can be achieved by substituting out t_1^1 in the z-equation using

$$t_1^1 = 1 - t_1^2 - t_1^3 - t_1^4 - t_1^5$$

	t_1^1	t_1^2	t_1^3	t_1^4	t_1^5	t_2^1	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6	t_2^7	S_1	
Z	0	-.1	-.37	-.72	-.91	0	-1.25	-3	-5.25	-8	-11.25	-15	0	Z
S_1	0	.25	1	2.25	3	0	.5	1	1.5	2	2.5	3	1	3
t_1^1	1	1	1	1	1	0	0	0	0	0	0	0	0	1
t_2^1	0	0	0	0	0	1	1	1	1	1	1	1	0	1
Z	0	-.1	-.37	-.72	-.91	15	13.75	12	9.75	7	3.75	0	0	17
S_1	0	.25	1	2.25	3	-3	-2.5	-2	-1.5	-1	-.5	0	1	0
t_1^1	1	1	1	1	1	0	0	0	0	0	0	0	0	1
t_2^7	0	0	0	0	0	1	1	1	1	1	1	1	0	1
Z	0	0	.03	.18	.29	13.8	12.75	11.2	9.15	6.6	3.55	0	.4	17
t_1^2	0	1	4	9	12	-12	-10	-8	-6	-4	-2	0	4	0
t_1^1	1	0	-3	-8	-11	12	10	8	6	4	2	0	-4	1
t_2^7	0	0	0	0	0	1	1	1	1	1	1	1	0	1

$t_1^1 = 1,\quad t_2^7 = 1$

Optimal solution: $X_1 = 0,\ X_2 = 3,\ Z = 17$

3

Let $y = x_1x_2x_3$. Because this is a maximization problem, $y > 0$.

$\ln y = \ln x_1 + \ln x_2 + \ln x_3$

Maximize $z = y$

subject to

$$\begin{aligned}-\ln y + \ln x_1 + \ln x_2 + \ln x_3 &= 0 \\ x_1^2 + x_2 + x_3 &\le 4\end{aligned}$$

Which is separable.

$f_1(y) = y \qquad g_1(y_1) = -\ln y$

$g_1^1(x_1) = \ln x_1,\ g_1^2(x_2) = \ln x_2$

$g_2^1(X_1) = x_1^2,\ g_2^2(x_2) = x_2$

$g_1^3(x_3) = \ln x_3,\ g_2^3(x_3) = x_3$

Use $0 \le y \le 7$ and $0 \le x_i \le 4$ to determine the breaking points; then solve using restricted basis

Set 19.2a

4

Separability requires using the ln function to separate the products into single-variable functions. That is, $y_1 = x_1 x_2$ and $y_2 = x_1 x_3$. However, to ensure that ln(0) will not be encountered, we use the substitution

$$\left.\begin{aligned} w_1 &= x_1 + 1 \\ w_2 &= x_2 + 1 \\ w_3 &= x_3 + 1 \end{aligned}\right\} \Rightarrow w_1, w_2, w_3 > 0$$

Thus,

$$x_1 x_2 = w_1 w_2 - w_1 - w_2 + 1$$
$$x_1 x_3 = w_1 w_3 - w_1 - w_3 + 1$$

Let $v_1 = w_1 w_2$, $v_2 = w_1 w_3$. Hence,

$$x_1 x_2 = v_1 - w_1 - w_2 + 1$$
$$x_1 x_3 = v_2 - w_1 - w_3 + 1$$

where $\ln(v_1) = \ln(w_1) + \ln(w_2)$

$\ln(v_3) = \ln(w_1) + \ln(w_3)$

The problem is expressed as

Maximize $z = v_1 + v_3 - 2w_1 - w_2 + 1$

subject to

$$v_1 + v_2 - 2w_1 - w_2 \leq 9$$
$$\ln(v_1) - \ln w_1 - \ln w_2 = 0$$
$$\ln v_2 - \ln w_1 - \ln w_3 = 0$$
$$v_1, v_2, w_1, w_2, w_3 \geq 0$$

5

Let $y = e^{2x_1 + x_2^2} > 0$

$\ln y = 2x_1 + x_2^2$

Maximize $z = y + (x_3 - 2)^2$

subject to

$$\ln y - 2x_1 - x_2^2 = 0$$
$$x_1 + x_2 + x_3 \leq 6$$
$$y, x_1, x_2, x_3 \geq 0$$

6

$$w_1 = x_1 + 1$$
$$w_2 = x_2 + 1$$
$$w_3 = x_3 + 1$$

Next, $y_1 = e^{x_1 x_2}$

$\ln y_1 = x_1 x_2$

Now,

$$x_1 x_2 = w_1 w_2 - w_1 - w_2 + 1$$
$$= y_2 - w_1 - w_2 + 1$$

where

$\ln y_2 = \ln w_1 + \ln w_2$

Thus,

$$\left.\begin{aligned} \ln y_1 &= y_2 - w_1 - w_2 + 1 \\ \ln y_2 &= \ln w_1 + \ln w_2 \end{aligned}\right\} \quad ①$$

Next,

$$x_2^2 x_3 = (w_2 - 1)^2 (w_3 - 1)$$
$$= w_2^2 w_3 + w_3 - 2w_2 w_3 - w_2^2 + 2w_2 + 1$$

Let

$y_3 = w_2^2 w_3$, $y_4 = w_2 w_3$

Then $\ln y_3 = 2 \ln w_2 + \ln w_3$

$\ln y_4 = \ln w_2 + \ln w_3$

and

$$\left.\begin{aligned} x_2^2 x_3 &= y_3 + w_3 - 2y_4 - w_2^2 + 2w_2 + 1 \\ \ln y_3 &= 2 \ln w_2 + \ln w_3 \\ \ln y_4 &= \ln w_2 + \ln w_3 \end{aligned}\right\} \quad ②$$

also,

$$\left.\begin{aligned} x_2 x_3 &= w_2 w_3 - w_2 - w_2 + 1 \\ &= y_3 - w_2 - w_3 + 1 \\ \ln y_5 &= \ln w_2 + \ln w_3 \end{aligned}\right\} \quad ③$$

Finally,

$x_3 x_4 = x_3 x_4^+ - x_3 x_4^-$, $x_4^+, x_4^- \geq 0$

Put $y_6 = x_3 x_4^+$ and $y_7 = x_3 x_4^-$

and let $w_4^+ = 1 + x^+$

$w_4^- = 1 + x^-$

Thus,

$$\left.\begin{aligned} x_3 x_4^+ &= y_8 - w_3 - w_4^+ + 1 \\ \ln y_8 &= \ln w_3 + \ln w_4^+ \end{aligned}\right\} \quad ④$$

continued...

$$\left.\begin{array}{l} x_3 x_4^- = y_9 - w_3 - w_4^- + 1 \\ \ln y_9 = \ln w_3 + \ln w_4^- \end{array}\right\} \quad (5)$$

From (1) through (5), the problem becomes:

$$\text{Maximize } z = y_1 + y_3 + w_2 - 2y_4 - w_2^2 + 2w_1 + 1 + w_4^+ - w_4^-$$

Subject to

$$\ln y_1 = y_2 - w_1 - w_2 + 1$$
$$\ln y_2 = \ln w_1 + \ln w_2$$
$$\ln y_3 = 2\ln w_2 + \ln w_3$$
$$\left.\begin{array}{l} \ln y_4 = \ln w_2 + \ln w_3 \\ \ln y_5 = \ln w_2 + \ln w_3 \end{array}\right\} \text{same}$$
$$\ln y_8 = \ln w_3 + \ln w_4^+$$
$$\ln y_9 = \ln w_3 + \ln w_4^-$$
$$w_1 + y_5 - w_2 - w_3 + y_8 - y_9 - w_4^+ - w_4^- \leq 10$$
$$y_i \geq 0,\ w_i \geq 0, \text{ all } i \text{ and } j$$

7

$$b = a_{k-1,i} - a_{k-2,i}$$

$$\delta = \min\{b - x_{k-1,i},\ x_{ki}\}$$

It is feasible to subtract δ from x_{ki} and add it to $x_{k-1,i}$. The net change in the value of the objective function is

$$\Delta = \delta(\rho_{k-1,i} - \rho_{ki}) > 0$$

Because $\rho_{k-1,i} < \rho_{ki}$ (minimization), $\Delta < 0$. Thus, adding δ to $x_{k-1,i}$ leads to a smaller value of the objective function.

The end result is that it is never optimal to have positive x_{ki} if $x_{k-1,i}$ has not attained its upper limit $a_{k-1,i} - a_{ki}$

8

$$\text{Minimize } z = x_1^4 + 2x_2^+ - 2x_2^- + x_3^2$$

Subject to

$$x_1^2 + x_2^+ - x_2^- + x_3^2 \leq 4$$
$$x_1 + x_2^+ - x_2^- \leq 3$$
$$-x_1 - x_2^+ + x_2^- \leq 3$$
$$x_1, x_2^+, x_2^-, x_3 \geq 0$$

$f_1(x_1) = x_1^4$; $g_1^1(x_1) = x_1^2$, $g_1^2(x_1) = x_1$

$g_1^3(x_1) = -x_1$

$f_3(x_3) = x_3^2$: $g_3^1(x_3)\ x_3^2$

k_1	a_{k_1}	$f_1(a_k)$	ρ_{k1}	g_1^1	ρ_{k1}^1	g_1^2	$\rho_{k_1}^2$	g_1^3	ρ_{k1}^3
0	0	0	–	0	–	0	–	0	–
1	1	1	1	1	1	1	1	1	1
2	2	16	15	4	3	2	1	2	1
3	3	81	65	90	5	3	1	3	1

k_3	a_{k_3}	$f_3(a_{k_3})$	ρ_{k3}	g_3^1	ρ_{k3}^1
0	0	0	0	0	0
1	1	1	1	1	1
2	2	4	3	4	3
3	3	9	5	9	5

$$\text{Min } z = x_{11} + 15x_{12} + 65x_{13} + 2x_2 + x_{13} + 3x_{23} + 5x_{33}$$

Subject to

$$x_{11} + 3x_{12} + 5x_{13} + x_2^+ - x_2^- + x_{13} + 3x_{23} + 5x_{33} \leq 4$$
$$x_{11} + x_{12} + x_{13} + x_2^+ - x_2^- \leq 3$$
$$-x_{11} - x_{12} - x_{13} - x_2^+ - x_2^- \leq 3$$
$$0 \leq x_{ij} \leq 1 \quad ; i = 1, 3, j = 1, 2, 3$$
$$x_2^+, x_2^- \geq 0$$

Use simplex with upper bounding to determine the approximate optimum solution.

1

$$Z = (6,3)\begin{pmatrix}x_1\\x_2\end{pmatrix} + (x_1, x_2)\begin{pmatrix}-2 & -2\\-2 & -3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$D = \begin{pmatrix}-2 & -2\\-2 & -3\end{pmatrix}$$

Principal minor determinants : $-2, +2$

Negative definite $\Rightarrow$ Z is concave

<u>Constraints</u>:

$$\begin{pmatrix}1 & 1\\2 & 3\\-1 & 0\\0 & -1\end{pmatrix}\underline{X} - \begin{pmatrix}1\\4\\0\\0\end{pmatrix} \le \underline{0}, \quad \lambda S = UX = 0$$

X^T		λ^T		U^T		S^T		RHS
4	4	1	2	-1	0	0	0	6
4	6	1	3	0	-1	0	0	3
1	1	0	0	0	0	1	0	1
2	3	0	0	0	0	0	1	4

Basis	r	x_1	x_3	λ_1	λ_2	u_1	u_2	R_1	R_2	S_1	S_2	Sol'n
r	①	8	10	2	5	-1	-1	0	0	0	0	9
R_1	0	4	4	1	2	-1	0	①	0	0	0	6
R_2	0	4	6	1	3	0	-1	0	①	0	0	3
S_1	0	1	1	0	0	0	0	0	0	①	0	1
S_2	0	2	3	0	0	0	0	0	0	0	①	4
r	①	4/3	0	1/3	0	-1	2/3	0	-5/3	0	0	4
R_1	0	4/3	0	1/3	0	-1	2/3	①	-2/3	0	0	4
X_2	0	2/3	①	1/6	1/2	0	-1/6	0	1/6	0	0	1/2
S_1	0	1/3	0	-1/6	-1/2	0	1/6	0	-1/6	①	0	1/2
S_2	0	0	0	-1/2	-3/2	0	1/2	0	-1/2	0	①	5/2
r	①	0	-2	0	-1	-1	1	0	-2	0	0	3
R_1	0	0	-2	0	-1	-1	1	①	-1	0	0	3
x_1	0	①	3/2	1/4	3/4	0	-1/4	0	1/4	0	0	3/4
S_1	0	0	-1/2	-1/4	-3/4	0	1/4	0	-1/4	①	0	1/4
S_2	0	0	0	-1/2	-3/2	0	1/2	0	-1/2	0	①	5/2
r	①	0	0	1	2	-1	0	0	-1	-4	0	2
R_1	0	0	0	1	2	-1	0	①	0	-4	0	2
x_1	0	①	1	0	0	0	0	0	0	1	0	1
u_2	0	0	-2	-1	-3	0	①	0	-1	4	0	1
S_2	0	0	1	0	0	0	0	0	0	-2	①	2
r	①	0	0	0	0	0	0	-1	-1	0	0	0
λ_1	0	0	0	①	2	-1	0	1	0	-4	0	2
x_1	0	①	1	0	0	0	0	0	0	1	0	1
u_2	0	0	-2	0	-1	-1	①	1	-1	0	0	3
S_2	0	0	1	0	0	0	0	0	0	-2	①	2

continued...

1 continued

Optimum solution:

$x_1 = 1, \quad \lambda_1 = 2, \quad \mu_1 = 0, \quad S_1 = 0$

$x_2 = 0, \quad \lambda_2 = 0, \quad \mu_2 = 3, \quad S_2 = 0$

$z = 4$

2

Let $w = -z$. Then, the problem becomes

maximize

$$w = (-1, 3, 5)X + X^T\begin{pmatrix}-2 & -1 & 0\\-1 & -2 & -1\\0 & -1 & -3\end{pmatrix}X$$

Subject to

$$\begin{pmatrix}-1 & -1 & -1\\3 & 2 & 1\end{pmatrix}X \le \begin{pmatrix}-1\\6\end{pmatrix}$$

$$D = \begin{pmatrix}-2 & -1 & 0\\-1 & -2 & -1\\0 & -1 & -3\end{pmatrix}$$

Principal minor determinants = $-2, 3, -7 \Rightarrow$ negative definite $\Rightarrow w$ is concave

Necessary conditions:

$$\begin{bmatrix}4 & 2 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0\\2 & 4 & 2 & -1 & 2 & 0 & -1 & 0 & 0 & 0\\0 & 2 & 6 & -1 & 1 & 0 & 0 & -1 & 0 & 0\\-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X\\\lambda\\U\\S\end{bmatrix} = \begin{bmatrix}-1\\3\\5\\-1\\6\end{bmatrix}$$

$\lambda S = 0 \qquad UX = 0$

Optimal solution:

$x_1 = 0, \quad x_2 = .4, \quad x_3 = .7$

Set 19.2c

1

Transformed problem:

Maximize $z = x_1 + 2x_2 + 5x_3$

Subject to

$$2x_1 + 3x_2 + 5x_3 + 1.28\,y \le 10$$

$$9x_1^2 + 16x_3^2 - y^2 = 0$$

$$7x_1 + 5x_2 + x_3 \le 12.4$$

$$x_1, x_2, x_3, y \ge 0$$

2

Transformed problem:

Maximize $z = x_1 + x_2^2 + x_3$

Subject to

$$x_1^2 + 5x_2^2 + 2\sqrt{x_3} + 1.28y \le 10$$

$$16x_2^2 + 25x_3 - y^2 = 0$$

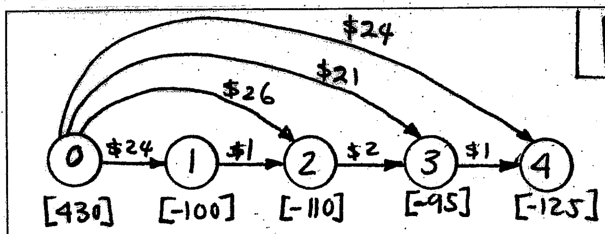
$$x_1, x_2, x_3, y \ge 0$$

Chapter 20

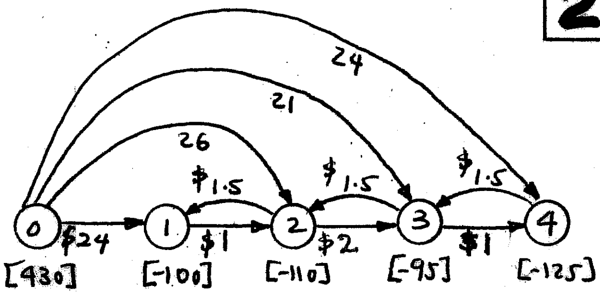
Additional Network and LP algorithms

Set 20.1a

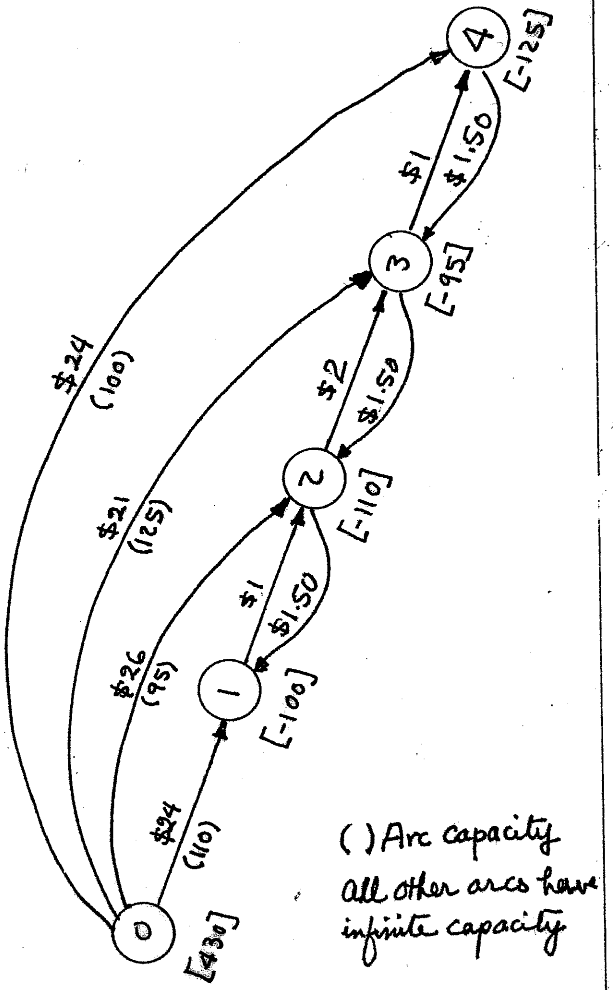
1



2

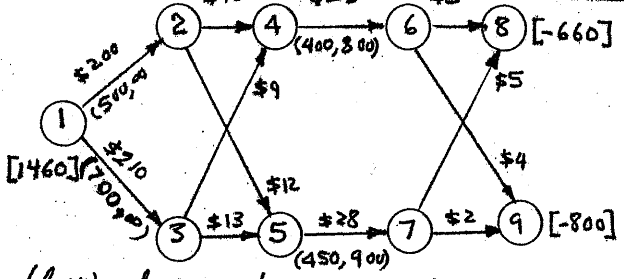


3



( ) Arc capacity
All other arcs have infinite capacity

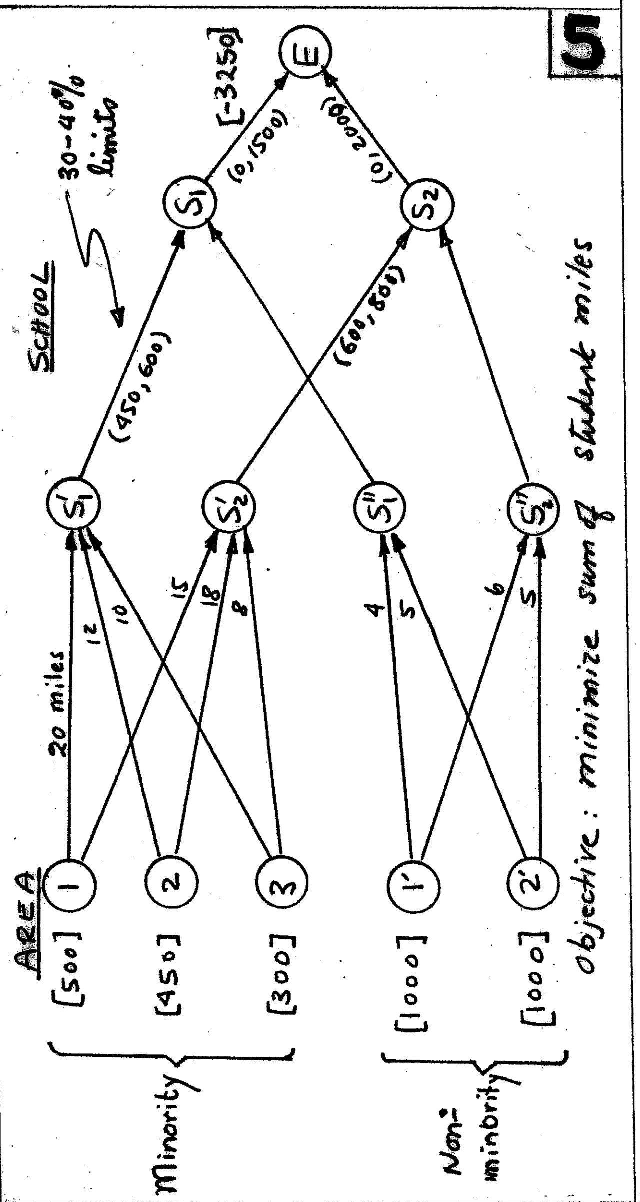
4



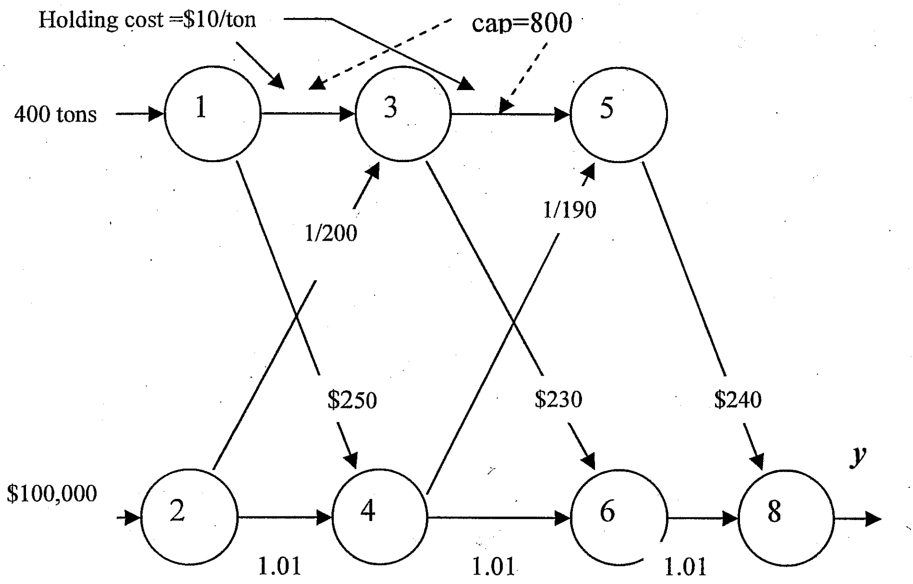
(l,u) = lower and upper bounds of arc flow

All other arcs have (0,∞) bounds.

5



objective: minimize sum of student miles



Let

$$x_{ij} = \begin{cases} \text{Tons from node i to node} j,\ i = 1, 3, 5, j = 4, 6, 8 \\ \text{Dollars from node i to node} j,\ i = 2, 4, 6, 8, j = 3, 5 \end{cases}$$

y = Total revenue

The associated LP is

$$\text{Maximize } z = y - 10(x_{13} + x_{35} + x_{57})$$

subjectto

$$x_{13} + x_{14} = 400$$

$$x_{13} + x_{23} / 200 = x_{35} + x_{36}$$

$$x_{35} + x_{45} / 190 = x_{58}$$

continued...

$$x_{23} + x_{24} = 100000$$
$$1.01x_{24} + 250x_{14} = x_{45}/190 + x_{46}$$
$$1.01x_{46} + 230x_{36} = x_{68}$$
$$1.01x_{68} + 240x_{58} = y$$
$$x_{13} \leq 800$$
$$x_{35} \leq 800$$
$$x_{ij} \geq 0, \text{ for all } i \text{ and } j$$

Optimum solution: $z = \$48{,}240{,}000$
$x_{14} = 400$ tons, x_{58}=20,100 tons
$x_{24} = \$100{,}000$, $x_{45} = \$38{,}190{,}000$

1

x_{ij} = Flow amount from node i to node j

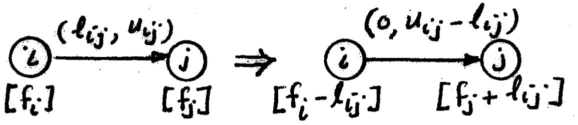
Case 1: No substitution of lower bounds

	x_{12}	x_{13}	x_{24}	x_{32}	x_{34}	
min Z	1	5	3	4	6	
node 1:	1	1				= 50
node 2:	-1		1	-1		= -40
node 3:		-1		1	1	= 20
node 4:			-1		-1	= -30
lower bd:	0	30	10	10	0	
upper bd:	∞	40	∞	∞	∞	

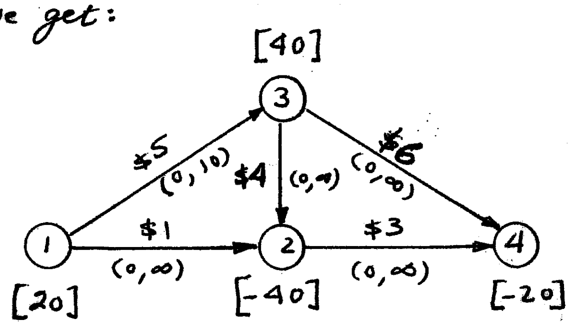
Optimum: $x_{12} = 20$, $x_{13} = 30$, $x_{24} = 10$, $x_{32} = 30$, $x_{34} = 20$

$z = \$440$

Case 2: Lower bounds substituted directly on network using the following rule:



Applying this rule to the network we get:



	x'_{12}	x'_{13}	x'_{24}	x'_{32}	x'_{34}	
min Z	1	5	3	4	6	
node 1	1	1				= 20
node 2	-1		1	-1		= -40
node 3		-1		1	1	= 40
node 4			-1		-1	= -20
upper bd	∞	10	∞	∞	∞	

where $x_{12} = x'_{12}$ $\quad x_{13} = x'_{13} + 30$

$x_{24} = x'_{24} + 10$ $\quad x_{32} = x'_{32} + 10$

$x_{34} = x'_{34}$

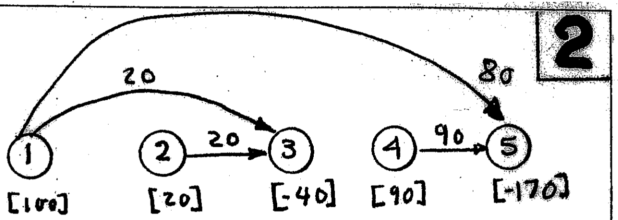
Optimum: $z = \$440$

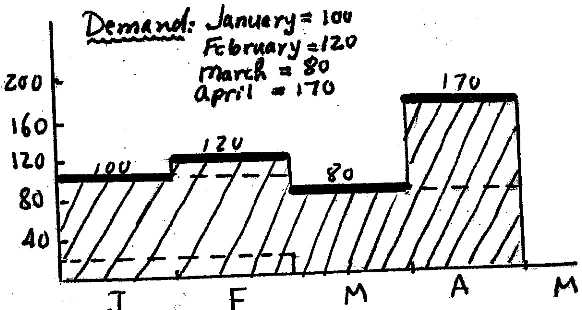
$x_{12} = x'_{12} = 20$ $\quad x_{32} = x'_{32} + 10 = 10 + 10 = 20$

$x_{13} = x'_{13} + 30 = 30$ $\quad x_{34} = x'_{34} = 20$

$x_{24} = x'_{24} + 10 = 10$

2





$x_{13} = 20$: hire 20 at the start of January for termination at start of March

$x_{23} = 20$: hire 20 at the start of February for termination at the start of March

$x_{15} = 80$: hire 80 at the start of January for termination at the start of May

$x_{45} = 90$: hire 90 at the start of April for termination at the start of May.

3

x_{ij} = # hired at the start of i and terminated at the start of j

$j \geq i+1$

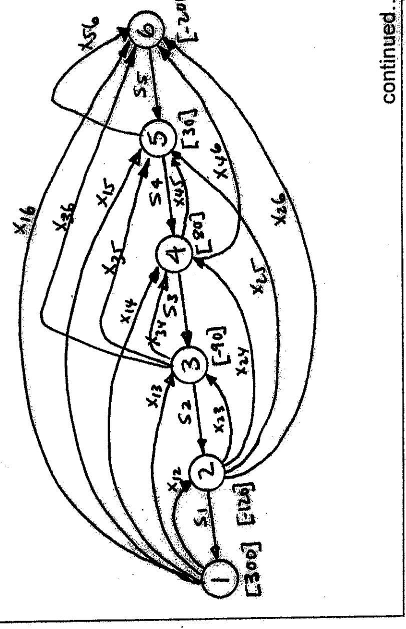
	x_{13}	x_{14}	x_{15}	x_{24}	x_{25}	x_{35}	S_1	S_2	S_3	S_4	
	130	180	220	130	180	130					
Jan	1	1	1				-1				= 100
Feb	1	1	1	1	1			-1			= 120
Mar		1	1	1	1	1			-1		= 80
Apr.			1		1	1				-1	= 170
Jan	1	1	1				-1				= 100
Feb				1	1		1	-1			= 20
Mar	-1					1		1	-1		= -40
Apr.		-1		-1					1	-1	= 90
May			-1		-1					1	= -170

Set 20.1b

4

(a)

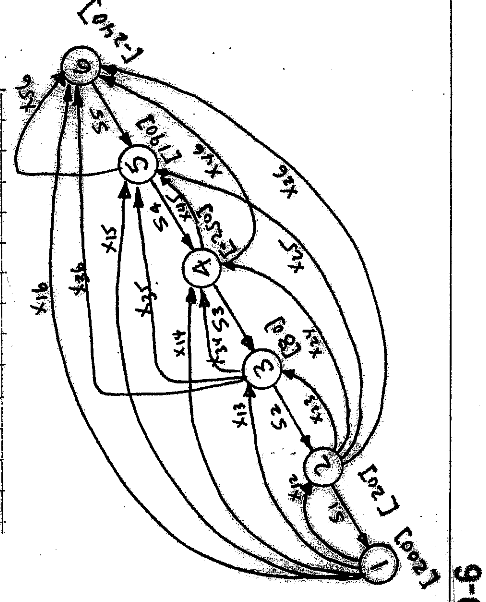
	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	S_1	S_2	S_3	S_4	S_5	
Min z =	50	70	85	100	130	50	70	85	100	50	70	85	50	70	50						
①	1	1	1	1	1											-1					300
②		1	1	1	1	1	1	1	1								-1				180
③			1	1	1		1	1	1	1	1	1						-1			90
④				1	1			1	1		1	1	1	1					-1		170
⑤					1				1			1		1	1					-1	200
	50	70	85	100	130	50	70	85	100	50	70	85	50	70	50						
①	1	1	1	1	1											-1					300
②	-1					1	1	1	1							1	-1				-120
③		-1				-1				1	1	1					1	-1			-90
④			-1				-1			-1			1	1				1	-1		80
⑤				-1				-1			-1		-1		1				1	-1	30
⑥					-1				-1			-1		-1	-1					1	-200



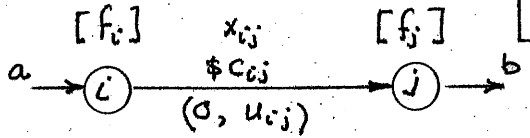
continued...

(b)

	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	S_1	S_2	S_3	S_4	S_5	
Min z =	50	70	85	100	130	50	70	85	100	50	70	85	50	70	50						
①	1	1	1	1	1											-1					200
②		1	1	1	1	1	1	1	1								-1				220
③			1	1	1		1	1	1	1	1	1						-1			300
④				1	1			1	1		1	1	1	1					-1		50
⑤					1				1			1		1	1					-1	240
	50	70	85	100	130	50	70	85	100	50	70	85	50	70	50						
①	1	1	1	1	1											-1					200
②	-1					1	1	1	1							1	-1				20
③		-1				-1				1	1	1					1	-1			80
④			-1				-1			-1			1	1				1	-1		-250
⑤				-1				-1			-1		-1		1				1	-1	190
⑥					-1				-1			-1		-1	-1					1	-240



5



Let a represent the total flow of incoming arcs at node i and b the total flow of outgoing arcs at node j.

$$x_{ij} \le u_{ij} \Rightarrow x_{ij} + x'_{ij} = u_{ij}\ ,\ x'_{ij} \ge 0 \quad ①$$

Node i:

$$a + f_i = x_{ij} \quad ②$$

Node j:

$$x_{ij} + f_j = b \quad ③$$

Thus,

$$u_{ij} - x'_{ij} - b = -f_j$$

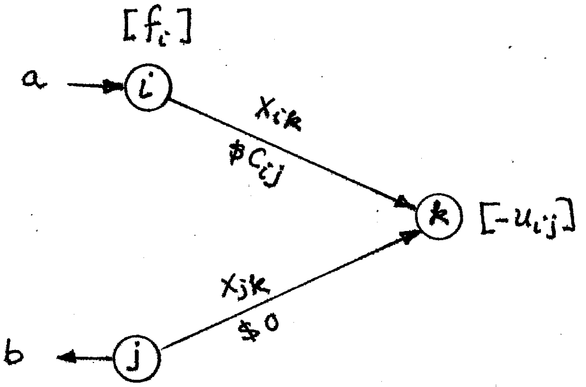
or

$$x'_{ij} + b = u_{ij} + f_j \quad ④$$

Letting

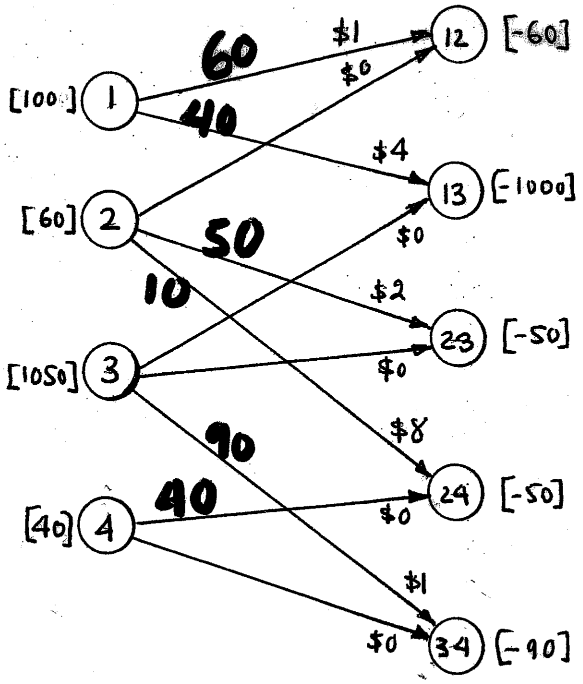
$$x_{ik} = x_{ij}$$
$$x_{jk} = x'_{ij}$$

equations ①, ②, and ④ produce the following equivalent network

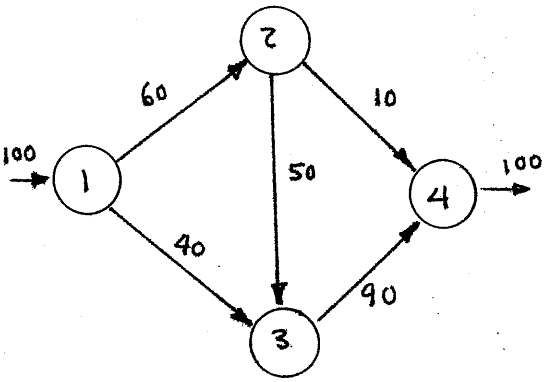


continued...

Application of the transformation to the network in Figure 6-42, we get



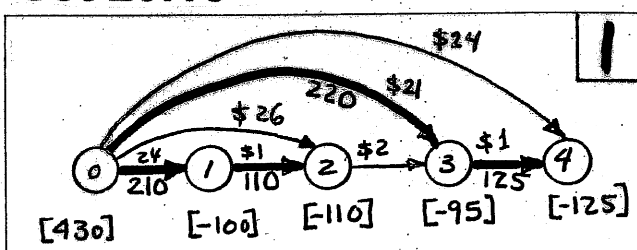
Optimum solution is obtained by using TORA's transportation model, and is shown in bold on the arcs. This solution is translated in terms of the original network as follows:



Total cost = \$490

Set 20.1c

1



The spanning tree shown by heavy arcs gives a starting basic feasible solution. We compute the dual values w_i, $i = 0, 1, \cdots, 4$ as follows:

$$w_0 \equiv 0$$

$$w_0 - w_1 = 24 \Rightarrow w_1 = -24$$

$$w_1 - w_2 = 1 \Rightarrow w_2 = -25$$

$$w_0 - w_3 = 21 \Rightarrow w_3 = -21$$

$$w_3 - w_4 = 1 \Rightarrow w_4 = -22$$

Evaluation of the nonbasic arcs:

$$z_{02} - c_{02} = w_0 - w_2 - c_{02}$$

$$= 0 - (-25) - 26 = -1$$

$$z_{04} - c_{04} = 0 - (-22) - 24 = -2$$

$$z_{23} - c_{23} = -25 - (-21) - 2 = -6$$

The given spanning tree solution is optimal.

Transshipment solution:

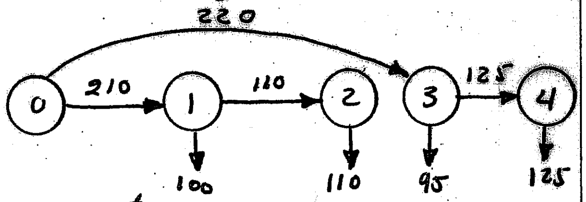
Since there are no finite upper bounds, the problem can be solved directly as a transshipment model

$\boxed{B = 430}$

	1	2	3	4	
0	24 210	26	21 220	24	430
1	0 320	1 110	M	M	B
2	M	0 430	2	m	B
3	m	m	0 305	1 125	B
	100+B	110 + B	95+B	125	

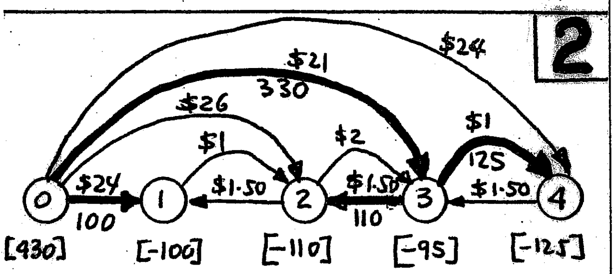
continued...

Solution Summary:



cost = $9,895

2



$$w_0 \equiv 0$$

$$w_1 = -24, \; w_2 = -22.5, \; w_3 = -21, \; w_4 = -22$$

$$z_{02} - c_{02} = 0 - (-22.5) - 26 = -3.5$$

$$z_{04} - c_{04} = 0 - (-22) - 24 = -2$$

$$z_{12} - c_{12} = -24 - (-22.5) - 1 \doteq -2.5$$

$$z_{21} - c_{21} = -22.5 - (-24) - 1.5 = 0$$

$$z_{23} - c_{23} = -22.5 - (-21) - 2 = -3.5$$

optimum!

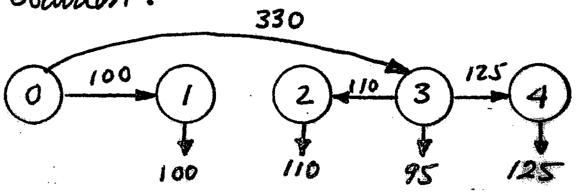
Transshipment Solution:

	1	2	3	4	
0	24 100	26	21 330	24	430
1	0 430	1	M	M	B
2	1.5	0 430	2	M	B
3	M	1.5 110	0 195	1 125	B
4	M	M	1.5	0 430	B
	100+B	110+B	95+B	125+B	

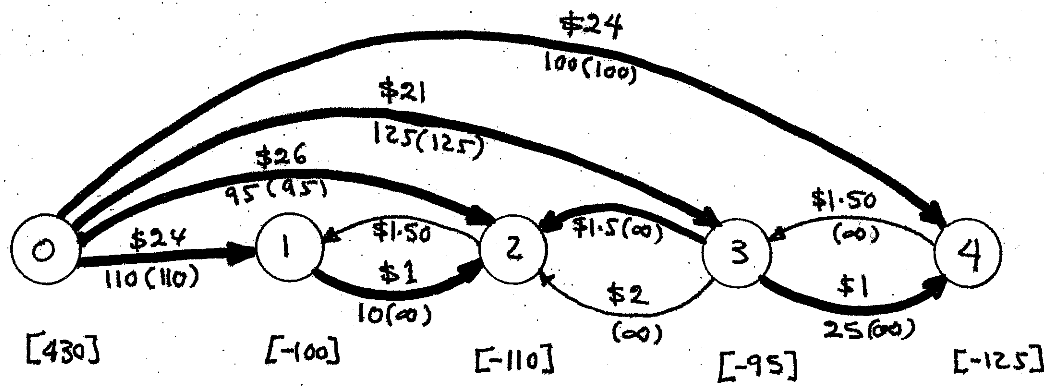
$\boxed{B = 430}$

Cost = $9,620

Summary of the optimum solution:



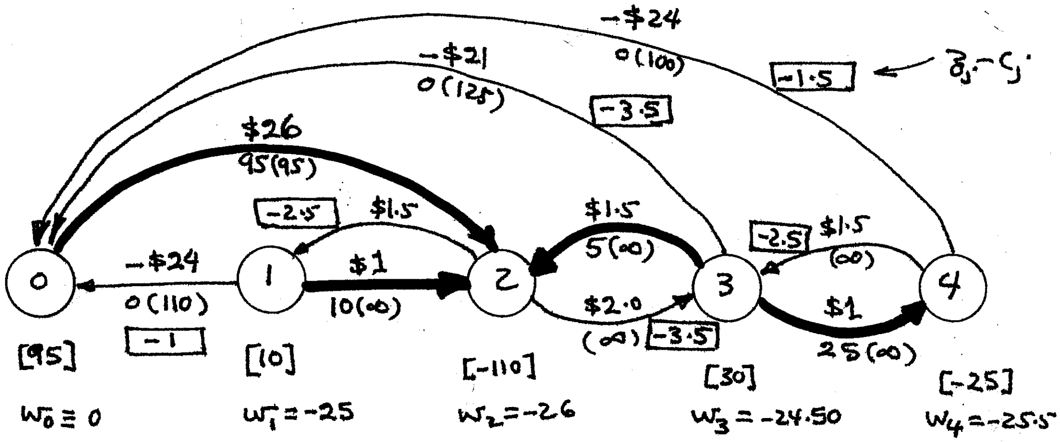
3



This solution is not basic because it does not comprise a spanning tree. To convert it into a spanning tree, substitute out arcs 0-1, 0-3, and 0-4 at upper bound—that is,

$$x_{01} = 110 - x_{10}, \quad x_{03} = 125 - x_{30}, \quad x_{04} = 100 - x_{40}$$

Arcs x_{10}, x_{30} & x_{40} are now nonbasic at zero level.



Optimum solution because all $z_j - c_j \le 0$

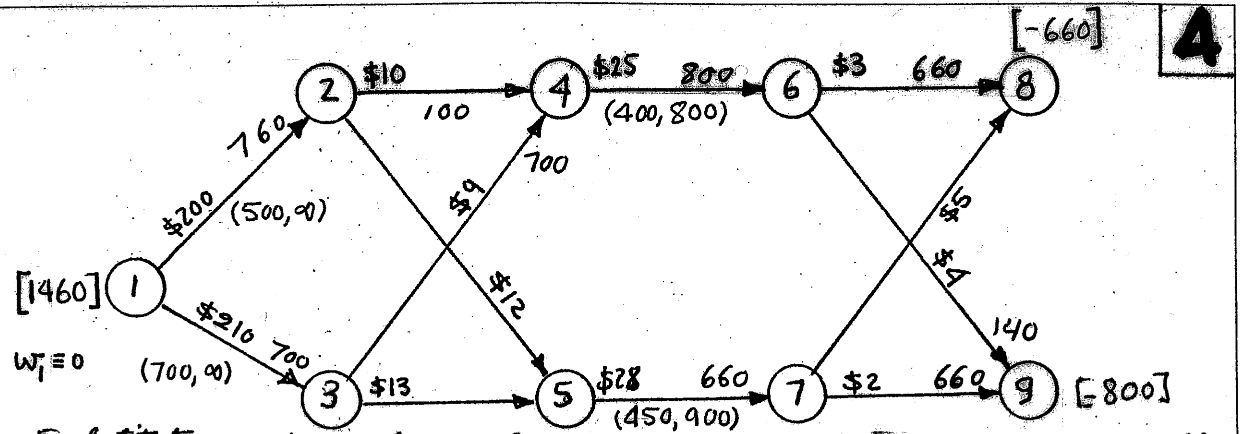
Optimum solution summary:

Period	Production	Demand		Surplus
1	110	100	10	10
2	95	110	5	-15
3	125	95	25	30
4	100	125		-25

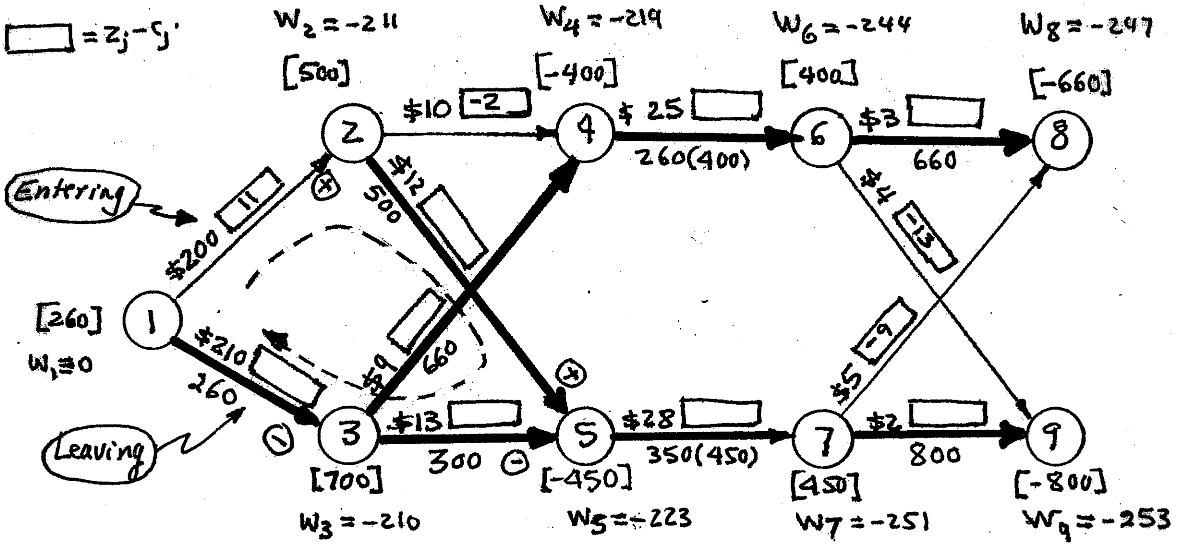
Total cost = $10,177.50

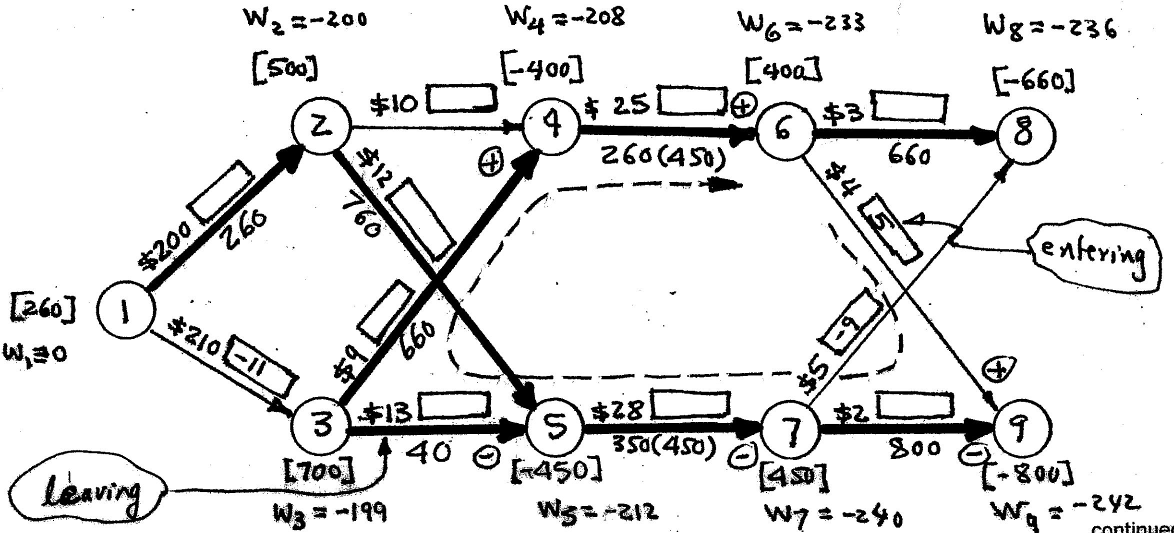
Set 20.1c

4



Substituting out the lower bounds, the following iterations will result:





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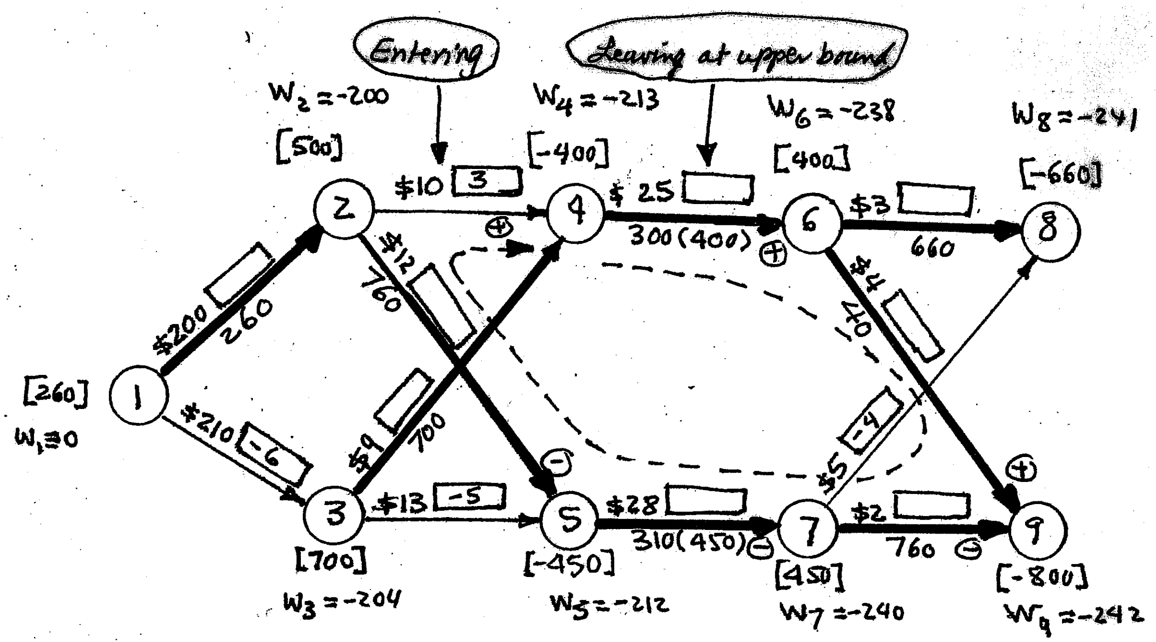
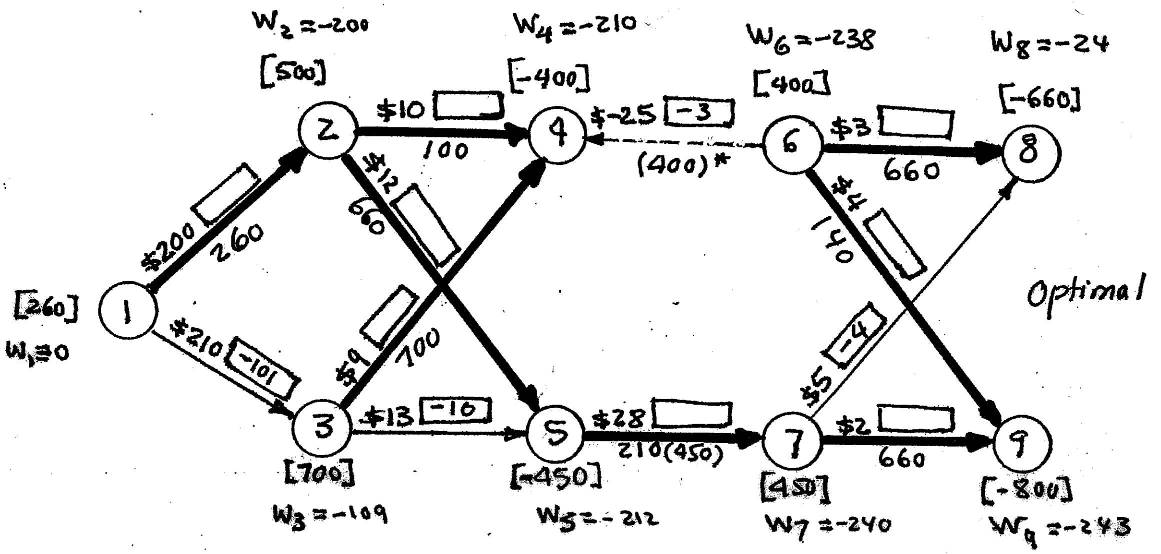




Optimal Solution:

$x_{12} = 260 + 500 = 760$, $x_{13} = 0 + 700 = 700$

$x_{24} = 100$, $x_{25} = 660$

$x_{34} = 700$, $x_{35} = 0$

$x_{46} = 400 + 400 = 800$, $x_{56} = 210$, $x_{57} = 210 + 450 = 660$

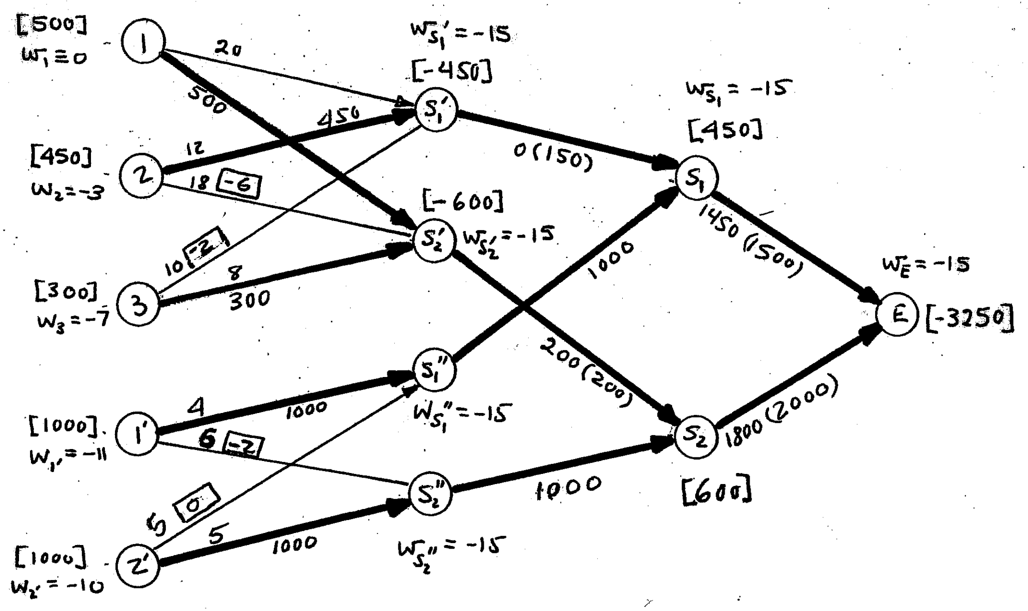
$x_{68} = 660$, $x_{69} = 140$

$x_{78} = 0$, $x_{79} = 660$

Total cost = \$356,560

5

Network after substituting out lower bounds:

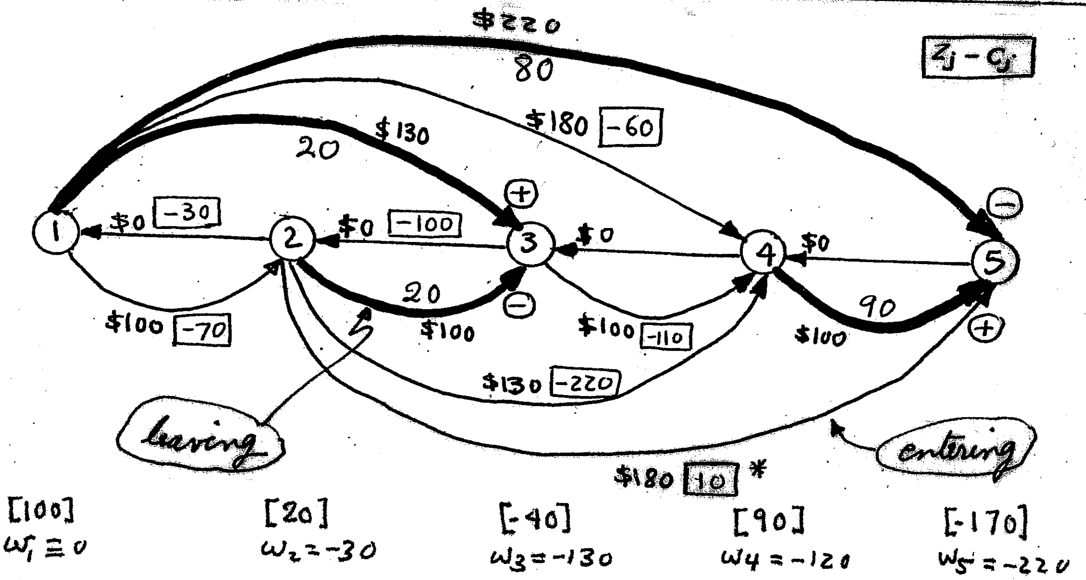


The given solution is optimal. Total student miles = 24,300

From area	To school	number of students
1	S2	500
2	S1	450
3	S2	300
1'	S1	1000
2'	S2	1000

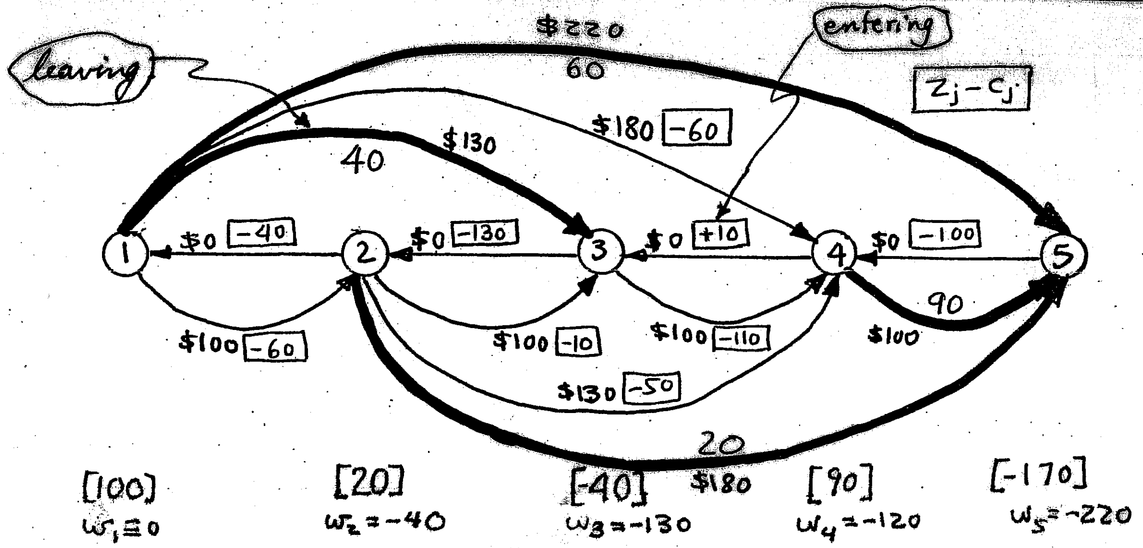
Problem has alternative optima

6

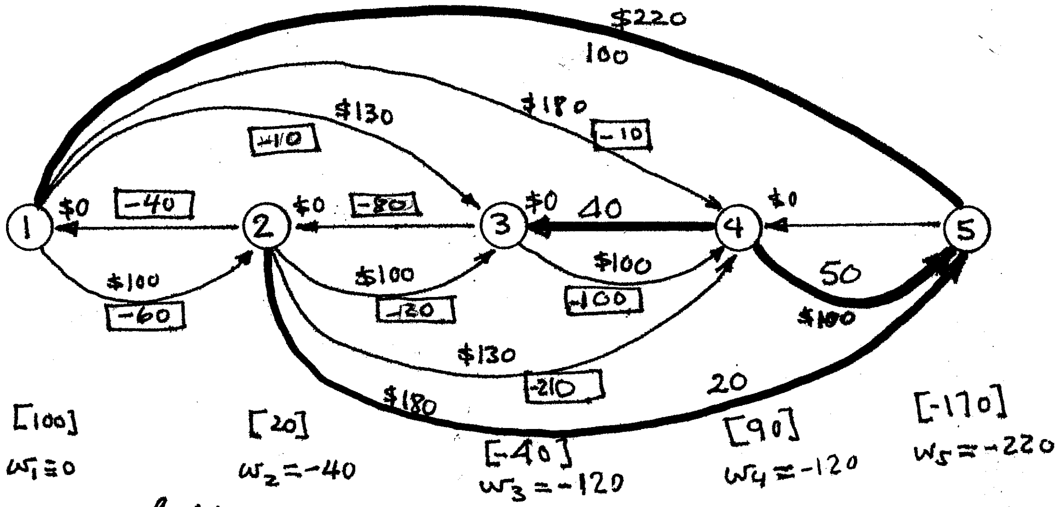


Loop:	(2,5) →	(1,5) →	(1,3) →	(2,3) ← leaving
Sign:	+	−	+	−
Flow:	Entering	−80	20	$\boxed{-20}$

continued...



Loop:	(4,3) →	(1,3) →	(1,5) →	(4,5)	
Sign	+	−	+	−	
flow	Entering	㊵	60	90	(1,3)-leaves



Optimum solution:

$X_{15} = 100$: Hire 100 at the start of period 1 for the entire horizon

$X_{25} = 20$: Hire 20 at the start of period 2 for the end of the horizon

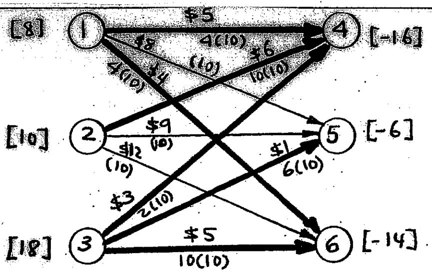
$X_{45} = 50$ Hire 50 at the start of period 4 for one period only

$X_{43} = 40$ means that period 4 (March) will carry a surplus of 40 workers (=120−80=40).

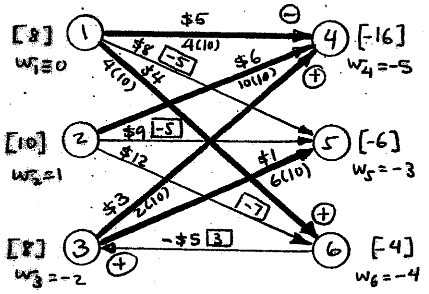
Total cost = $30,600

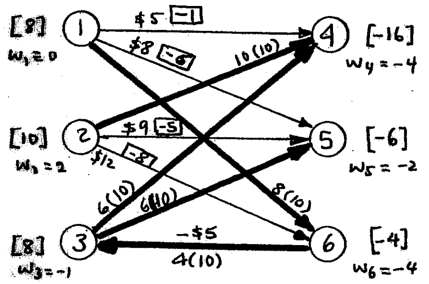
Set 20.1c

7



Substitute $x_{36} = 10 - x_{63}$ to produce a spanning tree. $C_{63} = -\$5$





Optimum solution: Cost \$146

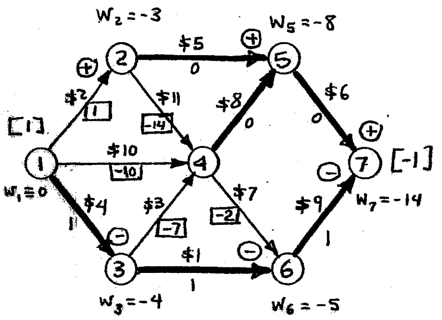
$x_{16} = 8$,

$x_{24} = 10$

$x_{34} = 6$

$x_{35} = 6$, $x_{36} = 10 - x_{63} = 10 - 4 = 6$

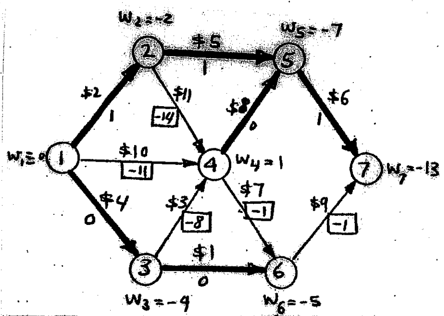
8



x_{12} enters the basis

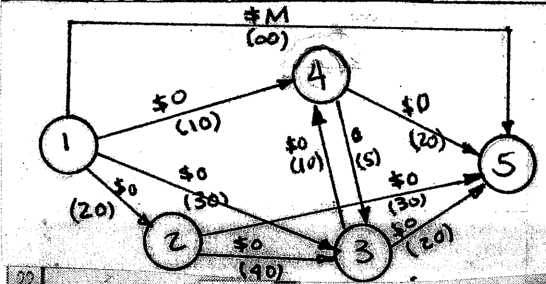
x_{13}, x_{36}, or x_{67} can leave. Select x_{67}

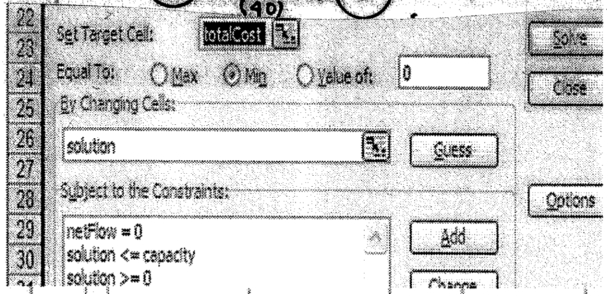
continued...



Optimum: Route 1-2-5-7 Distance = 13

9





Set Target Cell: totalCost

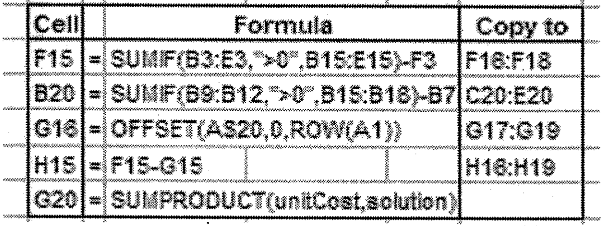
Equal To: ○ Max ● Min ○ Value of: 0

By Changing Cells:

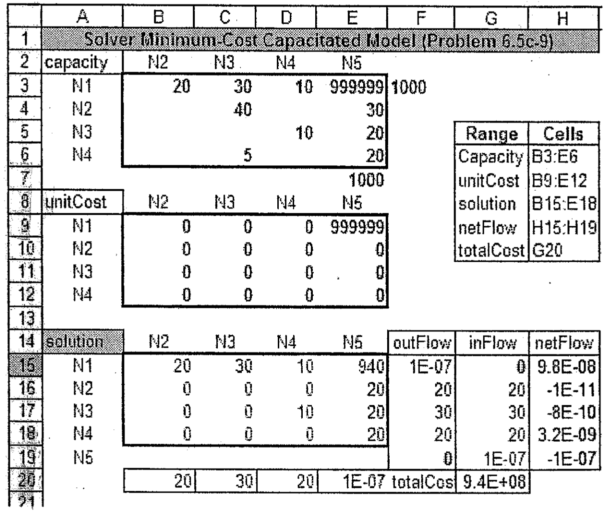
solution

Subject to the Constraints:

netFlow = 0
solution <= capacity
solution >= 0



Cell		Formula	Copy to
F15	=	SUMIF(B3:E3,">0",B15:E15)-F3	F16:F18
B20	=	SUMIF(B9:B12,">0",B15:B18)-B7	C20:E20
G16	=	OFFSET(A$20,0,ROW(A1))	G17:G19
H15	=	F15-G15	H16:H19
G20	=	SUMPRODUCT(unitCost,solution)	



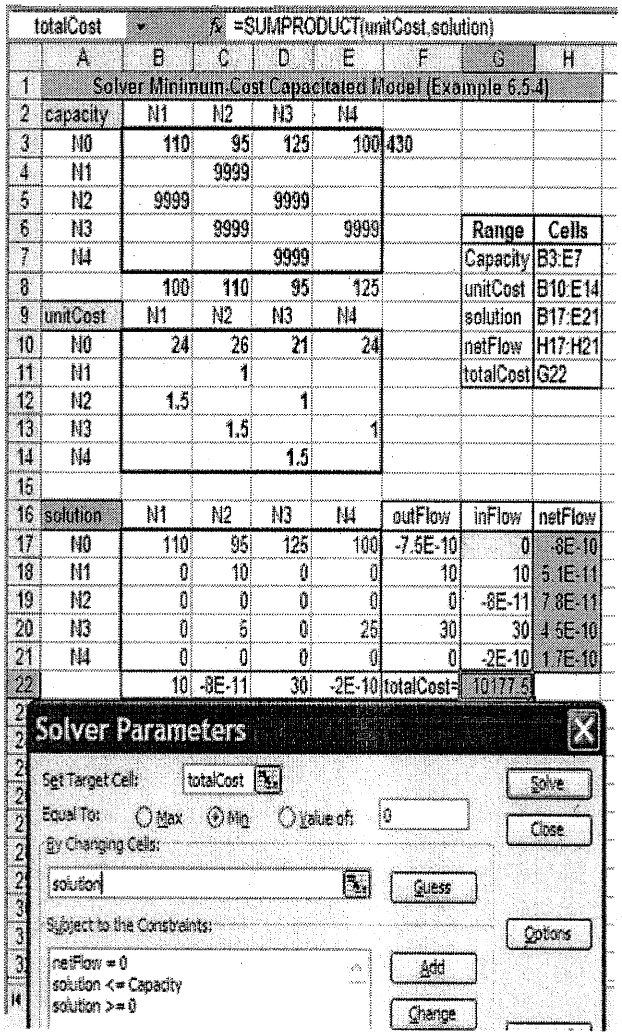
	A	B	C	D	E	F	G	H
1	Solver Minimum-Cost Capacitated Model (Problem 6.5c-9)							
2	capacity	N2	N3	N4	N5			
3	N1	20	30	10	999999	1000		
4	N2		40		30			
5	N3			10	20		Range	Cells
6	N4		5		20		Capacity	B3:E6
7					1000		unitCost	B9:E12
8	unitCost	N2	N3	N4	N5		solution	B15:E18
9	N1	0	0	0	999999		netFlow	H15:H19
10	N2	0	0	0	0		totalCost	G20
11	N3	0	0	0	0			
12	N4	0	0	0	0			
13								
14	solution	N2	N3	N4	N5	outFlow	inFlow	netFlow
15	N1	20	30	10	940	1E-07	0	9.8E-08
16	N2	0	0	0	20	20	20	-1E-11
17	N3	0	0	10	20	30	30	-8E-10
18	N4	0	0	0	20	20	20	3.2E-09
19	N5					0	1E-07	-1E-07
20		20	30	20	1E-07	totalCos	9.4E+08	
21								

continued...

Optimal solution:
N1-N2 = 20, N1-N3 = 30, N1-N4 = 10,
N2-N5 = 20, N3-N4 = 10,N3-N5 = 20,
N4-N5 = 20. Maximum flow = 60

10

(a) AMPL: See file amplProb6.5c-10a.txt.
Solver:



totalCost fx =SUMPRODUCT(unitCost,solution)

	A	B	C	D	E	F	G	H
1	Solver Minimum-Cost Capacitated Model (Example 6.5-4)							
2	capacity	N1	N2	N3	N4			
3	N0	110	95	125	100	430		
4	N1		9999					
5	N2	9999		9999				
6	N3		9999		9999		Range	Cells
7	N4			9999			Capacity	B3:E7
8		100	110	95	125		unitCost	B10:E14
9	unitCost	N1	N2	N3	N4		solution	B17:E21
10	N0	24	26	21	24		netFlow	H17:H21
11	N1		1				totalCost	G22
12	N2	1.5		1				
13	N3		1.5		1			
14	N4			1.5				
15								
16	solution	N1	N2	N3	N4	outFlow	inFlow	netFlow
17	N0	110	95	125	100	-7.5E-10	0	-8E-10
18	N1	0	10	0	0	10	10	5.1E-11
19	N2	0	0	0	0	0	-8E-11	7.8E-11
20	N3	0	5	0	25	30	30	4.5E-10
21	N4	0	0	0	0	0	-2E-10	1.7E-10
22		10	-8E-11	30	-2E-10	totalCost=	10177.5	

Solver Parameters

Set Target Cell: totalCost

Equal To: ○ Max ● Min ○ Value of: 0

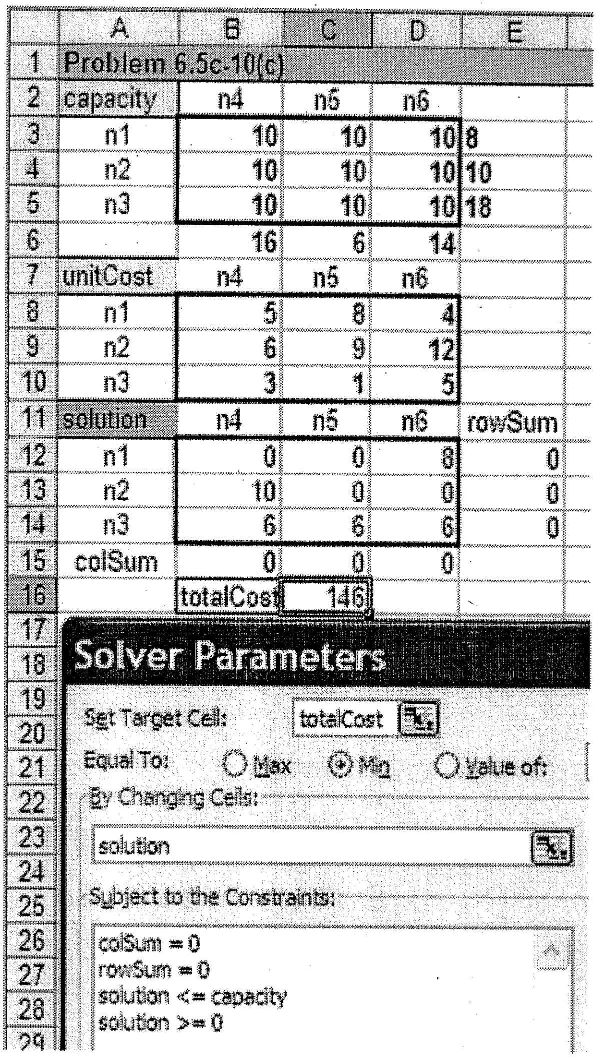
By Changing Cells: solution

Subject to the Constraints:
netFlow = 0
solution <= Capacity
solution >= 0

Optimum solution:
Period 1: Produce 110, surplus 10
Period 2: Produce 95, shortage 15
period 3: Produce 125, surplus 30
Period 4: Produce 100, shortage 25
Total shortage = 15+25=40
Total surplus = 10+30=40
Cost = $10,177.50

continued...

(c) Solver



	A	B	C	D	E
1	Problem 6.5c-10(c)				
2	capacity	n4	n5	n6	
3	n1	10	10	10	8
4	n2	10	10	10	10
5	n3	10	10	10	18
6		16	6	14	
7	unitCost	n4	n5	n6	
8	n1	5	8	4	
9	n2	6	9	12	
10	n3	3	1	5	
11	solution	n4	n5	n6	rowSum
12	n1	0	0	8	0
13	n2	10	0	0	0
14	n3	6	6	6	0
15	colSum	0	0	0	
16		totalCost	146		

Solver Parameters

Set Target Cell: totalCost

Equal To: ○ Max ● Min ○ Value of:

By Changing Cells: solution

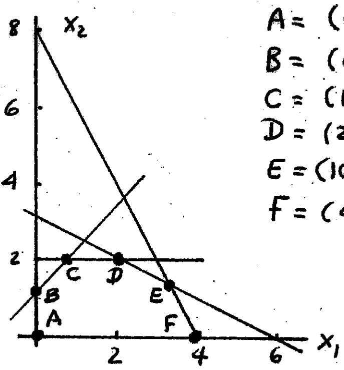
Subject to the Constraints:
colSum = 0
rowSum = 0
solution <= capacity
solution >= 0

Optimum solution:
x16 = 8
x24 = 10
x34 = 6
x35 = 6
x36 = 6
Cost = $146

Set 20.2a

1

(a)



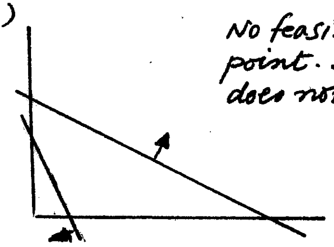
A = (0, 0)
B = (0, 1)
C = (1, 2)
D = (2, 2)
E = (10/3, 4/3)
F = (4, 0)

$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(0,1) + \alpha_3(1,2) + \alpha_4(2,2) + \alpha_5(10/3, 4/3) + \alpha_6(4,0)$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_6 = 1$$

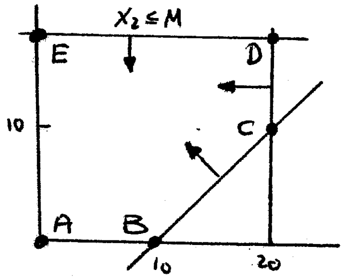
$$\alpha_j \geq 0, \quad j = 1, 2, \ldots, 6$$

(b)



No feasible extreme point. Solution space does not exist.

(c)



A = (0, 0)
B = (10, 0)
C = (20, 10)
D = (20, M)
E = (0, M)

M >>

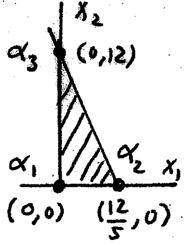
$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(10,0) + \alpha_3(20,10) + \alpha_4(20,M) + \alpha_5(0,M)$$

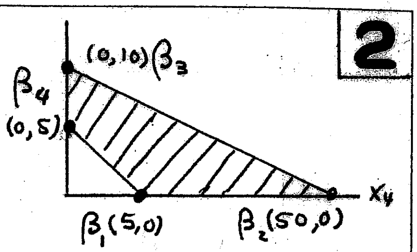
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$$

$$\alpha_j \geq 0, \quad j = 1, 2, \ldots, 5$$

continued...

2





$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(12/5, 0) + \alpha_3(0,12) = (12/5\,\alpha_2 + 12\alpha_3)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_1, \alpha_2, \alpha_3 \geq 0$$

$$(x_4, x_5) = \beta_1(5,0) + \beta_2(50,0) + \beta_3(0,10) + \beta_4(0,5)$$

$$= (5\beta_1 + 50\beta_2,\ 10\beta_3 + 5\beta_4)$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1, \quad \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$$

Thus, the original problem can be expressed in terms of the α's and the β's as

$$\text{Max } z = (0, \tfrac{36}{5}, 60, 5, 50, 10, 5)(\alpha_1, \alpha_2, \alpha_3, \beta_1, \ldots, \beta_4)^T$$

subject to

$$\begin{pmatrix} 0 & 12/5 & 12 & 5 & 50 & 10 & 5 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \vdots \\ \beta_4 \\ S_1 \end{pmatrix} = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$$

$$\alpha_i, \beta_j, S_1 \geq 0$$

M-method: M = 1000

Iteration 1:

Iterat	alfa1	alfa2	alfa3	beta1	beta2	beta3	beta4				
Basic	x1	x2	x3	x4	x5	x6	x7	sx8	Rx9	Rx10	Solution
z (max)	-1000.00	-1007.20	-1060.00	-1005.00	-1050.00	-1010.00	-1005.00	0.00	0.00	0.00	-2000.00
sx8	0.00	2.40	12.00	5.00	50.00	10.00	5.00	1.00	0.00	0.00	40.00
Rx9	1.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	1.00
Rx10	0.00	0.00	0.00	1.00	1.00	1.00	1.00	0.00	0.00	1.00	1.00

$$\min_{\alpha_i}\{z_j - c_j\} = -1060 = -60 - M, \text{ corresponds to } \alpha_3$$

$$\min_{\beta_j}\{z_j - c_j\} = -1050 = -50 - M, \text{ corresponds to } \beta_2$$

Thus, α_3 enters solution (its extreme pt is (0,12))

Iteration 2:

Iterat	alfa1	alfa2	alfa3	beta1	beta2	beta3	beta4				
Basic	x1	x2	x3	x4	x5	x6	x7	sx8	Rx9	Rx10	Solution
z (max)	60.00	52.80	0.00	-1005.00	-1050.00	-1010.00	-1005.00	0.00	1060.00	0.00	-940.00
sx8	-12.00	-9.60	0.00	5.00	50.00	10.00	5.00	1.00	-12.00	0.00	28.00
x3	1.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	1.00
Rx10	0.00	0.00	0.00	1.00	1.00	1.00	1.00	0.00	0.00	1.00	1.00

$$\min_{\alpha_i}\{z_j - c_j\} = 0, \text{ corresponds to } \alpha_3$$

continued...

$\min_{\beta_j}\{z_j - c_j\} = -1050 = -50-M$, Corresponds to β_2

Thus, β_2 enters solution (its extreme point is $(50,0)$)

Iteration 3:

Iterat	alfa1	alfa2	alfa3	beta1	beta2	beta3
Basic	x1	x2	x3	x4	x5	x6
z (max)	-192.00	-148.80	0.00	-900.00	0.00	-800.00
x5	-0.24	-0.19	0.00	0.10	1.00	0.20
x3	1.00	1.00	1.00	0.00	0.00	0.00
Rx10	0.24	0.19	0.00	0.90	0.00	0.80

	beta4				
Basic	x7	sx8	Rx9	Rx10	Solution
z (max)	-900.00	21.00	808.00	0.00	-352.00
x5	0.10	0.02	-0.24	0.00	0.56
x3	0.00	0.00	1.00	0.00	1.00
Rx10	0.90	-0.02	0.24	1.00	0.44

$\min_{\alpha_i}\{z_j - c_j\} = -192 = -\frac{12M}{50} + 48$, Corresponds to α_1

$\min_{\beta_j}\{z_j - c_j\} = -900 = \frac{-9M}{10}$, corresponds to β_1

$(z_j - c_j)$ for $x_8 = 21$

Iteration 4: optimum.

Iterat	alfa1	alfa2	alfa3	beta1	beta2	beta3
Basic	x1	x2	x3	x4	x5	x6
z (max)	48.00	43.20	0.00	0.00	0.00	0.00
x5	-0.27	-0.21	0.00	0.00	1.00	0.11
x3	1.00	1.00	1.00	0.00	0.00	0.00
x4	0.27	0.21	0.00	1.00	0.00	0.89

	beta4				
Basic	x7	sx8	Rx9	Rx10	Solution
z (max)	0.00	1.00	1048.00	1000.00	88.00
x5	0.00	0.02	-0.27	-0.11	0.51
x3	0.00	0.00	1.00	0.00	1.00
x4	1.00	-0.02	0.27	1.11	0.49

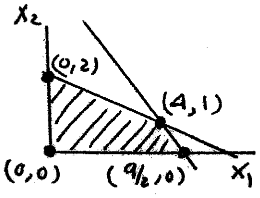
Optimum solution: $Z = 88$

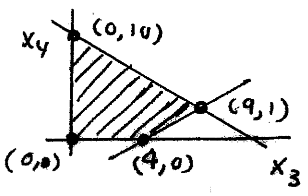
	Variable	Associated extreme point	
$x_1 =$	$\alpha_1 = 0$	$(0,0)$	$\Rightarrow x_1 = 0,\ x_2 = 12$
$x_2 =$	$\alpha_2 = 0$	$(12/5, 0)$	
$x_3 =$	$\alpha_3 = 1$	$(0,12)$	
$x_4 =$	$\beta_1 = .4889$	$(5,0)$	$\Rightarrow x_4 = 28,\ x_5 = 0$
$x_5 =$	$\beta_2 = .5111$	$(50,0)$	
$x_6 =$	$\beta_3 = 0$	$(0,10)$	
$x_7 =$	$\beta_4 = 0$	$(0,5)$	

3

$$D_1 = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix},\ \underline{X}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix},\ \underline{X}_2 = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix},\ b = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$





Continued...

$(x_1, x_2) = \alpha_2(9/2, 0) + \alpha_3(4,1) + \alpha_4(0,2)$
$= (9/2\,\alpha_2 + 4\alpha_3,\ \alpha_3 + 2\alpha_4)$

$(x_3, x_4) = \beta_2(4,0) + \beta_3(9,1) + \beta_4(0,10)$
$= (4\beta_2 + 9\beta_3,\ \beta_3 + 10\beta_4)$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	
$z =$	0	9/2	7	6	0	20	47	20	
	0	$22\frac{1}{2}$	23	6	0	16	36	0	≥ 10
	1	1	1	1					$= 1$
					1	1	1	1	$= 1$

$\alpha_i \geq 0,\ \beta_i \geq 0,\quad i = 1,2,3,4$

TORA optimum solution:

$\alpha_3 = 1,\qquad \beta_3 = 1$

all other variables $= 0$

Thus,

$(x_1, x_2) = (4,1)$

$(x_3, x_4) = (9,1)\qquad z = 54$

4

Subproblem $j=1$:

$\underline{X}_1 = (x_1, x_2),\ \underline{C}_1 = (1,3),\ A_1 = (5,3)$

$D_1 = (1\ \ 4),\quad b_1 = \binom{8}{9}$

Subproblem $j=2$:

$\underline{X}_2 = (x_3, x_4),\ \underline{C}_2 = (5,2),\ A_2 = (4,0)$

Starting solution:

Use R_1, R_2, R_3 as starting artificial vars.

$\underline{X}_B = (R_1, R_2, R_3)^T = (10, 1, 1)^T$

$B = B^{-1} = I,\ C_B = (-M, -M, -M)$

Iteration 1:

$j=1$: Min $w_1 = -(5M+1)x_1 - (3M+3)x_2 - M$

Optimum: $\hat{\underline{X}}_{11} = (4,1),\ w_1^* = -24M-7$

$j=2$: Min $w_2 = -(4M+5)x_3 - 2x_4 - M$

Optimum: $\hat{\underline{X}}_{21} = (9,1),\ w_2^* = -37M-47$

$z_5 - c_5 = (-M,-M,-M)\,I\begin{pmatrix}-1\\0\\0\end{pmatrix} - 0 = M > 0$

Continued...

Thus, β_3 associated with $\hat{X}_{21} = (9, 1)$ enters the solution:

$$P_{21} = \begin{pmatrix} A_2\hat{X}_{21} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 36 \\ 0 \\ 1 \end{pmatrix}, \; B^{-1}P_{21} = \begin{pmatrix} 36 \\ 0 \\ 1 \end{pmatrix}$$

$\theta = \min\left\{\frac{10}{36}, -, \frac{1}{1}\right\} = \frac{10}{36}$, R_1 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1/36 & 0 & 0 \\ 0 & 1 & 0 \\ -1/36 & 0 & 1 \end{pmatrix}$$

$$X_B = (\beta_3, R_2, R_3)^T = B^{-1}_{next}(10, 1, 1)^T$$
$$= (10/36, 1, 26/36)^T$$

$$C_{21} = \underline{C}_2\hat{\underline{X}}_{21} = (5, 2)\begin{pmatrix} 9 \\ 1 \end{pmatrix} = 47$$

<u>Iteration 2:</u>

$C_B = (47, -M, -M)$

<u>j=1:</u>

$\text{Min } w_1 = \left(\frac{199}{36} + \frac{5M}{36}\right)x_1 + \left(\frac{33}{36} + \frac{3M}{36}\right)x_2 - M$

Optimum: $\hat{\underline{X}}_{12} = (0, 0)$, $w_1^* = -M$

<u>j=2:</u>

$\text{Min } w_2 = \left(\frac{8}{36} + \frac{4M}{36}\right)x_3 - 2x_4$

Optimum: $\hat{\underline{X}}_{22} = (0, 10)$, $w_2^* = -20 - M$

$$Z_5 - C_5 = (47, -M, -M)B^{-1}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -\frac{47}{36} - \frac{M}{36}$$

Thus, β_4 associated with $\hat{\underline{X}}_{22} = (0, 10)$ enters the solution.

$$P_{22} = \begin{pmatrix} A_2\hat{\underline{X}}_{22} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; B^{-1}P_{22} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\theta = \min\left\{-, -, \frac{26/36}{1}\right\} = \frac{26}{36}$, R_3 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1/36 & 0 & 0 \\ 0 & 1 & 0 \\ -1/36 & 0 & 1 \end{pmatrix}$$

$$X_B = (\beta_3, R_2, \beta_4)^T = B^{-1}\begin{pmatrix} 10 \\ 1 \\ 1 \end{pmatrix} = \left(\frac{10}{36}, 1, \frac{26}{36}\right)^T$$

$$C_{22} = \underline{C}_2\hat{\underline{X}}_{22} = (5, 2)\begin{pmatrix} 0 \\ 10 \end{pmatrix} = 20$$

<u>Iteration 3:</u>

$C_B = (47, -M, 20)$

<u>j=1:</u> $\text{Min } w_1 = \frac{99}{36}x_1 - \frac{27}{36}x_2 - M$

Optimum: $\hat{\underline{X}}_{13} = (0, 2)$, $w_1^* = \frac{3}{2} - M$

<u>j=2:</u> $\text{Min } w_2 = -2x_3 - 2x_4 + 20$

Optimum: $\hat{X}_{23} = (9, 1)^T$, $w_2^* = 0$

continued...

$$Z_5 - C_5 = (47, -M, 20)B^{-1}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -3/4$$

Thus, α_3 associated with $\hat{\underline{X}}_{13} = (0, 2)$ enters the solution:

$$P_{13} = \begin{pmatrix} A_1\hat{X}_{13} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (5,3)\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$$

$$B^{-1}P_{13} = (1/6, 1, -1/6)^T$$

$\theta = \min\left\{\frac{10/36}{1/6}, \frac{1}{1}, -\right\} = 1$, R_2 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1/36 & -1/6 & 0 \\ 0 & 1 & 0 \\ -1/36 & 1/6 & 1 \end{pmatrix}$$

$$X_B = (\beta_3, \alpha_4, \beta_4)^T = \left(\frac{4}{36}, 1, \frac{32}{36}\right)^T$$

$$C_{13} = \underline{C}_1\hat{X}_{13} = (1, 3)\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 6$$

<u>Iteration 4:</u>

$C_B = (47, 6, 20)$

<u>j=1:</u> $\text{Min } w_1 = \frac{11}{4}x_1 - \frac{3}{4}x_2 + \frac{3}{}$

Optimum: $\hat{\underline{X}}_{14} = (0, 2)$, $w_1^* = 0$

<u>j=2:</u> $\text{Min } w_2 = -2x_3 - 2x_4 + 20$

Optimum: $\hat{\underline{X}}_{24} = (0, 10)$, $w_2^* = 0$

$$Z_5 - C_5 = (47, 6, 20)B^{-1}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -3/4$$

Thus, S enters the solution.

$$B^{-1}P_S = B^{-1}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \left(-\frac{1}{36}, 0, \frac{1}{36}\right)^T$$

$\theta = \min\left\{-, -, \frac{32/36}{1/36}\right\} = 32$, β_4 leaves

$$B^{-1}_{next} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 6 & 36 \end{pmatrix}$$

$$X_B = (\beta_3, \alpha_4, S)^T = B^{-1}\begin{pmatrix} 10 \\ 1 \\ 1 \end{pmatrix} = (1, 1, 32)^T$$

<u>Iteration 5:</u>

$C_B = (47, 6, 0)$

<u>j=1:</u> $\text{Min } w_1 = -x_1 - 3x_2 + 6$

Optimum: $\hat{\underline{X}}_{15} = (4, 1)^T$, $w_1^* = -1$

<u>j=2:</u> $\text{Min } w_2 = -5x_3 - 2x_4 + 47$

Optimum: $\hat{\underline{X}}_{25} = (9, 1)$, $w_2^* = 0$

Thus, α_3 associated with $\hat{\underline{X}}_{15} = (4, 1)^T$ enters the solution:

$$P_{15} = \begin{pmatrix} (5,3)\begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 23 \\ 1 \\ 0 \end{pmatrix}$$

$$B^{-1}P_{15} = (0, 1, -17)T$$

continued...

$\theta = \min\{-, \frac{1}{1}, -\} = 1$, α_4 leaves

$$B^{-1}_{next} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 23 & 36 \end{pmatrix}, \; c_{15} = (1,3)\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 7$$

$$X_B = (\beta_3, \alpha_3, s)^T = B^{-1}\begin{pmatrix} 10 \\ 1 \\ 1 \end{pmatrix} = (1, 1, 49)$$

Iteration 6:

$C_B = (47, 7, 0)$

j=1: $\min w_1 = -x_1 - 3x_2 + 7$

Optimum: $\hat{X}_{16} = (4,1)$, $w_1^* = 0$

j=2: $\min w_2 = -5x_3 - 2x_4 + 47$

Optimum: $\hat{X}_{26} = (9,1)$, $w_2^* = 0$

Optimum is reached!

$(\beta_3, \alpha_3, s) = (1, 1, 49)$ translates to $(x_1, x_2) = (4,1)$ and $(x_3, x_4) = (9,1)$

$Z = 54$

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j=1:

$\underline{X}_1 = (x_1, x_2)^T$

$\underline{C}_1 = (6, 7)$ $\quad\underline{A}_1 = (1, 1)$

$\underline{D}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\quad\underline{b}_1 = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$

j=2:

$\underline{X}_2 = (x_3, x_4)$

$\underline{C}_2 = (3,5)$ $\quad\underline{A}_2 = (1, 1)$

$\underline{D}_2 = (5, 1)$ $\quad\underline{b}_2 = (12)$

j=3:

$\underline{X}_3 = (x_5, x_6)$

$\underline{C}_3 = (1, 1)$, $\quad\underline{A}_3 = (1, 1)$

$D_3 = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$, $\quad b_3 = \begin{pmatrix} 5 \\ 50 \end{pmatrix}$

Starting solution:

$X_B = (S_1, R_1, R_2, R_3)^T = (50, 1, 1, 1)^T$

S_1 is the slack of the common constraint.

Iteration 0:

$C_B = (0, -M, -M, -M)$

$B = B^{-1} = I$

Continued...

Iteration 1:

j=1:

minimize $w_1 = -6x_1 - 7x_2 - M$

Solution: $\hat{X}_{11} = (2, 8)^T$

$w_1^* = -68 - M$

j=2:

minimize $w_2 = -3x_3 - 5x_4 - M$

Solution: $\hat{X}_{21} = (0, 12)^T$

$w_2^* = -60 - M$

j=3:

minimize $w_3 = -x_5 - x_6 - M$

Solution: $\hat{X}_{31} = (50, 0)^T$

$w_3^* = -50 - M$

β_{11} associated with $\hat{Y}_{11}$ enters the solution

$$P_{11} = \begin{pmatrix} (1,1)(2,8) \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$B^{-1}P_{11} = (10, 1, 0, 0)^T$

$\theta = \min\left\{\frac{50}{10}, \frac{1}{1}, -, -\right\} = 1$, R_1 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1 & -10 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$X_B = (S_1, \beta_{11}, R_2, R_3) = (40, 1, 1, 1)$

$c_{11} = \underline{C}_1 \hat{\underline{X}}_{11} = 68$

Iteration 2: $C_B = (0, 68, -M, -M)$

j=1:

minimize $w_1 = -6x_1 - 7x_2 - M$

Solution: $\hat{X}_{12} = (2, 8)^T$

$w_1^* = 0$

j=2:

minimize $w_2 = -3x_3 - 5x_4 - M$

Continued...

Set 20.2a

Solution: $\hat{Y}_{22} = (0, 12)^T$

$w_2^* = -60 - M$

j = 3:

minimize $w_3 = -x_5 - x_6 - M$

Solution: $\hat{X}_{32} = (50, 0)^T$

$w_3^* = -50 - M$

β_{22} associated with $\hat{Y}_{22}$ enters the solution.

$$P_{22} = \begin{pmatrix} (1,1)(0,12)^T \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$B^{-1}P_{22} = (12, 0, 1, 0)^T$

$\theta = \min\left\{\frac{40}{12}, -, \frac{1}{1}, -\right\}$; R_2 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1 & -10 & -12 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$X_B = (s_1, \beta_{11}, \beta_{22}, R_3)^T = (28, 1, 1, 1)^T$

$c_{22} = \underline{C}_2 \hat{X}_{22} = 60$

Iteration 3: $C_B = (0, 68, 60, -M)$

j = 1:

minimize $w_1 = -6x_1 - 7x_2 - M$

Solution: $\hat{X}_{13} = (2, 8)^T$

$w_1^* = 0$

j = 2:

minimize $w_2 = -3x_3 - 5x_4 - M$

Solution: $\hat{X}_{23} = (0, 12)^T$

$w_2^* = 0$

j = 3:

minimize $w_3 = -x_5 - x_6 - M$

Solution: $\hat{X}_{33} = (50, 0)^T$

$w_3^* = -50 - M$

β_{33} associated with $\hat{Y}_{33}$ enter solution

continued...

$$P_{33} = \begin{pmatrix} (1,1)(50,0)^T \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$B^{-1}P_{33} = (50, 0, 0, 1)^T$

$\theta = \min\left\{\frac{28}{50}, -, -, \frac{1}{1}\right\}$, s_1 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1/50 & -10/50 & -12/50 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/50 & 10/50 & 12/50 & 1 \end{pmatrix}$$

$X_B = (\beta_{33}, \beta_{11}, \beta_{22}, R_3)^T = \left(\frac{14}{25}, 1, 1, \frac{11}{25}\right)$

$c_{33} = \underline{C}_3 \hat{\underline{X}}_{33} = 50$

Iteration 4: $C_B = (50, 68, 60, -M)$

j = 1:

minimize $w_1 = \left(\frac{M}{50} - 5\right)x_1 + \left(\frac{M}{50} - 6\right)x_2 + 50 - \frac{M}{5}$

Solution: $\hat{X}_{41} = (0, 0)^T$

$w_1^* = 50 - M/5$

j = 2:

minimize $w_2 = \frac{50+M}{50}(x_3 + x_4) - 540$

Solution: $\hat{X}_{42} = (0, 0)^T$

$w_2^* = -540$

j = 3:

minimize $w_3 = \frac{M}{50}(x_5 + x_6) - M$

Solution: $\hat{X}_{43} = (5, 0)^T$

$w_3^* = -.9M$

s_1:

$z_{s_1} - c_{s_1} = C_B B^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0 = 1 + M/50$

β_{43} associated with $\hat{\underline{X}}_{43}$ enters solution

$$P_{43} = \begin{pmatrix} (1,1)(5,0)^T \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$B^{-1}P_{43} = \left(\frac{1}{10}, 0, 0, \frac{9}{10}\right)^T$

continued...

$\theta = \min\left\{\frac{28/50}{1/10}, -, -, \frac{22/50}{9/10}\right\}$

$= 22/45$, R_3 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1/45 & -10/45 & -12/45 & -5/45 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/45 & 10/45 & 12/45 & 50/45 \end{pmatrix}$$

$X_B = (\beta_{33}, \beta_{11}, \beta_{22}, \beta_{43})$

$= (23/45, 1, 1, 22/45)$

$C_{43} = (1,1,0,0)(5,0;0,45)^T = 5$

It can be shown that iteration 5 will prove optimality.

Optimum Solution:

$(x_1, x_2) = 1(2,8) = (2,8)$

$(x_3, x_4) = 1(0,12) = (0,12)$

$(x_5, x_6) = \frac{23}{45}(50,0) + \frac{22}{45}(5,0) = (28,0)$

$Z = C_B X_B = (50, 68, 60, 5)\left(\frac{23}{45}, 1, 1, \frac{22}{45}\right) = 156$

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Since the original problem is minimization, we must maximize w_j for each subproblem j.

$\underline{j=1}$:

$\underline{X}_1 = (x_1, x_2)^T$

$\underline{C}_1 = (5, 3)$ $\quad \underline{A}_1 = (1, 1)$

$D_1 = \begin{pmatrix} 5 & 1 \\ 5 & -1 \end{pmatrix}$ $\quad b_1 = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$

$\underline{j=2}$:

$\underline{X}_2 = (X_3, X_4)$

$\underline{C}_2 = (8, -5)$, $A_2 = (1, 1)$

$D_2 = (1, 1)$, $b_2 = 20$

Iteration 0:

$X_B = (R_1, R_2, R_3)^T = (25, 1, 1)^T$

$B = B^{-1} = I$, $C_B = (M, M, M)$

Iteration 1:

$\underline{j=1}$:

maximize $w_1 = (-5+M)x_1 + (M-3)x_2 + M$

Solution: $\hat{X}_{11} = (5/2, 15/2)^T$, $w_1^* = 11M - 35$

Continued...

$\underline{j=2}$:

maximize $w_2 = (M-8)X_3 + (M+5)X_4 + M$

Solution: $\hat{X}_{21} = (0, 20)^T$

$w_2^* = 21M + 100$

$z_{S_1} - c_S = (M, M, M) I \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -M$

β_{21} associated with $\hat{X}_{21}$ enters solution

$P_{21} = \begin{pmatrix} (1,1)(0,20)^T \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 1 \end{pmatrix}$

$B^{-1} P_{21} = (20, 0, 1)^T$

$\theta = \min\left\{\frac{25}{20}, -, \frac{1}{1}\right\} = 1$, R_3 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1 & 0 & -20 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$X_B = B^{-1}(25, 1, 1)^T = (5, 1, 1)^T$

$= (R_1, R_2, \beta_{21})^T$

$C_{21} = \underline{C}_2 \underline{\hat{X}}_{21} = (8, -5)(0, 20)^T = -100$

Iteration 2: $C_B = (M, M, -100)$

$\underline{j=1}$:

maximize $w_1 = (-5+M)X_1 + (M-3)X_2 + M$

Solution: $\hat{X}_{12} = (5/2, 15/2)^T$

$w_1^* = 11M - 35$

$\underline{j=2}$:

maximize $w_2 = (M-8)X_3 + (M+5)X_4 - 20M - 100$

Solution: $\hat{X}_{22} = (0, 20)$

$w_2^* = 0$

$z_{S_1} - c_{S_1} = -M$

β_{12} associated with $\hat{X}_{12}\left(\frac{5}{2}, \frac{15}{2}\right)$ enters the solution.

$\underline{P}_{12} = \begin{pmatrix} (1, 1)\begin{pmatrix} 5/2 \\ 15/2 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}$

$\underline{B}^{-1} \underline{P}_{12} = (10, 1, 0)^T$

$\theta = \min\left\{\frac{5}{10}, \frac{1}{1}, -\right\} = \frac{1}{2}$, R_1 leaves

Continued...

$$B^{-1}_{next} = \begin{pmatrix} 1/10 & 0 & -2 \\ -1/10 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (\beta_{12}, R_2, \beta_{21})^T = (1/2, 1/2, 1)$$

$$c_{12} = C_1 \hat{X}_{12} = (5, 3)(5/2, 15/2)^T = 35$$

<u>Iteration 3</u>: $C_B = (35, M, -100)$

<u>j=1</u>:

maximize $w_1 = -(\frac{M}{10} + \frac{3}{2})X_1 - (\frac{M}{10} - \frac{1}{2})X_2 + M$

<u>Solution</u>: $\hat{X}_{13} = (1, 0)^T$

$w_1^* = .9M - 3/2$

<u>j=2</u>:

maximize $w_2 = -(\frac{9}{2} + \frac{M}{10})X_3 - (\frac{M}{10} - \frac{17}{2})X_4 - 800$

<u>Solution</u>: $\hat{X}_{23} = (0, 20)$

$w_2^* = -630 - 2M$

$$z_{s_1} - c_{s_1} = (35, M, -100)B^{-1}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0$$

$$= M/10 - 7/2$$

β_{13} associated with $\hat{X}_{13}$ enters solution

$$P_{13} = \begin{pmatrix} (1,1)(1,0)^T \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$B^{-1}P_{13} = (1/10, 9/10, 0)^T$$

$\theta = \min\{\frac{1/2}{1/10}, \frac{1/2}{9/10}, -\} = 1/9$, R_2 leaves

$$B^{-1}_{next} = \begin{pmatrix} 1/9 & -1/9 & -20/9 \\ -1/9 & 10/9 & 20/9 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (\beta_{12}, \beta_{13}, \beta_{21})^T = B^{-1}(25, 1, 1)^T$$

$$= (4/9, 5/9, 1)^T$$

$$c_{13} = C_1 \hat{X}_{13} = (5, 3)(1, 0)^T = 5$$

<u>Iteration 4</u>: $C_B = (35, 5, -100)$

<u>j=1</u>:

maximize $w_1 = -5/3 X_1 + 1/3 X_2 + 5$

<u>Solution</u>: $\hat{X}_{14} = (5/2, 15/2)^T$, $w_1^* = 10/3$

continued...

<u>j=2</u>:

maximize $w_2 = -\frac{14}{3}X_3 + \frac{25}{3}X_4 - 800$

<u>Solution</u>: $\hat{X}_{24} = (0, 20)^T$

$w_2^* = -633\frac{1}{3}$

$$z_{s_1} - c_{s_1} = (35, 5, -100)B^{-1}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -\frac{30}{9}$$

β_{14} associated with $\hat{X}_{14}$ enters the solution

After one iteration, we get.

$(x_1, x_2) = 1(\frac{5}{2}, \frac{15}{2}) + 0(1, 0) = (\frac{5}{2}, \frac{15}{2})$

$(x_3, x_4) = 1(0, 20) = (0, 20)$, $Z = 195$

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Dual Problem:

Maximize $z = 8x_1 + 2x_2 + 4x_3 + 10x_4$

s.t.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\le 10 \\ 4x_1 + x_2 &\le 2 \\ -x_1 + x_2 &\le 4 \\ x_3 + 2x_4 &\le 8 \\ x_3 - x_4 &\le 1 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \ge 0$$

<u>j=1</u>:

$\underline{X}_1 = (x_1, x_2)^T$

$\underline{C}_1 = (8, 2)$, $\underline{A}_1 = (1, 2)$

$\underline{D}_1 = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}$ $\underline{b}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

<u>j=2</u>:

$\underline{X}_2 = (x_3, x_4)^T$

$\underline{C}_2 = (4, 10)$ $\underline{A}_2 = (3, 1)$

$\underline{D}_2 = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ $b_2 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

<u>Iteration 0</u>:

$X_B = (S_1, R_1, R_2)^T = (10, 1, 1)$

$B = B^{-1} = I$

<u>Iteration 1</u>: $C_B = (0, -M, -M)$

<u>j=1</u>:

minimize $w_1 = -8x_1 - 2x_2 - M$

<u>Solution</u>: $\hat{X}_{11} = (1/2, 0)$, or

$\hat{X}_{11} = (0, 2)$

$w_1^* = -4 - M$

continued...

$j=2$:

minimize $w_2 = -4X_3 - 10X_4 - M$

Solution: $\hat{X}_{21} = (0,4,0,5)^T$

$w_2^* = -40 - M$

β_{21} associated with $\hat{Y}_{21}$ enters

$$P_{21} = \begin{pmatrix} (3,1)(0,4)^T \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$B^{-1}P_{21} = (4,0,1)^T$

$\theta = \min\left\{\frac{10}{4}, -, \frac{1}{1}\right\} = 1$, R_2 leaves

$$B_{next}^{-1} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$X_B = B^{-1}(10,1,1)^T = (6,1,1)^T = (S_1, R_1, \beta_{21})$

$c_{21} = \underline{C}_2 X_{21} = (4,10,0,0)(0,4,0,5) = 40$

Iteration 2: $C_B = (0, -M, 40)$

$j=1$:

minimize $w_1 = -8X_1 - 2X_2 - M$

solution: $\hat{X}_{12} = (1/2, 0)^T$ or

$\hat{X}_{12} = (0,2)^T$

$w_1^* = -4 - M$

$j=2$:

minimize $w_2 = -4X_3 - 10X_4 + 40$

Solution: $\hat{X}_{22} = (0,4)^T$

$w_2^* = 0$

β_{12} associated with $\hat{Y}_{12} = (0,2)^T$ [or $(1/2, 0, 0, 9/2)$] enters the solution.

$$P_{12} = \begin{pmatrix} (1,2)(0,2)^T \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$B^{-1}P_{12} = (4,1,0)^T$

$\theta = \min\left\{\frac{6}{4}, \frac{1}{1}, -\right\} = 1$, R_1 leaves

Continued...

$$B_{nex}^{-1} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$X_B = (S_1, \beta_{12}, \beta_{21}) = (2,1,1)^T$

$c_{12} = \underline{C}_1 \hat{X}_{12} = (8, 2)(0, 2)^T = 4$

Iteration 3: $C_B = (0, 4, 40)$

$j=1$:

minimize $w_1 = -8X_1 - 2X_2 + 4$

Solution: $\hat{X}_{13} = (1/2, 0)^T$

$w_1^* = 0$

$j=2$:

minimize $w_2 = -4X_3 - 10X_4 + 40$

Solution: $\hat{X}_{23} = (0,4)^T$

$w_2^* = 0$

Solution is optimum!

$(X_1, X_2) = 1(0,2) = (0,2)$

$(X_3, X_4) = 1(0,4) = (0,4)$

The problem has an alternative solution, which can be determined using $\hat{Y}_{12} = (1/2, 0)$ in place of $(0,2)$ in iteration 2. The alternative solution is

$X_1 = 1/2,\ X_2 = 0,\ X_3 = 0,\ X_4 = 4$

To determine the primal solution, note that the basic dual variables as given above are

$(X_2, X_4, S_1, S_3, S_5)$

or $(X_1, X_4, S_1, S_3, S_5)$

where S_1 is the slack associated with the common constraint, S_3 is the slack for constraint 3, and S_5 is the slack for constraint 5. Thus,

Dual variable	Primal constraint equation
X_2	$2y_1 + y_2 + y_3 = 2$
X_4	$y_1 + 2y_4 - y_5 = 10$
S_1	$y_1 = 0$
S_3	$y_3 = 0$
S_5	$y_5 = 0$

Continued...

Solution: $y_1 = 0, y_2 = 2, y_3 = 0, y_4 = 5, y_5 = 0$

Consider the second alternative solution

Dual variable	Primal constraint equation
x_1	$y_1 + 4y_2 - y_3 = 8$
x_4	$y_1 + 2y_4 - y_5 = 10$
s_1	$y_1 = 0$
s_3	$y_3 = 0$
s_5	$y_5 = 0$

Solution: $y_1 = 0, y_2 = 2, y_3 = 0, y_4 = 5, y_5 = 0$

objective value $= 2 \times 2 + 5 \times 8$

$= 44$

8

Let $\mathbf{B}$ be the current basis of the *master* problem and $\mathbf{C}_B$ the vector of the corresponding coefficients in the objective function. Thus, according to the revised simplex method, the current solution is optimal if for all nonbasic $\mathbf{P}_j^k$,

$$z_j^k - c_j^k = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j^k - c_j^k \geq 0$$

where, from the definition of the master problem,

$$c_j^k = \mathbf{C}_j\hat{\mathbf{X}}_j^k \qquad \text{and} \qquad \mathbf{P}_j^k = \begin{array}{l} r_0\{ \\ \\ n\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \end{array} \begin{pmatrix} \mathbf{A}_j\hat{\mathbf{X}}_j^k \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow (r_0 + j)\text{th place} \\ \\ \\ \end{array}$$

The expression for $z_j^k - c_j^k$ can be simplified as follows. Let

$$\mathbf{B}^{-1} = (\overbrace{\mathbf{R}_0}^{r_0} \mid \overbrace{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_j, \ldots, \mathbf{V}_n}^{n})$$

where $\mathbf{R}_0$ is the matrix of size $(r_0 + n) \times r_0$ consisting of the first r_0 columns of $\mathbf{B}^{-1}$, and $\mathbf{V}_j$ is the $(r_0 + j)$th column of the same matrix $\mathbf{B}^{-1}$. Thus

$$\begin{aligned} z_j^k - c_j^k &= (\mathbf{C}_B\mathbf{R}_0\mathbf{A}_j\hat{\mathbf{X}}_j^k + \mathbf{C}_B\mathbf{V}_j) - \mathbf{C}_j\hat{\mathbf{X}}_j^k \\ &= (\mathbf{C}_B\mathbf{R}_0\mathbf{A}_j - \mathbf{C}_j)\hat{\mathbf{X}}_j^k + \mathbf{C}_B\mathbf{V}_j \end{aligned}$$

1

TORA optimal solution : $(M = 10)$

```
Final Iteration No.: 10
Objective Value =  0
---------------------------------------
Variable                          Value
---------------------------------------
y {x1:                          0.18182
  {x2:                          0.00000
  {x3:                          0.00000
w {x4:                          0.09091
  {x5:                          0.00000
   x6:                          0.63636
   x7:                          0.09091
   x8:                          0.00000
---------------------------------------
```

$$y_1 = (M+1)X_1 = 11 \times .18182 \cong 2$$
$$y_2 = (M+1)X_2 = \quad \times 0 = 0$$
$$y_3 = (M+1)X_3 = 11 \times 0 = 0$$
$$w_1 = (M+1)X_4 = 11 \times .090909 \cong 1$$
$$w_2 = (M+1)X_5 = 11 \times 0 = 0$$

2

Primal:

Maximize $z = 2y_1 + y_2$

Subject to

$$y_1 - y_2 \leq 2$$
$$y_1 + 2y_2 \leq 4$$
$$y_1, y_2 \geq 0$$

Dual:

Minimize $w = 2w_1 + 4w_2$

Subject to

$$w_1 + w_2 \geq 2$$
$$-w_1 + 2w_2 \geq 1$$
$$w_1, w_2 \geq 0$$

Thus, Conversion of the primal and dual constraint to equations yields:

$$2y_1 + y_2 - 2w_1 - 4w_2 = 0$$
$$y_1 - y_2 + y_3 = 2 \qquad w_1 + w_2 - w_3 = 2$$
$$y_1 + 2y_2 + y_4 = 4 \qquad -w_1 + 2w_2 - w_4 = 1$$

all variables ≥ 0

Next,

$$y_1 - y_2 + y_3 - 2S_2 = 0 \qquad w_1 + w_2 - w_3 - 2S_2 = 0$$
$$y_1 + 2y_2 + y_4 - 4S_2 = 0 \qquad -w + 2w_2 + w_4 - S_2 = 0$$
$$2y_1 + y_2 - 2w_1 - 4w_2 = 0$$
$$y_1 + y_2 + y_3 + y_4 + w_1 + w_2 + w_3 + w_4 - MS_2 + S_1 = 0$$
$$y_1 + y_2 + y_3 + y_4 + w_1 + w_2 + w_3 + w_4 + S_1 + S_2 = M+1$$

Continued...

Let

$$y_j = (M+1)x_j, \quad j = 1, 2, 3, 4$$
$$w_j = (M+1)x_j, \quad j = 5, 6, 7, 8$$
$$S_1 = (M+1)X_9$$
$$S_2 = (M+1)X_{10}$$

Thus, the equations become

$$X_1 - X_2 + X_3 - 2X_{10} = 0$$
$$X_1 + 2X_2 + X_4 - 4X_{10} = 0$$
$$X_5 + X_6 - X_7 - 2X_{10} = 0$$
$$-X_5 + 2X_6 - X_8 - X_{10} = 0$$
$$2X_1 + X_2 - 2X_5 - 4X_6 = 0$$
$$X_1 + X_2 + \cdots + X_9 - MX_{10} = 0$$
$$X_1 + X_2 + \cdots + X_{10} = 1$$

all $x_j \geq 0$

The complete problem thus becomes:

minimize $z = X_{11}$

Subject to

$$X_1 - X_2 + X_3 - 2X_{10} + X_{11} = 0$$
$$X_1 + 2X_2 + X_4 - 4X_{10} + 0X_{11} = 0$$
$$X_5 + X_6 - X_7 - 2X_{10} + X_{11} = 0$$
$$-X_5 + 2X_6 - X_8 - X_{10} + X_{11} = 0$$
$$2X_1 + X_2 - 2X_5 - 4X_6 + 3X_{11} = 0$$
$$X_1 + X_2 + \cdots + X_9 - MX_{10} + (M-9)X_{11} = 0$$
$$X_1 + X_2 + \cdots + X_{11} = 1$$

all $x_j \geq 0$

Chapter 21

Forecasting models

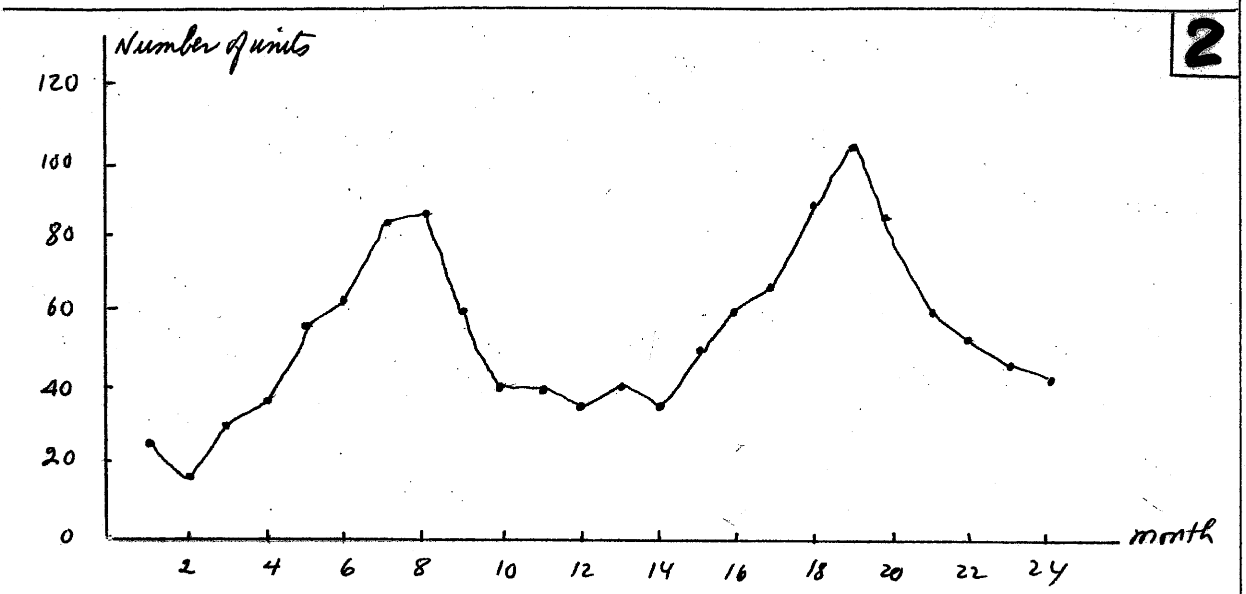
Set 21.1a

1

$$y^*_{25} = \frac{54+42+64+60+70+66+57+55+52+62+70+72}{12} = 60.33$$

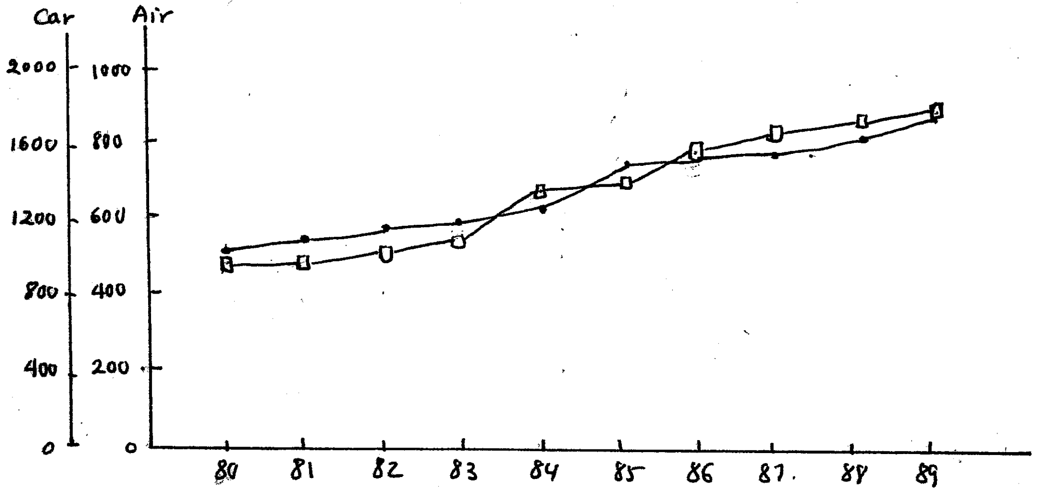
Larger n suppresses the flactuations in data

2



The seasonal nature of the data makes the moving average unsuitable as a prediction model. We need to select n small; e.g., $n=3$ yields $y^*_{25} = 50$.

3

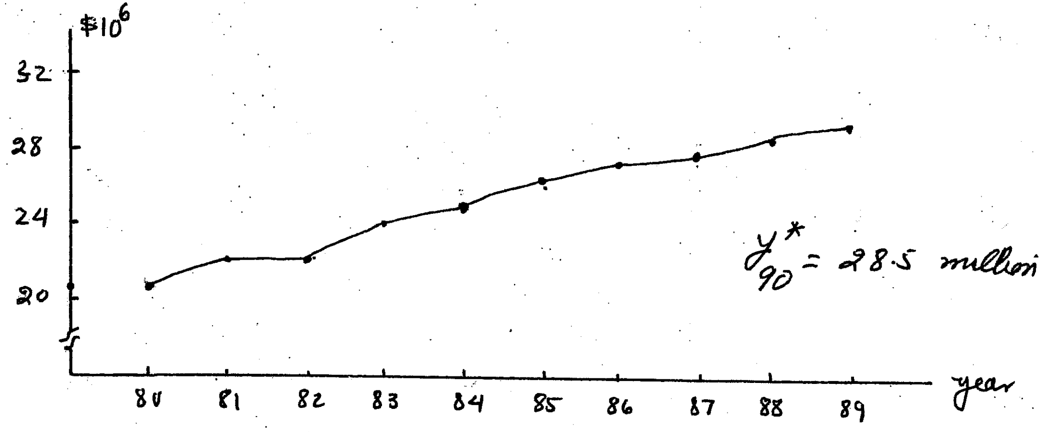


Data show an upward trend. Select small $n=3$ for the moving average.

Car : $y^*_{90} = 1791.3$ individuals

Air : $y^*_{90} = 938.33$ individuals

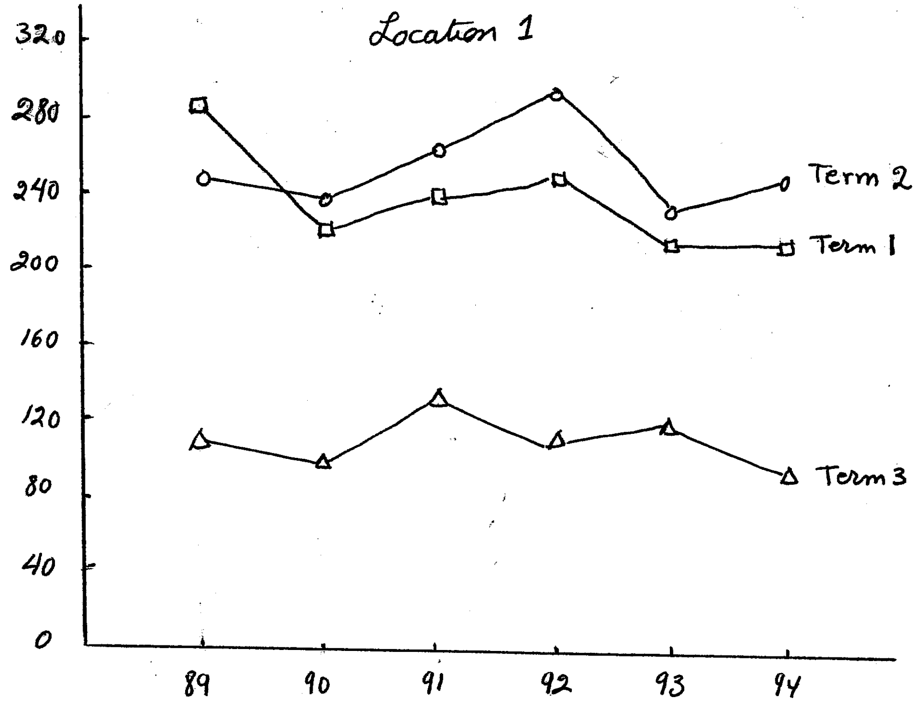
4



$y^*_{90} = 28.5$ million $

Data show linear trend. Use small $n = 3$ for the moving average.

5



The data appear stable. The moving average should apply nicely to this case. Similar analysis can be carried out for the remaining locations. Use $n = 5$

<u>At location 1</u>:
Term 1 : $y^*_{95} = 238.6$ students
Term 2 : $y^*_{95} = 260.2$ students
Term 3 : $y^*_{95} = 117$ students

Set 21.2a

1 $\alpha = .2$, $y^*_{25} = 59.63$

2 Car: $\alpha = .2$, $y^*_{90} = 1577.71$ individuals

Air: $\alpha = .2$, $y^*_{90} = 797.75$ individuals

3 $\alpha = .2$ $y^*_{90} = \$26.27$ million

4 <u>For location 1</u>: $\alpha = .2$.

Term 1: $y^*_{95} = 254.33$

Term 2: $y^*_{95} = 256.13$

Term 3: $y^*_{95} = 116.38$

1

The data have both seasonal variations and a trend. Regression analysis can be used to detect the trend.

$y = 39.23 + 1.262\,X$, $\quad r = .394$

$y^*_{25} = 70.77$ units

2

Car: $y = 977.4 + 92.69\,X$, $\quad r = .9928$

$y^*_{90} = 1997$ individuals

Air: $y = 407.73 + 59.21\,X$, $\quad r = .9895$

$y^*_{90} = 1059.07$ individuals

3

$y = 20.6 + .873\,X$, $\quad r = .991$

$y^*_{90} = \$30.2$ million

4

Location 1:

Term 1: $y = 272.73 - 7.4X$, $\quad r = -.538$

Term 2: $y = 247.4 + 3.03X$, $\quad r = .247$

Term 3: $y = 119.2 - .6286X$, $\quad r = -.0836$

5

$$\sum_{i=1}^{n}(y_i - bx_i - a) = \sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i - na$$

$$= n\bar{y} - nb\bar{x} - na$$

$$= n(\bar{y} - b\bar{x} - a)$$

$$= n(\bar{y} - b\bar{x} - \bar{y} + b\bar{x})$$

$$= 0$$

Chapter 22

Probabilistic Dynamic Programming

Set 22.1a

$\boxed{1}$

$$f_6(j) = 2j$$

$$f_i(j) = \max \begin{cases} \text{end: } 2j \\ \text{spin: } \frac{1}{8}\sum_{k=1}^{8} f_{i+1}(k) \end{cases} \quad i = 2, 3, 4, 5$$

$$f_1(0) = \frac{1}{8}\sum_{k=1}^{8} f_2(k)$$

Stage 6:

Spin 5 outcome	Opt. Soln.	
	$f_6(j)$	Decision
1	2	End
2	4	End
3	6	End
4	8	End
5	10	End
6	12	End
7	14	End
8	16	

Stage 5: $f_5(j) = \max\{2j, \frac{1}{8}(f_6(1) + f_6(2) + \cdots + f_6(8)\}$

$= \max\{2j, \frac{72}{8}\}$

$= \max\{2j, 9\}$

Spin 4 outcome			Opt. Sol.	
		Spin	$f_5(j)$	Decision
1	2	9	9	Spin
2	4	9	9	Spin
3	6	9	9	Spin
4	8	9	9	Spin
5	10	9	10	End
6	12	9	12	End
7	14	9	14	End
8	16	9	16	End

Stage 4:

$f_4(j) = \max\{2j, \frac{1}{8}(9+9+9+9+10+12+14+16)\}$

$= \max\{2j, 11\}$

continued...

Spin 3 outcome			Opt. Sol.	
	End	Spin	$f_4(j)$	Dension
1	2	11	11	Spin
2	4	11	11	Spin
3	6	11	11	Spin
4	8	11	11	Spin
5	10	11	11	Spin
6	12	11	12	End
7	14	11	14	End
8	16	11	16	End

Stage 3: $f_3(j) = \max\{2j, 12.125\}$

Spin 2 outcome			Opt. Sol.	
	End	Spin	$f_3(j)$	Decision
$1 \le j \le 6$	$2j$	12.125	12.185	Spin
$j = 7, 8$	$2j$	12.125	$2j$	End

Stage 2: $f_2(j) = \max\{2j, 12.84375\}$

Spin 1 outcome			Opt. Sol.	
	End	Spin	$f_2(j)$	Decision
$1 \le j \le 6$	$2j$	12.84375	12.84375	Spin
$j = 7, 8$	$2j$	12.84375	$2j$	End

Stage 1: $f_1(0) = \frac{1}{8}(6 \times 12.84375 + 14 + 16) = 13.38$

Solution:

Spin #	Strategy
1	Continue to spin
2	Continue if #1 produces 1→6, else end
3	Continue if #2 produces 1→6, else end
4	Continue if #3 produces 1→5, else end.
5	Continue if #4 produces 1→4, else end
6	End

Expected return = \$13.38

2

Let O_j represent the best offer at the end of day i, where

$$j = \begin{cases} 1, \text{ high offer} \\ 2, \text{ medium offer} \\ 3, \text{ low offer} \end{cases}$$

$i = 1, 2, 3$

$$f_4(j) = O_j$$

$$f_i(j) = \max \begin{cases} \text{accept}: O_j \\ \text{continue}: \frac{1}{3}\left(f_{i+1}(1) + f_{i+1}(2) + f_{i+1}(3)\right) \end{cases}$$

$$f_1(0) = \frac{1}{3}\left\{f_2(1) + f_2(2) + f_3(2)\right\}$$

Stage 4:

Day 3 best offer j	Opt. Sol. $f_4(j)$	Opt. Sol. Decision
1	1050	Accept
2	1900	Accept
3	2500	Accept

Stage 3:

$$f_3(j) = \max\left\{O_j, \frac{1}{3}\left(f_4(1) + f_4(2) + f_4(3)\right)\right\}$$
$$= \max\left\{O_j, 1816.67\right\}$$

Day 2 best offer j	Accept	Continue	Opt. Sol. $f_3(j)$	Opt. Sol. Decision
1	1050	1816.67	1816.67	Continue
2	1900	1816.67	1900	Accept
3	2500	1816.67	2500	Accept

Stage 2:

$$f_2(j) = \max\left\{O_j, \frac{1}{3}\left(f_3(1) + f_3(2) + f_3(3)\right)\right\}$$
$$= \max\left\{O_j, 2072.33\right\}$$

Day 1 best offer j	Accept	Continue	Opt. Sol. $f_2(j)$	Opt. Sol. Decision
1	1050	2072.33	2072.33	Continue
2	1900	2072.33	2072.33	Continue
3	2500	2072.33	2072.33	Accept

Stage 1:

$$f_1(0) = \frac{1}{3}(2 \times 2072.33 + 2500)$$
$$= \$2214.82$$

Solution:

Day 1: Accept if offer is high

Day 2: Accept if offer is medium or high

Day 3: Accept any offer.

Set 22.2a

1

Stage 4:

$$f_4(x_4) = x_4(1+.8\times.6+.4\times.2+.2\times.2)$$
$$= 1.6x_4$$

State	Opt. Sol.	
	$f_4(x_4)$	y_4
x_4	$1.6\,x_4$	x_4 (invest all)

Stage 3:

$$f_3(x_3) = \max_{0\le y_3\le x_3}\{.2\times1.6(x_3+4y_3) + .4\times1.6(x_3-y_3) + .4\times1.6(x_3-y_3)\}$$
$$= \max_{0\le y_3\le x_3}\{1.6x_3\}$$

State	Optimum	
	$f_3(x_3)$	y_3
x_3	$1.6x_3$	$0\le y_3\le x_3$

Stage 2:

$$f_2(x_2) = \max_{0\le y_2\le x_2}\{.4\times1.6(x_2+y_2)+.4\times1.6x_2 + .2\times1.6(x_2-y_2)\}$$
$$= \max_{0\le y_2\le x_2}\{1.6x_2+.32y_2\}$$

State	Opt. Sol.	
	$f_2(x_2)$	y_2
x_2	$1.92\,x_2$	x_2

Stage 1: $f_1(x_1) = \max_{0\le y_1\le x_1}\{.1\times1.92(x_1+2y_1) + .4\times1.92(x_1+y_1)+.5\times1.92(x_1+.5y_1)\}$
$$= \max_{0\le y_1\le x_1}\{1.92x_1+1.632y_1\}$$

State	Opt. Sol.	
	$f_1(x_1)$	y_1
x_1	$3.552\,x_1$	x_1

Solution: accumulation = \$35,520
Invest \$10,000 in year 1, all in year 2, none in year 3, and all in year 4.

2

c_i = penalty cost/shortage unit of item i
z_i = number of units of item i
v_i = volume per unit of item i
x_i = m^3 assigned to items $i,\dots,n$
p_{ij} = probability of j demand units of item i

$f_i(x_i)$ = minimum expected penalty cost for items $i, i+1, \dots,$ and n, given x_i

$$f_i(x_i) = \min_{0\le z_i\le\left[\frac{x_i}{v_i}\right]}\left\{c_i\sum_{j>z_i}(j-z_i)p_{ij} + f_{i+1}(x_i-z_iv_i)\right\}$$
$$i = 1, 2, \dots, n$$

$$f_{n+1}(\cdot)\equiv 0$$

Table for exp. shortage cost:

item i	$z_i=1$	$z_i=2$	$z_i=3$
1	8(1x.5) = 4	0	0
2	10(1x.4+2x.2+3x.1)=11	10(1x.2+2x.1)=4	10(1x.1)=1
3	15(1x.2+2x.5)= 18	15(1x.5)=7.5	0

stage 3: $f_3(x_3) = \min_{z_3\le\left[\frac{x_3}{v_3}\right]}\left\{15\sum_{j>z_3}(j-z_3)p_{3j}\right\}$

x_3	$v_3=3$			Opt Sol.	
	$z_3=1$	$z_3=2$	$z_3=3$	$f_3(x_3)$	z_3
3	18	—	—	18	1
4	18	—	—	18	1
5	18	—	—	18	1
6	18	7.5	—	7.5	2
7	18	7.5	—	7.5	2
8	18	7.5	—	7.5	2
9	18	7.5	0	0	3
10	18	7.5	0	0	3

stage 2: $v_2=1$

$$f_2(x_2) = \min_{z_2\le\left[\frac{x_2}{v_2}\right]}\left\{10\sum_{j>z_2}(j-z_2)p_{2j}+f_3(x_2-z_2)\right\}$$

2 continued

x_2	$z_2 = 1$	2	3	4	5	6	7	f_3	z_3
4	11+18=29	—	—	—	—	—	—	29	1
5	11+18=29	4+18=22	—	—	—	—	—	22	2
6	11+18=29	4+18=22	1+18=19	—	—	—	—	19	3
7	11+7.5=18.5	4+18=22	1+18=19	0+18=18	—	—	—	18	4
8	11+7.5=18.5	4+7.5=11.5	1+18=19	0+18=18	0+18=18	—	—	11.5	2
9	11+7.5=18.5	4+7.5=11.5	1+7.5=8.5	0+18=18	0+18=18	0+18=18	—	8.5	3
10	11+0=11	4+7.5=11.5	1+7.5=8.5	0+7.5=7.5	0+18=18	0+18=18	0+18=18	7.5	4

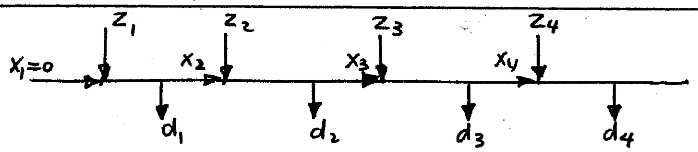
Stage 1: $f_1(x_1) = \min_{z_1 \le [\frac{x_1}{v_1}]} \{ 8 \sum_{j > z_1} (j - z_1) P_{1j} + f_2(x_1 - 2z_1)$

x_1	$z_1 = 1$	$z_1 = 2$	$z_1 = 3$	Opt. Sol. f_1	z_1
10	4+11.5=15.5	0+19=19	0+29=29	15.5	1

Solution:

$(x_1 = 10) \rightarrow z_1 = 1 \rightarrow (x_2 = 8) \rightarrow z_2 = 2 \rightarrow (x_3 = 6) \rightarrow z_3 = 2$

3



x = inventory
z = production
d = demand

$$f_n(x_n) = \min_{z_n} \{ C(x_n) \}$$

$$f_i(x_i) = \min_{z_i} \{ C(x_i) + \sum_{d=0}^{3} f_{i+1}(x_i + z_i - d_i)\, p(d_i) \}, \quad C(x_i) = \begin{cases} x_i, & x_i \ge 0 \\ -2x_i, & x_i \le 0 \end{cases}$$

$$i = 1, 2, \dots, n-1$$

Stage 4:

x_4	$z_4 = 0$	1	2	3	Opt. Sol. f_4	z_4
-3	—	—	—	6	6	3
-2	—	—	4	—	4	2
-1	—	2	—	—	2	1
0	0	—	—	—	0	0
1	1	—	—	—	1	0
2	2	—	—	—	2	0
3	3	—	—	—	3	0

Notice that negative x_4 allows for the possibility of backordering by producing for year 3 in period 4.

continued...

3 continued

Stage 3: $f_3(x_3) = \min_{z_3} \{C(x_3) + .5 f_4(x_3+z_3-1) + .3 f_4(x_3+z_3-2) + .2 f_4(x_3+z_3-3)\}$

x_3	$z_3 = 0$	1	2	3	Opt. Sol. f_3	z_3
-3	—	—	—	6+.5x2+.3x4+.2x6=9.4	9.4	3
-2	—	—	4+.5x2+.3x4+.2x6=7.4	4+.5x0+.3x2+.2x4=5.4	5.4	3
-1	—	2+.5x2+.3x4+.2x6=5.4	2+.5x0+.3x2+.2x4=3.4	2+.5x1+.3x0+.2x2=2.9	2.9	3
0	0+.5x2+.3x4+.2x6=3.4	0+.5x0+.3x2+.2x4=1.4	0+.5x1+.3x0+.2x2=.9	0+.5x2+.3x1+.2x0=1.3	.9	2
1	1+.5x0+.3x2+.2x4=2.4	1+.5x1+.3x0+.2x2=1.9	1+.5x2+.3x1+.2x0=2.3	—	1.9	1
2	2+.5x1+.3x0+.2x2=2.9	2+.5x2+.3x1+2x0=3.3	—	—	2.9	0
3	3+.5x2+.3x1+.2x0=4.3	—	—	—	4.3	0

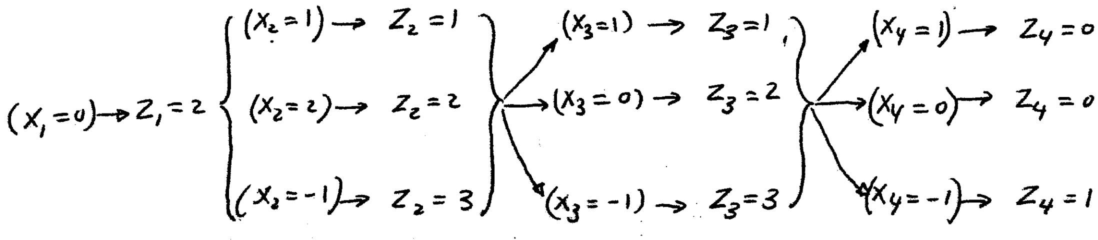
Stage 2: $f_2(x_2) = \min_{z_2} \{C(x_2) + .5 f_3(x_2+z_2-1) + .3 f_3(x_2+z_2-2) + .2 f_3(x_2+z_2-3)\}$

x_2	$z_2 = 0$	1	2	3	Opt. Sol. f_2	z_2
-3	—	—	—	6+.5x2.9+.3x5.4 +.2x9.4=10.94	10.94	3
-2	—	—	4+.5x2.9+.3x5.4 +.2x9.4=8.94	4+.5x.9+.3x2.9+ .2x5.4=6.4	6.4	3
-1	—	2+.5x2.9+.3x5.4+ .2x9.4=6.94	2+.5x.9+.3x2.9+ .2x5.4=4.4	2+.5x1.9+.3x.9 +2x2.9=3.8	3.8	3
0	0+.5x2.9+.3x5.4+ .2x9.4=4.94	0+.5x.9+.3x2.9+ .2x5.4=2.4	0+.5x1.9+.3x.9+ .2x2.9=1.8	0+.5x2.9+.3x1.9 +.2x.9=2.2	1.8	2
1	1+.5x.9+.3x2.9+ .2x5.4=3.4	1+.5x1.9+.3x.9 +2x2.9=2.8	1+.5x2.9+.3x1.9 +.2x.9=3.2	—	2.8	1
2	2+.5x1.9+.3x.9+ .2x2.9=3.8	2+.5x2.9+3x1.9 +.2x.9=4.2	2+.5x4.3+.3x2.9+ .2x1.9=5.4	—	3.8	0
3	3+.5x2.9+.3x1.9+ .2x.9=5.2	3+.5x4.3+.3x2.9+ .2x1.9=6.4	—	—	5.2	0

Stage 1: $f_1(x_1) = \min_{z_1} \{C(x_1) + .5 f_2(x_1+z_1-1) + .3 f_2(x_1+z_1-2) + .2 f_2(x_2+z_2-3)\}$

x_1	$z_1 = 0$	1	2	3	Opt. Sol. f_1	z_1
0	0+.5x3.8+.3x6.4 +.2x10.94=6.008	0+.5x1.8+.3x3.8+ .2x6.4=3.32	0+.5x2.8+.3x1.8+ .2x3.8=2.7	0+.5x3.8+.3x2.8 +.2x1.8=3.1	2.7	2

Solution:



continued...

Stage i = center i

alternative y_i = number of bikes assigned to center i

State x_i = number of bikes assigned to centers $i, i+1, \dots,$ and n

d_i = demand in center i

$f_i(x_i)$ = maximum expected revenue for stages $i, i+1, \dots,$ and n given x_i.

$$f_n(x_n) = \max_{y_n \le x_n} \{ C_n E\{d_n \mid y_n\} \}$$

$$f_i(x_i) = \max_{y_i \le x_i} \{ C_i E\{d_i \mid y_i\} + f_{i+1}(x_i - y_i) \}, \quad i = 1, 2, \dots, n-1$$

where

$E\{d_i \mid y_i\}$ = Average demand at center i given y_i bikes are allocated to center i

$$= 0p_0 + 1p_1 + \dots + y_{i-1} p_{y_{i-1}} + y_i (p_{y_i} + p_{y_{i+1}} + \dots + p_8)$$

Example calculations:

$$E\{d_1 \mid y_1 = 2\} = 0p_1 + 1p_1 + 2(p_2 + p_3 + \dots + p_8)$$

$$= 0 + 1 \times 1.5 + 2 \times .85 = 1.85$$

Table for $C_i E\{d_i \mid y_i\}$:

i	C_i	$y_i = 0$	1	2	3	4	5	6	7	8
1	5	0	5.40	9.60	12.00	13.20	13.80	13.80	13.80	13.8
2	7	0	6.86	13.51	19.46	23.66	25.76	26.81	27.51	27.86
3	6	0	5.00	9.25	18.25	19.75	20.50	20.75	20.875	20.875

Stage 3: $f_3(x_3) = \max_{y_3 \le x_3} \{ C_3 E\{d_3 \mid y_3\} \}$

x_3	$y_3 = 0$	1	2	3	4	5	6	7	8	Opt. Sol.	
										f_3	y_3
0	0	—								0	0
1	0	5	—							5.00	1
2	0	5	9.25	—						9.25	2
3	0	5	9.25	18.25	—					18.25	3
4	0	5	9.25	18.25	19.75	—				19.75	4
5	0	5	9.25	18.25	19.75	20.50	—			20.50	5
6	0	5	9.25	18.25	19.75	20.50	20.75	—		20.75	6
7	0	5	9.25	18.25	19.75	20.50	20.75	20.875	—	20.875	7
8	0	5	9.25	18.25	19.75	20.50	20.75	20.875	20.875	20.85	8

Continued...

<u>Stage 2</u>: $f_2(x_2) = \max\limits_{y_2 \le x_2} \{ C_2 E\{d_2 | y_2\} + f_3(x_2 - y_2) \}$

										Opt^{m}	
x_2	$y_2 = 0$	1	2	3	4	5	6	7	8	f_2	y_2
0	0 + 0=0	—								0	0
1	0+5=5	6.86+ 0 =6.86	—							6.86	1
2	0+9.25 = 9.25	6.86 +5 =11.86	13.51+0 =13.51	—						13.51	2
3	0+18.25 = 18.25	686+9.25 = 16.11	13.51+5 =18.51	19.46+0 =19.46	—					19.46	3
4	0+19.75 =19.75	6.86+18.25 = 25.11	13.51+9.25 =22.76	19.46+5 =24.46	23.66 +0 = 23.66	—				25.11	1
5	0+ 20.50 =20.50	6.86+19.75 = 26.61	13.51+18.25 =31.76	19.46+9.25 =28.71	23.66+5 =28.66	25.76+0 =25.76	—			31.76	2
6	0+20.75 = 20.75	6.86+20.5 = 27.36	13.51+ 19.75 =33.26	19.46+18.25 =37.71	23.66+9.25 =32.91	25.76+5 =30.76	26.81+0 =26.81	—		37.71	3
7	0+20.875 =20.875	6.86+ 20.75 =27.61	13.51+ 20.5 34.01	19.46+19.75 =39.21	23.66+18.25 = 41.91	25.76+9.25 = 35.01	26.81+5 = 31.81	27.51+0 = 27.51	—	41.91	4
8	0+ 20.875 =20.875	6.86+21.875 =27.735	1351+20.75 =34.20	19.46+20.5 39.96	23.66+19.75 =43.41	25.76+18.25 = 44.01	26.81+9.25 =36.06	27.51+5 =32.51	27+.86 27.86	44.01	5

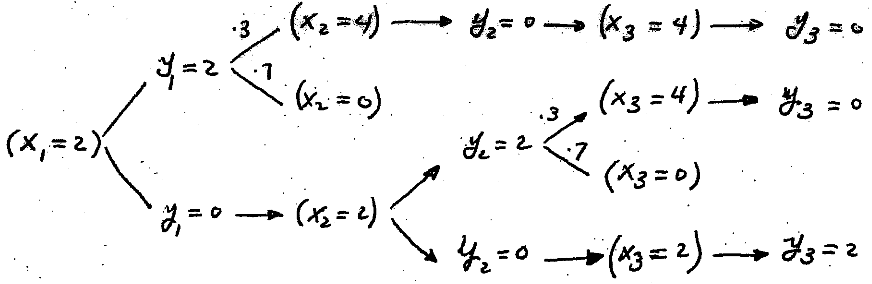
<u>Stage 1</u>: $f_1(x_1) = \max\limits_{y_1 \le x_1} \{ C_1 E\{ d_1 | y_1 \} + f_2(x_1 - y_1) \}$

										Opt	
x_1	$y_1 = 0$	1	2	3	4	5	6	7	8	f_1	y_1
8	0+ 44.01 = 44.01	5.4 + 41.9 = 47.3	9.6 +37.71 = 47.31	12+31.76 =43.76	13.7+ 25.11 =38.31	13.8+ 19.46 = 33.26	13.8+13.5 =27.3	13.8+6.86 = 20.66	13.8 +0 13.8	47.31	2

Optimum solution:

$(x_1 = 8) \rightarrow y_1 = 2 \rightarrow (x_2 = 6) \rightarrow y_2 = 3 \rightarrow (x_3 = 3) \rightarrow y_3 = 3$

1



2

Stage 3:

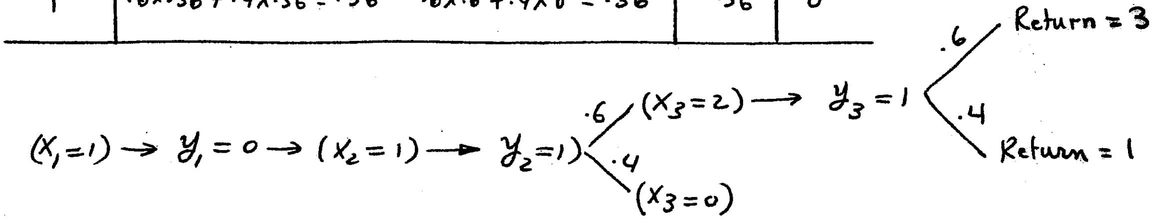
x_3	$.6\,P\{x_3+y_3\geq3\}+.4\,P\{x_3-y_3\geq3\}$					Opt. Sol.	
	$y_3=0$	1	2	3	4	f_3	y_3
0	$.6\times0+.4\times0=0$	—	—	—	—	0	0
1	$.6\times0+.4\times0=0$	$.6\times0+.4\times0=0$	—	—	—	0	0
2	$.6\times0+.4\times0=0$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	—	—	.6	1
3	$.6\times1+.4\times1=1$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	—	1	0
4	$.6\times1+.4\times1=1$	$.6\times1+.4\times1=1$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	1	0

Stage 2:

x_2	$.6\,f_3(x_2+y_2)+.4\,f_3(x_2-y_2)$			Opt. Sol.	
	$y_2=0$	1	2	f_2	y_2
0	$.6\times0+.4\times0=0$	—		0	0
1	$.6\times0+.4\times0=0$	$.6\times.6+.4\times0=.36$	—	.36	1
2	$.6\times0+.4\times0=0$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	.6	1

Stage 1:

x_1	$.6f_2(x_1+y_1)+.4f_1(x_1-y_1)$		Opt. Sol.	
	$y_1=0$	$y_1=1$	f_1	y_1
1	$.6\times.36+.4\times.36=.36$	$.6\times.6+.4\times0=.36$	.36	0



3

Stage 3: $f_3(x_3) = \max\limits_{y_3 \le x_3} \{.25\,P\{x_3+2y_3 \ge 4\} + .75\,P\{x_3 - y_3 \ge 4\}\}$

x_3	$y_3=0$	1	2	3	4	5	6	7	8	9	Opt. Sol. f_3	y_3
0	0	—									0	0
1	0	0	—								0	0
2	0	.25	.25	—							.25	1,2
3	0	.25	.25	.25	—						.25	1,2,3
4	1	.25	.25	.25	.25	—					1	0
5	1	1	.25	.25	.25	.25	—				1	0,1
6	1	1	1	.25	.25	.25	.25	—			1	0,1,2
7	1	1	1	1	.25	.25	.25	.25	—		1	0,1,2,3
8	1	1	1	1	1	.25	.25	.25	.25	—	1	0 → 4
9	1	1	1	1	1	1	.25	.25	.25	.25	1	0 → 5

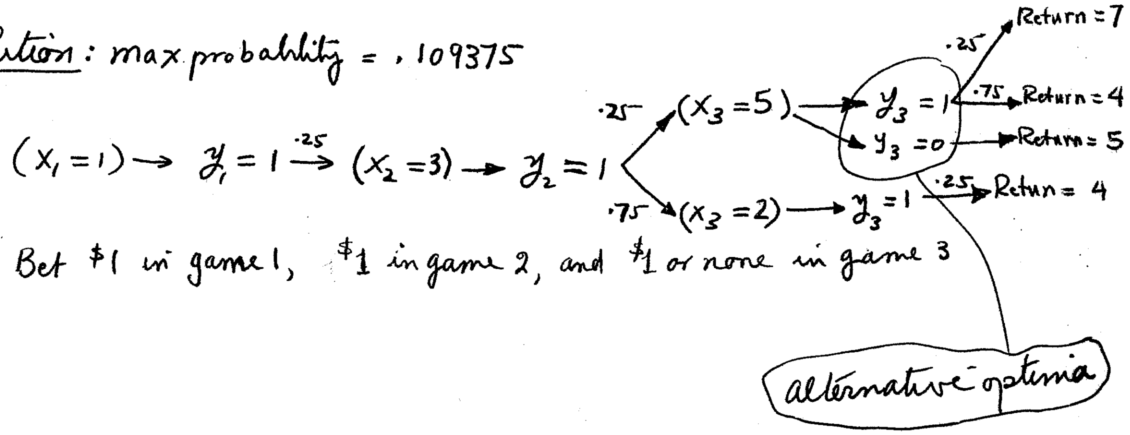
Stage 2: $f_2(x_2) = \max\limits_{y_2 \le x_2} \{.25\,f_3(x_2+2y_2) + .75\,f_3(x_2-y_2)\}$

x_2	$y_2=0$	1	2	3	Opt. Sol. f_2	y_2
0	0	—			0	0
1	0	$.25 \times .25 = .0625$	—		.0625	1
2	.25	$.25 \times 1 + .75 \times 0 = .25$	$.25 \times 1 + .75 \times 0 = .25$	—	.25	0 → 2
3	.25	$.25 \times 1 + .75 \times .25 = .4375$	$.25 \times 1 + .75 \times 0 = .25$	$.25 \times 1 + .75 \times 0 = .25$	.4375	1

Stage 1: $f_1(x_1) = \max\limits_{y_1 \le x_1} \{.25\,f_2(x_1+2y_1) + .75\,f_2(x_1-y_1)\}$

x_1	$y_1=0$	1	Opt. Sol. f_1	y_1
1	$.25 \times .0625 + .75 \times .0625 = .0625$	$.25 \times .4375 + .75 \times 0 =$	.109375	1

Solution: max probability = .109375



Bet \$1 in game 1, \$1 in game 2, and \$1 or none in game 3

Chapter 23

Markovian Decision Process

Set 23.1a

1

$$P = \begin{bmatrix} .2 & .5 & .3 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{bmatrix}$$

$$R = \begin{bmatrix} 7 & 6 & 3 \\ 7 & 4 & 0 \\ 6 & 3 & -2 \end{bmatrix}$$

2

Stationary policy	action
1	Do not fertilize
2	Fertilize when in state 1
3	Fertilize when in state 2
4	Fertilize when in state 3
5	Fertilize when in 1 or 2
6	Fertilize when in 1 or 3
7	Fertilize when in 2 or 3
8	Fertilize regardless of state

1

$v_1^1 = .9\times 2 + .1\times -1 = 1.7$

$v_2^1 = .6\times 1 + .4\times -3 = -.6$

$v_1^2 = .7\times 4 + .3\times 1 = 3.1$

$v_2^2 = .2\times 2 + .8\times -1 = -.4$

$$f_N(i) = \max_k \{v_i^k\}$$

$$f_n(i) = \max_k \{v_i^k + \sum_j p_{ij} f_{n+1}(j)\}$$

Stage 3:

i	v_i^k		Opt^m	
	$k=1$	$k=2$	$f_3(i)$	k^*
1	1.7	3.1	3.1	2
2	−.6	−.4	−.4	2

Stage 2:

i	$v_i^k + p_{i1}^k f_3(1) + p_{i2}^k f_3(2)$		Opt^m	
	$k=1$	$k=2$	$f_2(i)$	k^*
1	$1.7+.9\times 3.1+.1\times -.4 = 4.45$	$3.1+.7\times 3.1+.3\times -.4 = 5.15$	5.15	2
2	$-.6+.6\times 3.1+.4\times -.4 = 1.1$	$-.4+.2\times 3.1+.8\times -.4 = -.1$	1.1	1

Stage 1:

i	$v_i^k + p_{i1}^k f_2(1) + p_{i2}^k f_2(2)$		Opt^m	
	$k=1$	$k=2$	$f_1(i)$	k^*
1	$1.7+.9\times 5.15+.1\times 1.1 = 6.445$	$3.1+.7\times 5.15+.3\times 1.1 = 7.035$	7.85	2
2	$-.6+.6\times 5.15+.4\times 1.1 = 2.93$	$-.4+.2\times 5.15+.8\times 1.1 = 1.51$	2.93	1

Solution:

Years 1&2: Don't advertise if product is successful; Otherwise, advertise

Year 3 : Don't advertise

2

$v_1^R = 400\times .4 + 520\times .5 + 600\times .1 - 200 = \$280$

$v_2^R = 300\times .1 + 400\times .7 + 700\times .2 - 200 = \$250$

$v_3^R = 200\times .1 + 250\times .2 + 500\times .7 - 200 = \$220$

$v_1^T = 1000\times .7 + 1300\times .2 + 1600\times .1 - 900 = \$220$

$v_2^T = 800\times .3 + 1000\times .6 + 1700\times .1 - 900 = \$110$

$v_3^T = 600\times .1 + 700\times .7 + 1100\times .2 - 900 = -\$130$

continued...

$v_1^N = 400\times .2 + 530\times .5 + .3\times 710 - 300 = \$258$

$v_2^N = 450\times .7 + 800\times .3 - 300 = \$255$

$v_3^N = 400\times .2 + 650\times .8 - 300 = \$300$

Stage 3:

i	v_i			Opt^m	
	R	T	N	$f_3(i)$	Dec^n
1	280	220	258	280	R
2	250	110	255	255	N
3	220	−130	300	300	N

Stage 2:

i				Opt^m	
	R	T	N	$f_2(i)$	Dec^n
1	$280+.4\times 280+.5\times 255+.1\times 300 = 549.5$	$220+.7\times 280+.2\times 255+.1\times 300 = 497$	$258+.2\times 280+.5\times 255+.3\times 300 = 531.5$	549.5	R
2	$250+.1\times 280+.7\times 255+.2\times 300 = 516.5$	$110+.3\times 280+.6\times 255+.1\times 300 = 377$	$255+.7\times 255+.3\times 300 = 523.5$	523.5	N
3	$220+.1\times 280+.3\times 255+.7\times 300 = 534.5$	$-130+.1\times 280+.7\times 255+.2\times 300 = 136.5$	$300+0\times 280+.2\times 255+.8\times 300 = 591$	591	N

Stage 1:

i				Opt^m	
	R	T	N	$f_1(i)$	Dec^n
1	$280+.4\times 549.5+.5\times 523.5+.1\times 591 = 820.65$	$220+.7\times 549.5+.2\times 523.5+.1\times 591 = 768.45$	$258+.2\times 549.5+.5\times 523.5+.3\times 591 = 806.95$	820.65	R
2	$250+.1\times 549.5+.3\times 523.5+.7\times 591 = 845.7$	$110+.3\times 549.5+.6\times 523.5+.1\times 591 = 648.05$	$255+0\times 549.5+.7\times 523.5+.3\times 591 = 798.75$	798.75	N
3	$220+.1\times 549.5+.3\times 523.5+.7\times 591 = 845.7$	$-130+.1\times 549.5+.7\times 523.5+.2\times 591 = 409.6$	$300+0\times 549.5+.2\times 523.5+.8\times 591 = 877.5$	877.5	N

Optimum solution:

Use radio advertisement if the sales volume is poor; Otherwise use newspaper advertisement

Set 23.2a

3

(a) P^i = transition matrix given i refrig. on order. State $j \le 2-i$

$$P^0 = \begin{bmatrix} 1 & 0 & 0 \\ .8 & .2 & 0 \\ .3 & .5 & .2 \end{bmatrix} \quad P^1 = \begin{bmatrix} .8 & .2 & 0 \\ .3 & .5 & .2 \\ - & - & - \end{bmatrix}$$

$$P^2 = \begin{bmatrix} .3 & .5 & .2 \\ - & - & - \\ - & - & - \end{bmatrix}$$

(b)

Let

d_i^k = expected inventory cost/month given state i and decision k.

$d_0^0 = (0\times.2+1\times.5+2\times.3)\times150 = \$165$

$d_1^0 = 5(1\times.2+0\times.5)+(0\times.5+1\times.3)\times150 = \$46$

$d_2^0 = 5(2\times.2+1\times.5+0\times.3)+(0\times.3)\times150 = \$4.5$

$d_0^1 = 100+(1\times.2+0\times.5)\times5+1\times.3\times150 = \$146$

$d_1^1 = 100+(2\times.2+1\times.5+0\times.3)\times5 + 0\times.3\times150 = \$104.5$

$d_0^2 = 100+(2\times.2+1\times.5+0\times.3)\times50 = \$104.5$

(c)

Stage 3:

				Opt^m	
i	$k=0$	$k=1$	$k=2$	$f_3(i)$	k^*
0	165	146	104.5	104.5	2
1	46	104.5	—	46	0
2	4.5	—	—	4.5	0

Stage 2:

	$d_i^k + P_{i0}^k f_3(0)+P_{i1}^k f_3(1)+P_{i2}^k f_3(2)$			Opt^m	
	$k=0$	$k=1$	$k=2$	$f_2(i)$	k^*
0	$165+1\times104.5 = 269.5$	$146+.8\times104.5 + .2\times46 = 238.8$	$104.5 + .3\times104.5 + .5\times46 + .2\times4.5 = 159.75$	159.75	2
1	$46+.8\times104.5 + .2\times46 = 138.8$	$104.5+.3\times104.5 + .5\times46 + .2\times4.5 = 159.75$	—	138.8	0
2	$4.5+.3\times104.5 + .5\times46 + .2\times4.5 = 59.75$	—	—	59.75	0

continued...

Stage 1:

	$d_i^k + P_{i0}^k f_2(0)+P_{i1}^k f_2(1)+P_{i2}^k f_2(2)$				
	$k=0$	$k=1$	$k=2$	$f_1(i)$	k^*
0	$165+1\times159.75 = 324.75$	$146+.8\times159.75 + .2\times138.8 = 301.56$	$104.5 + .3\times159.75 + .5\times138.8 + .2\times59.75 = 233.75$	233.75	2
1	$46+.8\times159.75 + .2\times138.8 = 201.56$	$104.5+.3\times 159.75 + .5\times 138.8 + .3\times 59.75 = 239.75$	—	201.56	0
2	$4.5+.3\times159.75 + .5\times138.8 + .2\times59.75 = 13.775$	—	—	133.78	0

Optimum solution: If stock level at beginning of month is zero, order two refrigerators; otherwise order none.

4

Let

$d_{i,j}^k$ = expected inventory cost given state i, decision k and month j

P_j^k = transition matrix given month j and decision alternative k.

$$P_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ .9 & .1 & 0 \\ .5 & .4 & .1 \end{bmatrix}$$

$d_{0,1}^0 = (1\times.4+2\times.5)\times150 = 210$

$d_{1,1}^0 = (1\times.1)\times5+(1\times.5)\times150 = 75.5$

$d_{2,1}^0 = (2\times.1+1\times.4)\times5 + (0\times.5)\times150 = 3$

$$P_1^1 = \begin{bmatrix} .9 & .1 & 0 \\ .5 & .4 & .1 \\ - & - & - \end{bmatrix}$$

$d_{0,1}^1 = 100+(1\times.1)\times5 + (1\times.5)\times150 = 175.5$

$d_{1,1}^1 = 100+(2\times.1+1\times.4)\times5 = 103$

$$P_1^2 = \begin{bmatrix} .5 & .4 & .1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$d_{0,1}^2 = 100+(2\times.1+1\times.4)\times5 = 103$

$$P_2^0 = \begin{bmatrix} 1 & 0 & 0 \\ .7 & .3 & 0 \\ .2 & .5 & .3 \end{bmatrix}$$

$d_{0,2}^0 = (1\times.5+2\times.2)\times150 = 135$

$d_{1,2}^0 = (1\times.3)\times5+(1\times.2)\times150 = 31.5$

$d_{2,2}^0 = (2\times.3+1\times.5)\times5 = 5.5$

$$P_2^1 = \begin{bmatrix} .7 & .3 & 0 \\ .2 & .5 & .3 \\ - & - & - \end{bmatrix}$$

$d_{0,2}^1 = 100+(1\times3)\times5+((1\times.2)\times150 = 131.5$

$d_{1,2}^1 = 100+(2\times.3+1\times.5)\times5 = 105.5$

$$P_2^2 = \begin{bmatrix} .2 & .5 & .3 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$d_{0,2}^2 = 100+(2\times.3+1\times.5)\times5 = 105.5$

continued...

$$P_3^0 = \begin{bmatrix} 1 & 0 & 0 \\ .8 & .2 & 0 \\ .4 & .4 & .2 \end{bmatrix} \quad \begin{aligned} d_{0,3}^0 &= (1\times.5+2\times.4)\times150 = 195 \\ d_{1,3}^0 &= (1\times.2)\times5+1\times.4\times150 = 61 \\ d_{2,3}^0 &= (2\times.2+1\times.4)\times5 = 4 \end{aligned}$$

$$P_3^1 = \begin{bmatrix} .8 & .2 & 0 \\ .4 & .4 & .2 \\ - & - & - \end{bmatrix} \quad \begin{aligned} d_{0,3}^1 &= 100+(1\times.2)\times5 + (1\times.4)\times150 = 161 \\ d_{1,3}^1 &= 100+(2\times.2+1\times.4)\times5 = 104 \end{aligned}$$

$$P_3^2 = \begin{bmatrix} .4 & .4 & .2 \\ - & - & - \\ - & - & - \end{bmatrix} \quad d_{0,3}^2 = 100+(2\times.2+1\times.4)\times5 = 104$$

Stage 3:

	$d_{i,3}^k$			Opt^m	
i	$k=0$	$k=1$	$k=2$	$f_3(i)$	k^*
0	210	175.5	103	103	2
1	75.5	103	—	75.5	0
2	3	—	—	3	0

Stage 2:

	$d_{i,2}^k + p_{0,2,i}^k f_3(0) + p_{i1,2}^k f_3(1) + p_{i2,2}^k f_3(2)$			Opt^m	
i	$k=0$	$k=1$	$k=2$	$f_2(i)$	k^*
0	$135+103$ $=238$	$131.5+.7\times103$ $+.3\times75.5$ $=226.25$	$105.5+.2\times103$ $+.5\times75.5$ $+.3\times3$ $=164.75$	164.75	2
1	$31.5+.7\times103$ $+.3\times75.5$ $=126.25$	$105.5+.2\times103$ $+.5\times75.5+$ $.3\times3=164.75$	—	126.75	0
2	$5.5+.2\times103$ $+.5\times75.5$ $+.3\times3$ $=64.75$	—	—	64.75	0

Stage 1:

	$d_{i,1}^k + p_{i0,1}^k f_2(0) + p_{i1,1}^k f_2(1) + p_{i2,1}^k f_2(2)$			Opt^m	
i	$k=0$	$k=1$	$k=2$	$f_1(i)$	k^*
0	$195+103$ $=298$	$161+.8\times164.75$ $+.2\times126.25$ $=318.05$	$104+.4\times164.75$ $+.4\times126.25$ $+.2\times64.75$ $=233.35$	23	2
1	$61+.8\times164.75$ $+.2\times126.25$ $=218.05$	$104+.4\times164.75$ $+.4\times126.25$ $+.2\times64.75$ $=233.35$	—	218.05	0
2	$4+.4\times164.75$ $+.4\times126.25$ $+.2\times64.75$ $=133.35$	—	—	133.35	0

Solution: Order 2 if in state 0; otherwise, don't order.

Set 23.3a

I

Let

$s=1$: do not advertise

$s=2$: advertise regardless of stock level.

$s=3$: Advertise whenever in state 1

$s=4$: Advertise whenever in state 2

$$P^1=\begin{bmatrix} .9 & .1 \\ .6 & .4 \end{bmatrix} \quad R^1=\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$$

$$P^2=\begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} \quad R^2=\begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

$$P^3=\begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix} \quad R^3=\begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

$$P^4=\begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \quad R^4=\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

s	v_i^s	
	$i=1$	$i=2$
1	1.7	$-.6$
2	3.1	$-.4$
3	3.1	$-.6$
4	1.7	$-.4$

<u>$s=1$</u>: $\left.\begin{array}{l} .9\pi_1+.6\pi_2=\pi_1 \\ \pi_1+\pi_2=1 \end{array}\right\} \begin{array}{l} \pi_1=6/7 \\ \pi_2=1/7 \end{array}$

<u>$s=2$</u>: $\left.\begin{array}{l} .7\pi_1+.2\pi_2=\pi_1 \\ \pi_1+\pi_2=1 \end{array}\right\} \begin{array}{l} \pi_1=2/5 \\ \pi_2=3/5 \end{array}$

<u>$s=3$</u>: $\left.\begin{array}{l} .7\pi_1+.6\pi_2=\pi_1 \\ \pi_1+\pi_2=1 \end{array}\right\} \begin{array}{l} \pi_1=2/3 \\ \pi_2=1/3 \end{array}$

<u>$s=4$</u>: $\left.\begin{array}{l} .9\pi_1+.2\pi_2=\pi_1 \\ \pi_1+\pi_2=1 \end{array}\right\} \begin{array}{l} \pi_1=2/3 \\ \pi_2=1/3 \end{array}$

s	π_1^s	π_2^s	E^s
1	6/7	1/7	1.3714
2	2/5	3/5	1.
3	2/3	1/3	$\boxed{1.8667}$
4	2/3	1/3	1.

<u>Optimum decision</u>:

Advertise whenever in state 1.

Appendix A

AMPL Modeling Language

Set A.2a

1

```
#-----------------------------------------------sets
set paint;
set resource;
#-----------------------------------------parameters
param  unitprofit{paint};
param  rhs {resource};
param  aij {resource,paint};
#------------------------------------------variables
var product{paint} >= 0;
#----------------------------------------------model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to
 limit{i in resource}:
     sum{j in paint} aij[i,j]*product[j] <= rhs[i];
 marineDemand1: product["marine"]<=1.5;
 marineDemand2: product["marine"]>=.5;

data;
set paint := exterior interior marine;
set resource := m1 m2 demand market;
param unitprofit :=
                    exterior 5
                    interior 4
                    marine   3.5;
param rhs:=
                    m1       24
                    m2       6
                    demand   1
                    market   2;

param aij:  exterior  interior marine :=
          m1       6        4        .5
          m2       1        2        .75
          demand  -1        1        0
          market   0        1        0;
solve;
#-------------------------------------output results
display profit, product;
```

2

```
#-----------------------------------------------sets
set paint;
#-----------------------------------------parameters
param m;
param  unitprofit{paint};
param  rhs {1..m};
param  aij {1..m,paint};
#------------------------------------------variables
var product{paint} >= 0;
#----------------------------------------------model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to limit{i in 1..m}:
           sum{j in paint} aij[i,j]*product[j] <= rhs[i];
```

continued...

```
data;
param m:=4;
set paint := exterior  interior;
param unitprofit :=
                    exterior 5
                    interior 4;
param rhs:=
                    1        24
                    2        6
                    3        1
                    4        2;
param aij:  exterior  interior :=
          1       6         4
          2       1         2
          3      -1         1
          4       0         1;
solve;
#-------------------------------------output results
display profit, product;
```

3

```
#-----------------------------------------------sets
set paint;
set resource;
#-----------------------------------------parameters
param upper{paint};
param lower{paint};
param  unitprofit{paint};
param  rhs {resource};
param  aij {resource,paint};
#------------------------------------------variables
var product{i in paint} >= lower[i],<=upper[i];
#----------------------------------------------model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to limit{i in resource}:
           sum{j in paint} aij[i,j]*product[j] <= rhs[i];
data;
set paint := exterior interior;
set resource := m1 m2 demand market;
param upper:="exterior" 2 "interior" 2.5;
param lower:="exterior" 1 "interior" 0;
param unitprofit :=
                    exterior 5
                    interior 4;
param rhs:=
                    m1       24
                    m2       6
                    demand   1
                    market   2;

param aij:  exterior  interior :=
          m1       6        4
          m2       1        2
          demand  -1        1
          market   0        1;

solve;
#-------------------------------------output results
display profit, product;
```

4

```
#----------------------------sets
set paint;
set resource;
#-------------------------------------------parameters
param  unitProfit{paint};
param  rhs {resource};
param  aij {resource,paint};
#--------------------------------------------variables
var product{paint} >=0;
var varProfit{j in paint}=unitProfit[j]*product[j];
var resourceUse{i in resource,j in paint}=aij[i,j]*product[j];
#-----------------------------------------------model
maximize profit: sum{j in paint} varProfit[j];
subject to limit{i in resource}:
              sum{j in paint} resourceUse[i,j] <= rhs[i];
data;
set paint := exterior interior;
set resource := m1 m2 demand market;
param unitProfit :=
                        exterior 5
                        interior 4;
param rhs:=
                        m1        24
                        m2        6
                        demand    1
                        market    2;

param aij:  exterior  interior :=
            m1        6         4
            m2        1         2
            demand    -1        1
            market    0         1;
solve;
#---------------------------------------output results
display profit, product,varProfit,resourceUse;
display {j in paint}(resourceUse["m1",j],resourceUse["m2",j]);
```

5

```
#-----------------------------------------------sets
set input;
set output;
#-----------------------------------------parameters
param unitCost{input};
param yield{output,input};
param specs{output};
param minNeeds;
#------------------------------------------variables
var feedStuff{input} >=0;
var farmUse=sum{j in input}feedStuff[j];
#---------------------------------------------model
minimize cost:sum{j in input}unitCost[j]*feedStuff[j];
subject to
 aa: farmUse>=minNeeds;
 bb{i in output}:
   sum{j in input}yield[i,j]*feedStuff[j]<=specs[i]*farmUse;
data;
set input := corn soy;
set output := protein fiber;
param minNeeds:=800;
param unitCost := corn .3 soy .9;
param specs:= protein -.3 fiber .05; #negative because of <=
param yield:   corn    soy :=
          protein   -.09      -.6
          fiber      .02      .06;


solve;
#-------------------------------------output results
display cost,feedStuff, feedStuff.rc>a.txt;
display aa.dual,bb.dual>a.txt;
```

OUTPUT

```
cost = 437.647

:   feedStuff  feedStuff.rc   :=
corn   470.588   8.32667e-17
soy    329.412   -1.11022e-16
;

aa.dual = 0.547059

bb.dual [*] :=
  fiber  -2.05116e-15
protein  -1.17647
```

Reduced cost shows that both corn and soy assume positive values in the optimum solution.

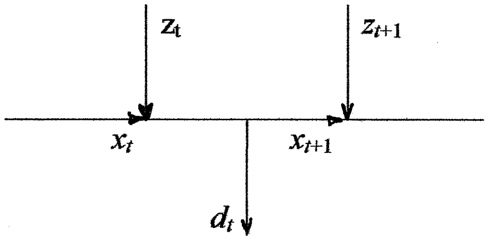
Dual price for constraint aa shows that a 1 unit increase in minNeeds increases the total cost by $.55, approximately.

Set A.3a

1

```
param n;
param c{1..n};
var x{1..n};
rest{i in 1..n}:(if i<=n-1 then x[i]+x[i+1] else
     x[1]+x[n])>=c[i];
```

2



$x_1 = c,\ x_{T+1} = 0$

```
param T;
param c{1..T};
var x{1..T};
subject to
Period{t in 1..T}:
 (if  t=1 then c else x[t]) + z[t] - d[t] -
                         (if t<T then x[t+1] else 0)=0;
```

Set A.4a

1

(a)

```
param m;
param n;
param k;
param p;
param q;
param c
#.......................method 1
set S1={1..m union m+k..n union n+p..q}
var x{S1};
subject to limit: sum{j in S1}x[j]>=c;
#.......................method 2
set S2={1..q diff (m+1..m+k-1 union
n+1..n+p-1)}
var x{S2};
subject to limit: sum{j in S2}x[j]>=c;
```

(b)

```
para m;
param n;
param c;
param k;
var x{i in m..2*n+k};
#.......................method 1
subject to CC:
   sum{i in m..2*n+k diff n+1..n+k-1}
   x[i]<=c;
#.......................method 2
subject to CC:
   sum{i in m..2*n+k: i<=n or i>=n+k}x[i]
   <=c;
```

2

(See file a.4a-2.txt)

```
set productsUsingComp{1..5};
param c{1..5};      #component cost
param a{1..5};      #min availability
param d; #maximum demand for each product
var x{1..10,1..5}>=0; # units of product i that use component j

minimize z: sum{j in 1..5}(c[j]*(sum{i in
productsUsingComp[j]}x[i,j]));
subject to
  C{j in 1..5}:sum{i in productsUsingComp[j]}x[i,j]>=a[j];
  D{i in 1..10}: sum{j in 1..5}x[i,j]<=d;
data;
set productsUsingComp[1]:=1 2 5 10;
set productsUsingComp[2]:=3 6 7 8 9;
set productsUsingComp[3]:=1 2 3 5 6 7 9;
set productsUsingComp[4]:=2 4 6 8 10;
set productsUsingComp[5]:=1 3 4 5 6 7 9 10;
param a:=1 500 2 400 3 900 4 700 5 100;
param c:=1 9  2 4  3 6  4 5  5 8;
param d:=300;
display productsUsingComp;
solve;display x;
```

3

In the following code, the indexed set `componentsInProduct` is determined directly from the original data, which precludes the need to determine the elements of componentsInProduct[i], i = 1, 2, ..., 10, manually.

```
set productsUsingComp{1..5};
set componentsInProduct{i in 1..10}=
            {j in 1..5:i in productsUsingComp[j]};
param c{1..10};     #component installation cost
param a{1..5};      #min availability
param d; #maximum demand for each product
var x{1..10,1..5}>=0;# units of product i that use component j

minimize z: sum{i in 1..10}c[i]*(sum{j in
componentsInProduct[i]}x[i,j]);
subject to
  C{j in 1..5}:sum{i in productsUsingComp[j]}x[i,j]>=a[j];
  D{i in 1..10}: sum{j in 1..5}x[i,j]<=d;
data;
set productsUsingComp[1]:=1 2 5 10;
set productsUsingComp[2]:=3 6 7 8 9;
set productsUsingComp[3]:=1 2 3 5 6 7 9;
set productsUsingComp[4]:=2 4 6 8 10;
set productsUsingComp[5]:=1 3 4 5 6 7 9 10;

param a:=1 500 2 400 3 900 4 700 5 100;
param c:=1 1  2 3  3 2  4 6  5 4  6 9  7 2  8 5  9 10  10 7;
param d:=300;
display productsUsingComp,componentsInProduct;
solve;display x;
```

Set A.5a

1

File RM3x.dat: The first row gives `unitprofit`. The first column in the remaining 4 rows defines `rhs`, and the second and third columns give `aij`.

```
5   4
24  6   4
6   1   2
1  -1   1
2   0   1
```

2

File RM3xx.dat: Column 1 gives `rhs`. Coulmn 2 repeats `unitprofit[1]` as many times as the number of constraints. Coulmn 3 repeats `unitprofit[2]` as many times as the number of constraints. Columns 3 and 5 give `aij`. Convoluted data file!

```
24 5  6  4  4
6  5  1  4  2
1  5 -1  4  1
2  5  0  4  1
```

Set A.5b

1

```
#-----------------------------------------------sets
set paint;
set resource;
#------------------------------------------parameters
param  unitprofit{paint};
param  rhs {resource};
param  aij {resource,paint};
#-------------------------------------------variables
var product{paint} >= 0;
#-----------------------------------------------model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to limit{i in resource}:
          sum{j in paint} aij[i,j]*product[j]<=rhs[i];
data;
set paint := exterior interior;
set resource := m1 m2 demand market;
param unitprofit :=
                    exterior 5
                    interior 4;
param rhs:=
                    m1        24
                    m2        6
                    demand    1
                    market    2;

param aij:  exterior  interior :=
          m1        6         4
          m2        1         2
          demand    -1        1
          market    0         1;
solve;
#--------------------------------------output results
printf "Objective value = %5.2f\n",profit>a1.txt;
printf "-------------------------------------\n">a1.txt;
printf "Product     Quantity     Profit($)\n">a1.txt;
printf "-------------------------------------\n">a1.txt;
printf {j in
paint}"%s%11.2f%15.2f\n",j,product[j],unitprofit[j]*product[j]>a1.txt;
printf "-------------------------------------\n">a1.txt;
printf "Constraint   Slack amount   Dual price\n">a1.txt;
printf "-------------------------------------\n">a1.txt;
printf {i in
resource}"%s%12.2f%15.2f\n",i,limit[i].slack,limit[i].dual>a1.txt;
printf "-------------------------------------\n">a1.txt;
```

OUTPUT:

```
Objective value = 21.00
-------------------------------------
Product       Quantity         Profit($)
-------------------------------------
exterior        3.00            15.00
interior        1.50             6.00
-------------------------------------
Constraint    Slack amount    Dual price
-------------------------------------
m1          0.00            0.75
m2          0.00            0.50
demand          2.50            0.00
market          0.50            0.00
-------------------------------------
```

Set A.5c

1

Sets `paint` and `resource` cannot be read from the double-subscripted table `RM4aij`, and hence will not be defined for `unitprofit` and `rhs`.

2

```
#--------------------------------------------------sets
set resource;
set paint;
#---------------------------------------------parameters
param  unitprofit{paint};
param  rhs {resource};
param  aij {resource,paint};
#----------------------------------------------variables
var product{paint} >= 0;
#-----------------------------------------model objective
maximize profit: sum {j in paint}
unitprofit[j]*product[j];
#---------------------------------------model constraints
subject to limit {i in resource}:
          sum {j in paint} aij[i,j]*product[j] <= rhs[i];
#--------------------------------------------read tables
table RM4profit IN: paint<-[COL1], unitprofit~COL2;
table RM4rhs IN: resource<-[constrName],
rhs~availability;
table RM4aij IN: [resource,paint], aij;
#table RM4arrayAij IN:[i~resource],{j in
paint}<aij[i,j]~(j)>;
#--------------------------------------------write tables
table varData OUT:[paint],product,product.rc;
table conData
OUT:[resource],limit.slack~slack,limit.dual~DUal;

read table RM4profit;
read table RM4rhs;
read table RM4aij;

#read table RM4arrayAij;
#---------------------------------------Solution command
solve;
#--------------------------------------write table files
write table varData;
write table conData;
#-----------------------------------------output results
display profit, product, limit.dual, product.rc;
#------------------------------------------end of model
```

Set A.7a

1

(a)

```
let rhs["m1"]:=20;
for {i in 1..100000}
   {
   solve;
   display rhs["m1"],product;
   if rhs["m1"]=35 then break;
   let rhs["m1"]:=rhs["m1"]+5;
   }
```

(b)

```
let rhs["m1"]:=20;
repeat while rhs["m1"]<=35
         {
         solve;
         display rhs["m1"],product;
         let rhs["m1"]:=rhs["m1"]+5;
         }
```

(c)

```
let rhs["m1"]:=20;
repeat until rhs["m1"]>35
         {
         solve;
         display rhs["m1"],product;
         let rhs["m1"]:=rhs["m1"]+5;
         }
```

Chapter 2 Cases

2-1

The cost distribution as figured out by the Accounting Department is not suitable for economic analysis. The purchasing cost of 3,000,000 lb of oranges (= .19 x 3,000,000 = \$570,000 is fixed, and hence plays no role in the development of the model. The variable cost per 5-gallon can should be recomputed to exclude the fixed cost of purchasing the oranges--that is,

For jam:

Sales price/can = \$15.50
Variable cost/can = 9.85 - 5 x (25.61/100)
= \$8.57
Gross profit per lb = (15.50 - 8.57)/5
= \$1.39

For concentrate:

Sales price/can = \$30.25
Variable cost/can = 21.05 - 30x(21.22/100)
= \$14.68
Gross profit per lb = (30.25 - 14.68)/30
= \$0.52

For juice:

Sales price/can = \$20.75
Variable cost/can = 13.28 - 30x(21.22/100)
= \$10.10
Gross profit per lb = (20.75 - 10.10)/15
= \$0.71

LP Model:

x_1 = lb oranges used in jam
x_2 = lb oranges used in concentrate
x_3 = lb oranges used in juice

Maximize $z = 1.39x_1 + .52x_2 + .71x_3 - \$570,000$
subject to

$$\begin{aligned}
x_1 &\le .3 \text{ x } 3,000,00\\
x_1/5 &\le 10,000\\
x_2/30 &\le 12,000\\
x_3/15 &\le 40,000\\
x_2/30 &\ge 2(x_3/15)\\
x_2 + x_3 &\le .6 \text{ x } 3,000,000\\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

The second constriant dominates the first constraint. This means that there is a definite surplus of at least 900,000 - 5 x 10,000 = 850,000 lb of Grade I oranges. Because concentrate and juice can use Grade I, the last constraint should be changed to

$$x_2 + x_3 \le 1,800,000 + 850,000 = 2,650,000$$

This means that the extra 850,000 lb of grade I can be used to produce concentrate and juice, if necessary.

Solution:

```
Final Iteration No: 5
Objective value (max) =320603.4688
-----------------------------------------------------------------
Variable           Value        Obj Coeff     Obj Val Contrib
-----------------------------------------------------------------
x1              50000.0000       1.3900         69500.0000
x2             360003.5938       0.5200        187201.8594
x3              90002.2500       0.7100         63901.5938
```

2-2

Let

x_{ij} = Rolls produced of type i in month j
s_{ij} = Rolls purchased of type i in month j

The objective function can be formulated to minimize the total cost of producing the rolls internally and/or purchasing them externally.

The maximum demand from the two mills can be summarized as:

	Roll 1	Roll 2	Roll 3
Month 1	700	300	400
Month 2	300	500	700
Month 3	100	400	500

LP Model:

Maximize $z = 90(x_{11} + x_{12} + x_{13}) + 130(x_{21} + x_{22} + x_{23}) + 180(x_{31} + x_{32} + x_{33}) + 108(s_{11} + s_{12} + s_{13}) + 145(s_{21} + s_{22} + s_{23}) + 194(s_{31} + s_{32} + s_{33})$

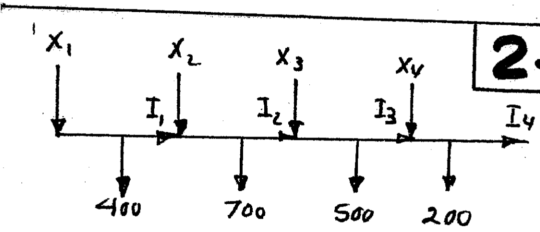
subject to

$$\begin{aligned}
&x_{11} + x_{12} + x_{13} + 5(x_{21} + x_{22} + x_{23}) + 7(x_{31} + x_{32} + x_{33}) \le 3 \text{ x } 320 \text{ x } 10\\
&4(x_{21} + x_{22} + x_{23}) + 6(x_{31} + x_{32} + x_{33}) \le 3 \text{ x } 310 \text{ x } 8\\
&6(x_{11} + x_{12} + x_{13}) + 3(x_{21} + x_{22} + x_{23}) \le 3 \text{ x } 300 \text{ x } 9\\
&3(x_{11} + x_{12} + x_{13}) + 6(x_{21} + x_{22} + x_{23}) + 9(x_{31} + x_{32} + x_{33}) \le 3 \text{ x } 320 \text{ x } 10\\
&x_{11} + s_{11} = 700,\ x_{12} + s_{12} = 300,\ x_{13} + s_{13} = 400\\
&x_{21} + s_{21} = 300,\ x_{22} + s_{22} = 500,\ x_{23} + s_{23} = 700\\
&x_{31} + s_{31} = 100,\ x_{32} + s_{32} = 400,\ x_{33} + s_{33} = 500\\
&\text{all } x_{ij}\ s_{ij} \ge 0
\end{aligned}$$

The solution of this model by TORA will show that it is cheaper to purchase all the rolls from outside source than to produce them locally. On the other hand, if we try to limit outside purchases, s_{ij}, to 5% of the total demand, as specified by the company, the problem will have no feasible solution.

These results point to the possibility that the company should be studying whether or not its present operation is economically viable.

2-3



$$\begin{aligned}
\text{minimize } Z = &\ 100(I_1 + I_2 + I_3 + I_4)\\
&+ 60(y_{12}^- + y_{23}^- + y_{34}^-)\\
&+ 50(y_{12}^+ + y_{23}^+ + y_{34}^+)
\end{aligned}$$

Subject to

$$X_1 - I_1 = 400$$

continued...

2-4

$$x_2 + I_1 - I_2 = 700$$
$$x_3 + I_2 - I_3 = 500$$
$$x_4 + I_3 - I_4 = 200$$
$$x_1 - x_2 - y_{12}^{+} + y_{12}^{-} = 0$$
$$x_2 - x_3 - y_{23}^{+} + y_{23}^{-} = 0$$
$$x_3 - x_4 - y_{34}^{+} + y_{34}^{-} = 0$$

all variables ≥ 0

```
---------------------------------------------------------------------
Title: Comprehensive Problem 2-3
Final iteration No: 8
Objective value (min) =32500.0000
--------------------------------------------------------------
Variable              Value      Obj Coeff    Obj Val Contrib
--------------------------------------------------------------
x1  x1              550.0000       0.0000          0.0000
x2  x2              550.0000       0.0000.         0.0000
x3  x3              200.0000       0.0000          0.0000
x4  x4              200.0000       0.0000          0.0000
x5  I1              150.0000     100.0000      15000.0000
x6  I2                0.0000     100.0000          0.0000
x7  I3                0.0000     100.0000          0.0000
x8  I4                0.0000     100.0000          0.0000
x9  y12+              0.0000      50.0000          0.0000
x10 y12-              0.0000      60.0000          0.0000
x11 y23+            350.0000      50.0000      17500.0000
x12 y23-              0.0000      60.0000          0.0000
x13 y34+              0.0000      50.0000          0.0000
x14 y34-              0.0000      60.0000          0.0000

-----------------------------------------------------------
Constraint            RHS           Slack(-)/Surplus(+)
-----------------------------------------------------------
1 (=)               400.0000          0.0000
2 (=)               700.0000          0.0000
3 (=)               200.0000          0.0000
4 (=)               200.0000          0.0000
5 (=)                 0.0000          0.0000
6 (=)                 0.0000          0.0000
7 (=)                 0.0000          0.0000
```

$$i = \begin{cases} 1 & \text{represent chairs} \\ 2 & \text{represent tables} \\ 3 & \text{represent bookshelves} \end{cases}$$

x_{ij} = units of type i produced in period j

S_{ij} = units of type i sold in period j

z_{ij} = inventory units of type i left at the end of period j

The objective function includes three components:

1. Sales revenue
2. Production cost
3. Inventory cost

The constraints include

1. Production capacity
2. Demand limit
3. Labor requirement
4. Inventory balance equations

$$\text{Maximize } z = \sum_{j=1}^{3} (45 S_{1j} + 100 S_{2j} + 20 S_{3j})$$
$$- (25 x_{1j} + 65 x_{2j} + 10 x_{3j})$$
$$- .02 (25 z_{1j} + 65 z_{2j} + 10 z_{3j})$$

Subject to

$$x_{1j} \leq 3000, \quad x_{2j} \leq 1000, \quad x_{3j} \leq 580, \quad j = 1,2,3$$
$$S_{11} \leq 2800, \quad S_{12} \leq 2300, \quad S_{13} \leq 3350$$
$$S_{21} \leq 500, \quad S_{22} \leq 800, \quad S_{23} \leq 1400$$
$$S_{31} \leq 320, \quad S_{32} \leq 300, \quad S_{33} \leq 600$$

$$20 x_{1j} + 40 x_{2j} + 15 x_{3j} \leq \underline{150} \times 5 \times 4 \times 2 \times 8 \times 60$$
$$j = 1,2,3$$

$$z_{ij} = z_{i,j-1} + x_{ij} - S_{ij}, \quad i = 1,2,3$$
$$j = 1,2,3$$

$$z_{10} = 30, \quad z_{20} = 100, \quad z_{30} = 50$$

all variables ≥ 0

The effect of leaves can be investigated by changing the right-hand side of labor requirement constraint.

Chapter 3 Cases

3-1

Although LP theory says that in a simplex method solution the number of positive variables cannot exceed the number of constraints, the real issue in the presentation by the OR analyst is that the model is not complete. The fact that the manager insists that <u>all</u> five products must be produced indicates that some important restrictions are missing. In particular, the impact of the competition appears to be important, at least from the manager's standpoint. Such restrictions must then be taken into account.

Although the analyst is correct in stating that LP theory requires more constraints to produce a desired product mix, the argument should be made from the standpoint of formulating the model properly and realistically. Once this task is done, it would not be necessary to "bargain" with the manager about the need to add more constraints. Rather, the realistic model will indicate whether or not the production system is operating optimally.

Of course, when all the restrictions of the problem are taken into account, it may well be that the resulting model would not be a linear program at all.

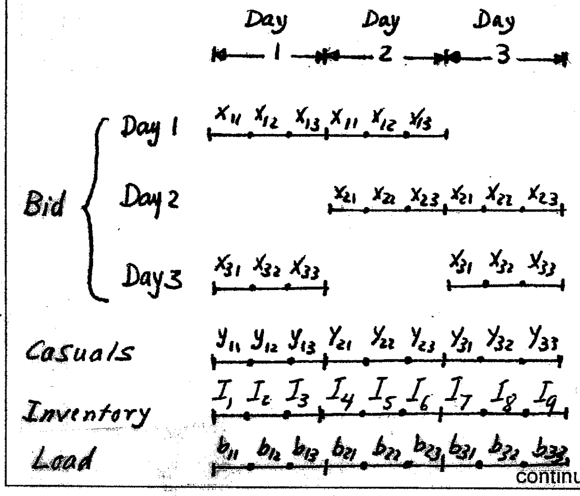
The conclusion from the analysis of this situation is that one must always aim at formulating the model to represent the original system as realistically as possible.

3-2

Consider a scaled-down model:

1. 3-day week
2. Each worker works 2 consecutive days per 3-day week
3. Loads are assumed constant over each 8-hour shift

The situation can be depicted as follows:



continued...

The situation assumes that the number of casuals that can be hired is limited; that is,

$$y_{ij} \le C_{ij} \text{ for all } i \text{ and } j$$

The inventory variables I_i represent is the load amount left unfinished from the preceding period $i-1$

For the purpose of defining the objective function, we want to minimize the number of bid and casual workers used, we will assign the bid worker half the weight we assign to the casual workers. Thus, the objective function is

$$\text{Minimize } z = \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij} + 2\sum_{i=1}^{3}\sum_{j=1}^{3} y_{ij}$$

The constraints take into account the fact that a load is allowed to stay on the dock up to 16 hours (or two shifts). Thus, (See the chart)

$$(x_{11}+x_{12})+(x_{31}+x_{32})+y_{11} \ge b_{11}$$
$$(x_{12}+x_{13})+(x_{32}+x_{33})+y_{12} \ge b_{12}$$
$$(x_{11}+x_{33})+(x_{13}+x_{21})+y_{13} \ge b_{13}$$
$$(x_{11}+x_{12})+(x_{21}+x_{22})+y_{21} \ge b_{21}$$
$$(x_{12}+x_{13})+(x_{22}+x_{23})+y_{22} \ge b_{22}$$
$$(x_{13}+x_{21})+(x_{23}+x_{31})+y_{23} \ge b_{23}$$
$$(x_{21}+x_{22})+(x_{31}+x_{32})+y_{31} \ge b_{31}$$
$$(x_{22}+x_{23})+(x_{32}+x_{33})+y_{32} \ge b_{32}$$
$$(x_{11}+x_{33})+(x_{23}+x_{31})+y_{3} \ge b_{33}$$
$$x_{11}+x_{31}+y_{11}-I_1+I_2 = b_{11}$$
$$x_{12}+x_{32}+y_{12}-I_2+I_3 = b_{12}$$
$$x_{13}+x_{33}+y_{13}-I_3+I_4 = b_{13}$$
$$x_{11}+x_{21}+y_{21}-I_4+I_5 = b_{21}$$

continued...

$x_{12} + x_{22} + y_{22} - I_5 + I_6 = b_{22}$

$x_{13} + x_{23} + y_{23} - I_6 + I_7 = b_{23}$

$x_{21} + x_{31} + y_{31} - I_7 + I_8 = b_{31}$

$x_{22} + x_{32} + y_{32} - I_8 + I_9 = b_{32}$

$x_{23} + x_{33} + y_{33} - I_9 + I_1 = b_{33}$

$y_{ij} \leq c_{ij}$

all variables are nonnegative.

The model has been tested for a 7-day week, three shifts per day, with the loads held constant over 4-hour intervals (instead of 8). The results were successfully implemented.

3-3

x_{ij} = bbl from line item i allocated to bidder j

b_{ij} = Bid for line item i by bidder j

C_i = capacity (bbl) of line item i

Maximize $z = \sum_{i=1}^{6} \sum_{j=1}^{8} b_{ij} x_{ij}$

s.t. $\sum_{i=1}^{6} x_{ij} \leq 36{,}000, \; j = 1, 2, \cdots, 8$ ①

$\sum_{j=1}^{8} x_{ij} \leq c_i, \; i = 1, 2, \cdots, 6$ ②

$x_{ij} \geq 0$

Solution: (See file ampCase3-3.txt)

	1	2	3	4	5	6	7	8	
1			20						20
2					13		17		30
3						6	19		25
4	4							36	40
5		19	16						35
6						30			30
Totals	4	19	36	0	13	36	36	36	

Revenue = \$202,180

continued...

Ranking solution:

	1	2	3	4	5	6	7	8
1			20					
2			16				14	
3						3	22	
4	4							36
5		35						
6	30							
Total	34	35	36	0	0	3	36	36

Revenue = \$201,550

LP solution is better than the ranking solution by \$630.

To investigate the effect of setting the 20% limit, the associated constraints ① yield the following dual prices:

Constraint	Dual price
1	0
2	0
3	.03
4	0
5	0
6	.01
7	.02
8	.02

The fact that some of the dual prices are positive shows that there are advantages in raising the limit above 20% as this will increase the total revenue. Indeed, the following table shows the change in revenue with the limit:

Percentage	Revenue(\$)
20	202,180
30	203,110
40	203,320
≥ 48	203,450

Thus, it is advantageous in this case to raise the limit to 48% as this will raise the revenue from \$202,180 at 20% to \$203,450. The best percentage is data dependent.

Chapter 4 Cases

4-1

Maximize $z = 3x_1 + 2x_2 + 5x_3$

subject to

$$
\begin{array}{llr}
(F1) & x_1 + 2x_2 + x_3 \le 430 & ① \\
(F2) & 3x_1 + 2x_3 \le 460 & ② \\
(R1) & x_1 + 4x_2 \le 420 & ③ \\
(R2) & x_1 + x_2 + x_3 \le 300 & ④ \\
 & x_3 \le 240 & ⑤
\end{array}
$$

$$x_1, x_2, x_3 \ge 0$$

Optimum Solution (excluding ⑤)

```
-----------------------------------------------------------------
Title: Comprehensive Problem 4-1
Final iteration No: 3
Objective value (max) = 1290.0000
-----------------------------------------------------------
Variable              Value       Obj Coeff    Obj Val Contrib
-----------------------------------------------------------
x1 P1                0.0000        3.0000          0.0000
x2 P2               70.0000        2.0000        140.0000
x3 P3              230.0000        5.0000       1150.0000

---------------------------------------------------------
Constraint             RHS           Slack(-)/Surplus(+)
---------------------------------------------------------
1 (<)              430.0000           60.0000-
2 (<)              460.0000            0.0000-
3 (<)              420.0000          140.0000-
4 (<)              300.0000            0.0000-

                     *** SENSITIVITY ANALYSIS ***
Objective coefficients -- Single Changes:
-----------------------------------------------------------------
Variable       Current Coeff     Min Coeff     Max Coeff    Reduced Cost
-----------------------------------------------------------------
x1 P1               3.0000       -infinity        6.5000          3.5000
x2 P2               2.0000          0.0000        5.0000          0.0000
x3 P3               5.0000          2.6667      infinity          0.0000

Right-hand Side -- Single Changes:
-----------------------------------------------------------------
Constraint       Current RHS       Min RHS       Max RHS      Dual Price
-----------------------------------------------------------------
1 (<)              430.0000        370.0000      infinity          0.0000
2 (<)              460.0000        390.0000      600.0000          1.5000
3 (<)              420.0000        280.0000      infinity          0.0000
4 (<)              300.0000        230.0000      330.0000          2.0000
```

The given optimum solution satisfies constraint ⑤. Hence ⑤ is redundant.

The associated optimal inverse B^{-1} is

$$X_B = \begin{pmatrix} x_4 \\ x_3 \\ x_6 \\ x_2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 1/2 & 0 & -2 \\ 0 & 1/2 & 0 & 0 \\ 0 & 2 & 1 & -4 \\ 0 & -1/2 & 0 & 1 \end{pmatrix}$$

Question 1:

The additional constraint

$$x_3 \le 210$$

is not satisfied the given optimal solution. We use the dual simplex method to recover the new feasible

continued...

solution. Then, we study the effect of the 20% increase in profitability on the optimality of the resulting feasible solution.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Solution
z	7/2	0	0	0	3/2	0	2	0	1290
x_4	1/2	0	0	1	1/2	0	-2	0	60
x_3	3/2	0	1	0	1/2	0	0	0	230
x_6	3	0	0	0	2	1	-4	0	140
x_2	-1/2	1	0	0	-1/2	0	1	0	70
x_8	0	0	1	0	0	0	0	1	210
z	7/2	0	0	0	3/2	0	2	0	1290
x_4	1/2	0	0	1	1/2	0	-2	0	60
x_3	3/2	0	1	0	1/2	0	0	0	230
x_6	3	0	0	0	2	1	-4	0	140
x_2	-1/2	1	0	0	-1/2	0	1	0	70
x_8	-3/2	0	0	0	-1/2	0	0	1	-20
z	0	0	0	0	1/3	0	2	7/3	1243 1/3
x_4	0	0	0	1	1/3	0	-2	1/3	53 1/3
x_3	0	0	1	0	0	0	0	1	210
x_6	0	0	0	0	1	1	-4	2	100
x_2	0	1	0	0	-1/3	0	1	-1/3	76 2/3
x_1	1	0	0	0	1/3	0	0	-2/3	13 1/3

Next, we study the effect of changing the unit profit by +20% for variable x_3

New dual values

$$Y = (0, \mathbf{6}, 0, 2, 3)\begin{bmatrix} 1 & 1/3 & 0 & -2 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -4 & 2 \\ 0 & -1/3 & 0 & 1 & -1/3 \\ 0 & 1/3 & 0 & 0 & -2/3 \end{bmatrix}$$

$$= (0, 1/3, 0, 2, 10/3)$$

continued...

The new objective row thus reads as

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
z	0	0	0	0	1/3	0	2	10/3	$1453\frac{1}{3}$

Thus, the last basic solution remains optimal.

Conclusions:

1. The new constraint $x_3 \leq 210$ reduces the profit from \$1290 to \$1243.33.
2. The 20% increase in the unit profit of x_3 increases the total profit to \$1453.33.
3. The proposal is acceptable because, in the end, the total profit increases from \$1290 to \$1453.33

Question 2:

From TORA's printout,
worth per unit of R2 = \$2

Because the unit price of additional units of R2 is \$3 higher than the present supplier, the additional cost is not justifiable economically.

Question 3:

```
Objective Coefficients -- Simultaneous Changes d:
---------------------------------------------------------------------------
Nonbasic Var  Optimality Condition
---------------------------------------------------------------------------
x1 P1          3.5000 +      1.5000 d3 +     -0.5000 d2 - d1 >= 0
sx5             1.5000 +      0.5000 d3 +     -0.5000 d2 >= 0
sx7             2.0000 +      1.0000 d2 >= 0

Right-hand Side Ranging -- Simultaneous Changes D:
---------------------------------------------------------------------------
Basic Var    Value/Feasibilty Condition
---------------------------------------------------------------------------
sx4            60.0000 +      1.0000 D1 +     0.5000 D2 +     -2.0000 D4
               >= 0
x3 P3         230.0000 +     0.5000 D2 >= 0
sx6           140.0000 +      2.0000 D2 +     1.0000 D3 +     -4.0000 D4
               >= 0
x2 P2          70.0000 +    -0.5000 D2 +     1.0000 D4 >= 0
```

TORA printout provides the effect of simultaneous changes in the availability of the resources. We can get the same result by considering

continued...

$$\begin{pmatrix} x_4 \\ x_3 \\ x_6 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 0 & -2 \\ 0 & 1/2 & 0 & 0 \\ 0 & 2 & 1 & -4 \\ 0 & -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 430+D_1 \\ 460+D_2 \\ 420 \\ 300 \end{pmatrix}$$

$$= \begin{pmatrix} 60 + D_1 + .5D_2 \\ 230 + .5D_2 \\ 140 + 2D_2 \\ 70 - .5D_2 \end{pmatrix}$$

Thus, for $D_1 = D_2 = 40$, the new Solution is

$x_1 = 0$ $x_2 = 50$, $x_3 = 250$,
$x_4 = 120$, $x_5 = 0$, $x_6 = 220$, $x_7 = 0$
which is feasible.

New profit = $3\times0 + 2\times50 + 5\times250$
= \$1350

Increase in profit = 1350 - 1290 = \$60

This result can also be computed from the dual objective function as

$$\Delta Z = D_1 y_1 + D_2 y_2$$
$$= 40\times0 + 40\times\frac{3}{2} = \$60$$

Because the worth per unit of F1 is zero, it means that resource F1 is already abundant (indeed, $x_4 = 60$ minutes). Hence, we need to increase F2 only by $D_2 = 40$ min at the cost of \$35, and the proposal is justifiable economically.

Question 4:

This question can be analyzed by adding the constraint $x_2 \geq 100$ and applying the dual simplex method.

The new optimum tableau is

continued...

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
z	2	0	0	3	0	0	0	5	1200
x_5	0	0	0	1	1	0	0	-1	30
x_6	1	0	0	-2	0	1	0	-2	60
x_7	1	0	0	4	0	0	1	0	20
x_3	1	0	1	1	0	0	0	1	200
x_2	0	1	0	-1	0	0	0	0	100

The new restriction reduces the profit by $90.

We should have expected this result even before the new tableau is computed. The reason is that the present solution does not satisfy the new constraint. Hence, the value of the objective function must deteriorate.

Question 5

Decrease in the unit processing time of P1 on F2 will produce the following reduced cost:

$$1y_1 + (3-1)y_2 + 1y_3 + 1y_4 - \$3$$

$$= 1\times 0 + 2\times\tfrac{3}{2} + 1\times 0 + 1\times 2 - 3$$

$$= 2$$

Thus, the reduction in the processing still would not make P1 profitable

4-2

The first proposal should not produce the desired results because it is based on an averaging procedure that does not have a valid theoretical basis. The second proposal may produce the desired result provided that the remaining two constraints are not violated.

We can check both proposals by computing

continued...

$$X_B = B^{-1}b^*$$

where b^* is the new right-hand side; that is

$$b^* = \begin{pmatrix} 32.4 \\ 14.4 \\ 1 \\ 2 \end{pmatrix} \text{ for proposal 1}$$

$$= \begin{pmatrix} 30 \\ 7.5 \\ 1 \\ 2 \end{pmatrix} \text{ for proposal 2}$$

Proposal 1:

$$X_B = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 32.4 \\ 14.4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} .9 \\ 6.75 \\ -4.85 \\ -4.75 \end{pmatrix}$$

The proposal does not result in 25% increase in x_1 and x_2. Moreover, the resulting solution is infeasible.

Proposal 2:

$$X_B = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 7.5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3.75 \\ 1.875 \\ 2.875 \\ .125 \end{pmatrix}$$

New solution is feasible. It also results in the desired 25% increase in x_1 and x_2

5-1

First, we compute the percent distribution of commodities to the different sites:

Site	Percent	
N1	.6 x .85 =	.51
N2	.6 x .15 =	.09
C1	.15 x .6 =	.09
C2	.15 x .4 =	.06
S1	.25 x .8 =	.20
S2	.25 x .2 =	.05

Next, instead of dealing with three different types of products, we convert all them to equivalent returnables bottles by using the given transportation cost factors of 60% for cans and 70% for nonreturnables. The supply amount at the new factory equals the difference between the demand and supply at the existing plant.

Yr		Returnables	Cans	Non-returnables	Equivalent returnables
1	Demand	2400	1750	490	3795
	Supply				
	current	1800	1250	350	2795
	new	600	500	140	998
2	Demand	2450	2000	500	4000
	Supply				
	current	1850	1350	380	2926
	new	600	650	120	1074
3	Demand	2600	2300	600	4400
	Supply				
	current	1900	1400	400	3020
	new	700	900	200	1380
4	Demand	2800	2650	650	4845
	Supply				
	current	2050	1500	400	3230
	new	750	1150	250	1615
5	Demand	3100	3050	720	5434
	Supply				
	current	2150	1800	450	3545
	new	950	1250	270	1889

continued...

5-1 continued

Let P_1, P_2, and P_3 represent the locations of the existing plant, the central plant, and the south plant. The generic transportation model for each period is given as

	N1	N2	C1	C2	S1	S2	
P_1	.8	1.2	1.5	1.6	2.9	2.1	a_1
P_2	1.3	1.9	1.05	.8	1.5	1.7	a_2
P_3	1.9	2.9	1.2	1.6	.9	.8	a_3
	b_{N1}	b_{N2}	b_{C1}	b_{C2}	b_{S1}	b_{S2}	

a_1 = Supply of equivalent returnables at exisiting plant

a_2 = supply of equivalent returnables at new central plant
= 0, if new plant is located south

a_3 = Supply of equivalent returnables at new south plant
= 0, if plant is located center

b_j = Total (equivalent returnables) demand for the year x allocated proportion for the site. (j = N1, N2, C1, C2, S1, S2)

For example, for year 1, we have

a_1 = 2795

$$a_2 = \begin{cases} 998, & \text{if new plant is in center} \\ 0, & \text{if new plant is in south} \end{cases}$$

$$a_3 = \begin{cases} 998, & \text{if new plant is in south} \\ 0, & \text{if new plant is in center} \end{cases}$$

$b_{N1} = .51 \times 3795 \cong 1939$

$b_{N2} = .09 \times 3795 \cong 342$

$\vdots$

$b_{S_2} = .05 \times 3795 \simeq 190$

The following table gives all a_i and b_j

continued...

Appendix E

Case Studies

5-1 continued

Yr	a_1	a_2 or a_3
1	2795	998
2	2926	1074
3	3020	1380
4	3230	1615
5	3545	1889

Yr	Total demand	b_{N1}	b_{N2}	b_{C1}	b_{C2}	b_{S1}	b_{S2}
1	3793	1934	341	341	228	759	190
2	4000	2040	360	360	240	800	200
3	4400	2244	396	396	264	880	220
4	4845	2471	436	436	291	969	242
5	5434	2771	489	489	326	1087	272

From the transportation model, we obtain the following summary:

Yr	Minimum cost ($) given P_2	Minimum cost ($) given P_3
1	4182.85	3653.10
2	3828.80	4039.20
3	4632.60	4341.20
4	5005.60	4665.70
5	4807.40	4922.80
Totals	22,457.25	21,622.00

By locating the plant in the south, we save

22,457.25 − 21,622.00 = $835.25

over a period of 5 years, or approximately $167.05 per year.

The result shows that the transportation cost is not an important factor in the selection of the location (a saving of $167.05 is not significant). Thus, other factors must be considered in the determination of the site of the new plant.

5-2

(a) Optimum tableau

	1	2	3	4	5	6	7	8	9	
1		192	444	216					108	960
2	62					7		90	42	201
3		21			50					71
4		4					20			24
5				99						99
	62	217	444	315	50	7	20	90	150	

The optimum solution corresponds to 1,886,300 ($m^3 \times 100 m$). Thus,

Cost savings = (2,495,000 − 1,886,300) × $.65
= $395,655

(b) Divide the model into two phases: Phase 1 is dedicated to building the perimeter road, and Phase 2 is used to build the roads that can be constructed only after the perimeter road is built.

We cannot use the transportation model, but must use a regular linear program that permits building the perimeter road in Phase 1 and the cross-roads in Phase 2.

Phase 1 distances (d_{ij1})

	1	2	3	4	5	6	7	8	9
1			12	10		18	11	8.5	20
2			14	12		20	13	10	22
3	16	20		20	1.5		6	22	18
4	20	22		22	6		14		18
5			10	4				14	21

All empty squares have a distance M = 999999

continued...

Phase 2 distances (d_{ij2})

5-2 continued

	1	2	3	4	5	6	7	8	9
1	22	26	12	10	18	18	11	8.5	
2	20	28	14	12	20	20	13	10	
3	16	20	26	20	1.5	28	6	22	
4	20	22	26	22	6		2	21	
5	22	26	10	4	16		24	14	

The LP model is thus developed as follows: Define

x_{ijk} = amount in ($m^3 \times 100m$) moved from source i to destination j during phase k

$$\text{Minimize } z = \sum_i \sum_j \sum_k d_{ijk} \, x_{ijk}$$

Subject to

$$\sum_j \sum_k x_{ijk} = a_i , \quad i = 1, 2, \ldots, 5$$

$$\sum_i \sum_k x_{ijk} = b_j , \quad j = 1, 2, \ldots, 9$$

$$x_{ijk} \geq 0 \text{ for all } i, j \,\&\, k$$

Optimum Solution (Phase 1 movements)

	1	2	3	4	5	6	7	8	9	
1			444	216		7			150	917
2								90		90
3		21			50					71
4		4					20			24
5				99						99
	0	25	444	315	50	7	20	90	150	

continued...

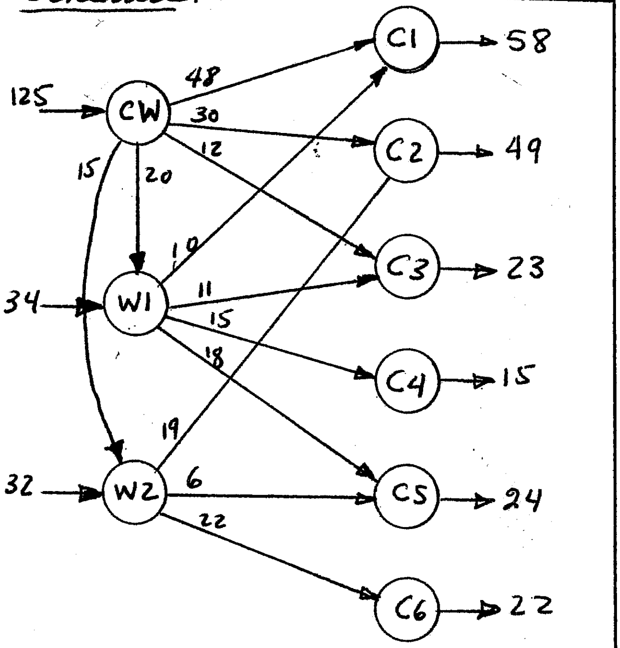
Optimum Solution (Phase II)

5-2 continued

1		143								143
2	62	49								111
3										
4										
5										
	62	192								

5-3

Current transportation Schedule:



Associated van-miles

$$= 20 \times 5 + 15 \times 45 + 48 \times 50 + 30 \times 30 + 12 \times 30 + 10 \times 38 + 11 \times 30 + 15 \times 8 + 18 \times 10 + 19 \times 35 + 6 \times 7 + 22 \times 90$$

$$= 8132$$

Next, we solve the problem as a transshipment model, using the same supply/demand amounts.

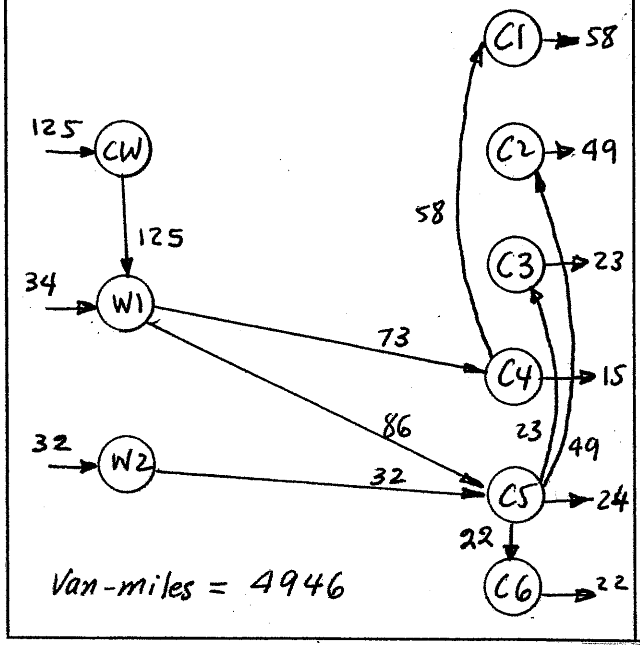
continued...

5-3 continued

$B \equiv 191$

				C1	C2	C3	C4	C5	C6	
	1	2	3	4	5	6	7	8	9	
1	0	5	45	50	30	30	60	75	80	125 + B
2	5	0	80	38	70	30	8	10	60	34 + B
3	45	80	0	85	35	60	55	7	90	32 + B
4	50	38	85	0	20	40	25	30	70	B
5	30	70	35	20	0	40	90	15	10	B
6	30	30	60	40	40	0	10	6	90	B
7	60	8	55	25	90	10	0	80	40	B
8	75	10	7	30	15	6	80	0	15	B
9	80	60	90	70	10	90	40	15	0	B
	B	B	B	58 + B	49 + B	23 + B	15 + B	24 + B	22 + B	

Optimum Solution from TORA



Van-miles = 4946

Layover times are computed depending on whether the roundtrip starts from A or from B. If the departure time from one city is not at least 90 minutes later than the arrival time of the crew at the same city, the crew must wait till the next day. For example, flight 1 arrives in City B at 8:30 and flight 10 leaves B at 7:30 If the crew of flight 1 is to return on flight 10, it must have a layover of 23 hours.

1. Layover time when roundtrip starts at A

Flight	10	20	30	40
1	23	24.75	8.00	11.50
2	20.75	22.50	5.75	9.25
3	15.50	17.25	24.50	4.00
4	14.00	15.75	23.00	2.50

2. Layover time when roundtrip starts at B

	10	20	30	40
1	20.5	18.75	11.50	8.00
2	22.75	21	.15	
3	4.00	1.75	19	15.50
4	5.50	3.75	20.50	17.00

The two tables are combined such that the base with the smaller layover time is used. The result is the following combined matrix:

continued...

5-4 continued

▨ = roundtrip starts in B

	10	20	30	40	
1	**20.5**	**18.75**	8	**8**	1
2	20.75	**21**	5.75	9.25	1
3	**4**	**1.75**	**19**	4	1
4	**5.5**	**3.75**	**20.5**	2.5	1
	1	1	1	1	

TORA optimum solution:

1-40, 2-30, 3-20, 4-10

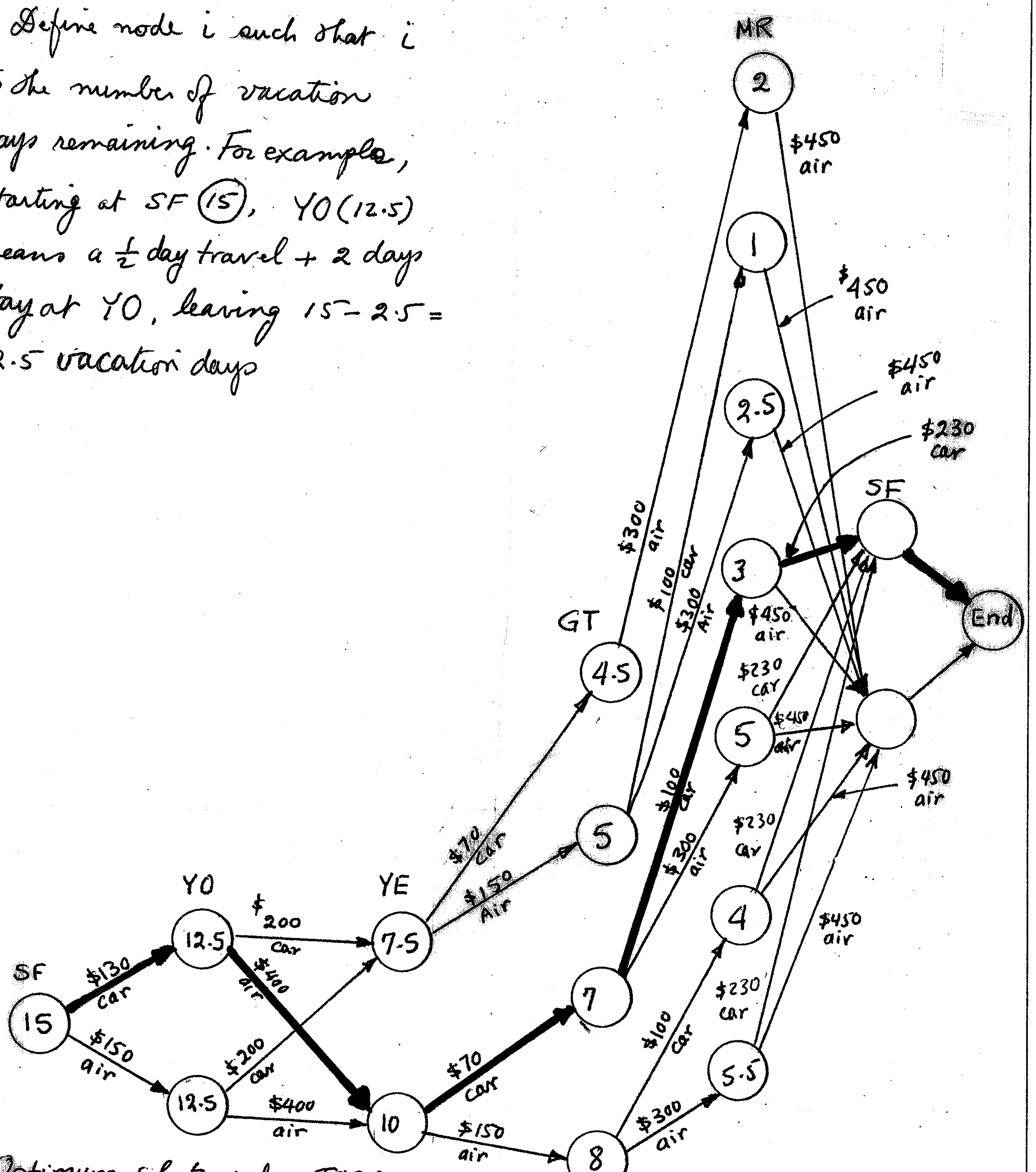
The solution is interpreted as

$B \rightarrow A(40)$, $A \rightarrow B(1)$
$A \rightarrow B(2)$, $B \rightarrow A(30)$
$B \rightarrow A(20)$, $A \rightarrow B(3)$
$B \rightarrow A(10)$, $A \rightarrow B(4)$

The optimum solution calls for stationing 1 crew in A and 3 crews in B.

6-1

Define node i such that i is the number of vacation days remaining. For example, Starting at SF ⑮, YO(12.5) means a ½ day travel + 2 days stay at YO, leaving 15 − 2.5 = 12.5 vacation days



Optimum solution form TORA:

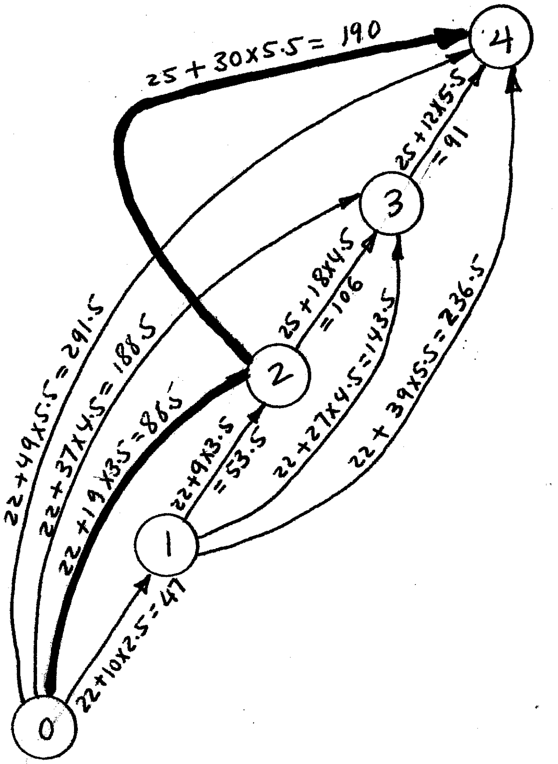
$SF \xrightarrow{car} YO \xrightarrow{air} YE \xrightarrow{car} GT \xrightarrow{car} MR \xrightarrow{car} SF$

Trip time = (.5 + .5 + 1 + 2 + 3) + 4X2 = 15 days

Trip cost = $930

6-2

Arrange the books in ascending order so that node 1 represent the 6" books and node 4 represent the 12" books. The network starts from node 0. An arc from node i to node j, $i<j$, signifies that all the books of height h_i, h_{i+1}, ..., and h_j are placed in a shelf of height h_j. The length of the arc equals the associated fixed plus variable costs. The optimum solution is given by the shortest route from node 0 to node 4.



Total cost = $278.50

Solution:

Produce 19 ft of height 8" and 30ft of height 12"

6-3

Shipment	Shipping Route	Delivery Date
1	A to D	10
2	A to E	15
3	B to D	4
4	B to E	5
5	C to E	18

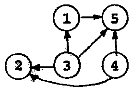
	A	B	C	D	E
A	▨	▨	▨	3	4
B	▨	▨	▨	3	2
C	▨	▨	▨	3	5
D	2	2	2	▨	▨
E	3	1	4	▨	▨

The ships schedules can be summarized as shown below.

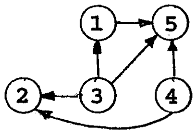
```
0--1--2--3--4--5--6--7--8--9--10--11--12--13--14--15--16--17--18--19--20--21--22

              1> A---------D------A,B,C
3> B--------D-----A,B,C       2> A---------------E--B--------A--C
    4> B-----E--B-----A--C      5> C------------------------E--B--------A--C
```

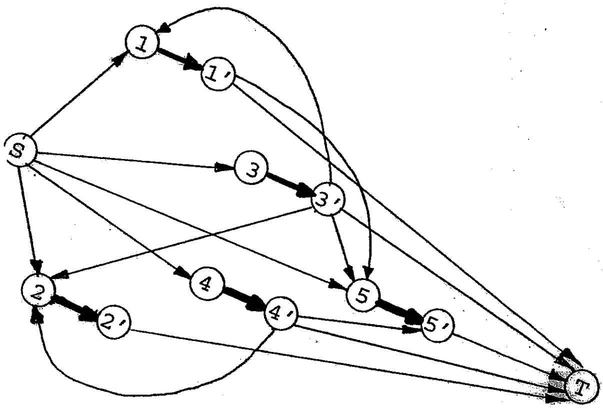
Precedence Relationships



Precedence Relationships



Flow Model



continued...

In the flow model, the flow in arcs (i – i'), i = 1, 2, ..., 5 must equal 1 to realize a feasible solution. The different arcs of the model represent the precedence relationships of the feasible schedule. The minimum flow from node S to node T will provide the minimum number of ships required to meet the proposed schedule.

The procedure for determing the minimum flow in the network consists of the following steps:

Step 1: Determine a feasible solution to the flow model from S → E

Step 2: Determine the residue network of the feasible solution

Step 3: Determine the maximum flow in the residue network from E → S; that is, from the end node E to the start node S.

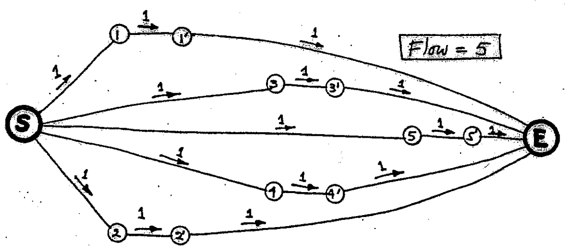
Step 4: Determine the minimum flow from S → E as

feasible flow S → E − maxflow E → S

Step 1:

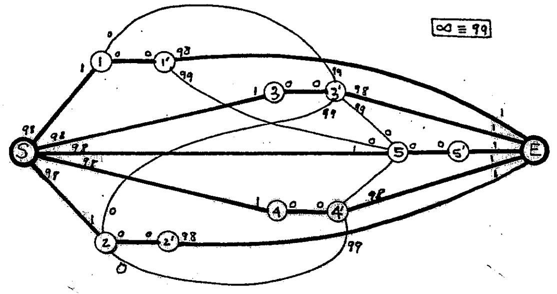
The easiest way to finding a feasible solution is to assume that each route is served by a separate ship. The network below provides such a solution.

continued...

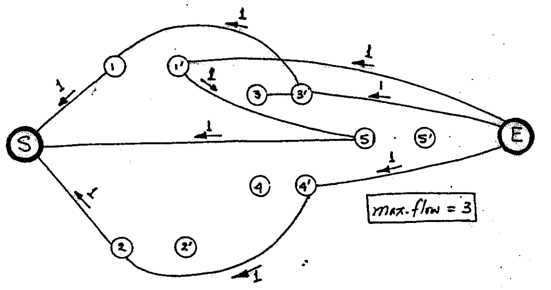


Step 2:

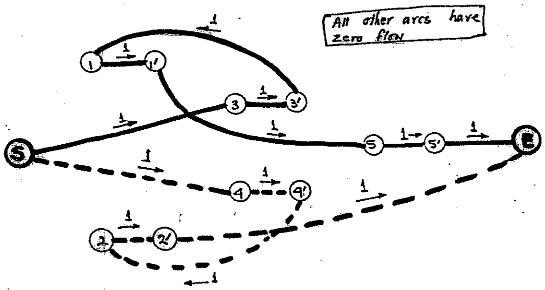
Residue Network



Step 3: Maximum flow in residue network (all missing arcs have zero flow) from E → S



Step 4:



Solution:
Two ships are needed.

6-4

Consider the case of four brokers. The actual situation for which the problem was analyzed included a total of 254 brokers

Let

P_i = payables by broker i, $i=1,2,3,4$

R_{ij} = receivables by i from j, $i,j=1,2,3,4$

A_i = assets of broker i, $i=1,2,3,4$

The data of the problem may be summarized as

j

		1	2	3	4	assets
	1	P_1	R_{12}	R_{13}	R_{14}	A_1
	2	R_{21}	P_2	R_{23}	R_{24}	A_2
i	3	R_{31}	R_{32}	P_3	R_{34}	A_3
	4	R_{41}	R_{42}	R_{43}	P_4	A_4

A broker is solvent if its net debt does not exceed its assets – that is, broker i is solvent if

$$P_i \le A_i + \sum_{i \neq j} R_{ij}$$

The problem deals with the brokers whose total assets are less than their debt – that is,

$$P_i > A_i + \sum_{j \neq i} R_{ij}$$

The proposed solution calls for prorating all debts such that for broker i

prorated payables – prorated receivables = assets

Let

α_i = prorating factor for broker i

continued...

6-4 continued

The value of α_i may be determined by solving

$$P_i \alpha_i - \sum_{i \neq j} R_{ij} \alpha_j = A_i \text{, for all } i$$

Consider the following hypothetical illustration:

	α_1	α_2	α_3	α_4	assets	Deficit
1	35	-5	-4	-1	20	5
2	-10	40	-5	-7	15	3
3	-3	-4	20	-1	5	7
4	-5	-5	-5	30	10	5
	17	26	6	21	← Payables	

<u>Equations</u>:

$$\begin{pmatrix} 35 & -5 & -4 & -1 \\ -10 & 40 & -5 & -7 \\ -3 & -4 & 20 & -1 \\ -5 & -5 & -5 & 30 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 20 \\ 15 \\ 5 \\ 10 \end{pmatrix}$$

<u>Solution</u>: $\alpha_1 = .76$, $\alpha_2 = .75$, $\alpha_3 = .55$, $\alpha_4 = .68$

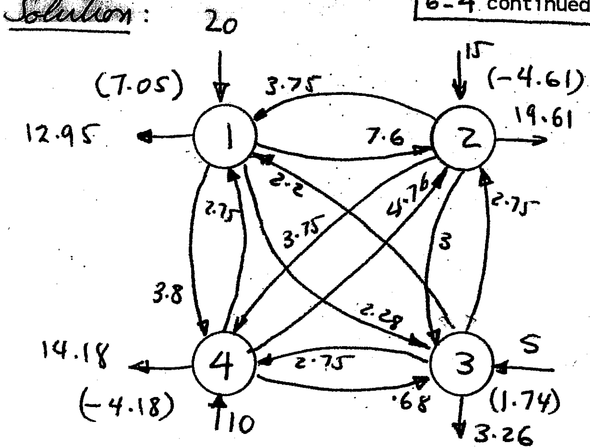
The net result of the prorating is that the total assets $(= 20+15+5+10 = 50)$ will be distributed to investors

The following table gives the prorated cash:

	.76	.75	.55	.68	Assets	Deficit
1	26.6	-3.75	-2.2	-.68	20	0
2	-7.6	30	-2.75	-4.76	15	0
3	-2.28	-3	11	-.68	5	0
4	-3.8	-3.75	-2.75	20.4	10	0
	12.95	19.61	3.26	14.28	← Payables	

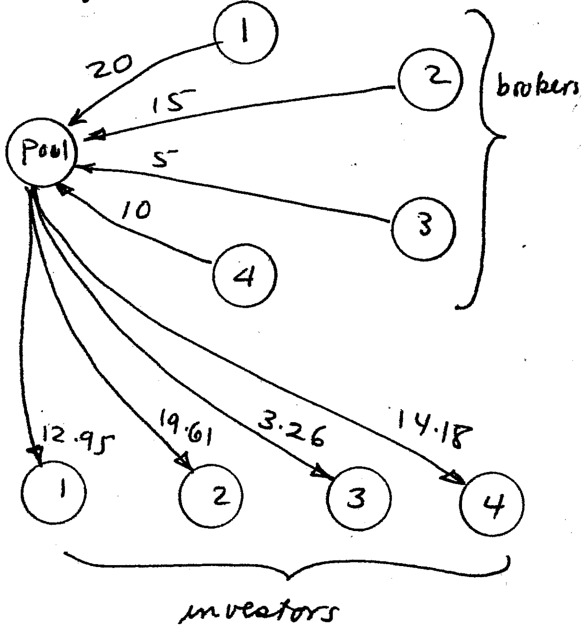
continued...

Solution:



The next phase of the analysis calls for eliminating the "loops" from the given solution. For example, 1 ows 2 \$7.6 and 2 ows 1 \$3.75. The net result is that 1 ows 2 \$7.6-3.75 = \$3.85. The idea is to eliminate all higher level loops

I first proposed solving the problem of the "loops" by pooling all the assets and distributing them to outside investors based on the determined value of α_i — that is



continued...

The proposed solution did not satisfy the legal requirements of the case. As a result, I proposed the following network-based LP.

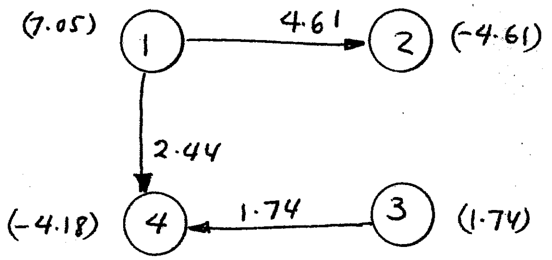
Let

x_{ij} = Flow from node i to node j.

The minimization of $\sum x_{ij}$ subject to the following flow constraint should produce the minimum number of loops (see the network in opposite column):

	X12	X13	X14	X21	X23	X24	X31	X32	X34	X41	X42	X43	RHSVALUE
OBJ (Min)	1	1	1	1	1	1	1	1	1	1	1	1	
NODE1	1	1	1	-1			-1			-1			7.05
NODE2	-1			1	1	1		-1			-1		-4.61
NODE3		-1			-1		1	1	1			-1	1.74
NODE4			-1						-1	1	1	1	-4.18
BOUND	7.6	2.28	3.8	3.75	3.	3.75	2.2	2.75	2.75	.68	4.76	.68	

The optimum LP solution is summarized below.



Comprehensive Problems

7-1

The problem reduces to finding a feasible solution of

$$\alpha_1 A + \alpha_2 B + \alpha_3 C = b$$
$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$
$$\alpha_1, \alpha_2, \alpha_3 \geq 0$$

(a) maximize $z = 0\alpha_1 + 0\alpha_2 + 0\alpha_3$

subject to

$$\alpha_1 \begin{pmatrix} 6 \\ 4 \\ 6 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4 \\ 12 \\ -4 \\ 8 \end{pmatrix} + \alpha_3 \begin{pmatrix} -4 \\ 0 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geq 0 \text{ all } j$$

Solution: $\alpha_1 = 1/2, \alpha_2 = 1/4, \alpha_3 = 1/4$

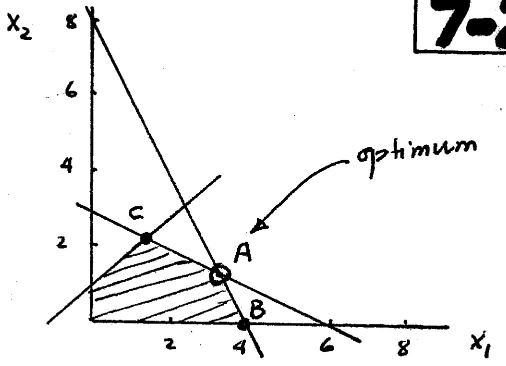
(b) maximize $z = 0\alpha_1 + 0\alpha_2 + 0\alpha_3$

subject to

$$\alpha_1 \begin{pmatrix} 6 \\ 4 \\ 6 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4 \\ 12 \\ -4 \\ 8 \end{pmatrix} + \alpha_3 \begin{pmatrix} -4 \\ 0 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 4 \\ 9 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geq 0 \text{ all } j$$

7-2



	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	1/3	4/3	0	38/3
x_2	0	1	2/3	-1/3	0	4/3
x_1	1	0	-1/3	2/3	0	10/3
x_5	0	0	-1	1	1	3

The optimum solution occurs at A. The adjacent extreme points B and C are determined by making x_3 and x_4 basic, one at a time

continued...

Adjacent extreme pt B: Introduce x_3 into the basic vector

	x_1	x_2	x_3	x_4	x_5	
z	0	-1/2	0	3/2	0	12
x_3	0	3/2	1	-1/2	0	2
x_1	1	1/2	0	1/2	0	4
x_5	0	3/2	0	1/2	1	5

Adjacent extreme point C: Introduce x_4 into the basic vector

	x_1	x_2	x_3	x_4	x_5	
z	0	0	5/3	0	-4/3	26/3
x_2	0	1	1/3	0	1/3	7/3
x_1	1	0	1/3	0	-2/3	4/3
x_4	0	0	-1	1	1	3

The next best extreme point is B with $z = 12$.

7-3

Iteration 0:

$\mathbf{B}_0 = \mathbf{I}$

$\mathbf{B}_0^{-1} = \mathbf{I}$

$\mathbf{x}_B = (y_1, y_2, y_3)^T = (150, 200, 300)^T$

$\mathbf{c}_B = (1,1,1)$

$d = \mathbf{c}_B \mathbf{B}_0^{-1} = (1,1,1)\mathbf{I} = (1,1,1)$

Interactive AMPL solution (amplProb7-3.txt):

```
solve;display zj,a;
zj = 4
a [*] :=
1  4
2  0
3  0
```

Iteration 1:

$\mathbf{P}_1 = (4,0,0)^T$ -- associated variable x_1

$$\mathbf{B}_0^{-1}\mathbf{P}_1 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y_1 \text{ leaves}$$

$$\mathbf{B}_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{B}_1^{-1} = \begin{pmatrix} .25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{x}_B = (x_1, y_2, y_3)^T = (37.5, 200, 300)^T$

$\mathbf{c}_B = (1,1,1)$

$d = \mathbf{c}_B \mathbf{B}_1^{-1} = (.25,1,1)$

Interactive AMPL solution:

```
let d[1]:=.25;
solve;display zj,a;
zj = 2.25
a [*] :=
1  1
2  2
3  0
```

continued...

Iteration 2:

$\mathbf{P}_2=(1,2,0)^T$ -- associated variable x_2

$$\mathbf{B}_1^{-1}\mathbf{P}_2=\begin{pmatrix}.25\\2\\0\end{pmatrix}\Rightarrow y_2 \text{ leaves}$$

$$\mathbf{B}_2=\begin{pmatrix}4&1&0\\0&2&0\\0&0&1\end{pmatrix},\mathbf{B}_2^{-1}=\begin{pmatrix}.25&-.125&0\\0&.5&0\\0&0&1\end{pmatrix}$$

$\mathbf{x}_B=(x_1,x_2,y_3)^T=(150, 200, 300)^T$

$\mathbf{c}_B=(1,1,1)$

$d=\mathbf{c}_B\mathbf{B}_2^{-1}=(.25, .375, 1)$

Interactive AMPL solution:

```
let d[2]:=.375;
solve;display zj,a;
zj = 2
a [*] :=
1  0
2  0
3  2
```

Iteration 3:

$\mathbf{P}_2=(0,0,2)^T$ -- associated variable x_3

$$\mathbf{B}_2^{-1}\mathbf{P}_3=\begin{pmatrix}0\\0\\2\end{pmatrix}\Rightarrow y_3 \text{ leaves}$$

$$\mathbf{B}_3=\begin{pmatrix}4&1&0\\0&2&0\\0&0&2\end{pmatrix},\mathbf{B}_3^{-1}=\begin{pmatrix}.25&-.125&0\\0&.5&0\\0&0&.5\end{pmatrix}$$

$\mathbf{x}_B=(x_1,x_2,x_3)^T=(12.5, 100, 150)^T$

$\mathbf{c}_B=(1,1,1)$

$d=\mathbf{c}_B\mathbf{B}_3^{-1}=(.25, .375, .5)$

Interactive AMPL solution:

```
let d[3]:= .5;
solve;display zj,a;
```

No feasible solution — Process ends

Optimal solution: Cut 12.5 rolls using setting (4,0,0), 100 rolls using (1,2,0), and 150 rolls using (0,0,2).

7-4

Maximize $z = CX$

subject to

$$AX \geq L$$
$$AX \leq U$$
$$X \geq 0$$

Let $Y = U - AX \geq 0$, the problem becomes

Maximize $z = CX$

subject to

$$AX + Y = U$$
$$Y \leq U - L$$
$$X \geq 0 \quad Y \geq 0$$

continued...

maximize $z = 5x_1 - 4x_2 + 6x_3$

Subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + y_1 \quad\quad &= 46\\ 3x_1 - x_2 + x_3 \quad + y_2 \quad &= 20\\ 2x_1 + 3x_2 - x_3 \quad\quad + y_3 &= 35 \end{aligned}$$

$$y_1 \leq 26,\ y_2 \leq 20,\ y_3 \leq 17$$

$$x_1, x_2, x_3, y_1, y_2, y_3 \geq 0$$

. Optimum solution:

$x_1 = 6.18,\ x_2 = 3.55,\ x_3 = 5$

$z = 46.73$

7-5

$$\begin{pmatrix}x_2\\x_1\\x_5\end{pmatrix}=\begin{pmatrix}4/3\\10/3+\theta\\3\end{pmatrix}\geq\begin{pmatrix}0\\0\\0\end{pmatrix}$$

The increase in x_1 can be translated to increasing the right-hand side of the constraints by D_1, D_2, and D_3. The values of D_1, D_2, and D_3 can be computed from $B^{-1}b = X_B$; that is

$$\begin{pmatrix}x_2\\x_1\\x_5\end{pmatrix}=\begin{pmatrix}2/3&-1/3&0\\-1/3&2/3&0\\-1&1&1\end{pmatrix}\begin{pmatrix}6+D_1\\8+D_2\\1+D_3\end{pmatrix}=\begin{pmatrix}4/3\\10/3+\theta\\3\end{pmatrix}$$

Thus,

$$\begin{pmatrix}2/3&-1/3&0\\-1/3&2/3&0\\-1&1&1\end{pmatrix}\begin{pmatrix}D_1\\D_2\\D_3\end{pmatrix}=\begin{pmatrix}0\\\theta\\0\end{pmatrix}$$

or

$$\begin{pmatrix}D_1\\D_2\\D_3\end{pmatrix}=\begin{pmatrix}2&1&0\\1&2&0\\1&-1&1\end{pmatrix}\begin{pmatrix}0\\\theta\\0\end{pmatrix}=\begin{pmatrix}\theta\\2\theta\\-\theta\end{pmatrix}$$

$\theta = -10/3$:

	x_1	x_2	x_3	x_4	x_5	Sol$^{\underline{n}}$
z	0	0	1/3	4/3	0	8/3
x_2	0	1	2/3	-1/3	0	4/3
x_1	1	0	$\boxed{-1/3}$	2/3	0	0
x_5	0	0	-1	1	1	3
z	1	0	0	2	0	8/3
x_2	2	1	0	1	0	4/3
x_3	-3	0	1	-2	0	0
x_5	-3	0	0	-1	1	3

continued...

$$\begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 6+\theta \\ 8+2\theta \\ 1-\theta \end{pmatrix} = \begin{pmatrix} 8+2\theta \\ -10-3\theta \\ -7-3\theta \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4 \leq \theta \leq -10/3$$

For $\theta < -4$, no feasible solution exists

<u>Summary</u>:

$-\infty \leq \theta < -4$: No feasible solution

$-4 \leq \theta \leq -10/3$: $x_1 = 0,\ x_2 = 8+2\theta$
$Z = 16+4\theta$

$-10/3 \leq \theta < \infty$: $x_1 = 10/3+\theta,\ x_2 = 4/3$
$Z = \frac{38}{3} + 3\theta$

7-6

at $t = 2$:

$$B = (P_2\ P_3) = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix},\ B^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix}$$

$$C_B = (4t-8, 0)$$

$$C_B B^{-1} = (4t-8, 0)\begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix} = (0, 2t-4)$$

$$z_1 - c_1 = (0, 2t-4)\begin{pmatrix} 2 \\ 4 \end{pmatrix} - (10t-4)$$
$$= -2t - 12$$

$$z_4 - c_4 = (0, 2t-4)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0$$
$$= 2t - 4$$

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 8 \\ 6-2t \end{pmatrix} = \begin{pmatrix} 3-t \\ 3+2t \end{pmatrix}$$

$$\left.\begin{array}{l} \text{at } t=2: \ z_4 - c_4 = 0 \\ \text{at } t>2: \ z_4 - c_4 > 0 \end{array}\right\} x_4 \text{ enters basis}$$

P_4 replaces P_2:

$$B = (P_3\ P_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$$

$$C_B = (0,0) \qquad C_B B^{-1} = (0,0)$$

$$z_1 - c_1 = (0,0)\,P_1 - (10t-4) = -10t - 4 \leq 0$$

$$z_2 - c_2 = (0,0)\,P_2 - (4t-8) = -4t + 8 \leq 0$$
$$t \geq 5/2$$

$$\begin{pmatrix} x_4 \\ x_4 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 8 \\ 6-2t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow t \leq 3$$

continued...

Solution becomes infeasible for $t > 3$. However, no feasible solution exists for $t > 3$

<u>Summary</u>:

$2 \leq t \leq 3$: optimal basis $B = (P_3, P_4)$

$t > 3$: no feasible solution exists

7-7

Let

$$J^+(J^-) = \{j \in NB \mid \alpha_{ij} > 0\ (<0)\}$$

Then $x_i \leq e_i$ yields

$$x_i = e_i + f_i - \sum_{j\in J^+} \alpha_{ij} x_j - \sum_{j \in J^-} \alpha_{ij} x_j$$

Because $x_i - e_i \leq 0$, it follows that

$$-\sum_{j\in J^+} \alpha_{ij} x_j - \sum_{j\in J^-} \alpha_{ij} x_j \leq -f_i$$

Adding this inequality to the simplex tableau and applying the simplex method, then, under the assumption of no change in basis, the decrease in the value of z is at least

$$P_d = \min_{j\in J^+} \left\{ \frac{(z_j - c_j) f_i}{\alpha_{ij}} \right\}$$

The corresponding upper bound on the value of z is $C_0 - P_d$.

In a similar manner, $x_i \geq d_i$ gives

$$x_i - d_i = f_i - 1 - \sum_{j\in J^+} \alpha_{ij} x_j - \sum_{j\in J^-} \alpha_{ij} x_j$$

Thus,

$$\sum_{j\in J^+} \alpha_{ij} x_j + \sum_{j \in J^-} \alpha_{ij} x_j \leq f_i - 1$$

and

$$P_u = \min_{j\in J^-} \left\{ \frac{(z_j - c_j)(f_i - 1)}{\alpha_{ij}} \right\}$$

The associated upper bound on z is $C_0 - P_u$

Chapter 8 Cases

8-1

X_{ij} = Acres from site i using alternative j

Sawlogs constraint:

$$7X_{15} + 6X_{16} + 5X_{17} +$$
$$5X_{25} + 4X_{26} +$$
$$4X_{33} + 3X_{34} + 5X_{35} + S_1^- - S_1^+ = 350{,}000$$

Plywood constraint:

$$6X_{13} + 7X_{14} + 5X_{23} + 4X_{24} + 4X_{32} + S_2^- - S_2^+ = 150{,}000$$

Pulpwood constraint:

$$1\ X_{11} + 10X_{12} + 5X_{13} + 4X_{14} + 3X_{15} + 2X_{16} +$$
$$3\ X_{17} + 9X_{21} + 8X_{22} + 2X_{23} + 3X_{24} + 2X_{25} +$$
$$2X_{26} + 7X_{31} + 6X_{32} + 2X_{33} + 2X_{34} +$$
$$X_{35} + S_3^- - S_3^+ = 200{,}000$$

Reforestation Constraint:

$$1000X_{11} + 800X_{12} + \cdots + 1500X_{17} +$$
$$1000X_{21} + 800X_{22} + \cdots + 1200X_{26} +$$
$$1000X_{31} + 800X_{32} + 1500X_{33} +$$
$$1200X_{34} + 1300X_{35} + S_4^- - S_4^+ = 2{,}500{,}000$$

Rotation Constraints:

$$20X_{11} + 25X_{12} + 40X_{13} + 15X_{14}$$
$$+ 40X_{15} + 40X_{16} + 50X_{1} \le 100{,}000$$
$$20X_{21} + 25X_{22} + 40X_{23} + 15X_{24}$$
$$+ 40X_{25} + 40X_{26} \le 180{,}000$$
$$30X_{31} + 25X_{32} + 40X_{33} + 15X_{34}$$
$$+ 40X_{35} \le 200{,}000$$

Goal for total return from stumpage

$$= 100(1000{,}000 + 180{,}000 + 200{,}000)$$
$$= \$48{,}000{,}000$$

Total return constraint:

$$(20\times160)X_{11} + (117\times25)X_{12} + (140\times40)X_{13} +$$
$$(195\times15)X_{14} + (182\times40)X_{15} + (180\times40)X_{16} +$$
$$(135\times50)X_{17} + (102\times20)X_{21} + (55\times25)X_{22} +$$
$$(95\times40)X_{23} + (120\times15)X_{24} + (100\times40)X_{25} +$$
$$(90\times40)X_{26} + (60\times\ 0)X_{31} + (48\times25)X_{32} +$$
$$(60\times40)X_{33} + (65\times15)X_{34} + (35\times40)X_{35}$$
$$+ S_5^- - S_5^+ = 48{,}000{,}000$$

continued...

Goals:

min S_1^-

min S_2^-

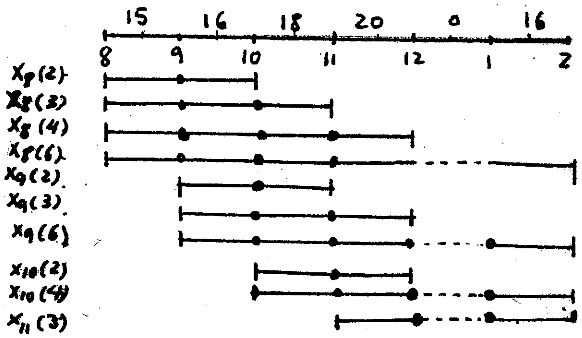
min S_3^-

min S_4^+

min S_5^-

8-2

X_{hj} = number of volunteers starting at hour h and continuing for j successive hours



Constraints:

$$X_8(2) + X_8(3) + X_8(4) + X_8(6) + S_1^- - S_1^+ = 15$$
$$X_8(2) + X_8(3) + X_8(4) + X_8(6) +$$
$$X_9(2) + X_9(3) + X_9(5) + S_2^- - S_2^+ = 16$$
$$X_8(3) + X_8(4) + X_8(6) + X_9(2) +$$
$$X_9(3) + X_9(5) + X_{10}(2) + X_{10}(4) + S_3^- - S_3^+ = 18$$
$$X_8(4) + X_8(6) + X_9(3) + X_9(5)$$
$$+ X_{10}(2) + X_{10}(4) + X_{11}(3) + S_4^- - S_4^+ = 20$$
$$X_8(6) + X_9(3) + X_{10}(4) + X_{11}(3) + S_5^- - S_5^+ = 16$$

Objective function:

$$\text{Minimize } Z = S_1^- + S_2^- + S_3^- + S_4^- + S_5^-$$

continued...

Chapter 9 Cases

9-1

$$x_{iy} = \begin{cases} 1, & \text{building } i \text{ opens in yr } y \\ 0, & \text{otherwise} \end{cases}$$

R_{iy} = ft^2 of building i rented in year y and thereafter

I_{iy} = operating income per ft^2 of building i in year y

C_{iy} = construction cost of building i in year y

θ = inflation rate applied to operating income and construction cost

r = dis-count rate

D_{1y} = demand for high-rise ft^2 in yr y

D_{2y} = demand for garden-space ft^2 in year y

K = capitalization rate used for determining property value in the year of sale

B_i = maximum capacity of building i

The sale value of building i at the end of 7 years is estimated as

$$R_{iy} \times \frac{I_{i7}}{K}$$

This means that the sale is estimated based on the net operating income for year 7.

Model:

$$\text{Maximize } z = \sum_{y=1}^{7}\left\{\left(\frac{1}{1+r}\right)^{y}\left(\sum_{i=1}^{7}\left(I_{iy}R_{iy} - C_{iy}x_{iy}\right)\right)\right\} + \sum_{y=1}^{7}\sum_{i=1}^{7}\left(\frac{1}{1+r}\right)^{7}\left(\frac{I_{i7}}{K}\right)R_{iy}$$

Subject to

continued...

$$\sum_{i=1}^{3} R_{iy} \le D_{1y}, \quad y = 1, 2, \ldots, 7$$

$$\sum_{i=4}^{7} R_{iy} \le D_{2y}, \quad y = 1, 2, \ldots, 7$$

$$\sum_{j=1}^{y} D_{1j}x_{ij} \ge R_{iy}, \quad i = 1, 2, 3; \; y = 1, 2, \ldots, 7$$

$$\sum_{j=1}^{y} D_{2j}x_{ij} \ge R_{iy}, \quad i = 3, 4, \ldots, 7; \; y = 1, 2, \ldots, 7$$

$$\sum_{y=1}^{7} R_{iy} \le B_i, \quad i = 1, 2, \ldots, 7$$

$$\sum_{y=1}^{7} x_{iy} = 1, \quad i = 1, 2, \ldots, 7$$

The rental income and expenses as given in the problem apply to year 1 of the planning horizon. These values must be adjusted for year y to allow for inflation. Assuming an inflation rate θ, the amount for year y is determined for year y by multiplying the values for year 1 by $(1+\theta)^{y-1}$

9-2

S_{ij} = expected score of gymnast i in event j, $i = 1, 2, \ldots, N$, $j = 1, 2, 3, 4$.

$$x_{ij} = \begin{cases} 1, & \text{if gymnast } i \text{ is assigned to event } j \\ 0, & \text{if otherwise} \end{cases}$$

$$y_i = \begin{cases} 1, & \text{if gymnast } i \text{ is an all-rounder} \\ 0, & \text{if otherwise} \end{cases}$$

Model:

$$\text{Maximize } z = \sum_{i=1}^{N}\sum_{j=1}^{4} S_{ij}x_{ij} + \sum_{i=1}^{N}\left(\sum_{j=1}^{4} S_{ij}\right)y_i$$

Subject to

$$\sum_{i=1}^{N} x_{ij} + y_i \le 6, \quad j = 1, 2, 3, 4$$

$$x_{ij} + y_i \le 1, \quad i = 1, 2, \ldots, N; \; j = 1, 2, 3, 4$$

$$\sum_{i=1}^{N} y_i \ge 4,$$

continued...

$$\sum_{j=1}^{4} x_{ij} \le 3, \quad i = 1, 2, \dots, N$$

$y_i, x_{ij} = (0,1)$ for all i and j

9-3

x_{ij} = fraction of traffic originating from area code i and handled by center j, $i = 1, 2, \dots, 8$; $j = 1, 2, \dots, 7$

$$y_j = \begin{cases} 1, & \text{if center } j \text{ is chosen} \\ 0, & \text{if otherwise} \end{cases}$$

c_{ij} = Communication cost/hr between area i and area j

Define:

i	1	2	3	4	5	6	7	8
Area	501	918	316	417	314	816	502	606

j	1	2	3	4	5	6	7
center	Dallas	Atlanta	L'ville	Denver	LR	Memphis	St. Luis

Communication traffic constraints

$x_{11} + x_{14} + x_{16} = 1$ (area 501)

$x_{21} + x_{23} = 1$ (918)

$x_{31} + x_{33} + x_{36} = 1$ (316)

$x_{41} + x_{43} + x_{45} + x_{46} = 1$ (417)

$x_{52} + x_{53} + x_{55} + x_{57} = 1$ (314)

$x_{62} + x_{63} + x_{65} + x_{67} = 1$ (816)

$x_{72} + x_{74} + x_{75} + x_{77} = 1$ (502)

$x_{82} + x_{84} + x_{86} + x_{87} = 1$ (606)

Centers constraints:

$x_{11} + x_{21} + x_{31} + x_{41} \le My_1$

$x_{52} + x_{62} + x_{72} + x_{82} \le My_2$

$x_{23} + x_{33} + x_{43} + x_{53} + x_{63} \le My_3$

$x_{14} + x_{74} + x_{84} \le My_4$

$x_{45} + x_{55} + x_{65} + x_{75} \le My_5$

$x_{16} + x_{36} + x_{46} + x_{86} \le My_6$

$x_{57} + x_{67} + x_{77} + x_{87} \le My_7$

continued...

Limit on number of centers

$$3 \le y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \le 4$$

Objective function: Minimize $z =$

$500000\, y_1 + 800{,}000\, y_2 + \cdots + 550{,}000\, y_7$

$+ 14x_{11} + 24x_{14} + 19x_{16}$

$+ 35x_{21} + 25x_{23}$

$+ \cdots$

$+ 15x_{82} + 30x_{84} + 12x_{86} + 22x_{87}$

9-4

Let

$$x_{ij} = \begin{cases} 1, & \text{if cluster } i \text{ is served by CSL in location } j \\ 0, & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1, & \text{if candidate loc. } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

p = number of facilities

w_i = number of customers in cluster i

d_{ij} = distance between cluster i and CSL location j.

Model: Given p, determine

$$\text{Min } z = \sum_{i=1}^{5} \sum_{j=1}^{5} w_i \cdot d_{ij} \cdot x_{ij}.$$

Subject to

$$\sum_{j=1}^{5} y_j = p$$

$$\sum_{j=1}^{5} x_{ij} = 1, \quad i = 1, 2, \dots, 5$$

$$\sum_{i=1}^{5} x_{ij} \le M y_j, \quad j = 1, 2, \dots, 5$$

y_j and $x_{ij} = (0, 1)$

The idea of the algorithm is to specify a value for $p = 1, 2, \dots,$ or 5. Then the model is optimized to determine where the specified p CSL centers should be located.

continued...

For example, if $p=1$, the optimum solution of the model (using TORA) will specify that the CSL should be located at $j=4$. This means that all 5 clusters will be served by the CSL located in location $j=4$. For this arrangement, the average traveled distance from $j=4$ to all 5 clusters is

$$\bar{D} = \frac{50+30+80+60+110}{5}$$

$$= \frac{330 \text{ miles}}{5}$$

$$= 66 \text{ miles}$$

Given that the truck travels at 45 miles per hour, the average time to reach a customer will be $\frac{66}{45} = 1.47$ hour = 88 minutes, which is less than the desired 90-minutes response time.

Another way of looking at the solution is to consider the maximum travel distance from location $j=4$; namely,

$$D_4 = \max\{50, 30, 80, 60, 110\} = 110 \text{ miles}$$

The associated truck travel time is 2.44 hours or 147 minutes. Because it exceeds the limit of 90 minutes, the new

continued...

criterion calls for trying $p=2$. TORA will give two locations:

$j=3$ serving clusters 1 and 5

$j=4$ serving clusters 2,3, and 4

Thus, $D_3 = \max\{20, 40\} = 40$ and $D_4 = \max\{30, 80, 60\} = 80$ miles. The new solution is within the desired 90-mile limit.

9-5

18 possible configurations:

1 4C-A-4D
2 4C-A-C
3 4C-A-W
4 4C-S-4D
5 4C-S-C
6 4C-S-W
7 6C-A-4D
8 6C-A-C
9 6C-A-W
10 6C-S-4D
11 6C-S-C
12 6C-S-W
13 8C-A-4D
14 8C-A-C
15 8C-A-W
16 8C-S-4D
17 8C-S-C
18 8C-S-W

	Testers configurations
1	4C-S
2	8C-C
3	6C-W
4	8C
5	S-W
6	6C-A

Let T_i = Set of testers using configuration i, $i = 1, 2, \ldots, 18$

$T_1 = T_2 = T_3 = \emptyset$

$T_4 = T_5 = \{1\}$

$T_6 = \{1, 5\}$

$T_7 = T_8 = \{6\}$

$T_9 = \{3, 6\}$

$T_{10} = T_{11} = \emptyset$

$T_{12} = \{3, 5\}$

$T_{13} = \{4\}$

$T_{14} = \{2, 4\}$

$T_{15} = T_{16} = \{4\}$

$T_{17} = \{2, 4\}$

$T_{18} = \{4, 5\}$

continued...

P_i = Set of prototypes covering tester i, $i=1,2,\dots,6$

$P_1 = \{4,5,6\}$, $P_2 = \{14,17\}$, $P_3 = \{9,12\}$,
$P_4 = \{13,14,15,16,17,18\}$
$P_5 = \{6,12,18\}$, $P_6 = \{7,8,9\}$

$$x_{ij} = \begin{cases} 1, & \text{if tester } i \text{ is covered by prototype } j \\ 0, & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1, & \text{if any tester uses prototype } j \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Minimize } z = \sum_{i=1}^{18} y_i$$

s.t.

$$\sum_{j \in P_i} x_{ij} = 1, \quad i=1,2,\dots,6$$

$$\sum_{i \in T_j} x_{ij} \le M * y_j, \quad j=1,2,\dots,18$$

Solution: See file amplCase9-5.txt

Prototype	Nbr. made	testers
6	2	1,5
9	2	3,6
14	2	2,4

9-6

F_{ij} = feasible pairing j of crew i expressed in flight numbers

Examples: Pairing (C3, C6, C4, C8, C3) of crew 1 is expressed as

$F_{11} = \{10, 15, 12, 18\}$

Pairing (C3, C2, C8, C3) of crew 1 is expressed as

$F_{12} = \{9, 7, 18\}$

$$x_{ij} = \begin{cases} 1, & \text{if pairing } j \text{ of crew } i \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

y_k = Nbr. of crews overallocated to flight k (≥ 0)

Card(F_{ij}) = Nbr. of elements of F_{ij}

N_i = Nbr. of pairings for crew i

continued...

$$\text{Minimize } z = \sum_{(i,j)} \text{Card}(F_{ij})\, x_{ij}$$

s.t.

$$\sum_{j=1}^{N_i} x_{ij} \le 1, \quad i=1,2,\dots,10 \qquad (1)$$

$$\sum_{\substack{\text{defined}(i,j) \\ k \in F_{ij}}} x_{ij} - y_k = 1, \quad k=1,2,\dots,18 \qquad (2)$$

Constraints (1) allow at most one pairing per crew and constraints (2) will give $y_k \ge 0$ if flight k is covered by at least one crew. If flight k cannot be covered by a crew, (2) is infeasible.

Solution: See file amplCase9-6.txt.

crew	pairing
1	None
2	1 (C3, C6, C4, C8, C3)
3	1 (C4, C8, C3, C2, C4)
4	1 (C1, C8, C3, C6, C4, C1)
5	3 (C2, C7, C4, C1, C2)
6	None
7	1 (C5, C2, C8, C3, C1, C5)
8	1 (C6, C1, C3, C6)
9	None
10	None

Flight, k	overallocation (y_k)
3	1
10	1
11	2
17	1
18	2

All other flights are allocated one crew each.

If the pairings does not produce at least one crew allocation/flight, the problem will not have a feasible solution

Chapter 9 Cases

9-7

D_k = Demand for module k, $k = 1, 2, 3$

I_j = initial inventory of device j, $j = 1, 2, .., 5$

$$C_{ij} = \begin{cases} 1, \text{ if device } i \text{ can be used in module } j \\ 0, \text{ otherwise} \end{cases}$$

$$\|C_{ij}\| = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 5 & 1 & 1 & 1 \end{array}$$

P = Total number of wafers produced

$x_j = r_j \cdot P_j$, Nbr of binned devices of type j

y_{jk} = Nbr. of units of device j in module k

Minimize $z = P$

s.t.

$$I_j + x_j - \sum_{k=1}^{3} C_{jk} y_{jk} \geq 0, \quad j = 1, 2, .., 5$$

$$\sum_{j=1}^{5} C_{jk} y_{jk} \geq D_k, \quad k = 1, 2, 3$$

Solution: See file amplCase9-7.txt

```
PRODUCTION SCHEDULE:
Produced wafers =  85  units

Device  Initial inventory  Produced units  Total available  End inventory
  1          10                 17              27               0
  2           4                 16              20               0
  3           8                  8              16              13
  4           0                 25              25               0
  5           3                 17              20               0

Module   Demand units   Devices used   Nbr. of units
  1          20
                             2              20
  2          30
                             1              27
                             3               3
  3          45
                             4              25
                             5              20
```

9-8

$$x_i = \begin{cases} 1, \text{ if check } i \text{ is cleared} \\ 0, \text{ otherwise} \end{cases}$$

Constraints:

$$200x_1 + 75x_2 + 900x_3 + 25x_4 + 525x_5 + 100x_6 + 675x_7 \leq 1200 \quad (1)$$

If residual balance $\geq$ amt of check j *then* check j can be cleared (2)

Conotraints (2) translate mathematically to

$$\text{If } 1200 - \sum_{i=1}^{7} c_i x_i \geq c_j \text{ then } x_j = 1$$

or

$$\text{If } 1200 - \sum_{i=1}^{7} c_i \cdot x_i - c_j \geq 0 \text{ then } x_j \geq 1$$

or

$$\text{If } 1200 - \sum_{i=1}^{7} c_i \cdot x_i - c_j \geq 0 \text{ then } -x_j + 1 \leq 0$$

or

$$1200 - \sum_{i=1}^{7} c_i \cdot x_j - c_j \leq M x_j - .0001 \quad (2a)$$

$$-x_j + 1 \leq M(1 - x_j) \quad (2b)$$

Actually (2a) implies (2b) in this case because (2a) requires x_j to equal 1 whenever the left-hand side allows it.

(a) Minimize $z = \sum_{j=1}^{7} x_j$

Solution: See file amplCase9-8.tx1

clear checks 5 and 7 (= 525 + 675 = \$1200)

(b) Maximize $z = \sum_{j=1}^{7} x_j$

Solution: see same file

Clear checks 1, 2, 4, 5, and 6 (= 200 + 75 + 25 + 525 + 100 = \$925) Remaining balance = 1200 − 925 = \$275, which is less than the amount of any of the uncleared checks (3 and 7).

9-9

C_i = capacity of line i (1000 bbl)

X_{ij} = 1000 bbl allocated to bidder j from line i

r_{ij} = min 1000 bbl from line i by bidder j

$y_{ij} = (0,1)$

b_{ij} = bonus bid by bidder i on line j

$$\text{Maximize } z = \sum_{i=1}^{6} \sum_{j=1}^{8} b_{ij} \cdot x_{ij}$$

s.t.

$$\sum_{i=1} x_{ij} \le .2 \sum_{k=1}^{m} c_k, \quad j = 1, 2, \ldots, 8$$

$$\sum_{j=1}^{8} x_{ij} \le C_i, \quad i = 1, 2, \ldots, 6$$

$$\left.\begin{aligned} x_{ij} &\le M y_{ij} \\ x_{ij} &\ge r_{ij} y_{ij} \end{aligned}\right\} i = 1, 2, \ldots, 6, \; j = 1, 3 \ldots 8$$

<u>Solution</u>: See file amplCase 9-9.txt

z = \$201,750

<u>Allocation</u>:

	Bidder 1	2	3	4	5	6	7	8
1			20					
2			12				18	
3						25	17	12
4	11							
5		35						
6						11		19

all quantities are in 1000 bbl.

9-10

S_{ij} = intensity measure for manger i working on project j
$= (t_{ij} + 1) \times 6 \times \log(c_j) + 1$

where

t_{ij} = travel time in hours by manager i to project j

c_j = cost in 10^6\$ of project j

$$x_{ij} = \begin{cases} 1, & \text{if manager } i \text{ is assigned to project } j \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Minimize } z = \sum_{i=1}^{5} \sum_{j=1}^{8} s_{ij} \cdot x_{ij}$$

s.t.

$$\sum_{i=1}^{5} x_{ij} = 1, \quad j = 1, 2, \ldots, 8$$

$$\sum_{j=1}^{8} x_{ij} \ge 1, \quad i = 1, 2, \ldots, 6$$

Each manager is assigned <u>at least</u> one project.

<u>Solution</u>: See file amplCase 9-10.txt, or file solverCase 9-10.xls

Manager	Assigned projects
a	6
b	3, 4
c	2, 7, 8
d	1
e	5

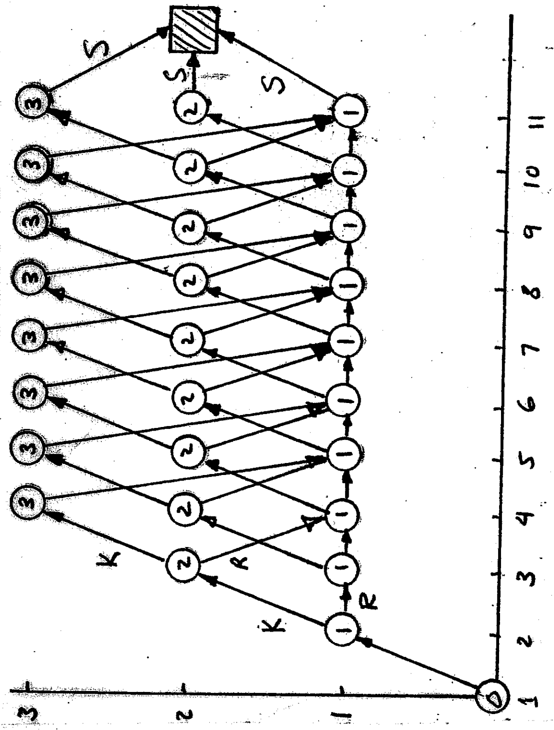
<u>Alternative solution from Solver</u>:

b	3
d	1, 4

10-1

$$f_i(t) = \min \begin{cases} C(t) + f_{i+1}(t+1), & K \\ I(i) + C(0) - S(t) + f_{i+1}(1), & R \end{cases}$$

$$f_N(t) = \min \begin{cases} C(t) + S(t+1), & K \\ I(N) + C(0) - S(t), & R \end{cases}$$



Stage 10:

t	K	R	Optimum	
	C(t)−S(t+1)	I(10)+C(0)−S(t)	f_{10}	Dec
1	.71 −15 = −14.29	16+.5−15.8 = .7	−14.29	K
2	.72 −14.5= −13.78	16+.5 −15 = 1.5	−13.78	K
3	Replace	16+.5−14.5 = 2.0	2.0	R

Stage 9:

t	K	R	Opt^m	
	C(t) + f_{10}(t+1)	I(9)+C(0)−S(t)+f_{10}(1)	f_9	Dec
1	.7 − 13.78 = −13.08	15.5+.45−15.5 − 14.29 = −13.84	−13.84	R
2	.73 + 2 = 2.73	15.5 +.45−14.5− 14.29 = −12.84	−12.84	R
3	must replace	15.5 +.45 − 13.8−14.29 = −12.14	−12.14	R

Stage 8:

t	K	R	Opt^m	
	C(t) + f_9 (t+1)	I(8)+C(0)−S(t)+f_9(1)	f_8	Dec
1	.67−12.84 = −12.17	15.2+.43−14−13.84 = −12.21	−12.21	R
2	.7−12.14= −12.07	15.2 +.43−13.2 −13.84 = −11.41	−12.07	K
3	must replace	15.2 +.43 −12−13.84 = −10.21	−10.21	R

Stage 7:

t	K	R	Opt^m	
	C(t)+ f_8(t+1)	I(7)+C(0)−S(t)+f_8 (1)	f_7	Dec
1	.6 −12.07 = −11.47	14.8 +.41 − 13.5 −12.21 = −10.5	−11.47	K
2	.62 −10.21 = −9.59	14.8+.41−12.9−12.21 = −9.9	−9.9	R
3	must replace	14.8+.41 −11.9 −12.21 = −8.9	−8.9	R

Stage 6:

t	K	R	Opt^m	
	C(t) + f_7(t+1)	I(6)+C(0)−S(t)+f_7(1)	f_6	Dec
1	.62 − 9.9 = −9.28	14.2 +.39 −12.5 −11.47 = −9.38	−9.38	R
2	.7−8.9 = −8.2	14.2 +.39−12.0 − 9.9 = −7.31	−8.2	K
3	must replace	14.2+.39−11.2 −8.9= −5.51	−5.51	R

Stage 5

t	K	R	Opt^m	
	C(t)+ f_6(t+1)	I(5)+C(0)−S(t) + f_6(1)	f_5	Dec
1	.59− 8.2 = −7.61	13.8+.35 −12 − 9.38 = −7.23	−7.61	K
2	.63 − 5.51= −4.88	13.8 +.35−11.8−9.38 = −7.03	−7.03	R
3	—	13.8+.35 −11.2 −9.38 = −6.43	−6.43	R

Stage 4:

t	K	R	Opt^m	
	C(t) + f_5(t+1)	I(4)+C(0)−S(t) + f_5(1)	f_4	Dec
1	.65−7.03 = −6.38	13.5+.32−12−7.61 = −5.79	−6.38	K
2	.7 − 6.46= −5.76	13.5 +.32 −11.5−7.61 = − 5.29	−5.76	K
3	—	13.5+.32 −11 −7.61 = −4.79	−4.79	R

continued...

continued...

Chapter 10 Cases

Stage 3

	K	R	$Opt^{\underline{m}}$	
t	$c(t)+f_4(t+1)$	$I(3)+c(0)-S(t)+f_4(1)$	f_3	Dec
1	$.55-5.76$ $=-5.21$	$13+.28-12-6.38$ $=-5.1$	-5.21	K
2	$.60-4.79$ $=-4.19$	$13+.28-11-6.38=-4.1$	-4.19	K
3	—	$13+.28-10-6.38=-3.1$	-3.1	R

Stage 2:

	K	R	$Opt^{\underline{m}}$	
t	$c(t)+f_3(t+1)$	$I(2)+c(0)-S(t)+f_3(1)$	f_2	Dec
1	$.6-4.19=-3.59$	$12+.25-11-5.21=-3.96$	-3.96	R
2	$.68-3.1=-2.42$	$12-9.5-5.21=-2.46$	-2.46	R
3	—	$12+.25-8-5.21=-.96$	$-.96$	R

Stage 1:

$t=0$,

$I(1)+c(t)+f_2(t+1)=10+.2-3.96=6.26$

Decision: K

Policy:

K → R → K → K → R → R →
K → R → K → S

11-1

Economic lot size formula:

$$y = \sqrt{\frac{2KD}{h}}$$

h = annual holding cost / unit
K = Setup cost
D = annual demand

Given h is a fixed proportion of the unit cost c, we have

$$y = \sqrt{\frac{2KD}{ic}}$$

Let

T = average time period needed to consume the average supply on hand, $y/2$

S = annual dollar usage of the item.

Then,

$$T = \frac{y/2}{D} = \frac{y}{2D}$$

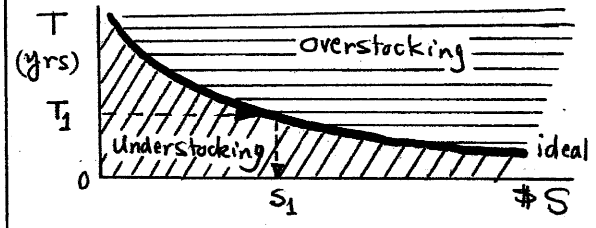
$$S = Dc$$

Under optimal conditions, we have

$$T = \sqrt{\frac{2KD}{4D^2ic}} = \sqrt{\frac{K}{2i}}\sqrt{\frac{1}{Dc}} = \alpha\sqrt{\frac{1}{S}}$$

where α is a constant.

The relationship between T and S for a typical inventory can be graphed as



Policy: If the annual dollar usage is S_1, order the quantity y every $2T_1$ time units

11-2

Inventory control should be based on the data for the final product, because the demand for the purchased component is dependent on the demand for the final product. Separate treatments of the two parts may result in shortage.

For the final product, we have

$$y = \sqrt{\frac{2KD(p+h)}{ph}}$$

$$= \sqrt{\frac{2\times100\times20(5+8)}{5\times8}}$$

$$\simeq 36 \text{ units}$$

time betn. orders = $\frac{30}{20}$ = 1.8 weeks
$\simeq$ 12 days

Ordering policy:

Order $36\times2 = 72$ units of purchased components every 12 days. This policy leads to producing 36 units of the final product every 12 days

11-3

Month	5-yr Av. Demand (rounded)
1	11
2	53
3	10
4	107
5	111
6	100
7	129
8	76
9	52
10	146
11	254
12	42

$\bar{\bar{X}} \simeq 91$

$S_{\bar{X}} = 67.6$

The fluctuations in the average demand per month suggests that the

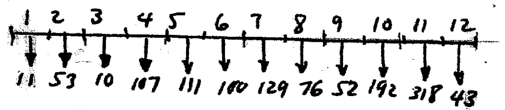
continued...

use of the EOQ based on the average monthly demand for the past 2 years may lead to gross underestimation or overestimation of demand. The given data yields $\bar{\bar{x}} = 91$ units with $S_{\bar{x}} = 67.6$ units, which reflect extreme variations in demand.

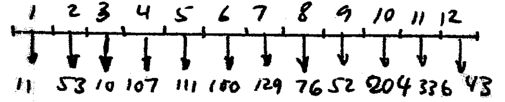
A study of the data shows that with the exception of Nov. and Dec. (and possibly April), the average monthly demand taken over the 5 year span is a good representation of the demand during the month. As for Nov. and Dec., there is a trend for increase in demand approximately equal to 12 units/year for Nov. and 18 units/year for Dec. A planning horizon of 12 months may thus be used to solve the problem. For each new year, the demand per month is taken equal to the averages given in the preceding table. In the cases of Nov. and Dec., the demands are increased by approximately 12 and 18 units, respectively, for each new year.

The following charts apply to the next two years

Next year:

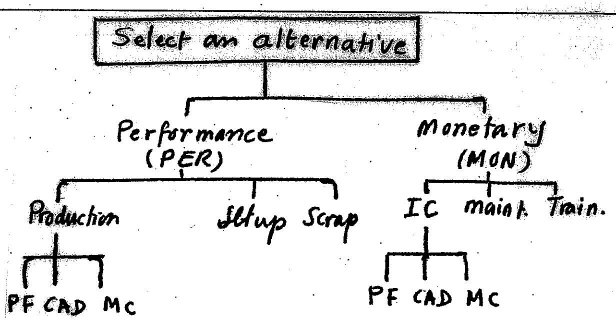


Year after:



The problem can be solved by DP

13-1



Comparison matrices

$$\begin{array}{c|cc} & PER & MON \\ \hline PER & 1 & 2/3 \\ MON & 3/2 & 1 \end{array}$$

Performance:

$$\begin{array}{c|ccc} & PRO & SET & SCR \\ \hline PRO & 1 & 2 & 3 \\ SET & 1/2 & 1 & 4 \\ SCR & 1/3 & 1/4 & 1 \end{array}$$

Monetary:

$$\begin{array}{c|ccc} & I & M & T \\ \hline I & 1 & 2 & 2 \\ M & .5 & 1 & 1 \\ T & .5 & 1 & 1 \end{array}$$

Production (PRO):

$$\begin{array}{c|ccc} & PF & CAD & MC \\ \hline PF & 1 & 8/14 & 8/40 \\ CAD & 14/8 & 1 & 14/40 \\ MC & 40/8 & 40/14 & 1 \end{array}$$

Note: The higher is the production, the more desirable is the alternative

Scrap (SCR):

$$\begin{array}{c|ccc} & PF & CAD & MC \\ \hline PF & 1 & 165/440 & 44/440 \\ CAD & 440/165 & 1 & 44/165 \\ MC & 440/44 & 165/44 & 1 \end{array}$$

Setup (SET):

$$\begin{array}{c|ccc} & PF & CAD & MC \\ \hline PF & 1 & 20/30 & 3/30 \\ CAD & 30/20 & 1 & 3/20 \\ MC & 30/3 & 20/3 & 1 \end{array}$$

continued...

13-1 continued

Initial cost (IC):

$$\begin{array}{c|ccc} & PF & CAD & MC \\ \hline PF & 1 & 25/12 & 120/12 \\ CAD & 12/25 & 1 & 120/25 \\ MC & 12/120 & 25/120 & 1 \end{array}$$

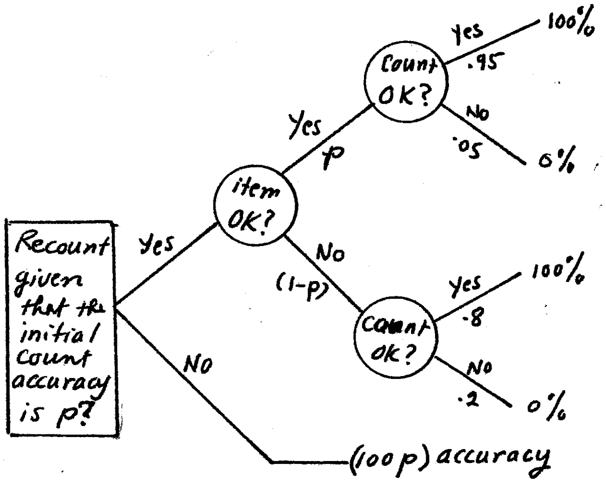
Maintenance cost (MA):

$$\begin{array}{c|ccc} & PF & CAD & MC \\ \hline PF & 1 & 4/2 & 15/2 \\ CAD & 2/4 & 1 & 15/4 \\ MC & 2/15 & 4/15 & 1 \end{array}$$

Training (TR)

$$\begin{array}{c|ccc} & PF & CAD & MC \\ \hline PF & 1 & 8/3 & 20/3 \\ CAD & 3/8 & 1 & 20/8 \\ MC & 3/20 & 8/20 & 1 \end{array}$$

13-2



Expected accuracy given "recount" is made $= (10 \times .95 + 0 \times .05) \times p$
$+ (100 \times .8 + 0 \times .2)(1-p)$
$= 15p + 80$

Expected accuracy given "recount" is not made $= 100p$

continued...

13-2 continued

Thus, "recount" is warrented if

$$15p + 80 > 100p$$

or $p \leq\ .12\%$

13-3

The present policy of using 2-707 or 1-747 crews between 5:00 and 17:00 and between 11:00 and 23:00 represents an overlap between 11:00 and 17:00. We can define the periods per day as

Period	Number of crews
1 (5:00–11:00)	2 (service 4-707 departure)
2 (11:00–17:00)	4 (Service 2-707 & 2-747)
3 (17:00–23:00)	2 (Service 1-747)

The corresponding probability of overnight delay is computed as follows. Let

C_i = number of crews called in period i, $i = 1, 2, 3$

Period 1:

$P\{C_1 > 4\} = 0$, because there are 4 departures only

Period 2:

$$\sum_{x=0}^{4} P\{C_2 > 4 - x \mid C_1 = x\}\, P\{C_1 = x\}$$

Examples of computations:

$P\{C_2 > 0 \mid C_1 = 4\}$
$= 1 - [P\{0 \text{ call from 707 category 2}\} \times$
$P\{0 \text{ call from 707 category 4}\} \times$
$P\{0 \text{ call from 747 category 4}\} \times$
$P\{0 \text{ call from 747 category 6}\}]$
$= 1 - [(1-.019)(1-.006)(1-.016)(1-0)]$
$= .042406848$

$P\{C_1 = 4\} = P\{1 \text{ call } 707\text{-}3\} \times P\{1 \text{ call } 707\text{-}6\}$
$\times P\{1 \text{ call } 707\text{-}2\} \times P\{1 \text{ call } 707\text{-}3\}$

continued...

13 -3 continued

$= .006 \times .003 \times .019 \times .006$

$= .000000002052$

$P\{C_2 > 1 | C_1 = 3\}$

$= P\{C_2 = 1 | C_1 = 3\} + P\{C_2 > 0 | C_1 = 4\}$

$= P\{1, 707\text{-}2 \cap 0, 707\text{-}4 \cap 0, 707\text{-}6 \cap 0, 747\text{-}4\}$

$+ P\{0, 707\text{-}2 \cap 1, 707\text{-}4 \cap 0, 747\text{-}6 \cap 0, 747\text{-}4\}$

$+ P\{C_2 > 0 | C_1 = 4\}$

$= (.019 \times .994 \times .998 \times .984) +$
$(.006 \times .981 \times .998 \times .984) +$
$.042406848$

$\cong .01808$

$P\{C_1 = 3\} = P\{1, 707\text{-}3 \cap 1, 707\text{-}6 \cap 1, 707\text{-}2 \cap 0, 707\text{-}3\}$

$+ P\{1, 707\text{-}3 \cap 1, 707\text{-}6 \cap 0, 707\text{-}2 \cap 1, 707\text{-}3\}$

$+ P\{1, 707\text{-}3 \cap 0, 707\text{-}6 \cap 1, 707\text{-}2 \cap 1, 707\text{-}3\}$

$+ P\{0, 707\text{-}3 \cap 1, 707\text{-}6 \cap 1, 707\text{-}2 \cap 1, 707\text{-}3\}$

$= (.006 \times .003 \times .019 \times .994) +$
$(.006 \times .003 \times .981 \times .006) +$
$(.006 \times .997 \times .019 \times .006) +$
$(.994 \times .003 \times .019 \times .006) \simeq .000002$

The probabilities for period (conservatively rounded up) are given as

x	$P\{C_2 > 4-x \| C_1 = x\}$	$P\{C_1 = x\}$	Product
0	.000001	.966356	.00000097
1	.000035	.033291	.00000116
2	.000478	.000351	.00000017
3	.01808	.000002	.00000004
4	.042407	.000000002	.0000000008
		Total	.0000234

continued...

13 -3 continued

In a similar manner, we compute the probabilities for period 3, which must depend now on what happens in both periods 1 and 2. These computations yield

$P\{\text{overnight delay for period } 3)$

$= .00001513$

Thus,

Total overnight delay probability

$= 0 + .0000234 + .00001513$

$= .000017$

Average number of delays per year

$= 365 \times .000017 = .006$

This is equivalent to having one delay every 166 years.

Associated annual cost of reserve crews

= \$30,000 x 28 members

= \$840,000 per year

Expected cost of delay per year

= \$50,000 x .006 = \$350/year

These costs indicate that the use of four reserve crews is perhaps unwarrented. The idea is to attempt to reduce the number of crews with possible reallocation to the hours of the day.

The following policies are typical of the new proposals made for this situation:

continued...

Chapter 14 Cases

	Present Policy	Policy A1	Policy A2	Policy A3
Crews	4-B707	3-B707	2-B707	3B707
Allocation	2(5:00-17:00) 2(11:00-23:00)	1(5:00-17:00) 2(11:00-23:00)	2(10:00-22:00)	1(6:30-14:30) 2(14:30-22:30)
Cost/year	$840,000	$618,000	$412,000	$490,000
Av. nbr delays	1 day/166 years	1 day/9 years	1 day/6 years	1 day/9 years
Delay cost	$350	$5,500	$8,500	$5,500
Total cost	$840,350	$623,500	**$420,500**	$495,500

Policy A2 has the least total expected cost. the decision is based on adopting a ling-term policy. If the data of the situation are changed, computations must be revised to see if the optimal policy changes.

14-4

Let

$P(S)$ = probabilty of an idividual being schizophrenic

$P(\bar{S})$ = probabilty of an idividual not being schizophrenic

$P(S \mid BA)$ = probability of schizophrenia given brain atrophy

$P(BA \mid S)$ = probability of brain atrophy given schizophrenia

$P(BA \mid \bar{S})$ = probability of brain atrophy given no schizophrenia

In terms of the data, we have

$$P(S) = .015$$

$$P(\bar{S}) = .985$$

$$P(BA \mid S) = .3$$

$$P(BA \mid \bar{S}) = .02$$

It thus follows that $P(S \mid BA) = \frac{.3\times.015}{.3\times.015+.02\times.985} = .186$

The result shows that, even though Hinkley's CAT scan shows brain atrophy, there is less than 20% chance that he is schizophrenic. This is not a strong argument in support of Hinkley's claim of mental illness.

14-5

Probability tree:

```
                                 { yes, probability q
           { H (real question)|
           |                      { no, probability (1-q)
coin       |
tossing --|
           |                       { yes, probability (20/35)
           {T (decoy question)|
                                  { no, probability  (15/35)
```

Per the results of the experiment, we have P{yes vote}=18/35
From the probability tree, we have

P{yes vote} = (1/2) × q + (1/2) × (20/35)
Thus,
(1/2) × q + (1/2)× (20/35) = (18/35)

Solving for q, we get

q = [(18/35) - (1/2) × (20/35)} × 2 = 16/35 =.457

Chapter 14 Case

14-1

The frequency histogram for the demand is given below. To be on the conservative side, we ignore the frequency of zero demand.

Nbr. of units	Frequency	Relative frequency	Cumulative relative frequency
1	89	.7807	.7807
2	20	.17544	**.9561**
3	4	.3509	.9912
4	1	.00877	1.00

Assuming that the demand stays stationary for at least the next year (that is, no appreciable trend), the company's requirement that the demand be met 95% of the time is satisfied with two units in stock.

Chapter 15 Cases

15-1

Because the teller is busy only 40% of the time, it is possible that one teller could attend to more than one customer. In fact, arriving customers may be served b a pool of tellers. The problem with this proposal is that a teller will not have a fixed station, which may create administrative problems in the bank operation.

15-2

The data show that the number of calls reaches a peak between 12:00 and 17:00 daily. The design of the system should based on this extreme condition, rather than on the overall average number of arrivals per day. Thus, for the daily period from 12:00 to 17:00, we have

$$\bar{x} = 9.11 \text{ arrival/hr}, \; s^2 = 7.81$$

There is not reason to believe that arriving calls will follow anything but a Poisson distribution. Notice that $\bar{x} = 9.11$ arrival/hr and $s^2 = 7.81$ are approximately equal, which supports the Poisson claim. (In general, we should use the chi-square to validate the Poisson assumption.)

Regarding the service time distribution (length of calls), the lack of data together with the principle of insufficient reason suggest once again that the service time distribution may also be exponential with mean 7 minutes.

We are now dealing with a Poisson queue with lambda = 9.11 calls per hour and mu = 60/7 = 8.75 phone answers per hour. The telephone lines represent the servers. Given lambda/mu = 1.06, the system needs at least 2 lines. We know, however, that lambda must be larger than 9.11 because the available data do not reflect the calls that are lost when the lines are busy. We thus need to run a type of sensitivity analysis to give us some idea about the "adequacy" of the telephone service under extreme conditions. We must remember that the number of lost calls must be reduced to an absolute minimum because the facility deals with situations that could affect the sell-being of a abused child.

The following table provides the measures of performance given lambda = 9.11, 13.7, and 18.22 calls per hour. These values represent 100%, 150%, and 200% of the estimated arrival rate.

Lambda	9.11				13.7				18.22			
Nbr. of lines	2	3	4	5	2	3	4	5	2	3	4	5
L_q	.4	.06	.009	.001	2.8	.3	.06	.01	1.2	.23	.05	.01
W_q (sec)	162	21.6	3.6	.36	720	72	14.4	3.6	216	36	10.8	2.52
$P\{n > c\}$	.2	.03	.01	.002	.6	.15	.04	.01	.36	.11	.04	.01

L_q could not be used as proper measure in this case. For example, for lambda = 9.11, L_q = .4 waiting calls for c = 2. This may appear quite small, but if we examine W_q, the average

continued...

waiting time until a call is acknowledged is 162 sec (about 3 min). This is a long waiting time for an anxious person reporting an abuse case. A waiting time of about 10 seconds is the most that can be tolerated in these situations. For example, for lambda = 18.22 calls per hour, 5 lines are needed.

15-3

An initial analysis of the situation can be made by comparing the rate of arrival of calls for truck service with the service rate. From the data

lambda = (0x30 + 1x90 + ... + 12x4)/(0 + 1 + 2 + … + 12) = 4.1 calls per hr

The average service time per call is computed from the second table as

tBar = (5x61 + 15x34 + … + 95x2)/(61 + 34 + 15 + … +2) = 20.2 min per call

Thus, the service rate is

mu = 1/20.2 = .05 services per min per truck = 2.97 services per hr per ruck

Given three trucks are in service, the total service rate is 3x2.97 = 8.9 services per hr. Thus, the utilization of the trucks is computed as

utilization = lambda/(3mu) = 4.1/8.9 = .46

The low utilization shows that the three trucks are sufficient to service the six departments adequately. The main drawback with the current setup is that the trucks do not have a "home" station, a basic assumption is calculation the utilization factor. In other words, the 46% utilization assumes that the trucks are available in one service pool. This difficulty is rectified by placing all calls for service to a common dispatcher who is in constant contact with the drivers of the trucks.

15-4

From the data of the problem, the average rate of breakdown per machine per hour is computed as

lambda = (7 + 8 + 8)/(3 mach x 8 hr) = .9583 per machine per hr

The difference between the failure time and the completion of repair gives the anount of time a broken machine spends in the repair system. Thus,

W_s = [(10+12+10+13+10+12+9) + (8+8+13+8+9+13+12+10) + (13+11+10+12+8+11+8+10)]/24
= 10.46 min

We can also estimate the number of machines in the system L_s from the information in the second table. For simplicity, we take the average of all the given data points. Normally, we should treat L_s as a time-based variable. However, this would require a complete history of the number of broken machines at all hours of the day.

L_s = (6 + 6 + 9 + 6 + … + 8 + 8 +6)/(8 data points x 5 days) = 6.73 machines

If the data are correct, and if the situation behaves per the Poisson assumptions, then L_s and W_s must satisfy the formulas

$$\begin{aligned} L_s &= \text{lambda}_{\text{eff}}\, W_s \\ &= \text{lambda}(N - L_s)W_s \end{aligned}$$

From the data, we have

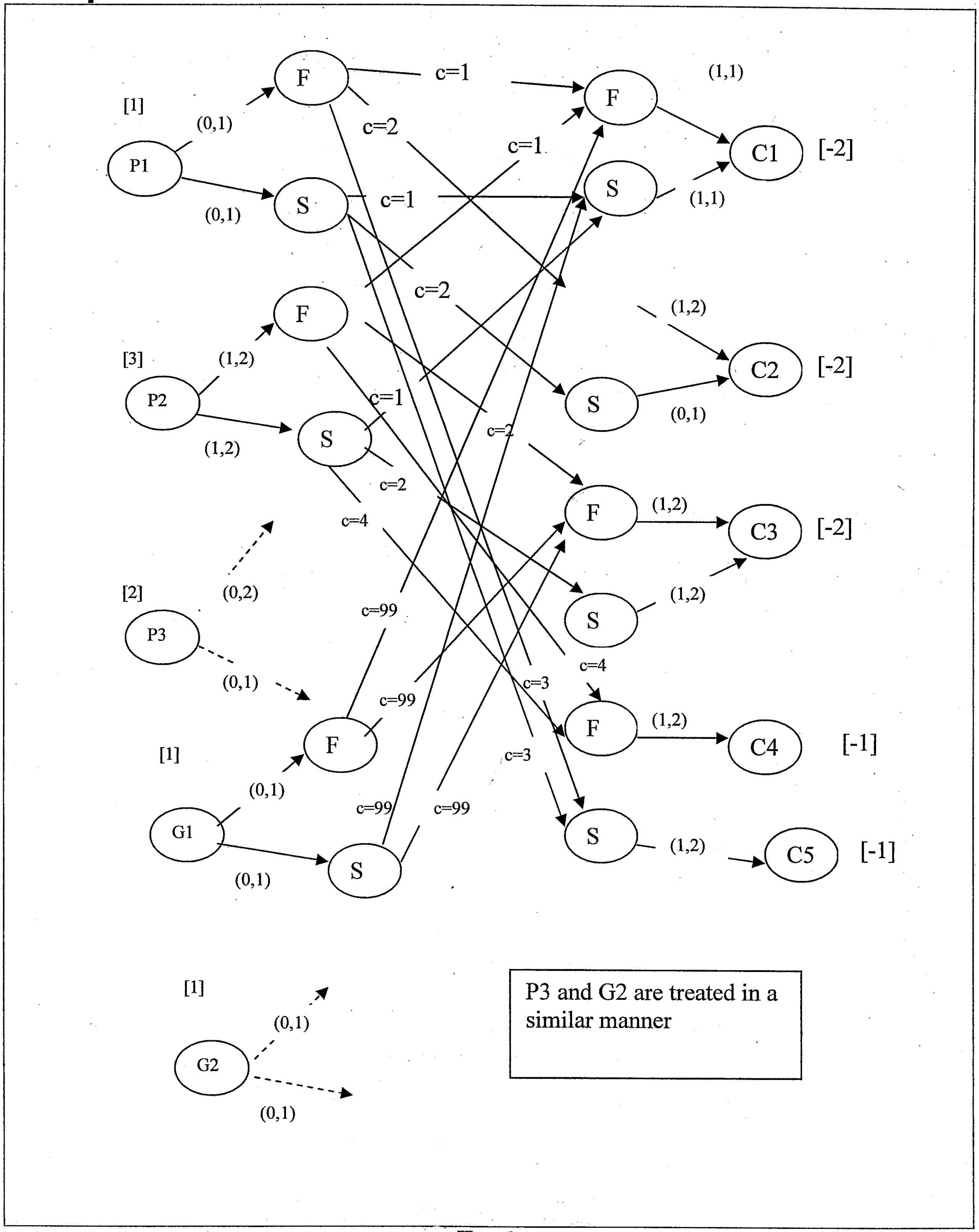
lambda = .9583 calls/machine/hr
$N = 30$ machines
$L_s = 6.73$ machines
$W_s = 10.46/60 = .1743$ hr

Thus,

$$\begin{aligned} \text{lambda}(N - L_s)W_s &= .9583(30 - 6.73)(.1743) \\ &= 3.887 \text{ machines} \end{aligned}$$

This result shows that the data for computing lambda and L_s are not consistent. Hence is the conclusion reached by the manager.

Chapter 20 Case



P3 and G2 are treated in a similar manner

Chapter 22 Case

22-1

Assume that the machine starts new, and define,

n = planning horizon (= 6 years)

I = initial purchase price

TW_i = trade-in value of a working machine whose age just turned i years

TF_i = trade-in value of a failed machine whose age just turned i years

P_i = probability that an i-year old machine in working order at the start of a year fails at the end of the year.

SW_i = Salvage value at the end of the planning horizon of a working machine of age i.

SF_i = Salvage value at the end of the planning horizon of a failed machine of age i

$f_k(i)$ = Minimum expected cost of the remaining periods of the horizon given that we start year k with a machine of age i and in working order

$k = 1, 2, \dots, n; \quad i = 1, 2, \dots, k-1$

C_i = expected operating cost of a working machine of age i

that a working machine

$$f_k(i) = \min \begin{cases} R: I - TW_i + C_0 + P_0\{I - TW_1 + f_{k+1}(0)\} \\ \qquad + (1-P_0)\, f_{k+1}(1) \\ K: C_i + P_i\{I - TW_{i+1} + f_{k+1}(0)\} \\ \qquad + (1-P_i)\, f_{k+1}(i+1) \end{cases}$$

$k = 1, 2, \dots, n-1$

$i > 0$

$$f_k(0) = C_0 + P_0\{I - TW_1 + f_{k+1}(0)\} + (1-P_0)\, f_{k+1}(1)$$

$$f_n(i) = \min \begin{cases} R: I - TW_i + C_0 - P_0 TF_1 \\ \qquad - (1-P_0)\, TW_1 \\ K: C_i - P_i TF_{i+1} - (1-P_i)\, TW_{i+1} \end{cases}$$

$$f_n(0) = C_0 - P_0 TF_1 - (1-P_0) TW_1$$

continued...