# Using Hamiltonian Mechanics

**0**
- Determine $T(q, \dot{q})$
- Determine $V(q, \dot{q})$

$\Rightarrow L = T - V$

$q_k$ cyclic if $\frac{\partial L}{\partial q_k} = 0$

$\Rightarrow \frac{\partial L}{\partial \dot{q}_k} =$ constant of motion

$p_i$

**1** Generalized Momentae : $p_i(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}_i}$

**2** Transform for $\dot{q}_i$'s : $\dot{q}_i = \dot{q}_i(\underline{q}, \underline{p}, t)$

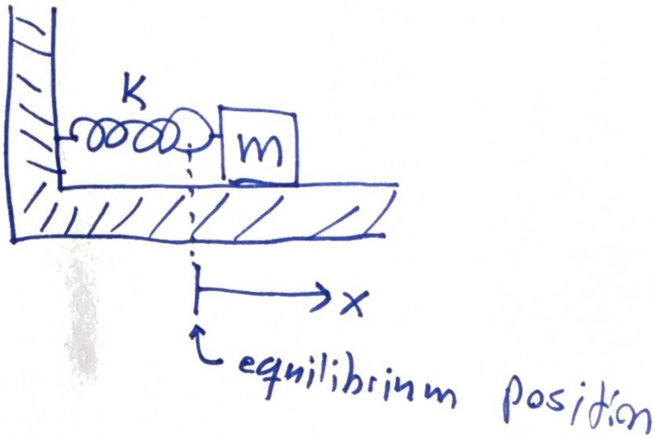
**3** The Hamiltonian : $\boxed{H(\underline{q}, \underline{p}, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L}$

**4** Hamilton's Equation of Motion :

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} , \quad i = 1, .., n \quad 2n \text{ 1st order ODE}$$

# Hamiltonian Mechanics : Harmonic Oscillator

⓪ Generalized coordinate : $x$




$T = \frac{1}{2} m \dot{x}^2$

$V = \frac{1}{2} k x^2$

$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$

① Generalized momentum : $p_x(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}}$

$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

② Transform for $\dot{x}$ :

$\dot{x}(x, p) = \frac{p_x}{m}$

③ Hamiltonian : $H(\underline{q}, \underline{p}, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L$

$\rightarrow H = \frac{p_x}{m} \cdot \cancel{m\dot{x}}\, p_x - L$

$\cancel{= p_x \dot{x} - \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2}$ $= \frac{p_x^2}{m} - \frac{1}{2} \frac{p_x^2}{m} + \frac{1}{2} k x^2$

$\cancel{= p_x \dot{x} - \frac{1}{2} p_x \dot{x}}$ $= \frac{p_x^2}{2m} + \frac{1}{2} k x^2$

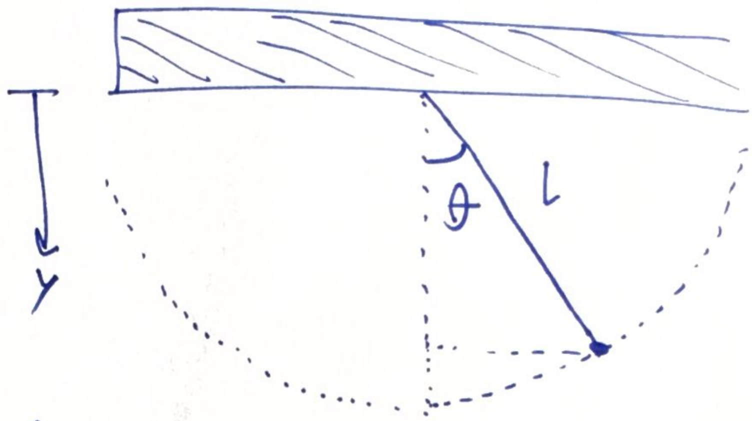
④ EOM : $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$

$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx$

# Hamiltonian Mechanics: Pendulum

## ⓪ Lagrange

Generalized Coordinate: $\theta$




$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = -mgl \cos\theta$$

$$L(\theta, \dot{\theta}) = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta$$

## ① Generalized momentum:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$|\bar{r} \times p| = |\bar{r} \times m\bar{v}|$$
$$= |\bar{r}| \cdot |m\bar{v}|$$
$$= l \cdot mv$$
$$= l\, m l \dot{\theta}$$
$$= m l^2 \dot{\theta} \rightarrow \text{impulsmoment}$$

## ② Transform for $\dot{\theta}$:

$$\dot{\theta}(\theta, p_\theta) = \frac{p_\theta}{m L^2}$$

## ③ Hamiltonian: $H = \sum_{i=1}^{n} p_i \dot{q}_i - L$

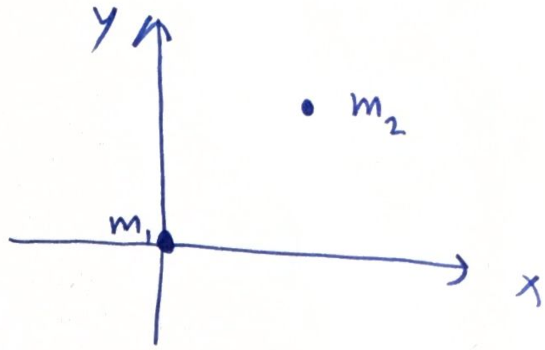
$$H = \frac{p_\theta \cdot p_\theta}{m L^2} - \frac{1}{2} m l^2 \dot{\theta}^2 - mgl \cos\theta$$

$$= \frac{p_\theta^2}{m L^2} - \frac{1}{2} m L^2 \cdot \frac{p_\theta^2}{m^2 L^4} - mgl \cos\theta$$

$$= \boxed{\frac{p_\theta^2}{2 m L^2} - mgl \cos\theta} \Rightarrow \boxed{\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m L^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mgl \sin\theta \end{aligned}}$$

# Hamiltonian Mechanics: Two-body problem, $m_1 \gg m_2$ (cartesian)

⓪ Generalized coordinates: $x, y$




$$T = \frac{1}{2} m \dot{x} + \frac{1}{2} m \dot{y}$$

$$V = -G \frac{m_1 m_2}{\sqrt{x^2+y^2}}$$

$$L = T - V = \frac{1}{2} m_2 \dot{x} + \frac{1}{2} m_2 \dot{y} + G \frac{m_1 m_2}{\sqrt{x^2+y^2}} = \frac{p_x^2 + p_y^2}{2m_2} + G \frac{m_1 m_2}{\sqrt{x^2+y^2}}$$

① Generalized momentae:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$p_x = m_2 \dot{x} \quad ; \quad p_y = m_2 \dot{y}$$

$$\boxed{\frac{\partial H}{\partial x} = G \frac{m_1 m_2 \cdot 2x}{2(x^2+y^2)^{3/2}} = G \frac{m_1 m_2}{(x^2+y^2)^{3/2}} \cdot x}$$

② Transform for $\dot{q}_i$'s:

$$\dot{x} = \frac{p_x}{m_2} \quad ; \quad \dot{y} = \frac{p_y}{m_2}$$

③ Hamiltonian:

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L$$

$$\rightarrow H = \frac{p_x^2 + p_y^2}{m_2} - \frac{p_x^2 + p_y^2}{2} - G \frac{m_1 m_2}{(x^2+y^2)^{1/2}}$$

$$\Leftrightarrow H = \frac{p_x^2 + p_y^2}{2m_2} - G \frac{m_1 m_2}{(x^2+y^2)^{1/2}}$$

④ EOM's

$\longrightarrow$

$$\boxed{\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m_2}} \quad \boxed{\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m_2}}$$

$$\boxed{\dot{p}_x = -\frac{\partial H}{\partial x} = G \frac{m_1 m_2 x}{(x^2+y^2)^{3/2}}} \quad \boxed{\dot{p}_y = -\frac{\partial H}{\partial y} = G \frac{m_1 m_2 y}{(x^2+y^2)^{3/2}}}$$