Montclair State University
Department of Mathematical Sciences

Comprehensive Exam Study Guide – Analysis

Real Variable

Topics

Real Number System: Ordered sets, fields, real and complex fields, Euclidean spaces, least upper-bound property, construction of $\mathbb{R}$ from $\mathbb{Q}$.

Basic Topology: countable and uncountable sets, metric spaces, open and closed sets, compact sets, perfect sets, connected sets.

Sequences and Series: Convergent sequences, Cauchy sequences, completeness, series, convergence criteria for series, absolute convergence, addition and multiplication of series.

Continuity: Limits of functions, continuous functions, continuity and compactness, continuity and connectedness, discontinuities, monotonic functions, infinite limits and limits at infinity.

Differentiation: The derivative of a function, Mean-Value Theorem, continuity of derivatives, L'Hopital's rule, derivatives of higher order, Taylor's Theorem.

Riemann-Stieljes Integral: Definition and existence of the Riemann-Stieljes integral, properties of the Riemann-Stieljes integral, Fundamental theorem of Calculus

References

Apostol, Tom M., Mathematical Analysis, Addison-Wesley
Bartle, R. G ., The Elements of Real Analysis, Wiley
DePree and Swartz, Introduction to Real Analysis, W iley
Fitzpatrick, Advanced Calculus, Pws Pub.
Parzynski and Zipse, Introduction to Analysis
Royden, H. L., Real Analysis, Prentice Hall
Rudin, Walter, Principles of Mathematical Analysis, McGraw Hill
Shilov, George E., Elementary Real and Complex Analysis, Dover

Complex Variable

Topics

Complex Number System: algebra of complex numbers, geometry of the complex plane, Euler's formula, polar representation of complex numbers, roots of unity

Elementary Functions of a Complex Variable: complex polynomials, exponential and trigonometric functions in the complex plane, logarithm in the complex plane, branch points and branch cuts, elementary Riemann surfaces

Derivative: limits in the complex plane, definition of derivative in the complex plane, Cauchy-Riemann Equations

Integration: parameterization of curves, integration on a parameterized curve, Cauchy's Theorem, Cauchy's Integral Formula, Maximum-Modulus Theorem, Morera's Theorem, Liouville's Theorem

Series: complex power series, Taylor-Series expansion of an analytic function, Laurent series, isolated singularities of analytic functions

Calculus of residues: Residue Theorem, evaluation of integrals by Calculus of Residues, evaluation of real integrals as contour integrals in the complex plane, Argument Principle, Rouche's Theorem

References

Ablowitz, M. J. and Fokas A. S. *Complex Variables and Applications* 2nd Ed., Cabridge University Press (2003)
Silverman, R. *Complex Analysis with Applications*, Dover (2010)
Ahlfors, L. *Complex Analysis* 3d Ed., McGraw-Hill (1979)

Sample Questions

1. Prove or provide a counter-example to the following statment:

   If the functions $\{f_n\}_{n=1}^{\infty}$ are continuous on $[a,b]$ and

   $$f_n(x) \to 0 \quad \text{as} \quad n \to \infty$$

   for all $x \in [ab]$. Then,

   $$\int_a^b f_n(x)\,dx \to 0\ .$$

2. Give an example of a function that is continuous and bounded, but not *uniformly* continuous on its domain.

3. Prove or provide a counter-example to the following statment:

   If $f : X \to Y$ is continuous and $D \in Y$ is compact, then $f^{-1}(D)$ is compact.

4. Prove or provide a counter-example to the following statment:

   The set of all continuous functions on $[0,1]$ with

   $$\rho(f,g) = \int_0^1 |f-g|$$

   is a metric space.

5. Let

   $$f(z) = 2(x^3 + iy^3) + (y^2 + x^2)$$

   where $z = x + iy$.

   Determine $f'(z)$ for points z where the derivative is well-defined. Determine the set z for which $f(z)$ is analytic.

6. Compute the integral

   $$\oint_C \frac{e^{\beta z}}{1+z^2}\,dz$$

   where $\beta > 0$ and C is the circle of radius $\frac{3}{2}$ centered on $z = 0$ with a counter-clockwise orientation.

7. Evaluate

   $$\int_C |z|\,dz$$

   where C is

   (a) the straight line from $-i$ to i

   (b) semicircular arc on $|z| = 1$ from $-i$ to i with $\operatorname{Re} z \geq 0$

   and explain the significance of your results.

8. Prove or provide a counter-example to the following statment:

   If $f(z)$ is an entire function such that $\operatorname{Im} f(z) = 0$, then $f(z)$ is constant.

9. $f(z) = \frac{z^2}{z-1}$

   (a) Find a series expansion for $f(z)$ centered on $z = 0$ that is valid at $z = 2$ (e.g., converges to $f(2) = 4$).

   (b) Describe the set on the complex plane where your series converges.

10. Evaluate the integral

$$\int_0^\pi \frac{1}{\sin^2\theta + 1}\, d\theta.$$