Preface

This solution manual was prepared as an aid for instructors who will benefit by having solutions available. In addition to providing detailed answers to most of the problems in the book, this manual can help the instructor determine which of the problems are most appropriate for the class.

The vast majority of the problems have been solved with the help of available computer software (SAS, S-Plus, Minitab). A few of the problems have been solved with hand calculators. The reader should keep in mind that round-off errors can occur—particularly in those problems involving long chains of arithmetic calculations.

We would like to take this opportunity to acknowledge the contribution of many students, whose homework formed the basis for many of the solutions. In particular, we would like to thank Jorge Achcar, Sebastiao Amorim, W. K. Cheang, S. S. Cho, S. G. Chow, Charles Fleming, Stu Janis, Richard Jones, Tim Kramer, Dennis Murphy, Rich Raubertas, David Steinberg, T. J. Tien, Steve Verrill, Paul Whitney and Mike Wincek. Dianne Hall compiled most of the material needed to make this current solutions manual consistent with the sixth edition of the book.

The solutions are numbered in the same manner as the exercises in the book. Thus, for example, 9.6 refers to the 6th exercise of chapter 9.

We hope this manual is a useful aid for adopters of our *Applied Multivariate Statistical Analysis*, 6th edition, text. The authors have taken a little more active role in the preparation of the current solutions manual. However, it is inevitable that an error or two has slipped through so please bring remaining errors to our attention. Also, comments and suggestions are always welcome.

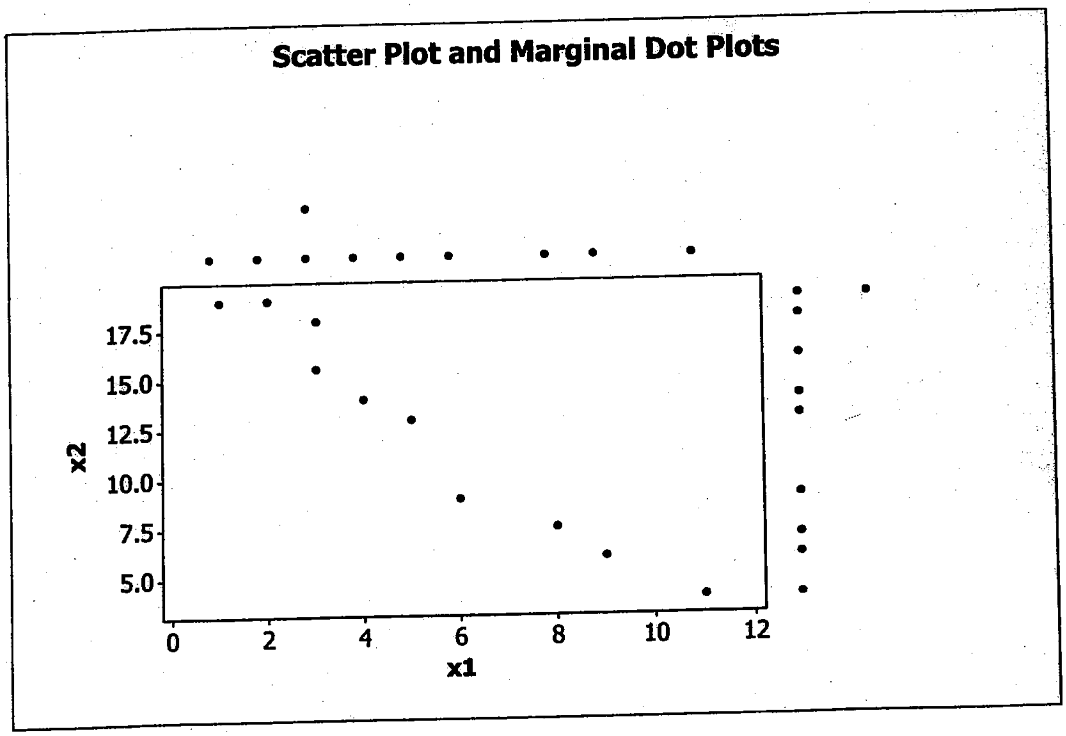
Richard A. Johnson
Dean W. Wichern

Chapter 1

1.1 $\bar{x}_1 = 4.29$ $\bar{x}_2 = 6.29$

$s_{11} = 4.20$ $s_{22} = 3.56$ $s_{12} = 3.70$

1.2 a)



b) s_{12} is negative

c)

$\bar{x}_1 = 5.20$ $\bar{x}_2 = 12.48$ $s_{11} = 3.09$ $s_{22} = 5.27$

$s_{12} = -15.94$ $r_{12} = -.98$

Large x_1 occurs with small x_2 and vice versa.

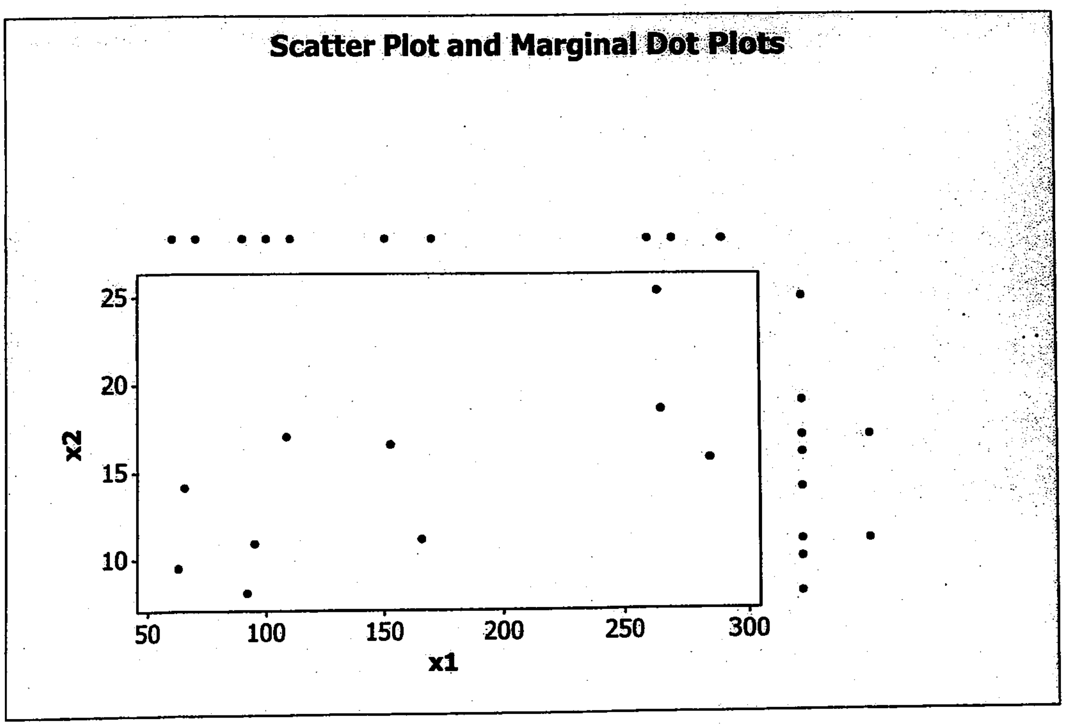
d)

$$\bar{x} = \begin{bmatrix} 5.20 \\ 12.48 \end{bmatrix} \quad S_n = \begin{bmatrix} 3.09 & -15.94 \\ -15.94 & 5.27 \end{bmatrix} \quad R = \begin{bmatrix} 1 & -.98 \\ -.98 & 1 \end{bmatrix}$$

1.3

$$\bar{\mathbf{x}} = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix} \qquad S_n = \begin{bmatrix} 6 & 4 & -1.4 \\ & 8 & 1.2 \\ \text{(symmetric)} & & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & .577 & -.404 \\ & 1 & .300 \\ \text{(symmetric)} & & 1 \end{bmatrix}$$

1.4 a) There is a positive correlation between x_1 and x_2. Since sample size is small, hard to be definitive about nature of marginal distributions. However, marginal distribution of x_1 appears to be skewed to the right. The marginal distribution of x_2 seems reasonably symmetric.



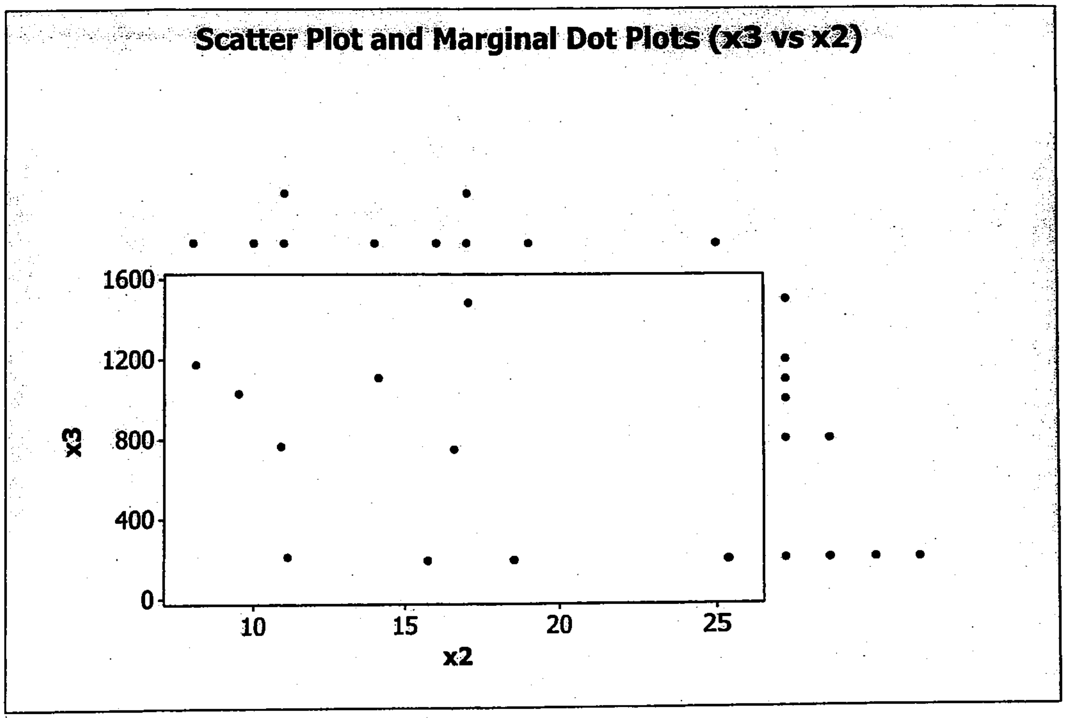
b)

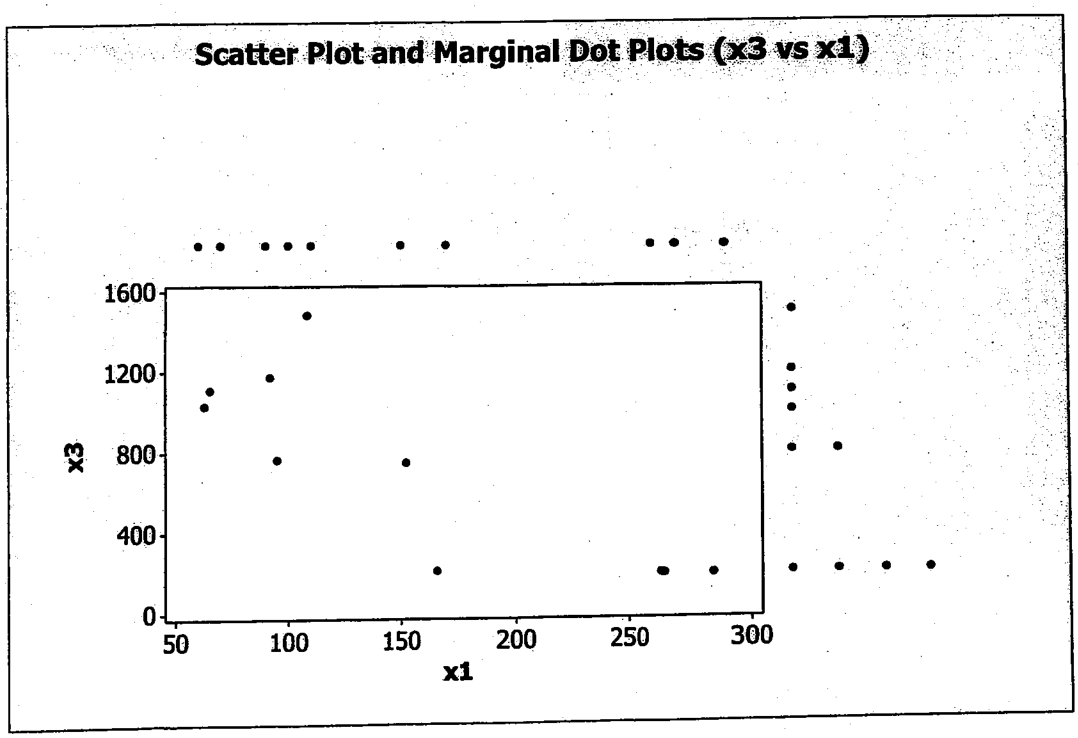
$\bar{x}_1 = 155.60 \quad \bar{x}_2 = 14.70 \quad s_{11} = 82.03 \quad s_{22} = 4.85$

$s_{12} = 273.26 \quad r_{12} = .69$

Large profits (x_2) tend to be associated with large sales (x_1); small profits with small sales.

1.5 a) There is negative correlation between x_2 and x_3 and negative correlation between x_1 and x_3. The marginal distribution of x_1 appears to be skewed to the right. The marginal distribution of x_2 seems reasonably symmetric. The marginal distribution of x_3 also appears to be skewed to the right.





1.5 b)

$$\bar{x} = \begin{bmatrix} 155.60 \\ 14.70 \\ 710.91 \end{bmatrix} \qquad S_n = \begin{bmatrix} 82.03 & 273.26 & -32018.36 \\ 273.26 & 4.85 & -948.45 \\ -32018.36 & -948.45 & 461.90 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & .69 & -.85 \\ .69 & 1 & -.42 \\ -.85 & -.42 & 1 \end{bmatrix}$$

1.6 a) Histograms

X_1

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
5.	5	*****
6.	8	********
7.	7	*******
8.	11	***********
9.	5	*****
10.	6	******

X_2

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
30.	1	*
40.	3	***
50.	2	**
60.	3	***
70.	10	**********
80.	12	************
90.	8	********
100.	2	**
110.	1	*

X_3

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
2.	1	*
3.	5	*****
4.	19	*******************
5.	9	*********
6.	3	***
7.	5	*****

X_4

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
1.	13	*************
2.	15	***************
3.	8	********
4.	5	*****
5.	1	*

X_5

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
5.	2	**
6.	3	***
7.	5	*****
8.	5	*****
9.	6	******
10.	4	****
11.	4	****
12.	5	*****
13.	4	****
14.	1	*
15.	0	
16.	1	*
17.	0	
18.	1	*
19.	0	
20.	0	
21.	1	*

X_6

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
2.	3	***
4.	4	****
6.	7	*******
8.	7	*******
10.	8	********
12.	5	*****
14.	2	**
16.	2	**
18.	1	*
20.	0	
22.	0	
24.	2	**
26.	1	*

X_7

MIDDLE OF INTERVAL	NUMBER OF OBSERVATIONS	
2.	7	*******
3.	25	*************************
4.	9	*********
5.	1	*

1.6 **b)**

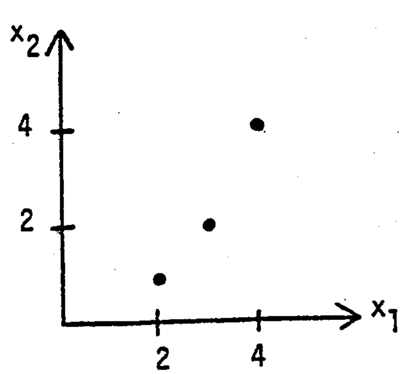
$$\bar{x} = \begin{bmatrix} 7.5 \\ 73.857 \\ 4.548 \\ 2.191 \\ 10.048 \\ 9.405 \\ 3.095 \end{bmatrix} \quad S_n = \begin{bmatrix} 2.440 & -2.714 & -.369 & -.452 & -.571 & -2.179 & .167 \\ & 293.360 & 3.816 & -1.354 & 6.602 & 30.058 & .609 \\ & & 1.486 & .658 & 2.260 & 2.755 & .138 \\ & & & 1.154 & 1.062 & -.791 & .172 \\ & & & & 11.093 & 3.052 & 1.019 \\ & & & & & 30.241 & .580 \\ \text{(symmetric)} & & & & & & .467 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -.101 & -.194 & -.270 & -.110 & -.254 & .156 \\ & 1 & .183 & -.074 & .116 & .319 & .052 \\ & & 1 & .502 & .557 & .411 & .166 \\ & & & 1 & .297 & -.134 & .235 \\ & & & & 1 & .167 & .448 \\ & & & & & 1 & .154 \\ \text{(symmetric)} & & & & & & 1 \end{bmatrix}$$

The pair x_3, x_4 exhibits a small to moderate positive correlation and so does the pair x_3, x_5. Most of the entries are small.

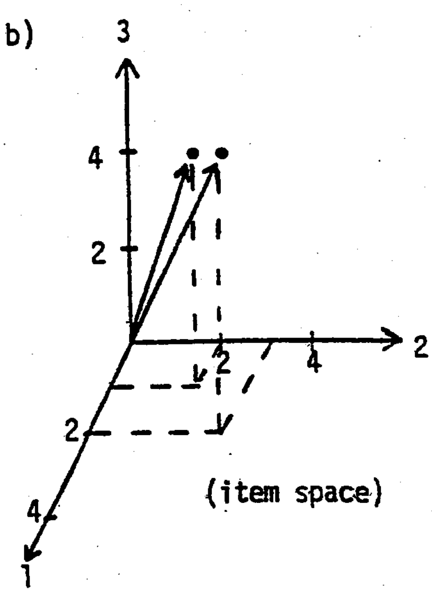
1.7

a)



Scatter plot
(variable space)

b)



(item space)

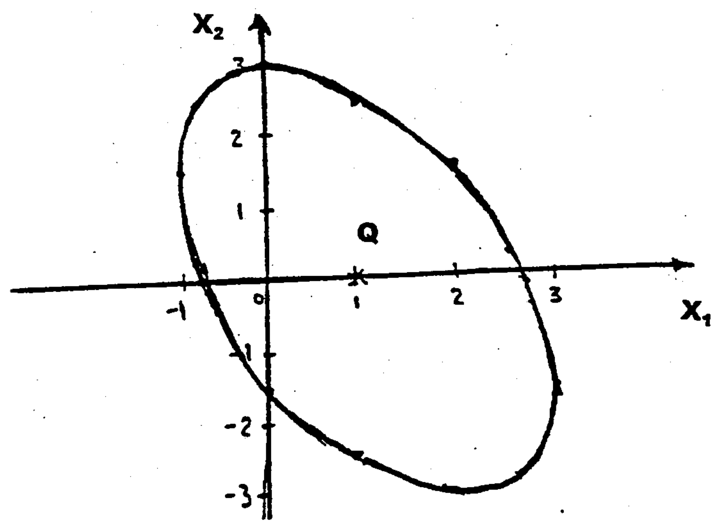
1.8 Using (1-12) $d(P,Q) = \sqrt{(-1-1)^2+(-1-0)^2} = \sqrt{5} = 2.236$

Using (1-20) $d(P,Q) = \sqrt{\frac{1}{3}(-1-1)^2+2(\frac{1}{9})(-1-1)(-1-0)+\frac{4}{27}(-1-0)^2} = \sqrt{\frac{52}{27}} = 1.388$

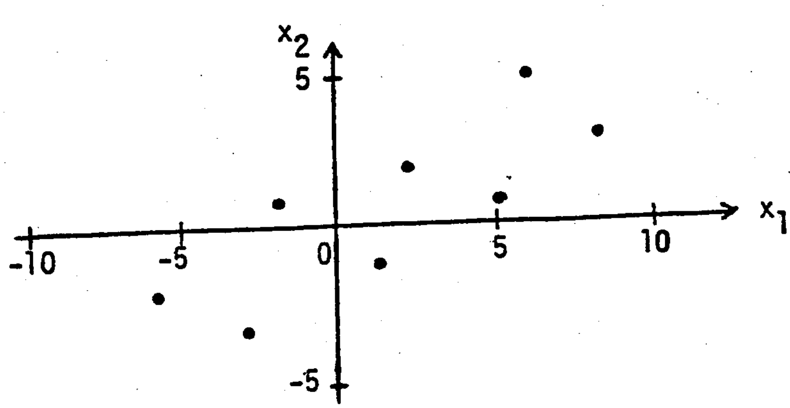
Using (1-20) the locus of points a constant squared distance 1 from Q = (1,0) is given by the expression $\frac{1}{3}(x_1-1)^2+\frac{2}{9}(x_1-1)x_2+\frac{4}{27}x_2^2 = 1$. To sketch the locus of points defined by this equation, we first obtain the coordinates of some points satisfying the equation:

(-1,1.5), (0,-1.5), (0,3), (1,-2.6), (1,2.6), (2,-3), (2,1.5), (3,-1.5)

The resulting ellipse is:



1.9 a) $s_{11} = 20.48$ $s_{22} = 6.19$ $s_{12} = 9.09$



1.9 b)

$\tilde{x}_1$	-6.20	-4.10	-1.23	.37	2.73	4.83	7.70	8.43
$\tilde{x}_2$	1.27	-1.10	1.87	-1.37	.73	-1.63	1.33	-1.40

c)

$\tilde{s}_{11} = 24.90$ $\tilde{s}_{22} = 1.77$ (Note $\tilde{s}_{12} = .00$)

d)

$(\tilde{x}_1, \tilde{x}_2) = (2.72, -3.55)$

$d(0,P) = 2.72$ using (1-17).

e) $d(0,P) = 2.72$ using (1-19).

1.10 a) This equation is of the form (1-19) with $a_{11} = 1$, $a_{12} = \frac{1}{2}$ and $a_{22} = 4$. Therefore this is a distance for correlated variables if it is non-negative for all values of x_1, x_2. But this follows easily if we write
$x_1^2 + 4x_2^2 + x_1x_2 = (x_1 + \frac{1}{2}x_2)^2 + \frac{15}{4}x_2^2 \geq 0.$

b) In order for this expression to be a distance it has to be non-negative for all values x_1, x_2. Since, for $(x_1,x_2) = (0,1)$ we have $x_1^2 - 2x_2^2 = -2$, we conclude that this is not a valid distance function.

1.11

$$d(P,Q) = \sqrt{4(x_1-y_1)^2 + 2(-1)(x_1-y_1)(x_2-y_2) + (x_2-y_2)^2}$$

$$= \sqrt{4(y_1-x_1)^2 + 2(-1)(y_1-x_1)(y_2-x_2) + (x_2-y_2)^2} = d(Q,P)$$

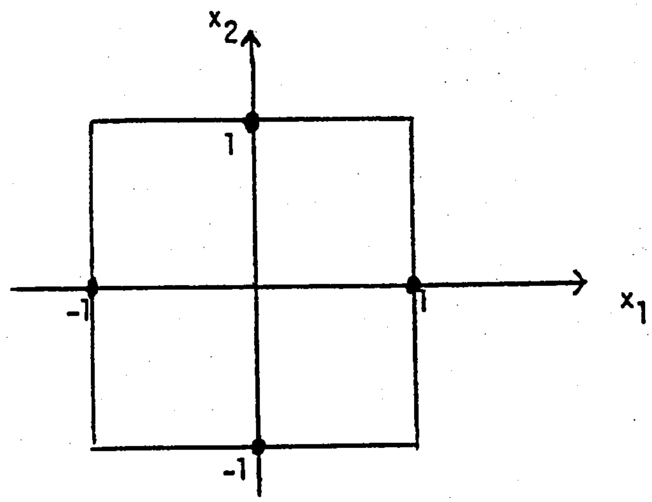
Next, $4(x_1-y_1)^2 - 2(x_1-y_1)(x_2-y_2) + (x_2-y_2)^2 =$

$$= (x_1-y_1-x_2+y_2)^2 + 3(x_1-y_1)^2 \geq 0 \quad \text{so} \quad d(P,Q) \geq 0.$$

The second term is zero in this last expression only if $x_1 = y_1$ and then the first is zero only if $x_2 = y_2$.

1.12 a) If $P = (-3,4)$ then $d(0,P) = \max(|-3|,|4|) = 4$

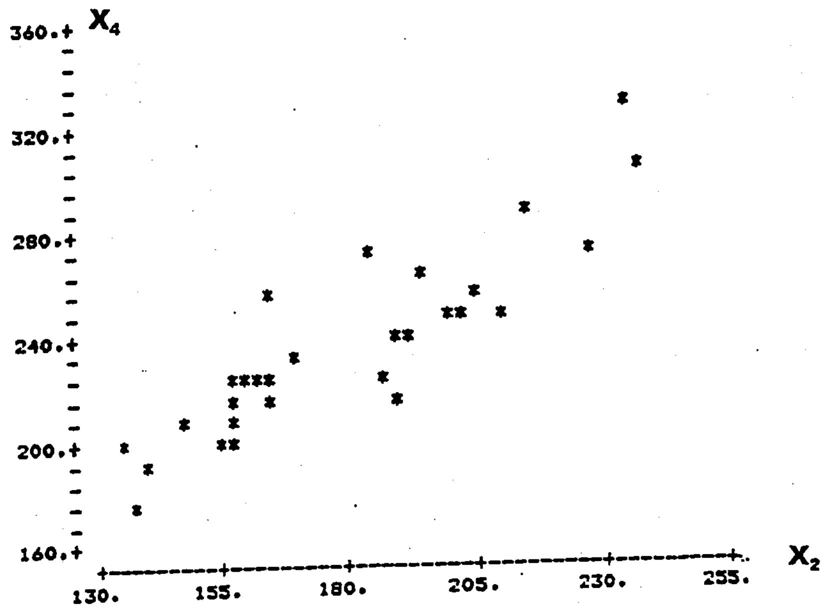
b) The locus of points whose squared distance from (0,0) is 1 is



c) The generalization to p-dimensions is given by $d(0,P) = \max(|x_1|,|x_2|,\ldots,|x_p|)$.

1.13 Place the facility at C-3.

1.14 a)



Strong positive correlation. No obvious "unusual" observations.

b) Multiple-sclerosis group.

$$\bar{\mathbf{x}} = \begin{pmatrix} 42.07 \\ 179.64 \\ 12.31 \\ 236.62 \\ 13.16 \end{pmatrix}$$

$$S_n = \begin{pmatrix} 116.91 & 61.78 & -20.10 & 61.13 & -27.65 \\ & 812.72 & 218.35 & 865.32 & 90.48 \\ & & 305.94 & 221.93 & 286.60 \\ & & & 1146.38 & 82.53 \\ \text{(symmetric)} & & & & 337.80 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & .200 & -.106 & .167 & -.139 \\ & 1 & .438 & .896 & .173 \\ & & 1 & .375 & .892 \\ & & & 1 & .133 \\ \text{(symmetric)} & & & & 1 \end{pmatrix}$$

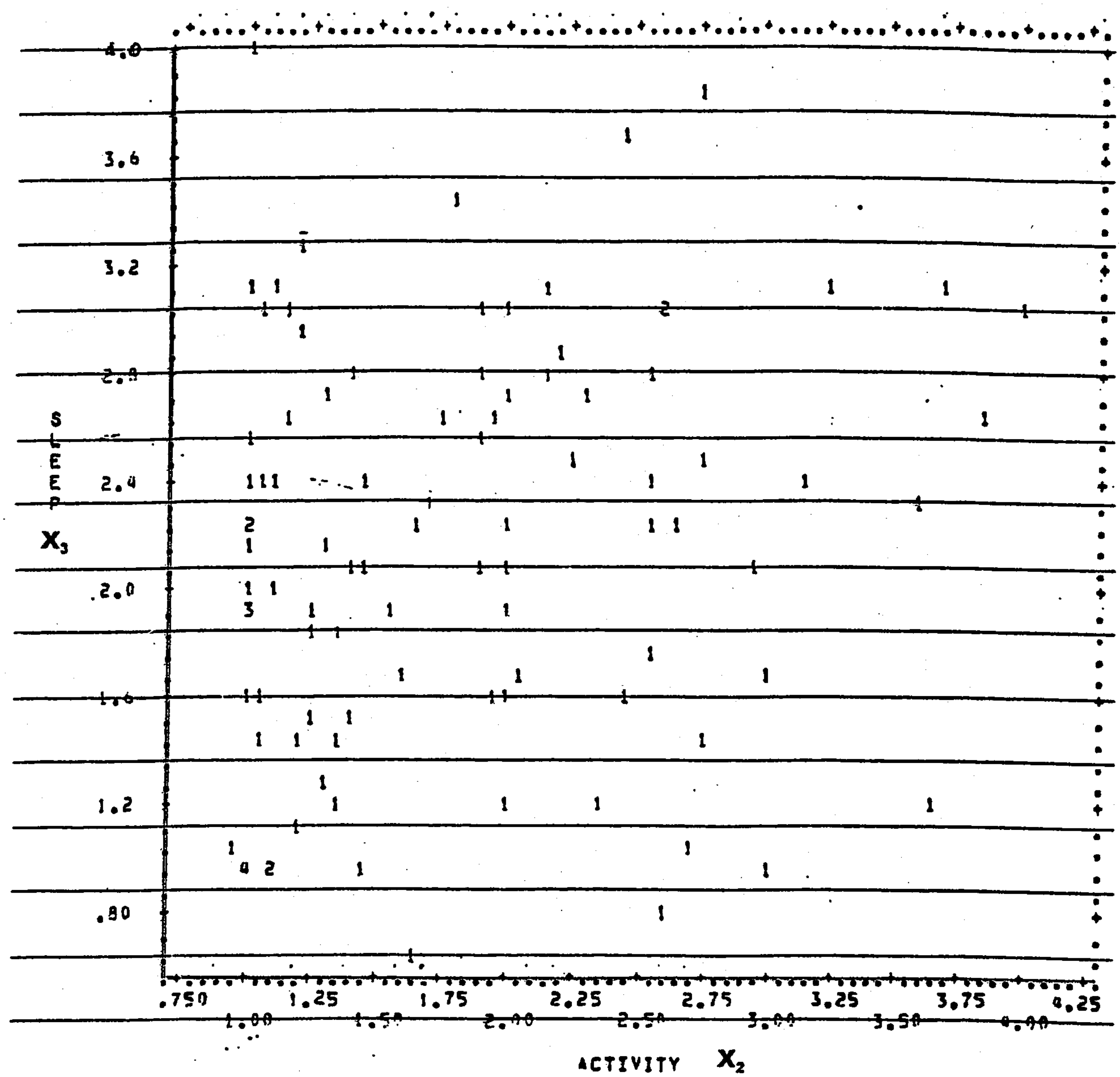
<u>Non multiple-sclerosis group</u>.

$$\bar{\underset{\sim}{x}} = \begin{pmatrix} 37.99 \\ 147.21 \\ 1.56 \\ 195.57 \\ 1.62 \end{pmatrix}$$

$$S_n = \begin{pmatrix} 273.61 & 95.08 & 5.28 & 101.67 & 3.20 \\ & 110.13 & 1.84 & 103.28 & 2.15 \\ & & 1.78 & 2.22 & .49 \\ & & & 183.04 & 2.35 \\ \text{(symmetric)} & & & & 2.32 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & .548 & .239 & .454 & .127 \\ & 1 & .132 & .727 & .134 \\ & & 1 & .123 & .244 \\ & & & 1 & .114 \\ \text{(symmetric)} & & & & 1 \end{pmatrix}$$

1.15 a) Scatterplot of x_2 and x_3.



b)

$$\bar{\underset{\sim}{x}} = \begin{pmatrix} 3.54 \\ 1.81 \\ 2.14 \\ 2.21 \\ 2.58 \\ 1.27 \end{pmatrix}$$

1.15

$$S_n = \begin{pmatrix} 4.61 & .92 & .58 & .27 & 1.06 & .15 \\ & .61 & .11 & .12 & .39 & -.02 \\ & & .57 & .09 & .34 & .11 \\ & & & .11 & .21 & .02 \\ & & & & .85 & -.01 \\ & \text{(symmetric)} & & & & .85 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & .551 & .362 & .386 & .537 & .077 \\ & 1 & .187 & .455 & .535 & -.035 \\ & & 1 & .346 & .496 & .156 \\ & & & 1 & .704 & .071 \\ & & & & 1 & -.010 \\ & \text{(symmetric)} & & & & 1 \end{pmatrix}$$

The largest correlation is between appetite and amount of food eaten. Both activity and appetite have moderate positive correlations with symptoms. Also, appetite and activity have a moderate positive correlation.

1.16

There are significant positive correlations among all variables. The lowest correlation is 0.4420 between Dominant humerus and Ulna, and the highest correlation is 0.89365 bewteen Dominant hemerus and Hemerus.

$$\bar{\mathbf{x}} = \begin{pmatrix} 0.8438 \\ 0.8183 \\ 1.7927 \\ 1.7348 \\ 0.7044 \\ 0.6938 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1.00000 & 0.85181 & 0.69146 & 0.66826 & 0.74369 & 0.67789 \\ 0.85181 & 1.00000 & 0.61192 & 0.74909 & 0.74218 & 0.80980 \\ 0.69146 & 0.61192 & 1.00000 & 0.89365 & 0.55222 & 0.44020 \\ 0.66826 & 0.74909 & 0.89365 & 1.00000 & 0.62555 & 0.61882 \\ 0.74369 & 0.74218 & 0.55222 & 0.62555 & 1.00000 & 0.72889 \\ 0.67789 & 0.80980 & 0.44020 & 0.61882 & 0.72889 & 1.00000 \end{pmatrix},$$

$$\mathbf{S}_n = \begin{pmatrix} 0.0124815 & 0.0099633 & 0.0214560 & 0.0192822 & 0.0087559 & 0.0076395 \\ 0.0099633 & 0.0109612 & 0.0177938 & 0.0202555 & 0.0081886 & 0.0085522 \\ 0.0214560 & 0.0177938 & 0.0771429 & 0.0641052 & 0.0161635 & 0.0123332 \\ 0.0192822 & 0.0202555 & 0.0641052 & 0.0667051 & 0.0170261 & 0.0161219 \\ 0.0087559 & 0.0081886 & 0.0161635 & 0.0170261 & 0.0111057 & 0.0077483 \\ 0.0076395 & 0.0085522 & 0.0123332 & 0.0161219 & 0.0077483 & 0.0101752 \end{pmatrix}.$$

1.17

There are large positive correlations among all variables. Particularly large correlations occur between running events that are "similar", for example, the 100m and 200m dashes, and the 1500m and 3000m runs.

$$\mathbf{x} = \begin{bmatrix} 11.36 \\ 23.12 \\ 51.99 \\ 2.02 \\ 4.19 \\ 9.08 \\ 153.62 \end{bmatrix} \quad \mathbf{S}_n = \begin{bmatrix} .152 & .338 & .875 & .027 & .082 & .230 & 4.254 \\ .338 & .847 & 2.152 & .065 & .199 & .544 & 10.193 \\ .875 & 2.152 & 6.621 & .178 & .500 & 1.400 & 28.368 \\ .027 & .065 & .178 & .007 & .021 & .060 & 1.197 \\ .082 & .199 & .500 & .021 & .073 & .212 & 3.474 \\ .230 & .544 & 1.400 & .060 & .212 & .652 & 10.508 \\ 4.254 & 10.193 & 28.368 & 1.197 & 3.474 & 10.508 & 265.265 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1.000 & .941 & .871 & .809 & .782 & .728 & .669 \\ .941 & 1.000 & .909 & .820 & .801 & .732 & .680 \\ .871 & .909 & 1.000 & .806 & .720 & .674 & .677 \\ .809 & .820 & .806 & 1.000 & .905 & .867 & .854 \\ .782 & .801 & .720 & .905 & 1.000 & .973 & .791 \\ .728 & .732 & .674 & .867 & .973 & 1.000 & .799 \\ .669 & .680 & .677 & .854 & .791 & .799 & 1.000 \end{bmatrix}$$

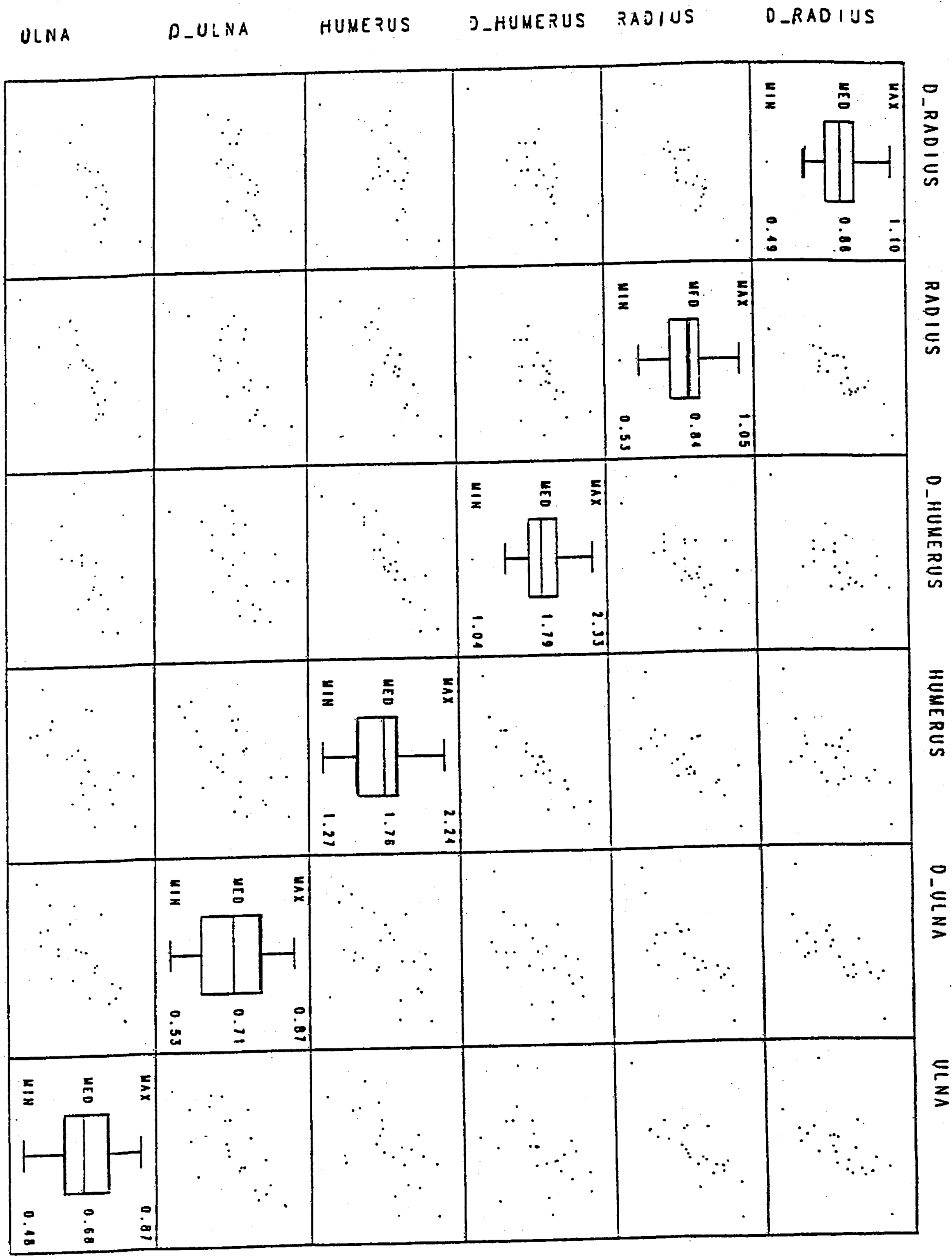
1.18

There are positive correlations among all variables. Notice the correlations decrease as the distances between pairs of running events increase (see the first column of the correlation matrix **R**). The correlation matrix for running events measured in meters per second is very similar to the correlation matrix for the running event times given in Exercise 1.17.

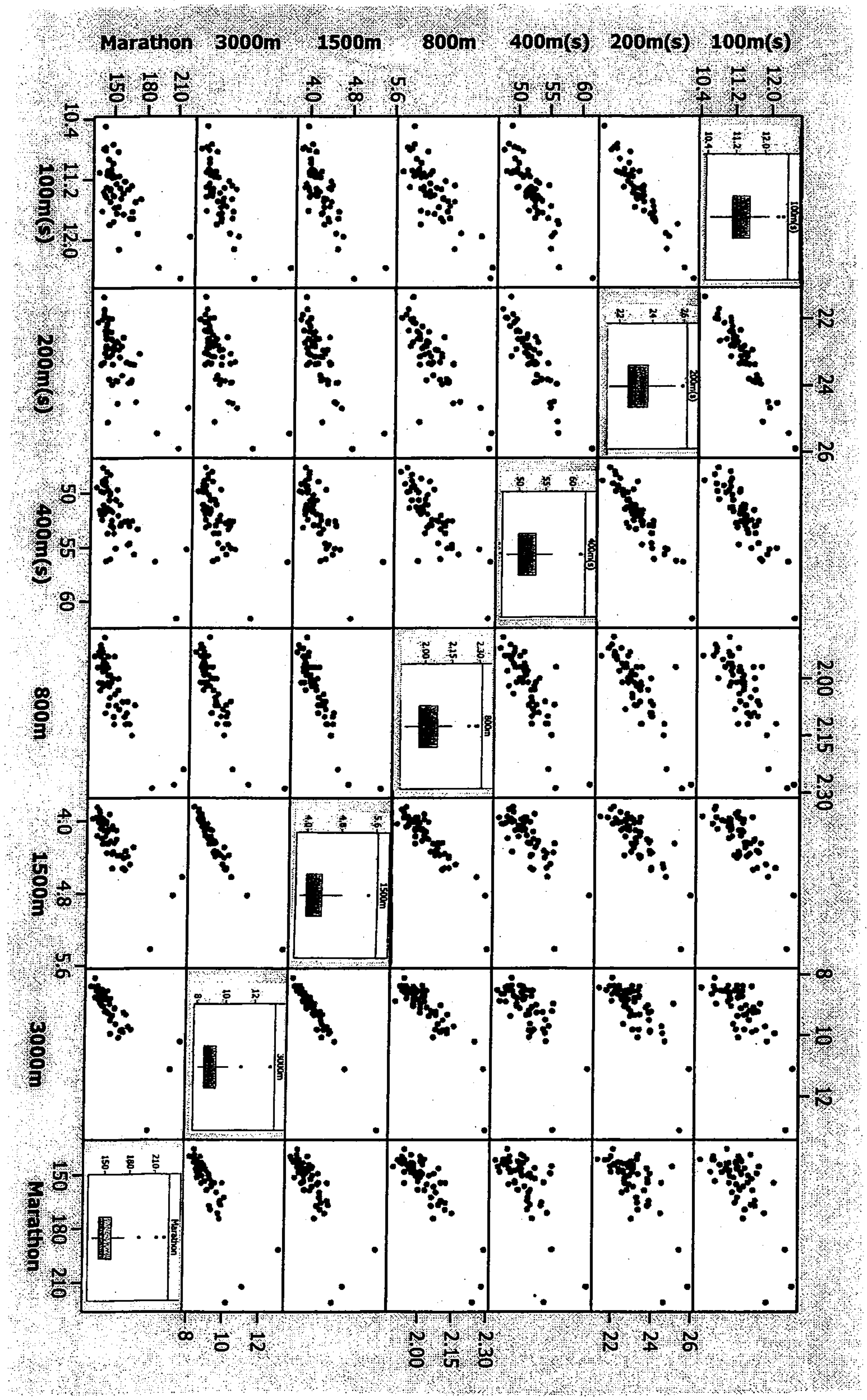
$$\bar{\mathbf{x}} = \begin{bmatrix} 8.81 \\ 8.66 \\ 7.71 \\ 6.60 \\ 5.99 \\ 5.54 \\ 4.62 \end{bmatrix} \qquad \mathbf{S}_n = \begin{bmatrix} .091 & .096 & .097 & .065 & .082 & .092 & .081 \\ .096 & .115 & .114 & .075 & .096 & .105 & .093 \\ .097 & .114 & .138 & .081 & .095 & .108 & .102 \\ .065 & .075 & .081 & .074 & .086 & .100 & .094 \\ .082 & .096 & .095 & .086 & .124 & .144 & .118 \\ .092 & .105 & .108 & .100 & .144 & .177 & .147 \\ .081 & .093 & .102 & .094 & .118 & .147 & .167 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1.000 & .938 & .866 & .797 & .776 & .729 & .660 \\ .938 & 1.000 & .906 & .816 & .806 & .741 & .675 \\ .866 & .906 & 1.000 & .804 & .731 & .694 & .672 \\ .797 & .816 & .804 & 1.000 & .906 & .875 & .852 \\ .776 & .806 & .731 & .906 & 1.000 & .972 & .824 \\ .729 & .741 & .694 & .875 & .972 & 1.000 & .854 \\ .660 & .675 & .672 & .852 & .824 & .854 & 1.000 \end{bmatrix}$$

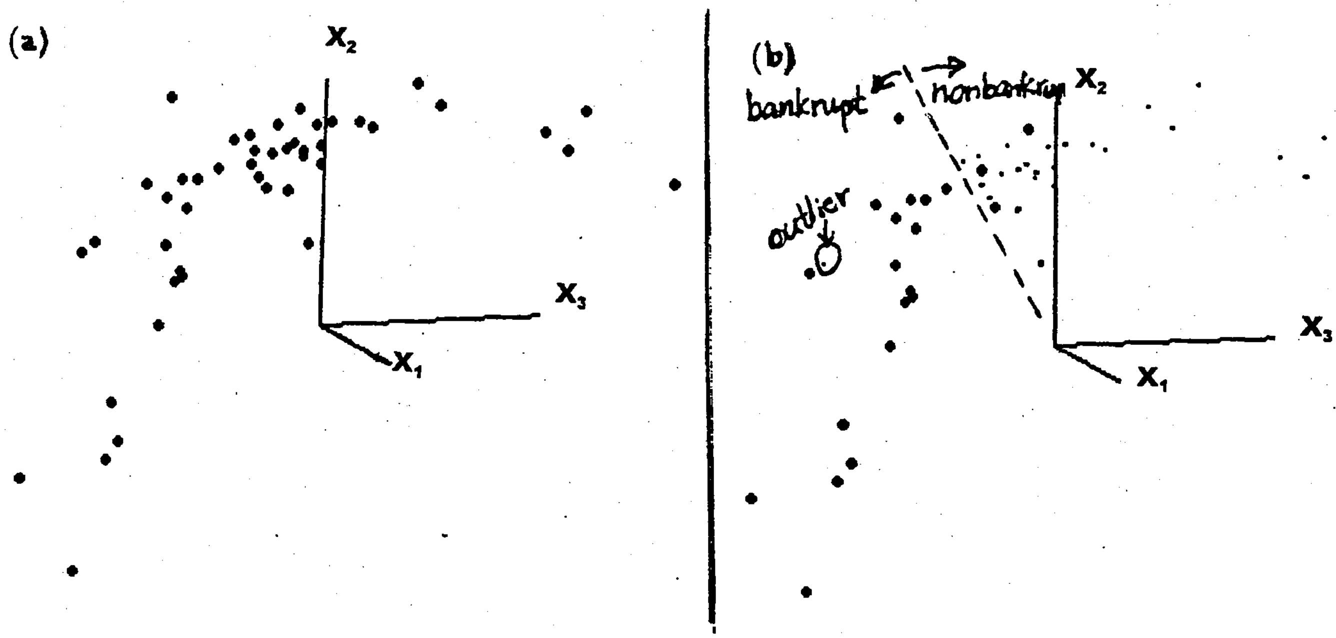
1.19 (a)


ULNA
D_ULNA
HUMERUS
D_HUMERUS
RADIUS
D_RADIUS
MIN
MED
MAX
0.49
0.86
1.10
0.53
0.84
1.05
1.04
1.79
2.33
1.27
1.76
2.24
0.53
0.71
0.87
0.48
0.68
0.87

1.19 (b)



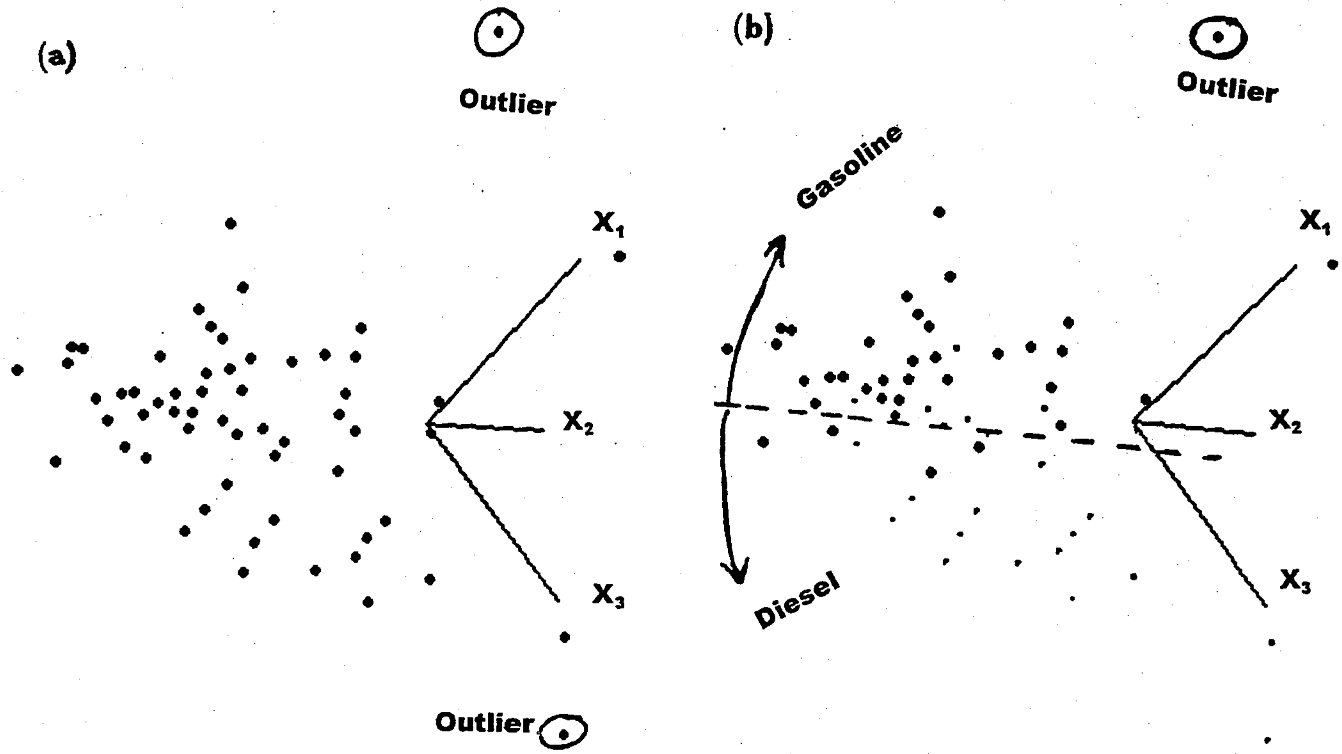
1.20



(a) The plot looks like a cigar shape, but bent. Some observations in the lower left hand part could be outliers. From the highlighted plot in (b) (actually non-bankrupt group not highlighted), there is one outlier in the nonbankrupt group, which is apparently located in the bankrupt group, besides the strung out pattern to the right.

(b) The dotted line in the plot would be an orientation for the classification.

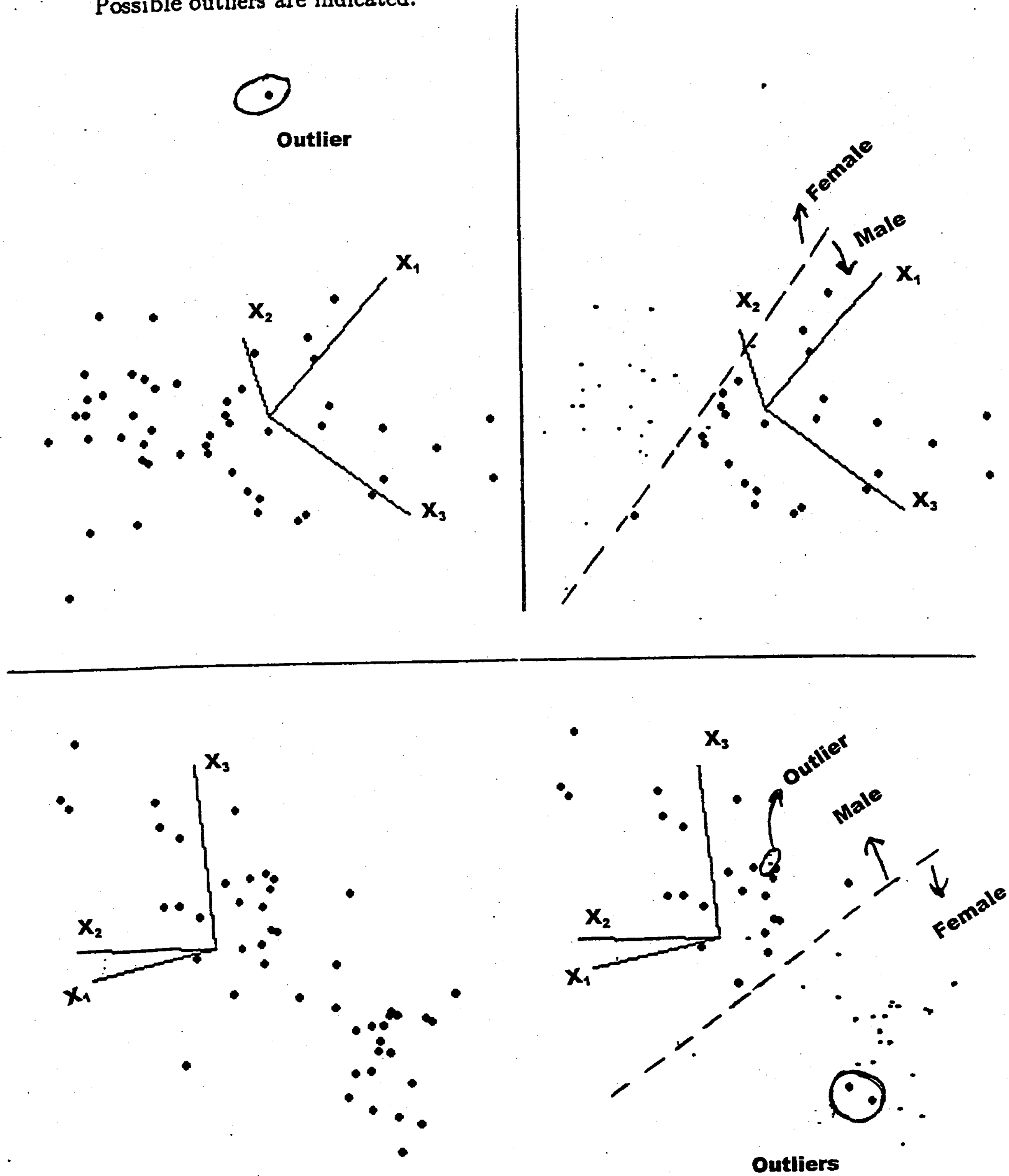
1.21



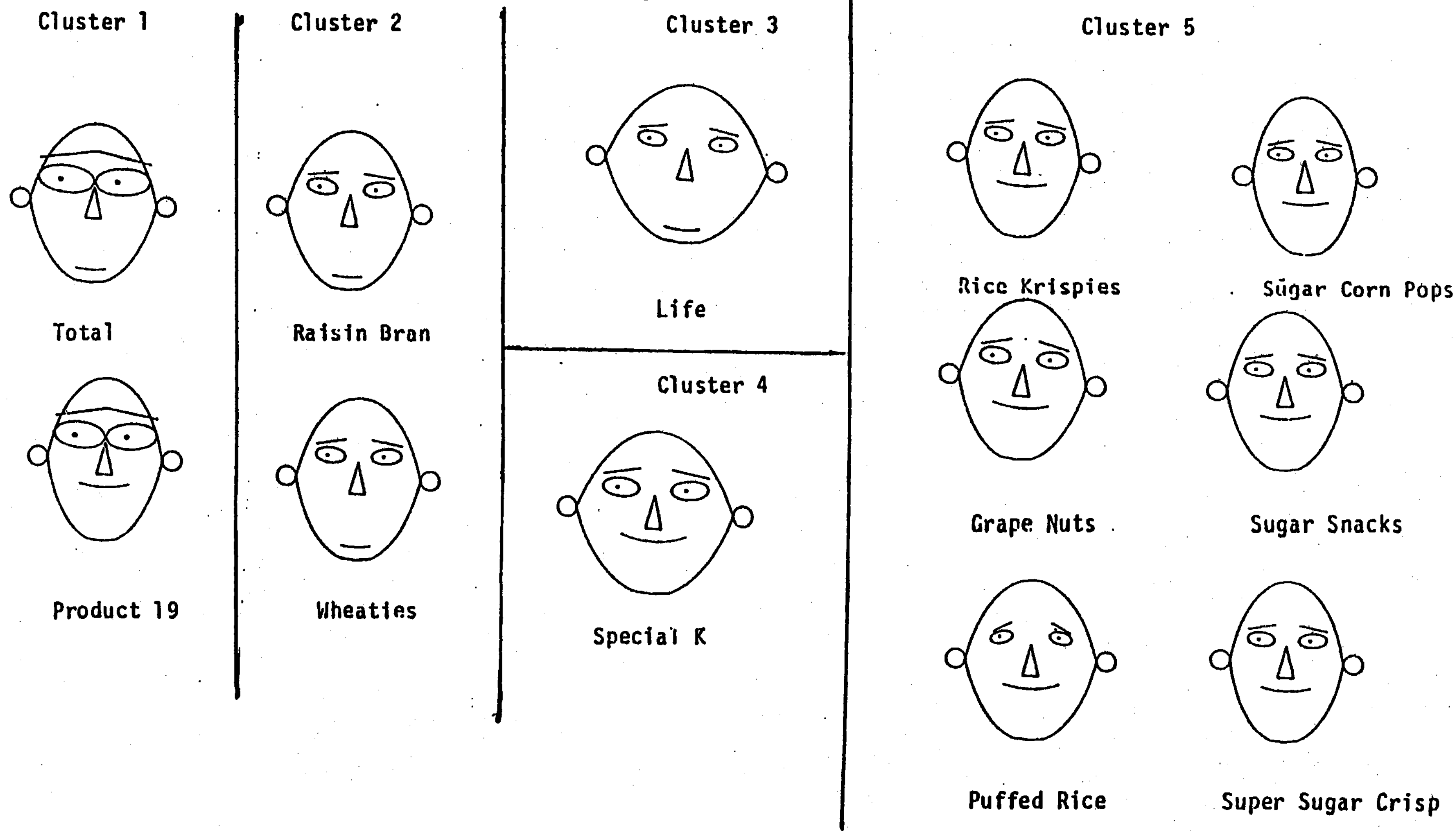
(a) There are two outliers in the upper right and lower right corners of the plot.

(b) Only the points in the gasoline group are highlighted. The observation in the upper right is the outlier. As indicated in the plot, there is an orientation to classify into two groups.

1.22 Possible outliers are indicated.

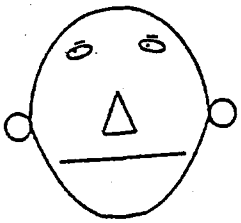


1.23 b) A visual clustering using Chernoff faces is given below.

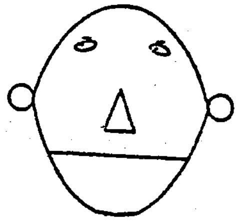


1.24

Cluster 1



20



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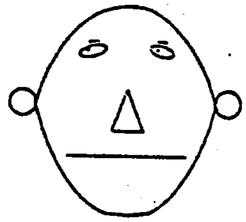

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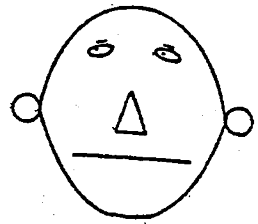
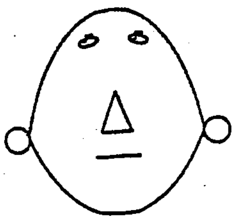
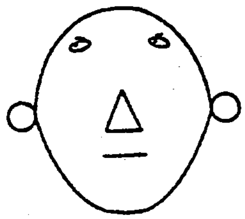


4

Cluster 2



14



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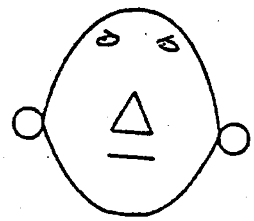

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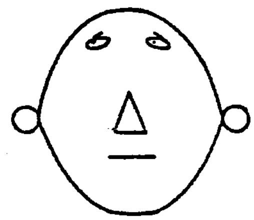

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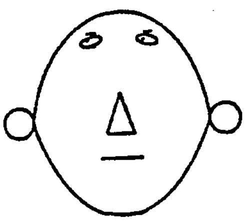

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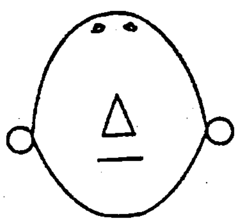
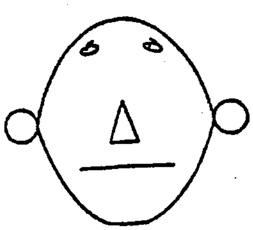


19

Cluster 3



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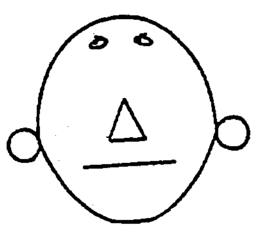

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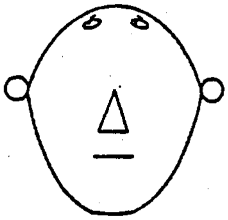

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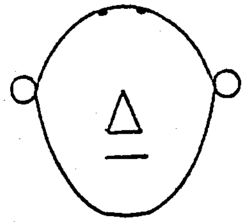

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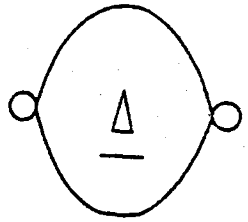

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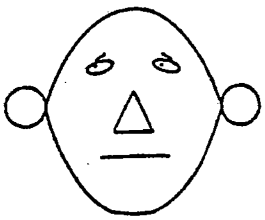
Cluster 5



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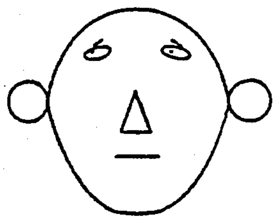
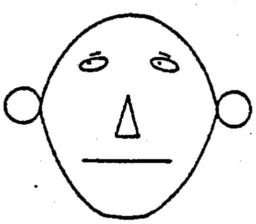


21

Cluster 6



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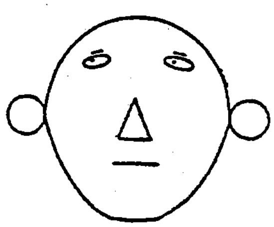

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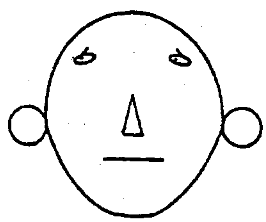
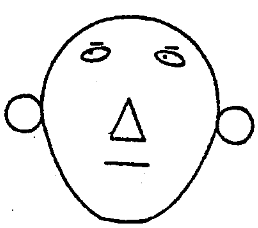


17

Cluster 7



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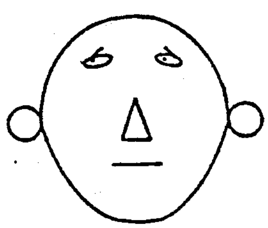
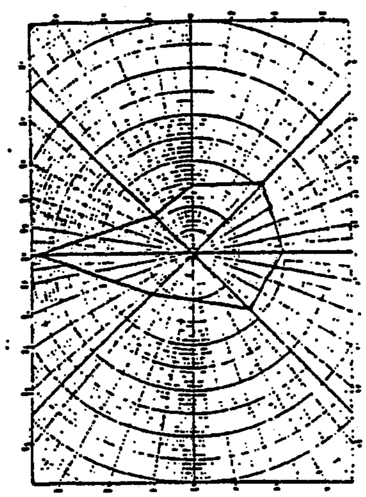


15

We have clustered these faces in the same manner as those in Example 1.12. Note, however, other groupings are equally plausible. For instance, utilities 9 and 18 might be switched from Cluster 2 to Cluster 3 and so forth.

1.25 We illustrate one cluster of "stars". The remaining stars (not shown) can be grouped in 3 or 4 additional clusters.



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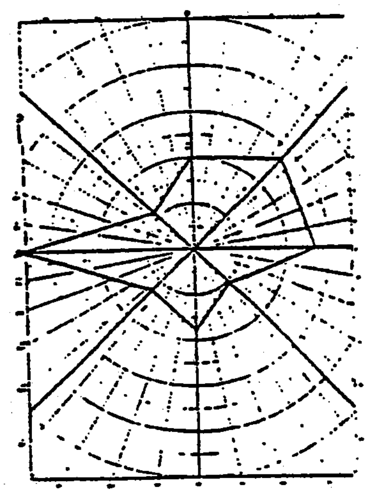

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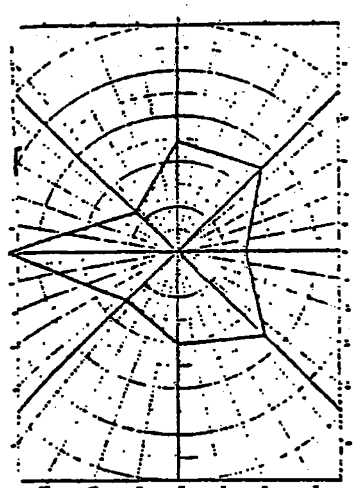

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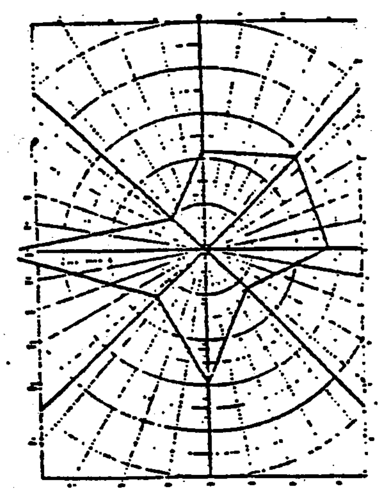


13

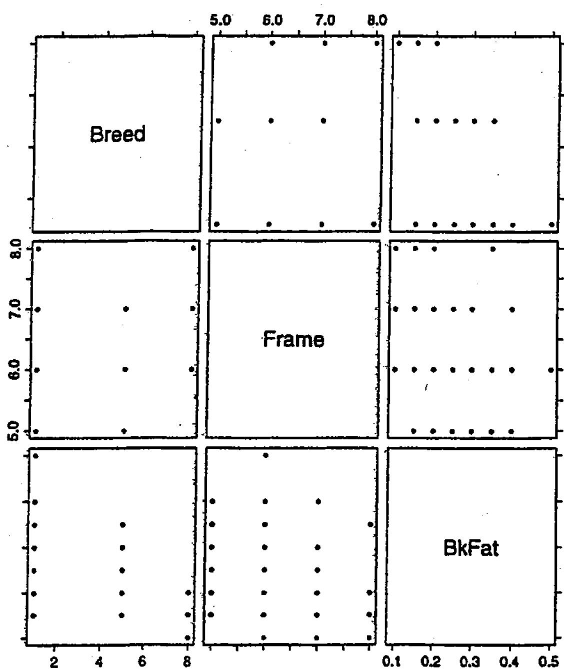
1.26 Bull data

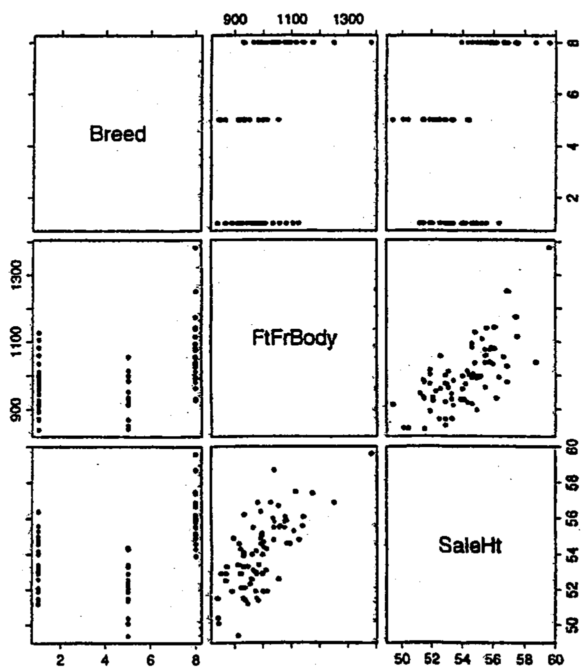
(a)

XBAR	R: Breed	SalePr	YrHgt	FtFrBody	PrctFFB	Frame	BkFat	SaleHt	SaleWt
4.3816	1.000	-0.224	0.525	0.409	0.472	0.434	-0.615	0.487	0.116
1742.4342	-0.224	1.000	0.423	0.102	-0.113	0.479	0.277	0.390	0.317
50.5224	0.525	0.423	1.000	0.624	0.523	0.940	-0.344	0.860	0.368
995.9474	0.409	0.102	0.624	1.000	0.691	0.605	-0.168	0.699	0.555
70.8816	0.472	-0.113	0.523	0.691	1.000	0.482	-0.488	0.521	0.198
6.3158	0.434	0.479	0.940	0.605	0.482	1.000	-0.260	0.801	0.368
0.1967	-0.615	0.277	-0.344	-0.168	-0.488	-0.260	1.000	-0.282	0.208
54.1263	0.487	0.390	0.860	0.699	0.521	0.801	-0.282	1.000	0.566
1555.2895	0.116	0.317	0.368	0.555	0.198	0.368	0.208	0.566	1.000

Sn

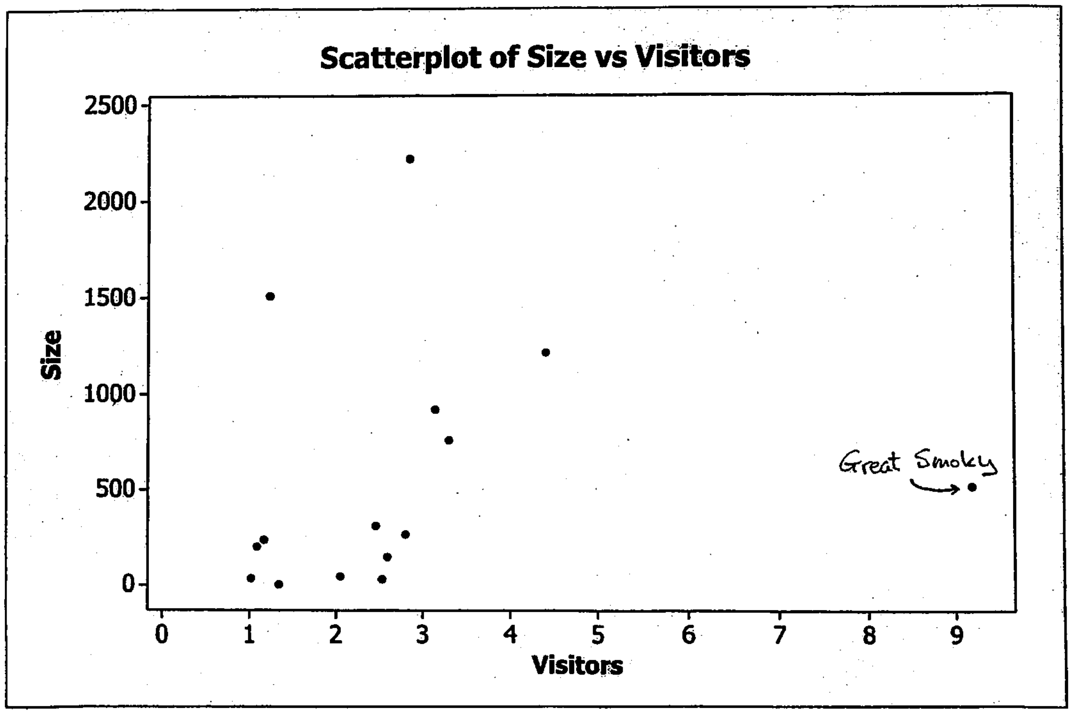
Breed	SalePr	YrHgt	FtFrBody	PrctFFB	Frame	BkFat	SaleHt	SaleWt
9.55	-429.02	2.79	116.28	4.73	1.23	-0.17	3.00	46.32
-429.02	383026.64	450.47	5813.09	-226.46	272.78	15.24	480.56	25308.44
2.79	450.47	2.96	98.81	2.92	1.49	-0.05	2.94	81.72
116.28	5813.09	98.81	8481.26	206.75	51.27	-1.38	128.23	6592.41
4.73	-226.46	2.92	206.75	10.55	1.44	-0.14	3.37	82.82
1.23	272.78	1.49	51.27	1.44	0.85	-0.02	1.47	43.74
-0.17	15.24	-0.05	-1.38	-0.14	-0.02	0.01	-0.05	2.38
3.00	480.56	2.94	128.23	3.37	1.47	-0.05	3.97	145.35
46.32	25308.44	81.72	6592.41	82.82	43.74	2.38	145.35	16628.94





1.27

(a) Correlation $r = .173$

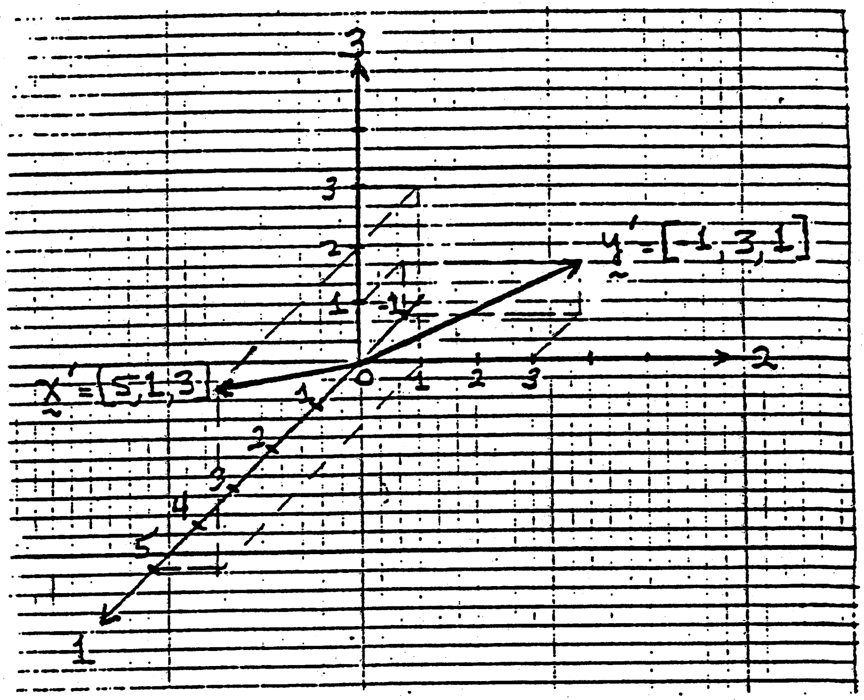


(b) Great Smoky is unusual park. Correlation with this park removed is $r = .391$. This single point has reasonably large effect on correlation reducing the positive correlation by more than half when added to the national park data set.

(c) The correlation coefficient is a dimensionless measure of association. The correlation in (b) would not change if size were measured in square miles instead of acres.

Chapter 2

2.1 a)



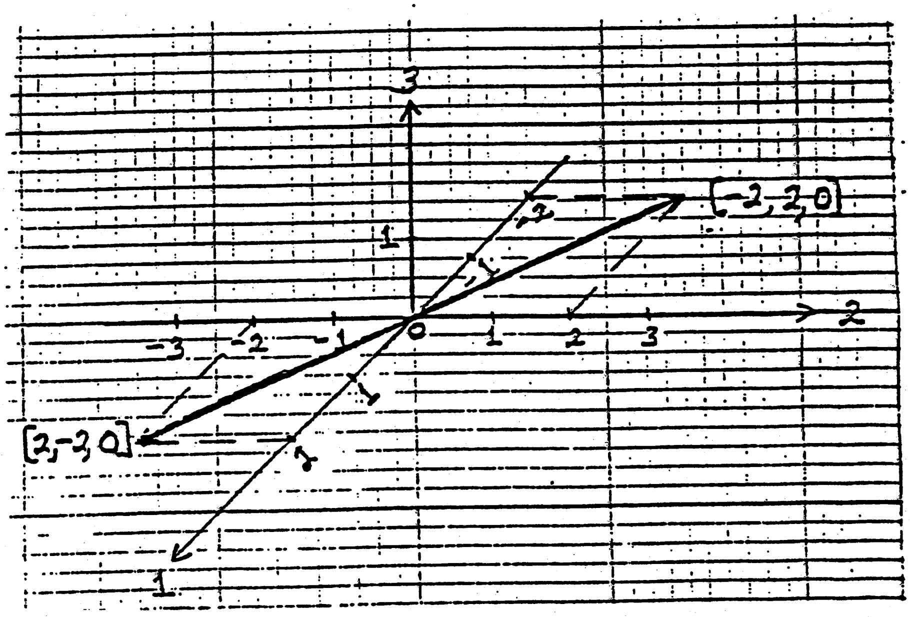
b) i) $L_{\underset{\sim}{x}} = \sqrt{\underset{\sim}{x}'\underset{\sim}{x}} = \sqrt{35} = 5.916$

ii) $\cos(\theta) = \dfrac{\underset{\sim}{x}'\underset{\sim}{y}}{L_{\underset{\sim}{x}} L_{\underset{\sim}{y}}} = \dfrac{1}{19.621} = .051$

$\theta = \text{arc cos}\ (.051) \doteq 87°$

iii) projection of $\underset{\sim}{y}$ on $\underset{\sim}{x}$ is $\left|\dfrac{\underset{\sim}{y}'\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}}\right| \underset{\sim}{x} = \dfrac{1}{35}\underset{\sim}{x} = \left[\dfrac{1}{7}, \dfrac{1}{35}, \dfrac{3}{35}\right]'$

c)



2.2 a) $5A = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$ b) $BA = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$

c) $A'B' = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$ d) $C'B = [12, -7]$

e) No.

2.3 a) $A' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A$ so $(A')' = A' = A$

b) $C' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $(C')^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix}$

$C^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$; $(C^{-1})' = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix} = (C')^{-1}$

c)

$(AB)' = \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix} = (AB)'$

d) AB has (i,j)th entry

$$a_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{\ell=1}^{k} a_{i\ell}b_{\ell j}$$

Consequently, (AB)' has $(i,j)^{th}$ entry

$$c_{ji} = \sum_{\ell=1}^{k} a_{j\ell}b_{\ell i} .$$

Next B' has i^{th} row $[b_{1i}, b_{2i}, \cdots, b_{ki}]$ and A' has j^{th}

column $[a_{j1}, a_{j2}, \cdots, a_{jk}]'$ so $B'A'$ has $(i,j)^{th}$ entry

$$b_{1i}a_{j1} + b_{2i}b_{j2} + \cdots + b_{ki}a_{jk} = \sum_{\ell=1}^{k} a_{j\ell}b_{\ell i} = c_{ji}$$

Since i and j were arbitrary choices, $(AB)' = B'A'$.

2.4 a) $I = I'$ and $AA^{-1} = I = A^{-1}A$. Thus $I' = I = (AA^{-1})' = (A^{-1})'A'$ and $I = (A^{-1}A)' = A'(A^{-1})'$. Consequently, $(A^{-1})'$ is the inverse of A' or $(A')^{-1} = (A^{-1})'$.

b) $(B^{-1}A^{-1})AB = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$ so AB has inverse $(AB)^{-1} = B^{-1}A^{-1}$. It was sufficient to check for a left inverse but we may also verify $AB(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{I}A^{-1} = AA^{-1} = I$.

2.5

$$QQ' = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} = \begin{bmatrix} \frac{169}{169} & 0 \\ 0 & \frac{169}{169} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q'Q \; .$$

2.6 a) Since $A = A'$, A is symmetric.

b) Since the quadratic form

$$\underset{\sim}{x}'A\underset{\sim}{x} = [x_1, x_2]\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 4x_1x_2 + 6x_2^2$$

$$= (2x_1 - x_2)^2 + 5(x_1^2 + x_2^2) > 0 \quad \text{for} \quad [x_1, x_2] \neq [0,0]$$

we conclude that A is positive definite.

2.7 a) Eigenvalues: $\lambda_1 = 10$, $\lambda_2 = 5$.

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, -1/\sqrt{5}] = [.894, -.447]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, 2/\sqrt{5}] = [.447, .894]$

b) $A = \begin{bmatrix} 9 & -2 \\ -2 & 9 \end{bmatrix} = 10 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ -1/\sqrt{5}] + 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, \ 2/\sqrt{5}]$

c) $A^{-1} = \frac{1}{9(6)-(-2)(-2)} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} .12 & .04 \\ .04 & .18 \end{bmatrix}$

d) Eigenvalues: $\lambda_1 = .2, \ \lambda_2 = .1$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [1/\sqrt{5}, \ 2/\sqrt{5}]$

$\underset{\sim}{e}_2' = [2/\sqrt{5}, \ -1/\sqrt{5}]$

2.8 Eigenvalues: $\lambda_1 = 2, \ \lambda_2 = -3$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, \ 1/\sqrt{5}]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, -2/\sqrt{5}]$

$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ 1/\sqrt{5}] - 3 \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, \ -2/\sqrt{5}]$

2.9 a) $A^{-1} = \frac{1}{1(-2)-2(2)} \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$

b) Eigenvalues: $\lambda_1 = 1/2, \ \lambda_2 = -1/3$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, \ 1/\sqrt{5}]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, \ -2/\sqrt{5}]$

c) $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ 1/\sqrt{5}] - \frac{1}{3} \begin{bmatrix} 1/\sqrt{5} \\ -2/\ 5 \end{bmatrix} [1/\sqrt{5}, \ -2/\sqrt{5}]$

2.10

$$B^{-1} = \frac{1}{4(4.002001)-(4.001)^2}\begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$= 333,333\begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4(4.002)-(4.001)^2}\begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$= -1,000,000\begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

Thus $A^{-1} \doteq (-3)B^{-1}$

2.11 With $p = 1$, $|a_{11}| = a_{11}$ and with $p = 2$

$$\begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22} - 0(0) = a_{11}a_{22}$$

Proceeding by induction, we assume the result holds for any $(p-1)\times(p-1)$ diagonal matrix A_{11}. Then writing

$$\underset{(p\times p)}{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{11} & \\ 0 & & & \end{bmatrix}.$$

we expand $|A|$ according to Definition 2A.24 to find $|A| = a_{11}\ |A_{11}| + 0 + \cdots + 0$. Since $|A_{11}| = a_{22}a_{33} \cdots a_{pp}$ by the induction hypothesis, $|A| = a_{11}(a_{22}a_{33} \cdots a_{pp}) = a_{11}a_{22}a_{33} \cdots a_{pp}$.

2.12 By (2-20), $A = P\Lambda P'$ with $PP' = P'P = I$. From Result 2A.11(e) $|A| = |P|\ |\Lambda|\ |P'| = |\Lambda|$. Since Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_p$, we can apply Exercise 2.11 to get $|A| = |\Lambda| = \prod_{i=1}^{p} \lambda_i$.

2.14 Let λ be an eigenvalue of A. Thus $0 = |A-\lambda I|$. If Q is orthogonal, $QQ' = I$ and $|Q||Q'| = 1$ by Exercise 2.13. Using Result 2A.11(e) we can then write

$$0 = |Q|\ |A-\lambda I|\ |Q'| = |QAQ'-\lambda I|$$

and it follows that λ is also an eigenvalue of QAQ' if Q is orthogonal.

2.16 $(A'A)' = A'(A')' = A'A$ showing $A'A$ is symmetric.

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = A\underset{\sim}{x}. \text{ Then } 0 \le y_1^2+y_2^2+ \cdots + y_p^2 = \underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'A'A\underset{\sim}{x}$$

and $A'A$ is non-negative definite by definition.

2.18 Write $c^2 = \underset{\sim}{x}'A\underset{\sim}{x}$ with $A = \begin{bmatrix} 4 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix}$. The eigenvalue-normalized eigenvector pairs for A are:

$$\lambda_1 = 2, \quad \underset{\sim}{e}_1' = [.577,\ .816]$$

$$\lambda_2 = 5, \quad \underset{\sim}{e}_2' = [.816,\ -.577]$$

For $c^2 = 1$, the half lengths of the major and minor axes of the ellipse of constant distance are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} = .707 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{5}} = .447$$

respectively. These axes lie in the directions of the vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ respectively.

For $c^2 = 4$, the half lengths of the major and minor axes are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{2}{\sqrt{2}} = 1.414 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{5}} = .894 \ .$$

As c^2 increases the lengths of the major and minor axes increase.

2.20 Using matrix A in Exercise 2.3, we determine

$$\lambda_1 = 1.382, \quad \mathbf{e}_1 = [.8507, \quad -.5257]'$$

$$\lambda_2 = 3.618, \quad \mathbf{e}_2 = [.5257, \quad .8507]'$$

We know

$$A^{1/2} = \sqrt{\lambda_1}\, \mathbf{e}_1 \mathbf{e}_1' + \sqrt{\lambda_2}\, \mathbf{e}_2 \mathbf{e}_2' = \begin{bmatrix} 1.376 & .325 \\ .325 & 1.701 \end{bmatrix}$$

$$A^{-1/2} = \frac{1}{\sqrt{\lambda_1}} \mathbf{e}_1 \mathbf{e}_1' + \frac{1}{\sqrt{\lambda_2}} \mathbf{e}_2 \mathbf{e}_2' = \begin{bmatrix} .7608 & -.1453 \\ -.1453 & .6155 \end{bmatrix}$$

We check

$$A^{1/2} A^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1/2} A^{1/2}$$

2.21 (a)

$$A'A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

$0 = |A'A - \lambda\, I\,| = (9-\lambda)^2 - 1 = (10-\lambda)(8-\lambda)$, so $\lambda_1 = 10$ and $\lambda_2 = 8$. Next,

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(b)

$$AA' = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

$$0 = |AA' - \lambda I\,| = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 8-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{vmatrix}$$

$= (2-\lambda)(8-\lambda)^2 - 4^2(8-\lambda) = (8-\lambda)(\lambda-10)\lambda$ so $\lambda_1 = 10, \lambda_2 = 8$, and $\lambda_3 = 0$.

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{matrix} 4e_3 & = & 8e_1 \\ 8e_2 & = & 10e_2 \end{matrix} \quad \text{so} \quad e_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{matrix} 4e_3 & = & 6e_1 \\ 4e_1 & = & 0 \end{matrix} \quad \text{so} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Also, $e_3 = [-2/\sqrt{5}, 0, 1/\sqrt{5}\,]'$.

(c)

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \sqrt{10}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \left[\ \frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\ \right] + \sqrt{8}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \left[\ \frac{1}{\sqrt{2}}\ ,\ -\frac{1}{\sqrt{2}}\ \right]$$

2.22 **(a)**

$$\mathbf{AA}' = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}$$

$0 = |\mathbf{AA}' - \lambda\,\mathbf{I}\,| = (144-\lambda)(126-\lambda) - (12)^2 = (150-\lambda)(120-\lambda)$, so $\lambda_1 = 150$ and $\lambda_2 = 120$. Next,

$$\begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 150 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

and $\lambda_2 = 120$ gives $e_2 = [1/\sqrt{5}, 2/\sqrt{5}\,]'$.

(b)

$$\mathbf{A}'\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

$$0 = |\mathbf{A}'\mathbf{A} - \lambda\mathbf{I}\,| = \begin{vmatrix} 25-\lambda & 50 & 5 \\ 50 & 100-\lambda & 10 \\ 5 & 10 & 145-\lambda \end{vmatrix} = (150-\lambda)(\lambda-120)\lambda$$

so $\lambda_1 = 150$, $\lambda_2 = 120$, and $\lambda_3 = 0$. Next,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 150 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{matrix} -120e_1 & + & 60e_2 & = 0 \\ -25e_1 & + & 5e_3 & = 0 \end{matrix} \quad \text{or } e_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 120 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{array}{rcrl} 60e_1 & + & 60e_3 & = 0 \\ -120e_2 & + & -240e_3 & = 0 \end{array} \quad \text{or} \quad e_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Also, $e_3 = [2/\sqrt{5}, -1/\sqrt{5}, 0]'$.

(c)

$$\begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

$$= \sqrt{150}\begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}\left[\begin{array}{ccc} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{array}\right] + \sqrt{120}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}\left[\begin{array}{ccc} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{array}\right]$$

2.24

a) $\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\lambda_1 = 4, \quad \mathbf{e}_1 = [1,0,0]'$
$\lambda_2 = 9, \quad \mathbf{e}_2 = [0,1,0]'$
$\lambda_3 = 1, \quad \mathbf{e}_3 = [0,0,1]'$

c) For $\boldsymbol{\Sigma}^{-1}$: $\lambda_1 = 1/4, \quad \mathbf{e}_1' = [1,0,0]'$
$\lambda_2 = 1/9, \quad \mathbf{e}_2' = [0,1,0]'$
$\lambda_3 = 1, \quad \mathbf{e}_3' = [0,0,1]'$

2.25

a) $$V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \quad \boldsymbol{\rho} = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -.2 & .267 \\ -.2 & 1 & .167 \\ .267 & .167 & 1 \end{bmatrix}$$

b) $V^{1/2} \boldsymbol{\rho} V^{1/2} =$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \boldsymbol{\Sigma}$$

2.26

a) $\rho_{13} = \sigma_{13}/\sigma_{11}^{1/2}\sigma_{22}^{1/2} = 4/\sqrt{25}\,\sqrt{9} = 4/15 = .267$

b) Write $X_1 = 1\cdot X_1 + 0\cdot X_2 + 0\cdot X_3 = \mathbf{c}_1'\mathbf{X}$ with $\mathbf{c}_1' = [1,0,0]$

$$\frac{1}{2}X_2 + \frac{1}{2}X_3 = \mathbf{c}_2'\mathbf{X} \quad \text{with} \quad \mathbf{c}_2' = [0, \tfrac{1}{2}, \tfrac{1}{2}]$$

Then $\text{Var}(X_1) = \sigma_{11} = 25$. By (2-43),

$$\text{Var}(\tfrac{1}{2}X_2 + \tfrac{1}{2}X_3) = \mathbf{c}_2' \boldsymbol{\Sigma} \mathbf{c}_2 = \frac{1}{4}\sigma_{22} + \frac{2}{4}\sigma_{23} + \frac{1}{4}\sigma_{33} = 1 + \frac{1}{2} + \frac{9}{4}$$

$$= \frac{15}{4} = 3.75$$

By (2-45), (see also hint to Exercise 2.28),

$$\text{Cov}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \mathbf{c}_1' \boldsymbol{\Sigma} \mathbf{c}_2 = \frac{1}{2}\sigma_{12} + \frac{1}{2}\sigma_{13} = -1 + 2 = 1$$

so

$$\text{Corr}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \frac{\text{Cov}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2)}{\sqrt{\text{Var}(X_1)}\,\sqrt{\text{Var}(\tfrac{1}{2}X_1 + \tfrac{1}{2}X_2)}} = \frac{1}{5\sqrt{3.75}} = .103$$

2.27

a) $\mu_1 - 2\mu_2$, $\sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$

b) $-\mu_1 + 3\mu_2$, $\sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $\mu_1 + \mu_2 + \mu_3$, $\sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23}$

d) $\mu_1 + 2\mu_2 - \mu_3$, $\sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}$

e) $3\mu_1 - 4\mu_2$, $9\sigma_{11} + 16\sigma_{22}$ since $\sigma_{12} = 0$.

2.29

$$\Sigma = \left[\begin{array}{cc|ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \hline \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right]$$

2.31 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{(b)} \quad \mathrm{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1$$

(c)

$$\mathrm{Cov}(\boldsymbol{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$\mathrm{Cov}(\mathrm{A}\boldsymbol{X}^{(1)}) = \mathrm{A}\boldsymbol{\Sigma}_{11}\mathrm{A}' = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{(f)} \quad \mathrm{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(g)

$$\mathrm{Cov}(\boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$$

(h)

$$\mathrm{Cov}(\mathrm{B}\boldsymbol{X}^{(2)}) = \mathrm{B}\boldsymbol{\Sigma}_{22}\mathrm{B}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 48 & -8 \\ -8 & 4 \end{bmatrix}$$

(i)

$$\mathrm{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(j)

$$\mathrm{Cov}(\mathrm{A}\boldsymbol{X}^{(1)}, \mathrm{B}\boldsymbol{X}^{(2)}) = \mathrm{A}\boldsymbol{\Sigma}_{12}\mathrm{B}' = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

2.32 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{(b)}\ \ \mathbf{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(c)

$$\text{Cov}(\boldsymbol{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \quad \text{(f)}\ \ \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

(g)

$$\text{Cov}(\boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}'$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix}$$

(i)

$$\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}, \mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}'$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.33 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \quad \text{(b)}\ \mathbf{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(c)

$$\text{Cov}(\boldsymbol{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}'$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 4 \\ 4 & 63 \end{bmatrix}$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{(f)}\ \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

(g)

$$\text{Cov}(\boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$$

(i)

$$\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}, \mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{A}\Sigma_{12}\mathbf{B}'$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -4.5 & 4.5 \end{bmatrix}$$

2.34 $\underset{\sim}{b}'\underset{\sim}{b} = 4 + 1 + 16 + 0 = 21$, $\underset{\sim}{d}'\underset{\sim}{d} = 15$ and $\underset{\sim}{b}'\underset{\sim}{d} = -2-3-8+0 = -13$

$$(\underset{\sim}{b}'\underset{\sim}{d})^2 = 169 \leq 21(15) = 315$$

2.35 $\underset{\sim}{b}'\underset{\sim}{d} = -4+3 = -1$

$$\underset{\sim}{b}'B\underset{\sim}{b} = [-4,\ 3]\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = [-14\ \ 23]\begin{bmatrix} -4 \\ 3 \end{bmatrix} = 125$$

$$\underset{\sim}{d}'B^{-1}\underset{\sim}{d} = [1,1]\begin{bmatrix} 5/6 & 2/6 \\ 2/6 & 2/6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 11/6$$

so $1 = (\underset{\sim}{b}'\underset{\sim}{d})^2 \leq 125\,(11/6) = 229.17$

2.36
$$4x_1^2 + 4x_2^2 + 6x_1x_2 = \mathbf{x}'A\mathbf{x} \quad \text{where } A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.$$

$(4-\lambda)^2 - 3^2 = 0$ gives $\lambda_1 = 7, \lambda_2 = 1$. Hence the maximum is 7 and the minimum is 1.

2.37 From (2-51),
$$\max_{\underset{\sim}{x}'\underset{\sim}{x}=1} \underset{\sim}{x}'A\underset{\sim}{x} = \max_{\underset{\sim}{x}\neq\underset{\sim}{0}} \frac{\underset{\sim}{x}'A\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}} = \lambda_1$$

where λ_1 is the largest eigenvalue of A. For A given in Exercise 2.6, we have from Exercise 2.7, $\lambda_1 = 10$ and $\underset{\sim}{e}_1' = [.894, -.447]$. Therefore $\max_{\underset{\sim}{x}'\underset{\sim}{x}=1} \underset{\sim}{x}'A\underset{\sim}{x} = 10$ and this maximum is attained for $\underset{\sim}{x} = \underset{\sim}{e}_1$.

2.38
Using computer, $\lambda_1 = 18$, $\lambda_2 = 9$, $\lambda_3 = 9$. Hence the maximum is 18 and the minimum is 9.

2.41 (a) $$E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(b) $$Cov(\mathbf{AX}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_X\mathbf{A}' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

(c) All pairs of linear combinations have zero covariances.

2.42 (a) $$E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(b) $$Cov(\mathbf{AX}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_X\mathbf{A}' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

(c) All pairs of linear combinations have zero covariances.

Chapter 3

3.1

a) $\bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

b) $\mathbf{e}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1} = [4, 0, -4]'$

$\mathbf{e}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1} = [-1, 1, 0]'$

c) $L_{\mathbf{e}_1} = \sqrt{32}$; $L_{\mathbf{e}_2} = \sqrt{2}$

Let θ be the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\cos(\theta) = -4/\sqrt{32 \times 2} = -.5$

Therefore $n\, s_{11} = L^2_{\mathbf{e}_1}$ or $s_{11} = 32/3$; $n\, s_{22} = L^2_{\mathbf{e}_2}$ or $s_{22} = 2/3$; $n\, s_{12} = \mathbf{e}_1' \mathbf{e}_2$ or $s_{12} = -4/3$. Also, $r_{12} = \cos(\theta) = -.5$. Consequently $S_n = \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & -.5 \\ -.5 & 1 \end{bmatrix}$.

3.2

a) $\bar{\mathbf{x}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

b) $\mathbf{e}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1} = [-1, 2, -1]'$

$\mathbf{e}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1} = [3, -3, 0]'$

c) $L_{\mathbf{e}_1} = \sqrt{6}$; $L_{\mathbf{e}_2} = \sqrt{18}$

Let θ be the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\cos(\theta) = -9/\sqrt{6 \times 18} = -.866$.

Therefore $n\, s_{11} = L^2_{\mathbf{e}_1}$ or $s_{11} = 6/3 = 2$; $n\, s_{22} = L^2_{\mathbf{e}_2}$ or $s_{22} = 18/3 = 6$; $n\, s_{12} = \mathbf{e}_1' \mathbf{e}_2$ or $s_{12} = -9/3 = -3$. Also, $r_{12} = \cos(\theta) = -.866$. Consequently $S_n = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & -.866 \\ -.866 & 1 \end{bmatrix}$

3.3 $\underset{\sim}{y}_1 = [1, 4, 4]'$; $\bar{x}_1 \underset{\sim}{1} = [3, 3, 3]$; $\underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1} = [-2, 1, 1]'$

Thus

$$\underset{\sim}{y}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \bar{x}_1 \underset{\sim}{1} + (\underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1})$$

3.5 a) $\mathbf{X}' = \begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}$; $\underset{\sim}{\bar{x}} \underset{\sim}{1}' = \begin{bmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \end{bmatrix}$

$$2\,S = (\mathbf{X} - \underset{\sim}{\bar{x}} \underset{\sim}{1}')(\mathbf{X} - \underset{\sim}{\bar{x}} \underset{\sim}{1}')' = \begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix}$$

so $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$ and $|S| = 12$

b) $\mathbf{X}' = \begin{bmatrix} 3 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix}$; $\underset{\sim}{\bar{x}} \underset{\sim}{1}' = \begin{bmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$$2\,S = (\mathbf{X} - \underset{\sim}{1}\, \underset{\sim}{\bar{x}}')'(\mathbf{X} - \underset{\sim}{1}\, \underset{\sim}{\bar{x}}') = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 18 \end{bmatrix}$$

so $S = \begin{bmatrix} 3 & -9/2 \\ -9/2 & 9 \end{bmatrix}$ and $|S| = 27/4$

3.6 a) $\mathbf{X}' - \underset{\sim}{1}\, \underset{\sim}{\bar{x}}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$. Thus $\underset{\sim}{d}'_1 = [-3, 0, -3]$,

$\underset{\sim}{d}'_2 = [0, 1, -1]$ and $\underset{\sim}{d}'_3 = [-3, 1, 2]$.

Since $\underset{\sim}{d}_1 = \underset{\sim}{d}_2 = \underset{\sim}{d}_3$, the matrix of deviations is not of full rank.

b)

$$2\,S = (\underset{\sim}{X} - \underset{\sim}{1}\,\bar{\underset{\sim}{x}}')'(\underset{\sim}{X} - \underset{\sim}{1}\,\bar{\underset{\sim}{x}}') = \begin{bmatrix} 18 & -3 & 15 \\ -3 & 2 & -1 \\ 15 & -1 & 14 \end{bmatrix}$$

So

$$S = \begin{bmatrix} 9 & -3/2 & 15/2 \\ -3/2 & 1 & -1/2 \\ 15/2 & -1/2 & 7 \end{bmatrix}$$

$|S| = 0$ (Verify). The 3 deviation vectors lie in a 2-dimensional subspace. The 3-dimensional volume enclosed by the deviation vectors is zero.

c) Total sample variance = 9 + 1 + 7 = 17 .

3.7 All ellipses are centered at $\bar{\underset{\sim}{x}}$.

i) For $S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix}$

Eigenvalue-normalized eigenvector pairs for S^{-1} are:

$$\lambda_1 = 1, \quad \underset{\sim}{e}_1' = [.707, \quad -.707]$$

$$\lambda_2 = 1/9, \quad \underset{\sim}{e}_2' = [.707, \quad .707]$$

Half lengths of axes of ellipse $(\underset{\sim}{x} - \bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x} - \bar{\underset{\sim}{x}}) \le 1$ are $1/\sqrt{\lambda_1} = 1$ and $1/\sqrt{\lambda_2} = 3$ respectively. The major axis of ellipse lies in the direction of $\underset{\sim}{e}_2$; the minor axis lies in the direction of $\underset{\sim}{e}_1$.

ii) For $S = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{bmatrix}$

Eigenvalue-normalized eigenvectors for S^{-1} are:

$$\lambda_1 = 1, \quad \underset{\sim}{e}_1' = [.707, \quad .707]$$

$$\lambda_2 = 1/9, \quad \underset{\sim}{e}_2' = [.707, \quad -.707]$$

Half lengths of axes of ellipse $(\underset{\sim}{x}-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}-\bar{\underset{\sim}{x}}) \le 1$ are, again, $1/\sqrt{\lambda_1} = 1$ and $1/\sqrt{\lambda_2} = 3$. The major axes of the ellipse lies in the direction of $\underset{\sim}{e}_2$; the minor axis lies in the direction of $\underset{\sim}{e}_1$. Note that $\underset{\sim}{e}_2$ here is $\underset{\sim}{e}_1$ in part (i) above and $\underset{\sim}{e}_1$ here is $\underset{\sim}{e}_2$ in part (i) above.

iii) For $S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$

Eigenvalue-normalized eigenvector pairs for S^{-1} are:

$$\lambda_1 = 1/3; \quad \underset{\sim}{e}_1' = [1, 0]$$

$$\lambda_2 = 1/3, \quad \underset{\sim}{e}_2' = [0, 1]$$

Half lengths of axes of ellipse $(\underset{\sim}{x}-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}-\bar{\underset{\sim}{x}}) \le 1$ are equal and given by $1/\sqrt{\lambda_1} = 1/\sqrt{\lambda_2} = \sqrt{3}$. Major and minor axes of ellipse can be taken to lie in the directions of the coordinate axes. Here, the solid ellipse is, in fact, a solid sphere.

Notice for all three cases $|S| = 9$.

3.8 a) Total sample variance in both cases is 3.

b) For $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|S| = 1$

For $S = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}$, $|S| = 0$

3.9 (a) We calculate $\overline{x} = [16, 18, 34\ ]'$ and

$$X_c = \begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and we notice } \text{col}_1(\ X_c)+\ \text{col}_2(\ X_c) = \text{col}_1(\ X_c)$$

so $a = [1, 1, -1\ ]'$ gives $X_c a = 0$.

(b)

$$\text{S} = \begin{bmatrix} 10 & 3 & 13 \\ 3 & 2.5 & 5.5 \\ 13 & 5.5 & 18.5 \end{bmatrix} \quad \text{so } |\,\text{S}\,| = \begin{matrix} 10(2.5)(18.5) & + & 39(15.5) & + & 39(15.5) \\ -(13)^2(2.5) & - & 9(18.5) & - & 55(5.5) \end{matrix} = 0$$

As above in a)

$$\text{S}a = \begin{bmatrix} 10 + 3 - 13 \\ 3 + 2.5 - 5.5 \\ 13 + 5.5 - 18.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Check.

3.10 (a) We calculate $\overline{x} = [5, 2, 3\ ]'$ and

$$X_c = \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and we notice } \text{col}_1(\ X_c)+\ \text{col}_2(\ X_c) = \text{col}_1(\ X_c)$$

so $a = [1, 1, -1\ ]'$ gives $X_c a = 0$.

(b)

$$S = \begin{bmatrix} 2.5 & 0 & 2.5 \\ 0 & 2.5 & 2.5 \\ 2.5 & 2.5 & 5 \end{bmatrix} \quad \text{so } |\,\text{S}\,| = \begin{matrix} 5(2.5)^2 & + & 0 & + & 0 \\ -(2.5)^3 & - & 0 & - & (2.5)^3 \end{matrix} = 0$$

Using the save coefficient vector a as in Part a) $\text{S}a = 0$.

(c) Setting $\boldsymbol{X}\boldsymbol{a} = 0$,

$$\begin{array}{lllll} 3a_1 & + & a_2 & & = 0 \\ 7a_1 & & & + \; 3a_3 & = 0 \\ 5a_1 & + & 3a_2 & + \; 4a_3 & = 0 \end{array} \quad \text{so} \quad \begin{array}{lll} a_1 & = & -\frac{3}{7}a_3 \\ 5a_1 & - & 3(3a_1) + 4a_3 = 0 \end{array}$$

so we must have $a_1 = a_3 = 0$ but then, by the first equation in the first set, $a_2 = 0$. The columns of the data matrix are linearly independent.

3.11

$$S = \begin{bmatrix} 14808 & 14213 \\ 14213 & 15538 \end{bmatrix} .$$

Consequently

$$R = \begin{bmatrix} 1 & .9370 \\ .9370 & 1 \end{bmatrix}; \quad D^{1/2} = \begin{bmatrix} 121.6881 & 0 \\ 0 & 124.6515 \end{bmatrix}$$

and $$D^{-1/2} = \begin{bmatrix} .0082 & 0 \\ 0 & .0080 \end{bmatrix}$$

The relationships $R = D^{-1/2} S D^{-1/2}$ and $S = D^{1/2} R D^{1/2}$ can now be verified by direct matrix multiplication.

3.14 a) From first principles we have

$$\underset{\sim}{b}'\,\underset{\sim}{x}_1 = [2 \quad 3]\begin{bmatrix} 9 \\ 1 \end{bmatrix} = 21$$

Similarly $\underset{\sim}{b}'\,\underset{\sim}{x}_2 = 19$ and $\underset{\sim}{b}'\,\underset{\sim}{x}_3 = 8$ so

$$\text{sample mean} = \frac{21+19+8}{3} = 16$$

$$\text{sample variance} = \frac{(21-16)^2+(19-16)^2+(8-16)^2}{2} = 49$$

Also $\underset{\sim}{c}'\,\underset{\sim}{x}_1 = [-1 \quad 2]\begin{bmatrix} 9 \\ 1 \end{bmatrix} = -7$; $\underset{\sim}{c}'\,\underset{\sim}{x}_2 = 1$ and $\underset{\sim}{c}'\,\underset{\sim}{x}_3 = 3$

so

sample mean = -1

sample variance = 28

Finally sample covariance = $\dfrac{(21-16)(-7+1)+(19-16)(1+1)+(8-16)(3+1)}{2} =$

-28.

b) $\underset{\sim}{\bar{x}}' = [5 \quad 2]$ and $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$

Using (3-36)

sample mean of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}'\bar{\underset{\sim}{x}} = [2 \quad 3]\begin{bmatrix}5\\2\end{bmatrix} = 16$

sample mean of $\underset{\sim}{c}'\underset{\sim}{X} = [-1 \quad 2]\begin{bmatrix}5\\2\end{bmatrix} = -1$

sample variance of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}'S\underset{\sim}{b} = [2 \quad 3]\begin{bmatrix}16 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = 49$

sample variance of $\underset{\sim}{c}'\underset{\sim}{X} = \underset{\sim}{c}'S\underset{\sim}{c} = [-1 \quad 2]\begin{bmatrix}16 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = 28$

sample covariance of $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X}$

$= \underset{\sim}{b}'S\underset{\sim}{c} = [2 \quad 3]\begin{bmatrix}16 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = -28$

Results same as those in part (a).

3.15

$$\bar{\underset{\sim}{x}} = \begin{bmatrix}5\\3\\4\end{bmatrix}, \quad S = \begin{bmatrix}13 & -2.5 & 1.5\\-2.5 & 1 & -1.5\\1.5 & -1.5 & 3\end{bmatrix}$$

sample mean of $\underset{\sim}{b}'\underset{\sim}{X} = 12$

sample mean of $\underset{\sim}{c}'\underset{\sim}{X} = -1$

sample variance of $\underset{\sim}{b}'\underset{\sim}{X} = 12$

sample variance of $\underset{\sim}{c}'\underset{\sim}{X} = 43$

sample covariance of $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X} = -3$

3.16 Since

$$\Sigma_V = E(\underset{\sim}{V}-\underset{\sim}{\mu}_V)(\underset{\sim}{V}-\underset{\sim}{\mu}_V)'$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}' - \underset{\sim}{V}\underset{\sim}{\mu}_V' - \underset{\sim}{\mu}_V\underset{\sim}{V}' + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V')$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}') - E(\underset{\sim}{V})\underset{\sim}{\mu}_V' - \underset{\sim}{\mu}_V E(\underset{\sim}{V}') + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V'$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}') - \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V' - \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V' + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V'$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}') - \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V' ,$$

we have $E(\underset{\sim}{V}\underset{\sim}{V}') = \Sigma + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V'$.

3.18 **(a)** Let $y = x_1+x_2+x_3+x_4$ be the total energy consumption. Then

$$\bar{y} = [1 \quad 1 \quad 1 \quad 1]\bar{\mathbf{x}} = 1.873$$

$$s_y^2 = [1 \quad 1 \quad 1 \quad 1]\mathbf{S}[1 \quad 1 \quad 1 \quad 1]' = 3.913$$

(b) Let $y = x_1-x_2$ be the excess of petroleum consumption over natural gas consumption. Then

$$\bar{y} = [1 \quad -1 \quad 0 \quad 0]\bar{\mathbf{x}} = .258$$

$$s_y^2 = [1 \quad -1 \quad 0 \quad 0]\mathbf{S}[1 \quad -1 \quad 0 \quad 0]' = .154$$

Chapter 4

4.1 (a) We are given $p = 2$, $\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 2 & -.8 \times \sqrt{2} \\ -.8 \times \sqrt{2} & 1 \end{bmatrix}$ so $|\boldsymbol{\Sigma}| = .72$ and

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{.72} & \frac{\sqrt{2}}{.9} \\ \frac{\sqrt{2}}{.9} & \frac{2}{.72} \end{bmatrix}$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)\sqrt{.72}} \exp\left(-\frac{1}{2}\left[\frac{1}{.72}(x_1-1)^2 + \frac{2\sqrt{2}}{.9}(x_1-1)(x_2-3) + \frac{2}{.72}(x_2 - 3)^2\right]\right)$$

(b)

$$\frac{1}{.72}(x_1-1)^2 + \frac{2\sqrt{2}}{.9}(x_1-1)(x_2-3) + \frac{2}{.72}(x_2 - 3)^2$$

4.2 (a) We are given $p = 2$, $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 2 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$ so $|\boldsymbol{\Sigma}| = 3/2$

and

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{2}{\sqrt{3}} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

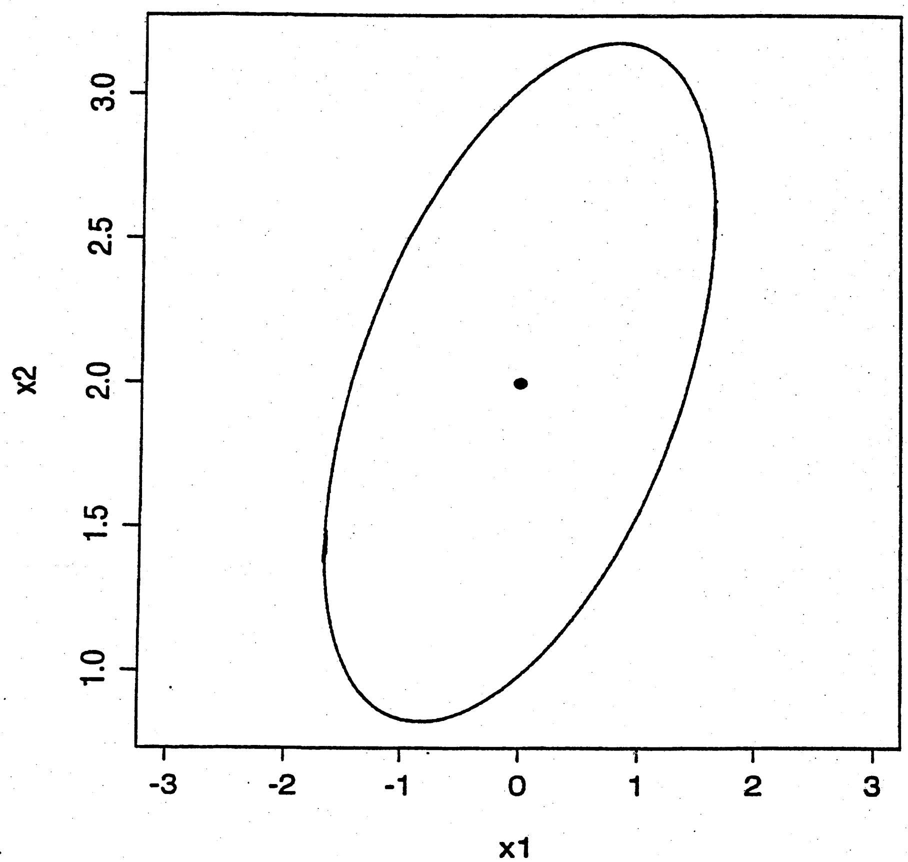
$$f(\boldsymbol{x}) = \frac{1}{(2\pi)\sqrt{3/2}} \exp\left(-\frac{1}{2}\left[\frac{2}{3}x_1^2 - \frac{2\sqrt{2}}{3}x_1(x_2-2) + \frac{4}{3}(x_2 - 2)^2\right]\right)$$

(b)

$$\frac{2}{3}x_1^2 - \frac{2\sqrt{2}}{3}x_1(x_2-2) + \frac{4}{3}(x_2 - 2)^2$$

(c) $c^2 = \chi_2^2(.5) = 1.39$. Ellipse centered at $[0,2]'$ with the major axis having half-length $\sqrt{\lambda_1}\, c = \sqrt{2.366}\sqrt{1.39} = 1.81$. The major axis lies in the direction $\boldsymbol{e} = [.888, .460]'$. The minor axis lies in the direction $\boldsymbol{e} = [-.460\,, .888]'$ and has half-length $\sqrt{\lambda_2}\, c = \sqrt{.634}\sqrt{1.39} = .94$.

Constant density contour that contains
50% of the probability



4.3 We apply Result 4.5 that relates zero covariance to statistical independence

a) No, $\sigma_{12} \neq 0$

b) Yes, $\sigma_{23} = 0$

c) Yes, $\sigma_{13} = \sigma_{23} = 0$

d) Yes, by Result 4.3, $(X_1+X_2)/2$ and X_3 are jointly normal and their covariance is $\frac{1}{2}\sigma_{13}+\frac{1}{2}\sigma_{23} = 0$.

e) No, by Result 4.3 with $A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$, form $A \Sigma A'$ to see that the covariance is 10 and not 0.

4.4 a) $3X_1 - 2X_2 + X_3$ is $N(13,9)$

b) Require $\text{Cov}(X_2, X_2 - a_1X_1 - a_3X_3) = 3 - a_1 - 2a_3 = 0$. Thus any $\underset{\sim}{a}' = [a_1, a_3]$ of the form $\underset{\sim}{a}' = [3-2a_3, a_3]$ will meet the requirement. As an example, $\underset{\sim}{a}' = [1,1]$.

4.5 a) $X_1|x_2$ is $N(\frac{1}{\sqrt{2}}(x_2-2), \frac{3}{2})$

b) $X_2|x_1,x_3$ is $N(-2x_1-5, 1)$

c) $X_3|x_1,x_2$ is $N(\frac{1}{2}(x_1+x_2+3), \frac{1}{2})$

4.6 (a) X_1 and X_2 are independent since they have a bivariate normal distribution with covariance $\sigma_{12} = 0$.

(b) X_1 and X_3 are dependent since they have nonzero covariance $\sigma_{13} = -1$.

(c) X_2 and X_3 are independent since they have a bivariate normal distribution with covariance $\sigma_{23} = 0$.

(d) X_1, X_3 and X_2 are independent since they have a trivariate normal distribution where $\sigma_{12} = 0$ and $\sigma_{32} = 0$.

(e) X_1 and $X_1 + 2X_2 - 3X_3$ are dependent since they have nonzero covariance

$$\sigma_{11} + 2\sigma_{12} - 3\sigma_{13} = 4 + 2(0) - 3(-1) = 7$$

4.7 (a) $X_1|x_3$ is $N(1 + .5(x_3 - 2)\ , 3.5)$

(b) $X_1|x_2, x_3$ is $N(1 + .5(x_3 - 2)\ , 3.5)$. Since X_2 is independent of X_1, conditioning further on x_2 does not change the answer from Part a).

4.15 First,

$$\sum_{j=1}^{n} (\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})' = (\bar{\mathbf{x}}-\boldsymbol{\mu})[\sum_{j=1}^{n} (\mathbf{x}_j-\bar{\mathbf{x}})']$$

$$= (\bar{\mathbf{x}}-\boldsymbol{\mu})(\sum_{j=1}^{n} \mathbf{x}_j-n\bar{\mathbf{x}})'$$

$$= (\bar{\mathbf{x}}-\boldsymbol{\mu})(n\bar{\mathbf{x}}-n\bar{\mathbf{x}})'$$

$$= \mathbf{0}$$

Also,

$$\sum_{j=1}^{n} (\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})' = [\sum_{j=1}^{n} (\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})']' = \mathbf{0}' = \mathbf{0}.$$

4.16 (a) By Result 4.8, with $c_1 = c_3 = 1/4$, $c_2 = c_4 = -1/4$ and $\boldsymbol{\mu}_j = \boldsymbol{\mu}$ for $j = 1,...,4$ we have $\sum_{j=1}^{4} c_j\boldsymbol{\mu}_j = \mathbf{0}$ and $(\sum_{j=1}^{4} c_j^2)\ \boldsymbol{\Sigma} = \frac{1}{4}\boldsymbol{\Sigma}$. Consequently, $\mathbf{V}_1$ is $N(\mathbf{0},\ \frac{1}{4}\boldsymbol{\Sigma}\ )$. Similarly, setting $b_1 = b_2 = 1/4$ and $b_3 = b_4 = -1/4$, we find that $\mathbf{V}_2$ is $N(\mathbf{0},\ \frac{1}{4}\boldsymbol{\Sigma}\ )$.

(b) Again by Result 4.8, we know that $\mathbf{V}_1$ and $\mathbf{V}_2$ are jointly multivariate normal with covariance

$$(\sum_{j=1}^{4} b_jc_j\ )\ \boldsymbol{\Sigma} = \left(\frac{1}{4}(\frac{1}{4}\ ) + \frac{-1}{4}(\frac{1}{4}\ ) + \frac{1}{4}(\frac{-1}{4}\ ) + \frac{-1}{4}(\frac{-1}{4}\ ) \right)\ \boldsymbol{\Sigma} = \mathbf{0}$$

That is,

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \quad \text{is distributed} \quad N_{2p}\left(\mathbf{0}, \begin{bmatrix} \frac{1}{4}\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \frac{1}{4}\boldsymbol{\Sigma} \end{bmatrix}\right)$$

so the joint density of the $2p$ variables is

$$f(\mathbf{v}_1,\ \mathbf{v}_2\ ) = \frac{1}{(2\pi)^p|\ \frac{1}{4}\boldsymbol{\Sigma}\ |} \exp\ \left(-\frac{1}{2}[\ \mathbf{v}_1',\ \mathbf{v}_2'\ ] \begin{bmatrix} \frac{1}{4}\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \frac{1}{4}\boldsymbol{\Sigma} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_1 \end{bmatrix} \right)$$

$$= \frac{1}{(2\pi)^p|\ \frac{1}{4}\boldsymbol{\Sigma}\ |} \exp\ \left(-\frac{1}{8}(\ \mathbf{v}_1'\ \boldsymbol{\Sigma}^{-1}\mathbf{v}_1\ +\ \mathbf{v}_2'\ \boldsymbol{\Sigma}^{-1}\mathbf{v}_2\ ) \right)$$

4.17 By Result 4.8, with $c_1 = c_2 = c_3 = c_4 = c_5 = 1/5$ and $\boldsymbol{\mu}_j = \boldsymbol{\mu}$ for $j = 1,...,5$ we find that $\mathbf{V}_1$ has mean $\sum_{j=1}^{5} c_j\boldsymbol{\mu}_j = \boldsymbol{\mu}$ and covariance matrix $(\ \sum_{j=1}^{5} c_j^2\ )\ \boldsymbol{\Sigma} = \frac{1}{5}\boldsymbol{\Sigma}$.

Similarly, setting $b_1 = b_3 = b_5 = 1/5$ and $b_2 = b_4 = -1/5$ we find that $\mathbf{V}_2$ has mean $\sum_{j=1}^{5} b_j\boldsymbol{\mu}_j = \frac{1}{5}\boldsymbol{\mu}$ and covariance matrix $(\ \sum_{j=1}^{5} b_j^2\ )\ \boldsymbol{\Sigma} = \frac{1}{5}\boldsymbol{\Sigma}$.

Again by Result 4.8, we know that $\mathbf{V}_1$ and $\mathbf{V}_2$ have covariance

$$(\ \sum_{i=1}^{4} b_j\, c_j\ )\ \boldsymbol{\Sigma} = \left(\frac{1}{5}(\frac{1}{5}\ ) + \frac{-1}{5}\ (\frac{1}{5}\ ) + \frac{1}{5}(\frac{1}{5}\ ) + \frac{-1}{5}(\frac{1}{5}\ ) + \frac{1}{5}(\frac{1}{5}\ ) \right) \boldsymbol{\Sigma} = \frac{1}{25}\boldsymbol{\Sigma}$$

4.18 By Result 4.11 we know that the maximum likelihood estimates of $\underset{\sim}{\mu}$ and Σ are $\bar{\underset{\sim}{x}} = [4,6]'$ and

$$\frac{1}{n}\sum_{j=1}^{n}(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})' = \frac{1}{4}\left\{\left(\begin{bmatrix}3\\6\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}3\\6\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'+\left(\begin{bmatrix}4\\4\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}4\\4\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'\right.$$

$$\left.+\left(\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'+\left(\begin{bmatrix}4\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}4\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'\right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix}+\begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}0 & -2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix}2 & 1\\1 & 6\end{bmatrix}$$

4.19 a) By Result 4.7 we know that $(\underset{\sim}{X}_1-\underset{\sim}{\mu})'\,\Sigma^{-1}(\underset{\sim}{X}_1-\underset{\sim}{\mu}) \sim \chi^2_6$

b) From (4-23), $\bar{\underset{\sim}{X}} \sim N_6(\underset{\sim}{\mu},\frac{1}{20}\Sigma)$. Then $\bar{\underset{\sim}{X}}-\underset{\sim}{\mu} \sim N_6(\underset{\sim}{0},\frac{1}{20}\Sigma)$ and finally $\sqrt{20}\,(\bar{\underset{\sim}{X}}-\underset{\sim}{\mu}) \sim N_6(\underset{\sim}{0},\Sigma)$

c) From (4-23), 19S has a Wishart distribution with 19 d.f.

4.20 $B(19S)B'$ is a 2×2 matrix distributed as $W_{19}(\cdot\,|\,B\Sigma B')$ with 19 d.f. where

a) $B\Sigma B'$ has

(1,1) entry $= \sigma_{11} + \frac{1}{4}\sigma_{22} + \frac{1}{4}\sigma_{33} - \sigma_{12} - \sigma_{13} + \frac{1}{2}\sigma_{23}$

(1,2) entry $= -\frac{1}{2}\sigma_{14} + \frac{1}{4}\sigma_{24} + \frac{1}{4}\sigma_{34} - \frac{1}{2}\sigma_{15} + \frac{1}{4}\sigma_{25} + \frac{1}{4}\sigma_{35} + \sigma_{16} - \frac{1}{2}\sigma_{26} - \frac{1}{2}\sigma_{36}$

(2,2) entry $= \sigma_{66} + \frac{1}{4}\sigma_{55} + \frac{1}{4}\sigma_{44} - \sigma_{46} - \sigma_{56} + \frac{1}{2}\sigma_{45}$

b)

$$B\Sigma B' = \begin{bmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{31} & \sigma_{33}\end{bmatrix}$$

4.21 (a) $\overline{\boldsymbol{X}}$ is distributed $N_4(\boldsymbol{\mu}, n^{-1}\Sigma)$

(b) $\boldsymbol{X}_1 - \boldsymbol{\mu}$ is distributed $N_4(\mathbf{0}, \Sigma)$ so $(\boldsymbol{X}_1 - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}_1 - \boldsymbol{\mu})$ is distributed as chi-square with p degrees of freedom.

(c) Using Part a),

$$(\overline{\boldsymbol{X}} - \boldsymbol{\mu})'(n^{-1}\Sigma)^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{\mu}) = n(\overline{\boldsymbol{X}} - \boldsymbol{\mu})'\Sigma^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{\mu})$$

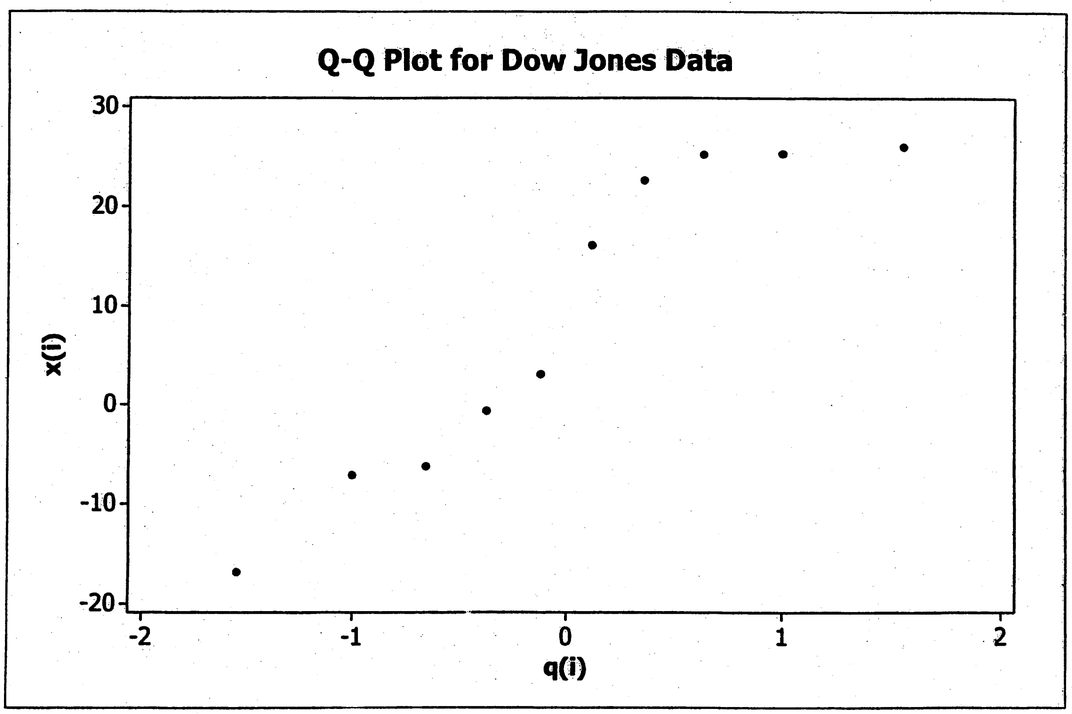
is distributed as chi-square with p degrees of freedom.

(d) Approximately distributed as chi-square with p degrees of freedom. Since the sample size is large, Σ can be replaced by S.

4.22 a) We see that $n = 75$ is a sufficiently large sample (compared with p) and apply Result 4.13 to get $\sqrt{n}(\mathbf{X}-\boldsymbol{\mu})$ is approximately $N_p(\mathbf{0},\Sigma)$ and that $\bar{\mathbf{X}}$ is approximately $N_p(\boldsymbol{\mu},\frac{1}{n}\Sigma)$.

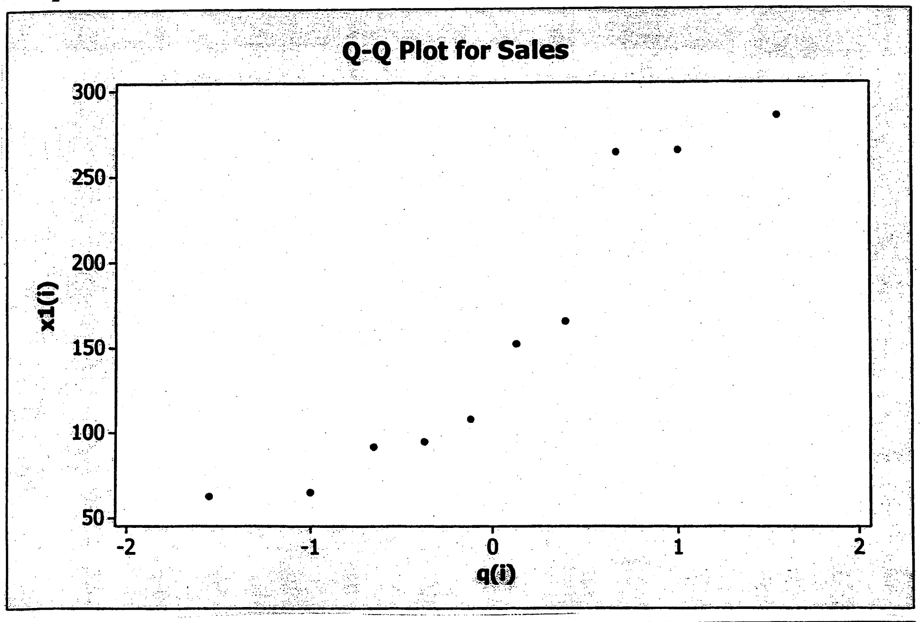
b) By (4-28) we conclude that $\sqrt{n}(\mathbf{X}-\boldsymbol{\mu})'S^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu})$ is approximately χ^2_p.

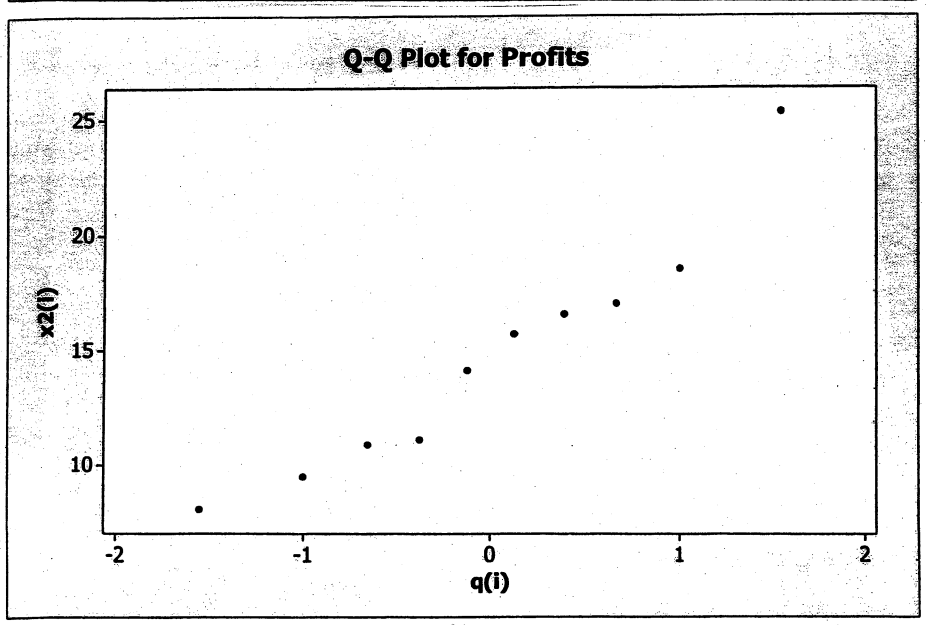
4.23 **(a)** The *Q-Q* plot shown below is not particularly straight, but the sample size $n = 10$ is small. Difficult to determine if data are normally distributed from the plot.



(b) $r_Q = .95$ and $n = 10$. Since $r_Q = .95 > .9351$ (see Table 4.2), cannot reject hypothesis of normality at the 10% level.

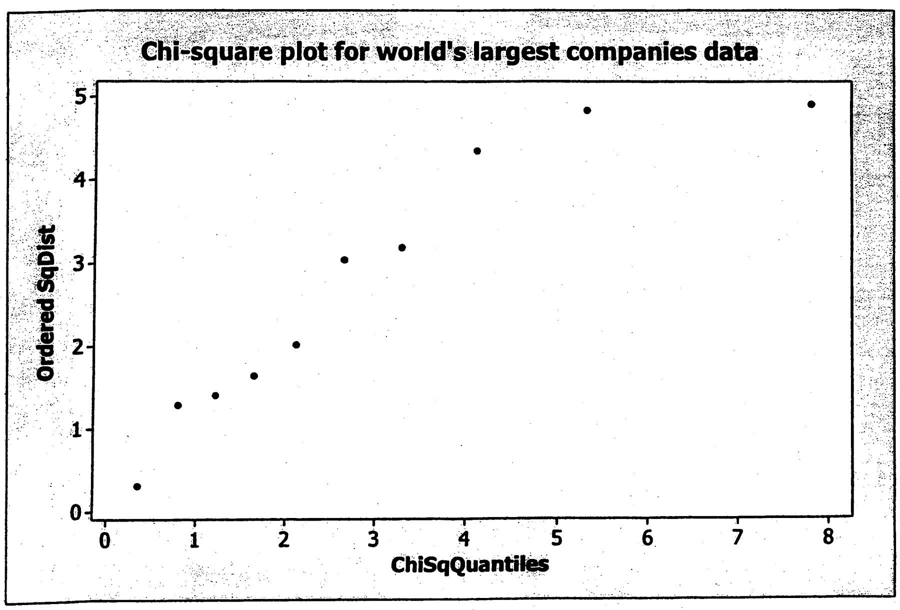
4.24 **(a)** *Q-Q* plots for sales and profits are given below. Plots not particularly straight, although *Q-Q* plot for profits appears to be "straighter" than plot for sales. Difficult to assess normality from plots with such a small sample size ($n = 10$).





(b) The critical point for $n = 10$ when $\alpha = .10$ is .9351. For sales, $r_Q = .940$ and for profits, $r_Q = .968$. Since the values for both of these correlations are greater than .9351, we cannot reject normality in either case.

4.25 The chi-square plot for the world's largest companies data is shown below. The plot is reasonably straight and it would be difficult to reject multivariate normality given the small sample size of $n = 10$. Information leading to the construction of this plot is also displayed.



$$\bar{\mathbf{x}} = \begin{bmatrix} 155.6 \\ 14.7 \\ 710.9 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 7476.5 & 303.6 & -35576 \\ 303.6 & 26.2 & -1053.8 \\ -35576 & -1053.8 & 237054 \end{bmatrix}$$

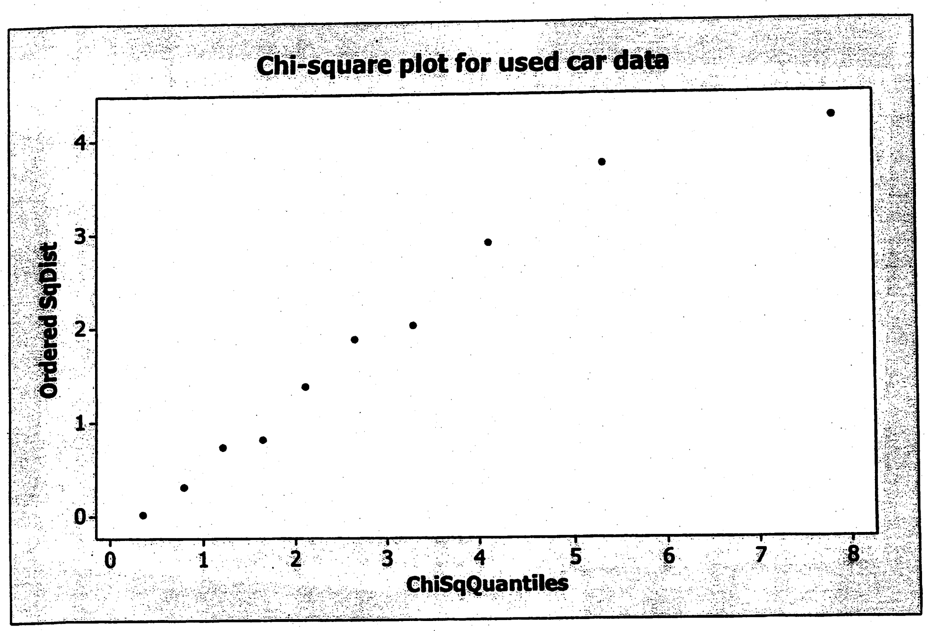
Ordered SqDist	Chi-square Quantiles
.3142	.3518
1.2894	.7978
1.4073	1.2125
1.6418	1.6416
2.0195	2.1095
3.0411	2.6430
3.1891	3.2831
4.3520	4.1083
4.8365	5.3170
4.9091	7.8147

4.26 **(a)** $\bar{\mathbf{x}} = \begin{bmatrix} 5.20 \\ 12.48 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 10.6222 & -17.7102 \\ -17.7102 & 30.8544 \end{bmatrix}$, $\mathbf{S}^{-1} = \begin{bmatrix} 2.1898 & 1.2569 \\ 1.2569 & .7539 \end{bmatrix}$

Thus d_j^2 = 1.8753, 2.0203, 2.9009, .7353, .3105, .0176, 3.7329, .8165, 1.3753, 4.2153

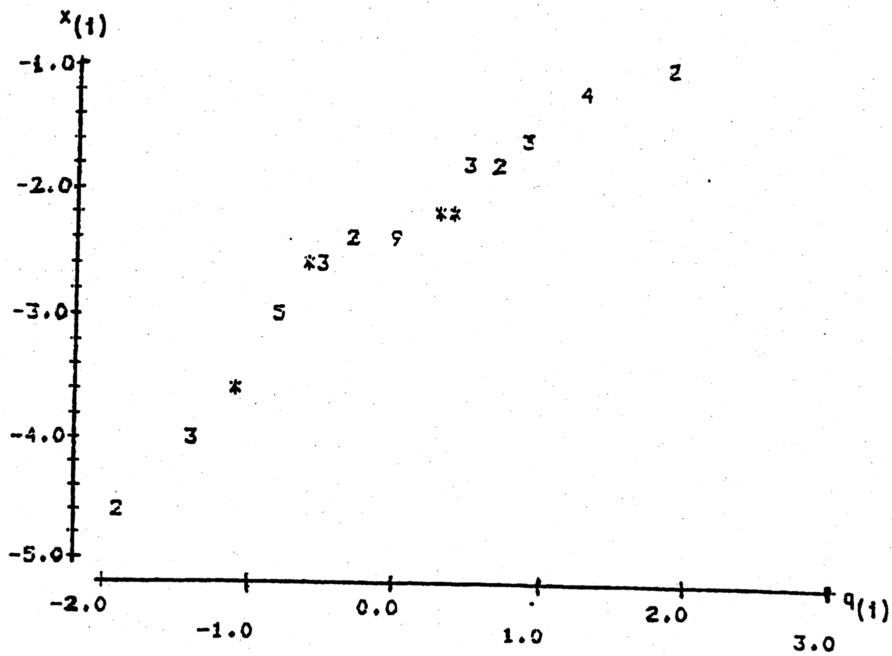
(b) Since $\chi_2^2(.5) = 1.39$, 5 observations (50%) are within the 50% contour.

(c) The chi-square plot is shown below.



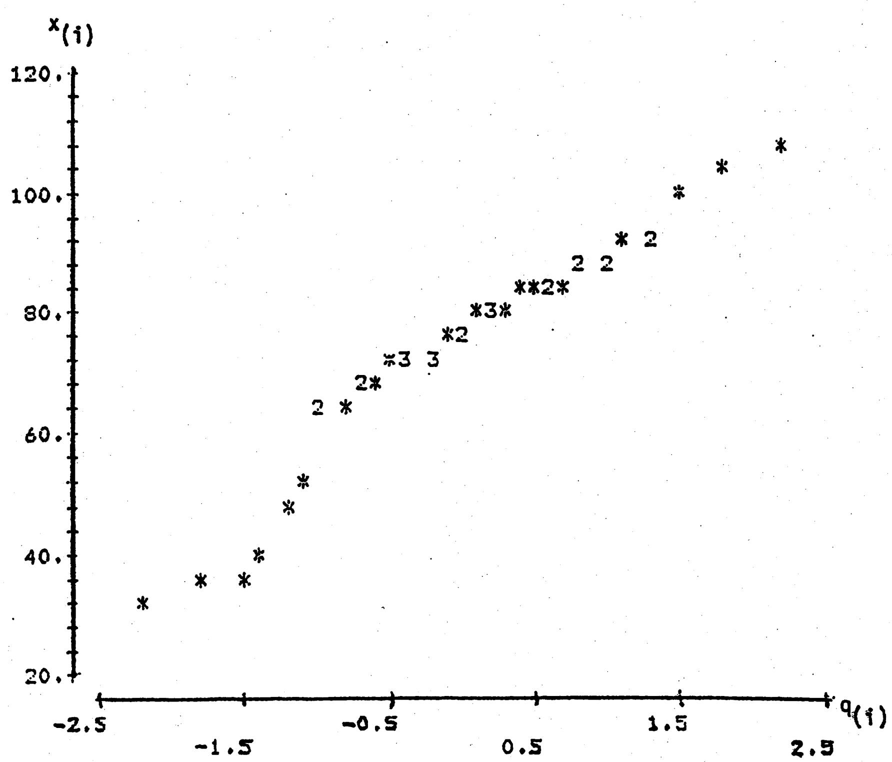
(d) Given the results in parts (b) and (c) and the small number of observations (n = 10), it is difficult to reject bivariate normality.

4.27 Q-Q plot is shown below.



The Q-Q plot is reasonably straight. $r_Q = .978$ ($\lambda=0$)
For $\lambda = 1/4$, $r_Q = .993$ so $\lambda = 1/4$ is a little better choice for the normalizing transformation.

4.28 Q-Q plot is shown below.



Since $r_Q = .970 < .973$ (See Table 4.2 for $n = 40$ and $\alpha = .05$), we would reject the hypothesis of normality at the 5% level.

4.29

(a).

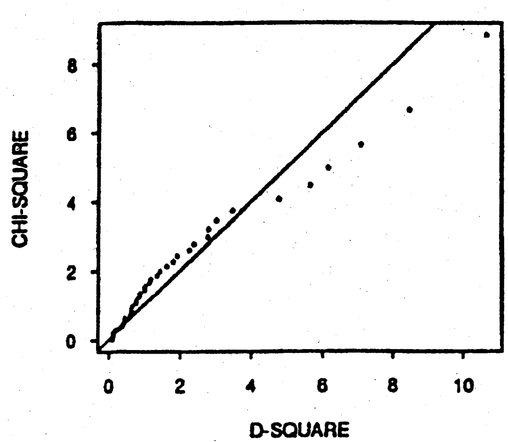
$$\bar{\mathbf{x}} = \begin{pmatrix} 10.046719 \\ 9.4047619 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 11.363531 & 3.126597 \\ & 30.978513 \end{pmatrix}.$$

Generalized distances are as follows;

0.4607	0.6592	2.3771	1.6283	0.4135	0.4761	1.1849
10.6392	0.1388	0.8162	1.3566	0.6228	5.6494	0.3159
0.4135	0.1225	0.8988	4.7647	3.0089	0.6592	2.7741
1.0360	0.7874	3.4438	6.1489	1.0360	0.1388	0.8856
0.1380	2.2489	0.1901	0.4607	1.1472	7.0857	1.4584
0.1225	1.8985	2.7783	8.4731	0.6370	0.7032	1.8014

(b). The number of observations whose generalized distances are less than $\chi_2(0.5) = 1.39$ is 26. So the proportion is 26/42=0.6190.

(c). CHI-SQUARE PLOT FOR (X1 X2)

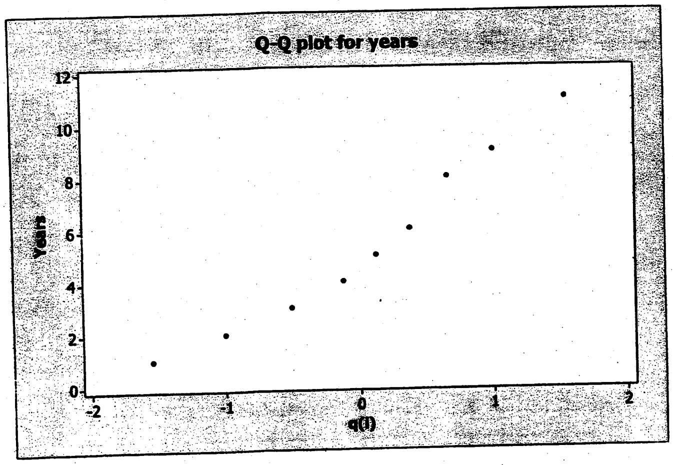


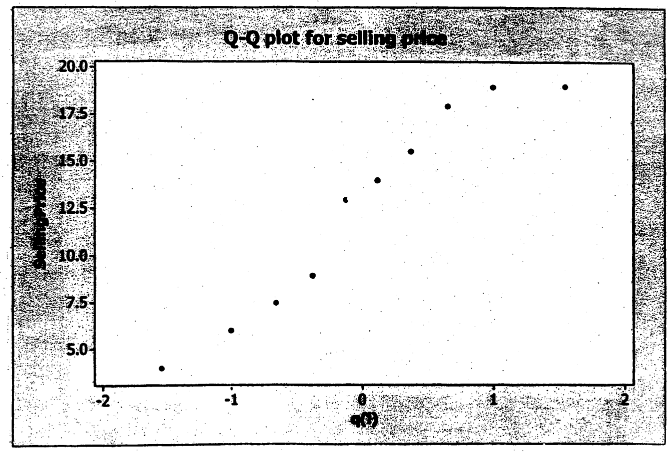
4.30 **(a)** $\hat{\lambda}_1 = 0.5$ but $\hat{\lambda}_1 = 1$ (i.e. no transformation) not ruled out by data. For $\hat{\lambda}_1 = 1$, $r_Q = .981 > .9351$ the critical point for testing normality with $n = 10$ and $\alpha = .10$. We cannot reject the hypothesis of normality at the 10% level (and, consequently, not at the 5% level).

(b) $\hat{\lambda}_1 = 1$ (i.e. no transformation). For $\hat{\lambda}_1 = 1$, $r_Q = .971 > .9351$ the critical point for testing normality with $n = 10$ and $\alpha = .10$. We cannot reject the hypothesis of normality at the 10% level (and, consequently, not at the 5% level).

(c) The likelihood function $l(\lambda_1, \lambda_2)$ is fairly flat in the region of $\lambda_1 = 1, \lambda_2 = 1$ so these values are not ruled out by the data. These results are consistent with those in parts (a) and (b).

Q-Q plots follow.





4.31

The non-multiple-sclerosis group:

	X_1	X_2	X_3	X_4	X_5
r_Q	0.94482*	0.96133*	0.95585*	0.97574*	0.94446*
Transformation	$X_1^{-0.5}$	$X_2^{-3.5}$	$(X_3+0.005)^{0.4}$	$X_4^{-3.4}$	$(X_5+0.005)^{0.32}$

*: significant at 5 % level (the critical point = 0.9826 for n=69).

The multiple-sclerosis group:

	X_1	X_2	X_3	X_4	X_5
r_Q	0.97137	0.97209	0.79523*	0.97869	0.84135*
Transformation	–	–	$(X_3+0.005)^{0.26}$	–	$(X_5+0.005)^{0.21}$

*: significant at 5 % level (the critical point = 0.9640 for n=29).

Transformations of X_3 and X_4 do not improve the approximation to normality very much because there are too many zeros.

4.32

	X_1	X_2	X_3	X_4	X_5	X_6
r_Q	0.98464*	0.94526*	0.9970	0.98098*	0.99057	0.92779*
Transformation	$(X_1+0.005)^{-0.59}$	$X_2^{-0.49}$	–	$X_4^{0.25}$	–	$(X_5+0.005)^{0.51}$

*: significant at 5 % level (the critical point = 0.9870 for n=98).

4.33

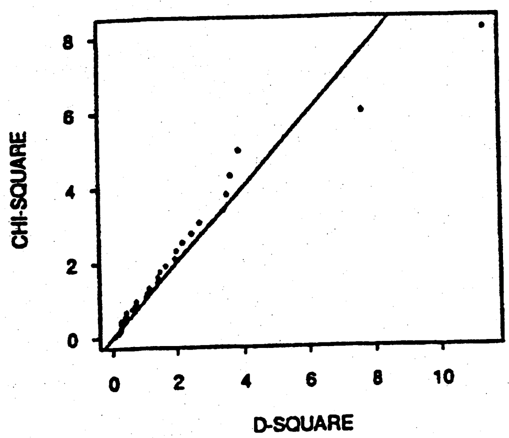
Marginal Normality:

	X_1	X_2	X_3	X_4
r_Q	0.95986*	0.95039*	0.96341	0.98079

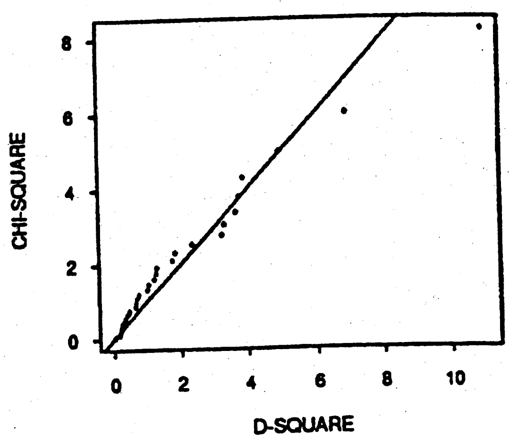
*: significant at 5 % level (the critical point = 0.9652 for n=30).

Bivariate Normality: the χ^2 plots are given in the next page. Those for (X_1, X_2), (X_1, X_3), (X_3, X_4) appear reasonably straight.

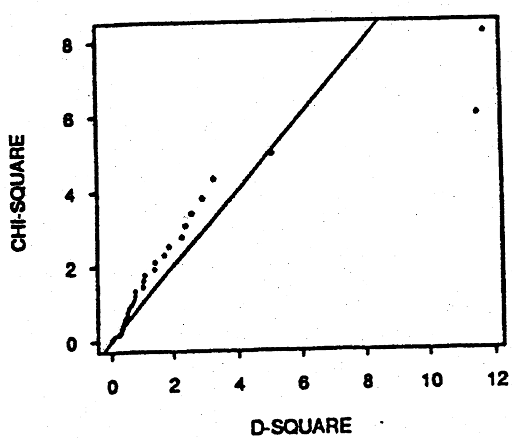
CHI-SQUARE PLOT FOR (X1,X2)



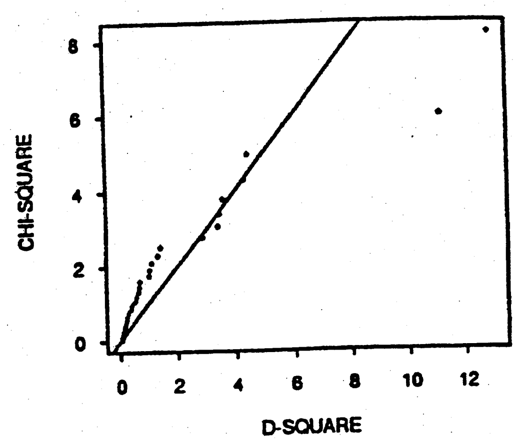
CHI-SQUARE PLOT FOR (X1,X3)



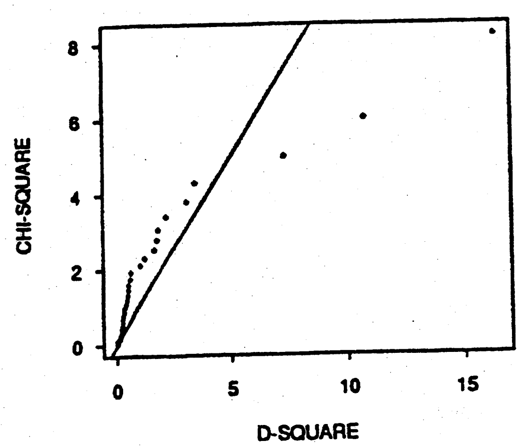
CHI-SQUARE PLOT FOR (X1,X4)



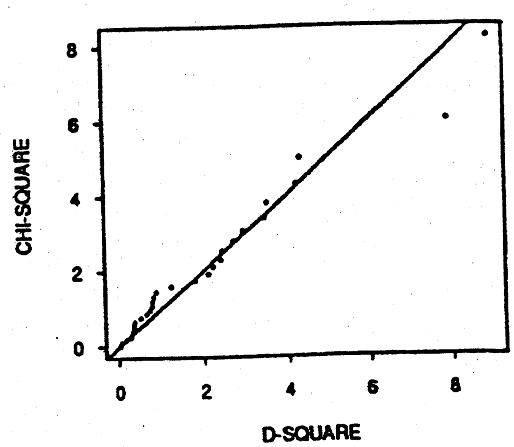
CHI-SQUARE PLOT FOR (X2,X3)



CHI-SQUARE PLOT FOR (X2,X4)



CHI-SQUARE PLOT FOR (X3,X4)



4.34

Marginal Normality:

	X_1	X_2	X_3	X_4	X_5	X_6
r_Q	0.95162*	0.97209	0.98421	0.99011	0.98124	0.99404

*: significant at 5 % level (the critical point = 0.9591 for n=25).

Bivariate Normality: Omitted.

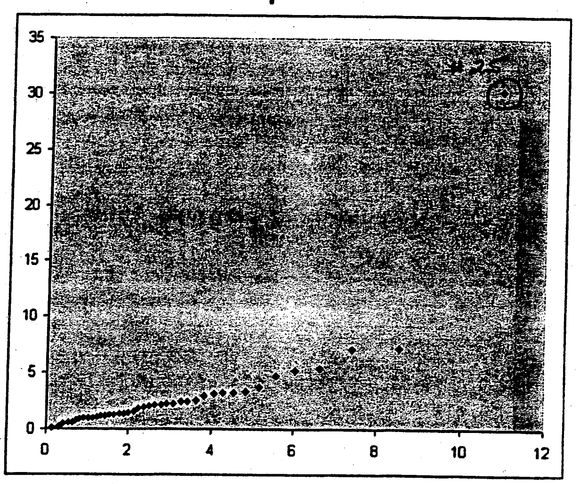
4.35 Marginal normality:

	X_1 (Density)	X_2 (MachDir)	X_3 (CrossDir)
r_Q	.897*	.991	.924*

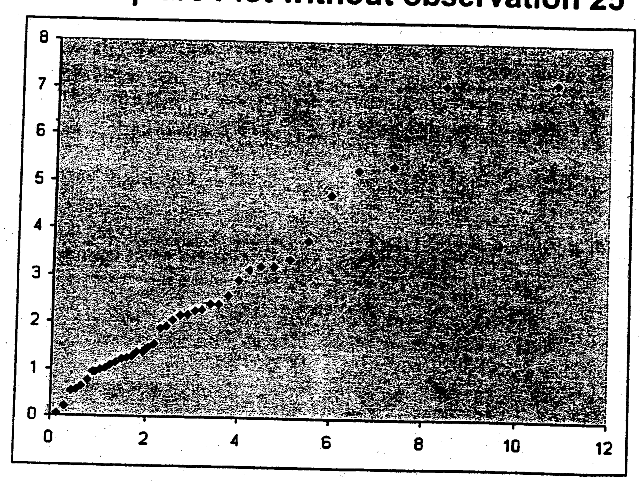
* significant at the 5% level; critical point = .974 for $n = 41$

From the chi-square plot (see below), it is obvious that observation #25 is a multivariate outlier. If this observation is removed, the chi-square plot is considerably more "straight line like" and it is difficult to reject a hypothesis of multivariate normality. Moreover, r_Q increases to .979 for density, it is virtually unchanged (.992) for machine direction and cross direction (.926).

Chi-square Plot



Chi-square Plot without observation 25



4.36 Marginal normality:

	100m	200m	400m	800m	1500m	3000m	Marathon
r_Q	.983	.976*	.969*	.952*	.909*	.866*	.859*

* significant at the 5% level; critical point = .978 for $n = 54$

Notice how the values of r_Q decrease with increasing distance. As the distance increases, the distribution of times becomes increasingly skewed to the right.

The chi-square plot is not consistent with multivariate normality. There are several multivariate outliers.

4.37 Marginal normality:

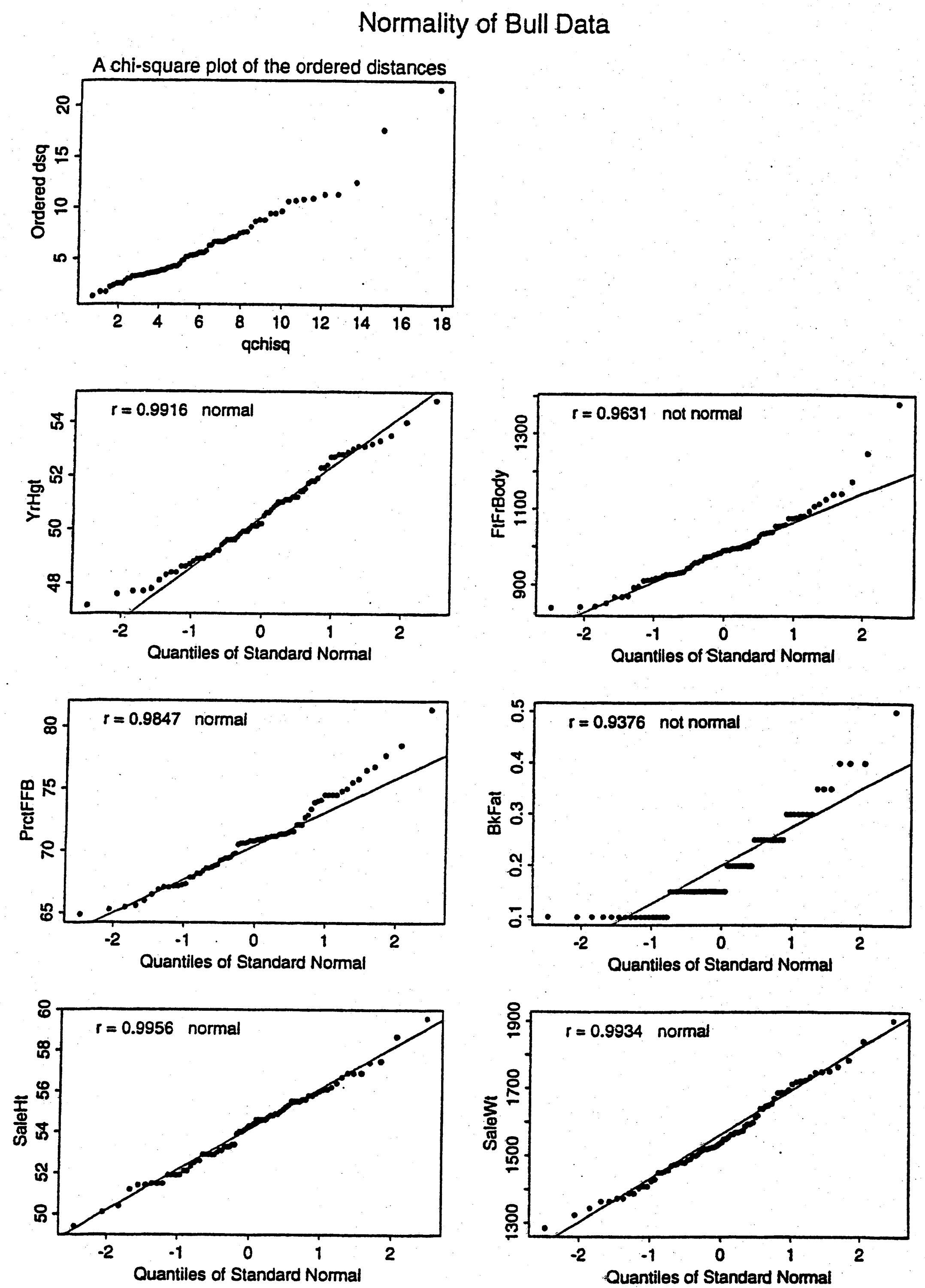
	100m	200m	400m	800m	1500m	3000m	Marathon
r_Q	.989	.985	.984	.968*	.947*	.929*	.921*

* significant at the 5% level; critical point = .978 for $n = 54$

As measured by r_Q, times measured in meters/second for the various distances are more nearly marginally normal than times measured in seconds or minutes (see Exercise 4.36). Notice the values of r_Q decrease with increasing distance. In this case, as the distance increases the distribution of times becomes increasingly skewed to the left.

The chi-square plot is not consistent with multivariate normality. There are several multivariate outliers.

4.38. Marginal and multivariate normality of bull data



XBAR	S					
	YrHgt	FtFrBody	PrctFFB	BkFat	SaleHt	SaleWt
50.5224	2.9980	100.1305	2.9600	-0.0534	2.9831	82.8108
995.9474	100.1305	8594.3439	209.5044	-1.3982	129.9401	6680.3088
70.8816	2.9600	209.5044	10.6917	-0.1430	3.4142	83.9254
0.1967	-0.0534	-1.3982	-0.1430	0.0080	-0.0506	2.4130
54.1263	2.9831	129.9401	3.4142	-0.0506	4.0180	147.2896
1555.2895	82.8108	6680.3088	83.9254	2.4130	147.2896	16850.6618

	Ordered dsq	qchisq		Ordered dsq	qchisq		Ordered dsq	qchisq
1	1.3396	0.7470	26	3.8618	4.0902	51	6.6693	6.8439
2	1.7751	1.1286	27	3.8667	4.1875	52	6.6748	6.9836
3	1.7762	1.3793	28	3.9078	4.2851	53	6.6751	7.1276
4	2.2021	1.5808	29	4.0413	4.3830	54	6.8168	7.2763
5	2.3870	1.7551	30	4.1213	4.4812	55	6.9863	7.4301
6	2.5512	1.9118	31	4.1445	4.5801	56	7.1405	7.5896
7	2.5743	2.0560	32	4.2244	4.6795	57	7.1763	7.7554
8	2.5906	2.1911	33	4.2522	4.7797	58	7.4577	7.9281
9	2.7604	2.3189	34	4.2828	4.8806	59	7.5816	8.1085
10	3.0189	2.4411	35	4.4599	4.9826	60	7.6287	8.2975
11	3.0495	2.5587	36	4.7603	5.0855	61	8.0873	8.4963
12	3.2679	2.6725	37	4.8587	5.1896	62	8.6430	8.7062
13	3.2766	2.7832	38	5.1129	5.2949	63	8.7748	8.9286
14	3.3115	2.8912	39	5.1876	5.4017	64	8.7940	9.1657
15	3.3470	2.9971	40	5.2891	5.5099	65	9.3973	9.4197
16	3.3669	3.1011	41	5.3004	5.6197	66	9.3989	9.6937
17	3.3721	3.2036	42	5.3518	5.7313	67	9.6524	9.9917
18	3.4141	3.3048	43	5.4024	5.8449	68	10.6254	10.3191
19	3.5279	3.4049	44	5.5938	5.9605	69	10.6958	10.6829
20	3.5453	3.5041	45	5.6060	6.0783	70	10.8037	11.0936
21	3.6097	3.6027	46	5.6333	6.1986	71	10.9273	11.5665
22	3.6485	3.7007	47	5.7754	6.3215	72	11.3006	12.1263
23	3.6681	3.7983	48	6.2524	6.4472	73	11.3216	12.8160
24	3.7236	3.8957	49	6.3264	6.5760	74	12.4744	13.7225
25	3.7395	3.9929	50	6.6491	6.7081	75	17.6149	15.0677
						76	21.5751	17.8649

From Table 4.2, with $\alpha = 0.05$ and $n = 76$, the critical point for the $Q-Q$ plot correlation coefficient test for normality is 0.9839. We reject the hypothesis of multivariate normality at $\alpha = 0.05$, because some marginals are not normal.

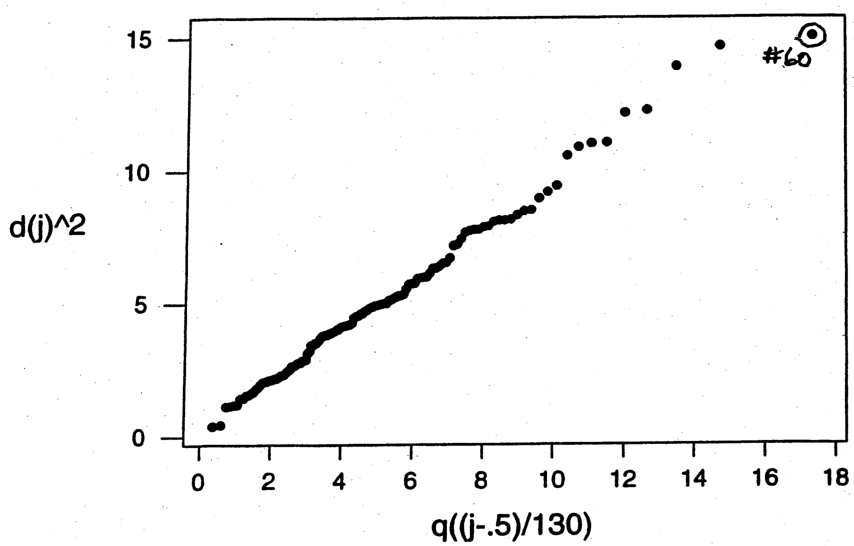
4.39 **(a)** Marginal normality:

	independence	support	benevolence	conformity	leadership
r_Q	.991	.993	.997	.997	.984*

* significant at the 5% level; critical point = .990 for n = 130

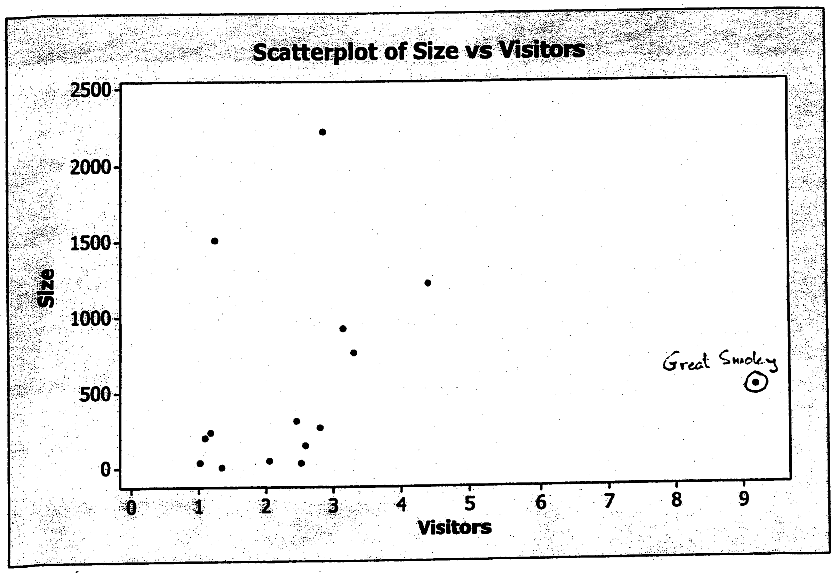
(b) The chi-square plot is shown below. Plot is straight with the exception of observation #60. Certainly if this observation is deleted would be hard to argue against multivariate normality.

Chi-square plot for indep, supp, benev, conform, leader

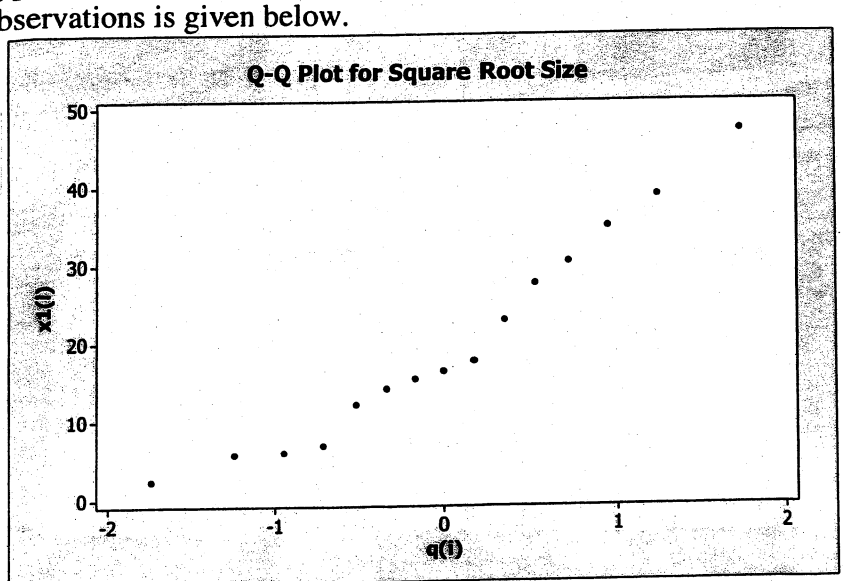


(c) Using the r_Q statistic, normality is rejected at the 5% level for leadership. If leadership is transformed by taking the square root (i.e. $\hat{\lambda} = 0.5$), r_Q = .998 and we cannot reject normality at the 5% level.

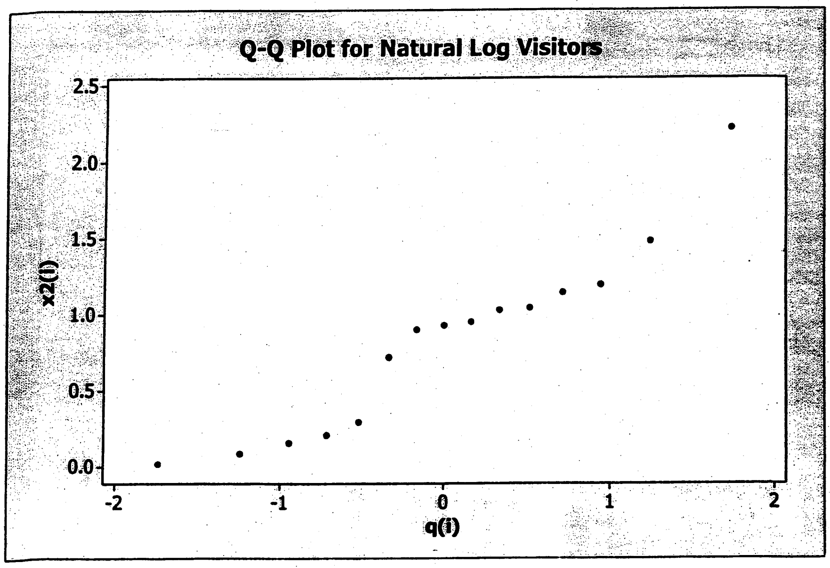
4.40 **(a)** Scatterplot is shown below. Great Smoky park is an outlier.



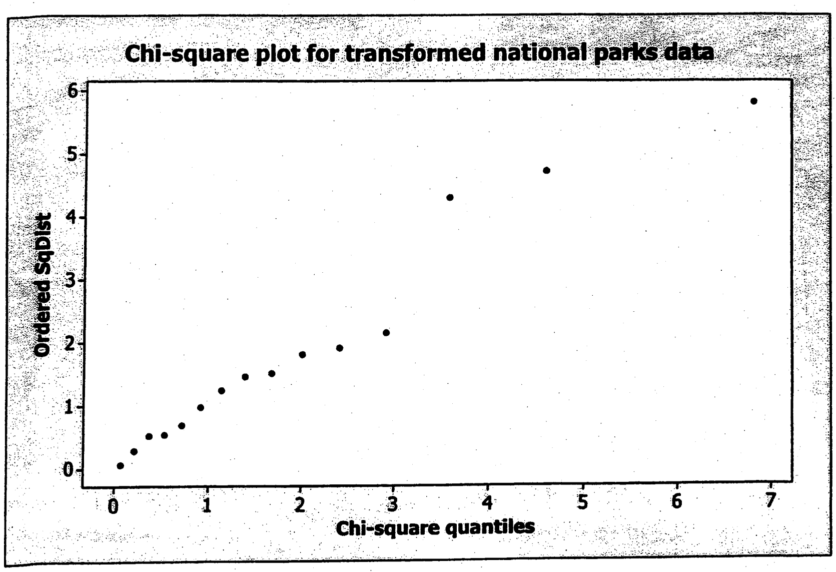
(b) The power transformation $\hat{\lambda}_1 = 0.5$ (i.e. square root) makes the size observations more nearly normal. $r_Q = .904$ before transformation and $r_Q = .975$ after transformation. The 5% critical point with $n = 15$ for the hypothesis of normality is .9389. The Q-Q plot for the transformed observations is given below.



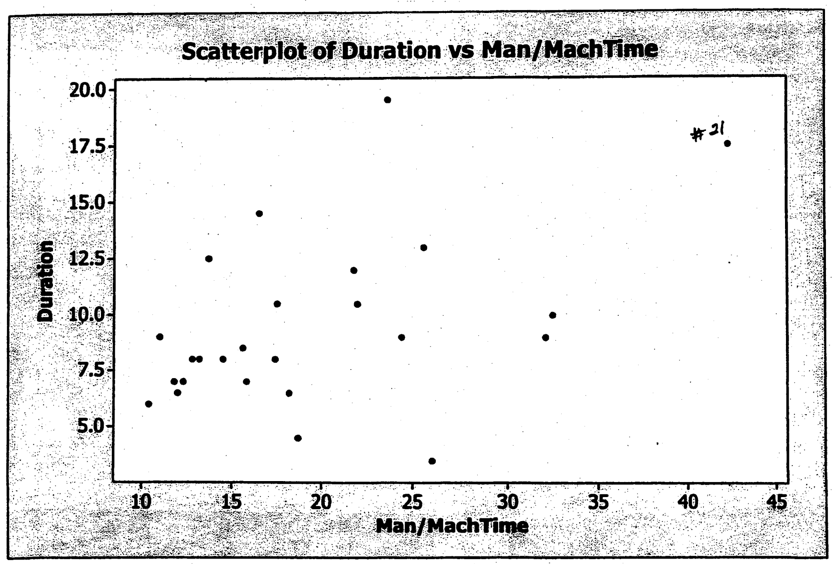
(c) The power transformation $\hat{\lambda}_2 = 0$ (i.e. logarithm) makes the visitor observations more nearly normal. $r_Q = .837$ before transformation and $r_Q = .960$ after transformation. The 5% critical point with $n = 15$ for the hypothesis of normality is .9389. The Q-Q plot for the transformed observations is given next.



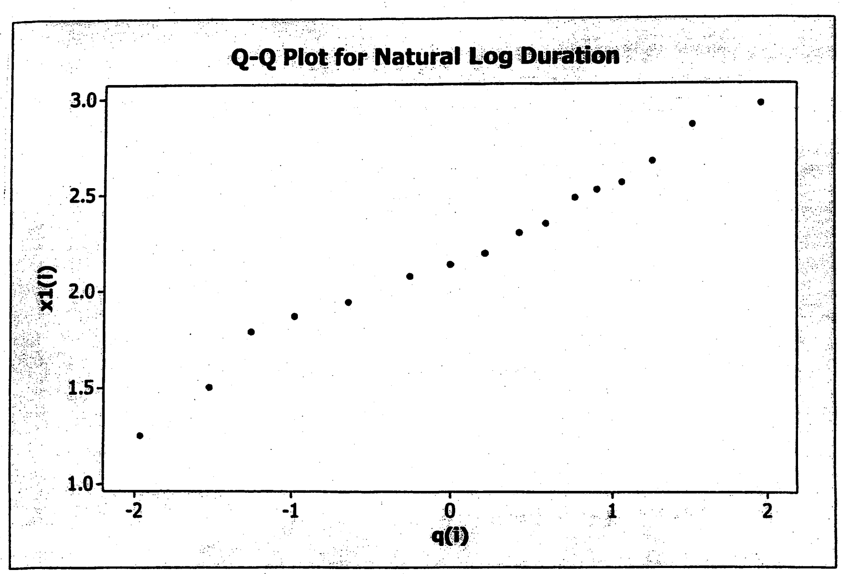
(d) A chi-square plot for the transformed observations is shown below. Given the small sample size ($n = 15$), the plot is reasonably straight and it would be hard to reject bivariate normality.



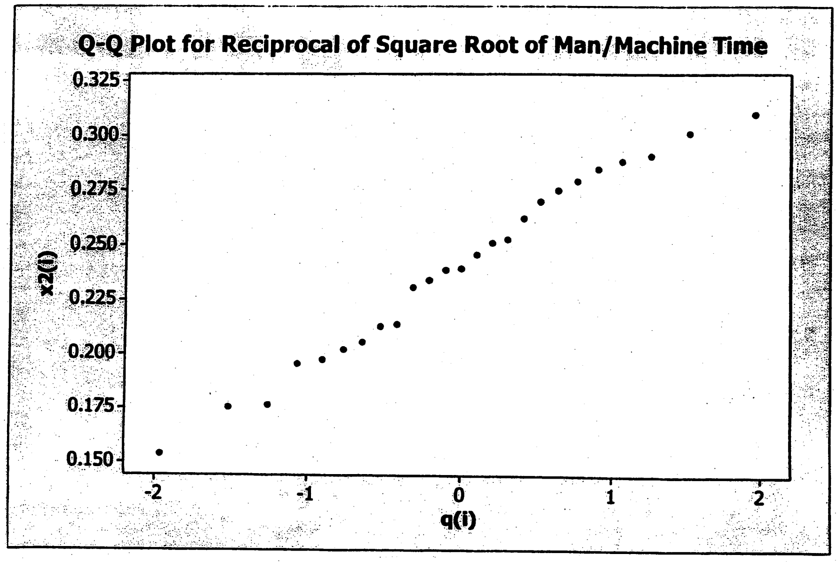
4.41 **(a)** Scatterplot is shown below. There do not appear to be any outliers with the possible exception of observation #21.



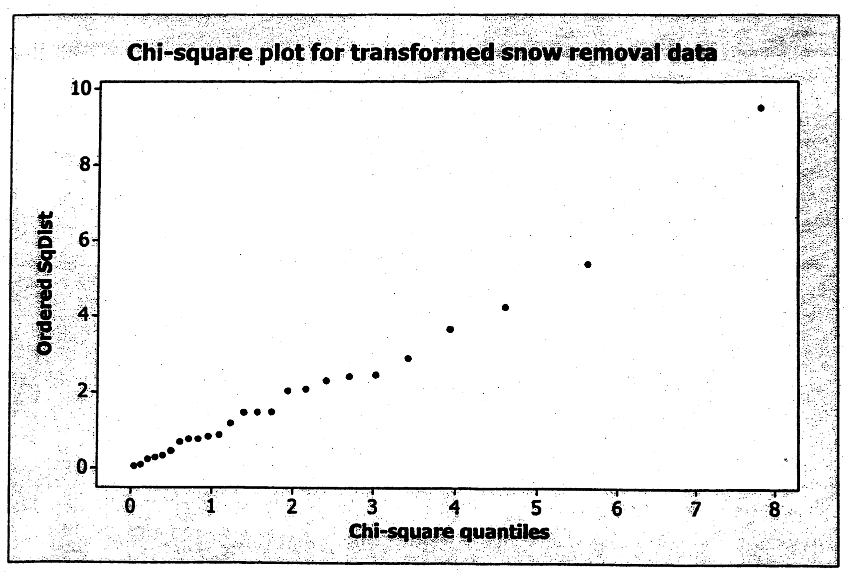
(b) The power transformation $\hat{\lambda}_1 = 0$ (i.e. logarithm) makes the duration observations more nearly normal. $r_Q = .958$ before transformation and $r_Q = .989$ after transformation. The 5% critical point with $n = 25$ for the hypothesis of normality is .9591. The *Q-Q* plot for the transformed observations is given below.



(c) The power transformation $\hat{\lambda}_2 = -0.5$ (i.e. reciprocal of square root) makes the man/machine time observations more nearly normal. $r_Q = .939$ before transformation and $r_Q = .991$ after transformation. The 5% critical point with $n = 25$ for the hypothesis of normality is .9591. The *Q-Q* plot for the transformed observations is given next.



(d) A chi-square plot for the transformed observations is shown below. The plot is straight and it would be difficult to reject bivariate normality.



Chapter 5

5.1 a) $\underset{\sim}{\bar{x}} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$; $S = \begin{bmatrix} 8 & -10/3 \\ -10/3 & 2 \end{bmatrix}$

$T^2 = 150/11 = 13.64$

b) T^2 is $3F_{2,2}$ (see (5-5))

c) $H_0: \underset{\sim}{\mu}' = [7,11]$

$\alpha = .05$ so $F_{2,2}(.05) = 19.00$

Since $T^2 = 13.64 < 3F_{2,2}(.05) = 3(19) = 57$; do not reject H_0 at the $\alpha = .05$ level

5.3 a) $$T^2 = \frac{(n-1)\left|\sum_{j=1}^{n}(\underset{\sim}{x}_j-\underset{\sim}{\mu}_0)(\underset{\sim}{x}_j-\underset{\sim}{\mu}_0)'\right|}{\left|\sum_{j=1}^{n}(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})'\right|} - (n-1) = \frac{3(244)}{44} - 3 = 13.64$$

b) $$\Lambda = \left(\frac{\left|\sum_{j=1}^{n}(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})'\right|}{\left|\sum_{j=1}^{n}(\underset{\sim}{x}_j-\underset{\sim}{\mu}_0)(\underset{\sim}{x}_j-\underset{\sim}{\mu}_0)'\right|}\right)^{n/2} = \left(\frac{44}{244}\right)^2 = .0325$$

Wilks' lambda = $\Lambda^{2/n} = \Lambda^{1/2} = \sqrt{.0325} = .1803$

5.5 $H_0: \underset{\sim}{\mu}' = [.55,.60]$; $T^2 = 1.17$

$\alpha = .05$; $F_{2,40}(.05) = 3.23$

Since $T^2 = 1.17 < \frac{2(41)}{40} F_{2,40}(.05) = 2.05(3.23) = 6.62$,

we do not reject H_0 at the $\alpha = .05$ level. The result is consistent with the 95% confidence ellipse for $\underset{\sim}{\mu}$ pictured in Figure 5.1 since $\underset{\sim}{\mu}' = [.55,.60]$ is <u>inside</u> the ellipse.

5.8 $\underset{\sim}{a} = S^{-1}(\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}_0) = \begin{bmatrix} 227.273 & -181.818 \\ -181.818 & 212.121 \end{bmatrix} \left(\begin{bmatrix} .564 \\ .603 \end{bmatrix} - \begin{bmatrix} .55 \\ .60 \end{bmatrix} \right)$

$$= \begin{bmatrix} 2.636 \\ -1.909 \end{bmatrix}$$

$$t^2 = \frac{n(\underset{\sim}{a}'(\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}_0))^2}{\underset{\sim}{a}' S \underset{\sim}{a}} = \frac{42\left([2.636 \;\; -1.909] \begin{bmatrix} .014 \\ .003 \end{bmatrix}\right)^2}{[2.636 \;\; -1.909] \begin{bmatrix} .0144 & .0117 \\ .0117 & .0146 \end{bmatrix} \begin{bmatrix} 2.636 \\ -1.909 \end{bmatrix}} = 1.31 = T^2$$

5.9 a) Large sample 95% T^2 simultaneous confidence intervals:

Weight:	(69.56, 121.48)	Girth:	(83.49, 103.29)
Body length:	(152.17, 176.59)	Head length:	(16.55, 19.41)
Neck:	(49.61, 61.77)	Head width:	(29.04, 33.22)

b) 95% confidence region determined by all μ_1, μ_4 such that

$$(95.52 - \mu_1, 93.39 - \mu_4) \begin{bmatrix} .002799 & -.006927 \\ -.006927 & .019248 \end{bmatrix} \begin{pmatrix} 95.52 - \mu_1 \\ 93.39 - \mu_4 \end{pmatrix} \le 12.59/61 = .2064$$

Beginning at the center $\underset{\sim}{\bar{x}}' = (95.52, 93.39)$, the axes of the 95% confidence ellipsoid are:

major axis $\pm\sqrt{3695.52}\sqrt{12.59}\begin{pmatrix} .939 \\ .343 \end{pmatrix}$

minor axis $\pm\sqrt{45.92}\sqrt{12.59}\begin{pmatrix} -.343 \\ .939 \end{pmatrix}$

(See confidence ellipsoid in part d.)

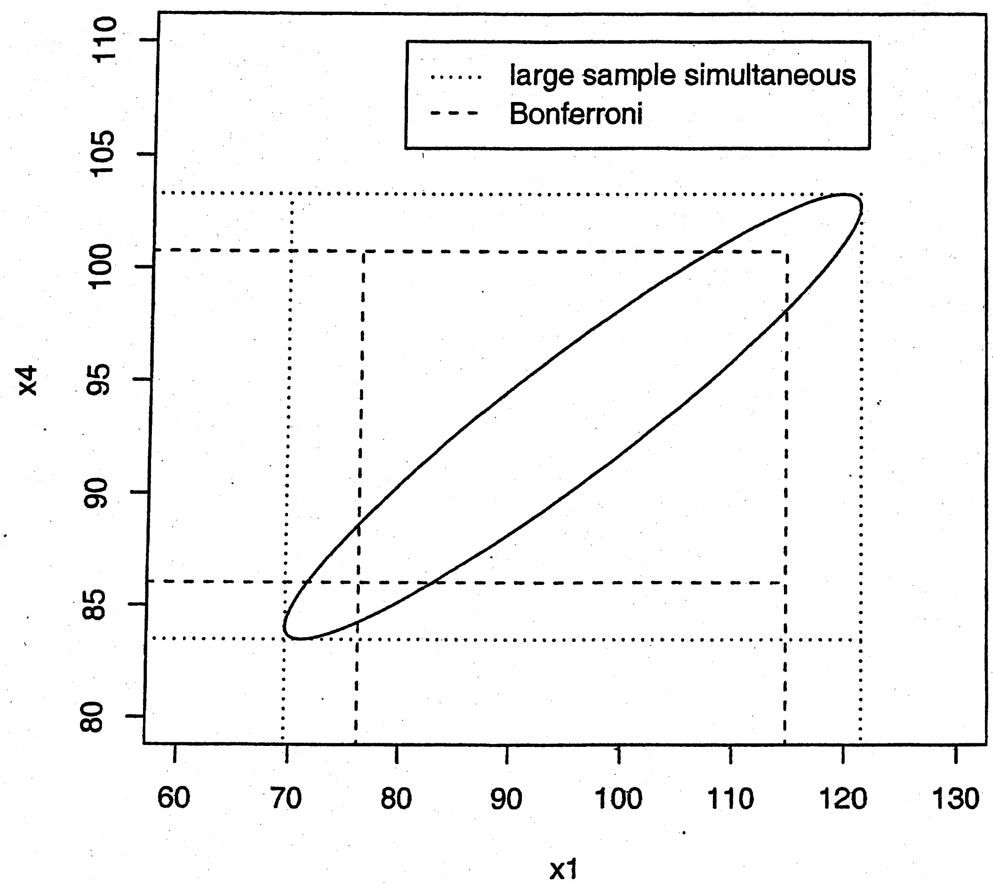
c) Bonferroni 95% simultaneous confidence intervals ($m = 6$):
$t_{60}(.025/6) = 2.728$ (Alternative multiplier is $z(.025/6) = 2.638$)

Weight:	(75.56, 115.48)	Girth:	(86.27, 100.51)
Body length:	(155.00, 173.76)	Head length:	(16.88, 19.08)
Neck:	(51.01, 60.37)	Head width:	(29.52, 32.74)

d) Because of the high positive correlation between weight (X_1) and girth (X_4), the 95% confidence ellipse is smaller, more informative, than the 95% Bonferroni rectangle.

5.9 (Continued)

Large sample 95% confidence regions.



e) Bonferroni 95% simultaneous confidence interval for difference between mean head width and mean head length ($\mu_6 - \mu_5$) follows.
($m = 7$ to allow for new statement and statements about individual means):
$t_{60}(.025/7) = 2.783$ (Alternative multiplier is $z(.025/7) = 2.690$)

$$\bar{x}_6 - \bar{x}_5 \pm t_{60}(.0036)\sqrt{\frac{s_{66} - 2s_{56} + s_{55}}{n}} = (31.13 - 17.98) \pm 2.783\sqrt{\frac{21.26 - 2(13.88) + 9.95}{61}}$$

or $$12.49 \le \mu_6 - \mu_5 \le 13.81$$

5.10 **a)** 95% T^2 simultaneous confidence intervals:

Lngth2: (130.65, 155.93) Lngth4: (160.33, 185.95)
Lngth3: (127.00, 191.58) Lngth5: (155.37, 198.91)

b) 95% T^2 simultaneous intervals for change in length (ΔLngth):

ΔLngth2-3: (–21.24, 53.24)
ΔLngth3-4: (–22.70, 50.42)
ΔLngth4-5: (–20.69, 28.69)

c) 95% confidence region determined by all μ_{2-3}, μ_{4-5} such that

$$(16-\mu_{2-3}, 4-\mu_{4-5})\begin{bmatrix} .011024 & .009386 \\ .009386 & .025135 \end{bmatrix}\begin{pmatrix} 16-\mu_{2-3} \\ 4-\mu_{4-5} \end{pmatrix} \le 72.96/7 = 10.42$$

where μ_{2-3} is the mean increase in length from year 2 to 3, and μ_{4-5} is the mean increase in length from year 4 to 5.

Beginning at the center $\underset{\sim}{\bar{x}}' = (16,4)$, the axes of the 95% confidence ellipsoid are:

major axis $\pm\sqrt{157.8}\sqrt{72.96}\begin{pmatrix} .895 \\ -.447 \end{pmatrix}$

minor axis $\pm\sqrt{33.53}\sqrt{72.96}\begin{pmatrix} .447 \\ .895 \end{pmatrix}$

(See confidence ellipsoid in part e.)

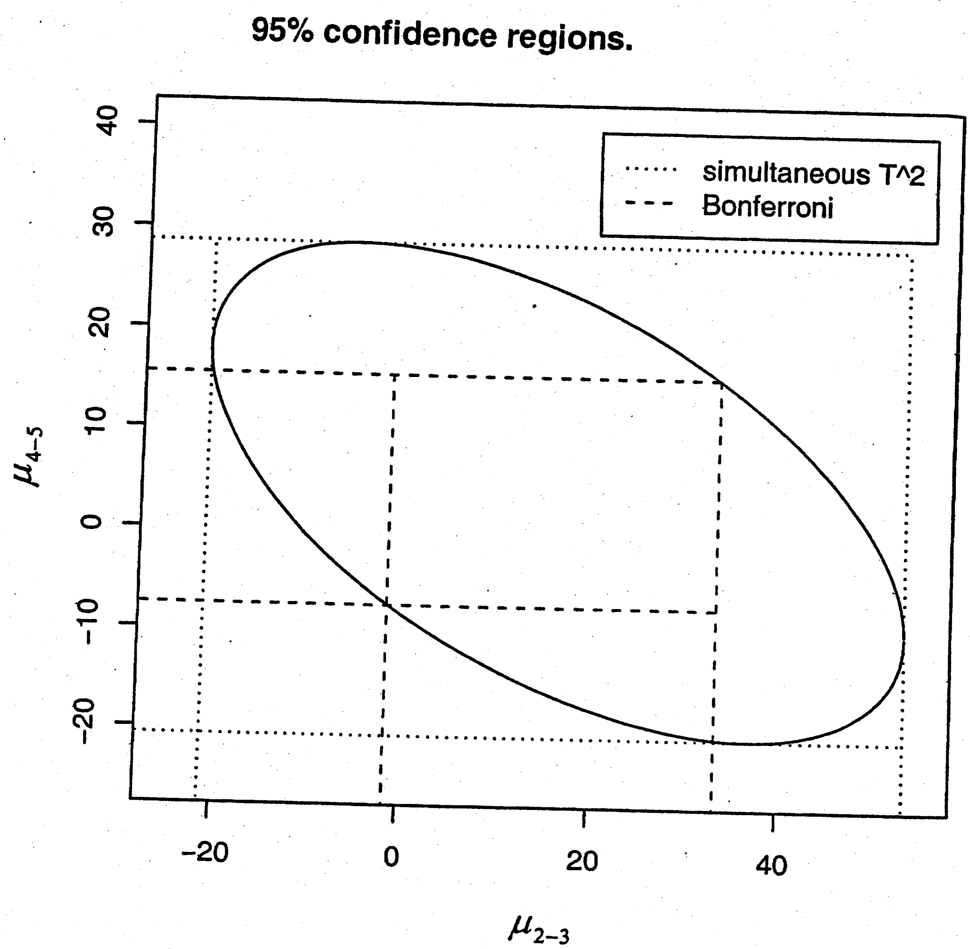
d) Bonferroni 95% simultaneous confidence intervals ($m = 7$):

Lngth2: (137.37, 149.21) Lngth4: (167.14, 179.14)
Lngth3: (144.18, 174.40) Lngth5: (166.95, 187.33)

ΔLngth2-3: (–1.43, 33.43) ΔLngth4-5: (–7.55, 15.55)
ΔLngth3-4: (–3.25, 30.97)

5.10 (Continued)

e) The Bonferroni 95% confidence rectangle is much smaller and more informative than the 95% confidence ellipse.



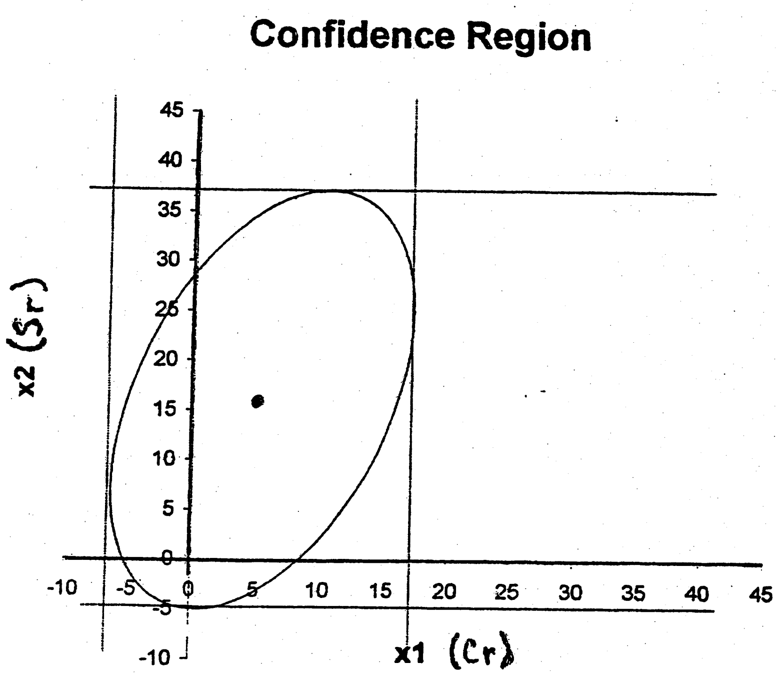
5.11 a) $\bar{\mathbf{x}}' = [5.1856,\ 16.0700]$

$$S = \begin{bmatrix} 176.0042 & 287.2412 \\ 287.2412 & 527.8493 \end{bmatrix}; \quad S^{-1} = \begin{bmatrix} .0508 & -.0276 \\ -.0276 & .0169 \end{bmatrix}$$

Eigenvalues and eigenvectors of S:

$$\hat{\lambda}_1 = 688.759 \qquad \hat{\mathbf{e}}_1' = (.49, .87)$$
$$\hat{\lambda}_2 = 15.094 \qquad \hat{\mathbf{e}}_2' = (.87, -.49)$$

$$\frac{(n-1)p}{(n-p)} F_{p,n-p}(.10) = \frac{8(2)}{7} F_{2,7}(.10) = \frac{16}{7}(3.26) = 7.45$$



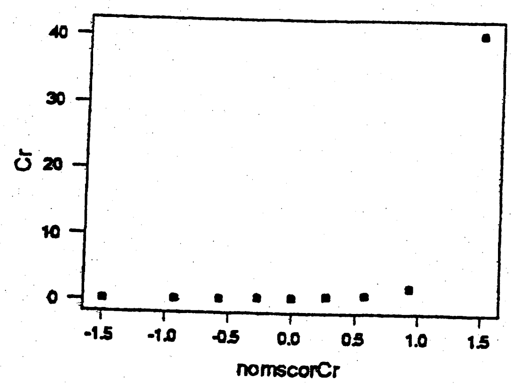
b) 90% T^2 intervals for the full data set:

Cr: (-6.88, 17.25) Sr: (-4.83, 36.97)

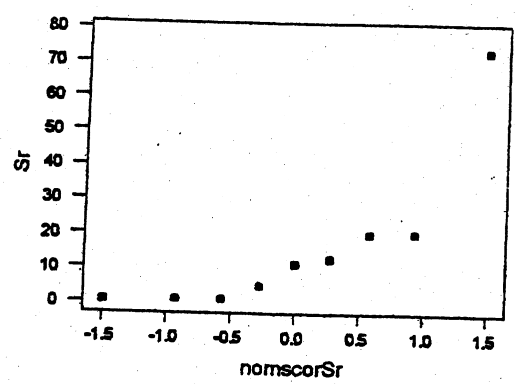
$[.30, 10]'$ is a plausible value for $\boldsymbol{\mu}$

5.11 (Continued)

c) Q-Q plots for the marginal distributions of both variables



Since r = 0.627 we reject the hypothesis of normality for this variable at $\alpha = 0.01$



Since r = 0.818 we reject the hypothesis of normality for this variable at $\alpha = 0.01$

d) With data point (40.53, 73.68) removed,

$$\bar{\mathbf{x}}' = [.7675, 8.8688]; \qquad S = \begin{bmatrix} .3786 & 1.0303 \\ 1.0303 & 69.8598 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 2.7518 & -.0406 \\ -.0406 & .0149 \end{bmatrix}$$

$$\frac{(n-1)p}{(n-p)} F_{p,n-p}(.10) = \frac{7(2)}{6} F_{2,6}(.10) = \frac{14}{6}(3.46) = 8.07$$

90% T^2 intervals: Cr: (.15, 1.39) Sr: (.47, 17.27)

5.12 Initial estimates are

$$\tilde{\mu} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} 0.5 & 0.0 & 0.5 \\ & 2.0 & 0.0 \\ & & 1.5 \end{bmatrix}.$$

The first revised estimates are

$$\tilde{\mu} = \begin{bmatrix} 4.0833 \\ 6.0000 \\ 2.2500 \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} 0.6042 & 0.1667 & 0.8125 \\ & 2.500 & 0.0 \\ & & 1.9375 \end{bmatrix}.$$

5.13 The χ^2 distribution with 3 degrees of freedom.

5.14 Length of one-at-a time t-interval / Length of Bonferroni interval $= t_{n-1}(\alpha/2)/t_{n-1}(\alpha/2m)$.

	m		
n	2	4	10
15	0.8546	0.7489	0.6449
25	0.8632	0.7644	0.6678
50	0.8691	0.7749	0.6836
100	0.8718	0.7799	0.6910
∞	0.8745	0.7847	0.6983

5.15

(a).

$$\begin{aligned} E(X_{ij}) &= (1)p_i + (0)(1-p_i) = p_i. \\ Var(X_{ij}) &= (1-p_i)^2 p_i + (0-p_i)^2(1-p_i) = p_i(1-p_i) \end{aligned}$$

(b). $Cov(X_{ij}, X_{kj}) = E(X_{ij}X_{ik}) - E(X_{ij})E(X_{kj}) = 0 - p_ip_k = -p_ip_k$.

5.16

(a). Using $\hat{p}_j \pm \sqrt{\chi_4^2(0.05)}\sqrt{\hat{p}_j(1-\hat{p}_j)/n}$, the 95 % confidence intervals for p_1, p_2, p_3, p_4, p_5 are
(0.221, 0.370), (0.258, 0.412), (0.098, 0.217), (0.029, 0.112), (0.084, 0.198) respectively.

(b). Using $\hat{p}_1 - \hat{p}_2 \pm \sqrt{\chi_4^2(0.05)}\sqrt{(\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2) - 2\hat{p}_1\hat{p}_2)/n}$, the 95 % confidence interval for $p_1 - p_2$ is $(-0.118, 0.0394)$. There is no significant difference in two proportions.

5.17

$\hat{p}_1 = 0.585$, $\hat{p}_2 = 0.310$, $\hat{p}_3 = 0.105$. Using $\hat{p}_j \pm \sqrt{\chi_3^2(0.05)}\sqrt{\hat{p}_j(1-\hat{p}_j)/n}$, the 95 % confidence intervals for p_1, p_2, p_3 are (0.488, 0.682), (0.219, 0.401), (0.044, 0.166), respectively.

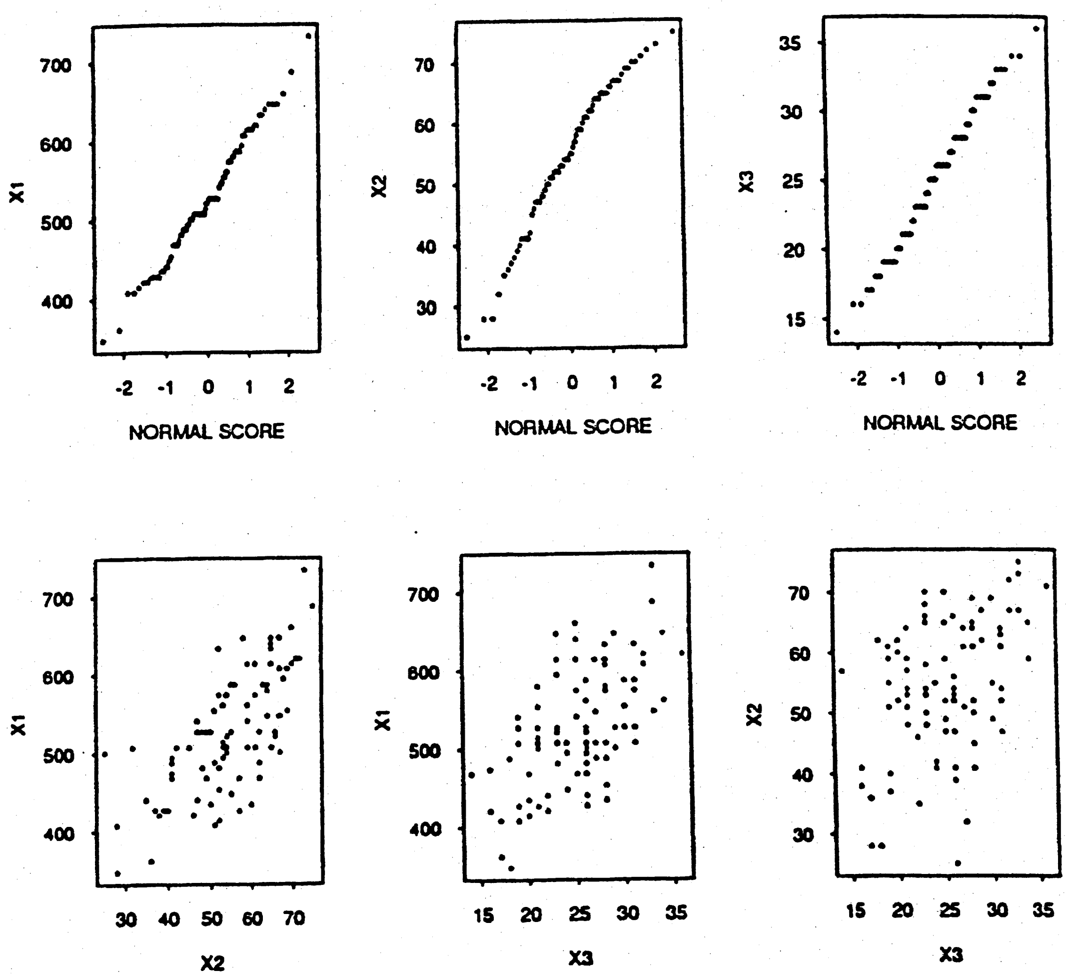
5.18

(a). Hotelling's $T^2 = 223.31$. The critical point for the statistic ($\alpha = 0.05$) is 8.33. We reject $H_0 : \mu = (500, 50, 30)'$. That is, The group of students represented by scores are significantly different from average college students.

(b). The lengths of three axes are 23.730, 2.473, 1.183. And directions of corresponding axes are

$$\begin{pmatrix} 0.994 \\ 0.103 \\ 0.038 \end{pmatrix}, \quad \begin{pmatrix} -0.104 \\ 0.995 \\ 0.006 \end{pmatrix}, \quad \begin{pmatrix} -0.037 \\ -0.010 \\ 0.999 \end{pmatrix}.$$

(c). Data look fairly normal.



5.19 a) The summary statistics are:

$$n = 30, \quad \bar{\mathbf{x}} = \begin{bmatrix} 1860.50 \\ 8354.13 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 124055.17 & 361621.03 \\ 361621.03 & 3486330.90 \end{bmatrix}$$

where S has eigenvalues and eigenvectors

$\lambda_1 = 3407292 \qquad \underset{\sim}{e}_1' = [.105740,\ .994394]$

$\lambda_2 = 82748 \qquad \underset{\sim}{e}_2' = [.994394,-.105740]$

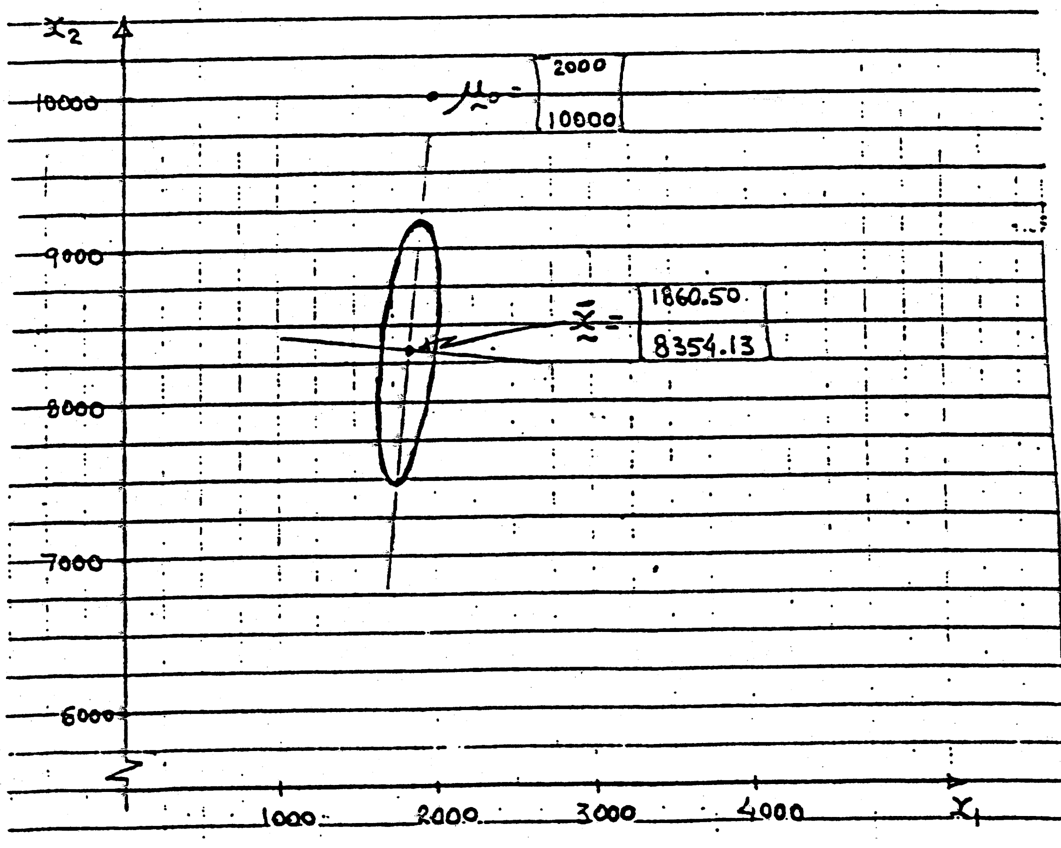
Then, since $\frac{1}{n}\frac{p(n-1)}{n-p}F_{p,n-p}(\alpha) = \frac{1}{30}\ \frac{2(29)}{28}F_{2,28}(.05) = .2306$,

a 95% confidence region for $\underset{\sim}{\mu}$ is given by the set of $\underset{\sim}{\mu}$

$$[1860.50-\mu_1,\ 8354.13-\mu_2]\begin{bmatrix}124055.17 & 361621.03\\ 361621.03 & 3486330.90\end{bmatrix}^{-1}\begin{bmatrix}1860.50-\mu_1\\ 8354.13-\mu_2\end{bmatrix}$$

$$\leq .2306$$

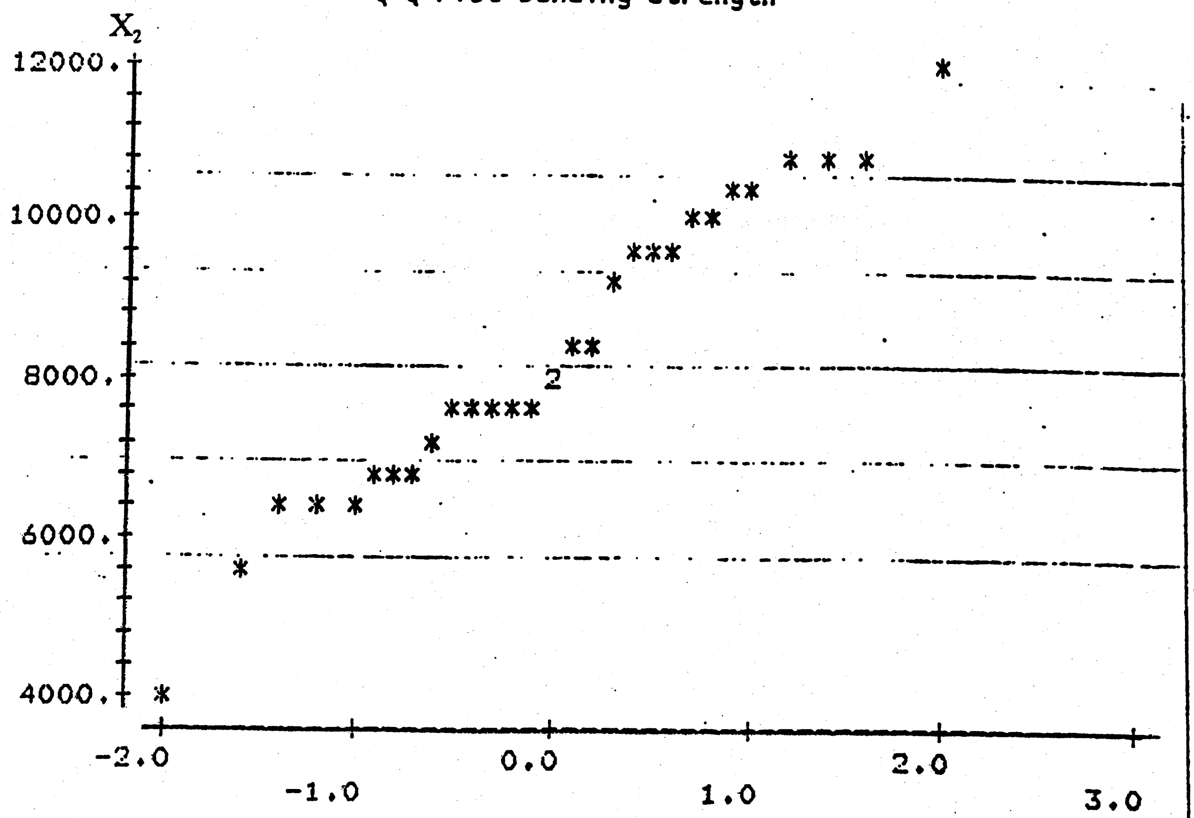
The half lengths of the axes of this ellipse are $\sqrt{.2306}\,\sqrt{\lambda_1} = 886.4$ and $\sqrt{.2306}\,\sqrt{\lambda_2} = 138.1$. Therefore the ellipse has the form



b) Since $\underset{\sim}{\mu}_0 = [2000, 10000]'$ does not fall within the 95% confidence ellipse, we would reject the hypothesis $H_0: \underset{\sim}{\mu} = \underset{\sim}{\mu}_0$ at the 5% level. Thus, the data analyzed are <u>not</u> consistent with these values.

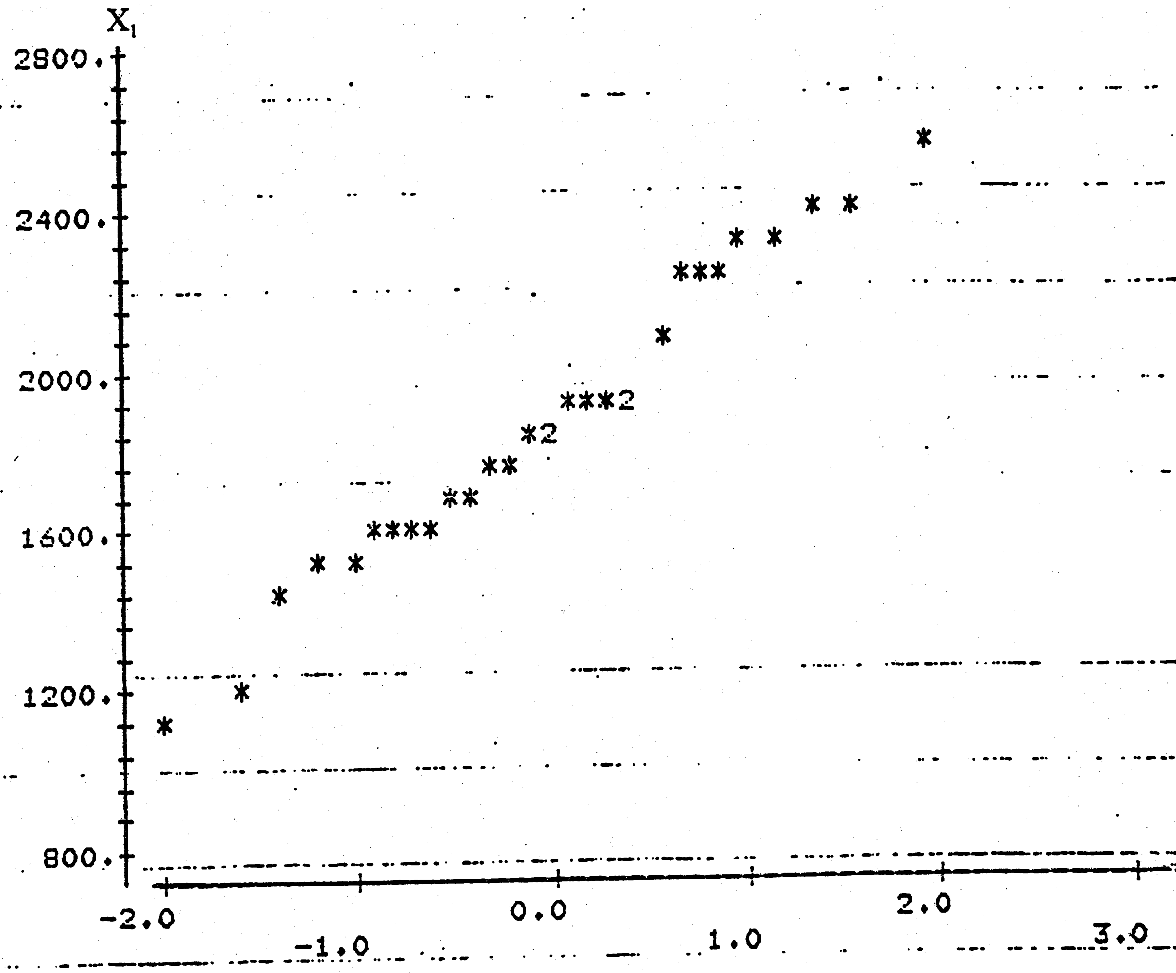
c) The Q-Q plots for both stiffness and bending strength (see below) show that the marginal normality is not seriously violated. Also the correlation coefficients for the test of normality are .989 and .990 respectively so that we fail to reject even at the 1% significance level. Finally, the scatter diagram (see below) does not indicate departure from bivariate normality. So, the bivariate normal distribution is a plausible probability model for these data.

Q-Q Plot-Bending Strength



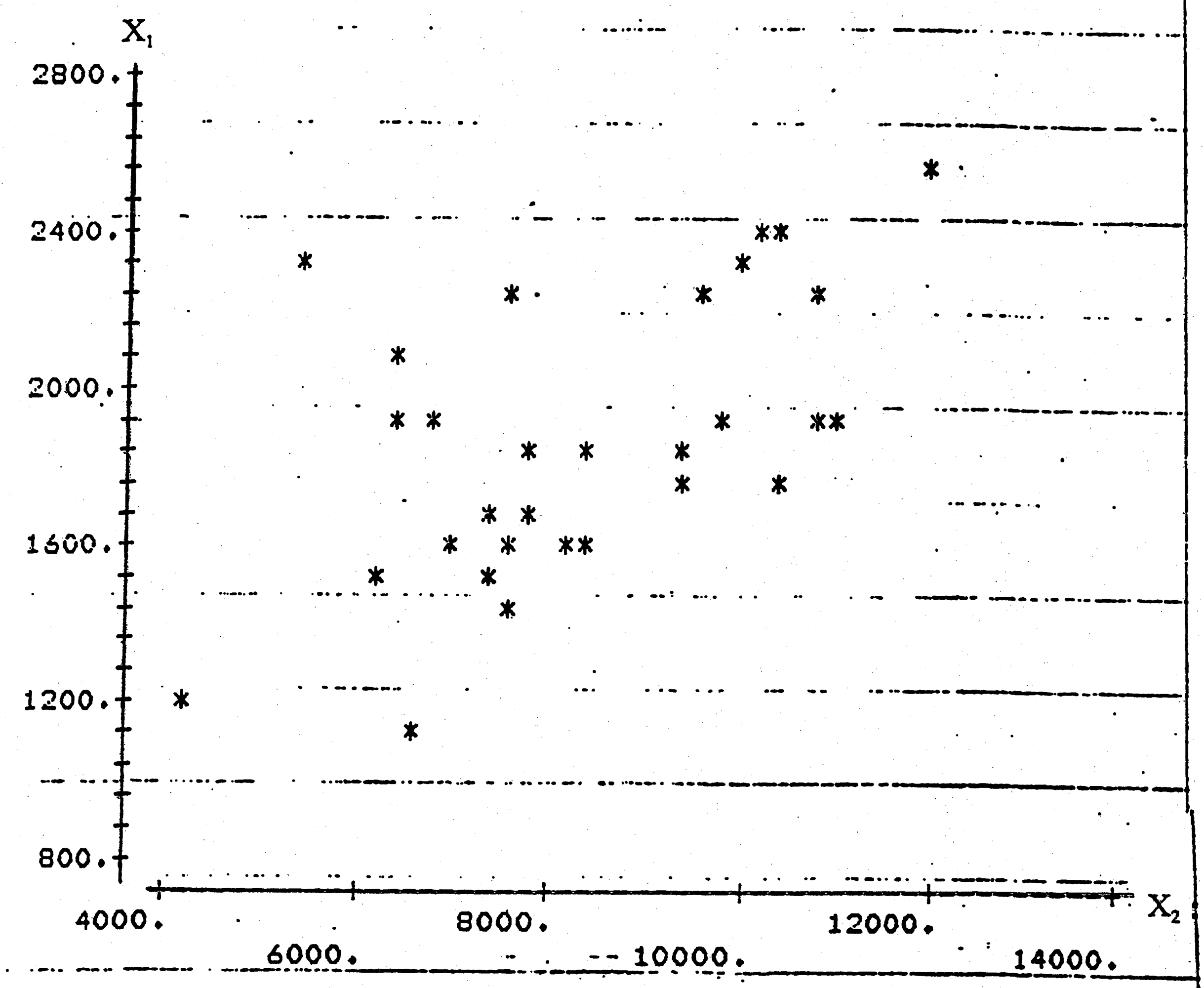
Correlation .989

Q-Q Plot-Stiffness

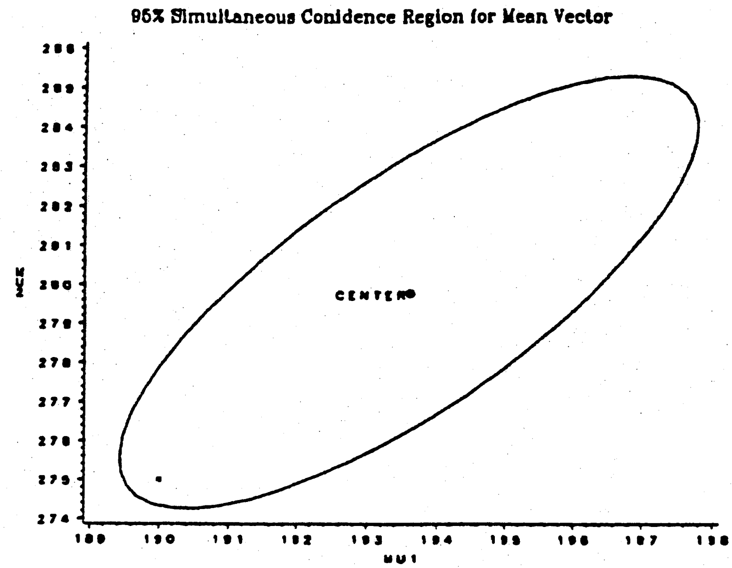


Correlation = .990

Scatter Diagram



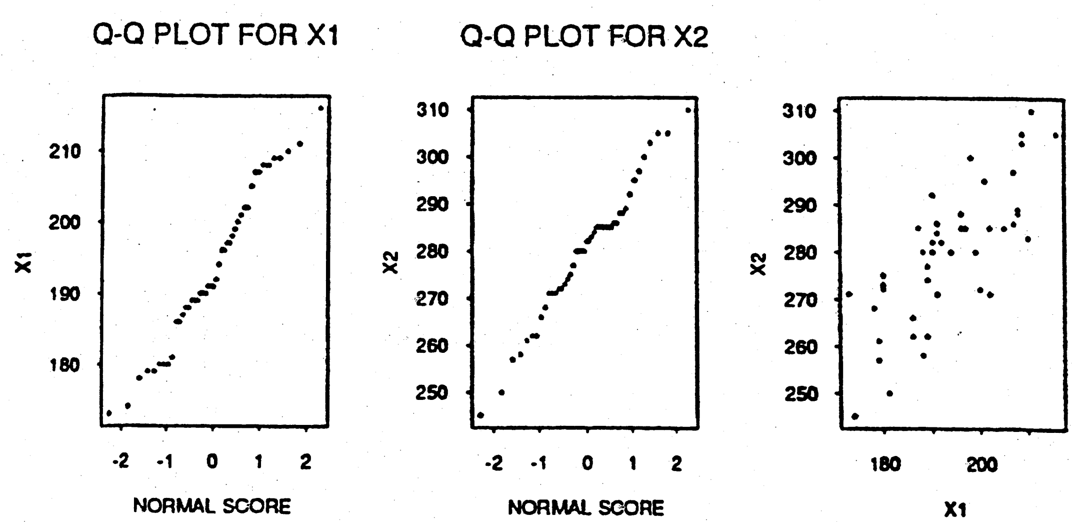
5.20 (a). Yes, they are plausible since the hypothesized vector $\underset{\sim}{\mu}_0$ (denoted as $*$ in the plot) is inside the 95% confidence region.



(b).

	LOWER	UPPER
Bonferroni C. I.:	189.822	197.423
	274.782	284.774
Simultaneous C. I.:	189.422	197.823
	274.256	285.299

Simultaneous confidence intervals are larger than Bonferroni's confidence intervals. Simultaneous confidence intervals will touch the simultaneous confidence region from outside.
(c). Q-Q plots suggests non-normality of (X_1, X_2). Could try transforming X_1.



5.21

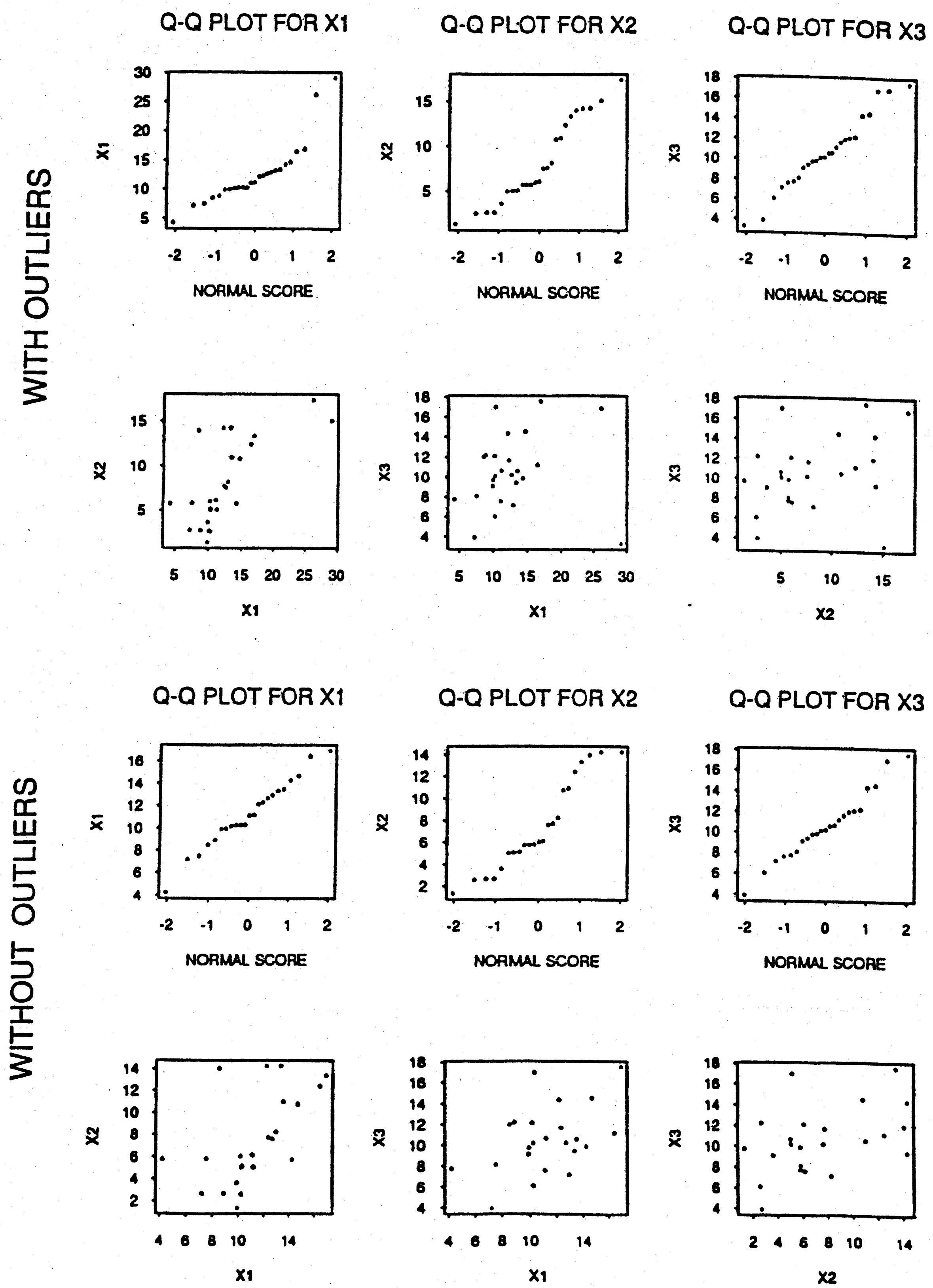
HOTELLING T SQUARE = 9.0218
P-VALUE 0.3616

	N	MEAN	STDEV	T2 INTERVAL	TO	BONFERRONI	TO
x1	25	0.84380	0.11402	.742	.946	.778	.909
x2	25	0.81832	0.10685	.723	.914	.757	.880
x3	25	1.79268	0.28347	1.540	2.046	1.629	1.956
x4	25	1.73484	0.26360	1.499	1.970	1.583	1.887
x5	25	0.70440	0.10756	.608	.800	.642	.766
x6	25	0.69384	0.10295	.602	.786	.635	.753

The Bonferroni intervals use t (.00417) = 2.88 and

the T2 intevals use the constant 4.465.

(a). After eliminating outliers, the approximation to normality is improved.



(b) Outliers removed:

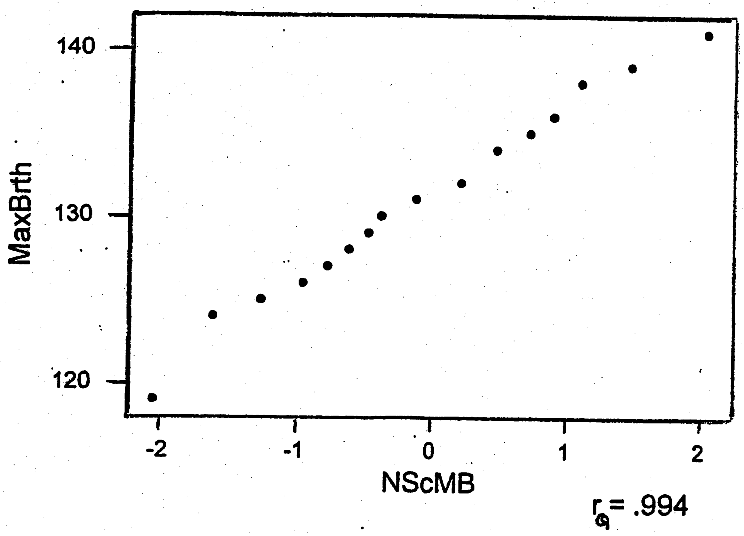
	LOWER	UPPER
Bonferroni C. I.:	9.63	12.87
	5.24	9.67
	8.82	12.34
Simultaneous C. I.:	9.25	13.24
	4.72	10.19
	8.41	12.76

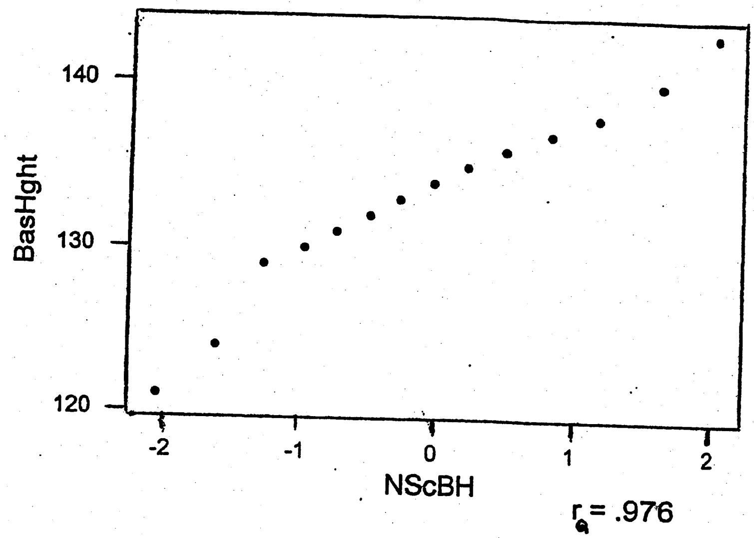
Simultaneous confidence intervals are larger than Bonferroni's confidence intervals.

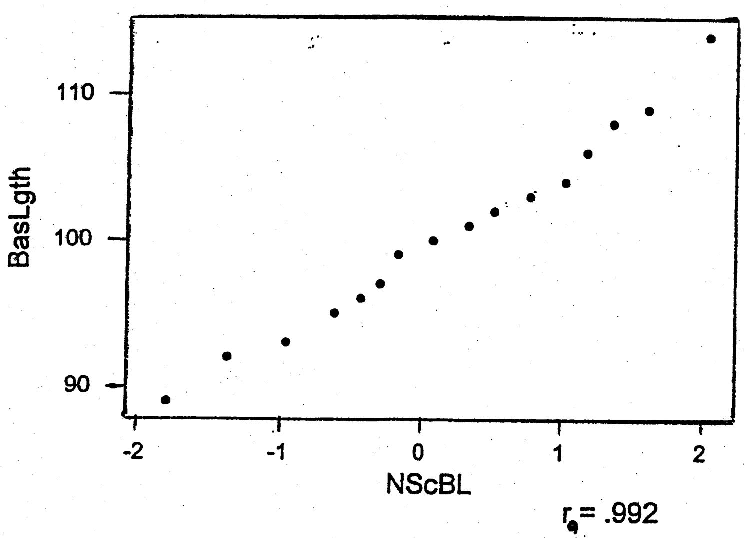
(b) Full data set:

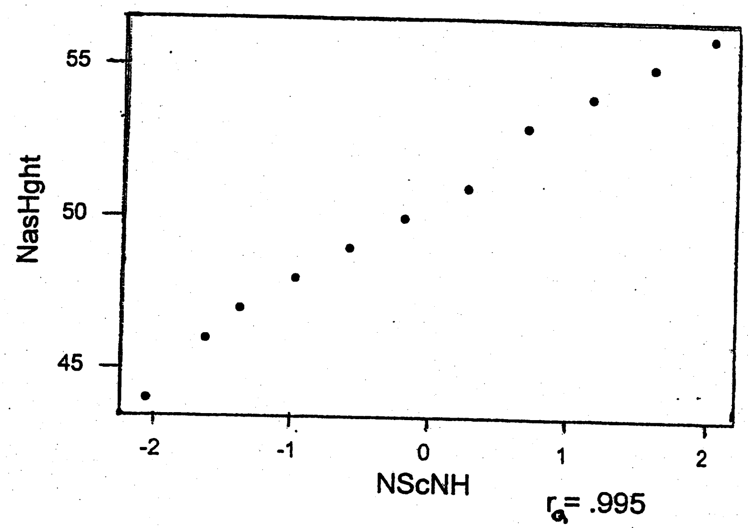
	Lower	Upper
Bonferroni C. I.:	9.79	15.33
	5.78	10.55
	8.65	12.44
Simultaneous C. I.:	9.16	15.96
	5.23	11.09
	8.21	12.87

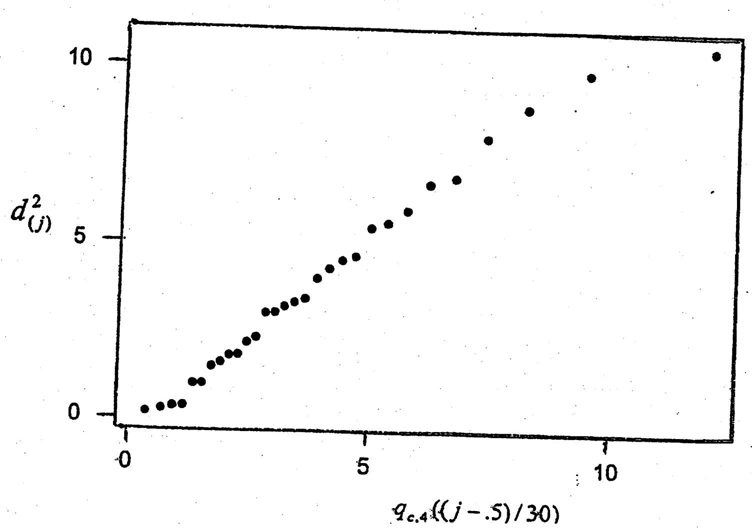
5.23 a) The data appear to be multivariate normal as shown by the "straightness" of the Q-Q plots and chi-square plot below.











5.23 (Continued)

b) Bonferroni 95% simultaneous confidence intervals ($m = p = 4$):
$t_{29}(.05/8) = 2.663$

MaxBrth:	(128.87, 133.87)
BasHgth:	(131.42, 135.78)
BasLngth:	(96.32, 102.02)
NasHgth:	(49.17, 51.89)

95% T^2 simultaneous confidence intervals:

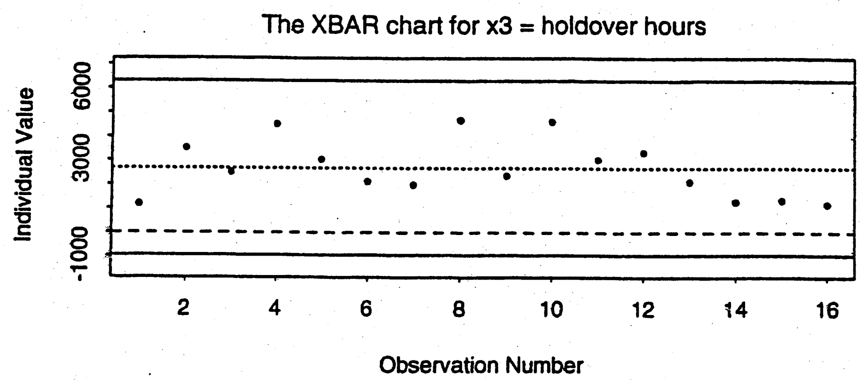
$$\sqrt{\frac{4(29)}{26}F_{4,26}(.05)} = 3.496$$

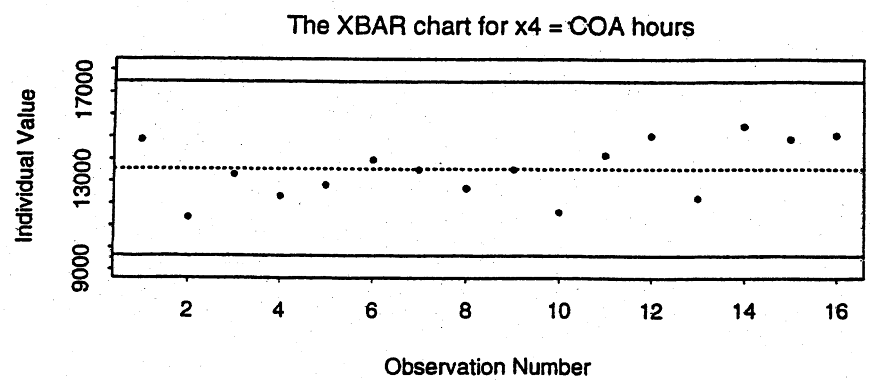
MaxBrth:	(128.08, 134.66)
BasHgth:	(130.73, 136.47)
BasLngth:	(95.43, 102.91)
NasHgth:	(48.75, 52.31)

The Bonferroni intervals are slightly shorter than the T^2 intervals.

5.24 Individual $\overline{X}$ charts for the Madison, Wisconsin, Police Department data

	xbar	s	LCL	UCL	
LegalOT	3557.8	606.5	1738.1	5377.4	
ExtraOT	1478.4	1182.8	-2070.0	5026.9	use LCL = 0
Holdover	2676.9	1207.7	-946.2	6300.0	use LCL = 0
COA	13563.6	1303.2	9654.0	17473.2	
MeetOT	800.0	474.0	-622.1	2222.1	use LCL = 0

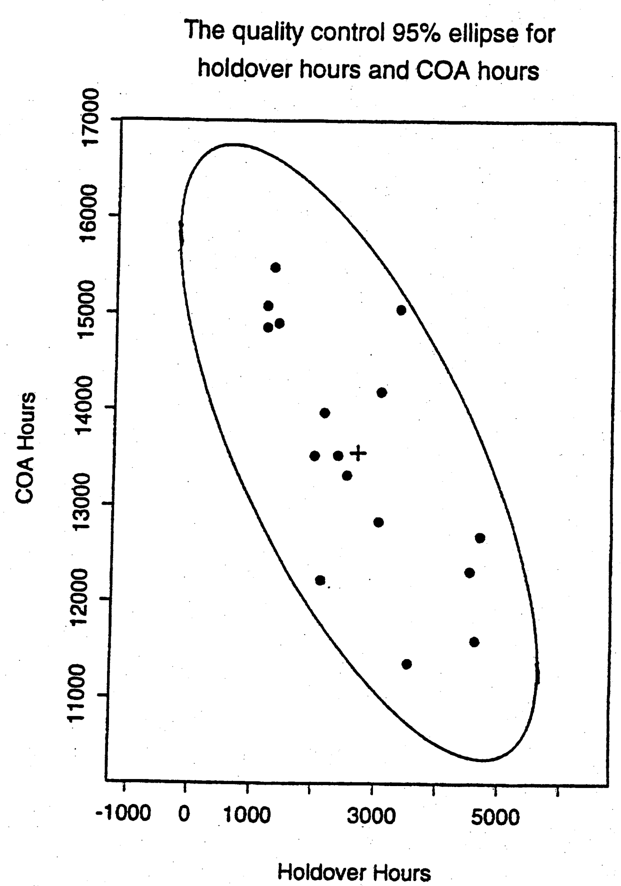


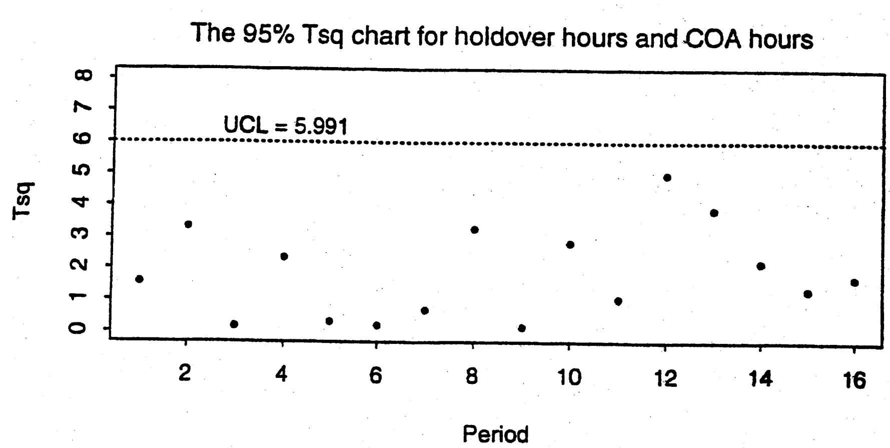


Both holdover and COA hours are stable and in control.

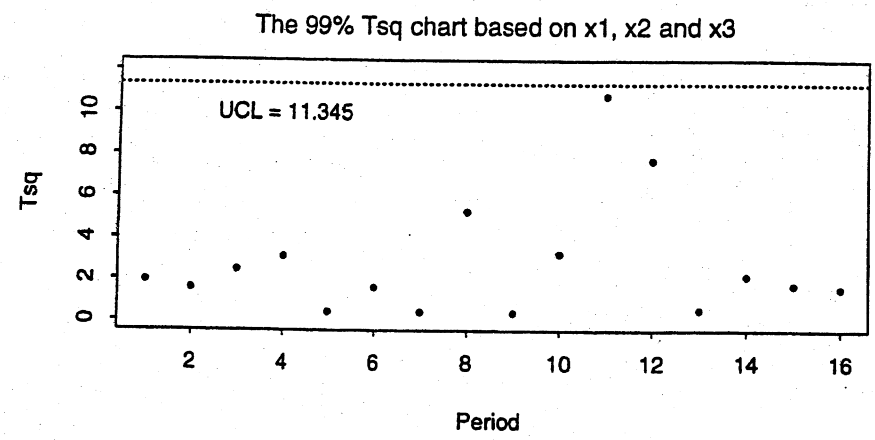
5.25 Quality ellipse and T^2 chart for the holdover and COA overtime hours. All points are in control. The quality control 95% ellipse is

$$1.37\times10^{-6}(x_3-2677)^2+1.18\times10^{-6}(x_4-13564)^2$$
$$+1.80\times10^{-6}(x_3-2677)(x_4-13564)=5.99.$$



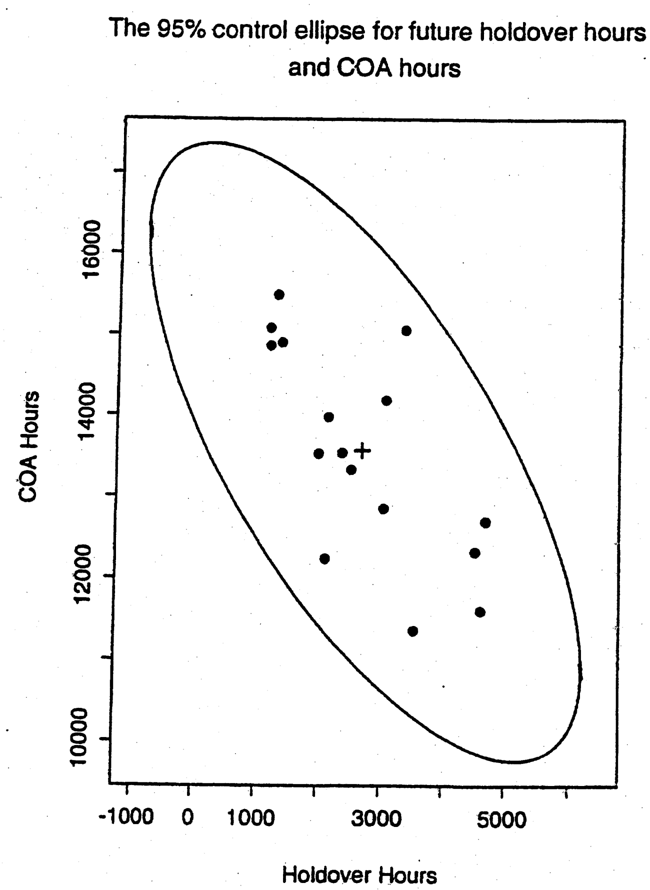


5.26 T^2 chart using the data on x_1 = legal appearances overtime hours, x_2 = extraordinary event overtime hours, and x_3 = holdover overtime hours. All points are in control.



5.27 The 95% prediction ellipse for x_3 = holdover hours and x_4 = COA hours is

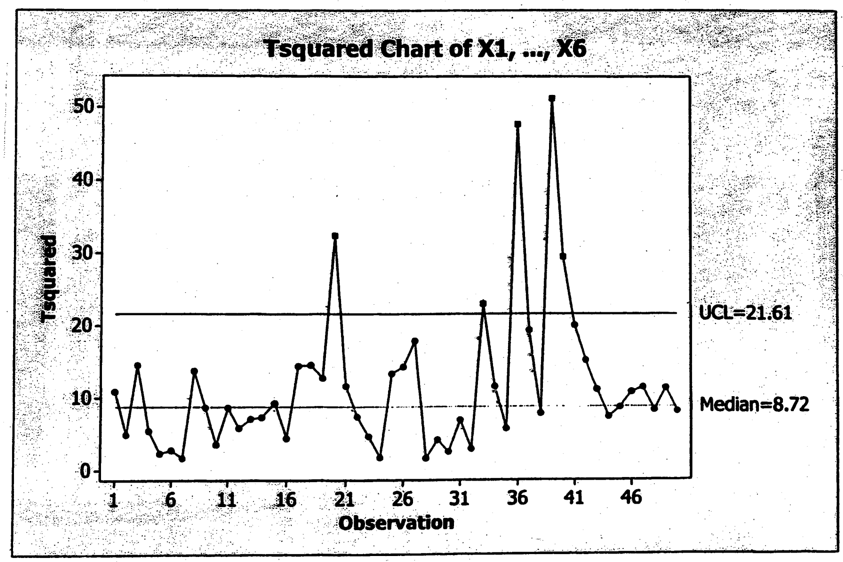
$$1.37\times10^{-6}(x_3-2677)^2+1.18\times10^{-6}(x_4-13564)^2$$
$$+1.80\times10^{-6}(x_3-2677)(x_4-13564)=8.51.$$



5.28 (a)

$$\bar{\mathbf{x}} = \begin{bmatrix} -.506 \\ -.207 \\ -.062 \\ -.032 \\ .698 \\ -.065 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} .0626 & .0616 & .0474 & .0083 & .0197 & .0031 \\ .0616 & .0924 & .0268 & -.0008 & .0228 & .0155 \\ .0474 & .0268 & .1446 & .0078 & .0211 & -.0049 \\ .0083 & -.0008 & .0078 & .1086 & .0221 & .0066 \\ .0197 & .0228 & .0211 & .0221 & .3428 & .0146 \\ .0031 & .0155 & -.0049 & .0066 & .0146 & .0366 \end{bmatrix}$$

The T^2 chart follows.



(b) Multivariate observations 20, 33, 36, 39 and 40 exceed the upper control limit. The individual variables that contribute significantly to the out of control data points are indicated in the table below.

	Point	Variable	P-Value
Greater Than UCL	20	X1	0.0000
		X2	0.0001
		X3	0.0000
		X4	0.0105
		X5	0.0210
		X6	0.0032
	33	X4	0.0088
		X6	0.0000
	36	X1	0.0000
		X2	0.0000
		X3	0.0000
		X4	0.0343
	39	X2	0.0198
		X4	0.0001
		X5	0.0054
		X6	0.0000
	40	X1	0.0000
		X2	0.0088
		X3	0.0114
		X4	0.0013

5.29 $T^2 = 12.472$. Since $T^2 = 12.472 < \frac{29(6)}{24} F_{6,24}(.05) = 7.25(2.51) = 18.2$, we do not reject $H_0 : \mu = \mathbf{0}$ at the 5% level.

5.30 **(a)** Large sample 95% Bonferroni intervals for the indicated means follow. Multiplier is $t_{49}(.05/2(6)) \approx z(.0042) = 2.635$

Petroleum: $.766 \pm 2.635(.925/\sqrt{50}) = .766 \pm .345 \rightarrow (.421,\ 1.111)$

Natural Gas: $.508 \pm 2.635(.753/\sqrt{50}) = .508 \pm .282 \rightarrow (.226,\ .790)$

Coal: $.438 \pm 2.635(.414/\sqrt{50}) = .438 \pm .155 \rightarrow (.283,\ .593)$

Nuclear: $.161 \pm 2.635(.207/\sqrt{50}) = .161 \pm .076 \rightarrow (.085,\ .237)$

Total: $1.873 \pm 2.635(1.978/\sqrt{50}) = 1.873 \pm .738 \rightarrow (1.135,\ 2.611)$

Petroleum – Natural Gas: $.258 \pm 2.635(.392/\sqrt{50}) = .258 \pm .146 \rightarrow (.112,\ .404)$

(b) Large sample 95% simultaneous T^2 intervals for the indicated means follow. Multiplier is $\sqrt{\chi_4^2(.05)} = \sqrt{9.49} = 3.081$

Petroleum: $.766 \pm 3.081(.925/\sqrt{50}) = .766 \pm .404 \rightarrow (.362,\ 1.170)$

Natural Gas: $.508 \pm 3.081(.753/\sqrt{50}) = .508 \pm .330 \rightarrow (.178,\ .838)$

Coal: $.438 \pm 3.081(.414/\sqrt{50}) = .438 \pm .182 \rightarrow (.256,\ .620)$

Nuclear: $.161 \pm 3.081(.207/\sqrt{50}) = .161 \pm .089 \rightarrow (.072,\ .250)$

Total: $1.873 \pm 3.081(1.978/\sqrt{50}) = 1.873 \pm .863 \rightarrow (1.010,\ 2.736)$

Petroleum – Natural Gas: $.258 \pm 3.081(.392/\sqrt{50}) = .258 \pm .171 \rightarrow (.087,\ .429)$

Since the multiplier, 3.081, for the 95% simultaneous T^2 intervals is larger than the multiplier, 2.635, for the Bonferroni intervals and everything else for a given interval is the same, the T^2 intervals will be wider than the Bonferroni intervals.

5.31 (a) The power transformation $\hat{\lambda}_1 = 0$ (i.e. logarithm) makes the duration observations more nearly normal. The power transformation $\hat{\lambda}_2 = -0.5$ (i.e. reciprocal of square root) makes the man/machine time observations more nearly normal. (See Exercise 4.41.) For the transformed observations, say $y_1 = \ln x_1$, $y_2 = 1/\sqrt{x_2}$ where x_1 is duration and x_2 is man/machine time,

$$\bar{\mathbf{y}} = \begin{bmatrix} 2.171 \\ .240 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} .1513 & -.0058 \\ -.0058 & .0018 \end{bmatrix} \quad \mathbf{S}^{-1} = \begin{bmatrix} 7.524 & 23.905 \\ 23.905 & 624.527 \end{bmatrix}$$

The eigenvalues for **S** are $\lambda_1 = .15153$, $\lambda_2 = .00160$ with corresponding eigenvectors $\mathbf{e}_1' = [.99925 \quad -.03866]$, $\mathbf{e}_2' = [.03866 \quad .99925]$. Beginning at center $\bar{\mathbf{y}}$, the axes of the 95% confidence ellipsoid are

major axis: $\pm\sqrt{\lambda_1}\sqrt{\dfrac{2(24)}{25(23)}F_{2,23}(.05)}\,\mathbf{e}_1 = \pm.208\mathbf{e}_1$

minor axis: $\pm\sqrt{\lambda_2}\sqrt{\dfrac{2(24)}{25(23)}F_{2,23}(.05)}\,\mathbf{e}_2 = \pm.021\mathbf{e}_2$

The ratio of the lengths of the major and minor axes, .416/.042 = 9.9, indicates the confidence ellipse is elongated in the $\mathbf{e}_1$ direction.

(b) $t_{24}(.05/2(2)) = 2.391$, so the 95% confidence intervals for the two component means (of the transformed observations) are:

$$\bar{y}_1 \pm t_{24}(.0125)\sqrt{s_{11}} = 2.171 \pm 2.391\sqrt{.1513} = 2.171 \pm .930 \rightarrow (1.241,\ 3.101)$$

$$\bar{y}_2 \pm t_{24}(.0125)\sqrt{s_{22}} = .240 \pm 2.391\sqrt{.0018} = .240 \pm .101 \rightarrow (.139,\ .341)$$

Chapter 6

6.1 Eigenvalues and eigenvectors of S_d are:

$$\lambda_1 = 449.778, \quad \underset{\sim}{e}_1' = [.333, \ .943]$$
$$\lambda_2 = 168.082, \quad \underset{\sim}{e}_2' = [.943, -.333]$$

Ellipse centered at $\bar{\underset{\sim}{d}}' = [-9.36, \ 13.27]$. Half length of major axis is 20.57 units. Half length of minor axis is 12.58 units. Major and minor axes lie in $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ directions, respectively.

Yes, the test answers the question: Is $\underset{\sim}{\delta} = \underset{\sim}{0}$ inside the 95% confidence ellipse?

6.2 Using a critical value $t_{n-1}(\alpha/2p) = t_{10}(0.0125) = 2.6338$,

	LOWER	UPPER
Bonferroni C. I.:	-20.57	1.85
	-2.97	29.52
Simultaneous C. I.:	-22.45	3.73
	-5.70	32.25

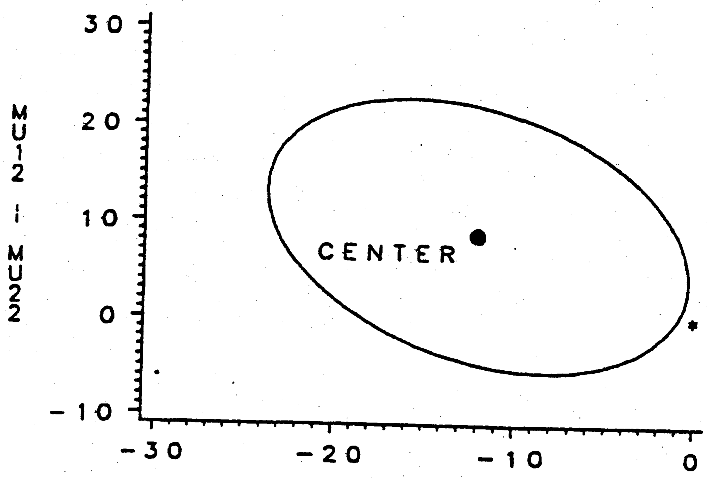
Simultaneous confidence intervals are larger than Bonferroni's confidence intervals.

6.3 The 95% Bonferroni intervals are

	LOWER	UPPER
Bonferroni C. I.:	-21.92	-2.08
	-3.36	20.56
Simultaneous C. I.:	-23.70	-0.30
	-5.50	22.70

Since the hypothesized vector $\delta = 0$ (denoted as $*$ in the plot) is outside the joint confidence region, we reject $H_0 : \delta = 0$. Bonferroni C.I. are consistent with this result. After the elimination of the outlier, the difference between pairs became significant.

95% Simultaneous Conidence Region for Delta Vector



Problem 6.3

6.4

(a). Hotelling's $T^2 = 10.215$. Since the critical point with $\alpha = 0.05$ is 9.459, we reject $H_0 : \underset{\sim}{\delta} = \underset{\sim}{0}$.

(b).

	Lower	Upper
Bonferroni C. I.:	−1.09	−0.02
	−0.04	0.64
T^2 Simultaneous C. I.:	−1.18	0.07
	−0.10	0.69

95% Confidence Ellipse About the Mean Vector

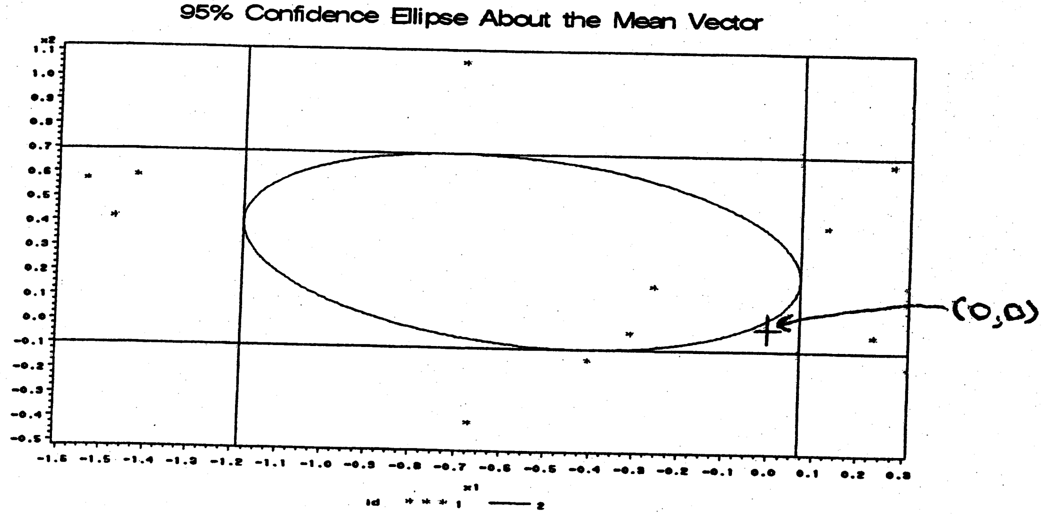
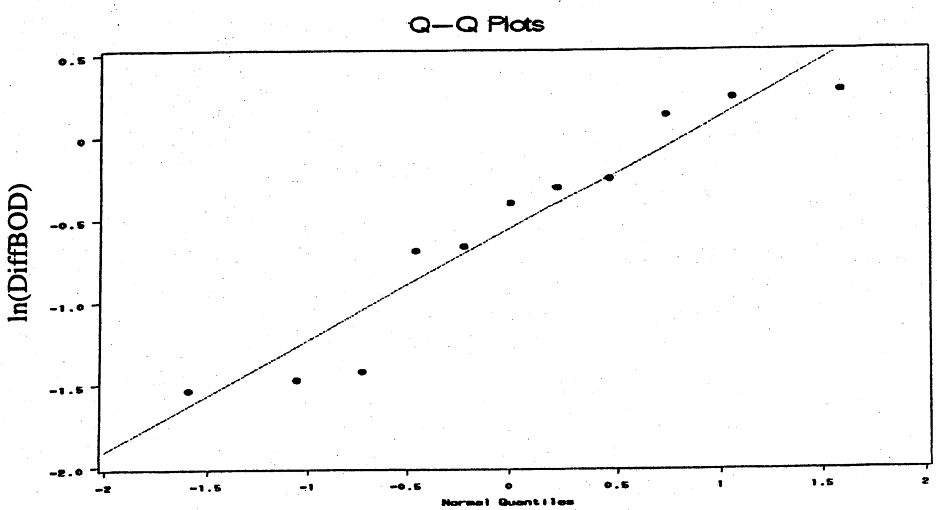
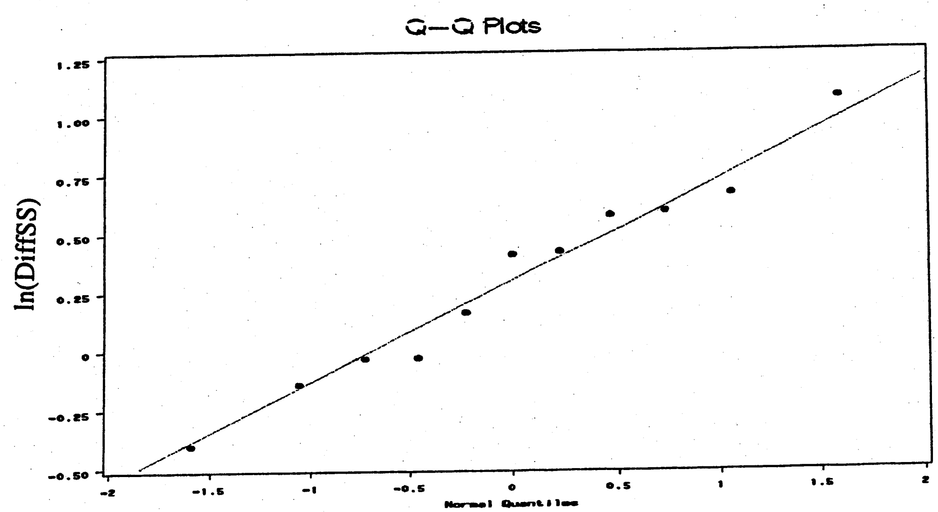
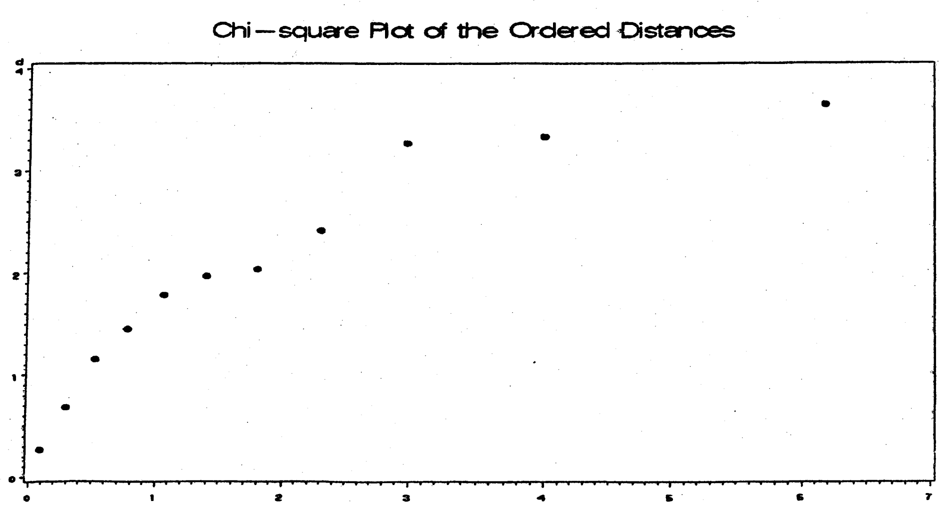


Figure 1: 95% Confidence Ellipse and Simultaneous T^2 Intervals for the Mean Difference

(c) The Q-Q plots for ln(DiffBOD) and ln(DiffSS) are shown below. Marginal normality cannot be rejected for either variable. The χ^2 plot is not straight (with at least one apparent bivariate outlier) and, although the sample size (n =11) is small, it is difficult to argue for bivariate normality.







6.5 a) $H_0: C\underset{\sim}{\mu} = \underset{\sim}{0}$ where $C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $\underset{\sim}{\mu}' = [\mu_1, \mu_2, \mu_3]$.

$$C\bar{\underset{\sim}{x}} = \begin{bmatrix} -11.2 \\ 6.9 \end{bmatrix}, \quad CSC' = \begin{bmatrix} 55.5 & -32.6 \\ -32.6 & 66.4 \end{bmatrix}$$

$T^2 = n(C\bar{\underset{\sim}{x}})'(CSC')^{-1}(C\bar{\underset{\sim}{x}}) = 90.4; \quad n = 40; \quad q = 3$

$\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(.05) = \frac{(39)2}{38}(3.25) = 6.67$

Since $T^2 = 90.4 > 6.67$ reject $H_0: C\underset{\sim}{\mu} = \underset{\sim}{0}$

b) 95% simultaneous confidence intervals:

$\mu_1 - \mu_2$: $(46.1 - 57.3) \pm \sqrt{6.67}\sqrt{\frac{55.5}{40}} = -11.2 \pm 3.0$

$\mu_2 - \mu_3$: 6.9 ± 3.3

$\mu_1 - \mu_3$: -4.3 ± 3.3

The means are all different from one another.

6.6 a) Treatment 2: Sample mean vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$; sample covariance matrix

$$\begin{bmatrix} 1 & -3/2 \\ -3/2 & 3 \end{bmatrix}$$

Treatment 3: Sample mean vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$; sample covariance matrix

$$\begin{bmatrix} 2 & -4/3 \\ -4/3 & 4/3 \end{bmatrix}$$

$$S_{pooled} = \begin{bmatrix} 1.6 \\ -1.4 \end{bmatrix}$$

b) $T^2 = [2-3,\ 4-2] \left[\left(\frac{1}{3} + \frac{1}{4}\right) \begin{bmatrix} 1.6 & -1.4 \\ -1.4 & 2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2-3 \\ 4-2 \end{bmatrix} = 3.88$

$$\frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,n_1+n_2-p-1}(.01) = \frac{(5)2}{4}(18) = 45$$

Since $T^2 = 3.88 < 45$ do not reject $H_0: \underset{\sim}{\mu}_2 - \underset{\sim}{\mu}_3 = \underset{\sim}{0}$ at the $\alpha = .01$ level.

c). 99% simultaneous confidence intervals:

$$\mu_{21} - \mu_{31}: (2-3) \pm \sqrt{45}\ \sqrt{\left(\frac{1}{3}+\frac{1}{4}\right)1.6} = -1 \pm 6.5$$

$$\mu_{22} - \mu_{32}: 2 \pm 7.2$$

6.7 $T^2 = [74.4 \quad 201.6] \left[\left(\frac{1}{45} + \frac{1}{55}\right) \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix} \right]^{-1} \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix} = 16.1$

$$\frac{(n_1+n_2-2)p}{n_1+n_2-p-1} F_{p,n_1+n_2-p-1}(.05) = 6.26$$

Since $T^2 = 16.1 > 6.26$ reject $H_0: \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$ at the $\alpha = .05$ level.

$$\hat{\mathbf{a}} \propto S_{pooled}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = \begin{bmatrix} .0017 \\ .0026 \end{bmatrix}$$

6.8 a) For first variable:

observation = mean + treatment effect + residual

$$\begin{bmatrix} 6 & 5 & 8 & 4 & 7 \\ 3 & 1 & 2 & & \\ 2 & 5 & 3 & 2 & \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & & \\ 4 & 4 & 4 & 4 & \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ -2 & -2 & -2 & & \\ -1 & -1 & -1 & -1 & \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 & -2 & 1 \\ 1 & -1 & 0 & & \\ -1 & 2 & 0 & -1 & \end{bmatrix}$$

$SS_{obs} = 246$ $SS_{mean} = 192$ $SS_{tr} = 36$ $SS_{res} = 18$

For second variable:

$$\begin{bmatrix} 7 & 9 & 6 & 9 & 9 \\ 3 & 6 & 3 & & \\ 3 & 1 & 1 & 3 & \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & & \\ 5 & 5 & 5 & 5 & \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & & \\ -3 & -3 & -3 & -3 & \end{bmatrix} + \begin{bmatrix} -1 & 1 & -2 & 1 & 1 \\ -1 & 2 & -1 & & \\ 1 & -1 & -1 & 1 & \end{bmatrix}$$

$SS_{obs} = 402$ $SS_{mean} = 300$ $SS_{tr} = 84$ $SS_{res} = 18$

Cross product contributions:

275 240 48 -13

b) MANOVA table:

Source of Variation	SSP	d.f.
Treatment	$B = \begin{bmatrix} 36 & 48 \\ 48 & 84 \end{bmatrix}$	$3 - 1 = 2$
Residual	$W = \begin{bmatrix} 18 & -13 \\ -13 & 18 \end{bmatrix}$	$5 + 3 + 4 - 3 = 9$
Total (corrected)	$\begin{bmatrix} 54 & 35 \\ 35 & 102 \end{bmatrix}$	11

c) $\Lambda^* = \frac{|W|}{|B+W|} = \frac{155}{4283} = .0362$

Using Table 6.3 with $p = 2$ and $g = 3$

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\left(\frac{\Sigma n_\ell - g - 1}{g-1}\right) = 17.02 .$$

Since $F_{4,16}(.01) = 4.77$ we conclude that treatment differences exist at $\alpha = .01$ level.

Alternatively, using Bartlett's procedure,

$$-(n-1-\frac{(p+g)}{2})\,\ell n\,\Lambda^* = -(12-1-\frac{5}{2})\ell n(.0362) = 28.209$$

Since $\chi^2_4(.01) = 13.28$ we again conclude treatment differences exist at $\alpha = .01$ level.

6.9 For any matrix C

$$\bar{\underset{\sim}{d}} = \frac{1}{n}\Sigma\, \underset{\sim}{d}_j = C(\frac{1}{n}\Sigma\, \underset{\sim}{x}_j) = C\,\bar{\underset{\sim}{x}}$$

and $\underset{\sim}{d}_j - \bar{\underset{\sim}{d}} = C(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})$

so $S_d = \frac{1}{n-1}\Sigma(\underset{\sim}{d}_j-\bar{\underset{\sim}{d}})(\underset{\sim}{d}_j-\bar{\underset{\sim}{d}})' = C(\frac{1}{n-1}\Sigma(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})')C' = CSC'$

6.10 $(\bar{x}\,\underset{\sim}{1})'[(\bar{x}_1-\bar{x})\underset{\sim}{u}_1 + \dots + (\bar{x}_g-\bar{x})\underset{\sim}{u}_g]$

$$= \bar{x}[(\bar{x}_1-\bar{x})n_1 + \dots + (\bar{x}_g-\bar{x})n_g]$$

$$= \bar{x}[n_1\bar{x}_1 + \dots + n_g\bar{x}_g - \bar{x}(n_1 + \dots + n_g)]$$

$$= \bar{x}[(n_1 + \dots + n_g)\bar{x} - \bar{x}(n_1 + \dots + n_g)] = 0$$

6.11 $L(\underset{\sim}{\mu}_1, \underset{\sim}{\mu}_2, \Sigma) = L(\underset{\sim}{\mu}_1, \Sigma) L(\underset{\sim}{\mu}_2, \Sigma)$

$$= \left[\frac{1}{(2\pi)^{\frac{(n_1+n_2)p}{2}} |\Sigma|^{\frac{n_1+n_2}{2}}} \right] \exp\left\{ -\frac{1}{2} \left(\text{tr}\ \Sigma^{-1}[(n_1-1)S_1 + (n_2-1)S_2] \right. \right.$$

$$\left. \left. + n_1(\bar{\underset{\sim}{x}}_1 - \underset{\sim}{\mu}_1)' \Sigma^{-1} (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}_1) + n_2(\bar{\underset{\sim}{x}}_2 - \underset{\sim}{\mu}_2)' \Sigma^{-1} (\bar{\underset{\sim}{x}}_2 - \underset{\sim}{\mu}_2) \right) \right\}$$

using (4-16) and (4-17). The likelihood is maximized with respect to $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ at $\hat{\underset{\sim}{\mu}}_1 = \bar{\underset{\sim}{x}}_1$ and $\hat{\underset{\sim}{\mu}}_2 = \bar{\underset{\sim}{x}}_2$ respectively and with respect to Σ at

$$\hat{\Sigma} = \frac{1}{n_1+n_2} [(n_1 - 1)S_1 + (n_2 - 2)S_2] = \left(\frac{n_1+n_2-2}{n_1+n_2} \right) S_{pooled}$$

(For the maximization with respect to Σ see Result 4.10 with $b = \frac{n_1+n_2}{2}$ and $B = (n_1 - 1)S_1 + (n_2 - 2)S_2$)

6.13 a) and b) For first variable:

$$\text{Observation} = \text{mean} + \text{factor 1 effect} + \text{factor 2 effect} + \text{residual}$$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 3 & -3 & 4 & -4 \\ -3 & -4 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 & 4 \\ -1 & -1 & -1 & -1 \\ -3 & -3 & -3 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 4 & -3 \\ 1 & -2 & 4 & -3 \\ 1 & -2 & 4 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 & 0 \\ 2 & -1 & 0 & -1 \\ -2 & 0 & 1 & 1 \end{bmatrix}$$

$SS_{tot} = 220$ $SS_{mean} = 12$ $SS_{fac\,1} = 104$ $SS_{fac\,2} = 90$ $SS_{res} = 14$

For second variable:

$$\begin{bmatrix} 8 & 6 & 12 & 6 \\ 8 & 2 & 3 & 3 \\ 2 & -5 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 \\ -6 & -6 & -6 & -6 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 & -2 \\ 3 & -2 & 1 & -2 \\ 3 & -2 & 1 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 3 & 0 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & -1 & -1 \end{bmatrix}$$

$SS_{tot} = 440$ $SS_{mean} = 108$ $SS_{fac\,1} = 248$ $SS_{fac\,2} = 54$ $SS_{res} = 30$

Sum of cross products:

$$SCP_{tot} = SCP_{mean} + SCP_{fac\,1} + SCP_{fac\,2} + SCP_{res}$$

$$227 = 36 + 148 + 51 - 8$$

c) MANOVA table:

Source of Variation	SSP	d.f.
Factor 1	$\begin{bmatrix} 104 & 148 \\ 148 & 248 \end{bmatrix}$	$g - 1 = 3 - 1 = 2$
Factor 2	$\begin{bmatrix} 90 & 51 \\ 51 & 54 \end{bmatrix}$	$b - 1 = 4 - 1 = 3$
Residual	$\begin{bmatrix} 14 & -8 \\ -8 & 30 \end{bmatrix}$	$(g-1)(b-1) = 6$
Total (Corrected)	$\begin{bmatrix} 208 & 191 \\ 191 & 332 \end{bmatrix}$	$gb - 1 = 11$

d) We reject $H_0: \underset{\sim}{\tau}_1 = \underset{\sim}{\tau}_2 = \underset{\sim}{\tau}_3 = \underset{\sim}{0}$ at $\alpha = .05$ level since

$$-\left[(g-1)(b-1) - \left(\frac{p+1-(g-1)}{2}\right)\right]\ln\Lambda^* = -\left[6 - \frac{3-2}{2}\right]\ln\left(\frac{|SS_{res}|}{|SSP_{fac\,1} + SSP_{res}|}\right)$$

$$= -5.5\,\ln\left(\frac{356}{13204}\right) = 19.87 > \chi^2_4(.05) = 9.49$$

and conclude there are factor 1 effects.

We also reject $H_0: \underset{\sim}{\beta}_1 = \underset{\sim}{\beta}_2 = \underset{\sim}{\beta}_3 = \underset{\sim}{\beta}_4 = \underset{\sim}{0}$ at the $\alpha = .05$ level since

$$-[(g-1)(b-1)-(\frac{p+1-(b-1)}{2})\ell n\Lambda^* = -[6-\frac{3-3}{2}]\ell n\left(\frac{|SSP_{res}|}{|SSP_{fac\,2}+SSP_{res}|}\right)$$

$$= -6\,\ell n\left(\frac{356}{6887}\right) = 17.77 > \chi_6^2(.05) = 12.59$$

and conclude there are factor 2 effects.

6.14 b) MANOVA Table:

Source of Variation	SSP	d.f.
Factor 1	$\begin{bmatrix} 496 & 184 \\ 184 & 208 \end{bmatrix}$	2
Factor 2	$\begin{bmatrix} 36 & 24 \\ 24 & 36 \end{bmatrix}$	3
Interaction	$\begin{bmatrix} 32 & 0 \\ 0 & 44 \end{bmatrix}$	6
Residual	$\begin{bmatrix} 312 & -84 \\ -84 & 400 \end{bmatrix}$	12
Total (Corrected)	$\begin{bmatrix} 876 & 124 \\ 124 & 688 \end{bmatrix}$	23

c) Since $-[gb(n-1)-(p+1-(g-1)(b-1))/2]\ell n\Lambda^* = -13.5\ell n\left(\frac{|SSP_{res}|}{|SSP_{int}+SSP_{res}|}\right)$

$= -13.5\ell n(.808) = 2.88 < \chi_{12}^2(.05) = 21.03$ we <u>do not reject</u>

$H_0: \underset{\sim}{\gamma}_{11} = \underset{\sim}{\gamma}_{12} = \cdots = \underset{\sim}{\gamma}_{34} = \underset{\sim}{0}$ (no interaction effects) at the

$\alpha = .05$ level.

Since

$$-[gb(n-1)-(p+1-(g-1))/2]\ell n\Lambda^* = -11.5\ell n\left(\frac{|SSP_{res}|}{|SSP_{fac\,1} + SSP_{res}|}\right)$$

$$= -11.5\ell n(.2447) = 16.19 > \chi^2_4(.05) = 9.49$$ we reject

$H_0: \underset{\sim}{\tau}_1 = \underset{\sim}{\tau}_2 = \underset{\sim}{\tau}_3 = \underset{\sim}{0}$ (no factor 1 effects) at the $\alpha = .05$ level.

Since

$$-[gb(n-1)-(p+1-(b-1))/2]\ell n\Lambda^* = -12\ell n\left(\frac{|SSP_{res}|}{|SSP_{fac\,2} + SSP_{res}|}\right)$$

$$= -12\ell n(.7949) = 2.76 < \chi^2_6(.05) = 12.59$$ we do not reject

$H_0: \underset{\sim}{\beta}_1 = \underset{\sim}{\beta}_2 = \underset{\sim}{\beta}_3 = \underset{\sim}{\beta}_4 = \underset{\sim}{0}$ (no factor 2 effects) at the $\alpha = .05$ level.

6.15 Example 6.11, $g = b = 2$, $n = 5$;

a) For $H_0: \underset{\sim}{\tau}_1 = \underset{\sim}{\tau}_2 = \underset{\sim}{0}$, $\Lambda^* = .3819$

Since

$$-[gb(n-1)-(p+1-(g-1))/2]\ln \Lambda^* = -14.5\ln(.3819) =$$

$$= 13.96 > \chi_3^2(.05) = 7.81,$$

we <u>reject</u> H_0 at $\alpha = .05$ level. For $H_0: \underset{\sim}{\beta}_1 = \underset{\sim}{\beta}_2 = \underset{\sim}{0}$. $\Lambda^* = .5230$ and $-14.5\ln(.5230) = 9.40$. Again we reject H_0 at $\alpha = .05$ level. These results are consistent with the exact F tests.

6.16 $H_0: C\underset{\sim}{\mu} = \underset{\sim}{0}$; $H_1: C\underset{\sim}{\mu} \neq \underset{\sim}{0}$ where $C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Summary statistics:

$$\bar{\underset{\sim}{x}} = \begin{bmatrix} 1906.1 \\ 1749.5 \\ 1509.1 \\ 1725.0 \end{bmatrix}; \qquad S = \begin{bmatrix} 105625 & 94759 & 87249 & 94268 \\ & 101761 & 76166 & 81193 \\ & & 91809 & 90333 \\ & & & 104329 \end{bmatrix}$$

$$T^2 = n(C\bar{\underset{\sim}{x}})'(CSC')^{-1}(C\bar{\underset{\sim}{x}}) = 254.7$$

$$\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha) = \frac{(30-1)(4-1)}{(30-4+1)} F_{3,27}(.05) = 9.54$$

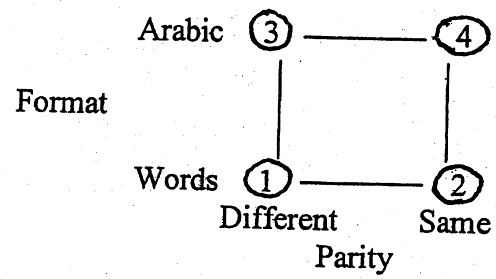
Since $T^2 = 254.7 > 9.54$ we reject H_0 at $\alpha = .05$ level.

95% simultaneous confidence interval for "dynamic" versus "static" means $(\mu_1 + \mu_2) - (\mu_3 + \mu_4)$ is, with $\underset{\sim}{c}' = [1 \quad 1 \quad -1 \quad -1]$,

$$\underset{\sim}{c}'\bar{\underset{\sim}{x}} \pm \sqrt{\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha)} \sqrt{\frac{\underset{\sim}{c}'S\underset{\sim}{c}}{n}}$$

$$= 421.5 \pm 174.5 \longrightarrow (247, 596)$$

6.17 (a)



Effects	Contrast
Parity main:	$(\mu_2+\mu_4)-(\mu_1+\mu_3)$
Format main:	$(\mu_3+\mu_4)-(\mu_1+\mu_2)$
Interaction:	$(\mu_2+\mu_3)-(\mu_1+\mu_4)$

Contrast matrix:

$$C=\begin{pmatrix}-1 & 1 & -1 & 1\\ -1 & -1 & 1 & 1\\ -1 & 1 & 1 & -1\end{pmatrix}$$

Since $T^2=135.9>\frac{31(3)}{29}(2.93)=9.40$, reject $H_0:\mathbf{C}\mu=\mathbf{0}$ (no treatment effects) at the 5% level.

(b) 95% simultaneous T^2 intervals for the contrasts:

Parity main effect: $-206.4\pm\sqrt{9.40}\sqrt{\frac{20{,}598.6}{32}}\rightarrow(-280.3,\ -125.1)$

Format main effect: $-307\pm\sqrt{9.40}\sqrt{\frac{42{,}939.5}{32}}\rightarrow(-411.4,\ -186.9)$

Interaction effect: $22.4\pm\sqrt{9.40}\sqrt{\frac{9{,}818.5}{32}}\rightarrow(-32.3,\ 75.0)$

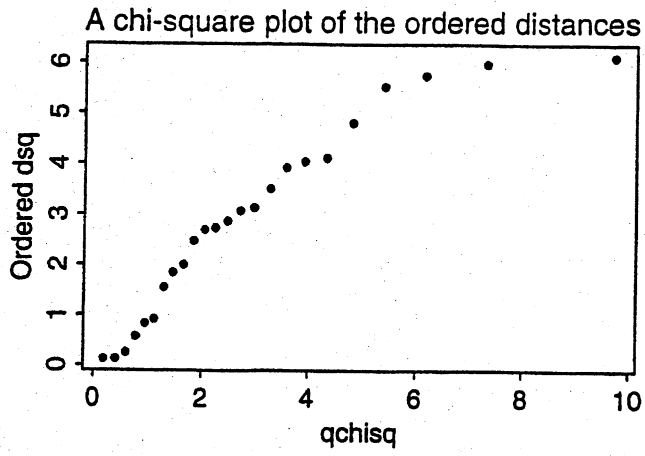
No interaction effect. Parity effect—"different" responses slower than "same" responses. Format effect—"words" slower than "Arabic".

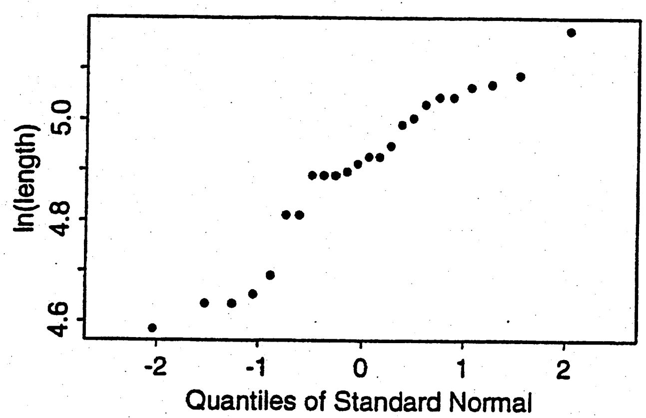
(c) The M model of numerical cognition is a reasonable population model for the scores.

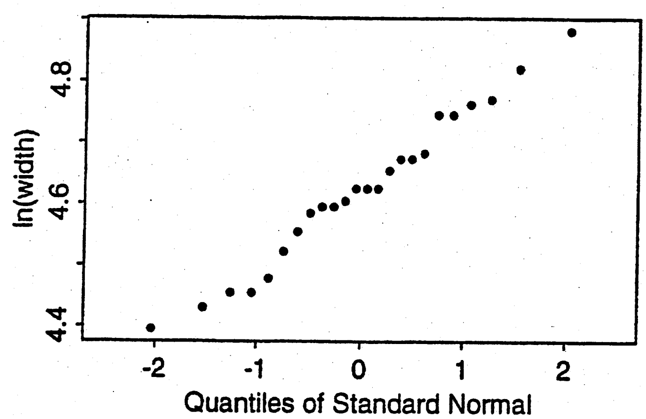
(d) The multivariate normal model is a reasonable model for the scores corresponding to the parity contrast, the format contrast and the interaction contrast.

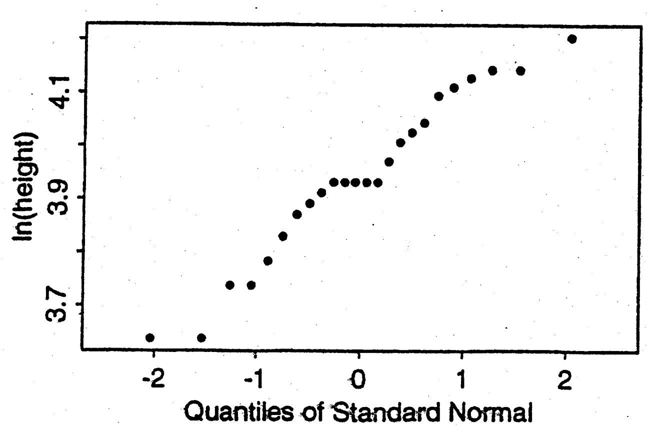
6.18

Female turtle

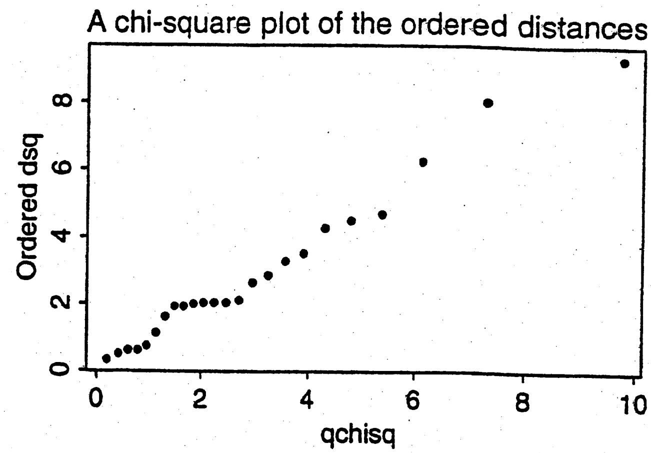


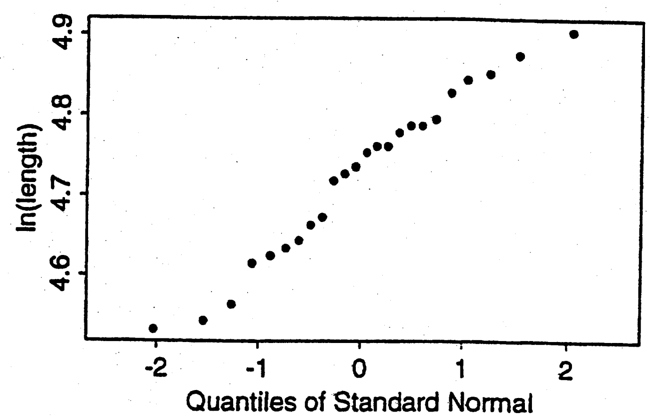


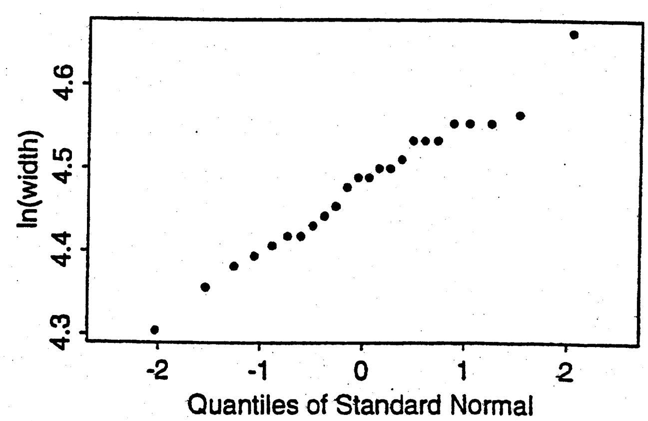


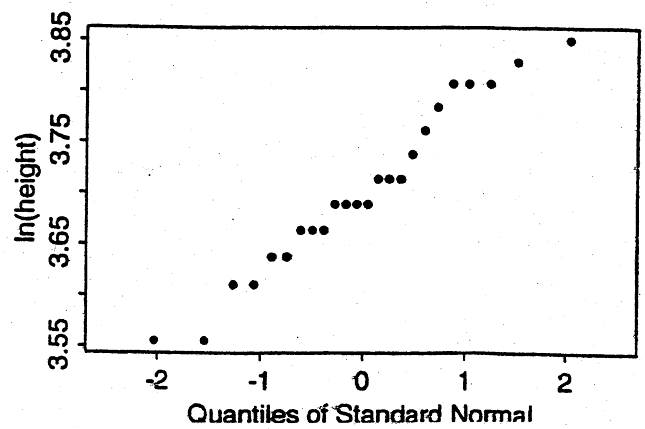


Male turtle









mean vector for females:

X1BAR
4.9006593
4.6229089
3.9402858

mean vector for males:

X2BAR
4.7254436
4.4775738
3.7031858

SPOOLED			
	0.0187388	0.0140655	0.0165386
	0.0140655	0.0113036	0.0127148
	0.0165386	0.0127148	0.0158563

TSQ	CVTSQ	F	CVF	PVALUE
85.052001	8.833461	27.118029	2.8164658	4.355E-10

linear combination most responsible for rejection

of HO has coefficient vector:

COEFFVEC
-43.72677
-8.710687
67.546415

95% simultaneous CI for the difference

in female and male means

LOWER	UPPER
0.0577676	0.2926638
0.0541167	0.2365537
0.1290622	0.3451377

Bonferroni CI

LOWER	UPPER
0.0768599	0.2735714
0.0689451	0.2217252
0.1466248	0.3275751

6.19

a) $\bar{\underset{\sim}{x}}_1 = \begin{bmatrix} 12.219 \\ 8.113 \\ 9.590 \end{bmatrix}$; $\bar{\underset{\sim}{x}}_2 = \begin{bmatrix} 10.106 \\ 10.762 \\ 18.168 \end{bmatrix}$;

$$S_1 = \begin{bmatrix} 223.0134 & 12.3664 & 2.9066 \\ & 17.5441 & 4.7731 \\ & & 13.9633 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 4.3623 & .7599 & 2.3621 \\ & 25.8512 & 7.6857 \\ & & 46.6543 \end{bmatrix};$$

$$S_{pooled} = \begin{bmatrix} 15.8112 & 7.8550 & 2.6959 \\ & 20.7458 & 5.8960 \\ & & 26.5750 \end{bmatrix}$$

$$[(\frac{1}{n_1} + \frac{1}{n_2})S_{pooled}]^{-1} = \begin{bmatrix} 1.0939 & -.4084 & -.0203 \\ & .8745 & -.1525 \\ & & .5640 \end{bmatrix}$$

H_0: $\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$

Since $T^2 = (\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2)'[(\frac{1}{n_1} + \frac{1}{n_2})S_{pooled}]^{-1}(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2) = 50.92$

$$> \frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,n_1+n_2-p-1}(.01) = \frac{(57)(3)}{55} F_{3,55}(.01) = 13,$$

we reject H_0 at the $\alpha = .01$ level. There is a difference in the (mean) cost vectors between gasoline trucks and diesel trucks.

b) $\hat{\underset{\sim}{a}} = S^{-1}_{pooled}(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2) = \begin{bmatrix} 3.58 \\ -1.88 \\ -4.48 \end{bmatrix}$

c) 99% simultaneous confidence intervals are:

$$\mu_{11} - \mu_{21}: \ 2.113 \pm 3.790$$

$$\mu_{12} - \mu_{22}: \ -2.650 \pm 4.341$$

$$\mu_{13} - \mu_{23}: \ -8.578 \pm 4.913$$

d) Assumption $\Sigma_1 = \Sigma_2$.

Since S_1 and S_2 are quite different, it may not be reasonable to pool. However, using "large sample" theory ($n_1 = 36$, $n_2 = 23$) we have, by Result 6.4,

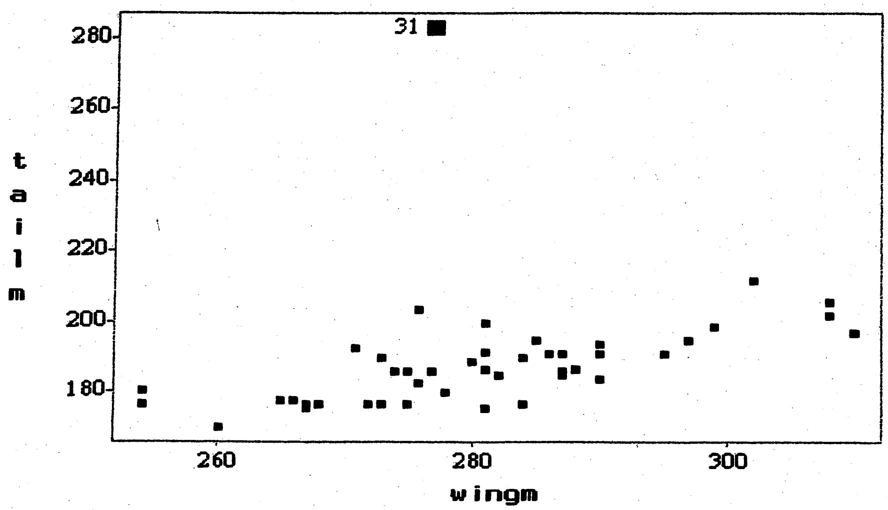
$$(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2 - (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2))'[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2]^{-1}(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2 - (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)) \sim \chi_p^2$$

Since

$$(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2)'[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2]^{-1}(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2) = 43.15 > \chi_3^2(.01) = 11.34$$

we reject $H_0: \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$ at the $\alpha = .01$ level. This is consistent with the result in part (a).

6.20 (a)



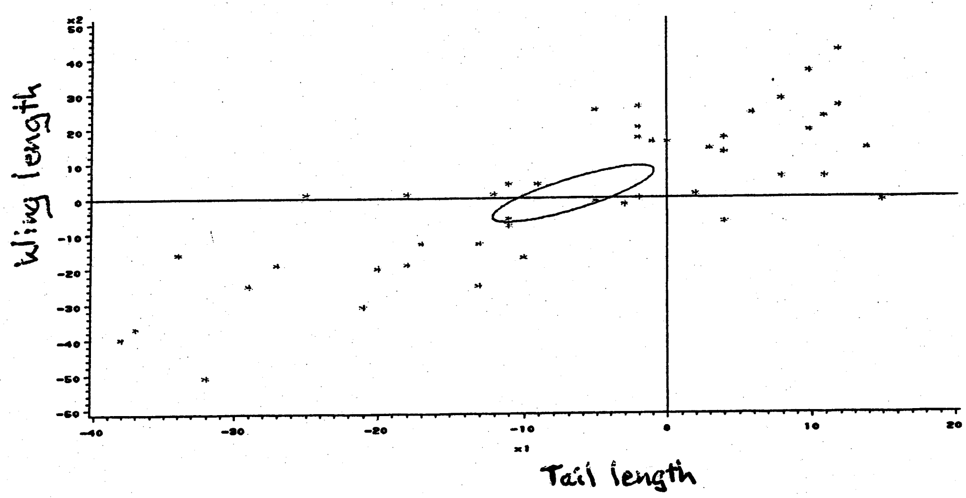
(b) The output below shows that the analysis does not differ when we delete the observation 31 or when we consider it equals 184. Both tests reject the null hypothesis of equal mean difference. The most critical linear combination leading to the rejection of H_0 has coefficient vector $[-3.490238; 2.07955]'$ and the the linear combination most responsible for the rejection of H_0 is the Tail difference.

(c) Results below.

```
Comparing Mean Vectors from Two Populations

          [Obs. 31 Deleted]

          T2          C
    25.005014 5.9914645

    Reject HO. There is mean difference

 95% simultaneous confidence intervals:

 LABELCI              LICIMD    LSCIMD

 Mean Diff. 1:   -11.76436 -1.161905  (Tail difference)
 Mean Diff. 2:   -5.985685 8.3392202  (Wing difference)

 RESULT                    COEF

 Coefficient Vector:   -3.490238
                        2.07955
```

```
Comparing Mean Vectors from Two Populations
          Tail
        [Obs. 31 = 184]

        T2         C
   25.662531 5.9914645

   Reject H0. There is mean difference

  95% simultaneous confidence intervals:

  LABELCI            LICIMD    LSCIMD

  Mean Diff. 1:   -11.78669  -1.27998
  Mean Diff. 2:   -6.003431 8.1812088

  RESULT                    COEF

  Coefficient Vector:   -3.574268
                         2.1220203
```

95% Confidence Ellipse About the Mean Vector



(d) Female birds are generally larger, since the confidence interval bounds for difference in Tails (Male - Female) are negative and the confidence interval for difference in Wings includes zero, indicating no significance difference.

6.21 (a) The (4,2) and (4,4) entries in S_1 and S_2 differ considerably. However, $n_1 = n_2$ so the large sample approximation amounts to pooling.

(b) H_0: $\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$ and H_1: $\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 \neq \underset{\sim}{0}$

$$T^2 = 15.830 > \frac{(38)(4)}{35} F_{4,35}(.05) = 11.47$$

so we reject H_0 at the $\alpha = .05$ level.

(c) $$\hat{\underset{\sim}{a}} \propto S^{-1}_{pooled}(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2) = \begin{bmatrix} -.24 \\ .16 \\ -3.74 \\ .01 \end{bmatrix}$$

(d) Looking at the coefficients $\hat{a}_i\sqrt{s_{ii,\text{pooled}}}$, which apply to the standardized variables, we see that X_2: long term interest rate has the largest coefficient and therefore might be useful in classifying a bond as "high" or "medium" quality.

(e) From (b), $T^2 = 15.830$. Have $p = 4$ and $\nu = \dfrac{4+16}{.53556} = 37.344$ so, at the 5% level, the critical value is

$$\frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}(.05) = \frac{37.344(4)}{37.344 - 4 + 1} F_{4,37.344-4+1}(.05) = \frac{149.376}{34.344}(2.647) = 11.513$$

Since $T^2 = 15.830 > 11.513$, reject $H_0 : \mu_1 - \mu_2 = \mathbf{0}$, the same conclusion reached in (b). Notice the critical value here is only slightly larger than the critical value in (b).

6.22 (a) The sample means for female and male are :

$$\bar{x}_F = \begin{bmatrix} 0.3136 \\ 5.1788 \\ 2.3152 \\ 38.1548 \end{bmatrix}, \quad \bar{x}_M = \begin{bmatrix} 0.3972 \\ 5.3296 \\ 3.6876 \\ 49.3404 \end{bmatrix}.$$

The Hotelling's $T^2 = 96.487 > 11.00$ where 11.00 is a critical point corresponding to $\alpha = 0.05$. Therefore, we reject $H_0 : \mu_1 - \mu_2 = 0$. The coefficient of the linear combination of most responsible for rejection is $(-95.600, 6.145, 5.737, -0.762)'$.

(b) The 95% simultaneous C. I. for female mean $-$male mean:

$$\begin{bmatrix} -0.1697234, & 0.00252336 \\ -1.4650835, & 1.16348346 \\ -1.8760572, & -0.8687428 \\ -17.032834, & -5.3383659 \end{bmatrix}$$

(c) We cannot extend the obtained result to the population of persons in their mid-twenties. Firstly this was a self selected sample of volunteers (friends) and is not even a random sample of graduate students. Further, graduate students are probably more sedentary than the typical persons of their age.

6.23 $n_1 = n_2 = n_3 = 50$; $p = 2$, $g = 3$ (sepal width and petal width responses only!)

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 3.428 \\ .306 \end{bmatrix}; \quad S_1 = \begin{bmatrix} .14364 & -.00474 \\ & .18576 \end{bmatrix}$$

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} 2.770 \\ 1.326 \end{bmatrix}; \quad S_2 = \begin{bmatrix} .09860 & .04128 \\ & .03920 \end{bmatrix}$$

$$\bar{\mathbf{x}}_3 = \begin{bmatrix} 2.974 \\ 2.026 \end{bmatrix}; \quad S_3 = \begin{bmatrix} .10368 & .04764 \\ & .07563 \end{bmatrix}$$

MANOVA Table:

Source	SSP	d.f.
Treatment	$B = \begin{bmatrix} 11.344 & -21.820 \\ & 75.352 \end{bmatrix}$	2
Residual	$W = \begin{bmatrix} 16.950 & 4.125 \\ & 14.729 \end{bmatrix}$	147
Total	$B+W = \begin{bmatrix} 28.294 & -17.695 \\ & 90.081 \end{bmatrix}$	149

$$\Lambda^* = \frac{|W|}{|B+W|} = \frac{232.64}{2235.64} = .104$$

Since $\left(\frac{\Sigma n_\ell - p - 2}{p}\right)\left(\frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) = 153.3 > 2.37 \doteq F_{4,292}(.05)$

we reject H_0: $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \boldsymbol{\tau}_3$ at the $\alpha = .05$ level.

6.24 Wilks' lambda: $\Lambda^* = .8301$. Since $g = 3$, $\left(\frac{90-4-2}{4}\right)\left(\frac{1-\sqrt{.8301}}{\sqrt{.8301}}\right) = 2.049$ is an F value with 8 and 168 degrees of freedom. Since p-value $= P(F > 2.049) = .044$, we would just reject the null hypothesis $H_0 : \underset{\sim}{\tau}_1 = \underset{\sim}{\tau}_2 = \underset{\sim}{\tau}_3 = \underset{\sim}{0}$ at the 5% level implying there is a time period effect.

F statistics and p-values for ANOVA's:

	F	p-value
MaxBrth:	3.66	.030
BasHght:	0.47	.629
BasLgth:	3.84	.025
NasHght:	0.10	.901

Any differences over time periods are probably due to changes in maximum breath of skull (MaxBrth) and basialveolar length of skull (BasLgth).

95% Bonferroni simultaneous intervals: $m = pg(g-1)/2 = 12$,
$t_{87}(.05/24) = 2.94$

BasBrth

$\tau_{11} - \tau_{21}$: $-1 \pm 2.94\sqrt{\frac{1785.4}{87}\left(\frac{1}{30}+\frac{1}{30}\right)} \rightarrow -1 \pm 3.44$

$\tau_{11} - \tau_{31}$: -3.1 ± 3.44

$\tau_{21} - \tau_{31}$: -2.1 ± 3.44

BasHght

$\tau_{12} - \tau_{22}$: $0.9 \pm 2.94\sqrt{\frac{1924.3}{87}\left(\frac{1}{30}+\frac{1}{30}\right)} \rightarrow 0.9 \pm 3.57$

$\tau_{12} - \tau_{32}$: -0.2 ± 3.57

$\tau_{22} - \tau_{32}$: -1.1 ± 3.57

BasLgth

$\tau_{13} - \tau_{23}$: $0.10 \pm 2.94\sqrt{\frac{2153}{87}\left(\frac{1}{30}+\frac{1}{30}\right)} \rightarrow 0.10 \pm 3.78$

$\tau_{13} - \tau_{33}$: 3.14 ± 3.78

$\tau_{23} - \tau_{33}$: 3.03 ± 3.78

NasHgth

$\tau_{14} - \tau_{24}$: $0.30 \pm 2.94\sqrt{\frac{840.2}{87}\left(\frac{1}{30}+\frac{1}{30}\right)} \rightarrow 0.30 \pm 2.36$

$\tau_{14} - \tau_{34}$: -0.03 ± 2.36

$\tau_{24} - \tau_{34}$: -0.33 ± 2.36

All the simultaneous intervals include 0. Evidence for changes in skull size over time is marginal. If changes exist, then these changes might be in maximum breath and basialveolar length of skull from time periods 1 to 3.

The usual MANOVA assumptions appear to be satisfied for these data.

6.25

Without transforming the data, $\Lambda^* = \frac{|W|}{|B+W|} = .1159$ and $F = 18.98$.

After transformation, $\Lambda^* = .1198$ and $F = 18.52 > F_{10,98}(.05) = 1.93$

There is a clear need for transforming the data to make the hypothesis tenable.

6.26 To test for parallelism, consider H_{01}: $C\underline{\mu}_1 = C\underline{\mu}_2$ with C given by (6-61).

$$C(\bar{\underline{x}}_1 - \bar{\underline{x}}_2) = \begin{bmatrix} -.413 \\ -.167 \\ -.036 \end{bmatrix}; \quad (CS_{pooles}C')^{-1} = \begin{bmatrix} 1.674 & .947 & .616 \\ & 2.014 & 1.144 \\ & & 2.341 \end{bmatrix}$$

$T^2 = 9.58 > c^2 = 8.0$, we reject H_0 at the $\alpha = .05$ level. The excess electrical usage of the test group was much lower than that of the control group for the 11 A.M., 1 P.M. and 3 P.M. hours. The similar 9 A.M. usage for the two groups contradicts the parallelism hypothesis.

6.27 a) Plots of the husband and wife profiles look similar but seem disparate for the level of "companionate love that you feel for your partner".

b) Parallelism hypothesis H_0: $C\underline{\mu}_1 = C\underline{\mu}_2$ with C given by (6-61).

$$C(\bar{\underline{x}}_1 - \bar{\underline{x}}_2) = \begin{bmatrix} -.13 \\ -.17 \\ .33 \end{bmatrix}; \quad CS_{pooled}C' = \begin{bmatrix} .685 & .733 & .029 \\ & .870 & -.028 \\ & & .095 \end{bmatrix}$$

For $\alpha = .05$, $c^2 = 8.7$ (see (6-62)). Since $T^2 = 19.58 > c^2 = 8.7$ we reject H_0 at the $\alpha = .05$ level.

6.28 $T^2 = 106.13 > 16.59$. We reject $H_0 : \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$ at 5% significance level. There is a significant difference in the two species.

Sample Mean for L.torrens and L.carteri:

L.torrens	L.carteri	Difference
96.457	99.343	-2.886
42.914	43.743	-0.829
35.371	39.314	-3.943
14.514	14.657	-0.143
25.629	30.000	-4.371
9.571	9.657	-0.086
9.714	9.371	0.343

Pooled Sample Covariance Matrix:

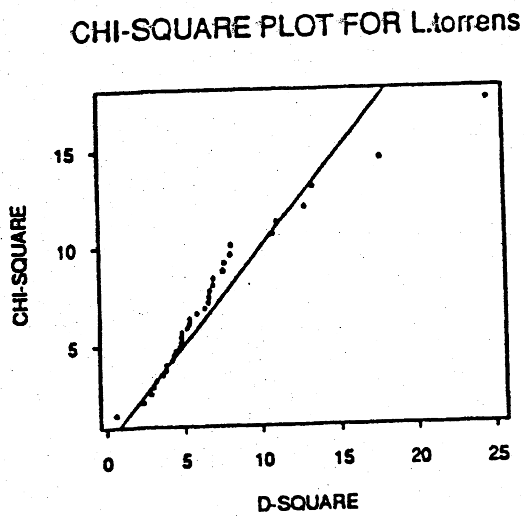
36.008	14.595	6.078	3.675	9.573	2.426	2.649
	16.639	2.764	2.992	6.101	1.053	0.934
		6.437	0.692	1.615	0.211	0.671
			3.039	2.407	0.274	0.229
				13.767	0.565	0.637
					1.213	0.914
						0.990

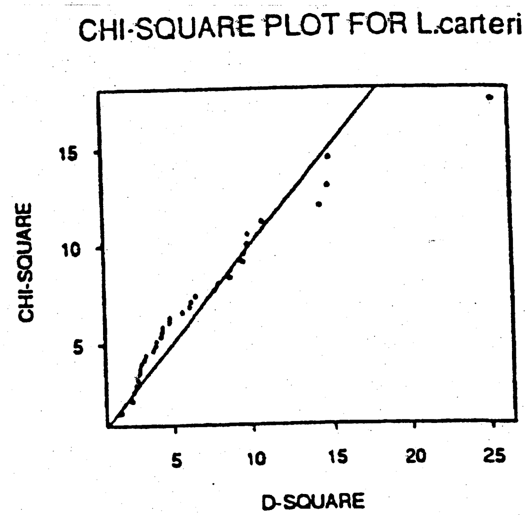
Linear Combination of most responsible for rejection
of Ho: L.torrens mean - L.carteri mean = 0 is :
(0.006, 0.151, -0.854, 0.268, -0.383, -2.187, 2.971)'

95% Simultaneous C. I. for L.torrens mean - L.carteri mean:

LOWER	UPPER
-8.73	2.96
-4.80	3.14
-6.41	-1.47
-1.84	1.55
-7.98	-0.76
-1.16	0.99
-0.63	1.31

The third and fifth components are most responsible for rejecting H_0. The χ^2 plots look fairy straight.





6.29

(a).

	XBAR	S		
Summary Statistics:	0.02548	0.00366259	0.00482862	0.00154159
	0.05784	0.00482862	0.01628931	0.00304801
	0.01056	0.00154159	0.00304801	0.00602526

Hotelling's $T^2 = 5.946$. The critical point is 9.979 and we fail to reject $H_0 : \mu_1 - \mu_2 = 0$ at 5% significance level.

(b). (c).

	LOWER	UPPER
Bonferroni C. I.:	-0.0057	0.0566
	-0.0079	0.1235
	-0.0294	0.0505
Simultaneous C. I.:	-0.0128	0.0637
	-0.0228	0.1385
	-0.0385	0.0596

6.30

HOTELLING T SQUARE = 9.0218
P-VALUE 0.3616

	N	MEAN	STDEV	T2 INTERVAL	TO	BONFERRONI	TO
x1	24	0.00012	0.04817	-.0443	.0445	-.0283	.0285
x2	24	-0.00325	0.02751	-.0286	.0221	-.0195	.0130
x3	24	-0.0072	0.1030	-.1020	.0876	-.0679	.0535
x4	24	-0.0123	0.0625	-.0701	.0455	-.0493	.0247
x5	24	0.01513	0.03074	-.0130	.0436	-.0030	.0333
x6	24	0.00017	0.04689	-.0430	.0434	-.0275	.0278

The Bonferroni intervals use t (.00417) = 2.89 and

the T intevals use the constant 4.516.

6.31 **(a)** Two-factor MANOVA of peanuts data

E = Error SS&CP Matrix

	X1	X2	X3
X1	104.205	49.365	76.48
X2	49.365	352.105	121.995
X3	76.48	121.995	94.835

H = Type III SS&CP Matrix for FACTOR1 (Location)

	X1	X2	X3
X1	0.7008333333	-10.6575	7.1291666667
X2	-10.6575	162.0675	-108.4125
X3	7.1291666667	-108.4125	72.520833333

Manova Test Criteria and Exact F Statistics for
the Hypothesis of no Overall FACTOR1 Effect
H = Type III SS&CP Matrix for FACTOR1   E = Error SS&CP Matrix

S=1   M=0.5   N=1

Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.10651620	11.1843	3	4	0.0205
Pillai's Trace	0.89348380	11.1843	3	4	0.0205
Hotelling-Lawley Trace	8.38824348	11.1843	3	4	0.0205
Roy's Greatest Root	8.38824348	11.1843	3	4	0.0205

H = Type III SS&CP Matrix for FACTOR2 (Variety)

	X1	X2	X3
X1	196.115	365.1825	42.6275
X2	365.1825	1089.015	414.655
X3	42.6275	414.655	284.10166667

Manova Test Criteria and F Approximations for
the Hypothesis of no Overall FACTOR2 Effect
H = Type III SS&CP Matrix for FACTOR2   E = Error SS&CP Matrix

S=2   M=0   N=1

Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.01244417	10.6191	6	8	0.0019
Pillai's Trace	1.70910921	9.7924	6	10	0.0011
Hotelling-Lawley Trace	21.37567504	10.6878	6	6	0.0055
Roy's Greatest Root	18.18761127	30.3127	3	5	0.0012

H = Type III SS&CP Matrix for FACTOR1*FACTOR2

	X1	X2	X3
X1	205.10166667	363.6675	107.78583333
X2	363.6675	780.695	254.22
X3	107.78583333	254.22	85.951666667

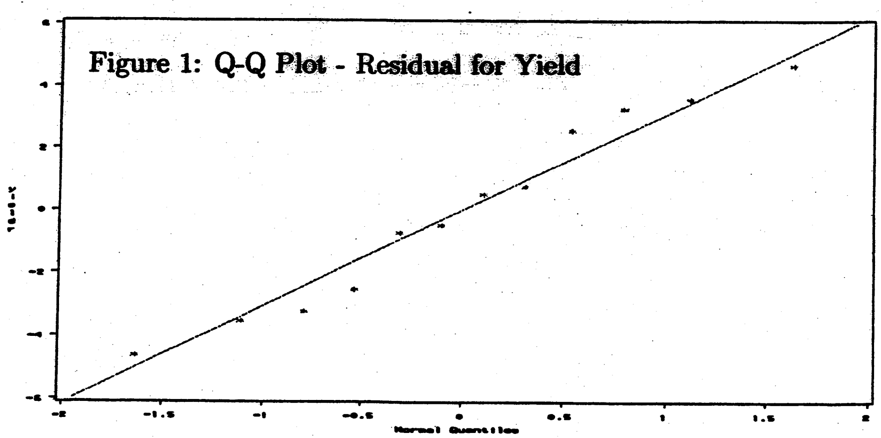
```
Manova Test Criteria and F Approximations for
the Hypothesis of no Overall FACTOR1*FACTOR2 Effect
H = Type III SS&CP Matrix for FACTOR1*FACTOR2    E = Error SS&CP Matrix

S=2     M=0     N=1
```

Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.07429984	3.5582	6	8	0.0508
Pillai's Trace	1.29086073	3.0339	6	10	0.0587
Hotelling-Lawley Trace	7.54429038	3.7721	6	6	0.0655
Roy's Greatest Root	6.82409388	11.3735	3	5	0.0113

(b) The residuals for X_2 at location 2 for variety 5 seem large in absolute value, but Q-Q plots of residuals indicate that univariate normality cannot be rejected for all three variables.

CODE	FACTOR1	FACTOR2	PRED1	RES1	PRED2	RES2	PRED3	RES3
a	1	5	194.80	0.50	160.40	-7.30	52.55	-1.15
a	1	5	194.80	-0.50	160.40	7.30	52.55	1.15
b	2	5	185.05	4.65	130.30	9.20	49.95	5.55
b	2	5	185.05	-4.65	130.30	-9.20	49.95	-5.55
c	1	6	199.45	3.55	161.40	-4.60	47.80	2.00
c	1	6	199.45	-3.55	161.40	4.60	47.80	-2.00
d	2	6	200.15	2.55	163.95	2.15	57.25	3.15
d	2	6	200.15	-2.55	163.95	-2.15	57.25	-3.15
e	1	8	190.25	3.25	164.80	-0.30	58.20	-0.40
e	1	8	190.25	-3.25	164.80	0.30	58.20	0.40
f	2	8	200.75	0.75	170.30	-3.50	66.10	-1.10
f	2	8	200.75	-0.75	170.30	3.50	66.10	1.10

Figure 1: Q-Q Plot - Residual for Yield



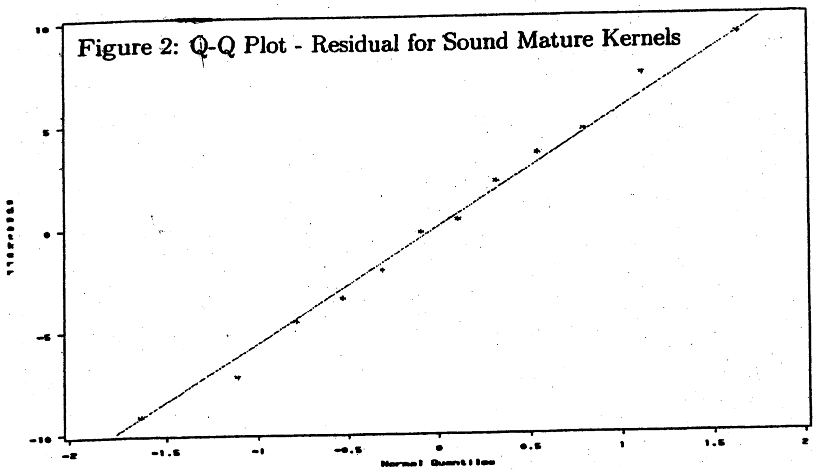


Figure 2: Q-Q Plot - Residual for Sound Mature Kernels

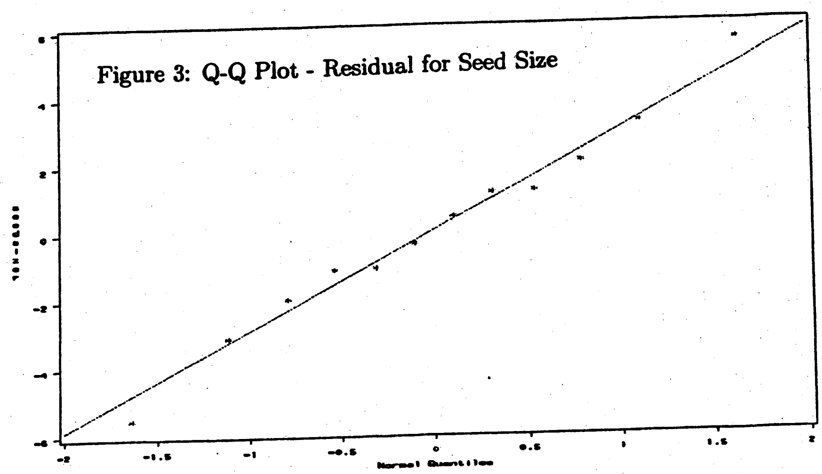


Figure 3: Q-Q Plot - Residual for Seed Size

(c) Univariate two factor ANOVAs follow. Evidence of variety effect and, for X_1 = yield and X_2 = sound mature kernel, a location*variety interaction.

Dependent Variable: yield

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	401.9175000	80.3835000	4.63	0.0446
Error	6	104.2050000	17.3675000		
Corrected Total	11	506.1225000			

R-Square	Coeff Var	Root MSE	yield Mean
0.794111	2.136324	4.167433	195.0750

Source	DF	Type III SS	Mean Square	F Value	Pr > F
location	1	0.7008333	0.7008333	0.04	0.8474
variety	2	196.1150000	98.0575000	5.65	0.0418
location*variety	2	205.1016667	102.5508333	5.90	0.0382

Dependent Variable: sdmatker

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	2031.777500	406.355500	6.92	0.0177
Error	6	352.105000	58.684167		
Corrected Total	11	2383.882500			

R-Square	Coeff Var	Root MSE	sdmatker Mean
0.852298	4.832398	7.660559	158.5250

Source	DF	Type III SS	Mean Square	F Value	Pr > F
location	1	162.067500	162.067500	2.76	0.1476
variety	2	1089.015000	544.507500	9.28	0.0146
location*variety	2	780.695000	390.347500	6.65	0.0300

The GLM Procedure

Dependent Variable: seedsize

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	442.5741667	88.5148333	5.60	0.0292
Error	6	94.8350000	15.8058333		
Corrected Total	11	537.4091667			

R-Square	Coeff Var	Root MSE	seedsize Mean
0.823533	7.188166	3.975655	55.30833

Source	DF	Type III SS	Mean Square	F Value	Pr > F
location	1	72.5208333	72.5208333	4.59	0.0759
variety	2	284.1016667	142.0508333	8.99	0.0157
location*variety	2	85.9516667	42.9758333	2.72	0.1443

(d) Bonferroni simultaneous comparisons of variety.
Only varieties 5 and 8 differ, and they differ only on X_3.

Bonferroni (Dunn) T tests for variable: X1
Alpha= 0.05 Confidence= 0.95 df= 8 MSE= 38.66333
Critical Value of T= 3.01576
Minimum Significant Difference= 13.26
Comparisons significant at the 0.05 level are indicated by '***'.

FACTOR2 Comparison	Simultaneous Lower Confidence Limit	Difference Between Means	Simultaneous Upper Confidence Limit
6 - 8	-8.960	4.300	17.560
6 - 5	-3.385	9.875	23.135
8 - 6	-17.560	-4.300	8.960
8 - 5	-7.685	5.575	18.835
5 - 6	-23.135	-9.875	3.385
5 - 8	-18.835	-5.575	7.685

Bonferroni (Dunn) T tests for variable: X2
Alpha= 0.05 Confidence= 0.95 df= 8 MSE= 141.6
Critical Value of T= 3.01576
Minimum Significant Difference= 25.375
Comparisons significant at the 0.05 level are indicated by '***'.

FACTOR2 Comparison	Simultaneous Lower Confidence Limit	Difference Between Means	Simultaneous Upper Confidence Limit
8 - 6	-20.500	4.875	30.250
8 - 5	-3.175	22.200	47.575
6 - 8	-30.250	-4.875	20.500
6 - 5	-8.050	17.325	42.700
5 - 8	-47.575	-22.200	3.175
5 - 6	-42.700	-17.325	8.050

Bonferroni (Dunn) T tests for variable: X3
Alpha= 0.05 Confidence= 0.95 df= 8 MSE= 22.59833
Critical Value of T= 3.01576
Minimum Significant Difference= 10.137
Comparisons significant at the 0.05 level are indicated by '***'.

FACTOR2 Comparison	Simultaneous Lower Confidence Limit	Difference Between Means	Simultaneous Upper Confidence Limit	
8 - 6	-0.512	9.625	19.762	
8 - 5	0.763	10.900	21.037	***
6 - 8	-19.762	-9.625	0.512	
6 - 5	-8.862	1.275	11.412	
5 - 8	-21.037	-10.900	-0.763	***
5 - 6	-11.412	-1.275	8.862	

6.32 (a) MANOVA for Species: Wilks' lambda $\Lambda_1^* = .00823$

$F = 5.011$; $p\text{-value} = P(F > 5.011) = .173$

$F_{4,2}(.05) = 19.25$

Do not reject H_0: No species effects

MANOVA for Nutrient: Wilks' lambda $\Lambda_2^* = .31599$

$F = 1.082$; $p\text{-value} = P(F > 1.082) = .562$

$F_{2,1}(.05) = 199.5$

Do not reject H_0: No nutrient effects

(b) Minitab output for the two-way ANOVA's:

560CM

Analysis of Variance for 560CM

Source	DF	SS	MS	F	P
Spec	2	47.476	23.738	10.06	0.090
Nutrient	1	8.260	8.260	3.50	0.202
Error	2	4.722	2.361		
Total	5	60.458			

720CM

Analysis of Variance for 720CM

Source	DF	SS	MS	F	P
Spec	2	262.239	131.119	28.82	0.034
Nutrient	1	4.489	4.489	0.99	0.425
Error	2	9.099	4.550		
Total	5	275.827			

The ANOVA results are mostly consistent with the MANOVA results. The exception is for 720CM where there appears to be Species effects. A look at the data suggests the spectral reflectance of Japanese larch (JL) at 720 nanometers is somewhat larger than the reflectance of the other two species (SS and LP) regardless of nutrient level. This difference is not as apparent at 560 nanometers.

For MANOVA, the value of Wilks' lambda statistic does not indicate Species effects. However, Pillai's trace statistic, 1.6776 with $F = 5.203$ and $p\text{-value} = .07$, suggests there may be Species effects. (For Nutrient, Wilks' lambda and Pillai's trace statistic give the same F value.) For larger sample sizes, Wilks' lambda and Pillai's trace statistic would give essentially the same result for all factors.

6.33 (a) MANOVA for Species: Wilks' lambda $\Lambda_1^* = .06877$

$F = 36.571$; p-value $= P(F > 36.571) = .000$

$F_{4,52}(.05) = 2.55$

Reject H_0: No species effects

MANOVA for Time: Wilks' lambda $\Lambda_2^* = .04917$

$F = 45.629$; p-value $= P(F > 45.629) = .000$

$F_{4,52}(.05) = 2.55$

Reject H_0: No time effects

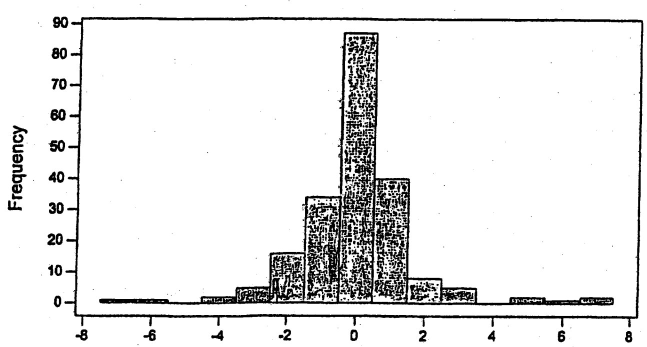
MANOVA for Species*Time: Wilks' lambda $\Lambda_{12}^* = .08707$

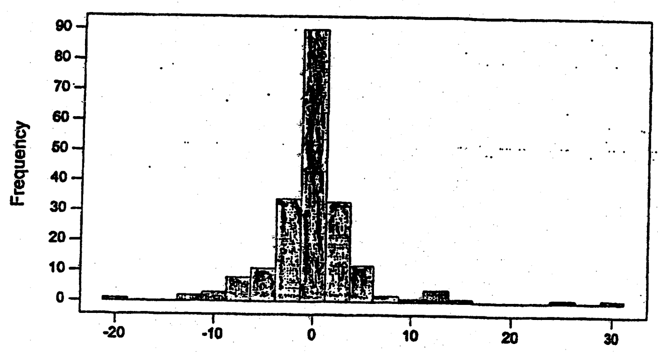
$F = 15.528$; p-value $= P(F > 15.528) = .000$

$F_{8,52}(.05) = 2.12$

Reject H_0: No interaction effects

(b) A few outliers but, in general, residuals approximately normally distributed (see histograms below). Observations are likely to be positively correlated over time. Observations are not independent.





(c) Interaction shows up for the 560nm wavelength but not for the 720nm wavelength. See the Minitab ANOVA output below.

Analysis of Variance for 560nm

Source	DF	SS	MS	F	P
Species	2	965.18	482.59	169.97	0.000
Time	2	1275.25	637.62	224.58	0.000
Species*Time	4	795.81	198.95	70.07	0.000
Error	27	76.66	2.84		
Total	35	3112.90			

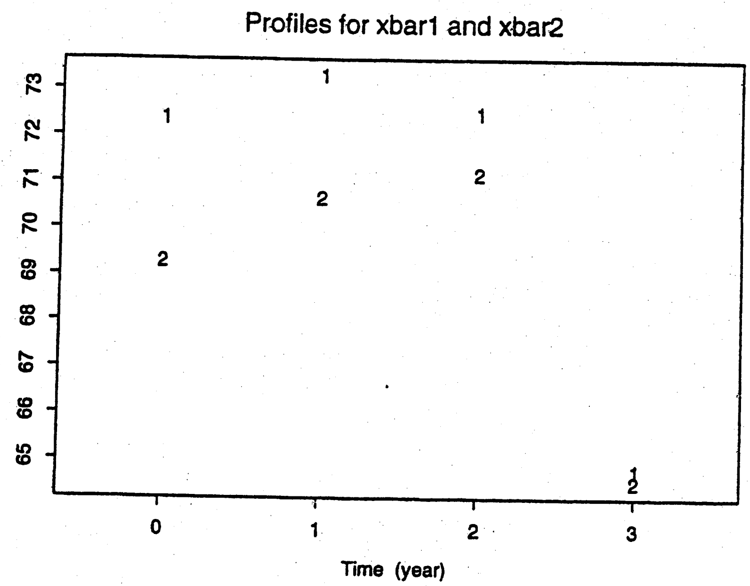
Analysis of Variance for 720nm

Source	DF	SS	MS	F	P
Species	2	2026.86	1013.43	15.46	0.000
Time	2	5573.81	2786.90	42.52	0.000
Species*Time	4	193.55	48.39	0.74	0.574
Error	27	1769.64	65.54		
Total	35	9563.85			

6.33 (Continued)

(d) The data might be analyzed using the growth curve methodology discussed in Section 6.9. The data might also be analyzed assuming species are "nested" within date. In this case, an interesting question is: Is spectral reflectance the same for all species for each date?

6.34 Fitting a linear growth curve to calcium measurements on the dominant ulna



XBAR

72.3800	69.2875
73.2933	70.6562
72.4733	71.1812
64.7867	64.5312

Grand mean

71.1939
71.8273
72.1848
65.2667

MLE of beta

73.4707	70.5049
-1.9035	-0.9818

[B'Sp^(-1)B]^(-1)

93.1313	-5.2393
-5.2393	1.2948

S1

92.1189	86.1106	73.3623	74.5890
86.1106	89.0764	72.9555	71.7728
73.3623	72.9555	71.8907	63.5918
74.5890	71.7728	63.5918	75.4441

S2

98.1745	97.0134	89.4824	86.1111
97.0134	100.5960	88.1425	88.2095
89.4824	88.1425	86.3496	80.5506
86.1111	88.2095	80.5506	81.4156

Spooled

95.2511	91.7500	81.7003	80.5487
91.7500	95.0348	80.8108	80.2745
81.7003	80.8108	79.3694	72.3636
80.5487	80.2745	72.3636	78.5328

W = (N-g)*Spooled

2762.282	2660.749	2369.308	2335.912
2660.749	2756.009	2343.514	2327.961
2369.308	2343.514	2301.714	2098.544
2335.912	2327.961	2098.544	2277.452

Estimated covariance matrix

7.1816	-0.4040	0.0000	0.0000
-0.4040	0.0998	0.0000	0.0000
0.0000	0.0000	6.7328	-0.3788
0.0000	0.0000	-0.3788	0.0936

W1

2803.839	2610.438	2271.920	2443.549
2610.438	2821.243	2464.120	2196.065
2271.920	2464.120	2531.625	1845.313
2443.549	2196.065	1845.313	2556.818

Lambda = |W|/|W1| = 0.201

Since, with $\alpha = 0.01$, $-\left[N - \frac{1}{2}(p-q+g)\right]\log(\Lambda) = 45.72 > \chi^2_{(4-1-1)2}(0.01) = 13.28$, we reject the null hypothesis of a linear fit at $\alpha = 0.01$.

6.35 Fitting a quadratic growth curve to calcium measurements on the dominant ulna, treating all 31 subjects as a single group.

```
XBAR            MLE of beta        [B'Sp^(-1)B]^(-1)
70.7839         71.6039            92.2789 -5.9783  0.0799
71.9323          3.8673            -5.9783  9.3020 -2.9033
71.8065         -1.9404             0.0799 -2.9033  1.0760
64.6548

S                                  W = (n-1)*S
94.5441 90.7962 80.0081 78.0676    2836.322 2723.886 2400.243 2342.027
90.7962 93.6616 78.9965 77.7725    2723.886 2809.848 2369.894 2333.175
80.0081 78.9965 77.1546 70.0366    2400.243 2369.894 2314.639 2101.099
78.0676 77.7725 70.0366 75.9319    2342.027 2333.175 2101.099 2277.957

Estimated covariance matrix        W2
 3.1894 -0.2066  0.0028            2857.167 2764.522 2394.410 2369.674
-0.2066  0.3215 -0.1003            2764.522 2889.063 2358.522 2387.070
 0.0028 -0.1003  0.0372            2394.410 2358.522 2316.271 2093.362
                                   2369.674 2387.070 2093.362 2314.625

Lambda = |W|/|W2| = 0.7653
```

Since, with $\alpha = 0.01$, $-\left[n - \frac{1}{2}(p-q+1)\right] \log(\Lambda) = 7.893 > \chi^2_{4-2-1}(0.01) = 6.635$, we reject the null hypothesis of a quadratic fit at $\alpha = 0.01$.

6.36 Here

$$p = 2, n_1 = 45, n_2 = 55, \ln|S_1| = 19.90948, \ln|S_2| = 18.40324, \ln|S_{pooled}| = 19.27712$$

so
$$u = \left[\frac{1}{44} + \frac{1}{54} - \frac{1}{44+54}\right]\left[\frac{2(4)+3(2)-1}{6(2+1)(2-1)}\right] = .02242$$

and
$$C = (1 - .02242)(98(19.27712) - 44(19.90948) - 54(18.40324)) = 18.93$$

The chi-square degrees of freedom $\nu = \frac{1}{2}2(3)(1) = 3$ and $\chi^2_3(.05) = 7.81$. Since $C = 18.93 > \chi^2_3(.05) = 7.83$, reject $H_0 : \Sigma_1 = \Sigma_2 = \Sigma$ at the 5% level.

6.37 Here

$$p = 3, n_1 = 24, n_2 = 24, \ln|S_1| = 9.48091, \ln|S_2| = 6.67870, \ln|S_{pooled}| = 8.62718$$

so $$u = \left[\frac{1}{23} + \frac{1}{23} - \frac{1}{23+23}\right]\left[\frac{2(9)+3(3)-1}{6(3+1)(2-1)}\right] = .07065$$

and

$$C = (1 - .07065)(46(8.62718) - 23(9.48091) - 23(6.67870)) = 23.40$$

The chi-square degrees of freedom $\nu = \frac{1}{2}3(4)(1) = 6$ and $\chi_6^2(.05) = 12.59$. Since $C = 23.40 > \chi_6^2(.05) = 12.59$, reject $H_0 : \Sigma_1 = \Sigma_2 = \Sigma$ at the 5% level.

6.38 Working with the transformed data, X_1 = vanadium, $X_2 = \sqrt{\text{iron}}$, $X_3 = \sqrt{\text{beryllium}}$, $X_4 = 1/[\text{saturated hydrocarbons}]$, X_5 = aromatic hydrocarbons, we have
$p = 5, n_1 = 7, n_2 = 11, n_3 = 38, \ln|S_1| = -17.81620, \ln|S_2| = -7.24900,$
$\ln|S_3| = -7.09274, \ln|S_{pooled}| = -7.11438$

so $$u = \left[\frac{1}{6} + \frac{1}{10} + \frac{1}{37} - \frac{1}{6+10+37}\right]\left[\frac{2(25)+3(5)-1}{6(5+1)(3-1)}\right] = .24429$$

and

$$C = (1 - .24429)(53(-7.11438) - 6(-17.81620) - 10(-7.24900) - 37(-7.09274)) = 48.94$$

The chi-square degrees of freedom $\nu = \frac{1}{2}5(6)(2) = 30$ and $\chi_{30}^2(.05) = 43.77$. Since $C = 48.94 > \chi_{30}^2(.05) = 43.77$, reject $H_0 : \Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma$ at the 5% level.

6.39 **(a)** Following Example 6.5, we have $(\bar{\mathbf{x}}_F - \bar{\mathbf{x}}_M)' = (119.55, 29.97)$,

$$\left[\frac{1}{28}\mathbf{S}_F + \frac{1}{28}\mathbf{S}_M\right]^{-1} = \begin{bmatrix} .033186 & -.108533 \\ -.108533 & .423508 \end{bmatrix}$$ and $T^2 = 76.97$. Since

$T^2 = 76.97 > \chi_2^2(.05) = 5.99$, we reject $H_0 : \mu_F - \mu_M = \mathbf{0}$ at the 5% level.

(b) With equal sample sizes, the large sample procedure is essentially the same as the procedure based on the pooled covariance matrix.

(c) Here p=2, $t_{54}(.05/2(2)) \approx z(.0125) = 2.24$, $\left[\frac{1}{28}\mathbf{S}_F + \frac{1}{28}\mathbf{S}_M\right] = \begin{bmatrix} 186.148 & 47.705 \\ 47.705 & 14.587 \end{bmatrix}$, so

$\mu_{F1} - \mu_{M1}$: $119.55 \pm 2.24\sqrt{186.148} \rightarrow (88.99,\ 150.11)$

$\mu_{F2} - \mu_{M2}$: $29.97 \pm 2.24\sqrt{14.587} \rightarrow (21.41,\ 38.52)$

Female Anacondas are considerably longer and heavier than males.

6.41 Three factors: (Problem) Severity, (Problem) Complexity and (Engineer) Experience, each at two levels. Two responses: Assessment time, Implementation time. MANOVA results for significant (at the 5% level) effects.

Effect	Wilks' lambda	*F*	*P*-value
Severity	.06398	73.1	.000
Complexity	.01852	265.0	.000
Experience	.03694	130.4	.000
Severity*Complexity	.33521	9.9	.004

Individual ANOVA's for each of the two responses, Assessment time and Implementation time, show only the same three main effects and two factor interaction as significant with p-values for the appropriate F statistics less than .01 in all cases. We see that both assessment time and implementation time is affected by problem severity, problem complexity and engineer experience as well as the interaction between severity and complexity. Because of the interaction effect, the main effects severity and complexity are not additive and do not have a clear interpretation. For this reason, we do not calculate simultaneous confidence intervals for the magnitudes of the mean differences in times across the two levels of each of these main effects. There is no interaction term associated with experience however. Since there are only two levels of experience, we can calculate ordinary t intervals for the mean difference in assessment time and the mean difference in implementation time for gurus (G) and novices (N). Relevant summary statistics and calculations are given below.

$$\text{Error sum of squares and crossproducts matrix} = \begin{bmatrix} 2.222 & 1.217 \\ 1.217 & 2.667 \end{bmatrix}$$

Error deg. of freedom: 11

Assessment time: $\bar{x}_G = 3.68, \bar{x}_N = 5.39$

95% confidence interval for mean difference in experience:

$$3.68 - 5.39 \pm 2.201\sqrt{\frac{2.222}{11}\frac{2}{8}} = -1.71 \pm .49 \rightarrow (-2.20, -1.22)$$

Implementation time: $\bar{x}_G = 6.80, \bar{x}_N = 10.96$

95% confidence interval for mean difference in experience:

$$6.80 - 10.96 \pm 2.201\sqrt{\frac{2.667}{11}\frac{2}{8}} = -4.16 \pm .54 \rightarrow (-4.70, -3.62)$$

The decrease in mean assessment time for gurus relative to novices is estimated to be between 1.22 and 2.20 hours. Similarly the decrease in mean implementation time for gurus relative to novices is estimated to be between 3.62 and 4.70 hours.

Chapter 7

7.1

$$\hat{\boldsymbol{\beta}} = (Z'Z)^{-1}Z'\mathbf{y} = \frac{1}{120}\begin{bmatrix}120 & -10\\ -10 & 1\end{bmatrix}\begin{bmatrix}72\\ 872\end{bmatrix} = \frac{1}{15}\begin{bmatrix}-10\\ 19\end{bmatrix} = \begin{bmatrix}-.667\\ 1.267\end{bmatrix}$$

$$\hat{\mathbf{y}} = Z\hat{\boldsymbol{\beta}} = \frac{1}{15}\begin{bmatrix}180\\ 85\\ 123\\ 351\\ 199\\ 142\end{bmatrix} = \begin{bmatrix}12.000\\ 5.667\\ 8.200\\ 23.400\\ 13.267\\ 9.467\end{bmatrix}\ ; \quad \hat{\boldsymbol{\varepsilon}} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix}15\\ 9\\ 3\\ 25\\ 9\\ 13\end{bmatrix} - \begin{bmatrix}12.000\\ 5.667\\ 8.200\\ 23.400\\ 13.267\\ 9.467\end{bmatrix} = \begin{bmatrix}3.000\\ 3.333\\ -5.200\\ 1.600\\ -6.267\\ 3.533\end{bmatrix}$$

Residual sum of squares: $\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = 101.467$

Fitted equation: $\hat{y} = -.667 + 1.267\, z_1$

7.2 Standardized variables

z_1	z_2	y
-.292	-1.088	.391
-1.166	-.726	-.391
-.817	-.726	-1.174
1.283	.363	1.695
-.117	.726	-.652
1.108	1.451	.130

Fitted equation:

$$\hat{y} = 1.33z_1 - .79z_2$$

Also, prior to standardizing the variables, $\bar{z}_1 = 11.667$, $\bar{z}_2 = 5.000$ and $\bar{y} = 12.000$; $\sqrt{s_{z_1z_1}} = 5.716$, $\sqrt{s_{z_2z_2}} = 2.757$ and $\sqrt{s_{yy}} = 7.667$.

The fitted equation for the original variables is

$$\frac{\hat{y} - 12}{7.667} = 1.33\left(\frac{z_1 - 11.667}{5.716}\right) - .79\left(\frac{z_2 - 5}{2.757}\right)$$

$$\hat{y} = .43 + 1.78z_1 - 2.19z_2$$

7.3 Follow hint and note that $\hat{\boldsymbol{\varepsilon}}^* = \mathbf{Y}^* - \hat{\mathbf{Y}}^* = V^{-1/2}\mathbf{Y} - V^{-1/2}Z\boldsymbol{\beta}_W$ and $(n-r-1)\sigma^2 = \hat{\boldsymbol{\varepsilon}}^{*\prime}\hat{\boldsymbol{\varepsilon}}^*$ is distributed as χ^2_{n-r-1}.

7.4 a) $V = I$ so $\hat{\beta}_W = (z'z)^{-1}z'y = (\sum_{j=1}^{n} z_j y_j)/(\sum_{j=1}^{n} z_j^2)$.

b) V^{-1} is diagonal with $j^{\underline{th}}$ diagonal element $1/z_j$ so

$$\hat{\beta}_W = (z'V^{-1}z)^{-1} z'V^{-1}y = (\sum_{j=1}^{n} y_j)/(\sum_{j=1}^{n} z_j)$$

c) V^{-1} is diagonal with $j^{\underline{th}}$ diagonal element $1/z_j^2$ so

$$\hat{\beta}_W = (z'V^{-1}z)^{-1}z'V^{-1}y = (\sum_{j=1}^{n} (y_j/z_j))/n$$

7.5 Solution follows from Hint.

7.6 a) First note that $\Lambda^- = \text{diag}[\lambda_1^{-1},\ldots,\lambda_{r_1+1}^{-1},0,\ldots 0]$ is a generalized inverse of Λ since

$$\Lambda\Lambda^- = \begin{bmatrix} I_{r_1+1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad \Lambda\Lambda^-\Lambda = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_{r_1+1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix} = \Lambda$$

Since $$Z'Z = \sum_{i=1}^{p} \lambda_i e_i e_i' = P\Lambda P'$$

$$(Z'Z)^- = \sum_{i=1}^{r_1+1} \lambda_i^{-1} e_i e_i' = P\Lambda^- P'$$

with $PP' = P'P = I_p$, we check that the defining relation holds

$$\begin{aligned}(Z'Z)(Z'Z)^-(Z'Z) &= P\Lambda \underbrace{P'(P}\Lambda^- \underbrace{P')P}\Lambda P' \\ &= P\Lambda\Lambda^-\Lambda P' \\ &= P\Lambda P' = Z'Z\end{aligned}$$

b) By the hint, if $Z\hat{\beta}$ is the projection, $0 = Z'(y - Z\hat{\beta})$ or $Z'Z\hat{\beta} = Z'y$. In c), we show that $Z\hat{\beta}$ is the projection of y .

c) Consider $\underset{\sim}{q}_i = \lambda_i^{-1/2} Z\underset{\sim}{e}_i$ for $i = 1,2,\ldots,r_1+1$. Then

$$Z(Z'Z)^{-}Z' = Z\left(\sum_{i=1}^{r_1+1} \lambda_i^{-1} \underset{\sim}{e}_i \underset{\sim}{e}_i'\right)Z' = \sum_{i=1}^{r_1+1} \underset{\sim}{q}_i \underset{\sim}{q}_i'$$

The $\{\underset{\sim}{q}_i\}$ are r_1+1 mutually perpendicular unit length vectors that span the space of all linear combinations of the columns of Z. The projection of $\underset{\sim}{y}$ is then (see Result 2A.2 and Definition 2A.12)

$$\sum_{i=1}^{r_1+1} (\underset{\sim}{q}_i'\underset{\sim}{y})\underset{\sim}{q}_i = \sum_{i=1}^{r_1+1} \underset{\sim}{q}_i(\underset{\sim}{q}_i'\underset{\sim}{y}) = \left(\sum_{i=1}^{r_1+1} \underset{\sim}{q}_i\underset{\sim}{q}_i'\right)\underset{\sim}{y} = Z(Z'Z)^{-} Z'\underset{\sim}{y}$$

d) See Hint.

7.7 Write $\underset{\sim}{\beta} = \left[\begin{array}{c}\underset{\sim}{\beta}_{(1)} \\ \hline \underset{\sim}{\beta}_{(2)}\end{array}\right]$ and $Z = \left[Z_1 \;\vdots\; Z_2\right]$.

Recall from Result 7.4 that $\hat{\underset{\sim}{\beta}} = \left[\begin{array}{c}\hat{\underset{\sim}{\beta}}_{(1)} \\ \hline \hat{\underset{\sim}{\beta}}_{(2)}\end{array}\right] = (Z'Z)^{-1}Z'\underset{\sim}{y}$ is distributed as $N_{r+1}(\underset{\sim}{\beta},\sigma^2(Z'Z)^{-1})$ independently of $n\hat{\sigma}^2 = (n-r-1)s^2$ which is distributed as $\sigma^2 \chi^2_{n-r-1}$. From the Hint, $(\hat{\underset{\sim}{\beta}}_{(2)}-\underset{\sim}{\beta}_{(2)})'(C^{22})^{-1}(\hat{\underset{\sim}{\beta}}_{(2)}-\underset{\sim}{\beta}_{(2)})$ is $\sigma^2\chi^2_{r-q}$ and this is distributed independently of s^2. (The latter follows because the full random vector $\hat{\underset{\sim}{\beta}}$ is distributed independently of s^2). The result follows from the definition of a F random variable as the ratio of two independent χ^2 random variables divided by their degrees of freedom.

7.8 (a) $H^2 = Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z' = Z(Z'Z)^{-1}Z' = H$.

(b) Since $I - H$ is an idempotent matrix, it is positive semidefinite. Let a be an $n \times 1$ unit vector with j th element 1. Then $0 \leq a'(I - H)a = (1 - h_{jj})$. That is, $h_{jj} \leq 1$. On the other hand, $(Z'Z)^{-1}$ is positive definite. Hence $h_{jj} = b_j'(Z'Z)^{-1}b_j > 0$ where b_j is the j th row of Z.
$\sum_{i=1}^{r+1} h_{jj} = tr(Z(Z'Z)^{-1}Z') = tr((Z'Z)^{-1}Z'Z) = tr(I_{r+1}) = r + 1$.

(c) Using

$$(Z'Z)^{-1} = \frac{1}{n\sum_{i=1}^{n}(z_j - \bar{z})^2}\begin{bmatrix} \sum_{i=1}^{n} z_i^2 & -\sum_{i=1}^{n} z_i \\ -\sum_{i=1}^{n} z_i & n \end{bmatrix},$$

we obtain

$$\begin{aligned} h_{jj} &= (1 \ z_j)(Z'Z)^{-1}\begin{pmatrix} 1 \\ z_j \end{pmatrix} \\ &= \frac{1}{n\sum_{i=1}^{n}(z_j - \bar{z})^2}\left(\sum_{i=1}^{n} z_i^2 - 2z_j\sum_{i=1}^{n} z_i + nz_j^2\right) \\ &= \frac{1}{n} + \frac{(z_j - \bar{z})^2}{\sum_{i=1}^{n}(z_j - \bar{z})^2} \end{aligned}$$

7.9

$$Z' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix}; \quad (Z'Z)^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix}$$

$$\hat{\beta}_{(1)} = (Z'Z)^{-1}Z'y_{(1)} = \begin{bmatrix} 3 \\ -.9 \end{bmatrix}; \quad \hat{\beta}_{(2)} = (Z'Z)^{-1}Z'y_{(2)} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

$$\hat{\beta} = \left[\hat{\beta}_{(1)} \,\vdots\, \hat{\beta}_{(2)}\right] = \begin{bmatrix} 3 & 0 \\ -.9 & 1.5 \end{bmatrix}$$

Hence

$$\hat{Y} = Z\hat{\beta} = \begin{bmatrix} 4.8 & -3.0 \\ 3.9 & -1.5 \\ 3.0 & 0 \\ 2.1 & 1.5 \\ 1.2 & 3.0 \end{bmatrix};$$

$$\hat{\varepsilon} = Y - \hat{Y} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \\ 4 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4.8 & -3.0 \\ 3.9 & -1.5 \\ 3.0 & 0 \\ 2.1 & 1.5 \\ 1.2 & 3.0 \end{bmatrix} = \begin{bmatrix} .2 & 0 \\ -.9 & .5 \\ 1.0 & -1.0 \\ -.1 & .5 \\ -.2 & 0 \end{bmatrix}$$

$$Y'Y = \hat{Y}'\hat{Y} + \hat{\varepsilon}'\hat{\varepsilon}$$

$$\begin{bmatrix} 55 & -15 \\ -15 & 24 \end{bmatrix} = \begin{bmatrix} 53.1 & -13.5 \\ -13.5 & 22.5 \end{bmatrix} + \begin{bmatrix} 1.9 & -1.5 \\ -1.5 & 1.5 \end{bmatrix}$$

7.10 **a)** Using Result 7.7, the 95% confidence interval for the mean reponse is given by

$$[1, .5]\begin{bmatrix}3.0\\-.9\end{bmatrix} \pm 3.18\sqrt{[1, .5]\begin{bmatrix}.2 & 0\\0 & .1\end{bmatrix}\begin{bmatrix}1\\.5\end{bmatrix}\left(\frac{1.9}{3}\right)} \quad \text{or}$$

(1.35, 3.75).

b) Using Result 7.8, the 95% prediction interval for the actual Y is given by

$$[1, -.5]\begin{bmatrix}3.0\\-.9\end{bmatrix} \pm 3.18\sqrt{\left\{1 + [1, .5]\begin{bmatrix}.2 & 0\\0 & .1\end{bmatrix}\begin{bmatrix}1\\.5\end{bmatrix}\right\}\left(\frac{1.9}{3}\right)} \quad \text{or}$$

(-.25, 5.35) .

c) Using (7-42) a 95% prediction ellipse for the actual Y's is given by

$$[y_{01} - 2.55, y_{02} - .75]\begin{bmatrix}7.5 & 7.5\\7.5 & 9.5\end{bmatrix}\begin{bmatrix}y_{01} - 2.55\\y_{02} - .75\end{bmatrix}$$

$$\le (1 + .225)\left(\frac{(2)(3)}{2}\right)(19) = 69.825$$

7.11 The proof follows the proof of Result 7.10 with Σ^{-1} replaced by **A**.

$$(Y-ZB)'(Y-Z'B) = \sum_{j=1}^{n} (\underset{\sim}{Y}_j - B\underset{\sim}{z}_j)(\underset{\sim}{Y}_j - B\underset{\sim}{z}_j)'$$

and

$$\Sigma_{j=1}^{n} d_j^2(B) = \text{tr}[A^{-1}(Y-ZB)'(Y-ZB)] .$$

Next,

$$(Y-ZB)'(Y-ZB) = (Y-Z\hat{\beta}+Z\hat{\beta}-ZB)'(Y-Z\hat{\beta}+Z\hat{\beta}-ZB) = \hat{\varepsilon}'\hat{\varepsilon} + (\hat{\beta}-B)'Z'Z(\hat{\beta}-B)]$$

so

$$\Sigma_{j=1}^{n} d_j^2(B) = \text{tr}[A^{-1}\hat{\varepsilon}'\hat{\varepsilon}] + \text{tr}[A^{-1}(\hat{\beta}-B)'Z'Z(\hat{\beta}-B)]$$

The first term does not depend on the choice of B. Using Result 2A.12(c)

$$\begin{aligned} \text{tr}[A^{-1}(\hat{\beta}-B)'Z'Z(\hat{\beta}-B) &= \text{tr}[(\hat{\beta}-B)'Z'Z(\hat{\beta}-B)A] \\ &= \text{tr}[Z'Z(\hat{\beta}-B)A(\hat{\beta}-B)'] \\ &= \text{tr}[Z(\hat{\beta}-B)A(\hat{\beta}-B)'Z'] \\ &\geq \underset{\sim}{c}'A\underset{\sim}{c} > 0 \end{aligned}$$

where $\underset{\sim}{c}$ is any non-zero row of $\underset{\sim}{Z}(\hat{\beta}-B)$. Unless $B = \hat{\beta}$, $Z(\hat{\beta}-B)$ will have a non-zero row. Thus $\hat{\beta}$ is the best choice for any positive definite A.

7.12 (a) best linear predictor $= -4 + 2Z_1 - Z_2$

(b) mean square error $= \sigma_{YY} - \boldsymbol{\sigma}'_{ZY} \boldsymbol{\Sigma}^{-1}_{ZZ} \boldsymbol{\sigma}_{ZY} = 4$

(c) $$\rho_{Y(X)} = \sqrt{\frac{\boldsymbol{\sigma}'_{ZY} \boldsymbol{\Sigma}^{-1}_{ZZ} \boldsymbol{\sigma}_{ZY}}{\sigma_{YY}}} = \frac{\sqrt{5}}{3} = .745$$

(d) Following equation (7-56), we partition $\boldsymbol{\Sigma}$ as

$$\boldsymbol{\Sigma} = \left[\begin{array}{cc|c} 9 & 3 & 1 \\ 3 & 2 & 1 \\ \hline 1 & 1 & 1 \end{array}\right]$$

and determine covariance of $\begin{bmatrix} Y \\ Z_1 \end{bmatrix}$ given Z_2 to be

$$\begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1)^{-1} [1, 1] = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix}. \text{ Therefore}$$

$$\rho_{YZ_1 \cdot Z_2} = \frac{2}{\sqrt{8}\sqrt{1}} = \frac{\sqrt{2}}{2} = .707$$

7.13 (a) By Result 7.13, $\hat{\boldsymbol{\beta}} = S^{-1}_{ZZ} \mathbf{s}_{ZY} = \begin{bmatrix} 3.73 \\ 5.57 \end{bmatrix}$

(b) Let $\mathbf{Z}'_{(2)} = [Z_2, Z_3]$ $$R_{Z_1(Z_2Z_3)} = \sqrt{\frac{\mathbf{s}'_{Z_{(2)}Z_1} S^{-1}_{Z_{(2)}Z_{(2)}} \mathbf{s}_{Z_{(2)}Z_1}}{s_{Z_1Z_1}}}$$

$$= \sqrt{\frac{3452.33}{5691.34}} = .78$$

(c) Partition $\mathbf{Z} = \left[\begin{array}{c} \mathbf{Z}_{(1)} \\ \hline Z_3 \end{array}\right]$ so

$$S = \left[\begin{array}{cc|c} 5691.34 & & \\ 600.51 & 126.05 & \\ \hline 217.25 & 23.37 & 23.11 \end{array}\right] = \left[\begin{array}{c|c} S_{z_{(1)}z_{(1)}} & \underset{\sim}{s}'_{z_3 z_{(1)}} \\ \hline \underset{\sim}{s}_{z_3 z_{(1)}} & s_{z_3 z_3} \end{array}\right]$$

and

$$S_{z_{(1)}z_{(1)}} - \underset{\sim}{s}'_{z_3 z_{(1)}} s^{-1}_{z_3 z_3} \underset{\sim}{s}_{z_3 z_{(1)}} = \begin{bmatrix} 3649.04 & 380.82 \\ 380.82 & 102.42 \end{bmatrix}$$

Thus

$$r_{z_1 z_2 \cdot z_3} = \frac{380.82}{\sqrt{3649.04}\,\sqrt{102.42}} = .62$$

7.14 (a) The large positive correlation between a manager's experience and achieved rate of return on portfolio indicates an apparent advantage for managers with experience. The negative correlation between attitude toward risk and achieved rate of return indicates an apparent advantage for conservative managers.

(b) From (7-57)

$$r_{yz_1 \cdot z_2} = \frac{s_{yz_1 \cdot z_2}}{\sqrt{s_{yy \cdot z_2}}\,\sqrt{s_{z_1 z_1 \cdot z_2}}} = \frac{s_{yz_1} - \dfrac{s_{yz_2} s_{z_1 z_2}}{s_{z_2 z_2}}}{\sqrt{s_{yy} - \dfrac{s^2_{yz_2}}{s_{z_2 z_2}}}\,\sqrt{s_{z_1 z_1} - \dfrac{s^2_{z_1 z_2}}{s_{z_2 z_2}}}}$$

$$= \frac{r_{yz_1} - r_{yz_2} r_{z_1 z_2}}{\sqrt{1 - r^2_{yz_2}}\,\sqrt{1 - r^2_{z_1 z_2}}} = .31$$

Removing "years of experience" from consideration, we now have a positive correlation between "attitude toward risk" and "achieved return". After adjusting for years of experience, there is an apparent advantage to managers who take risks.

7.15 (a) MINITAB computer output gives: $\hat{y} = 11{,}870 + 2634z_1 + 45.2z_2$; residual sum of squares = 204995012 with 17 degrees of freedom. Thus $s = 3473$. Now for example, the estimated standard deviation of $\hat{\beta}_0$ is $\sqrt{1.9961s^2} = 4906$. Similar calculations give the estimated standard deviations of $\hat{\beta}_1$ and $\hat{\beta}_2$.

(b) An analysis of the residuals indicate there are no apparent model inadequacies.

(c) The 95% prediction interval is (\$51,228; \$66,239)

(d) Using (7-14), $F = \dfrac{(45.2)(.0067)^{-1}(45.2)}{12058533} = .025$

Since $F_{1,17}(.05) = 4.45$ we cannot reject $H_0{:}\beta_2 = 0$. It appears as if Z_2 is not needed in the model provided Z_1 is included in the model.

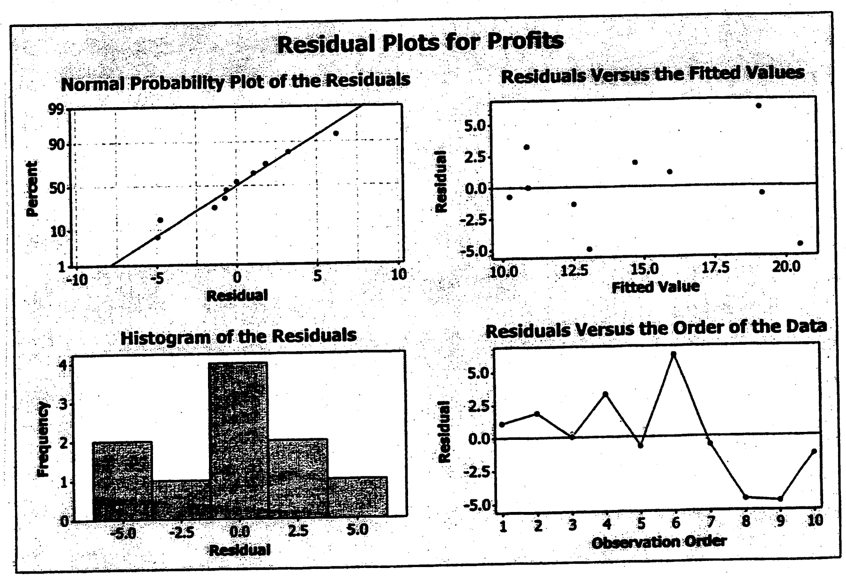
7.16

Predictors	$p = r + 1$	C_p
Z_1	2	1.025
Z_2	2	12.24
Z_1, Z_2	3	3

7.17 (a) Minitab output for the regression of profits on sales and assets follows.

```
Profits = 0.01 + 0.0681 Sales + 0.00577 Assets


Predictor       Coef     SE Coef     T      P
Constant       0.013       7.641  0.00  0.999
Sales        0.06806     0.02785  2.44  0.045
Assets      0.005768    0.004946  1.17  0.282


S = 3.86282   R-Sq = 55.7%    R-Sq(adj) = 43.0%


Analysis of Variance

Source          DF      SS     MS     F      P
Regression       2  131.26  65.63  4.40  0.058
Residual Error   7  104.45  14.92
Total            9  235.71
```

(b) Given the small sample size, the residual plots below are consistent with the usual regression assumptions. The leverages do not indicate any unusual observations. All leverages are less than $3p/n=3(3)/10=.9$.



Obs	1	2	3	4	5	6	7	8	9	10
Lev	.6257	.1011	.2433	.2222	.2513	.2746	.2785	.3642	.2029	.4362

(c) With sales = 100 and assets = 500, a 95% prediction interval for profits is: (−1.55, 20.95).

(d) The t-value for testing $H_0 : \beta_2 = 0$ is $t = 1.17$ with a p value of .282. We cannot reject H_0 at any reasonable significance level. The model should be refit after dropping assets as a predictor variable. That is, consider the simple linear regression model relating profits to sales.

7.18 **(a)** The calculations for the C_p plot are given below. Note that p is the number of model parameters including the intercept.

p (predictor)	2 (sales)	2 (assets)	3 (sales, assets)
C_p	2.4	7.0	3.0

(b) The AIC values are shown below.

p (predictor)	2 (sales)	2 (assets)	3 (sales, assets)
AIC	29.24	33.63	29.46

7.19 **(a)** The "best" regression equation involving ln(y) and $z_1, z_2, \ldots, z_5$ is

$$\widehat{\ln}(y) = 2.756 - .322z_2 + .114z_4$$

with s = 1.058 and R^2 = .60. It may be possible to find a better model using first and second order predictor variable terms.

(b) A plot of the residuals versus the predicted values indicates no apparent problems. A Q-Q plot of the residuals is a bit wavy but the sample size is not large. Perhaps a transformation other than the logarithmic transformation would produce a better model.

7.20 Eigenvalues of the correlation matrix of the predictor variables z_1, $z_2,\ldots,z_5$ are 1.4465, 1.1435, .8940, .8545, .6615. The corresponding eigenvectors give the coefficients of z_1, $z_2,\ldots,z_5$ in the principle component. For example, the first principal component, written in terms of standardized predictor variables, is

$$\hat{x}_1 = .6064z_1^* - .3901z_2^* - .6357z_3^* - .2755z_4^* - .0045z_5^* .$$

A regression of $\ln(y)$ on the first principle component gives

$$\widehat{\ln}(y) = 1.7371 - .0701\hat{x}_1$$

with $s = .701$ and $R^2 = .015$.

A regression of $\ln(y)$ on the fourth principle component produces the best of the one principle component predictor variable regressions. In this case $\widehat{\ln}(y) = 1.7371 + .3604\hat{x}_4$ and $s = .618$ and $R^2 = .235$.

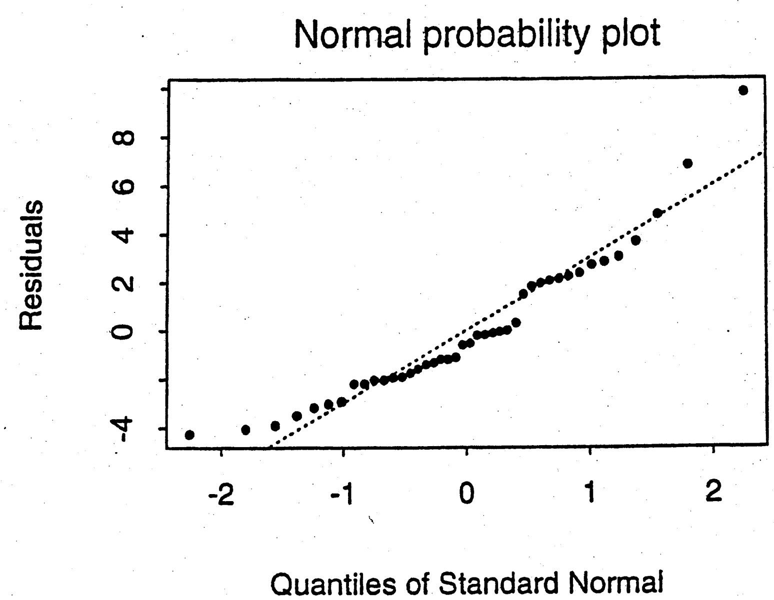
7.21 This data set doesn't appear to yield a regression relationship which explains a large proportion of the variation in the responses.

(a) (i) One reader, starting with a full quadratic model in the predictors z_1 and z_2, suggested the fitted regression equation:

$$\hat{y}_1 = -7.3808 + .5281z_2 - .0038z_2^2$$

with $s = 3.05$ and $R^2 = .22$. (Can you do better than this?)

(ii) A plot of the residuals versus the fitted values suggests the response may not have constant variance. Also a Q-Q plot of the residuals has the following general appearance:



Therefore the normality assumption may also be suspect. Perhaps a better regression can be obtained after the responses have been transformed or re-expressed in a different metric.

(iii) Using the results in (a)(i), a 95% prediction interval of $z_1 = 10$ (not needed) and $z_2 = 80$ is

$$10.84 \pm 2.02\sqrt{7.47} \quad \text{or} \quad (5.32, 16.37).$$

7.22 (a) The full regression model relating the dominant radius bone to the four predictor variables is shown below along with the "best" model after eliminating non-significant predictors. A residual analysis for the best model indicates there is no reason to doubt the standard regression assumptions although observations 19 and 23 have large standardized residuals.

(i)

```
The regression equation is
DomRadius = 0.103 + 0.276 DomHumerus - 0.165 Humerus + 0.357 DomUlna
            + 0.407 Ulna
```

Predictor	Coef	SE Coef	T	P
Constant	0.1027	0.1064	0.97	0.346
DomHumerus	0.2756	0.1147	2.40	0.026
Humerus	-0.1652	0.1381	-1.20	0.246
DomUlna	0.3566	0.1985	1.80	0.088
Ulna	0.4068	0.2174	1.87	0.076

```
S = 0.0663502   R-Sq = 71.8%   R-Sq(adj) = 66.1%
```

```
The regression equation is
DomRadius = 0.164 + 0.162 DomHumerus + 0.552 DomUlna
```

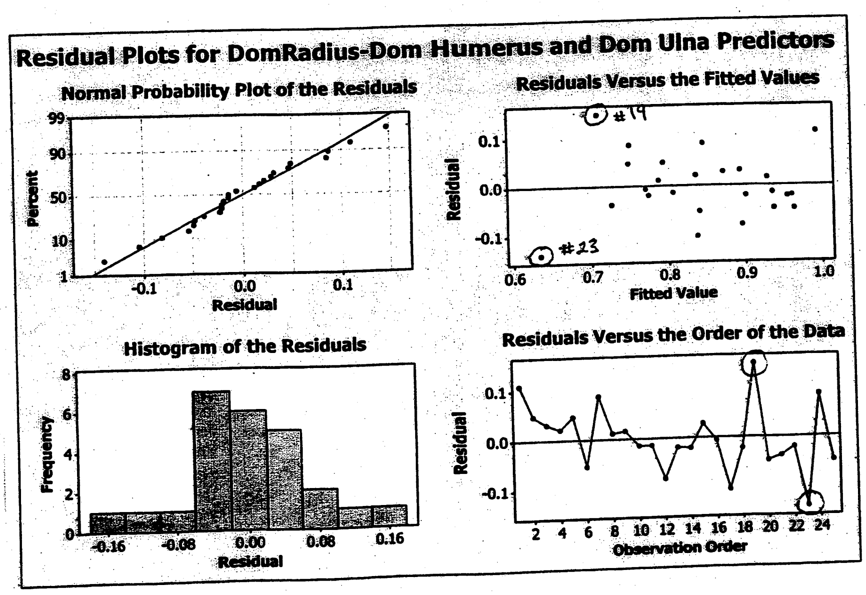
Predictor	Coef	SE Coef	T	P
Constant	0.1637	0.1035	1.58	0.128
DomHumerus	0.16249	0.05940	2.74	0.012
DomUlna	0.5519	0.1566	3.53	0.002

```
S = 0.0687763   R-Sq = 66.7%   R-Sq(adj) = 63.6%
```

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	0.20797	0.10399	21.98	0.000
Residual Error	22	0.10406	0.00473		
Total	24	0.31204			

(ii)



(b) The full regression model relating the radius bone to the four predictor variables is shown below. This fitted model along with the fitted model for the dominant radius bone using four predictors shown in part (a) (i) and the error sum of squares and cross products matrix constitute the multivariate multiple regression model. It appears as if a multivariate regression model with only one or two predictors will represent the data well. Using Result 7.11, a multivariate regression model with predictors dominant ulna and ulna may be reasonable. The results for these predictors follow.

```
The regression equation is
Radius = 0.114 - 0.0110 DomHumerus + 0.152 Humerus + 0.198 DomUlna + 0.462 Ulna
```

Predictor	Coef	SE Coef	T	P
Constant	0.11423	0.08971	1.27	0.217
DomHumerus	-0.01103	0.09676	-0.11	0.910
Humerus	0.1520	0.1165	1.31	0.207
DomUlna	0.1976	0.1674	1.18	0.252
Ulna	0.4625	0.1833	2.52	0.020

S = 0.0559501   R-Sq = 77.2%   R-Sq(adj) = 72.6%

Error sum of squares and cross products matrix: $\begin{bmatrix} .088047 & .050120 \\ .050120 & .062608 \end{bmatrix}$

```
The regression equation is
DomRadius = 0.223 + 0.564 DomUlna + 0.321 Ulna
```

Predictor	Coef	SE Coef	T	P
Constant	0.2235	0.1120	2.00	0.059
DomUlna	0.5645	0.2108	2.68	0.014
Ulna	0.3209	0.2202	1.46	0.159

S = 0.0760309   R-Sq = 59.2%   R-Sq(adj) = 55.5%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	0.184863	0.092431	15.99	0.000
Residual Error	22	0.127175	0.005781		
Total	24	0.312038			

```
The regression equation is
Radius = 0.178 + 0.322 DomUlna + 0.595 Ulna
```

Predictor	Coef	SE Coef	T	P	VIF
Constant	0.17846	0.08931	2.00	0.058	
DomUlna	0.3220	0.1680	1.92	0.068	2.1
Ulna	0.5953	0.1755	3.39	0.003	2.1

S = 0.0606160   R-Sq = 70.5%   R-Sq(adj) = 67.8%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	0.193195	0.096597	26.29	0.000
Residual Error	22	0.080835	0.003674		
Total	24	0.274029			

Error sum of squares and cross products matrix: $\begin{bmatrix} .127175 & .064903 \\ .064903 & .080835 \end{bmatrix}$

7.23. (a) Regression analysis using the response Y_1 = SalePr.

Summary of Backward Elimination Procedure for Dependent Variable X2

Step	Variable Removed	Number In	Partial R**2	Model R**2	C(p)	F	Prob>F
1	X9	7	0.0041	0.5826	7.6697	0.6697	0.4161
2	X3	6	0.0043	0.5782	6.3735	0.7073	0.4033
3	X5	5	0.0127	0.5655	6.4341	2.0795	0.1538

Dependent Variable: X2 SalePr

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Prob>F
Model	5	16462859.832	3292571.9663	18.224	0.0001
Error	70	12647164.839	180673.78342		
C Total	75	29110024.671			

Root MSE 425.05739 R-square 0.5655

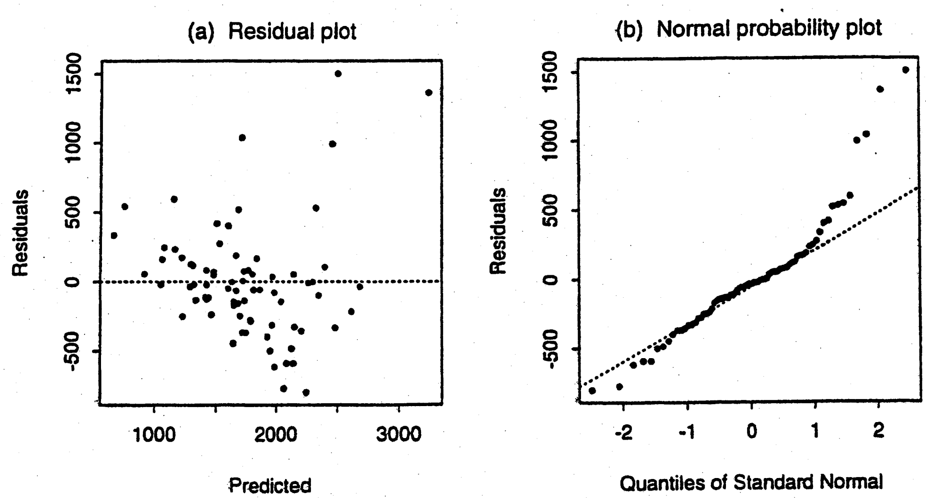
Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
INTERCEP	1	-5605.823664	1929.3986440	-2.905	0.0049
X1	1	-77.633612	22.29880197	-3.482	0.0009
X4	1	-2.332721	0.75490590	-3.090	0.0029
X6	1	389.364490	89.17300145	4.366	0.0001
X7	1	1749.420733	701.21819165	2.495	0.0150
X8	1	133.177529	46.66673277	2.854	0.0057

The 95% prediction interval for SalePr for z_0 is

$$z_0'\hat{\beta} \pm t_{70}(0.025)\sqrt{(425.06)^2(1 + z_0'(Z'Z)^{-1}z_0)}.$$

SalePr=f(Breed, FtFrBody, Frame, BkFat, SaleHt)



(b) Regression analysis using the response $Y_1 = \ln(\text{SalePr})$.

Summary of Backward Elimination Procedure for Dependent Variable LOGX2

Step	Variable Removed	Number In	Partial R**2	Model R**2	C(p)	F	Prob>F
1	X3	7	0.0033	0.6368	7.6121	0.6121	0.4368
2	X7	6	0.0057	0.6311	6.6655	1.0594	0.3070
3	X9	5	0.0122	0.6189	6.9445	2.2902	0.1348
4	X4	4	0.0081	0.6108	6.4537	1.4890	0.2265

Dependent Variable: LOGX2
Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Prob>F
Model	4	4.02968	1.00742	27.854	0.0001
Error	71	2.56794	0.03617		
C Total	75	6.59762			

Root MSE 0.19018 R-square 0.6108

Parameter Estimates

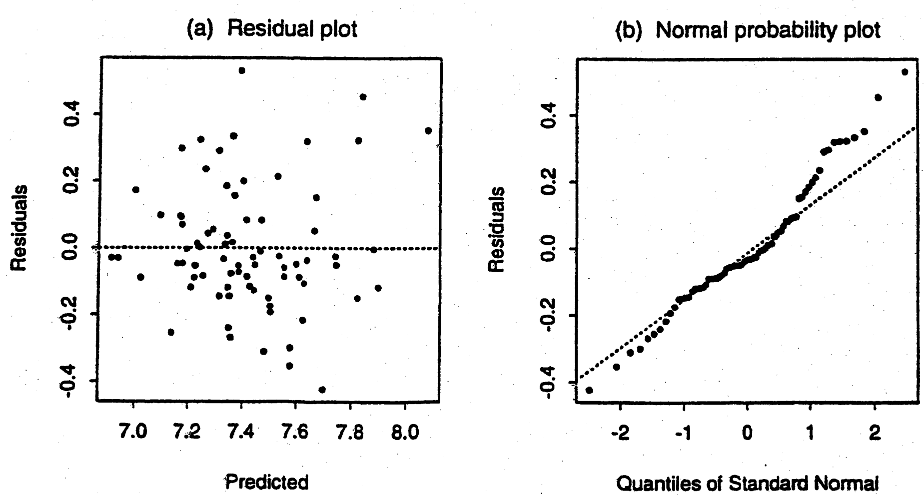
Variable	DF	Parameter Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
INTERCEP	1	5.235773	0.91286786	5.736	0.0001
X1	1	-0.049418	0.00846029	-5.841	0.0001
X5	1	-0.027613	0.00827438	-3.337	0.0013
X6	1	0.183611	0.03992448	4.599	0.0001
X8	1	0.058996	0.01927655	3.060	0.0031

The 95% prediction interval for ln(SalePr) for z_0 is

$$z_0'\hat{\beta} \pm t_{70}(0.025)\sqrt{(0.1902)^2(1 + z_0'(\mathbf{Z}'\mathbf{Z})^{-1}z_0)}.$$

The few outliers among these latter residuals are not so pronounced.

ln(SalePr)=f(Breed, PrctFFB, Frame, SaleHt)



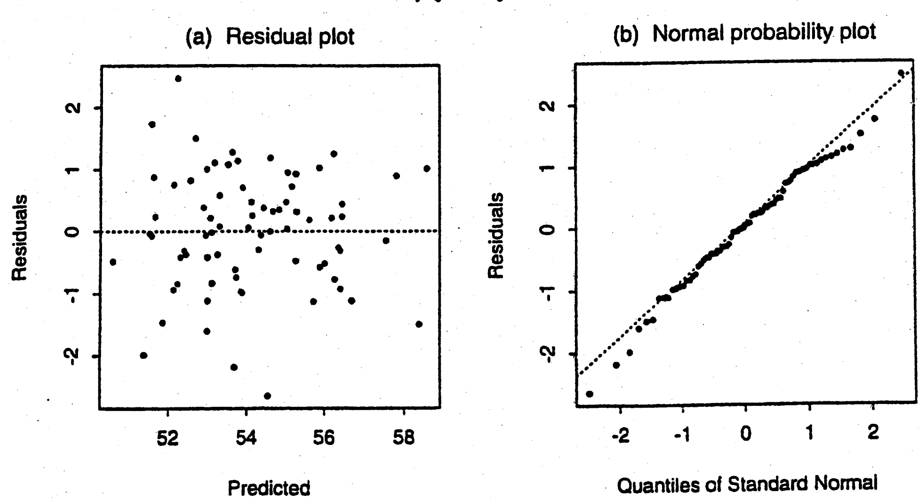
7.24. (a) Regression analysis using the response Y_1 = SaleHt and the predictors Z_1 = YrHgt and Z_2 = FtFrBody.

```
Dependent Variable: X8          SaleHt
Analysis of Variance
                         Sum of          Mean
Source          DF      Squares        Square       F Value       Prob>F
Model            2    235.74533     117.87267       131.165       0.0001
Error           73     65.60204       0.89866
C Total         75    301.34737

    Root MSE        0.94798       R-square        0.7823
Parameter Estimates
                      Parameter        Standard     T for H0:
Variable   DF          Estimate           Error   Parameter=0    Prob > |T|
INTERCEP    1          7.846281      3.36221288         2.334        0.0224
X3          1          0.802235      0.08088562         9.918        0.0001
X4          1          0.005773      0.00151072         3.821        0.0003
```

The 95% prediction interval for SaleHt for $z_0' = (1, 50.5, 970)$ is

$$53.96 \pm t_{73}(0.025)\sqrt{0.8987(1.0148)} \;=\; (52.06, 55.86).$$

SaleHt=f(YrHgt, FtFrBody)

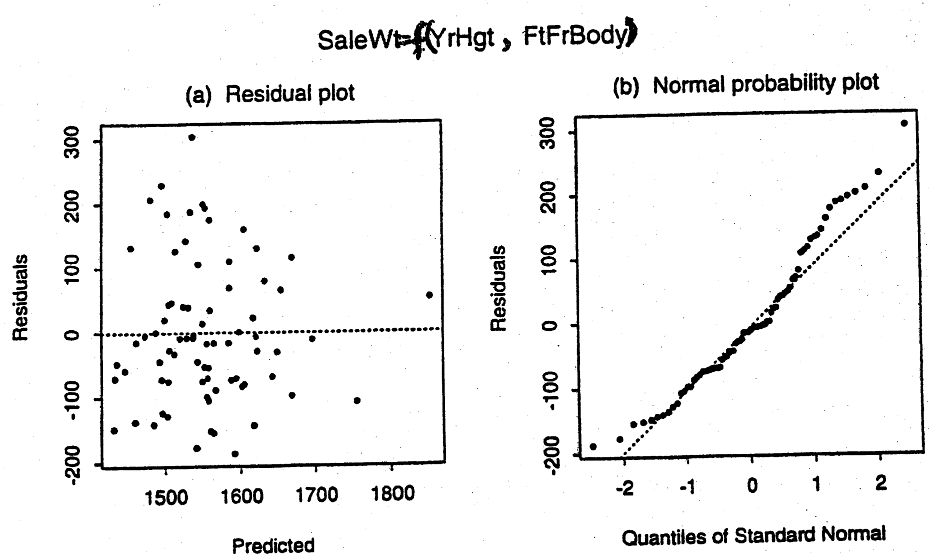


(b) Regression analysis using the response Y_2 = SaleWt and the predictors Z_1 = YrHgt and Z_2 = FtFrBody.

```
Dependent Variable: X9            SaleWt
Analysis of Variance
                          Sum of          Mean
Source          DF       Squares        Square      F Value      Prob>F
Model            2  390456.63614  195228.31807       16.319      0.0001
Error           73  873342.99544   11963.60268
C Total         75  1263799.6316

    Root MSE      109.37826      R-square          0.3090
Parameter Estimates
                    Parameter        Standard    T for H0:
Variable   DF       Estimate           Error   Parameter=0    Prob > |T|
INTERCEP    1     675.316794    387.93499836         1.741        0.0859
X3          1       2.719286      9.33265244         0.291        0.7716
X4          1       0.745610      0.17430765         4.278        0.0001
```

The 95% prediction interval for SaleWt for $z_0' = (1, 50.5, 970)$ is

$$1535.9 \pm t_{73}(0.025)\sqrt{11963.6(1.0148)} = (1316.3, 1755.5).$$



Multivariate regression analysis using the responses Y_1 = SaleHt and Y_2 = SaleWt and the predictors Z_1 = YrHgt and Z_2 = FtFrBody.

```
Multivariate Test:  H0: YrHgt = 0
Multivariate Statistics and Exact F Statistics
S=1     M=0     N=35
```

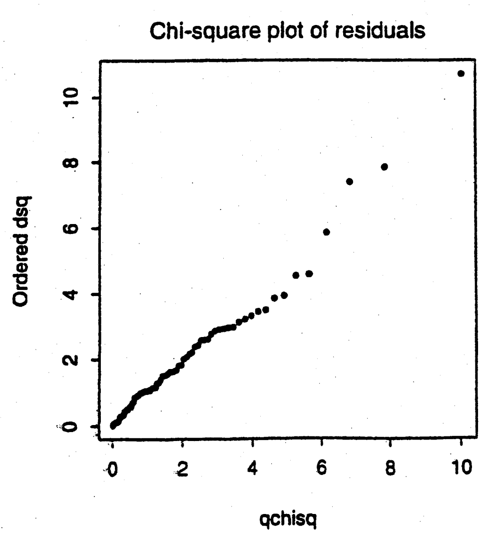
Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.38524567	57.4469	2	72	0.0001
Pillai's Trace	0.61475433	57.4469	2	72	0.0001
Hotelling-Lawley Trace	1.59574625	57.4469	2	72	0.0001
Roy's Greatest Root	1.59574625	57.4469	2	72	0.0001

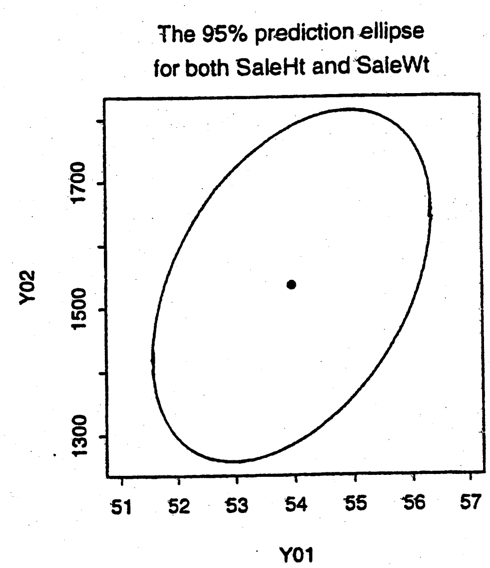
```
Multivariate Test:  H0: FtFrBody = 0
Multivariate Statistics and Exact F Statistics
S=1     M=0     N=35
```

Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.75813396	11.4850	2	72	0.0001
Pillai's Trace	0.24186604	11.4850	2	72	0.0001
Hotelling-Lawley Trace	0.31902811	11.4850	2	72	0.0001
Roy's Greatest Root	0.31902811	11.4850	2	72	0.0001

The theory requires using x_3 (YrHgt) to predict both SaleHt and SaleWt, even though this term could be dropped in the prediction equation for SaleWt. The 95% prediction ellipse for both SaleHt and SaleWt for $z_0' = (1, 50.5, 970)$ is

$$1.3498(Y_{01} - 53.96)^2 + 0.0001(Y_{02} - 1535.9)^2 - 0.0098(Y_{01} - 53.96)(Y_{02} - 1535.9)$$
$$= 1.0148\frac{2(73)}{72}F_{2,72}(0.05) = 6.4282.$$





7.25. (a) Regression analysis using the first response Y_1. The backward elimination procedure gives $Y_1 = \beta_{01} + \beta_{11}Z_1 + \beta_{21}Z_2$. All variables left in the model are significant at the 0.05 level. (It is possible to drop the intercept but we retain it.)

Dependent Variable: Y1 TOT

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Prob>F
Model	2	5905583.8728	2952791.9364	22.962	0.0001
Error	14	1800356.3625	128596.88303		
C Total	16	7705940.2353			

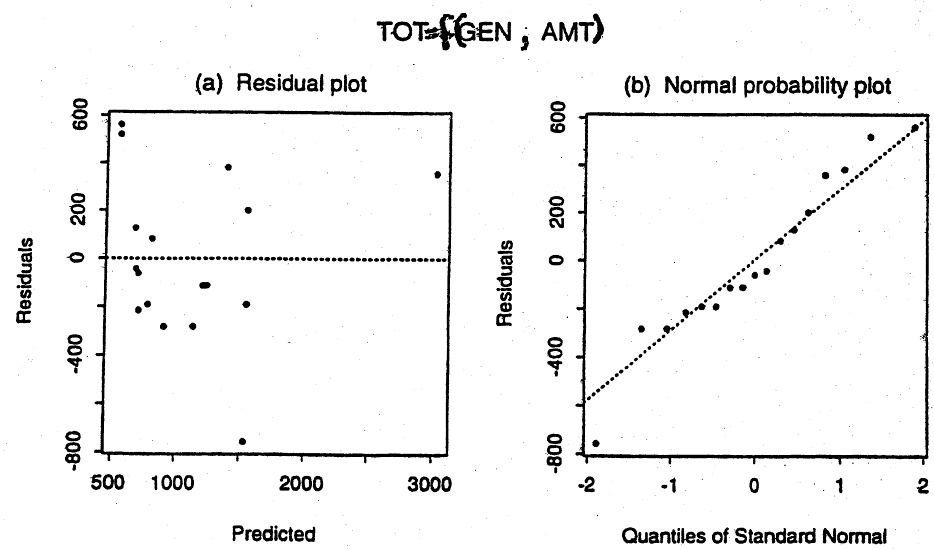
Root MSE	358.60408	R-square	0.7664

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
INTERCEP	1	56.720053	206.70336862	0.274	0.7878
Z1	1	507.073084	193.79082471	2.617	0.0203
Z2	1	0.328962	0.04977501	6.609	0.0001

The 95% prediction interval for $Y_1 = \text{TOT}$ for $z_0' = (1, 1, 1200)$ is

$$958.5 \pm t_{14}(0.025)\sqrt{128596.9(1.0941)} \;=\; (154.0, 1763.1).$$



(b) Regression analysis using the second response Y_2. The backward elimination procedure gives $Y_2 = \beta_{02} + \beta_{12}Z_1 + \beta_{22}Z_2$. All variables left in the model are significant at the 0.05 level.

Dependent Variable: Y2 AMI

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Prob>F
Model	2	5989720.5384	2994860.2692	25.871	0.0001
Error	14	1620657.344	115761.23886		
C Total	16	7610377.8824			

Root MSE 340.23703 R-square 0.7870

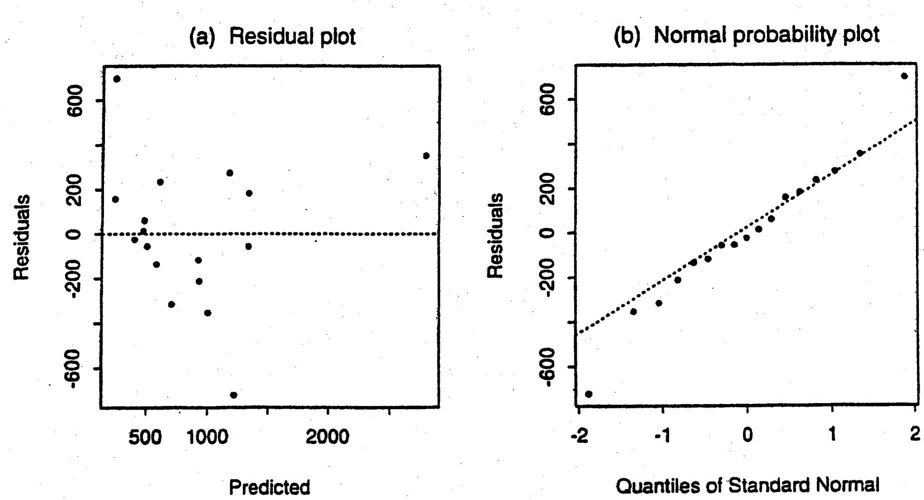
Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
INTERCEP	1	-241.347910	196.11640164	-1.231	0.2387
Z1	1	606.309666	183.86521452	3.298	0.0053
Z2	1	0.324255	0.04722563	6.866	0.0001

The 95% prediction interval for Y_2 = AMI for $z_0' = (1, 1, 1200)$ is

$$754.1 \pm t_{14}(0.025)\sqrt{115761.2(1.0941)} = (-9.234, 1517.4).$$

AMI=f(GEN, AMT)



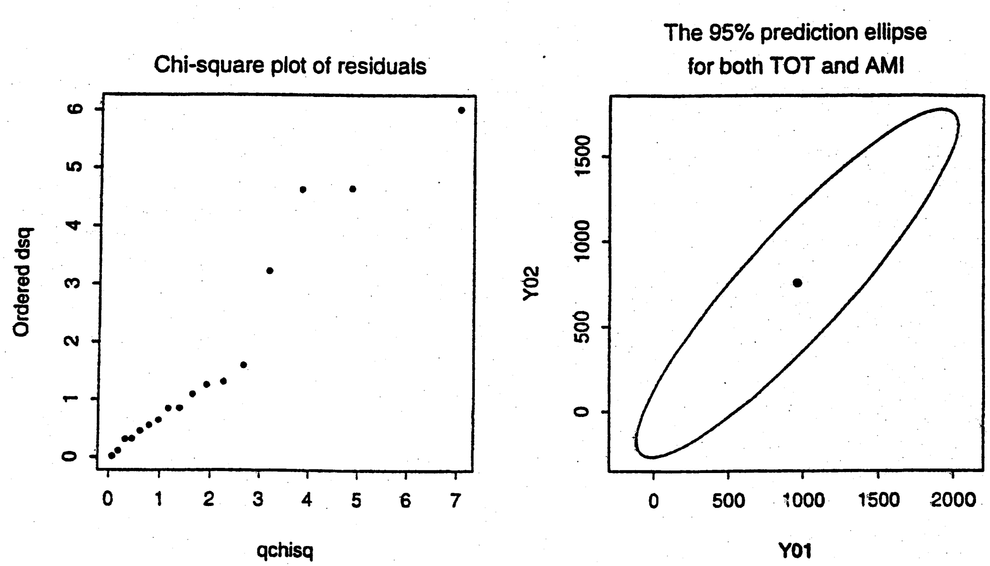
(c) Multivariate regression analysis using Y_1 and Y_2.

```
Multivariate Test:  HO: PR=0, DIAP=0, QRS=0
Multivariate Statistics and F Approximations
S=2     M=0     N=4
```

Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.44050214	1.6890	6	20	0.1755
Pillai's Trace	0.60385990	1.5859	6	22	0.1983
Hotelling-Lawley Trace	1.16942861	1.7541	6	18	0.1657
Roy's Greatest Root	1.07581808	3.9447	3	11	0.0391

Based on Wilks' Lambda, the three variables Z_3, Z_4 and Z_5 are not significant. The 95% prediction ellipse for both TOT and AMI for $z_0' = (1, 1, 1200)$ is

$$4.305 \times 10^{-5}(Y_{01} - 958.5)^2 + 4.782 \times 10^{-5}(Y_{02} - 754.1)^2$$
$$- 8.214 \times 10^{-5}(Y_{01} - 958.5)(Y_{02} - 754.1) = 1.0941\frac{2(14)}{13}F_{2,13}(0.05) = 8.968.$$



7.26 (a) (i) The table below summarizes the results of the "best" individual regressions. Each predictor variable is significant at the 5% level.

Fitted model	R^2	s
$\hat{y}_1 = -70.1 + .0593z_2 + .0555z_3 + 82.53z_4$	73.6%	1.5192
$\hat{y}_2 = -21.6 - .9640z_1 + 27.04z_4$ $\hat{y}_2 = -20.92 + .0117z_3 + 26.12z_4$	76.5% 75.4%	.3530 .3616
$\hat{y}_3 = -43.8 + .0288z_2 + .0282z_3 + 44.59z_4$	80.7%	.6595
$\hat{y}_4 = -17.0 + .0224z_2 + .0120z_3 + 15.77z_4$	75.7%	.3504

(ii) Observations with large standardized residuals (outliers) include #51, #52 and #56. Observations with high leverage include #57, #58, #60 and #61. Apart from the outliers, the residuals plots look good.

(iii) 95% prediction interval for Y_3 is: (1.077, 4.239)

(b) (i) Using all four predictor variables, the estimated coefficient matrix and estimated error covariance matrix are

$$B = \begin{bmatrix} -74.232 & -24.015 & -45.763 & -17.727 \\ -3.120 & -1.185 & -1.486 & -.550 \\ .098 & .009 & .047 & .029 \\ .049 & .008 & .025 & .011 \\ 85.076 & 28.755 & 45.798 & 16.220 \end{bmatrix}$$

$$\hat{\Sigma} = \begin{bmatrix} 2.244 & .398 & .914 & .511 \\ .398 & .118 & .193 & .089 \\ .914 & .193 & .419 & .210 \\ .511 & .089 & .210 & .122 \end{bmatrix}$$

A multivariate regression model using only the three predictors z_2, z_3 and z_4 will adequately represent the data.

(ii) The same outliers and leverage points indicated in (a) (ii) are present. Otherwise the residual analysis suggests the usual regression assumptions are reasonable.

(iii) The simultaneous prediction interval for Y_3 will be wider than the individual interval in (a) (iii).

7.27 The table below summarizes the results of the "best" individual regressions. Each predictor variable is significant at the 5% level. (The levels of Severity are coded: Low=1, High=2; the levels of Complexity are coded: Simple=1, Complex=2; the levels of Exper are coded: Novice=1, Guru=2, Experienced=3.) There are no significant interaction terms in either model.

Fitted model	R^2	s
$\widehat{Assessment} = -1.834 + 1.270 Severity + 3.003 Complexity$	74.1%	.9853
$\widehat{Implementation} = -4.919 + 3.477 Severity + 5.827 Complexity$	71.9%	2.1364

For the multivariate regression with the two predictor variables Severity and Complexity, the estimated coefficient matrix and estimated error covariance matrix are

$$\mathrm{B} = \begin{bmatrix} -1.834 & -4.919 \\ 1.270 & 3.477 \\ 3.003 & 5.827 \end{bmatrix}$$

$$\hat{\Sigma} = \begin{bmatrix} .9707 & 1.9162 \\ 1.9162 & 4.5643 \end{bmatrix}$$

A residual analysis suggests there is no reason to doubt the standard regression assumptions.

Chapter 8

8.1 Eigenvalues of $\boldsymbol{\Sigma}$ are $\lambda_1 = 6$, $\lambda_2 = 1$. The principal components are

$$Y_1 = .894X_1 + .447X_2$$
$$Y_2 = .447X_1 - .894X_2$$

$\text{Var}(Y_1) = \lambda_1 = 6$. Therefore, proportion of total population variance explained by Y_1 is $6/(6+1) = .86$.

8.2

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & .6325 \\ .6325 & 1 \end{bmatrix}$$

(a) $Y_1 = .707Z_1 + .707Z_2$ $\qquad$ $\text{Var}(Y_1) = \lambda_1 = 1.6325$

$Y_2 = .707Z_1 - .707Z_2$ $\qquad$ Proportion of total population variance explained by Y_1 is $1.6325/(1+1) = .816$

(b) No. The two (standardized) variables contribute equally to the principal components in 8.2(a). The two variables contribute unequally to the principal components in 8.1 because of their unequal variances.

(c) $\rho_{Y_1 Z_1} = .903$; $\quad \rho_{Y_1 Z_2} = .903$; $\quad \rho_{Y_2 Z_1} = .429$

8.3 Eigenvalues of $\boldsymbol{\Sigma}$ are 2, 4, 4. Eigenvectors associated with the eigenvalues 4, 4 are not unique. One choice is $\mathbf{e}_2' = [0 \;\; 1 \;\; 0]$ and $\mathbf{e}_3' = [0 \;\; 0 \;\; 1]$. With these assignments the principal components are $Y_1 = X_1$, $Y_2 = X_2$ and $Y_3 = X_3$.

8.4 Eigenvalues of $\boldsymbol{\Sigma}$ are solutions of $|\boldsymbol{\Sigma}-\lambda\mathbf{I}| = (\sigma^2-\lambda)^3-2(\sigma^2-\lambda)(\sigma^2\rho)^2 = 0$
Thus $(\sigma^2-\lambda)[(\sigma^2-\lambda)^2-2\sigma^4\rho^2] = 0$ so $\lambda = \sigma^2$ or $\lambda = \sigma^2(1\pm\rho\sqrt{2})$. For $\lambda_1 = \sigma^2, \mathbf{e}_1' = [1/\sqrt{2}, 0, -1/\sqrt{2}]$. For $\lambda_2 = \sigma^2(1+\rho\sqrt{2})$; $\mathbf{e}_2' = [1/2, 1/\sqrt{2}, 1/2]$. For $\lambda_3 = \sigma^2(1-\rho\sqrt{2})$, $\mathbf{e}_3' = [1/2, -1/\sqrt{2}, 1/\sqrt{2}]$

Principal Component	Variance	Proportion of Total Variance Explained
$Y_1 = \frac{1}{\sqrt{2}} X_1 - \frac{1}{\sqrt{2}} X_3$	σ^2	1/3
$Y_2 = \frac{1}{2} X_1 + \frac{1}{\sqrt{2}} X_2 + \frac{1}{2} X_3$	$\sigma^2(1+\rho\sqrt{2})$	$\frac{1}{3}(1+\rho\sqrt{2})$
$Y_3 = \frac{1}{2} X_1 - \frac{1}{\sqrt{2}} X_2 + \frac{1}{2} X_3$	$\sigma^2(1-\rho\sqrt{2})$	$\frac{1}{3}(1-\rho\sqrt{2})$

8.5 (a) Eigenvalues of $\underset{\sim}{\rho}$ satisfy

$$|\underset{\sim}{\rho}-\lambda I| = (1-\lambda)^3 + 2\rho^3 - 3(1-\lambda)\rho^2 = 0$$

or $(1+2\rho-\lambda)(1-\rho-\lambda)^2 = 0$. Hence $\lambda_1 = 1+2\rho$; $\lambda_2 = \lambda_3 = 1-\rho$ and results are consistent with (8-16) for $p = 3$.

(b) By direct multiplication

$$\underset{\sim}{\rho}\left(\frac{1}{\sqrt{p}}\underset{\sim}{1}\right) = (1+(p-1)\rho)\left(\frac{1}{\sqrt{p}}\underset{\sim}{1}\right)$$

thus varifying the first eigenvalue-eigenvector pair. Further $\underset{\sim}{\rho}\,\underset{\sim}{e}_i = (1-\rho)\underset{\sim}{e}_i$, $i = 2,3,\ldots,p$.

8.6 (a)

$\hat{y}_1 = .999x_1 + .041x_2$ Sample variance of $\hat{y}_1 = \hat{\lambda}_1 = 7488.8$

$\hat{y}_2 = -.041x_1 + .999x_2$ Sample variance of $\hat{y}_2 = \hat{\lambda}_2 = 13.8$

(b) Proportion of total sample variance explained by $\hat{y}_1$ is $\hat{\lambda}_1/(\hat{\lambda}_1 + \hat{\lambda}_2) = .9982$

(c) Center of constant density ellipse is (155.60, 14.70). Half length of major axis is 102.4 in direction of $\hat{y}_1$. Half length of perpendicular minor axis is 4.4 in direction of $\hat{y}_2$.

(d) $r_{\hat{y}_1,x_1} = 1.000$, $r_{\hat{y}_1,x_2} = .687$ The first component is almost completely determined by x_1 = sales since its variance is approximately 285 times that of x_2 = profits. This is confirmed by the correlation coefficient $r_{\hat{y}_1,x_1} = 1.000$.

8.7 (a)

$\hat{y}_1 = .707z_1 + .707z_2$ Sample variance of $\hat{y}_1 = \hat{\lambda}_1 = 1.6861$

$\hat{y}_2 = .707z_1 - .707z_2$ Sample variance of $\hat{y}_2 = \hat{\lambda}_2 = .3139$

(b) Proportion of total sample variance explained by $\hat{y}_1$ is $\hat{\lambda}_1/(\hat{\lambda}_1 + \hat{\lambda}_2) = .8431$

(c) $r_{\hat{y}_1,z_1} = .918$, $r_{\hat{y}_1,z_2} = .918$ The standardized "sales" and "profits" contribute equally to the first sample principal component.

(d) The sales numbers are much larger than the profits numbers and consequently, sales, with the larger variance, will dominate the first principal component obtained from the sample covariance matrix. Obtaining the principal components from the sample correlation matrix (the covariance matrix of the standardized variables) typically produces components where the importance of the variables, as measured by correlation coefficients, is more nearly equal. It is usually best to use the correlation matrix or equivalently, to put the all the variables on similar numerical scales.

8.8 (a) $r_{\hat{y}_i, z_k} = \hat{e}_{ik}\sqrt{\hat{\lambda}_i} \quad i = 1,2 \;\; k = 1,2,\ldots,5$

Correlations:

i\k	1	2	3	4	5
1	.732	.831	.726	.604	.564
2	-.437	-.280	-.374	.694	.719

The correlations seem to reinforce the interpretations given in Example 8.5.

(b) Using (8-34) and (8-35) we have

k	$\bar{r}_k$
1	.353
2	.435
3	.354
4	.326
5	.299

$\bar{r} = .353$ $\qquad \hat{\gamma} = 2.485$

$T = 103.1 > \chi^2_9(.01) = 21.67$ so would reject H_0 at the 1% level. This test assumes a large random sample and a multivariate normal parent population.

8.9 (a) By (5-10)

$$\max_{\underset{\sim}{\mu},\Sigma} L(\underset{\sim}{\mu},\Sigma) = \frac{e^{-\frac{np}{2}}}{(2\pi)^{\frac{pn}{2}} \left(\frac{n-1}{n}\right)^{\frac{pn}{2}} |S|^{\frac{n}{2}}}$$

The same result applied to each variable independently gives

$$\max_{\mu_i,\sigma_{ii}} L(\mu_i,\sigma_{ii}) = \frac{e^{-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}} \left(\frac{n-1}{n}\right)^{\frac{n}{2}} s_{ii}^{\frac{n}{2}}}$$

Under H_0, $\max_{\underset{\sim}{\mu},\Sigma_0} L(\underset{\sim}{\mu},\Sigma_0) = \prod_{i=1}^{p} L(\mu_i,\sigma_{ii})$

and the likelihood ratio statistic becomes

$$\Lambda = \frac{\max_{\underset{\sim}{\mu},\Sigma_0} L(\underset{\sim}{\mu},\Sigma_0)}{\max_{\underset{\sim}{\mu},\Sigma} L(\underset{\sim}{\mu},\Sigma)} = \frac{|S|^{\frac{n}{2}}}{\prod_{i=1}^{p} s_{ii}^{\frac{n}{2}}}$$

(b) When $\Sigma = \sigma^2 I$, using (4-16) and (4-17) we get

$$\max_{\underset{\sim}{\mu}} L(\underset{\sim}{\mu},\sigma^2 I) = \frac{1}{(2\pi)^{\frac{np}{2}} (\sigma^2)^{\frac{np}{2}}} e^{-\frac{1}{2\sigma^2}\{tr[(n-1)S]\}}$$

8.9 (Continued)

so

$$\max_{\underset{\sim}{\mu},\sigma^2} L(\underset{\sim}{\mu},\sigma^2 I) = \frac{(np)^{np/2}\, e^{-np/2}}{(2\pi)^{np/2}(n-1)^{np/2}(tr[S])^{np/2}}$$

$$= \frac{e^{-np/2}}{(2\pi)^{np/2}\,(\frac{n-1}{n})^{np/2}\,(\frac{1}{p}\,tr\,(S))^{np/2}}$$

and the result follows. Under H_0 there are p μ_i's and one variance so the dimension of the parameter space is $\gamma_0 = p + 1$. The unrestricted case has dimension $p + p(p+1)/2$ so the χ^2 has $p(p+1)/2 - 1 = (p+2)(p-1)/2$ d.f.

8.10 (a) **Covariances: JPMorgan, CitiBank, WellsFargo, RoyDutShell, ExxonMobil**

	JPMorgan	CitiBank	WellsFargo	RoyDutShell	ExxonMobil
JPMorgan	0.00043327				
CitiBank	0.00027566	0.00043872			
WellsFargo	0.00015903	0.00017999	0.00022398		
RoyDutShell	0.00006410	0.00018144	0.00007341	0.00072251	
ExxonMobil	0.00008897	0.00012325	0.00006055	0.00050828	0.00076568

Principal Component Analysis: JPMorgan, CitiBank, WellsFargo, RoyDutShell, Exxon

Eigenanalysis of the Covariance Matrix
103 cases used

Eigenvalue	0.0013677	0.0007012	0.0002538	0.0001426	0.0001189
Proportion	0.529	0.271	0.098	0.055	0.046
Cumulative	0.529	0.801	0.899	0.954	1.000

Variable	PC1	PC2	PC3	PC4	PC5
JPMorgan	0.223	-0.625	-0.326	0.663	-0.118
CitiBank	0.307	-0.570	0.250	-0.414	0.589
WellsFargo	0.155	-0.345	0.038	-0.497	-0.780
RoyDutShell	0.639	0.248	0.642	0.309	-0.149
ExxonMobil	0.651	0.322	-0.646	-0.216	0.094

(b) From part (a),

$\hat{\lambda}_1 = .00137 \quad \hat{\lambda}_2 = .00070 \quad \hat{\lambda}_3 = .00025 \quad \hat{\lambda}_4 = .00014 \quad \hat{\lambda}_5 = .00012$,

so the total sample variance is $\sum_{i=1}^{5} \hat{\lambda}_i = .00258$ and the proportion of total variance explained by the first three components is $\sum_{i=1}^{3} \hat{\lambda}_i / \sum_{i=1}^{5} \hat{\lambda}_i = .899$. As in Example 8.5, the first component might be interpreted as a market component, the second component as an industry component, and the third component is difficult to interpret.

(c) Using (8-33), Bonferroni 90% simultaneous confidence intervals for $\lambda_1 \ \lambda_2 \ \lambda_3$ are

λ_1: (.00106, .00195)
λ_2: (.00054, .00100)
λ_3: (.00019, .00036)

(d) Stock returns are probably best summarized in two dimensions with 80% of the total variation accounted for by a "market" component and an "industry" component.

8.11 (a)

$$\mathbf{S} = \begin{bmatrix} 3.397 & -1.102 & 4.306 & -2.078 & .270 \\ & 9.673 & -1.513 & 10.953 & 12.030 \\ & & 55.626 & -28.937 & -.440 \\ & & & 89.067 & 9.570 \\ & \text{(Symmetric)} & & & 31.900 \end{bmatrix}$$

(b)

$\hat{\lambda}_1 = 108.27 \quad \hat{\lambda}_2 = 43.15 \quad \hat{\lambda}_3 = 31.29 \quad \hat{\lambda}_4 = 4.60 \quad \hat{\lambda}_5 = 2.35$

$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$	$\hat{e}_4$	$\hat{e}_5$
-0.037630	-0.062264	0.040076	0.554515	0.828018
0.118931	-0.249442	-0.259861	-0.769147	0.514314
-0.479670	-0.759246	0.431404	-0.027909	-0.081081
0.858905	-0.315978	0.393975	0.068822	-0.049884
0.128991	-0.507549	-0.767815	0.308887	-0.202000

$$\hat{y}_1 = -.038x_1 + .119x_2 - .480x_3 + .859x_4 + .129x_5$$
$$\hat{y}_2 = -.062x_1 - .249x_2 - .759x_3 - .316x_4 - .508x_5$$

(c) Correlations between variables and components:

	x_1	x_2	x_3	x_4	x_5
$r_{\hat{y}_1,x_i}$	-.212	.398	-.669	.947	.238
$r_{\hat{y}_2,x_i}$	-.222	-.527	-.669	-.220	-.590

The proportion of total sample variance explained by the first two principal Components is (108.27+43.15)/(108.27+43.15+31/29+4.60+2.35)=.80.

The first component appears to be a weighted difference between percent total employment and percent employed by government. We might call this component an employment contrast. The second component appears to be influenced most by roughly equal contributions from percent with professional degree (x_2), percent employment (x_3) and median home value (x_5). We might call this an achievement component. The change in scale for x_5 did not appear to have much affect on the first sample principal component (see Example 8.3) but did change the nature of the second component. Variable x_5 now has much more influence in the second principal component.

8.12

$$S = \begin{bmatrix} 2.500 & -2.768 & -.378 & -.464 & -.586 & -2.235 & .171 \\ & 300.516 & 3.914 & -1.395 & 6.779 & 30.779 & .624 \\ & & 1.522 & .673 & 2.316 & 2.822 & .142 \\ & & & 1.182 & 1.089 & -.811 & .177 \\ & & & & 11.364 & 3.133 & 1.045 \\ & & & & & 30.978 & .593 \\ \text{(Symmetric)} & & & & & & .479 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.0 & -.101 & -.194 & -.270 & -.110 & -.254 & .156 \\ & 1.0 & .183 & -.074 & .116 & .319 & .052 \\ & & 1.0 & .502 & .557 & .411 & .166 \\ & & & 1.0 & .297 & -.134 & .235 \\ & & & & 1.0 & .167 & .448 \\ & & & & & 1.0 & .154 \\ \text{(Symmetric)} & & & & & & 1.0 \end{bmatrix}$$

Using S:

$\hat{\lambda}_1 = 304.26$; $\hat{\lambda}_2 = 28.28$; $\hat{\lambda}_3 = 11.46$; $\hat{\lambda}_4 = 2.52$; $\hat{\lambda}_5 = 1.28$;

$\hat{\lambda}_6 = .53$; $\hat{\lambda}_7 = .21$

The first sample principal component

$$\hat{y}_1 = -.010x_1 + .993x_2 + .014x_3 - .005x_4 + .024x_5 + .112x_6 + .002x_7$$

accounts for 87% of the total sample variance. The first component is essentially "solar radiation". (Note the large sample variance for x_2 in S).

Using R:

$\hat{\lambda}_1 = 2.34$; $\hat{\lambda}_2 = 1.39$; $\hat{\lambda}_3 = 1.20$; $\hat{\lambda}_4 = .73$; $\hat{\lambda}_5 = .65$;

$\hat{\lambda}_6 = .54$; $\hat{\lambda}_7 = .16$

The first three sample principle components are

$$\hat{y}_1 = .237z_1 - .205z_2 - .551z_3 - .378z_4 - .498z_5 - .324z_6 - .319z_7$$

$$\hat{y}_2 = -.278z_1 + .527z_2 + .007z_3 - .435z_4 - .199z_5 + .567z_6 - .308z_7$$

$$\hat{y}_3 = .644z_1 + .225z_2 - .113z_3 - .407z_4 + .197z_5 + .159z_6 + .541z_7$$

These components account for 70% of the total sample variance.

The first component contrasts "wind" with the remaining variables. It might be some general measure of the pollution level. The second component is largely composed of "solar radiation" and the pollutants "NO" and "O_3". It might represent the effects of solar radiation since solar radiation is involved in the production of NO and O_3 from the other pollutants. The third component is composed largely of "wind" and certain pollutants (e.g. "NO" and "HC"). It might represent a wind transport effect. A "better" interpretation of the components would depend on more extensive subject matter knowledge.

The data can be effectively summarized in three or fewer dimensions. The choice of S or R makes a difference.

8.13

(a) Covariance Matrix

	X1	X2	X3
X1	4.654750889	0.931345370	0.589699088
X2	0.931345370	0.612821160	0.110933412
X3	0.589699088	0.110933412	0.571428861
X4	0.276915309	0.118469052	0.087004959
X5	1.074885659	0.388886434	0.347989910
X6	0.158150852	-0.024851988	0.110131391

	X4	X5	X6
X1	0.276915309	1.074885659	0.158150852
X2	0.118469052	0.388886434	-0.024851988
X3	0.087004959	0.347989910	0.110131391
X4	0.110409072	0.217405649	0.021814433
X5	0.217405649	0.862172372	-0.008817694
X6	0.021814433	-0.008817694	0.861455923

Correlation Matrix

	X1	X2	X3	X4	X5	X6
X1	1.0000	0.5514	0.3616	0.3863	0.5366	0.0790
X2	0.5514	1.0000	0.1875	0.4554	0.5350	-.0342
X3	0.3616	0.1875	1.0000	0.3464	0.4958	0.1570
X4	0.3863	0.4554	0.3464	1.0000	0.7046	0.0707
X5	0.5366	0.5350	0.4958	0.7046	1.0000	-.0102
X6	0.0790	-.0342	0.1570	0.0707	-.0102	1.0000

(b) We will work with **R** since the sample variance of x1 is approximately 40 times larger than that of x4.

Eigenvalues of the Correlation Matrix

	Eigenvalue	Difference	Proportion	Cumulative
PRIN1	2.86431	1.78786	0.477385	0.47738
PRIN2	1.07645	0.29881	0.179408	0.65679
PRIN3	0.77764	0.12733	0.129607	0.78640
PRIN4	0.65031	0.26228	0.108386	0.89479
PRIN5	0.38803	0.14478	0.064672	0.95946
PRIN6	0.24326		0.040543	1.00000

Eigenvectors

	PRIN1	PRIN2	PRIN3	PRIN4	PRIN5	PRIN6
X1	0.444858	-.026660	0.339330	-.551149	-.600851	0.146492
X2	0.429300	-.291738	0.498607	-.061367	0.687297	0.076408
X3	0.358773	0.380135	-.628157	-.421060	0.331839	0.211635
X4	0.462854	-.020959	-.124585	0.665604	-.207413	0.532689
X5	0.521276	-.073690	-.203339	0.200526	-.103175	-.794127
X6	0.055877	0.873960	0.429880	0.178715	0.053090	-.116262

(c) It is not possible to summarize the radiotherapy data with a single component. We need the first four components to summarize the data.

(d) Correlations between principal components and $X1 - X6$ are

	PRIN1	PRIN2	PRIN3	PRIN4
X1	0.75289	-0.02766	0.29923	-0.44446
X2	0.72656	-0.30268	0.43969	-0.04949
X3	0.60720	0.39440	-0.55393	-0.33955
X4	0.78335	-0.02175	-0.10986	0.53676
X5	0.88222	-0.07646	-0.17931	0.16171
X6	0.09457	0.90675	0.37909	0.14412

8.14 S is given in Example 5.2.

$$\hat{\lambda}_1 = 200.5, \quad \hat{\lambda}_2 = 4.5, \quad \hat{\lambda}_3 = 1.3$$

The first sample principal component explains a proportion

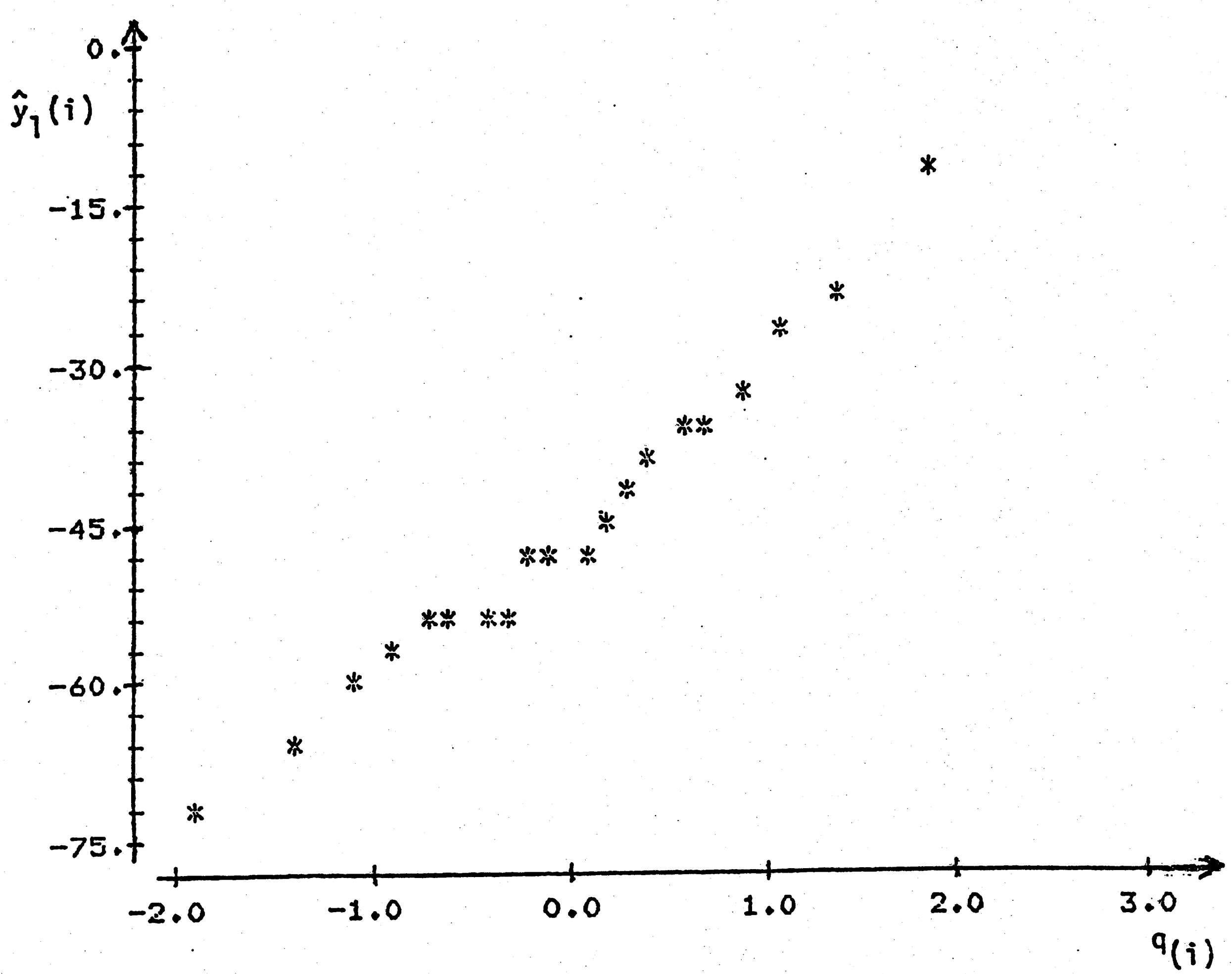
200.5/(200.5 + 4.5 + 1.3) = .97 of the total sample variance.

Also,

$$\hat{\mathbf{e}}_1' = [-.051, -.998, .029]$$

Hence $\hat{y}_1 = -.051x_1 - .998x_2 + .029x_3$

The first principal component is essentially X_2 = sodium content. (Note the (relatively) large sample variance for sodium in S). A Q-Q plot of the $\hat{y}_1$ values is shown below. These data appear to be approximately normal with no suspect observations.



Q-Q plot for $\hat{y}_1$.

8.15

$$S = \begin{bmatrix} 1088.40 & 831.28 & 763.23 & 784.09 \\ & 1128.41 & 850.32 & 926.73 \\ & & 1336.15 & 904.53 \\ \text{(Symmetric)} & & & 1395.15 \end{bmatrix}$$

$$\hat{\lambda}_1 = 3779.01; \quad \hat{\lambda}_2 = 468.25; \quad \hat{\lambda}_3 = 452.13; \quad \hat{\lambda}_4 = 248.72$$

Consequently, the first sample principal component accounts for a proportion $3779.01/4948.11 = .76$ of the total sample variance. Also,

$$\hat{\underset{\sim}{e}}_1' = [.45, \ .49, \ .51, \ .53]$$

Consequently,

$$\hat{y}_1 = .45x_1 + .49x_2 + .51x_3 + .53x_4$$

The interpretation of the first component is the same as the interpretation of the first component, obtained from **R**, in Example 8.6. (Note the sample variances in S are nearly equal).

8.16. Principal component analysis of Wisconsin fish data

(a) All are positively correlated.

(b) Principal component analysis using $x1 - x4$

```
Eigenvalues of R
2.1539 0.7875 0.6157 0.4429

Eigenvectors of R
0.7032  0.4295  0.1886 -0.7071
0.6722  0.3871 -0.4652  0.4702
0.5914 -0.7126 -0.2787 -0.3216
0.6983 -0.2016  0.4938  0.5318

                     pc1    pc2    pc3    pc4
         St. Dev. 1.4676 0.8874 0.7846 0.6655
    Prop. of Var. 0.5385 0.1969 0.1539 0.1107
Cumulative Prop. 0.5385 0.7354 0.8893 1.0000
```

The first principal component is essentially a total of all four. The second contrasts the Bluegill and Crappie with the two bass.

(c) Principal component analysis using $x1 - x6$

```
Eigenvalues of R
2.3549 1.0719 0.9842 0.6644 0.5004 0.4242

Eigenvectors of R
-0.6716  0.0114  0.5284 -0.0471  0.3765 -0.7293
-0.6668 -0.0100  0.2302 -0.7249 -0.1863  0.5172
-0.5555 -0.2927 -0.2911  0.1810 -0.6284 -0.3081
-0.7013 -0.0403  0.0355  0.6231  0.3407  0.5972
 0.3621 -0.4203  0.0143 -0.2250  0.5074  0.0872
-0.4111  0.0917 -0.8911 -0.2530  0.4021 -0.1731

                     pc1    pc2    pc3    pc4    pc5    pc6
         St. Dev. 1.5346 1.0353 0.9921 0.8151 0.7074 0.6513
    Prop. of Var. 0.3925 0.1786 0.1640 0.1107 0.0834 0.0707
Cumulative Prop. 0.3925 0.5711 0.7352 0.8459 0.9293 1.0000
```

The Walleye is contrasted with all the others in the first principal component (look at the covariance pattern). The second principal component is essentially the Walleye and somewhat the largemouth bass. The third principal component is nearly a contrast between Northern pike and Bluegill.

8.17

COVARIANCE MATRIX

x1	.0130016					
x2	.0103784	.0114179				
x3	.0223500	.0185352	.0803572			
x4	.0200857	.0210995	.0667762	.0694845		
x5	.0912071	.0085298	.0168369	.0177355	.0115684	
x6	.0079578	.0089085	.0128470	.0167936	.0080712	.0105991

The eigenvalues are

0.164 0.018 0.008 0.003 0.002 0.001

and the first two principal components are

$$[\ .218\ ,\ .204\ ,\ .673\ ,\ .633\ ,\ .181\ ,\ .159\ ]\ \underset{\sim}{x}$$

$$[\ .337\ ,\ .432\ ,\ -.500\ ,\ .024\ ,\ .430\ ,\ .514\ ]\ \underset{\sim}{x}$$

8.18 (a) & (b) Principal component analysis of the correlation matrix follows.

Correlations: 100m(s), 200m(s), 400m(s), 800m, 1500m, 3000m, Marathon

	100m(s)	200m(s)	400m(s)	800m	1500m	3000m
200m(s)	0.941					
400m(s)	0.871	0.909				
800m	0.809	0.820	0.806			
1500m	0.782	0.801	0.720	0.905		
3000m	0.728	0.732	0.674	0.867	0.973	
Marathon	0.669	0.680	0.677	0.854	0.791	0.799

Eigenanalysis of the Correlation Matrix

Eigenvalue	5.8076	0.6287	0.2793	0.1246	0.0910	0.0545	0.0143
Proportion	0.830	0.090	0.040	0.018	0.013	0.008	0.002
Cumulative	0.830	0.919	0.959	0.977	0.990	0.998	1.000

Variable	PC1	PC2	PC3	PC4	PC5	PC6	PC7
100m(s)	0.378	-0.407	0.141	-0.587	0.167	-0.540	0.089
200m(s)	0.383	-0.414	0.101	-0.194	-0.094	0.745	-0.266
400m(s)	0.368	-0.459	-0.237	0.645	-0.327	-0.240	0.127
800m	0.395	0.161	-0.148	0.295	0.819	0.017	-0.195
1500m	0.389	0.309	0.422	0.067	-0.026	0.189	0.731
3000m	0.376	0.423	0.406	0.080	-0.352	-0.240	-0.572
Marathon	0.355	0.389	-0.741	-0.321	-0.247	0.048	0.082

$$\hat{y}_1 = .378z_1 + .383z_2 + .368z_3 + .395z_4 + .389z_5 + .376z_6 + .355z_7$$
$$\hat{y}_2 = -.407z_1 - .414z_2 - .459z_3 + .161z_4 + .309z_5 + .423z_6 + .389z_7$$

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
$r_{\hat{y}_1, z_i}$	.911	.923	.887	.952	.937	.906	.856
$r_{\hat{y}_2, z_i}$	-.323	-.328	-.364	.128	.245	.335	.308

Cumulative proportion of total sample variance explained by the first two components is .919.

(c) All track events contribute about equally to the first component. This component might be called a track index or track excellence component. The second component contrasts the times for the shorter distances (100m, 200m 400m) with the times for the longer distances (800m, 1500m, 3000m, marathon) and might be called a distance component.

(d) The "track excellence" rankings for the first 10 and very last countries follow. These rankings appear to be consistent with intuitive notions of athletic excellence.

1. USA 2. Germany 3. Russia 4. China 5. France 6. Great Britain
7. Czech Republic 8. Poland 9. Romania 10. Australia .... 54. Somoa

8.19 Principal component analysis of the covariance matrix follows.

Covariances: 100m/s, 200m/s, 400m/s, 800m/s, 1500m/s, 3000m/s, Marm/s

	100m/s	200m/s	400m/s	800m/s	1500m/s	3000m/s
100m/s	0.0905383					
200m/s	0.0956063	0.1146714				
400m/s	0.0966724	0.1138699	0.1377889			
800m/s	0.0650640	0.0749249	0.0809409	0.0735228		
1500m/s	0.0822198	0.0960189	0.0954430	0.0864542	0.1238405	
3000m/s	0.0921422	0.1054364	0.1083164	0.0997547	0.1437148	0.1765843
Marm/s	0.0810999	0.0933103	0.1018807	0.0943056	0.1184578	0.1465604

	Marm/s
Marm/s	0.1667141

Eigenanalysis of the Covariance Matrix

Eigenvalue	0.73215	0.08607	0.03338	0.01498	0.00885	0.00617	0.00207
Proportion	0.829	0.097	0.038	0.017	0.010	0.007	0.002
Cumulative	0.829	0.926	0.964	0.981	0.991	0.998	1.000

Variable	PC1	PC2	PC3	PC4	PC5	PC6	PC7
100m/s	0.310	-0.376	0.098	-0.585	-0.046	-0.624	0.138
200m/s	0.357	-0.434	0.089	-0.323	-0.030	0.689	-0.311
400m/s	0.379	-0.519	-0.274	0.667	-0.187	-0.124	0.132
800m/s	0.299	0.053	-0.053	0.128	0.894	-0.136	-0.265
1500m/s	0.391	0.211	0.435	0.055	0.127	0.236	0.734
3000m/s	0.460	0.396	0.427	0.184	-0.357	-0.199	-0.499
Marm/s	0.423	0.445	-0.730	-0.237	-0.136	0.081	0.095

$$\hat{y}_1 = .310x_1 + .357x_2 + .379x_3 + .299x_4 + .391x_5 + .460x_6 + .423x_7$$
$$\hat{y}_2 = -.376x_1 - .434x_2 - .519x_3 + .053x_4 + .211x_5 + .396x_6 + .445x_7$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$r_{\hat{y}_1,x_i}$	.882	.902	.874	.944	.951	.937	.886
$r_{\hat{y}_2,x_i}$	-.367	-.376	-.410	.057	.176	.276	.320

Cumulative proportion of total sample variance explained by the first two components is .926.

The interpretation of the sample component is similar to the interpretation in Exercise 8.18. All track events contribute about equally to the first component. This component might be called a track index or track excellence component. The second component contrasts times in m/s for the shorter distances (100m, 200m 400m) with the times for the longer distances (800m, 1500m, 3000m, marathon) and might be called a distance component.

The "track excellence" rankings for the countries are very similar to the rankings for the countries obtained in Exercise 8.18.

8.20 (a) & (b) Principal component analysis of the correlation matrix follows.

```
Eigenanalysis of the Correlation Matrix

Eigenvalue  6.7033  0.6384  0.2275  0.2058  0.0976  0.0707  0.0469  0.0097
Proportion   0.838   0.080   0.028   0.026   0.012   0.009   0.006   0.001
Cumulative   0.838   0.918   0.946   0.972   0.984   0.993   0.999   1.000

Variable     PC1     PC2     PC3     PC4     PC5     PC6     PC7     PC8
100m       0.332   0.529   0.344  -0.381   0.300  -0.362   0.348  -0.066
200m       0.346   0.470  -0.004  -0.217  -0.541   0.349  -0.440   0.061
400m       0.339   0.345  -0.067   0.851   0.133   0.077   0.114  -0.003
800m       0.353  -0.089  -0.783  -0.134  -0.227  -0.341   0.259  -0.039
1500m      0.366  -0.154  -0.244  -0.233   0.652   0.530  -0.147  -0.040
5000m      0.370  -0.295   0.183   0.055   0.072  -0.359  -0.328   0.706
10,000m    0.366  -0.334   0.244   0.087  -0.061  -0.273  -0.351  -0.697
Marathon   0.354  -0.387   0.335  -0.018  -0.338   0.375   0.594   0.069
```

$$\hat{y}_1 = .332z_1 + .346z_2 + .339z_3 + .353z_4 + .366z_5 + .370z_6 + .366z_7 + .354z_8$$
$$\hat{y}_2 = .529z_1 + .470z_2 + .345z_3 - .089z_4 - .154z_5 - .295z_6 - .334z_7 - .387z_8$$

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
$r_{\hat{y}_1, z_i}$	.860	.896	.878	.914	.948	.958	.948	.917
$r_{\hat{y}_2, z_i}$	.423	.376	.276	-.071	-.123	-.236	-.267	-.309

Cumulative proportion of total sample variance explained by the first two components is .918.

(c) All track events contribute about equally to the first component. This component might be called a track index or track excellence component. The second component contrasts the times for the shorter distances (100m, 200m 400m) with the times for the longer distances (800m, 1500m, 5000m, 10,000m, marathon) and might be called a distance component.

(d) The male "track excellence" rankings for the first 10 and very last countries follow. These rankings appear to be consistent with intuitive notions of athletic excellence.

1. USA 2. Great Britain 3. Kenya 4. France 5. Australia 6. Italy
7. Brazil 8. Germany 9. Portugal 10. Canada .... 54. Cook Islands

The principal component analysis of the men's track data is consistent with that for the women.

8.21 Principal component analysis of the covariance matrix follows.

Covariances: 100m/s, 200m/s, 400m/s, 800m/s, 1500m/s, 5000m/s, 10,000m/s, Marathonm/s

	100m/s	200m/s	400m/s	800m/s	1500m/s
100m/s	0.0434979				
200m/s	0.0482772	0.0648452			
400m/s	0.0434632	0.0558678	0.0688217		
800m/s	0.0314951	0.0432334	0.0428221	0.0468840	
1500m/s	0.0425034	0.0535265	0.0537207	0.0523058	0.0729140
5000m/s	0.0469252	0.0587731	0.0617664	0.0571560	0.0766388
10,000m/s	0.0448325	0.0572512	0.0599354	0.0553945	0.0745719
Marathonm/s	0.0431256	0.0562945	0.0567342	0.0541911	0.0736518

	5000m/s	10,000m/s	Marathonm/s
5000m/s	0.0959398		
10,000m/s	0.0937357	0.0942894	
Marathonm/s	0.0905819	0.0909952	0.0979276

Eigenanalysis of the Covariance Matrix

Eigenvalue	0.49405	0.04622	0.01391	0.01332	0.00752	0.00575	0.00322
Proportion	0.844	0.079	0.024	0.023	0.013	0.010	0.006
Cumulative	0.844	0.923	0.947	0.970	0.983	0.993	0.998

Eigenvalue	0.00112
Proportion	0.002
Cumulative	1.000

Variable	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8
100m/s	0.244	-0.432	0.173	-0.450	-0.390	0.119	0.584	-0.119
200m/s	0.311	-0.523	0.235	-0.318	0.341	-0.247	-0.535	0.096
400m/s	0.317	-0.469	-0.684	0.420	0.046	0.177	0.039	-0.008
800m/s	0.278	-0.033	0.436	0.543	0.332	-0.368	0.432	-0.070
1500m/s	0.364	0.063	0.439	0.317	-0.303	0.608	-0.327	-0.044
5000m/s	0.428	0.261	-0.111	-0.016	-0.374	-0.334	-0.006	0.696
10,000m/s	0.421	0.310	-0.187	-0.100	-0.215	-0.352	-0.180	-0.693
Marathonm/s	0.416	0.387	-0.128	-0.339	0.584	0.391	0.215	0.074

$$\hat{y}_1 = .244x_1 + .311x_2 + .317x_3 + .278x_4 + .364x_5 + .428x_6 + .421x_7 + .416x_8$$

$$\hat{y}_2 = -.432x_1 - .523x_2 - .469x_3 - .033x_4 + .063x_5 + .261x_6 + .310x_7 + .387x_8$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$r_{\hat{y}_1, x_i}$	.822	.858	.849	.902	.948	.971	.964	.934
$r_{\hat{y}_2, x_i}$	-.445	-.442	-.384	-.033	.050	.181	.217	.266

Cumulative proportion of total sample variance explained by the first two components is .923.

The interpretation of the sample component is similar to the interpretation in Exercise 8.20. All track events contribute about equally to the first component. This component might be called a track index or track excellence component. The second component contrasts times in m/s for the shorter distances (100m, 200m 400m, 800m) with the times for the longer distances (1500m, 5000m, 10,000m, marathon) and might be called a distance component.

The "track excellence" rankings for the countries are very similar to the rankings for the countries obtained in Exercise 8.20.

8.22 Using S

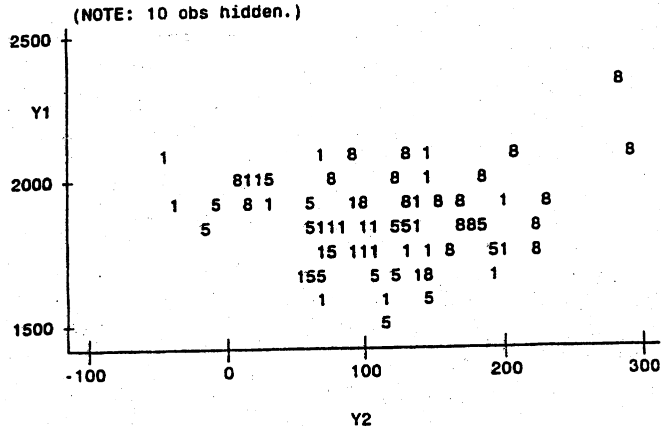
Eigenvalues of the Covariance Matrix

	Eigenvalue	Difference	Proportion	Cumulative
PRIN1	20579.6	15704.9	0.808198	0.80820
PRIN2	4874.7	4869.2	0.191437	0.99964
PRIN3	5.4	2.1	0.000213	0.99985
PRIN4	3.3	2.8	0.000130	0.99998
PRIN5	0.5	0.4	0.000018	1.00000
PRIN6	0.1	0.1	0.000003	1.00000
PRIN7	0.0	.	0.000000	1.00000

Eigenvectors

	PRIN1	PRIN2	PRIN3	PRIN4	PRIN5	PRIN6	PRIN7	
X3	0.005887	0.009680	0.286337	0.608787	0.535569	-.509727	0.024592	yrhgt
X4	0.487047	0.872697	-.034277	-.003227	0.000444	-.000457	-.000253	ftfrbody
X5	0.008526	0.029196	0.904389	-.425175	0.008388	0.010389	0.014293	prctffb
X6	0.003112	0.004886	0.133267	0.311194	0.390573	0.855204	-.037984	frame
X7	0.000069	-.000493	-.018864	-.005278	0.011906	0.043786	0.998778	bkfat
X8	0.009330	0.008577	0.284215	0.593037	-.748598	0.082331	0.013820	saleht
X9	0.873259	-.487193	0.004847	-.005597	0.002665	-.000341	-.000256	salewt

Plot of Y1*Y2. Symbol is value of X1.
(NOTE: 10 obs hidden.)



8.22 (Continued)

Using *R*

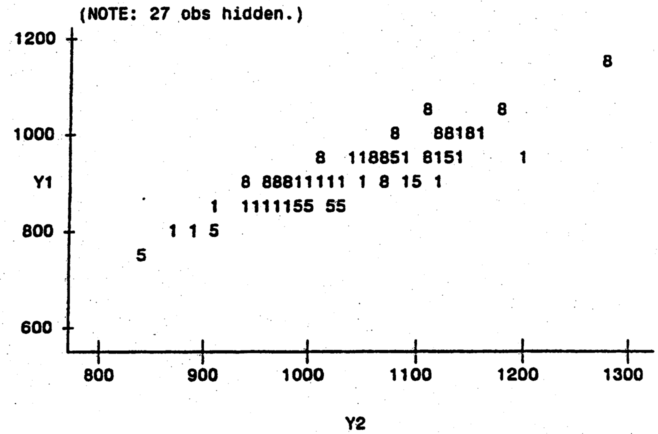
Eigenvalues of the Correlation Matrix

	Eigenvalue	Difference	Proportion	Cumulative
PRIN1	4.12070	2.78357	0.588671	0.58867
PRIN2	1.33713	0.59575	0.191018	0.77969
PRIN3	0.74138	0.31996	0.105912	0.88560
PRIN4	0.42143	0.23562	0.060204	0.94580
PRIN5	0.18581	0.03930	0.026544	0.97235
PRIN6	0.14650	0.09945	0.020929	0.99328
PRIN7	0.04706	.	0.006722	1.00000

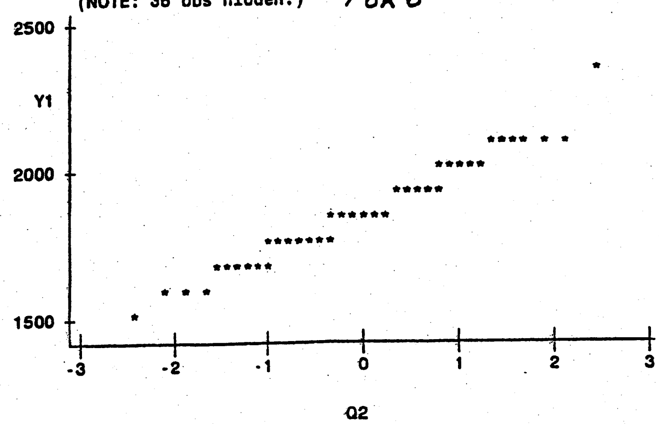
Eigenvectors

	PRIN1	PRIN2	PRIN3	PRIN4	PRIN5	PRIN6	PRIN7	
X3	0.449931	-.042790	-.415709	0.113356	0.065871	-.072234	0.774926	yrhgt
X4	0.412326	0.129837	0.450292	0.247479	-.719343	-.177061	0.017768	ftfrbody
X5	0.355562	-.315508	0.568273	0.314787	0.579367	0.127800	-.002397	prctffb
X6	0.433957	0.007728	-.452345	0.242818	0.142995	-.434144	-.582337	frame
X7	-.186705	0.714719	-.038732	0.618117	0.160238	0.208017	0.042442	bkfat
X8	0.452854	0.101315	-.176650	-.215769	-.109535	0.799288	-.236723	saleht
X9	0.269947	0.600515	0.253312	-.582433	0.290547	-.276561	0.047036	salewt

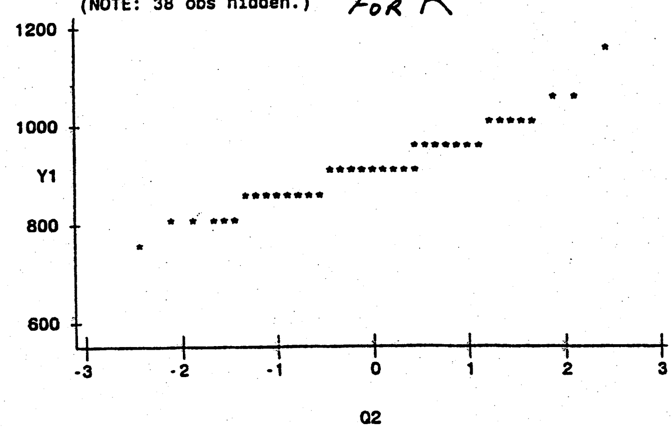
Plot of Y1*Y2. Symbol is value of X1.
(NOTE: 27 obs hidden.)



Plot of Y1*Q2. Symbol used is '*'.
(NOTE: 36 obs hidden.) FOR S



Plot of Y1*Q2. Symbol used is '*'.
(NOTE: 38 obs hidden.) FOR R



8.23 a) Using *S*

Eigenvalues of S

4478.87 152.47 32.32 8.12 1.52 0.54

Eigenvectors of S (in columns)

-0.849339	0.470832	-0.226606	0.074260	-0.008692	-0.000202
-0.368552	-0.846078	-0.368132	0.012754	-0.110784	-0.019105
-0.194132	-0.058127	0.303143	-0.928388	-0.012289	-0.070597
-0.314678	-0.216748	0.848576	0.355060	-0.082353	0.032666
-0.043918	-0.060354	0.001815	-0.060162	0.440119	0.892805
-0.064458	-0.092026	0.033880	0.052267	0.887138	-0.443264

The first component might be identified as a "size" component. It is dominated by Weight, Body length and Girth, those variables with the largest sample variances. The first component explains 4478.87/4673.84 = .958 or 95.8% of the total sample variance. The second component essentially contrasts Weight with the remaining body size variables, Body length, Neck, Girth, Head length, and Head width, although the sample correlation between the second component and Neck is small (–.05). The first two components explain 99.1% of the total sample variance.

These body measurement data can be effectively summarized in one dimension.

b) Using *R*

R

1.0000	0.8752	0.9559	0.9437	0.9025	0.9045
0.8752	1.0000	0.9013	0.9177	0.9461	0.9503
0.9559	0.9013	1.0000	0.9635	0.9270	0.9200
0.9437	0.9177	0.9635	1.0000	0.9271	0.9439
0.9025	0.9461	0.9270	0.9271	1.0000	0.9544
0.9045	0.9503	0.9200	0.9439	0.9544	1.0000

Eigenvalues of R

5.6447 0.1758 0.0565 0.0492 0.0473 0.0266

Eigenvectors of R (in columns)

-0.403672	-0.558334	0.286817	0.261937	-0.598371	0.128024
-0.404313	0.532348	-0.186741	0.719785	0.004276	0.012490
-0.409938	-0.389366	0.035396	0.073950	0.561034	-0.599053
-0.411999	-0.222694	-0.581252	-0.228969	0.231095	0.580499
-0.409162	0.318718	0.695916	-0.291938	0.251473	0.313431
-0.410333	0.319513	-0.243840	-0.519785	-0.458838	-0.435168

8.23 (Continued)

Again, the first principal component is a "size" component. All variables contribute equally to the first component. This component explains 5.6447/6 = .941 or 94.1% of the total sample variance. The second principal component contrasts Weight, Neck and Girth with Body length, Head length and Head width. The first two components explain 97% of the total sample variance.

These data can be effectively summarized in one dimension.

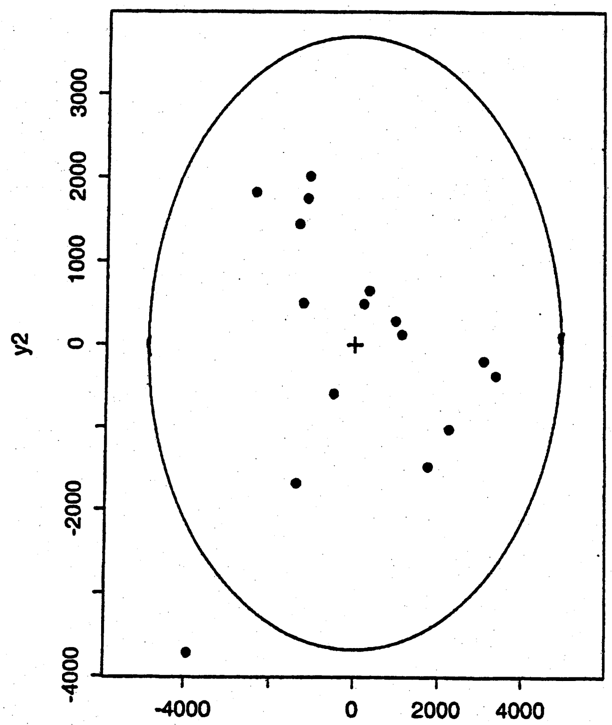
c) The results are similar for both the covariance matrix S and the correlation matrix R. The first component in each analysis is a "size" component and almost all of the variation in the data. The analyses differ a bit with respect to the second and remaining components, but these latter components explain very little of the total sample variance.

8.24 An ellipse format chart based on the first two principal components of the Madison, Wisconsin, Police Department data

```
XBAR
3557.8 1478.4 2676.9 13563.6 800 7141

S
367884.7    -72093.8     85714.8    222491.4   -44908.3    101312.9
-72093.8   1399053.1     43399.9    139692.2   110517.1   1161018.3
 85714.8     43399.9   1458543.0  -1113809.8   330923.8   1079573.3
222491.4    139692.2  -1113809.8   1698324.4  -244785.9   -462615.6
-44908.3    110517.1    330923.8   -244785.9   224718.0    427767.5
101312.9   1161018.3   1079573.3   -462615.6   427767.5   2488728.4

Eigenvalues of S
4045921.9 2265078.9  761592.1  288919.3  181437.0   94302.6

Eigenvectors of S
-0.0008 -0.0567 -0.5157  0.6122  0.4311 -0.4126
-0.3092 -0.5541  0.5615  0.4932 -0.1796 -0.0810
-0.4821  0.3862 -0.3270  0.3404 -0.5696  0.2667
 0.3675 -0.6415 -0.4898 -0.0642 -0.4308  0.1543
-0.1544  0.0359 -0.0316 -0.3071 -0.4062 -0.8453
-0.7163 -0.3575 -0.2662 -0.4094  0.3269  0.1173

Principal components
        y1      y2      y3     y4     y5     y6
 1  1745.4 -1479.3   618.7  222.6    7.2  178.1
 2 -1096.6  2011.8   652.5  -69.5  636.9  560.2
 3   210.6   490.6   365.8 -899.8 -293.5  -15.2
 4 -1360.1  1448.1   420.1  523.5 -972.2   88.5
 5 -1255.9   502.1  -422.4 -893.8  359.9 -273.7
 6   971.6   284.7  -316.9 -942.8  -83.5  -70.1
 7  1118.5   123.7   572.9  319.9  -60.8 -598.5
 8 -1151.6  1752.0 -1322.1  700.2 -242.2 -158.8
 9  -497.3  -593.0   209.5 -149.2  101.6 -586.2
10 -2397.1  1819.6    -9.5 -147.6 -109.9  207.8
11 -3931.9 -3715.7   924.1   35.1 -274.2  152.9
12 -1392.4 -1688.0 -2285.1  372.1  444.0   85.2
13   326.8   650.8  1251.6  728.8  809.5 -140.0
14  3371.4  -379.1  -499.9 -114.6 -324.3  286.9
15  3076.6  -199.1  -105.7  419.8 -122.3    3.4
16  2261.9 -1029.3   -53.7 -104.5  123.8  279.6
```

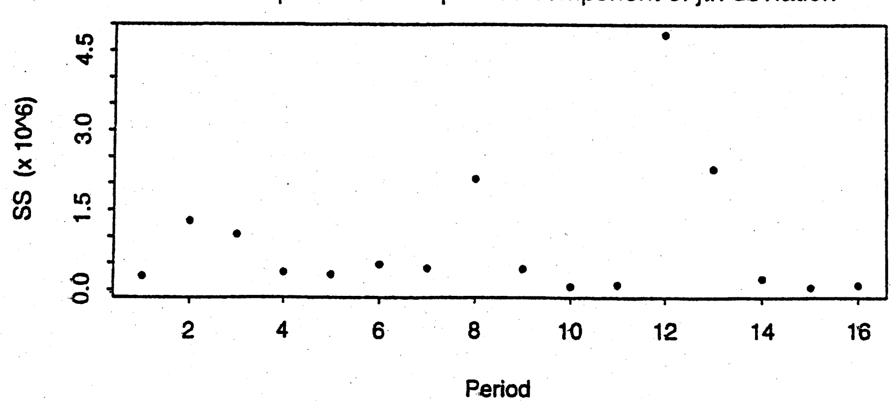
$$2.5 \times 10^{-7} y_1^2 + 4.4 \times 10^{-7} y_2^2 = 5.99$$

The 95% control ellipse based on the
first two principal components of overtime hours



8.25 A control chart based on the sum of squares d_{Uj}^2. Period 12 looks unusual.

Sum of squares of unexplained component of jth deviation



8.26 (a)-(c) Principal component analysis of the correlation matrix **R**.

Correlations: Indep, Supp, Benev, Conform, Leader

	Indep	Supp	Benev	Conform
Supp	-0.173			
Benev	-0.561	0.018		
Conform	-0.471	-0.327	0.298	
Leader	0.187	-0.401	-0.492	-0.333

Cell Contents: Pearson correlation

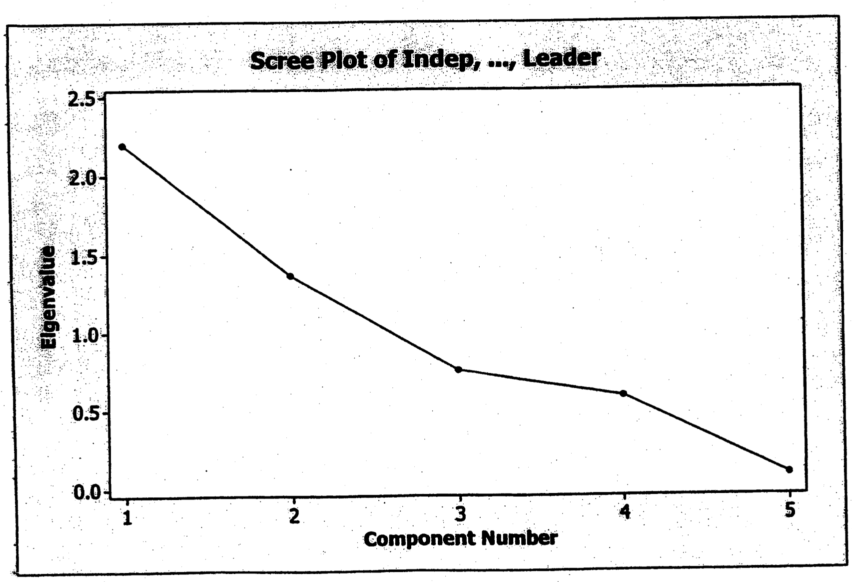
Principal Component Analysis: Indep, Supp, Benev, Conform, Leader

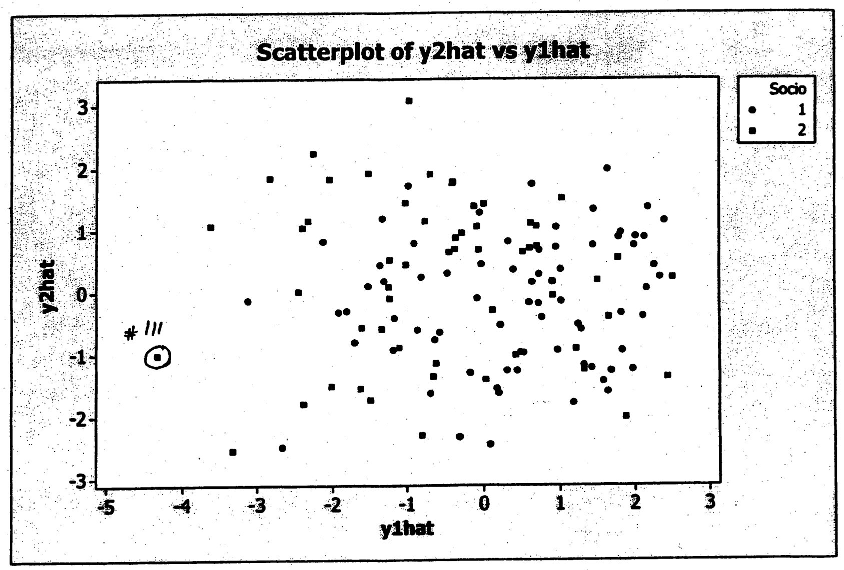
Eigenanalysis of the Correlation Matrix

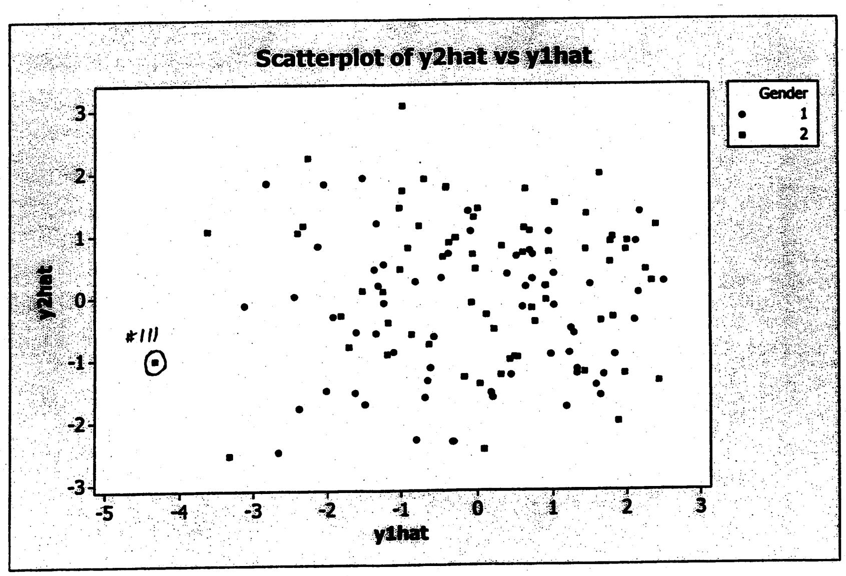
Eigenvalue	2.1966	1.3682	0.7559	0.5888	0.0905
Proportion	0.439	0.274	0.151	0.118	0.018
Cumulative	0.439	0.713	0.864	0.982	1.000

Variable	PC1	PC2	PC3	PC4	PC5
Indep	-0.521	0.087	-0.667	-0.253	-0.460
Supp	0.121	0.788	0.187	0.351	-0.454
Benev	0.548	-0.008	0.115	-0.733	-0.386
Conform	0.439	-0.491	-0.295	0.525	-0.451
Leader	-0.469	-0.361	0.648	0.007	-0.480

Using the scree plot and the proportion of variance explained, it appears as if 4 components should be retained. These components explain almost all (98%) of the variability. It is difficult to provide an interpretation of the components without knowing more about the subject matter. All four of the components represent contrasts of some form. The first component contrasts independence and leadership with benevolence and conformity. The second component contrasts support with conformity and leadership and so on.







The two dimensional plot of the scores on the first two components suggests that the two socioeconomic levels cannot be distinguished from one another nor can the two genders be distinguished. Observation #111 is a bit removed from the rest and might be called an outlier.

(a)-(d) Principal component analysis of the covariance matrix **S**.

Covariances: Indep, Supp, Benev, Conform, Leader

	Indep	Supp	Benev	Conform	Leader
Indep	34.7502				
Supp	-4.2767	17.5134			
Benev	-18.0718	0.4198	29.8447		
Conform	-15.9729	-7.8682	9.3488	33.0426	
Leader	5.7165	-8.7233	-13.9422	-9.9419	26.9580

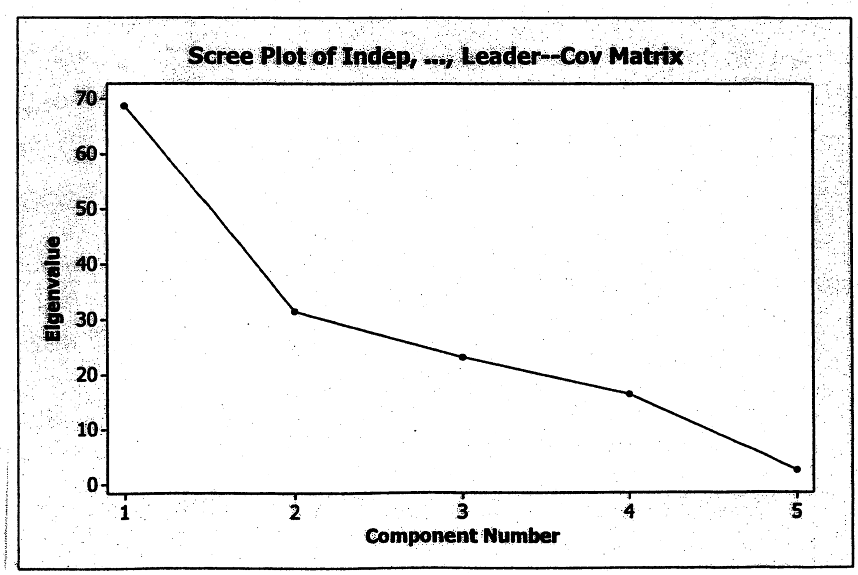
Principal Component Analysis: Indep, Supp, Benev, Conform, Leader

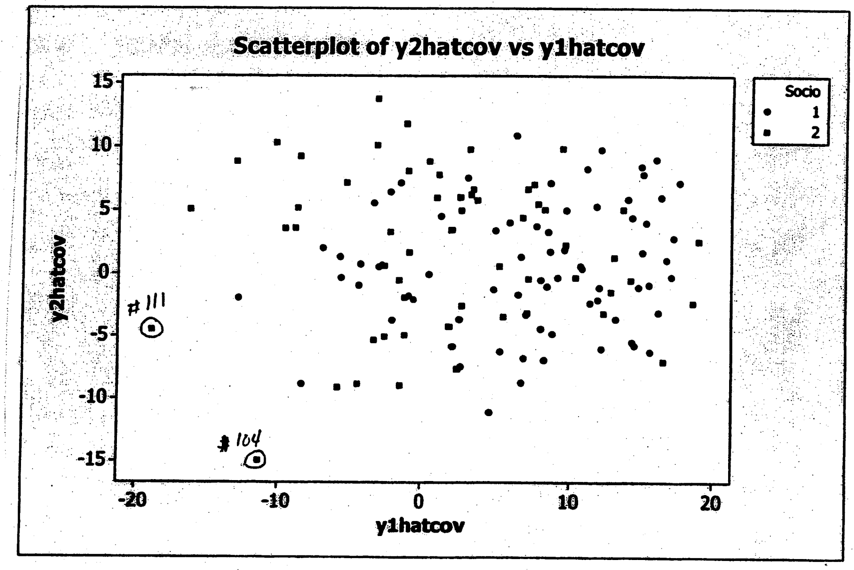
Eigenanalysis of the Covariance Matrix

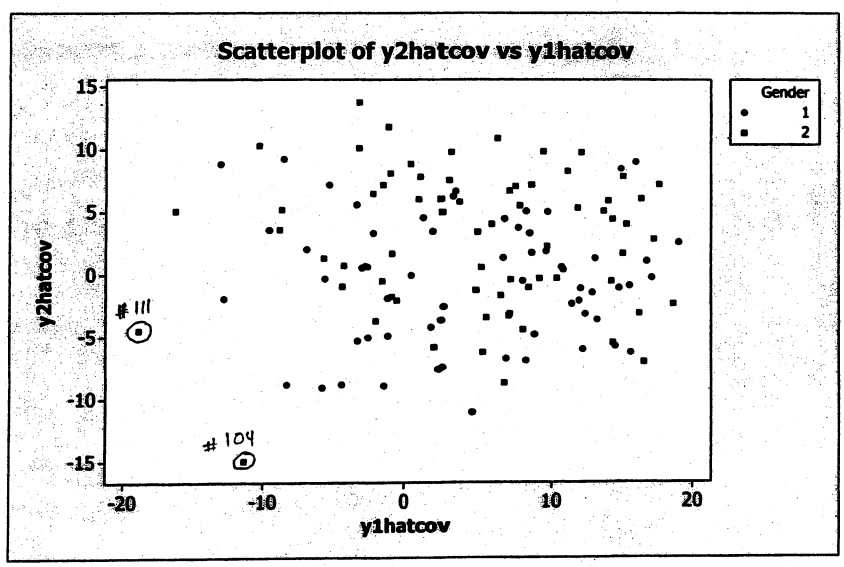
Eigenvalue	68.752	31.509	23.101	16.354	2.392
Proportion	0.484	0.222	0.163	0.115	0.017
Cumulative	0.484	0.706	0.868	0.983	1.000

Variable	PC1	PC2	PC3	PC4	PC5
Indep	-0.579	0.079	-0.643	0.309	0.386
Supp	0.042	0.612	0.140	-0.515	0.583
Benev	0.524	0.219	0.119	0.734	0.352
Conform	0.493	-0.572	-0.422	-0.304	0.398
Leader	-0.380	-0.494	0.612	0.090	0.478

Using the scree plot and the proportion of variance explained, it appears as if 4 components should be retained. These components explain almost all (98%) of the variability. The components are very similar to those obtained from the correlation matrix **R**. All four of the components represent contrasts of some form. The first component contrasts independence and leadership with benevolence and conformity. The second component contrasts support with conformity and leadership and so on. In this case, it makes little difference whether the components are obtained from the sample correlation matrix or the sample covariance matrix.







The two dimensional plot of the scores on the first two components suggests that the two socioeconomic levels cannot be distinguished from one another nor can the two genders be distinguished. Observations #111 and #104 are a bit removed from the rest and might be labeled outliers.

Large sample 95% confidence interval for λ_1:

$$\left(\frac{68.752}{(1+1.96\sqrt{2/130}}, \frac{68.752}{(1-1.96\sqrt{2/130}}\right) = (55.31,\ 90.83)$$

8.27 (a)-(d) Principal component analysis of the correlation matrix **R**.

Correlations: BL, EM, SF, BS

	BL	EM	SF
EM	0.914		
SF	0.984	0.942	
BS	0.988	0.875	0.975

Cell Contents: Pearson correlation

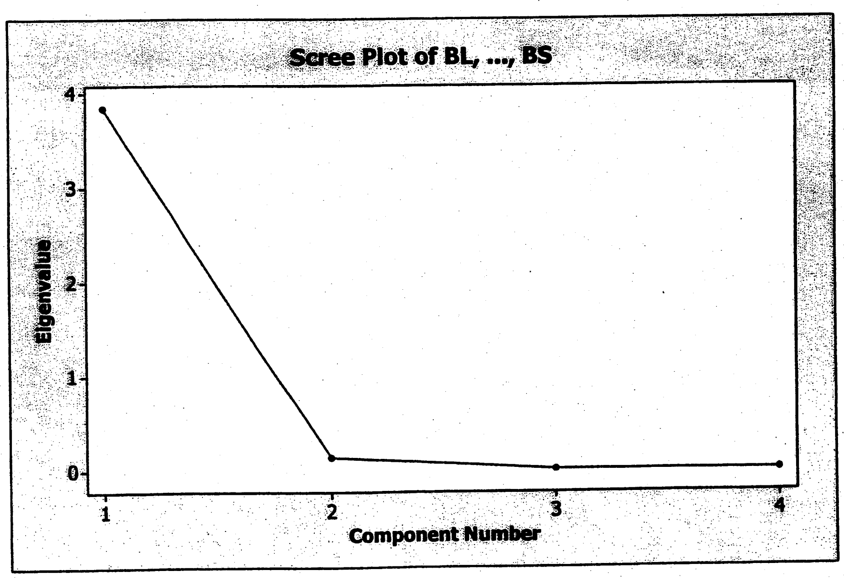
Principal Component Analysis: BL, EM, SF, BS

Eigenanalysis of the Correlation Matrix

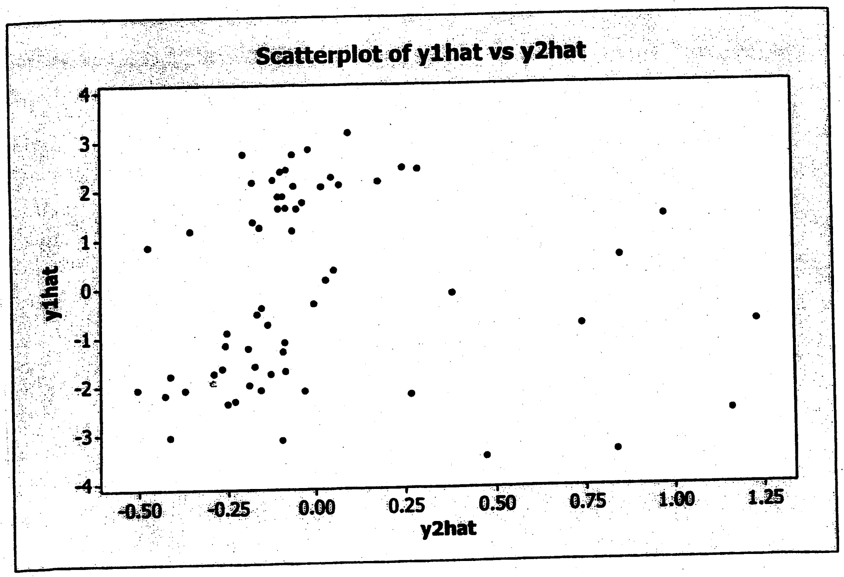
Eigenvalue	3.8395	0.1403	0.0126	0.0076
Proportion	0.960	0.035	0.003	0.002
Cumulative	0.960	0.995	0.998	1.000

Variable	PC1	PC2	PC3	PC4
BL	0.506	-0.261	-0.565	0.597
EM	0.485	0.819	-0.194	-0.237
SF	0.508	-0.020	0.800	0.318
BS	0.500	-0.510	-0.053	-0.698

The proportion of variance explained and the scree plot below suggest that one principal component effectively summarizes the paper properties data. All the variables load about equally on this component so it might be labeled an index of paper strength.



The plot below of the scores on the first two sample principal components does not indicate any obvious outliers.



(a)-(d) Principal component analysis of the covariance matrix **S**.

Covariances: BL, EM, SF, BS

	BL	EM	SF	BS
BL	8.302871			
EM	1.886636	0.513359		
SF	4.147318	0.987585	2.140046	
BS	1.972056	0.434307	0.987966	0.480272

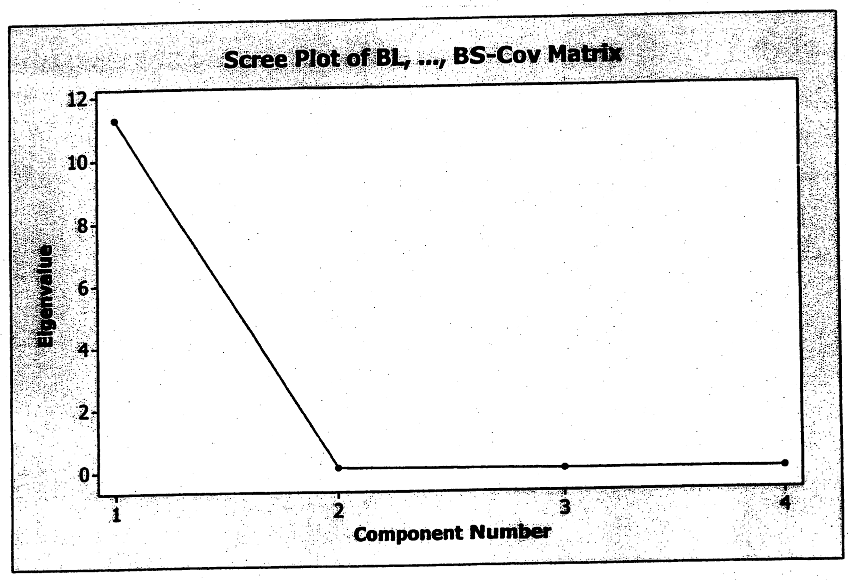
Principal Component Analysis: BL, EM, SF, BS

Eigenanalysis of the Covariance Matrix

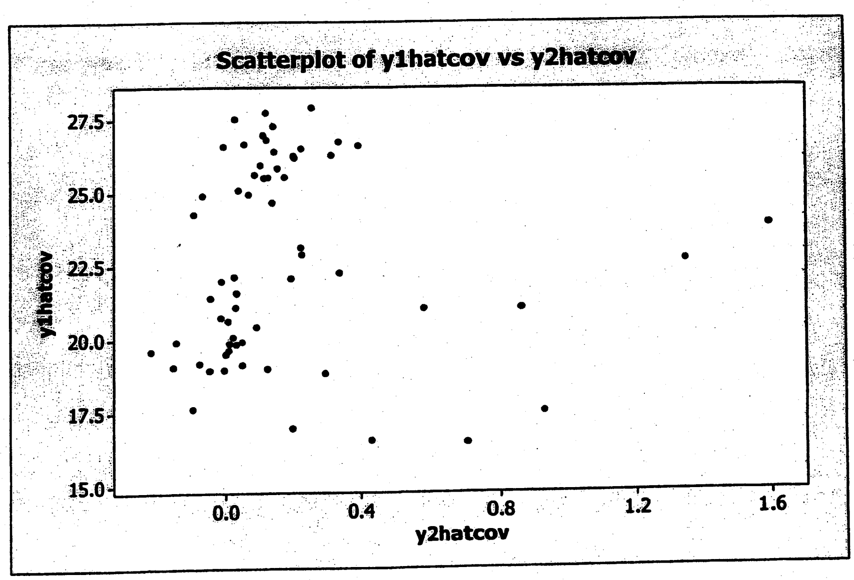
Eigenvalue	11.295	0.104	0.032	0.006
Proportion	0.988	0.009	0.003	0.001
Cumulative	0.988	0.997	0.999	1.000

Variable	PC1	PC2	PC3	PC4
BL	0.856	-0.364	-0.332	0.155
EM	0.198	0.786	-0.497	-0.310
SF	0.431	0.458	0.733	0.259
BS	0.204	-0.201	0.325	-0.901

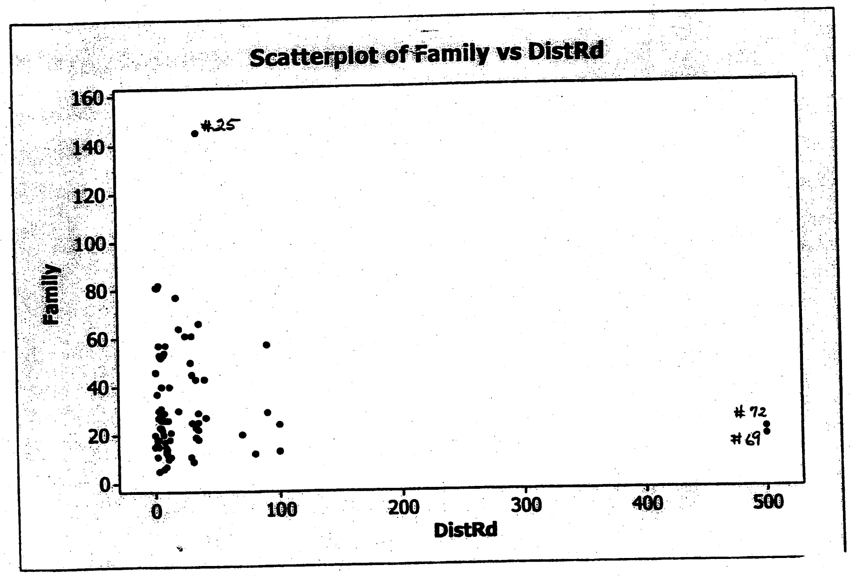
The proportion of variance explained and the scree plot that follows suggest that one principal component effectively summarizes the paper properties data. The loadings of the variables on the first component are all positive, but there are some differences in magnitudes. However, the correlations of the variables with the first component are .998, .928, .990 and .989 for BL, EM, SF and BS respectively. Again, this component might be labeled an index of paper strength.

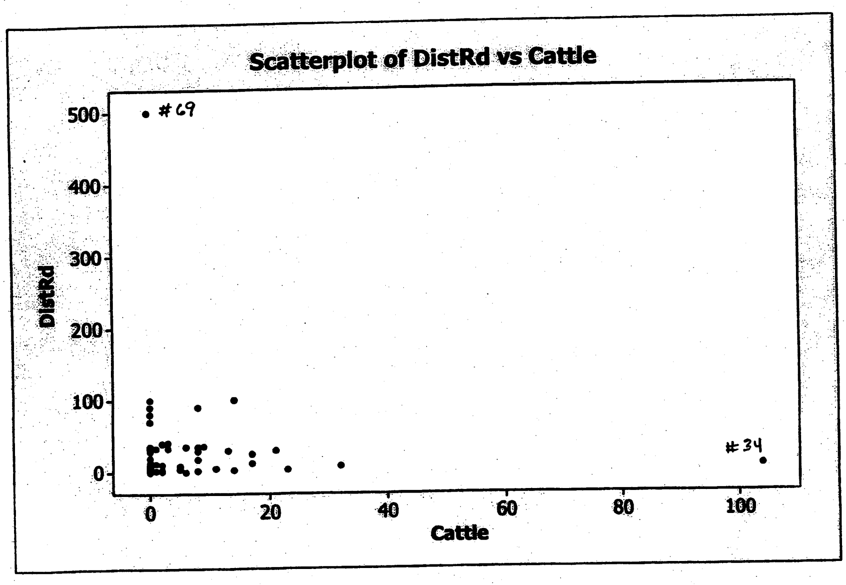


The plot below of the scores on the first two sample principal components does not indicate any obvious outliers.

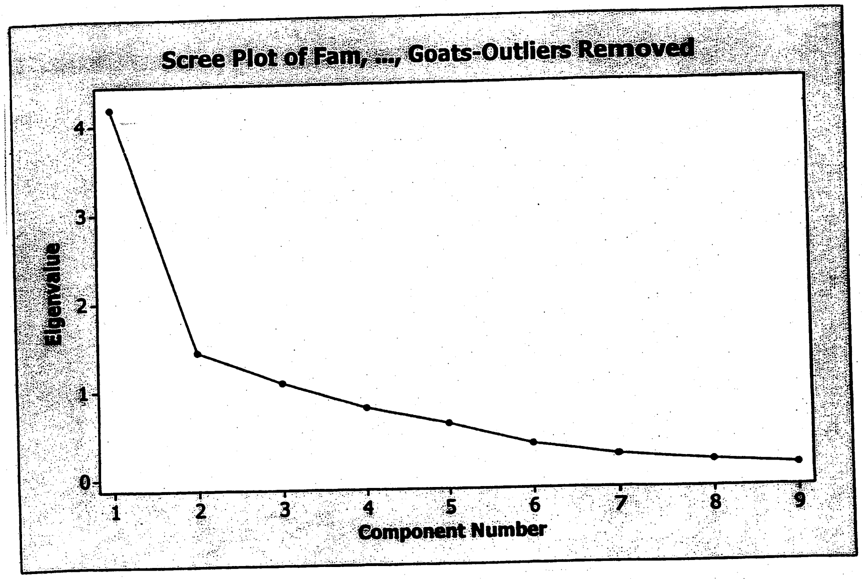


8.28 (a) See scatter plots below. Observations 25, 34, 69 and 72 are outliers.





(b) Principal component analysis of **R** follows. Removing the outliers has some but relatively little effect on the analysis. Five components explain about 90% of the total variability in the data set and seems a reasonable number given the scree plot.



Principal Component Analysis: AdjFam, AdjDistRd, AdjCotton, AdjMaize, AdjSorg,..(Outliers 25,34,69,72 removed)

Eigenanalysis of the Correlation Matrix

Eigenvalue	4.1851	1.4381	1.0845	0.7918	0.6043	0.3661	0.2400	0.1718
Proportion	0.465	0.160	0.121	0.088	0.067	0.041	0.027	0.019
Cumulative	0.465	0.625	0.745	0.833	0.900	0.941	0.968	0.987

Eigenvalue	0.1182
Proportion	0.013
Cumulative	1.000

Variable	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9
AdjFam	0.434	0.065	0.098	0.171	0.011	-0.040	-0.797	-0.263	-0.249
AdjDistRd	0.008	-0.497	-0.569	0.496	-0.378	0.187	0.021	-0.048	-0.065
AdjCotton	0.446	-0.009	0.132	-0.027	-0.219	-0.200	0.361	0.329	-0.675
AdjMaize	0.352	-0.353	0.388	0.240	-0.079	-0.273	-0.024	0.363	0.574
AdjSorg	0.204	0.604	-0.111	-0.059	-0.645	0.246	-0.021	0.126	0.293
AdjMillet	0.240	0.415	-0.116	0.616	0.527	0.181	0.241	0.077	0.048
AdjBull	0.445	-0.068	-0.030	-0.146	-0.028	-0.134	0.396	-0.751	0.190
AdjCattle	0.355	-0.284	0.014	-0.373	0.218	0.759	-0.011	0.169	0.038
AdjGoats	0.255	0.049	-0.687	-0.351	0.249	-0.402	-0.131	0.274	0.149

Principal Component Analysis: Family, DistRd, Cotton, Maze, Sorg, Millet, Bull, ...

Eigenanalysis of the Correlation Matrix

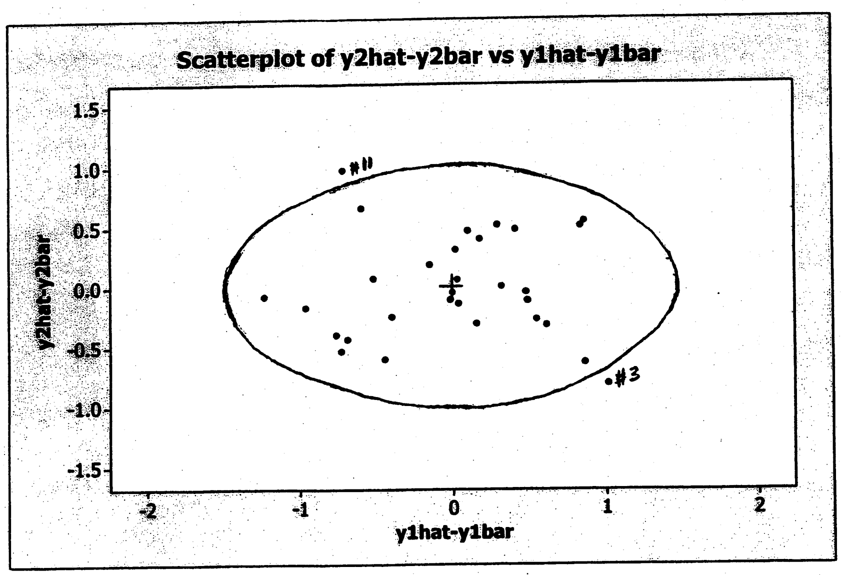
Eigenvalue	4.1443	1.2364	1.0581	0.9205	0.6058	0.5044	0.2720	0.1470
Proportion	0.460	0.137	0.118	0.102	0.067	0.056	0.030	0.016
Cumulative	0.460	0.598	0.715	0.818	0.885	0.941	0.971	0.988

Eigenvalue	0.1114
Proportion	0.012
Cumulative	1.000

Variable	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9
Family	0.444	-0.100	-0.002	-0.123	-0.089	-0.127	-0.579	0.454	-0.461
DistRd	-0.033	-0.072	-0.831	0.502	-0.194	-0.051	-0.045	0.082	0.041
Cotton	0.411	-0.342	-0.068	0.030	0.100	-0.216	0.509	-0.372	-0.504
Maze	0.337	-0.554	0.170	0.164	-0.134	0.053	-0.352	-0.360	0.499
Sorg	0.311	0.452	-0.069	-0.229	-0.361	-0.632	0.055	-0.139	0.300
Millet	0.269	0.043	-0.385	-0.606	-0.182	0.594	0.089	-0.097	0.077
Bull	0.440	-0.029	0.122	0.197	0.129	0.110	0.458	0.621	0.357
Cattle	0.247	0.458	0.278	0.486	-0.392	0.407	-0.012	-0.215	-0.225
Goats	0.309	0.379	-0.173	0.100	0.770	0.043	-0.242	-0.242	0.095

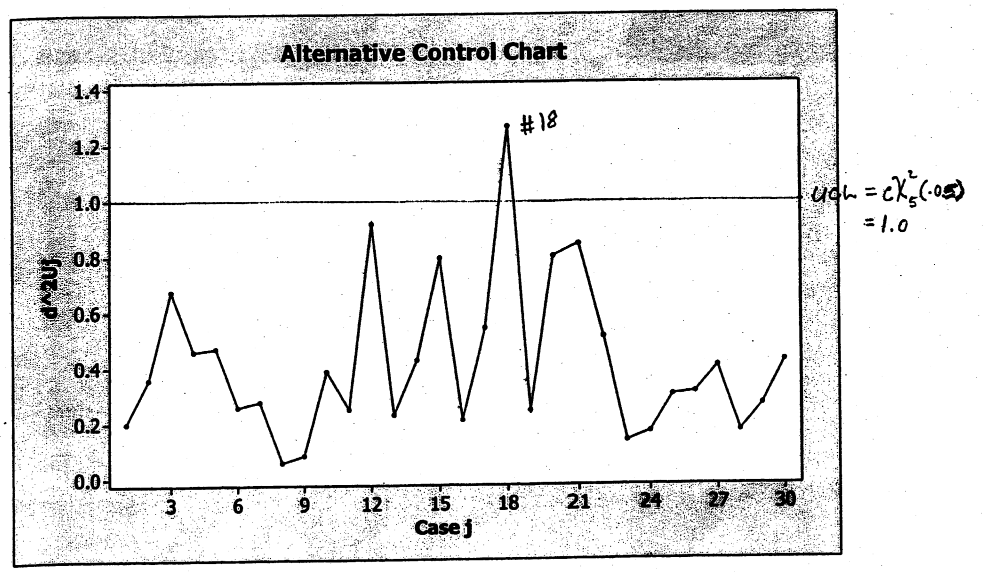
(c) All the variables (all crops, all livestock, family) except for distance to road (DistRd) load about equally on the first component. This component might be called a farm size component. Millet and sorghum load positively and distance to road and maize load negatively on the second component. Without additional subject matter knowledge, this component is difficult to interpret. The third component is essentially a distance to the road and goats component. This component might represent subsistence farms. The fourth component appears to be a contrast between distance to road and millet versus cattle and goats. Again, this component is difficult to interpret. The fifth component appears to contrast sorghum with millet.

8.29 (a) The 95% ellipse format chart using the first two principal components from the covariance matrix **S** (for the first 30 cases of the car body assembly data) is shown below. The ellipse consists of all $\hat{y}_1, \hat{y}_2$ such that $\frac{\hat{y}_1^2}{\hat{\lambda}_1} + \frac{\hat{y}_2^2}{\hat{\lambda}_2} \leq \chi_2^2(.05) = 5.99$ where $\hat{\lambda}_1 = .354, \hat{\lambda}_2 = .186$. Observations 3 and 11 lie outside the ellipse.



(b) To construct the alternative control chart based upon unexplained components of the observations we note that $\bar{d}_U^2 = .4137$, $s_{d^2}^2 = .0782$ so

$c = \frac{.0782}{2(.4137)} = .0946$, $\upsilon = 2\frac{(.4137)^2}{.0782} = 4.4$. Conservatively, we set the chi-squared degrees of freedom to $\upsilon = 5$ and the UCL becomes $c\chi_5^2(.05) = .0946(11.07) = 1.05$ or approximately 1.0. The alternative control chart is plotted on the next page and it appears as if multivariate observation 18 is out of control. For observation 18, $\hat{y}_4^2$ makes the largest contribution to d_{U18}^2 and the variables getting the most weight in $\hat{y}_4$ are the thickness measurements x_1 and x_2. Car body #18 could be examined at locations 1 and 2 to determine the cause of the unusual deviations in thickness from the nominal levels.



Chapter 9

9.1

$$L' = [.9 \quad .7 \quad .5]; \qquad LL' = \begin{bmatrix} .81 & .63 & .45 \\ .63 & .49 & .35 \\ .45 & .35 & .25 \end{bmatrix}$$

so $\boldsymbol{\rho} = LL' + \Psi$

9.2 a) For $m=1$

$$h_1^2 = \ell_{11}^2 = .81$$
$$h_2^2 = \ell_{21}^2 = .49$$
$$h_3^2 = \ell_{31}^2 = .25$$

The communalities are those parts of the variances of the variables explained by the single factor.

b) $\text{Corr}(Z_i,F_1) = \text{Cov}(Z_i,F_1)$, $i = 1,2,3$. By (9-5) $\text{Cov}(Z_i,F_1) = \ell_{i1}$. Thus $\text{Corr}(Z_1,F_1) = \ell_{11} = .9$; $\text{Corr}(Z_2,F_1) = \ell_{21} = .7$; $\text{Corr}(Z_3,F_1) = \ell_{31} = .5$. The first variable, Z_1, has the largest correlation with the factor and therefore will probably carry the most weight in naming the factor.

9.3 a) $L = \sqrt{\lambda_1}\,\underset{\sim}{e}_1 = \sqrt{1.96}\begin{bmatrix} .625 \\ .593 \\ .507 \end{bmatrix} = \begin{bmatrix} .876 \\ .831 \\ .711 \end{bmatrix}$. Slightly different from result in Exercise 9.1.

b) Proportion of total variance explained $= \dfrac{\lambda_1}{p} = \dfrac{1.96}{3} = .65$

9.4

$$\boldsymbol{\rho} = \boldsymbol{\rho} - \Psi = LL' = \begin{bmatrix} .81 & .63 & .45 \\ .63 & .49 & .35 \\ .45 & .35 & .25 \end{bmatrix}$$

$$L = \sqrt{\lambda_1}\,\underset{\sim}{e}_1 = \sqrt{1.55}\begin{bmatrix} .7229 \\ .5623 \\ .4016 \end{bmatrix} = \begin{bmatrix} .9 \\ .7 \\ .5 \end{bmatrix}$$

Result is consistent with results in Exercise 9.1. It should be since $m=1$ common factor completely determines $\tilde{\rho} = \rho - \Psi$.

9.5

Since $\tilde{\Psi}$ is diagonal and $S - \tilde{L}\tilde{L}' - \tilde{\Psi}$ has zeros on the diagonal, (sum of squared entries of $S - \tilde{L}\tilde{L}' - \tilde{\Psi}$) $\le$ (sum of squared entries of $S - \tilde{L}\tilde{L}$). By the hint, $S - \tilde{L}\tilde{L} = \hat{P}_{(2)}\hat{\Lambda}_{(2)}\hat{P}'_{(3)}$ which has sum of squared entries

$$\text{tr}[\hat{P}_{(2)}\hat{\Lambda}_{(2)}\hat{P}'_{(2)}(\hat{P}_{(2)}\hat{\Lambda}_{(2)}\hat{P}'_{(2)})'] = \text{tr}[\hat{P}_{(2)}\hat{\Lambda}_{(2)}\hat{\Lambda}'_{(2)}\hat{P}'_{(2)}]$$

$$= \text{tr}[\hat{\Lambda}_{(2)}\hat{\Lambda}'_{(2)}\hat{P}'_{(2)}\hat{P}_{(2)}] = \text{tr}[\hat{\Lambda}_{(2)}\hat{\Lambda}'_{(2)}]$$

$$= \hat{\lambda}^2_{m+1} + \hat{\lambda}^2_{m+2} + \cdots + \hat{\lambda}^2_p$$

Therefore,

(sum of squared entries of $S - \tilde{L}\tilde{L}' - \tilde{\Psi}$) $\le \hat{\lambda}^2_{m+1} + \hat{\lambda}^2_{m+2} + \cdots + \hat{\lambda}^2_p$

9.6

a) Follows directly from hint.

b) Using the hint, we post multiply by $(LL' + \Psi)$ to get

$$I = (\Psi^{-1} - \Psi^{-1}L(I + L'\Psi^{-1}L)^{-1}L'\Psi^{-1})(LL' + \Psi)$$

$$= \Psi^{-1}(LL' + \Psi) - \Psi^{-1}\underbrace{L(I + L'\Psi^{-1}L)^{-1}L'\Psi^{-1}(L}_{\text{(use part (a))}}L' + \Psi)$$

$$= \Psi^{-1}(LL' + \Psi) - \Psi^{-1}L(I - (I + L'\Psi^{-1}L)^{-1})L'$$

$$- \Psi^{-1}L(I + L'\Psi^{-1}L)^{-1}L'$$

$$= \Psi^{-1}LL' + I - \Psi^{-1}LL' + \Psi^{-1}L(I + L'\Psi^{-1}L)^{-1}L'$$

$$- \Psi^{-1}L(I + L'\Psi^{-1}L)^{-1}L' = I$$

Note all these multiplication steps are reversible.

c) Multiplying the result in (b) by L we get

$$(LL' + \Psi)^{-1}L = \Psi^{-1}L - \Psi^{-1}\underbrace{L(I + L'\Psi^{-1}L)^{-1}L'\Psi^{-1}L}_{\text{(use part (a))}}$$

$$= \Psi^{-1}L - \Psi^{-1}L(I - (I + L'\Psi^{-1}L)^{-1}) = \Psi^{-1}L(I + L'\Psi^{-1}L)^{-1}$$

Result follows by taking the transpose of both sides of the final equality.

9.7 From the equation $\Sigma = LL' + \Psi$, $m = 1$, we have

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \ell_{11}^2 + \psi_1 & \ell_{11}\ell_{21} \\ \ell_{11}\ell_{21} & \ell_{21}^2 + \psi_2 \end{bmatrix}$$

so $\sigma_{11} = \ell_{11}^2 + \psi_1$, $\sigma_{22} = \ell_{21}^2 + \psi_2$ and $\sigma_{12} = \ell_{11}\ell_{21}$. Let $\rho = \sigma_{12}/\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$. Then, for any choice $|\rho|\sqrt{\sigma_{22}} \le \ell_{21} \le \sqrt{\sigma_{22}}$, set $\ell_{11} = \sigma_{12}/\ell_{21}$ and check $\sigma_{12} = \ell_{11}\ell_{21}$. We obtain $\psi_1 = \sigma_{11} - \ell_{11}^2 = \sigma_{11} - \frac{\sigma_{12}^2}{\ell_{21}^2} \ge \sigma_{11} - \frac{\sigma_{12}^2}{\rho^2\sigma_{22}} = \sigma_{11} - \sigma_{11} = 0$ and $\psi_2 = \sigma_{22} - \ell_{21}^2 \ge \sigma_{22} - \sigma_{22} = 0$. Since ℓ_{21} was arbitrary within a suitable interval, there are an infinite number of solutions to the factorization.

9.8 $\Sigma = LL' + \Psi$ for $m = 1$ implies

$$\begin{pmatrix} 1 = \ell_{11}^2 + \psi_1 & .4 = \ell_{11}\ell_{21} & .9 = \ell_{11}\ell_{31} \\ & 1 = \ell_{21}^2 + \psi_2 & .7 = \ell_{21}\ell_{31} \\ & & 1 = \ell_{31}^2 + \psi_3 \end{pmatrix}$$

Now $\frac{\ell_{11}}{\ell_{21}} = \frac{.9}{.7}$ and $\ell_{11}\ell_{21} = .4$, so $\ell_{11}^2 = (\frac{.9}{.7})(.4)$ and $\ell_{11} = \pm .717$. Thus $\ell_{21} = \pm .558$. Finally, from $.9 = \ell_{11}\ell_{31}$, we have $\ell_{31} = \pm .9/.717 = \pm 1.255$.

Note all the loadings must be of the same sign because all the covariances are positive. We have

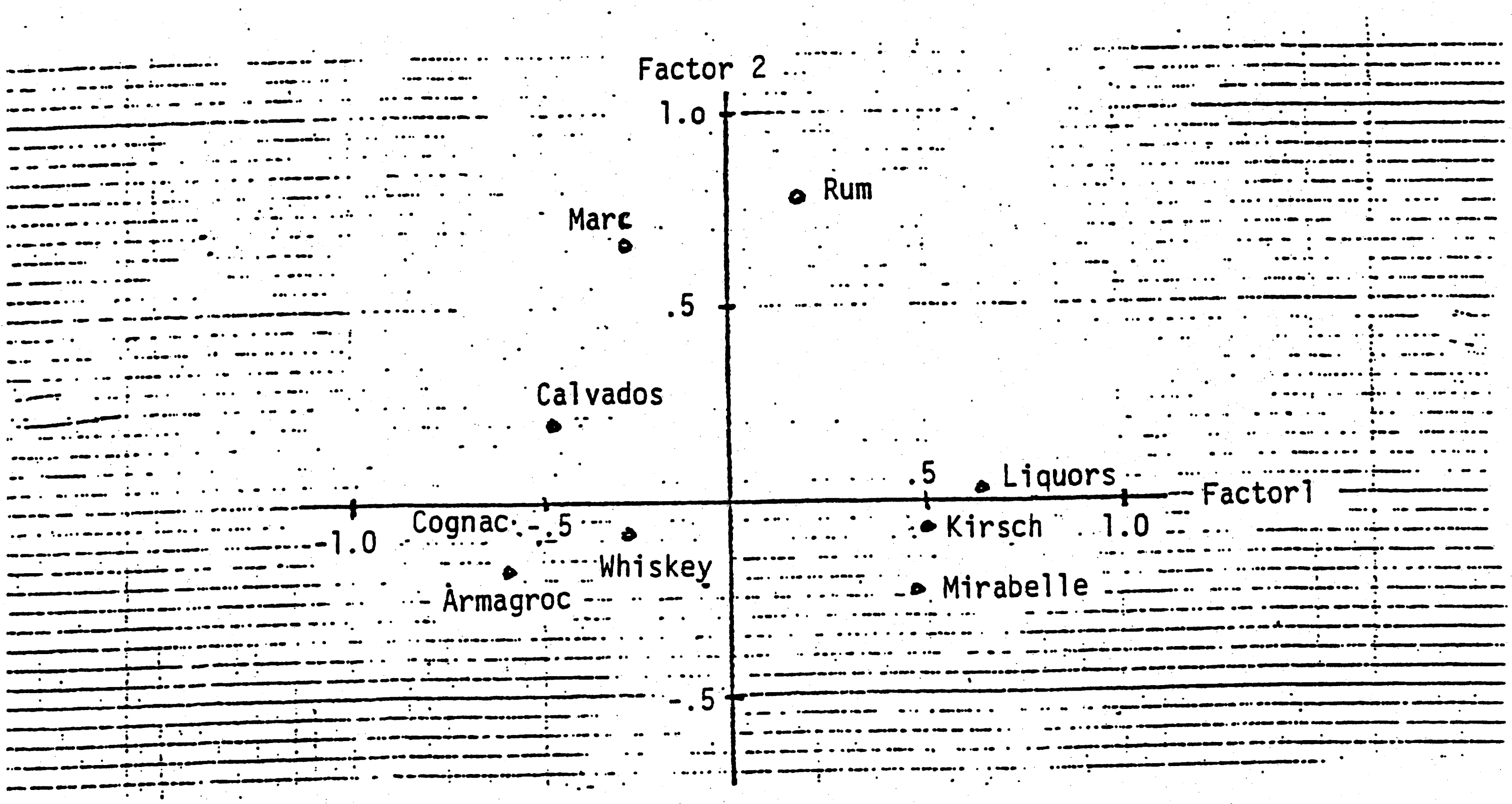
$$LL' = \begin{bmatrix} .717 \\ .558 \\ 1.255 \end{bmatrix} [.717 \ .558 \ 1.255] = \begin{bmatrix} .514 & .4 & .9 \\ .4 & .3111 & .7 \\ .9 & .7 & 1.575 \end{bmatrix}$$

so $\psi_3 = 1 - 1.575 = -.575$, which is inadmissible as a variance.

9.9

(a) Stoetzel's interpretation seems reasonable. The first factor seems to contrast sweet with strong liquors.

(b)



It doesn't appear as if rotation of the factor axes is necessary.

(a) & (b)

The specific variances and communalities based on the unrotated factors, are given in the following table:

Variable	Specific Variance	Communality
Skull length	.5976	.4024
Skull breadth	.7582	.2418
Femur length	.1221	.8779
Tibia length	.0000	1.0000
Humerus length	.0095	.9905
Ulna length	.0938	.9062

(c) The proportion of variance explained by each factor is:

$$\text{Factor 1}: \quad \frac{1}{6}\sum_{i=1}^{p} \ell_{1i}^2 = \frac{4.0001}{6} \quad \text{or} \quad 66.7\%$$

$$\text{Factor 2}: \quad \frac{1}{6}\sum_{i=1}^{p} \ell_{2i}^2 = \frac{.4177}{6} \quad \text{or} \quad 6.7\%$$

(d) $R - \hat{L}_z \hat{L}_z' - \hat{\Psi} =$

$$\begin{bmatrix} 0 & & & & & \\ .193 & 0 & & & & \\ -.017 & -.032 & 0 & & & \\ .000 & .000 & .000 & 0 & & \\ -.000 & .001 & .000 & .000 & 0 & \\ -.001 & -.018 & .003 & .000 & .000 & 0 \end{bmatrix}$$

9.11 Substituting the factor loadings given in the table (Exercise 9.10) into equation (9-45) gives.

$$V \text{ (unrotated)} = .01087$$

$$V \text{ (rotated)} = .04692$$

Although the rotated loadings are to be preferred by the varimax ("simple structure") criterion, interpretation of the factors seems clearer with the unrotated loadings.

9.12

The covariance matrix for the logarithms of turtle measurements is:

$$\mathbf{S} = 10^{-3} \times \begin{bmatrix} 11.0720040 & 8.0191419 & 8.1596480 \\ 8.0191419 & 6.4167255 & 6.0052707 \\ 8.1596480 & 6.0052707 & 6.7727585 \end{bmatrix}$$

The maximum likelihood estimates of the factor loadings for an m=1 model are

Variable	Estimated factor loadings F_1
1. ln(length)	0.1021632
2. ln(width)	0.0752017
3. ln(height)	0.0765267

Therefore,

$$\hat{L} = \begin{bmatrix} 0.1021632 \\ 0.0752017 \\ 0.0765267 \end{bmatrix}, \quad \hat{L}\hat{L}' = 10^{-3} \times \begin{bmatrix} 10.4373 & 7.6828 & 7.8182 \\ 7.6828 & 5.6553 & 5.7549 \\ 7.8182 & 5.7549 & 5.8563 \end{bmatrix}$$

(b) Since $\hat{h}_i^2 = \hat{l}_{i1}^2$ for an m=1 model, the communalities are

$$\hat{h}_1^2 = 0.0104373, \quad \hat{h}_2^2 = 0.0056553, \quad \hat{h}_3^2 = 0.0058563$$

(a) To find specific variances ψ_i's, we use the equation

$$\hat{\psi}_i = s_{ii} - \hat{h}_i^2$$

Note that in this case, we should use S_n to get s_{ii}, not S because the maximum likelihood estimation method is used.

$$S_n = \frac{n-1}{n}S = \frac{23}{24}S = 10^{-3} \times \begin{bmatrix} 10.6107 & 7.685 & 7.8197 \\ 7.685 & 6.1494 & 5.7551 \\ 7.8197 & 5.7551 & 6.4906 \end{bmatrix}$$

Thus we get

$$\hat{\psi}_1 = 0.0001734, \quad \hat{\psi}_2 = 0.0004941, \quad \hat{\psi}_3 = 0.0006342$$

(c) The proportion explained by the factor is

$$\frac{\hat{h}_1^2 + \hat{h}_2^2 + \hat{h}_3^2}{s_{11} + s_{22} + s_{33}} = \frac{0.0219489}{0.0232507} = .9440$$

(d) From (a)-(c), the residual matrix is:

$$S_n - \hat{L}\hat{L}' - \hat{\Psi} = 10^{-6} \times \begin{bmatrix} 0 & 2.1673 & 1.4474 \\ 2.1673 & 0 & 0.112497 \\ 1.4474 & 0.112497 & 0 \end{bmatrix}.$$

9.13

Equation (9-40) requires $m < \frac{1}{2}(2p+1 - \sqrt{8p+1})$. Here we have $m = 1$, $p = 3$ and the strict inequality does not hold.

9.14 Since

$$\hat{\Psi}^{\frac{1}{2}}\hat{\Psi}^{-1}\hat{\Psi}^{\frac{1}{2}} = I, \quad \hat{\Delta}^{\frac{1}{2}}\hat{\Delta}^{\frac{1}{2}} = \hat{\Delta} \quad \text{and} \quad \hat{E}'\hat{E} = I,$$

$$\hat{L}'\hat{\Psi}^{-1}\hat{L} = \hat{\Delta}^{\frac{1}{2}}\hat{E}'\hat{\Psi}^{\frac{1}{2}}\hat{\Psi}^{-1}\hat{\Psi}^{\frac{1}{2}}\hat{E}\hat{\Delta}^{\frac{1}{2}} = \hat{\Delta}^{\frac{1}{2}}\hat{E}'\hat{E}\hat{\Delta}^{\frac{1}{2}} = \hat{\Delta}^{\frac{1}{2}}\hat{\Delta}^{\frac{1}{2}} = \hat{\Delta}.$$

9.15

(a)

variable	variance	communality
HRA	0.188966	0.811034
HRE	0.133955	0.866045
HRS	0.068971	0.931029
RRA	0.100611	0.899389
RRE	0.079682	0.920318
RRS	0.096522	0.903478
Q	0.02678	0.97322
REV	0.039634	0.960366

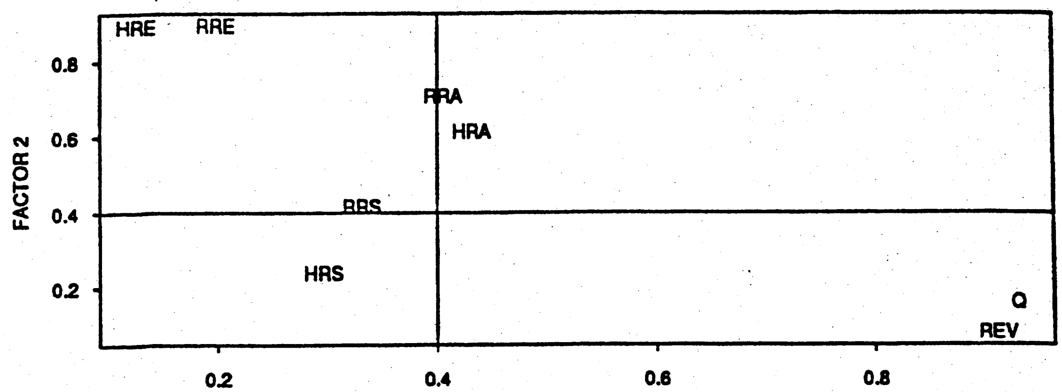
(b) **Residual Matrix**

0	0.021205	0.014563	-0.022111	-0.093691	-0.078402	-0.02145	-0.015523
0.021205	0	0.063146	-0.107308	-0.058312	-0.052289	-0.005516	0.035712
0.014563	0.063146	0	-0.065101	-0.009639	-0.070351	0.005454	0.013953
-0.022111	-0.107308	-0.065101	0	0.036263	0.058415	0.00695	-0.033857
-0.093691	-0.058312	-0.009639	0.036263	0	0.032645	0.008854	0.00065
-0.078402	-0.052289	-0.070351	0.058415	0.032645	0	0.002626	-0.004011
-0.02145	-0.005516	0.005454	0.00695	0.008854	0.002626	0	-0.02449
-0.015523	0.035712	0.013953	-0.033857	0.00065	-0.004011	-0.02449	0

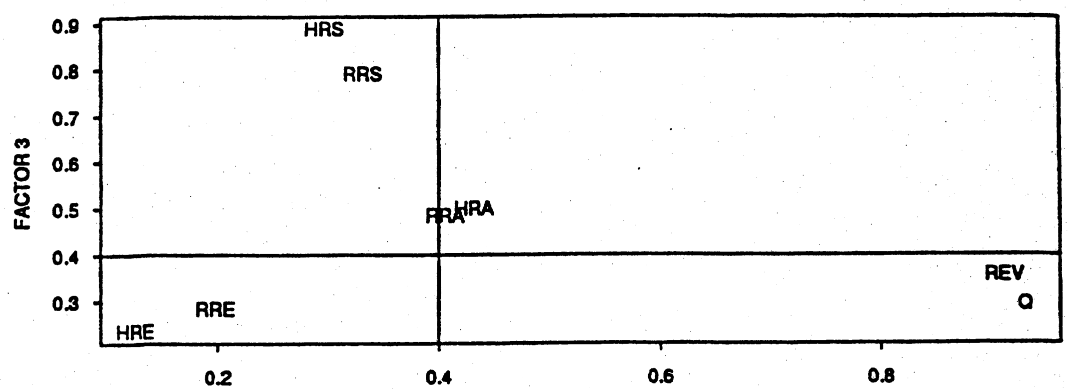
The m=3 factor model appears appropriate.

(c) The first factor is related to market-value measures (Q, REV). The second factor is related to accounting historical measures on equity (HRE, RRE). The third factor is related to accounting historical measures on sales (HRS, RRS). Accounting historical measures on assets (HRA,RRA) are weakly related to all factors. Therefore, market-value measures provide evidence of profitability distinct from that provided by the accounting measures. However, we cannot separate accounting historical measures of profitability from accounting replacement measures.

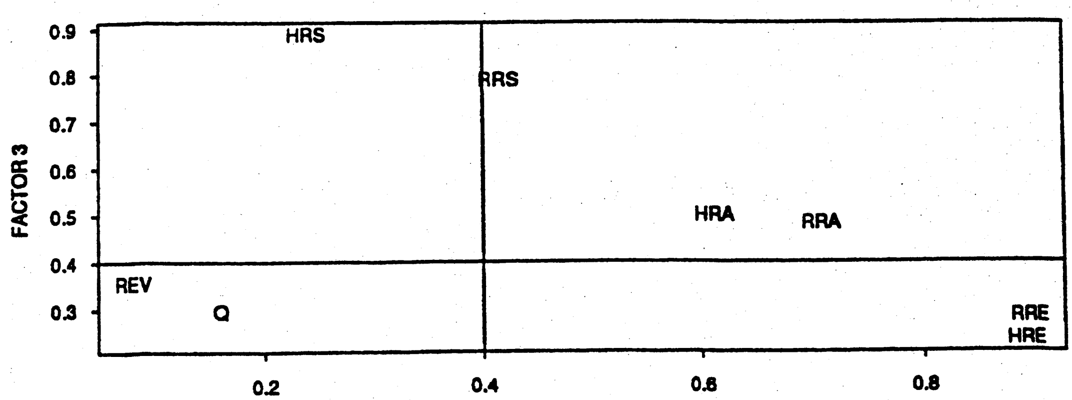
PROBLEM 9.15



FACTOR 1
Rotated Factor Pattern



FACTOR 1
Rotated Factor Pattern



FACTOR 2
Rotated Factor Pattern

9.16 From (9-50) $\hat{\underset{\sim}{f}}_j = \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-1}(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})$ and

$$\sum_{j=1}^{n} \hat{\underset{\sim}{f}}_j = \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-1} \sum_{j=1}^{n} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}}) = \underset{\sim}{0} \ .$$

Since $$\hat{\underset{\sim}{f}}_j\hat{\underset{\sim}{f}}_j' = \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-1}(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})'\hat{\Psi}^{-1}\hat{L}\hat{\Delta}^{-1} ,$$

$$\sum_{j=1}^{n} \hat{\underset{\sim}{f}}_j\hat{\underset{\sim}{f}}_j' = \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-1} \sum_{j=1}^{n} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})'\hat{\Psi}^{-1}\hat{L}\hat{\Delta}^{-1}$$

$$= n\, \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-1} S_n \hat{\Psi}^{-1}\hat{L}\hat{\Delta}^{-1} .$$

Using (9A-1),

$$\sum_{j=1}^{n} \underset{\sim}{f}_j\underset{\sim}{f}_j' = n\, \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-\frac{1}{2}}\hat{\Psi}^{-\frac{1}{2}}\hat{L}(I + \hat{\Delta})\hat{\Delta}^{-1}$$

$$= n\, \hat{\Delta}^{-1}\hat{\Delta}(I + \hat{\Delta})\hat{\Delta}^{-1} = n(I + \hat{\Delta}^{-1}),$$

a diagonal matrix. Consequently, the factor scores have sample mean vector $\underset{\sim}{0}$ and zero sample covariances.

9.17 Using the information in Example 9.12, we have

$(\hat{\mathbf{L}}_z{}'\hat{\Psi}_z^{-1}\hat{\mathbf{L}}_z)^{-1} = \begin{pmatrix} .2220 & -.0283 \\ -.0283 & .0137 \end{pmatrix}$ which, apart from rounding error, is a diagonal matrix. Since the number in the (1,1) position, .2220, is appreciably different from 0, and the observations have been standardized, equation (9-57) suggests the regression and generalized least squares methods for computing factor scores could give somewhat different results.

9.18. Factor analysis of Wisconsin fish data

(a) Principal component solution using $x_1 - x_4$

Initial Factor Method: Principal Components

	1	2	3	4
Eigenvalue	2.1539	0.7876	0.6157	0.4429
Difference	1.3663	0.1719	0.1728	
Proportion	0.5385	0.1969	0.1539	0.1107
Cumulative	0.5385	0.7354	0.8893	1.0000

Factor Pattern (m = 1)

	FACTOR1
BLUEGILL	0.77273
BCRAPPIE	0.73867
SBASS	0.64983
LBASS	0.76738

Factor Pattern (m = 2)

	FACTOR1	FACTOR2
BLUEGILL	0.77273	-0.40581
BCRAPPIE	0.73867	-0.36549
SBASS	0.64983	0.67309
LBASS	0.76738	0.19047

(b) Maximum likelihood solution using $x_1 - x_4$

Initial Factor Method: Maximum Likelihood

Factor Pattern (m = 1)

	FACTOR1
BLUEGILL	0.70812
BCRAPPIE	0.63002
SBASS	0.48544
LBASS	0.65312

Factor Pattern (m = 2)

	FACTOR1	FACTOR2
BLUEGILL	0.98748	-0.02251
BCRAPPIE	0.50404	0.25907
SBASS	0.28186	0.65863
LBASS	0.48073	0.41799

(c) Varimax rotation. Note that rotation is not possible with 1 factor.

Principal Components
Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2
BLUEGILL	0.85703	0.16518
BCRAPPIE	0.80526	0.17543
SBASS	0.08767	0.93147
LBASS	0.48072	0.62774

Maximum Likelihood
Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2
BLUEGILL	0.96841	0.19445
BCRAPPIE	0.43501	0.36324
SBASS	0.13066	0.70439
LBASS	0.37743	0.51319

For both solutions, Bluegill and Crappie load heavily on the first factor, while largemouth and smallmouth bass load heavily on the second factor.

(d) Factor analysis using $x_1 - x_6$

Initial Factor Method: Principal Components

	1	2	3	4	5	6
Eigenvalue	2.3549	1.0719	0.9843	0.6644	0.5004	0.4242
Difference	1.2830	0.0876	0.3199	0.1640	0.0762	
Proportion	0.3925	0.1786	0.1640	0.1107	0.0834	0.0707
Cumulative	0.3925	0.5711	0.7352	0.8459	0.9293	1.0000

Factor Pattern (m = 3)

	FACTOR1	FACTOR2	FACTOR3
BLUEGILL	0.72944	-0.02285	-0.47611
BCRAPPIE	0.72422	0.01989	-0.20739
SBASS	0.60333	0.58051	0.26232
LBASS	0.76170	0.07998	-0.03199
WALLEYE	-0.39334	0.83342	-0.01286
NPIKE	0.44657	-0.18156	0.80285

Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2	FACTOR3
BLUEGILL	0.85090	-0.12720	-0.13806
BCRAPPIE	0.74189	0.11256	-0.06957
SBASS	0.51192	0.46222	0.54231
LBASS	0.71176	0.28458	0.00311
WALLEYE	-0.24459	-0.21480	0.86227
NPIKE	0.05282	0.92348	-0.14613

Initial Factor Method: Maximum Likelihood

Factor Pattern

	FACTOR1	FACTOR2	FACTOR3
BLUEGILL	0.00000	1.00000	0.00000
BCRAPPIE	0.18979	0.49190	0.23481
SBASS	0.96466	0.26350	0.00000
LBASS	0.29875	0.46530	0.29435
WALLEYE	0.12927	-0.22770	-0.49746
NPIKE	0.24062	0.06520	0.46665

Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2	FACTOR3
BLUEGILL	0.99637	0.06257	0.05767
BCRAPPIE	0.46485	0.21097	0.26931
SBASS	0.20017	0.97853	0.04905
LBASS	0.42801	0.31567	0.33099
WALLEYE	-0.20771	0.13392	-0.50492
NPIKE	0.02359	0.22600	0.47779

The first principal component factor influences the Bluegill, Crappie and the Bass. The Northern Pike alone loads heavily on the second factor, and the Walleye and smallmouth bass on the third factor. The MLE solution is different.

9.19 (a), (b) and (c) Maximum Likelihood (m = 3)

UNROTATED FACTOR LOADINGS (PATTERN)
FOR MAXIMUM LIKELIHOOD CANONICAL FACTORS

		Factor 1	Factor 2	Factor 3
Growth	1	0.772	0.295	0.527
Profits	2	0.570	0.347	0.721
Newaccts	3	0.774	0.433	0.355
Creative	4	0.389	0.921	0.000
Mechanic	5	0.509	0.426	0.334
Abstract	6	0.968	-0.250	0.000
Math	7	0.632	0.181	0.729
	VP	3.267	1.520	1.566

ROTATED FACTOR LOADINGS (PATTERN)

		Factor 1	Factor 2	Factor 3
Growth	1	0.794	0.374	0.437
Profits	2	0.912	0.316	0.184
Newaccts	3	0.653	0.544	0.437
Creative	4	0.255	0.967	0.019
Mechanic	5	0.541	0.464	0.208
Abstract	6	0.390	0.054	0.953
Math	7	0.919	0.179	0.295
	VP	3.180	1.720	1.454

		Communalities	Specific Variances
1	Growth	0.9615	.0385
2	Profits	0.9648	.0352
3	Newaccts	0.9124	.0876
4	Creative	1.0000	.0000
5	Mechanic	0.5519	.4481
6	Abstract	1.0000	.0000
7	Math	0.9631	.0369

$$R = \begin{bmatrix} 1.0 & .926 & .884 & .572 & .708 & .674 & .927 \\ & 1.0 & .843 & .542 & .746 & .465 & .944 \\ & & 1.0 & .700 & .637 & .641 & .853 \\ & & & 1.0 & .591 & .147 & .413 \\ & & & & 1.0 & .386 & .575 \\ \text{(Symmetric)} & & & & & 1.0 & .566 \\ & & & & & & 1.0 \end{bmatrix}$$

$$\hat{L}\hat{L}' + \hat{\Psi} = \begin{bmatrix} 1.0 & .923 & .912 & .572 & .694 & .674 & .925 \\ & 1.0 & .848 & .542 & .679 & .465 & .948 \\ & & 1.0 & .700 & .696 & .641 & .826 \\ & & & 1.0 & .591 & .147 & .413 \\ & & & & 1.0 & .386 & .646 \\ & & & & & 1.0 & .566 \\ \text{(Symmetric)} & & & & & & 1.0 \end{bmatrix}$$

It is clear from an examination of the residual matrix $R - (\hat{L}\hat{L}' + \hat{\Psi})$ that an $m = 3$ factor solution represents the observed correlations quite well. However, it is difficult to provide interpretations for the factors. If we consider the rotated loadings, we see that the last two factors are dominated by the single variables "creative" and "abstract" respectively. The first factor links the salespeople performance variables with math ability.

(d) Using (9-39) with $n = 50$, $p = 7$, $m = 3$ we have

$$43.833 \ln\left(\frac{.000075933}{.000018427}\right) = 62.1 > \chi_3^2(.01) = 11.3$$

so we reject $H_0: \Sigma = LL' + \Psi$ for $m = 3$. Neither of the $m = 2$, $m = 3$ factor models appear to fit by the χ^2 criterion. We note that the matrices R, $\hat{L}\hat{L}' + \hat{\Psi}$ have small determinants and rounding error could affect the calculation of the test statistic. Again, the residual matrix above indicates a good fit for $m = 3$.

(e) $z' = [1.522, -.852, .465, .957, 1.129, .673, .497]$

Using the regression method for computing factor scores, we have; with $f = \hat{L}_z R_z^{-1}$:

Principal components (m = 3)	Maximum likelihood (m = 3)
$f' = [.686, .271, 1.395]$	$f' = [-.702, .679, -.751]$

Factor scores using weighted least squares can only be computed for the principal component solutions since $\hat{\Psi}^{-1}$ cannot be computed for the maximum likelihood solutions. ($\hat{\Psi}$ has zeros on the main diagonal for the maximum likelihood solutions). Using (9-50),

Principal components (m = 3)

$f' = [.344, .233, 1.805]$

9.20

$$S = \begin{bmatrix} & X_1 & X_2 & X_5 & X_6 \\ & 2.50 & -2.77 & -.59 & -2.23 \\ & & 300.52 & 6.78 & 30.78 \\ & & & 11.36 & 3.13 \\ & \text{(symmetric)} & & & 31.98 \end{bmatrix}$$

(a) Principal components ($m = 2$)

	Factor 1 loadings	Factor 2 loadings
X_1 (wind)	-.17	-.37
X_2 (solar rad.)	17.32	-.61
X_5 (NO_2)	.42	.74
X_6 (O_3)	1.96	5.19

(b) Maximum likelihood estimates of the loadings are obtained from $\hat{L} = \hat{V}^{1/2}\hat{L}_z$ where $\hat{L}_z$ are the loadings obtained from the sample correlation matrix R. (For $\hat{L}_z$ see problem 9.23). Note: Maximum likelihood estimates of the loadings for $m = 2$ may be difficult to obtain for some computer packages without good estimates of the communalities. One choice for initial estimates of the communalities are the communalities from the $m = 2$ principal components solution.

(c) Maximum likelihood estimation (with $m = 2$) does a better job of accounting for the covariances in S than the $m = 2$ principal component solution. On the other hand, the principal component solution generally produces uniformly smaller estimates of the specific variances. For the unrotated $m = 2$ solution, the first factor is dominated by X_2 = solar radiation and $X_6 = O_3$. The second factor seems to be a contrast between the pair X_1 = wind; X_2 = solar radiation and the pair $X_5 = NO_2$ and $X_6 = O_3$.

9.21 Principal components ($m = 2$)

	Rotated loadings	
	Factor 1	Factor 2
X_1 (wind)	.10	-.46
X_2 (solar rad.)	2.00	.05
X_5 (NO_2)	.05	.87
X_6 (O_3)	.71	5.49

Again the first factor is dominated by solar radiation and, to some extent, ozone. The second factor might be interpreted as a contrast between wind and the pair of pollutants NO_2 and O_3. Recall solar radiation and ozone have the largest sample variances. This will affect the estimated loadings obtained by the principal component method.

9.22 (a) Since, for maximum likelihood estimates, $\hat{L} = D^{\frac{1}{2}}\hat{L}_z$ and $S = D^{\frac{1}{2}}RD^{\frac{1}{2}}$, the factor scores generated by the equations for $\underset{\sim}{f}_j$ in (9-58) will be identical. Similarly, the factor scores generated by the weighted least squares formulas in (9-50) will be identical.

The factor scores generated by the regression method with maximum likelihood estimates ($m = 2$; see problem 9.23.) are given below for the first 10 cases.

Case	$\hat{f}_1$	$\hat{f}_2$
1	0.316	-0.544
2	0.252	-0.546
3	0.129	-0.509
4	0.332	-0.790
5	0.492	-0.012
6	0.515	-0.370
7	0.530	-0.456
8	1.070	0.724
9	0.384	-0.023
10	-0.179	0.105

(b) Factor scores using principal component estimates ($m = 2$) and (9-51) for the first 10 cases are given below:

Case	$\hat{f}_1$	$\hat{f}_2$
1	1.203	-0.368
2	1.646	-1.029
3	1.447	-0.937
4	0.717	0.795
5	0.856	-0.049
6	0.811	0.394
7	0.518	0.950
8	-0.083	1.168
9	0.410	0.259
10	-0.492	0.072

(c) The sets of factor scores are quite different. Factor scores depend heavily on the method used to estimate loadings and specific variances as well as the method used to generate them.

9.23

Principal components ($m = 2$)

	Factor 1 loadings	Factor 2 loadings	Rotated loadings Factor 1	Rotated loadings Factor 2
X_1 (wind)	-.56	-.24	-.31	[-.53]
X_2 (solar rad.)	.65	-.52	[.83]	-.04
X_5 (NO_2)	.48	.74	-.05	[.88]
X_6 (O_3)	.77	-.20	[.74]	.30

Maximum likelihood (m = 2)

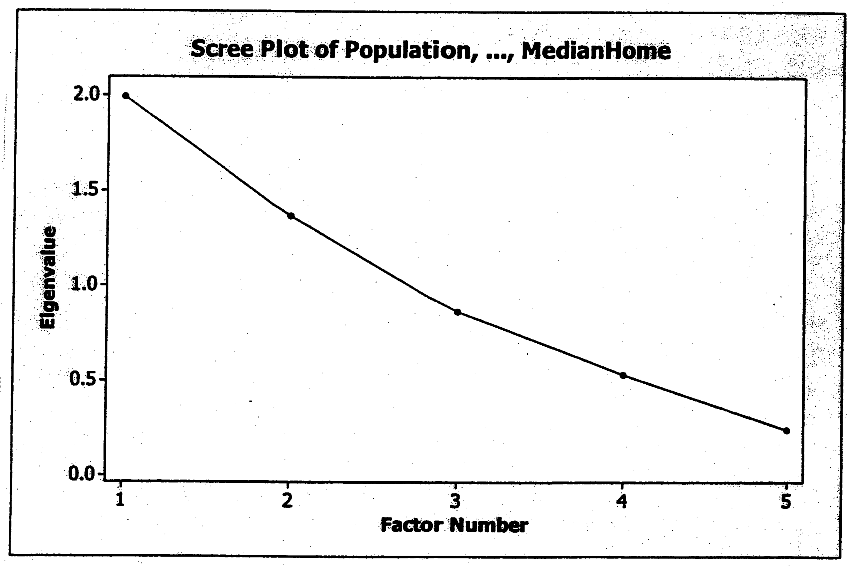
	Factor 1 loadings	Factor 2 loadings	Rotated loadings Factor 1	Factor 2
X_1 (wind)	-.38	.32	-.09	.49
X_2 (solar rad.)	.50	.27	.56	-.10
X_5 (NO_2)	.25	-.04	.17	-.19
X_6 (O_3)	.65	-.03	.49	-.43

Examining the rotated loadings, we see that both solution methods yield similar estimated loadings for the first factor. It might be called a "ozone pollution factor". There are some differences for the second factor. However, the second factor appears to compare one of the pollutants with wind. It might be called a "pollutant transport" factor. We note that the interpretations of the factors might differ depending upon the choice of R or S (see problems 9.20 and 9.21) for analysis. Also the two solution methods give somewhat different results indicating the solution is not very stable. Some of the observed correlations between the variables are very small implying that a $m = 1$ or $m = 2$ factor model for these four variables will not be a completely satisfactory description of the underlying structure. We may need about as many factors as variables. If this is the case, there is nothing to be gained by proposing a factor model.

9.24

$$\mathbf{R}=\begin{pmatrix} 1.0 & -.192 & .313 & -.119 & .026 \\ -.192 & 1.0 & -.065 & .373 & .685 \\ .313 & -.065 & 1.0 & -.411 & -.010 \\ -.119 & .373 & -.411 & 1.0 & .180 \\ .026 & .685 & -.010 & .180 & 1.0 \end{pmatrix}$$

The correlations are relatively small with the possible exception of .685, the correlation between Percent Professional Degree and Median Home Value. Consequently, a factor analysis with fewer than 4 or 5 factors may be problematic. The scree plot, shown below, reinforces this conjecture. The scree plot falls off almost linearly, there is no sharp elbow. However, we present a factor analysis with $m = 3$ factors for both the principal components and maximum likelihood solutions.



Principal Component Factor Analysis ($m = 3$)

Unrotated Factor Loadings and Communalities

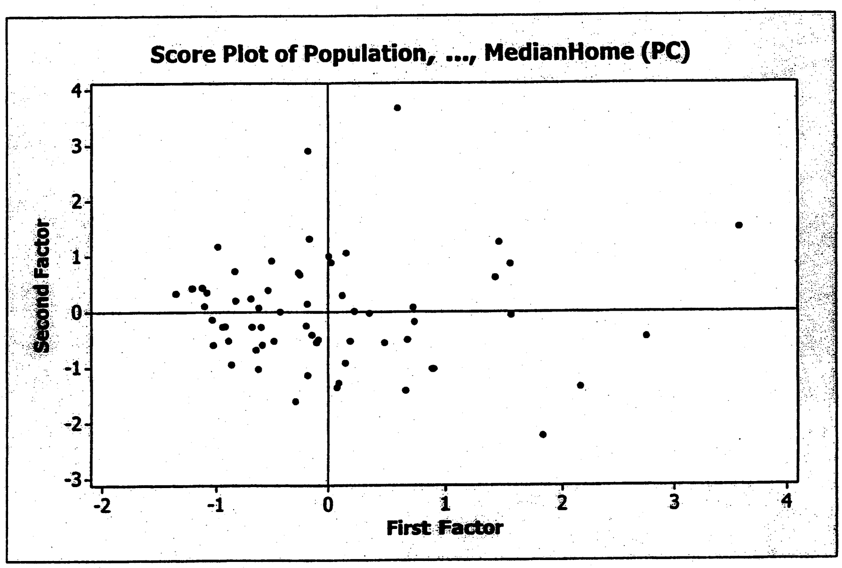
Variable	Factor1	Factor2	Factor3	Communality
Population	-0.371	-0.541	-0.729	0.962
PerCentProDeg	0.837	-0.381	0.153	0.870
PerCentEmp>16	-0.460	-0.708	0.209	0.756
PerCentGovEmp	0.676	0.295	-0.512	0.807
MedianHome	0.696	-0.584	0.064	0.830
Variance	1.9919	1.3675	0.8642	4.2236
% Var	0.398	0.274	0.173	0.845

```
Rotated Factor Loadings and Communalities
Varimax Rotation
```

Variable	Factor1	Factor2	Factor3	Communality
Population	-0.059	-0.118	-0.972	0.962
PerCentProDeg	0.907	0.160	0.147	0.870
PerCentEmp>16	0.102	-0.801	-0.321	0.756
PerCentGovEmp	0.277	0.850	-0.082	0.807
MedianHome	0.908	0.009	-0.068	0.830
Variance	1.7382	1.4050	1.0803	4.2236
% Var	0.348	0.281	0.216	0.845

```
Factor Score Coefficients
```

Variable	Factor1	Factor2	Factor3
Population	-0.019	0.138	-0.940
PerCentProDeg	0.522	-0.028	0.109
PerCentEmp>16	0.169	-0.577	-0.135
PerCentGovEmp	0.052	0.658	-0.278
MedianHome	0.544	-0.099	-0.070

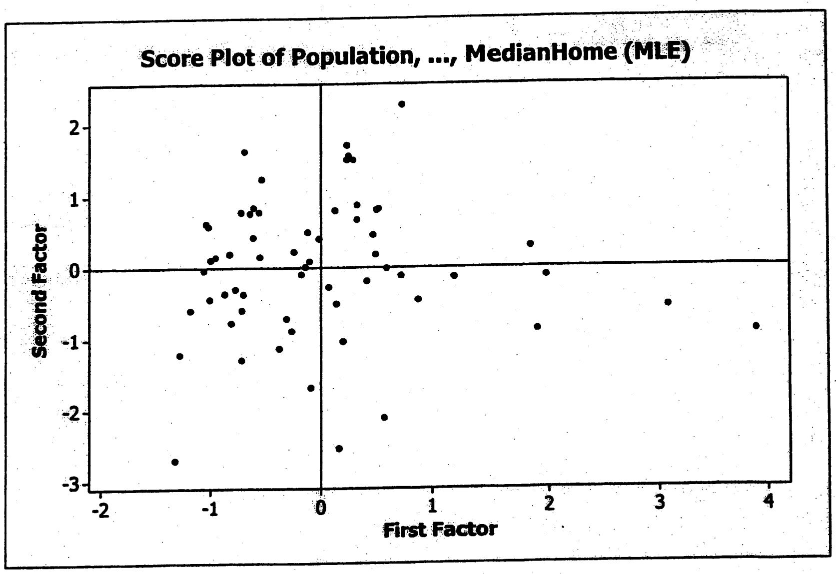


Maximum Likelihood Factor Analysis (m = 3)

```
* NOTE * Heywood case

Unrotated Factor Loadings and Communalities
```

Variable	Factor1	Factor2	Factor3	Communality
Population	-0.047	-0.999	-0.000	1.000
PerCentProDeg	0.989	0.146	-0.000	1.000
PerCentEmp>16	-0.020	-0.313	0.941	0.984
PerCentGovEmp	0.362	0.103	-0.395	0.298
MedianHome	0.701	-0.059	-0.015	0.496
Variance	1.6043	1.1310	1.0419	3.7772
% Var	0.321	0.226	0.208	0.755

```
Rotated Factor Loadings and Communalities
Varimax Rotation

Variable         Factor1  Factor2  Factor3  Communality
Population        -0.036    0.155   -0.987        1.000
PerCentProDeg      0.985   -0.090    0.145        1.000
PerCentEmp>16      0.047    0.977   -0.165        0.984
PerCentGovEmp      0.333   -0.430    0.041        0.298
MedianHome         0.699   -0.054   -0.061        0.496

Variance          1.5750   1.1740   1.0282       3.7772
% Var              0.315    0.235    0.206        0.755


Factor Score Coefficients

Variable         Factor1  Factor2  Factor3
Population         0.137   -0.177   -1.046
PerCentProDeg      1.017   -0.053   -0.046
PerCentEmp>16      0.070    1.025    0.159
PerCentGovEmp     -0.001   -0.010   -0.002
MedianHome        -0.000   -0.001   -0.000
```



A $m = 3$ factor solution explains from 75% to 85% of the variance depending on the solution method. Using the rotated loadings, the first factor in both methods has large loadings on Percent Professional Degree and Median Home Value. It is difficult to label this factor but since income is probably somewhere in this mix, it might be labeled an "affluence" or "white collar" factor. The second and third factors from the two solutions are similar as well. The second factor is a bipolar factor with large loadings (in absolute value) on Percent Employed over 16 and Percent Government Employment. We call this factor an "employment" factor. The third factor is clearly a "population" factor. Factor scores for the first two factors from the two solutions methods are similar.

9.25

$$S = \begin{bmatrix} 105{,}625 & 94{,}734 & 87{,}242 & 94{,}280 \\ & 101{,}761 & 76{,}186 & 81{,}204 \\ & & 91{,}809 & 90{,}343 \\ (\text{Symmetric}) & & & 104{,}329 \end{bmatrix}$$

A $m = 1$ factor model appears to represent these data quite well.

	Principal Components	Maximum Likelihood
	Factor 1 loadings	Factor 1 loadings
Shock wave	317.	320.
Vibration	293.	291.
Static test 1	287.	275.
Static test 2	307.	297.
Proportion Variance Explained	90.1%	86.9%

Factor scores ($m = 1$) using the regression method for the first few cases are:

Principal Components	Maximum Likelihood
-.009	-.033
1.530	1.524
.808	.719
-.804	-.802

The factor scores produced from the two solution methods are very similar. The correlation between the two sets of scores is .992.

The outliers, specimens 9 and 16, were identified in Example 4.15.

9.26

a) Principal Components

	m = 1		m = 2		
	Factor 1 loadings	$\tilde{\Psi}_i$	Factor 1 loadings	Factor 2 loadings	$\tilde{\Psi}_i$
Litter 1	27.9	309.0	27.9	-6.2	271.2
Litter 2	30.4	205.7	30.4	-4.9	182.2
Litter 3	31.5	344.3	31.5	18.5	1.7
Litter 4	32.9	310.0	32.9	-8.0	245.8
Percentage Variance Explained	76.4%		76.4%	9.4%	

b) Maximum Likelihood

m = 1

	Factor loadings	$\hat{\Psi}_i$
Litter 1	26.8	370.2
Litter 2	30.5	198.2
Litter 3	28.4	529.6
Litter 4	30.4	471.0
Percentage Variance Explained	68.8%	

The maximum likelihood estimates of the factor loadings for $m = 2$ were not obtained due to convergence difficulties in the computer program.

c) It is only necessary to rotate the $m = 2$ solution.

Principal Components (m = 2)

	Rotated loadings Factor 1	Factor 2
Litter 1	26.2	11.4
Litter 2	27.5	13.8
Litter 3	14.7	33.4
Litter 4	31.4	12.8
Percentage Variance Explained	53.5%	32.4%

9.27

Principal Components (m = 2)

	Factor 1 loadings	Factor 2 loadings	$\tilde{\Psi}_i$	Rotated loadings Factor 1	Factor 2
Litter 1	.86	.44	.06	.33	.91
Litter 2	.91	.12	.15	.59	.71
Litter 3	.85	-.36	.14	.87	.32
Litter 4	.87	-.21	.20	.78	.44
Percentage Variance Explained	76.5%	9.5%		45.4%	40.6%

Maximum Likelihood (m = 1)

	Factor 1 loadings	$\hat{\Psi}_i$
Litter 1	.81	.34
Litter 2	.91	.17
Litter 3	.78	.39
Litter 4	.81	.34
Percentage Variance Explained	68.8%	

$$\hat{f} = \hat{L}_z R^{-1} \underset{\sim}{z} = .297$$

9.28 The covariance matrix **S** (see below) is dominated by the marathon since the marathon times are given in minutes. It is unlikely that a factor analysis will be useful; however, the principal component solution with m = 2 is given below. Using the unrotated loadings, the first factor explains about 98% of the variance and the largest factor loading is associated with the marathon. Using the rotated loadings, the first factor explains about 87% of the variance and again the largest loading is associated with the marathon. The second factor, with either unrotated or rotated loadings, explains relatively little of the remaining variance and can be ignored. The first factor might be labeled a "running endurance" factor but this factor provides us with little insight into the nature of the running events. It is better to factor analyze the correlation matrix **R** in this case.

Covariances: 100m(s), 200m(s), 400m(s), 800m, 1500m, 3000m, Marathon

	100m(s)	200m(s)	400m(s)	800m	1500m	3000m
100m(s)	0.15532					
200m(s)	0.34456	0.86309				
400m(s)	0.89130	2.19284	6.74546			
800m	0.02770	0.06617	0.18181	0.00755		
1500m	0.08389	0.20276	0.50918	0.02141	0.07418	
3000m	0.23388	0.55435	1.42682	0.06138	0.21616	0.66476
Marathon	4.33418	10.38499	28.90373	1.21965	3.53984	10.70609

	Marathon
Marathon	270.27015

Principal Component Factor Analysis of **S** ($m = 2$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Factor2	Communality
100m(s)	0.267	-0.230	0.124
200m(s)	0.640	-0.582	0.749
400m(s)	1.785	-1.881	6.725
800m	0.075	-0.027	0.006
1500m	0.217	-0.073	0.052
3000m	0.654	-0.158	0.453
Marathon	16.438	0.238	270.270
Variance	274.36	4.02	278.38
% Var	0.984	0.014	0.999

Rotated Factor Loadings and Communalities
Varimax Rotation

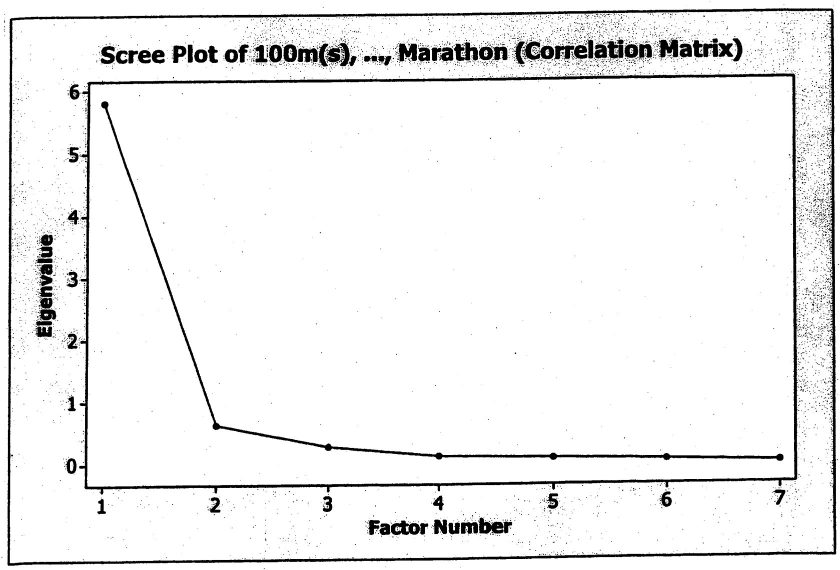
Variable	Factor1	Factor2	Communality
100m(s)	0.172	-0.308	0.124
200m(s)	0.401	-0.767	0.749
400m(s)	1.030	-2.380	6.725
800m	0.061	-0.051	0.006
1500m	0.178	-0.143	0.052
3000m	0.560	-0.373	0.453
Marathon	15.517	-5.431	270.270
Variance	242.38	36.00	278.38
% Var	0.869	0.129	0.999

The correlation matrix **R** for the women's track records follows.

Correlations: 100m(s), 200m(s), 400m(s), 800m, 1500m, 3000m, Marathon

	100m(s)	200m(s)	400m(s)	800m	1500m	3000m
200m(s)	0.941					
400m(s)	0.871	0.909				
800m	0.809	0.820	0.806			
1500m	0.782	0.801	0.720	0.905		
3000m	0.728	0.732	0.674	0.867	0.973	
Marathon	0.669	0.680	0.677	0.854	0.791	0.799

The scree plot below suggests at most a $m = 2$ factor solution.



Principal Component Factor Analysis of **R** (m =2)

Unrotated Factor Loadings and Communalities

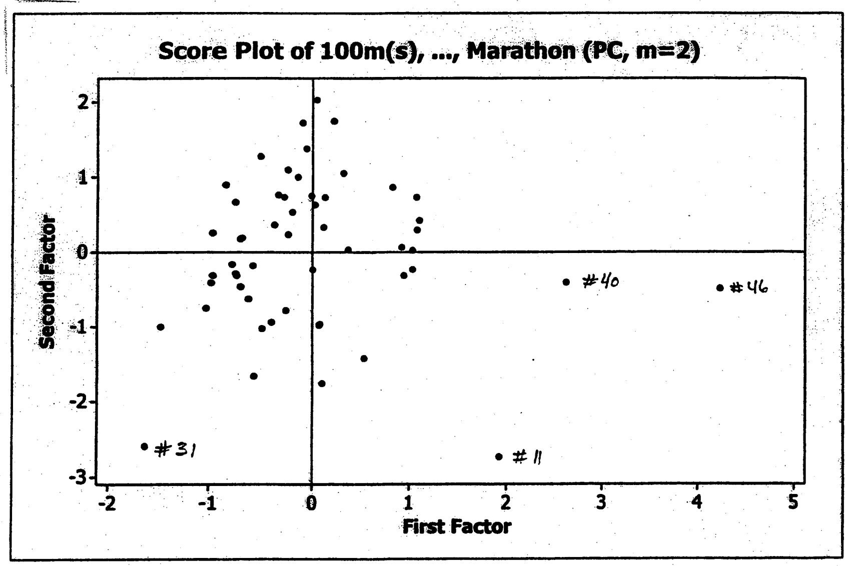
Variable	Factor1	Factor2	Communality
100m(s)	0.910	-0.323	0.933
200m(s)	0.923	-0.328	0.960
400m(s)	0.887	-0.364	0.919
800m	0.951	0.128	0.921
1500m	0.938	0.245	0.940
3000m	0.906	0.336	0.934
Marathon	0.856	0.309	0.828
Variance	5.8076	0.6287	6.4363
% Var	0.830	0.090	0.919

```
Rotated Factor Loadings and Communalities
Varimax Rotation
```

Variable	Factor1	Factor2	Communality
100m(s)	0.438	-0.861	0.933
200m(s)	0.444	-0.874	0.960
400m(s)	0.393	-0.875	0.919
800m	0.778	-0.562	0.921
1500m	0.849	-0.468	0.940
3000m	0.888	-0.381	0.934
Marathon	0.833	-0.365	0.828
Variance	3.3530	3.0833	6.4363
% Var	0.479	0.440	0.919

```
Factor Score Coefficients
```

Variable	Factor1	Factor2
100m(s)	-0.240	-0.480
200m(s)	-0.244	-0.488
400m(s)	-0.288	-0.525
800m	0.259	0.035
1500m	0.386	0.172
3000m	0.481	0.280
Marathon	0.445	0.255



Maximum Likelihood Factor Analysis of $\mathbf{R}$ ($m = 2$)

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Unrotated Factor Loadings and Communalities
```

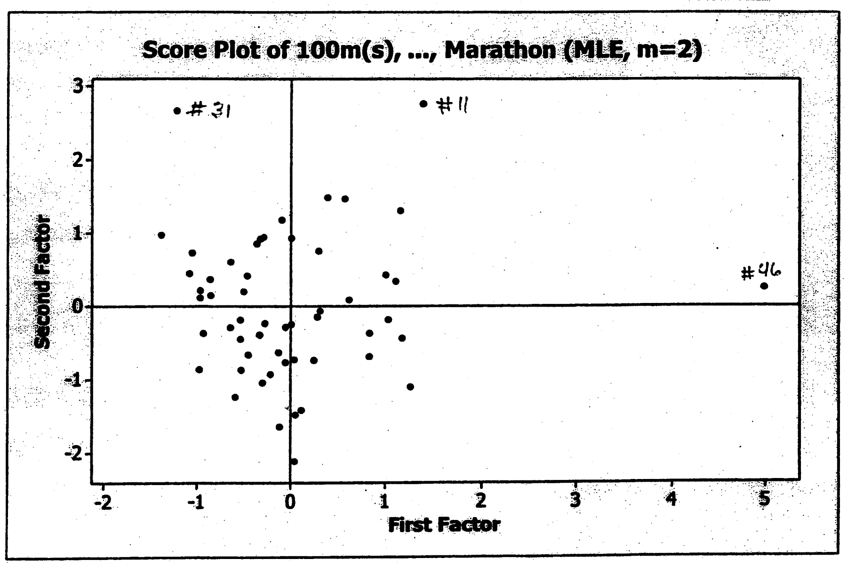
Variable	Factor1	Factor2	Communality
100m(s)	0.876	0.371	0.906
200m(s)	0.899	0.410	0.976
400m(s)	0.827	0.405	0.848
800m	0.925	-0.006	0.856
1500m	0.974	-0.187	0.984
3000m	0.945	-0.282	0.972
Marathon	0.809	-0.091	0.662
Variance	5.6104	0.5927	6.2032
% Var	0.801	0.085	0.886

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m(s)	0.455	0.836	0.906
200m(s)	0.449	0.880	0.976
400m(s)	0.395	0.832	0.848
800m	0.728	0.571	0.856
1500m	0.879	0.460	0.984
3000m	0.915	0.367	0.972
Marathon	0.690	0.432	0.662
Variance	3.1806	3.0225	6.2032
% Var	0.454	0.432	0.886

Factor Score Coefficients

Variable	Factor1	Factor2
100m(s)	-0.107	0.237
200m(s)	-0.481	1.019
400m(s)	-0.077	0.157
800m	0.036	0.025
1500m	0.772	-0.317
3000m	0.595	-0.369
Marathon	0.024	-0.003



The results from the two solution methods are very similar. Using the unrotated loadings, the first factor might be identified as a "running excellence" factor. All the running events load highly on this factor. The second factor appears to contrast the shorter running events (100m, 200m, 400m) with the longer events (800m, 1500m, 3000m, marathon). This bipolar factor might be called a "running speed-running endurance" factor. After rotation the overall excellence factor disappears and the first factor appears to represent "running endurance" since the running events 800m through the marathon load highly on this factor. The second factor might be classified as a "running speed" factor. Note, for both factors, the remaining running events in each case have moderately large loadings on the factor. The two factor solution accounts for 89%-92% (depending on solution method) of the total variance. The plots of the factor scores indicate that observations #46 (Samoa), #11 (Cook Islands) and #31 (North Korea) are outliers.

9.29 The covariance matrix **S** for the running events measured in meters/second is given below. Since all the running event variables are now on a commensurate measurement scale, it is likely a factor analysis of **S** will produce nearly the same results as a factor analysis of the correlation matrix **R**. The results for a $m = 2$ factor analysis of **S** using the principal component method are shown below. A factor analysis of **R** follows.

Covariances: 100m/s, 200m/s, 400m/s, 800m/s, 1500m/s, 3000m/s, Marm/s

	100m/s	200m/s	400m/s	800m/s	1500m/s	3000m/s
100m/s	0.0905383					
200m/s	0.0956063	0.1146714				
400m/s	0.0966724	0.1138699	0.1377889			
800m/s	0.0650640	0.0749249	0.0809409	0.0735228		
1500m/s	0.0822198	0.0960189	0.0954430	0.0864542	0.1238405	
3000m/s	0.0921422	0.1054364	0.1083164	0.0997547	0.1437148	0.1765843
Marm/s	0.0810999	0.0933103	0.1018807	0.0943056	0.1184578	0.1465604

	Marm/s
Marm/s	0.1667141

Principal Component Factor Analysis of **S** ($m = 2$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Factor2	Communality
100m/s	0.265	-0.110	0.083
200m/s	0.306	-0.127	0.110
400m/s	0.324	-0.152	0.128
800m/s	0.256	0.016	0.066
1500m/s	0.335	0.062	0.116
3000m/s	0.393	0.116	0.168
Marm/s	0.362	0.130	0.148
Variance	0.73215	0.08607	0.81822
% Var	0.829	0.097	0.926

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m/s	0.128	-0.257	0.083
200m/s	0.147	-0.297	0.110
400m/s	0.145	-0.327	0.128
800m/s	0.204	-0.156	0.066
1500m/s	0.293	-0.173	0.116
3000m/s	0.373	-0.170	0.168
Marm/s	0.359	-0.139	0.148
Variance	0.45423	0.36399	0.81822
% Var	0.514	0.412	0.926

Factor Score Coefficients

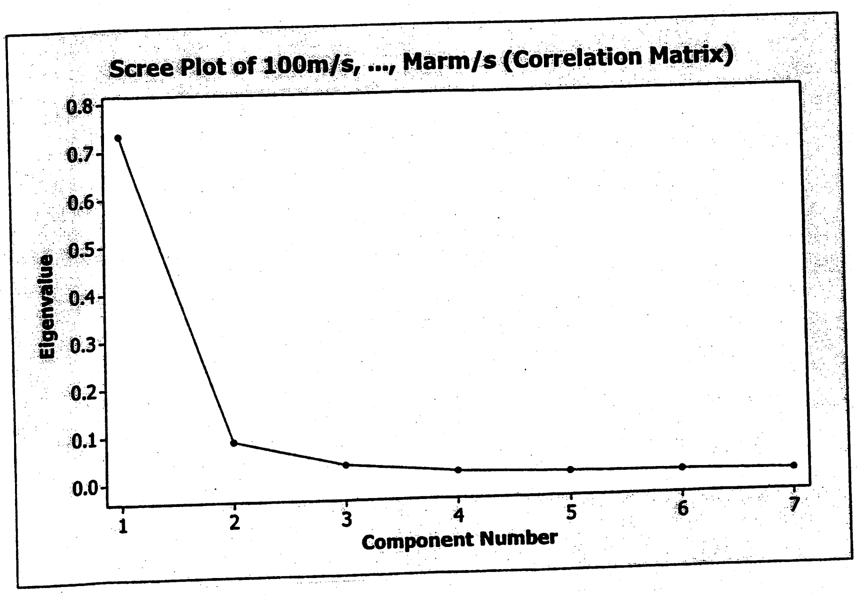
Variable	Factor1	Factor2
100m/s	-0.171	-0.363
200m/s	-0.222	-0.471
400m/s	-0.306	-0.603
800m/s	0.104	-0.025
1500m/s	0.287	0.085
3000m/s	0.542	0.280
Marm/s	0.558	0.335

Using the unrotated loadings, the first factor might be identified as a "running excellence" factor. All the running events have roughly the same size loadings on this factor. The second factor appears to contrast the shorter running events (100m, 200m, 400m) with the longer events (800m, 1500m, 3000m, marathon). This bipolar factor might be called a "running speed-running endurance" factor. After rotation the overall excellence factor disappears and the first factor appears to represent "running endurance" since the running events 800m through the marathon have higher loadings on this factor. The second factor might be classified as a "running speed" factor. Note, for both factors, the remaining running events in each case have moderate and roughly equal loadings on the factor. The two factor solution accounts for 93% of the variance.

The correlation matrix **R** is shown below along with the scree plot. A two factor solution seems warranted.

Correlations: 100m/s, 200m/s, 400m/s, 800m/s, 1500m/s, 3000m/s, Marm/s

	100m/s	200m/s	400m/s	800m/s	1500m/s	3000m/s
200m/s	0.938					
400m/s	0.866	0.906				
800m/s	0.797	0.816	0.804			
1500m/s	0.776	0.806	0.731	0.906		
3000m/s	0.729	0.741	0.694	0.875	0.972	
Marm/s	0.660	0.675	0.672	0.852	0.824	0.854



Principal Component Factor Analysis of R ($m = 2$)

Unrotated Factor Loadings and Communalities

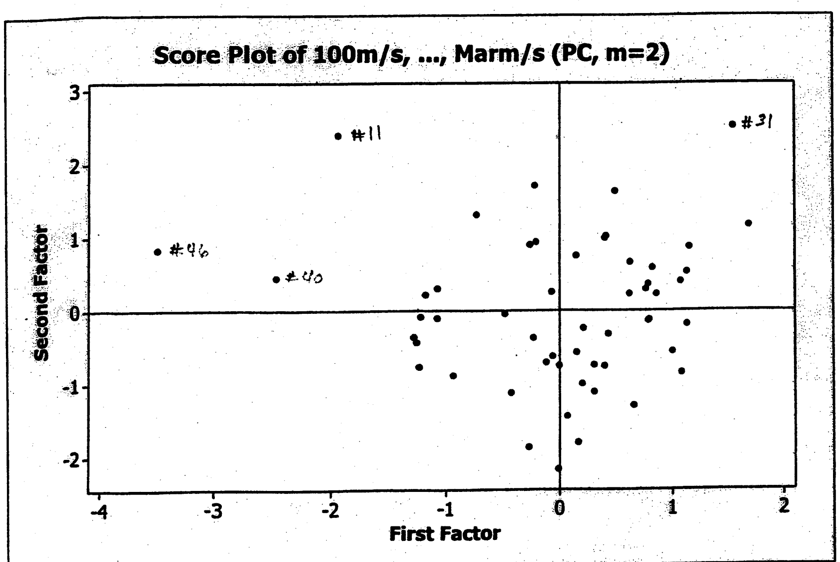
Variable	Factor1	Factor2	Communality
100m/s	0.903	-0.342	0.932
200m/s	0.921	-0.335	0.960
400m/s	0.887	-0.352	0.911
800m/s	0.948	0.123	0.914
1500m/s	0.943	0.227	0.941
3000m/s	0.919	0.320	0.947
Marm/s	0.866	0.354	0.875
Variance	5.8323	0.6477	6.4799
% Var	0.833	0.093	0.926

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m/s	0.418	-0.870	0.932
200m/s	0.436	-0.878	0.960
400m/s	0.400	-0.867	0.911
800m/s	0.771	-0.565	0.914
1500m/s	0.839	-0.486	0.941
3000m/s	0.886	-0.402	0.947
Marm/s	0.871	-0.341	0.875
Variance	3.3675	3.1125	6.4799
% Var	0.481	0.445	0.926

Factor Score Coefficients

Variable	Factor1	Factor2
100m/s	-0.252	-0.489
200m/s	-0.243	-0.484
400m/s	-0.265	-0.499
800m/s	0.248	0.025
1500m/s	0.358	0.142
3000m/s	0.455	0.249
Marm/s	0.484	0.293



Maximum Likelihood Factor Analysis of R ($m = 2$)

Unrotated Factor Loadings and Communalities

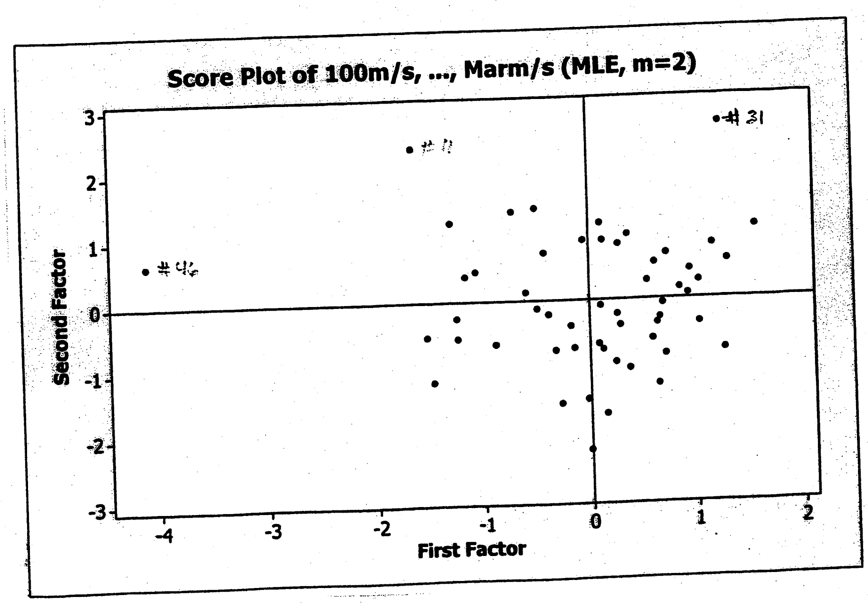
Variable	Factor1	Factor2	Communality
100m/s	0.880	-0.349	0.896
200m/s	0.910	-0.393	0.983
400m/s	0.844	-0.352	0.836
800m/s	0.921	0.042	0.850
1500m/s	0.966	0.193	0.971
3000m/s	0.945	0.302	0.984
Marm/s	0.834	0.203	0.737
Variance	5.6844	0.5716	6.2560
% Var	0.812	0.082	0.894

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m/s	0.441	-0.838	0.896
200m/s	0.435	-0.891	0.983
400m/s	0.412	-0.816	0.836
800m/s	0.726	-0.568	0.850
1500m/s	0.859	-0.482	0.971
3000m/s	0.914	-0.386	0.984
Marm/s	0.765	-0.389	0.737
Variance	3.2395	3.0165	6.2560
% Var	0.463	0.431	0.894

Factor Score Coefficients

Variable	Factor1	Factor2
100m/s	-0.073	-0.167
200m/s	-0.521	-1.122
400m/s	-0.048	-0.106
800m/s	0.039	-0.014
1500m/s	0.379	0.124
3000m/s	0.949	0.518
Marm/s	0.041	0.017



The results from the two solution methods are very similar and very similar to the principal component factor analysis of the covariance matrix S. Using the unrotated loadings, the first factor might be identified as a "running excellence" factor. All the running events load highly on this factor. The second factor appears to contrast the shorter running events (100m, 200m, 400m) with the longer events (800m, 1500m, 3000m, marathon). This bipolar factor might be called a "running speed-running endurance" factor. After rotation the overall excellence factor disappears and the first factor appears to represent "running endurance" since the running events 800m through the marathon load highly on this factor. The second factor might be classified as a "running speed" factor. Note, for both factors, the remaining running events in each case have moderately large loadings on the factor. The two factor solution accounts for 89%-93% (depending on solution method) of the total variance. The plots of the factor scores indicate that observations #46 (Samoa), #11 (Cook Islands) and #31 (North Korea) are outliers.

The results of the $m = 2$ factor analysis of women's track records when time is measured in meters per second are very much the same as the results for the $m = 2$ factor analysis of **R** presented in Exercise 9.28. If the correlation matrix **R** is factor analyzed, it makes little difference whether running event time is measured in seconds (or minutes) as in Exercise 9.28 or in meters per second. It does make a difference if the covariance matrix **S** is factor analyzed, since the measurement scales in Exercise 9.28 are quite different from the meters/second scale.

9.30 The covariance matrix **S** (see below) is dominated by the marathon since the marathon times are given in minutes. It is unlikely that a factor analysis will be useful; however, the principal component solution with m = 2 is given below. Using the unrotated loadings, the first factor explains about 98% of the variance and the largest factor loading is associated with the marathon. Using the rotated loadings, the first factor explains about 83% of the variance and again the largest loading is associated with the marathon. The second factor, with either unrotated or rotated loadings, explains relatively little of the remaining variance and can be ignored. The first factor might be labeled a "running endurance" factor but this factor provides us with little insight into the nature of the running events. It is better to factor analyze the correlation matrix **R** in this case.

Covariances: 100m, 200m, 400m, 800m, 1500m, 5000m, 10,000m, Marathon

	100m	200m	400m	800m	1500m	5000m
100m	0.048973					
200m	0.111044	0.300903				
400m	0.256022	0.666818	2.069956			
800m	0.008264	0.022929	0.057938	0.002751		
1500m	0.025720	0.066193	0.168473	0.007131	0.023034	
5000m	0.124575	0.317734	0.853486	0.034348	0.105833	0.578875
10,000m	0.265613	0.688936	1.849941	0.074257	0.229701	1.262533
Marathon	1.340139	3.541038	9.178857	0.378905	1.192564	6.430489

	10,000m	Marathon
10,000m	2.819569	
Marathon	14.342538	80.135356

Principal Component Factor Analysis of **S** ($m = 2$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Factor2	Communality
100m	0.152	-0.107	0.034
200m	0.401	-0.270	0.234
400m	1.044	-0.979	2.049
800m	0.043	-0.015	0.002
1500m	0.134	-0.033	0.019
5000m	0.722	-0.125	0.537
10,000m	1.610	-0.223	2.643
Marathon	8.950	0.179	80.130
Variance	84.507	1.141	85.649
% Var	0.983	0.013	0.996

Rotated Factor Loadings and Communalities
Varimax Rotation

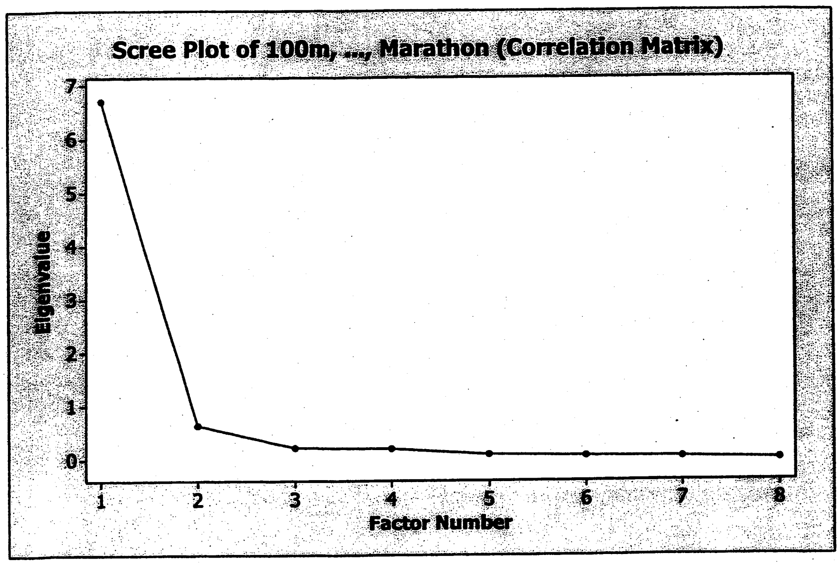
Variable	Factor1	Factor2	Communality
100m	0.097	-0.158	0.034
200m	0.262	-0.406	0.234
400m	0.573	-1.312	2.049
800m	0.033	-0.031	0.002
1500m	0.110	-0.083	0.019
5000m	0.615	-0.399	0.537
10,000m	1.392	-0.841	2.643
Marathon	8.294	-3.367	80.130
Variance	71.529	14.119	85.649
% Var	0.832	0.164	0.996

The correlation matrix **R** for the men's track records follows.

Correlations: 100m, 200m, 400m, 800m, 1500m, 5000m, 10,000m, Marathon

	100m	200m	400m	800m	1500m	5000m	10,000m
200m	0.915						
400m	0.804	0.845					
800m	0.712	0.797	0.768				
1500m	0.766	0.795	0.772	0.896			
5000m	0.740	0.761	0.780	0.861	0.917		
10,000m	0.715	0.748	0.766	0.843	0.901	0.988	
Marathon	0.676	0.721	0.713	0.807	0.878	0.944	0.954

The scree plot below suggests at most a $m = 2$ factor solution.



Principal Component Factor Analysis of **R** (m =2)

Unrotated Factor Loadings and Communalities

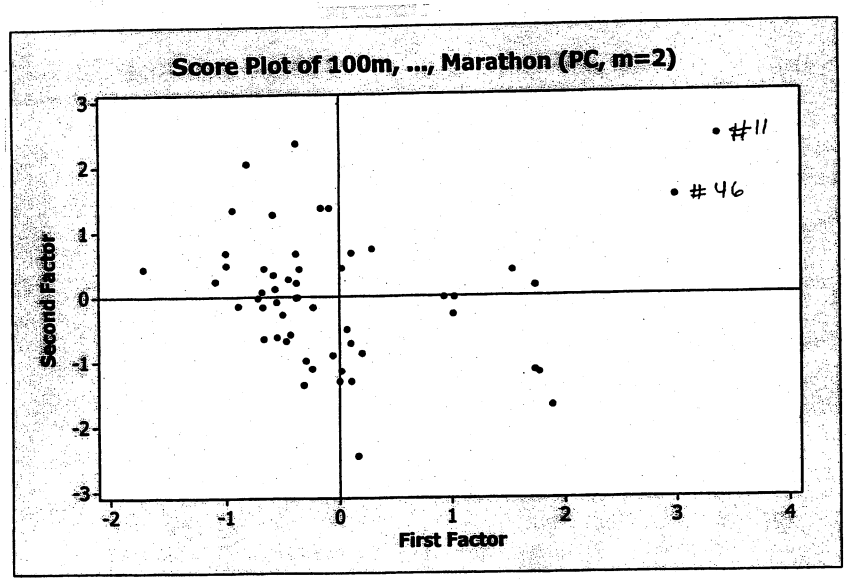
Variable	Factor1	Factor2	Communality
100m	0.861	0.423	0.920
200m	0.896	0.376	0.944
400m	0.878	0.276	0.847
800m	0.914	-0.071	0.840
1500m	0.948	-0.123	0.913
5000m	0.957	-0.236	0.972
10,000m	0.947	-0.267	0.969
Marathon	0.917	-0.309	0.937
Variance	6.7033	0.6384	7.3417
% Var	0.838	0.080	0.918

```
Rotated Factor Loadings and Communalities
Varimax Rotation
```

Variable	Factor1	Factor2	Communality
100m	0.375	0.882	0.920
200m	0.433	0.870	0.944
400m	0.485	0.782	0.847
800m	0.739	0.543	0.840
1500m	0.798	0.526	0.913
5000m	0.879	0.447	0.972
10,000m	0.892	0.417	0.969
Marathon	0.896	0.365	0.937
Variance	4.1168	3.2249	7.3417
% Var	0.515	0.403	0.918

```
Factor Score Coefficients
```

Variable	Factor1	Factor2
100m	-0.335	0.586
200m	-0.283	0.533
400m	-0.183	0.413
800m	0.176	0.004
1500m	0.233	-0.053
5000m	0.349	-0.186
10,000m	0.380	-0.224
Marathon	0.420	-0.277



Maximum Likelihood Factor Analysis of **R** ($m = 2$)

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Unrotated Factor Loadings and Communalities
```
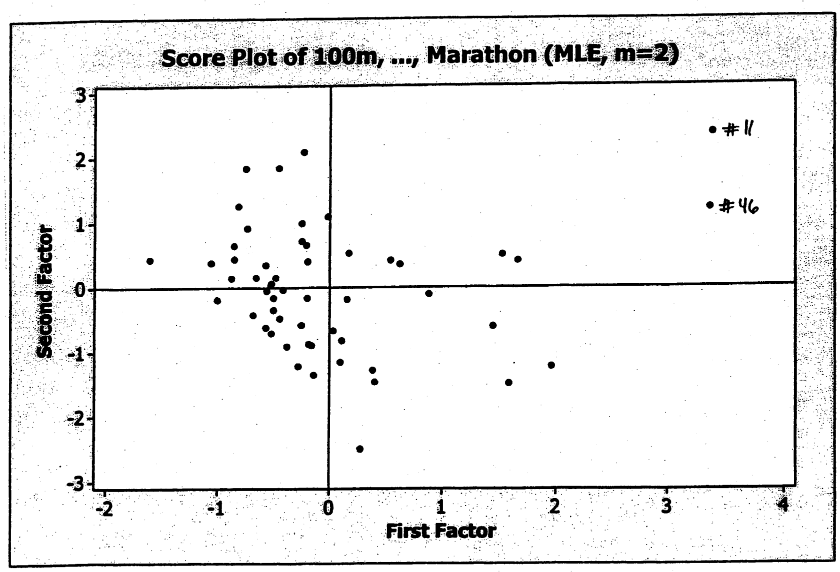

Variable	Factor1	Factor2	Communality
100m	0.780	0.507	0.866
200m	0.814	0.548	0.963
400m	0.810	0.339	0.772
800m	0.875	0.147	0.788
1500m	0.927	0.083	0.866
5000m	0.991	-0.077	0.988
10,000m	0.989	-0.106	0.989
Marathon	0.949	-0.104	0.912
Variance	6.4134	0.7299	7.1432
% Var	0.802	0.091	0.893

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m	0.401	0.839	0.866
200m	0.409	0.892	0.963
400m	0.515	0.712	0.772
800m	0.671	0.581	0.788
1500m	0.748	0.554	0.866
5000m	0.886	0.450	0.988
10,000m	0.900	0.424	0.989
Marathon	0.865	0.405	0.912
Variance	3.9446	3.1986	7.1432
% Var	0.493	0.400	0.893

Factor Score Coefficients

Variable	Factor1	Factor2
100m	-0.125	0.256
200m	-0.490	0.994
400m	-0.044	0.104
800m	-0.011	0.054
1500m	0.003	0.056
5000m	0.558	-0.209
10,000m	0.761	-0.423
Marathon	0.089	-0.051



The results from the two solution methods are very similar. Using the unrotated loadings, the first factor might be identified as a "running excellence" factor. All the running events load highly on this factor. The second factor appears to contrast the shorter running events with the longer events although the nature of the contrast is a bit different for the two methods. For the principal component method, the 100m, 200m and 400m events have positive loadings and the 800m, 1500m, 5000m, 10,000m and marathon events have negative loadings. For the maximum likelihood method, the 100m, 200m, 400m, 800m and 1500m events are in one group (positive loadings) and the 5000, 10,000m and marathon are in the other group (negative loadings). Nevertheless, this bipolar factor might be called a "running speed-running endurance" factor. After rotation the overall excellence factor disappears and the first factor appears to represent "running endurance" since the running events 800m through the marathon load highly on this factor. The second factor might be classified as a "running speed" factor. Note, for both factors, the remaining running events in each case have moderately large loadings on the factor. The two factor solution accounts for 89%-92% (depending on solution method) of the total variance. The plots of the factor scores indicate that observations #46 (Samoa) and #11 (Cook Islands) are outliers. The factor analysis of the men's track records is very much the same as that for the women's track records in Exercise 9.28.

9.31 The covariance matrix **S** for the running events measured in meters/second is given below. Since all the running event variables are now on a commensurate measurement scale, it is likely a factor analysis of **S** will produce nearly the same results as a factor analysis of the correlation matrix **R**. The results for a $m = 2$ factor analysis of **S** using the principal component method are shown below. A factor analysis of **R** follows.

Covariances: 100m/s, 200m/s, 400m/s, 800m/s, 1500m/s, 5000m/s, 10,000m/s, ...

	100m/s	200m/s	400m/s	800m/s	1500m/s
100m/s	0.0434979				
200m/s	0.0482772	0.0648452			
400m/s	0.0434632	0.0558678	0.0688217		
800m/s	0.0314951	0.0432334	0.0428221	0.0468840	
1500m/s	0.0425034	0.0535265	0.0537207	0.0523058	0.0729140
5000m/s	0.0469252	0.0587731	0.0617664	0.0571560	0.0766388
10,000m/s	0.0448325	0.0572512	0.0599354	0.0553945	0.0745719
Marathonm/s	0.0431256	0.0562945	0.0567342	0.0541911	0.0736518

	5000m/s	10,000m/s	Marathonm/s
5000m/s	0.0959398		
10,000m/s	0.0937357	0.0942894	
Marathonm/s	0.0905819	0.0909952	0.0979276

Principal Component Factor Analysis of **S** ($m = 2$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Factor2	Communality
100m/s	0.171	-0.093	0.038
200m/s	0.219	-0.113	0.061
400m/s	0.223	-0.101	0.060
800m/s	0.195	-0.007	0.038
1500m/s	0.256	0.014	0.066
5000m/s	0.301	0.056	0.094
10,000m/s	0.296	0.067	0.092
Marathonm/s	0.293	0.083	0.093
Variance	0.49405	0.04622	0.54027
% Var	0.844	0.079	0.923

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m/s	0.080	-0.178	0.038
200m/s	0.105	-0.222	0.061
400m/s	0.116	-0.215	0.060
800m/s	0.151	-0.124	0.038
1500m/s	0.212	-0.145	0.066
5000m/s	0.273	-0.138	0.094
10,000m/s	0.275	-0.127	0.092
Marathonm/s	0.283	-0.112	0.093
Variance	0.32860	0.21168	0.54027
% Var	0.562	0.362	0.923

Factor Score Coefficients

Variable	Factor1	Factor2
100m/s	-0.197	-0.377
200m/s	-0.287	-0.561
400m/s	-0.254	-0.526
800m/s	0.048	-0.078
1500m/s	0.159	-0.022
5000m/s	0.379	0.184
10,000m/s	0.415	0.240
Marathonm/s	0.489	0.334

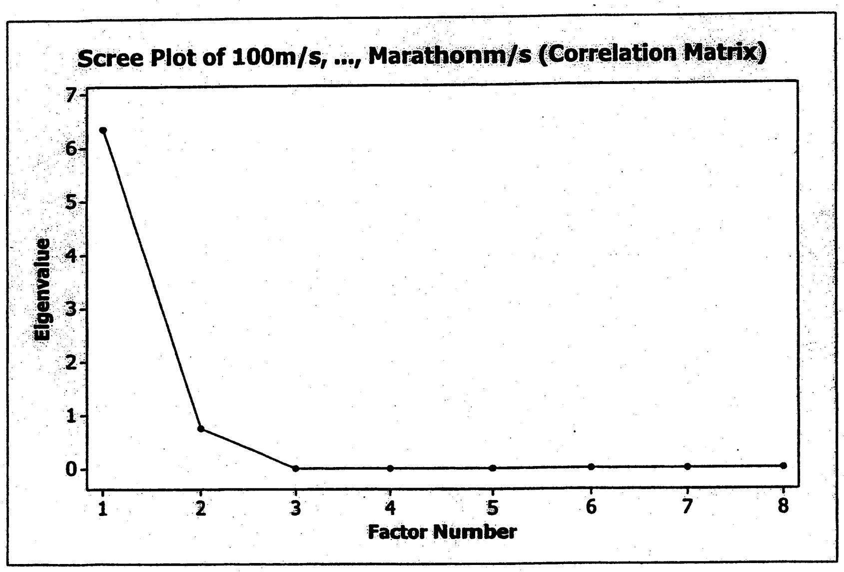
Using the unrotated loadings, the first factor might be identified as a "running excellence" factor. All the running events have roughly the same size loadings on this factor. The second factor appears to contrast the shorter running events (100m, 200m, 400m, 800m) with the longer events (1500m, 5000m, 10,000, marathon). This bipolar factor might be called a "running speed-running endurance" factor. After rotation the overall excellence factor disappears and the first factor appears to represent "running endurance" since the running events 1500m through the marathon have higher loadings on this factor. The second factor might be classified as a "running speed" factor. Note, the 800m run has about equal (in absolute value) loadings on both factors and the remaining running events in each case have moderate and roughly equal loadings on the factor. The two factor solution accounts for 92% of the variance.

The correlation matrix **R** is shown next along with the scree plot. A two factor solution seems warranted.

Correlations: 100m/s, 200m/s, 400m/s, 800m/s, 1500m/s, 5000m/s, 10,000m/s, ...

	100m/s	200m/s	400m/s	800m/s	1500m/s
200m/s	0.909				
400m/s	0.794	0.836			
800m/s	0.697	0.784	0.754		
1500m/s	0.755	0.778	0.758	0.895	
5000m/s	0.726	0.745	0.760	0.852	0.916
10,000m/s	0.700	0.732	0.744	0.833	0.899
Marathonm/s	0.661	0.706	0.691	0.800	0.872

	5000m/s	10,000m/s
10,000m/s	0.986	
Marathonm/s	0.935	0.947



Principal Component Factor Analysis of **R** ($m = 2$)

Unrotated Factor Loadings and Communalities

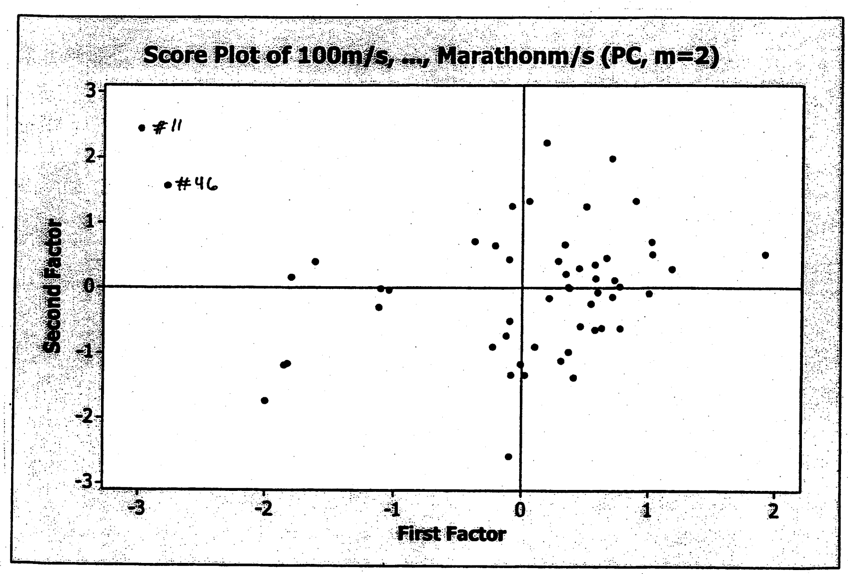
Variable	Factor1	Factor2	Communality
100m/s	0.854	-0.430	0.913
200m/s	0.888	-0.387	0.939
400m/s	0.868	-0.297	0.841
800m/s	0.910	0.076	0.834
1500m/s	0.947	0.133	0.914
5000m/s	0.954	0.242	0.968
10,000m/s	0.943	0.274	0.965
Marathonm/s	0.912	0.312	0.929
Variance	6.6258	0.6765	7.3023
% Var	0.828	0.085	0.913

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m/s	0.369	-0.881	0.913
200m/s	0.423	-0.872	0.939
400m/s	0.466	-0.790	0.841
800m/s	0.741	-0.534	0.834
1500m/s	0.805	-0.515	0.914
5000m/s	0.882	-0.437	0.968
10,000m/s	0.895	-0.405	0.965
Marathonm/s	0.896	-0.355	0.929
Variance	4.1116	3.1907	7.3023
% Var	0.514	0.399	0.913

Factor Score Coefficients

Variable	Factor1	Factor2
100m/s	-0.315	-0.566
200m/s	-0.270	-0.522
400m/s	-0.186	-0.418
800m/s	0.178	-0.004
1500m/s	0.236	0.056
5000m/s	0.341	0.178
10,000m/s	0.371	0.215
Marathonm/s	0.405	0.261



Maximum Likelihood Factor Analysis of **R** ($m = 2$)

Unrotated Factor Loadings and Communalities

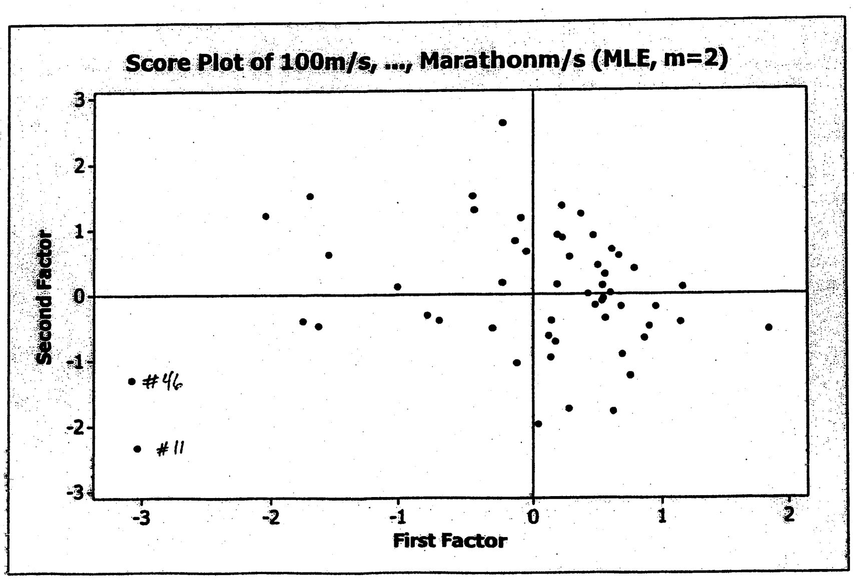
Variable	Factor1	Factor2	Communality
100m/s	0.773	0.511	0.859
200m/s	0.806	0.554	0.957
400m/s	0.797	0.351	0.758
800m/s	0.870	0.140	0.777
1500m/s	0.928	0.067	0.865
5000m/s	0.989	-0.088	0.985
10,000m/s	0.986	-0.117	0.986
Marathonm/s	0.942	-0.108	0.899
Variance	6.3380	0.7485	7.0865
% Var	0.792	0.094	0.886

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
100m/s	0.394	0.839	0.859
200m/s	0.400	0.893	0.957
400m/s	0.497	0.715	0.758
800m/s	0.670	0.573	0.777
1500m/s	0.757	0.540	0.865
5000m/s	0.890	0.440	0.985
10,000m/s	0.903	0.413	0.986
Marathonm/s	0.860	0.398	0.899
Variance	3.9325	3.1540	7.0865
% Var	0.492	0.394	0.886

Factor Score Coefficients

Variable	Factor1	Factor2
100m/s	-0.128	0.268
200m/s	-0.457	0.951
400m/s	-0.046	0.111
800m/s	-0.008	0.055
1500m/s	0.012	0.055
5000m/s	0.570	-0.219
10,000m/s	0.711	-0.388
Marathonm/s	0.089	-0.047



The results from the two solution methods are very similar and very similar to the principal component factor analysis of the covariance matrix S. Using the unrotated loadings, the first factor might be identified as a "running excellence" factor. All the running events load highly on this factor. The second factor appears to contrast the shorter running events with the longer events although there is some difference in the groupings depending on the solution method. The 800m and 1500m runs are in the longer group for the principal component method and in the shorter group for the maximum likelihood method. Nevertheless, this bipolar factor might be called a "running speed-running endurance" factor. After rotation the overall excellence factor disappears and the first factor appears to represent "running endurance" since the running events 800m through the marathon load highly on this factor. The second factor might be classified as a "running speed" factor. Note, for both factors, the remaining running events in each case have moderately large loadings on the factor. The two factor solution accounts for 89%-91% (depending on solution method) of the total variance. The plots of the factor scores indicate that observations #46 (Samoa) and #11 (Cook Islands) are outliers.

The results of the $m = 2$ factor analysis of men's track records when time is measured in meters per second are very much the same as the results for the $m = 2$ factor analysis of **R** presented in Exercise 9.30. If the correlation matrix **R** is factor analyzed, it makes little difference whether running event time is measured in seconds (or minutes) as in Exercise 9.30 or in meters per second. It does make a difference if the covariance matrix **S** is factor analyzed, since the measurement scales in Exercise 9.30 are quite different from the meters/second scale.

9.32. Factor analysis of data on bulls

Factor analysis using sample covariance matrix S

Initial Factor Method: Principal Components

	1	2	3	4	5	6	7
Eigenvalue	20579.6126	4874.6748	5.4292	3.3163	0.4688	0.0741	0.0045
Difference	15704.9378	4869.2456	2.1129	2.8475	0.3948	0.0695	
Proportion	0.8082	0.1914	0.0002	0.0001	0.0000	0.0000	0.0000
Cumulative	0.8082	0.9996	0.9998	1.0000	1.0000	1.0000	1.0000

Factor Pattern

	FACTOR1	FACTOR2	FACTOR3	
X3	0.48777	0.39033	0.38532	YrHgt
X4	0.75367	0.65725	-0.00086	FtFrBody
X5	0.37408	0.62342	0.64446	PrctFFB
X6	0.48170	0.36809	0.33505	Frame
X7	0.11083	-0.38394	-0.49074	BkFat
X8	0.66769	0.29875	0.33038	SaleHt
X9	0.96506	-0.26204	0.00009	SaleWt

SAS scales the loadings obtained from a covariance matrix and then rotates the scaled loadings. The scaling is $\hat{l}_{ij} / \sqrt{s_{ii}}$.

Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2	FACTOR3	
X3	0.50195	0.42460	0.32637	YrHgt
X4	0.25853	0.90600	0.33514	FtFrBody
X5	0.83816	0.45576	0.18354	PrctFFB
X6	0.44716	0.42166	0.31943	Frame
X7	-0.60974	-0.06913	0.15478	BkFat
X8	0.40890	0.46689	0.50894	SaleHt
X9	-0.13508	0.30219	0.94363	SaleWt

Initial Factor Method: Maximum Likelihood

Factor Pattern

	FACTOR1	FACTOR2	FACTOR3	
X3	0.00000	1.00000	0.00000	YrHgt
X4	0.42819	0.62380	0.39838	FtFrBody
X5	0.85244	0.52282	0.00000	PrctFFB
X6	-0.01180	0.94025	0.03120	Frame
X7	-0.36162	-0.34428	0.39308	BkFat
X8	0.08393	0.85951	0.28992	SaleHt
X9	0.00598	0.36843	0.83599	SaleWt

Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2	FACTOR3	
X3	0.94438	0.28442	0.16509	YrHgt
X4	0.41219	0.50159	0.55648	FtFrBody
X5	0.23003	0.94883	0.21635	PrctFFB
X6	0.88812	0.25026	0.18382	Frame
X7	-0.25711	-0.51405	0.27102	BkFat
X8	0.75340	0.26667	0.43720	SaleHt
X9	0.25282	-0.05273	0.87634	SaleWt

Factor analysis using sample correlation matrix R

Initial Factor Method: Principal Components

	1	2	3	4	5	6	7
Eigenvalue	4.1207	1.3371	0.7414	0.4214	0.1858	0.1465	0.0471
Difference	2.7836	0.5957	0.3200	0.2356	0.0393	0.0994	
Proportion	0.5887	0.1910	0.1059	0.0602	0.0265	0.0209	0.0067
Cumulative	0.5887	0.7797	0.8856	0.9458	0.9723	0.9933	1.0000

Factor Pattern

	FACTOR1	FACTOR2	FACTOR3	
X3	0.91334	-0.04948	-0.35794	YrHgt
X4	0.83700	0.15014	0.38772	FtFrBody
X5	0.72177	-0.36484	0.48930	PrctFFB
X6	0.88091	0.00894	-0.38949	Frame
X7	-0.37900	0.82646	-0.03335	BkFat
X8	0.91927	0.11715	-0.15210	SaleHt
X9	0.54798	0.69440	0.21811	SaleWt

Varimax Rotated Factor Pattern

	FACTOR1	FACTOR2	FACTOR3	
X3	0.94188	0.27085	-0.06532	YrHgt
X4	0.44792	0.78354	0.24262	FtFrBody
X5	0.26505	0.87071	-0.25513	PrctFFB
X6	0.93812	0.21799	-0.01382	Frame
X7	-0.23541	-0.37460	0.79502	BkFat
X8	0.83365	0.41206	0.13094	SaleHt
X9	0.34932	0.39692	0.74194	SaleWt

Initial Factor Method: Maximum Likelihood

Factor Pattern

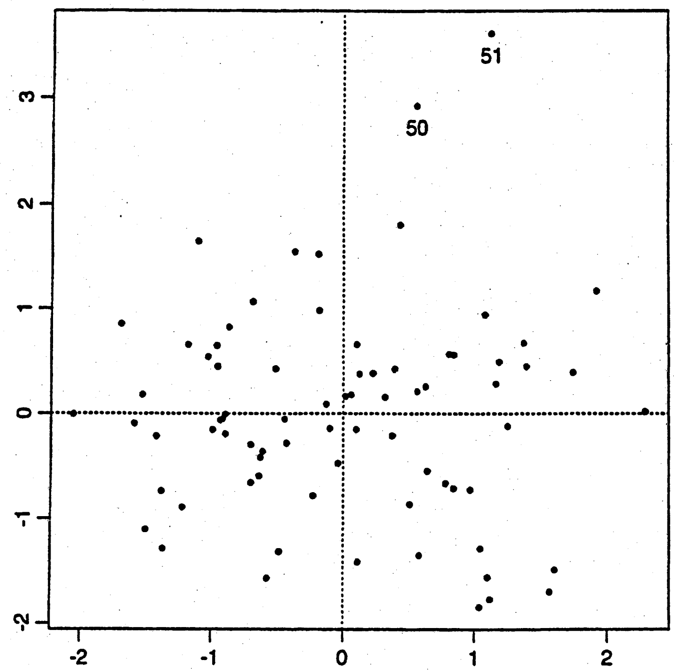
	FACTOR1	FACTOR2	FACTOR3	
X3	0.00000	1.00000	0.00000	YrHgt
X4	0.42819	0.62380	0.39838	FtFrBody
X5	0.85244	0.52282	0.00000	PrctFFB
X6	-0.01180	0.94025	0.03120	Frame
X7	-0.36162	-0.34428	0.39308	BkFat
X8	0.08393	0.85951	0.28992	SaleHt
X9	0.00598	0.36843	0.83599	SaleWt

Varimax Rotated Factor Pattern

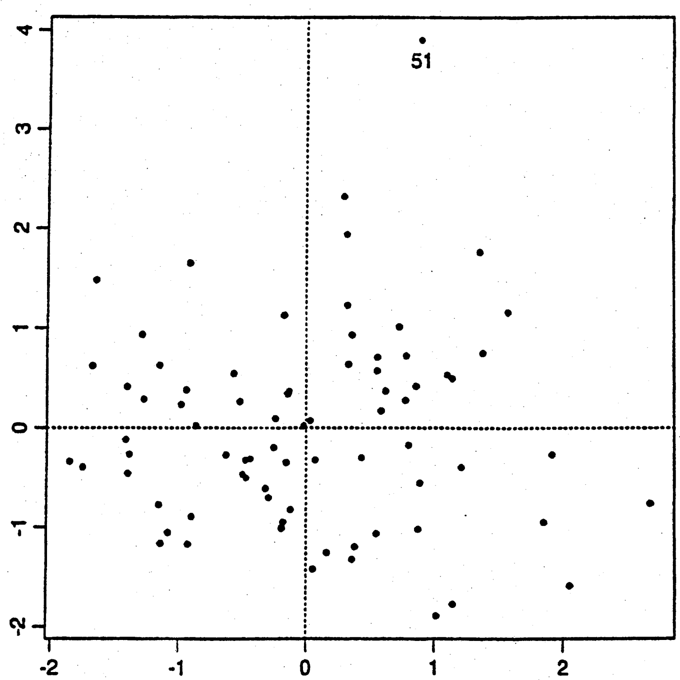
	FACTOR1	FACTOR2	FACTOR3	
X3	0.94438	0.28442	0.16509	YrHgt
X4	0.41219	0.50159	0.55648	FtFrBody
X5	0.23003	0.94883	0.21635	PrctFFB
X6	0.88812	0.25026	0.18382	Frame
X7	-0.25711	-0.51405	0.27102	BkFat
X8	0.75340	0.26667	0.43720	SaleHt
X9	0.25282	-0.05273	0.87634	SaleWt

The interpretation of factors from R is different of the interpretation of factors from S.

Factor scores for the first two factors using S
and varimax rotated PC estimates of factor loadings



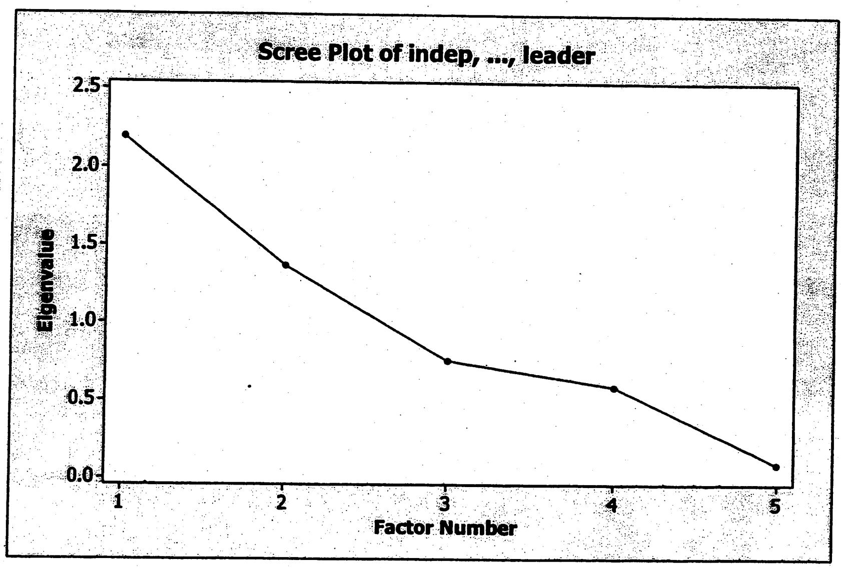
Factor scores for the first two factors using R
and varimax rotated PC estimates of factor loadings



9.33 The correlation matrix **R** and the scree plot follow. The correlations are relatively modest. These correlations and the scree plot suggest $m = 2$ factors is probably too few. An initial factor analysis with $m = 2$ confirms this conjecture. Consequently, we give a $m = 3$ factor solution.

Correlations: indep, supp, benev, conform, leader

	indep	supp	benev	conform
supp	-0.173			
benev	-0.561	0.018		
conform	-0.471	-0.327	0.298	
leader	0.187	-0.401	-0.492	-0.333



Principal Component Factor Analysis of **R** ($m = 3$)

Unrotated Factor Loadings and Communalities

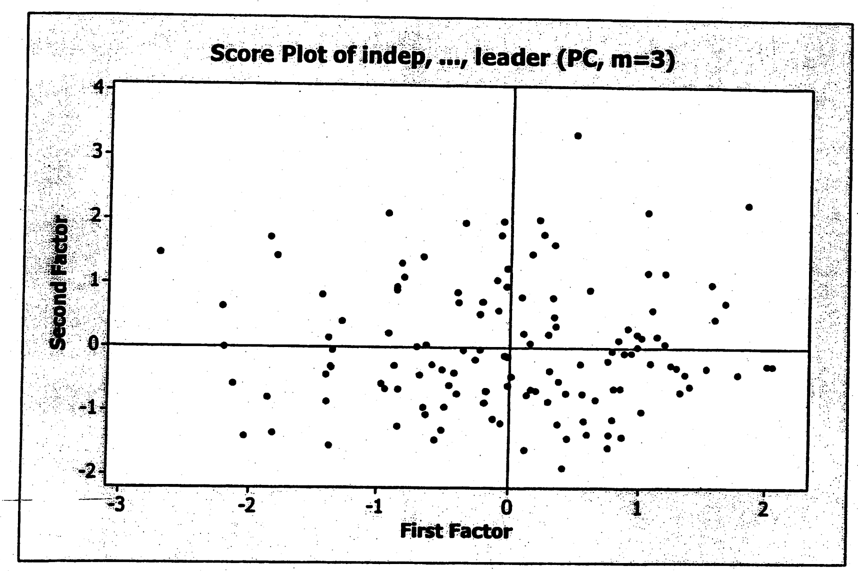
Variable	Factor1	Factor2	Factor3	Communality
indep	-0.772	0.101	-0.580	0.943
supp	0.180	0.922	0.163	0.909
benev	0.813	-0.009	0.100	0.670
conform	0.651	-0.574	-0.256	0.819
leader	-0.696	-0.422	0.563	0.979
Variance	2.1966	1.3682	0.7559	4.3207
% Var	0.439	0.274	0.151	0.864

```
Rotated Factor Loadings and Communalities
Varimax Rotation
```

Variable	Factor1	Factor2	Factor3	Communality
indep	-0.971	0.018	-0.003	0.943
supp	0.136	-0.312	0.890	0.909
benev	0.700	-0.418	-0.081	0.670
conform	0.419	-0.379	-0.707	0.819
leader	-0.155	0.971	-0.111	0.979
Variance	1.6506	1.3587	1.3114	4.3207
% Var	0.330	0.272	0.262	0.864

```
Factor Score Coefficients
```

Variable	Factor1	Factor2	Factor3
indep	-0.752	-0.362	-0.147
supp	0.119	-0.129	0.690
benev	0.372	-0.127	-0.010
conform	0.073	-0.277	-0.545
leader	0.240	0.832	0.008

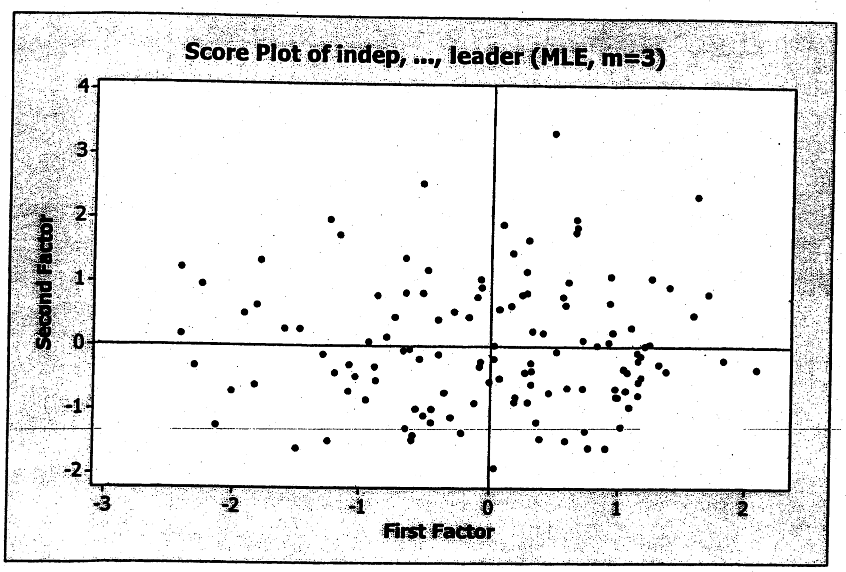


Maximum Likelihood Factor Analysis of R $(m = 3)$

```
* NOTE * Heywood case

Unrotated Factor Loadings and Communalities
```

Variable	Factor1	Factor2	Factor3	Communality
indep	-0.788	0.187	0.587	1.000
supp	-0.464	-0.401	-0.790	1.000
benev	0.532	-0.492	-0.086	0.532
conform	0.664	-0.333	0.194	0.589
leader	0.000	1.000	0.000	1.000
Variance	1.5591	1.5486	1.0133	4.1211
% Var	0.312	0.310	0.203	0.824

```
Rotated Factor Loadings and Communalities
Varimax Rotation

Variable  Factor1  Factor2  Factor3  Communality
indep      -0.992    0.034    0.122        1.000
supp        0.048   -0.192   -0.980        1.000
benev       0.562   -0.454    0.098        0.532
conform     0.515   -0.371    0.432        0.589
leader     -0.129    0.968    0.213        1.000

Variance   1.5842   1.3199   1.2170       4.1211
% Var       0.317    0.264    0.243        0.824


Factor Score Coefficients

Variable  Factor1  Factor2  Factor3
indep      -1.016   -0.130   -0.024
supp       -0.123    0.219   -1.069
benev      -0.000    0.000    0.000
conform    -0.000    0.000   -0.000
leader      0.011    1.081   -0.211
```



Using the unrotated loadings and including moderate loadings of magnitudes .4—.5, the factors are all bipolar and appear to be difficult to interpret. Moreover, the arrangement of relatively large loadings on each factor is quite different for the two solution methods. The rotated loadings are consistent with one another for the two solution methods and, although all the factors are bipolar, may be easier to interpret. The first factor is a contrast between Independence and the pair Benevolence and Conformity. Perhaps this factor could be classified as a "conforming—not conforming" factor. The second factor is essentially a "leadership" factor although if moderate loadings are included, this factor is a contrast between Leadership and Benevolence. Teenagers with above average scores on Leadership tend to be above average on this factor, while those with above average scores on Benevolence tend to be below average on this factor. Perhaps we could label this factor a "lead—follow" factor. The third factor is essentially a "support" factor although, again, if moderate loadings are used, this factor is a contrast between Support and Conformity. To our minds however, the latter is difficult to interpret. The factor scores for the first two factors are similar for the two solutions methods. No outliers are immediately evident.

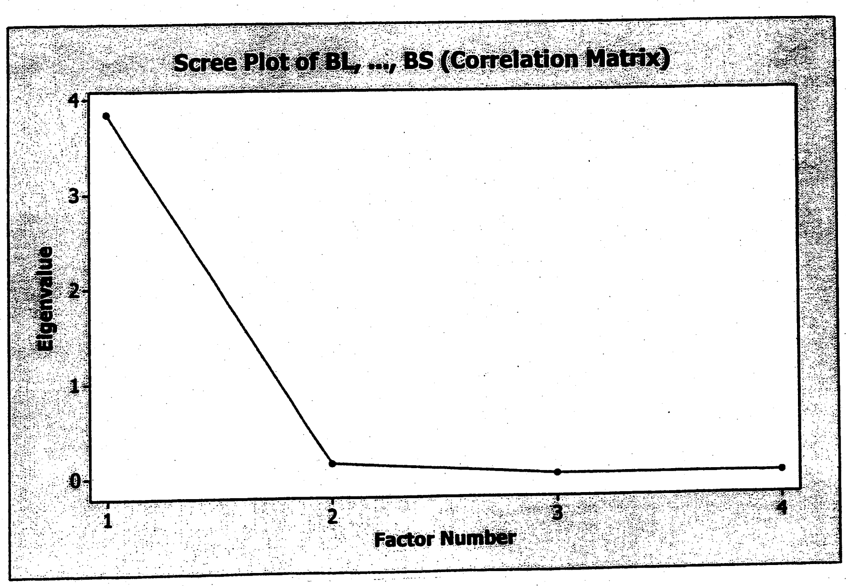
9.34 A factor analysis of the paper property variables with either **S** or **R** suggests a $m = 1$ factor solution is reasonable. All variables load highly on a single factor. The covariance matrix **S** and correlation matrix **R** follow along with a scree plot using **R**. For completeness, the results for a $m = 2$ factor solution using both solution methods is also given. Plots of factor scores from the two factor model suggest that observations 58, 59, 60 and 61 may be outliers.

Covariances: BL, EM, SF, BS

	BL	EM	SF	BS
BL	8.302871			
EM	1.886636	0.513359		
SF	4.147318	0.987585	2.140046	
BS	1.972056	0.434307	0.987966	0.480272

Correlations: BL, EM, SF, BS

	BL	EM	SF
EM	0.914		
SF	0.984	0.942	
BS	0.988	0.875	0.975



Principal Component Factor Analysis of **S** ($m = 1$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Communality
BL	2.878	8.285
EM	0.664	0.441
SF	1.449	2.101
BS	0.684	0.468
Variance	11.295	11.295
% Var	0.988	0.988

Factor Score Coefficients

Variable	Factor1
BL	0.734
EM	0.042
SF	0.188
BS	0.042

The first factor explains 99% of the total variance. All variables, given their measurement scales, load highly on this factor. Note: There is no factor rotation with one factor.

Principal Component Factor Analysis of **R** ($m = 1$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Communality
BL	0.992	0.984
EM	0.951	0.905
SF	0.996	0.991
BS	0.980	0.960
Variance	3.8395	3.8395
% Var	0.960	0.960

Factor Score Coefficients

Variable	Factor1
BL	0.258
EM	0.248
SF	0.259
BS	0.255

The first factor explains 96% of the variance. All variables load highly and about equally on this factor. This factor might be called a "paper properties index."

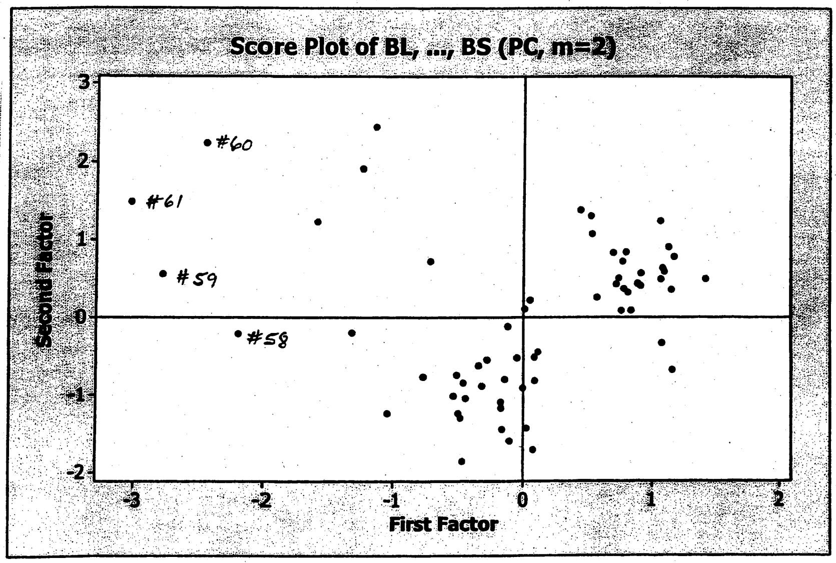
Maximum Likelihood Factor Analysis of **R** ($m = 1$)

```
* NOTE * Heywood case

Unrotated Factor Loadings and Communalities

Variable  Factor1  Communality
BL          1.000        1.000
EM          0.914        0.835
SF          0.984        0.968
BS          0.988        0.975

Variance   3.7784       3.7784
% Var       0.945        0.945


Factor Score Coefficients

Variable  Factor1
BL          1.000
EM          0.000
SF          0.000
BS          0.000
```

The results are similar to the results for the principal component method. The first factor explains about 95% of the variance and all variables load highly and about equally on this factor. Again, the factor might be called a "paper properties index."

Principal Component Factor Analysis of **R** ($m = 2$)

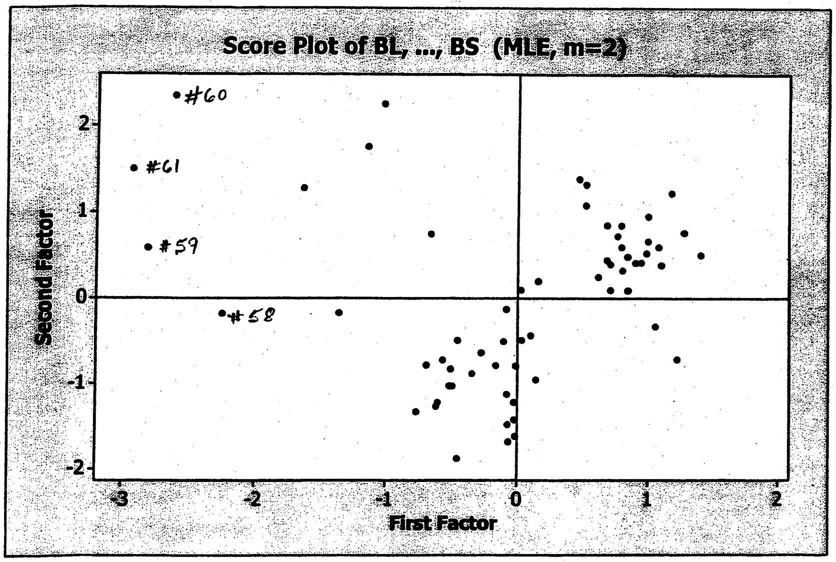
```
Unrotated Factor Loadings and Communalities

Variable  Factor1  Factor2  Communality
BL          0.992   -0.098        0.993
EM          0.951    0.307        0.999
SF          0.996   -0.008        0.991
BS          0.980   -0.191        0.996

Variance   3.8395   0.1403       3.9798
% Var       0.960    0.035        0.995


Rotated Factor Loadings and Communalities
Varimax Rotation

Variable  Factor1  Factor2  Communality
BL          0.817    0.571        0.993
EM          0.522    0.852        0.999
SF          0.761    0.642        0.991
BS          0.868    0.493        0.996

Variance   2.2717   1.7082       3.9798
% Var       0.568    0.427        0.995


Factor Score Coefficients

Variable  Factor1  Factor2
BL          0.650   -0.361
EM         -1.235    1.821
SF          0.232    0.128
BS          1.081   -0.868
```



Using the unrotated loadings, the second factor explains very little of the variance beyond that of the first factor and is not needed. Since the unrotated loadings provide a clear interpretation of the first factor there is no need to consider the rotated loadings. The potential outliers are evident in the plot of factor scores.

Maximum Likelihood Factor Analysis of **R** ($m = 2$)

```
* NOTE * Heywood case

Unrotated Factor Loadings and Communalities

Variable  Factor1  Factor2  Communality
BL          0.988    0.103        0.986
EM          0.875    0.485        1.000
SF          0.975    0.185        0.984
BS          1.000    0.000        1.000

Variance   3.6900   0.2800       3.9700
% Var       0.922    0.070        0.992


Rotated Factor Loadings and Communalities
Varimax Rotation

Variable  Factor1  Factor2  Communality
BL          0.809    0.576        0.986
EM          0.523    0.853        1.000
SF          0.757    0.641        0.984
BS          0.870    0.492        1.000

Variance   2.2572   1.7128       3.9700
% Var       0.564    0.428        0.992


Factor Score Coefficients

Variable  Factor1  Factor2
BL         -0.000   -0.000
EM         -1.016    1.795
SF         -0.000   -0.000
BS          1.759   -1.078
```



The results are similar to the results for the principal component method. Using the unrotated loadings, the first factor explains 92% of the total variance and the second factor explains very little of the remaining variance. Since the unrotated loadings provide a clear interpretation of the first factor (paper properties index) there is no need to consider the rotated loadings. The same potential outliers are evident in the plot of factor scores.

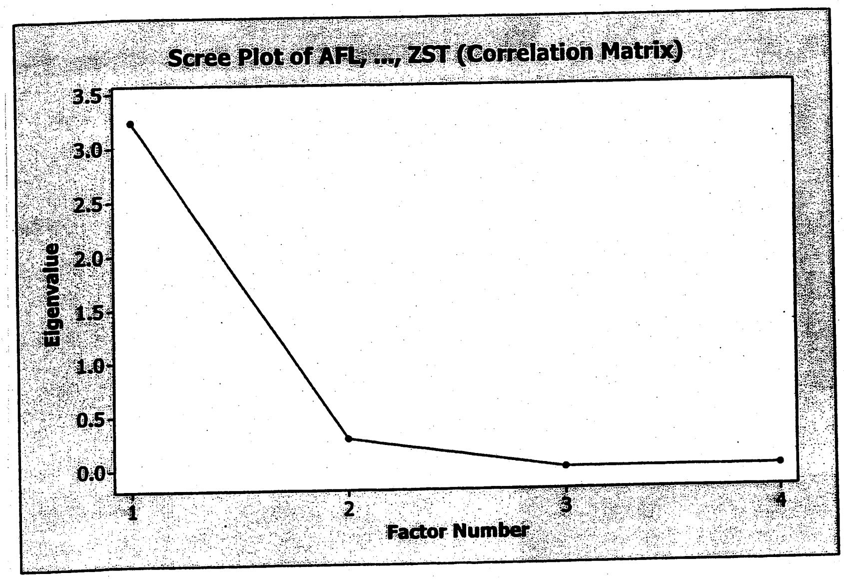
9.35 A factor analysis of the pulp fiber characteristic variables with **S** and **R** for $m = 1$ and $m = 2$ factors is summarized below. The covariance matrix **S** and correlation matrix **R** follow along with a scree plot using **R**. Plots of factor scores from the two factor model suggest that observations 60 and 61 and possibly observations 57, 58 and 59 may be outliers. A $m = 1$ factor solution using **R** appears to be the best choice.

Covariances: AFL, LFF, FFF, ZST

	AFL	LFF	FFF	ZST
AFL	0.06227			
LFF	3.35980	221.05161		
FFF	-3.21404	-185.63707	308.39989	
ZST	0.00577	0.34760	-0.40633	0.00087

Correlations: AFL, LFF, FFF, ZST

	AFL	LFF	FFF
LFF	0.906		
FFF	-0.733	-0.711	
ZST	0.784	0.793	-0.785



Principal Component Factor Analysis of **S** ($m = 1$)

```
Unrotated Factor Loadings and Communalities

Variable  Factor1  Communality
AFL         0.216        0.047
LFF        13.250      175.573
FFF       -16.729      279.858
ZST         0.025        0.001

Variance   455.48       455.48
% Var       0.860        0.860


Factor Score Coefficients

Variable  Factor1
AFL         0.000
LFF         0.433
FFF        -0.645
ZST         0.000
```

The first factor explains 86% of the total variance and represents a contrast between FFF (with a negative loading) and the AFL, LFF and ZST group, all with positive loadings. AFL (average fiber length), LFF (long fiber fraction) and ZST (zero span tensile strength) may all have to do with paper strength while FFF (fine fiber fraction) may have something to do with paper quality. Perhaps we could label this factor a "strength—quality" factor.

Principal Component Factor Analysis of **R** ($m = 1$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Communality
AFL	0.936	0.877
LFF	0.933	0.870
FFF	-0.878	0.770
ZST	0.917	0.841
Variance	3.3577	3.3577
% Var	0.839	0.839

Factor Score Coefficients

Variable	Factor1
AFL	0.279
LFF	0.278
FFF	-0.261
ZST	0.273

The first factor explains 84% of the variance and the pattern of loadings is consistent with that of the $m = 1$ factor analysis of the covariance matrix **S**. Again, we might label this bi polar factor a "strength—quality" factor.

Maximum Likelihood Factor Analysis of **R** ($m = 1$)

Unrotated Factor Loadings and Communalities

Variable	Factor1	Communality
AFL	0.949	0.900
LFF	0.945	0.894
FFF	-0.784	0.614
ZST	0.846	0.717
Variance	3.1245	3.1245
% Var	0.781	0.781

Factor Score Coefficients

Variable	Factor1
AFL	0.422
LFF	0.394
FFF	-0.090
ZST	0.132

The first factor explains 78% of the variance and the pattern of loadings is consistent with that of the $m = 1$ factor analysis of the covariance matrix **R** using the principal component method. Again, we might label this bi polar factor a "strength—quality" factor.

Because the different measurement scales make the factor loadings obtained from the covariance matrix difficult to interpret, we continue with a factor analysis of the correlation matrix **R** with $m = 2$.

Principal Component Factor Analysis of **R** ($m = 2$)

Unrotated Factor Loadings and Communalities

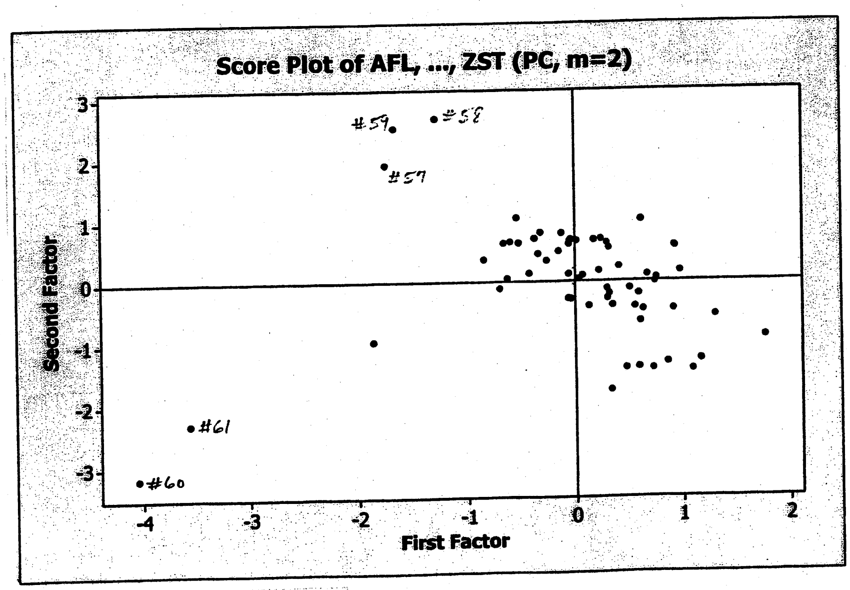
Variable	Factor1	Factor2	Communality
AFL	0.936	0.256	0.942
LFF	0.933	0.288	0.953
FFF	-0.878	0.423	0.949
ZST	0.917	-0.150	0.863
Variance	3.3577	0.3493	3.7070
% Var	0.839	0.087	0.927

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
AFL	0.868	-0.434	0.942
LFF	0.887	-0.408	0.953
FFF	-0.372	0.900	0.949
ZST	0.583	-0.723	0.863
Variance	2.0176	1.6893	3.7070
% Var	0.504	0.422	0.927

Factor Score Coefficients

Variable	Factor1	Factor2
AFL	0.696	0.359
LFF	0.757	0.429
FFF	0.613	1.075
ZST	-0.082	-0.501



Maximum Likelihood Factor Analysis of **R** ($m = 2$)

Unrotated Factor Loadings and Communalities

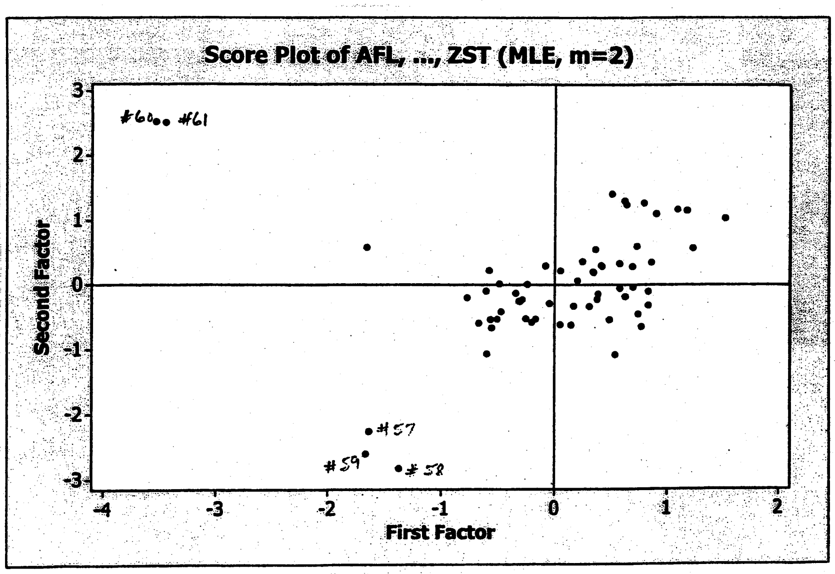
Variable	Factor1	Factor2	Communality
AFL	0.913	-0.205	0.876
LFF	0.926	-0.292	0.943
FFF	-0.890	-0.388	0.944
ZST	0.866	0.033	0.752
Variance	3.2351	0.2796	3.5146
% Var	0.809	0.070	0.879

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Communality
AFL	0.819	0.454	0.876
LFF	0.886	0.397	0.943
FFF	-0.407	-0.882	0.944
ZST	0.625	0.601	0.752
Variance	2.0124	1.5023	3.5146
% Var	0.503	0.376	0.879

Factor Score Coefficients

Variable	Factor1	Factor2
AFL	0.336	-0.101
LFF	0.922	-0.423
FFF	0.534	-1.197
ZST	0.049	0.076

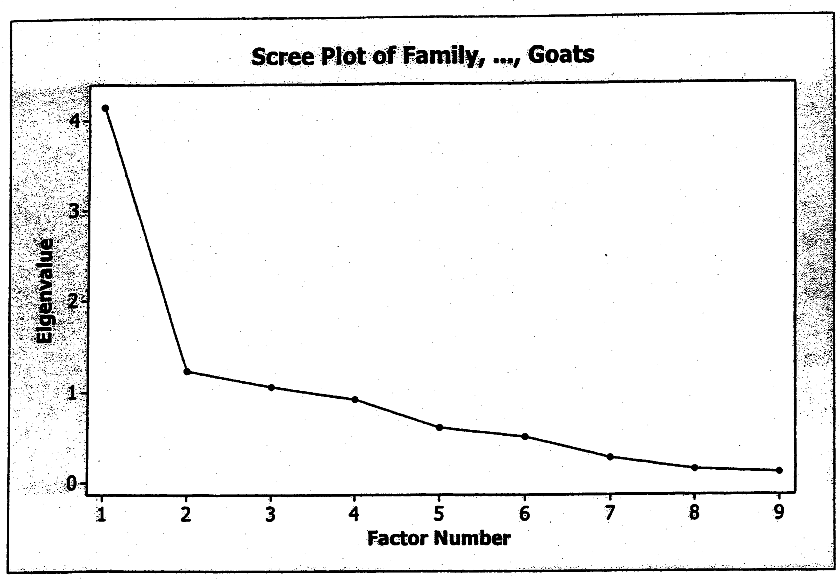


Examining the unrotated loadings for both solution methods, we see that the second factor explains little (about 7%-8%) of the remaining variance. Also, this factor has moderate to very small loadings on all the variables with the possible exception of variable FFF. If retained, this factor might be called a "fine fiber" of "quality" factor. Using the rotated loadings, the second factor looks much like the first factor for both solution methods. That is, this factor appears to be a contrast between variable FFF and the group of variables AFL, LFF and ZST. To summarize, there seems to be no gain in understanding from adding a second factor to the model. A one factor model appears be sufficient in this case. However, plots of the factor scores for m = 2 suggest observations 60, 61 and, perhaps, observations 57, 58 and 59 may be outliers.

9.36 The correlation matrix **R** and the scree plot is shown below. After $m = 2$ there is no sharp elbow in the scree plot and the plot falls off almost linearly. Potential choices for m are 2, 3, 4 and 5. We give the results for $m = 4$ but, to our minds, here is a case where a factor model is not particularly well defined.

Correlations: Family, DistRd, Cotton, Maze, Sorg, Millet, Bull, Cattle, Goats

	Family	DistRd	Cotton	Maze	Sorg	Millet	Bull	Cattle
DistRd	-0.084							
Cotton	0.724	0.028						
Maze	0.679	-0.054	0.730					
Sorg	0.568	-0.071	0.383	0.109				
Millet	0.506	0.022	0.389	0.217	0.382			
Bull	0.727	-0.088	0.765	0.623	0.443	0.353		
Cattle	0.336	-0.063	0.175	0.197	0.404	0.081	0.520	
Goats	0.484	0.031	0.399	0.136	0.424	0.305	0.560	0.357



Principal Component Factor Analysis of **R** ($m = 4$)

Unrotated Factor Loadings and Communalities

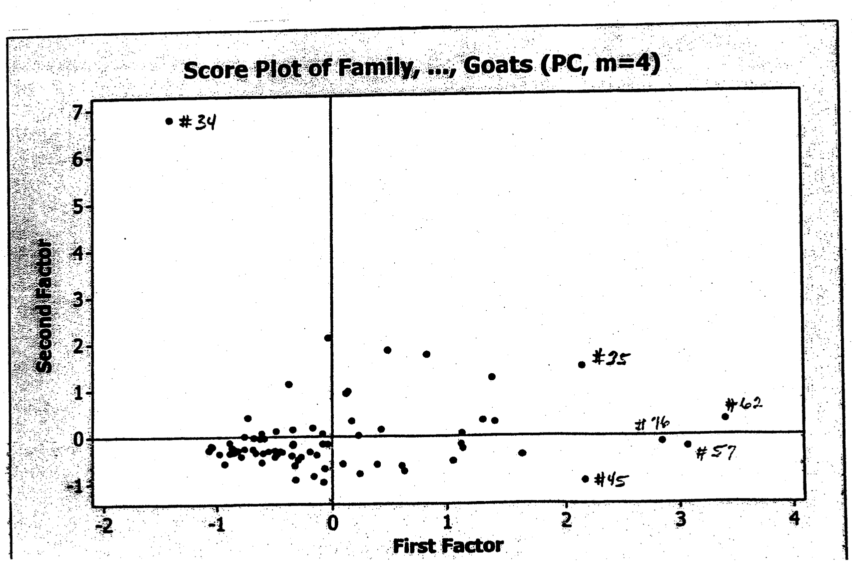
Variable	Factor1	Factor2	Factor3	Factor4	Communality
Family	0.903	-0.111	-0.002	-0.118	0.842
DistRd	-0.068	-0.080	-0.855	0.482	0.974
Cotton	0.837	-0.380	-0.070	0.028	0.851
Maze	0.687	-0.616	0.175	0.158	0.907
Sorg	0.633	0.503	-0.070	-0.219	0.706
Millet	0.547	0.048	-0.396	-0.582	0.798
Bull	0.896	-0.033	0.125	0.189	0.856
Cattle	0.502	0.510	0.286	0.466	0.811
Goats	0.629	0.421	-0.178	0.096	0.614
Variance	4.1443	1.2364	1.0581	0.9205	7.3593
% Var	0.460	0.137	0.118	0.102	0.818

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Factor3	Factor4	Communality
Family	0.714	0.320	-0.473	0.080	0.842
DistRd	-0.026	-0.022	0.006	-0.986	0.974
Cotton	0.856	0.150	-0.301	-0.076	0.851
Maze	0.951	0.008	0.032	0.047	0.907
Sorg	0.092	0.564	-0.606	0.112	0.706
Millet	0.226	-0.026	-0.863	-0.029	0.798
Bull	0.724	0.535	-0.210	0.043	0.856
Cattle	0.148	0.879	0.108	0.074	0.811
Goats	0.180	0.629	-0.406	-0.145	0.614
Variance	2.7840	1.8985	1.6476	1.0291	7.3593
% Var	0.309	0.211	0.183	0.114	0.818

Factor Score Coefficients

Variable	Factor1	Factor2	Factor3	Factor4
Family	0.197	-0.013	-0.171	0.063
DistRd	0.014	0.042	0.030	-0.963
Cotton	0.344	-0.115	-0.024	-0.090
Maze	0.494	-0.165	0.247	0.023
Sorg	-0.199	0.246	-0.374	0.100
Millet	-0.078	-0.260	-0.697	-0.001
Bull	0.224	0.204	0.110	0.005
Cattle	-0.063	0.633	0.329	0.019
Goats	-0.114	0.338	-0.156	-0.164



Maximum Likelihood Factor Analysis of **R** $(m = 4)$

Unrotated Factor Loadings and Communalities

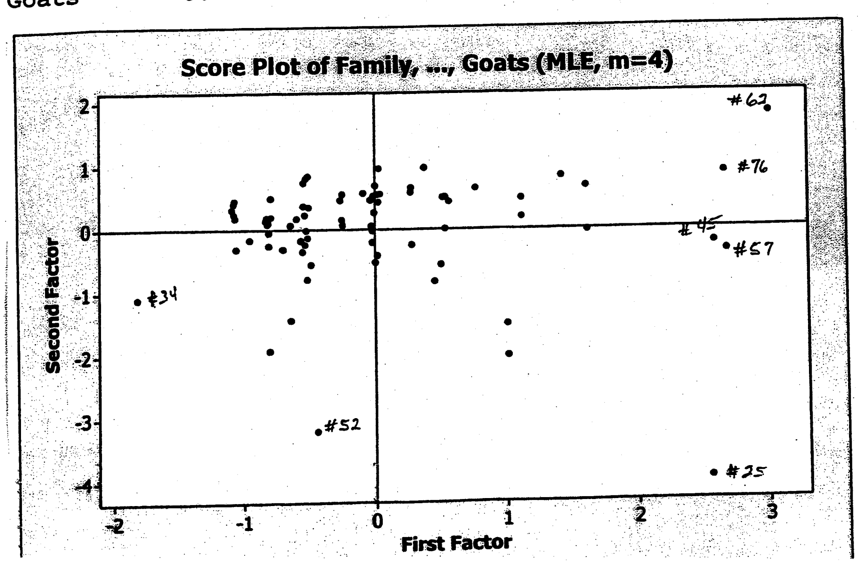
Variable	Factor1	Factor2	Factor3	Factor4	Communality
Family	0.752	-0.324	-0.162	-0.374	0.837
DistRd	-0.064	0.056	-0.044	-0.003	0.009
Cotton	0.794	-0.238	-0.307	-0.044	0.782
Maze	0.980	0.170	0.025	-0.002	0.990
Sorg	0.211	-0.567	-0.071	-0.527	0.649
Millet	0.276	-0.269	-0.301	-0.361	0.369
Bull	0.746	-0.616	-0.096	0.131	0.962
Cattle	0.290	-0.608	0.640	-0.074	0.869
Goats	0.249	-0.607	-0.151	-0.109	0.465
Variance	2.9824	1.7047	0.6610	0.5841	5.9322
% Var	0.331	0.189	0.073	0.065	0.659

Rotated Factor Loadings and Communalities
Varimax Rotation

Variable	Factor1	Factor2	Factor3	Factor4	Communality
Family	0.630	-0.605	0.229	-0.148	0.837
DistRd	-0.040	0.017	-0.081	0.025	0.009
Cotton	0.713	-0.362	0.075	-0.370	0.782
Maze	0.980	-0.034	0.166	-0.016	0.990
Sorg	0.034	-0.740	0.303	-0.089	0.649
Millet	0.206	-0.558	-0.028	-0.120	0.369
Bull	0.540	-0.324	0.437	-0.612	0.962
Cattle	0.039	-0.154	0.915	-0.079	0.869
Goats	0.072	-0.466	0.268	-0.414	0.465
Variance	2.2098	1.7035	1.2850	0.7340	5.9322
% Var	0.246	0.189	0.143	0.082	0.659

Factor Score Coefficients

Variable	Factor1	Factor2	Factor3	Factor4
Family	0.013	-0.606	-0.078	0.247
DistRd	0.001	-0.002	-0.009	-0.002
Cotton	0.033	-0.161	-0.162	-0.113
Maze	0.995	0.440	0.109	0.681
Sorg	-0.023	-0.404	0.017	0.206
Millet	0.003	-0.185	-0.062	0.052
Bull	-0.026	0.215	0.103	-1.426
Cattle	-0.141	0.091	0.896	0.385
Goats	-0.009	-0.093	-0.010	-0.023



The two solution methods for $m = 4$ factors produce somewhat different results. The patterns for unrotated loadings on the first two factors are similar but not identical. The patterns of loadings for the two solution methods on the third and fourth factors are quite different. Notice that DistRd does not load on any factor in the maximum likelihood solution. The factor loading patterns are more alike for the two solution methods using the rotated loadings, although factors 2 & 3 in the principal component solution appear to be reversed in the maximum likelihood solution. The rotated loadings on factor 4 for the two methods are quite different. Again, DistRd does not load on any factor in the maximum likelihood solution, it appears to define factor 4 in the principal component solution. (From **R** we see that DistRd is not correlated with any of the other variables.) Variables Family, Cotton, Maze, and Bullocks load highly on the first factor. The variables Family, Sorghum, Millet and Goats load highly on the second factor (maximum likelihood solution) or the third factor (principal component solution). Growing cotton and maze is labor intensive and bullocks are helpful. The first factor might be called a "family farm—row crop" factor. Millet and sorghum are grasses and may provide feed for goats. Consequently, the second (or third in the case of the principal component solution) factor might be called a "family farm—grass crop" factor. The third factor in the maximum likelihood solution (second factor in the principal component solution) may have different interpretations depending on the solution method but in both cases, Bullocks and Cattle load highly on this factor. Perhaps this factor could be called a "livestock" factor. The rotated loadings are considerably different on the fourth factor. This factor is clearly a "distance to the road" factor in the principal component solution. The interpretation is not clear in the maximum likelihood solution. The fact that the two solution methods produce somewhat different results and explain quite different proportions of the total variation (82% for principal components, 66% for maximum likelihood) reinforces the notion that a linear factor model is not particularly well defined for this problem. Plots of factor scores for the first two factors indicated there are several potential outliers. If these observations are removed, the results could change.

Chapter 10

10.1. $\rho_{11}^{-1/2}\rho_{12}\rho_{22}^{-1}\rho_{21}\rho_{11}^{-1/2} = \begin{bmatrix} 0 & 0 \\ 0 & (.95)^2 \end{bmatrix}$

which has eigenvalues $\rho_1^{*2} = (.95)^2$ and $\rho_2^{*2} = 0$.

The normalized eigenvectors are $\underset{\sim}{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\underset{\sim}{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus

$$U_1 = \underset{\sim}{e}_1' \rho_{11}^{1/2} \underset{\sim}{X}^{(1)} = [0\ 1] \begin{bmatrix} .1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix} = X_2^{(1)}$$

Since $\underset{\sim}{f}_1' \rho_{22}^{-1/2} = [1\ \ 0]$, $V_1 = X_1^{(2)}$.

Thus $U_1 = X_2^{(1)}$, $V_1 = X_1^{(2)}$ and $\rho_1^* = .95$.

10.2 a) $\rho_1^* = .55$, $\rho_2^* = .49$

b) $U_1 = .32X_1^{(1)} - .36X_2^{(1)}$

$V_1 = .36X_1^{(2)} - .10X_2^{(2)}$

$U_2 = .20X_1^{(1)} + .30X_2^{(1)}$

$V_2 = .23X_1^{(2)} + .30X_2^{(1)}$

c)

$$E\begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -1.675 \\ .015 \\ -.095 \\ .386 \end{bmatrix}$$

$$\text{Cov}\begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{V} \end{bmatrix} = \begin{bmatrix} 1 & 0 & .55 & 0 \\ 0 & 1 & 0 & .49 \\ .55 & 0 & 1 & 0 \\ 0 & .49 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \rho_1^* & 0 \\ 0 & 1 & 0 & \rho_2^* \\ \rho_1^* & 0 & 1 & 0 \\ 0 & \rho_2^* & 0 & 1 \end{bmatrix}$$

10.5 a) $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \boldsymbol{\rho}_{11}^{-1}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21} = \begin{bmatrix} .45189 & .28919 \\ .14633 & .17361 \end{bmatrix}$

$$\begin{vmatrix} .45189-\lambda & .28919 \\ .14633 & .17361-\lambda \end{vmatrix} = \lambda^2 - .5467\lambda + .0005$$

$$= (\lambda - .5457)(\lambda - .0009)$$

The characteristic equation is the same as that of ${}_{11}^{-1/2}\ {}_{12}\ {}_{22}^{-1}\ {}_{21}\ {}_{11}^{-1/2}$ (see Example 10.1) and consequently the eigenvalues are the same.

b) $U_2 = -.677Z_1^{(1)} + 1.055Z_2^{(1)}$

$V_2 = -.863Z_1^{(2)} + .706Z_2^{(2)}$

$\text{Var}(U_2) = (-0.677)^2 + (1.055)^2 - 2(.677)(1.055)(.4) = 1.0$

$\text{Var}(V_2) = 1.0$

$\text{Corr}(U_2, V_2) = (-.677)(-.863)(.5) + (-.863)(1.055)(.3)$

$+ (.706)(-.677)(.6) + (.706)(1.055)(.4) = .03 = \rho_2^*$

10.7 a) $\rho_1^* = \frac{2\rho}{1+\rho} \quad 0 < \rho < 1$

$$U_1 = \frac{1}{\sqrt{2(1+\rho)}} (X_1^{(1)} + X_2^{(1)})$$

$$V_1 = \frac{1}{\sqrt{2(1+\rho)}} (X_1^{(2)} + X_2^{(2)})$$

10.8 c) $\hat{\rho}_1^* = .72$

$\hat{V}_1 = .20X_1^{(2)} + .70X_2^{(2)}$

$\hat{\beta} = 45° \equiv \frac{\pi}{4}$ radians

d) $\hat{\rho}_1^* = .57$

$\hat{U}_1 = 1.03 \cos \theta_1 + .46 \text{ Sin } \theta_1$

$V_1 = .49 \cos \theta_2 + .78 \text{ Sin } \theta_2$

10.9 a) $\hat{\rho}_1^* = .39 \quad ; \quad \hat{\rho}_2^* = .07$

$\hat{U}_1 = 1.26Z_1^{(1)} - 1.03Z_2^{(1)}; \quad \hat{U}_2 = .30Z_1^{(1)} + .79Z_2^{(1)}$

$\hat{V}_1 = 1.10Z_1^{(2)} - .45Z_2^{(2)} \quad ; \quad V_2 = -.02Z_1^{(2)} + 1.01Z_2^{(2)}$

b) $n = 140,\ p=2,\ q=2,\ n-1-\frac{1}{2}(p+q+1) = 136.5$

Null hypothesis	Value of test statistic	Degrees of Freedom	Upper 5% point of χ^2 distribution
$H_0: \Sigma_{12} = \rho_{12} = 0$	$-136.5\,\ell n(.8444)(.9953) = 23.74$	4	9.48
$H_0^{(1)}: \rho_1^* \neq 0,\ \rho_2^* = 0$	$-136.5\,\ell n(.9953) = .65$	1	3.84

Therefore, reject H_0 but do not reject $H_0^{(1)}$. Reading ability (summarized by $\hat{U}_1$) does correlate with arithmetic ability (summarized by $\hat{V}_1$) but the correlation (represented by $\rho_1 = .39$) is not particularly strong.

10.10 a) $\hat{\rho}_1^* = .33, \; \hat{\rho}_2^* = .17$

b) $\hat{U}_1 = 1.002Z_1^{(1)} - .003Z_2^{(1)}$

$\hat{V}_1 = -.602Z_1^{(2)} - .977Z_2^{(2)}$

$\hat{U}_1 \doteq Z_1^{(1)}$ = 1973 nonprimary homicides (standardized)

$\hat{V}_1 \doteq \frac{3}{5} Z_1^{(2)} + Z_2^{(2)}$ = a "punishment index"

Punishment appears to be correlated with nonprimary homicides but not primary homicides.

10.11 Using the correlation matrix **R** and standardized variables, the canonical correlations and canonical variables follow. The $Z^{(1)}$'s are the banks, the $Z^{(2)}$'s are the oil companies.

$\hat{\rho}_1^* = .348, \quad \hat{\rho}_2^* = .130$

$$\hat{U}_1 = -.539z_1^{(1)} + 1.209z_2^{(1)} + .079z_3^{(1)}, \qquad \hat{U}_2 = 1.142z_1^{(1)} - .410z_2^{(1)} + .142z_3^{(1)}$$
$$\hat{V}_1 = 1.160z_1^{(2)} - .261z_2^{(2)} \qquad\qquad \hat{V}_2 = -.728z_1^{(2)} + 1.345z_2^{(2)}$$

Additional correlations:

$$R_{U_1,z^{(1)}} = (.266 \;\; .913 \;\; .498), \quad R_{V_1,z^{(2)}} = (.982 \;\; .532)$$
$$R_{U_1,z^{(2)}} = (.342 \;\; .185), \quad R_{V_1,z^{(1)}} = (.093 \;\; .318 \;\; .174)$$

Here $H_0 : \Sigma_{12} (\rho_{12}) = 0$ is rejected at the 5% level and $H_0^{(1)} : \rho_1^* \neq 0, \rho_2^* = 0$ is not rejected at the 5% level. The first canonical correlation, although relatively small, is significant. The second canonical correlation is not significant.

Focusing attention on the first pair of canonical variables, $\hat{U}_1$ is dominated by Citibank, $\hat{V}_1$ is dominated by Royal Dutch Shell. The canonical correlation (.348) between $\hat{U}_1$ and $\hat{V}_1$ suggests there is not much co-movement between the rates of return for the banks on one hand and the oil companies on the other. Moreover, $\hat{U}_1$ is not highly correlated with any of the $Z^{(2)}$'s (oil companies) and $\hat{V}_1$ is not highly correlated with any of the $Z^{(1)}$'s (banks). The first canonical variables differentiate stocks in different industries with some, but not much, overlap.

10.12 a) $\hat{\rho}_1^* = .69, \quad \hat{\rho}_2^* = .19$

Reject H_0: $\rho_{12} = 0$ at the 5% level but do not reject $H_0^{(1)} = \rho_1^* \neq 0, \ \rho_2^* = 0$ at the 5% level.

b) $\hat{U}_1 = .77Z_1^{(1)} + .27Z_2^{(1)}$

$\hat{V}_1 = .05Z_1^{(2)} + .90Z_2^{(2)} + .19Z_3^{(2)}$

c) Sample Correlations Between Original Variables and Canonical Variables

$X^{(1)}$ Variables	$\hat{U}_1$	$\hat{V}_1$	$X^{(2)}$ Variables	$\hat{U}_1$	$\hat{V}_1$
1. annual frequency of restaurant dining	.99	.68	1. age of head of household	.29	.42
2. annual frequency of attending movies	.89	.61	2. annual family income	.68	.98
			3. educational level of head of household	.35	.51

d) $\hat{U}_1$ is a measure of family entertainment outside the home. $\hat{V}_1$ may be considered a measure of family "status" which is dominated by family income. Essentially, family entertainment outside the home is positively associated with family income.

10.13 a) $\hat{\rho}_1^* = .909, \quad \hat{\rho}_2^* = .636, \quad \hat{\rho}_3^* = .256, \quad \hat{\rho}_4^* = .094$

	Null hypothesis	Value of test statistic	Degrees of freedom	Conclusion at 1% level
1.	H_0: $\Sigma_{12} = \rho_{12} = 0$	309.98	20	Reject H_0
2	H_0: $\rho_1 \neq 0, \ \rho_2 = \ldots = \rho_4 = 0$	78.63	12	Reject H_0
3.	H_0: $\rho_1 \neq 0, \ \rho_2 \neq 0, \ \rho_3 = 0, \ \rho_4 = 0$	16.81	6	Do not reject H_0.

$$\begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = \begin{bmatrix} .21 & .17 & -.33 & -.26 & .30 \\ .92 & -.58 & .65 & .34 & .55 \end{bmatrix} \begin{bmatrix} Z_1^{(1)} \\ Z_2^{(1)} \\ Z_3^{(1)} \\ Z_4^{(1)} \\ Z_5^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix} = \begin{bmatrix} -.54 & -.29 & .46 & .03 \\ 1.01 & .03 & .98 & -.18 \end{bmatrix} \begin{bmatrix} Z_1^{(2)} \\ Z_2^{(2)} \\ Z_3^{(2)} \\ Z_4^{(2)} \end{bmatrix}$$

b) $\hat{U}_1$ appears to measure quality of wheat as a "contrast" between "good" aspects ($Z_1^{(1)}$, $Z_2^{(1)}$ and $Z_5^{(1)}$) and "bad" aspects ($Z_3^{(1)}$, $Z_4^{(1)}$).

$\hat{V}_1$ is harder to interpret. It appears to measure the quality of the flour as represented by $Z_1^{(2)}$, $Z_2^{(2)}$ and $Z_3^{(2)}$.

10.14

a) $\hat{\rho}_1^* = 0.7520$, $\hat{\rho}_2^* = 0.5395$. And the sample canonical variates are

Raw Canonical Coefficients for the Accounting measures of profitability

	U1	U2
HRA	-0.494697741	1.9655018549
HRE	0.2133051339	-0.794353012
HRS	0.7228315515	-0.538822808
RRA	2.7749354333	-4.38345956
RRE	-1.383659039	1.6471230054
RRS	-1.032933813	2.5190103052

Raw Canonical Coefficients for the Market measures of profitability

	V1	V2
Q	1.3930601511	-2.500804367
REV	-0.431692979	2.8298904995

U_1 is most highly correlated with RRA and HRA and also HRS and RRS. V_1 is highly correlated with both of its components. The second pair does not correlate well with their respective components.

b) Standardized Variance of the Accounting measures of profitability Explained by

	Their Own Canonical Variables			The Opposite Canonical Variables	
	Proportion	Cumulative Proportion	Canonical R-Squared	Proportion	Cumulative Proportion
1	0.5041	0.5041	0.5655	0.2851	0.2851
2	0.0905	0.5946	0.2910	0.0263	0.3114

Standardized Variance of the Market measures of profitability Explained by

	Their Own Canonical Variables			The Opposite Canonical Variables	
	Proportion	Cumulative Proportion	Canonical R-Squared	Proportion	Cumulative Proportion
1	0.8702	0.8702	0.5655	0.4921	0.4921
2	0.1298	1.0000	0.2910	0.0378	0.5299

Market measures can be well explained by its canonical variate $\hat{V}_1$. However, accounting measures cannot be well explained. In fact, from the correlation between measures and canonical variates, accounting measures on equity have weak correlation with $\hat{U}_1$.

Correlations Between the Accounting measures of profitability and Their Canonical Variables

	U1	U2
HRA	0.8110	0.2711
HRE	0.4225	0.0968
HRS	0.7184	0.5526
RRA	0.8605	-0.0089

RRE	0.5741	-0.0959
RRS	0.7761	0.3814

Correlations Between the Market measures of profitability and Their Canonical Variables

	V1	V2
Q	0.9886	0.1508
REV	0.8736	0.4866

10.15

$\hat{\rho}_1^* = 0.9129$, $\hat{\rho}_2^* = 0.0681$. And the sample canonical variates are

Raw Canonical Coefficients for the dynamic measure

	U1	U2
X1	0.0036015621	-0.006663216
X2	-0.000595735	0.0077029513

Raw Canonical Coefficients for the static measures

	V1	V2
X3	0.0013448038	0.008471035
X4	0.0018933921	-0.007828962

Standardized Variance of the dynamic measure Explained by

	Their Own Canonical Variables			The Opposite Canonical Variables	
	Proportion	Cumulative Proportion	Canonical R-Squared	Proportion	Cumulative Proportion
1	0.8840	0.8840	0.8334	0.7367	0.7367
2	0.1160	1.0000	0.0046	0.0005	0.7373

Standardized Variance of the static measures Explained by

	Their Own Canonical Variables			The Opposite Canonical Variables	
	Proportion	Cumulative Proportion	Canonical R-Squared	Proportion	Cumulative Proportion
1	0.9601	0.9601	0.8334	0.8002	0.8002
2	0.0399	1.0000	0.0046	0.0002	0.8003

Static measures can be well explained by its canonical variate $\hat{U}_1$. Also, dynamic measures can be well explained by its canonical variate $\hat{V}_1$.

10.16 From the computer output below, the first two canonical correlations are $\hat{\rho}_1^* = 0.517345$ and $\hat{\rho}_2^* = 0.125508$. The large sample tests

$$-(n-1-\frac{1}{2}(p+q-1)\,)\ln[(1-\hat{\rho*}_1^2)(1-\hat{\rho*}_1^2)\,] \;\geq\; \chi^2_{pq}(.05)$$

or

$$-(46-1-\frac{1}{2}(3+2-1)\,)\ln[(1-(.517345)^2)(1-(.125508)^2)\,] \;=\; 13.50 \;\geq\; \chi^2_6(.05) = 12.59$$

and

$$-(n-1-\frac{1}{2}(p+q-1)\,)\ln[(1-\hat{\rho*}_1^2)\,] \;\geq\; \chi^2_{(p-1)(q-1)}(.05)$$

or

$$-(46-1-\frac{1}{2}(3+2-1)\,)\ln[(1-(.125508)^2)\,] \;=\; 0.667 \;\geq\; \chi^2_2(.05) = 5.99$$

suggest that only the first pair of canonical variables are important. Even if the variables means were given, we prefer to interpret the canonical variables obtained from **S** in terms of coefficients of standardized variables.

$$\begin{aligned}\hat{U}_1 &= .4357z_1^{(1)} - .7047z_2^{(1)} + 1.0815z_3^{(1)} \\ \hat{V}_1 &= 1.020z_1^{(2)} - .1609z_2^{(2)}\end{aligned}$$

The two insulin responses dominate $\hat{U}_1$ while $\hat{V}_1$ consists primarily of the relative weight variable.

```
                    Canonical Correlation Analysis
                                Adjusted         Approx         Squared
                 Canonical      Canonical       Standard       Canonical
                Correlation    Correlation        Error       Correlation
            1     0.517345       0.517145       0.007324        0.267646
            2     0.125508       0.125158       0.009843        0.015752

                    Canonical Correlation Analysis
           Raw Canonical Coefficients for the Glucose and Insulin
                GLUCOSE         0.0131006541        0.0247524811
                INSULIN        -0.014438254        -0.009317525
                INSULRES        0.023399723        -0.008667216

           Raw Canonical Coefficients for the Weight and Fasting
                WEIGHT         8.0655750801        -0.375167814
                FASTING       -0.019159052          0.1200675138
```

Standardized Canonical Coefficients for the Glucose and Insulin

GLUCOSE	0.4357	0.8232
INSULIN	-0.7047	-0.4547
INSULRES	1.0815	-0.4006

Standardized Canonical Coefficients for the Weight and Fasting

	SECONDA1	SECONDA2
WEIGHT	1.0202	-0.0475
FASTING	-0.1609	1.0086

Correlations Between the Glucose and Insulin and Their Canonical Variables

	PRIMARY1	PRIMARY2
GLUCOSE	0.3397	0.6838
INSULIN	-0.0502	-0.4565
INSULRES	0.7551	-0.5729

Correlations Between the Weight and Fasting and Their Canonical Variables

	SECONDA1	SECONDA2
WEIGHT	0.9875	0.1576
FASTING	0.0465	0.9989

10.17 The computer output below suggests maybe two canonical pairs of variables. the canonical correlations are 0.521594, 0.375256, 0.242181 and 0.136568. $\hat{U}_1$ ignores the first smoking question and $\hat{U}_2$ ignores the third. $\hat{V}_1$ depends heavily on the difference of annoyance and tenseness.

Even the second pairs do not explain their own variances very well. $R^2_{\boldsymbol{z}^{(1)}|\hat{U}_2} = .1249$ and $R^2_{\boldsymbol{z}^{(1)}|\hat{V}_2} = 0.0879$

Canonical Correlation Analysis

	Canonical Correlation	Adjusted Canonical Correlation	Approx Standard Error	Squared Canonical Correlation
1	0.521594	0.520771	0.007280	0.272060
2	0.375256	0.374364	0.008592	0.140817
3	0.242181	0.241172	0.009414	0.058652
4	0.136568	0.135586	0.009814	0.018651

Standardized Canonical Coefficients for the Smoking

	SMOKING1	SMOKING2	SMOKING3	SMOKING4
SMOK1	-0.0430	1.0898	1.1161	-1.0092
SMOK2	1.1622	0.6988	-1.4170	0.1732
SMOK3	-1.3753	0.2081	0.0156	1.6899
SMOK4	0.8909	-1.6506	0.8325	-0.2630

Standardized Canonical Coefficients for the Psych and Physical State

	STATE1	STATE2	STATE3	STATE4
CONCEN	0.4733	-0.8141	0.4946	-0.1604
ANNOY	-0.7806	-0.4510	0.5909	-0.7193
SLEEP	0.2567	-0.6052	0.6981	0.6246
TENSE	0.6919	0.3800	-0.4190	0.4376
ALERT	-0.1451	-0.1840	-1.5191	-0.7253
IRRITAB	-0.0704	0.6255	-0.3343	0.8760
TIRED	0.3127	0.5898	0.2276	0.1861
CONTENT	0.3364	0.4869	0.8334	-0.6557

Canonical Structure

Correlations Between the Smoking and Their Canonical Variables

	SMOKING1	SMOKING2	SMOKING3	SMOKING4
SMOK1	0.4458	0.5278	0.6615	0.2917
SMOK2	0.7305	0.3822	0.1487	0.5461
SMOK3	0.2910	0.2664	0.4668	0.7915
SMOK4	0.6403	-0.0620	0.5586	0.5236

Correlations Between the Psychological and Physical State and Their Canonical Variables

	STATE1	STATE2	STATE3	STATE4
CONCEN	0.7199	-0.3579	0.0125	-0.3137
ANNOY	0.3035	0.1365	0.3906	-0.4058
SLEEP	0.5995	-0.3490	0.3709	0.2586
TENSE	0.7015	0.3305	0.0053	-0.1861
ALERT	0.7290	-0.1539	-0.1459	-0.3681
IRRITAB	0.4585	0.3342	0.1211	-0.0805
TIRED	0.6905	-0.0267	0.2544	0.0749
CONTENT	0.5323	0.4350	0.3207	-0.5601

Canonical Redundancy Analysis

Raw Variance of the Smoking Explained by

	Their Own Canonical Variables			The Opposite Canonical Variables	
	Proportion	Cumulative Proportion	Canonical R-Squared	Proportion	Cumulative Proportion
1	0.3068	0.3068	0.2721	0.0835	0.0835
2	0.1249	0.4316	0.1408	0.0176	0.1010
3	0.2474	0.6790	0.0587	0.0145	0.1155
4	0.3210	1.0000	0.0187	0.0060	0.1215

Raw Variance of the Psychological and Physical State Explained by

	Their Own Canonical Variables			The Opposite Canonical Variables	
	Proportion	Cumulative Proportion	Canonical R-Squared	Proportion	Cumulative Proportion
1	0.3705	0.3705	0.2721	0.1008	0.1008
2	0.0879	0.4583	0.1408	0.0124	0.1132
3	0.0617	0.5201	0.0587	0.0036	0.1168
4	0.1032	0.6233	0.0187	0.0019	0.1187

10.18 The canonical correlation analysis expressed in terms of standardized variables follows. The $Z^{(1)}$'s are the paper characteristic variables, the $Z^{(2)}$'s are the pulp fiber characteristic variables.

Canonical correlations:

$$\hat{\rho}_1^* = .917, \quad \hat{\rho}_2^* = .817, \quad \hat{\rho}_3^* = .265, \quad \hat{\rho}_4^* = .092$$

First three canonical variate pairs:

$$\hat{U}_1 = -1.505z_1^{(1)} - .212z_2^{(1)} + 1.998z_3^{(1)} + .676z_4^{(1)}$$
$$\hat{V}_1 = -.159z_1^{(2)} + .633z_2^{(2)} + .325z_3^{(2)} + .818z_4^{(2)}$$

$$\hat{U}_2 = -3.496z_1^{(1)} - 1.543z_2^{(1)} + 1.076z_3^{(1)} + 3.768z_4^{(1)}$$
$$\hat{V}_2 = .689z_1^{(2)} + 1.003z_2^{(2)} + .005z_3^{(2)} - 1.562z_4^{(2)}$$

$$\hat{U}_3 = -5.702z_1^{(1)} + 3.525z_2^{(1)} - 4.714z_3^{(1)} + 7.153z_4^{(1)}$$
$$\hat{V}_3 = -.513z_1^{(2)} + .077z_2^{(2)} - 1.663z_3^{(2)} - .779z_4^{(2)}$$

Additional correlations:

$$R_{U_1,z^{(1)}} = (.935\ .887\ .977\ .952),\quad R_{V_1,z^{(2)}} = (.817\ .906\ -.650\ .940)$$

$$R_{U_1,z^{(2)}} = (.749\ .831\ -.596\ .862),\quad R_{V_1,z^{(1)}} = (.858\ .814\ .896\ .873)$$

Here $H_0 : \boldsymbol{\Sigma}_{12}\,(\boldsymbol{\rho}_{12}) = \mathbf{0}$ is rejected at the 5% level and $H_0^{(1)} : \rho_1^* \neq 0, \rho_2^* = 0$ is rejected at the 5% level. $H_0^{(2)} : \rho_1^* \neq 0, \rho_2^* \neq 0, \rho_3^* = \rho_4^* = 0$ is not rejected at the 5% level. The first two canonical correlations are significantly different from 0. The last two canonical correlations are not significant.

The first canonical variable $\hat{U}_1$ explains 88% of the total standardized variance of it's set, the $Z^{(1)}$'s. The first canonical variable $\hat{V}_1$ explains 70% of the total standardized variance of it's set, the $Z^{(2)}$'s. The first canonical variates are good summary measures of their respective sets of variables. Moreover, the first canonical variates, which might be labeled a "paper characteristic index" and "a pulp fiber strength—quality index", are highly correlated. There is a strong association between an index of pulp fiber characteristics and an index of the characteristics of paper made from them.

The second canonical variable $\hat{U}_2$ appears to be a contrast between the first two variables, breaking length and elastic modulus, and the last two variables, stress at failure and burst strength. However, the only moderately large (in absolute value) correlation between the canonical variate and it's component variables is the correlation (–.428) between $\hat{U}_2$ and $Z_2^{(1)}$, elastic modulus. The remaining correlations are small. This canonical variable might be a "paper stretch" measure. The canonical variable $\hat{V}_2$ appears to be determined by all variables except $Z_3^{(2)}$, fine fiber fraction. This canonical variable might be a "fiber length/strength" measure. The second pair of canonical variates is also highly correlated.

10.19 The correlation matrix **R** and the canonical analysis for the standardized variables follows. The $Z^{(1)}$'s are the running speed events (100m, 400m, long jump), the $Z^{(2)}$'s are the arm strength events (discus, javelin, shot put).

$$\mathbf{R}_{11} = \begin{pmatrix} 1.0 & .5520 & .6386 \\ .5520 & 1.0 & .4706 \\ .6386 & .4706 & 1.0 \end{pmatrix} \qquad \mathbf{R}_{22} = \begin{pmatrix} 1.0 & .4179 & .7926 \\ .4179 & 1.0 & .4682 \\ .7926 & .4682 & 1.0 \end{pmatrix}$$

$$\mathbf{R}_{12} = \mathbf{R}_{21}' = \begin{pmatrix} .3509 & .1821 & .4752 \\ .2100 & .2116 & .2539 \\ .3998 & .3102 & .4953 \end{pmatrix}$$

Canonical correlations:

$$\hat{\rho}_1^* = .540, \quad \hat{\rho}_2^* = .212, \quad \hat{\rho}_3^* = .014$$

Canonical variables:

$$\hat{U}_1 = .540z_1^{(1)} - .120z_2^{(1)} + .633z_3^{(1)} \qquad \hat{U}_2 = 1.277z_1^{(1)} - .768z_2^{(1)} - .773z_3^{(1)}$$

$$\hat{V}_1 = -.057z_1^{(2)} + .043z_2^{(2)} + 1.024z_3^{(2)} \qquad \hat{V}_2 = -.422z_1^{(2)} - 1.0685z_2^{(2)} + .859z_3^{(2)}$$

$$\hat{U}_3 = .399z_1^{(1)} + .940z_2^{(1)} - .866z_3^{(1)}$$

$$\hat{V}_3 = 1.590z_1^{(2)} - .384z_2^{(2)} - 1.038z_3^{(2)}$$

Additional correlations:

$$R_{U_1,z^{(1)}} = (.662 \;\; .160 \;\; .732), \quad R_{V_1,z^{(2)}} = (.772 \;\; .498 \;\; .999)$$

Here $H_0 : \Sigma_{12}\,(\boldsymbol{\rho}_{12}) = \mathbf{0}$ is rejected at the 5% level and $H_0^{(1)} : \rho_1^* \neq 0, \rho_2^* = \rho_3^* = 0$ is rejected at the 5% level. $H_0^{(2)} : \rho_1^* \neq 0, \rho_2^* \neq 0, \rho_3^* = 0$ is not rejected at the 5% level. The first and second canonical correlations are significant. The third canonical correlation is not significant.

We might identify $\hat{U}_1$ as a "running speed" measure since the 100m run and the long jump receive the greatest weight in this canonical variate and also are each highly correlated with $\hat{U}_1$. We might call $\hat{V}_1$ a "strength" or "arm strength" measure since the shot put has a large coefficient in this canonical variate and the discuss, javelin and shot put are each highly correlated with $\hat{V}_1$.

Chapter 11

11.1 **(a)** The linear discriminant function given in (11-19) is

$$\hat{y} = (\overline{x}_1 - \overline{x}_2)' S_{\text{pooled}}^{-1} x = \hat{a}' x$$

where

$$S_{\text{pooled}}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

so the the linear discriminant function is

$$\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 8 \end{bmatrix} \right)' \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} x = [-2 \quad 0] x = -2x_1$$

(b)

$$\hat{m} = \frac{1}{2}(\hat{y}_1 + \hat{y}_2) = \frac{1}{2}(\hat{a}'\overline{x}_1 + \hat{a}'\overline{x}_2) = -8$$

Assign x_0' to π_1 if

$$\hat{y}_0 = [2 \quad 7] x_0 \geq \hat{m} = -8$$

and assign x_0 to π_2 otherwise.

Since $[-2 \quad 0] x_0 = -4$ is greater than $\hat{m} = -8$, assign x_0' to population π_1.

11.2 **(a)** $\pi_1 \equiv$ Riding-mower owners; $\pi_2 \equiv$ Nonowners

Here are some summary statistics for the data in Example 11.1:

$$\bar{x}_1 = \begin{bmatrix} 109.475 \\ 20.267 \end{bmatrix}, \qquad \bar{x}_2 = \begin{bmatrix} 87.400 \\ 17.633 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 352.644 & -11.818 \\ -11.818 & 4.082 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 200.705 & -2.589 \\ -2.589 & 4.464 \end{bmatrix}$$

$$S_{\text{pooled}} = \begin{bmatrix} 276.675 & -7.204 \\ -7.204 & 4.273 \end{bmatrix}, \qquad S_{\text{pooled}}^{-1} = \begin{bmatrix} .00378 & .00637 \\ .00637 & .24475 \end{bmatrix}$$

The linear classification function for the data in Example 11.1 using (11-19) is

$$\left(\begin{bmatrix} 109.475 \\ 20.267 \end{bmatrix} - \begin{bmatrix} 87.400 \\ 17.633 \end{bmatrix}\right)' \begin{bmatrix} .00378 & .00637 \\ .00637 & .24475 \end{bmatrix} x = \begin{bmatrix} .100 & .785 \end{bmatrix} x$$

where

$$\hat{m} = \frac{1}{2}(\bar{y}_1 + \bar{y}_2) = \frac{1}{2}(\hat{a}'\bar{x}_1 + \hat{a}'\bar{x}_2) = 24.719$$

(b) Assign an observation x to π_1 if

$$0.100x_1 + 0.785x_2 \geq 24.72$$

Otherwise, assign x to π_2

Here are the observations and their classifications:

Owners			Nonowners		
Observation	$a'x_0$	Classification	Observation	$a'x_0$	Classification
1	23.444	nonowner	1	25.886	owner
2	24.738	owner	2	24.608	nonowner
3	26.436	owner	3	22.982	nonowner
4	25.478	owner	4	23.334	nonowner
5	30.226	owner	5	25.216	owner
6	29.082	owner	6	21.736	nonowner
7	27.616	owner	7	21.500	nonowner
8	28.864	owner	8	24.044	nonowner
9	25.600	owner	9	20.614	nonowner
10	28.628	owner	10	21.058	nonowner
11	25.370	owner	11	19.090	nonowner
12	26.800	owner	12	20.918	nonowner

From this, we can construct the confusion matrix:

		Predicted Membership		
		π_1	π_2	Total
Actual membership	π_1	11	1	12
	π_2	2	10	12

(c) The apparent error rate is $\frac{1+2}{12+12} = 0.125$

(d) The assumptions are that the observations from π_1 and π_2 are from multivariate normal distributions with equal covariance matrices, $\Sigma_1 = \Sigma_2 = \Sigma$.

11.3 We need to show that the regions R_1 and R_2 that minimize the ECM are defined by the values x for which the following inequalities hold:

$$R_1: \frac{f_1(x)}{f_2(x)} \geq \left(\frac{c(1|2)}{c(2|1)}\right)\left(\frac{p_2}{p_1}\right)$$

$$R_2: \frac{f_1(x)}{f_2(x)} < \left(\frac{c(1|2)}{c(2|1)}\right)\left(\frac{p_2}{p_1}\right)$$

Substituting the expressions for $P(2|1)$ and $P(1|2)$ into (11-5) gives

$$\text{ECM} = c(2|1)p_1 \int_{R_2} f_1(x)dx + c(1|2)p_2 \int_{R_1} f_2(x)dx$$

And since $\Omega = R_1 \cup R_2$,

$$1 = \int_{R_1} f_1(x)dx + \int_{R_2} f_1(x)dx$$

and thus,

$$\text{ECM} = c(2|1)p_1 \left[1 - \int_{R_1} f_1(x)dx\right] + c(1|2)p_2 \int_{R_1} f_2(x)dx$$

Since both of the integrals above are over the same region, we have

$$\text{ECM} = \int_{R_1} [c(1|2)p_2 f_2(x)dx - c(2|1)p_1 f_1(x)]dx + c(2|1)p_1$$

The minimum is obtained when R_1 is chosen to be the region where the term in brackets is less than or equal to 0. So choose R_1 so that

$$c(2|1)p_1 f_1(x) \geq c(1|2)p_2 f_2(x) \quad \text{or}$$

$$\frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} \geq \left(\frac{c(1|2)}{c(2|1)}\right)\left(\frac{p_2}{p_1}\right)$$

11.4 **(a)** The minimum ECM rule is given by assigning an observation $\boldsymbol{x}$ to π_1 if

$$\frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} \geq \left(\frac{c(1|2)}{c(2|1)}\right)\left(\frac{p_2}{p_1}\right) = \left(\frac{100}{50}\right)\left(\frac{.2}{.8}\right) = .5$$

and assigning $\boldsymbol{x}$ to π_2 if

$$\frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} < \left(\frac{c(1|2)}{c(2|1)}\right)\left(\frac{p_2}{p_1}\right) = \left(\frac{100}{50}\right)\left(\frac{.2}{.8}\right) = .5$$

(b) Since $f_1(\boldsymbol{x}) = .3$ and $f_2(\boldsymbol{x}) = .5$,

$$\frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} = .6 \geq .5$$

and assign $\boldsymbol{x}$ to π_1.

11.5 $$-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_1)'\Sigma^{-1}(\bar{x}-\underset{\sim}{\mu}_1) + \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_2)'\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}_2) =$$

$$-\frac{1}{2}[\underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{x}-2\underset{\sim}{\mu}_1'\Sigma^{-1}\underset{\sim}{x}+\underset{\sim}{\mu}_1'\Sigma^{-1}\underset{\sim}{\mu}_1-\underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{x}+2\underset{\sim}{\mu}_2'\Sigma^{-1}\underset{\sim}{x}-\underset{\sim}{\mu}_2'\Sigma^{-1}\underset{\sim}{\mu}_2]$$

$$= -\frac{1}{2}[-2(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'\Sigma^{-1}\underset{\sim}{x}+\underset{\sim}{\mu}_1'\Sigma^{-1}\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2'\Sigma^{-1}\underset{\sim}{\mu}_2]$$

$$= (\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'\Sigma^{-1}\underset{\sim}{x} - \frac{1}{2}(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'\Sigma^{-1}(\underset{\sim}{\mu}_1+\underset{\sim}{\mu}_2) .$$

11.6 **a)** $E(\underset{\sim}{a}'\underset{\sim}{X}|\pi_1) - m = \underset{\sim}{a}'\underset{\sim}{\mu}_1 - m = \underset{\sim}{a}'\underset{\sim}{\mu}_1 - \frac{1}{2}\underset{\sim}{a}'(\underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_2)$

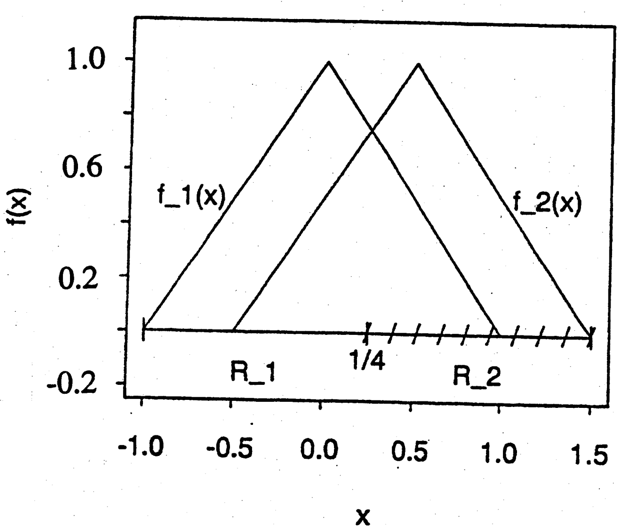
$= \frac{1}{2}\underset{\sim}{a}'(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2) = \frac{1}{2}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2) > 0$ since

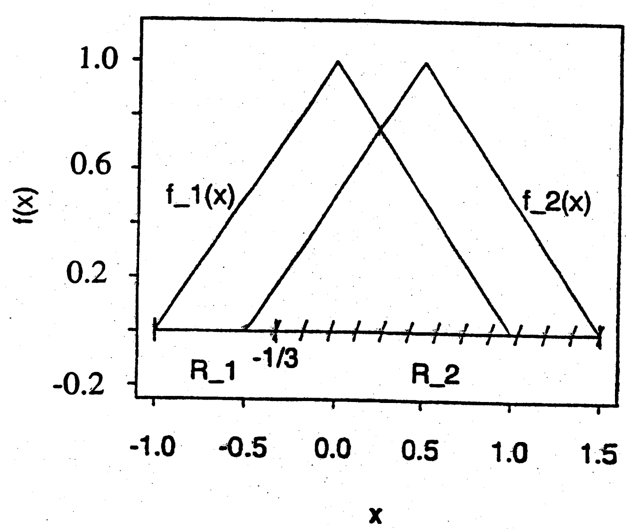
$\underset{\sim}{\Sigma}^{-1}$ is positive definite.

b) $E(\underset{\sim}{a}'\underset{\sim}{X}|\pi_2) - m = \underset{\sim}{a}'\underset{\sim}{\mu}_2 - m = \frac{1}{2}\underset{\sim}{a}'(\underset{\sim}{\mu}_2 - \underset{\sim}{\mu}_1)$

$= -\frac{1}{2}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2) < 0.$

11.7 **(a)** Here are the densities:





(b) When $p_1 = p_2$ and $c(1|2) = c(2|1)$, the classification regions are

$$R_1 : \frac{f_1(x)}{f_2(x)} \geq 1 \qquad R_2 : \frac{f_1(x)}{f_2(x)} < 1$$

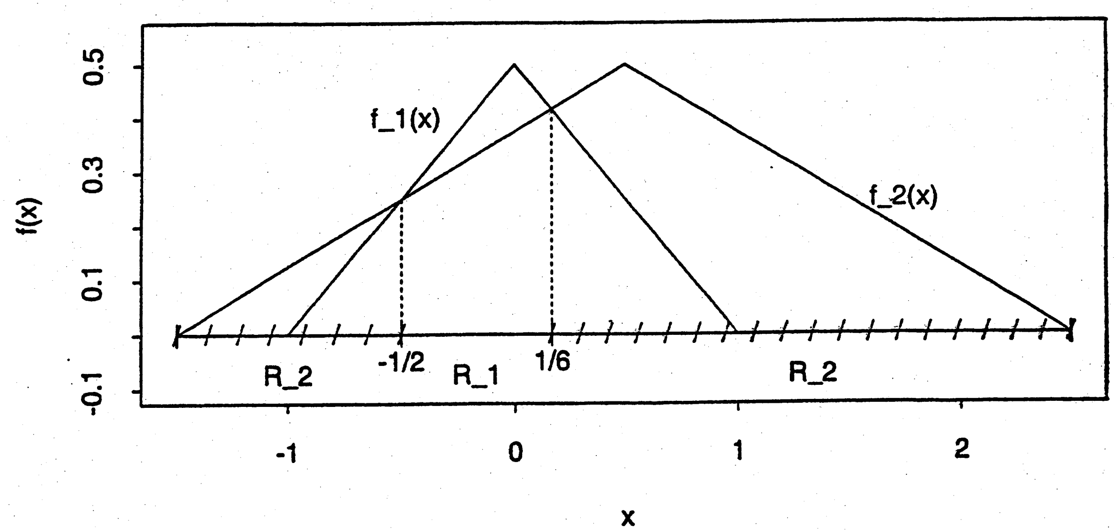
These regions are given by $R_1 : -1 \leq x \leq .25$ and $R_2 : .25 < x \leq 1.5$.

(c) When $p_1 = .2$, $p_2 = .8$, and $c(1|2) = c(2|1)$, the classification regions are

$$R_1 : \frac{f_1(x)}{f_2(x)} \geq \frac{p_2}{p_1} = .4 \qquad R_2 : \frac{f_1(x)}{f_2(x)} < .4$$

These regions are given by $R_1 : -1 \leq x \leq -1/3$ and $R_2 : -1/3 < x \leq 1.5$.

11.8 **(a)** Here are the densities:



(b) When $p_1 = p_2$ and $c(1|2) = c(2|1)$, the classification regions are

$$R_1 : \frac{f_1(x)}{f_2(x)} \geq 1 \qquad R_2 : \frac{f_1(x)}{f_2(x)} < 1$$

These regions are given by

$$R_1 : -1/2 \le x < 1/6 \quad \text{and} \quad R_2 = -1.5 \le x < -1/2, \quad 1/6 \le x \le 2.5$$

11.9

$$\frac{\mathbf{a}'\mathbf{B}_\mu\mathbf{a}}{\mathbf{a}'\Sigma\mathbf{a}} = \frac{\underset{\sim}{a}'[(\underset{\sim}{\mu}_1-\bar{\underset{\sim}{\mu}})(\underset{\sim}{\mu}_1-\bar{\underset{\sim}{\mu}})' + (\underset{\sim}{\mu}_2-\bar{\underset{\sim}{\mu}})(\underset{\sim}{\mu}_2-\bar{\underset{\sim}{\mu}})']\underset{\sim}{a}}{\underset{\sim}{a}'\Sigma\underset{\sim}{a}}$$

where $\bar{\underset{\sim}{\mu}} = \frac{1}{2}(\underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_2)$. Thus $\underset{\sim}{\mu}_1 - \bar{\underset{\sim}{\mu}} = \frac{1}{2}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)$ and $\underset{\sim}{\mu}_2 - \bar{\underset{\sim}{\mu}} = \frac{1}{2}(\underset{\sim}{\mu}_2 - \underset{\sim}{\mu}_1)$ so

$$\frac{\mathbf{a}'\mathbf{B}_\mu\mathbf{a}}{\mathbf{a}'\Sigma\mathbf{a}} = \frac{\frac{1}{2}\,\underset{\sim}{a}'(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'\underset{\sim}{a}}{\underset{\sim}{a}'\Sigma\underset{\sim}{a}} .$$

11.10 **(a)** Hotelling's two-sample T^2-statistic is

$$\begin{aligned} T^2 &= (\bar{x}_1 - \bar{x}_2)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2} \right) S_{\text{pooled}} \right]^{-1} (\bar{x}_1 - \bar{x}_2) \\ &= [-3 \quad -2] \left[\left(\frac{1}{11} + \frac{1}{12} \right) \begin{bmatrix} 7.3 & -1.1 \\ -1.1 & 4.8 \end{bmatrix} \right]^{-1} \begin{bmatrix} -3 \\ -2 \end{bmatrix} = 14.52 \end{aligned}$$

Under $H_0 : \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2$,

$$T^2 \sim \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1} F_{p, n_1 + n_2 - p - 1}$$

Since $T^2 = 14.52 \geq \frac{(11+12-2)2}{11+12-2-1} F_{2,20}(.1) = 5.44$, we reject the null hypothesis $H_0 : \mu_1 = \mu_2$ at the $\alpha = 0.1$ level of significance.

(b) Fisher's linear discriminant function is

$$\hat{y}_0 = \hat{a}' x_0 = -.49 x_1 - .53 x_2$$

(c) Here, $\hat{m} = -.25$. Assign x_0' to π_1 if $-.49x_1 - .53x_2 + .25 \geq 0$. Otherwise assign x_0' to π_2.

For $x_0' = [0 \quad 1], \hat{y}_0 = -.53(1) = -.53$ and $\hat{y}_0 - \hat{m} = -.28 < 0$. Thus, assign x_0 to π_2.

11.11 Assuming equal prior probabilities $p_1 = p_2 = \frac{1}{2}$, and equal misclassification costs $c(2|1) = c(1|2) = \$10$:

c	$P(B1\|A2)$	$P(B2\|A1)$	$P(A2 \text{ and } B1)$	$P(A1 \text{ and } B2)$	$P(\text{error})$	Expected cost
9	.006	.691	.346	.003	.349	3.49
10	.023	.500	.250	.011	.261	2.61
11	.067	.309	.154	.033	.188	1.88
12	.159	.159	.079	.079	.159	1.59
13	.309	.067	.033	.154	.188	1.88
14	.500	.023	.011	.250	.261	2.61

Using (11- 5) , the expected cost is minimized for $c = 12$ and the minimum expected cost is \$1.59.

11.12 Assuming equal prior probabilities $p_1 = p_2 = \frac{1}{2}$, and misclassification costs $c(2|1) =$ \$5 and $c(1|2) = \$10$,

expected cost $= \$5P(A1 \text{ and } B2) + \$15P(A2 \text{ and } B1)$.

c	$P(B1\|A2)$	$P(B2\|A1)$	$P(A2 \text{ and } B1)$	$P(A1 \text{ and } B2)$	$P(\text{error})$	Expected cost
9	0.006	0.691	0.346	0.003	0.349	1.78
10	0.023	0.500	0.250	0.011	0.261	1.42
11	0.067	0.309	0.154	0.033	0.188	1.27
12	0.159	0.159	0.079	0.079	0.159	1.59
13	0.309	0.067	0.033	0.154	0.188	2.48
14	0.500	0.023	0.011	0.250	0.261	3.81

Using (11- 5) , the expected cost is minimized for $c = 10.90$ and the minimum expected cost is \$1.27.

11.13 Assuming prior probabilities $P(A1) = 0.25$ and $P(A2) = 0.75$, and misclassification costs $c(2|1) = \$5$ and $c(1|2) = \$10$,

expected cost $= \$5P(B2|A1)(.25) + \$15P(B1|A2)(.75)$.

c	$P(B1\|A2)$	$P(B2\|A1)$	$P(A2 \text{ and } B1)$	$P(A1 \text{ and } B2)$	$P(\text{error})$	Expected cost
9	0.006	0.691	0.173	0.005	0.178	0.93
10	0.023	0.500	0.125	0.017	0.142	0.88
11	0.067	0.309	0.077	0.050	0.127	1.14
12	0.159	0.159	0.040	0.119	0.159	1.98
13	0.309	0.067	0.017	0.231	0.248	3.56
14	0.500	0.023	0.006	0.375	0.381	5.65

Using (11-5) , the expected cost is minimized for $c = 9.80$ and the minimum expected cost is \$0.88.

11.14 Using (11-21),

$$\hat{a}_1^* = \frac{\hat{a}}{\sqrt{\hat{a}'\hat{a}}} = \begin{bmatrix} .79 \\ -.61 \end{bmatrix} \quad \text{and} \quad \hat{m}_1^* = -0.10$$

Since $\hat{a}_1^* x_0 = -0.14 < \hat{m}_1^* = -0.1$, classify x_0 as π_2.

Using (11-22),

$$\hat{a}_2^* = \frac{\hat{a}}{\hat{a}_1} = \begin{bmatrix} 1.00 \\ -.77 \end{bmatrix} \quad \text{and} \quad \hat{m}_2^* = -0.12$$

Since $\hat{a}_2^* x_0 = -0.18 < \hat{m}_2^* = -0.12$, classify x_0 as π_2.

These results are consistent with the classification obtained for the case of equal prior probabilities in Example 11.3. These two classification results should be identical to those of Example 11.3.

11.15 $\frac{f_1(\underset{\sim}{x})}{f_2(\underset{\sim}{x})} \geq \left[\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right]$ defines the same region as

$\ln f_1(\underset{\sim}{x}) - \ln f_2(\underset{\sim}{x}) \geq \ln \left[\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right]$. For a multivariate normal distribution

$$\ln f_i(\underset{\sim}{x}) = -\frac{1}{2} \ln |\Sigma_i| - \frac{p}{2} \ln 2\pi - \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_i)'\Sigma_i^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}_i), \quad i = 1,2$$

so

$$\ln f_1(\underset{\sim}{x}) - \ln f_2(\underset{\sim}{x}) = -\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_1)'\Sigma_1^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}_1)$$

$$+ \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_2)'\Sigma_2^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}_2) - \frac{1}{2} \ln \left(\frac{|\Sigma_1|}{|\Sigma_2|}\right)$$

$$= -\frac{1}{2}[\underset{\sim}{x}'\Sigma_1^{-1}\underset{\sim}{x} - 2\underset{\sim}{\mu}_1'\Sigma_1^{-1}\underset{\sim}{x} + \underset{\sim}{\mu}_1'\Sigma_1^{-1}\underset{\sim}{\mu}_1$$

$$- \underset{\sim}{x}'\Sigma_2^{-1}\underset{\sim}{x} + 2\underset{\sim}{\mu}_2'\Sigma_2^{-1}\underset{\sim}{x} - \underset{\sim}{\mu}_2'\Sigma_2^{-1}\underset{\sim}{\mu}_2] - \frac{1}{2} \ln \left(\frac{|\Sigma_1|}{|\Sigma_2|}\right)$$

$$= -\frac{1}{2}\underset{\sim}{x}'(\Sigma_1^{-1} - \Sigma_2^{-1})\underset{\sim}{x} + (\underset{\sim}{\mu}_1'\Sigma_1^{-1} - \underset{\sim}{\mu}_2'\Sigma_2^{-1})\underset{\sim}{x} - k$$

where $k = \frac{1}{2} \ln \left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) + \frac{1}{2}(\underset{\sim}{\mu}_1'\Sigma_1^{-1}\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2'\Sigma_2^{-1}\underset{\sim}{\mu}_2)$.

11.16

$$Q = \ln\left[\frac{f_1(\underset{\sim}{x})}{f_2(\underset{\sim}{x})}\right] = -\frac{1}{2}\ln|\Sigma_1| - \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_1)'\Sigma_1^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}_1)$$

$$+\frac{1}{2}\ln|\Sigma_2| + \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu}_2)'\Sigma_1^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}_2)$$

$$= -\frac{1}{2}\underset{\sim}{x}'(\Sigma_1^{-1}-\Sigma_2^{-1})\underset{\sim}{x} + \underset{\sim}{x}'\Sigma_1^{-1}\underset{\sim}{\mu}_1 - \underset{\sim}{x}'\Sigma_2^{-1}\underset{\sim}{\mu}_2 - k$$

where $k = \frac{1}{2}\left[\ln\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) + \underset{\sim}{\mu}_1'\Sigma_1^{-1}\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2'\Sigma_2^{-1}\underset{\sim}{\mu}_2\right]$.

When $\Sigma_1 = \Sigma_2 = \Sigma$,

$$Q = \underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{\mu}_1 - \underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{\mu}_2 - \frac{1}{2}(\underset{\sim}{\mu}_1'\Sigma^{-1}\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2'\Sigma^{-1}\underset{\sim}{\mu}_2)$$

$$= \underset{\sim}{x}'\Sigma^{-1}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2) - \frac{1}{2}(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)'\Sigma^{-1}(\underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_2)$$

11.17 Assuming equal prior probabilities and misclassification costs $c(2|1) = \$10$ and $c(1|2) = \$73.89$. In the table below,

$$\begin{aligned} Q &= -\frac{1}{2}\boldsymbol{x}_0'(\Sigma_1^{-1} - \Sigma_2^{-1})\boldsymbol{x}_0 + (\boldsymbol{\mu}_1'\Sigma_1^{-1} - \boldsymbol{\mu}_2'\Sigma_2^{-1})\boldsymbol{x}_0 \\ &\quad -\frac{1}{2}\ln\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) - \frac{1}{2}(\boldsymbol{\mu}_1'\Sigma_1^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2'\Sigma_2^{-1}\boldsymbol{\mu}_2) \end{aligned}$$

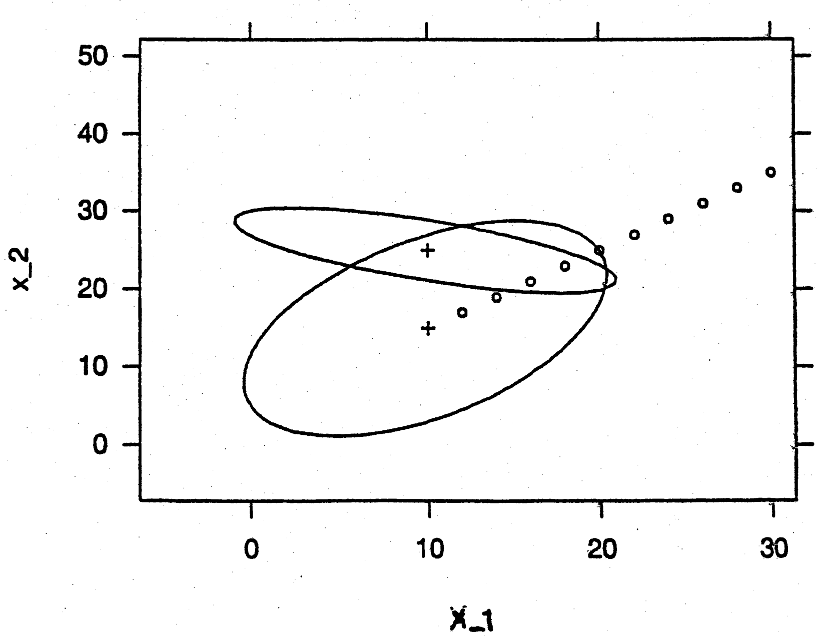
x	$P(\pi_1 \vert x)$	$P(\pi_2 \vert x)$	Q	Classification
[10, 15]'	1.00000	0	18.54	π_1
[12, 17]'	0.99991	0.00009	9.36	π_1
[14, 19]'	0.95254	0.04745	3.00	π_1
[16, 21]'	0.36731	0.63269	-0.54	π_2
[18, 23]'	0.21947	0.78053	-1.27	π_2
[20, 25]'	0.69517	0.30483	0.87	π_2
[22, 27]'	0.99678	0.00322	5.74	π_1
[24, 29]'	1.00000	0.00000	13.46	π_1
[26, 31]'	1.00000	0.00000	24.01	π_1
[28, 33]'	1.00000	0.00000	37.38	π_1
[30, 35]'	1.00000	0.00000	53.56	π_1

The quadratic discriminator was used to classify the observations in the above table. An observation x is classified as π_1 if

$$Q \geq \ln\left[\left(\frac{c(1|2)}{c(2|1)}\right)\left(\frac{p_2}{p_1}\right)\right] = \ln\left(\frac{73.89}{10}\right) = 2.0$$

Otherwise, classify x as π_2.

For (a), (b), (c) and (d), see the following plot.



11.18 The vector $\underset{\sim}{e}$ is an (unscaled) eigenvector of $\Sigma^{-1}B$ since

$$\Sigma^{-1}B\underset{\sim}{e} = \Sigma^{-1}c(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'c\Sigma^{-1}(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)$$

$$= c^2\Sigma^{-1}(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'\Sigma^{-1}(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)$$

$$= \lambda\,\Sigma^{-1}(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2) = \lambda\,\underset{\sim}{e}$$

where $\lambda = c^2\,(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)'\Sigma^{-1}(\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2)$.

11.19 (a) The calculated values agree with those in Example 11.7.

(b) Fisher's linear discriminant function is

$$\hat{y}_0 = \hat{\boldsymbol{a}}'\boldsymbol{x}_0 = -\frac{1}{3}x_1 + \frac{2}{3}x_2$$

where

$$\bar{y}_1 = \frac{17}{3};\ \bar{y}_2 = \frac{10}{3};\ \hat{m} = \frac{27}{6} = 4.5$$

Assign $\boldsymbol{x}_0'$ to π_1 if $-\frac{1}{3}x_1 + \frac{2}{3}x_2 - 4.5 \geq 0$

Otherwise assign $\boldsymbol{x}_0'$ to π_2.

π_1

Observation	$\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m}$	Classification
1	2.83	π_1
2	0.83	π_1
3	-0.17	π_2

π_2

Observation	$\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m}$	Classification
1	-1.50	π_2
2	0.50	π_1
3	-2.50	π_2

The results from this table verify the confusion matrix given in Example 11.7.

(c) This is the table of squared distances $\hat{D}_i^2(\boldsymbol{x})$ for the observations, where

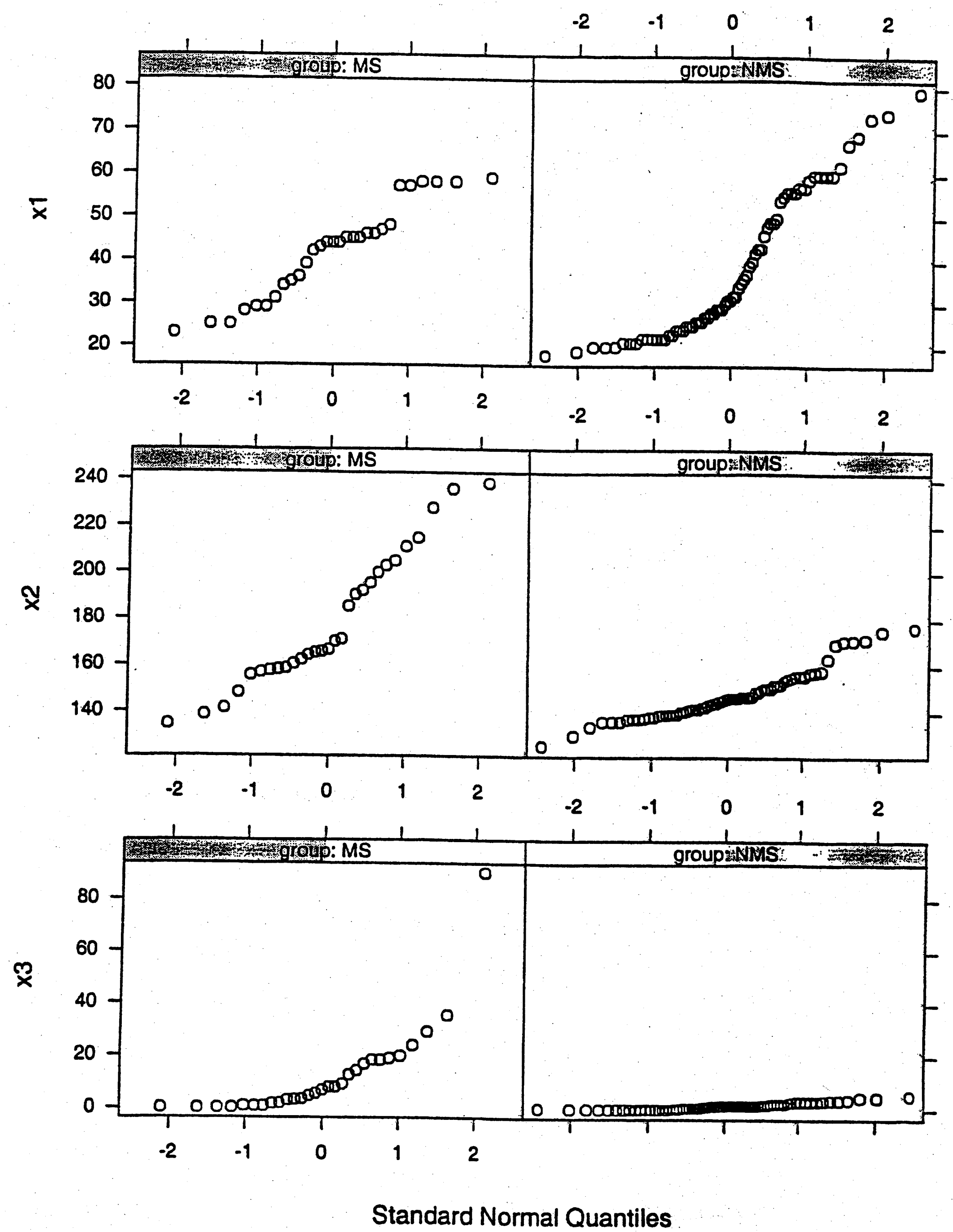
$$D_i^2(\boldsymbol{x}) = (\boldsymbol{x} - \overline{\boldsymbol{x}}_i)' \boldsymbol{S}_{\text{pooled}}^{-1} (\boldsymbol{x} - \overline{\boldsymbol{x}}_i)$$

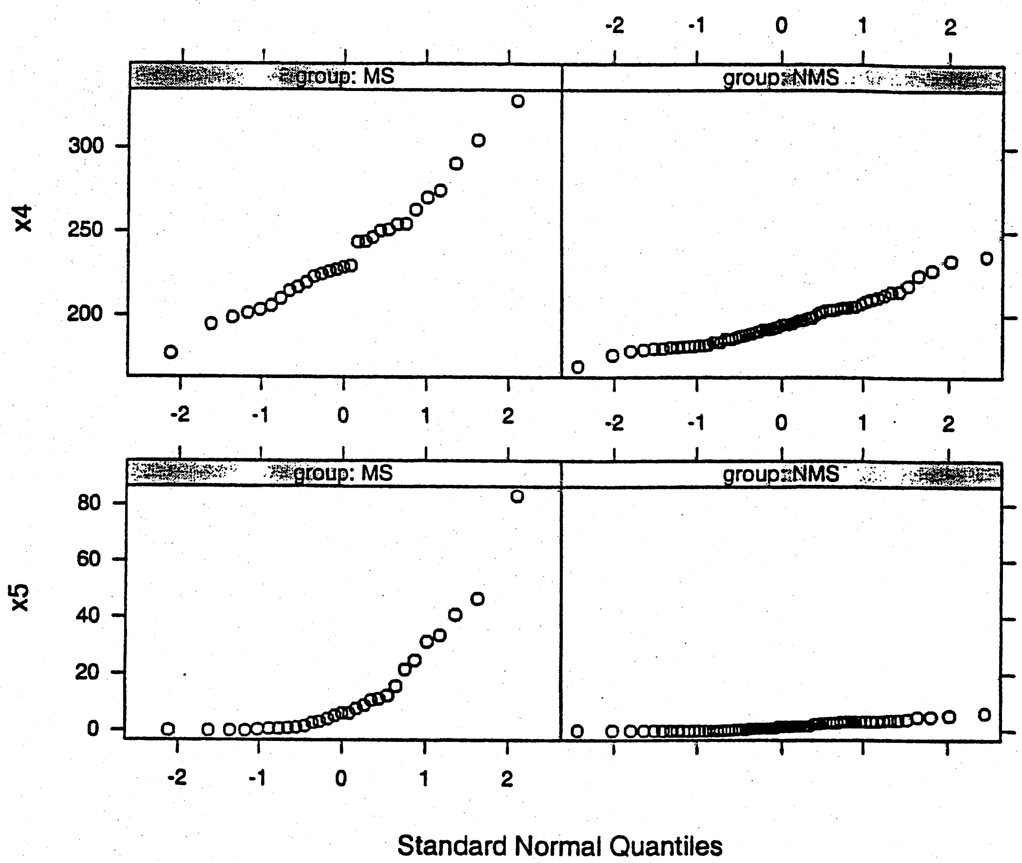
π_1				π_2			
Obs.	$\hat{D}_1^2(\boldsymbol{x})$	$\hat{D}_2^2(\boldsymbol{x})$	Classification	Obs.	$\hat{D}_1^2(\boldsymbol{x})$	$\hat{D}_2^2(\boldsymbol{x})$	Classification
1	$\frac{4}{3}$	$\frac{21}{3}$	π_1	1	$\frac{13}{3}$	$\frac{4}{3}$	π_2
2	$\frac{4}{3}$	$\frac{9}{3}$	π_1	2	$\frac{1}{3}$	$\frac{4}{3}$	π_1
3	$\frac{4}{3}$	$\frac{3}{3}$	π_2	3	$\frac{19}{3}$	$\frac{4}{3}$	π_2

The classification results are identical to those obtained in **(b)**

11.20 The result obtained from this matrix identity is identical to the result of Example 11.7.

11.23 (a) Here are the normal probability plots for each of the variables x_1, x_2, x_3, x_4, x_5


group: MS
group: NMS
x1
x2
x3
Standard Normal Quantiles



Variables x_1, x_3, and x_5 appear to be nonnormal. The transformations $\ln(x_1), \ln(x_3+1)$, and $\ln(x_5+1)$ appear to slightly improve normality.

(b) Using the original data, the linear discriminant function is:

$$\hat{y} = \hat{\boldsymbol{a}}'\boldsymbol{x} = 0.023x_1 - 0.034x_2 + 0.21x_3 - 0.08x_4 - 0.25x_5$$

where

$$\hat{m} = -23.23$$

Thus, we allocate x_0 to π_1 (NMS group) if

$$\hat{a}x_0 - \hat{m} = 0.023x_1 - 0.034x_2 + 0.21x_3 - 0.08x_4 - 0.25x_5 + 23.23 \geq 0$$

Otherwise, allocate x_0 to π_2 (MS group).

(c) Confusion matrix:

		Predicted Membership		
		π_1	π_2	Total
Actual membership	π_1	66	3	69
	π_2	7	22	29

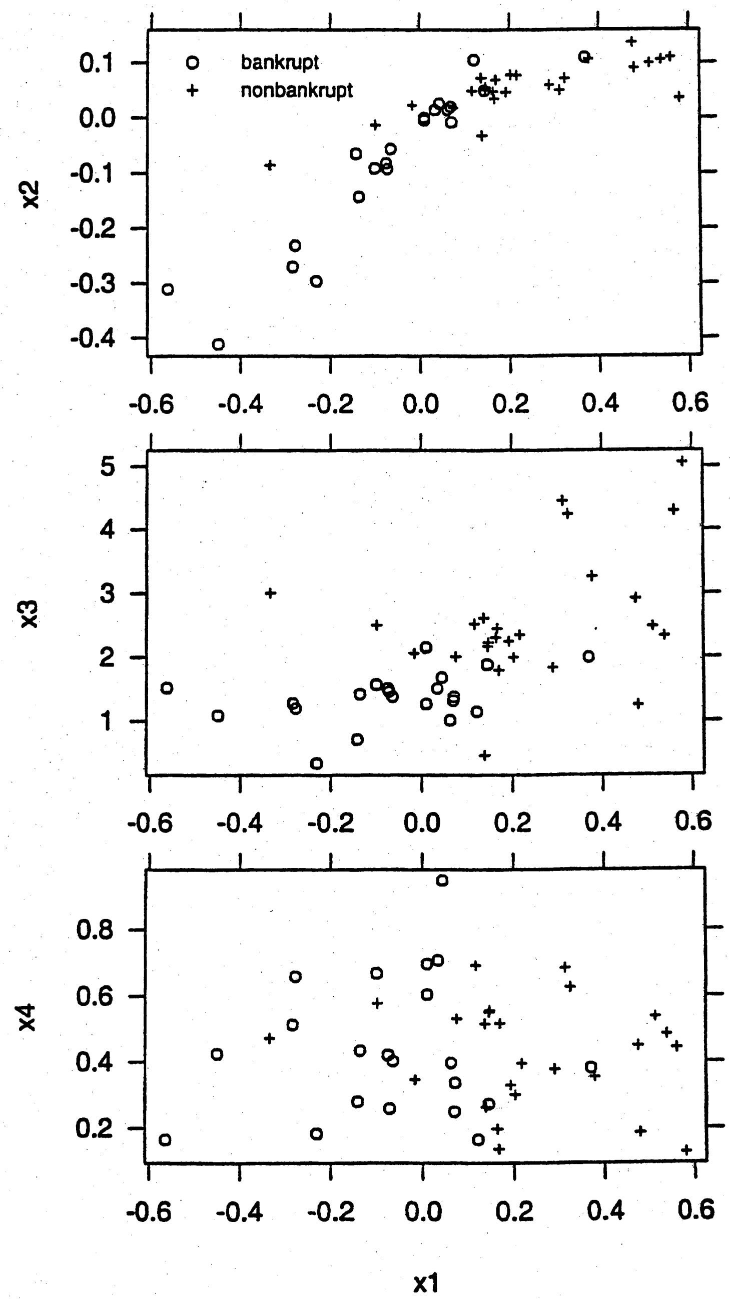
APER= $\frac{3+7}{69+29} = .102$

This is the holdout confusion matrix:

		Predicted Membership		
		π_1	π_2	Total
Actual membership	π_1	64	5	69
	π_2	8	21	29

$\hat{E}(\text{AER}) = \frac{5+8}{69+29} = .133$

11.24 (a) Here are the scatterplots for the pairs of observations (x_1, x_2), (x_1, x_3), and (x_1, x_4):



The data in the above plot appear to form fairly elliptical shapes, so bivariate normality does not seem like an unreasonable assumption.

(b) $\pi_1 \equiv$ bankrupt firms, $\pi_2 \equiv$ nonbankrupt firms. For (x_1, x_2):

$$\bar{\boldsymbol{x}}_1 = \begin{bmatrix} -0.0688 \\ -0.0819 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 0.04424 & 0.02847 \\ 0.02847 & 0.02092 \end{bmatrix}$$

$$\bar{\boldsymbol{x}}_2 = \begin{bmatrix} 0.2354 \\ 0.0551 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 0.04735 & 0.00837 \\ 0.00837 & 0.00231 \end{bmatrix}$$

(c), (d), (e) See the tables of part (g)

(f)

$$\boldsymbol{S}_{\text{pooled}} = \begin{bmatrix} 0.04594 & 0.01751 \\ 0.01751 & 0.01077 \end{bmatrix}$$

Fisher's linear discriminant function is

$$\hat{y} = \hat{\boldsymbol{a}}'\boldsymbol{x} = -4.67x_1 - 5.12x_2$$

where

$$\hat{m} = -.32$$

Thus, we allocate $\boldsymbol{x}_0$ to π_1 (Bankrupt group) if

$$\hat{\boldsymbol{a}}\boldsymbol{x}_0 - \hat{m} = -4.67x_1 - 5.12x_2 + .32 \geq 0$$

Otherwise, allocate $\boldsymbol{x}_0$ to π_2 (Nonbankrupt group).

APER= $\frac{9}{46}$ = .196.

Since S_1 and S_2 look quite different, Fisher's linear discriminant function may not be appropriate. However the performance of this linear discriminant function is as good as that of the quadratic discriminant function, based on the APER criterion.

(g) For (x_1, x_3),

$$\overline{x}_1 = \begin{bmatrix} -0.0688 \\ 1.3675 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.04424 & 0.03428 \\ 0.03428 & 0.16455 \end{bmatrix}$$

$$\overline{x}_2 = \begin{bmatrix} 0.2354 \\ 2.5939 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.04735 & 0.07543 \\ 0.07543 & 1.04596 \end{bmatrix}$$

For (x_1, x_4),

$$\overline{x}_1 = \begin{bmatrix} -0.0688 \\ 0.4368 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.04424 & 0.00431 \\ 0.00431 & 0.04441 \end{bmatrix}$$

$$\overline{x}_2 = \begin{bmatrix} 0.2354 \\ 0.4264 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.04735 & -0.00662 \\ -0.00662 & 0.02618 \end{bmatrix}$$

For the various classification rules and error rates for these variable pairs, see the following tables.

This is the table of quadratic functions for the variable pairs $(x_1, x_2), (x_1, x_3)$, and (x_1, x_5), both with $p_1 = 0.5$ and $p_1 = 0.05$. The classification rule for any of these functions is to classify a new observation into π_1 (bankrupt firms) if the quadratic function is ≥ 0, and to classify the new observation into

π_2 (nonbankrupt firms) otherwise. Notice in the table below that only the constant term changes when the prior probabilities change.

Variables	Prior	Quadratic function		
(x_1, x_2)	$p_1 = 0.5$	$-61.77x_1^2 + 35.84x_1x_2 + 407.20x_2^2 + 5.64x_1 - 30.60x_2$	$-$	0.17
	$p_1 = 0.05$		$-$	3.11
(x_1, x_3)	$p_1 = 0.5$	$-1.55x_1^2 + 3.89x_1x_3 - 3.08x_3^2 - 10.69x_1 + 7.90x_3$	$-$	3.14
	$p_1 = 0.05$		$-$	6.08
(x_1, x_4)	$p_1 = 0.5$	$-0.46x_1^2 + 7.75x_1x_4 + 8.43x_4^2 - 10.05x_1 - 8.11x_4$	$+$	2.23
	$p_1 = 0.05$		$-$	0.71

Here is a table of the APER and $\hat{E}$(AER) for the various variable pairs and prior probabilities.

	APER		$\hat{E}$(APR)	
Variables	$p_1 = 0.5$	$p_1 = 0.05$	$p_1 = 0.5$	$p_1 = 0.05$
(x_1, x_2)	0.20	0.26	0.22	0.26
(x_1, x_3)	0.11	0.37	0.13	0.39
(x_1, x_4)	0.17	0.39	0.22	0.46

For equal priors, it appears that the (x_1, x_3) variable pair is the best classifier, as it has the lowest APER. For unequal priors, $p_1 = 0.05$ and $p_2 = 0.95$, the variable pair (x_1, x_2) has the lowest APER.

(h) When using all four variables (X_1, X_2, X_3, X_4),

$$\bar{x}_1 = \begin{bmatrix} -0.0688 \\ -0.0819 \\ 1.3675 \\ 0.4368 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.04424 & 0.02847 & 0.03428 & 0.00431 \\ 0.02847 & 0.02092 & 0.02580 & 0.00362 \\ 0.03428 & 0.02580 & 0.16455 & 0.03300 \\ 0.00431 & 0.00362 & 0.03300 & 0.04441 \end{bmatrix}$$

$$\bar{x}_2 = \begin{bmatrix} 0.2354 \\ 0.0551 \\ 2.5939 \\ 0.4264 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.04735 & 0.00837 & 0.07543 & -0.00662 \\ 0.00837 & 0.00231 & 0.00873 & 0.00031 \\ 0.07543 & 0.00873 & 1.04596 & 0.03177 \\ -0.00662 & 0.00031 & 0.03177 & 0.02618 \end{bmatrix}$$

Assign a new observation x_0 to π_1 if its quadratic function given below is less than 0:

Prior	Quadratic function
$p_1 = 0.5$	$x_0' \begin{bmatrix} -49.232 & -20.657 & -2.623 & 14.050 \\ -20.657 & 526.336 & 11.412 & -52.493 \\ -2.623 & 11.412 & -3.748 & 1.4337 \\ 14.050 & -52.493 & 1.434 & 11.974 \end{bmatrix} x_0 + \begin{bmatrix} 4.91 \\ -28.42 \\ 8.65 \\ -11.80 \end{bmatrix}' x_0 - 2.69$
$p_1 = 0.05$	$- 5.64$

For $p_1 = 0.5$: APER $= \frac{3}{46} = .07$, $\quad \hat{E}(\text{AER}) = \frac{5}{46} = .11$

For $p_1 = 0.05$: APER $= \frac{9}{46} = .20$, $\quad \hat{E}(\text{AER}) = \frac{11}{46} = .24$

11.25 **(a)** Fisher's linear discriminant function is

$$\hat{y}_0 = a'x_0 - \hat{m} = -4.80x_1 - 1.48x_3 + 3.33$$

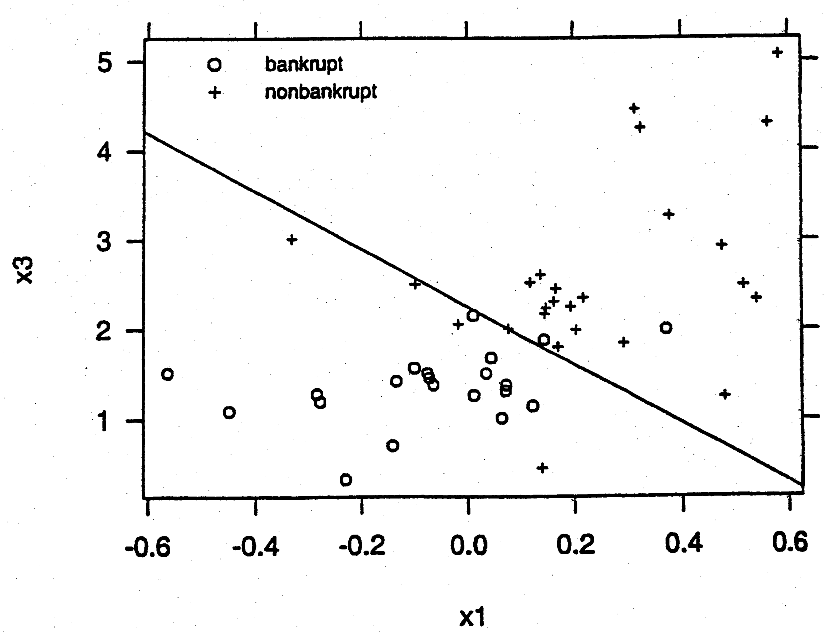
Classify x_0 to π_1 (bankrupt firms) if

$$a'x_0 - \hat{m} \geq 0$$

Otherwise classify x_0 to π_2 (nonbankrupt firms).

The APER is $\frac{2+4}{46} = .13$.

This is the scatterplot of the data in the (x_1, x_3) coordinate system, along with the discriminant line.



(b) With data point 16 for the bankrupt firms deleted, Fisher's linear discriminant function is given by

$$\hat{y}_0 = \boldsymbol{a}'\boldsymbol{x}_0 - \hat{m} = -5.93x_1 - 1.46x_3 + 3.31$$

Classify $\boldsymbol{x}_0$ to π_1 (bankrupt firms) if

$$\boldsymbol{a}'\boldsymbol{x}_0 - \hat{m} \geq 0$$

Otherwise classify $\boldsymbol{x}_0$ to π_2 (nonbankrupt firms).

The APER is $\frac{1+4}{45} = .11$.

With data point 13 for the nonbankrupt firms deleted, Fisher's linear discriminant function is given by

$$\hat{y}_0 = \boldsymbol{a}'\boldsymbol{x}_0 - \hat{m} = -4.35x_1 - 1.97x_3 + 4.36$$

Classify $\boldsymbol{x}_0$ to π_1 (bankrupt firms) if

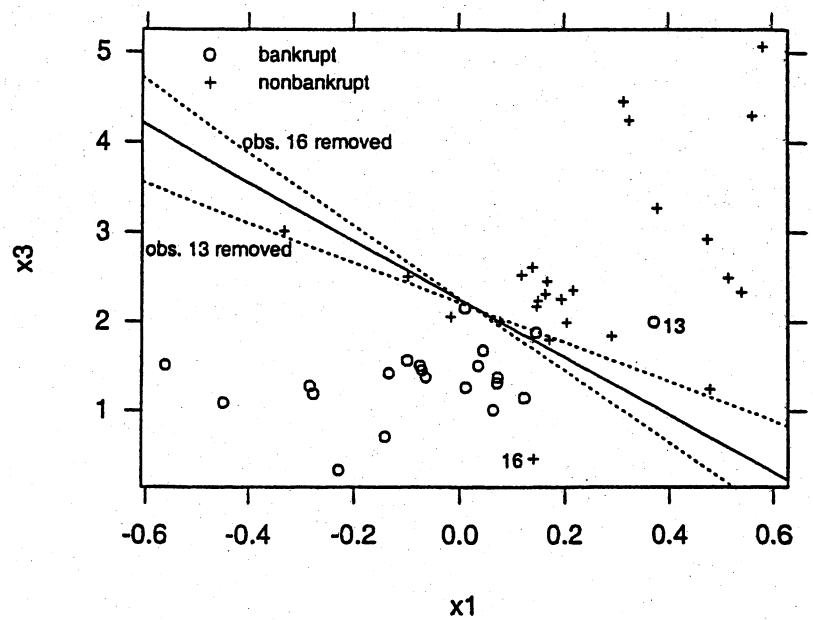
$$\boldsymbol{a}'\boldsymbol{x}_0 - \hat{m} \geq 0$$

Otherwise classify $\boldsymbol{x}_0$ to π_2 (nonbankrupt firms).

The APER is $\frac{1+3}{45} = .089$.

This is the scatterplot of the observations in the (x_1, x_3), coordinate system with the discriminant lines for the three linear discriminant functions given above. Also labelled are observation 16 for bankrupt firms and observation

13 for nonbankrupt firms.



It appears that deleting these observations has changed the line significantly.

11.26 (a) The least squares regression results for the X, Z data are:

```
Parameter Estimates

                   Parameter      Standard    T for H0:
Variable  DF        Estimate         Error   Parameter=0    Prob > |T|

INTERCEP   1       -0.081412    0.13488497        -0.604        0.5492
X3         1        0.307221    0.05956685         5.158        0.0001
```

Here are the dot diagrams of the fitted values for the bankrupt firms and for the nonbankrupt firms:

```
                .  :
     .    .  ::::::.  .  ..
+---------+---------+---------+---------+---------+-------Bankrupt
                  .  ..  .
   .        .  .   : :  ::.:.   ..   .          :  .     .
+---------+---------+---------+---------+---------+-------Nonbankrupt
0.00      0.30      0.60      0.90      1.20      1.50
```

This table summarizes the classification results using the fitted values:

OBS	GROUP	FITTED	CLASSIFICATION
13	bankrupt	0.57896	misclassify
16	bankrupt	0.53122	misclassify
31	nonbankr	0.47076	misclassify
34	nonbankr	0.06025	misclassify
38	nonbankr	0.48329	misclassify
41	nonbankr	0.30089	misclassify

The confusion matrix is:

		Predicted Membership π_1	Predicted Membership π_2	Total
Actual membership	π_1	19	2	21
	π_2	4	21	25

Thus, the APER is $\frac{2+4}{46} = .13$.

(b) The least squares regression results using all four variables X_1, X_2, X_3, X_4 are:

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
INTERCEP	1	0.208915	0.18615284	1.122	0.2683
X1	1	0.156317	0.46653100	0.335	0.7393
X2	1	1.149093	0.90606395	1.268	0.2119
X3	1	0.225972	0.07030479	3.214	0.0026
X4	1	-0.305175	0.32336357	-0.944	0.3508

Here are the dot diagrams of the fitted values for the bankrupt firms and for the nonbankrupt firms:

```
                         .:
         .  .   .. .   ... :: ... .  .  .
  -+---------+---------+---------+---------+---------+-----Bankrupt
                                  .
                                  :  . .
                   .        .::: :: :    ..  :   .   .
  -+---------+---------+---------+---------+---------+-----Nonbankrupt
-0.35      0.00      0.35      0.70      1.05      1.40
```

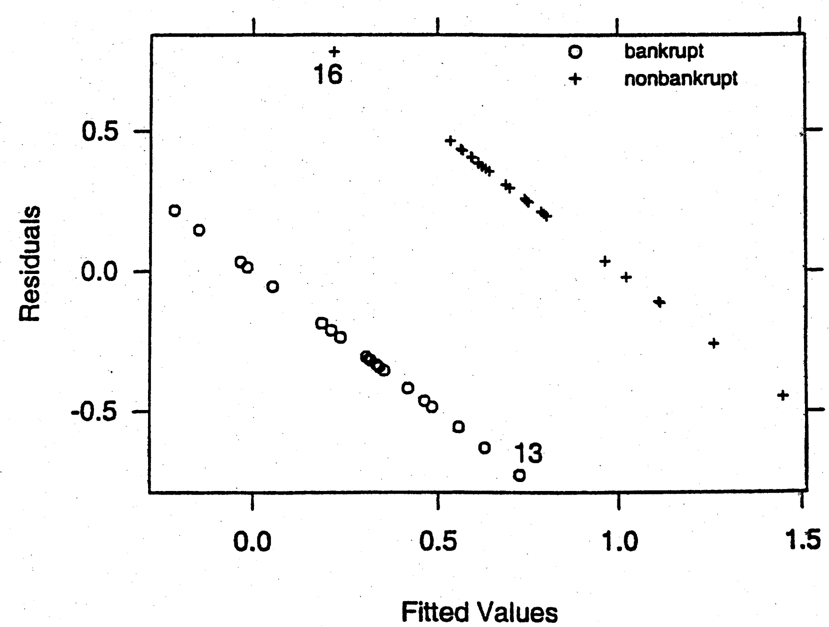
This table summarizes the classification results using the fitted values:

OBS	GROUP	FITTED	CLASSIFICATION
15	bankrupt	0.62997	misclassify
16	bankrupt	0.72676	misclassify
20	bankrupt	0.55719	misclassify
34	nonbankr	0.21845	misclassify

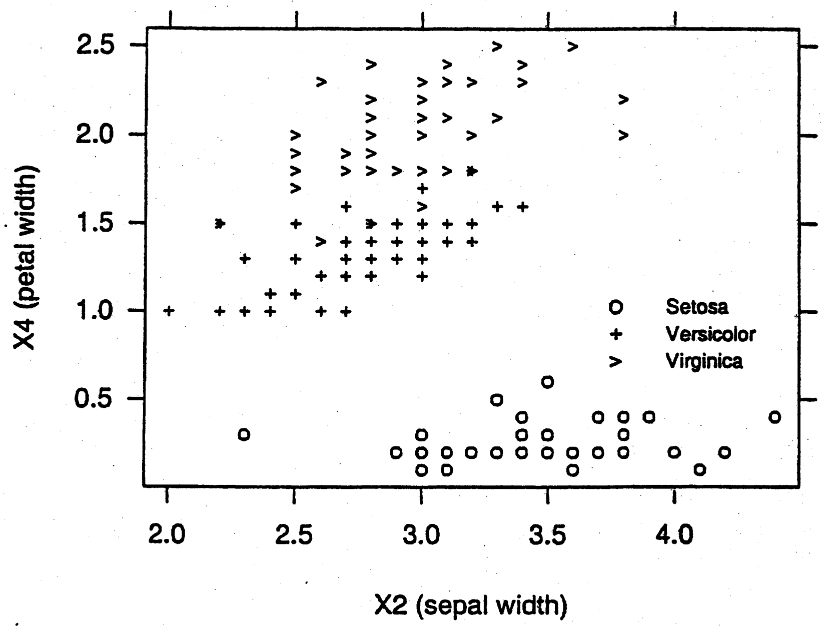
The confusion matrix is:

		Predicted Membership		
		π_1	π_2	Total
Actual membership	π_1	18	3	21
	π_2	1	24	25

Thus, the APER is $\frac{3+1}{46} = .087$. Here is a scatterplot of the residuals against the fitted values, with points 16 of the bankrupt firms and 13 of the non-bankrupt firms labelled. It appears that point 16 of the bankrupt firms is an outlier.



11.27 (a) Plot of the data in the (x_2, x_4) variable space:



The points from all three groups appear to form an elliptical shape. However, it appears that the points of π_1 *(Iris setosa)* form an ellipse with a different orientation than those of π_2 *(Iris versicolor)* and π_3 *(Iris virginica)*. This indicates that the observations from π_1 may have a different covariance matrix from the observations from π_2 and π_3.

(b) Here are the results of a test of the null hypothesis $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \boldsymbol{\mu}_3$ versus H_1 : at least one of the $\boldsymbol{\mu}_i$'s is different from the others at the $\alpha = 0.05$ level of significance:

Statistic	Value	F	Num DF	Den DF	Pr > F
Wilks' Lambda	0.02343863	199.145	8	288	0.0001

Thus, the null hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$ is rejected at the $\alpha = 0.05$ level of significance. As discussed earlier, the plots give us reason to doubt the assumption of equal covariance matrices for the three groups.

(c) $\pi_1 \equiv$ *Iris setosa*; $\pi_2 \equiv$ *Iris versicolor* $\pi_3 \equiv$ *Iris virginica*

The quadratic discriminant scores $\hat{d}_i^Q(\boldsymbol{x})$ given by (11-47) with $p_1 = p_2 = p_3 = \frac{1}{3}$ are:

population	$\hat{d}_i^Q(\boldsymbol{x}) = -\frac{1}{2}\ln\lvert \boldsymbol{S}_i\rvert - \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}}_i)'\boldsymbol{S}_i^{-1}(\boldsymbol{x} - \bar{\boldsymbol{x}}_i)$
π_1	$-3.68x_2^2 + 6.16x_2x_4 - 47.60x_4^2 + 23.71x_2 + 2.30x_4 - 37.67$
π_2	$-9.09x_2^2 + 19.57x_2x_4 - 22.87x_4^2 + 24.94x_2 + 7.63x_4 - 36.53$
π_3	$-6.76x_2^2 + 8.54x_2x_4 - 9.32x_4^2 + 22.92x_2 + 12.38x_4 - 44.04$

To classify the observation $\boldsymbol{x}_0' = [3.5 \quad 1.75]$, compute $\hat{d}_i^Q(\boldsymbol{x}_0)$ for $i = 1, 2, , 3$, and classify $\boldsymbol{x}_0$ to the population for which $\hat{d}_i^Q(\boldsymbol{x}_0)$ is the largest.

$$\hat{d}_1^Q(\boldsymbol{x}_0) = -103.77$$

$$\hat{d}_2^Q(\boldsymbol{x}_0) = 0.043$$

$$\hat{d}_3^Q(\boldsymbol{x}_0) = -1.23$$

So classify $\boldsymbol{x}_0$ to π_2 (*Iris versicolor*).

(d) The linear discriminant scores $\hat{d}_i(\boldsymbol{x})$ are:

population	$\hat{d}_i(\boldsymbol{x}) = \bar{\boldsymbol{x}}_i'\boldsymbol{S}_{\text{pooled}}\boldsymbol{x} - \frac{1}{2}\bar{\boldsymbol{x}}_i'\boldsymbol{S}_{\text{pooled}}\bar{\boldsymbol{x}}_i$	$\hat{d}_i(\boldsymbol{x}_0)$
π_1	$36.02x_2 - 22.26x_4 - 59.00$	28.12
π_2	$19.31x_2 + 16.58x_4 - 37.73$	58.86
π_3	$15.49x_2 + 36.28x_4 - 59.78$	57.92

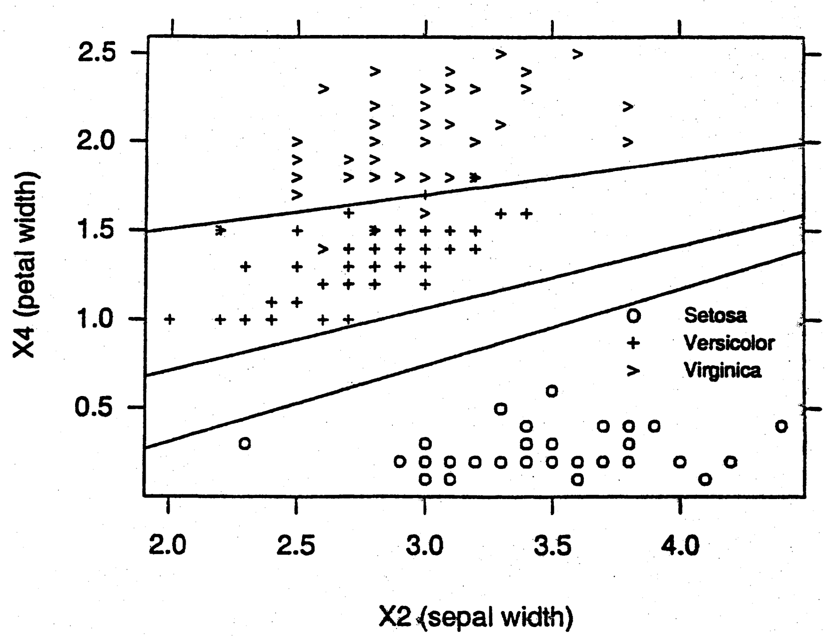
Since $\hat{d}_i(\boldsymbol{x}_0)$ is the largest for $i = 2$, we classify the new observation $\boldsymbol{x}_0' = [3.5 \quad 1.75]$ to π_1 according to (11-52). The results are the same for (c) and (d).

(e) To use rule (11-56), construct $\hat{d}_{ki}(\boldsymbol{x}) = \hat{d}_k(\boldsymbol{x}) - \hat{d}_i(\boldsymbol{x})$ for all $i \neq k$. Then classify $\boldsymbol{x}$ to π_k if $\hat{d}_{ki}(\boldsymbol{x}) \geq 0$ for all $i = 1, 2, 3$. Here is a table of $\hat{d}_{ki}(\boldsymbol{x}_0)$ for $i, k = 1, 2, 3$:

		i 1	2	3
	1	0	-30.74	-29.80
j	2	30.74	0	0.94
	1	29.80	-0.94	0

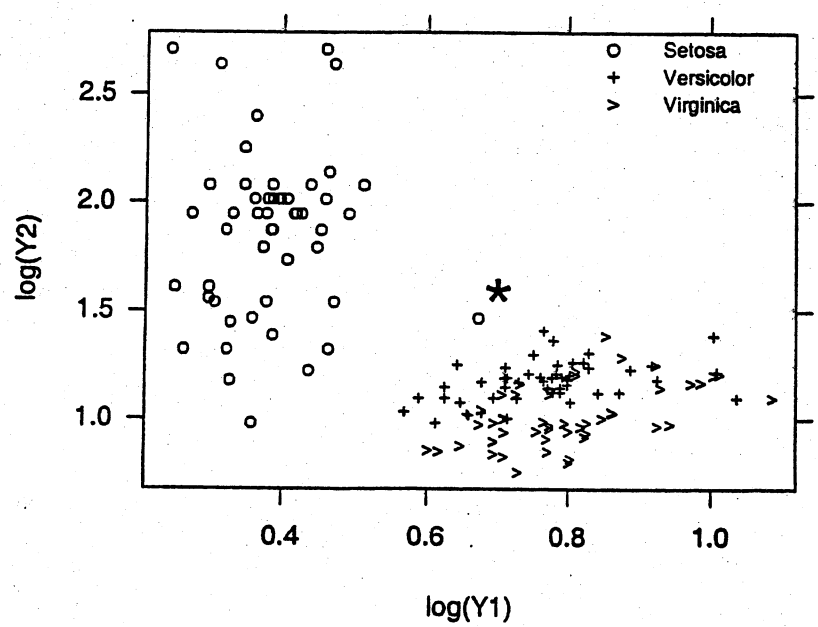
Since $\hat{d}_{ki}(\boldsymbol{x}_0) \geq 0$ for all $i \neq 2$, we allocate $\boldsymbol{x}_0$ to π_2, using (11-52)

Here is the scatterplot of the data in the (x_2, x_4) variable space, with the classification regions $\hat{R}_1, \hat{R}_2$, and $\hat{R}_3$ delineated.



(f) The APER $= \frac{1+4}{150} = .033$. $\hat{E}(\text{AER}) = \frac{4+2}{150} = .04$

11.28 (a) This is the plot of the data in the $(\log Y_1, \log Y_2)$ variable space:



The points of all three groups appear to follow roughly an ellipse-like pattern. However, the orientation of the ellipse appears to be different for the observations from π_1 (*Iris setosa*), from the observations from π_2 and π_3. In π_1, there also appears to be an outlier, labelled with a "*".

(b), (c) Assuming equal covariance matrices and ivariate normal populations, these are the linear discriminant scores $\hat{d}_i(x)$ for $i = 1, 2, 3$.

For both variables $\log Y_1$, and $\log Y_2$:

population	$\hat{d}_i(\boldsymbol{x}) = \bar{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \boldsymbol{x} - \frac{1}{2} \bar{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \bar{\boldsymbol{x}}_i$
π_1	$26.81 \log Y_1 + 28.90 \log Y_2 - 31.97$
π_2	$75.10 \log Y_1 + 13.82 \log Y_2 - 36.83$
π_3	$79.94 \log Y_1 + 10.80 \log Y_2 - 37.30$

For variable $\log Y_1$ only:

population	$\hat{d}_i(\boldsymbol{x}) = \bar{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \boldsymbol{x} - \frac{1}{2} \bar{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \bar{\boldsymbol{x}}_i$
π_1	$40.90 \log Y_1 - 7.82$
π_2	$81.84 \log Y_1 - 31.30$
π_3	$85.20 \log Y_1 - 33.93$

For variable $\log Y_2$ only:

population	$\hat{d}_i(\boldsymbol{x}) = \bar{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \boldsymbol{x} - \frac{1}{2} \bar{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \bar{\boldsymbol{x}}_i$
π_1	$30.93 \log Y_2 - 28.73$
π_2	$19.52 \log Y_2 - 11.44$
π_3	$16.87 \log Y_2 + 8.54$

Variables	APER	E(AER)
$\log Y_1, \log Y_2$	$\frac{26}{150} = .17$	$\frac{27}{150} = .18$
$\log Y_1$	$\frac{49}{150} = .33$	$\frac{49}{150} = .33$
$\log Y_2,$	$\frac{34}{150} = .23$	$\frac{34}{150} = .23$

The preceeding misclassification rates are not nearly as good as those in Example 11.12. Using "shape" is effective in discriminating π_1 (*iris versicolor*) from π_2 and π_3. It is not as good at discriminating π_2 from π_3, because of the overlap of π_1 and π_2 in both shape variables. Therefore, shape is not an effective discriminator of all three species of iris.

(d) Given the bivariate normal-like scatter and the relatively large samples, we do not expect the error rates in parts (b) and (c) to differ much.

11.29 (a) The calculated values of $\overline{x}_1, \overline{x}_1, \overline{x}_3, \overline{x}$, and S_{pooled} agree with the results for these quantities given in Example 11.11

(b)

$$W^{-1} = \begin{bmatrix} 0.348899 & 0.000193 \\ 0.000193 & .000003 \end{bmatrix}, \quad B = \begin{bmatrix} 12.50 & 1518.74 \\ 1518.74 & 258471.12 \end{bmatrix}$$

The eigenvalues and scaled eigenvectors of $W^{-1}B$ are

$$\hat{\lambda}_1 = 5.646, \quad \hat{a}_1' = \begin{bmatrix} 5.009 \\ 0.009 \end{bmatrix}$$

$$\hat{\lambda}_2 = 0.191, \quad \hat{a}_2' = \begin{bmatrix} 0.207 \\ -0.014 \end{bmatrix}$$

To classify $x_0' = [3.21 \quad 497]$, use (11-67) and compute

$$\Sigma_{j=1}^{2}[\hat{a}_j'(x - \overline{x}_i)]^2 \qquad i = 1, 2, 3$$

Allocate x_0' to π_k if

$$\Sigma_{j=1}^{2}[\hat{a}_j'(x - \overline{x}_k)]^2 \leq \Sigma_{j=1}^{2}[\hat{a}_j'(x - \overline{x}_i)]^2 \qquad \text{for all} i \neq k$$

For x_0,

k	$\Sigma_{j=1}^{2}[\hat{a}_j'(x - \overline{x}_k)]^2$
1	2.63
2	16.99
3	2.43

Thus, classify x_0 to π_3 This result agrees with the classification given in Example 11.11. Any time there are three populations with only two discriminants, classification results using Fisher's Discriminants will be identical to those using the sample distance method of Example 11.11.

11.30 **(a)** Assuming normality and equal covariance matrices for the three populations π_1, π_2, and π_3, the minimum TPM rule is given by:

Allocate $\boldsymbol{x}$ to π_k if the linear discriminant score $\hat{d}_k(\boldsymbol{x})$ = the largest of $\hat{d}_1(\boldsymbol{x}), \hat{d}_2(\boldsymbol{x}), \hat{d}_3(\boldsymbol{x})$

where $\hat{d}_i(\boldsymbol{x})$ is given in the following table for $i = 1, 2, 3$.

population	$\hat{d}_i(\boldsymbol{x}) = \overline{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \boldsymbol{x} - \frac{1}{2}\overline{\boldsymbol{x}}_i' \boldsymbol{S}_{\text{pooled}} \overline{\boldsymbol{x}}_i$
π_1	$0.70x_1 + 0.58x_2 - 13.52x_3 + 6.93x_4 + 1.44x_5 - 44.78$
π_2	$1.85x_1 + 0.32x_2 - 12.78x_3 + 8.33x_4 - 0.14x_5 - 35.20$
π_3	$2.64x_1 + 0.20x_2 - 2.16x_3 + 5.39x_4 - 0.08x_5 - 23.61$

(b) Confusion matrix is:

		Predicted Membership			
		π_1	π_2	π_3	Total
Actual membership	π_1	7	0	0	7
	π_2	1	10	0	11
	π_3	0	3	35	38

And the APER $\frac{0+1+3}{56} = .071$

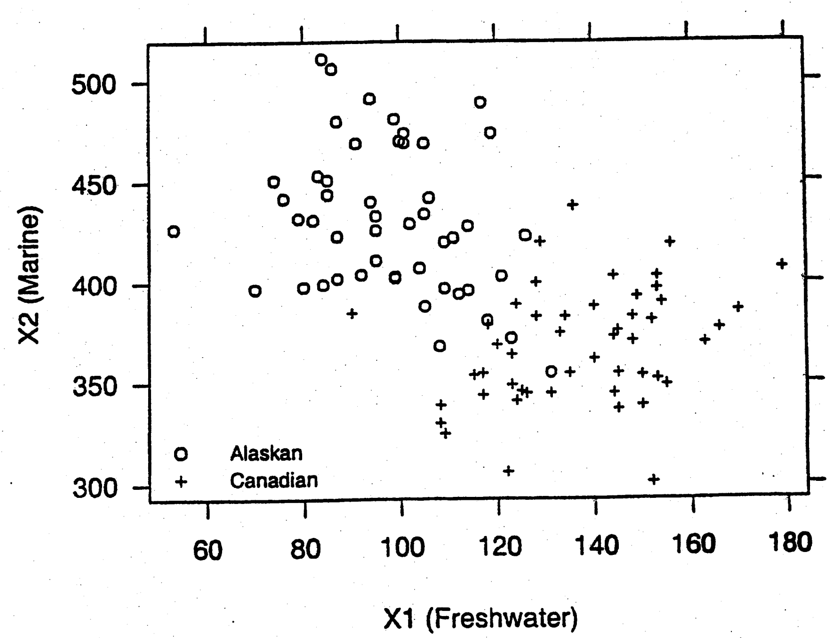
The holdout confusion matrix is:

		Predicted Membership			
		π_1	π_2	π_3	Total
Actual membership	π_1	7	0	0	7
	π_2	2	7	2	11
	π_3	0	3	35	38

$\hat{E}(\text{AER}) = \frac{2+2+3}{56} = .125$

(c) One choice of transformations, $x_1, \log x_2, \sqrt{x_3}, \log x_4, \sqrt{x_5}$ appears to improve the normality of the data but the classification rule from these data has slightly higher error rates than the rule derived from the original data. The error rates (APER, $\hat{E}$(AER)) for the linear discriminants in Example 11.14 are also slightly higher than those for the original data.

11.31 (a) The data look fairly normal.



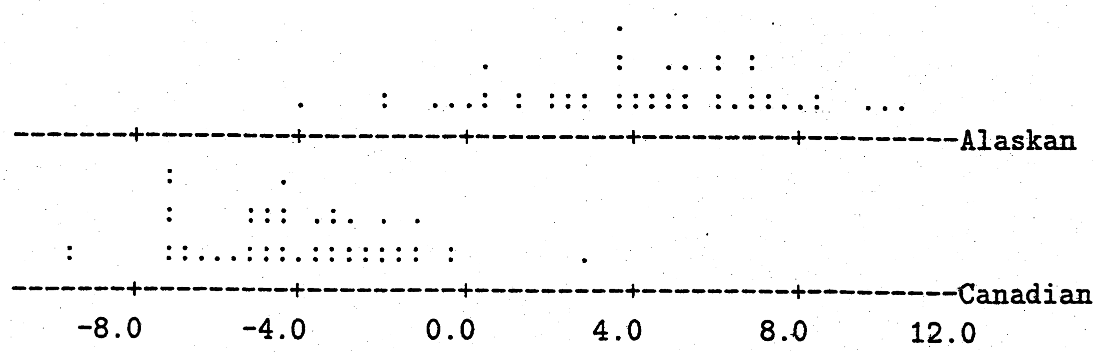
Although the covariances have different signs for the two groups, the correlations are small. Thus the assumption of bivariate normal distributions with equal covariance matrices does not seem unreasonable.

(b) The linear discriminant function is

$$\hat{\boldsymbol{a}}'\boldsymbol{x} - \hat{m} = -0.13x_1 + 0.052x_2 - 5.54$$

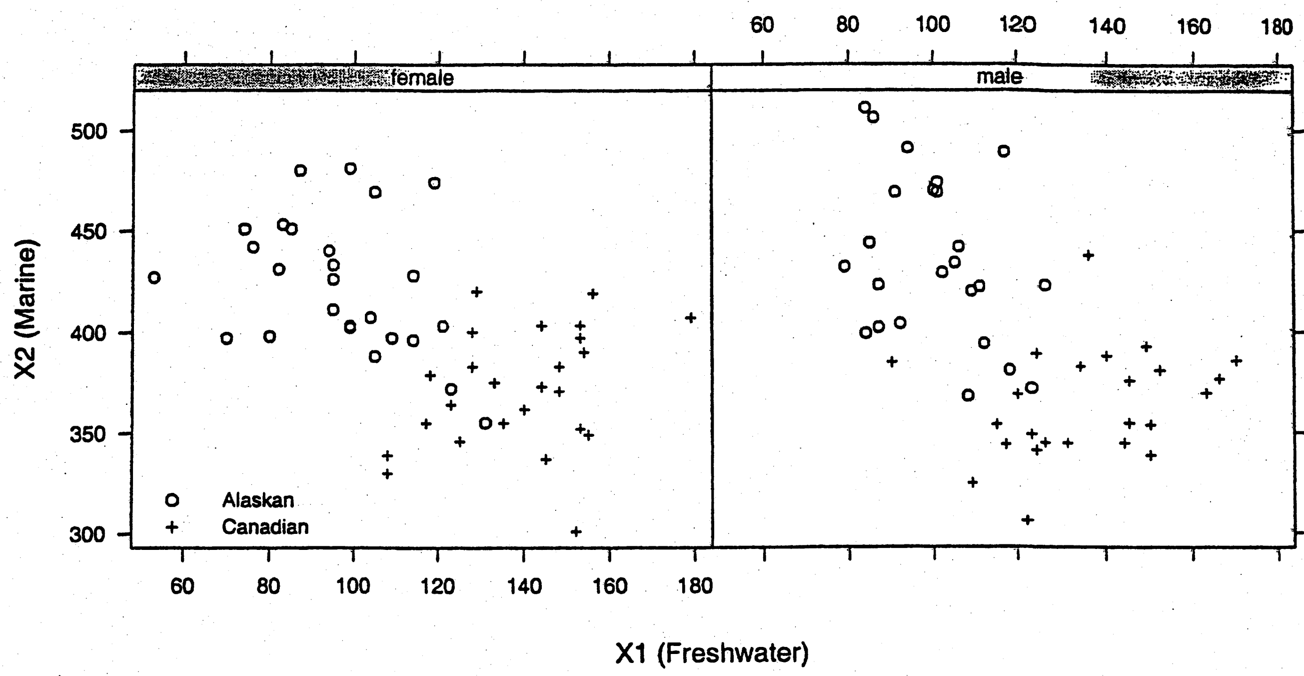
Classify an observation $\boldsymbol{x}_0$ to π_1 (Alaskan salmon) if $\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m} \geq 0$ and classify $\boldsymbol{x}_0$ to π_2 (Canadian salmon) otherwise.

Dot diagrams of the discriminant scores:



It does appear that growth ring diameters separate the two groups reasonably well, as APER= $\frac{6+1}{100} = .07$ and E(AER)= $\frac{6+1}{100} = .07$

(c) Here are the bivariate plots of the data for male and female salmon separately.



For the male salmon, these are some summary statistics

$$\overline{\boldsymbol{x}}_1 = \begin{bmatrix} 100.3333 \\ 436.1667 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 181.97101 & -197.71015 \\ -197.71015 & 1702.31884 \end{bmatrix}$$

$$\overline{\boldsymbol{x}}_2 = \begin{bmatrix} 135.2083 \\ 364.0417 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 370.17210 & 141.64312 \\ 141.64312 & 760.65036 \end{bmatrix}$$

The linear discriminant function for the male salmon only is

$$\hat{\boldsymbol{a}}'\boldsymbol{x} - \hat{m} = -0.12x_1 + 0.056x_2 - 8.12$$

Classify an observation $\boldsymbol{x}_0$ to π_1 (Alaskan salmon) if $\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m} \geq 0$ and classify $\boldsymbol{x}_0$ to π_2 (Canadian salmon) otherwise.

Using this classification rule, APER= $\frac{3+1}{48} = .08$ and E(AER)= $\frac{3+2}{48} = .10$.

For the female salmon, these are some summary statistics

$$\overline{\boldsymbol{x}}_1 = \begin{bmatrix} 96.5769 \\ 423.6539 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 336.33385 & -210.23231 \\ -210.23231 & 1097.91539 \end{bmatrix}$$

$$\overline{\boldsymbol{x}}_2 = \begin{bmatrix} 139.5385 \\ 369.0000 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 289.21846 & 120.64000 \\ 120.64000 & 1038.72000 \end{bmatrix}$$

The linear discriminant function for the female salmon only is

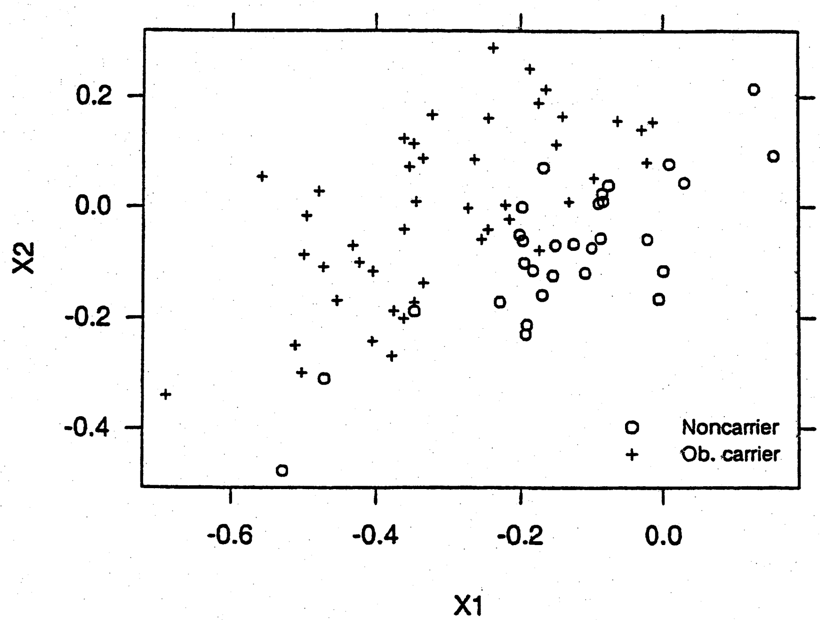
$$\hat{\boldsymbol{a}}'\boldsymbol{x} - \hat{m} = -0.13x_1 + 0.05x_2 - 2.66$$

Classify an observation $\boldsymbol{x}_0$ to π_1 (Alaskan salmon) if $\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m} \geq 0$ and classify $\boldsymbol{x}_0$ to π_2 (Canadian salmon) otherwise.

Using this classification rule, APER= $\frac{3+0}{52} = .06$ and E(AER)= $\frac{3+0}{52} = .06$.

It is unlikely that gender is a useful discriminatory variable, as splitting the data into female and male salmon did not improve the classification results greatly.

11.32 **(a)** Here is the bivariate plot of the data for the two groups:



Because the points for both groups form fairly elliptical shapes, the bivariate normal assumption appears to be a reasonable one. Normal score plots for each group confirm this.

(b) Assuming equal prior probabilities, the sample linear discriminant function is

$$\hat{\boldsymbol{a}}'\boldsymbol{x} - \hat{m} = 19.32x_1 - 17.12x_2 + 3.56$$

Classify an observation $\boldsymbol{x}_0$ to π_1 (Noncarriers) if $\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m} \geq 0$ and classify $\boldsymbol{x}_0$ to π_2 (Obligatory carriers) otherwise.

The holdout confusion matrix is

		Predicted Membership π_1	π_2	Total
Actual membership	π_1	26	4	30
	π_2	8	37	45

$\hat{E}(\text{AER}) = \frac{4+8}{75} = .16$

(c) The classification results for the 10 new cases using the discriminant function in part (b):

Case	x_1	x_2	$\hat{a}'x - \hat{m}$	Classification
1	-0.112	-0.279	6.17	π_1
2	-0.059	-0.068	3.58	π_1
3	0.064	0.012	4.59	π_1
4	-0.043	-0.052	3.62	π_1
5	-0.050	-0.098	4.27	π_1
6	-0.094	-0.113	3.68	π_1
7	-0.123	-0.143	3.63	π_1
8	-0.011	-0.037	3.98	π_1
9	-0.210	-0.090	1.04	π_1
10	-0.126	-0.019	1.45	π_1

(d) Assuming that the prior probability of obligatory carriers is $\frac{1}{4}$ and that of noncarriers is $\frac{3}{4}$, the sample linear discriminant function is

$$\hat{\boldsymbol{a}}'\boldsymbol{x} - \hat{m} = 19.32x_1 - 17.12x_2 + 4.66$$

Classify an observation $\boldsymbol{x}_0$ to π_1 (Noncarriers) if $\hat{\boldsymbol{a}}'\boldsymbol{x}_0 - \hat{m} \geq 0$ and classify $\boldsymbol{x}_0$ to π_2 (Obligatory carriers) otherwise.

The holdout confusion matrix is

		Predicted Membership π_1	π_2	Total
Actual membership	π_1	30	0	30
	π_2	18	27	45

$\hat{E}(\text{AER}) = \frac{18+0}{75} = 0.24$

The classification results for the 10 new cases using the discriminant function in part (b):

Case	x_1	x_2	$\hat{a}'x - \hat{m}$	Classification
1	-0.112	-0.279	7.27	π_1
2	-0.059	-0.068	4.68	π_1
3	0.064	0.012	5.69	π_1
4	-0.043	-0.052	4.72	π_1
5	-0.050	-0.098	5.37	π_1
6	-0.094	-0.113	4.78	π_1
7	-0.123	-0.143	4.73	π_1
8	-0.011	-0.037	5.08	π_1
9	-0.210	-0.090	2.14	π_1
10	-0.126	-0.019	2.55	π_1

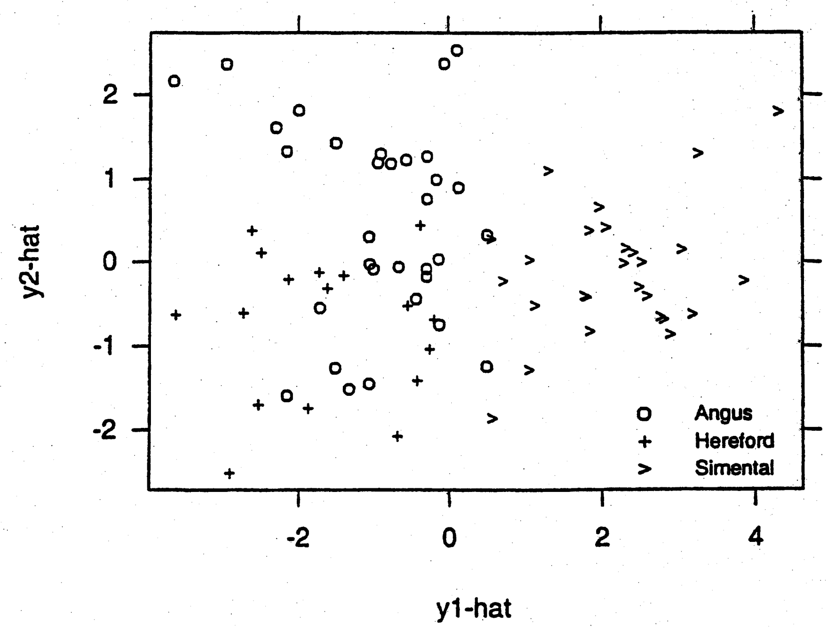
11.33 Let $x_3 \equiv$ YrHgt, $x_4 \equiv$ FtFrBody, $x_6 \equiv$ Frame, $x_7 \equiv$ BkFat, $x_8 \equiv$ SaleHt, and $x_9 \equiv$ SaleWt.

(a) For $\pi_1 \equiv$ Angus, $\pi_2 \equiv$ Hereford, and $\pi_3 \equiv$ Simental, here are Fisher's linear discriminants

$$\begin{aligned}
\hat{d}_1 &= -3737 + 126.88x_3 - 0.48x_4 + 19.08x_5 - 205.22x_6 \\
&\quad +275.84x_7 + 28.15x_8 - 0.03x_9 \\
\hat{d}_2 &= -3686 + 127.70x_3 - 0.47x_4 + 18.65x_5 - 206.18x_6 \\
&\quad +265.33x_7 + 26.80x_8 - 0.03x_9 \\
\hat{d}_1 &= -3881 + 128.08x_3 - 0.48x_4 + 19.39x_5 - 206.36x_6 \\
&\quad +245.50x_7 + 29.47x_8 - 0.03x_9
\end{aligned}$$

When $x_0' = [50, 1000, 73, 7, .17, 54, 1525]$ we obtain $\hat{d}_1 = 3596.31$, $\hat{d}_2 = 3593.32$, and $\hat{d}_3 = 3594.13$, so assign the new observation to π_2, Hereford.

This is the plot of the discriminant scores in the two-dimensional discriminant space:



(b) Here is the APER and $\hat{E}$(AER) for different subsets of the variables:

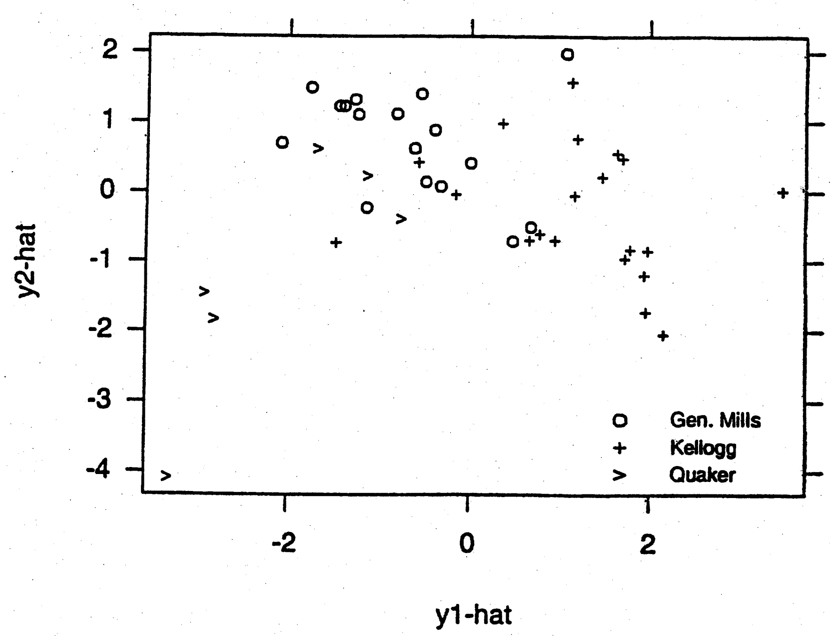
Subset	APER	$\hat{E}$(AER)
$x_3, x_4, x_5, x_6, x_7, x_8, x_9$	.13	.25
x_4, x_5, x_7, x_8	.14	.20
x_5, x_7, x_8	.21	.24
x_4, x_5	.43	.46
x_4, x_7	.36	.39
x_4, x_8	.32	.36
x_7, x_8	.22	.22
x_5, x_7	.25	.29
x_5, x_8	.28	.32

11.34 For $\pi_1 \equiv$ General Mills, $\pi_2 \equiv$ Kellogg, and $\pi_3 \equiv$ Quaker and assuming multivariate normal data with a common covariance matrix, equal costs, and equal priors, these are Fisher's linear discriminant functions:

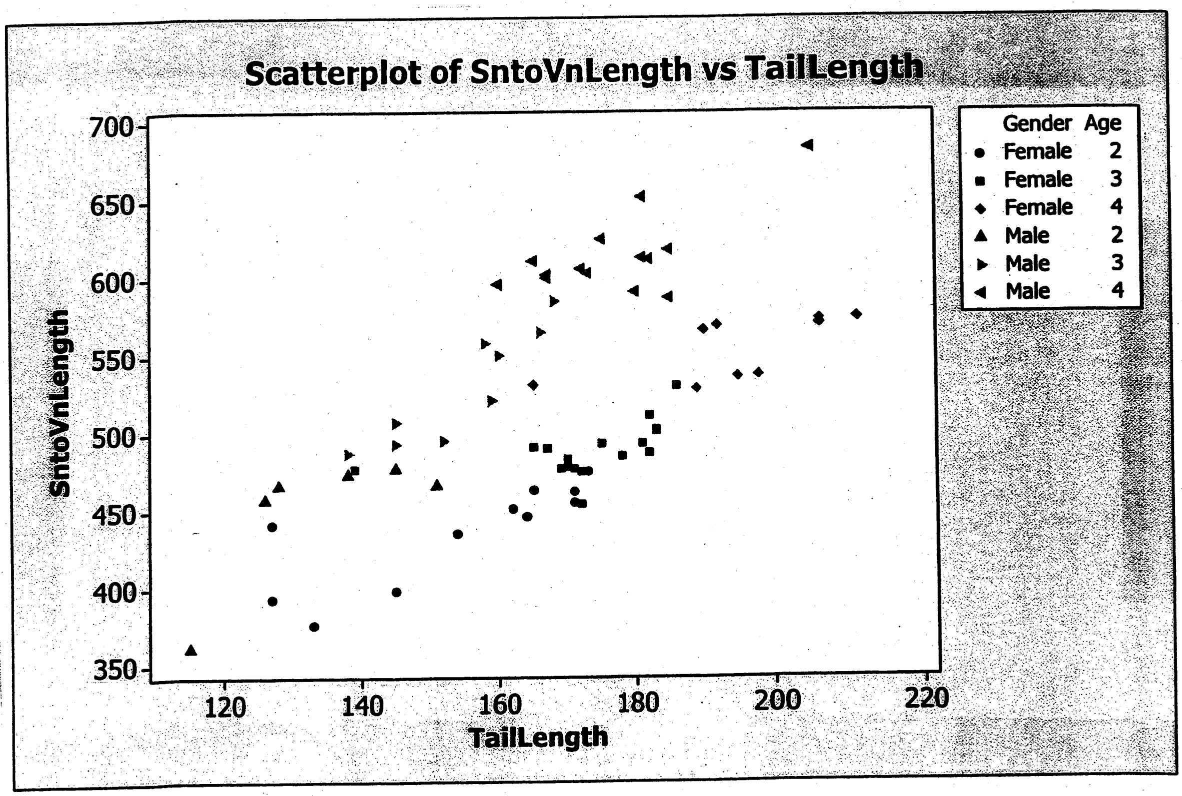
$$
\begin{aligned}
\hat{d}_1 &= .23x_3 + 3.79x_4 - 1.69x_5 - .01x_6 5.53x_7 \\
& \quad 1.90x_8 + 1.36x_9 - 0.12x_{10} - 33.14 \\
\hat{d}_2 &= .32x_3 + 4.15x_4 - 3.62x_5 - .02x_6 9.20x_7 \\
& \quad 2.07x_8 + 1.50x_9 - 0.20x_{10} - 43.07 \\
\hat{d}_3 &= .29x_3 + 2.64x_4 - 1.20x_5 - .02x_6 5.43x_7 \\
& \quad 1.22x_8 + .65x_9 - 0.13x_{10}
\end{aligned}
$$

The Kellogg cereals appear to have high protein, fiber, and carbohydrates, and low fat. However, they also have high sugar. The Quaker cereals appear to have low sugar, but also have low protein and carbohydrates.

Here is a plot of the cereal data in two-dimension discriminant space:



11.35 (a) Scatter plot of tail length and snout to vent length follows. It appears as if these variables will effectively discriminate gender but will be less successful in discriminating the age of the snakes.



(b)

```
Linear Discriminant Function for Groups

                 Female     Male
Constant        -36.429  -41.501
SntoVnLength      0.039    0.163
TailLength        0.310   -0.046


Summary of Classification with Cross-validation

                     True Group
Put into Group    Female   Male
Female                34      2
Male                   3     27
Total N               37     29
N correct             34     27
Proportion         0.919  0.931

N = 66            N Correct = 61          Proportion Correct = 0.924
```

$E(\text{AER}) = 1 - .924 = .076 \rightarrow 7.6\%$

(c) Linear Discriminant Function for Groups

	2	3	4
Constant	-112.44	-145.76	-193.14
SntoVnLength	0.33	0.38	0.45
TailLength	0.53	0.60	0.65

Summary of Classification with Cross-validation

	True Group		
Put into Group	2	3	4
2	13	2	0
3	4	21	2
4	0	3	21
Total N	17	26	23
N correct	13	21	21
Proportion	0.765	0.808	0.913

N = 66 N Correct = 55 Proportion Correct = 0.833

$E(\text{AER}) = 1 - .833 = .167 \rightarrow 16.7\%$

(d) Linear Discriminant Function for Groups

	2	3	4
Constant	-79.11	-102.76	-141.94
SntoVnLength	0.36	0.41	0.48

Summary of Classification with Cross-validation

	True Group		
Put into Group	2	3	4
2	14	1	0
3	3	21	4
4	0	4	19
Total N	17	26	23
N correct	14	21	19
Proportion	0.824	0.808	0.826

N = 66 N Correct = 54 Proportion Correct = 0.818

$E(\text{AER}) = 1 - .818 = .182 \rightarrow 18.2\%$

Using only snout to vent length to discriminate the ages of the snakes is about as effective as using both tail length and snout to vent length. Although in both cases, there is a reasonably high proportion of misclassifications.

11.36 Logistic Regression Table

Predictor	Coef	SE Coef	Z	P	Odds Ratio	95% CI Lower	95% CI Upper
Constant	3.92484	6.31500	0.62	0.534			
Freshwater	0.126051	0.0358536	3.52	0.000	1.13	1.06	1.22
Marine	-0.0485441	0.0145240	-3.34	0.001	0.95	0.93	0.98

```
Log-Likelihood = -19.394
Test that all slopes are zero: G = 99.841, DF = 2, P-Value = 0.000
```

The regression is significant (p-value = 0.000) and retaining the constant term the fitted function is

$$\ln\left(\frac{\hat{p}(z)}{1-\hat{p}(z)}\right) = 3.925 + .126(\text{freshwater growth}) - .049(\text{marine growth})$$

Consequently:

Assign z to population 2 (Canadian) if $\ln\left(\frac{\hat{p}(z)}{1-\hat{p}(z)}\right) \geq 0$; otherwise assign z to population 1 (Alaskan).

The confusion matrix follows.

		Predicted 1	Predicted 2	Total
Actual	1	46	4	50
	2	3	47	50

$\text{APER} = \frac{7}{100} = .07 \rightarrow 7\%$ This is the same APER produced by the linear classification function in Example 11.8.

Chapter 12

12.1 a) Codes:

1 →	South	Yes	Democrat	Yes	Yes
0 →	non-South	No	Republican	No	No

e.g. Reagan - Carter:

	1	0
1	1	0
0	2	2

, $\frac{a+d}{p} = 3/5 = .60$

Pair	Coefficient (a+d)/p
R-C	.6
R-F	.4
R-N	.6
R-J	0
R-K	.6
C-F	0
C-N	.2
C-J	.4
C-K	.6
F-N	.8
F-J	.6
F-K	.4
N-J	.4
N-K	.6
J-K	.4

12.1 b)

Pair	Coefficient 1	2	3	Rank Order 1	2	3
R-C	.6	.75	.429	4.5	4.5	4.5
R-F	.4	.571	.25	10	10	10
R-N	.6	.75	.429	4.5	4.5	4.5
R-J	0	0	0	14.5	14.5	14.5
R-K	.6	.75	.429	4.5	4.5	4.5
C-F	0	0	0	14.5	14.5	14.5
C-N	.2	.333	.111	13	13	13
C-J	.4	.571	.25	10	10	10
C-K	.6	.75	.429	4.5	4.5	4.5
F-N	.8	.889	.667	1	1	1
F-J	.6	.75	.429	4.5	4.5	4.5
F-K	.4	.571	.25	10	10	10
N-J	.4	.571	.25	10	10	10
N-K	.6	.75	.429	4.5	4.5	4.5
J-K	.4	.571	.25	10	10	10

12.2

Pair	Coefficient 5	6	7	Rank Order 5	6	7
R-C	.333	.5	.2	9	9	9
R-F	0	0	0	14	14	14
R-N	.333	.5	.2	9	9	9
R-J	0	0	0	14	14	14
R-K	.333	.5	.2	9	9	9
C-F	0	0	0	14	14	14
C-N	.2	.333	.111	12	12	12
C-J	.4	.571	.25	6	6	6
C-K	.5	.667	.333	3	3	3
F-N	.667	.8	.5	1	1	1
F-J	.5	.667	.333	3	3	3
F-K	.25	.4	.143	11	11	11
N-J	.4	.571	.25	6	6	6
N-K	.5	.667	.333	3	3	3
J-K	.4	.571	.25	6	6	6

12.3

x\y	1	0	Total
1	a	b	a+b
0	c	d	c+d
Total	a+c	b+d	p=a+b+c+d

$$r = \frac{\sum_{i=1}^{p}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{p}(x_i-\bar{x})^2 \sum_{i=1}^{p}(y_i-\bar{y})^2}}$$

$$\bar{x} = (a+b)/p; \quad \bar{y} = (a+c)/p$$

$$\Sigma(x_i-\bar{x})^2 = (a+b)(1-(a+b)/p)^2 + (c+d)(0-(a+b)/p)^2 = \frac{(c+d)(a+b)}{p}$$

$$\Sigma(y_i-\bar{y})^2 = (a+c)(1-(a+c)/p)^2 + (b+d)(0-(a+c)/p)^2 = \frac{(a+c)(b+d)}{p}$$

$$\Sigma(x_i-\bar{x})(y_i-\bar{y}) = \Sigma(x_iy_i-y_i\bar{x}-x_i\bar{y}+\bar{x}\bar{y})$$

$$= a - \frac{(a+c)(a+b)}{p} - \frac{(a+b)(a+c)}{p} + p\,\frac{(a+b)(a+c)}{p^2}$$

$$= \frac{a(a+b+c+d)-(a+c)(a+b)}{p} = \frac{ad-bc}{p}$$

Therefore

$$r = \frac{(ad-bc)/p}{\left[\frac{(c+d)(a+b)(a+c)(b+d)}{p^2}\right]^{\frac{1}{2}}} = \frac{ad-bc}{[(a+b)(c+d)(a+c)(b+d)]^{\frac{1}{2}}}$$

12.4 Let $c_1 = \frac{a+d}{p}$, $c_2 = \frac{2(a+d)}{2(a+d)+(b+c)}$ and $c_3 = \frac{a+d}{(a+d)+2(b+c)}$

then $c_3 = \frac{1}{1+2(c_1^{-1}-1)}$ so c_3 increases as c_1 increases

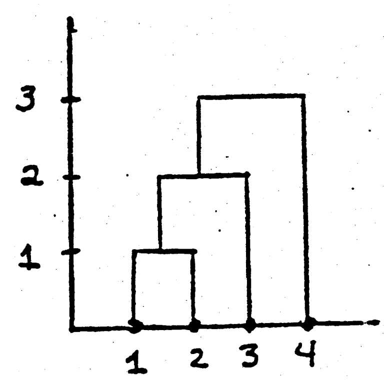
Also, $c_2 = \frac{2}{c_1^{-1}+1}$ so c_2 increases as c_1 increases

Finally, $c_2 = \frac{4}{c_3^{-1}+3}$ so c_2 increases as c_3 increases

12.5 a) Single linkage

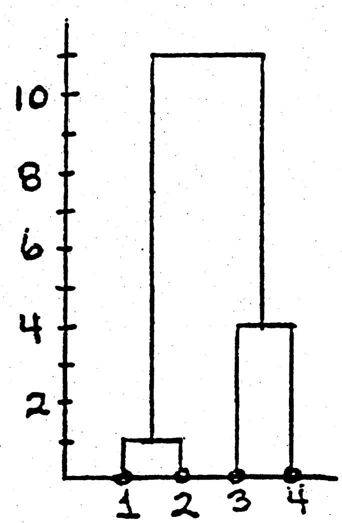
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & & & \\ \textcircled{1} & 0 & & \\ 11 & 2 & 0 & \\ 5 & 3 & 4 & 0 \end{array}\right] \end{array} \rightarrow \begin{array}{c} \\ (12) \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} (12) & 3 & 4 \end{array} \\ \left[\begin{array}{ccc} 0 & & \\ \textcircled{2} & 0 & \\ 3 & 4 & 0 \end{array}\right] \end{array} \rightarrow \begin{array}{c} \\ (123) \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} (123) & 4 \end{array} \\ \left[\begin{array}{cc} 0 & \\ 3 & 0 \end{array}\right] \end{array}$$

Dendogram



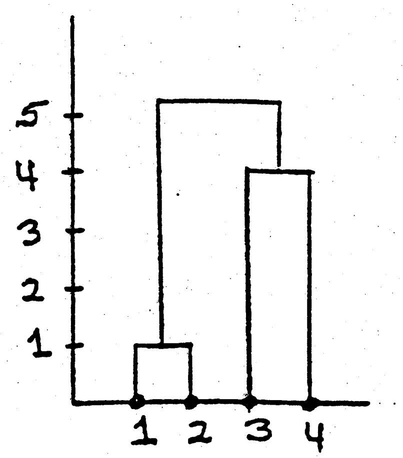
12.5 b) Complete Linkage

Dendogram

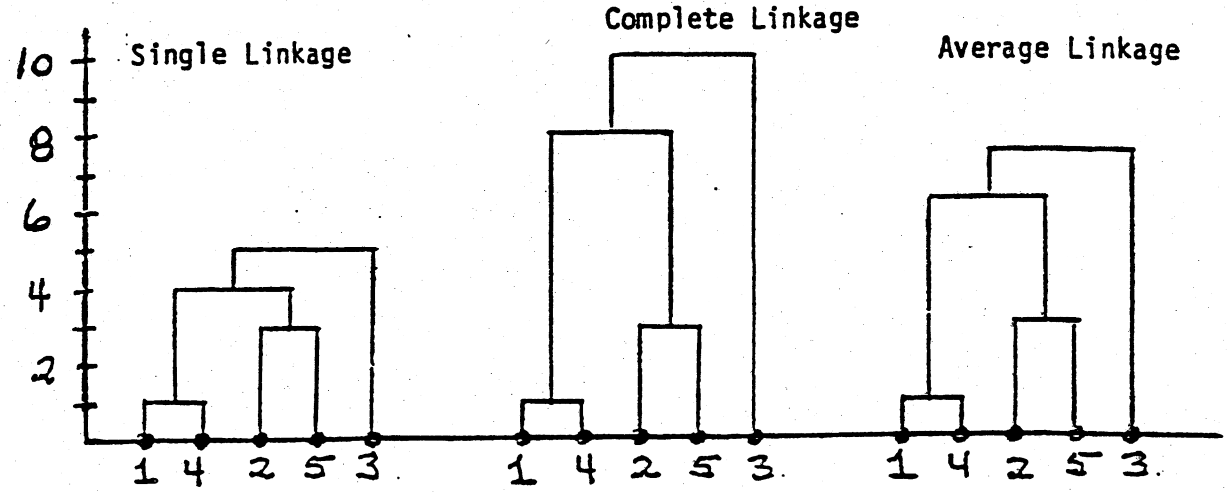


c) Average Linkage

Dendogram



12.6 Dendograms



All three methods produce the same hierarchical arrangements. Item 3 is somewhat different from the other items.

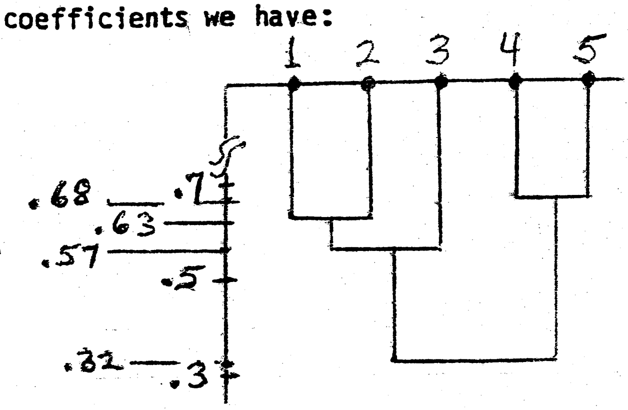
12.7 Treating correlations as similarity coefficients we have:

Single linkage

$S_{45} = .68$

$S_{(45)1} = \max(S_{41}, S_{51}) = .16$

$S_{(45)2} = .32, \quad S_{(45)3} = .18,$ and so forth.

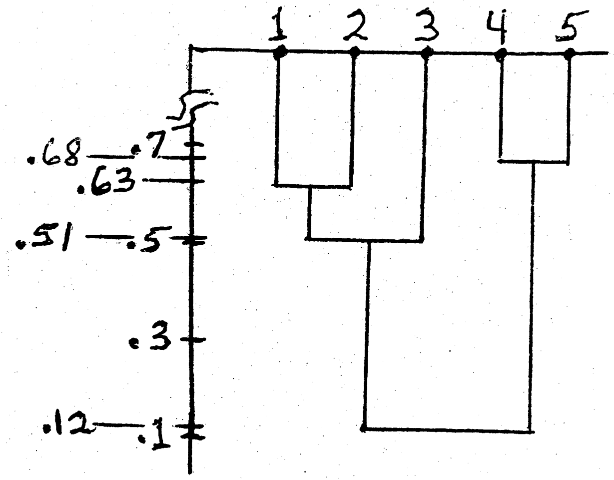


<u>Complete linkage</u>

$S_{45} = .68$

$S_{(45)1} = \min (S_{41}, S_{51}) = .12$

$S_{(45)2} = .21, \quad S_{(45)3} = .15$, and so forth.



Both methods arrive at nearly the same clustering.

12.8

	1	2	3	4	5
1	0				
2	9	0			
3	3	7	0		
4	6	5	9	0	
5	11	10	(2)	8	0

→

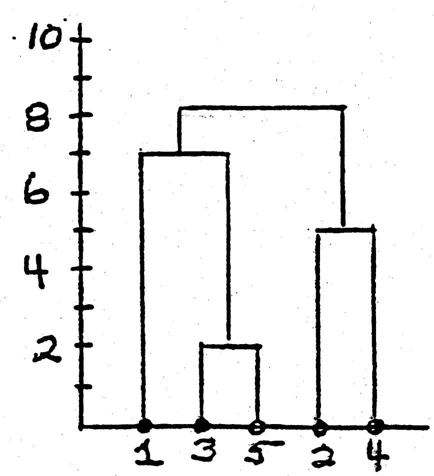
	1	2	(35)	4
1	0			
2	9	0		
(35)	7	8.5	0	
4	6	(5)	8.5	0

→

	1	(35)	(24)
1	0		
(35)	(7)	0	
(24)	7.5	8.5	0

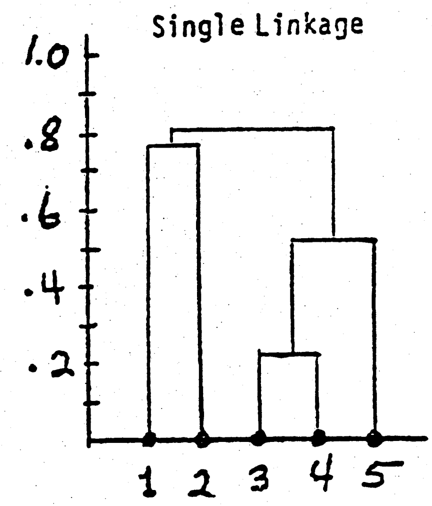
→

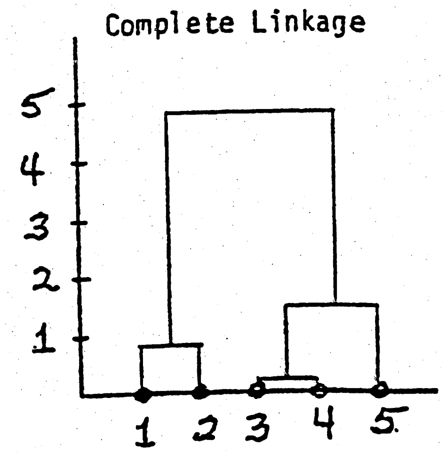
	(135)	(24)
(135)	0	
(24)	(8.167)	0

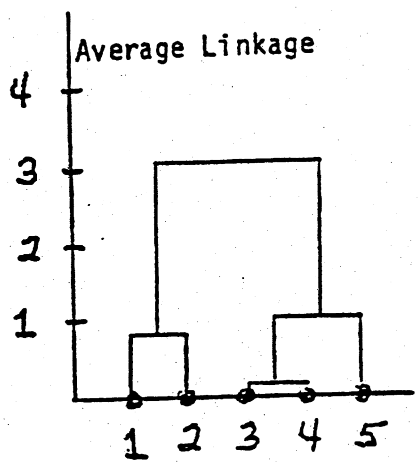


Average linkage produces results similar to single linkage.

12.9 Dendograms







Although the vertical scales are different, all three linkage methods produce the same groupings. (Note different vertical scales.)

12.10 (a) $ESS_1 = (2-2)^2 = 0$, $ESS_2 = (1-1)^2 = 0$, $ESS_3 = (5-5)^2 = 0$, and $ESS_4 = (8-8)^2 = 0$.

(b) At step 2

Clusters			Increase in ESS
{12}	{3}	{4}	.5
{13}	{2}	{4}	4.5
{14}	{2}	{3}	18.0
{1}	{23}	{4}	8.0
{1}	{24}	{3}	24.5
{1}	{2}	{34}	4.5

(c) At step 3

Clusters		Increase in ESS
{12}	{34}	5.0
{123}	{4}	8.7

Finally all four together have

$ESS = (2-4)^2 + (1-4)^2 + (5-4)^2 + (8-4)^2 = 30$

12.11 K = 2 initial clusters (AB) and (CD)

	$\bar{x}_1$	$\bar{x}_2$
(AB)	3	1
(CD)	1	1

Final clusters (AD) and (BC)

	$\bar{x}_1$	$\bar{x}_2$
(AD)	4	2.5
(BC)	0	−.5

Cluster	Squared distance to group centroids: A	B	C	D
(AD)	3.25	29.25	27.25	3.25
(BC)	45.25	3.25	3.25	11.25

12.12 K = 2 initial clusters (AC) and (BD)

	$\bar{x}_1$	$\bar{x}_2$
(AC)	3	.5
(BD)	−2	−.5

Final clusters (A) and (BCD)

	$\bar{x}_1$	$\bar{x}_2$
(A)	5	3
(BCD)	−1	−1

	Squared distance to group centroids			
Cluster	A	B	C	D
(A)	0	40	41	89
(BCD)	52	4	5	5

As expected, this result is the same as the result in Example 12.11. A graph of the items supports the (A) and (BCD) groupings.

12.13 K = 2 initial clusters (AB) and (CD)

	$\bar{x}_1$	$\bar{x}_2$
(AB)	2	2
(CD)	−1	−2

Final clusters (A) and (BCD)

	$\bar{x}_1$	$\bar{x}_2$
(A)	5	3
(BCD)	−1	−1

	Squared distance to group centroids			
Cluster	A	B	C	D
A	0 ✓	40	41	89
(BCD)	52	4✓	5✓	5✓

The final clusters (A) and (BCD) are the same as they are in Example 12.11. In this case we start with the same initial groups and the first, and only, reassignment is the same. It makes no difference if you start at the top or bottom of the list of items.

12.14. (a) The Euclidean distances between pairs of cereal brands

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
C1	0.0											
C2	116.0	0.0										
C3	15.5	121.7	0.0									
C4	6.4	117.9	10.0	0.0								
C5	103.2	61.6	100.6	102.1	0.0							
C6	72.8	44.1	78.4	74.4	54.3	0.0						
C7	86.4	71.9	82.5	84.9	22.3	52.4	0.0					
C8	15.3	121.5	1.4	10.1	100.6	78.3	82.4	0.0				
C9	46.2	72.6	54.7	48.9	75.8	32.1	65.2	54.5	0.0			
C10	54.9	123.0	68.9	59.5	134.7	87.8	122.5	68.8	65.7	0.0		
C11	81.3	154.7	94.7	85.8	169.6	121.3	157.0	94.6	94.5	47.1	0.0	
C12	42.3	114.2	31.3	38.5	81.1	75.3	60.2	31.0	59.8	92.9	121.9	0.0
C13	163.2	163.4	177.9	168.1	208.0	155.4	205.1	177.9	148.9	112.4	110.7	198.0
C14	46.7	90.8	60.4	51.5	103.8	55.4	92.9	60.3	28.5	44.3	67.5	75.9
C15	60.3	170.5	50.0	56.6	141.5	127.8	121.5	50.0	103.8	101.7	115.6	62.0
C16	46.9	90.8	60.5	51.6	103.8	55.5	92.9	60.3	28.5	44.3	67.6	75.8
C17	23.1	101.0	21.6	21.6	81.4	58.5	63.6	21.4	37.5	70.1	100.7	26.0
C18	265.7	221.1	280.0	270.6	278.9	233.9	283.3	280.0	235.6	227.7	218.6	294.5
C19	68.2	181.9	60.5	65.2	155.9	138.7	136.2	60.5	113.2	102.7	111.7	76.6
C20	116.6	71.0	113.2	115.3	19.7	69.9	32.1	113.1	89.3	150.5	183.5	90.6
C21	103.0	217.7	96.6	100.6	191.7	174.7	171.6	96.6	148.1	129.7	130.5	111.7
C22	98.6	160.1	112.6	103.4	181.3	130.5	170.2	112.6	106.9	54.1	22.5	139.2
C23	58.0	102.8	49.1	54.9	62.4	68.1	41.3	48.9	61.2	105.4	136.9	20.7
C24	68.1	181.8	60.4	65.2	155.8	138.7	136.1	60.4	113.1	102.7	111.6	76.5
C25	49.4	121.0	36.2	44.8	82.5	82.1	62.8	36.2	68.9	101.7	130.2	14.7
C26	182.8	290.3	186.0	183.8	285.6	250.4	267.2	185.9	220.2	173.8	145.7	210.7
C27	134.7	99.9	148.2	139.1	150.9	101.1	152.2	148.2	104.2	99.6	113.7	160.9
C28	16.1	128.3	14.2	14.2	111.1	85.7	92.3	13.7	59.2	63.5	86.3	39.4
C29	107.5	159.0	120.3	111.6	180.7	132.1	170.7	120.3	116.0	54.1	64.6	144.1
C30	33.5	120.1	21.2	29.2	90.7	78.8	71.2	21.0	61.7	83.1	113.7	17.2
C31	78.9	80.5	90.9	82.8	108.5	59.2	103.1	90.8	56.9	52.6	90.6	101.7
C32	32.1	122.6	43.5	36.0	120.8	83.1	105.0	43.3	51.3	50.9	60.0	65.9
C33	143.1	68.0	141.3	142.4	42.0	84.5	61.1	141.2	109.8	170.6	203.8	120.8
C34	173.0	157.7	187.8	177.9	207.5	155.6	206.8	187.8	151.8	127.0	123.8	205.9
C35	116.2	70.4	112.7	114.9	16.9	69.2	30.4	112.6	89.9	148.8	183.8	90.0
C36	114.1	230.0	111.1	112.9	210.2	186.9	190.8	111.1	158.8	129.8	122.7	131.2
C37	53.1	78.2	51.4	52.4	51.6	41.3	34.2	51.1	38.1	91.1	124.5	36.6
C38	54.2	100.4	45.8	51.0	61.8	63.5	43.5	45.8	59.0	99.2	133.6	25.8
C39	48.3	93.5	42.5	45.9	61.0	55.1	43.3	42.5	49.6	90.7	125.9	27.3
C40	40.6	140.9	51.6	44.3	139.8	100.7	123.8	51.4	70.3	44.1	46.2	79.4
C41	197.8	309.6	194.3	196.6	288.1	268.0	268.1	194.3	237.8	215.5	194.4	209.9
C42	191.1	301.3	190.3	190.8	286.6	260.4	267.3	190.2	229.3	200.8	174.0	209.7
C43	185.2	290.7	189.2	186.6	288.1	251.4	270.2	189.2	221.4	173.6	143.7	214.8

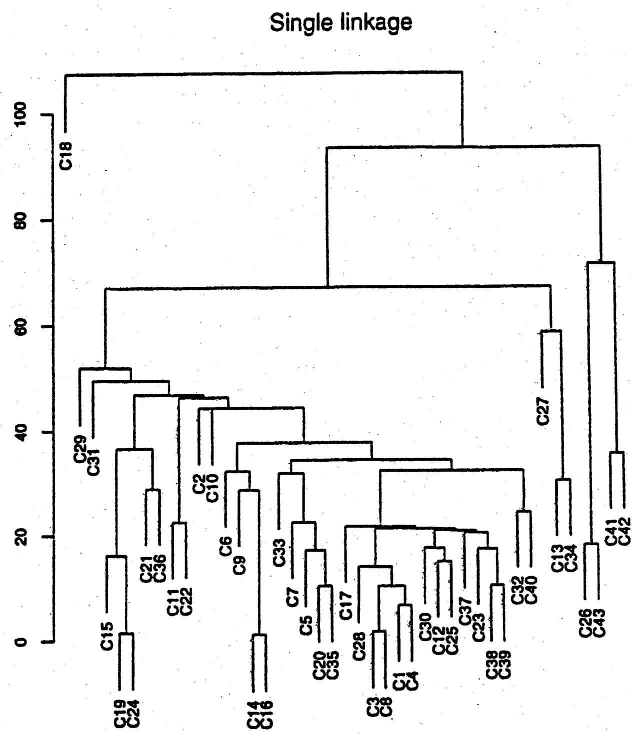
	C13	C14	C15	C16	C17	C18	C19	C20	C21	C22	C23	C24
C13	0.0											
C14	127.4	0.0										
C15	213.2	105.0	0.0									
C16	127.4	1.0	105.0	0.0								
C17	173.1	51.3	69.7	51.3	0.0							
C18	134.4	220.7	321.2	220.8	270.1	0.0						
C19	212.5	110.8	16.2	110.9	81.2	322.6	0.0					

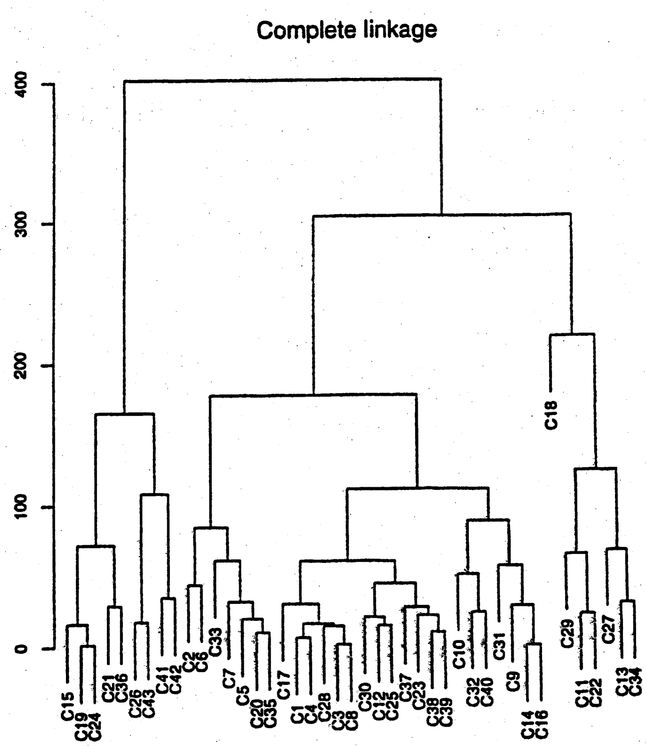
C20	223.2	117.3	151.2	117.3	94.3	288.6	166.1	0.0				
C21	234.6	142.8	50.3	142.8	117.2	347.4	36.5	201.2	0.0			
C22	91.5	79.1	135.2	79.2	116.8	204.1	131.1	195.9	148.8	0.0		
C23	204.9	83.3	81.1	83.2	36.8	295.9	96.2	70.9	130.9	153.2	0.0	
C24	212.5	110.7	16.0	110.8	81.1	322.6	1.4	166.0	36.5	131.1	96.1	0.0
C25	207.5	86.0	60.0	86.1	35.2	303.9	75.3	91.8	110.1	147.9	23.2	75.3
C26	233.8	200.3	159.3	200.3	204.2	342.0	143.8	297.3	121.0	152.7	231.2	143.8
C27	67.1	92.1	193.3	92.2	136.5	141.1	197.4	164.6	227.0	105.1	162.0	197.4
C28	174.0	59.3	46.7	59.3	30.1	278.3	55.0	123.1	89.7	104.7	58.5	54.9
C29	83.1	93.3	144.4	93.3	122.6	214.5	141.7	197.4	160.4	51.8	156.3	141.7
C30	191.2	73.8	53.3	73.8	24.6	293.2	66.8	102.5	102.5	130.6	34.3	66.8
C31	104.8	49.4	135.7	49.3	78.9	207.0	141.7	124.7	173.2	91.2	104.5	141.7
C32	150.5	37.5	75.3	37.5	47.4	248.1	78.9	132.4	108.8	79.4	80.7	78.7
C33	230.0	136.6	181.8	136.5	121.5	283.5	196.3	31.7	231.9	214.1	101.6	196.3
C34	30.1	132.2	226.4	132.3	180.7	107.3	226.8	221.3	250.8	107.0	210.8	226.8
C35	221.6	117.8	150.9	117.7	93.7	289.9	165.8	10.1	201.0	195.7	70.2	165.7
C36	226.8	148.7	71.8	148.7	131.9	341.0	56.0	221.0	28.8	139.2	151.3	56.0
C37	182.4	63.6	95.5	63.6	31.1	270.0	108.7	64.4	144.7	138.6	27.7	108.6
C38	198.4	80.8	81.3	80.9	34.1	292.4	95.7	74.1	131.3	148.9	17.1	95.7
C39	188.6	71.5	83.1	71.6	27.4	282.6	96.8	74.6	132.8	140.6	21.8	96.7
C40	146.6	52.5	71.8	52.6	62.1	252.4	70.9	152.7	96.8	66.6	96.6	70.8
C41	301.1	227.1	153.1	227.1	213.8	401.5	140.2	295.1	108.9	210.5	228.7	140.1
C42	277.2	214.8	154.9	214.9	209.3	375.5	140.8	294.9	112.9	188.1	229.2	140.7
C43	229.1	200.6	165.0	200.7	207.1	335.7	149.7	300.2	128.8	149.4	235.2	149.6

	C25	C26	C27	C28	C29	C30	C31	C32	C33	C34	C35	C36
C25	0.0											
C26	213.9	0.0										
C27	170.1	257.2	0.0									
C28	46.5	175.0	148.2	0.0								
C29	152.5	172.5	103.0	113.8	0.0							
C30	20.8	200.3	158.2	30.2	132.8	0.0						
C31	111.4	225.7	66.9	91.2	79.1	97.2	0.0					
C32	75.0	170.7	126.2	36.4	101.6	62.2	81.5	0.0				
C33	122.5	324.8	167.2	151.1	214.1	131.9	137.3	157.0	0.0			
C34	215.5	253.2	58.3	184.8	107.8	201.1	112.6	158.5	225.1	0.0		
C35	91.3	297.5	163.7	122.7	194.6	101.0	121.9	133.6	33.3	220.7	0.0	
C36	131.0	93.2	227.1	102.7	152.9	120.7	178.1	114.7	250.8	244.4	220.8	0.0
C37	43.5	234.6	136.1	60.4	141.6	44.5	81.7	72.4	91.2	186.6	63.7	161.4
C38	24.7	230.4	156.4	57.3	148.9	30.7	97.7	81.1	103.2	205.3	72.0	150.5
C39	30.1	227.7	146.5	53.6	140.6	30.7	87.9	74.5	102.6	195.3	72.6	150.5
C40	86.9	150.1	132.6	41.9	88.9	71.1	88.4	24.1	177.4	158.4	153.0	98.1
C41	209.3	98.9	305.4	186.0	236.3	204.2	264.3	190.2	325.4	315.9	297.0	96.8
C42	210.6	71.2	286.8	180.8	216.6	203.0	251.2	179.4	324.1	292.0	296.8	94.0
C43	218.2	17.7	254.4	178.3	170.3	204.2	225.5	172.3	327.1	248.4	300.5	100.9

	C37	C38	C39	C40	C41	C42	C43
C37	0.0						
C38	27.0	0.0					
C39	20.2	10.1	0.0				
C40	90.2	94.6	88.5	0.0			
C41	241.1	232.1	233.1	177.4	0.0		
C42	237.9	231.7	231.2	164.5	35.2	0.0	
C43	237.2	233.9	230.8	151.2	108.2	78.7	0

(b) Complete linkage produces results similar to single linkage.





12.15. In K-means method, we use the means of the clusters identified by average linkage as the initial cluster centers.

Final cluster centers for K = 4

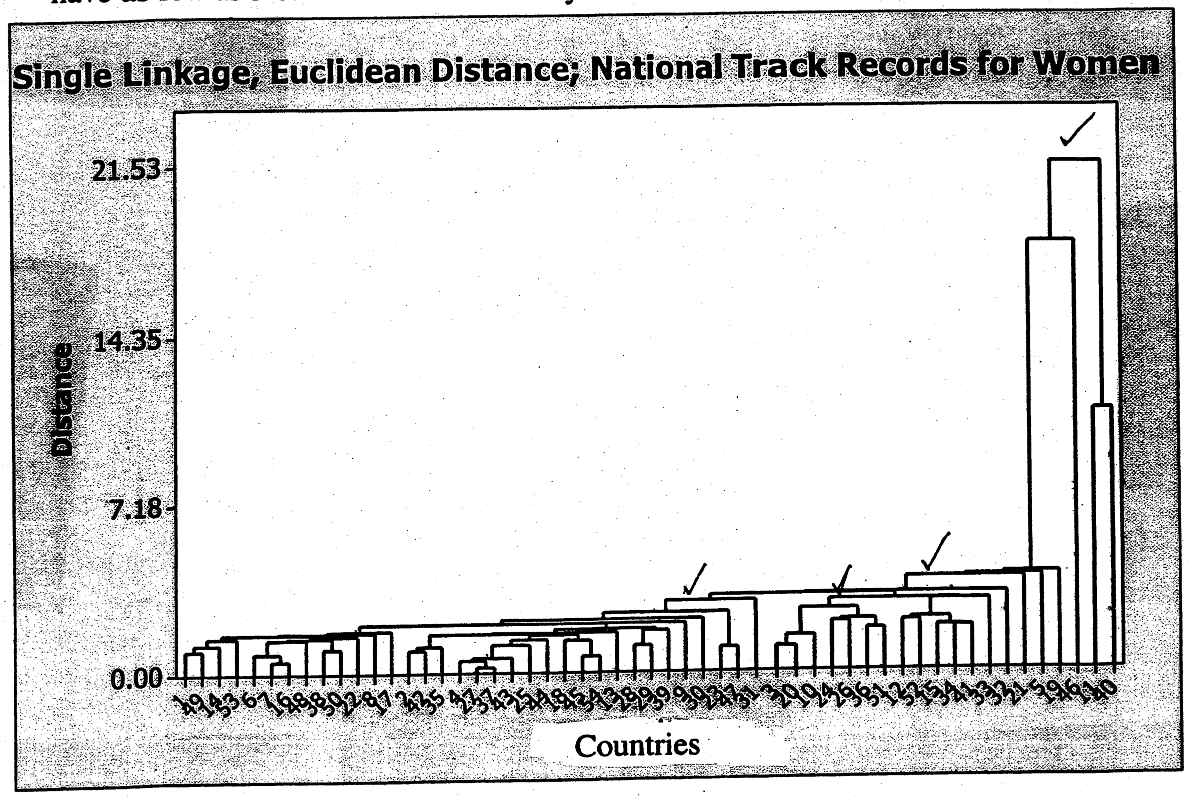
	1	2	3	4	5	6	7	8
1	110.0	2.1	0.9	215.0	0.7	15.3	7.9	50.0
2	114.4	3.1	1.7	171.1	2.8	15.0	6.6	123.9
3	86.7	2.3	0.5	26.7	1.4	10.0	5.8	55.8
4	112.5	3.2	0.8	225.0	5.8	12.5	10.8	245.0

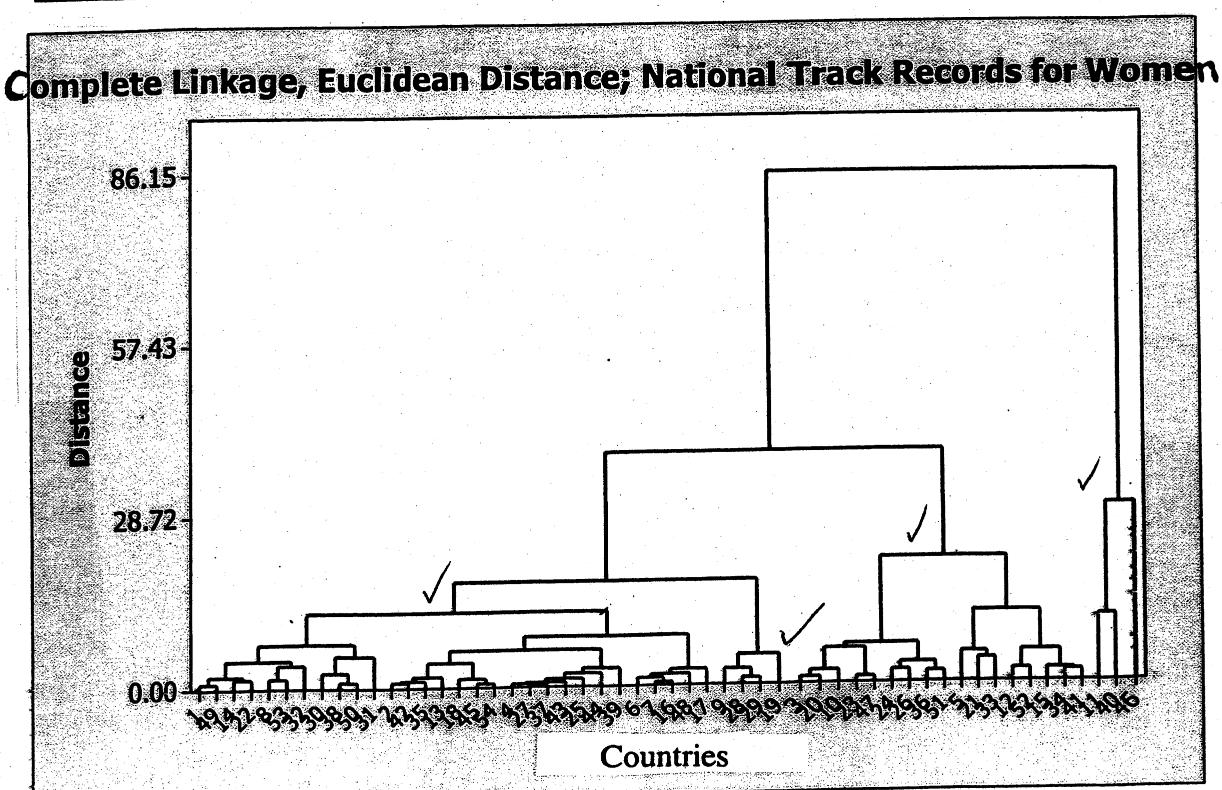
Distances between centers

	1	2	3	4
1	0.0			
2	86.1	0.0		
3	190.0	162.2	0.0	
4	195.4	132.7	275.4	0.0

	K-means						4 clusters			
	K = 2		K = 3		K = 4		Single		Complete	
1	C1	1	C1	1	C1	1	C1	1	C1	1
2	C2	1	C2	1	C2	1	C2	1	C2	1
3	C3	1	C3	1	C3	1	C3	1	C3	1
4	C4	1	C4	1	C4	1	C4	1	C4	1
5	C5	1	C5	1	C5	1	C5	1	C5	1
6	C6	1	C6	1	C6	1	C6	1	C6	1
7	C7	1	C7	1	C7	1	C7	1	C7	1
8	C8	1	C8	1	C8	1	C8	1	C8	1
9	C9	1	C9	1	C9	1	C9	1	C9	1
10	C10	1	C10	1	C12	1	C10	1	C10	1
11	C12	1	C12	1	C15	1	C11	1	C12	1
12	C14	1	C14	1	C17	1	C12	1	C14	1
13	C15	1	C15	1	C19	1	C13	1	C16	1
14	C16	1	C16	1	C20	1	C14	1	C17	1
15	C17	1	C17	1	C23	1	C15	1	C20	1
16	C19	1	C19	1	C24	1	C16	1	C23	1
17	C20	1	C20	1	C25	1	C17	1	C25	1
18	C21	1	C23	1	C28	1	C19	1	C28	1
19	C23	1	C24	1	C30	1	C20	1	C30	1
20	C24	1	C25	1	C33	1	C21	1	C31	1
21	C25	1	C28	1	C35	1	C22	1	C32	1
22	C26	1	C30	1	C37	1	C23	1	C33	1
23	C28	1	C31	1	C38	1	C24	1	C35	1
24	C30	1	C32	1	C39	1	C25	1	C37	1
25	C32	1	C33	1	C10	2	C27	1	C38	1
26	C33	1	C35	1	C11	2	C28	1	C39	1
27	C35	1	C37	1	C14	2	C29	1	C40	1
28	C36	1	C38	1	C16	2	C30	1	C11	11
29	C37	1	C39	1	C22	2	C31	1	C13	11
30	C38	1	C40	1	C29	2	C32	1	C22	11
31	C39	1	C21	2	C31	2	C33	1	C27	11
32	C40	1	C26	2	C32	2	C34	1	C29	11
33	C41	1	C36	2	C40	2	C35	1	C34	11
34	C42	1	C41	2	C21	3	C36	1	C15	15
35	C43	1	C42	2	C26	3	C37	1	C19	15
36	C11	2	C43	2	C36	3	C38	1	C21	15
37	C13	2	C11	3	C41	3	C39	1	C24	15
38	C18	2	C13	3	C42	3	C40	1	C26	15
39	C22	2	C18	3	C43	3	C18	18	C36	15
40	C27	2	C22	3	C13	4	C26	26	C41	15
41	C29	2	C27	3	C18	4	C43	26	C42	15
42	C31	2	C29	3	C27	4	C41	41	C43	15
43	C34	2	C34	3	C34	4	C42	41	C18	18

12.16 (a), (b) Dendrograms for single linkage and complete linkage follow. The dendrograms are similar; as examples, in both procedures, countries 11, 40 and 46 form a group at a relatively high level of distance, and countries 4, 27, 37, 43, 25 and 44 form a group at a relatively small distance. The clusters are more apparent in the complete linkage dendrogram and, depending on the distance level, might have as few as 3 or 4 clusters or as many as 6 or 7 clusters.





(c) The results for K = 4 and K = 6 clusters are displayed below. The results seem reasonable and are consistent with the results for the linkage procedures. Depending on use, K = 4 may be an adequate number of clusters.

Data Display

Country	ClustMemK=6	ClustMemK=4
1	6	2
2	2	4
3	1	2
4	4	4
5	3	1
6	6	2
7	6	2
8	1	2
9	4	4
10	1	2
11	5	3
12	3	1
13	2	4
14	6	2
15	3	1
16	6	2
17	6	2
18	2	4
19	4	4
20	1	2
21	3	1
22	6	2
23	1	2
24	1	1
25	4	4
26	1	2
27	4	4
28	4	4
29	4	4
30	6	2
31	6	2
32	6	2
33	3	1
34	3	1
35	2	4
36	1	1
37	4	4
38	6	2
39	4	4
40	5	3
41	3	1
42	2	4
43	4	4
44	2	4
45	2	4
46	5	3
47	1	2
48	6	4
49	6	2
50	6	4
51	1	1
52	3	1
53	6	2
54	2	4

Number of clusters: 4

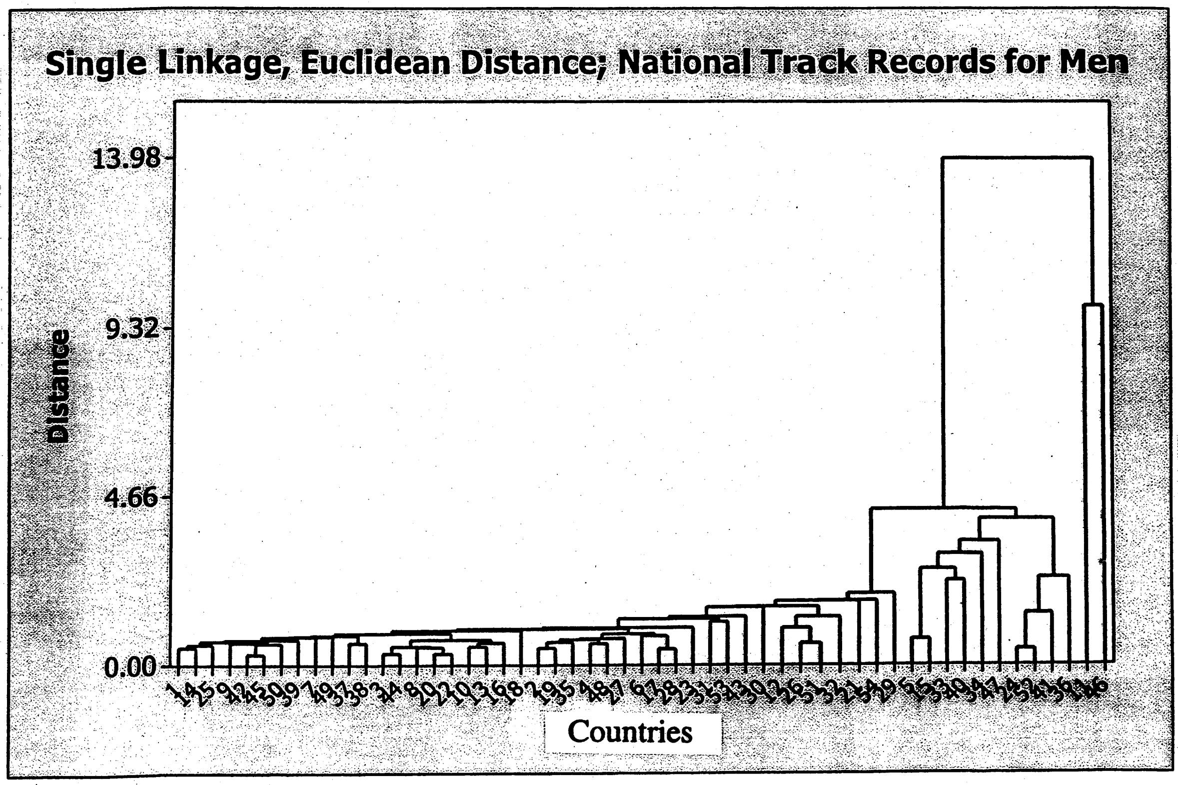
	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
Cluster1	11	298.660	4.494	9.049
Cluster2	20	318.294	3.613	6.800
✓Cluster3	3	490.251	11.895	16.915
Cluster4	20	182.870	2.681	7.024

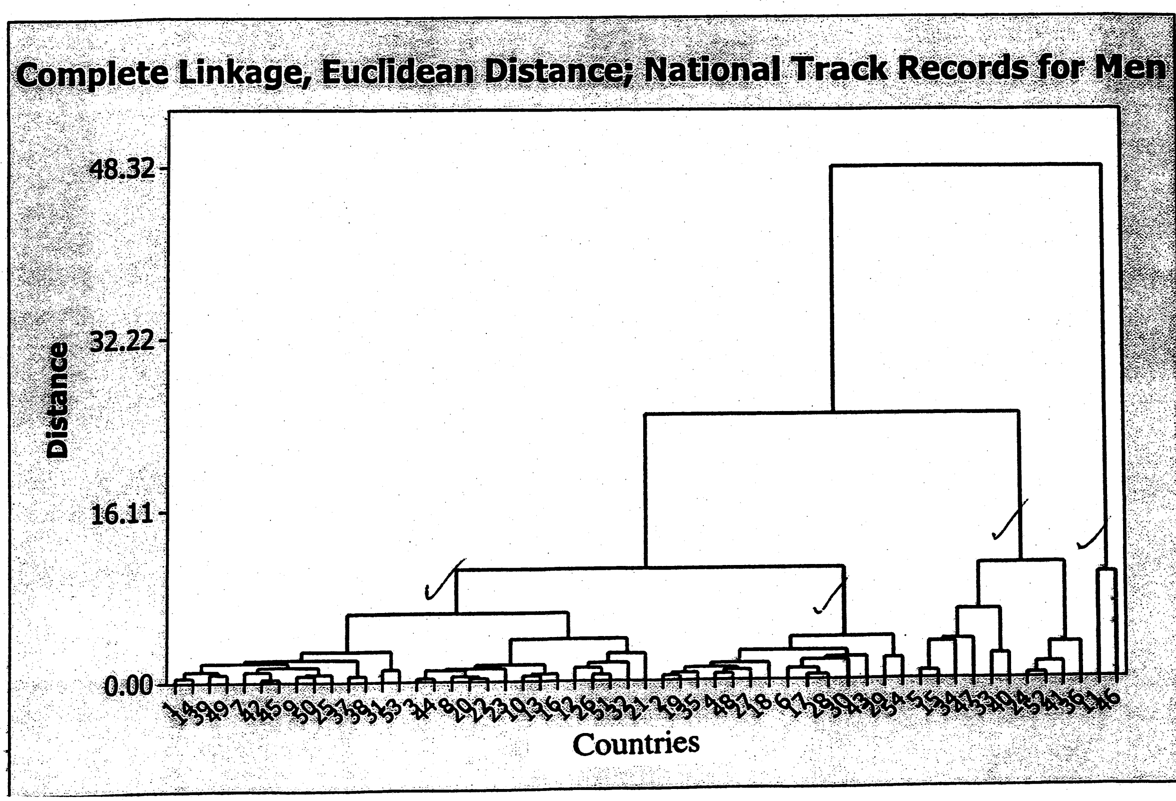
Number of clusters: 6

	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
Cluster1	10	90.154	2.884	4.008
Cluster2	8	22.813	1.613	2.428
Cluster3	8	116.518	3.346	6.651
Cluster4	10	78.508	2.513	5.977
✓Cluster5	3	490.251	11.895	16.915
Cluster6	15	128.783	2.669	5.521

✓ Identical

12.17 (a), (b) Dendrograms for single linkage and complete linkage follow. The dendrograms are similar; as examples, in both procedures, countries 11 and 46 form a group at a relatively high level of distance, and countries 2, 19, 35, 4, 48 and 27 form a group at a relatively small distance. The clusters are more apparent in the complete linkage dendrogram and, depending on the distance level, might have as few as 3 or 4 clusters or as many as 6 or 7 clusters.





(c) The results for K = 4 and K = 6 clusters are displayed below. The results seem reasonable and are consistent with the results for the linkage procedures. Depending on use, K = 4 may be an adequate number of clusters. The results for the men are similar to the results for the women.

Data Display

Country	ClustMemK=4	ClustMemK=6
1	2	2
2	4	4
3	2	1
4	4	4
5	1	3
6	4	6
7	2	2
8	2	1
9	4	2
10	2	2
11	3	5
12	2	1
13	2	1
14	2	2
15	1	3
16	2	2
17	4	6
18	4	4
19	4	4
20	2	1
21	2	1
22	2	1
23	2	1
24	1	3
25	4	2
26	2	1
27	4	4
28	4	6
29	4	6
30	4	4
31	2	2
32	2	1
33	1	3
34	1	3
35	4	4
36	1	3
37	4	4
38	4	2
39	2	2
40	1	3
41	1	3
42	4	2
43	4	4
44	2	1
45	4	2
46	3	5
47	1	3
48	4	4
49	2	2
50	4	2
51	2	1
52	1	3
53	2	2
54	4	6

Number of clusters: 4

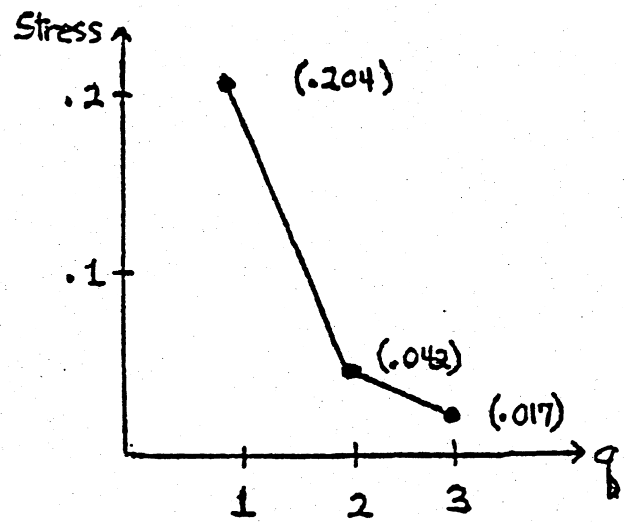
	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
Cluster1	10	169.042	3.910	5.950
Cluster2	21	73.281	1.684	3.041
✓ Cluster3	2	49.174	4.959	4.959
Cluster4	21	56.295	1.481	3.249

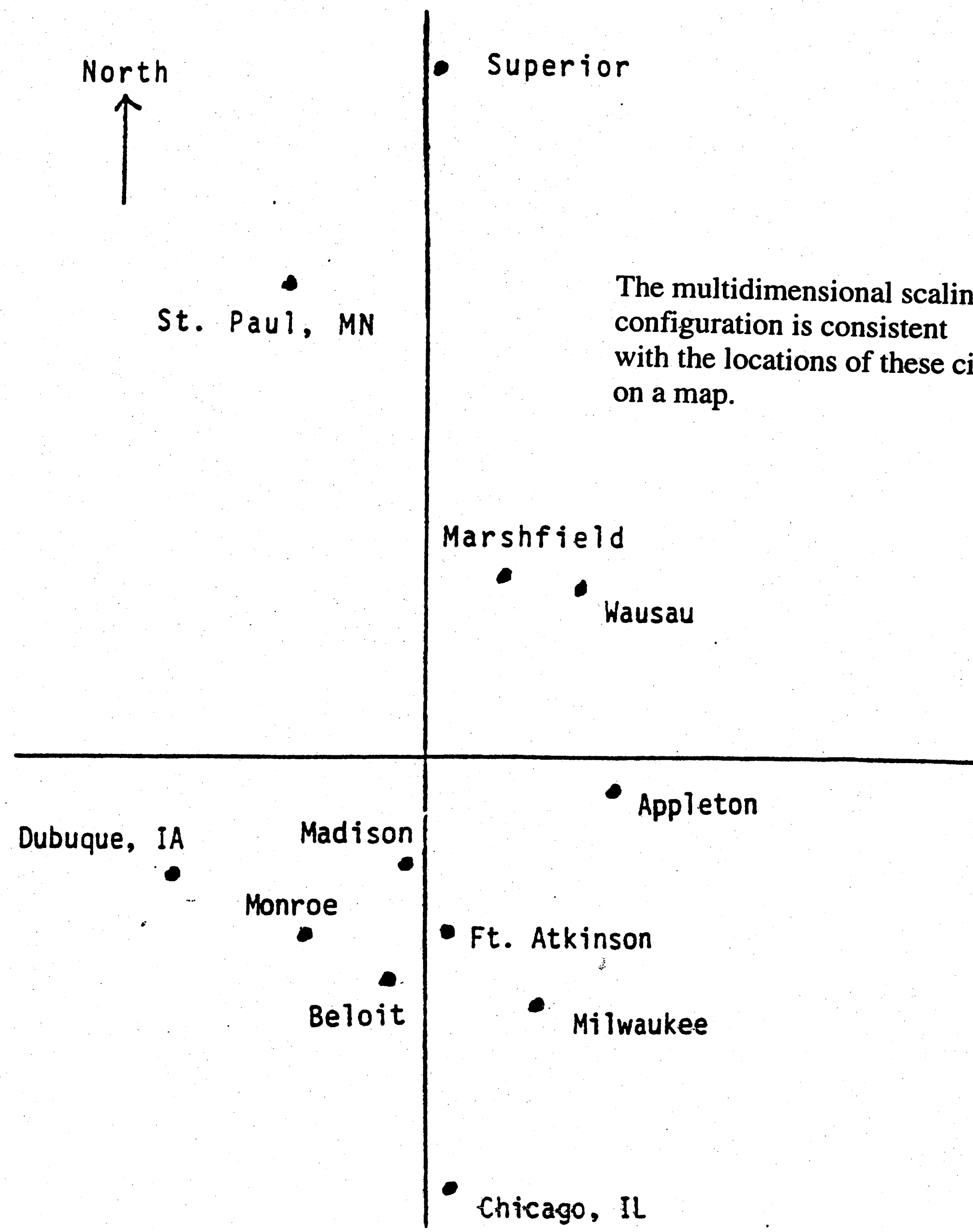
Number of clusters: 6

	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
Cluster1	12	26.806	1.418	2.413
Cluster2	15	18.764	1.048	1.844
Cluster3	10	169.042	3.910	5.950
Cluster4	10	10.137	0.935	1.559
✓ Cluster5	2	49.174	4.959	4.959
Cluster6	5	6.451	1.092	1.606

✓ Identical

12.18.





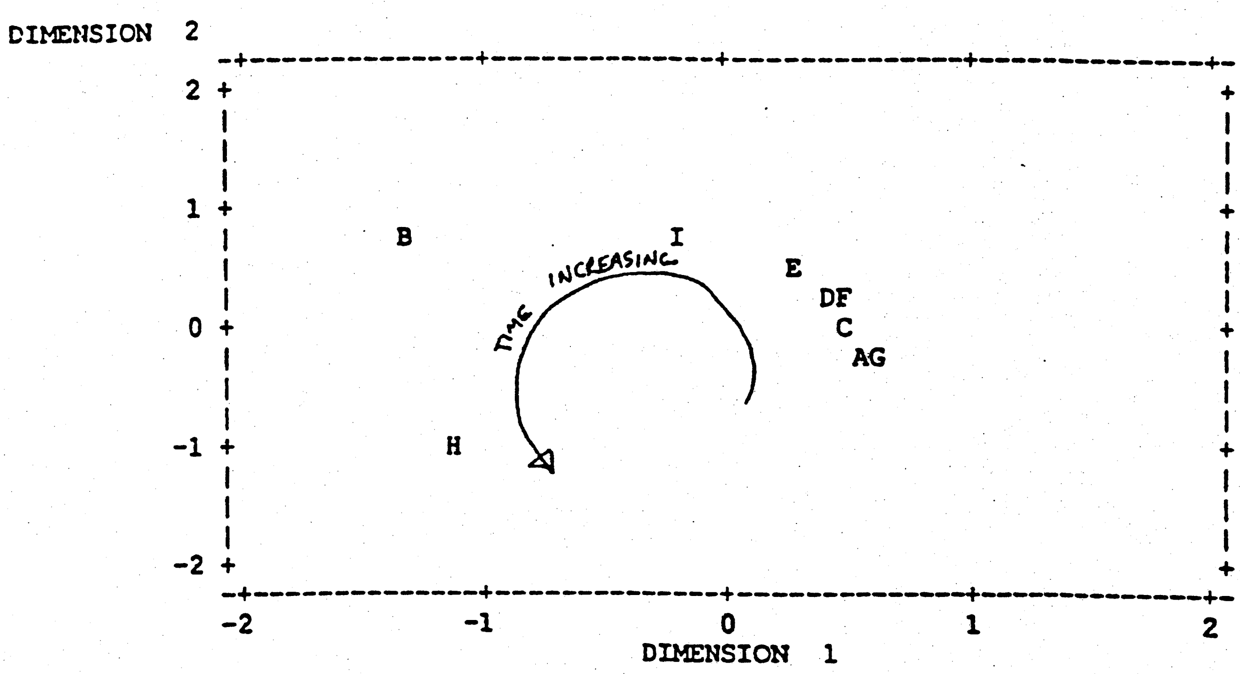
The multidimensional scaling configuration is consistent with the locations of these cities on a map.

12.19.

The stress of final configuration for q=5 is 0.000. The sites in 5 dimensions and the plot of the sites in two dimensions are

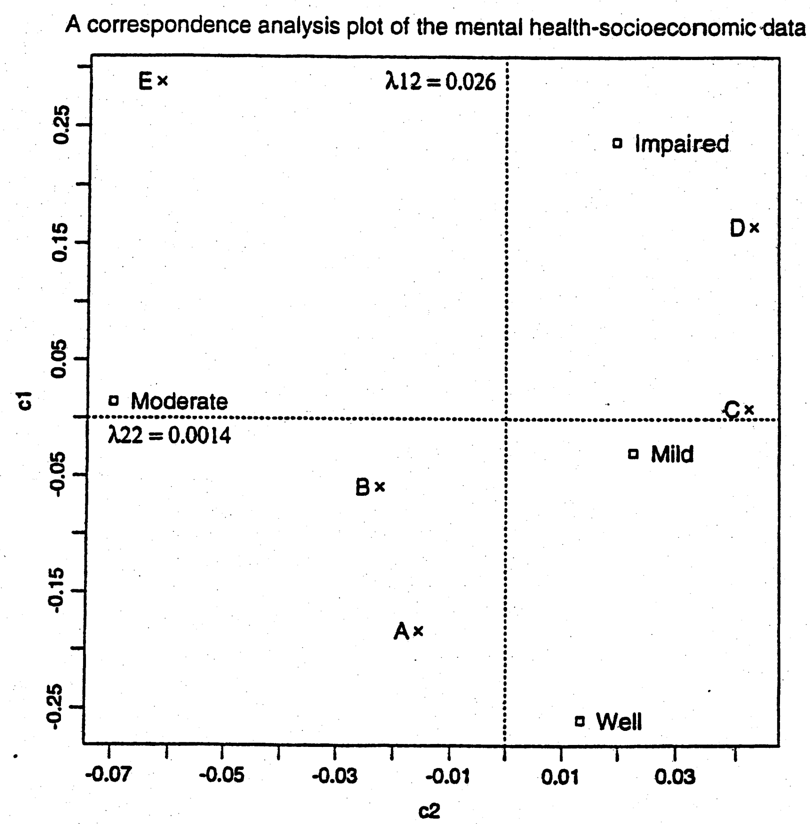
COORDINATES IN 5 DIMENSIONS

VARIABLE	PLOT	DIMENSION 1	2	3	4	5
P1980918	A	.51	-.28	.24	-.68	.12
P1931131	B	-1.32	.69	.62	-.05	-.02
P1550960	C	.47	-.07	.19	.30	.06
P1530987	D	.39	.09	.05	.34	.10
P1361024	E	.23	.30	-.32	.05	.12
P1351005	F	.47	.14	-.22	-.14	-.28
P1340945	G	.58	-.35	.46	.18	-.10
P1311137	H	-1.12	-1.12	-.31	.05	-.01
P1301062	I	-.22	.61	-.70	-.06	.01



The results show a definite time pattern (where time of site is frequently determined by C-14 and tree ring (lumber in great houses) dating).

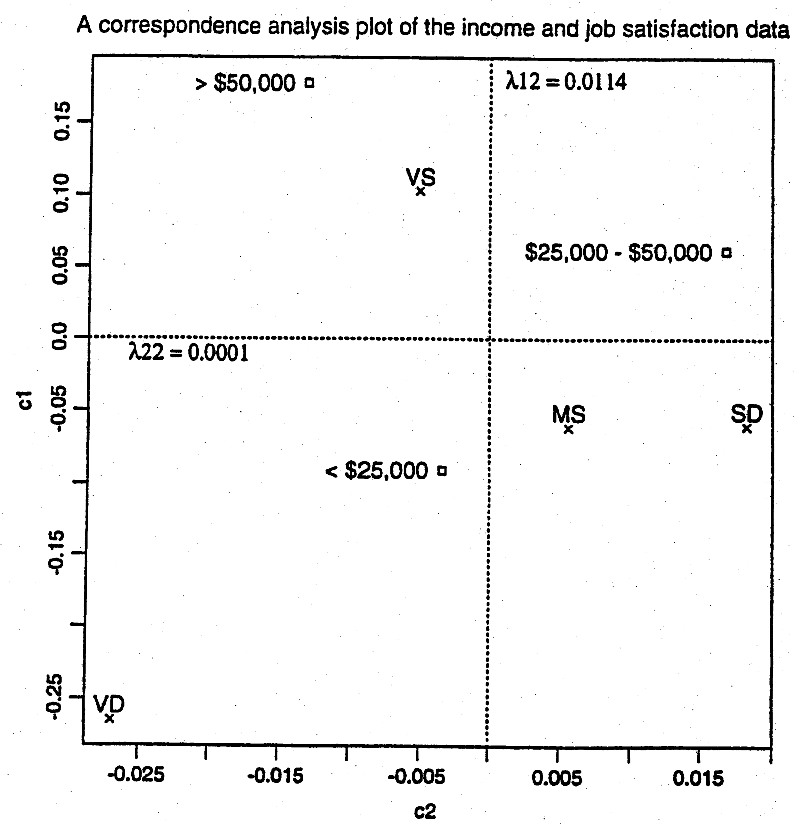
12.20. A correspondence analysis of the mental health-socioeconomic data




```
U                                      V
-0.6922  0.1539  0.5588 0.4300        -0.6266 -0.2313  0.0843 -0.3341
-0.1100  0.3665 -0.7007 0.6022        -0.1521 -0.2516 -0.5109 -0.6407
 0.0411 -0.8809 -0.0659 0.4670         0.0265  0.5490  0.5869 -0.5756
 0.7121  0.2570  0.4388 0.4841         0.4097  0.4668 -0.5519 -0.2297
                                       0.6448 -0.6032  0.2879 -0.3062

lambda
0.1613 0.0371 0.0082 0.0000

Cumulative inertia
0.0260 0.0274 0.0275

Cumulative proportion
0.9475 0.9976 1.0000
```

The lowest economic class is located between moderate and impaired. The next lowest class is closest to impaired.

12.21. A correspondence analysis of the income and job satisfaction data



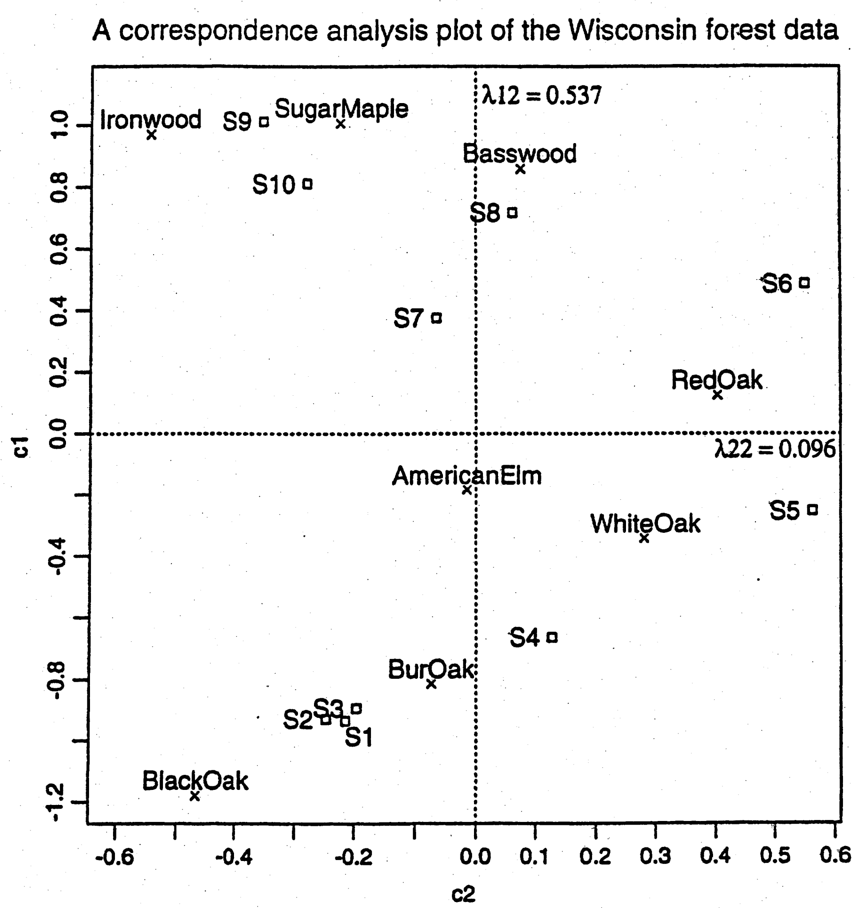

```
U
-0.6272 -0.2392 0.7412
 0.2956  0.8073 0.5107
 0.7206 -0.5394 0.4356
```

```
V
-0.6503 -0.6661 -0.3561
-0.1944  0.5933 -0.7758
-0.3400  0.3159  0.2253
 0.6510 -0.3233 -0.4696
```

```
lambda
0.1069 0.0106 0.0000

Cumulative inertia
0.0114 0.0116

Cumulative proportion
0.9902 1.0000
```

Very satisfied is closest to the highest income group, and very dissatisfied is below the lowest income group. Satisfaction appears to increase with income.

12.22. A correspondence analysis of the Wisconsin forest data



U

-0.3877	-0.2108	-0.0616	0.4029	-0.0582	0.3269	0.4247	-0.1590
-0.3856	-0.2428	-0.0106	0.4345	-0.1950	-0.1968	-0.2635	-0.3835
-0.3495	-0.1821	0.4079	-0.5718	0.2343	-0.1167	0.3294	-0.1272
-0.3006	0.1355	0.0540	-0.2646	0.0006	-0.0826	-0.6644	-0.3192
-0.1108	0.5817	-0.4856	-0.1598	-0.2333	0.1607	0.0772	-0.0518
0.2022	0.5400	0.4626	0.2687	-0.0978	-0.3943	0.2668	-0.3606
0.1852	-0.0756	-0.5090	-0.0291	0.6026	-0.1955	0.1520	-0.5154
0.3140	0.0644	0.3394	0.1567	0.3366	0.6573	-0.2507	-0.2267
0.4200	-0.3484	-0.0394	0.1165	-0.0625	-0.3772	-0.1456	0.1381
0.3549	-0.2897	-0.0345	-0.3393	-0.5994	0.2002	0.1262	-0.4907

V

-0.3904	-0.0831	-0.4781	0.4562	-0.0377	0.3369	0.4071	-0.3511
-0.5327	-0.4985	0.4080	0.0925	-0.0738	-0.3420	-0.2464	-0.3310
-0.1999	0.3889	0.4089	-0.3622	0.4391	0.3217	0.1808	-0.4260
0.0698	0.5382	-0.1726	0.3181	-0.0544	-0.1596	-0.6122	-0.4138
-0.0820	-0.0151	-0.4271	-0.7086	-0.4160	-0.1685	0.0307	-0.3258
0.4005	0.0831	0.1478	0.1866	-0.0042	-0.5895	0.5587	-0.3412
0.3634	-0.4850	-0.3232	-0.0937	0.6298	0.0164	-0.2172	-0.2745
0.4689	-0.2476	0.3150	0.0726	-0.4771	0.5142	-0.0763	-0.3412

lambda

0.7326 0.3101 0.2685 0.2134 0.1052 0.0674 0.0623 0.0000

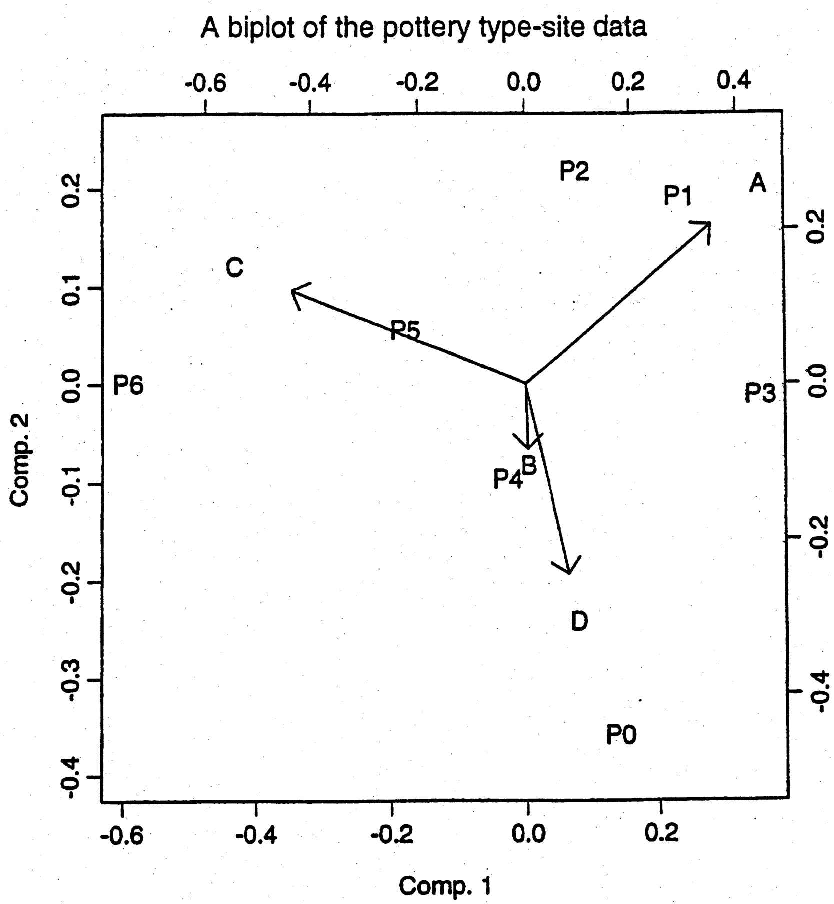
Cumulative inertia

0.5367 0.6329 0.7050 0.7506 0.7616 0.7662 0.7700

Cumulative proportion

0.6970 0.8219 0.9155 0.9747 0.9891 0.9950 1.0000

12.23. We construct biplot of the pottery type-site data, with row proportions as variables.



S

0.0511	-0.0059	-0.0390	-0.0061
-0.0059	0.0084	-0.0051	0.0025
-0.0390	-0.0051	0.0628	-0.0187
-0.0061	0.0025	-0.0187	0.0223

Eigenvectors of S

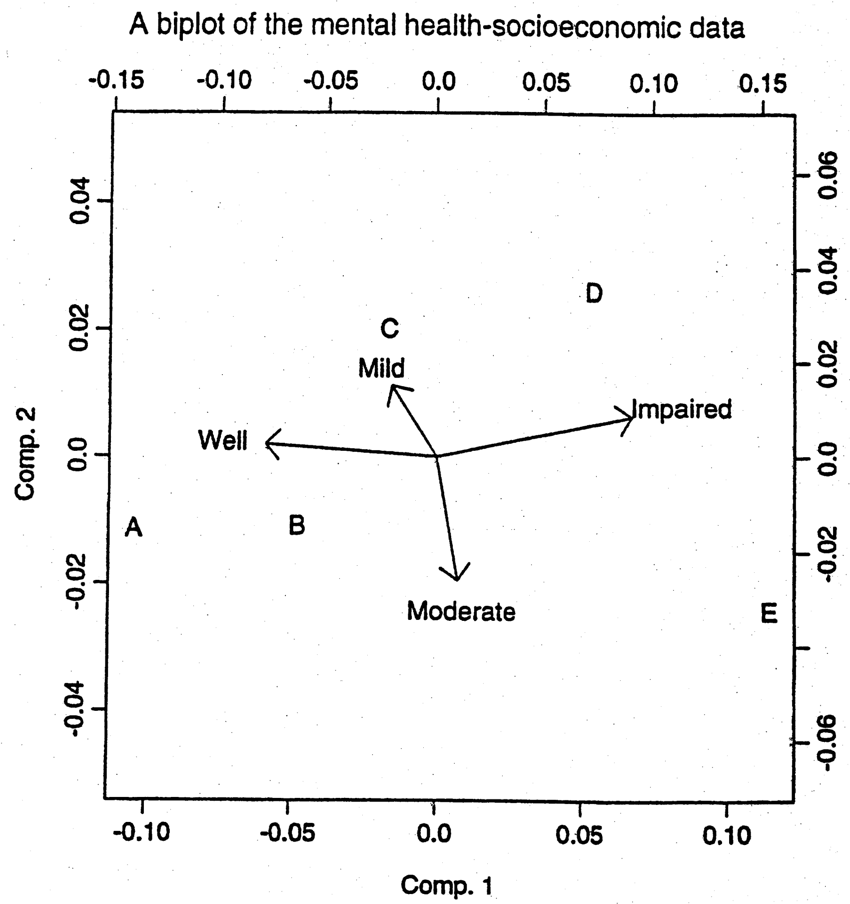
0.6233	0.5853	0.1374	-0.5
0.0064	-0.2385	-0.8325	-0.5
-0.7694	0.3464	0.1951	-0.5
0.1396	-0.6932	0.5000	-0.5

Eigenvalues of S
0.0978 0.0376 0.0091 0.0000

	pc1	pc2	pc3	pc4
St. Dev.	0.3128	0.1940	0.0952	0
Prop. of Var.	0.6769	0.2604	0.0627	0
Cumulative Prop.	0.6769	0.9373	1.0000	1

As in the correspondence analysis.

12.24. We construct biplot of the mental health-socioeconomic data, with column proportions as variables.



S

0.003089	0.000809	-0.000413	-0.003485
0.000809	0.000329	-0.000284	-0.000853
-0.000413	-0.000284	0.000379	0.000318
-0.003485	-0.000853	0.000318	0.004021

Eigenvectors of S

-0.6487	0.0837	-0.5676	0.5
-0.1685	0.4764	0.7033	0.5
0.0794	-0.8320	0.2270	0.5
0.7379	0.2719	-0.3628	0.5

Eigenvalues of S

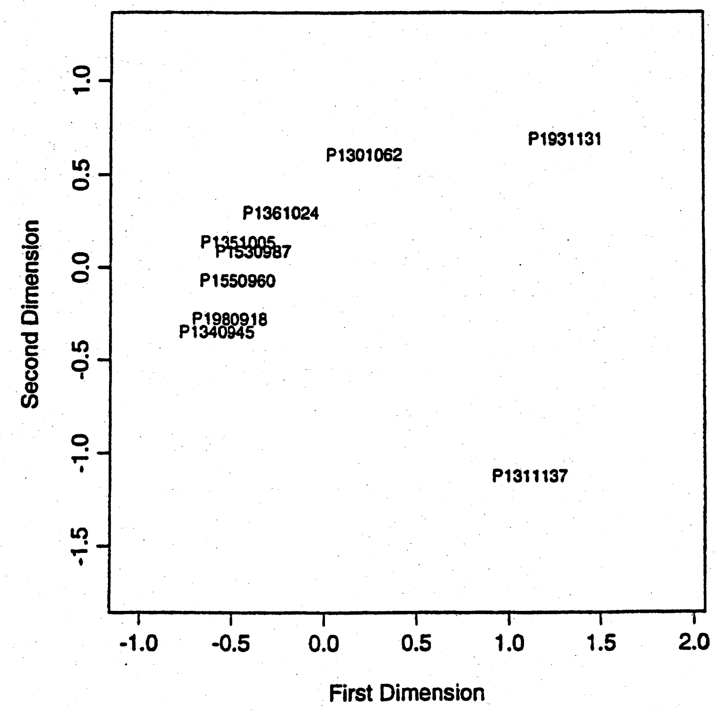
0.007314	0.000480	0.000024	0.000000

	pc1	pc2	pc3	pc4
St. Dev.	0.0855	0.0219	0.0049	0
Prop. of Var.	0.9355	0.0614	0.0031	0
Cumulative Prop.	0.9355	0.9969	1.0000	1

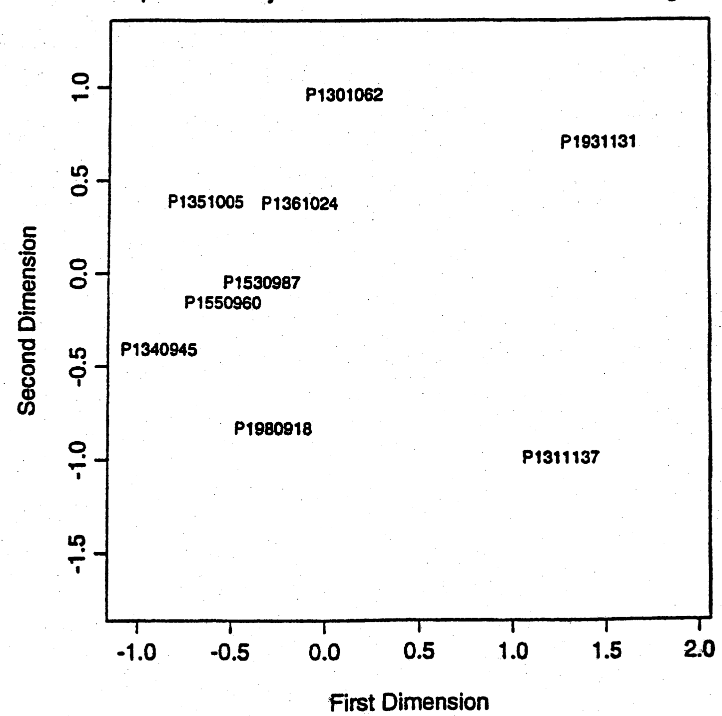
The biplot gives similar locations for health and socioeconomic status. A reflection about the 45 degree line would make them appear more alike.

12.25. A Procrustes analysis of archaeological data

A two-dimensional representation of archaeological sites produced by metric multidimensional scaling



A two-dimensional representation of archaeological sites produced by nonmetric multidimensional scaling



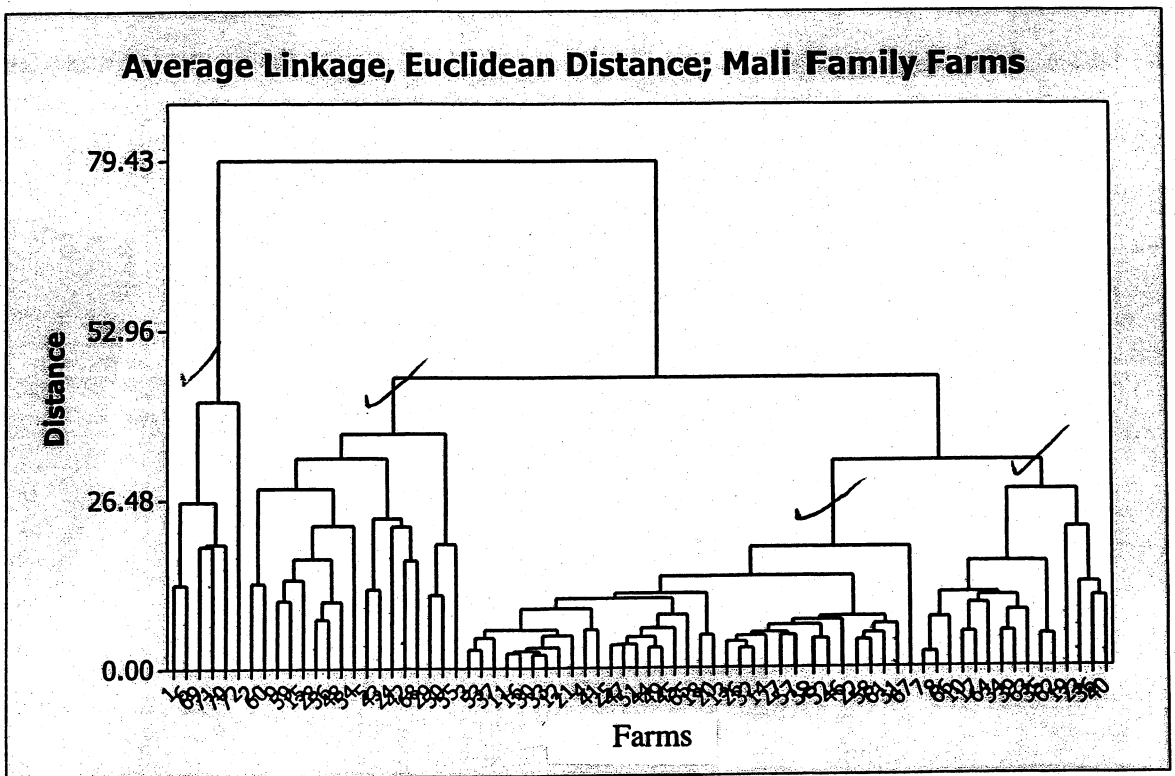
Site	Metric MDS		Nonmetric MDS	
P1980918	-0.512	-0.278	-0.276	-0.829
P1931131	1.318	0.692	1.469	0.703
P1550960	-0.470	-0.071	-0.545	-0.156
P1530987	-0.387	0.088	-0.338	-0.048
P1361024	-0.234	0.296	-0.137	0.379
P1351005	-0.469	0.137	-0.642	0.387
P1340945	-0.581	-0.349	-0.889	-0.409
P1311137	1.118	-1.122	1.262	-0.989
P1301062	0.216	0.608	0.096	0.963

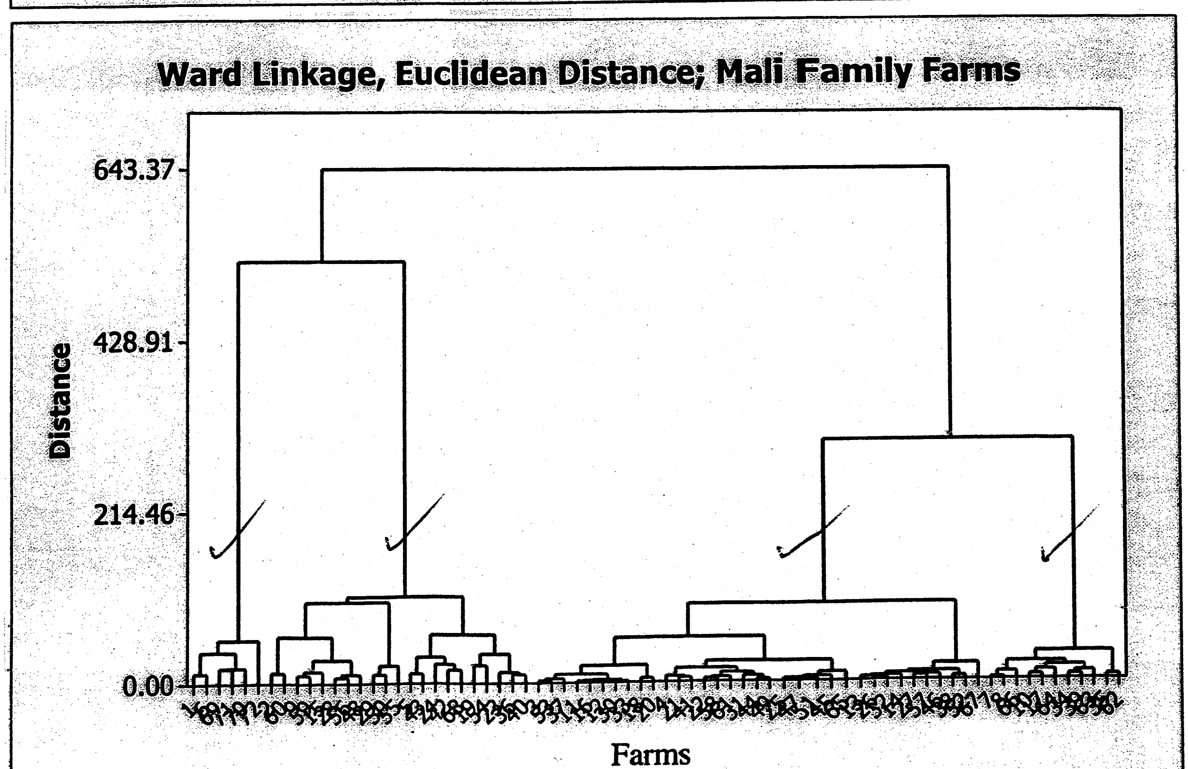
U		V	
-0.9893	-0.1459	-0.9977	-0.0679
-0.1459	0.9893	-0.0679	0.9977

Q		Lambda	
0.9969	0.0784	4.7819	0.000
-0.0784	0.9969	0.0000	2.715

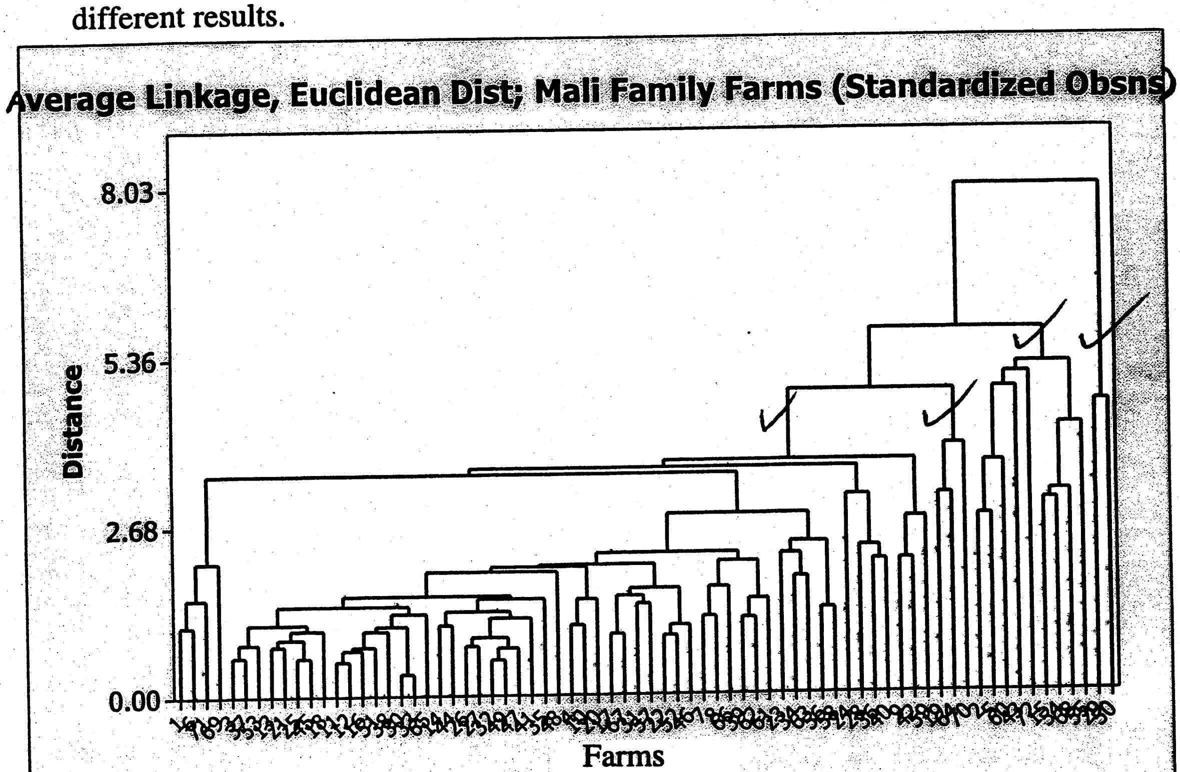
To better align the metric and nonmetric solutions, we multiply the nonmetric scaling solution by the orthogonal matrix $\hat{\mathbf{Q}}$. This corresponds to clockwise rotation of the nonmetric solution by 4.5 degrees. After rotation, the sum of squared distances, 0.803, is reduced to the Procrustes measure of fit $PR^2 = 0.756$.

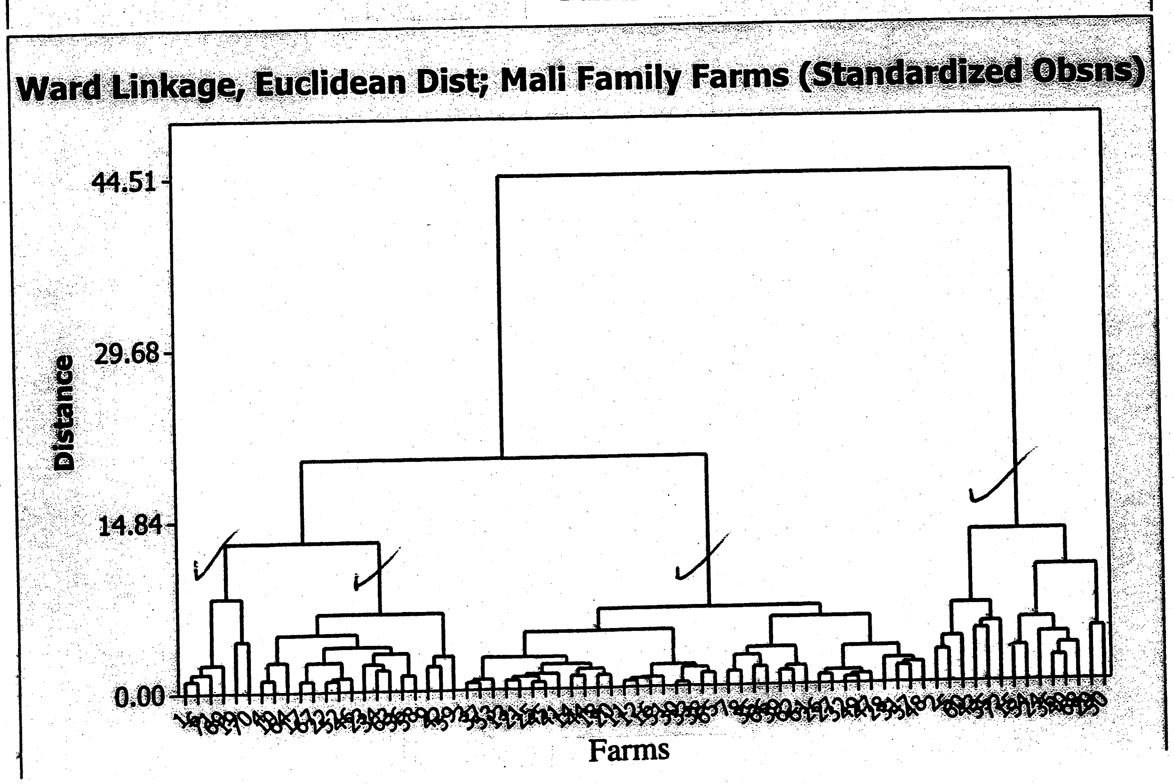
12.26 The dendrograms for clustering Mali Family Farms are given below for average linkage and Ward's method. The dendrograms are similar but a moderate number of distinct clusters is more apparent in the Ward's method dendrogram than the average linkage dendrogram. Both dendrograms suggest there may be as few as 4 clusters (indicated by the checkmarks in the figures) or perhaps as many as 7 or 8 clusters. Reading the "right" number of clusters from either dendrogram would depend on the use and require some subject matter knowledge.





12.27 If average linkage and Ward's method clustering is used with the standardized Mali Family Farm observations, the results are somewhat different from those using the original observations and different from one another. The dendrograms follow. There could be as few as 4 clusters (indicated by the checkmarks in the figures) or there could be as many as 8 or 9 clusters or more. The distinct clusters are more clearly delineated in the Ward's method dendrogram and if we focus attention on the 4 marked clusters, we see the two procedures produce quite different results.





12.28 The results for K = 5 and K = 6 clusters follow. The results seem reasonable and are similar to the results for Ward's method considered in Exercise 12.26. Note as the number of clusters increases from 5 to 6, cluster 1 in the K = 5 solution is partitioned into two clusters, 1 and 6, in the K = 6 solution, there is no change in the other clusters. Although not shown, K = 4 is a reasonable solution as well.

Data Display

Farm	ClustMemK=5	ClustMemK=6
1	1	1
2	2	2
3	3	3
4	3	3
5	5	5
6	1	6
7	4	4
8	4	4
9	2	2
10	4	4
11	3	3
12	3	3
13	3	3
14	3	3
15	2	2
16	3	3
17	4	4
18	3	3
19	5	5
20	3	3
21	3	3
22	3	3
23	5	5
24	5	5
25	3	3
26	3	3
27	3	3
28	3	3
29	2	2
30	3	3
31	3	3
32	3	3
33	3	3
34	4	4
35	4	4
36	5	5
37	3	3
38	3	3
39	4	4
40	5	5
41	3	3
42	5	5
43	5	5
44	3	3
45	3	3
46	3	3
47	3	3
48	2	2
49	3	3
50	2	2
51	3	3
52	3	3
53	3	3
54	2	2
55	2	2
56	3	3
57	3	3
58	3	3
59	2	2
60	2	2
61	3	3
62	4	4
63	4	4
64	4	4
65	3	3
66	4	4
67	4	4
68	2	2
69	1	1
70	1	1
71	1	1
72	1	6

Number of clusters: 5

	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
Cluster1	6	2431.094	18.498	33.076
✓Cluster2	11	4440.330	19.511	24.647
✓Cluster3	35	3298.539	8.878	21.053
✓Cluster4	12	1129.083	9.072	16.024
✓Cluster5	8	1943.156	15.030	19.619

Number of clusters: 6

	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
Cluster1	4	696.609	13.005	15.474
✓Cluster2	11	4440.330	19.511	24.647
✓Cluster3	35	3298.539	8.878	21.053
✓Cluster4	12	1129.083	9.072	16.024
✓Cluster5	8	1943.156	15.030	19.619
Cluster6	2	1005.125	22.418	22.418

✓ Identical for two choices of K

12.29 The results for K = 5 and K = 6 clusters follow. The results seem reasonable and are similar to the results for Ward's method considered in Exercise 12.27. Note as the number of clusters increases from 5 to 6, clusters 3 and 4 in the K = 5 solution lose 1 and 2 farms respectively to form cluster 6 in the K = 6 solution, there is no change in the other clusters. These results using standardized observations are somewhat different from the corresponding results using the original data. It makes a difference whether standardized or un-standardized observations are used.

Data Display

Farm	SdClusMemK=5	SdClusMemK=6
1	1	1
2	5	5
3	3	3
4	3	3
5	5	5
6	1	1
7	3	3
8	3	3
9	5	5
10	3	4
11	3	3
12	3	3
13	3	3
14	3	3
15	4	4
16	3	3
17	4	4
18	3	3
19	2	2
20	3	3
21	3	3
22	4	4
23	4	4
24	4	6
25	3	3
26	3	3
27	3	3
28	4	4
29	2	2
30	3	3
31	4	4
32	3	3
33	3	3
34	4	4
35	3	3
36	4	6
37	3	3
38	3	3
39	3	3
40	4	6
41	3	3
42	4	4
43	5	5
44	3	3
45	3	3
46	4	4
47	3	3
48	3	3
49	3	3
50	2	2
51	3	3
52	4	4
53	3	3
54	2	2
55	5	5
56	4	4
57	4	4
58	4	4
59	4	4
60	5	5
61	3	3
62	3	3
63	4	4
64	4	4
65	3	3
66	3	3
67	4	4
68	2	2
69	1	1
70	1	1
71	1	1
72	5	5

Number of clusters: 5

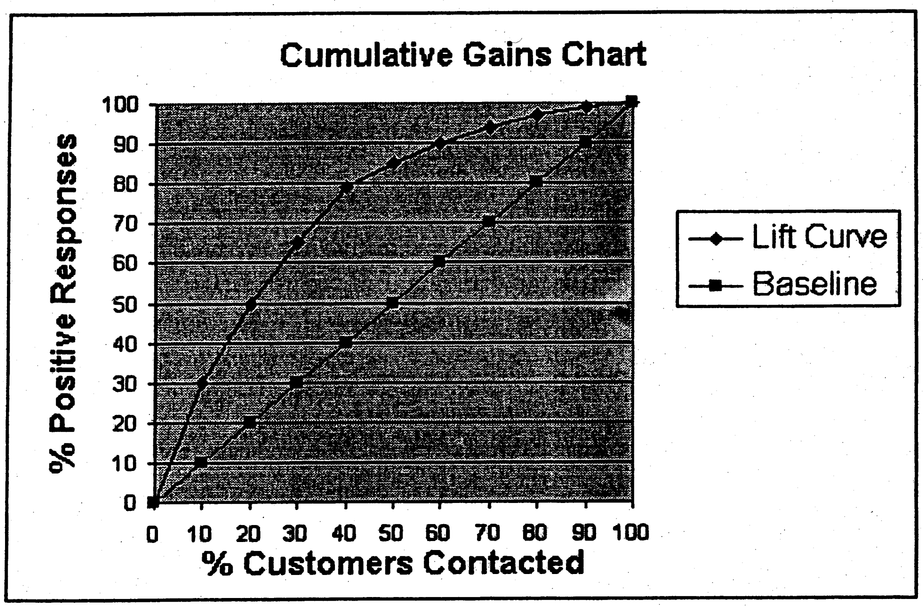
	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
✓Cluster1	5	14.050	1.568	2.703
✓Cluster2	5	56.727	3.288	4.259
Cluster3	35	55.318	1.211	1.993
Cluster4	20	84.099	1.954	3.172
✓Cluster5	7	63.071	2.970	3.482

Number of clusters: 6

	Number of observations	Within cluster sum of squares	Average distance from centroid	Maximum distance from centroid
✓Cluster1	5	14.050	1.568	2.703
✓Cluster2	5	56.727	3.288	4.259
Cluster3	34	51.228	1.183	1.951
Cluster4	18	65.501	1.806	3.195
✓Cluster5	7	63.071	2.970	3.482
Cluster6	3	7.960	1.604	1.954

✓ Identical for two choices of K

12.30 The cumulative lift (gains) chart is shown below. The y-axis shows the percentage of positive responses. This is the percentage of the total possible positive responses (20,000). The x-axis shows the percentage of customers contacted, which is a fraction of the 100,000 total customers. With no model, if we contact 10% of the customers we would expect 10%, or 2,000 = .1×20,000, of the positive responses. Our response model predicts 6,000 or 30% of the positive responses if we contact the top 10,000 customers. Consequently, the y-values at $x = 10\%$ shown in the chart are 10% for baseline (no model) and 30% for the gain (lift) provided by the model. Continuing this argument for other choices of x (% customers contacted) and cumulating the results produces the lift (gains) chart shown. We see, for example, if we contact the top 40% of the customers determined by the model, we expect to get 80% of the positive responses.



12.31 **(a)** The Mclust function, which selects the best overall model according to the BIC criterion, selects a mixture with four multivariate normal components. The four estimated centers are:

$$\hat{\underset{\sim}{\mu}}_1 = \begin{bmatrix} 3.3188 \\ 6.7044 \\ 0.3526 \\ 0.1418 \\ 11.9742 \end{bmatrix} \quad \hat{\underset{\sim}{\mu}}_2 = \begin{bmatrix} 5.1806 \\ 5.2871 \\ 0.5910 \\ 0.1794 \\ 5.5369 \end{bmatrix}, \quad \hat{\underset{\sim}{\mu}}_3 = \begin{bmatrix} 7.2454 \\ 4.8099 \\ 0.3290 \\ 0.2431 \\ 3.2834 \end{bmatrix}, \quad \hat{\underset{\sim}{\mu}}_4 = \begin{bmatrix} 8.6893 \\ 4.1730 \\ 0.5158 \\ 0.2445 \\ 7.4846 \end{bmatrix}$$

and the estimated covariance matrices turn out to be restricted to be of the form $\eta_k\mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix.

The estimated

$$\widehat{\mathbf{D}} = \text{diag}(11.2598, 2.7647, 0.3355, 0.0053, 18.0295)$$

and the estimated scale factors are $\widehat{\eta}_1 = 0.0319$, $\widehat{\eta}_2 = 0.3732$, $\widehat{\eta}_3 = 0.0909$, $\widehat{\eta}_4 = 0.1073$.

The estimated proportions are $\widehat{p}_1 = 0.1059$, $\widehat{p}_2 = 0.4986$, $\widehat{p}_3 = 0.1322, \widehat{p}_4 = 0.2633$.

This minimum BIC model has $BIC = -547.1408$.

(b) The model chosen above has 4 multivariate normal components. These four components are shown in the matrix scatter plot where the observations have been classified into one of the four populations.

The matrix scatter plot of the true classification, is given in the next figure.

Comparing the matrix scatter plot of the four group classification with the matrix scatter plot of the true classification, we see how the oil samples from the Upper sandstone are essentially split into two groups. This is clear from comparing the two scatter plots for (x_1, x_2).

We also repeat the analysis using the *me* function to select mixture distribution with $K = 3$ components. We further restrict the covariance matrices to satisfy $\mathbf{\Sigma}_k = \eta_k\mathbf{D}$. The $K = 3$ groups selected by this function have estimated centers

$$\hat{\underset{\sim}{\mu}}_1 = \begin{bmatrix} 5.3395 \\ 5.2467 \\ 0.5485 \\ 0.1862 \\ 5.2465 \end{bmatrix}, \quad \hat{\underset{\sim}{\mu}}_2 = \begin{bmatrix} 8.5343 \\ 4.2762 \\ 0.4988 \\ 0.2453 \\ 6.6993 \end{bmatrix}, \quad \hat{\underset{\sim}{\mu}}_3 = \begin{bmatrix} 3.3228 \\ 6.7093 \\ 0.3511 \\ 0.1418 \\ 11.9780 \end{bmatrix},$$

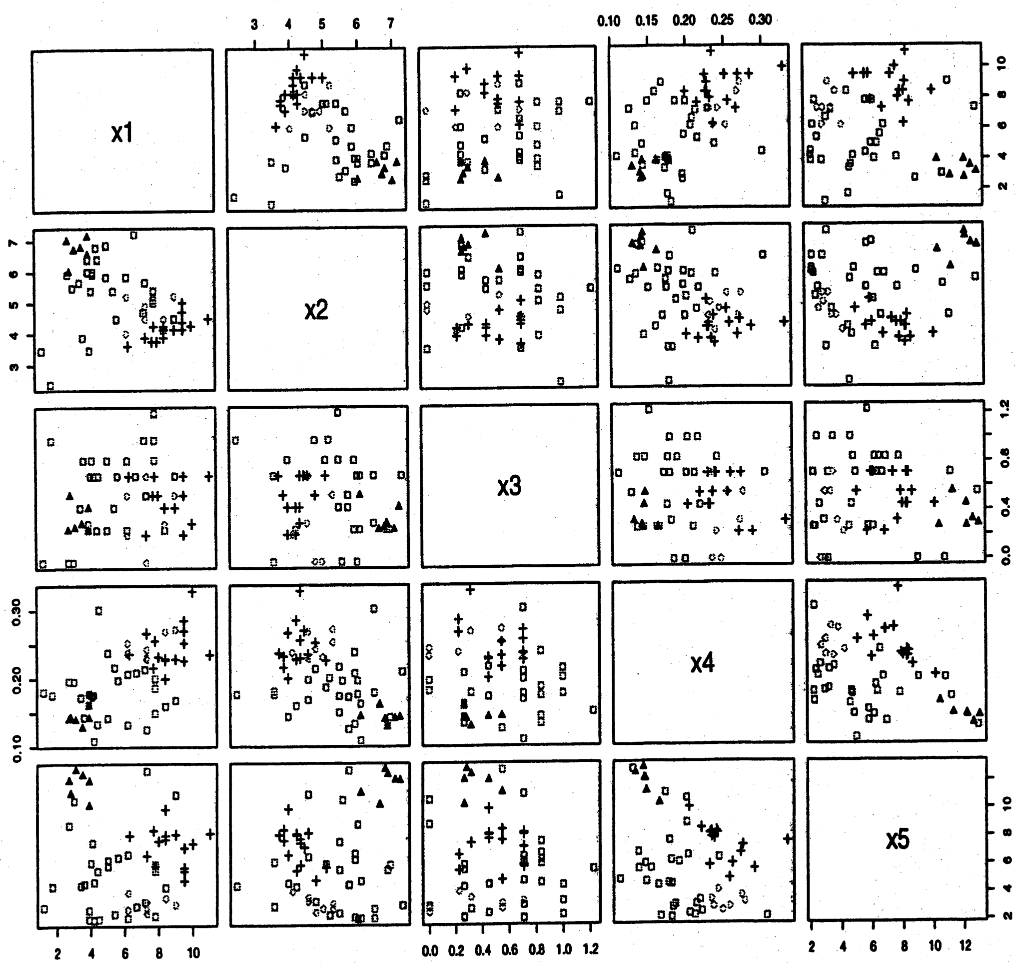


Figure 1: Classification into four groups using Mclust

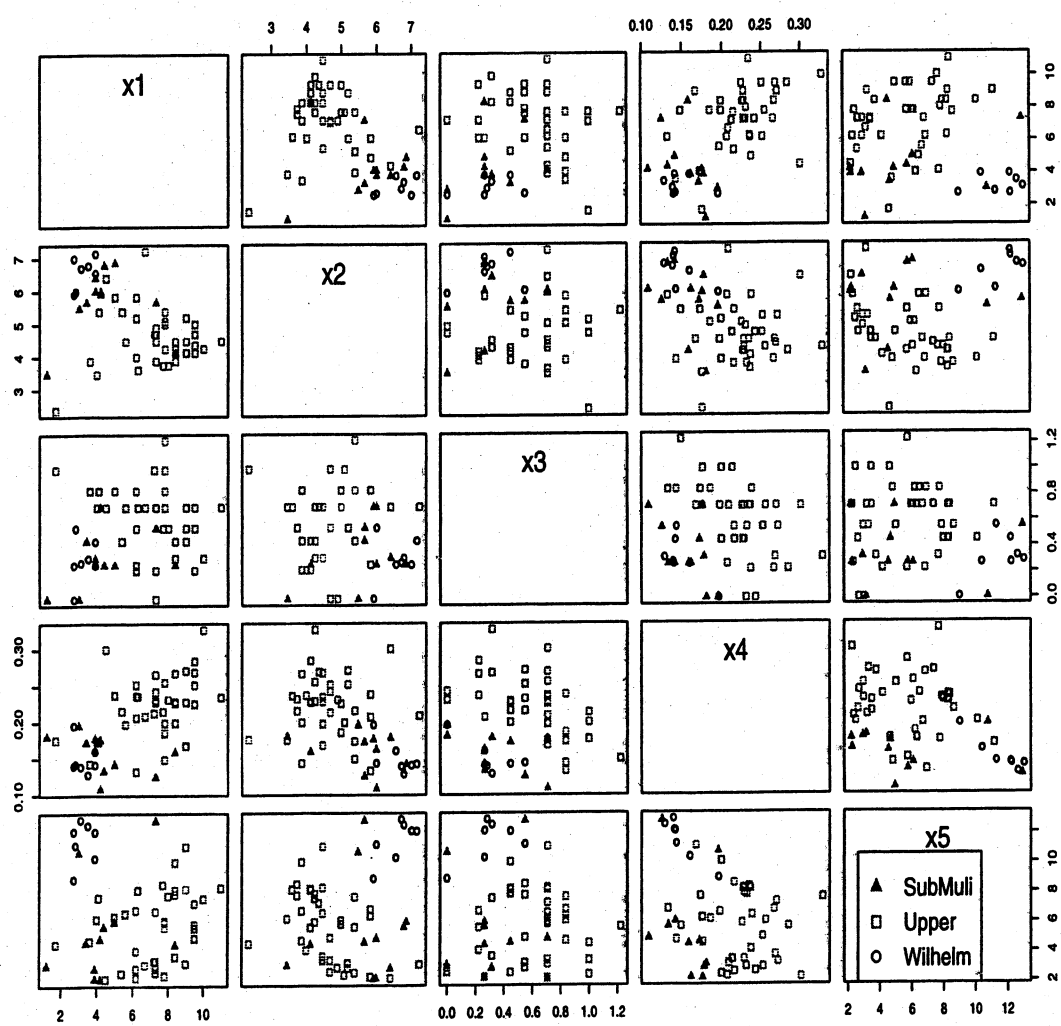


Figure 2: True classification into sandstone strata

the estimated diagonal matrix

$$\widehat{\mathbf{D}} = \text{diag}(10.1535, 2.6295, 0.2969, 0.0052, 24.0955)$$

with estimated scale parameters $\widehat{\eta}_1 = 0.3702$, $\widehat{\eta}_2 = 0.1315$, $\widehat{\eta}_3 = 0.0314$, with resulting $BIC = -534.0949$.

The estimated proportions are $\widehat{p}_1 = 0.5651$, $\widehat{p}_2 = 0.3296$, $\widehat{p}_3 = 0.1052$.

If we use this method to classify the oil samples, the following samples are misclassified:

```
 7  19  22  25  26  27  28  29  30  31
32  33  34  35  39  44  45  46  49
```

and the misclassification error rate is 33.93%.

Proofs of Results in Chapter 7

Proof of Result 7.2 Before the response $\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ is observed, it is a random vector. Now,

$$\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) = \boldsymbol{\beta} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}$$
$$\begin{aligned}\hat{\boldsymbol{\varepsilon}} &= [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\mathbf{Y} \\ &= [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'][\mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}] = [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon}\end{aligned}$$

since $[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\mathbf{Z} = \mathbf{Z} - \mathbf{Z} = \mathbf{0}$. From (2-24) and (2-45),

$$\begin{aligned}E(\hat{\boldsymbol{\beta}}) &= \boldsymbol{\beta} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'E(\boldsymbol{\varepsilon}) = \boldsymbol{\beta} \\ \operatorname{Cov}(\hat{\boldsymbol{\beta}}) &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\operatorname{Cov}(\boldsymbol{\varepsilon})\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1} = \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1} \\ &= \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1} \\ E(\hat{\boldsymbol{\varepsilon}}) &= [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']E(\boldsymbol{\varepsilon}) = \mathbf{0} \\ \operatorname{Cov}(\hat{\boldsymbol{\varepsilon}}) &= [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\operatorname{Cov}(\boldsymbol{\varepsilon})[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']' \\ &= \sigma^2[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\end{aligned}$$

where the last equality follows from (7-6). Also,

$$\begin{aligned}\operatorname{Cov}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\varepsilon}}) &= E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\hat{\boldsymbol{\varepsilon}}'] = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] \\ &= \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] = \mathbf{0}\end{aligned}$$

because $\mathbf{Z}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] = \mathbf{0}$. From the definition of $\hat{\boldsymbol{\varepsilon}}$ above, (7-6) and Result 4.9,

$$\begin{aligned}\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} &= \boldsymbol{\varepsilon}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'][\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon} \\ &= \boldsymbol{\varepsilon}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon} \\ &= \operatorname{tr}[\boldsymbol{\varepsilon}'(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\boldsymbol{\varepsilon}] \\ &= \operatorname{tr}([\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\end{aligned}$$

Now, for an arbitrary $n \times n$ random matrix $\mathbf{W}$,

$$\begin{aligned}E(\operatorname{tr}(\mathbf{W})) &= E(W_{11} + W_{22} + \cdots + W_{nn}) \\ &= E(W_{11}) + E(W_{22}) + \cdots + E(W_{nn}) = \operatorname{tr}[E(\mathbf{W})]\end{aligned}$$

Thus, using Result 2A.12, we obtain

$$\begin{aligned}E(\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}) &= \operatorname{tr}([\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')) \\ &= \sigma^2 \operatorname{tr}[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] \\ &= \sigma^2 \operatorname{tr}(\mathbf{I}) - \sigma^2 \operatorname{tr}[\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] \\ &= \sigma^2 n - \sigma^2 \operatorname{tr}[(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Z}] \\ &= n\sigma^2 - \sigma^2 \operatorname{tr}\Big[\underset{(r+1)\times(r+1)}{\mathbf{I}}\Big] \\ &= \sigma^2(n - r - 1)\end{aligned}$$

and the result for $s^2 = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}/(n - r - 1)$ follows. ■

Proof of Result 7.4 Given the data and the normal assumption for the errors, the likelihood function for $\boldsymbol{\beta}, \sigma^2$ is

$$L(\boldsymbol{\beta}, \sigma^2) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\varepsilon_j^2/2\sigma^2} = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}/2\sigma^2}$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-(\mathbf{y}-\mathbf{Z}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{Z}\boldsymbol{\beta})/2\sigma^2}$$

For a fixed value σ^2, the likelihood is maximized by minimizing $(\mathbf{y} - \mathbf{Z}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{Z}\boldsymbol{\beta})$. But this minimization yields the least squares estimate $\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$, which does not depend upon σ^2. Therefore, under the normal assumption, the maximum likelihood and least squares approaches provide the same estimator $\hat{\boldsymbol{\beta}}$. Next, maximizing $L(\hat{\boldsymbol{\beta}}, \sigma^2)$ over σ^2 [see (4-18)] gives

$$L(\hat{\boldsymbol{\beta}}, \hat{\sigma}^2) = \frac{1}{(2\pi)^{n/2}(\hat{\sigma}^2)^{n/2}} e^{-n/2} \quad \text{where} \quad \hat{\sigma}^2 = \frac{(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})}{n}$$

As shown in the proof of Result 7.2, we can express $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\varepsilon}}$ as linear combinations of the normal variables $\boldsymbol{\varepsilon}$. Specifically,

$$\begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hline \hat{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon} \\ \hline [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta} \\ \hline \mathbf{0} \end{bmatrix} + \begin{bmatrix} (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\ \hline \mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \end{bmatrix} \boldsymbol{\varepsilon} = \boldsymbol{\alpha} + \mathbf{A}\boldsymbol{\varepsilon}$$

Because $\mathbf{Z}$ is fixed, Result 4.3 implies the joint normality of $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\varepsilon}}$. Their mean vectors and covariance matrices were obtained in Result 7.2. Again, using (7-6), we get

$$\text{Cov}\left(\begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hline \hat{\boldsymbol{\varepsilon}} \end{bmatrix}\right) = \mathbf{A}\,\text{Cov}(\boldsymbol{\varepsilon})\mathbf{A}' = \sigma^2 \left[\begin{array}{c|c} (\mathbf{Z}'\mathbf{Z})^{-1} & \mathbf{0} \\ \hline \mathbf{0}' & \mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \end{array}\right]$$

Since $\text{Cov}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\varepsilon}}) = \mathbf{0}$ for the normal random vectors $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\varepsilon}}$, these vectors are independent. (See Result 4.5.)

Next, let $(\lambda, \mathbf{e})$ be any eigenvalue–eigenvector pair for $\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$. Then, by (7-6), $[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']^2 = [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']$ so

$$\lambda\mathbf{e} = [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\mathbf{e} = [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']^2\mathbf{e} = \lambda[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\mathbf{e} = \lambda^2\mathbf{e}$$

That is, $\lambda = 0$ or 1. Now, $\text{tr}[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] = n - r - 1$ (see the proof of Result 7.2), and from Result 4.9, $\text{tr}[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'.]$ Consequently, exactly $n - r - 1$ values of λ_i equal one, and the rest are zero. It then follows from the spectral decomposition that

$$[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] = \mathbf{e}_1\mathbf{e}_1' + \mathbf{e}_2\mathbf{e}_2' + \cdots + \mathbf{e}_{n-r-1}\mathbf{e}_{n-r-1}'$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-r-1}$ are the normalized eigenvectors associated with the eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-r-1} = 1$. Let

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{n-r-1} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1' \\ \hline \mathbf{e}_2' \\ \hline \vdots \\ \hline \mathbf{e}_{n-r-1}' \end{bmatrix} \boldsymbol{\varepsilon}$$

Then $\mathbf{V}$ is normal with mean vector $\mathbf{0}$ and

$$\text{Cov}(V_i, V_k) = \begin{cases} \mathbf{e}_i'\sigma^2\mathbf{I}\mathbf{e}_k = \sigma^2\mathbf{e}_i'\mathbf{e}_k = \sigma^2, & i = k \\ 0, & \text{otherwise} \end{cases}$$

That is, the V_i are independent $N(0, \sigma^2)$ and

$$n\hat{\sigma}^2 = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon} = V_1^2 + V_2^2 + \cdots + V_{n-r-1}^2$$

is distributed $\sigma^2\chi^2_{n-r-1}$. ■

Proof of Result 7.10 According to the regression model, the likelihood is determined from the data $\mathbf{Y} = [\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n]'$ whose rows are independent, with $\mathbf{Y}_j$ distributed as $N_m(\boldsymbol{\beta}'\mathbf{z}_j, \boldsymbol{\Sigma})$. We first note that $\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta} = [\mathbf{Y}_1 - \boldsymbol{\beta}'\mathbf{z}_1, \mathbf{Y}_2 - \boldsymbol{\beta}'\mathbf{z}_2, \ldots, \mathbf{Y}_n - \boldsymbol{\beta}'\mathbf{z}_n]'$ so

$$(\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta}) = \sum_{j=1}^{n} (\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)(\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)'$$

and

$$\begin{aligned}\sum_{j=1}^{n} (\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j) &= \sum_{j=1}^{n} \operatorname{tr}[(\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)] \\ &= \sum_{j=1}^{n} \operatorname{tr}[\boldsymbol{\Sigma}^{-1}(\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)(\mathbf{Y}_j - \boldsymbol{\beta}'\mathbf{z}_j)'] \\ &= \operatorname{tr}[\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta})]\end{aligned}$$

Another preliminary calculation will enable us to express the likelihood in a simple form. Since $\hat{\boldsymbol{\varepsilon}} = \mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}$ satisfies $\mathbf{Z}'\hat{\boldsymbol{\varepsilon}} = \mathbf{0}$ [(see 7-29)],

$$\begin{aligned}&(\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{Z}\boldsymbol{\beta}) \\ &\quad = [\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}} + \mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]'[\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}} + \mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] \\ &\quad = (\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}})'(\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}) + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}'\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \\ &\quad = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}'\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\end{aligned}$$

Using the results above, we obtain the likelihood

$$\begin{aligned}L(\boldsymbol{\beta}, \boldsymbol{\Sigma}) &= \prod_{j=1}^{n} \frac{1}{(2\pi)^{m/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{y}_j - \boldsymbol{\beta}'\mathbf{z}_j)'\boldsymbol{\Sigma}^{-1}(\mathbf{y}_j - \boldsymbol{\beta}'\mathbf{z}_j)} \\ &= \frac{1}{(2\pi)^{mn/2}} \frac{1}{|\boldsymbol{\Sigma}|^{n/2}} e^{-\frac{1}{2}\operatorname{tr}[\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}'\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}))]} \\ &= \frac{1}{(2\pi)^{mn/2}} \frac{1}{|\boldsymbol{\Sigma}|^{n/2}} e^{-\frac{1}{2}\operatorname{tr}[\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}] - \frac{1}{2}\operatorname{tr}[\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}']}\end{aligned}$$

The matrix $\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}'$ is the form $\mathbf{A}'\mathbf{A}$, with $\mathbf{A} = \boldsymbol{\Sigma}^{-1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}'$, and, from Exercise 2.16, it is nonnegative definite. Therefore, its eigenvalues are nonnegative also. Since, by Result 4.9, $\operatorname{tr}[\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}']$ is the sum of its eigenvalues, this trace will equal its minimum value, zero, if $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$. This choice is unique because $\mathbf{Z}$ is of full rank and $\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)} \neq \mathbf{0}$, implies that $\mathbf{Z}(\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)}) \neq \mathbf{0}$, in which case $\operatorname{tr}[\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}'] \geq \mathbf{c}'\boldsymbol{\Sigma}^{-1}\mathbf{c} > 0$, where $\mathbf{c}'$ is any nonzero row of $\mathbf{Z}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$. Applying Result 4.10 with $\mathbf{B} = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}$, $b = n/2$, and $p = m$, we find that $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\Sigma}} = n^{-1}\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}$ are the maximum likelihood estimators of $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$, respectively, and

$$L(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\Sigma}}) = \frac{1}{(2\pi)^{mn/2}} \frac{(n)^{mn/2}}{|\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}|^{n/2}} e^{-nm/2} = \frac{e^{-nm/2}}{(2\pi)^{mn/2}|\hat{\boldsymbol{\Sigma}}|^{n/2}}$$

It remains to establish the distributional results. From (7-33), we know that $\hat{\boldsymbol{\beta}}_{(i)}$ and $\hat{\boldsymbol{\varepsilon}}_{(i)}$ are linear combinations of the elements of $\boldsymbol{\varepsilon}$. Specifically,

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{(i)} &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}_{(i)} + \boldsymbol{\beta}_{(i)} \\ \hat{\boldsymbol{\varepsilon}}_{(i)} &= [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon}_{(i)}, \qquad i = 1, 2, \ldots, m\end{aligned}$$

Therefore, by Result 4.3, $\hat{\boldsymbol{\beta}}_{(1)}, \hat{\boldsymbol{\beta}}_{(2)}, \ldots, \hat{\boldsymbol{\beta}}_{(m)}, \hat{\boldsymbol{\varepsilon}}_{(1)}, \hat{\boldsymbol{\varepsilon}}_{(2)}, \ldots, \hat{\boldsymbol{\varepsilon}}_{(m)}$ are jointly normal. Their mean vectors and covariance matrices are given in Result 7.9. Since $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\beta}}$ have a zero covariance matrix, by Result 4.5 they are independent. From the proof of Result 7.4, $[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'] = \sum_{\ell=1}^{n-r-1} \mathbf{e}_\ell \mathbf{e}_\ell'$, where $\mathbf{e}_\ell' \mathbf{e}_k = 0$, $\ell \neq k$, and $\mathbf{e}_\ell' \mathbf{e}_\ell = 1$. Set $\mathbf{V}_\ell = \boldsymbol{\varepsilon}' \mathbf{e}_\ell = [\boldsymbol{\varepsilon}_{(1)}' \mathbf{e}_\ell, \boldsymbol{\varepsilon}_{(2)}' \mathbf{e}_\ell, \ldots, \boldsymbol{\varepsilon}_{(m)}' \mathbf{e}_\ell]' = e_{\ell 1}\boldsymbol{\varepsilon}_1 + e_{\ell 2}\boldsymbol{\varepsilon}_2 + \cdots + e_{\ell n}\boldsymbol{\varepsilon}_n$. Because $\mathbf{V}_\ell$, $\ell = 1, 2, \ldots, n - r - 1$, are linear combinations of the elements of $\boldsymbol{\varepsilon}$, they have a joint normal distribution with $E(\mathbf{V}_\ell) = E(\boldsymbol{\varepsilon}')\,\mathbf{e}_\ell = \mathbf{0}$. Also, by Result 4.8, $\mathbf{V}_\ell$ and $\mathbf{V}_k$ have covariance matrix $(\mathbf{e}_\ell' \mathbf{e}_k)\boldsymbol{\Sigma} = (0)\boldsymbol{\Sigma} = \mathbf{0}$ if $\ell \neq k$. Consequently, the $\mathbf{V}_\ell$ are independently distributed as $N_m(\mathbf{0}, \boldsymbol{\Sigma})$. Finally,

$$\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon} = \sum_{\ell=1}^{n-r-1} \boldsymbol{\varepsilon}' \mathbf{e}_\ell \mathbf{e}_\ell' \boldsymbol{\varepsilon} = \sum_{\ell=1}^{n-r-1} \mathbf{V}_\ell \mathbf{V}_\ell'$$

which has the $W_{p,n-r-1}(\boldsymbol{\Sigma})$ distribution, by (4-22). ■

Proof of Result in Chapter 10

Proof of Result 10.1 We assume that Σ_{11} and Σ_{22} are nonsingular.[1] Introduce the symmetric square-root matrices $\Sigma_{11}^{1/2}$ and $\Sigma_{22}^{1/2}$ with $\Sigma_{11} = \Sigma_{11}^{1/2}\Sigma_{11}^{1/2}$ and $\Sigma_{11}^{-1} = \Sigma_{11}^{-1/2}\Sigma_{11}^{-1/2}$. [See (2-22).] Set $\mathbf{c} = \Sigma_{11}^{1/2}\mathbf{a}$ and $\mathbf{d} = \Sigma_{22}^{1/2}\mathbf{b}$, so $\mathbf{a} = \Sigma_{11}^{-1/2}\mathbf{c}$ and $\mathbf{b} = \Sigma_{22}^{-1/2}\mathbf{d}$. Then

$$\text{Corr}(\mathbf{a}'\mathbf{X}^{(1)}, \mathbf{b}'\mathbf{X}^{(2)}) = \frac{\mathbf{a}'\Sigma_{12}\mathbf{b}}{\sqrt{\mathbf{a}'\Sigma_{11}\mathbf{a}}\sqrt{\mathbf{b}'\Sigma_{22}\mathbf{b}}} = \frac{\mathbf{c}'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}\mathbf{d}}{\sqrt{\mathbf{c}'\mathbf{c}}\sqrt{\mathbf{d}'\mathbf{d}}} \qquad (1)$$

By the Cauchy–Schwarz inequality (2-48),

$$\mathbf{c}'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}\mathbf{d} \le (\mathbf{c}'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{c})^{1/2}(\mathbf{d}'\mathbf{d})^{1/2} \qquad (2)$$

Since $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$ is a $p \times p$ symmetric matrix, the maximization result (2-51) yields

$$\mathbf{c}'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{c} \le \lambda_1\mathbf{c}'\mathbf{c} \qquad (3)$$

where λ_1 is the largest eigenvalue of $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$. Equality occurs in (3) for $\mathbf{c} = \mathbf{e}_1$, a normalized eigenvalue associated with λ_1. Equality also holds in (2) if $\mathbf{d}$ is proportional to $\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{e}_1$. Thus,

$$\max_{\mathbf{a},\mathbf{b}} \text{Corr}(\mathbf{a}'\mathbf{X}^{(1)}, \mathbf{b}'\mathbf{X}^{(2)}) = \sqrt{\lambda_1} \qquad (4)$$

with equality occurring for $\mathbf{a} = \Sigma_{11}^{-1/2}\mathbf{c} = \Sigma_{11}^{-1/2}\mathbf{e}_1$ and with $\mathbf{b}$ proportional to $\Sigma_{22}^{-1/2}\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{e}_1$, where the sign is selected to give positive correlation. We take $\mathbf{b} = \Sigma_{22}^{-1/2}\mathbf{f}_1$. This last correspondence follows by multiplying both sides of

$$(\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2})\mathbf{e}_1 = \lambda_1\mathbf{e}_1$$

by $\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}$, yielding

$$\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}(\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{e}_1) = \lambda_1(\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{e}_1) \qquad (5)$$

Thus, if $(\lambda_1, \mathbf{e}_1)$ is an eigenvalue–eigenvector pair for $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$, then $(\lambda_1, \mathbf{f}_1)$—with $\mathbf{f}_1$ the normalized form of $\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{e}_1$—is an eigenvalue–eigenvector pair for $\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}$. The sign for $\mathbf{f}_1$ is chosen to give a positive correlation. We have demonstrated that $U_1 = \mathbf{e}_1'\Sigma_{11}^{-1/2}\mathbf{X}^{(1)}$ and $V_1 = \mathbf{f}_1'\Sigma_{22}^{-1/2}\mathbf{X}^{(2)}$ are the first pair of canonical variables and that their correlation is $\rho_1^* = \sqrt{\lambda_1}$. Also, $\text{Var}(U_1) = \mathbf{e}_1'\Sigma_{11}^{-1/2}\Sigma_{11}\Sigma_{11}^{-1/2}\mathbf{e}_1 = \mathbf{e}_1'\mathbf{e}_1 = 1$, and similarly, $\text{Var}(V_1) = 1$.

Continuing, we note that U_1 and an arbitrary linear combination $\mathbf{a}'\mathbf{X}^{(1)} = \mathbf{c}'\Sigma_{11}^{-1/2}\mathbf{X}^{(1)}$ are uncorrelated if

$$0 = \text{Cov}(U_1, \mathbf{c}'\Sigma_{11}^{-1/2}\mathbf{X}^{(1)}) = \mathbf{e}_1'\Sigma_{11}^{-1/2}\Sigma_{11}\Sigma_{11}^{-1/2}\mathbf{c} = \mathbf{e}_1'\mathbf{c}, \quad \text{or} \quad \mathbf{c} \perp \mathbf{e}_1$$

At the kth stage, we require that $\mathbf{c} \perp \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{k-1}$. The maximization result (2-52) then yields

$$\mathbf{c}'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{c} \le \lambda_k\mathbf{c}'\mathbf{c} \quad \text{for } \mathbf{c} \perp \mathbf{e}_1, \ldots, \mathbf{e}_{k-1}$$

[1] If Σ_{11} or Σ_{22} is singular, one or more variables may be deleted from the appropriate set, and the linear combinations $\mathbf{a}'\mathbf{X}^{(1)}$ and $\mathbf{b}'\mathbf{X}^{(2)}$ can be expressed in terms of the reduced set. If $p > \text{rank}(\Sigma_{12}) = p_1$, then the nonzero canonical correlations are $\rho_1^*, \ldots, \rho_{p_1}^*$.

and by (1),

$$\text{Corr}(\mathbf{a}'\mathbf{X}^{(1)}, \mathbf{b}'\mathbf{X}^{(2)}) = \frac{\mathbf{c}'\boldsymbol{\Sigma}_{11}^{-1/2}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1/2}\mathbf{d}}{\sqrt{\mathbf{c}'\mathbf{c}}\sqrt{\mathbf{d}'\mathbf{d}}} \leq \sqrt{\lambda_k}$$

with equality for $\mathbf{c} = \mathbf{e}_k$ or $\mathbf{a} = \boldsymbol{\Sigma}_{11}^{-1/2}\mathbf{e}_k$ and $\mathbf{b} = \boldsymbol{\Sigma}_{22}^{-1/2}\mathbf{f}_k$, as before. Thus, $U_k = \mathbf{e}_k'\boldsymbol{\Sigma}_{11}^{-1/2}\mathbf{X}^{(1)}$ and $V_k = \mathbf{f}_k'\boldsymbol{\Sigma}_{22}^{-1/2}\mathbf{X}^{(2)}$, are the kth canonical pair, and they have correlation $\sqrt{\lambda_k} = \rho_k^*$.

Although we did not explicitly require the V_k to be uncorrelated,

$$\text{Cov}(V_k, V_\ell) = \mathbf{f}_k'\boldsymbol{\Sigma}_{22}^{-1/2}\boldsymbol{\Sigma}_{22}\boldsymbol{\Sigma}_{22}^{-1/2}\mathbf{f}_\ell = \mathbf{f}_k'\mathbf{f}_\ell = 0, \qquad \text{if } k \neq \ell \leq p$$

Also,

$$\text{Cov}(U_k, V_\ell) = \mathbf{e}_k'\boldsymbol{\Sigma}_{11}^{-1/2}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1/2}\mathbf{f}_\ell = 0, \qquad \text{if } k \neq \ell \leq p$$

since $\mathbf{f}_k'$ is a multiple of $\mathbf{e}_k'\boldsymbol{\Sigma}_{11}^{-1/2}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1/2}$ by (5). ■