Chapter 2
Simple Comparative Experiments
Solutions

2-1 The breaking strength of a fiber is required to be at least 150 psi. Past experience has indicated that the standard deviation of breaking strength is $\sigma = 3$ psi. A random sample of four specimens is tested. The results are y_1=145, y_2=153, y_3=150 and y_4=147.

(a) State the hypotheses that you think should be tested in this experiment.

H_0: $\mu = 150$ H_1: $\mu > 150$

(b) Test these hypotheses using $\alpha = 0.05$. What are your conclusions?

$n = 4$, $\sigma = 3$, $\bar{y} = 1/4\ (145 + 153 + 150 + 147) = 148.75$

$$z_o = \frac{\bar{y} - \mu_o}{\frac{\sigma}{\sqrt{n}}} = \frac{148.75 - 150}{\frac{3}{\sqrt{4}}} = \frac{-1.25}{\frac{3}{2}} = -0.8333$$

Since $z_{0.05} = 1.645$, do not reject.

(c) Find the *P*-value for the test in part (b).

From the *z*-table: $P \cong 1 - \left[0.7967 + (2/3)(0.7995 - 0.7967)\right] = 0.2014$

(d) Construct a 95 percent confidence interval on the mean breaking strength.

The 95% confidence interval is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$
$$148.75 - (1.96)(3/2) \le \mu \le 148.75 + (1.96)(3/2)$$

$$145.81 \le \mu \le 151.69$$

2-2 The viscosity of a liquid detergent is supposed to average 800 centistokes at 25°C. A random sample of 16 batches of detergent is collected, and the average viscosity is 812. Suppose we know that the standard deviation of viscosity is $\sigma = 25$ centistokes.

(a) State the hypotheses that should be tested.

H_0: $\mu = 800$ H_1: $\mu \neq 800$

(b) Test these hypotheses using $\alpha = 0.05$. What are your conclusions?

$$z_o = \frac{\bar{y} - \mu_o}{\frac{\sigma}{\sqrt{n}}} = \frac{812 - 800}{\frac{25}{\sqrt{16}}} = \frac{12}{\frac{25}{4}} = 1.92$$

Since $z_{\alpha/2} = z_{0.025} = 1.96$, do not reject.

(c) What is the *P*-value for the test? $P = 2(0.0274) = 0.0549$

(d) Find a 95 percent confidence interval on the mean.

The 95% confidence interval is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$812 - (1.96)(25/4) \le \mu \le 812 + (1.96)(25/4)$$

$$812 - 12.25 \le \mu \le 812 + 12.25$$

$$799.75 \le \mu \le 824.25$$

2-3 The diameters of steel shafts produced by a certain manufacturing process should have a mean diameter of 0.255 inches. The diameter is known to have a standard deviation of $\sigma = 0.0001$ inch. A random sample of 10 shafts has an average diameter of 0.2545 inches.

(a) Set up the appropriate hypotheses on the mean μ.

H_0: $\mu = 0.255$ H_1: $\mu \ne 0.255$

(b) Test these hypotheses using $\alpha = 0.05$. What are your conclusions?

$$n = 10, \quad \sigma = 0.0001, \quad \bar{y} = 0.2545$$

$$z_o = \frac{\bar{y} - \mu_o}{\frac{\sigma}{\sqrt{n}}} = \frac{0.2545 - 0.255}{\frac{0.0001}{\sqrt{10}}} = -15.81$$

Since $z_{0.025} = 1.96$, reject H_0.

(c) Find the *P*-value for this test. $P = 2.6547\text{x}10^{-56}$

(d) Construct a 95 percent confidence interval on the mean shaft diameter.

The 95% confidence interval is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$0.2545 - (1.96)\left(\frac{0.0001}{\sqrt{10}}\right) \le \mu \le 0.2545 + (1.96)\left(\frac{0.0001}{\sqrt{10}}\right)$$

$$0.254438 \le \mu \le 0.254562$$

2-4 A normally distributed random variable has an unknown mean μ and a known variance $\sigma^2 = 9$. Find the sample size required to construct a 95 percent confidence interval on the mean, that has total width of 1.0.

Since $y \sim N(\mu,9)$, a 95% two-sided confidence interval on μ is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\bar{y} - (1.96)\frac{3}{\sqrt{n}} \le \mu \le \bar{y} + (1.96)\frac{3}{\sqrt{n}}$$

If the total interval is to have width 1.0, then the half-interval is 0.5. Since $z_{\square/2} = z_{0.025} = 1.96$,

$$(1.96)\left(3/\sqrt{n}\right) = 0.5$$

$$\sqrt{n} = (1.96)(3/0.5) = 11.76$$

$$n = (11.76)^2 = 138.30 \cong 139$$

2-5 The shelf life of a carbonated beverage is of interest. Ten bottles are randomly selected and tested, and the following results are obtained:

Days	
108	138
124	163
124	159
106	134
115	139

(a) We would like to demonstrate that the mean shelf life exceeds 120 days. Set up appropriate hypotheses for investigating this claim.

H_0: $\mu = 120$ $\quad$ H_1: $\mu > 120$

(b) Test these hypotheses using $\alpha = 0.01$. What are your conclusions?

$\bar{y} = 131$

$s^2 = [\ (108 - 131)^2 + (124 - 131)^2 + (124 - 131)^2 + (106 - 131)^2 + (115 - 131)^2 + (138 - 131)^2 + (163 - 131)^2 + (159 - 131)^2 + (134 - 131)^2 + (\ 139 - 131)^2\ ] / (10 - 1)$

$s^2 = 3438 / 9 = 382$

$s = \sqrt{382} = 19.54$

$$t_o = \frac{\bar{y} - \mu_o}{s/\sqrt{n}} = \frac{131 - 120}{19.54/\sqrt{10}} = 1.78$$

since $t_{0.01,9} = 2.821$; do not reject H_0

Minitab Output

T-Test of the Mean

```
Test of mu = 120.00 vs mu > 120.00

Variable      N      Mean     StDev   SE Mean        T          P
Shelf Life   10    131.00     19.54      6.18     1.78      0.054
```

T Confidence Intervals

```
Variable      N       Mean     StDev  SE Mean        99.0 % CI
Shelf Life   10     131.00     19.54     6.18   (  110.91,   151.09)
```

(c) Find the *P*-value for the test in part (b). *P*=0.054

(d) Construct a 99 percent confidence interval on the mean shelf life.

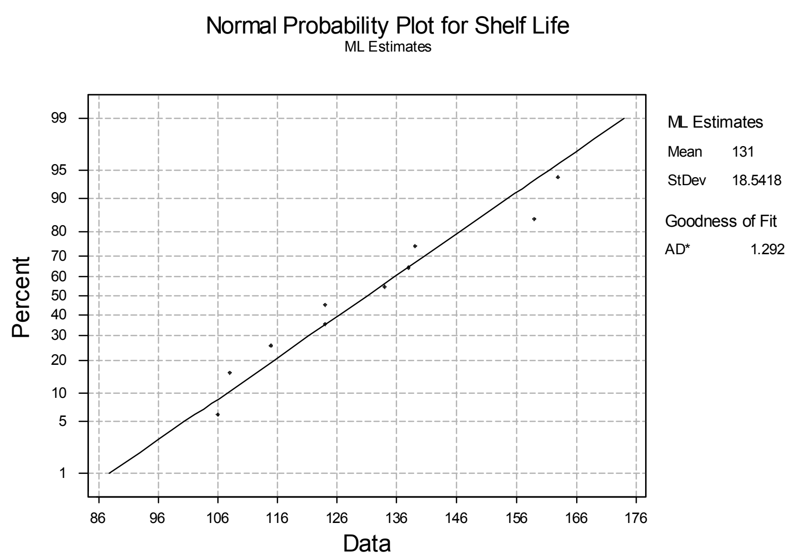
The 95% confidence interval is $\bar{y} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \le \mu \le \bar{y} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$

$$131-(3.250)\left(\frac{1954}{\sqrt{10}}\right) \le \mu \le 131+(3.250)\left(\frac{1954}{\sqrt{10}}\right)$$

$$110.91 \le \mu \le 151.09$$

2-6 Consider the shelf life data in Problem 2-5. Can shelf life be described or modeled adequately by a normal distribution? What effect would violation of this assumption have on the test procedure you used in solving Problem 2-5?

A normal probability plot, obtained from Minitab, is shown. There is no reason to doubt the adequacy of the normality assumption. If shelf life is not normally distributed, then the impact of this on the *t*-test in problem 2-5 is not too serious unless the departure from normality is severe.



2-7 The time to repair an electronic instrument is a normally distributed random variable measured in hours. The repair time for 16 such instruments chosen at random are as follows:

Hours			
159	280	101	212
224	379	179	264
222	362	168	250
149	260	485	170

(a) You wish to know if the mean repair time exceeds 225 hours. Set up appropriate hypotheses for investigating this issue.

$$H_0\text{: } \mu = 225 \qquad H_1\text{: } \mu > 225$$

(b) Test the hypotheses you formulated in part (a). What are your conclusions? Use $\alpha = 0.05$.

$$\bar{y} = 247.50$$

$$s^2 = 146202 / (16 - 1) = 9746.80$$

$$s = \sqrt{9746.8} = 98.73$$

$$t_o = \frac{\bar{y} - \mu_o}{\frac{s}{\sqrt{n}}} = \frac{241.50 - 225}{\frac{98.73}{\sqrt{16}}} = 0.67$$

since $t_{0.05,15} = 1.753$; do not reject H_0

Minitab Output

T-Test of the Mean

```
Test of mu = 225.0 vs mu > 225.0

Variable     N      Mean    StDev   SE Mean        T          P
Hours       16     241.5     98.7      24.7     0.67       0.26
```

T Confidence Intervals

```
Variable     N      Mean    StDev  SE Mean       95.0 % CI
Hours       16     241.5     98.7     24.7  (   188.9,   294.1)
```

(c) Find the *P*-value for this test. *P*=0.26

(d) Construct a 95 percent confidence interval on mean repair time.

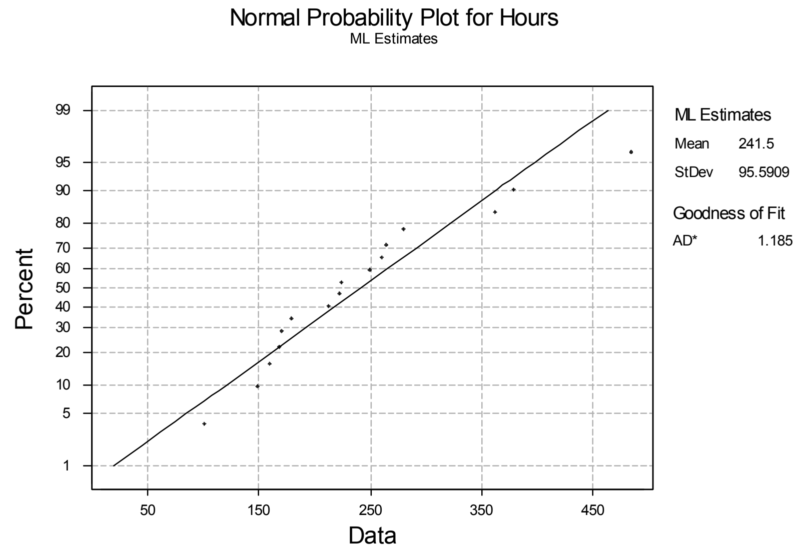
The 95% confidence interval is $\bar{y} - t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} \le \mu \le \bar{y} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$

$$241.50 - (2.131)\left(\frac{98.73}{\sqrt{16}}\right) \le \mu \le 241.50 + (2.131)\left(\frac{98.73}{\sqrt{16}}\right)$$

$$188.9 \le \mu \le 294.1$$

2-8 Reconsider the repair time data in Problem 2-7. Can repair time, in your opinion, be adequately modeled by a normal distribution?

The normal probability plot below does not reveal any serious problem with the normality assumption.



2-9 Two machines are used for filling plastic bottles with a net volume of 16.0 ounces. The filling processes can be assumed to be normal, with standard deviation of $\sigma_1 = 0.015$ and $\sigma_2 = 0.018$. The quality engineering department suspects that both machines fill to the same net volume, whether or not this volume is 16.0 ounces. An experiment is performed by taking a random sample from the output of each machine.

Machine 1		Machine 2	
16.03	16.01	16.02	16.03
16.04	15.96	15.97	16.04
16.05	15.98	15.96	16.02
16.05	16.02	16.01	16.01
16.02	15.99	15.99	16.00

(a) State the hypotheses that should be tested in this experiment.

H_0: $\mu_1 = \mu_2$ $\qquad$ H_1: $\mu_1 \neq \mu_2$

(b) Test these hypotheses using α=0.05. What are your conclusions?

$$\bar{y}_1 = 16.015 \qquad \bar{y}_2 = 16.005$$
$$\sigma_1 = 0.015 \qquad \sigma_2 = 0.018$$
$$n_1 = 10 \qquad n_2 = 10$$

$$z_o = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{16.015 - 16.018}{\sqrt{\frac{0.015^2}{10} + \frac{0.018^2}{10}}} = 1.35$$

$z_{0.025} = 1.96$; do not reject

(c) What is the P-value for the test? P=0.1770

(d) Find a 95 percent confidence interval on the difference in the mean fill volume for the two machines.

The 95% confidence interval is

$$\bar{y}_1 - \bar{y}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{y}_1 - \bar{y}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

$$(16.015-16.005)-(19.6)\sqrt{\frac{0.015^2}{10}+\frac{0.018^2}{10}} \le \mu_1 - \mu_2 \le (16.015-16.005)+(19.6)\sqrt{\frac{0.015^2}{10}+\frac{0.018^2}{10}}$$

$$-0.0045 \le \mu_1 - \mu_2 \le 0.0245$$

2-10 Two types of plastic are suitable for use by an electronic calculator manufacturer. The breaking strength of this plastic is important. It is known that $\sigma_1 = \sigma_2 = 1.0$ psi. From random samples of $n_1 = 10$ and $n_2 = 12$ we obtain $\bar{y}_1 = 162.5$ and $\bar{y}_2 = 155.0$. The company will not adopt plastic 1 unless its breaking strength exceeds that of plastic 2 by at least 10 psi. Based on the sample information, should they use plastic 1? In answering this questions, set up and test appropriate hypotheses using $\alpha = 0.01$. Construct a 99 percent confidence interval on the true mean difference in breaking strength.

H_0: $\mu_1 - \mu_2 = 10$ H_1: $\mu_1 - \mu_2 > 10$

$\bar{y}_1 = 162.5$ $\bar{y}_2 = 155.0$

$\sigma_1 = 1$ $\sigma_2 = 1$

$n_1 = 10$ $n_2 = 10$

$$z_o = \frac{\bar{y}_1 - \bar{y}_2 - 10}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} = \frac{162.5-155.0-10}{\sqrt{\frac{1^2}{10}+\frac{1^2}{12}}} = -5.85$$

$z_{0.01} = 2.225$; do not reject

The 99 percent confidence interval is

$$\bar{y}_1 - \bar{y}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{y}_1 - \bar{y}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

$$(162.5-155.0)-(2.575)\sqrt{\frac{1^2}{10}+\frac{1^2}{12}} \le \mu_1 - \mu_2 \le (162.5-155.0)+(2.575)\sqrt{\frac{1^2}{10}+\frac{1^2}{12}}$$

$$6.40 \le \mu_1 - \mu_2 \le 8.60$$

2-11 The following are the burning times of chemical flares of two different formulations. The design engineers are interested in both the means and variance of the burning times.

Type 1		Type 2	
65	82	64	56
81	67	71	69
57	59	83	74
66	75	59	82
82	70	65	79

(a) Test the hypotheses that the two variances are equal. Use $\alpha = 0.05$.

$$H_0 : \sigma_1^2 = \sigma_2^2 \qquad S_1 = 9.264$$
$$H1 : \sigma_1^2 \neq \sigma_2^2 \qquad S_2 = 9.367$$
$$F_0 = \frac{S_1^2}{S_2^2} = \frac{85.82}{87.73} = 0.98$$
$$F_{0.025,9,9} = 4.03 \qquad F_{0.975,9,9} = \frac{1}{F_{0.025,9,9}} = \frac{1}{4.03} = 0.248 \quad \text{Do not reject.}$$

(b) Using the results of (a), test the hypotheses that the mean burning times are equal. Use α = 0.05. What is the *P*-value for this test?

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{1561.95}{18} = 86.775$$
$$S_p = 9.32$$
$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{70.4-70.2}{9.32\sqrt{\frac{1}{10}+\frac{1}{10}}} = 0.048$$
$$t_{0.025,18} = 2.101 \quad \text{Do not reject.}$$

From the computer output, *t*=0.05; do not reject. Also from the computer output *P*=0.96

Minitab Output

Two Sample T-Test and Confidence Interval

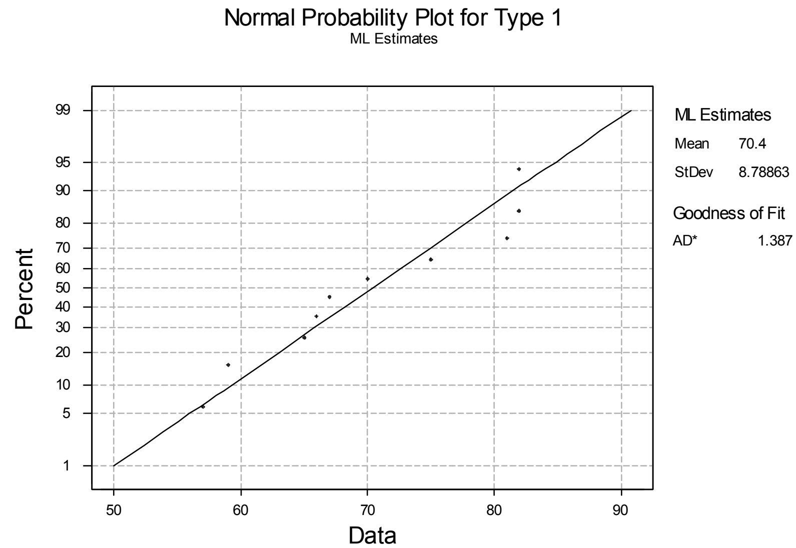
```
Two sample T for Type 1 vs Type 2

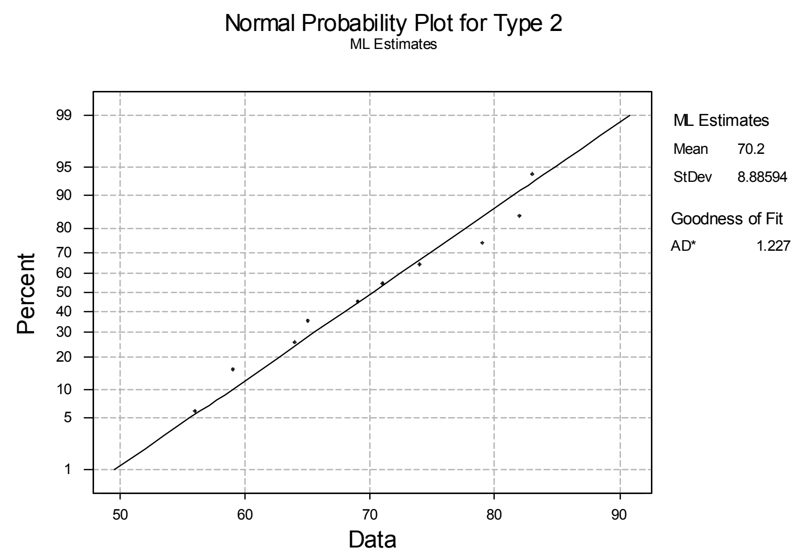
         N      Mean     StDev   SE Mean
Type 1  10     70.40      9.26       2.9
Type 2  10     70.20      9.37       3.0

95% CI for mu Type 1 - mu Type 2: ( -8.6,  9.0)
T-Test mu Type 1 = mu Type 2 (vs not =): T = 0.05  P = 0.96  DF = 18
Both use Pooled StDev = 9.32
```

(c) Discuss the role of the normality assumption in this problem. Check the assumption of normality for both types of flares.

The assumption of normality is required in the theoretical development of the *t*-test. However, moderate departure from normality has little impact on the performance of the *t*-test. The normality assumption is more important for the test on the equality of the two variances. An indication of nonnormality would be of concern here. The normal probability plots shown below indicate that burning time for both formulations follow the normal distribution.





2-12 An article in *Solid State Technology*, "Orthogonal Design of Process Optimization and Its Application to Plasma Etching" by G.Z. Yin and D.W. Jillie (May, 1987) describes an experiment to determine the effect of C_2F_6 flow rate on the uniformity of the etch on a silicon wafer used in integrated circuit manufacturing. Data for two flow rates are as follows:

C_2F_6	Uniformity Observation					
(SCCM)	1	2	3	4	5	6
125	2.7	4.6	2.6	3.0	3.2	3.8
200	4.6	3.4	2.9	3.5	4.1	5.1

(a) Does the C_2F_6 flow rate affect average etch uniformity? Use $\alpha = 0.05$.

No, C_2F_6 flow rate does not affect average etch uniformity.

Minitab Output

Two Sample T-Test and Confidence Interval

```
Two sample T for Uniformity

Flow Rat    N      Mean     StDev   SE Mean
125         6     3.317     0.760      0.31
200         6     3.933     0.821      0.34

95% CI for mu (125) - mu (200): ( -1.63,  0.40)
T-Test mu (125) = mu (200) (vs not =): T = -1.35  P = 0.21  DF = 10
Both use Pooled StDev = 0.791
```

(b) What is the *P*-value for the test in part (a)? From the computer printout, *P*=0.21

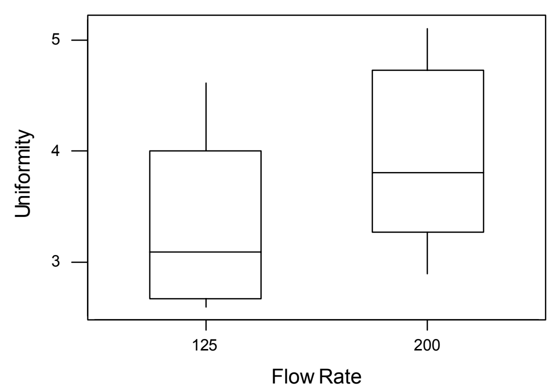
(c) Does the C_2F_6 flow rate affect the wafer-to-wafer variability in etch uniformity? Use $\alpha = 0.05$.

$$H_0 : \sigma_1^2 = \sigma_2^2$$
$$H1 : \sigma_1^2 \neq \sigma_2^2$$
$$F_{0.05,5,5} = 5.05$$
$$F_0 = \frac{0.5776}{0.6724} = 0.86$$

Do not reject; C_2F_6 flow rate does not affect wafer-to-wafer variability.

(d) Draw box plots to assist in the interpretation of the data from this experiment.

The box plots shown below indicate that there is little difference in uniformity at the two gas flow rates. Any observed difference is not statistically significant. See the *t*-test in part (a).



2-13 A new filtering device is installed in a chemical unit. Before its installation, a random sample yielded the following information about the percentage of impurity: $\bar{y}_1 = 12.5$, $S_1^2 = 101.17$, and $n_1 = 8$. After installation, a random sample yielded $\bar{y}_2 = 10.2$, $S_2^2 = 94.73$, $n_2 = 9$.

(a) Can you concluded that the two variances are equal? Use $\alpha = 0.05$.

$$H_0 : \sigma_1^2 = \sigma_2^2$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2$$
$$F_{0.025,7,8} = 4.53$$
$$F_0 = \frac{S_1^2}{S_2^2} = \frac{101.17}{94.73} = 1.07$$

Do Not Reject. Assume that the variances are equal.

(b) Has the filtering device reduced the percentage of impurity significantly? Use $\alpha = 0.05$.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$
$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{(8-1)(101.17) + (9-1)(94.73)}{8+9-2} = 97.74$$
$$S_p = 9.89$$
$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{12.5 - 10.2}{9.89\sqrt{\frac{1}{8} + \frac{1}{9}}} = 0.479$$
$$t_{0.05,15} = 1.753$$

Do not reject. There is no evidence to indicate that the new filtering device has affected the mean

2-14 Twenty observations on etch uniformity on silicon wafers are taken during a qualification experiment for a plasma etcher. The data are as follows:

5.34	6.65	4.76	5.98	7.25
6.00	7.55	5.54	5.62	6.21
5.97	7.35	5.44	4.39	4.98
5.25	6.35	4.61	6.00	5.32

(a) Construct a 95 percent confidence interval estimate of σ^2.

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{(1-\alpha/2),n-1}}$$
$$\frac{(20-1)(0.88907)^2}{32.852} \leq \sigma^2 \leq \frac{(20-1)(0.88907)^2}{8.907}$$
$$0.457 \leq \sigma^2 \leq 1.686$$

(b) Test the hypothesis that $\sigma^2 = 1.0$. Use $\alpha = 0.05$. What are your conclusions?

$$H_0 : \sigma^2 = 1$$
$$H_1 : \sigma^2 \neq 1$$
$$\chi_0^2 = \frac{SS}{\sigma_0^2} = 15.019$$
$$\chi^2_{0.025,19} = 32.852 \qquad \chi^2_{0.975,19} = 8.907$$

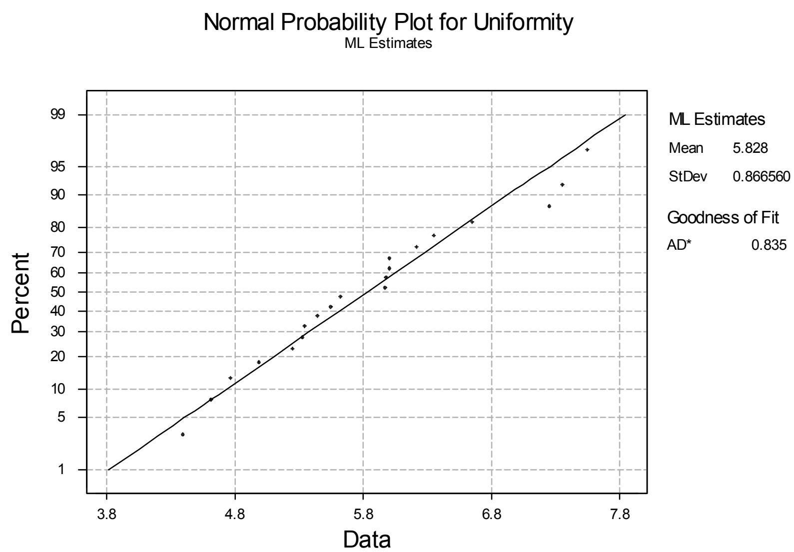
Do not reject. There is no evidence to indicate that $\sigma_1^2 \neq 1$

(c) Discuss the normality assumption and its role in this problem.

The normality assumption is much more important when analyzing variances then when analyzing means. A moderate departure from normality could cause problems with both statistical tests and confidence intervals. Specifically, it will cause the reported significance levels to be incorrect.

(d) Check normality by constructing a normal probability plot. What are your conclusions?

The normal probability plot indicates that there is not any serious problem with the normality assumption.



2-15 The diameter of a ball bearing was measured by 12 inspectors, each using two different kinds of calipers. The results were:

Inspector	Caliper 1	Caliper 2	Difference	Difference^2
1	0.265	0.264	.001	.000001
2	0.265	0.265	.000	0
3	0.266	0.264	.002	.000004
4	0.267	0.266	.001	.000001
5	0.267	0.267	.000	0
6	0.265	0.268	-.003	.000009
7	0.267	0.264	.003	.000009
8	0.267	0.265	.002	.000004
9	0.265	0.265	.000	0
10	0.268	0.267	.001	.000001
11	0.268	0.268	.000	0
12	0.265	0.269	-.004	.000016
			$\sum = 0.003$	$\sum = 0.000045$

(a) Is there a significant difference between the means of the population of measurements represented by the two samples? Use $\alpha = 0.05$.

$$H_0: \mu_1 = \mu_2 \quad \text{or equivalently} \quad H_0: \mu_d = 0$$
$$H_1: \mu_1 \neq \mu_2 \qquad\qquad\qquad\quad H_1: \mu_d \neq 0$$

Minitab Output

Paired T-Test and Confidence Interval

```
Paired T for Caliper 1 - Caliper 2

                  N      Mean     StDev   SE Mean
Caliper          12  0.266250  0.001215  0.000351
Caliper          12  0.266000  0.001758  0.000508
Difference       12  0.000250  0.002006  0.000579

95% CI for mean difference: (-0.001024, 0.001524)
T-Test of mean difference = 0 (vs not = 0): T-Value = 0.43  P-Value = 0.674
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(b) Find the *P*-value for the test in part (a). *P*=0.674

(c) Construct a 95 percent confidence interval on the difference in the mean diameter measurements for the two types of calipers.

$$\bar{d} - t_{\alpha/2, n-1}\frac{S_d}{\sqrt{n}} \le \mu_D\left(= \mu_1 - \mu_2\right) \le \bar{d} + t_{\alpha/2, n-1}\frac{S_d}{\sqrt{n}}$$
$$0.00025 - 2.201\frac{0.002}{\sqrt{12}} \le \mu_d \le 0.00025 + 2.201\frac{0.002}{\sqrt{12}}$$
$$-0.00102 \le \mu_d \le 0.00152$$

2-16 An article in the *Journal of Strain Analysis* (vol.18, no. 2, 1983) compares several procedures for predicting the shear strength for steel plate girders. Data for nine girders in the form of the ratio of predicted to observed load for two of these procedures, the Karlsruhe and Lehigh methods, are as follows:

Girder	Karlsruhe Method	Lehigh Method	Difference	Difference^2
S1/1	1.186	1.061	0.125	0.015625
S2/1	1.151	0.992	0.159	0.025281
S3/1	1.322	1.063	0.259	0.067081
S4/1	1.339	1.062	0.277	0.076729
S5/1	1.200	1.065	0.135	0.018225
S2/1	1.402	1.178	0.224	0.050176
S2/2	1.365	1.037	0.328	0.107584
S2/3	1.537	1.086	0.451	0.203401
S2/4	1.559	1.052	0.507	0.257049
		Sum =	2.465	0.821151
		Average =	0.274	

(a) Is there any evidence to support a claim that there is a difference in mean performance between the two methods? Use $\alpha = 0.05$.

$$H_0: \mu_1 = \mu_2 \quad \text{or equivalently} \quad H_0: \mu_d = 0$$
$$H_1: \mu_1 \neq \mu_2 \qquad\qquad\qquad\quad H_1: \mu_d \neq 0$$

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{9}\left(2.465\right) = 0.274$$

$$s_d = \left[\frac{\sum_{i=1}^{n} d_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2}{n-1}\right]^{\frac{1}{2}} = \left[\frac{0.821151 - \frac{1}{9}(2.465)^2}{9-1}\right]^{\frac{1}{2}} = 0.135$$

$$t_0 = \frac{\bar{d}}{\frac{S_d}{\sqrt{n}}} = \frac{0.274}{\frac{0.135}{\sqrt{9}}} = 6.08$$

$$t_{\alpha/2,n-1} = t_{0.025,9} = 2.306 \text{, reject the null hypothesis.}$$

Minitab Output

Paired T-Test and Confidence Interval

Paired T for Karlsruhe - Lehigh

	N	Mean	StDev	SE Mean
Karlsruh	9	1.3401	0.1460	0.0487
Lehigh	9	1.0662	0.0494	0.0165
Difference	9	0.2739	0.1351	0.0450

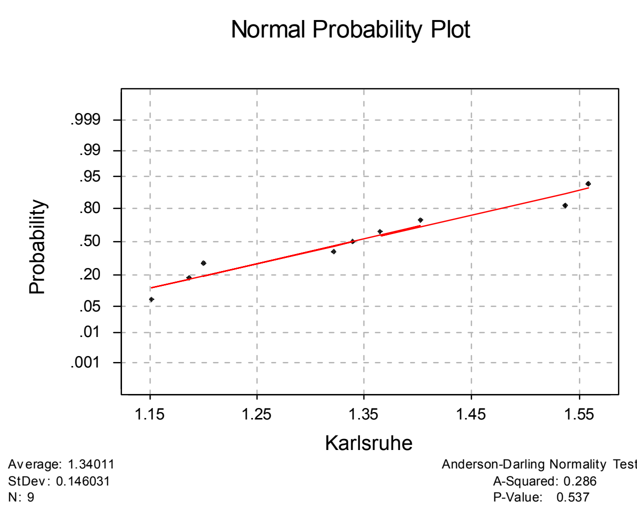
95% CI for mean difference: (0.1700, 0.3777)
T-Test of mean difference = 0 (vs not = 0): T-Value = 6.08 P-Value = 0.000

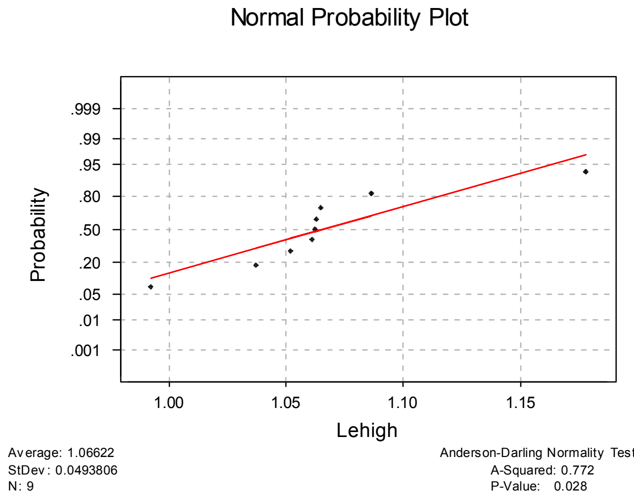
(b) What is the *P*-value for the test in part (a)? *P*=0.0002

(c) Construct a 95 percent confidence interval for the difference in mean predicted to observed load.

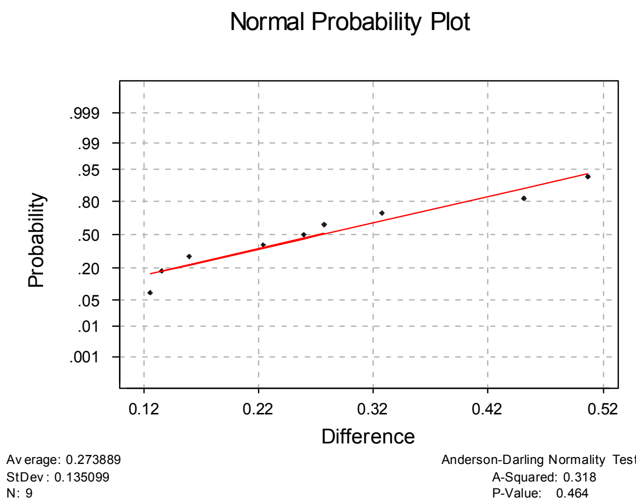
$$\bar{d} - t_{\alpha/2,n-1}\frac{S_d}{\sqrt{n}} \le \mu_d \le \bar{d} + t_{\alpha/2,n-1}\frac{S_d}{\sqrt{n}}$$
$$0.274 - 2.306\frac{0.135}{\sqrt{9}} \le \mu_d \le 0.274 + 2.306\frac{0.135}{\sqrt{9}}$$
$$0.17023 \le \mu_d \le 0.37777$$

(d) Investigate the normality assumption for both samples.





(e) Investigate the normality assumption for the difference in ratios for the two methods.



(f) Discuss the role of the normality assumption in the paired *t*-test.

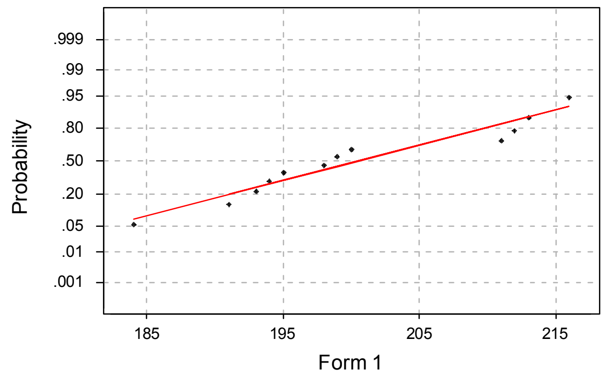
As in any *t*-test, the assumption of normality is of only moderate importance. In the paired *t*-test, the assumption of normality applies to the distribution of the differences. That is, the individual sample measurements do not have to be normally distributed, only their difference.

2-17 The deflection temperature under load for two different formulations of ABS plastic pipe is being studied. Two samples of 12 observations each are prepared using each formulation, and the deflection temperatures (in °F) are reported below:

Formulation 1			Formulation 2		
212	199	198	177	176	198
194	213	216	197	185	188
211	191	200	206	200	189
193	195	184	201	197	203

(a) Construct normal probability plots for both samples. Do these plots support assumptions of normality and equal variance for both samples?

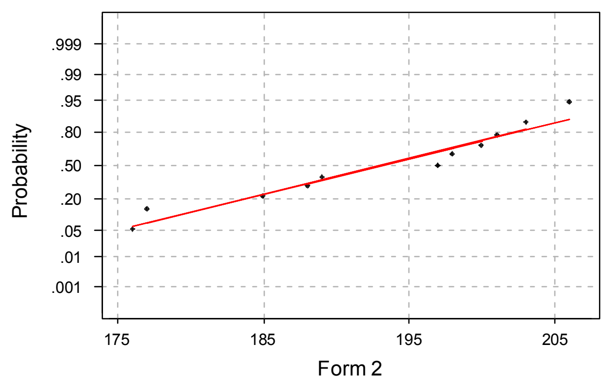
Normal Probability Plot



Average: 200.5
StDev: 10.1757
N: 12

Anderson-Darling Normality Test
A-Squared: 0.450
P-Value: 0.227

Normal Probability Plot



Average: 193.083
StDev: 9.94949
N: 12

Anderson-Darling Normality Test
A-Squared: 0.443
P-Value: 0.236

(b) Do the data support the claim that the mean deflection temperature under load for formulation 1 exceeds that of formulation 2? Use $\alpha = 0.05$.

Minitab Output

Two Sample T-Test and Confidence Interval

```
Two sample T for Form 1 vs Form 2

         N      Mean     StDev   SE Mean
Form 1  12     200.5      10.2       2.9
Form 2  12    193.08      9.95       2.9

95% CI for mu Form 1 - mu Form 2: ( -1.1,  15.9)
T-Test mu Form 1 = mu Form 2 (vs >): T = 1.81  P = 0.042  DF = 22
Both use Pooled StDev = 10.1
```

(c) What is the *P*-value for the test in part (a)? $P = 0.042$

2-18 Refer to the data in problem 2-17. Do the data support a claim that the mean deflection temperature under load for formulation 1 exceeds that of formulation 2 by at least 3 °F?
Yes, formulation 1 exceeds formulation 2 by at least 3 °F.

Minitab Output

Two-Sample T-Test and CI: Form1, Form2

```
Two-sample T for Form1 vs Form2

        N      Mean      StDev   SE Mean
Form1  12     200.5      10.2       2.9
Form2  12    193.08      9.95       2.9r
Difference = mu Form1 - mu Form2
Estimate for difference:  7.42
95% lower bound for difference: 0.36
T-Test of difference = 3 (vs >): T-Value = 1.08  P-Value = 0.147  DF = 22
Both use Pooled StDev = 10.1
```

2-19 In semiconductor manufacturing, wet chemical etching is often used to remove silicon from the backs of wafers prior to metalization. The etch rate is an important characteristic of this process. Two different etching solutionsare being evaluated. Eight randomly selected wafers have been etched in each solution and the observed etch rates (in mils/min) are shown below:

Solution 1		Solution 2	
9.9	10.6	10.2	10.6
9.4	10.3	10.0	10.2
10.0	9.3	10.7	10.4
10.3	9.8	10.5	10.3

(a) Do the data indicate that the claim that both solutions have the same mean etch rate is valid? Use α = 0.05 and assume equal variances.

See the Minitab output below.

Minitab Output

Two Sample T-Test and Confidence Interval

```
Two sample T for Solution 1 vs Solution 2

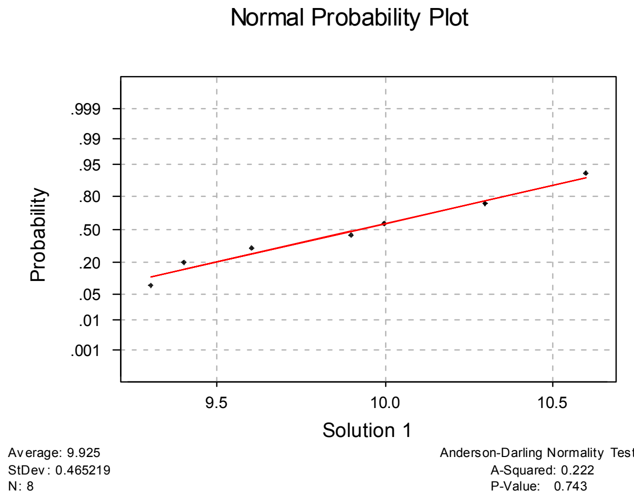
           N      Mean      StDev   SE Mean
Solution  8     9.925     0.465      0.16
Solution  8    10.362     0.233     0.082

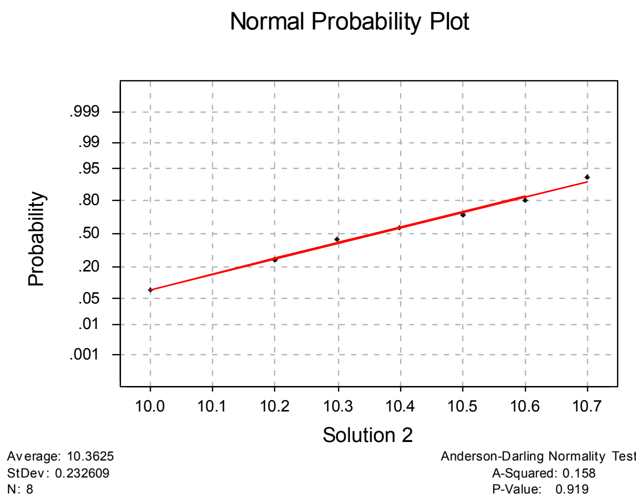
95% CI for mu Solution - mu Solution: ( -0.83,  -0.043)
T-Test mu Solution = mu Solution (vs not =):T = -2.38 P = 0.032 DF = 14
Both use Pooled StDev = 0.368
```

(b) Find a 95% confidence interval on the difference in mean etch rate.

From the Minitab output, -0.83 to –0.043.

(c) Use normal probability plots to investigate the adequacy of the assumptions of normality and equal variances.





Both the normality and equality of variance assumptions are valid.

2-20 Two popular pain medications are being compared on the basis of the speed of absorption by the body. Specifically, tablet 1 is claimed to be absorbed twice as fast as tablet 2. Assume that σ_1^2 and σ_2^2 are known. Develop a test statistic for

$$H_0\text{: } 2\mu_1 = \mu_2$$
$$H_1\text{: } 2\mu_1 \neq \mu_2$$

$$2\bar{y}_1 - \bar{y}_2 \sim N\left(2\mu_1 - \mu_2, \frac{4\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$, assuming that the data is normally distributed.

The test statistic is: $z_o = \dfrac{2\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{4\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$, reject if $|z_o| > z_{\alpha/2}$

2-21 Suppose we are testing

$$H_0\text{: } \mu_1 = \mu_2$$
$$H_1\text{: } \mu_1 \neq \mu_2$$

where σ_1^2 and σ_2^2 are known. Our sampling resources are constrained such that $n_1 + n_2 = N$. How should we allocate the N observations between the two populations to obtain the most powerful test?

The most powerful test is attained by the n_1 and n_2 that maximize z_o for given $\bar{y}_1 - \bar{y}_2$.
Thus, we chose n_1 and n_2 to $\max z_o = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$, subject to $n_1 + n_2 = N$.

This is equivalent to min $L = \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2} = \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{N - n_1}$, subject to $n_1 + n_2 = N$.

Now $\dfrac{dL}{dn_1} = \dfrac{-\sigma_1^2}{n_1^2} + \dfrac{\sigma_2^2}{(N - n_1)^2} = 0$, implies that $n_1 / n_2 = \sigma_1 / \sigma_2$.

Thus n_1 and n_2 are assigned proportionally to the ratio of the standard deviations. This has intuitive appeal, as it allocates more observations to the population with the greatest variability.

2-22 Develop Equation 2-46 for a 100(1 - α) percent confidence interval for the variance of a normal distribution.

$\dfrac{SS}{\sigma^2} \sim \chi^2_{n-1}$. Thus, $P\left\{\chi^2_{1-\frac{\alpha}{2},n-1} \le \dfrac{SS}{\sigma^2} \le \chi^2_{\frac{\alpha}{2},n-1}\right\} = 1 - \alpha$. Therefore,

$P\left\{\dfrac{SS}{\chi^2_{\frac{\alpha}{2},n-1}} \le \sigma^2 \le \dfrac{SS}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right\} = 1 - \alpha$,

so $\left[\dfrac{SS}{\chi^2_{\frac{\alpha}{2},n-1}}, \dfrac{SS}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$ is the 100(1 - α)% confidence interval on σ^2.

2-23 Develop Equation 2-50 for a 100(1 - α) percent confidence interval for the ratio σ_1^2 / σ_2^2, where σ_1^2 and σ_2^2 are the variances of two normal distributions.

$$\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \sim F_{n_2-1,n_1-1}$$

$$P\left\{F_{1-\alpha/2,n_2-1,n_1-1} \le \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \le F_{\alpha/2,n_2-1,n_1-1}\right\} = 1 - \alpha \quad \text{or}$$

$$P\left\{\frac{S_1^2}{S_2^2} F_{1-\alpha/2,n_2-1,n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\alpha/2,n_2-1,n_1-1}\right\} = 1 - \alpha$$

2-24 Develop an equation for finding a 100(1 - α) percent confidence interval on the difference in the means of two normal distributions where $\sigma_1^2 \neq \sigma_2^2$. Apply your equation to the portland cement experiment data, and find a 95% confidence interval.

$$\frac{(\bar{y}_1-\bar{y}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} \sim t_{\alpha/2,\upsilon}$$

$$t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \le (\bar{y}_1-\bar{y}_2)-(\mu_1-\mu_2) \le t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

$$(\bar{y}_1-\bar{y}_2)-t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \le (\mu_1-\mu_2) \le (\bar{y}_1-\bar{y}_2)+t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

$$\text{where} \quad \upsilon = \frac{\left(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1-1}+\frac{\left(\frac{S_2^2}{n_2}\right)^2}{n_2-1}}$$

Using the data from Table 2-1

$$\begin{array}{ll} n_1 = 10 & n_2 = 10 \\ \bar{y}_1 = 16.764 & \bar{y}_2 = 17.343 \\ S_1^2 = 0.100138 & S_2^2 = 0.0614622 \end{array}$$

$$(16.764-17.343)-2.110\sqrt{\frac{0.100138}{10}+\frac{0.0614622}{10}} \le (\mu_1-\mu_2) \le$$

$$(16.764-17.343)+2.110\sqrt{\frac{0.100138}{10}+\frac{0.0614622}{10}}$$

$$\text{where} \quad \upsilon = \frac{\left(\frac{0.100138}{10}+\frac{0.0614622}{10}\right)^2}{\frac{\left(\frac{0.100138}{10}\right)^2}{10-1}+\frac{\left(\frac{0.0614622}{10}\right)^2}{10-1}} = 17.024 \cong 17$$

$$-1.426 \le (\mu_1-\mu_2) \le -0.889$$

This agrees with the result in Table 2-2.

2-25 Construct a data set for which the paired *t*-test statistic is very large, but for which the usual two-sample or pooled *t*-test statistic is small. In general, describe how you created the data. Does this give you any insight regarding how the paired *t*-test works?

A	B	delta
7.1662	8.2416	1.07541
2.3590	2.4555	0.09650
19.9977	21.1018	1.10412
0.9077	2.3401	1.43239
-15.9034	-15.0013	0.90204
-6.0722	-5.5941	0.47808

9.9501	10.6910	0.74085
-1.0944	-0.1358	0.95854
-4.6907	-3.3446	1.34615
-6.6929	-5.9303	0.76256

Minitab Output

Paired T-Test and Confidence Interval

```
Paired T for A - B
                  N      Mean     StDev   SE Mean
A                10      0.59     10.06      3.18
B                10      1.48     10.11      3.20
Difference       10    -0.890     0.398     0.126

95% CI for mean difference: (-1.174, -0.605)
T-Test of mean difference = 0 (vs not = 0): T-Value = -7.07  P-Value = 0.000
```

Two Sample T-Test and Confidence Interval

```
Two sample T for A vs B

    N      Mean     StDev   SE Mean
A  10       0.6      10.1       3.2
B  10       1.5      10.1       3.2

95% CI for mu A - mu B: ( -10.4,  8.6)
T-Test mu A = mu B (vs not =): T = -0.20  P = 0.85  DF = 18
Both use Pooled StDev = 10.1
```

These two sets of data were created by making the observation for *A* and *B* moderately different within each pair (or block), but making the observations between pairs very different. The fact that the difference between pairs is large makes the pooled estimate of the standard deviation large and the two-sample *t*-test statistic small. Therefore the fairly small difference between the means of the two treatments that is present when they are applied to the same experimental unit cannot be detected. Generally, if the blocks are very different, then this will occur. Blocking eliminates the variabiliy associated with the nuisance variable that they represent.

Chapter 3
Experiments with a Single Factor: The Analysis of Variance
Solutions

3-1 The tensile strength of portland cement is being studied. Four different mixing techniques can be used economically. The following data have been collected:

Mixing Technique	Tensile Strength (lb/in^2)			
1	3129	3000	2865	2890
2	3200	3300	2975	3150
3	2800	2900	2985	3050
4	2600	2700	2600	2765

(a) Test the hypothesis that mixing techniques affect the strength of the cement. Use $\alpha = 0.05$.

Design Expert Output

Response: Tensile Strengthin lb/in^2
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	4.897E+005	3	1.632E+005	12.73	0.0005	significant
A	*4.897E+005*	*3*	*1.632E+005*	*12.73*	*0.0005*	
Residual	1.539E+005	12	12825.69			
Lack of Fit	*0.000*	*0*				
Pure Error	*1.539E+005*	*12*	*12825.69*			
Cor Total	6.436E+005	15				

The Model F-value of 12.73 implies the model is significant. There is only a 0.05% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

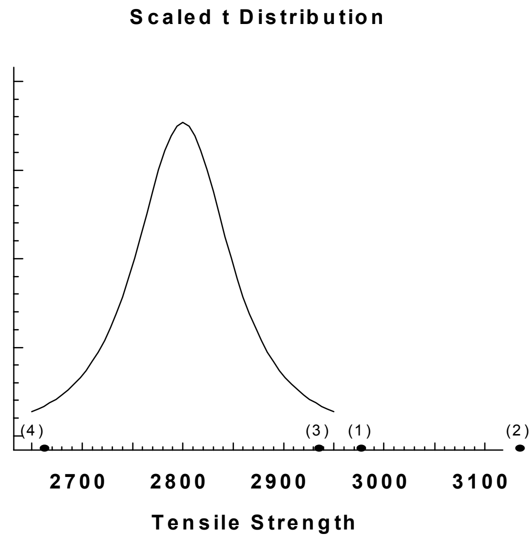
	Estimated Mean	Standard Error
1-1	2971.00	56.63
2-2	3156.25	56.63
3-3	2933.75	56.63
4-4	2666.25	56.63

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-185.25	1	80.08	-2.31	0.0392
1 vs 3	37.25	1	80.08	0.47	0.6501
1 vs 4	304.75	1	80.08	3.81	0.0025
2 vs 3	222.50	1	80.08	2.78	0.0167
2 vs 4	490.00	1	80.08	6.12	< 0.0001
3 vs 4	267.50	1	80.08	3.34	0.0059

The *F*-value is 12.73 with a corresponding *P*-value of .0005. Mixing technique has an effect.

(b) Construct a graphical display as described in Section 3-5.3 to compare the mean tensile strengths for the four mixing techniques. What are your conclusions?

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{12825.7}{4}} = 56.625$$



Based on examination of the plot, we would conclude that μ_1 and μ_3 are the same; that μ_4 differs from μ_1 and μ_3, that μ_2 differs from μ_1 and μ_3, and that μ_2 and μ_4 are different.

(c) Use the Fisher LSD method with α=0.05 to make comparisons between pairs of means.

$$LSD = t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}}$$

$$LSD = t_{0.025,16-4}\sqrt{\frac{2(12825.7)}{4}}$$

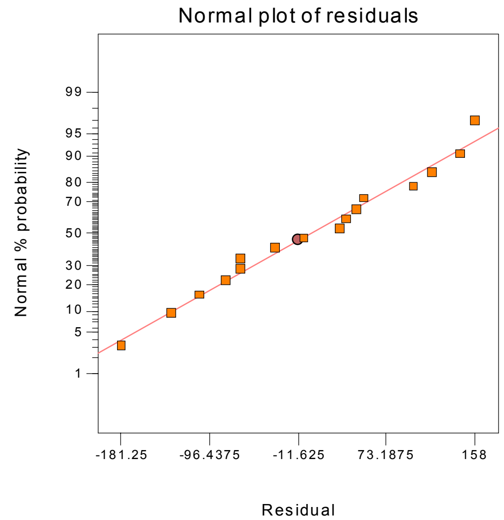
$$LSD = 2.179\sqrt{6412.85} = 174.495$$

Treatment 2 vs. Treatment 4 = 3156.250 - 2666.250 = 490.000 > 174.495
Treatment 2 vs. Treatment 3 = 3156.250 - 2933.750 = 222.500 > 174.495
Treatment 2 vs. Treatment 1 = 3156.250 - 2971.000 = 185.250 > 174.495
Treatment 1 vs. Treatment 4 = 2971.000 - 2666.250 = 304.750 > 174.495
Treatment 1 vs. Treatment 3 = 2971.000 - 2933.750 = 37.250 < 174.495
Treatment 3 vs. Treatment 4 = 2933.750 - 2666.250 = 267.500 > 174.495

The Fisher LSD method is also presented in the Design-Expert computer output above. The results agree with graphical method for this experiment.

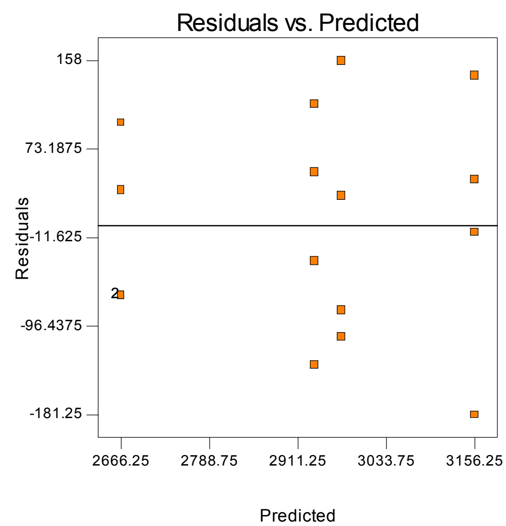
(d) Construct a normal probability plot of the residuals. What conclusion would you draw about the validity of the normality assumption?

There is nothing unusual about the normal probability plot of residuals.



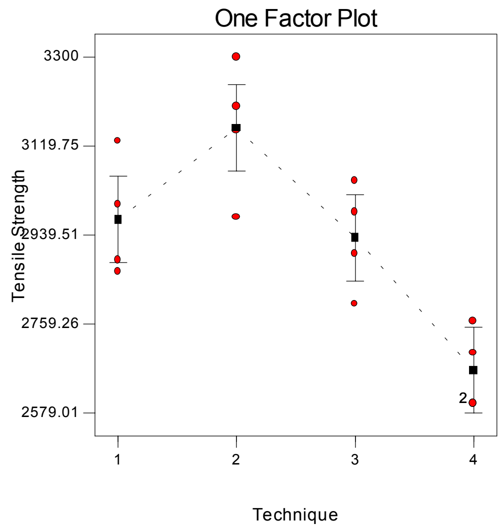
(e) Plot the residuals versus the predicted tensile strength. Comment on the plot.

There is nothing unusual about this plot.



(f) Prepare a scatter plot of the results to aid the interpretation of the results of this experiment.

Design-Expert automatically generates the scatter plot. The plot below also shows the sample average for each treatment and the 95 percent confidence interval on the treatment mean.



3-2 Rework part (b) of Problem 3-1 using the Duncan's multiple range test. Does this make any difference in your conclusions?

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{12825.7}{4}} = 56.625$$
$$R_2 = r_{0.05}(2,12)S_{\bar{y}_{i.}} = 3.08(56.625) = 174.406$$
$$R_3 = r_{0.05}(3,12)S_{\bar{y}_{i.}} = 3.23(56.625) = 182.900$$
$$R_4 = r_{0.05}(4,12)S_{\bar{y}_{i.}} = 3.33(56.625) = 188.562$$

Treatment 2 vs. Treatment 4 = 3156.250 - 2666.250 = 490.000 > 188.562 (R_4)
Treatment 2 vs. Treatment 3 = 3156.250 - 2933.750 = 222.500 > 182.900 (R_3)
Treatment 2 vs. Treatment 1 = 3156.250 - 2971.000 = 185.250 > 174.406 (R_2)
Treatment 1 vs. Treatment 4 = 2971.000 - 2666.250 = 304.750 > 182.900 (R_3)
Treatment 1 vs. Treatment 3 = 2971.000 - 2933.750 = 37.250 < 174.406 (R_2)
Treatment 3 vs. Treatment 4 = 2933.750 - 2666.250 = 267.500 > 174.406 (R_2)

Treatment 3 and Treatment 1 are not different. All other pairs of means differ. This is the same result obtained from the Fisher LSD method and the graphical method.

(b) Rework part (b) of Problem 3-1 using Tukey's test with $\alpha = 0.05$. Do you get the same conclusions from Tukey's test that you did from the graphical procedure and/or Duncan's multiple range test?

Minitab Output

```
Tukey's pairwise comparisons

   Family error rate = 0.0500
Individual error rate = 0.0117
Critical value = 4.20

Intervals for (column level mean) - (row level mean)

             1          2          3

     2      -423
```

	53		
3	-201 275	-15 460	
4	67 543	252 728	30 505

No, the conclusions are not the same. The mean of Treatment 4 is different than the means of Treatments 1, 2, and 3. However, the mean of Treatment 2 is not different from the means of Treatments 1 and 3 according to the Tukey method, they were found to be different using the graphical method and Duncan's multiple range test.

(c) Explain the difference between the Tukey and Duncan procedures.

A single critical value is used for comparison with the Tukey procedure where $a - 1$ critical values are used with the Duncan procedure. Tukey's test has a type I error rate of α for all pairwise comparisons where Duncan's test type I error rate varies based on the steps between the means. Tukey's test is more conservative and has less power than Duncan's test.

3-3 Reconsider the experiment in Problem 3-1. Find a 95 percent confidence interval on the mean tensile strength of the portland cement produced by each of the four mixing techniques. Also find a 95 percent confidence interval on the difference in means for techniques 1 and 3. Does this aid in interpreting the results of the experiment?

$$\bar{y}_{i.} - t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}} \le \mu_i \le \bar{y}_{i.} + t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}}$$

$$\text{Treatment 1: } 2971 \pm 2.179\sqrt{\frac{1282837}{4}}$$

$$2971 \pm 123.387$$

$$2847.613 \le \mu_1 \le 3094.387$$

$$\text{Treatment 2: } 3156.25 \pm 123.387$$

$$3032.863 \le \mu_2 \le 3279.637$$

$$\text{Treatment 3: } 2933.75 \pm 123.387$$

$$2810.363 \le \mu_3 \le 3057.137$$

$$\text{Treatment 4: } 2666.25 \pm 123.387$$

$$2542.863 \le \mu_4 \le 2789.637$$

$$\text{Treatment 1 - Treatment 3: } \bar{y}_{i.} - \bar{y}_{j.} - t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i.} - \bar{y}_{j.} + t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}}$$

$$2971.00 - 2933.75 \pm 2.179\sqrt{\frac{2(12825.7)}{4}}$$

$$-137.245 \le \mu_1 - \mu_3 \le 211.745$$

3-4 An experiment was run to determine whether four specific firing temperatures affect the density of a certain type of brick. The experiment led to the following data:

Temperature	Density				
100	21.8	21.9	21.7	21.6	21.7
125	21.7	21.4	21.5	21.4	
150	21.9	21.8	21.8	21.6	21.5
175	21.9	21.7	21.8	21.4	

(a) Does the firing temperature affect the density of the bricks? Use $\alpha = 0.05$.

No, firing temperature does not affect the density of the bricks. Refer to the Design-Expert output below.

Design Expert Output

Response: Density
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.16	3	0.052	2.02	0.1569	not significant
A	*0.16*	*3*	*0.052*	*2.02*	*0.1569*	
Residual	0.36	14	0.026			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.36*	*14*	*0.026*			
Cor Total	0.52	17				

The "Model F-value" of 2.02 implies the model is not significant relative to the noise. There is a 15.69 % chance that a "Model F-value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

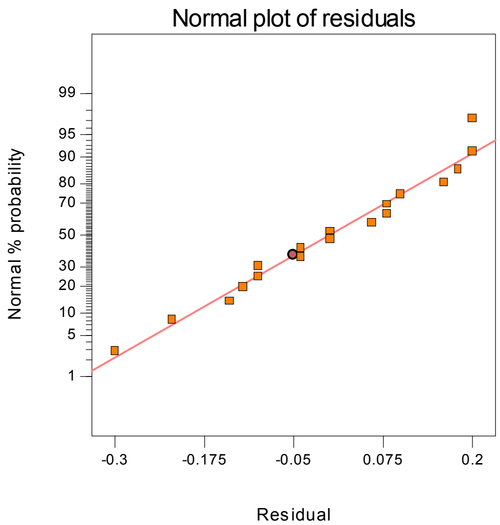
	Estimated Mean	Standard Error
1-100	21.74	0.072
2-125	21.50	0.080
3-150	21.72	0.072
4-175	21.70	0.080

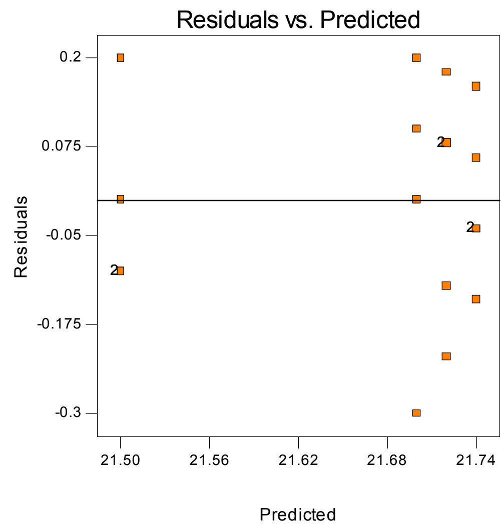
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.24	1	0.11	2.23	0.0425
1 vs 3	0.020	1	0.10	0.20	0.8465
1 vs 4	0.040	1	0.11	0.37	0.7156
2 vs 3	-0.22	1	0.11	-2.05	0.0601
2 vs 4	-0.20	1	0.11	-1.76	0.0996
3 vs 4	0.020	1	0.11	0.19	0.8552

(b) Is it appropriate to compare the means using Duncan's multiple range test in this experiment?

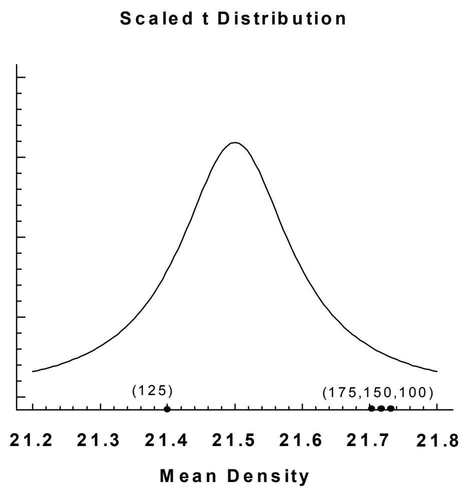
The analysis of variance tells us that there is no difference in the treatments. There is no need to proceed with Duncan's multiple range test to decide which mean is difference.

(c) Analyze the residuals from this experiment. Are the analysis of variance assumptions satisfied?
There is nothing unusual about the residual plots.





(d) Construct a graphical display of the treatments as described in Section 3-5.3. Does this graph adequately summarize the results of the analysis of variance in part (b). Yes.



3-5 Rework Part (d) of Problem 3-4 using the Fisher LSD method. What conclusions can you draw? Explain carefully how you modified the procedure to account for unequal sample sizes.

When sample sizes are unequal, the appropriate formula for the LSD is

$$LSD = t_{\alpha/2, N-a}\sqrt{MS_E\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$$

Treatment 1	vs. Treatment 2	$= 21.74 - 21.50 = 0.24 > 0.2320$
Treatment 1	vs. Treatment 3	$= 21.74 - 21.72 = 0.02 < 0.2187$
Treatment 1	vs. Treatment 4	$= 21.74 - 21.70 = 0.04 < 0.2320$
Treatment 3	vs. Treatment 2	$= 21.72 - 21.50 = 0.22 < 0.2320$
Treatment 4	vs. Treatment 2	$= 21.70 - 21.50 = 0.20 < 0.2446$
Treatment 3	vs. Treatment 4	$= 21.72 - 21.70 = 0.02 < 0.2320$

Treatment 1, temperature of 100, is different than Treatment 2, temperature of 125. All other pairwise comparisons do not identify differences. Notice something very interesting has happened here. The analysis of variance indicated that there were no differences between treatment means, yet the LSD procedure found a difference; in fact, the Design-Expert output indicates that the *P*-value if slightly less that 0.05. This illustrates a danger of using multiple comparison procedures without relying on the results from the analysis of variance. Because we could not reject the hypothesis of equal means using the analysis of variance, we should **never** have performed the Fisher LSD (or any other multiple comparison procedure, for that matter). If you ignore the analysis of variance results and run multiple comparisons, you will likely make type I errors.

The LSD calculations utilized Equation 3-32, which accommodates different sample sizes. Equation 3-32 simplifies to Equation 3-33 for a balanced design experiment.

3-6 A manufacturer of television sets is interested in the effect of tube conductivity of four different types of coating for color picture tubes. The following conductivity data are obtained:

Coating Type	Conductivity			
1	143	141	150	146
2	152	149	137	143
3	134	136	132	127
4	129	127	132	129

(a) Is there a difference in conductivity due to coating type? Use $\alpha = 0.05$.

Yes, there is a difference in means. Refer to the Design-Expert output below..

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	844.69	3	281.56	14.30	0.0003	significant
A	*844.69*	*3*	*281.56*	*14.30*	*0.0003*	
Residual	236.25	12	19.69			
Lack of Fit	*0.000*	*0*				
Pure Error	*236.25*	*12*	*19.69*			
Cor Total	1080.94	15				

The Model F-value of 14.30 implies the model is significant. There is only a 0.03% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	145.00	2.22
2-2	145.25	2.22
3-3	132.25	2.22
4-4	129.25	2.22

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-0.25	1	3.14	-0.080	0.9378
1 vs 3	12.75	1	3.14	4.06	0.0016
1 vs 4	15.75	1	3.14	5.02	0.0003
2 vs 3	13.00	1	3.14	4.14	0.0014
2 vs 4	16.00	1	3.14	5.10	0.0003
3 vs 4	3.00	1	3.14	0.96	0.3578

(b) Estimate the overall mean and the treatment effects.

$$\hat{\mu} = 2207/16 = 137.9375$$
$$\hat{\tau}_1 = \bar{y}_{1.} - \bar{y}_{..} = 145.00 - 137.9375 = 7.0625$$
$$\hat{\tau}_2 = \bar{y}_{2.} - \bar{y}_{..} = 145.25 - 137.9375 = 7.3125$$
$$\hat{\tau}_3 = \bar{y}_{3.} - \bar{y}_{..} = 132.25 - 137.9375 = -5.6875$$
$$\hat{\tau}_4 = \bar{y}_{4.} - \bar{y}_{..} = 129.25 - 137.9375 = -8.6875$$

(c) Compute a 95 percent interval estimate of the mean of coating type 4. Compute a 99 percent interval estimate of the mean difference between coating types 1 and 4.

$$\text{Treatment 4: } 129.25 \pm 2.179\sqrt{\frac{19.69}{4}}$$
$$124.4155 \le \mu_4 \le 134.0845$$
$$\text{Treatment 1 - Treatment 4: } (145 - 129.25) \pm 3.055\sqrt{\frac{(2)19.69}{4}}$$
$$6.164 \le \mu_1 - \mu_4 \le 25.336$$

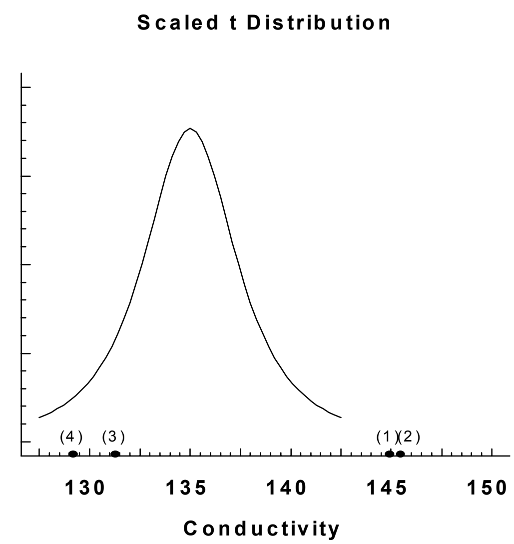
(d) Test all pairs of means using the Fisher LSD method with α=0.05.

Refer to the Design-Expert output above. The Fisher LSD procedure is automatically included in the output.

The means of Coating Type 2 and Coating Type 1 are not different. The means of Coating Type 3 and Coating Type 4 are not different. However, Coating Types 1 and 2 produce higher mean conductivity that does Coating Types 3 and 4.

(e) Use the graphical method discussed in Section 3-5.3 to compare the means. Which coating produces the highest conductivity?

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{16.96}{4}} = 2.219 \text{ Coating types 1 and 2 produce the highest conductivity.}$$

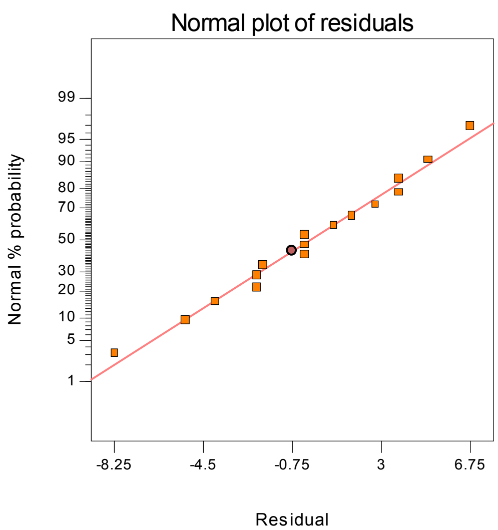


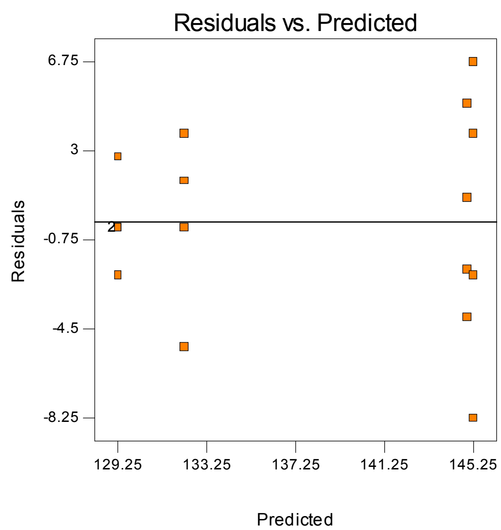
(f) Assuming that coating type 4 is currently in use, what are your recommendations to the manufacturer? We wish to minimize conductivity.

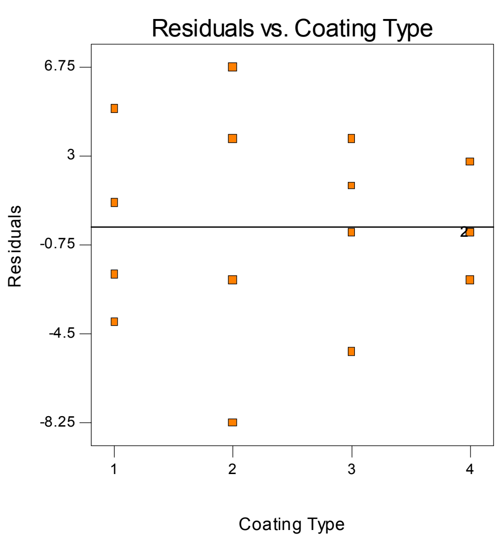
Since coatings 3 and 4 do not differ, and as they both produce the lowest mean values of conductivity, use either coating 3 or 4. As type 4 is currently being used, there is probably no need to change.

3-7 Reconsider the experiment in Problem 3-6. Analyze the residuals and draw conclusions about model adequacy.

There is nothing unusual in the normal probability plot. A funnel shape is seen in the plot of residuals versus predicted conductivity indicating a possible non-constant variance.







3-8 An article in the *ACI Materials Journal* (Vol. 84, 1987. pp. 213-216) describes several experiments investigating the rodding of concrete to remove entrapped air. A 3" x 6" cylinder was used, and the number of times this rod was used is the design variable. The resulting compressive strength of the concrete specimen is the response. The data are shown in the following table.

Rodding Level	Compressive Strength		
10	1530	1530	1440
15	1610	1650	1500
20	1560	1730	1530
25	1500	1490	1510

(a) Is there any difference in compressive strength due to the rodding level? Use $\alpha = 0.05$.

There are no differences.

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	28633.33	3	9544.44	1.87	0.2138	not significant
A	*28633.33*	*3*	*9544.44*	*1.87*	*0.2138*	
Residual	40933.33	8	5116.67			
Lack of Fit	*0.000*	*0*				
Pure Error	*40933.33*	*8*	*5116.67*			
Cor Total	69566.67	11				

The "Model F-value" of 1.87 implies the model is not significant relative to the noise. There is a 21.38 % chance that a "Model F-value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

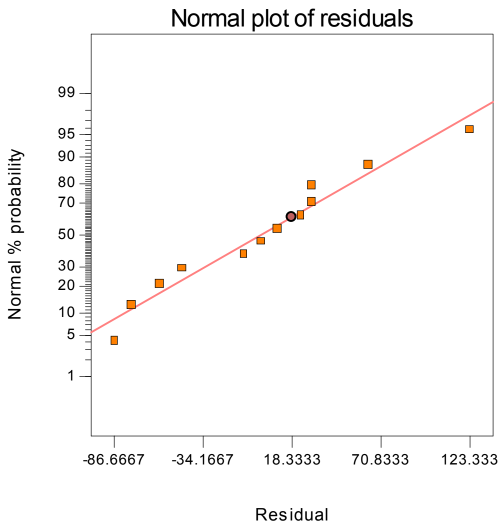
Estimated Mean	Standard Error	
1-10	1500.00	41.30
2-15	1586.67	41.30
3-20	1606.67	41.30
4-25	1500.00	41.30

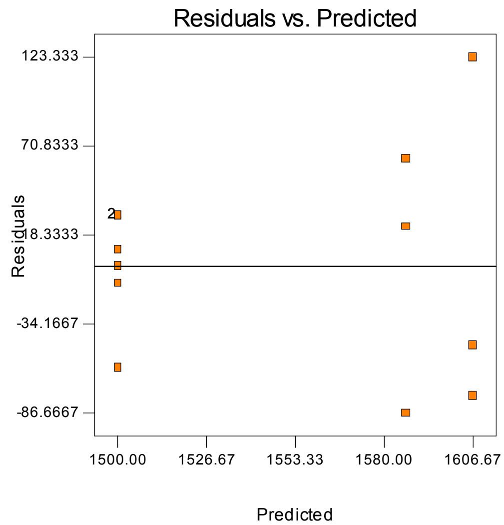
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-86.67	1	58.40	-1.48	0.1761
1 vs 3	-106.67	1	58.40	-1.83	0.1052
1 vs 4	0.000	1	58.40	0.000	1.0000
2 vs 3	-20.00	1	58.40	-0.34	0.7408
2 vs 4	86.67	1	58.40	1.48	0.1761
3 vs 4	106.67	1	58.40	1.83	0.1052

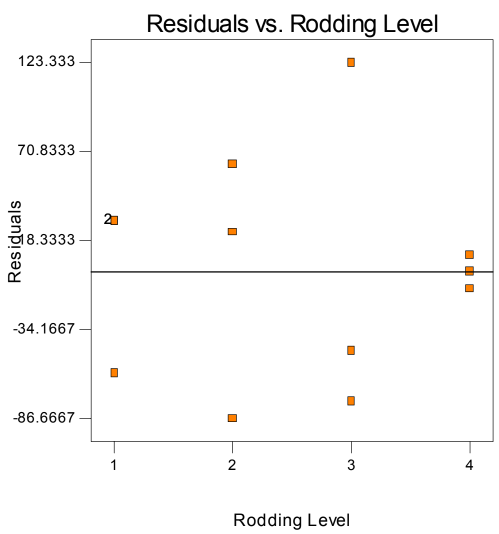
(b) Find the *P*-value for the *F* statistic in part (a). From computer output, P=0.2138.

(c) Analyze the residuals from this experiment. What conclusions can you draw about the underlying model assumptions?

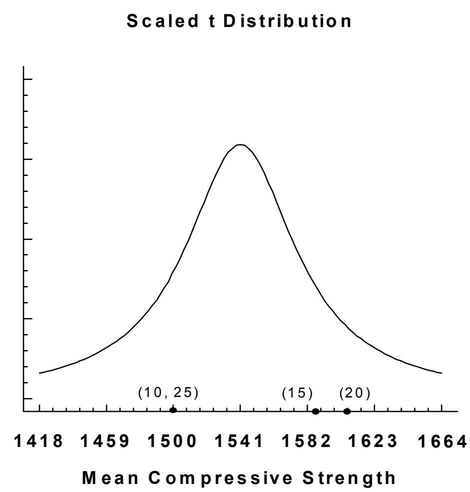
There is nothing unusual about the residual plots.







(d) Construct a graphical display to compare the treatment means as describe in Section 3-5.3.



3-9 An article in *Environment International* (Vol. 18, No. 4, 1992) describes an experiment in which the amount of radon released in showers was investigated. Radon enriched water was used in the experiment and six different orifice diameters were tested in shower heads. The data from the experiment are shown in the following table.

Orifice Dia.	Radon Released (%)			
0.37	80	83	83	85
0.51	75	75	79	79
0.71	74	73	76	77
1.02	67	72	74	74
1.40	62	62	67	69
1.99	60	61	64	66

(a) Does the size of the orifice affect the mean percentage of radon released? Use $\alpha = 0.05$.

Yes. There is at least one treatment mean that is different.

Design Expert Output

Response: Radon Released in %
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1133.38	5	226.68	30.85	< 0.0001	significant
A	*1133.38*	*5*	*226.68*	*30.85*	*< 0.0001*	
Residual	132.25	18	7.35			
Lack of Fit	*0.000*	*0*				
Pure Error	*132.25*	*18*	*7.35*			
Cor Total	1265.63	23				

The Model F-value of 30.85 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

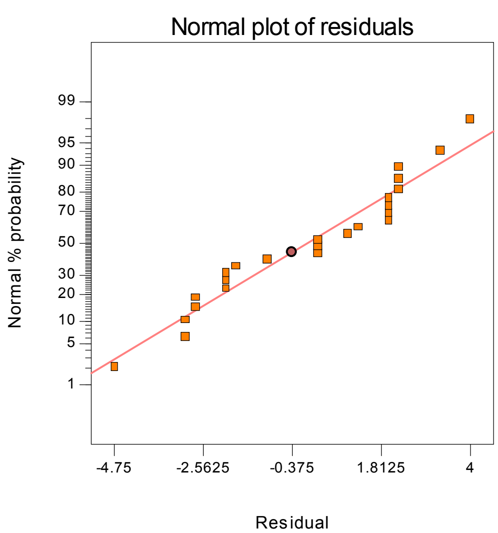
	Estimated Mean	Standard Error
1-0.37	82.75	1.36
2-0.51	77.00	1.36
3-0.71	75.00	1.36
4-1.02	71.75	1.36
5-1.40	65.00	1.36

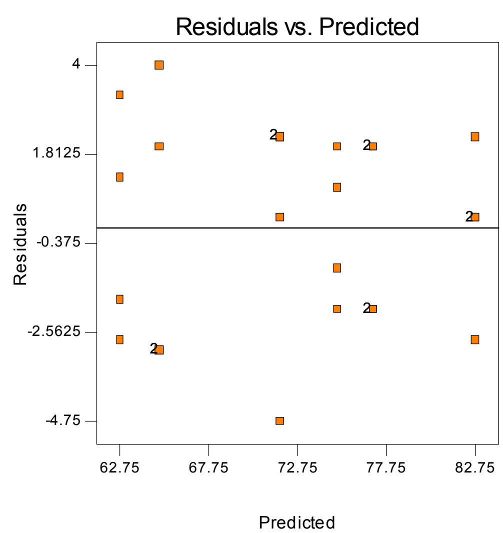
6-1.99	62.75	1.36			
Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	5.75	1	1.92	3.00	0.0077
1 vs 3	7.75	1	1.92	4.04	0.0008
1 vs 4	11.00	1	1.92	5.74	< 0.0001
1 vs 5	17.75	1	1.92	9.26	< 0.0001
1 vs 6	20.00	1	1.92	10.43	< 0.0001
2 vs 3	2.00	1	1.92	1.04	0.3105
2 vs 4	5.25	1	1.92	2.74	0.0135
2 vs 5	12.00	1	1.92	6.26	< 0.0001
2 vs 6	14.25	1	1.92	7.43	< 0.0001
3 vs 4	3.25	1	1.92	1.70	0.1072
3 vs 5	10.00	1	1.92	5.22	< 0.0001
3 vs 6	12.25	1	1.92	6.39	< 0.0001
4 vs 5	6.75	1	1.92	3.52	0.0024
4 vs 6	9.00	1	1.92	4.70	0.0002
5 vs 6	2.25	1	1.92	1.17	0.2557

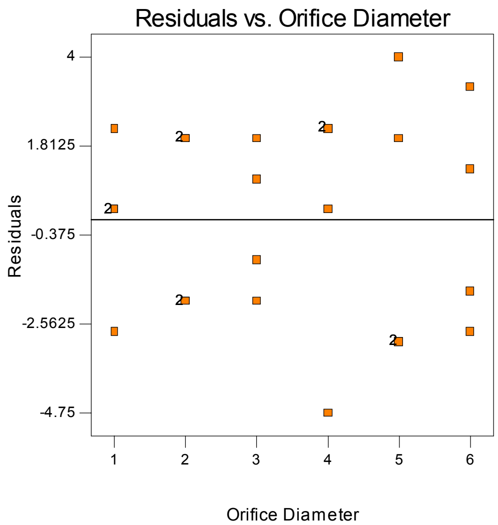
(b) Find the P-value for the F statistic in part (a). P=3.161 x 10^{-8}

(c) Analyze the residuals from this experiment.

There is nothing unusual about the residuals.





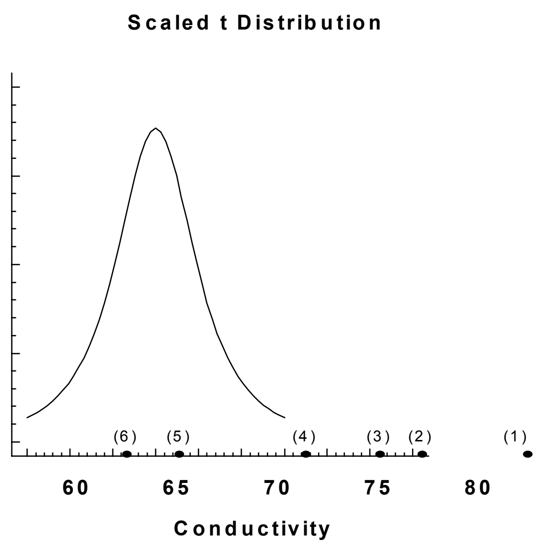


(d) Find a 95 percent confidence interval on the mean percent radon released when the orifice diameter is 1.40.

$$\text{Treatment 5 (Orifice =1.40):} \quad 6 \pm 2.101\sqrt{\frac{7.35}{4}}$$

$$62.152 \le \mu \le 67.848$$

(e) Construct a graphical display to compare the treatment means as describe in Section 3-5.3. What conclusions can you draw?



Treatments 5 and 6 as a group differ from the other means; 2, 3, and 4 as a group differ from the other means, 1 differs from the others.

3-10 The response time in milliseconds was determined for three different types of circuits that could be used in an automatic valve shutoff mechanism. The results are shown in the following table.

Circuit Type	Response Time				
1	9	12	10	8	15
2	20	21	23	17	30
3	6	5	8	16	7

(a) Test the hypothesis that the three circuit types have the same response time. Use $\alpha = 0.01$.

From the computer printout, F=16.08, so there is at least one circuit type that is different.

Design Expert Output

Response: Response Time in ms
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	543.60	2	271.80	16.08	0.0004	significant
A	*543.60*	*2*	*271.80*	*16.08*	*0.0004*	
Residual	202.80	12	16.90			
Lack of Fit	*0.000*	*0*				
Pure Error	*202.80*	*12*	*16.90*			
Cor Total	746.40	14				

The Model F-value of 16.08 implies the model is significant. There is only a 0.04% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	10.80	1.84
2-2	22.20	1.84
3-3	8.40	1.84

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-11.40	1	2.60	-4.38	0.0009
1 vs 3	2.40	1	2.60	0.92	0.3742
2 vs 3	13.80	1	2.60	5.31	0.0002

(b) Use Tukey's test to compare pairs of treatment means. Use $\alpha = 0.01$.

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{1690}{5}} = 1.8385$$

$$q_{0.01,(3,12)} = 5.04$$

$$t_0 = 1.8385(5.04) = 9.266$$

$$1 \text{ vs. } 2: \ |10.8\text{-}22.2| = 11.4 > 9.266$$

$$1 \text{ vs. } 3: \ |10.8\text{-}8.4| = 2.4 < 9.266$$

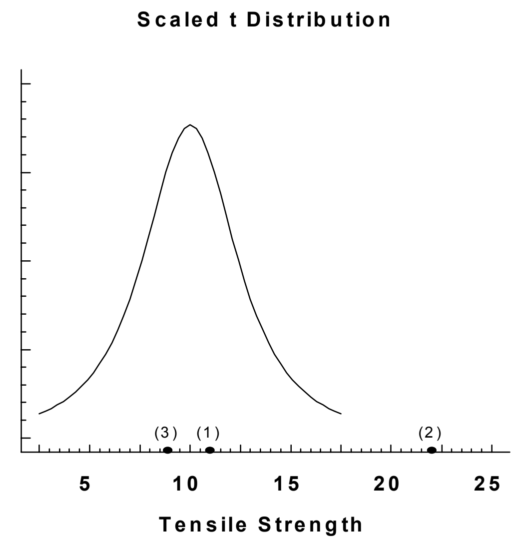
$$2 \text{ vs. } 3: \ |22.2\text{-}8.4| = 13.8 > 9.266$$

1 and 2 are different. 2 and 3 are different.

Notice that the results indicate that the mean of treatment 2 differs from the means of both treatments 1 and 3, and that the means for treatments 1 and 3 are the same. Notice also that the Fisher LSD procedure (see the computer output) gives the same results.

(c) Use the graphical procedure in Section 3-5.3 to compare the treatment means. What conclusions can you draw? How do they compare with the conclusions from part (a).

The scaled-*t* plot agrees with part (b). In this case, the large difference between the mean of treatment 2 and the other two treatments is very obvious.



(d) Construct a set of orthogonal contrasts, assuming that at the outset of the experiment you suspected the response time of circuit type 2 to be different from the other two.

$$H_0 = \mu_1 - 2\mu_2 + \mu_3 = 0$$
$$H_1 = \mu_1 - 2\mu_2 + \mu_3 \neq 0$$
$$C_1 = y_{1.} - 2y_{2.} + y_{3.}$$
$$C_1 = 54 - 2(111) + 42 = -126$$
$$SS_{C1} = \frac{(-126)^2}{5(6)} = 529.2$$
$$F_{C1} = \frac{529.2}{16.9} = 31.31$$

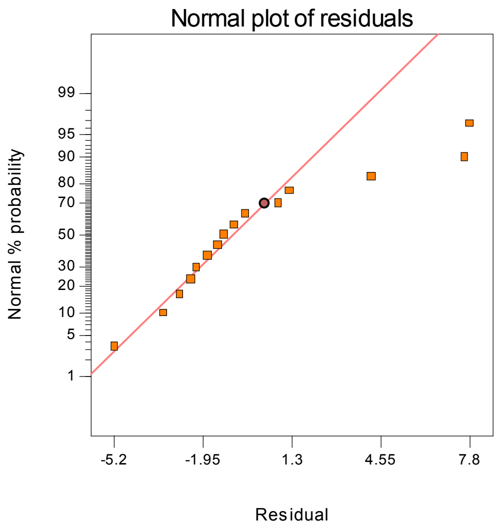
Type 2 differs from the average of type 1 and type 3.

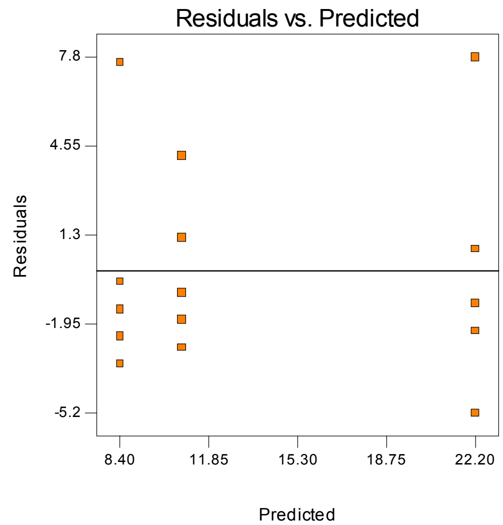
(e) If you were a design engineer and you wished to minimize the response time, which circuit type would you select?

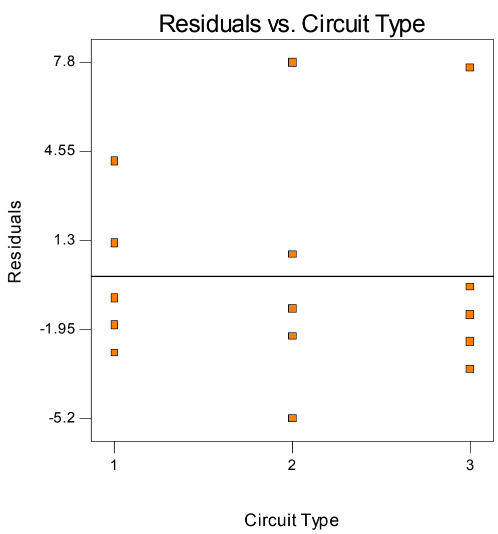
Either type 1 or type 3 as they are not different from each other and have the lowest response time.

(f) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied?

The normal probability plot has some points that do not lie along the line in the upper region. This may indicate potential outliers in the data.







3-11 The effective life of insulating fluids at an accelerated load of 35 kV is being studied. Test data have been obtained for four types of fluids. The results were as follows:

Fluid Type	Life (in h) at 35 kV Load					
1	17.6	18.9	16.3	17.4	20.1	21.6
2	16.9	15.3	18.6	17.1	19.5	20.3
3	21.4	23.6	19.4	18.5	20.5	22.3
4	19.3	21.1	16.9	17.5	18.3	19.8

(a) Is there any indication that the fluids differ? Use $\alpha = 0.05$.

At $\alpha = 0.05$ there are no difference, but at since the *P*-value is just slightly above 0.05, there is probably a difference in means.

Design Expert Output

Response:	**Life**	**in in h**

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	30.17	3	10.06	3.05	0.0525	not significant
A	*30.16*	*3*	*10.05*	*3.05*	*0.0525*	
Residual	65.99	20	3.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*65.99*	*20*	*3.30*			
Cor Total	96.16	23				

The Model F-value of 3.05 implies there is a 5.25% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

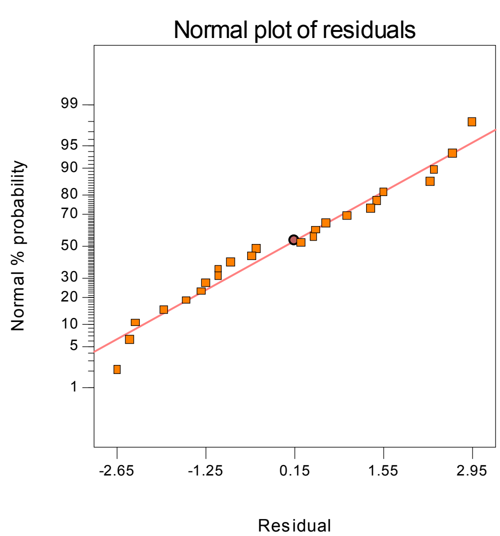
	Estimated Mean	Standard Error
1-1	18.65	0.74
2-2	17.95	0.74
3-3	20.95	0.74
4-4	18.82	0.74

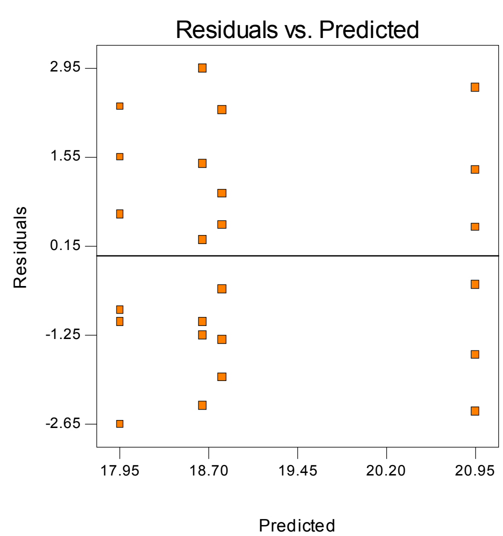
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.70	1	1.05	0.67	0.5121
1 vs 3	-2.30	1	1.05	-2.19	0.0403
1 vs 4	-0.17	1	1.05	-0.16	0.8753
2 vs 3	-3.00	1	1.05	-2.86	0.0097
2 vs 4	-0.87	1	1.05	-0.83	0.4183
3 vs 4	2.13	1	1.05	2.03	0.0554

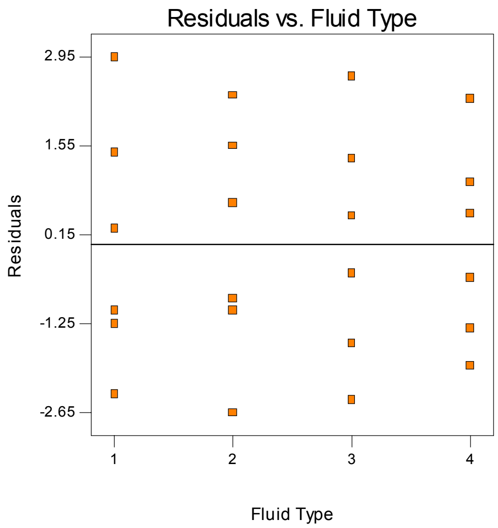
(b) Which fluid would you select, given that the objective is long life?

Treatment 3. The Fisher LSD procedure in the computer output indicates that the fluid 3 is different from the others, and it's average life also exceeds the average lives of the other three fluids.

(c) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied? There is nothing unusual in the residual plots.







3-12 Four different designs for a digital computer circuit are being studied in order to compare the amount of noise present. The following data have been obtained:

Circuit Design	Noise Observed				
1	19	20	19	30	8
2	80	61	73	56	80
3	47	26	25	35	50
4	95	46	83	78	97

(a) Is the amount of noise present the same for all four designs? Use $\alpha = 0.05$.

No, at least one treatment mean is different.

Design Expert Output

Response: Noise
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	12042.00	3	4014.00	21.78	< 0.0001	significant
A	*12042.00*	*3*	*4014.00*	*21.78*	*< 0.0001*	
Residual	2948.80	16	184.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*2948.80*	*16*	*184.30*			
Cor Total	14990.80	19				

The Model F-value of 21.78 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

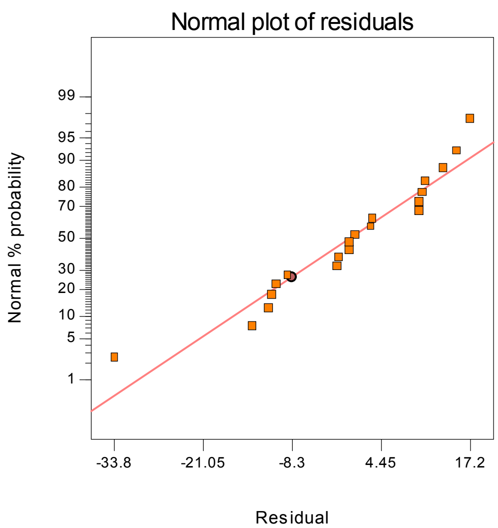
Treatment Means (Adjusted, If Necessary)

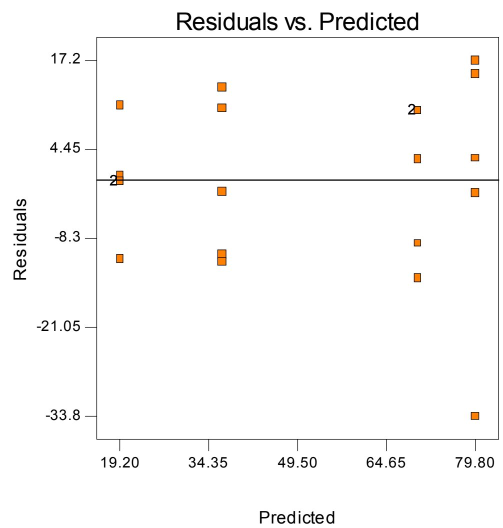
	Estimated Mean	Standard Error
1-1	19.20	6.07
2-2	70.00	6.07
3-3	36.60	6.07
4-4	79.80	6.07

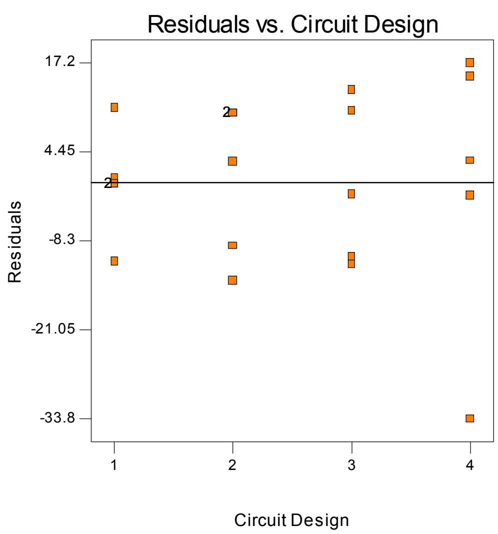
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-50.80	1	8.59	-5.92	< 0.0001

1 vs 3	-17.40	1	8.59	-2.03	0.0597
1 vs 4	-60.60	1	8.59	-7.06	< 0.0001
2 vs 3	33.40	1	8.59	3.89	0.0013
2 vs 4	-9.80	1	8.59	-1.14	0.2705
3 vs 4	-43.20	1	8.59	-5.03	0.0001

(b) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied?
There is nothing unusual about the residual plots.







(c) Which circuit design would you select for use? Low noise is best.

From the Design Expert Output, the Fisher LSD procedure comparing the difference in means identifies Type 1 as having lower noise than Types 2 and 4. Although the LSD procedure comparing Types 1 and 3 has a *P*-value greater than 0.05, it is less than 0.10. Unless there are other reasons for choosing Type 3, Type 1 would be selected.

3-13 Four chemists are asked to determine the percentage of methyl alcohol in a certain chemical compound. Each chemist makes three determinations, and the results are the following:

Chemist	Percentage of Methyl Alcohol		
1	84.99	84.04	84.38
2	85.15	85.13	84.88
3	84.72	84.48	85.16
4	84.20	84.10	84.55

(a) Do chemists differ significantly? Use $\alpha = 0.05$.

There is no significant difference at the 5% level, but chemists differ significantly at the 10% level.

Design Expert Output

Response: Methyl Alcohol in %
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.04	3	0.35	3.25	0.0813	not significant
A	*1.04*	*3*	*0.35*	*3.25*	*0.0813*	
Residual	0.86	8	0.11			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.86*	*8*	*0.11*			
Cor Total	1.90	11				

The Model F-value of 3.25 implies there is a 8.13% chance that a "Model F-Value" this large could occur due to noise.

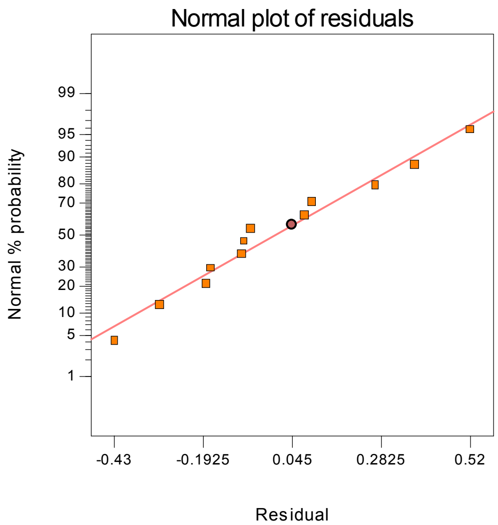
Treatment Means (Adjusted, If Necessary)

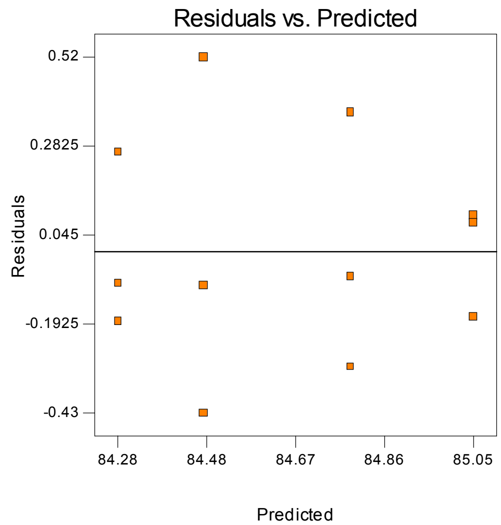
	Estimated Mean	Standard Error
1-1	84.47	0.19
2-2	85.05	0.19
3-3	84.79	0.19
4-4	84.28	0.19

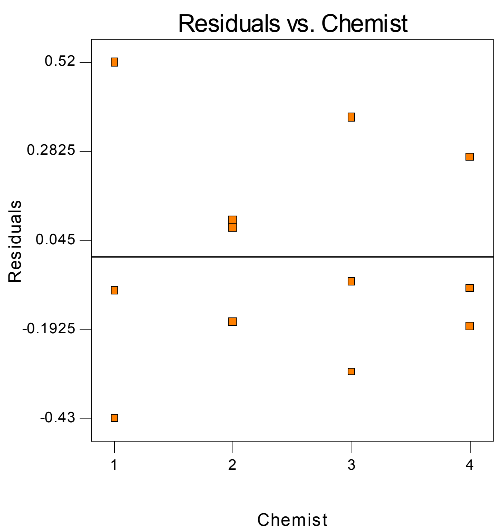
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-0.58	1	0.27	-2.18	0.0607
1 vs 3	-0.32	1	0.27	-1.18	0.2703
1 vs 4	0.19	1	0.27	0.70	0.5049
2 vs 3	0.27	1	0.27	1.00	0.3479
2 vs 4	0.77	1	0.27	2.88	0.0205
3 vs 4	0.50	1	0.27	1.88	0.0966

(b) Analyze the residuals from this experiment.

There is nothing unusual about the residual plots.







(c) If chemist 2 is a new employee, construct a meaningful set of orthogonal contrasts that might have been useful at the start of the experiment.

Chemists	Total	C1	C2	C3
1	253.41	1	-2	0
2	255.16	-3	0	0
3	254.36	1	1	-1
4	252.85	1	1	1
	Contrast Totals:	-4.86	0.39	-1.51

$$SS_{C1} = \frac{(-4.86)^2}{3(12)} = 0.656 \quad F_{C1} = \frac{0.656}{0.10727} = 6.115*$$
$$SS_{C2} = \frac{(0.39)^2}{3(6)} = 0.008 \quad F_{C2} = \frac{0.008}{0.10727} = 0.075$$
$$SS_{C3} = \frac{(-1.51)^2}{3(2)} = 0.380 \quad F_{C3} = \frac{0.380}{0.10727} = 3.54$$

Only contrast 1 is significant at 5%.

3-14 Three brands of batteries are under study. It is s suspected that the lives (in weeks) of the three brands are different. Five batteries of each brand are tested with the following results:

Weeks of Life

Brand 1	Brand 2	Brand 3
100	76	108
96	80	100
92	75	96
96	84	98
92	82	100

(a) Are the lives of these brands of batteries different?

Yes, at least one of the brands is different.

Design Expert Output

Response: Life in Weeks
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1196.13	2	598.07	38.34	< 0.0001	significant
A	*1196.13*	*2*	*598.07*	*38.34*	*< 0.0001*	
Residual	187.20	12	15.60			
Lack of Fit	*0.000*	*0*				
Pure Error	*187.20*	*12*	*15.60*			
Cor Total	1383.33	14				

The Model F-value of 38.34 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

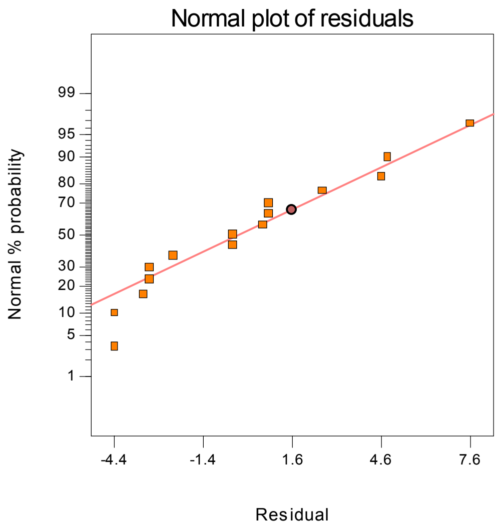
Treatment Means (Adjusted, If Necessary)

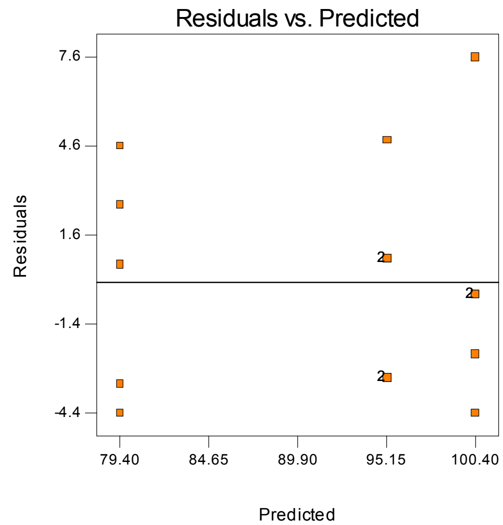
	Estimated Mean	Standard Error
1-1	95.20	1.77
2-2	79.40	1.77
3-3	100.40	1.77

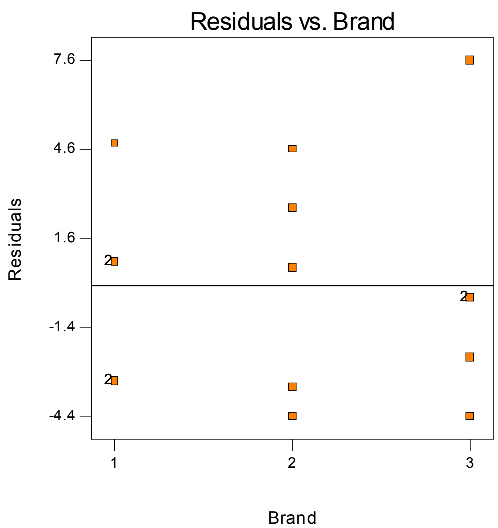
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	15.80	1	2.50	6.33	< 0.0001
1 vs 3	-5.20	1	2.50	-2.08	0.0594
2 vs 3	-21.00	1	2.50	-8.41	< 0.0001

(b) Analyze the residuals from this experiment.

There is nothing unusual about the residuals.







(c) Construct a 95 percent interval estimate on the mean life of battery brand 2. Construct a 99 percent interval estimate on the mean difference between the lives of battery brands 2 and 3.

$$\bar{y}_{i.} \pm t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}}$$

$$\text{Brand 2: } 79.4 \pm 2.179\sqrt{\frac{15.60}{5}}$$

$$79.40 \pm 3.849$$

$$75.551 \le \mu_2 \le 83.249$$

$$\text{Brand 2 - Brand 3: } \bar{y}_{i.} - \bar{y}_{j.} \pm t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}}$$

$$79.4 - 100.4 \pm 3.055\sqrt{\frac{2(15.60)}{5}}$$

$$-28.631 \le \mu_2 - \mu_3 \le -13.369$$

(d) Which brand would you select for use? If the manufacturer will replace without charge any battery that fails in less than 85 weeks, what percentage would the company expect to replace?

Chose brand 3 for longest life. Mean life of this brand in 100.4 weeks, and the variance of life is estimated by 15.60 (*MSE*). Assuming normality, then the probability of failure before 85 weeks is:

$$\Phi\left(\frac{85-100.4}{\sqrt{15.60}}\right)=\Phi(-3.90)=0.00005$$

That is, about 5 out of 100,000 batteries will fail before 85 week.

3-15 Four catalysts that may affect the concentration of one component in a three component liquid mixture are being investigated. The following concentrations are obtained:

Catalyst			
1	2	3	4
58.2	56.3	50.1	52.9
57.2	54.5	54.2	49.9
58.4	57.0	55.4	50.0
55.8	55.3		51.7
54.9			

(a) Do the four catalysts have the same effect on concentration?

No, their means are different.

Design Expert Output

Response: Concentration
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	85.68	3	28.56	9.92	0.0014	significant
A	*85.68*	*3*	*28.56*	*9.92*	*0.0014*	
Residual	34.56	12	2.88			
Lack of Fit	*0.000*	*0*				
Pure Error	*34.56*	*12*	*2.88*			
Cor Total	120.24	15				

The Model F-value of 9.92 implies the model is significant. There is only a 0.14% chance that a "Model F-Value" this large could occur due to noise.

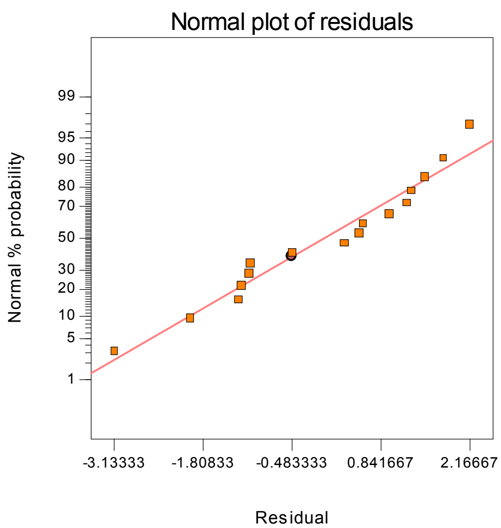
Treatment Means (Adjusted, If Necessary)

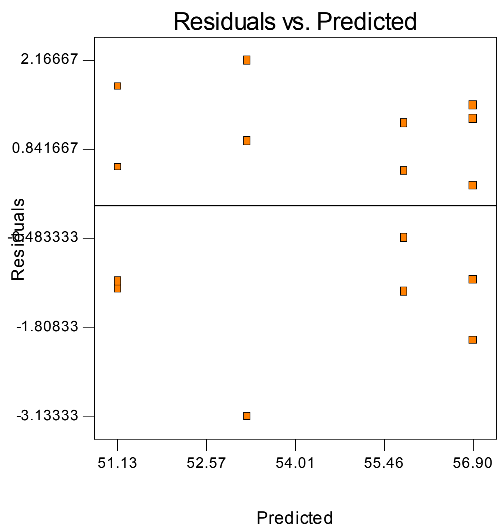
	Estimated Mean	**Standard Error**
1-1	56.90	0.76
2-2	55.77	0.85
3-3	53.23	0.98
4-4	51.13	0.85

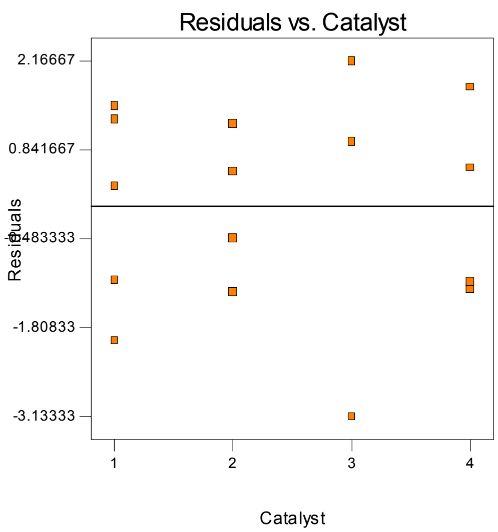
Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	1.13	1	1.14	0.99	0.3426
1 vs 3	3.67	1	1.24	2.96	0.0120
1 vs 4	5.77	1	1.14	5.07	0.0003
2 vs 3	2.54	1	1.30	1.96	0.0735
2 vs 4	4.65	1	1.20	3.87	0.0022
3 vs 4	2.11	1	1.30	1.63	0.1298

(b) Analyze the residuals from this experiment.

There is nothing unusual about the residual plots.







(c) Construct a 99 percent confidence interval estimate of the mean response for catalyst 1.

$$\bar{y}_{i.} \pm t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}}$$

$$\text{Catalyst 1:} \quad 56.9 \pm 3.055\sqrt{\frac{2.88}{5}}$$

$$56.9 \pm 2.3186$$

$$54.5814 \le \mu_1 \le 59.2186$$

3-16 An experiment was performed to investigate the effectiveness of five insulating materials. Four samples of each material were tested at an elevated voltage level to accelerate the time to failure. The failure times (in minutes) is shown below.

Material	Failure Time (minutes)			
1	110	157	194	178
2	1	2	4	18
3	880	1256	5276	4355
4	495	7040	5307	10050
5	7	5	29	2

(a) Do all five materials have the same effect on mean failure time?

No, at least one material is different.

Design Expert Output

Response: Failure Timein Minutes
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.032E+008	4	2.580E+007	6.19	0.0038	significant
A	*1.032E+008*	*4*	*2.580E+007*	*6.19*	*0.0038*	
Residual	6.251E+007	15	4.167E+006			
Lack of Fit	*0.000*	*0*				
Pure Error	*6.251E+007*	*15*	*4.167E+006*			
Cor Total	1.657E+008	19				

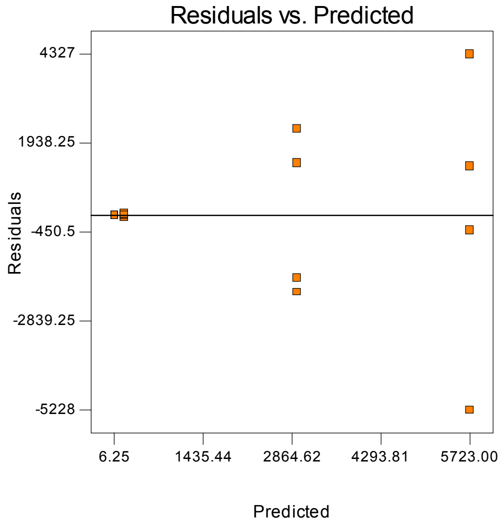
The Model F-value of 6.19 implies the model is significant. There is only a 0.38% chance that a "Model F-Value" this large could occur due to noise.

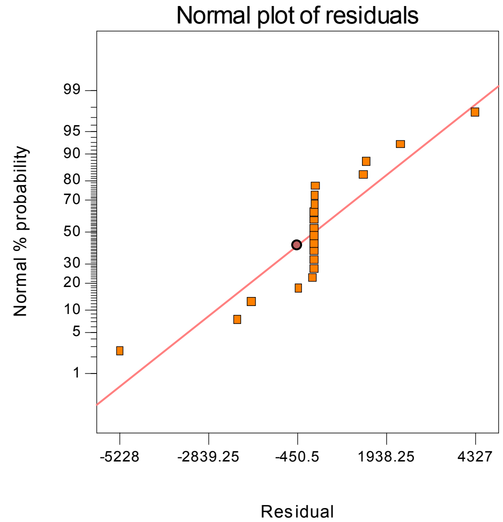
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	159.75	1020.67
2-2	6.25	1020.67
3-3	2941.75	1020.67
4-4	5723.00	1020.67
5-5	10.75	1020.67

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	153.50	1	1443.44	0.11	0.9167
1 vs 3	-2782.00	1	1443.44	-1.93	0.0731
1 vs 4	-5563.25	1	1443.44	-3.85	0.0016
1 vs 5	149.00	1	1443.44	0.10	0.9192
2 vs 3	-2935.50	1	1443.44	-2.03	0.0601
2 vs 4	-5716.75	1	1443.44	-3.96	0.0013
2 vs 5	-4.50	1	1443.44	-3.118E-003	0.9976
3 vs 4	-2781.25	1	1443.44	-1.93	0.0732
3 vs 5	2931.00	1	1443.44	2.03	0.0604
4 vs 5	5712.25	1	1443.44	3.96	0.0013

(b) Plot the residuals versus the predicted response. Construct a normal probability plot of the residuals. What information do these plots convey?





The plot of residuals versus predicted has a strong outward-opening funnel shape, which indicates the variance of the original observations is not constant. The residuals plotted in the normal probability plot also imply that the normality assumption is not valid. A data transformation is recommended.

(c) Based on your answer to part (b) conduct another analysis of the failure time data and draw appropriate conclusions.

A natural log transformation was applied to the failure time data. The analysis identifies that there exists at least one difference in treatment means.

Design Expert Output

Response: Failure Timein Minutes Transform: Natural log Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	165.06	4	41.26	37.66	< 0.0001	significant
A	*165.06*	*4*	*41.26*	*37.66*	*< 0.0001*	
Residual	16.44	15	1.10			
Lack of Fit	*0.000*	*0*				
Pure Error	*16.44*	*15*	*1.10*			
Cor Total	181.49	19				

The Model F-value of 37.66 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

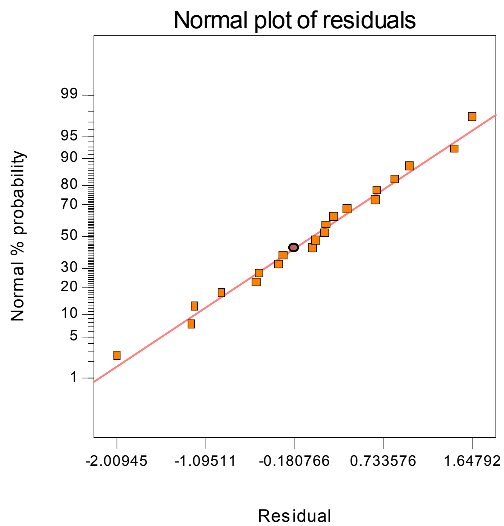
Treatment Means (Adjusted, If Necessary)

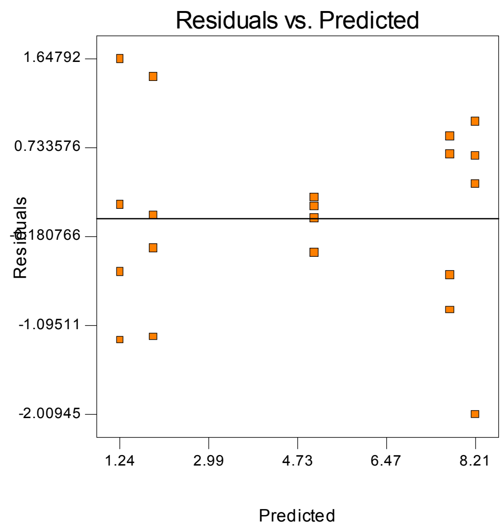
	Estimated Mean	**Standard Error**
1-1	5.05	0.52
2-2	1.24	0.52
3-3	7.72	0.52
4-4	8.21	0.52
5-5	1.90	0.52

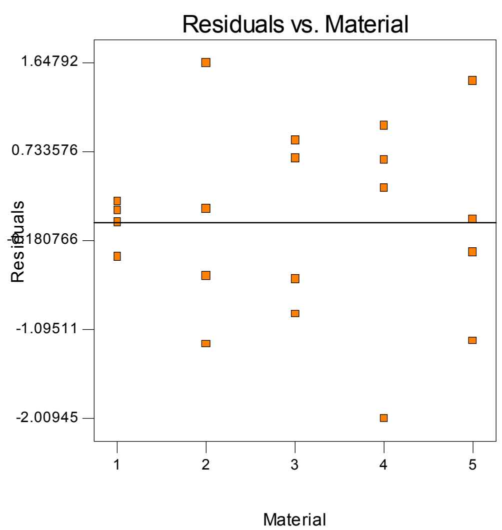
Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	3.81	1	0.74	5.15	0.0001
1 vs 3	-2.66	1	0.74	-3.60	0.0026
1 vs 4	-3.16	1	0.74	-4.27	0.0007
1 vs 5	3.15	1	0.74	4.25	0.0007
2 vs 3	-6.47	1	0.74	-8.75	< 0.0001

2 vs 4	-6.97	1	0.74	-9.42	< 0.0001
2 vs 5	-0.66	1	0.74	-0.89	0.3856
3 vs 4	-0.50	1	0.74	-0.67	0.5116
3 vs 5	5.81	1	0.74	7.85	< 0.0001
4 vs 5	6.31	1	0.74	8.52	< 0.0001

There is nothing unusual about the residual plots when the natural log transformation is applied.







3-17 A semiconductor manufacturer has developed three different methods for reducing particle counts on wafers. All three methods are tested on five wafers and the after-treatment particle counts obtained. The data are shown below.

Method	Count				
1	31	10	21	4	1
2	62	40	24	30	35
3	58	27	120	97	68

(a) Do all methods have the same effect on mean particle count?

No, at least one method has a different effect on mean particle count.

Design Expert Output

Response: Count
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	8963.73	2	4481.87	7.91	0.0064	significant
A	*8963.73*	*2*	*4481.87*	*7.91*	*0.0064*	
Residual	6796.00	12	566.33			
Lack of Fit	*0.000*	*0*				
Pure Error	*6796.00*	*12*	*566.33*			
Cor Total	15759.73	14				

The Model F-value of 7.91 implies the model is significant. There is only a 0.64% chance that a "Model F-Value" this large could occur due to noise.

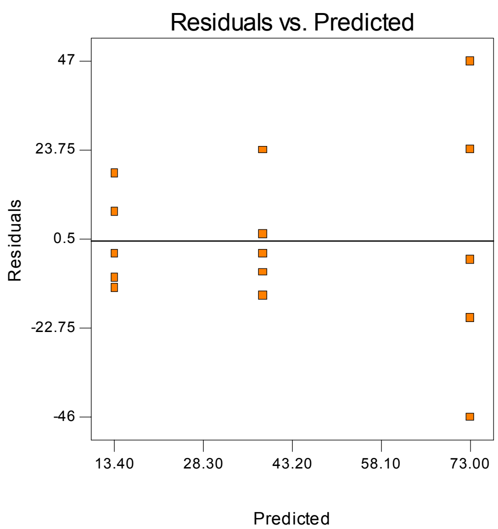
Treatment Means (Adjusted, If Necessary)

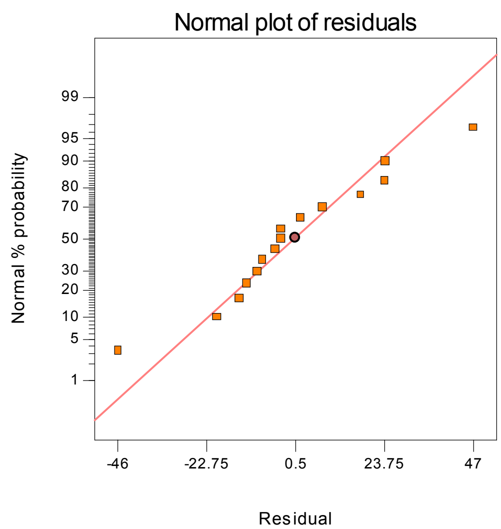
	Estimated Mean	Standard Error
1-1	13.40	10.64
2-2	38.20	10.64
3-3	73.00	10.64

Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	-24.80	1	15.05	-1.65	0.1253
1 vs 3	-59.60	1	15.05	-3.96	0.0019
2 vs 3	-34.80	1	15.05	-2.31	0.0393

(b) Plot the residuals versus the predicted response. Construct a normal probability plot of the residuals. Are there potential concerns about the validity of the assumptions?

The plot of residuals versus predicted appears to be funnel shaped. This indicates the variance of the original observations is not constant. The residuals plotted in the normal probability plot do not fall along a straight line, which suggests that the normality assumption is not valid. A data transformation is recommended.





(c) Based on your answer to part (b) conduct another analysis of the particle count data and draw appropriate conclusions.

For count data, a square root transformation is often very effective in resolving problems with inequality of variance. The analysis of variance for the transformed response is shown below. The difference between methods is much more apparent after applying the square root transformation.

Design Expert Output

Response: Count Transform: Square root Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	63.90	2	31.95	9.84	0.0030	significant
A	*63.90*	*2*	*31.95*	*9.84*	*0.0030*	
Residual	38.96	12	3.25			
Lack of Fit	*0.000*	*0*				
Pure Error	*38.96*	*12*	*3.25*			
Cor Total	102.86	14				

The Model F-value of 9.84 implies the model is significant. There is only a 0.30% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	3.26	0.81
2-2	6.10	0.81
3-3	8.31	0.81

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	-2.84	1	1.14	-2.49	0.0285
1 vs 3	-5.04	1	1.14	-4.42	0.0008
2 vs 3	-2.21	1	1.14	-1.94	0.0767

3-18 Consider testing the equality of the means of two normal populations, where the variances are unknown but are assumed to be equal. The appropriate test procedure is the pooled *t* test. Show that the pooled *t* test is equivalent to the single factor analysis of variance.

$$t_0 = \frac{\bar{y}_{1.} - \bar{y}_{2.}}{S_p\sqrt{\frac{2}{n}}} \sim t_{2n-2} \quad \text{assuming } n_1 = n_2 = n$$

$$S_p = \frac{\sum_{j=1}^{n}\left(y_{1j} - \bar{y}_{1.}\right)^2 + \sum_{j=1}^{n}\left(y_{2j} - \bar{y}_{2.}\right)^2}{2n-2} = \frac{\sum_{i=1}^{2}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{1.}\right)^2}{2n-2} \equiv MS_E \quad \text{for a=2}$$

Furthermore, $\left(\bar{y}_{1.} - \bar{y}_{2.}\right)^2\left(\frac{n}{2}\right) = \sum_{i=1}^{2}\frac{y_{i.}^2}{n} - \frac{y_{..}^2}{2n}$, which is exactly the same as $SS_{\text{Treatments}}$ in a one-way classification with a=2. Thus we have shown that $t_0^2 = \frac{SS_{Treatments}}{MS_E}$. In general, we know that $t_u^2 = F_{1,u}$ so that $t_0^2 \sim F_{1,2n-2}$. Thus the square of the test statistic from the pooled *t*-test is the same test statistic that results from a single-factor analysis of variance with a=2.

3-19 Show that the variance of the linear combination $\sum_{i=1}^{a} c_i y_{i.}$ is $\sigma^2 \sum_{i=1}^{a} n_i c_i^2$.

$$V\left[\sum_{i=1}^{a} c_i y_{i.}\right] = \sum_{i=1}^{a} V(c_i y_{i.}) = \sum_{i=1}^{a} c_i^2 V\left[\sum_{j=1}^{n_i} y_{ij}\right] = \sum_{i=1}^{a} c_i^2 \sum_{j=1}^{n_i} V(y_{ij.}),\ V(y_{ij}) = \sigma^2$$

$$= \sum_{i=1}^{a} c_i^2 n_i \sigma^2$$

3-20 In a fixed effects experiment, suppose that there are *n* observations for each of four treatments. Let Q_1^2, Q_2^2, Q_3^2 be single-degree-of-freedom components for the orthogonal contrasts. Prove that $SS_{Treatments} = Q_1^2 + Q_2^2 + Q_3^2$.

$$C_1 = 3y_{1.} - y_{2.} - y_{3.} - y_{4.} \qquad SS_{C1} = Q_1^2$$
$$C_2 = 2y_{2.} - y_{3.} - y_{4.} \qquad SS_{C2} = Q_2^2$$
$$C_3 = y_{3.} - y_{4.} \qquad SS_{C3} = Q_3^2$$

$$Q_1^2 = \frac{(3y_{1.} - y_{2.} - y_{3.} - y_{4.})^2}{12n}$$

$$Q_2^2 = \frac{(2y_{2.} - y_{3.} - y_{4.})^2}{6n}$$

$$Q_3^2 = \frac{(y_{3.} - y_{4.})^2}{2n}$$

$$Q_1^2 + Q_2^2 + Q_3^2 = \frac{9\sum_{i=1}^{4} y_{i.}^2 - 6\left(\sum\sum_{i<j} y_{i.} y_{j.}\right)}{12n} \text{ and since}$$

$$\sum\sum_{i<j} y_{i.} y_{j.} = \frac{1}{2}\left(y_{..}^2 - \sum_{i=1}^{4} y_{i.}^2\right), \text{ we have } Q_1^2 + Q_2^2 + Q_3^2 = \frac{12\sum_{i=1}^{4} y_{i.}^2 - 3y_{..}^2}{12n} = \sum_{i=1}^{4} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{4n} = SS_{Treatments}$$

for a=4.

3-21 Use Bartlett's test to determine if the assumption of equal variances is satisfied in Problem 3-14. Use $\alpha = 0.05$. Did you reach the same conclusion regarding the equality of variance by examining the residual plots?

$$\chi_0^2 = 2.3026 \frac{q}{c}, \text{ where}$$

$$q = (N-a)log_{10}\ S_p^2 - \sum_{i=1}^{a}(n_i - 1)log_{10}\ S_i^2$$

$$c = 1 + \frac{1}{3(a-1)}\left(\sum_{i=1}^{a}(n_i - 1)^{-1} - (N-a)^{-1}\right)$$

$$S_p^2 = \frac{\sum_{i=1}^{a}(n_i - 1)S_i^2}{N-a}$$

$$S_1^2 = 11.2 \qquad S_p^2 = \frac{(5-1)11.2 + (5-1)14.8 + (5-1)20.8}{15-3}$$

$$S_2^2 = 14.8$$

$$S_3^2 = 20.8 \qquad S_p^2 = \frac{(5-1)11.2 + (5-1)14.8 + (5-1)20.8}{15-3} = 15.6$$

$$c = 1 + \frac{1}{3(3-1)}\left(\sum_{i=1}^{a}(5-1)^{-1} - (15-3)^{-1}\right)$$

$$c = 1 + \frac{1}{3(3-1)}\left(\frac{3}{4} + \frac{1}{12}\right) = 1.1389$$

$$q = (N-a)log_{10}\ S_p^2 - \sum_{i=1}^{a}(n_i - 1)log_{10}\ S_i^2$$

$$q = (15-3)log_{10}\ 15.6 - 4(log_{10}\ 11.2 + log_{10}\ 14.8 + log_{10}\ 20.8)$$

$$q = 14.3175 - 14.150 = 0.1675$$

$$\chi_0^2 = 2.3026\frac{q}{c} = 2.3026\frac{0.1675}{1.1389} = 0.3386 \qquad \chi_{0.05,4}^2 = 9.49$$

Cannot reject null hypothesis; conclude that the variance are equal. This agrees with the residual plots in Problem 3-16.

3-22 Use the modified Levene test to determine if the assumption of equal variances is satisfied on Problem 3-14. Use α = 0.05. Did you reach the same conclusion regarding the equality of variances by examining the residual plots?

The absolute value of Battery Life – brand median is:

$$\left|y_{ij} - \tilde{y}_i\right|$$

Brand 1	Brand 2	Brand 3
4	4	8
0	0	0
4	5	4
0	4	2
4	2	0

The analysis of variance indicates that there is not a difference between the different brands and therefore the assumption of equal variances is satisfired.

Design Expert Output

Response: Mod Levine
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	0.93	2	0.47	0.070	0.9328
A	*0.93*	*2*	*0.47*	*0.070*	*0.9328*
Pure Error	80.00	12	6.67		
Cor Total	80.93	14			

3-23 Refer to Problem 3-10. If we wish to detect a maximum difference in mean response times of 10 milliseconds with a probability of at least 0.90, what sample size should be used? How would you obtain a preliminary estimate of σ^2?

$$\Phi^2 = \frac{nD^2}{2a\sigma^2}$$, use MS_E from Problem 3-10 to estimate σ^2.

$$\Phi^2 = \frac{n(10)^2}{2(3)(16.9)} = 0.986n$$

Letting $\alpha = 0.05$, P(accept) = 0.1 , $\upsilon_1 = a - 1 = 2$

Trial and Error yields:

n	υ_2	Φ	P(accept)
5	12	2.22	0.17
6	15	2.43	0.09
7	18	2.62	0.04

Choose n ≥ 6, therefore N ≥ 18

Notice that we have used an estimate of the variance obtained from the present experiment. This indicates that we probably didn't use a large enough sample (n was 5 in problem 3-10) to satisfy the criteria specified in this problem. However, the sample size *was* adequate to detect differences in one of the circuit types.

When we have no prior estimate of variability, sometimes we will generate sample sizes for a range of possible variances to see what effect this has on the size of the experiment. Often a knowledgeable expert will be able to bound the variability in the response, by statements such as "the standard deviation is going to be *at least*…" or "the standard deviation shouldn't be larger than…".

3-24 Refer to Problem 3-14.

(a) If we wish to detect a maximum difference in mean battery life of 0.5 percent with a probability of at least 0.90, what sample size should be used? Discuss how you would obtain a preliminary estimate of σ^2 for answering this question.

Use the MS_E from Problem 3-14.

$$\Phi^2 = \frac{nD^2}{2a\sigma^2} \qquad \Phi^2 = \frac{n(0.005 \times 91.6667)^2}{2(3)(15.60)} = 0.002244n$$

Letting $\alpha = 0.05$, P(accept) = 0.1 , $\upsilon_1 = a - 1 = 2$

Trial and Error yields:

n	υ_2	Φ	*P(accept)*
40	117	1.895	0.18
45	132	2.132	0.10
50	147	2.369	0.05

Choose n ≥ 45, therefore N ≥ 135

See the discussion from the previous problem about the estimate of variance.

(b) If the difference between brands is great enough so that the standard deviation of an observation is increased by 25 percent, what sample size should be used if we wish to detect this with a probability of at least 0.90?

$$\upsilon_1 = a-1 = 2 \qquad \upsilon_2 = N-a = 15-3 = 12 \qquad \alpha = 0.05 \qquad P(accept) \le 0.1$$
$$\lambda = \sqrt{1+n\left[(1+0.01P)^2-1\right]} = \sqrt{1+n\left[(1+0.01(25))^2-1\right]} = \sqrt{1+0.5625n}$$

Trial and Error yields:

n	υ_2	λ	*P(accept)*
40	117	4.84	0.13
45	132	5.13	0.11
50	147	5.40	0.10

Choose n ≥ 50, therefore N ≥ 150

3-25 Consider the experiment in Problem 3-16. If we wish to construct a 95 percent confidence interval on the difference in two mean battery lives that has an accuracy of ±2 weeks, how many batteries of each brand must be tested?

$$\alpha = 0.05 \qquad MS_E = 15.6$$
$$width = t_{0.025,N-a}\sqrt{\frac{2MS_E}{n}}$$

Trial and Error yields:

n	υ_2	*t*	width
5	12	2.179	5.44
10	27	2.05	3.62
31	90	1.99	1.996
32	93	1.99	1.96

Choose n ≥ 31, therefore N ≥ 93

3-26 Suppose that four normal populations have means of μ_1=50, μ_2=60, μ_3=50, and μ_4=60. How many observations should be taken from each population so that the probability or rejecting the null hypothesis of equal population means is at least 0.90? Assume that α=0.05 and that a reasonable estimate of the error variance is σ^2=25.

$$\mu_i = \mu + \tau_i, i = 1,2,3,4$$

$$\mu = \frac{\sum_{i=1}^{4} \mu_i}{4} = \frac{220}{4} = 55$$

$$\tau_1 = -5, \tau_2 = 5, \tau_3 = -5, \tau_4 = 5$$

$$\sum_{i=1}^{4} \tau_i^2 = 100$$

$$\Phi^2 = \frac{n\sum \tau_i^2}{a\sigma^2} = \frac{100n}{4(25)} = n$$

$$\Phi = \sqrt{n}$$

$\upsilon_1 = 3, \upsilon_2 = 4(n-1), \alpha = 0.05$, From the O.C. curves we can construct the following:

n	Φ	υ_2	β	1-β
4	2.00	12	0.18	0.82
5	2.24	16	0.08	0.92

Therefore, select n=5

3-27 Refer to Problem 3-26.

(a) How would your answer change if a reasonable estimate of the experimental error variance were σ^2 = 36?

$$\Phi^2 = \frac{n\sum \tau_i^2}{a\sigma^2} = \frac{100n}{4(36)} = 0.6944n$$

$$\Phi = \sqrt{0.6944n}$$

$\upsilon_1 = 3, \upsilon_2 = 4(n-1), \alpha = 0.05$, From the O.C. curves we can construct the following:

n	Φ	υ_2	β	1-β
5	1.863	16	0.24	0.76
6	2.041	20	0.15	0.85
7	2.205	24	0.09	0.91

Therefore, select n=7

(b) How would your answer change if a reasonable estimate of the experimental error variance were σ^2 = 49?

$$\Phi^2 = \frac{n\sum \tau_i^2}{a\sigma^2} = \frac{100n}{4(49)} = 0.5102n$$

$$\Phi = \sqrt{0.5102n}$$

$\upsilon_1 = 3, \upsilon_2 = 4(n-1), \alpha = 0.05$, From the O.C. curves we can construct the following:

n	Φ	υ_2	β	$1-\beta$
7	1.890	24	0.16	0.84
8	2.020	28	0.11	0.89
9	2.142	32	0.09	0.91

Therefore, select n=9

(c) Can you draw any conclusions about the sensitivity of your answer in the particular situation about how your estimate of σ affects the decision about sample size?

As our estimate of variability increases the sample size must increase to ensure the same power of the test.

(d) Can you make any recommendations about how we should use this general approach to choosing *n* in practice?

When we have no prior estimate of variability, sometimes we will generate sample sizes for a range of possible variances to see what effect this has on the size of the experiment. Often a knowledgeable expert will be able to bound the variability in the response, by statements such as "the standard deviation is going to be *at least*…" or "the standard deviation shouldn't be larger than…".

3-28 Refer to the aluminum smelting experiment described in Section 4-2. Verify that ratio control methods do not affect average cell voltage. Construct a normal probability plot of residuals. Plot the residuals versus the predicted values. Is there an indication that any underlying assumptions are violated?

Design Expert Output

Response: Cell Average

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.746E-003	3	9.153E-004	0.20	0.8922	not significant
A	*2.746E-003*	*3*	*9.153E-004*	*0.20*	*0.8922*	
Residual	0.090	20	4.481E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.090*	*20*	*4.481E-003*			
Cor Total	0.092	23				

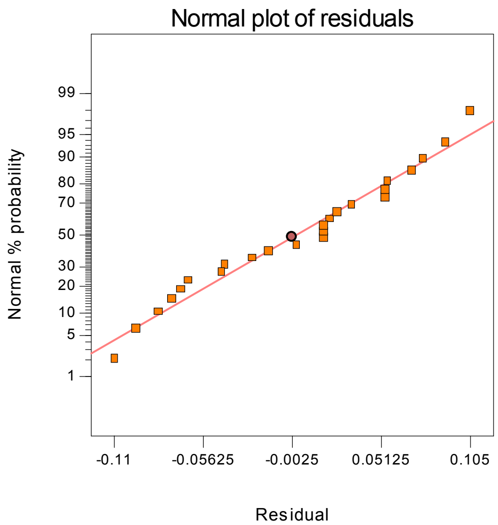
The "Model F-value" of 0.20 implies the model is not significant relative to the noise. There is a 89.22 % chance that a "Model F-value" this large could occur due to noise.

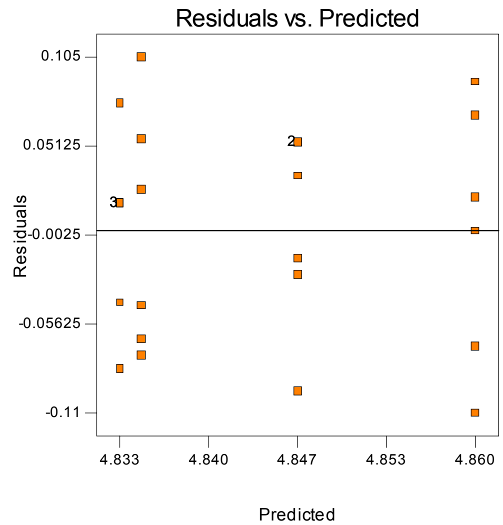
Treatment Means (Adjusted, If Necessary)

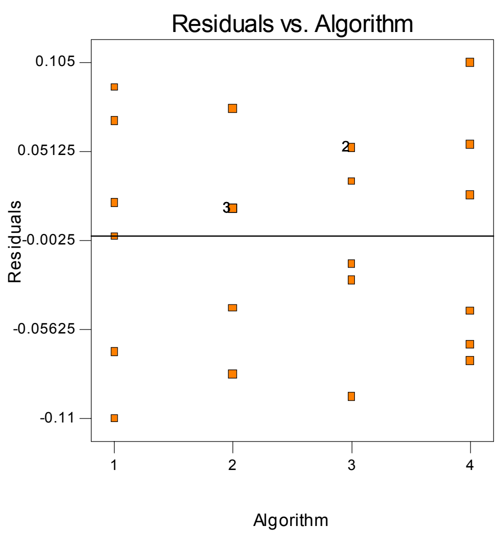
	Estimated Mean	Standard Error
1-1	4.86	0.027
2-2	4.83	0.027
3-3	4.85	0.027
4-4	4.84	0.027

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.027	1	0.039	0.69	0.4981
1 vs 3	0.013	1	0.039	0.35	0.7337
1 vs 4	0.025	1	0.039	0.65	0.5251
2 vs 3	-0.013	1	0.039	-0.35	0.7337
2 vs 4	-1.667E-003	1	0.039	-0.043	0.9660
3 vs 4	0.012	1	0.039	0.30	0.7659

The following residual plots are satisfactory.







3-29 Refer to the aluminum smelting experiment in Section 3-8. Verify the analysis of variance for pot noise summarized in Table 3-13. Examine the usual residual plots and comment on the experimental validity.

Design Expert Output

Response: Cell StDev Transform: Natural log Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	6.17	3	2.06	21.96	< 0.0001	significant
A	*6.17*	*3*	*2.06*	*21.96*	*< 0.0001*	
Residual	1.87	20	0.094			
Lack of Fit	*0.000*	*0*				
Pure Error	*1.87*	*20*	*0.094*			
Cor Total	8.04	23				

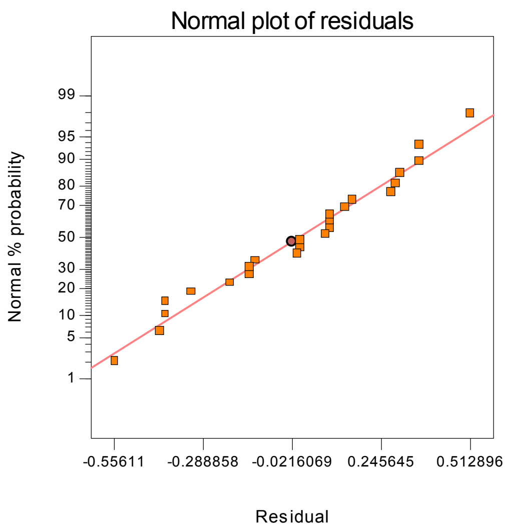
The Model F-value of 21.96 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

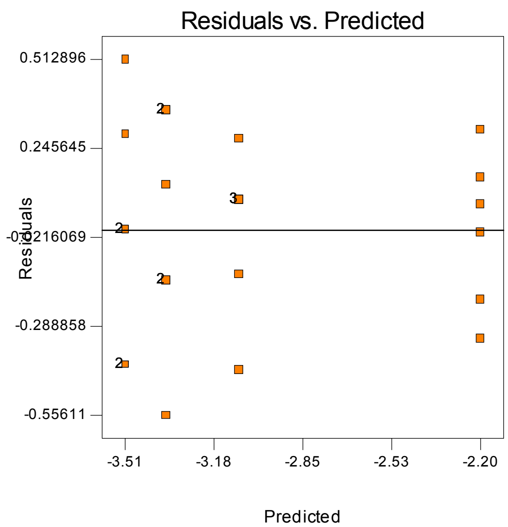
Treatment Means (Adjusted, If Necessary)

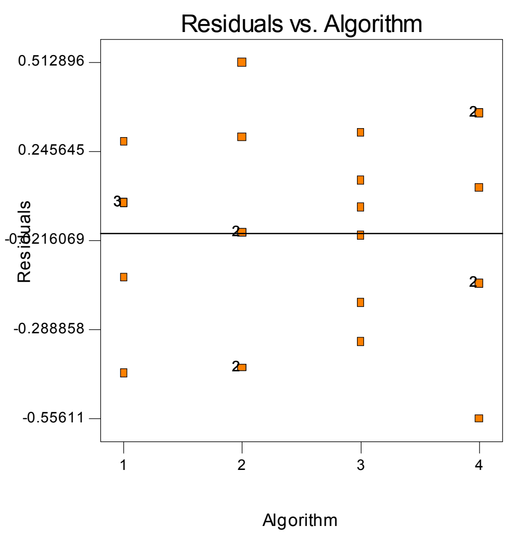
	Estimated Mean	Standard Error
1-1	-3.09	0.12
2-2	-3.51	0.12
3-3	-2.20	0.12
4-4	-3.36	0.12

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.42	1	0.18	2.38	0.0272
1 vs 3	-0.89	1	0.18	-5.03	< 0.0001
1 vs 4	0.27	1	0.18	1.52	0.1445
2 vs 3	-1.31	1	0.18	-7.41	< 0.0001
2 vs 4	-0.15	1	0.18	-0.86	0.3975
3 vs 4	1.16	1	0.18	6.55	< 0.0001

The following residual plots identify the residuals to be normally distributed, randomly distributed through the range of prediction, and uniformly distributed across the different algorithms. This validates the assumptions for the experiment.







3-30 Four different feed rates were investigated in an experiment on a CNC machine producing a component part used in an aircraft auxiliary power unit. The manufacturing engineer in charge of the experiment knows that a critical part dimension of interest may be affected by the feed rate. However, prior experience has indicated that only dispersion effects are likely to be present. That is, changing the feed rate does not affect the average dimension, but it could affect dimensional variability. The engineer makes five production runs at each feed rate and obtains the standard deviation of the critical dimension (in 10^{-3} mm). The data are shown below. Assume that all runs were made in random order.

Feed Rate (in/min)	Production Run 1	2	3	4	5
10	0.09	0.10	0.13	0.08	0.07
12	0.06	0.09	0.12	0.07	0.12
14	0.11	0.08	0.08	0.05	0.06
16	0.19	0.13	0.15	0.20	0.11

(a) Does feed rate have any effect on the standard deviation of this critical dimension?

Because the residual plots were not acceptable for the non-transformed data, a square root transformation was applied to the standard deviations of the critical dimension. Based on the computer output below, the feed rate has an effect on the standard deviation of the critical dimension.

Design Expert Output

Response: Run StDev Transform: Square root Constant: 0.000
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.040	3	0.013	7.05	0.0031	significant
A	*0.040*	*3*	*0.013*	*7.05*	*0.0031*	
Residual	0.030	16	1.903E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.030*	*16*	*1.903E-003*			
Cor Total	0.071	19				

The Model F-value of 7.05 implies the model is significant. There is only a 0.31% chance that a "Model F-Value" this large could occur due to noise.

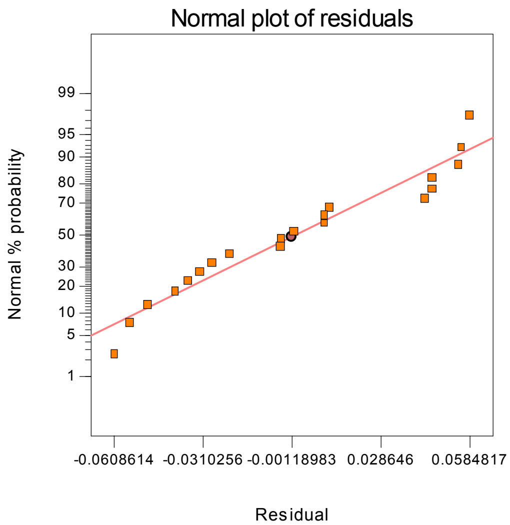
Treatment Means (Adjusted, If Necessary)

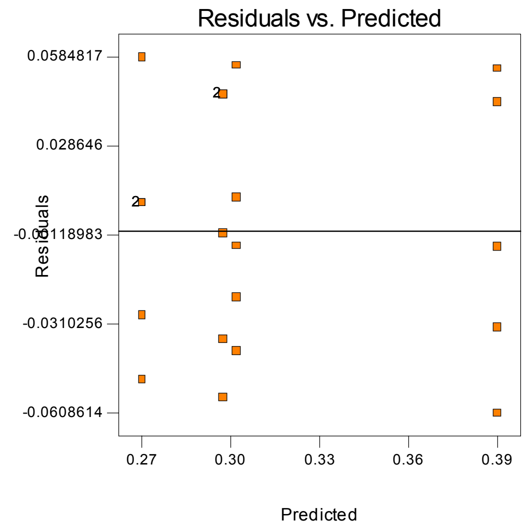
	Estimated Mean	Standard Error
1-10	0.30	0.020
2-12	0.30	0.020
3-14	0.27	0.020
4-16	0.39	0.020

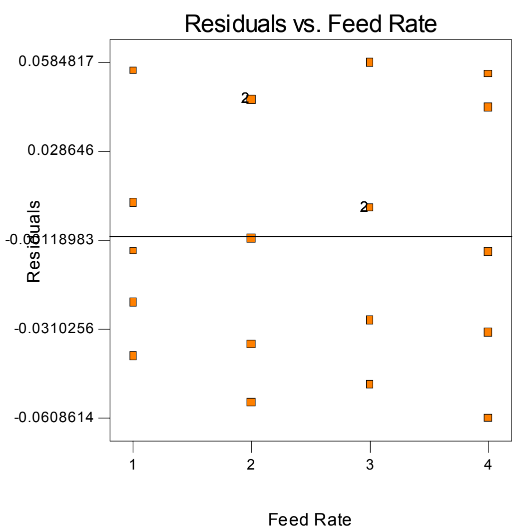
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	4.371E-003	1	0.028	0.16	0.8761
1 vs 3	0.032	1	0.028	1.15	0.2680
1 vs 4	-0.088	1	0.028	-3.18	0.0058
2 vs 3	0.027	1	0.028	0.99	0.3373
2 vs 4	-0.092	1	0.028	-3.34	0.0042
3 vs 4	-0.12	1	0.028	-4.33	0.0005

(b) Use the residuals from this experiment of investigate model adequacy. Are there any problems with experimental validity?

The residual plots are satisfactory.







3-31 Consider the data shown in Problem 3-10.

(a) Write out the least squares normal equations for this problem, and solve them for $\hat{\mu}$ and $\hat{\tau}_i$, using the usual constraint $\left(\sum_{i=1}^{3} \hat{\tau}_i = 0\right)$. Estimate $\tau_1 - \tau_2$.

$$
\begin{array}{rcccl}
15\hat{\mu} & +5\hat{\tau}_1 & +5\hat{\tau}_2 & +5\hat{\tau}_3 & =207 \\
5\hat{\mu} & +5\hat{\tau}_1 & & & =54 \\
5\hat{\mu} & & +5\hat{\tau}_2 & & =111 \\
15\hat{\mu} & & & +5\hat{\tau}_3 & =42
\end{array}
$$

Imposing $\sum_{i=1}^{3} \hat{\tau}_i = 0$, therefore $\hat{\mu} = 13.80$, $\hat{\tau}_1 = -3.00$, $\hat{\tau}_2 = 8.40$, $\hat{\tau}_3 = -5.40$

$$\hat{\tau}_1 - \hat{\tau}_2 = -3.00 - 8.40 = -11.40$$

(b) Solve the equations in (a) using the constraint $\hat{\tau}_3 = 0$. Are the estimators $\hat{\tau}_i$ and $\hat{\mu}$ the same as you found in (a)? Why? Now estimate $\tau_1 - \tau_2$ and compare your answer with that for (a). What statement can you make about estimating contrasts in the τ_i?

Imposing the constraint, $\hat{\tau}_3 = 0$ we get the following solution to the normal equations: $\hat{\mu} = 8.40$, $\hat{\tau}_1 = 2.40$, $\hat{\tau}_2 = 13.8$, and $\hat{\tau}_3 = 0$. These estimators are not the same as in part (a). However, $\hat{\tau}_1 - \hat{\tau}_2 = 2.40 - 13.80 = -11.40$, is the same as in part (a). The contrasts are estimable.

(c) Estimate $\mu + \tau_1$, $2\tau_1 - \tau_2 - \tau_3$ and $\mu + \tau_1 + \tau_2$ using the two solutions to the normal equations. Compare the results obtained in each case.

	Contrast	Estimated from Part (a)	Estimated from Part (b)
1	$\mu + \tau_1$	10.80	10.80
2	$2\tau_1 - \tau_2 - \tau_3$	-9.00	-9.00
3	$\mu + \tau_1 + \tau_2$	19.20	24.60

Contrasts 1 and 2 are estimable, 3 is not estimable.

3-32 Apply the general regression significance test to the experiment in Example 3-1. Show that the procedure yields the same results as the usual analysis of variance.

From Table 3-3:

$$y_{..} = 376$$

from Example 3-1, we have:

$$\hat{\mu} = 15.04 \quad \hat{\tau}_1 = -5.24 \quad \hat{\tau}_2 = 0.36$$
$$\hat{\tau}_3 = -2.56 \quad \hat{\tau}_4 = 6.56 \quad \hat{\tau}_5 = -4.24$$
$$\sum_{i=1}^{5}\sum_{j=1}^{5} y_{ij}^2 = 6292, \text{ with 25 degrees of freedom.}$$

$$\begin{aligned} R(\mu,\tau) &= \hat{\mu}y_{..} + \sum_{i=1}^{5}\hat{\tau}y_{i.} \\ &= (15.04)(376) + (-5.24)(49) + (0.36)(77) + (2.56)(88)) + (6.56)(108) + (-4.24)(54) \\ &= 6{,}130.80 \end{aligned}$$
with 5 degrees of freedom.

$$SS_E = \sum_{i=1}^{5}\sum_{j=1}^{5} y_{ij}^2 - R(\mu,\tau) = 6292 - 6130.8 = 161.20$$
with 25-5 degrees of freedom.

This is identical to the SS_E found in Example 3-1.

The reduced model:

$$R(\mu) = \hat{\mu} y_{..} = (15.04)(376) = 5655.04 \text{ , with 1 degree of freedom.}$$

$$R(\tau|\mu) = R(\mu,\tau) - R(\mu) = 6130.8 - 5655.04 = 475.76 \text{ , with 5-1=4 degrees of freedom.}$$

Note: $R(\tau|\mu) = SS_{Treatment}$ from Example 3-1.

Finally,

$$F_0 = \frac{\frac{R(\tau t|\mu)}{4}}{\frac{SS_E}{20}} = \frac{118.94}{8.06} = 14.76$$

which is the same as computed in Example 3-1.

3-33 Use the Kruskal-Wallis test for the experiment in Problem 3-11. Are the results comparable to those found by the usual analysis of variance?

From Design Expert Output of Problem 3-11

Response: Life in in h
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	30.17	3	10.06	3.05	0.0525	not significant
A	*30.16*	*3*	*10.05*	*3.05*	*0.0525*	
Residual	65.99	20	3.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*65.99*	*20*	*3.30*			
Cor Total	96.16	23				

$$H = \frac{12}{N(N+1)}\left[\sum_{i=1}^{a}\frac{R_{i.}^2}{n_i}\right] - 3(N+1) = \frac{12}{24(24+1)}[4040.5] - 3(24+1) = 5.81$$

$$\chi^2_{0.05,3} = 7.81$$

Accept the null hypothesis; the treatments are not different. This agrees with the analysis of variance.

3-34 Use the Kruskal-Wallis test for the experiment in Problem 3-12. Compare conclusions obtained with those from the usual analysis of variance?

From Design Expert Output of Problem 3-12

Response: Noise
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	12042.00	3	4014.00	21.78	< 0.0001	significant
A	*12042.00*	*3*	*4014.00*	*21.78*	*< 0.0001*	
Residual	2948.80	16	184.30			
Lack of Fit	*0.000*	*0*				

Pure Error	*2948.80*	*16*	*184.30*
Cor Total	14990.80	19	

$$H=\frac{12}{N(N+1)}\left[\sum_{i=1}^{a}\frac{R_{i.}^{2}}{n_i}\right]-3(N+1)=\frac{12}{20(20+1)}[2691.6]-3(20+1)=13.90$$

$$\chi^2_{0.05,4}=12.84$$

Reject the null hypothesis because the treatments are different. This agrees with the analysis of variance.

3-35 Consider the experiment in Example 3-1. Suppose that the largest observation on tensile strength is incorrectly recorded as 50. What effect does this have on the usual analysis of variance? What effect does is have on the Kruskal-Wallis test?

The incorrect observation reduces the analysis of variance F_0 from 14.76 to 5.44. It does not change the value of the Kruskal-Wallis test.

Chapter 4
Randomized Blocks, Latin Squares, and Related Designs
Solutions

4-1 A chemist wishes to test the effect of four chemical agents on the strength of a particular type of cloth. Because there might be variability from one bolt to another, the chemist decides to use a randomized block design, with the bolts of cloth considered as blocks. She selects five bolts and applies all four chemicals in random order to each bolt. The resulting tensile strengths follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw appropriate conclusions.

	Bolt				
Chemical	1	2	3	4	5
1	73	68	74	71	67
2	73	67	75	72	70
3	75	68	78	73	68
4	73	71	75	75	69

Design Expert Output

Response: Strength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	157.00	4	39.25			
Model	12.95	3	4.32	2.38	0.1211	not significant
A	*12.95*	*3*	*4.32*	*2.38*	*0.1211*	
Residual	21.80	12	1.82			
Cor Total	191.75	19				

The "Model F-value" of 2.38 implies the model is not significant relative to the noise. There is a 12.11 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	1.35	R-Squared	0.3727
Mean	71.75	Adj R-Squared	0.2158
C.V.	1.88	Pred R-Squared	-0.7426
PRESS	60.56	Adeq Precision	10.558

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	70.60	0.60
2-2	71.40	0.60
3-3	72.40	0.60
4-4	72.60	0.60

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-0.80	1	0.85	-0.94	0.3665
1 vs 3	-1.80	1	0.85	-2.11	0.0564
1 vs 4	-2.00	1	0.85	-2.35	0.0370
2 vs 3	-1.00	1	0.85	-1.17	0.2635
2 vs 4	-1.20	1	0.85	-1.41	0.1846
3 vs 4	-0.20	1	0.85	-0.23	0.8185

There is no difference among the chemical types at $\alpha = 0.05$ level.

4-2 Three different washing solutions are being compared to study their effectiveness in retarding bacteria growth in five-gallon milk containers. The analysis is done in a laboratory, and only three trials can be run on any day. Because days could represent a potential source of variability, the experimenter decides to use a randomized block design. Observations are taken for four days, and the data are shown here. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Days			
Solution	1	2	3	4
1	13	22	18	39
2	16	24	17	44
3	5	4	1	22

Design Expert Output

Response: Growth

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	1106.92	3	368.97			
Model	703.50	2	351.75	40.72	0.0003	significant
A	*703.50*	*2*	*351.75*	*40.72*	*0.0003*	
Residual	51.83	6	8.64			
Cor Total	1862.25	11				

The Model F-value of 40.72 implies the model is significant. There is only a 0.03% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.94	R-Squared	0.9314
Mean	18.75	Adj R-Squared	0.9085
C.V.	15.68	Pred R-Squared	0.7255
PRESS	207.33	Adeq Precision	19.687

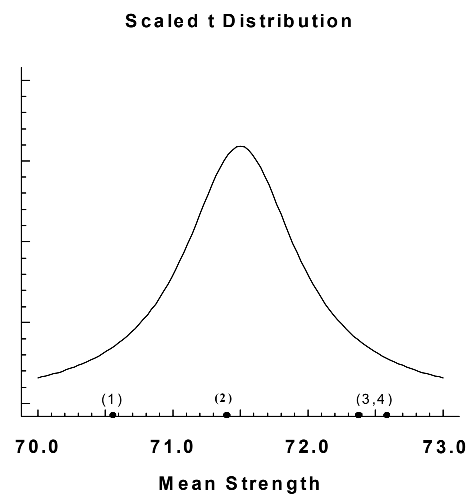
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	23.00	1.47
2-2	25.25	1.47
3-3	8.00	1.47

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	-2.25	1	2.08	-1.08	0.3206
1 vs 3	15.00	1	2.08	7.22	0.0004
2 vs 3	17.25	1	2.08	8.30	0.0002

There is a difference between the means of the three solutions. The Fisher LSD procedure indicates that solution 3 is significantly different than the other two.

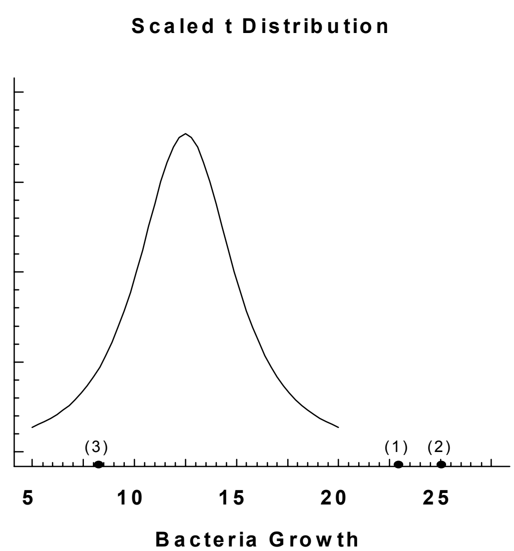
4-3 Plot the mean tensile strengths observed for each chemical type in Problem 4-1 and compare them to a scaled *t* distribution. What conclusions would you draw from the display?



$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{1.82}{5}} = 0.603$$

There is no obvious difference between the means. This is the same conclusion given by the analysis of variance.

4-4 Plot the average bacteria counts for each solution in Problem 4-2 and compare them to an appropriately scaled *t* distribution. What conclusions can you draw?



$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{8.64}{4}} = 1.47$$

There is no difference in mean bacteria growth between solutions 1 and 2. However, solution 3 produces significantly lower mean bacteria growth. This is the same conclusion reached from the Fisher LSD procedure in Problem 4-4.

4-5 An article in the *Fire Safety Journal* ("The Effect of Nozzle Design on the Stability and Performance of Turbulent Water Jets," Vol. 4, August 1981) describes an experiment in which a shape factor was determined for several different nozzle designs at six levels of efflux velocity. Interest focused on potential differences between nozzle designs, with velocity considered as a nuisance variable. The data are shown below:

Nozzle	Jet Efflux Velocity (m/s)					
Design	11.73	14.37	16.59	20.43	23.46	28.74
1	0.78	0.80	0.81	0.75	0.77	0.78
2	0.85	0.85	0.92	0.86	0.81	0.83
3	0.93	0.92	0.95	0.89	0.89	0.83
4	1.14	0.97	0.98	0.88	0.86	0.83
5	0.97	0.86	0.78	0.76	0.76	0.75

(a) Does nozzle design affect the shape factor? Compare nozzles with a scatter plot and with an analysis of variance, using $\alpha = 0.05$.

Design Expert Output

Response: Shape
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	0.063	5	0.013			
Model	0.10	4	0.026	8.92	0.0003	significant
A	*0.10*	*4*	*0.026*	*8.92*	*0.0003*	
Residual	0.057	20	2.865E-003			
Cor Total	0.22	29				

The Model F-value of 8.92 implies the model is significant. There is only a 0.03% chance that a "Model F-Value" this large could occur due to noise.

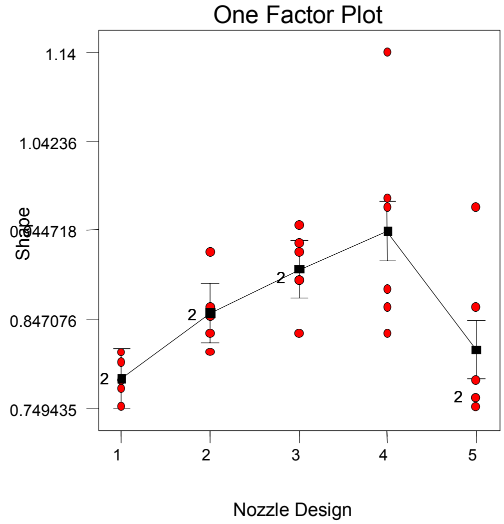
Std. Dev.	0.054	R-Squared	0.6407
Mean	0.86	Adj R-Squared	0.5688
C.V.	6.23	Pred R-Squared	0.1916
PRESS	0.13	Adeq Precision	9.438

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	0.78	0.022
2-2	0.85	0.022
3-3	0.90	0.022
4-4	0.94	0.022
5-5	0.81	0.022

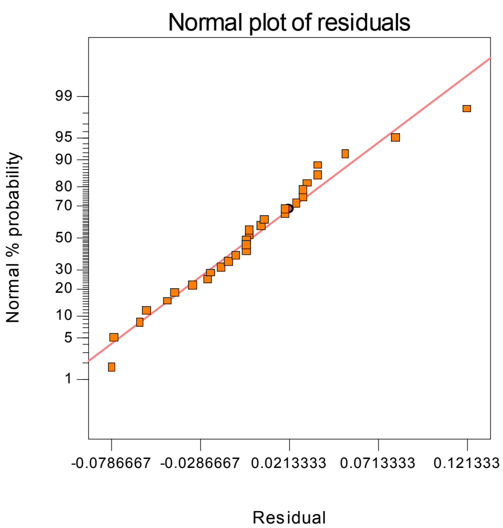
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-0.072	1	0.031	-2.32	0.0311
1 vs 3	-0.12	1	0.031	-3.88	0.0009
1 vs 4	-0.16	1	0.031	-5.23	< 0.0001
1 vs 5	-0.032	1	0.031	-1.02	0.3177
2 vs 3	-0.048	1	0.031	-1.56	0.1335
2 vs 4	-0.090	1	0.031	-2.91	0.0086
2 vs 5	0.040	1	0.031	1.29	0.2103
3 vs 4	-0.042	1	0.031	-1.35	0.1926
3 vs 5	0.088	1	0.031	2.86	0.0097
4 vs 5	0.13	1	0.031	4.21	0.0004

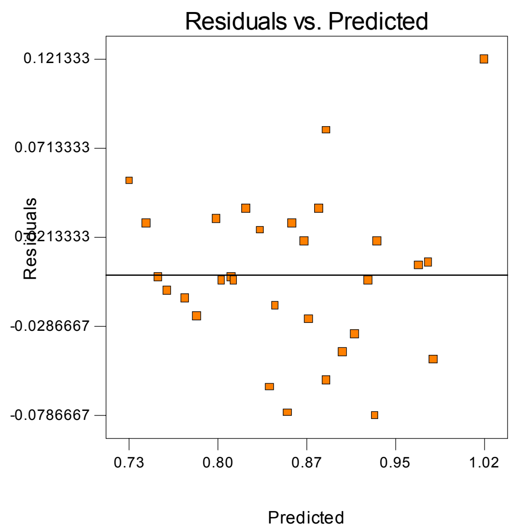
Nozzle design has a significant effect on shape factor.

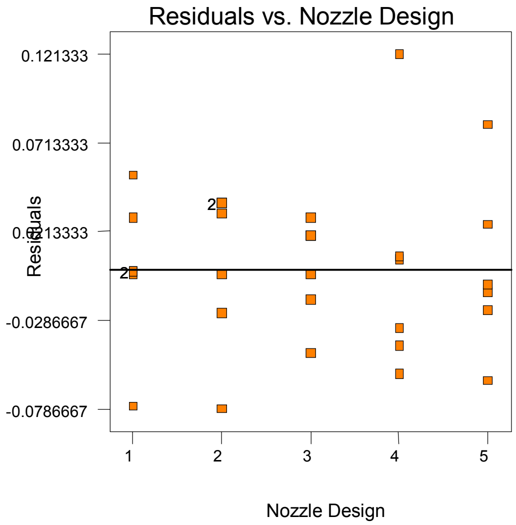


(b) Analyze the residual from this experiment.

The plots shown below do not give any indication of serious problems. Thre is some indication of a mild outlier on the normal probability plot and on the plot of residualks versus the predicted velocity.





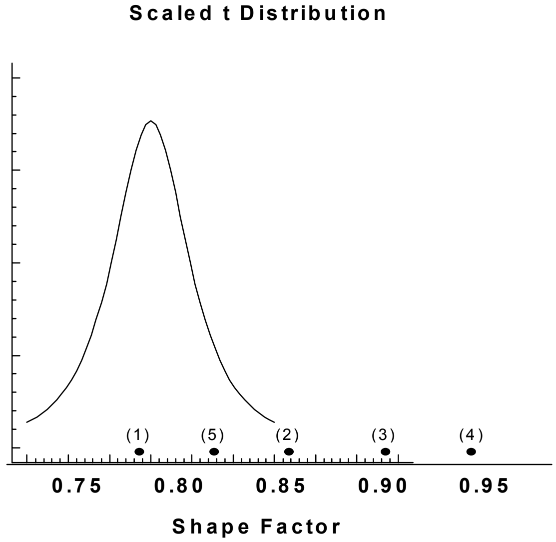


(c) Which nozzle designs are different with respect to shape factor? Draw a graph of average shape factor for each nozzle type and compare this to a scaled *t* distribution. Compare the conclusions that you draw from this plot to those from Duncan's multiple range test.

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{0.002865}{6}} = 0.021852$$

$R_2=$	$r_{0.05}(2,20)\ S_{\bar{y}_{i.}} =$	(2.95)(0.021852)=	0.06446
$R_3=$	$r_{0.05}(3,20)\ S_{\bar{y}_{i.}} =$	(3.10)(0.021852)=	0.06774
$R_4=$	$r_{0.05}(4,20)\ S_{\bar{y}_{i.}} =$	(3.18)(0.021852)=	0.06949
$R_5=$	$r_{0.05}(5,20)\ S_{\bar{y}_{i.}} =$	(3.25)(0.021852)=	0.07102

	Mean Difference		R	
1 vs 4	0.16167	>	0.07102	different
1 vs 3	0.12000	>	0.06949	different
1 vs 2	0.07167	>	0.06774	different
1 vs 5	0.03167	<	0.06446	
5 vs 4	0.13000	>	0.06949	different
5 vs 3	0.08833	>	0.06774	different
5 vs 2	0.04000	<	0.06446	
2 vs 4	0.09000	>	0.06774	different
2 vs 3	0.04833	<	0.06446	
3 vs 4	0.04167	<	0.06446	



4-6 Consider the ratio control algorithm experiment described in Chapter 3, Section 3-8. The experiment was actually conducted as a randomized block design, where six time periods were selected as the blocks, and all four ratio control algorithms were tested in each time period. The average cell voltage and the standard deviation of voltage (shown in parentheses) for each cell as follows:

Ratio Control	Time Period					
Algorithms	1	2	3	4	5	6
1	4.93 (0.05)	4.86 (0.04)	4.75 (0.05)	4.95 (0.06)	4.79 (0.03)	4.88 (0.05)
2	4.85 (0.04)	4.91 (0.02)	4.79 (0.03)	4.85 (0.05)	4.75 (0.03)	4.85 (0.02)
3	4.83 (0.09)	4.88 (0.13)	4.90 (0.11)	4.75 (0.15)	4.82 (0.08)	4.90 (0.12)
4	4.89 (0.03)	4.77 (0.04)	4.94 (0.05)	4.86 (0.05)	4.79 (0.03)	4.76 (0.02)

(a) Analyze the average cell voltage data. (Use α = 0.05.) Does the choice of ratio control algorithm affect the cell voltage?

Design Expert Output

Response: Average
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.017	5	3.487E-003			
Model	2.746E-003	3	9.153E-004	0.19	0.9014	not significant
A	*2.746E-003*	*3*	*9.153E-004*	*0.19*	*0.9014*	
Residual	0.072	15	4.812E-003			
Cor Total	0.092	23				

The "Model F-value" of 0.19 implies the model is not significant relative to the noise. There is a 90.14 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	0.069	R-Squared	0.0366
Mean	4.84	Adj R-Squared	-0.1560
C.V.	1.43	Pred R-Squared	-1.4662
PRESS	0.18	Adeq Precision	2.688

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	4.86	0.028

2-2	4.83	0.028
3-3	4.85	0.028
4-4	4.84	0.028

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.027	1	0.040	0.67	0.5156
1 vs 3	0.013	1	0.040	0.33	0.7438
1 vs 4	0.025	1	0.040	0.62	0.5419
2 vs 3	-0.013	1	0.040	-0.33	0.7438
2 vs 4	-1.667E-003	1	0.040	-0.042	0.9674
3 vs 4	0.012	1	0.040	0.29	0.7748

The ratio control algorithm does not affect the mean cell voltage.

(b) Perform an appropriate analysis of the standard deviation of voltage. (Recall that this is called "pot noise.") Does the choice of ratio control algorithm affect the pot noise?

Design Expert Output

Response: StDev Transform: Natural log Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	0.94	5	0.19			
Model	6.17	3	2.06	33.26	< 0.0001	significant
A	*6.17*	*3*	*2.06*	*33.26*	*< 0.0001*	
Residual	0.93	15	0.062			
Cor Total	8.04	23				

The Model F-value of 33.26 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.25	R-Squared	0.8693
Mean	-3.04	Adj R-Squared	0.8432
C.V.	-8.18	Pred R-Squared	0.6654
PRESS	2.37	Adeq Precision	12.446

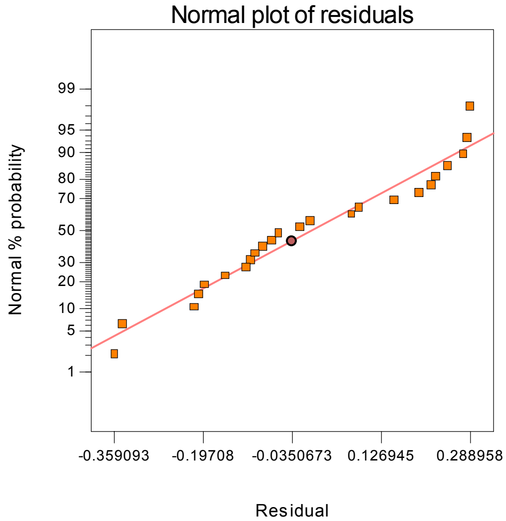
Treatment Means (Adjusted, If Necessary)

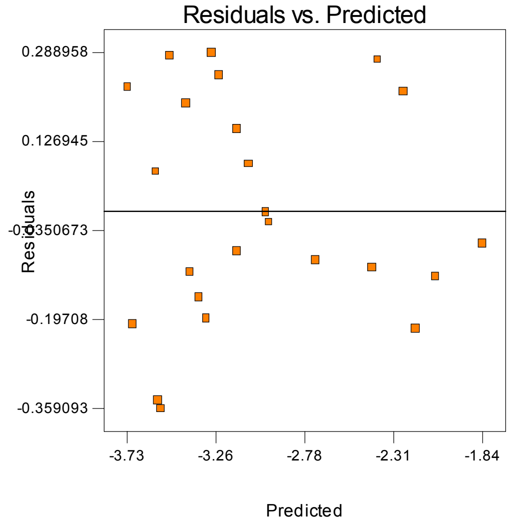
	Estimated Mean	Standard Error
1-1	-3.09	0.10
2-2	-3.51	0.10
3-3	-2.20	0.10
4-4	-3.36	0.10

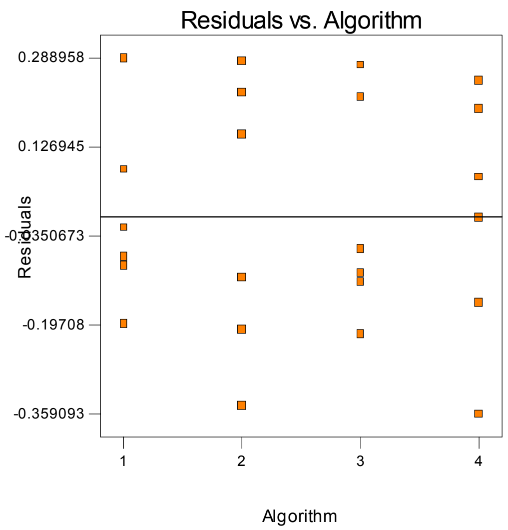
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.42	1	0.14	2.93	0.0103
1 vs 3	-0.89	1	0.14	-6.19	< 0.0001
1 vs 4	0.27	1	0.14	1.87	0.0813
2 vs 3	-1.31	1	0.14	-9.12	< 0.0001
2 vs 4	-0.15	1	0.14	-1.06	0.3042
3 vs 4	1.16	1	0.14	8.06	< 0.0001

A natural log transformatio was applied to the pot noise data. The ratio control algorithm does affect the pot noise.

(c) Conduct any residual analyses that seem appropriate.







The normal probability plot shows slight deviations from normality; however, still acceptable.

(d) Which ratio control algorithm would you select if your objective is to reduce both the average cell voltage and the pot noise?

Since the ratio control algorithm has little effect on average cell voltage, select the algorithm that minimizes pot noise, that is algorithm #2.

4-7 An aluminum master alloy manufacturer produces grain refiners in ingot form. This company produces the product in four furnaces. Each furnace is known to have its own unique operating characteristics, so any experiment run in the foundry that involves more than one furnace will consider furnace a nuisance variable. The process engineers suspect that stirring rate impacts the grain size of the product. Each furnace can be run at four different stirring rates. A randomized block design is run for a particular refiner and the resulting grain size data is shown below.

Furnace

Stirring Rate	1	2	3	4
5	8	4	5	6
10	14	5	6	9
15	14	6	9	2
20	17	9	3	6

(a) Is there any evidence that stirring rate impacts grain size?

Design Expert Output

Response: Grain Size
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	165.19	3	55.06			
Model	22.19	3	7.40	0.85	0.4995	not significant
A	*22.19*	*3*	*7.40*	*0.85*	*0.4995*	
Residual	78.06	9	8.67			
Cor Total	265.44	15				

The "Model F-value" of 0.85 implies the model is not significant relative to the noise. There is a 49.95 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	2.95	R-Squared	0.2213
Mean	7.69	Adj R-Squared	-0.0382
C.V.	38.31	Pred R-Squared	-1.4610
PRESS	246.72	Adeq Precision	5.390

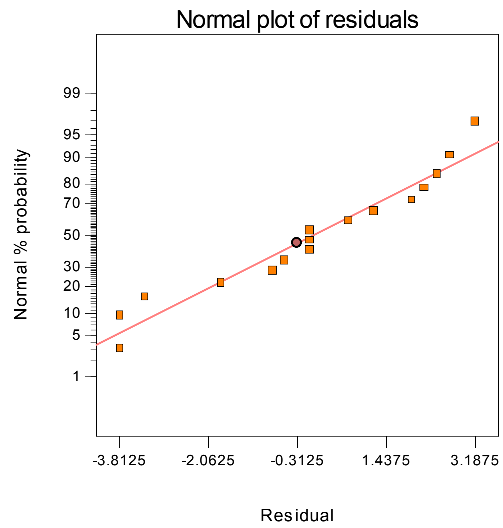
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-5	5.75	1.47
2-10	8.50	1.47
3-15	7.75	1.47
4-20	8.75	1.47

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-2.75	1	2.08	-1.32	0.2193
1 vs 3	-2.00	1	2.08	-0.96	0.3620
1 vs 4	-3.00	1	2.08	-1.44	0.1836
2 vs 3	0.75	1	2.08	0.36	0.7270
2 vs 4	-0.25	1	2.08	-0.12	0.9071
3 vs 4	-1.00	1	2.08	-0.48	0.6425

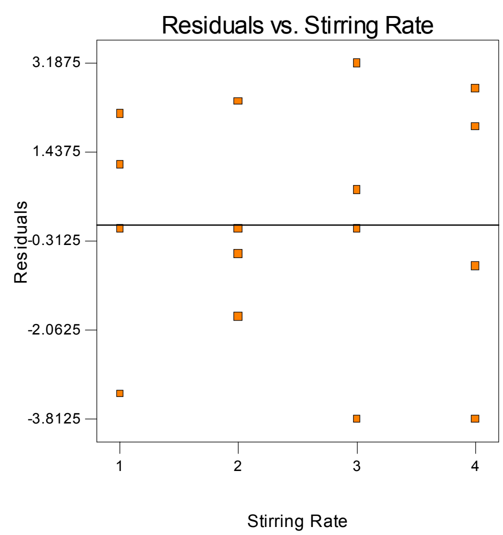
The analysis of variance shown above indicates that there is no difference in mean grain size due to the different stirring rates.

(b) Graph the residuals from this experiment on a normal probability plot. Interpret this plot.



The plot indicates that normality assumption is valid.

(c) Plot the residuals versus furnace and stirring rate. Does this plot convey any useful information?



The variance is consistent at different stirring rates. Not only does this validate the assumption of uniform variance, it also identifies that the different stirring rates do not affect variance.

(d) What should the process engineers recommend concerning the choice of stirring rate and furnace for this particular grain refiner if small grain size is desirable?

There really isn't any effect due to the stirring rate.

4-8 Analyze the data in Problem 4-2 using the general regression significance test.

$$
\begin{array}{llllllllll}
\mu: & 12\hat{\mu} & +4\hat{\tau}_1 & +4\hat{\tau}_2 & +4\hat{\tau}_3 & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =225 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =92 \\
\tau_2: & 4\hat{\mu} & & +4\hat{\tau}_2 & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =101
\end{array}
$$

$$
\begin{array}{llllllllll}
\tau_3: & 4\hat{\mu} & & & +4\hat{\tau}_3 & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =32 \\
\beta_1: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +3\hat{\beta}_1 & & & & =34 \\
\beta_2: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & & +3\hat{\beta}_2 & & & =50 \\
\beta_3: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & & & +3\hat{\beta}_3 & & =36 \\
\beta_4: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & & & & +3\hat{\beta}_4 & =105
\end{array}
$$

Applying the constraints $\sum \hat{\tau}_i = \sum \hat{\beta}_j = 0$, we obtain:

$$\hat{\mu} = \frac{225}{12},\ \hat{\tau}_1 = \frac{51}{12},\ \hat{\tau}_2 = \frac{78}{12},\ \hat{\tau}_3 = \frac{-129}{12},\ \hat{\beta}_1 = \frac{-89}{12},\ \hat{\beta}_2 = \frac{-25}{12},\ \hat{\beta}_3 = \frac{-81}{12},\ \hat{\beta}_4 = \frac{195}{12}$$

$$R(\mu,\tau,\beta) = \left(\frac{225}{12}\right)(225) + \left(\frac{51}{12}\right)(92) + \left(\frac{78}{12}\right)(101) + \left(\frac{-129}{12}\right)(32) + \left(\frac{-89}{12}\right)(34) + \left(\frac{-25}{12}\right)(50) +$$

$$\left(\frac{-81}{12}\right)(36) + \left(\frac{195}{12}\right)(105) = 6029.17$$

$$\sum\sum y_{ij}^2 = 6081,\ SS_E = \sum\sum y_{ij}^2 - R(\mu,\tau,\beta) = 6081 - 6029.17 = 51.83$$

Model Restricted to $\tau_i = 0$:

$$
\begin{array}{lllllll}
\mu: & 12\hat{\mu} & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =225 \\
\beta_1: & 3\hat{\mu} & +3\hat{\beta}_1 & & & & =34 \\
\beta_2: & 3\hat{\mu} & & +3\hat{\beta}_2 & & & =50 \\
\beta_3: & 3\hat{\mu} & & & +3\hat{\beta}_3 & & =36 \\
\beta_4: & 3\hat{\mu} & & & & +3\hat{\beta}_4 & =105
\end{array}
$$

Applying the constraint $\sum \hat{\beta}_j = 0$, we obtain:

$$\hat{\mu} = \frac{225}{12},\ \hat{\beta}_1 = -89/12,\ \hat{\beta}_2 = \frac{-25}{12},\ \hat{\beta}_3 = \frac{-81}{12},\ \hat{\beta}_4 = \frac{195}{12}.\ \text{Now:}$$

$$R(\mu,\beta) = \left(\frac{225}{12}\right)(225) + \left(\frac{-89}{12}\right)(34) + \left(\frac{-25}{12}\right)(50) + \left(\frac{-81}{12}\right)(36) + \left(\frac{195}{12}\right)(105) = 5325.67$$

$$R(\tau|\mu,\beta) = R(\mu,\tau,\beta) - R(\mu,\beta) = 6029.17 - 5325.67 = 703.50 = SS_{Treatments}$$

Model Restricted to $\beta_j = 0$:

$$
\begin{array}{llllll}
\mu: & 12\hat{\mu} & +4\hat{\tau}_1 & +4\hat{\tau}_2 & +4\hat{\tau}_3 & =225 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & =92 \\
\tau_2: & 4\hat{\mu} & & +4\hat{\tau}_2 & & =101 \\
\tau_3: & 4\hat{\mu} & & & +4\hat{\tau}_3 & =32
\end{array}
$$

Applying the constraint $\sum \hat{\tau}_i = 0$, we obtain:

$$\hat{\mu} = \frac{225}{12},\ \hat{\tau}_1 = \frac{51}{12},\ \hat{\tau}_2 = \frac{78}{12},\ \hat{\tau}_3 = \frac{-129}{12}$$

$$R(\mu,\tau)=\left(\frac{225}{12}\right)(225)+\left(\frac{51}{12}\right)(92)+\left(\frac{78}{12}\right)(101)+\left(\frac{-129}{12}\right)(32)=4922.25$$
$$R(\beta|\mu,\tau)=R(\mu,\tau,\beta)-R(\mu,\tau)=6029.17-4922.25=1106.92=SS_{Blocks}$$

4-9 Assuming that chemical types and bolts are fixed, estimate the model parameters τ_i and β_j in Problem 4-1.

Using Equations 4-14, Applying the constraints, we obtain:

$$\hat{\mu}=\frac{35}{20},\ \hat{\tau}_1=\frac{-23}{20},\ \hat{\tau}_2=\frac{-7}{20},\ \hat{\tau}_3=\frac{13}{20},\hat{\tau}_4=\frac{17}{20},\ \hat{\beta}_1=\frac{35}{20},\ \hat{\beta}_2=\frac{-65}{20},\ \hat{\beta}_3=\frac{75}{20},\ \hat{\beta}_4=\frac{20}{20},\hat{\beta}_5=\frac{-65}{20}$$

4-10 Draw an operating characteristic curve for the design in Problem 4-2. Does this test seem to be sensitive to small differences in treatment effects?

Assuming that solution type is a fixed factor, we use the OC curve in appendix V. Calculate

$$\Phi^2=\frac{b\sum\tau_i^2}{a\sigma^2}=\frac{4\sum\tau_i^2}{3(8.69)}$$

using MS_E to estimate σ^2. We have:

$$\upsilon_1=a-1=2 \qquad \upsilon_2=(a-1)(b-1)=(2)(3)=6\ .$$

If $\sum\hat{\tau}^2_{\ i}=\sigma^2=MS_E$, then:

$$\Phi=\sqrt{\frac{4}{3(1)}}=1.15 \text{ and } \beta\cong 0.70$$

If $\sum\hat{\tau}_i=2\sigma^2=2MS_E$, then:

$$\Phi=\sqrt{\frac{4}{3(2)}}=1.63 \text{ and } \beta\cong 0.55\text{ , etc.}$$

This test is not very sensitive to small differences.

4-11 Suppose that the observation for chemical type 2 and bolt 3 is missing in Problem 4-1. Analyze the problem by estimating the missing value. Perform the exact analysis and compare the results.

$$y_{23}\text{ is missing. }\quad \hat{y}_{23}=\frac{ay_{2.}'+by_{.3}'-y_{..}'}{(a-1)(b-1)}=\frac{4(282)+5(227)-1360}{(4)(3)}=75.25$$

Thus, $y_{2.}$=357.25, $y_{.3}$=3022.25, and $y_{..}$=1435.25

Source	SS	DF	MS	F_0
Chemicals	12.7844	3	4.2615	2.154

Bolts	158.8875	4	
Error	21.7625	11	1.9784
Total	193.4344	18	

$F_{0.10,3,11}$=2.66, Chemicals are not significant.

4-12 ***Two missing values in a randomized block.*** Suppose that in Problem 4-1 the observations for chemical type 2 and bolt 3 and chemical type 4 and bolt 4 are missing.

(a) Analyze the design by iteratively estimating the missing values as described in Section 4-1.3.

$$\hat{y}_{23} = \frac{4y'_{2.} + 5y'_{.3} - y'_{..}}{12} \text{ and } \hat{y}_{44} = \frac{4y'_{4.} + 5y'_{.4} - y'_{..}}{12}$$

Data is coded y-70. As an initial guess, set y^0_{23} equal to the average of the observations available for chemical 2. Thus, $y^0_{23} = \frac{2}{4} = 0.5$. Then ,

$$\hat{y}^0_{44} = \frac{4(8)+5(6)-25.5}{12} = 3.04$$
$$\hat{y}^1_{23} = \frac{4(2)+5(17)-28.04}{12} = 5.41$$
$$\hat{y}^1_{44} = \frac{4(8)+5(6)-30.41}{12} = 2.63$$
$$\hat{y}^2_{44} = \frac{4(2)+5(17)-27.63}{12} = 5.44$$
$$\hat{y}^2_{44} = \frac{4(8)+5(6)-30.44}{12} = 2.63$$
$$\therefore \hat{y}_{23} = 5.44 \quad \hat{y}_{44} = 2.63$$

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	156.83	4	39.21			
Model	9.59	3	3.20	2.08	0.1560	not significant
A	*9.59*	*3*	*3.20*	*2.08*	*0.1560*	
Residual	18.41	12	1.53			
Cor Total	184.83	19				

(b) Differentiate SS_E with respect to the two missing values, equate the results to zero, and solve for estimates of the missing values. Analyze the design using these two estimates of the missing values.

$$SS_E = \sum\sum y_{ij}^2 - \tfrac{1}{5}\sum y_{i.}^2 - \tfrac{1}{4}\sum y_{.j}^2 + \tfrac{1}{20}\sum y_{..}^2$$
$$SS_E = 0.6y_{23}^2 + 0.6y_{44}^2 - 6.8y_{23} - 3.7y_{44} + 0.1y_{23}y_{44} + R$$

From $\frac{\partial SS_E}{\partial y_{23}} = \frac{\partial SS_E}{\partial y_{44}} = 0$, we obtain:

$$\begin{matrix} 1.2\hat{y}_{23} + 0.1\hat{y}_{44} = 6.8 \\ 0.1\hat{y}_{23} + 1.2\hat{y}_{44} = 3.7 \end{matrix} \quad \Rightarrow \hat{y}_{23} = 5.45\,,\ \hat{y}_{44} = 2.63$$

These quantities are almost identical to those found in part (a). The analysis of variance using these new data does not differ substantially from part (a).

(c) Derive general formulas for estimating two missing values when the observations are in *different* blocks.

$$SS_E = y_{iu}^2 + y_{kv}^2 - \frac{\left(y'_{i.} + y'_{iu}\right)^2 + \left(y'_{k.} + y_{kv}\right)^2}{b} - \frac{\left(y'_{.u} + y_{iu}\right)^2 + \left(y'_{.v} + y_{kv}\right)^2}{a} + \frac{\left(y'_{..} + y_{iu} + y\right)_{kv}^{\ 2}}{ab}$$

From $\dfrac{\partial SS_E}{\partial y_{23}} = \dfrac{\partial SS_E}{\partial y_{44}} = 0$, we obtain:

$$\hat{y}_{iu}\left[\frac{(a-1)(b-1)}{ab}\right] = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{ab} - \frac{\hat{y}_{kv}}{ab}$$

$$\hat{y}_{kv}\left[\frac{(a-1)(b-1)}{ab}\right] = \frac{ay'_{k.} + by'_{.v} - y'_{..}}{ab} - \frac{\hat{y}_{iu}}{ab}$$

whose simultaneous solution is:

$$\hat{y}_{iu} = \frac{y'_{i.}\, a\left[1-(a-1)^2(b-1)^2 - ab\right] + y'_{.u}\, b\left[1-(a-1)^2(b-1)^2 - ab\right] - y'_{..}\left[1 - ab(a-1)^2(b-1)^2\right]}{(a-1)(b-1)\left[1-(a-1)^2(b-1)^2\right]} + \frac{ab\left[ay'_{k.} + by'_{.v} - y'_{..}\right]}{\left[1-(a-1)^2(b-1)^2\right]}$$

$$\hat{y}_{kv} = \frac{ay'_{i.} + by'_{.u} - y'_{..} - (b-1)(a-1)\left[ay'_{k.} + by'_{.v} - y'_{..}\right]}{\left[1-(a-1)^2(b-1)^2\right]}$$

(d) Derive general formulas for estimating two missing values when the observations are in the *same* block. Suppose that two observations y_{ij} and y_{kj} are missing, $i \neq k$ (same block j).

$$SS_E = y_{ij}^2 + y_{kj}^2 - \frac{\left(y'_{i.} + y_{ij}\right)^2 + \left(y'_{k.} + y_{kj}\right)^2}{b} - \frac{\left(y'_{.j} + y_{ij} + y_{kj}\right)^2}{a} + \frac{\left(y'_{..} + y_{ij} + y_{kj}\right)^2}{ab}$$

From $\dfrac{\partial SS_E}{\partial y_{23}} = \dfrac{\partial SS_E}{\partial y_{44}} = 0$, we obtain

$$\hat{y}_{ij} = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} + \hat{y}_{kj}(a-1)(b-1)^2$$

$$\hat{y}_{kj} = \frac{ay'_{k.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} + \hat{y}_{ij}(a-1)(b-1)^2$$

whose simultaneous solution is:

$$\hat{y}_{ij} = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} + \frac{(b-1)\left[ay'_{k.} + by'_{.j} - y'_{..} + (a-1)(b-1)^2\left(ay'_{i.} + by'_{.j} - y'_{..}\right)\right]}{\left[1-(a-1)^2(b-1)\right]^2}$$

$$\hat{y}_{kj} = \frac{ay'_{k.} + by'_{.j} - y'_{..} - (b-1)^2(a-1)\left[ay'_{i.} + by'_{.j} - y'_{..}\right]}{(a-1)(b-1)\left[1-(a-1)^2(b-1)^4\right]}$$

4-13 An industrial engineer is conducting an experiment on eye focus time. He is interested in the effect of the distance of the object from the eye on the focus time. Four different distances are of interest. He has five subjects available for the experiment. Because there may be differences among individuals, he decides to conduct the experiment in a randomized block design. The data obtained follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw appropriate conclusions.

	Subject				
Distance (ft)	1	2	3	4	5
4	10	6	6	6	6
6	7	6	6	1	6
8	5	3	3	2	5
10	6	4	4	2	3

Design Expert Output

Response: Focus Time
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	36.30	4	9.07			
Model	32.95	3	10.98	8.61	0.0025	significant
A	*32.95*	*3*	*10.98*	*8.61*	*0.0025*	
Residual	15.30	12	1.27			
Cor Total	84.55	19				

The Model F-value of 8.61 implies the model is significant. There is only a 0.25% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.13	R-Squared	0.6829
Mean	4.85	Adj R-Squared	0.6036
C.V.	23.28	Pred R-Squared	0.1192
PRESS	42.50	Adeq Precision	10.432

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-4	6.80	0.50
2-6	5.20	0.50
3-8	3.60	0.50
4-10	3.80	0.50

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	1.60	1	0.71	2.24	0.0448
1 vs 3	3.20	1	0.71	4.48	0.0008
1 vs 4	3.00	1	0.71	4.20	0.0012
2 vs 3	1.60	1	0.71	2.24	0.0448
2 vs 4	1.40	1	0.71	1.96	0.0736
3 vs 4	-0.20	1	0.71	-0.28	0.7842

Distance has a statistically significant effect on mean focus time.

4-14 The effect of five different ingredients (*A, B, C, D, E*) on reaction time of a chemical process is being studied. Each batch of new material is only large enough to permit five runs to be made. Furthermore, each runs requires approximately 1 1/2 hours, so only five runs can be made in one day. The experimenter decides to run the experiment as a Latin square so that day and batch effects can be systematically controlled. She obtains the data that follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Day				
Batch	1	2	3	4	5
1	*A*=8	*B*=7	*D*=1	*C*=7	*E*=3
2	*C*=11	*E*=2	*A*=7	*D*=3	*B*=8
3	*B*=4	*A*=9	*C*=10	*E*=1	*D*=5
4	*D*=6	*C*=8	*E*=6	*B*=6	*A*=10
5	*E*=4	*D*=2	*B*=3	*A*=8	*C*=8

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Batch    random      5 1 2 3 4 5
Day      random      5 1 2 3 4 5
Catalyst  fixed      5 A B C D E

Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Catalyst    4    141.440    141.440     35.360   11.31  0.000
Batch       4     15.440     15.440      3.860    1.23  0.348
Day         4     12.240     12.240      3.060    0.98  0.455
Error      12     37.520     37.520      3.127
Total      24    206.640
```

4-15 An industrial engineer is investigating the effect of four assembly methods (*A, B, C, D*) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is, a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin square design shown below. Analyze the data from this experiment ($\alpha = 0.05$) draw appropriate conclusions.

Order of	Operator			
Assembly	1	2	3	4
1	*C*=10	*D*=14	*A*=7	*B*=8
2	*B*=7	*C*=18	*D*=11	*A*=8
3	*A*=5	*B*=10	*C*=11	*D*=9
4	*D*=10	*A*=10	*B*=12	*C*=14

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Order    random      4 1 2 3 4
Operator random      4 1 2 3 4
Method    fixed      4 A B C D


Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
```

Method	3	72.500	72.500	24.167	13.81	0.004
Order	3	18.500	18.500	6.167	3.52	0.089
Operator	3	51.500	51.500	17.167	9.81	0.010
Error	6	10.500	10.500	1.750		
Total	15	153.000				

4-16 Suppose that in Problem 4-14 the observation from batch 3 on day 4 is missing. Estimate the missing value from Equation 4-24, and perform the analysis using this value.

y_{354} is missing. $\hat{y}_{354} = \dfrac{p\left[y'_{i..} + y'_{.j.} + y'_{..k}\right] - 2y'_{...}}{(p-2)(p-1)} = \dfrac{5[28+15+24]-2(146)}{(3)(4)} = 3.58$

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Batch    random      5 1 2 3 4 5
Day      random      5 1 2 3 4 5
Catalyst  fixed      5 A B C D E

Analysis of Variance for Time, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS       F      P
Catalyst     4    128.676    128.676     32.169   11.25  0.000
Batch        4     16.092     16.092      4.023    1.41  0.290
Day          4      8.764      8.764      2.191    0.77  0.567
Error       12     34.317     34.317      2.860
Total       24    187.849
```

4-17 Consider a *p x p* Latin square with rows (α_i), columns (β_k), and treatments (τ_j) fixed. Obtain least squares estimates of the model parameters α_i, β_k, τ_j.

$$\mu : p^2\hat{\mu} + p\sum_{i=1}^{p}\hat{\alpha}_i + p\sum_{j=1}^{p}\hat{\tau}_j + p\sum_{k=1}^{p}\hat{\beta}_k = y_{...}$$

$$\alpha_i : p\hat{\mu} + p\hat{\alpha}_i + p\sum_{j=1}^{p}\hat{\tau}_j + p\sum_{k=1}^{p}\hat{\beta}_k = y_{i..} \quad , \ i = 1,2,\ldots,p$$

$$\tau_j : p\hat{\mu} + p\sum_{i=1}^{p}\hat{\alpha}_i + p\hat{\tau}_j + p\sum_{k=1}^{p}\hat{\beta}_k = y_{.j.} , \ j = 1,2,\ldots,p$$

$$\beta_k : p\hat{\mu} + p\sum_{i=1}^{p}\hat{\alpha}_i + p\sum_{j=1}^{p}\hat{\tau}_j + p\hat{\beta}_k = y_{..k} , \ k = 1,2,\ldots,p$$

There are 3*p*+1 equations in 3*p*+1 unknowns. The rank of the system is 3*p*-2. Three side conditions are necessary. The usual conditions imposed are: $\sum_{i=1}^{p}\hat{\alpha}_i = \sum_{j=1}^{p}\hat{\tau}_j = \sum_{k=1}^{p}\hat{\beta}_k = 0$. The solution is then:

$$\hat{\mu} = \frac{y_{...}}{p^2} = \bar{y}_{...}$$
$$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}, i = 1,2,\ldots,p$$

$$\hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{...},\ j = 1,2,...,p$$
$$\hat{\beta}_k = \bar{y}_{i..} - \bar{y}_{...},\ k = 1,2,...,p$$

4-18 Derive the missing value formula (Equation 4-24) for the Latin square design.

$$SS_E = \sum\sum\sum y_{ijk}^2 - \sum\frac{y_{i..}^2}{p} - \sum\frac{y_{.j.}^2}{p} - \sum\frac{y_{..k}^2}{p} + 2\left(\frac{y_{...}^2}{p^2}\right)$$

Let y_{ijk} be missing. Then

$$SS_E = y_{ijk}^2 - \frac{\left(y'_{i..} + y_{ijk}\right)^2}{p} - \frac{\left(y'_{.j.} + y_{ijk}\right)^2}{p} - \frac{\left(y'_{..k} + y_{ijk}\right)^2}{p} + \frac{2\left(y'_{...} + y_{ijk}\right)}{p^2} + R$$

where R is all terms without y_{ijk}.. From $\frac{\partial SS_E}{\partial y_{ijk}} = 0$, we obtain:

$$y_{ijk}\frac{(p-1)(p-2)}{p^2} = \frac{p\left(y'_{i..} + y'_{.j.} + y'_{..k}\right) - 2y'_{...}}{p^2}, \text{ or } y_{ijk} = \frac{p\left(y'_{i..} + y'_{.j.} + y'_{..k}\right) - 2y'_{...}}{(p-1)(p-2)}$$

4-19 ***Designs involving several Latin squares***. [See Cochran and Cox (1957), John (1971).] The p x p Latin square contains only p observations for each treatment. To obtain more replications the experimenter may use several squares, say n. It is immaterial whether the squares used are the same are different. The appropriate model is

$$y_{ijkh} = \mu + \rho_h + \alpha_{i(h)} + \tau_j + \beta_{k(h)} + (\tau\rho)_{jh} + \varepsilon_{ijkh} \quad \begin{cases} i = 1,2,\ldots,p \\ j = 1,2,\ldots,p \\ k = 1,2,\ldots,p \\ h = 1,2,\ldots,n \end{cases}$$

where y_{ijkh} is the observation on treatment j in row i and column k of the hth square. Note that $\alpha_{i(h)}$ and $\beta_{k(h)}$ are row and column effects in the hth square, and ρ_h is the effect of the hth square, and $(\tau\rho)_{jh}$ is the interaction between treatments and squares.

(a) Set up the normal equations for this model, and solve for estimates of the model parameters. Assume that appropriate side conditions on the parameters are $\sum_h \hat{\rho}_h = 0$, $\sum_i \hat{\alpha}_{i(h)} = 0$, and $\sum_k \hat{\beta}_{k(h)} = 0$ for each h, $\sum_j \hat{\tau}_j = 0$, $\sum_j \left(\hat{\tau\rho}\right)_{jh} = 0$ for each h, and $\sum_h \left(\hat{\tau\rho}\right)_{jh} = 0$ for each j.

$$\hat{\mu} = \bar{y}_{....}$$
$$\hat{\rho}_h = \bar{y}_{...h} - \bar{y}_{....}$$
$$\hat{\tau}_j = \bar{y}_{.j..} - \bar{y}_{....}$$
$$\hat{\alpha}_{i(h)} = \bar{y}_{i..h} - \bar{y}_{...h}$$
$$\hat{\beta}_{k(h)} = \bar{y}_{..kh} - \bar{y}_{...h}$$
$$\left(\hat{\tau\rho}\right)_{jh} = \bar{y}_{.j.h} - \bar{y}_{.j..} - \bar{y}_{...h} + \bar{y}_{....}$$

(b) Write down the analysis of variance table for this design.

Source	SS	DF
Treatments	$\sum \frac{y_{.j..}^2}{np} - \frac{y_{....}^2}{np^2}$	p-1
Squares	$\sum \frac{y_{...h}^2}{p^2} - \frac{y_{....}^2}{np^2}$	n-1
Treatment x Squares	$\sum \frac{y_{.j.h}^2}{p} - \frac{y_{....}^2}{np^2} - SS_{Treatments} - SS_{Squares}$	(p-1)(n-1)
Rows	$\sum \frac{y_{i..h}^2}{p} - \frac{y_{...h}^2}{np^2}$	n(p-1)
Columns	$\sum \frac{y_{..kh}^2}{p} - \frac{y_{...h}^2}{np^2}$	n(p-1)
Error	subtraction	n(p-1)(p-2)
Total	$\sum\sum\sum\sum y_{ijkh}^2 - \frac{y_{....}^2}{np^2}$	np^2-1

4-20 Discuss how the operating characteristics curves in the Appendix may be used with the Latin square design.

For the fixed effects model use:

$$\Phi^2 = \frac{\sum p\tau_j^2}{p\sigma^2} = \sum \frac{\tau_j^2}{\sigma^2}, \; \upsilon_1 = p-1 \quad \upsilon_2 = (p-2)(p-1)$$

For the random effects model use:

$$\lambda = \sqrt{1 + \frac{p\sigma_\tau^2}{\sigma^2}}, \; \upsilon_1 = p-1 \quad \upsilon_2 = (p-2)(p-1)$$

4-21 Suppose that in Problem 4-14 the data taken on day 5 were incorrectly analyzed and had to be discarded. Develop an appropriate analysis for the remaining data.

Two methods of analysis exist: (1) Use the general regression significance test, or (2) recognize that the design is a Youden square. The data can be analyzed as a balanced incomplete block design with a=b=5, r=k=4 and λ=3. Using either approach will yield the same analysis of variance.

Minitab Output

```
                              General Linear Model

Factor     Type Levels Values
Catalyst  fixed      5 A B C D E
Batch    random      5 1 2 3 4 5
Day      random      4 1 2 3 4

Analysis of Variance for Time, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS        F      P
Catalyst     4    119.800    120.167     30.042     7.48  0.008
Batch        4     11.667     11.667      2.917     0.73  0.598
Day          3      6.950      6.950      2.317     0.58  0.646
Error        8     32.133     32.133      4.017
Total       19    170.550
```

4-22 The yield of a chemical process was measured using five batches of raw material, five acid concentrations, five standing times, (*A, B, C, D, E*) and five catalyst concentrations ($\alpha, \beta, \gamma, \delta, \varepsilon$). The Graeco-Latin square that follows was used. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Acid Concentration				
Batch	1	2	3	4	5
1	$A\alpha$=26	$B\beta$=16	$C\gamma$=19	$D\delta$=16	$E\varepsilon$=13
2	$B\gamma$=18	$C\delta$=21	$D\varepsilon$=18	$E\alpha$=11	$A\beta$=21
3	$C\varepsilon$=20	$D\alpha$=12	$E\beta$=16	$A\gamma$=25	$B\delta$=13
4	$D\beta$=15	$E\gamma$=15	$A\delta$=22	$B\varepsilon$=14	$C\alpha$=17
5	$E\delta$=10	$A\varepsilon$=24	$B\alpha$=17	$C\beta$=17	$D\gamma$=14

```
                              General Linear Model

Factor     Type Levels Values
Time      fixed      5 A B C D E
Catalyst random      5 a b c d e
Batch    random      5 1 2 3 4 5
Acid     random      5 1 2 3 4 5

Analysis of Variance for Yield, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS        F      P
Time         4    342.800    342.800     85.700    14.65  0.001
Catalyst     4     12.000     12.000      3.000     0.51  0.729
Batch        4     10.000     10.000      2.500     0.43  0.785
Acid         4     24.400     24.400      6.100     1.04  0.443
Error        8     46.800     46.800      5.850
Total       24    436.000
```

4-23 Suppose that in Problem 4-15 the engineer suspects that the workplaces used by the four operators may represent an additional source of variation. A fourth factor, workplace ($\alpha, \beta, \gamma, \delta$) may be introduced and another experiment conducted, yielding the Graeco-Latin square that follows. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

Order of	Operator			
Assembly	1	2	3	4
1	$C\beta$=11	$B\gamma$=10	$D\delta$=14	$A\alpha$=8
2	$B\alpha$=8	$C\delta$=12	$A\gamma$=10	$D\beta$=12
3	$A\delta$=9	$D\alpha$=11	$B\beta$=7	$C\gamma$=15

4	$D\gamma$=9	$A\beta$=8	$C\alpha$=18	$B\delta$=6

Minitab Output

General Linear Model

```
Factor      Type Levels Values
Method     fixed      4 A B C D
Order     random      4 1 2 3 4
Operator random       4 1 2 3 4
Workplac random       4 a b c d

Analysis of Variance for Time, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS        F       P
Method       3     95.500     95.500     31.833     3.47   0.167
Order        3      0.500      0.500      0.167     0.02   0.996
Operator     3     19.000     19.000      6.333     0.69   0.616
Workplac     3      7.500      7.500      2.500     0.27   0.843
Error        3     27.500     27.500      9.167
Total       15    150.000
```

However, there are only three degrees of freedom for error, so the test is not very sensitive.

4-24 Construct a 5 x 5 hypersquare for studying the effects of five factors. Exhibit the analysis of variance table for this design.

Three 5 x 5 orthogonal Latin Squares are:

ABCDE	$\alpha\beta\gamma\delta\varepsilon$	12345
BCDEA	$\gamma\delta\varepsilon\alpha\beta$	45123
CDEAB	$\varepsilon\alpha\beta\gamma\delta$	23451
DEABC	$\beta\gamma\delta\varepsilon\alpha$	51234
EABCD	$\delta\varepsilon\alpha\beta\gamma$	34512

Let rows = factor 1, columns = factor 2, Latin letters = factor 3, Greek letters = factor 4 and numbers = factor 5. The analysis of variance table is:

Source	DF
Rows	4
Columns	4
Latin Letters	4
Greek Letters	4
Numbers	4
Error	4
Total	24

4-25 Consider the data in Problems 4-15 and 4-23. Suppressing the Greek letters in 4-23, analyze the data using the method developed in Problem 4-19.

Square 1 - Operator

Batch	1	2	3	4	Row Total
1	*C*=10	*D*=14	*A*=7	*B*=8	(39)
2	*B*=7	*C*=18	*D*=11	*A*=8	(44)
3	*A*=5	*B*=10	*C*=11	*D*=9	(35)
4	*D*=10	*A*=10	*B*=12	*C*=14	(46)
	(32)	(52)	(41)	(36)	164=$y_{...1}$

Square 2 - Operator

Batch	1	2	3	4	Row Total
1	*C*=11	*B*=10	*D*=14	*A*=8	(43)
2	*B*=8	*C*=12	*A*=10	*D*=12	(42)
3	*A*=9	*D*=11	*B*=7	*C*=15	(42)
4	*D*=9	*A*=8	*C*=18	*B*=6	(41)
	(37)	(41)	(49)	(41)	168=$y_{...2}$

Assembly Methods	Totals
A	$y_{.1..}$=65
B	$y_{.2..}$=68
C	$y_{.3..}$=109
D	$y_{.4..}$=90

Source	SS	DF	MS	F_0
Assembly Methods	159.25	3	53.08	14.00*
Squares	0.50	1	0.50	
A x *S*	8.75	3	2.92	0.77
Assembly Order (Rows)	19.00	6	3.17	
Operators (columns)	70.50	6	11.75	
Error	45.50	12	3.79	
Total	303.50	31		

Significant at 1%.

4-26 Consider the randomized block design with one missing value in Table 4-7. Analyze this data by using the exact analysis of the missing value problem discussed in Section 4-1.4. Compare your results to the approximate analysis of these data given in Table 4-8.

$$
\begin{array}{llllllllllr}
\mu: & 15\hat{\mu} & +4\hat{\tau}_1 & +3\hat{\tau}_2 & +4\hat{\tau}_3 & +4\hat{\tau}_4 & +4\hat{\beta}_1 & +4\hat{\beta}_2 & +3\hat{\beta}_3 & +4\hat{\beta}_4 & =17 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =3 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & +\hat{\beta}_1 & +\hat{\beta}_2 & & +\hat{\beta}_4 & =1 \\
\tau_3: & 4\hat{\mu} & & & +4\hat{\tau}_3 & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =-2 \\
\tau_4: & 4\hat{\mu} & & & & +4\hat{\tau}_4 & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =15 \\
\beta_1: & 4\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & +4\hat{\beta}_1 & & & & =-4 \\
\beta_2: & 4\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & & +3\hat{\beta}_2 & & & =-3 \\
\beta_3: & 3\hat{\mu} & +\hat{\tau}_1 & & +\hat{\tau}_3 & +\hat{\tau}_4 & & & +4\hat{\beta}_3 & & =6 \\
\beta_4: & 4\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & & & & +4\hat{\beta}_4 & =19
\end{array}
$$

Applying the constraints $\sum \hat{\tau}_i = \sum \hat{\beta}_j = 0$, we obtain:

$$\hat{\mu} = \frac{41}{36},\ \hat{\tau}_1 = \frac{-14}{36},\ \hat{\tau}_2 = \frac{-24}{36},\ \hat{\tau}_3 = \frac{-59}{36}, \hat{\tau}_4 = \frac{94}{36},\ \hat{\beta}_1 = \frac{-77}{36},\ \hat{\beta}_2 = \frac{-68}{36},\ \hat{\beta}_3 = \frac{24}{36},\ \hat{\beta}_4 = \frac{121}{36}$$

$$R(\mu,\tau,\beta)=\hat{\mu}y_{..}+\sum_{i=1}^{4}\hat{\tau}_i y_{i.}+\sum_{j=1}^{4}\hat{\beta}_j y_{.j}=138.78$$

With 7 degrees of freedom.

$$\sum\sum y_{ij}^2=145.00\,,\; SS_E=\sum\sum y_{ij}^2-R(\mu,\tau,\beta)=145.00-138.78=6.22$$

which is identical to SS_E obtained in the approximate analysis. In general, the SS_E in the exact and approximate analyses will be the same.

To test H$_o$: $\tau_i=0$ the reduced model is $y_{ij}=\mu+\beta_j+\varepsilon_{ij}$. The normal equations used are:

$$\begin{array}{lllllll}
\mu: & 15\hat{\mu} & +4\hat{\beta}_1 & +4\hat{\beta}_2 & +3\hat{\beta}_3 & +4\hat{\beta}_4 & =17 \\
\beta_1: & 4\hat{\mu} & +4\hat{\beta}_1 & & & & =-4 \\
\beta_2: & 4\hat{\mu} & & +4\hat{\beta}_2 & & & =-3 \\
\beta_3: & 3\hat{\mu} & & & +3\hat{\beta}_3 & & =6 \\
\beta_4: & 4\hat{\mu} & & & & +4\hat{\beta}_4 & =18
\end{array}$$

Applying the constraint $\sum\hat{\beta}_j=0$, we obtain:

$$\hat{\mu}=\frac{19}{16}\,,\;\hat{\beta}_1=\frac{-35}{16}\,,\;\hat{\beta}_2=\frac{-31}{16}\,,\;\hat{\beta}_3=\frac{13}{16}\,,\;\hat{\beta}_4=\frac{53}{16}.\;\text{Now}\;\; R(\mu,\beta)=\hat{\mu}y_{..}+\sum_{j=1}^{4}\hat{\beta}_j y_{.j}=99.25$$

with 4 degrees of freedom.

$$R(\tau|\mu,\beta)=R(\mu,\tau,\beta)-R(\mu,\beta)=138.78-99.25=39.53=SS_{Treatments}$$

with 7-4=3 degrees of freedom. $R(\tau|\mu,\beta)$ is used to test H$_o$: $\tau_i=0$.

The sum of squares for blocks is found from the reduced model $y_{ij}=\mu+\tau_i+\varepsilon_{ij}$. The normal equations used are:

Model Restricted to $\beta_j=0$:

$$\begin{array}{lllllll}
\mu: & 15\hat{\mu} & +4\hat{\tau}_1 & +3\hat{\tau}_2 & +4\hat{\tau}_3 & +4\hat{\tau}_4 & =17 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & & =3 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & =1 \\
\tau_3: & 4\hat{\mu} & & & +4\hat{\tau}_3 & & =-2 \\
\tau_4: & 4\hat{\mu} & & & & +4\hat{\tau}_4 & =15
\end{array}$$

Applying the constraint $\sum\hat{\tau}_i=0$, we obtain:

$$\hat{\mu}=\frac{13}{12},\;\hat{\tau}_1=\frac{-4}{12}\,,\;\hat{\tau}_2=\frac{-9}{12}\,,\;\hat{\tau}_3=\frac{-19}{12}\,,\;\hat{\tau}_4=\frac{32}{12}$$

$$R(\mu,\tau) = \hat{\mu}y_{..} + \sum_{i=1}^{4} \hat{\tau}_i y_{i.} = 59.83$$

with 4 degrees of freedom.

$$R(\beta|\mu,\tau) = R(\mu,\tau,\beta) - R(\mu,\tau) = 138.78 - 59.83 = 78.95 = SS_{Blocks}$$

with 7-4=3 degrees of freedom.

Source	DF	SS(exact)	SS(approximate)
Tips	3	39.53	39.98
Blocks	3	78.95	79.53
Error	8	6.22	6.22
Total	14	125.74	125.73

Note that for the exact analysis, $SS_T \neq SS_{Tips} + SS_{Blocks} + SS_E$.

4-27 An engineer is studying the mileage performance characteristics of five types of gasoline additives. In the road test he wishes to use cars as blocks; however, because of a time constraint, he must use an incomplete block design. He runs the balanced design with the five blocks that follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Car				
Additive	1	2	3	4	5
1		17	14	13	12
2	14	14		13	10
3	14		13	14	9
4	13	11	11	12	
5	11	12	10		8

There are several computer software packages that can analyze the incomplete block designs discussed in this chapter. The Minitab General Linear Model procedure is a widely available package with this capability. The output from this routine for Problem 4-27 follows. The adjusted sums of squares are the appropriate sums of squares to use for testing the difference between the means of the gasoline additives.

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Additive  fixed      5 1 2 3 4 5
Car      random      5 1 2 3 4 5

Analysis of Variance for Mileage, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS       F      P
Additive     4    31.7000    35.7333     8.9333    9.81  0.001
Car          4    35.2333    35.2333     8.8083    9.67  0.001
Error       11    10.0167    10.0167     0.9106
Total       19    76.9500
```

4-28 Construct a set of orthogonal contrasts for the data in Problem 4-27. Compute the sum of squares for each contrast.

One possible set of orthogonal contrasts is:

$$H_0 : \mu_4 + \mu_5 = \mu_1 + \mu_2 \qquad (1)$$
$$H_0 : \mu_1 = \mu_2 \qquad (2)$$
$$H_0 : \mu_4 = \mu_5 \qquad (3)$$
$$H_0 : 4\mu_3 = \mu_4 + \mu_5 + \mu_1 + \mu_2 \qquad (4)$$

The sums of squares and *F*-tests are:

Brand ->	1	2	3	4	5			
Q_i	33/4	11/4	-3/4	-14/4	-27/4	$\sum c_i Q_i$	SS	F_0
(1)	-1	-1	0	1	1	-85/4	30.10	39.09
(2)	1	-1	0	0	0	-22/4	4.03	5.23
(3)	0	0	0	-1	1	-13/4	1.41	1.83
(4)	-1	-1	4	-1	-1	-15/4	0.19	0.25

Contrasts (1) and (2) are significant at the 1% and 5% levels, respectively.

4-29 Seven different hardwood concentrations are being studied to determine their effect on the strength of the paper produced. However the pilot plant can only produce three runs each day. As days may differ, the analyst uses the balanced incomplete block design that follows. Analyze this experiment (use $\alpha = 0.05$) and draw conclusions.

Hardwood				Days			
Concentration (%)	1	2	3	4	5	6	7
2	114				120		117
4	126	120				119	
6		137	114				134
8	141		129	149			
10		145		150	143		
12			120		118	123	
14				136		130	127

There are several computer software packages that can analyze the incomplete block designs discussed in this chapter. The Minitab General Linear Model procedure is a widely available package with this capability. The output from this routine for Problem 4-29 follows. The adjusted sums of squares are the appropriate sums of squares to use for testing the difference between the means of the hardwood concentrations.

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Concentr  fixed      7  2  4  6  8 10 12 14
Days     random      7 1 2 3 4 5 6 7

Analysis of Variance for Strength, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS       F      P
Concentr     6    2037.62    1317.43     219.57   10.42  0.002
Days         6     394.10     394.10      65.68    3.12  0.070
Error        8     168.57     168.57      21.07
Total       20    2600.29
```

4-30 Analyze the data in Example 4-6 using the general regression significance test.

$$
\begin{array}{llllllllllll}
\mu: & 12\hat{\mu} & +3\hat{\tau}_1 & +3\hat{\tau}_2 & +3\hat{\tau}_3 & +3\hat{\tau}_4 & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =870 \\
\tau_1: & 3\hat{\mu} & +3\hat{\tau}_1 & & & & +\hat{\beta}_1 & & +\hat{\beta}_3 & +\hat{\beta}_4 & =218 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =214 \\
\tau_3: & 3\hat{\mu} & & & +3\hat{\tau}_3 & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & & =216 \\
\tau_4: & 3\hat{\mu} & & & & +3\hat{\tau}_4 & +\hat{\beta}_1 & +\hat{\beta}_2 & & +\hat{\beta}_4 & =222 \\
\beta_1: & 3\hat{\mu} & +\hat{\tau}_1 & & +\hat{\tau}_3 & +\hat{\tau}_4 & +3\hat{\beta}_1 & & & & =221 \\
\beta_2: & 3\hat{\mu} & & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & & +3\hat{\beta}_2 & & & =207 \\
\beta_3: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & & & & +3\hat{\beta}_3 & & =224 \\
\beta_4: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & & +\hat{\tau}_4 & & & & +3\hat{\beta}_4 & =218
\end{array}
$$

Applying the constraints $\sum \hat{\tau}_i = \sum \hat{\beta}_j = 0$, we obtain:

$$
\hat{\mu} = 870/12,\ \hat{\tau}_1 = -9/8,\ \hat{\tau}_2 = -7/8,\ \hat{\tau}_3 = -4/8,\ \hat{\tau}_4 = 20/8,
$$
$$
\hat{\beta}_1 = 7/8,\ \hat{\beta}_2 = -31/8,\ \hat{\beta}_3 = 24/8,\ \hat{\beta}_4 = 0/8
$$

with 7 degrees of freedom.

$$
\sum\sum y_{ij}^2 = 63{,}156.00
$$
$$
SS_E = \sum\sum y_{ij}^2 - R(\mu,\tau,\beta) = 63156.00 - 63152.75 = 3.25 .
$$

To test H$_o$: $\tau_i = 0$ the reduced model is $y_{ij} = \mu + \beta_j + \varepsilon_{ij}$. The normal equations used are:

$$
\begin{array}{lllllll}
\mu: & 12\hat{\mu} & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =870 \\
\beta_1: & 3\hat{\mu} & +3\hat{\beta}_1 & & & & =221 \\
\beta_2: & 3\hat{\mu} & & +3\hat{\beta}_2 & & & =207 \\
\beta_3: & 3\hat{\mu} & & & +3\hat{\beta}_3 & & =224 \\
\beta_4: & 3\hat{\mu} & & & & +3\hat{\beta}_4 & =218
\end{array}
$$

Applying the constraint $\sum \hat{\beta}_j = 0$, we obtain:

$$
\hat{\mu} = \frac{870}{12},\ \hat{\beta}_1 = \frac{7}{6},\ \hat{\beta}_2 = \frac{-21}{6},\ \hat{\beta}_3 = \frac{13}{6},\ \hat{\beta}_4 = \frac{1}{6}
$$
$$
R(\mu,\beta) = \hat{\mu}y_{..} + \sum_{j=1}^{4} \hat{\beta}_j y_{.j} = 63{,}130.00
$$

with 4 degrees of freedom.

$$R(\tau|\mu,\beta) = R(\mu,\tau,\beta) - R(\mu,\beta) = 63152.75 - 63130.00 = 22.75 = SS_{Treatments}$$

with 7-4=3 degrees of freedom. $R(\tau|\mu,\beta)$ is used to test H$_o$: $\tau_i = 0$.

The sum of squares for blocks is found from the reduced model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$. The normal equations used are:

Model Restricted to $\beta_j = 0$:

μ:	$12\hat{\mu}$	$+3\hat{\tau}_1$	$+3\hat{\tau}_2$	$+3\hat{\tau}_3$	$+3\hat{\tau}_4$	=870
τ_1:	$3\hat{\mu}$	$+3\hat{\tau}_1$				=218
τ_2:	$3\hat{\mu}$		$+3\hat{\tau}_2$			=214
τ_3:	$3\hat{\mu}$			$+3\hat{\tau}_3$		=216
τ_4:	$3\hat{\mu}$				$+3\hat{\tau}_4$	=222

The sum of squares for blocks is found as in Example 4-6. We may use the method shown above to find an adjusted sum of squares for blocks from the reduced model, $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$.

4-31 Prove that $\dfrac{k\sum_{i=1}^{a} Q_i^2}{(\lambda a)}$ is the adjusted sum of squares for treatments in a BIBD.

We may use the general regression significance test to derive the computational formula for the adjusted treatment sum of squares. We will need the following:

$$\hat{\tau}_i = \frac{kQ_i}{(\lambda a)},\ kQ_i = ky_{i.} - \sum_{i=1}^{b} n_{ij} y_{.j}$$

$$R(\mu,\tau,\beta) = \hat{\mu}y_{..} + \sum_{i=1}^{a} \hat{\tau}_i y_{i.} + \sum_{j=1}^{b} \hat{\beta}_j y_{.j}$$

and the sum of squares we need is:

$$R(\tau|\mu,\beta) = \hat{\mu}y_{..} + \sum_{i=1}^{a} \hat{\tau}_i y_{i.} + \sum_{j=1}^{b} \hat{\beta}_j y_{.j} - \sum_{j=1}^{b} \frac{y_{.j}^2}{k}$$

The normal equation for β is, from equation (4-35),

$$\beta : k\hat{\mu} + \sum_{i=1}^{a} n_{ij}\hat{\tau}_i + k\hat{\beta}_j = y_{.j}$$

and from this we have:

$$ky_{.j}\hat{\beta}_j = y_{.j}^2 - ky_{.j}\hat{\mu} - y_{.j}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i$$

therefore,

$$R(\tau|\mu,\beta) = \hat{\mu}y_{..} + \sum_{i=1}^{a}\hat{\tau}_i y_{i.} + \sum_{j=1}^{b}\left[\frac{y_{.j}^2}{k} - \frac{k\hat{\mu}y_{.j}}{k} - \frac{y_{.j}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i}{k} - \frac{y_{.j}^2}{k}\right]$$

$$R(\tau|\mu,\beta) = \sum_{i=1}^{a}\hat{\tau}_i\left(y_{i.} - \frac{1}{k}\sum_{i=1}^{a} n_{ij}y_{.j}\right) = \sum_{i=1}^{a} Q_i\left(\frac{kQ}{\lambda a}\right) = k\sum_{i=1}^{a}\left(\frac{Q_i^2}{\lambda a}\right) \equiv SS_{Treatments(adjusted)}$$

4-32 An experimenter wishes to compare four treatments in blocks of two runs. Find a BIBD for this experiment with six blocks.

Treatment	Block 1	Block 2	Block 3	Block 4	Block 5	Block 6
1	X	X	X			
2	X			X	X	
3		X		X		X
4			X		X	X

Note that the design is formed by taking all combinations of the 4 treatments 2 at a time. The parameters of the design are $\lambda = 1$, a=4, b=6, k=3, and r=2

4-33 An experimenter wishes to compare eight treatments in blocks of four runs. Find a BIBD with 14 blocks and $\lambda = 3$.

The design has parameters a=8, b=14, $\lambda = 3$, r=2 and k=4. It may be generated from a 2^3 factorial design confounded in two blocks of four observations each, with each main effect and interaction successively confounded (7 replications) forming the 14 blocks. The design is discussed by John (1971, pg. 222) and Cochran and Cox (1957, pg. 473). The design follows:

Blocks	1=(I)	2=a	3=b	4=ab	5=c	6=ac	7=bc	8=abc
1	X		X		X		X	
2		X		X		X		X
3	X		X			X		X
4		X		X	X		X	
5	X	X			X	X		
6			X	X			X	X
7	X	X					X	X
8			X	X	X	X		
9	X	X	X	X				
10					X	X	X	X
11	X			X		X	X	
12		X	X		X			X
13	X			X	X			X
14		X	X			X	X	

4-34 Perform the interblock analysis for the design in Problem 4-27.

The interblock analysis for Problem 4-27 uses $\hat{\sigma}^2 = 0.77$ and $\hat{\sigma}_\beta^2 = 2.14$. A summary of the interblock, intrablock and combined estimates is:

Parameter	Intrablock	Interblock	Combined
τ_1	2.20	-1.80	2.18
τ_2	0.73	0.20	0.73
τ_3	-0.20	-5.80	-0.23
τ_4	-0.93	9.20	-0.88
τ_5	-1.80	-1.80	-1.80

4-35 Perform the interblock analysis for the design in Problem 4-29. The interblock analysis for problem 4-29 uses $\hat{\sigma}^2 = 21.07$ and $\sigma_\beta^2 = \frac{\left[MS_{Blocks(adj)} - MS_E\right](b-1)}{a(r-1)} = \frac{[65.68 - 21.07](6)}{7(2)} = 19.12$. A summary of the interblock, intrablock, and combined estimates is give below

Parameter	Intrablock	Interblock	Combined
τ_1	-12.43	-11.79	-12.38
τ_2	-8.57	-4.29	-7.92
τ_3	2.57	-8.79	1.76
τ_4	10.71	9.21	10.61
τ_5	13.71	21.21	14.67
τ_6	-5.14	-22.29	-6.36
τ_7	-0.86	10.71	-0.03

4-36 Verify that a BIBD with the parameters a = 8, r = 8, k = 4, and b = 16 does not exist. These conditions imply that $\lambda = \frac{r(k-1)}{a-1} = \frac{8(3)}{7} = \frac{24}{7}$, which is not an integer, so a balanced design with these parameters cannot exist.

4-37 Show that the variance of the intra block estimators $\{\hat{\tau}_i\}$ is $\frac{k((a-1))\sigma^2}{(\lambda a^2)}$.

Note that $\hat{\tau}_i = \frac{kQ_i}{(\lambda a)}$, and $Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{.j}$, and $kQ_i = ky_{i.} - \sum_{j=1}^{b} n_{ij} y_{.j} = (k-1)y_{i.} - \left(\sum_{j=1}^{b} n_{ij} y_{.j} - y_{i.}\right)$

$y_{i.}$ contains r observations, and the quantity in the parenthesis is the sum of r(k-1) observations, not including treatment i. Therefore,

$$V(kQ_i) = k^2 V(Q_i) = r(k-1)^2\sigma^2 + r(k-1)\sigma^2$$

or

$$V(Q_i) = \frac{1}{k^2}\left[r(k-1)\sigma^2\{(k-1)+1\}\right] = \frac{r(k-1)\sigma^2}{k}$$

To find $V(\hat{\tau}_i)$, note that:

$$V(\hat{\tau}_i) = \left(\frac{k}{\lambda a}\right)^2 V(Q)_i = \left(\frac{k}{\lambda a}\right)^2 \frac{r(k-1)}{k}\sigma^2 = \frac{kr(k-1)}{(\lambda a)^2}\sigma^2$$

However, since $\lambda(a-1) = r(k-1)$, we have:

$$V(\hat{\tau}_i) = \frac{k(a-1)}{\lambda a^2}\sigma^2$$

Furthermore, the $\{\hat{\tau}_i\}$ are not independent, this is required to show that $V(\hat{\tau}_i - \hat{\tau}_j) = \frac{2k}{\lambda a}\sigma^2$

4-38 ***Extended incomplete block designs.*** Occasionally the block size obeys the relationship $a < k < 2a$. An extended incomplete block design consists of a single replicate or each treatment in each block along with an incomplete block design with $k^* = k\text{-}a$. In the balanced case, the incomplete block design will have parameters $k^* = k\text{-}a$, $r^* = r\text{-}b$, and λ^*. Write out the statistical analysis. (Hint: In the extended incomplete block design, we have $\lambda = 2r\text{-}b+\lambda^*$.)

As an example of an extended incomplete block design, suppose we have a=5 treatments, b=5 blocks and k=9. A design could be found by running all five treatments in each block, plus a block from the balanced incomplete block design with $k^* = k\text{-}a$=9-5=4 and λ^*=3. The design is:

Block	Complete Treatment	Incomplete Treatment
1	1,2,3,4,5	2,3,4,5
2	1,2,3,4,5	1,2,4,5
3	1,2,3,4,5	1,3,4,5
4	1,2,3,4,5	1,2,3,4
5	1,2,3,4,5	1,2,3,5

Note that r=9, since the augmenting incomplete block design has r*=4, and $r = r^* + b$ = 4+5=9, and $\lambda = 2r\text{-}b+\lambda^*$=18-5+3=16. Since some treatments are repeated in each block it is possible to compute an error sum of squares between repeat observations. The difference between this and the residual sum of squares is due to interaction. The analysis of variance table is shown below:

Source	SS	DF
Treatments (adjusted)	$k\sum \frac{Q_i^2}{a\lambda}$	a-1
Blocks	$\sum \frac{y_{.j}^2}{k} - \frac{y_{..}^2}{N}$	b-1
Interaction	Subtraction	(a-1)(b-1)
Error	[SS between repeat observations]	b(k-a)
Total	$\sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$	N-1

Chapter 5
Introduction to Factorial Designs
Solutions

5-1 The yield of a chemical process is being studied. The two most important variables are thought to be the pressure and the temperature. Three levels of each factor are selected, and a factorial experiment with two replicates is performed. The yield data follow:

	Pressure		
Temperature	200	215	230
150	90.4	90.7	90.2
	90.2	90.6	90.4
160	90.1	90.5	89.9
	90.3	90.6	90.1
170	90.5	90.8	90.4
	90.7	90.9	90.1

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

Both pressure (A) and temperature (B) are significant, the interaction is not.

Design Expert Output

Response:Surface Finish
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

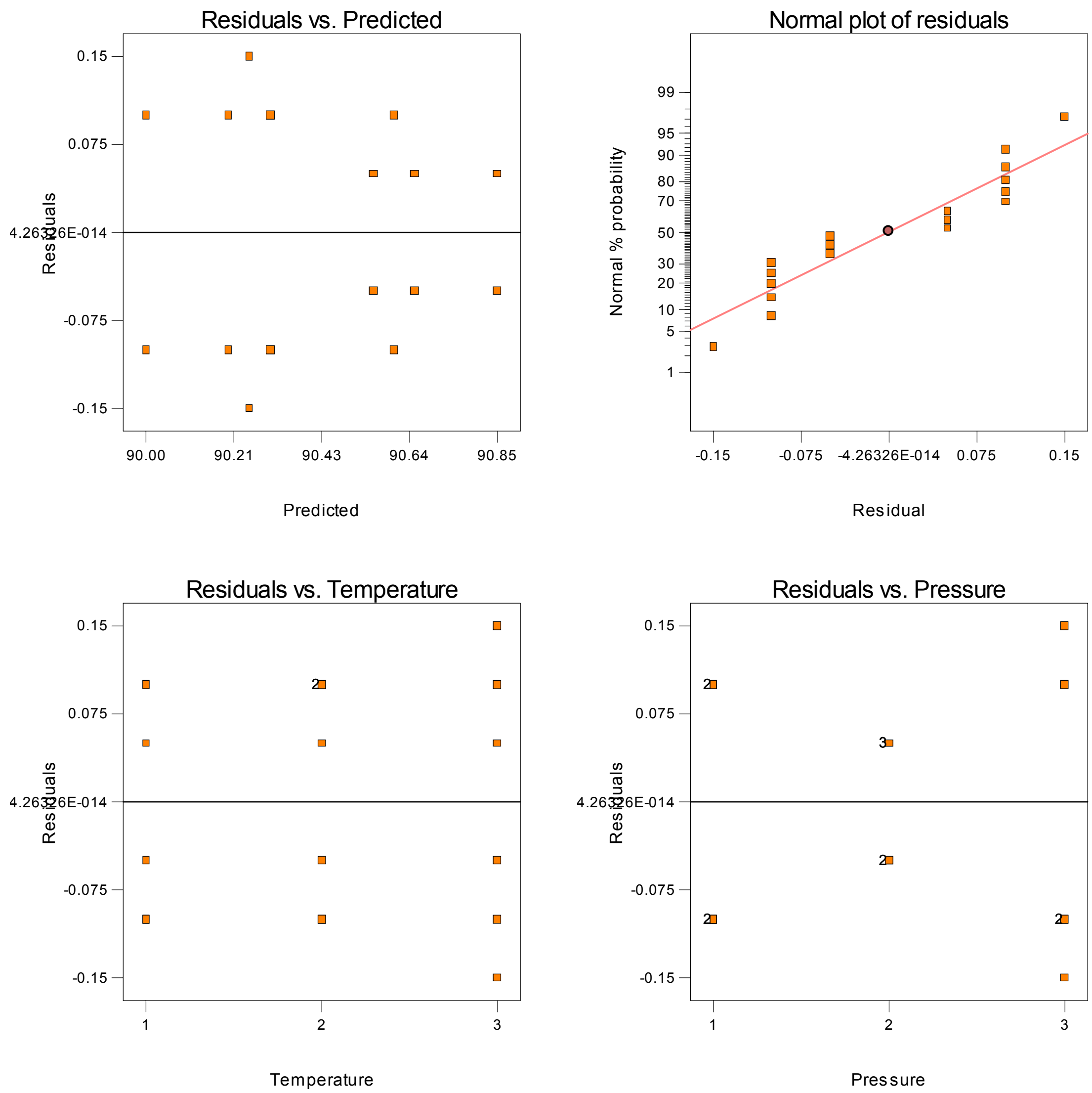
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.14	8	0.14	8.00	0.0026	significant
A	*0.77*	*2*	*0.38*	*21.59*	*0.0004*	
B	*0.30*	*2*	*0.15*	*8.47*	*0.0085*	
AB	*0.069*	*4*	*0.017*	*0.97*	*0.4700*	
Residual	0.16	9	0.018			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.16*	*9*	*0.018*			
Cor Total	1.30	17				

The Model F-value of 8.00 implies the model is significant. There is only a 0.26% chance that a "Model F-Value" this large could occur due to noise.

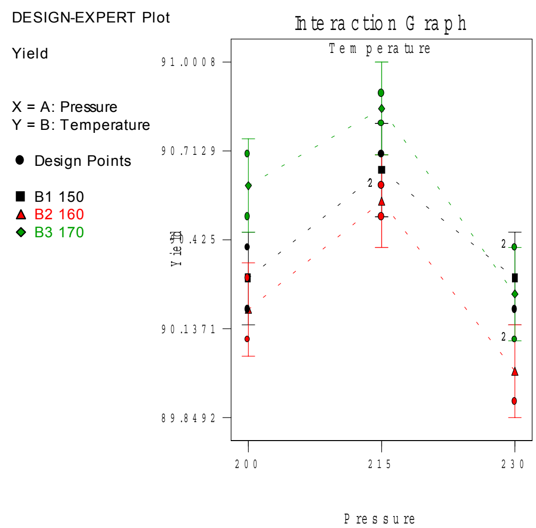
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

(b) Prepare appropriate residual plots and comment on the model's adequacy.

The residuals plot show no serious deviations from the assumptions.



(c) Under what conditions would you operate this process?



Pressure set at 215 and Temperature at the high level, 170 degrees C, give the highest yield.

The standard analysis of variance treats all design factors as if they were qualitative. In this case, both factors are quantitative, so some further analysis can be performed. In Section 5-5, we show how response curves and surfaces can be fit to the data from a factorial experiment with at least one quantative factor. Since both factors in this problem are quantitative and have three levels, we can fit linear and quadratic effects of both temperature and pressure, exactly as in Example 5-5 in the text. The Design-Expert output, including the response surface plots, now follows.

Design Expert Output

Response:Surface Finish
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1.13	5	0.23	16.18	< 0.0001	significant
A	*0.10*	*1*	*0.10*	*7.22*	*0.0198*	
B	*0.067*	*1*	*0.067*	*4.83*	*0.0483*	
A2	*0.67*	*1*	*0.67*	*47.74*	*< 0.0001*	
B2	*0.23*	*1*	*0.23*	*16.72*	*0.0015*	
AB	*0.061*	*1*	*0.061*	*4.38*	*0.0582*	
Residual	0.17	12	0.014			
Lack of Fit	*7.639E-003*	*3*	*2.546E-003*	*0.14*	*0.9314*	*not significant*
Pure Error	*0.16*	*9*	*0.018*			
Cor Total1.30	17					

The Model F-value of 16.18 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, A^2, B^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy),
model reduction may improve your model.

Std. Dev.	0.12	R-Squared	0.8708
Mean	90.41	Adj R-Squared	0.8170
C.V.	0.13	Pred R-Squared	0.6794
PRESS	0.42	Adeq Precision	11.968

Coefficient **Standard** **95% CI** **95% CI**

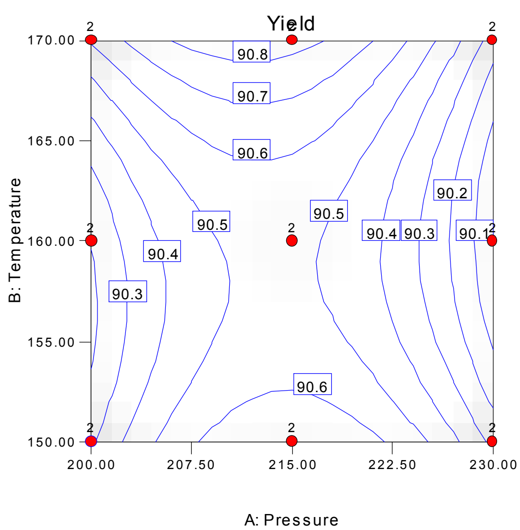
Factor	Estimate	DF	Error	Low	High	VIF
Intercept	90.52	1	0.062	90.39	90.66	
A-Pressure	-0.092	1	0.034	-0.17	-0.017	1.00
B-Temperature	0.075	1	0.034	6.594E-004	0.15	1.00
A^2	-0.41	1	0.059	-0.54	-0.28	1.00
B^2	0.24	1	0.059	0.11	0.37	1.00
AB	-0.087	1	0.042	-0.18	3.548E-003	1.00

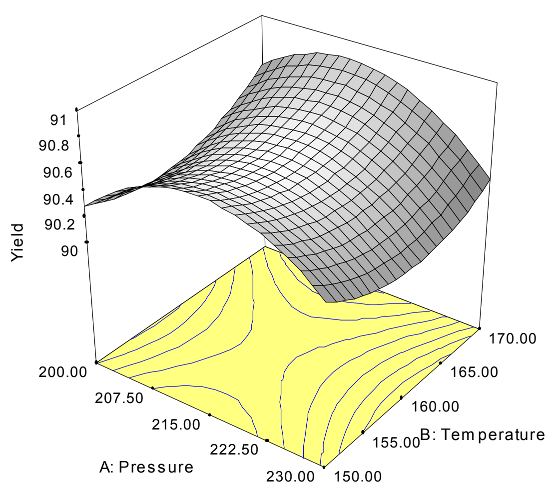
Final Equation in Terms of Coded Factors:

Yield	=
+90.52	
-0.092	* A
+0.075	* B
-0.41	* A^2
+0.24	* B^2
-0.087	* A * B

Final Equation in Terms of Actual Factors:

Yield	=
+48.54630	
+0.86759	* Pressure
-0.64042	* Temperature
-1.81481E-003	* Pressure^2
+2.41667E-003	* Temperature^2
-5.83333E-004	* Pressure * Temperature





5-2 An engineer suspects that the surface finish of a metal part is influenced by the feed rate and the depth of cut. She selects three feed rates and four depths of cut. She then conducts a factorial experiment and obtains the following data:

	Depth of Cut (in)			
Feed Rate (in/min)	0.15	0.18	0.20	0.25
	74	79	82	99
0.20	64	68	88	104
	60	73	92	96
	92	98	99	104

0.25	86	104	108	110
	88	88	95	99
	99	104	108	114
0.30	98	99	110	111
	102	95	99	107

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

The depth (A) and feed rate (B) are significant, as is the interaction (AB).

Design Expert Output

Response: Surface Finish
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

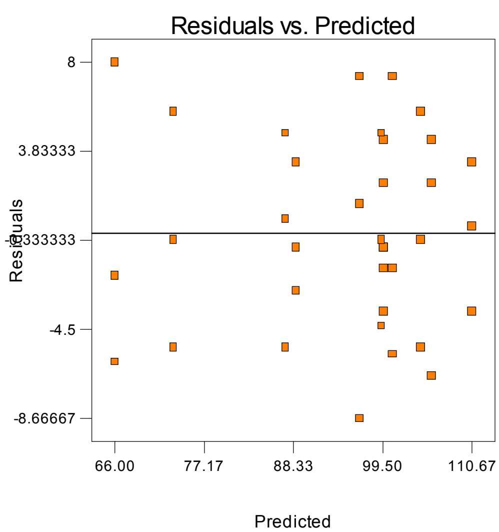
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	5842.67	11	531.15	18.49	< 0.0001	significant
A	*2125.11*	*3*	*708.37*	*24.66*	*< 0.0001*	
B	*3160.50*	*2*	*1580.25*	*55.02*	*< 0.0001*	
AB	*557.06*	*6*	*92.84*	*3.23*	*0.0180*	
Residual	689.33	24	28.72			
Lack of Fit	*0.000*	*0*				
Pure Error	*689.33*	*24*	*28.72*			
Cor Total	6532.00	35				

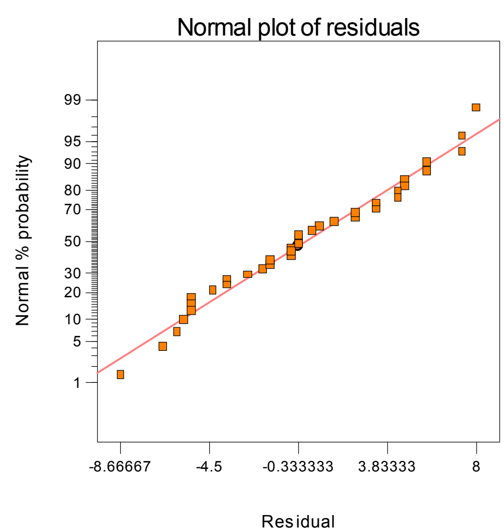
The Model F-value of 18.49 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

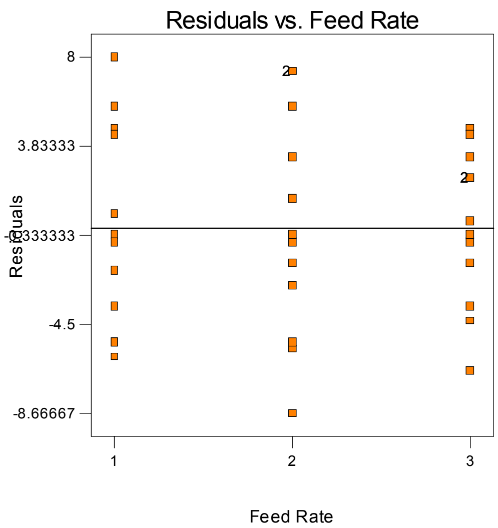
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.

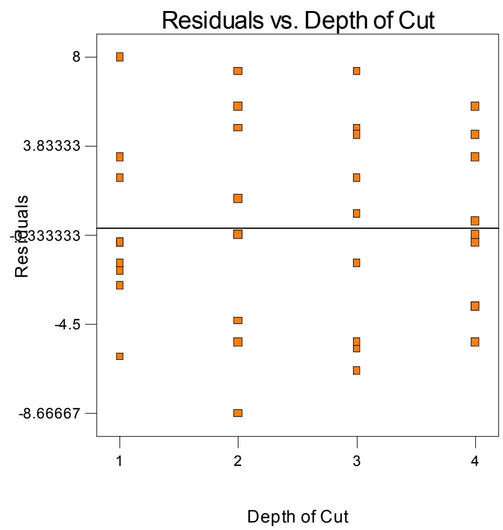
(b) Prepare appropriate residual plots and comment on the model's adequacy.

The residual plots shown indicate nothing unusual.



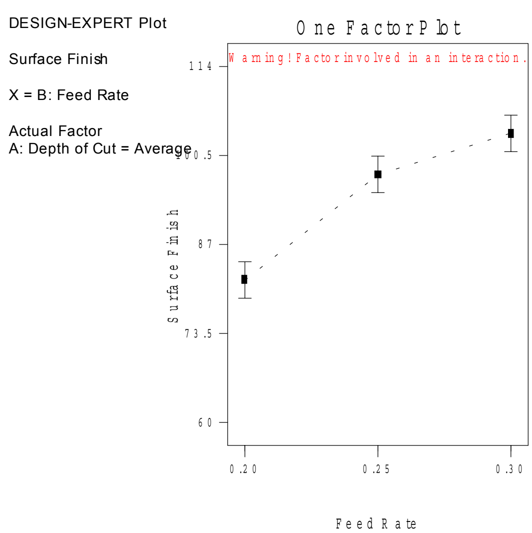






(c) Obtain point estimates of the mean surface finish at each feed rate.

Feed Rate	Average
0.20	81.58
0.25	97.58
0.30	103.83



(d) Find *P*-values for the tests in part (a).

The *P*-values are given in the computer output in part (a).

5-3 For the data in Problem 5-2, compute a 95 percent interval estimate of the mean difference in response for feed rates of 0.20 and 0.25 in/min.

We wish to find a confidence interval on $\mu_1 - \mu_2$, where μ_1 is the mean surface finish for 0.20 in/min and μ_2 is the mean surface finish for 0.25 in/min.

$$\bar{y}_{1..} - \bar{y}_{2..} - t_{\alpha/2,ab(n-1)}\sqrt{\frac{2MS_E}{n}} \le \mu_1 - \mu_2 \le \bar{y}_{1..} - \bar{y}_{2..} + t_{\alpha/2,ab(n-1))}\sqrt{\frac{2MS_E}{n}}$$

$$(81.5833 - 97.5833) \pm (2.064)\sqrt{\frac{2(28.7222)}{3}} = -16 \pm 9.032$$

Therefore, the 95% confidence interval for $\mu_1 - \mu_2$ is -16.000 ± 9.032.

5-4 An article in *Industrial Quality Control* (1956, pp. 5-8) describes an experiment to investigate the effect of the type of glass and the type of phosphor on the brightness of a television tube. The response variable is the current necessary (in microamps) to obtain a specified brightness level. The data are as follows:

Glass	Phosphor Type		
Type	1	2	3
	280	300	290
1	290	310	285
	285	295	290
	230	260	220
2	235	240	225
	240	235	230

(a) Is there any indication that either factor influences brightness? Use $\alpha = 0.05$.

Both factors, phosphor type (*A*) and Glass type (*B*) influence brightness.

Design Expert Output

Response: Current in microamps
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	15516.67	5	3103.33	58.80	< 0.0001	significant
A	*933.33*	*2*	*466.67*	*8.84*	*0.0044*	
B	*14450.00*	*1*	*14450.00*	*273.79*	*< 0.0001*	
AB	*133.33*	*2*	*66.67*	*1.26*	*0.3178*	
Residual	633.33	12	52.78			
Lack of Fit	*0.000*	*0*				
Pure Error	*633.33*	*12*	*52.78*			
Cor Total	16150.00	17				

The Model F-value of 58.80 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

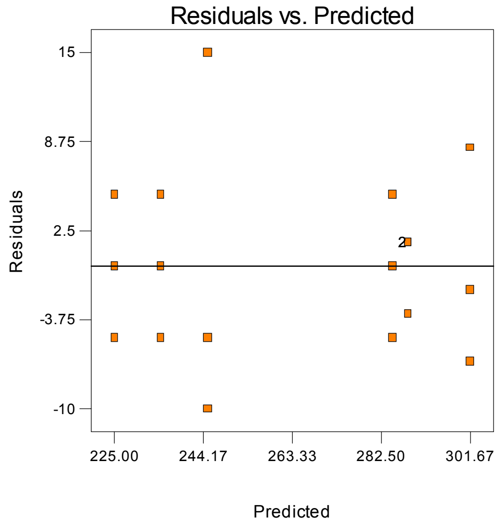
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

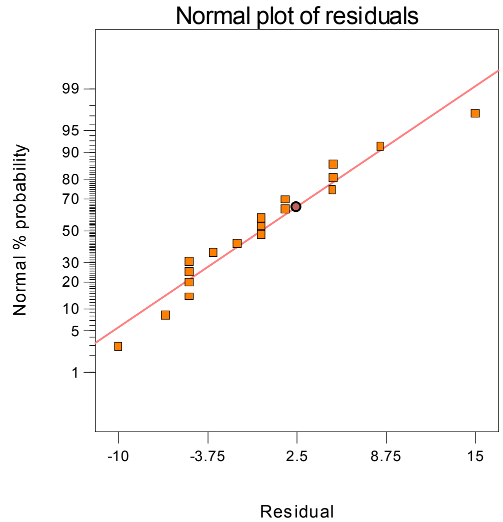
(b) Do the two factors interact? Use $\alpha = 0.05$.

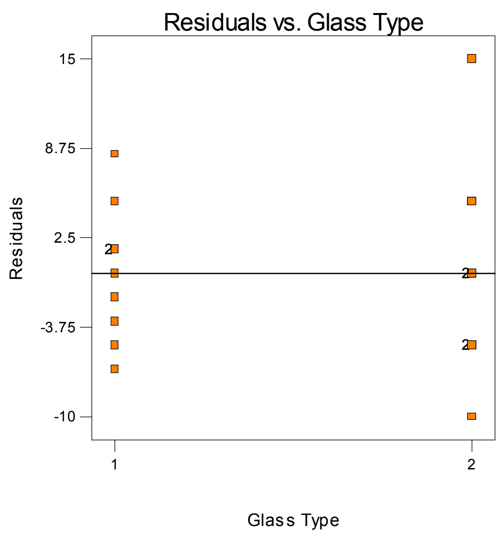
There is no interaction effect.

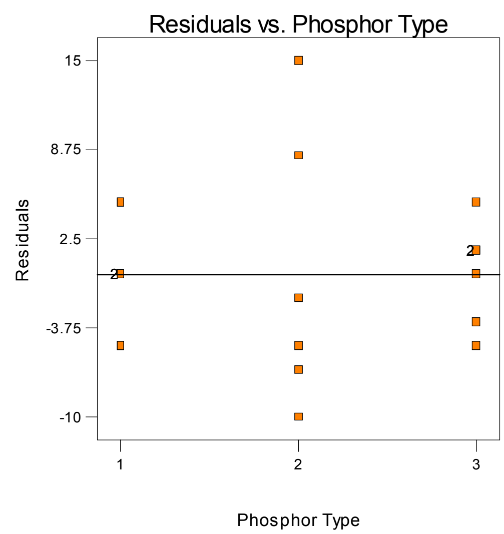
(c) Analyze the residuals from this experiment.

The residual plot of residuals versus phosphor content indicates a very slight inequality of variance. It is not serious enough to be of concern, however.









5-5 Johnson and Leone (*Statistics and Experimental Design in Engineering and the Physical Sciences*, Wiley 1977) describe an experiment to investigate the warping of copper plates. The two factors studies were the temperature and the copper content of the plates. The response variable was a measure of the amount of warping. The data were as follows:

	Copper Content (%)			
Temperature (°C)	40	60	80	100
50	17,20	16,21	24,22	28,27
75	12,9	18,13	17,12	27,31
100	16,12	18,21	25,23	30,23
125	21,17	23,21	23,22	29,31

(a) Is there any indication that either factor affects the amount of warping? Is there any interaction between the factors? Use $\alpha = 0.05$.

Both factors, copper content (A) and temperature (B) affect warping, the interaction does not.

Design Expert Output

Response: Warping
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

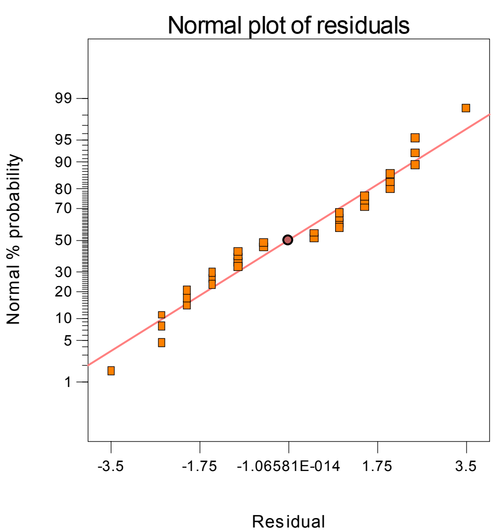
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	968.22	15	64.55	9.52	< 0.0001	significant
A	*698.34*	*3*	*232.78*	*34.33*	*< 0.0001*	
B	*156.09*	*3*	*52.03*	*7.67*	*0.0021*	
AB	*113.78*	*9*	*12.64*	*1.86*	*0.1327*	
Residual	108.50	16	6.78			
Lack of Fit	*0.000*	*0*				
Pure Error	*108.50*	*16*	*6.78*			
Cor Total	1076.72	31				

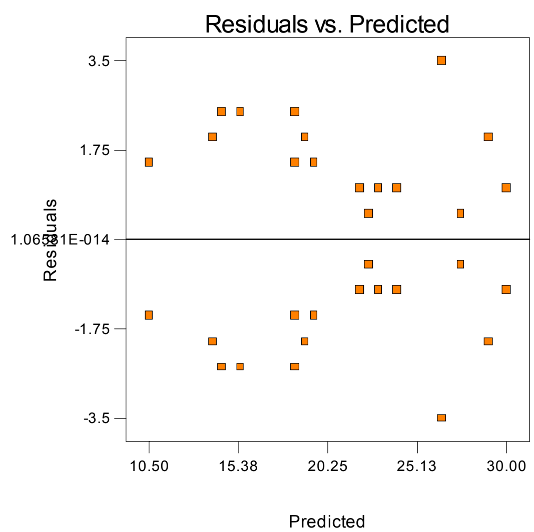
The Model F-value of 9.52 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

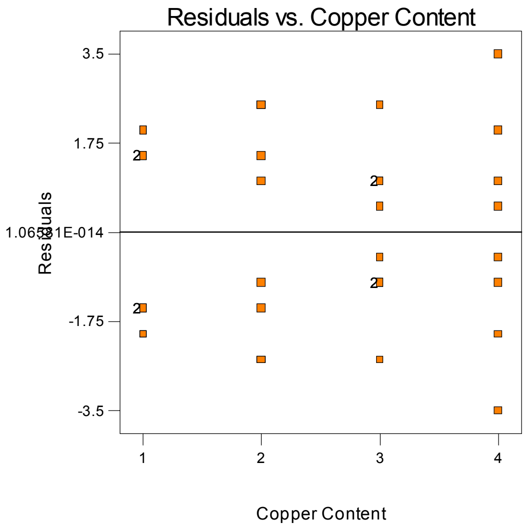
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

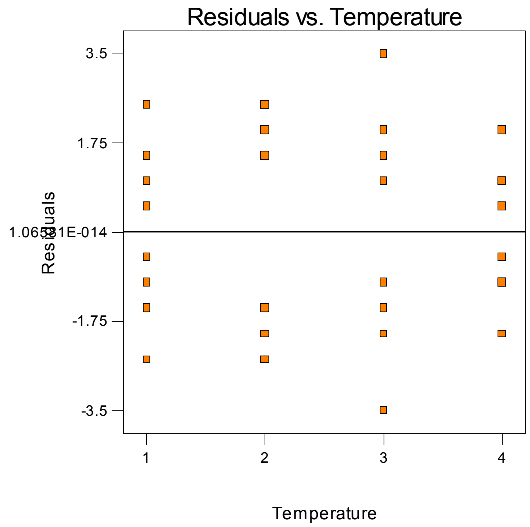
(b) Analyze the residuals from this experiment.

There is nothing unusual about the residual plots.









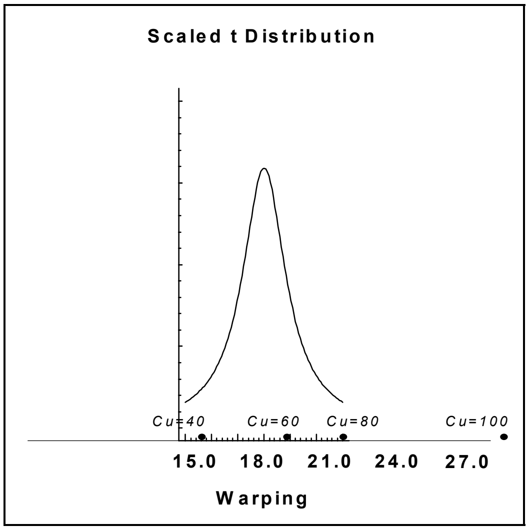
(c) Plot the average warping at each level of copper content and compare them to an appropriately scaled *t* distribution. Describe the differences in the effects of the different levels of copper content on warping. If low warping is desirable, what level of copper content would you specify?

Design Expert Output

Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**40**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	15.50	1.84	11.60	19.40	3.19	8.74	22.26
Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**60**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	18.88	1.84	14.97	22.78	3.19	12.11	25.64
Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**80**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	21.00	1.84	17.10	24.90	3.19	14.24	27.76
Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**100**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	28.25	1.84	24.35	32.15	3.19	21.49	35.01

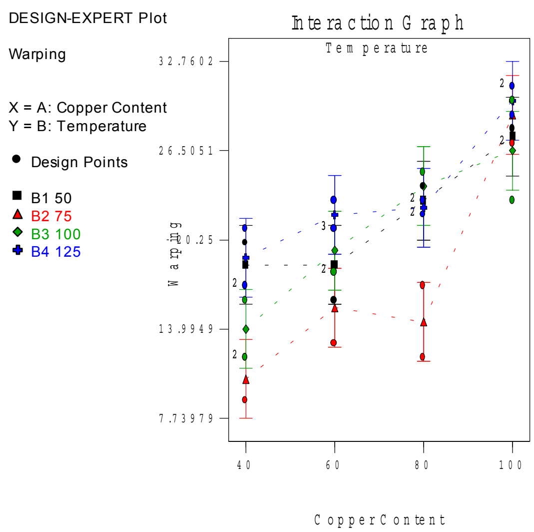
Use a copper content of 40 for the lowest warping.

$$S = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{6.78125}{4}} = 1.3$$



(d) Suppose that temperature cannot be easily controlled in the environment in which the copper plates are to be used. Does this change your answer for part (c)?

Use a copper of content of 40. This is the same as for part (c).



5-6 The factors that influence the breaking strength of a synthetic fiber are being studied. Four production machines and three operators are chosen and a factorial experiment is run using fiber from the same production batch. The results are as follows:

	Machine			
Operator	1	2	3	4
1	109	110	108	110
	110	115	109	108
2	110	110	111	114
	112	111	109	112
3	116	112	114	120

114	115	119	117

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

Only the Operator (A) effect is significant.

Design Expert Output

Response:Stength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

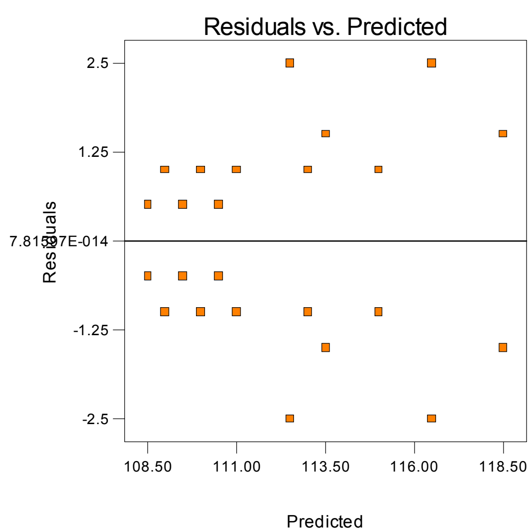
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	217.46	11	19.77	5.21	0.0041	significant
A	*160.33*	*2*	*80.17*	*21.14*	*0.0001*	
B	*12.46*	*3*	*4.15*	*1.10*	*0.3888*	
AB	*44.67*	*6*	*7.44*	*1.96*	*0.1507*	
Residual	45.50	12	3.79			
Lack of Fit	*0.000*	*0*				
Pure Error	*45.50*	*12*	*3.79*			
Cor Total	262.96	23				

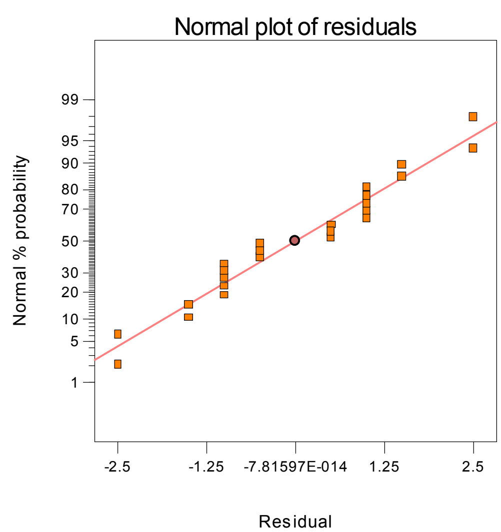
The Model F-value of 5.21 implies the model is significant.
There is only a 0.41% chance that a "Model F-Value" this large could occur due to noise.

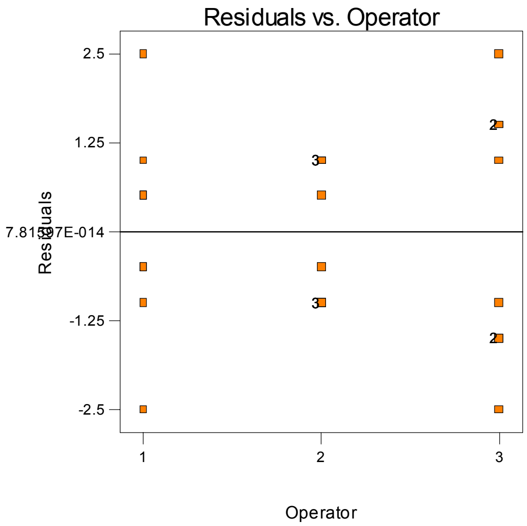
Values of "Prob > F" less than 0.0500 indicate model terms aresignificant.
In this case A are significant model terms.

(b) Prepare appropriate residual plots and comment on the model's adequacy.

The residual plot of residuals versus predicted shows that variance increases very slightly with strength. There is no indication of a severe problem.







5-7 A mechanical engineer is studying the thrust force developed by a drill press. He suspects that the drilling speed and the feed rate of the material are the most important factors. He selects four feed rates and uses a high and low drill speed chosen to represent the extreme operating conditions. He obtains the following results. Analyze the data and draw conclusions. Use $\alpha = 0.05$.

(A)		Feed	Rate (B)	
Drill Speed	0.015	0.030	0.045	0.060
125	2.70	2.45	2.60	2.75
	2.78	2.49	2.72	2.86
200	2.83	2.85	2.86	2.94
	2.86	2.80	2.87	2.88

Design Expert Output

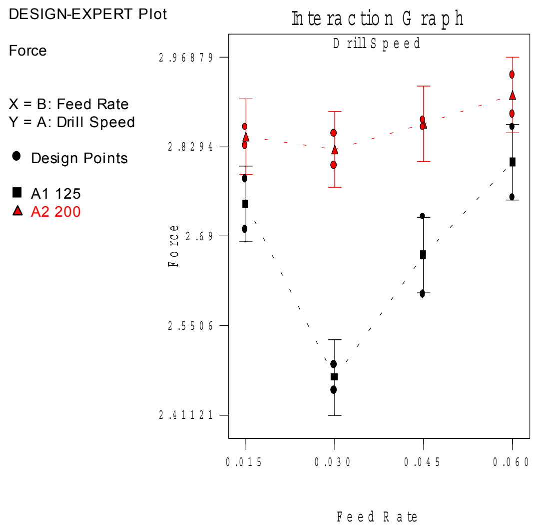
Response: Force
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.28	7	0.040	15.53	0.0005	significant
A	*0.15*	*1*	*0.15*	*57.01*	*< 0.0001*	
B	*0.092*	*3*	*0.031*	*11.86*	*0.0026*	
AB	*0.042*	*3*	*0.014*	*5.37*	*0.0256*	
Residual	0.021	8	2.600E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.021*	*8*	*2.600E-003*			
Cor Total	0.30	15				

The Model F-value of 15.53 implies the model is significant.
There is only a 0.05% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

The factors speed and feed rate, as well as the interaction is important.



The standard analysis of variance treats all design factors as if they were qualitative. In this case, both factors are quantitative, so some further analysis can be performed. In Section 5-5, we show how response curves and surfaces can be fit to the data from a factorial experiment with at least one quantative factor. Since both factors in this problem are quantitative and have three levels, we can fit linear and quadratic effects of both temperature and pressure, exactly as in Example 5-5 in the text. The Design-Expert output, including the response surface plots, now follows.

Design Expert Output

Response: Force
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.23	4	0.057	8.05	0.0027	significant
A	*0.15*	*1*	*0.15*	*21.11*	*0.0008*	
B	*0.019*	*1*	*0.019*	*2.74*	*0.1262*	
B2	*0.058*	*1*	*0.058*	*8.20*	*0.0154*	
AB	*1.125E-003*	*1*	*1.125E-003*	*0.16*	*0.6966*	
Residual	0.077	11	7.021E-003			
Lack of Fit	*0.056*	*3*	*0.019*	*7.23*	*0.0115*	*significant*
Pure Error	*0.021*	*8*	*2.600E-003*			
Cor Total	0.30	15				

The Model F-value of 8.05 implies the model is significant. There is only
a 0.27% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy),
model reduction may improve your model.

Std. Dev.	0.084	R-Squared	0.7455
Mean	2.77	Adj R-Squared	0.6529
C.V.	3.03	Pred R-Squared	0.4651
PRESS	0.16	Adeq Precision	7.835

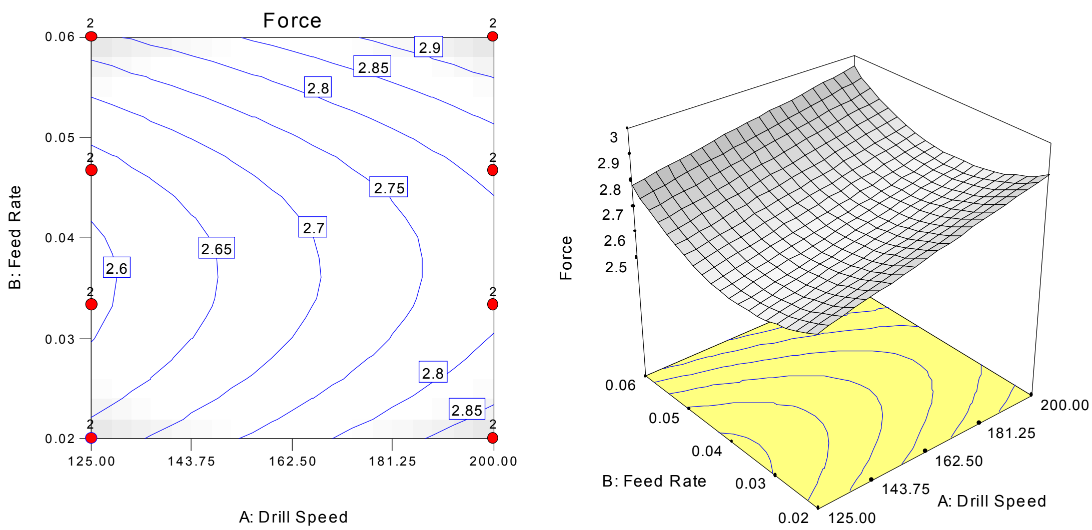
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	2.69	1	0.034	2.62	2.76	
A-Drill Speed	0.096	1	0.021	0.050	0.14	1.00
B-Feed Rate	0.047	1	0.028	-0.015	0.11	1.00
B2	0.13	1	0.047	0.031	0.24	1.00
AB	-0.011	1	0.028	-0.073	0.051	1.00

Final Equation in Terms of Coded Factors:

Force	=
+2.69	
+0.096	* A
+0.047	* B
+0.13	* B^2
-0.011	* A * B

Final Equation in Terms of Actual Factors:

Force	=
+2.48917	
+3.06667E-003	* Drill Speed
-15.76667	* Feed Rate
+266.66667	* Feed Rate^2
-0.013333	* Drill Speed * Feed Rate



5-8 An experiment is conducted to study the influence of operating temperature and three types of face-plate glass in the light output of an oscilloscope tube. The following data are collected:

	Temperature		
Glass Type	100	125	150
	580	1090	1392
1	568	1087	1380
	570	1085	1386
	550	1070	1328
2	530	1035	1312
	579	1000	1299
	546	1045	867
3	575	1053	904
	599	1066	889

Use α = 0.05 in the analysis. Is there a significant interaction effect? Does glass type or temperature affect the response? What conclusions can you draw? Use the method discussed in the text to partition the temperature effect into its linear and quadratic components. Break the interaction down into appropriate components.

Design Expert Output

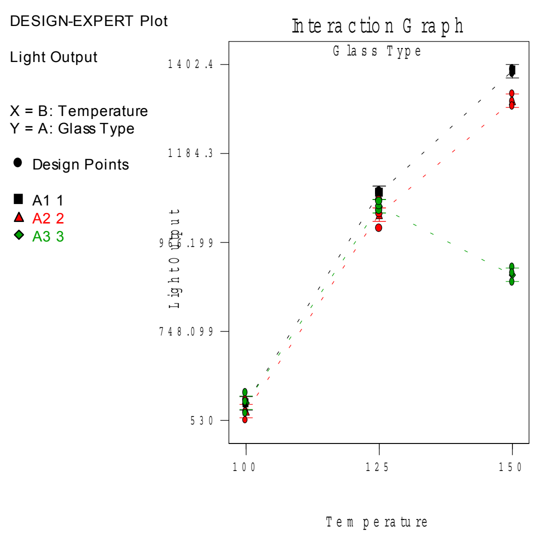
Response: Light Output
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.412E+006	8	3.015E+005	824.77	< 0.0001	significant
A	*1.509E+005*	*2*	*75432.26*	*206.37*	*< 0.0001*	
B	*1.970E+006*	*2*	*9.852E+005*	*2695.26*	*< 0.0001*	
AB	*2.906E+005*	*4*	*72637.93*	*198.73*	*< 0.0001*	
Residual	6579.33	18	365.52			
Lack of Fit	*0.000*	*0*				
Pure Error	*6579.33*	*18*	*365.52*			
Cor Total	2.418E+006	26				

The Model F-value of 824.77 implies the model is significant.
There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

Both factors, Glass Type (*A*) and Temperature (*B*) are significant, as well as the interaction (*AB*). For glass types 1 and 2 the response is fairly linear, for glass type 3, there is a quadratic effect.



Design Expert Output

Response: Light Output
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.412E+006	8	3.015E+005	824.77	< 0.0001	significant
A	*1.509E+005*	*2*	*75432.26*	*206.37*	*< 0.0001*	
B	*1.780E+006*	*1*	*1.780E+006*	*4869.13*	*< 0.0001*	
B2	*1.906E+005*	*1*	*1.906E+005*	*521.39*	*< 0.0001*	
AB	*2.262E+005*	*2*	*1.131E+005*	*309.39*	*< 0.0001*	
AB2	*64373.93*	*2*	*32186.96*	*88.06*	*< 0.0001*	
Pure Error	*6579.33*	*18*	*365.52*			

Cor Total 2.418E+006 26

The Model F-value of 824.77 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, B^2, AB, AB^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy),
model reduction may improve your model.

Std. Dev.	19.12	R-Squared	0.9973
Mean	940.19	Adj R-Squared	0.9961
C.V.	2.03	Pred R-Squared	0.9939
PRESS	14803.50	Adeq Precision	75.466

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	1059.00	1	6.37	1045.61	1072.39	
A[1]	28.33	1	9.01	9.40	47.27	
A[2]	-24.00	1	9.01	-42.93	-5.07	
B-Temperature	314.44	1	4.51	304.98	323.91	1.00
B2	-178.22	1	7.81	-194.62	-161.82	1.00
A[1]B	92.22	1	6.37	78.83	105.61	
A[2]B	65.56	1	6.37	52.17	78.94	
A[1]B2	70.22	1	11.04	47.03	93.41	
A[2]B2	76.22	1	11.04	53.03	99.41	

Final Equation in Terms of Coded Factors:

Light Output	=
+1059.00	
+28.33	* A[1]
-24.00	* A[2]
+314.44	* B
-178.22	* B2
+92.22	* A[1]B
+65.56	* A[2]B
+70.22	* A[1]B2
+76.22	* A[2]B2

Final Equation in Terms of Actual Factors:

Glass Type	1
Light Output	=
-3646.00000	
+59.46667	* Temperature
-0.17280	* Temperature2

Glass Type	2
Light Output	=
-3415.00000	
+56.00000	* Temperature
-0.16320	* Temperature2

Glass Type	3
Light Output	=
-7845.33333	
+136.13333	* Temperature
-0.51947	* Temperature2

5-9 Consider the data in Problem 5-1. Use the method described in the text to compute the linear and quadratic effects of pressure.

See the alternative analysis shown in Problem 5-1 part (c).

5-10 Use Duncan's multiple range test to determine which levels of the pressure factor are significantly different for the data in Problem 5-1.

$$\bar{y}_{.3.} = 90.18 \qquad \bar{y}_{.1.} = 90.37 \qquad \bar{y}_{.2.} = 90.68$$

$$S_{\bar{y}_{.j.}} = \sqrt{\frac{MS_E}{an}} = \sqrt{\frac{0.01777}{(3)(2)}} = 0.0543$$

$$r_{0.01}(2,9) = 4.60 \qquad r_{0.01}(3,9) = 4.86$$

$$R_2 = (4.60)(0.0543) = 0.2498 \qquad R_3 = (4.86)(0.0543) = 0.2640$$

2 vs. 3 = 0.50 > 0.2640 (R_3)
2 vs. 1 = 0.31 > 0.2498 (R_2)
1 vs. 3 = 0.19 < 0.2498 (R_2)
Therefore, 2 differs from 1 and 3.

5-11 An experiment was conducted to determine if either firing temperature or furnace position affects the baked density of a carbon anode. The data are shown below.

	Temperature (°C)		
Position	800	825	850
	570	1063	565
1	565	1080	510
	583	1043	590
	528	988	526
2	547	1026	538
	521	1004	532

Suppose we assume that no interaction exists. Write down the statistical model. Conduct the analysis of variance and test hypotheses on the main effects. What conclusions can be drawn? Comment on the model's adequacy.

The model for the two-factor, no interaction model is $y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}$. Both factors, furnace position (A) and temperature (B) are significant. The residual plots show nothing unusual.

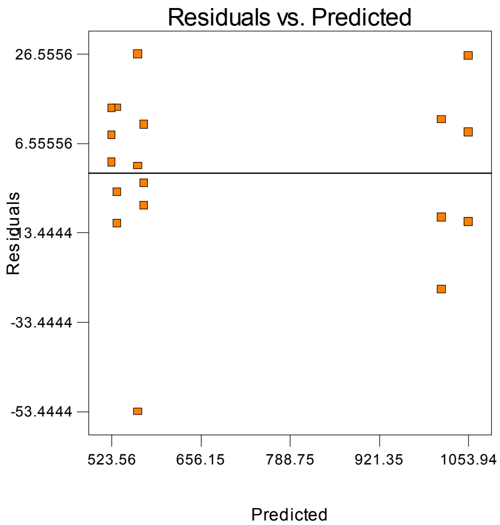
Design Expert Output

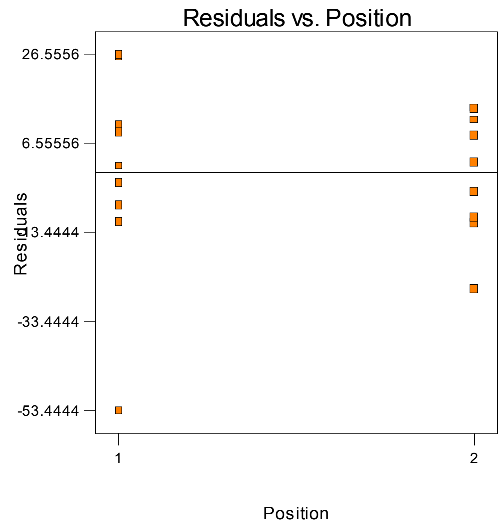
Response: Density
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

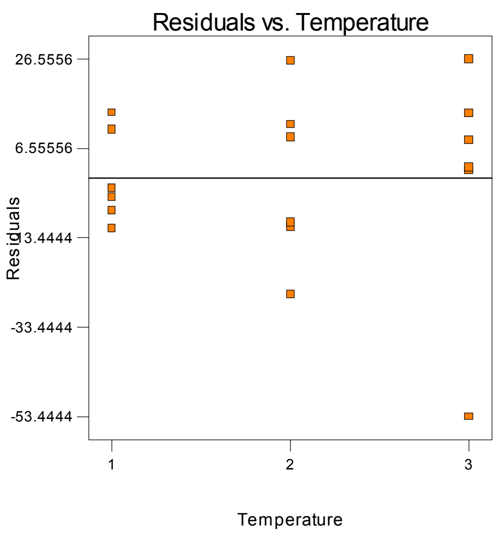
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	9.525E+005	3	3.175E+005	718.24	< 0.0001	significant
A	*7160.06*	*1*	*7160.06*	*16.20*	*0.0013*	
B	*9.453E+005*	*2*	*4.727E+005*	*1069.26*	*< 0.0001*	
Residual	6188.78	14	442.06			
Lack of Fit	*818.11*	*2*	*409.06*	*0.91*	*0.4271*	*not significant*
Pure Error	*5370.67*	*12*	*447.56*			
Cor Total	9.587E+005	17				

The Model F-value of 718.24 implies the model is significant.
There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.







5-12 Derive the expected mean squares for a two-factor analysis of variance with one observation per cell, assuming that both factors are fixed.

	Degrees of Freedom
$E(MS_A)=\sigma^2+b\sum_{i=1}^{a}\frac{\tau_i^2}{(a-1)}$	a-1
$E(MS_B)=\sigma^2+a\sum_{j=1}^{b}\frac{\beta_j^2}{(b-1)}$	b-1
$E(MS_{AB})=\sigma^2+\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{(a-1)(b-1)}{ab-1}$

5-13 Consider the following data from a two-factor factorial experiment. Analyze the data and draw conclusions. Perform a test for nonadditivity. Use $\alpha = 0.05$.

	Column	Factor		
Row Factor	1	2	3	4
1	36	39	36	32
2	18	20	22	20
3	30	37	33	34

Design Expert Output

Response: **data**
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	609.42	5	121.88	25.36	0.0006	significant
A	*580.50*	*2*	*290.25*	*60.40*	*0.0001*	
B	*28.92*	*3*	*9.64*	*2.01*	*0.2147*	
Residual	28.83	6	4.81			
Cor Total	638.25	11				

The Model F-value of 25.36 implies the model is significant. There is only a 0.06% chance that a "Model F-Value" this large could occur due to noise.

The row factor (A) is significant.

The test for nonadditivity is as follows:

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_ASS_B}$$

$$SS_N = \frac{\left[4010014 - (357)\left(580.50 + 28.91667 + \frac{357^2}{(4)(3)}\right)\right]^2}{(4)(3)(580.50)(28.91667)}$$

$$SS_N = 3.54051$$

$$SS_{Error} = SS_{Residual} - SS_N = 28.8333 - 3.54051 = 25.29279$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Row	580.50	2	290.25	57.3780
Column	28.91667	3	9.63889	1.9054
Nonadditivity	3.54051	1	3.54051	0.6999
Error	25.29279	5	5.058558	
Total	638.25	11		

5-14 The shear strength of an adhesive is thought to be affected by the application pressure and temperature. A factorial experiment is performed in which both factors are assumed to be fixed. Analyze the data and draw conclusions. Perform a test for nonadditivity.

	Temperature (°F)

Pressure (lb/in2)	250	260	270
120	9.60	11.28	9.00
130	9.69	10.10	9.57
140	8.43	11.01	9.03
150	9.98	10.44	9.80

Design Expert Output

Response: Strength

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	5.24	5	1.05	2.92	0.1124	not significant
A	*0.58*	*3*	*0.19*	*0.54*	*0.6727*	
B	*4.66*	*2*	*2.33*	*6.49*	*0.0316*	
Residual	2.15	6	0.36			
Cor Total	7.39	11				

The "Model F-value" of 2.92 implies the model is not significant relative to the noise.
There is a 11.24 % chance that a "Model F-value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B are significant model terms.

Temperature (B) is a significant factor.

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_ASS_B}$$

$$SS_N = \frac{\left[415113.777 - (117.93)\left(0.5806917 + 4.65765 + \frac{117.93^2}{(4)(3)}\right)\right]^2}{(4)(3)(0.5806917)(4.65765)}$$

$$SS_N = 0.48948$$

$$SS_{Error} = SS_{Re\,sidual} - SS_N = 2.1538833 - 0.48948 = 1.66440$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Row	0.5806917	3	0.1935639	0.5815
Column	4.65765	2	2.328825	6.9960
Nonadditivity	0.48948	1	0.48948	1.4704
Error	1.6644	5	0.33288	
Total	7.392225	11		

5-15 Consider the three-factor model

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \varepsilon_{ijk} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,c \end{cases}$$

Notice that there is only one replicate. Assuming the factors are fixed, write down the analysis of variance table, including the expected mean squares. What would you use as the "experimental error" in order to test hypotheses?

Source	Degrees of Freedom	Expected Mean Square
A	a-1	$\sigma^2 + bc\sum_{i=1}^{a}\frac{\tau_i^2}{(a-1)}$
B	b-1	$\sigma^2 + ac\sum_{j=1}^{b}\frac{\beta_j^2}{(b-1)}$
C	c-1	$\sigma^2 + ab\sum_{k=1}^{c}\frac{\gamma_k^2}{(c-1)}$
AB	(a-1)(b-1)	$\sigma^2 + c\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\tau(\beta)_{ij}^2}{(a-1)(b-1)}$
BC	(b-1)(c-1)	$\sigma^2 + a\sum_{j=1}^{b}\sum_{k=1}^{c}\frac{(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
Error (AC + ABC)	b(a-1)(c-1)	σ^2
Total	abc-1	

5-16 The percentage of hardwood concentration in raw pulp, the vat pressure, and the cooking time of the pulp are being investigated for their effects on the strength of paper. Three levels of hardwood concentration, three levels of pressure, and two cooking times are selected. A factorial experiment with two replicates is conducted, and the following data are obtained:

Percentage of Hardwood	Cooking Time 3.0 Hours Pressure			Cooking Time 4.0 Hours Pressure		
Concentration	400	500	650	400	500	650
2	196.6	197.7	199.8	198.4	199.6	200.6
	196.0	196.0	199.4	198.6	200.4	200.9
4	198.5	196.0	198.4	197.5	198.7	199.6
	197.2	196.9	197.6	198.1	198.0	199.0
8	197.5	195.6	197.4	197.6	197.0	198.5
	196.6	196.2	198.1	198.4	197.8	199.8

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

Design Expert Output

Response: strength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	59.73	17	3.51	9.61	< 0.0001	significant
A	*7.76*	*2*	*3.88*	*10.62*	*0.0009*	
B	*20.25*	*1*	*20.25*	*55.40*	*< 0.0001*	
C	*19.37*	*2*	*9.69*	*26.50*	*< 0.0001*	
AB	*2.08*	*2*	*1.04*	*2.85*	*0.0843*	
AC	*6.09*	*4*	*1.52*	*4.17*	*0.0146*	

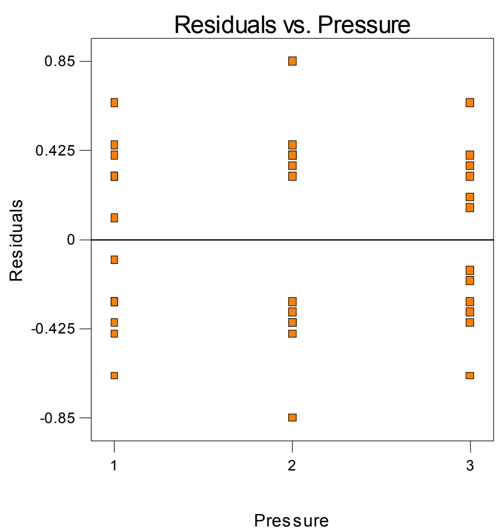
BC	*2.19*	*2*	*1.10*	*3.00*	*0.0750*
ABC	*1.97*	*4*	*0.49*	*1.35*	*0.2903*
Residual	6.58	18	0.37		
Lack of Fit	*0.000*	*0*			
Pure Error	*6.58*	*18*	*0.37*		
Cor Total	66.31	35			

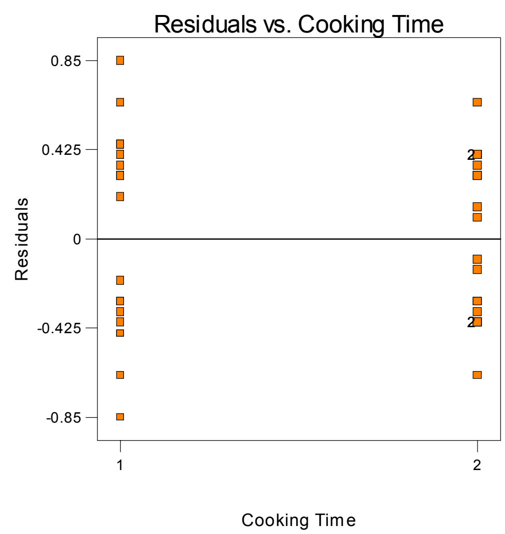
The Model F-value of 9.61 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

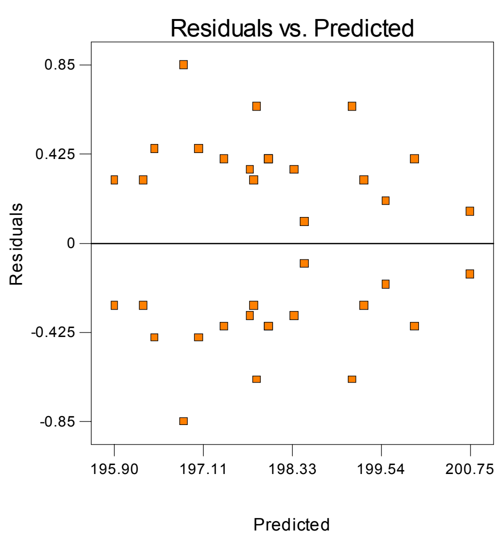
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AC are significant model terms.

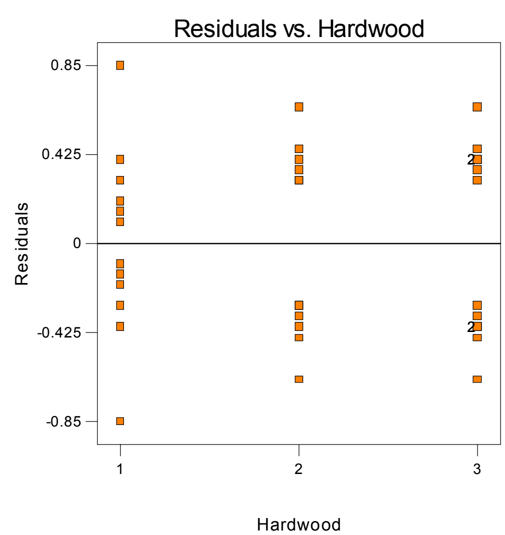
All three main effects, concentration (*A*), pressure (*C*) and time (*B*), as well as the concentration x pressure interaction (*AC*) are significant at the 5% level. The concentration x time (*AB*) and pressure x time interactions (*BC*) are significant at the 10% level.

(b) Prepare appropriate residual plots and comment on the model's adequacy.



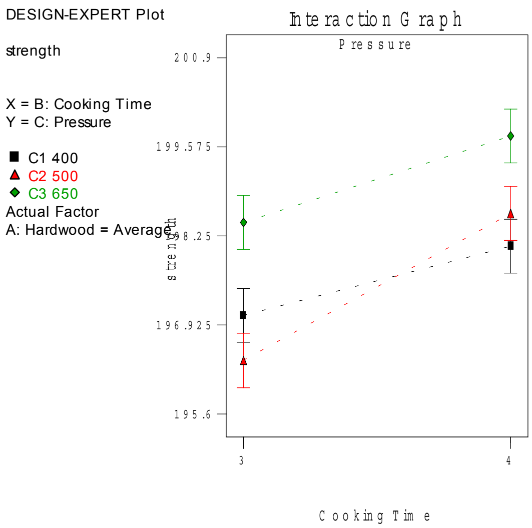


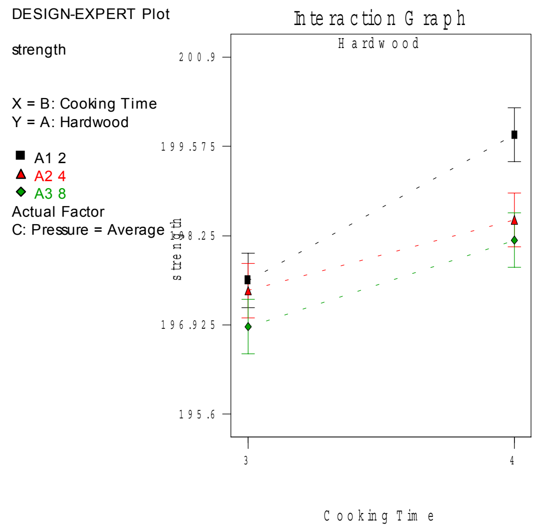


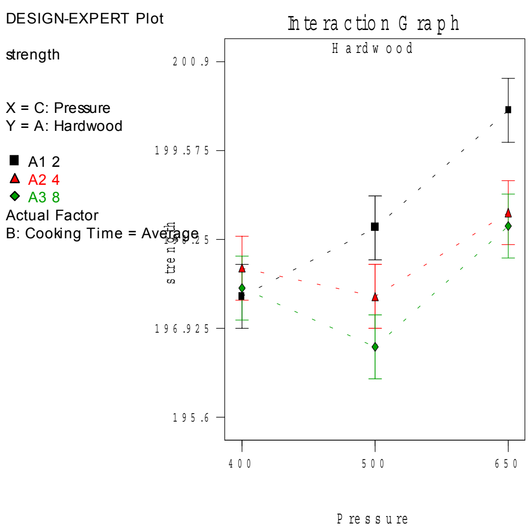


There is nothing unusual about the residual plots.

(c) Under what set of conditions would you run the process? Why?







For the highest strength, run the process with the percentage of hardwood at 2, the pressure at 650, and the time at 4 hours.

The standard analysis of variance treats all design factors as if they were qualitative. In this case, all three factors are quantitative, so some further analysis can be performed. In Section 5-5, we show how response curves and surfaces can be fit to the data from a factorial experiment with at least one quantative factor. Since the factors in this problem are quantitative and two of them have three levels, we can fit linear and quadratic. The Design-Expert output, including the response surface plots, now follows.

Design Expert Output

Response: Strength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	58.02	13	4.46	11.85	< 0.0001	significant

A	*7.15*	*1*	*7.15*	*18.98*	*0.0003*	
B	*3.42*	*1*	*3.42*	*9.08*	*0.0064*	
C	*0.22*	*1*	*0.22*	*0.58*	*0.4559*	
A2	*1.09*	*1*	*1.09*	*2.88*	*0.1036*	
C2	*4.43*	*1*	*4.43*	*11.77*	*0.0024*	
AB	*1.06*	*1*	*1.06*	*2.81*	*0.1081*	
AC	*3.39*	*1*	*3.39*	*9.01*	*0.0066*	
BC	*0.15*	*1*	*0.15*	*0.40*	*0.5350*	
A2B	*1.30*	*1*	*1.30*	*3.46*	*0.0763*	
A2C	*2.19*	*1*	*2.19*	*5.81*	*0.0247*	
AC2	*1.65*	*1*	*1.65*	*4.38*	*0.0482*	
BC2	*2.18*	*1*	*2.18*	*5.78*	*0.0251*	
ABC	*0.40*	*1*	*0.40*	*1.06*	*0.3136*	
Residual	8.29	22	0.38			
Lack of Fit	*1.71*	*4*	*0.43*	*1.17*	*0.3576*	*not significant*
Pure Error	*6.58*	*18*	*0.37*			
Cor Total	66.31	35				

The Model F-value of 11.85 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C^2, AC, A^2C, AC^2, BC^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

Std. Dev.	0.61	R-Squared	0.8750
Mean	198.06	Adj R-Squared	0.8011
C.V.	0.31	Pred R-Squared	0.6657
PRESS	22.17	Adeq Precision	14.071

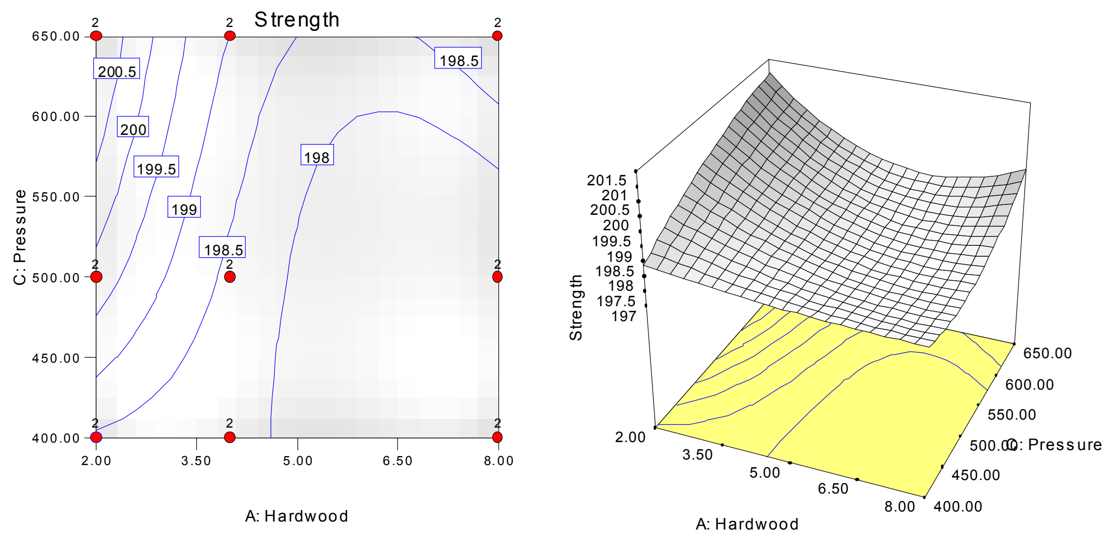
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	197.21	1	0.26	196.67	197.74	
A-Hardwood	-0.98	1	0.23	-1.45	-0.51	3.36
B-Cooking Time	0.78	1	0.26	0.24	1.31	6.35
C-Pressure	0.19	1	0.25	-0.33	0.71	4.04
A2	0.42	1	0.25	-0.094	0.94	1.04
C2	0.79	1	0.23	0.31	1.26	1.03
AB	-0.21	1	0.13	-0.47	0.050	1.06
AC	-0.46	1	0.15	-0.78	-0.14	1.08
BC	0.080	1	0.13	-0.18	0.34	1.04
A2B	0.46	1	0.25	-0.053	0.98	3.96
A2C	0.73	1	0.30	0.10	1.36	3.97
AC2	0.57	1	0.27	4.979E-003	1.14	3.32
BC2	-0.55	1	0.23	-1.02	-0.075	3.30
ABC	0.15	1	0.15	-0.16	0.46	1.02

Final Equation in Terms of Coded Factors:

Strength =
+197.21
-0.98 * A
+0.78 * B
+0.19 * C
+0.42 * A2
+0.79 * C2
-0.21 * A * B
-0.46 * A * C
+0.080 * B * C
+0.46 * A2 * B
+0.73 * A2 * C
+0.57 * A * C2
-0.55 * B * C2
+0.15 * A * B * C

Final Equation in Terms of Actual Factors:

Strength	=
+229.96981	
+12.21654	* Hardwood
-12.97602	* Cooking Time
-0.21224	* Pressure
-0.65287	* Hardwood2
+2.34333E-004	* Pressure2
-1.60038	* Hardwood * Cooking Time
-0.023415	* Hardwood * Pressure
+0.070658	* Cooking Time * Pressure
+0.10278	* Hardwood2 * Cooking Time
+6.48026E-004	* Hardwood2 * Pressure
+1.22143E-005	* Hardwood * Pressure2
-7.00000E-005	* Cooking Time * Pressure2
+8.23308E-004	* Hardwood * Cooking Time * Pressure



Cooking Time: B = 4.00

5-17 The quality control department of a fabric finishing plant is studying the effect of several factors on the dyeing of cotton-synthetic cloth used to manufacture men's shirts. Three operators, three cycle times, and two temperatures were selected, and three small specimens of cloth were dyed under each set of conditions. The finished cloth was compared to a standard, and a numerical score was assigned. The results follow. Analyze the data and draw conclusions. Comment on the model's adequacy.

	Temperature					
	300°			350°		
	Operator			Operator		
Cycle Time	1	2	3	1	2	3
	23	27	31	24	38	34
40	24	28	32	23	36	36
	25	26	29	28	35	39
	36	34	33	37	34	34
50	35	38	34	39	38	36
	36	39	35	35	36	31
	28	35	26	26	36	28
60	24	35	27	29	37	26

	27	34	25	25	34	24

All three main effects, and the *AB, AC*, and *ABC* interactions are significant. There is nothing unusual about the residual plots.

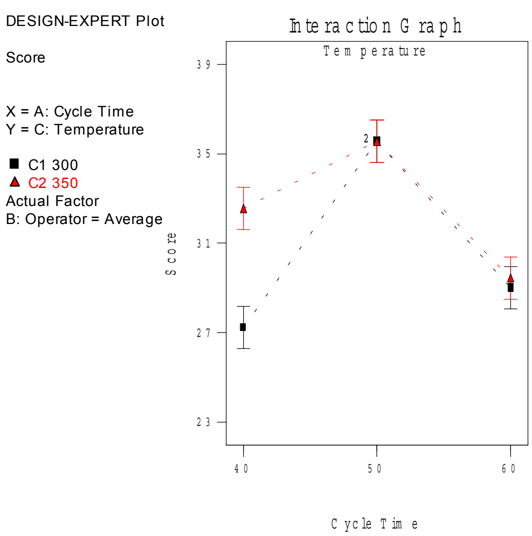
Design Expert Output

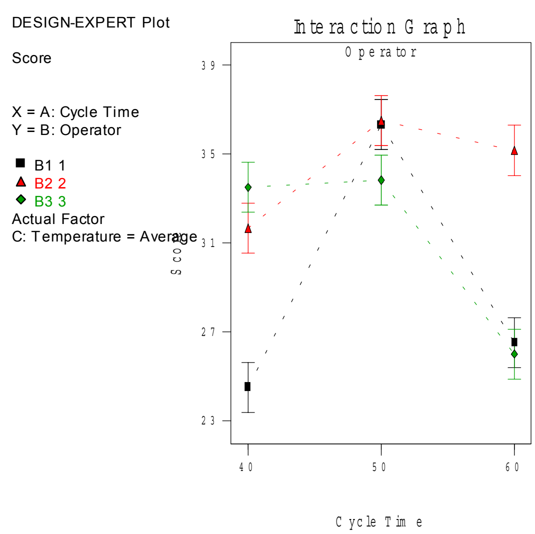
Response: Score
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

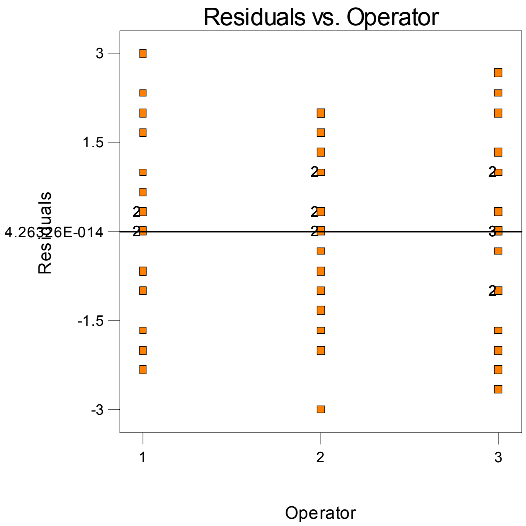
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1239.33	17	72.90	22.24	< 0.0001	significant
A	*436.00*	*2*	*218.00*	*66.51*	*< 0.0001*	
B	*261.33*	*2*	*130.67*	*39.86*	*< 0.0001*	
C	*50.07*	*1*	*50.07*	*15.28*	*0.0004*	
AB	*355.67*	*4*	*88.92*	*27.13*	*< 0.0001*	
AC	*78.81*	*2*	*39.41*	*12.02*	*0.0001*	
BC	*11.26*	*2*	*5.63*	*1.72*	*0.1939*	
ABC	*46.19*	*4*	*11.55*	*3.52*	*0.0159*	
Residual	118.00	36	3.28			
Lack of Fit	*0.000*	*0*				
Pure Error	*118.00*	*36*	*3.28*			
Cor Total	1357.33	53				

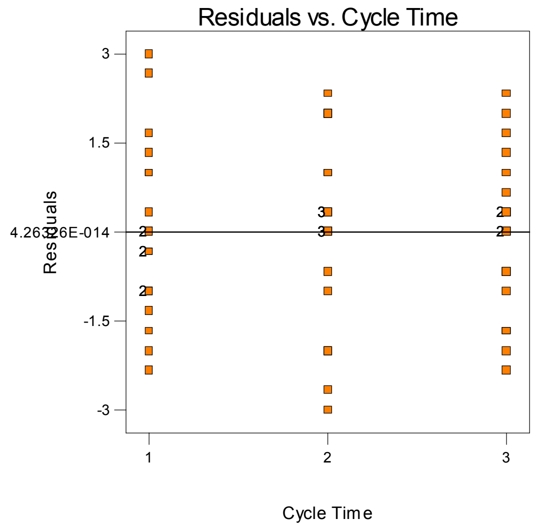
The Model F-value of 22.24 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

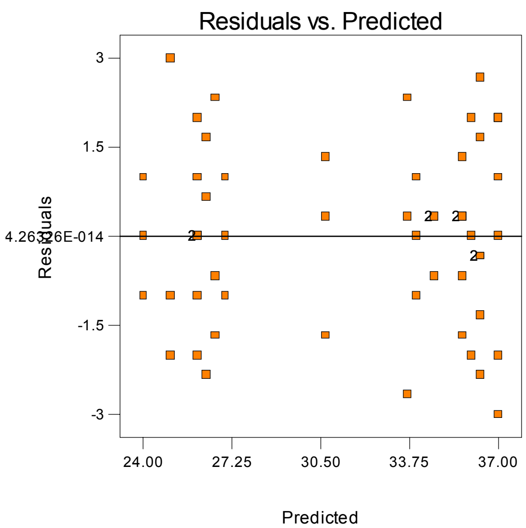
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AB, AC, ABC are significant model terms.

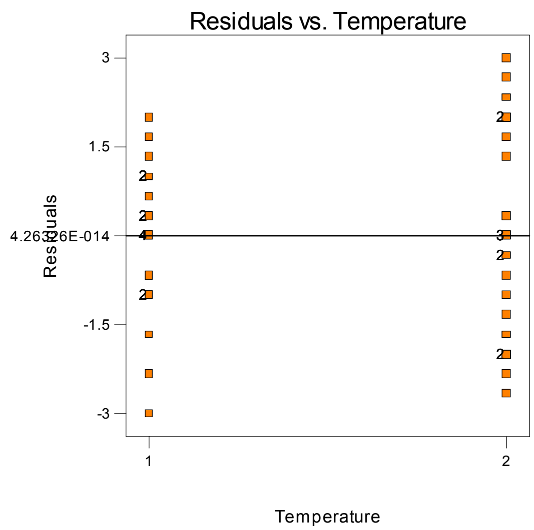












5-18 In Problem 5-1, suppose that we wish to reject the null hypothesis with a high probability if the difference in the true mean yield at any two pressures is as great as 0.5. If a reasonable prior estimate of the standard deviation of yield is 0.1, how many replicates should be run?

$$\Phi^2 = \frac{naD^2}{2b\sigma^2} = \frac{n(3)(0.5)^2}{2(3)(0.1)^2} = 12.5n$$

n	Φ^2	Φ	$\upsilon_1 = (b-1)$	$\upsilon_2 = ab(n-1)$	β
2	25	5	2	(3)(3)(1)	0.014

2 replications will be enough to detect the given difference.

5-19 The yield of a chemical process is being studied. The two factors of interest are temperature and pressure. Three levels of each factor are selected; however, only 9 runs can be made in one day. The experimenter runs a complete replicate of the design on each day. The data are shown in the following table. Analyze the data assuming that the days are blocks.

	Day 1			Day 2		
	Pressure			Pressure		
Temperature	250	260	270	250	260	270
Low	86.3	84.0	85.8	86.1	85.2	87.3
Medium	88.5	87.3	89.0	89.4	89.9	90.3
High	89.1	90.2	91.3	91.7	93.2	93.7

Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	13.01	1	13.01			
Model	109.81	8	13.73	25.84	< 0.0001	significant
A	*5.51*	*2*	*2.75*	*5.18*	*0.0360*	
B	*99.85*	*2*	*49.93*	*93.98*	*< 0.0001*	
AB	*4.45*	*4*	*1.11*	*2.10*	*0.1733*	
Residual	4.25	8	0.53			
Cor Total	127.07	17				

The Model F-value of 25.84 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

Both main effects, temperature and pressure, are significant.

5-20 Consider the data in Problem 5-5. Analyze the data, assuming that replicates are blocks.

Design Expert Output

Response: Warping
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	11.28	1	11.28			
Model	968.22	15	64.55	9.96	< 0.0001	significant
A	*698.34*	*3*	*232.78*	*35.92*	*< 0.0001*	
B	*156.09*	*3*	*52.03*	*8.03*	*0.0020*	
AB	*113.78*	*9*	*12.64*	*1.95*	*0.1214*	
Residual	97.22	15	6.48			
Cor Total	1076.72	31				

The Model F-value of 9.96 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

Both temperature and copper content are significant. This agrees with the analysis in Problem 5-5.

5-21 Consider the data in Problem 5-6. Analyze the data, assuming that replicates are blocks.

Design-Expert Output

Response: Stength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	1.04	1	1.04			
Model	217.46	11	19.77	4.89	0.0070	significant
A	*160.33*	*2*	*80.17*	*19.84*	*0.0002*	
B	*12.46*	*3*	*4.15*	*1.03*	*0.4179*	
AB	*44.67*	*6*	*7.44*	*1.84*	*0.1799*	
Residual	44.46	11	4.04			
Cor Total	262.96	23				

The Model F-value of 4.89 implies the model is significant. There is only a 0.70% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A are significant model terms.

Only the operator factor (A) is significant. This agrees with the analysis in Problem 5-6.

5-22 An article in the *Journal of Testing and Evaluation* (Vol. 16, no.2, pp. 508-515) investigated the effects of cyclic loading and environmental conditions on fatigue crack growth at a constant 22 MPa stress for a particular material. The data from this experiment are shown below (the response is crack growth rate).

	Environment		
Frequency	Air	H_2O	Salt H_2O
	2.29	2.06	1.90
10	2.47	2.05	1.93
	2.48	2.23	1.75
	2.12	2.03	2.06
	2.65	3.20	3.10
1	2.68	3.18	3.24
	2.06	3.96	3.98
	2.38	3.64	3.24
	2.24	11.00	9.96
0.1	2.71	11.00	10.01
	2.81	9.06	9.36
	2.08	11.30	10.40

(a) Analyze the data from this experiment (use $\alpha = 0.05$).

Design Expert Output

Response: Crack Growth
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	376.11	8	47.01	234.02	< 0.0001	significant
A	*209.89*	*2*	*104.95*	*522.40*	*< 0.0001*	
B	*64.25*	*2*	*32.13*	*159.92*	*< 0.0001*	
AB	*101.97*	*4*	*25.49*	*126.89*	*< 0.0001*	
Residual	5.42	27	0.20			
Lack of Fit	*0.000*	*0*				

Pure Error	*5.42*	*27*	*0.20*
Cor Total	381.53	35	

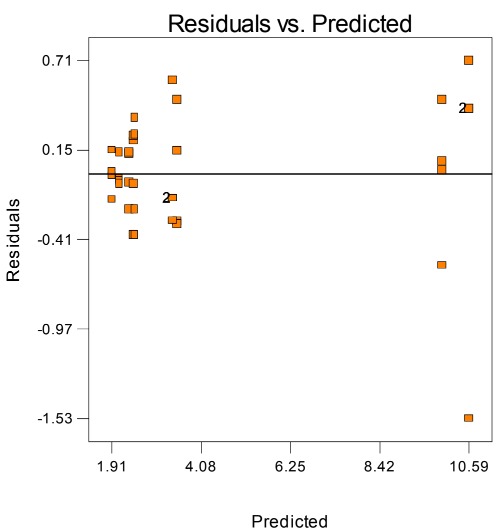
The Model F-value of 234.02 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

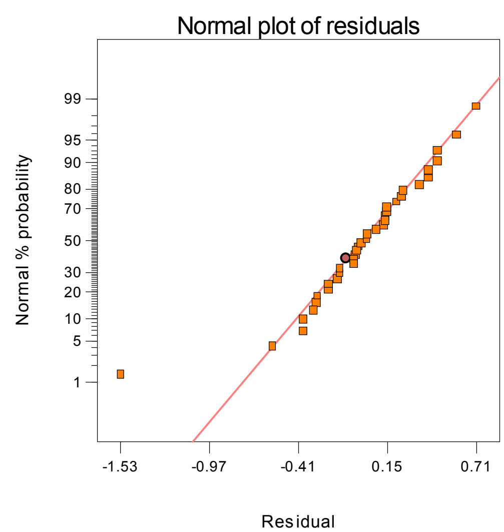
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.

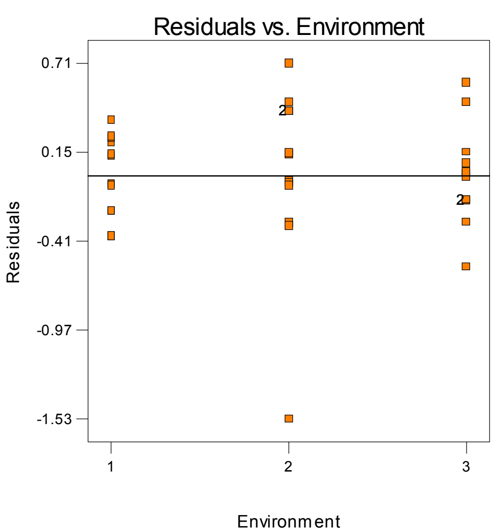
Both frequency and environment, as well as their interaction are significant.

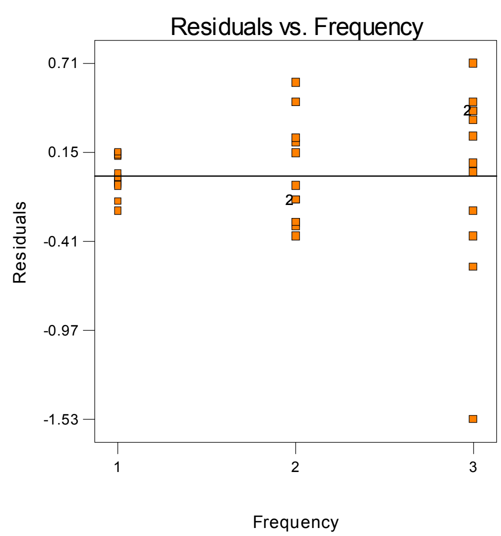
(b) Analyze the residuals.

The residual plots indicate that there may be some problem with inequality of variance. This is particularly noticable on the plot of residuals versus predicted response and the plot of residuals versus frequency.









(c) Repeat the analyses from parts (a) and (b) using $\ln(y)$ as the response. Comment on the results.

Design Expert Output

Response: Crack Growth **Transform: Natural log** **Constant: 0.000**

ANOVA for Selected Factorial Model

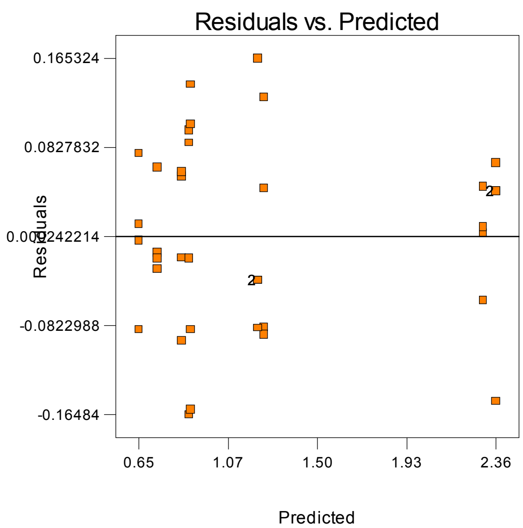
Analysis of variance table [Partial sum of squares]

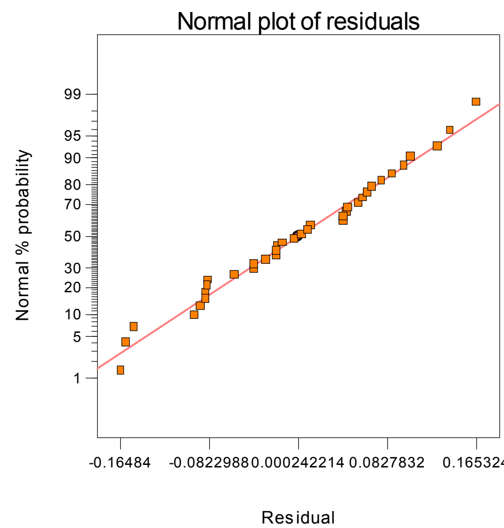
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	13.46	8	1.68	179.57	< 0.0001	significant
A	*7.57*	*2*	*3.79*	*404.09*	*< 0.0001*	
B	*2.36*	*2*	*1.18*	*125.85*	*< 0.0001*	
AB	*3.53*	*4*	*0.88*	*94.17*	*< 0.0001*	
Residual	0.25	27	9.367E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.25*	*27*	*9.367E-003*			
Cor Total	13.71	35				

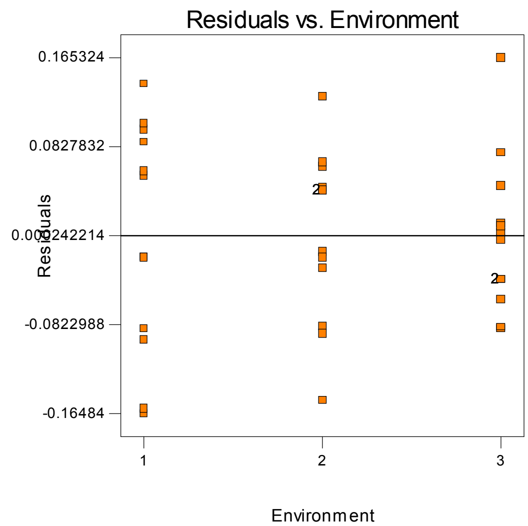
The Model F-value of 179.57 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

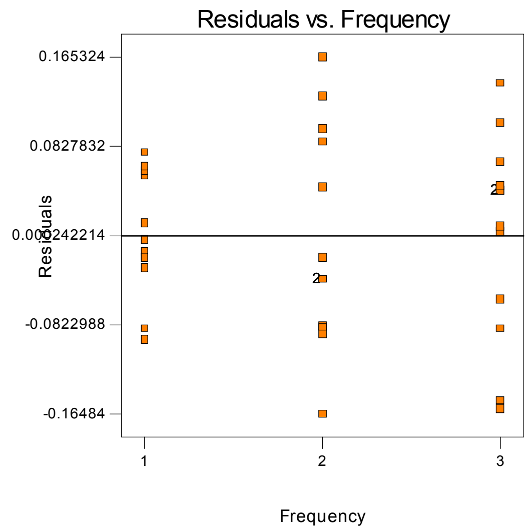
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

Both frequency and environment, as well as their interaction are significant. The residual plots of the based on the transformed data look better.









5-23 An article in the *IEEE Transactions on Electron Devices* (Nov. 1986, pp. 1754) describes a study on polysilicon doping. The experiment shown below is a variation of their study. The response variable is base current.

Polysilicon	Anneal Temperature (°C)		
Doping (ions)	900	950	1000
1×10^{20}	4.60	10.15	11.01
	4.40	10.20	10.58
2×10^{20}	3.20	9.38	10.81
	3.50	10.02	10.60

(a) Is there evidence (with $\alpha = 0.05$) indicating that either polysilicon doping level or anneal temperature affect base current?

Design Expert Output

Response: Base Current
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

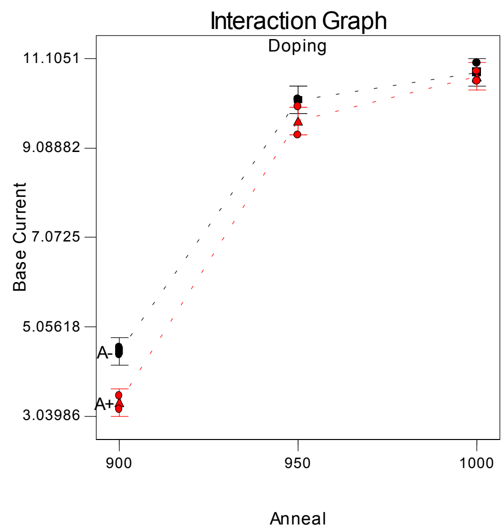
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	112.74	5	22.55	350.91	< 0.0001	significant
A	*0.98*	*1*	*0.98*	*15.26*	*0.0079*	
B	*111.19*	*2*	*55.59*	*865.16*	*< 0.0001*	
AB	*0.58*	*2*	*0.29*	*4.48*	*0.0645*	
Residual	0.39	6	0.064			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.39*	*6*	*0.064*			
Cor Total	113.13	11				

The Model F-value of 350.91 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

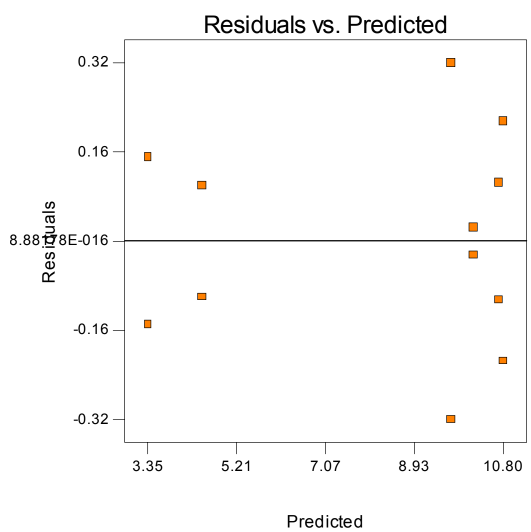
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

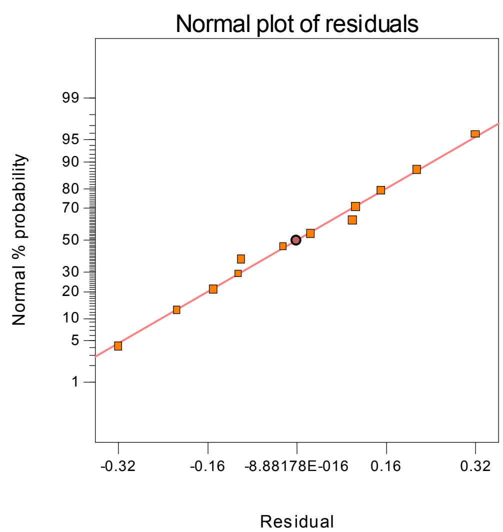
Both factors, doping and anneal are significant. Their interaction is significant at the 10% level.

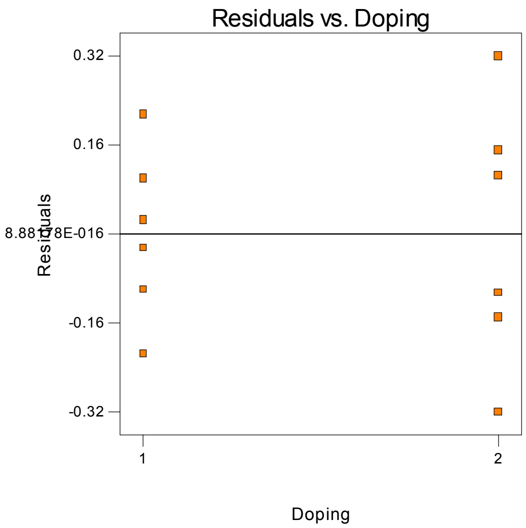
(b) Prepare graphical displays to assist in interpretation of this experiment.

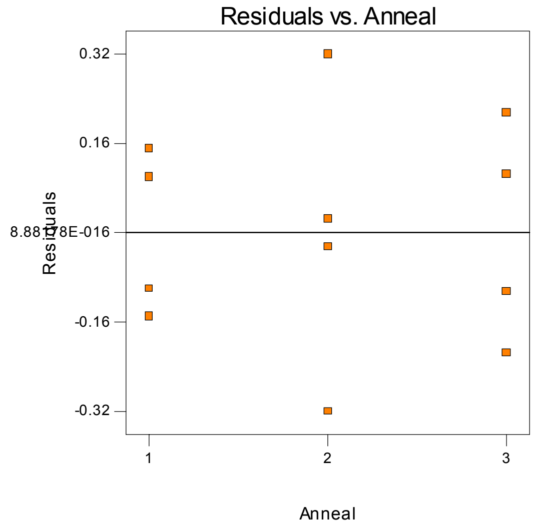


(c) Analyze the residuals and comment on model adequacy.









There is a funnel shape in the plot of residuals versus predicted, indicating some inequality of variance.

(d) Is the model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \varepsilon$ supported by this experiment (x_1 = doping level, x_2 = temperature)? Estimate the parameters in this model and plot the response surface.

Design Expert Output

Response: Base Current
ANOVA for Response Surface Reduced Quadratic Model
Analysis of variance table [Partial sum of squares]

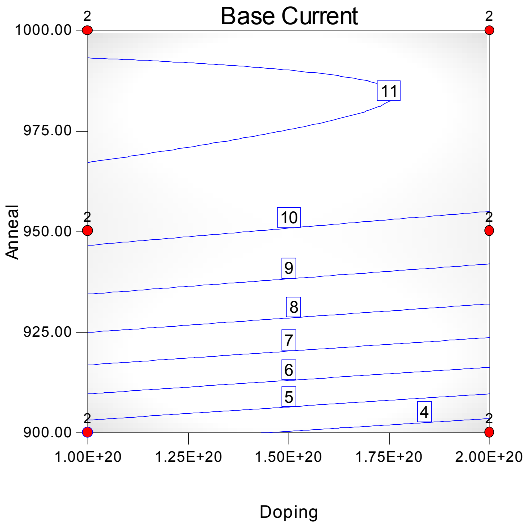
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	112.73	4	28.18	493.73	< 0.0001	significant
A	*0.98*	*1*	*0.98*	*17.18*	*0.0043*	
B	*93.16*	*1*	*93.16*	*1632.09*	*< 0.0001*	
B²	*18.03*	*1*	*18.03*	*315.81*	*< 0.0001*	
AB	*0.56*	*1*	*0.56*	*9.84*	*0.0164*	
Residual	0.40	7	0.057			
Lack of Fit	*0.014*	*1*	*0.014*	*0.22*	*0.6569*	*not significant*
Pure Error	*0.39*	*6*	*0.064*			
Cor Total	113.13	11				

The Model F-value of 493.73 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, B^2, AB are significant model terms.

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	9.94	1	0.12	9.66	10.22	
A-Doping	-0.29	1	0.069	-0.45	-0.12	1.00
B-Anneal	3.41	1	0.084	3.21	3.61	1.00
B²	-2.60	1	0.15	-2.95	-2.25	1.00
AB	0.27	1	0.084	0.065	0.46	1.00

All of the coefficients in the assumed model are significant. The quadratic effect is easily observable in the response surface plot.


Base Current
Anneal
Doping
1000.00
975.00
950.00
925.00
900.00
1.00E+20
1.25E+20
1.50E+20
1.75E+20
2.00E+20
11
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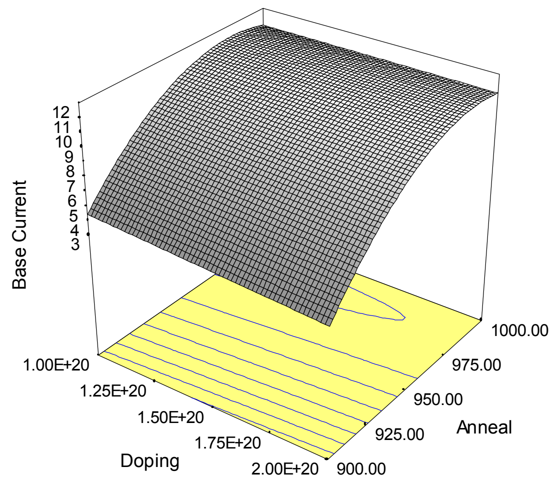
9
8
7
6
5
4


Base Current
Doping
Anneal
12
11
10
9
8
7
6
5
4
3
1.00E+20
1.25E+20
1.50E+20
1.75E+20
2.00E+20
900.00
925.00
950.00
975.00
1000.00

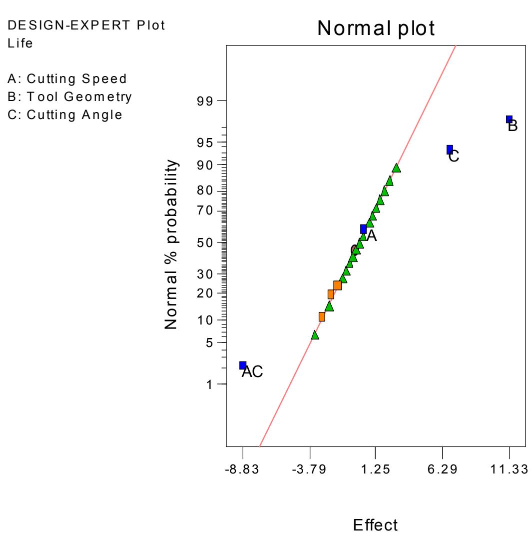
Chapter 6
The 2^k Factorial Design
Solutions

6-1 An engineer is interested in the effects of cutting speed (A), tool geometry (B), and cutting angle on the life (in hours) of a machine tool. Two levels of each factor are chosen, and three replicates of a 2^3 factorial design are run. The results follow:

			Treatment	Replicate		
A	*B*	*C*	Combination	I	II	III
-	-	-	(1)	22	31	25
+	-	-	*a*	32	43	29
-	+	-	*b*	35	34	50
+	+	-	*ab*	55	47	46
-	-	+	*c*	44	45	38
+	-	+	*ac*	40	37	36
-	+	+	*bc*	60	50	54
+	+	+	*abc*	39	41	47

(a) Estimate the factor effects. Which effects appear to be large?

From the normal probability plot of effects below, factors B, C, and the AC interaction appear to be significant.



(b) Use the analysis of variance to confirm your conclusions for part (a).

The analysis of variance confirms the significance of factors B, C, and the AC interaction.

Design Expert Output

Response: Life in hours

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1612.67	7	230.38	7.64	0.0004	significant
A	0.67	1	0.67	0.022	0.8837	
B	*770.67*	*1*	*770.67*	*25.55*	*0.0001*	
C	*280.17*	*1*	*280.17*	*9.29*	*0.0077*	
AB	*16.67*	*1*	*16.67*	*0.55*	*0.4681*	
AC	*468.17*	*1*	*468.17*	*15.52*	*0.0012*	
BC	*48.17*	*1*	*48.17*	*1.60*	*0.2245*	
ABC	*28.17*	*1*	*28.17*	*0.93*	*0.3483*	
Pure Error	*482.67*	*16*	*30.17*			
Cor Total	2095.33	23				

The Model F-value of 7.64 implies the model is significant. There is only a 0.04% chance that a "Model F-Value" this large could occur due to noise.

The reduced model ANOVA is shown below. Factor A was included to maintain hierarchy.

Design Expert Output

Response: Life in hours
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1519.67	4	379.92	12.54	< 0.0001	significant
A	*0.67*	*1*	*0.67*	*0.022*	*0.8836*	
B	*770.67*	*1*	*770.67*	*25.44*	*< 0.0001*	
C	*280.17*	*1*	*280.17*	*9.25*	*0.0067*	
AC	*468.17*	*1*	*468.17*	*15.45*	*0.0009*	
Residual	575.67	19	30.30			
Lack of Fit	*93.00*	*3*	*31.00*	*1.03*	*0.4067*	*not significant*
Pure Error	*482.67*	*16*	*30.17*			
Cor Total	2095.33	23				

The Model F-value of 12.54 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Effects *B, C* and *AC* are significant at 1%.

(c) Write down a regression model for predicting tool life (in hours) based on the results of this experiment.

$$y_{ijk} = 40.8333 + 0.1667x_A + 5.6667x_B + 3.4167x_C + 4.4167x_Ax_C$$

Design Expert Output

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	40.83	1	1.12	38.48	43.19	
A-Cutting Speed	0.17	1	1.12	-2.19	2.52	1.00
B-Tool Geometry	5.67	1	1.12	3.31	8.02	1.00
C-Cutting Angle	3.42	1	1.12	1.06	5.77	1.00
AC	-4.42	1	1.12	-6.77	-2.06	1.00

Final Equation in Terms of Coded Factors:

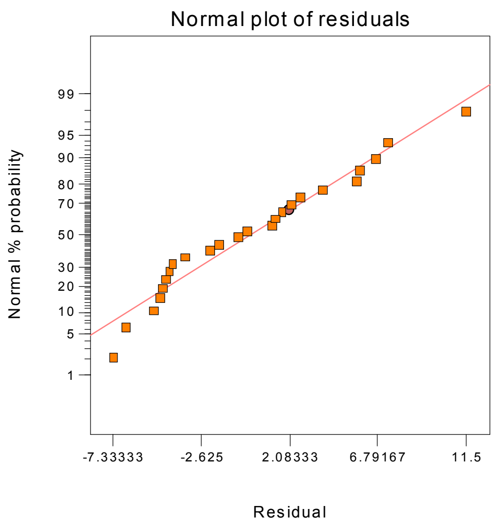
Life	=
+40.83	
+0.17	* A
+5.67	* B
+3.42	* C
-4.42	* A * C

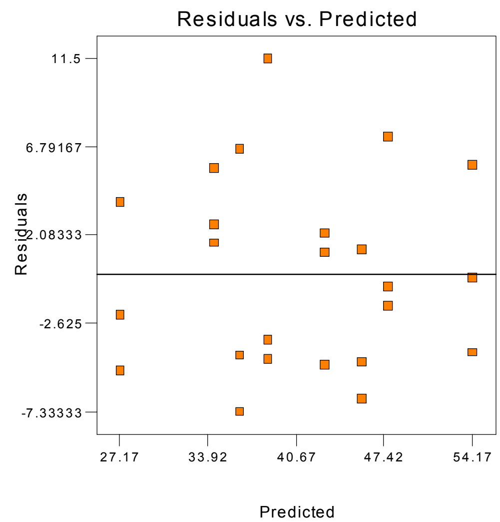
Final Equation in Terms of Actual Factors:

Life	=
+40.83333	
+0.16667	* Cutting Speed
+5.66667	* Tool Geometry
+3.41667	* Cutting Angle
-4.41667	* Cutting Speed * Cutting Angle

The equation in part (c) and in the given in the computer output form a "hierarchial" model, that is, if an interaction is included in the model, then all of the main effects referenced in the interaction are also included in the model.

(d) Analyze the residuals. Are there any obvious problems?

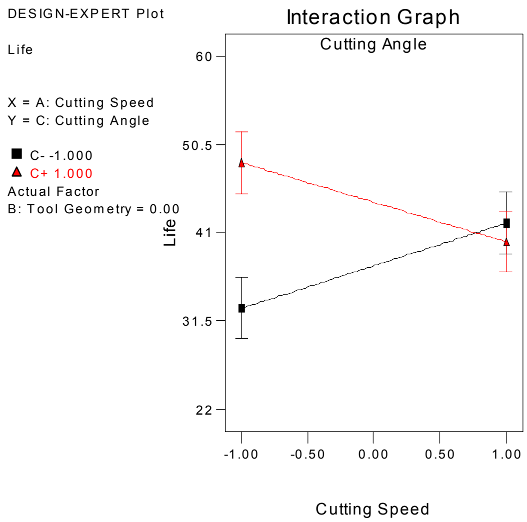


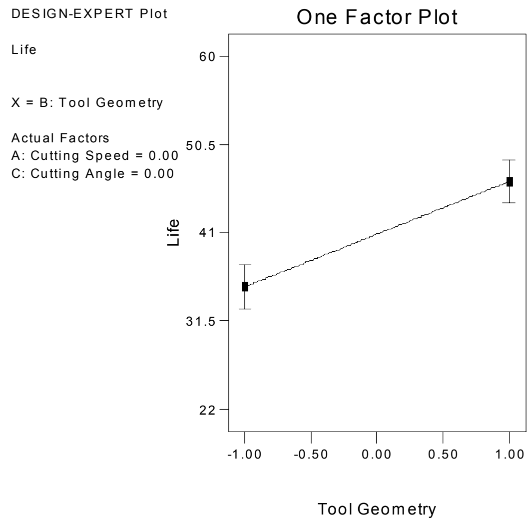


There is nothing unusual about the residual plots.

(e) Based on the analysis of main effects and interaction plots, what levels of *A*, *B*, and *C* would you recommend using?

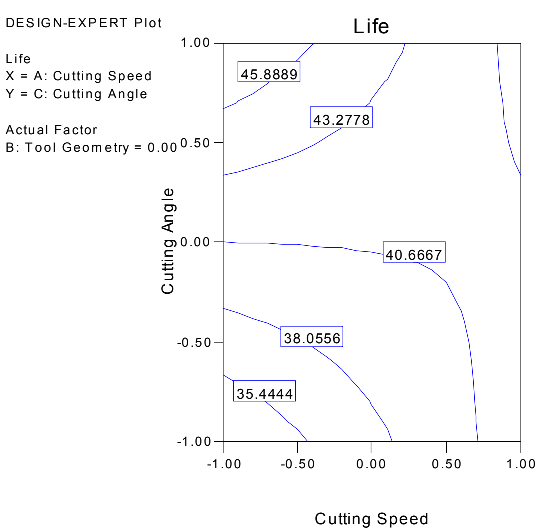
Since *B* has a positive effect, set *B* at the high level to increase life. The *AC* interaction plot reveals that life would be maximized with *C* at the high level and *A* at the low level.

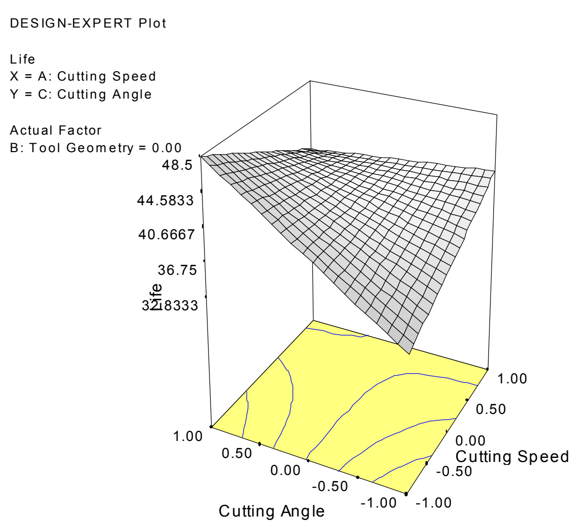




6-2 Reconsider part (c) of Problem 6-1. Use the regression model to generate response surface and contour plots of the tool life response. Interpret these plots. Do they provide insight regarding the desirable operating conditions for this process?

The response surface plot and the contour plot in terms of factors *A* and *C* with *B* at the high level are shown below. They show the curvature due to the *AC* interaction. These plots make it easy to see the region of greatest tool life.





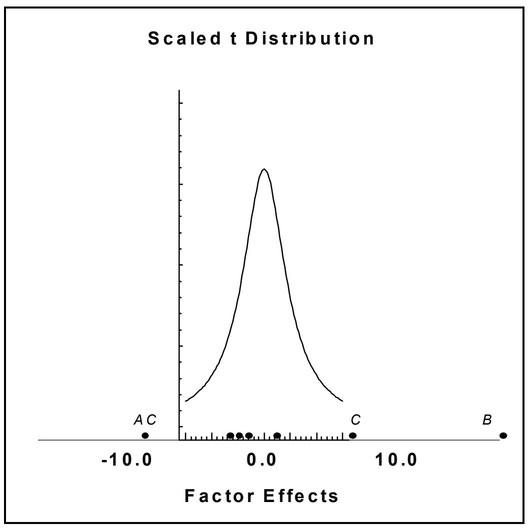
6-3 Find the standard error of the factor effects and approximate 95 percent confidence limits for the factor effects in Problem 6-1. Do the results of this analysis agree with the conclusions from the analysis of variance?

$$SE_{(effect)} = \sqrt{\frac{1}{n2^{k-2}} S^2} = \sqrt{\frac{1}{(3)2^{3-2}} 30.17} = 2.24$$

Variable	Effect	C.I.	
A	0.333	±4.395	
B	11.333	±4.395	*
AB	-1.667	±4.395	
C	6.833	±4.395	*
AC	-8.833	±4.395	*
BC	-2.833	±4.395	
ABC	-2.167	±4.395	

The 95% confidence intervals for factors *B, C* and *AC* do not contain zero. This agrees with the analysis of variance approach.

6-4 Plot the factor effects from Problem 6-1 on a graph relative to an appropriately scaled *t* distribution. Does this graphical display adequately identify the important factors? Compare the conclusions from this plot with the results from the analysis of variance. $S = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{30.17}{3}} = 3.17$



This method identifies the same factors as the analysis of variance.

6-5 A router is used to cut locating notches on a printed circuit board. The vibration level at the surface of the board as it is cut is considered to be a major source of dimensional variation in the notches. Two factors are thought to influence vibration: bit size (*A*) and cutting speed (*B*). Two bit sizes (1/16 and 1/8 inch) and two speeds (40 and 90 rpm) are selected, and four boards are cut at each set of conditions shown below. The response variable is vibration measured as a resultant vector of three accelerometers (*x*, *y*, and *z*) on each test circuit board.

		Treatment	Replicate			
A	*B*	Combination	I	II	III	IV
-	-	(1)	18.2	18.9	12.9	14.4
+	-	*a*	27.2	24.0	22.4	22.5
-	+	*b*	15.9	14.5	15.1	14.2

+	+	*ab*	41.0	43.9	36.3	39.9

(a) Analyze the data from this experiment.

Design Expert Output

Response: Vibration

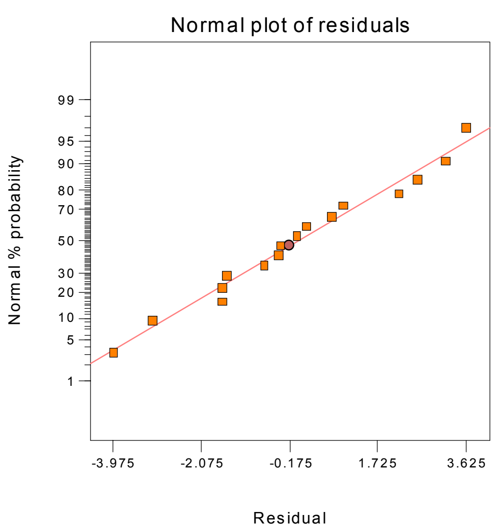
ANOVA for Selected Factorial Model

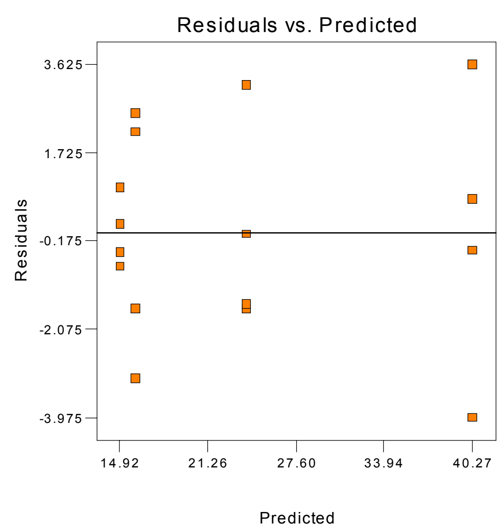
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1638.11	3	546.04	91.36	< 0.0001	significant
A	*1107.23*	*1*	*1107.23*	*185.25*	*< 0.0001*	
B	*227.26*	*1*	*227.26*	*38.02*	*< 0.0001*	
AB	*303.63*	*1*	*303.63*	*50.80*	*< 0.0001*	
Residual	71.72	12	5.98			
Lack of Fit	*0.000*	*0*				
Pure Error	*71.72*	*12*	*5.98*			
Cor Total	1709.83	15				

The Model F-value of 91.36 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

(b) Construct a normal probability plot of the residuals, and plot the residuals versus the predicted vibration level. Interpret these plots.

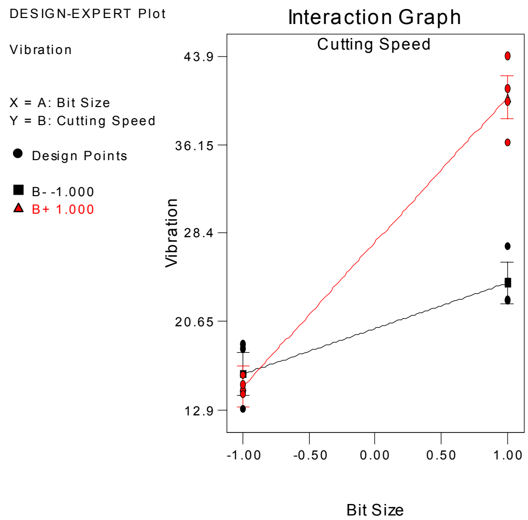




There is nothing unusual about the residual plots.

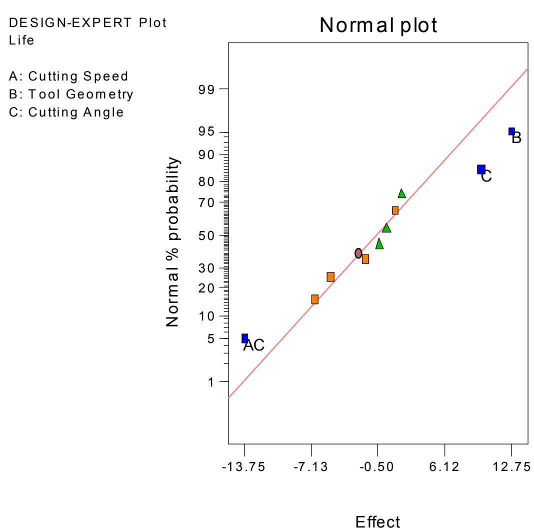
(c) Draw the *AB* interaction plot. Interpret this plot. What levels of bit size and speed would you recommend for routine operation?

To reduce the vibration, use the smaller bit. Once the small bit is specified, either speed will work equally well, because the slope of the curve relating vibration to speed for the small tip is approximately zero. The process is robust to speed changes if the small bit is used.



6-6 Reconsider the experiment described in Problem 6-1. Suppose that the experimenter only performed the eight trials from replicate I. In addition, he ran four center points and obtained the following response values: 36, 40, 43, 45.

(a) Estimate the factor effects. Which effects are large?



Effects *B, C,* and *AC* appear to be large.

(b) Perform an analysis of variance, including a check for pure quadratic curvature. What are your conclusions? $SS_{PureQuadratic} = \dfrac{n_F n_C\left(\bar{y}_F - \bar{y}_C\right)^2}{n_F + n_C} = \dfrac{(8)(4)(40.875 - 41.000)^2}{8+4} = 0.0417$

Design Expert Output

Response:	**Life**	**in hours**

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1048.88	7	149.84	9.77	0.0439	significant
A	*3.13*	*1*	*3.13*	*0.20*	*0.6823*	
B	*325.13*	*1*	*325.13*	*21.20*	*0.0193*	
C	*190.12*	*1*	*190.12*	*12.40*	*0.0389*	
AB	*6.13*	*1*	*6.13*	*0.40*	*0.5722*	
AC	*378.12*	*1*	*378.12*	*24.66*	*0.0157*	
BC	*55.12*	*1*	*55.12*	*3.60*	*0.1542*	
ABC	*91.12*	*1*	*91.12*	*5.94*	*0.0927*	
Curvature	0.042	1	0.042	2.717E-003	0.9617	not significant
Pure Error	46.00	3	15.33			
Cor Total	1094.92	11				

The Model F-value of 9.77 implies the model is significant. There is only a 4.39% chance that a "Model F-Value" this large could occur due to noise.

The "Curvature F-value" of 0.00 implies the curvature (as measured by difference between the average of the center points and the average of the factorial points) in the design space is not significant relative to the noise. There is a 96.17% chance that a "Curvature F-value" this large could occur due to noise.

Design Expert Output

Response: Life in hours
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	896.50	4	224.13	7.91	0.0098	significant
A	*3.13*	*1*	*3.13*	*0.11*	*0.7496*	
B	*325.12*	*1*	*325.12*	*11.47*	*0.0117*	
C	*190.12*	*1*	*190.12*	*6.71*	*0.0360*	
AC	*378.12*	*1*	*378.12*	*13.34*	*0.0082*	
Residual	198.42	7	28.35			
Lack of Fit	*152.42*	*4*	*38.10*	*2.49*	*0.2402*	*not significant*
Pure Error	*46.00*	*3*	*15.33*			
Cor Total	1094.92	11				

The Model F-value of 7.91 implies the model is significant. There is only a 0.98% chance that a "Model F-Value" this large could occur due to noise.

Effects *B, C* and *AC* are significant at 5%. There is no effect of curvature.

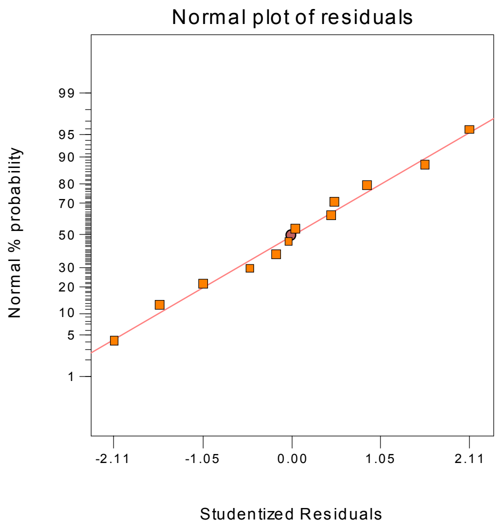
(c) Write down an appropriate model for predicting tool life, based on the results of this experiment. Does this model differ in any substantial way from the model in Problem 7-1, part (c)?

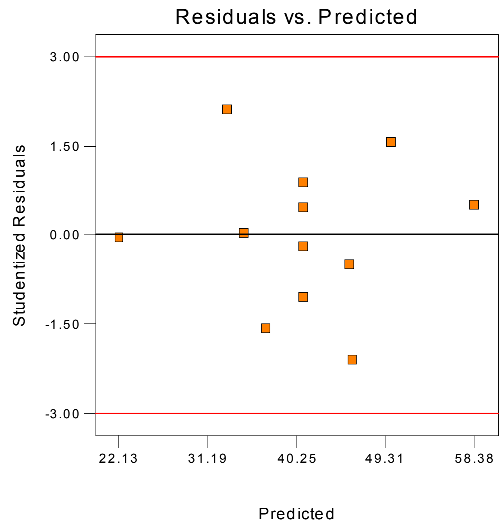
Design Expert Output

Final Equation in Terms of Coded Factors:

Life	=
+40.88	
+0.62	* A
+6.37	* B
+4.87	* C
-6.88	* A * C

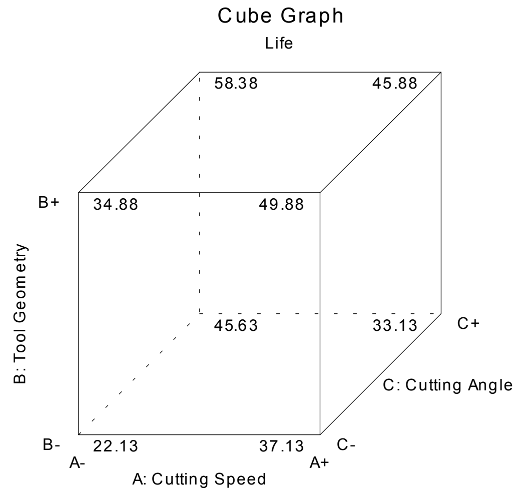
(d) Analyze the residuals.





(e) What conclusions would you draw about the appropriate operating conditions for this process?

To maximize life run with *B* at the high level, *A* at the low level and *C* at the high level



6-7 An experiment was performed to improve the yield of a chemical process. Four factors were selected, and two replicates of a completely randomized experiment were run. The results are shown in the following table:

Treatment Combination	Replicate I	Replicate II	Treatment Combination	Replicate I	Replicate II
(1)	90	93	*d*	98	95
a	74	78	*ad*	72	76
b	81	85	*bd*	87	83

ab	83	80	*abd*	85	86
c	77	78	*cd*	99	90
ac	81	80	*acd*	79	75
bc	88	82	*bcd*	87	84
abc	73	70	*abcd*	80	80

(a) Estimate the factor effects.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-9.0625	657.031	40.3714
Error	B	-1.3125	13.7812	0.84679
Error	C	-2.6875	57.7813	3.55038
Error	D	3.9375	124.031	7.62111
Error	AB	4.0625	132.031	8.11267
Error	AC	0.6875	3.78125	0.232339
Error	AD	-2.1875	38.2813	2.3522
Error	BC	-0.5625	2.53125	0.155533
Error	BD	-0.1875	0.28125	0.0172814
Error	CD	1.6875	22.7812	1.3998
Error	ABC	-5.1875	215.281	13.228
Error	ABD	4.6875	175.781	10.8009
Error	ACD	-0.9375	7.03125	0.432036
Error	BCD	-0.9375	7.03125	0.432036
Error	ABCD	2.4375	47.5313	2.92056

(b) Prepare an analysis of variance table, and determine which factors are important in explaining yield.

Design Expert Output

Response: yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1504.97	15	100.33	13.10	< 0.0001	significant
A	*657.03*	*1*	*657.03*	*85.82*	*< 0.0001*	
B	*13.78*	*1*	*13.78*	*1.80*	*0.1984*	
C	*57.78*	*1*	*57.78*	*7.55*	*0.0143*	
D	*124.03*	*1*	*124.03*	*16.20*	*0.0010*	
AB	*132.03*	*1*	*132.03*	*17.24*	*0.0007*	
AC	*3.78*	*1*	*3.78*	*0.49*	*0.4923*	
AD	*38.28*	*1*	*38.28*	*5.00*	*0.0399*	
BC	*2.53*	*1*	*2.53*	*0.33*	*0.5733*	
BD	*0.28*	*1*	*0.28*	*0.037*	*0.8504*	
CD	*22.78*	*1*	*22.78*	*2.98*	*0.1038*	
ABC	*215.28*	*1*	*215.28*	*28.12*	*< 0.0001*	
ABD	*175.78*	*1*	*175.78*	*22.96*	*0.0002*	
ACD	*7.03*	*1*	*7.03*	*0.92*	*0.3522*	
BCD	*7.03*	*1*	*7.03*	*0.92*	*0.3522*	
ABCD	*47.53*	*1*	*47.53*	*6.21*	*0.0241*	
Residual	122.50	16	7.66			
Lack of Fit	*0.000*	*0*				
Pure Error	*122.50*	*16*	*7.66*			
Cor Total	1627.47	31				

The Model F-value of 13.10 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, C, D, AB, AD, ABC, ABD, ABCD are significant model terms.

$F_{0.01,1,16} = 8.53$, and $F_{0.025,1,16} = 6.12$ therefore, factors *A* and *D* and interactions *AB, ABC*, and *ABD* are significant at 1%. Factor *C* and interactions *AD* and *ABCD* are significant at 2.5%.

(b) Write down a regression model for predicting yield, assuming that all four factors were varied over the range from -1 to +1 (in coded units).

Model with hierarchy maintained:

Design Expert Output

Final Equation in Terms of Coded Factors:

```
          yield   =
         +82.78
          -4.53   * A
          -0.66   * B
          -1.34   * C
          +1.97   * D
          +2.03   * A * B
          +0.34   * A * C
          -1.09   * A * D
          -0.28   * B * C
         -0.094   * B * D
          +0.84   * C * D
          -2.59   * A * B * C
          +2.34   * A * B * D
          -0.47   * A * C * D
          -0.47   * B * C * D
          +1.22   * A * B * C * D
```

Model without hierarchy terms:

Design Expert Output

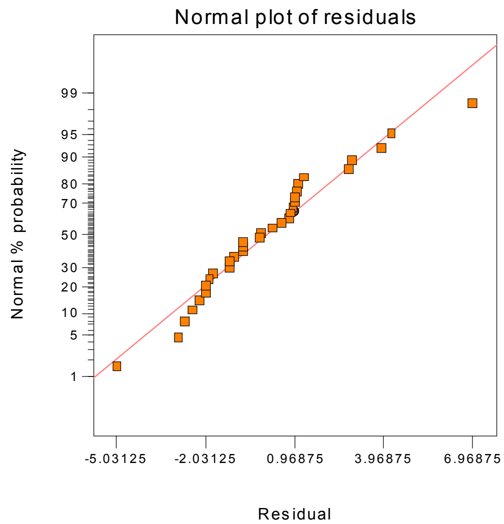
Final Equation in Terms of Coded Factors:

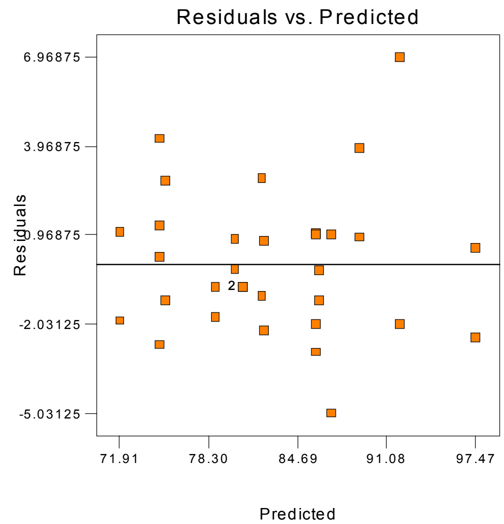
```
          yield   =
         +82.78
          -4.53   * A
          -1.34   * C
          +1.97   * D
          +2.03   * A * B
          -1.09   * A * D
          -2.59   * A * B * C
          +2.34   * A * B * D
          +1.22   * A * B * C * D
```

Confirmation runs might be run to see if the simpler model without hierarchy is satisfactory.

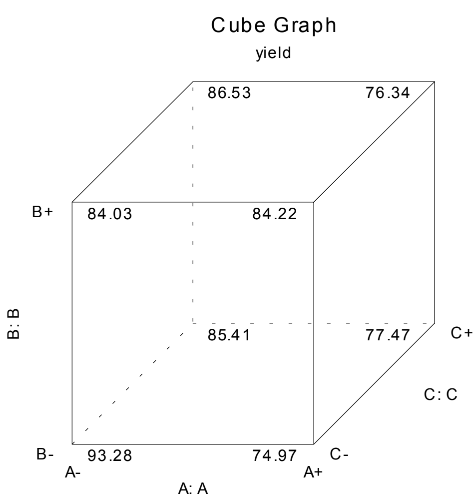
(d) Plot the residuals versus the predicted yield and on a normal probability scale. Does the residual analysis appear satisfactory?

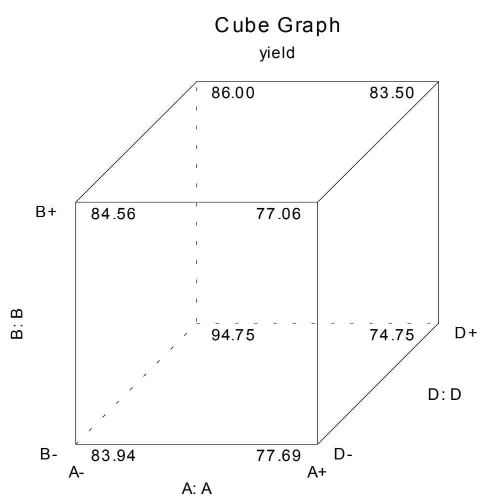
There appears to be one large residual both in the normal probability plot and in the plot of residuals versus predicted.





(e) Two three-factor interactions, *ABC* and *ABD*, apparently have large effects. Draw a cube plot in the factors *A*, *B*, and *C* with the average yields shown at each corner. Repeat using the factors *A*, *B*, and *D*. Do these two plots aid in data interpretation? Where would you recommend that the process be run with respect to the four variables?





Run the process at *A* low *B* low, *C* low and *D* high.

6-8 A bacteriologist is interested in the effects of two different culture media and two different times on the growth of a particular virus. She performs six replicates of a 2^2 design, making the runs in random order. Analyze the bacterial growth data that follow and draw appropriate conclusions. Analyze the residuals and comment on the model's adequacy.

	Culture Medium	
Time	1	2

	21	22	25	26
12 hr	23	28	24	25
	20	26	29	27
	37	39	31	34
18 hr	37	39	31	34
	35	36	30	35

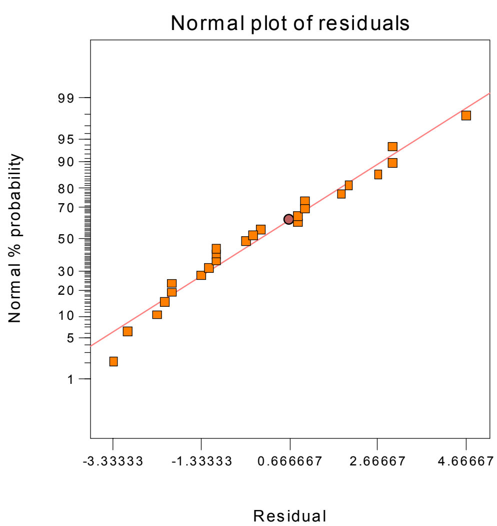
Design Expert Output

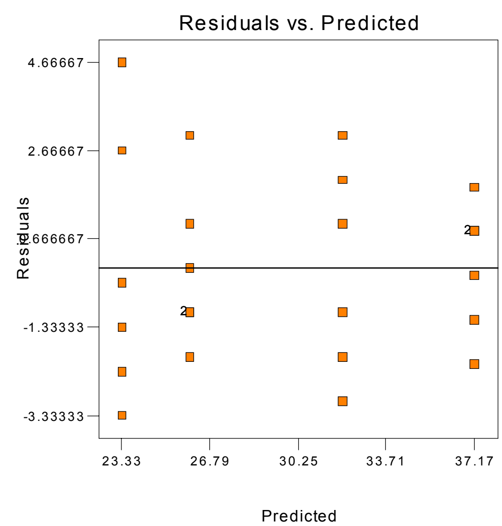
Response: Virus growth
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	691.46	3	230.49	45.12	< 0.0001	significant
A	*9.38*	*1*	*9.38*	*1.84*	*0.1906*	
B	*590.04*	*1*	*590.04*	*115.51*	*< 0.0001*	
AB	*92.04*	*1*	*92.04*	*18.02*	*0.0004*	
Residual	102.17	20	5.11			
Lack of Fit	*0.000*	*0*				
Pure Error	*102.17*	*20*	*5.11*			
Cor Total	793.63	23				

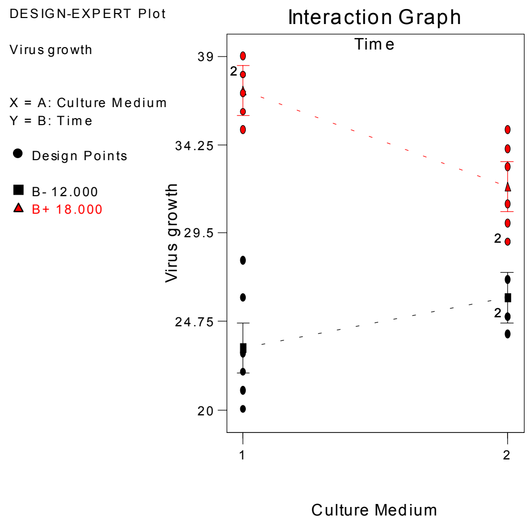
The Model F-value of 45.12 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B, AB are significant model terms.





Growth rate is affected by factor *B* (Time) and the *AB* interaction (Culture medium and Time). There is some very slight indication of inequality of variance shown by the small decreasing funnel shape in the plot of residuals versus predicted.



6-9 An industrial engineer employed by a beverage bottler is interested in the effects of two different typed of 32-ounce bottles on the time to deliver 12-bottle cases of the product. The two bottle types are glass and plastic. Two workers are used to perform a task consisting of moving 40 cases of the product 50 feet on a standard type of hand truck and stacking the cases in a display. Four replicates of a 2^2 factorial design are performed, and the times observed are listed in the following table. Analyze the data and draw the appropriate conclusions. Analyze the residuals and comment on the model's adequacy.

	Worker			
Bottle Type	1	1	2	2
Glass	5.12	4.89	6.65	6.24
	4.98	5.00	5.49	5.55
Plastic	4.95	4.43	5.28	4.91
	4.27	4.25	4.75	4.71

Design Expert Output

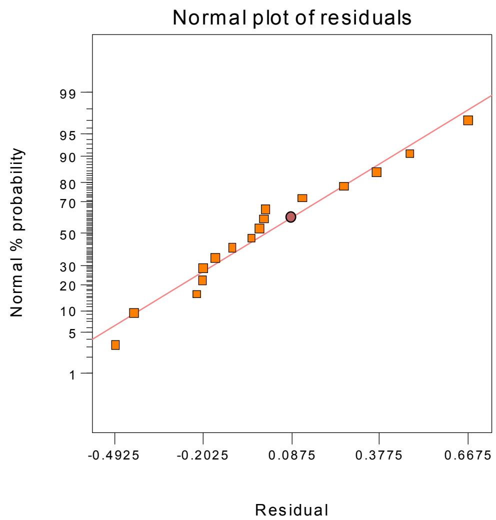
Response: Times
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

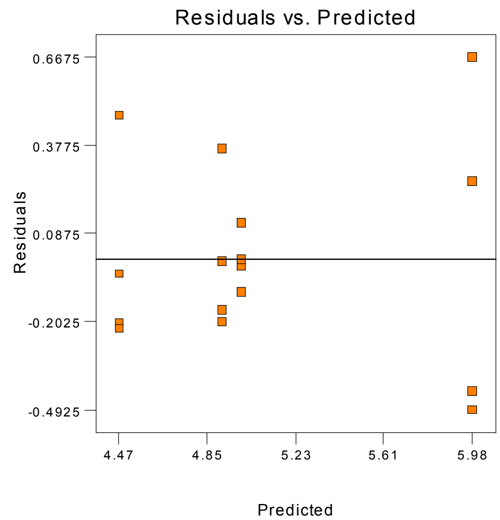
	Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	4.86	3	1.62	13.04	0.0004		significant
A	*2.02*	*1*	*2.02*	*16.28*	*0.0017*		
B	*2.54*	*1*	*2.54*	*20.41*	*0.0007*		
AB	*0.30*	*1*	*0.30*	*2.41*	*0.1463*		
Residual	1.49	12	0.12				
Lack of Fit	*0.000*	*0*					
Pure Error	*1.49*	*12*	*0.12*				
Cor Total	6.35	15					

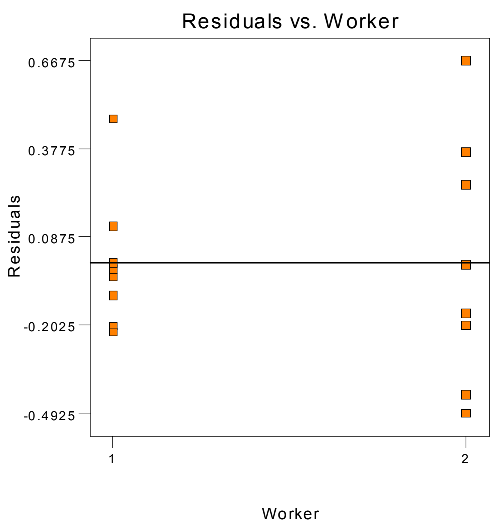
The Model F-value of 13.04 implies the model is significant. There is only a 0.04% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

There is some indication of non-constant variance in this experiment.







6-10 In problem 6-9, the engineer was also interested in potential fatigue differences resulting from the two types of bottles. As a measure of the amount of effort required, he measured the elevation of heart rate (pulse) induced by the task. The results follow. Analyze the data and draw conclusions. Analyze the residuals and comment on the model's adequacy.

	Worker			
Bottle Type	1	1	2	2
Glass	39	45	20	13
	58	35	16	11
Plastic	44	35	13	10
	42	21	16	15

Design Expert Output

Response: Pulse

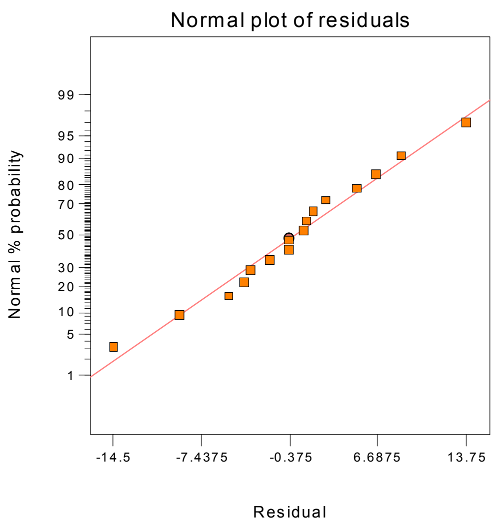
ANOVA for Selected Factorial Model

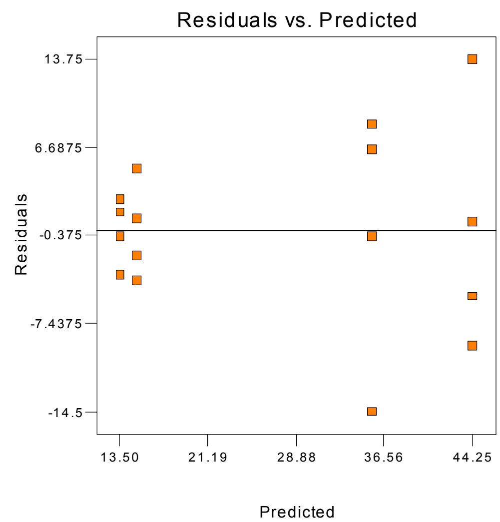
Analysis of variance table [Partial sum of squares]

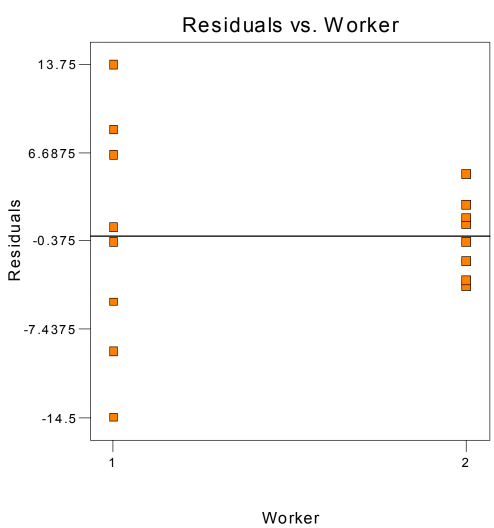
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2784.19	3	928.06	16.03	0.0002	significant
A	*2626.56*	*1*	*2626.56*	*45.37*	*< 0.0001*	
B	*105.06*	*1*	*105.06*	*1.81*	*0.2028*	
AB	*52.56*	*1*	*52.56*	*0.91*	*0.3595*	
Residual	694.75	12	57.90			
Lack of Fit	*0.000*	*0*				
Pure Error	*694.75*	*12*	*57.90*			
Cor Total	3478.94	15				

The Model F-value of 16.03 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A are significant model terms.







There is an indication that one worker exhibits greater variability than the other.

6-11 Calculate approximate 95 percent confidence limits for the factor effects in Problem 6-10. Do the results of this analysis agree with the analysis of variance performed in Problem 6-10?

$$SE_{(effect)} = \sqrt{\frac{1}{n2^{k-2}}S^2} = \sqrt{\frac{1}{(4)2^{2-2}}57.90} = 3.80$$

Variable	Effect	C.I.
A	-25.625	±3.80(1.96)= ±7.448
B	-5.125	±3.80(1.96)= ±7.448
AB	-7.25	±3.80(1.96)= ±7.448

The 95% confidence intervals for factors *A* does not contain zero. This agrees with the analysis of variance approach.

6-12 An article in the AT&T Technical Journal (March/April 1986, Vol. 65, pp. 39-50) describes the application of two-level factorial designs to integrated circuit manufacturing. A basic processing step is to grow an epitaxial layer on polished silicon wafers. The wafers mounted on a susceptor are positioned inside a bell jar, and chemical vapors are introduced. The susceptor is rotated and heat is applied until the epitaxial layer is thick enough. An experiment was run using two factors: arsenic flow rate (*A*) and deposition time (*B*). Four replicates were run, and the epitaxial layer thickness was measured (in mm). The data are shown below:

			Replicate				Factor	Levels
A	*B*	I	II	III	IV		Low (-)	High (+)
-	-	14.037	16.165	13.972	13.907	*A*	55%	59%
+	-	13.880	13.860	14.032	13.914			
-	+	14.821	14.757	14.843	14.878	*B*	Short	Long
+	+	14.888	14.921	14.415	14.932		(10 min)	(15 min)

(a) Estimate the factor effects.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-0.31725	0.40259	6.79865
Error	B	0.586	1.37358	23.1961
Error	AB	0.2815	0.316969	5.35274
Error	Lack Of Fit		0	0
Error	Pure Error		3.82848	64.6525

(b) Conduct an analysis of variance. Which factors are important?

Design Expert Output

Response: Thickness
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2.09	3	0.70	2.19	0.1425	not significant
A	*0.40*	*1*	*0.40*	*1.26*	*0.2833*	
B	*1.37*	*1*	*1.37*	*4.31*	*0.0602*	

AB	*0.32*	*1*	*0.32*	*0.99*	*0.3386*
Residual	3.83	12	0.32		
Lack of Fit	*0.000*	*0*			
Pure Error	*3.83*	*12*	*0.32*		
Cor Total	5.92	15			

The "Model F-value" of 2.19 implies the model is not significant relative to the noise. There is a 14.25 % chance that a "Model F-value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case there are no significant model terms.

(c) Write down a regression equation that could be used to predict epitaxial layer thickness over the region of arsenic flow rate and deposition time used in this experiment.

Design Expert Output

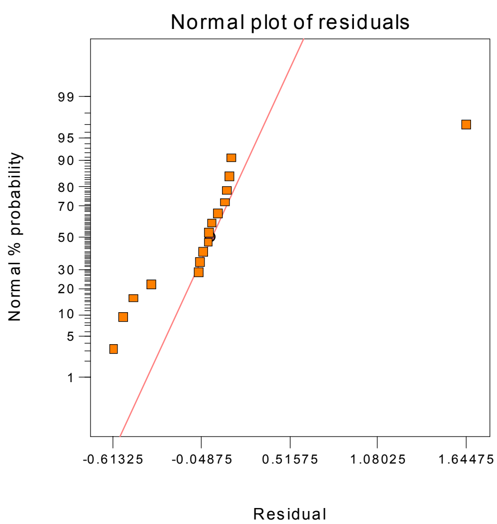
Final Equation in Terms of Coded Factors:

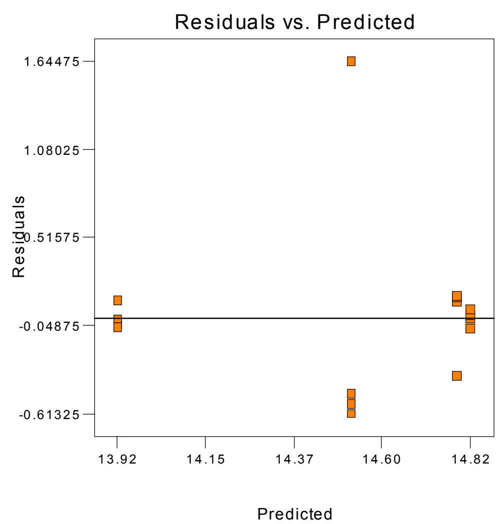
Thickness =
+14.51
-0.16 * A
+0.29 * B
+0.14 * A * B

Final Equation in Terms of Actual Factors:

Thickness =
+37.62656
-0.43119 * Flow Rate
-1.48735 * Dep Time
+0.028150 * Flow Rate * Dep Time

(d) Analyze the residuals. Are there any residuals that should cause concern? Observation #2 falls outside the groupings in the normal probability plot and the plot of residual versus predicted.





(e) Discuss how you might deal with the potential outlier found in part (d).

One approach would be to replace the observation with the average of the observations from that experimental cell. Another approach would be to identify if there was a recording issue in the original data. The first analysis below replaces the data point with the average of the other three. The second analysis assumes that the reading was incorrectly recorded and should have been 14.165.

Analysis with the run associated with standard order 2 replaced with the average of the remaining three runs in the cell, 13.972:

Design Expert Output

Response: Thickness

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.97	3	0.99	53.57	< 0.0001	significant
A	*7.439E-003*	*1*	*7.439E-003*	*0.40*	*0.5375*	
B	*2.96*	*1*	*2.96*	*160.29*	*< 0.0001*	
AB	*2.176E-004*	*1*	*2.176E-004*	*0.012*	*0.9153*	
Pure Error	*0.22*	*12*	*0.018*			
Cor Total	3.19	15				

The Model F-value of 53.57 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

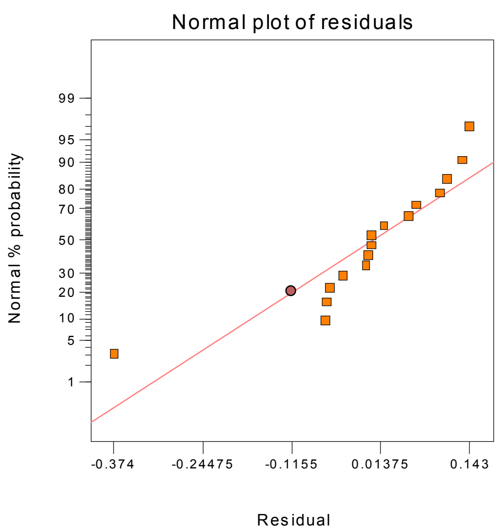
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case B are significant model terms.

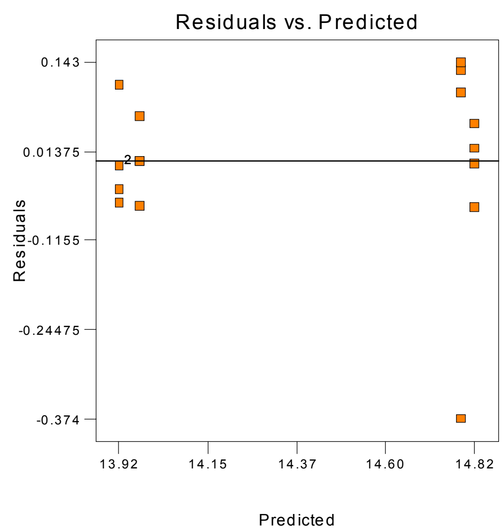
Final Equation in Terms of Coded Factors:

Thickness	=
+14.38	
-0.022	* A
+0.43	* B
+3.688E-003	* A * B

Final Equation in Terms of Actual Factors:

Thickness	=
+13.36650	
-0.020000	* Flow Rate
+0.12999	* Dep Time
+7.37500E-004	* Flow Rate * Dep Time





A new outlier is present and should be investigated.

Analysis with the run associated with standard order 2 replaced with the value 14.165:

Design Expert Output

Response: Thickness

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.82	3	0.94	45.18	< 0.0001	significant
A	*0.018*	*1*	*0.018*	*0.87*	*0.3693*	
B	*2.80*	*1*	*2.80*	*134.47*	*< 0.0001*	
AB	*3.969E-003*	*1*	*3.969E-003*	*0.19*	*0.6699*	
Pure Error	*0.25*	*12*	*0.021*			
Cor Total	3.07	15				

The Model F-value of 45.18 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

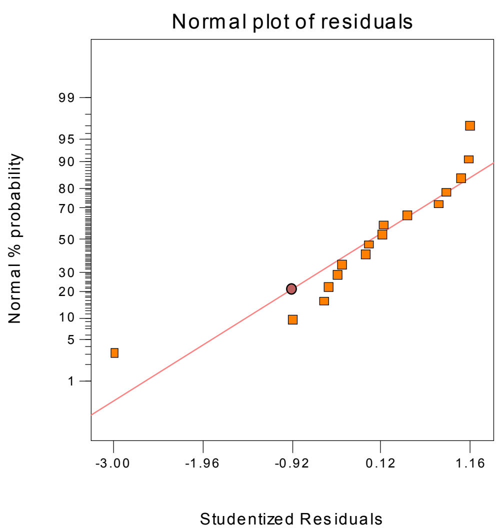
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case B are significant model terms.

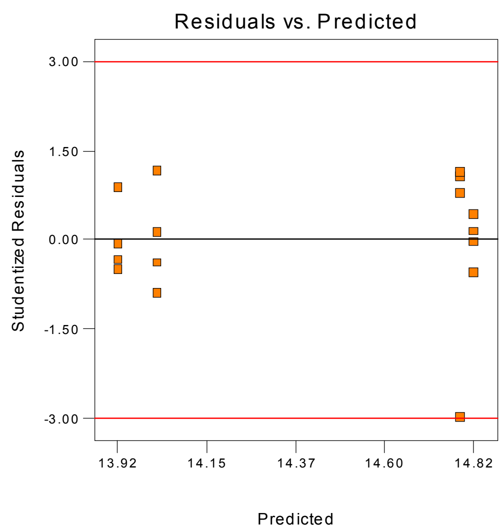
Final Equation in Terms of Coded Factors:

Thickness	=
+14.39	
-0.034	* A
+0.42	* B
+0.016	* A * B

Final Equation in Terms of Actual Factors:

Thickness	=
+15.50156	
-0.056188	* Flow Rate
-0.012350	* Dep Time
+3.15000E-003	* Flow Rate * Dep Time

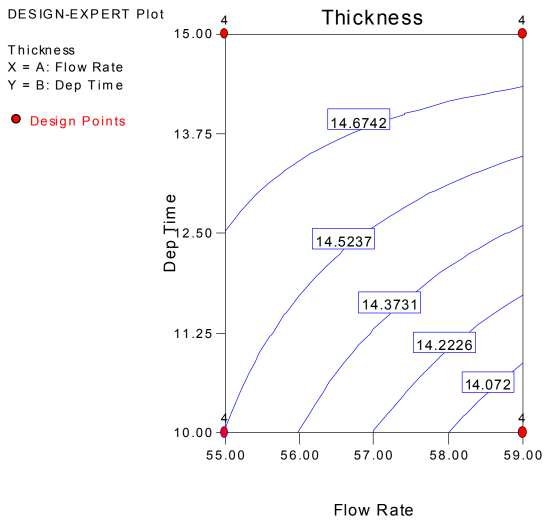


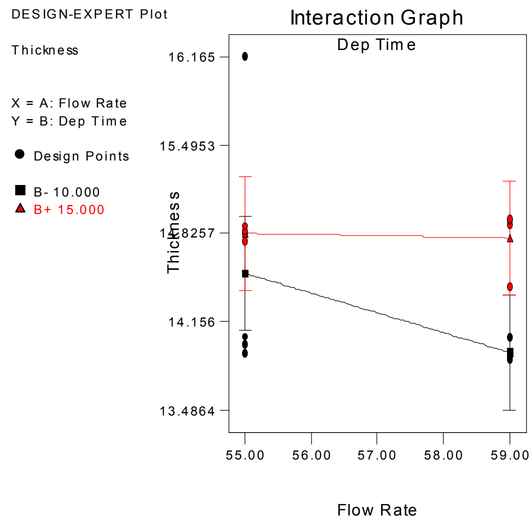


Another outlier is present and should be investigated.

6-13 Continuation of Problem 6-12. Use the regression model in part (c) of Problem 6-12 to generate a response surface contour plot for epitaxial layer thickness. Suppose it is critically important to obtain layer thickness of 14.5 mm. What settings of arsenic flow rate and deposition time would you recommend?

Arsenic flow rate may be set at any of the experimental levels, while the deposition time should be set at 12.4 minutes.





6-14 Continuation of Problem 6-13. How would your answer to Problem 6-13 change if arsenic flow rate was more difficult to control in the process than the deposition time?

Running the process at a high level of Deposition Time there is no change in thickness as flow rate changes.

6-15 A nickel-titanium alloy is used to make components for jet turbine aircraft engines. Cracking is a potentially serious problem in the final part, as it can lead to non-recoverable failure. A test is run at the parts producer to determine the effects of four factors on cracks. The four factors are pouring temperature (*A*), titanium content (*B*), heat treatment method (*C*), and the amount of grain refiner used (*D*). Two replicated of a 2^4 design are run, and the length of crack (in μm) induced in a sample coupon subjected to a standard test is measured. The data are shown below:

A	*B*	*C*	*D*	Treatment Combination	Replicate I	Replicate II
-	-	-	-	(1)	7.037	6.376
+	-	-	-	*a*	14.707	15.219
-	+	-	-	*b*	11.635	12.089
+	+	-	-	*ab*	17.273	17.815
-	-	+	-	*c*	10.403	10.151
+	-	+	-	*ac*	4.368	4.098
-	+	+	-	*bc*	9.360	9.253
+	+	+	-	*abc*	13.440	12.923
-	-	-	+	*d*	8.561	8.951

+	-	-	+	*ad*	16.867	17.052
-	+	-	+	*bd*	13.876	13.658
+	+	-	+	*abd*	19.824	19.639
-	-	+	+	*cd*	11.846	12.337
+	-	+	+	*acd*	6.125	5.904
-	+	+	+	*bcd*	11.190	10.935
+	+	+	+	*abcd*	15.653	15.053

(a) Estimate the factor effects. Which factors appear to be large?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	3.01888	72.9089	12.7408
Model	B	3.97588	126.461	22.099
Model	C	-3.59625	103.464	18.0804
Model	D	1.95775	30.6623	5.35823
Model	AB	1.93412	29.9267	5.22969
Model	AC	-4.00775	128.496	22.4548
Error	AD	0.0765	0.046818	0.00818145
Error	BC	0.096	0.073728	0.012884
Error	BD	0.04725	0.0178605	0.00312112
Error	CD	-0.076875	0.0472781	0.00826185
Model	ABC	3.1375	78.7512	13.7618
Error	ABD	0.098	0.076832	0.0134264
Error	ACD	0.019125	0.00292613	0.00051134
Error	BCD	0.035625	0.0101531	0.00177426
Error	ABCD	0.014125	0.00159613	0.000278923

(b) Conduct an analysis of variance. Do any of the factors affect cracking? Use α=0.05.

Design Expert Output

Response: Crack Lengthin mm x 10^-2

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	570.95	15	38.06	468.99	< 0.0001	significant
A	*72.91*	*1*	*72.91*	*898.34*	*< 0.0001*	
B	*126.46*	*1*	*126.46*	*1558.17*	*< 0.0001*	
C	*103.46*	*1*	*103.46*	*1274.82*	*< 0.0001*	
D	*30.66*	*1*	*30.66*	*377.80*	*< 0.0001*	
AB	*29.93*	*1*	*29.93*	*368.74*	*< 0.0001*	
AC	*128.50*	*1*	*128.50*	*1583.26*	*< 0.0001*	
AD	*0.047*	*1*	*0.047*	*0.58*	*0.4586*	
BC	*0.074*	*1*	*0.074*	*0.91*	*0.3547*	
BD	*0.018*	*1*	*0.018*	*0.22*	*0.6453*	
CD	*0.047*	*1*	*0.047*	*0.58*	*0.4564*	
ABC	*78.75*	*1*	*78.75*	*970.33*	*< 0.0001*	
ABD	*0.077*	*1*	*0.077*	*0.95*	*0.3450*	
ACD	*2.926E-003*	*1*	*2.926E-003*	*0.036*	*0.8518*	
BCD	*0.010*	*1*	*0.010*	*0.13*	*0.7282*	
ABCD	*1.596E-003*	*1*	*1.596E-003*	*0.020*	*0.8902*	
Residual	1.30	16	0.081			
Lack of Fit	*0.000*	*0*				
Pure Error	*1.30*	*16*	*0.081*			
Cor Total	572.25	31				

The Model F-value of 468.99 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, D, AB, AC, ABC are significant model terms.

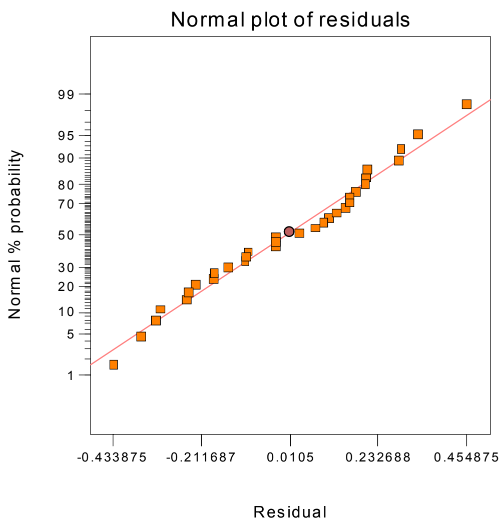
(c) Write down a regression model that can be used to predict crack length as a function of the significant main effects and interactions you have identified in part (b).

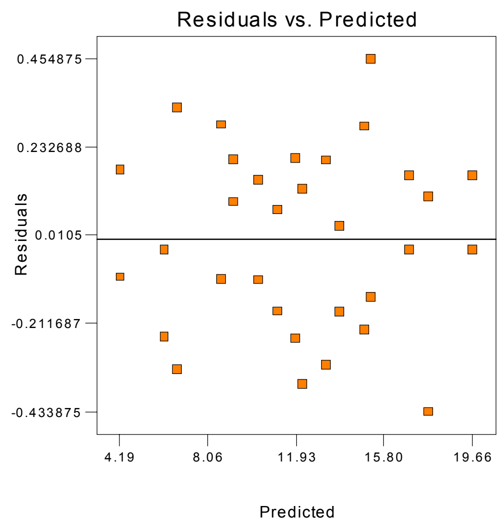
Design Expert Output

Final Equation in Terms of Coded Factors:

```
Crack Length=
        +11.99
        +1.51  *A
        +1.99  *B
        -1.80  *C
        +0.98  *D
        +0.97  *A*B
        -2.00  *A*C
        +1.57  * A * B * C
```

(d) Analyze the residuals from this experiment.

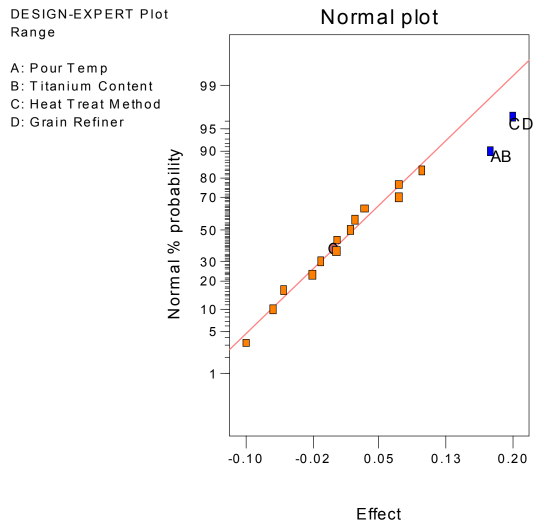




There is nothing unusual about the residuals.

(e) Is there an indication that any of the factors affect the variability in cracking?

By calculating the range of the two readings in each cell, we can also evaluate the effects of the factors on variation. The following is the normal probability plot of effects:



It appears that the AB and CD interactions could be significant. The following is the ANOVA for the range data:

Design Expert Output

Response: Range

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.29	2	0.14	11.46	0.0014	significant
AB	0.13	1	0.13	9.98	0.0075	
CD	0.16	1	0.16	12.94	0.0032	
Residual	0.16	13	0.013			
Cor Total	0.45	15				

The Model F-value of 11.46 implies the model is significant. There is only a 0.14% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case AB, CD are significant model terms.

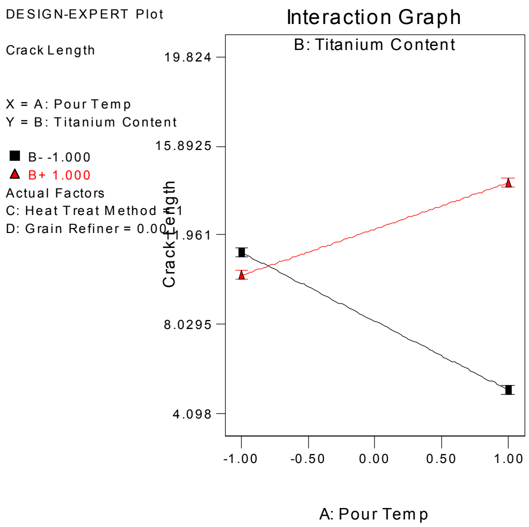
Final Equation in Terms of Coded Factors:

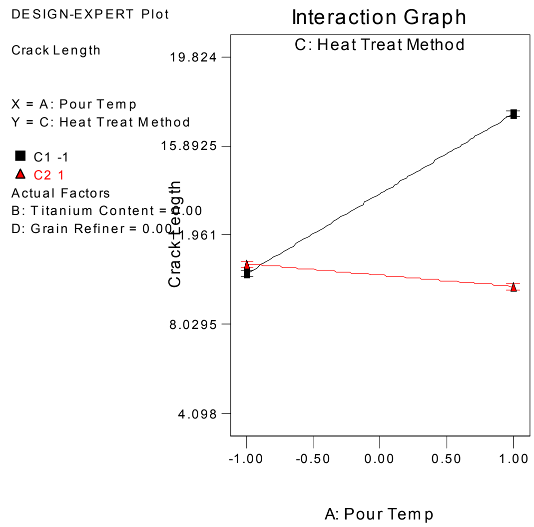
Range =
+0.37
+0.089 * A * B
+0.10 * C * D

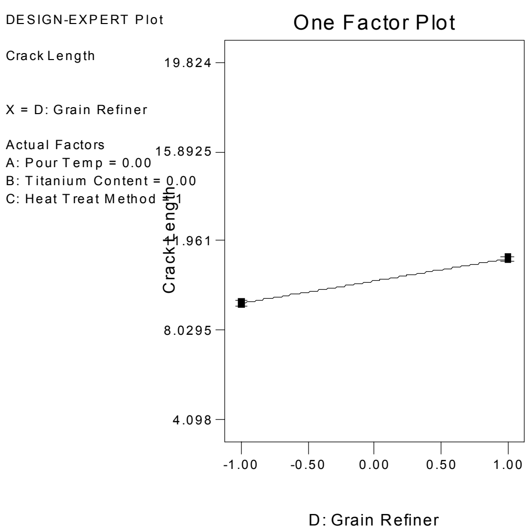
(f) What recommendations would you make regarding process operations?

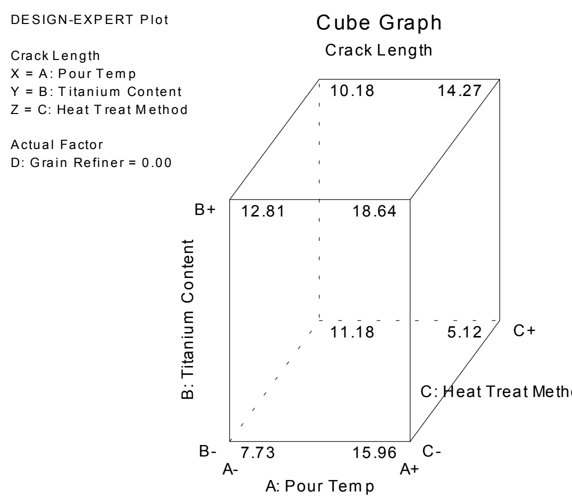
Use interaction and/or main effect plots to assist in drawing conclusions. From the interaction plots, choose *A* at the high level and *B* at the high level. In each of these plots, *D* can be at either level. From the main effects plot of *C*, choose *C* at the high level. Based on the range analysis, with *C* at the high level, *D* should be set at the low level.

From the analysis of the crack length data:

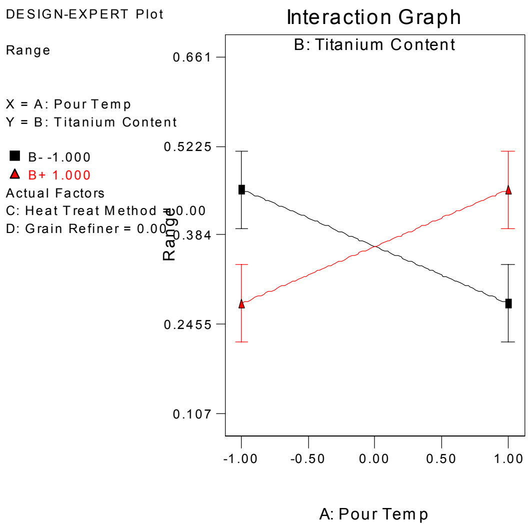


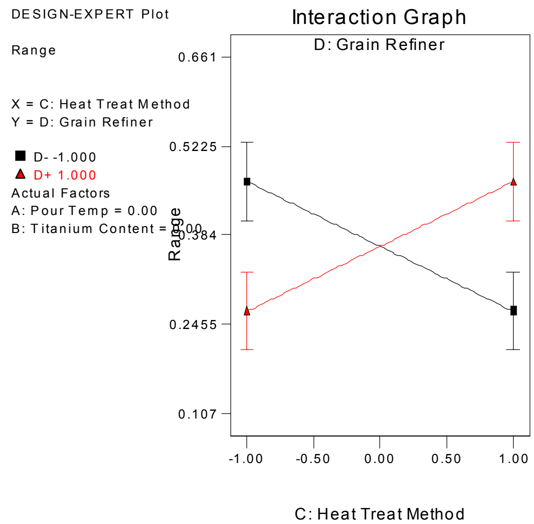






From the analysis of the ranges:





6-16 Continuation of Problem 6-15. One of the variables in the experiment described in Problem 6-15, heat treatment method (c), is a categorical variable. Assume that the remaining factors are continuous.

(a) Write two regression models for predicting crack length, one for each level of the heat treatment method variable. What differences, if any, do you notice in these two equations?

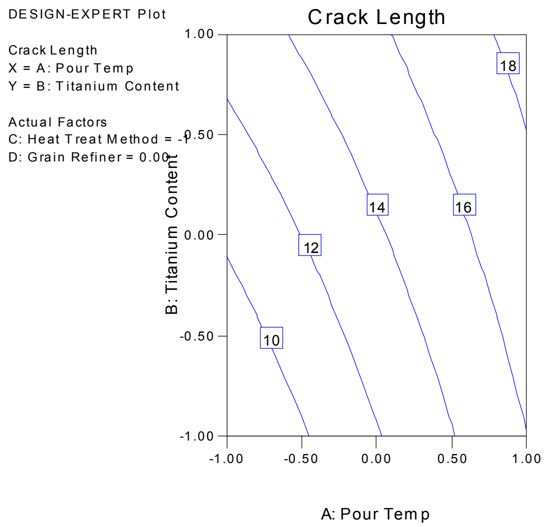
Design Expert Output

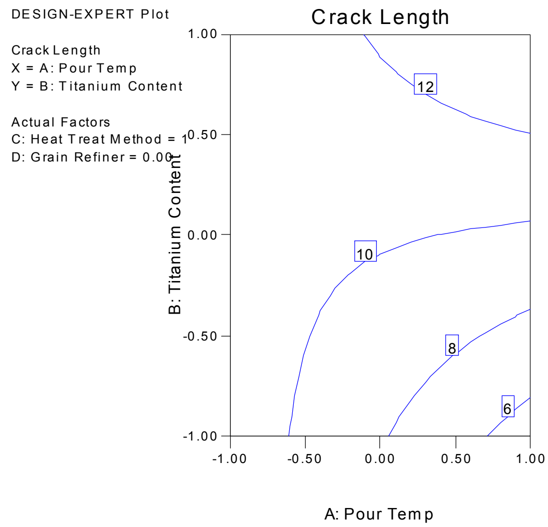
```
Final Equation in Terms of Coded Factors

        Heat Treat Method  -1
              Crack Length  =
                 +13.78619
                  +3.51331   * Pour Temp
                  +1.93994   * Titanium Content
                  +0.97888   * Grain Refiner
                  -0.60169   * Pour Temp * Titanium Content

        Heat Treat Method   1
              Crack Length  =
                 +10.18994
                  -0.49444   * Pour Temp
                  +2.03594   * Titanium Content
                  +0.97888   * Grain Refiner
                  +2.53581   * Pour Temp * Titanium Content
```

(b) Generate appropriate response surface contour plots for the two regression models in part (a).





(c) What set of conditions would you recommend for the factors *A*, *B* and *D* if you use heat treatment method *C*=+?

High level of *A*, low level of *B*, and low level of *D*.

(d) Repeat part (c) assuming that you wish to use heat treatment method *C*=-.

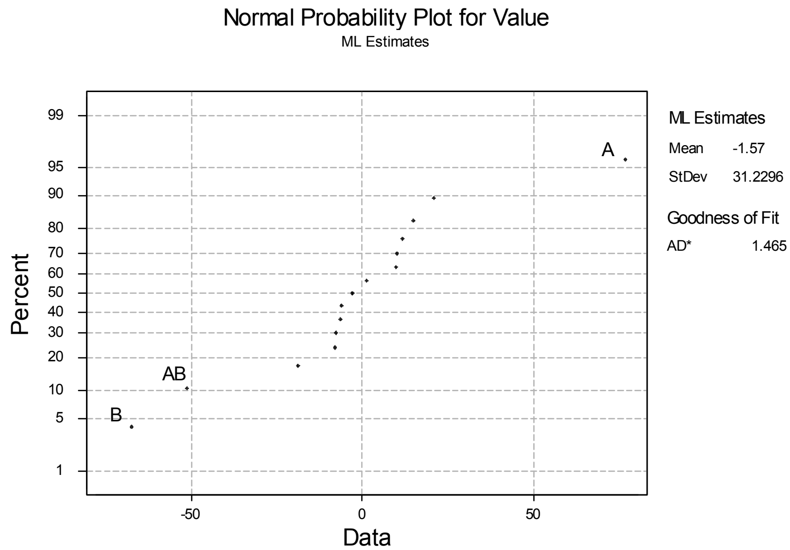
Low level of *A*, low level of *B*, and low level of *D*.

6-17 An experimenter has run a single replicate of a 2^4 design. The following effect estimates have been calculated:

$A = 76.95$	$AB = -51.32$	$ABC = -2.82$
$B = -67.52$	$AC = 11.69$	$ABD = -6.50$
$C = -7.84$	$AD = 9.78$	$ACD = 10.20$
$D = -18.73$	$BC = 20.78$	$BCD = -7.98$
	$BD = 14.74$	$ABCD = -6.25$
	$CD = 1.27$	

(a) Construct a normal probability plot of these effects.

The plot from Minitab follows.



(b) Identify a tentative model, based on the plot of the effects in part (a).

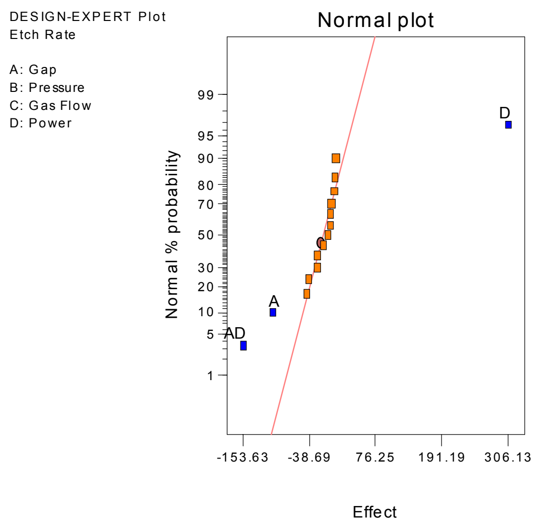
$$\hat{y} = Intercept + 38.475x_A - 33.76x_B - 25.66x_Ax_B$$

6-18 An article in *Solid State Technology* ("Orthogonal Design for Process Optimization and Its Application in Plasma Etching," May 1987, pp. 127-132) describes the application of factorial designs in developing a nitride etch process on a single-wafer plasma etcher. The process uses C_2F_6 as the reactant gas. Four factors are of interest: anode-cathode gap (*A*), pressure in the reactor chamber (*B*), C_2F_6 gas flow (*C*), and power applied to the cathode (*D*). The response variable of interest is the etch rate for silicon nitride. A single replicate of a 2^4 design in run, and the data are shown below:

Run Number	Actual Run Order	*A*	*B*	*C*	*D*	Etch Rate (A/min)		Factor Low (-)	Levels High (+)
1	13	-	-	-	-	550	*A* (cm)	0.80	1.20
2	8	+	-	-	-	669	*B* (mTorr)	4.50	550
3	12	-	+	-	-	604	*C* (SCCM)	125	200
4	9	+	+	-	-	650	*D* (W)	275	325
5	4	-	-	+	-	633			
6	15	+	-	+	-	642			
7	16	-	+	+	-	601			
8	3	+	+	+	-	635			
9	1	-	-	-	+	1037			
10	14	+	-	-	+	749			
11	5	-	+	-	+	1052			
12	10	+	+	-	+	868			
13	11	-	-	+	+	1075			
14	2	+	-	+	+	860			
15	7	-	+	+	+	1063			

16	6	+	+	+	+	729

(a) Estimate the factor effects. Construct a normal probability plot of the factor effects. Which effects appear large?



(b) Conduct an analysis of variance to confirm your findings for part (a).

Design Expert Output

Response: Etch Rate in A/min
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	5.106E+005	3	1.702E+005	97.91	< 0.0001	significant
A	*41310.56*	*1*	*41310.56*	*23.77*	*0.0004*	
D	*3.749E+005*	*1*	*3.749E+005*	*215.66*	*< 0.0001*	
AD	*94402.56*	*1*	*94402.56*	*54.31*	*< 0.0001*	
Residual	20857.75	12	1738.15			
Cor Total	5.314E+005	15				

The Model F-value of 97.91 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, D, AD are significant model terms.

(c) What is the regression model relating etch rate to the significant process variables?

Design Expert Output

Final Equation in Terms of Coded Factors:

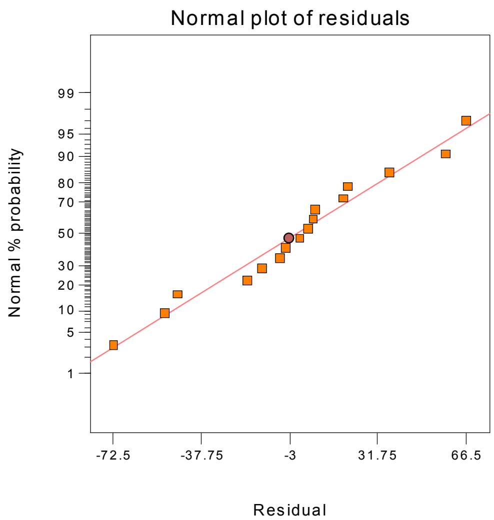
Etch Rate =
+776.06
-50.81 * A
+153.06 * D
-76.81 * A * D

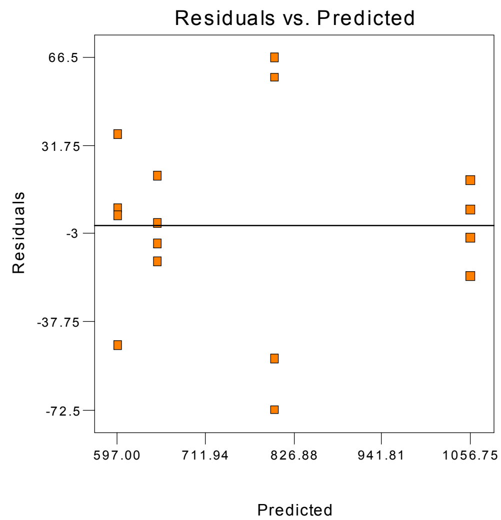
Final Equation in Terms of Actual Factors:

Etch Rate =
-5415.37500

+4354.68750	* Gap
+21.48500	* Power
-15.36250	* Gap * Power

(d) Analyze the residuals from this experiment. Comment on the model's adequacy.





The residual versus predicted plot shows a slight football shape indicating very mild inequality of variance.

(e) If not all the factors are important, project the 2^4 design into a 2^k design with k<4 and conduct that analysis of variance. The analysis of variance table is the same as in part (b).

Design Expert Output

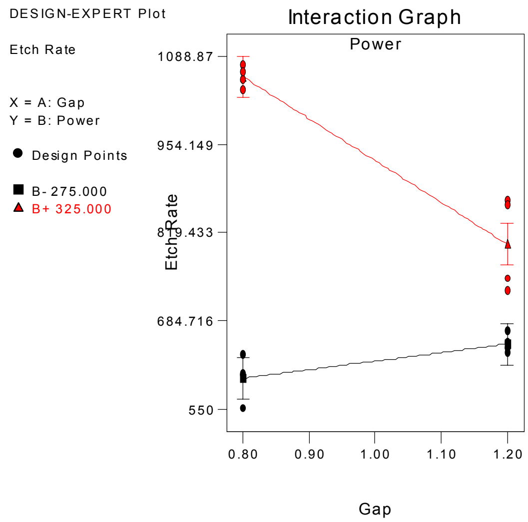
Response: Etch Rate in A/min
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	5.106E+005	3	1.702E+005	97.91	< 0.0001	significant
A	*41310.56*	*1*	*41310.56*	*23.77*	*0.0004*	
B	*3.749E+005*	*1*	*3.749E+005*	*215.66*	*< 0.0001*	
AB	*94402.56*	*1*	*94402.56*	*54.31*	*< 0.0001*	
Residual	20857.75	12	1738.15			
Lack of Fit	*0.000*	*0*				
Pure Error	*20857.75*	*12*	*1738.15*			
Cor Total	5.314E+005	15				

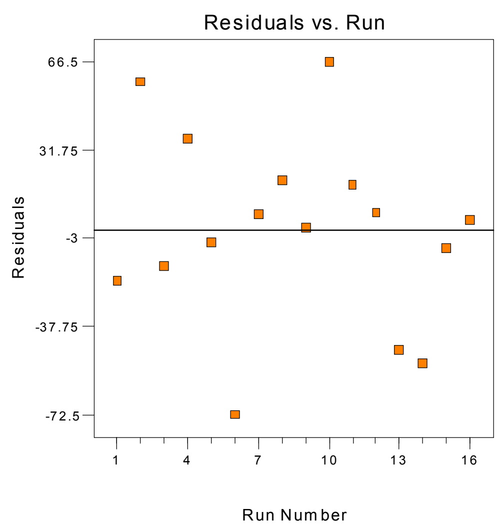
The Model F-value of 97.91 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.

(f) Draw graphs to interpret any significant interactions.



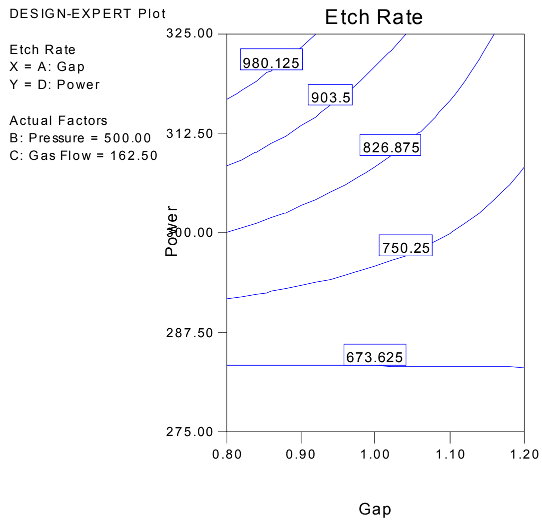
(g) Plot the residuals versus the actual run order. What problems might be revealed by this plot?



The plot of residuals versus run order can reveal trends in the process over time, inequality of variance with time, and possibly indicate that there may be factors that were not included in the original experiment.

6-19 Continuation of Problem 6-18. Consider the regression model obtained in part (c) of Problem 6-18.

(a) Construct contour plots of the etch rate using this model.



(b) Suppose that it was necessary to operate this process at an etch rate of 800 Å/min. What settings of the process variables would you recommend?

Run at the low level of anode-cathode gap (0.80 cm) and at a cathode power level of about 286 watts. The curve is flatter (more robust) on the low end of the anode-cathode variable.

6-20 Consider the single replicate of the 2^4 design in Example 6-2. Suppose we had arbitrarily decided to analyze the data assuming that all three- and four-factor interactions were negligible. Conduct this analysis and compare your results with those obtained in the example. Do you think that it is a good idea to arbitrarily assume interactions to be negligible even if they are relatively high-order ones?

Design Expert Output

Response: Etch Rate in A/min
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	5.212E+005	10	52123.41	25.58	0.0011	significant
A	*41310.56*	*1*	*41310.56*	*20.28*	*0.0064*	
B	*10.56*	*1*	*10.56*	*5.184E-003*	*0.9454*	
C	*217.56*	*1*	*217.56*	*0.11*	*0.7571*	
D	*3.749E+005*	*1*	*3.749E+005*	*183.99*	*< 0.0001*	
AB	*248.06*	*1*	*248.06*	*0.12*	*0.7414*	
AC	*2475.06*	*1*	*2475.06*	*1.21*	*0.3206*	
AD	*94402.56*	*1*	*94402.56*	*46.34*	*0.0010*	
BC	*7700.06*	*1*	*7700.06*	*3.78*	*0.1095*	
BD	*1.56*	*1*	*1.56*	*7.669E-004*	*0.9790*	
CD	*18.06*	*1*	*18.06*	*8.866E-003*	*0.9286*	
Residual	10186.81	5	2037.36			
Cor Total	5.314E+005	15				

The Model F-value of 25.58 implies the model is significant. There is only a 0.11% chance that a "Model F-Value" this large could occur due to noise.

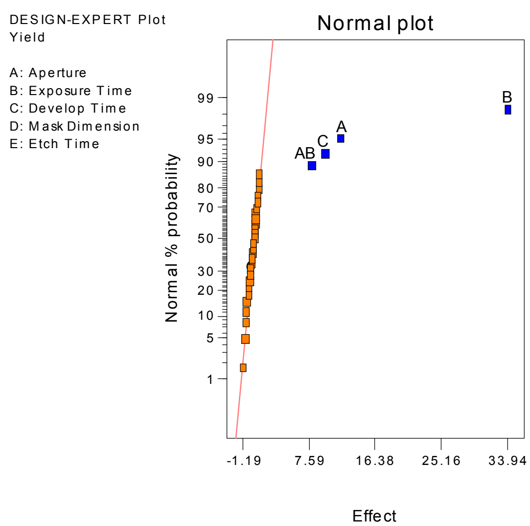
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, D, AD are significant model terms.

This analysis of variance identifies the same effects as the normal probability plot of effects approach used in Example 6-2. In general, it is not a good idea to arbitrarily pool interactions. Use the normal probability plot of effect estimates as a guide in the choice of which effects to tentatively include in the model.

6-21 An experiment was run in a semiconductor fabrication plant in an effort to increase yield. Five factors, each at two levels, were studied. The factors (and levels) were A = aperture setting (small, large), B = exposure time (20% below nominal, 20% above nominal), C = development time (30 s, 45 s), D = mask dimension (small, large), and E = etch time (14.5 min, 15.5 min). The unreplicated 2^5 design shown below was run.

(1) =	7	*d* =	8	*e* =	8	*de* =	6
a =	9	*ad* =	10	*ae* =	12	*ade* =	10
b =	34	*bd* =	32	*be* =	35	*bde* =	30
ab =	55	*abd* =	50	*abe* =	52	*abde* =	53
c =	16	*cd* =	18	*ce* =	15	*cde* =	15
ac =	20	*acd* =	21	*ace* =	22	*acde* =	20
bc =	40	*bcd* =	44	*bce* =	45	*bcde* =	41
abc =	60	*abcd* =	61	*abce* =	65	*abcde* =	63

(a) Construct a normal probability plot of the effect estimates. Which effects appear to be large?



(b) Conduct an analysis of variance to confirm your findings for part (a).

Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	11585.13	4	2896.28	991.83	< 0.0001	significant
A	*1116.28*	*1*	*1116.28*	*382.27*	*< 0.0001*	
B	*9214.03*	*1*	*9214.03*	*3155.34*	*< 0.0001*	
C	*750.78*	*1*	*750.78*	*257.10*	*< 0.0001*	
AB	*504.03*	*1*	*504.03*	*172.61*	*< 0.0001*	
Residual	78.84	27	2.92			
Cor Total	11663.97	31				

```
The Model F-value of 991.83 implies the model is significant.  There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, AB are significant model terms.
```

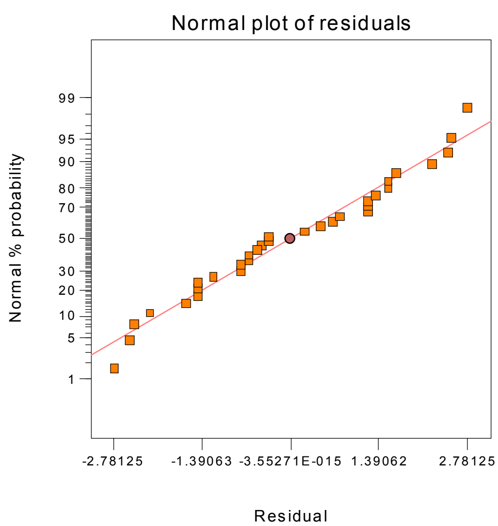
(c) Write down the regression model relating yield to the significant process variables.

Design Expert Output

```
Final Equation in Terms of Actual Factors:
        Aperture     small
                      Yield       =
                     +0.40625
                     +0.65000     * Exposure Time
                     +0.64583     * Develop Time

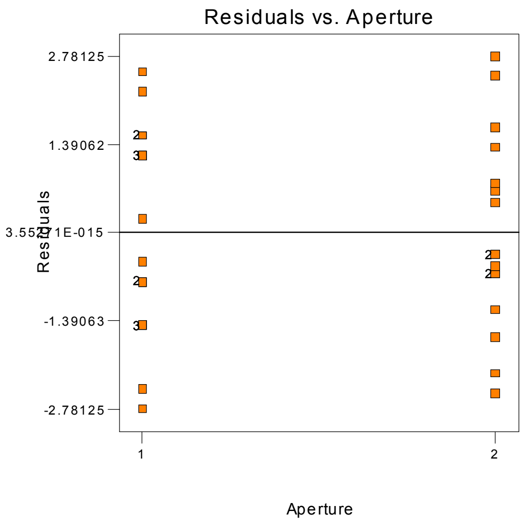
        Aperture     large
                      Yield       =
                     +12.21875
                     +1.04688     * Exposure Time
                     +0.64583     * Develop Time
```

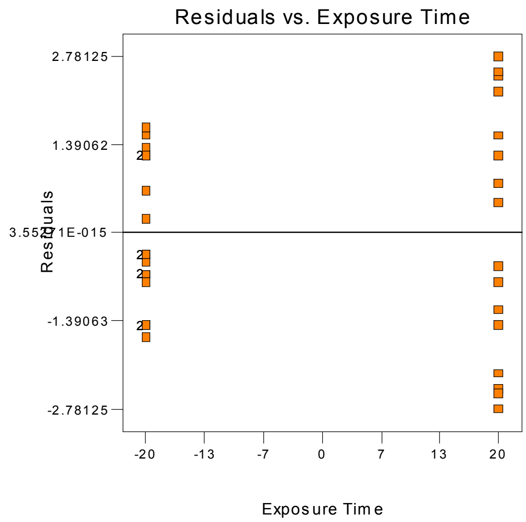
(d) Plot the residuals on normal probability paper. Is the plot satisfactory?

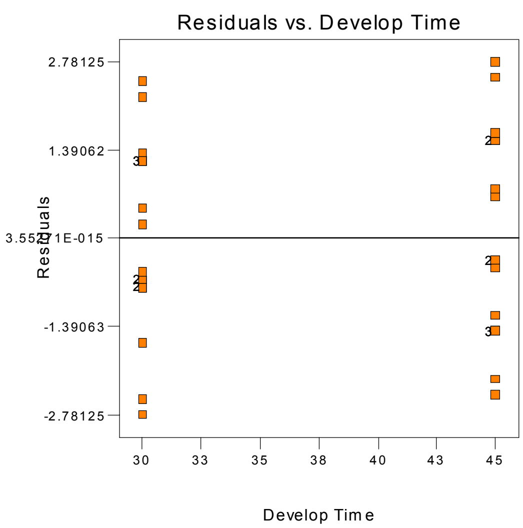


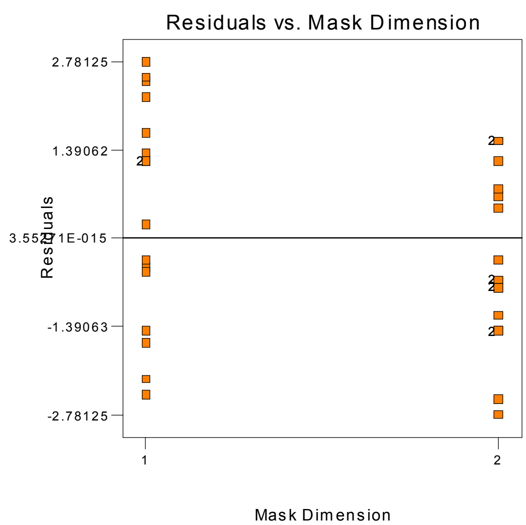
There is nothing unusual about this plot.

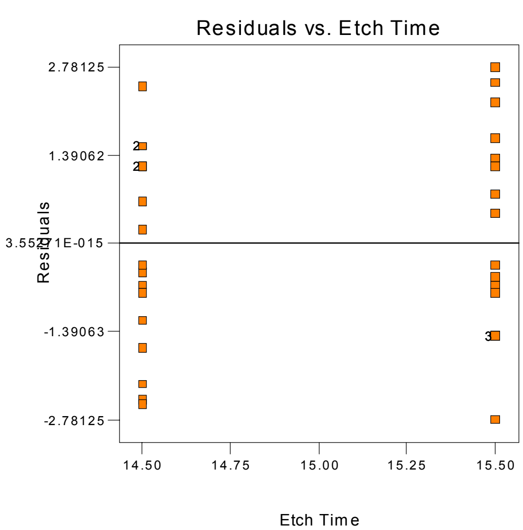
(e) Plot the residuals versus the predicted yields and versus each of the five factors. Comment on the plots.


Residuals vs. Aperture
2.78125
1.39062
3.55271E-015
-1.39063
-2.78125
Residuals
1
2
Aperture


Residuals vs. Exposure Time
2.78125
1.39062
3.55271E-015
-1.39063
-2.78125
Residuals
-20
-13
-7
0
7
13
20
Exposure Time

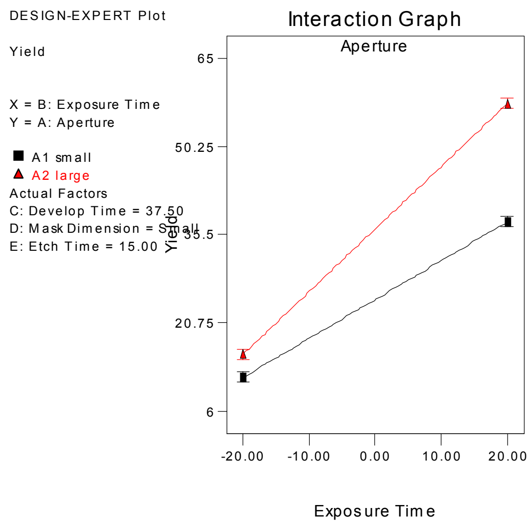

Residuals vs. Develop Time
2.78125
1.39062
3.55271E-015
-1.39063
-2.78125
Residuals
30
33
35
38
40
43
45
Develop Time


Residuals vs. Mask Dimension
2.78125
1.39062
3.55271E-015
-1.39063
-2.78125
Residuals
1
2
Mask Dimension


Residuals vs. Etch Time
2.78125
1.39062
3.55271E-015
-1.39063
-2.78125
Residuals
14.50
14.75
15.00
15.25
15.50
Etch Time

The plot of residual versus exposure time shows some very slight inequality of variance. There is no strong evidence of a potential problem.

(f) Interpret any significant interactions.

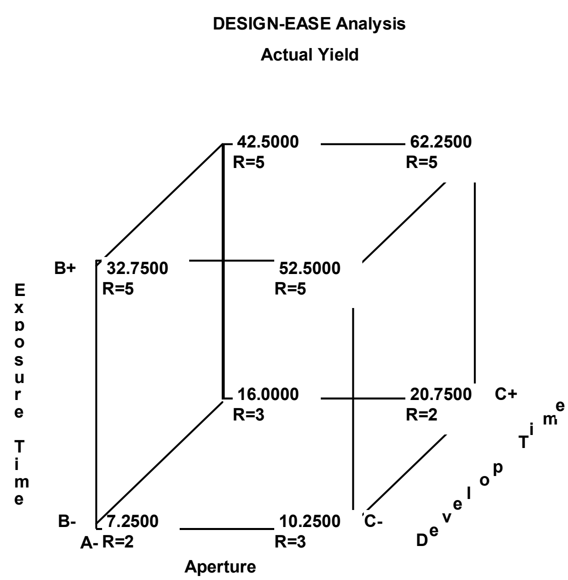


Factor A does not have as large an effect when B is at its low level as it does when B is at its high level.

(g) What are your recommendations regarding process operating conditions?

For the highest yield, run with B at the high level, A at the high level and C at the high level.

(h) Project the 2^5 design in this problem into a 2^k design in the important factors. Sketch the design and show the average and range of yields at each run. Does this sketch aid in interpreting the results of this experiment?



This cube plot aids in interpretation. The strong *AB* interaction and the large positive effect of *C* are clearly evident.

6-22 Continuation of Problem 6-21. Suppose that the experimenter had run four runs at the center points in addition to the 32 trials in the original experiment. The yields obtained at the center point runs were 68, 74, 76, and 70.

(a) Reanalyze the experiment, including a test for pure quadratic curvature.

$$SS_{PureQuadratic} = \frac{n_F n_C \left(\bar{y}_F - \bar{y}_C\right)^2}{n_F + n_C} = \frac{(32)(4)(30.53125 - 72)^2}{32 + 4} = 6114.337$$

Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	11461.09	4	2865.27	353.92	< 0.0001	significant
A	*992.25*	*1*	*992.25*	*122.56*	*< 0.0001*	
B	*9214.03*	*1*	*9214.03*	*1138.12*	*< 0.0001*	
C	*750.78*	*1*	*750.78*	*92.74*	*< 0.0001*	
AB	*504.03*	*1*	*504.03*	*62.26*	*< 0.0001*	
Curvature	6114.34	1	6114.34	755.24	< 0.0001	significant
Residual	242.88	30	8.10			
Cor Total	17818.31	35				

The Model F-value of 353.92 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AB are significant model terms.

(b) Discuss what your next step would be.

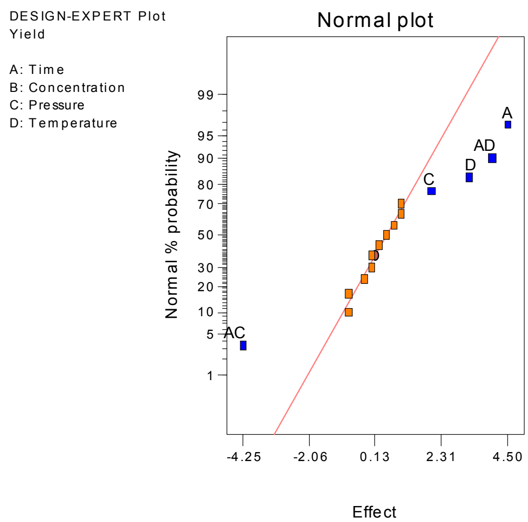
Add axial points and fit a second-order model.

6-23 In a process development study on yield, four factors were studied, each at two levels: time (*A*), concentration (*B*), pressure (*C*), and temperature (*D*). A single replicate of a 2^4 design was run, and the resulting data are shown in the following table:

Run Number	Actual Run Order	*A*	*B*	*C*	*D*	Yield (lbs)		Factor Low (-)	Levels High (+)
1	5	-	-	-	-	12	*A* (h)	2.5	3.0
2	9	+	-	-	-	18	*B* (%)	14	18
3	8	-	+	-	-	13	*C* (psi)	60	80
4	13	+	+	-	-	16	*D* (°C)	225	250
5	3	-	-	+	-	17			
6	7	+	-	+	-	15			
7	14	-	+	+	-	20			
8	1	+	+	+	-	15			
9	6	-	-	-	+	10			

10	11	+	-	-	+	25
11	2	-	+	-	+	13
12	15	+	+	-	+	24
13	4	-	-	+	+	19
14	16	+	-	+	+	21
15	10	-	+	+	+	17
16	12	+	+	+	+	23

(a) Construct a normal probability plot of the effect estimates. Which factors appear to have large effects?



A, C, D and the *AC* and *AD* interactions.

(b) Conduct an analysis of variance using the normal probability plot in part (a) for guidance in forming an error term. What are your conclusions?

Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	275.50	5	55.10	33.91	< 0.0001	significant
A	*81.00*	*1*	*81.00*	*49.85*	*< 0.0001*	
C	*16.00*	*1*	*16.00*	*9.85*	*0.0105*	
D	*42.25*	*1*	*42.25*	*26.00*	*0.0005*	
AC	*72.25*	*1*	*72.25*	*44.46*	*< 0.0001*	
AD	*64.00*	*1*	*64.00*	*39.38*	*< 0.0001*	
Residual	16.25	10	1.62			
Cor Total	291.75	15				

The Model F-value of 33.91 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, C, D, AC, AD are significant model terms.

(c) Write down a regression model relating yield to the important process variables.

Design Expert Output

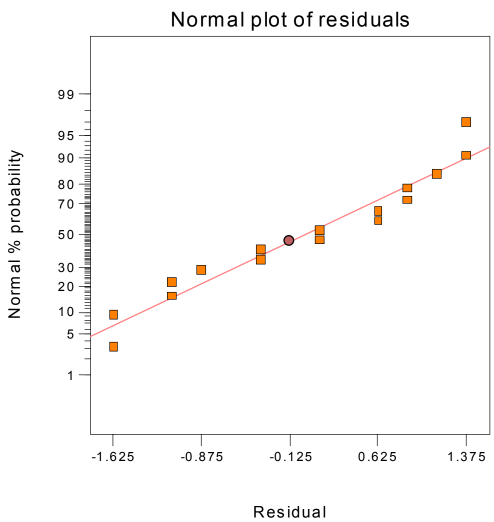
Final Equation in Terms of Coded Factors:

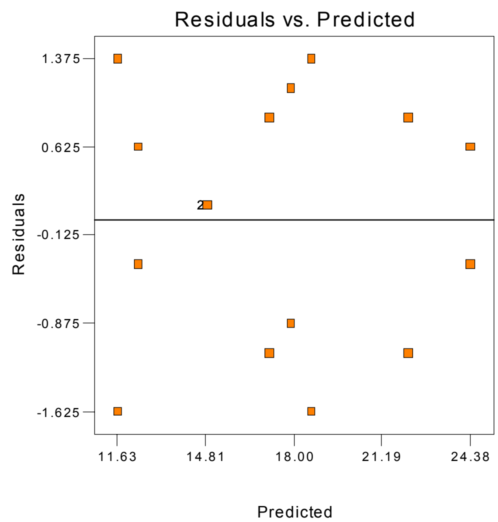
Yield	=
+17.38	
+2.25	*A
+1.00	*C
+1.63	*D
-2.13	*A*C
+2.00	*A*D

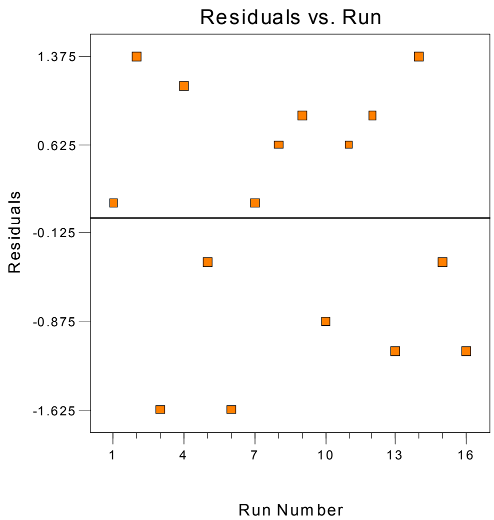
Final Equation in Terms of Actual Factors:

Yield	=
+209.12500	
-83.50000	* Time
+2.43750	* Pressure
-1.63000	* Temperature
-0.85000	* Time * Pressure
+0.64000	* Time * Temperature

(d) Analyze the residuals from this experiment. Does your analysis indicate any potential problems?

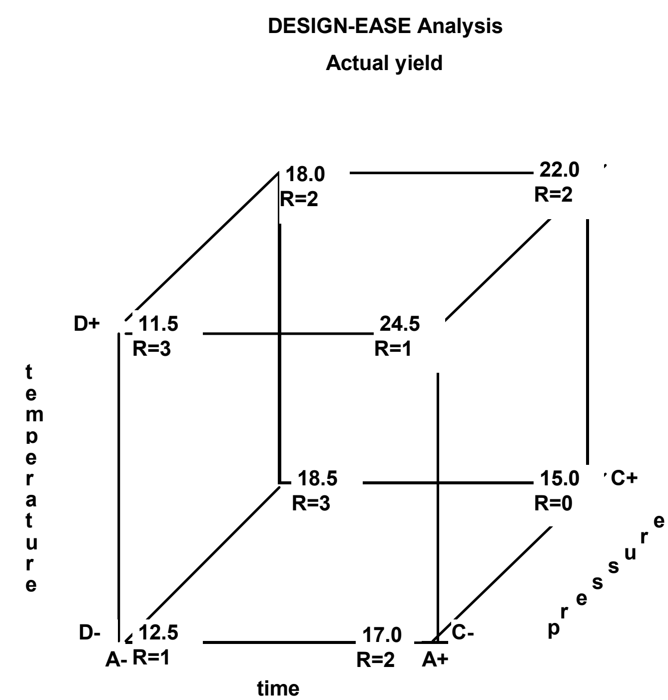






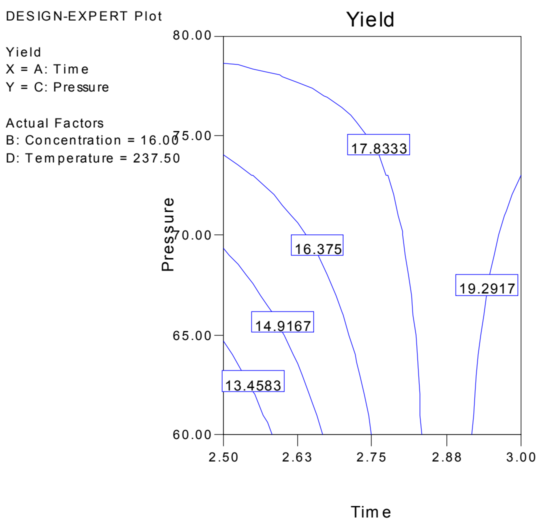
There is nothing unusual about the residual plots.

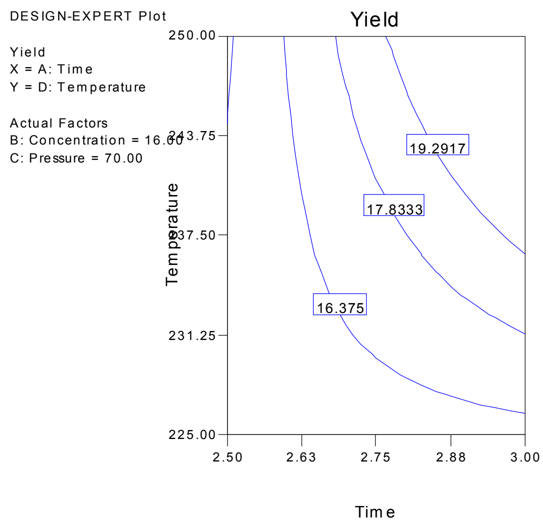
(e) Can this design be collapsed into a 2^3 design with two replicates? If so, sketch the design with the average and range of yield shown at each point in the cube. Interpret the results.



6-24 Continuation of Problem 6-23. Use the regression model in part (c) of Problem 6-23 to generate a response surface contour plot of yield. Discuss the practical purpose of this response surface plot.

The response surface contour plot shows the adjustments in the process variables that lead to an increasing or decreasing response. It also displays the curvature of the response in the design region, possibly indicating where robust operating conditions can be found. Two response surface contour plots for this process are shown below. These were formed from the model written in terms of the original design variables.





6-25 The scrumptious brownie experiment. The author is an engineer by training and a firm believer in learning by doing. I have taught experimental design for many years to a wide variety of audiences and have always assigned the planning, conduct, and analysis of an actual experiment to the class participants. The participants seem to enjoy this practical experience and always learn a great deal from it. This problem uses the results of an experiment performed by Gretchen Krueger at Arizona State University.

There are many different ways to bake brownies. The purpose of this experiment was to determine how the pan material, the brand of brownie mix, and the stirring method affect the scrumptiousness of brownies. The factor levels were

Factor	Low (-)	High (+)
A = pan material	Glass	Aluminum
B = stirring method	Spoon	Mixer
C = brand of mix	Expensive	Cheap

The response variable was scrumptiousness, a subjective measure derived from a questionnaire given to the subjects who sampled each batch of brownies. (The questionnaire dealt with such issues as taste, appearance, consistency, aroma, and so forth.) An eight-person test panel sampled each batch and filled out the questionnaire. The design matrix and the response data are shown below:

Brownie Batch	*A*	*B*	*C*	1	Test 2	Panel 3	Results 4	5	6	7	8
1	-	-	-	11	9	10	10	11	10	8	9
2	+	-	-	15	10	16	14	12	9	6	15
3	-	+	-	9	12	11	11	11	11	11	12
4	+	+	-	16	17	15	12	13	13	11	11
5	-	-	+	10	11	15	8	6	8	9	14

6	+	-	+	12	13	14	13	9	13	14	9
7	-	+	+	10	12	13	10	7	7	17	13
8	+	+	+	15	12	15	13	12	12	9	14

(a) Analyze the data from this experiment as if there were eight replicates of a 2^3 design. Comment on the results.

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	93.25	7	13.32	2.20	0.0475	significant
A	*72.25*	*1*	*72.25*	*11.95*	*0.0010*	
B	*18.06*	*1*	*18.06*	*2.99*	*0.0894*	
C	*0.063*	*1*	*0.063*	*0.010*	*0.9194*	
AB	*0.062*	*1*	*0.062*	*0.010*	*0.9194*	
AC	*1.56*	*1*	*1.56*	*0.26*	*0.6132*	
BC	*1.00*	*1*	*1.00*	*0.17*	*0.6858*	
ABC	*0.25*	*1*	*0.25*	*0.041*	*0.8396*	
Residual	338.50	56	6.04			
Lack of Fit	*0.000*	*0*				
Pure Error	*338.50*	*56*	*6.04*			
Cor Total	431.75	63				

The Model F-value of 2.20 implies the model is significant. There is only a 4.75% chance that a "Model F-Value" this large could occur due to noise.

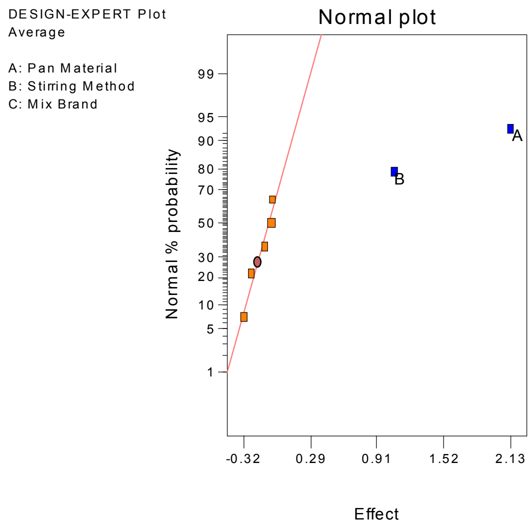
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A are significant model terms.

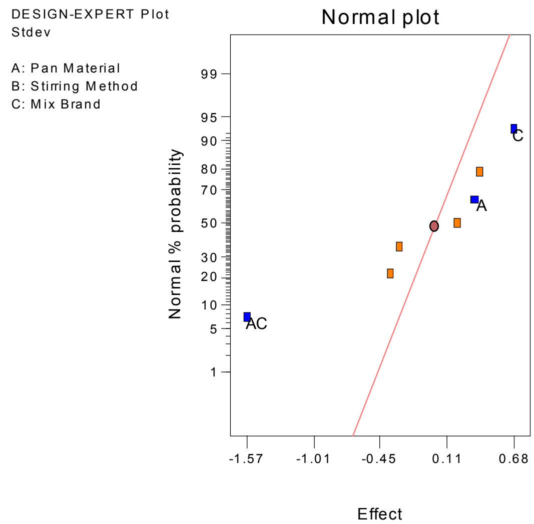
In this analysis, *A*, the pan material and *B*, the stirring method, appear to be significant. There are 56 degrees of freedom for the error, yet only eight batches of brownies were cooked, one for each recipe.

(b) Is the analysis in part (a) the correct approach? There are only eight batches; do we really have eight replicates of a 2^3 factorial design?

The different rankings by the taste-test panel are not replicates, but repeat observations by differenttesters on the same batch of brownies. It is not a good idea to use the analysis in part (a) because the estimate of error may not reflect the batch-to-batch variation.

(c) Analyze the average and standard deviation of the scrumptiousness ratings. Comment on the results. Is this analysis more appropriate than the one in part (a)? Why or why not?





Design Expert Output

Response: Average
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	11.28	2	5.64	76.13	0.0002	significant
A	*9.03*	*1*	*9.03*	*121.93*	*0.0001*	
B	*2.25*	*1*	*2.25*	*30.34*	*0.0027*	
Residual	0.37	5	0.074			
Cor Total	11.65	7				

The Model F-value of 76.13 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

Design Expert Output

Response: Stdev
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	6.05	3	2.02	9.77	0.0259	significant
A	*0.24*	*1*	*0.24*	*1.15*	*0.3432*	
C	*0.91*	*1*	*0.91*	*4.42*	*0.1034*	
AC	*4.90*	*1*	*4.90*	*23.75*	*0.0082*	
Residual	0.82	4	0.21			
Cor Total	6.87	7				

The Model F-value of 9.77 implies the model is significant. There is only a 2.59% chance that a "Model F-Value" this large could occur due to noise.

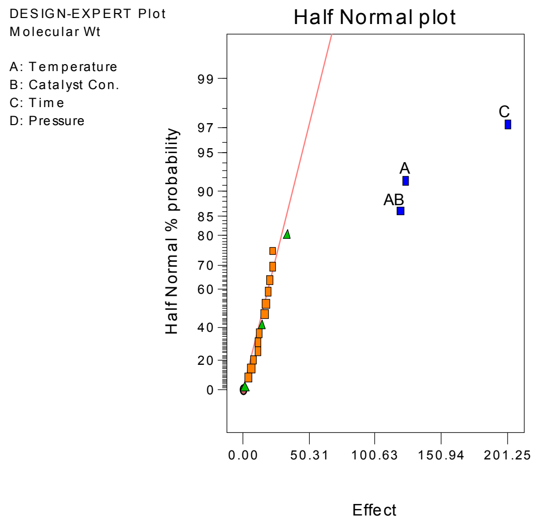
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case AC are significant model terms.

Variables A and B affect the mean rank of the brownies. Note that the AC interaction affects the standard deviation of the ranks. This is an indication that both factors A and C have some effect on the variability in the ranks. It may also indicate that there is some inconsistency in the taste test panel members. For the analysis of both the average of the ranks and the standard deviation of the ranks, the mean square error is now determined by pooling apparently unimportant effects. This is a more estimate of error than obtained assuming that all observations were replicates.

6-26 An experiment was conducted on a chemical process that produces a polymer. The four factors studied were temperature (*A*), catalyst concentration (*B*), time (*C*), and pressure (*D*). Two responses, molecular weight and viscosity, were observed. The design matrix and response data are shown below:

Run Number	Actual Run Order	*A*	*B*	*C*	*D*	Molecular Weight	Viscosity		Factor Low (-)	Levels High (+)
1	18	-	-	-	-	2400	1400	*A* (°C)	100	120
2	9	+	-	-	-	2410	1500	*B* (%)	4	8
3	13	-	+	-	-	2315	1520	*C* (min)	20	30
4	8	+	+	-	-	2510	1630	*D* (psi)	60	75
5	3	-	-	+	-	2615	1380			
6	11	+	-	+	-	2625	1525			
7	14	-	+	+	-	2400	1500			
8	17	+	+	+	-	2750	1620			
9	6	-	-	-	+	2400	1400			
10	7	+	-	-	+	2390	1525			
11	2	-	+	-	+	2300	1500			
12	10	+	+	-	+	2520	1500			
13	4	-	-	+	+	2625	1420			
14	19	+	-	+	+	2630	1490			
15	15	-	+	+	+	2500	1500			
16	20	+	+	+	+	2710	1600			
17	1	0	0	0	0	2515	1500			
18	5	0	0	0	0	2500	1460			
19	16	0	0	0	0	2400	1525			
20	12	0	0	0	0	2475	1500			

(a) Consider only the molecular weight response. Plot the effect estimates on a normal probability scale. What effects appear important?



A, C and the *AB* interaction.

(b) Use an analysis of variance to confirm the results from part (a). Is there an indication of curvature? *A,C* and the *AB* interaction are significant. While the main effect of *B* is not significant, it could be included to preserve hierarchy in the model. There is no indication of quadratic curvature.

Design Expert Output

Response: Molecular Wt
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.809E+005	3	93620.83	73.00	< 0.0001	significant
A	*61256.25*	*1*	*61256.25*	*47.76*	*< 0.0001*	
C	*1.620E+005*	*1*	*1.620E+005*	*126.32*	*< 0.0001*	
AB	*57600.00*	*1*	*57600.00*	*44.91*	*< 0.0001*	
Curvature	3645.00	1	3645.00	2.84	0.1125	not significant
Residual	19237.50	15	1282.50			
Lack of Fit	*11412.50*	*12*	*951.04*	*0.36*	*0.9106*	*not significant*
Pure Error	*7825.00*	*3*	*2608.33*			
Cor Total	3.037E+005	19				

The Model F-value of 73.00 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, C, AB are significant model terms.

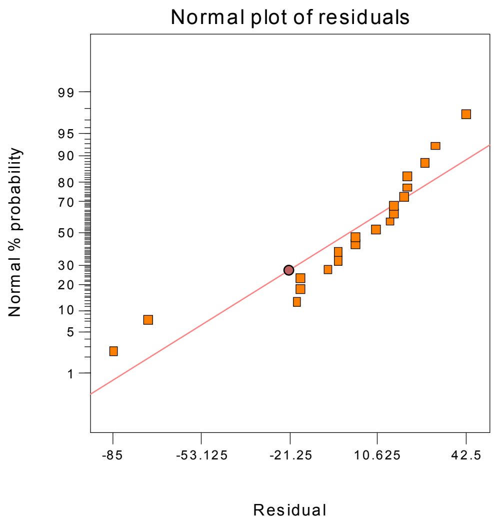
(c) Write down a regression model to predict molecular weight as a function of the important variables.

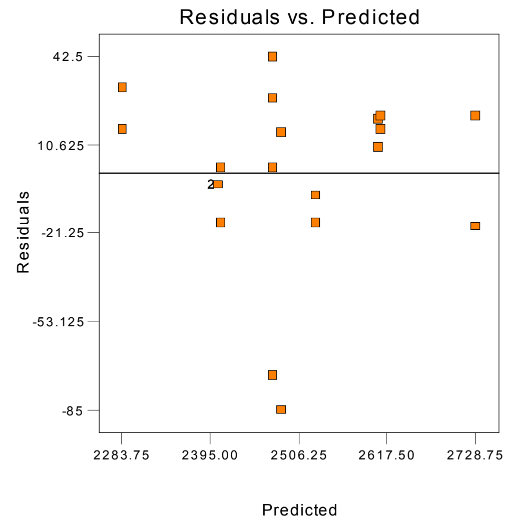
Design Expert Output

Final Equation in Terms of Coded Factors:

Molecular Wt	=
+2506.25	
+61.87	* A
+100.63	* C
+60.00	* A * B

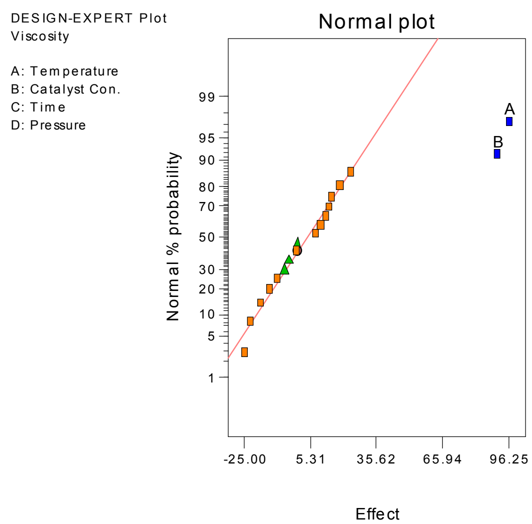
(d) Analyze the residuals and comment on model adequacy.





There are two residuals that appear to be large and should be investigated.

(e) Repeat parts (a) - (d) using the viscosity response.



Design Expert Output

Response: Viscosity

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

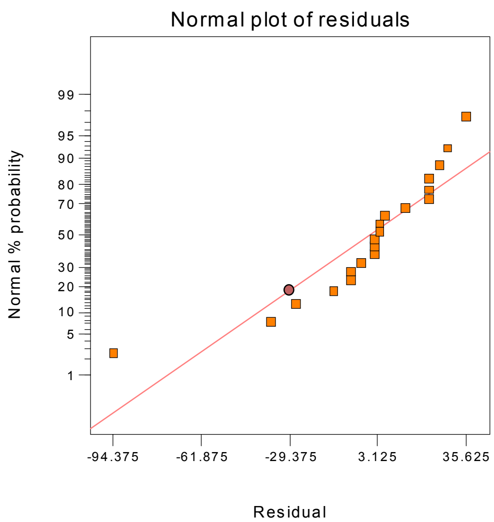
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	70362.50	2	35181.25	35.97	< 0.0001	significant
A	*37056.25*	*1*	*37056.25*	*37.88*	*< 0.0001*	
B	*33306.25*	*1*	*33306.25*	*34.05*	*< 0.0001*	
Curvature	61.25	1	61.25	0.063	0.8056	not significant
Residual	15650.00	16	978.13			
Lack of Fit	*13481.25*	*13*	*1037.02*	*1.43*	*0.4298*	*not significant*
Pure Error	*2168.75*	*3*	*722.92*			
Cor Total	86073.75	19				

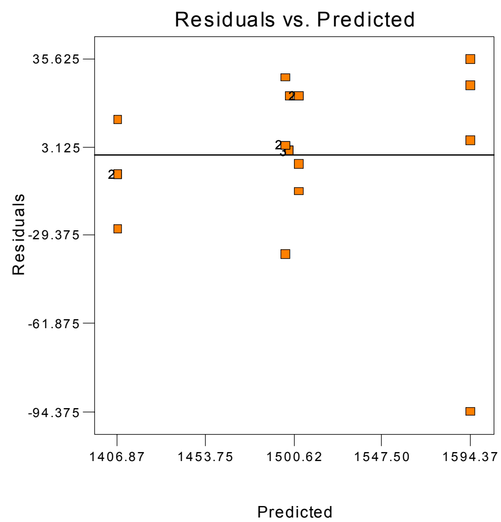
The Model F-value of 35.97 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

Final Equation in Terms of Coded Factors:

Viscosity	=	
+1500.62		
+48.13	* A	
+45.63	* B	

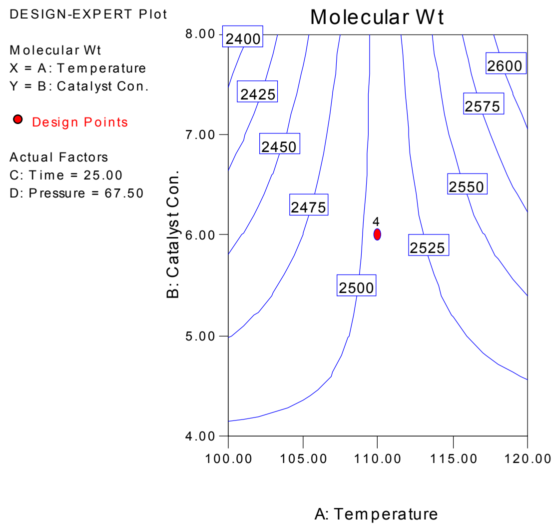




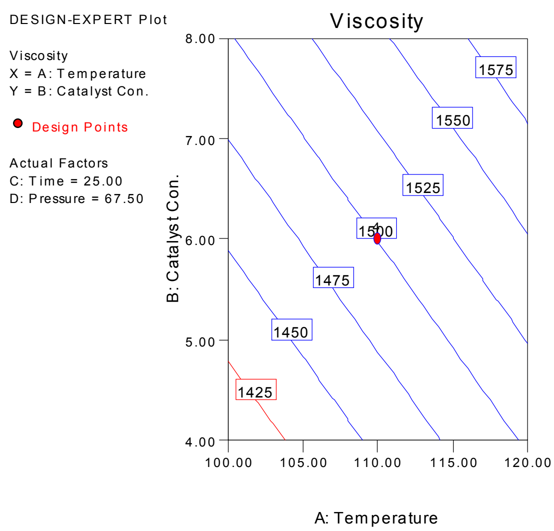
There is one large residual that should be investigated.

6-27 Continuation of Problem 6-26. Use the regression models for molecular weight and viscosity to answer the following questions.

(a) Construct a response surface contour plot for molecular weight. In what direction would you adjust the process variables to increase molecular weight? Increase temperature, catalyst and time.

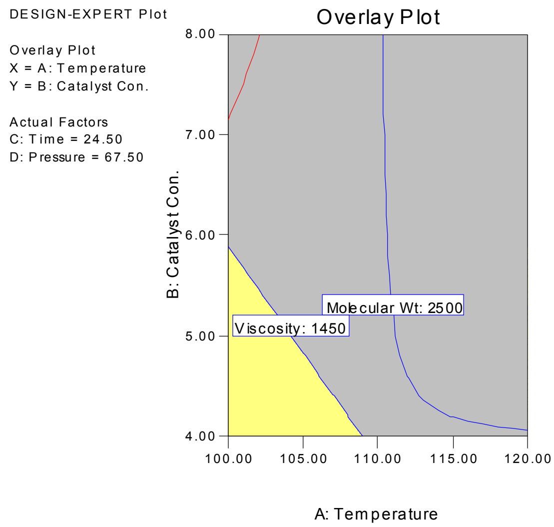


(a) Construct a response surface contour plot for viscosity. In what direction would you adjust the process variables to decrease viscosity?



Decrease temperature and catalyst.

(c) What operating conditions would you recommend if it was necessary to produce a product with a molecular weight between 2400 and 2500, and the lowest possible viscosity?



Set the temperature between 100 and 105, the catalyst between 4 and 5%, and the time at 24.5 minutes. The pressure was not significant and can be set at conditions that may improve other results of the process such as cost.

6-28 Consider the single replicate of the 2^4 design in Example 6-2. Suppose that we ran five points at the center (0,0,0,0) and observed the following responses: 73, 75, 71, 69, and 76. Test for curvature in this experiment. Interpret the results.

Design Expert Output

Response: Filtration Rate

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	5535.81	5	1107.16	68.01	< 0.0001	significant
A	*1870.56*	*1*	*1870.56*	*114.90*	*< 0.0001*	
C	*390.06*	*1*	*390.06*	*23.96*	*0.0002*	
D	*855.56*	*1*	*855.56*	*52.55*	*< 0.0001*	
AC	*1314.06*	*1*	*1314.06*	*80.71*	*< 0.0001*	
AD	*1105.56*	*1*	*1105.56*	*67.91*	*< 0.0001*	
Curvature	28.55	1	28.55	1.75	0.2066	not significant
Residual	227.93	14	16.28			
Lack of Fit	*195.13*	*10*	*19.51*	*2.38*	*0.2093*	*not significant*
Pure Error	*32.80*	*4*	*8.20*			
Cor Total	5792.29	20				

The Model F-value of 68.01 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, C, D, AC, AD are significant model terms.

The "Curvature F-value" of 1.75 implies the curvature (as measured by difference between the average of the center points and the average of the factorial points) in the design space is not significant relative to the noise. There is a 20.66% chance that a "Curvature F-value" this large could occur due to noise.

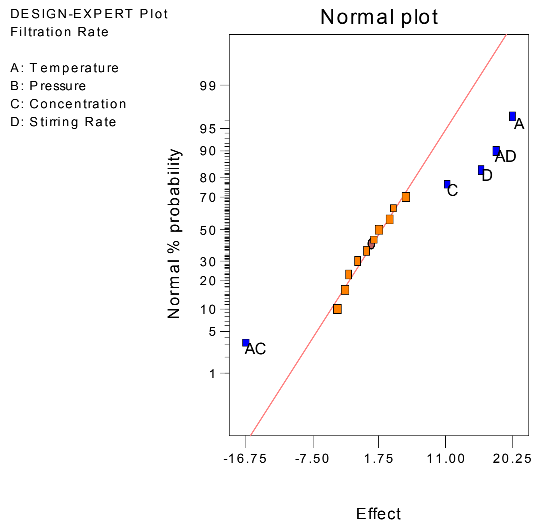
There is no indication of curvature.

6-29 A missing value in a 2^k factorial. It is not unusual to find that one of the observations in a 2^k design is missing due to faulty measuring equipment, a spoiled test, or some other reason. If the design is replicated *n* times (*n*>1) some of the techniques discussed in Chapter 14 can be employed, including estimating the missing observations. However, for an unreplicated factorial (*n*-1) some other method must be used. One logical approach is to estimate the missing value with a number that makes the highest-order interaction contrast zero. Apply this technique to the experiment in Example 6-2 assuming that run *ab* is missing. Compare the results with the results of Example 6-2.

Treatment Combination	Response	Response * *ABCD*	*ABCD*	*A*	*B*	*C*	*D*
(1)	45	45	1	-1	-1	-1	-1
a	71	-71	-1	1	-1	-1	-1
b	48	-48	-1	-1	1	-1	-1
ab	missing	missing * 1	1	1	1	-1	-1
c	68	-68	-1	-1	-1	1	-1
ac	60	60	1	1	-1	1	-1
bc	80	80	1	-1	1	1	-1
abc	65	-65	-1	1	1	1	-1
d	43	-43	-1	-1	-1	-1	1
ad	100	100	1	1	-1	-1	1
bd	45	45	1	-1	1	-1	1
abd	104	-104	-1	1	1	-1	1
cd	75	75	1	-1	-1	1	1
acd	86	-86	-1	1	-1	1	1
bcd	70	-70	-1	-1	1	1	1
abcd	96	96	1	1	1	1	1
	Contrast (*ABCD*) = missing - 54 = 0						
		missing = 54					

Substitute the value 54 for the missing run at *ab*.

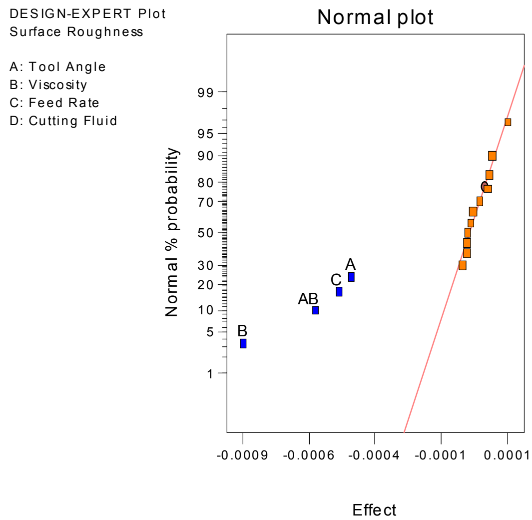
	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	20.25	1640.25	27.5406
Model	B	1.75	12.25	0.205684
Model	C	11.25	506.25	8.50019
Model	D	16	1024	17.1935
Model	AB	-1.25	6.25	0.104941
Model	AC	-16.75	1122.25	18.8431
Model	AD	18	1296	21.7605
Model	BC	3.75	56.25	0.944465
Model	BD	1	4	0.067162
Model	CD	-2.5	25	0.419762
Model	ABC	3.25	42.25	0.709398
Model	ABD	5.5	121	2.03165
Model	ACD	-3	36	0.604458
Model	BCD	-4	64	1.07459
Model	ABCD	0	0	0
	Lenth's ME	11.5676		
	Lenth's SME	23.4839		



6-30 An engineer has performed an experiment to study the effect of four factors on the surface roughness of a machined part. The factors (and their levels) are A = tool angle (12 degrees, 15 degrees), B = cutting fluid viscosity (300, 400), C = feed rate (10 in/min, 15 in/min), and D = cutting fluid cooler used (no, yes). The data from this experiment (with the factors coded to the usual -1, +1 levels) are shown below.

Run	*A*	*B*	*C*	*D*	Surface Roughness
1	-	-	-	-	0.00340
2	+	-	-	-	0.00362
3	-	+	-	-	0.00301
4	+	+	-	-	0.00182
5	-	-	+	-	0.00280
6	+	-	+	-	0.00290
7	-	+	+	-	0.00252
8	+	+	+	-	0.00160
9	-	-	-	+	0.00336
10	+	-	-	+	0.00344
11	-	+	-	+	0.00308
12	+	+	-	+	0.00184
13	-	-	+	+	0.00269
14	+	-	+	+	0.00284
15	-	+	+	+	0.00253
16	+	+	+	+	0.00163

(a) Estimate the factor effects. Plot the effect estimates on a normal probability plot and select a tentative model.



(b) Fit the model identified in part (a) and analyze the residuals. Is there any indication of model inadequacy?

Design Expert Output

Response: Surface Roughness

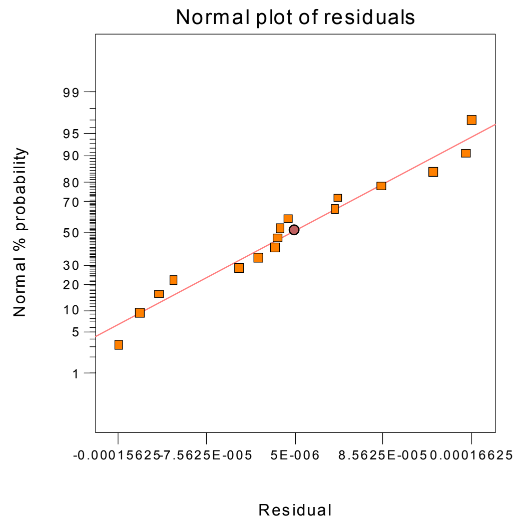
ANOVA for Selected Factorial Model

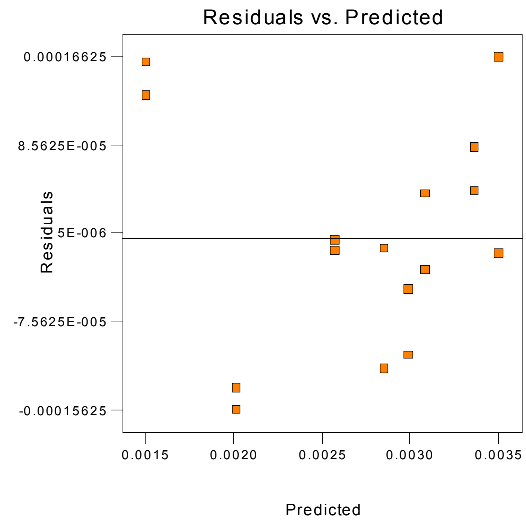
Analysis of variance table [Partial sum of squares]

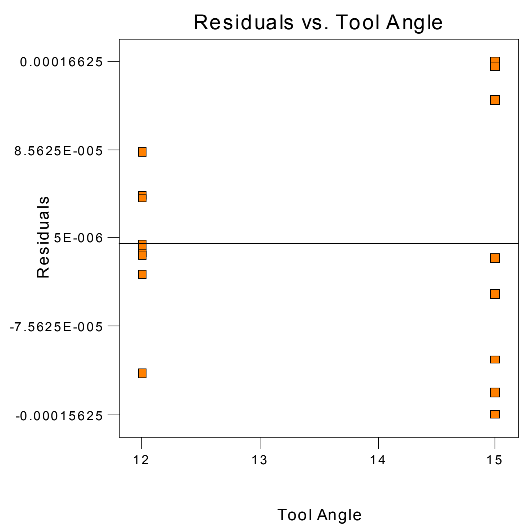
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	6.406E-006	4	1.601E-006	114.97	< 0.0001	significant
A	*8.556E-007*	*1*	*8.556E-007*	*61.43*	*< 0.0001*	
B	*3.080E-006*	*1*	*3.080E-006*	*221.11*	*< 0.0001*	
C	*1.030E-006*	*1*	*1.030E-006*	*73.96*	*< 0.0001*	
AB	*1.440E-006*	*1*	*1.440E-006*	*103.38*	*< 0.0001*	
Residual	1.532E-007	11	1.393E-008			
Cor Total	6.559E-006	15				

The Model F-value of 114.97 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AB are significant model terms.



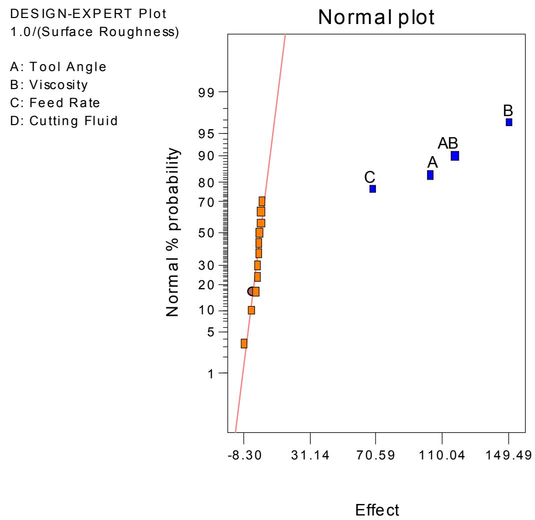




The plot of residuals versus predicted shows a slight "u-shaped" appearance in the residuals, and the plot of residuals versus tool angle shows an outward-opening funnel.

(c) Repeat the analysis from parts (a) and (b) using $1/y$ as the response variable. Is there and indication that the transformation has been useful?

The plots of the residuals are more representative of a model that does not violate the constant variance assumption.

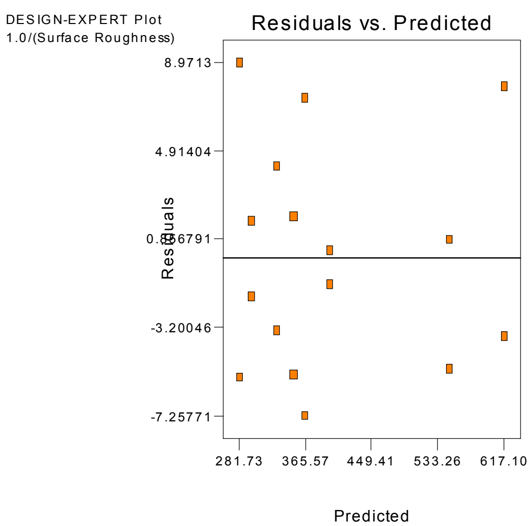


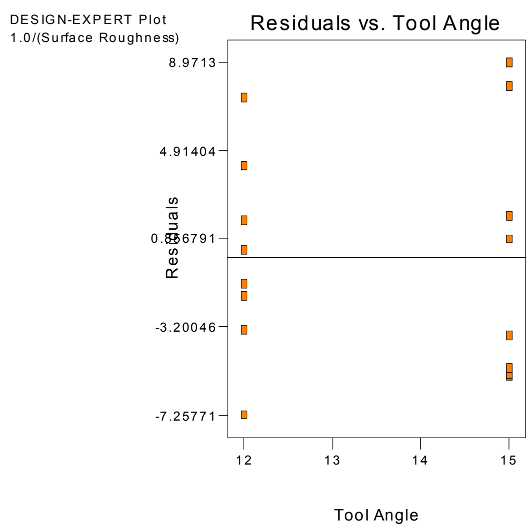
Design Expert Output

Response:	**Surface RoughnessTransform:Inverse**					
ANOVA for Selected Factorial Model						
Analysis of variance table [Partial sum of squares]						
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2.059E+005	4	51472.28	1455.72	< 0.0001	significant
A	*42610.92*	*1*	*42610.92*	*1205.11*	*< 0.0001*	
B	*89386.27*	*1*	*89386.27*	*2527.99*	*< 0.0001*	
C	*18762.29*	*1*	*18762.29*	*530.63*	*< 0.0001*	
AB	*55129.62*	*1*	*55129.62*	*1559.16*	*< 0.0001*	
Residual	388.94	11	35.36			
Cor Total	2.063E+005	15				

The Model F-value of 1455.72 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AB are significant model terms.





(d) Fit a model in terms of the coded variables that can be used to predict the surface roughness. Convert this prediction equation into a model in the natural variables.

Design Expert Output

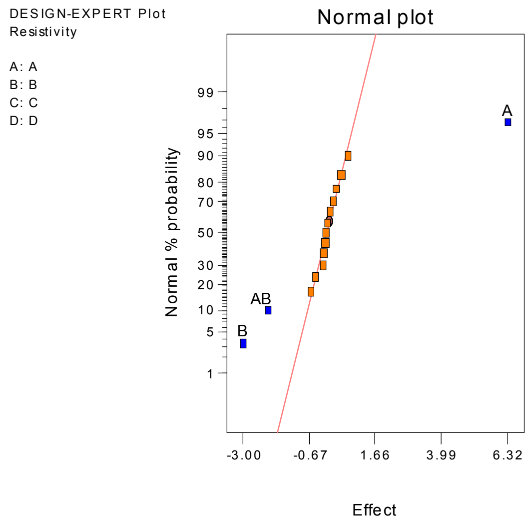
Final Equation in Terms of Coded Factors:

```
1.0/(Surface Roughness)    =
+397.81
+51.61     * A
+74.74     * B
+34.24     * C
+58.70     * A * B
```

6-31 Resistivity on a silicon wafer is influenced by several factors. The results of a 2^4 factorial experiment performed during a critical process step is shown below.

Run	*A*	*B*	*C*	*D*	Resistivity
1	-	-	-	-	1.92
2	+	-	-	-	11.28
3	-	+	-	-	1.09
4	+	+	-	-	5.75
5	-	-	+	-	2.13
6	+	-	+	-	9.53
7	-	+	+	-	1.03
8	+	+	+	-	5.35
9	-	-	-	+	1.60
10	+	-	-	+	11.73
11	-	+	-	+	1.16
12	+	+	-	+	4.68
13	-	-	+	+	2.16
14	+	-	+	+	9.11
15	-	+	+	+	1.07
16	+	+	+	+	5.30

(a) Estimate the factor effects. Plot the effect estimates on a normal probability plot and select a tentative model.



(b) Fit the model identified in part (a) and analyze the residuals. Is there any indication of model inadequacy?

The normal probability plot of residuals is not satisfactory. The plots of residual versus predicted, residual versus factor *A*, and the residual versus factor *B* are funnel shaped indicating non-constant variance.

Design Expert Output

Response: Resistivity

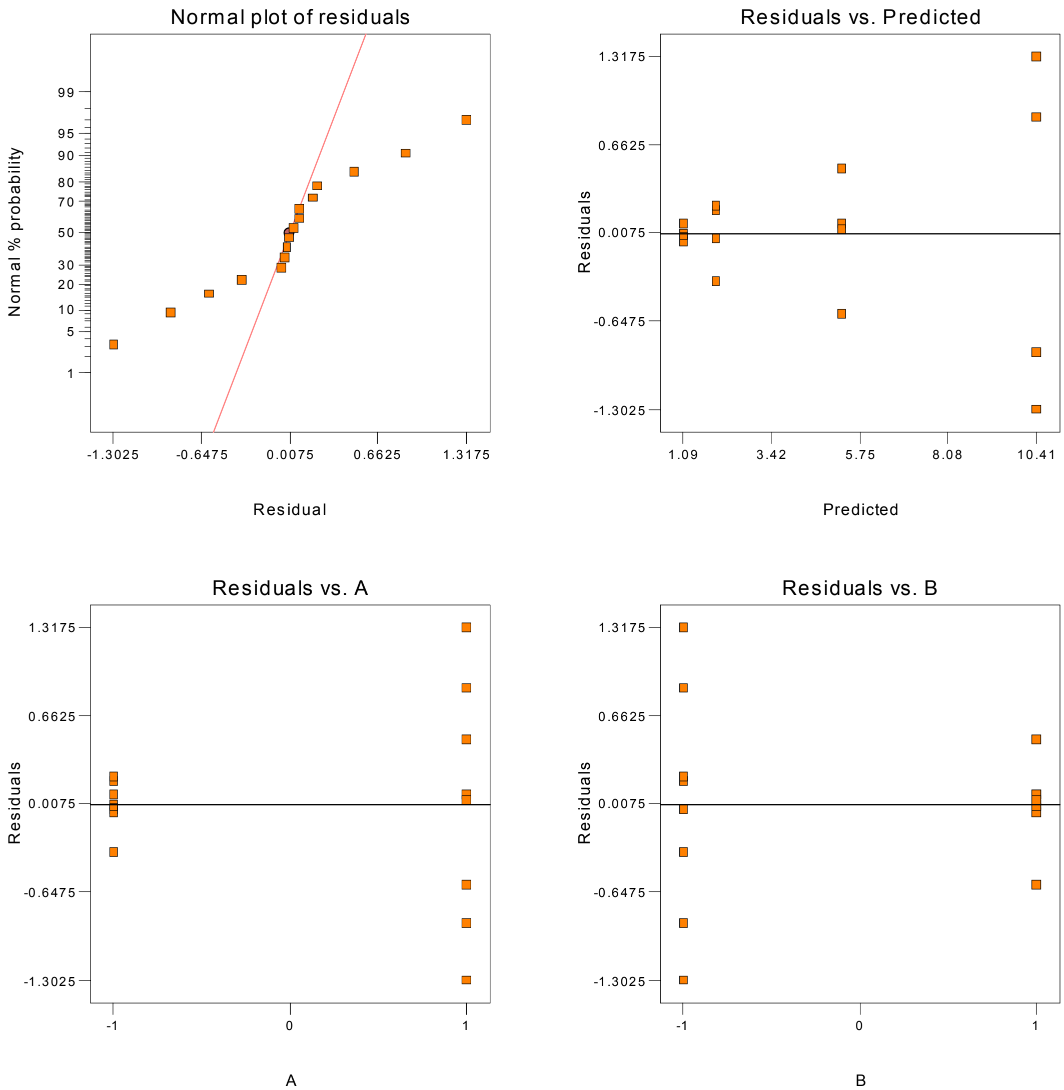
ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

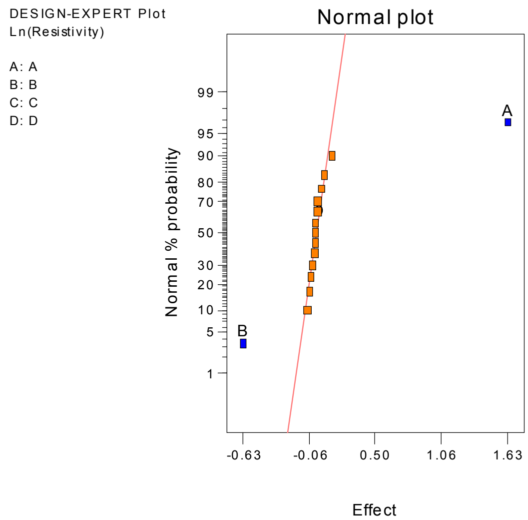
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	214.22	3	71.41	148.81	< 0.0001	significant
A	*159.83*	*1*	*159.83*	*333.09*	*< 0.0001*	
B	*36.09*	*1*	*36.09*	*75.21*	*< 0.0001*	
AB	*18.30*	*1*	*18.30*	*38.13*	*< 0.0001*	
Residual	5.76	12	0.48			
Cor Total	219.98	15				

The Model F-value of 148.81 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.



(c) Repeat the analysis from parts (a) and (b) using ln(y) as the response variable. Is there any indication that the transformation has been useful?



Design Expert Output

Response: **Resistivity** **Transform:** **Natural log** **Constant:** **0.000**

ANOVA for Selected Factorial Model

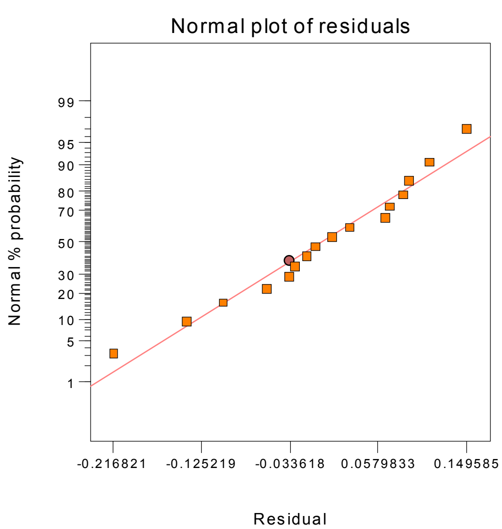
Analysis of variance table [Partial sum of squares]

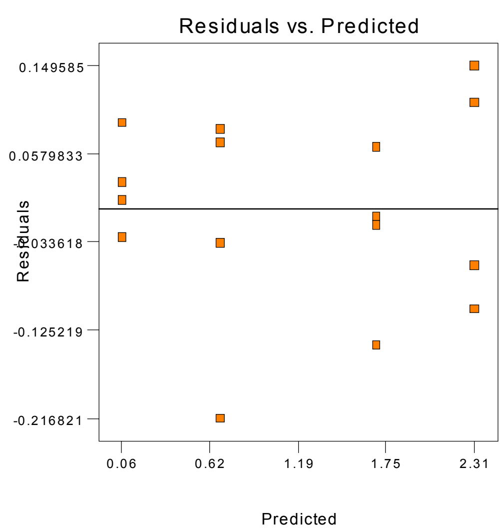
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	12.15	2	6.08	553.44	< 0.0001	significant
A	*10.57*	*1*	*10.57*	*962.95*	*< 0.0001*	
B	*1.58*	*1*	*1.58*	*143.94*	*< 0.0001*	
Residual	0.14	13	0.011			
Cor Total	12.30	15				

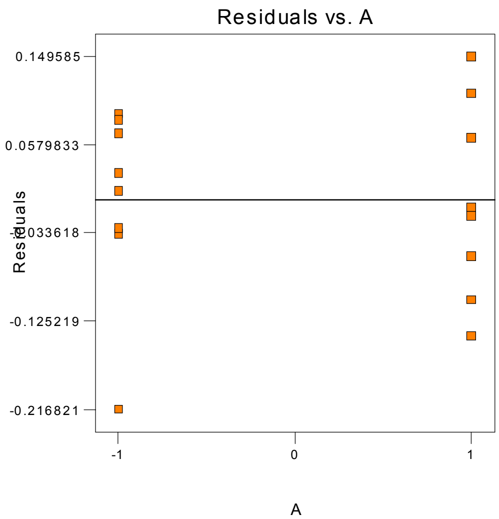
The Model F-value of 553.44 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

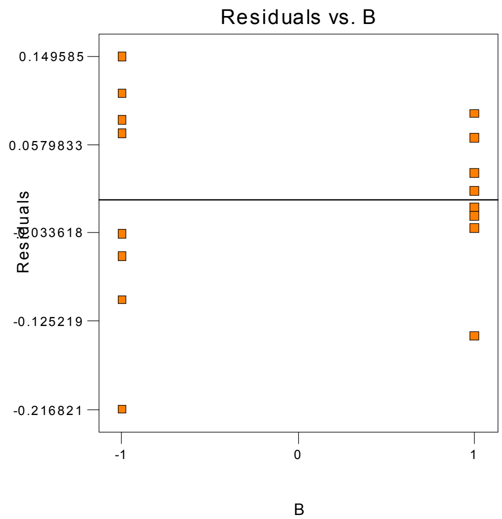
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

The transformed data no longer indicates that the *AB* interaction is significant. A simpler model has resulted from the log transformation.









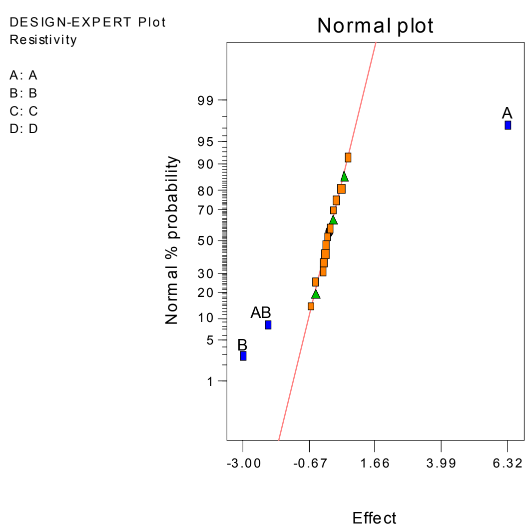
The residual plots are much improved.

(d) Fit a model in terms of the coded variables that can be used to predict the resistivity.

Design Expert Output

Final Equation in Terms of Coded Factors:
Ln(Resistivity) =
+1.19
+0.81 * A
-0.31 * B

6.32 Continuation of Problem 6-31. Suppose that the experiment had also run four center points along with the 16 runs in Problem 6-31. The resistivity measurements at the center points are: 8.15, 7.63, 8.95, 6.48. Analyze the experiment again incorporating the center points. What conclusions can you draw now?



Design Expert Output

Response: Resistivity

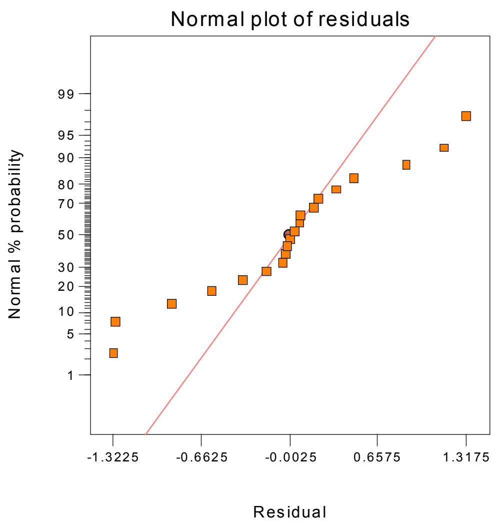
ANOVA for Selected Factorial Model

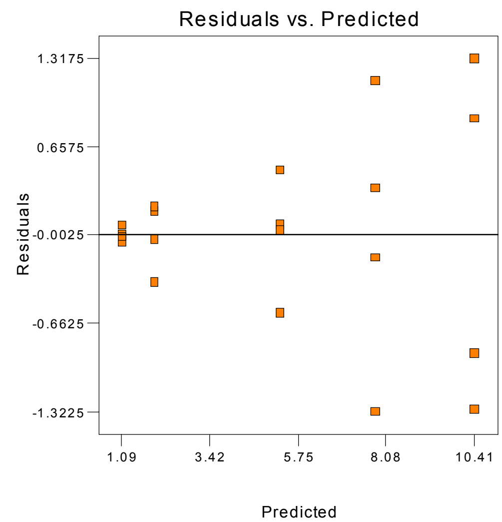
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	214.22	3	71.41	119.35	< 0.0001	significant
A	*159.83*	*1*	*159.83*	*267.14*	*< 0.0001*	
B	*36.09*	*1*	*36.09*	*60.32*	*< 0.0001*	
AB	*18.30*	*1*	*18.30*	*30.58*	*< 0.0001*	
Curvature	31.19	1	31.19	52.13	< 0.0001	significant
Residual	8.97	15	0.60			
Lack of Fit	*5.76*	*12*	*0.48*	*0.45*	*0.8632*	*not significant*
Pure Error	*3.22*	*3*	*1.07*			
Cor Total	254.38	19				

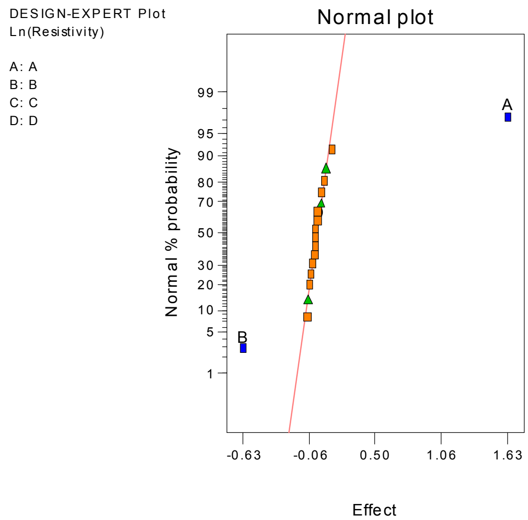
The Model F-value of 119.35 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.





Repeated analysis with the natural log transformation.



Design Expert Output

Response: Resistivity Transform: Natural log Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	12.15	2	6.08	490.37	< 0.0001	significant
A	*10.57*	*1*	*10.57*	*853.20*	*< 0.0001*	
B	*1.58*	*1*	*1.58*	*127.54*	*< 0.0001*	
Curvature	2.38	1	2.38	191.98	< 0.0001	significant
Residual	0.20	16	0.012			
Lack of Fit	*0.14*	*13*	*0.011*	*0.59*	*0.7811*	*not significant*
Pure Error	*0.056*	*3*	*0.019*			
Cor Total	14.73	19				

The Model F-value of 490.37 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

The "Curvature F-value" of 191.98 implies there is significant curvature (as measured by difference between the average of the center points and the average of the factorial points) in the design space. There is only a 0.01% chance that a "Curvature F-value" this large could occur due to noise.

The curvature test indicates that the model has significant pure quadratic curvature.

6.33 Often the fitted regression model from a 2^k factorial design is used to make predictions at points of interest in the design space.

(a) Find the variance of the predicted response $\hat{y}$ at the point $x_1, x_2, \ldots, x_k$ in the design space. Hint: Remember that the x's are coded variables, and assume a 2^k design with an equal number of replicates n at each design point so that the variance of a regression coefficient $\hat{\beta}$ is $\dfrac{\sigma^2}{n2^k}$ and that the covariance between any pair of regression coefficients is zero.

Let's assume that the model can be written as follows:

$$\hat{y}(\mathbf{x}) = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \ldots + \hat{\beta}_p x_p$$

where $\mathbf{x}' = [x_1, x_2, ..., x_k]$ are the values of the original variables in the design at the point of interest where a prediction is required , and the variables in the model $x_1, x_2, ..., x_p$ potentially include interaction terms among the original k variables. Now the variance of the predicted response is

$$\begin{aligned} V[\hat{y}(\mathbf{x})] &= V(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \ldots + \hat{\beta}_p x_p) \\ &= V(\hat{\beta}_0) + V(\hat{\beta}_1 x_1) + V(\hat{\beta}_2 x_2) + \ldots + V(\hat{\beta}_p x_p) \\ &= \frac{\sigma^2}{n2^k}\left(1 + \sum_{i=1}^{p} x_i^2\right) \end{aligned}$$

This result follows because the design is orthogonal and all model parameter estimates have the same variance. Remember that some of the x's involved in this equation are potentially interaction terms.

(b) Use the result of part (a) to find an equation for a 100(1-α)% confidence interval on the true mean response at the point $x_1, x_2, \ldots, x_k$ in the design space.

The confidence interval is

$$\hat{y}(\mathbf{x}) - t_{\alpha/2, df_E}\sqrt{V[\hat{y}(\mathbf{x})]} \le y(\mathbf{x}) \le \hat{y}(\mathbf{x}) + t_{\alpha/2, df_E}\sqrt{V[\hat{y}(\mathbf{x})]}$$

where df_E is the number of degrees of freedom used to estimate σ^2 and the estimate of σ^2 has been used in computing the variance of the predicted value of the response at the point of interest.

6.34 Hierarchical Models. Several times we have utilized the hierarchy principal in selecting a model; that is, we have included non-significant terms in a model because they were factors involved in significant higher-order terms. Hierarchy is certainly not an absolute principle that must be followed in all cases. To illustrate, consider the model resulting from Problem 6-1, which required that a non-significant main effect be included to achieve hierarchy. Using the data from Problem 6-1:

(a) Fit both the hierarchical model and the non-hierarchical model.

Design Expert Output for Hierarchial Model

Response: Life in hours
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1519.67	4	379.92	12.54	< 0.0001	significant
A	*0.67*	*1*	*0.67*	*0.022*	*0.8836*	
B	*770.67*	*1*	*770.67*	*25.44*	*< 0.0001*	
C	*280.17*	*1*	*280.17*	*9.25*	*0.0067*	
AC	*468.17*	*1*	*468.17*	*15.45*	*0.0009*	
Residual	575.67	19	30.30			
Lack of Fit	*93.00*	*3*	*31.00*	*1.03*	*0.4067*	*not significant*
Pure Error	*482.67*	*16*	*30.17*			
Cor Total	2095.33	23				

The Model F-value of 12.54 implies the model is significant. There is only

a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B, C, AC are significant model terms.

Std. Dev.	5.50	R-Squared	0.7253
Mean	40.83	Adj R-Squared	0.6674
C.V.	13.48	Pred R-Squared	0.5616
PRESS	918.52	Adeq Precision	10.747

The "Pred R-Squared" of 0.5616 is in reasonable agreement with the "Adj R-Squared" of 0.6674. A difference greater than 0.20 between the "Pred R-Squared" and the "Adj R-Squared" indicates a possible problem with your model and/or data.

Design Expert Output for Non-Hierarchical Model

Response: Life in hours
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1519.00	3	506.33	17.57	< 0.0001	significant
B	*770.67*	*1*	*770.67*	*26.74*	*< 0.0001*	
C	*280.17*	*1*	*280.17*	*9.72*	*0.0054*	
AC	*468.17*	*1*	*468.17*	*16.25*	*0.0007*	
Residual	576.33	20	28.82			
Lack of Fit	*93.67*	*4*	*23.42*	*0.78*	*0.5566*	*not significant*
Pure Error	*482.67*	*16*	*30.17*			
Cor Total	2095.33	23				

The Model F-value of 17.57 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B, C, AC are significant model terms.

The "Lack of Fit F-value" of 0.78 implies the Lack of Fit is not significant relative to the pure error. There is a 55.66% chance that a "Lack of Fit F-value" this large could occur due to noise. Non-significant lack of fit is good -- we want the model to fit.

Std. Dev.	5.37	R-Squared	0.7249
Mean	40.83	Adj R-Squared	0.6837
C.V.	13.15	Pred R-Squared	0.6039
PRESS	829.92	Adeq Precision	12.320

The "Pred R-Squared" of 0.6039 is in reasonable agreement with the "Adj R-Squared" of 0.6837. A difference greater than 0.20 between the "Pred R-Squared" and the "Adj R-Squared" indicates a possible problem with your model and/or data.

(b) Calculate the PRESS statistic, the adjusted R^2 and the mean square error for both models.

The PRESS and R^2 are in the Design Expert Output above. The PRESS is smaller for the non-hierarchical model than the hierarchical model suggesting that the non-hierarchical model is a better predictor.

(c) Find a 95 percent confidence interval on the estimate of the mean response at a cube corner ($x_1 = x_2 = x_3 = \pm 1$). Hint: Use the result of Problem 6-33.

Design Expert Output

	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Life	27.45	2.18	22.91	31.99	5.79	15.37	39.54
Life	36.17	2.19	31.60	40.74	5.80	24.07	48.26
Life	38.67	2.19	34.10	43.24	5.80	26.57	50.76
Life	47.50	2.19	42.93	52.07	5.80	35.41	59.59
Life	43.00	2.19	38.43	47.57	5.80	30.91	55.09
Life	34.17	2.19	29.60	38.74	5.80	22.07	46.26

Life	54.33	2.19	49.76	58.90	5.80	42.24	66.43
Life	45.50	2.19	40.93	50.07	5.80	33.41	57.59

(d) Based on the analyses you have conducted, which model would you prefer?

Notice that PRESS is smaller and the adjusted R^2 is larger for the non-hierarchical model. This is an indication that strict adherence to the hierarchy principle isn't always necessary. Note also that the confidence interval is shorter for the non-hierarchical model.

Chapter 7
Blocking and Confounding in the 2^k Factorial Design Solutions

7-1 Consider the experiment described in Problem 6-1. Analyze this experiment assuming that each replicate represents a block of a single production shift.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Cutting Speed (*A*)	0.67	1	0.67	<1
Tool Geometry (*B*)	770.67	1	770.67	22.38*
Cutting Angle (*C*)	280.17	1	280.17	8.14*
AB	16.67	1	16.67	<1
AC	468.17	1	468.17	13.60*
BC	48.17	1	48.17	1.40
ABC	28.17	1	28.17	<1
Blocks	0.58	2	0.29	
Error	482.08	14	34.43	
Total	2095.33	23		

Design Expert Output

Response: Life in hours
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	0.58	2	0.29			
Model	1519.67	4	379.92	11.23	0.0001	significant
A	*0.67*	*1*	*0.67*	*0.020*	*0.8900*	
B	*770.67*	*1*	*770.67*	*22.78*	*0.0002*	
C	*280.17*	*1*	*280.17*	*8.28*	*0.0104*	
AC	*468.17*	*1*	*468.17*	*13.84*	*0.0017*	
Residual	575.08	17	33.83			
Cor Total	2095.33	23				

The Model F-value of 11.23 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B, C, AC are significant model terms.

These results agree with the results from Problem 6-1. Tool geometry, cutting angle and the cutting speed x cutting angle factors are significant at the 5% level. The Design Expert program also includes *A*, speed, in the model to preserve hierarchy.

7-2 Consider the experiment described in Problem 6-5. Analyze this experiment assuming that each one of the four replicates represents a block.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Bit Size (*A*)	1107.23	1	1107.23	364.22*

Cutting Speed (*B*)	227.26	1	227.26	74.76*
AB	303.63	1	303.63	99.88*
Blocks	44.36	3	14.79	
Error	27.36	9	3.04	
Total	1709.83	15		

These results agree with those from Problem 6-5. Bit size, cutting speed and their interaction are significant at the 1% level.

Design Expert Output

Response: Vibration

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	44.36	3	14.79			
Model	1638.11	3	546.04	179.61	< 0.0001	significant
A	*1107.23*	*1*	*1107.23*	*364.21*	*< 0.0001*	
B	*227.26*	*1*	*227.26*	*74.75*	*< 0.0001*	
AB	*303.63*	*1*	*303.63*	*99.88*	*< 0.0001*	
Residual	27.36	9	3.04			
Cor Total	1709.83	15				

The Model F-value of 179.61 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.

7-3 Consider the alloy cracking experiment described in Problem 6-15. Suppose that only 16 runs could be made on a single day, so each replicate was treated as a block. Analyze the experiment and draw conclusions.

The analysis of variance for the full model is as follows:

Design Expert Output

Response: Crack Lengthin mm x 10^-2

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	0.016	1	0.016			
Model	570.95	15	38.06	445.11	< 0.0001	significant
A	*72.91*	*1*	*72.91*	*852.59*	*< 0.0001*	
B	*126.46*	*1*	*126.46*	*1478.83*	*< 0.0001*	
C	*103.46*	*1*	*103.46*	*1209.91*	*< 0.0001*	
D	*30.66*	*1*	*30.66*	*358.56*	*< 0.0001*	
AB	*29.93*	*1*	*29.93*	*349.96*	*< 0.0001*	
AC	*128.50*	*1*	*128.50*	*1502.63*	*< 0.0001*	
AD	*0.047*	*1*	*0.047*	*0.55*	*0.4708*	
BC	*0.074*	*1*	*0.074*	*0.86*	*0.3678*	
BD	*0.018*	*1*	*0.018*	*0.21*	*0.6542*	
CD	*0.047*	*1*	*0.047*	*0.55*	*0.4686*	
ABC	*78.75*	*1*	*78.75*	*920.92*	*< 0.0001*	
ABD	*0.077*	*1*	*0.077*	*0.90*	*0.3582*	
ACD	*2.926E-003*	*1*	*2.926E-003*	*0.034*	*0.8557*	
BCD	*0.010*	*1*	*0.010*	*0.12*	*0.7352*	
ABCD	*1.596E-003*	*1*	*1.596E-003*	*0.019*	*0.8931*	
Residual	1.28	15	0.086			
Cor Total	572.25	31				

The Model F-value of 445.11 implies the model is significant. There is only

a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, D, AB, AC, ABC are significant model terms.

The analysis of variance for the reduced model based on the significant factors is shown below. The BC interaction was included to preserve hierarchy.

Design Expert Output

Response: Crack Lengthin mm x 10^-2
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.016	1	0.016			
Model	570.74	8	71.34	1056.10	< 0.0001	significant
A	*72.91*	*1*	*72.91*	*1079.28*	*< 0.0001*	
B	*126.46*	*1*	*126.46*	*1872.01*	*< 0.0001*	
C	*103.46*	*1*	*103.46*	*1531.59*	*< 0.0001*	
D	*30.66*	*1*	*30.66*	*453.90*	*< 0.0001*	
AB	*29.93*	*1*	*29.93*	*443.01*	*< 0.0001*	
AC	*128.50*	*1*	*128.50*	*1902.15*	*< 0.0001*	
BC	*0.074*	*1*	*0.074*	*1.09*	*0.3075*	
ABC	*78.75*	*1*	*78.75*	*1165.76*	*< 0.0001*	
Residual	1.49	22	0.068			
Cor Total	572.25	31				

The Model F-value of 1056.10 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

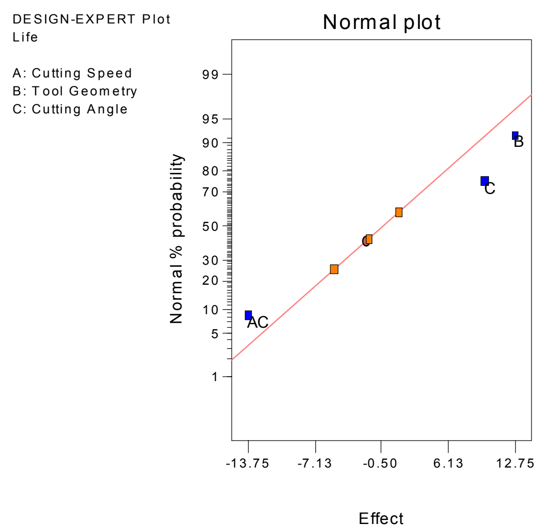
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, D, AB, AC, ABC are significant model terms.

Blocking does not change the results of Problem 6-15.

7-4 Consider the data from the first replicate of Problem 6-1. Suppose that these observations could not all be run using the same bar stock. Set up a design to run these observations in two blocks of four observations each with *ABC* confounded. Analyze the data.

Block 1	Block 2
(1)	*a*
ab	*b*
ac	*c*
bc	*abc*

From the normal probability plot of effects, *B*, *C*, and the *AC* interaction are significant. Factor *A* was included in the analysis of variance to preserve hierarchy.



Design Expert Output

Response: Life in hours

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	91.13	1	91.13			
Model	896.50	4	224.13	7.32	0.1238	not significant
A	*3.13*	*1*	*3.13*	*0.10*	*0.7797*	
B	*325.12*	*1*	*325.12*	*10.62*	*0.0827*	
C	*190.12*	*1*	*190.12*	*6.21*	*0.1303*	
AC	*378.13*	*1*	*378.13*	*12.35*	*0.0723*	
Residual	61.25	2	30.62			
Cor Total	1048.88	7				

The "Model F-value" of 7.32 implies the model is not significant relative to the noise. There is a 12.38 % chance that a "Model F-value" this large could occur due to noise.

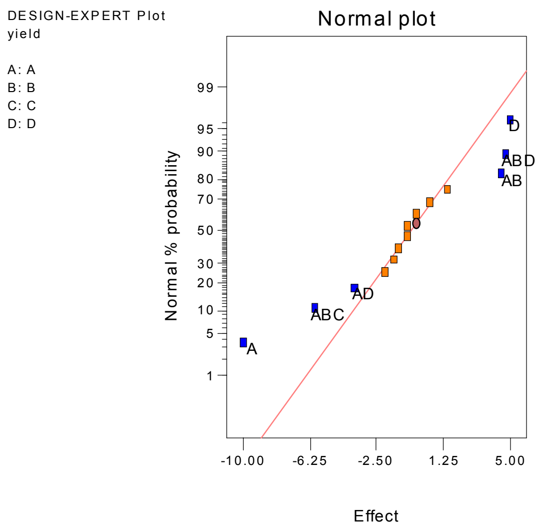
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case there are no significant model terms.

This design identifies the same significant factors as Problem 6-1.

7-5 Consider the data from the first replicate of Problem 6-7. Construct a design with two blocks of eight observations each with *ABCD* confounded. Analyze the data.

Block 1	Block 2
(1)	*a*
ab	*b*
ac	*c*
bc	*d*
ad	*abc*
bd	*abd*
cd	*acd*
abcd	*bcd*

The significant effects are identified in the normal probability plot of effects below:



AC, *BC*, and *BD* were included in the model to preserve hierarchy.

Design Expert Output

Response: yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	42.25	1	42.25			
Model	892.25	11	81.11	9.64	0.0438	significant
A	*400.00*	*1*	*400.00*	*47.52*	*0.0063*	
B	*2.25*	*1*	*2.25*	*0.27*	*0.6408*	
C	*2.25*	*1*	*2.25*	*0.27*	*0.6408*	
D	*100.00*	*1*	*100.00*	*11.88*	*0.0410*	
AB	*81.00*	*1*	*81.00*	*9.62*	*0.0532*	
AC	*1.00*	*1*	*1.00*	*0.12*	*0.7531*	
AD	*56.25*	*1*	*56.25*	*6.68*	*0.0814*	
BC	*6.25*	*1*	*6.25*	*0.74*	*0.4522*	
BD	*9.00*	*1*	*9.00*	*1.07*	*0.3772*	
ABC	*144.00*	*1*	*144.00*	*17.11*	*0.0256*	
ABD	*90.25*	*1*	*90.25*	*10.72*	*0.0466*	
Residual	25.25	3	8.42			
Cor Total	959.75	15				

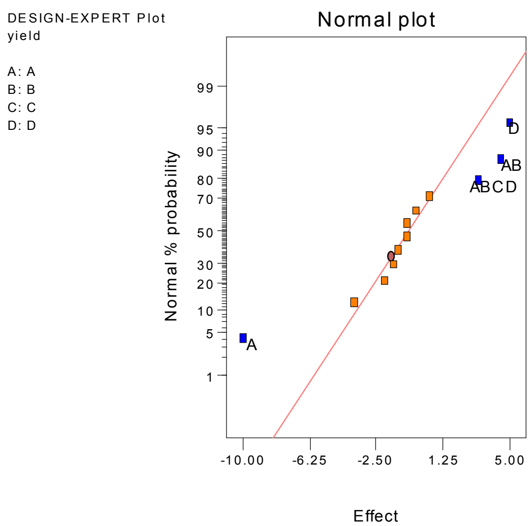
The Model F-value of 9.64 implies the model is significant. There is only a 4.38% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, D, ABC, ABD are significant model terms.

7-6 Repeat Problem 7-5 assuming that four blocks are required. Confound *ABD* and *ABC* (and consequently *CD*) with blocks.

Block 1	Block 2	Block 3	Block 4
(1)	*ac*	*c*	*a*
ab	*bc*	*abc*	*b*
acd	*d*	*ad*	*cd*

bcd	*abd*	*bd*	*abcd*



Design Expert Output

Response: **yield**

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	243.25	3	81.08			
Model	623.25	4	155.81	13.37	0.0013	significant
A	*400.00*	*1*	*400.00*	*34.32*	*0.0004*	
D	*100.00*	*1*	*100.00*	*8.58*	*0.0190*	
AB	*81.00*	*1*	*81.00*	*6.95*	*0.0299*	
ABCD	*42.25*	*1*	*42.25*	*3.62*	*0.0934*	
Residual	93.25	8	11.66			
Cor Total	959.75	15				

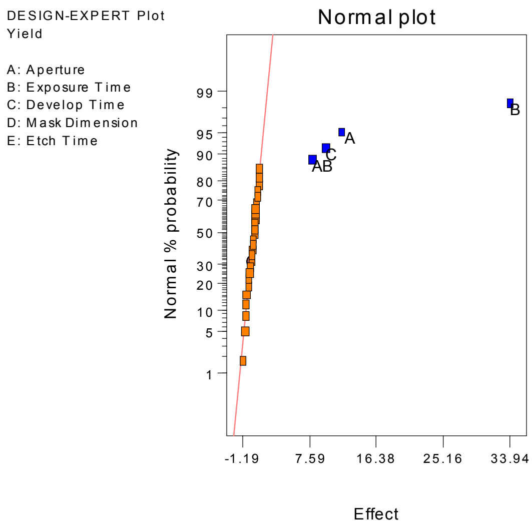
The Model F-value of 13.37 implies the model is significant. There is only a 0.13% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, D, AB are significant model terms.

7-7 Using the data from the 2^5 design in Problem 6-21, construct and analyze a design in two blocks with *ABCDE* confounded with blocks.

Block 1	Block 1	Block 2	Block 2
(1)	*ae*	*a*	*e*
ab	*be*	*b*	*abe*
ac	*ce*	*c*	*ace*
bc	*abce*	*abc*	*bce*
ad	*de*	*d*	*ade*
bd	*abde*	*abd*	*bde*
cd	*acde*	*acd*	*cde*
abcd	*bcde*	*bcd*	*abcde*

The normal probability plot of effects identifies factors A, B, C, and the AB interaction as being significant. This is confirmed with the analysis of variance.



Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.28	1	0.28			
Model	11585.13	4	2896.28	958.51	< 0.0001	significant
A	*1116.28*	*1*	*1116.28*	*369.43*	*< 0.0001*	
B	*9214.03*	*1*	*9214.03*	*3049.35*	*< 0.0001*	
C	*750.78*	*1*	*750.78*	*248.47*	*< 0.0001*	
AB	*504.03*	*1*	*504.03*	*166.81*	*< 0.0001*	
Residual	78.56	26	3.02			
Cor Total	11663.97	31				

The Model F-value of 958.51 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, AB are significant model terms.

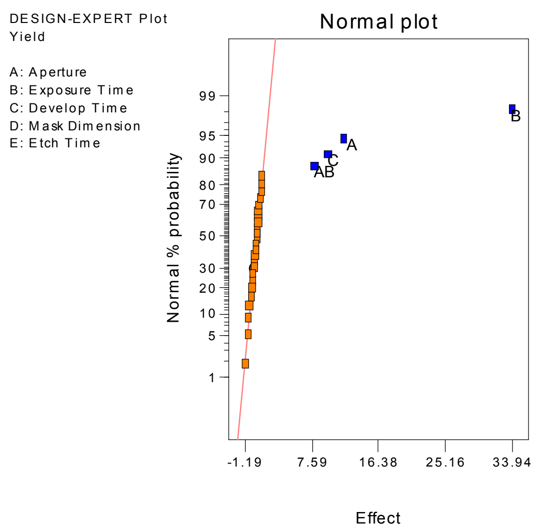
7-8 Repeat Problem 7-7 assuming that four blocks are necessary. Suggest a reasonable confounding scheme.

Use *ABC, CDE*, confounded with *ABDE*. The four blocks follow.

Block 1	Block 2	Block 3	Block 4
(1)	*a*	*ac*	*c*
ab	*b*	*bc*	*abc*
acd	*cd*	*d*	*ad*
bcd	*abcd*	*abd*	*bd*
ace	*ce*	*e*	*ae*
bce	*abce*	*abe*	*be*

de	*ade*	*acde*	*cde*
abde	*bde*	*bcde*	*abcde*

The normal probability plot of effects identifies the same significant effects as in Problem 7-7.



Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	13.84	3	4.61			
Model	11585.13	4	2896.28	1069.40	< 0.0001	significant
A	*1116.28*	*1*	*1116.28*	*412.17*	*< 0.0001*	
B	*9214.03*	*1*	*9214.03*	*3402.10*	*< 0.0001*	
C	*750.78*	*1*	*750.78*	*277.21*	*< 0.0001*	
AB	*504.03*	*1*	*504.03*	*186.10*	*< 0.0001*	
Residual	65.00	24	2.71			
Cor Total	11663.97	31				

The Model F-value of 1069.40 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

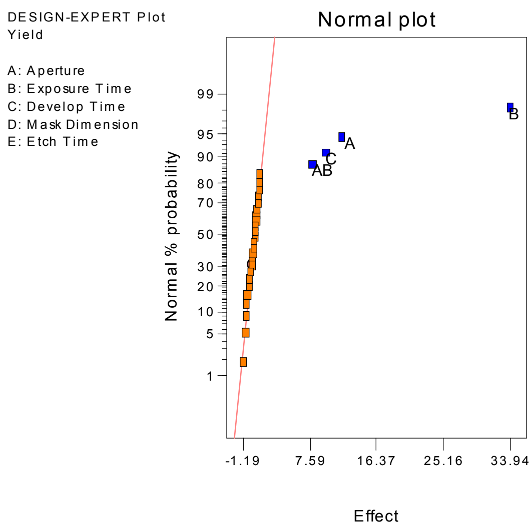
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, AB are significant model terms.

7-9 Consider the data from the 2^5 design in Problem 6-21. Suppose that it was necessary to run this design in four blocks with *ACDE* and *BCD* (and consequently *ABE*) confounded. Analyze the data from this design.

Block 1	Block 2	Block 3	Block 4
(1)	*a*	*b*	*c*
ae	*e*	*abe*	*ace*
cd	*acd*	*bcd*	*d*
abc	*bc*	*ac*	*ab*
acde	*cde*	*abcde*	*ade*

bce	*abce*	*ce*	*be*
abd	*bd*	*ab*	*abcd*
bde	*abde*	*de*	*bcde*

Even with four blocks, the same effects are identified as significant per the normal probability plot and analysis of variance below:



Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	2.59	3	0.86			
Model	11585.13	4	2896.28	911.62	< 0.0001	significant
A	*1116.28*	*1*	*1116.28*	*351.35*	*< 0.0001*	
B	*9214.03*	*1*	*9214.03*	*2900.15*	*< 0.0001*	
C	*750.78*	*1*	*750.78*	*236.31*	*< 0.0001*	
AB	*504.03*	*1*	*504.03*	*158.65*	*< 0.0001*	
Residual	76.25	24	3.18			
Cor Total	11663.97	31				

The Model F-value of 911.62 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C, AB are significant model terms.

7-10 Design an experiment for confounding a 2^6 factorial in four blocks. Suggest an appropriate confounding scheme, different from the one shown in Table 7-8.

We choose ABCE and ABDF. Which also confounds with CDEF

Block 1	Block 2	Block 3	Block 4
a	*c*	*ac*	(1)
b	*abc*	*bc*	*ab*

cd	*ad*	*d*	*acd*
abcd	*bd*	*abd*	*bcd*
ace	*e*	*ae*	*ce*
bce	*abe*	*be*	*abce*
de	*acde*	*cde*	*ade*
abde	*bcde*	*abcde*	*bde*
cf	*af*	*f*	*acf*
abcf	*bf*	*abf*	*bcf*
adf	*cdf*	*acdf*	*df*
bdf	*abcdf*	*bcdf*	*abdf*
ef	*acef*	*cef*	*aef*
abef	*bcef*	*abcef*	*bef*
acdef	*def*	*adef*	*cdef*
bcdef	*abdef*	*bdef*	*abcdef*

7-11 Consider the 2^6 design in eight blocks of eight runs each with *ABCD*, *ACE*, and *ABEF* as the independent effects chosen to be confounded with blocks. Generate the design. Find the other effects confound with blocks.

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
b	*abc*	*a*	*c*	*ac*	(1)	*bc*	*ab*
acd	*d*	*bcd*	*abd*	*bd*	*abcd*	*ad*	*cd*
ce	*ae*	*abce*	*be*	*abe*	*bce*	*e*	*ace*
abde	*bcde*	*de*	*acde*	*cde*	*ade*	*abcde*	*bde*
abcf	*bf*	*cf*	*af*	*f*	*acf*	*abf*	*bcf*
de	*acdf*	*abdf*	*bcdf*	*abcdf*	*bdf*	*cdf*	*adf*
aef	*cef*	*def*	*abcef*	*bcef*	*abef*	*acef*	*ef*
bcdef	*abdef*	*acdef*	*def*	*adef*	*cdef*	*bdef*	*abcdef*

The factors that are confounded with blocks are *ABCD, ABEF, ACE, BDE, CDEF, BCF*, and *ADF*.

7-12 Consider the 2^2 design in two blocks with *AB* confounded. Prove algebraically that $SS_{AB} = SS_{\text{Blocks}}$.

If *AB* is confounded, the two blocks are:

Block 1	Block 2
(1)	*a*
ab	*b*
(1) + *ab*	*a* + *b*

$$SS_{Blocks} = \frac{[(1)+ab]^2 + [a+b]^2}{2} - \frac{[(1)+ab+a+b]^2}{4}$$

$$SS_{Blocks} = \frac{(1)^2 + ab^2 + 2(1)ab + a^2 + b^2 + 2ab}{2}$$

$$-\frac{(1)^2+ab^2+a^2+b^2+2(1)ab+2(1)a+2(1)b+2a(ab)+2b(ab)+2ab}{4}$$

$$SS_{Blocks}=\frac{(1)^2+ab^2+a^2+b^2+2(1)ab+2ab-2(1)a-2(1)b-2a(ab)-2b(ab)}{4}$$

$$SS_{Blocks}=\frac{1}{4}[(1)+ab-a-b]^2=SS_{AB}$$

7-13 Consider the data in Example 7-2. Suppose that all the observations in block 2 are increased by 20. Analyze the data that would result. Estimate the block effect. Can you explain its magnitude? Do blocks now appear to be an important factor? Are any other effect estimates impacted by the change you made in the data?

$$\text{Block Effect}=\bar{y}_{Block1}-\bar{y}_{Block2}=\frac{406}{8}-\frac{715}{8}=\frac{-309}{8}=-38.625$$

This is the block effect estimated in Example 7-2 plus the additional 20 units that were added to each observation in block 2. All other effects are the same.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	1870.56	1	1870.56	89.93
C	390.06	1	390.06	18.75
D	855.56	1	855.56	41.13
AC	1314.06	1	1314.06	63.18
AD	1105.56	1	1105.56	53.15
Blocks	5967.56	1	5967.56	
Error	187.56	9	20.8	
Total	11690.93	15		

Design Expert Output

Response: Filtration in gal/hr
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	5967.56	1	5967.56			
Model	5535.81	5	1107.16	53.13	< 0.0001	significant
A	*1870.56*	*1*	*1870.56*	*89.76*	*< 0.0001*	
C	*390.06*	*1*	*390.06*	*18.72*	*0.0019*	
D	*855.56*	*1*	*855.56*	*41.05*	*0.0001*	
AC	*1314.06*	*1*	*1314.06*	*63.05*	*< 0.0001*	
AD	*1105.56*	*1*	*1105.56*	*53.05*	*< 0.0001*	
Residual	187.56	9	20.84			
Cor Total	11690.94	15				

The Model F-value of 53.13 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, C, D, AC, AD are significant model terms.

7-14 Suppose that the data in Problem 7-1 we had confounded *ABC* in replicate I, *AB* in replicate II, and *BC* in replicate III. Construct the analysis of variance table.

	Replicate I (*ABC* Confounded)		Replicate II (*AB* Confounded)		Replicate III (*BC* Confounded)	
Block->	1	2	1	2	1	2
	(1)	*a*	(1)	*a*	(1)	*b*
	ab	*b*	*ab*	*b*	*bc*	*c*
	ac	*c*	*abc*	*ac*	*abc*	*ab*
	bc	*abc*	*c*	*bc*	*a*	*ac*

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	0.67	1	0.67	<1
B	770.67	1	770.67	20.77
C	280.17	1	280.17	7.55
AB (reps 1 and III)	25.00	1	25.00	<1
AC	468.17	1	468.17	12.62
BC (reps I and II)	22.56	1	22.56	<1
ABC (reps II and III)	0.06	1	0.06	<1
Blocks within replicates	119.83	3	15.87	
Replicates	0.58	2		
Error	408.21	11	37.11	
Total	2095.33	23		

7-15 Repeat Problem 7-1 assuming that *ABC* was confounded with blocks in each replicate.

	Replicate I, II, and III (*ABC* Confounded)	
Block->	1	2
	(1)	*a*
	ab	*b*
	ac	*c*
	bc	*abc*

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	0.67	1	0.67	<1
B	770.67	1	770.67	22.15
C	280.17	1	280.17	8.05
AB	16.67	1	16.67	<1
AC	468.17	1	468.17	13.46
BC	48.17	1	48.17	1.38
Blocks (or *ABC*)	119.83	1	119.83	
Replicates/Lack of Fit	64.83	4		

Error	471.50	12	34.79
Total	2095.33	23	

7-16 Suppose that in Problem 7-7 *ABCD* was confounded in replicate I and *ABC* was confounded in replicate II. Perform the statistical analysis of variance.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	657.03	1	657.03	84.89
B	13.78	1	13.78	1.78
C	57.78	1	57.78	7.46
D	124.03	1	124.03	16.02
AB	132.03	1	132.03	17.06
AC	3.78	1	3.78	<1
AD	38.28	1	38.28	4.95
BC	2.53	1	2.53	<1
BD	0.28	1	0.28	<1
CD	22.78	1	22.78	2.94
ABC	144.00	1	144.00	18.64
ABD	175.78	1	175.78	22.71
ACD	7.03	1	7.03	<1
BCD	7.03	1	7.03	<1
ABCD	10.56	1	10.56	1.36
Replicates	11.28	1	11.28	
Blocks	118.81	2	15.35	
Error	100.65	13	7.74	
Total	1627.47	31		

7-17 Construct a 2^3 design with *ABC* confounded in the first two replicates and *BC* confounded in the third. Outline the analysis of variance and comment on the information obtained.

	Replicate I (*ABC* Confounded)		Replicate II (*ABC* Confounded)		Replicate III (*BC* Confounded)	
Block->	1	2	1	2	1	2
	(1)	*a*	(1)	*a*	(1)	*b*
	ab	*b*	*ab*	*b*	*bc*	*c*
	ac	*c*	*ac*	*c*	*abc*	*ab*
	bc	*abc*	*bc*	*abc*	*a*	*ac*

Source of Variation	Degrees of Freedom
A	1
B	1
C	1

AB	1
AC	1
BC	1
ABC	1
Replicates	2
Blocks	3
Error	11
Total	23

This design provides "two-thirds" information on *BC* and "one-third" information on *ABC*.

Chapter 8
Two-Level Fractional Factorial Designs
Solutions

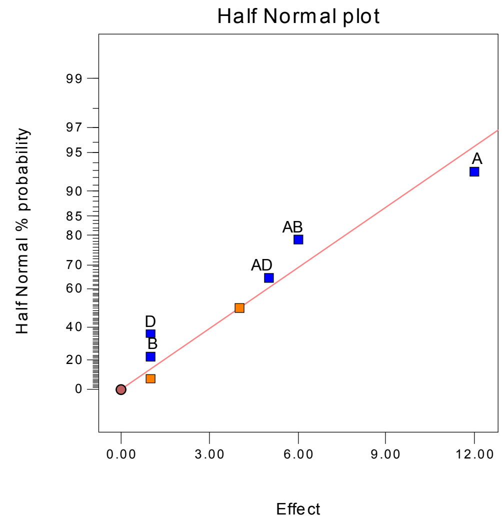
8-1 Suppose that in the chemical process development experiment in Problem 6-7, it was only possible to run a one-half fraction of the 2^4 design. Construct the design and perform the statistical analysis, using the data from replicate 1.

The required design is a 2^{4-1} with $I=ABCD$.

A	*B*	*C*	*D=ABC*		
-	-	-	-	(1)	90
+	-	-	+	*ad*	72
-	+	-	+	*bd*	87
+	+	-	-	*ab*	83
-	-	+	+	*cd*	99
+	-	+	-	*ac*	81
-	+	+	-	*bc*	88
+	+	+	+	*abcd*	80

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	-12	288	64.2857
Model	B	-1	2	0.446429
Model	C	4	32	7.14286
Model	D	-1	2	0.446429
Model	AB	6	72	16.0714
Model	AC	-1	2	0.446429
Model	AD	-5	50	11.1607
Error	BC	Aliased		
Error	BD	Aliased		
Error	CD	Aliased		
Error	ABC	Aliased		
Error	ABD	Aliased		
Error	ACD	Aliased		
Error	BCD	Aliased		
Error	ABCD	Aliased		
	Lenth's ME		22.5856	
	Lenth's SME	54.0516		



The largest effect is *A*. The next largest effects are the *AB* and *AD* interactions. A plausible tentative model would be *A*, *AB* and *AD*, along with *B* and *D* to preserve hierarchy.

Design Expert Output

Response: yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	414.00	5	82.80	4.87	0.1791	not significant
A	*288.00*	*1*	*288.00*	*16.94*	*0.0543*	
B	*2.00*	*1*	*2.00*	*0.12*	*0.7643*	
D	*2.00*	*1*	*2.00*	*0.12*	*0.7643*	
AB	*72.00*	*1*	*72.00*	*4.24*	*0.1758*	
AD	*50.00*	*1*	*50.00*	*2.94*	*0.2285*	
Residual	34.00	2	17.00			
Cor Total	448.00	7				

The "Model F-value" of 4.87 implies the model is not significant relative to the noise. There is a 17.91 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	4.12	R-Squared	0.9241
Mean	85.00	Adj R-Squared	0.7344
C.V.	4.85	Pred R-Squared	-0.2143
PRESS	544.00	Adeq Precision	6.441

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	85.00	1	1.46	78.73	91.27	
A-A	-6.00	1	1.46	-12.27	0.27	1.00
B-B	-0.50	1	1.46	-6.77	5.77	1.00
D-D	-0.50	1	1.46	-6.77	5.77	1.00
AB	3.00	1	1.46	-3.27	9.27	1.00
AD	-2.50	1	1.46	-8.77	3.77	1.00

Final Equation in Terms of Coded Factors:

yield	=
+85.00	
-6.00	* A
-0.50	* B
-0.50	* D
+3.00	* A * B
-2.50	* A * D

Final Equation in Terms of Actual Factors:

yield	=
+85.00000	
-6.00000	* A
-0.50000	* B
-0.50000	* D
+3.00000	* A * B
-2.50000	* A * D

The Design-Expert output indicates that we really only need the main effect of factor *A*. The updated analysis is shown below:

Design Expert Output

Response: **yield**

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	288.00	1	288.00	10.80	0.0167	significant
A	*288.00*	*1*	*288.00*	10.80	0.0167	
Residual	160.00	6	26.67			
Cor Total	448.00	7				

The Model F-value of 10.80 implies the model is significant. There is only a 1.67% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	5.16	R-Squared	0.6429
Mean	85.00	Adj R-Squared	0.5833
C.V.	6.08	Pred R-Squared	0.3651
PRESS	284.44	Adeq Precision	4.648

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	85.00	1	1.83	80.53	89.47	
A-A	-6.00	1	1.83	-10.47	-1.53	1.00

Final Equation in Terms of Coded Factors:

yield	=
+85.00	
-6.00	* A

Final Equation in Terms of Actual Factors:

yield	=
+85.00000	
-6.00000	* A

8-2 Suppose that in Problem 6-15, only a one-half fraction of the 2^4 design could be run. Construct the design and perform the analysis, using the data from replicate I.

The required design is a 2^{4-1} with $I=ABCD$.

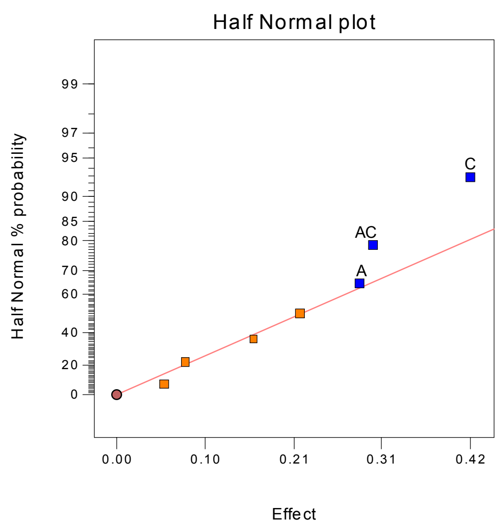
A	*B*	*C*	*D=ABC*		
-	-	-	-	(1)	1.71
+	-	-	+	*ad*	1.86
-	+	-	+	*bd*	1.79
+	+	-	-	*ab*	1.67
-	-	+	+	*cd*	1.81
+	-	+	-	*ac*	1.25
-	+	+	-	*bc*	1.46

+	+	+	+	*abcd*	0.85

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	-0.285	0.16245	19.1253
Error	B	-0.215	0.09245	10.8842
Model	C	-0.415	0.34445	40.5522
Error	D	0.055	0.00605	0.712267
Error	AB	-0.08	0.0128	1.50695
Model	AC	-0.3	0.18	21.1914
Error	AD	-0.16	0.0512	6.02778
Error	BC	Aliased		
Error	BD	Aliased		
Error	CD	Aliased		
Error	ABC	Aliased		
Error	ABD	Aliased		
Error	ACD	Aliased		
Error	BCD	Aliased		
Error	ABCD	Aliased		
	Lenth's ME	1.21397		
	Lenth's SME	2.90528		

C, A and $AC + BD$ are the largest three effects. Now because the main effects of A and C are large, the large effect estimate for the $AC + BD$ alias chain probably indicates that the AC interaction is important.



Design Expert Output

Response: Crack Lengthin mm x 10^-2
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.69	3	0.23	5.64	0.0641	not significant
A	*0.16*	*1*	*0.16*	*4.00*	*0.1162*	
C	*0.34*	*1*	*0.34*	*8.48*	*0.0436*	
AC	*0.18*	*1*	*0.18*	*4.43*	*0.1031*	
Residual	0.16	4	0.041			
Cor Total	0.85	7				

The Model F-value of 5.64 implies there is a 6.41% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.20	R-Squared	0.8087

Mean	1.55	Adj R-Squared	0.6652
C.V.	13.00	Pred R-Squared	0.2348
PRESS	0.65	Adeq Precision	5.017

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	1.55	1	0.071	1.35	1.75	
A-Pour Temp	-0.14	1	0.071	-0.34	0.055	1.00
C-Heat Tr Mtd	-0.21	1	0.071	-0.41	-9.648E-003	1.00
AC	-0.15	1	0.071	-0.35	0.048	1.00

Final Equation in Terms of Coded Factors:

Crack Length	=
+1.55	
-0.14	* A
-0.21	* C
-0.15	* A * C

Final Equation in Terms of Actual Factors:

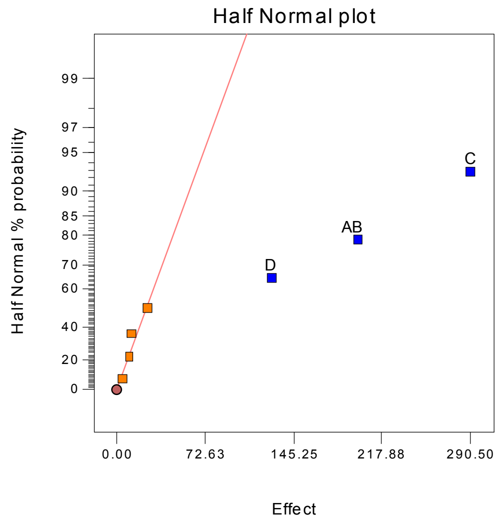
Crack Length	=
+1.55000	
-0.14250	* Pour Temp
-0.20750	* Heat Treat Method
-0.15000	* Pour Temp * Heat Treat Method

8-3 Consider the plasma etch experiment described in Problem 6-18. Suppose that only a one-half fraction of the design could be run. Set up the design and analyze the data.

A	*B*	*C*	*D=ABC*	Etch Rate (A/min)		Factor Low (-)	Levels High (+)
-	-	-	-	550	*A* (cm)	0.80	1.20
+	+	-	-	650	*B* (mTorr)	4.50	550
+	-	+	-	642	*C* (SCCM)	125	200
-	+	+	-	601	*D* (W)	275	325
+	-	-	+	749			
-	+	-	+	1052			
-	-	+	+	1075			
+	+	+	+	729			

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	4	32	0.0113941
Error	B	11.5	264.5	0.0941791
Model	C	290.5	168780	60.0967
Model	D	-127	32258	11.4859
Error	AB	-197.5	78012.5	27.7775
Error	AC	-25.5	1300.5	0.463062
Error	AD	-10	200	0.0712129
Error	BC	Aliased		
Error	BD	Aliased		
Model	CD	Aliased		
Error	ABC	Aliased		
Error	ABD	Aliased		
Error	ACD	Aliased		
Error	BCD	Aliased		
Error	ABCD	Aliased		
	Lenth's ME		60.6987	
	Lenth's SME		145.264	



The large $AB + CD$ alias chain is most likely the CD interaction.

Design Expert Output

Response: Etch Rate in A/min
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.791E+005	3	93017.00	207.05	< 0.0001	significant
C	*1.688E+005*	*1*	*1.688E+005*	*375.69*	*< 0.0001*	
D	*32258.00*	*1*	*32258.00*	*71.80*	*0.0011*	
CD	*78012.50*	*1*	*78012.50*	*173.65*	*0.0002*	
Residual	1797.00	4	449.25			
Cor Total	2.808E+005	7				

The Model F-value of 207.05 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	21.20	R-Squared	0.9936
Mean	756.00	Adj R-Squared	0.9888
C.V.	2.80	Pred R-Squared	0.9744
PRESS	7188.00	Adeq Precision	32.560

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	756.00	1	7.49	735.19	776.81	
C-Gas Flow	145.25	1	7.49	124.44	166.06	1.00
D-Power	-63.50	1	7.49	-84.31	-42.69	1.00
CD	-98.75	1	7.49	-119.56	-77.94	1.00

Final Equation in Terms of Coded Factors:

Etch Rate =
+756.00
+145.25 * C
-63.50 * D
-98.75 * C * D

Final Equation in Terms of Actual Factors:

Etch Rate =
-4246.41667
+35.47333 * Gas Flow
+14.57667 * Power
-0.10533 * Gas Flow * Power

8-4 Problem 6-21describes a process improvement study in the manufacturing process of an integrated circuit. Suppose that only eight runs could be made in this process. Set up an appropriate 2^{5-2} design and find the alias structure. Use the appropriate observations from Problem 6-21 as the observations in this design and estimate the factor effects. What conclusions can you draw?

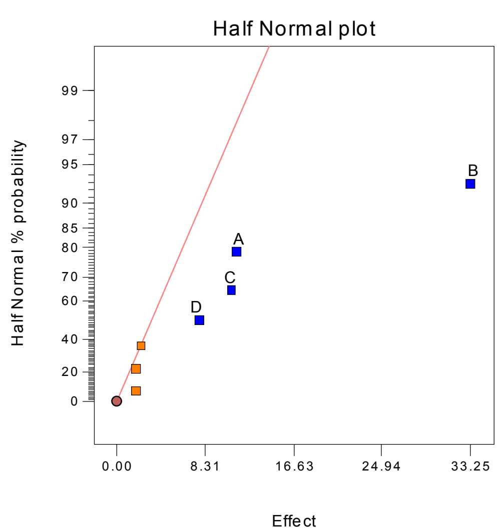
$$I = ABD = ACE = BCDE$$

A	*(ABD)*	*=BD*	*A*	*(ACE)*	*=CE*	*A*	*(BCDE)*	*=ABCDE*	*A=BD=CE=ABCDE*
B	*(ABD)*	*=AD*	*B*	*(ACE)*	*=ABCE*	*B*	*(BCDE)*	*=CDE*	*B=AD=ABCE=CDE*
C	*(ABD)*	*=ABCD*	*C*	*(ACE)*	*=AE*	*C*	*(BCDE)*	*=BDE*	*C=ABCD=AE=BDE*
D	*(ABD)*	*=AB*	*D*	*(ACE)*	*=ACDE*	*D*	*(BCDE)*	*=BCE*	*D=AB=ACDE=BCE*
E	*(ABD)*	*=ABDE*	*E*	*(ACE)*	*=AC*	*E*	*(BCDE)*	*=BCD*	*E=ABDE=AC=BCD*
BC	*(ABD)*	*=ACD*	*BC*	*(ACE)*	*=ABE*	*BC*	*(BCDE)*	*=DE*	*BC=ACD=ABE=DE*
BE	*(ABD)*	*=ADE*	*BE*	*(ACE)*	*=ABC*	*BE*	*(BCDE)*	*=CD*	*BE=ADE=ABC=CD*

A	*B*	*C*	*D=AB*	*E=AC*		
-	-	-	+	+	*de*	6
+	-	-	-	-	*a*	9
-	+	-	-	+	*be*	35
+	+	-	+	-	*abd*	50
-	-	+	+	-	*cd*	18
+	-	+	-	+	*ace*	22
-	+	+	-	-	*bc*	40
+	+	+	+	+	*abcde*	63

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	11.25	253.125	8.91953
Model	B	33.25	2211.13	77.9148
Model	C	10.75	231.125	8.1443
Model	D	7.75	120.125	4.23292
Error	E	2.25	10.125	0.356781
Error	BC	-1.75	6.125	0.215831
Error	BE	1.75	6.125	0.215831
	Lenth's ME	28.232		
	Lenth's SME	67.5646		



The main *A*, *B*, *C*, and *D* are large. However, recall that you are really estimating *A+BD+CE*, *B+AD*, *C+DE* and *D+AD*. There are other possible interpretations of the experiment because of the aliasing.

Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2815.50	4	703.88	94.37	0.0017	significant
A	*253.13*	*1*	*253.13*	*33.94*	*0.0101*	
B	*2211.12*	*1*	*2211.12*	*296.46*	*0.0004*	
C	*231.13*	*1*	*231.13*	*30.99*	*0.0114*	
D	*120.13*	*1*	*120.13*	*16.11*	*0.0278*	
Residual	22.38	3	7.46			
Cor Total	2837.88	7				

The Model F-value of 94.37 implies the model is significant. There is only a 0.17% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.73	R-Squared	0.9921
Mean	30.38	Adj R-Squared	0.9816
C.V.	8.99	Pred R-Squared	0.9439
PRESS	159.11	Adeq Precision	25.590

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	30.38	1	0.97	27.30	33.45	
A-Aperture	5.63	1	0.97	2.55	8.70	1.00
B-Exposure Time	16.63	1	0.97	13.55	19.70	1.00
C-Develop Time	5.37	1	0.97	2.30	8.45	1.00
D-Mask Dimension	3.87	1	0.97	0.80	6.95	1.00

Final Equation in Terms of Coded Factors:

Yield =
+30.38
+5.63 * A
+16.63 * B
+5.37 * C
+3.87 * D

Final Equation in Terms of Actual Factors:

Aperture small
Mask Dimension Small
Yield =
-6.00000
+0.83125 * Exposure Time
+0.71667 * Develop Time

Aperture large
Mask Dimension Small
Yield =
+5.25000
+0.83125 * Exposure Time
+0.71667 * Develop Time

Aperture small
Mask Dimension Large
Yield =
+1.75000
+0.83125 * Exposure Time
+0.71667 * Develop Time

Aperture large
Mask Dimension Large
Yield =
+13.00000

```
            +0.83125     * Exposure Time
            +0.71667     * Develop Time
```

8-5 Continuation of Problem 8-4. Suppose you have made the eight runs in the 2^{5-2} design in Problem 8-4. What additional runs would be required to identify the factor effects that are of interest? What are the alias relationships in the combined design?

We could fold over the original design by changing the signs on the generators *D* = *AB* and *E* = *AC* to produce the following new experiment.

A	*B*	*C*	*D=-AB*	*E=-AC*		
-	-	-	-	-	(1)	7
+	-	-	+	+	*ade*	12
-	+	-	+	-	*bd*	32
+	+	-	-	+	*abe*	52
-	-	+	-	+	*ce*	15
+	-	+	+	-	*acd*	21
-	+	+	+	+	*bcde*	41
+	+	+	-	-	*abc*	60

A	(-ABD)	=-BD	A	(-ACE)	=-CE	A	(BCDE)	=ABCDE	A=-BD=-CE=ABCDE
B	(-ABD)	=-AD	B	(-ACE)	=-ABCE	B	(BCDE)	=CDE	B=-AD=-ABCE=CDE
C	(-ABD)	=-ABCD	C	(-ACE)	=-AE	C	(BCDE)	=BDE	C=-ABCD=-AE=BDE
D	(-ABD)	=-AB	D	(-ACE)	=-ACDE	D	(BCDE)	=BCE	D=-AB=-ACDE=BCE
E	(-ABD)	=-ABDE	E	(-ACE)	=-AC	E	(BCDE)	=BCD	E=-ABDE=-AC=BCD
BC	(-ABD)	=-ACD	BC	(-ACE)	=-ABE	BC	(BCDE)	=DE	BC=-ACD=-ABE=DE
BE	(-ABD)	=-ADE	BE	(-ACE)	=-ABC	BE	(BCDE)	=CD	BE=-ADE=-ABC=CD

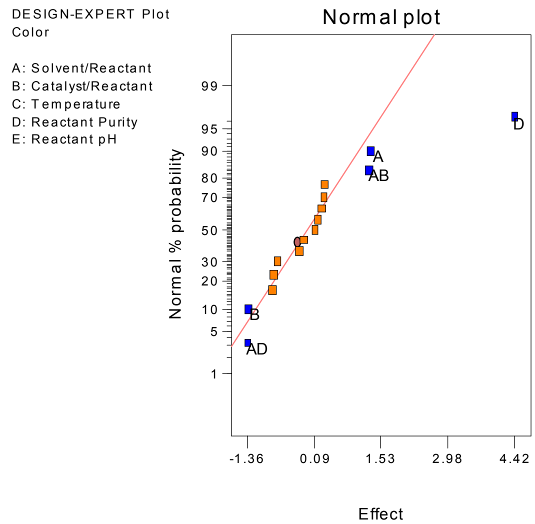
Assuming all three factor and higher interactions to be negligible, all main effects can be separated from their two-factor interaction aliases in the combined design.

8-6 R.D. Snee ("Experimenting with a Large Number of Variables," in *Experiments in Industry: Design, Analysis and Interpretation of Results*, by R.D. Snee, L.B. Hare, and J.B. Trout, Editors, ASQC, 1985) describes an experiment in which a 2^{5-1} design with *I=ABCDE* was used to investigate the effects of five factors on the color of a chemical product. The factors are *A* = solvent/reactant, *B* = catalyst/reactant, *C* = temperature, *D* = reactant purity, and *E* = reactant pH. The results obtained were as follows:

e =	-0.63	*d* =	6.79
a =	2.51	*ade* =	5.47
b =	-2.68	*bde* =	3.45
abe =	1.66	*abd* =	5.68
c =	2.06	*cde* =	5.22
ace =	1.22	*acd* =	4.38
bce =	-2.09	*bcd* =	4.30
abc =	1.93	*abcde* =	4.05

(a) Prepare a normal probability plot of the effects. Which effects seem active?

Factors *A*, *B*, *D*, and the *AB*, *AD* interactions appear to be active.



Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	1.31	6.8644	5.98537
Model	B	-1.34	7.1824	6.26265
Error	C	-0.1475	0.087025	0.0758809
Model	D	4.42	78.1456	68.1386
Error	E	-0.8275	2.73902	2.38828
Model	AB	1.275	6.5025	5.66981
Error	AC	-0.7875	2.48062	2.16297
Model	AD	-1.355	7.3441	6.40364
Error	AE	0.3025	0.366025	0.319153
Error	BC	0.1675	0.112225	0.0978539
Error	BD	0.245	0.2401	0.209354
Error	BE	0.2875	0.330625	0.288286
Error	CD	-0.7125	2.03063	1.77059
Error	CE	-0.24	0.2304	0.200896
Error	DE	0.0875	0.030625	0.0267033
	Lenth's ME	1.95686		
	Lenth's SME	3.9727		

Design Expert Output

Response: Color
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	106.04	5	21.21	24.53	< 0.0001	significant
A	*6.86*	*1*	*6.86*	*7.94*	*0.0182*	
B	*7.18*	*1*	*7.18*	*8.31*	*0.0163*	
D	*78.15*	*1*	*78.15*	*90.37*	*< 0.0001*	
AB	*6.50*	*1*	*6.50*	*7.52*	*0.0208*	
AD	*7.34*	*1*	*7.34*	*8.49*	*0.0155*	
Residual	8.65	10	0.86			
Cor Total	114.69	15				

The Model F-value of 24.53 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.93	R-Squared	0.9246
Mean	2.71	Adj R-Squared	0.8869
C.V.	34.35	Pred R-Squared	0.8070
PRESS	22.14	Adeq Precision	14.734

	Coefficient		**Standard**	**95% CI**	**95% CI**

Factor	Estimate	DF	Error	Low	High	VIF
Intercept	2.71	1	0.23	2.19	3.23	
A-Solvent/Reactant	0.66	1	0.23	0.14	1.17	1.00
B-Catalyst/Reactant	-0.67	1	0.23	-1.19	-0.15	1.00
D-Reactant Purity	2.21	1	0.23	1.69	2.73	1.00
AB	0.64	1	0.23	0.12	1.16	1.00
AD	-0.68	1	0.23	-1.20	-0.16	1.00

Final Equation in Terms of Coded Factors:

Color =
+2.71
+0.66 * A
-0.67 * B
+2.21 * D
+0.64 * A * B
-0.68 * A * D

Final Equation in Terms of Actual Factors:

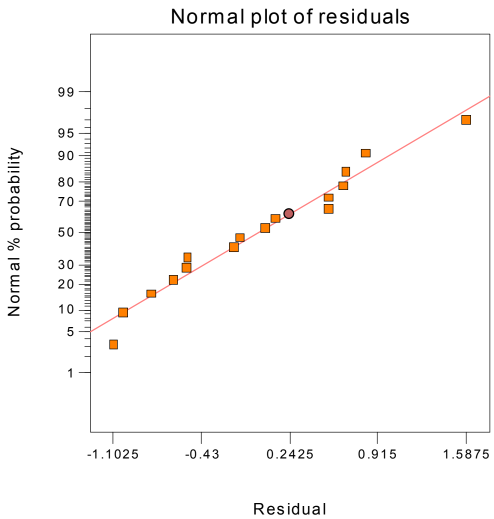
Color =
+2.70750
+0.65500 * Solvent/Reactant
-0.67000 * Catalyst/Reactant
+2.21000 * Reactant Purity
+0.63750 * Solvent/Reactant * Catalyst/Reactant
-0.67750 * Solvent/Reactant * Reactant Purity

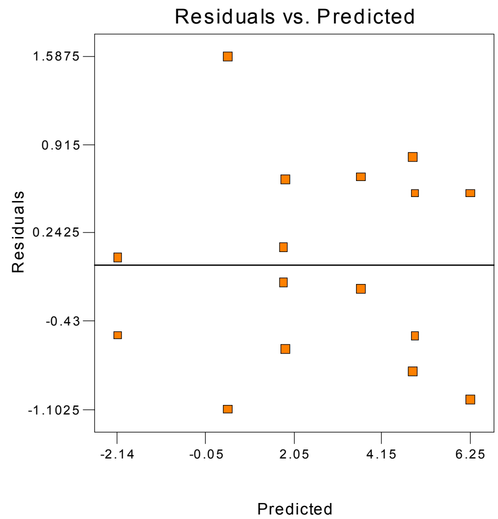
(b) Calculate the residuals. Construct a normal probability plot of the residuals and plot the residuals versus the fitted values. Comment on the plots.

Design Expert Output

Diagnostics Case Statistics

Standard Order	Actual Value	Predicted Value	Residual	Leverage	Student Residual	Cook's Distance	Outlier t	Run Order
1	-0.63	0.47	-1.10	0.375	-1.500	0.225	-1.616	2
2	2.51	1.86	0.65	0.375	0.881	0.078	0.870	6
3	-2.68	-2.14	-0.54	0.375	-0.731	0.053	-0.713	14
4	1.66	1.80	-0.14	0.375	-0.187	0.003	-0.178	11
5	2.06	0.47	1.59	0.375	2.159	0.466	2.804	8
6	1.22	1.86	-0.64	0.375	-0.874	0.076	-0.863	15
7	-2.09	-2.14	0.053	0.375	0.071	0.001	0.068	10
8	1.93	1.80	0.13	0.375	0.180	0.003	0.171	3
9	6.79	6.25	0.54	0.375	0.738	0.054	0.720	4
10	5.47	4.93	0.54	0.375	0.738	0.054	0.720	5
11	3.45	3.63	-0.18	0.375	-0.248	0.006	-0.236	16
12	5.68	4.86	0.82	0.375	1.112	0.124	1.127	12
13	5.22	6.25	-1.03	0.375	-1.398	0.195	-1.478	9
14	4.38	4.93	-0.55	0.375	-0.745	0.055	-0.727	1
15	4.30	3.63	0.67	0.375	0.908	0.082	0.899	13
16	4.05	4.86	-0.81	0.375	-1.105	0.122	-1.119	7





The residual plots are satisfactory.

(c) If any factors are negligible, collapse the 2^{5-1} design into a full factorial in the active factors. Comment on the resulting design, and interpret the results.

The design becomes two replicates of a 2^3 in the factors *A*, *B* and *D*. When re-analyzing the data in three factors, *D* becomes labeled as *C*.

Design Expert Output

Response: Color

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	106.51	7	15.22	14.89	0.0005	significant
A	*6.86*	*1*	*6.86*	*6.72*	*0.0320*	
B	*7.18*	*1*	*7.18*	*7.03*	*0.0292*	
C	*78.15*	*1*	*78.15*	*76.46*	*< 0.0001*	
AB	*6.50*	*1*	*6.50*	*6.36*	*0.0357*	
AC	*7.34*	*1*	*7.34*	*7.19*	*0.0279*	
BC	*0.24*	*1*	*0.24*	*0.23*	*0.6409*	
ABC	*0.23*	*1*	*0.23*	*0.23*	*0.6476*	
Residual	8.18	8	1.02			
Lack of Fit	*0.000*	*0*				
Pure Error	*8.18*	*8*	*1.02*			
Cor Total	114.69	15				

The Model F-value of 14.89 implies the model is significant. There is only a 0.05% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.01	R-Squared	0.9287
Mean	2.71	Adj R-Squared	0.8663
C.V.	37.34	Pred R-Squared	0.7148
PRESS	32.71	Adeq Precision	11.736

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	2.71	1	0.25	2.12	3.29	
A-Solvent/Reactant	0.66	1	0.25	0.072	1.24	1.00
B-Catalyst/Reactant	-0.67	1	0.25	-1.25	-0.087	1.00
C-Reactant Purity	2.21	1	0.25	1.63	2.79	1.00
AB	0.64	1	0.25	0.055	1.22	1.00
AC	-0.68	1	0.25	-1.26	-0.095	1.00

BC	0.12	1	0.25	-0.46	0.71	1.00
ABC	-0.12	1	0.25	-0.70	0.46	1.00

Final Equation in Terms of Coded Factors:

Color	=
+2.71	
+0.66	* A
-0.67	* B
+2.21	* C
+0.64	* A * B
-0.68	* A * C
+0.12	* B * C
-0.12	* A * B * C

Final Equation in Terms of Actual Factors:

Color	=
+2.70750	
+0.65500	* Solvent/Reactant
-0.67000	* Catalyst/Reactant
+2.21000	* Reactant Purity
+0.63750	* Solvent/Reactant * Catalyst/Reactant
-0.67750	* Solvent/Reactant * Reactant Purity
+0.12250	* Catalyst/Reactant * Reactant Purity
-0.12000	* Solvent/Reactant * Catalyst/Reactant * Reactant Purity

8-7 An article by J.J. Pignatiello, Jr. And J.S. Ramberg in the *Journal of Quality Technology*, (Vol. 17, 1985, pp. 198-206) describes the use of a replicated fractional factorial to investigate the effects of five factors on the free height of leaf springs used in an automotive application. The factors are A = furnace temperature, B = heating time, C = transfer time, D = hold down time, and E = quench oil temperature. The data are shown below:

A	*B*	*C*	*D*	*E*	Free Height		
-	-	-	-	-	7.78	7.78	7.81
+	-	-	+	-	8.15	8.18	7.88
-	+	-	+	-	7.50	7.56	7.50
+	+	-	-	-	7.59	7.56	7.75
-	-	+	+	-	7.54	8.00	7.88
+	-	+	-	-	7.69	8.09	8.06
-	+	+	-	-	7.56	7.52	7.44
+	+	+	+	-	7.56	7.81	7.69
-	-	-	-	+	7.50	7.25	7.12
+	-	-	+	+	7.88	7.88	7.44
-	+	-	+	+	7.50	7.56	7.50
+	+	-	-	+	7.63	7.75	7.56
-	-	+	+	+	7.32	7.44	7.44
+	-	+	-	+	7.56	7.69	7.62
-	+	+	-	+	7.18	7.18	7.25
+	+	+	+	+	7.81	7.50	7.59

(a) Write out the alias structure for this design. What is the resolution of this design?

I=*ABCD*, Resolution IV

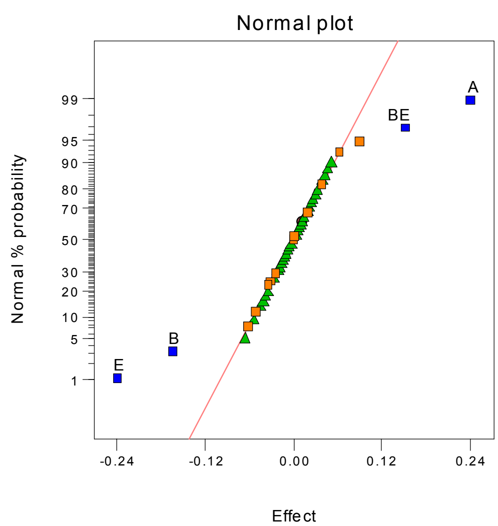
A	*(ABCD)=*	*BCD*
B	*(ABCD)=*	*ACD*
C	*(ABCD)=*	*ABD*
D	*(ABCD)=*	*ABC*
E	*(ABCD)=*	*ABCDE*
AB	*(ABCD)=*	*CD*
AC	*(ABCD)=*	*BD*
AD	*(ABCD)=*	*BC*

AE	*(ABCD)=*	*BCDE*
BE	*(ABCD)=*	*ACDE*
CE	*(ABCD)=*	*ABDE*
DE	*(ABCD)=*	*ABCE*

(b) Analyze the data. What factors influence the mean free height?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	0.242083	0.703252	24.3274
Model	B	-0.16375	0.321769	11.1309
Model	C	-0.0495833	0.0295021	1.02056
Model	D	0.09125	0.0999188	3.45646
Model	E	-0.23875	0.684019	23.6621
Model	AB	-0.0295833	0.0105021	0.363296
Model	AC	0.00125	1.875E-005	0.000648614
Model	AD	-0.0229167	0.00630208	0.218006
Model	AE	0.06375	0.0487687	1.68704
Error	BC	Aliased		
Error	BD	Aliased		
Model	BE	0.152917	0.280602	9.70679
Error	CD	Aliased		
Model	CE	-0.0329167	0.0130021	0.449777
Model	DE	0.0395833	0.0188021	0.650415
Error	Pure Error		0.627067	21.6919
	Lenth's ME	0.088057		
	Lenth's SME	0.135984		



Design Expert Output

Response:Free Height
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1.99	4	0.50	23.74	< 0.0001	significant
A	*0.70*	*1*	*0.70*	*33.56*	*< 0.0001*	
B	*0.32*	*1*	*0.32*	*15.35*	*0.0003*	
E	*0.68*	*1*	*0.68*	*32.64*	*< 0.0001*	
BE	*0.28*	*1*	*0.28*	*13.39*	*0.0007*	
Residual	0.90	43	0.021			
Lack of Fit	*0.27*	*11*	*0.025*	*1.27*	*0.2844 not significant*	

Pure Error	*0.63*	*32*	*0.020*
Cor Total	2.89	47	

The Model F-value of 23.74 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.14	R-Squared	0.6883
Mean	7.63	Adj R-Squared	0.6593
C.V.	1.90	Pred R-Squared	0.6116
PRESS	1.12	Adeq Precision	13.796

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	7.63	1	0.021	7.58	7.67	
A-Furnace Temp	0.12	1	0.021	0.079	0.16	1.00
B-Heating Time	-0.082	1	0.021	-0.12	-0.040	1.00
E-Quench Temp	-0.12	1	0.021	-0.16	-0.077	1.00
BE	0.076	1	0.021	0.034	0.12	1.00

Final Equation in Terms of Coded Factors:

Free Height	=
+7.63	
+0.12	* A
-0.082	* B
-0.12	* E
+0.076	* B * E

Final Equation in Terms of Actual Factors:

Free Height	=
+7.62562	
+0.12104	* Furnace Temp
-0.081875	* Heating Time
-0.11937	* Quench Temp
+0.076458	* Heating Time * Quench Temp

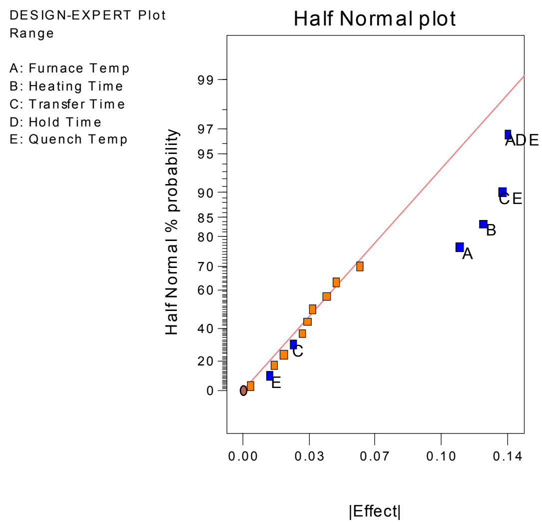
(c) Calculate the range and standard deviation of the free height for each run. Is there any indication that any of these factors affects variability in the free height?

Design Expert Output (Range)

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	0.11375	0.0517563	16.2198
Model	B	-0.12625	0.0637563	19.9804
Model	C	0.02625	0.00275625	0.863774
Error	D	0.06125	0.0150063	4.70277
Model	E	-0.01375	0.00075625	0.236999
Error	AB	0.04375	0.00765625	2.39937
Error	AC	-0.03375	0.00455625	1.42787
Error	AD	0.03625	0.00525625	1.64724
Error	AE	-0.00375	5.625E-005	0.017628
Model	BC	Aliased		
Error	BD	Aliased		
Model	BE	0.01625	0.00105625	0.331016
Error	CD	Aliased		
Model	CE	-0.13625	0.0742562	23.271
Error	DE	-0.02125	0.00180625	0.566056
Error	ABC	Aliased		
Error	ABD	Aliased		
Error	ABE	0.03125	0.00390625	1.22417
Error	ACD	Aliased		
Error	ACE	0.04875	0.00950625	2.97914
Error	ADE	0.13875	0.0770062	24.1328
Error	BCD	Aliased		
Model	BCE	Aliased		
Error	BDE	Aliased		
Error	CDE	Aliased		
	Lenth's ME	0.130136		

Lenth's SME	0.264194

Interaction ADE is aliased with BCE. Although the plot below identifies ADE, BCE was included in the analysis.



Design Expert Output (Range)

Response: Range
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.28	8	0.035	5.70	0.0167	significant
A	0.052	1	0.052	8.53	0.0223	
B	0.064	1	0.064	10.50	0.0142	
C	2.756E-003	1	2.756E-003	0.45	0.5220	
E	7.562E-004	1	7.562E-004	0.12	0.7345	
BC	5.256E-003	1	5.256E-003	0.87	0.3831	
BE	1.056E-003	1	1.056E-003	0.17	0.6891	
CE	0.074	1	0.074	12.23	0.0100	
BCE	0.077	1	0.077	12.69	0.0092	
Residual	0.042	7	6.071E-003			
Cor Total	0.32	15				

The Model F-value of 5.70 implies the model is significant. There is only
a 1.67% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.078	R-Squared	0.8668
Mean	0.22	Adj R-Squared	0.7146
C.V.	35.52	Pred R-Squared	0.3043
PRESS	0.22	Adeq Precision	7.166

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	0.22	1	0.019	0.17	0.27	
A-Furn Temp	0.057	1	0.019	0.011	0.10	1.00
B-Heat Time	-0.063	1	0.019	-0.11	-0.017	1.00
C-Transfer Time	0.013	1	0.019	-0.033	0.059	1.00
E-Qnch Temp	-6.875E-003	1	0.019	-0.053	0.039	1.00
BC	0.018	1	0.019	-0.028	0.064	1.00
BE	8.125E-003	1	0.019	-0.038	0.054	1.00
CE	-0.068	1	0.019	-0.11	-0.022	1.00
BCE	0.069	1	0.019	0.023	0.12	1.00

Final Equation in Terms of Coded Factors:

```
              Range     =
              +0.22
              +0.057     * A
              -0.063     * B
              +0.013     * C
         -6.875E-003     * E
              +0.018     * B * C
         +8.125E-003     * B * E
              -0.068     * C * E
              +0.069     * B * C * E
```

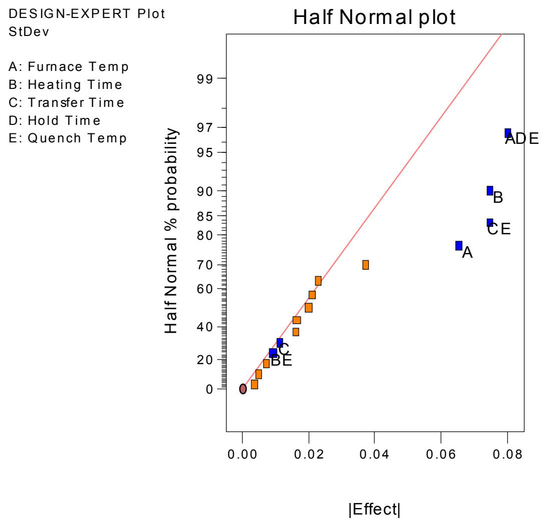
Final Equation in Terms of Actual Factors:

```
              Range     =
           +0.21937
          +0.056875     * Furnace Temp
          -0.063125     * Heating Time
          +0.013125     * Transfer Time
     -6.87500E-003     * Quench Temp
          +0.018125     * Heating Time * Transfer Time
     +8.12500E-003     * Heating Time * Quench Temp
          -0.068125     * Transfer Time * Quench Temp
          +0.069375     * Heating Time * Transfer Time * Quench Temp
```

Design Expert Output (StDev)

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	0.0625896	0.0156698	16.873
Model	B	-0.0714887	0.0204425	22.0121
Model	C	0.010567	0.000446646	0.48094
Error	D	0.0353616	0.00500176	5.3858
Model	E	-0.00684034	0.000187161	0.201532
Error	AB	0.0153974	0.000948317	1.02113
Error	AC	-0.0218505	0.00190978	2.05641
Error	AD	0.0190608	0.00145326	1.56484
Error	AE	-0.00329035	4.33057E-005	0.0466308
Model	BC	Aliased		
Error	BD	Aliased		
Model	BE	0.0087666	0.000307413	0.331017
Error	CD	Aliased		
Model	CE	-0.0714816	0.0204385	22.0078
Error	DE	-0.00467792	8.75317E-005	0.0942525
Error	ABC	Aliased		
Error	ABD	Aliased		
Error	ABE	0.0155599	0.000968437	1.0428
Error	ACD	Aliased		
Error	ACE	0.0199742	0.00159587	1.7184
Error	ADE	Aliased		
Error	BCD	Aliased		
Model	BCE	0.0764346	0.023369	25.1633
Error	BDE	Aliased		
Error	CDE	Aliased		
	Lenth's ME	0.0596836		
	Lenth's SME	0.121166		

Interaction ADE is aliased with BCE. Although the plot below identifies ADE, BCE was included in the analysis.



Design Expert Output (StDev)

Response: **StDev**

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.082	8	0.010	6.82	0.0101	significant
A	*0.016*	*1*	*0.016*	*10.39*	*0.0146*	
B	*0.020*	*1*	*0.020*	*13.56*	*0.0078*	
C	*4.466E-004*	*1*	*4.466E-004*	*0.30*	*0.6032*	
E	*1.872E-004*	*1*	*1.872E-004*	*0.12*	*0.7350*	
BC	*1.453E-003*	*1*	*1.453E-003*	*0.96*	*0.3589*	
BE	*3.074E-004*	*1*	*3.074E-004*	*0.20*	*0.6653*	
CE	*0.020*	*1*	*0.020*	*13.55*	*0.0078*	
BCE	*0.023*	*1*	*0.023*	*15.50*	*0.0056*	
Residual	0.011	7	1.508E-003			
Cor Total	0.093	15				

The Model F-value of 6.82 implies the model is significant. There is only
a 1.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.039	R-Squared	0.8863
Mean	0.12	Adj R-Squared	0.7565
C.V.	33.07	Pred R-Squared	0.4062
PRESS	0.055	Adeq Precision	7.826

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	0.12	1	9.708E-003	0.094	0.14	
A-Furnace Temp	0.031	1	9.708E-003	8.340E-003	0.054	1.00
B-Heating Time	-0.036	1	9.708E-003	-0.059	-0.013	1.00
C-Transfer Time	5.283E-003	1	9.708E-003	-0.018	0.028	1.00
E-Quench Temp	-3.420E-003	1	9.708E-003	-0.026	0.020	1.00
BC	9.530E-003	1	9.708E-003	-0.013	0.032	1.00
BE	4.383E-003	1	9.708E-003	-0.019	0.027	1.00
CE	-0.036	1	9.708E-003	-0.059	-0.013	1.00
BCE	0.038	1	9.708E-003	0.015	0.061	1.00

Final Equation in Terms of Coded Factors:

StDev =
+0.12
+0.031 * A
-0.036 * B
+5.283E-003 * C

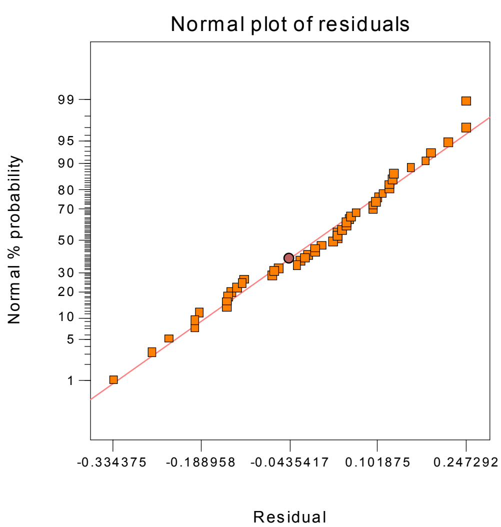
-3.420E-003	* E
+9.530E-003	* B * C
+4.383E-003	* B * E
-0.036	* C * E
+0.038	* B * C * E

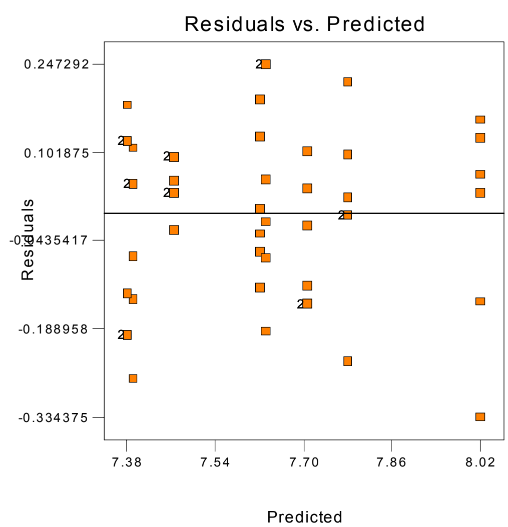
Final Equation in Terms of Actual Factors:

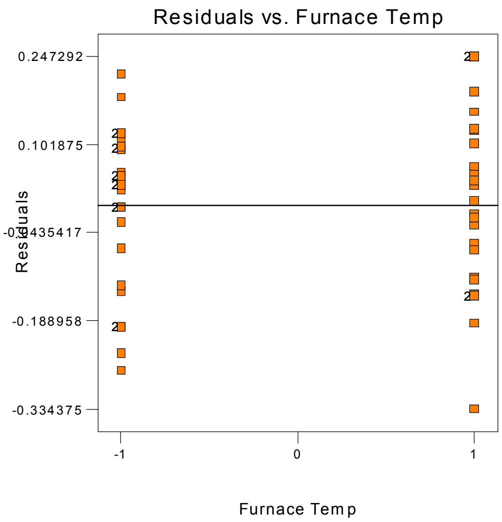
StDev	=
+0.11744	
+0.031295	* Furnace Temp
-0.035744	* Heating Time
+5.28350E-003	* Transfer Time
-3.42017E-003	* Quench Temp
+9.53040E-003	* Heating Time * Transfer Time
+4.38330E-003	* Heating Time * Quench Temp
-0.035741	* Transfer Time * Quench Temp
+0.038217	* Heating Time * Transfer Time * Quench Temp

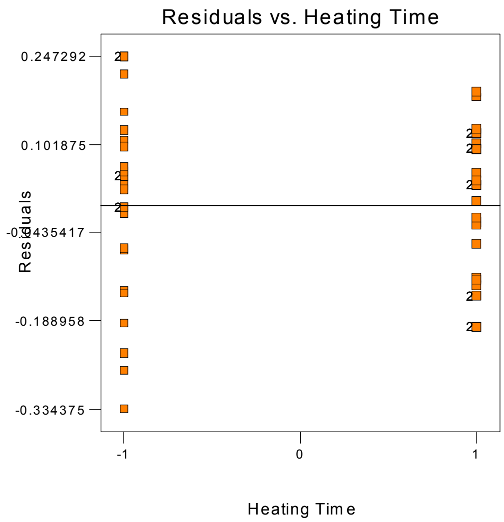
(d) Analyze the residuals from this experiment, and comment on your findings.

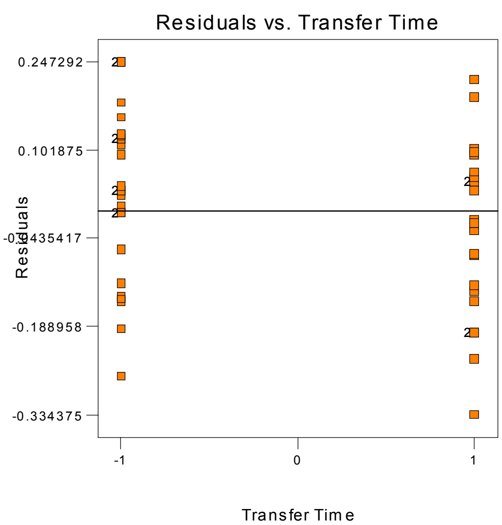
The residual plot follows. All plots are satisfactory.

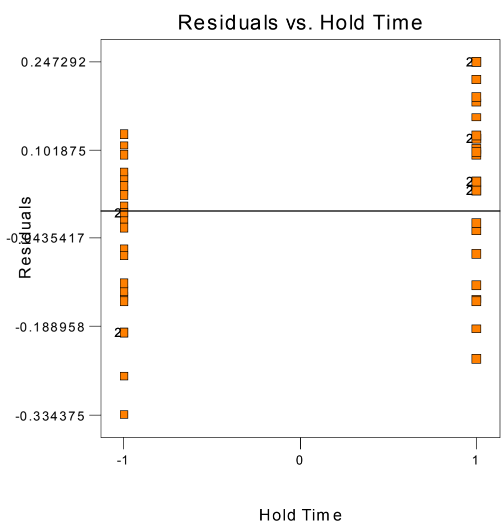


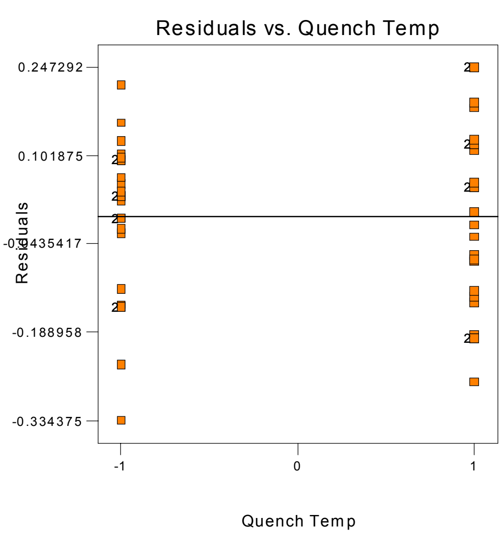



Residuals vs. Furnace Temp
0.247292
0.101875
-0.0435417
-0.188958
-0.334375
Residuals
-1
0
1
Furnace Temp


Residuals vs. Heating Time
0.247292
0.101875
-0.0435417
-0.188958
-0.334375
Residuals
-1
0
1
Heating Time


Residuals vs. Transfer Time
0.247292
0.101875
-0.0435417
-0.188958
-0.334375
Residuals
-1
0
1
Transfer Time


Residuals vs. Hold Time
0.247292
0.101875
-0.0435417
-0.188958
-0.334375
Residuals
-1
0
1
Hold Time


Residuals vs. Quench Temp
0.247292
0.101875
-0.0435417
-0.188958
-0.334375
Residuals
-1
0
1
Quench Temp

(e) Is this the best possible design for five factors in 16 runs? Specifically, can you find a fractional design for five factors in 16 runs with a higher resolution than this one?

This was not the best design. A resolution V design is possible by setting the generator equal to the highest order interaction, *ABCDE*.

8-8 An article in *Industrial and Engineering Chemistry* ("More on Planning Experiments to Increase Research Efficiency," 1970, pp. 60-65) uses a 2^{5-2} design to investigate the effect of A = condensation, B = amount of material 1, C = solvent volume, D = condensation time, and E = amount of material 2 on yield. The results obtained are as follows:

e =	23.2	*ad* =	16.9	*cd* =	23.8	*bde* =	16.8
ab =	15.5	*bc* =	16.2	*ace* =	23.4	*abcde* =	18.1

(a) Verify that the design generators used were I = *ACE* and I = *BDE*.

A	*B*	*C*	*D=BE*	*E=AC*	
-	-	-	-	+	*e*
+	-	-	+	-	*ad*
-	+	-	+	+	*bde*
+	+	-	-	-	*ab*
-	-	+	+	-	*cd*
+	-	+	-	+	*ace*
-	+	+	-	-	*bc*
+	+	+	+	+	*abcde*

(b) Write down the complete defining relation and the aliases for this design.

I=*BDE*=*ACE*=*ABCD*.

A	*(BDE)*	*=ABDE*	*A*	*(ACE)*	*=CE*	*A*	*(ABCD)*	*=BCD*	*A=ABDE=CE=BCD*
B	*(BDE)*	*=DE*	*B*	*(ACE)*	*=ABCE*	*B*	*(ABCD)*	*=ACD*	*B=DE=ABCE=ACD*
C	*(BDE)*	*=BCDE*	*C*	*(ACE)*	*=AE*	*C*	*(ABCD)*	*=ABD*	*C=BCDE=AE=ABD*
D	*(BDE)*	*=BE*	*D*	*(ACE)*	*=ACDE*	*D*	*(ABCD)*	*=ABC*	*D=BE=ACDE=ABC*
E	*(BDE)*	*=BD*	*E*	*(ACE)*	*=AC*	*E*	*(ABCD)*	*=ABCDE*	*E=BD=AC=ABCDE*
AB	*(BDE)*	*=ADE*	*AB*	*(ACE)*	*=BCE*	*AB*	*(ABCD)*	*=CD*	*AB=ADE=BCE=CD*
AD	*(BDE)*	*=ABE*	*AD*	*(ACE)*	*=CDE*	*AD*	*(ABCD)*	*=BC*	*AD=ABE=CDE=BC*

(c) Estimate the main effects.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	-1.525	4.65125	5.1831
Model	B	-5.175	53.5613	59.6858
Model	C	2.275	10.3512	11.5349
Model	D	-0.675	0.91125	1.01545
Model	E	2.275	10.3513	11.5349

(d) Prepare an analysis of variance table. Verify that the *AB* and *AD* interactions are available to use as error.

The analysis of variance table is shown below. Part (b) shows that *AB* and *AD* are aliased with other factors. If all two-factor and three factor interactions are negligible, then *AB* and *AD* could be pooled as an estimate of error.

Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	79.83	5	15.97	3.22	0.2537	not significant
A	*4.65*	*1*	*4.65*	*0.94*	*0.4349*	
B	*53.56*	*1*	*53.56*	*10.81*	*0.0814*	
C	*10.35*	*1*	*10.35*	*2.09*	*0.2853*	
D	*0.91*	*1*	*0.91*	*0.18*	*0.7098*	
E	*10.35*	*1*	*10.35*	*2.09*	*0.2853*	
Residual	9.91	2	4.96			
Cor Total	89.74	7				

The "Model F-value" of 3.22 implies the model is not significant relative to the noise. There is a 25.37 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	2.23	R-Squared	0.8895
Mean	19.24	Adj R-Squared	0.6134
C.V.	11.57	Pred R-Squared	-0.7674
PRESS	158.60	Adeq Precision	5.044

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	19.24	1	0.79	15.85	22.62	
A-Condensation	-0.76	1	0.79	-4.15	2.62	1.00
B-Material 1	-2.59	1	0.79	-5.97	0.80	1.00
C-Solvent	1.14	1	0.79	-2.25	4.52	1.00
D-Time	-0.34	1	0.79	-3.72	3.05	1.00
E-Material 2	1.14	1	0.79	-2.25	4.52	1.00

Final Equation in Terms of Coded Factors:

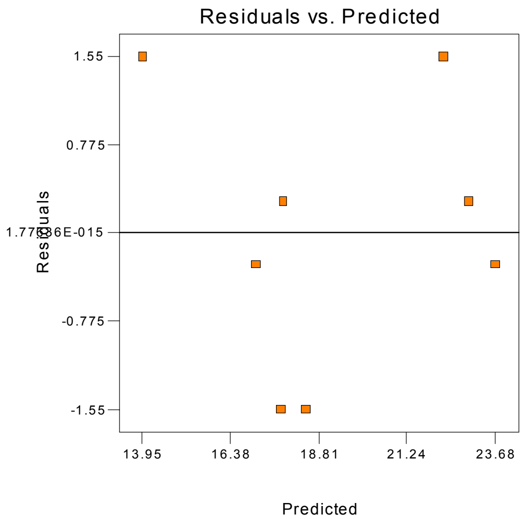
Yield =
+19.24
-0.76 * A
-2.59 * B
+1.14 * C
-0.34 * D
+1.14 * E

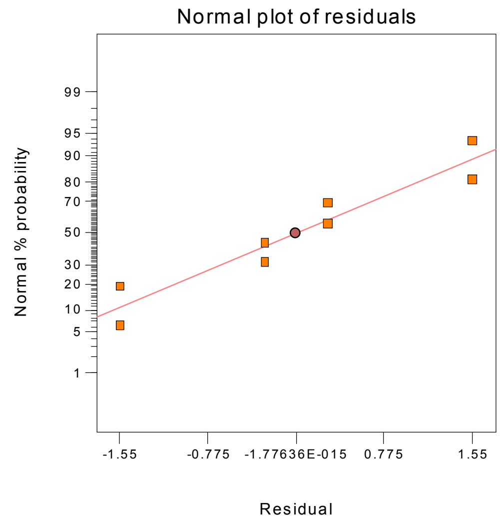
Final Equation in Terms of Actual Factors:

Yield =
+19.23750
-0.76250 * Condensation
-2.58750 * Material 1
+1.13750 * Solvent
-0.33750 * Time
+1.13750 * Material 2

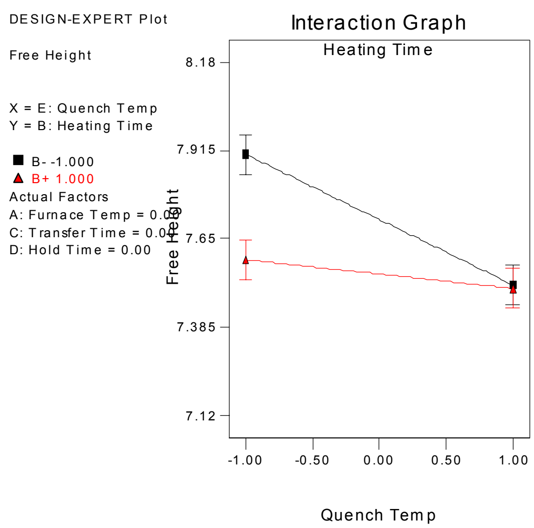
(e) Plot the residuals versus the fitted values. Also construct a normal probability plot of the residuals. Comment on the results.

The residual plots are satisfactory.





8-9 Consider the leaf spring experiment in Problem 8-7. Suppose that factor *E* (quench oil temperature) is very difficult to control during manufacturing. Where would you set factors *A, B, C* and *D* to reduce variability in the free height as much as possible regardless of the quench oil temperature used?



Run the process with *A* at the high level, *B* at the low level, *C* at the low level and *D* at either level (the low level of *D* may give a faster process).

8-10 Construct a 2^{7-2} design by choosing two four-factor interactions as the independent generators. Write down the complete alias structure for this design. Outline the analysis of variance table. What is the resolution of this design?

I=*CDEF*=*ABCG*=*ABDEFG,* Resolution IV

A	B	C	D	E	F=CDE	G=ABC

1	-	-	-	-	-	-	-	(1)
2	+	-	-	-	-	-	+	*ag*
3	-	+	-	-	-	-	+	*bg*
4	+	+	-	-	-	-	-	*ab*
5	-	-	+	-	-	+	+	*cfg*
6	+	-	+	-	-	+	-	*acf*
7	-	+	+	-	-	+	-	*bcf*
8	+	+	+	-	-	+	+	*abcfg*
9	-	-	-	+	-	+	-	*df*
10	+	-	-	+	-	+	+	*adfg*
11	-	+	-	+	-	+	+	*bdfg*
12	+	+	-	+	-	+	-	*abdf*
13	-	-	+	+	-	-	+	*cdg*
14	+	-	+	+	-	-	-	*acd*
15	-	+	+	+	-	-	-	*bcd*
16	+	+	+	+	-	-	+	*abcdg*
17	-	-	-	-	+	+	-	*ef*
18	+	-	-	-	+	+	+	*aefg*
19	-	+	-	-	+	+	+	*befg*
20	+	+	-	-	+	+	-	*abef*
21	-	-	+	-	+	-	+	*ceg*
22	+	-	+	-	+	-	-	*ace*
23	-	+	+	-	+	-	-	*bce*
24	+	+	+	-	+	-	+	*abceg*
25	-	-	-	+	+	-	-	*de*
26	+	-	-	+	+	-	+	*adeg*
27	-	+	-	+	+	-	+	*bdeg*
28	+	+	-	+	+	-	-	*abde*
29	-	-	+	+	+	+	+	*cdefg*
30	+	-	+	+	+	+	-	*acdef*
31	-	+	+	+	+	+	-	*bcdef*
32	+	+	+	+	+	+	+	*abcdefg*

Alias Structure

A(CDEF)= ACDEF	*A(ABCG)= BCG*	*A(ABDEFG)= BDEFG*	*A=ACDEF=BCG=BDEFG*
B(CDEF)= BCDEF	*B(ABCG)= ACG*	*B(ABDEFG)= ADEFG*	*B=BCDEF=ACG=ADEFG*
C(CDEF)= DEF	*C(ABCG)= ABG*	*C(ABDEFG)= ABCDEFG*	*C=DEF=ABG=ABCDEFG*
D(CDEF)= CEF	*D(ABCG)= ABCDG*	*D(ABDEFG)= ABEFG*	*D=CEF=ABCDG=ABEFG*
E(CDEF)= CDF	*E(ABCG)= ABCEG*	*E(ABDEFG)= ABDFG*	*E=CDF=ABCEG=ABDFG*
F(CDEF)= CDE	*F(ABCG)= ABCFG*	*F(ABDEFG)= ABDEG*	*F=CDE=ABCFG=ABDEG*
G(CDEF)= CDEFG	*G(ABCG)= ABC*	*G(ABDEFG)= ABDEF*	*G=CDEFG=ABC=ABDEF*
AB(CDEF)= ABCDEF	*AB(ABCG)= CG*	*AB(ABDEFG)= DEFG*	*AB=ABCDEF=CG=DEFG*
AC(CDEF)= ADEF	*AC(ABCG)= BG*	*AC(ABDEFG)= BCDEFG*	*AC=ADEF=BG=BCDEFG*
AD(CDEF)= ACEF	*AD(ABCG)= BCDG*	*AD(ABDEFG)= BEFG*	*AD=ACEF=BCDG=BEFG*
AE(CDEF)= ACDF	*AE(ABCG)= BCEG*	*AE(ABDEFG)= BDFG*	*AE=ACDF=BCEG=BDFG*
AF(CDEF)= ACDE	*AF(ABCG)= BCFG*	*AF(ABDEFG)= BDEG*	*AF=ACDE=BCFG=BDEG*
AG(CDEF)= ACDEFG	*AG(ABCG)= BC*	*AG(ABDEFG)= BDEF*	*AG=ACDEFG=BC=BDEF*
BD(CDEF)= BCEF	*BD(ABCG)= ACDG*	*BD(ABDEFG)= AEFG*	*BD=BCEF=ACDG=AEFG*
BE(CDEF)= BCDF	*BE(ABCG)= ACEG*	*BE(ABDEFG)= ADFG*	*BE=BCDF=ACEG=ADFG*
BF(CDEF)= BCDE	*BF(ABCG)= ACFG*	*BF(ABDEFG)= ADEG*	*BF=BCDE=ACFG=ADEG*
CD(CDEF)= EF	*CD(ABCG)= ABDG*	*CD(ABDEFG)= ABCEFG*	*CD=EF=ABDG=ABCEFG*
CE(CDEF)= DF	*CE(ABCG)= ABEG*	*CE(ABDEFG)= ABCDFG*	*CE=DF=ABEG=ABCDFG*
CF(CDEF)= DE	*CF(ABCG)= ABFG*	*CF(ABDEFG)= ABCDEG*	*CF=DE=ABFG=ABCDEG*
DG(CDEF)= CEFG	*DG(ABCG)= ABCD*	*DG(ABDEFG)= ABEF*	*DG=CEFG=ABCD=ABEF*
EG(CDEF)= CDFG	*EG(ABCG)= ABCE*	*EG(ABDEFG)= ABDF*	*EG=CDFG=ABCE=ABDF*
FG(CDEF)= CDEG	*FG(ABCG)= ABCF*	*FG(ABDEFG)= ABDE*	*FG=CDEG=ABCF=ABDE*

Analysis of Variance Table

Source	Degrees of Freedom
A	1
B	1
C	1
D	1
E	1
F	1
G	1
AB=CG	1

AC=BG	1
AD	1
AE	1
AF	1
AG=BC	1
BD	1
BE	1
CD=EF	1
CE=DF	1
CF=DE	1
DG	1
EG	1
FG	1
Error	9
Total	31

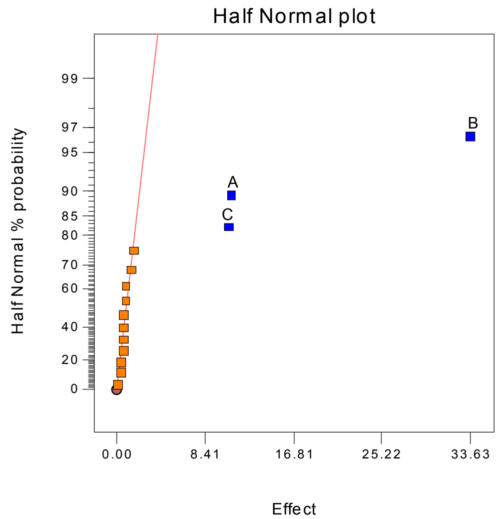
8-11 Consider the 2^5 design in Problem 6-21. Suppose that only a one-half fraction could be run. Furthermore, two days were required to take the 16 observations, and it was necessary to confound the 2^{5-1} design in two blocks. Construct the design and analyze the data.

A	*B*	*C*	*D*	*E=ABCD*		Data	Blocks = *AB*	Block
-	-	-	-	+	*e*	8	+	1
+	-	-	-	-	*a*	9	-	2
-	+	-	-	-	*b*	34	-	2
+	+	-	-	+	*abe*	52	+	1
-	-	+	-	-	*c*	16	+	1
+	-	+	-	+	*ace*	22	-	2
-	+	+	-	+	*bce*	45	-	2
+	+	+	-	-	*abc*	60	+	1
-	-	-	+	-	*d*	8	+	1
+	-	-	+	+	*ade*	10	-	2
-	+	-	+	+	*bde*	30	-	2
+	+	-	+	-	*abd*	50	+	1
-	-	+	+	+	*cde*	15	+	1
+	-	+	+	-	*acd*	21	-	2
-	+	+	+	-	*bcd*	44	-	2
+	+	+	+	+	*abcde*	63	+	1

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	10.875	473.063	8.6343
Model	B	33.625	4522.56	82.5455
Model	C	10.625	451.562	8.24188
Error	D	-0.625	1.5625	0.0285186
Error	E	0.375	0.5625	0.0102667
Error	AB	Aliased		
Error	AC	0.625	1.5625	0.0285186
Error	AD	0.875	3.0625	0.0558965
Error	AE	1.375	7.5625	0.13803
Error	BC	0.875	3.0625	0.0558965
Error	BD	-0.375	0.5625	0.0102667
Error	BE	0.125	0.0625	0.00114075
Error	CD	0.625	1.5625	0.0285186
Error	CE	0.625	1.5625	0.0285186
Error	DE	-1.625	10.5625	0.192786
	Lenth's ME	2.46263		
	Lenth's SME	5.0517		

The *AB* interaction in the above table is aliased with the three-factor interaction *BCD*, and is also confounded with blocks.



Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	203.06	1	203.06			
Model	5447.19	3	1815.73	630.31	< 0.0001	significant
A	*473.06*	*1*	*473.06*	*164.22*	*< 0.0001*	
B	*4522.56*	*1*	*4522.56*	*1569.96*	*< 0.0001*	
C	*451.56*	*1*	*451.56*	*156.76*	*< 0.0001*	
Residual	31.69	11	2.88			
Cor Total	5681.94	15				

The Model F-value of 630.31 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.70	R-Squared	0.9942
Mean	30.44	Adj R-Squared	0.9926
C.V.	5.58	Pred R-Squared	0.9878
PRESS	67.04	Adeq Precision	58.100

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	30.44	1	0.42	29.50	31.37	
Block 1	3.56	1				
Block 2	-3.56					
A-Aperture	5.44	1	0.42	4.50	6.37	1.00
B-Exposure Time	16.81	1	0.42	15.88	17.75	1.00
C-Develop Time	5.31	1	0.42	4.38	6.25	1.00

Final Equation in Terms of Coded Factors:

Yield =
+30.44
+5.44 * A
+16.81 * B
+5.31 * C

Final Equation in Terms of Actual Factors:

Aperture small
Yield =
-1.56250
+0.84063 * Exposure Time

```
        +0.70833      * Develop Time

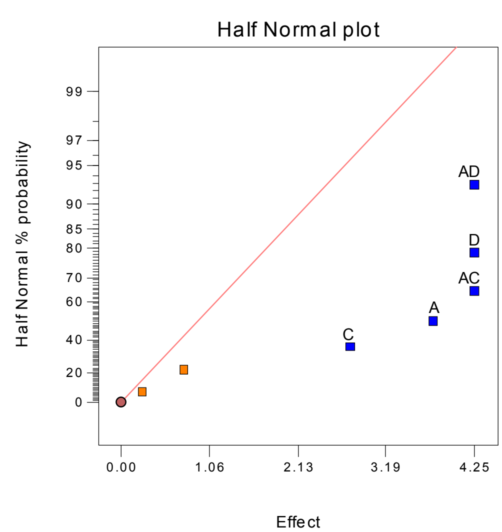
         Aperture    large
            Yield     =
        +9.31250
        +0.84063      * Exposure Time
        +0.70833      * Develop Time
```

8-12 Analyze the data in Problem 6-23 as if it came from a 2_{IV}^{4-1} design with I = *ABCD*. Project the design into a full factorial in the subset of the original four factors that appear to be significant.

Run Number	*A*	*B*	*C*	*D=ABC*		Yield (lbs)		Factor Low (-)	Levels High (+)
1	-	-	-	-	(1)	12	*A* (h)	2.5	3.0
2	+	-	-	+	*ad*	25	*B* (%)	14	18
3	-	+	-	+	*bd*	13	*C* (psi)	60	80
4	+	+	-	-	*ab*	16	*D* (°C)	225	250
5	-	-	+	+	*cd*	19			
6	+	-	+	-	*ac*	15			
7	-	+	+	-	*bc*	20			
8	+	+	+	+	*abcd*	23			

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	3.75	28.125	18.3974
Error	B	0.25	0.125	0.0817661
Model	C	2.75	15.125	9.8937
Model	D	4.25	36.125	23.6304
Error	AB	-0.75	1.125	0.735895
Model	AC	-4.25	36.125	23.6304
Model	AD	4.25	36.125	23.6304
	Lenth's ME	21.174		
	Lenth's SME	50.6734		



Design Expert Output

Response: Yield in lbs
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	151.63	5	30.32	48.52	0.0203	significant
A	*28.13*	*1*	*28.13*	*45.00*	*0.0215*	
C	*15.13*	*1*	*15.13*	*24.20*	*0.0389*	
D	*36.12*	*1*	*36.12*	*57.80*	*0.0169*	
AC	*36.12*	*1*	*36.12*	*57.80*	*0.0169*	
AD	*36.13*	*1*	*36.13*	*57.80*	*0.0169*	
Residual	1.25	2	0.62			
Cor Total	152.88	7				

The Model F-value of 48.52 implies the model is significant. There is only a 2.03% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.79	R-Squared	0.9918
Mean	17.88	Adj R-Squared	0.9714
C.V.	4.42	Pred R-Squared	0.8692
PRESS	20.00	Adeq Precision	17.892

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	17.88	1	0.28	16.67	19.08	
A-Time	1.87	1	0.28	0.67	3.08	1.00
C-Pressure	1.37	1	0.28	0.17	2.58	1.00
D-Temperature	2.13	1	0.28	0.92	3.33	1.00
AC	-2.13	1	0.28	-3.33	-0.92	1.00
AD	2.13	1	0.28	0.92	3.33	1.00

Final Equation in Terms of Coded Factors:

```
       Yield   =
      +17.88
       +1.87    * A
       +1.37    * C
       +2.13    * D
       -2.13    * A * C
       +2.13    * A * D
```

Final Equation in Terms of Actual Factors:

```
       Yield   =
  +227.75000
   -94.50000    * Time
    +2.47500    * Pressure
    -1.70000    * Temperature
    -0.85000    * Time * Pressure
    +0.68000    * Time * Temperature
```

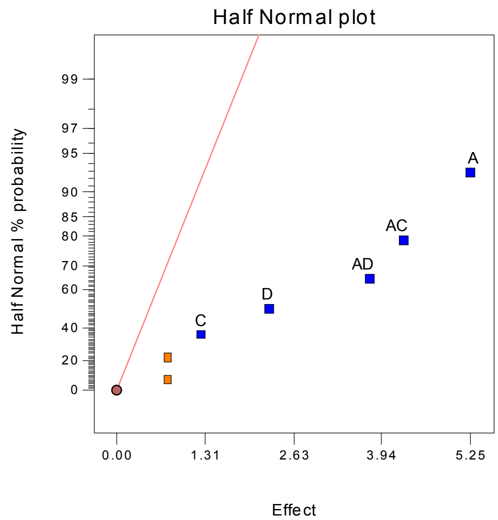
8-13 Repeat Problem 8-12 using I = *-ABCD*. Does use of the alternate fraction change your interpretation of the data?

Run Number	*A*	*B*	*C*	*D=ABC*		Yield (lbs)		Factor Low (-)	Levels High (+)
1	-	-	-	+	*d*	10	*A* (h)	2.5	3.0
2	+	-	-	-	*a*	18	*B* (%)	14	18
3	-	+	-	-	*b*	13	*C* (psi)	60	80
4	+	+	-	+	*abd*	24	*D* (°C)	225	250
5	-	-	+	-	*c*	17			
6	+	-	+	+	*acd*	21			
7	-	+	+	+	*bcd*	17			
8	+	+	+	-	*abc*	15			

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			

Model	A	5.25	55.125	40.8712
Error	B	0.75	1.125	0.834106
Model	C	1.25	3.125	2.31696
Model	D	2.25	10.125	7.50695
Error	AB	-0.75	1.125	0.834106
Model	AC	-4.25	36.125	26.7841
Model	AD	3.75	28.125	20.8526
	Lenth's ME	12.7044		
	Lenth's SME	30.404		



Design Expert Output

Response: Yield in lbs

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	132.63	5	26.52	23.58	0.0412	significant
A	*55.13*	*1*	*55.13*	*49.00*	*0.0198*	
C	*3.13*	*1*	*3.13*	*2.78*	*0.2375*	
D	*10.13*	*1*	*10.13*	*9.00*	*0.0955*	
AC	*36.13*	*1*	*36.13*	*32.11*	*0.0298*	
AD	*28.13*	*1*	*28.13*	*25.00*	*0.0377*	
Residual	2.25	2	1.12			
Cor Total	134.88	7				

The Model F-value of 23.58 implies the model is significant. There is only a 4.12% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.06	R-Squared	0.9833
Mean	16.88	Adj R-Squared	0.9416
C.V.	6.29	Pred R-Squared	0.7331
PRESS	36.00	Adeq Precision	14.425

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	16.88	1	0.37	15.26	18.49	
A-Time	2.63	1	0.37	1.01	4.24	1.00
C-Pressure	0.63	1	0.37	-0.99	2.24	1.00
D-Temperature	1.13	1	0.37	-0.49	2.74	1.00
AC	-2.13	1	0.37	-3.74	-0.51	1.00
AD	1.88	1	0.37	0.26	3.49	1.00

Final Equation in Terms of Coded Factors:

Yield =

```
      +16.88
       +2.63     * A
       +0.63     * C
       +1.13     * D
       -2.13     * A * C
       +1.88     * A * D
```

Final Equation in Terms of Actual Factors:

```
        Yield  =
   +190.50000
    -72.50000    * Time
     +2.40000    * Pressure
     -1.56000    * Temperature
     -0.85000    * Time * Pressure
     +0.60000    * Time * Temperature
```

8-14 Project the 2_{IV}^{4-1} design in Example 8-1 into two replicates of a 2^2 design in the factors A and B. Analyze the data and draw conclusions.

Design Expert Output

Response: Filtration Rate

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	728.50	3	242.83	0.41	0.7523	not significant
A	*722.00*	*1*	*722.00*	*1.23*	*0.3291*	
B	*4.50*	*1*	*4.50*	*7.682E-003*	*0.9344*	
AB	*2.00*	*1*	*2.00*	*3.414E-003*	*0.9562*	
Residual	2343.00	4	585.75			
Lack of Fit	*0.000*	*0*				
Pure Error	*2343.00*	*4*	*585.75*			
Cor Total	3071.50	7				

The "Model F-value" of 0.41 implies the model is not significant relative to the noise. There is a 75.23 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	24.20	R-Squared	0.2372
Mean	70.75	Adj R-Squared	-0.3349
C.V.	34.21	Pred R-Squared	-2.0513
PRESS	9372.00	Adeq Precision	1.198

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	70.75	1	8.56	46.99	94.51	
A-Temperature	9.50	1	8.56	-14.26	33.26	1.00
B-Pressure	0.75	1	8.56	-23.01	24.51	1.00
AB	-0.50	1	8.56	-24.26	23.26	1.00

Final Equation in Terms of Coded Factors:

```
Filtration Rate  =
       +70.75
        +9.50    * A
        +0.75    * B
        -0.50    * A * B
```

Final Equation in Terms of Actual Factors:

```
Filtration Rate  =
    +70.75000
     +9.50000    * Temperature
     +0.75000    * Pressure
     -0.50000    * Temperature * Pressure
```

8-15 Construct a 2_{III}^{6-3} design. Determine the effects that may be estimated if a second fraction of this design is run with all signs reversed.

A	*B*	*C*	*D=AB*	*E=AC*	*F=BC*	
-	-	-	+	+	+	*def*
+	-	-	-	-	+	*af*
-	+	-	-	+	-	*be*
+	+	-	+	-	-	*abd*
-	-	+	+	-	-	*cd*
+	-	+	-	+	-	*ace*
-	+	+	-	-	+	*bcf*
+	+	+	+	+	+	*abcdef*

Principal Fraction	Second Fraction
$\ell_A=A+BD+CE$	$\ell^*_A=A-BD-CE$
$\ell_B=B+AD+CF$	$\ell^*_B=B-AD-CF$
$\ell_C=C+AE+BF$	$\ell^*_C=C-AE-BF$
$\ell_D=D+AB+EF$	$\ell^*_D=D-AB-EF$
$\ell_E=E+AC+DF$	$\ell^*_E=E-AC-DF$
$\ell_F=F+BC+DE$	$\ell^*_F=F-BC-DE$
$\ell_{BE}=BE+CD+AF$	$\ell^*_{BE}=BE+CD+AF$

By combining the two fractions we can estimate the following:

$(\ell_i+\ell^*_I)/2$	$(\ell_i-\ell^*_I)/2$
A	*BD+CE*
B	*AD+CF*
C	*AE+BF*
D	*AB+EF*
E	*AC+DF*
F	*BC+DE*
BE+CD+AF	

8-16 Consider the 2_{III}^{6-3} design in Problem 8-15. Determine the effects that may be estimated if a second fraction of this design is run with the signs for factor *A* reversed.

Principal Fraction	Second Fraction
$\ell_A=A+BD+CE$	$\ell^*_A=-A+BD+CE$
$\ell_B=B+AD+CF$	$\ell^*_B=B-AD+CF$
$\ell_C=C+AE+BF$	$\ell^*_C=C-AE+BF$
$\ell_D=D+AB+EF$	$\ell^*_D=D-AB+EF$
$\ell_E=E+AC+DF$	$\ell^*_E=E-AC+DF$
$\ell_F=F+BC+DE$	$\ell^*_F=F+BC+DE$
$\ell_{BE}=BE+CD+AF$	$\ell^*_{BE}=BE+CD-AF$

By combining the two fractions we can estimate the following:

$(\ell_i-\ell^*_I)/2$	$(\ell_i+\ell^*_I)/2$
A	*BD+CE*
AD	*B+CF*

AE	*C+BF*
AB	*D+EF*
AC	*E+DF*
	F+BC+DE
AF	

8-17 Fold over the 2_{III}^{7-4} design in Table 8-19 to produce a eight-factor design. Verify that the resulting design is a 2_{IV}^{8-4} design. Is this a minimal design?

	H	*A*	*B*	*C*	*D=AB*	*E=AC*	*F=BC*	*G=ABC*
	+	-	-	-	+	+	+	-
Original	+	+	-	-	-	-	+	+
Design	+	-	+	-	-	+	-	+
	+	+	+	-	+	-	-	-
	+	-	-	+	+	-	-	+
	+	+	-	+	-	+	-	-
	+	-	+	+	-	-	+	-
	+	+	+	+	+	+	+	+
	-	+	+	+	-	-	-	+
Second	-	-	+	+	+	+	-	-
Set of	-	+	-	+	+	-	+	-
Runs w/	-	-	-	+	-	+	+	+
all Signs	-	+	+	-	-	+	+	-
Switched	-	-	+	-	+	-	+	+
	-	+	-	-	+	+	-	+
	-	-	-	-	-	-	-	-

After folding the original design over, we add a new factor *H*, and we have a design with generators *D=ABH, E=ACH, F=BCH*, and *G=ABC*. This is a 2_{IV}^{8-4} design. It is a minimal design, since it contains 2*k*=2(8)=16 runs.

8-18 Fold over a 2_{III}^{5-2} design to produce a six-factor design. Verify that the resulting design is a 2_{IV}^{6-2} design. Compare this 2_{IV}^{6-2} design to the in Table 8-10.

	F	*A*	*B*	*C*	*D=AB*	*E=BC*
	+	-	-	-	+	+
Original	+	+	-	-	-	+
Design	+	-	+	-	-	-
	+	+	+	-	+	-
	+	-	-	+	+	-
	+	+	-	+	-	-
	+	-	+	+	-	+
	+	+	+	+	+	+
	-	+	+	+	-	-
Second	-	-	+	+	+	-
Set of	-	+	-	+	+	+
Runs w/	-	-	-	+	-	+
all Signs	-	+	+	-	-	+
Switched	-	-	+	-	+	+
	-	+	-	-	+	-
	-	-	-	-	-	-

If we relabel the factors from left to right as *A, B, C, D, E, F*, then this design becomes 2_{IV}^{6-2} with generators I=ABDF and I=BCEF. It is not a minimal design, since 2*k*=2(6)=12 runs, and the design contains 16 runs.

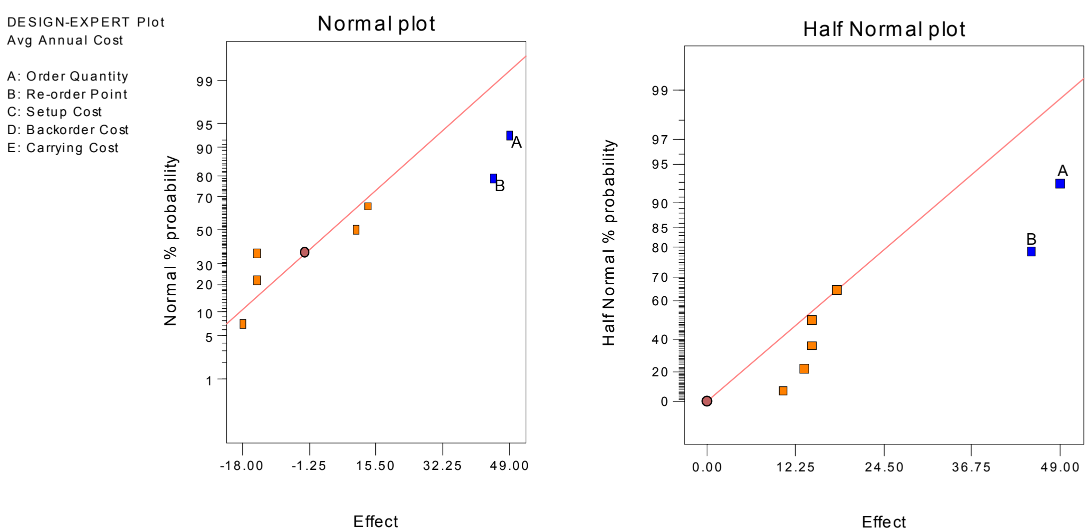
8-19 An industrial engineer is conducting an experiment using a Monte Carlo simulation model of an inventory system. The independent variables in her model are the order quantity (*A*), the reorder point (B), the setup cost (*C*), the backorder cost (*D*), and the carrying cost rate (*E*). The response variable is average annual cost. To conserve computer time, she decides to investigate these factors using a 2_{III}^{5-2} design with I = *ABD* and I = *BCE*. The results she obtains are *de* = 95, *ae* = 134, *b* = 158, *abd* = 190, *cd* = 92, *ac* = 187, *bce* = 155, and *abcde* = 185.

(a) Verify that the treatment combinations given are correct. Estimate the effects, assuming three-factor and higher interactions are negligible.

A	*B*	*C*	*D=AB*	*E=BC*	
-	-	-	+	+	*de*
+	-	-	-	+	*ae*
-	+	-	-	-	*b*
+	+	-	+	-	*abd*
-	-	+	+	-	*cd*
+	-	+	-	-	*ac*
-	+	+	-	+	*bce*
+	+	+	+	+	*abcde*

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	49	4802	43.9502
Model	B	45	4050	37.0675
Error	C	10.5	220.5	2.01812
Error	D	-18	648	5.93081
Error	E	-14.5	420.5	3.84862
Error	AC	13.5	364.5	3.33608
Error	AE	-14.5	420.5	3.84862
	Lenth's ME	81.8727		
	Lenth's SME	195.937		



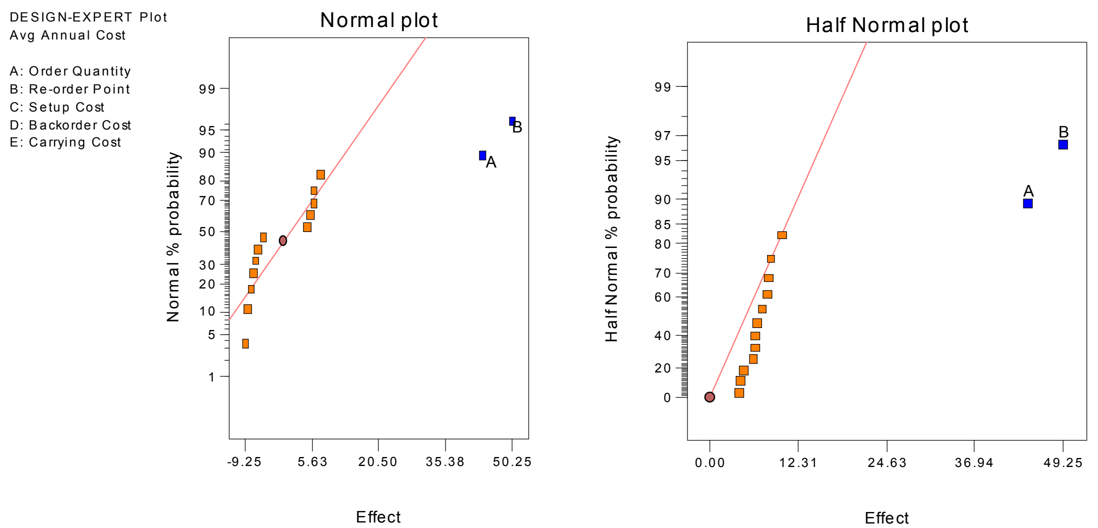
(b) Suppose that a second fraction is added to the first, for example *ade* = 136, *e* = 93, *ab* = 187, *bd* = 153, *acd* = 139, *c* = 99, *abce* - 191, and *bcde* = 150. How was this second fraction obtained? Add this data to the original fraction, and estimate the effects.

This second fraction is formed by reversing the signs of factor A.

A	*B*	*C*	*D=AB*	*E=BC*	
+	-	-	+	+	*ade*
-	-	-	-	+	*e*
+	+	-	-	-	*ab*
-	+	-	+	-	*bd*
+	-	+	+	-	*acd*
-	-	+	-	-	*c*
+	+	+	-	+	*abce*
-	+	+	+	+	*bcde*

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	44.25	7832.25	39.5289
Model	B	49.25	9702.25	48.9666
Error	C	6.5	169	0.852932
Error	D	-8	256	1.29202
Error	E	-8.25	272.25	1.37403
Error	AB	-10	400	2.01877
Error	AC	7.25	210.25	1.06112
Error	AD	-4.25	72.25	0.364641
Error	AE	-6	144	0.726759
Error	BD	4.75	90.25	0.455486
Error	CD	-8.5	289	1.45856
Error	DE	6.25	156.25	0.788584
Error	ACD	-6.25	156.25	0.788584
Error	ADE	4	64	0.323004
	Lenth's ME	25.1188		
	Lenth's SME	51.5273		



(c) Suppose that the fraction *abc* = 189, *ce* = 96, *bcd* = 154, *acde* = 135, *abe* = 193, *bde* = 152, *ad* = 137, and (1) = 98 was run. How was this fraction obtained? Add this data to the original fraction and estimate the effects.

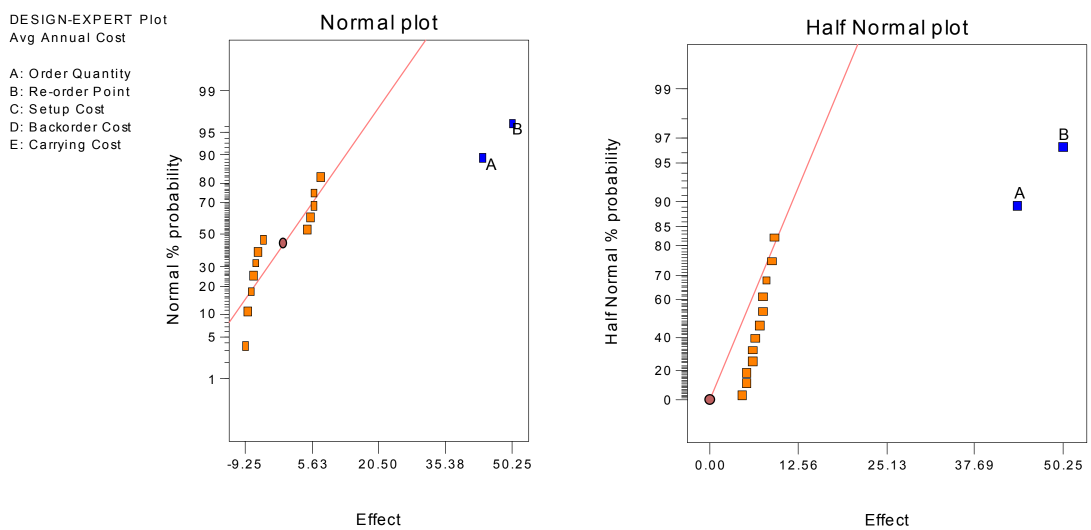
This second fraction is formed by reversing the signs of all factors.

A	*B*	*C*	*D=AB*	*E=BC*	
+	+	+	-	-	*abc*

-	+	+	+	-	*bcd*
+	-	+	+	+	*acde*
-	-	+	-	+	*ce*
+	+	-	-	+	*abe*
-	+	-	+	+	*bde*
+	-	-	+	-	*ad*
-	-	-	-	-	(1)

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	43.75	7656.25	38.1563
Model	B	50.25	10100.3	50.3364
Error	C	4.5	81	0.403678
Error	D	-8.75	306.25	1.52625
Error	E	-7.5	225	1.12133
Error	AB	-9.25	342.25	1.70566
Error	AC	6	144	0.71765
Error	AD	-5.25	110.25	0.549451
Error	AE	-6.5	169	0.842242
Error	BC	-7	196	0.976801
Error	BD	5.25	110.25	0.549451
Error	BE	6	144	0.71765
Error	ABC	-8	256	1.27582
Error	ABE	7.5	225	1.12133
	Lenth's ME	26.5964		
	Lenth's SME	54.5583		



8-20 Construct a 2^{5-1} design. Show how the design may be run in two blocks of eight observations each. Are any main effects or two-factor interactions confounded with blocks?

A	*B*	*C*	*D*	*E=ABCD*		Blocks = *AB*	Block
-	-	-	-	+	*e*	+	1
+	-	-	-	-	*a*	-	2
-	+	-	-	-	*b*	-	2
+	+	-	-	+	*abe*	+	1
-	-	+	-	-	*c*	+	1
+	-	+	-	+	*ace*	-	2

-	+	+	-	+	*bce*	-	2
+	+	+	-	-	*abc*	+	1
-	-	-	+	-	*d*	+	1
+	-	-	+	+	*ade*	-	2
-	+	-	+	+	*bde*	-	2
+	+	-	+	-	*abd*	+	1
-	-	+	+	+	*cde*	+	1
+	-	+	+	-	*acd*	-	2
-	+	+	+	-	*bcd*	-	2
+	+	+	+	+	*abcde*	+	1

Blocks are confounded with *AB* and *CDE.*

8-21 Construct a 2^{7-2} design. Show how the design may be run in four blocks of eight observations each. Are any main effects or two-factor interactions confounded with blocks?

	A	*B*	*C*	*D*	*E*	*F=CDE*	*G=ABC*		Block=ACE	Block=BFG	Block assignment
1	-	-	-	-	-	-	-	(1)	-	-	1
2	+	-	-	-	-	-	+	ag	+	+	4
3	-	+	-	-	-	-	+	bg	-	-	1
4	+	+	-	-	-	-	-	ab	+	+	4
5	-	-	+	-	-	+	+	cfg	+	-	3
6	+	-	+	-	-	+	-	acf	-	+	2
7	-	+	+	-	-	+	-	bcf	+	-	3
8	+	+	+	-	-	+	+	abcfg	-	+	2
9	-	-	-	+	-	+	-	df	-	+	2
10	+	-	-	+	-	+	+	adfg	+	-	3
11	-	+	-	+	-	+	+	bdfg	-	+	2
12	+	+	-	+	-	+	-	abdf	+	-	3
13	-	-	+	+	-	-	+	cdg	+	+	4
14	+	-	+	+	-	-	-	acd	-	-	1
15	-	+	+	+	-	-	-	bcd	+	+	4
16	+	+	+	+	-	-	+	abcdg	-	-	1
17	-	-	-	-	+	+	-	ef	+	+	4
18	+	-	-	-	+	+	+	aefg	-	-	1
19	-	+	-	-	+	+	+	befg	+	+	4
20	+	+	-	-	+	+	-	abef	-	-	1
21	-	-	+	-	+	-	+	ceg	-	+	2
22	+	-	+	-	+	-	-	ace	+	-	3
23	-	+	+	-	+	-	-	bce	-	+	2
24	+	+	+	-	+	-	+	abceg	+	-	3
25	-	-	-	+	+	-	-	de	+	-	3
26	+	-	-	+	+	-	+	adeg	-	+	2
27	-	+	-	+	+	-	+	bdeg	+	-	3
28	+	+	-	+	+	-	-	abde	-	+	2
29	-	-	+	+	+	+	+	cdefg	-	-	1
30	+	-	+	+	+	+	-	acdef	+	+	4
31	-	+	+	+	+	+	-	bcdef	-	-	1
32	+	+	+	+	+	+	+	abcdefg	+	+	4

Blocks are confounded with *ACE, BFG,* and *ABCEFG*.

8-22 Irregular fractions of the 2^k [John (1971)]. Consider a 2^4 design. We must estimate the four main effects and the six two-factor interactions, but the full 2^4 factorial cannot be run. The largest possible block contains 12 runs. These 12 runs can be obtained from the four one-quarter fractions defined by $I = \pm AB = \pm ACD = \pm BCD$ by omitting the principal fraction. Show how the remaining three 2^{4-2} fractions can be combined to estimate the required effects, assuming that three-factor and higher interactions are negligible. This design could be thought of as a three-quarter fraction.

The four 2^{4-2} fractions are as follows:

(1) I=+*AB*=+*ACD*=+*BCD*
Runs: *c,d,ab,abcd*

(2) I=+*AB*=-*ACD*=-*BCD*
Runs: (1), *cd, abc, abd*

(3) I=-*AB*=+*ACD*=-*BCD*
Runs: *a, bc, bd, acd*

(4) I=-*AB*=-*ACD*=+*BCD*
Runs: *b, ac, ad, bcd*

If we do not run the principal fraction (1), then we can combine the remaining 3 fractions to from 3 one-half fractions of the 2^4 as follows:

Fraction 1: (2) + (3) implies I=-*BCD*. This fraction estimates: *A, AB, AC*, and *AD*
Fraction 2: (2) + (4) implies I=-*ACD*. This fraction estimates: *B, BC, BD*, and *AB*
Fraction 3: (3) + (4) implies I=-*AB*. This fraction estimates: *C, D*, and *CD*

In estimating these effects we assume that all three-factor and higher interactions are negligible. Note that *AB* is estimated in two of the one-half fractions: 1 and 2. We would average these quantities and obtain a single estimate of *AB*. John (1971, pp. 161-163) discusses this design and shows that the estimates obtained above are also the least squares estimates. John also derives the variances and covariances of these estimators.

8-23 Carbon anodes used in a smelting process are baked in a ring furnace. An experiment is run in the furnace to determine which factors influence the weight of packing material that is stuck to the anodes after baking. Six variables are of interest, each at two levels: *A* = pitch/fines ratio (0.45, 0.55); *B* = packing material type (1, 2); *C* = packing material temperature (ambient, 325 C); *D* = flue location (inside, outside); *E* = pit temperature (ambient, 195 C); and *F* = delay time before packing (zero, 24 hours). A 2^{6-3} design is run, and three replicates are obtained at each of the design points. The weight of packing material stuck to the anodes is measured in grams. The data in run order are as follows: *abd* = (984, 826, 936); *abcdef* = (1275, 976, 1457); *be* = (1217, 1201, 890); *af* = (1474, 1164, 1541); *def* = (1320, 1156, 913); *cd* = (765, 705, 821); *ace* = (1338, 1254, 1294); and *bcf* = (1325, 1299, 1253). We wish to minimize the amount stuck packing material.

(a) Verify that the eight runs correspond to a 2_{III}^{6-3} design. What is the alias structure?

A	*B*	*C*	*D=AB*	*E=AC*	*F=BC*	
-	-	-	+	+	+	*def*
+	-	-	-	-	+	*af*
-	+	-	-	+	-	*be*
+	+	-	+	-	-	*abd*
-	-	+	+	-	-	*cd*
+	-	+	-	+	-	*ace*
-	+	+	-	-	+	*bcf*
+	+	+	+	+	+	*abcdef*

I=*ABD*=*ACE*=*BCF*=*BCDE*=*ACDF*=*ABEF*=*DEF*, Resolution III

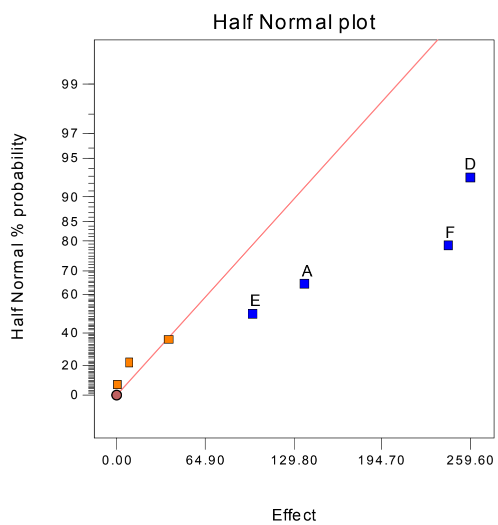
A=*BD*=*CE*=*CDF*=*BEF*
B=*AD*=*CF*=*CDE*=*AEF*
C=*AE*=*BF*=*BDE*=*ADF*

$$D=AB=EF=BCE=ACF$$
$$E=AC=DF=BCD=ABF$$
$$F=BC=DE=ACD=ABE$$
$$CD=BE=AF=ABC=ADE=BDF=CEF$$

(b) Use the average weight as a response. What factors appear to be influential?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	137.9	37996.1	12.0947
Error	B	-8.9	156.056	0.049675
Error	C	0.221108	2094.02	0.666559
Model	D	-259.6	136168	43.3443
Model	E	99.7667	27246.7	8.67305
Model	F	243.567	107863	34.3345
Error	BC	-38.0306	2629.69	0.837072
	Lenth's ME	563.322		
	Lenth's SME	1348.14		



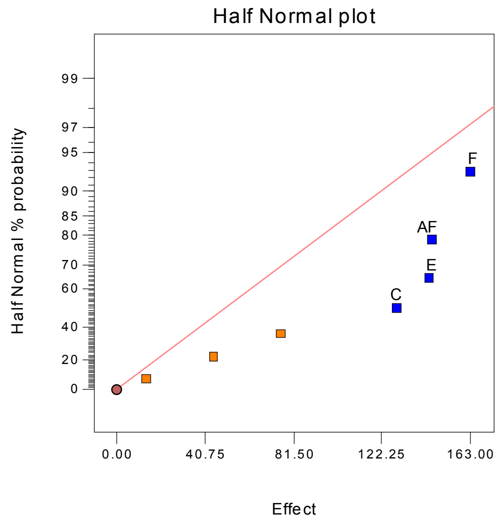
Factors *A, D, E* and *F* (and their aliases) are apparently important.

(c) Use the range of the weights as a response. What factors appear to be influential?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	44.5	3960.5	2.13311
Error	B	13.5	364.5	0.196319
Model	C	-129	33282	17.9256
Error	D	75.5	11400.5	6.14028
Model	E	144	41472	22.3367
Model	F	163	53138	28.62
Model	AF	145	42050	22.648
	Lenth's ME	728.384		
	Lenth's SME	1743.17		

Factors *C, E, F* and the *AF* interaction (and their aliases) appear to be large.



(d) What recommendations would you make to the process engineers?

It is not known exactly what to do here, since *A, D, E* and *F* are large effects, and because the design is resolution III, the main effects are aliased with two-factor interactions. Note, for example, that *D* is aliased with *EF* and the main effect could really be a *EF* interaction. If the main effects are really important, then setting all factors at the low level would minimize the amount of material stuck to the anodes. It would be necessary to run additional experiments to confirm these findings.

8-24 A 16-run experiment was performed in a semiconductor manufacturing plant to study the effects of six factors on the curvature or camber of the substrate devices produced. The six variables and their levels are shown below:

Run	Lamination Temperature (c)	Lamination Time (s)	Lamination Pressure (tn)	Firing Temperature (c)	Firing Cycle Time (h)	Firing Dew Point (c)
1	55	10	5	1580	17.5	20
2	75	10	5	1580	29	26
3	55	25	5	1580	29	20
4	75	25	5	1580	17.5	26
5	55	10	10	1580	29	26
6	75	10	10	1580	17.5	20
7	55	25	10	1580	17.5	26
8	75	25	10	1580	29	20
9	55	10	5	1620	17.5	26
10	75	10	5	1620	29	20
11	55	25	5	1620	29	26
12	75	25	5	1620	17.5	20
13	55	10	10	1620	29	20
14	75	10	10	1620	17.5	26
15	55	25	10	1620	17.5	20
16	75	25	10	1620	29	26

Each run was replicated four times , and a camber measurement was taken on the substrate. The data are shown below:

Run	Camber 1	for 2	Replicate 3	(in/in) 4	Total (10^{-4} in/in)	Mean (10^{-4} in/in)	Standard Deviation
1	0.0167	0.0128	0.0149	0.0185	629	157.25	24.418
2	0.0062	0.0066	0.0044	0.0020	192	48.00	20.976

3	0.0041	0.0043	0.0042	0.0050	176	44.00	4.083
4	0.0073	0.0081	0.0039	0.0030	223	55.75	25.025
5	0.0047	0.0047	0.0040	0.0089	223	55.75	22.410
6	0.0219	0.0258	0.0147	0.0296	920	230.00	63.639
7	0.0121	0.0090	0.0092	0.0086	389	97.25	16.029
8	0.0255	0.0250	0.0226	0.0169	900	225.00	39.420
9	0.0032	0.0023	0.0077	0.0069	201	50.25	26.725
10	0.0078	0.0158	0.0060	0.0045	341	85.25	50.341
11	0.0043	0.0027	0.0028	0.0028	126	31.50	7.681
12	0.0186	0.0137	0.0158	0.0159	640	160.00	20.083
13	0.0110	0.0086	0.0101	0.0158	455	113.75	31.120
14	0.0065	0.0109	0.0126	0.0071	371	92.75	29.510
15	0.0155	0.0158	0.0145	0.0145	603	150.75	6.750
16	0.0093	0.0124	0.0110	0.0133	460	115.00	17.450

(a) What type of design did the experimenters use?

The 2_{IV}^{6-2}, a 16-run design.

(b) What are the alias relationships in this design? The defining relation is I=*ABCE*=*ACDF*=*BDEF*

A(ABCE)=	*BCE*	*A(ACDF)*=	*CDF*	*A(BDEF)*=	*ABCDEF*	*A=BCE=CDF=ABDEF*
B(ABCE)=	*ACE*	*B(ACDF)*=	*ABCDF*	*B(BDEF)*=	*DEF*	*B=ACE=ABCDF=DEF*
C(ABCE)=	*ABE*	*C(ACDF)*=	*ADF*	*C(BDEF)*=	*BCDEF*	*C=ABE=ADF=BCDEF*
D(ABCE)=	*ABCDE*	*D(ACDF)*=	*ACF*	*D(BDEF)*=	*BEF*	*D=ABCDE=ACF=BEF*
E(ABCE)=	*ABC*	*E(ACDF)*=	*ACDEF*	*E(BDEF)*=	*BDF*	*E=ABC=ABDEF=BDF*
F(ABCE)=	*ABCEF*	*F(ACDF)*=	*ACD*	*F(BDEF)*=	*BDE*	*F=ABCEF=ACD=BDE*
AB(ABCE)=	*CE*	*AB(ACDF)*=	*BCDF*	*AB(BDEF)*=	*ADEF*	*AB=CE=BCDF=ADEF*
AC(ABCE)=	*BE*	*AC(ACDF)*=	*DF*	*AC(BDEF)*=	*ABCDEF*	*AC=BE=DF=ABCDEF*
AD(ABCE)=	*BCDE*	*AD(ACDF)*=	*CF*	*AD(BDEF)*=	*ABEF*	*AD=BCDE=CF=ABEF*
AE(ABCE)=	*BC*	*AE(ACDF)*=	*CDEF*	*AE(BDEF)*=	*ABDF*	*AE=BC=CDEF=ABDF*
AF(ABCE)=	*BCEF*	*AF(ACDF)*=	*CD*	*AF(BDEF)*=	*ABDE*	*AF=BCEF=CD=ABDE*
BD(ABCE)=	*ACDE*	*BD(ACDF)*=	*ABCF*	*BD(BDEF)*=	*EF*	*BD=ACDE=ABCF=EF*
BF(ABCE)=	*ACEF*	*BF(ACDF)*=	*ABCD*	*BF(BDEF)*=	*DE*	*BF=ACEF=ABCD=DE*

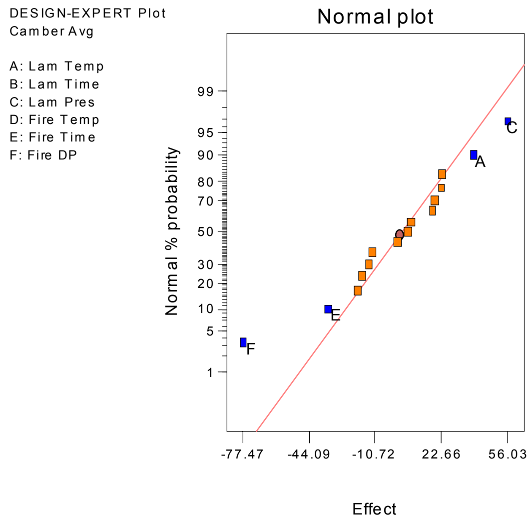
(c) Do any of the process variables affect average camber?

Yes, per the analysis below, variables *A*, *C*, *D*, and *F* affect average camber.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	38.9063	6054.79	10.2962
Error	B	5.78125	133.691	0.227344
Model	C	56.0313	12558	21.355
Error	D	-14.2188	808.691	1.37519
Model	E	-34.4687	4752.38	8.08148
Model	F	-77.4688	24005.6	40.8219
Error	AB	19.1563	1467.85	2.49609
Error	AC	22.4063	2008.16	3.4149
Error	AD	-12.2188	597.191	1.01553
Error	AE	18.1563	1318.6	2.24229
Error	AF	-19.7187	1555.32	2.64483
Error	BC	Aliased		
Error	BD	23.0313	2121.75	3.60807
Error	BE	Aliased		
Error	BF	7.40625	219.41	0.37311
Error	CD	Aliased		
Error	CE	Aliased		
Error	CF	Aliased		
Error	DE	Aliased		
Error	DF	Aliased		
Error	EF	Aliased		
Error	ABC	Aliased		
Error	ABD	0.53125	1.12891	0.00191972

Error	ABE	Aliased		
Error	ABF	-17.3438	1203.22	2.04609
	Lenth's ME	71.9361		
	Lenth's SME	146.041		



Design Expert Output

Response: Camber Avg in in/in
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	47370.80	4	11842.70	11.39	0.0007	significant
A	*6054.79*	*1*	*6054.79*	*5.82*	*0.0344*	
C	*12558.00*	*1*	*12558.00*	*12.08*	*0.0052*	
E	*4752.38*	*1*	*4752.38*	*4.57*	*0.0558*	
F	*24005.63*	*1*	*24005.63*	*23.09*	*0.0005*	
Residual	11435.01	11	1039.55			
Cor Total	58805.81	15				

The Model F-value of 11.39 implies the model is significant. There is only a 0.07% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	32.24	R-Squared	0.8055
Mean	107.02	Adj R-Squared	0.7348
C.V.	30.13	Pred R-Squared	0.5886
PRESS	24193.08	Adeq Precision	11.478

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	107.02	1	8.06	89.27	124.76	
A-Lam Temp	19.45	1	8.06	1.71	37.19	1.00
C-Lam Pres	28.02	1	8.06	10.27	45.76	1.00
E-Fire Time	-17.23	1	8.06	-34.98	0.51	1.00
F-Fire DP	-38.73	1	8.06	-56.48	-20.99	1.00

Final Equation in Terms of Coded Factors:

Camber Avg =
+107.02
+19.45 * A
+28.02 * C
-17.23 * E
-38.73 * F

Final Equation in Terms of Actual Factors:

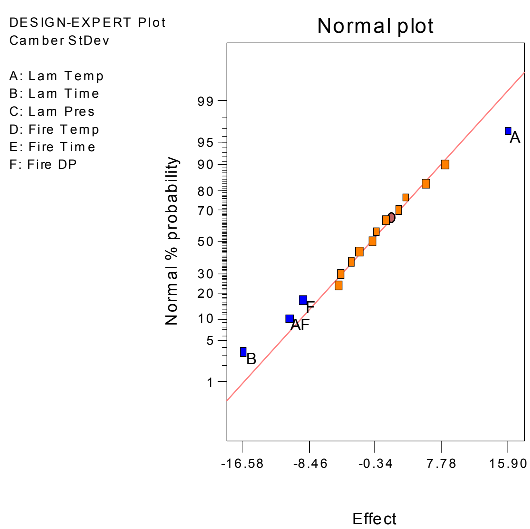
Camber Avg	=
+263.17380	
+1.94531	* Lam Temp
+11.20625	* Lam Pres
-2.99728	* Fire Time
-12.91146	* Fire DP

(d) Do any of the process variables affect the variability in camber measurements?

Yes, *A*, *B*, *F*, and *AF* interaction affect the variability in camber measurements.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	15.9035	1011.69	27.6623
Model	B	-16.5773	1099.22	30.0558
Error	C	5.8745	138.039	3.77437
Error	D	-3.2925	43.3622	1.18564
Error	E	-2.33725	21.851	0.597466
Model	F	-9.256	342.694	9.37021
Error	AB	0.95525	3.65001	0.0998014
Error	AC	2.524	25.4823	0.696757
Error	AD	-4.6265	85.618	2.34103
Error	AE	-0.18025	0.12996	0.00355347
Model	AF	-10.8745	473.019	12.9337
Error	BC	Aliased		
Error	BD	-4.85575	94.3132	2.57879
Error	BE	Aliased		
Error	BF	8.21825	270.159	7.38689
Error	CD	Aliased		
Error	CE	Aliased		
Error	CF	Aliased		
Error	DE	Aliased		
Error	DF	Aliased		
Error	EF	Aliased		
Error	ABC	Aliased		
Error	ABD	-0.68125	1.85641	0.0507593
Error	ABE	Aliased		
Error	ABF	3.39825	46.1924	1.26303
	Lenth's ME	17.8392		
	Lenth's SME	36.2162		



Response: Camber StDev
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2926.62	4	731.65	11.02	0.0008	significant
A	*1011.69*	*1*	*1011.69*	*15.23*	*0.0025*	
B	*1099.22*	*1*	*1099.22*	*16.55*	*0.0019*	
F	*342.69*	*1*	*342.69*	*5.16*	*0.0442*	
AF	*473.02*	*1*	*473.02*	*7.12*	*0.0218*	
Residual	730.65	11	66.42			
Cor Total	3657.27	15				

The Model F-value of 11.02 implies the model is significant. There is only a 0.08% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	8.15	R-Squared	0.8002
Mean	25.35	Adj R-Squared	0.7276
C.V.	32.15	Pred R-Squared	0.5773
PRESS	1545.84	Adeq Precision	9.516

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	25.35	1	2.04	20.87	29.84	
A-Lam Temp	7.95	1	2.04	3.47	12.44	1.00
B-Lam Time	-8.29	1	2.04	-12.77	-3.80	1.00
F-Fire DP	-4.63	1	2.04	-9.11	-0.14	1.00
AF	-5.44	1	2.04	-9.92	-0.95	1.00

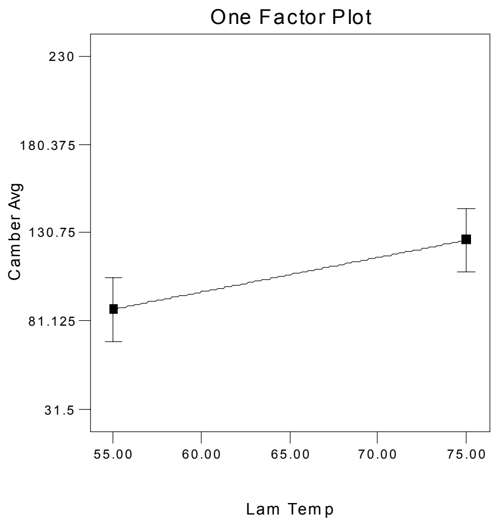
Final Equation in Terms of Coded Factors:

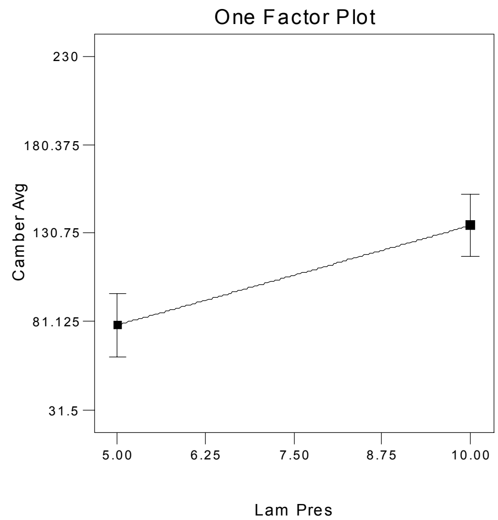
Camber StDev =
+25.35
+7.95 * A
-8.29 * B
-4.63 * F
-5.44 * A * F

Final Equation in Terms of Actual Factors:

Camber StDev =
-242.46746
+4.96373 * Lam Temp
-1.10515 * Lam Time
+10.23804 * Fire DP
-0.18124 * Lam Temp * Fire DP

(e) If it is important to reduce camber as much as possible, what recommendations would you make?


One Factor Plot
230
180.375
130.75
81.125
31.5
Camber Avg
55.00
60.00
65.00
70.00
75.00
Lam Temp


One Factor Plot
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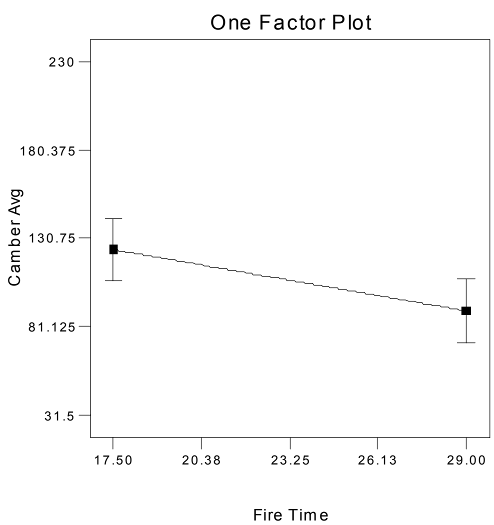
130.75
81.125
31.5
Camber Avg
5.00
6.25
7.50
8.75
10.00
Lam Pres


One Factor Plot
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81.125
31.5
Camber Avg
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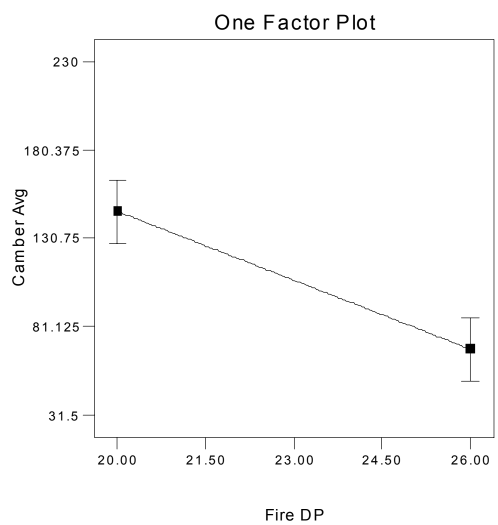
20.38
23.25
26.13
29.00
Fire Time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81.125
31.5
Camber Avg
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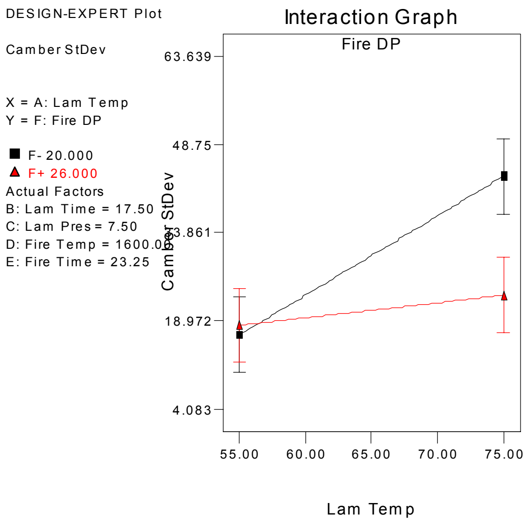
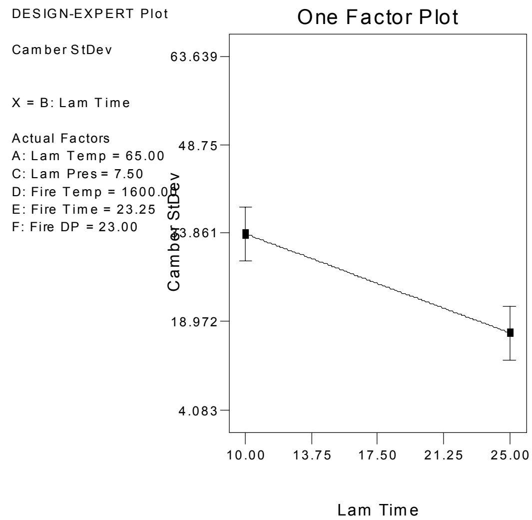
21.50
23.00
24.50
26.00
Fire DP





Run A and *C* at the low level and E and *F* at the high level. B at the low level enables a lower variation without affecting the average camber.

8-25 A spin coater is used to apply photoresist to a bare silicon wafer. This operation usually occurs early in the semiconductor manufacturing process, and the average coating thickness and the variability in the coating thickness has an important impact on downstream manufacturing steps. Six variables are used in the experiment. The variables and their high and low levels are as follows:

Factor	Low Level	High Level
Final Spin Speed	7350 rpm	6650 rpm
Acceleration Rate	5	20
Volume of Resist Applied	3 cc	5 cc
Time of Spin	14 s	6 s
Resist Batch Variation	Batch 1	Batch 2
Exhaust Pressure	Cover Off	Cover On

The experimenter decides to use a 2^{6-1} design and to make three readings on resist thickness on each test wafer. The data are shown in table 8-29.

Table 8-29

	A	*B*	*C*	*D*	*E*	*F*		Resist Thickness			
Run	Volume	Batch	Time	Speed	Acc.	Cover	Left	Center	Right	Avg.	Range
1	5	2	14	7350	5	Off	4531	4531	4515	4525.7	16
2	5	1	6	7350	5	Off	4446	4464	4428	4446	36
3	3	1	6	6650	5	Off	4452	4490	4452	4464.7	38
4	3	2	14	7350	20	Off	4316	4328	4308	4317.3	20
5	3	1	14	7350	5	Off	4307	4295	4289	4297	18
6	5	1	6	6650	20	Off	4470	4492	4495	4485.7	25
7	3	1	6	7350	5	On	4496	4502	4482	4493.3	20
8	5	2	14	6650	20	Off	4542	4547	4538	4542.3	9
9	5	1	14	6650	5	Off	4621	4643	4613	4625.7	30
10	3	1	14	6650	5	On	4653	4670	4645	4656	25
11	3	2	14	6650	20	On	4480	4486	4470	4478.7	16
12	3	1	6	7350	20	Off	4221	4233	4217	4223.7	16
13	5	1	6	6650	5	On	4620	4641	4619	4626.7	22
14	3	1	6	6650	20	On	4455	4480	4466	4467	25
15	5	2	14	7350	20	On	4255	4288	4243	4262	45
16	5	2	6	7350	5	On	4490	4534	4523	4515.7	44
17	3	2	14	7350	5	On	4514	4551	4540	4535	37
18	3	1	14	6650	20	Off	4494	4503	4496	4497.7	9

19	5	2	6	7350	20	Off	4293	4306	4302	4300.3	13
20	3	2	6	7350	5	Off	4534	4545	4512	4530.3	33
21	5	1	14	6650	20	On	4460	4457	4436	4451	24
22	3	2	6	6650	5	On	4650	4688	4656	4664.7	38
23	5	1	14	7350	20	Off	4231	4244	4230	4235	14
24	3	2	6	7350	20	On	4225	4228	4208	4220.3	20
25	5	1	14	7350	5	On	4381	4391	4376	4382.7	15
26	3	2	6	6650	20	Off	4533	4521	4511	4521.7	22
27	3	1	14	7350	20	On	4194	4230	4172	4198.7	58
28	5	2	6	6650	5	Off	4666	4695	4672	4677.7	29
29	5	1	6	7350	20	On	4180	4213	4197	4196.7	33
30	5	2	6	6650	20	On	4465	4496	4463	4474.7	33
31	5	2	14	6650	5	On	4653	4685	4665	4667.7	32
32	3	2	14	6650	5	Off	4683	4712	4677	4690.7	35

(a) Verify that this is a 2^{6-1} design. Discuss the alias relationships in this design.

I=*ABCDEF*. This is a resolution VI design where main effects are aliased with five-factor interactions and two-factor interactions are aliased with four-factor interactions.

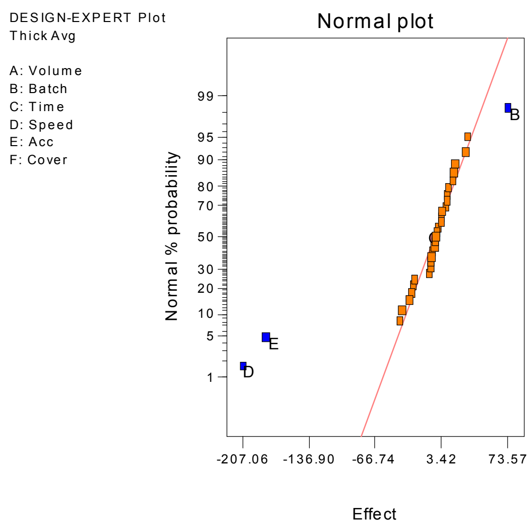
(b) What factors appear to affect average resist thickness?

Factors *B*, *D*, and *E* appear to affect the average resist thickness.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	9.925	788.045	0.107795
Model	B	73.575	43306.2	5.92378
Error	C	3.375	91.125	0.0124648
Model	D	-207.062	342999	46.9182
Model	E	-182.925	267692	36.6172
Error	F	-5.6625	256.511	0.0350877
Error	AB	-9	648	0.0886387
Error	AC	-7.3	426.32	0.0583155
Error	AD	-3.8625	119.351	0.0163258
Error	AE	-7.1	403.28	0.0551639
Error	AF	-26.9875	5826.6	0.79701
Error	BC	10.875	946.125	0.129419
Error	BD	18.1125	2624.5	0.359001
Error	BE	-28.35	6429.78	0.879518
Error	BF	-30.2375	7314.45	1.00053
Error	CD	-24.9875	4995	0.683257
Error	CE	8.2	537.92	0.0735811
Error	CF	-6.7875	368.561	0.0504148
Error	DE	-38.5375	11881.1	1.6252
Error	DF	-3.2	81.92	0.0112057
Error	EF	-41.1625	13554.8	1.85414
Error	ABC	0.375	1.125	0.000153887
Error	ABD	Aliased		
Error	ABE	16.5	2178	0.297925
Error	ABF	31.4125	7893.96	1.0798
Error	ACD	15.5875	1943.76	0.265883
Error	ACE	Aliased		
Error	ACF	Aliased		
Error	ADE	9.5375	727.711	0.0995423
Error	ADF	Aliased		
Error	AEF	Aliased		
Error	BCD	29.0875	6768.66	0.925873
Error	BCE	-1.625	21.125	0.00288965
Error	BCF	Aliased		
Error	BDE	-1.8875	28.5013	0.00389863
Error	BDF	3.95	124.82	0.0170739
Error	BEF	Aliased		
Error	CDE	Aliased		
Error	CDF	Aliased		
Error	CEF	3.1375	78.7512	0.0107722

Error	DEF	Aliased
	Lenth's ME	28.6178
	Lenth's SME	54.4118



Design Expert Output

Response: Thick Avg

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	6.540E+005	3	2.180E+005	79.21	< 0.0001	significant
B	*43306.24*	*1*	*43306.24*	*15.74*	*0.0005*	
D	*3.430E+005*	*1*	*3.430E+005*	*124.63*	*< 0.0001*	
E	*2.677E+005*	*1*	*2.677E+005*	*97.27*	*< 0.0001*	
Residual	77059.83	28	2752.14			
Cor Total	7.311E+005	31				

The Model F-value of 79.21 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	52.46	R-Squared	0.8946
Mean	4458.51	Adj R-Squared	0.8833
C.V.	1.18	Pred R-Squared	0.8623
PRESS	1.006E+005	Adeq Precision	24.993

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	4458.51	1	9.27	4439.52	4477.51	
B-Batch	36.79	1	9.27	17.79	55.78	1.00
D-Speed	-103.53	1	9.27	-122.53	-84.53	1.00
E-Acc	-91.46	1	9.27	-110.46	-72.47	1.00

Final Equation in Terms of Coded Factors:

Thick Avg =
+4458.51
+36.79 * B
-103.53 * D
-91.46 * E

Final Equation in Terms of Actual Factors:

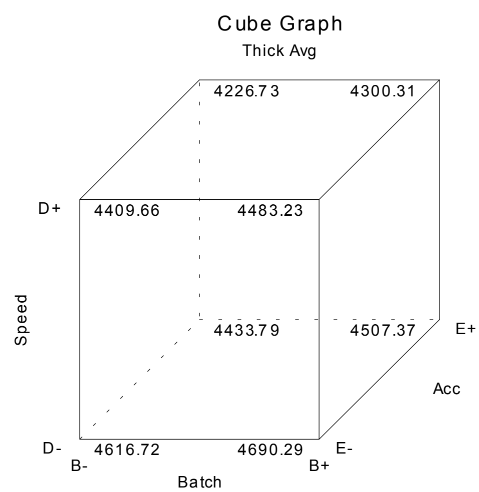
Batch Batch 1
Thick Avg =
+6644.78750

-0.29580	* Speed
-12.19500	* Acc
Batch	Batch 2
Thick Avg	=
+6718.36250	
-0.29580	* Speed
-12.19500	* Acc

(c) Since the volume of resist applied has little effect on average thickness, does this have any important practical implications for the process engineers?

Yes, less material could be used.

(d) Project this design into a smaller design involving only the significant factors. Graphically display the results. Does this aid in interpretation?



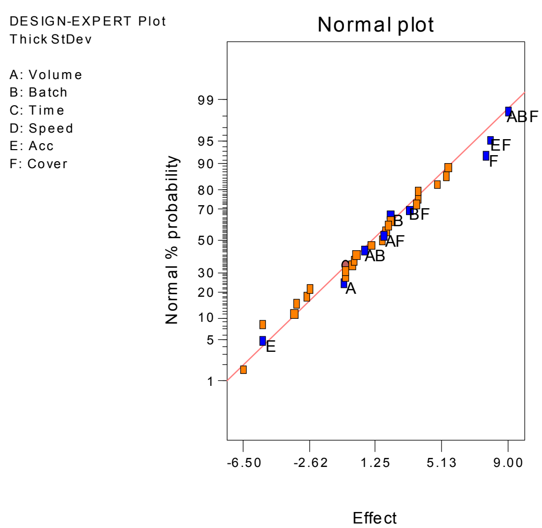
The cube plot usually assists the experimenter in drawing conclusions.

(e) Use the range of resist thickness as a response variable. Is there any indication that any of these factors affect the variability in resist thickness?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	-0.625	3.125	0.0777387
Model	B	2.125	36.125	0.89866
Error	C	-2.75	60.5	1.50502
Error	D	1.625	21.125	0.525514
Model	E	-5.375	231.125	5.74956
Model	F	7.75	480.5	11.9531
Model	AB	0.625	3.125	0.0777387
Error	AC	-3.5	98	2.43789
Error	AD	-0.125	0.125	0.00310955
Error	AE	1.875	28.125	0.699649
Model	AF	1.75	24.5	0.609472
Error	BC	0	0	0
Error	BD	0.125	0.125	0.00310955
Error	BE	-5.375	231.125	5.74956
Model	BF	3.25	84.5	2.10206
Error	CD	3.75	112.5	2.79859

Error	CE	3.75	112.5	2.79859
Error	CF	4.875	190.125	4.72962
Error	DE	5.375	231.125	5.74956
Error	DF	5.5	242	6.02009
Model	EF	8	512	12.7367
Error	ABC	Aliased		
Error	ABD	Aliased		
Error	ABE	3.625	105.125	2.61513
Model	ABF	9	648	16.1199
Error	ACD	-6.5	338	8.40822
Error	ACE	Aliased		
Error	ACF	Aliased		
Error	ADE	-3.375	91.125	2.26686
Error	ADF	-0.5	2	0.0497528
Error	AEF	1	8	0.199011
Error	BCD	Aliased		
Error	BCE	Aliased		
Error	BCF	Aliased		
Error	BDE	-2.625	55.125	1.37131
Error	BDF	-0.5	2	0.0497528
Error	BEF	Aliased		
Error	CDE	Aliased		
Error	CDF	Aliased		
Error	CEF	2.125	36.125	0.89866
Error	DEF	2	32	0.796045
	Lenth's ME	9.15104		
	Lenth's SME	17.3991		



Design Expert Output

Response: Thick StDev
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2023.00	9	224.78	2.48	0.0400	significant
A	*3.13*	*1*	*3.13*	*0.034*	*0.8545*	
B	*36.13*	*1*	*36.13*	*0.40*	*0.5346*	
E	*231.12*	*1*	*231.12*	*2.55*	*0.1248*	
F	*480.50*	*1*	*480.50*	*5.29*	*0.0313*	
AB	*3.12*	*1*	*3.12*	*0.034*	*0.8545*	
AF	*24.50*	*1*	*24.50*	*0.27*	*0.6086*	
BF	*84.50*	*1*	*84.50*	*0.93*	*0.3451*	
EF	*512.00*	*1*	*512.00*	*5.64*	*0.0267*	
ABF	*648.00*	*1*	*648.00*	*7.14*	*0.0139*	

Residual	1996.88	22	90.77
Cor Total	4019.88	31	

The Model F-value of 2.48 implies the model is significant. There is only a 4.00% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	9.53	R-Squared	0.5032
Mean	26.56	Adj R-Squared	0.3000
C.V.	35.87	Pred R-Squared	-0.0510
PRESS	4224.79	Adeq Precision	5.586

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	26.56	1	1.68	23.07	30.06	
A-Volume	-0.31	1	1.68	-3.81	3.18	1.00
B-Batch	1.06	1	1.68	-2.43	4.56	1.00
E-Acc	-2.69	1	1.68	-6.18	0.81	1.00
F-Cover	3.88	1	1.68	0.38	7.37	1.00
AB	0.31	1	1.68	-3.18	3.81	1.00
AF	0.88	1	1.68	-2.62	4.37	1.00
BF	1.63	1	1.68	-1.87	5.12	1.00
EF	4.00	1	1.68	0.51	7.49	1.00
ABF	4.50	1	1.68	1.01	7.99	1.00

Final Equation in Terms of Coded Factors:

Thick StDev =
+26.56
-0.31 * A
+1.06 * B
-2.69 * E
+3.88 * F
+0.31 * A * B
+0.88 * A * F
+1.63 * B * F
+4.00 * E * F
+4.50 * A * B * F

Final Equation in Terms of Actual Factors:

Batch Batch 1
Cover Off
Thick StDev =
+22.39583
+3.00000 * Volume
-0.89167 * Acc

Batch Batch 2
Cover Off
Thick StDev =
+54.77083
-5.37500 * Volume
-0.89167 * Acc

Batch Batch 1
Cover On
Thick StDev =
+42.56250
-4.25000 * Volume
+0.17500 * Acc

Batch Batch 2
Cover On
Thick StDev =
+9.43750
+5.37500 * Volume
+0.17500 * Acc

The model here for variability isn't very strong. Notice the small value of R^2, and in particular, the adjusted R^2. Often we find that obtaining a good model for a response that expresses variability isn't as easy as finding a satisfactory model for a response that essentially measures the mean.

(f) Where would you recommend that the process engineers run the process?

Considering only the average thickness results, the engineers could use factors *B, D* and *E* to put the process mean at target. Then the engineer could consider the other factors on the range model to try to set the factors to reduce the variation in thickness at that mean.

8-26 Harry and Judy Peterson-Nedry (two friends of the author) own a vineyard in Oregon. They grow several varieties of grapes and manufacture wine. Harry and Judy have used factorial designs for process and product development in the winemaking segment of their business. This problem describes the experiment conducted for their 1985 Pinot Noir. Eight variables, shown below, were originally studied in this experiment:

	Variable	Low Level	High Level
A	Pinot Noir Clone	Pommard	Wadenswil
B	Oak Type	Allier	Troncais
C	Age of Barrel	Old	New
D	Yeast/Skin Contact	Champagne	Montrachet
E	Stems	None	All
F	Barrel Toast	Light	Medium
G	Whole Cluster	None	10%
H	Fermentation Temperature	Low (75 F Max)	High (92 F Max)

Harry and Judy decided to use a 2_{IV}^{8-4} design with 16 runs. The wine was taste-tested by a panel of experts on 8 March 1986. Each expert ranked the 16 samples of wine tasted, with rank 1 being the best. The design and taste-test panel results are shown in Table 8-30.

Table 8-30

Run	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	HPN	JPN	CAL	DCM	RGB	y_{bar}	*s*
1	-	-	-	-	-	-	-	-	12	6	13	10	7	9.6	3.05
2	+	-	-	-	-	+	+	+	10	7	14	14	9	10.8	3.11
3	-	+	-	-	+	-	+	+	14	13	10	11	15	12.6	2.07
4	+	+	-	-	+	+	-	-	9	9	7	9	12	9.2	1.79
5	-	-	+	-	+	+	+	-	8	8	11	8	10	9.0	1.41
6	+	-	+	-	+	-	-	+	16	12	15	16	16	15.0	1.73
7	-	+	+	-	-	+	-	+	6	5	6	5	3	5.0	1.22
8	+	+	+	-	-	-	+	-	15	16	16	15	14	15.2	0.84
9	-	-	-	+	+	+	-	+	1	2	3	3	2	2.2	0.84
10	+	-	-	+	+	-	+	-	7	11	4	7	6	7.0	2.55
11	-	+	-	+	-	+	+	-	13	3	8	12	8	8.8	3.96
12	+	+	-	+	-	-	-	+	3	1	5	1	4	2.8	1.79
13	-	-	+	+	-	-	+	+	2	10	2	4	5	9.6	3.29
14	+	-	+	+	-	+	-	-	4	4	1	2	1	2.4	1.52
15	-	+	+	+	+	-	-	-	5	15	9	6	11	9.2	4.02
16	+	+	+	+	+	+	+	+	11	14	12	13	13	12.6	1.14

(a) What are the alias relationships in the design selected by Harry and Judy?

$$E = BCD,\ F = ACD,\ G = ABC,\ H = ABD$$

Defining Contrast : $I = BCDE = ACDF = ABEF = ABCG = ADEG = BDFG = CEFG = ABDH$
$= ACEH = BCFH = DEFH = CDGH = BEGH = AFGH = ABCDEFGH$

Aliases:

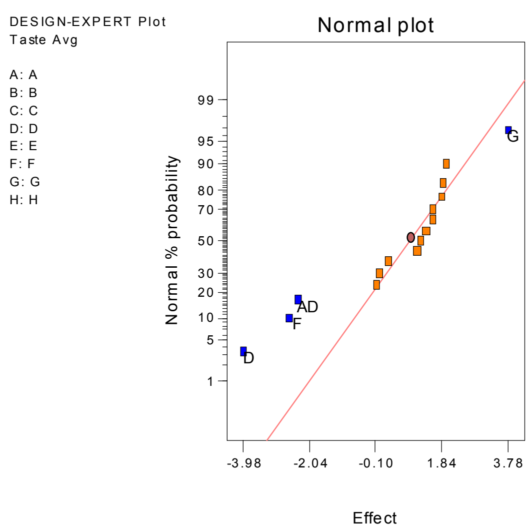
$$A = BCG = BDH = BEF = CDF = CEH = DEG = FGH$$

$$B = ACG = ADH = AEF = CDE = CFH = DFG = EGH$$
$$C = ABG = ADF = AEH = BDE = BFH = DGH = EFG$$
$$D = ABH = ACF = AEG = BCE = BFG = CGH = EFH$$
$$E = ABF = ACH = ADG = BCD = BGH = CFG = DFH$$
$$F = ABE = ACD = AGH = BCH = BDG = CEG = DEH$$
$$G = ABC = ADE = AFH = BDF = BEH = CDH = CEF$$
$$H = ABD = ACE = AFG = BCF = BEG = CDG = DEF$$
$$AB = CG = DH = EF$$
$$AC = BG = DF = EH$$
$$AD = BH = CF = EG$$
$$AE = BF = CH = DG$$
$$AF = BE = CD = GH$$
$$AG = BC = DE = FH$$
$$AH = BD = CE = FG$$

(b) Use the average ranks ($\bar{y}$) as a response variable. Analyze the data and draw conclusions. You will find it helpful to examine a normal probability plot of effect estimates.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	1.125	5.0625	2.00799
Error	B	1.225	6.0025	2.38083
Error	C	1.875	14.0625	5.57776
Model	D	-3.975	63.2025	25.0687
Error	E	1.575	9.9225	3.93566
Model	F	-2.625	27.5625	10.9324
Model	G	3.775	57.0025	22.6095
Error	H	0.025	0.0025	0.000991601
Error	AB	-0.075	0.0225	0.00892441
Error	AC	1.975	15.6025	6.18858
Model	AD	-2.375	22.5625	8.9492
Error	AE	1.575	9.9225	3.93566
Error	AF	1.375	7.5625	2.99959
Error	AG	0.275	0.3025	0.119984
Error	AH	1.825	13.3225	5.28424
	Lenth's ME	6.073		
	Lenth's SME	12.3291		



Design Expert Output

Response: Taste Avg

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	175.39	5	35.08	4.57	0.0198	significant
A	*5.06*	*1*	*5.06*	*0.66*	*0.4355*	
D	*63.20*	*1*	*63.20*	*8.24*	*0.0167*	
F	*27.56*	*1*	*27.56*	*3.59*	*0.0873*	
G	*57.00*	*1*	*57.00*	*7.43*	*0.0214*	
AD	*22.56*	*1*	*22.56*	*2.94*	*0.1171*	
Residual	76.72	10	7.67			
Cor Total	252.12	15				

The Model F-value of 4.57 implies the model is significant. There is only a 1.98% chance that a "Model F-Value" this large could occur due to noise.

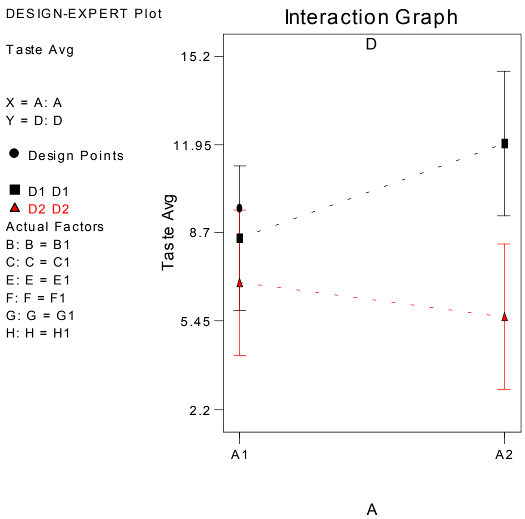
Std. Dev.	2.77	R-Squared	0.6957
Mean	8.81	Adj R-Squared	0.5435
C.V.	31.43	Pred R-Squared	0.2209
PRESS	196.42	Adeq Precision	7.517

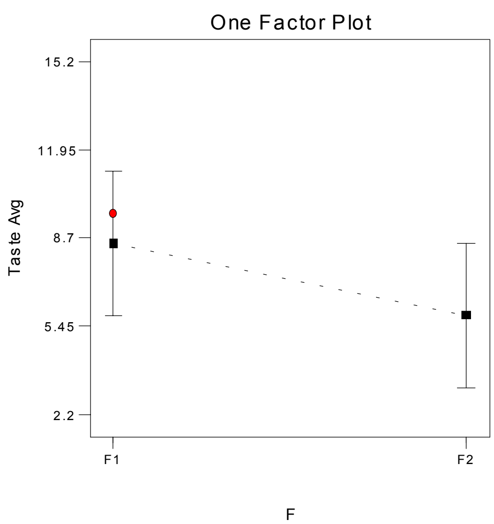
Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	8.81	1	0.69	7.27	10.36	
A-A	0.56	1	0.69	-0.98	2.11	1.00
D-D	-1.99	1	0.69	-3.53	-0.44	1.00
F-F	-1.31	1	0.69	-2.86	0.23	1.00
G-G	1.89	1	0.69	0.34	3.43	1.00
AD	-1.19	1	0.69	-2.73	0.36	1.00

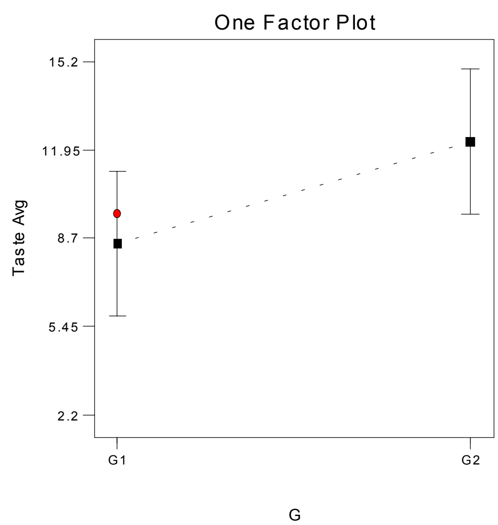
Final Equation in Terms of Coded Factors:

Taste Avg =
+8.81
+0.56 * A
-1.99 * D
-1.31 * F
+1.89 * G
-1.19 * A * D

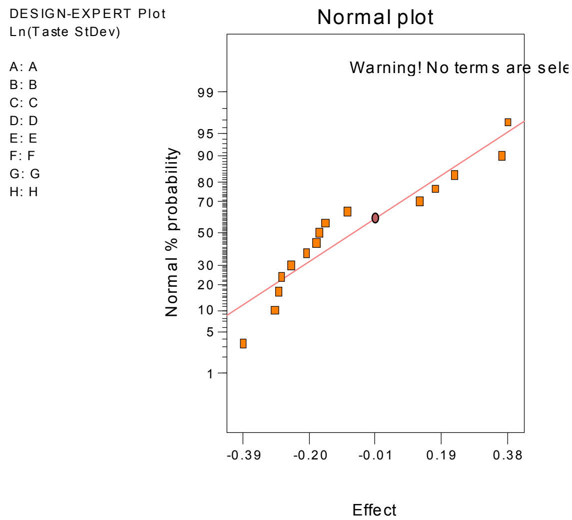
Factors *D, F, G* and the *AD* interaction are important. Factor *A* is added to the model to preserve hierarchy. Notice that the *AD* interaction is aliased with other two-factor interactions that could also be important. So the interpretation of the two-factor interaction is somewhat uncertain. Normally, we would add runs to the design to isolate the significant interactions, but that won't work very well here because each experiment requires a full growing season. In other words, it would require a very long time to add runs to dealias the alias chain of interest.







(c) Use the standard deviation of the ranks (or some appropriate transformation such as log *s*) as a response variable. What conclusions can you draw about the effects of the eight variables on variability in wine quality?

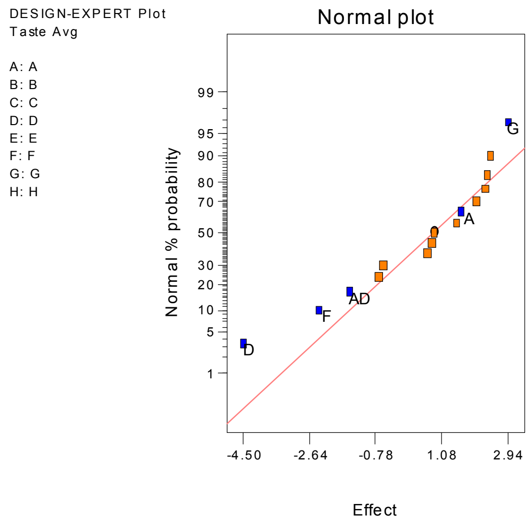


There do not appear to be any significant factors.

(d) After looking at the results, Harry and Judy decide that one of the panel members (DCM) knows more about beer than he does about wine, so they decide to delete his ranking. What affect would this have on the results and on conclusions from parts (b) and (c)?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	1.625	10.5625	4.02957
Error	B	2.0625	17.0156	6.49142
Error	C	1.5	9	3.43348
Model	D	-4.5	81	30.9013
Error	E	2.4375	23.7656	9.06652
Model	F	-2.375	22.5625	8.60753
Model	G	2.9375	34.5156	13.1676
Error	H	-0.6875	1.89063	0.721268
Error	AB	-0.5625	1.26562	0.482833
Error	AC	2.375	22.5625	8.60753
Model	AD	-1.5	9	3.43348
Error	AE	0.6875	1.89062	0.721268
Error	AF	0.875	3.0625	1.16834
Error	AG	0.8125	2.64062	1.00739
Error	AH	2.3125	21.3906	8.16047
	Lenth's ME	6.26579		
	Lenth's SME	12.7205		



Design Expert Output

Response: Taste Avg

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	157.64	5	31.53	3.02	0.0646	not significant
A	*10.56*	*1*	*10.56*	*1.01*	*0.3384*	
D	*81.00*	*1*	*81.00*	*7.75*	*0.0193*	
F	*22.56*	*1*	*22.56*	*2.16*	*0.1724*	
G	*34.52*	*1*	*34.52*	*3.30*	*0.0992*	
AD	*9.00*	*1*	*9.00*	*0.86*	*0.3752*	
Residual	104.48	10	10.45			
Cor Total	262.13	15				

The Model F-value of 3.02 implies there is a 6.46% chance that a "Model F-Value" this large could occur due to noise.

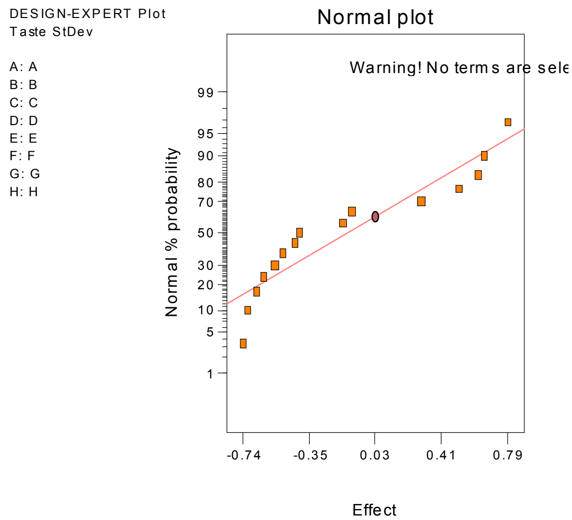
Std. Dev.	3.23	R-Squared	0.6014
Mean	8.50	Adj R-Squared	0.4021
C.V.	38.03	Pred R-Squared	-0.0204
PRESS	267.48	Adeq Precision	5.778

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	8.50	1	0.81	6.70	10.30	
A-A	0.81	1	0.81	-0.99	2.61	1.00
D-D	-2.25	1	0.81	-4.05	-0.45	1.00
F-F	-1.19	1	0.81	-2.99	0.61	1.00
G-G	1.47	1	0.81	-0.33	3.27	1.00
AD	-0.75	1	0.81	-2.55	1.05	1.00

Final Equation in Terms of Coded Factors:

Taste Avg	=
+8.50	
+0.81	* A
-2.25	* D
-1.19	* F
+1.47	* G
-0.75	* A * D

The results are the same for average taste without DCM as they were with DCM.



The standard deviation response is much the same with or without DCM's responses. Again, there are no significant factors.

(e) Suppose that just before the start of the experiment, Harry and Judy discovered that the eight new barrels they ordered from France for use in the experiment would not arrive in time, and all 16 runs would have to be made with old barrels. If Harry and Judy just drop column C from their design, what does this do to the alias relationships? Do they need to start over and construct a new design?

The resulting design is a 2_{IV}^{7-3} with defining relations: I = *ABEF* = *ADEG* = *BDFG* = *ABDH* = *DEFH* = *BEGH* = *AFGH*.

(f) Harry and Judy know from experience that some treatment combinations are unlikely to produce good results. For example, the run with all eight variables at the high level generally results in a poorly rated wine. This was confirmed in the 8 March 1986 taste test. They want to set up a new design to make the run with all eight factors at the high level. What design would you suggest?

By changing the sign of any of the design generators, a design that does not include the principal fraction will be generated. This will give a design without an experimental run combination with all of the variables at the high level.

8-27 In an article in *Quality Engineering* ("An Application of Fractional Factorial Experimental Designs," 1988, Vol. 1 pp. 19-23) M.B. Kilgo describes an experiment to determine the effect of CO_2 pressure (*A*), CO_2 temperature (*B*), peanut moisture (*C*), CO_2 flow rate (*D*), and peanut particle size (*E*) on the total yield of oil per batch of peanuts (*y*). The levels she used for these factors are as follows:

Coded Level	*A* Pressure (bar)	*B* Temp (C)	*C* Moisture (% by weight)	*D* Flow (liters/min)	*E* Particle Size (mm)
-1	415	25	5	40	1.28
1	550	95	15	60	4.05

She conducted the 16-run fractional factorial experiment shown below:

	A	*B*	*C*	*D*	*E*	y
1	415	25	5	40	1.28	63

2	550	25	5	40	4.05	21
3	415	95	5	40	4.05	36
4	550	95	5	40	1.28	99
5	415	25	15	40	4.05	24
6	550	25	15	40	1.28	66
7	415	95	15	40	1.28	71
8	550	95	15	40	4.05	54
9	415	25	5	60	4.05	23
10	550	25	5	60	1.28	74
11	415	95	5	60	1.28	80
12	550	95	5	60	4.05	33
13	415	25	15	60	1.28	63
14	550	25	15	60	4.05	21
15	415	95	15	60	4.05	44
16	550	95	15	60	1.28	96

(a) What type of design has been used? Identify the defining relation and the alias relationships.

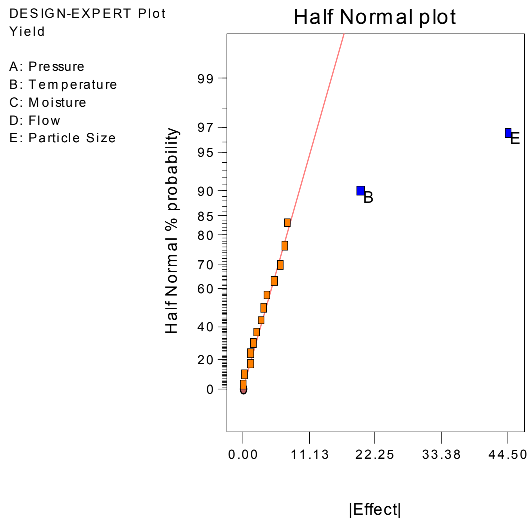
A 2_V^{5-1}, 16-run design, with I= *-ABCDE*.

$A(-ABCDE) = -BCDE$	$A = -BCDE$
$B(-ABCDE) = -ACDE$	$B = -ACDE$
$C(-ABCDE) = -ABDE$	$C = -ABDE$
$D(-ABCDE) = -ABCE$	$D = -ABCE$
$E(-ABCDE) = -ABCD$	$E = -ABCD$
$AB(-ABCDE) = -CDE$	$AB = -CDE$
$AC(-ABCDE) = -BDE$	$AC = -BDE$
$AD(-ABCDE) = -BCE$	$AD = -BCE$
$AE(-ABCDE) = -BCD$	$AE = -BCD$
$BC(-ABCDE) = -ADE$	$BC = -ADE$
$BD(-ABCDE) = -ACE$	$BD = -ACE$
$BE(-ABCDE) = -ACD$	$BE = -ACD$
$CD(-ABCDE) = -ABE$	$CD = -ABE$
$CE(-ABCDE) = -ABD$	$CE = -ABD$
$DE(-ABCDE) = -ABC$	$DE = -ABC$

(b) Estimate the factor effects and use a normal probability plot to tentatively identify the important factors.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	7.5	225	2.17119
Model	B	19.75	1560.25	15.056
Error	C	1.25	6.25	0.0603107
Error	D	0	0	0
Model	E	44.5	7921	76.4354
Error	AB	5.25	110.25	1.06388
Error	AC	1.25	6.25	0.0603107
Error	AD	-4	64	0.617582
Error	AE	7	196	1.89134
Error	BC	3	36	0.34739
Error	BD	-1.75	12.25	0.118209
Error	BE	0.25	0.25	0.00241243
Error	CD	2.25	20.25	0.195407
Error	CE	-6.25	156.25	1.50777
Error	DE	3.5	49	0.472836
	Lenth's ME	11.5676		
	Lenth's SME	23.4839		



(c) Perform an appropriate statistical analysis to test the hypothesis that the factors identified in part above have a significant effect on the yield of peanut oil.

Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	9481.25	2	4740.63	69.89	< 0.0001	significant
B	*1560.25*	*1*	*1560.25*	*23.00*	*0.0003*	
E	*7921.00*	*1*	*7921.00*	*116.78*	*< 0.0001*	
Residual	881.75	13	67.83			
Cor Total	10363.00	15				

The Model F-value of 69.89 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	8.24	R-Squared	0.9149
Mean	54.25	Adj R-Squared	0.9018
C.V.	15.18	Pred R-Squared	0.8711
PRESS	1335.67	Adeq Precision	18.017

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	54.25	1	2.06	49.80	58.70	
B-Temperature	9.88	1	2.06	5.43	14.32	1.00
E-Particle Size	22.25	1	2.06	17.80	26.70	1.00

(d) Fit a model that could be used to predict peanut oil yield in terms of the factors that you have identified as important.

Design Expert Output

Final Equation in Terms of Coded Factors:

Yield =
+54.25
+9.88 * B
+22.25 * E

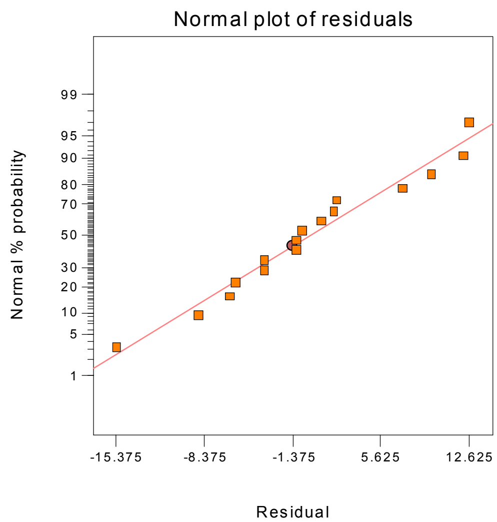
Final Equation in Terms of Actual Factors:

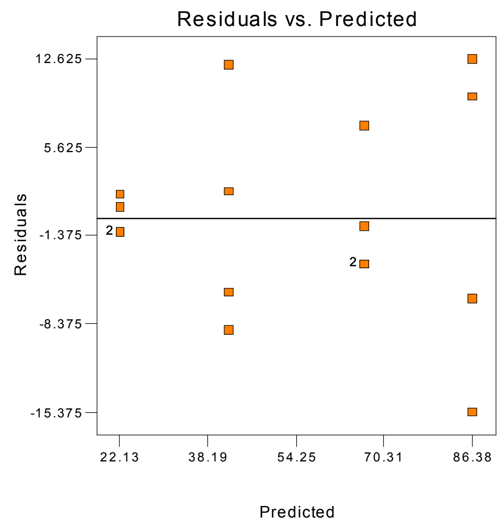
Yield =

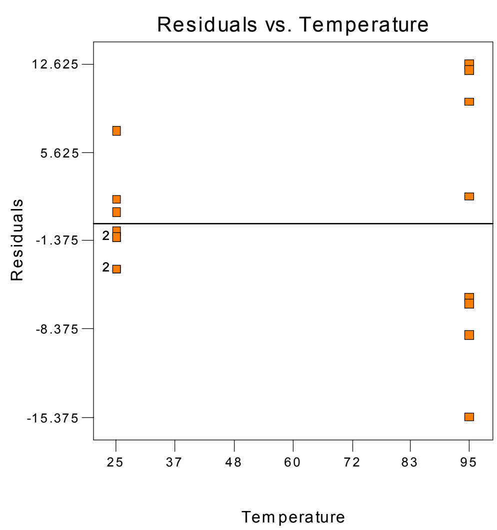
-5.49175	
+0.28214	* Temperature
+16.06498	* Particle Size

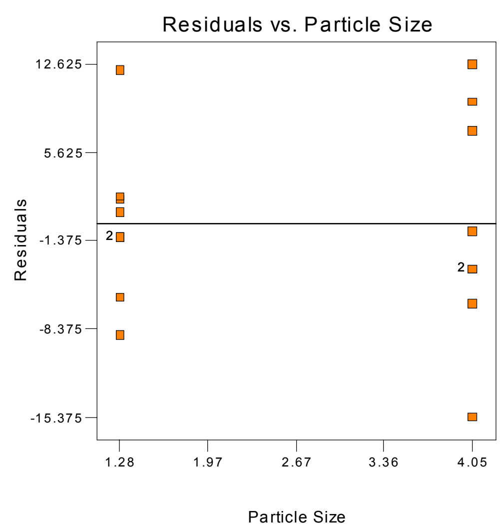
(e) Analyze the residuals from this experiment and comment on model adequacy.

The residual plots are satisfactory. There is a slight tendency for the variability of the residuals to increase with the predicted value of y.









8-28 A 16-run fractional factorial experiment in 10 factors on sand-casting of engine manifolds was conducted by engineers at the Essex Aluminum Plant of the Ford Motor Company and described in the article "Evaporative Cast Process 3.0 Liter Intake Manifold Poor Sandfill Study," by D. Becknell (*Fourth Symposium on Taguchi Methods*, American Supplier Institute, Dearborn, MI, 1986, pp. 120-130). The purpose was to determine which of 10 factors has an effect on the proportion of defective castings. The design and the resulting proportion of nondefective castings p observed on each run are shown below.

This is a resolution III fraction with generators *E=CD, F=BD, G=BC, H=AC, J=AB*, and *K=ABC*. Assume that the number of castings made at each run in the design is 1000.

Run	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*J*	*K*	p	arcsin	F&T's Modification
1	-	-	-	-	+	+	+	+	+	-	0.958	1.364	1.363
2	+	-	-	-	+	+	+	-	-	+	1.000	1.571	1.555
3	-	+	-	-	+	-	-	+	-	+	0.977	1.419	1.417
4	+	+	-	-	+	-	-	-	+	-	0.775	1.077	1.076
5	-	-	+	-	-	+	-	-	+	+	0.958	1.364	1.363
6	+	-	+	-	-	+	-	+	-	-	0.958	1.364	1.363
7	-	+	+	-	-	-	+	-	-	-	0.813	1.124	1.123
8	+	+	+	-	-	-	+	+	+	+	0.906	1.259	1.259
9	-	-	-	+	-	-	+	+	+	-	0.679	0.969	0.968
10	+	-	-	+	-	-	+	-	-	+	0.781	1.081	1.083
11	-	+	-	+	-	+	-	+	-	+	1.000	1.571	1.556
12	+	+	-	+	-	+	-	-	+	-	0.896	1.241	1.242
13	-	-	+	+	+	-	-	-	+	+	0.958	1.364	1.363
14	+	-	+	+	+	-	-	+	-	-	0.818	1.130	1.130
15	-	+	+	+	+	+	+	-	-	-	0.841	1.161	1.160
16	+	+	+	+	+	+	+	+	+	+	0.955	1.357	1.356

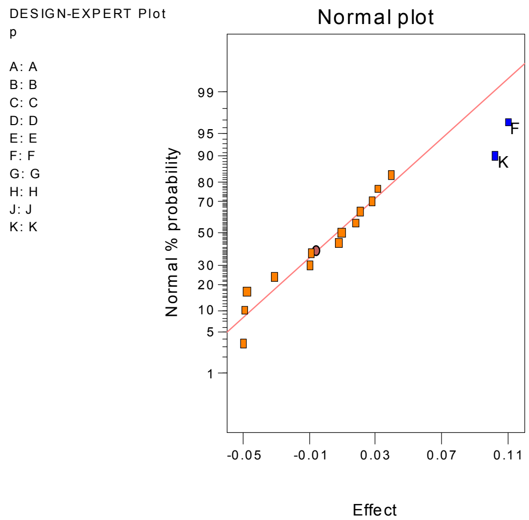
(a) Find the defining relation and the alias relationships in this design.

I=*CDE=BDF=BCG=ACH=ABJ=ABCK=BCEF=BDEG=ADEH=ABCDEJ=ABDEK=CDFG=ABCDFH = ADFJ=ACDFK=ABGH=ACGJ=AGK=BCHJ=BHK=CKJ*

(b) Estimate the factor effects and use a normal probability plot to tentatively identify the important factors.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-0.011875	0.000564063	0.409171
Error	B	0.006625	0.000175562	0.127353
Error	C	0.017625	0.00124256	0.901355
Error	D	-0.052125	0.0108681	7.88369
Error	E	0.036375	0.00529256	3.83923
Model	F	0.107375	0.0461176	33.4537
Error	G	-0.050875	0.0103531	7.51011
Error	H	0.028625	0.00327756	2.37754
Error	J	-0.012875	0.000663062	0.480986
Model	K	0.099625	0.0397006	28.7988
Error	AB	Aliased		
Error	AC	Aliased		
Error	AD	0.004875	9.50625E-005	0.0689584
Error	AE	-0.034625	0.00479556	3.4787
Error	AF	0.024875	0.00247506	1.79541
Error	BE	-0.053125	0.0112891	8.18909
Error	DK	0.015375	0.000945563	0.685911
	Lenth's ME	0.103145		
	Lenth's SME	0.209399		



(c) Fit an appropriate model using the factors identified in part (b) above.

Design Expert Output

Response: p
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.086	2	0.043	10.72	0.0018	significant
F	*0.046*	*1*	*0.046*	*11.52*	*0.0048*	
K	*0.040*	*1*	*0.040*	*9.92*	*0.0077*	
Residual	0.052	13	4.003E-003			
Cor Total	0.14	15				

The Model F-value of 10.72 implies the model is significant. There is only a 0.18% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.063	R-Squared	0.6225
Mean	0.89	Adj R-Squared	0.5645
C.V.	7.09	Pred R-Squared	0.4282
PRESS	0.079	Adeq Precision	7.556

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	0.89	1	0.016	0.86	0.93	
F-F	0.054	1	0.016	0.020	0.088	1.00
K-K	0.050	1	0.016	0.016	0.084	1.00

Final Equation in Terms of Coded Factors:

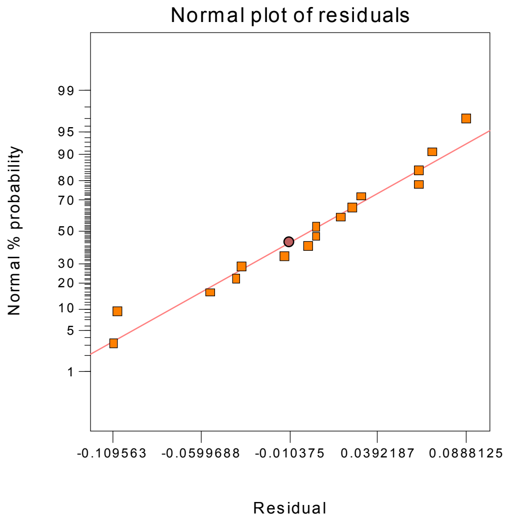
p =
+0.89
+0.054 * F
+0.050 * K

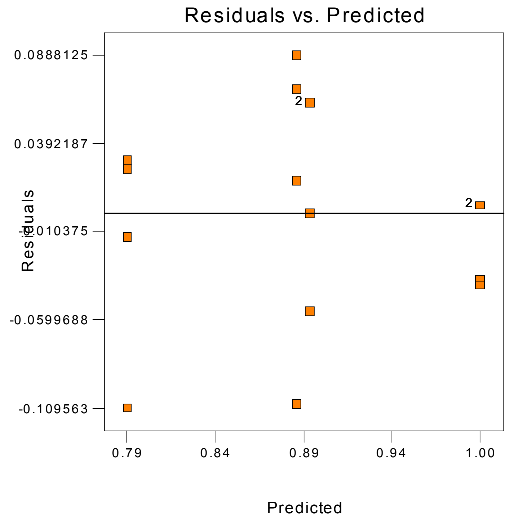
Final Equation in Terms of Actual Factors:

p =
+0.89206
+0.053688 * F
+0.049812 * K

(d) Plot the residuals from this model versus the predicted proportion of nondefective castings. Also prepare a normal probability plot of the residuals. Comment on the adequacy of these plots.

The residual versus predicted plot identifies an inequality of variances. This is likely caused by the response variable being a proportion. A transformation could be used to correct this.



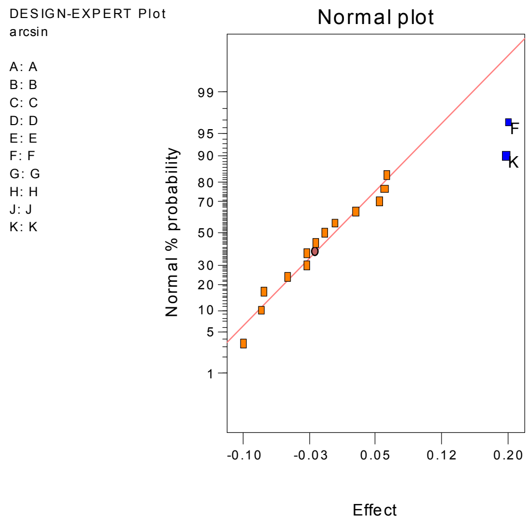


(e) In part (d) you should have noticed an indication that the variance of the response is not constant (considering that the response is a proportion, you should have expected this). The previous table also shows a transformation on P, the arcsin square root, that is a widely used variance stabilizing transformation for proportion data (refer to the discussion of variance stabilizing transformations is Chapter 3). Repeat parts (a) through (d) above using the transformed response and comment on your results. Specifically, are the residuals plots improved?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-0.032	0.004096	0.884531
Error	B	0.00025	2.5E-007	5.39875E-005
Error	C	-0.02125	0.00180625	0.39006
Error	D	-0.0835	0.027889	6.02263
Error	E	0.05875	0.0138062	2.98146
Model	F	0.19625	0.154056	33.2685
Error	G	-0.0805	0.025921	5.59764
Error	H	0.05625	0.0126562	2.73312
Error	J	-0.05325	0.0113422	2.44936
Model	K	0.1945	0.151321	32.6778
Error	AD	-0.032	0.004096	0.884531
Error	AF	0.05025	0.0101003	2.18115
Error	BE	-0.104	0.043264	9.34286
Error	DH	-0.01125	0.00050625	0.109325
Error	DK	0.0235	0.002209	0.477034
	Lenth's ME	0.205325		
	Lenth's SME	0.41684		

As with the original analysis, factors *F* and *K* remain significant with a slight increase with the R^2.



Design Expert Output

Response: arcsin

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.31	2	0.15	12.59	0.0009	significant
F	*0.15*	*1*	*0.15*	*12.70*	*0.0035*	
K	*0.15*	*1*	*0.15*	*12.47*	*0.0037*	
Residual	0.16	13	0.012			
Cor Total	0.46	15				

The Model F-value of 12.59 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.11	R-Squared	0.6595
Mean	1.28	Adj R-Squared	0.6071
C.V.	8.63	Pred R-Squared	0.4842
PRESS	0.24	Adeq Precision	8.193

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	1.28	1	0.028	1.22	1.34	
F-F	0.098	1	0.028	0.039	0.16	1.00
K-K	0.097	1	0.028	0.038	0.16	1.00

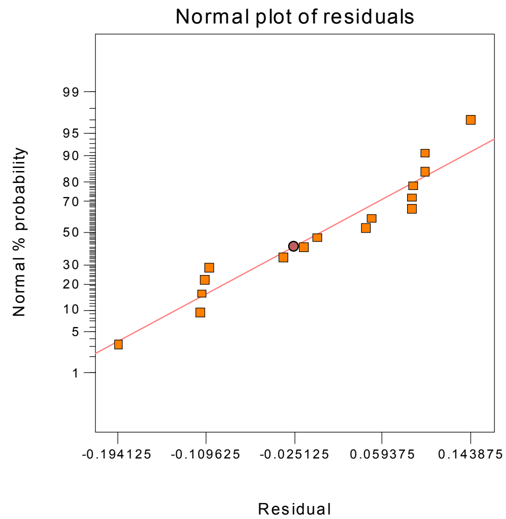
Final Equation in Terms of Coded Factors:

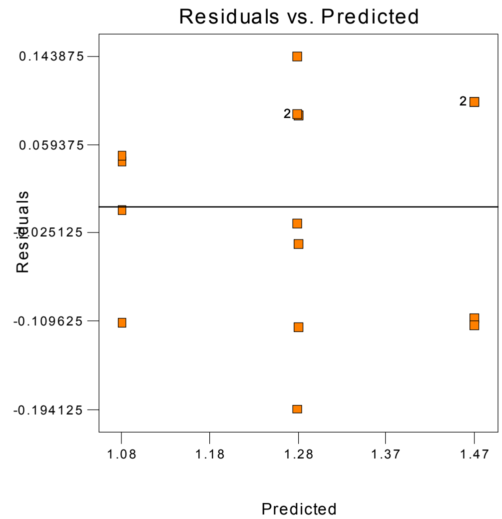
arcsin =
+1.28
+0.098 * F
+0.097 * K

Final Equation in Terms of Actual Factors:

arcsin =
+1.27600
+0.098125 * F
+0.097250 * K

The inequality of variance has improved; however, there remain hints of inequality in the residuals versus predicted plot and the normal probability plot now appears to be irregular.





(f) There is a modification to the arcsin square root transformation, proposed by Freeman and Tukey ("Transformations Related to the Angular and the Square Root," *Annals of Mathematical Statistics*, Vol. 21, 1950, pp. 607-611) that improves its performance in the tails. F&T's modification is:

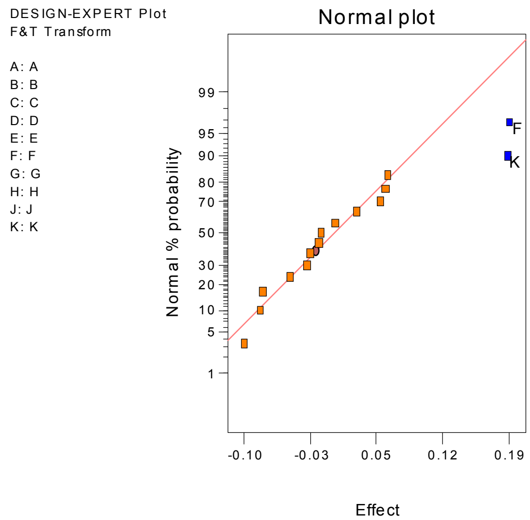
$$\frac{1}{2}\left[arcsin\sqrt{\frac{n\hat{p}}{(n+1)}} + arcsin\sqrt{\frac{(n\hat{p}+1)}{(n+1)}} \right]$$

Rework parts (a) through (d) using this transformation and comment on the results. (For an interesting discussion and analysis of this experiment, refer to "Analysis of Factorial Experiments with Defects or Defectives as the Response," by S. Bisgaard and H.T. Fuller, *Quality Engineering*, Vol. 7, 1994-5, pp. 429-443.)

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-0.031125	0.00387506	0.871894
Error	B	0.000125	6.25E-008	1.40626E-005
Error	C	-0.017875	0.00127806	0.287566
Error	D	-0.082625	0.0273076	6.14424
Error	E	0.057875	0.0133981	3.01458
Model	F	0.192375	0.148033	33.3075
Error	G	-0.080375	0.0258406	5.81416
Error	H	0.055875	0.0124881	2.80983
Error	J	-0.049625	0.00985056	2.21639
Model	K	0.190875	0.145733	32.7901
Error	AD	-0.027875	0.00310806	0.699318
Error	AF	0.049625	0.00985056	2.21639
Error	BE	-0.100625	0.0405016	9.1129
Error	DH	-0.015375	0.000945563	0.212753
Error	DK	0.023625	0.00223256	0.502329
	Lenth's ME	0.191348		
	Lenth's SME	0.388464		

As with the prior analysis, factors *F* and *K* remain significant.



Design Expert Output

Response: F&T Transform
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.29	2	0.15	12.67	0.0009	significant
F	*0.15*	*1*	*0.15*	*12.77*	*0.0034*	
K	*0.15*	*1*	*0.15*	*12.57*	*0.0036*	
Residual	0.15	13	0.012			
Cor Total	0.44	15				

The Model F-value of 12.67 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.11	R-Squared	0.6610
Mean	1.27	Adj R-Squared	0.6088
C.V.	8.45	Pred R-Squared	0.4864
PRESS	0.23	Adeq Precision	8.221

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	1.27	1	0.027	1.22	1.33	
F-F	0.096	1	0.027	0.038	0.15	1.00
K-K	0.095	1	0.027	0.037	0.15	1.00

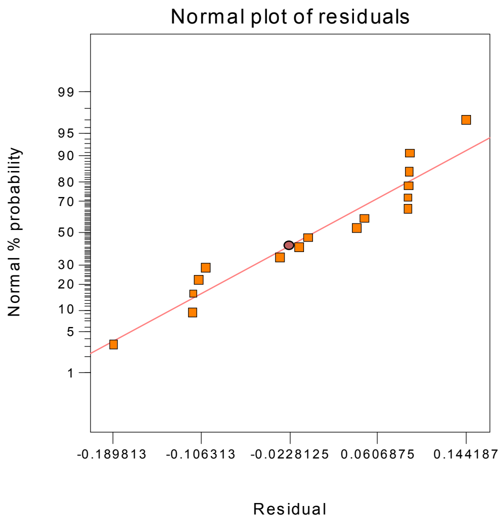
Final Equation in Terms of Coded Factors:

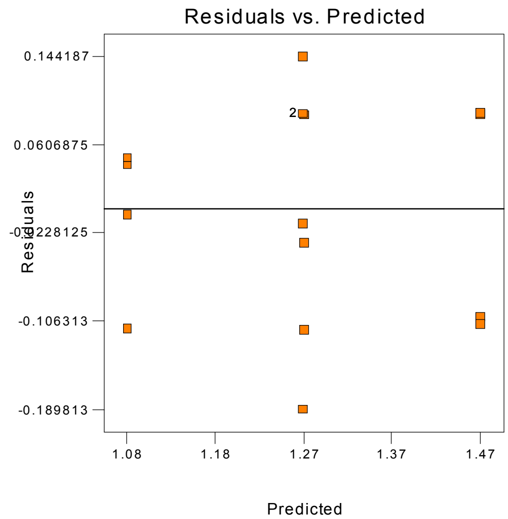
F&T Transform =
+1.27
+0.096 * F
+0.095 * K

Final Equation in Terms of Actual Factors:

F&T Transform =
+1.27356
+0.096188 * F
+0.095437 * K

The residual plots appears as they did with the arcsin square root transformation.





8-29 A 16-run fractional factorial experiment in 9 factors was conducted by Chrysler Motors Engineering and described in the article "Sheet Molded Compound Process Improvement," by P.I. Hsieh and D.E. Goodwin (*Fourth Symposium on Taguchi Methods*, American Supplier Institute, Dearborn, MI, 1986, pp. 13-21). The purpose was to reduce the number of defects in the finish of sheet-molded grill opening panels. The design, and the resulting number of defects, c, observed on each run, is shown below. This is a resolution III fraction with generators *E=BD, F=BCD, G=AC, H=ACD*, and *J=AB.*

Run	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*J*	c	$\sqrt{c}$	F&T's Modification
1	-	-	-	-	+	-	+	-	+	56	7.48	7.52
2	+	-	-	-	+	-	-	+	-	17	4.12	4.18
3	-	+	-	-	-	+	+	-	-	2	1.41	1.57
4	+	+	-	-	-	+	-	+	+	4	2.00	2.12
5	-	-	+	-	+	+	-	+	+	3	1.73	1.87
6	+	-	+	-	+	+	+	-	-	4	2.00	2.12
7	-	+	+	-	-	-	-	+	-	50	7.07	7.12
8	+	+	+	-	-	-	+	-	+	2	1.41	1.57
9	-	-	-	+	-	+	+	+	+	1	1.00	1.21
10	+	-	-	+	-	+	-	-	-	0	0.00	0.50
11	-	+	-	+	+	-	+	+	-	3	1.73	1.87
12	+	+	-	+	+	-	-	-	+	12	3.46	3.54
13	-	-	+	+	-	-	-	-	+	3	1.73	1.87
14	+	-	+	+	-	-	+	+	-	4	2.00	2.12
15	-	+	+	+	+	+	-	-	-	0	0.00	0.50
16	+	+	+	+	+	+	+	+	+	0	0.00	0.50

(a) Find the defining relation and the alias relationships in this design.

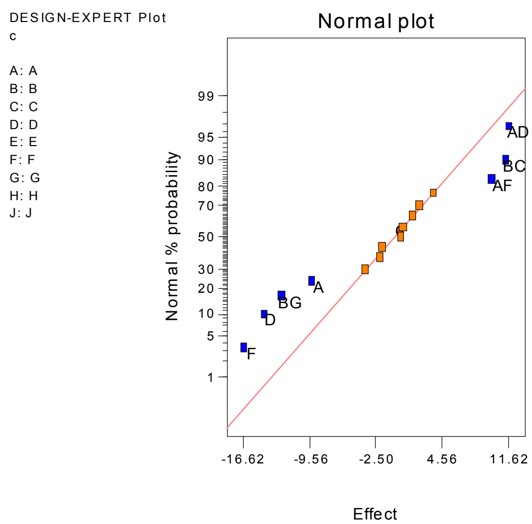
I=*BDE=BCDF=CEF=ACG=ABCDEG=ABDEG=AEFG=ACDH=ABCEH=ABFH=ADEFH=DGH=*
BEGH=BCRG=CDEFGH=ABJ=ADEJ=ACDFJ=ABCEFJ=BCGJ=CDEGJ=DEGJ=BEFGJ=BCDHJ=
CEHJ=FHJ=BDEFHJ=ABDGHJ=AEGHJ=ACEGJ=ABCDEFGHJ

(b) Estimate the factor effects and use a normal probability plot to tentatively identify the important factors.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			

Model	A	-9.375	351.562	7.75573
Model	B	-1.875	14.0625	0.310229
Model	C	-3.625	52.5625	1.15957
Model	D	-14.375	826.562	18.2346
Error	E	3.625	52.5625	1.15957
Model	F	-16.625	1105.56	24.3895
Model	G	-2.125	18.0625	0.398472
Error	H	0.375	0.5625	0.0124092
Error	J	0.125	0.0625	0.0013788
Model	AD	11.625	540.563	11.9252
Error	AE	2.125	18.0625	0.398472
Model	AF	9.875	390.063	8.60507
Error	AH	1.375	7.5625	0.166834
Model	BC	11.375	517.563	11.4178
Model	BG	-12.625	637.562	14.0651
	Lenth's ME	13.9775		
	Lenth's SME	28.3764		



(c) Fit an appropriate model using the factors identified in part (b) above.

Design Expert Output

Response: c
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	4454.13	10	445.41	28.26	0.0009	significant
A	*351.56*	*1*	*351.56*	*22.30*	*0.0052*	
B	*14.06*	*1*	*14.06*	*0.89*	*0.3883*	
C	*52.56*	*1*	*52.56*	*3.33*	*0.1274*	
D	*826.56*	*1*	*826.56*	*52.44*	*0.0008*	
F	*1105.56*	*1*	*1105.56*	*70.14*	*0.0004*	
G	*18.06*	*1*	*18.06*	*1.15*	*0.3333*	
AD	*540.56*	*1*	*540.56*	*34.29*	*0.0021*	
AF	*390.06*	*1*	*390.06*	*24.75*	*0.0042*	
BC	*517.56*	*1*	*517.56*	*32.84*	*0.0023*	
BG	*637.56*	*1*	*637.56*	*40.45*	*0.0014*	
Residual	78.81	5	15.76			
Cor Total	4532.94	15				

The Model F-value of 28.26 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	3.97	R-Squared	0.9826

Mean	10.06	Adj R-Squared	0.9478
C.V.	39.46	Pred R-Squared	0.8220
PRESS	807.04	Adeq Precision	17.771

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	10.06	1	0.99	7.51	12.61	
A-A	-4.69	1	0.99	-7.24	-2.14	1.00
B-B	-0.94	1	0.99	-3.49	1.61	1.00
C-C	-1.81	1	0.99	-4.36	0.74	1.00
D-D	-7.19	1	0.99	-9.74	-4.64	1.00
F-F	-8.31	1	0.99	-10.86	-5.76	1.00
G-G	-1.06	1	0.99	-3.61	1.49	1.00
AD	5.81	1	0.99	3.26	8.36	1.00
AF	4.94	1	0.99	2.39	7.49	1.00
BC	5.69	1	0.99	3.14	8.24	1.00
BG	-6.31	1	0.99	-8.86	-3.76	1.00

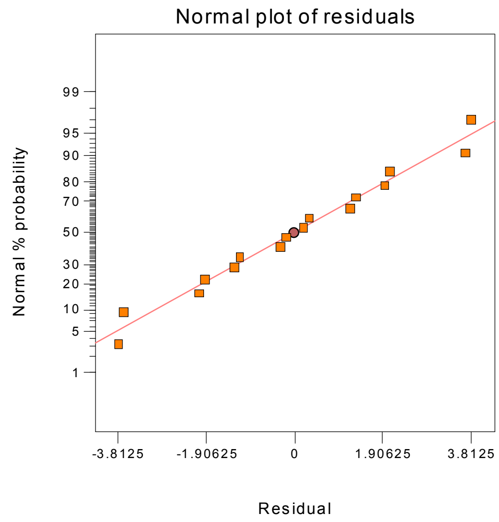
Final Equation in Terms of Coded Factors:

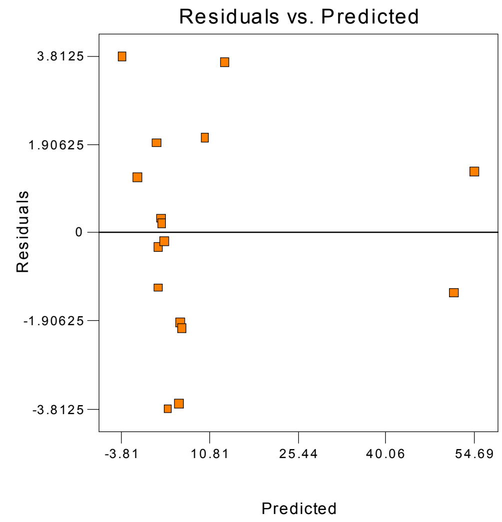
c	=
+10.06	
-4.69	* A
-0.94	* B
-1.81	* C
-7.19	* D
-8.31	* F
-1.06	* G
+5.81	* A * D
+4.94	* A * F
+5.69	* B * C
-6.31	* B * G

Final Equation in Terms of Actual Factors:

c	=
+10.06250	
-4.68750	* A
-0.93750	* B
-1.81250	* C
-7.18750	* D
-8.31250	* F
-1.06250	* G
+5.81250	* A * D
+4.93750	* A * F
+5.68750	* B * C
-6.31250	* B * G

(d) Plot the residuals from this model versus the predicted number of defects. Also, prepare a normal probability plot of the residuals. Comment on the adequacy of these plots.





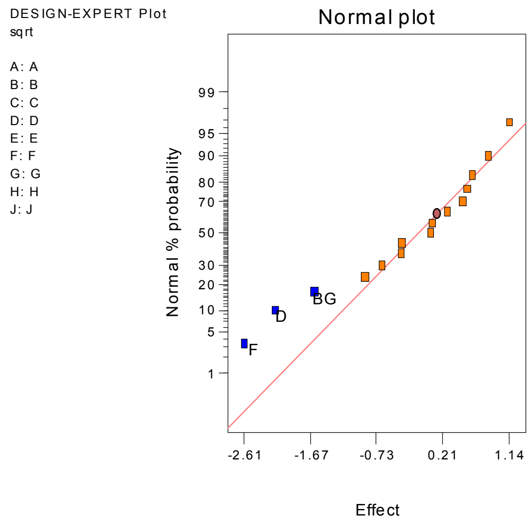
There is a significant problem with inequality of variance. This is likely caused by the response variable being a count. A transformation may be appropriate.

(e) In part (d) you should have noticed an indication that the variance of the response is not constant (considering that the response is a count, you should have expected this). The previous table also shows a transformation on c, the square root, that is a widely used variance stabilizing transformation for count data (refer to the discussion of variance stabilizing transformations in Chapter 3). Repeat parts (a) through (d) using the transformed response and comment on your results. Specifically, are the residual plots improved?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-0.895	3.2041	4.2936
Model	B	-0.3725	0.555025	0.743752
Error	C	-0.6575	1.72922	2.31722
Model	D	-2.1625	18.7056	25.0662
Error	E	0.4875	0.950625	1.27387
Model	F	-2.6075	27.1962	36.4439
Model	G	-0.385	0.5929	0.794506
Error	H	0.27	0.2916	0.390754
Error	J	0.06	0.0144	0.0192965
Error	AD	1.145	5.2441	7.02727
Error	AE	0.555	1.2321	1.65106
Error	AF	0.86	2.9584	3.96436
Error	AH	0.0425	0.007225	0.00968175
Error	BC	0.6275	1.57502	2.11059
Model	BG	-1.61	10.3684	13.894
	Lenth's ME	2.27978		
	Lenth's SME	4.62829		

The analysis of the data with the square root transformation identifies only *D*, *F*, the *BG* interaction as being significant. The original analysis identified factor *A* and several two factor interactions as being significant.



Design Expert Output

Response: sqrt

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	57.42	5	11.48	6.67	0.0056	significant
B	*0.56*	*1*	*0.56*	*0.32*	*0.5826*	
D	*18.71*	*1*	*18.71*	*10.87*	*0.0081*	
F	*27.20*	*1*	*27.20*	*15.81*	*0.0026*	
G	*0.59*	*1*	*0.59*	*0.34*	*0.5702*	
BG	*10.37*	*1*	*10.37*	*6.03*	*0.0340*	
Residual	17.21	10	1.72			
Cor Total	74.62	15				

The Model F-value of 6.67 implies the model is significant. There is only a 0.56% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.31	R-Squared	0.7694
Mean	2.32	Adj R-Squared	0.6541
C.V.	56.51	Pred R-Squared	0.4097
PRESS	44.05	Adeq Precision	8.422

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	2.32	1	0.33	1.59	3.05	
B-B	-0.19	1	0.33	-0.92	0.54	1.00
D-D	-1.08	1	0.33	-1.81	-0.35	1.00
F-F	-1.30	1	0.33	-2.03	-0.57	1.00
G-G	-0.19	1	0.33	-0.92	0.54	1.00
BG	-0.80	1	0.33	-1.54	-0.074	1.00

Final Equation in Terms of Coded Factors:

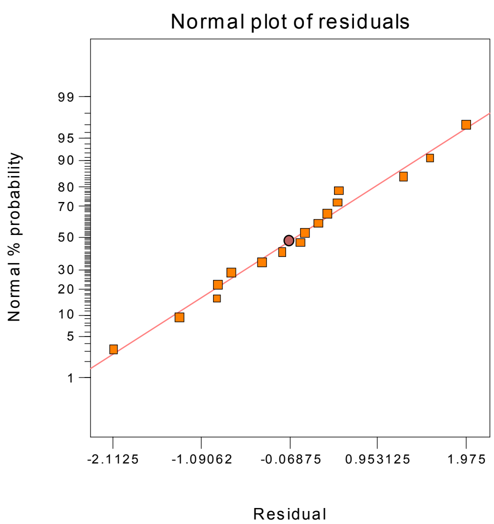
sqrt =
+2.32
-0.19 * B
-1.08 * D
-1.30 * F
-0.19 * G
-0.80 * B * G

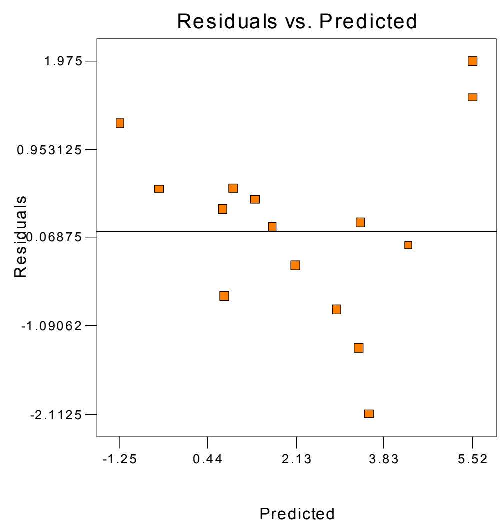
Final Equation in Terms of Actual Factors:

sqrt =

+2.32125	
-0.18625	* B
-1.08125	* D
-1.30375	* F
-0.19250	* G
-0.80500	* B * G

The residual plots are acceptable; although, there appears to be a slight "u" shape to the residuals versus predicted plot.





(f) There is a modification to the square root transformation proposed by Freeman and Tukey ("Transformations Related to the Angular and the Square Root," *Annals of Mathematical Statistics*, Vol. 21, 1950, pp. 607-611) that improves its performance. F&T's modification to the square root transformation is:

$$\frac{1}{2}\left[\sqrt{c}+\sqrt{c+1}\right]$$

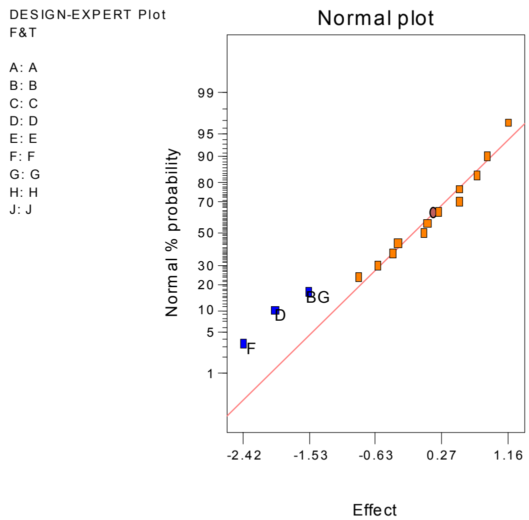
Rework parts (a) through (d) using this transformation and comment on the results. (For an interesting discussion and analysis of this experiment, refer to "Analysis of Factorial Experiments with Defects or Defectives as the Response," by S. Bisgaard and H.T. Fuller, *Quality Engineering*, Vol. 7, 1994-5, pp. 429-443.)

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Error	A	-0.86	2.9584	4.38512
Model	B	-0.325	0.4225	0.626255
Error	C	-0.605	1.4641	2.17018
Model	D	-1.995	15.9201	23.5977
Error	E	0.5025	1.01002	1.49712
Model	F	-2.425	23.5225	34.8664
Model	G	-0.4025	0.648025	0.960541
Error	H	0.225	0.2025	0.300158
Error	J	0.0275	0.003025	0.00448383
Error	AD	1.1625	5.40562	8.01254
Error	AE	0.505	1.0201	1.51205
Error	AF	0.8825	3.11523	4.61757
Error	AH	0.0725	0.021025	0.0311645

Error	BC	0.7525	2.26503	3.35735
Model	BG	-1.54	9.4864	14.0613
	Lenth's ME	2.14001		
	Lenth's SME	4.34453		

As with the square root transformation, factors *D*, *F*, and the *BG* interaction remain significant.



Design Expert Output

Response: F&T

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	50.00	5	10.00	5.73	0.0095	significant
B	*0.42*	*1*	*0.42*	*0.24*	*0.6334*	
D	*15.92*	*1*	*15.92*	*9.12*	*0.0129*	
F	*23.52*	*1*	*23.52*	*13.47*	*0.0043*	
G	*0.65*	*1*	*0.65*	*0.37*	*0.5560*	
BG	*9.49*	*1*	*9.49*	*5.43*	*0.0420*	
Residual	17.47	10	1.75			
Cor Total	67.46	15				

The Model F-value of 5.73 implies the model is significant. There is only a 0.95% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.32	R-Squared	0.7411
Mean	2.51	Adj R-Squared	0.6117
C.V.	52.63	Pred R-Squared	0.3373
PRESS	44.71	Adeq Precision	7.862

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	2.51	1	0.33	1.78	3.25	
B-B	-0.16	1	0.33	-0.90	0.57	1.00
D-D	-1.00	1	0.33	-1.73	-0.26	1.00
F-F	-1.21	1	0.33	-1.95	-0.48	1.00
G-G	-0.20	1	0.33	-0.94	0.53	1.00
BG	-0.77	1	0.33	-1.51	-0.034	1.00

Final Equation in Terms of Coded Factors:

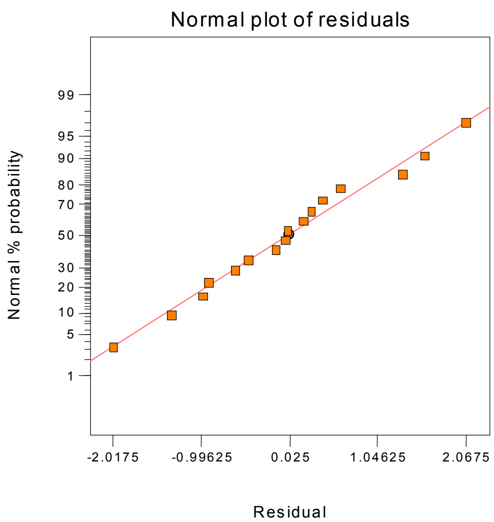
F&T =
+2.51
-0.16 * B

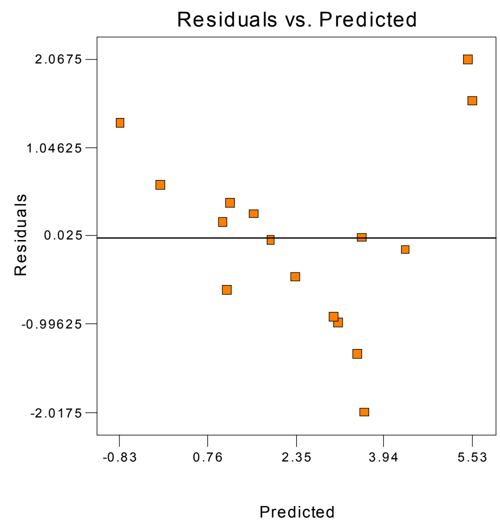
```
              -1.00     * D
              -1.21     * F
              -0.20     * G
              -0.77     * B * G

     Final Equation in Terms of Actual Factors:

               F&T    =
          +2.51125
          -0.16250     * B
          -0.99750     * D
          -1.21250     * F
          -0.20125     * G
          -0.77000     * B * G
```

The following interaction plots appear as they did with the square root transformation; a slight "u" shape is observed in the residuals versus predicted plot.





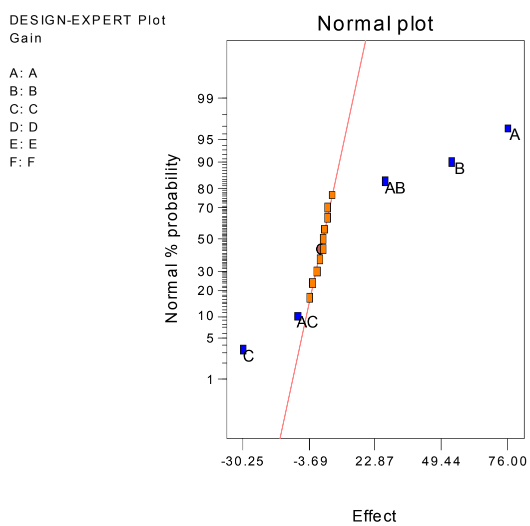
8-30 An experiment is run in a semiconductor factory to investigate the effect of six factors on transistor gain. The design selected is the 2_{IV}^{6-2} shown below.

Standard Order	Run Order	*A*	*B*	*C*	*D*	*E*	*F*	Gain
1	2	-	-	-	-	-	-	1455
2	8	+	-	-	-	+	-	1511
3	5	-	+	-	-	+	+	1487
4	9	+	+	-	-	-	+	1596
5	3	-	-	+	-	+	+	1430
6	14	+	-	+	-	-	+	1481
7	11	-	+	+	-	-	-	1458
8	10	+	+	+	-	+	-	1549
9	15	-	-	-	+	-	+	1454
10	13	+	-	-	+	+	+	1517
11	1	-	+	-	+	+	-	1487
12	6	+	+	-	+	-	-	1596
13	12	-	-	+	+	+	-	1446
14	4	+	-	+	+	-	-	1473
15	7	-	+	+	+	-	+	1461
16	16	+	+	+	+	+	+	1563

(a) Use a normal plot of the effects to identify the significant factors.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	76	23104	55.2714
Model	B	53.75	11556.2	27.6459
Model	C	-30.25	3660.25	8.75637
Error	D	3.75	56.25	0.134566
Error	E	2	16	0.0382766
Error	F	1.75	12.25	0.0293055
Model	AB	26.75	2862.25	6.84732
Model	AC	-8.25	272.25	0.6513
Error	AD	-0.75	2.25	0.00538265
Error	AE	-3.5	49	0.117222
Error	AF	5.25	110.25	0.26375
Error	BD	0.5	1	0.00239229
Error	BF	2.5	25	0.0598072
Error	ABD	3.5	49	0.117222
Error	ABF	-2.5	25	0.0598072
	Lenth's ME	9.63968		
	Lenth's SME	19.57		



(b) Conduct appropriate statistical tests for the model identified in part (a).

Design Expert Output

Response: Gain

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	41455.00	5	8291.00	239.62	< 0.0001	significant
A	*23104.00*	*1*	*23104.00*	*667.75*	*< 0.0001*	
B	*11556.25*	*1*	*11556.25*	*334.00*	*< 0.0001*	
C	*3660.25*	*1*	*3660.25*	*105.79*	*< 0.0001*	
AB	*2862.25*	*1*	*2862.25*	*82.72*	*< 0.0001*	
AC	*272.25*	*1*	*272.25*	*7.87*	*0.0186*	
Residual	346.00	10	34.60			
Cor Total	41801.00	15				

The Model F-value of 239.62 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	5.88	R-Squared	0.9917
Mean	1497.75	Adj R-Squared	0.9876
C.V.	0.39	Pred R-Squared	0.9788
PRESS	885.76	Adeq Precision	44.419

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	1497.75	1	1.47	1494.47	1501.03	
A-A	38.00	1	1.47	34.72	41.28	1.00
B-B	26.87	1	1.47	23.60	30.15	1.00
C-C	-15.13	1	1.47	-18.40	-11.85	1.00
AB	13.38	1	1.47	10.10	16.65	1.00
AC	-4.12	1	1.47	-7.40	-0.85	1.00

Final Equation in Terms of Coded Factors:

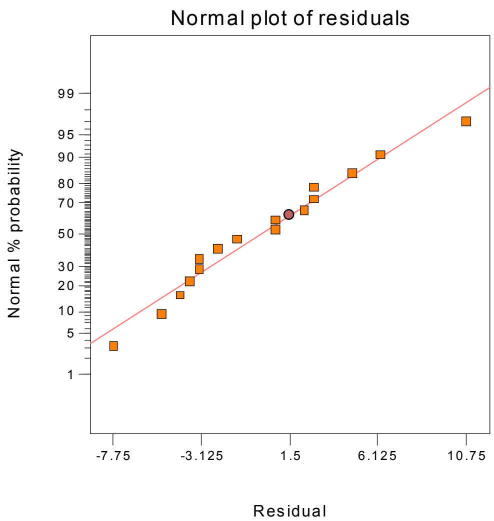
Gain	=
+1497.75	
+38.00	* A
+26.87	* B
-15.13	* C
+13.38	* A * B
-4.12	* A * C

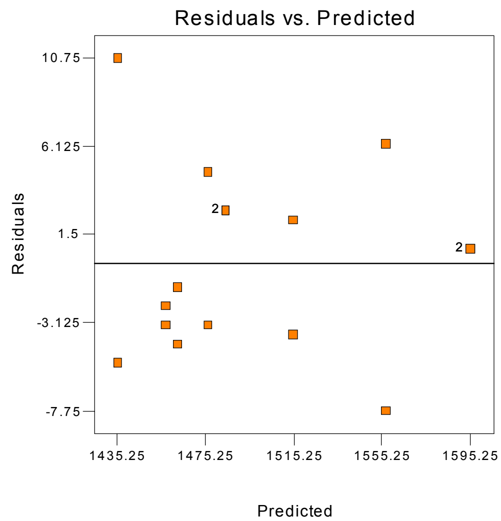
Final Equation in Terms of Actual Factors:

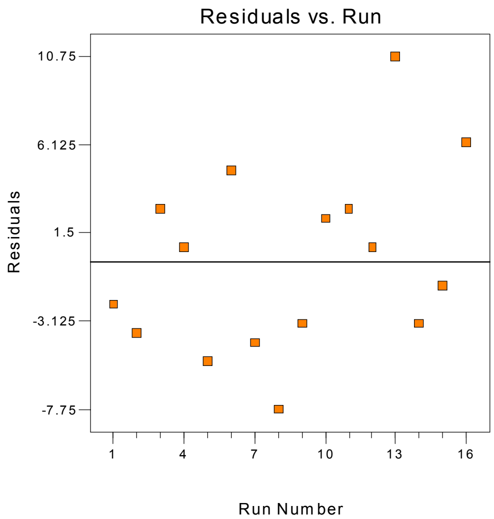
Gain	=
+1497.75000	
+38.00000	* A
+26.87500	* B
-15.12500	* C
+13.37500	* A * B
-4.12500	* A * C

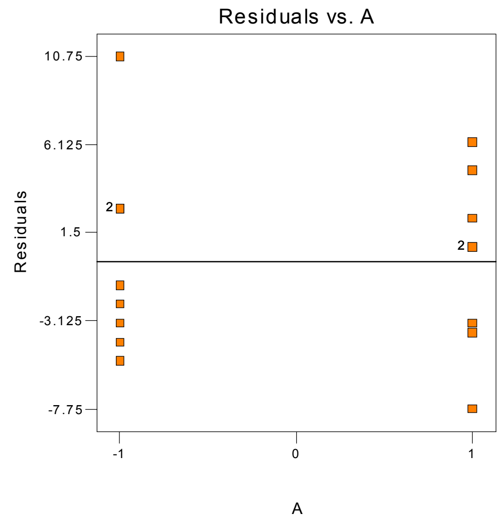
(c) Analyze the residuals and comment on your findings.

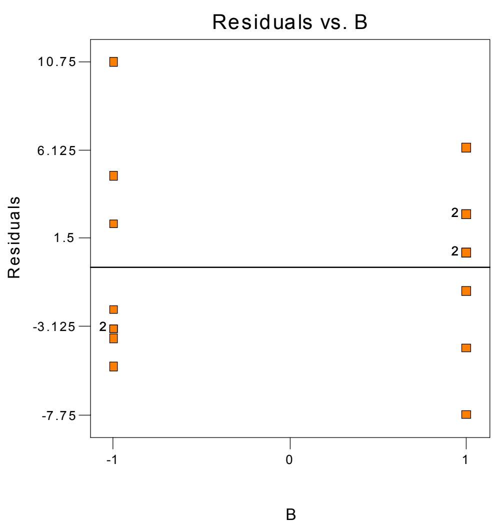
The residual plots are acceptable. The normality and equality of variance assumptions are verified. There does not appear to be any trends or interruptions in the residuals versus run order plot.

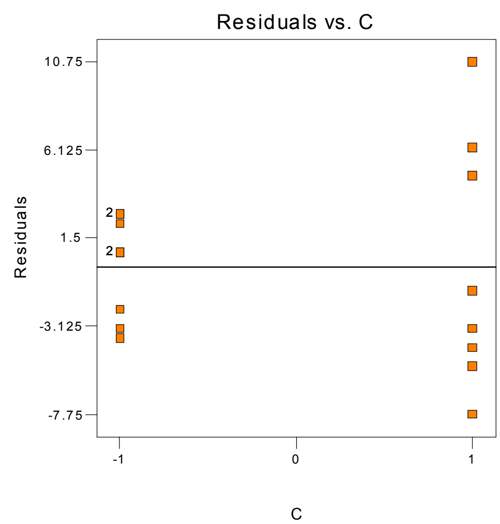


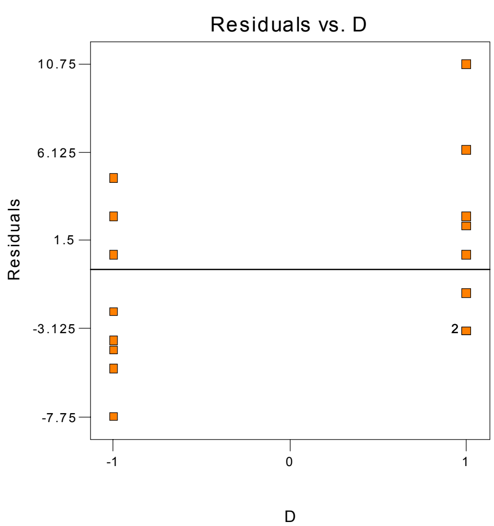


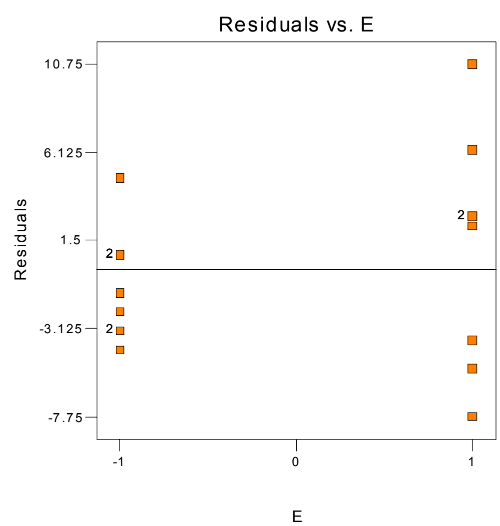

Residuals vs. Run
Residuals
10.75
6.125
1.5
-3.125
-7.75
1
4
7
10
13
16
Run Number

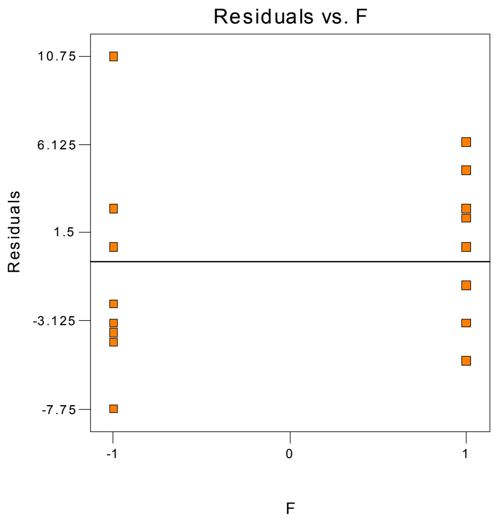

Residuals vs. A
Residuals
10.75
6.125
1.5
-3.125
-7.75
2
2
-1
0
1
A


Residuals vs. B
Residuals
10.75
6.125
1.5
-3.125
-7.75
2
2
2
-1
0
1
B


Residuals vs. C
Residuals
10.75
6.125
1.5
-3.125
-7.75
2
2
-1
0
1
C

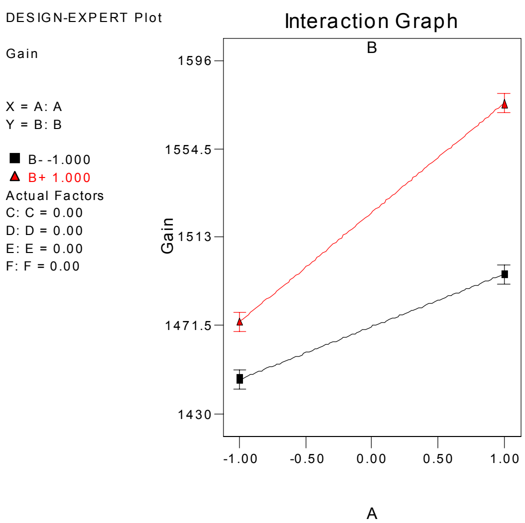

Residuals vs. D
Residuals
10.75
6.125
1.5
-3.125
-7.75
2
-1
0
1
D

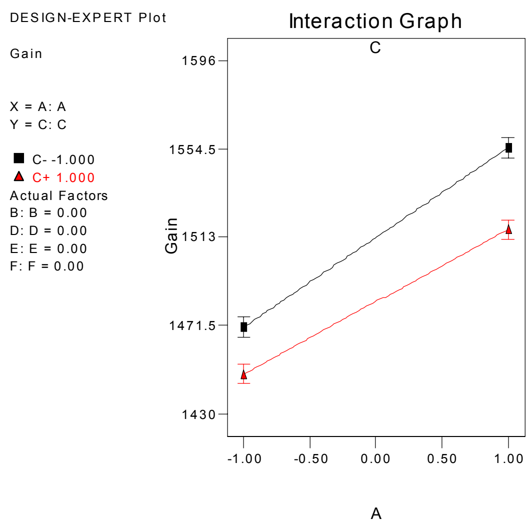

Residuals vs. E
Residuals
10.75
6.125
1.5
-3.125
-7.75
2
2
2
-1
0
1
E

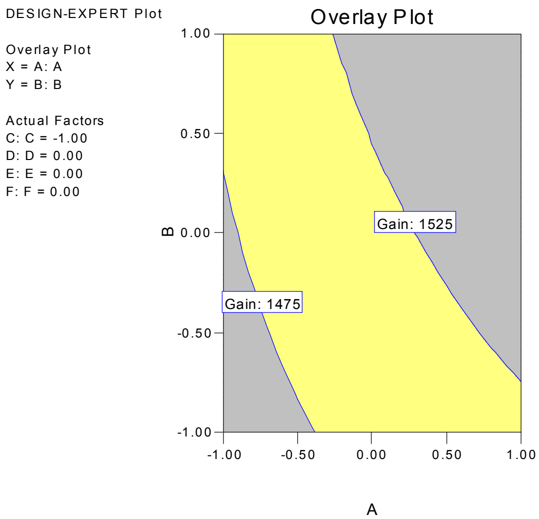


(d) Can you find a set of operating conditions that produce gain of 1500 ± 25 ?

Yes, see the graphs below.







8-31 Heat treating is often used to carbonize metal parts, such as gears. The thickness of the carbonized layer is a critical output variable from this process, and it is usually measured by performing a carbon analysis on the gear pitch (top of the gear tooth). Six factors were studied on a 2_{IV}^{6-2} design: A = furnace temperature, B = cycle time, C = carbon concentration, D = duration of the carbonizing cycle, E = carbon concentration of the diffuse cycle, and F = duration of the diffuse cycle. The experiment is shown below:

Standard Order	Run Order	*A*	*B*	*C*	*D*	*E*	*F*	Pitch
1	5	-	-	-	-	+	-	74
2	7	+	-	-	-	-	-	190
3	8	-	+	-	-	-	+	133
4	2	+	+	-	-	+	+	127
5	10	-	-	+	-	-	+	115
6	12	+	-	+	-	+	+	101
7	16	-	+	+	-	+	-	54
8	1	+	+	+	-	-	-	144
9	6	-	-	-	+	+	+	121
10	9	+	-	-	+	-	+	188
11	14	-	+	-	+	-	-	135
12	13	+	+	-	+	+	-	170
13	11	-	-	+	+	-	-	126
14	3	+	-	+	+	+	-	175
15	15	-	+	+	+	+	+	126
16	4	+	+	+	+	-	+	193

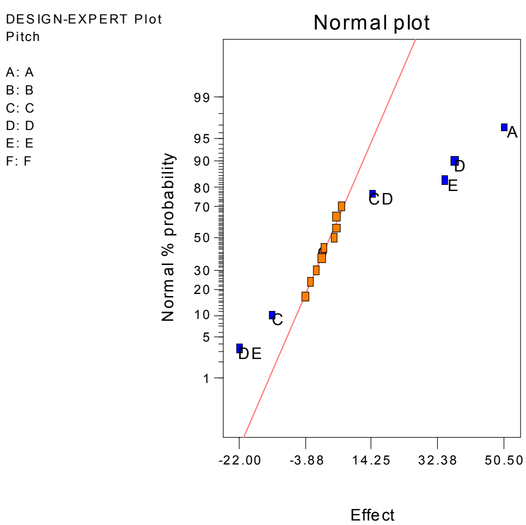
(a) Estimate the factor effects and plot them on a normal probability plot. Select a tentative model.

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	50.5	10201	41.8777
Error	B	-1	4	0.016421
Model	C	-13	676	2.77515
Model	D	37	5476	22.4804
Model	E	34.5	4761	19.5451
Error	F	4.5	81	0.332526
Error	AB	-4	64	0.262737
Error	AC	-2.5	25	0.102631
Error	AD	4	64	0.262737
Error	AE	1	4	0.016421

Error	BD	4.5	81	0.332526
Model	CD	14.5	841	3.45252
Model	DE	-22	1936	7.94778
Error	ABD	0.5	1	0.00410526
Error	ABF	6	144	0.591157
	Lenth's ME	15.4235		
	Lenth's SME	31.3119		

Factors *A*, *C*, *D*, *E* and the two factor interactions *CD* and *DE* appear to be significant. The model can be found in the Design Expert Output below.



(b) Perform appropriate statistical tests on the model.

Design Expert Output

Response: Pitch
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	23891.00	6	3981.83	76.57	< 0.0001	significant
A	*10201.00*	*1*	*10201.00*	*196.17*	*< 0.0001*	
C	*676.00*	*1*	*676.00*	*13.00*	*0.0057*	
D	*5476.00*	*1*	*5476.00*	*105.31*	*< 0.0001*	
E	*4761.00*	*1*	*4761.00*	*91.56*	*< 0.0001*	
CD	*841.00*	*1*	*841.00*	*16.17*	*0.0030*	
DE	*1936.00*	*1*	*1936.00*	*37.23*	*0.0002*	
Residual	468.00	9	52.00			
Cor Total	24359.00	15				

The Model F-value of 76.57 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	7.21	R-Squared	0.9808
Mean	135.75	Adj R-Squared	0.9680
C.V.	5.31	Pred R-Squared	0.9393
PRESS	1479.11	Adeq Precision	28.618

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	135.75	1	1.80	131.67	139.83	
A-A	25.25	1	1.80	21.17	29.33	1.00
C-C	-6.50	1	1.80	-10.58	-2.42	1.00
D-D	18.50	1	1.80	14.42	22.58	1.00

E-E	17.25	1	1.80	13.17	21.33	1.00
CD	7.25	1	1.80	3.17	11.33	1.00
DE	-11.00	1	1.80	-15.08	-6.92	1.00

Final Equation in Terms of Coded Factors:

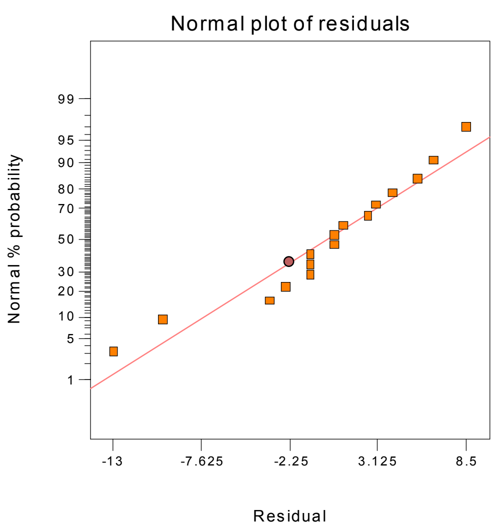
Pitch	=
+135.75	
+25.25	* A
-6.50	* C
+18.50	* D
+17.25	* E
+7.25	* C * D
-11.00	* D * E

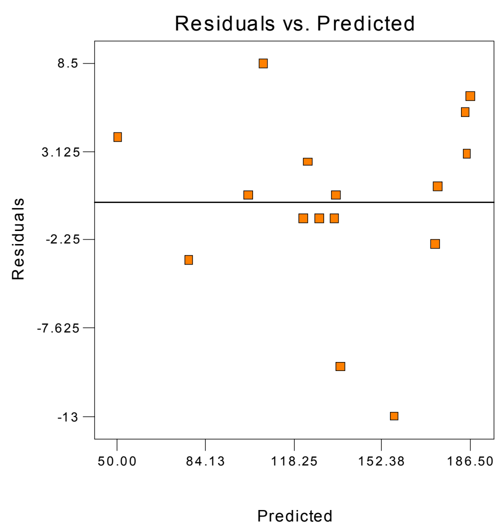
Final Equation in Terms of Actual Factors:

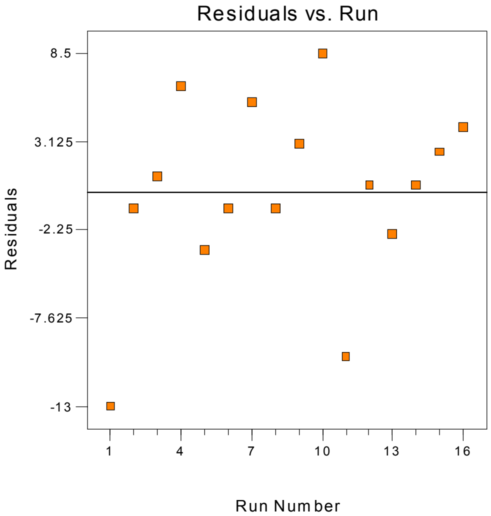
Pitch	=
+135.75000	
+25.25000	* A
-6.50000	* C
+18.50000	* D
+17.25000	* E
+7.25000	* C * D
-11.00000	* D * E

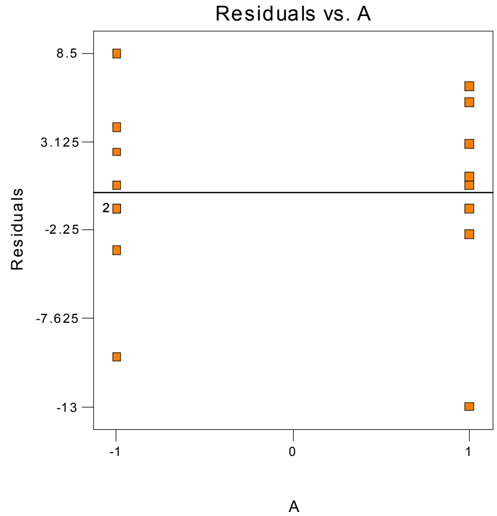
(c) Analyze the residuals and comment on model adequacy.

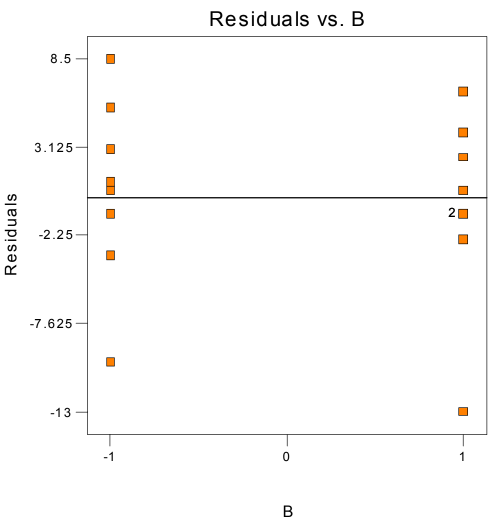
The residual plots are acceptable. The normality and equality of variance assumptions are verified. There does not appear to be any trends or interruptions in the residuals versus run order plot. The plots of the residuals versus factors *C* and *E* identify reduced variation at the lower level of both variables while the plot of residuals versus factor *F* identifies reduced variation at the upper level. Because *C* and *E* are significant factors in the model, this might not affect the decision on the optimum solution for the process. However, factor *F* is not included in the model and may be set at the upper level to reduce variation.

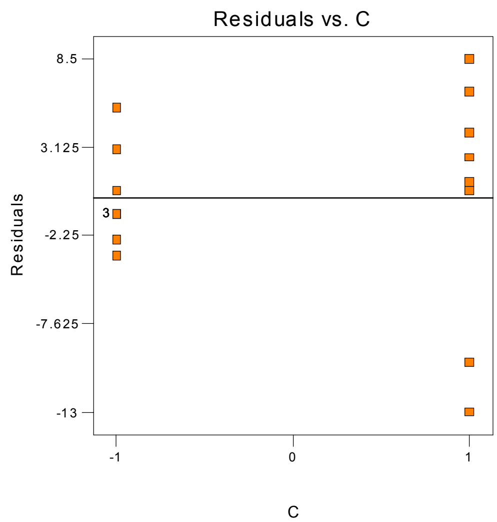


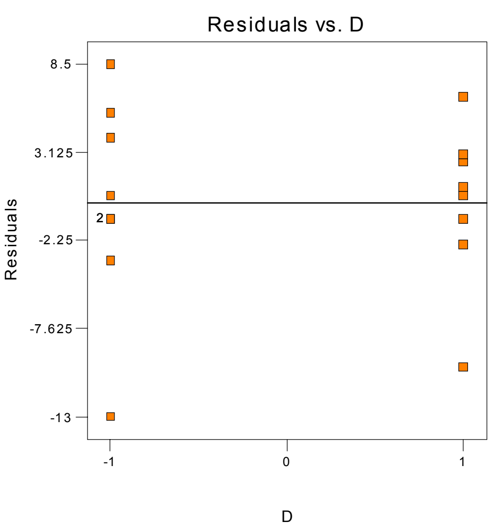


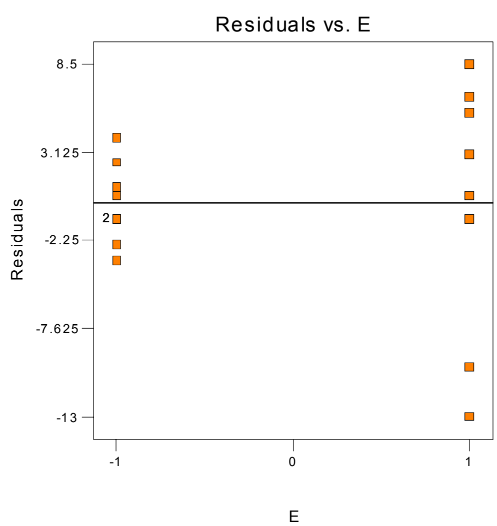

Residuals vs. Run
Residuals
8.5
3.125
-2.25
-7.625
-13
1
4
7
10
13
16
Run Number

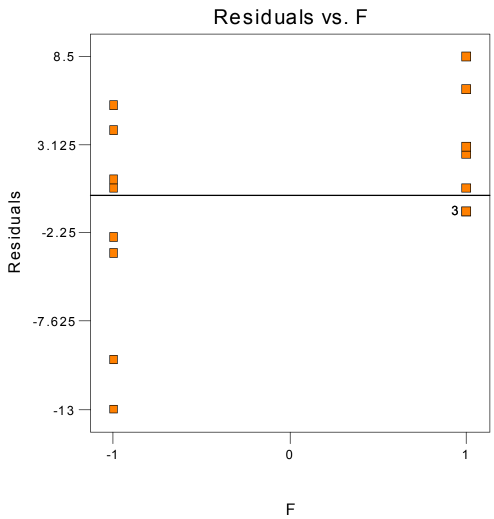

Residuals vs. A
Residuals
8.5
3.125
-2.25
-7.625
-13
2
-1
0
1
A


Residuals vs. B
Residuals
8.5
3.125
-2.25
-7.625
-13
2
-1
0
1
B


Residuals vs. C
Residuals
8.5
3.125
-2.25
-7.625
-13
3
-1
0
1
C

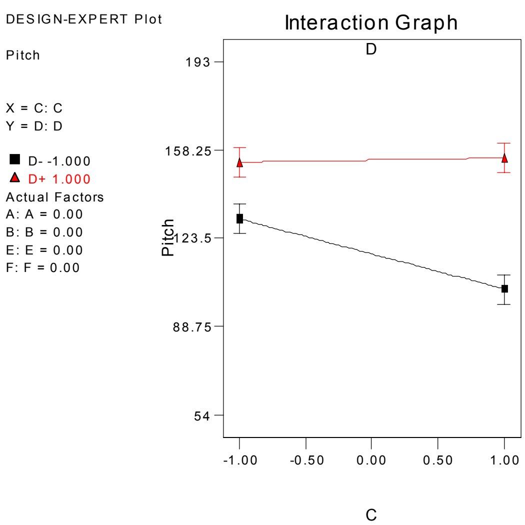

Residuals vs. D
Residuals
8.5
3.125
-2.25
-7.625
-13
2
-1
0
1
D

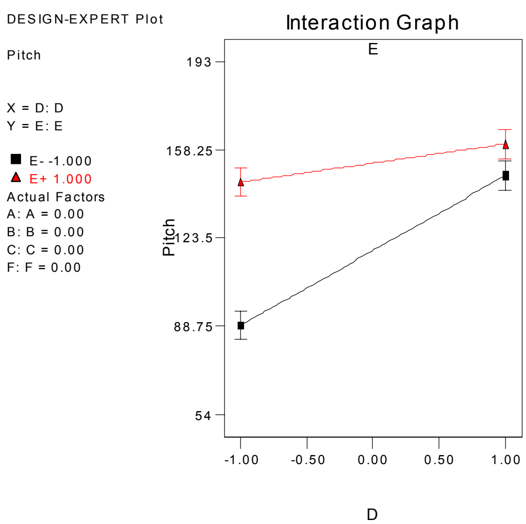

Residuals vs. E
Residuals
8.5
3.125
-2.25
-7.625
-13
2
-1
0
1
E

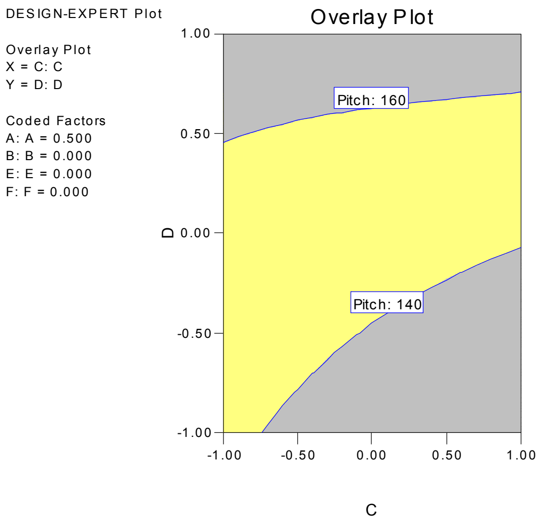


(d) Interpret the results of this experiment. Assume that a layer thickness of between 140 and 160 is desirable.

The graphs below identify a region that is acceptable between 140 and 160.







8-32 Five factors are studied in the irregular fractional factorial design of resolution V shown below.

Standard Order	Run Order	*A*	*B*	*C*	*D*	*E*	Gain
1	1	-	-	-	-	-	16.33
2	10	-	+	-	-	-	18.43
3	5	+	+	-	-	-	27.07
4	4	-	-	+	-	-	16.95
5	15	+	-	+	-	-	14.58
6	19	-	+	+	-	-	19.12
7	16	-	-	-	+	-	18.96
8	7	+	-	-	+	-	23.56
9	8	+	+	-	+	-	29.15
10	3	+	-	+	+	-	15.74
11	13	-	+	+	+	-	20.73
12	11	+	+	+	+	-	21.52
13	12	-	-	-	-	+	15.58
14	20	+	-	-	-	+	21.03
15	9	+	+	-	-	+	26.78
16	22	+	-	+	-	+	13.39
17	21	-	+	+	-	+	18.63
18	6	+	+	+	-	+	19.01
19	23	-	-	-	+	+	17.96
20	18	-	+	-	+	+	20.49
21	24	+	+	-	+	+	29.31
22	17	-	-	+	+	+	17.62
23	2	+	-	+	+	+	16.03
24	14	-	+	+	+	+	21.42

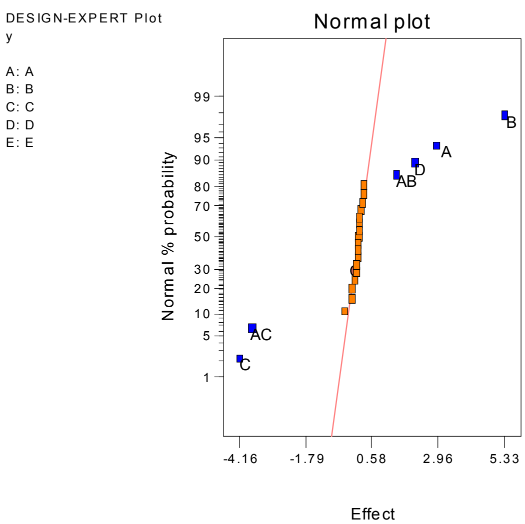
(a) Analyze the data from this experiment. What factors influence the response *y*?

Design Expert Output

	Term	Effect	SumSqr	% Contribtn
Model	Intercept			
Model	A	2.9125	50.8959	11.2736
Model	B	5.3275	170.294	37.7207
Model	C	-4.15917	103.792	22.9903
Model	D	2.1325	27.2853	6.04381
Error	E	-0.4075	0.996338	0.220693
Model	AB	1.45428	12.6896	2.8108
Model	AC	-3.71585	82.8451	18.3505
Error	AD	-0.0282843	0.0048	0.00106322

Error	AE	0.113137	0.0768	0.0170115
Error	BC	0.142887	7.5E-005	1.66128E-005
Error	BD	0.133172	0.102704	0.0227494
Error	BE	0.281664	0.710704	0.157424
Error	CD	-0.128458	0.0990083	0.0219307
Error	CE	0.0294628	0.00520833	0.00115367
Error	DE	0.291898	0.511225	0.113238
Error	ABC	-0.130639	0.264033	0.0584844
Error	ABD	0.067361	0.027225	0.00603044
Error	ABE	Aliased		
Error	ACD	0.189835	0.216225	0.0478947
Error	ACE	Aliased		
Error	ADE	0.102062	0.0625	0.013844
Error	BCD	0.155134	0.1444	0.0319852
Error	BCE	0.0898146	0.0484	0.0107208
Error	BDE	0.0408248	0.01	0.00221504
Error	CDE	0.251073	0.378225	0.0837783
	Lenth's ME	0.455325		
	Lenth's SME	0.881839		

Factors *A*, *B*, *C*, *D*, and the *AB* and *AC* interactions appear to be significant.



Design Expert Output

Response: y
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	447.80	6	74.63	346.86	< 0.0001	significant
A	*50.90*	*1*	*50.90*	*236.54*	*< 0.0001*	
B	*85.92*	*1*	*85.92*	*399.32*	*< 0.0001*	
C	*70.86*	*1*	*70.86*	*329.32*	*< 0.0001*	
D	*27.29*	*1*	*27.29*	*126.81*	*< 0.0001*	
AB	*12.69*	*1*	*12.69*	*58.98*	*< 0.0001*	
AC	*82.85*	*1*	*82.85*	*385.02*	*< 0.0001*	
Residual	3.66	17	0.22			
Cor Total	451.46	23				

The Model F-value of 346.86 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.46	R-Squared	0.9919
Mean	19.97	Adj R-Squared	0.9890
C.V.	2.32	Pred R-Squared	0.9832

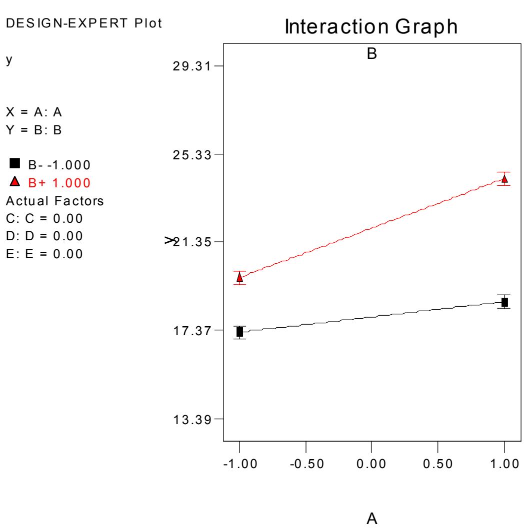
PRESS	7.60	Adeq Precision	60.974			
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	19.97	1	0.095	19.77	20.17	
A-A	1.46	1	0.095	1.26	1.66	1.00
B-B	2.01	1	0.10	1.79	2.22	1.13
C-C	-1.82	1	0.10	-2.03	-1.61	1.13
D-D	1.07	1	0.095	0.87	1.27	1.00
AB	0.77	1	0.10	0.56	0.98	1.12
AC	-1.97	1	0.10	-2.18	-1.76	1.12

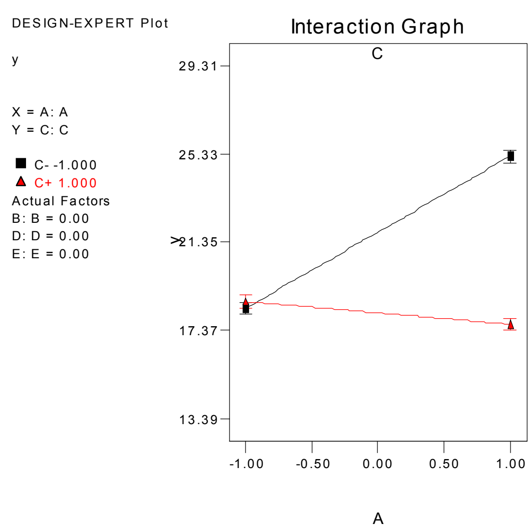
Final Equation in Terms of Coded Factors:

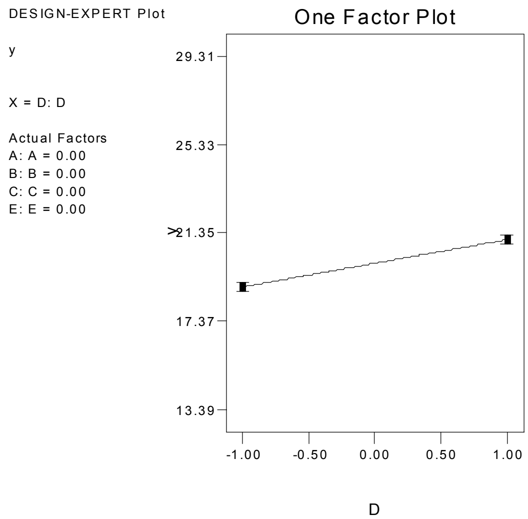
y	=
+19.97	
+1.46	* A
+2.01	* B
-1.82	* C
+1.07	* D
+0.77	* A * B
-1.97	* A * C

Final Equation in Terms of Actual Factors:

y	=
+19.97458	
+1.45625	* A
+2.00687	* B
-1.82250	* C
+1.06625	* D
+0.77125	* A * B
-1.97062	* A * C

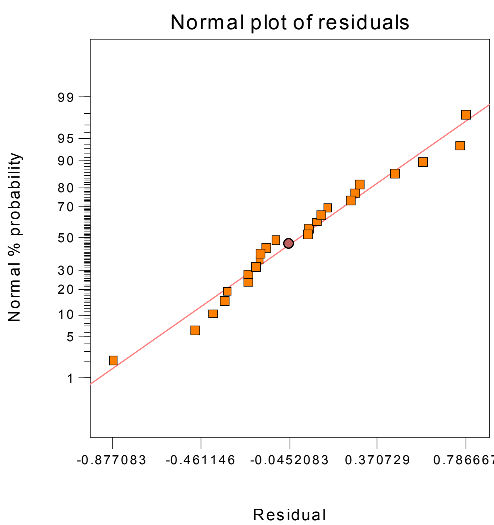


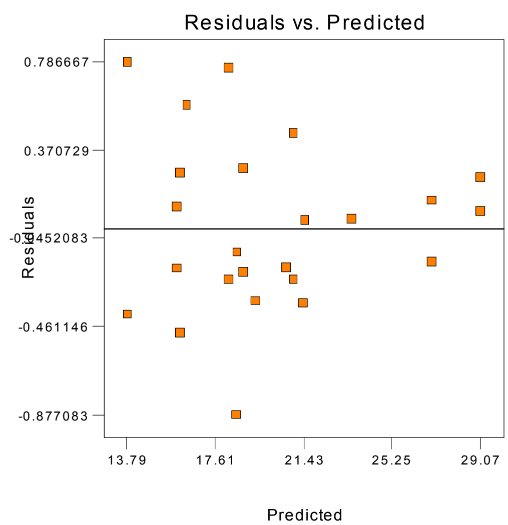


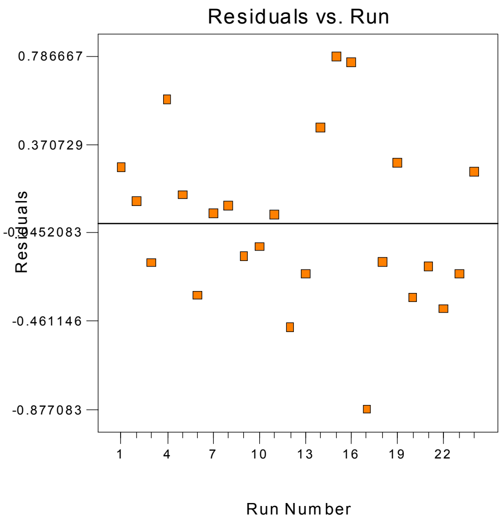


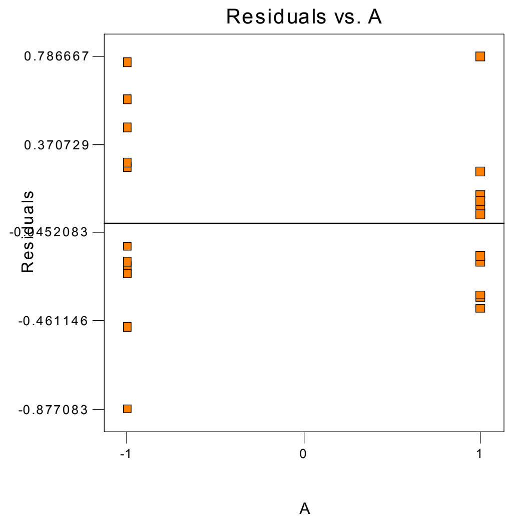
(b) Analyze the residuals. Comment on model adequacy.

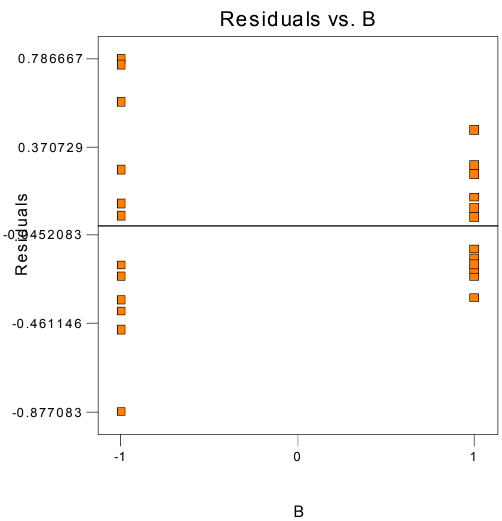
The residual plots are acceptable. The normality and equality of variance assumptions are verified. There does not appear to be any trends or interruptions in the residuals versus run order plot.

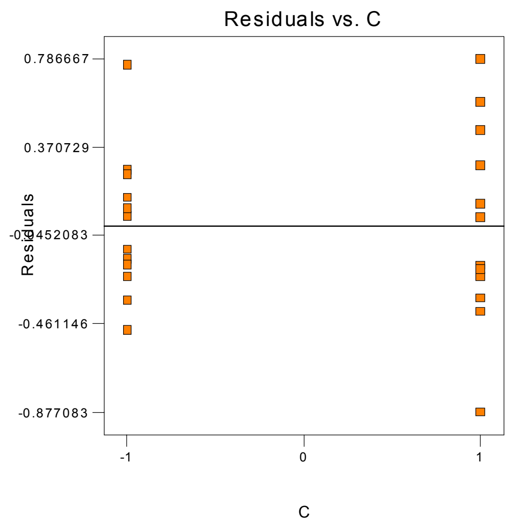


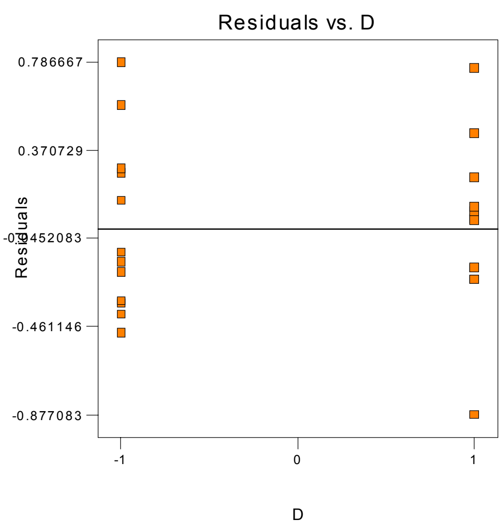


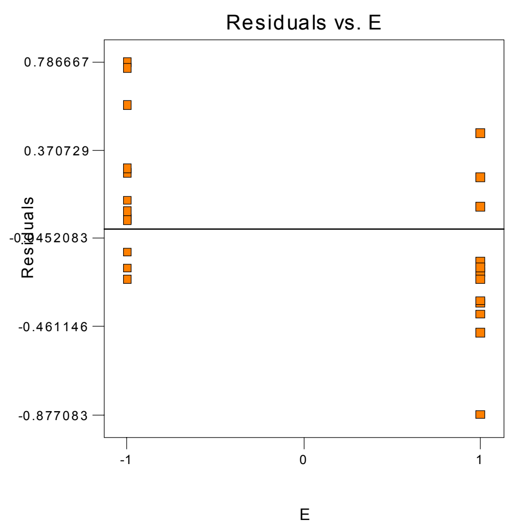

Residuals vs. Run
Residuals
0.786667
0.370729
-0.0452083
-0.461146
-0.877083
1
4
7
10
13
16
19
22
Run Number


Residuals vs. A
Residuals
0.786667
0.370729
-0.0452083
-0.461146
-0.877083
-1
0
1
A


Residuals vs. B
Residuals
0.786667
0.370729
-0.0452083
-0.461146
-0.877083
-1
0
1
B


Residuals vs. C
Residuals
0.786667
0.370729
-0.0452083
-0.461146
-0.877083
-1
0
1
C


Residuals vs. D
Residuals
0.786667
0.370729
-0.0452083
-0.461146
-0.877083
-1
0
1
D


Residuals vs. E
Residuals
0.786667
0.370729
-0.0452083
-0.461146
-0.877083
-1
0
1
E

Chapter 9
Three-Level and Mixed-Level Factorial and Fractional Factorial Design Solutions

9-1 The effects of developer concentration (A) and developer time (B) on the density of photographic plate film are being studied. Three strengths and three times are used, and four replicates of a 3^2 factorial experiment are run. The data from this experiment follow. Analyze the data using the standard methods for factorial experiments.

	Development Time (minutes)					
Developer Concentration	10		14		18	
10%	0	2	1	3	2	5
	5	4	4	2	4	6
12%	4	6	6	8	9	10
	7	5	7	7	8	5
14%	7	10	10	10	12	10
	8	7	8	7	9	8

Design Expert Output

Response: Data
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	224.22	8	28.03	10.66	< 0.0001	significant
A	*198.22*	*2*	*99.11*	*37.69*	*< 0.0001*	
B	*22.72*	*2*	*11.36*	*4.32*	*0.0236*	
AB	*3.28*	*4*	*0.82*	*0.31*	*0.8677*	
Residual	71.00	27	2.63			
Lack of Fit	*0.000*	*0*				
Pure Error	*71.00*	*27*	*2.63*			
Cor Total	295.22	35				

The Model F-value of 10.66 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

Concentration and time are significant. The interaction is not significant. By letting both A and B be treated as numerical factors, the analysis can be performed as follows:

Design Expert Output

Response: Data
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	221.01	5	44.20	17.87	< 0.0001	significant
A	*192.67*	*1*	*192.67*	*77.88*	*< 0.0001*	
B	*22.04*	*1*	*22.04*	*8.91*	*0.0056*	
A2	*5.56*	*1*	*5.56*	*2.25*	*0.1444*	
B2	*0.68*	*1*	*0.68*	*0.28*	*0.6038*	
AB	*0.062*	*1*	*0.062*	*0.025*	*0.8748*	
Residual	74.22	30	2.47			
Lack of Fit	*3.22*	*3*	*1.07*	*0.41*	*0.7488*	*not significant*
Pure Error	*71.00*	*27*	*2.63*			
Cor Total	295.22	35				

The Model F-value of 17.87 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

9-2 Compute the *I* and *J* components of the two-factor interaction in Problem 9-1.

	B		
A	11	10	17
	22	28	32
	32	35	39

$$AB \text{ Totals} = 77,\ 78,\ 71;\quad SS_{AB} = \frac{77^2 + 78^2 + 71^2}{12} - \frac{226^2}{36} = 2.39 = I(AB)$$

$$AB^2 \text{ Totals} = 78,\ 74,\ 74;\quad SS_{AB^2} = \frac{78^2 + 74^2 + 74^2}{12} - \frac{226^2}{36} = 0.89 = J(AB)$$

$$SS_{AB} = I(AB) + J(AB) = 3.28$$

9-3 An experiment was performed to study the effect of three different types of 32-ounce bottles (A) and three different shelf types (B) -- smooth permanent shelves, end-aisle displays with grilled shelves, and beverage coolers -- on the time it takes to stock ten 12-bottle cases on the shelves. Three workers (factor C) were employed in this experiment, and two replicates of a 3^3 factorial design were run. The observed time data are shown in the following table. Analyze the data and draw conclusions.

		Replicate I			Replicate 2		
Worker	Bottle Type	Permanent	EndAisle	Cooler	Permanent	EndAisle	Cooler
1	Plastic	3.45	4.14	5.80	3.36	4.19	5.23
	28-mm glass	4.07	4.38	5.48	3.52	4.26	4.85
	38-mm glass	4.20	4.26	5.67	3.68	4.37	5.58
2	Plastic	4.80	5.22	6.21	4.40	4.70	5.88
	28-mm glass	4.52	5.15	6.25	4.44	4.65	6.20
	38-mm glass	4.96	5.17	6.03	4.39	4.75	6.38
3	Plastic	4.08	3.94	5.14	3.65	4.08	4.49
	28-mm glass	4.30	4.53	4.99	4.04	4.08	4.59
	38-mm glass	4.17	4.86	4.85	3.88	4.48	4.90

Design Expert Output

Response: Time
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

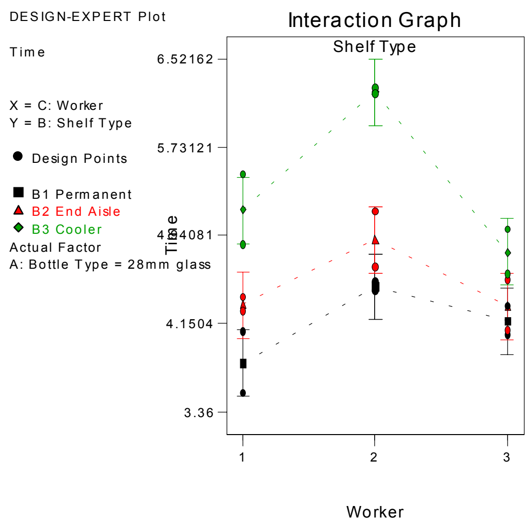
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	28.38	26	1.09	13.06	< 0.0001	significant
A	*0.33*	*2*	*0.16*	*1.95*	*0.1618*	
B	*17.91*	*2*	*8.95*	*107.10*	*< 0.0001*	
C	*7.91*	*2*	*3.96*	*47.33*	*< 0.0001*	
AB	*0.11*	*4*	*0.027*	*0.33*	*0.8583*	
AC	*0.11*	*4*	*0.027*	*0.32*	*0.8638*	
BC	*1.59*	*4*	*0.40*	*4.76*	*0.0049*	
ABC	*0.43*	*8*	*0.053*	*0.64*	*0.7380*	
Residual	2.26	27	0.084			
Lack of Fit	*0.000*	*0*				
Pure Error	*2.26*	*27*	*0.084*			
Cor Total	30.64	53				

The Model F-value of 13.06 implies the model is significant. There is only

a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B, C, BC are significant model terms.

Factors *B* and *C*, shelf type and worker, and the *BC* interaction are significant. For the shortest time regardless of worker chose the permanent shelves. This can easily be seen in the interaction plot below.



9-4 A medical researcher is studying the effect of lidocaine on the enzyme level in the heart muscle of beagle dogs. Three different commercial brands of lidocaine (*A*), three dosage levels (*B*), and three dogs (*C*) are used in the experiment, and two replicates of a 3^3 factorial design are run. The observed enzyme levels follow. Analyze the data from this experiment.

		Replicate I			Replicate 2		
Lidocaine	Dosage	Dog			Dog		
Brand	Strength	1	2	3	1	2	3
1	1	86	84	85	84	85	86
	2	94	99	98	95	97	90
	3	101	106	98	105	104	103
2	1	85	84	86	80	82	84
	2	95	98	97	93	99	95
	3	108	114	109	110	102	100
3	1	84	83	81	83	80	79
	2	95	97	93	92	96	93
	3	105	100	106	102	111	108

Design Expert Output

Response: Enzyme Level
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	4490.33	26	172.71	16.99	< 0.0001	significant
A	*31.00*	*2*	*15.50*	*1.52*	*0.2359*	
B	*4260.78*	*2*	*2130.39*	*209.55*	*< 0.0001*	
C	*28.00*	*2*	*14.00*	*1.38*	*0.2695*	
AB	*69.56*	*4*	*17.39*	*1.71*	*0.1768*	
AC	*3.33*	*4*	*0.83*	*0.082*	*0.9872*	
BC	*36.89*	*4*	*9.22*	*0.91*	*0.4738*	
ABC	*60.78*	*8*	*7.60*	*0.75*	*0.6502*	

Residual	274.50	27	10.17
Lack of Fit	*0.000*	*0*	
Pure Error	*274.50*	*27*	*10.17*
Cor Total	4764.83	53	

The Model F-value of 16.99 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case B are significant model terms.

The dosage is significant.

9-5 Compute the *I* and *J* components of the two-factor interactions for Example 9-1.

	A		
B	134	188	44
	-155	-348	-289
	176	127	288

I totals = 74,75,16 J totals = -128,321,-28

I(*AB*) = 126.78 J(*AB*) = 6174.12

$SS_{AB} = 6300.90$

	A		
C	-190	-58	-211
	399	230	394
	6	-205	-140

I totals = -100,342,-77 J totals = 25,141,-1

I(*AC*) = 6878.78 J(*AC*) = 635.12

$SS_{AC} = 7513.90$

	B		
C	-93	-350	-16
	-155	-133	533
	-104	-309	74

I totals = -152,79,238 J totals =-253,287,131

I(*BC*) = 4273.00 J(*BC*) = 8581.34

$SS_{BC} = 12854.34$

9-6 An experiment is run in a chemical process using a 3^2 factorial design. The design factors are temperature and pressure, and the response variable is yield. The data that result from this experiment are shown below.

	Pressure, psig		
Temperature, °C	100	120	140
80	47.58, 48.77	64.97, 69.22	80.92, 72.60
90	51.86, 82.43	88.47, 84.23	93.95, 88.54
100	71.18, 92.77	96.57, 88.72	76.58, 83.04

(a) Analyze the data from this experiment by conducting an analysis of variance. What conclusions can you draw?

Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3187.13	8	398.39	4.37	0.0205	significant
A	*1096.93*	*2*	*548.47*	*6.02*	*0.0219*	
B	*1503.56*	*2*	*751.78*	*8.25*	*0.0092*	
AB	*586.64*	*4*	*146.66*	*1.61*	*0.2536*	
Pure Error	819.98	9	91.11			
Cor Total	4007.10	17				

The Model F-value of 4.37 implies the model is significant. There is only
a 2.05% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

Temperature and pressure are significant. Their interaction is not. An alternate analysis is performed below with the *A* and *B* treated as numeric factors:

Design Expert Output

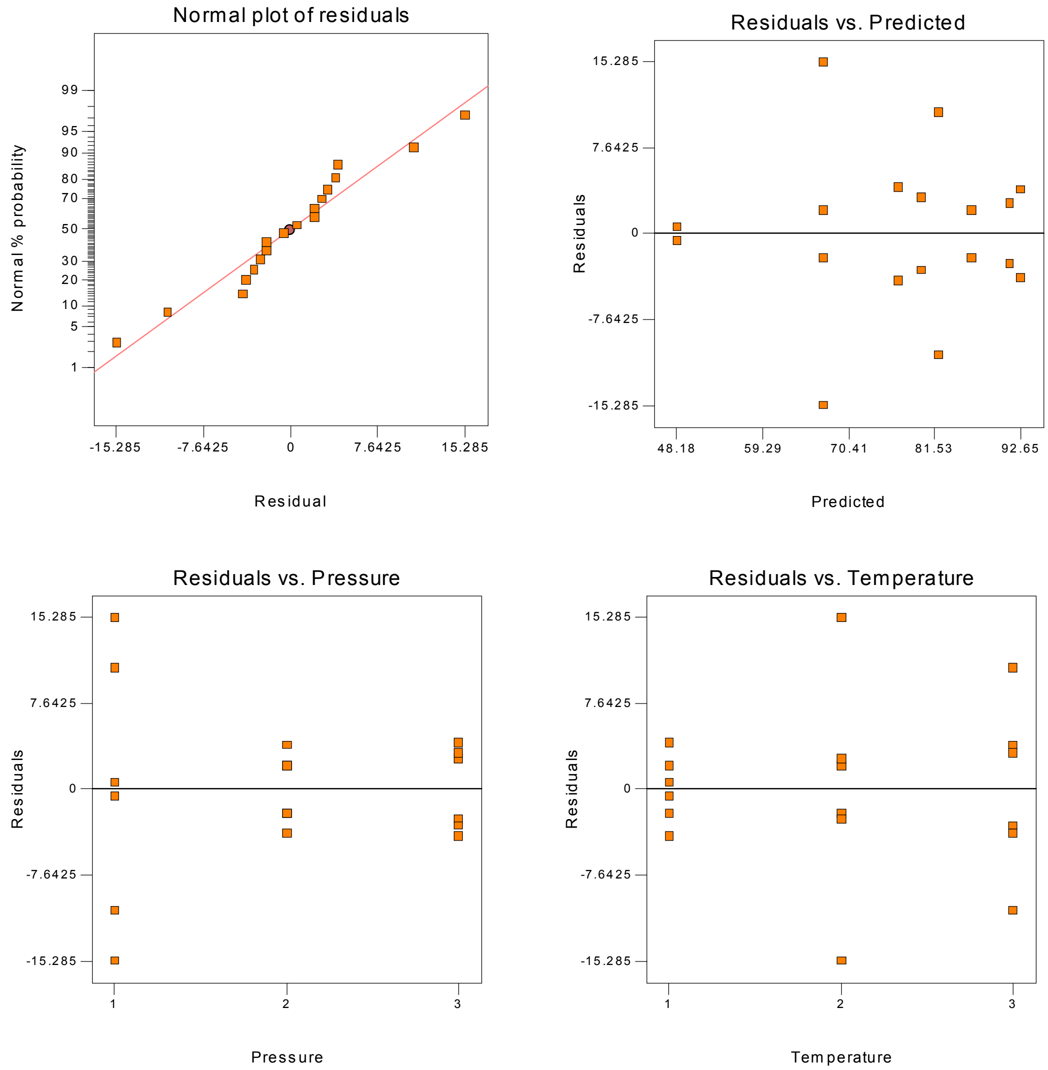
Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3073.27	5	614.65	7.90	0.0017	significant
A	*850.76*	*1*	*850.76*	*10.93*	*0.0063*	
B	*1297.92*	*1*	*1297.92*	*16.68*	*0.0015*	
A2	*246.18*	*1*	*246.18*	*3.16*	*0.1006*	
B2	*205.64*	*1*	*205.64*	*2.64*	*0.1300*	
AB	*472.78*	*1*	*472.78*	*6.08*	*0.0298*	
Residual	933.83	12	77.82			
Lack of Fit	*113.86*	*3*	*37.95*	*0.42*	*0.7454*	*not significant*
Pure Error	*819.98*	*9*	*91.11*			
Cor Total	4007.10	17				

The Model F-value of 7.90 implies the model is significant. There is only
a 0.17% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

(b) Graphically analyze the residuals. Are there any concerns about underlying assumptions or model adequacy?



The plot of residuals versus pressure shows a decreasing funnel shape indicating a non-constant variance.

(c) Verify that if we let the low, medium and high levels of both factors in this experiment take on the levels -1, 0, and +1, then a least squares fit to a second order model for yield is

$$\hat{y} = 86.81 + 10.4x_1 + 8.42x_2 - 7.17x_1^2 - 7.86x_2^2 - 7.69x_1x_2$$

The coefficients can be found in the following table of computer output.

Design Expert Output

Final Equation in Terms of Coded Factors:

Yield =	
+86.81	
+8.42	* A
+10.40	* B
-7.84	* A^2
-7.17	* B^2
-7.69	* A * B

(d) Confirm that the model in part (c) can be written in terms of the natural variables temperature (T) and pressure (P) as

$$\hat{y} = -1335.63 + 18.56T + 8.59P - 0.072T^2 - 0.0196P^2 - 0.0384TP$$

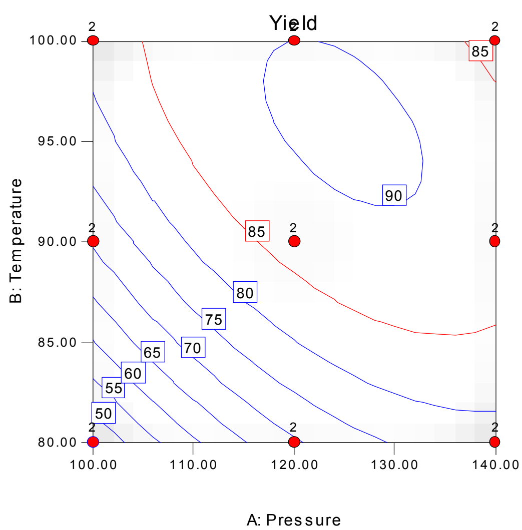
The coefficients can be found in the following table of computer output.

Design Expert Output

Final Equation in Terms of Actual Factors:

Yield	=
-1335.62500	
+8.58737	* Pressure
+18.55850	* Temperature
-0.019612	* Pressure^2
-0.071700	* Temperature^2
-0.038437	* Pressure * Temperature

(e) Construct a contour plot for yield as a function of pressure and temperature. Based on the examination of this plot, where would you recommend running the process.



Run the process in the oval region indicated by the yield of 90.

9-7

(a) Confound a 3^3 design in three blocks using the ABC^2 component of the three-factor interaction. Compare your results with the design in Figure 9-7.

$$L = X_1 + X_2 + 2X_3$$

Block 1	Block 2	Block 3
000	100	200
112	212	012
210	010	110

120	220	020
022	122	222
202	002	102
221	021	121
101	201	001
011	111	211

The new design is a 180° rotation around the Factor *B* axis.

(b) Confound a 3^3 design in three blocks using the AB^2C component of the three-factor interaction. Compare your results with the design in Figure 9-7.

$$L = X_1 + 2X_2 + X_3$$

Block 1	Block 2	Block 3
000	210	112
022	202	120
011	221	101
212	100	010
220	122	002
201	111	021
110	012	200
102	020	222
121	001	211

The new design is a 180° rotation around the Factor *C* axis.

(c) Confound a 3^3 design three blocks using the *ABC* component of the three-factor interaction. Compare your results with the design in Figure 9-7.

$$L = X_1 + X_2 + X_3$$

Block 1	Block 2	Block 3
000	112	221
210	022	101
120	202	011
021	100	212
201	010	122
111	220	002
012	121	200
222	001	110
102	211	020

The new design is a 90° rotation around the Factor *C* axis along with switching layer 0 and layer 1 in the *C* axis.

(d) After looking at the designs in parts (a), (b), and (c) and Figure 9-7, what conclusions can you draw?

All four designs are relatively the same. The only differences are rotations and swapping of layers.

9-8 Confound a 3^4 design in three blocks using the AB^2CD component of the four-factor interaction.

$$L = X_1 + 2X_2 + X_3 + X_4$$

Block 1								
0000	1100	0110	0101	2200	0220	0202	1210	1201
0211	1222	2212	2221	0122	2111	1121	1112	2010
2102	0021	2001	2120	1011	2022	0012	1002	1020

Block 2								
1021	1110	1202	0001	0120	0212	1012	1101	1220
0200	0022	0111	2002	2121	2210	0010	0102	0221
1000	1122	1211	2112	2201	2020	2011	2100	2222

Block 3								
2012	2101	2220	1022	1111	1200	2000	2121	2211
1221	1010	1102	0020	0112	0201	1001	1120	1212
2021	2110	2202	0100	0222	0011	0002	0121	0210

9-9 Consider the data from the first replicate of Problem 9-3. Assuming that all 27 observations could not be run on the same day, set up a design for conducting the experiment over three days with AB^2C confounded with blocks. Analyze the data.

		Block 1			Block 2			Block 3
000	=	3.45	100	=	4.07	200	=	4.20
110	=	4.38	210	=	4.26	010	=	4.14
011	=	5.22	111	=	4.14	211	=	5.17
102	=	4.30	202	=	4.17	002	=	4.08
201	=	4.96	001	=	4.80	101	=	4.52
212	=	4.86	012	=	3.94	112	=	4.53
121	=	6.25	221	=	4.99	021	=	6.21
022	=	5.14	122	=	6.03	222	=	4.85
220	=	5.67	020	=	5.80	120	=	5.48
Totals->	=	44.23			43.21			43.18

Design Expert Output

Response: Time

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.23	2	0.11			
Model	13.17	18	0.73	4.27	0.0404	significant
A	*0.048*	*2*	*0.024*	*0.14*	*0.8723*	
B	*8.92*	*2*	*4.46*	*26.02*	*0.0011*	
C	*1.57*	*2*	*0.78*	*4.57*	*0.0622*	
AB	*1.31*	*4*	*0.33*	*1.91*	*0.2284*	
AC	*0.87*	*4*	*0.22*	*1.27*	*0.3774*	
BC	*0.45*	*4*	*0.11*	*0.66*	*0.6410*	
Residual	1.03	6	0.17			
Cor Total	14.43	26				

The Model F-value of 4.27 implies the model is significant. There is only a 4.04% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case B are significant model terms.

9-10 Outline the analysis of variance table for the 3^4 design in nine blocks. Is this a practical design?

Source	DF
A	2
B	2
C	2
D	2
AB	4
AC	4
AD	4
BC	4
BD	4
CD	4
$ABC\ (AB^2C, ABC^2, AB^2C^2)$	6
$ABD\ (ABD, AB^2D, ABD^2)$	6
$ACD\ (ACD, ACD^2, AC^2D^2)$	6
$BCD\ (BCD, BC^2D, BCD^2)$	6
$ABCD$	16
Blocks ($ABC, AB^2C^2, AC^2D, BC^2D^2$)	8
Total	80

Any experiment with 81 runs is large. Instead of having three full levels of each factor, if two levels of each factor could be used, then the overall design would have 16 runs plus some center points. This two-level design could now probably be run in 2 or 4 blocks, with center points in each block. Additional curvature effects could be determined by augmenting the experiment with the axial points of a central composite design and additional enter points. The overall design would be less than 81 runs.

9-11 Consider the data in Problem 9-3. If ABC is confounded in replicate I and ABC^2 is confounded in replicate II, perform the analysis of variance.

$L_1 = X_1 + X_2 + X_3$			$L_2 = X_1 + X_2 + 2X_2$		
Block 1	Block 2	Block 3	Block 1	Block 2	Block 3
000 = 3.45	001 = 4.80	002 = 4.08	000 = 3.36	100 = 3.52	200 = 3.68
111 = 5.15	112 = 4.53	110 = 4.38	101 = 4.44	201 = 4.39	001 = 4.40
222 = 4.85	220 = 5.67	221 = 6.03	011 = 4.70	111 = 4.65	211 = 4.75
120 = 5.48	121 = 6.25	122 = 4.99	221 = 6.38	021 = 5.88	121 = 6.20
102 = 4.30	100 = 4.07	101 = 4.52	202 = 3.88	002 = 3.65	102 = 4.04
210 = 4.26	211 = 5.17	212 = 4.86	022 = 4.49	122 = 4.59	222 = 4.90
201 = 4.96	202 = 4.17	200 = 4.20	120 = 4.85	220 = 5.58	020 = 5.23
012 = 3.94	010 = 4.14	011 = 5.22	210 = 4.37	010 = 4.19	110 = 4.26
021 = 6.21	022 = 5.14	020 = 5.80	112 = 4.08	212 = 4.48	012 = 4.08

The sums of squares for A, B, C, AB, AC, and BC are calculated as usual. The only sums of squares presenting difficulties with calculations are the four components of the ABC interaction (ABC, ABC^2, AB^2C, and AB^2C^2). ABC is computed using replicate I and ABC^2 is computed using replicate II. AB^2C and AB^2C^2 are computed using data from both replicates.

We will show how to calculate AB^2C and AB^2C^2 from both replicates. Form a two-way table of A x B at each level of C. Find the I(AB) and J(AB) totals for each third of the A x B table.

		A				
C	B	0	1	2	I	J
	0	6.81	7.59	7.88	26.70	27.55

0	1	8.33	8.64	8.63		27.25	27.17
	2	11.03	10.33	11.25		26.54	25.77
	0	9.20	8.96	9.35		31.41	31.25
1	1	9.92	9.80	9.92		30.97	31.29
	2	12.09	12.45	12.41		31.72	31.57
	0	7.73	8.34	8.05		26.09	26.29
2	1	8.02	8.61	9.34		27.31	26.11
	2	9.63	9.58	9.75		25.65	26.65

The I and J components for each third of the above table are used to form a new table of diagonal totals.

C		I(*AB*)			J(*AB*)	
0	2.670	27.25	26.54	27.55	27.17	25.77
1	31.41	30.97	31.72	31.25	31.29	31.57
2	26.09	27.31	25.65	26.29	26.11	26.65

I Totals: 85.06,85.26,83.32	I Totals: 85.99,85.03,83.12
J Totals: 85.73,83.60,84.31	J Totals: 83.35,85.06,85.23

Now, $AB^2C^2 = I[C \text{ x } I(AB)] = \dfrac{(85.06)^2 + (85.26)^2 + (83.32)^2}{18} - \dfrac{(253.64)^2}{54} = 0.1265$

and, $AB^2C = J[C \text{ x } I(AB)] = \dfrac{(85.73)^2 + (83.60)^2 + (84.31)^2}{18} - \dfrac{(253.64)^2}{54} = 0.1307$

If it were necessary, we could find ABC^2 as $ABC^2 = I[C \text{ x } J(AB)]$ and ABC as $J[C \text{ x } J(AB)]$. However, these components must be computed using the data from the appropriate replicate.

The analysis of variance table:

Source	SS	DF	MS	F_0
Replicates	1.06696	1		
Blocks within Replicates	0.2038	4		
A	0.4104	2	0.2052	5.02
B	17.7514	2	8.8757	217.0
C	7.6631	2	3.8316	93.68
AB	0.1161	4	0.0290	<1
AC	0.1093	4	0.0273	<1
BC	1.6790	4	0.4198	10.26
ABC (rep I)	0.0452	2	0.0226	<1
ABC^2 (rep II)	0.1020	2	0.0510	1.25
AB^2C	0.1307	2	0.0754	1.60
AB^2C^2	0.1265	2	0.0633	1.55
Error	0.8998	22	0.0409	
Total	30.3069	53		

9-12 Consider the data from replicate I in Problem 9-3. Suppose that only a one-third fraction of this design with $I=ABC$ is run. Construct the design, determine the alias structure, and analyze the design.

The design is 000, 012, 021, 102, 201, 111, 120, 210, 222.

The alias structure is: $A = BC = AB^2C^2$

$B = AC = AB^2C$

$C = AB = ABC^2$

$AB^2 = AC^2 = BC^2$

			C	
A	B	0	1	2
	0	3.45		
0	1			5.48
	2		4.26	
	0			6.21
1	1		5.15	
	2	4.96		
	0		3.94	
2	1	4.30		
	2			4.85

Source	SS	DF
A	2.25	2
B	0.30	2
C	2.81	2
AB^2	0.30	2
Total	5.66	8

9-13 From examining Figure 9-9, what type of design would remain if after completing the first 9 runs, one of the three factors could be dropped?

A full 3^2 factorial design results.

9-14 Construct a 3_{IV}^{4-1} design with $I=ABCD$. Write out the alias structure for this design.

The 27 runs for this design are as follows:

0000	1002	2001
0012	1011	2010
0021	1020	2022
0102	1101	2100
0111	1110	2112
0120	1122	2121
0201	1200	2202
0210	1212	2211
0222	1221	2220

$A = AB^2C^2D^2 = BCD$ $\quad B = AB^2CD = ACD$ $\quad C = ABC^2D = ABD$ $\quad D = ABCD^2 = ABC$

$AB = ABC^2D^2 = CD$ $\quad AB^2 = AC^2D^2 = BC^2D^2$ $\quad AC = AB^2CD^2 = BD$ $\quad AC^2 = AB^2D^2 = BC^2D$

$BC = AB^2C^2D = AD$ $\quad BC^2 = AB^2D = AC^2D$ $\quad BD^2 = AB^2C = ACD^2$ $\quad CD^2 = ABC^2 = ABCD^2$

$AD^2 = AB^2C^2 = BCD^2$

9-15 Verify that the design in Problem 9-14 is a resolution IV design.

The design in Problem 9-14 is a Resolution IV design because no main effect is aliased with a component of a two-factor interaction, but some two-factor interaction components are aliased with each other.

9-16 Construct a 3^{5-2} design with $I=ABC$ and $I=CDE$. Write out the alias structure for this design. What is the resolution of this design?

The complete defining relation for this design is : $I = ABC = CDE = ABC^2DE = ABD^2E^2$
This is a resolution III design. The defining contrasts are $L_1 = X_1 + X_2 + X_3$ and $L_2 = X_3 + X_4 + X_5$.

00000	11120	20111
00012	22111	22222
00022	21021	01210
01200	02111	12000
02100	01222	20120
10202	12012	11111
20101	02120	22201
11102	10210	21012
21200	12021	10222

To find the alias of any effect, multiply the effect by I and I^2. For example, the alias of A is:

$A = AB^2C^2 = ACDE = AB^2CDE = AB^2DE = BC = AC^2D^2E^2 = BC^2DE = BD^2E^2$

9-17 Construct a 3^{9-6} design, and verify that is a resolution III design.

Use the generators $I = AC^2D^2$, $I = AB^2C^2E$, $I = BC^2F^2$, $I = AB^2CG$, $I = ABCH^2$, and $I = ABJ^2$

000000000	021201102	102211001
022110012	212012020	001212210
011220021	100120211	211100110
221111221	122200220	020022222
210221200	010011111	222020101
202001212	201122002	200210122
112222112	002121120	121021010
101002121	111010202	110101022
120112100	220202011	012102201

To find the alias of any effect, multiply the effect by I and I^2. For example, the alias of C is:

$C = C(BC^2F^2) = BF^2$, At least one main effect is aliased with a component of a two-factor interaction.

9-18 Construct a 4 x 2^3 design confounded in two blocks of 16 observations each. Outline the analysis of variance for this design.

Design is a 4 x 2^3, with ABC at two levels, and Z at 4 levels. Represent Z with two pseudo-factors D and E as follows:

Factor	Pseudo-	Factors
Z	*D*	*E*
Z_1	0	0 = (1)
Z_2	1	0 = *d*
Z_3	0	1 = *e*
Z_4	1	1 = *de*

The 4 x 2^3 is now a 2^5 in the factors *A, B, C, D* and *E*. Confound *ABCDE* with blocks. We have given both the letter notation and the digital notation for the treatment combinations.

Block 1			Block 2		
(1)	=	000	*a*	=	1000
ab	=	1100	*b*	=	0100
ac	=	1010	*c*	=	0010
bc	=	0110	*abc*	=	1110
abcd	=	1111	*bcd*	=	0111
abce	=	1112	*bce*	=	0112
cd	=	0011	*acd*	=	1011
ce	=	0012	*ace*	=	1012
de	=	0003	*ade*	=	1003
abde	=	1103	*bde*	=	0103
bcde	=	0113	*abcde*	=	1113
be	=	0102	*abd*	=	1101
ad	=	1001	*abe*	=	1102
ae	=	1002	*d*	=	0001
acde	=	1013	*e*	=	0002
bd	=	0101	*cde*	=	0013

Source	DF
A	1
B	1
C	1
Z (D+E+DE)	3
AB	1
AC	1
AZ (AD+AE+ADE)	3
BC	1
BZ (BD+BE+BDE)	3
CZ (CD+CE+CDE)	3
ABC	1
ABZ (ABD+ABE+ABDE)	3
ACZ (ACD+ACE+ACDE)	3
BCZ (BCD+BCE+BCDE)	3
ABCZ (ABCD+ABCE)	2
Blocks (or *ABCDE*)	1
Total	31

9-19 Outline the analysis of variance table for a $2^2 3^2$ factorial design. Discuss how this design may be confounded in blocks.

Suppose we have *n* replicates of a $2^2 3^2$ factorial design. *A* and *B* are at 2 levels, and *C* and *D* are at 3 levels.

Source	DF	Components for Confounding
A	1	*A*
B	1	*B*
C	2	*C*
D	2	*D*
AB	1	*AB*
AC	2	*AC*
AD	2	*AD*
BC	2	*BD*
BD	2	CD, CD^2
CD	4	*ABC*
ABC	2	*ABD*
ABD	2	ACD, ACD^2
ACD	4	BCD, BCD^2
BCD	4	$ABCD, ABCD^2$
ABCD	4	
Error	36(n-1)	
Total	36n-1	

Confounding in this series of designs is discussed extensively by Margolin (1967). The possibilities for a single replicate of the $2^2 3^2$ design are:

2 blocks of 18 observations	6 blocks of 6 observations
3 blocks of 12 observations	9 blocks of 4 observations
4 blocks of 9 observations	

For example, one component of the four-factor interaction, say $ABCD^2$, could be selected to confound the design in 3 blocks for 12 observations each, while to confound the design in 2 blocks of 18 observations 3 each we would select the *AB* interaction. Cochran and Cox (1957) and Anderson and McLean (1974) discuss confounding in these designs.

9-20 Starting with a 16-run 2^4 design, show how two three-level factors can be incorporated in this experiment. How many two-level factors can be included if we want some information on two-factor interactions?

Use column *A* and *B* for one three-level factor and columns *C* and *D* for the other. Use the *AC* and *BD* columns for the two, two-level factors. The design will be of resolution V.

9-21 Starting with a 16-run 2^4 design, show how one three-level factor and three two-level factors can be accommodated and still allow the estimation of two-factor interactions.

Use columns *A* and *B* for the three-level factor, and columns *C* and *D* and *ABCD* for the three two-level factors. This design will be of resolution V.

9-22 In Problem 9-26, you met Harry and Judy Peterson-Nedry, two friends of the author who have a winery and vineyard in Newberg, Oregon. That problem described the application of two-level fractional factorial designs to their 1985 Pinor Noir product. In 1987, they wanted to conduct another Pinot Noir experiment. The variables for this experiment were

Variable	Levels
Clone of Pinot Noir	Wadenswil, Pommard
Berry Size	Small, Large

Fermentation temperature	80F, 85F, 90/80F, 90F
Whole Berry	None, 10%
Maceration Time	10 days, 21 days
Yeast Type	Assmanhau, Champagne
Oak Type	Troncais, Allier

Harry and Judy decided to use a 16-run two-level fractional factorial design, treating the four levels of fermentation temperature as two two-level variables. As in Problem 8-26, they used the rankings from a taste-test panel as the response variable. The design and the resulting average ranks are shown below:

Run	Clone	Berry Size	Ferm. Temp.		Whole Berry	Macer. Time	Yeast Type	Oak Type	Average Rank
1	-	-	-	-	-	-	-	-	4
2	+	-	-	-	-	+	+	+	10
3	-	+	-	-	+	-	+	+	6
4	+	+	-	-	+	+	-	-	9
5	-	-	+	-	+	+	+	-	11
6	+	-	+	-	+	-	-	+	1
7	-	+	+	-	-	+	-	+	15
8	+	+	+	-	-	-	+	-	5
9	-	-	-	+	+	+	-	+	12
10	+	-	-	+	+	-	+	-	2
11	-	+	-	+	-	+	+	-	16
12	+	+	-	+	-	-	-	+	3
13	-	-	+	+	-	-	+	+	8
14	+	-	+	+	-	+	-	-	14
15	-	+	+	+	+	-	-	-	7
16	+	+	+	+	+	+	+	+	13

(a) Describe the aliasing in this design.

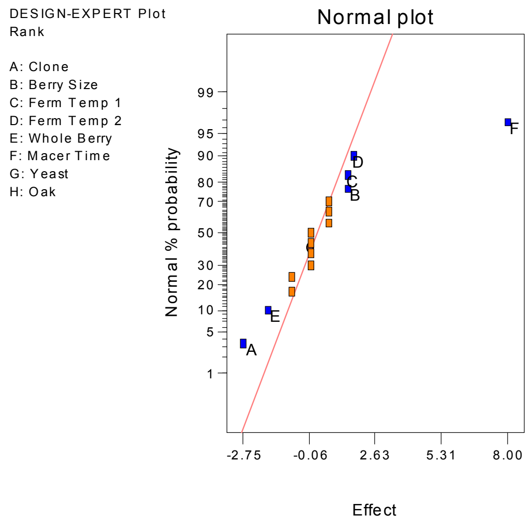
The design is a resolution IV design such that the main effects are aliased with three factor interactions.

Design Expert Output

Term	Aliases
Intercept	ABCG ABDH ABEF ACDF ACEH ADEG AFGH BCDE BCFH BDFG BEGH CDGH CEFG DEFH
A	BCG BDH BEF CDF CEH DEG FGH ABCDE
B	ACG ADH AEF CDE CFH DFG EGH
C	ABG ADF AEH BDE BFH DGH EFG
D	ABH ACF AEG BCE BFG CGH EFH
E	ABF ACH ADG BCD BGH CFG DFH
F	ABE ACD AGH BCH BDG CEG DEH
G	ABC ADE AFH BDF BEH CDH CEF
H	ABD ACE AFG BCF BEG CDG DEF
AB	CG DH EF ACDE ACFH ADFG AEGH BCDF BCEH BDEG BFGH
AC	BG DF EH ABDE ABFH ADGH AEFG BCDH BCEF CDEG CFGH
AD	BH CF EG ABCE ABFG ACGH AEFH BCDG BDEF CDEH DFGH
AE	BF CH DG ABCD ABGH ACFG ADFH BCEG BDEH CDEF EFGH
AF	BE CD GH ABCH ABDG ACEG ADEH BCFG BDFH CEFH DEFG
AG	BC DE FH ABDF ABEH ACDH ACEF BDGH BEFG CDFG CEGH
AH	BD CE FG ABCF ABEG ACDG ADEF BCGH BEFH CDFH DEGH

(b) Analyze the data and draw conclusions.

All of the main effects except Yeast and Oak are significant. The Macer Time is the most significant. None of the interactions were significant.



Design Expert Output

Response: **Rank**

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	328.75	6	54.79	43.83	< 0.0001	significant
A	*30.25*	*1*	*30.25*	*24.20*	*0.0008*	
B	*9.00*	*1*	*9.00*	*7.20*	*0.0251*	
C	*9.00*	*1*	*9.00*	*7.20*	*0.0251*	
D	*12.25*	*1*	*12.25*	*9.80*	*0.0121*	
E	*12.25*	*1*	*12.25*	*9.80*	*0.0121*	
F	*256.00*	*1*	*256.00*	*204.80*	*< 0.0001*	
Residual	11.25	9	1.25			
Cor Total	340.00	15				

The Model F-value of 43.83 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.12	R-Squared	0.9669
Mean	8.50	Adj R-Squared	0.9449
C.V.	13.15	Pred R-Squared	0.8954
PRESS	35.56	Adeq Precision	19.270

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	8.50	1	0.28	7.87	9.13	
A-Clone	-1.38	1	0.28	-2.01	-0.74	1.00
B-Berry Size	0.75	1	0.28	0.12	1.38	1.00
C-Ferm Temp 1	0.75	1	0.28	0.12	1.38	1.00
D-Ferm Temp 2	0.88	1	0.28	0.24	1.51	1.00
E-Whole Berry	-0.87	1	0.28	-1.51	-0.24	1.00
F-Macer Time	4.00	1	0.28	3.37	4.63	1.00

Final Equation in Terms of Coded Factors:

Rank =
+8.50
-1.38 * A
+0.75 * B
+0.75 * C
+0.88 * D
-0.87 * E
+4.00 * F

(c) What comparisons can you make between this experiment and the 1985 Pinot Noir experiment from Problem 8-26?

The experiment from Problem 8-26 indicates that yeast, barrel, whole cluster and the clone x yeast interactions were significant. This experiment indicates that maceration time, whole berry, clone and fermentation temperature are significant.

9-23 An article by W.D. Baten in the 1956 volume of *Industrial Quality Control* described an experiment to study the effect of three factors on the lengths of steel bars. Each bar was subjected to one of two heat treatment processes, and was cut on one of four machines at one of three times during the day (8 am, 11 am, or 3 pm). The coded length data are shown below

(a) Analyze the data from this experiment assuming that the four observations in each cell are replicates.

The Machine effect (C) is significant, the Heat Treat Process (B) is also significant, while the Time of Day (A) is not significant. None of the interactions are significant.

		Machine							
Time of Day	Heat Treat Process	1		2		3		4	
8am	1	6	9	7	9	1	2	6	6
		1	3	5	5	0	4	7	3
	2	4	6	6	5	-1	0	4	5
		0	1	3	4	0	1	5	4
11 am	1	6	3	8	7	3	2	7	9
		1	-1	4	8	1	0	11	6
	2	3	1	6	4	2	0	9	4
		1	-2	1	3	-1	1	6	3
3 pm	1	5	4	10	11	-1	2	10	5
		9	6	6	4	6	1	4	8
	2	6	0	8	7	0	-2	4	3
		3	7	10	0	4	-4	7	0

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	590.33	23	25.67	4.13	< 0.0001	significant
A	*26.27*	*2*	*13.14*	*2.11*	*0.1283*	
B	*42.67*	*1*	*42.67*	*6.86*	*0.0107*	
C	*393.42*	*3*	*131.14*	*21.10*	*< 0.0001*	
AB	*23.77*	*2*	*11.89*	*1.91*	*0.1552*	
AC	*42.15*	*6*	*7.02*	*1.13*	*0.3537*	
BC	*13.08*	*3*	*4.36*	*0.70*	*0.5541*	
ABC	*48.98*	*6*	*8.16*	*1.31*	*0.2623*	
Pure Error	447.50	72	6.22			
Cor Total	1037.83	95				

The Model F-value of 4.13 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.49	R-Squared	0.5688
Mean	3.96	Adj R-Squared	0.4311

C.V.	62.98	Pred R-Squared	0.2334
PRESS	795.56	Adeq Precision	7.020

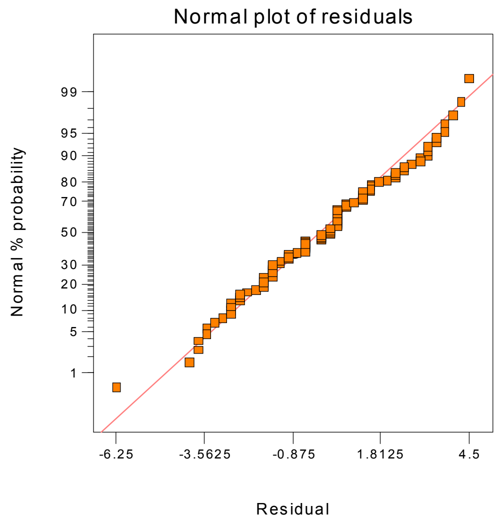
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	3.96	1	0.25	3.45	4.47	
A[1]	0.010	1	0.36	-0.71	0.73	
A[2]	-0.65	1	0.36	-1.36	0.071	
B-Process	-0.67	1	0.25	-1.17	-0.16	1.00
C[1]	-0.54	1	0.44	-1.42	0.34	
C[2]	1.92	1	0.44	1.04	2.80	
C[3]	-3.08	1	0.44	-3.96	-2.20	
A[1]B	0.010	1	0.36	-0.71	0.73	
A[2]B	0.60	1	0.36	-0.11	1.32	
A[1]C[1]	0.32	1	0.62	-0.92	1.57	
A[2]C[1]	-1.27	1	0.62	-2.51	-0.028	
A[1]C[2]	-0.39	1	0.62	-1.63	0.86	
A[2]C[2]	-0.10	1	0.62	-1.35	1.14	
A[1]C[3]	0.24	1	0.62	-1.00	1.48	
A[2]C[3]	0.77	1	0.62	-0.47	2.01	
BC[1]	-0.25	1	0.44	-1.13	0.63	
BC[2]	-0.46	1	0.44	-1.34	0.42	
BC[3]	0.46	1	0.44	-0.42	1.34	
A[1]BC[1]	-0.094	1	0.62	-1.34	1.15	
A[2]BC[1]	-0.44	1	0.62	-1.68	0.80	
A[1]BC[2]	0.11	1	0.62	-1.13	1.36	
A[2]BC[2]	-1.10	1	0.62	-2.35	0.14	
A[1]BC[3]	-0.43	1	0.62	-1.67	0.82	
A[2]BC[3]	0.60	1	0.62	-0.64	1.85	

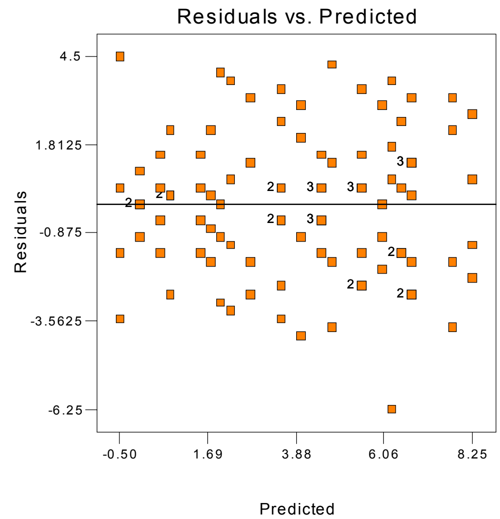
Final Equation in Terms of Coded Factors:

Length	=
+3.96	
+0.010	* A[1]
-0.65	* A[2]
-0.67	* B
-0.54	* C[1]
+1.92	* C[2]
-3.08	* C[3]
+0.010	* A[1]B
+0.60	* A[2]B
+0.32	* A[1]C[1]
-1.27	* A[2]C[1]
-0.39	* A[1]C[2]
-0.10	* A[2]C[2]
+0.24	* A[1]C[3]
+0.77	* A[2]C[3]
-0.25	* BC[1]
-0.46	* BC[2]
+0.46	* BC[3]
-0.094	* A[1]BC[1]
-0.44	* A[2]BC[1]
+0.11	* A[1]BC[2]
-1.10	* A[2]BC[2]
-0.43	* A[1]BC[3]
+0.60	* A[2]BC[3]

(b) Analyze the residuals from this experiment. Is there any indication that there is an outlier in one cell? If you find an outlier, remove it and repeat the analysis from part (a). What are your conclusions?

Standard Order 84, Time of Day at 3:00pm, Heat Treat #2, Machine #2, and length of 0, appears to be an outlier.





The following analysis was performed with the outlier described above removed. As with the original analysis, Machine is significant and Heat Treat Process is also significant, but now Time of Day, factor *A*, is also significant with an *F*-value of 3.05 (the *P*-value is just above 0.05).

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	626.58	23	27.24	4.89	< 0.0001	significant
A	*34.03*	*2*	*17.02*	*3.06*	*0.0533*	
B	*33.06*	*1*	*33.06*	*5.94*	*0.0173*	
C	*411.89*	*3*	*137.30*	*24.65*	*< 0.0001*	
AB	*16.41*	*2*	*8.20*	*1.47*	*0.2361*	
AC	*50.19*	*6*	*8.37*	*1.50*	*0.1900*	
BC	*8.38*	*3*	*2.79*	*0.50*	*0.6824*	
ABC	*67.00*	*6*	*11.17*	*2.01*	*0.0762*	
Pure Error	395.42	71	5.57			
Cor Total	1022.00	94				

The Model F-value of 4.89 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.36	R-Squared	0.6131
Mean	4.00	Adj R-Squared	0.4878
C.V.	59.00	Pred R-Squared	0.3100
PRESS	705.17	Adeq Precision	7.447

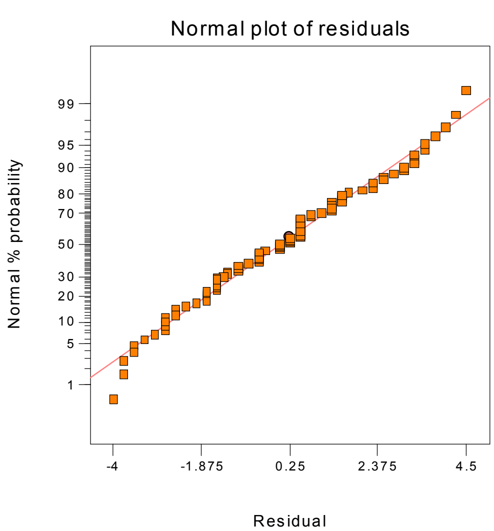
Term	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	4.05	1	0.24	3.z56	4.53	
A[1]	-0.076	1	0.34	-0.76	0.61	
A[2]	-0.73	1	0.34	-1.41	-0.051	
B-Process	-0.58	1	0.24	-1.06	-0.096	1.00
C[1]	-0.63	1	0.42	-1.46	0.21	
C[2]	2.18	1	0.43	1.33	3.03	
C[3]	-3.17	1	0.42	-4.00	-2.34	
A[1]B	-0.076	1	0.34	-0.76	0.61	
A[2]B	0.52	1	0.34	-0.16	1.20	
A[1]C[1]	0.41	1	0.59	-0.77	1.59	
A[2]C[1]	-1.18	1	0.59	-2.36	-6.278E-003	
A[1]C[2]	-0.65	1	0.60	-1.83	0.54	
A[2]C[2]	-0.36	1	0.60	-1.55	0.82	

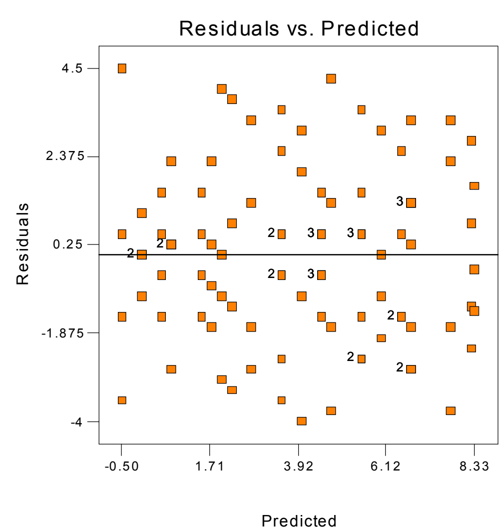
A[1]C[3]	0.33	1	0.59	-0.85	1.50
A[2]C[3]	0.86	1	0.59	-0.32	2.04
BC[1]	-0.34	1	0.42	-1.17	0.50
BC[2]	-0.20	1	0.43	-1.05	0.65
BC[3]	0.37	1	0.42	-0.46	1.21
A[1]BC[1]	-6.944E-003	1	0.59	-1.18	1.17
A[2]BC[1]	-0.35	1	0.59	-1.53	0.83
A[1]BC[2]	-0.15	1	0.60	-1.33	1.04
A[2]BC[2]	-1.36	1	0.60	-2.55	-0.18
A[1]BC[3]	-0.34	1	0.59	-1.52	0.84
A[2]BC[3]	0.69	1	0.59	-0.49	1.87

Final Equation in Terms of Coded Factors:

Length	=
+4.05	
-0.076	* A[1]
-0.73	* A[2]
-0.58	* B
-0.63	* C[1]
+2.18	* C[2]
-3.17	* C[3]
-0.076	* A[1]B
+0.52	* A[2]B
+0.41	* A[1]C[1]
-1.18	* A[2]C[1]
-0.65	* A[1]C[2]
-0.36	* A[2]C[2]
+0.33	* A[1]C[3]
+0.86	* A[2]C[3]
-0.34	* BC[1]
-0.20	* BC[2]
+0.37	* BC[3]
-6.944E-003	* A[1]BC[1]
-0.35	* A[2]BC[1]
-0.15	* A[1]BC[2]
-1.36	* A[2]BC[2]
-0.34	* A[1]BC[3]
+0.69	* A[2]BC[3]

The following residual plots are acceptable. Both the normality and constant variance assumptions are satisfied





(c) Suppose that the observations in the cells are the lengths (coded) of bars processed together in heat treating and then cut sequentially (that is, in order) on the three machines. Analyze the data to determine the effects of the three factors on mean length.

The analysis with all effects and interactions included:

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	147.58	23	6.42		
A	*6.57*	*2*	*3.28*		
B	*10.67*	*1*	*10.67*		
C	*98.35*	*3*	*32.78*		
AB	*5.94*	*2*	*2.97*		
AC	*10.54*	*6*	*1.76*		
BC	*3.27*	*3*	*1.09*		
ABC	*12.24*	*6*	*2.04*		
Pure Error	0.000	0			
Cor Total	147.58	23			

The by removing the three factor interaction from the model and applying it to the error, the analysis identifies factor *C* as being significant and *B* as being mildly significant.

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	135.34	17	7.96	3.90	0.0502	not significant
A	*6.57*	*2*	*3.28*	*1.61*	*0.2757*	
B	*10.67*	*1*	*10.67*	*5.23*	*0.0623*	
C	*98.35*	*3*	*32.78*	*16.06*	*0.0028*	
AB	*5.94*	*2*	*2.97*	*1.46*	*0.3052*	
AC	*10.54*	*6*	*1.76*	*0.86*	*0.5700*	
BC	*3.27*	*3*	*1.09*	*0.53*	*0.6756*	
Residual	12.24	6	2.04			
Cor Total	147.58	23				

When removing the remaining insignificant factors from the model, *C*, Machine, is the most significant factor while *B*, Heat Treat Process, is moderately significant. Factor *A*, Time of Day, is not significant.

Design Expert Output

Response: Avg
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	109.02	4	27.26	13.43	< 0.0001	significant
B	*10.67*	*1*	*10.67*	*5.26*	*0.0335*	
C	*98.35*	*3*	*32.78*	*16.15*	*< 0.0001*	
Residual	38.56	19	2.03			
Cor Total	147.58	23				

The Model F-value of 13.43 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

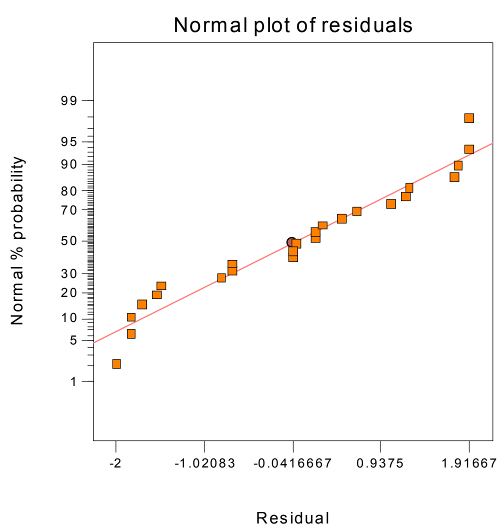
Std. Dev.	1.42	R-Squared	0.7387
Mean	3.96	Adj R-Squared	0.6837
C.V.	35.99	Pred R-Squared	0.5831
PRESS	61.53	Adeq Precision	9.740

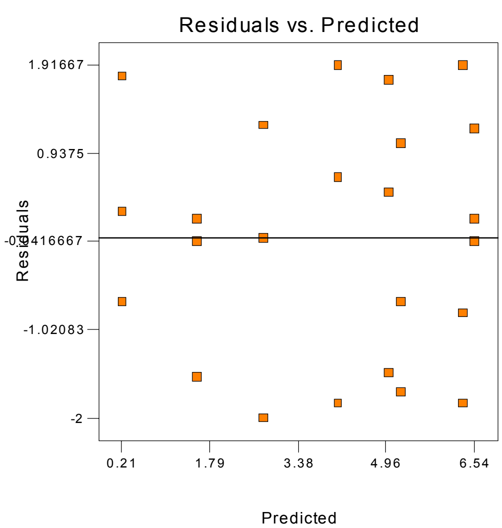
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	3.96	1	0.29	3.35	4.57	
B-Process	-0.67	1	0.29	-1.28	-0.058	1.00
C[1]	-0.54	1	0.50	-1.60	0.51	
C[2]	1.92	1	0.50	0.86	2.97	
C[3]	-3.08	1	0.50	-4.14	-2.03	

Final Equation in Terms of Coded Factors:

Avg =
+3.96
-0.67 * B
-0.54 * C[1]
+1.92 * C[2]
-3.08 * C[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





(d) Calculate the log variance of the observations in each cell. Analyze the average length and the log variance of the length for each of the 12 bars cut at each machine/heat treatment process combination. What conclusions can you draw?

Factor *B*, Heat Treat Process, has an affect on the log variance of the observations while Factor *A*, Time of Day, and Factor *C*, Machine, are not significant at the 5 percent level. However, *A* is significant at the 10 percent level, so Tome of Day has some effect on the variance.

Design Expert Output

Response: **Log(Var)**
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.79	11	0.25	2.51	0.0648	not significant
A	*0.58*	*2*	*0.29*	*2.86*	*0.0966*	
B	*0.50*	*1*	*0.50*	*4.89*	*0.0471*	
C	*0.59*	*3*	*0.20*	*1.95*	*0.1757*	
AB	*0.49*	*2*	*0.24*	*2.40*	*0.1324*	
BC	*0.64*	*3*	*0.21*	*2.10*	*0.1538*	
Residual	1.22	12	0.10			

Cor Total	4.01	23

The Model F-value of 2.51 implies there is a 6.48% chance that a "Model F-Value" this large could occur due to noise.

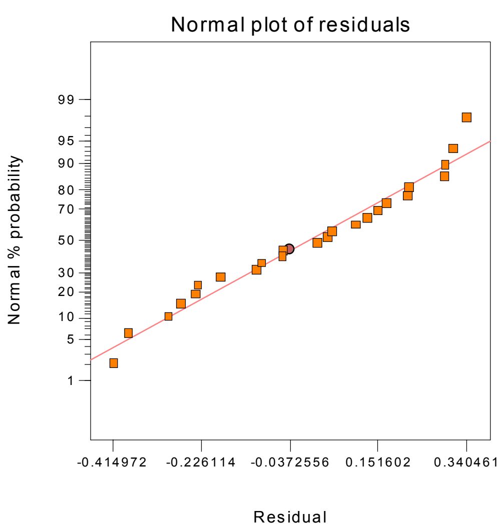
Std. Dev.	0.32	R-Squared	0.6967
Mean	0.65	Adj R-Squared	0.4186
C.V.	49.02	Pred R-Squared	-0.2133
PRESS	4.86	Adeq Precision	5.676

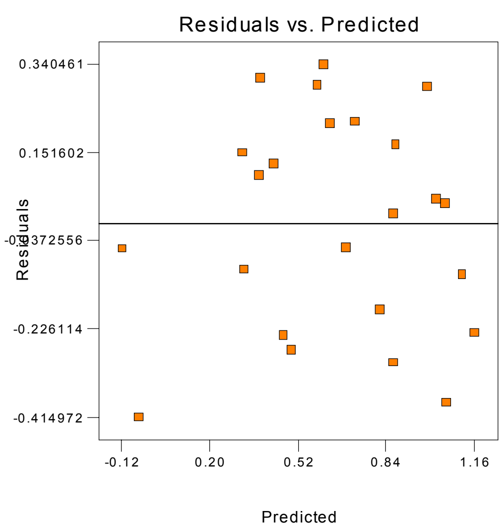
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	0.65	1	0.065	0.51	0.79	
A[1]	-0.054	1	0.092	-0.25	0.15	
A[2]	-0.16	1	0.092	-0.36	0.043	
B-Process	0.14	1	0.065	2.181E-003	0.29	1.00
C[1]	0.22	1	0.11	-0.025	0.47	
C[2]	0.066	1	0.11	-0.18	0.31	
C[3]	-0.19	1	0.11	-0.44	0.052	
A[1]B	-0.20	1	0.092	-0.40	3.237E-003	
A[2]B	0.14	1	0.092	-0.065	0.34	
BC[1]	-0.18	1	0.11	-0.42	0.068	
BC[2]	-0.15	1	0.11	-0.39	0.098	
BC[3]	0.14	1	0.11	-0.10	0.39	

Final Equation in Terms of Coded Factors:

Log(Var)	=
+0.65	
-0.054	* A[1]
-0.16	* A[2]
+0.14	* B
+0.22	* C[1]
+0.066	* C[2]
-0.19	* C[3]
-0.20	* A[1]B
+0.14	* A[2]B
-0.18	* BC[1]
-0.15	* BC[2]
+0.14	* BC[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





(e) Suppose the time at which a bar is cut really cannot be controlled during routine production. Analyze the average length and the log variance of the length for each of the 12 bars cut at each machine/heat treatment process combination. What conclusions can you draw?

The analysis of the average length is as follows:

Design Expert Output

Response: Avg
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	37.43	7	5.35		
A	3.56	1	3.56		
B	32.78	3	10.93		
AB	1.09	3	0.36		
Pure Error	0.000	0			
Cor Total	37.43	7			

Because the Means Square of the AB interaction is much less than the main effects, it is removed from the model and placed in the error. The average length is strongly affected by Factor *B*, Machine, and moderately affected by Factor *A*, Heat Treat Process. The interaction effect was small and removed from the model.

Design Expert Output

Response: Avg
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	36.34	4	9.09	25.00	0.0122	significant
A	*3.56*	*1*	*3.56*	*9.78*	*0.0522*	
B	*32.78*	*3*	*10.93*	*30.07*	*0.0097*	
Residual	1.09	3	0.36			
Cor Total	37.43	7				

The Model F-value of 25.00 implies the model is significant. There is only a 1.22% chance that a "Model F-Value" this large could occur due to noise.

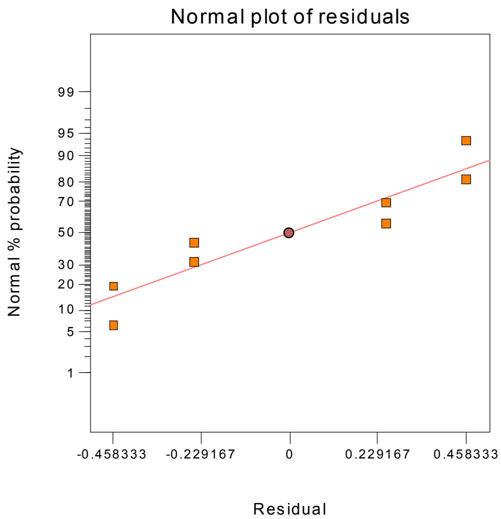
Std. Dev.	0.60	R-Squared	0.9709
Mean	3.96	Adj R-Squared	0.9320
C.V.	15.23	Pred R-Squared	0.7929
PRESS	7.75	Adeq Precision	13.289

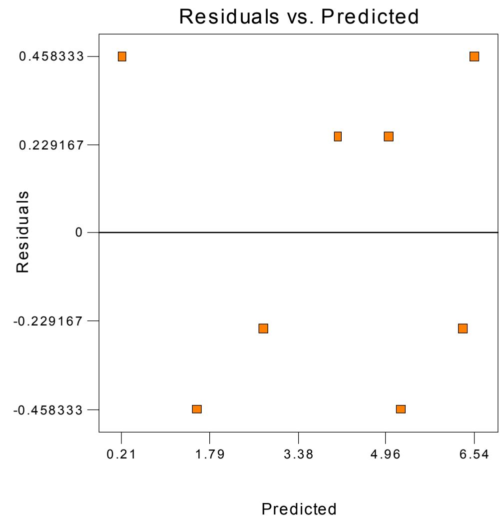
Term	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	3.96	1	0.21	3.28	4.64	
A-Process	-0.67	1	0.21	-1.34	0.012	1.00
B[1]	-0.54	1	0.37	-1.72	0.63	
B[2]	1.92	1	0.37	0.74	3.09	
B[3]	-3.08	1	0.37	-4.26	-1.91	

Final Equation in Terms of Coded Factors:

Avg	=
+3.96	
-0.67	* A
-0.54	* B[1]
+1.92	* B[2]
-3.08	* B[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





The Log(Var) is analyzed below:

Design Expert Output

Response: Log(Var)
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	0.32	7	0.046		
A	*0.091*	*1*	*0.091*		
B	*0.13*	*3*	*0.044*		
AB	*0.098*	*3*	*0.033*		
Pure Error	0.000	0			
Cor Total	0.32	7			

Because the *AB* interaction has the smallest Mean Square, it was removed from the model and placed in the error. From the following analysis of variance, neither Heat Treat Process, Machine, nor the interaction affect the log variance of the length.

Design Expert Output

Response: Log(Var)
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.22	4	0.056	1.71	0.3441	not significant
A	*0.091*	*1*	*0.091*	*2.80*	*0.1926*	
B	*0.13*	*3*	*0.044*	*1.34*	*0.4071*	
Residual	0.098	3	0.033			
Cor Total	0.32	7				

The "Model F-value" of 1.71 implies the model is not significant relative to the noise. There is a 34.41 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	0.18	R-Squared	0.6949
Mean	0.79	Adj R-Squared	0.2882
C.V.	22.90	Pred R-Squared	-1.1693
PRESS	0.69	Adeq Precision	3.991

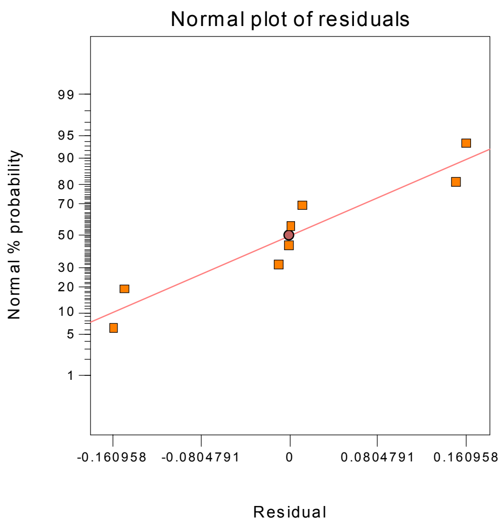
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	0.79	1	0.064	0.59	0.99	
A-Process	0.11	1	0.064	-0.096	0.31	1.00

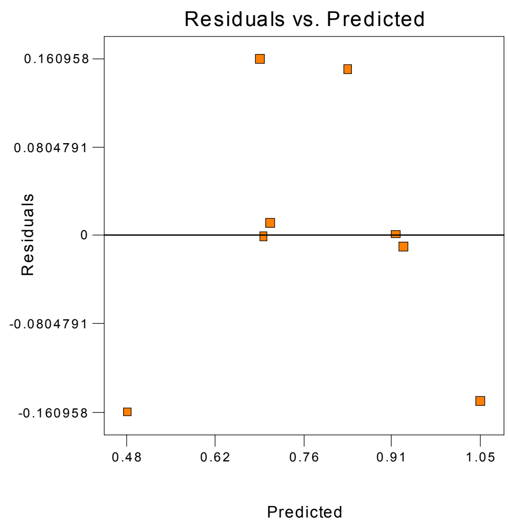
B[1]	0.15	1	0.11	-0.20	0.51
B[2]	0.030	1	0.11	-0.32	0.38
B[3]	-0.20	1	0.11	-0.55	0.15

Final Equation in Terms of Coded Factors:

Log(Var)	=
+0.79	
+0.11	* A
+0.15	* B[1]
+0.030	* B[2]
-0.20	* B[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





Chapter 10
Fitting Regression Models
Solutions

10-1 The tensile strength of a paper product is related to the amount of hardwood in the pulp. Ten samples are produced in the pilot plant, and the data obtained are shown in the following table.

Strength	Percent Hardwood	Strength	Percent Hardwood
160	10	181	20
171	15	188	25
175	15	193	25
182	20	195	28
184	20	200	30

(a) Fit a linear regression model relating strength to percent hardwood.

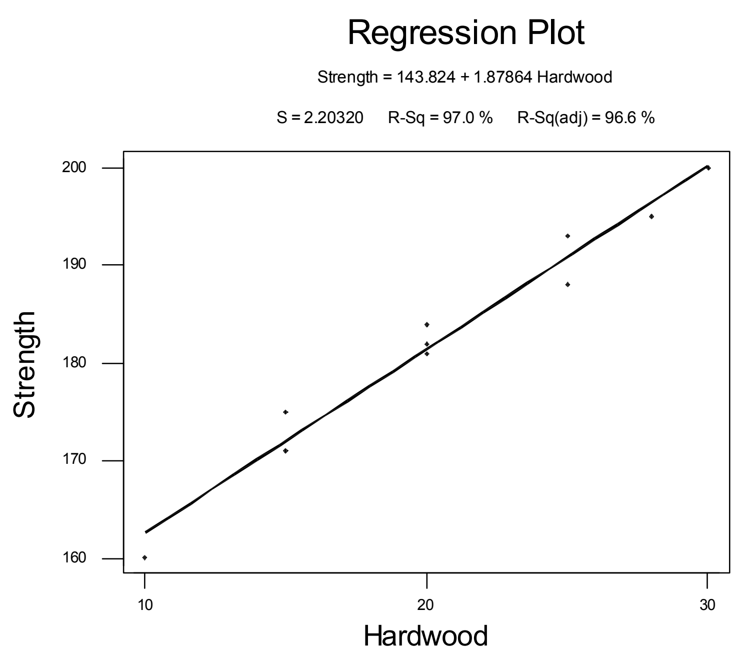
Minitab Output

Regression Analysis: Strength versus Hardwood

```
The regression equation is
Strength = 144 + 1.88 Hardwood

Predictor        Coef     SE Coef          T        P
Constant      143.824       2.522      57.04    0.000
Hardwood       1.8786      0.1165      16.12    0.000

S = 2.203              R-Sq = 97.0%         R-Sq(adj) = 96.6%
PRESS = 66.2665        R-Sq(pred) = 94.91%
```



(b) Test the model in part (a) for significance of regression.

Minitab Output

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      1262.1      1262.1    260.00    0.000
Residual Error     8        38.8         4.9
  Lack of Fit      4        13.7         3.4      0.54    0.716
  Pure Error       4        25.2         6.3
Total              9      1300.9

3 rows with no replicates

No evidence of lack of fit (P > 0.1)
```

(c) Find a 95 percent confidence interval on the parameter β_1.

The 95 percent confidence interval is:

$$\hat{\beta}_1 - t_{\alpha/2,n-p}se\left(\hat{\beta}_1\right) \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2,n-p}se\left(\hat{\beta}_1\right)$$
$$1.8786 - 2.3060(0.1165) \le \beta_1 \le 1.8786 + 2.3060(0.1165)$$
$$1.6900 \le \beta_1 \le 2.1473$$

10-2 A plant distills liquid air to produce oxygen , nitrogen, and argon. The percentage of impurity in the oxygen is thought to be linearly related to the amount of impurities in the air as measured by the "pollution count" in part per million (ppm). A sample of plant operating data is shown below.

Purity(%)	93.3	92.0	92.4	91.7	94.0	94.6	93.6	93.1	93.2	92.9	92.2	91.3	90.1	91.6	91.9
Pollution count (ppm)	1.10	1.45	1.36	1.59	1.08	0.75	1.20	0.99	0.83	1.22	1.47	1.81	2.03	1.75	1.68

(a) Fit a linear regression model to the data.

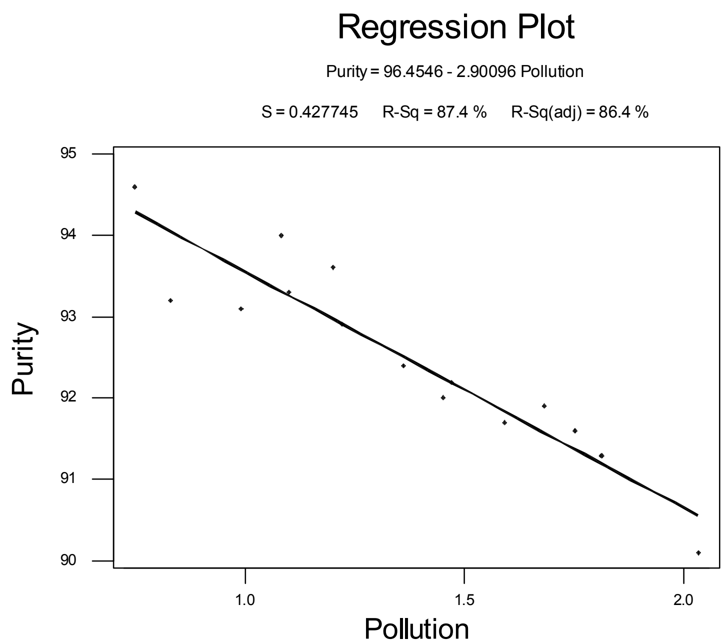
Minitab Output

Regression Analysis: Purity versus Pollution

```
The regression equation is
Purity = 96.5 - 2.90 Pollution

Predictor        Coef     SE Coef          T        P
Constant      96.4546      0.4282     225.24    0.000
Pollutio      -2.9010      0.3056      -9.49    0.000

S = 0.4277             R-Sq = 87.4%         R-Sq(adj) = 86.4%
PRESS = 3.43946        R-Sq(pred) = 81.77%
```



(b) Test for significance of regression.

Minitab Output

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      16.491      16.491     90.13    0.000
Residual Error    13       2.379       0.183
Total             14      18.869

No replicates. Cannot do pure error test.

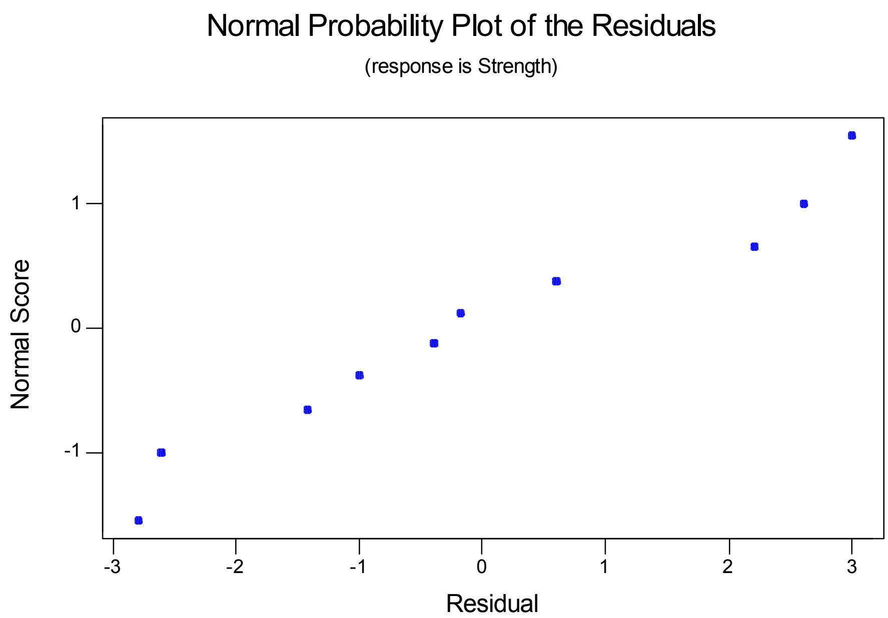
No evidence of lack of fit (P > 0.1)
```

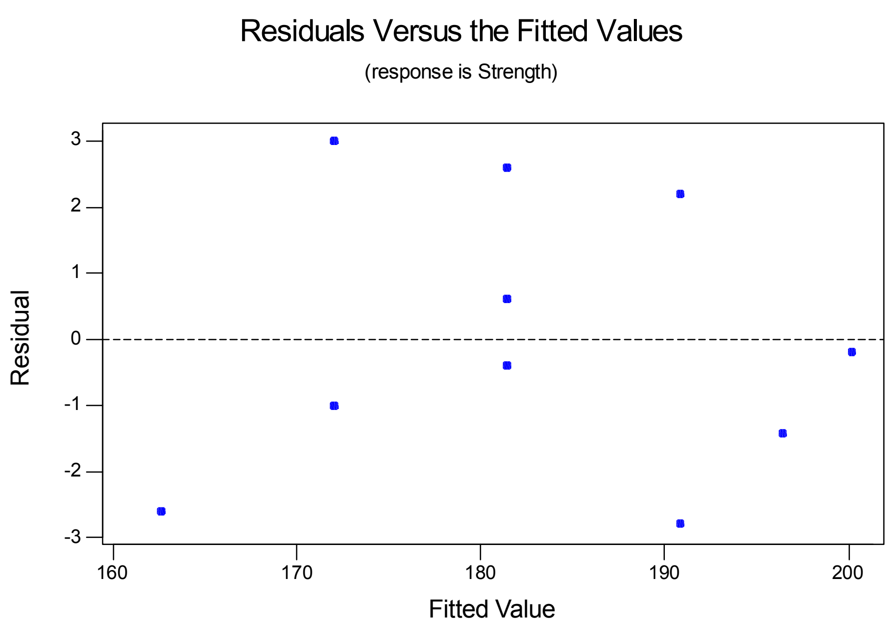
(c) Find a 95 percent confidence interval on β_1.

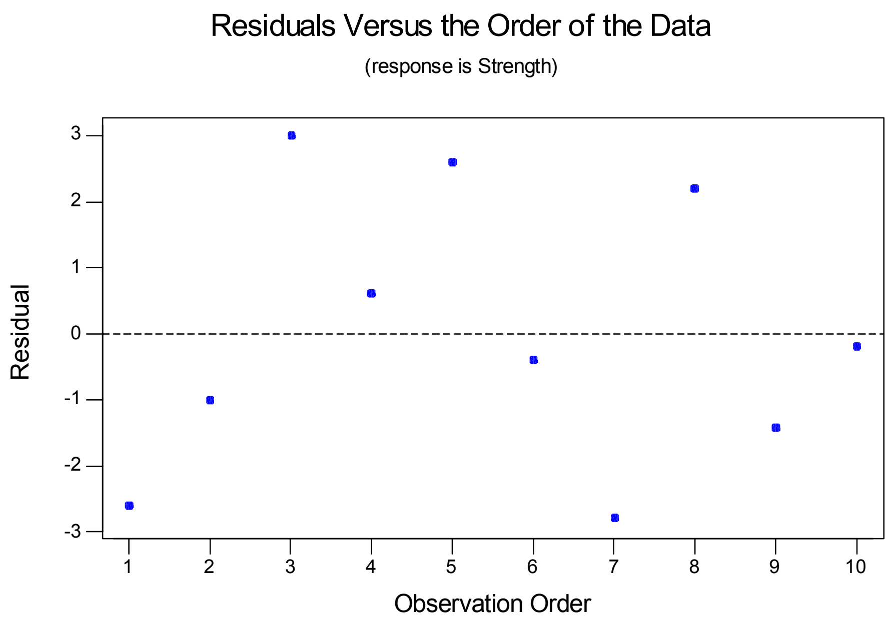
The 95 percent confidence interval is:

$$\hat{\beta}_1 - t_{\alpha/2,n-p}se(\hat{\beta}_1) \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2,n-p}se(\hat{\beta}_1)$$
$$-2.9010 - 2.1604(0.3056) \le \beta_1 \le -2.9010 + 2.1604(0.3056)$$
$$-3.5612 \le \beta_1 \le -2.2408$$

10-3 Plot the residuals from Problem 10-1 and comment on model adequacy.

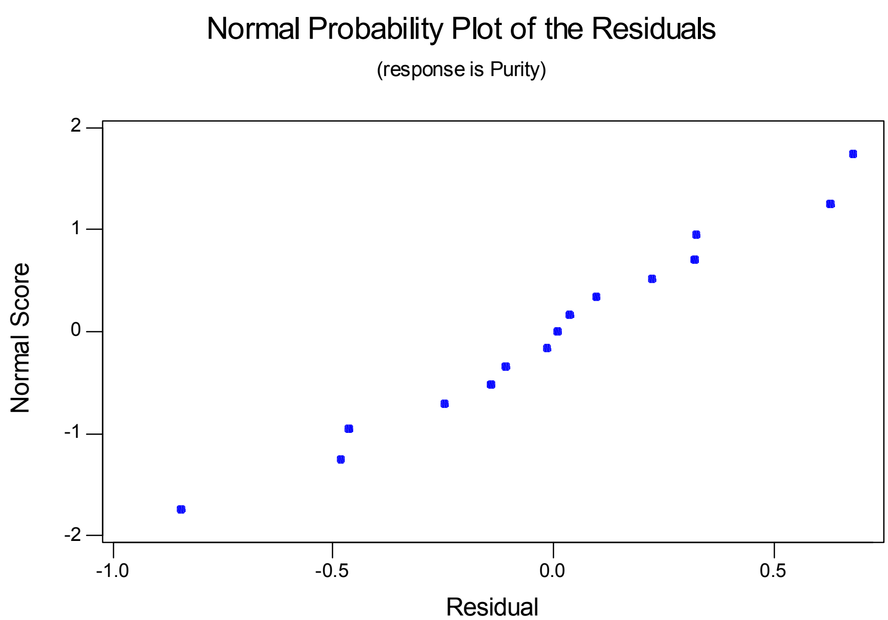

Normal Probability Plot of the Residuals
(response is Strength)
Normal Score
1
0
-1
-3
-2
-1
0
1
2
3
Residual

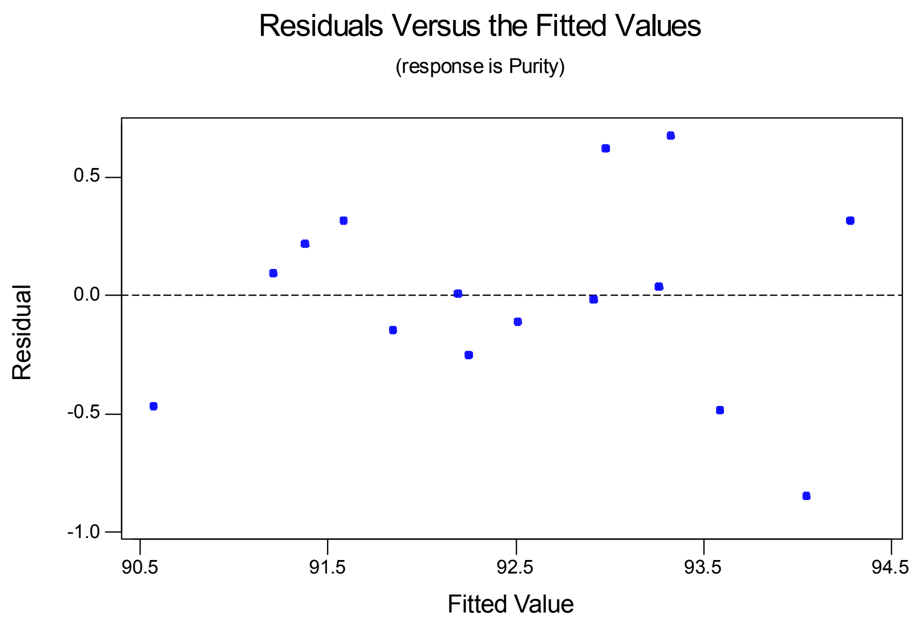

Residuals Versus the Fitted Values
(response is Strength)
Residual
3
2
1
0
-1
-2
-3
160
170
180
190
200
Fitted Value

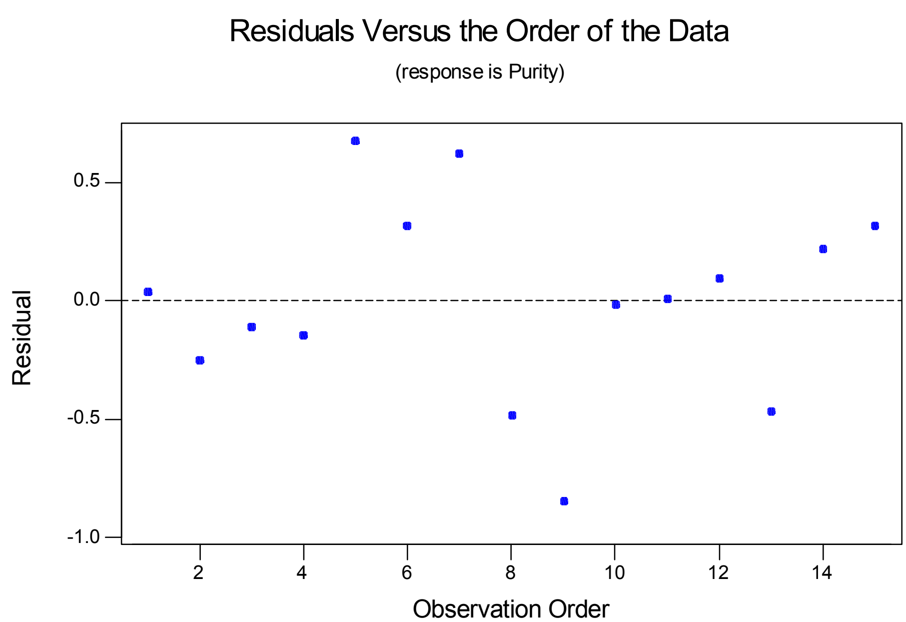


There is nothing unusual about the residual plots. The underlying assumptions have been met.

10-4 Plot the residuals from Problem 10-2 and comment on model adequacy.







There is nothing unusual about the residual plots. The underlying assumptions have been met.

10-5 Using the results of Problem 10-1, test the regression model for lack of fit.

Minitab Output
```
Analysis of Variance

Source            DF        SS        MS        F        P
Regression         1    1262.1    1262.1   260.00    0.000
Residual Error     8      38.8       4.9
```

```
  Lack of Fit      4        13.7         3.4      0.54    0.716
  Pure Error       4        25.2         6.3
Total              9      1300.9

3 rows with no replicates

No evidence of lack of fit (P > 0.1)
```

10-6 A study was performed on wear of a bearing y and its relationship to x_1 = oil viscosity and x_2 = load. The following data were obtained.

y	x_1	x_2
193	1.6	851
230	15.5	816
172	22.0	1058
91	43.0	1201
113	33.0	1357
125	40.0	1115

(a) Fit a multiple linear regression model to the data.

Minitab Output

Regression Analysis: Wear versus Viscosity, Load

```
The regression equation is
Wear = 351 - 1.27 Viscosity - 0.154 Load

Predictor        Coef     SE Coef          T        P       VIF
Constant       350.99       74.75       4.70    0.018
Viscosit       -1.272       1.169      -1.09    0.356       2.6
Load         -0.15390     0.08953      -1.72    0.184       2.6

S = 25.50              R-Sq = 86.2%         R-Sq(adj) = 77.0%
PRESS = 12696.7        R-Sq(pred) = 10.03%
```

(b) Test for significance of regression.

Minitab Output

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2     12161.6      6080.8      9.35    0.051
Residual Error     3      1950.4       650.1
Total              5     14112.0

No replicates. Cannot do pure error test.

Source       DF      Seq SS
Viscosit      1     10240.4
Load          1      1921.2

* Not enough data for lack of fit test
```

(c) Compute t statistics for each model parameter. What conclusions can you draw?

Minitab Output

Regression Analysis: Wear versus Viscosity, Load

```
The regression equation is
Wear = 351 - 1.27 Viscosity - 0.154 Load

Predictor        Coef     SE Coef          T        P       VIF
Constant       350.99       74.75       4.70    0.018
```

```
Viscosit       -1.272       1.169      -1.09    0.356       2.6
Load         -0.15390     0.08953      -1.72    0.184       2.6

S = 25.50               R-Sq = 86.2%          R-Sq(adj) = 77.0%
PRESS = 12696.7         R-Sq(pred) = 10.03%
```

The *t*-tests are shown in part (a). Notice that overall regression is significant (part(b)), but neither variable has a large *t*-statistic. This could be an indicator that the regressors are nearly linearly dependent.

10-7 The brake horsepower developed by an automobile engine on a dynomometer is thought to be a function of the engine speed in revolutions per minute (rpm), the road octane number of the fuel, and the engine compression. An experiment is run in the laboratory and the data that follow are collected.

Brake Horsepower	rpm	Road Octane Number	Compression
225	2000	90	100
212	1800	94	95
229	2400	88	110
222	1900	91	96
219	1600	86	100
278	2500	96	110
246	3000	94	98
237	3200	90	100
233	2800	88	105
224	3400	86	97
223	1800	90	100
230	2500	89	104

(a) Fit a multiple linear regression model to the data.

Minitab Output

Regression Analysis: Horsepower versus rpm, Octane, Compression

```
The regression equation is
Horsepower = - 266 + 0.0107 rpm + 3.13 Octane + 1.87 Compression

Predictor        Coef     SE Coef          T        P       VIF
Constant      -266.03       92.67      -2.87    0.021
rpm          0.010713    0.004483       2.39    0.044       1.0
Octane         3.1348      0.8444       3.71    0.006       1.0
Compress       1.8674      0.5345       3.49    0.008       1.0

S = 8.812               R-Sq = 80.7%          R-Sq(adj) = 73.4%
PRESS = 2494.05         R-Sq(pred) = 22.33%
```

(b) Test for significance of regression. What conclusions can you draw?

Minitab Output

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3     2589.73      863.24     11.12    0.003
Residual Error     8      621.27       77.66
Total             11     3211.00
r No replicates. Cannot do pure error test.

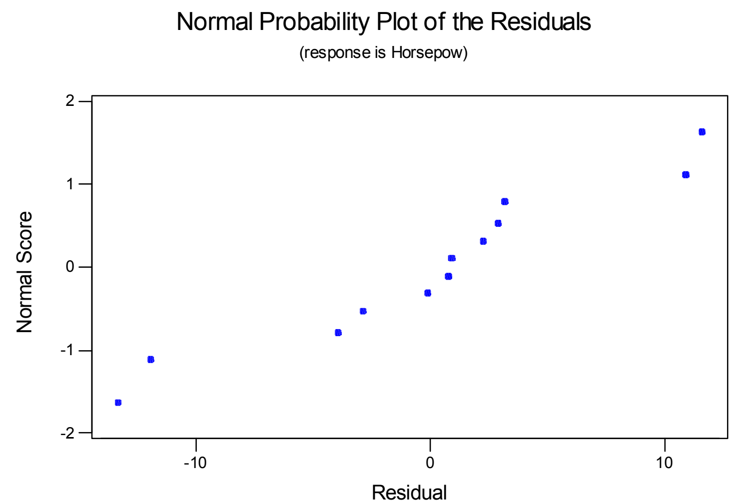
Source       DF      Seq SS
rpm           1      509.35
Octane        1     1132.56
Compress      1      947.83

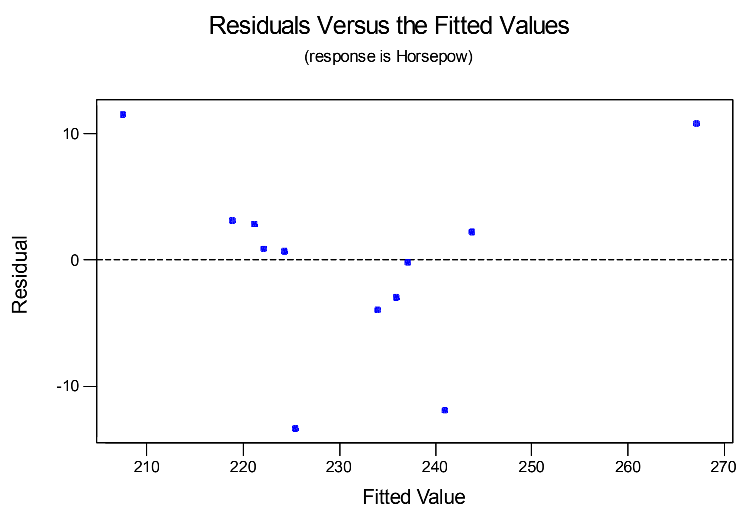
Lack of fit test
Possible interactions with variable Octane (P-Value = 0.028)
Possible lack of fit at outer X-values       (P-Value = 0.000)
Overall lack of fit test is significant at P = 0.000
```

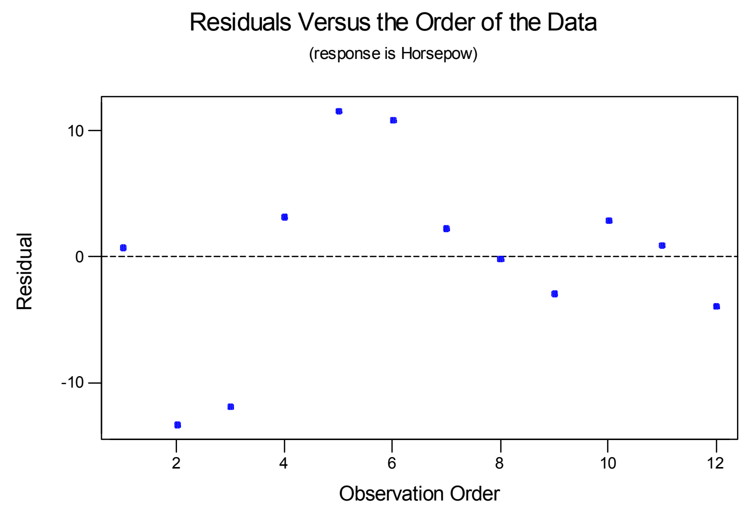
(c) Based on *t* tests, do you need all three regressor variables in the model?

Yes, all of the regressor variables are important.

10-8 Analyze the residuals from the regression model in Problem 10-7. Comment on model adequacy.







The normal probability plot is satisfactory, as is the plot of residuals versus run order (assuming that observation order is run order). The plot of residuals versus predicted response exhibits a slight "bow" shape. This could be an indication of lack of fit. It might be useful to consider adding some ineraction terms to the model.

10-9 The yield of a chemical process is related to the concentration of the reactant and the operating temperature. An experiment has been conducted with the following results.

Yield	Concentration	Temperature
81	1.00	150
89	1.00	180
83	2.00	150
91	2.00	180
79	1.00	150
87	1.00	180
84	2.00	150
90	2.00	180

(a) Suppose we wish to fit a main effects model to this data. Set up the **X'X** matrix using the data exactly as it appears in the table.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1.00 & 1.00 & 2.00 & 2.00 & 1.00 & 1.00 & 2.00 & 2.00 \\ 150 & 180 & 150 & 180 & 150 & 180 & 150 & 180 \end{bmatrix} \begin{bmatrix} 1 & 1.00 & 150 \\ 1 & 1.00 & 180 \\ 1 & 2.00 & 150 \\ 1 & 2.00 & 180 \\ 1 & 1.00 & 150 \\ 1 & 1.00 & 180 \\ 1 & 2.00 & 150 \\ 1 & 2.00 & 180 \end{bmatrix} = \begin{bmatrix} 8 & 12 & 1320 \\ 12 & 20 & 1980 \\ 1320 & 1980 & 219600 \end{bmatrix}$$

(b) Is the matrix you obtained in part (a) diagonal? Discuss your response.

The **X'X** is not diagonal, even though an orthogonal design has been used. The reason is that we have worked with the natural factor levels, not the orthogonally coded variables.

(c) Suppose we write our model in terms of the "usual" coded variables

$$x_1 = \frac{Conc - 1.5}{0.5},\ x_2 = \frac{Temp - 165}{15}$$

Set up the **X'X** matrix for the model in terms of these coded variables. Is this matrix diagonal? Discuss your response.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

The **X'X** matrix is diagonal because we have used the orthogonally coded variables.

(d) Define a new set of coded variables

$$x_1 = \frac{Conc - 1.0}{1.0},\ x_2 = \frac{Temp - 150}{30}$$

Set up the **X'X** matrix for the model in terms of this set of coded variables. Is this matrix diagonal? Discuss your response.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 4 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

The **X'X** is not diagonal, even though an orthogonal design has been used. The reason is that we have not used orthogonally coded variables.

(e) Summarize what you have learned from this problem about coding the variables.

If the design is orthogonal, use the orthogonal coding. This not only makes the analysis somewhat easier, but it also results in model coefficients that are easier to interpret because they are both dimensionless and uncorrelated.

10-10 Consider the 2^4 factorial experiment in Example 6-2. Suppose that the last observation in missing. Reanalyze the data and draw conclusions. How do these conclusions compare with those from the original example?

The regression analysis with the one data point missing indicates that the same effects are important.

Minitab Output

Regression Analysis: Rate versus A, B, C, D, AB, AC, AD, BC, BD, CD

```
The regression equation is
Rate = 69.8 + 10.5 A + 1.25 B + 4.63 C + 7.00 D - 0.25 AB - 9.38 AC + 8.00 AD
           + 0.87 BC - 0.50 BD - 0.87 CD

Predictor        Coef     SE Coef          T        P       VIF
Constant       69.750       1.500      46.50    0.000
A              10.500       1.500       7.00    0.002       1.1
B               1.250       1.500       0.83    0.452       1.1
C               4.625       1.500       3.08    0.037       1.1
D               7.000       1.500       4.67    0.010       1.1
AB             -0.250       1.500      -0.17    0.876       1.1
AC             -9.375       1.500      -6.25    0.003       1.1
AD              8.000       1.500       5.33    0.006       1.1
BC              0.875       1.500       0.58    0.591       1.1
BD             -0.500       1.500      -0.33    0.756       1.1
CD             -0.875       1.500      -0.58    0.591       1.1

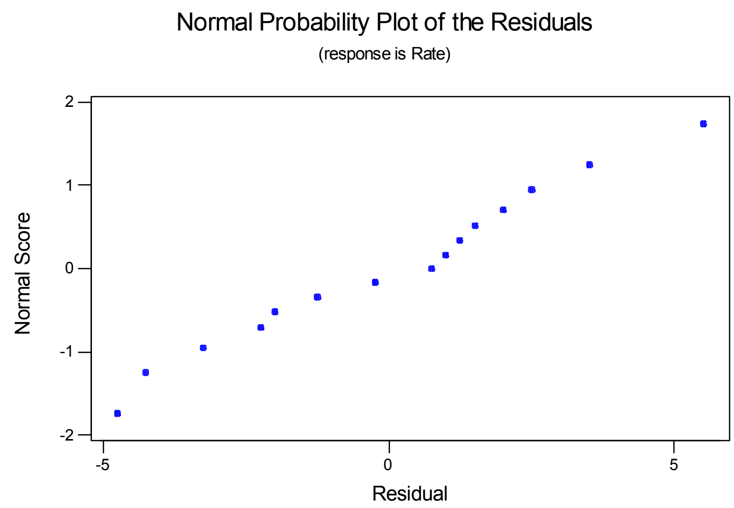
S = 5.477               R-Sq = 97.6%          R-Sq(adj) = 91.6%
PRESS = 1750.00         R-Sq(pred) = 65.09%

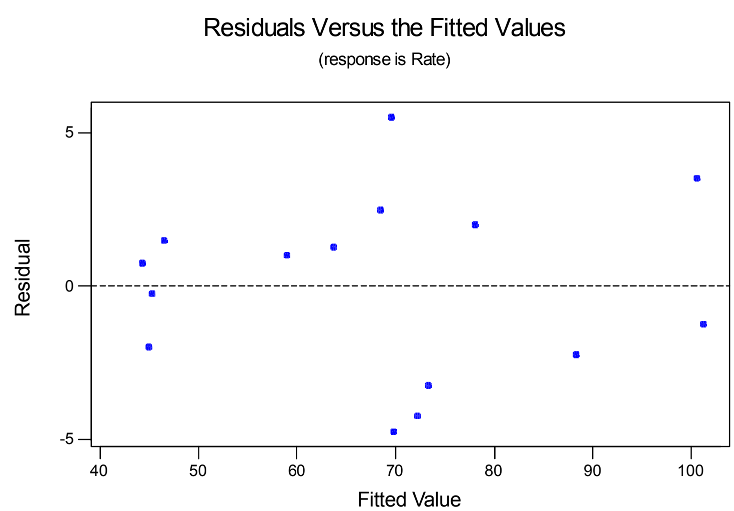
Analysis of Variance

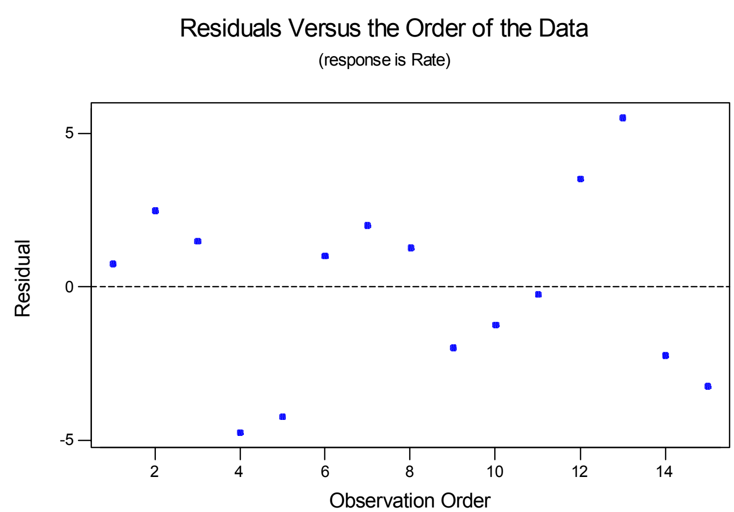
Source            DF          SS          MS         F        P
Regression        10     4893.33      489.33     16.31    0.008
Residual Error     4      120.00       30.00
Total             14     5013.33

No replicates. Cannot do pure error test.

Source       DF      Seq SS
A             1     1414.40
B             1        4.01
C             1      262.86
D             1      758.88
AB            1        0.06
AC            1     1500.63
AD            1      924.50
BC            1       16.07
BD            1        1.72
CD            1       10.21
```


Normal Probability Plot of the Residuals
(response is Rate)
Normal Score
Residual
2
1
0
-1
-2
-5
0
5


Residuals Versus the Fitted Values
(response is Rate)
Residual
Fitted Value
5
0
-5
40
50
60
70
80
90
100


Residuals Versus the Order of the Data
(response is Rate)
Residual
Observation Order
5
0
-5
2
4
6
8
10
12
14

The residual plots are acceptable; therefore, the underlying assumptions are valid.

10-11 Consider the 2^4 factorial experiment in Example 6-2. Suppose that the last two observations are missing. Reanalyze the data and draw conclusions. How do these conclusions compare with those from the original example?

The regression analysis with the one data point missing indicates that the same effects are important.

Minitab Output

Regression Analysis: Rate versus A, B, C, D, AB, AC, AD, BC, BD, CD

```
The regression equation is
Rate = 71.4 + 10.1 A + 2.87 B + 6.25 C + 8.62 D - 0.66 AB - 9.78 AC + 7.59 AD
           + 2.50 BC + 1.12 BD + 0.75 CD

Predictor        Coef     SE Coef          T        P       VIF
Constant       71.375       1.673      42.66    0.000
A              10.094       1.323       7.63    0.005       1.1
B               2.875       1.673       1.72    0.184       1.7
C               6.250       1.673       3.74    0.033       1.7
D               8.625       1.673       5.15    0.014       1.7
AB             -0.656       1.323      -0.50    0.654       1.1
AC             -9.781       1.323      -7.39    0.005       1.1
AD              7.594       1.323       5.74    0.010       1.1
BC              2.500       1.673       1.49    0.232       1.7
BD              1.125       1.673       0.67    0.549       1.7
CD              0.750       1.673       0.45    0.684       1.7

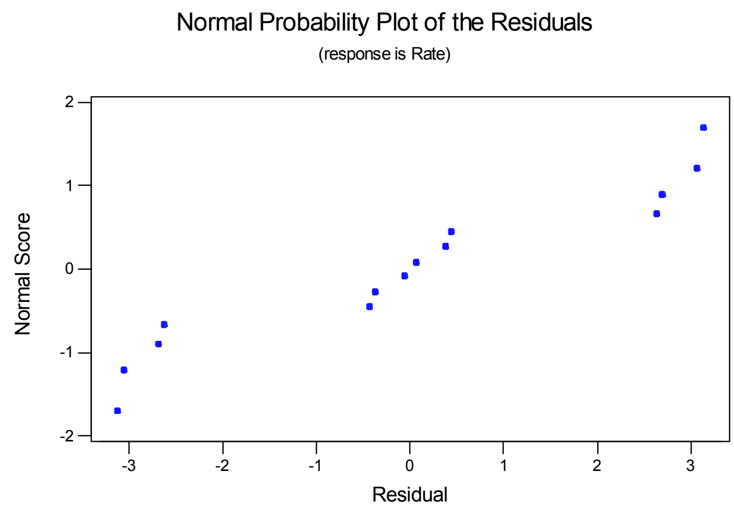
S = 4.732               R-Sq = 98.7%         R-Sq(adj) = 94.2%
PRESS = 1493.06         R-Sq(pred) = 70.20%

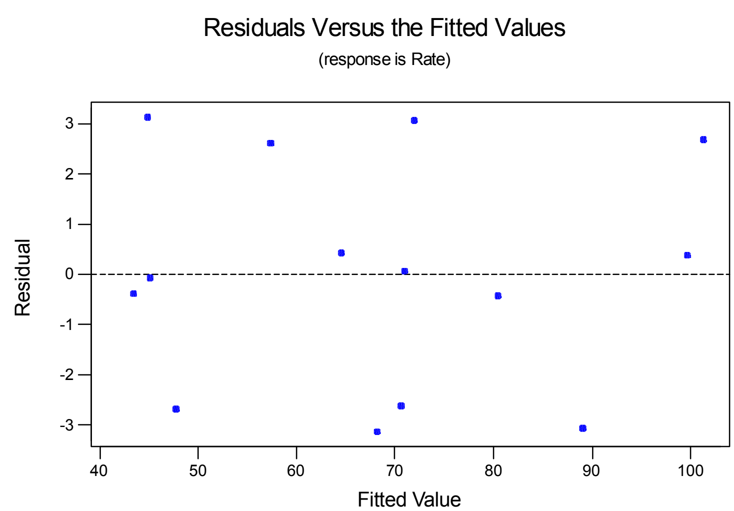
Analysis of Variance

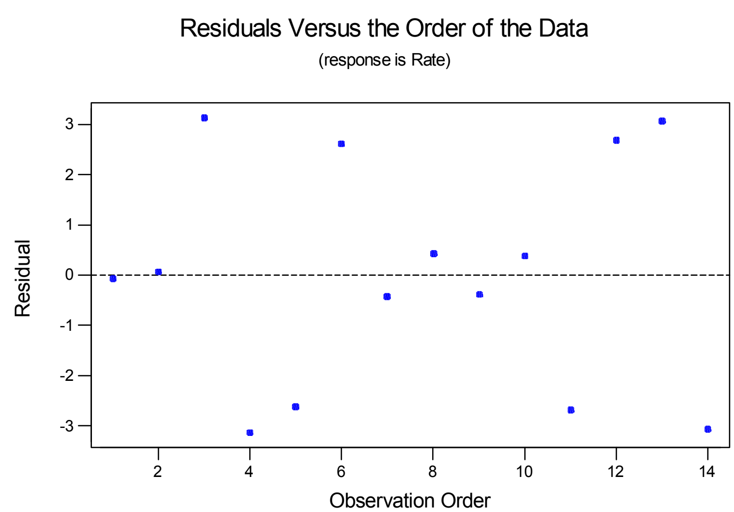
Source            DF          SS          MS         F        P
Regression        10     4943.17      494.32     22.07    0.014
Residual Error     3       67.19       22.40
Total             13     5010.36

No replicates. Cannot do pure error test.

Source       DF      Seq SS
A             1     1543.50
B             1        1.52
C             1      177.63
D             1      726.01
AB            1        1.17
AC            1     1702.53
AD            1      738.11
BC            1       42.19
BD            1        6.00
CD            1        4.50
```


Normal Probability Plot of the Residuals
(response is Rate)
Normal Score
Residual
2
1
0
-1
-2
-3
-2
-1
0
1
2
3


Residuals Versus the Fitted Values
(response is Rate)
Residual
Fitted Value
3
2
1
0
-1
-2
-3
40
50
60
70
80
90
100


Residuals Versus the Order of the Data
(response is Rate)
Residual
Observation Order
3
2
1
0
-1
-2
-3
2
4
6
8
10
12
14

The residual plots are acceptable; therefore, the underlying assumptions are valid.

10-12 Given the following data, fit the second-order polynomial regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \varepsilon$$

y	x_1	x_2
26	1.0	1.0
24	1.0	1.0
175	1.5	4.0
160	1.5	4.0
163	1.5	4.0
55	0.5	2.0
62	1.5	2.0
100	0.5	3.0
26	1.0	1.5
30	0.5	1.5
70	1.0	2.5
71	0.5	2.5

After you have fit the model, test for significance of regression.

Minitab Output

Regression Analysis: y versus x1, x2, x1^2, x2^2, x1x2

```
The regression equation is
y = 24.4 - 38.0 x1 + 0.7 x2 + 35.0 x1^2 + 11.1 x2^2 - 9.99 x1x2

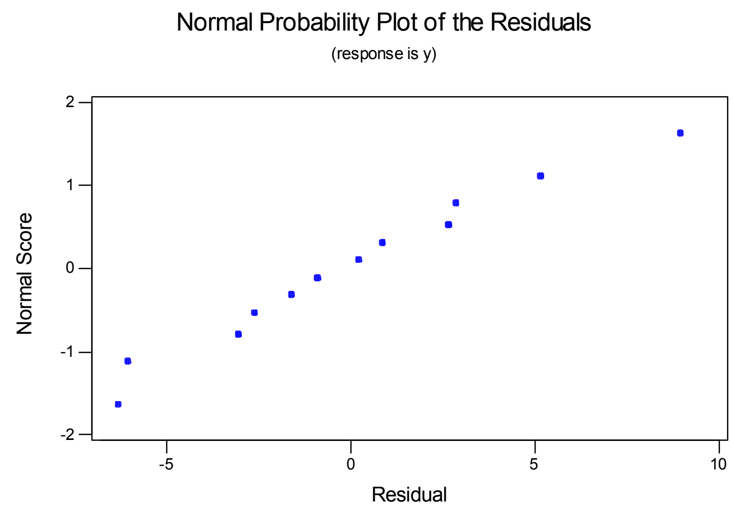
Predictor        Coef     SE Coef          T        P       VIF
Constant        24.41       26.59       0.92    0.394
x1             -38.03       40.45      -0.94    0.383      89.6
x2               0.72       11.69       0.06    0.953      52.1
x1^2            34.98       21.56       1.62    0.156     103.9
x2^2           11.066       3.158       3.50    0.013     104.7
x1x2           -9.986       8.742      -1.14    0.297     105.1

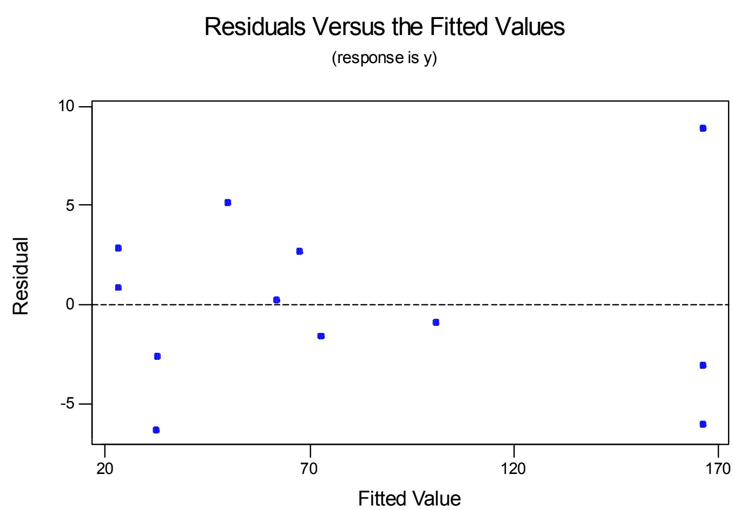
S = 6.042              R-Sq = 99.4%         R-Sq(adj) = 98.9%
PRESS = 1327.71        R-Sq(pred) = 96.24%
r Analysis of Variance

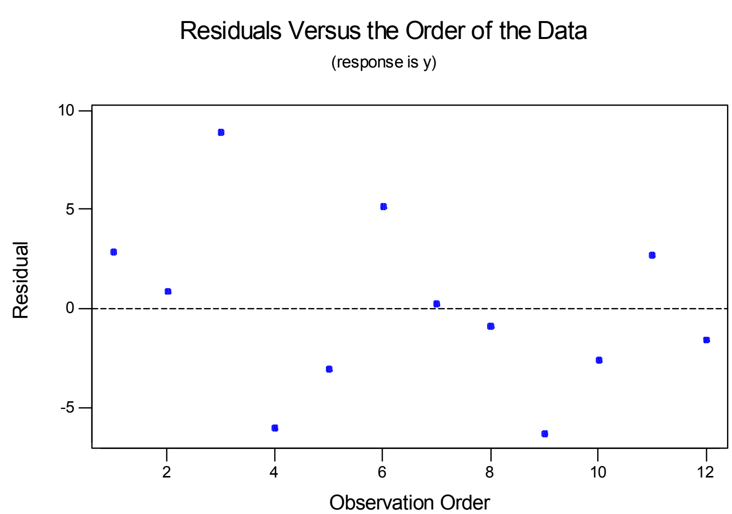
Source            DF          SS          MS         F        P
Regression         5     35092.6      7018.5    192.23    0.000
Residual Error     6       219.1        36.5
  Lack of Fit      3        91.1        30.4      0.71    0.607
  Pure Error       3       128.0        42.7
Total             11     35311.7

7 rows with no replicates

Source       DF      Seq SS
x1            1     11552.0
x2            1     22950.3
x1^2          1        21.9
x2^2          1       520.8
x1x2          1        47.6
```


Normal Probability Plot of the Residuals
(response is y)
Normal Score
Residual
2
1
0
-1
-2
-5
0
5
10


Residuals Versus the Fitted Values
(response is y)
Residual
Fitted Value
10
5
0
-5
20
70
120
170


Residuals Versus the Order of the Data
(response is y)
Residual
Observation Order
10
5
0
-5
2
4
6
8
10
12

10-13

(a) Consider the quadratic regression model from Problem 10-12. Compute t statistics for each model parameter and comment on the conclusions that follow from the quantities.

Minitab Output

Predictor	Coef	SE Coef	T	P	VIF
Constant	24.41	26.59	0.92	0.394	
x1	-38.03	40.45	-0.94	0.383	89.6
x2	0.72	11.69	0.06	0.953	52.1
x1^2	34.98	21.56	1.62	0.156	103.9
x2^2	11.066	3.158	3.50	0.013	104.7
x1x2	-9.986	8.742	-1.14	0.297	105.1

x_2^2 is the only model parameter that is statistically significant with a t-value of 3.50. A logical model might also include x_2 to preserve model hierarchy.

(b) Use the extra sum of squares method to evaluate the value of the quadratic terms, x_1^2, x_2^2 and $x_1 x_2$ to the model.

The extra sum of squares due to β_2 is

$$SS_R(\boldsymbol{\beta}_2|\boldsymbol{\beta}_0,\boldsymbol{\beta}_1) = SS_R(\boldsymbol{\beta}_0,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2) - SS_R(\boldsymbol{\beta}_0,\boldsymbol{\beta}_1) = SS_R(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2|\boldsymbol{\beta}_0) - SS_R(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0)$$

$SS_R(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2|\boldsymbol{\beta}_0)$ sum of squares of regression for the model in Problem 10-12 = 35092.6

$$SS_R(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0)=34502.3$$

$$SS_R(\boldsymbol{\beta}_2|\boldsymbol{\beta}_0,\boldsymbol{\beta}_1) = 35092.6 - 34502.3 = 590.3$$

$$F_0 = \frac{SS_R(\boldsymbol{\beta}_2|\boldsymbol{\beta}_0,\boldsymbol{\beta}_1)/3}{MS_E} = \frac{590.3/3}{36.511} = 5.3892$$

Since $F_{0.05,3,6} = 4.76$, then the addition of the quadratic terms to the model is significant. The P-values indicate that it's probably the term x_2^2 that is responsible for this.

10-14 ***Relationship between analysis of variance and regression.*** Any analysis of variance model can be expressed in terms of the general linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where the $\mathbf{X}$ matrix consists of zeros and ones. Show that the single-factor model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, i=1,2,3, j=1,2,3,4 can be written in general linear model form. Then

(a) Write the normal equations $(\mathbf{X}'\mathbf{X})\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ and compare them with the normal equations found for the model in Chapter 3.

The normal equations are $(\mathbf{X}'\mathbf{X})\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$

$$\begin{bmatrix} 12 & 4 & 4 & 4 \\ 4 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \end{bmatrix} = \begin{bmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{bmatrix}$$

which are in agreement with the results of Chapter 3.

(b) Find the rank of $\mathbf{X}'\mathbf{X}$. Can $(\mathbf{X}'\mathbf{X})^{-1}$ be obtained?

$\mathbf{X}'\mathbf{X}$ is a 4 x 4 matrix of rank 3, because the last three columns add to the first column. Thus $(\mathbf{X'X})^{-1}$ does not exist.

(c) Suppose the first normal equation is deleted and the restriction $\sum_{i=1}^{3} n\hat{\tau}_i = 0$ is added. Can the resulting system of equations be solved? If so, find the solution. Find the regression sum of squares $\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y}$, and compare it to the treatment sum of squares in the single-factor model.

Imposing $\sum_{i=1}^{3} n\hat{\tau}_i = 0$ yields the normal equations

$$\begin{bmatrix} 0 & 4 & 4 & 4 \\ 4 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \end{bmatrix} = \begin{bmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{bmatrix}$$

The solution to this set of equations is

$$\hat{\mu} = \frac{y_{..}}{12} = \bar{y}_{..}$$
$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$$

This solution was found be solving the last three equations for $\hat{\tau}_i$, yielding $\hat{\tau}_i = \bar{y}_{i.} - \hat{\mu}$, and then substituting in the first equation to find $\hat{\mu} = \bar{y}_{..}$

The regression sum of squares is

$$SS_R(\boldsymbol{\beta}) = \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} = \bar{y}_{..}y_{..} + \sum_{i=1}^{a}(\bar{y}_{i.} - y_{..})^2 = \frac{y_{..}^2}{an} + \sum_{i=1}^{a}\frac{\bar{y}_{i.}^2}{n} - \frac{y_{..}^2}{an} = \sum_{i=1}^{a}\frac{\bar{y}_{i.}^2}{n}$$

with a degrees of freedom. This is the same result found in Chapter 3. For more discussion of the relationship between analysis of variance and regression, see Montgomery and Peck (1992).

10-15 Suppose that we are fitting a straight line and we desire to make the variance of as small as possible. Restricting ourselves to an even number of experimental points, where should we place these points so as to minimize $V(\hat{\beta}_1)$? (Note: Use the design called for in this exercise with great caution because, even though it minimized $V(\hat{\beta}_1)$, it has some undesirable properties; for example, see Myers and

Montgomery (1995). Only if you are *very sure* the true functional relationship is linear should you consider using this design.

Since $V\left(\hat{\beta}_1\right)=\dfrac{\sigma^2}{S_{xx}}$, we may minimize $V\left(\hat{\beta}_1\right)$by making S_{xx} as large as possible. S_{xx} is maximized by spreading out the x_j's as much as possible. The experimenter usually has a "region of interest" for x. If n is even, $n/2$ of the observations should be run at each end of the "region of interest". If n is odd, then run one of the observations in the center of the region and the remaining $(n-1)/2$ at either end.

10-16 ***Weighted least squares.*** Suppose that we are fitting the straight line $y=\beta_0+\beta_1 x+\varepsilon$, but the variance of the y's now depends on the level of x; that is,

$$V\left(y|x_i\right)=\sigma^2=\frac{\sigma^2}{w_i}, i=1,2,\ldots,n$$

where the w_i are known constants, often called weights. Show that if we choose estimates of the regression coefficients to minimize the weighted sum of squared errors given by $\sum_{i=1}^{n} w_i\left(y_i-\beta_0+\beta_1 x_i\right)^2$, the resulting least squares normal equations are

$$\hat{\beta}_0\sum_{i=1}^{n} w_i+\hat{\beta}_1\sum_{i=1}^{n} w_i x_i=\sum_{i=1}^{n} w_i y_i$$
$$\hat{\beta}_0\sum_{i=1}^{n} w_i x_i+\hat{\beta}_1\sum_{i=1}^{n} w_i x_i^2=\sum_{i=1}^{n} w_i x_i y_i$$

The least squares normal equations are found:

$$L=\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1 x_1\right)^2 w_i$$
$$\frac{\partial L}{\partial\beta_0}=-2\sum_{i=1}^{n}\left(y_i-\hat{\beta}_0-\hat{\beta}_1 x_1\right)w_i=0$$
$$\frac{\partial L}{\partial\beta_1}=-2\sum_{i=1}^{n}\left(y_i-\hat{\beta}_0-\hat{\beta}_1 x_1\right)x_1 w_i=0$$

which simplify to

$$\hat{\beta}_0\sum_{i=1}^{n} w_i+\hat{\beta}_1\sum_{i=1}^{n} x_1 w_i=\sum_{i=1}^{n} w_i y_i$$
$$\hat{\beta}_0\sum_{i=1}^{n} x_1 w_i+\hat{\beta}_1\sum_{i=1}^{n} x_1^2 w_i=\sum_{i=1}^{n} w_i x_1 y_i$$

10-17 Consider the 2_{IV}^{4-1} design discussed in Example 10-5.

(a) Suppose you elect to augment the design with the single run selected in that example. Find the variances and covariances of the regression coefficients in the model (ignoring blocks):

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_{12} x_1 x_2 + \beta_{34} x_3 x_4 + \varepsilon$$

$$\mathbf{X'X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 9 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 9 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 9 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 9 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 9 & 7 \\ -1 & 1 & 1 & 1 & -1 & 7 & 9 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} 0.125 & 0 & 0 & 0 & 0 & -0.0625 & 0.0625 \\ 0 & 0.125 & 0 & 0 & 0 & 0.0625 & -0.0625 \\ 0 & 0 & 0.125 & 0 & 0 & 0.0625 & -0.0625 \\ 0 & 0 & 0 & 0.125 & 0 & 0.0625 & -0.0625 \\ 0 & 0 & 0 & 0 & 0.125 & -0.0625 & 0.0625 \\ -0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.0625 & 0.4375 & -0.375 \\ 0.0625 & -0.0625 & -0.0625 & -0.0625 & 0.0625 & -0.375 & 0.4375 \end{bmatrix}$$

(b) Are there any other runs in the alternate fraction that

Any other run from the alternate fraction will dealias *AB* from *CD.*

(c) Suppose you augment the design with four runs suggested in Example 10-5. Find the variance and the covariances of the regression coefficients (ignoring blocks) for the model in part (a).

Choose 4 runs that are one of the quarter fractions not used in the principal half fraction.

$$\mathbf{X'X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{X'X} = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 12 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 12 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 12 & 4 & 0 \\ 0 & 4 & 0 & 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} 0.0833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1071 & 0 & 0 & -0.0179 & -0.0536 & 0.0357 \\ 0 & 0 & 0.0938 & 0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0.0313 & 0.0938 & 0 & 0 & 0 \\ 0 & -0.0179 & 0 & 0 & 0.1071 & -0.0536 & 0.0357 \\ 0 & -0.0536 & 0 & 0 & -0.0536 & 0.2142 & -0.1429 \\ 0 & 0.0357 & 0 & 0 & 0.0357 & -0.1429 & 0.1785 \end{bmatrix}$$

(d) Considering parts (a) and (c), which augmentation strategy would you prefer and why?

If you only have the resources to run one more run, then choose the one-run augmentation. But if resources are not scarce, then augment the design in multiples of two runs, to keep the design orthogonal. Using four runs results in smaller variances of the regression coefficients and a simpler covariance structure.

10-18 Consider the 2_{III}^{7-4}. Suppose after running the experiment, the largest observed effects are *A* + *BD*, *B* + *AD*, and *D* + *AB*. You wish to augment the original design with a group of four runs to dealias these effects.

(a) Which four runs would you make?

Take the first four runs of the original experiment and change the sign on *A*.

Design Expert Output

			Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
Std	Run	Block	A:x1	B:x2	C:x3	D:x4	E:x5	F:x6	G:x7
1	1	Block 1	-1.00	-1.00	-1.00	1.00	1.00	1.00	-1.00
2	2	Block 1	1.00	-1.00	-1.00	-1.00	-1.00	1.00	1.00
3	3	Block 1	-1.00	1.00	-1.00	-1.00	1.00	-1.00	1.00
4	4	Block 1	1.00	1.00	-1.00	1.00	-1.00	-1.00	-1.00
5	5	Block 1	-1.00	-1.00	1.00	1.00	-1.00	-1.00	1.00
6	6	Block 1	1.00	-1.00	1.00	-1.00	1.00	-1.00	-1.00
7	7	Block 1	-1.00	1.00	1.00	-1.00	-1.00	1.00	-1.00
8	8	Block 1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	9	Block 2	1.00	1.00	1.00	-1.00	-1.00	-1.00	-1.00
10	10	Block 2	1.00	-1.00	-1.00	1.00	-1.00	-1.00	-1.00
11	11	Block 2	-1.00	-1.00	1.00	1.00	-1.00	-1.00	-1.00
12	12	Block 2	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00

Main effects and interactions of interest are:

x1	x2	x4	x1x2	x1x4	x2x4
-1	-1	1	1	-1	-1

1	-1	-1	-1	-1	1
-1	1	-1	-1	1	-1
1	1	1	1	1	1
-1	-1	1	1	-1	-1
1	-1	-1	-1	-1	1
-1	1	-1	-1	1	-1
1	1	1	1	1	1
1	**-1**	**1**	**-1**	**1**	**-1**
-1	**-1**	**-1**	**1**	**1**	**1**
1	**1**	**-1**	**1**	**-1**	**-1**
-1	**1**	**1**	**-1**	**-1**	**1**

(b) Find the variances and covariances of the regression coefficients in the model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_4 x_4 + \beta_{12} x_1 x_2 + \beta_{14} x_1 x_4 + \beta_{24} x_2 x_4 + \varepsilon$$

$$\mathbf{X'X} = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 12 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 12 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 12 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 12 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} 0.0833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1071 & -0.0178 & 0 & 0 & 0.0536 & 0.0714 \\ 0 & -0.0179 & 0.1071 & 0 & 0 & 0.0714 & -0.0536 \\ 0 & 0 & 0 & 0.0938 & 0.0313 & 0 & 0 \\ 0 & 0 & 0 & 0.0313 & 0.0938 & 0 & 0 \\ 0 & -0.0536 & 0.0714 & 0 & 0 & 0.2143 & -0.1607 \\ 0 & 0.0714 & -0.0536 & 0 & 0 & -0.1607 & 0.2143 \end{bmatrix}$$

(c) Is it possible to dealias these effects with fewer than four additional runs?

It is possible to dealias these effects in only two runs. By utilizing Design Expert's design augmentation function, the runs 9 and 10 (Block 2) were generated as follows:

Design Expert Output

			Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
Std	Run	Block	A:x1	B:x2	C:x3	D:x4	E:x5	F:x6	G:x7
1	1	Block 1	-1.00	-1.00	-1.00	1.00	1.00	1.00	-1.00
2	2	Block 1	1.00	-1.00	-1.00	-1.00	-1.00	1.00	1.00
3	3	Block 1	-1.00	1.00	-1.00	-1.00	1.00	-1.00	1.00
4	4	Block 1	1.00	1.00	-1.00	1.00	-1.00	-1.00	-1.00
5	5	Block 1	-1.00	-1.00	1.00	1.00	-1.00	-1.00	1.00
6	6	Block 1	1.00	-1.00	1.00	-1.00	1.00	-1.00	-1.00
7	7	Block 1	-1.00	1.00	1.00	-1.00	-1.00	1.00	-1.00
8	8	Block 1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	9	Block 2	-1.00	1.00	-1.00	1.00	-1.00	-1.00	-1.00
10	10	Block 2	1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00

Chapter 11
Response Surface Methods and Other Approaches to Process Optimization
Solutions

11-1 A chemical plant produces oxygen by liquefying air and separating it into its component gases by fractional distillation. The purity of the oxygen is a function of the main condenser temperature and the pressure ratio between the upper and lower columns. Current operating conditions are temperature $(\xi_1) =$ -220°C and pressure ratio $(\xi_2) = 1.2$. Using the following data find the path of steepest ascent.

Temperature (x_1)	Pressure Ratio (x_2)	Purity
-225	1.1	82.8
-225	1.3	83.5
-215	1.1	84.7
-215	1.3	85.0
-220	1.2	84.1
-220	1.2	84.5
-220	1.2	83.9
-220	1.2	84.3

Design Expert Output

Response: Purity
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3.14	2	1.57	26.17	0.0050	significant
A	*2.89*	*1*	*2.89*	*48.17*	*0.0023*	
B	*0.25*	*1*	*0.25*	*4.17*	*0.1108*	
Curvature	0.080	1	0.080	1.33	0.3125	not significant
Residual	0.24	4	0.060			
Lack of Fit	*0.040*	*1*	*0.040*	*0.60*	*0.4950*	*not significant*
Pure Error	*0.20*	*3*	*0.067*			
Cor Total	3.46	7				

The Model F-value of 26.17 implies the model is significant. There is only a 0.50% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.24	R-Squared	0.9290
Mean	84.10	Adj R-Squared	0.8935
C.V.	0.29	Pred R-Squared	0.7123
PRESS	1.00	Adeq Precision	12.702

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	84.00	1	0.12	83.66	84.34	
A-Temperature	0.85	1	0.12	0.51	1.19	1.00
B-Pressure Ratio	0.25	1	0.12	-0.090	0.59	1.00
Center Point	0.20	1	0.17	-0.28	0.68	1.00

Final Equation in Terms of Coded Factors:

Purity =
+84.00
+0.85 * A
+0.25 * B

Final Equation in Terms of Actual Factors:

Purity =
+118.40000

```
                    +0.17000   * Temperature
                    +2.50000   * Pressure Ratio
```

From the computer output use the model $\hat{y} = 84 + 0.85x_1 + 0.25x_2$ as the equation for steepest ascent. Suppose we use a one degree change in temperature as the basic step size. Thus, the path of steepest ascent passes through the point (x_1=0, x_2=0) and has a slope 0.25/0.85. In the coded variables, one degree of temperature is equivalent to a step of Δx_1 = 1/5=0.2. Thus, Δx_2 = (0.25/0.85)0.2=0.059. The path of steepest ascent is:

	Coded	Variables	Natural	Variables
	x_1	x_2	ξ_1	ξ_2
Origin	0	0	-220	1.2
Δ	0.2	0.059	1	0.0059
Origin + Δ	0.2	0.059	-219	1.2059
Origin +5 Δ	1.0	0.295	-215	1.2295
Origin +7 Δ	1.40	0.413	-213	1.2413

11-2 An industrial engineer has developed a computer simulation model of a two-item inventory system. The decision variables are the order quantity and the reorder point for each item. The response to be minimized is the total inventory cost. The simulation model is used to produce the data shown in the following table. Identify the experimental design. Find the path of steepest descent.

Item 1		Item 2		
Order Quantity (x1)	Reorder Point (x2)	Order Quantity (x3)	Reorder Point (x4)	Total Cost
100	25	250	40	625
140	45	250	40	670
140	25	300	40	663
140	25	250	80	654
100	45	300	40	648
100	45	250	80	634
100	25	300	80	692
140	45	300	80	686
120	35	275	60	680
120	35	275	60	674
120	35	275	60	681

The design is a 2^{4-1} fractional factorial with generator *I=ABCD*, and three center points.

Design Expert Output

Response: Total Cost
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	3880.00	6	646.67	63.26	0.0030	significant
A	*684.50*	*1*	*684.50*	*66.96*	*0.0038*	
C	*1404.50*	*1*	*1404.50*	*137.40*	*0.0013*	
D	*450.00*	*1*	*450.00*	*44.02*	*0.0070*	
AC	*392.00*	*1*	*392.00*	*38.35*	*0.0085*	
AD	*264.50*	*1*	*264.50*	*25.88*	*0.0147*	
CD	*684.50*	*1*	*684.50*	*66.96*	*0.0038*	
Curvature	815.52	1	815.52	79.78	0.0030	significant
Residual	30.67	3	10.22			
Lack of Fit	*2.00*	*1*	*2.00*	*0.14*	*0.7446*	*not significant*
Pure Error	*28.67*	*2*	*14.33*			
Cor Total	4726.18	10				

The Model F-value of 63.26 implies the model is significant. There is only

a 0.30% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	3.20	R-Squared	0.9922
Mean	664.27	Adj R-Squared	0.9765
C.V.	0.48	Pred R-Squared	0.9593
PRESS	192.50	Adeq Precision	24.573

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	659.00	1	1.13	655.40	662.60	
A-Item 1 QTY	9.25	1	1.13	5.65	12.85	1.00
C-Item 2 QTY	13.25	1	1.13	9.65	16.85	1.00
D-Item 2 Reorder	7.50	1	1.13	3.90	11.10	1.00
AC	-7.00	1	1.13	-10.60	-3.40	1.00
AD	-5.75	1	1.13	-9.35	-2.15	1.00
CD	9.25	1	1.13	5.65	12.85	1.00
Center Point	19.33	1	2.16	12.44	26.22	1.00

Final Equation in Terms of Coded Factors:

Total Cost	=
+659.00	
+9.25	* A
+13.25	* C
+7.50	* D
-7.00	* A * C
-5.75	* A * D
+9.25	* C * D

Final Equation in Terms of Actual Factors:

Total Cost	=
+175.00000	
+5.17500	* Item 1 QTY
+1.10000	* Item 2 QTY
-2.98750	* Item 2 Reorder
-0.014000	* Item 1 QTY * Item 2 QTY
-0.014375	* Item 1 QTY * Item 2 Reorder
+0.018500	* Item 2 QTY * Item 2 Reorder
+0.019	* Item 2 QTY * Item 2 Reorder

The equation used to compute the path of steepest ascent is $\hat{y} = 659 + 9.25x_1 + 13.25x_3 + 7.50x_4$. Notice that even though the model contains interaction, it is relatively common practice to ignore the interactions in computing the path of steepest ascent. This means that the path constructed is only an approximation to the path that would have been obtained if the interactions were considered, but it's usually close enough to give satisfactory results.

It is helpful to give a general method for finding the path of steepest ascent. Suppose we have a first-order model in k variables, say

$$\hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i$$

The path of steepest ascent passes through the origin, **x**=**0**, and through the point on a hypersphere of radius, R where $\hat{y}$ is a maximum. Thus, the x's must satisfy the constraint

$$\sum_{i=1}^{k} x_i^2 = R^2$$

To find the set of x's that maximize $\hat{y}$ subject to this constraint, we maximize

$$L = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i - \lambda \left[\sum_{i=1}^{k} x_i^2 - R^2 \right]$$

where λ is a LaGrange multiplier. From $\partial L / \partial x_i = \partial L / \partial \lambda = 0$, we find

$$x_i = \frac{\hat{\beta}_i}{2\lambda}$$

It is customary to specify a basic step size in one of the variables, say Δx_j, and then calculate 2λ as $2\lambda = \hat{\beta}_j / \Delta x_j$. Then this value of 2λ can be used to generate the remaining coordinates of a point on the path of steepest ascent.

We demonstrate using the data from this problem. Suppose that we use -10 units in ξ_1 as the basic step size. Note that a decrease in ξ_1 is called for, because we are looking for a path of steepest decent. Now -10 units in ξ_1 is equal to -10/20 = -0.5 units change in x_1.

Thus, $2\lambda = \hat{\beta}_1 / \Delta x_1 = 9.25/(-0.5) = -18.50$

Consequently,

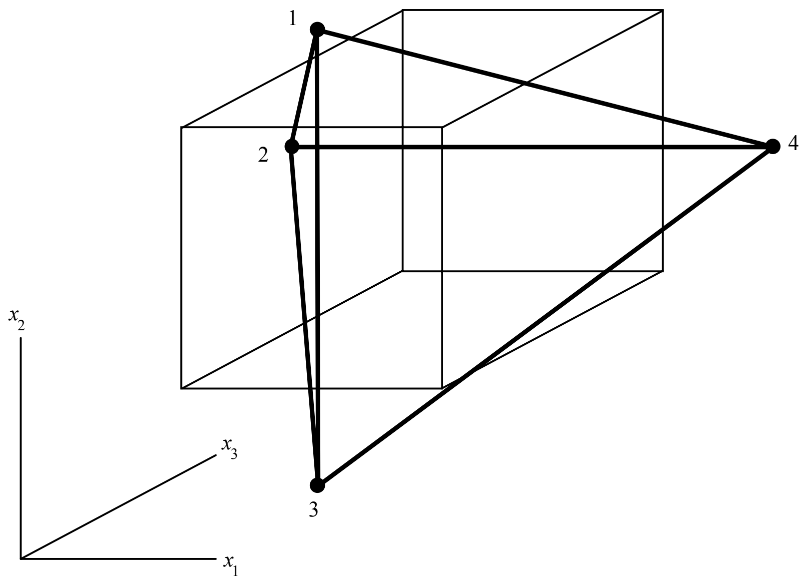
$$\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda} = \frac{13.25}{-18.50} = -0.716$$
$$\Delta x_4 = \frac{\hat{\beta}_4}{2\lambda} = \frac{7.50}{-18.50} = -0.705$$

are the remaining coordinates of points along the path of steepest decent, in terms of the coded variables. The path of steepest decent is shown below:

	Coded	Variables			Natural	Variables		
	x_1	x_2	x_3	x_4	ξ_1	ξ_2	ξ_3	ξ_4
Origin	0	0	0	0	120	35	275	60
Δ	-0.50	0	-0.716	-0.405	-10	0	-17.91	-8.11
Origin + Δ	-0.50	0	-0.716	-0.405	110	35	257.09	51.89
Origin +2 Δ	-1.00	0	-1.432	-0.810	100	35	239.18	43.78

11-3 Verify that the following design is a simplex. Fit the first-order model and find the path of steepest ascent.

Position	x_1	x_2	x_3	y
1	0	$\sqrt{2}$	-1	18.5
2	$-\sqrt{2}$	0	1	19.8
3	0	$-\sqrt{2}$	-1	17.4
4	$\sqrt{2}$	0	1	22.5



The graphical representation of the design identifies a tetrahedron; therefore, the design is a simplex.

Design Expert Output

Response: y
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	14.49	3	4.83		
A	*3.64*	*1*	*3.64*		
B	*0.61*	*1*	*0.61*		
C	*10.24*	*1*	*10.24*		
Pure Error	0.000	0			
Cor Total	14.49	3			

Std. Dev.		R-Squared	1.0000
Mean	19.55	Adj R-Squared	
C.V.		Pred R-Squared	N/A
PRESS	N/A	Adeq Precision	0.000

Case(s) with leverage of 1.0000: Pred R-Squared and PRESS statistic not defined

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	19.55	1				
A-x1	1.35	1				1.00
B-x2	0.55	1				1.00
C-x3	1.60	1				1.00

Final Equation in Terms of Coded Factors:

y =
+19.55
+1.35 * A
+0.55 * B
+1.60 * C

Final Equation in Terms of Actual Factors:

y =
+19.55000
+0.95459 * x1
+0.38891 * x2
+1.60000 * x3

The first order model is $\hat{y} = 19.55 + 1.35x_1 + 0.55x_2 + 1.60x_3$.

To find the path of steepest ascent, let the basic step size be $\Delta x_3 = 1$. Then using the results obtained in the previous problem, we obtain

$$\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda} \quad \text{or} \quad 1.0 = \frac{1.60}{2\lambda}$$

which yields $2\lambda = 1.60$. Then the coordinates of points on the path of steepest ascent are defined by

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} = \frac{0.96}{1.60} = 0.60$$
$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{0.24}{1.60} = 0.24$$

Therefore, in the coordinated variables we have:

	Coded	Variables	
	x_1	x_2	x_3
Origin	0	0	0
Δ	0.60	0.24	1.00
Origin + Δ	0.60	0.24	1.00
Origin +2 Δ	1.20	0.48	2.00

11-4 For the first-order model $\hat{y} = 60 + 1.5x_1 - 0.8x_2 + 2.0x_3$ find the path of steepest ascent. The variables are coded as $-1 \le x_i \le 1$.

Let the basic step size be $\Delta x_3 = 1$. $\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda}$ or $1.0 = \frac{2.0}{2\lambda}$. Then $2\lambda = 2.0$

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} = \frac{1.50}{2.0} = 0.75$$
$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{-0.8}{2.0} = -0.40$$

Therefore, in the coded variables we have

	Coded	Variables	
	x_1	x_2	x_3
Origin	0	0	0
Δ	0.75	-0.40	1.00
Origin + Δ	0.75	-0.40	1.00
Origin +2 Δ	1.50	-0.80	2.00

11-5 The region of experimentation for three factors are time ($40 \le T_1 \le 80$ min), temperature ($200 \le T_2 \le 300$ °C), and pressure ($20 \le P \le 50$ psig). A first-order model in coded variables has been fit to yield data from a 2^3 design. The model is

$$\hat{y} = 30 + 5x_1 + 2.5x_2 + 3.5x_3$$

Is the point $T_1 = 85$, $T_2 = 325$, P=60 on the path of steepest ascent?

The coded variables are found with the following:

$$x_1 = \frac{T_1 - 60}{20} \qquad x_2 = \frac{T_2 - 250}{50} \qquad x_3 \frac{P_1 - 35}{15}$$

$$\Delta T_1 = 5 \quad \Delta x_1 = \frac{5}{20} = 0.25$$

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} \text{ or } 0.25 = \frac{20}{2\lambda} \quad 2\lambda = 20$$

$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{2.5}{20} = 0.125$$

$$\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda} = \frac{3.5}{20} = 0.175$$

	Coded	Variables		Natural	Variables	
	x_1	x_2	x_3	T_1	T_2	P
Origin	0	0	0	60	250	35
Δ	0.25	0.125	0.175	5	6.25	2.625
Origin + Δ	0.25	0.125	0.175	65	256.25	37.625
Origin +5 Δ	1.25	0.625	0.875	85	281.25	48.125

The point T_1=85, T_2=325, and P=60 is not on the path of steepest ascent.

11-6 The region of experimentation for two factors are temperature ($100 \le T \le 300°$ F) and catalyst feed rate ($10 \le C \le 30$ lb/h). A first order model in the usual ± 1 coded variables has been fit to a molecular weight response, yielding the following model.

$$\hat{y} = 2000 + 125x_1 + 40x_2$$

(a) Find the path of steepest ascent.

$$x_1 = \frac{T - 200}{100} \qquad x_2 = \frac{C - 20}{10}$$

$$\Delta T = 100 \quad \Delta x_1 = \frac{100}{100} = 1$$

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} \text{ or } 1 = \frac{125}{2\lambda} \quad 2\lambda = 125$$

$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{40}{125} = 0.32$$

	Coded	Variables	Natural	Variables
	x_1	x_2	T	C
Origin	0	0	200	20
Δ	1	0.32	100	3.2

Origin + Δ	1	0.32	300	23.2
Origin +5 Δ	5	1.60	700	36.0

(a) It is desired to move to a region where molecular weights are above 2500. Based on the information you have from the experiment, in this region, about how may steps along the path of steepest ascent might be required to move to the region of interest?

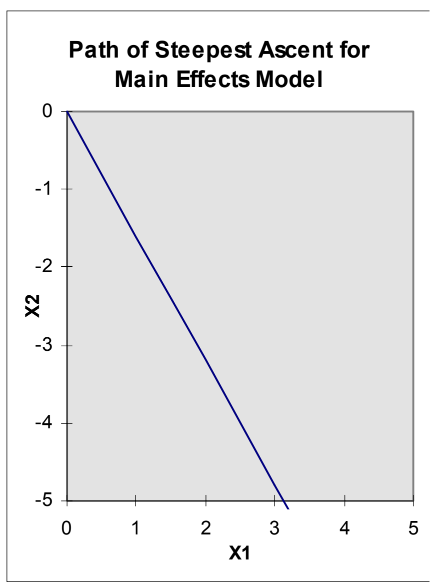
$$\Delta\hat{y} = \Delta x_1 \hat{\beta}_1 + \Delta x_2 \hat{\beta}_2 = (1)(125) + (0.32)(40) = 137.8$$
$$\# \, Steps = \frac{2500 - 2000}{137.8} = 3.63 \rightarrow 4$$

11-7 The path of steepest ascent is usually computed assuming that the model is truly first-order.; that is, there is no interaction. However, even if there is interaction, steepest ascent ignoring the interaction still usually produces good results. To illustrate, suppose that we have fit the model

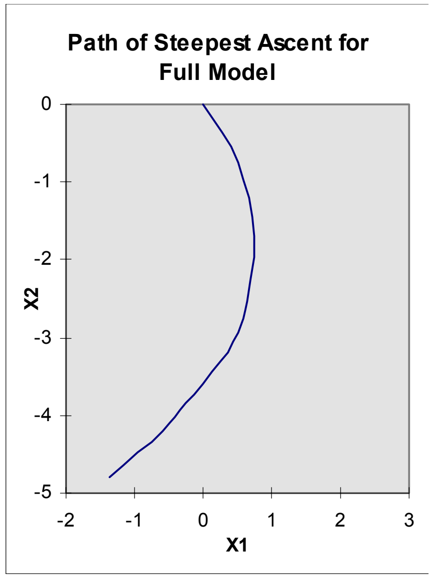
$$\hat{y} = 20 + 5x_1 - 8x_2 + 3x_1 x_2$$

using coded variables ($-1 \le x_1 \le +1$)

(a) Draw the path of steepest ascent that you would obtain if the interaction were ignored.



(b) Draw the path of steepest ascent that you would obtain with the interaction included in the model. Compare this with the path found in part (a).



11-8 The data shown in the following table were collected in an experiment to optimize crystal growth as a function of three variables x_1, x_2, and x_3. Large values of y (yield in grams) are desirable. Fit a second order model and analyze the fitted surface. Under what set of conditions is maximum growth achieved?

x_1	x_2	x_3	y
-1	-1	-1	66
-1	-1	1	70
-1	1	-1	78
-1	1	1	60
1	-1	-1	80
1	-1	1	70
1	1	-1	100
1	1	1	75
-1.682	0	0	100
1.682	0	0	80
0	-1.682	0	68
0	1.682	0	63
0	0	-1.682	65
0	0	1.682	82
0	0	0	113
0	0	0	100
0	0	0	118
0	0	0	88
0	0	0	100
0	0	0	85

Design Expert Output

Response: Yield
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3662.00	9	406.89	2.19	0.1194	not significant
A	*22.08*	*1*	*22.08*	*0.12*	*0.7377*	
B	*25.31*	*1*	*25.31*	*0.14*	*0.7200*	
C	*30.50*	*1*	*30.50*	*0.16*	*0.6941*	

A^2	*204.55*	*1*	*204.55*	*1.10*	*0.3191*	
B^2	*2226.45*	*1*	*2226.45*	*11.96*	*0.0061*	
C^2	*1328.46*	*1*	*1328.46*	*7.14*	*0.0234*	
AB	*66.12*	*1*	*66.12*	*0.36*	*0.5644*	
AC	*55.13*	*1*	*55.13*	*0.30*	*0.5982*	
BC	*171.13*	*1*	*171.13*	*0.92*	*0.3602*	
Residual	1860.95	10	186.09			
Lack of Fit	*1001.61*	*5*	*200.32*	*1.17*	*0.4353*	*not significant*
Pure Error	*859.33*	*5*	*171.87*			
Cor Total	5522.95	19				

The "Model F-value" of 2.19 implies the model is not significant relative to the noise. There is a 11.94 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	13.64	R-Squared	0.6631
Mean	83.05	Adj R-Squared	0.3598
C.V.	16.43	Pred R-Squared	-0.6034
PRESS	8855.23	Adeq Precision	3.882

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	100.67	1	5.56	88.27	113.06	
A-x1	1.27	1	3.69	-6.95	9.50	1.00
B-x2	1.36	1	3.69	-6.86	9.59	1.00
C-x3	-1.49	1	3.69	-9.72	6.73	1.00
A^2	-3.77	1	3.59	-11.77	4.24	1.02
B^2	-12.43	1	3.59	-20.44	-4.42	1.02
C^2	-9.60	1	3.59	-17.61	-1.59	1.02
AB	2.87	1	4.82	-7.87	13.62	1.00
AC	-2.63	1	4.82	-13.37	8.12	1.00
BC	-4.63	1	4.82	-15.37	6.12	1.00

Final Equation in Terms of Coded Factors:

Yield	=
+100.67	
+1.27	* A
+1.36	* B
-1.49	* C
-3.77	* A^2
-12.43	* B^2
-9.60	* C^2
+2.87	* A * B
-2.63	* A * C
-4.63	* B * C

Final Equation in Terms of Actual Factors:

Yield	=
+100.66609	
+1.27146	* x1
+1.36130	* x2
-1.49445	* x3
-3.76749	* $x1^2$
-12.42955	* $x2^2$
-9.60113	* $x3^2$
+2.87500	* x1 * x2
-2.62500	* x1 * x3
-4.62500	* x2 * x3

There are so many nonsignificant terms in this model that we should consider eliminating some of them. A reasonable reduced model is shown below.

Design Expert Output

Response: Yield

ANOVA for Response Surface Reduced Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3143.00	4	785.75	4.95	0.0095	significant
B	25.31	1	25.31	0.16	0.6952	
C	30.50	1	30.50	0.19	0.6673	
B2	2115.31	1	2115.31	13.33	0.0024	
C2	1239.17	1	1239.17	7.81	0.0136	
Residual	2379.95	15	158.66			
Lack of Fit	*1520.62*	*10*	*152.06*	*0.88*	*0.5953*	*not significant*
Pure Error	*859.33*	*5*	*171.87*			
Cor Total	5522.95	19				

The Model F-value of 4.95 implies the model is significant. There is only a 0.95% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	12.60	R-Squared	0.5691
Mean	83.05	Adj R-Squared	0.4542
C.V.	15.17	Pred R-Squared	0.1426
PRESS	4735.52	Adeq Precision	5.778

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	97.58	1	4.36	88.29	106.88	
B-x2	1.36	1	3.41	-5.90	8.63	1.00
C-x3	-1.49	1	3.41	-8.76	5.77	1.00
B2	-12.06	1	3.30	-19.09	-5.02	1.01
C2	-9.23	1	3.30	-16.26	-2.19	1.01

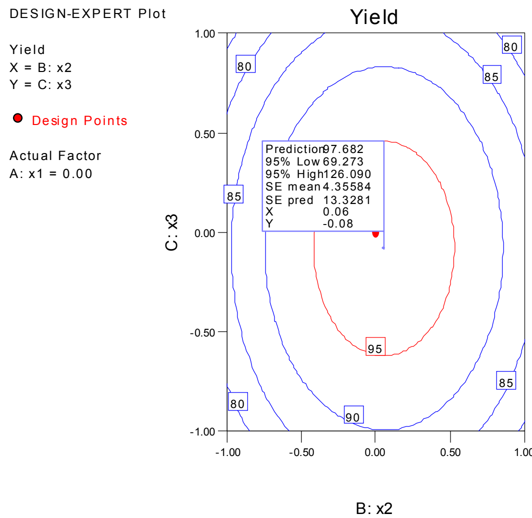
Final Equation in Terms of Coded Factors:

$$\begin{aligned} \text{Yield} &= \\ &+97.58 \\ &+1.36 * B \\ &-1.49 * C \\ &-12.06 * B^2 \\ &-9.23 * C^2 \end{aligned}$$

Final Equation in Terms of Actual Factors:

$$\begin{aligned} \text{Yield} &= \\ &+97.58260 \\ &+1.36130 * x2 \\ &-1.49445 * x3 \\ &-12.05546 * x2^2 \\ &-9.22703 * x3^2 \end{aligned}$$

The contour plot identifies a maximum near the center of the design space.



11-9 The following data were collected by a chemical engineer. The response y is filtration time, x_1 is temperature, and x_2 is pressure. Fit a second-order model.

x_1	x_2	y
-1	-1	54
-1	1	45
1	-1	32
1	1	47
-1.414	0	50
1.414	0	53
0	-1.414	47
0	1.414	51
0	0	41
0	0	39
0	0	44
0	0	42
0	0	40

Design Expert Output

Response: **y**
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	264.22	4	66.06	2.57	0.1194	not significant
A	13.11	1	13.11	0.51	0.4955	
B	25.72	1	25.72	1.00	0.3467	
A^2	81.39	1	81.39	3.16	0.1132	
AB	144.00	1	144.00	5.60	0.0455	
Residual	205.78	8	25.72			
Lack of Fit	*190.98*	*4*	*47.74*	*12.90*	*0.0148*	*significant*
Pure Error	*14.80*	*4*	*3.70*			
Cor Total	470.00	12				

The "Model F-value" of 2.57 implies the model is not significant relative to the noise. There is a 11.94 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev. 5.07 R-Squared 0.5622

Mean	45.00	Adj R-Squared	0.3433
C.V.	11.27	Pred R-Squared	-0.5249
PRESS	716.73	Adeq Precision	4.955

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	42.91	1	1.83	38.69	47.14	
A-Temperature	1.28	1	1.79	-2.85	5.42	1.00
B-Pressure	-1.79	1	1.79	-5.93	2.34	1.00
A^2	3.39	1	1.91	-1.01	7.79	1.00
AB	6.00	1	2.54	0.15	11.85	1.00

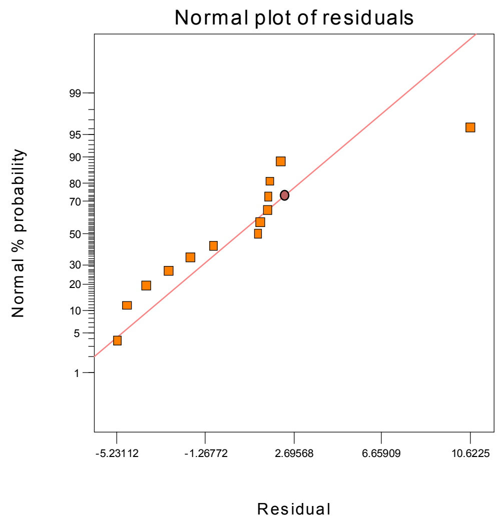
Final Equation in Terms of Coded Factors:

$$\text{Time} = +42.91 + 1.28 * A - 1.79 * B + 3.39 * A^2 + 6.00 * A * B$$

Final Equation in Terms of Actual Factors:

$$\text{Time} = +42.91304 + 1.28033 * \text{Temperature} - 1.79289 * \text{Pressure} + 3.39130 * \text{Temperature}^2 + 6.00000 * \text{Temperature} * \text{Pressure}$$

The lack of fit test in the above analysis is significant. Also, the residual plot below identifies an outlier which happens to be standard order number 8.



We chose to remove this run and re-analyze the data.

Design Expert Output

Response: y

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	407.34	4	101.84	30.13	0.0002	significant
A	13.11	1	13.11	3.88	0.0895	

B	132.63	1	132.63	39.25	0.0004	
A^2	155.27	1	155.27	45.95	0.0003	
AB	144.00	1	144.00	42.61	0.0003	
Residual	23.66	7	3.38			
Lack of Fit	*8.86*	*3*	*2.95*	*0.80*	*0.5560*	*not significant*
Pure Error	*14.80*	*4*	*3.70*			
Cor Total	431.00	11				

The Model F-value of 30.13 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.84	R-Squared	0.9451
Mean	44.50	Adj R-Squared	0.9138
C.V.	4.13	Pred R-Squared	0.8129
PRESS	80.66	Adeq Precision	18.243

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	40.68	1	0.73	38.95	42.40	
A-Temperature	1.28	1	0.65	-0.26	2.82	1.00
B-Pressure	-4.82	1	0.77	-6.64	-3.00	1.02
A^2	4.88	1	0.72	3.18	6.59	1.02
AB	6.00	1	0.92	3.83	8.17	1.00

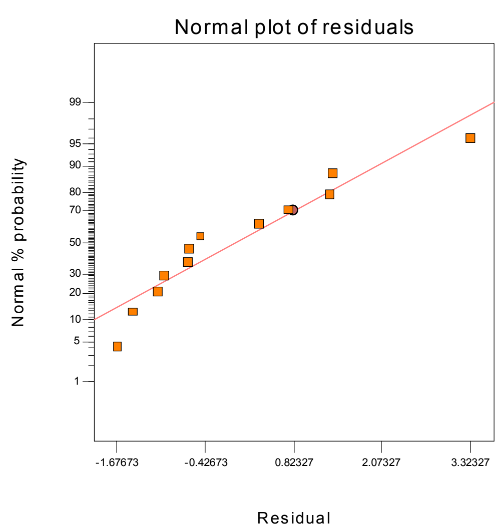
Final Equation in Terms of Coded Factors:

$$\text{Time} = +40.68 + 1.28 * A - 4.82 * B + 4.88 * A^2 + 6.00 * A * B$$

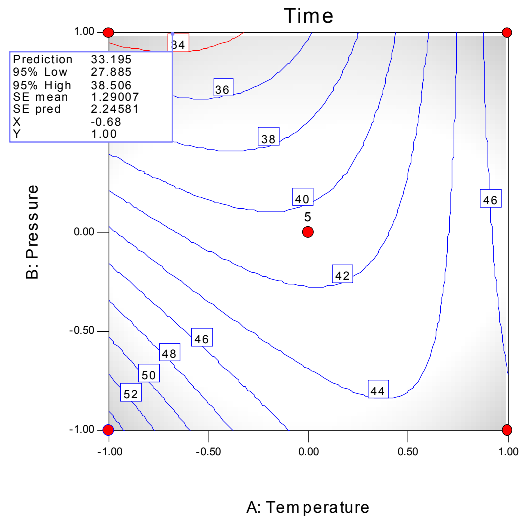
Final Equation in Terms of Actual Factors:

$$\text{Time} = +40.67673 + 1.28033 * \text{Temperature} - 4.82374 * \text{Pressure} + 4.88218 * \text{Temperature}^2 + 6.00000 * \text{Temperature} * \text{Pressure}$$

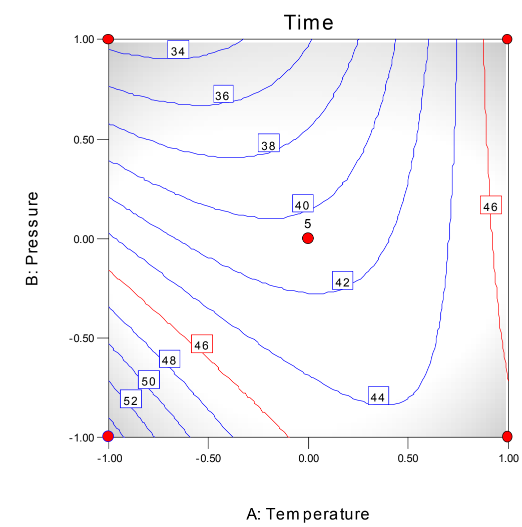
The lack of fit test is satisfactory as well as the following normal plot of residuals:



(a) What operating conditions would you recommend if the objective is to minimize the filtration time?



(b) What operating conditions would you recommend if the objective is to operate the process at a mean filtration time very close to 46?



There are two regions that enable a filtration time of 46. Either will suffice; however, higher temperatures and pressures typically have higher operating costs. We chose the operating conditions at the lower pressure and temperature as shown.

11-10 The hexagon design that follows is used in an experiment that has the objective of fitting a second-order model.

x_1	x_2	y
1	0	68

0.5	$\sqrt{0.75}$	74
-0.5	$\sqrt{0.75}$	65
-1	0	60
-0.5	$-\sqrt{0.75}$	63
0.5	$-\sqrt{0.75}$	70
0	0	58
0	0	60
0	0	57
0	0	55
0	0	69

(a) Fit the second-order model.

Design Expert Output

Response: y

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	245.26	5	49.05	1.89	0.2500	not significant
A	*85.33*	*1*	*85.33*	*3.30*	*0.1292*	
B	*9.00*	*1*	*9.00*	*0.35*	*0.5811*	
A^2	*25.20*	*1*	*25.20*	*0.97*	*0.3692*	
B^2	*129.83*	*1*	*129.83*	*5.01*	*0.0753*	
AB	*1.00*	*1*	*1.00*	*0.039*	*0.8519*	
Residual	129.47	5	25.89			
Lack of Fit	*10.67*	*1*	*10.67*	*0.36*	*0.5813*	*not significant*
Pure Error	*118.80*	*4*	*29.70*			
Cor Total	374.73	10				

The "Model F-value" of 1.89 implies the model is not significant relative to the noise. There is a 25.00 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	5.09	R-Squared	0.6545
Mean	63.55	Adj R-Squared	0.3090
C.V.	8.01	Pred R-Squared	-0.5201
PRESS	569.63	Adeq Precision	3.725

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	59.80	1	2.28	53.95	65.65	
A-x1	5.33	1	2.94	-2.22	12.89	1.00
B-x2	1.73	1	2.94	-5.82	9.28	1.00
A^2	4.20	1	4.26	-6.74	15.14	1.00
B^2	9.53	1	4.26	-1.41	20.48	1.00
AB	1.15	1	5.88	-13.95	16.26	1.00

Final Equation in Terms of Coded Factors:

$$
\begin{aligned}
y &= \\
&+59.80 \\
&+5.33 \; * A \\
&+1.73 \; * B \\
&+4.20 \; * A^2 \\
&+9.53 \; * B^2 \\
&+1.15 \; * A * B
\end{aligned}
$$

(a) Perform the canonical analysis. What type of surface has been found?

The full quadratic model is used in the following analysis because the reduced model is singular.

Solution

Variable	Critical Value
X1	-0.627658
X2	-0.052829
Predicted Value at Solution	58.080492

Eigenvalues and Eigenvectors

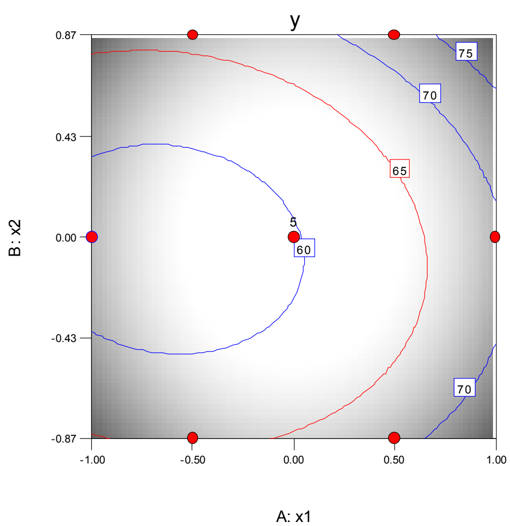
Variable	9.5957	4.1382
X1	0.10640	0.99432
X2	0.99432	-0.10640

Since both eigenvalues are positive, the response is a minimum at the stationary point.

(c) What operating conditions on x_1 and x_2 lead to the stationary point?

The stationary point is (x_1,x_2) = (-0.62766, -0.05283)

(d) Where would you run this process if the objective is to obtain a response that is as close to 65 as possible?



Any value of x_1 and x_2 that give a point on the contour with value of 65 would be satisfactory.

11-11 An experimenter has run a Box-Behnken design and has obtained the results below, where the response variable is the viscosity of a polymer.

Level	Temp.	Agitation Rate	Pressure	x_1	x_2	x_3
High	200	10.0	25	+1	+1	+1
Middle	175	7.5	20	0	0	0
Low	150	5.0	15	-1	-1	-1

Run	x_1	x_2	x_3	y_1
1	-1	-1	0	535
2	1	-1	0	580
3	-1	1	0	596
4	1	1	0	563
5	-1	0	-1	645
6	1	0	-1	458
7	-1	0	1	350
8	1	0	1	600
9	0	-1	-1	595
10	0	1	-1	648
11	0	-1	1	532
12	0	1	1	656
13	0	0	0	653
14	0	0	0	599
15	0	0	0	620

(a) Fit the second-order model.

Design Expert Output

Response: Viscosity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	89652.58	9	9961.40	9.54	0.0115	significant
A	*703.12*	*1*	*703.12*	*0.67*	*0.4491*	
B	*6105.12*	*1*	*6105.12*	*5.85*	*0.0602*	
C	*5408.00*	*1*	*5408.00*	*5.18*	*0.0719*	
A^2	*20769.23*	*1*	*20769.23*	*19.90*	*0.0066*	
B^2	*1404.00*	*1*	*1404.00*	*1.35*	*0.2985*	
C^2	*4719.00*	*1*	*4719.00*	*4.52*	*0.0868*	
AB	*1521.00*	*1*	*1521.00*	*1.46*	*0.2814*	
AC	*47742.25*	*1*	*47742.25*	*45.74*	*0.0011*	
BC	*1260.25*	*1*	*1260.25*	*1.21*	*0.3219*	
Residual	5218.75	5	1043.75			
Lack of Fit	*3736.75*	*3*	*1245.58*	*1.68*	*0.3941*	*not significant*
Pure Error	*1482.00*	*2*	*741.00*			
Cor Total	94871.33	14				

The Model F-value of 9.54 implies the model is significant. There is only a 1.15% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	32.31	R-Squared	0.9450
Mean	575.33	Adj R-Squared	0.8460
C.V.	5.62	Pred R-Squared	0.3347
PRESS	63122.50	Adeq Precision	10.425

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	624.00	1	18.65	576.05	671.95	
A-Temperatue	9.37	1	11.42	-19.99	38.74	1.00
B-Agitation Rate	27.62	1	11.42	-1.74	56.99	1.00
C-Pressure	-26.00	1	11.42	-55.36	3.36	1.00
A^2	-75.00	1	16.81	-118.22	-31.78	1.01
B^2	19.50	1	16.81	-23.72	62.72	1.01
C^2	-35.75	1	16.81	-78.97	7.47	1.01
AB	-19.50	1	16.15	-61.02	22.02	1.00
AC	109.25	1	16.15	67.73	150.77	1.00
BC	17.75	1	16.15	-23.77	59.27	1.00

Final Equation in Terms of Coded Factors:

Viscosity	=
+624.00	
+9.37	* A
+27.62	* B
-26.00	* C
-75.00	* A^2
+19.50	* B^2
-35.75	* C^2
-19.50	* A * B
+109.25	* A * C
+17.75	* B * C

Final Equation in Terms of Actual Factors:

Viscosity	=
-629.50000	
+27.23500	* Temperatue
-9.55000	* Agitation Rate
-111.60000	* Pressure
-0.12000	* Temperatue^2
+3.12000	* Agitation Rate^2
-1.43000	* Pressure^2
-0.31200	* Temperatue * Agitation Rate
+0.87400	* Temperatue * Pressure
+1.42000	* Agitation Rate * Pressure

(b) Perform the canonical analysis. What type of surface has been found?

Solution

	Variable	Critical Value
	X1	2.1849596
	X2	-0.871371
	X3	2.7586015
Predicted Value at Solution		586.34437

Eigevalues and Eigevectors

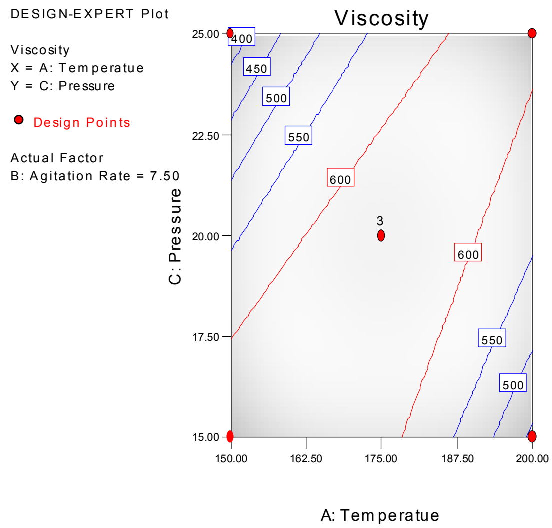
Variable	20.9229	2.5208	-114.694
X1	-0.02739	0.58118	0.81331
X2	0.99129	-0.08907	0.09703
X3	0.12883	0.80888	-0.57368

The system is a saddle point.

(c) What operating conditions on x_1, x_2, and x_3 lead to the stationary point?

The stationary point is (x_1, x_2, x_3) = (2.18496, -0.87167, 2.75860). This is outside the design region. It would be necessary to either examine contour plots or use numerical optimization methods to find desired operating conditions.

(d) What operating conditions would you recommend if it is important to obtain a viscosity that is as close to 600 as possible?



Any point on either of the contours showing a viscosity of 600 is satisfactory.

11-12 Consider the three-variable central composite design shown below. Analyze the data and draw conclusions, assuming that we wish to maximize conversion (y_1) with activity (y_2) between 55 and 60.

Run	Time (min)	Temperature (°C)	Catalyst (%)	Conversion (%) y_1	Activity y_2
1	-1.000	-1.000	-1.000	74.00	53.20
2	1.000	-1.000	-1.000	51.00	62.90
3	-1.000	1.000	-1.000	88.00	53.40
4	1.000	1.000	-1.000	70.00	62.60
5	-1.000	-1.000	1.000	71.00	57.30
6	1.000	-1.000	1.000	90.00	67.90
7	-1.000	1.000	1.000	66.00	59.80
8	1.000	1.000	1.000	97.00	67.80
9	0.000	0.000	0.000	81.00	59.20
10	0.000	0.000	0.000	75.00	60.40
11	0.000	0.000	0.000	76.00	59.10
12	0.000	0.000	0.000	83.00	60.60
13	-1.682	0.000	0.000	76.00	59.10
14	1.682	0.000	0.000	79.00	65.90
15	0.000	-1.682	0.000	85.00	60.00
16	0.000	1.682	0.000	97.00	60.70
17	0.000	0.000	-1.682	55.00	57.40
18	0.000	0.000	1.682	81.00	63.20
19	0.000	0.000	0.000	80.00	60.80
20	0.000	0.000	0.000	91.00	58.90

Quadratic models are developed for the Conversion and Activity response variables as follows:

Design Expert Output

Response: Conversion

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2555.73	9	283.97	12.76	0.0002	significant
A	*14.44*	*1*	*14.44*	*0.65*	*0.4391*	
B	*222.96*	*1*	*222.96*	*10.02*	*0.0101*	
C	*525.64*	*1*	*525.64*	*23.63*	*0.0007*	
A^2	*48.47*	*1*	*48.47*	*2.18*	*0.1707*	
B^2	*124.48*	*1*	*124.48*	*5.60*	*0.0396*	
C^2	*388.59*	*1*	*388.59*	*17.47*	*0.0019*	
AB	*36.13*	*1*	*36.13*	*1.62*	*0.2314*	
AC	*1035.13*	*1*	*1035.13*	*46.53*	*< 0.0001*	
BC	*120.12*	*1*	*120.12*	*5.40*	*0.0425*	
Residual	222.47	10	22.25			
Lack of Fit	*56.47*	*5*	*11.29*	*0.34*	*0.8692*	*not significant*
Pure Error	*166.00*	*5*	*33.20*			
Cor Total	287.28	19				

The Model F-value of 12.76 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	4.72	R-Squared	0.9199
Mean	78.30	Adj R-Squared	0.8479
C.V.	6.02	Pred R-Squared	0.7566
PRESS	676.22	Adeq Precision	14.239

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	81.09	1	1.92	76.81	85.38	
A-Time	1.03	1	1.28	-1.82	3.87	1.00
B-Temperature	4.04	1	1.28	1.20	6.88	1.00
C-Catalyst	6.20	1	1.28	3.36	9.05	1.00
A2	-1.83	1	1.24	-4.60	0.93	1.02
B2	2.94	1	1.24	0.17	5.71	1.02
C2	-5.19	1	1.24	-7.96	-2.42	1.02
AB	2.13	1	1.67	-1.59	5.84	1.00
AC	11.38	1	1.67	7.66	15.09	1.00
BC	-3.87	1	1.67	-7.59	-0.16	1.00

Final Equation in Terms of Coded Factors:

Conversion =
+81.09
+1.03 * A
+4.04 * B
+6.20 * C
-1.83 * A2
+2.94 * B2
-5.19 * C2
+2.13 * A * B
+11.38 * A * C
-3.87 * B * C

Final Equation in Terms of Actual Factors:

Conversion =
+81.09128
+1.02845 * Time
+4.04057 * Temperature
+6.20396 * Catalyst
-1.83398 * Time2
+2.93899 * Temperature2
-5.19274 * Catalyst2
+2.12500 * Time * Temperature
+11.37500 * Time * Catalyst

-3.87500 * Temperature * Catalyst

Design Expert Output

Response: Activity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	256.20	9	28.47	9.16	0.0009	significant
A	*175.35*	*1*	*175.35*	*56.42*	*< 0.0001*	
B	*0.89*	*1*	*0.89*	*0.28*	*0.6052*	
C	*67.91*	*1*	*67.91*	*21.85*	*0.0009*	
A^2	*10.05*	*1*	*10.05*	*3.23*	*0.1024*	
B^2	*0.081*	*1*	*0.081*	*0.026*	*0.8753*	
C^2	*0.047*	*1*	*0.047*	*0.015*	*0.9046*	
AB	*1.20*	*1*	*1.20*	*0.39*	*0.5480*	
AC	*0.011*	*1*	*0.011*	*3.620E-003*	*0.9532*	
BC	*0.78*	*1*	*0.78*	*0.25*	*0.6270*	
Residual	31.08	10	3.11			
Lack of Fit	*27.43*	*5*	*5.49*	*7.51*	*0.0226*	*significant*
Pure Error	*3.65*	*5*	*0.73*			
Cor Total	287.28	19				

The Model F-value of 9.16 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.76	R-Squared	0.8918
Mean	60.51	Adj R-Squared	0.7945
C.V.	2.91	Pred R-Squared	0.2536
PRESS	214.43	Adeq Precision	10.911

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	59.85	1	0.72	58.25	61.45	
A-Time	3.58	1	0.48	2.52	4.65	1.00
B-Temperature	0.25	1	0.48	-0.81	1.32	1.00
C-Catalyst	2.23	1	0.48	1.17	3.29	1.00
A^2	0.83	1	0.46	-0.20	1.87	1.02
B^2	0.075	1	0.46	-0.96	1.11	1.02
C^2	0.057	1	0.46	-0.98	1.09	1.02
AB	-0.39	1	0.62	-1.78	1.00	1.00
AC	-0.038	1	0.62	-1.43	1.35	1.00
BC	0.31	1	0.62	-1.08	1.70	1.00

Final Equation in Terms of Coded Factors:

Conversion =
+59.85
+3.58 * A
+0.25 * B
+2.23 * C
+0.83 * A^2
+0.075 * B^2
+0.057 * C^2
-0.39 * A * B
-0.038 * A * C
+0.31 * B * C

Final Equation in Terms of Actual Factors:

Conversion =
+59.84984
+3.58327 * Time
+0.25462 * Temperature
+2.22997 * Catalyst
+0.83491 * Time^2
+0.074772 * Temperature^2

+0.057094	* Catalyst^2
-0.38750	* Time * Temperature
-0.037500	* Time * Catalyst
+0.31250	* Temperature * Catalyst

Because many of the terms are insignificant, the reduced quadratic model is fit as follows:

Design Expert Output

Response: Activity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	253.20	3	84.40	39.63	< 0.0001	significant
A	*175.35*	*1*	*175.35*	*82.34*	*< 0.0001*	
C	*67.91*	*1*	*67.91*	*31.89*	*< 0.0001*	
A^2	*9.94*	*1*	*9.94*	*4.67*	*0.0463*	
Residual	34.07	16	2.13			
Lack of Fit	*30.42*	*11*	*2.77*	*3.78*	*0.0766*	*not significant*
Pure Error	*3.65*	*5*	*0.73*			
Cor Total	287.28	19				

The Model F-value of 39.63 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.46	R-Squared	0.8814
Mean	60.51	Adj R-Squared	0.8591
C.V.	2.41	Pred R-Squared	0.6302
PRESS	106.24	Adeq Precision	20.447

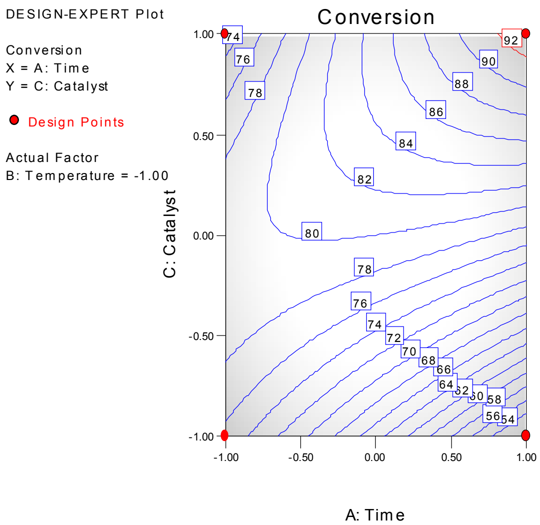
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	59.95	1	0.42	59.06	60.83	
A-Time	3.58	1	0.39	2.75	4.42	1.00
C-Catalyst	2.23	1	0.39	1.39	3.07	1.00
A^2	0.82	1	0.38	0.015	1.63	1.00

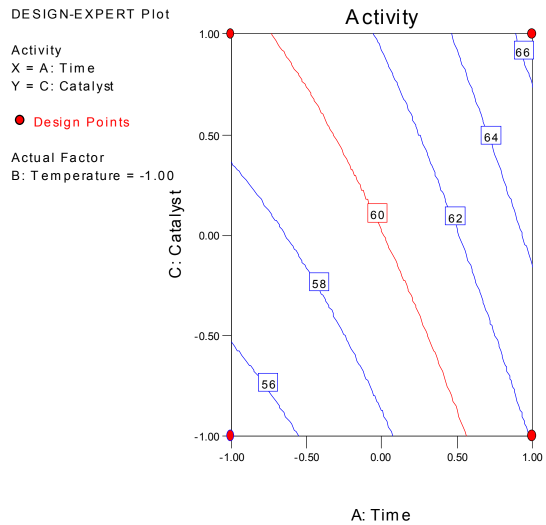
Final Equation in Terms of Coded Factors:

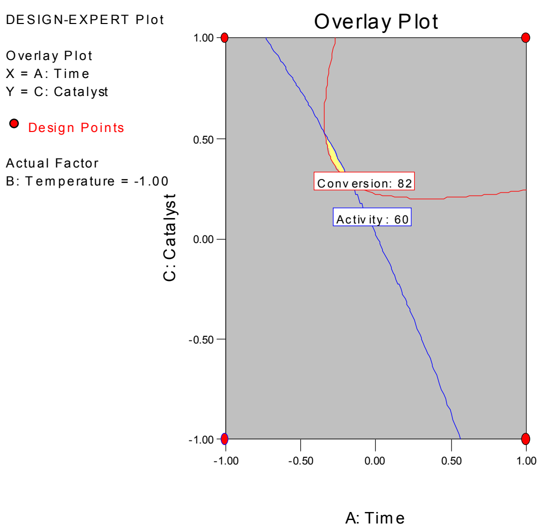
Activity	=
+59.95	
+3.58	* A
+2.23	* C
+0.82	* A2

Final Equation in Terms of Actual Factors:

Activity	=
+59.94802	
+3.58327	* Time
+2.22997	* Catalyst
+0.82300	* Time2







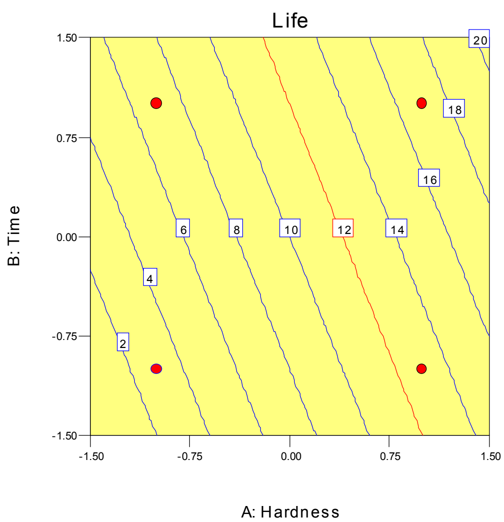
The contour plots visually describe the models while the overlay plots identifies the acceptable region for the process.

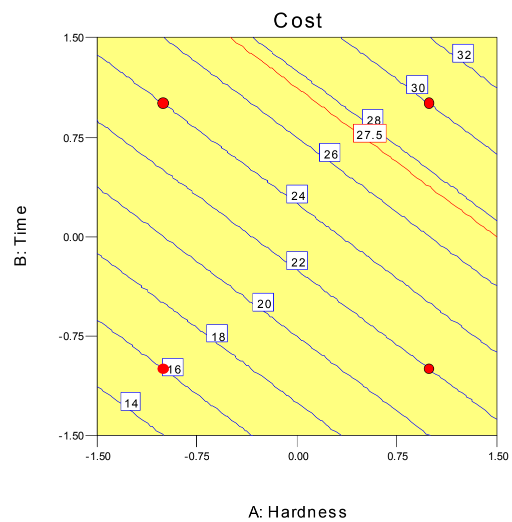
11-13 A manufacturer of cutting tools has developed two empirical equations for tool life in hours (y_1) and for tool cost in dollars (y_2). Both models are linear functions of steel hardness (x_1) and manufacturing time (x_2). The two equations are

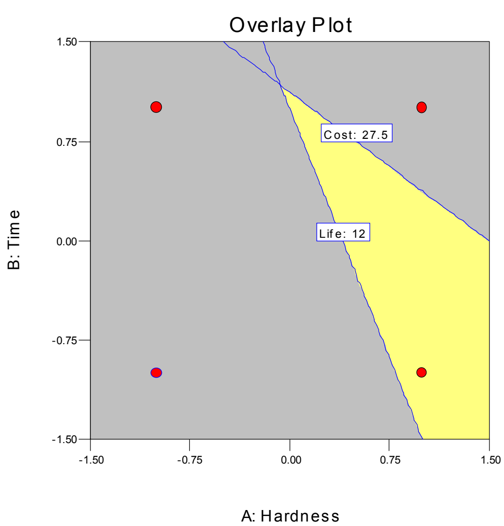
$$\hat{y}_1 = 10 + 5x_1 + 2x_2$$
$$\hat{y}_2 = 23 + 3x_1 + 4x_2$$

and both equations are valid over the range $-1.5 \le x_1 \le 1.5$. Unit tool cost must be below \$27.50 and life must exceed 12 hours for the product to be competitive. Is there a feasible set of operating conditions for this process? Where would you recommend that the process be run?

The contour plots below graphically describe the two models. The overlay plot identifies the feasible operating region for the process.







$$10 + 5x_1 + 2x_2 \ge 12$$
$$23 + 3x_1 + 4x_2 \le 27.50$$

11-14 A central composite design is run in a chemical vapor deposition process, resulting in the experimental data shown below. Four experimental units were processed simultaneously on each run of the design, and the responses are the mean and variance of thickness, computed across the four units.

x_1	x_2	$\bar{y}$	s^2
-1	-1	360.6	6.689
-1	1	445.2	14.230
1	-1	412.1	7.088
1	1	601.7	8.586
1.414	0	518.0	13.130
-1.414	0	411.4	6.644
0	1.414	497.6	7.649
0	-1.414	397.6	11.740
0	0	530.6	7.836
0	0	495.4	9.306
0	0	510.2	7.956
0	0	487.3	9.127

(a) Fit a model to the mean response. Analyze the residuals.

Design Expert Output

Response: Mean Thick
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	47644.26	5	9528.85	16.12	0.0020	significant
A	*22573.36*	*1*	*22573.36*	*38.19*	*0.0008*	
B	*15261.91*	*1*	*15261.91*	*25.82*	*0.0023*	
A^2	*2795.58*	*1*	*2795.58*	*4.73*	*0.0726*	
B^2	*5550.74*	*1*	*5550.74*	*9.39*	*0.0221*	
AB	*2756.25*	*1*	*2756.25*	*4.66*	*0.0741*	
Residual	3546.83	6	591.14			
Lack of Fit	*2462.04*	*3*	*820.68*	*2.27*	*0.2592*	*not significant*
Pure Error	*1084.79*	*3*	*361.60*			
Cor Total	51191.09	11				

The Model F-value of 16.12 implies the model is significant. There is only a 0.20% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	24.31	R-Squared	0.9307
Mean	472.31	Adj R-Squared	0.8730
C.V.	5.15	Pred R-Squared	0.6203
PRESS	19436.37	Adeq Precision	11.261

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	505.88	1	12.16	476.13	535.62	
A-x1	53.12	1	8.60	32.09	74.15	1.00
B-x2	43.68	1	8.60	22.64	64.71	1.00
A^2	-20.90	1	9.61	-44.42	2.62	1.04
B^2	-29.45	1	9.61	-52.97	-5.93	1.04
AB	26.25	1	12.16	-3.50	56.00	1.00

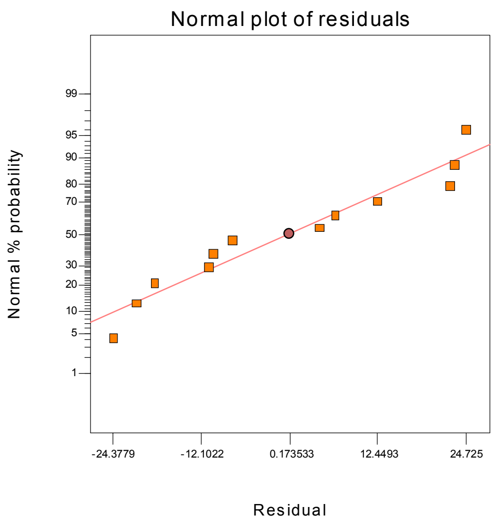
Final Equation in Terms of Coded Factors:

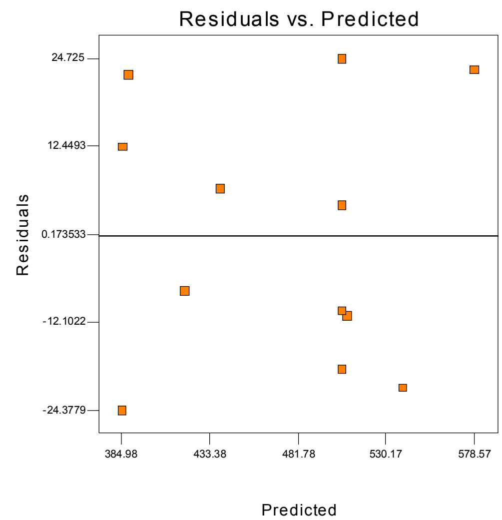
Mean Thick =
+505.88
+53.12 * A
+43.68 * B

```
          -20.90   * A^2
          -29.45   * B^2
          +26.25   * A * B

Final Equation in Terms of Actual Factors:

     Mean Thick   =
     +505.87500
      +53.11940   * x1
      +43.67767   * x2
      -20.90000   * x1^2
      -29.45000   * x2^2
      +26.25000   * x1 * x2
```





A modest deviation from normality can be observed in the Normal Plot of Residuals; however, not enough to be concerned.

(b) Fit a model to the variance response. Analyze the residuals.

Design Expert Output

Response: Var Thick
ANOVA for Response Surface 2FI Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	65.80	3	21.93	35.86	< 0.0001	significant
A	*41.46*	*1*	*41.46*	*67.79*	*< 0.0001*	
B	*15.21*	*1*	*15.21*	*24.87*	*0.0011*	
AB	*9.13*	*1*	*9.13*	*14.93*	*0.0048*	
Residual	4.89	8	0.61			
Lack of Fit	*3.13*	*5*	*0.63*	*1.06*	*0.5137*	*not significant*
Pure Error	*1.77*	*3*	*0.59*			
Cor Total	70.69	11				

The Model F-value of 35.86 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.78	R-Squared	0.9308
Mean	9.17	Adj R-Squared	0.9048
C.V.	8.53	Pred R-Squared	0.8920
PRESS	7.64	Adeq Precision	18.572

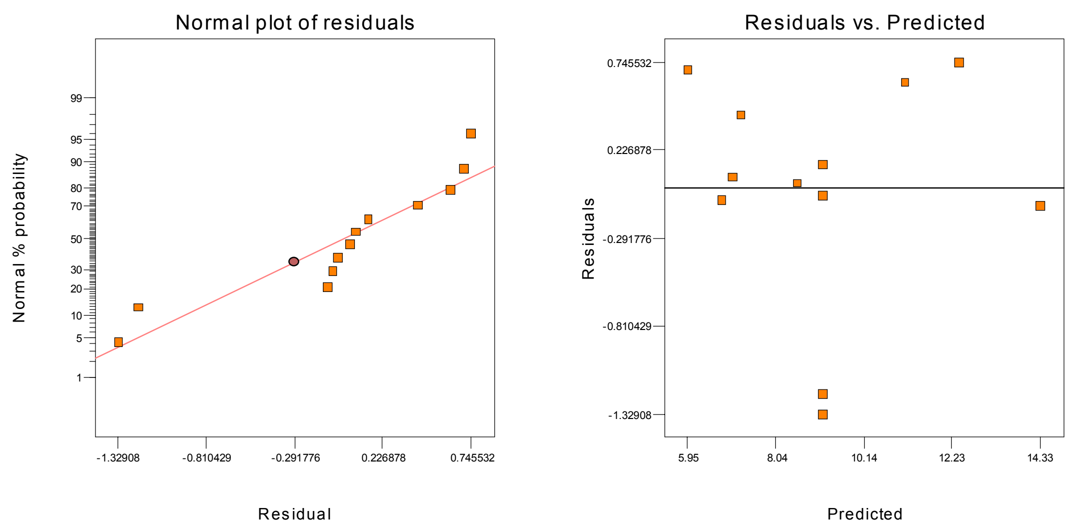
Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	9.17	1	0.23	8.64	9.69	
A-x1	2.28	1	0.28	1.64	2.91	1.00
B-x2	-1.38	1	0.28	-2.02	-0.74	1.00
AB	-1.51	1	0.39	-2.41	-0.61	1.00

Final Equation in Terms of Coded Factors:

Var Thick =
+9.17
+2.28 * A
-1.38 * B
-1.51 * A * B

Final Equation in Terms of Actual Factors:

Var Thick =
+9.16508
+2.27645 * x1
-1.37882 * x2
-1.51075 * x1 * x2



The residual plots are not acceptable. A transformation should be considered. If not successful at correcting the residual plots, further investigation into the two apparently unusual points should be made.

(c) Fit a model to the $\ln(s^2)$. Is this model superior to the one you found in part (b)?

Design Expert Output

Response: Var Thick Transform: Natural log Constant: 0
ANOVA for Response Surface 2FI Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.67	3	0.22	36.94	< 0.0001	significant
A	*0.46*	*1*	*0.46*	*74.99*	*< 0.0001*	
B	*0.14*	*1*	*0.14*	*22.80*	*0.0014*	
AB	*0.079*	*1*	*0.079*	*13.04*	*0.0069*	
Residual	0.049	8	6.081E-003			
Lack of Fit	*0.024*	*5*	*4.887E-003*	*0.61*	*0.7093*	*not significant*
Pure Error	*0.024*	*3*	*8.071E-003*			
Cor Total	0.72	11				

The Model F-value of 36.94 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.078	R-Squared	0.9327
Mean	2.18	Adj R-Squared	0.9074
C.V.	3.57	Pred R-Squared	0.8797
PRESS	0.087	Adeq Precision	18.854

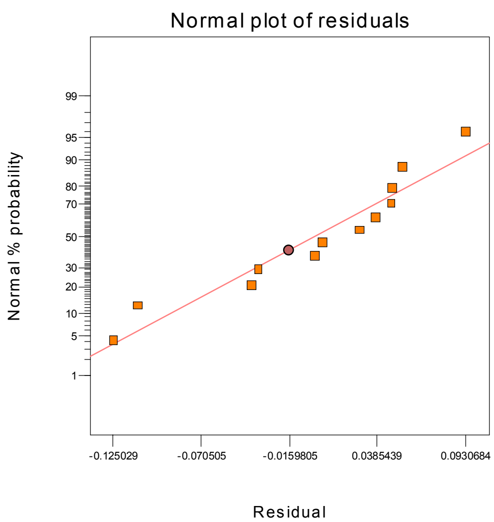
Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	2.18	1	0.023	2.13	2.24	
A-x1	0.24	1	0.028	0.18	0.30	1.00
B-x2	-0.13	1	0.028	-0.20	-0.068	1.00
AB	-0.14	1	0.039	-0.23	-0.051	1.00

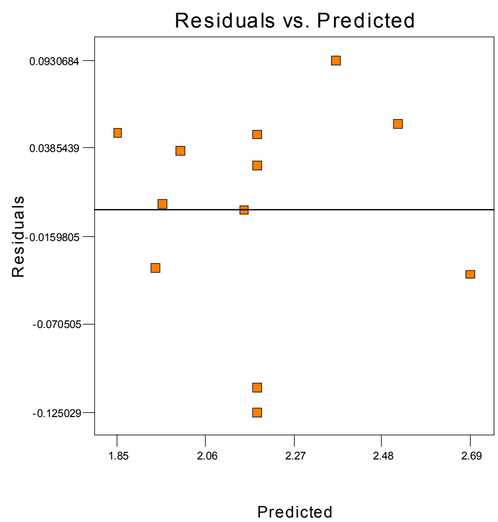
Final Equation in Terms of Coded Factors:

Ln(Var Thick)	=
+2.18	
+0.24	* A
-0.13	* B
-0.14	* A * B

Final Equation in Terms of Actual Factors:

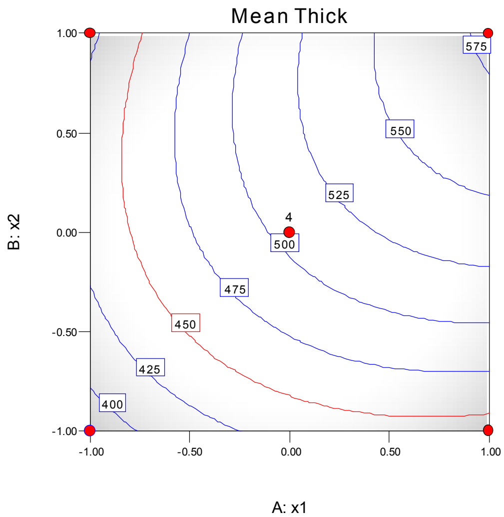
Ln(Var Thick)	=
+2.18376	
+0.23874	* x1
-0.13165	* x2
-0.14079	* x1 * x2

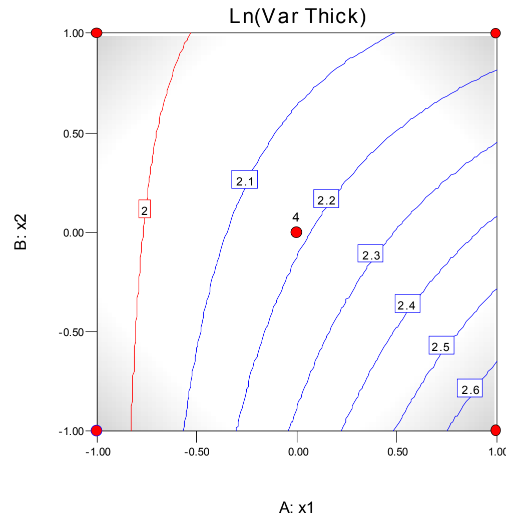


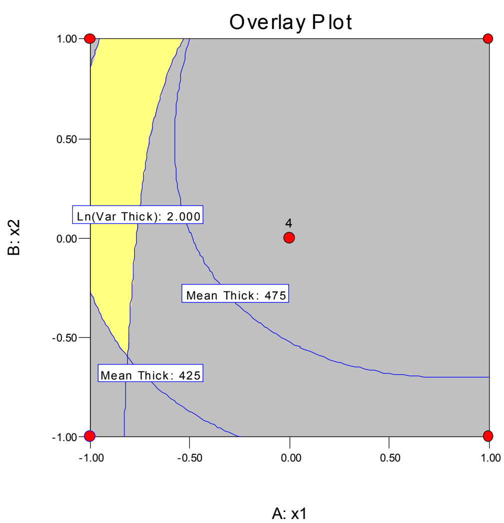


The residual plots are much improved following the natural log transformation; however, the two runs still appear to be somewhat unusual and should be investigated further. They will be retained in the analysis.

(d) Suppose you want the mean thickness to be in the interval 450±25. Find a set of operating conditions that achieve the objective and simultaneously minimize the variance.







The contour plots describe the two models while the overlay plot identifies the acceptable region for the process.

(e) Discuss the variance minimization aspects of part (d). Have you minimized total process variance?

The within run variance has been minimized; however, the run-to-run variation has not been minimized in the analysis. This may not be the most robust operating conditions for the process.

11-15 Verify that an orthogonal first-order design is also first-order rotatable.

To show that a first order orthogonal design is also first order rotatable, consider

$$V(\hat{y}) = V(\hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i) = V(\hat{\beta}_0) + \sum_{i=1}^{k} x_i^2 V(\hat{\beta}_i)$$

since all covariances between $\hat{\beta}_i$ and $\hat{\beta}_j$ are zero, due to design orthogonality. Furthermore, we have:

$$V\left(\hat{\beta}_0\right) = V(\hat{\beta}_1) = V(\hat{\beta}_2) = \ldots = V(\hat{\beta}_k) = \frac{\sigma^2}{n} \text{, so}$$

$$V(\hat{y}) = \frac{\sigma^2}{n} + \frac{\sigma^2}{n}\sum_{i=1}^{k} x_i^2$$

$$V(\hat{y}) = \frac{\sigma^2}{n}\left(1 + \frac{\sigma^2}{n}\sum_{i=1}^{k} x_i^2\right)$$

which is a function of distance from the design center (i.e. **x**=**0**), and not direction. Thus the design must be rotatable. Note that n is, in general, the number of points in the exterior portion of the design. If there are n_c centerpoints, then $V(\hat{\beta}_0) = \dfrac{\sigma^2}{(n+n_c)}$.

11-16 Show that augmenting a 2^k design with n_c center points does not affect the estimates of the β_i (i=1, 2, . . . , k), but that the estimate of the intercept β_0 is the average of all $2^k + n_c$ observations.

In general, the **X** matrix for the 2^k design with n_c center points and the **y** vector would be:

$$\mathbf{X} = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta k \\ 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ --- & --- & --- & --- & --- \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

← The upper half of the matrix is the usual ± 1 notation of the 2^k design

← The lower half of the matrix represents the center points (n_c rows)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2^k} \\ --- \\ n_{0_1} \\ n_{0_2} \\ \vdots \\ n_{0_c} \end{bmatrix}$$

← 2^k+n_c observations
.

$$\mathbf{X'X} = \begin{bmatrix} 2^k + n_c & 0 & \cdots & 0 \\ & 2^k & \cdots & 0 \\ & & \ddots & \vdots \\ & & & 2^k \end{bmatrix} \qquad \mathbf{X'y} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}$$

← Grand total of all 2^k+n_c observations

← usual contrasts from 2^k

Therefore, $\hat{\beta}_0 = \dfrac{g_0}{2^k + n_c}$, which is the average of all $\left(2^k + n_c\right)$observations, while $\hat{\beta}_i = \dfrac{g_i}{2^k}$, which does not depend on the number of center points, since in computing the contrasts g_i, all observations at the center are multiplied by zero.

11-17 ***The rotatable central composite design.*** It can be shown that a second-order design is rotatable if $\sum_{u=1}^{n} x_{iu}^a x_{ju}^b = 0$ if a or b (or both) are odd and if $\sum_{u=1}^{n} x_{iu}^4 = 3\sum_{u=1}^{n} x_{iu}^2 x_{ju}^2$. Show that for the central composite design these conditions lead to $\alpha = \left(n_f\right)^{1/4}$ for rotatability, where n_f is the number of points in the factorial portion.

The balance between +1 and -1 in the factorial columns and the orthogonality among certain column in the X matrix for the central composite design will result in all odd moments being zero. To solve for α use the following relations:

$$\sum_{u=1}^{n} x_{iu}^4 = n_f + 2\alpha^4, \qquad \sum_{u=1}^{n} x_{iu}^2 x_{ju}^2 = n_f$$

then

$$\begin{aligned}
&\sum_{u=1}^{n} x_{iu}^4 = 3\sum_{u=1}^{n} x_{iu}^2 x_{ju}^2 \\
&n_f + 2\alpha^4 = 3(n_f) \\
&2\alpha^4 = 2n_f \\
&\alpha^4 = n_f \\
&\alpha = \sqrt[4]{n_f}
\end{aligned}$$

11-18 Verify that the central composite design shown below blocks orthogonally.

Block 1			Block 2			Block 3		
x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
0	0	0	0	0	0	-1.633	0	0
0	0	0	0	0	0	1.633	0	0
1	1	1	1	1	-1	0	-1.633	0
1	-1	-1	1	-1	1	0	1.633	0
-1	-1	1	-1	1	1	0	0	-1.633
-1	1	-1	-1	-1	-1	0	0	1.633
						0	0	0
						0	0	0

Note that each block is an orthogonal first order design, since the cross products of elements in different columns add to zero for each block. To verify the second condition, choose a column, say column x_2. Now

$$\sum_{u=1}^{k} x_{2u}^2 = 13.334 \text{ , and } n=20$$

For blocks 1 and 2,

$$\sum_m x_{2m}^2 = 4 \text{ , } n_{\mathrm{m}}=6$$

So

$$\frac{\sum_m x_{2m}^2}{\sum_{u=1}^{n} x_{2u}^2} = n_m = 6$$

$$\frac{4}{13.334} = \frac{6}{20}$$

$$0.3 = 0.3$$

and condition 2 is satisfied by blocks 1 and 2. For block 3, we have

$$\sum_m x_{2m}^2 = 5.334 \text{ , } n_{\mathrm{m}} = 8, \text{ so}$$

$$\frac{\sum_m x_{2m}^2}{\sum_{u=1}^{n} x_{2u}^2} = \frac{n_m}{n}$$

$$\frac{5.334}{13.334} = \frac{8}{20}$$

$$0.4 = 0.4$$

And condition 2 is satisfied by block 3. Similar results hold for the other columns.

11-19 *Blocking in the central composite design*. Consider a central composite design for k = 4 variables in two blocks. Can a rotatable design always be found that blocks orthogonally?

To run a central composite design in two blocks, assign the n_f factorial points and the n_{01} center points to block 1 and the 2^k axial points plus n_{02} center points to block 2. Both blocks will be orthogonal first order designs, so the first condition for orthogonal blocking is satisfied.

The second condition implies that

$$\frac{\sum_m x_{im}^2(block1)}{\sum_m x_{im}^2(block2)} = \frac{n_f + n_{c1}}{2k + n_{c2}}$$

However, $\sum_m x_{im}^2 = n_f$ in block 1 and $\sum_m x_{im}^2 = 2\alpha^2$ in block 2, so

$$\frac{n_f}{2\alpha^2} = \frac{n_f + n_{c1}}{2k + n_{c2}}$$

Which gives:

$$\alpha = \left[\frac{n_f\left(2k + n_{c2}\right)}{2\left(n_f + n_{c1}\right)}\right]^{\frac{1}{2}}$$

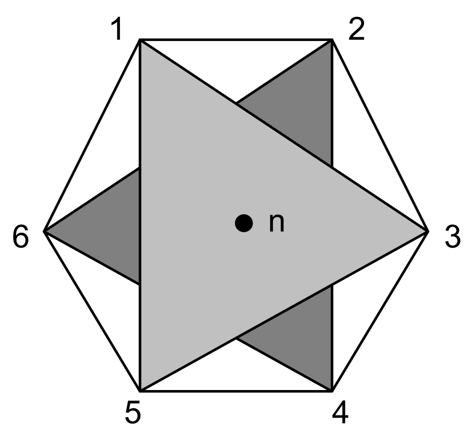
Since $\alpha = \sqrt[4]{n_f}$ if the design is to be rotatable, then the design must satisfy

$$n_f = \left[\frac{n_f\left(2k + n_{c2}\right)}{2\left(n_f + n_{c1}\right)}\right]^2$$

It is not possible to find rotatable central composite designs which block orthogonally for all k. For example, if k=3, the above condition cannot be satisfied. For k=2, there must be an equal number of center points in each block, i.e. $n_{c1} = n_{c2}$. For k=4, we must have $n_{c1} = 4$ and $n_{c2} = 2$.

11-20 How could a hexagon design be run in two orthogonal blocks?

The hexagonal design can be blocked as shown below. There are $n_{c1} = n_{c2} = n_c$ center points with n_c even.



Put the points 1,3,and 5 in block 1 and 2,4,and 6 in block 2. Note that each block is a simplex.

11-21 Yield during the first four cycles of a chemical process is shown in the following table. The variables are percent concentration (x_1) at levels 30, 31, and 32 and temperature (x_2) at 140, 142, and 144°F. Analyze by EVOP methods.

	Conditions				
Cycle	(1)	(2)	(3)	(4)	(5)

1	60.7	59.8	60.2	64.2	57.5
2	59.1	62.8	62.5	64.6	58.3
3	56.6	59.1	59.0	62.3	61.1
4	60.5	59.8	64.5	61.0	60.1

Cycle: n=1 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum						Previous Sum S=
(ii)	Previous Cycle Average						Previous Average =
(iii)	New Observation	60.7	59.8	60.2	64.2	57.5	New S=Range x $f_{k,n}$
(iv)	Differences						Range=
(v)	New Sums	60.7	59.8	60.2	64.2	57.5	New Sum S=
(vi)	New Averages	60.7	59.8	60.2	64.2	57.5	New average S = New Sum S/(n-1)=

Calculation of Effects		Calculation of Error Limits	
$A=\frac{1}{2}(\bar{y}_3+\bar{y}_4-\bar{y}_2-\bar{y}_5)=$	3.55	For New Average: $\left(\frac{2}{\sqrt{n}}\right)S=$	
$B=\frac{1}{2}(\bar{y}_3-\bar{y}_4-\bar{y}_2+\bar{y}_5)=$	-3.55	For New Effects: $\left(\frac{2}{\sqrt{n}}\right)S=$	
$AB=\frac{1}{2}(\bar{y}_3-\bar{y}_4+\bar{y}_2-\bar{y}_5)=$	-0.85	For CIM: $\left(\frac{1.78}{\sqrt{n}}\right)S=$	
$CIM=\frac{1}{2}(\bar{y}_3+\bar{y}_4+\bar{y}_2+\bar{y}_5-4\bar{y}_1)=$	-0.22		

Cycle: n=2 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum	60.7	59.8	60.2	64.2	57.5	Previous Sum S=
(ii)	Previous Cycle Average	60.7	59.8	60.2	64.2	57.5	Previous Average =
(iii)	New Observation	59.1	62.8	62.5	64.6	58.3	New S=Range x $f_{k,n}$=1.38
(iv)	Differences	1.6	-3.0	-2.3	-0.4	-0.8	Range=4.6
(v)	New Sums	119.8	122.6	122.7	128.8	115.8	New Sum S=1.38
(vi)	New Averages	59.90	61.30	61.35	64.40	57.90	New average S = New Sum S/(n-1)=1.38

Calculation of Effects		Calculation of Error Limits	
$A=\frac{1}{2}(\bar{y}_3+\bar{y}_4-\bar{y}_2-\bar{y}_5)=$	3.28	For New Average: $\left(\frac{2}{\sqrt{n}}\right)S=$	1.95
$B=\frac{1}{2}(\bar{y}_3-\bar{y}_4-\bar{y}_2+\bar{y}_5)=$	-3.23	For New Effects: $\left(\frac{2}{\sqrt{n}}\right)S=$	1.95
$AB=\frac{1}{2}(\bar{y}_3-\bar{y}_4+\bar{y}_2-\bar{y}_5)=$	0.18	For CIM: $\left(\frac{1.78}{\sqrt{n}}\right)S=$	1.74
$CIM=\frac{1}{2}(\bar{y}_3+\bar{y}_4+\bar{y}_2+\bar{y}_5-4\bar{y}_1)=$	1.07		

Cycle: n=3 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum	119.8	122.6	122.7	128.8	115.8	Previous Sum S=1.38

(ii)	Previous Cycle Average	59.90	61.30	61.35	64.40	57.90	Previous Average =1.38
(iii)	New Observation	56.6	59.1	59.0	62.3	61.1	New S=Range x $f_{k,n}$=2.28
(iv)	Differences	3.30	2.20	2.35	2.10	-3.20	Range=6.5
(v)	New Sums	176.4	181.7	181.7	191.1	176.9	New Sum S=3.66
(vi)	New Averages	58.80	60.57	60.57	63.70	58.97	New average S = New Sum S/(n-1)=1.38

Calculation of Effects		Calculation of Error Limits	
$A=\frac{1}{2}(\bar{y}_3+\bar{y}_4-\bar{y}_2-\bar{y}_5)=$	2.37	For New Average: $\left(\frac{2}{\sqrt{n}}\right)S=$	2.11
$B=\frac{1}{2}(\bar{y}_3-\bar{y}_4-\bar{y}_2+\bar{y}_5)=$	-2.37	For New Effects: $\left(\frac{2}{\sqrt{n}}\right)S=$	2.11
$AB=\frac{1}{2}(\bar{y}_3-\bar{y}_4+\bar{y}_2-\bar{y}_5)=$	-0.77	For CIM: $\left(\frac{1.78}{\sqrt{n}}\right)S=$	1.74
$CIM=\frac{1}{2}(\bar{y}_3+\bar{y}_4+\bar{y}_2+\bar{y}_5-4\bar{y}_1)=$	1.72		

Cycle: n=4 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum	176.4	181.7	181.7	191.1	176.9	Previous Sum S=3.66
(ii)	Previous Cycle Average	58.80	60.57	60.57	63.70	58.97	Previous Average =1.83
(iii)	New Observation	60.5	59.8	64.5	61.0	60.1	New S=Range x $f_{k,n}$=2.45
(iv)	Differences	-1.70	0.77	-3.93	2.70	-1.13	Range=6.63
(v)	New Sums	236.9	241.5	245.2	252.1	237.0	New Sum S=6.11
(vi)	New Averages	59.23	60.38	61.55	63.03	59.25	New average S = New Sum S/(n-1)=2.04

Calculation of Effects		Calculation of Error Limits	
$A=\frac{1}{2}(\bar{y}_3+\bar{y}_4-\bar{y}_2-\bar{y}_5)=$	2.48	For New Average: $\left(\frac{2}{\sqrt{n}}\right)S=$	2.04
$B=\frac{1}{2}(\bar{y}_3-\bar{y}_4-\bar{y}_2+\bar{y}_5)=$	-1.31	For New Effects: $\left(\frac{2}{\sqrt{n}}\right)S=$	2.04
$AB=\frac{1}{2}(\bar{y}_3-\bar{y}_4+\bar{y}_2-\bar{y}_5)=$	-0.18	For CIM: $\left(\frac{1.78}{\sqrt{n}}\right)S=$	1.82
$CIM=\frac{1}{2}(\bar{y}_3+\bar{y}_4+\bar{y}_2+\bar{y}_5-4\bar{y}_1)=$	1.46		

From studying cycles 3 and 4, it is apparent that *A* (and possibly *B*) has a significant effect. A new phase should be started following cycle 3 or 4.

11-22 Suppose that we approximate a response surface with a model of order d_1, such as $\mathbf{y}=\mathbf{X}_1\boldsymbol{\beta}_1+\boldsymbol{\varepsilon}$, when the true surface is described by a model of order $d_2>d_1$; that is $E(\mathbf{y})=\mathbf{X}_1\boldsymbol{\beta}_1+\mathbf{X}_2\boldsymbol{\beta}_2$.

(a) Show that the regression coefficients are biased, that is, that $E(\hat{\boldsymbol{\beta}}_1)=\boldsymbol{\beta}_1+\mathbf{A}\boldsymbol{\beta}_2$, where $\mathbf{A}=(\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_1\mathbf{X}_2$. $\mathbf{A}$ is usually called the alias matrix.

$$
\begin{aligned}
E\left[\hat{\boldsymbol{\beta}}_1\right] &= E\left[\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{y}\right] \\
&= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'E[\mathbf{y}] \\
&= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\left(\mathbf{X}_1\boldsymbol{\beta}_1+\mathbf{X}_2\boldsymbol{\beta}_2\right) \\
&= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_1\boldsymbol{\beta}_1+\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2 \\
&= \boldsymbol{\beta}_1+\mathbf{A}\boldsymbol{\beta}_2
\end{aligned}
$$

where $\mathbf{A}=\left(\mathbf{X_1'}\mathbf{X}_1\right)^{-1}\mathbf{X_1'}\mathbf{X_2}$

(a) If d_1=1 and d_2=2, and a full 2^k is used to fit the model, use the result in part (a) to determine the alias structure.

In this situation, we have assumed the true surface to be first order, when it is really second order. If a full factorial is used for k=2, then

$$
\mathbf{X}_1=\begin{array}{c}\begin{array}{ccc}\beta_0 & \beta_1 & \beta_2\end{array}\\ \begin{bmatrix}1 & -1 & -1\\ 1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\end{array}\quad
\mathbf{X}_2=\begin{array}{c}\begin{array}{ccc}\beta_{11} & \beta_{22} & \beta_{12}\end{array}\\ \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\end{array}\quad\text{and}\quad
\mathbf{A}=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}
$$

$$
\text{Then, }\mathbf{E}\left[\hat{\boldsymbol{\beta}}_1\right]=\mathbf{E}\begin{bmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix}\begin{matrix}=\\=\\=\end{matrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix}+\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\beta_{11}\\ \beta_{22}\\ \beta_{12}\end{bmatrix}\begin{matrix}=\\=\\=\end{matrix}\begin{bmatrix}\beta_0+\beta_{11}+\beta_{22}\\ \beta_1\\ \beta_2\end{bmatrix}
$$

The pure quadratic terms bias the intercept.

(b) If d_1=1, d_2=2 and k=3, find the alias structure assuming that a 2^{3-1} design is used to fit the model.

$$
\mathbf{X}_1=\begin{array}{c}\begin{array}{cccc}\beta_0 & \beta_1 & \beta_2 & \beta_3\end{array}\\ \begin{bmatrix}1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & 1 & 1 & 1\end{bmatrix}\end{array}\quad
\mathbf{X}_2=\begin{array}{c}\begin{array}{cccccc}\beta_{11} & \beta_{22} & \beta_{33} & \beta_{12} & \beta_{13} & \beta_{23}\end{array}\\ \begin{bmatrix}1 & 1 & 1 & 1 & -1 & -1\\ 1 & 1 & 1 & -1 & -1 & 1\\ 1 & 1 & 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}\end{array}\quad\text{and}\quad
\mathbf{A}=\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}
$$

$$
\text{Then, }\mathbf{E}\left[\hat{\boldsymbol{\beta}}_1\right]=\mathbf{E}\begin{bmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\end{bmatrix}\begin{matrix}=\\=\\=\end{matrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix}+\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\beta_{11}\\ \beta_{22}\\ \beta_{33}\\ \beta_{12}\\ \beta_{13}\\ \beta_{23}\end{bmatrix}\begin{matrix}=\\=\\=\end{matrix}\begin{bmatrix}\beta_0+\beta_{11}+\beta_{22}+\beta_{22}\\ \beta_1+\beta_{23}\\ \beta_2+\beta_{13}\\ \beta_3+\beta_{12}\end{bmatrix}
$$

(d) If d_1=1, d_2=2, k=3, and the simplex design in Problem 11-3 is used to fit the model, determine the alias structure and compare the results with part (c).

$$\mathbf{X}_1 = \begin{array}{c} \begin{array}{cccc} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad \mathbf{X}_2 = \begin{array}{c} \begin{array}{cccccc} \beta_{11} & \beta_{22} & \beta_{33} & \beta_{12} & \beta_{13} & \beta_{23} \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & -\sqrt{2} \\ 2 & 0 & 1 & 0 & -\sqrt{2} & 0 \\ 0 & 2 & 1 & 0 & 0 & -\sqrt{2} \\ 2 & 0 & 1 & 0 & -\sqrt{2} & 0 \end{bmatrix} \end{array} \quad \text{and } \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Then, } \mathbf{E}\left[\hat{\boldsymbol{\beta}}_1\right] = \mathbf{E}\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{12} \\ \beta_{13} \\ \beta_{23} \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} \beta_0 + \beta_{11} + \beta_{22} + \beta_{22} \\ \beta_1 + \beta_{13} \\ \beta_2 - \beta_{23} \\ \beta_3 + \beta_{11} - \beta_{22} \end{bmatrix}$$

Notice that the alias structure is different from that found in the previous part for the 2^{3-1} design. In general, the **A** matrix will depend on which simplex design is used.

11-23 In an article ("Let's All Beware the Latin Square," *Quality Engineering*, Vol. 1, 1989, pp. 453-465) J.S. Hunter illustrates some of the problems associated with 3^{k-p} fractional factorial designs. Factor *A* is the amount of ethanol added to a standard fuel and factor *B* represents the air/fuel ratio. The response variable is carbon monoxide (CO) emission in g/m^2. The design is shown below.

Design				Observations	
A	*B*	x_1	x_2	*y*	*y*
0	0	-1	-1	66	62
1	0	0	-1	78	81
2	0	1	-1	90	94
0	1	-1	0	72	67
1	1	0	0	80	81
2	1	1	0	75	78
0	2	-1	1	68	66
1	2	0	1	66	69
2	2	1	1	60	58

Notice that we have used the notation system of 0, 1, and 2 to represent the low, medium, and high levels for the factors. We have also used a "geometric notation" of -1, 0, and 1. Each run in the design is replicated twice.

(a) Verify that the second-order model

$$\hat{y} = 78.5 + 4.5x_1 - 7.0x_2 - 4.5x_1^2 - 4.0x_2^2 - 9.0x_1x_2$$

is a reasonable model for this experiment. Sketch the CO concentration contours in the x_1, x_2 space.

In the computer output that follows, the "coded factors" model is in the -1, 0, +1 scale.

Design Expert Output

Response: CO Emis
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1624.00	5	324.80	50.95	< 0.0001	significant
A	*243.00*	*1*	*243.00*	*38.12*	*< 0.0001*	

B	*588.00*	*1*	*588.00*	*92.24*	*< 0.0001*	
A^2	*81.00*	*1*	*81.00*	*12.71*	*0.0039*	
B^2	*64.00*	*1*	*64.00*	*10.04*	*0.0081*	
AB	*648.00*	*1*	*648.00*	*101.65*	*< 0.0001*	
Residual	76.50	12	6.37			
Lack of Fit	*30.00*	*3*	*10.00*	*1.94*	*0.1944*	*not significant*
Pure Error	*46.50*	*9*	*5.17*			
Cor Total	1700.50	17				

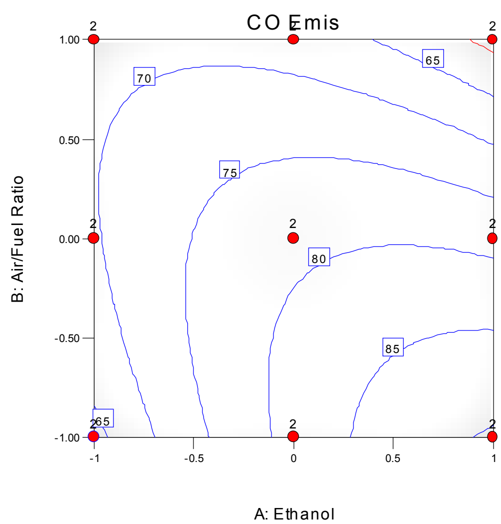
The Model F-value of 50.95 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.52	R-Squared	0.9550
Mean	72.83	Adj R-Squared	0.9363
C.V.	3.47	Pred R-Squared	0.9002
PRESS	169.71	Adeq Precision	21.952

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	78.50	1	1.33	75.60	81.40	
A-Ethanol	4.50	1	0.73	2.91	6.09	1.00
B-Air/Fuel Ratio	-7.00	1	0.73	-8.59	-5.41	1.00
A^2	-4.50	1	1.26	-7.25	-1.75	1.00
B^2	-4.00	1	1.26	-6.75	-1.25	1.00
AB	-9.00	1	0.89	-10.94	-7.06	1.00

Final Equation in Terms of Coded Factors:

CO Emis	=
+78.50	
+4.50	* A
-7.00	* B
-4.50	* A^2
-4.00	* B^2
-9.00	* A * B



(b) Now suppose that instead of only two factors, we had used *four* factors in a 3^{4-2} fractional factorial design and obtained *exactly* the same data in part (a). The design would be as follows:

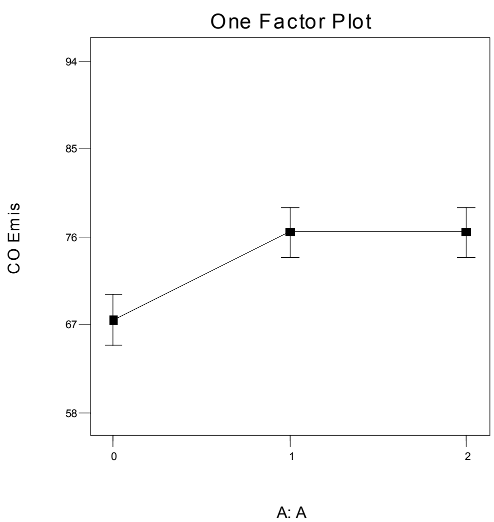
Design								Observations	
A	*B*	*C*	*D*	x_1	x_2	x_3	x_4	*y*	*y*

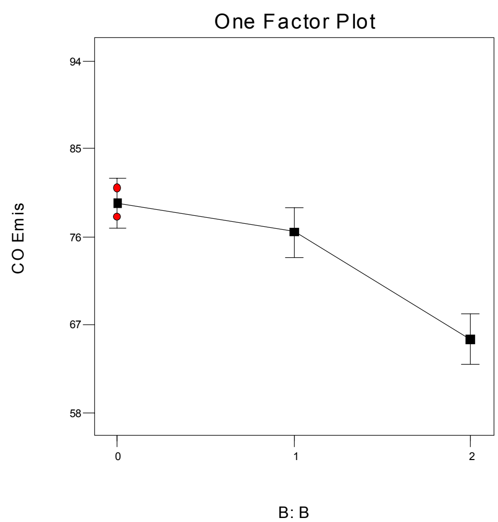
0	0	0	0	-1	-1	-1	-1	66	62
1	0	1	1	0	-1	0	0	78	81
2	0	2	2	+1	-1	+1	+1	90	94
0	1	2	1	-1	0	+1	0	72	67
1	1	0	2	0	0	-1	+1	80	81
2	1	1	0	+1	0	0	-1	75	78
0	2	1	2	-1	+1	0	+1	68	66
1	2	2	0	0	+1	+1	-1	66	69
2	2	0	1	+1	+1	-1	0	60	58

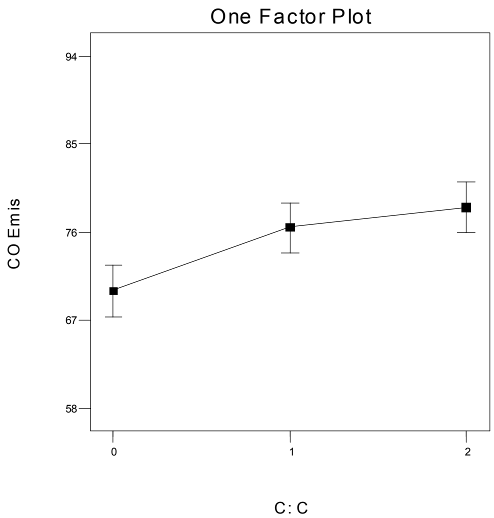
Confirm that this design is an L_9 orthogonal array.

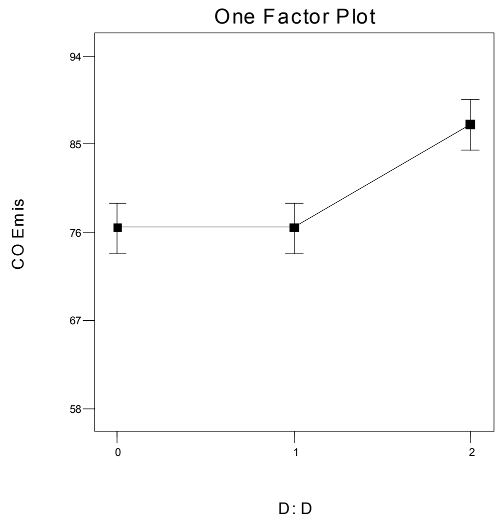
This is the same as the design in Table 11-22.

(c) Calculate the marginal averages of the CO response at each level of the four factors *A, B, C,* and *D*. Construct plots of these marginal averages and interpret the results. Do factors *C* and *D* appear to have strong effects? Do these factors really have any effect on CO emission? Why is their apparent effect strong?









Both Factors C and D appear to have an effect on CO emission. This is probably because both C and D are aliased with components of interaction involving A and B, and there is a strong AB interaction.

(a) The design in part (b) allows the model

$$y = \beta_0 + \sum_{i=1}^{4} \beta_i x_i + \sum_{i=1}^{4} \beta_{ii} x_i^2 + \varepsilon$$

to be fitted. Suppose that the *true* model is

$$y = \beta_0 + \sum_{i=1}^{4} \beta_i x_i + \sum_{i=1}^{4} \beta_{ii} x_i^2 + \sum_{i<j}\sum \beta_{ij} x_i x_j + \varepsilon$$

Show that if $\hat{\beta}_j$ represents the least squares estimates of the coefficients in the fitted model, then

$$\begin{aligned}
E\left(\hat{\beta}_0\right) &= \beta_0 - \beta_{13} - \beta_{14} - \beta_{34} \\
E\left(\hat{\beta}_1\right) &= \beta_1 - \left(\beta_{23} + \beta_{24}\right)/2 \\
E\left(\hat{\beta}_2\right) &= \beta_2 - \left(\beta_{13} + \beta_{14} + \beta_{34}\right)/2 \\
E\left(\hat{\beta}_3\right) &= \beta_3 - \left(\beta_{12} + \beta_{24}\right)/2 \\
E\left(\hat{\beta}_4\right) &= \beta_4 - \left(\beta_{12} + \beta_{23}\right)/2 \\
E\left(\hat{\beta}_{11}\right) &= \beta_{11} - \left(\beta_{23} - \beta_{24}\right)/2 \\
E\left(\hat{\beta}_{22}\right) &= \beta_{22} + \left(\beta_{13} + \beta_{14} + \beta_{34}\right)/2 \\
E\left(\hat{\beta}_{33}\right) &= \beta_{33} - \left(\beta_{24} - \beta_{12}\right)/2 + \beta_{14} \\
E\left(\hat{\beta}_{44}\right) &= \beta_{44} - \left(\beta_{12} - \beta_{23}\right)/2 + \beta_{13}
\end{aligned}$$

Let $\mathbf{X}_1 = \begin{array}{c} \begin{array}{ccccccccc} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_{11} & \beta_{22} & \beta_{33} & \beta_{44} \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$ and $\mathbf{X}_2 = \begin{array}{c} \begin{array}{cccccc} \beta_{12} & \beta_{13} & \beta_{14} & \beta_{23} & \beta_{24} & \beta_{34} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 0 & 0 \end{bmatrix} \end{array}$

$$\text{Then, } \mathbf{A} = \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2 = \mathbf{A} = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & -1/2 & 0 & 0 & -1/2 \\ -1/2 & 0 & 0 & 0 & -1/2 & 0 \\ -1/2 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1 & 0 & -1/2 & 0 \\ -1/2 & 1 & 0 & 1/2 & 0 & 0 \end{bmatrix}$$

$$E\begin{bmatrix} \hat\beta_0 \\ \hat\beta_1 \\ \hat\beta_2 \\ \hat\beta_3 \\ \hat\beta_4 \\ \hat\beta_{11} \\ \hat\beta_{22} \\ \hat\beta_{33} \\ \hat\beta_{44} \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{44} \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & -1/2 & 0 & 0 & -1/2 \\ -1/2 & 0 & 0 & 0 & -1/2 & 0 \\ -1/2 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1 & 0 & -1/2 & 0 \\ -1/2 & 1 & 0 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_{12} \\ \beta_{13} \\ \beta_{14} \\ \beta_{23} \\ \beta_{24} \\ \beta_{34} \end{bmatrix} = \begin{bmatrix} \beta_0 - \beta_{13} - \beta_{14} - \beta_{34} \\ \beta_1 - 1/2\beta_{23} - 1/2\beta_{24} \\ \beta_2 - 1/2\beta_{13} - 1/2\beta_{14} - 1/2\beta_{34} \\ \beta_3 - 1/2\beta_{12} - 1/2\beta_{24} \\ \beta_4 - 1/2\beta_{12} - 1/2\beta_{23} \\ \beta_{11} - 1/2\beta_{23} + 1/2\,\beta_{24} \\ \beta_{22} + 1/2\,\beta_{13} + 1/2\beta_{14} + 1/2\beta_{34} \\ \beta_{33} + 1/2\beta_{12} + \beta_{14} - 1/2\beta_{24} \\ \beta_{44} - 1/2\beta_{12} + \beta_{13} + 1/2\beta_{23} \end{bmatrix}$$

11-24 Suppose that you need to design an experiment to fit a quadratic model over the region $-1 \le x_i \le +1$, i=1,2 subject to the constraint $x_1 + x_2 \le 1$. If the constraint is violated, the process will not work properly. You can afford to make no more than n=12 runs. Set up the following designs:

(a) An "inscribed" CCD with center points at $x_1 = x_2 = 0$

x_1	x_2
-0.5	-0.5
0.5	-0.5
-0.5	0.5
0.5	0.5
-0.707	0
0.707	0

0	-0.707
0	0.707
0	0
0	0
0	0
0	0

(a)* An "inscribed" CCD with center points at $x_1 = x_2 = -0.25$ so that a larger design could be fit within the constrained region

x_1	x_2
-1	-1
0.5	-1
-1	0.5
0.5	0.5
-1.664	-0.25
1.164	-0.25
-0.25	-1.664
-0.25	1.164
-0.25	-0.25
-0.25	-0.25
-0.25	-0.25
-0.25	-0.25

(a) An "inscribed" 3^2 factorial with center points at $x_1 = x_2 - 0.25$

x_1	x_2
-1	-1
-0.25	-1
0.5	-1
-1	-0.25
-0.25	-0.25
0.5	-0.25
-1	0.5
-0.25	0.5
0.5	0.5
-0.25	-0.25
-0.25	-0.25
-0.25	-0.25

(a) A D-optimal design.

x_1	x_2
-1	-1
1	-1
-1	1
1	0
0	1
0	0
-1	0
0	-1

0.5	0.5
-1	-1
1	-1
-1	1

(a) A modified D-optimal design that is identical to the one in part (c), but with all replicate runs at the design center.

x_1	x_2
1	0
0	0
0	1
-1	-1
1	-1
-1	1
-1	0
0	-1
0.5	0.5
0	0
0	0
0	0

(a) Evaluate the $\left|(\mathbf{X'X})^{-1}\right|$ criteria for each design.

	(a)	(a)*	(b)	(c)	(d)
$\left|(\mathbf{X'X})^{-1}\right|$	0.5	0.00005248	0.007217	0.0001016	0.0002294

(a) Evaluate the D-efficiency for each design relative to the D-optimal design in part (c).

	(a)	(a)*	(b)	(c)	(d)
D-efficiency	24.25%	111.64%	49.14%	100.00%	87.31%

(a) Which design would you prefer? Why?

The offset CCD, (a)*, is the preferred design based on the D-efficiency. Not only is it better than the D-optimal design, (c), but it maintains the desirable design features of the CCD.

11-25 Consider a 2^3 design for fitting a first-order model.

(a) Evaluate the D-criterion $\left|(\mathbf{X'X})^{-1}\right|$ for this design.

$$\left|(\mathbf{X'X})^{-1}\right| = 2.441\text{E-}4$$

(b) Evaluate the A-criterion $tr(\mathbf{X'X})^{-1}$ for this design.

$$tr(\mathbf{X'X})^{-1} = 0.5$$

(c) Find the maximum scaled prediction variance for this design. Is this design G-optimal?

$$v(\mathbf{x}) = \frac{NVar(\hat{y}(\mathbf{x}))}{\sigma^2} = N\mathbf{x}'^{(1)}(\mathbf{X'X})^{-1}\mathbf{x}^{(1)} = 4$$. Yes, this is a G-optimal design.

11-26 Repeat Problem 11-25 using a first order model with the two-factor interaction.

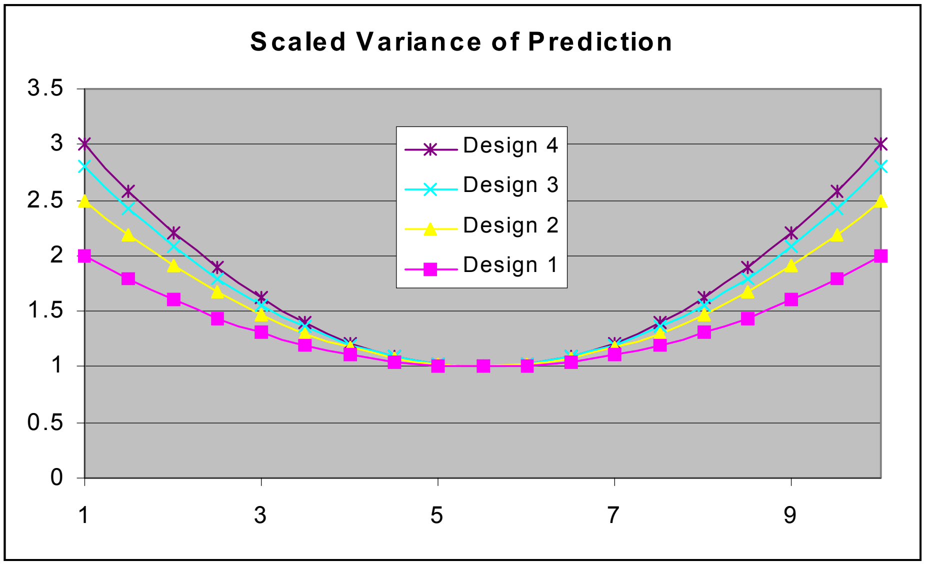
$$\left|(\mathbf{X'X})^{-1}\right| = 4.768\text{E-}7$$

$$tr(\mathbf{X'X})^{-1} = 0.875$$

$$v(\mathbf{x}) = \frac{NVar(\hat{y}(\mathbf{x}))}{\sigma^2} = N\mathbf{x}'^{(1)}(\mathbf{X'X})^{-1}\mathbf{x}^{(1)} = 7$$. Yes, this is a G-optimal design.

11-27 A chemical engineer wishes to fit a calibration curve for a new procedure used to measure the concentration of a particular ingredient in a product manufactured in his facility. Twelve samples can be prepared, having known concentration. The engineer's interest is in building a model for the measured concentrations. He suspects that a linear calibration curve will be adequate to model the measured concentration as a function of the known concentrations; that is, where x is the actual concentration. Four experimental designs are under consideration. Design 1 consists of 6 runs at known concentration 1 and 6 runs at known concentration 10. Design 2 consists of 4 runs at concentrations 1, 5.5, and 10. Design 3 consists of 3 runs at concentrations 1, 4, 7, and 10. Finally, design 4 consists of 3 runs at concentrations 1 and 10 and 6 runs at concentration 5.5.

(a) Plot the scaled variance of prediction for all four designs on the same graph over the concentration range. Which design would be preferable, in your opinion?



Because it has the lowest scaled variance of prediction at all points in the design space with the exception of 5.5, Design 1 is preferred.

(b) For each design calculate the determinant of $(\mathbf{X'X})^{-1}$. Which design would be preferred according to the "D" criterion?

Design	$\lvert(\mathbf{X'X})^{-1}\rvert$
1	0.000343
2	0.000514
3	0.000617
4	0.000686

Design 1 would be preferred.

(c) Calculate the D-efficiency of each design relative to the "best" design that you found in part b.

Design	D-efficiency
1	100.00%
2	81.65%
3	74.55%
4	70.71%

(a) For each design, calculate the average variance of prediction over the set of points given by $x = 1, 1.5, 2, 2.5, \ldots, 10$. Which design would you prefer according to the V-criterion?

Average Variance of Prediction		
Design	Actual	Coded
1	1.3704	0.1142
2	1.5556	0.1296
3	1.6664	0.1389
4	1.7407	0.1451

Design 1 is still preferred based on the V-criterion.

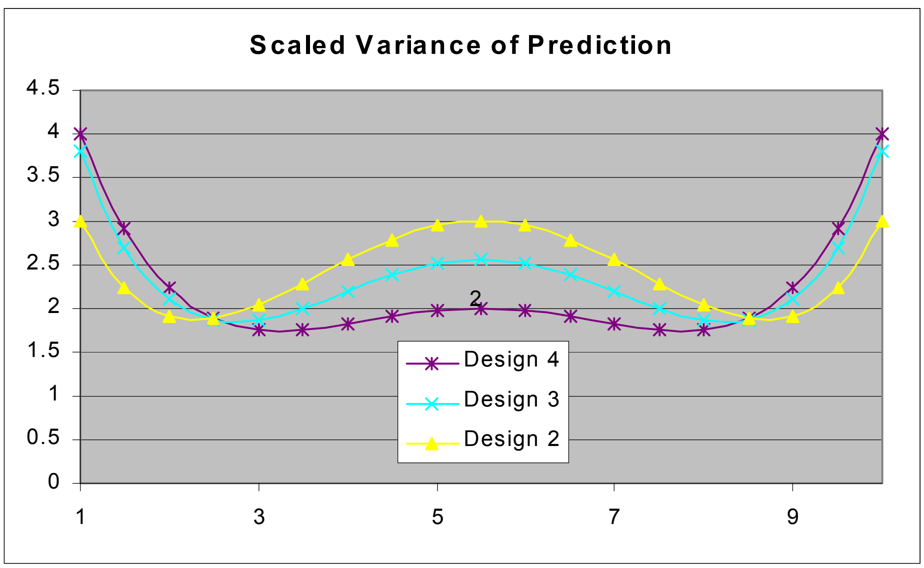
(e) Calculate the V-efficiency of each design relative to the best design you found in part (d).

Design	V-efficiency
1	100.00%
2	88.10%
3	82.24%
4	78.72%

(f) What is the G-efficiency of each design?

Design	G-efficiency
1	100.00%
2	80.00%
3	71.40%
4	66.70%

11-28 Rework Problem 11-27 assuming that the model the engineer wishes to fit is a quadratic. Obviously, only designs 2, 3, and 4 can now be considered.



Based on the plot, the preferred design would depend on the region of interest. Design 4 would be preferred if the center of the region was of interest; otherwise, Design 2 would be preferred.

Design	$\left\|(\mathbf{X}'\mathbf{X})^{-1}\right\|$
2	4.704E-07
3	6.351E-07
4	5.575E-07

Design 2 is preferred based on $\left|(\mathbf{X}'\mathbf{X})^{-1}\right|$.

Design	D-efficiency
2	100.00%
3	90.46%
4	94.49%

Average Variance of Prediction

Design	Actual	Coded
2	2.441	0.2034
3	2.393	0.1994
4	2.242	0.1869

Design 4 is preferred.

Design	V-efficiency
2	91.89%
3	93.74%
4	100.00%

Design	G-efficiency
2	100.00%
3	79.00%
4	75.00%

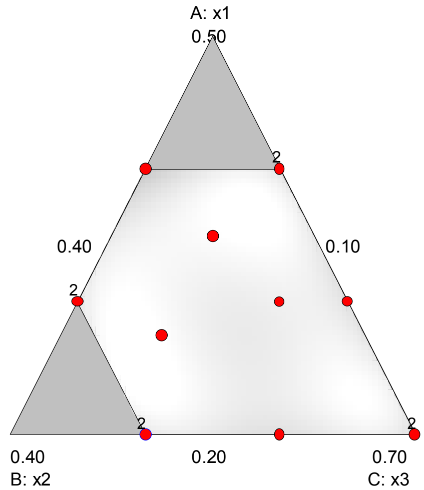
11-29 An experimenter wishes to run a three-component mixture experiment. The constraints are the components proportions are as follows:

$$0.2 \le x_1 \le 0.4$$
$$0.1 \le x_2 \le 0.3$$
$$0.4 \le x_3 \le 0.7$$

(a) Set up an experiment to fit a quadratic mixture model. Use n=14 runs, with 4 replicates. Use the D-criteria.

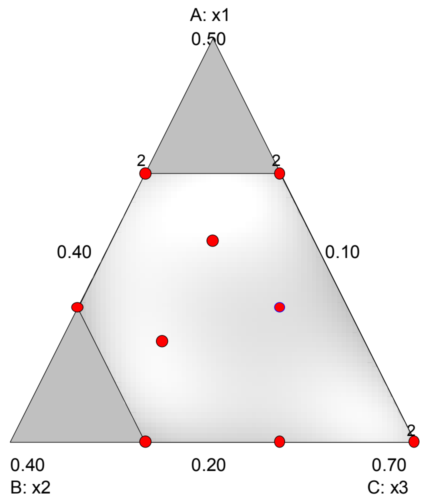
Std	x1	x2	x3
1	0.2	0.3	0.5
2	0.3	0.3	0.4
3	0.3	0.15	0.55
4	0.2	0.1	0.7
5	0.4	0.2	0.4
6	0.4	0.1	0.5
7	0.2	0.2	0.6
8	0.275	0.25	0.475
9	0.35	0.175	0.475
10	0.3	0.1	0.6
11	0.2	0.3	0.5
12	0.3	0.3	0.4
13	0.2	0.1	0.7
14	0.4	0.1	0.5

(a) Draw the experimental design region.



(c) Set up an experiment to fit a quadratic mixture model with n=12 runs, assuming that three of these runs are replicated. Use the D-criterion.

Std	x_1	x_2	x_3
1	0.3	0.15	0.55
2	0.2	0.3	0.5
3	0.3	0.3	0.4
4	0.2	0.1	0.7
5	0.4	0.2	0.4
6	0.4	0.1	0.5
7	0.2	0.2	0.6
8	0.275	0.25	0.475
9	0.35	0.175	0.475
10	0.2	0.1	0.7
11	0.4	0.1	0.5
12	0.4	0.2	0.4



(d) Comment on the two designs you have found.

The design points are the same for both designs except that the edge center on the x1-x3 edge is not included in the second design. None of the replicates for either design are in the center of the experimental region. The experimental runs are fairly uniformly spaced in the design region.

11-30 Myers and Montgomery (1995) describe a gasoline blending experiment involving three mixture components. There are no constraints on the mixture proportions, and the following 10 run design is used.

Design Point	x_1	x_2	x_3	y(mpg)
1	1	0	0	24.5, 25.1
2	0	1	0	24.8, 23.9
3	0	0	1	22.7, 23.6
4	½	½	0	25.1
5	½	0	½	24.3
6	0	½	½	23.5
7	1/3	1/3	1/3	24.8, 24.1
8	2/3	1/6	1/6	24.2
9	1/6	2/3	1/6	23.9
10	1/6	1/6	2/3	23.7

(a) What type of design did the experimenters use?

A simplex centroid design was used.

(b) Fit a quadratic mixture model to the data. Is this model adequate?

Design Expert Output

Response: y
ANOVA for Mixture Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F

Model	4.22	5	0.84	3.90	0.0435	significant
Linear Mixture	*3.92*	*2*	*1.96*	*9.06*	*0.0088*	
AB	*0.15*	*1*	*0.15*	*0.69*	*0.4289*	
AC	*0.081*	*1*	*0.081*	*0.38*	*0.5569*	
BC	*0.077*	*1*	*0.077*	*0.36*	*0.5664*	
Residual	1.73	8	0.22			
Lack of Fit	*0.50*	*4*	*0.12*	*0.40*	*0.8003*	*not significant*
Pure Error	*1.24*	*4*	*0.31*			
Cor Total	5.95	13				

The Model F-value of 3.90 implies the model is significant. There is only a 4.35% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.47	R-Squared	0.7091
Mean	24.16	Adj R-Squared	0.5274
C.V.	1.93	Pred R-Squared	0.1144
PRESS	5.27	Adeq Precision	5.674

Component	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**
A-x1	24.74	1	0.32	24.00	25.49
B-x2	24.31	1	0.32	23.57	25.05
C-x3	23.18	1	0.32	22.43	23.92
AB	1.51	1	1.82	-2.68	5.70
AC	1.11	1	1.82	-3.08	5.30
BC	-1.09	1	1.82	-5.28	3.10

Final Equation in Terms of Pseudo Components:

y =
+24.74 * A
+24.31 * B
+23.18 * C
+1.51 * A * B
+1.11 * A * C
-1.09 * B * C

Final Equation in Terms of Real Components:

y =
+24.74432 * x1
+24.31098 * x2
+23.17765 * x3
+1.51364 * x1 * x2
+1.11364 * x1 * x3
-1.08636 * x2 * x3

The quadratic terms appear to be insignificant. The analysis below is for the linear mixture model:

Design Expert Output

Response: y
ANOVA for Mixture Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	3.92	2	1.96	10.64	0.0027	significant
Linear Mixture	*3.92*	*2*	*1.96*	*10.64*	*0.0027*	
Residual	2.03	11	0.18			
Lack of Fit	*0.79*	*7*	*0.11*	*0.37*	*0.8825*	*not significant*
Pure Error	*1.24*	*4*	*0.31*			
Cor Total	5.95	13				

The Model F-value of 10.64 implies the model is significant. There is only a 0.27% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.43	R-Squared	0.6591
Mean	24.16	Adj R-Squared	0.5972
C.V.	1.78	Pred R-Squared	0.3926
PRESS	3.62	Adeq Precision	8.751

Component	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High
A-x1	24.93	1	0.25	24.38	25.48
B-x2	24.35	1	0.25	23.80	24.90
C-x3	23.19	1	0.25	22.64	23.74

Component	Adjusted Effect	DF	Adjusted Std Error	Approx t for H0 Effect=0	Prob > |t|
A-x1	1.16	1	0.33	3.49	0.0051
B-x2	0.29	1	0.33	0.87	0.4021
C-x3	-1.45	1	0.33	-4.36	0.0011

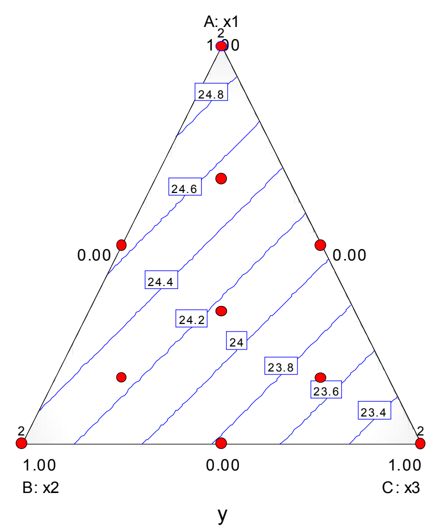
Final Equation in Terms of Pseudo Components:

y =
+24.93 * A
+24.35 * B
+23.19 * C

Final Equation in Terms of Real Components:

y =
+24.93048 * x1
+24.35048 * x2
+23.19048 * x3

(c) Plot the response surface contours. What blend would you recommend to maximize the MPG?



To maximize the miles per gallon, the recommended blend is $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$.

11-31 Consider the bottle filling experiment in Example 6-1. Suppose that the percent carbonation (*A*) is a noise variable (in coded units $\sigma_z^2 = 1$).

(a) Fit the response model to these data. Is there a robust design problem?

From the analysis below, the *AB* interaction appears to have some importance. Because of this, there is opportunity for improvement in the robustness of the process.

Design Expert Output

Response: Fill Height
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	73.00	7	10.43	16.69	0.0003	significant
A	*36.00*	*1*	*36.00*	*57.60*	*< 0.0001*	
B	*20.25*	*1*	*20.25*	*32.40*	*0.0005*	
C	*12.25*	*1*	*12.25*	*19.60*	*0.0022*	
AB	*2.25*	*1*	*2.25*	*3.60*	*0.0943*	
AC	*0.25*	*1*	*0.25*	*0.40*	*0.5447*	
BC	*1.00*	*1*	*1.00*	*1.60*	*0.2415*	
ABC	*1.00*	*1*	*1.00*	*1.60*	*0.2415*	
Pure Error	5.00	8	0.63			
Cor Total	78.00	15				

The Model F-value of 16.69 implies the model is significant. There is only
a 0.03% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.79	R-Squared	0.9359
Mean	1.00	Adj R-Squared	0.8798
C.V.	79.06	Pred R-Squared	0.7436
PRESS	20.00	Adeq Precision	13.416

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	1.00	1	0.20	0.54	1.46	
A-Carbination	1.50	1	0.20	1.04	1.96	1.00
B-Pressure	1.13	1	0.20	0.67	1.58	1.00
C-Speed	0.88	1	0.20	0.42	1.33	1.00
AB	0.38	1	0.20	-0.081	0.83	1.00
AC	0.13	1	0.20	-0.33	0.58	1.00
BC	0.25	1	0.20	-0.21	0.71	1.00
ABC	0.25	1	0.20	-0.21	0.71	1.00

Final Equation in Terms of Coded Factors:

Fill Height	=
+1.00	
+1.50	* A
+1.13	* B
+0.88	* C
+0.38	* A * B
+0.13	* A * C
+0.25	* B * C
+0.25	* A * B * C

Final Equation in Terms of Actual Factors:

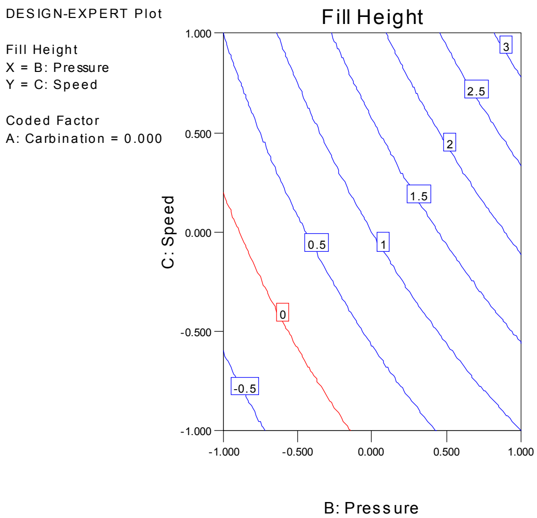
Fill Height	=
-225.50000	
+21.00000	* Carbination
+7.80000	* Pressure
+1.08000	* Speed
-0.75000	* Carbination * Pressure
-0.10500	* Carbination * Speed
-0.040000	* Pressure * Speed
+4.00000E-003	* Carbination * Pressure * Speed

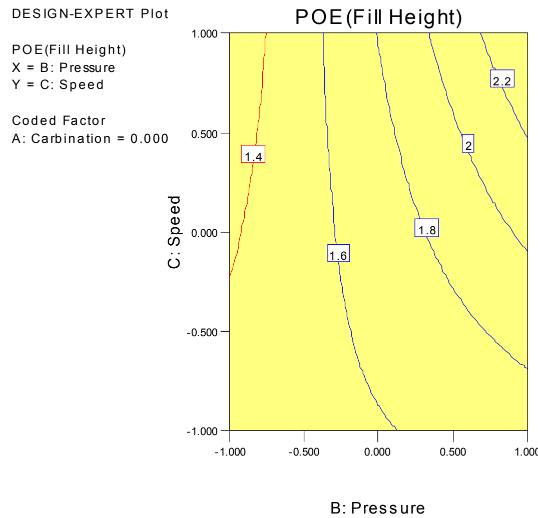
(b) Find the mean model and either the variance model or the POE.

The mean model in coded terms is:

$$E_z[y(\mathbf{x}, z_1)] = 1.00 + 1.13B + 0.88C + 0.25BC$$

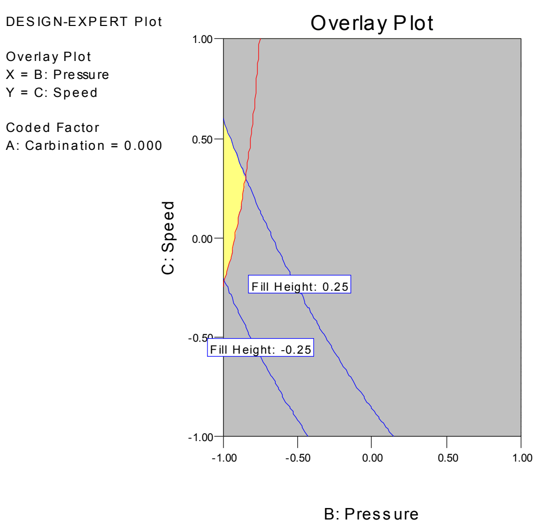
Contour plots of the mean model and POE are shown below:





(c) Find a set of conditions that result in mean fill deviation as close to zero as possible with minimum transmitted variability from carbonation.

The overlay plot below identifies a region that meets these requirements. The Pressure should be set at its low level and the Speed should be set between approximately 0.0 and 0.5 in coded terms.



11-32 Consider the experiment in Problem 11-12. Suppose that temperature is a noise variable ($\sigma_z^2 = 1$ in coded units). Fit response models for both responses. Is there a robust design problem with respect to both responses? Find a set of conditions that maximize conversion with activity between 55 and 60, and that minimize the variability transmitted from temperature.

The following is the analysis of variance for the Conversion response. Because of a significant *BC* interaction, there is some opportunity for improvement in the robustness of the process with regards to Conversion.

Design Expert Output

Response: Conversion
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2555.73	9	283.97	12.76	0.0002	significant
A	*14.44*	*1*	*14.44*	*0.65*	*0.4391*	
B	*222.96*	*1*	*222.96*	*10.02*	*0.0101*	
C	*525.64*	*1*	*525.64*	*23.63*	*0.0007*	
A^2	*48.47*	*1*	*48.47*	*2.18*	*0.1707*	
B^2	*124.48*	*1*	*124.48*	*5.60*	*0.0396*	
C^2	*388.59*	*1*	*388.59*	*17.47*	*0.0019*	
AB	*36.13*	*1*	*36.13*	*1.62*	*0.2314*	
AC	*1035.13*	*1*	*1035.13*	*46.53*	*< 0.0001*	
BC	*120.12*	*1*	*120.12*	*5.40*	*0.0425*	
Residual	222.47	10	22.25			
Lack of Fit	*56.47*	*5*	*11.29*	*0.34*	*0.8692*	*not significant*
Pure Error	*166.00*	*5*	*33.20*			
Cor Total	287.28	19				

The Model F-value of 12.76 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	4.72	R-Squared	0.9199
Mean	78.30	Adj R-Squared	0.8479
C.V.	6.02	Pred R-Squared	0.7566
PRESS	676.22	Adeq Precision	14.239

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	81.09	1	1.92	76.81	85.38	
A-Time	1.03	1	1.28	-1.82	3.87	1.00
B-Temperature	4.04	1	1.28	1.20	6.88	1.00
C-Catalyst	6.20	1	1.28	3.36	9.05	1.00
A2	-1.83	1	1.24	-4.60	0.93	1.02
B2	2.94	1	1.24	0.17	5.71	1.02
C2	-5.19	1	1.24	-7.96	-2.42	1.02
AB	2.13	1	1.67	-1.59	5.84	1.00
AC	11.38	1	1.67	7.66	15.09	1.00
BC	-3.87	1	1.67	-7.59	-0.16	1.00

Final Equation in Terms of Coded Factors:

Conversion	=
+81.09	
+1.03	* A
+4.04	* B
+6.20	* C
-1.83	* A2
+2.94	* B2
-5.19	* C2
+2.13	* A * B
+11.38	* A * C
-3.87	* B * C

Final Equation in Terms of Actual Factors:

Conversion	=
+81.09128	
+1.02845	* Time
+4.04057	* Temperature
+6.20396	* Catalyst
-1.83398	* Time2
+2.93899	* Temperature2
-5.19274	* Catalyst2
+2.12500	* Time * Temperature
+11.37500	* Time * Catalyst
-3.87500	* Temperature * Catalyst

The following is the analysis of variance for the Activity response. Because there is not a significant interaction term involving temperature, there is no opportunity for improvement in the robustness of the process with regards to Activity.

Design Expert Output

Response: Activity
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	256.20	9	28.47	9.16	0.0009	significant
A	*175.35*	*1*	*175.35*	*56.42*	*< 0.0001*	
B	*0.89*	*1*	*0.89*	*0.28*	*0.6052*	
C	*67.91*	*1*	*67.91*	*21.85*	*0.0009*	
A^2	*10.05*	*1*	*10.05*	*3.23*	*0.1024*	
B^2	*0.081*	*1*	*0.081*	*0.026*	*0.8753*	
C^2	*0.047*	*1*	*0.047*	*0.015*	*0.9046*	
AB	*1.20*	*1*	*1.20*	*0.39*	*0.5480*	
AC	*0.011*	*1*	*0.011*	*3.620E-003*	*0.9532*	
BC	*0.78*	*1*	*0.78*	*0.25*	*0.6270*	
Residual	31.08	10	3.11			
Lack of Fit	*27.43*	*5*	*5.49*	*7.51*	*0.0226*	*significant*
Pure Error	*3.65*	*5*	*0.73*			
Cor Total	287.28	19				

The Model F-value of 9.16 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.76	R-Squared	0.8918
Mean	60.51	Adj R-Squared	0.7945
C.V.	2.91	Pred R-Squared	0.2536
PRESS	214.43	Adeq Precision	10.911

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	59.85	1	0.72	58.25	61.45	
A-Time	3.58	1	0.48	2.52	4.65	1.00
B-Temperature	0.25	1	0.48	-0.81	1.32	1.00
C-Catalyst	2.23	1	0.48	1.17	3.29	1.00
A^2	0.83	1	0.46	-0.20	1.87	1.02
B^2	0.075	1	0.46	-0.96	1.11	1.02
C^2	0.057	1	0.46	-0.98	1.09	1.02
AB	-0.39	1	0.62	-1.78	1.00	1.00
AC	-0.038	1	0.62	-1.43	1.35	1.00
BC	0.31	1	0.62	-1.08	1.70	1.00

Final Equation in Terms of Coded Factors:

Conversion	=
+59.85	
+3.58	* A
+0.25	* B
+2.23	* C
+0.83	* A^2
+0.075	* B^2
+0.057	* C^2
-0.39	* A * B
-0.038	* A * C
+0.31	* B * C

Final Equation in Terms of Actual Factors:

Conversion	=
+59.84984	
+3.58327	* Time
+0.25462	* Temperature
+2.22997	* Catalyst
+0.83491	* $Time^2$

```
                  +0.074772   * Temperature^2
                  +0.057094   * Catalyst^2
                  -0.38750    * Time * Temperature
                  -0.037500   * Time * Catalyst
                  +0.31250    * Temperature * Catalyst
```

Because many of the terms are insignificant, the reduced quadratic model is fit as follows:

Design Expert Output

Response: Activity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	253.20	3	84.40	39.63	< 0.0001	significant
A	*175.35*	*1*	*175.35*	*82.34*	*< 0.0001*	
C	*67.91*	*1*	*67.91*	*31.89*	*< 0.0001*	
A^2	*9.94*	*1*	*9.94*	*4.67*	*0.0463*	
Residual	34.07	16	2.13			
Lack of Fit	*30.42*	*11*	*2.77*	*3.78*	*0.0766*	*not significant*
Pure Error	*3.65*	*5*	*0.73*			
Cor Total	287.28	19				

The Model F-value of 39.63 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.46	R-Squared	0.8814
Mean	60.51	Adj R-Squared	0.8591
C.V.	2.41	Pred R-Squared	0.6302
PRESS	106.24	Adeq Precision	20.447

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	59.95	1	0.42	59.06	60.83	
A-Time	3.58	1	0.39	2.75	4.42	1.00
C-Catalyst	2.23	1	0.39	1.39	3.07	1.00
A^2	0.82	1	0.38	0.015	1.63	1.00

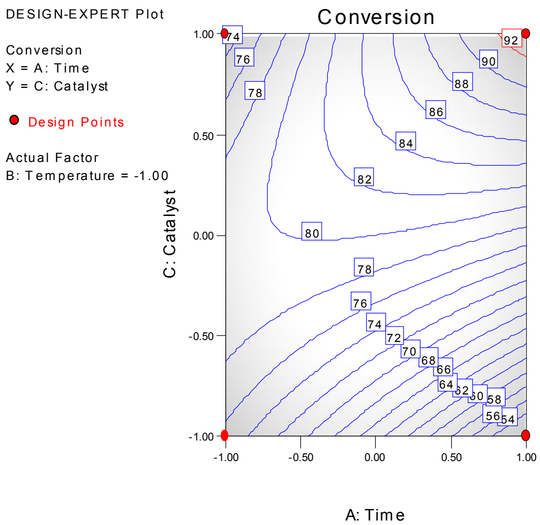
Final Equation in Terms of Coded Factors:

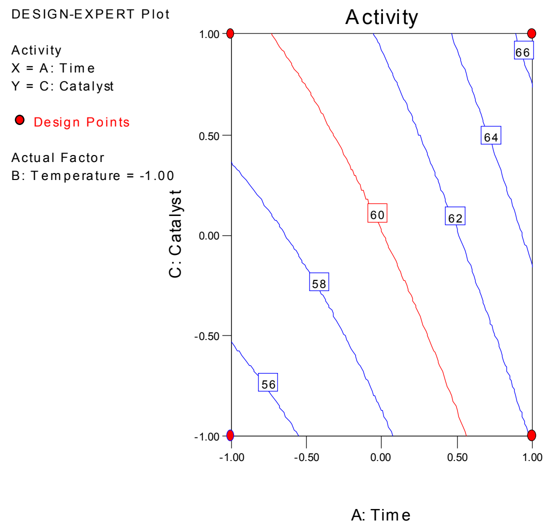
```
        Activity  =
          +59.95
           +3.58  * A
           +2.23  * C
           +0.82  * A2
```

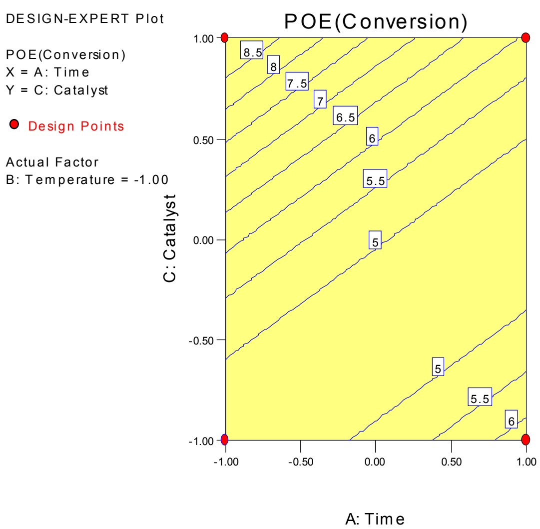
Final Equation in Terms of Actual Factors:

```
        Activity  =
       +59.94802
        +3.58327  * Time
        +2.22997  * Catalyst
        +0.82300  * Time2
```

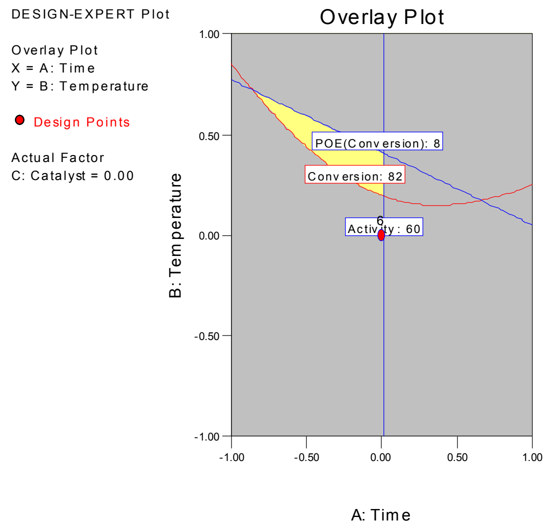
Contour plots of the mean models for the responses along with POE for Conversion are shown below:







The overlay plot shown below identifies a region near the center of the design space that meets the constraints for the process.



11-33 An experiment has been run in a process that applies a coating material to a wafer. Each run in the experiment produced a wafer, and the coating thickness was measured several times at different locations on the wafer. Then the mean y_1, and standard deviation y_2 of the thickness measurement was obtained. The data [adapted from Box and Draper (1987)] are shown in the table below.

Run	Speed	Pressure	Distance	Mean (y_1)	Std Dev (y_2)
1	-1.000	-1.000	-1.000	24.0	12.5
2	0.000	-1.000	-1.000	120.3	8.4
3	1.000	-1.000	-1.000	213.7	42.8
4	-1.000	0.000	-1.000	86.0	3.5
5	0.000	0.000	-1.000	136.6	80.4
6	1.000	0.000	-1.000	340.7	16.2
7	-1.000	1.000	-1.000	112.3	27.6
8	0.000	1.000	-1.000	256.3	4.6
9	1.000	1.000	-1.000	271.7	23.6
10	-1.000	-1.000	0.000	81.0	0.0
11	0.000	-1.000	0.000	101.7	17.7
12	1.000	-1.000	0.000	357.0	32.9
13	-1.000	0.000	0.000	171.3	15.0
14	0.000	0.000	0.000	372.0	0.0
15	1.000	0.000	0.000	501.7	92.5
16	-1.000	1.000	0.000	264.0	63.5
17	0.000	1.000	0.000	427.0	88.6
18	1.000	1.000	0.000	730.7	21.1
19	-1.000	-1.000	1.000	220.7	133.8
20	0.000	-1.000	1.000	239.7	23.5
21	1.000	-1.000	1.000	422.0	18.5
22	-1.000	0.000	1.000	199.0	29.4
23	0.000	0.000	1.000	485.3	44.7
24	1.000	0.000	1.000	673.7	158.2
25	-1.000	1.000	1.000	176.7	55.5
26	0.000	1.000	1.000	501.0	138.9
27	1.000	1.000	1.000	1010.0	142.4

(a) What type of design did the experimenters use? Is this a good choice of design for fitting a quadratic model?

The design is a 3^3. A better choice would be a 2^3 central composite design. The CCD gives more information over the design region with fewer points.

(b) Build models of both responses.

The model for the mean is developed as follows:

Design Expert Output

Response: Mean

ANOVA for Response Surface Reduced Cubic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.289E+006	7	1.841E+005	60.45	< 0.0001	significant
A	*5.640E+005*	*1*	*5.640E+005*	*185.16*	*< 0.0001*	
B	*2.155E+005*	*1*	*2.155E+005*	*70.75*	*< 0.0001*	
C	*3.111E+005*	*1*	*3.111E+005*	*102.14*	*< 0.0001*	
AB	*52324.81*	*1*	*52324.81*	*17.18*	*0.0006*	
AC	*68327.52*	*1*	*68327.52*	*22.43*	*0.0001*	
BC	*22794.08*	*1*	*22794.08*	*7.48*	*0.0131*	
ABC	*54830.16*	*1*	*54830.16*	*18.00*	*0.0004*	
Residual	57874.57	19	3046.03			
Cor Total	1.347E+006	26				

The Model F-value of 60.45 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	55.19	R-Squared	0.9570
Mean	314.67	Adj R-Squared	0.9412
C.V.	17.54	Pred R-Squared	0.9056
PRESS	1.271E+005	Adeq Precision	33.333

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	314.67	1	10.62	292.44	336.90	
A-Speed	177.01	1	13.01	149.78	204.24	1.00
B-Pressure	109.42	1	13.01	82.19	136.65	1.00
C-Distance	131.47	1	13.01	104.24	158.70	1.00
AB	66.03	1	15.93	32.69	99.38	1.00
AC	75.46	1	15.93	42.11	108.80	1.00
BC	43.58	1	15.93	10.24	76.93	1.00
ABC	82.79	1	19.51	41.95	123.63	1.00

Final Equation in Terms of Coded Factors:

```
      Mean  =
   +314.67
   +177.01   * A
   +109.42   * B
   +131.47   * C
    +66.03   * A * B
    +75.46   * A * C
    +43.58   * B * C
    +82.79   * A * B * C
```

Final Equation in Terms of Actual Factors:

```
      Mean  =
 +314.67037
 +177.01111   * Speed
 +109.42222   * Pressure
 +131.47222   * Distance
  +66.03333   * Speed * Pressure
  +75.45833   * Speed * Distance
  +43.58333   * Pressure * Distance
  +82.78750   * Speed * Pressure * Distance
```

The model for the Std. Dev. response is as follows. A square root transformation was applied to correct problems with the normality assumption.

Design Expert Output

Response: Std. Dev. Transform: Square root Constant: 0
ANOVA for Response Surface Linear Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	116.75	3	38.92	4.34	0.0145	significant
A	*16.52*	*1*	*16.52*	*1.84*	*0.1878*	
B	*26.32*	*1*	*26.32*	*2.94*	*0.1001*	
C	*73.92*	*1*	*73.92*	*8.25*	*0.0086*	
Residual	206.17	23	8.96			
Cor Total	322.92	26				

The Model F-value of 4.34 implies the model is significant. There is only
a 1.45% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.99	R-Squared	0.3616
Mean	6.00	Adj R-Squared	0.2783
C.V.	49.88	Pred R-Squared	0.1359
PRESS	279.05	Adeq Precision	7.278

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	6.00	1	0.58	4.81	7.19	
A-Speed	0.96	1	0.71	-0.50	2.42	1.00
B-Pressure	1.21	1	0.71	-0.25	2.67	1.00
C-Distance	2.03	1	0.71	0.57	3.49	1.00

Final Equation in Terms of Coded Factors:

Sqrt(Std. Dev.) =
+6.00
+0.96 * A
+1.21 * B
+2.03 * C

Final Equation in Terms of Actual Factors:

Sqrt(Std. Dev.) =
+6.00273
+0.95796 * Speed
+1.20916 * Pressure
+2.02643 * Distance

Because Factor *A* is insignificant, it is removed from the model. The reduced linear model analysis is shown below:

Design Expert Output

Response: Std. Dev. Transform: Square root Constant: 0
ANOVA for Response Surface Reduced Linear Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	100.23	2	50.12	5.40	0.0116	significant
B	*26.32*	*1*	*26.32*	*2.84*	*0.1051*	
C	*73.92*	*1*	*73.92*	*7.97*	*0.0094*	
Residual	222.68	24	9.28			
Cor Total	322.92	26				

The Model F-value of 5.40 implies the model is significant. There is only
a 1.16% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	3.05	R-Squared	0.3104
Mean	6.00	Adj R-Squared	0.2529
C.V.	50.74	Pred R-Squared	0.1476
PRESS	275.24	Adeq Precision	6.373

	Coefficient		Standard	95% CI	95% CI

Factor	Estimate	DF	Error	Low	High	VIF
Intercept	6.00	1	0.59	4.79	7.21	
B-Pressure	1.21	1	0.72	-0.27	2.69	1.00
C-Distance	2.03	1	0.72	0.54	3.51	1.00

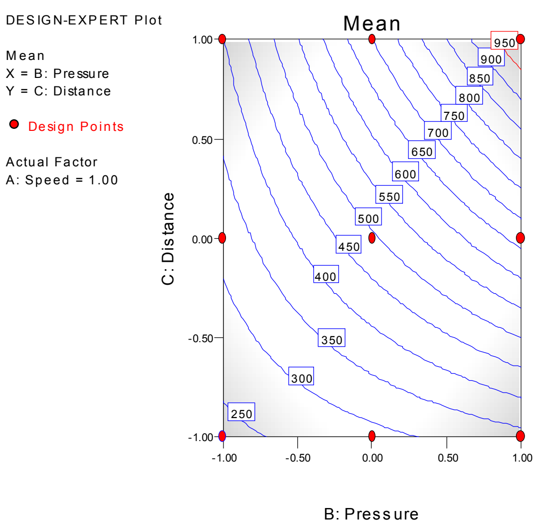
Final Equation in Terms of Coded Factors:

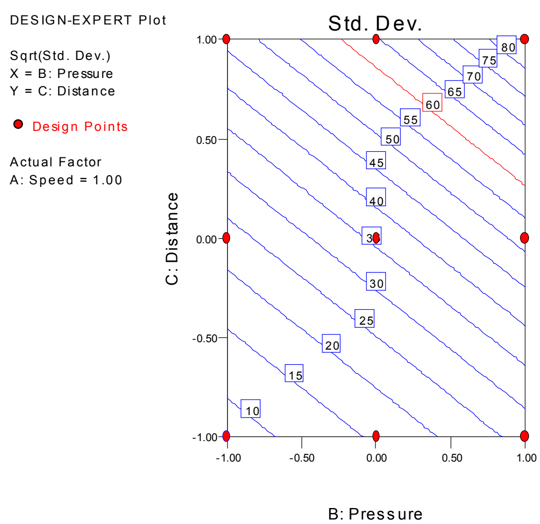
Sqrt(Std. Dev.) =
+6.00
+1.21 * B
+2.03 * C

Final Equation in Terms of Actual Factors:

Sqrt(Std. Dev.) =
+6.00273
+1.20916 * Pressure
+2.02643 * Distance

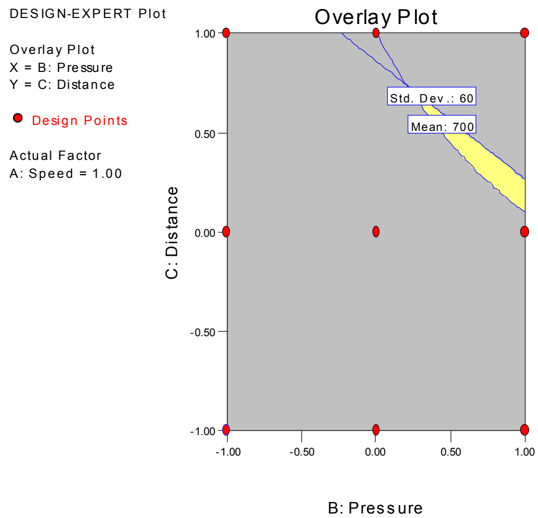
The following contour plots graphically represent the two models:





(c) Find a set of optimum conditions that result in the mean as large as possible with the standard deviation less than 60.

The overlay plot identifies a region that meets the criteria of the mean as large as possible with the standard deviation less than 60. The optimum conditions in coded terms are approximately Speed = 1.0, Pressure = 0.75 and Distance = 0.25.



11-34 A variation of Example 6-2. In example 6-2 we found that one of the process variables (B=pressure) was not important. Dropping this variable produced two replicates of a 2^3 design. The data are shown below.

C	D	$A(+)$	$A(-)$	$\bar{y}$	s^2
-	-	45, 48	71, 65	57.75	121.19
+	-	68, 80	60, 65	68.25	72.25
-	+	43, 45	100, 104	73.00	1124.67
+	+	75, 70	86, 96	81.75	134.92

Assume that C and D are controllable factors and that A is a noise factor.

(a) Fit a model to the mean response.

The following is the analysis of variance with all terms in the model:

Design Expert Output

Response: Mean
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	300.05	3	100.02		
A	*92.64*	*1*	*92.64*		
B	*206.64*	*1*	*206.64*		
AB	*0.77*	*1*	*0.77*		
Pure Error	0.000	0			
Cor Total	300.05	3			

Based on the above analysis, the AB interaction is removed from the model and used as error.

Design Expert Output

Response: Mean
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**

Model	299.28	2	149.64	195.45	0.0505	not significant
A	*92.64*	*1*	*92.64*	*121.00*	*0.0577*	
B	*206.64*	*1*	*206.64*	*269.90*	*0.0387*	
Residual	0.77	1	0.77			
Cor Total	300.05	3				

The Model F-value of 195.45 implies there is a 5.05% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.87	R-Squared	0.9974
Mean	70.19	Adj R-Squared	0.9923
C.V.	1.25	Pred R-Squared	0.9592
PRESS	12.25	Adeq Precision	31.672

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	70.19	1	0.44	64.63	75.75	
A-Concentration	4.81	1	0.44	-0.75	10.37	1.00
B-Stir Rate	7.19	1	0.44	1.63	12.75	1.00

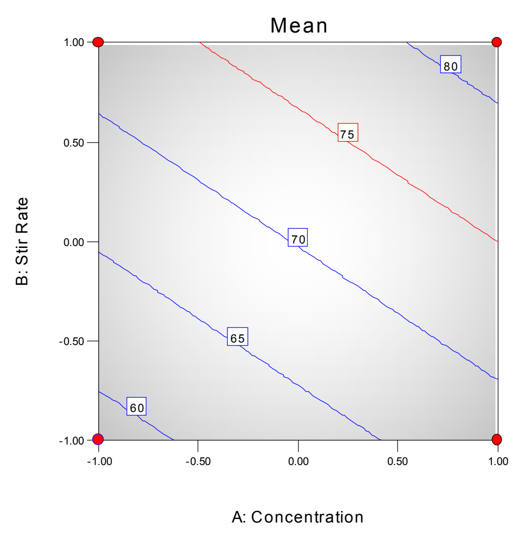
Final Equation in Terms of Coded Factors:

Mean =
+70.19
+4.81 * A
+7.19 * B

Final Equation in Terms of Actual Factors:

Mean =
+70.18750
+4.81250 * Concentration
+7.18750 * Stir Rate

The following is a contour plot of the mean model:



(b) Fit a model to the $ln(s^2)$ response.

The following is the analysis of variance with all terms in the model:

Design Expert Output

Response:	**Variance**	**Transform:**	**Natural log**	**Constant:**	**0**	

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	4.42	3	1.47		
A	*1.74*	*1*	*1.74*		
B	*2.03*	*1*	*2.03*		
AB	*0.64*	*1*	*0.64*		
Pure Error	0.000	0			
Cor Total	4.42	3			

Based on the above analysis, the AB interaction is removed from the model and applied to the residual error.

Design Expert Output

Response: Variance Transform: Natural log Constant: 0
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3.77	2	1.89	2.94	0.3815	not significant
A	*1.74*	*1*	*1.74*	*2.71*	*0.3477*	
B	*2.03*	*1*	*2.03*	*3.17*	*0.3260*	
Residual	0.64	1	0.64			
Cor Total	4.42	3				

The "Model F-value" of 2.94 implies the model is not significant relative to the noise. There is a 38.15 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	0.80	R-Squared	0.8545
Mean	5.25	Adj R-Squared	0.5634
C.V.	15.26	Pred R-Squared	-1.3284
PRESS	10.28	Adeq Precision	3.954

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	5.25	1	0.40	0.16	10.34	
A-Concentration	-0.66	1	0.40	-5.75	4.43	1.00
B-Stir Rate	0.71	1	0.40	-4.38	5.81	1.00

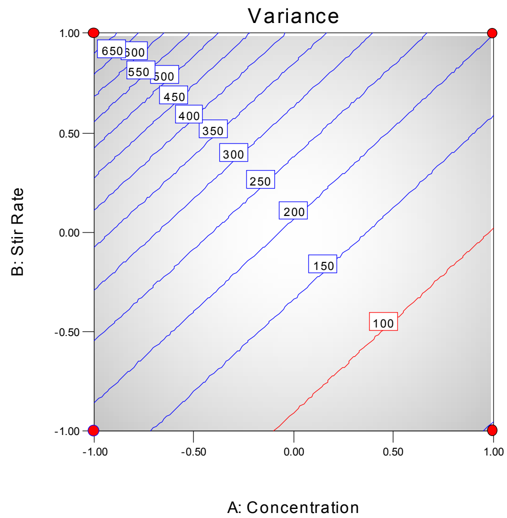
Final Equation in Terms of Coded Factors:

Ln(Variance) =
+5.25
-0.66 * A
+0.71 * B

Final Equation in Terms of Actual Factors:

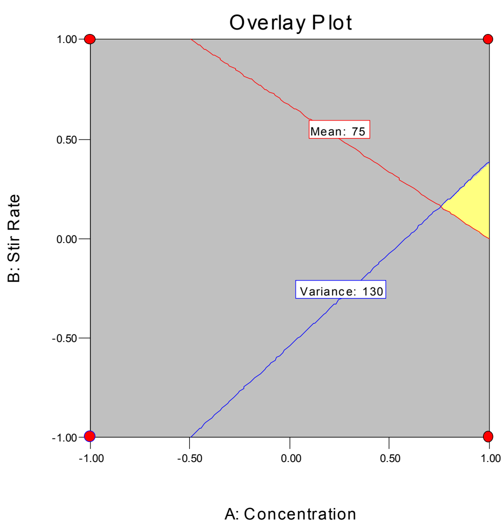
Ln(Variance) =
+5.25185
-0.65945 * Concentration
+0.71311 * Stir Rate

The following is a contour plot of the variance model in the untransformed form:



(c) Find operating conditions that result in the mean filtration rate response exceeding 75 with minimum variance.

The overlay plot shown below identifies the region required by the process:



(d) Compare your results with those from Example 11-6 which used the transmission of error approach. How similar are the two answers.

The results are very similar. Both require the Concentration to be held at the high level while the stirring rate is held near the middle.

Chapter 12
Experiments with Random Factors
Solutions

12-1 A textile mill has a large number of looms. Each loom is supposed to provide the same output of cloth per minute. To investigate this assumption, five looms are chosen at random and their output is noted at different times. The following data are obtained:

Loom	Output (lb/min)				
1	14.0	14.1	14.2	14.0	14.1
2	13.9	13.8	13.9	14.0	14.0
3	14.1	14.2	14.1	14.0	13.9
4	13.6	13.8	14.0	13.9	13.7
5	13.8	13.6	13.9	13.8	14.0

(a) Explain why this is a random effects experiment. Are the looms equal in output? Use $\alpha = 0.05$.

The looms used in the experiment are a random sample of all the looms in the manufacturing area. The following is the analysis of variance for the data:

Minitab Output

ANOVA: Output versus Loom

```
Factor     Type Levels Values
Loom     random      5     1     2     3     4     5

Analysis of Variance for Output

Source      DF         SS         MS       F      P
Loom         4    0.34160    0.08540    5.77  0.003
Error       20    0.29600    0.01480
Total       24    0.63760

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Loom      0.01412   2    (2) + 5(1)
 2 Error     0.01480        (2)
```

(b) Estimate the variability between looms.

$$\hat{\sigma}_\tau^2 = \frac{MS_{Model} - MS_E}{n} = \frac{0.0854 - 0.0148}{5} = 0.01412$$

(c) Estimate the experimental error variance.

$$\hat{\sigma}^2 = MS_E = 0.0148$$

(d) Find a 95 percent confidence interval for $\sigma_\tau^2 / \left(\sigma_\tau^2 + \sigma^2\right)$.

$$L = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{\alpha/2,a-1,n-a}} - 1\right] = 0.1288$$

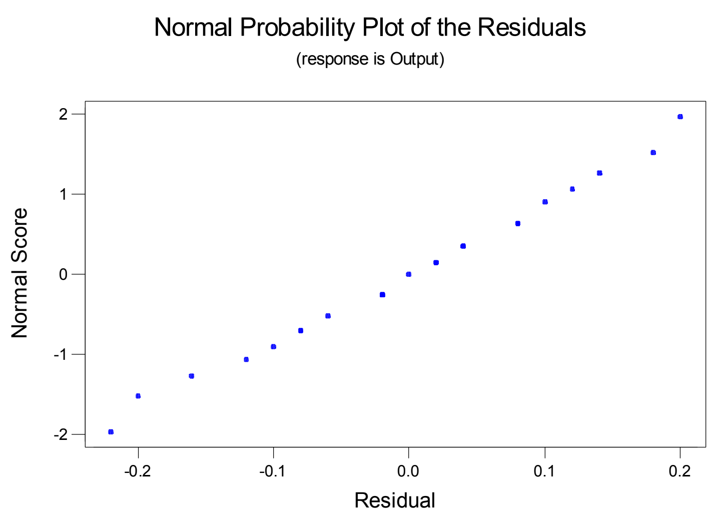
$$U = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{1-\alpha/2,a-1,n-a}} - 1\right] = 3.851$$

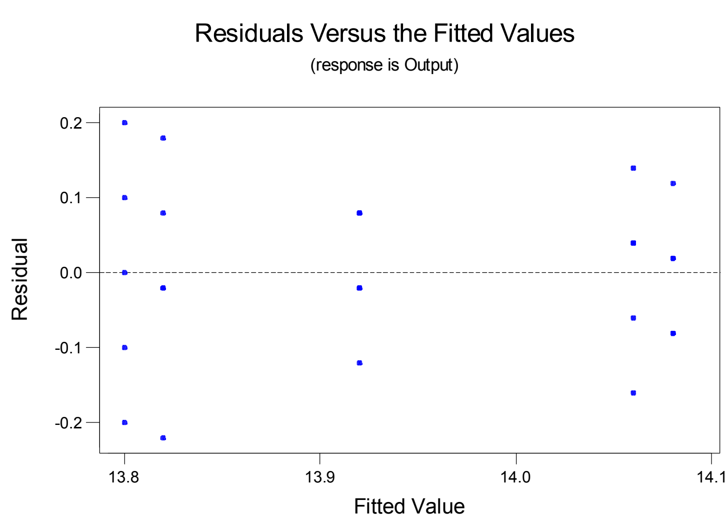
$$\frac{L}{L+1} \le \frac{\sigma_\tau^2}{\sigma_\tau^2 + \sigma^2} \le \frac{U}{U+1}$$

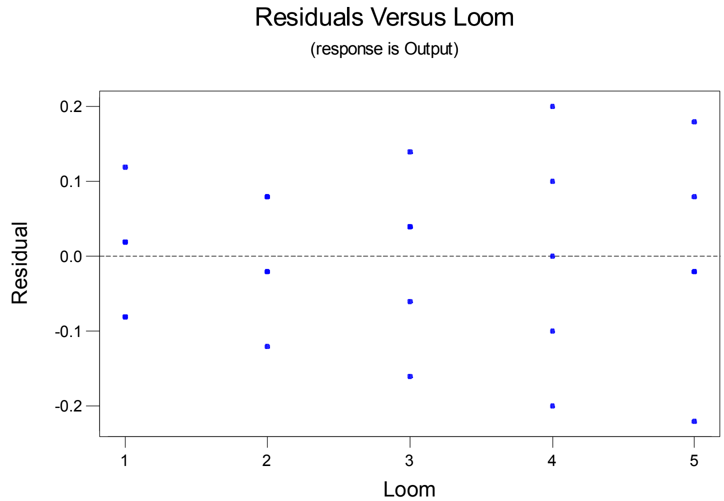
$$0.144 \le \frac{\sigma_\tau^2}{\sigma_\tau^2 + \sigma^2} \le 0.794$$

(e) Analyze the residuals from this experiment. Do you think that the analysis of variance assumptions are satisfied?

There is nothing unusual about the residual plots; therefore, the analysis of variance assumptions are satisfied.







12-2 A manufacturer suspects that the batches of raw material furnished by her supplier differ significantly in calcium content. There are a large number of batches currently in the warehouse. Five of these are randomly selected for study. A chemist makes five determinations on each batch and obtains the following data:

Batch 1	Batch 2	Batch 3	Batch 4	Batch 5
23.46	23.59	23.51	23.28	23.29
23.48	23.46	23.64	23.40	23.46
23.56	23.42	23.46	23.37	23.37
23.39	23.49	23.52	23.46	23.32
23.40	23.50	23.49	23.39	23.38

(a) Is there significant variation in calcium content from batch to batch? Use $\alpha = 0.05$.

Yes, as shown in the Minitab Output below, there is a difference.

Minitab Output

ANOVA: Calcium versus Batch

```
Factor      Type Levels Values
Batch     random      5      1      2      3      4      5

Analysis of Variance for Calcium

Source      DF         SS         MS        F      P
Batch        4   0.096976   0.024244     5.54  0.004
Error       20   0.087600   0.004380
Total       24   0.184576

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Batch     0.00397   2    (2) + 5(1)
 2 Error     0.00438        (2)
```

(b) Estimate the components of variance.

$$\hat{\sigma}_{\tau}^{2} = \frac{MS_{Model} - MS_{E}}{n} = \frac{.024244 - .004380}{5} = 0.00397$$

$$\hat{\sigma}^2 = MS_E = 0.004380$$

(c) Find a 95 percent confidence interval for $\sigma_\tau^2 / \left(\sigma_\tau^2 + \sigma^2\right)$.

$$L = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{\alpha/2,a-1,n-a}} - 1\right] = 0.1154$$

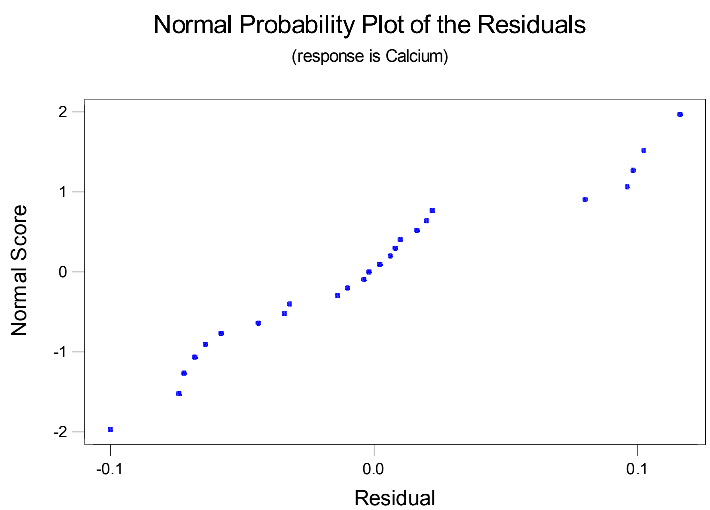
$$U = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{1-\alpha/2,a-1,n-a}} - 1\right] = 9.276$$

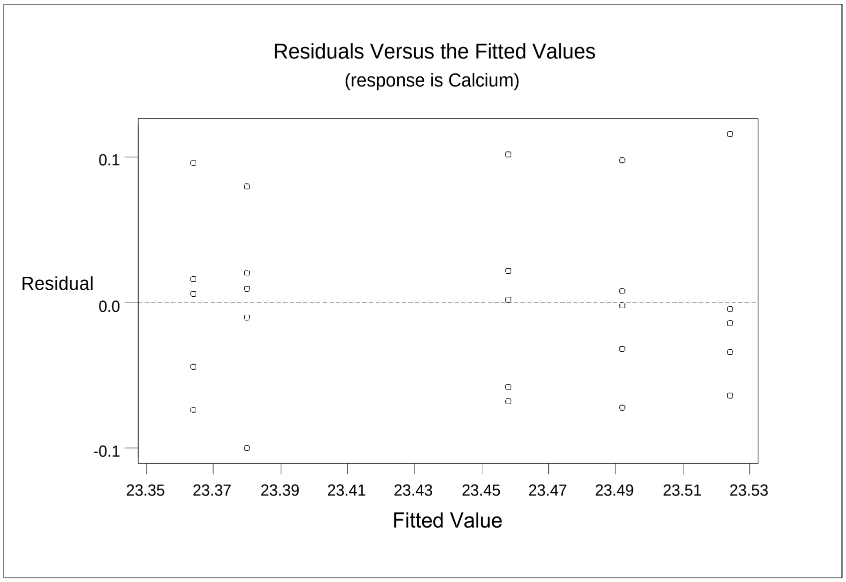
$$\frac{L}{L+1} \le \frac{\sigma_\tau^2}{\sigma_\tau^2 + \sigma^2} \le \frac{U}{U+1}$$

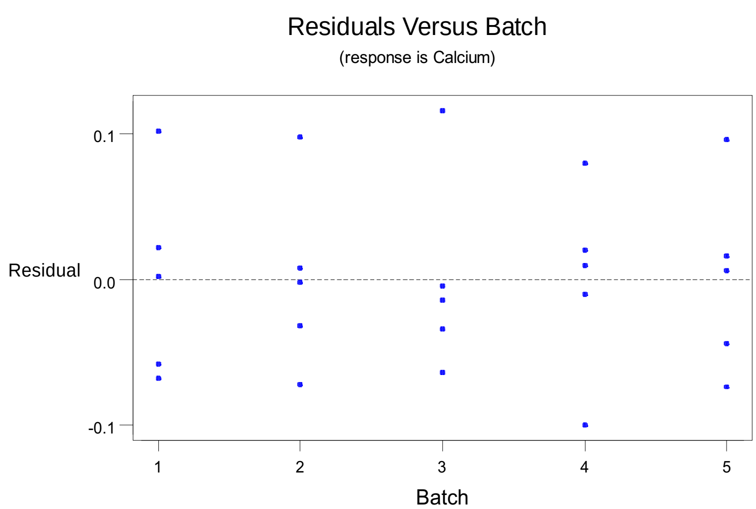
$$0.1035 \le \frac{\sigma_\tau^2}{\sigma_\tau^2 + \sigma^2} \le 0.9027$$

(d) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied?

There are five residuals that stand out in the normal probability plot. From the Residual vs. Batch plot, we see that one point per batch appears to stand out. A natural log transformation was applied to the data but did not change the results of the residual analysis. Further investigation should probably be performed to determine if these points are outliers.







12-3 Several ovens in a metal working shop are used to heat metal specimens. All the ovens are supposed to operate at the same temperature, although it is suspected that this may not be true. Three ovens are selected at random and their temperatures on successive heats are noted. The data collected are as follows:

Oven	Temperature					
1	491.50	498.30	498.10	493.50	493.60	
2	488.50	484.65	479.90	477.35		
3	490.10	484.80	488.25	473.00	471.85	478.65

(a) Is there significant variation in temperature between ovens? Use $\alpha = 0.05$.

The analysis of variance shown below identifies significant variation in temperature between the ovens.

Minitab Output

General Linear Model: Temperature versus Oven

```
Factor     Type Levels Values
```

```
Oven      random      3 1 2 3

Analysis of Variance for Temperat, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS       F      P
Oven         2     594.53     594.53     297.27    8.62  0.005
Error       12     413.81     413.81      34.48
Total       14    1008.34

Expected Mean Squares, using Adjusted SS

Source       Expected Mean Square for Each Term
 1 Oven      (2) +  4.9333(1)
 2 Error     (2)

Error Terms for Tests, using Adjusted SS

Source       Error DF  Error MS  Synthesis of Error MS
 1 Oven         12.00     34.48  (2)

Variance Components, using Adjusted SS

Source    Estimated Value
Oven                53.27
Error               34.48
```

(b) Estimate the components of variance.

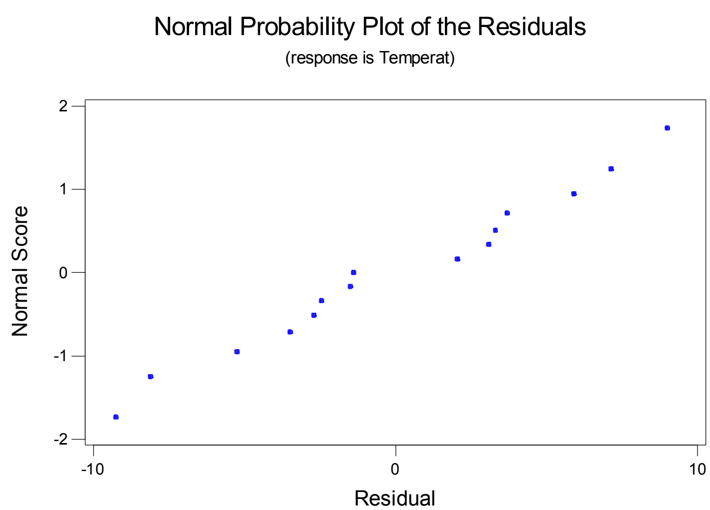
$$n_0 = \frac{1}{a-1}\left[\sum n_i - \frac{\sum n_i^2}{\sum n_i}\right] = \frac{1}{2}\left[15 - \frac{25+16+36}{15}\right] = 4.93$$

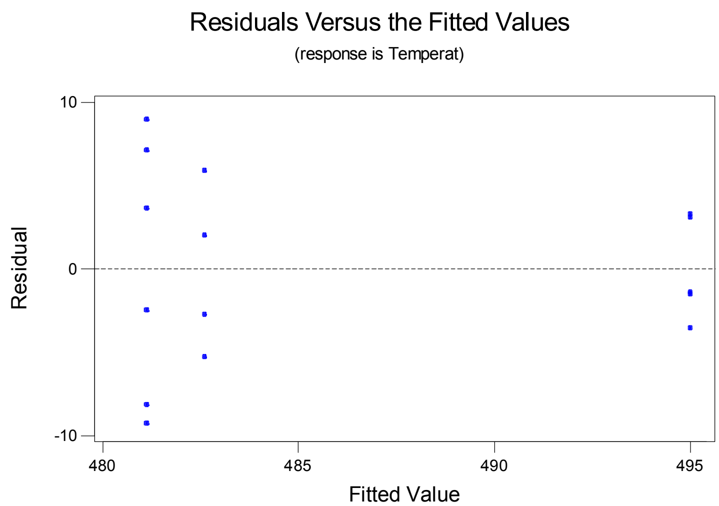
$$\hat{\sigma}_\tau^2 = \frac{MS_{Model} - MS_E}{n} = \frac{297.27 - 34.48}{4.93} = 53.30$$

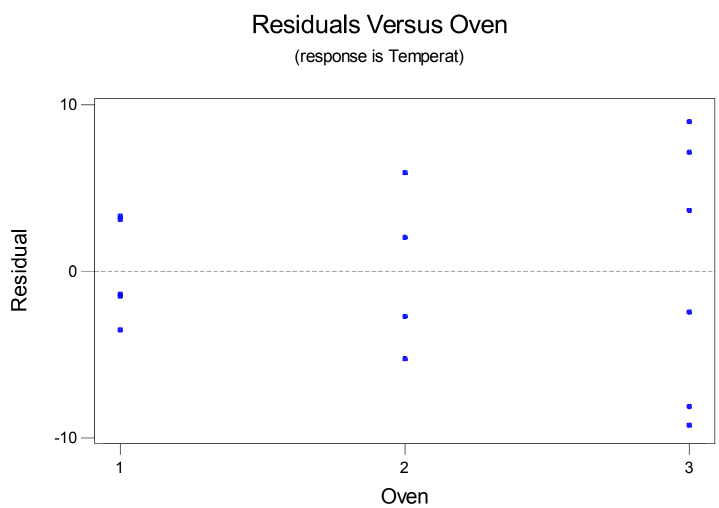
$$\hat{\sigma}^2 = MS_E = 34.48$$

(c) Analyze the residuals from this experiment. Draw conclusions about model adequacy.

There is a funnel shaped appearance in the plot of residuals versus predicted value indicating a possible non-constant variance. There is also some indication of non-constant variance in the plot of residuals versus oven. The inequality of variance problem is not severe.







12-4 An article in the *Journal of the Electrochemical Society* (Vol. 139, No. 2, 1992, pp. 524-532) describes an experiment to investigate the low-pressure vapor deposition of polysilicon. The experiment was carried out in a large-capacity reactor at Sematech in Austin, Texas. The reactor has several wafer positions, and four of these positions are selected at random. The response variable is film thickness uniformity. Three replicates of the experiments were run, and the data are as follows:

Wafer Position	Uniformity		
1	2.76	5.67	4.49
2	1.43	1.70	2.19
3	2.34	1.97	1.47
4	0.94	1.36	1.65

(a) Is there a difference in the wafer positions? Use $\alpha = 0.05$.

Yes, there is a difference.

Minitab Output

```
ANOVA: Uniformity versus Wafer Position

Factor      Type Levels Values
Wafer Po   fixed      4     1     2     3     4

Analysis of Variance for Uniformi

Source       DF         SS         MS       F      P
Wafer Po      3    16.2198     5.4066    8.29  0.008
Error         8     5.2175     0.6522
Total        11    21.4373

Source       Variance Error Expected Mean Square for Each Term
            component term (using restricted model)
 1 Wafer Po             2   (2) + 3Q[1]
 2 Error       0.6522       (2)
```

(b) Estimate the variability due to wafer positions.

$$\hat{\sigma}_\tau^2 = \frac{MS_{Treatment} - MS_E}{n}$$

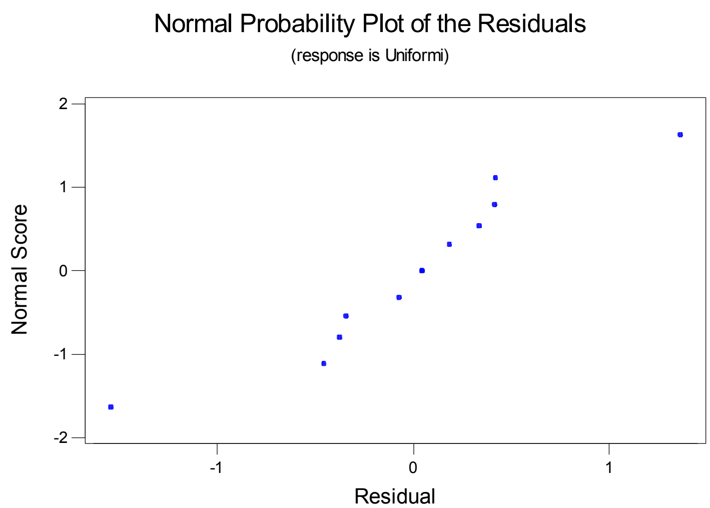
$$\hat{\sigma}_\tau^2 = \frac{5.4066 - 0.6522}{3} = 1.5844$$

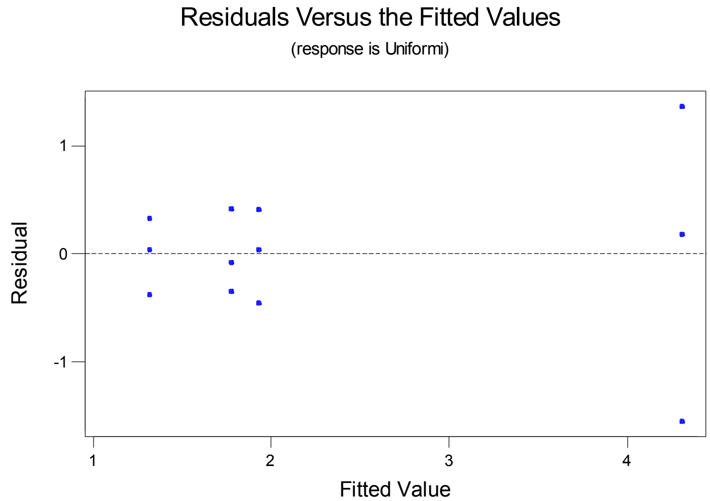
(c) Estimate the random error component.

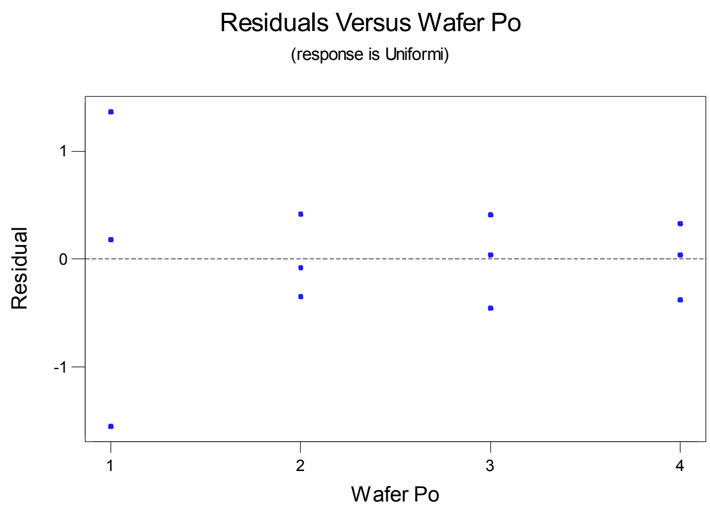
$$\hat{\sigma}^2 = 0.6522$$

(d) Analyze the residuals from this experiment and comment on model adequacy.

Variability in film thickness seems to depend on wafer position. These observations also show up as outliers on the normal probability plot. Wafer position number 1 appears to have greater variation in uniformity than the other positions.







12-5 Consider the vapor deposition experiment described in Problem 12-4.

(a) Estimate the total variability in the uniformity response.

$$\hat{\sigma}_\tau^2 + \hat{\sigma}^2 = 1.5848 + 0.6522 = 2.2370$$

(b) How much of the total variability in the uniformity response is due to the difference between positions in the reactor?

$$\frac{\hat{\sigma}_\tau^2}{\hat{\sigma}^2 + \hat{\sigma}_\tau^2} = \frac{1.5848}{2.2370} = 0.70845$$

(c) To what level could the variability in the uniformity response be reduced, if the position-to-position variability in the reactor could be eliminated? Do you believe this is a significant reduction?

The variability would be reduced from 2.2370 to $\hat{\sigma}^2 = 0.6522$ which is a reduction of approximately:

$$\frac{2.2370 - 0.6522}{2.2370} = 71\%$$

12-6 An article in the *Journal of Quality Technology* (Vol. 13, No. 2, 1981, pp. 111-114) describes and experiment that investigates the effects of four bleaching chemicals on pulp brightness. These four chemicals were selected at random from a large population of potential bleaching agents. The data are as follows:

Chemical	Pulp Brightness				
1	77.199	74.466	92.746	76.208	82.876
2	80.522	79.306	81.914	80.346	73.385
3	79.417	78.017	91.596	80.802	80.626
4	78.001	78.358	77.544	77.364	77.386

(a) Is there a difference in the chemical types? Use $\alpha = 0.05$.

The computer output shows that the null hypothesis cannot be rejected. Therefore, there is no evidence that there is a difference in chemical types.

Minitab Output

ANOVA: Brightness versus Chemical

```
Factor     Type Levels Values
Chemical random      4     1     2     3     4

Analysis of Variance for Brightne

Source      DF         SS         MS       F      P
Chemical     3      53.98      17.99    0.75  0.538
Error       16     383.99      24.00
Total       19     437.97

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Chemical   -1.201   2    (2) + 5(1)
 2 Error      23.999        (2)
```

(b) Estimate the variability due to chemical types.

$$\hat{\sigma}_\tau^2 = \frac{MS_{Treatment} - MS_E}{n}$$

$$\hat{\sigma}_\tau^2 = \frac{17.994 - 23.999}{5} = -1.201$$

which agrees with the Minitab output.

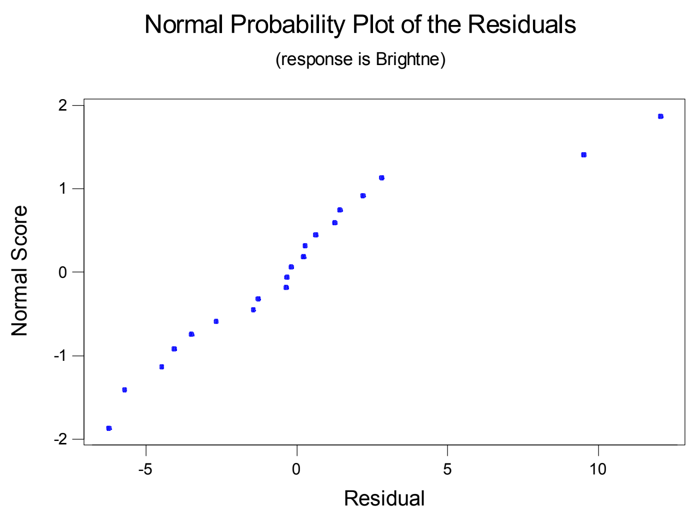
Because the variance component cannot be negative, this likely means that the variability due to chemical types is zero.

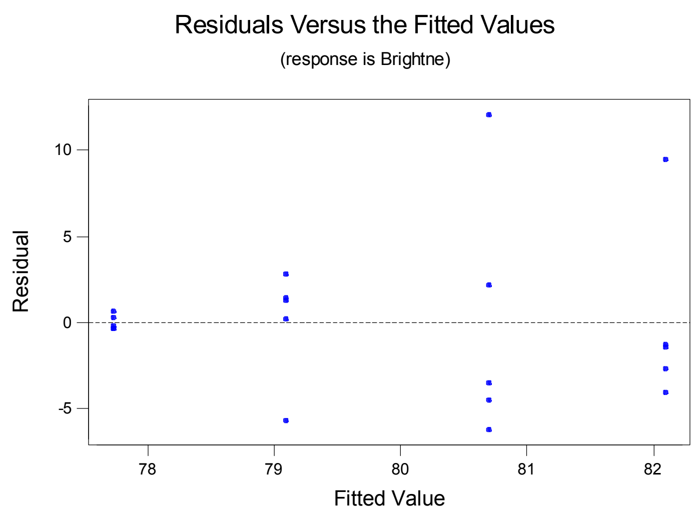
(c) Estimate the variability due to random error.

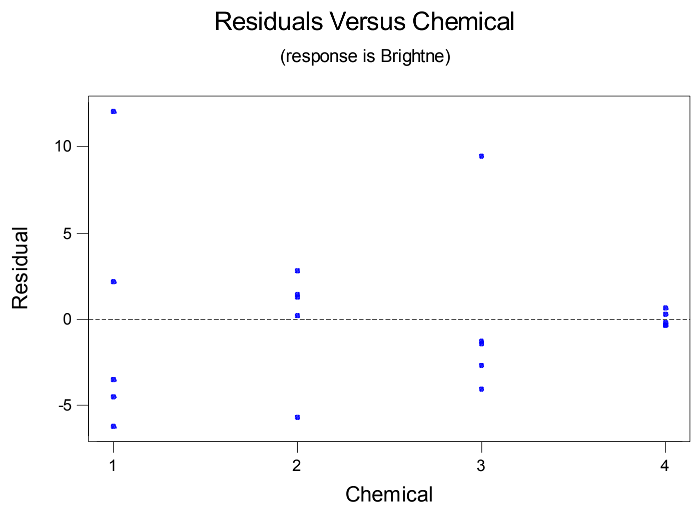
$$\hat{\sigma}^2 = 23.999$$

(d) Analyze the residuals from this experiment and comment on model adequacy.

Two data points appear to be outliers in the normal probability plot of effects. These outliers belong to chemical types 1 and 3 and should be investigated. There seems to be much less variability in brightness with chemical type 4.







12-7 Consider the one-way balanced, random effects method. Develop a procedure for finding a 100(1-α) percent confidence interval for $\sigma^2/(\sigma_\tau^2+\sigma^2)$.

We know that $P\left[L \le \dfrac{\sigma_\tau^2}{\sigma^2} \le U\right] = 1-\alpha$

$$P\left[L+1 \le \frac{\sigma_\tau^2}{\sigma^2}+\frac{\sigma^2}{\sigma^2} \le U+1\right] = 1-\alpha$$

$$P\left[L+1 \le \frac{\sigma_\tau^2+\sigma^2}{\sigma^2} \le U+1\right] = 1-\alpha$$

$$P\left[\frac{L}{1+L} \ge \frac{\sigma^2}{\sigma_\tau^2+\sigma^2} \ge \frac{U}{1+U}\right] = 1-\alpha$$

12-8 Refer to Problem 12-1.

(a) What is the probability of accepting H_0 if σ_τ^2 is four times the error variance σ^2?

$$\lambda = \sqrt{1+\frac{n\sigma_\tau^2}{\sigma^2}} = \sqrt{1+\frac{5(4\sigma^2)}{\sigma^2}} = \sqrt{21} = 4.6$$

$\upsilon_1 = a-1 = 4 \qquad \upsilon_2 = N-a = 25-5 = 20 \qquad \beta \approx 0.035$, from the OC curve.

(b) If the difference between looms is large enough to increase the standard deviation of an observation by 20 percent, we wish to detect this with a probability of at least 0.80. What sample size should be used?

$$\upsilon_1 = a-1 = 4 \qquad \upsilon_2 = N-a = 25-5 = 20 \qquad \alpha = 0.05 \qquad P(accept) \le 0.2$$

$$\lambda = \sqrt{1+n\left[(1+0.01P)^2-1\right]} = \sqrt{1+n\left[(1+0.01(20))^2-1\right]} = \sqrt{1+0.44n}$$

Trial and Error yields:

n	υ_2	λ	P(accept)
5	20	1.79	0.6
10	45	2.32	0.3
14	65	2.67	0.2

Choose $n \geq 14$, therefore $N \geq 70$

12-9 An experiment was performed to investigate the capability of a measurement system. Ten parts were randomly selected, and two randomly selected operators measured each part three times. The tests were made in random order, and the data below resulted.

Part	Operator 1 Measurements			Operator 2 Measurements		
Number	1	2	3	1	2	3
1	50	49	50	50	48	51
2	52	52	51	51	51	51
3	53	50	50	54	52	51
4	49	51	50	48	50	51
5	48	49	48	48	49	48
6	52	50	50	52	50	50
7	51	51	51	51	50	50
8	52	50	49	53	48	50
9	50	51	50	51	48	49
10	47	46	49	46	47	48

(a) Analyze the data from this experiment.

Minitab Output

ANOVA: Measurement versus Part, Operator

```
Factor     Type Levels Values
Part     random     10      1      2      3      4      5      6      7
                            8      9     10
Operator random      2      1      2

Analysis of Variance for Measurem

Source           DF         SS         MS        F        P
Part              9     99.017     11.002    18.28    0.000
Operator          1      0.417      0.417     0.69    0.427
Part*Operator     9      5.417      0.602     0.40    0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Part           1.73333   3   (4) + 3(3) + 6(1)
 2 Operator      -0.00617   3   (4) + 3(3) + 30(2)
 3 Part*Operator -0.29938   4   (4) + 3(3)
 4 Error          1.50000       (4)
```

(b) Find point estimates of the variance components using the analysis of variance method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 1.5$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^2 = \frac{0.6018519 - 1.5000000}{3} < 0 \text{, assume } \hat{\sigma}_{\tau\beta}^2 = 0$$

$$\hat{\sigma}_{\beta}^2 = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}_{\beta}^2 = \frac{11.001852 - 0.6018519}{2(3)} = 1.7333$$

$$\hat{\sigma}_{\tau}^2 = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}_{\tau}^2 = \frac{0.416667 - 0.6018519}{10(3)} < 0 \text{, assume } \hat{\sigma}_{\tau}^2 = 0$$

All estimates agree with the Minitab output.

12-10 Reconsider the data in Problem 5-6. Suppose that both factors, machines and operators, are chosen at random.

(a) Analyze the data from this experiment.

	Machine			
Operator	1	2	3	4
1	109	110	108	110
	110	115	109	108
2	110	110	111	114
	112	111	109	112
3	116	112	114	120
	114	115	119	117

The following Minitab output contains the analysis of variance and the variance component estimates:

Minitab Output

ANOVA: Strength versus Operator, Machine

```
Factor     Type Levels Values
Operator random      3     1     2     3
Machine  random      4     1     2     3     4


Analysis of Variance for Strength

Source             DF        SS         MS        F      P
Operator            2   160.333     80.167    10.77  0.010
Machine             3    12.458      4.153     0.56  0.662
Operator*Machine    6    44.667      7.444     1.96  0.151
Error              12    45.500      3.792
Total              23   262.958


Source             Variance Error Expected Mean Square for Each Term
                  component term (using restricted model)
 1 Operator           9.0903   3   (4) + 2(3) + 8(1)
 2 Machine           -0.5486   3   (4) + 2(3) + 6(2)
 3 Operator*Machine   1.8264   4   (4) + 2(3)
 4 Error              3.7917       (4)
```

(b) Find point estimates of the variance components using the analysis of variance method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.79167$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^2 = \frac{7.44444 - 3.79167}{2} = 1.82639$$

$$\hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}_\beta^2 = \frac{4.15278 - 7.44444}{3(2)} < 0\text{, assume } \hat{\sigma}_\beta^2 = 0$$

$$\hat{\sigma}_\tau^2 = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}_\tau^2 = \frac{80.16667 - 7.44444}{4(2)} = 9.09028$$

These results agree with the Minitab variance component analysis.

12-11 Reconsider the data in Problem 5-13. Suppose that both factors are random.

(a) Analyze the data from this experiment.

	Column	Factor		
Row Factor	1	2	3	4
1	36	39	36	32
2	18	20	22	20
3	30	37	33	34

Minitab Output

General Linear Model: Response versus Row, Column

```
Factor     Type Levels Values
Row      random      3 1 2 3
Column   random      4 1 2 3 4

Analysis of Variance for Response, using Adjusted SS for Tests

Source       DF     Seq SS     Adj SS     Adj MS       F      P
Row           2    580.500    580.500    290.250   60.40     **
Column        3     28.917     28.917      9.639    2.01     **
Row*Column    6     28.833     28.833      4.806      **
Error         0      0.000      0.000      0.000
Total        11    638.250

** Denominator of F-test is zero.

Expected Mean Squares, using Adjusted SS

Source          Expected Mean Square for Each Term
 1 Row          (4) +         (3) +  4.0000(1)
 2 Column       (4) +         (3) +  3.0000(2)
 3 Row*Column   (4) +         (3)
 4 Error        (4)

Error Terms for Tests, using Adjusted SS

Source          Error DF  Error MS  Synthesis of Error MS
 1 Row                *     4.806  (3)
 2 Column             *     4.806  (3)
 3 Row*Column         *         *  (4)

Variance Components, using Adjusted SS

Source      Estimated Value
Row                71.3611
Column              1.6111
Row*Column          4.8056
Error               0.0000
```

(b) Estimate the variance components.

Because the experiment is unreplicated and the interaction term was included in the model, there is no estimate of MS_E, and therefore, no estimate of σ^2.

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{4.8056 - 0}{1} = 4.8056$$

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}^2_{\beta} = \frac{9.6389 - 4.8056}{3(1)} = 1.6111$$

$$\hat{\sigma}^2_{\tau} = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}^2_{\tau} = \frac{290.2500 - 4.8056}{4(1)} = 71.3611$$

These estimates agree with the Minitab output.

12-12 Suppose that in Problem 5-11 the furnace positions were randomly selected, resulting in a mixed model experiment. Reanalyze the data from this experiment under this new assumption. Estimate the appropriate model components.

	Temperature (°C)		
Position	800	825	850
	570	1063	565
1	565	1080	510
	583	1043	590
	528	988	526
2	547	1026	538
	521	1004	532

The following analysis assumes a restricted model:

Minitab Output

ANOVA: Density versus Position, Temperature

```
Factor     Type Levels Values
Position random      2     1     2
Temperat  fixed      3   800   825   850

Analysis of Variance for Density

Source               DF         SS         MS       F       P
Position              1       7160       7160   16.00  0.002
Temperat              2     945342     472671 1155.52  0.001
Position*Temperat     2        818        409    0.91  0.427
Error                12       5371        448
Total                17     958691

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Position            745.83   4   (4) + 9(1)
 2 Temperat                     3   (4) + 3(3) + 6Q[2]
 3 Position*Temperat   -12.83   4   (4) + 3(3)
 4 Error               447.56       (4)
```

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 447.56$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{409 - 448}{3} < 0 \text{ assume } \hat{\sigma}^2_{\tau\beta} = 0$$

$$\hat{\sigma}_\tau^2 = \frac{MS_A - MS_E}{bn} \qquad \hat{\sigma}_\tau^2 = \frac{7160 - 448}{3(3)} = 745.83$$

These results agree with the Minitab output.

12-13 Reanalyze the measurement systems experiment in Problem 12-9, assuming that operators are a fixed factor. Estimate the appropriate model components.

The following analysis assumes a restricted model:

Minitab Output

ANOVA: Measurement versus Part, Operator

```
Factor     Type Levels Values
Part     random     10     1     2     3     4     5     6     7
                           8     9    10
Operator  fixed      2     1     2

Analysis of Variance for Measurem

Source           DF         SS         MS       F      P
Part              9     99.017     11.002    7.33  0.000
Operator          1      0.417      0.417    0.69  0.427
Part*Operator     9      5.417      0.602    0.40  0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Part            1.5836   4   (4) + 6(1)
 2 Operator                 3   (4) + 3(3) + 30Q[2]
 3 Part*Operator  -0.2994   4   (4) + 3(3)
 4 Error           1.5000       (4)
```

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 1.5000$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^2 = \frac{0.60185 - 1.5000}{3} < 0 \text{ assume } \hat{\sigma}_{\tau\beta}^2 = 0$$

$$\hat{\sigma}_\tau^2 = \frac{MS_A - MS_E}{bn} \qquad \hat{\sigma}_\tau^2 = \frac{11.00185 - 1.50000}{2(3)} = 1.58364$$

These results agree with the Minitab output.

12-14 In problem 5-6, suppose that there are only four machines of interest, but the operators were selected at random.

(a) What type of model is appropriate?

A mixed model is appropriate.

(b) Perform the analysis and estimate the model components.

The following analysis assumes a restricted model:

Minitab Output

ANOVA: Strength versus Operator, Machine

```
Factor     Type Levels Values
```

```
Operator random      3     1     2     3
Machine   fixed      4     1     2     3     4

Analysis of Variance for Strength

Source              DF         SS         MS       F      P
Operator             2    160.333     80.167   21.14  0.000
Machine              3     12.458      4.153    0.56  0.662
Operator*Machine     6     44.667      7.444    1.96  0.151
Error               12     45.500      3.792
Total               23    262.958

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Operator             9.547   4    (4) + 8(1)
 2 Machine                      3    (4) + 2(3) + 6Q[2]
 3 Operator*Machine     1.826   4    (4) + 2(3)
 4 Error                3.792        (4)
```

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.792$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^2 = \frac{7.444 - 3.792}{2} = 1.826$$

$$\hat{\sigma}_{\tau}^2 = \frac{MS_A - MS_E}{bn} \qquad \hat{\sigma}_{\tau}^2 = \frac{80.167 - 3.792}{4(2)} = 9.547$$

These results agree with the Minitab output.

12-15 By application of the expectation operator, develop the expected mean squares for the two-factor factorial, mixed model. Use the restricted model assumptions. Check your results with the expected mean squares given in Table 12-11 to see that they agree.

The sums of squares may be written as

$$SS_A = bn\sum_{i=1}^{a}\left(\bar{y}_{i..} - \bar{y}_{...}\right)^2, \qquad SS_B = an\sum_{j=1}^{b}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2$$

$$SS_{AB} = n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right)^2, \qquad SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \bar{y}_{...}\right)^2$$

Using the model $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$, we may find that

$$\begin{aligned}
\bar{y}_{i..} &= \mu + \tau_i + \left(\overline{\tau\beta}\right)_{i.} + \bar{\varepsilon}_{i..} \\
\bar{y}_{.j.} &= \mu + \beta_j + \bar{\varepsilon}_{.j.} \\
\bar{y}_{ij.} &= \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \bar{\varepsilon}_{ij.} \\
\bar{y}_{...} &= \mu + \bar{\beta}_{.} + \bar{\varepsilon}_{...}
\end{aligned}$$

Using the assumptions for the restricted form of the mixed model, $\tau_{.} = 0$, $(\tau\beta)_{.j} = 0$, which imply that $(\tau\beta)_{..} = 0$. Substituting these expressions into the sums of squares yields

$$SS_A = bn\sum_{i=1}^{a}\left(\tau + (\tau\beta)_{i.} + \bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...}\right)^2$$

$$SS_B = an\sum_{j=1}^{b}\left(\beta_j + \bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)^2$$

$$SS_{AB} = n\sum_{i=1}^{a}\sum_{j=1}^{b}\left((\tau\beta)_{ij} - (\tau\beta)_{i.} + \bar{\varepsilon}_{ij.} - \bar{\varepsilon}_{i..} - \bar{\varepsilon}_{.j.} + \bar{\varepsilon}_{...}\right)^2$$

$$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\varepsilon_{ijk} - \bar{\varepsilon}_{ij.}\right)^2$$

Using the assumption that $E(\varepsilon_{ijk}) = 0$, $V(\varepsilon_{ijk}) = 0$, and $E(\varepsilon_{ijk}\cdot\varepsilon_{i'j'k'}) = 0$, we may divide each sum of squares by its degrees of freedom and take the expectation to produce

$$E(MS_A) = \sigma^2 + \left[\frac{bn}{(a-1)}\right]E\sum_{i=1}^{a}\left(\tau_i + (\overline{\tau\beta})_{i.}\right)^2$$

$$E(MS_B) = \sigma^2 + \left[\frac{an}{(b-1)}\right]\sum_{j=1}^{b}\beta_j^2$$

$$E(MS_{AB}) = \sigma^2 + \left[\frac{n}{(a-1)(b-1)}\right]E\sum_{i=1}^{a}\sum_{j=1}^{b}\left((\tau\beta)_{ij} - (\overline{\tau\beta})_{i.}\right)^2$$

$$E(MS_E) = \sigma^2$$

Note that $E(MS_B)$ and $E(MS_E)$ are the results given in Table 8-3. We need to simplify $E(MS_A)$ and $E(MS_{AB})$. Consider $E(MS_A)$

$$E(MS_A) = \sigma^2 + \frac{bn}{a-1}\left[\sum_{i=1}^{a}E(\tau_i)^2 + \sum_{i=1}^{a}E(\tau\beta)_{i.}^2 + (crossproducts = 0)\right]$$

$$E(MS_A) = \sigma^2 + \frac{bn}{a-1}\left[\sum_{i=1}^{a}\tau_i^2 + a\frac{\left[\frac{(a-1)}{a}\right]}{b}\sigma_{\tau\beta}^2\right]$$

$$E(MS_A) = \sigma^2 + n\sigma_{\tau\beta}^2 + \frac{bn}{a-1}\sum_{i=1}^{a}\tau_i^2$$

since $(\tau\beta)_{ij}$ is $NID\left(0, \frac{a-1}{a}\sigma_{\tau\beta}^2\right)$. Consider $E(MS_{AB})$

$$E(MS_{AB}) = \sigma^2 + \frac{n}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}E\left((\tau\beta)_{ij} - (\overline{\tau\beta})_{i.}\right)^2$$

$$E(MS_{AB}) = \sigma^2 + \frac{n}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\frac{b-1}{b}\right)\left(\frac{a-1}{a}\right)\sigma_{\tau\beta}^2$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2$$

Thus $E(MS_A)$and $E(MS_{AB})$agree with table 12-8.

12-16 Consider the three-factor factorial design in Example 12-6. Propose appropriate test statistics for all main effects and interactions. Repeat for the case where *A* and *B* are fixed and *C* is random.

If all three factors are random there are no exact tests on main effects. We could use the following:

$$A : F = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}}$$
$$B : F = \frac{MS_B + MS_{ABC}}{MS_{AB} + MS_{BC}}$$
$$C : F = \frac{MS_C + MS_{ABC}}{MS_{AC} + MS_{BC}}$$

If *A* and *B* are fixed and *C* is random, the expected mean squares are (assuming the restricted for m of the model):

Factor	F *a* *i*	F *b* *j*	R *c* *k*	R *n* *l*	E(*MS*)
τ_i	0	*b*	*c*	*n*	$\sigma^2 + bn\sigma^2_{\tau\gamma} + bcn\sum\frac{\tau_i^2}{(a-1)}$
β_j	*a*	0	*c*	*n*	$\sigma^2 + an\sigma^2_{\beta\gamma} + acn\sum\frac{\beta_j^2}{(b-1)}$
γ_k	*a*	*b*	1	*n*	$\sigma^2 + abn\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	0	0	*c*	*n*	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sum\sum\frac{(\tau\beta)^2_{ji}}{(a-1)(b-1)}$
$(\tau\gamma)_{ik}$	0	*b*	1	*n*	$\sigma^2 + bn\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	*a*	0	1	*n*	$\sigma^2 + an\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	0	0	1	*n*	$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

These are exact tests for all effects.

12-17 Consider the experiment in Example 12-7. Analyze the data for the case where *A*, *B*, and *C* are random.

Minitab Output

ANOVA: Drop versus Temp, Operator, Gauge

```
Factor      Type Levels Values
Temp     random       3    60    75    90
Operator random       4     1     2     3     4
Gauge    random       3     1     2     3

Analysis of Variance for Drop

Source                 DF          SS          MS        F       P
Temp                    2     1023.36      511.68     2.30   0.171 x
```

```
Operator               3     423.82     141.27    0.63  0.616 x
Gauge                  2       7.19       3.60    0.06  0.938 x
Temp*Operator          6    1211.97     202.00   14.59  0.000
Temp*Gauge             4     137.89      34.47    2.49  0.099
Operator*Gauge         6     209.47      34.91    2.52  0.081
Temp*Operator*Gauge   12     166.11      13.84    0.65  0.788
Error                 36     770.50      21.40
Total                 71    3950.32

x Not an exact F-test.

Source                 Variance Error Expected Mean Square for Each Term
                      component term (using restricted model)
 1 Temp                  12.044   *    (8) + 2(7) + 8(5) + 6(4) + 24(1)
 2 Operator              -4.544   *    (8) + 2(7) + 6(6) + 6(4) + 18(2)
 3 Gauge                 -2.164   *    (8) + 2(7) + 6(6) + 8(5) + 24(3)
 4 Temp*Operator         31.359   7    (8) + 2(7) + 6(4)
 5 Temp*Gauge             2.579   7    (8) + 2(7) + 8(5)
 6 Operator*Gauge         3.512   7    (8) + 2(7) + 6(6)
 7 Temp*Operator*Gauge   -3.780   8    (8) + 2(7)
 8 Error                 21.403        (8)

* Synthesized Test.

Error Terms for Synthesized Tests

Source                  Error DF  Error MS  Synthesis of Error MS
 1 Temp                     6.97    222.63   (4) + (5) - (7)
 2 Operator                 7.09    223.06   (4) + (6) - (7)
 3 Gauge                    5.98     55.54   (5) + (6) - (7)
```

Since all three factors are random there are no exact tests on main effects. Minitab uses an approximate F test for the these factors.

12-18 Derive the expected mean squares shown in Table 12-14.

Factor	F a i	R b j	R c k	R n l	E(MS)
τ_i	0	b	c	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma} + cn\sigma^2_{\tau\beta} + bcn\sum\frac{\tau_i^2}{(a-1)}$
β_j	a	1	c	n	$\sigma^2 + an\sigma^2_{\beta\gamma} + acn\sigma^2_{\beta}$
γ_k	a	b	1	n	$\sigma^2 + an\sigma^2_{\beta\gamma} + abn\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	0	1	c	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sigma^2_{\tau\beta}$
$(\tau\gamma)_{ik}$	0	b	1	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	a	1	1	n	$\sigma^2 + an\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	0	1	1	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$
ε_{ijkl}	1	1	1	1	σ^2

12-19 Consider a four-factor factorial experiment where factor A is at a levels, factor B is at b levels, factor C is at c levels, factor D is at d levels, and there are n replicates. Write down the sums of squares, the degrees of freedom, and the expected mean squares for the following cases. Do exact tests exist for all effects? If not, propose test statistics for those effects that cannot be directly tested. Assume the restricted model on all cases. You may use a computer package such as Minitab.

The four factor model is:

$$y_{ijklh} = \mu + \tau_i + \beta_j + \gamma_k + \delta_l + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\tau\delta)_{il} + (\beta\gamma)_{jk} + (\beta\delta)_{jl} + (\gamma\delta)_{kl} + (\tau\beta\gamma)_{ijk} + (\tau\beta\delta)_{ijl} + (\beta\gamma\delta)_{jkl} + (\tau\gamma\delta)_{ikl} + (\tau\beta\gamma\delta)_{ijkl} + \varepsilon_{ijklh}$$

To simplify the expected mean square derivations, let capital Latin letters represent the factor effects or variance components. For example, $A = \frac{bcdn\sum \tau_i^2}{a-1}$, or $B = acdn\sigma_\beta^2$.

(a) *A, B, C*, and *D* are fixed factors.

Factor	F *a* *i*	F *b* *j*	F *c* *k*	F *d* *l*	R *n* *h*	E(*MS*)
τ_i	0	b	c	d	n	$\sigma^2 + A$
β_j	a	0	c	d	n	$\sigma^2 + B$
γ_k	a	b	0	d	n	$\sigma^2 + C$
δ_l	a	b	c	0	n	$\sigma^2 + D$
$(\tau\beta)_{ij}$	0	0	c	d	n	$\sigma^2 + AB$
$(\tau\gamma)_{ik}$	0	b	0	d	n	$\sigma^2 + AC$
$(\tau\delta)_{il}$	0	b	c	0	n	$\sigma^2 + AD$
$(\beta\gamma)_{jk}$	a	0	0	d	n	$\sigma^2 + BC$
$(\beta\delta)_{jl}$	a	0	c	0	n	$\sigma^2 + BD$
$(\gamma\delta)_{kl}$	a	b	0	0	n	$\sigma^2 + CD$
$(\tau\beta\gamma)_{ijk}$	0	0	0	d	n	$\sigma^2 + ABC$
$(\tau\beta\delta)_{ijl}$	0	0	c	0	n	$\sigma^2 + ABD$
$(\beta\gamma\delta)_{jkl}$	a	0	0	0	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	0	0	n	$\sigma^2 + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	0	0	0	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

There are exact tests for all effects. The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor      Type Levels Values
A          fixed      2     H     L
B          fixed      2     H     L
C          fixed      2     H     L
D          fixed      2     H     L

Analysis of Variance for y

Source       DF        SS         MS       F      P
A             1      6.13       6.13    0.49  0.492
B             1      0.13       0.13    0.01  0.921
C             1      1.13       1.13    0.09  0.767
D             1      0.13       0.13    0.01  0.921
A*B           1      3.13       3.13    0.25  0.622
A*C           1      3.13       3.13    0.25  0.622
A*D           1      3.13       3.13    0.25  0.622
B*C           1      3.13       3.13    0.25  0.622
B*D           1      3.13       3.13    0.25  0.622
C*D           1      3.13       3.13    0.25  0.622
A*B*C         1      3.13       3.13    0.25  0.622
A*B*D         1     28.13      28.13    2.27  0.151
```

```
A*C*D        1       3.13       3.13    0.25  0.622
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                 16   (16) + 16Q[1]
 2 B                 16   (16) + 16Q[2]
 3 C                 16   (16) + 16Q[3]
 4 D                 16   (16) + 16Q[4]
 5 A*B               16   (16) + 8Q[5]
 6 A*C               16   (16) + 8Q[6]
 7 A*D               16   (16) + 8Q[7]
 8 B*C               16   (16) + 8Q[8]
 9 B*D               16   (16) + 8Q[9]
10 C*D               16   (16) + 8Q[10]
11 A*B*C             16   (16) + 4Q[11]
12 A*B*D             16   (16) + 4Q[12]
13 A*C*D             16   (16) + 4Q[13]
14 B*C*D             16   (16) + 4Q[14]
15 A*B*C*D           16   (16) + 2Q[15]
16 Error       12.38        (16)
```

(b) *A, B, C*, and *D* are random factors.

Factor	R *a* *i*	R *b* *j*	R *c* *k*	R *d* *l*	R *n* *h*	E(*MS*)
τ_i	1	b	c	d	n	$\sigma^2 + ABCD + ACD + ABD + ABC + AD + AC + AB + A$
β_j	a	1	c	d	n	$\sigma^2 + ABCD + BCD + ABD + ABC + BD + BC + AB + B$
γ_k	a	b	1	d	n	$\sigma^2 + ABCD + ACD + BCD + ABC + AB + BC + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + ABCD + ACD + BCD + ABD + BD + AD + CD + D$
$(\tau\beta)_{ij}$	1	1	c	d	n	$\sigma^2 + ABCD + ABC + ABD + AB$
$(\tau\gamma)_{ik}$	1	b	1	d	n	$\sigma^2 + ABCD + ABC + ACD + AC$
$(\tau\delta)_{il}$	1	b	c	1	n	$\sigma^2 + ABCD + ABD + ACD + AD$
$(\beta\gamma)_{jk}$	a	1	1	d	n	$\sigma^2 + ABCD + ABC + BCD + BC$
$(\beta\delta)_{jl}$	a	1	c	1	n	$\sigma^2 + ABCD + ABD + BCD + BD$
$(\gamma\delta)_{kl}$	a	b	1	1	n	$\sigma^2 + ABCD + ACD + BCD + CD$
$(\tau\beta\gamma)_{ijk}$	1	1	1	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	1	1	c	1	n	$\sigma^2 + ABCD + ABD$
$(\beta\gamma\delta)_{jkl}$	a	1	1	1	n	$\sigma^2 + ABCD + BCD$
$(\tau\gamma\delta)_{ikl}$	1	b	1	1	n	$\sigma^2 + ABCD + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	1	1	1	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

No exact tests exist on main effects or two-factor interactions. For main effects use statistics such as:

$$A{:}F = \frac{MS_A + MS_{ABC} + MS_{ABD} + MS_{ACD}}{MS_{AB} + MS_{AC} + MS_{AD} + MS_{ABCD}}$$

For testing two-factor interactions use statistics such as: $AB{:}F = \dfrac{MS_{AB} + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$

The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A        random      2     H     L
B        random      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13      **
B            1       0.13       0.13      **
C            1       1.13       1.13    0.36  0.843 x
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.667 x
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A          1.7500   *   (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                            + 8(5) + 16(1)
 2 B          1.3750   *   (16) + 2(15) + 4(14) + 4(12) + 4(11) + 8(9) + 8(8)
                            + 8(5) + 16(2)
 3 C         -0.1250   *   (16) + 2(15) + 4(14) + 4(13) + 4(11) + 8(10) + 8(8)
                            + 8(6) + 16(3)
 4 D          1.3750   *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                            + 8(7) + 16(4)
 5 A*B       -3.1250   *   (16) + 2(15) + 4(12) + 4(11) + 8(5)
 6 A*C        0.0000   *   (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250   *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000   *   (16) + 2(15) + 4(14) + 4(11) + 8(8)
 9 B*D       -3.1250   *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D        0.0000   *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C      0.0000  15   (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15   (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15   (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.56         *   (5) + (6) + (7) - (11) - (12) - (13) + (15)
 2 B             0.56         *   (5) + (8) + (9) - (11) - (12) - (14) + (15)
 3 C             0.14      3.13   (6) + (8) + (10) - (11) - (13) - (14) + (15)
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
```

```
 8 B*C          0.33      3.13   (11) + (14) - (15)
 9 B*D          0.98     28.13   (12) + (14) - (15)
10 C*D          0.33      3.13   (13) + (14) - (15)
```

(c) *A* is fixed and *B, C,* and *D* are random.

	F	R	R	R	R	
	a	b	c	d	n	
Factor	i	j	k	l	h	E(*MS*)
τ_i	0	b	c	d	n	$\sigma^2 + ABCD + ACD + ABD + ABC + AD + AC + AB + A$
β_j	a	1	c	d	n	$\sigma^2 + BCD + ABD + BC + B$
γ_k	a	b	1	d	n	$\sigma^2 + BCD + BC + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + BCD + BD + CD + D$
$(\tau\beta)_{ij}$	0	1	c	d	n	$\sigma^2 + ABCD + ABC + ABD + AB$
$(\tau\gamma)_{ik}$	0	b	1	d	n	$\sigma^2 + ABCD + ABC + ACD + AC$
$(\tau\delta)_{il}$	0	b	c	1	n	$\sigma^2 + ABCD + ABD + ACD + AD$
$(\beta\gamma)_{jk}$	a	1	1	d	n	$\sigma^2 + BCD + BC$
$(\beta\delta)_{jl}$	a	1	c	1	n	$\sigma^2 + BCD + BD$
$(\gamma\delta)_{kl}$	a	b	1	1	n	$\sigma^2 + BCD + CD$
$(\tau\beta\gamma)_{ijk}$	0	1	1	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	0	1	c	1	n	$\sigma^2 + ABCD + ABD$
$(\beta\gamma\delta)_{jkl}$	a	1	1	1	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	1	1	n	$\sigma^2 + ABCD + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	1	1	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

No exact tests exist on main effects or two-factor interactions involving the fixed factor *A*. To test the fixed factor *A* use

$$A{:}F = \frac{MS_A + MS_{ABC} + MS_{ABD} + MS_{ACD}}{MS_{AB} + MS_{AC} + MS_{AD} + MS_{ABCD}}$$

Random main effects could be tested by, for example: $D{:}F = \dfrac{MS_D + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$

For testing two-factor interactions involving *A* use: $AB{:}F = \dfrac{MS_{AB} + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$

The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B        random      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS        F      P
A            1       6.13       6.13       **
B            1       0.13       0.13    0.04  0.907 x
```

```
C            1       1.13       1.13    0.36  0.761 x
D            1       0.13       0.13    0.04  0.907 x
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.500
B*D          1       3.13       3.13    1.00  0.500
C*D          1       3.13       3.13    1.00  0.500
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                  *    (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                             + 8(5) + 16Q[1]
 2 B         -0.1875   *    (16) + 4(14) + 8(9) + 8(8) + 16(2)
 3 C         -0.1250   *    (16) + 4(14) + 8(10) + 8(8) + 16(3)
 4 D         -0.1875   *    (16) + 4(14) + 8(10) + 8(9) + 16(4)
 5 A*B       -3.1250   *    (16) + 2(15) + 4(12) + 4(11) + 8(5)
 6 A*C        0.0000   *    (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250   *    (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000  14    (16) + 4(14) + 8(8)
 9 B*D        0.0000  14    (16) + 4(14) + 8(9)
10 C*D        0.0000  14    (16) + 4(14) + 8(10)
11 A*B*C      0.0000  15    (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15    (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15    (16) + 2(15) + 4(13)
14 B*C*D     -2.3125  16    (16) + 4(14)
15 A*B*C*D   -4.6250  16    (16) + 2(15)
16 Error     12.3750        (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.56         *   (5) + (6) + (7) - (11) - (12) - (13) + (15)
 2 B             0.33      3.13   (8) + (9) - (14)
 3 C             0.33      3.13   (8) + (10) - (14)
 4 D             0.33      3.13   (9) + (10) - (14)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
```

(d) *A* and *B* are fixed and *C* and *D* are random.

Factor	F *a* *i*	F *b* *j*	R *c* *k*	R *d* *l*	R *n* *h*	E(*MS*)
τ_i	0	b	c	d	n	$\sigma^2 + ACD + AD + AC + A$
β_j	a	0	c	d	n	$\sigma^2 + BCD + BC + BD + B$
γ_k	a	b	1	d	n	$\sigma^2 + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + CD + D$
$(\tau\beta)_{ij}$	0	0	c	d	n	$\sigma^2 + ABCD + ABC + ABD + AB$
$(\tau\gamma)_{ik}$	0	b	1	d	n	$\sigma^2 + ACD + AC$
$(\tau\delta)_{il}$	0	b	c	1	n	$\sigma^2 + ACD + AD$

$(\beta\gamma)_{jk}$	a	0	1	d	n	$\sigma^2 + BCD + BC$
$(\beta\delta)_{jl}$	a	0	c	1	n	$\sigma^2 + BCD + BD$
$(\gamma\delta)_{kl}$	a	b	1	1	n	$\sigma^2 + CD$
$(\tau\beta\gamma)_{ijk}$	0	0	1	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	0	0	c	1	n	$\sigma^2 + ABCD + ABD$
$(\beta\gamma\delta)_{jkl}$	a	0	1	1	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	1	1	n	$\sigma^2 + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	0	1	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

There are no exact tests on the fixed factors *A* and *B*, or their two-factor interaction *AB*. The appropriate test statistics are:

$$A: F = \frac{MS_A + MS_{ACD}}{MS_{AC} + MS_{AD}}$$
$$B: F = \frac{MS_B + MS_{BCD}}{MS_{BC} + MS_{BD}}$$
$$AB: F = \frac{MS_{AB} + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$$

The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF        SS        MS       F       P
A            1      6.13      6.13    1.96  0.604 x
B            1      0.13      0.13    0.04  0.907 x
C            1      1.13      1.13    0.36  0.656
D            1      0.13      0.13    0.04  0.874
A*B          1      3.13      3.13    0.11  0.796 x
A*C          1      3.13      3.13    1.00  0.500
A*D          1      3.13      3.13    1.00  0.500
B*C          1      3.13      3.13    1.00  0.500
B*D          1      3.13      3.13    1.00  0.500
C*D          1      3.13      3.13    0.25  0.622
A*B*C        1      3.13      3.13    1.00  0.500
A*B*D        1     28.13     28.13    9.00  0.205
A*C*D        1      3.13      3.13    0.25  0.622
B*C*D        1      3.13      3.13    0.25  0.622
A*B*C*D      1      3.13      3.13    0.25  0.622
Error       16    198.00     12.38
Total       31    264.88

x Not an exact F-test.

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                  *   (16) + 4(13) + 8(7) + 8(6) + 16Q[1]
 2 B                  *   (16) + 4(14) + 8(9) + 8(8) + 16Q[2]
 3 C         -0.1250  10   (16) + 8(10) + 16(3)
```

```
 4 D         -0.1875  10   (16) + 8(10) + 16(4)
 5 A*B                 *   (16) + 2(15) + 4(12) + 4(11) + 8Q[5]
 6 A*C        0.0000  13   (16) + 4(13) + 8(6)
 7 A*D        0.0000  13   (16) + 4(13) + 8(7)
 8 B*C        0.0000  14   (16) + 4(14) + 8(8)
 9 B*D        0.0000  14   (16) + 4(14) + 8(9)
10 C*D       -1.1563  16   (16) + 8(10)
11 A*B*C      0.0000  15   (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D     -2.3125  16   (16) + 4(13)
14 B*C*D     -2.3125  16   (16) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.33      3.13   (6) + (7) - (13)
 2 B             0.33      3.13   (8) + (9) - (14)
 5 A*B           0.98     28.13   (11) + (12) - (15)
```

(e) *A, B* and *C* are fixed and *D* is random.

	F	F	F	R	R	
	a	b	c	d	n	
Factor	i	j	k	l	h	E(MS)
τ_i	0	b	c	d	n	$\sigma^2 + AD + A$
β_j	a	0	c	d	n	$\sigma^2 + BD + B$
γ_k	a	b	0	d	n	$\sigma^2 + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + D$
$(\tau\beta)_{ij}$	0	0	c	d	n	$\sigma^2 + ABD + AB$
$(\tau\gamma)_{ik}$	0	b	0	d	n	$\sigma^2 + ACD + AC$
$(\tau\delta)_{il}$	0	b	c	1	n	$\sigma^2 + AD$
$(\beta\gamma)_{jk}$	a	0	0	d	n	$\sigma^2 + BCD + BC$
$(\beta\delta)_{jl}$	a	0	c	1	n	$\sigma^2 + BD$
$(\gamma\delta)_{kl}$	a	b	0	1	n	$\sigma^2 + CD$
$(\tau\beta\gamma)_{ijk}$	0	0	0	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	0	0	c	1	n	$\sigma^2 + ABD$
$(\beta\gamma\delta)_{jkl}$	a	0	0	1	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	0	1	n	$\sigma^2 + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	0	0	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

There are exact tests for all effects. The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C         fixed      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source       DF          SS          MS        F       P
```

```
A            1       6.13       6.13    1.96  0.395
B            1       0.13       0.13    0.04  0.874
C            1       1.13       1.13    0.36  0.656
D            1       0.13       0.13    0.01  0.921
A*B          1       3.13       3.13    0.11  0.795
A*C          1       3.13       3.13    1.00  0.500
A*D          1       3.13       3.13    0.25  0.622
B*C          1       3.13       3.13    1.00  0.500
B*D          1       3.13       3.13    0.25  0.622
C*D          1       3.13       3.13    0.25  0.622
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    2.27  0.151
A*C*D        1       3.13       3.13    0.25  0.622
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                  7    (16) + 8(7) + 16Q[1]
 2 B                  9    (16) + 8(9) + 16Q[2]
 3 C                 10    (16) + 8(10) + 16Q[3]
 4 D        -0.7656  16    (16) + 16(4)
 5 A*B               12    (16) + 4(12) + 8Q[5]
 6 A*C               13    (16) + 4(13) + 8Q[6]
 7 A*D      -1.1563  16    (16) + 8(7)
 8 B*C               14    (16) + 4(14) + 8Q[8]
 9 B*D      -1.1563  16    (16) + 8(9)
10 C*D      -1.1563  16    (16) + 8(10)
11 A*B*C             15    (16) + 2(15) + 4Q[11]
12 A*B*D     3.9375  16    (16) + 4(12)
13 A*C*D    -2.3125  16    (16) + 4(13)
14 B*C*D    -2.3125  16    (16) + 4(14)
15 A*B*C*D  -4.6250  16    (16) + 2(15)
16 Error    12.3750        (16)
```

12-20 Reconsider cases (c), (d) and (e) of Problem 12-19. Obtain the expected mean squares assuming the unrestricted model. You may use a computer package such as Minitab. Compare your results with those for the restricted model.

A is fixed and *B, C,* and *D* are random.

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B        random      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13      **
B            1       0.13       0.13      **
C            1       1.13       1.13    0.36  0.843 x
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.667 x
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
```

```
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term  (using unrestricted model)
 1 A                   *    (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                             + 8(5) + Q[1]
 2 B          1.3750   *    (16) + 2(15) + 4(14) + 4(12) + 4(11) + 8(9) + 8(8)
                             + 8(5) + 16(2)
 3 C         -0.1250   *    (16) + 2(15) + 4(14) + 4(13) + 4(11) + 8(10) + 8(8)
                             + 8(6) + 16(3)
 4 D          1.3750   *    (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                             + 8(7) + 16(4)
 5 A*B       -3.1250   *    (16) + 2(15) + 4(12) + 4(11) + 8(5)
 6 A*C        0.0000   *    (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250   *    (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000   *    (16) + 2(15) + 4(14) + 4(11) + 8(8)
 9 B*D       -3.1250   *    (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D        0.0000   *    (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C      0.0000  15    (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15    (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15    (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15    (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16    (16) + 2(15)
16 Error     12.3750        (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.56         *   (5) + (6) + (7) - (11) - (12) - (13) + (15)
 2 B             0.56         *   (5) + (8) + (9) - (11) - (12) - (14) + (15)
 3 C             0.14      3.13   (6) + (8) + (10) - (11) - (13) - (14) + (15)
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
 8 B*C           0.33      3.13   (11) + (14) - (15)
 9 B*D           0.98     28.13   (12) + (14) - (15)
10 C*D           0.33      3.13   (13) + (14) - (15)
```

A and *B* are fixed and *C* and *D* are random.

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13    1.96  0.604 x
B            1       0.13       0.13    0.04  0.907 x
C            1       1.13       1.13    0.36  0.843 x
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
```

```
B*C          1       3.13       3.13    1.00  0.667 x
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                  *   (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                           + Q[1,5]
 2 B                  *   (16) + 2(15) + 4(14) + 4(12) + 4(11) + 8(9) + 8(8)
                           + Q[2,5]
 3 C         -0.1250  *   (16) + 2(15) + 4(14) + 4(13) + 4(11) + 8(10) + 8(8)
                           + 8(6) + 16(3)
 4 D          1.3750  *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                           + 8(7) + 16(4)
 5 A*B                *   (16) + 2(15) + 4(12) + 4(11) + Q[5]
 6 A*C        0.0000  *   (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250  *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000  *   (16) + 2(15) + 4(14) + 4(11) + 8(8)
 9 B*D       -3.1250  *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D        0.0000  *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C      0.0000  15  (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15  (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15  (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15  (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16  (16) + 2(15)
16 Error     12.3750      (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.33      3.13   (6) + (7) - (13)
 2 B             0.33      3.13   (8) + (9) - (14)
 3 C             0.14      3.13   (6) + (8) + (10) - (11) - (13) - (14) + (15)
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
 8 B*C           0.33      3.13   (11) + (14) - (15)
 9 B*D           0.98     28.13   (12) + (14) - (15)
10 C*D           0.33      3.13   (13) + (14) - (15)
```

(e) *A, B* and *C* are fixed and *D* is random.

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C         fixed      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13    1.96  0.395
```

```
B           1       0.13       0.13    0.04  0.874
C           1       1.13       1.13    0.36  0.656
D           1       0.13       0.13      **
A*B         1       3.13       3.13    0.11  0.795
A*C         1       3.13       3.13    1.00  0.500
A*D         1       3.13       3.13    0.11  0.796 x
B*C         1       3.13       3.13    1.00  0.500
B*D         1       3.13       3.13    0.11  0.796 x
C*D         1       3.13       3.13    1.00  0.667 x
A*B*C       1       3.13       3.13    1.00  0.500
A*B*D       1      28.13      28.13    9.00  0.205
A*C*D       1       3.13       3.13    1.00  0.500
B*C*D       1       3.13       3.13    1.00  0.500
A*B*C*D     1       3.13       3.13    0.25  0.622
Error      16     198.00      12.38
Total      31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                  7   (16) + 2(15) + 4(13) + 4(12) + 8(7) + Q[1,5,6,11]
 2 B                  9   (16) + 2(15) + 4(14) + 4(12) + 8(9) + Q[2,5,8,11]
 3 C                 10   (16) + 2(15) + 4(14) + 4(13) + 8(10) + Q[3,6,8,11]
 4 D          1.3750   *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                            + 8(7) + 16(4)
 5 A*B               12   (16) + 2(15) + 4(12) + Q[5,11]
 6 A*C               13   (16) + 2(15) + 4(13) + Q[6,11]
 7 A*D       -3.1250   *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C               14   (16) + 2(15) + 4(14) + Q[8,11]
 9 B*D       -3.1250   *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D        0.0000   *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C             15   (16) + 2(15) + Q[11]
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15   (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15   (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 4 D            0.56         *    (7) + (9) + (10) - (12) - (13) - (14) + (15)
 7 A*D          0.98     28.13    (12) + (13) - (15)
 9 B*D          0.98     28.13    (12) + (14) - (15)
10 C*D          0.33      3.13    (13) + (14) - (15)
```

12-21 In Problem 5-17, assume that the three operators were selected at random. Analyze the data under these conditions and draw conclusions. Estimate the variance components.

Minitab Output

ANOVA: Score versus Cycle Time, Operator, Temperature

```
Factor     Type Levels Values
Cycle Ti  fixed      3    40    50    60
Operator random      3     1     2     3
Temperat  fixed      2   300   350

Analysis of Variance for Score

Source                        DF         SS         MS       F      P
Cycle Ti                       2    436.000    218.000    2.45  0.202
Operator                       2    261.333    130.667   39.86  0.000
Temperat                       1     50.074     50.074    8.89  0.096
```

```
Cycle Ti*Operator               4    355.667     88.917   27.13  0.000
Cycle Ti*Temperat               2     78.815     39.407    3.41  0.137
Operator*Temperat               2     11.259      5.630    1.72  0.194
Cycle Ti*Operator*Temperat      4     46.185     11.546    3.52  0.016
Error                          36    118.000      3.278
Total                          53   1357.333

Source                         Variance Error Expected Mean Square for Each Term
                              component term (using restricted model)
 1 Cycle Ti                              4   (8) + 6(4) + 18Q[1]
 2 Operator                      7.0772  8   (8) + 18(2)
 3 Temperat                              6   (8) + 9(6) + 27Q[3]
 4 Cycle Ti*Operator           14.2731  8   (8) + 6(4)
 5 Cycle Ti*Temperat                     7   (8) + 3(7) + 9Q[5]
 6 Operator*Temperat            0.2613  8   (8) + 9(6)
 7 Cycle Ti*Operator*Temperat   2.7562  8   (8) + 3(7)
 8 Error                        3.2778      (8)
```

The following calculations agree with the Minitab results:

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.27778$$

$$\hat{\sigma}^2_{\tau\beta\gamma} = \frac{MS_{ABC} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta\gamma} = \frac{11.546296 - 3.277778}{3} = 2.7562$$

$$\hat{\sigma}^2_{\beta\gamma} = \frac{MS_{BC} - MS_E}{an} \qquad \hat{\sigma}^2_{\beta\gamma} = \frac{88.91667 - 3.277778}{2(3)} = 14.27315$$

$$\hat{\sigma}^2_{\tau\gamma} = \frac{MS_{AC} - MS_E}{bn} \qquad \hat{\sigma}^2_{\tau\gamma} = \frac{5.629630 - 3.277778}{3(3)} = 0.26132$$

$$\hat{\sigma}^2_{\gamma} = \frac{MS_C - MS_E}{abn} \qquad \hat{\sigma}^2_{\gamma} = \frac{130.66667 - 3.277778}{2(3)(3)} = 7.07716$$

12-22 Consider the three-factor model

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \varepsilon_{ijk}$$

Assuming that all the factors are random, develop the analysis of variance table, including the expected mean squares. Propose appropriate test statistics for all effects.

Source	DF	E(MS)
A	*a*-1	$\sigma^2 + c\sigma^2_{\tau\beta} + bc\sigma^2_{\tau}$
B	*b*-1	$\sigma^2 + c\sigma^2_{\tau\beta} + a\sigma^2_{\beta\gamma} + ac\sigma^2_{\beta}$
C	*c*-1	$\sigma^2 + a\sigma^2_{\beta\gamma} + ab\sigma^2_{\gamma}$
AB	(*a*-1)(*b*-1)	$\sigma^2 + c\sigma^2_{\tau\beta}$
BC	(*b*-1)(*c*-1)	$\sigma^2 + a\sigma^2_{\beta\gamma}$
Error (*AC* + *ABC*)	*b*(*a*-1)(*c*-1)	σ^2
Total	*abc*-1	

There are exact tests for all effects except B. To test B, use the statistic $F = \dfrac{MS_B + MS_E}{MS_{AB} + MS_{BC}}$

12-23 The three-factor model for a single replicate is

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + (\tau\gamma)_{ik} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijk}$$

If all the factors are random, can any effects be tested? If the three-factor interaction and the $(\tau\beta)_{ij}$ interaction do not exist, can all the remaining effects be tested.

The expected mean squares are found by referring to Table 12-9, deleting the line for the error term $\varepsilon_{(ijk)l}$ and setting *n*=1. The three-factor interaction now cannot be tested; however, exact tests exist for the two-factor interactions and approximate *F* tests can be conducted for the main effects. For example, to test the main effect of *A*, use

$$F = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}}$$

If $(\tau\beta\gamma)_{ijk}$ and $(\tau\beta)_{ij}$ can be eliminated, the model becomes

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + (\tau\gamma)_{ik} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijk}$$

For this model, the analysis of variance is

Source	DF	E(MS)
A	*a*-1	$\sigma^2 + b\sigma_{\tau\gamma}^2 + bc\sigma_\tau^2$
B	*b*-1	$\sigma^2 + a\sigma_{\beta\gamma}^2 + ac\sigma_\beta^2$
C	*c*-1	$\sigma^2 + a\sigma_{\beta\gamma}^2 + b\sigma_{\tau\gamma}^2 + ab\sigma_\gamma^2$
AC	(*a*-1)(*c*-1)	$\sigma^2 + b\sigma_{\tau\gamma}^2$
BC	(*b*-1)(*c*-1)	$\sigma^2 + a\sigma_{\beta\gamma}^2$
Error (*AB* + *ABC*)	*c*(*a*-1)(*b*-1)	σ^2
Total	*abc*-1	

There are exact tests for all effect except *C*. To test the main effect of *C*, use the statistic:

$$F = \frac{MS_C + MS_E}{MS_{BC} + MS_{AC}}$$

12-24 In Problem 5-6, assume that both machines and operators were chosen randomly. Determine the power of the test for detecting a machine effect such that $\sigma_\beta^2 = \sigma^2$, where σ_β^2 is the variance component for the machine factor. Are two replicates sufficient?

$$\lambda = \sqrt{1 + \frac{an\sigma_\beta^2}{\sigma^2 + n\sigma_{\tau\beta}^2}}$$

If $\sigma_\beta^2 = \sigma^2$, then an estimate of $\sigma^2 = \sigma_\beta^2 = 3.79$, and an estimate of $\sigma^2 = n\sigma_{\tau\beta}^2 = 7.45$, from the analysis of variance table. Then

$$\lambda = \sqrt{1 + \frac{(3)(2)(3.79)}{7.45}} = \sqrt{2.22} = 1.49$$

and the other OC curve parameters are $\upsilon_1 = 3$ and $\upsilon_2 = 6$. This results in $\beta \approx 0.75$ approximately, with $\alpha = 0.05$, or $\beta \approx 0.9$ with $\alpha = 0.01$. Two replicates does not seem sufficient.

12-25 In the two-factor mixed model analysis of variance, show that Cov$\left[(\tau\beta)_{ij}, (\tau\beta)_{i'j}\right] = -(1/a)^2_{\tau\beta\sigma}$ for $i \neq i'$.

Since $\sum_{i=1}^{a} (\tau\beta)_{ij} = 0$ (constant) we have $V\left[\sum_{i=1}^{a} (\tau\beta)_{ij}\right] = 0$, which implies that

$$\sum_{i=1}^{a} V(\tau\beta)_{ij} + 2\binom{a}{2} Cov\left[(\tau\beta)_{ij}, (\tau\beta)_{i'j}\right] = 0$$

$$a\left[\frac{a-1}{a}\right]\sigma^2_{\tau\beta} + \frac{a!}{2!(a-2)!}(2) Cov\left[(\tau\beta)_{ij}, (\tau\beta)_{i'j}\right] = 0$$

$$(a-1)\sigma^2_{\tau\beta} + a(a-1) Cov\left[(\tau\beta)_{ij}, \tau(\beta)_{i'j}\right] = 0$$

$$Cov\left[\tau(\beta)_{ij}, (\tau\beta)_{i'j}\right] = -\left(\frac{1}{a}\right)\sigma^2_{\tau\beta}$$

12-26 Show that the method of analysis of variance always produces unbiased point estimates of the variance component in any random or mixed model.

Let **g** be the vector of mean squares from the analysis of variance, chosen so that $E(\mathbf{g})$ does not contain any fixed effects. Let $\boldsymbol{\sigma}^2$ be the vector of variance components such that $E(\mathbf{g}) = \mathbf{A}\boldsymbol{\sigma}^2$, where **A** is a matrix of constants. Now in the analysis of variance method of variance component estimation, we equate observed and expected mean squares, i.e.

$$\mathbf{g} = \mathbf{A}\mathbf{s}^2 \Rightarrow \hat{\mathbf{s}}^2 = \mathbf{A}^{-1}\mathbf{g}$$

Since $\mathbf{A}^{-1}$ always exists then,

$$E(\mathbf{s}^2) = E(\mathbf{A}^{-1})\mathbf{g} = \mathbf{A}^{-1}E(\mathbf{g}) = \mathbf{A}^{-1}(\mathbf{A}\mathbf{s}^2) = \mathbf{s}^2$$

Thus $\hat{\boldsymbol{\sigma}}^2$ is an unbiased estimator of $\boldsymbol{\sigma}^2$. This and other properties of the analysis of variance method are discussed by Searle (1971a).

12-27 Invoking the usual normality assumptions, find an expression for the probability that a negative estimate of a variance component will be obtained by the analysis of variance method. Using this result, write a statement giving the probability that $\hat{\sigma}^2_\tau < 0$ in a one-factor analysis of variance. Comment on the usefulness of this probability statement.

Suppose $\hat{\sigma}^2 = \dfrac{MS_1 - MS_2}{c}$, where MS_i for i=1,2 are two mean squares and c is a constant. The probability that $\hat{\sigma}_\tau^2 < 0$ (negative) is

$$P\left\{\hat{\sigma}^2 < 0\right\} = P\left\{MS_1 - MS_2 < 0\right\} = P\left\{\frac{MS_1}{MS_2} < 1\right\} = P\left\{\frac{\dfrac{MS_1}{E(MS_1)}}{\dfrac{MS_2}{E(MS_2)}} < \frac{E(MS_1)}{E(MS_2)}\right\} = P\left\{F_{u,v} < \frac{E(MS_1)}{E(MS_2)}\right\}$$

where u is the number of degrees of freedom for MS_1 and v is the number of degrees of freedom for MS_2. For the one-way model, this equation reduces to

$$P\left\{\hat{\sigma}^2 < 0\right\} = P\left\{F_{a-1,N-a} < \frac{\sigma^2}{\sigma^2 + n\sigma_\tau^2}\right\} = P\left\{F_{a-1,N-a} < \frac{1}{1+nk}\right\}$$

where $k = \dfrac{\sigma_\tau^2}{\sigma^2}$. Using arbitrary values for some of the parameters in this equation will give an experimenter some idea of the probability of obtaining a negative estimate of $\hat{\sigma}_\tau^2 < 0$.

12-28 Analyze the data in Problem 12-9, assuming that the operators are fixed, using both the unrestricted and restricted forms of the mixed models. Compare the results obtained from the two models.

The restricted model is as follows:

Minitab Output

ANOVA: Measurement versus Part, Operator

```
Factor      Type Levels Values
Part      random     10      1      2      3      4      5      6      7
                             8      9     10
Operator  fixed       2      1      2

Analysis of Variance for Measurem

Source           DF         SS         MS       F      P
Part              9     99.017     11.002    7.33  0.000
Operator          1      0.417      0.417    0.69  0.427
Part*Operator     9      5.417      0.602    0.40  0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Part            1.5836   4   (4) + 6(1)
 2 Operator                 3   (4) + 3(3) + 30Q[2]
 3 Part*Operator  -0.2994   4   (4) + 3(3)
 4 Error           1.5000       (4)
```

The second approach is the unrestricted mixed model.

Minitab Output

ANOVA: Measurement versus Part, Operator

```
Factor      Type Levels Values
Part      random     10      1      2      3      4      5      6      7
                             8      9     10
Operator  fixed       2      1      2
```

```
Analysis of Variance for Measurem

Source           DF        SS        MS       F      P
Part              9    99.017    11.002   18.28  0.000
Operator          1     0.417     0.417    0.69  0.427
Part*Operator     9     5.417     0.602    0.40  0.927
Error            40    60.000     1.500
Total            59   164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using unrestricted model)
 1 Part            1.7333   3   (4) + 3(3) + 6(1)
 2 Operator                 3   (4) + 3(3) + Q[2]
 3 Part*Operator  -0.2994   4   (4) + 3(3)
 4 Error           1.5000       (4)
```

Source	Sum of Squares	DF	Mean Square	E(MS)	*F*-test	*F*
A	0.416667	*a*-1=1	0.416667	$\sigma^2+n\sigma^2_{\tau\beta}+bn\frac{\sum_{i=1}^{a}\tau_i^2}{a-1}$	$F=\frac{MS_A}{MS_{AB}}$	0.692
B	99.016667	*b*-1=9	11.00185	$\sigma^2+n\sigma^2_{\tau\beta}+an\sigma^2_{\beta}$	$F=\frac{MS_B}{MS_{AB}}$	18.28
AB	5.416667	(*a*-1)(*b*-1)=9	0.60185	$\sigma^2+n\sigma^2_{\tau\beta}$	$F=\frac{MS_{AB}}{MS_E}$	0.401
Error	60.000000	40	1.50000	σ^2		
Total	164.85000	*nabc*-1=59				

In the unrestricted model, the *F*-test for *B* is different. The *F*-test for *B* in the unrestricted model should generally be more conservative, since MS_{AB} will generally be larger than MS_E. However, this is not the case with this particular experiment.

12-29 Consider the two-factor mixed model. Show that the standard error of the fixed factor mean (e.g. *A*) is $\left[MS_{AB}/bn\right]^{1/2}$.

The standard error is often used in Duncan's Multiple Range test. Duncan's Multiple Range Test requires the variance of the difference in two means, say

$$V\left(\bar{y}_{i..}-\bar{y}_{m..}\right)$$

where rows are fixed and columns are random. Now, assuming all model parameters to be independent, we have the following:

$$\left(\bar{y}_{i..}-\bar{y}_{m..}\right)=\tau_i-\tau_m+\frac{1}{b}\sum_{j=1}^{b}(\tau\beta)_{ij}-\frac{1}{b}\sum_{j=1}^{b}(\tau\beta)_{mj}+\frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk}-\frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{mjk}$$

and

$$V\left(\bar{y}_{i..}-\bar{y}_{m..}\right)=\left(\frac{1}{b}\right)^2 b\sigma^2_{\tau\beta}+\left(\frac{1}{b}\right)^2 b\sigma^2_{\tau\beta}+\left(\frac{1}{bn}\right)^2 bn\sigma^2+\left(\frac{1}{bn}\right)^2 bn\sigma^2=\frac{2\left(\sigma^2+n\sigma^2_{\tau\beta}\right)}{bn}$$

Since MS_{AB} estimates $\sigma^2 + n\sigma_{\tau\beta}^2$, we would use

$$\frac{2\,MS_{AB}}{bn}$$

as the standard error to test the difference. However, the table of ranges for Duncan's Multiple Range test already include the constant 2.

12-30 Consider the variance components in the random model from Problem 12-9.

(a) Find an exact 95 percent confidence interval on σ^2.

$$\frac{f_E MS_E}{\chi^2_{\alpha/2,f_E}} \le \sigma^2 \le \frac{f_E MS_E}{\chi^2_{1-\alpha/2,f_E}}$$
$$\frac{(40)(1.5)}{59.34} \le \sigma^2 \le \frac{(40)(1.5)}{24.43}$$
$$1.011 \le \sigma^2 \le 2.456$$

(b) Find approximate 95 percent confidence intervals on the other variance components using the Satterthwaite method.

$\hat{\sigma}_{\tau\beta}^2$ and $\hat{\sigma}_{\tau}^2$ are negative, and the Satterthwaithe method does not apply. The confidence interval on $\hat{\sigma}_{\beta}^2$ is

$$\hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}_\beta^2 = \frac{11.001852 - 0.6018519}{2(3)} = 1.7333$$
$$r = \frac{(MS_B - MS_{AB})^2}{\frac{MS_B^2}{(b-1)} + \frac{MS_{AB}^2}{(a-1)(b-1)}} = \frac{(11.001852 - 0.6018519)^2}{\frac{1.001852^2}{(9)} + \frac{0.6018519^2}{(1)(9)}} = 8.01826$$
$$\frac{r\hat{\sigma}_O^2}{\chi^2_{\alpha/2,r}} \le \sigma_\beta^2 \le \frac{r\hat{\sigma}_\beta^2}{\chi^2_{1-\alpha/2,r}}$$
$$\frac{(8.01826)(1.7333)}{17.55752} \le \sigma_\beta^2 \le \frac{(8.01826)(1.7333)}{2.18950}$$
$$0.79157 \le \sigma_\beta^2 \le 6.34759$$

12-31 Use the experiment described in Problem 5-6 and assume that both factor are random. Find an exact 95 percent confidence interval on σ^2. Construct approximate 95 percent confidence interval on the other variance components using the Satterthwaite method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.79167$$
$$\frac{f_E MS_E}{\chi^2_{\alpha/2,f_E}} \le \sigma^2 \le \frac{f_E MS_E}{\chi^2_{1-\alpha/2,f_E}}$$
$$\frac{(12)(3.79167)}{23.34} \le \sigma^2 \le \frac{(12)(3.79167)}{4.40}$$

$$1.9494 \le \sigma^2 \le 10.3409$$

Satterthwaite Method:

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{7.44444 - 3.79167}{2} = 1.82639$$

$$r = \frac{(MS_{AB} - MS_E)^2}{\frac{MS^2_{AB}}{(a-1)(b-1)} + \frac{MS^2_E}{df_E}} = \frac{(7.44444 - 3.79167)^2}{\frac{7.44444^2}{(2)(3)} + \frac{3.79167^2}{(12)}} = 2.2940$$

$$\frac{r\hat{\sigma}^2_\beta}{\chi^2_{\alpha/2,r}} \le \sigma^2_\beta \le \frac{r\hat{\sigma}^2_\beta}{\chi^2_{1-\alpha/2,r}}$$

$$\frac{(2.2940)(1.82639)}{7.95918} \le \sigma^2_\beta \le \frac{(2.2940)(1.82639)}{0.09998}$$

$$0.52640 \le \sigma^2_\beta \le 41.90577$$

$\hat{\sigma}^2_\beta < 0$, this variance component does not have a confidence interval using Satterthwaite's Method.

$$\hat{\sigma}^2_\tau = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}^2_\tau = \frac{80.16667 - 7.44444}{4(2)} = 9.09028$$

$$r = \frac{(MS_A - MS_{AB})^2}{\frac{MS^2_A}{(a-1)} + \frac{MS^2_{AB}}{(a-1)(b-1)}} = \frac{(80.16667 - 7.44444)^2}{\frac{80.16667^2}{(2)} + \frac{7.44444^2}{(2)(3)}} = 1.64108$$

$$\frac{r\hat{\sigma}^2_\tau}{\chi^2_{\alpha/2,r}} \le \sigma^2_\tau \le \frac{r\hat{\sigma}^2_\tau}{\chi^2_{1-\alpha/2,r}}$$

$$\frac{(1.64108)(9.09028)}{6.53295} \le \sigma^2_\tau \le \frac{(1.64108)(9.09028)}{0.03205}$$

$$2.28348 \le \sigma^2_\tau \le 465.45637$$

12-32 Consider the three-factor experiment in Problem 5-17 and assume that operators were selected at random. Find an approximate 95 percent confidence interval on the operator variance component.

$$\hat{\sigma}^2_\gamma = \frac{MS_C - MS_E}{abn} \qquad \hat{\sigma}^2_\gamma = \frac{130.66667 - 3.277778}{2(3)(3)} = 7.07716$$

$$r = \frac{(MS_C - MS_E)^2}{\frac{MS^2_C}{(c-1)} + \frac{MS^2_E}{df_E}} = \frac{(130.66667 - 3.27778)^2}{\frac{130.66667^2}{(2)} + \frac{3.27778^2}{(36)}} = 1.90085$$

$$\frac{r\hat{\sigma}^2_\gamma}{\chi^2_{\alpha/2,r}} \le \sigma^2_\gamma \le \frac{r\hat{\sigma}^2_\gamma}{\chi^2_{1-\alpha/2,r}}$$

$$\frac{(1.90085)(7.07716)}{9.15467} \le \sigma^2_\gamma \le \frac{(1.90085)(7.07716)}{0.04504}$$

$$1.46948 \le \sigma^2_\gamma \le 4298.66532$$

12-33 Rework Problem 12-30 using the modified large-sample approach described in Section 12-7.2. Compare the two sets of confidence intervals obtained and discuss.

$$\hat{\sigma}_O^2 = \hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}_O^2 = \frac{11.001852 - 0.6018519}{2(3)} = 1.7333$$

$$G_1 = 1 - \frac{1}{F_{0.05,9,\infty}} = 1 - \frac{1}{1.88} = 0.46809$$

$$H_1 = \frac{1}{F_{.95,9_i,\infty}} - 1 = \frac{1}{\frac{\chi^2_{.95,9}}{9}} - 1 = .\frac{1}{0.370} - 1 = 1.7027$$

$$G_{ij} = \frac{\left(F_{\alpha,f_i,f_j} - 1\right)^2 - G_1^2 F_{\alpha,f_i,f_j} - H_1^2}{F_{\alpha,f_i,f_j}} = \frac{(3.18-1)^2 - (0.46809)^2(3.18) - 1.7027^2}{3.18} = 0.36366$$

$$V_L = G_1^2 c_1^2 MS_B^2 + H_1^2 c_2^2 MS_{AB}^2 + G_{11} c_1 c_2 MS_B MS_{AB}$$

$$V_L = (0.46809)^2\left(\frac{1}{6}\right)^2(11.00185)^2 + (1.7027)^2\left(\frac{1}{6}\right)^2(0.60185)^2 + (0.36366)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)(11.00185)(0.60185)$$

$$V_L = 0.83275$$

$$L = \hat{\sigma}_\beta^2 - \sqrt{V_L} = 1.7333 - \sqrt{0.83275} = 0.82075$$

12-34 Rework Problem 12-32 using the modified large-sample method described in Section 12-7.2. Compare this confidence interval with he one obtained previously and discuss.

$$\hat{\sigma}_\gamma^2 = \frac{MS_C - MS_E}{abn} \qquad \hat{\sigma}_\gamma^2 = \frac{130.66667 - 3.277778}{2(3)(3)} = 7.07716$$

$$G_1 = 1 - \frac{1}{F_{0.05,3,\infty}} = 1 - \frac{1}{2.60} = 0.61538$$

$$H_1 = \frac{1}{F_{.95,36,\infty}} - 1 = \frac{1}{\frac{\chi^2_{.95,36}}{36}} - 1 = .\frac{1}{0.64728} - 1 = 0.54493$$

$$G_{ij} = \frac{\left(F_{\alpha,f_i,f_j} - 1\right)^2 - G_1^2 F_{\alpha,f_i,f_j} - H_1^2}{F_{\alpha,f_i,f_j}} = \frac{(2.88-1)^2 - (0.61538)^2(2.88) - 0.54493^2}{2.88} = 0.74542$$

$$V_L = G_1^2 c_1^2 MS_B^2 + H_1^2 c_2^2 MS_{AB}^2 + G_{11} c_1 c_2 MS_B MS_{AB}$$

$$V_L = (0.61538)^2\left(\frac{1}{18}\right)^2(130.66667)^2 + (0.54493)^2\left(\frac{1}{18}\right)^2(3.27778)^2 + (0.74542)\left(\frac{1}{18}\right)\left(\frac{1}{18}\right)(130.66667)(3.27778)$$

$$V_L = 20.95112$$

$$L = \hat{\sigma}_\gamma^2 - \sqrt{V_L} = 7.07716 - \sqrt{20.95112} = 2.49992$$

Chapter 13
Nested and Split-Plot Designs
Solutions

In this chapter we have not shown residual plots and other diagnostics to conserve space. A complete analysis would, of course, include these model adequacy checking procedures.

13-1 A rocket propellant manufacturer is studying the burning rate of propellant from three production processes. Four batches of propellant are randomly selected from the output of each process and three determinations of burning rate are made on each batch. The results follow. Analyze the data and draw conclusions.

	Process 1				Process 2				Process 3			
Batch	1	2	3	4	1	2	3	4	1	2	3	4
	25	19	15	15	19	23	18	35	14	35	38	25
	30	28	17	16	17	24	21	27	15	21	54	29
	26	20	14	13	14	21	17	25	20	24	50	33

Minitab Output

ANOVA: Burn Rate versus Process, Batch

```
Factor           Type Levels Values
Process         fixed      3     1     2     3
Batch(Process) random      4     1     2     3     4

Analysis of Variance for Burn Rat

Source            DF         SS         MS       F      P
Process            2     676.06     338.03    1.46  0.281
Batch(Process)     9    2077.58     230.84   12.20  0.000
Error             24     454.00      18.92
Total             35    3207.64

Source            Variance Error Expected Mean Square for Each Term
                 component term (using restricted model)
 1 Process                  2   (3) + 3(2) + 12Q[1]
 2 Batch(Process)    70.64  3   (3) + 3(2)
 3 Error             18.92      (3)
```

There is no significant effect on mean burning rate among the different processes; however, different batches from the same process have significantly different burning rates.

13-2 The surface finish of metal parts made on four machines is being studied. An experiment is conducted in which each machine is run by three different operators and two specimens from each operator are collected and tested. Because of the location of the machines, different operators are used on each machine, and the operators are chosen at random. The data are shown in the following table. Analyze the data and draw conclusions.

	Machine 1			Machine 2			Machine 3			Machine 4		
Operator	1	2	3	1	2	3	1	2	3	1	2	3
	79	94	46	92	85	76	88	53	46	36	40	62
	62	74	57	99	79	68	75	56	57	53	56	47

Minitab Output

ANOVA: Finish versus Machine, Operator

```
Factor              Type Levels Values
Machine            fixed      4     1     2     3     4
Operator(Machine) random      3     1     2     3

Analysis of Variance for Finish

Source               DF         SS         MS       F      P
Machine               3    3617.67    1205.89    3.42  0.073
Operator(Machine)     8    2817.67     352.21    4.17  0.013
Error                12    1014.00      84.50
Total                23    7449.33

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Machine                     2   (3) + 2(2) + 6Q[1]
 2 Operator(Machine)   133.85   3   (3) + 2(2)
 3 Error                84.50       (3)
```

There is a slight effect on surface finish due to the different processes; however, the different operators running the same machine have significantly different surface finish.

13-3 A manufacturing engineer is studying the dimensional variability of a particular component that is produced on three machines. Each machine has two spindles, and four components are randomly selected from each spindle. These results follow. Analyze the data, assuming that machines and spindles are fixed factors.

	Machine 1		Machine 2		Machine 3	
Spindle	1	2	1	2	1	2
	12	8	14	12	14	16
	9	9	15	10	10	15
	11	10	13	11	12	15
	12	8	14	13	11	14

Minitab Output

ANOVA: Variability versus Machine, Spindle

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Spindle(Machine)  fixed      2     1     2

Analysis of Variance for Variabil

Source              DF         SS         MS       F      P
Machine              2     55.750     27.875   18.93  0.000
Spindle(Machine)     3     43.750     14.583    9.91  0.000
Error               18     26.500      1.472
Total               23    126.000
```

There is a significant effects on dimensional variability due to the machine and spindle factors.

13-4 To simplify production scheduling, an industrial engineer is studying the possibility of assigning one time standard to a particular class of jobs, believing that differences between jobs is negligible. To see if this simplification is possible, six jobs are randomly selected. Each job is given to a different group of three operators. Each operator completes the job twice at different times during the week, and the following results were obtained. What are your conclusions about the use of a common time standard for all jobs in this class? What value would you use for the standard?

Job	Operator 1		Operator 2		Operator 3	
1	158.3	159.4	159.2	159.6	158.9	157.8

2	154.6	154.9	157.7	156.8	154.8	156.3
3	162.5	162.6	161.0	158.9	160.5	159.5
4	160.0	158.7	157.5	158.9	161.1	158.5
5	156.3	158.1	158.3	156.9	157.7	156.9
6	163.7	161.0	162.3	160.3	162.6	161.8

Minitab Output

ANOVA: Time versus Job, Operator

```
Factor          Type Levels Values
Job           random      6     1     2     3     4     5     6
Operator(Job) random      3     1     2     3

Analysis of Variance for Time

Source           DF         SS         MS       F      P
Job               5    148.111     29.622   27.89  0.000
Operator(Job)    12     12.743      1.062    0.69  0.738
Error            18     27.575      1.532
Total            35    188.430

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Job             4.7601   2   (3) + 2(2) + 6(1)
 2 Operator(Job)  -0.2350   3   (3) + 2(2)
 3 Error           1.5319       (3)
```

The jobs differ significantly; the use of a common time standard would likely not be a good idea.

13-5 Consider the three-stage nested design shown in Figure 13-5 to investigate alloy hardness. Using the data that follow, analyze the design, assuming that alloy chemistry and heats are fixed factors and ingots are random.

Alloy Chemistry	1						2					
Heats	1		2		3		1		2		3	
Ingots	1	2	1	2	1	2	1	2	1	2	1	2
	40	27	95	69	65	78	22	23	83	75	61	35
	63	30	67	47	54	45	10	39	62	64	77	42

Minitab Output

ANOVA: Hardness versus Alloy, Heat, Ingot

```
Factor              Type Levels Values
Alloy              fixed      2     1     2
Heat(Alloy)        fixed      3     1     2     3
Ingot(Alloy Heat) random      2     1     2

Analysis of Variance for Hardness

Source               DF         SS         MS       F      P
Alloy                 1      315.4      315.4    0.85  0.392
Heat(Alloy)           4     6453.8     1613.5    4.35  0.055
Ingot(Alloy Heat)     6     2226.3      371.0    2.08  0.132
Error                12     2141.5      178.5
Total                23    11137.0

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Alloy                        3   (4) + 2(3) + 12Q[1]
 2 Heat(Alloy)                  3   (4) + 2(3) + 4Q[2]
 3 Ingot(Alloy Heat)    96.29   4   (4) + 2(3)
 4 Error               178.46       (4)
```

Alloy hardness differs significantly due to the different heats within each alloy.

13-6 Reanalyze the experiment in Problem 13-5 using the unrestricted form of the mixed model. Comment on any differences you observe between the restricted and unrestricted model results. You may use a computer software package.

Minitab Output

```
ANOVA: Hardness versus Alloy, Heat, Ingot

Factor              Type Levels Values
Alloy              fixed      2     1     2
Heat(Alloy)        fixed      3     1     2     3
Ingot(Alloy Heat) random      2     1     2

Analysis of Variance for Hardness

Source               DF         SS         MS       F      P
Alloy                 1      315.4      315.4    0.85  0.392
Heat(Alloy)           4     6453.8     1613.5    4.35  0.055
Ingot(Alloy Heat)     6     2226.3      371.0    2.08  0.132
Error                12     2141.5      178.5
Total                23    11137.0

Source               Variance Error Expected Mean Square for Each Term
                    component term (using unrestricted model)
 1 Alloy                        3   (4) + 2(3) + Q[1,2]
 2 Heat(Alloy)                  3   (4) + 2(3) + Q[2]
 3 Ingot(Alloy Heat)    96.29   4   (4) + 2(3)
 4 Error               178.46       (4)
```

13-7 Derive the expected means squares for a balanced three-stage nested design, assuming that *A* is fixed and that *B* and *C* are random. Obtain formulas for estimating the variance components.

	F	R	R	R	
	a	*b*	*c*	*n*	
Factor	*i*	*j*	*k*	*l*	E(*MS*)
τ_i	0	*b*	*c*	*n*	$\sigma^2 + n\sigma_\gamma^2 + cn\sigma_\beta^2 + \frac{bcn}{a-1}\sum \tau_i^2$
$\beta_{j(i)}$	1	1	*c*	*n*	$\sigma^2 + n\sigma_\gamma^2 + cn\sigma_\beta^2$
$\gamma_{k(ij)}$	1	1	1	*n*	$\sigma^2 + n\sigma_\gamma^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}_\gamma^2 = \frac{MS_{C(B)} - MS_E}{n} \qquad \hat{\sigma}_\beta^2 = \frac{MS_{B(A)} - MS_{C(B)}}{cn}$$

The expected mean squares can be generated in Minitab as follows:

Minitab Output

```
ANOVA: y versus A, B, C

Factor     Type Levels Values
A         fixed      2    -1     1
B(A)     random      2    -1     1
C(A B)   random      2    -1     1

Analysis of Variance for y

Source      DF         SS         MS        F       P
```

```
A            1      0.250      0.250    0.06  0.831
B(A)         2      8.500      4.250    0.35  0.726
C(A B)       4     49.000     12.250    2.13  0.168
Error        8     46.000      5.750
Total       15    103.750

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                  2   (4) + 2(3) + 4(2) + 8Q[1]
 2 B(A)       -2.000  3   (4) + 2(3) + 4(2)
 3 C(A B)      3.250  4   (4) + 2(3)
 4 Error       5.750      (4)
```

13-8 Repeat Problem 13-7 assuming the unrestricted form of the mixed model. You may use a computer software package. Comment on any differences you observe between the restricted and unrestricted model analysis and conclusions.

Minitab Output

ANOVA: y versus A, B, C

```
Factor     Type Levels Values
A         fixed      2    -1     1
B(A)     random      2    -1     1
C(A B)   random      2    -1     1

Analysis of Variance for y

Source      DF         SS         MS       F       P
A            1      0.250      0.250    0.06  0.831
B(A)         2      8.500      4.250    0.35  0.726
C(A B)       4     49.000     12.250    2.13  0.168
Error        8     46.000      5.750
Total       15    103.750

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                  2   (4) + 2(3) + 4(2) + Q[1]
 2 B(A)       -2.000  3   (4) + 2(3) + 4(2)
 3 C(A B)      3.250  4   (4) + 2(3)
 4 Error       5.750      (4)
```

In this case there is no difference in results between the restricted and unrestricted models.

13-9 Derive the expected means squares for a balanced three-stage nested design if all three factors are random. Obtain formulas for estimating the variance components. Assume the restricted form of the mixed model.

	R	R	R	R	
	a	b	c	n	
Factor	i	j	k	l	E(MS)
τ_i	1	b	c	n	$\sigma^2 + n\sigma_\gamma^2 + cn\sigma_\beta^2 + bcn\sigma_\tau^2$
$\beta_{j(i)}$	1	1	c	n	$\sigma^2 + n\sigma_\gamma^2 + cn\sigma_\beta^2$
$\gamma_{k(ij)}$	1	1	1	n	$\sigma^2 + n\sigma_\gamma^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}_\gamma^2 = \frac{MS_{C(B)} - MS_E}{n} \qquad \hat{\sigma}_\beta^2 = \frac{MS_{B(A)} - MS_{C(B)}}{cn} \qquad \hat{\sigma}_\gamma^2 = \frac{MS_A - MS_{B(A)}}{bcn}$$

The expected mean squares can be generated in Minitab as follows:

Minitab Output

```
ANOVA: y versus A, B, C

Factor     Type Levels Values
A        random      2    -1     1
B(A)     random      2    -1     1
C(A B)   random      2    -1     1

Analysis of Variance for y

Source      DF        SS         MS       F      P
A            1     0.250      0.250    0.06  0.831
B(A)         2     8.500      4.250    0.35  0.726
C(A B)       4    49.000     12.250    2.13  0.168
Error        8    46.000      5.750
Total       15   103.750

Source      Variance Error Expected Mean Square for Each Term
           component term  (using unrestricted model)
 1 A        -0.5000   2   (4) + 2(3) + 4(2) + 8(1)
 2 B(A)     -2.0000   3   (4) + 2(3) + 4(2)
 3 C(A B)    3.2500   4   (4) + 2(3)
 4 Error     5.7500       (4)
```

13-10 Verify the expected mean squares given in Table 13-1.

Factor	F a i	F b j	R n l	E(MS)
τ_i	0	b	n	$\sigma^2 + \frac{bn}{a-1}\sum \tau_i^2$
$\beta_{j(i)}$	1	0	n	$\sigma^2 + \frac{n}{a(b-1)}\sum\sum \beta_{j(i)}^2$
$\varepsilon_{(ijk)l}$	1	1	1	σ^2

Factor	R a i	R b j	R n l	E(MS)
τ_i	1	b	n	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$\beta_{j(i)}$	1	1	n	$\sigma^2 + n\sigma_\beta^2$
$\varepsilon_{(ijk)l}$	1	1	1	σ^2

Factor	F a i	R b j	R n l	E(MS)
τ_i	0	b	n	$\sigma^2 + n\sigma_\beta^2 + \frac{bn}{a-1}\sum \tau_i^2$
$\beta_{j(i)}$	1	1	n	$\sigma^2 + n\sigma_\beta^2$
$\varepsilon_{(ijk)l}$	1	1	1	σ^2

13-11 ***Unbalanced designs.*** Consider an unbalanced two-stage nested design with b_j levels of B under the ith level of A and n_{ij} replicates in the ijth cell.

(a) Write down the least squares normal equations for this situation. Solve the normal equations.

The least squares normal equations are:

$$\mu = n_{..}\hat{\mu} + \sum_{i=1}^{a} n_{i.}\hat{\tau}_i + \sum_{i=1}^{a}\sum_{j=1}^{b_i} n_{ij}\hat{\beta}_{j(i)} = y_{...}$$

$$\tau_i = n_{i.}\hat{\mu} + n_{i.}\hat{\tau}_i + \sum_{j=1}^{b_i} n_{ij}\hat{\beta}_{j(i)} = y_{i..}\text{ , for } i = 1,2,\ldots,a$$

$$\beta_{j(i)} = n_{ij}\hat{\mu} + n_{ij}\hat{\tau}_i + n_{ij}\hat{\beta}_{j(i)} = y_{ij.}\text{, for } i = 1,2,\ldots,a \text{ and } j = 1,2,\ldots,b_i$$

There are 1+a+b equations in 1+a+b unknowns. However, there are a+1linear dependencies in these equations, and consequently, a+1 side conditions are needed to solve them. Any convenient set of a+1 linearly independent equations can be used. The easiest set is $\hat{\mu} = 0$, $\hat{\tau}_i = 0$, for i=1,2,…,a. Using these conditions we get

$$\hat{\mu} = 0,\ \hat{\tau}_i = 0,\ \hat{\beta}_{j(i)} = \bar{y}_{ij.}$$

as the solution to the normal equations. See Searle (1971) for a full discussion.

(b) Construct the analysis of variance table for the unbalanced two-stage nested design.

The analysis of variance table is

Source	SS	DF
A	$\sum_{i-1}^{a}\frac{y_{i..}^2}{n_{i.}} - \frac{y_{...}^2}{n_{..}}$	a-1
B	$\sum_{i=1}^{a}\sum_{j=1}^{b_i}\frac{y_{ij.}^2}{n_{ij}} - \sum_{i-1}^{a}\frac{y_{i..}^2}{n_{i.}}$	$b.-a$
Error	$\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{n_{ij}} y_{ijk}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b_i}\frac{y_{ij.}^2}{n_{ij}}$	$n..-b$
Total	$\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{n_{ij}} y_{ijk}^2 - \frac{y_{...}^2}{n_{..}}$	$n..$-1

(c) Analyze the following data, using the results in part (b).

Factor A	1		2		
Factor B	1	2	1	2	3
	6	-3	5	2	1
	4	1	7	4	0
	8		9	3	-3
			6		

Note that a=2, b_1=2, b_2=3, $b.=b_1+b_2$=5, n_{11}=3, n_{12}=2, n_{21}=4, n_{22}=3 and n_{23}=3

Source	SS	DF	MS

A	0.13	1	0.13
B	153.78	3	51.26
Error	35.42	10	3.54
Total	189.33	14	

The analysis can also be performed in Minitab as follows. The adjusted sum of squares is utilized by Minitab's general linear model routine.

Minitab Output

General Linear Model: y versus A, B

```
Factor     Type Levels Values
A         fixed      2 1 2
B(A)      fixed      5 1 2 1 2 3

Analysis of Variance for y, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F       P
A           1      0.133      0.898      0.898    0.25  0.625
B(A)        3    153.783    153.783     51.261   14.47  0.001
Error      10     35.417     35.417      3.542
Total      14    189.333
```

13-12 ***Variance components in the unbalanced two-stage nested design.*** Consider the model

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{k(ij)} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,n_{ij} \end{cases}$$

where *A* and *B* are random factors. Show that

$$E(MS_A) = \sigma^2 + c_1\sigma_\beta^2 + c_2\sigma_\tau^2$$
$$E(MS_{B(A)}) = \sigma^2 + c_0\sigma_\beta^2$$
$$E(MS_E) = \sigma^2$$

where

$$c_0 = \frac{N - \sum_{i=1}^{a}\left(\sum_{j=1}^{b_i}\frac{n_{ij}^2}{n_{i.}}\right)}{b-a}$$

$$c_1 = \frac{\sum_{i=1}^{a}\left(\sum_{j=1}^{b_i}\frac{n_{ij}^2}{n_{i.}}\right) - \sum_{i=1}^{a}\sum_{j=1}^{b_i}\frac{n_{ij}^2}{N}}{a-1}$$

$$c_2 = \frac{N - \frac{\sum_{i=1}^{a} n_{i.}^2}{N}}{a-1}$$

See "Variance Component Estimation in the 2-way Nested Classification," by S.R. Searle, *Annals of Mathematical Statistics*, Vol. 32, pp. 1161-1166, 1961. A good discussion of variance component estimation from unbalanced data is in Searle (1971a).

13-13 A process engineer is testing the yield of a product manufactured on three machines. Each machine can be operated at two power settings. Furthermore, a machine has three stations on which the product is formed. An experiment is conducted in which each machine is tested at both power settings, and three observations on yield are taken from each station. The runs are made in random order, and the results follow. Analyze this experiment, assuming all three factors are fixed.

	Machine 1			Machine 2			Machine 3		
Station	1	2	3	1	2	3	1	2	3
Power Setting 1	34.1	33.7	36.2	32.1	33.1	32.8	32.9	33.8	33.6
	30.3	34.9	36.8	33.5	34.7	35.1	33.0	33.4	32.8
	31.6	35.0	37.1	34.0	33.9	34.3	33.1	32.8	31.7
Power Setting 2	24.3	28.1	25.7	24.1	24.1	26.0	24.2	23.2	24.7
	26.3	29.3	26.1	25.0	25.1	27.1	26.1	27.4	22.0
	27.1	28.6	24.9	26.3	27.9	23.9	25.3	28.0	24.8

The linear model is $y_{ijkl} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_{k(j)} + (\tau\gamma)_{ik(j)} + \varepsilon_{(ijk)l}$

Minitab Output

ANOVA: Yield versus Machine, Power, Station

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Power             fixed      2     1     2
Station(Machine)  fixed      3     1     2     3

Analysis of Variance for Yield

Source                     DF         SS         MS        F      P
Machine                     2     21.143     10.572     6.46  0.004
Power                       1    853.631    853.631   521.80  0.000
Station(Machine)            6     32.583      5.431     3.32  0.011
Machine*Power               2      0.616      0.308     0.19  0.829
Power*Station(Machine)      6     28.941      4.824     2.95  0.019
Error                      36     58.893      1.636
Total                      53    995.808

Source                     Variance Error Expected Mean Square for Each Term
                          component term (using restricted model)
 1 Machine                            6   (6) + 18Q[1]
 2 Power                              6   (6) + 27Q[2]
 3 Station(Machine)                   6   (6) + 6Q[3]
 4 Machine*Power                      6   (6) + 9Q[4]
 5 Power*Station(Machine)             6   (6) + 3Q[5]
 6 Error                      1.636       (6)
```

13-14 Suppose that in Problem 13-13 a large number of power settings could have been used and that the two selected for the experiment were chosen randomly. Obtain the expected mean squares for this situation and modify the previous analysis appropriately.

	R	F	F	R	
	2	3	3	3	
Factor	i	j	k	l	E(MS)
τ_i	1	3	3	3	$\sigma^2 + 27\sigma_\tau^2$
β_j	2	0	3	3	$\sigma^2 + 9\sigma_{\tau\beta}^2 + 9\sum \beta_j^2$

$(\tau\beta)_{ij}$	1	0	3	3	$\sigma^2 + 9\sigma^2_{\tau\beta}$
$\gamma_{k(j)}$	2	1	0	3	$\sigma^2 + 3\sigma^2_{\tau\gamma} + \sum\sum\gamma^2_{k(j)}$
$(\tau\gamma)_{ik(j)}$	1	1	0	3	$\sigma^2 + 3\sigma^2_{\tau\gamma}$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

The analysis of variance and the expected mean squares can be completed in Minitab as follows:

Minitab Output

ANOVA: Yield versus Machine, Power, Station

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Power            random      2     1     2
Station(Machine)  fixed      3     1     2     3

Analysis of Variance for Yield

Source                    DF          SS          MS       F      P
Machine                    2      21.143      10.572   34.33  0.028
Power                      1     853.631     853.631  521.80  0.000
Station(Machine)           6      32.583       5.431    1.13  0.445
Machine*Power              2       0.616       0.308    0.19  0.829
Power*Station(Machine)     6      28.941       4.824    2.95  0.019
Error                     36      58.893       1.636
Total                     53     995.808

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using restricted model)
 1 Machine                           4   (6) + 9(4) + 18Q[1]
 2 Power                   31.5554   6   (6) + 27(2)
 3 Station(Machine)                  5   (6) + 3(5) + 6Q[3]
 4 Machine*Power           -0.1476   6   (6) + 9(4)
 5 Power*Station(Machine)   1.0625   6   (6) + 3(5)
 6 Error                    1.6359       (6)
```

13-15 Reanalyze the experiment in Problem 13-14 assuming the unrestricted form of the mixed model. You may use a computer software program to do this. Comment on any differences between the restricted and unrestricted model analysis and conclusions.

ANOVA: Yield versus Machine, Power, Station

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Power            random      2     1     2
Station(Machine)  fixed      3     1     2     3

Analysis of Variance for Yield

Source                    DF          SS          MS       F      P
Machine                    2      21.143      10.572   34.33  0.028
Power                      1     853.631     853.631 2771.86  0.000
Station(Machine)           6      32.583       5.431    1.13  0.445
Machine*Power              2       0.616       0.308    0.06  0.939
Power*Station(Machine)     6      28.941       4.824    2.95  0.019
Error                     36      58.893       1.636
Total                     53     995.808

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using unrestricted model)
 1 Machine                           4   (6) + 3(5) + 9(4) + Q[1,3]
 2 Power                   31.6046   4   (6) + 3(5) + 9(4) + 27(2)
 3 Station(Machine)                  5   (6) + 3(5) + Q[3]
 4 Machine*Power           -0.5017   5   (6) + 3(5) + 9(4)
 5 Power*Station(Machine)   1.0625   6   (6) + 3(5)
 6 Error                    1.6359       (6)
```

There are differences between several of the expected mean squares. However, the conclusions that could be drawn do not differ in any meaningful way from the restricted model analysis.

13-16 A structural engineer is studying the strength of aluminum alloy purchased from three vendors. Each vendor submits the alloy in standard-sized bars of 1.0, 1.5, or 2.0 inches. The processing of different sizes of bar stock from a common ingot involves different forging techniques, and so this factor may be important. Furthermore, the bar stock if forged from ingots made in different heats. Each vendor submits two tests specimens of each size bar stock from the three heats. The resulting strength data follow. Analyze the data, assuming that vendors and bar size are fixed and heats are random.

	Vendor 1			Vendor 2			Vendor 3		
Heat	1	2	3	1	2	3	1	2	3
Bar Size: 1 inch	1.230	1.346	1.235	1.301	1.346	1.315	1.247	1.275	1.324
	1.259	1.400	1.206	1.263	1.392	1.320	1.296	1.268	1.315
1 1/2 inch	1.316	1.329	1.250	1.274	1.384	1.346	1.273	1.260	1.392
	1.300	1.362	1.239	1.268	1.375	1.357	1.264	1.265	1.364
2 inch	1.287	1.346	1.273	1.247	1.362	1.336	1.301	1.280	1.319
	1.292	1.382	1.215	1.215	1.328	1.342	1.262	1.271	1.323

$$y_{ijkl} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_{k(j)} + (\tau\gamma)_{ik(j)} + \varepsilon_{(ijk)l}$$

Minitab Output

ANOVA: Strength versus Vendor, Bar Size, Heat

```
Factor         Type Levels Values
Vendor        fixed      3     1     2     3
Heat(Vendor) random      3     1     2     3
Bar Size      fixed      3   1.0   1.5   2.0

Analysis of Variance for Strength

Source                  DF         SS         MS       F      P
Vendor                   2  0.0088486  0.0044243    0.26  0.776
Heat(Vendor)             6  0.1002093  0.0167016   41.32  0.000
Bar Size                 2  0.0025263  0.0012631    1.37  0.290
Vendor*Bar Size          4  0.0023754  0.0005939    0.65  0.640
Bar Size*Heat(Vendor)   12  0.0110303  0.0009192    2.27  0.037
Error                   27  0.0109135  0.0004042
Total                   53  0.1359034

Source                  Variance Error Expected Mean Square for Each Term
                       component term (using restricted model)
 1 Vendor                          2   (6) + 6(2) + 18Q[1]
 2 Heat(Vendor)          0.00272   6   (6) + 6(2)
 3 Bar Size                        5   (6) + 2(5) + 18Q[3]
 4 Vendor*Bar Size                 5   (6) + 2(5) + 6Q[4]
 5 Bar Size*Heat(Vendor) 0.00026   6   (6) + 2(5)
 6 Error                 0.00040       (6)
```

13-17 Reanalyze the experiment in Problem 13-16 assuming the unrestricted form of the mixed model. You may use a computer software program to do this. Comment on any differences between the restricted and unrestricted model analysis and conclusions.

Minitab Output

ANOVA: Strength versus Vendor, Bar Size, Heat

```
Factor         Type Levels Values
Vendor        fixed      3     1     2     3
Heat(Vendor) random      3     1     2     3
```

```
Bar Size      fixed      3   1.0   1.5   2.0

Analysis of Variance for Strength

Source                   DF         SS         MS       F      P
Vendor                    2  0.0088486  0.0044243    0.26  0.776
Heat(Vendor)              6  0.1002093  0.0167016   18.17  0.000
Bar Size                  2  0.0025263  0.0012631    1.37  0.290
Vendor*Bar Size           4  0.0023754  0.0005939    0.65  0.640
Bar Size*Heat(Vendor)    12  0.0110303  0.0009192    2.27  0.037
Error                    27  0.0109135  0.0004042
Total                    53  0.1359034

Source                   Variance Error Expected Mean Square for Each Term
                        component term (using unrestricted model)
 1 Vendor                          2   (6) + 2(5) + 6(2) + Q[1,4]
 2 Heat(Vendor)           0.00263  5   (6) + 2(5) + 6(2)
 3 Bar Size                        5   (6) + 2(5) + Q[3,4]
 4 Vendor*Bar Size                 5   (6) + 2(5) + Q[4]
 5 Bar Size*Heat(Vendor)  0.00026  6   (6) + 2(5)
 6 Error                  0.00040      (6)
```

There are some differences in the expected mean squares. However, the conclusions do not differ from those of the restricted model analysis.

13-18 Suppose that in Problem 13-16 the bar stock may be purchased in many sizes and that the three sizes are actually used in experiment were selected randomly. Obtain the expected mean squares for this situation and modify the previous analysis appropriately. Use the restricted form of the mixed model.

Minitab Output

ANOVA: Strength versus Vendor, Bar Size, Heat

```
Factor         Type Levels Values
Vendor        fixed      3     1     2     3
Heat(Vendor) random      3     1     2     3
Bar Size     random      3   1.0   1.5   2.0

Analysis of Variance for Strength

Source                   DF         SS         MS       F      P
Vendor                    2  0.0088486  0.0044243    0.27  0.772 x
Heat(Vendor)              6  0.1002093  0.0167016   18.17  0.000
Bar Size                  2  0.0025263  0.0012631    1.37  0.290
Vendor*Bar Size           4  0.0023754  0.0005939    0.65  0.640
Bar Size*Heat(Vendor)    12  0.0110303  0.0009192    2.27  0.037
Error                    27  0.0109135  0.0004042
Total                    53  0.1359034

x Not an exact F-test.

Source                   Variance Error Expected Mean Square for Each Term
                        component term (using restricted model)
 1 Vendor                          *   (6) + 2(5) + 6(4) + 6(2) + 18Q[1]
 2 Heat(Vendor)           0.00263  5   (6) + 2(5) + 6(2)
 3 Bar Size               0.00002  5   (6) + 2(5) + 18(3)
 4 Vendor*Bar Size       -0.00005  5   (6) + 2(5) + 6(4)
 5 Bar Size*Heat(Vendor)  0.00026  6   (6) + 2(5)
 6 Error                  0.00040      (6)

* Synthesized Test.

Error Terms for Synthesized Tests

Source                    Error DF  Error MS  Synthesis of Error MS
 1 Vendor                     5.75 0.0163762   (2) + (4) - (5)
```

Notice that a Satterthwaite type test is used for vendor.

13-19 Steel in normalized by heating above the critical temperature, soaking, and then air cooling. This process increases the strength of the steel, refines the grain, and homogenizes the structure. An experiment is performed to determine the effect of temperature and heat treatment time on the strength of normalized steel. Two temperatures and three times are selected. The experiment is performed by heating the oven to a randomly selected temperature and inserting three specimens. After 10 minutes one specimen is removed, after 20 minutes the second specimen is removed, and after 30 minutes the final specimen is removed. Then the temperature is changed to the other level and the process is repeated. Four shifts are required to collect the data, which are shown below. Analyze the data and draw conclusions, assume both factors are fixed.

		Temperature (F)	
Shift	Time(minutes)	1500	1600
1	10	63	89
	20	54	91
	30	61	62
2	10	50	80
	20	52	72
	30	59	69
3	10	48	73
	20	74	81
	30	71	69
4	10	54	88
	20	48	92
	30	59	64

This is a split-plot design. Shifts correspond to blocks, temperature is the whole plot treatment, and time is the subtreatments (in the subplot or split-plot part of the design). The expected mean squares and analysis of variance are shown below.

	R	F	F	R	
	4	2	3	1	
Factor	*i*	*j*	*k*	*l*	E(*MS*)
τ_i (blocks)	1	2	3	1	$\sigma^2 + 6\sigma_\tau^2$
β_j (temp)	4	0	3	1	$\sigma^2 + 3\sigma_{\tau\beta}^2 + (12/3)\sum \beta_j^2$
$(\tau\beta)_{ij}$	1	0	3	1	$\sigma^2 + 2\sigma_{\tau\beta}^2$
γ_k (time)	4	2	0	1	$\sigma^2 + 2\sigma_{\tau\gamma}^2 + (8/2)\sum \gamma_k^2$
$(\tau\gamma)_{ik}$	1	2	0	1	$\sigma^2 + 2\sigma_{\tau\gamma}^2$
$(\beta\gamma)_{jk}$	4	0	0	1	$\sigma^2 + \sigma_{\tau\beta\gamma}^2 + (12/3)\sum\sum (\beta\gamma)_{jk}^2$
$(\tau\beta\gamma)_{ijk}$	1	0	0	1	$\sigma^2 + \sigma_{\tau\beta\gamma}^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2 (not estimable)

The following Minitab Output has been modified to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Notice that the Error term in the analysis of variance is actually the three factor interaction.

Minitab Output

ANOVA: Strength versus Shift, Temperature, Time

```
Factor      Type Levels Values
Shift    random      4     1     2     3     4
Temperat  fixed      2  1500  1600
Time      fixed      3    10    20    30

Analysis of Variance for Strength
```

```
                                               Standard     Split Plot
Source            DF         SS          MS       F       P       F       P
Shift              3     145.46       48.49    1.19  0.390
Temperat           1    2340.38     2340.38   29.20  0.012   29.21  0.012
Shift*Temperat     3     240.46       80.15    1.97  0.220
Time               2     159.25       79.63    1.00  0.422    1.00  0.422
Shift*Time         6     478.42       79.74    1.96  0.217
Temperat*Time      2     795.25      397.63    9.76  0.013    9.76  0.013
Error              6     244.42       40.74
Total             23    4403.63

Source            Variance Error Expected Mean Square for Each Term
                 component term (using restricted model)
 1 Shift            1.292   7   (7) + 6(1)
 2 Temperat                 3   (7) + 3(3) + 12Q[2]
 3 Shift*Temperat   13.139   7   (7) + 3(3)
 4 Time                     5   (7) + 2(5) + 8Q[4]
 5 Shift*Time       19.500   7   (7) + 2(5)
 6 Temperat*Time            7   (7) + 4Q[6]
 7 Error            40.736       (7)
```

13-20 An experiment is designed to study pigment dispersion in paint. Four different mixes of a particular pigment are studied. The procedure consists of preparing a particular mix and then applying that mix to a panel by three application methods (brushing, spraying, and rolling). The response measured is the percentage reflectance of the pigment. Three days are required to run the experiment, and the data obtained follow. Analyze the data and draw conclusions, assuming that mixes and application methods are fixed.

		Mix			
Day	App Method	1	2	3	4
1	1	64.5	66.3	74.1	66.5
	2	68.3	69.5	73.8	70.0
	3	70.3	73.1	78.0	72.3
2	1	65.2	65.0	73.8	64.8
	2	69.2	70.3	74.5	68.3
	3	71.2	72.8	79.1	71.5
3	1	66.2	66.5	72.3	67.7
	2	69.0	69.0	75.4	68.6
	3	70.8	74.2	80.1	72.4

This is a split plot design. Days correspond to blocks, mix is the whole plot treatment, and method is the subtreatment (in the subplot or split plot part of the design). The expected mean squares are:

	R	F	F	R	
	3	4	3	1	
Factor	i	j	k	l	E(MS)
τ_i (blocks)	1	4	3	1	$\sigma^2 + 12\sigma_\tau^2$
β_j (temp)	3	0	3	1	$\sigma^2 + 3\sigma_{\tau\beta}^2 + (9/3)\sum \beta_j^2$
$(\tau\beta)_{ij}$	1	0	3	1	$\sigma^2 + 3\sigma_{\tau\beta}^2$
γ_k (time)	3	4	0	1	$\sigma^2 + 4\sigma_{\tau\gamma}^2 + (12/2)\sum \gamma_k^2$
$(\tau\gamma)_{ik}$	1	4	0	1	$\sigma^2 + 4\sigma_{\tau\gamma}^2$
$(\beta\gamma)_{jk}$	3	0	0	1	$\sigma^2 + \sigma_{\tau\beta\gamma}^2 + (3/6)\sum\sum (\beta\gamma)_{jk}^2$
$(\tau\beta\gamma)_{ijk}$	1	0	0	1	$\sigma^2 + \sigma_{\tau\beta\gamma}^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2 (not estimable)

The following Minitab Output has been modified to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Notice that the Error term in the analysis of variance is actually the three factor interaction.

Minitab Output

ANOVA: Reflectance versus Day, Mix, Method

```
Factor     Type Levels Values
Day      random      3     1     2     3
Mix       fixed      4     1     2     3     4
Method    fixed      3     1     2     3

Analysis of Variance for Reflecta
                                           Standard     Split Plot
Source        DF         SS         MS       F      P        F      P
Day            2      2.042      1.021    1.39  0.285
Mix            3    307.479    102.493  135.77  0. 000 135.75  0.000
Day*Mix        6      4.529      0.755    1.03  0.451
Method         2    222.095    111.047  226.24  0.000  226.16  0.000
Day*Method     4      1.963      0.491    0.67  0.625
Mix*Method     6     10.036      1.673    2.28  0.105    2.28  0.105
Error         12      8.786      0.732
Total         35    556.930

Source        Variance Error Expected Mean Square for Each Term
             component term (using restricted model)
 1 Day         0.02406   7   (7) + 12(1)
 2 Mix                   3   (7) + 3(3) + 9Q[2]
 3 Day*Mix     0.00759   7   (7) + 3(3)
 4 Method                5   (7) + 4(5) + 12Q[4]
 5 Day*Method -0.06032   7   (7) + 4(5)
 6 Mix*Method            7   (7) + 3Q[6]
 7 Error       0.73213        (7)
```

13-21 Repeat Problem 13-20, assuming that the mixes are random and the application methods are fixed.

The expected mean squares are:

	R	R	F	R	
	3	4	3	1	
Factor	i	j	k	l	E(*MS*)
τ_i (blocks)	1	4	3	1	$\sigma^2 + 3\sigma^2_{\tau\beta} + 12\sigma^2_\tau$
β_j (temp)	3	1	3	1	$\sigma^2 + 3\sigma^2_{\tau\beta} + 19\sigma^2_\beta$
$(\tau\beta)_{ij}$	1	1	3	1	$\sigma^2 + 3\sigma^2_{\tau\beta}$
γ_k (time)	3	4	0	1	$\sigma^2 + \sigma^2_{\tau\beta\gamma} + 4\sigma^2_{\tau\gamma} + (12/2)\sum\gamma_k^2$
$(\tau\gamma)_{ik}$	1	4	0	1	$\sigma^2 + \sigma^2_{\tau\beta\gamma} + 4\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	3	1	0	1	$\sigma^2 + \sigma^2_{\tau\beta\gamma} + 3\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	1	1	0	1	$\sigma^2 + \sigma^2_{\tau\beta\gamma}$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2 (not estimable)

The F-tests are the same as those in Problem 13-20. The following Minitab Output has been edited to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Again, the Error term in the analysis of variance is actually the three factor interaction.

Minitab Output

ANOVA: Reflectance versus Day, Mix, Method

```
Factor     Type Levels Values
Day      random      3     1     2     3
Mix      random      4     1     2     3     4
Method    fixed      3     1     2     3

Analysis of Variance for Reflecta
                                            Standard        Split Plot
Source        DF         SS         MS       F       P           F      P
Day            2      2.042      1.021    1.35  0.328
Mix            3    307.479    102.493  135.77  0.000     135.75  0.000
Day*Mix        6      4.529      0.755    1.03  0.451
Method         2    222.095    111.047   77.58  0.001 x   226.16  0.000
Day*Method     4      1.963      0.491    0.67  0.625
Mix*Method     6     10.036      1.673    2.28  0.105       2.28  0.105
Error         12      8.786      0.732
Total         35    556.930

x Not an exact F-test.

Source        Variance Error Expected Mean Square for Each Term
             component term (using restricted model)
 1 Day          0.0222   3   (7) + 3(3) + 12(1)
 2 Mix         11.3042   3   (7) + 3(3) + 9(2)
 3 Day*Mix      0.0076   7   (7) + 3(3)
 4 Method                *   (7) + 3(6) + 4(5) + 12Q[4]
 5 Day*Method  -0.0603   7   (7) + 4(5)
 6 Mix*Method   0.3135   7   (7) + 3(6)
 7 Error        0.7321       (7)

* Synthesized Test.

Error Terms for Synthesized Tests

Source         Error DF  Error MS  Synthesis of Error MS
 4 Method          3.59     1.431   (5) + (6) - (7)
```

13-22 Consider the split-split-plot design described in example 13-3. Suppose that this experiment is conducted as described and that the data shown below are obtained. Analyze and draw conclusions.

		Technician								
		1			2			3		
Blocks	Dose Strengths	1	2	3	1	2	3	1	2	3
	Wall Thickness									
1	1	95	71	108	96	70	108	95	70	100
	2	104	82	115	99	84	100	102	81	106
	3	101	85	117	95	83	105	105	84	113
	4	108	85	116	97	85	109	107	87	115
2	1	95	78	110	100	72	104	92	69	101
	2	106	84	109	101	79	102	100	76	104
	3	103	86	116	99	80	108	101	80	109
	4	109	84	110	112	86	109	108	86	113
3	1	96	70	107	94	66	100	90	73	98
	2	105	81	106	100	84	101	97	75	100
	3	106	88	112	104	87	109	100	82	104
	4	113	90	117	121	90	117	110	91	112
4	1	90	68	109	98	68	106	98	72	101
	2	100	84	112	102	81	103	102	78	105
	3	102	85	115	100	85	110	105	80	110
	4	114	88	118	118	85	116	110	95	120

Using the computer output, the F-ratios were calculated by hand using the expected mean squares found in Table 13-18. The following Minitab Output has been edited to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Notice that the Error term in the analysis of variance is actually the four factor interaction.

Minitab Output

ANOVA: Time versus Day, Tech, Dose, Thick

Factor	Type	Levels	Values			
Day	random	4	1	2	3	4
Tech	fixed	3	1	2	3	
Dose	fixed	3	1	2	3	
Thick	fixed	4	1	2	3	4

Analysis of Variance for Time

				Standard		Split Plot	
Source	DF	SS	MS	F	P	F	P
Day	3	48.41	16.14	3.38	0.029		
Tech	2	248.35	124.17	4.62	0.061	4.62	0.061
Day*Tech	6	161.15	26.86	5.62	0.000		
Dose	2	20570.06	10285.03	550.44	0.000	550.30	0.000
Day*Dose	6	112.11	18.69	3.91	0.004		
Tech*Dose	4	125.94	31.49	3.32	0.048	3.32	0.048
Day*Tech*Dose	12	113.89	9.49	1.99	0.056		
Thick	3	3806.91	1268.97	36.47	0.000	36.48	0.000
Day*Thick	9	313.12	34.79	7.28	0.000		
Tech*Thick	6	126.49	21.08	2.26	0.084	2.26	0.084
Day*Tech*Thick	18	167.57	9.31	1.95	0.044		
Dose*Thick	6	402.28	67.05	17.13	0.000	17.15	0.000
Day*Dose*Thick	18	70.44	3.91	0.82	0.668		
Tech*Dose*Thick	12	205.89	17.16	3.59	0.001	3.59	0.001
Error	36	172.06	4.78				
Total	143	26644.66					

Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1 Day	0.3155	15	(15) + 36(1)
2 Tech		3	(15) + 12(3) + 48Q[2]
3 Day*Tech	1.8400	15	(15) + 12(3)
4 Dose		5	(15) + 12(5) + 48Q[4]
5 Day*Dose	1.1588	15	(15) + 12(5)
6 Tech*Dose		7	(15) + 4(7) + 16Q[6]
7 Day*Tech*Dose	1.1779	15	(15) + 4(7)
8 Thick		9	(15) + 9(9) + 36Q[8]
9 Day*Thick	3.3346	15	(15) + 9(9)
10 Tech*Thick		11	(15) + 3(11) + 12Q[10]
11 Day*Tech*Thick	1.5100	15	(15) + 3(11)
12 Dose*Thick		13	(15) + 3(13) + 12Q[12]
13 Day*Dose*Thick	-0.2886	15	(15) + 3(13)
14 Tech*Dose*Thick		15	(15) + 4Q[14]
15 Error	4.7793		(15)

13-23 Rework Problem 13-22, assuming that the dosage strengths are chosen at random. Use the restricted form of the mixed model.

The following Minitab Output has been edited to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Again, the Error term in the analysis of variance is actually the four factor interaction.

Minitab Output

ANOVA: Time versus Day, Tech, Dose, Thick

Factor	Type	Levels	Values			
Day	random	4	1	2	3	4
Tech	fixed	3	1	2	3	
Dose	random	3	1	2	3	
Thick	fixed	4	1	2	3	4

Analysis of Variance for Time

				Standard		Split Plot	
Source	DF	SS	MS	F	P	F	P
Day	3	48.41	16.14	0.86	0.509		
Tech	2	248.35	124.17	2.54	0.155	4.62	0.061
Day*Tech	6	161.15	26.86	2.83	0.059		
Dose	2	20570.06	10285.03	550.44	0.000	550.30	0.000

```
Day*Dose            6     112.11      18.69    3.91  0.004
Tech*Dose           4     125.94      31.49    3.32  0.048        3.32  0.048
Day*Tech*Dose      12     113.89       9.49    1.99  0.056
Thick               3    3806.91    1268.97   12.96  0.001 x    36.48  0.000
Day*Thick           9     313.12      34.79    8.89  0.000
Tech*Thick          6     126.49      21.08    0.97  0.475 x     2.26  0.084
Day*Tech*Thick     18     167.57       9.31    1.95  0.044
Dose*Thick          6     402.28      67.05   17.13  0.000      17.15  0.000
Day*Dose*Thick     18      70.44       3.91    0.82  0.668
Tech*Dose*Thick    12     205.89      17.16    3.59  0.001       3.59  0.001
Error              36     172.06       4.78
Total             143   26644.66

x Not an exact F-test.

Source              Variance Error Expected Mean Square for Each Term
                   component term  (using restricted model)
 1 Day                -0.071   5   (15) + 12(5) + 36(1)
 2 Tech                        *   (15) + 4(7) + 16(6) + 12(3) + 48Q[2]
 3 Day*Tech            1.447   7   (15) + 4(7) + 12(3)
 4 Dose              213.882   5   (15) + 12(5) + 48(4)
 5 Day*Dose            1.159  15   (15) + 12(5)
 6 Tech*Dose           1.375   7   (15) + 4(7) + 16(6)
 7 Day*Tech*Dose       1.178  15   (15) + 4(7)
 8 Thick                       *   (15) + 3(13) + 12(12) + 9(9) + 36Q[8]
 9 Day*Thick           3.431  13   (15) + 3(13) + 9(9)
10 Tech*Thick                  *   (15) + 4(14) + 3(11) + 12Q[10]
11 Day*Tech*Thick      1.510  15   (15) + 3(11)
12 Dose*Thick          5.261  13   (15) + 3(13) + 12(12)
13 Day*Dose*Thick     -0.289  15   (15) + 3(13)
14 Tech*Dose*Thick     3.095  15   (15) + 4(14)
15 Error               4.779       (15)

* Synthesized Test.

Error Terms for Synthesized Tests

Source               Error DF  Error MS  Synthesis of Error MS
 2 Tech                  6.35     48.85   (3) + (6) - (7)
 8 Thick                10.84     97.92   (9) + (12) - (13)
10 Tech*Thick           15.69     21.69   (11) + (14) - (15)
```

The expected mean squares can also be shown as follows:

	R	F	R	F	R	
	4	3	3	4	1	
Factor	i	j	k	h	l	E(*MS*)
τ_i	1	3	3	4	1	$\sigma^2 + 12\sigma^2_{\tau\gamma} + 36\sigma^2_{\tau}$
β_j	4	0	3	4	1	$\sigma^2 + 4\sigma^2_{\tau\beta\gamma} + 16\sigma^2_{\beta\gamma} + 12\sigma^2_{\tau\beta} + (48/2)\sum \beta_j^2$
$(\tau\beta)_{ij}$	1	0	3	4	1	$\sigma^2 + 4\sigma^2_{\tau\beta\gamma} + 12\sigma^2_{\tau\beta}$
γ_k	4	3	1	4	1	$\sigma^2 + 3\sigma^2_{\tau\gamma\delta} + 12\sigma^2_{\gamma\delta} + 12\sigma^2_{\tau\gamma} + 48\sigma^2_{\gamma}$
$(\tau\gamma)_{ik}$	1	3	1	4	1	$\sigma^2 + 12\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	4	0	1	4	1	$\sigma^2 + 4\sigma^2_{\tau\beta\gamma} + 16\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	1	0	1	4	1	$\sigma^2 + 4\sigma^2_{\tau\beta\gamma}$
δ_h	4	3	3	0	1	$\sigma^2 + 3\sigma^2_{\tau\gamma\delta} + 12\sigma^2_{\gamma\delta} + 9\sigma^2_{\tau\delta} + (36/3)\sum \delta_h^2$
$(\tau\delta)_{ih}$	1	3	3	0	1	$\sigma^2 + 3\sigma^2_{\tau\gamma\delta} + 9\sigma^2_{\tau\delta}$
$(\beta\delta)_{jh}$	4	0	3	0	1	$\sigma^2 + \sigma^2_{\tau\beta\gamma\delta} + 4\sigma^2_{\beta\gamma\delta} + 3\sigma^2_{\tau\beta\delta} + (12/6)\sum\sum \beta\delta_{jh}^2$
$(\tau\beta\delta)_{ijh}$	1	0	3	0	1	$\sigma^2 + \sigma^2_{\tau\beta\gamma\delta} + 3\sigma^2_{\tau\beta\delta}$
$(\gamma\delta)_{kh}$	4	3	1	0	1	$\sigma^2 + 3\sigma^2_{\tau\gamma\delta} + 12\sigma^2_{\gamma\delta}$

$(\tau\gamma\delta)_{ikh}$	1	3	1	0	1	$\sigma^2+3\sigma^2_{\tau\gamma\delta}$
$(\beta\gamma\delta)_{jkh}$	4	0	1	0	1	$\sigma^2+\sigma^2_{\tau\beta\gamma\delta}+4\sigma^2_{\beta\gamma\delta}$
$(\tau\beta\gamma\delta)_{ijkh}$	1	0	1	0	1	$\sigma^2+\sigma^2_{\tau\beta\gamma\delta}$
$\varepsilon_{(ijk)l}$	1	1	1	1	1	σ^2

There are no exact tests on technicians β_j, dosage strengths γ_k, wall thickness δ_h, or the technician x wall thickness interaction $(\beta\delta)_{jh}$. The approximate *F*-tests are as follows:

H$_0$: $\beta_j=0$

$$F=\frac{MS_B+MS_{ABC}}{MS_{AB}+MS_{BC}}=\frac{124.174+9.491}{26.859+31.486}=2.291$$
$$p=\frac{(MS_B+MS_{ABC})^2}{\frac{MS_B^2}{2}+\frac{MS_{ABC}^2}{12}}=\frac{(124.174+9.491)^2}{\frac{124.174^2}{2}+\frac{9.491^2}{12}}=2.315$$
$$q=\frac{(MS_{AB}+MS_{BC})^2}{\frac{MS_{AB}^2}{6}+\frac{MS_{BC}^2}{4}}=\frac{(26.859+31.486)^2}{\frac{26.859^2}{6}+\frac{31.486^2}{4}}=9.248$$

Do not reject H$_0$: $\beta_j=0$

H$_0$: $\gamma_k=0$

$$F=\frac{MS_C+MS_{ACD}}{MS_{CD}+MS_{AD}}=\frac{10285.028+3.914}{67.046+34.791}=101.039$$
$$p=\frac{(MS_C+MS_{ACD})^2}{\frac{MS_C^2}{2}+\frac{MS_{ACD}^2}{18}}=\frac{(10285.028+3.914)^2}{\frac{10285.028^2}{2}+\frac{3.914^2}{18}}=2.002$$
$$q=\frac{(MS_{CD}+MS_{AD})^2}{\frac{MS_{CD}^2}{6}+\frac{MS_{AD}^2}{9}}=\frac{(67.046+34.791)^2}{\frac{67.046^2}{6}+\frac{34.791^2}{9}}=11.736$$

Reject H$_0$: $\gamma_k=0$

H$_0$: $\delta_h=0$

$$F=\frac{MS_D+MS_{ACD}}{MS_{CD}+MS_{AD}}=\frac{1268.970+3.914}{67.046+34.791}=12.499$$
$$p=\frac{(MS_D+MS_{ACD})^2}{\frac{MS_D^2}{3}+\frac{MS_{ACD}^2}{18}}=\frac{(1268.970+3.914)^2}{\frac{1268.970^2}{3}+\frac{3.914^2}{18}}=3.019$$

$$q=\frac{\left(MS_{CD}+MS_{AD}\right)^2}{\frac{MS_{CD}^2}{6}+\frac{MS_{AD}^2}{9}}=\frac{\left(67.046+34.791\right)^2}{\frac{67.046^2}{6}+\frac{34.791^2}{9}}=11.736$$

Reject H_0: $\delta_h=0$

H_0: $\left(\beta\delta\right)_{jh}=0$

$$F=\frac{MS_{BD}+MS_{ABCD}}{MS_{BCD}+MS_{ABD}}=\frac{21.081+4.779}{17.157+9.309}=0.977$$

F<1, Do not reject H_0: $\left(\beta\delta\right)_{jh}=0$

13-24 Suppose that in Problem 13-22 four technicians had been used. Assuming that all the factors are fixed, how many blocks should be run to obtain an adequate number of degrees of freedom on the test for differences among technicians?

The number of degrees of freedom for the test is (a-1)(4-1)=3(a-1), where a is the number of blocks used.

Number of Blocks (a)	DF for test
2	3
3	6
4	9
5	12

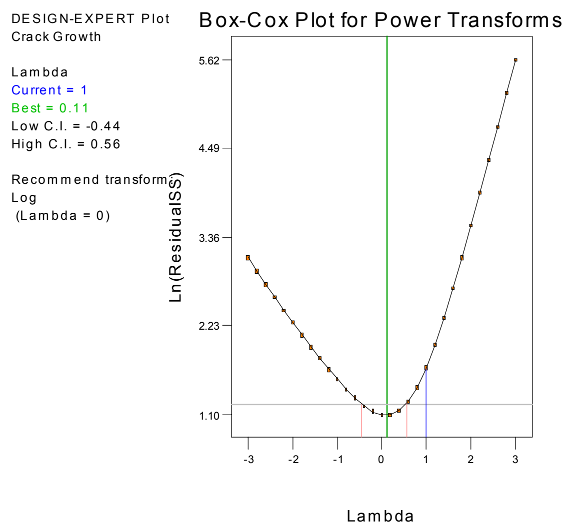
At least three blocks should be run, but four would give a better test.

13-25 Consider the experiment described in Example 13-3. Demonstrate how the order in which the treatments combinations are run would be determined if this experiment were run as (a) a split-split-plot, (b) a split-plot, (c) a factorial design in a randomized block, and (d) a completely randomized factorial design.

(a) Randomization for the split-split plot design is described in Example 13-3.
(b) In the split-plot, within a block, the technicians would be the main treatment and within a block-technician plot, the 12 combinations of dosage strength and wall thickness would be run in random order. The design would be a two-factor factorial in a split-plot.
(c) To run the design in a randomized block, the 36 combinations of technician, dosage strength, and wall thickness would be ran in random order within each block. The design would be a three factor factorial in a randomized block.
(d) The blocks would be considered as replicates, and all 144 observations would be 4 replicates of a three factor factorial.

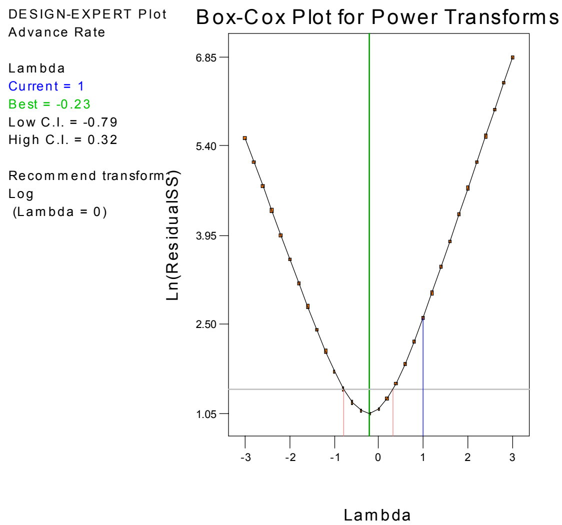
Chapter 14
Other Design and Analysis Topics
Solutions

14-1 Reconsider the experiment in Problem 5-22. Use the Box-Cox procedure to determine if a transformation on the response is appropriate (or useful) in the analysis of the data from this experiment.



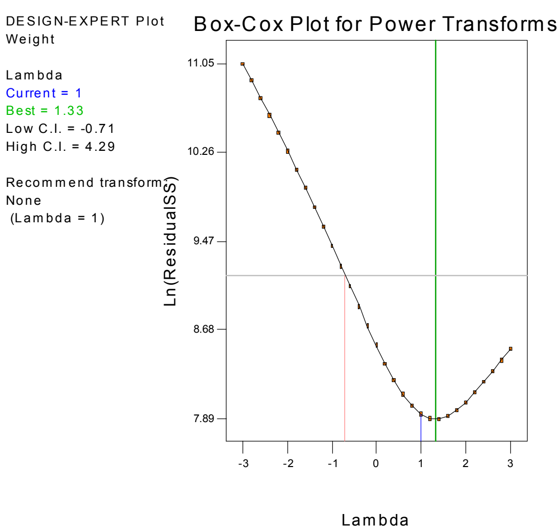
With the value of lambda near zero, and since the confidence interval does not include one, a natural log transformation would be appropriate.

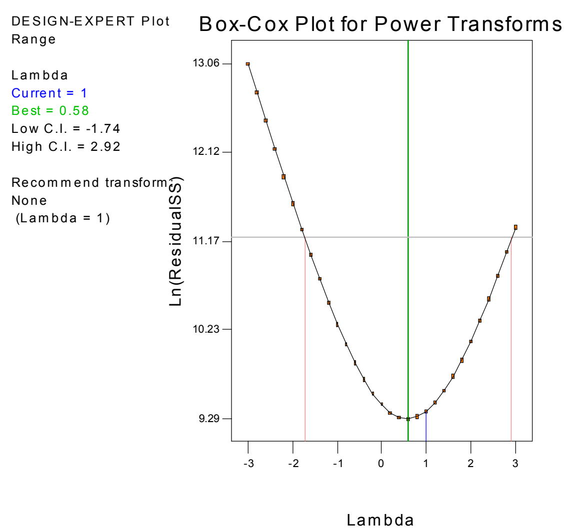
14-2 In example 6-3 we selected a log transformation for the drill advance rate response. Use the Box-Cox procedure to demonstrate that this is an appropriate data transformation.



Because the value of lambda is very close to zero, and the confidence interval does not include one, the natural log was the correct transformation chosen for this analysis.

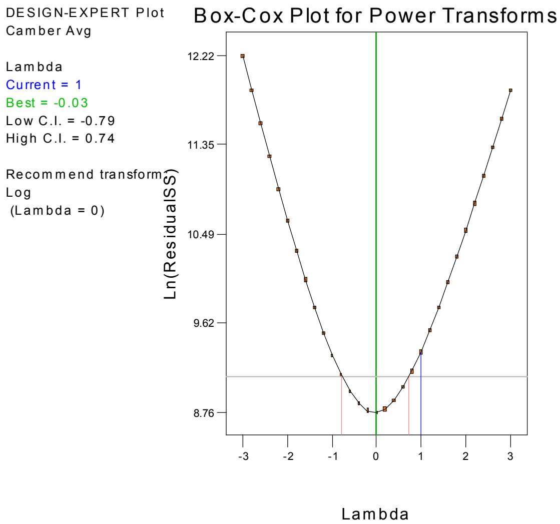
14-3 Reconsider the smelting process experiment in Problem 8-23, where a 2^{6-3} fractional factorial design was used to study the weight of packing material stuck to carbon anodes after baking. Each of the eight runs in the design was replicated three times and both the average weight and the range of the weights at each test combination were treated as response variables. Is there any indication that that a transformation is required for either response?

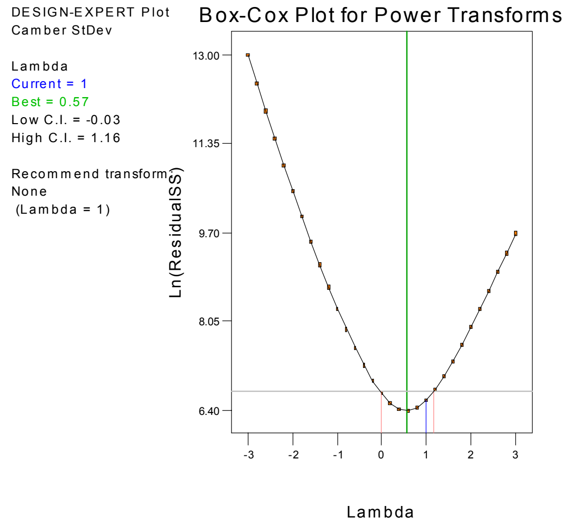




There is no indication that a transformation is required for either response.

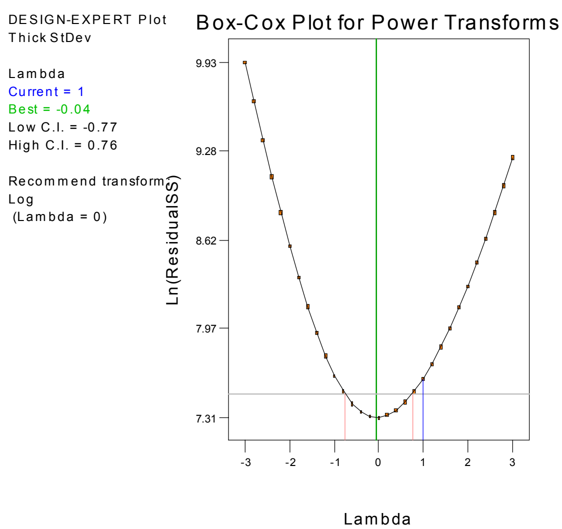
14-4 In Problem 8-24 a replicated fractional factorial design was used to study substrate camber in semiconductor manufacturing. Both the mean and standard deviation of the camber measurements were used as response variables. Is there any indication that a transformation is required for either response?





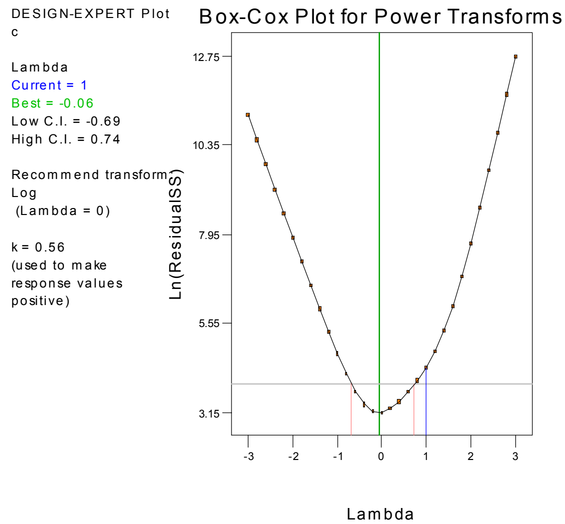
The Box-Cox plot for the Camber Average suggests a natural log transformation should be applied. This decision is based on the confidence interval for lambda not including one and the point estimate of lambda being very close to zero. With a lambda of approximately 0.5, a square root transformation could be considered for the Camber Standard Deviation; however, the confidence interval indicates that no transformation is needed.

14-5 Reconsider the photoresist experiment in Problem 8-25. Use the variance of the resist thickness at each test combination as the response variable. Is there any indication that a transformation is required?



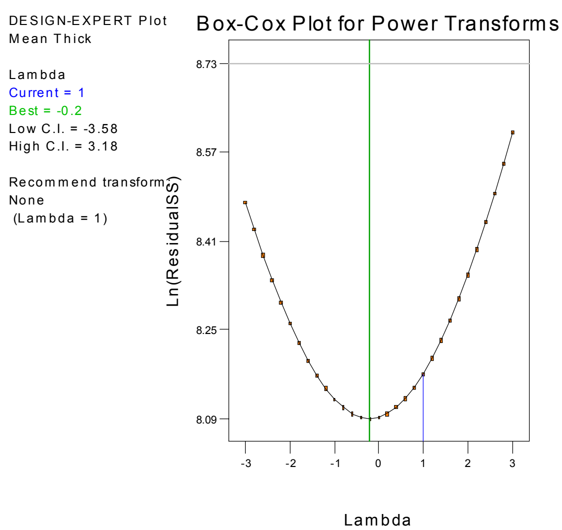
With the point estimate of lambda near zero, and the confidence interval for lambda not inclusive of one, a natural log transformation would be appropriate.

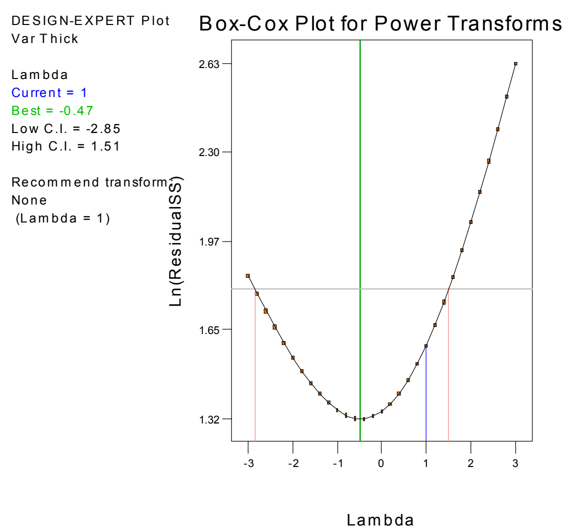
14-6 In the grill defects experiment described in Problem 8-29 a variation of the square root transformation was employed in the analysis of the data. Use the Box-Cox method to determine if this is the appropriate transformation.



Because the confidence interval for the minimum lambda does not include one, the decision to use a transformation is correct. Because the lambda point estimate is close to zero, the natural log transformation would be appropriate. This is a stronger transformation than the square root.

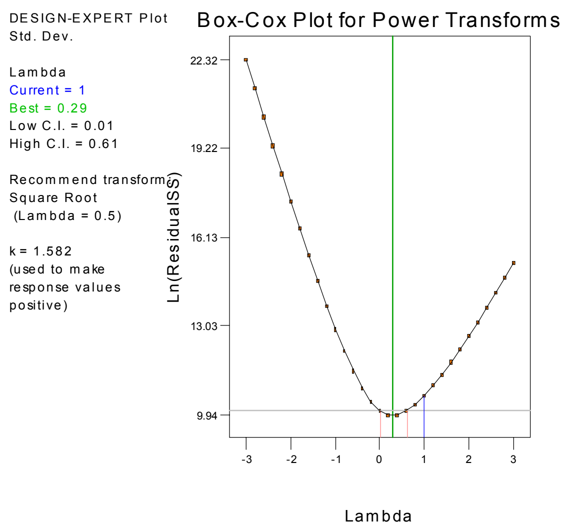
14-7 In the central composite design of Problem 11-14, two responses were obtained, the mean and variance of an oxide thickness. Use the Box-Cox method to investigate the potential usefulness of transformation for both of these responses. Is the log transformation suggested in part (c) of that problem appropriate?





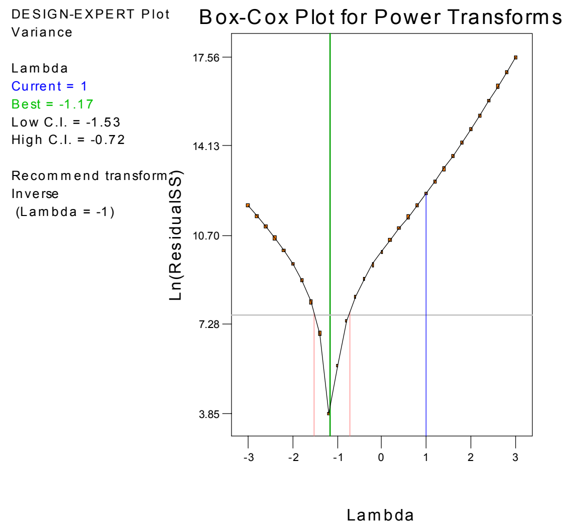
The Box-Cox plot for the Mean Thickness model suggests that a natural log transformation could be applied; however, the confidence interval for lambda includes one. Therefore, a transformation would have a minimal effect. The natural log transformation applied to the Variance of Thickness model appears to be acceptable; however, again the confidence interval for lambda includes one.

14-8 In the 3^3 factorial design of Problem 11-33 one of the responses is a standard deviation. Use the Box-Cox method to investigate the usefulness of transformations for this response. Would your answer change if we used the variance of the response?



Because the confidence interval for lambda does not include one, a transformation should be applied. The natural log transformation should not be considered due to zero not being included in the confidence interval. The square root transformation appears to be acceptable. However, notice that the value of zero is very close to the lower confidence limit, and the minimizing value of lambda is between 0 and 0.5. It is likely that either the natural log or the square root transformation would work reasonably well.

14-9 Problem 11-34 suggests using the $ln(s^2)$ as the response (refer to part b). Does the Box-Cox method indicate that a transformation is appropriate?



Because the confidence interval for lambda does not include one, a transformation should be applied. The confidence interval does not include zero; therefore, the natural log transformation is inappropriate. With the point estimate of lambda at –1.17, the reciprocal transformation is appropriate.

14-10 A soft drink distributor is studying the effectiveness of delivery methods. Three different types of hand trucks have been developed, and an experiment is performed in the company's methods engineering laboratory. The variable of interest is the delivery time in minutes (y); however, delivery time is also strongly related to the case volume delivered (x). Each hand truck is used four times and the data that follow are obtained. Analyze the data and draw the appropriate conclusions. Use α=0.05.

		Hand	Truck	Type	
1	1	2	2	3	3
y	x	y	x	y	x
27	24	25	26	40	38
44	40	35	32	22	26
33	35	46	42	53	50
41	40	26	25	18	20

From the analysis performed in Minitab, hand truck does not have a statistically significant effect on delivery time. Volume, as expected, does have a significant effect.

Minitab Output

General Linear Model: Time versus Truck

```
Factor      Type Levels Values
Truck      fixed       3 1 2 3

Analysis of Variance for Time, using Adjusted SS for Tests

Source      DF      Seq SS      Adj SS      Adj MS        F       P
Volume       1     1232.07     1217.55     1217.55   232.20   0.000
Truck        2       11.65       11.65        5.82     1.11   0.375
Error        8       41.95       41.95        5.24
Total       11     1285.67

Term           Coef    SE Coef          T        P
Constant     -4.747      2.638      -1.80    0.110
Volume      1.17326    0.07699      15.24    0.000
```

14-11 Compute the adjusted treatment means and the standard errors of the adjusted treatment means for the data in Problem 14-10.

$$\text{adj } \bar{y}_{i.} = \bar{y}_{i.} - \hat{\beta}\left(\bar{x}_{i.} - \bar{x}_{..}\right)$$

$$\text{adj } \bar{y}_{1.} = \frac{145}{4} - (1.173)\left(\frac{139}{4} - \frac{398}{12}\right) = 34.39$$

$$\text{adj } \bar{y}_{2.} = \frac{132}{4} - (1.173)\left(\frac{125}{4} - \frac{398}{12}\right) = 35.25$$

$$\text{adj } \bar{y}_{3.} = \frac{133}{4} - (1.173)\left(\frac{134}{4} - \frac{398}{12}\right) = 32.86$$

$$S_{adj.\bar{y}_{i.}} = \left[MS_E\left\{\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right\}\right]^{\frac{1}{2}}$$

$$S_{adj.\bar{y}_{1.}} = \left[5.24\left\{\frac{1}{4} + \frac{(34.75 - 33.17)^2}{884.50}\right\}\right]^{\frac{1}{2}} = 1.151$$

$$S_{adj.\bar{y}_{2.}} = \left[5.24\left\{\frac{1}{4} + \frac{(31.25 - 33.17)^2}{884.50}\right\}\right]^{\frac{1}{2}} = 1.154$$

$$S_{adj.\bar{y}_{3.}} = \left[5.24\left\{\frac{1}{4} + \frac{(33.50 - 33.17)^2}{884.50}\right\}\right]^{\frac{1}{2}} = 1.145$$

The solutions can also be obtained with Minitab as follows:

Minitab Output
```
Least Squares Means for Time

Truck      Mean   SE Mean
1         34.39     1.151
2         35.25     1.154
3         32.86     1.145
```

14-12 The sums of squares and products for a single-factor analysis of covariance follow. Complete the analysis and draw appropriate conclusions. Use $\alpha = 0.05$.

Source of Variation	Degrees of Freedom	Sums of Squares and Products x	xy	x
Treatment	3	1500	1000	650
Error	12	6000	1200	550
Total	15	7500	2200	1200

		Sums of	Squares &	Products	Adjusted			
Source	df	x	xy	y	y	df	MS	F_0
Treatment	3	1500	1000	650	-	-		
Error	12	6000	1200	550	310	11	28.18	
Total	15	7500	2200	1200	559.67	14		

Adjusted	Treat.	244.67	3	81.56	2.89

Treatments differ only at 10%.

14-13 Find the standard errors of the adjusted treatment means in Example 14-4.

From Example 14-4 $\bar{y}_{1.} = 40.38$, adj $\bar{y}_{2.} = 41.42$, adj $\bar{y}_{3.} = 37.78$

$$S_{adj.\bar{y}_{1.}} = \left[2.54\left\{\frac{1}{5} + \frac{(25.20 - 24.13)^2}{195.60}\right\}\right]^{\frac{1}{2}} = 0.7231$$

$$S_{adj.\bar{y}_{2.}} = \left[2.54\left\{\frac{1}{5} + \frac{(26.00 - 24.13)^2}{195.60}\right\}\right]^{\frac{1}{2}} = 0.7439$$

$$S_{adj.\bar{y}_{3.}} = \left[2.54\left\{\frac{1}{5} + \frac{(21.20 - 24.13)^2}{195.60}\right\}\right]^{\frac{1}{2}} = 0.7871$$

14-14 Four different formulations of an industrial glue are being tested. The tensile strength of the glue when it is applied to join parts is also related to the application thickness. Five observations on strength (y) in pounds and thickness (x) in 0.01 inches are obtained for each formulation. The data are shown in the following table. Analyze these data and draw appropriate conclusions.

			Glue	Formulation			
1	1	2	2	3	3	4	4
y	x	y	x	y	x	y	x
46.5	13	48.7	12	46.3	15	44.7	16
45.9	14	49.0	10	47.1	14	43.0	15
49.8	12	50.1	11	48.9	11	51.0	10
46.1	12	48.5	12	48.2	11	48.1	12
44.3	14	45.2	14	50.3	10	48.6	11

From the analysis performed in Minitab, glue formulation does not have a statistically significant effect on strength. As expected, glue thickness does affect strength.

Minitab Output

General Linear Model: Strength versus Glue

```
Factor     Type Levels Values
Glue      fixed      4 1 2 3 4

Analysis of Variance for Strength, using Adjusted SS for Tests

Source      DF      Seq SS     Adj SS     Adj MS        F      P
Thick        1      68.852     59.566     59.566    42.62  0.000
Glue         3       1.771      1.771      0.590     0.42  0.740
Error       15      20.962     20.962      1.397
Total       19      91.585

Term          Coef   SE Coef        T       P
Constant    60.089     1.944    30.91   0.000
Thick      -1.0099    0.1547    -6.53   0.000

Unusual Observations for Strength
```

```
Obs  Strength       Fit      SE Fit  Residual   St Resid
  3   49.8000   47.5299      0.5508    2.2701      2.17R

R denotes an observation with a large standardized residual.

Expected Mean Squares, using Adjusted SS

Source       Expected Mean Square for Each Term
 1 Thick     (3) + Q[1]
 2 Glue      (3) + Q[2]
 3 Error     (3)

Error Terms for Tests, using Adjusted SS

Source       Error DF  Error MS  Synthesis of Error MS
 1 Thick        15.00     1.397  (3)
 2 Glue         15.00     1.397  (3)

Variance Components, using Adjusted SS

Source    Estimated Value
Error               1.397
```

14-15 Compute the adjusted treatment means and their standard errors using the data in Problem 14-14.

$$\text{adj } \bar{y}_{i.} = \bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$$

$$\text{adj } \bar{y}_{1.} = 46.52 - (-1.0099)(13.00 - 12.45) = 47.08$$

$$\text{adj } \bar{y}_{2.} = 48.30 - (-1.0099)(11.80 - 12.45) = 47.64$$

$$\text{adj } \bar{y}_{3.} = 48.16 - (-1.0099)(12.20 - 12.45) = 47.91$$

$$\text{adj } \bar{y}_{4.} = 47.08 - (-1.0099)(12.80 - 12.45) = 47.43$$

$$S_{adj.\bar{y}_{i.}} = \left[MS_E \left\{ \frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}} \right\} \right]^{\frac{1}{2}}$$

$$S_{adj.\bar{y}_{1.}} = \left[1.40 \left\{ \frac{1}{5} + \frac{(13.00 - 12.45)^2}{58.40} \right\} \right]^{\frac{1}{2}} = 0.5360$$

$$S_{adj.\bar{y}_{2.}} = \left[1.40 \left\{ \frac{1}{5} + \frac{(11.80 - 12.45)^2}{58.40} \right\} \right]^{\frac{1}{2}} = 0.5386$$

$$S_{adj.\bar{y}_{3.}} = \left[1.40 \left\{ \frac{1}{5} + \frac{(12.20 - 12.45)^2}{58.40} \right\} \right]^{\frac{1}{2}} = 0.5306$$

$$S_{adj.\bar{y}_{4.}} = \left[1.40 \left\{ \frac{1}{5} + \frac{(12.80 - 12.45)^2}{58.40} \right\} \right]^{\frac{1}{2}} = 0.5319$$

The adjusted treatment means can also be generated in Minitab as follows:

Minitab Output

```
Least Squares Means for Strength

Glue       Mean   SE Mean
1         47.08    0.5355
2         47.64    0.5382
3         47.91    0.5301
```

```
4        47.43    0.5314
```

14-16 An engineer is studying the effect of cutting speed on the rate of metal removal in a machining operation. However, the rate of metal removal is also related to the hardness of the test specimen. Five observations are taken at each cutting speed. The amount of metal removed (y) and the hardness of the specimen (x) are shown in the following table. Analyze the data using and analysis of covariance. Use α=0.05.

		Cutting	Speed	(rpm)	
1000	1000	1200	1200	1400	1400
y	x	y	x	y	x
68	120	112	165	118	175
90	140	94	140	82	132
98	150	65	120	73	124
77	125	74	125	92	141
88	136	85	133	80	130

As shown in the analysis performed in Minitab, there is no difference in the rate of removal between the three cutting speeds. As expected, the hardness does have an impact on rate of removal.

Minitab Output

General Linear Model: Removal versus Speed

```
Factor     Type Levels Values
Speed     fixed      3 1000 1200 1400

Analysis of Variance for Removal, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS       F      P
Hardness     1     3075.7     3019.3     3019.3  347.96  0.000
Speed        2        2.4        2.4        1.2    0.14  0.872
Error       11       95.5       95.5        8.7
Total       14     3173.6

Term          Coef   SE Coef        T      P
Constant   -41.656     6.907    -6.03  0.000
Hardness   0.93426   0.05008    18.65  0.000
Speed
1000         0.478     1.085     0.44  0.668
1200         0.036     1.076     0.03  0.974

Unusual Observations for Removal

Obs   Removal       Fit      SE Fit  Residual   St Resid
  8    65.000    70.491       1.558    -5.491     -2.20R

R denotes an observation with a large standardized residual.

Expected Mean Squares, using Adjusted SS

Source       Expected Mean Square for Each Term
 1 Hardness  (3) + Q[1]
 2 Speed     (3) + Q[2]
 3 Error     (3)

Error Terms for Tests, using Adjusted SS

Source       Error DF  Error MS  Synthesis of Error MS
 1 Hardness     11.00       8.7  (3)
 2 Speed        11.00       8.7  (3)

Variance Components, using Adjusted SS

Source    Estimated Value
```

```
Error              8.677

Means for Covariates

Covariate       Mean      StDev
Hardness       137.1      15.94

Least Squares Means for Removal

Speed       Mean   SE Mean
1000       86.88     1.325
1200       86.44     1.318
1400       85.89     1.328
```

14-17 Show that in a single factor analysis of covariance with a single covariate a 100(1-α) percent confidence interval on the i_{th} adjusted treatment mean is

$$\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \pm t_{\alpha/2,a(n-1)-1}\left[MS_E\left(\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right)\right]^{\frac{1}{2}}$$

Using this formula, calculate a 95 percent confidence interval on the adjusted mean of machine 1 in Example 14-4.

The 100(1-α) percent interval on the i_{th} adjusted treatment mean would be

$$\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \pm t_{\alpha/2,a(n-1)-1} S_{adj\bar{y}_{i.}}$$

since $\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$ is an estimator of the i_{th} adjusted treatment mean. The standard error of the adjusted treatment mean is found as follows:

$$V(adj.\bar{y}_{i.}) = V\left[\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})\right] = V(\bar{y}_{i.}) + (\bar{x}_{i.} - \bar{x}_{..})^2 V(\hat{\beta})$$

Since the $\{\bar{y}_{i.}\}$ and $\hat{\beta}$ are independent. From regression analysis, we have $V(\hat{\beta}) = \frac{\sigma^2}{E_{xx}}$. Therefore,

$$V(adj.\bar{y}_{i.}) = \frac{\sigma^2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2\sigma^2}{E_{xx}} = \sigma^2\left[\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right]$$

Replacing σ^2 by its estimator MS_E, yields

$$\hat{V}(adj.\bar{y}_{i.}) = MS_E\left[\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right] \text{ or}$$

$$S(adj.\bar{y}_{i.}) = \left\{MS_E\left[\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right]\right\}^{\frac{1}{2}}$$

Substitution of this result into $\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \pm t_{\alpha/2,a(n-1)-1} S_{adj\bar{y}_{i.}}$ will produce the desired confidence interval. A 95% confidence interval on the mean of machine 1 would be found as follows:

$$\begin{gathered} adj.\bar{y}_{i.} = \bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) = 40.38 \\ S(adj.\bar{y}_{i.}) = 0.7231 \\ [40.38 \pm t_{0.025,11}(0.7231)] \\ [40.38 \pm (2.20)(0.7231)] \\ [40.38 \pm 1.59] \end{gathered}$$

Therefore, $38.79 \le \mu_1 \le 41.96$, where μ_1 denotes the true adjusted mean of treatment one.

14-18 Show that in a single-factor analysis of covariance with a single covariate, the standard error of the difference between any two adjusted treatment means is

$$S_{Adj\bar{y}_{i.} - Adj\bar{y}_{j.}} = \left[MS_E \left(\frac{2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}} \right) \right]^{\frac{1}{2}}$$

$$adj.\bar{y}_{i.} - adj.\bar{y}_{j.} = \bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) - [\bar{y}_{j.} - \hat{\beta}(\bar{x}_{j.} - \bar{x}_{..})]$$

$$adj.\bar{y}_{i.} - adj.\bar{y}_{j.} = \bar{y}_{i.} - \bar{y}_{j.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{j.})$$

The variance of this statistic is

$$V[\bar{y}_{i.} - \bar{y}_{j.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{j.})] = V(\bar{y}_{i.}) + V(\bar{y}_{j.}) + (\bar{x}_{i.} - \bar{x}_{j.})^2 V(\hat{\beta})$$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{j.})^2 \sigma^2}{E_{xx}} = \sigma^2 \left[\frac{2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{j.})^2}{E_{xx}} \right]$$

Replacing σ^2 by its estimator MS_E, , and taking the square root yields the standard error

$$S_{Adj\bar{y}_{i.} - Adj\bar{y}_{j.}} = \left[MS_E \left(\frac{2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}} \right) \right]^{\frac{1}{2}}$$

14-19 Discuss how the operating characteristic curves for the analysis of variance can be used in the analysis of covariance.

To use the operating characteristic curves, fixed effects case, we would use as the parameter Φ^2,

$$\Phi^2 = \frac{a \sum \tau_i^2}{n\sigma^2}$$

The test has *a*-1 degrees of freedom in the numerator and *a*(*n*-1)-1 degrees of freedom in the denominator.