Solution Manual
Game Theory: An Introduction

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ABSTRACT This Solution Manual includes only the even numbered questions and is available for public access. It is still incomplete. It will be updated every 2-3 weeks to add the solutions to problems as they become available. A complete version is expected by March 15, 2013.

Contents

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Part I

Rational Decision Making

1

The Single-Person Decision Problem

1.

2. **Going to the Movies:** There are two movie theatres in your neighborhood: Cineclass, which is located one mile from your home, and Cineblast, located 3 miles from your home, each showing three films. Cineclass is showing *Casablanca, Gone with the Wind* and *Dr. Strangelove*, while Cineblast is showing *The Matrix, Blade Runner* and *Aliens*. Your problem is to decide which movie to go to.

    (a) Draw a decision tree that represents this problem without assigning payoff values.
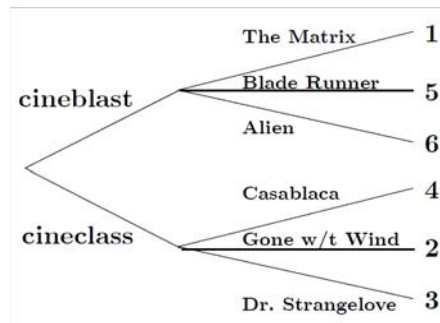
       Answer:
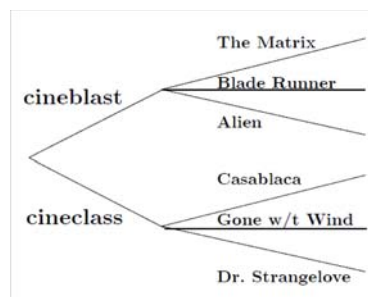
4   1. The Single-Person Decision Problem

■

(b) Imagine that you don't care about distance and that your preferences for movies is alphabetic (i.e., you like *Aliens* the most and *The Matrix* the least.) Using payoff values 1 through 6 complete the decision tree you drew in part (a). What option would you choose?

Answer:

■

(c) Now imagine that your car is in the shop, and the cost of walking each mile is equal to one unit of payoff. Update the payoffs in the decision tree. Would your choice change?

Answer:

■
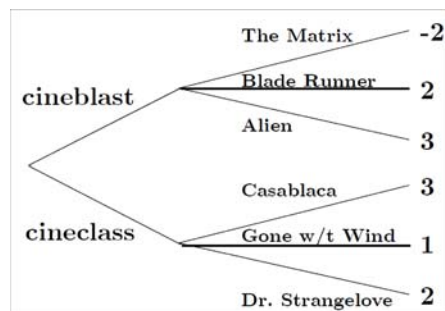
3.

4. **Alcohol Consumption:** Recall the example in which you needed to choose how much to drink. Imagine that your payoff function is given by $\theta a - 4a^2$

where θ is a parameter that depends on your physique. Every person may have a different value of θ, and it is known that in the population (i) the smallest θ is 0.2; (i) the largest θ is 6; and (iii) larger people have higher θ's than smaller people.

(a) Can you find an amount of drinking that no person should drink?

Answer: The utility from drinking 0 is equal to 0. If a decision maker drinks $a = 2$ then, if he has the largest $\theta = 6$, his payoff is $v = 6 \times 2 - 4 \times (2)^2 = -4$ and it is easy to see that decision makers with smaller values of θ will obtain an even more negative payoff from consuming $a = 2$. Hence, no person should choose $a = 2$. ■

(b) How much should you drink if your $\theta = 1$? If $\theta = 4$?

Answer: The optimal solution is obtained by maximizing the payoff function $v(a) = \theta a - 4a^2$. The first-order maximization condition is $\theta - 8a = 0$ implying that $a = \frac{\theta}{8}$ is the optimal solution. For $\theta = 1$ the solution is $a = \frac{1}{8}$ and for $\theta = 4$ it is $a = \frac{1}{2}$. ■

(c) Show that in general, smaller people should drink less than larger people.

Answer: This follows from the solution in part (b) above. For every type of person θ, the solution is $a(\theta) = \frac{\theta}{8}$ which is increasing in θ, and larger people have higher values of θ. ■

(d) Should any person drink more than one bottle of wine?

Answer: No. Even the largest type of person with $\theta = 6$ should only consume $a = \frac{3}{4}$ of a bottle of wine. ■

5.

6. **Fruit Trees:** You have room for up to two fruit bearing trees in your garden. The fruit trees that can grow in your garden are either apple, orange or pear. The cost of maintenance is $100 for an apple tree, $70 for an orange tree and $120 for a pear tree. Your food bill will be reduced by $130 for each apple

tree you plant, by \$145 for each pear tree you plant and by \$90 for each orange tree you plant. You care only about your total expenditure in making any planting decisions.

(a) What is the set of possible actions and related outcomes?

Answer: You have two "slots" that can be left empty, or have one of 3 possible trees planted in each slot. Hence, you have 10 possible choices.[1] The outcomes will just be the choices of what to plant. ∎

(b) What is the payoff of each action/outcome?

Answer: To calculate the payoffs from each choice it is convenient to use a table as follows:

Choice	cost	food savings	net payoff
nothing	0	0	0
one apple tree	100	130	30
one orange tree	70	90	20
one pear tree	120	145	25
two apple trees	200	260	60
two orange trees	140	180	40
two pear trees	240	290	50
apple and orange	170	220	50
apple and pear	220	275	55
pear and orange	190	235	45

(c) Which actions are dominated?

Answer: All but choosing two apple trees are dominated. ∎

(d) Draw the associated decision tree. What will a rational player choose?

Answer: The tree will have ten branches with the payoffs associated with the table above, and the optimal choice is two apple trees. ∎

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[1] This is a probem of choosing 2 items out of 4 possibilities with replacement, which is equal to $\binom{4+2-1}{2} = \frac{(4+2-1)!}{2!(4-1)!} = \frac{5\times4}{2} = 10$.

(e) Now imagine that the food bill reduction is half for the second tree of the same kind (you like variety). That is, the first apple still reduces your food bill by \$130, but if you plant two apple trees your food bill will be reduced by \$130 + \$65 = \$195. (Similarly for pear and orange trees.) What will a rational player choose now?

Answer: An apple tree is still the best choice for the first tree, but now the second tree should be a pear tree. ■

8   1. The Single-Person Decision Problem

2

Introducing Uncertainty and Time

1.

2. **Recreation Choices:** A player has three possible venues to choose from: going to a football game, going to a boxing match, or going for a hike. The payoff from each of these alternatives will depend on the weather. The following table gives the agent's payoff in each of the two relevant weather events:

Alternative	payoff if Rain	payoff if Shine
Football game	1	2
Boxing Match	3	0
Hike	0	1

For Let p denote the probability of rain.

(a) Is there an alternative that a rational player will never take regardless of p? (i.e., it is dominated for any $p \in [0, 1]$.)

Answer: For this decision maker choosing the hike is always worse (dominated) by going to the football game, and he should never go on a hike. ■

(b) What is the optimal decision, or best response, as a function of p.

Answer: The expected payoffs from each of the remaining two choices are given by,

$$
\begin{aligned}
v(\text{Football}) &= p \times 1 + (1 - p) \times 2 = 2 - p\,, \\
v(\text{Boxing}) &= p \times 3 + (1 - p) \times 0 = 3p\,,
\end{aligned}
$$

which implies that football is a better choice if and only if

$$2 - p \geq 3p\,,$$

or, $p \leq \frac{1}{2}$, and boxing is better otherwise. ■

3.

4. **Drilling for Oil:** An oil drilling company must decide whether or not to engage in a new drilling activity before regulators pass a law that bans drilling at that site. The cost of drilling is $1,000,000. After drilling is completed and the drilling costs are incurred, then the company will learn if there is oil or not. If there is oil, operating profits generated are estimated at $4,000,000. If there is no oil, there will be no future profits.

   (a) Using p to denote the likelihood that drilling results in oil, draw the decision tree of this problem.

   Answer: Two decision branches: drill or not drill. Following drilling, Nature chooses oil with probability p, with the payoff of $3 million (4 minus the initial investment). With probability $1 - p$ Nature chooses no-oil with a payoff $\$ - 1$ million. ■

   (b) The company estimates that $p = 0.6$. What is the expected value of drilling? Should the company go ahead and drill?

   Answer: The expected payoff (in millions) from drilling is $p \times 3 - (1 - p) \times 1 = 4p - 1 = 0.6$, which means that the company should drill. ■

(c) To be on the safe side, the company hires a specialist to come up with a more accurate estimate of p. What is the minimum vale of p for which it would be the company's best response to go ahead and drill?

Answer: The minimum value of p for which drilling causes no expected loss is calculated by solving $p \times 3 - (1 - p) \times 1 \geq 0$, or $p \geq \frac{1}{4}$. ■

5.

6. **Real Estate Development:** A real estate developer wishes to build a new development. Regulations impose an environmental impact study that will yield an "impact score," which is an index number based on the impact the development will likely have on traffic, air quality, sewage and water usage, etc. The developer, who has lots of experience, knows that the score will be no less than 40, and no more than 70. Furthermore, he knows that any score between 40 and 70 is as likely as any other score between 40 and 70 (use continuous values). The local government's past behavior implies that there is a 35% chance that it will approve the development if the impact score is less than 50, a 5% chance that it will approve the development if the score is between 50 and 55, and if the score is greater than 55 then the project will surely be halted. The value of the development to the developer is $20,000,000. Assuming that the developer is risk neutral, what is the maximum cost of the impact study such that it is still worthwhile for the developer to have it conducted?

Answer: Observe that there is a $\frac{1}{3}$ probability of getting a score between 40 and 50 given that 40 to 50 is one-third of the range 40 to 70. There is a $\frac{1}{6}$ probability of getting a score between 50 and 55 given that 50 to 55 is one-sixth of the range 40 to 70. Hence, the expected value of doing a study is

$$
\frac{1}{3} \times .35 \times \$20,000,000 + \frac{1}{6} \times .05 \times \$20,000,000 + \frac{1}{2} \times 0 \times \$20,000,000
$$
$$
= \$2,500,000
$$

Hence, the most the developer should pay for the study is $2,500,000. ■

7.

8. **Juice:** Bozoni is a renowned Swiss maker of fruit and vegetable juice, whose products are sold at specialty stores around Western Europe. Bozoni is considering whether to add cherimoya juice to its line of products. "It would be one of our more difficult varieties to produce and distribute," observes Johann Ziffenboeffel, Bozoni's CEO. "The cherimoya would be flown in from New Zealand in firm, unripe form, and it would need its own dedicated ripening facility here in Europe." Three successful steps are absolutely necessary for the new cherimoya variety to be worth producing. The industrial ripening process must be shown to allow the delicate flavors of the cherimoya to be preserved; the testing of the ripening process requires the building of a small-scale ripening facility. Market research in selected small regions around Europe must show that there is sufficient demand among consumers for cherimoya juice. And cherimoya juice must be shown to withstand the existing tiny gaps in the cold chain between the Bozoni plant and the end consumers (these gaps would be prohibitively expensive to fix). Once these three steps have been completed, there are about €2,500,000 worth of expenses in launching the new variety of juice. A successful new variety will then yield profits, in expected present-value terms, of €42.5 million.

The three absolutely necessary steps can be done in parallel or sequentially in any order. Data about these three steps is given in Table 1. "Probability of success" refers to how likely it is that the step will be successful. If it is not successful, then that means that cherimoya juice cannot be sold at a profit. All probabilities are independent of each other (i.e., whether a given step is successful or not does not affect the probabilities that the other steps will be successful). "Cost" refers to the cost of doing this step (regardless of whether it is successful or not).

   (a) Suppose Mr. Ziffenboeffel calls you and asks your advice about the project. In particular, he wants to know (i) should he do the three necessary steps in parallel (i.e., all at once) or should he do them sequentially; and (ii) if sequentially, what's the right order for the steps

to be done? What answers do you give him?

Answer: Bozoni should do the steps sequentially in this order: first test the cold chain, then the ripening process, then do the test-marketing. The expected value of profits is 1.84 million. Observe that it would not be profitable to launch the product if Bozoni had to do all the steps simultaneously. This is an example of real options—by sequencing the steps, Bozoni creates options to switch out of a doomed project before too much money gets spent. ■

(b) Mr. Ziffenboeffel calls you back. Since Table 1 was produced (see below), Bozoni has found a small research firm that can perform the necessary tests for the ripening process at a lower cost than Bozoni's in-house research department.

Table 1: Data on launching the Cherimoya juice

Step	Probability of success	Cost
Ripening process	0.7	€1,000,000
Test marketing	0.3	€5,000,000
Cold chain	0.6	€500,000

At the same time, the EU has raised the criteria for getting approval for new food producing facilities, which raises the costs of these tests. Mr. Ziffenboeffel would, therefore, like to know how your answer to (a) changes as a function of the cost of the ripening test. What do you tell him?

Answer: This is sensitivity analysis for the cost of testing the ripening process. This can be done by varying the cost for ripening, and seeing which expected payoff (highlighted yellow) is highest for which values of the cost. For example, whenever we set the cost below 375,000 it turns out that the payoff from the sequence $R \to C \to T$ gives the highest payoff among the six possible sequences. (Excel's GoalSeek is a particularly handy way for finding the threshold values quickly).
Specifically, the optimal sequence is

i) $R \to C \to T$ if the cost of $R \le 375,000$

ii) $C \to R \to T$ if the cost of $375,000 \le R \le 2,142,857$

iii) $C \to T \to R$ if the cost of $2,142,857 \le R \le 8,640,000$

iv) don't launch if R costs more than $8,640,000$

where "R" stands for the ripening process, "C" stands for the cold chain, and "T" stands for test marketing. ■

(c) Mr. Ziffenboeffel calls you back yet again. The good news is the EU regulations and the outsourcing of the ripening tests "balance" each other out, so the cost of the test remains €1,000,000. Now the problem is that his marketing department is suggesting that the probability that the market research will result in good news about the demand could be different in light of some recent data on the sales of other subtropical fruit products. He would, therefore, like to know how your answer to (a) changes as a function of the probability of a positive result from the market research. What do you tell him?

Answer: This can be found by varying the probability of success for test marketing (highlighted by blue in the excel sheet) between 0 and 1. The optimal sequence turns out to be

i) don't launch if $p \le 0.1905$

ii) $C \to R \to T$ if $p > 0.1905$

where p is the probability that the test marketing will be successful. ■

9.

10. **Surgery:** A patient is very sick, and will die in 6 months if he goes untreated. The only available treatment is risky surgery. The patient is expected to live for 12 months if the surgery is successful, but the probability that the surgery fails and the patient dies immediately is 0.3.

(a) Draw a decision tree for this decision problem.

Answer: Using $v(t)$ to denote the value of living t more months, the following is the decision tree:
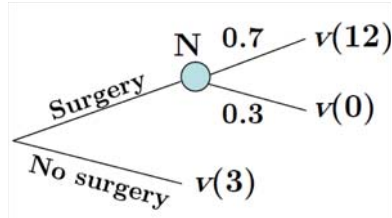
(b) Let $v(x)$ be the patient's payoff function, where x is the number of months till death. Assuming that $v(12) = 1$ and $v(0) = 0$, what is the lowest payoff the patient can have for living 3 months so that having surgery is a best response?

Answer: The expected value of the surgery given the payoffs above is

$$E[v(\text{surgery})] = 0.7v(12) + 0.3v(0) = 0.7$$

which implies that if $v(3) < 0.7$ then the surgery should be performed. ■

For the rest of the problem, assume that $v(3) = 0.8$.

(c) A test is available that will provide some information that predicts whether or not surgery will be successful. A positive test implies an increased likelihood that the patient will survive the surgery as follows:
True-positive rate: The probability that the results of this test will be positive if surgery is to be successful is 0.90.
False-positive rate: The probability that the results of this test will be positive if the patient will not survive the operation is 0.10.
What is the probability of a successful surgery if the test is positive?

Answer: The easiest way to think about this is to imagine that the original 0.7 probability of success is true because for 70% of the sick

population, call these the "treatable" patients, the surgery is successful, while for the other 30% ("untreatable") it is not, and previously the patient did not know which population he belongs to. The test can be thought of as detecting which population the patient belongs to. The above description means that if the patient is treatable then the test will claim he is treatable with probability 0.9, while if the patient is untreatable then the test will claim he is treatable with probability 0.1. Hence, 63% of the population are treatable and detected as such (0.7×0.9), while 3% of the population are *untreatable* but are detected as treatable (0.3×0.1). Hence, of the population of people for whom the test is positive, the probability of successful surgery is $\frac{63}{63+3} = 0.955$. ∎

(d) Assuming that the patient has the test done, at no cost, and the result is positive, should surgery be performed?

Answer: The value from not having surgery is $v(3) = 0.8$, and a positive test updates the probability of success to 0.955 with the expected payoff being 0.955×1 so the patient should have surgery done. ∎

(e) It turns out that the test may have some fatal complications, i.e., the patient may die during the test. Draw a decision tree for this revised problem.

Answer: Given the data above, we know that without taking the test the patient will not have surgery because the expected value of surgery is 0.7 while the value of living 3 months is 0.8. Also, we showed above that after a positive test the patient will choose to have surgery, and it is easy to show that after a negative test he won't (the probability of a successful outcome is $\frac{7}{7+27} = 0.206$.) Hence, the decision tree can be collapsed as follows 9the decision to have surgery have been collapsed to the relevant payoffs):

■
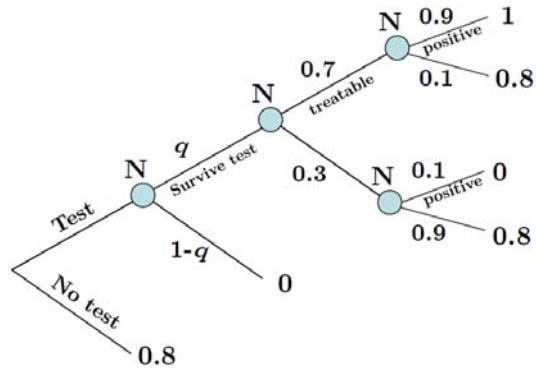
(f) If the probability of death during the test is 0.005, should the patient
opt to have the test prior to deciding on the operation?

Answer: From the decision tree in part (e), the expected value conditional on surviving the test is equal to

$$0.7(0.9 \times 1 + 0.1 \times 0.8) + 0.3(0.1 \times 0 + 0.9 \times 0.8) = 0.902$$

which implies that if the test succeeds with probability 0.995 then the
expected payoff from taking the test is

$$0.995 \times 0.902 + 0.005 \times 0 = 0.897$$

which implies that the test should be taken because $0.897 > 0.8$. ■

11.

12. **More Oil:** Chevron, the No. 2 US oil company, is facing a tough decision.
The new oil project dubbed "Tahiti" is scheduled to produce its first commercial oil in mid-2008, yet it is still unclear how productive it will be. "Tahiti
is one of Chevron's five big projects," told Peter Robertson, vice chairman
of the company's board to the *Wall Street Journal*.[1] Still, it was unclear

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[1] "Chevron's   Tahiti   Facility   Bets   Big   on   Gulf   Oil   Boom."   Jun   27,   2007.   pg.   B5C.

http://proquest.umi.com/pqdweb?did=1295308671&sid=1&Fmt=3&clientId=1566&RQT=309&VName=PQD

whether the project will result in the blockbuster success Chevron is hoping for. As of June 2007, $4-billion has been invested in the high-tech deep sea platform, which suffices to perform early well tests. Aside from offering information on the type of reservoir, the tests will produce enough oil to just cover the incremental costs of the testing (beyond the $4 billion investment). Following the test wells, Chevron predicts one of three possible scenarios. The optimistic one is that Tahiti sits on one giant, easily accessible oil reservoir, in which case the company expects to extract 200,000 barrels a day after expending another $5 billion in platform setup costs, with a cost of extraction at about $10 a barrel. This will continue for 10 years, after which the field will have no more economically recoverable oil. Chevron believes this scenario has a 1 in 6 chance of occurring. A less rosy scenario, that is twice as likely as the optimistic one, is that Chevron would have to drill two more wells at an additional cost of $0.5 billion each (above and beyond the $5 billion set-up costs), and in which case production will be around 100,000 barrels a day with a cost of extraction at about $30 a barrel, and the field will still be depleted after 10 years. The worst case scenario involves the oil tucked away in numerous pockets, requiring expensive water injection techniques which would include up-front costs of another $4 billion (above and beyond the $5 billion set-up costs), extraction costs of $50 a barrel, and production is estimated to be at about 60,000 barrels a day, for 10 years. Bill Varnado, Tahiti's project manager, was quoted giving this least desirable outcome odds of 50-50.

The current price of oil is $70 a barrel. For simplicity, assume that the price of oil and all costs will remain constant (adjusted for inflation) and that Chevron's faces a 0% cost of capital (also adjusted for inflation).

(a) If the test-wells would not produce information about which one of three possible scenarios will result, should Chevron invest the set-up costs of $5 billion to be prepared to produce at whatever scenario is realized?

Answer: We start by noticing that the $2 billion that were invested are a sunk cost and hence irrelevant. Also, since the cost of capital is just about the same as the projected increase in oil prices, we do not need to

discount future oil revenues to get the net present value (NPV) sine the two effects (price increase and time discounting) will cancel each other out. If the company invests the $2.5 billion dollars, then they will be prepared to act upon whatever scenario arises (great with probability $\frac{1}{6}$, ok with probability $\frac{1}{3}$, or bad with probability $\frac{1}{2}$). Notice from the table below that in each scenario the added costs of extraction that Chevron needs to invest (once it becomes clear which scenario it is) is worthwhile (e.g., even in the bad scenario, the profits are $2.19 billion, which covers the added drilling costs of $2 billion in this case.) Hence, Chevron would proceed to drill in each of the three scenarios, and the expected profits including the initial $2.5 billion investment would be,

$$E\pi = \frac{1}{6}\times(\$21B)+\frac{1}{3}\times(\$7.3B-\$0.5B)+\frac{1}{2}\times(\$2.19B-\$2B)-\$2.5B = \$3$$

(b) If the test-wells do produce accurate information about which of three possible scenarios is true, what is the added value of performing these tests?

Answer: Now, if the test drilling will reveal the scenario ahead of time, then in the event of the bad scenario the revenues would not cover the total investment of $4.5 billion ($2.5 billion initially, and another $2 billion for the bad scenario.) In the great and ok scenarios, however, the revenues cover all the costs. Hence, with the information Chevron would not proceed with the investments at all when the bad scenario happens (probability $\frac{1}{2}$), and proceed only when the scenario is great or ok, yielding an expected profit of

$$E\pi = \frac{1}{6}\times(\$21B-2.5B)+\frac{1}{3}\times(\$7.3B-\$2.5B-\$0.5B)+\frac{1}{2}\times0 = \$4,666,666$$

Hence, the added value of performing the tests is,

$$v_{\text{info}} = \$4,666,666,667 - \$3,511,666,667 = \$1,155,000,000$$

	great	ok	bad
quantity/day	200000	100000	60000
price	$70	$70	$70
extr. Costs	$10	$30	$50
gross profits/day	$12,000,000	$4,000,000	$1,200,000
gross profits 5 yr	$21,900,000,000	$7,300,000,000	$2,190,000,000
added drilling costs	$0	$500,000,000	$2,000,000,000
net profits from extraction	$21,900,000,000	$6,800,000,000	$190,000,000
probability	0.166666667	0.333333333	0.5

Expected profits w/o initial investment:	$6,011,666,667
initial investment:	$2,500,000,000
Expected net pofits	$3,511,666,667
Expected net profits with added info	$4,666,666,667
value of information:	$1,155,000,000

■

13.

Part II

Static Games of Complete Information

22

3

Preliminaries

1.

2. **Penalty Kicks:** Imagine a kicker and a goalie who confront each other in a penalty kick that will determine the outcome of the game. The kicker can kick the ball left or right, while the goalie can choose to jump left or right. Because of the speed of the kick, the decisions need to be made simultaneously. If the goalie jumps in the same direction as the kick, then the goalie wins and the kicker loses. If the goalie jumps in the opposite direction of the kick then the kicker wins and the goalie loses. Model this as a normal form game and write down the matrix that represents the game you modeled.

   Answer: There are two players, 1 (kicker) and 2 (goalie). Each has two actions, $a_i \in \{L, R\}$ to denote left or right. The kicker wins when they choose opposite directions while the goalie wins if they choose the same direction. Using 1 to denote a win and -1 to denote a loss, we can write $v_1(L, R) = v_1(R, L) = v_2(L, L) = v_2(R, R) = 1$ and $v_1(L, L) = v_1(R, R) =$

$v_2(L, R) = v_2(R, L) = -1$. The matrix is therefore,

Player 2

		L	R
	L	$-1, 1$	$1, -1$
Player 1	R	$1, -1$	$-1, 1$

■

3.

4. **Hunting:** Two hunters, players 1 and 2, can each choose to hunt a stag, which provides a rather large and tasty meal, or hunt a hare, also tasty, but much less filling. Hunting stags is challenging and requires mutual coopera-tion. If either hunts a stag alone, then the stag will get away, while hunting the stag together guarantees that the stag is caught. Hunting hares is an individualistic enterprise that is not done in pairs, and whoever chooses to hunt a hare will catch one. The payoff from hunting a hare is 1, while the payoff to each from hunting a stag together is 3. The payoff from an unsuc-cessful stag-hunt is 0. Represent this game as a matrix.

Answer: This is the famous "stag hunt" game. Using S for stag and H for hare, the matrix is,

Player 2

		S	H
	S	$3, 3$	$0, 1$
Player 1	H	$1, 0$	$1, 1$

■

5.

6. **Price Competition:** Imagine a market with demand $p(q) = 100 - q$. There are two firms, 1 and 2, and each firm i has to simultaneously choose it's price p_i. If $p_i < p_j$, then firm i gets all of the market while no one demands the good of firm j. If the prices are the same then both firms equally split the

market demand. Imagine that there are no costs to produce any quantity of the good. (These are two large dairy farms, and the product is manure.) Write down the normal form of this game.

Answer: The players are $N = \{1, 2\}$ and the strategy sets are $S_i = [0, \infty$ for $i \in \{1, 2\}$ and firms choose prices $p_i \in S_i$. To calculate payoffs, we need to know what the quantities will be for each firm given prices (p_1, p_2). Given the assumption on ties, the quantities are given by,

$$q_i(p_i, p_j) = \begin{cases} 100 - p_i \text{ if } p_i < p_j \\ 0 \text{ if } p_i > p_j \\ \frac{100 - p_i}{2} \text{ if } p_i = p_j \end{cases}$$

which in turn means that the payoff function is given by quantity times price (there are no costs):

$$v_i(p_i, p_j) = \begin{cases} (100 - p_i)p_i & \text{if } p_i < p_j \\ 0 & \text{if } p_i > p_j \\ \frac{100 - p_i}{2}p_i & \text{if } p_i = p_j \end{cases}$$

■

7.

26   3. Preliminaries

4

Rationality and Common Knowledge

1.

2. **Weak dominance.** We call the strategy profile $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$.

   (a) Provide an example of a game in which there is no weakly dominant strategy equilibrium.

      Answer:

		Player 2	
		H	T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

      ■

   (b) Provide an example of a game in which there is more than one weakly dominant strategy equilibrium.

      Answer: In the following game each player is indifferent between his

strategies and so each one is weakly dominated by the other. This means that any outcome is a weakly dominant strategy equilibrium.

Player 2

		H	T
	H	$1,1$	$1,1$
Player 1	T	$1,1$	$1,1$

■

3.

4. **eBay's recommendation:** It is hard to imagine that anyone is not familiar with eBay©, the most popular auction website by far. The way a typical eBay auction works is that a good is placed for sale, and each bidder places a "proxy bid", which eBay keeps in memory. If you enter a proxy bid that is lower than the current highest bid, then your bid is ignored. If, however, it is higher, then the current bid increases up to one increment (say, 1 cent) above the *second highest* proxy bid. For example, imagine that three people placed bids on a used laptop of $55, $98 and $112. The current price will be at $98.01, and if the auction ended the player who bid $112 would win at a price of $98.01. If you were to place a bid of $103.45 then the who bid $112 would still win, but at a price of $103.46, while if your bid was $123.12 then you would win at a price of $112.01.

Now consider eBay's historical recommendation that you think hard about your value of the good, and that you enter your true value as your bid, no more, no less. Assume that the value of the good for each potential bidder is independent of how much other bidders value it.

(a) Argue that bidding more than your valuation is weakly dominated by actually bidding your valuation.

Answer: If you put in a bid $b_i = b'_i > v_i$ where v_i is your valuation, then only the three following cases can happen: (i) <u>All other bids are below v</u> In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll win

at the same price. (ii) <u>Some bid is above b'_i.</u> In this case bidding $b_i = v$
will yield the exact same outcome: you'll lose to a higher bid. (iii) No
bids are above b'_i and some bid b^*_j is in between v_i and b'_i. In this case
bidding b'_i will cause you to win in and pay $b^*_j > v_i$ which means that
your payoff is negative, while if you would have bid $b_i = v_i$ then you
would lose and get nothing. Hence, in cases (i) and (ii) bidding v_i would
do as well as bidding b'_i, and in case (iii) it would do strictly better,
implying that bidding more than your valuation is weakly dominated
by actually bidding your valuation. ∎

(b) Argue that bidding less than your valuation is weakly dominated by
actually bidding your valuation.

Answer: If you put in a bid $b_i = b'_i < v_i$ where v_i is your valuation, then
only the three following cases can happen: (i) <u>Some other bid are above</u>
In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll
lose to a higher bid. (ii) <u>All other bids are below b'_i.</u> In this case bidding
$b_i = v_i$ will yield the exact same outcome: you'll win at the same price.
(iii) No bids are above v_i and some bid b^*_j is in between b'_i and v_i. In
this case bidding b'_i will cause you to lose and get nothing, while if you
would have bid $b_i = v_i$ then you would win and get a positive payoff of
$v_i - b^*_j$. Hence, in cases (i) and (ii) bidding v_i would do as well as bidding
b'_i, and in case (iii) it would do strictly better, implying that bidding
less than your valuation is weakly dominated by actually bidding your
valuation. ∎

(c) Use your analysis above to make sense of eBay's recommendation. Would
you follow it?

Answer: The recommendation is indeed supported by an analysis of
rational behavior.[1]

---

[1] Those familiar with eBay know about sniping, which is bidding in the last minute. It still is a weakly dominated
strategy to bid your valuation at that time, and waiting for the last minute may be a "best response" if you believe
other people may respond to an early bid. More on this is discussed in chapter 13.

5.

6. **Roommates**: Two roommates need to each choose to clean their apartment, and each can choose an amount of time $t_i \geq 0$ to clean. If their choices are t and t_j, then player i's payoff is given by $(10 - t_j)t_i - t_i^2$. (This payoff function implies that the more one roommate cleans, the less valuable is cleaning for the other roommate.)

(a) What is the best response correspondence of each player i?

Answer: Player i maximizes $(10 - t_j)t_i - t_i^2$ given a belief about t_j, and the first-order optimality condition is $10 - t_j - 2t_i = 0$ implying that the best response is $t_i = \frac{10-t_j}{2}$. ∎

(b) Which choices survive one round of IESDS?

Answer: The most player i would choose is $t_i = 5$, which is a BR to $t_j = 0$. Hence, any $t_i > 5$ is dominated by $t_i = 5$.[2] Hence, $t_i \in [0, 5]$ are the choices that survive one round of IESDS.

(c) Which choices survive IESDS?

Answer: The analysis follows the same ideas that were used for the Cournot duopoly in section 4.2.2. In the second round of elimination, because $t_2 \leq 5$, the best response $t_i = \frac{10-t_j}{2}$ implies that firm 1 will choose $t_1 \geq 2.5$, and a symmetric argument applies to firm 2. Hence, the second round of elimination implies that the surviving strategy sets are $t_i \in [2.5, 5]$ for $i \in \{1, 2\}$. If this process were to converge to an interval, and not to a single point, then by the symmetry between both players, the resulting interval for each firm would be $[t_{\min}, t_{\max}]$ that simultaneously satisfy two equations with two unknowns: $t_{\min} = \frac{10-t_{\max}}{2}$ and $t_{\max} = \frac{10-t_{\min}}{2}$. However, the only solution to these two equations is

---

[2] This can be shown directly: The payoff from choosing $t_i = 5$ when the opponent is choosing t_j is $v(5, t_j) = (10 - t_j)5 - 25 = 25 - 5t_j$. The payoff from choosing $t_i = 5 + k$ where $k > 0$ when the opponent is choosing t_j is $v(5+k, t_j) = (10-t_j)(5+k) - (5+k)^2 = 25 - 5t - k^2 - t_j k_j$, and because $k > 0$ it follows that $v(5+k, t_j) < v(5, t_j$

.

$t_{\min} = t_{\max} = \frac{10}{3}$. Hence, the unique pair of choices that survive IESDS for this game are $t_1 = t_2 = \frac{10}{3}$. ∎

7.

8. Consider the p-Beauty contest presented in section 4.3.5.

(a) Show that if player i believes that everyone else is choosing 20 then 19 is not the only best response for any number of players n.

Answer: If everyone else is choosing 20 and if player i chooses 19 then $\frac{3}{4}$ of the average will be somewhere below 15, and 19 is closer to that number, and therefore is a best response. But the same argument holds for any choice of player i that is between 15 and and 20 regardless of the number of players. (In fact, you should be able to convince yourself that this will be true for any choice of i between 10 and 20.) ∎

(b) Show that the set of best response strategies to everyone else choosing the number 20 depends on the number of players n.

Answer: Imagine that $n = 2$. If one player j is choosing 20, then any number s_i between 0 and 19 will beat 20. This follows because the target number ($\frac{3}{4}$ of the average) is equal to $\frac{3}{4} \times \frac{20+s_i}{2} = \frac{15}{2} + \frac{3}{8}s_i$, the distance between 20 and the target number is $\frac{25}{2} - \frac{3}{8}s_i$ (this will always be positive because the target number is less than 20) while the distance between s_i and the target number is $\left|\frac{5}{8}s_i - \frac{15}{2}\right|$. The latter will be smaller than the former if and only if $\left|\frac{5}{8}s_i - \frac{15}{2}\right| < \frac{25}{2} - \frac{3}{8}s_i$, or $-20 < s_i < 20$. Given the constraints on the choices, $BR_i \in \{0, 1, ...19\}$. Now imagine that $n = 5$. The target number is equal to $\frac{3}{4} \times \frac{80+s_i}{5} = 12 + \frac{3}{20}s_i$, the distance between 20 and the target number is $8 - \frac{3}{20}s_i$ while the distance between s_i and the target number is $\left|\frac{17}{20}s_i - 12\right|$. The latter will be smaller than the former if and only if $\left|\frac{17}{20}s - 12\right| < 8 - \frac{3}{20}s$, or $\frac{40}{7} < s_i < 20$. Hence, $BR_i = \{6, 7, ..., 19\}$. You should be able to convince yourself that as $n \to \infty$, if everyone but i chooses 20 then i's best response will converge to $BR_i = \{10, 11, ..., 19\}$. ∎

9.

10. **Popsicle stands:** There are five lifeguard towers lined along a beach, where the left-most tower is number 1 and the right most tower is number 5. Two vendors, players 1 and 2, each have a popsicle stand that can be located next to one of five towers. There are 25 people located next to each tower, and each person will purchase a popsicle from the stand that is closest to him or her. That is, if player 1 locates his stand at tower 2 and player 2 at tower 3, then 50 people (at towers 1 and 2) will purchase from player 1, while 75 (from towers 3,4 and 5) will purchase from vendor 2. Each purchase yields a profit of $1.

    (a) Specify the strategy set of each player. Are there any strictly dominated strategies?

    Answer: The strategy sets for each player are $S_i = \{1, 2, ..., 5\}$ where each choice represents a tower. To see whether there are any strictly dominated strategies it is useful to construct the matrix representation of this game. Assume that if a group of people are indifferent between the two places (equidistant) then they will split between the two vendors (e.g., if the vendors are at the same tower then their payoffs will be 62.5 each, while if they are located at towers 1 and 3 then they split the people from tower 2 and their payoffs are 37.5 and 87.5 respectively.) Otherwise they get the people closest to them, so payoffs are:

    Player 2

		1	2	3	4	5
	1	$62.5, 62.5$	$25, 100$	$37.5, 87.5$	$50, 75$	$62.5, 62.5$
Player 1	2	$100, 25$	$62.5, 62.5$	$50, 75$	$62.5, 62.5$	$75, 50$
	3	$87.5, 37.5$	$75, 50$	$62.5, 62.5$	$75, 50$	$87.5, 37.5$
	4	$75, 50$	$62.5, 62.5$	$50, 75$	$62.5, 62.5$	$100, 25$
	5	$62.5, 62.5$	$50, 75$	$37.5, 87.5$	$25, 100$	$62.5, 62.5$

    Notice that the choices of 1 and 5 are strictly dominated by any other choice for both players 1 and 2. ∎

(b) Find the set of strategies that survive Rationalizability.

Answer: Because the strategies 1 and 5 are strictly dominated then they cannot be a best response to any belief (Proposition 4.3). In the reduced game in which these strategies are removed, both strategies 2 and 4 are dominated by 3, and therefore cannot be a best response in this second stage. Hence, only the choice $\{3\}$ is rationalizable. ■

34   4. Rationality and Common Knowledge

5

Pinning Down Beliefs: Nash Equilibrium

1.

2. A strategy $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i^W, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$. Provide an example of a game for which there is a weakly dominant strategy equilibrium, as well as another Nash equilibrium.

   Answer: Consider the following game:

   Player 2

		L	R
Player 1	U	$1, 1$	$1, 1$
	D	$1, 1$	$2, 2$

   In this game, (D, R) is a weakly dominant strategy equilibrium (and of course, a Nash equilibrium), yet (U, L) is a Nash equilibrium that is not a weakly dominant strategy equilibrium. ∎

3.

4. **Splitting Pizza:** You and a friend are in an Italian restaurant, and the owner offers both of you an 8-slice pizza for free under the following condition. Each of you must simultaneously announce how many slices you would like; that is, each player $i \in \{1,2\}$ names his desired amount of pizza, $0 \leq s_i \leq 8$ If $s_1 + s_2 \leq 8$ then the players get their demands (and the owner eats any leftover slices). If $s_1 + s_2 > 8$, then the players get nothing. Assume that you each care only about how much pizza you individually consume, and the more the better.

   (a) Write out or graph each player's best-response correspondence.

   Answer: Restrict attention to integer demands (more on continuous demands is below). If player j demands $s_j \in \{0, 1, ..., 7\}$ then i's best response is to demand the complement to 8 slices. If i asks for more then both get nothing while if i asks for less then he is leaving some slices unclaimed. If instead player j demands $s_j = 8$ then player i gets nothing regardless of his request so any demand is a best response. In summary,

   $$BR_i(s_j) = \begin{cases} 8 - s_j & \text{if } s_j \in \{0, 1, ..., 7\} \\ \{0, 1, ..., 8\} & \text{if } s_j = 8 \end{cases}.$$

   Note: if the players can ask for amounts that are not restricted to integers then the same logic applies and the best response is

   $$BR_i(s_j) = \begin{cases} 8 - s_j & \text{if } s_j \in [0, 8) \\ [0, 8] & \text{if } s_j = 8 \end{cases}.$$

   ∎

   (b) What outcomes can be supported as pure-strategy Nash equilibria?

   Answer: It is easy to see from the best response correspondence that any pair of demands that add up to 8 will be a Nash equilibrium, i.e., $(0, 8), (1, 7), ..., (8, 0)$. However, there is another Nash equilibrium: $(8,8)$ in which both players get nothing. It is a Nash equilibrium because

given that each player is asking for 8 slices, the other player gets nothing *regardless* of his request, hence he is indifferent between all of his requests including 8.

Note: The pair $s_j = 8$ and $s_i = s$ where $s \in \{1, 2, ..., 7\}$ is not a Nash equilibrium because even though player i is playing a best response to s_j, player j is not playing a best response to s_i because by demanding 8 player j received nothing, but if he instead demanded $8 - s > 0$ then he would get those amount of slices and get something. ■

5.

6. **Hawk-Dove:** The following game has been widely used in evolutionary biology to understand how "fighting" and "display" strategies by animals could coexist in a population. For a typical Hawk-Dove game there are resources to be gained (i.e. food, mates, territories, etc.) denoted as v. Each of two players can chooses to be aggressive, called "Hawk" (H), or can be compromising, called "Dove" (D). If both players choose H then they split the resources, but loose some payoff from injuries, denoted as k. Assume that $k > \frac{v}{2}$. If both choose D then they split the resources, but engage in some display of power that a display cost d, with $d < \frac{v}{2}$. Finally, if player i chooses H while j chooses D, then i gets all the resources while j leaves with no benefits and no costs.

(a) Describe this game in a matrix

Answer:

Player 2

		H	D
Player 1	H	$\frac{v}{2} - k, \frac{v}{2} - k$	$v, 0$
	D	$0, v$	$\frac{v}{2} - d, \frac{v}{2} - d$

■

(b) Assume that $v = 10$, $k = 6$ and $d = 4$. What outcomes can be supported
as pure-strategy Nash equilibria?[1]

Answer: The game is:

Player 2

		H	D
Player 1	H	$-1, -1$	$10, 0$
	D	$0, 10$	$1, 1$

and the two strategy profiles that can be supported as pure strategy
Nash equilibria are (H, D) and (D, H), leading to outcomes $(10, 0)$ and
$(0, 10)$ respectively. ■

7.

8. **The n firm Cournot Model:** Suppose there are n firms in the Cournot
oligopoly model. Let q_i denote the quantity produced by firm i, and let
$Q = q_i + \cdots + q_n$ denote the aggregate production. Let $P(Q)$ denote the market
clearing price (when demand equals Q) and assume that inverse demand
function is given by $P(Q) = a - Q$ (where $Q < a$). Assume that firms have
no fixed cost, and the cost of producing quantity q_i is cq_i (all firms have the
same marginal cost, and assume that $c < a$).

(a) Model this as a Normal form game

Answer: The players are $N = \{1, 2, ..., n\}$, each player chooses $q_i \in S$
where the strategy sets are $S_i = [0, \infty)$ for all $i \in N$, and the payoffs of
each player are given by,

$$
v_i(q_i, q_{-i}) = \begin{cases} (a - \sum_{j=1}^{n} q_j)q_i - cq_i & \text{if } \sum_{j=1}^{n} q_j < a \\ -cq_i & \text{if } \sum_{j=1}^{n} q_j \geq a \end{cases}
$$

---

[1] In the evolutionary biology literature, the analysis performed is of a very different nature. Instead of considering
the Nash equilibrium analysis of a static game, the analysis is a dynamic analysis where successful strategies
"replicate" in a large population. This analysis is part of a methodology called "evolutionary game theory." For
more on this see Gintis (2000).

■

(b) What is the Nash (Cournot) Equilibrium of the game where firms choose their quantities simultaneously?

Answer: Let's begin by assuming that there is a symmetric "interior solution" where each firm chooses the same positive quantity as a Nash equilibrium, and then we will show that this is the only possible Nash equilibrium. Because each firm maximizes

$$v_i(q_i, q_{-i}) = (a - \sum_{j=1}^{n} q_j)q_i - cq_i \ ,$$

the first order condition is

$$a - \sum_{j \neq i} q_j - 2q_i - c = 0 \ ,$$

which yields the best response of player i to be

$$BR_i(q_{-i}) = \frac{a - \sum_{j \neq i} q_j - c}{2} \ .$$

Imposing symmetry in equilibrium implies that all n best response conditions will hold with the same values $q_i^* = q^*$ for all $i \in N$, and can be solved using the best response function as follows,

$$q^* = \frac{a - (n-1)q^* - c}{2} \ ,$$

which yields

$$q^* = \frac{a - c}{n + 1} \ .$$

It is more subtle to show that there cannot be other Nash equilibria. To show this we will show that conditional on whatever is chosen by all but two players, the two players must choose the same amount in a Nash equilibrium. Assume that there is another asymmetric Nash equilibrium in which two players, i and j, choose two different equilibrium quantities $q_i^* \neq q_j^*$. Let $\bar{q} = \sum_{m \neq i,j} q_m^*$ be the sum of all the other equilibrium

quantity choices of the players who are not i or l. Because we assumed that this is a Nash equilibrium, the best response function of both i and j must hold simultaneously, that is,

$$q_i^* = \frac{a - \overline{q} - q_j^* - c}{2} \ ,$$

(5.1)

and

$$q_j^* = \frac{a - \overline{q} - q_i^* - c}{2} \ .$$

(5.2)

If we substitute (5.2) into (5.1) we obtain,

$$q_i^* = \frac{a - \overline{q} - \frac{a - \overline{q} - q_i^* - c}{2} - c}{2} \ ,$$

which implies that $q_i^* = \frac{a - \overline{q} - c}{3}$. If we substitute this back into (5.2) we obtain,

$$q_j^* = \frac{a - \overline{q} - \frac{a - \overline{q} - c}{3} - c}{2} = \frac{a - \overline{q} - c}{3} = k_i^* \ ,$$

which contradicts the assumption we started with, that $q_i^* \neq q_j^*$. Hence, the unique Nash equilibrium has all the players choosing the same level $q^* = \frac{a - c}{n + 1}$. ■

(c) What happens to the equilibrium price as n approaches infinity? Is this familiar?

Answer: First consider the total quantity in the Nash equilibrium as a function of n,

$$Q^* = nq^* = \frac{n(a - c)}{n + 1}$$

and the resulting limit price is

$$\lim_{n \to \infty} P(Q^*) = \lim_{n \to \infty} \left(a - \frac{n(a - c)}{n + 1} \right) = c.$$

This means that as the number of firms grow, the Nash equilibrium price will also fall and will approach the marginal costs of the firms as the number of firms grows to infinity. Those familiar with a standard economics class know that in perfect competition price will equal marginal costs, which is what happens here when n approaches infinity. ■

9.

10. **Synergies:** Two division managers can invest time and effort in creating a
better working relationship. Each invests $e_i \geq 0$, and if both invest more then
both are better off, but it is costly for each manager to invest. In particular,
the payoff function for player i from effort levels (e_i, e_j) is $v_i(e_i, e_j) = (a +
e_j)e_i - e_i^2$.

(a) What is the best response correspondence of each player?

Answer: If player i believes that player j chooses e_j then i's first order
optimality condition for maximizing his payoff is,

$$a + e_j - 2e_i = 0 \ ,$$

yielding the best response function,

$$BR_i(e_j) = \frac{a + e_j}{2} \text{ for all } e_j \geq 0.$$

■

(b) In what way are the best response correspondences different from those
in the Cournot game? Why?

Answer: Here the best response function of player i is *increasing* in
the choice of player j whereas in the Cournot model it is *decreasing* in
the choice of player j. This is because in this game the choices of the
two players are strategic complements while in the Cournot game they
are strategic substitutes. ■

(c) Find the Nash equilibrium of this game and argue that it is unique.

Answer: We solve two equations with two unknowns,

$$e_1 = \frac{a + e_2}{2} \text{ and } e_2 = \frac{a + e_1}{2},$$

which yield the solution $e_1 = e_2 = a$. It is easy to see that it is unique
because it is the only point at which these two best response functions
cross. ■

11.

12. **Asymmetric Bertrand:** Consider the Bertrand game with $c_1(q_1) = q_1$ and $c_2(q_2) = 2q_2$, demand equal to $p = 100 - q$, and where firms must choose prices in increments of one cent. We have seen in section **??** that one possible Nash equilibrium is $(p_1^*, p_2^*) = (1.99, 2.00)$.

(a) Show that there are other Nash equilibria for this game.

Answer: Another Nash equilibrium is $(p_1', p_2') = (1.50, 1.51)$.In this equilibrium firm 1 fulfills market demand at a price of 1.50 and has no incentive to change the price in either direction. Firm 2 is indifferent between the current price and any higher price, and strictly prefers it to lower prices. ∎

(b) How many Nash equilibria does this game have?

Answer: There are 100 Nash equilibria of this game starting with $(p_1, p_2) = (1.00, 1.01)$ and going all the way up with one-cent increases to $(p_1^*, p_2^*) = (1.99, 2.00)$. The same logic explains why each of these is a Nash equilibrium. ∎

13.

14. **Negative Ad Campaigns:** Each one of two political parties can choose to buy time on commercial radio shows to broadcast negative ad campaigns against their rival. These choices are made simultaneously. Due to government regulation it is forbidden to buy more than 2 hours of negative campaign time so that each party cannot choose an amount of negative campaigning above 2 hours. Given a pair of choices (a_1, a_2), the payoff of party i is given by the following function: $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$.

(a) What is the normal form representation of this game?

Answer: Two players $N = \{1, 2\}$, for each player the strategy space is $S_i = [0, 2]$ and the payoff of player i is given by $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$. ∎

(b) What is the best response function for each party?

Answer: Each player maximizes $v_i(a_1, a_2)$ resulting in the first order optimality condition $1 + a_j - 2a_i = 0$ resulting in the best response function,

$$a_i(a_j) = \frac{1 + a_j}{2} \, .$$

■

(c) What is the pure strategy Nash equilibrium? is it unique?

Answer: Solving the two best response functions simultaneously,

$$a_1 = \frac{1 + a_2}{2} \text{ and } a_2 = \frac{1 + a_1}{2}$$

yields the Nash equilibrium $a_1 = a_2 = 1$, and this is the unique solution to these equations implying that this is the unique equilibrium. ■

(d) If the parties could sign a binding agreement on how much to campaign, what levels would they choose?

Answer: Both parties would be better off if they can choose not to spend money on negative campaigns. The payoffs for each player from the Nash equilibrium solved in part (c) are $v_i(1, 1) = -1$ while of they agreed not to spend anything they each would obtain zero. This is a variant of the Prisoners' Dilemma. ■

15.

16. **Hotelling's Price Competition:** Imagine a continuum of potential buyers, located on the line segment $[0, 1]$, with uniform distribution. (Hence, the "mass" or quantity of buyers in the interval $[a, b]$ is equal to $b - a$.) Imagine two firms, players 1 and 2 who are located at each end of the interval (player 1 at the 0 point and player 2 at the 1 point.) Each player i can choose its price p_i, and each customer goes to the vendor who offers them the highest value. However, price alone does not determine the value, but distance is important as well. In particular, each buyer who buys the product from player i has

a net value of $v - p_i - d_i$ where d_i is the distance between the buyer and vendor i, and represents the transportation costs of buying from vendor i Thus, buyer $a \in [0, 1]$ buys from 1 and not 2 if $v - p_1 - d_1 > v - p_2 - d_2$, and if buying is better than getting zero. (Here $d_1 = a$ and $d_2 = 1 - a$. The buying choice would be reversed if the inequality is reversed.) Finally, assume that the cost of production is zero.

(a) Assume that v is very large so that all the customers will be served by at least one firm, and that some customer $x^* \in [0, 1]$ is indifferent between the two firms. What is the best response function of each player?

Answer: Because customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, his indifference implies that

$$v - p_1 - x^* = v - p_2 - (1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1 + p_2 - p_1}{2}.$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$q_1(p_1, p_2) = x^* = \frac{1 + p_2 - p_1}{2},$$
$$q_1(p_1, p_2) = 1 - x^* = \frac{1 + p_1 - p_2}{2}.$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1 + p_2 - p_1}{2} \right) p_1$$

which yields the first order condition

$$1 + p_2 - 2p_1 = 0,$$

implying the best response function

$$p_1 = \frac{1}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{2} + \frac{p_1}{2}.$$

■

(b) Assume that $v = 1$. What is the Nash equilibrium? Is it unique?

Answer: If we use the best response functions calculated in part (a) above then we obtain a unique Nash equilibrium $p_1 = p_2 = 1$, and this implies that $x^* = \frac{1}{2}$ so that each firm gets half the market. However, when $v = 1$ then the utility of customer $x^* = \frac{1}{2}$ is $v - p_1 - \frac{1}{2} = 1 - 1 - \frac{1}{2} = -\frac{1}{2}$, implying that he would prefer not to buy, and by continuity, an interval of customers around x^* would also prefer not to buy. his violated the assumptions we used to calculate the best response functions.[2] So, the analysis in part (a) is invalid when $v = 1$. It is therefore useful to start with the monopoly case when $v = 1$ and see how each firm would have priced if the other is absent. Firm 1 maximizes

$$\max_{p_1} \ (1 - p_1)p_1$$

which yields the solution $p_1 = \frac{1}{2}$ so that everyone in the interval $x \in [0, \frac{1}{2}]$ wished to buy from firm 1 and no other customer would buy. By symmetry, if firm 2 were a monopoly then the solution would be $p_2 = \frac{1}{2}$ so that everyone in the interval $x \in [\frac{1}{2}, 1]$ would buy from firm 2 and no other customer would buy. But this implies that if both firms set their monopoly prices $p_1 = p_2 = \frac{1}{2}$ then each would maximize profits ignoring the other firm, and hence this is the (trivially) unique Nash equilibrium.

■

(c) Now assume that $v = 1$ and that the transportation costs are $\frac{1}{2}d_i$, so that a buyer buys from 1 if and only if $v - p_1 - \frac{1}{2}d_1 > v - p_2 - \frac{1}{2}d_2$ Write the best response function of each player and solve for the Nash

---

[2] We need $v \geq 1.5$ for customer $x^* = \frac{1}{2}$ to be just indifferent between buying and not buying when $p_1 = p_2 = 1$ All the other customers will strictly prefer buying.

Equilibrium.

Answer: Like in part (a), assume that customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, and he is indifferent between buying from either, so his indifference implies that

$$v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + p_2 - p_1 \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$q_1(p_1, p_2) \quad = \quad x^* = \frac{1}{2} + p_2 - p_1 \ ,$$

$$q_1(p_1, p_2) \quad = \quad 1 - x^* = \frac{1}{2} + p_1 - p_2 \ .$$

Firm 1's maximization problem is

$$\max_{p_1} \ \left(\frac{1}{2} + p_2 - p_1 \right) p_1$$

which yields the first order condition

$$\frac{1}{2} + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{4} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{4} + \frac{p_1}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \frac{1}{2}$, and

$x^* = \frac{1}{2}$. To verify that this is a Nash equilibrium notice that for customer x^*, the utility form buying from firm 1 is $v - p_1 - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$ implying that he is indeed indifferent between buying or not, which in turn implies that every other customer prefer buying over not buying. ∎

(d) Following your analysis in (c) above, imagine that transportation costs are αd_i, with $\alpha \in [0, \frac{1}{2}]$. What happens to the Nash equilibrium as $\alpha \to 0$? What is the intuition for this result?

Answer: Using the assumed indifferent customer x^*, his indifference implies that

$$v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*)$$

$$v - p_1 - \alpha x = v - p_2 - \alpha(1 - x)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$q_1(p_1, p_2) = x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ ,$$
$$q_1(p_1, p_2) = 1 - x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_1 - p_2) \ .$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \right) p_1$$

which yields the first order condition

$$\frac{1}{2} + \frac{p_2}{2\alpha} - \frac{p_1}{\alpha} = 0 \ ,$$

implying the best response function

$$p_1 = \frac{\alpha}{2} + \frac{p_2}{2} .$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{\alpha}{2} + \frac{p_1}{2}.$$

$$p_2 = \frac{\alpha}{2} + \frac{\frac{\alpha}{2} + \frac{p_2}{2}}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \alpha$, and $x^* = \frac{1}{2}$. From the analysis in (c) above we know that for any $\alpha \in [0, \frac{1}{2})$ customer x^* will strictly prefer to buy over not buying and so will every other customer. We see that as α decreases, so do the equilibrium prices, so that at the limit of $\alpha = 0$ the prices will be zero. The intuition is that the transportation costs d cause firms 1 and 2 to be differentiated, and this "softens" the Bertrand competition between the two firms. When the transportation costs are higher this implies that competition is less fierce and prices are higher, and the opposite holds for lower transportation costs.■

17.

18. **Political Campaigning:** Two candidates are competing in a political race. Each candidate i can spend $s_i \geq 0$ on adds that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability $\frac{1}{2}$ Each candidate values winning at a payoff of $v > 0$, and the cost of spending s_i is just s_i.

(a) Given two spend levels (s_1, s_2), write the expected payoff of a candidate i.

Answer: Player i's payoff function is

$$v_i(s_1, s_2) = \frac{s_i v}{s_1 + s_2} - s_i .$$

■

(b) What is the function that represents each player's best response function?

Answer: Player 1 maximizes his payoff $v_1(s_1, s_2)$ shown in (a) above and the first order optimality condition is,

$$\frac{v(s_1 + s_2) - s_1 v}{(s_1 + s_2)^2} - 1 = 0$$

and if we use $s_1(s_2)$ to denote player 1's best response function then it explicitly solves the following equality that is derived from the first-order condition,

$$[s_1(s_2)]^2 + 2s_1(s_2)s_2 + (s_2)^2 - vs_2 = 0 .$$

Because this is a quadratic equation we cannot write an explicit best response function (or correspondence). However, if we can graph $s_1(s_2)$ as shown in the following figure (the values correspond for the case of $v = 1$).
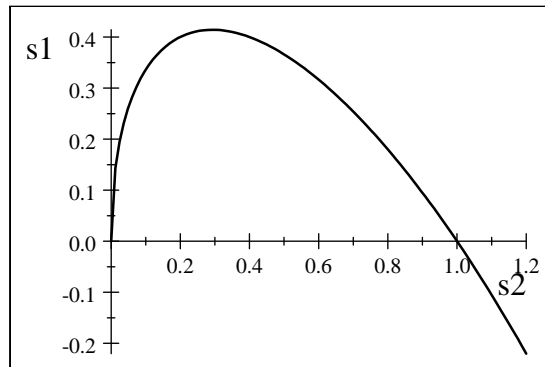
Similarly we can derive the symmetric function for player 2. ■

(c) Find the unique Nash equilibrium.

Answer: The best response functions are symmetric mirror images and have a symmetric solution where $s_1 = s_2$ in the unique Nash equilibrium. We can therefore use any one of the two best response functions and replace both variables with a single variable s,

$$s^2 + 2s^2 + s^2 - vs = 0 ,$$

or,

$$s = \frac{v}{4}$$

so that the unique Nash equilibrium has $s_1^* = s_2^* = \frac{v}{4}$. ∎

(d) What happens to the Nash equilibrium spending levels if v increases?

Answer: It is easy to see from part (c) that higher values of v cause the players to spend more in equilibrium. As the stakes of the prize rise, it is more valuable to fight over it. ∎

(e) What happens to the Nash equilibrium levels if player 1 still values winning at v, but player 2 values winning at kv where $k > 1$?

Answer: Now the two best response functions are not symmetric. The best response function of player 1 remains as above, but that of player 2 will now have kv instead of v,

$$(s_1)^2 + 2s_1 s_2 + (s_2)^2 - vs_2 = 0 \ . \tag{BR1}$$

and

$$(s_2)^2 + 2s_1 s_2 + (s_1)^2 - kvs_1 = 0 \ . \tag{BR2}$$

Subtracting (BR2) from (BR1) we obtain,

$$ks_1 = s_2,$$

which implies that the solution will no longer be symmetric and, moreover, $s_2 > s_1$, which is intuitive because now player 2 cares more about the prize. Using $ks_1 = s_2$ we substitute for s_2 in (BR1) to obtain,

$$(s_1)^2 + 2k(s_1)^2 + k^2(s_1)^2 - kvs_1 = 0$$

which results in,

$$s_1 = \frac{kv}{1 + 2k + k^2} < \frac{v}{1 + 2k + k^2} < \frac{v}{4}$$

where both inequalities follow from the fact that $k > 1$. From $ks_1 = s_2$ above we have

$$s_2 = \frac{k^2 v}{1 + 2k + k^2} > \frac{k^2 v}{k^2 + 2k^2 + k^2} = \frac{v}{4}$$

where the inequality follows from $k > 1$. ∎

6

Mixed Strategies

1.

2. Let σ_i be a mixed strategy of player i that puts positive weight on one strictly dominated pure strategy. Show that there exists a mixed strategy σ_i' that puts no weight on any dominated pure strategy and that dominates σ_i.

    Answer: Let player i have L pure strategies $S_i = \{s_{i1}, s_{i2}, ..., s_{iL}\}$ and let s_{ik} be a pure strategy which is strictly dominated by $s_{ik'}$, that is, $v_i(s_{ik'}, s_{-i}) > v_i(s_{ik'}, s_{-i})$ for any strategy profile of i's opponents s_{-i}. Let $\sigma_i = (\sigma_{i1}, \sigma_{i2}, ..., \sigma$ be a mixed strategy that puts some positive weight $\sigma_{ik} > 0$ on s_{ik} and let σ be identical to σ_i except that it puts weight 0 on s_{ik} and diverts that weight over to $s_{ik'}$. That is, $\sigma_{ik}' = 0$ and $\sigma_{ik'}' = \sigma_{ik'} + \sigma_{ik}$, and $\sigma_{il}' = \sigma_{il}$ for all $l \neq k$ and $l \neq k'$. It follows that for all s_{-i},

    $$v_i(\sigma_i', s_{-i}) = \sum_{l=1}^{L} \sigma_{il}' v_i(s_{il}, s_{-i}) > \sum_{l=1}^{L} \sigma_{il} v_i(s_{il}, s_{-i}) = v_i(\sigma_i', s_{-i})$$

    because $v_i(s_{ik'}, s_{-i}) > v_i(s_{ik'}, s_{-i})$ and the way in which σ_i' was constructed. Hence, σ_i is strictly dominated by σ_i'. ∎

3.

4. **Monitoring:** An employee (player 1) who works for a boss (player 2) can either work (W) or shirk (S), while his boss can either monitor the employee (M) or ignore him (I). Like most employee-boss relationships, if the employee is working then the boss prefers not to monitor, but if the boss is not monitoring then the employee prefers to shirk. The game is represented in the following matrix:

Player 2

		M	I
player 1	W	1, 1	1, 2
	S	0, 2	2, 1

(a) Draw the best response functions of each player.

Answer: Let p be the probability that player 1 chooses W and q the probability that player 2 chooses M. It follows that $v_1(W, q) > v_1(S, q)$ if and only if $1 > 2(1 - q)$, or $q > \frac{1}{2}$, and $v_2(p, M) > v_2(p, I)$ if and only if $p + 2(1 - p) > 2p + (1 - p)$, or $p < \frac{1}{2}$. It follows that for player 1,

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0, 1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

and for player 2,

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0, 1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases}.$$

Notice that these are identical to the best response functions for the matching pennies game (see Figure 6.3). ■

(b) Find the Nash equilibrium of this game. What kind of game does this game remind you of?

Answer: From the two best response correspondences the unique Nash equilibrium is $(p, q) = (\frac{1}{2}, \frac{1}{2})$ and the game's strategic forces are identical to those in the Matching Pennies game. ■

5.

6. **Declining Industry:** Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Every quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are $(-1, 1)$ because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) Write down this game in matrix form.

Answer: Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter.

<div align="center">

Player 2

		E	T	N
Player 1	E	$0, 0$	$0, 2$	$0, 4$
	T	$2, 0$	$-1, -1$	$-1, 1$
	N	$4, 0$	$1, -1$	$-2, -2$

</div>

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer: There are no strictly dominated strategies but there is a weakly dominated one: T. To see this note that choosing both E and N with probability $\frac{1}{2}$ each yields the same expected payoff as choosing T against E or N, and a higher expected payoff against T, and hence $\sigma_i = (\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{2}, 0, \frac{1}{2})$ weakly dominates T. The reason there is no strictly dominated strategy is that, starting with σ_i increasing the weight on E causes the mixed strategy to be worse than

T against E, while increasing the weight on N causes the mixed strategy to be worse than T against N, implying it is impossible to find a mixed strategy that strictly dominates T. ∎

(c) Find the pure strategy Nash equilibria.

Answer: Because T is weakly dominated, it is suspect of never being a best response. A quick observation should convince you that this is indeed the case: it is never a best response to any of the pure strategies, and hence cannot be part of a pure strategy Nash equilibrium. Removing T from consideration results in the reduced game:

Player 2

		E	N
Player 1	E	$0, 0$	$0, 4$
	N	$4, 0$	$-2, -2$

for which there are two pure strategy Nash equilibria, (E, N) and (N, E)
∎

(d) Find the unique mixed strategy Nash equilibrium (hint: you can use your answer to (b) to make things easier.)

Answer: We start by ignoring T and using the reduced game in part (c) by assuming that the weakly dominated strategy T will never be part of a Nash equilibrium. We need to find a pair of mixed strategies, $(\sigma_1(E), \sigma_1(N))$ and $(\sigma_2(E), \sigma_2(N))$ that make both players indifferent between E and N. For player 1 the indifference equation is,

$$0 = 4\sigma_2(E) - 2(1 - \sigma_2(E))$$

which results in $\sigma_2(E) = \frac{1}{3}$, and for player 2 the indifference equation is symmetric, resulting in $\sigma_1(E) = \frac{1}{3}$. Hence, the mixed strategy Nash equilibrium of the original game is $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$ Notice that at this Nash equilibrium, each player is not only indifferent between E and N, but choosing T gives the same expected payoff of zero. However, choosing T with positive probability cannot be part of

a mixed strategy Nash equilibrium. To prove this let player 2 play the mixed strategy $\sigma_2 = (\sigma_2(E), \sigma_2(T), \sigma_2(N)) = (\sigma_{2E}, \sigma_{2T}, 1 - \sigma_{2E} - \sigma_{2T})$ The strategy T for player 1 is at least as good as E if and only if,

$$0 \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2E} \geq \frac{1}{3}$. The strategy T for player 1 is at least as good as N if and only if,

$$4\sigma_{2E} - \sigma_{2T} - 2(1 - \sigma_{2E} - \sigma_{2T}) \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2T} \leq 1 - 3\sigma_{2E}$. But if $\sigma_{2E} \geq \frac{1}{3}$ (when T is as good as E) then $\sigma_{2T} \leq 1 - 3\sigma_{2E}$ reduces to $\sigma_{2T} \leq 0$, which can only hold when $\sigma_{2E} = \frac{1}{3}$ and $\sigma_{2T} = 0$ (which is the Nash equilibrium we found above). A symmetric argument holds to conclude that $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$ is the unique mixed strategy Nash equilibrium. ∎

7.

8. **Market entry:** There are 3 firms that are considering entering a new market. The payoff for each firm that enters is $\frac{150}{n}$ where n is the number of firms that enter. The cost of entering is 62.

   (a) Find all the pure strategy Nash equilibria.

   Answer: The costs of entry are 62 so the benefits of entry must be at least that for a firm to choose to enter. Clearly, if a firm believes the other two are not entering then it wants to enter, and if it believes that the other firms are entering then it would stay out (it would only get 50). If a firm believes that only one other firm is entering then it prefers to enter and get 75. Hence, there are three pure strategy Nash equilibria in which two of the three firms enter and one stays out. ∎

   (b) Find the symmetric mixed strategy equilibrium where all three players enter with the same probability.

   Answer: Let p be the probability that a firm enters. In order to be

willing to mix the expected payoff of entering must be equal to zero. If a firm enters then with probability p^2 it will face two other entrants and receive $v_i = 50 - 62 = -12$, with probability $(1-p)^2$ it will face no other entrants and receive $v_i = 150 - 62 = 88$, and with probability $2(1-p)p$ it will face one other entrant and receive $v_i = 75 - 62 = 13$ Hence, to be willing to mix the expected payoff must be zero,$p^2 + 1 - p^2$

$$(1-p)^2 88 + 2(1-p)p13 - p^2 12 = 0$$

which results in the quadratic equation $25p^2 - 75p + 44 = 0$, and the relevant solution (between 0 and 1) is $p = \frac{4}{5}$. ∎

9.

10. **Continuous all pay auction:** Consider an all-pay auction for a good worth 1 to each of the two bidders. Each bidder can choose to offer a bid from the unit interval so that $S_i = [0, 1]$. Players only care about the expected value they will end up with at the end of the game (i.e., if a player bids 0.4 and expects to win with probability 0.7 then his payoff is $0.7 \times 1 - 0.4$).

(a) Model this auction as a normal-form game.

Answer: There are two players, $N = \{1, 2\}$, each has a strategy set $S_i = [0, 1]$, and assuming that the players are equally likely to get the good in case of a tie, the payoff to player i is given by

$$v_i(s_i, s_j) = \begin{cases} 1 - s_i & \text{if } s_i > s_j \\ \frac{1}{2} - s_i & \text{if } s_i = s_j \\ -s_i & \text{if } s_i < s_j \end{cases}$$

(b) Show that this game has no pure strategy Nash Equilibrium.

Answer: First, it cannot be the case that $s_i = s_j < 1$ because then each player would benefit from raising his bid by a tiny amount ε in order to win the auction and receive a higher payoff $1 - \varepsilon - s_i > \frac{1}{2} - s_i$. Second,

it cannot be the case that $s_i = s_j = 1$ because each player would prefer to bid nothing and receive $0 > -\frac{1}{2}$. Last, it cannot be the case that $s_i > s_j \geq 0$ because then player i would prefer to lower his bid by ε while still beating player j and paying less money. Hence, there cannot be a pure strategy Nash equilibrium. ■

(c) Show that this game cannot have a Nash Equilibrium in which each player is randomizing over a finite number of bids.

Answer: Assume that a Nash equilibrium involves player 1 mixing between a finite number of bids, $\{s_{11}, s_{12}, ..., s_{1K}\}$ where $s_{11} \geq 0$ is the lowest bid, $s_{1K} \leq 1$ is the highest, $s_{1k} < s_{1(k+1)}$ and each bid s_{1k} is being played with some positive probability σ_{1k}. Similarly assume that player 2 is mixing between a finite number of bids, $\{s_{21}, s_{22}, ..., s_{2L}\}$ and each bid s_{2l} is being played with some positive probability σ_{2l}. (i) First observe that it cannot be true that $s_{1K} < s_{2L}$ (or the reverse by symmetry). If it were the case then player 2 will win for sure when he bids s_{2L} and pay his bid, while if he reduces his bid by some ε such that $s_{1K} < s_{2L} - \varepsilon$ then he will still win for sure and pay less, contradicting that playing s_{2L} was part of a Nash equilibrium. (ii) Second observe that when $s_{1K} = s_{2L}$ then the expected payoff of player 2 from bidding s_{2L} is

$$
\begin{aligned}
Ev_2 &= \Pr\{s_{1k} < s_{2L}\}(1 - s_{2L}) + \Pr\{s_{1k} = s_{2L}\}(\frac{1}{2} - s_{2L}) \\
&= (1 - \sigma_{1K})(1 - s_{2L}) + \sigma_{1K}(\frac{1}{2} - s_{2L}) \\
&= 1 - s_{2L} - \frac{1}{2}\sigma_K \geq 0
\end{aligned}
$$

where the last inequality follows from the fact that $\sigma_{2L} > 0$ (he would not play it with positive probability if the expected payoff were negative.) Let $s'_{2L} = s_{2L} + \varepsilon$ where $\varepsilon = \frac{1}{4}\sigma_K$. If instead of bidding s_{2L} player 2 bids s'_{2L} then he wins for sure and his utility is

$$
v_2 = 1 - s'_{2L} = 1 - s_{2L} - \frac{1}{4}\sigma_K > 1 - s_{2L} - \frac{1}{2}\sigma_K
$$

contradicting that playing s_{2L} was part of a Nash equilibrium. ■

(d) Consider mixed strategies of the following form: Each player i chooses and interval, $[\underline{x}_i, \overline{x}_i]$ with $0 \leq \underline{x}_i < \overline{x}_i \leq 1$ together with a cumulative distribution $F_i(x)$ over the interval $[\underline{x}_i, \overline{x}_i]$. (Alternatively you can think of each player choosing $F_i(x)$ over the interval $[0, 1]$, with two values $\underline{x}$ and $\overline{x}_i$ such that $F_i(\underline{x}_i) = 0$ and $F_i(\overline{x}_i) = 1$.)

i. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2$ and $\overline{x}_1 = \overline{x}_2$.

Answer: Assume not. There are two cases: (a) $\underline{x}_1 \neq \underline{x}_2$: Without loss assume that $\underline{x}_1 < \underline{x}_2$. This means that there are values of $s'_1 \in (\underline{x}_1, \underline{x}_2)$ for which $s'_1 > 0$ but for which player 1 loses with probability 1. This implies that the expected payoff from this bid is negative, and player would be better off bidding 0 instead. Hence, $\underline{x}_1 = \underline{x}_2$ must hold. (b) $\overline{x}_1 \neq \overline{x}_2$: Without loss assume that $\overline{x}_1 < \overline{x}_2$. This means that there are values of $s'_2 \in (\overline{x}_1, \overline{x}_2)$ for which $\overline{x}_1 < s'_2 < 1$ but for which player 2 wins with probability 1. But then player 2 could replace s'_2 with $s''_2 = s'_2 - \varepsilon$ with ε small enough such that $\overline{x}_1 < s''_2 < s'_2 < 1$, he will win with probability 1 and pay less than he would pay with s'_2. Hence, $\overline{x}_1 = \overline{x}_2$ must hold. ∎

ii. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2 = 0$.

Answer: Assume not so that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$. This means that when player i bids $\underline{x}$ then he loses with probability 1, and get an expected payoff of $-\underline{x} < 0$. But instead of bidding $\underline{x}$ player i can bid 0 and receive 0 which is better than $-\underline{x}$, implying that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$ cannot be an equilibrium. ∎

iii. Using the above, argue that if two such strategies are a mixed strategy Nash equilibrium then both players must be getting an expected payoff of zero.

Answer: As proposition 6.1 states, if a player is randomizing between two alternatives then he must be indifferent between them. Because both players are including 0 in the support of their mixed

strategy, their payoff from 0 is zero, and hence their expected payoff from any choice in equilibrium must be zero. ■

iv. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\overline{x}_1 = \overline{x}_2 = 1$.

Answer: Assume not so that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$. From (iii) above the expected payoff from any bid in $[0, \overline{x}]$ is equal to zero. If one of the players deviates from this strategy and choose to bid $\overline{x} + \varepsilon < 1$ then he will win with probability 1 and receive a payoff of $1 - (\overline{x} + \varepsilon) > 0$ contradicting that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$ is an equilibrium. ■

v. Show that $F_i(x)$ being uniform over $[0, 1]$ is a symmetric Nash equilibrium of this game.

Answer: Imagine that player 2 is playing according to the proposed strategy $F_2(x)$ uniform over $[0, 1]$. If player 1 bids some value $s_1 \in [0, 1]$ then his expected payoff is

$$\Pr\{s_1 > s_2\}(1-s_1)+\Pr\{s_1 < s_2\}(-s_1) = s_1(1-s_1)+(1-s_1)(-s_1) = 0$$

implying that player 1 is willing to bid any value in the $[0, 1]$ interval, and in particular, choosing a bid according to $F_1(x)$ uniform over $[0, 1]$. Hence, this is a symmetric Nash equilibrium. ■

11.

12. **The Tax Man:** A citizen (player 1) must choose whether or not to file taxes honestly or whether to cheat. The tax man (player 2) decides how much effort to invest in auditing and can choose $a \in [0, 1]$, and the cost to the tax man of investing at a level a is $c(a) = 100a^2$. If the citizen is honest then he receives the benchmark payoff of 0, and the tax man pays the auditing costs without any benefit from the audit, yielding him a payoff of $(-100a^2)$. If the citizen cheats then his payoff depends on whether he is caught. If he is caught then his payoff is (-100) and the tax man's payoff is $100 - 100a^2$. If he is not caught then his payoff is 50 while the tax man's payoff is $(-100a^2)$. If the

citizen cheats and the tax man chooses to audit at level a then the citizen is caught with probability a and is not caught with probability $(1 - a)$.

(a) If the tax man believes that the citizen is cheating for sure, what is his best response level of a?

Answer: The tax man maximizes $a(100 - 100a^2) + (1 - a)(0 - 100a^2) = 100a - 100a^2$. The first-order optimality condition is $100 - 200a = 0$ yielding $a = \frac{1}{2}$. ∎

(b) If the tax man believes that the citizen is honest for sure, what is his best response level of a?

Answer: The tax man maximizes $-100a^2$ which is maximized at $a = 0$ ∎

(c) If the tax man believes that the citizen is honest with probability p what is his best response level of a as a function of p?

Answer: The tax man maximizes $p(-100a^2) + (1 - p)(100a - 100a^2) = 100(1 - p)a - 100a^2$. The first-order optimality condition is $100(1 - p) - 200a = 0$, yielding the best response function $a^*(p) = \frac{1-p}{2}$. ∎

(d) Is there a pure strategy Nash equilibrium of this game? Why or why not?

Answer: There is no pure strategy Nash equilibrium. To see this, consider the best response of player 1 who believes that player 2 chooses some level $a \in [0, 1]$. His payoff from being honest is 0 while his payoff from cheating is $a(-100) + (1 - a)50 = 50 - 150a$. Hence, he prefers to be honest if and only if $0 \geq 50 - 150a$, or $a \geq \frac{1}{3}$. Letting $p^*(a)$ denote the best response correspondence of player 1 as the probability that he is honest, we have that

$$p^*(a) = \begin{cases} 1 & \text{if } a > \frac{1}{3} \\ [0, 1] & \text{if } a = \frac{1}{3} \\ 0 & \text{if } a < \frac{1}{3} \end{cases}$$

and it is easy to see that there are no values of a and p for which both players are playing mutual best responses. ∎

(e) Is there a mixed strategy Nash equilibrium of this game? Why or why not?

Answer: From (d) above we know that player 1 is willing to mix if and only if $a = \frac{1}{3}$, which must therefore hold true in a mixed strategy Nash equilibrium. For player 2 to be willing to play $a = \frac{1}{3}$ we use his best response from part (c), $\frac{1}{3} = \frac{1-p}{2}$, which yields, $p = \frac{1}{3}$. Hence, the unique mixed strategy Nash equilibrium has player 1 being honest with probability $\frac{1}{3}$ and player 2 choosing $a = \frac{1}{3}$. ∎

Part III

Dynamic Games of Complete Information

7

Preliminaries

1.

2. **Strategies and equilibrium:** Consider a two player game in which player 1 can choose A or B. The game ends if he chooses A while it continues to player 2 if he chooses B. Player 2 can then choose C or D, with he game ending after C and continuing again with player 1 after D. Player 1 then can choose E or F, and the game ends after each of these choices.

   (a) Model this as an extensive form game tree. Is it a game of perfect or imperfect information?
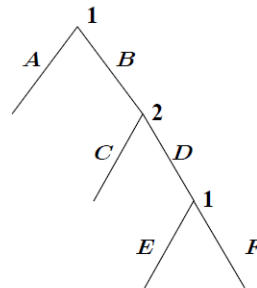
      Answer:

      This game is a game of perfect information. ■

(b) How many terminal nodes does the game have? How many information
sets?

Answer: The game has 4 terminal nodes (after choices A, C, E and F)
and 3 information sets (one for each player. ■

(c) How many pure strategies does each player have?

Answer: Player 1 has 4 pure strategies and player 2 has 2. ■

(d) Imagine that the payoffs following choice A by player 1 are $(2,0)$, fol-
lowing C by player 2 are $(3,1)$, following E by player 1 are $(0,0)$ and
following F by player 1 are $(1,2)$. What are the Nash equilibria of this
game? Does one strike you as more "appealing" than the other? If so,
explain why.

Answer: We can write down the matrix form of this game as follows
(xy denotes a strategy for player 1 where $x \in \{A, B\}$ is what he does
in his first information set and $y \in \{E, F\}$ in his second one),

<center>Player 2</center>

		C	D
	AE	$\overline{2,0}$	$\overline{2,0}$
Player 1	AF	$2,0$	$\overline{2,0}$
	BE	$\underline{3,\overline{1}}$	$0,0$
	BF	$\underline{3,1}$	$\overline{1,\overline{2}}$

It's easy to see that there are three pure strategy Nash equilibria:
$(AE, D), (AF, D)$ and (BE, C). The equilibria $(AE, D), (AF, D)$ are
Pareto dominated by the equilibrium (BE, C), and hence it would be
tempting to argue that (BE, C) is the more "appealing" equilibrium.
As we will see in Chapter 8 it is actually (AF, D) that has properties
that are more appealing (sequential rationality). ■

3.

4. **Centipedes**: Imagine a two player game that proceeds as follows. A pot of money is created with \$6 in it initially. Player 1 moves first, then player 2, then player 1 again and finally player 2 again. At each player's turn to move, he has two possible actions: grab (G) or share (S). If he grabs, he gets $\frac{2}{3}$ of the current pot of money, the other player gets $\frac{1}{3}$ of the pot and the game ends. If he shares then the size of the current pot is multiplied by $\frac{3}{2}$ and the next player gets to move. At the last stage in which player 2 moves, if he chooses share then the pot is still multiplied by $\frac{3}{2}$, player 2 gets $\frac{1}{3}$ of the pot and player 1 gets $\frac{2}{3}$ of the pot.

(a) Model this as an extensive form game tree. Is it a game of perfect or imperfect information?

Answer:
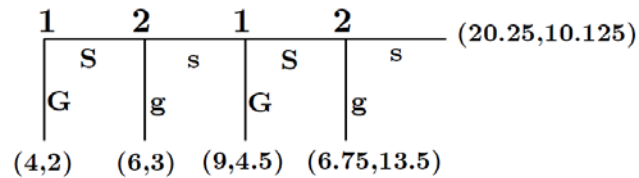
This is a game of perfect information. Note that we draw the game from left to right (which is the common convention for "centipede games" of this sort.) We use capital letters for player 1 and lower case for player 2. ∎

(b) How many terminal nodes does the game have? How many information sets?

Answer: The game has five terminal nodes and four information sets. ∎

(c) How many pure strategies does each player have?

Answer: Each player has four pure strategies (2 actions in each of his 2 information sets). ∎

(d) Find the Nash equilibria of this game. How many outcomes can be supported in equilibrium?

Answer: Using the convention of xy to denote a strategy of player where he chooses x in his first information set and y in his second, we can draw the following matrix representation of this game,

Player 2

		gg	gs	sg	ss
	GG	$\underline{\underline{4,2}}$	$\underline{\underline{4,2}}$	$\overline{4,2}$	$\overline{4,2}$
Player 1	GS	$\underline{\underline{4,2}}$	$\underline{\underline{4,2}}$	$\overline{4,2}$	$\overline{4,2}$
	SG	$\overline{3,6}$	$\overline{3,6}$	$\underline{9,4.5}$	$9,4.5$
	SS	$3,6$	$3,6$	$\overline{6.75,13.5}$	$\underline{20.25,10.125}$

We see that only one outcome can be supported as a Nash equilibrium: player 1 grabs immediately and the players' payoffs are $(4,2)$. ∎

(e) Now imagine that at the last stage in which player 2 moves, if he chooses to share then the pot is equally split among the players. Does your answer to part (d) above change?

Answer: The answer does change because the payoffs from the pair of strategies (SS, ss) changes from $(20.25, 10.125)$ to $(15.1875, 15.1875)$ in which case player 2's best response to SS will be ss, and player 1's best response to ss remains SS, so that (SS, ss) is another Nash equilibrium in which they split 30.375 equally (the previous Nash equilibria are still equilibria). ∎

5.

6. **Entering an Industry:** A firm (player 1) is considering entering an established industry with one incumbent firm (player 2). Player 1 must choose whether to enter or to not enter the industry. If player 1 enters the industry then player 2 can either accommodate the entry, or fight the entry with a price war. Player 1's most preferred outcome is entering with player 2 not fighting, and his least preferred outcome is entering with player 2 fighting. Player 2's most preferred outcome is player 1 not entering, and his least preferred outcome is player 1 entering with player 2 fighting.

(a) Model this as an extensive form game tree (choose payoffs that represent the preferences).

Answer:
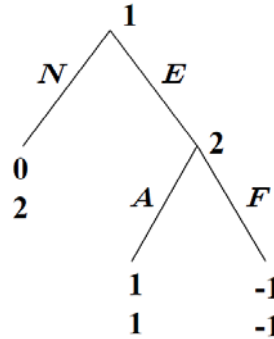
(b) How many pure strategies does each player have?

Answer: Each player has two pure strategies. ■

(c) Find all the Nash equilibria of this game.

Answer: There are two Nash equilibria which can be seen in the matrix,

Player 2

Player 1		A	F
	N	$\overline{0,2}$	$\overline{0,2}$
	E	$\overline{1,1}$	$-1,-1$

Both (N, F) and (E, A) are Nash equilibria of this game. ■

7.

8. **Brothers**: Consider the following game that proceeds in two steps: In the first stage one brother (player 2) has two $10 bills and can choose one of two options: he can give his younger brother (player 1) $20, or give him one of the $10 bills (giving nothing is inconceivable given the way they were raised.) This money will be used to buy snacks at the show they will see, and each

one dollar of snack yields one unit of payoff for a player who uses it. The
show they will see is determined by the following "battle of the sexes" game:

Player 2

	O	F
O	16,12	0,0
F	0,0	12,16

Player 1

(a) Present the *entire* game in extensive form (a game tree).

Answer: Let the choices of player 1 first be S **for spliting the**
$20 and G for giving it all away. The entire game will have the payoffs
from the choice of how to split the money added to the payoffs from the
Battle of the Sexes part of the game as follows,
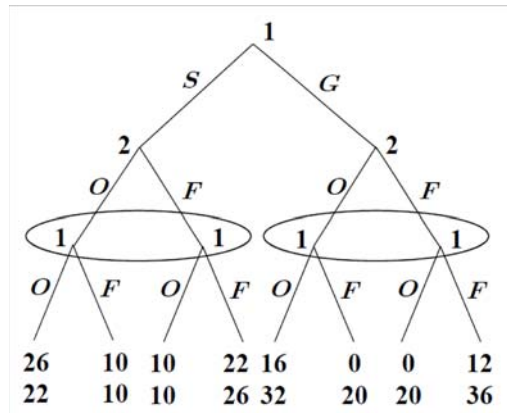
Because the latter is simultaneous, it does not mater which player moves
after player 1 as long as the last player cannot distinguish between the
choice of the player who moves just before him. ∎

(b) Write the (pure) strategy sets for both players.

Answer: Both players can condition their choice in the Battle of the
Sexes game on the initial split/give choice of player 1. For player 2,
$S_2 = \{OO, OF, FO, FF\}$ where $s_2 = xy$ means that player 2 chooses
$x \in \{O, F\}$ after player 1 chose S while player 2 chooses $y \in \{O, F\}$
after player 1 chose G. For player 1, however, even though he chooses
first between S or G, he must specify his action for each information

set even if he knows it will not happen (e.g., what he will do following S even when he plans to play G). Hence, he has 8 pure strategies, $S_1 = \{SOO, SOF, SFO, SFF, GOO, GOF, GFO, GFF\}$ where $s_1 = xyz$ means that player 1 first chooses $x \in \{S, G\}$ and then chooses $y \in \{O, F\}$ if he played S and $z \in \{O, F\}$ if he played G. ■

(c) Present the *entire* game in one matrix.

Answer: This will be a 8×4 matrix as follows,

Player 2

	OO	OF	FO	FF
SOO	26, 22	26, 22	10, 10	10, 10
SOF	26, 22	26, 22	10, 10	10, 10
SFO	10, 10	10, 10	22, 26	22, 26
SFF	10, 10	10, 10	22, 26	22, 26
GOO	16, 32	0, 20	16, 32	0, 20
GOF	0, 20	12, 36	0, 20	12, 36
GFO	16, 32	0, 20	16, 32	0, 20
GFF	0, 20	12, 36	0, 20	12, 36

Player 1 labels the rows.

(d) Find the Nash equilibria of the *entire* game (pure and mixed strategies).

Answer: First note that for player 1, mixing equally between SOO and SFO will strictly dominate the four strategies GOO, GOF, GFO and GFF. Hence, we can consider the reduced 4×4 game,

Player 2

	OO	OF	FO	FF
SOO	26, 22	26, 22	10, 10	10, 10
SOF	26, 22	26, 22	10, 10	10, 10
SFO	10, 10	10, 10	22, 26	22, 26
SFF	10, 10	10, 10	22, 26	22, 26

The simple overline-underline method shows that we have eight pure strategy Nash equilibria, four yielding the payoffs $(26, 22)$ and the other

four yielding $(22, 26)$. Because of each players indifference between the ways in which the payoffs are reached, there are infinitely many mixed strategies that yield the same payoffs. For example, any profile where player 1 mixes between SOO and SOF and where player 2 mixes between OO and OF will be a Nash equilibrium that yields $(26, 22)$. Similarly, any profile where player 1 mixes between SFO and SFF and where player 2 mixes between FO and FF will be a Nash equilibrium that yields $(22, 26)$. There is, however, one more class of mixed strategy Nash equilibria that are similar to the one found in section 6.2.3. To see this, focus on an even simpler game where we eliminate the duplicate payoffs as follows,

		Player 2	
		OO	FF
Player 1	SOO	$26, 22$	$10, 10$
	SFF	$10, 10$	$22, 26$

which preserve the nature of the game. For player 1 to be indifferent between SOO and SFF it must be that player 2 chooses OO with probability q such that

$$26q + 10(1 - q) = 10q + 22(1 - q)$$

which yields $q = \frac{3}{7}$. Similarly, for player 2 to be indifferent between OO and FF it must be that player 1 chooses SOO with probability p such that

$$22p + 10(1 - p) = 10p + 26(1 - p)$$

which yields $p = \frac{4}{7}$. Hence, we found a mixed strategy Nash equilibrium that results in each player getting an expected payoff of $26 \times \frac{3}{7} + 10 \times \frac{4}{7} = 16\frac{6}{7}$. Notice, however, that player 1 is always indifferent between SOO and SOF, as well as between SFO and SFF so there are infinitely many ways to achieve this kind of mixed strategy, and similarly for player 2 because of his indifference between OO and OF as well as FO and FF. ∎
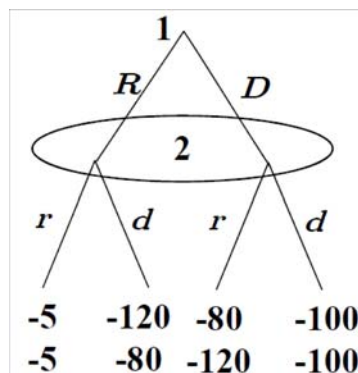
8

Credibility and Sequential Rationality

1.

2. **Mutually Assured Destruction (revisited):** Consider the game in section **??**.

   (a) Find the mixed strategy equilibrium of the war stage game and argue that it is unique.

      Answer: The war-game in the text has a weakly dominated Nash equilibrium (D, d) and hence does not have an equilibrium in which any player is mixing. This exercise should have replaced the war-stage game with the following game:

The subgame we called the war-stage game is given in the following matrix:

Player 2

		r	d
Player 1	R	$-5, -5$	$-120, -80$
	D	$-80, -120$	$-100, -100$

Let player 1 choose R with probability p and player 2 choose r with probability q. For player 2 to be indifferent it must be that $\frac{4}{19}(-5) + (1 - \frac{4}{19})(-120) = -\frac{1820}{19}$

$$p(-5) + (1 - p)(-120) = p(-80) + (1 - p)(-100)$$

and the solution is $p = \frac{4}{19}$. By symmetry, for player 1 to be indifferent it must be that $q = \frac{4}{19}$. Hence, $(p, q) = (\frac{4}{19}, \frac{4}{19})$ is the unique mixed strategy Nash equilibrium of this subgame with expected payoffs of $(v_1, v_2) = (-95.78, -95.79)$. ∎

(b) What is the unique subgame perfect equilibrium that includes the mixed strategy you found above?

Answer: Working backward, player 2 would prefer to choose B over N and player 1 would prefer E over I.

3.

4. **The Industry Leader:** Three oligopolists operate in a market with inverse demand given by $P(Q) = a - Q$, where $Q = q_1 + q_2 + q_3$, and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, c and no fixed cost. The firms choose their quantities dynamically as follows: (1) Firm 1, who is the industry leader, chooses $q_1 \geq 0$; (2) Firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3 respectively.

(a) How many proper subgames does this dynamic game have? Explain Briefly.

Answer: There are infinitely many proper subgames because every quantity choice of payer 1 results in a proper subgame. ∎

(b) Is it a game of perfect or imperfect information? Explain Briefly.

Answer: This is a game of imperfect information because players 2 and 3 make their choice without observing each other's choice first. ∎

(c) What is the subgame perfect equilibrium of this game? Show that it is unique.

Answer: first we solve for the Nash equilibrium of the simultaneous move stage in which players 2 and 3 make their choices as a function of the choice made first by player 1. Given a choice of q_1 and a belief about q_3, player 2 maximizes

$$\max_{q_2} \ (a - (q_1 + q_2 + q_3) - c)q_2$$

which leads to the first order condition

$$a - q_1 - q_3 - c - 2q_2 = 0$$

yielding the best response function

$$q_2 = \frac{a - q_1 - q_3 - c}{2},$$

and symmetrically, the best response function of player 3 is

$$q_3 = \frac{a - q_1 - q_2 - c}{2}.$$

Hence, following any choice of q_1 by player 1, the unique Nash equilibrium in the resulting subgame is the solution to the two best response functions, which yields

$$q_2^*(q_1) = q_3^*(q_1) = \frac{a - c - q_1}{3}.$$

Moving back to player 1's decision node, he will choose q_1 knowing that q_2 and q_3 be be chosen using the best response function above, and hence player 1 maximizes,

$$\max_{q_1} \ (a - (q_1 + \frac{a - c - q_1}{3} + \frac{a - c - q_1}{3}) - c)q_1$$

which leads to the first order equation

$$\frac{1}{3}(a - c - 2q_1) = 0$$

resulting in a unique solution $q_1 = \frac{a-c}{2}$. Hence, the unique subgame perfect equilibrium dictates that $q_1^* = \frac{a-c}{2}$, and $q_2^*(q_1) = q_3^*(q_1) = \frac{a-c-q_1}{3}$ ∎

(d) Find a Nash equilibrium that is not a subgame perfect equilibrium.

Answer: There are infinitely many Nash equilibria of the form "if player 1 plays q_1' then players 2 and 3 play $q_2^*(q_1') = q_3^*(q_1') = \frac{a-c-q_1'}{3}$, and otherwise they play $q_2 = q_3 = a$." In any such Nash equilibrium, players 2 and 3 are playing a Nash equilibrium on the equilibrium path (following q_1') while they are flooding the market and casing the price to be zero off the equilibrium path. One example would be $q_1' = 0$. In this case, following $q_1' = 0$ the remaining two players play the duopoly Nash equilibrium, and player 1 gets zero profits. If player 1 were to choose any positive quantity, his belief is that players 2 and 3 will flood the market and he will earn $-cq_1 < 0$, so he would prefer to choose $q_1' = 0$ given those beliefs. Of course, the threats of players 2 and 3 are not sequentially rational, which is the reason that this Nash equilibrium is not a subgame perfect equilibrium. ∎

5.

6. **Investment in the Future:** Consider two firms that play a Cournot competition game with demand $p = 100 - q$, and costs for each firm given by $c_i(q_i) = 10q_i$. Imagine that before the two firms play the Cournot game, firm 1 can invest in cost reduction. If it invests, the costs of firm 1 will drop to $c_1(q_1) = 5q_1$. The cost of investment is $F > 0$. Firm 2 does not have this investment opportunity.

(a) Find the value F^* for which the unique subgame perfect equilibrium involves firm 1 investing.

Answer: If firm 1 does not invest then they are expected to play the Cournot Nash equilibrium where both firms have costs of $10q_i$. Each firm solves,

$$\max_{q_i} \ (100 - (q_i + q_j) - 10)q_i$$

which leads to the first order condition

$$90 - q_j - 2q_i = 0$$

yielding the best response function

$$q_i(q_j) = \frac{90 - q_j}{2} \ ,$$

and the unique Cournot Nash equilibrium is $q_1 = q_2 = 30$ with profits $v_1 = v_2 = 900$. If firm 1 does invest then for firm 1 the problem becomes

$$\max_{q_1} \ (100 - (q_1 + q_2) - 5)q_1$$

which leads to the best response function

$$q_1(q_2) = \frac{95 - q_2}{2} \ .$$

For firm 2 the best response function remains the same as solved earlier with costs $10q_2$, so the unique Cournot Nash equilibrium is now solved using both equations,

$$q_1 = \frac{95 - \frac{90 - q_1}{2}}{2} \ ,$$

which yields $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and profits are $v_1 = 1{,}111\frac{1}{9}$ while $v_2 = 802\frac{7}{9}$. Hence, the increase in profits from the equilibrium with investment for player 1 are $F^* = 1{,}111\frac{1}{9} - 900 = 211\frac{1}{9}$, which is the most that player 1 would be willing to pay for the investment anticipating that they will play the Cournot Nash equilibrium after any choice of player 1 regarding investment. If $F < F^*$ then the unique subgame perfect equilibrium is that first, player 1 invests, then they players choose $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and if player 1 did not invest the payers choose $q_1 = q_2 = 30$ (Note that if $F > F^*$ then the unique subgame perfect equilibrium is that first, player 1 does not invest, then they players choose $q_1 = q_2 = 30$, and if player 1 did invest the payers choose $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$.) ∎

(b) Assume that $F > F^*$. Find a Nash equilibrium of the game that is not subgame perfect.

Answer: We construct a Nash equilibrium in which player 1 will invest despite $F > F^*$. Player 2's strategy will be, play $q_2 = \frac{85}{3}$ if player 1 invests, and $q_2 = 100$ if he does not invest. With this belief, if player 1 does not invest then he expects the price to be 0, and his best response is $q_1 = 0$ leading to profits $v_1 = 0$. If he invests then his best response to $q_2 = \frac{85}{3}$ is $q_1 = \frac{100}{3}$, which together are a Nash equilibrium in the Cournot game after investment. For any $F < 1,111\frac{1}{9}$ this will lead to positive profits, and hence, for $F^* < F < 1,111\frac{1}{9}$ the strategy of player 2 described above, together with player 1 choosing to invest, play $q_1 = \frac{100}{3}$ if he invests and $q_1 = 0$ if he does not is a Nash equilibrium. It is not subgame perfect because in the subgame following no investment, the players are not playing a Nash equilibrium. ∎

7.

8. **Entry Deterrence 1:** NSG is considering entry into the local phone market in the Bay Area. The incumbent S&P, predicts that a price war will result if NSG enters. If NSG stays out, S&P earns monopoly profits valued at $10 million (net present value, or NPV of profits), while NSG earns zero. If NSG enters, it must incur irreversible entry costs of $2 million. If there is a price war, each firm earns $1 million (NPV). S&P always has the option of accommodating entry (i.e., not starting a price war). In such a case, both firms earn $4 million (NPV). Suppose that the timing is such that NSG first has to choose whether or not to enter the market. Then S&P decides whether to "accommodate entry" or "engage in a price war." What is the subgame perfect equilibrium outcome to this sequential game? (Set up a game tree.)

Answer: Letting NSG be player 1 and S&P be player 2,
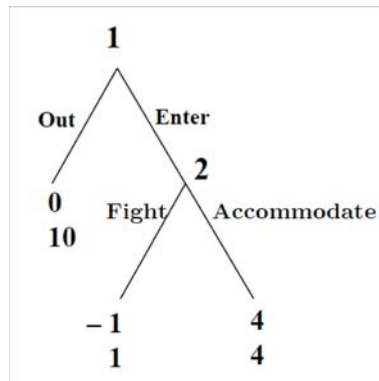
Backward induction implies that player 2 will Accommodate, and player 1 will therefore enter. Hence, the unique subgame perfect equilibrium is (Enter,Accommodate). ∎

9.

10. **Playing it safe:** Consider the following dynamic game: Player 1 can choose to play it safe (denote this choice by S), in which case both he and player 2 get a **payoff of 3 each**, or he can risk playing a game with player 2 (denote this choice by R). If he chooses R, then they play the following **simultaneous move** game:

Player 2

		A	B
player 1	C	$8,0$	$0,2$
	D	$6,6$	$2,2$

(a) Draw a game tree that represents this game. How many proper subgames does it have?

Answer:
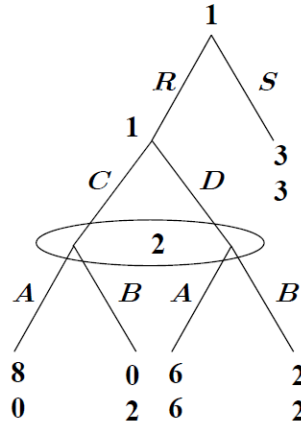
The game has two proper subgames: the whole game and the subgame starting at the node where 1 chooses between C and D. ∎

(b) Are there other game trees that would work? Explain briefly.

Answer: Yes - it is possible to have player 2 move after 1's initial move, and then have player 1 with an information set as follows:
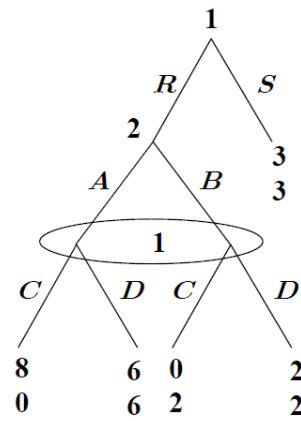
(c) Construct the matrix representation of the normal form of this dynamic game.

Answer: The game can be represented by the following matrix,

Player 2

		A	B
	RC	8, 0	0, 2
Player 1	RD	6, 6	2, 2
	SC	3, 3	3, 3
	SD	3, 3	3, 3

■

(d) Find all the Nash and subgame perfect equilibria of the dynamic game.

Answer: It is easy to see that there are two pure strategy Nash equilibria: (SC, B) and (SD, B). It follows immediately that there are infinitely many mixed strategy Nash equilibria in which player 1 is mixing between SC and SD in any arbitrary way and player 2 chooses B. It is also easy to see that following a choice of R, there is no pure strategy Nash equilibrium in the resulting subgame. To find the mixed strategy Nash equilibrium in that subgame, let player 1 choose RC with probability p and RD with probability $(1-p)$, and let player 2 choose A with probability q. For player 2 to be indifferent it must be that

$$p(0) + (1 - p)(6) = p(2) + (1 - p)(2)$$

and the solution is $p = \frac{2}{3}$. Similarly, for player 1 to be indifferent it must be that

$$q(8) + (1 - q)(0) = q(6) + (1 - q)(2)$$

and the solution is $q = \frac{1}{2}$. Hence, $(p, q) = (\frac{1}{2}, \frac{3}{5})$ is a mixed strategy Nash equilibrium of the subgame after player 1 chooses R, yielding expected payoffs of $(v_1, v_2) = (4, 2)$. In any subgame perfect equilibrium the players will have to play this mixed strategy equilibrium following R, and because $4 > 3$ player 1 will prefer R over S. Hence, choosing R followed by the mixed strategy computed above is the unique subgame perfect equilibrium. ■

11.

12. **Agenda Setting:** An agenda-setting game is described as follows. The "issue space" (set of possible policies) is an interval $X = [0, 5]$. An Agenda Setter (player 1) proposes an alternative $x \in X$ against the status quo $q = 4$ After player 1 proposes x, the Legislator (player 2) observes the proposal and selects between the proposal x and the status quo q. Player 1's most preferred policy is 1, and for any final policy $y \in X$, his payoff is given by

$$v_1(y) = 10 - |y - 1|,$$

where $|y - 1|$ denotes the absolute value of $(y - 1)$. Player 2's most preferred policy is 3, and for any final policy $y \in X$, her payoff is given by

$$v_2(y) = 10 - |y - 3|.$$

That is, each player prefers policies that are closer to their most preferred policy.

(a) Write the game down as a normal form game. Is this a game of perfect or imperfect information?

   Answer: There are two players, $i \in \{1, 2\}$ with strategy sets $S_1 = X = [0, 5]$ and $S_2 = \{A, R\}$ where A denotes accepting the proposal $x \in X$ and R means rejecting it and adopting the status quo $q = 4$. The payoffs are given by

$$v_1(s_1, s_2) = \begin{cases} 10 - |s_1 - 1| & \text{if } s_2 = A \\ 7 & \text{if } s_2 = R \end{cases}$$

and

$$v_2(s_1, s_2) = \begin{cases} 10 - |s_1 - 3| & \text{if } s_2 = A \\ 9 & \text{if } s_2 = R \end{cases}$$

(b) Find a subgame perfect equilibrium of this game. Is it unique?

   Answer: Player 2 can guarantee himself a payoff of 9 by choosing R

implying that his best response is to choose A if and only if $10-|s_1-3| \geq 9$, which will hold for any $s_1 \in [2, 4]$. Player 1 would like to have an alternative adopted that is closest to 1, which implies that his best response to player 2's sequentially rational strategy is to choose $s_1 = 2$ This is the unique subgame perfect equilibrium which results in the payoffs of $(v_1, v_2) = (9, 9)$. ∎

(c) Find a Nash equilibrium that is not subgame perfect. Is it unique? If yes, explain. If not, show all the Nash equilibria of this game.

Answer: One Nash equilibrium is where player 2 adopts the strategy "I will reject anything except $s_1 = 3$." If player 1 chooses $s_1 = 3$ then his payoff is 8, while any other choice of s_1 is expected to yield player 1 a payoff of 7. Hence, player 1s best response to player 2's proposed strategy is indeed to choose $s_1 = 3$ and the payoffs from this Nash equilibrium are $(v_1, v_2) = (8, 10)$. Since player 2 can guarantee himself a payoff of 9, there are infinitely many Nash equilibria that are not subgame perfect and that follow a similar logic: player 2 adopts the strategy "I will reject anything except $s_1 = x$" for some value $x \in (2, 4)$ Player 1 would strictly prefer the adoption of x over 4, and hence would indeed propose x, and player 2 would accept the proposal. For $x = 4$ both players are indifferent so it would also be supported as a Nash equilibrium. ∎

13.

14. **Hyperbolic Discounting:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods and with discount factors $0 < \delta < 1$ and $0 < \beta < 1$

(a) Solve the optimal choice of player 2, the second period self, as a function of his budget K_2, δ and β.

Answer: Player 2's optimization problem is given by

$$\max_{x_2} v_2(x_2, K - x_2) = \ln(x_2) + \beta\delta \ln(K - x_2),$$

for which the first order condition is

$$\frac{dv_2}{dx_2} = \frac{1}{x_2} - \frac{\beta\delta}{K_2 - x_2} = 0 \ ,$$

which in turn implies that player 2's best response function is,

$$x_2(K_2) = \frac{K_2}{\beta\delta + 1} \ ,$$

which leaves $x_3 = K_2 - x_2(K_2) = \frac{\beta\delta K_2}{\beta\delta+1}$ for consumption in the third period. ∎

(b) Solve the optimal choice of player 1, the first period self, as a function of K, δ and β.

Answer: Player 1 decides how much to allocate between his own consumption and that of player 2 taking into account that $x_2(K_2) = \frac{K_2}{\beta\delta+1}$ hence player 1 solves the following problem,

$$\max_{x_1} v_1(x_1, \frac{K - x_1}{\beta\delta + 1}, \frac{\beta\delta(K - x_1)}{\beta\delta + 1}) = \ln(x_1) + \beta\delta \ln(\frac{K - x_1}{\beta\delta + 1}) + \beta\delta^2 \ln(\frac{\beta\delta(K}{\beta\delta}$$

for which the first order condition is,

$$\frac{dv_1}{dx_1} = \frac{1}{x_1} - \frac{\beta\delta}{K - x_1} - \frac{\beta\delta^2}{K - x_1} = 0 \ ,$$

which in turn implies that player 1's best response function is,

$$x_1(K) = \frac{K}{\beta\delta + \beta\delta^2 + 1} \ .$$

∎

15.

16. **The Value of Commitment:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods and with discount factors $\delta = 1$ and $\beta = \frac{1}{2}$. We solved the optimal consumption plan of a sophisticated player 1.

(a) Imagine that an external entity can enforce any plan of action that player 1 chooses in $t = 1$ and will prevent player 2 from modifying it. What is the plan that player 1 would choose to enforce?

Answer: Player 1 wants to maximize,

$$\max_{x_2, x_3} v(K - x_2 - x_3, x_2, x_3) = \ln(K - x_2 - x_3) + \beta\delta \ln(x_2) + \beta\delta^2 \ln(x$$

$$= \ln(K - x_2 - x_3) + \frac{1}{2}\ln(x_2) + \frac{1}{2}\ln(x_3)$$

when $\beta = \frac{1}{2}$ and $\delta = 1$. The two fist order conditions are,

$$\frac{\partial v}{\partial x_2} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_2} = 0 \ ,$$

and,

$$\frac{\partial v}{\partial x_3} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_3} = 0 \ .$$

Solving these two equations yields the solution

$$x_2 = x_3 = \frac{K}{4},$$

and using $x_1 = K - x_2 - x_3$ gives,

$$x_1 = \frac{K}{2} \ .$$

Thus, player 1 would choose to enforce $x_1 = \frac{K}{2}$ and $x_2 = x_3 = \frac{K}{4}$. ∎

(b) Assume that $K = 90$. Up to how much of his initial budget K will player 1 be willing to pay the external entity in order to enforce the plan you found in part (a)?

Answer: If the external entity does not enforce the plan, then from the analysis on pages 168-169 we know that player 2 will choose $x_2 = \frac{K}{3} = 30$ and $x_3 = \frac{K}{6} = 15$, and player 1 will choose $x_1 = \frac{K}{2} = 45$. The discounted value of the stream of payoffs for player 1 from this outcome is therefore,

$$\ln(45) + \frac{1}{2}\ln(30) + \frac{1}{2}\ln(15) \approx 6.86 \ .$$

If, however, player 1 can have the plan in part (a) above enforced then his discounted value of the stream of payoffs is

$$\ln(45) + \frac{1}{2}\ln(22.5) + \frac{1}{2}\ln(22.5) \approx 6.92 \ .$$

We can therefore solve for the amount m of budget $K = 90$ that player 1 would be willing to give up which is found by the following equality,

$$\ln(45 - m) + \frac{1}{2}\ln(22.5) + \frac{1}{2}\ln(22.5) = 6.86 \ ,$$

which yields $m \approx 2.63$. Hence, player 1 will be willing to give up to 2.63 of his initial budget $K = 90$ in order to enforce the plan $x_2 = x_3 = \frac{K}{4} = 22.5$. ∎