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ORIGINAL ARTICLE

Problem solving in Chinese mathematics education: research and practice

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Abstract This paper is an attempt to paint a picture of problem solving in Chinese mathematics education, where problem solving has been viewed both as an instructional goal and as an instructional approach. In discussing problem-solving research from four perspectives, it is found that the research in China has been much more content and experience-based than cognitive and empirical-based. We also describe several problem-solving activities in the Chinese classroom, including "one problem multiple solutions," "multiple problems one solution," and "one problem multiple changes." Unfortunately, there are no empirical investigations that document the actual effectiveness and reasons for the effectiveness of those problem-solving activities. Nevertheless, these problem-solving activities should be useful references for helping students make sense of mathematics.

1 Introduction

In China, there is a long history of interest in integrating problem solving into school mathematics (Siu, 2004; Stanic & Kilpatrick, 1989). *Jiuzhang Suanshu* or *Nine chapters on the mathematical art* can be regarded as the earliest Chinese mathematical textbook. This classical book of the problems and its extensions and variations were widely used as mathematical curriculum in Ancient China (Fischer, 2006; Siu, 2004). While it is not exactly clear when it was compiled, researchers generally agree that it was compiled during the Han Dynasty (206 BC to 220 AD) (Martzloff, 1997; Siu, 2004). This practical handbook includes 246 problems with answers, and general rules about how to solve the problems. The problems are embedded in real-life situations and grouped into nine chapters: (1) Survey of land, (2) Millet and rice, (3) Distribution by progressions, (4) Diminishing breadth, (5) Consultation on engineering works, (6) Imperial taxation, (7) Excess and deficiency, (8) Calculating by tabulation, (9) Gou-gu (right triangles) (Siu, 2004).

The tradition of integrating mathematical problems into the school curriculum extends to the present. While there has been a series of reforms/revisions of school mathematical curriculum in modern China (e.g., Basic Education Curriculum Material Development Center, 2001; Ma, Wang, Sun, & Wang, 1991; Office of School Teaching Materials and Institute of Curriculum and Teaching Materials, 1986), the development of students' abilities to solve problems as one of the fundamental goals in school mathematics has remained the same over the years.

During the period of 1949–1957, the Chinese mathematics education was adopting the Soviet mathematics curriculum, as well as the models of learning and teaching mathematics. However, Chinese mathematics education did not experience dramatic changes like "New Math" in 1960s and "Back to basics" in 1970s in the West. The three critical properties of mathematics in Soviet school mathematics, rigorousness, abstractness and application, influenced greatly the development of Chinese school mathematics curriculum. Thus mathematics education paid more attention to mathematical deductions using formal and rigorous mathematical language. In the late 1970s, the Chinese syllabus of mathematics teaching for elementary

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and secondary schools required students to apply mathematical knowledge to solve real-life problems (Chinese National Ministry of Education, 1978). The syllabus in 1988 explicitly indicated that students should not only be able to calculate correctly, but also understand the principles of mathematical operations, and use appropriate strategies to solve problems (Chinese State Education Commission, 1988).

In this paper, we will discuss research about and the practice of problem solving in China from various perspectives. In particular, we will discuss research about problem solving in Chinese mathematics education from four perspectives: (1) international comparative perspective, (2) mathematical methodology perspective, (3) classroom instructional perspective, and (4) teaching and curricular materials perspective. We will discuss four aspects of practices of problem solving in Chinese mathematics education: (1) one problem, multiple solutions; (2) one problem, multiple changes; (3) multiple problems, one solution; and (4) the "examination culture" and problem solving in the Chinese classroom.

2 Research

2.1 Research from international comparative perspective

People first started to learn about Chinese mathematics education in general and problem solving in particular because of international comparative studies. Chinese students from Mainland China, Hong Kong, and Taiwan participated in both large-scale international comparative studies and small scale, in-depth international comparative studies. The large-scale studies include the first international mathematical study (FIMS) (e.g., Husen, 1967), the second international mathematical study (SIMS) (e.g., Robitaille & Travers, 1992), and the third international mathematics and science study (TIMSS) (U.S. Department of Education, 1996, 1997, 1998), the international assessment of educational progress (IAEP) (Lapointe, Mead, & Askew, 1992; Lapointe, Mead, & Phillips, 1989), and the program for international student assessment (PISA) (Organization for Economic Cooperation and Development, 2004). The small-scale, in-depth studies include those by An (2004), Cai and his associates (e.g., Cai, 2000, 2004; Cai & Hwang, 2002; Cai & Lester, 2005), Clarke and his associates (e.g., Clarke, Keitel, & Shimizu, 2006), Geary and his associates (e.g., Geary, Bow-Thomas, Liu, & Siegler, 1996), Leung (e.g., 1995), Ma (1999), and Stevenson, Stigler, and their associates (e.g., Stevenson & Lee, 1990; Stevenson, Chen, & Lee, 1993; Stigler, Lee, & Stevenson, 1990).

A general finding from almost all existing international comparative studies in mathematics was that Chinese students consistently outperformed US students across grade levels and mathematical topics. For example, Geary et al. (1996) examined the development of arithmetical competencies of Chinese and US children in kindergarten, and first, second, and third grades. They found that Chinese children had higher success rates than did US children in each grade level. The results of TIMSS showed that Chinese students from Hong Kong outperformed US students in overall mathematics achievement at both 8th and 12th grades (U.S. Department of Education, 1996, 1998). In 1990 the second IAEP surveyed 13-year-old students from 20 countries and 9-year-old students from 14 of the 20 countries. Results from the second IAEP study again showed that 9- and 13-year-old Chinese students from Mainland and Taiwan performed better than did their US counterparts (Lapointe et al., 1992).

Chinese students not only outperformed their Western counterparts across grade levels, but also outperformed them across mathematical topics. For example, the second IAEP surveyed students' performance on the following topic areas: numbers and operations, measurement, geometry, data analysis, statistics and probability, and algebra and functions (Lapointe et al., 1992). Results show that Chinese students performed better than did US students on each of the topics. Stevenson et al. (1990) used samples of first- and fifth-grade students from Chicago (US) and Beijing (China) to compare their achievement on mathematical topic areas including word problems, number concepts, mathematical operations, measurement and scaling, graphs and tables, spatial relations, visualization, estimation, and speed tests. They found that there were almost no areas in which the children in Chicago performed as well as children in Beijing. The second international mathematics study measured eighth grade students' abilities to solve arithmetic, algebra, geometry, statistics, and measurement problems and also measured 12th grade students' abilities to solve algebra, geometry, elementary functions and calculus, probability and statistics, sets and relations, and number-system problems. In each of the topic areas tested, Chinese students from Hong Kong outperformed the US students (Robitaille et al., 1992).

These cross-national studies provided sufficient evidence showing that Chinese students are better performers on tasks measuring basic mathematical knowledge and skills in mathematics. This general finding from cross-national studies is echoed by the heavy emphasis of basic skills and knowledge in Chinese education (Gardner, 1989; Zhang, 2006b). However, Chinese students are not necessarily higher performers on complex, open-ended tasks measuring creativity, problem posing, and non-routine problem solving (Cai, 2000; Cai & Hwang, 2002; Chen et al., 2002). In one of the cross-national studies, it was found that the success rates for Chinese students on the computation tasks and simple problem-solving tasks are much higher than for those on the complex problem solving tasks (See a description in Cai & Cifiralli, 2004). Such a performance pattern for the Chinese students was also shown in the following Division Problem, which was from a slight modification of a problem in the 1983 US National Assessment of Educational Progress: (Messick et al., 1983)

> Students and teachers at Gunming elementary school will go by bus for Spring sightseeing. There are a total of 1128 students and teachers. Each bus holds 36 people. How many buses are needed?

In solving the division problem, one not only needs to correctly apply and execute division computation (computation phase), but also to interpret correctly the computational results with respect to the given situation (sense-making phase). Chinese students outperformed US students on the computation phase, but not on the sense-making phase. Moreover, Chinese students' success rate in the computation phase was much higher than that in the sense-making phase. Based on the available evidence from cross-national studies, Cai and Cifarelli (2004) identified the following six characteristics of Chinese students' mathematical problem solving:

(a) Chinese students perform unevenly on various tasks—better on tasks assessing computation skills and basic knowledge than on tasks assessing open-ended complex problem solving.
(b) Chinese students are more likely to use generalized strategies and symbolic representations.
(c) Chinese students usually provide more conventional solutions.
(d) Chinese students can generate more solutions if they are asked for them.
(e) Chinese students frequently commit errors involving unjustified symbol manipulations.
(f) Chinese students are less willing to take risks in problem solving.

The detailed description for each of the characteristics can be found in Cai and Cifarelli (2004). In this paper, we briefly describe the second characteristic: Chinese students are more likely to use generalized strategies and symbolic representations to solve problems. In one of the studies (Cai & Hwang, 2002), we used the following *Doorbell problem* to examine Chinese and US students' thinking:

> Sally is having a party. The first time the doorbell rings, one guest enters.
>
> The second time the doorbell rings, three guests enter.
>
> The third time the doorbell rings, five guests enter.
>
> The fourth time the doorbell rings, seven guests enter.
>
> Keep on going in the same way. On the next ring a group enters that has two more persons than the group that entered on the previous ring.
>
> 1. How many guests will enter on the 10th ring? Explain how you found your answer.
>
> 2. In the space below, write a rule or describe in words how to find the number of guests that entered on each ring.
>
> 3. Ninety nine guests entered on one of the rings. What ring was it? Explain or show how you found your answer.

Chinese and US students had almost identical success rates when they were asked to find the number of guests who entered on the 10th ring. However, the success rate for Chinese students (43%) was significantly higher than that of the US students (24%) when they were asked to find the ring number at which 99 guests would enter the room. The success of the Chinese students in answering the last part of the question was due to the fact that a larger percentage of the Chinese students used generalized strategies than did US students. Students who chose a generalized strategy to solve the *Doorbell problem* followed one of the two paths. One of the paths is to notice that the number of guests who entered on a particular ring of the doorbell was equal to twice the ring number minus one (i.e., $y = 2n - 1$, where y represents the number of guests and n represents the ring number). The other path is to recognize that the number of guests who entered on a particular ring equaled the ring number plus the ring number minus one [i.e., $y = n + (n - 1)$]. If we exclude those Chinese students who used generalized strategies from the analysis, the success rates between the two samples are almost identical. Therefore, the Chinese students' preference for generalized strategies seems to help them outperform the US students on problems amenable to generalized strategies.

2.2 Research from the mathematical methodology perspective

Historically, mathematics education in China put a heavy emphasis on mathematical methodologies in problem solving. In studying ancient mathematics education in China, researchers (e.g., Lam, 1977; Martzloff, 1997; Siu, 2004; Zhou, 1990) found that students were required to master problem-solving methods through repetitive learning. According to Siu (2004), the first paper on mathematics education in China was *Cheng Chu Tong Bian Ben Mo* (Alpha and omega of variations on multiplication and division) written by the Song mathematician Yang Hui in

1274. The following quotation shows the importance of a method to solve problems.

> "The working of a problem is selected from various methods, and the method should suit the problem. In order that a method is to be clearly understood, it should be illustrated by an example. If one meets a problem, its method must be carefully chosen. If numerical exercises are performed daily, this establishes a quicker insight into analyzing a problem and hence is beneficial to all." (Book I, Chap. 3) (cited in Siu, 2004, p.164)

The tradition for emphasizing the methods of solving mathematical problems continued throughout the history of mathematics education in China (Ma et al., 1991; Siu, 1995; Zhou, 1990). One of the goals for teaching mathematics continues to be to help students learn methods, which can be transferred and applied to other problems. In this direction, students are encouraged to acquire generalized methodology for solving problems.

This tradition of emphasizing the methods of solving problems has been expanded by research mathematicians who systematically study mathematical methodology. The earliest representative work on the methodology of mathematics is the *Selected lectures of mathematical methodology* by Xu and published in 1983. In this book, Xu discussed ten topics for developing mathematical ideas, including the axiomatic method and modeling method. Xu situated his study of the mathematical methodology in the broad context of history and philosophy of mathematics, including an in-depth discussion of formalism, intuitionism, and logicism.

While Xu's book on mathematical methodology is influential and seminal, the contents and examples are quite advanced and not accessible for many mathematics teachers. However, during the same period of time, in 1984, George Polya's classical book, (Polya, 1945) *How to solve it* was translated into Chinese and published by several publishing houses in China. Thus the direct impact of Xu's earlier work on mathematics education is not as great as that of Polya's. However, in 1985, Xu's student, Yuxin Zheng, published *Introduction to mathematical methodology*, which is much more accessible for school mathematics teachers and students. Since then, Mathematical methodology has been listed as one of the required courses for mathematics teachers' pre-service and in-service training and graduate programs in mathematics education in China.

There are at least three unique features for this aspect of research. First of all, this aspect of research can be characterized as the study of methodological principles of problem solving (Zheng, 2001). They can also be labeled as general problem solving strategies (Dai, 1996; Luo, 2001). However, different books included different principles or strategies. In some cases, different authors used the same name to describe a problem-solving strategy, but the meaning behind it may not be the same. The following is a partial list of the strategies: looking for patterns, decomposition and combination, working forwards, working backwards, means-end analysis, using counter-examples, examining a problem situation dynamically in a static process, drawing a diagram, setting-up an equation, analysis of finite differences, making a systematic list, using sub-goals, examining individual cases, making generalizations, and using transformations. Following the suggestions of the Song mathematician Yang Hui in 1274 (Zhou, 1990), these authors usually describe a problem-solving principle or strategy, and then use a few examples to illustrate how the principle or strategy can be used to solve mathematical problems.

Second, many secondary mathematics teachers joined the effort of studying mathematical methodology, with a focus on studying mathematical problem-solving strategies. Many of these teachers shared their experiences of teaching mathematical problem solving strategies in various mathematics education journals in China. It was estimated that in 1985 there were 3,625 such articles involving problem-solving strategies published in 28 secondary mathematics education journals (Dai, 1996). According to a survey of articles published in three popular secondary mathematics education journals, a total of 665 articles were published in the three journals in 1991. Among the 665 articles, 546 of them (or 82%) involved mathematical problem-solving strategies (Dai, 1996). The focus of teachers' sharing teaching experience on problem solving is to develop students' specific problem-solving strategies as well as to develop general problem-solving heuristics. Currently, there are at least 30 national and regional mathematics education journals in China. These national journals are similar in nature as *Mathematics teacher* and *Teaching children mathematics* published by the National Council of Teachers of Mathematics in the US as well as *Mathematics teaching* published by the Association of Teachers of Mathematics in the United Kingdom. The regional journals in China are similar to *Ohio journal of school mathematics* published by Ohio Council of Teachers of Mathematics. Chinese mathematics teachers frequently use those national and regional journals to plan lessons (Cai, 2005).

Third, with the increased interest in studying the methodological aspect of problem solving and the emphasis on an "examination culture", the study of problem solving gradually shifted from studying general heuristics, to problem solving strategies, and then to very specific problem solving techniques. Many teachers require students to memorize specific techniques for solving different problems, so that students can quickly recognize the types of problems and solve them immediately (Dai, 1996; Lee, Zhang, & Zheng, 1997; Zheng, 2001). The focus of teaching these techniques is to gain high scores in high-stake examinations, rather than to develop students' thinking skills for analyzing problem situations and solving problems. As a result, some students committed unjustified symbol manipulation errors when trying to solve mathematical problems without understanding the meaning of the symbol manipulations (Cai & Cifiralli, 2004). Lee et al. (1997) documented a dramatic example when a group of Chinese fourth graders, seventh graders, eighth graders, and twelfth graders were asked to solve an absurd problem: *There are 26 sheep and ten goats in a ship. How old is the captain*? About 90% of the Chinese fourth graders, 82% of the seventh and eighth graders, and 34% of the twelfth graders "solved" this problem by combining numbers in it without realizing the absurd nature of the problem (Lee et al., 1997). When these Chinese students were asked why they did not recognize that the problem was meaningless, many students responded that "any problem assigned by a teacher always has a solution."

2.3 Research from classroom instructional perspective

Teaching with variation is a typical feature of teaching mathematics in the Chinese classroom (Marton, Runesson, & Tsui, 2004), and this teaching approach has been used by Chinese teachers widely and frequently (Gu, Huang, & Marton, 2004; Nie, 2004). Therefore, to discuss the research about problem solving from a classroom instructional perspective in China, we decided to focus on the discussion of the teaching with variation.

Let us first use a music metaphor to understand the meaning of teaching with variation. In music, variation means "one of several short tunes which are based on the same simple tune, but are different from it and from the others" (Walter & Woodford, 2005). While variation includes "changing" and "being different," it also has the essential features from the "same original" and "same simple tune." The teaching with variation just grasps those essential features of the "changing" and "being different."

In teaching with variation, a series of related problems are presented to help students understand a concept, master a problem solving method, or make a generalization. When learning mathematical concepts, students are provided a series of problems in which essential features of mathematical concepts are kept unchanged, but the nonessential features of mathematical concepts are changed. In the case of this concept formation, the use of problem variations is designed to help students discern what the essential features of mathematical concepts are, and what they are not (Gu, 1994; Marton et al., 2004; Wang, 1995). For example, after the introduction of the positive nature of the arithmetic square root, students are provided with the following variations of the problems to solidify the understanding of their features (Nie, 2004):

$$\sqrt{9} = ?; \quad (1)$$

$$\sqrt{3^2} = ?; \quad (2)$$

$$\sqrt{(-3)^2} = ?; \quad (3)$$

$$\sqrt{x^2} = ?; \quad (4)$$

$$\sqrt{(x-3)^2} = ?; \quad (5)$$

$$\sqrt{(x-y)^2} = ? \quad (6)$$

According to the meaning of arithmetic square roots, the answer to each of the above problems has to be positive. These numbers and expressions are chosen and arranged according to increasing the complexity and cognitive requirements of problem solving. The different numbers and expressions under the radical signs help students understand the positive nature of the arithmetic square root in different problem situations. Marton et al. (2004) pointed out that arranging for learning implies arranging for developing learners' ways of seeing and experiencing. It is impossible to discern a certain feature without experiencing variations in a dimension corresponding to the feature (Marton et al., 2004). An example of this point is the recognition of colors. If there were no variations of colors, it would be impossible for us to know what red, green, or yellow was. The idea of "no variation, no discernment" is the answer to the question "why teaching with variation" makes sense.

The use of variations is not only an instructional approach in the Chinese classroom, but also it is an effective way to solve mathematical problems. According to Gu et al. (2004), using variations is a crucial phase in helping students solve problems through the scaffolding of transferring problems in China. Gu et al. (2004) used the diagram shown in Fig. 1 to demonstrate the scaffolding process in problem solving by using variations. The variations of problems serve as means to connect the known problem to the unknown problem. The series of varied problems paves the way to move gradually from the problems students have solved to the problem to be solved. In this process, "students' experience in solving problems is manifested by the richness of varying problems and the variety of transferring strategies." (Gu et al., 2004, p. 322)

There is a growing consensus among researchers, educators, and teachers in China that teaching with variation offers considerable promise (Gu et al., 2004; Nie, 2004). Theoretically, this approach makes sense to foster students' learning and problem solving. In teaching with variation,

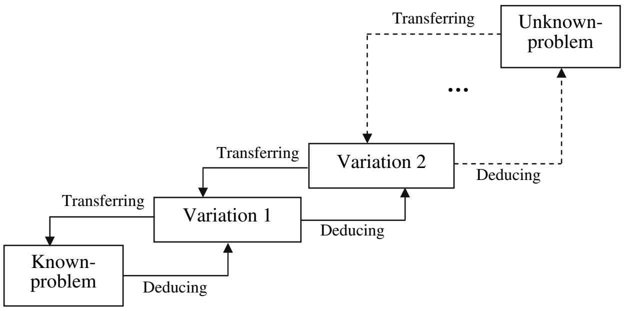


Fig. 1 Variation for solving problems

students develop mathematical ideas and learn a skill through solving a series of related mathematical problems. The variations of problems help students make meaningful connections. However, there is little empirical data available to confirm the promise of teaching with variation (Wong, Lam, & Sun, 2006; Sun, 2006). Therefore, to actually realize this promise, research about and development and refinement of this practice must take place.

It should be indicated that in teaching with variation, the lessons are usually well structured. In a recent study, Mok, Cai, and Fung (2007) analyzed a well-structured lesson to examine the opportunities and missed opportunities for students' learning. In the lesson, the teacher selected variations of instructional tasks to compare fractions. The follow-up interviews with students and teachers clearly showed that students mastered the procedures for comparing fractions well. However, the analysis of the lesson showed that students also missed many opportunities for independent exploration. Because of using variations of problems, the difference between one variation and another was so small that students had little room to think independently. Instead, students mainly followed the teacher's "planned frame" to learn what was earlier determined by the teacher. The instructional activities in the lesson that show a very systematic choice of variation and clear focus may serve well the goal of teaching a specific basic skill. However, the type of engagement the teacher created in the lesson using variations of problems is less ideal for fostering students' higher-order thinking skills. While teaching with variation does not necessarily lead to the development of basic skills and missed opportunities for fostering students' higher-order thinking skills, the findings from the study by Mok et al. (2007) suggest the need for investigating ways to teach effectively with variation.

2.4 Research from the teaching and curricular materials perspective

In early 1950s, Chinese mathematics curriculum was adopted from that of the Soviet Union. Since then, the mathematics curriculum has gone through a number of revisions. One of the major criticisms of the traditional mathematics curriculum in China has been its exclusive use of close-ended routine problems (21st Century Mathematics education project, 1992, 1995; Ding & Zhang, 1989; Wong, Lam, & Chan, 2002; Yan, 1994). Most problems are presented without contexts. Even if some problems are presented with contexts, the contexts are usually artificial and disconnected from students' lives. Chinese mathematics education has an extremely heavy emphasis on the basic knowledge and basic skills (known as the "two basics"). Many researchers and educators in China worry that overemphasis on the two basics is done at the cost of sacrificing the development students' higher-order thinking and problem solving skills.

Currently, there is a mathematics curriculum reform movement in China. One of the important focuses of the curriculum reform is to include more real-life and open-ended problems. The mathematics curriculum standards for the 9 years of compulsory education calls for providing students opportunities to pose problems, understand problems, and apply the knowledge and skills learned to solve real-life problems (Basic Education Curriculum Material Development Centre, 2001). Similarly, the curriculum standards for senior high school mathematics also calls for developing students' abilities to pose, analyze, and solve problems from mathematics and real-life (Basic Education Curriculum Material Development Centre, 2003). That is, school mathematics should enable students to experience the mathematical modeling process. The curriculum standards for senior high school suggested the modeling process shown in Fig. 2.

To date, there have been a number of research efforts related to school mathematics from the teaching and curricular materials perspectives. In this section, we briefly introduce some of the efforts.

2.4.1 Comparative studies about school mathematics

In a study (Cai, Lo, & Watanabe, 2002), a Chinese curriculum was compared to two US National Science Foundation-funded curricula including Mathematics in Context (MiC) (National Center for Research in Mathematical Sciences Education at the University of Wisconsin/Madison and Freudenthal Institute at the University of Utrecht, 1997–1998) and Connected Mathematics Program (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). Part of the study involved the analysis of worked-out examples and practice problems in these curricula. Worked-out examples and practice problems were classified into three categories. Table 1 describes these categories, and within each category different types of problems were identified. It clearly shows that the two US series focus more on the statistical aspect of the concept of the average (as a representative of a data set);

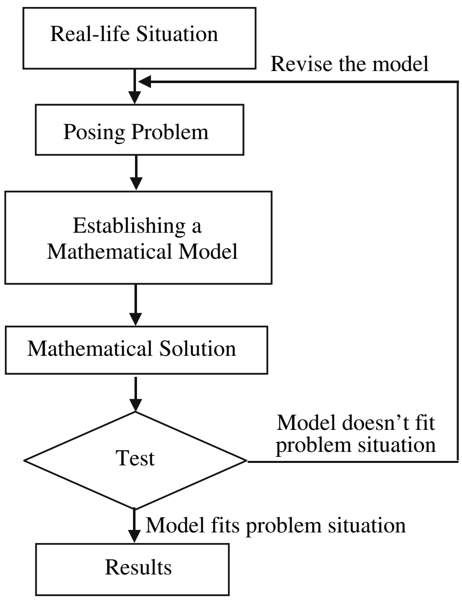


Fig. 2 A modeling process suggested in the Chinese curriculum standards

Table 1 Types of worked-out examples and practice problems

(A) Direct application of average algorithm
A1. Given data set represented in table or graph, find average. *China*, *CMP*, *MiC*
A2. Given average and number of quantities, find the total. *China*, *MiC*
(B) Flexible application of average algorithm
B1. Given average and a data set with one or more missing numbers, find the missing numbers. *China*
B2. Create a data set with a given average and number of data. *China*, *CMP*, *MiC*
B3. Find an overall average from multiple data sets. *China*
B4. Weighted average problem. *China*
(C) Appropriate interpretation and use of mean in statistical context
C1. Using the average to compare two data sets with unequal items. *China*, *CMP*
C2. The mean does not have to equal to any data. *CMP*, *MiC*
C3. Using the mean together with range will give a fuller picture of the data set. *CMP*, *MiC*
C4. Compare the mean, median and mode and decide the appropriate usage for them. *CMP*, *MiC*

while the Chinese series focus more on the concept as an algorithm. In the Chinese series, all of the worked-out examples or practice problems in the fourth grade involve direct application of the averaging algorithm (A1 and A2), and the majority of the worked-out examples or practice problems in the fifth grade involve reverse or flexible application of the averaging algorithm (B1, B2, B3 and B4). In particular, the Chinese fifth grade textbook contains a considerable number of weighted mean problems in various contexts. Only several problems in the China series require understanding the concept of the average as a representative of a set of data in the fifth grade textbook (C1).

2.4.2 Longitudinal investigation of the Standards-based mathematics curriculum on students' learning

Since the release of the mathematics curriculum standards for 9 years of compulsory education in China, a number of *Standards*-based school mathematics curricula were developed and implemented to align with the recommendations in the *Standards*. The implementation of *Standards*-based instructional materials requires change not only in how mathematics is viewed but also in how mathematics is taught and learned. As in the US, there have been heated debates over the mathematics education reform movement in general and *Standards*-based curricula in particular in China in the past several years. The process and issues involved are very similar to those in the US. The ultimate goal of the reform effort is to bring about changes in classroom practice and consequently to improve students' learning. As the debates continue, there is an increasing demand for data that show how well *Standards*-based school mathematics curricula work. In particular, the effectiveness of *Standards*-based curriculum depends critically on how teachers understand and implement the curriculum. Currently, there is an on-going research project entitled "*Has curriculum reform made a difference? Looking for changes in classroom practice*." In this project, a group of independent investigators is investigating longitudinally whether the curriculum reform initiatives in China have or have not influenced classroom practices and student learning (Ni, Cai, Hau, & Li, 2004). While it is too soon to share results because the project is still in its early stage, it is expected that the findings from this project should provide insightful information for the curriculum reform in China and around the world.

2.4.3 Collection of mathematical problems

Some researchers have collected and edited books to include open-ended or real-life problems. *A collection of mathematical problems for secondary school* by Zhang and Dai (1993) is one of the earliest books. Subsequently, Dai edited and published a series of open-ended problem collections by grades and content topics (e.g., Dai, 2000a, b). While these problem collections are not meant to replace the curriculum and textbooks, they have been widely used by teachers and integrated into their regular curriculum. The open-ended problems were used to supplement regular textbook problems. In these problem collection books, there are also discussions about the knowledge and skills involved in each of the problems. Some books even include recommendations about when and where to use a particular problem and what knowledge students would learn through solving the problem. The intention is to continue the focus of the two basics (basic knowledge and basic skills), but at the same time, it is expected that the inclusion of open-ended and real-life problems can foster students' development of higher-order thinking skills, since open-ended and real-life problems facilitate students' sense-making of mathematics in context and support meaningful connections of different mathematical ideas and representations. Although there is no specific documentation of the actual impact of the problem collections on students' learning and teachers' teaching, the impact might be greater than what was expected since, over the years (especially between 1995 and 2005), a national movement of developing and using open-ended problems in China has occurred (Zhang, 2006a).

2.4.4 Designing mathematical situations and problem posing

Originating from an international comparative study (Cai & Hwang, 2002), Lu and Wang and their associates (Lu & Wang, 2006; Wang & Lu, 2000) launched a project on mathematical situations and problem posing. The project has two interrelated key components. The first is the systematic development of teaching materials about mathematical situations and problem-posing tasks. Similar to the collections of open-ended mathematical problems, the teaching materials—including mathematical situations and problem-posing tasks—are not intended to replace textbooks; instead, they are used to supplement regular textbook problems. The second is the systematic implementation of teaching materials, including mathematical situations and problem-posing tasks. By 2006, more than 300 schools in ten provinces in China have participated in the project. Teachers have been receiving training to use mathematical situations and problem-posing tasks along with regular curriculum. Most importantly, teachers have been receiving training about how to develop mathematical situations and to pose problems (Lu et al., 2006). As supplementary materials for the regular mathematical curriculum, a series of teaching cases has been developed by mathematics educators and teachers across grade levels and across content areas. Here is a sample teaching case about *Making a billboard* from Lu et al. (2006, p.359).

Mathematical content. Linear equation with one unknown (for junior high school students).

Situation. A factory is planning to make a billboard. A master worker and his apprentice are employed to do the job. It will take 4 days by the master worker alone to complete the job, but it takes 6 days for the apprentice alone to complete the job.

Please create problems based on the situation. Students may add conditions for problems they create.

Posed problems:

1. How many days will it take the two workers to complete the job together?
2. If the master joins the work after the apprentice has worked for 1 day, how many additional days will it take the master and the apprentice to complete the job together?
3. After the master has worked for 2 days, the apprentice joins the master to complete the job. How many days in total will the master have to work to complete the job?
4. If the master has to leave for other business after the two workers have worked together on the job for 1 day, how many additional days will it take the apprentice to complete the remaining part of the job?
5. If the apprentice has to leave for other business after the two workers have worked together for 1 day, how many additional days will it take the master to complete the remaining part of the job?
6. The master and the apprentice are paid 450 Yuans after they completed the job. How much should the master and the apprentice each receive if each worker's payment is determined by the proportion of the job the worker completed?

(Note. These are the posed sample problems in the teaching case. Students may pose more than the six problems listed).

Problem solving. (Note. After students pose various problems, a teacher would show students how to solve these problems. Here is a sample solution to the posed problem 3).

Solution for problem 3: suppose the two workers worked together for x days, the master worked $(x + 2)$ days.

$$\frac{1}{4}(x+2)+\frac{1}{6}x=1, \text{ and } x=\frac{6}{5};$$

$$\text{So the master worked: } \quad x+2=2+\frac{6}{5}=\frac{16}{5} \text{ days.}$$

(Note. Once students solved each of the posed problems, they may pose new problems. Here are two sample problems).

Additional problems posed:

7. The apprentice started the work by himself for 1 day, and then the master joined the effort, and they completed the remaining part of the job together. Finally, they received 490 Yuans in total for completing the job. How much should the master and the apprentice

each receive if each worker's payment is determined by the proportion of the job the worker completed?

8. The master started the work by himself for 1 day, and then the apprentice joined the effort, and they completed the remaining part of the job together. Finally, they received 450 Yuans in total for completing the job. How much should the master and the apprentice each receive if each worker's payment is determined by the proportion of the job the worker completed?

More problem solving. (Note. After students pose additional problems, the teacher would show students how to solve these problems. Here is a sample solution to posed problem 8).

Solution for problem 8: suppose the two workers worked together for x days, the master worked $(x + 1)$ days.

$$\frac{1}{4}(x+1)+\frac{1}{6}x=1, \text{ and } x=\frac{9}{5}\text{days.}$$

The master's payment: $\frac{1}{4}\left(\frac{9}{5}+1\right)450=315$ Yuans;

The apprentice's payment: 450–315 = 135 Yuans.

3 Practice

When Chinese teachers were asked about problem solving in the classroom, most of them highlighted the following three activities: one problem multiple solutions (OPMS), multiple problems one solution (MPOS), and one problem multiple changes (OPMC). In a survey, a group of teachers were asked to point out how often they use each of these problem-solving activities in classrooms (Nie, 2004). The results in Table 2 showed that all of the teachers used these problem-solving activities, although they used them to different degrees. Over half of the teachers surveyed used OPMS, MPOS, and OPMC very often in their instruction. In particular, over 82% of the teachers used "one problem multiple solutions" very often.

Table 2 The frequency of teaching with variation in the Chinese classroom

	Used them very often	Used them occasionally	Never used them
One problem multiple solutions (n = 102)	84	18	0
One problem multiple changes (n = 102)	69	33	0
Multiple problems one solution (n = 100)	52	48	0

In this section, we discuss each of these problem-solving activities. In addition, since classroom instruction is quite examination-driven in China, we also discuss the "examination culture" and problem solving in the Chinese classroom in this section.

3.1 One problem multiple solutions

All too often students hold the misconception that there is only one "right" way to approach and solve a problem and, therefore, they fail to develop flexibility in inventing and selecting appropriate strategies and finding solutions. This misconception might be largely due to their lack of experience of using multiple ways to approach a problem. In Chinese classrooms, teachers usually present a problem and provide opportunities for students to solve it in different ways. For example, in upper elementary schools, teachers usually encourage students to use both arithmetic and algebraic approaches to solve the following problem (Cai & Moyer, 2007):

> Liming elementary school had funds to buy 12 basketballs that cost 24 Yuans each. Before buying the basketballs, the school spent 144 Yuans of the funds for some soccer balls. How many basketballs can the school buy with the remaining funds?

Solution 1 Begin by computing the original funding and subtracting the money spent on soccer balls: $(24 \times 12 - 144)/24 = 144/24 = 6$ basketballs.

Solution 2 Begin by computing the number of basketballs that can no longer be bought: $12 - (144/24) = 6$ basketballs.

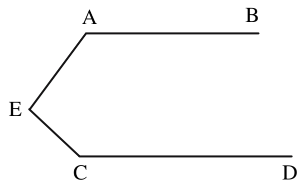
Solution 3 Assume that the school can still buy x basketballs: $(24 \times 12 - 144) = 24x$. Therefore, $x = 6$ basketballs.

Solution 4 Assume that the school can still buy x basketballs: $24 \times 12 = 24x + 144$. Therefore, $x = 6$ basketballs.

Solution 5 Assume that the school can still buy x basketballs; $12 = (144/24) + x$. Therefore, $x = 6$ basketballs.

There are three objectives in teaching students to solve problems both arithmetically and algebraically: (1) to help students attain an in-depth understanding of quantitative relationships by representing them both arithmetically and algebraically; (2) to guide students to discover the similarities and differences between arithmetic and algebraic approaches, so they can make a smooth transition from arithmetic to algebraic thinking; and, (3) to develop students' thinking skills as well as flexibility in using appropriate approaches to solve problems.

In a junior high school geometry class, students are also provided opportunities to use the OPMS approach. Students can use more than ten different methods to solve the following geometry problem: *In the Figure below, AB//CD. What can you say about the summation of* A + E + C?



In addition to discussing the solutions one by one, Chinese teachers usually guide students to compare the solutions with respect to their similarities and differences. Through discussion, teachers help students make connections between theorems and how they were applied in order to solve the problem from different perspectives.

Theorem 1: *If a pair of parallel lines are intersected by a transversal, then*

(1) *The alternate interior angles are congruent;*
(2) *The corresponding angles are congruent;*
(3) *The alternate exterior angles are congruent;*
(4) *The interior angles on the same side of the transversal are supplementary.*

Theorem 2: *In any triangle, the sum of the measures of the three interior angles of the triangle is 180°.*

Theorem 3: (Generalization of Theorem 2). *In any polygon with n-side ($n \geq 3$) the sum of the measures of the n interior angles of the polygon is 180°. ($n - 2$).*

Theorem 4: *The measure of each exterior angle of a triangle equals to the sum of the measures of its two remote interior angles.*

The problem could be extended when the point E is placed in different locations. Figure 3 shows sample figures with different positions of point E.

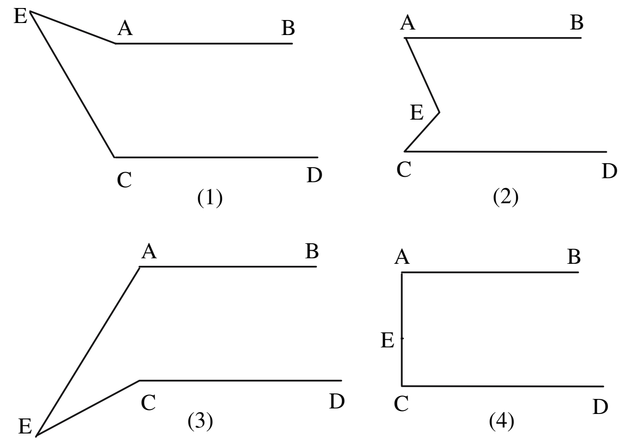


Fig. 3 Extensions of the problem with different locations of point E

3.2 One problem multiple changes

This problem-solving activity is to solve variations of a problem after the original problem is solved. The change of point E to different positions in Fig. 3 is one of the examples for OPMC. The variation of a problem could happen in the process of solving a problem, and it also could happen before or after solving a problem. Figure 4 shows a framework for variations of a problem in the process of solving the problem (Zheng, 2001, p.62).

The variations of a problem after solving the problem often show a series of extended problems by varying some aspects of the problem. The following example in Fig. 5 is a variation of a problem after students finished solving problem 1.

From problem 1 to problems 2, 3, 4, only one of the conditions of the problem 1 has been changed. However, from problem 1 to problems 5 and 6, different questions were asked. Solving problems 1–5 requires knowledge of congruent triangles, but solving problem 6 requires knowledge about similar shapes. The variation from congruent triangles to similar triangles is another way of seeing changes. So conditions, conclusions, or the knowledge required to solve the problem could be the dimensions of finding variations of a problem.

3.3 Multiple problems one solution

Using this approach, teachers guide students to use one solution method to solve a set of problems. This set of problems may have different surface features, but structurally, they all can be solved using one solution method. In most of the situations, a solution method refers to a specific

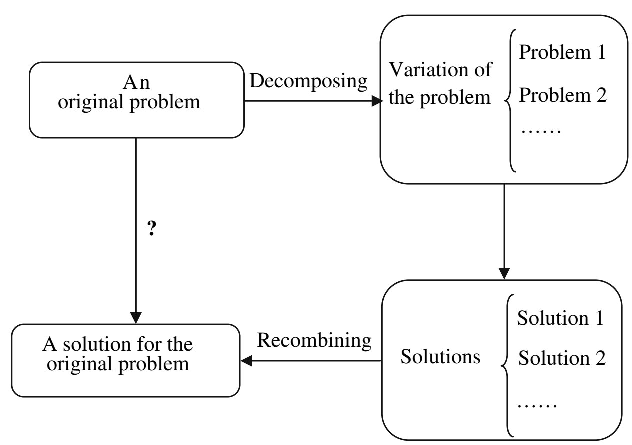
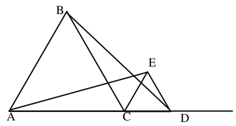
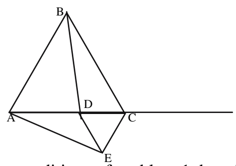


Fig. 4 A framework for varying a problem in the solving process

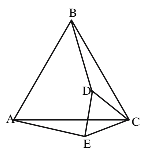
Problem 1: Point D is located on the extension line of line segment AC. △ ABC and △ CDE are equilateral triangles. Points A and E are connected to form line segment AE, and points B and D are connected to form line segment BD. Prove: *m*AE=*m*BD.



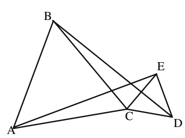
Problem 2: Keep the other conditions of problem 1, but change the position of point D to be between points A and C. △ ABC and △ CDE are equilateral triangles. Is the equality *m*AE=*m*BD still true?



Problem 3: Keep the other conditions of problem 1, but change the position of point D to the interior of △ ABC. △ ABC and △ CDE are equilateral triangles. Is the equality *m*AE=*m*BD still true?



Problem 4: Keep the other conditions of problem 1, but change the position of point D to the exterior of △ ABC and so that points A, C, and D are not on the same line. △ ABC and △ CDE are equilateral triangles. Is the equality *m*AE=*m*BD still true?



Problem 5: Can you find additional equalities about line segment or angles in the previous problems?

Problem 6: Keep the other conditions of problem 1, but connect points F and G. △ ABC and △ CDE are equilateral triangles. Prove: FG∥AD.

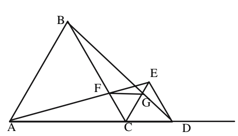


Fig. 5 An example of one problem multiple changes in the Chinese classroom

problem solving technique, rather than a big idea or a general solution strategy (e.g., "guess & check"). The following example (see Fig. 6) shows the same solution used in different problems. At the same time, this example also shows that the variations of the problem could be implemented with equivalent transformation (Nie, 2004).

The five problems in Fig. 6 involve a quadratic equation with one unknown, quadratic inequality, factorization of quadratic polynomial, and analytic geometry. They belong to different textbook units. However, the problems can be

Problem 1: What real number(s) could m be so that the equation $x^2-(m+2)x+4=0$ has no real roots?

Problem 2: What real number(s) could m have so that the graph of the quadratic function $f(x)=x^2-(m+2)x+4$ doesn't intersect with the x-axis?

Problem 3: What real number(s) could m be so that the solution set of the quadratic inequality $x^2-(m-2)x+4\le 0$ is an empty set?

Problem 4: What real number(s) could m be so that a quadratic polynomial $x^2-(m-2)x+4$ could not be factorized into two linear polynomials in real numbers domain?

Problem 5: What real number(s) could m be so that there is no intersection between the straight line $y=(m-2)x$ and the parabola $y=x^2+4$?

Fig. 6 An example of multiple problems one solution in the Chinese classroom

solved by the method of using the properties of solving a quadratic equation with one unknown. Namely, the solution for all five of the problems requires that the value of the discriminant should be less than zero. When those five problems are presented as a system of tasks for a comprehensive review, students are guided to explore the essential connections among the problems.

Unfortunately, there are no guidelines for what the "same solution method" should be to solve multiple problems. One of the downsides of this MPOS approach is that sometimes teachers require students to memorize the method so that it can be automatically applied to other problems, without understanding and analyzing each problem situation.

3.4 "Examination culture" and problem solving in the Chinese classroom

China is one of the countries in which, to a great extent, the scores on examinations can determine one's opportunity for additional education and even future careers. Many parents (and even teachers) believe that obtaining higher scores in examinations means being intellectually elite (Zhang, Tang, & Liu, 1991). At the same time, most students view examinations as competitions and filters for better opportunities. To a great degree, therefore, one of the main goals of classroom instruction is to prepare for examinations and to ensure high scores in examinations.

The vast majority of problems in any examinations are related to basic knowledge and skills. Thus, the principal purpose of instruction in problem solving is to help students grasp basic knowledge and skills, so that they can receive higher scores in examinations. There are usually three types of problems in examinations: (1) multiple-choice items, (2) short answer items, and (3) extended constructed-response items. Because of the competitive

(1) A multiple choice test item:

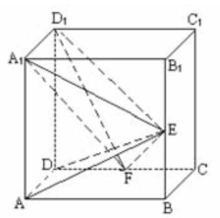
The minimum value of function $y = \cos^2 x - 3\cos x + 2$ is:

(A) 2 (B) 0 (C) $-\frac{1}{4}$ (D) 6

(2) A short answer test item:

If the coefficient of x^3 in the expansion of $(\frac{a}{x} - \sqrt{\frac{x}{2}})^9$ is $\frac{9}{4}$, the value of the constant a is:____.

(3) An extended constructed-response item:
In the cube $ABCD\text{-}A_1B_1C_1D$, points E and F are mid points of segments BB_1 and CD, respectively.
I. Prove: $AD \perp D_1F$;
II. Calculate the measure of the angle formed by segments AE and D_1F;
III. Prove: plane $AED \perp$ plane A_1FD_1;
IV. Given mAA_1=2, calculate the volume $V_{F-A_1ED_1}$ of the pyramid $F\text{-}A_1ED_1$.



(4) Another extended constructed-response item:

Given a quadratic function $f(x) = ax^2 + bx + c$ $(a > 0)$, x_1 and x_2 are the two roots of the equation $f(x) - x = 0$ and satisfy $0 < x_1 < x_2 < \frac{1}{a}$.

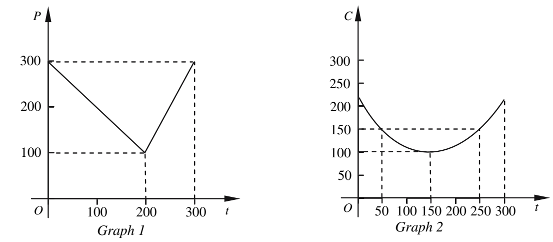
I. Prove: $x < f(x) < x_1$, where $x \in (0, x_1)$;
II. Suppose the graph of function $f(x)$ is symmetric with respect to the line $x = x_0$, prove: $x_0 < \frac{x_1}{2}$.

Fig. 7 Four sample problems from the college entrance examinations in 1997

nature of the exams, problems are usually very challenging, especially for those constructed-response items. Figure 7 shows four sample problems from the National College Entrance Mathematics Examinations (for majors in Agricultural, Engineering, Mathematics, Medicine, Science, and Technology) in 1997 (National Education Examinations Authority of Ministry of Education, 1997).

The examinations not only include challenging problems, but they also include a large quantity of problems. For example, in the 1997 National College Entrance Mathematics Examination (National Education Examinations Authority of Ministry of Education, 1997), there were 15 multiple-choice items, four short-answer items, and six extended constructed-response items (see Fig. 7 for a sample problem in each type). Students were only allowed 120 min to complete all of these problems. Therefore, speed is critical to solve these problems. To be able to solve problems in a speedy manner, students have to engage in huge amounts of practice in different techniques of problem solving for different types of problems. The difficult nature and large quantities of examinations are challenges to teaching problem solving in the Chinese classroom.

A farmer grows tomatoes. The historical record of the relationship between the market price of tomatoes and the time when the tomatoes go on the market is shown in Graph 1. Graph 1 contains the market price of tomatoes for the time up to 300 days from February 1. The relationship between the production cost and the time when the tomatoes go on the market is represented with a segment of a parabola in Graph 2.



(1) Find a function $P = f(t)$ to represent the relationship between the market price of tomatoes (P) and the time when the tomatoes go on the market (t) based on Graph 1. Find a function $C = g(t)$ to represent the relationship between the production cost (C) and the time when the tomatoes go on the market (t) based on Graph 2.

(2) Suppose the net profit is determined by the market price minus the production cost, when should the tomatoes go on the market to make the highest net profit?

(The unit of market price and production cost is $Yuan/10^2\,kg$; the unit of time is *day*.)

Fig. 8 An application problem from the national college entrance examinations

Starting the early 1990s, it was recommended to include some open-ended and real-life problems in both the College Entrance Examinations and Senior High School Entrance Examinations. The application problem (shown in Fig. 8) was from the National College Entrance Mathematics Examinations in 2000 (National Education Examinations Authority of Ministry of Education, 2000).

In addition to real-life application problems and open-ended problems, problem-posing tasks have also been recommended to assess students' learning (Basic Education Curriculum Material Development Centre, 2001). There are two reasons for integrating open-ended and real-life problems into College and Senior High School Entrance Examinations. First, the current mathematics curriculum reform in China emphasizes the development of students' abilities to pose, analyze, and solve problems. This new emphasis requires a corresponding emphasis on assessment and evaluation. Second, given the nature of examination-driven instruction in China, examinations can be used as the driving force to integrate more real-life and open-ended problems into school mathematics: that is, to use examinations to impact classroom instruction in a positive way.

While there are positive signs for the impact of including real-life and open-ended problems in examinations on classroom instruction in China, there are still a number of issues to be addressed. The most controversial issue is related to the scoring of real-life and open-ended problems. Using open-ended tasks results in much more diverse student responses. How the increased diverse responses can be scored reliably is critical, given the high-stakes of the College and Senior High School Entrance Examinations.

4 Conclusion

This paper is an attempt to paint a picture of problem solving in Chinese mathematics education. We discussed both research and practice from different perspectives. It is clear that in Chinese mathematics education there are two ways to integrate problem solving in school mathematics. First, the activity of mathematical problem solving in the classroom is viewed as an important focus of instruction that provides opportunities for students to enhance their flexible and independent mathematical thinking and reasoning abilities. That is, problem solving is viewed as a process that provides students the opportunity to experience the power of mathematics in the world around them. The purpose of teaching problem solving in the classroom is to develop students' problem solving skills, help them acquire ways of thinking, form habits of persistence, and build their confidence in dealing with unfamiliar situations. Second, problem-solving activities in the classroom are used as an instructional approach that provides a context for students to learn and understand mathematics. In this way, problem solving is valued not only for the purpose of learning mathematics but also as a means to achieve learning goals. These two emphases have deep historical roots and are still maintained in the Chinese mathematics classroom today.

In this paper, we particularly discussed problem-solving research from four perspectives. This discussion indicated that the problem-solving research in China has been much more content and experience-based than cognitive and empirical-based. Chinese teachers have accumulated extensive experience of teaching students how to solve mathematical problems, and they have a large body of knowledge about problem-solving strategies. Their experience and knowledge have been shared in various teaching journals and supplemental teaching materials. However, the experience and knowledge that have been accumulated by mathematics teachers have not been included in textbooks and teaching reference books. It might be desirable to systematically include problem-solving strategies in school mathematics textbooks, so that they can be taught systematically in the regular curriculum agenda.

We also listed several instructional activities in the Chinese classroom, including teaching with variation. In particular, we presented the following three problem-solving activities: one problem multiple solutions, multiple problems one solution, and one problem multiple changes. These are frequently used problem-solving activities in China. Pedagogically, these problem-solving activities make sense, since engaging in any of these activities helps students make connections. Unfortunately, there are no empirical investigations to document the actual effectiveness and reasons for the effectiveness. Nevertheless, these problem-solving activities should be useful references for teachers in other countries to help students make sense of mathematics.

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