1st December 2004

Munkres §26

Ex. 26.1 (Morten Poulsen).

(a). Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Suppose $\mathcal{T}' \supset \mathcal{T}$.

If $(X, \mathcal{T}')$ is compact then $(X, \mathcal{T})$ is compact: Clear, since every open covering if $(X, \mathcal{T})$ is an open covering in $(X, \mathcal{T}')$.

If $(X, \mathcal{T})$ is compact then $(X, \mathcal{T})$ is in general not compact: Consider $[0, 1]$ in the standard topology and the discrete topology.

(b).

Lemma 1. *If $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ are compact Hausdorff spaces then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or not comparable.*

Proof. If $(X, \mathcal{T})$ compact and $\mathcal{T}' \supset \mathcal{T}$ then the identity map $(X, \mathcal{T}') \to (X, \mathcal{T})$ is a bijective continuous map, hence a homeomorphism, by theorem 26.6. This proves the result. □

Finally note that the set of topologies on the set X is partially ordered, c.f. ex. 11.2, under inclusion. From the lemma we conclude that the compact Hausdorff topologies on X are minimal elements in the set of all Hausdorff topologies on X.

Ex. 26.2 (Morten Poulsen).

(a). The result follows from the following lemma.

Lemma 2. *If the set X is equipped with the finite complement topology then every subspace of X is compact.*

Proof. Suppose $A \subset X$ and let $\mathcal{A}$ be an open covering of A. Then any set $A_0 \in \mathcal{A}$ will covering all but a finite number of points. Now choose a finite number of sets from $\mathcal{A}$ covering $A - A_0$. These sets and A_0 is a finite subcovering, hence A compact. □

(b). Lets prove a more general result: Let X be an uncountable set. Let

$$\mathcal{T}_c = \{\, A \subset X \mid X - A \text{ countable or equal } X \,\}.$$

It is straightforward to check that $\mathcal{T}_c$ is a topology on X. This topology is called the countable complement topology.

Lemma 3. *The compact subspaces of X are exactly the finite subspaces.*

Proof. Suppose A is infinite. Let $B = \{b_1, b_2, \dots\}$ be a countable subset of A. Set

$$A_n = (X - B) \cup \{b_1, \dots, b_n\}.$$

Note that $\{A_n\}$ is an open covering of A with no finite subcovering. □

The lemma shows that $[0, 1] \subset \mathbf{R}$ in the countable complement topology is not compact.

Finally note that $(X, \mathcal{T}_c)$ is not Hausdorff, since no two nonempty open subsets A and B of X are disjoint: If $A \cap B = \emptyset$ then $X - (A \cap B) = (X - A) \cup (X - B)$, hence X countable, contradicting that X uncountable.

Ex. 26.3 (Morten Poulsen).

Theorem 4. *A finite union of compact subspaces of X is compact.*

Proof. Let $A_1, \dots, A_n$ be compact subspaces of X. Let $\mathcal{A}$ be an open covering of $\bigcup_{i=1}^n A_i$. Since $A_j \subset \bigcup_{i_1}^n A_i$ is compact, $1 \le j \le n$, there is a finite subcovering $\mathcal{A}_j$ of $\mathcal{A}$ covering A_j. Thus $\bigcup_{j=1}^n \mathcal{A}_j$ is a finite subcovering of $\mathcal{A}$, hence $\bigcup_{i=1}^n A_i$ is compact. □

Ex. 26.5. For each $a \in A$, choose [Lemma 26.4] disjoint open sets $U_a \in a$ and $V_a \supset B$. Since A is compact, A is contained in a finite union $U = U_1 \cup \cdots \cup U_n$ of the U_as. Let $V = V_1 \cap \cdots V_n$ be the intersection of the corresponding V_as. Then U is an open set containing A, V is an open set containing B, and U and V are disjoint as $U \cap V = \bigcup U_i \cap V \subset \bigcup U_i \cap V_i = \emptyset$.

Ex. 26.6. Since any closed subset A of the compact space X is compact [Thm 26.2], the image $f(A)$ is a compact [Thm 26.5], hence closed [Thm 26.3], subspace of the Hausdorff space Y.

Ex. 26.7. This is just reformulation of The tube lemma [Lemma 26.8]: Let C be a closed subset of $X \times Y$ and $x \in X$ a point such that the slice $\{x\} \times Y$ is disjoint from C. Then, since Y is compact, there is a neighborhood W of x such that the whole tube $W \times Y$ is disjoint from C.

In other words, if $x \notin \pi_1(C)$ then there is a neighborhood W of x which is disjoint from $\pi_1(C)$. Thus The tube lemma says that $\pi_1 \colon X \times Y \to X$ is closed when Y is compact (so that π_1 is an example of a perfect map [Ex 26.12]). On the other hand, projection maps are always open [Ex 16.4].

Ex. 26.8. Let $G \subset X \times Y$ be the graph of a function $f \colon X \to Y$ where Y is compact Hausdorff. Then

$$G \text{ is closed in } X \times Y \Leftrightarrow f \text{ is continuous}$$

$\Leftarrow$: (For this it suffices that Y be Hausdorff.) Let $(x, y) \in X \times Y$ be a point that is not in the graph of f. Then $y \neq f(x)$ so by the Hausdorff axiom there will be disjoint neighborhoods $V \ni y$ and $W \ni f(x)$. By continuity of f, $f(U) \subset W \subset Y - V$. This means that $(U \times V) \cap G = \emptyset$.
$\Rightarrow$: Let V be a neighborhood of $f(x)$ for some $x \in X$. Then $G \cap (X \times (Y - V))$ is closed in $X \times Y$ so [Ex 26.7] the projection $\pi_1(G \cap (X \times (Y - V)))$ is closed in X and does not contain x. Let U be a neighborhood of X such that $U \times Y$ does not intersect $G \cap (X \times (Y - V))$. Then $f(U)$ does not intersect $Y - V$, or $f(U) \subset V$. This shows that f is continuous at the arbitrary point $x \in X$.

Ex. 26.12. (Any perfect map is *proper*; see the January 2003 exam for more on proper maps.) Let $p \colon X \to Y$ be closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$. Then $p^{-1}(C)$ is compact for any compact subspace $C \subset Y$.

For this exercise we shall use the following lemma.

Lemma 5. *Let $p \colon X \to Y$ be a closed map.*

(1) *If $p^{-1}(y) \subset U$ where U is an open subspace of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of y.*
(2) *If $p^{-1}(B) \subset U$ for some subspace B of Y and some open subspace U of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of B.*

Proof. Note that

$$\begin{aligned} p^{-1}(W) \subset U &\Leftrightarrow \big[p(x) \in W \Rightarrow x \in U\big] \Leftrightarrow \big[x \notin U \Rightarrow p(x) \notin W\big] \Leftrightarrow p(X - U) \subset Y - W \\ &\Leftrightarrow p(X - U) \cap W = \emptyset \end{aligned}$$

(1) The point y does not belong to the closed set $p(X - U)$. Therefore a whole neighborhood $W \subset Y$ of y is disjoint from $p(X - U)$, i.e. $p^{-1}(W) \subset U$.
(2) Each point $y \in B$ has a neighborhood W_y such that $p^{-1}(W_y) \subset U$. The union $W = \bigcup W_y$ is then a neighborhood of B with $p^{-1}(W) \subset U$. $\square$

We shall not need point (2) here.

Let $C \subset Y$ be compact. Consider a collection $\{U_\alpha\}_{\alpha \in J}$ of open sets covering of $p^{-1}(C)$. For each $y \in C$, the compact space $p^{-1}(y)$ is contained in a the union of a finite subcollection $\{U_\alpha\}_{\alpha \in J(y)}$. There is neighborhood W_y of y such that $p^{-1}(W_y)$ is contained in this finite union. By compactness of C, finitely many $W_{y_1}, \ldots, W_{y_k}$ cover Y. Then the finite collection $\bigcup_{i=1}^{k} \{U_\alpha\}_{\alpha \in J(y_i)}$ cover $p^{-1}(C)$. This shows that $p^{-1}(C)$ is compact.

Ex. 26.13. Let G be a topological group and A and B subspaces of G.

(a). A closed and B compact $\Rightarrow AB$ closed
Assume $c \notin AB = \bigcup_{b\in B} Ab$. The regularity axiom for G [Suppl Ex 22.7] implies that there are disjoint open sets $W_b \ni c$ and $U_b \supset Ab$ separating c and Ab for each point $b \in B$. Then $A^{-1}U_b$ is an open neighborhood of b. Since B is compact, it can be covered by finitely many of these open sets $A^{-1}U_b$, say

$$B \subset A^{-1}U_1 \cup \cdots \cup A^{-1}U_k = A^{-1}U$$

where $U = U_1 \cup \cdots \cup U_k$. The corresponding open set $W = W_1 \cap \cdots \cap W_k$ is an open neighborhood of c that is disjoint from AB since $W \cap AB \subset \bigcup W \cap U_i \subset \bigcup W_i \cap U_i = \emptyset$.

(b). H compact subgroup of $G \Rightarrow p\colon G \to G/H$ is a closed map
The saturation AH of any closed subset $A \subset G$ is closed by (a).

(c). H compact subgroup of G and G/H compact $\Rightarrow G$ compact
The quotient map $p\colon G \to G/H$ is a perfect map because it is a closed map by (b) and has compact fibres $p^{-1}(gH) = gH$. Now apply [Ex 26.12].

References